ORTHOGONAL INTRINSIC MODE FUNCTIONS VIA OPTIMIZATION APPROACH

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Abstract. This paper proposes an optimization approach to find a set of orthogonal intrinsic mode functions (IMFs). In particular, an optimization problem is formulated in such a way that the total energy of the difference between the original IMFs and the corresponding obtained IMFs is minimized subject to both the orthogonal condition and the IMF conditions. This formulated optimization problem consists of an exclusive or constraint. This exclusive or constraint is further reformulated to an inequality constraint. Using the Lagrange multiplier approach, it is required to solve a linear matrix equation, a quadratic matrix equation and a highly nonlinear matrix equation only dependent on the orthogonal IMFs as well as a nonlinear matrix equation dependent on both the orthogonal IMFs and the Lagrange multipliers. To solve these matrix equations, the first three equations are considered. First, a new optimization problem is formulated in such a way that the error energy of the highly nonlinear matrix equation is minimized subject to the linear matrix equation and the quadratic matrix equation. By finding the nearly global optimal solution of this newly formulated optimization problem and checking whether the objective functional value evaluated at the obtained solution is close to zero or not, the orthogonal IMFs are found. Finally, by substituting the obtained orthogonal IMFs to the last matrix equation, this last matrix equation reduced to a linear matrix equation which is only dependent on the Lagrange multipliers. Therefore, the Lagrange multipliers can be found. Consequently, the solution of the original optimization problem is found. By repeating these procedures with different initial conditions, a nearly global optimal solution is obtained.

1. Introduction. Empirical mode decomposition (EMD) is a method to represent a signal by the sum of its IMFs and its residue. It is worth noting that the IMFs are obtained based on the extrema of the signal [6]. Since the total numbers of the extrema of different IMFs are different and the IMF with more extrema usually contains more higher frequency components, the EMD is a kind of time frequency representations of signals [4]. On the other hand, as the extrema of different signals are located at different positions and the positions of the extrema of a weighted sum of different signals are located differently compared with those of the individual signals, the EMD is a nonlinear representation of signals [1]. Furthermore, there

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is no predefined kernel for the EMD except the interpolation function. The IMFs are solely dependent on the signal itself. Hence, the EMD is a type of adaptive representations of signals [3]. Unlike the conventional linear and nonadaptive based time frequency representations of signals such as the wavelet based representations of signals, the nonlinear and adaptive properties of the EMD facilitate some science and engineering applications such as the denoising application [8] and the underlying trend extraction application [14], [15]. Therefore, the EMD plays a very important role in many science and engineering disciplines.

However, as the IMFs are obtained based on the extrema of the signal, the IMFs are in general not orthogonal to each others [7]-[13]. Since the projection of any IMF to other IMFs may not be equal to zero, the reconstructed signal by using some of the IMFs may still have some components on the removed IMFs. This results in the leakage of the energy. For the EMD based denoising application [8] and the EMD based underlying trend extraction application [14], [15], the processed signal is obtained based on the reconstruction of some of the IMFs. Therefore, the EMD is not efficient for these applications. As a result, it would be useful to find a set of IMFs which are exactly orthogonal to each others [2],[13]. This paper aims to address this issue.

The most common approach to address the above issue is to apply the Gram Schmidt orthogonalization to the obtained IMFs in the time domain [13]. On the other hand, a frequency domain approach is employed to obtain the orthogonal components. In particular, the IMFs are processed by different filters where the frequency responses of these filters are orthogonal to each others. However, these approaches do not guarantee that the obtained components satisfy the IMF conditions.

In this paper, an optimization approach is proposed for tackling this difficulty. The orthogonal condition and the IMF conditions are formulated as the constraints of the optimization problem. By putting the orthogonal vectors into the columns of a matrix, we have an orthogonal matrix. In fact, the orthogonal matrices are on the surface of a generalized unit energy hyperspace. It is worth noting that a convex combination of any two different orthogonal matrices is in general not on the surface of that generalized unit energy hyperspace. In other words, a convex combination of any two different orthogonal matrices is not an orthogonal matrix. Therefore, the set of orthogonal matrices is nonconvex. As a result, the formulated optimization problem is a nonconvex optimization problem [10]. In general, it is very difficult to find a nearly global optimal solution of such kind of nonconvex optimization problems [9]. Besides, one of the IMF condition is defined on both the total number of the extrema and the total number of the zero crossing points. This condition is defined based on the corresponding properties and it is difficult to be characterized via an analytical formulae. Therefore, in general it is very difficult to formulate an optimization problem subject to this IMF condition. Moreover, the difference between the total number of the extrema and the total number of the zero crossing points of each IMF should be either equal to zero or equal to one. Here, there are two conditions. However, one and only one condition is required to be satisfied. This is equivalent to the satisfaction of the corresponding exclusive or constraint. (The exclusive or constrint consists of two constraints where one and only one of these two constraints is satisfied.) In other words, this problem is an exclusive or constrained optimization problem. Nevertheless, existing optimization algorithms mainly deal with the satisfaction of all the constraints [12]. Therefore, the existing algorithms do
not apply for finding the solution of this type of optimization problems \cite{12}. In fact, handling the exclusive or constraint is challenging. Furthermore, the optimization problem subject to both the orthogonal condition and the IMF conditions involves a nonsmooth function. Nevertheless, existing gradient descent based optimization algorithms \cite{11} do not apply for finding the solutions of such kind of nonsmooth optimization problems \cite{5}. Therefore, finding a set of orthogonal IMFs is very challenging.

The outline of this paper is as follows. In Section 2, finding the orthogonal IMFs is formulated as an optimization problem. The method for finding the solution of this optimization problem is presented in Section 3. Computer numerical simulation results are presented in Section 4. Finally, a conclusion is drawn in Section 5.

2. Problem formulation.

2.1. Notations and preliminaries. Denote \( x(t) \) as a real valued continuous time signal. Suppose that the EMD of \( x(t) \) exists. Let \( M \) be the total number of the IMFs. Denote these IMFs as \( \mu_i(t) \) for \( i = 0, \cdots, M - 1 \) and \( r(t) \) as the residue. That is,

\[
x(t) = r(t) + \sum_{i=0}^{M-1} \hat{u}_i(t)
\]

Suppose that \( x(t), r(t) \) and \( \hat{u}_i(t) \) for \( i = 0, \cdots, M - 1 \) are sampled in the time domain. Let \( N \) be the total number of sampling points in \( x(t) \). This is similar for \( r(t) \) and \( \hat{u}_i(t) \) for \( i = 0, \cdots, M - 1 \). Denote the vectors containing the sampling points of \( x(t), r(t) \) and \( \hat{u}_i(t) \) for \( i = 0, \cdots, M - 1 \) as \( \mathbf{x}, \mathbf{r} \) and \( \mathbf{\hat{u}}_i \) for \( i = 0, \cdots, M - 1 \), respectively. In this paper, it is assumed that \( N \gg M + 1 \) and \( \mathbf{x}, \mathbf{r} \) and \( \pi_i \) for \( i = 0, \cdots, M - 1 \) are good representations of \( x(t), r(t) \) and \( \hat{u}_i(t) \) for \( i = 0, \cdots, M - 1 \), respectively. Denote \( \iota = [1 \cdots 1]^T \) and \( \mathbf{U} = [\mathbf{\hat{u}}_0 \cdots \mathbf{\hat{u}}_{M-1}] \). Here, there are \( M \) elements in \( \iota \). Then, we have \( \mathbf{U} \mathbf{1} + \mathbf{r} = \mathbf{x} \). Since \( \mathbf{\hat{u}}_i \) for \( i = 0, \cdots, M - 1 \) are in general not the unit energy vectors, define \( \mathbf{\hat{u}}_i \) for \( i = 0, \cdots, M - 1 \) as the normalized vectors of \( \mathbf{\hat{u}}_i \) for \( i = 0, \cdots, M - 1 \) such that \( \mathbf{\hat{u}}_i \) for \( i = 0, \cdots, M - 1 \) are the unit energy vectors. That is, \( \mathbf{\hat{u}}_i = \frac{\mathbf{\hat{u}}_i}{\sqrt{\mathbf{\hat{u}}_i^T \mathbf{\hat{u}}_i}} \) for \( i = 0, \cdots, M - 1 \). Denote

\[
\mathbf{w} = \left[ \sqrt{\mathbf{\hat{u}}_0^T \mathbf{\hat{u}}_0} \cdots \sqrt{\mathbf{\hat{u}}_{M-1}^T \mathbf{\hat{u}}_{M-1}} \right]^T \text{ and } \tilde{\mathbf{U}} = \left[ \mathbf{\hat{u}}_0 \cdots \mathbf{\hat{u}}_{M-1} \right].
\]

Then, we have

\[
\tilde{\mathbf{U}} \mathbf{w} + \mathbf{r} = \mathbf{x}
\]

2.2. Orthogonal condition. In general, these IMFs are not orthogonal to each others. That is,

\[
\int_{-\infty}^{+\infty} \hat{u}_i(t)\hat{u}_j(t)dt \neq \delta(i - j)
\]

for some \( i \in \{0, \cdots, M - 1\} \) and for some \( j \in \{0, \cdots, M - 1\} \). Here, \( \delta(\cdot) \) denotes the continuous time Dirac delta function. Let \( \mathbf{u}_i \) for \( i = 0, \cdots, M - 1 \) be the approximations of \( \hat{u}_i \) for \( i = 0, \cdots, M - 1 \) such that \( \mathbf{u}_i \) for \( i = 0, \cdots, M - 1 \) are orthogonal to each others. Denote \( \mathbf{U} = \left[ \mathbf{u}_0 \cdots \mathbf{u}_{M-1} \right] \). Then, we have

\[
\mathbf{U}^T \mathbf{U} = \mathbf{I}_M
\]

Here, \( \mathbf{I}_M \) denotes the \( M \times M \) identity matrix.
2.3. IMF conditions. Denote \( \tau = [1 \cdots 1]^T \). Here, there are \( N \) elements in \( \tau \). Then, \( \frac{\tau^T u_p}{N} \) is the mean of all the elements in \( u_p \) for \( p = 0, \cdots, M - 1 \). As one of the IMF condition is that its local mean is equal to zero, we have

\[
\frac{\tau^T u_p}{N} = 0 \text{ for } p = 0, \cdots, M - 1
\]

Denote \( \imath_n = [0 \cdots 0 -1 1 0 \cdots 0]^T \) for \( n = 0, \cdots, N - 2 \). Here, there are \( n \) zeros before the element \(-1\) and \( N - 2 - n \) zeros after the element \( 1 \) in \( \imath_n \) for \( n = 0, \cdots, N - 2 \). Let \( u_p[n] \) and \( \hat{u}_p[n] \) for \( n = 0, \cdots, N - 1 \) and for \( p = 0, \cdots, M - 1 \) be the \((n + 1)\)th element of \( u_p \) and \( \hat{u}_p \) for \( p = 0, \cdots, M - 1 \), respectively. Then, \( \imath_n^T u_p \) is equivalent to compute \( u_p[n + 1] - u_p[n] \) for \( n = 0, \cdots, N - 2 \) and \( \hat{u}_p[n] \) for \( p = 0, \cdots, M - 1 \). Therefore, \( \imath_n^T \hat{u}_p \) is equivalent to compute \((u_p[n + 1] - u_p[n]) (u_p[n + 2] - u_p[n + 1])\) for \( n = 0, \cdots, N - 3 \) and for \( p = 0, \cdots, M - 1 \). As a result, \( \hat{u}_p^T \imath_{n+1} \imath_n^T u_p < 0 \) is equivalent to \( u_p[n+1] \) being an extremum. Denote \( \text{sgn}(z) = \begin{cases} 1 & z \geq 0 \\ -1 & z < 0 \end{cases} \) Then, \( \sum_{n=0}^{N-3} \frac{1}{2} (1 - \text{sgn}(\hat{u}_p^T \imath_{n+1} \imath_n^T u_p)) \) is the total number of the extrema in \( u_p \) for \( p = 0, \cdots, M - 1 \).

Denote \( \iota_n = [0 \cdots 0 1 0 \cdots 0]^T \) for \( n = 0, \cdots, N - 1 \). Here, there are \( n \) zeros and \( N - 1 - n \) zeros before and after the element \( 1 \) in \( \iota_n \) for \( n = 0, \cdots, N - 1 \), respectively. Then, \( \iota_n^T u_p \) is equivalent to extract the \((n + 1)\)th element out from \( u_p \) for \( n = 0, \cdots, N - 1 \) and for \( p = 0, \cdots, M - 1 \). Therefore, \( \iota_n^T \iota_{n+1} \iota_n^T u_p \) is equivalent to compute \( u_p[n]u_p[n+1] \) for \( n = 0, \cdots, N - 2 \) and for \( p = 0, \cdots, M - 1 \). As a result, \( \hat{u}_p^T \iota_{n+1} \iota_n^T u_p < 0 \) is equivalent to \( u_p[n] \) being a zero crossing point. Then, \( \sum_{n=0}^{N-2} \frac{1}{2} (1 - \text{sgn}(\hat{u}_p^T \iota_{n+1} \iota_n^T u_p)) \) is the total number of the zero crossing points in \( u_p \) for \( p = 0, \cdots, M - 1 \).

As one of the IMF condition is that the difference between the total number of the extrema and the total number of the zero crossing points is either equal to zero or equal to one, we have

\[
\sum_{n=0}^{N-3} \frac{1}{2} (1 - \text{sgn}(\hat{u}_p^T \imath_{n+1} \imath_n^T u_p)) = \sum_{n=0}^{N-2} \frac{1}{2} (1 - \text{sgn}(\hat{u}_p^T \iota_{n+1} \iota_n^T u_p))
\]

for \( p = 0, \cdots, M - 1 \) \hspace{1cm} (6a)

or

\[
\left| \sum_{n=0}^{N-3} \frac{1}{2} (1 - \text{sgn}(\hat{u}_p^T \imath_{n+1} \imath_n^T u_p)) - \sum_{n=0}^{N-2} \frac{1}{2} (1 - \text{sgn}(\hat{u}_p^T \iota_{n+1} \iota_n^T u_p)) \right| = 1
\]

for \( p = 0, \cdots, M - 1 \) \hspace{1cm} (6b)

for \( p = 0, \cdots, M - 1 \). As discussed in the above, there are two conditions. However, one and only one condition is required to be satisfied. Therefore, this problem is an exclusive or constrained optimization problem.

2.4. Definition of the optimization problem. Our objective is to find \( U \) such that \( U \) is the closest to \( \hat{U} \) subject to both the orthogonal condition among the vectors in \( U \) and the above IMF conditions. In this paper, the closeness between \( U \) and \( \hat{U} \) is defined using the \( L_2 \) sense. That is, \( \sum_{i=0}^{M-1} |u_i - \hat{u}_i|^2 \) is minimized. The \( L_2 \) criterion is employed for the problem formulation because it is the most common criterion used in various science and engineering applications. Therefore, the determination of \( U \) can be formulated as the following optimization problem (P):
min \sum_{i=0}^{M-1} \|u_i - \hat{u}_i\|^2 \quad (7a)
\text{subject to } U^T U = I_M \quad (7b)
\frac{t^T u_p}{N} = 0 \quad \text{for } p = 0, \cdots, M - 1 \quad (7c)
\begin{align*}
(6 - a) \quad \text{or } (6 - b).
\end{align*}

As it is discussed in Section 1 that the convex combination of any two different unitary matrices is not unitary, the feasible set corresponding to the constraint \( U^T U = I_M \) is nonconvex. As a result, Problem(P) is a nonconvex optimization problem. In general, it is very challenging to finding a nearly global optimal solution of such kind of nonconvex optimization problems. Furthermore, as \( \text{sgn}(z) \) is a discontinuous function, the conventional gradient descent based methods do not apply for finding the solution of Problem(P). Besides, Problem(P) consists of an exclusive or constraint. As most of existing optimization algorithms only deal with the satisfaction of all the constraints, the existing optimization algorithms do not apply for finding the solution of Problem(P) too [12].

3. Solution method.

3.1. Handling the exclusive or constraint. Although the constraint (7-c) is an exclusive or constraint, actually it is equivalent to the following inequality constraint:

\[
\sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T \theta_{n+1} \theta_n^T u_p \right) \right) - \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T i_n i_n^T u_p \right) \right) \leq 1
\quad \text{for } p = 0, \cdots, M - 1 \quad (8)
\]

This is because both the total number of the extrema and the total number of the zero crossing points of \( u_p \) for \( p = 0, \cdots, M - 1 \) are integers.

3.2. Converting the constrained problem to an unconstrained problem. To find a solution of Problem(P), a Lagrange multiplier approach is applied. Define the Lagrange multipliers corresponding to the constraint \( U^T U = I_M \) as \( \lambda_{i,j} \) for \( i = 0, \cdots, M - 1 \) and for \( j = 0, \cdots, M - 1 \). Let \( \lambda = [ \lambda_0 \cdots \lambda_{M-1} ]^T \). Also, let the Lagrange multipliers corresponding to the inequality constraint equation (8) for \( p = 0, \cdots, M - 1 \). Define \( \lambda = [ \lambda_0 \cdots \lambda_{M-1} ]^T \). Similarly, let the Lagrange multipliers corresponding to the equality constraint equation (5) for \( p = 0, \cdots, M - 1 \) as \( \text{lambda}_p \) for \( p = 0, \cdots, M - 1 \). Define \( \text{lambda}_p = [ \hat{\lambda}_0 \cdots \hat{\lambda}_{M-1} ]^T \). Denote the Lagrange function as

\[
J(U, \lambda, \lambda, \bar{\lambda}, \hat{\lambda}) = \sum_{i=0}^{M-1} \|u_i - \hat{u}_i\|^2 + \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \lambda_{i,j} \left( u_i^T u_j - \delta[i-j] \right) + \sum_{p=0}^{M-1} \bar{\lambda}_p \frac{t^T u_p}{N} + \sum_{p=0}^{M-1} \sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T \theta_{n+1} \theta_n^T u_p \right) \right) - \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T i_n i_n^T u_p \right) \right) - 1)
\]
The above objective function is continuously differentiable, so the conventional gradient descent approach can be applied for finding the solution of this approximated problem. It is worth noting that

\[
\frac{\partial J(U, \lambda, \bar{\lambda})}{\partial u_i} = 2u_i - 2u_i + \sum_{j=0}^{M-1} (\lambda_{i,j} + \lambda_{j,i}) u_j + \frac{\lambda \tau}{N} \left( \sum_{n=0}^{N-3} \frac{1}{2} (1 - \frac{2}{\pi} \tan^{-1} (Du_{n+1t}^T u_1)) - \sum_{n=0}^{N-2} \frac{1}{2} (1 - \frac{2}{\pi} \tan^{-1} (Du_{n+1t}^T u_1)) \right)
\]

\[
+ \bar{\lambda} \left( \sum_{n=0}^{N-3} \frac{1}{2} (1 - \frac{2}{\pi} \tan^{-1} (Du_{n+1t}^T u_1)) - \sum_{n=0}^{N-2} \frac{1}{2} (1 - \frac{2}{\pi} \tan^{-1} (Du_{n+1t}^T u_1)) \right)
\]
Define

\[ \lambda_i \partial f_\delta \left( z_i \right) \left( \frac{1}{\pi} \sum_{n=0}^{N-3} \frac{1}{1 + \left( Du_i^T \ell_{n+1}^T u_i \right)^2} + \frac{1}{\pi} \sum_{n=0}^{N-2} \frac{1}{1 + \left( Du_i^T i_{n+1}^T u_i \right)^2} \right) \]
\[ = 2u_i - \lambda_i \frac{\lambda_i}{N} \]
for \( i = 0, \ldots, M-1 \).

Here,

\[ z_i = \sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} \left( Du_i^T \ell_{n+1}^T u_i \right) \right) \]
\[ - \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} \left( Du_i^T i_{n+1}^T u_i \right) \right) \]
for \( i = 0, \ldots, M-1 \).

Define

\[ L(U) = \begin{bmatrix} l_{0,0} & \cdots & l_{0,M-1} \\ \vdots & \ddots & \vdots \\ l_{N-1,0} & \cdots & l_{N-1,M-1} \end{bmatrix} \]
\[
\frac{\partial f_b(z_0)}{\partial z_0} \left( -\frac{1}{\pi} \sum_{n=0}^{N-3} \frac{\bar{\jmath}_{n+1}\bar{\jmath}_n u_0}{1 + (Du_0^T \bar{\jmath}_{n+1}\bar{\jmath}_n u_0)^2} + \frac{1}{\pi} \sum_{n=0}^{N-2} \frac{i_{n+1}i_n u_0}{1 + (Du_0^T i_{n+1}i_n u_0^2)^2} \right) \ldots
\]
\[
\frac{\partial f_b(z_{M-1})}{\partial z_{M-1}} \left( -\frac{1}{\pi} \sum_{n=0}^{N-3} \frac{\bar{\jmath}_{n+1}\bar{\jmath}_n u_{M-1}}{1 + (Du_{M-1}^T \bar{\jmath}_{n+1}\bar{\jmath}_n u_{M-1})^2} \\
+ \frac{1}{\pi} \sum_{n=0}^{N-2} \frac{i_{n+1}i_n u_{M-1}}{1 + (Du_{M-1}^T i_{n+1}i_n u_{M-1})^2} \right)
\]
\[
(16)
\]

Then, we have
\[
2U + U(\lambda + \lambda^T) + L(U) \text{diag}(\bar{\lambda}) + \frac{[\tau \cdots \tau] \text{diag}(\lambda)}{N} = 2\hat{U}
\]
where refers to a diagonal matrix with its diagonal elements being the elements of the vector \( \lambda \). Besides, as
\[
\frac{\partial}{\partial \lambda_{i,j}} \hat{J}(U, \lambda, \bar{\lambda}, \hat{\lambda}) = u_i^T u_j - \delta[i,j]
\]
for \( i = 0, \ldots, M - 1 \) and for \( j = 0, \ldots, M - 1 \)

we have
\[
U^T U = I_M
\]

Moreover, as
\[
\frac{\partial}{\partial \lambda_i} \hat{J}(U, \lambda, \bar{\lambda}, \hat{\lambda}) = f_b \left( \sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} (Du_0^T \bar{\jmath}_{n+1}\bar{\jmath}_n u_i) \right) \\
- \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} (Du_0^T i_{n+1}i_n u_i) \right) \right) - 1
\]
\[
(20)
\]

for \( i = 0, \ldots, M - 1 \)

we have
\[
f(U) = \iota
\]

Here,
\[
f(U) = \left[ f_b \left( \sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} (Du_0^T \bar{\jmath}_{n+1}\bar{\jmath}_n u_0) \right) \\
- \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} (Du_0^T i_{n+1}i_n u_0) \right) \right) \right]
\]
\[
\ldots \left[ f_b \left( \sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} (Du_{M-1}^T \bar{\jmath}_{n+1}\bar{\jmath}_n u_{M-1}) \right) \\
- \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} (Du_{M-1}^T i_{n+1}i_n u_{M-1}) \right) \right) \right]
\]
\[
(22)
\]

Finally, as
\[
\frac{\partial}{\partial \lambda_i} \hat{J}(U, \lambda, \bar{\lambda}, \hat{\lambda}) = \frac{\tau_i^T u_i}{N} \text{ for } i = 0, \ldots, M - 1
\]
\[
(23)
\]
we have
\[
\frac{U^T N}{T} = 0_{M \times 1}
\]  
where \(0_{M+1}\) denotes the \(M \times 1\) zero vector. Now, finding the solution of the original optimization problem becomes finding the roots of these four matrix equations.

3.5. Solving the required equations. To find the roots of the above four matrix equations, we first find \(U\) using the last three equations. That is, \(U^T U = I_M\), \(U^T \xi_N = 0_{M \times 1}\) and \(f(U) = \iota\). We define the following optimization problem: Problem (P)

\[
\begin{align*}
\min_U & \|f(U) - \iota\|^2 \\
\text{subject to} & \quad U^T U = I_M \\
& \quad \frac{U^T \xi_N}{N} = 0_{M \times 1}
\end{align*}
\]

It is worth noting that if the feasible set of Problem \((P)\) is nonempty, then \(\exists U\) such that \(f(U) = 1\), \(U^T U = I_M\) and \(\frac{U^T \xi_N}{N} = 0_{M \times 1}\). This implies that the feasible set of Problem \((P')\) is also nonempty and any point in the feasible set of Problem \((P)\) is the nearly global optimal solution of Problem \((P')\). This is because the minimum objective functional value of Problem \((P')\) is equal to zero in which the objective functional values of Problem \((P)\) evaluated at any point in the feasible set of Problem \((P)\) are also equal to zero. Therefore, the matrix \(U\) satisfying \(U^T U = I_M\), \(\frac{U^T \xi_N}{N} = 0_{M \times 1}\) and \(f(U) = \iota\) can be found by finding the nearly global optimal solution of Problem \((P')\).

To achieve this goal, it is worth noting that \(U\) satisfying \(U^T U = I_M\) and \(\frac{U^T \xi_N}{N} = 0_{M \times 1}\) implies that the columns of \(U\) is in the orthonormal null space of \(\tau\). Therefore, we first initialize a set of matrices \(U\) with their columns being in the orthonormal null space of \(\tau\). Then, the locally optimal solution of Problem \((P')\) is found via the conventional gradient descent approach by using each \(U\) obtained in the above as the initial condition. The matrix \(U\) that nearly global minimizes \(\|f(U) - \iota\|^2\) is chosen. If \(\|f(U) - \iota\|^2\) evaluated at the obtained nearly global optimal solution is close to zero, then this nearly global optimal solution is employed for performing the following procedures. Otherwise, a new set of \(U\) with the columns of \(U\) being in the orthonormal null space of \(\tau\) is reinitialized. The above procedures are repeated until \(\|f(U) - \iota\|^2\) evaluated at the obtained \(U\) is close to zero.

Once \(U\) is found, \(\lambda, \tilde{X}\) and \(\tilde{\lambda}\) can be found using the last equation (17) as follows. since this is an \(N \times M\) matrix equation, we have

\[
2u_q[p] + \sum_{k=0}^{M-1} (\lambda_{qk} + \lambda_{kq}) u_k[p] + l_{pq} \tilde{\lambda}_q + \frac{\lambda_q}{N} = 2\tilde{u}_q[p]
\]

according to the \((p+1)^{th}\) row and the \((q+1)^{th}\) column of this matrix equation for \(p = 0, \cdots, N - 1\) and for \(q = 0, \cdots, M - 1\). Denote \(A(U)\) as a \(MN \times M^2\) matrix, \(B(U)\) as a \(MN \times 2M\) matrix and \(C(U)\) as a \(MN\) column vector. In particular, denote the \((Mp + q + 1)^{th}\) row of \(A(U)\) as \([a_{Mp+q,0}^T \cdots a_{Mp+q,M-1}^T]\) for

\[
a_{Mp+q,i} = \begin{bmatrix} 0 & \cdots & 0 & u_q[p] & 0 & \cdots & 0 \end{bmatrix}^T, \quad i \in \{0, \cdots, M - 1\} \setminus \{q\}
\]
for $p = 0, \ldots, N - 1$ and for $q = 0, \ldots, M - 1$, where there are $q$ zeros and $M - q - 1$ zeros before and after $u_q[p]$ in $a_{Mp+q,i}$. Also

$$a_{Mp+q,q} = [u_0[p] \cdots u_{q-1}[p] 2u_q[p] u_{q+1}[p] \cdots u_{N-1}[p]]^T$$

(28)

for $p = 0, \ldots, N - 1$ and for $q = 0, \ldots, M - 1$.

On the other hand, denote the $(Mp + q + 1)^{th}$ row of $B(U)$ as

$$[0 \cdots 0 l_{p,q} 0 \cdots 0 \frac{1}{N} 0 \cdots 0]$$

for $p = 0, \ldots, N - 1$ and for $q = 0, \ldots, M - 1$. Here, there are $q$ zeros before $l_{pq}$, $M - q - 1$ zeros after $\frac{1}{N}$ and $M - 1$ zeros between $l_{pq}$ and $\frac{1}{N}$ and in $B(U)$. Also, denote the $(Mp + q + 1)^{th}$ element of $C(U)$ as $2u_q[p] - 2u_q[p]$ for $p = 0, \ldots, N - 1$ and for $q = 0, \ldots, M - 1$. Now, the $N \times M$ matrix equation is equivalent to the following linear matrix equation:

$$[\mathbf{A}(U) \quad \mathbf{B}(U)] \begin{bmatrix} \lambda_0^T \\ \vdots \\ \lambda_{M-1}^T \\ \bar{X}^T \\ \bar{\lambda}^T \end{bmatrix}^T = C(U)$$

(29)

From here, we can see that there are $MN$ linear equations with $M^2 + 2M$ independent variables in this linear matrix equation. Therefore, we have $M^2 - M(N - 2)$ degrees of freedoms on choosing $\lambda, \bar{X}$ and $\bar{\lambda}$ if $M^2 - M(N - 2) > 0$. There is no solution if $M^2 - M(N - 2) < 0$. $\lambda, \bar{X}$ and $\bar{\lambda}$ are uniquely defined if $M^2 - M(N - 2) = 0$.

3.6. **Handing the nonconvex issue.** It is worth noting that the above method can only find a locally optimal solution of Problem(P). To find a nearly global optimal solution, the above procedures are repeated and the objective functional value is evaluated at each obtained locally optimal solution. These iterative procedures are terminated when the functional values evaluated at consecutive several locally optimal solutions are no longer smaller than the smallest objective functional value evaluated using the previous obtained locally optimal solutions. Finally, the locally optimal solution corresponding to the smallest objective functional value evaluated using the previous obtained locally optimal solutions is taken as a nearly global optimal solution.

4. **Numerical computer simulation results.**

4.1. **Methods for the comparisons.** It is worth noting that it is very difficult to have a fair comparison. This is because the existing methods cannot guarantee that the obtained components satisfy both the orthogonal condition and the conditions of the IMFs. In order to demonstrate the effectiveness of our proposed method, our obtained IMFs are compared to the unit energy components obtained by performing the normalization on the original IMFs. Here, the normalized components are guaranteed to satisfy the conditions of the IMFs. However, they are not orthogonal to each others. On the other hand, our obtained IMFs are also compared to the orthonormal components obtained by applying the Gram Schmidt orthogonalization method [8] to the original IMFs. Here, the components are guaranteed to satisfy the orthogonal condition. However, the conditions of the IMFs are not satisfied.

4.2. **Metrics for the comparisons.** Let

$$E = \begin{bmatrix} E_{0,0} & \cdots & E_{0,u-1} \\
\vdots & \ddots & \vdots \\
E_{M-1,0} & \cdots & E_{M-1,M-1} \end{bmatrix} = U^T U - I_M$$

(30)
Define

\[ err_{\text{orth}} = \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} |E_{m,n}| \]  

(31)

Obviously, \( err_{\text{orth}} \geq 0 \) and \( err_{\text{orth}} = 0 \) if the columns of \( U \) are orthogonal to each other. Therefore, \( err_{\text{orth}} \) is used to measure the orthogonal error of all the IMFs. On the other hand, define

\[
g(u_p) = \left\{ \begin{array}{ll}
\sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T i_{n+1}^T i_n^T u_p \right) \right) - \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T i_{n+1}^T i_n^T u_p \right) \right) & \geq 0 \\
\sum_{n=0}^{N-3} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T i_{n+1}^T i_n^T u_p \right) \right) - \sum_{n=0}^{N-2} \frac{1}{2} \left( 1 - \text{sgn} \left( u_p^T i_{n+1}^T i_n^T u_p \right) \right) & < 1
\end{array} \right.
\]

(32)

\[
\text{for } p = 0, \ldots, M - 1
\]

and

\[
err_{\text{ex-zc}} = \| [ \ g(U_0) \ \cdots \ g(U_{M-1}) ]^T \|^2
\]

(33)

It can be seen that \( err_{\text{ex-zc}} \geq 0 \) and \( err_{\text{ex-zc}} = 0 \) if the differences between the total number of the extrema and the total number of the zero crossing points of all the IMFs are either equal to zero or equal to one. Therefore, \( err_{\text{ex-zc}} \) is used to measure the error between the total number of the extrema and the total number of the zero crossing points of the IMFs. Finally, define

\[
err_{\text{mean}} = \left\| \frac{\tau^T u_0}{N} \cdots \frac{\tau^T u_{M-1}}{N} \right\|^2
\]

(34)

It can be seen that \( err_{\text{mean}} \geq 0 \) and \( err_{\text{mean}} = 0 \) if \( \frac{\tau^T u_p}{N} = 0 \ \forall p \in \{0, \ldots, M - 1\} \). Therefore, \( err_{\text{mean}} \) is used to measure the zero mean error of all the IMFs.

4.3. Selections of the parameters in the proposed algorithm. For the approximation functions \( \frac{2}{\delta} \tan^{-1}Dz \) and \( f_k(z) \) used in our proposed algorithm, \( D = 10^3 \) and \( \delta = 10^{-3} \) are chosen as they are large and small enough for most applications. On the other hand, is dependent on both the duration and the bandwidth of \( x(t) \). In this paper, the demonstrated signal is with 5 seconds and it is a low frequency signal. Therefore, the sampling rate is chosen as Hz. Hence, \( N = 501 \).

4.4. Illustrations. It is worth noting that if a test signal consists of two different cosine components with their frequencies and amplitudes satisfy the well separated condition of the IMF, then the mode mixing phenomenon does not occur. In this case, the IMF condition is satisfied. On the other hand, as two different cosine components are orthogonal to each other, the orthogonal condition is also satisfied. Therefore, in order to illustrate the effectiveness of the proposed method, a signal with two components where they neither satisfy the IMF condition nor the orthogonal condition is employed. After applying the proposed method, the obtained components will satisfy both the IMF condition and the orthogonal condition. To achieve this objective, a signal with two components with the same frequencies but modulated by two different exponential decay carriers is chosen for the illustration.
In particular, the signal is chosen as 
\[ x(t) = \sin(2\pi t)e^{t^2} + e^{-t}\cos(2\pi t). \]
Figure 1 shows the impulse response of this signal. When a cosine component is modulated by an exponential decay carrier, the positive functional values at the peaks are not equal to the corresponding negative functional values at the peaks. As a result, the mean values of the components are not equal to zero. Hence, the IMF condition does not satisfy. In this case, a set of IMFs different from these two components will be obtained by applying our proposed method. Besides, due to the similar reason, these two components are not orthogonal to each other. Hence, the orthogonal condition is not satisfied. In this case, a new set of components will be obtained after applying our proposed method. By using this example, a sharp contrast can be demonstrated.

This signal consists of two components \( \sin(2\pi t)e^{t^2} \) and \( e^{-t}\cos(2\pi t) \). They are shown in Figure 2. As it is discussed in the above that these two components are neither satisfy the IMF condition nor the orthogonal condition. Hence, a different set of IMFs will be obtained after applying the EMD algorithm. In particular, there are three IMFs and one residue after applying the EMD algorithm. The normalized IMFs are shown in Figure 3. The orthogonalized components obtained by applying the Gram Schmidt orthogonalization method [8] and the new set of IMFs obtained by our proposed method are shown in Figure 4 and Figure 5, respectively. As discussed in the above that the normalized IMFs are not orthogonal to each others. Therefore, the orthogonalized components obtained by applying the Gram Schmidt orthogonalization method [7] and the new set of IMFs obtained by our proposed method will be different from the normalized IMFs. Table 1 demonstrates the orthogonal errors of these three methods. The components obtained by both our proposed method and the Gram Schmidt orthogonalization method [7] are guaranteed to be orthogonal. Therefore, these errors are exactly equal to zero. On the other hand, the orthogonal error of the components obtained by the normalization method is large. This is because the normalization method does not guarantee that the components are orthogonal to each others. Table 2 demonstrates the errors between the total numbers of the extrema and the total numbers of the zero crossing points of all the IMFs of these three methods. The components obtained by both our proposed method and the normalization method are guaranteed to be the IMFs. Therefore, the errors are exactly equal to zero. On the other hand, the error obtained by the Gram Schmidt orthogonalization method [7] are large. This is because the components obtained by the Gram Schmidt orthogonalization method [7] are not the IMFs. Table 3 demonstrates the zero mean errors of all the IMFs of these three methods. Theoretically, the zero mean errors of all the IMFs based on our proposed method and the normalization method are equal to zero. Here, very small errors are found. This is due to the numerical rounding errors introduced by the computer aided design tool. From the above results, it can be concluded that our proposed method outperforms the existing methods in term of the guarantee of the satisfaction of both the orthogonal condition and the conditions of the IMFs.

5. Conclusions. This paper aims to find a set of orthogonal IMFs via an optimization approach. Here, each IMF is sampled and represented as a vector. To formulate the IMF conditions, first the gradient at the current sampling point of each IMF is evaluated by computing the difference between the current sampling point and the previous sampling point. Similarly, the gradient at the next sampling point of each IMF is evaluated by computing the difference between the next sampling point and the current sampling point. Then, an extremum of each IMF can
Figure 1. Original signal

Figure 2. Signal components

Figure 3. Normalized IMFs
Figure 4. Components obtained by the Gram Schmidt orthogonalization method

Figure 5. Components obtained by our proposed method

Table 1. Orthogonal errors of all the IMFs $err_{orth}$

| Signal | Our proposed method | Gram Schmidt orthogonalization method [8] | Normalization method |
|--------|---------------------|------------------------------------------|---------------------|
| $x(t) = \sin(2\pi t)e^{-\frac{t^2}{2}} + e^{-t}\cos(2\pi t)$ | 0 | 0 | 1.3545 |

Table 2. Errors between the total number of the extrema and the total number of the zero crossing points of all the IMFs $err_{ex-zx}$

| Signal | Our proposed method | Gram Schmidt orthogonalization method [8] | Normalization method |
|--------|---------------------|------------------------------------------|---------------------|
| $x(t) = \sin(2\pi t)e^{-\frac{t^2}{2}} + e^{-t}\cos(2\pi t)$ | 0 | 25 | 0 |
be detected by testing whether the product of these two gradients is negative or not. On the other hand, a zero crossing point of each IMF can also be detected by testing whether the product of the current sampling point and the next sampling point is negative or not. Second, by computing the sums of the above products, the total number of the extrema and the total number of the zero crossing points of each IMF are obtained. As a result, one of the IMF condition can be formulated accordingly. To formulate another IMF condition, the mean of each IMF is evaluated and tested whether it is zero or not. To formulate the orthogonal condition, the product of the matrix containing the IMFs and its transpose is computed and tested whether it is equal to the identity matrix or not. Finally, an optimization problem is formulated in such a way that the energy of the difference between the original IMFs and the corresponding obtained IMFs is minimized subject to both the orthogonal condition and the IMF conditions.

To satisfy the IMF condition, the difference between the total number of the extrema and the total number of the zero crossing points of each IMF should be either equal to zero or equal to one. Therefore, the optimization problem is subject to an exclusive or constraint. This constraint is further reformulated to an inequality constraint. Besides, the constrained optimization problem is converted to an unconstrained optimization problem using the Lagrange multiplier approach. However, it requires to satisfy a linear matrix equation, a quadratic matrix equation and a highly nonlinear matrix equation only dependent on the orthogonal IMFs as well as a nonlinear matrix equation dependent on both the orthogonal IMFs and the Lagrange multipliers. To solve these matrix equations, the first three equations are considered. First, a new optimization problem is formulated in such a way that the error energy of the highly nonlinear matrix equation is minimized subject to the linear matrix equation and the quadratic matrix equation. By finding the nearly global optimal solution of this newly formulated optimization problem and checking whether the objective functional value evaluated at the obtained solution is close to zero or not, the orthogonal IMFs are found. Finally, by substituting the obtained orthogonal IMFs to the last matrix equation, this last matrix equation reduced to a linear matrix equation which is only dependent on the Lagrange multipliers. Therefore, the Lagrange multipliers can be found. Consequently, the solution of the original optimization problem is found. By repeating these procedures with different initial conditions, a nearly global optimal solution is obtained.

Computer numerical simulation results show that our proposed method can find the components which satisfy both the orthogonal condition and the IMF conditions. On the other hand, the existing methods fail to do so.

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