MODULAR REPRESENTATIONS AND THE HOMOTOPY OF LOW RANK
*p*-LOCAL CW-COMPLEXES

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Abstract. Fix an odd prime $p$ and let $X$ be the $p$-localization of a finite suspended CW-complex. Given certain conditions on the reduced mod-$p$ homology $\tilde{H}_*(X; \mathbb{Z}_p)$ of $X$, we use a decomposition of $\Omega \Sigma X$ due to the second author and computations in modular representation theory to show there are arbitrarily large integers $i$ such that $\Omega \Sigma^i X$ is a homotopy retract of $\Omega \Sigma X$. This implies the stable homotopy groups of $\Sigma X$ are in a certain sense retracts of the unstable homotopy groups, and by a result of Stanley, one can confirm the Moore conjecture for $\Sigma X$. Under additional assumptions on $\tilde{H}_*(X; \mathbb{Z}_p)$, we generalize a result of Cohen and Neisendorfer to produce a homotopy decomposition of $\Omega \Sigma X$ that has infinitely many finite $H$-spaces as factors.

1. Introduction

Finding homotopy decompositions of loop spaces is of sizeable and broad interest in homotopy theory. One of the most fundamental examples is Serre’s odd $p$-primary decomposition $\Omega S^{2m} \simeq S^{2m-1} \times \Omega S^{4m-1}$ [17], implying the $p$-components of homotopy groups for even dimensional spheres are determined by those of the odd dimensional spheres. Selick’s odd $p$-primary decomposition [10] of the homotopy fiber for the $p$-power map $\Omega^2 S^{2p+1} \overset{p^p}{\to} \Omega^2 S^{2p+1}$ allowed him to find the $p$-exponent of the sphere $S^3$, and in combination with an odd $p$-primary decomposition of looped Moore spaces, one is lead to the computation for $p$-exponents of higher dimensional spheres (Cohen, Moore, Neisendorfer [3, 4]).

One would like to move beyond these initial successes, and find $p$-primary homotopy decompositions of general $H$-spaces, or loop suspensions $\Omega \Sigma X$ for $X$ a CW-complex in particular. Ideally such a decomposition would be in terms of indecomposable spaces, and one says that the decomposition is the finest possible. The decomposition problem has been tackled for particular spaces using prior knowledge of self-maps (see [8] for example) or universal properties of certain constructed $H$-spaces [22], using techniques related to the Ganea fibration [12], and when localized at sufficiently large primes $p$ [7, 8, 2, 20]. Most progress to date has involved attempts at geometrically realizing some given coalgebra decomposition of the tensor algebra $H_*(\Omega \Sigma X; \mathbb{Z}_{(p)}) \cong T(H_*(X; \mathbb{Z}_{(p)}))$. In this direction, Cohen and Neisendorfer [5], as well as Cooke, Harper, and Zabrodsky [6], extended Serre’s decomposition to where $X$ is a certain low rank CW-complex, and in turn, Selick, Theriault, and Wu [14, 16, 13] further generalized this to when any simply connected co-$H$-space is taken in place

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of $\Sigma X$. However, these decompositions were the finest possible only in a functorial sense, and unlike those of Cohen and Neisendorfer, they were not explicit constructions. The computation of the mod-$p$ homology of the factors in the decomposition of Selick, Theriault, and Wu is dependent on open problems in modular representation theory, and as such these decompositions remain rather mysterious.

Fortunately, there do exist fairly general decompositions that are also explicit. In [24] the second author constructed a homotopy decomposition of $\Omega \Sigma X$ whenever $X$ is a suspension by analysing composites of James-Hopf maps and Samelson products. This turned out to be a partial geometric realization of the Poincaré-Birkhoff-Witt coalgebra decomposition

$$T(H_*(X;\mathbb{Z}(p))) \cong \bigotimes_{i=1}^{\infty} S(L_i(H_*(X;\mathbb{Z}(p)))),$$

where $L_i(V)$ is the is the $\mathbb{Z}_p$-submodule of length $i$ Lie brackets in $V$, and $S(V)$ is the free commutative algebra generated by $V$. Our main interest in this paper is to give more information concerning the factors in this decomposition, which is stated in Theorem (5.1).

Fix $p$ to be an odd prime throughout. We will be working with $CW$-complexes that have been localized at $p$. A cell structure on a $p$-local space is taken to be in the $p$-local sense. As we will mostly be using reduced mod-$p$ homology, it will be convenient to simply denote it by $\tilde{H}_*(X)$ for any space $X$, without indicating coefficients. Our main result is as follows:

**Theorem 1.1.** Let $X$ be the $p$-localization of a suspended $CW$-complex. Set $V = \tilde{H}_*(X)$, let $M$ denote the sum of the degrees of the generators of $V$, and define the sequence of integers $b_i$ recursively, with $b_0 = 0$ and

$$b_i = (1 + \dim V)b_{i-1} + M.$$

Suppose either $V_{\text{odd}} = 0$ or $V_{\text{even}} = 0$, and $1 < \dim V \leq p$.

(i) If $\dim V < p-1$, then $\Omega \Sigma^{b_i+1}X$ is a homotopy retract of $\Omega \Sigma X$ for each $i \geq 1$;

(ii) if $\dim V = p$, there exist spaces $Y_i$ such that $\Omega \Sigma Y_i$ is a homotopy retract of $\Omega \Sigma X$, and $\tilde{H}_*(Y_i) \cong \tilde{H}_*(\Sigma^{b_i}X)$ for $i \geq 1$.

**Remark 1.2.** Notice the case $\dim V = p-1$ is missing in Theorem (1.1). Proposition (5.2) in Section (5) does not hold when $\dim V = p-1$, while Theorem (5.1) would not be applicable even if it did hold.

Theorem (1.1) will depend on some computational work dealing with representations of symmetric groups in Section (2), as well as Wu's decomposition (Theorem (5.1)). A consequence is that the stable homotopy groups of the spaces defined are, in a precise sense, retracts of their regular homotopy groups. This property leads us to a small application towards the Moore conjecture at the end of this paper.
For the case \( \dim V = 2 \) we prove a stronger version of Theorem 1.1. We show the retracts in Theorem 1.1 are indeed factors in a decomposition of \( \Omega \Sigma X \), and that the connectivity of these factors grows at a slower rate.

**Theorem 1.3.** Fix \( p \geq 5 \). Let \( X \) be any any suspended \( p \)-local CW-complex with \( \dim V = 2 \), and either \( V_{\text{odd}} = 0 \) or \( V_{\text{even}} = 0 \). Let \( M \) denoting the sum of the degrees of the two generators of \( V \).

Suppose \( 0 < k_1 < k_2 < \cdots \) is any sequence satisfying the following properties:

1. \( 2k_i + 1 \) is prime to \( p \);
2. \( 2k_i + 1 \) is not a multiple of \( 2k_j + 1 \) whenever \( i > j \).

Then there exists a decomposition

\[
\Omega \Sigma X \simeq \prod_j (\Omega \Sigma^{k_i M + 1} X) \times (\text{Some other space}).
\]

Another application is to say a bit more about Cohen’s and Neisendorfer’s \([5]\) decompositions involving low rank CW-complexes. Recall this is an extension Serre’s decomposition in precisely the following sense:

**Theorem 1.4** (Cohen, Neisendorfer). Let \( X \) be the \( p \)-localization of a suspended CW-complex. Set \( V = \tilde{H}_*(X) \), and assume \( V_{\text{even}} = 0 \), and \( 1 \leq \dim V < p - 1 \). Then there exists a functorial decomposition

\[
\Omega \Sigma X \simeq A(X) \times \Omega Q(X)
\]

such that \( A(X) \) is a finite \( H \)-space with

\[
H_*(A(X)) \cong \Lambda(V)
\]

as primitively generated algebras, and

\[
H_*(\Omega Q(X)) \cong \mathbb{S}([L(V), L(V)]) = \bigotimes_{i=2}^{\infty} \mathbb{S}(L_i(V)),
\]

where \( L(V) \) is the free Lie algebra generated by \( V \), and \([L(V), L(V)]\) is the sub Lie algebra of \( L(V) \) generated by Lie brackets of length greater than one. \( \square \)

From the perspective of obtaining finite \( H \)-spaces, we strengthen the above decomposition when \( \dim V \) is even as follows:

**Theorem 1.5.** Let \( X \) be the \( p \)-localization of a suspended CW-complex. Set \( V = \tilde{H}_*(X) \), and assume \( V_{\text{even}} = 0 \), \( 1 < \dim V < p - 1 \), and \( \dim V \) is even.

Let \( M \) denote the sum of the degrees of the generators of \( V \), and define the sequence of integers \( b_i \) recursively, with \( b_0 = 0 \) and \( b_i = (1 + \dim V)b_{i-1} + M \). Then there exists a decomposition

\[
\Omega \Sigma X \simeq \prod_{i=0}^{\infty} A(\Sigma^{b_i} X) \times (\text{Some other space}),
\]
where (as in Theorem 1.4) \( \text{A}(\Sigma^n X) \) is a finite \( H \)-space that is a homotopy retract of \( \Omega \Sigma^{n+1} X \), with
\[
H_*(\text{A}(\Sigma^n X)) \cong \Lambda(\Sigma^n V).
\]

2. Preliminary

In this section, and in Section 3, \( R \) denotes either the field \( \mathbb{Z}_p \), or the ring of \( p \)-local integers \( \mathbb{Z}(p) \). The symmetric group on \( k \) letters is denoted by \( S_k \), and \( R[S_k] \) is the group ring over \( R \) generated by \( S_k \). The sequence \((a_1, a_2, \cdots, a_k)\) is used to denote the permutation \( \sigma \in S_k \) satisfying \( \sigma(i) = a_i \).

Let \( V \) be any graded \( R \)-module, and \( V^{\otimes k} \) denote the \( k \)-fold tensor product. Consider the (right) action of \( R[S_k] \) on \( V^{\otimes k} \) that is defined by permuting factors in a graded sense. In this case, the action of a single element \( \sigma \in R[S_k] \) on \( V^{\otimes k} \) induces a self-map
\[
V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k},
\]
which we also denote by \( \sigma \) for the sake of convenience. More generally, when \( j_1 + j_2 + \cdots + j_l = k \), we have the natural pairing
\[
R[S_{j_1}] \otimes R[S_{j_2}] \otimes \cdots \otimes R[S_{j_l}] \rightarrow R[S_k],
\]
which defines an action of \( R[S_{j_1}] \otimes R[S_{j_2}] \otimes \cdots \otimes R[S_{j_l}] \) on \( V^{\otimes k} \).

We will label by \( \hat{s}_k \) and \( \bar{s}_k \) the elements in \( R[S_k] \) defined as the sums
\[
\hat{s}_k = \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma,
\]
\[
\bar{s}_k = \sum_{\sigma \in S_k} \sigma.
\]
One sees that the action of \( \hat{s}_k \) and \( \bar{s}_k \) on \( V^{\otimes k} \) sends a tensor in \( V^{\otimes k} \) to a linear combination of all permutations of that tensor. In particular, for any \( x_1 \otimes \cdots \otimes x_k \in V^{\otimes k} \) and \( \sigma \in S_k \), we have
\[
\hat{s}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) = \text{sgn}(\sigma)\hat{s}_k(x_1 \otimes \cdots \otimes x_k),
\]
when each \( x_i \) has even degree, and when each \( x_i \) has odd degree,
\[
\bar{s}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) = \text{sgn}(\sigma)\bar{s}_k(x_1 \otimes \cdots \otimes x_k).
\]
An easy consequence is that \( \hat{s}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) = 0 \) if \( x_{\sigma(i)} \) has even degree and is a multiple of \( x_{\sigma(j)} \) for some \( i \neq j \). Likewise, \( \bar{s}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) = 0 \) if \( x_{\sigma(i)} \) has odd degree and is a multiple of \( x_{\sigma(j)} \) for some \( i \neq j \). We will make frequent use of these facts.

The Dynkin-Specht-Wever elements \( \beta_k \in \Sigma_k \) are defined inductively starting with \( \beta_2 = 1 - (2,1) \) and
\[
\beta_k = (1 \otimes \beta_{k-1})(1 - (2,3,\ldots,k,1)).
\]
The action of $\beta_k$ on $V^\otimes k$ is given by sending a tensor $x_1 \otimes \cdots \otimes x_k \in V^\otimes k$ to the commutator
\[ [x_1, [x_2, ..., [x_{k-1}, x_k]] \cdots ] \in V^\otimes k. \]
That is, $\beta_2(x \otimes y) = x \otimes y - (-1)^{|x||y|} y \otimes x,$ and
\[ \beta_k(x_1 \otimes \cdots \otimes x_1) = x_k \otimes \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) - (-1)^{|x_k||x_{k-1}\otimes \cdots \otimes x_1|} \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k. \]

In the following theorem we define certain integers $c_{n,\ell}$ and $d_{n,\ell}$, which will be referred to throughout this paper.

**Theorem 2.1.** There exist and integer $c_{n,\ell} \geq 0$ such that for any graded free $R$-module $V$ with $\dim V = \ell > 1$ and $V_{odd} = 0$, and each $x \in V^{\otimes(n\ell+1)}$, we have
\[ (\hat{s}_n^\otimes \otimes 1) \circ \beta_{\ell n+1} \circ (\hat{s}_n^\otimes \otimes 1)(x) = \pm c_{n,\ell} (\hat{s}_n^\otimes \otimes 1)(x). \]
On the other hand, if $V_{even} = 0$, there exists an integer $d_{n,\ell} \geq 0$ such that
\[ (\hat{s}_n^\otimes \otimes 1) \circ \beta_{\ell n+1} \circ (\hat{s}_n^\otimes \otimes 1)(x) = \pm d_{n,\ell} (\hat{s}_n^\otimes \otimes 1)(x). \]
These integers are independent of our choice of $V.$

**Proof.** Let $V$ be a free graded $R$-module such that $\ell = \dim V > 1$ and $V_{odd} = 0$, and let $g$ denote the self-map $(\hat{s}_n^\otimes \otimes 1) : V^{\otimes(n\ell+1)} \to V^{\otimes(n\ell+1)}$. Take a basis $\{v_1, ..., v_\ell\}$ of $V$, and let $y = v_1 \otimes \cdots \otimes v_\ell \in V^\otimes \ell$, and $z_i = y^\otimes n \otimes v_i \in V^{\otimes(n\ell+1)}$. Observe that $V_{odd} = 0$ and $\ell = \dim V$ implies that for every $x \in V^{\otimes(n\ell+1)}$, we can write $g(x)$ as a linear combination of the elements $g(z_i)$ for each $i$, and in turn each $g(z_i)$ is a linear combination of tensors $\sigma_1(y) \otimes \cdots \otimes \sigma_n(y) \otimes v_i$ for all choices of $\sigma_1, ..., \sigma_n \in S_n$.

Let $\gamma \in R[S_{\ell n+1}]$ be any element. Since the factor $v_1$ occurs in $y$,

\[ g \circ \gamma (y^\otimes n \otimes v_1) = \pm c_\gamma g(y^\otimes n \otimes v_1) \quad (1) \]

for some integer $c_\gamma \geq 0$. If we let $y_j \in V^\otimes \ell$ be the permutation of the tensor $y$ such that the first and $j^{th}$ factors are interchanged, it is clear that $g \circ \gamma (y_j^\otimes n \otimes v_j) = \pm c_\gamma g(y_j^\otimes n \otimes v_j)$ for each $j$, as this is the same as replacing $v_1$ with $v_j$ and $v_j$ with $v_1$ in equation (1), and both $v_1$ and $v_j$ have either even degree or odd degree. Then since $s_n(y) = \pm s_n(y_j)$ and $s_n(\sigma(y)) = sgn(\sigma) s_n(y)$ for any $\sigma \in S_n$, $g \circ \gamma (\sigma_1(y) \otimes \cdots \otimes \sigma_n(y) \otimes v_j) = \pm c_\gamma g \circ \gamma (\sigma_1(y) \otimes \cdots \otimes \sigma_n(y) \otimes v_j)$ for each $j$ and $\sigma_1, ..., \sigma_n \in S_n$.

Since $g(x)$ takes the form of a linear combination as stated above, thus

\[ g \circ \gamma \circ g(x) = \pm c_\gamma g(x) \]

for any $x \in V^{\otimes(n\ell+1)}$.

Next, let $W$ be any free grade $R$-module such that $\dim W = \dim V$ and $W_{odd} = 0$. If $\{\omega_1, ..., \omega_\ell\}$ is a basis of $W$, there is an isomorphism $V \xrightarrow{\theta} W$ of ungraded $R$-modules defined by sending $v_i$ to $\omega_i$. Since both $W_{odd} = 0$ and $V_{odd} = 0$, the isomorphism $V^{\otimes(n\ell+1)} \xrightarrow{\theta^{\otimes(n\ell+1)}} W^{\otimes(n\ell+1)}$ of ungraded

$R$-modules is equivariant with respect to the (graded) action of $R[S_{n\ell+1}]$. Thus $c_\gamma$ is independent of $V$. We finish by setting $\gamma = \beta_{n\ell+1}$, and $c_{n,\ell} = c_\gamma$. The proof for the $V_{\text{even}} = 0$ case is identical. 

The values for several instances of the integers $c_{n,\ell}$ and $d_{n,\ell}$ are given in the following theorem.

**Theorem 2.2.**

(i) $c_{1,\ell} = d_{1,\ell} = (\ell + 1)((\ell - 1)!)$ for $\ell > 1$;

(ii) $c_{n,2} = d_{n,2} = 3^n$;

**Remark 2.3.** Computer calculations suggest $c_{2,3} = 64$, $d_{2,3} = 32$; $c_{3,3} = 512$, $d_{3,3} = 64$; $c_{2,4} = 420$, $d_{2,4} = 900$. Though computational complexity requires $n$ and $\ell$ be kept small, by following the pattern one might guess that in general at least one of $c_{n,\ell}$ or $d_{n,\ell}$ is equal to $(\ell + 1)^n((\ell - 1)!)^n$.

3. Calculating $c_{1,\ell}$ and $d_{1,\ell}$ for $\ell > 1$

To show that $c_{1,\ell} = (\ell + 1)((\ell - 1)!)$, it will suffice to show

$$(\hat{s}_\ell \otimes 1) \circ \beta_{\ell+1} \circ (\hat{s}_\ell \otimes 1)(x) = \pm(\ell + 1)((\ell - 1)!)(\hat{s}_\ell \otimes 1)(x)$$

for any particular choice of graded free $R$-module $V$ satisfying $\ell = \dim V > 1$ and $V_{\text{odd}} = 0$, and any particular choice of $x \in V$ satisfying $(\hat{s}_\ell \otimes 1)(x) \neq 0$. We actually work in a slightly more general context. We will show that this equation holds for any graded $R$-module $V$ as long as $x$ is of the form $x_{k-1} \otimes \cdots \otimes x_1 \otimes x_i$ for some $1 \leq i \leq k - 1$, and each $x_i$ is a homogeneous element in $V$ of even degree. This is done in Proposition (3.5), together with an analogous calculation for $d_{1,\ell}$.

The following elements in $R[S_k]$ will be of use:

$$\hat{t}_{j,k} = \sum_{\sigma \in T_{j,k}} sgn(\sigma)\sigma,$$

$$\tilde{t}_{j,k} = \sum_{\sigma \in T_{j,k}} \sigma,$$

where $T_{j,k} \subseteq S_k$ consists of the $k$ elements switching the position of the $j^{th}$ letter:

$$T_{j,k} = \{(j,1,\cdots,j-1,j+1,k),(1,j,\cdots,j-1,j+1,k)\cdots,(1,\cdots,j-1,j+1,k,j)\}.$$ 

With this given, it is easy to see these elements satisfy the equations

$$\hat{s}_k = \hat{t}_{1,k} \circ (1 \otimes \hat{s}_{k-1}) = \hat{t}_{k,k} \circ (\hat{s}_{k-1} \otimes 1),$$

and

$$\tilde{s}_k = \tilde{t}_{1,k} \circ (1 \otimes \tilde{s}_{k-1}) = \tilde{t}_{k,k} \circ (\tilde{s}_{k-1} \otimes 1).$$

We begin with a few lemmas.
Lemma 3.1. Let $k > 2$, and $V$ be any graded $R$-module. Then for every $y \in V^\otimes k$

$$\hat{s}_k \circ \beta_k(y) = 0,$$

and

$$\bar{s}_k \circ \beta_k(y) = 0.$$

Proof. It is sufficient to show the lemma holds for homogeneous elements in $V^\otimes k$. We proceed by induction. By inspection we have $\hat{s}_3 \otimes \beta_3(x) = 0$ for any $x \in V^\otimes 3$. Let us assume $\hat{s}_{k-1} \circ \beta_{k-1}(x) = 0$ for every $x \in V^\otimes (k-1)$. Take any homogeneous element $x_k \otimes \cdots \otimes x_1 \in V^\otimes k$ and let $c = |x_{k-1} \otimes \cdots \otimes x_1|$. Then

\[
\hat{s}_k \circ \beta_k(x_k \otimes \cdots \otimes x_1) = \hat{s}_k(x_k \otimes \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1)) - (-1)^{|x_k|} \hat{s}_k(\beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k)
\]

\[
= t_{1,k}(1 \otimes \hat{s}_{k-1})(x_k \otimes \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1)) - (-1)^{|x_k|} t_{k,k}(\hat{s}_{k-1} \otimes 1)(\beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k)
\]

\[
= t_{1,k}(x_k \otimes (\hat{s}_{k-1} \circ \beta_{k-1})(x_{k-1} \otimes \cdots \otimes x_1)) - (-1)^{|x_k|} t_{k,k}(\hat{s}_{k-1} \circ \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k)
\]

\[
= t_{1,k}(0) - (-1)^{|x_k|} t_{k,k}(0)
\]

\[
= 0.
\]

The proof for $\bar{s}_k$ is identical. \qed

Lemma 3.2. Let $k > 2$, and $V$ be any graded $R$-module. Then for every $y \in V^\otimes k$ and each $3 \leq j \leq k$,

$$\hat{s}_k \circ (1^\otimes (k-j) \otimes \beta_j)(y) = 0,$$

and

$$\bar{s}_k \circ (1^\otimes (k-j) \otimes \beta_j)(y) = 0.$$

Proof. It suffices to show the statement holds for homogeneous elements in $V^\otimes k$. We proceed by induction, using two inductive assumptions at each step. For our first one, assume that for each $m, j$ such that $1 < j \leq m$ and $m < k$, and every $x \in V^\otimes m$, we have

$$\hat{s}_m \circ (1^\otimes (m-j-1) \otimes \beta_{j+1})(x) = 0.$$

We must show it also holds for $m = k$, and each $1 < j < k$. For our second inductive assumption, let us assume

$$\hat{s}_k \circ (1^\otimes (k-j-1) \otimes \beta_{j+1})(y) = 0.$$
holds for every \( y \in V^{\otimes k} \) and some \( j \leq k - 1 \). The base case \( j = k - 1 \) follows by Lemma (3.1).

Notice that by our first inductive assumption we have

\[
\hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1) \otimes x_{j+1})
= t_{k,k}(\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1)) \otimes x_{j+1})
= t_{k,k}(0x_{j+1})
= 0.
\]

Therefore

\[
0 = \hat{s}_k \circ (1^{\otimes (k-j-1)} \otimes \beta_{j+1})(x_k \otimes \cdots \otimes x_1)
= \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_{j+1}(x_{j+1} \otimes \cdots \otimes x_1))
= \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1))
\]

\[
(-1)^{c(x_j+1)}\hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1) \otimes x_{j+1})
= \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1))
= \hat{s}_k \circ (1^{\otimes (k-j)} \otimes \beta_j)(x_k \otimes \cdots \otimes x_1),
\]

where \( c = |x_{k-1} \otimes \cdots \otimes x_1| \). This completes the induction step. The proof for \( \bar{s}_k \) is identical. \( \square \)

**Lemma 3.3.** Let \( k > 2 \), and \( V \) be any graded \( R \)-module. Take any homogeneous element \( y = x_k \otimes \cdots \otimes x_1 \in V^{\otimes k} \). If \( |x_i| \) is even for each \( i \), then

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(y) = (\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes (k-3)} \otimes ((1, 2, 3) - (1, 3, 2) - 2(2, 3, 1)))(y),
\]

and if \( |x_i| \) is odd for each \( i \),

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(y) = (\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes (k-3)} \otimes ((1, 2, 3) + (1, 3, 2) - 2(2, 3, 1)))(y).
\]

**Proof.** Let \( c = |x_{k-1} \otimes \cdots \otimes x_1| \). Assume \( |x_i| \) is even for each \( i \). Using Lemma (3.2)

\[
\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1)) = 0
\]

for each \( 3 < j \leq k \), so

\[
(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1))
= (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes x_j \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1))
\]

\[
- (-1)^{c(x_j+1)}(\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1))) \otimes x_j
= (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes x_j \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1)).
\]

Then by induction

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1) = (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1))
\]
for each $3 \leq j \leq k$. In particular, when $j = 3$, the fact that
\[
\beta _3(x_3 \otimes x_2 \otimes x_1) = ((1, 2, 3) - (1, 3, 2) - (2, 3, 1) + (3, 2, 1))(x_3 \otimes x_2 \otimes x_1)
\]
implies
\[
\left( \hat{\alpha}_{k-1} \otimes 1 \right) \circ \beta _k(x_k \otimes \cdots \otimes x_1)
= (\hat{\alpha}_{k-1} \otimes 1) \circ (1 \otimes (k-3) \otimes ((1, 2, 3) - (1, 3, 2) - (2, 3, 1) + (3, 2, 1)))(x_k \otimes \cdots \otimes x_1)
\]
Since $(3, 2, 1)$ is an odd permutation of $(2, 3, 1)$ leaving the last letter fixed,
\[
(\hat{\alpha}_{k-1} \otimes 1) \circ (1 \otimes (k-3) \otimes (3, 2, 1))(x_k \otimes \cdots \otimes x_1)
= -(\hat{\alpha}_{k-1} \otimes 1) \circ (1 \otimes (k-3) \otimes (2, 3, 1))(x_k \otimes \cdots \otimes x_1).
\]
Hence
\[
(\hat{\alpha}_{k-1} \otimes 1) \circ \beta _k(x_k \otimes \cdots \otimes x_1)
= (\hat{\alpha}_{k-1} \otimes 1) \circ (1 \otimes (k-3) \otimes ((1, 2, 3) - (1, 3, 2) - 2(2, 3, 1)))(x_k \otimes \cdots \otimes x_1).
\]
The proof of the second case, when the degrees of each $x_i$ are odd, is similar.

As an immediate consequence of Lemma 3.3, we obtain the following lemma.

**Lemma 3.4.** Take any homogeneous element $x_k \otimes \cdots \otimes x_1 \in V^{\otimes k}$. If $|x_i|$ is even for each $i$,
\[
(\hat{\alpha}_{k-1} \otimes 1) \circ \beta _k(x_k \otimes \cdots \otimes x_1) = \begin{cases} 
(\hat{\alpha}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_i \text{ for some } i > 3 \\
0, & \text{if } x_1 = x_2 \\
3(\hat{\alpha}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_3
\end{cases}
\]
Similarly, if $|x_i|$ is odd for each $i$,
\[
(\hat{\alpha}_{k-1} \otimes 1) \circ \beta _k(x_k \otimes \cdots \otimes x_1) = \begin{cases} 
(\hat{\alpha}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_i \text{ for some } i > 3 \\
0, & \text{if } x_1 = x_2 \\
3(\hat{\alpha}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_3
\end{cases}
\]

**Proof.** Suppose each $x_i$ is of even degree, and $x_1 = x_3$. By Lemma 3.3
\[
(\hat{\alpha}_{k-1} \otimes 1) \circ \beta _k(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1)
= (\hat{\alpha}_{k-1} \otimes 1)(1 \otimes (k-3) \otimes ((1, 2, 3) - (1, 3, 2) - 2(2, 3, 1))(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1)
\]
which in turn is equal to
\[
(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1) - \\
(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2) - \\
2(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_2 \otimes x_1).
\]

Notice that
\[
(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2) = 0,
\]

and since \(\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_1 \otimes x_2) = -\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_1 \otimes x_2)\), we have
\[
(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_2 \otimes x_1 \otimes x_1) = - (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1).
\]

Therefore we have
\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1) = 3(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1).
\]

The proof for the rest of the cases is similar.

\[\square\]

Part (i) of Theorem 2.2 follows from the following proposition.

**Proposition 3.5.** Take any homogeneous element \(y = x_{k-1} \otimes \cdots \otimes x_1 \in V^\otimes k-1\), for \(V\) a graded \(R\)-module. If \(|x_i|\) is even for each \(i\), then for each \(1 \leq j \leq k-1\)
\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\hat{s}_{k-1} \otimes 1)(y \otimes x_j) = k((k-2)!)(\hat{s}_{k-1} \otimes 1)(y \otimes x_j),
\]

and if \(|x_i|\) is odd for each \(i\),
\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\hat{s}_{k-1} \otimes 1)(y \otimes x_j) = k((k-2)!)(\hat{s}_{k-1} \otimes 1)(y \otimes x_j).
\]

**Proof.** For each \(1 \leq j, n \leq k-1\), let \(S_{k-1}^{j,n} \subseteq S_{k-1}\) be the subset of all permutations \(\sigma \in S_{k-1}\) satisfying \(\sigma(j) = n\). Notice that
\[
|S_{k-1}^{j,n}| = (k-2)!,
\]

and for \(m \neq n\)
\[
|S_{k-1}^{j,n} \cap S_{k-1}^{j,m}| = 0,
\]

and
\[
|S_{k-1} \setminus (S_{k-1}^{j,n} \cup S_{k-1}^{j,m})| = (k-1)! - 2(k-2)!.
\]

Let \(A_j\) denote the set \(S_{k-1} \setminus (S_{k-1}^{j,1} \cup S_{k-1}^{j,2})\). For each \(1 \leq j \leq k-1\) we can write \(\hat{s}_k \in S_k\) as
\[
\hat{s}_k = \sum_{\sigma \in S_k} sgn(\sigma)\sigma \\
= \sum_{\sigma \in A_j} sgn(\sigma)\sigma + \sum_{\sigma \in S_{k-1}^{j,1}} sgn(\sigma)\sigma + \sum_{\sigma \in S_{k-1}^{j,2}} sgn(\sigma)\sigma.
\]
Notice that for every \( \sigma \in S_{k-1} \) we have

\[
\hat{s}_{k-1}\sigma = \text{sgn}(\sigma)\hat{s}_{k-1}.
\]

So for each \( \sigma \in S_{j,2}^{k-1} \), Lemma \((3.2)\) implies

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\sigma(y) \otimes x_j) = 3(\hat{s}_{k-1} \otimes 1)(\sigma(y) \otimes x_j) = 3\hat{s}_{k-1}(\sigma(y)) \otimes x_j = 3\text{sgn}(\sigma)(\hat{s}_{k-1}(y) \otimes x_j).
\]

Likewise Lemma \((3.2)\) implies that for every \( \sigma \in A_j \),

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\sigma(y) \otimes x_j) = sgn(\sigma)(\hat{s}_{k-1}(y) \otimes x_j)
\]

and for every \( \sigma \in S_{j,1}^{k-1} \) we have

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\sigma(y) \otimes x_j) = 0.
\]

Hence

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_{j,2}^{k-1}} sgn(\sigma)\sigma(y) \right) \otimes x_j = \sum_{\sigma \in S_{j,2}^{k-1}} (\hat{s}_{k-1} \otimes 1) \circ \beta_k(sgn(\sigma)\sigma(y) \otimes x_j) = \sum_{\sigma \in S_{j,2}^{k-1}} (3s\text{gn}(\sigma)^2)\hat{s}_{k-1}(y) \otimes x_j = 3(k-2)!\hat{s}_{k-1}(y) \otimes x_j).
\]

Similarly

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in A_j} sgn(\sigma)\sigma(y) \right) \otimes x_j = ((k-1)! - 2(k-2)!)(\hat{s}_{k-1}(y) \otimes x_j),
\]

and

\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_{j,1}^{k-1}} sgn(\sigma)\sigma(y) \right) \otimes x_j = 0.
\]
Putting these facts together,
\[
(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\hat{s}_{k-1} \otimes 1)(y \otimes x_j)
= (\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma(y) \right) \otimes x_j
= (\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma(y) + \sum_{\sigma \in S_{k-1}^1} \text{sgn}(\sigma)\sigma(y) + \sum_{\sigma \in S_{k-1}^2} \text{sgn}(\sigma)\sigma(y) \right) \otimes x_j
= ((k-1)! - 2(k-2)!)(\hat{s}_{k-1}(y) \otimes x_j) + 0 + 3(k-2)!((\hat{s}_{k-1}(y) \otimes x_j)
= k(k-2)!((\hat{s}_{k-1} \otimes 1)(y \otimes x_j).
\]

The proof for second part of the proposition is similar. □

4. Calculating \(c_{n,2}\) and \(d_{n,2}\) for \(n \geq 1\)

Like in the previous section we will work in the more general context of graded \(R\)-modules, and calculate \(c_{n,2}\) (and \(d_{n,2}\)) by proving the equality \((\hat{s}_2 \otimes 1) \circ \beta_{2n+1} \circ (\hat{s}_2 \otimes 1)(x) = (3^n)(\hat{s}_2 \otimes 1)(x)\) holds for certain homogeneous tensors \(x\) whose factors are of even degree (or odd degree for the calculation of \(d_{n,2}\)). We begin working our way towards a proof of this starting with a few technical lemmas.

**Lemma 4.1.** Let \(V\) be any graded \(R\)-module, and \(x_1, x_2 \in V\) any homogeneous elements. Let \(\sigma_1, ..., \sigma_k \in S_2\) be any \(k \geq 1\) choices of the two elements in \(S_2 = \{(12), (21)\}\), and take
\[
y = x_{\sigma_1(1)} \otimes (x_{\sigma_1(1)} \otimes x_{\sigma_1(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \in V^\otimes 2k,
\]
and
\[
z = (x_{\sigma_1(1)} \otimes x_{\sigma_1(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(1)} \otimes x_{\sigma_1(1)} \in V^\otimes 2k.
\]

(i) Suppose \(|x_1|\) and \(|x_2|\) are both odd, and let \(n \geq 0\) be the number of times \(\sigma_i = \sigma_1\) for \(i > 1\). Then
\[
\hat{s}_2^\otimes k \circ (1 \otimes \beta_{2k-1})(y) = (-1)^{k-1}(-2)^n(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes 2^k.
\]
Furthermore,
\[
\hat{s}_2^\otimes k \circ (\beta_{2k-1} \otimes 1)(z) = -(\hat{s}_2^\otimes k \circ (1 \otimes \beta_{2k-1})(y)).
\]

(ii) Suppose \(|x_1|\) and \(|x_2|\) are both even. If \(\sigma_{2i} = \sigma_1\) for some \(i\), then
\[
\hat{s}_2^\otimes k \circ (1 \otimes \beta_{2k-1})(y) = 0.
\]
Otherwise, if \(m \geq 0\) is the number of times \(\sigma_{2i+1} = \sigma_1\) for \(i > 0\),
\[
\hat{s}_2^\otimes k \circ (1 \otimes \beta_{2k-1})(y) = (-2)^m(3^14^2^36^4^5\cdots)(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes 2^k.
\]
Furthermore,
\[ \tilde{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k(\tilde{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1}))(y). \]

Proof of part (i). With our choice of \( y \in V^{\otimes 2k} \) defined in the statement of the lemma, it will be convenient to let \( y' \in V^{\otimes 2k-3} \) denote \( (x_{\sigma_{k-1}(1)} \otimes \cdots \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \) - that is, the tensor of the last \( 2k-3 \) factors of \( y \). Also, let \( n' \) be the number of choices of \( i \) such that \( 1 < i < k \) and \( \sigma_i = \sigma_1 \), and \( n \) be the number of choices of \( i \) such that \( 1 < i \leq k \) and \( \sigma_i = \sigma_1 \).

Part (i) holds for \( k = 2 \) by inspection. Assume it holds for some \( k - 1 \geq 2 \). In particular, our inductive assumptions are
\[ (s_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = -(s_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')), \]
and
\[ (s_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = (-1)^{k-2}(-2)^{n'}(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_2(2)}))^{\otimes k-1}. \]

Noting \( x_1 \) and \( x_2 \) are of odd degree, one has the following equality:
\[ (1 \otimes \beta_{2k-1})(y) = x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} - x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} \]
\[ + x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} - x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} \]
\[ - x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} - x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}, \]

Let us assume \( \sigma_1 \neq \sigma_k \). Then \( x_{\sigma_1(2)} = x_{\sigma_1(2)} \), \( x_{\sigma_k(1)} = x_{\sigma_2(2)} \), and \( x_{\sigma_2(2)} = x_{\sigma_1(2)} \) in equation (4). Since \( \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(1)}) = 0 \), and using equations (4) and (2),
\[ (s_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) \]
\[ = \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_2(2)}) \circ (s_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \]
\[ - \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_2(2)}) \circ (s_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \]
\[ - (s_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \circ \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_2(2)}) \]
\[ = -(s_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \circ \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_2(2)}). \]

Then by equation (3),
\[ (s_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = -(-1)^{k-2}(-2)^{n'}(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_2(2)}))^{\otimes k}, \]
and since our assumption is that \( \sigma_1 \neq \sigma_k \), then \( n' = n \), and so we are done.
Next, let us assume $\sigma_1 = \sigma_k$. Then $x_{\sigma_1(1)} = x_{\sigma_1(1)}$, and $x_{\sigma_k(2)} = x_{\sigma(2)}$. Thus we replace $x_{\sigma_k(1)}$ with $x_{\sigma_1(1)}$ and $x_{\sigma_k(2)}$ with $x_{\sigma(2)}$ in equation (4), and using the fact that $\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(1)}) = 0$, we obtain

\[
(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = -\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(2)}) \otimes \bar{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)})
\]

\[
= -\bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes \bar{s}_2(x_{\sigma(2)} \otimes x_{\sigma_1(1)})
\]

\[
= \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(2)}) \otimes \bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y'))
\]

\[
= -\bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes (-\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(2)}))
\]

So by equation (4),

\[
(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 2(-1)^{k-2}(-2)^{n'}(\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(2)}))^{\otimes k}
\]

\[
= (-1)^{k-1}(-2)^{n'+1}(\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(2)}))^{\otimes k}.
\]

Since $\sigma_1 = \sigma_k$, then $n' + 1 = n$, and we are done.

To complete the induction we have to show

\[
\bar{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = -\bar{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y),
\]

where $z$ is as defined in the statement of the lemma. Having found $(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y)$ above, this is the same as proving the equality

\[
\bar{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^{k}(-2)^{(n'}(\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma(2)}))^{\otimes k}.
\]

Thus we follow a similar argument for $z$ as we did for $y$ above. First, since $z = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes y' \otimes x_{\sigma_1(1)}$,

\[
(\beta_{2k-1} \otimes 1)(z) = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_1(1)}
\]

\[
+ x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_1(1)}
\]

\[
- x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \otimes x_{\sigma_1(1)}
\]

\[
- \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_1(1)}.
\]

As before, let us first assume $\sigma_1 \neq \sigma_k$. Then

\[
(\bar{s}_2^{\otimes k}) \circ (\beta_{2k-1} \otimes 1)(y) = (-\bar{s}_2(x_{\sigma_k(1)} \otimes x_{\sigma_k(2)})) \otimes \bar{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)})
\]

\[
- (\bar{s}_2^{\otimes k-1})(x_{\sigma_k(1)} \otimes \beta_{2k-3}(y')) \otimes (-\bar{s}_2(x_{\sigma_k(1)} \otimes x_{\sigma_k(2)}))
\]

\[
- (\bar{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_k(1)}) \otimes (-\bar{s}_2(x_{\sigma_k(1)} \otimes x_{\sigma_k(2)})).
\]

So by equations (2) and (3),

\[
(\bar{s}_2^{\otimes k}) \circ (\beta_{2k-1} \otimes 1)(y) = (-1)^{k}(-2)^{n'}(\bar{s}_2(x_{\sigma_k(1)} \otimes x_{\sigma_k(2)}))^{\otimes k},
\]

and since we are assuming $\sigma_1 \neq \sigma_k$, then $n' = n$, so we are done.
Next, assume \( \sigma_1 = \sigma_k \). By equation (6)

\[
(\hat{s}_2^\otimes k) \circ (\beta_{2k-1} \otimes 1)(y) = \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes \hat{s}_2^\otimes k-1(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) \\
+ \hat{s}_2^\otimes k-1(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes (-\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})�
\]

and so using equation (3) we obtain

\[
(\hat{s}_2^\otimes k) \circ (\beta_{2k-1} \otimes 1)(y) = (-1)^{k-1}(2)^{n'}+1(\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})^\otimes k).
\]

Since \( \sigma_1 = \sigma_k \), then \( n' + 1 = n \). This completes the induction for part (i). \( \square \)

Proof of part (ii). We follow along a similar line as the proof of part (i), with \( y \in V^{\otimes 2k} \) and \( y' \in V^{\otimes 2k-3} \) defined as before (but this time with \( x_1 \) and \( x_2 \) having both even degree). Let \( m' \) be the number of choices of \( i \) such that \( 1 < 2i + 1 < k \) and \( \sigma_{2i+1} = \sigma_1 \), and \( m \) be the number of choices of \( i \) such that \( 1 < 2i + 1 \leq k \) and \( \sigma_{2i+1} = \sigma_1 \).

Part (ii) holds for \( k = 2 \) by inspection. Let us assume part (ii) holds for some \( k - 1 \geq 2 \). In particular, our inductive assumptions are as follows: first,

\[
(\hat{s}_2^\otimes k-1)(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = (-1)^{k-1}(\tilde{s}_2^\otimes k-1)(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')),
\]

and whenever \( \sigma_1 = \sigma_{2i} \) for some \( i \) such that \( 2i < k \), we have

\[
(\hat{s}_2^\otimes k-1)(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = 0.
\]

Otherwise,

\[
(\hat{s}_2^\otimes k-1)(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = (-2)^{m'}(3^{3^{m'}}-1)(\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})^\otimes k-1).
\]

Next, observe the following equality:

\[
(1 \otimes \beta_{2k-1})(y) = x_{\sigma_1(1)} \otimes x_{\sigma_1(2)} \otimes \beta_{2k-2}(x_{\sigma_2(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \\
- x_{\sigma_1(1)} \otimes \beta_{2k-2}(x_{\sigma_2(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \otimes x_{\sigma_k(1)} \\
= x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \\
- x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \\
- x_{\sigma_1(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \\
+ x_{\sigma_1(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}.
\]

Let us assume \( \sigma_1 \neq \sigma_k \). Then \( x_{\sigma_k(2)} = x_{\sigma_1(1)} \), \( x_{\sigma_k(1)} = x_{\sigma_1(2)} \), and so we replace \( x_{\sigma_k(2)} \) with \( x_{\sigma_1(1)} \), and \( x_{\sigma_k(1)} \) with \( x_{\sigma_1(2)} \) in equation (10). Since \( \tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) = 0 \),

\[
\hat{s}_2^\otimes k(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)}) = 0.
\]
Therefore by equations (10) and (7) we have

\[(s_{2}^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y)\]
\[= \hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes (\hat{s}_{2}^{\otimes k-1})(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) + \hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes (\hat{s}_{2}^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma(1)})\]
\[= (1 - (-1)^{k-1})\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes (\hat{s}_{2}^{\otimes k-1})(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) + (\hat{s}_{2}^{\otimes k-1})(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) \otimes \hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}).\]

Now suppose \(\sigma_{2i} = \sigma_{1}\) for some \(2i < k\). Then \((\hat{s}_{2}^{\otimes k-1})(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) = 0\) by our inductive assumption, and so

\[(\hat{s}_{2}^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 0\]

by equation (11). On the other hand, suppose \(\sigma_{2i} \neq \sigma_{1}\) for all \(i\). Recall \(m'\) is the number of choices of \(i\) such that \(1 < 2i + 1 < k\) and \(\sigma_{2i+1} = \sigma_{1}\), and \(m\) is the number of choices of \(i\) such that \(1 < 2i + 1 \leq k\) and \(\sigma_{2i+1} = \sigma_{1}\). By equation (9)

\[(s_{2}^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y)\]
\[= (1 - (-1)^{k-1})(-2)^{m'}(3\lceil \frac{k-1}{2} \rceil)(\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}))^{\otimes k}\]
\[+ (-2)^{m'}(3\lceil \frac{k-1}{2} \rceil)(\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}))^{\otimes k}\]
\[= (2 + (-1)^{k})(-2)^{m'}(3\lceil \frac{k-1}{2} \rceil)(\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}))^{\otimes k}.\]

Since \(\sigma_{1} \neq \sigma_{k}\), \(m' = m\). So when \(k\) is odd, \(\lceil \frac{k-1}{2} \rceil = \lfloor \frac{k}{2} \rfloor\), and \((2 + (-1)^{k})(-2)^{m'}(3\lfloor \frac{k-1}{2} \rfloor) = (-2)^{m}(3\lfloor \frac{k}{2} \rfloor)\). Likewise, when \(k\) is even, \((2+(-1)^{k})(-2)^{m'}(3\lceil \frac{k-1}{2} \rceil) = 3(-2)^{m}(3\lceil \frac{k-1}{2} \rceil) = (-2)^{m}(3\lfloor \frac{k}{2} \rfloor)\).

So in any case

\[(\hat{s}_{2}^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = (-2)^{m}(3\lfloor \frac{k}{2} \rfloor)(\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}))^{\otimes k}.\]

This finishes the case \(\sigma_{1} \neq \sigma_{k}\).

Let us assume \(\sigma_{1} = \sigma_{k}\). Then we have \(x_{\sigma(1)} = x_{\sigma(1)}\), and \(x_{\sigma(2)} = x_{\sigma(2)}\). So replacing \(x_{\sigma(1)}\) with \(x_{\sigma(1)}\) and \(x_{\sigma(2)}\) with \(x_{\sigma(2)}\) in equation (10), using equation (7), and the fact that \(\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(1)}) = 0\), one obtains

\[(\hat{s}_{2}^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = -(-1)^{k-1}\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes \hat{s}_{2}^{\otimes k-1}(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) + \hat{s}_{2}^{\otimes k-1}(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) \otimes (-\hat{s}_{2}(x_{\sigma(1)} \otimes x_{\sigma(2)})).\]

Suppose \(\sigma_{1} = \sigma_{2i}\) for some choice of \(i\) such that \(2i < k\). Then \(\hat{s}_{2}^{\otimes k-1}(x_{\sigma(1)} \otimes \beta_{2k-3}(y')) = 0\) by equation (8), and so

\[(\hat{s}_{2}^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 0\]

using equation (11).
Suppose $\sigma_1 \neq \sigma_2$, for all choices of $i$ such that $2i < k$. Then by equations (9) and (15),

$$(s_2^\otimes k) \circ (1 \otimes \beta_{2k-1})(y) = ((-1)^k - 1)(-2)^m(3^{1 \frac{1}{k-1}})(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^\otimes k.$$  

Since we are assuming $\sigma_1 = \sigma_k$, by the statement of our lemma one would expect that $(s_2^\otimes k) \circ (1 \otimes \beta_{2k-1})(y) = 0$ whenever $k$ is even. By equation (12) this is indeed the case. On the other hand when $k$ is odd, since $\sigma_1 = \sigma_k$ we have $m = m' + 1$, and therefore by equation (12)

$$(s_2^\otimes k) \circ (1 \otimes \beta_{2k-1})(y) = (-2)^m(3^{1 \frac{1}{k}})(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^\otimes k.$$  

This finishes the $\sigma_1 = \sigma_k$ case.

To complete the induction we have to show

$$s_2^\otimes k \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k(s_2^\otimes k \circ (1 \otimes \beta_{2k-1})(y))$$

(where $z$ is defined in the statement of the lemma), as an equality of this form in equation (7) is assumed in our induction step. Having found $(s_2^\otimes k) \circ (1 \otimes \beta_{2k-1})(y)$ above, this is the same as proving the equality

$$s_2^\otimes k \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k(-2)^m(3^{1 \frac{1}{k}})(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^\otimes k.$$  

Here we have $z = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes y' \otimes x_{\sigma_1(1)}$, whereas before $y = x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes y'$. None-the-less, an argument similar to the one above shows that this equality is correct. \hfill $\Box$

The following is a consequence of Lemma (4.1).

**Lemma 4.2.** Let $V$ be any graded $R$-module, and $x_1, x_2 \in V$ any homogeneous elements. Let $\sigma_1, \ldots, \sigma_k \in S_2$ be any $k > 1$ choices of the two elements in $S_2 = \{(12), (21)\}$, and let

$$y = (x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \in V^\otimes 2k-1.$$  

(i) Suppose $|x_1|$ and $|x_2|$ are both odd, and let $n \geq 0$ be the number of times $\sigma_i = \sigma_1$ for $i > 1$. Then

$$(s_2^\otimes k-1 \otimes 1) \circ \beta_{2k-1}(y) = (-1)^{k-1}(-2)^n(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^\otimes k-1 \otimes x_{\sigma_1(2)}.$$  

(ii) Suppose $|x_1|$ and $|x_2|$ are both even. If $\sigma_{2i} = \sigma_1$ for some $i$, then

$$(s_2^\otimes k-1 \otimes 1) \circ \beta_{2k-1}(y) = 0.$$  

Otherwise, if $m \geq 0$ is the number of times $\sigma_{2i+1} = \sigma_1$ for $i > 0$, then

$$(s_2^\otimes k-1 \otimes 1) \circ \beta_{2k-1}(y) = (-1)^{k-1}(-2)^m(3^{1 \frac{1}{k-1}})(s_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^\otimes k-1 \otimes x_{\sigma_1(2)}.$$  

**Proof of part (i).** Part (i) holds for $k = 2$ by inspection. For our inductive assumption, assume part (i) holds for some $k - 1 > 2$. Let $y$ be a choice of tensor as defined in the statement of the lemma, and for convenience let $y' = x_{\sigma_{k-1}(1)} \otimes x_{\sigma_{k-1}(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} \in V^\otimes 2k-3$. Let $n$ be
the number of times \( \sigma_i = \sigma_1 \) for \( 1 < i \leq k \), and \( n' \) be the number of times \( \sigma_i = \sigma_1 \) for \( 1 < i < k \). Note the following equality:

\[
(\beta_{2k-1})(y) = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y')
\]

\[
+ x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)}
\]

\[
- x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)}
\]

\[
- \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}.
\]

Let us assume \( \sigma_k \neq \sigma_1 \), so we can replace \( x_{\sigma_k(2)} \) with \( x_{\sigma_1(1)} \) and \( x_{\sigma_k(1)} \) with \( x_{\sigma_1(2)} \) in equation (13). In this case, observe that the factor \( x_{\sigma_1(2)} \) occurs more often than the factor \( x_{\sigma_1(1)} \) in the tensor \( x_{\sigma_k(1)} \otimes y' \), thus

\[
\tilde{s}^{\otimes k-1}(x_{\sigma_k(1)} \otimes \beta_{2k-3}(y')) = 0.
\]

Also, if \( \sigma_1 \neq \sigma_{2i} \) for all \( 1 < 2i < k \), then by our inductive assumption

\[
(\tilde{s}^{\otimes k-2} \otimes 1) \circ \beta_{2k-3}(y') = (-1)^{k-2}(-2)^{n'}(\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-2} \otimes x_{\sigma_1(2)},
\]

and by Lemma (4.1),

\[
(\tilde{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = -(\tilde{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')).
\]

Therefore, by equation (13)

\[
(\tilde{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y)
\]

\[
= (-1)^{k-2}(-2)^{n'} \tilde{s}_2(x_{\sigma_1(2)} \otimes x_{\sigma_1(1)}) \otimes (\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-2} \otimes x_{\sigma_1(2)}
\]

\[
= -(-1)^{k-2}(-2)^{n'}(\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.
\]

Since \( \sigma_k \neq \sigma_1 \), \( n = n' \), and we are done.

Next we assume \( \sigma_k = \sigma_1 \). Replacing \( x_{\sigma_k(1)} \) with \( x_{\sigma_1(1)} \) and \( x_{\sigma_k(2)} \) with \( x_{\sigma_1(2)} \) in equation (13), the factor \( x_{\sigma_1(2)} \) occurs more often than the factor \( x_{\sigma_1(1)} \) in the tensors \( x_{\sigma_k(2)} \otimes y' \) and \( y' \otimes x_{\sigma_k(1)} \), implying \( \tilde{s}_2^{\otimes k-1}(x_{\sigma_k(2)} \otimes \beta_{2k-3}(y')) = 0 \), and \( \tilde{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_k(1)}) = 0 \). Also, by Lemma (4.1)

\[
(\tilde{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = (-1)^{k-2}(-2)^{n'}(\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1}.
\]

Following along a similar line as the previous case,

\[
(\tilde{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) = 2(-1)^{k-2}(-2)^{n'}(\tilde{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.
\]

Because we are assuming \( \sigma_k \neq \sigma_1 \), then \( n = n' + 1 \), and so \( 2(-1)^{k-2}(-2)^{n'} = (-1)^{k-1}(-2)^{n} \). This completes the induction. \( \square \)
Proof of part (ii). The structure of the proof is similar to that of part (i). Here, part (ii) holds for $k = 2$ by inspection, and our inductive assumption is that it holds for some $k - 1 > 2$. Let $y$ and $y'$ be as in the proof of part (i), and let $m$ be the number of times $\sigma_{2i+1} = \sigma_1$ for $0 < 2i + 1 \leq k$, and $m'$ be the number of times $\sigma_{2i+1} = \sigma_1$ for $0 < 2i + 1 < k$. Note the following equality:

\begin{equation}
(\hat{\beta}_{2k-1})(y) = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \\
- x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \\
- x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \\
+ \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}.
\end{equation}

Let us assume $\sigma_k \neq \sigma_1$, so we replace $x_{\sigma_k(2)}$ with $x_{\sigma_1(1)}$ and $x_{\sigma_k(1)}$ with $x_{\sigma_1(2)}$ in equation (14). In this case, observe that the factor $x_{\sigma_1(2)}$ occurs more often than the factor $x_{\sigma_1(1)}$ in the tensor $x_{\sigma_k(1)} \otimes y'$, thus

\[ \hat{s}_2 \otimes k^{-1}(x_{\sigma_k(1)} \otimes \beta_{2k-3}(y')) = 0. \]

Also, if $\sigma_1 \neq \sigma_{2i}$ for all $1 < 2i < k$, then by our inductive assumption

\begin{equation}
(\hat{s}_2 \otimes k^{-2} \otimes 1) \circ \beta_{2k-3}(y') = (-1)^{k-2}(-2)^m'(3\lfloor \frac{k-1}{2} \rfloor)(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes k^{-2} \otimes x_{\sigma_1(2)},
\end{equation}

and by Lemma (4.1),

\begin{equation}
(\hat{s}_2 \otimes k^{-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = (-1)^{k-1}(-2)^m'(3\lfloor \frac{k-1}{2} \rfloor)(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes k^{-1}.
\end{equation}

Therefore

\[ (\hat{s}_2 \otimes k^{-1} \otimes 1) \circ (\beta_{2k-1})(y) = (2(-1)^{k-1} - 1)(-2)^m'(3\lfloor \frac{k-1}{2} \rfloor)(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes k^{-1} \otimes x_{\sigma_1(2)}. \]

When $k$ is even, $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor + 1$ and $(2(-1)^{k-1} - 1) = 3 = (-1)^{k-1}$; when $k$ is odd, $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor$ and $(2(-1)^{k-1} - 1) = 1 = (-1)^{k-1}$. Since $\sigma_k \neq \sigma_1$, $m = m'$, and we are done. If $\sigma_1 = \sigma_{2i}$ for some $1 < 2i < k$, both equations (15) and (16) are zero by our inductive assumption and Lemma (4.1), and so $\hat{\beta}_{2k-1}(y) = 0$.

Finally, let us assume $\sigma_k = \sigma_1$. Replacing $x_{\sigma_k(1)}$ with $x_{\sigma_1(1)}$ and $x_{\sigma_2(1)}$ with $x_{\sigma_1(2)}$ in equation (14), notice that the factor $x_{\sigma_1(2)}$ occurs more often than the factor $x_{\sigma_1(1)}$ in the tensors $x_{\sigma_k(2)} \otimes y'$ and $y' \otimes x_{\sigma_k(2)}$, so $\hat{s}_2 \otimes k^{-1}(x_{\sigma_k(2)} \otimes \beta_{2k-3}(y')) = 0$, and $\hat{s}_2 \otimes k^{-1}(\beta_{2k-3}(y') \otimes x_{\sigma_k(2)}) = 0$. The rest being similar as before, we have $\hat{s}_2 \otimes k^{-1}(y) = 0$ when $\sigma_1 = \sigma_{2i}$ for some $1 < 2i < k$, and

\[ (\hat{s}_2 \otimes k^{-1} \otimes 1) \circ (\beta_{2k-1})(y) = ((-1)^{k-2} - 1)(-2)^m'(3\lfloor \frac{k-1}{2} \rfloor)(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes k^{-1} \otimes x_{\sigma_1(2)}. \]

whenever $\sigma_1 \neq \sigma_{2i}$ for all $1 < 2i < k$. When $k$ is even, by the statement of the lemma one would expect this equation to be zero, as we are assuming $\sigma_k \neq \sigma_1$. This is in fact the case since $((-1)^{k-2} - 1) = 0$ whenever $k$ is even. On the other hand, when $k$ is odd we have $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor$ and
$((-1)^{k-2} - 1) = -2 = (-1)^{k-1}(-2)$. Also, because $\sigma_k = \sigma_1$, then $m = m' + 1$. This completes the induction. 

Part (ii) of Theorem 4.2 follows from the following proposition.

**Proposition 4.3.** Let $V$ be any graded $R$-module, and $x_1, x_2 \in V$ any homogeneous elements. Let $\sigma_1, \ldots, \sigma_k \in S_2$ be any $k > 1$ choices of the two elements in $S_2 = \{(12), (21)\}$, and let

$$y = (x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \in V^{\otimes 2k-1}.$$ 

(i) If $|x_1|$ and $|x_2|$ are both odd, then

$$(s_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} \circ (s_2^{\otimes k-1} \otimes 1)(y) = \pm 3^{k-1}(s_2^{\otimes k-1} \otimes 1)(y).$$

(ii) If $|x_1|$ and $|x_2|$ are both even, then

$$(s_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} \circ (s_2^{\otimes k-1} \otimes 1)(y) = \pm 3^{k-1}(s_2^{\otimes k-1} \otimes 1)(y).$$

**Proof.** Without loss of generality assume $\sigma_1 = (21)$ (so $x_{\sigma_1(1)} = x_2$ and $x_{\sigma_1(2)} = x_1$), and take the tensor

$$x = (x_1 \otimes x_2)^{\otimes k-1} \otimes x_1 \in V^{\otimes 2k-1}.$$ 

Notice that $(s_2^{\otimes k-1} \otimes 1)(y) = (\text{sgn}(\sigma_2) \cdots \text{sgn}(\sigma_k))(s_2^{\otimes k-1} \otimes 1)(x)$ when $x_1$ and $x_2$ have odd degree, and $(s_2^{\otimes k-1} \otimes 1)(y) = (\text{sgn}(\sigma_2) \cdots \text{sgn}(\sigma_k))(s_2^{\otimes k-1} \otimes 1)(x)$ when $x_1$ and $x_2$ have even degree. Since $(\text{sgn}(\sigma_2) \cdots \text{sgn}(\sigma_k)) = \pm 1$, it is sufficient we prove that the proposition holds for $x$ in place of $y$.

In this case, for any collection $\bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1} \in S_2$ we shall write $((x_{\bar{\sigma}_{k-1}(1)} \otimes x_{\bar{\sigma}_{k-1}(2)}) \otimes \cdots \otimes (x_{\bar{\sigma}_1(1)} \otimes x_{\bar{\sigma}_1(2)}) \otimes x_1$ as $(\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x)$ for short.

Assume $x_1$ and $x_2$ have odd degree. Then

$$\sum_{\sigma_1, \ldots, \sigma_{k-1} \in S_2} (\text{sgn}(\bar{\sigma}_1) \cdots \text{sgn}(\bar{\sigma}_{k-1}))(\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x)$$

$$= \sum_{n=0}^{k-1} \sum_{\sigma_1, \ldots, \sigma_{k-1} \in S_2, \sigma_1 = (21)} (-1)^n(\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x).$$

Also, by part (i) of Lemma 4.2,

$$(s_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1}((\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x))$$

$$= (-1)^{k-1}(-2)^n(s_2(x_1 \otimes x_2))^{\otimes k-1} \otimes x_1$$

$$= (-1)^{k-1}(-2)^n(s_2^{\otimes k-1} \otimes 1)(x).$$
whenever \( \bar{\sigma}_i = (21) \) for exactly \( n \) choices of \( i \). Since each \( \bar{\sigma}_i \) can either be (12) or (21), there are \( \binom{k-1}{n} \) choices of \( \bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1} \in S_2 \) with the property that \( \bar{\sigma}_i = (21) \) for exactly \( n \) choices of \( i \). Therefore by equation (17)

\[
(\hat{s}_2 \otimes^{k-1} 1) \circ \beta_{2k-1} \circ (\hat{s}_2 \otimes^{k-1} 1) (x)
\]

\[
= (-1)^{k-1} \sum_{n=0}^{k-1} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1} \in S_2, \bar{\sigma}_i = (21) \text{ for } n \text{ choices of } i} (2^n)(\hat{s}_2 \otimes^{k-1} 1)(x)
\]

\[
= (-1)^{k-1} \sum_{n=0}^{k-1} \binom{k-1}{n} (2^n)(\hat{s}_2 \otimes^{k-1} 1)(x)
\]

\[
= (-1)^{k-1} 3^{k-1}(\hat{s}_2 \otimes^{k-1} 1)(x),
\]

where the last equality follows by the binomial formula.

On the other hand, assume \( x_1 \) and \( x_2 \) have even degree. Then

\[
(\hat{s}_2 \otimes^{k-1} 1)(x) = \sum_{m=0}^{k-1} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1} \in S_2, \bar{\sigma}_i = (21) \text{ for } m \text{ choices of } i} (-1)^m (\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x)
\]

By part (ii) of Lemma (4.2), if \( \bar{\sigma}_{2i} = (21) \) for some \( i \), \((\hat{s}_2 \otimes^{k-1} 1) \circ \beta_{2k-1}((\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x)) = 0. \) Otherwise if \( \bar{\sigma}_{2i+1} = (21) \) for exactly \( m \) choices of \( i \), then we have

\[
(\hat{s}_2 \otimes^{k-1} 1) \circ \beta_{2k-1}((\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x))
\]

\[
= (-1)^{k-1} (-2)^m (3^k \hat{s}_2)(\hat{s}_2 \otimes^{k-1} 1)(x).
\]

So because there are \( \left\lfloor \frac{k-1}{2} \right\rfloor \) choices of \( \bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1} \in S_2 \) with the property that \( \bar{\sigma}_{2i+1} = (21) \) for exactly \( m \) choices of \( i \), and \( \bar{\sigma}_{2i} \neq (21) \) for each \( i \), then

\[
(\hat{s}_2 \otimes^{k-1} 1) \circ \beta_{2k-1} \circ (\hat{s}_2 \otimes^{k-1} 1)(x)
\]

\[
= (-1)^{k-1} (3^k \hat{s}_2)(\hat{s}_2 \otimes^{k-1} 1)(x)
\]

\[
= (-1)^{k-1} 3^{k-1}(\hat{s}_2 \otimes^{k-1} 1)(x),
\]

where again the last equalities follow using the binomial formula. \( \square \)
5. Geometrically realizing Theorem 2.1

Let $X$ be the $p$-localization of a suspended $CW$-complex of finite type. Denote the reduced $\mathbb{Z}_p$-homology of $X$ by $V$. The $k$-fold self-smash of $X$ is written as $X^{(k)}$, is also the $p$-localization of a suspended $CW$-complex of finite type, and its reduced mod-$p$ homology given by

$$\tilde{H}_*(X^{(k)}) \cong V^{\otimes k}.$$  

Fix an element $\sigma \in \mathbb{Z}_p[S_k]$. The action of $\mathbb{Z}_p[S_k]$ on $V^{\otimes k}$ induces the self-map $V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}$. Note $X^{(k)}$ is a suspension since $X$ is one as well. By permuting factors of the smash product and taking $co$-$H$-space sums, one constructs a self-map $X^{(k)} \xrightarrow{f_{\sigma}} X^{(k)}$ with property that $(f_{\sigma})_* \in V^{\otimes k}$ on mod-$p$ homology. We will denote the mapping telescope

$$X^{(k)} \xrightarrow{f_{\sigma}} X^{(k)} \xrightarrow{f_{\sigma}} \cdots$$

by $T(f_{\sigma})$. If $\sigma \in \mathbb{Z}_p[S_k]$ happens to be an idempotent, $(1 - \sigma) \in \mathbb{Z}_p[S_k]$ is also an idempotent orthogonal to $\sigma$. We have isomorphisms

$$\tilde{H}_*(T(f_{\sigma})) \cong \text{Im}(\sigma : V^{\otimes k} \rightarrow V^{\otimes k})$$

and the canonical inclusions

$$X^{(k)} \xrightarrow{\iota_1} T(f_{\sigma})$$

$$X^{(k)} \xrightarrow{\iota_2} T(f_{1-\sigma})$$

induce the respective projections maps on mod-$p$ homology. Then the composite

$$X^{(k)} \xrightarrow{\text{pinch}} X^{(k)} \vee X^{(k)} \xrightarrow{1 \vee \iota_2} T(f_{\sigma}) \vee T(f_{1-\sigma})$$

induces an isomorphism on mod-$p$ homology. Since $X^{(k)}$ is the $p$-localization of a finite type $CW$-complex, this composite is a homotopy equivalence. In particular, the inclusion $\iota_1$ is a homotopy retraction, and its right homotopy inverse $T(f_{\sigma}) \xrightarrow{\iota_1} X^{(k)}$ can be chosen so that the composite $X^{(k)} \xrightarrow{\iota_2} T(f_{\sigma}) \xrightarrow{\iota_1} X^{(k)}$ is homotopic to $f_{\sigma}$.

It is a well known fact that $\beta_k \beta_k = k \beta_k \in \mathbb{Z}_p[S_k]$ [18, 23], and as such $\frac{1}{k} \beta_k \in \mathbb{Z}_p[S_k]$ is an idempotent whenever $k$ is prime to $p$. By taking mapping telescopes, one has $T(f_{\beta_k})$ a homotopy retract of $X^{(k)}$ when $k$ is prime to $p$. We shall denote $T(f_{\beta_k})$ by $L_k(X)$. The mod-$p$ homology of $L_k(X)$ is the image of $V^{\otimes k} \xrightarrow{\beta_k} V^{\otimes k}$, so it is the submodule of length $k$ Lie brackets in $V^{\otimes k}$, and is denoted by $L_k(V)$. These spaces $L_k(X)$ turn out to have some significance, as is apparent in the following homotopy decomposition (see [24]).

**Theorem 5.1.** Let $X$ be as above, and let $1 < k_1 < k_2 < \cdots$ be any sequence satisfying the following properties:

1. $k_i$ is prime to $p$;
(2) $k_i$ is not a multiple of $k_j$ whenever $i > j$.

Then there exists a homotopy decomposition

$$\Omega \Sigma X \simeq \prod_j \Omega \Sigma L_{k_j}(X) \times (\text{Some other space}).$$

\[
\Box
\]

The factors in this decomposition are of direct interest to the homotopy theory of $\Omega \Sigma X$. Ideally, these factors can be broken down into more familiar spaces. Since $L_k(X)$ is a homotopy retract of $X^{(k)}$, one can start by searching for splittings of $X^{(k)}$. A comprehensive look at the finest possible 2-primary splittings of $X^{(k)}$ for $X$ a 2-cell complex can be found in [15].

In the following theorem we give criteria for the existence of a certain homotopy retract of $L_{nt+1}(X)$.

**Proposition 5.2.** Fix $n > 0$ and $\ell > 1$ such that $n\ell + 1$ is prime to $p$, and take the integers $c_{n,\ell}$ and $d_{n,\ell}$ in Theorem (2.1).

Let $X$ be as above, and suppose $\dim V = \ell > 1$ (where $V$ denotes $H_*(X)$). Let $M$ denote the sum of the degrees of the generators of $V$. If $c_{n,\ell}$ is prime to $p$ and $V_{\text{odd}} = 0$, or $d_{n,\ell}$ is prime to $p$ and $V_{\text{even}} = 0$, then

1. there exist a space $Y$ that is a homotopy retract of $L_{nt+1}(X)$, and $H_*(Y) \cong H_*(\Sigma^{nM}X)$;
2. if $\ell \leq p - 1$, then $\Sigma^{nM}X$ is a homotopy retract of $L_{nt+1}(X)$.

**Proof of part (i).** Recall the elements $s_\ell, \delta_\ell \in \mathbb{Z}_p[\mathcal{S}_\ell]$ defined in Section (2). If $V_{\text{even}} = 0$, let $s_\ell = \delta_\ell$ and assume $c = c_{n,\ell}$ is prime to $p$. Otherwise if $V_{\text{odd}} = 0$, let $s_\ell = \delta_\ell$ and assume $c = d_{n,\ell}$ is prime to $p$. We have self-maps $X^{(\ell)} \xrightarrow{f_{\ell}} X^{(\ell)}$ and $X^{(n\ell+1)} \xrightarrow{f_{n\ell+1}} X^{(n\ell+1)}$ inducing $V \otimes \ell \xrightarrow{s_\ell} V \otimes \ell$ and $V \otimes (n\ell+1) \xrightarrow{\beta_{n\ell+1}} V \otimes (n\ell+1)$ on mod-$p$ homology. Consider the composite

$$g : X^{(n\ell+1)} \xrightarrow{f_{n\ell+1}} X^{(n\ell+1)} \xrightarrow{f_{n\ell+1}} X^{(n\ell+1)}$$

where $I$ is the identity map on $X$, and $f_{n\ell}^{(n)}$ is the $n$-fold self-smash of $f_{\ell}$. On mod-$p$ homology $g$ induces

$$g_* : V \otimes (n\ell+1) \xrightarrow{s_\ell \otimes 1} V \otimes (n\ell+1) \xrightarrow{\beta_{n\ell+1}} V \otimes (n\ell+1).$$

Let $T(g)$ be the telescope of $g$. By Theorem (2.1),

$$s_\ell \otimes 1 \circ \beta_{n\ell+1} \circ (s_\ell \otimes 1) = c(s_\ell \otimes 1).$$

Thus $g_* \circ g_* = c(g_*)$. Since $c$ is prime to $p$, this implies $H_*(T(g)) \cong \text{Im}(g_*)$. Notice $\text{Im}(g_*) \subseteq \text{Im}(s_\ell \otimes 1)$, and $\text{Im}(s_\ell \otimes 1) \circ g_* \subseteq \text{Im}(g_* \circ g_*) = \text{Im}(g_*)$. By equation (13), $\text{Im}((s_\ell \otimes 1) \circ g_*) = \text{Im}(c(s_\ell \otimes 1)) = \text{Im}(s_\ell \otimes 1)$. Therefore $\text{Im}(g_*) = \text{Im}(s_\ell \otimes 1)$. Also, $\text{Im}(s_\ell \otimes 1)$ is a submodule of
\( V^{\otimes n\ell} \) with dimension 1, whose single generator has degree \( nM \). So \( \text{Im}(s_{\ell}^{\otimes n} \otimes 1) \cong \Sigma^n M V \) as graded \( \mathbb{Z}_p \)-modules, where \( \Sigma^n M V \) is the \( nM \)-fold suspension of the graded \( \mathbb{Z}_p \)-module \( V \). Hence

\[
\tilde{H}_*(T(g)) \cong \text{Im}(g_*) \cong \Sigma^n M V \cong \tilde{H}_*(\Sigma^n M X).
\]

Let us also consider the composite

\[
h = (\xi - g) : X^{(n\ell + 1)} \xrightarrow{\psi} X^{(n\ell + 1)} \vee X^{(n\ell + 1)} \xrightarrow{\xi - g} X^{(n\ell + 1)} \xrightarrow{\psi} X^{(n\ell + 1)},
\]

where \( \psi \) is the pinch map, \( \xi \) is the degree \( c \) map on \( X^{(n\ell + 1)} \), \( -g \) is the composite of \( g \) and the degree -1 map on \( X^{(n\ell + 1)} \), and \( \psi \) is the fold map. On mod-\( p \) homology we have \( h_* = c - g_* \), and \( c \) is prime to \( p \). Since \( g_* \circ g_* = c(g_*) \), this implies \( \text{Im}(h_*) \cong (V^{\otimes (n\ell + 1)} - \text{Im}(g_*)) \) and \( \text{Im}(g_*) \cap \text{Im}(h_*) = \emptyset \).

Therefore \( V^{\otimes (n\ell + 1)} \) splits as a sum of \( \mathbb{Z}_p \)-submodules \( \text{Im}(g_*) \oplus \text{Im}(h_*) \). Notice that

\[
h_* \circ h_* = (c - g_*) \circ (c - g_*) = c^2 - 2c(g_*) + (g_* \circ g_*) = c^2 - 2c(g_*) + c(g_*) = c(h_*),
\]

so taking the telescope \( T(h) \), we have \( \tilde{H}_*(T(h)) \cong \text{Im}(h_*) \). Thus we have the following splitting of graded \( \mathbb{Z}_p \)-modules,

\[
\tilde{H}_*(X^{(n\ell + 1)}) = V^{\otimes (n\ell + 1)} = \text{Im}(g_*) \oplus \text{Im}(h_*) \cong \tilde{H}_*(T(g)) \oplus \tilde{H}_*(T(h)).
\]

As the inclusions \( X^{(n\ell + 1)} \xrightarrow{i_g} T(g) \) and \( X^{(n\ell + 1)} \xrightarrow{i_h} T(h) \) induce projections of \( \text{Im}(g_*) \) and \( \text{Im}(h_*) \) isomorphically onto \( \tilde{H}_*(T(g)) \) and \( \tilde{H}_*(T(h)) \) in mod-\( p \) homology, the map

\[
f : X^{(n\ell + 1)} \xrightarrow{\psi} X^{(n\ell + 1)} \vee X^{(n\ell + 1)} \xrightarrow{\xi - g \vee h} T(g) \vee T(h)
\]

induces an isomorphism on mod-\( p \) homology. Since \( X^{(n\ell + 1)} \) is the \( p \)-localization of a finite type CW-complex, \( f \) is a homotopy equivalence.

Let \( f^{-1} : T(g) \vee T(h) \rightarrow X^{(n\ell + 1)} \) denote the inverse homotopy equivalence of \( f \). Since \( f \circ f^{-1} : X^{(n\ell + 1)} \rightarrow X^{(n\ell + 1)} \) is homotopic to the identity, and \( X^{(n\ell + 1)} \xrightarrow{i_g} T(g) \) maps \( \text{Im}(g_*) \) isomorphically onto \( \tilde{H}_*(T(g)) \) on mod-\( p \) homology, the composite

\[
\kappa_g : T(g) \xrightarrow{f^{-1}} T(g) \vee T(h) \xrightarrow{f} X^{(n\ell + 1)}
\]

maps \( \tilde{H}_*(T(g)) \) isomorphically onto \( \text{Im}(g_*) \) in mod-\( p \) homology. Also, since \( n\ell + 1 \) is prime to \( p \) and \( \beta_{n\ell + 1} \circ \beta_{n\ell + 1} = (n\ell + 1)\beta_{n\ell + 1}, \frac{1}{n\ell + 1} \beta_{n\ell + 1} \in \mathbb{Z}_p[S_{n\ell + 1}] \) is an idempotent. So the inclusion \( X^{(n\ell + 1)} \xrightarrow{i} T(f\beta_{n\ell + 1}) = L_{n\ell + 1}(X) \) is a homotopy retraction, and we can take some left homotopy inverse \( \kappa \) such that \( X^{(n\ell + 1)} \xrightarrow{i} L_{n\ell + 1}(X) \xrightarrow{\kappa} X^{(n\ell + 1)} \) is homotopic to \( X^{(n\ell + 1)} \xrightarrow{f\beta_{n\ell + 1}} X^{(n\ell + 1)} \).

Now consider the composite

\[
\alpha : T(g) \xrightarrow{\kappa_g} X^{(n\ell + 1)} \xrightarrow{i} L_{n\ell + 1}(X) \xrightarrow{\kappa} X^{(n\ell + 1)} \xrightarrow{i_g} T(g).
\]

Recall \( g_* = \beta_{n\ell + 1} \circ (s_{\ell}^{\otimes n} \otimes 1) \) by definition. Then on mod-\( p \) homology \( \kappa_g \circ \iota_* \) sends \( \text{Im}(g_*) \) surjectively onto \( \beta_{n\ell + 1}(\text{Im}(g_*)) = \text{Im}(\beta_{n\ell + 1} \circ \beta_{n\ell + 1} \circ (s_{\ell}^{\otimes n} \otimes 1))) = \text{Im}((n\ell + 1)\beta_{n\ell + 1} \circ (s_{\ell}^{\otimes n} \otimes 1)) = \text{Im}(g_*) \).

Since \( (i_g)_* \) projects \( \text{Im}(g_*) \) isomorphically onto \( \tilde{H}_*(T(g)) \), and \( \kappa_g \) maps \( \tilde{H}_*(T(g)) \) isomorphically
onto $Im(g_\ast)$, $\alpha_\ast$ is an isomorphism on mod-$p$ homology. Since $T(g)$ is a summand in the above splitting of $X(n\ell+1)$, which is the $p$-localization of a finite type $CW$-complex, $\alpha$ must be a homotopy equivalence, and so $T(g)$ is a homotopy retract of $L_{n\ell+1}(X)$. 

Proof of part (ii). We continue where the proof of part (i) left off to avoid redefining things. This time we have the added assumption that $\ell \leq p-1$. Recall $s_\ell \in \mathbb{Z}_p[S_\ell]$ is either $\bar{s}_\ell$, or $\hat{s}_\ell$, depending on whether $V_{\text{even}} = 0$ or $V_{\text{odd}} = 0$. In either case it is well known (and not difficult to see) that

$$s_\ell s_\ell = \ell! s_\ell,$$

so one can take the idempotent $\frac{1}{\ell!} s_\ell$ when $\ell \leq p-1$. The inclusion $X^{(\ell)} \xrightarrow{\ell} T(f_{s_\ell})$ is therefore a homotopy retraction, and since $\tilde{H}_s(T(f_{s_\ell})) \cong Im(s_\ell)$ is a 1-dimensional submodule of $V^{\otimes n\ell}$ whose generator has degree $M$, $T(f_{s_\ell})$ is homotopy equivalent to the $M$-sphere $S^M$.

Let $\gamma : X^{(n\ell)} \xrightarrow{\gamma^{(\ell)}} S^nM$ be the $n$-fold smash of $\hat{i}$. On mod-$p$ homology $\gamma$ induces an isomorphism onto $Im(s_\ell^{\otimes n})$, and so the smash of $\gamma$ and the identity on $X$,

$$\gamma \wedge \mathbb{1} : X^{(n\ell)} \longrightarrow \Sigma^nM X,$$

induces an isomorphism onto $Im(s_\ell^{\otimes n} \otimes 1)$. Since the section map $T(g) \xrightarrow{\gamma} X^{(n\ell+1)}$ defined in the proof of part (i) induces an isomorphism onto $Im(s_\ell^{\otimes n} \otimes 1)$ on mod-$p$ homology, the composite $T(g) \xrightarrow{\gamma} X^{(n\ell)} \xrightarrow{\gamma \wedge \mathbb{1}} \Sigma^nM X$ is an isomorphism on mod-$p$ homology, so it is a homotopy equivalence. Thus part (ii) follows from part (i).

If the criteria in Proposition 5.2 are satisfied, Proposition 5.2 together with Theorem 5.1 imply $\Omega\Sigma Y$ is a homotopy retract of $\Omega\Sigma X$. Similarly, when $\ell \leq p-1$, $\Omega\Sigma^{n\ell+1}X$ is a homotopy retract of $\Omega\Sigma X$. If both $d_{n,\ell}$ and $c_{n,\ell}$ are prime to $p$, one can iterate Proposition 5.2.

**Proposition 5.3.** Fix $n > 0$ and $\ell > 1$ such that $n\ell+1$ is prime to $p$, and suppose both $c_{n,\ell}$ and $d_{n,\ell}$ are prime to $p$.

Let $X$ be any suspended $p$-local $CW$-complex with $\dim V = \ell > 1$ (where $V$ denotes $\tilde{H}_s(X)$), and either $V_{\text{odd}} = 0$ or $V_{\text{even}} = 0$. Let $M$ denote the sum of the degrees of the generators of $V$, and define the sequence of integers $b_{i,n}$ recursively with $b_{0,n} = 0$, and

$$b_{i,n} = (n\ell+1)b_{i-1,n} + nM.$$

Then

(i) there exist spaces $Y_i$ such that $\Omega\Sigma Y_i$ is a homotopy retract of $\Omega\Sigma X$, and $\tilde{H}_s(Y_i) \cong \tilde{H}_s(\Sigma^{b_i,n}X)$ for each $i \geq 1$;

(ii) if $\ell \leq p-1$, then $\Omega\Sigma^{b_i,n+1}X$ is a homotopy retract of $\Omega\Sigma X$ for each $i \geq 1$. 
Proof. We will prove part (ii) since part (i) is similar. This is done by induction, with the base case being $\Omega \Sigma^n X$ being a homotopy retract of $\Omega \Sigma X$. This base case holds true since $\Sigma^n X$ is a homotopy retract of $L_{n+1}(X)$ by Proposition (5.2), and by Theorem (5.1) $\Omega \Sigma L_{n+1}(X)$ is a homotopy retract of $\Omega \Sigma X$ when $n+1$ is prime to $p$.

For our inductive assumption, let us assume $\Omega \Sigma^{b_i,n+1} X$ is a homotopy retract of $\Omega \Sigma X$ for some $i \geq 1$, and let $M'$ be the sum of the degrees of the generators of $\tilde{H}_*(\Sigma^{b_i,n} X)$. Notice that $\dim \Sigma^{b_i,n} V = \dim V = \ell$, and since $V_{odd} = 0$ or $V_{even} = 0$, either $(\Sigma^{b_i,n} V)_{odd} = 0$ or $(\Sigma^{b_i,n} V)_{even} = 0$. Since we have an isomorphism $\tilde{H}_*(\Sigma^{b_i,n} X) \cong \Sigma^{b_i,n} V$ of graded $\mathbb{Z}_p$-modules, and since we are assuming that both $c_{n,\ell}$ and $d_{n,\ell}$ are prime to $p$, by Proposition (5.2) $\Sigma^{nM'}(\Sigma^{b_i,n} X)$ is a homotopy retract of $L_{n+1}(\Sigma^{b_i,n} X)$. Also, because $n+1$ is prime to $p$, by Theorem (5.1) $\Omega \Sigma L_{n+1}(\Sigma^{b_i,n} X)$ is a homotopy retract of $\Omega \Sigma(\Sigma^{b_i,n} X)$, so $\Omega \Sigma^{nM'}(\Sigma^{b_i,n} X)$ is also a homotopy retract of $\Omega \Sigma(\Sigma^{b_i,n} X)$. Then using our inductive assumption, $\Omega \Sigma^{nM'}(\Sigma^{b_i,n} X)$ is a homotopy retract of $\Omega \Sigma X$.

To check that $M'$ has the correct value, let $\{v_1, ..., v_\ell\}$ be a basis for $V$ and $M$ be the sum of the degrees of the generators in this basis. In this case

$$M' = \sum_{1 \leq i \leq \ell} (b_{i,n} + |v_i|) = \ell b_{i,n} + M.$$  

Thus $\Sigma^{nM'}(\Sigma^{b_i,n} X) = \Sigma^{b_i+1,n} X$. This completes the induction.

Proof of Theorem (1.1). In Theorem (2.2) we found that $c_{1,\ell} = d_{1,\ell} = (\ell + 1)!(\ell - 1)!$ for $\ell > 1$. Theorem (1.1) now follows as a direct consequence of Proposition (5.3).

Proof of Theorem (1.3). Fix $p \geq 5$, and let $X$ be any suspended $p$-local CW-complex with dim $V = 2$, and either $V_{odd} = 0$ or $V_{even} = 0$. Let $M$ denote the sum of the degrees of the two generators of $V$.

Let $0 < k_1 < k_2 < \cdots$ be any sequence satisfying $2k_i + 1$ is prime to $p$ and $2k_i + 1$ is not a multiple of $2k_j + 1$ whenever $i > j$. By Theorem (5.1) there is a decomposition

$$\Omega \Sigma X \cong \prod_j \Omega \Sigma L_{2k_j+1}(X) \times (\text{Some other space}).$$

Since $2k_j + 1$ and $c_{k_j,2} = d_{k_j,2} = 3k_j$ are prime to $p$ for all $j$, by Proposition (5.2) $\Sigma^{k_j M} X$ is a retract of $L_{2k_j+1}(X)$, so in turn $\Omega \Sigma^{k_j M+1} X$ is a retract of $\Omega \Sigma L_{2k_j+1}(X)$. This proves the theorem.

Remark 5.4. There is an analogous decomposition of the 2-cell complex $X$ in Theorem (1.3) in terms of spheres, on the condition that $V_{odd} = 0$. In this case it is easy to show $L_2(X) \cong S^M$ and $L_3(X) \cong \Sigma^M X$. Repeating this argument starting with $\Sigma^M X$ in place of $X$, one can show $\Omega \Sigma X \cong \prod_{i=1}^\infty \Omega(S^{b_i+1}) \times (\text{Some other space})$ by using Theorem (5.1) (where $b_{i,1}$ are the integers defined in Proposition (5.3)).
Proof of Theorem (1.5). Let $X$ be any suspended $p$-local CW-complex, and let $V$ denote $\tilde{H}_*(X)$, and $M$ be the sum of the degrees of the generators in $V$. Assume $V_{\text{even}} = 0$, $\ell = \text{dim } V$ is even, and $1 < \ell < p - 1$. Let $L(V)$ is the free Lie algebra generated by $V$, and $[L(V), L(V)]$ the sub Lie algebra of $L(V)$ generated by Lie brackets of length greater than one. By the Poincare–Birkhoff-Witt theorem, there is an isomorphism of coalgebras

$$T(V) \cong \Lambda(V) \otimes S([L(V), L(V)]).$$

This isomorphism is geometrically realized by Cohen’s and Neisendorfer’s decomposition (Theorem (1.4))

$$\Omega \Sigma X \cong A(X) \times \Omega Q(X),$$

with $H_*(A(X)) \cong \Lambda(V)$ and $H_*(\Omega Q(X)) \cong S([L(V), L(V)])$. By Theorem (5.1) $\Omega \Sigma L_{\ell+1}(X)$ is a homotopy retract of $\Omega \Sigma X$, and the proof of this in [24] indicates the section map $\Omega \Sigma L_{\ell+1}(X) \to \Omega \Sigma X$ for this retraction induces the natural inclusion

$$\tilde{H}_*(\Omega \Sigma L_{\ell+1}(X)) \cong T(L_{\ell+1}(V)) \cong \bigotimes_{i=1}^\infty S(L_i(L_{\ell+1}(V))) \subseteq \Lambda(V) \otimes S([L(V), L(V)])$$

into the right-hand factor (where $L_i(V)$ denotes the $\mathbb{Z}_p$-submodule of length $i$ Lie brackets in $L(V)$, and where the isomorphism follows by the Poincare–Birkhoff-Witt theorem). Therefore $\Omega \Sigma L_{\ell+1}(X)$ is also a homotopy retract of $\Omega Q(X)$. In turn, $\Omega \Sigma M+1 X$ is a homotopy retract of $\Omega \Sigma L_{\ell+1}(X)$ using Proposition (5.3), so we obtain a decomposition

$$\Omega \Sigma X \cong A(X) \times \Omega \Sigma M+1 X \times (\text{Some other space}).$$

Since $\ell = \text{dim } V$ is even and $V_{\text{even}} = 0$, $\tilde{H}_*(\Sigma^M X) \cong \Sigma^M V$ has only odd degree generators, so we can reapply Cohen’s and Neisendorfer’s decomposition to $\Omega \Sigma M+1 X$. Iterating this argument, starting by taking $\Sigma^M X$ in place of $X$, and using an induction similar to the proof of Proposition (5.3), we obtain the decomposition

$$\Omega \Sigma X \cong \prod_{i=0}^\infty A(\Sigma^{b_{i,1}} X) \times (\text{Some other space}),$$

where $b_{i,1}$ are the integers defined in Proposition (5.3).

\[\square\]

6. An Application to the Moore conjecture

The $p$-exponent $\exp_p(X)$ of a space $X$ is defined as the smallest power $p^t$ that annihilates the $p$-primary torsion of $\pi_i(X)$ for all $i > 0$. Spheres, finite $H$-spaces, and mod-$p$ Moore spaces are all examples of spaces that have finite $p$-exponents at odd primes $p$ [4, 19, 9]. In the other direction, a simply connected wedge $S^m \vee \Sigma X$ with $\Sigma X$ rationally non-trivial does not have a finite $p$-exponent [19]. These isolated examples aside, there is no known general set of criteria for distinguishing spaces that have a finite $p$-exponent from those that do not. However, a conjecture of Moore
suggests that the answer is very simple for finite simply connected CW-complexes $X$: $\exp_p(X)$ is finite at any prime $p$ if and only if $\pi_*(X) \otimes \mathbb{Q}$ is a finite dimensional vector space. When $X$ is a suspension it is known that $X$ is rational wedge of spheres, and so in this case the Moore conjecture says that $\exp_p(X)$ is finite if and only if $\dim(H_*(X; \mathbb{Q})) \leq 1$.

McGibbon and Wilkerson [7] were able to give a partial result in one direction of the Moore conjecture. They showed that a finite simply connected CW-complex has a finite $p$-exponent at sufficiently large primes $p$ when it has finite dimensional rational homotopy. This prime $p$ depends on the given space. In the other direction Selick [11] showed that a finite simply connected CW-complex $X$ has no finite $p$-exponent whenever $X$ is a suspension, and $H_*(X; \mathbb{Z})$ is torsion-free of rank greater than one. This result was in some sense extended by Stelzer [20, 21] to any finite simply connected CW-complex $X$, as long as one selects a sufficiently large prime $p$ that depends on the dimension and connectivity of $X$. By combining Stelzer’s result with that of McGibbon’s and Wilkerson’s, one sees that the Moore conjecture holds in the sense of sufficiently large primes.

There also happens to be a stable analogue of the Moore conjecture due to Stanley [19], which has the fortune of being much easier to prove.

**Theorem 6.1 (Stanley).** A finite simply connected CW-complex $X$ has a finite $p$-exponent on stable homotopy $\pi_*^s(X)$ if and only if $X$ is rationally trivial.

Now combining the following proposition with Stanley’s theorem, we can recover Selick’s work on the Moore conjecture when restricted to the spaces in Theorem (1.1).

**Proposition 6.2.** Take the integers $b_i$ and a suspended $p$-local CW-complex $X$ as in Theorem (1.1), letting $V = \tilde{H}_*(X)$, $1 < \dim V < p - 1$, and either $V_{\text{odd}} = 0$ or $V_{\text{even}} = 0$. Assume $X$ is $(m - 1)$-connected for some $m \geq 1$. Then for each $j$ the stable homotopy group $\pi_*^s(\Sigma X)$ is a homotopy retract of $\pi_{j+b_i}(\Sigma X)$ for all $i$ large enough such that $j \leq b_i + 2m$.

**Proof.** By Theorem (1.1) $\Omega \Sigma^{b_i+1}X$ is a homotopy retract of $\Omega \Sigma X$, so $\pi_{j+b_i}(\Sigma^{b_i+1}X)$ is a homotopy retract of $\pi_{j+b_i}(\Sigma X)$ for each $j$. By the Freudenthal suspension theorem $\pi_{j+b_i}(\Sigma^{b_i+1}X) \cong \pi_{j+b_i}(\Sigma^{b_i+1}X)$ for $j \leq b_i + 2m$, and $\pi_{j+b_i}(\Sigma^{b_i+1}X) \cong \pi_j(\Sigma X)$. Thus $\pi_j^s(\Sigma X)$ is a homotopy retract of $\pi_{j+b_i}(\Sigma X)$ when $j \leq b_i + 2m$. \hfill $\Box$

Thus we see that the stable homotopy groups $\pi_*^s(\Sigma X)$ of the space $\Sigma X$ in Proposition (6.2) are retracts of $\pi_*^s(\Sigma X)$. Since $\Sigma X$ is rationally nontrivial, $\exp_p(\pi_*^s(\Sigma X))$ is infinite by Stanley’s theorem. So $\exp_p(\Sigma X)$ must also be infinite.

One would hope for some sort of generalization of Theorem (1.1), beyond the restrictions $V_{\text{odd}} = 0$ or $V_{\text{even}} = 0$. There are unfortunately many examples where this is impossible. Let $X$ be a wedge $S^m \vee P^n(p')$, where the mod-$p$ Moore space $P^n(p')$ is the cofibre of the degree $p'$ map $S^{n-1} \xrightarrow{p'} S^{n-1}$. Then $\Sigma X$ has torsion in its integral homology, but is rationally nontrivial, and the mod-$p$ homology
V = H_*(X) satisfies V_{odd} \neq 0 and V_{even} \neq 0. If Theorem (1.1) applied to this space X, the p-exponent of the stable homotopy groups of \pi_*^s(\Sigma X) would be bounded above by the p-exponent of \pi_*^s(\Sigma X), and so using Stanley’s theorem, exp_p(\Sigma X) would be infinite. But by application of the Hilton-Milnor theorem to \Omega \Sigma X, and the fact that exp_p(P_j(p^r)) = p^{r+1} independently of j [9], exp_p(\Sigma X) is in fact finite, a contradiction.

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