Generations of solvable discrete-time dynamical systems

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Abstract

A technique is introduced which allows to generate—starting from any solvable discrete-time dynamical system involving \(N\) time-dependent variables—new, generally nonlinear, generations of discrete-time dynamical systems, also involving \(N\) time-dependent variables and being as well solvable by algebraic operations (essentially by finding the \(N\) zeros of explicitly known polynomials of degree \(N\)). The dynamical systems constructed using this technique may also feature large numbers of arbitrary constants, and they need not be autonomous. The solvable character of these models allows to identify special cases with remarkable time evolutions: for instance, isochronous or asymptotically isochronous discrete-time dynamical systems. The technique is illustrated by a few examples.

Keywords: discrete time, solvable discrete-time dynamical systems, difference equations.

1 Introduction

The investigation of the evolution in discrete-time of dynamical systems—and in particular the identification of such models which are amenable to exact treatments—has become an important area of mathematical physics in the last one-two decades, see for instance the following review papers and books: \([1,2,3,4,5]\). The models under consideration in this paper describe the evolution in the discrete-time variable \(\ell = 0, 1, 2, ...\) of an arbitrary number \(N\) of identical points moving in the complex plane, the positions of which are characterized by \(N\) complex coordinates, for instance \(x_n(\ell)\) or \(y_n(\ell)\). Both the equations of motion characterizing these models, and their solution, only involve the algebraic operation of finding the \(N\) zeros of an explicitly known \(\ell\)-dependent polynomial of degree \(N\) in \(z\). The technique to identify such models is analogous to, but more flexible than, the approach employed in \([6,7,8,9,10,11]\) to identify and discuss several solvable discrete-time many-body problems.
Notation 1.1. Above and hereafter (unless otherwise indicated) indices such as \( n, m, j \) run over the integers from 1 to \( N \), with \( N \) a given positive integer (\( N \geq 2 \)). All quantities—except those taking integer values, such as the indices and the discrete-time \( \ell \)—are complex numbers. Hereafter superimposed arrows denote \( N \)-vectors, for instance the \( N \)-vector \( \vec{y} = (y_1, y_2, ..., y_N) \) features the \( N \) components \( y_n \), while the underlined notation \( \vec{x} = \{x_1, x_2, ..., x_N\} \) denotes the unordered set of \( N \) (complex) numbers \( x_n \). In the following we will generally limit consideration to the generic case in which the \( N \) complex numbers \( y_m \) or \( x_n \) are all different among themselves.

In the following it will be important to associate to an unordered set \( \vec{x} \) the \( N \)-vectors the components of which correspond to a specific ordering assignment of the \( N \) elements \( x_n \) of \( \vec{x} \). There are of course generally \( N! \) different such vectors, and we will hereafter use for them the notation \( \vec{x}_\mu = \{x_{\mu,1}, x_{\mu,2}, ..., x_{\mu,N}\} \), with the index \( \mu \) identifying a specific ordering assignment of the \( N \) numbers \( x_n \), hence taking—as above—the \( N \) integer values from 1 to \( N! \) (since \( N \) different objects can of course be ordered in \( N! \) different ways).

We also introduce the notation \( \sum_{n_1, n_2, ..., n_m=1}^N \) (note the appended star!) to denote a sum ranging from 1 to \( N \) over each of the \( m \) indices \( n_1, n_2, ..., n_m \), with the restriction that these \( m \) indices be all different among themselves.

Finally, we adopt the usual convention according to which a void sum vanishes and a void product equals unity; note that this implies that the sum \( \sum_{n_1, n_2, ..., n_m=1}^N \) vanishes identically if \( m \) exceeds \( N, m > N \).

Remark 1.1. In the following we mainly focus on dynamical systems characterized by first-order equations of motion, say

\[
y_m(\ell + 1) = f_m(\vec{y}),
\]

but occasionally below it will also be of interest to consider higher-order equations of motion.

Remark 1.2. As in the previous models [6, 7, 8, 9, 10, 11], the solvable equations of motion identified below determine the \( N \) points characterized by the \( N \) coordinates, say, \( x_n \equiv x_n(\ell) \), as an unordered set \( \vec{x}(\ell) \equiv \{x_1(\ell), x_2(\ell), ..., x_N(\ell)\} \), indeed generally as the \( N \) zeros of an \( \ell \)-dependent polynomial \( P_N(z;\ell) \) of degree \( N \) in \( z \); hence these equations of motion are only deterministic inasmuch as they identify (uniquely) the unordered set \( \vec{x}(\ell + 1) \) in terms of the unordered set \( \vec{x}(\ell) \), but they do not associate each coordinate \( x_n(\ell) \) to a specific value of the index \( n \) labeling it. So these models describe \( N \) indistinguishable “particles”, the positions of which in the complex \( x \)-plane at the discrete-time \( \ell \) are characterized by the \( N \) coordinates \( x_n(\ell) \). A preferred association of these coordinates to their labels might of course be provided at each step of the discrete-time evolution by an argument of contiguity, but this is interesting only if the \( N \) values \( x_n(\ell + 1) \) are all adequately separated from each other—in the complex \( x \)-plane—and each of them is adequately close to one and only one of the \( N \) values \( x_m(\ell) \) being themselves well separated among each other; more about this below.

The main idea of the previous models [6, 7, 8, 9] was to identify a solvable discrete-time evolution of an \( N \times N \) matrix and to then focus on the evolution of
its $N$ eigenvalues. A more straightforward approach—already employed in [10] and [11]—focused directly on the solvable evolution of a polynomial $p_N(z;\ell)$ of degree $N$ in $z$ and of its $N$ zeros $x_n(\ell)$. The more convenient approach employed in the present paper focuses more directly on the discrete-time evolutions of the $N$ coefficients $y_m(\ell)$ respectively the $N$ zeros $x_n(\ell)$ of a time-dependent polynomial $p_N(z;\ell)$, by taking advantage of a key formula relating these evolutions, see below. The solvable discrete-time dynamical systems which can be identified via this approach are described in the following Section 2 (with a proof postponed to Appendix A in order to avoid interrupting the flow of the presentation), and several examples are discussed in the subsequent Section 3 and in Appendix B. In the last Section 4 we indicate to what extent the findings reported in the present paper go beyond previously reported results, and we tersely outline possible future developments.

2 Solvable discrete-time dynamical systems

A main protagonist of our treatment is the time-dependent monic polynomial

$$p_N(z;\ell) \equiv p_N(z;\bar{y}(\ell);\bar{x}(\ell)) = z^N + \sum_{m=1}^{N} [y_m(\ell) z^{N-m}] = \prod_{n=1}^{N} [z - x_n(\ell)] .$$

(2)

Remark 2.1. The notation $p_N(z;\bar{y}(\ell);\bar{x}(\ell))$ is of course redundant, since to define this monic polynomial of degree $N$ in the variable $z$ at any time $\ell$ it is clearly sufficient to assign either its $N$ coefficients $y_m(\ell)$ or its $N$ zeros $x_n(\ell)$: see (2). Indeed, the $N$ coefficients $y_m(\ell)$ are defined in terms of the $N$ zeros $x_n(\ell)$ by the standard formulas

$$y_m(\ell) = \frac{(-1)^m}{m!} \sigma_m(\bar{x}(\ell)) ,$$

(3a)

where, above and hereafter (see Notation 1.1),

$$\sigma_m(\bar{x}) = \sum_{n_1,n_2,\ldots,n_m=1}^{N} \star (x_{n_1} x_{n_2} \cdots x_{n_m}) ;$$

(3b)

and conversely the $N$ zeros $x_n(\ell)$ are uniquely determined (but only up to permutations) by the $N$ coefficients $y_m(\ell)$, although of course explicit formulas to this effect are only available for $N \leq 4$. ■

The main tool of our approach is the following key formula, implied by (2) (for a proof see Appendix A):

$$\prod_{j=1}^{N} [x_n(\ell+p) - x_j(\ell)] + \sum_{m=1}^{N} \left\{ [y_m(\ell+p) - y_m(\ell)] [x_n(\ell+p)]^{N-m} \right\} = 0 ,$$

(4a)
which holds for every positive integer $p$. Note that this formula entails that the $N$ values of the variables $x_n(\ell + p)$—for the $N$ values of the index $n$ in the range from 1 to $N$—are the $N$ zeros of the following polynomial of degree $N$ in the complex variable $z$:

$$
\hat{p}_N(z; \ell) = \prod_{j=1}^{N} [z - x_j(\ell)] + \sum_{m=1}^{N} \left\{ [y_m(\ell + p) - y_m(\ell)] (z)^{N-m} \right\}. \quad (4b)
$$

The merit of this formula—in either one of its equivalent versions $(4a)$ or $(4b)$—is to relate the discrete-time evolution of the $N$ zeros $x_n(\ell)$ to the discrete-time evolution of the $N$ coefficients $y_m(\ell)$, allowing to identify directly the equations of motion of new solvable dynamical systems from those of known solvable dynamical systems. Indeed let us assume that the $N$ quantities $y_m(\ell)$ evolve in the discrete-time variable $\ell$ according to the following dynamical system:

$$
y_m(\ell + 1) = f_m(\vec{y}(\ell), \ldots, \vec{y}(\ell + p - 1); \ell), \quad (5)$$

where the $N$ functions $f_m$ are conveniently assigned so that this system is solvable. For instance this solvable dynamical system might be any one of those treated in the papers $[6, 7, 8, 9, 10, 11]$. Note that, at this stage, we do not exclude that the functions $f_m$ might feature an explicit dependence on the discrete-time variable $\ell$, implying thereby that the corresponding solvable dynamical system (5) is not autonomous; there indeed exist nonautonomous discrete-time dynamical systems which are nevertheless solvable, for instance the nonlinear system treated in $[10]$ or the rather trivial decoupled linear system (with $p = 1$)

$$
y_m(\ell + 1) = g_m(\ell) y_m(\ell) + h_m(\ell) \quad (6a)
$$

with $g_m(\ell)$ and $h_m(\ell)$ arbitrarily assigned functions, the solution of which reads of course as follows (see Notation 1.1):

$$
y_m(\ell) = y_m(0) \prod_{\ell'=0}^{\ell-1} [g_m(\ell')] + \sum_{\ell'=0}^{\ell-1} \left\{ \prod_{\ell''=0}^{\ell-1} g_m(\ell'') \right\} h_m(\ell''), \quad (6b)
$$

It is then plain that the dynamical system characterized by the equations of motion implied by (4) and (5), the equations of motion of which read as follows,

$$
\prod_{j=1}^{N} [x_n(\ell + p) - x_j(\ell)] \\
+ \sum_{m=1}^{N} \left\{ [f_m(\vec{y}(\ell), \ldots, \vec{y}(\ell + p - 1); \ell) - y_m(\ell)] [x_n(\ell + p)]^{N-m} \right\} = 0 \quad (7a)
$$

is as well solvable. Note that $(7a)$ is equivalent to the prescription that the $N$ updated values $x_n(\ell + p)$ of the $N$ coordinates $x_n(\ell)$ coincide with the $N$ zeros
of the following polynomial equation of degree $N$ in $z$,

$$
\prod_{j=1}^{N} [z - x_j (\ell)] + \sum_{m=1}^{N} \left\{ \left[ f_m (\vec{y}(\ell)), \ldots, \vec{y}(\ell + p - 1); y_m (\ell) \right] z^{N-m} \right\} = 0.
$$

Of course in both of the last two formulas the $N$ components $y_m (k)$ of the $N$-vector $\vec{y}(k)$, where $k = \ell, \ell + 1, \ldots, \ell + p - 1$, should be replaced by their expressions, see (3), in terms of the $N$ components of the unordered set $\vec{x}(k)$, namely

$$
y_m (k) = \frac{(-1)^m}{m!} \sigma_m (\vec{x}(k)).
$$

The claim that this dynamical system (7) is solvable is of course validated by the fact that the coordinates $x_n (\ell)$, yielding its solution at time $\ell$, are the $N$ zeros of the polynomial (2), of degree $N$ in $z$—hence they are obtainable by an algebraic operation—while the coefficients $y_m (\ell)$ of this polynomial (2) are themselves obtainable by algebraic operations, since the dynamical system (5) characterizing the time-evolution of these quantities is by assumption itself solvable. In particular the initial-value problem for this dynamical system can be solved—for an arbitrary assignment of the initial data $x_n (0)$—via the following 3 steps: (i) compute the corresponding $N$ initial data $y_m (k)$ with $k = 0, 1, \ldots, p - 1$ of the dynamical system (5) via (7c); (ii) obtain the $N$ values $y_m (\ell)$ for $\ell \geq p$ via the solvable dynamical system (5) with the initial data computed in step (i); (iii) find the $N$ zeros $x_n (\ell)$ for $\ell \geq p$ of the polynomial $p_N(z;\ell)$, see (2), defined by its $N$ coefficients $y_m (\ell)$ as computed in step (ii).

Specific examples of solvable dynamical systems (7) will be exhibited and discussed in the following Section 3.

It is moreover plain that the usefulness of the key formula (4) is not limited to the identification of the new solvable dynamical system (7) associated to the previously known solvable dynamical system (5): it also opens the possibility to iterate—albeit up to a limitation that we explain below—the procedure we used above in order to obtain the new solvable dynamical system (7) from the known solvable system (5), by then using as known input in this approach just the newly identified solvable dynamical system (7). And clearly this approach can be iterated over and over again, yielding endless hierarchies of new solvable dynamical systems, in analogy to what was recently done for continuous-time dynamical systems, see [12].

But there is a significant difference with respect to the analogous treatment in the continuous-time case, see [12] and [13]. The difference originates from the fact that in the continuous-time case the time evolution is in fact completely deterministic, implying that in that context one is dealing with distinguishable particles: the assignment of the initial values $x_n (0)$ of the particle positions in the complex $x$-plane at the initial time $t = 0$ determines unambiguously—thanks to the continuity of the time-evolution of the particle coordinates $x_n (t)$ as functions of the continuous-time variable $t$—the particle positions $x_n (t)$ for all future time $t$: say, the coordinate $x_1 (t)$ is the one that has evolved continuously
over time from the initial datum \(x_1(0)\), and likewise for all values of the labels \(n\) in the range from 1 to \(N\) identifying the coordinates \(x_n(t)\). Note that this is the case even though the particle positions \(x_n(t)\) are, also in that context, identified with the zeros of a time-dependent polynomial \(p_N(z; t)\) of degree \(N\) in the complex variable \(z\).

In the discrete-time case treated in this paper one is instead dealing—as explained above—with indistinguishable particles. This does not cause any problem for the definition and solution of the discrete-time model \( (7) \) described just above, which characterizes the evolution in the discrete-time variable \(\ell\) of the unordered set of coordinates \(z(\ell)\). But of course it raises an issue when we try to use that model to describe the evolution of the coefficients of a polynomial, since this set of numbers is of course an ordered set.

To explain what we mean in the simplest setting—and thereby also clarify some relevant differences among the continuous-time case treated in \([12]\) and the discrete-time case treated in the present paper—let us now focus on equations of motion of first order, i.e. let us refer—in the remaining part of this Section 2—to the \(p = 1\) special case of the dynamical systems \( (5) \) and \( (7) \) and of the key formula \( (4) \). The interested reader will have no difficulty to extend the following treatment to values of the integer parameter \(p\) larger than unity.

In the following we will refer to the solvable dynamical system \( (5) \) as the seed dynamical system and to the new dynamical system \( (7) \)—the solvable character of which has been detailed just above—as the generation zero dynamical system. We will also refer to polynomial \( (2) \) as the generation zero polynomial.

Let us then try and iterate the process used to obtain the generation zero dynamical system \( (7) \) from the seed system \( (5) \). The idea—following the analogous treatment in the continuous-time case \([12]\)—is to introduce a new (monic, \(\ell\)-dependent) polynomial of degree \(N\) in the complex variable \(z\) characterized by the property that its \(N\) coefficients evolve in the discrete time \(\ell\) as the solutions of the solvable generation zero system \( (7) \). Specifically, we introduce the generation one polynomial

\[
p_N^{(\mu_1)}(z;\ell) \equiv p_N^{(\mu_1)}(z;\vec{y}^{(\mu_1)}(\ell);\vec{x}^{(\mu_1)}(\ell)) = z^N + \sum_{m=1}^{N} [y_m^{(\mu_1)}(\ell) \cdot z^{N-m}]
\]

\[
= \prod_{n=1}^{N} \left[ z - x_n^{(\mu_1)}(\ell) \right], \tag{8}
\]

such that its coefficient vector \(\vec{y}^{(\mu_1)}(\ell) = (y_1^{(\mu_1)}(\ell), \ldots, y_N^{(\mu_1)}(\ell))\) (see Notation 1.1) coincides with an appropriate ordering (prescribed by the index \(\mu_1\)) of the unordered solution set \(\underline{x}(\ell)\) of the generation zero dynamical system \( (7) \) (of course with \(p = 1\)). Here the index \(\mu_1\) labels the possible orderings of the \(N\) complex numbers \(x_1(\ell), x_2(\ell), \ldots, x_N(\ell)\), implying the definition of the \(N\)-vectors

\[
\underline{x}_{[\mu_1]}(\ell) = (x_{[\mu_1],1}(\ell), x_{[\mu_1],2}(\ell), \ldots, x_{[\mu_1],N}(\ell)).
\]

This index \(\mu_1\) takes the \(N!\) integer values from 1 to \(N!\) (see Notation 1.1), since
there are $N!$ different permutations of $N$ different objects, implying of course the existence of $N!$, generally different, $N$-vectors $\vec{x}_{\mu_1}(\ell)$. And the prescription for $\vec{y}^{(\mu_1)}(\ell)$ mentioned just above, characterizing the generation one polynomial \[8\], reads as follows:

$$\vec{y}^{(\mu_1)}(\ell) = \vec{x}_{\mu_1}(\ell), \quad y_m^{(\mu_1)}(\ell) = x_{\mu_1,m}(\ell). \quad (9)$$

**Remark 2.2.** Let us reiterate that throughout our discussion we focus for simplicity on the generic case of monic polynomials of degree $N$ in the complex variable $z$ the $N$ coefficients of which—and likewise the $N$ zeros of which—are all different among themselves, and correspondingly on dynamical systems describing the evolution in the discrete time $\ell$ of $N$ points moving in the complex plane the positions of which at the same time $\ell$ never coincide. It is indeed plain that this is the generic situation, and it will be clear from the following treatment that the occurrence of “particle collisions”—the coincidence of two particle positions at the same time—is an event that can only occur for a set of initial data having vanishing measure in the space of such data. ■

**Remark 2.3.** To allay any possible uneasiness about the notion that a specific permutation labeled by an index $\mu$ in the range $1 \leq \mu \leq N!$ identifies a specific order of the $N$ elements $x_n$ of an unordered set $x$ of $N$ complex numbers $x$, let us provide an example of a procedure to do so. One begins by defining a specific ordering assignment of the unordered set $x$ of $N$ different complex numbers $x_n$, for instance that characterized by the (“increasing”) lexicographic rule stipulating that, of two complex numbers with different real parts, the one with algebraically smaller real part comes first, and of two complex numbers with equal real parts, the one with algebraically smaller imaginary part comes first. After this ordering of the unordered set $x$ is thus established—defining the $N$-vector $\vec{x}[1]$ with components $x[1],n$—one can subsequently apply $N!$ standard sequential reorderings—labeled by the index $\mu$ ranging from 2 to $N!$—consisting in $N!$ permutations of the $N$ components $x[1],n$ of the $N$-vector $\vec{x}[1]$, these permutations being themselves labeled by the value of the index $\mu$ according to some standard rule, for instance the standard lexicographic ordering of the $N!$ permutations of $N$ different objects (for $N = 3$: $abc, acb, bac, bca, cab, cba$). Note that—see Remark 2.2—we always assume to deal with the generic case of (monic) polynomials of degree $N$ featuring $N$ different zeros, therefore defining new polynomials with $N$ different coefficients as detailed above, see (9). ■

The generation one dynamical systems characterize the evolution in the discrete time $\ell$ of the unordered set $\vec{x}^{(\mu_1)}(\ell) \equiv \{x_1^{(\mu_1)}(\ell), x_2^{(\mu_1)}(\ell), ..., x_N^{(\mu_1)}(\ell)\}$ of the $N$ zeros of the polynomial $p_N^{(\mu_1)}(z;\ell)$, see \[8\] with \[9\]. It is plain—via the key formula \[4\] (with $\ell = 1$) and \[9\]—that the equations of motion of these
the following polynomial equation of degree $N$

Here of course the coordinates $x$ unordered

must recall that, via (9), $x$

obtainable by algebraic operations since its

coefficients

by an algebraic operation. Of course, the generation one polynomial is itself

the solutions of the

generation zero dynamical system

new
dynamical systems read as follows:

$$
\prod_{j=1}^{N} \left[ x_n^{(\mu_1)} (\ell + 1) - x_j^{(\mu_1)} (\ell) \right] \\
+ \sum_{m=1}^{N} \left\{ [x_{[\mu_1],m} (\ell + 1) - x_{[\mu_1],m} (\ell)] \left[ x_n^{(\mu_1)} (\ell + 1) \right]^{N-m} \right\} = 0 , \quad (10a)
$$

i.e. they imply that the $N$ (updated) elements $x_n^{(\mu_1)} (\ell + 1)$ of the unordered set

$\mathcal{Z}^{(\mu_1)} (\ell + 1) \equiv \{ x_1^{(\mu_1)} (\ell + 1) , x_2^{(\mu_1)} (\ell + 1) , \ldots , x_N^{(\mu_1)} (\ell + 1) \}$ are the $N$ zeros of the following polynomial equation of degree $N$ in the complex variable $z$:

$$
\prod_{j=1}^{N} \left[ z - x_j^{(\mu_1)} (\ell) \right] + \sum_{m=1}^{N} \left\{ [x_{[\mu_1],m} (\ell + 1) - x_{[\mu_1],m} (\ell)] \right\} z^{N-m} = 0 . \quad (10b)
$$

Here of course the coordinates $x_{[\mu_1],m} (\ell)$ appearing in these equations (10) are the solutions of the generation zero dynamical system, see (7), ordered according to the prescription characterized by the value of the index $\mu_1$, as explained above (and see also below). Thus, the quantities $x_{[\mu_1],m} (\ell + 1)$ are the $N$ zeros—ordered according to the prescription identified by the value of the index $\mu_1$—of the polynomial

$$
\prod_{j=1}^{N} \left[ z - x_j (\ell) \right] + \sum_{m=1}^{N} \left\{ [f_m (\bar{y} (\ell); \ell) - y_m (\ell)] \right\} z^{N-m} = 0 \quad (10c)
$$

(see (7b) with $p = 1$) where the $N$ quantities $y_m (\ell)$ must of course be replaced by their expressions (7c) in terms of the $N$ coordinates $x_n (\ell)$ (the order of the labeling being in this context irrelevant, see (7d)).

To interpret (10) as the evolution law for the unordered set $x^{(\mu_1)} (\ell)$, we must recall that, via (9), $x_{[\mu_1],m} (\ell) = y_{m}^{(\mu_1)} (\ell)$ and, using relation (8), make the following replacement in (10):

$$
x_{[\mu_1],m} (\ell) = y_{m}^{(\mu_1)} (\ell) = \frac{(-1)^m}{m!} \sigma_m \left( x^{(\mu_1)} (\ell) \right) . \quad (10d)
$$

The new dynamical system (10) is solvable, since its solution is provided by the $N$ zeros of the generation one polynomial $p_N^{(\mu_1)} (z; \ell)$ given by (5)—i.e., by an algebraic operation. Of course, the generation one polynomial is itself obtainable by algebraic operations since its coefficients $y_{m}^{(\mu_1)} = x_{[\mu_1],m} (\ell)$ are given by the permutation of the unordered set $\mathcal{Z} (\ell)$ associated with the index $\mu_1$, itself the solution of the solvable dynamical system (7).

Because the evolution prescribed by the new dynamical system (10) depends on the assignment of the ordering of the zeros of the generation zero polynomial (2), see (9), we discuss several possibilities of making this assignment.

One possibility is to make a specific assignment for the ordering prescription, once and for all, corresponding to a specific assignment of the value of the index
\( \mu_1 \): for instance one might assume that all unordered sets \( x(\ell) \) be replaced by \( N \)-vectors \( \vec{x}(\ell) \) the \( N \) components of which are ordered, say, lexicographically. This has the following consequences: (i) consideration is then limited to only a specific one out of the \textit{a priori} possible \textit{generation one dynamical systems}; (ii) one is then dealing with a dynamical system describing the evolution of \textit{distinguishable} particles, the identity of which is identified by their relative positions in the \textit{complex} \( x \)-plane; (iii) the initial data for this specific dynamical system cannot be freely assigned: indeed, their values must be assigned not only consistently with their identities (which is always possible by adjusting the identities to the assigned values), they must moreover be the \( N \) zeros of a polynomial of degree \( N \) in its \textit{complex} variable \( z \) the \( N \) coefficients of which satisfy themselves the assigned prescription, and this of course entails a limitation on the corresponding \( N \) zeros, hence on these \textit{initial values}. Note that an equivalent way to describe this possibility is to let the \textit{initial data} be assigned \textit{arbitrarily} and then to adjust the ordering assignment of the \textit{unordered sets} of data consistently with this initial assignment: but then the very dynamics associated with this point of view would be dependent on the assignment of the \textit{initial data}, which is not consistent with what is usually meant by the definition of a dynamical system as a set of rules—not themselves dependent on the initial data—which determine how the initial data evolve over time...

Another possibility would be to assign an \( \ell \)-dependent ordering prescription based on \textit{contiguity over time}: given two sets of unordered data \( x(\ell) \) and \( x(\ell + 1) \) one might require that they be ordered so that the distance in the \textit{complex} \( x \)-plane between these coordinates at time \( \ell \) and \( \ell + 1 \) having the same label be less than the distance between the coordinates having different labels, 

\[
|x_n(\ell + 1) - x_n(\ell)| < |x_n(\ell + 1) - x_m(\ell)| \quad \text{if} \quad n \neq m.
\]

Clearly this prescription defines \textit{unambiguously} the label assignments (i.e., the particle identities) at time \( \ell + 1 \) corresponding to any label assignment at time \( \ell \), and viceversa, for any \textit{generic} configuration of \( 2N \) points \( x_n(\ell) \) and \( x_n(\ell + 1) \) in the \textit{complex} \( x \)-plane (i.e., for any \textit{arbitrary} configuration excluding a set of configurations having \textit{zero measure} in the set of all possible configurations); but it is plain that this is a reasonable prescription only if 

\[
|x_n(\ell + 1) - x_n(\ell)| < |x_n(\ell + 1) - x_m(\ell)| \quad \text{if} \quad n \neq m,
\]

namely when—with this assignment—the positions of every particle at time \( \ell \) and \( \ell + 1 \) are much closer to each other than the positions of any two different particles among themselves at time \( \ell \) and at time \( \ell + 1 \) (see examples below).

A third interesting possibility—not discussed any further in the present paper—is to assign in a \textit{random} manner—at every step of the \textit{discrete-time} evolution—the prescription to go from the \textit{unordered} set of the \( N \) zeros of the \textit{generation zero} polynomial to the \textit{ordered} set of the \( N \) coefficients of the \textit{generation one} polynomial, thereby producing a dynamical system featuring a \textit{random} evolution.

Up to now we have discussed the \textit{first} iteration of our approach—and this justified our notation \( \mu_1 \) rather than just \( \mu \) for the relevant parameter identifying the prescription characterizing the transition from the \( N \) zeros of the \textit{generation zero} polynomial to the \( N \) coefficients of the \textit{generation one} polynomial, yield-
ing the identification of solvable generation one dynamical systems. It is plain how further iterations could be performed, yielding new solvable discrete-time dynamical systems: these developments—which are not detailed in the present paper—are rather obvious given the analogies with the treatment provided in [12] in the continuous-time context and the new features of the discrete-time context discussed above.

Let us end this Section 2 with the following important remark, which is then illustrated by some of the examples discussed in the following Section 3.

Remark 2.4. A solvable dynamical system may well inherit some properties from the seed solvable dynamical system generating it. For instance, if the seed solvable dynamical system is isochronous respectively asymptotically isochronous with period $L_1 = pL$ if the contiguity rule was applied instead. Similarly, if the solution $\tilde{y}(\ell)$ of the seed system has the asymptotic isochronicity property $\tilde{y}(\ell + L) - \tilde{y}(\ell) \to 0$ as $\ell \to \infty$, then the solution $\tilde{x}(\ell)$ of the generation one system is asymptotically isochronous with period $L_1$ that is an integer multiple of $L$, at most $N!$. ■

The validity of this Remark 2.4 is justified by considerations sufficiently analogous to those made in the continuous-time context, see [12], not to warrant their repetition here. These considerations are also illustrated by examples reported in the following Section 3, in particular see Remark 3.1 in that section.

3 Examples

In this section we illustrate our findings via the treatment of some examples.

Notation 3.1. In this section we often omit to indicate explicitly the $\ell$-dependence of the quantities under consideration, and we use a superimposed tilde to denote a unit updating of the discrete-time variable $\ell$, hence, for instance, $\tilde{y}_m \equiv y_m (\ell + 1)$ and $\tilde{y}_m \equiv y_m (\ell + 2)$. ■

Plots of solutions of the dynamical systems considered in this Section 3 are given in Appendix B.

Example 1. As seed dynamical system take the simple first-order dynamical system

$$\tilde{y}_m = a_m y_m + b_m ,$$ (11a)
corresponding to (5) with $p = 1$ and

$$f_m(\bar{y}) = a_m \ y_m + b_m .$$

Equation (11b)

Note that this dynamical system is a simpler version of (6a) because the $2N$ parameters $a_m$ and $b_m$ are assumed to be $\ell$-independent.

It is easily seen that the solution of these equations of motion reads

$$y_m(\ell) = (a_m)^\ell \ y_m(0) + \left[ \frac{(a_m)^\ell - 1}{a_m - 1} \right] b_m .$$

The generation zero system constructed from the seed system (11) via the method described in Section 2, see (7a), reads

$$\prod_{j=1}^{N} [x_n(\ell + 1) - x_j(\ell)]
+ \sum_{m=1}^{N} \left\{ [(a_m - 1)y_m(\ell) + b_m] \ [x_n(\ell + p)]^{N - m} \right\} = 0 ,$$

Equation (13)

where $y_m(\ell)$ are expressed in terms of $x(\ell)$ via (3). Of course, this system describes the evolution of the unordered set $x(\ell)$ of the zeros of the monic polynomial with the coefficients $y_m(\ell)$ given by (12), see the discussion below for the case where $N = 2$.

Remark 3.1. It is plain that, if

$$a_m = \exp \left( \frac{2 \pi \ i \ q_m}{p_m} \right) ,$$

Equation (14)

with $p_m$ an arbitrary positive integer, $q_m$ an arbitrary integer (of course, coprime to $p_m$), and of course $i$, above and hereafter, the imaginary unit so that $i^2 = -1$, then $y_m(\ell)$ is isochronous with period $L_m = p_m$, namely, for any arbitrary initial datum $y_m(0)$, there holds the periodicity property $y_m(\ell + L_m) = y_m(\ell)$ .

Note that (14) implies $|a_m| = 1$. While if $|a_m| < 1$, then clearly—again, for any arbitrary initial datum $y_m(0)$—there obtains the asymptotic limit $\lim_{\ell \to \infty} y_m(\ell) = b_m/(1 - a_m)$.

And it is as well plain that the generic solutions of the generation zero model (13) obtained from this seed model (11) via the technique described in Section 2 have the following remarkable properties: (i) they are isochronous with period $L = P$, where $P$ is the Least Common Multiple of the positive integers $p_1, \ldots, p_N$, provided that all the parameters $a_1, \ldots, a_N$ satisfy condition (14); (ii) they are asymptotically isochronous with asymptotic period $L = P$ if some (at least one but not all) of the parameters $a_1, \ldots, a_N$, say, $a_j$, have modulus less than unity, $|a_j| < 1$, and the remaining parameters, say, $a_{m_1}, \ldots, a_{m_k}$, satisfy condition (14); in this case, $P$ is the the Least Common Multiple of the indices $p_{m_1}, \ldots, p_{m_k}$; (iii) they converge asymptotically to fixed values (independent of the initial data) if $|a_m| < 1$ for all $m = 1, \ldots, N$; (iv) if some among
the parameters $a_1, \ldots, a_N$, say, $a_j$, have modulus larger than unity, $|a_j| > 1$, for generic initial data some of the $N$ coordinates $x_n(\ell)$ diverge asymptotically as $\ell \to \infty$ and others converge to a finite value, as implied by the findings reported in Appendix G (“Asymptotic behavior of the zeros of a polynomial whose coefficients diverge exponentially”) of [14].

Let us discuss the generation zero system that stems from the seed system [11], for the case where $N = 2$. To construct this generation zero system, consider the generation zero polynomial given by

$$p_2(z) = z^2 + y_1(\ell)z + y_2(\ell) = [z - x_1(\ell)] [z - x_2(\ell)],$$

where $y_1(\ell), y_2(\ell)$ are given by [12] with $N = 2$. Note that the ordered pair of the coefficients $y(\ell) = (y_1(\ell), y_2(\ell))$ determines the unordered set of the zeros $\mathcal{Z}(\ell) = \{x_1(\ell), x_2(\ell)\}$. The evolution of the latter unordered set is given by the dynamical system

$$\mathcal{Z}(\ell + 1) = \{g_1(\mathcal{Z}(\ell)), g_2(\mathcal{Z}(\ell))\},$$

where

$$g_m(\mathcal{Z}) \equiv g_m(\{x_1, x_2\}) = \frac{1}{2} \left[a_1 (x_1 + x_2) - b_1 \right]$$

$$+ \frac{(-1)^n}{2} \left\{a_1 (x_1 + x_2) - b_1 \right\}^2 - 4 b_2 - 4 a_2 x_1 x_2 \right\}^{1/2}. \hspace{1cm} (16b)$$

Of course, this system [16] is the particular case of system [13] for $N = 2$. Its solution with the initial condition $\mathcal{Z}(0) = \{x_1(0), x_2(0)\}$ reads

$$\mathcal{Z}(\ell) = \{x_1(\ell), x_2(\ell)\},$$

where

$$x_m(\ell) = -\frac{1}{2} y_1(\ell) + (-1)^n \frac{1}{2} \left\{[y_1(\ell)]^2 - 4 y_2(\ell) \right\}^{1/2} \hspace{1cm} (17b)$$

and $y_m(\ell)$ are given by [12] with $y_1(0) = -x_1(0) - x_2(0)$ and $y_2(0) = x_1(0)x_2(0)$. In more expanded form,

$$x_m(\ell) = \frac{1}{2} \left\{ (a_1)^\ell [x_1(0) + x_2(0)] - \left(\frac{(a_1)^\ell - 1}{a_1 - 1}\right) b_1 \right\}$$

$$+ (-1)^n \frac{1}{2} \left\{ (a_2)^\ell [x_1(0) + x_2(0)] - \left(\frac{(a_1)^\ell - 1}{a_1 - 1}\right) b_2 \right\}^2$$

$$- 4 \left\{ (a_2)^\ell x_1(0) x_2(0) + \left(\frac{(a_2)^\ell - 1}{a_2 - 1}\right) b_2 \right\}^{1/2},$$

$\ell = 1, 2, 3, \ldots$ \hspace{1cm} (17c)

**Remark 3.2.** If the generic complex number $z$ is defined via its modulus and phase as follows, $z = |z| \exp(i \varphi)$ with $0 < \varphi \leq 2\pi$, its square-root $\sqrt{z}$
is of course defined up to a sign ambiguity, say $\sqrt{z} = \pm \sqrt{|z|} \exp (1 \varphi /2)$, with $\sqrt{|z|} > 0$. But, above and hereafter, we assume for definiteness—irrelevant as this is—that the notation $z^{1/2}$ indicates a specific determination of the square-root of $z$, say that with positive real part, and, if the real part vanishes, with positive imaginary part.

Most of the plots of the solutions $\mathcal{x}(\ell)$ of (16) in Appendix B correspond to the lexicographic ordering of the solution pair $\mathcal{x}(\ell) = \{x_1(\ell), x_2(\ell)\}$, see Examples 1a, 1b there. Example 1c uses instead the ordering by contiguity.

**Example 2.** Let us now report the generation one dynamical systems that stem from the seed system (11), again taking $N = 2$ for simplicity. Recall that for $N = 2$ the generation one polynomials are given by

$$p_{\mu_1}(z) = z^2 + y_1^{(\mu_1)}(\ell)z + y_2^{(\mu_1)}(\ell) = \left[ z - x_1^{(\mu_1)}(\ell) \right] \left[ z - x_2^{(\mu_1)}(\ell) \right], \quad \mu_1 = 1, 2,$$

(18)

see (8), where the ordered pair $\vec{\gamma}^{(\mu_1)}(\ell) = (y_1^{(\mu_1)}(\ell), y_2^{(\mu_1)}(\ell))$ equals an appropriately ordered pair $x_1(\ell), x_2(\ell)$. For example, we may choose the lexicographic order of $x_1(\ell), x_2(\ell)$ if $\mu_1 = 1$ and the other (anti-lexicographic) order if $\mu_1 = 2$.

The dynamical system for the unordered pair $\mathcal{x}^{(\mu_1)}(\ell) = \{x_1^{(\mu_1)}(\ell), x_2^{(\mu_1)}(\ell)\}$ is then given by

$$\mathcal{x}^{(\mu_1)}(\ell + 1) = \left\{ g_1^{(\mu_1)} \left( \mathcal{x}^{(\mu_1)}(\ell) \right), g_2^{(\mu_1)} \left( \mathcal{x}^{(\mu_1)}(\ell) \right) \right\},$$

(19a)

where

$$g_m^{(\mu_1)} (\mathcal{x}) \equiv g_m \left( \{x_1, x_2\} \right) = - \frac{1}{2} \gamma_+^{(\mu_1)} (\mathcal{x})$$

$$+ \frac{1}{2} (-1)^m \left\{ \left[ \gamma_-^{(\mu_1)} (\mathcal{x}) \right]^2 - 4 \gamma_+^{(\mu_1)} (\mathcal{x}) \right\}^{1/2}$$

$$m = 1, 2, \quad \mu_1 = 1, 2,$$

(19b)

with the pair

$$\left( \gamma_-^{(\mu_1)} (\mathcal{x}), \gamma_+^{(\mu_1)} (\mathcal{x}) \right)$$

(19c)

being equal to the ordering of the pair

$$\{ \alpha (\mathcal{x}) - (-1)^{\mu_1} \beta (\mathcal{x}), \alpha (\mathcal{x}) + (-1)^{\mu_1} \beta (\mathcal{x}) \}$$

(19d)

that corresponds to the index $\mu_1$, where

$$\alpha (\mathcal{x}) = \frac{1}{2} \left[ a_1 \left( -x_1 - x_2 + x_1 x_2 \right) - b_1 \right]$$

and

$$\beta (\mathcal{x}) = \left\{ \alpha (\mathcal{x}) \right\}^2 - b_2 + a_2 x_1 x_2 (x_1 + x_2) \right\}^{1/2}.$$
The solution of dynamical system (19) with the initial condition \( x(0) = \{x_1(0), x_2(0)\} \) is given by

\[
x^{(\mu_1)}(\ell) = \left\{ x_1^{(\mu_1)}(\ell), x_2^{(\mu_1)}(\ell) \right\},
\]

where

\[
x_m^{(\mu_1)}(\ell) = -\frac{1}{2} y_1^{(\mu_1)}(\ell) + (-1)^m \frac{1}{2} \left\{ [y_1^{(\mu_1)}(\ell)]^2 - 4 y_2^{(\mu_1)}(\ell) \right\}^{1/2}
\]

and \( y_m^{(\mu_1)}(\ell) \) are given by formulas (17c) for \( x_m(\ell) \), with \( x_m(0) \) replaced by \( y_m^{(\mu_1)}(0) \), where

\[
y_1^{(\mu_1)}(0) = -x_1(0) - x_2(0), \quad y_2^{(\mu_1)}(0) = x_1(0) x_2(0).
\]

As before, \( \mu_1 = 1 \) indicates the lexicographic order of the pair \( y_1^{(\mu_1)}(\ell), y_2^{(\mu_1)}(\ell) \), while \( \mu_1 = 2 \) indicates the other (anti-lexicographic) order.

In summary, given the initial condition \( \{x_1(0), x_2(0)\} \), we can solve dynamical system (19) as follows. First, we order the initial condition to ensure that the pair \( (x_1(0), x_2(0)) \) is in the lexicographic order. Second, we compute \( y_m^{(\mu_1)}(0) \) by formulas (20b) and assign \( \mu_1 = 1 \) if the pair \( (-x_1(0) - x_2(0), x_1(0) x_2(0)) \) turns out to be in the lexicographic order and \( \mu_1 = 2 \) otherwise. Third, we compute \( y_m^{(\mu_1)}(\ell) \) using formulas (17c) for \( x_m(\ell) \), with \( x_m(0) \) replaced by \( y_m^{(\mu_1)}(0) \), while ensuring that each pair \( (y_1^{(\mu_1)}(\ell), y_2^{(\mu_1)}(\ell)) \) is ordered according to the value of \( \mu_1 \) chosen in the previous step. Fourth, we compute \( x^{(\mu_1)}(\ell) \) using formulas (20). In the plots of Examples 2a, 2b in Appendix B the solutions \( \{x_1^{(\mu_1)}(\ell), x_2^{(\mu_1)}(\ell)\} \) are ordered lexicographically.

**Example 3.** We do not display the equations of motion of the subsequent model with \( k = 2 \), nor the formulas displaying its solutions, since they are not very illuminating and the interested readers can easily figure them out for themselves. We rather display a few representative plots of the solutions of the system in the \( k = 2 \) generations for several particular values of the parameters \( a_m, b_m \) in the seed system (11) with \( N = 2 \), see Appendix B.

**Example 4.** In this example we take the following solvable second-order discrete-time dynamical system as the seed system:

\[
y_m(\ell + 2) = a_m(\ell) \frac{y_m^2(\ell + 1)}{y_m(\ell)} + b_m(\ell) y_m(\ell + 1),
\]

where \( a_m(\ell) \) and \( b_m(\ell) \) are some functions of \( \ell \). Via the substitution

\[
u_m(\ell) = \frac{y_m(\ell + 1)}{y_m(\ell)}
\]

we...
we find that the solution of system (21) with the initial conditions \(y_m(0), y_m(1)\) is given by

\[
y_m(\ell) = y_m(0) \prod_{j=0}^{\ell-1} u_m(j), \tag{23a}
\]

where

\[
u_m(\ell) = \frac{y_m(1)}{y_m(0)} \prod_{j=0}^{\ell-1} a_m(j) + \sum_{k=0}^{\ell-1} \left\{ \prod_{j=k+1}^{\ell-1} a_m(j) \right\} b_m(k). \tag{23b}
\]

The evolution of the zeros of the polynomial \(z^N + \sum_{m=1}^{N} y_m(\ell) z^{N-m}\) is then described by the dynamical system

\[
\prod_{j=1}^{N} [x_n(\ell + 2) - x_j(\ell)] + \sum_{m=1}^{N} \left\{ a_m(\ell) \frac{[y_m(\ell + 1)]^2}{y_m(\ell)} + b_m(\ell) y_m(\ell + 1) - y_m(\ell) \right\} [x_n(\ell + 2)]^{N-m} = 0, \tag{24}
\]

where \(y_m(\ell)\) and \(y_m(\ell + 1)\) in the right-hand side must be replaced with the appropriate expressions depending on \(x(\ell)\) and \(x(\ell + 1)\), see (3).

Let us consider the autonomous case of system (21) with

\[
a_m(\ell) = a_m, \quad b_m(\ell) = b_m \tag{25}
\]

and

\[
a_m = \exp\left(2\pi i q_m/p_m\right), \tag{26}
\]

where \(q_m, p_m\) coprime integers and \(p_m > 0\). In this case system (21) reads

\[
y_m(\ell + 2) = a_m \frac{y_m(\ell + 1)^2}{y_m(\ell)} + b_m y_m(\ell + 1). \tag{27}
\]

and its solution is given by (23a), where

\[
u_m(\ell) = (a_m)^\ell u_m(0) + \frac{(a_m)^\ell - 1}{a_m - 1} b_m, \tag{28}
\]

and \(u_m(0) = y_m(1)/y_m(0)\). Because \(a_m\) are given by (26), all \(u_m(\ell)\) defined by the last formula are \(P\)-periodic, where \(P\) is the Least Common Multiple of the \(N\) integers \(p_m\). It is then easy to see from (23a) that

\[
y_m(\ell + P) = \alpha_m y_m(\ell), \tag{29a}
\]

15
where
\[ \alpha_m = \prod_{k=0}^{P-1} u_m(k). \tag{29b} \]

Suppose that the initial data \( x(0), x(1) \) for system (24) are such that \( u_m(\ell) \) given by (28) with \( u_m(0) = y_m(1)/y_m(0) = \sigma_m(\bar{x}(1))/\sigma_m(\bar{x}(0)) \) (see (3)) satisfy the condition
\[ \prod_{k=0}^{P-1} u_m(k) = \rho_m \exp (2\pi ir_m/s_m), \tag{30} \]
where \( r_m, s_m \) are coprime integers with \( s_m > 0 \) and \( \rho_m \) are positive real numbers such that \( \rho_m \leq 1 \) with at least one of them being equal to unity, say \( \rho_j = 1 \). That is, suppose that \( \bar{x}(0), \bar{x}(1) \) are such that
\[ \prod_{k=0}^{P-1} \left[ \left( a_m \right)^k \frac{\sigma_m(\bar{x}(1))}{\sigma_m(\bar{x}(0))} \right] + \left( a_m \right)^k \frac{b_m}{a_m-1} = \beta_m, \tag{31a} \]
where
\[ \beta_m = \rho_m \exp (2\pi ir_m/s_m). \tag{31b} \]

Then the solutions \( y_m(\ell) \) of system (27), (26) with the initial data \( y_m(k) = (-1)^m \sigma_m(\bar{x}(k))/m! \), \( k = 0, 1 \), are periodic or asymptotically periodic with period \( L \) that is the Least Common Multiple of the integers in the set \( \{ s_m P : \rho_m = 1 \} \) (of course, the periodic case corresponds to the situation where all \( \rho_m = 1 \)).

To summarize, we conclude that the generation zero system (24) in the autonomous case given by (25), (26) features a periodic or an asymptotically periodic solution with period \( L \) provided that the initial data \( \bar{x}(0), \bar{x}(1) \) satisfy conditions (31), see Remark 2.4. Plots of a periodic solution of (27) are given in Example 4 of Appendix B.

4 Looking backward and forward

In this last section we tersely indicate the extent to which the findings reported in the present paper go beyond previously reported results, and we outline possible future developments.

A basic idea underlies the identification of many, perhaps most, solvable dynamical systems characterizing the time-evolution of \( N \) points moving in the complex plane (or, equivalently, in the real Cartesian plane). The idea is to identify/manufacture dynamical systems solvable by algebraic operations—hence evolving nonchaotically—by taking advantage of the nonlinear but algebraic relations among the \( N \) zeros and the \( N \) coefficients of a time-dependent polynomial \( p_N(z; t) \) of degree \( N \) in the complex variable \( z \). Its exploitation is quite old—see for instance (15)—yet only quite recently a very useful tool to better take advantage of this approach has emerged (16), leading to the identification
of several new solvable dynamical systems [12, 16, 17, 18, 19, 20, 21, 22, 23]. These developments—which also led to the idea of generations of polynomials [12]—focussed up to now on evolutions in the continuous-time variable \( t \). The main novelty of the present paper is the extension of this approach to evolutions in the discrete-time variable \( \ell \).

To conclude this terse outline of previous developments, let us insert the following

**Remark 4.1.** A simple way to “generalize” any dynamical system is to perform a change of dependent and independent variables; but generally the “new” models obtained in such a way from a known model are not considered really new. It might therefore be inferred that the techniques described in all the papers referred to above, and in the present paper, are not really yielding new solvable dynamical systems, since the main tool employed—the relation between the coefficients and the zeros of a polynomial—may well be considered just a change of dependent variables for the dynamical systems under consideration. But this criticism conflicts with the observation that essentially all solvable dynamical system can be reduced, by appropriate changes of variables, to trivial evolutions. The rub is, of course, to identify the appropriate changes of variables. Hence the emergence of the “inverse” approach: to start from certain changes of variables—in particular, those relating the (time-dependent) coefficients and the (time-dependent) zeros of (time-dependent, monic) polynomials—and to then try and identify the dynamical systems solvable via this kind of transformation of dependent variables. To those who consider such an “inverse” approach a kind of cheating, we can only reply by begging them to ponder what is written in the Foreword (see, in particular, page VII) of the book [14] to justify this approach (indeed, amply practiced both in that book and in most other publications on solvable/integrable dynamical systems, including the present one).

Let us finally look forward and tersely list a possible future development. An analogous approach to that discussed in this paper—but leading to solvable dynamical systems in “\( q \)-discrete time”, that is, characterized by \( q \)-difference equations of motion rather than difference equations of motion—obtains by taking as point of departure, instead of the polynomial formula (2), the following definition:

\[
p_N(z; q; \ell) = z^N + \sum_{m=1}^{N} [y_m(q^{\ell}) z^{N-m}] = \prod_{n=1}^{N} [z - x_n(q^{\ell})] ,
\]

(32)

with \( q \) an arbitrary parameter (of course \( q \neq 1 \)). It is then easily seen—again, by a quite analogous treatment to that provided in Appendix A—that the key formula (4a) is replaced by the relation

\[
\prod_{j=1}^{N} [x_n(q^{\ell+p}) - x_j(q^{\ell})] + \sum_{m=1}^{N} \left\{ [y_m(q^{\ell+p}) - y_m(q^{\ell})] [x_n(q^{\ell+p})]^{N-m} \right\} = 0 ,
\]

(33)

where \( p \) is a positive integer. And a simple example of a first order seed system to generate other nonlinear solvable dynamical systems in “\( q \)-discrete time”
reads as follows:

\[ y_m (q^\ell + 1) = a_m y_m (q^\ell) + b_m , \quad \text{(34a)} \]

since the explicit solution of its initial-value problem clearly reads

\[ y_m (q^\ell) = (a_m)^\ell y_m (1) + \left[ \frac{(a_m)^\ell - 1}{a_m - 1} \right] b_m . \quad \text{(34b)} \]

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6 Appendix A: Proof of formula (4)

Our task in this Appendix A is to prove key formula (4), which is in fact a rather immediate consequence of definition (2) of the polynomial \( p_N (z; \ell) \). Let \( p \) be a positive integer. The first of equalities (2) implies

\[ p_N (z; \ell + p) - p_N (z; \ell) = \sum_{m=1}^{N} \left\{ [y_m (\ell + p) - y_m (\ell)] z^{N-m} \right\} , \quad \text{(35a)} \]

and, via the second of equalities (2), this formula reads

\[ \prod_{j=1}^{N} [z - x_j (\ell + p)] - \prod_{j=1}^{N} [z - x_j (\ell)] = \sum_{m=1}^{N} \left\{ [y_m (\ell + p) - y_m (\ell)] z^{N-m} \right\} . \quad \text{(35b)} \]

It is now plain that, by setting in this formula \( z = x_n (\ell + p) \), one obtains (4a).

Q. E. D.

Remark A.1. It is also plain that, by setting \( z = x_n (\ell) \) in (35b), one obtains the alternative formula

\[ \prod_{j=1}^{N} [x_n (\ell) - x_j (\ell + 1)] = \sum_{m=1}^{N} \left\{ [y_m (\ell + 1) - y_m (\ell)] [x_n (\ell)]^{N-m} \right\} . \quad \text{(36)} \]
7 Appendix B: Plots of solutions of systems treated in Section 3

In this Appendix B we provide several plots of the solutions $x_\ell = \{x_1(\ell), x_2(\ell)\}$ of the discrete-time dynamical systems treated in Section 3 for $N = 2$ and for various assignments of the parameters and of the initial conditions. Because the solution $x_\ell = \{x_1(\ell), x_2(\ell)\}$ is an unordered set, for the purpose of plotting it, we assume that the pair $(x_1(\ell), x_2(\ell))$ is ordered lexicographically, with the only exception of Example 1c, in which the solution set $x_\ell$ is ordered by contiguity, see Remark 2.3. In the following graphs, the (dashed or continuous; if any) line segments joining points are only visual aids having no other significance for the purpose of our discussion.

Example 1. Generation zero system with seed (11) 

Choose $N = 2$ in the seed system (11) to obtain, via the method described in Section 2, system (16) with solution (17). Let us recall that Remark 3.1 predicts the behavior of solutions (17) depending on the values of the parameters $a_1, a_2$.

Example 1a. We begin with an isochronous case of system (16) with the solution (17). In Figures 1 and 2 we plot the solution $\{x_1(\ell), x_2(\ell)\}$ of system (16) with

\[
\begin{align*}
a_1 &= \exp\left(\frac{2\pi i}{3}\right), \quad a_2 = \exp\left(\frac{4\pi i}{5}\right); \quad b_1 = 1, \quad b_2 = 2; \\
x_1(0) &= -1 - i, \quad x_2(0) = 1.
\end{align*}
\]

(37)

The solution of this system is periodic with period $L = 15$, the Least Common Multiple of 3 and 5, see Remark 3.1.

Example 1b. This is an asymptotically isochronous case of system (16) with the solution (17). In Figure 3 we plot the solution $\{x_1(\ell), x_2(\ell)\}$ of system (16) with

\[
\begin{align*}
a_1 &= \exp\left(\frac{2\pi i}{7}\right), \quad a_2 = 0.9 \exp\left(\frac{4\pi i}{5}\right); \quad b_1 = .1, \quad b_2 = .2; \\
x_1(0) &= -1 - 1, \quad x_2(0) = 1.
\end{align*}
\]

(38)

The solution of this system is asymptotically periodic with asymptotic period $L = 7$.

Example 1c. We give another example of an asymptotically isochronous case of system (16) with the solution (17). In this example we order the components of the solution by contiguity, see Remark 2.3. In Figures 4 and 5 we plot the solution $\{x_1(\ell), x_2(\ell)\}$ of system (16) with

\[
\begin{align*}
a_1 &= 0.1 \exp\left(\frac{2\pi i}{3}\right), \quad a_2 = \exp\left(\frac{2\pi i}{25}\right); \quad b_1 = 1, \quad b_2 = 1; \\
x_1(0) &= -1 - i, \quad x_2(0) = 1.
\end{align*}
\]

(39)
Figure 1: Example 1a. For each $\ell = 0, 1, \ldots, 15$, the position of $x_1(\ell)$ in the complex $x$-plane is indicated by a dot and the label $\ell$ that indicates the value of the discrete-time, while the position of $x_2(\ell)$ is indicated by a star and the analogous label $\ell$. The complex pairs $(x_1(\ell), x_2(\ell))$ are ordered lexicographically.
Figure 2: **Example 1a.** Top graph: the evolution of the real parts $\text{Re} \left[ x_1(\ell) \right]$ (dots) and $\text{Re} \left[ x_2(\ell) \right]$ (stars). Bottom graph: the evolution of the imaginary parts $\text{Im} \left[ x_1(\ell) \right]$ (dots) and $\text{Im} \left[ x_2(\ell) \right]$ (stars). The evolution is, of course, with respect to the discrete-time variable $\ell$, which corresponds to the horizontal axis. Note the periodicity with period $L = 15$. 
Figure 3: **Example 1b.** Top graph: the evolution of the real parts $\text{Re} \{x_1(\ell)\}$ (dots) and $\text{Re} \{x_2(\ell)\}$ (stars). Bottom graph: the evolution of the imaginary parts $\text{Im} \{x_1(\ell)\}$ (dots) and $\text{Im} \{x_2(\ell)\}$ (stars). The complex pairs $x_1(\ell), x_2(\ell)$ are ordered lexicographically. The evolution is, of course, with respect to the discrete-time variable $\ell$, which corresponds to the horizontal axis. Note the asymptotic periodicity with asymptotic period $L = 7$. Occasionally a dot and a start are too close for their separation to be actually visible.
Figure 4: **Example 1c.** For each $\ell = 0, 1, \ldots, 25$, the position of $x_1(\ell)$ in the complex $x$-plane is indicated by a dot and the label $\ell$ that indicates the value of the discrete-time, while the position of $x_2(\ell)$ is indicated by a star and the analogous label $\ell$. The pairs $(x_1(\ell), x_2(\ell))$ are ordered by contiguity.

The solution of this system is asymptotically periodic with asymptotic period $L = 25$.

**Example 2. Generation one system with seed (11)**

**Example 2a.** We begin with an isochronous case of generation one system (19) with the solution (20). In Figure 6 we plot the solution $\{x_1(\ell), x_2(\ell)\}$ of system (19) with

$$a_1 = \exp\left(\frac{2\pi i}{3}\right), \quad a_2 = \exp\left(\frac{4\pi i}{5}\right); \quad b_1 = 1, \quad b_2 = 2;$$

$$x_1(0) = -1 - i, \quad x_2(0) = 1. \quad (40)$$

The solution of this system is periodic with period $L = 15$, which is the Least Common Multiple of 3 and 5, see **Remark 3.1.**
Figure 5: **Example 1c.** Top graph: the evolution of the real parts $\text{Re} \left[ x_1(\ell) \right]$ (dots) and $\text{Re} \left[ x_2(\ell) \right]$ (stars). Bottom graph: the evolution of the imaginary parts $\text{Im} \left[ x_1(\ell) \right]$ (dots) and $\text{Im} \left[ x_2(\ell) \right]$ (stars). The evolution is, of course, with respect to the discrete-time variable $\ell$, which corresponds to the horizontal axis. The pairs $(x_1(\ell), x_2(\ell))$ are ordered by contiguity. Note the asymptotic periodicity with period $L = 25$. 
Figure 6: **Example 2a.** Top graph: the evolution of the real parts $\text{Re} \left [ x_1(\ell) \right ]$ (dots) and $\text{Re} \left [ x_2(\ell) \right ]$ (stars). Bottom graph: the evolution of the imaginary parts $\text{Im} \left [ x_1(\ell) \right ]$ (dots) and $\text{Im} \left [ x_2(\ell) \right ]$ (stars). The pairs $(x_1(\ell), x_2(\ell))$ are ordered lexicographically. The evolution is, of course, with respect to the discrete-time variable $\ell$, which corresponds to the horizontal axis. Note the periodicity with period $L = 15$. 
Figure 7: Example 2b. Top graph: the evolution of the real parts $\text{Re}\left[x_1(\ell)\right]$ (dots) and $\text{Re}\left[x_2(\ell)\right]$ (stars). Bottom graph: the evolution of the imaginary parts $\text{Im}\left[x_1(\ell)\right]$ (dots) and $\text{Im}\left[x_2(\ell)\right]$ (stars). The pairs $(x_1(\ell), x_2(\ell))$ are ordered lexicographically. The evolution is, of course, with respect to the discrete-time variable $\ell$, which corresponds to the horizontal axis. Note the asymptotic periodicity with asymptotic period $L = 7$.

**Example 2b.** This is an asymptotically isochronous case of generation one system (19) with the solution (20). In Figure 7 we plot the solution \{x_1(\ell), x_2(\ell)\} of system (19) with

$$a_1 = \exp\left(\frac{2\pi i}{7}\right), \quad a_2 = 0.9 \exp\left(\frac{4\pi i}{5}\right); \quad b_1 = .1, \quad b_2 = .2;$$

$$x_1(0) = -1 - i, \quad x_2(0) = 1.$$  \hspace{1cm} (41)

The solution of this system is asymptotically periodic with asymptotic period $L = 7$.

**Example 3. Generation two system with seed (11)**

**Example 3a.** We begin with an isochronous case of the generation two system derived from the seed system (11) according to the general procedure described in Section 2 and discussed in Section 3 for the case of the seed system (11) (with $N = 2$). In Figure 8 we plot the solution \{x_1(\ell), x_2(\ell)\} of the
Figure 8: **Example 3a.** Top graph: the evolution of the real parts \( \text{Re} [x_1(\ell)] \) (dots) and \( \text{Re} [x_2(\ell)] \) (stars). Bottom graph: the evolution of the imaginary parts \( \text{Im} [x_1(\ell)] \) (dots) and \( \text{Im} [x_2(\ell)] \) (stars). The evolution is, of course, with respect to the discrete-time variable \( \ell \), which corresponds to the horizontal axis. Note the periodicity with period \( L = 15 \).

The solution of this system is periodic with period \( L = 15 \), which is the Least Common Multiple of 3 and 5, see **Remark 3.1.**

**Example 3b.** This is an *asymptotically isochronous* case of the generation two system derived from the seed system (11) according to the general procedure described in Section 2 and discussed in Section 3 for the case of the seed system (11) (with \( N = 2 \)). In Figure 9 we plot the solution \( \{x_1(\ell), x_2(\ell)\} \) of the generation two system with

\[
a_1 = \exp \left( \frac{2\pi i}{3} \right) , \quad a_2 = \exp \left( \frac{4\pi i}{5} \right) ; \quad b_1 = 1 , \quad b_2 = 2 ;
\]

\[
x_1(0) = -1 - i , \quad x_2(0) = 1 .
\]

(42)

The solution of this system is periodic with period \( L = 15 \), which is the Least Common Multiple of 3 and 5, see **Remark 3.1.**

\[
a_1 = \exp \left( \frac{2\pi i}{7} \right) , \quad a_2 = 0.9 \exp \left( \frac{4\pi i}{5} \right) ; \quad b_1 = .1 , \quad b_2 = .2 ;
\]

\[
x_1(0) = -1 - i , \quad x_2(0) = 1 .
\]

(43)
Figure 9: **Example 3b.** Top graph: the evolution of the real parts $\text{Re}[x_1(\ell)]$ (dots) and $\text{Re}[x_2(\ell)]$ (stars). Bottom graph: the evolution of the imaginary parts $\text{Im}[x_1(\ell)]$ (dots) and $\text{Im}[x_2(\ell)]$ (stars). The evolution is, of course, with respect to the discrete-time variable $\ell$, which corresponds to the horizontal axis. The pair $(x_1(\ell), x_2(\ell))$ is ordered lexicographically. Note the asymptotic periodicity with asymptotic period $L = 7$.

The solution of this system is *asymptotically periodic* with asymptotic period $L = 7$.

**Example 4.** In Figure 10 we plot the solutions of system (24) with (25), (26), for the case where $N = 2$ and

\[
\begin{align*}
a_1 &= \exp(i \pi), \quad a_2 = \exp\left(\frac{i \pi}{2}\right); \quad b_1 = 1, \quad b_2 = 2. \\
\end{align*}
\]  

We used conditions (31) with $\beta_1 = \exp(i \pi)$ and $\beta_2 = \exp(i \pi)$ to find initial conditions for which system (24) with (44) has a periodic solution:

\[
\begin{align*}
x_1(0) &= -\frac{17^{1/4} + \gamma}{17^{1/4} + (1 + 2 i) \gamma - 2 (-3)^{1/4} \gamma}, \\
x_1(1) &= \left[1 + i - 3^{1/4} \exp\left(\frac{i \pi}{4}\right)\right] x_1(0), \\
x_2(0) &= 1, \quad x_2(1) = 1,
\end{align*}
\]  

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Figure 10: Example 4. Top graph: the evolution of the real parts \( \text{Re} [x_1(\ell)] \) (dots) and \( \text{Re} [x_2(\ell)] \) (stars). Bottom graph: the evolution of the imaginary parts \( \text{Im} [x_1(\ell)] \) (dots) and \( \text{Im} [x_2(\ell)] \) (stars). The pair \((x_1(\ell), x_2(\ell))\) is ordered lexicographically. The evolution is, of course, with respect to the discrete-time variable \( \ell \), which corresponds to the horizontal axis. Note the periodicity with period \( L = 8 \).

where \( \gamma = \exp \left[ i \arctan(4)/2 \right] \). The solution of system (24) with (44) with the initial conditions (45) is periodic with period 8.

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