A NONAUTONOMOUS DYNAMICS PERSPECTIVE FOR
THE JACOBIAN CONJECTURE

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Abstract. Firstly, we introduce a global problem on nonuniform asymptotic stability for nonautonomous differential systems, whose restriction to the autonomous framework is the classical Markus–Yamabe Conjecture. Secondly, we prove that this stability problem implies, after a certain initial time, the global injectivity of the map corresponding to the system. To cope with the above claim, we propose a notion of injectivity tailored to the nonautonomous context.

1. Introduction

An unpublished manuscript of G. Fournier and M. Martelli (see [11, p.175] and [21] for details) pointed out the existence of a relationship between the Jacobian Conjecture and the problem of the global stability of discrete and continuous autonomous dynamical systems on finite dimension, also known as the Markus-Yamabe Conjecture. The authors proved that if the Markus-Yamabe Conjecture is true for dynamical systems described by polynomial vector fields of $\mathbb{R}^n$ with degree $\leq 3$ and for all $n \geq 2$, then the Jacobian Conjecture is true.

The above mentioned conjectures have been widely studied. In fact, the Markus-Yamabe conjecture was proved to be false by A. Cima et al. in [7] thirty five years after that L. Markus and H. Yamabe stated this problem in 1960 [20]. We point out that this problem is true in dimension two, which was proved independently by C. Gutiérrez [15], R. Feßler [13] and A. A. Glutsyuk [14]; who used the fact that hypothesis of the Markus-Yamabe Conjecture, in dimension two, is equivalent to the map is injective (see [22]).

On the other hand, the Jacobian Conjecture is still open, even in dimension two, since its formulation by O. H. Keller in 1939 [17] and there exist remarkable efforts to address it as the reduction results by H. Bass et al. [3], A.V. Yagzhev [26], and M. de Bondt & A. van den Essen [4]. The results in [3, 26] establish that it is sufficient to focus on maps, for all dimension
\( n \geq 1 \), having the form \( X + H \) where \( H \) is homogeneous of degree 3 and its jacobian \( JH \) is nilpotent; this result is improved in \([4]\) by showing that it is sufficient to investigate the Jacobian conjecture for all maps of the form \((x_1 + f_{x_1}, \ldots, x_n + f_{x_n})\) where \( f \) is a homogeneous polynomial of degree 4, \( f_{x_i} \) denotes the partial derivatives of \( f \) with respect to \( x_i \) and \( n \geq 1 \).

Similarly to the Fournier–Martelli approach, there exist several conjectures implying the Jacobian Conjecture and we refer the reader to \([10, 29, 28]\). Our article is inscribed in this context and the main goal is to propose a new approach to address the Jacobian Conjecture from a nonautonomous dynamical systems perspective.

We point out that in the autonomous context, the Markus–Yamabe Conjecture is stated in terms of Hurwitz vector fields, which are defined by the eigenvalues of its linearization around any point, and the corresponding stability is known as uniform asymptotic stability. Contrarily, in the nonautonomous framework, the asymptotic stability is not determined by the eigenvalues; thus there exist several spectral theories based either on characteristic exponents or dichotomies \([12]\), which have associated a wide range of asymptotic stabilities, being a uniform one a particular case.

Taking in account the wide range of asymptotic stabilities above mentioned, in this article we work with the spectrum associated to nonuniform exponential dichotomy and its corresponding stability, namely the nonuniform asymptotic stability. Notice that in \([8\text{ p.540}]\) is stated that this dichotomy is admitted by any linear systems with nonzero Lyapunov exponents.

This article mimics the Markus–Yamabe problem in a nonuniform and nonautonomous framework, this requires an overview of nonuniform exponential dichotomy and their associated spectrum and stability. Once the Markus-Yamabe problem is well defined, we follow the Fournier-Martelli’s ideas and show that implies the Jacobian Conjecture (see \([11\text{ Proposition 8.1.8}]\)).

The article is organized as follows. In the Section 2, we recall the main tools on nonautonomous dynamical systems, such as the nonuniform exponential dichotomy and its corresponding spectrum in order to introduce the nonuniform nonautonomous Markus-Yamabe Conjecture, which is proved to be true in dimension one. In the Section 3, we introduce the concept of uniform injectivity in order to posed well the Nonautonomous Weak Markus-Yamabe Conjecture, which is a fundamental step to prove that the Nonuniform Nonautonomous Markus-Yamabe Conjecture implies, after a certain initial time, the Jacobian conjecture; this last result is established in the final section. Moreover, in that section we show an example that verifies every conditions that we impose through of this manuscript.

**Notations:** In this paper \( M_n(\mathbb{R}) \) denotes the set of \( n \times n \) matrices over \( \mathbb{R} \), \( I_n \) is the identity matrix and we will use \( \text{Diag}\{\lambda\} \) to denote \( \lambda I_n \), namely, the
diagonal matrix with diagonal terms \( \lambda \). Moreover, the matrix norm induced by the euclidean vector norm \( |\cdot| \) will be denoted by \( ||\cdot|| \).

2. Nonuniform Markus-Yamabe Conjecture

In this section we establish a nonuniform version of nonautonomous Markus-Yamabe Conjecture whose uniform version was previously introduced in [5]. Similarly as in the uniform case, the conjecture will be stated in terms of the spectrum associated to the nonuniform exponential dichotomy of linear systems combined with the property of global nonuniform asymptotical stability of nonlinear systems. For this purpose, we have to revisit basic properties and results related to these nonuniform topics.

2.1. Nonuniform exponential dichotomy spectrum. Let us consider the linear ODE system

\[
\dot{x} = A(t)x,
\]

where \( x \in \mathbb{R}^n \), \( A: \mathbb{R}^+ \mapsto M_n(\mathbb{R}) \) is a continuous matrix function. A fundamental matrix of (1) and its corresponding transition matrix will be respectively denoted by \( \Phi(t) \) and \( \Phi(t,s) = \Phi(t)\Phi^{-1}(s) \).

Definition 1. ([1], [8], [27]) The system (1) has a nonuniform exponential dichotomy on \( \mathbb{R}^+ \) if there exist an invariant projector \( P(\cdot) \), constants \( K \geq 1 \), \( \alpha > 0 \) and \( \varepsilon \geq 0 \) such that

\[
\begin{align*}
\|\Phi(t,s)P(s)\| &\leq K \exp(-\alpha(t-s) + \varepsilon s), & t \geq s, & t, s \in \mathbb{R}^+, \\
\|\Phi(t,s)(I_n - P(s))\| &\leq K \exp(\alpha(t-s) + \varepsilon s), & t \leq s, & t, s \in \mathbb{R}^+.
\end{align*}
\]  
(2)

Definition 2. ([8], [27]) The nonuniform spectrum (also called nonuniform exponential dichotomy spectrum) of (1) is the set \( \Sigma(A) \) of \( \lambda \in \mathbb{R} \) such that

\[
\dot{x} = [A(t) - \lambda I_n]x
\]

(3)
does not have a nonuniform exponential dichotomy on \( \mathbb{R}^+ \).

Remark 1. The above defined spectrum \( \Sigma(A) \) can be seen as a nonuniform version of the spectrum constructed by S. Siegmund [25], which is a friendly and simple reconstruction of the Sacker & Sell spectrum [24] for the linear system (1). We point out that the Sacker & Sell spectrum was originally established for linear skew product flows with compact base and can not be applied directly to linear systems (see [25, Remark 3.2]).

The following result establishes simple conditions ensuring that \( \Sigma(A) \) is a non empty and compact set.

Proposition 1. ([8], [19], [25], [27]), If the evolution operator \( \Phi(t,s) \) of (1) has a nonuniformly bounded growth, namely, there exist constants \( K_0 \geq 1, \alpha \geq 0 \) and \( \varepsilon \geq 0 \) such that

\[
\|\Phi(t,s)\| \leq K_0 \exp(\alpha |t-s| + \varepsilon s), \quad t, s \in \mathbb{R}^+,
\]
its nonuniform spectrum \( \Sigma(A) \) is the union of \( m \) compact intervals where \( 0 < m \leq n \), that is,
\[
\Sigma(A) = \bigcup_{i=1}^{m} [a_i, b_i],
\]
with \(-\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty\).

2.2. Nonuniform stability of nonlinear systems. Taking in account the nonuniform behaviour of the solutions of our systems, let us consider the nonlinear system
\[
\dot{x} = g(t, x) \tag{5}
\]
where \( g: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \) is such that the existence, uniqueness and unbounded forward continuability of the solutions is ensured. In addition, it will be assumed that the origin is an equilibrium, that is, \( g(t, 0) = 0 \) for any \( t \geq 0 \).

The stability of the origin in (5) is addressed with the comparison functions:
- A function \( \alpha: \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( K \) function if \( \alpha(0) = 0 \) and it is nondecreasing.
- A function \( \alpha: \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( K_\infty \) function if \( \alpha(0) = 0 \), \( \alpha(t) \to \infty \) as \( t \to \infty \) and it is strictly increasing.
- A function \( \alpha: \mathbb{R}^+ \to (0, \infty) \) is a \( N \) function if it is nondecreasing.
- A function \( \alpha(t, s): \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( KL \) function if \( \alpha(t, \cdot) \in K \) and \( \alpha(\cdot, s) \) is nondecreasing with respect to \( s \) and \( \lim_{s \to \infty} \alpha(t, s) = 0 \).

Definition 3. The system (5) is globally nonuniformly asymptotically stable if, for any \( \eta > 0 \), there exists a \( \delta(t_0, \eta) > 0 \) such that
\[
|x(t_0)| < \delta(t_0, \eta) \Rightarrow |x(t, t_0, x_0)| < \eta \quad \forall t \geq t_0
\]
and for any \( x(t_0) \in \mathbb{R}^n \) it follows that \( \lim_{t \to +\infty} x(t, t_0, x_0) = 0 \).

Proposition 2. [16, Prop. 2.5] The system (5) is globally nonuniformly asymptotically stable if and only if there exists \( \beta \in KL \) and \( \theta \in N \) (positive and nondecreasing) such that for any \( x(t_0) \in \mathbb{R}^n \) it follows that
\[
|x(t, t_0, x_0)| \leq \beta(\theta(t_0)|x(t_0)|, t - t_0) \quad \forall t \geq t_0. \tag{6}
\]

Definition 4. The linear system (1) is nonuniformly exponentially stable if and only if there exist constants \( K \geq 1, \alpha > 0 \) and \( \varepsilon > 0 \) such that
\[
||\Phi(t, t_0)|| \leq Ke^{-\alpha(t-t_0)+\varepsilon t_0} \quad \text{for any} \quad t \geq t_0 \geq 0.
\]

Lemma 1. If the linear system (1) is nonuniformly exponentially stable then it is globally nonuniformly asymptotically stable.

Proof. As the linear system is nonuniformly exponentially stable, that is
\[
|x(t, t_0, x(t_0))| = ||\Phi(t, t_0)|| \leq Ke^{\varepsilon t_0} e^{-\alpha(t-t_0)|x(t_0)|},
\]
clearly the inequality (5) is verified with the functions \( \theta(t_0) = e^{\varepsilon t_0} \) and \( \beta(r, s) = Ke^{-\alpha s} \) and the result is a consequence of Proposition 2. □
Remark 2. We recall that in the uniform framework, namely, when $\varepsilon = 0$, the above Lemma also has a converse statement and there exists an equivalence between uniform exponential stability and global uniform asymptotical stability. We refer the reader to [18, pp.156–157] for details.

2.3. Statement of the conjecture. As we have set forth the premises now we are able to state our main result of this section.

Nonuniform Nonautonomous Markus–Yamabe Conjecture (NNMYC):

Let us consider the nonlinear system

$$\dot{x} = f(t, x)$$

(7)

where $f: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is such that

(G1) $f$ is continuous with respect to $t$ and $C^1$ with respect to $x$. Its jacobian matrix will be denoted by $Jf(t, \cdot)$.

(G2) $f(t, x) = 0$ if $x = 0$ for all $t \geq 0$.

(G3) For any measurable function $t \mapsto \omega(t)$, the family of linear systems

$$\dot{\vartheta} = Jf(t, \omega(t))\vartheta$$

(8)

has a nonuniform exponential dichotomy spectrum satisfying

$$\Sigma(Jf(t, \omega(t))) \subset (-\infty, 0).$$

(G4) For any measurable function $t \mapsto \omega(t)$, each one of linear systems of the family [8] has an evolution operator associated $\Phi_{\omega(t)}(t, s)$ with nonuniformly bounded growth.

Is the trivial solution globally nonuniformly asymptotically stable for [11]?

We show that this conjecture is true for scalar equation.

**Theorem 1.** The NNMYC is verified for dimension $n = 1$.

**Proof.** Let us consider the nonlinear scalar equation

$$\dot{x} = g(t, x),$$

(9)

with the initial condition $x_0 \neq 0$ at time $t = t_0$. Without loss of generality, it will be supposed that $x_0 > 0$. By uniqueness of the solution it follows that $x(t) > 0$ for any $t \geq t_0$.

Notice that any solution of (9) can be written as follows

$$g(t, x(t)) - g(t, 0) = \frac{\partial g}{\partial x}(t, \theta_t)u$$

with $0 < \theta_t < x(t)$

$$g(t, x(t)) = \frac{\partial g}{\partial x}(t, \theta_t)x(t)$$

with $0 < \theta_t < x(t)$

$$\dot{x}(t) = \frac{\partial g}{\partial x}(t, \theta_t)x(t)$$

with $0 < \theta_t < x(t)$. 

Then we have that
\[
\int_{t_0}^{t} \dot{x}(\tau) d\tau = \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \theta_\tau) x(\tau) d\tau
\]
\[
x(t) = x_0 + \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \theta_\tau) x(\tau) d\tau,
\]
where \(0 < \theta_\tau < x(\tau)\). The above inequality is equivalent to
\[
\frac{d}{dt} \ln \left( x_0 + \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \theta_\tau) x(\tau) d\tau \right) = \frac{\partial g}{\partial x}(t, \theta_t) \quad \text{with} \quad 0 < \theta_t < x(t).
\]
We integrate between \(t_0\) and \(t\) obtaining that
\[
x(t) = x_0 \exp \left( \int_{t_0}^{t} \frac{\partial g}{\partial x}(\tau, \theta_\tau) d\tau \right) \quad \text{with} \quad 0 < \theta_\tau < x(\tau),
\]
that is, the solutions of (9) can be seen as solutions of the linear equation
\[
\dot{z} = \frac{\partial g}{\partial x}(t, \theta_t) z, \quad (10)
\]
for some measurable function \(t \mapsto \theta_t\). Now, the assumption
\[
\Sigma \left( \frac{\partial g}{\partial x}(t, \theta_t) \right) \subset (-\infty, 0) \quad (11)
\]
and the nonuniform asymptotic stability is a consequence of Lemma 1. □

Remark 3. The condition (G4) is established as consequence of (11) which is equivalent to the system (10) has nonuniform dichotomy with projector \(P(\cdot) = I_1\).

On the other hand, by using the Gronwall’s Lemma it can be proved that if \(t \mapsto g_x(t, \theta_t)\) is bounded or locally integrable for any measurable function \(\theta_t\) then (G4) is also verified.

3. Uniform Injectivity and Nonautonomous Weak Markus-Yamabe

In this section we will introduce a Weak Markus-Yamabe Conjecture in a nonautonomous context, which will allow us to connect the Nonuniform Nonautonomous Markus-Yamabe Conjecture and the Jacobian Conjecture. Firstly, given a function \(f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n\), for any \(t \geq 0\) fixed, let us define a family of maps \(F_t : \mathbb{R}^n \to \mathbb{R}^n\) indexed by \(t \in \mathbb{R}^+\) as \(F_t(x) = f(t, x)\).

Definition 5. A family of maps \(F_t : \mathbb{R}^n \to \mathbb{R}^n\) is uniformly injective after a certain time \(t_0 \geq 0\), if for all \(t \geq t_0\) such that \(F_t(x) = F_t(y)\), then \(x = y\).

Example 1. The map
\[
F_t(x, y, z) = (-x + e^{-t}(x + y)^3, -y + e^{-t}(x + y)^3, -z - e^{-t}(x + y)^3)
\]
is uniformly injective after $t_0 = 0$. In fact, we can find explicitly the inverse for this map for each $t \geq 0$. Namely, $F_t^{-1}(x, y, z) = (G_1, G_2, G_3)_t(x, y, z)$ where

\begin{align*}
G_1_t &= -x - e^{-t}(x + y)^3(1 + e^{-t}(x + y)^2)^3 \\
G_2_t &= -y - e^{-t}(x + y)^3 - (x + y)^3(1 + e^{-t}(x + y)^2)^3 \\
G_3_t &= -z + e^{-t}(x + y)^3(1 + e^{-t}(x + y)^2)^3.
\end{align*}

**Example 2.** Given $\lambda_0$ and $a$ such that $\lambda_0 < a < 0$, the family of maps $F_t : \mathbb{R} \to \mathbb{R}$ defined by $F_t(x) = [\lambda_0 + at \sin(t)]x$ is not uniformly injective after none $t_0 \geq 0$ due to the set

$$\{t \in \mathbb{R}^+ : \lambda_0 + at \sin(t) = 0\},$$

is upperly unbounded.

Now, by Definition 5 we introduce the following nonautonomous Weak Markus-Yamabe conjecture.

**Nonautonomous Weak Markus-Yamabe Conjecture:**
Let $F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. If $F$ satisfies (G1), (G3) and (G4), then the family of maps $F_t : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly injective after a certain time $t_0 \geq 0$.

The goal of the following proposition is to relate the Nonuniform Nonautonomous Markus-Yamabe Conjecture and Weak Markus-Yamabe Conjecture.

**Proposition 3.** If the Nonuniform Nonautonomous Markus-Yamabe Conjecture it satisfied then the Nonautonomous Weak Markus-Yamabe Conjecture is true.

**Proof.** The proof will be made by contradiction by assuming that the family of maps $F_t$ is not uniformly injective after a certain time $t_0$, this means there exist a couple $x, y \in \mathbb{R}^n$ such that

$$F_t(x) = F_t(y) \quad \text{for all} \quad t \geq t_0.$$

Now we establish, for all $t \geq 0$ fixed, a new map $G : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$G(t, z) := G_t(z) = F_t(z + x) - F_t(x).$$

Notice that $G(t, 0) = G_t(0) = 0$ for any $t \geq 0$. Moreover, we can verify that the ODE system $\dot{z} = G(t, z)$ satisfies (G1), (G3) and (G4) as a consequence of the nonlinear system

$$\dot{x} = F(t, x)$$

also satisfies them, thus the Nonuniform Nonautonomous Markus-Yamabe Conjecture assures that the origin is globally nonuniformly asymptotically stable.
On the other hand, note that the initial value problem
\[
\begin{aligned}
\dot{z} &= G(t, z) \\
z(t_0) &= z_0
\end{aligned}
\]
with \( z_0 = y - x \), has a constant solution \( z(t, t_0, z_0) = y - x \neq 0 \) for all \( t \geq t_0 \), which does not converge to 0 when \( t \to \infty \), therefore we obtain a contradiction. Finally the family of maps \( F_t : \mathbb{R}^n \to \mathbb{R}^n \) is uniformly injective after a certain time \( t_0 \).

**Remark 4.** The proof of this proposition is inspired by [11, p.177]. We stress that the initial time \( t_0 \) is not relevant in an autonomous context. Contrarily, \( t_0 \) plays a fundamental role in the nonautonomous framework because the solutions of an IVP define a continuous time process (see [19, Definition 2.1]) under suitable conditions. In consequence, we have introduced a definition of uniform injectivity tailored for our purposes.

\[\square\]

### 4. Jacobian Conjecture and main result

The following Theorem is the main result of this article, which states that an affirmative answer of the Nonuniform Nonautonomous Markus-Yamabe Conjecture implies, after a certain time \( t_0 \geq 0 \), the Jacobian Conjecture.

By [3], we know that it is suffices to prove the Jacobian Conjecture for all \( n \geq 1 \) and for all \( S : \mathbb{C}^n \to \mathbb{C}^n \) of the form \( S = x + H \) with \( JH \) nilpotent and each \( H_i \) homogeneous of degree 3 or \( H_i \equiv 0 \), where \( H_i \) represents the \( i \)-th coordinate of \( H \).

Thus, for our purpose and taking in account the nonuniform nonautonomous framework, from now on we consider maps \( M : \mathbb{R}^+ \times \mathbb{C}^n \to \mathbb{C}^n \) defined by

\[
(t, x) \mapsto (M_1(t, x), \ldots, M_n(t, x)) = (\lambda x_1 + H_1(t, x), \ldots, \lambda x_n + H_n(t, x)),
\]

with \( \lambda < 0 \) such that

(i) \( M \) is continuous with respect to \( t \),

(ii) for all \( t \geq 0 \) fixed, \( JH(t,x) \) is nilpotent and \( (H_i)_t \) is zero or homogeneous of degree 3 for \( i = 1, \ldots, n \),

(iii) for any bounded measurable map \( t \mapsto \omega(t) \)

\[
\|JH(t, \omega(t))\| \leq \delta e^{-\epsilon t} \quad \text{for any} \quad t \geq 0.
\]

**Remark 5.** The core of property (iii) is the Robustness's theorem. In particular we have in mind the [2] Theorem 1 which states that if \( \|B(t)\| \leq \delta e^{-\epsilon t}, t \in \mathbb{R}^+ \), with \( \delta < \alpha/K \) (see Definition 7) then the system \( \dot{x} = (A(t) + B(t))x \) has nonuniform exponential dichotomy on \( \mathbb{R}^+ \) with \( P(\cdot) = I_n \).

**Theorem 2.** If for all \( n \geq 1 \) the Nonuniform Nonautonomous Markus-Yamabe Conjecture is verified for the map \( M \) then, after a certain time \( t_0 \geq 0 \), the Jacobian Conjecture is true.
Proof. The result follows if we prove that nonautonomous vector field \( \overline{M}(t, x) : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) defined by

\[
\overline{M}(t, x) := (\text{Re } M_1(t, x), \text{Im } M_1(t, x), \ldots, \text{Re } M_n(t, x), \text{Im } M_n(t, x)),
\]

satisfies the Nonuniform Nonautonomous Markus-Yamabe Conjecture.

The conditions (G1)–(G2) are verified due to \( \overline{M} \) is a homogeneous polynomial map and it is continuous with respect to \( t \).

With respect to conditions (G3)–(G4), let us consider any bounded measurable function \( t \mapsto \omega(t) \) and the \( 2n \)-dimensional linear systems

\[
\dot{\vartheta} = (\text{Diag}\{\lambda\} + J\overline{M}(t, \omega(t))) \vartheta. 
\] (13)

It is clear that the system \( \dot{\vartheta} = \text{Diag}\{\lambda\} \vartheta \) has nonuniform exponential dichotomy on \( \mathbb{R}^+ \) with projector \( P(t) = I \) for any \( t \geq 0 \); thus \( \Sigma(\text{Diag}\{\lambda\}) = \{\lambda\} \subset (-\infty, 0) \) and the evolution operator \( \Phi_{\omega(t)}(t, s) \) of this system has nonuniformly bounded growth. Now, due to (12) and by using Robustness result [2, Theorem 1], we have that the family of linear systems (13) has nonuniform exponential dichotomy on \( \mathbb{R}^+ \) with projector \( P(t) = I \) for any \( t \geq 0 \), thus \( \Sigma(\text{Diag}\{\lambda\} + J\overline{M}(t, \omega(t))) \subset (-\infty, 0) \) and the evolution operator \( \Phi_{\omega(t)}(t, s) \) of the systems (13) also has nonuniformly bounded growth.

By Proposition 3, the map \( M(t) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) is uniformly injective after a certain time \( t_0 \geq 0 \), and therefore the map \( M_t : \mathbb{C}^n \to \mathbb{C}^n \) is also uniformly injective after a certain time \( t_0 \geq 0 \).

Finally, our result follows by using [9, Theorem 2.2] for each \( M_t \) with \( t \geq t_0 \) fixed.

\[ \square \]

Remark 6. Similarly as in Proposition 3, the proof of above theorem follows the steps done by van den Essen in [11], however we are using a nonautonomous spectral theory instead of the eigenvalues spectrum, which induces differences beyond the formal and requires the mastery of several tools which can be pigeonholed in the nonautonomous linear algebra (see [23, p.423]).

The following example shows a nonautonomous map that satisfies conditions of NN-MYC. Moreover, this map verifies conditions (i)–(iii), and therefore, for each \( t \geq 0 \) fixed, satisfies the Jacobian Conjecture; in particular we can find explicitly its inverse.

Example 3. Let us consider the nonautonomous map

\[
M(t, x, y, z) = (\lambda x + e^{-t}y^3, \lambda y + e^{-t}(x + z)^3, \lambda z - e^{-t}y^3), \lambda < 0.
\]

It is easy to see that \( M \) satisfies (G1) and (G2). In order to show that the map verifies (G3), we note that for bounded measurable map \( t \mapsto \omega(t) =
\]
$(x(t), y(t), z(t))$ we have that

$$JM(t, \omega(t)) = \text{Diag}\{\lambda\} + 3e^{-t} \begin{pmatrix}
0 & y(t)^2 & 0 \\
(x(t) + z(t))^2 & 0 & (x(t) + z(t))^2 \\
0 & -y(t)^2 & 0
\end{pmatrix},$$

thus $||JH|| = \sqrt{18e^{-2t}y(t)^2}$ or $||JH|| = \sqrt{18e^{-2t}(x(t) + z(t))^2}$. Now, taking account [2, Theorem 1], if we consider $K = 1, \varepsilon \geq 1$ and $\alpha \leq -\lambda$ with $-\lambda$ large enough; we can use the roughness theorem in order to verify $\Sigma JM(t, \omega(t)) \subset (-\infty, 0)$. The condition (G4) is verified due to $\theta = \text{Diag}\{\lambda\} \theta$ has a nonuniform exponential dichotomy on $\mathbb{R}^+$ with projector $P(t) = I_3$ for any $t \geq 0$; thus the evolution operator $\Phi_{\omega(t)}(t, s)$ of this system has nonuniformly bounded growth. Now, by using again the roughness theorem, we can deduce that the evolution operator $\Phi_{\omega(t)}(t, s)$ of the systems $\dot{\vartheta} = JM(t, \omega(t)) \vartheta$ also have nonuniformly bounded growth. Therefore, the map $M$ satisfies hypothesis of NN-MYC.

On the other hand, the map $M$ verifies the condition (i) clearly. The nilpotency condition (ii) for any fixed $t \geq 0$ follows from [3, Theorem 1]. Now, (iii) follows of the fact to have use previously Roughness Theorem.

Hence, $M_t$ is an example to Jacobian Conjecture for each $t \geq 0$ fixed.

Finally, we can find explicitly the inverse of $M_t(\cdot)$ for each $t \geq 0$ fixed. Namely, $M_t^{-1}(x, y, z) = (N_1, N_2, N_3)_t(x, y, z)$ where

$$N_1(x, y, z) = \frac{1}{\lambda} \left( x - e^{-t} \left[ \frac{1}{\lambda} (y - e^{-t}(\frac{x+z}{\lambda})^3) \right]^3 \right)$$

$$N_2(x, y, z) = \frac{1}{\lambda} \left( y - e^{-t}(\frac{x+z}{\lambda})^3 \right)$$

$$N_3(x, y, z) = \frac{1}{\lambda} \left( z + e^{-t} \left[ \frac{1}{\lambda} (y - e^{-t}(\frac{x+z}{\lambda})^3) \right]^3 \right).$$

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