NORMALIZED SOLUTIONS TO KIRCHHOFF TYPE EQUATIONS WITH A CRITICAL GROWTH NONLINEARITY

JIAN ZHANG, JIANJUN ZHANG, AND XUEXIU ZHONG

Abstract. In this paper, we are concerned with normalized solutions of the Kirchhoff type equation

\[ -M \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u = \lambda u + f(u) \quad \text{in} \quad \mathbb{R}^N \]

with \( u \in S_c := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 \, dx = c^2 \} \). When \( N = 2 \) and \( f \) has exponential critical growth at infinity, normalized mountain pass type solutions are obtained via the variational methods. When \( N \geq 4 \), \( M(t) = a + bt \) with \( a, b > 0 \) and \( f \) has Sobolev critical growth at infinity, we investigate the existence of normalized ground state solutions, mountain pass type solutions and local constraint minimizer with positive energy. Moreover, the non-existence of normalized solutions is also considered.

1. Introduction and results

1.1. Background. The Kirchhoff type problem appears as models of several physical phenomena. For example, it is related to the stationary analogue of the equation:

\[ \rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \]  

where \( u \) is the lateral displacement at \( x \) and \( t \), \( E \) is the Young modulus, \( \rho \) is the mass density, \( h \) is the cross-section area, \( L \) is the length, \( P_0 \) is the initial axial tension. For more background, see [2,30] and the references therein. Because of the presence of the nonlocal term, the Kirchhoff type equation is no longer a pointwise identity, which causes additional mathematical difficulties.

In this paper, we study solutions to the following Kirchhoff type equation with critical growth nonlinearities:

\[ -M \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u = \lambda u + f(u) \quad \text{in} \quad \mathbb{R}^N \]  

having prescribed mass

\[ \int_{\mathbb{R}^N} u^2 \, dx = c^2, \]  

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where $N = 2$ or $N \geq 4$, $\lambda \in \mathbb{R}$ is an unknown parameter appearing as a Lagrange multiplier and $f$ is of critical growth. For the case of $N = 2$, we consider the general Kirchhoff type case. While for $N \geq 4$, we consider the classical Kirchhoff case that $M(t) = a + bt$ with $a, b > 0$.

A solution $u$ to Eq. (1.2) satisfying (1.3) is called a normalized solution. One can search for normalized solutions of (1.2) by studying critical points of the functional

$$I(u) = \frac{1}{2} \tilde{M} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) - \int_{\mathbb{R}^N} F(u) dx,$$

constrained on the $S_c$, where $\tilde{M}(t) := \int_0^t M(s) ds$, $F(u) := \int_0^u f(s) ds$ and

$$S_c := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = c^2 \right\}. \quad (1.5)$$

For $c > 0$, if $I|_{S_c}(u_c) = 0$, then there exists some $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c)$ is a couple of solution to (1.2), where $\lambda_c$ is the associated Lagrange multiplier. For any $c > 0$, define

$$I_c := \inf_{u \in S_c} I(u). \quad (1.6)$$

We say that $u_0$ is a normalized ground state solution of (1.2) if

$$I(u_0) = \inf \{ I(u) : I|_{S_c}(u) = 0, u \in S_c \}.$$ 

In particular, if $u_0$ attains $I_c$, then $u_0$ is a normalized ground state solution of (1.2).

The normalized solution has important physical background and it has attracted attentions of many researchers in recent years. For the convenience, we refer the readers to the following papers, and the references therein, [1, 5, 7–10, 10–13, 18–24, 26–29, 32–34, 38–40]. In particular, L. Jeanjean [24] exploited the mountain pass lemma to study normalized solutions of Schrödinger equations for the $L^2$-supercritical case. N. Soave [39] considered normalized ground state solutions of Schrödinger equations with combined nonlinearities for the Sobolev critical case. Jeanjean and Lu [27] investigated normalized local minimizers and mountain pass type solutions for a coercive problem. Jeanjean, Zhang and Zhong [29] introduced a global branch approach to study normalized solutions of the Schrödinger equation. There are relatively few results about normalized solutions of Kirchhoff type equations. In [43, 44], the author considered normalized solutions of (1.2) for the case of $N \leq 3$, $M(t) = a + bt$ with $a, b > 0$ and $f(u) = |u|^{p-2}u$ with $p \in (2, 2^*)$. Precisely, for $p \in (2, \frac{2N+8}{N})$, the authors studied the existence and non-existence of global constraint minimizers; for $p \in (\frac{2N+4}{N}, \frac{2N+8}{N})$, the authors studied local constraint minimizers; for $p \in (\frac{2N+8}{N}, 2^*)$, the authors studied mountain pass type solutions. In [45], the authors improved the results in [43]. In [35], the authors investigated the exact number and expressions of positive normalized solutions for Kirchhoff equations. In [31, 47], the authors studied normalized solutions of Kirchhoff equations with critical growth nonlinearity in dimension three.

To the best of our knowledge, there are no results on normalized solutions of Kirchhoff type equations with exponential critical growth in dimension two. Here we recall that the nonlinearity $f$ has exponential subcritical growth if for any $\alpha > 0$,

$$\lim_{u \to +\infty} \frac{f(u)}{e^{\alpha u^2}} = 0.$$
and the nonlinearity $f$ has exponential critical growth if there exists $\alpha_0 > 0$ such that
\[
\lim_{u \to +\infty} \frac{f(u)}{e^{\alpha_0 u^2}} = \begin{cases} 
0, & \forall \alpha > \alpha_0, \\
+\infty, & \forall \alpha < \alpha_0.
\end{cases}
\]
We notice that in [3], the authors used the idea due to L. Jeanjean [24] to study normalized solutions of Schrödinger equations (i.e., $M(t) \equiv 1$ in (1.2)) for $N = 2$, where $f$ is of exponential critical growth satisfying the following technical condition.

(f) There exists $p > 4$ and $\mu > 0$ such that $\text{sgn}(t)f(t) \geq \mu|t|^{p-1}$ for $t \neq 0$, where $\text{sgn}(t) = 1$ if $t > 0$ and $\text{sgn}(t) = -1$ if $t < 0$.

We remark that in [3], the condition (f) with $\mu > 0$ being large is essential, which enables the upper bound on the energy being controlled. It is natural to ask if the restriction can be removed. This is one motivation of the present paper.

We recall that in [15], the authors studied positive solutions of the Dirichlet problem:
\[-\Delta u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]
In this classical paper, to estimate the upper bound on the energy, the author introduced the following more natural condition:

\[(f') \quad \text{There exists } \beta > \frac{4}{3\alpha_0 d^2} \text{ such that}
\lim_{u \to +\infty} \frac{f(x, u)u}{e^{\alpha_0 u^2}} \geq \beta,
\]
where $d$ is the radius of the largest open ball in $\Omega$.

This argument is based on the Moser sequence of functions and the proof by the contradiction. Similar results can be found in [17, 37]. However, the arguments cannot be applied directly to study normalized solutions. This is one of the difficulties. We give a direct argument, which enables us to get the desired upper bound estimate on the energy using the condition $(f_3)$.

Another difficulty is the presence of the nonlocal term, which brings an additional obstacle in proving the $PSP$-condition in Lemma 2.11. It is well known that the Trudinger-Moser inequality is essential to deal with the loss of the compactness caused by the exponential critical nonlinearity. When using this inequality, a key step is to give a suitable uniformly upper bound on the $H^1$-norm of the sequence. However, the nonlocal term prevents us from using the upper bound on the energy and the Ambrosetti-Rabinowitz type condition to deduce the suitable $H^1$-norm estimate. We firstly give a compactness result in Lemma 2.10, based on which we obtain the desired $H^1$-norm estimate. Then we use the Trudinger-Moser inequality to establish a Brézis-Lieb type result in Lemma 2.9 and solve the compactness of the Kirchhoff type problem through a brief argument. We believe that the Brézis-Lieb type result can also be used to study related compact problems in a non-radial setting.

1.2. Main results. Now we state our result in dimension two. We consider the so-called general Kirchhoff type equation (1.2)-(1.3). For the nonlocal term $M(t)$, we assume the following conditions:

\[(M_1) \quad M \in C([0, +\infty)) \cap C^1([R, +\infty)) \text{ for some } R > 0, \quad M_0 := M(0) > 0 \text{ and } M(t) \text{ increases respect to } t \in \mathbb{R}^+.
\]
Remark 1.1. We recall that in Theorem 1.1. Assume that (M₁)-(M₃) and (f₁)-(f₅) hold. Then problem (1.2)-(1.3) has a couple of solution \((u_ε, λ_ε) \in H^1(\mathbb{R}^2) \times \mathbb{R}\) such that \(λ_ε < 0\) and \(u_ε\) is a mountain pass type critical point of \(I|_{S_ε}\).

Remark 1.1. We recall that in [15], the compactness result in Lemma 2.1 plays an important role in proving the PS-condition of the Dirichlet problem with exponential critical growth in \(\mathbb{R}^2\). To get the compactness result, the authors introduced the following classical condition:

\((f_5')\) There exist \(u_0, L_0 > 0\) such that \(F(u) \leq L_0 f(u)\) for \(u \geq u_0\).

To the best of our knowledge, the condition \((f_5')\) was later used to study all critical problems in \(\mathbb{R}^2\). A natural question is if \((f_5')\) can be weaken or removed? In this paper, we introduce a new condition \((f_5)\), which can replace \((f_5')\) to study the compactness of the problem with exponential critical growth in \(\mathbb{R}^2\).

For the case of \(N \geq 4\), there are few results about normalized solutions to Kirchhoff type equations. We notice that in [46], the authors studied normalized solutions of (1.2)-(1.3), where \(M(t) = a + bt\) with \(a, b > 0\) and \(f\) satisfies the following conditions:

\((f_1')\) \(f \in C^1(\mathbb{R}^+, \mathbb{R}^+)\) and \(f(u) > 0\) for \(u > 0\).

\((f_2')\) There exist \(2 < \alpha, \beta < 2^* := \frac{2N}{N-2}\) such that \(\lim_{u \to 0^+} \frac{f''(u)}{u^{\alpha - 2}} = \mu_1(\alpha - 1) > 0\) and

\(\lim_{u \to +\infty} \frac{f''(u)}{u^{\beta - 2}} = \mu_2(\beta - 1) > 0\).

\((f_3')\) \(-\Delta u = f(u)\) has no positive radial decreasing classical solution in \(\mathbb{R}^N\).

Let \(U\) be the unique positive solutions of the following equation:

\[-\Delta U + U = \mu_1 U^{\alpha - 1} \quad \text{in} \ \mathbb{R}^N, \quad \lim_{|x| \to \infty} U(x) = 0.\]

By using the global branch approach developed by Jeanjean et al. in [29] and the Azzollini’s correspondence in [4], which provided a homeomorphism between ground
state solutions of autonomous Kirchhoff type equations and related local semilinear elliptic equations, the authors obtained the following results in [46].

**Theorem 1.2.** (The case \( N = 4 \)) Let \( M(t) = a + bt \) with \( a, b > 0 \) and \((f'_i)-(f'_3)\) hold.

1. If \( \alpha \in (3, 4) \), then there exists \( c^* > 0 \) such that for any \( c > c^* \), problem (1.2)-(1.3) has at least two distinct positive normalized solutions \((\lambda_i, u_{\lambda_i}) \in (-\infty, 0) \times H^1_i(\mathbb{R}^4), i = 1, 2.\)

2. If \( \alpha = 3 \), then for any \( c > a\|U\|_2 \), problem (1.2)-(1.3) has at least one positive normalized solution \((\lambda, u_\lambda) \in (-\infty, 0) \times H^1_i(\mathbb{R}^4).\)

3. If \( \alpha \in (2, 3) \), then for any \( c > 0 \), problem (1.2)-(1.3) has at least one positive normalized solution \((\lambda, u_\lambda) \in (-\infty, 0) \times H^1_i(\mathbb{R}^4).\)

**Theorem 1.3.** (The case \( N \geq 5 \)) Let \( M(t) = a + bt \) with \( a, b > 0 \) and \((f'_i)-(f'_3)\) hold.

1. If \( \alpha \in (2 + \frac{4}{N}, 2^*) \), then there exists \( c^* > 0 \) such that for any \( c > c^* \), problem (1.2)-(1.3) has at least two distinct positive normalized solutions \((\lambda_i, u_{\lambda_i}) \in (-\infty, 0) \times H^1_i(\mathbb{R}^N), i = 1, 2.\)

2. If \( \alpha = 2 + \frac{4}{N} \), then for any \( c > a\|U\|_2 \), problem (1.2)-(1.3) has at least one positive normalized solution \((\lambda, u_\lambda) \in (-\infty, 0) \times H^1_i(\mathbb{R}^N).\)

3. If \( \alpha \in (2, 2 + \frac{4}{N}) \), then for any \( c > 0 \), problem (1.2)-(1.3) has at least one positive normalized solution \((\lambda, u_\lambda) \in (-\infty, 0) \times H^1_i(\mathbb{R}^N).\)

**Theorem 1.4.** Let \( M(t) = a + bt \) with \( a, b > 0 \), \( N \geq 4 \) and \((f'_i)-(f'_3)\) hold. If \( \alpha \in [2 + \frac{4}{N}, 2^*) \), then there exists \( c_0 > 0 \) such that problem (1.2)-(1.3) has no positive solution for \( c < c_0 \).

For the critical case, we cannot use the Azzollini’s correspondence in [4] to study the normalized solution problem, since the mass is not preserved via the transformation. And we also can not use the global branch approach in [29] since the presence of the Sobolev critical term. Motivated by the above facts, we consider the following Kirchhoff type equation with combined nonlinearities

\[
-\left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u = \lambda u + |u|^{p-2}u + |u|^{2^*-2}u \quad \text{in} \quad \mathbb{R}^N, \tag{1.7}
\]

for some \( \lambda \in \mathbb{R} \) having prescribed mass

\[
\int_{\mathbb{R}^2} u^2 dx = c^2, \tag{1.8}
\]

in the Sobolev critical case, where \( N \geq 4, a, b > 0 \) and \( 2 < p < 2^* := \frac{2N}{N-2}. \)

The Sobolev critical term complicates the problem. As we will see, the interplay between Sobolev subcritical and Sobolev critical nonlinearities strongly affects the structure of the functional, the compactness of the problem and the existence of normalized solutions. For the case of \( N \geq 4 \), since \( 2^* := \frac{2N}{N-2} \leq 4 \), the critical term will be affected by the nonlocal term. We give a technical Lemma 3.1, which helps to deal with the structure of the functional and solve the loss of the compactness caused by the critical term.

The \( L^2 \)-critical exponent \( 2 + \frac{4}{N} \) plays an important role in studying normalized solutions. For the critical case, the nonlinearity involves a critical term \( |u|^{2^*-2}u \) growing
faster than $|u|^\frac{p}{4} u$ at infinity. Furthermore, for $N \geq 4$, we have that $2^* \leq 2 + \frac{8}{N}$, where $2 + \frac{8}{N}$ is the so called $L^2$-critical exponent for Kirchhoff equations, which is quite different from the case $N = 3$. In this paper, we study the problem for the case of $p > 2 + \frac{4}{N}$, $p = 2 + \frac{4}{N}$ and $p < 2 + \frac{4}{N}$ respectively. Firstly, we study the non-existence of normalized solutions. Secondly, we study the existence of normalized ground state solutions. Thirdly, we study normalized mountain pass type solutions and normalized local constraint minimizer with positive energy.

We introduce the following Gagliardo-Nirenberg inequality.

**Lemma 1.1.** ([41]) Let $p \in [2, \frac{2N}{N-2})$ if $N \geq 3$ and $p \geq 2$ if $N = 1, 2$. Then

$$
\|u\|_p \leq \left(\frac{p}{2\|Q\|_{L^2}^2}\right)^\frac{1}{p} \|\nabla u\|_{L^2}^{\frac{N(p-2)}{2p}} \|u\|_{L^2}^{\frac{1}{2} - \frac{N(p-2)}{4p}},
$$

(1.9)

with equality only for $u = Q$, where up to a translation, $Q$ is the unique ground state solution of

$$
-\frac{N(p-2)}{4} \Delta Q + \left(1 + \frac{(p-2)(2-N)}{4}\right) Q = |Q|^{p-2}Q, \quad x \in \mathbb{R}^N.
$$

(1.10)

Here comes our main results for $N \geq 5$.

**Theorem 1.5.** Let $N \geq 5$, $(\frac{2a}{4-2^*}) \frac{4-2^*}{2} (\frac{2b}{2^*-2}) \frac{2^*-2}{2} > \frac{1}{8N}$ and $p \in (2, 2^*)$.

(i) If $p \in \left(2 + \frac{4}{N}, 2^*\right)$, then

(i-1) there exists $c_1 \in (0, +\infty)$ such that $I_c = 0$ is not attained for $c \in (0, c_1)$, $I_c = 0$ is attained for $c = c_1$ and $I_c < 0$ is attained for $c > c_1$. In particular, for $c \geq c_1$, problem (1.7)-(1.8) has a normalized ground state solution.

(i-2) for $c \geq c_1$, problem (1.7)-(1.8) has a couple of solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ such that $\lambda_c < 0$ and $u_c$ is a mountain pass type critical point of $I|_{S_c}$.

(i-3) there exists $\bar{c} > 0$ such that for $c \in [c_1 - \bar{c}, c_1)$, problem (1.7)-(1.8) has two couples of solutions $\left(u^{(i)}_c, \lambda^{(i)}_c\right) \in H^1(\mathbb{R}^N) \times \mathbb{R}$, $i = 1, 2$ such that $\lambda^{(1)}_c < 0$ and $u^{(1)}_c$ is a mountain pass type critical point of $I|_{S_c}$; $\lambda^{(2)}_c < 0$ and $u^{(2)}_c$ is a local constraint minimizer of $I|_{S_c}$ with positive energy.

(i-4) there exists $c_0 \in (0, c_1 - \bar{c}]$ such that for $c \in (0, c_0)$, problem (1.7)-(1.8) has no normalized solution. In particular, $I|_{S_c}$ has no constraint critical points.

(ii) If $p = 2 + \frac{4}{N}$, then

(ii-1) there exists $c_1 > 0$ such that $I_c = 0$ is not attained for $c \in (0, c_1)$, $I_c = 0$ for $c = c_1$ and $I_c < 0$ is attained for $c > c_1$. In particular, for $c > c_1$, problem (1.7)-(1.8) has a normalized ground state solution.

(ii-2) there exists $c_0 \geq \|Q\|_2 \left[a - \frac{4-2^*}{2s \cdot \frac{2^*-2}{2b}} \left(\frac{2^*-2}{2b}\right)^{\frac{2^*-2}{2}}\right]^{\frac{2}{N-2}}$ such that problem (1.7)-(1.8) has no normalized solution provided $c \in (0, c_0)$. Consequently, $I|_{S_c}$ has no constraint critical points.

(iii) If $p \in \left(2, 2 + \frac{4}{N}\right)$, then $I_c < 0$ is attained for all $c > 0$. In particular, problem (1.7)-(1.8) has a normalized ground state solution.

For the case of $N = 4$, we have the following results.
Theorem 1.6. Let \( N = 4, b > \frac{1}{3p} \) and \( p \in (2, 4) \).

(i) If \( p \in (3, 4) \), then

(i-1) there exists \( c_1 \in (0, +\infty) \) such that \( I_c = 0 \) is not attained for \( c \in (0, c_1) \), \( I_c = 0 \) is attained for \( c = c_1 \) and \( I_c < 0 \) is attained for \( c > c_1 \). In particular, for \( c \geq c_1 \), problem (1.7)-(1.8) has a normalized ground state solution.

(i-2) for \( c \geq c_1 \), problem (1.7)-(1.8) has a couple of solution \( (u_c, \lambda_c) \in H^1(\mathbb{R}^4) \times \mathbb{R} \) such that \( I_c < 0 \) and \( u_c \) is a mountain pass type critical point of \( I_{\mid S_c} \).

(i-3) there exists \( \bar{c} > 0 \) such that for \( c \in [c_1 - \bar{c}, c_1] \), problem (1.7)-(1.8) has two couples of solutions \( (u_c^{(i)}, \lambda_c^{(i)}) \in H^1(\mathbb{R}^4) \times \mathbb{R} \) such that \( \lambda_c^{(1)} < 0 \) and \( u_c^{(1)} \) is a mountain pass type critical point of \( I_{\mid S_c} \); \( \lambda_c^{(2)} < 0 \) and \( u_c^{(2)} \) is a local constraint minimizer of \( I_{\mid S_c} \) with positive energy.

(i-4) there exists some \( c_0 \geq \frac{a\|Q\|_2}{(4-p)(p-2)\frac{p}{p-3}} \left[ \frac{1}{p-3} (b - \frac{1}{3p}) \right]^{\frac{2}{2-p}} \) such that problem (1.7)-(1.8) has no normalized solution provided \( c \in (0, c_0) \). In particular, \( I_{\mid S_c} \) has no constraint critical points.

(ii) If \( p = 3 \), then

(ii-1) \( I_c = 0 \) is not attained for \( c \in (0, a\|Q\|_2) \) and \( I_c < 0 \) is attained for \( c > a\|Q\|_2 \). In particular, for \( c > a\|Q\|_2 \), problem (1.7)-(1.8) has a normalized ground state solution.

(ii-2) for \( c \in (0, a\|Q\|_2) \), problem (1.7)-(1.8) has no normalized solution. In particular, \( I_{\mid S_c} \) has no constraint critical points.

(iii) If \( p \in (2, 3) \), then \( I_c < 0 \) is attained for all \( c > 0 \). In particular, problem (1.7)-(1.8) has a normalized ground state solution.

Remark 1.2. Compared with [46], we establish a more specific relationship between the range of \( c \) and the existence, non-existence, multiplicity of normalized solutions. Furthermore, we get types of normalized solutions.

The outline of this paper is as follows: in Section 2, we study the case \( N = 2 \); in Section 3, we study the case \( N \geq 5 \); in Section 4, we study the case \( N = 4 \).

Notations:

- Denote \( \|u\|_s := \left( \int_{\mathbb{R}^N} |u|^s dx \right)^{\frac{1}{s}} \), where \( 1 \leq s < \infty \).
- Denote \( H^1(\mathbb{R}^N) \) the Hilbert space with the norm
  \[ \|u\|_{H^1} := \left( \|\nabla u\|_2^2 + \|u\|_2^2 \right)^{\frac{1}{2}}. \]
- Denote \( D^{1,2}(\mathbb{R}^N) \) the Sobolev space with the norm
  \[ \|u\|_{D^{1,2}} := \|\nabla u\|_2. \]
- Denote \( C \) a universal positive constant (possibly different).

2. The case \( N = 2 \)

Let

\[ H^1_+(\mathbb{R}^N) := \{ u \in H^1(\mathbb{R}^N) : u(x) = u(|x|) \}. \]
It is well known that the embedding $H^1_r(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ is compact for all $t \in (2, 2^*)$, where $2^* = \infty$ if $N = 2$ and $2^* = \frac{2N}{N-2}$ if $N \geq 3$. To deal with the compactness of the problem, instead of $H^1(\mathbb{R}^N)$, we work in the subspace $H^1_r(\mathbb{R}^N)$.

Without loss of generality, we assume that $f(u) = 0$ for $u \leq 0$. Define the functional $I : H^1_r(\mathbb{R}^N) \rightarrow \mathbb{R}$ as follows:

$$I(u) = \frac{1}{2} \hat{M}(\|\nabla u\|_2^2) - \int_{\mathbb{R}^N} F(u)\,dx.$$  

Let $S_{r,c} := S_c \cap H^1_r(\mathbb{R}^N)$. Define

$$P_{r,c} := \{u \in S_{r,c} : G(u) = 0\},$$

where

$$G(u) = M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + N \int_{\mathbb{R}^N} F(u)\,dx - \frac{N}{2} \int_{\mathbb{R}^N} f(u)\,udx. \quad (2.1)$$

Define $H := H^1(\mathbb{R}^N) \times \mathbb{R}$ with the scalar product

$$\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_{H^1(\mathbb{R}^N)} + \langle \cdot, \cdot \rangle_\mathbb{R}.$$  

Then the norm is

$$\| \cdot \| = (\| \cdot \|_{H^1}^2 + | \cdot |_\mathbb{R}^2)^{\frac{1}{2}}.$$  

Define $T : H \rightarrow H^1(\mathbb{R}^N)$ as follows:

$$T(u, s) = e^{\frac{Ns}{2}}u(e^s x).$$

Then $\|T(u, s)\|_2 = \|u\|_2$ for all $u \in H^1(\mathbb{R}^N)$ and $s \in \mathbb{R}$.

We introduce the following Trudinger-Moser inequality.

**Lemma 2.1.** ([16, 36, 37]) If $u \in H^1(\mathbb{R}^2)$ and $\alpha > 0$, then

$$\int_{\mathbb{R}^2} \left( e^{\alpha u^2} - 1 \right) \,dx < \infty. \quad (2.2)$$

Moreover, for any fixed $\tau > 0$, there exists a constant $C > 0$ such that

$$\sup_{u \in H^1(\mathbb{R}^2) : \|\nabla u\|_2^2 + \tau \|u\|_2^2 \leq 1} \int_{\mathbb{R}^2} \left( e^{4\pi u^2} - 1 \right) \,dx \leq C. \quad (2.3)$$

**Lemma 2.2.** For any $\varepsilon > 0$, there exist $C_{1,\varepsilon}$, $C_{2,\varepsilon} > 0$ such that

$$M(t) \leq C_{1,\varepsilon} + C_{2,\varepsilon} t^{\theta + \varepsilon}, \quad \forall \ t \in \mathbb{R}^+.$$  

**Proof.** By $(M_1)$-$(M_2)$, we get

$$\frac{1}{\theta + 1} \leq \limsup_{t \to +\infty} \frac{\hat{M}(t)}{M(t)t} = \limsup_{t \to +\infty} \frac{M(t)}{M(t) + M'(t)t}. $$

Then for any $\varepsilon > 0$, there exists $T_{\varepsilon} > 0$ such that for $t \geq T_{\varepsilon}$,

$$\frac{1}{\varepsilon + \theta + 1} \leq \frac{M(t)}{M(t) + M'(t)t}.$$
from which we derive that
\[
\frac{\theta + \varepsilon}{t} \geq \frac{M'(t)}{M(t)}, \quad \forall \ t \geq T_\varepsilon.
\]
Thus,
\[
\int_{T_\varepsilon}^t \frac{\theta + \varepsilon}{s} ds \geq \int_{T_\varepsilon}^t \frac{M'(s)}{M(s)} ds.
\]
Let \( C_\varepsilon = \frac{M(T_\varepsilon)}{T_\varepsilon^{\theta+\varepsilon}} \), we have that
\[
M(t) \leq C_\varepsilon t^{\theta+\varepsilon}, \quad \forall t \geq T_\varepsilon.
\]
Together with \((M_1)\), we get the result. \(\square\)

2.1. Mountain pass structure. By the structure of \( I \), we get the following results.

**Lemma 2.3.** Let \( u \in S_{r,c} \). Then
\[
\begin{align*}
(1) & \ \|\nabla T(u,s)\|_2 \to 0 \text{ and } I(T(u,s)) \to 0 \text{ as } s \to -\infty; \\
(2) & \ \|\nabla T(u,s)\|_2 \to +\infty \text{ and } I(T(u,s)) \to -\infty \text{ as } s \to +\infty.
\end{align*}
\]

**Proof.** Obviously, \( \lim_{s \to -\infty} \|\nabla T(u,s)\|_2 = 0 \) and \( \lim_{s \to -\infty} \|T(u,s)\|_p = 0 \) for any \( p > 2 \).

Let \( \alpha > \alpha_0 \) and \( q > 4 \). By \((f_1)-(f_2)\), for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
|f(u)| \leq \varepsilon |u|^3 + C_\varepsilon |u|^q (e^{\alpha u^2} - 1), \quad \forall u \in \mathbb{R}. \tag{2.4}
\]
Then
\[
|F(u)| \leq \frac{\varepsilon}{4} |u|^4 + \frac{C_\varepsilon}{q} |u|^q (e^{\alpha u^2} - 1), \quad \forall u \in \mathbb{R}. \tag{2.5}
\]
By (2.5) and Lemma 2.1, we obtain that \( \lim_{s \to -\infty} I(T(u,s)) = 0 \). Let \( p > 2\theta + 3 \). By \((f_1)-(f_2)\), there exist \( c_1, c_2 > 0 \) such that
\[
f(u) \geq c_1 u^p - c_2 u^3, \quad \forall u \in \mathbb{R}^+. \tag{2.6}
\]
We note that \( \|\nabla T(u,s)\|^2 = e^{2s} \|\nabla u\|^2_2 \) and \( \|T(u,s)\|^{p+1} = e^{s(p-1)} \|u\|^{p+1}_{p+1} \). Then by (2.6) and Lemma 2.2, we get \( \lim_{s \to +\infty} \|\nabla T(u,s)\|_2 = +\infty \) and \( \lim_{s \to +\infty} I(T(u,s)) = -\infty \). \(\square\)

**Lemma 2.4.** There exists \( K_c > 0 \) small such that
\[
0 < \sup_{u \in A_c} I(u) < \inf_{u \in B_c} I(u),
\]
where
\[
A_c := \{u \in S_{r,c} : \|\nabla u\|^2_2 \leq K_c\}, \quad B_c := \{u \in S_{r,c} : \|\nabla u\|^2_2 = 2K_c\}.
\]
Moreover, \( I(u) > 0 \) for all \( u \in A_c \).

**Proof.** We note that
\[
I(v) - I(u) \geq \frac{1}{2} \left[ \frac{\tilde{M}(\|\nabla v\|^2_2)}{\tilde{M}(\|\nabla u\|^2_2)} - \tilde{M}(\|\nabla u\|^2_2) \right] - \int_{\mathbb{R}^2} F(v) dx, \quad \forall u \in A_c, \ \forall v \in B_c. \tag{2.7}
\]
Let \( g(t) = \tilde{M}(t + s) - \tilde{M}(t) - \tilde{M}(s) \), where \( t \in \mathbb{R}^+ \). By \((M_1)\), we get \( g'(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \). So
\[
\tilde{M}(t + s) \geq \tilde{M}(t) + \tilde{M}(s), \quad \forall \ t, s \in \mathbb{R}^+. \tag{2.8}
\]
Let $t > 1$ (close to 1), $\alpha > \alpha_0$ (close to $\alpha_0$) and $\tau \in (0,1)$ (close to 0) such that $t\alpha(\|\nabla v\|_2^2 + \tau\|v\|_2^2) < 4\pi$ for all $v \in B_c$. Let $K_c < \frac{2\pi}{\alpha_0}$ and $t' = \frac{t}{t-1}$. By Lemma 2.1, there exists $C > 0$ such that for all $v \in B_c$,

$$\int_{\mathbb{R}^2} |v|^q \left( e^{\alpha v^2} - 1 \right) \, dx \leq C\|v\|_{tv}^q. \tag{2.9}$$

By (2.5), (2.9) and Lemma 1.1, we get for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^2} F(v) \, dx \leq \varepsilon \|\nabla v\|_2^2 + C_\varepsilon \|\nabla v\|_2^{q-\frac{2}{2}}. \tag{2.10}$$

Then by (2.7)-(2.8), (2.10) and $(M_1)$, we obtain that $\forall u \in A_c, \forall v \in B_c$,

$$I(v) - I(u) \geq \frac{1}{2} M_0 K_c - 2\varepsilon K_c - C_c(2K_c)^{\frac{q}{2} - \frac{1}{2}}, \quad \forall u \in A_c, \forall v \in B_c. \tag{2.11}$$

And $\forall u \in A_c$,

$$I(u) \geq \left( \frac{1}{2} M_0 - \varepsilon \right) \|\nabla u\|_2^2 - C_c \|\nabla u\|_2^{q-\frac{2}{2}}. \tag{2.12}$$

Since $\frac{q}{2} - \frac{1}{2} > 1$, we can choose $\varepsilon$ and $K_c$ small such that $\inf_{u \in B_c} I(u) > \sup_{u \in A_c} I(u)$ and $I(u) > 0$ for all $u \in A_c$. \hfill $\square$

Let $I^0 := \{u \in S_{r,c} : I(u) \leq 0\}$. Define the mountain pass level

$$\gamma_c := \inf_{h \in \Gamma_c} \max_{t \in [0,1]} I(h(t)), \tag{2.13}$$

where

$$\Gamma_c := \left\{ h \in C([0,1], S_{r,c}) : h(0) \in A_c, h(1) \in I^0 \right\}.$$

Define the functional $J : H \to \mathbb{R}$ as follows:

$$J(u, s) = I(T(u, s)) = \frac{1}{2} M(\varepsilon^2 \|\nabla u\|_2^2) = \frac{1}{e^{Ns}} \int_{\mathbb{R}^N} F(e^{-\frac{Ns}{2}} u) \, dx.$$

Define

$$\tilde{\gamma}_c := \inf_{h \in \Gamma_c} \max_{t \in [0,1]} J(\tilde{h}(t)), \tag{2.14}$$

where

$$\tilde{\Gamma}_c := \left\{ \tilde{h} \in C([0,1], S_{r,c} \times \mathbb{R}) : \tilde{h}(0) \in (A_c, 0), \tilde{h}(1) \in (I^0, 0) \right\}.$$

Obviously, we have $\gamma_c = \tilde{\gamma}_c$. Similar to the argument of [24, Proposition 2.2 and Lemma 2.4], we can derive from Lemma 2.3 and Lemma 2.4 to get the following result.

**Lemma 2.5.** There exists $\{u_n\} \subset S_{r,c}$ such that

$$I(u_n) \to \gamma_c, \quad I'_c(u_n) \to 0, \quad G(u_n) \to 0. \tag{2.15}$$
2.2. Estimation of the mountain pass value. We consider the Moser sequence of functions

\[ \omega_n(x) = \begin{cases} 
(\log n)^{\frac{1}{2}}, & 0 \leq |x| \leq \frac{1}{n}, \\
\frac{\log |x|}{(\log n)^{\frac{1}{2}}}, & \frac{1}{n} \leq |x| \leq 1, \\
0, & |x| \geq 1.
\end{cases} \]

It is well known that \( \| \nabla \omega_n \|_2^2 = 1 \). By a direct calculation,

\[ \| \omega_n \|_2^2 = \frac{\log n}{2n^2} + \frac{1}{\log n} \int_0^1 x \log^2 x \, dx \]

\[ = \frac{\log n}{2n^2} + \frac{1}{\log n} \left( \frac{1}{4} - \frac{1}{4n^2} - \frac{\log n}{2n^2} - \frac{\log^2 n}{2n^2} \right) \]

\[ = \frac{1}{4\log n} + o \left( \frac{1}{\log^2 n} \right). \]

Define \( \omega_n = \frac{\omega_n}{\| \omega_n \|_2} \). Then we have that

\[ \| \nabla \omega_n \|_2^2 = 4c^2 \log n \left( 1 + o \left( \frac{1}{\log n} \right) \right), \tag{2.16} \]

and

\[ \omega_n(x) = \begin{cases} 
\sqrt{2}c \sqrt{\frac{n}{\pi}} \left( 1 + o \left( \frac{1}{\log n} \right) \right), & 0 \leq |x| \leq \frac{1}{n}, \\
\frac{1}{\log n} \left( 1 + o \left( \frac{1}{\log n} \right) \right), & \frac{1}{n} \leq |x| \leq 1, \\
0, & |x| \geq 1. \tag{2.17}
\]

Let

\[ g_n(t) = \frac{1}{2} \tilde{M}(t^2 \| \nabla \omega_n \|_2^2) - t^{-2} \int_{\mathbb{R}^2} F(t\omega_n) \, dx, \quad t \geq 0. \tag{2.18} \]

Fix \( n \in \mathbb{N} \). By Lemma 2.3, there exist \( s_n < -1 \) small and \( t_n > 1 \) large such that \( T(\omega_n, s_n) \in A_c \) and \( I(T(\omega_n, t_n)) < 0 \). Let \( h_n(t) := T(\omega_n, tt_n + (1-t)s_n) \), where \( t \in [0, 1] \). Then

\[ \gamma_c \leq \max_{t \in [0, 1]} I(h_n(t)) \leq \max_{t \in \mathbb{R}} I(T(\omega_n, t)) \leq \max_{t \geq 0} g_n(t). \tag{2.19} \]

Lemma 2.6. For any fixed \( n \in \mathbb{N} \), \( \max_{t \geq 0} g_n(t) > 0 \) is attained at some \( t_n > 0 \).

Proof. By (2.5) and Lemma 2.1, we have that \( g_n(0) = 0 \). Also, for \( t > 0 \) small,

\[ t^{-2} \int_{\mathbb{R}^2} F(t\omega_n) \, dx \leq \frac{\varepsilon}{4} t^2 \| \omega_n \|_4^2 + \frac{C_{\varepsilon}}{q} t^{q-2} \| \omega_n \|_2^2 \left( \int_{\mathbb{R}^2} (e^{2at \omega_n^2} - 1) \, dx \right)^{\frac{1}{2}} \]

\[ \leq C \left( \varepsilon t^2 + C_{\varepsilon} t^{q-2} \right), \tag{2.20} \]
where \( q > 4 \). By (\( M_1 \)), we get \( \hat{M}(s) \geq M_0 s \) for \( s \in \mathbb{R}^+ \). So \( g_n(t) > 0 \) for \( t > 0 \) small. By (2.6) and Lemma 2.2, we have that \( g_n(t) < 0 \) for \( t > 0 \) large. Thus, \( \max_{t \geq 0} g_n(t) > 0 \) is attained at some \( t_n > 0 \). □

**Lemma 2.7.** For \( n \) large, there holds \( \max_{t \geq 0} g_n(t) < \frac{1}{2} \hat{M} \left( \frac{4\pi}{\alpha_0} \right) \).

**Proof.** By Lemma 2.6, we get \( g_n'(t_n) = 0 \). Let \( \sigma = 2\theta + 4 \), by (\( f_4 \)),

\[
M(t_n^2 \| \nabla \omega_n \|^2_2) t_n^2 \| \nabla \omega_n \|^2_2 = -2t_n^{-2} \int_{\mathbb{R}^2} F(t_n \omega_n) dx + t_n^{-2} \int_{\mathbb{R}^2} f(t_n \omega_n) t_n \omega_n dx
\geq (\sigma - 2)t_n^{-2} \int_{B_1(0)} F(t_n \omega_n) dx. \tag{2.21}
\]

By (\( f_3 \)), we have that

\[
\lim_{t \to +\infty} \frac{F(t)}{t^{-2} e^{\alpha_0 t^2}} = \lim_{t \to +\infty} \frac{f(t)}{2\alpha_0 t^{-1} e^{\alpha_0 t^2}} \geq \frac{\beta}{2\alpha_0}.
\]

Hence, for any \( \delta > 0 \), there exists \( t_\delta > 0 \) such that for \( t \geq t_\delta \),

\[
f(t) t \geq (\beta - \delta) e^{\alpha_0 t^2}, \quad F(t) t^2 \geq \frac{\beta - \delta}{2\alpha_0} e^{\alpha_0 t^2}. \tag{2.22}
\]

If \( \lim_{n \to \infty}(t_n^2 \log n) = 0 \), by (2.16), we have that \( \frac{1}{2} \hat{M}(t_n^2 \| \nabla \omega_n \|^2_2) \to 0 \). Then \( \lim_{n \to \infty} g_n(t_n) = 0 \) and the conclusion holds. So we may assume that \( \lim_{n \to \infty}(t_n^2 \log n) = l \in (0, +\infty] \). In such a case, it is trivial that \( \lim_{n \to \infty}(t_n \log n) = +\infty \). By (2.16)-(2.17) and (2.21)-(2.22), we derive that

\[
M \left( 4c^2 \log n \left( 1 + o \left( \frac{1}{\log n} \right) \right) t_n^2 \right) \log n(1 + o(\frac{1}{\log n})) t_n^2
= M(t_n^2 \| \nabla \omega_n \|^2_2) t_n^2 \| \nabla \omega_n \|^2_2
\geq (\sigma - 2) t_n^{-2} \int_{B_1(0)} F(t_n \omega_n) dx
\geq \frac{\pi^2 (\beta - \delta)(\sigma - 2)}{4\alpha_0 c^2} e^{2\alpha_0 c^2 \pi (\log n)^2(1+o(\frac{1}{\log n}))} 2\log n \frac{1}{t_n^4 (\log n)^2(1 + o(\frac{1}{\log n}))}.
\]

If \( l = +\infty \), by Lemma 2.2, we get a contradiction by the inequality above. So \( l \in (0, +\infty) \). In particular, by the inequality above again and letting \( n \to +\infty \), we have that \( l \in \left( 0, \frac{\pi}{\alpha_0 c^2} \right] \). If \( l \in \left( 0, \frac{\pi}{\alpha_0 c^2} \right) \), then

\[
\lim_{n \to \infty} g_n(t_n) \leq \frac{1}{2} \lim_{n \to \infty} \hat{M}(t_n^2 \| \nabla \omega_n \|^2_2) < \frac{1}{2} \hat{M} \left( \frac{4\pi}{\alpha_0} \right). \tag{2.23}
\]

Now we consider the case \( l = \frac{\pi}{\alpha_0 c^2} \). Let

\[
A_n := \{ x \in B_1(0) : t_n \omega_n(x) \geq t_\delta \}.
\]
By (2.22), we have that
\[ \int_{\mathbb{R}^2} F(t_n \omega_n) dx \geq \frac{\beta - \delta}{2\alpha_0} \int_{A_n} t_n^{-2} \omega_n^{-2} e^{\alpha_0 t_n^2 \omega_n^2} dx. \]

Let \( s \in (0, \frac{1}{2}) \). By (2.17), one can see that
\[ t_n \omega_n(x) \geq t_{\delta}, \forall |x| \leq \frac{1}{n^s} \text{ uniformly for } n \text{ large enough.} \] (2.24)

So combining with (2.22), we have that
\[ t_n^{-2} \int_{\mathbb{R}^2} F(t_n \omega_n) dx \geq \frac{\beta - \delta}{2\alpha_0} \int_{B_{\frac{1}{n^s}}(0)} t_n^{-4} \omega_n^{-2} e^{\alpha_0 t_n^2 \omega_n^2} dx. \] (2.25)

By a direct calculation,
\[ \int_{B_{\frac{1}{n^s}}(0)} t_n^{-4} \omega_n^{-2} e^{\alpha_0 t_n^2 \omega_n^2} dx = \int_{|x| \leq \frac{1}{n^s}} \frac{\pi e^{2(\alpha_0 t_n^2 \log^2 n)} (1 + o((\frac{1}{\log n}))}{2 \pi t_n^4 \log^2 n(1 + o(\frac{1}{\log n}))} dx \]
\[ + \int_{\frac{1}{n} \leq |x| \leq \frac{1}{n^s}} \frac{\pi e^{2(\alpha_0 t_n^2 \log^2 n)} (1 + o((\frac{1}{\log n}))}{2 \pi t_n^4 \log^2 n(1 + o(\frac{1}{\log n}))} dx. \] (2.26)

Let \( C_n = \frac{2\alpha_0 c^2 t_n^2}{\pi} \) and \( t = C_n \log \frac{1}{x} \). Then
\[ \int_{\frac{1}{n}}^{\frac{1}{n^s}} \frac{\pi e^{2(\alpha_0 t_n^2 \log^2 n)} (1 + o((\frac{1}{\log n}))}{2 \pi t_n^4 \log^2 n(1 + o(\frac{1}{\log n}))} dx \]
\[ = C_n(1 + o(\frac{1}{\log n})) \int_{C_n \log n} e^{-2x \log n + C_n x^2 \log^2 n(1 + o(\frac{1}{\log n}))} x^{-2} dx \]
\[ \geq \frac{1 + o(\frac{1}{\log n})}{\log n} \int_{1}^{1} e^{-2x \log n + C_n x^2 \log^2 n(1 + o(\frac{1}{\log n}))} dx. \] (2.27)
Here

\[
\int_{s}^{1} e^{-2x \log n + C_n x^2 \log^2 n (1 + o(\frac{1}{\log n}))} \, dx
\]

\[
\geq \int_{c_n \log n}^{1} e^{\left(2C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n\right) x - C_n \log^2 n (1 + o(\frac{1}{\log n}))} \, dx
\]

\[
+ \int_{s}^{c_n \log n} e^{-2x \log n} \, dx
\]

\[
= e^{-C_n \log^2 n (1 + o(\frac{1}{\log n}))} - 2 \log n e^{2C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}
\]

\[
- e^{-C_n \log^2 n (1 + o(\frac{1}{\log n}))} - \frac{2}{c_n} \log n e^{2C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}
\]

\[
+ \frac{1}{2 \log n} \left( e^{-2s \log n} - e^{-\frac{2}{c_n}} \right).
\]

Moreover, by \( \lim_{n \to \infty} C_n \log n = 2 \), we obtain that for any \( \varepsilon \in (0, 1 - 2s) \), there exists \( N_1 \in \mathbb{N} \) such that for \( n > N_1 \),

\[
\int_{s}^{1} e^{-2x \log n + C_n x^2 \log^2 n (1 + o(\frac{1}{\log n}))} \, dx
\]

\[
\geq e^{C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}
\]

\[
\geq \frac{e^{-(-1+\varepsilon) \log n} + e^{-2s \log n} - e^{(-1+\varepsilon) \log n}}{2(1 - \varepsilon) \log n}
\]

\[
> \frac{e^{C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}}{2(1 + \varepsilon) \log n}.
\]

(2.28)

By (2.25)-(2.26) and (2.28), we derive that for \( n \) large,

\[
t_n^{-2} \int_{\mathbb{R}^2} F(t_n \omega_n) \, dx
\]

\[
\geq \frac{(\beta - \delta) \pi^2 e^{C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}}{4\alpha c t_n^4 (1 + \varepsilon) \log^2 n}
\]

\[
+ \frac{(\beta - \delta) \pi^2 (1 - \varepsilon) e^{C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}}{4\alpha c t_n^4 (1 + \varepsilon) \log^2 n}
\]

\[
= \frac{(2 - \varepsilon)(\beta - \delta) \pi^2 e^{C_n \log^2 n (1 + o(\frac{1}{\log n})) - 2 \log n}}{4(1 + \varepsilon) \alpha c t_n^4 \log^2 n}.
\]

(2.29)
Together with \((M_1)\), we obtain that for \(n\) large,

\[
g_n(t_n) \leq \frac{1}{2} \overline{M} \left( 4c^2 \left( 1 + o \left( \frac{1}{\log n} \right) \right) t_n^2 \log n \right) - \frac{(2 - \varepsilon)(\beta - \delta)\pi^2 e C_n \log^2 n (1 + o \left( \frac{1}{\log n} \right)) - 2 \log n}{4(1 + \varepsilon) \alpha_0 c^2 t_n^4 \log^2 n}
\]

\[
\leq \frac{1}{2} \overline{M} \left( 4c^2 t_n^2 \log n \right) + o \left( \frac{1}{\log n} \right)
\]

\[- \frac{(2 - \varepsilon)(\beta - \delta)(\alpha_0 c^2 - \varepsilon)e^{2 \alpha_0 c^2 t_n^4 \log^2 n (1 + o \left( \frac{1}{\log n} \right)) - 2 \log n}}{4(1 + \varepsilon)}.
\]

Let

\[
l_n(t) := \frac{1}{2} \overline{M} \left( 4c^2 t^2 \right) - \frac{(2 - \varepsilon)(\beta - \delta)(\alpha_0 c^2 - \varepsilon)n^{2 \alpha_0 c^2 t^2 (1 + o \left( \frac{1}{\log n} \right)) - 2}}{4(1 + \varepsilon)}.
\]

So \(g_n(t_n) \leq \sup_{t \geq 0} l_n(t) + o \left( \frac{1}{\log^2 n} \right)\). Moreover, there exists \(t_n^* > 0\) such that \(\sup_{t \geq 0} l_n(t) = l_n(t_n^*)\) and \(l_n'(t_n^*) = 0\), from which we get

\[
M \left( 4c^2(t_n^*)^2 \right)
= \frac{\alpha_0 (2 - \varepsilon)(\beta - \delta)(\alpha_0 c^2 - \varepsilon)(1 + o \left( \frac{1}{\log n} \right))}{4\pi(1 + \varepsilon)} (\log n)n^{2 \alpha_0 c^2 (t_n^*)^2 (1 + o \left( \frac{1}{\log n} \right)) - 2}.
\]

By (2.30)-(2.31), we have that

\[
g_n(t_n) \leq \frac{1}{2} \overline{M} \left( 4c^2(t_n^*)^2 \right) - \frac{\pi}{\alpha_0 \log n (1 + o \left( \frac{1}{\log n} \right))} M \left( 4c^2(t_n^*)^2 \right) + o \left( \frac{1}{\log n} \right).
\]

We claim that \(\lim_{n \to \infty} (t_n^*)^2 = \frac{\pi}{\alpha_0 c^2}\). If \(\lim_{n \to \infty} (t_n^*)^2 < \frac{\pi}{\alpha_0 c^2}\), by (2.31) and \((M_1)\), we get \(M_0 \leq 0\), a contradiction. If \(\lim_{n \to \infty} (t_n^*)^2 \in \left( \frac{\pi}{\alpha_0 c^2}, +\infty \right)\), by \((M_1)\), we get

\[
\lim_{n \to +\infty} \frac{\log n n^{2 \alpha_0 c^2 (t_n^*)^2 (1 + o \left( \frac{1}{\log n} \right)) - 2}}{M \left( 4c^2(t_n^*)^2 \right)} = +\infty,
\]

a contradiction with (2.31). Obviously, there exists \(N_0 \in \mathbb{N}\) such that for \(n \geq N_0\),

\[
n^{-\frac{\alpha_0 c^2 t_n^4}{2n}} \geq t^{2g+2}, \quad \forall \ t \geq 1.
\]

Thus, if \(\lim_{n \to \infty} (t_n^*)^2 = +\infty\), by Lemma 2.2, we get (2.33), a contradiction with (2.31). So \(\lim_{n \to \infty} (t_n^*)^2 = \frac{\pi}{\alpha_0 c^2}\). Let

\[
A_n := \frac{4\pi(1 + \varepsilon)M \left( 4c^2(t_n^*)^2 \right)}{\alpha_0 (2 - \varepsilon)(\beta - \delta)(\alpha_0 c^2 - \varepsilon)(1 + o \left( \frac{1}{\log n} \right))}.
\]
Then
\[
(t_n^*)^2 = \frac{\pi}{\alpha_0 c^2 (1 + o(\frac{1}{\log n}))} + \frac{\pi \log A_n}{2 \alpha_0 c^2 (1 + o(\frac{1}{\log n})) \log n}
\leq \frac{\pi}{\alpha_0 c^2} + o\left(\frac{1}{\log n}\right) + C \frac{\log A_n}{\log n}.
\] (2.34)

By (2.32), (2.34) and (M1), we obtain that there exists \( C' > 0 \) such that
\[
g_n(t_n) \leq \frac{1}{2} M \left( \frac{4\pi}{\alpha_0} \right) + o\left(\frac{1}{\log n}\right) - \frac{C'}{\log n}.
\] (2.35)

By choosing \( n \) large, we get \( g_n(t_n) < \frac{1}{2} M \left( \frac{4\pi}{\alpha_0} \right) \). \( \square \)

2.3. Brezis-Lieb type results.

**Lemma 2.8.** Assume that \( u_n \rightharpoonup u_0 \) weakly in \( H^1(\mathbb{R}^2) \) and \( \limsup_{n \to \infty} \|\nabla(u_n - u_0)\|^2_2 < \frac{4\pi}{\alpha_0} \).

Then for \( \alpha > \alpha_0 \) close to \( \alpha_0 \), we have that \( \{e^{\alpha u_n^2} - 1\} \) is bounded in \( L^r(\mathbb{R}^2) \) provided \( r > 1 \) close to 1.

**Proof.** We only need to prove the result for \( n \) large enough. For \( \alpha > \alpha_0 \) close to \( \alpha_0 \) and \( r > 1 \) close to 1, we still have that
\[
\limsup_{n \to \infty} \left(r\alpha \|\nabla(u_n - u_0)\|^2_2 \right) < 4\pi.
\] (2.36)

Let \( v_n = u_n - u_0 \). By choosing \( \sigma > 0 \) small enough, we have that
\[
\int_{\mathbb{R}^2} (e^{\alpha u_n^2} - 1)^r \, dx \leq \int_{\mathbb{R}^2} (e^{\alpha u_0^2} - 1) \, dx
= \int_{\mathbb{R}^2} (e^{\alpha (u_0 + v_n)^2} - 1) \, dx
\leq \int_{\mathbb{R}^2} (e^{\alpha [1 + (1 + \frac{1}{\sigma})u_0^2]} - 1) \, dx
= \int_{\mathbb{R}^2} (e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1) \left(e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1\right) \, dx + \int_{\mathbb{R}^2} (e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1) \, dx
+ \int_{\mathbb{R}^2} (e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1) \, dx.
\]

By choosing \( \eta > 1 \) close to 1, we derive from (2.36), Lemma 2.1 and the Hölder inequality to obtain some \( C > 0 \) independent of \( n \) such that
\[
\left\{ \int_{\mathbb{R}^2} (e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1) \left(e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1\right) \, dx \leq \|e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1\|_0 \|e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1\|_\eta' \leq C, \right.
\left\{ \int_{\mathbb{R}^2} (e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1) \, dx \leq C, \right.
\left\{ \int_{\mathbb{R}^2} (e^{\alpha (1 + \frac{1}{\sigma})u_0^2} - 1) \, dx \leq C, \right.
\]
where \( \eta' = \frac{\eta}{\eta - 1} \). Hence, \( \{e^{\alpha u_n^2} - 1\} \) is bounded in \( L^r(\mathbb{R}^2) \). \( \square \)

Then we have the following Brézis-Lieb type result.
Lemma 2.9. Assume that \( \{u_n\} \subset H^1(\mathbb{R}^2) \) such that \( u_n \rightharpoonup u_0 \) weakly in \( H^1(\mathbb{R}^2) \) and \( \limsup_{n \to \infty} \|\nabla(u_n - u_0)\|^2 \leq \frac{4\pi}{\alpha_0} \). Under the assumptions \((f_1)\) and \((f_2)\), we have that

\[
\int_{\mathbb{R}^2} F(u_n)dx = \int_{\mathbb{R}^2} F(u_0)dx + \int_{\mathbb{R}^2} F(u_n - u_0)dx + o_n(1). \tag{2.37}
\]

Suppose further that \( f \in C^1(\mathbb{R}) \), and for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
|f'(s)| \leq \varepsilon |s|^2 + C_\varepsilon |s|^{q-2} \left( e^{\alpha|s|^2} - 1 \right), \quad \forall \ s \in \mathbb{R}, \tag{2.38}
\]

where \( \alpha > \alpha_0 \) and \( q > 4 \), then we also have that

\[
\int_{\mathbb{R}^2} f(u_n)u_n dx = \int_{\mathbb{R}^2} f(u_0)u_0 dx + \int_{\mathbb{R}^2} f(u_n - u_0)(u_n - u_0)dx + o_n(1). \tag{2.39}
\]

Proof. We may assume that \( u_n \to u_0 \) a.e. in \( \mathbb{R}^2 \). By the mean value theorem, there exists \( \theta_n(x) \in [0, 1] \) such that

\[
\int_{\mathbb{R}^2} |F(u_n) - F(u_n - u_0)|dx = \int_{\mathbb{R}^2} |f(u_n - (1 - \theta_n)u_0)u_0|dx. \tag{2.40}
\]

Let \( \alpha > \alpha_0 \) and \( q > 3 \). By \((f_1)\)-\((f_2)\), for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
|f(u_n - (1 - \theta_n)u_0)u_0| \leq C_\varepsilon |u_n - (1 - \theta_n)u_0|^{q-1}\left(e^{\alpha|u_n - (1 - \theta_n)u_0|^2} - 1\right)|u_0| + \varepsilon |u_n - (1 - \theta_n)u_0|^2|u_0|. \tag{2.41}
\]

By Lemma 2.8, \( \{e^{au^2} - 1\} \) is bounded in \( L^r(\mathbb{R}^2) \) provided \( \alpha > \alpha_0 \) close to \( \alpha_0 \) and \( r \in (1, 2) \) close to 1. Then it is easy to see that \( \{e^{\alpha|u_n - (1 - \theta_n)u_0|^2} - 1\} \) is bounded in \( L^{r_1}(\mathbb{R}^2) \) provided \( 1 < r_1 < r \). We claim that \( \{f(u_n - (1 - \theta_n)u_0)\} \) is bounded in \( L^\tau(\mathbb{R}^2) \) for any \( 1 < \tau < r_1 \). Indeed, by (2.41), we only need to prove that \( \{u_n - (1 - \theta_n)u_0|^{q-1}\left(e^{\alpha|u_n - (1 - \theta_n)u_0|^2} - 1\right)\} \) is bounded in \( L^\tau(\mathbb{R}^2) \) for any \( 1 < \tau < r_1 \). By the Sobolev embedding \( H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2) \) for any \( p \geq 2 \) and the Hölder inequality, we have that

\[
\int_{\mathbb{R}^2} |u_n - (1 - \theta_n)u_0|^{(q-1)\tau}' (e^{\alpha|u_n - (1 - \theta_n)u_0|^2} - 1) \frac{dx}{x_1} \leq C \|e^{\alpha|u_n - (1 - \theta_n)u_0|^2} - 1\|_{r_1}^\tau < \infty,
\]

and the claim is proved.

Noting that \( u_0 \in L^{\tau'}(\mathbb{R}^2) \) with \( \tau' = \frac{r}{r-1} > 2 \), we have that \( \{f(u_n - (1 - \theta_n)u_0)u_0\} \) is bounded in \( L^1(\mathbb{R}^2) \). Define \( B^c_R := \{x \in \mathbb{R}^2 : |x| > R\} \). Then

\[
\int_{B^c_R} |f(u_n - (1 - \theta_n)u_0)u_0|dx \leq \|f(u_n - (1 - \theta_n)u_0)\|_r \|u_0\|_{L^{\tau'}(B^c_R)} \to 0, \tag{2.42}
\]

uniformly in \( n \) as \( R \to +\infty \). Furthermore, for any \( \Lambda \subset \mathbb{R}^2 \), we have that

\[
\int_{\Lambda} |f(u_n - (1 - \theta_n)u_0)u_0|dx \leq \|f(u_n - (1 - \theta_n)u_0)\|_{L^\tau(\Lambda)} \|u_0\|_{L^{\tau'}(\Lambda)} \to 0, \tag{2.43}
\]
uniformly in $n$ as $\text{meas}(\Lambda) \to 0$. So $\{F(u_n) - F(u_n - u_0)\}$ possesses the uniform integrability condition, and then by the well known Vitali’s convergence theorem, noting that $F(0) = 0$, we obtain that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} [F(u_n) - F(u_n - u)]dx = \int_{\mathbb{R}^2} \lim_{n \to \infty} [F(u_n) - F(u_n - u)]dx = \int_{\mathbb{R}^2} F(u)dx,$$  \hspace{1cm} (2.44)

and the result of (2.37) holds.

Similarly, if $f \in C^1$ satisfies (2.38), applying a similar argument as above, one can prove (2.39).

**Corollary 2.1.** Assume that $\{u_n\} \subset H^1_r(\mathbb{R}^2)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1_r(\mathbb{R}^2)$ and

$$\limsup_{n \to \infty} \|\nabla (u_n - u_0)\|^2 < \frac{4\pi}{\alpha_0}.\hspace{1cm} \text{Under the assumptions } (f_1) \text{ and } (f_2), \text{ we have that}$$

$$\int_{\mathbb{R}^2} f(u_n)u_n \, dx = \int_{\mathbb{R}^2} f(u_0)u_0 \, dx + o_n(1), \hspace{1cm} (2.45)$$

and

$$\int_{\mathbb{R}^2} F(u_n) \, dx = \int_{\mathbb{R}^2} F(u_0) \, dx + o_n(1). \hspace{1cm} (2.46)$$

**Proof.** Recalling (2.4), by Lemma 2.8, one can prove that $\{f(u_n)\}$ is bounded in $L^\tau(\mathbb{R}^2)$ provided $1 < \tau < \min\{r, 2\}$. By the radial compact embedding, we have that $u_n \rightharpoonup u_0$ in $L^{\tau'}(\mathbb{R}^2)$, where $\tau' = \frac{\tau}{\tau - 1} > 2$. Thus,

$$\int_{\mathbb{R}^2} |f(u_n)(u_n - u_0)| \, dx \leq \|f(u_n)\|_\tau \|u_n - u_0\|_{\tau'} = o_n(1). \hspace{1cm} (2.47)$$

On the other hand, up to a subsequence, $f(u_n) \rightharpoonup f(u_0)$ weakly in $L^\tau(\mathbb{R}^2)$, which implies that

$$\int_{\mathbb{R}^2} [f(u_n) - f(u_0)]u_0 \, dx = o_n(1). \hspace{1cm} (2.48)$$

Hence,

$$\left| \int_{\mathbb{R}^2} f(u_n)u_n - f(u_0)u_0 \, dx \right| \leq \left| \int_{\mathbb{R}^2} [f(u_n) - f(u_0)]u_0 \, dx \right| + \left| \int_{\mathbb{R}^2} f(u_n)(u_n - u_0) \, dx \right| = o_n(1)$$

and (2.45) is proved.

Basing on Lemma 2.9, to prove (2.46), it is sufficient to prove that

$$\int_{\mathbb{R}^2} F(u_n - u_0) \, dx = o_n(1).$$

Recalling (2.5), we only need to prove that

$$\int_{\mathbb{R}^2} |u_n - u_0|^q \left( e^{\alpha(u_n - u_0)^2} - 1 \right) \, dx = o_n(1). \hspace{1cm} (2.49)$$

Indeed, by (2.36), Lemma 2.1 and the Hölder inequality, there exists $C > 0$ independent of $n$ such that

$$\int_{\mathbb{R}^2} |u_n - u_0|^q \left( e^{\alpha(u_n - u_0)^2} - 1 \right) \, dx \leq \|u_n - u_0\|^q \|e^{\alpha(u_n - u_0)^2} - 1\|_r \leq C \|u_n - u_0\|^q \|\cdot\|_{r'},$$
where \( r' = \frac{r}{1 - r} \). By the radial compact embedding again, we have that \( u_n \to u_0 \) in \( L^{q'}(\mathbb{R}^2) \) and thus (2.49) holds. \( \square \)

2.4. PSP-condition.

**Lemma 2.10.** Assume that \( \{u_n\} \subset H^1_0(\mathbb{R}^2) \) such that \( u_n \to u_0 \) weakly in \( H^1_0(\mathbb{R}^2) \) and \( \int_{\mathbb{R}^2} f(u_n)u_n \, dx \) is bounded. Under the assumptions \((f_1)-(f_2)\) and \((f_5)\), we have that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} F(u_n) \, dx = \int_{\mathbb{R}^2} F(u_0) \, dx. \tag{2.50}
\]

**Proof.** By \((f_5)\), for any \( \varepsilon > 0 \), there exists \( M_\varepsilon > 0 \) such that

\[
F(s) \leq \varepsilon f(s) s, \quad \forall |s| \geq M_\varepsilon.
\]

Moreover, by \((f_1)\), we can find some \( C_\varepsilon > 0 \) such that

\[
F(s) \leq \varepsilon f(s) s + C_\varepsilon |s|^4, \quad \forall s \in \mathbb{R}. \tag{2.51}
\]

Then

\[
\int_{\mathbb{R}^2} F(u_n) \, dx \leq \varepsilon \int_{\mathbb{R}^2} f(u_n) u_n \, dx + C_\varepsilon \| u_n \|_4^4.
\]

So \( \{\int_{\mathbb{R}^2} F(u_n) \, dx\} \) is bounded. For any \( \Omega \subset \mathbb{R}^2 \), by \( u_n \to u_0 \) in \( L^4(\mathbb{R}^2) \),

\[
\int_{\Omega} F(u_n) \, dx \leq \int_{\Omega} \varepsilon f(u_n) u_n \, dx + C_\varepsilon \int_{\Omega} |u_n|^4 \, dx
\]

\[
\leq C_\varepsilon + o_n(1) + C_\varepsilon \int_{\Omega} |u_0|^4 \, dx.
\]

Then by the arbitrary of \( \varepsilon \), it is easy to see that \( \int_{\mathbb{R}^2} F(u_n) \, dx \) satisfies the uniform absolute continuity, and thus by the Vitali’s convergence theorem, we get (2.50). \( \square \)

We establish the following so-called PSP-condition.

**Lemma 2.11.** If \( \{u_n\} \subset S_{r,c} \) is a \((PSP)_d\) sequence with \( d < \frac{1}{2} \hat{M} \left( \frac{4\pi}{\omega_0} \right) \), \( d \neq 0 \), i.e.,

\[
I(u_n) \to d < \frac{1}{2} \hat{M} \left( \frac{4\pi}{\omega_0} \right), \quad d \neq 0, \quad I|_{S_{r,c}}'(u_n) \to 0 \quad \text{and} \quad G(u_n) \to 0,
\]

then \( \{u_n\} \) converges strongly in \( H^1_r(\mathbb{R}^2) \) up to a subsequence.

**Proof.** Define the functional \( \varphi(u) = \frac{1}{2} ||u||_2^2 \), where \( u \in H^1_r(\mathbb{R}^2) \). By \( I|_{S_{r,c}}'(u_n) \to 0 \), we get there exists \( \{\lambda_n\} \subset \mathbb{R} \) such that

\[
||I'(u_n) - \lambda_n \varphi'(u_n)|| \to 0. \tag{2.52}
\]

By \( I(u_n) \to d, G(u_n) \to 0 \) and \((f_4)\),

\[
d \geq \frac{1}{2} \hat{M}(||\nabla u_n||_2^2) - \frac{1}{2(\theta + 1)} M(\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 + o_n(1).
\]

So by \((M_2)\), we get \( ||\nabla u_n||_2 \) is bounded. Moreover, \( ||u_n||_{H^1_r}, \int_{\mathbb{R}^2} F(u_n) \, dx, \int_{\mathbb{R}^2} f(u_n) u_n \, dx \) and \( |\lambda_n| \) are bounded. Going to a subsequence, we may assume that \( u_n \to u_0 \) weakly in \( H^1_r(\mathbb{R}^2) \) and \( \lambda_n \to \lambda_0 \) as \( n \to \infty \).

We consider two cases.
Case 1: $u_n \rightharpoonup 0$ weakly in $H^1_r(\mathbb{R}^2)$.

By Lemma 2.10, we have that $\int_{\mathbb{R}^2} F(u_n)dx \rightarrow 0$. Then $d = \frac{1}{2} \overline{M} (\lim_{n \rightarrow \infty} \|\nabla u_n\|_2^2)$. Together with $(M_1)$, we get
\[
\lim_{n \rightarrow \infty} \|\nabla u_n\|_2^2 < \frac{4\pi}{\alpha_0}. \tag{2.53}
\]

By Corollary 2.1, we have that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} f(u_n)u_n dx = 0$. Moreover, by $G(u_n) \rightarrow 0$, we obtain that $\|\nabla u_n\|_2 \rightarrow 0$, a contradiction with $d \neq 0$.

Case 2: $u_n \rightharpoonup u_0 \neq 0$ weakly in $H^1_r(\mathbb{R}^2)$.

Define
\[
D = \lim_{n \rightarrow \infty} \frac{\overline{M}(\|\nabla u_n\|_2^2)}{\|\nabla u_n\|_2^2}, \quad E = \lim_{n \rightarrow \infty} M(\|\nabla u_n\|_2^2). \tag{2.54}
\]

Let $v_n = u_n - u_0$. By Lemma 2.10, we have that
\[
d \geq \frac{D}{2} \left( \lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2 + \|\nabla u_0\|_2^2 \right) - \int_{\mathbb{R}^2} F(u_0)dx. \tag{2.55}
\]

By (2.52), $G(u_n) \rightarrow 0$ and Lemma 2.10, we derive that in the weak sense,
\[
-E \Delta u_0 - \lambda_0 u_0 = f(u_0) \text{ in } \mathbb{R}^2, \tag{2.56}
\]

with
\[
\lambda_0 = -\frac{2}{c^2} \int_{\mathbb{R}^2} F(u_0)dx < 0. \tag{2.57}
\]

Then we have the Pohozaev identity: $\lambda_0 \|u_0\|_2^2 = -2 \int_{\mathbb{R}^2} F(u_0)dx$. So $u_0 \in S_{r,c}$ and thus $u_n \rightharpoonup u_0$ in $L^2(\mathbb{R}^2)$. Moreover,
\[
E\|\nabla u_0\|_2^2 + 2 \int_{\mathbb{R}^2} F(u_0)dx - \int_{\mathbb{R}^2} f(u_0)u_0 dx = 0. \tag{2.58}
\]

By (2.58), $(M_2)$ and $(f_4)$,
\[
\frac{D}{2} \|\nabla u_0\|_2^2 - \int_{\mathbb{R}^2} F(u_0)dx \geq \frac{D}{2E} \int_{\mathbb{R}^2} \left[ \frac{1}{2} f(u_0)u_0 - \left( 1 + \frac{E}{D} \right) F(u_0) \right] dx \\
\geq \frac{D}{2E} \int_{\mathbb{R}^2} [f(u_0)u_0 - (2\theta + 4)F(u_0)] dx \geq 0. \tag{2.59}
\]

By $(M_1)$, we get $\frac{\overline{G}(t)}{t}$ is increasing for $t > 0$. So by (2.55) and (2.59),
\[
d \geq \frac{D}{2} \lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2 \geq \lim_{n \rightarrow \infty} \overline{M}(\|\nabla v_n\|_2^2). \tag{2.60}
\]

Moreover,
\[
\lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2 < \frac{4\pi}{\alpha_0}. \tag{2.61}
\]

Then by Corollary 2.1, we have that
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} f(u_n)u_n dx = \int_{\mathbb{R}^2} f(u_0)u_0 dx. \tag{2.62}
\]
Now, by (2.50), (2.58), (2.62) and the fact of $G(u_n) \to 0$, we have that $\|\nabla u_n\|_2^2 \to \|\nabla u_0\|_2^2$, which implies that $u_n \to u_0$ in $D^{1,2}(\mathbb{R}^2)$. Hence, $u_n \to u_0$ in $H^1_0(\mathbb{R}^2)$. 

2.5. Proof of Theorem 1.1. By Lemma 2.7 and (2.19), we have that $\gamma_c < \frac{1}{2} \hat{M} \left( \frac{4\pi}{\alpha_0} \right)$. By Lemma 2.5, there exists a $(PSP)_{c_0}$-sequence. By Lemma 2.4, we have that $\gamma_c > 0$. Then by Lemma 2.11, we get the result. 

Remark 2.1. For any $0 < a \leq b < +\infty$, define

$$U_{r,a}^b := \{ u \in S_{r,c} : I(u) = \gamma_c, I'(u) = 0, c \in [a,b] \}.$$ (2.63)

which is compact in $H^1_0(\mathbb{R}^2)$. Indeed, it is standard to prove that $\gamma_c$ depends continuously on $c > 0$. On the other hand, by Lemma 2.7, $\gamma_c < \frac{1}{2} \hat{M} \left( \frac{4\pi}{\alpha_0} \right)$ for all $c \in [a,b]$. Then similar to the proof of Lemma 2.11, one can prove the conclusion.

3. The case $N \geq 5$

3.1. Some preliminaries. Define the best Sobolev constant:

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} |u|^{2^*} \, dx)^{\frac{2}{2^*}}}.$$ (3.1)

Consider the case of $M(t) = a + bt$, $a > 0$ and $f(u) = |u|^{p-2}u$, $p \in (2,2^*)$. In such a case, the corresponding energy functional given by (1.4) is

$$I(u) = \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{1}{p} \|u\|_p^p - \frac{1}{2^*} \|u\|_{2^*}^{2^*}.$$

By (3.1) and Lemma 1.1, we have that

$$I(u) \geq \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{c^{p-N(p-2)}}{2\|Q\|_{2^*}^2} \|\nabla u\|_2^{N(p-2)} - \frac{1}{2^* S^{\frac{2}{2^*}}} \|\nabla u\|_2^{2^*}$$ (3.2)

holds for all $u \in S_c$.

Lemma 3.1. Let $\kappa_1, \kappa_2 > 0$, $0 < p_1 < p_2 < +\infty$, $\kappa_3 \geq 0$, $\kappa_4 \geq 0$ and $(\kappa_3, \kappa_4) \neq (0,0)$. For $p_3, p_4 \in (p_1, p_2)$, we define

$$\Omega^p_{\kappa_1, \kappa_2, \kappa_3, \kappa_4} := \inf_{t > 0} \frac{\kappa_1 t^{p_1} + \kappa_2 t^{p_2}}{\kappa_3 t^{p_3} + \kappa_4 t^{p_4}}.$$ (3.3)

Then $\Omega^p_{\kappa_1, \kappa_2, \kappa_3, \kappa_4} > 0$ is attained. Furthermore,

(i) $\Omega^p_{\kappa_1, \kappa_2, \kappa_3, \kappa_4}$ is continuous with respect to $\kappa_i \in (0, +\infty)$, $i = 1, 2, 3, 4$.

(ii) $\Omega^p_{\kappa_1, \kappa_2, \kappa_3, \kappa_4}$ increases strictly for $\kappa_i \in (0, +\infty)$, $i = 1, 2$.

(iii) $\Omega^p_{\kappa_1, \kappa_2, \kappa_3, \kappa_4}$ decreases strictly for $\kappa_i \in (0, +\infty)$, $i = 3, 4$. In particular,

$$\lim_{\kappa_i \to +\infty} \Omega^p_{\kappa_1, \kappa_2, \kappa_3, \kappa_4} = 0, i = 3, 4.$$ (3.4)

Proof. By a direct calculation, we get the result. 

□
Remark 3.1. By the definition of $\Omega_{x_1,x_2,x_3,x_4}$, one can see that
\[ \kappa_1 t^{p_1} + \kappa_2 t^{p_2} \geq \Omega_{x_1,x_2,x_3,x_4} (\kappa_3 t^{p_3} + \kappa_4 t^{p_4}), \forall t \geq 0. \]
In particular, for $\kappa_1 = A > 0$, $\kappa_2 = B > 0$, $\kappa_3 = 0$, $\kappa_4 = 1$ and $p_1 = 2$, $p_2 = 4$, $p_3 = 2$, $p_4 = (2, 4)$, a direct computation shows that
\[ \Omega_{a,b,0,1}^{2,4,p,q} = 2(q - 2)^{-(q-1)}(4 - q)^{-(2-q)}A^{2-q}B^{2-q}. \] (3.5)
And we note that
\[ \Omega_{a,b,0,1}^{2,4,p,q} = \frac{1}{\kappa_4} \Omega_{a,b,0,1}^{2,4,p,q} = \frac{1}{\kappa_4} \left[ 2(q - 2)^{-(q-1)}(4 - q)^{-(2-q)}A^{2-q}B^{2-q} \right]. \] (3.6)
In the present paper, we take $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \left( \frac{a}{2}, \frac{b}{4}, 0, \frac{1}{2+\frac{2}{S_x}} \right)$ or $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (a, b, 0, S^{-2/2})$ as applications. We also note that
\[ \Omega_{a,b,0,1}^{2,4,p,q} > 1 \] since
\[ 2^{2-q-1}4^{2-q}2^q \geq 2^{2-q} > 1 \] for $N \geq 5$.
In particular, by (3.6), \( \Omega_{a,b,0,1}^{2,4,p,q} > 1 \) equivalents to
\[ \left( \frac{2a}{4} \right)^{4-2^q} \left( \frac{2b}{2^{q+2}} \right)^{2^q-2} > \frac{1}{S^{2^q}}. \] (3.7)
\[ \square \]

Lemma 3.2. For $c > 0$, let $I_c$ be defined by (1.6). If $N \geq 5$ and $p \in (2, 2^*)$, then
\begin{enumerate}
  \item $-\infty < I_c \leq 0$ for all $c > 0$;
  \item $I_c$ is non-increasing on $c \in (0, +\infty)$;
  \item $I_c$ is continuous on $c \in (0, +\infty)$.
\end{enumerate}

Proof. (i) Since $2^* < 2 + \frac{8}{N}$ for $N \geq 5$, we have that $\frac{N(p-2)}{2} < 4$. So by (3.2), we get
\[ \lim_{||\nabla u||_2 \to +\infty} I(u) = +\infty. \] (3.9)
Moreover, $I_c > -\infty$ for all $c > 0$. In particular, fix $u \in S_c$ and let $u^t(x) := t^\frac{N}{2}u(tx)$, $t > 0$. Then
\[ I_c \leq I(u^t) \leq \frac{a}{2} t^{2}\|\nabla u\|_2^2 + \frac{b}{4} t^4 \|\nabla u\|_2^4. \]
Let $t \to 0^+$, we get $I_c \leq 0$.
(ii) Let $c_2 > c_1$. By the definition of $I_{c_1}$, there exists $\{u_n\} \subset S_{c_1}$ such that
\[ I_{c_1} \leq I(u_n) \leq I_{c_1} + \frac{1}{n}. \]
Let
\[ v_n(x) := \left( \frac{c_1}{c_2} \right)^{\frac{N}{2}} u_n \left( \frac{c_1}{c_2} x \right). \]
Then \( v_n \in S_{c_2} \). In particular, it holds that
\[
\|\nabla v_n\|_2 = \|\nabla u_n\|_2, \quad \|v_n\|_{2^*} = \|u_n\|_{2^*} \quad \text{and} \quad \|v_n\|_p = \left( \frac{c_2}{c_1} \right)^{N-N-2,p} \|u_n\|_p.
\]
So
\[
I_{c_2} \leq I(v_n) = I(u_n) + \frac{1}{p} \left[ 1 - \left( \frac{c_2}{c_1} \right)^{N-N-2,p} \right] \|u_n\|_p < I(u_n) \leq I_{c_1} + \frac{1}{n}. \tag{3.10}
\]

By letting \( n \to +\infty \), we obtain that \( I_{c_2} \leq I_{c_1} \).

(iii) For any \( c > 0 \) and any sequence \( \{c_n\} \subset \mathbb{R}^+ \) with \( c_n \to c \) as \( n \to \infty \), by the definition of \( I_{c_n} \), there exists \( \{u_n\} \subset S_{c_n} \) such that \( I_{c_n} \leq I(u_n) \leq I_{c_n} + \frac{1}{n} \). Let
\[
v_n := u_n(\theta_n^\frac{2}{c_n^2} x), \quad \text{where} \quad \theta_n = \frac{c^2}{c_n^2} = 1 + o_n(1).
\]
Then \( v_n \in S_c \). Moreover,
\[
\|\nabla v_n\|_2 = \theta_n^\frac{2}{c_n^2} \|\nabla u_n\|_2, \quad \|v_n\|_p = \theta_n^2 \|u_n\|_p \quad \text{and} \quad \|v_n\|_{2^*} = \theta_n^2 \|u_n\|_{2^*}.
\]

By \( I_c \leq 0 \) for all \( c > 0 \) and (3.9), one can see that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). So
\[
I_c \leq I(v_n) = \frac{a}{2} \theta_n^{2-N} \|\nabla u_n\|_2^2 + \frac{b}{4} \theta_n^{4-N} \|\nabla u_n\|_4^4 - \frac{1}{p} \theta_n^2 \|u_n\|_p^p - \frac{1}{2^*} \theta_n^2 \|u_n\|_{2^*}^{2^*} = I(u_n) + o_n(1).
\]

Let \( n \to \infty \), we get \( I_c \leq \liminf_{n \to \infty} I_{c_n} \).

Similarly, for any \( n \in \mathbb{N} \), there exists some \( u_n \in S_c \) such that \( I_c \leq I(u_n) \leq I_c + \frac{1}{n} \). Then
\[
v_n(x) := u_n(\theta_n^\frac{2}{c_n^2} x) \in S_{c_n}, \quad \text{where} \quad \theta_n = \frac{c_n^2}{c^2} = 1 + o_n(1).
\]
So \( I_{c_n} \leq I(v_n) = I(u_n) + o_n(1) \), and thus \( \limsup_{n \to \infty} I_{c_n} \leq I_c \). \( \square \)

**Corollary 3.1.** Let \( N \geq 5 \) and \( p \in (2, 2^*) \). If there exists \( 0 < \underline{c} < \bar{c} < +\infty \) such that \( I_{\bar{c}} = I_c \), then \( I_c \) is not attained provided \( c \in [\underline{c}, \bar{c}] \).

**Proof.** By (ii) of Lemma 3.2, we have that \( I_c \equiv I_{\bar{c}}, \forall c \in [\underline{c}, \bar{c}] \). Suppose that there exists some \( c \in [\underline{c}, \bar{c}] \) which is attained by some \( u \in S_c \). That is, \( I(u) = I_c = I_{\bar{c}} \). Define
\[
v(x) := \left( \frac{\bar{c}}{c} \right)^{N-2} u \left( \frac{c}{\bar{c}} x \right),
\]
then \( v \in S_{\bar{c}} \) and thus
\[
I_{\bar{c}} \leq I(v) = I(u) + \frac{1}{p} \left[ 1 - \left( \frac{\bar{c}}{c} \right)^{N-N-2,p} \right] \|u\|_p^p < I(u) = I_c,
\]
a contradiction. \( \square \)

**Lemma 3.3.** Let \( N \geq 5 \) and \( p \in (2, 2^*) \). Then \( I_c < 0 \) for \( c > 0 \) large enough. And thus we can define that
\[
c_1 := \inf \{ c > 0 : I_c < 0 \} \in [0, +\infty). \tag{3.11}
\]
Furthermore, if \( c_1 > 0 \), then \( I_c \equiv 0 \) for \( c \in (0, c_1] \) and \( I_c \) is not attained provided \( c \in (0, c_1] \).
Proof. Recalling Lemma 1.1,
\[
\|\nabla Q\|_2^2 = \|Q\|_2^2 = \frac{2}{p}\|Q\|_p^p.
\]
(3.12)

Let \( Q_t(x) := \frac{a^t}{\|Q\|_2^2} \), then by (3.12),
\[
I(Q_t) = \frac{a}{2} c^2 t^2 + \frac{b}{4} c^4 t^4 - \frac{c^t t^{\frac{N_p}{2}} - N}{2\|Q\|_{p-2}^2} - \frac{c^2\|Q\|_{2}^{2} t^{2^*}}{2^*\|Q\|_{2}^{2^*}}.
\]
(3.13)

If \( 2 < p < 2 + \frac{4}{N} \), then \( 0 < \frac{N_p}{2} - N < 2 \). So for any \( c > 0 \), it is easy to see that \( I(Q_t) < 0 \) for \( t > 0 \) small enough, and thus \( I_c < 0 \) for any \( c > 0 \). Hence, \( c_1 = 0 \).

If \( p = 2 + \frac{4}{N} \), put \( s = ct \), then
\[
I(Q_t) = \left[ \frac{a}{2} - \frac{c^t t^{\frac{N_p}{2}}}{2\|Q\|_{p-2}^2} \right] s^2 + \frac{b}{4} s^4 - \frac{\|Q\|_{2}^{2^*} s^{2^*}}{2^*\|Q\|_{2}^{2^*}}.
\]

Then \( \frac{a}{2} - \frac{c^t t^{\frac{N_p}{2}}}{2\|Q\|_{p-2}^2} < 0 \) provided \( c > c_0 := \left( \frac{\|Q\|_{p-2}^2}{2} \right)^{\frac{1}{N_p}} \). So for \( t > 0 \) small enough, we have that \( s > 0 \) small enough and then \( I_c \leq I(Q_t) < 0 \) for \( c > c_0 \). Hence, \( c_1 \in [0, +\infty) \) is also well defined.

If \( p \in (2 + \frac{4}{N}, 2^*) \), we rewrite
\[
I(Q_t) = \frac{a}{2} s^2 + \frac{b}{4} s^4 - \frac{t^{-(N+p-\frac{N_p}{2})}}{2\|Q\|_{p-2}^2} s^p - \frac{\|Q\|_{2}^{2^*} s^{2^*}}{2^*\|Q\|_{2}^{2^*}}, \text{ with } s = ct.
\]
(3.14)

Noting that in such a case, Lemma 3.1 is applied and by (3.4), there exists \( \kappa_3 > 0 \) large enough such that
\[
\Omega_{\frac{a}{2}, \frac{b}{4}, \kappa_3, \frac{\|Q\|_{p-2}^2}{2^*\|Q\|_{2}^{2^*}}} < 1.
\]
(3.15)

In particular, by the definition of \( \Omega_{\frac{a}{2}, \frac{b}{4}, \kappa_3, \frac{\|Q\|_{p-2}^2}{2^*\|Q\|_{2}^{2^*}}} \), there exists some \( s_0 > 0 \) such that
\[
\frac{a}{2} s_0^2 + \frac{b}{4} s_0^4 - \kappa_3 s_0^p - \frac{\|Q\|_{2}^{2^*} s_0^{2^*}}{2^*\|Q\|_{2}^{2^*}} < 0.
\]
(3.16)

Since \( p < 2^* \), there exists some \( c_0 > 1 \) large enough such that \( \frac{(\frac{a}{2}) - (N+p-\frac{N_p}{2})}{2\|Q\|_{p-2}^2} > \kappa_3 \). Then
for any \( c > c_0 \), by \( t_0 = \frac{s_0}{c} < \frac{s_0}{c_0} \), we have that \( \frac{t_{0}^{-(N+p-\frac{N_p}{2})}}{2\|Q\|_{p-2}^2} > \kappa_3 \), and thus
\[
I_c \leq I(Q_{t_0}) < \frac{a}{2} s_0^2 + \frac{b}{4} s_0^4 - \kappa_3 s_0^p - \frac{\|Q\|_{2}^{2^*} s_0^{2^*}}{2^*\|Q\|_{2}^{2^*}} < 0, \quad \forall c > c_0.
\]

It is trivial that \( c_1 \geq 0 \). If \( c_1 > 0 \), by Lemma 3.2, one can see that
\[
I_c \begin{cases} 
0, & \text{if } c \in (0, c_1), \\
< 0, & \text{if } c > c_1.
\end{cases}
\]

And thus, by Corollary 3.1, \( I_c \) is not attained for \( c \in (0, c_1) \).
To establish the existence result, we need the following \(PSP\)-condition.

**Lemma 3.4.** Let \(N \geq 5\), \(p \in (2, 2^*)\) and \(\Omega^{2.4, p, 2^*}_{a,b,0, \frac{1}{s^2}} > 1\). Assume that \(\{u_n\} \subset \mathcal{S}_{r,c}\) is a \((PSP)\) sequence with \(d \neq 0\), i.e.,

\[
I(u_n) \to d \neq 0, \quad I'_{S_{r,c}}(u_n) \to 0 \quad \text{and} \quad G(u_n) \to 0,
\]

then \(\{u_n\}\) converges strongly in \(H^1_r(\mathbb{R}^N)\) up to a subsequence.

**Proof.** By (3.9), we get \(\|\nabla u_n\|_2\) is bounded. And thus, \(\{u_n\}\) is bounded in \(H^1_r(\mathbb{R}^N)\). Assume that \(u_n \rightharpoonup u_0\) weakly in \(H^1_r(\mathbb{R}^N)\). We assume that 

\[
E = \lim_{n \to \infty} \|\nabla u_n\|_2^2.
\]

Noting that there exists \(\{\lambda_n\} \subset \mathbb{R}\) such that

\[
I'(u_n) = I'_{S_{r,c}}(u_n) + \lambda_n u_n.
\]  

(3.17)

So by \(I'_{S_{r,c}}(u_n) \to 0\), we get

\[
I'(u_n)u_n = o_n(1)\|u_n\|_{H^1} + \lambda_n\|u_n\|_2^2.
\]  

(3.18)

Furthermore, by the boundedness of \(\{u_n\}\) in \(H^1_r(\mathbb{R}^N)\) and \(\{u_n\} \subset \mathcal{S}_{r,c}\), we obtain that \(\|\lambda_n\| \) is bounded. So we can assume that \(\lambda_n \to \lambda_0\). Now, by the facts of

\[
\left\{ \begin{array}{l}
I'(u_n) - \lambda_n u_n \to 0, \\
\lambda_n \to \lambda_0, \\
\|u_n\|_2^2 = c, \|\nabla u_n\|_2^2 \to E, \\
0 \to u_0 \text{ weakly in } H^1_r(\mathbb{R}^N) \text{ and } u_n \to u_0 \text{ in } L^p(\mathbb{R}^N),
\end{array} \right.
\]

(3.19)

it is easy to check that \(u_0 \in H^1_r(\mathbb{R}^N)\) weakly solves

\[
-(a + bE)\Delta u_0 - \lambda_0 u_0 = |u_0|^{p-2} u_0 + |u_0|^{2^*-2} u_0 \text{ in } \mathbb{R}^N.
\]  

(3.20)

Testing (3.20) by \(u_0\), we have that

\[
(a + bE)\|\nabla u_0\|_2^2 - \lambda_0\|u_0\|_2^2 = \|u_0\|_p^p + \|u_0\|_2^{2^*}. \]

(3.21)

Since \(u_0\) satisfies the following Pohozaev identity:

\[
(N - 2)(a + bE)\|\nabla u_0\|_2^2 - N\lambda_0\|u_0\|_2^2 = 2N \left[ \frac{1}{p}\|u_0\|_p^p + \frac{1}{2^*}\|u_0\|_2^{2^*} \right],
\]

(3.22)

we obtain that

\[
\lambda_0\|u_0\|_2^2 = \left[ \frac{N}{2} - \frac{N}{p} - 1 \right]\|u_0\|_p^p.
\]  

(3.23)

On the other hand, by \(I'_{S_{r,c}}(u_n) \to 0\) and \(G(u_n) \to 0\), we have that

\[
\left\{ \begin{array}{l}
(a + b\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 = \lambda_n\|u_n\|_2^2 + \|u_n\|_p^p + \|u_n\|_2^{2^*} + o_n(1), \\
(a + b\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 = \frac{(p-2)N}{2p}\|u_n\|_p^p + \|u_n\|_2^{2^*} + o_n(1).
\end{array} \right.
\]

(3.24)

So by the boundedness of \(\{u_n\}\) in \(H^1_r(\mathbb{R}^N)\) again, we obtain that

\[
\lambda_n c^2 = \lambda_n\|u_n\|_2^2 = \left[ \frac{N}{2} - \frac{N}{p} - 1 \right]\|u_n\|_p^p + o_n(1).
\]  

(3.25)
By $u_n \to u_0$ in $L^p(\mathbb{R}^N)$, (3.23) and (3.25), we get $\lambda_c^2 \to \lambda_0\|u_0\|_2^2$. Furthermore, by $\lambda_n \to \lambda_0$, we obtain that

$$\lambda_0\|u_0\|_2^2 - c^2 = 0. \tag{3.26}$$

If $u_0 = 0$, we have that $\lambda_0 = 0$ and $\|u_n\|_p^p \to 0$. So by (3.24) and the Sobolev inequality,

$$(a + b\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 = \|u_n\|_2^2 + o_n(1) \leq \frac{\|\nabla u_n\|_2^2}{\delta_n^{2\gamma}} + o_n(1). \tag{3.27}$$

By letting $n \to +\infty$ in (3.27), we obtain that

$$a(\sqrt{E})^2 + b(\sqrt{E})^2 \leq S^{-\frac{2\gamma}{2\gamma}}(\sqrt{E})^{2\gamma}. $$

So by $\Omega^{2,4,p,2\gamma}_{a,b,0,S^{-\frac{2\gamma}{2\gamma}}} > 1$, we obtain that $\|\nabla u_n\|_2^2 \to E = 0$. By the Sobolev inequality again, we also have that $\|u_n\|_2^2 \to 0$. Hence, $d = \lim_{n \to \infty} I(u_n) = 0$, a contradiction.

Now, by $u_0 \neq 0$, (3.25) implies that

$$\lambda_0 = \frac{1}{c^2} \left[ \frac{N}{2} - \frac{N}{p} - 1 \right] \|u_0\|_p^p. $$

In particular, by $p < 2^*$, we have that $\lambda_0 < 0$. And by (3.26), we obtain that $u_0 \in S_c$ and thus $u_n \to u_0$ in $L^2(\mathbb{R}^N)$. Let $v_n := u_n - u_0$. By (3.21), (3.24) and the Brezis-Lieb lemma in [42], we have that

$$(a + b\|\nabla v_n\|_2^2)\|\nabla v_n\|_2^2 \leq (a + bE)\|\nabla v_n\|_2^2 + o_n(1)$$

$$= \|v_n\|_2^2 + o_n(1) \leq \frac{\|v_n\|_2^2}{\delta_n^{2\gamma}} + o_n(1). \tag{3.28}$$

By $\Omega^{2,4,p,2\gamma}_{a,b,0,S^{-\frac{2\gamma}{2\gamma}}} > 1$ again, we obtain that $\|\nabla v_n\|_2 \to 0$. That is, $u_n \to u_0$ in $D^{1,2}(\mathbb{R}^N)$. Hence, we obtain that $u_n \to u_0$ in $H^1_c(\mathbb{R}^N)$.

We can also have the following non-existence result.

**Theorem 3.1.** Let $N \geq 5$, $\Omega^{2,4,p,2\gamma}_{a,b,0,S^{-\frac{2\gamma}{2\gamma}}} > 1$ and $p \in [2 + \frac{4}{N}, 2^*]$. Then there exists $c_0 > 0$ such that problem (1.7)-(1.8) has no normalized solution provided $c < c_0$.

**Proof.** If $u \in H^1(\mathbb{R}^N)$ solves (1.7)-(1.8), it holds the following Pohozaev identity:

$$2a\|\nabla u\|_2^2 + 2b\|\nabla u\|_2^4 - 2\|u\|_2^2 = (N - \frac{2N}{p})\|u\|_p^p. \tag{3.29}$$

Moreover, by (3.1) and Lemma 1.1,

$$2a\|\nabla u\|_2^2 + 2b\|\nabla u\|_2^4 \leq \frac{N(p - 2)c^p}{2\|Q\|_p^2} \|\nabla u\|_2^{\frac{N(p-2)}{2}} + \frac{2}{\delta_n^{2\gamma}}\|\nabla u\|_2^{2\gamma}. \tag{3.30}$$

For the case of $p > 2 + \frac{4}{N}$, by $\Omega^{2,4,p,2\gamma}_{a,b,0,S^{-\frac{2\gamma}{2\gamma}}} > 1$ and Lemma 3.1, there exists $\delta > 0$ small such that $\Omega^{2,4,p,2\gamma}_{a-\delta,b-\delta,0,S^{-\frac{2\gamma}{2\gamma}}} > 1$. So

$$(a - \delta)\|\nabla u\|_2^2 + (b - \delta)\|\nabla u\|_2^4 \geq \frac{1}{\delta_n^{2\gamma}}\|\nabla u\|_2^{2\gamma}. \tag{3.31}$$
By (3.30)-(3.31), we obtain that
\[
2\delta \|\nabla u\|_2^2 + 2\delta \|\nabla u\|_2^4 \leq \frac{N(p - 2)c^p - \frac{N(p - 2)}{2}}{2\|Q\|_2^{p-2}} \|\nabla u\|_2^{N(p-2)}.
\] (3.32)

However, by Lemma 3.1 again, we have that \(\Omega_{26,28,1,0}^{2,4,N(p-2),2^*} > 0\) and thus
\[
2\delta \|\nabla u\|_2^2 + 2\delta \|\nabla u\|_2^4 \geq \Omega_{26,28,1,0}^{2,4,N(p-2),2^*} \|\nabla u\|_2^{N(p-2)}.
\] (3.33)

By \(p < 2^*\), we have that \(p - \frac{N(p-2)}{2} > 0\), and thus \(\frac{N(p - 2)c^p - \frac{N(p - 2)}{2}}{2\|Q\|_2^{p-2}} < \Omega_{26,28,1,0}^{2,4,N(p-2),2^*}\)
for all \(c < c_0 := \left(\frac{2\|Q\|_2^{p-2} \Omega_{26,28,1,0}^{2,4,N(p-2),2^*}}{N(p - 2)}\right)^{\frac{2}{p-2-N(p-2)}}\). So by (3.32)-(3.33), we obtain that \(\|\nabla u\|_2 = 0\). Hence, \(u \equiv 0\), a contradiction.

For the case of \(p = 2 + \frac{4}{N}\), by (3.30), we have that
\[
\left(a - \frac{c^4}{\|Q\|_2^\frac{4}{N}}\right) \|\nabla u\|_2 + b\|\nabla u\|_2^4 \leq \frac{1}{S^{2^*}} \|\nabla u\|_2^2.
\] (3.34)

Let \(c_0 := \|Q\|_2 \left[a - \frac{4 - 2^*}{2S^{\frac{N}{N-4}}} \left(\frac{2^* - 2}{2b}\right) \frac{N^2}{N-4}\right]^{\frac{N}{N-4}}\). By Lemma 3.1, we get
\[
\frac{\Omega_{2,4,p,2^*}^{2,4,p,2^*}}{\|Q\|_2^{\frac{p-2}{p}}} > 1, \ \forall c \in (0, c_0).
\]

Then by the definition of \(\Omega_{2,4,p,2^*}^{2,4,p,2^*}\), the formula (3.34) implies that for \(c \in (0, c_0)\), \(\|\nabla u\|_2 = 0\) and thus \(u \equiv 0\), contradiction. \(\square\)

3.2. The case of \(p > 2 + \frac{4}{N}\).

3.2.1. The normalized ground state solution.

Lemma 3.5. Let \(N \geq 5\), \(\Omega_{2,4,p,2^*}^{2,4,N(p-2),2^*} > 1\) and \(p \in \left(2 + \frac{4}{N}, 2^*\right)\). Let \(c_1\) be defined by (3.11). Then \(c_1 > 0\).

Proof. We note that \(\frac{N(p-2)}{2} > 2\) for \(p > 2 + \frac{4}{N}\). Since \(\Omega_{2,4,p,2^*}^{2,4,N(p-2),2^*} > 1\), by Lemma 3.1, there exists \(\sigma > 0\) such that
\[
\Omega_{2,4,p,2^*}^{2,4,N(p-2),2^*} > 1, \ \forall \kappa \in [0, \sigma).
\]
By $p < 2^*$, we see that $p - \frac{N(p-2)}{2} > 0$. Let $c_0 := (2\|Q\|_{L^2}^{p-2})^{2p-N(p-2)/2}$. So
\[
\frac{c^p - \frac{N(p-2)}{2}}{2\|Q\|_{L^2}^{p-2}} < \sigma, \forall c \in (0, c_0).
\]
Hence,
\[
\Omega_{\frac{2,4}{2,1};0,\kappa,\frac{1}{2^{*}s}}^{a,b,0,2^{*}} > 1, \text{ with } \kappa = \frac{c^p - \frac{N(p-2)}{2}}{2\|Q\|_{L^2}^{p-2}} \text{ and } 0 < c < c_0.
\]
Then by (3.2) and the definition of $\Omega_{\frac{2,4}{2,1};0,\kappa,\frac{1}{2^{*}s}}^{a,b,0,2^{*}}$, we obtain that for $c \in (0, c_0)$, $I(u) > 0$ for all $u \in S_c$ and thus $I_c \geq 0$. Combining with Lemma 3.2-(i), we obtain that $I_c = 0$ provided $0 < c < c_0$. Hence, $c_1 \geq c_0 > 0$. □

Furthermore, we have the following property.

**Theorem 3.2.** Let $N \geq 5$, $\Omega_{\frac{2,4}{2,1};0,\kappa,\frac{1}{2^{*}s}}^{a,b,0,2^{*}} > 1$ and $p \in \left(2 + \frac{4}{N}, 2^{*}\right)$. Then
\[
c_1 = \sup\{c > 0 : I_c = 0\}. \tag{3.35}
\]
And $I_c = 0$ for $c \in (0, c_1]$ while $I_c < 0$ for $c > c_1$. In particular,

(i) $I_c = 0$ and it is not attained provided $0 < c < c_1$.

(ii) $I_c < 0$ and it is attained when $c > c_1$.

(iii) $I_{c_1} = 0$ and it is attained.

**Proof.** By (3.7), we have that $\Omega_{\frac{2,4}{2,1};0,\kappa,\frac{1}{2^{*}s}}^{a,b,0,2^{*}} > \Omega_{\frac{2,4}{2,1};0,\kappa,\frac{1}{2^{*}s}}^{a,b,0,2^{*}} > 1$. So by Lemma 3.3 and Lemma 3.5, we see that $c_1 \in (0, +\infty)$. By Lemma 3.2-(ii) and (iii), the monotonicity and continuity imply that
\[
I_c \begin{cases} 
= 0 & \text{if } 0 < c \leq c_1, \\
< 0 & \text{if } c > c_1.
\end{cases}
\]
So (3.35) holds. In particular, the conclusion of (i) has been stated out in Lemma 3.3. We also note that (ii) will be proved by Lemma 3.6 and (iii) will be proved by Lemma 3.7 in the following.

**Lemma 3.6.** Let $N \geq 5$, $\Omega_{\frac{2,4}{2,1};0,\kappa,\frac{1}{2^{*}s}}^{a,b,0,2^{*}} > 1$ and $p \in \left(2 + \frac{4}{N}, 2^{*}\right)$. Then $I_c$ is attained by some $u \in S_c$ if $c > c_1$.

**Proof.** Noting that we have proved that $-\infty < I_c < 0$ for $c > c_1$ in Lemma 3.2 and Theorem 3.2. Let $\{u_n\} \subset H^1(\mathbb{R}^N) \cap S_c$ be a minimizing sequence for $I_c$. That is, $\|u_n\|_{L^2} = c$ and $I(u_n) = I_c + o_n(1)$. Without loss of generality, we may assume that $u_n \in S_{r,c}$. Indeed, let $u_n^*$ be the Schwarz rearrangement of $u_n$. Then $\{u_n^*\} \subset H^1_+(\mathbb{R}^N)$ is also a minimizing sequence for $I_c$. Obviously, $I(u_n) < 0$ for $n$ large. Then there exists $t_n \in \mathbb{R}$ such that $T(u_n, t_n) \in P_{r,c}$ and
\[
I(T(u_n, t_n)) = \min\{I(T(u_n, t)) : t \in \mathbb{R}\} < 0. \tag{3.36}
\]
Let \( v_n = T(u_n, t_n) \). Then \( I(v_n) \leq I(u_n) \). Moreover, \( I(v_n) \to I_c \). By the Ekeland principle, there exists \( \{w_n\} \subset S_{r,c} \) such that \( I(w_n) \leq I(v_n), \|w_n - v_n\|_{H^1} \to 0 \) and for all \( v \in S_{r,c} \),

\[
I(w_n) \leq I(v) + \frac{1}{n}\|w_n - v\|_{H^1}. \tag{3.37}
\]

It follows from the implicit function theorem that \( S_{r,c} \) is a \( C^1 \)-manifold of codimension 1 and \( H^1_r(\mathbb{R}^N) = \mathbb{R}u \oplus T_u(S_{r,c}) \) for each \( u \in S_{r,c} \). Fix \( u \in S_{r,c} \), then for any \( \phi \in H^1_r(\mathbb{R}^N) \), we write \( \phi = \psi + s_{u,\phi}u \) with \( \psi \in T_u(S_{r,c}) \) and \( s_{u,\phi} \in \mathbb{R} \). Precisely, \( s_{u,\phi} \) can be computed by

\[
s_{u,\phi} = \frac{\int_{\mathbb{R}^N} \phi ud\mathbf{x}}{\|u\|_2^2} = \frac{1}{c^2} \int_{\mathbb{R}^N} \phi ud\mathbf{x}. \tag{3.38}
\]

Now, we fix \( u = w_n \) and consider \( v(t) = c\frac{w_n + t\phi}{\|w_n + t\phi\|_2}, t \in \mathbb{R} \). Then \( v(t) \) is a \( C^1 \) curve in \( S_{r,c} \). In particular, \( v(0) = w_n \) and \( v'(0) = \phi - \frac{\int_{\mathbb{R}^N} w_n \phi d\mathbf{x}}{c^2}w_n = \phi - s_{w_n,\phi}w_n \in T_{w_n}(S_{r,c}) \).

By a direct calculation, the formula (3.37) implies that

\[
(I'(w_n), \phi) - \frac{1}{c^2} (I'(w_n), w_n) \int_{\mathbb{R}^N} w_n \phi d\mathbf{x}
\]

\[
\geq - \frac{1}{n} \left\| \phi - \frac{\int_{\mathbb{R}^N} w_n \phi d\mathbf{x}}{c^2}w_n \right\|_{H^1_r} , \forall \phi \in H^1_r(\mathbb{R}^N). \tag{3.39}
\]

Thus

\[
(I'(w_n), \phi - s_{w_n,\phi}w_n) - \frac{1}{c^2} (I'(w_n), w_n) \int_{\mathbb{R}^N} w_n (\phi - s_{w_n,\phi}w_n) d\mathbf{x}
\]

\[
\geq - \frac{1}{n} \left\| \phi - s_{w_n,\phi}w_n \right\|_{H^1_r}. \tag{3.40}
\]

By the arbitrary of \( \phi \in H^1_r(\mathbb{R}^N) \), we obtain the arbitrary of \( \psi = \phi - s_{w_n,\phi}w_n \in T_{w_n}(S_{r,c}) \). Hence, there exists \( \{\lambda_n\} \subset \mathbb{R} \) with \( \lambda_n := \frac{1}{c^2} (I'(w_n), w_n) \) such that

\[
\left\| I'(w_n) - \lambda_n \varphi(w_n) \right\|_{(T_{w_n}S_{r,c})^*} \to 0, \tag{3.41}
\]

where \( \varphi(w_n) = \frac{1}{n} \|w_n\|_2^2 \). By the definition of \( \lambda_n \), we have that \( (I'(w_n) - \lambda_n \varphi(w_n), w_n) \equiv 0 \). So by \( H^1_r(\mathbb{R}^N) = \mathbb{R}u \oplus T_u(S_{r,c}) \), the formula (3.41) implies that \( I'(w_n) - \lambda_n \varphi(w_n) \to 0 \) as \( n \to +\infty \). That is, \( I'_{S_{r,c}}(w_n) \to 0 \) as \( n \to +\infty \).

Since \( v_n \in P_{r,c} \), by \( \|w_n - v_n\|_{H^1} \to 0 \), we have that \( G(w_n) \to 0 \). Together with \( I(w_n) \to I_c \) and \( I'_{S_{r,c}}(w_n) \to 0 \), we see that \( \{w_n\} \subset S_{r,c} \) is a \( (PSP)_{I_c} \) sequence. Since \( I_c < 0 \) for \( c > c_1 \), by Lemma 3.4, \( \{w_n\} \) converges strongly to some \( u \) in \( H^1_r(\mathbb{R}^N) \) up to a subsequence. Hence, \( I_c \) is attained by \( u \in S_{r,c} \).

**Lemma 3.7.** Let \( N \geq 5, \frac{N^2}{2} \frac{\frac{4}{N}+\frac{2}{p}}{s_N} > 1 \) and \( p \in (2 + \frac{4}{N}, 2^*) \). Then \( I_{c_1} = 0 \) is attained by some \( u \in S_{r,c_1} \).

**Proof.** Let \( c_n = c_1 + \frac{1}{n} \). By Lemma 3.6, there exists \( \{u_n\} \subset S_{r,c_n} \) such that \( I(u_n) = I_{c_n} < 0 \). By Lemma 3.2 and Theorem 3.2, we have that \( \lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} I_{c_n} = I_{c_1} = 0 \).

By (3.9), we obtain that \( \{u_n\} \) is bounded in \( H^1_r(\mathbb{R}^N) \). Up to a subsequence, we assume that \( u_n \rightharpoonup u_0 \) weakly in \( H^1_r(\mathbb{R}^N) \). If \( u_0 = 0 \), we have that \( \lim_{n \to \infty} \|u_n\|_p = 0 \). Thanks to
\( \Omega_{a,b,0}^{2,4,p,2^*} > 1 \), by a similar argument as in the proof of Lemma 3.4, we can get that
\[
\lim_{n \to \infty} \left\| \nabla u_n \right\|_2 = 0.
\]
However, by \( p > 2 + \frac{4}{N} \), we have that \( \frac{N(p-2)}{2} > 2 \). Then by (3.2), we have that \( I_{c_n} = I(u_n) > 0 \) for \( n \) large enough, a contradiction. So \( u_0 \neq 0 \). Since \( I_{c_n} \) is attained by \( u_n \), there exists \( \lambda_n \in \mathbb{R} \) such that in a weak sense,
\[
- (a + b\|\nabla u_n\|_2^2) \Delta u_n = \lambda_n u_n + |u_n|^{p-2}u_n + |u_n|^{2^*-2}u_n \text{ in } \mathbb{R}^N.
\]
(3.42)

Applying a similar argument in the proof of Lemma 3.4, we can have that
\[
\lambda_n c_n^2 = \left[ \frac{N}{2} - \frac{N}{p} - 1 \right] \|u_n\|_p^p \to \left[ \frac{N}{2} - \frac{N}{p} - 1 \right] \|u_0\|_p^p.
\]
(3.43)

Since \( \lim_{n \to +\infty} c_n = c_1 > 0 \), we get
\[
\lambda_n \to \frac{1}{c_1^2} \left[ \frac{N}{2} - \frac{N}{p} - 1 \right] \|u_0\|_p^p := \lambda_0 < 0.
\]
(3.44)

Let \( E = \lim_{n \to \infty} \|\nabla u_n\|_2^2 \). Then \( u_0 \) weakly solves (3.20) and the equalities (3.21) and (3.22) hold. Let \( v_n := u_n - u_0 \), we can deduce (3.28) again. Then \( \|\nabla v_n\|_2 \to 0 \), which means that \( u_n \to u_0 \) in \( D^{1,2}(\mathbb{R}^N) \). Hence, \( E = \|\nabla u_0\|_2^2 \). Then one can see that
\[
I(u_0) = \lim_{n \to +\infty} I(u_n) = I_{c_1}.
\]

In particular, similar to (3.26), we can prove that
\[
\lambda_0(\|u_0\|_2^2 - c_1^2) = 0.
\]

And thus \( \|u_0\|_2^2 = c_1^2 \). That is, \( u_0 \in S_{r,c_1} \) attains \( I_{c_1} = 0 \). \( \square \)

3.2.2. The mountain pass type solution.

Lemma 3.8. Let \( N \geq 5 \), \( \Omega_{a,b,0}^{2,4,p,2^*} > 1 \) and \( p \in (2 + \frac{4}{N}, 2^*) \). Then there exist \( \theta > 1 \) and \( \eta \in (0,1) \) small such that for any \( K_c \in (0, \eta) \), it holds that
\[
0 < \sup_{u \in A_c} I(u) < \inf_{u \in B_c} I(u),
\]
(3.45)

where
\[
A_c := \{ u \in S_{r,c} : \|\nabla u\|_2^2 \leq K_c \}, \quad B_c := \{ u \in S_{r,c} : \|\nabla u\|_2^2 = \theta K_c \}.
\]

Moreover, \( I(u) > 0 \) for all \( u \in A_c \).

Proof. By (3.7), we have that \( \Omega_{a,b,0}^{2,4,p,2^*} > \Omega_{a,b,0}^{2,4,p,2^*} > 1 \). By Lemma 3.1, there exists some \( \delta \in (0,1) \) small such that \( \Omega_{a,b,0}^{2,4,p,2^*} > 1 \). So
\[
\frac{a}{2}(1 - \delta)t^2 + \frac{b}{4}(1 - \delta^2)t^4 > \frac{1}{2^* S_2^{2^*}} t^{2^*}, \quad \forall t > 0.
\]
(3.46)
For any \( u \in A_c \), we have that
\[
I(u) \geq \frac{a}{2} \|
abla u\|_2^2 + \frac{b}{4} \|
abla u\|_2^4 - \frac{1}{p} \|u\|_p^p - \frac{1}{2s^2} \|
abla u\|_2^2 \quad \text{by (3.1)}
\]
\[
> \frac{a\delta}{2} \|
abla u\|_2^2 + \frac{b\delta^2}{4} \|
abla u\|_2^4 - \frac{1}{p} \|u\|_p^p \quad \text{by (3.46)}
\]
\[
\geq \frac{a\delta}{2} \|
abla u\|_2^2 + \frac{b\delta^2}{4} \|
abla u\|_2^4 - \frac{1}{2\|\nabla u\|_2^2} \|
abla u\|_2^N\frac{(p-2)}{2}\left( \frac{2}{\delta} \right) \frac{c^{p-N\frac{(p-2)}{2}}}{K_c^{N\frac{(p-2)}{4}}} \quad \text{by Lemma 1.1. (3.47)}
\]

Since \( p > 2 + \frac{4}{N} \), we see that \( \frac{N(p-2)}{4} > 2 \). Hence, there exists \( \eta_1 > 0 \) small such that for \( K_c \in (0, \eta_1) \), we have that \( I(u) > 0 \) for any \( u \in A_c \).

Furthermore, we take \( \theta = \frac{2}{\delta} \), then \( \theta > 1 \) and for any \( v \in B_c \) and \( u \in A_c \),
\[
I(v) - I(u) = \frac{a}{2} \left( \|
abla v\|_2^2 - \|
abla u\|_2^2 \right) + \frac{b}{4} \left( \|
abla v\|_2^4 - \|
abla u\|_2^4 \right)
\]
\[
- \frac{1}{2} \left( \|
abla v\|_2^2 - \|
abla u\|_2^2 \right) - \frac{1}{p} \left( \|v\|_p^p - \|u\|_p^p \right)
\]
\[
> \frac{a}{2} \left( \|
abla v\|_2^2 - \|
abla u\|_2^2 \right) + \frac{b}{4} \left( \|
abla v\|_2^4 - \|
abla u\|_2^4 \right) - \frac{1}{2} \left( \|
abla v\|_2^2 - \|
abla u\|_2^2 \right) - \frac{1}{p} \left( \|v\|_p^p - \|u\|_p^p \right)
\]
\[
= \frac{aK_c}{2} + \frac{3bK_c^2}{4} - \frac{1}{2\|\nabla v\|_2^2} \left( \frac{2}{\delta} \right) \frac{c^{p-N\frac{(p-2)}{2}}}{K_c^{N\frac{(p-2)}{4}}} \quad \text{by Lemma 1.1. (3.47)}
\]

Then by \( \frac{N(p-2)}{4} > 1 \), there exists \( \eta_2 \in (0, 1) \) small such that for \( K_c \in (0, \eta_2) \), we have that
\[
I(v) - I(u) > \Lambda(K_c, \delta) > 0, \forall v \in B_c, \forall u \in A_c.
\]

And thus \( \sup_{u \in A_c} I(u) < \inf_{u \in B_c} I(u) \). Hence, we can take \( \eta := \min\{\eta_1, \eta_2\} \) and the proof is finished. \( \square \)

**Remark 3.2.** By the details in the proof of Lemma 3.8, the number \( \eta \) indeed can be chosen uniformly for bounded \( c \). That is, for such a fixed \( \theta > 1 \), we can find \( \eta := \eta_M > 0 \) small such that
\[
0 < \sup_{u \in S_{c,\eta} \|\nabla u\|_2^2 \leq K_c} I(u) < \inf_{u \in S_{c,\eta} \|\nabla u\|_2^2 = \theta K_c} I(u), \forall c \leq M, \forall K_c \leq \eta. \quad (3.48)
\]

**Theorem 3.3.** Let \( N \geq 5 \), \( \Omega^{2,4,p,2^*}_{a,b,0,\frac{1}{s}} > 1 \) and \( p \in \left( 2 + \frac{4}{N}, 2^* \right) \). Then \( I|_{S_{c,\eta}} \) has a mountain pass type critical point for any \( c \geq c_1 \).
Proof. By Theorem 3.2, for any \( c \geq c_1 \), there exists \( u_0 \in S_{r,c} \) such that \( I(u_0) = I_c \leq 0 \). Let \( \theta > 1 \) and \( \eta > 0 \) be given by Lemma 3.8, we take
\[
K_c < \min \left\{ \frac{1}{\theta} \left\| \nabla u_0 \right\|_2^2, \eta \right\}.
\]
Then \( \left\| \nabla u_0 \right\|_2^2 > \theta K_c \). Noting that \( \lim_{s \to \infty} \left\| \nabla T(u_0, s) \right\|_2 = 0 \). We can take some \( s_1 < 0 \) and put \( u_1 := T(u_0, s_1) \) such that \( u_1 \in S_{r,c} \) and \( \left\| \nabla u_1 \right\|_2^2 \leq \frac{1}{2} K_c \). Define
\[
\gamma_c := \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} I(\gamma(t)),
\]
where \( \Gamma_c := \{ \gamma \in C([0,1], S_{r,c}) : \gamma(0) = u_1, \gamma(1) = u_0 \} \). By the choice of \( K_c \) and Lemma 3.8, we have that
\[
\gamma_c > \max\{ I(u_1), I(u_0) \}. \tag{3.50}
\]
Similar to the argument of [24, Proposition 2.2 and Lemma 2.4], we can derive the existence of \( (PSP)_{\gamma_c} \)-sequence, that is, \( \{ u_n \} \subset S_{r,c} \) satisfies
\[
I(u_n) \to \gamma_c, I|_{S_{r,c}}(u_n) \to 0 \text{ and } G(u_n) \to 0.
\]
Noting that \( \gamma_c > 0 \), by Lemma 3.4, there exists \( u \in H^1_{r}(\mathbb{R}^N) \) such that, up to a subsequence, \( u_n \to u \) in \( H^1_{r}(\mathbb{R}^N) \). So \( u \in S_{r,c} \), \( I(u) = \gamma_c \) and \( I|_{S_{r,c}}(u) = 0 \).

**Theorem 3.4.** Let \( N \geq 5 \), \( \Omega^{2,4,p,2^*}_{a,b,\bar{t}} > 1 \) and \( p \in \left( 2 + \frac{4}{N}, 2^* \right) \). Then there exists \( \bar{\eta} \in (0,1) \) small such that \( I|_{S_{r,c}} \) has a mountain pass type critical point for \( c \in [\bar{c}_1 - \bar{\eta}, \bar{c}_1] \).

**Proof.** By Remark 3.2, there exist \( \theta > 1 \) and \( \eta \in (0,1) \) small independent of \( c \leq c_1 \) such that
\[
0 < \sup_{u \in S_{r,c}, \left\| \nabla u \right\|_2^2 = \theta K_c} I(u) < \inf_{u \in S_{r,c}, \left\| \nabla u \right\|_2^2 = \theta K_c} I(u), \ \forall \ c \leq c_1, \forall K_c \leq \eta. \tag{3.51}
\]
By Lemma 3.7, there exists some \( u_0 \in S_{r,c_1} \) such that \( I(u_0) = I_{c_1}^2 = 0 \). By (3.47), we can choose \( K_{c_1} \in (0, \eta] \) small and \( \varepsilon_0 \in (0, 1) \) small such that \( \left\| \nabla u_0 \right\|_2^2 > \theta K_{c_1} \) and
\[
\inf_{u \in S_{r,(1-\varepsilon)_{c_1}}, \left\| \nabla u \right\|_2^2 = \theta(1-\varepsilon)^2 K_{c_1}} I(u) > I((1-\varepsilon)u_0), \ \forall \varepsilon \in [0, \varepsilon_0]. \tag{3.52}
\]
For \( c < c_1 \) close to \( c_1 \), we rewrite \( c = (1-\varepsilon)c_1 \) with \( \varepsilon > 0 \) small. By (3.51), we can take \( K_c := (1-\varepsilon)^2 K_{c_1} < K_{c_1} \leq \eta \) such that
\[
0 < \sup_{u \in S_{r,c}, \left\| \nabla u \right\|_2^2 = \theta K_c} I(u) < \inf_{u \in S_{r,c}, \left\| \nabla u \right\|_2^2 = \theta K_c} I(u). \tag{3.53}
\]
Put \( u_1 := (1-\varepsilon)u_0 \in S_{r,c} \), we note that \( \left\| \nabla u_1 \right\|_2^2 = (1-\varepsilon)^2 \left\| \nabla u_0 \right\|_2^2 > (1-\varepsilon)^2 \theta K_{c_1} = \theta K_c. \) Let \( u_2 := T(u_1, s_2) \) with some \( s_2 < 0 \) such that \( \left\| \nabla u_2 \right\|_2^2 \leq \frac{1}{2} K_c \). Then for \( \varepsilon \in [0, \varepsilon_0] \), it holds that
\[
\inf_{u \in S_{r,c}, \left\| \nabla u \right\|_2^2 = \theta K_c} I(u) > \max\{ I(u_1), I(u_2) \}. \tag{3.54}
\]
Hence, we have that
\[
\gamma_c := \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} I(\gamma(t)) \geq \inf_{u \in S_{r,c}, \left\| \nabla u \right\|_2^2 = \theta K_c} I(u) > \max\{ I(u_1), I(u_2) \}, \tag{3.55}
\]
where \( \Gamma_c := \{ \gamma \in C([0,1], S_{r,c}) : \gamma(0) = u_1, \gamma(1) = u_0 \} \). Similar to the argument of Theorem 3.3, we obtain the final result. \( \square \)
3.2.3. The local constraint minimizer.

**Theorem 3.5.** Let \( N \geq 5 \), \( \Omega^{2,0,0}_{a,b,0,\frac{1}{s+\frac{1}{2}}} > 1 \) and \( p \in (2 + \frac{4}{N}, 2^*) \). Then \( I|_{S_{r,c}} \) has a local constraint minimizer with positive energy for \( c \in [c_1 - \eta, c_1) \).

**Proof.** By Lemma 3.7, there exists some \( u_0 \in S_{r,c_1} \) such that \( I(u_0) = c_1 = 0 \). By (3.7), (3.47) and \( \Omega^{2,0,0}_{a,b,0,\frac{1}{s+\frac{1}{2}}} > 1 \), we can choose \( K_{c_1} \in (0, \eta] \) small and \( \varepsilon_0 \in (0, \frac{1}{2}) \) small such that \( \|\nabla u_0\|_2^2 > \theta K_{c_1} \),

\[
\inf_{u \in S_{r,(1-\varepsilon)c_1}} I(u) > I((1-\varepsilon)u_0), \quad \forall \varepsilon \in [0, \varepsilon_0], \tag{3.56}
\]

and

\[
\inf_{\|\nabla u\|_2^2 \geq \theta (1-\varepsilon)^2 K_{c_1}} \left( \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{2^* S^{2^*}_{2}} \|\nabla u\|_2^{2^*} \right) > I((1-\varepsilon)u_0), \quad \forall \varepsilon \in [0, \varepsilon_0]. \tag{3.57}
\]

Let \( c := (1-\varepsilon)c_1 \), \( u_1 := (1-\varepsilon)u_0 \in S_{r,c} \) and \( K_c := (1-\varepsilon)^2 K_{c_1} \). Define \( m_c := \inf_{u \in S_{r,c}, \|\nabla u\|_2^2 \geq \theta K_c} I(u) \).

By Theorem 3.2 and (3.56), we get

\[
0 \leq m_c \leq I(u_1) < \inf_{u \in S_{r,c}, \|\nabla u\|_2^2 = \theta K_c} I(u).
\]

Moreover, there exists \( \sigma \in (0, 1) \) small such that

\[
I(u_1) < \inf_{u \in S_{r,c}, \|\nabla u\|_2^2 \leq \theta K_c + \sigma} I(u).
\]

By the definition of \( m_c \), there exists \( \{u_n\} \subset S_{r,c} \) such that \( \|\nabla u_n\|_2^2 > \theta K_c + \sigma \) and \( I(u_n) \to m_c \). Similar to the argument of Lemma 3.6, there exists \( \{w_n\} \subset S_{r,c} \) such that \( \|w_n - u_n\|_{H^1} \to 0 \), \( I(w_n) \to m_c \) and \( I'(w_n) \to 0 \).

If \( w_n \rightharpoonup 0 \) weakly in \( H^1_r(\mathbb{R}^N) \), then

\[
I(u_1) \geq m_c \geq \frac{a}{2} \lim_{n \to \infty} \|\nabla w_n\|_2^2 + \frac{b}{4} \lim_{n \to \infty} \|\nabla w_n\|_2^4 - \frac{1}{2^* S^{2^*}_{2}} \lim_{n \to \infty} \|\nabla w_n\|_2^{2^*},
\]

a contradiction with \( \lim_{n \to \infty} \|\nabla w_n\|_2^2 \geq \theta K_c \) and (3.57). So \( w_n \rightharpoonup w \neq 0 \) weakly in \( H^1_r(\mathbb{R}^N) \). We assume that \( E = \lim_{n \to \infty} \|\nabla w_n\|_2^2 \). By \( I'(w_n) \to 0 \), we get

\[
I'(w_n)w_n = \lambda_n \|w_n\|_2^2 + o_n(1).
\]

So \( |\lambda_n| \) is bounded. Furthermore,

\[
\begin{cases}
I'(w_n) - \lambda_n w_n \to 0, \\
\lambda_n \to \lambda_0, \\
\|w_n\|_2^2 = c, \|\nabla w_n\|_2^2 \to E, \\
w_n \rightharpoonup w \text{ weakly in } H^1_r(\mathbb{R}^N) \text{ and } w_n \to w \text{ in } L^p(\mathbb{R}^N).
\end{cases}
\]

Then \( w \in H^1_r(\mathbb{R}^N) \) weakly solves

\[
-(a + bE)\Delta w - \lambda_0 w = |w|^{p-2}w + |w|^{2^*-2}w \text{ in } \mathbb{R}^N,
\]

(3.59)
from which we derive that
\[(a + bE)\|\nabla w\|^2 - \lambda_0\|w\|^2 = \|w\|^p + \|w\|^{2^*},\] (3.60)
and
\[(N - 2)(a + bE)\|\nabla w\|^2 - N\lambda_0\|w\|^2 = 2N \left[\frac{1}{p}\|w\|^p + \frac{1}{2^*}\|w\|^{2^*}\right].\] (3.61)
By (3.60)-(3.61), we obtain that
\[\lambda_0\|w\|^2 = \left[\frac{N}{2} - \frac{N}{p} - 1\right]\|w\|^p.\] (3.62)
So \(\lambda_0 < 0\). Let \(v_n = w_n - w\). By (3.58)-(3.59), we have that
\[(a + b\|\nabla v_n\|^2)\|\nabla v_n\|^2 - \lambda_0\|v_n\|^2 \leq \|v_n\|^{2^*} + o_n(1) \leq \frac{\|\nabla v_n\|^2}{S^{\frac{2^*}{2}}} + o_n(1).\] (3.63)
So by \(\Omega_{\alpha,b,0,s}^{2,4,p,2^*} > 1\), we obtain that \(\|\nabla v_n\| \to 0\). Moreover, \(\|v_n\| \to 0\). Hence, we have that \(w_n \to w\) in \(H^1_0(\mathbb{R}^N)\).

By \(I(w_n) \to m_c\), \(I'_{S,c}(w_n) \to 0\) and \(w_n \to w\) in \(H^1_0(\mathbb{R}^N)\), we get \(I(w) = m_c\) and \(I'_{S,c}(w) = 0\). By Theorem 3.2, we know that \(I_c = 0\) is not attained for \(c \in (0, c_1)\). Then \(m_c > 0\) for \(c \in [c_1 - \tilde{c}, c_1)\). \(\square\)

**Proof of Theorem 1.5-(i).** By Theorem 3.2, we get (i-1); by Theorem 3.3, we get (i-2); by Theorem 3.4 and Theorem 3.5, we get (i-3); by Theorem 3.1, we get (i-4). \(\square\)

### 3.3. The case of \(p = 2 + \frac{4}{N}\)

Put
\[c_* := \|Q\|_2 \left[ a - \frac{(4 - 2^*)}{(2^*)^{\frac{N-2}{N}}} \left(\frac{2(2^* - 2)}{b}\right)^{\frac{2^*-2}{N-4}}\right]^{\frac{N}{N-4}}.\] (3.64)
We note that \(\frac{2\sqrt{2}}{(2^*)^{\frac{N-2}{N}}} < \left(\frac{1}{2}\right)^{\frac{N-2}{4}}\) for \(N \geq 5\). Suppose that \(\Omega_{\alpha,b,0,s}^{2,4,p,2^*} > 1\), then by (3.8) in Remark 3.1, we can have that
\[\Omega_{\alpha,b,0,s}^{2,4,p,2^*} > 1\]
\[\Leftrightarrow \left(\frac{2a}{4 - 2^*}\right)^{\frac{4-2^*}{2}} \left(\frac{2b}{2^* - 2}\right)^{\frac{2^*-2}{N-4}} > \frac{1}{S^{\frac{2^*}{2}}},\]
\[\Leftrightarrow a > \left(\frac{1}{2}\right)^{\frac{N-2}{N-4}} \left(4 - 2^*\right) \left(\frac{2(2^* - 2)}{b}\right)^{\frac{2^*-2}{N-4}} \frac{1}{S^{\frac{N}{N-4}}}\]
\[\Rightarrow a > \left(4 - 2^*\right) \left(\frac{N-2}{N-4}S^{\frac{N}{N-4}}\right)^{\frac{N-2}{N-4}} \left(\frac{2(2^* - 2)}{b}\right)^{\frac{2^*-2}{N-4}}\]
\[\Rightarrow c_* > 0.\]

**Lemma 3.9.** Let \(N \geq 5\), \(\Omega_{\alpha,b,0,s}^{2,4,p,2^*} > 1\) and \(p = 2 + \frac{4}{N}\). Let \(c_1\) be defined by (3.11). It holds that
\[0 < c_* \leq c_1 \leq a^\frac{4}{N}\|Q\|_2.\] (3.65)
Proof. Recalling (3.2), by \( p = 2 + \frac{4}{N} \), we obtain that
\[
I(u) \geq \left( \frac{a}{2} - \frac{c^\frac{4}{N}}{2\|Q\|_2^\frac{4}{N}} \right) \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{1}{2} \frac{4^\frac{2}{p^*}}{S^{\frac{2}{p^*}}} \|\nabla u\|_2^{2^*} \tag{3.66}
\]

Let
\[
A := \frac{a}{2} - \frac{c^\frac{4}{N}}{2\|Q\|_2^\frac{4}{N}} > 0, \quad B := \frac{b}{4}.
\]

Consider the equation
\[
\Omega_{A,B,0}^{2,4,p,2^*} \frac{1}{2^* S^{\frac{2}{p^*}}} = 2^* S^{\frac{2}{p^*}} \Omega_{A,B,0,1}^{2,4,p,2^*} = 1.
\]

By (3.5), we have that
\[
2^* S^{\frac{2}{p^*}} 2(2^* - 2) \frac{2^{\frac{4}{N}}}{2^*} (4 - 2^*) \left( \frac{a}{2} - \frac{c^\frac{4}{N}}{2\|Q\|_2^\frac{4}{N}} \right) \left( \frac{b}{4} \right)^{2^* - 2} = 1. \tag{3.67}
\]

By solving (3.67), we obtain that \( c = c_* \). Hence, for \( c \leq c_* \), we have that \( \Omega_{A,B,0}^{2,4,p,2^*} \frac{1}{2^* S^{\frac{2}{p^*}}} \geq 1 \).

So by (3.66) and Lemma 3.2-(i), we obtain that \( I_c = 0, \forall c \in (0, c_*] \). Then by the definition of \( c_1 \), we have that \( c_1 \geq c_* > 0 \).

On the other hand, by (3.13),
\[
I(Q_t) \leq \frac{a}{2} c^2 t^2 + \frac{b}{4} c^4 t^4 - \frac{c^p t^{\frac{Np}{2} - N}}{2\|Q\|_2^{p-2}} = \left[ a - \frac{c^\frac{4}{N}}{\|Q\|_2^\frac{4}{N}} \right] \frac{1}{2} c^2 t^2 + \frac{b}{4} c^4 t^4.
\]

So if \( c > a^\frac{4}{\|Q\|_2} \), it holds that \( a - \frac{c^\frac{4}{N}}{\|Q\|_2^\frac{4}{N}} < 0 \). Hence, \( \min_{t \geq 0} I(Q_t) < 0 \), which implies that \( I_c < 0 \) for \( c > a^\frac{4}{\|Q\|_2} \). By the definition of \( c_1 \), we have that \( c_1 \leq a^\frac{4}{\|Q\|_2} \). \( \square \)

Proof of Theorem 1.5-(ii). By Lemma 3.9, we see that \( c_1 > 0 \) and thus \( I_c < 0 \) for \( c > c_1 \). By Lemma 3.3, we have that \( I_c = 0 \) for \( c \in (0, c_1] \) and \( I_c \) is not attained for \( c \in (0, c_1) \). Applying a similar argument as the proof of Lemma 3.6, we can get (ii-1); by Theorem 3.1, we get (ii-2). \( \square \)

3.4. The case of \( 2 < p < 2 + \frac{4}{N} \). By Lemma 3.2-(i), we get \( I_c > -\infty \) for any \( c > 0 \). Furthermore, for the case of \( p < 2 + \frac{4}{N} \), by (3.13),
\[
I(Q_t) \leq \frac{a}{2} c^2 t^2 + \frac{b}{4} c^4 t^4 - \frac{c^p t^{\frac{Np}{2} - N}}{2\|Q\|_2^{p-2}}.
\]

Then
\[
I_c \leq \min_{t > 0} I(Q_t) < 0.
\]

Proof of Theorem 1.5-(iii). In such a case, it holds that \( I_c \in (-\infty, 0) \) for any \( c > 0 \). Then apply a similar argument as that in Lemma 3.6, we can prove that \( I_c \) is attained. \( \square \)
4. The case \( N = 4 \)

4.1. Some preliminaries. When \( N = 4 \), by (3.2),
\[
I(u) \geq \frac{a}{2} \| \nabla u \|_2^2 + \frac{1}{4} \left( b - \frac{1}{S^2} \right) \| \nabla u \|_2^4 - \frac{c^{4-p}}{2 \| Q \|_2^{p-2}} \| \nabla u \|_2^{2(p-2)}
\]
holds for all \( u \in S_c \).

If \( b > \frac{1}{S^2} \) and \( 2 < p < 4 = 2^* \), similar to the arguments in subsection 3.1, we have the following results. Since the proofs are almost the same, we omit the details.

**Lemma 4.1.** Let \( N = 4 \), \( b > \frac{1}{S^2} \) and \( p \in (2, 4) \). Then

(i) \(-\infty < I_c \leq 0 \) for all \( c > 0 \);

(ii) \( I_c \) is non-increasing on \( c \in (0, +\infty) \);

(iii) \( I_c \) is continuous on \( c \in (0, +\infty) \).

**Corollary 4.1.** Let \( N = 4 \), \( b > \frac{1}{S^2} \) and \( p \in (2, 4) \). If there exists \( 0 < \underline{c} < \overline{c} < +\infty \) such that \( I_{\underline{c}} = I_{\overline{c}} \), then \( I_c \) is not attained provided \( c \in [\underline{c}, \overline{c}] \).

**Lemma 4.2.** Let \( N = 4 \), \( b > \frac{1}{S^2} \) and \( p \in (2, 4) \). Then \( I_c < 0 \) for \( c > 0 \) large enough. And thus we can define \( c_1 \) as in (3.11). Furthermore, if \( c_1 > 0 \), then \( I_c \equiv 0 \) for \( c \in (0, c_1] \) and \( I_c \) is not attained provided \( c \in (0, c_1) \).

Similar to Lemma 3.4, we also can establish the following compactness result.

**Lemma 4.3.** Let \( N = 4 \), \( b > \frac{1}{S^2} \) and \( p \in (2, 4) \). If \( \{u_n\} \subset S_{r,c} \) such that \( I(u_n) \to \gamma_c \neq 0 \), \( I'_{S_{r,c}}(u_n) \to 0 \) and \( G(u_n) \to 0 \), then \( \{u_n\} \) converges strongly in \( H^1_r(\mathbb{R}^4) \) up to a subsequence.

4.2. The case of \( 3 < p < 4 \)

**Lemma 4.4.** Let \( N = 4 \), \( b > \frac{1}{S^2} \) and \( p \in (3, 4) \). Then \( c_1 > 0 \).

**Proof.** Let \( A = \frac{a}{2} \), \( B = \frac{1}{2}(b - \frac{1}{S^2}) \), \( \kappa_4 = \frac{c^{4-p}}{2 \| Q \|_2^{p-2}} \), \( q = 2(p - 2) \) and \( p_3 \in (2, 4) \). By (3.6), we have that
\[
\Omega_{A,B,0,\kappa_4}^{2,4,p_3,q} \to +\infty \text{ as } c \to 0^+.
\]
Hence, by Lemma 3.1, the continuity implies that there exists some \( c_* > 0 \) such that \( \Omega_{A,B,0,\kappa_4}^{2,4,p_3,q} > 1 \) provided \( c \leq c_* \). Then by the definition of \( \Omega_{A,B,0,\kappa_4}^{2,4,p_3,q} \), the formula (4.1) implies that \( I(u) \geq 0 \) for any \( u \in S_c \) with \( c \leq c_* \). And thus \( I_c \geq 0, \forall c \in (0, c_*) \). Combining with Lemma 4.1-(i), we obtain that \( I_c = 0, \forall c \in (0, c_*) \). Then by the definition of \( c_1 \), we have that \( c_1 \geq c_* > 0 \).

Similar to Theorem 3.2, we can establish the following result for the normalized ground state solution.

**Theorem 4.1.** Let \( N = 4 \), \( b > \frac{1}{S^2} \) and \( p \in (3, 4) \). Then
\[
c_1 = \sup\{c > 0 : I_c = 0\}.
\]
And \( I_c = 0 \) for \( c \in (0, c_1] \) while \( I_c < 0 \) for \( c > c_1 \). In particular,

(i) \( I_c = 0 \) and it is not attained provided \( 0 < c < c_1 \).

(ii) \( I_c < 0 \) and it is attained when \( c > c_1 \).
(iii) $I_{c_1} = 0$ and it is attained.

Similar to Theorem 3.3 and Theorem 3.4, we get the following result for the normalized mountain pass type solution.

**Theorem 4.2.** Let $N = 4$, $b > \frac{1}{S^2}$ and $p \in (3, 4)$. Then there exists $\bar{\eta} > 0$ such that $I|_{S_{r,c}}$ has a mountain pass type critical point for $c \geq c_1 - \bar{\eta}$.

Similar to Theorem 3.5, we get the following result about local constraint minimizer.

**Theorem 4.3.** Let $N = 4$, $b > \frac{1}{S^2}$ and $p \in (3, 4)$. Then $I|_{S_{r,c}}$ has a local constraint minimizer with positive energy for $c \in [c_1 - \bar{\eta}, c_1)$.

We can also have the following non-existence result.

**Theorem 4.4.** Let $N = 4$, $b > \frac{1}{S^2}$ and $p \in (3, 4)$. Then there exists $c_0 > 0$ such that problem (1.7)-(1.8) has no normalized solution provided $c < c_0$. In particular, one can check that

$$c_0 \geq \frac{a\|Q\|_{L^2}^{p-2}}{(4-p)(p-2)^{\frac{1}{p-3}}} \left[ \frac{1}{p-3} \left( b - \frac{1}{S^2} \right) \right]^{\frac{4}{p-3}}. \quad (4.4)$$

**Proof.** If $u \in H^1(\mathbb{R}^4)$ solves (1.7)-(1.8), by (3.30), we have that

$$a\|\nabla u\|_2^2 + \left( b - \frac{1}{S^2} \right) \|\nabla u\|_4^4 \leq \frac{(p-2)c^{4-p}}{\|Q\|_2^{p-2}} \|\nabla u\|_2^{2(p-2)}. \quad (4.5)$$

By a similar argument in Lemma 4.4, there exists some $c_0 > 0$ such that

$$\Omega^{2,4,p_3,2(p-2)}_{a,b-\frac{1}{S^2},0,(p-2)c^{4-p},\|Q\|_2^{p-2}} > 1, \forall c < c_0.$$

Then by the definition of $\Omega^{2,4,p_3,2(p-2)}_{a,b-\frac{1}{S^2},0,(p-2)c^{4-p},\|Q\|_2^{p-2}}$, the formula (4.5) implies that $\|\nabla u\|_2 = 0$ and thus $u \equiv 0$, a contradiction. Furthermore, by (3.6), a direct calculation shows that (4.4) holds.

**Proof of Theorem 1.6-(i).** By Theorem 4.1, we get (i-1). By Theorem 4.2 and Theorem 4.3, we get (i-2) and (i-3). By Theorem 4.4, we get (i-4).

4.3. The case of $p = 3$.

**Lemma 4.5.** Let $N = 4$, $b > \frac{1}{S^2}$ and $p = 3$. Then $c_1 = a\|Q\|_2$.

**Proof.** By (4.1), we have that

$$I(u) \geq \left( \frac{a}{2} - \frac{c}{2\|Q\|_2} \right) \|\nabla u\|_2^2 + \frac{1}{4} \left( b - \frac{1}{S^2} \right) \|\nabla u\|_4^4. \quad (4.6)$$

Then for any $c < a\|Q\|_2$, the formula (4.6) implies that $I(u) > 0$ for any $u \in S_c$ and thus $I_c \geq 0$. Combining with Lemma 4.1-(i), we obtain that $I_c = 0$. Hence, by the definition of $c_1$, we have that $c_1 \geq a\|Q\|_2$.

On the other hand, by (3.13), we have that

$$I(Q_t) = \left( a - \frac{c}{\|Q\|_2} \right) \frac{1}{2} c^2 t^2 + \left( b - \frac{\|Q\|_4^4}{\|Q\|_2^2} \right) \frac{1}{4} c^4 t^4. \quad (4.7)$$
So for any \( c > a\|Q\|_2 \), one can see that \( I(Q_t) < 0 \) for \( t > 0 \) small enough and thus \( I_c \leq \min_{t>0} I(Q_t) < 0 \). So by the definition of \( c_1 \), we have that \( c_1 \leq a\|Q\|_2 \).

**Theorem 4.5.** Let \( N = 4, \ b > \frac{1}{S_2} \) and \( p = 3 \). Then \( I_c < 0 \) is attained for \( c > a\|Q\|_2 \).

**Proof.** By Lemma 4.5, we have that \( I_c < 0 \) for \( c > a\|Q\|_2 \). Thanks to the compactness result in Lemma 4.3, by a similar argument as Lemma 3.6, we can obtain the result.

Furthermore, we can prove the following non-existence result.

**Theorem 4.6.** Let \( N = 4, \ b > \frac{1}{S_2} \) and \( p = 3 \). Then problem (1.7)-(1.8) has no normalized solution provided \( c \leq a\|Q\|_2 \).

**Proof.** If \( u \in H^1(\mathbb{R}^4) \) solves (1.7)-(1.8), by (4.5), we obtain that
\[
\left(a - \frac{c}{\|Q\|_2}\right)\|\nabla u\|_2^2 + \left(b - \frac{1}{S_2^2}\right)\|\nabla u\|_2^4 \leq 0. \tag{4.8}
\]
So for \( c \leq a\|Q\|_2 \), the formula (4.8) implies that \( \|\nabla u\|_2 = 0 \) and thus \( u = 0 \), a contradiction.

**Proof of Theorem 1.6-(ii).** By Theorem 4.5 and Theorem 4.6, we get the result.

4.4. The case of \( 2 < p < 3 \).

**Lemma 4.6.** Let \( N = 4, \ b > \frac{1}{S_2} \) and \( p \in (2, 3) \). Then \( I_c < 0 \) is attained for \( c > 0 \).

**Proof.** For any \( c > 0 \), by (3.13), we have that
\[
I(Q_t) = \frac{a}{2}c^2t^2 + \left(b - \frac{\|Q\|_4^4}{\|Q\|_2^4}\right)\frac{1}{4}c^4t^4 - \frac{e^p t^{2p-4}}{2\|Q\|_2^{p-2}}. \tag{4.9}
\]
Noting that \( 2p - 4 < 2 \), we have that \( I(Q_t) < 0 \) for \( t > 0 \) small enough and thus \( I_c \leq \min_{t>0} I(Q_t) < 0 \). By a similar argument as Lemma 3.6, we can obtain that \( I_c \) is attained by some \( u \in S_{r,c} \).

**Proof of Theorem 1.6-(iii).** By Lemma 4.6, we get the result.

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(J. Zhang)
**College of Science, China University of Petroleum**
QINGDAO 266580, SHANDONG, PR CHINA

*Email address: zjianmath@163.com*

(J. J. Zhang)
**College of Mathematics and Statistics, Chongqing Jiaotong University**
CHONGQING 400074, PR CHINA

*Email address: zhangjianjun09@tsinghua.org.cn*

(X. X. Zhong)
**South China Research Center for Applied Mathematics and Interdisciplinary Studies**
**South China Normal University**
GUANGZHOU 510631, PR CHINA

*Email address: zhongxuexiu1989@163.com*