The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras

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Abstract
We show a number of properties of the commutator algebra of a nilpotent matrix over a field. In particular we determine the simple modules of the commutator algebra. Then the results are applied to prove that certain Artinian complete intersections have the strong Lefschetz property.

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1 Introduction

Suppose that $A = \bigoplus_{i=0}^{\infty} A_i$ is an Artinian graded algebra over a field $K$ and $z$ is a homogenous element of $A$. We denote by $\times z$ the linear map $A \rightarrow A$ defined by $\times z(a) = az$. We say that $A$ has the weak Lefschetz property if there is a linear form $g \in A$ such that the rank of $\times g$ is maximum that can be expected only from the Hilbert function of $A$. In the same sense the strong Lefschetz property means that the rank of $\times g^k$ is maximum possible for every $k \geq 0$. (For details see Definition 17). We use the definition of “strong Lefschetz property” in a restricted sense. For the reason see [6, Remark 4]. Consider the exact sequence

$$0 \rightarrow A/(0 : z) \rightarrow A \rightarrow A/(z) \rightarrow 0$$

where the first map is induced by the multiplication map $\times z$. It seems to be interesting to ask under what conditions the strong/weak Lefschetz property of $A$ can be deduced from that of $A/(0 : z)$ and $A/(z)$. Let $g$ be a general linear form (one independent of $z$). For the definition of “general linear form,” see Definition 17. To determine the rank of the linear map $\times g : A \rightarrow A$ from the knowledge of the two homomorphisms $\times g : A/(z) \rightarrow A/(z)$ and $\times g : A/(0 : z) \rightarrow A/(0 : z)$, one has to consider how a preimage of $A/(z)$ in $A$ is mapped to $(z)$ by the map $\times z$. Thus we are led to consider the exact sequence $0 \rightarrow (0 : z) \rightarrow A \rightarrow (z) \rightarrow 0$ as well. Now to determine the rank of the linear map

$$\times z : A/(0 : z) \rightarrow A/(0 : z)$$

we consider the exact sequences

$$0 \rightarrow A/((0 : z) : z) \rightarrow A/(0 : z) \rightarrow A/(0 : z) + (z) \rightarrow 0$$

and

$$0 \rightarrow (0 : z) : z \rightarrow A/(0 : z) \rightarrow (z) + (0 : z)/(0 : z) \rightarrow 0.$$ 

These sequences may be used repeatedly, so as to complete the calculation of the rank of $\times z$.

In our paper [7], we consider a family $\mathcal{F}$ of $K$-algebras such that if $A \in \mathcal{F}$ then there is an exact sequence $0 \rightarrow A' \rightarrow A \rightarrow B \rightarrow 0$, with some linear form $z$, where $A' \in \mathcal{F}$ and $B \in \overline{\mathcal{F}}$, and the rank of $\times g : A \rightarrow A$ for a general linear form can be determined from the knowledge of $A'$ and $B$. The purpose of this paper is to provide basic tools and framework for considering such a family of algebras, and to prove the strong/weak Lefschetz property for all members of the family altogether.

One of the basic tools is the commutator algebra of a nilpotent matrix. A famous lemma due to I. Schur states that the commutator algebra of an irreducible matrix set is a division ring. In some sense we are interested in the other extremal case, namely the commutator algebra of a
single nilpotent matrix. In our previous paper \[6\] the commutator algebra of some commuting family of matrices in a full matrix algebra played an important role in studying commutative Artinian algebras. In this paper we treat the idea more generally. For a nilpotent matrix \(J\) we denote by \(C(J)\) the commutator algebra of \(J\). A description of \(C(J)\) can be found in \([4, 11]\), and \([1]\). The algebra \(C(J)\) as a set of matrices is easily determined as it is the set of solutions of a system of linear equations. The way it is put as a set of matrices, however, is not quite adequate for our purposes. We rearrange the order of the basis elements, so that the matrices of the commutator algebra are “upper triangular” as much as possible. The resulting set of matrices is denoted by \(\hat{C}(J)\). The use of the transformation \(J \mapsto \hat{J}\) was suggested by \([13]\) and it worked quite effectively in \([6]\). We show the relation between \(C(\hat{J})\) and \(C(\hat{J}')\), where \(\hat{J}'\) is the submatrix of \(\hat{J}\) corresponding to the restricted map \(J : \text{im} J \rightarrow \text{im} J\). This is important in the inductive argument and it reveals the structure of the algebra \(C(J)\). Once this is done, we may determine the Jacobson radical of the commutator algebra; we also determine in several ways the simple modules of \(C(J)\) (Proposition \([12]\)). In our papers \([7]\) and \([8]\) we study further these central simple modules for Gorenstein algebras \(A\) with a linear form fixed.

Furthermore we show how the rank of \(\hat{M} + \hat{J}\) can be computed (Proposition \([16]\)) for a certain element \(\hat{M} \in C(\hat{J})\). In our application to commutative Artinian \(K\)-algebras we are going to choose two linear elements, \(l, z\) of \(A\), and show that, with certain conditions imposed on \(A\) and \(z \in A\), the rank of \(\times(l + \lambda z)\), for most \(\lambda \in K\), reaches the CoSperner number of \(A\), so the pair \((A, l + \lambda z)\) is weak Lefschetz.

The main results of this paper are Proposition \([20]\) and Theorem \([21]\) of Section 4. To explain the meaning of Theorem \([21]\) let \(y, z \in A\) be linearly independent linear forms of an Artinian algebra \(A\). The theorem gives a lower bound for the rank

\[
\times(y + \lambda z) \in \text{End}(A)
\]

in terms of \(\text{dim } A/(z)\) and the ranks of the diagonal blocks of the linear maps

\[
\times y^i : A/(z) \longrightarrow A/(z) \in \text{End}(A/(z)), \quad i = 1, 2, \ldots.
\]

Since the obvious upper bound for the rank \(\times(y + \lambda z)\) is the CoSperner number of \(A\), these considerations give us a sufficient condition for \(A\) to have the weak Lefschetz property (Theorem \([21](ii)\)). This is a direct consequence of Propositions \([16]\) and \([20]\) since \(\times y \in C(xz)\). To prove Proposition \([20]\) we consider the form ring

\[
\text{Gr}(x)(A) = A/(z) \oplus (z^2)/(z^2) \oplus (z^3)/(z^3) \oplus \cdots \oplus (z^{p-1})/(z^p)
\]

for a \(K\)-algebra \(A\) with respect to the principal ideal \((z)\). It is well known that this is endowed with algebra structure; in fact it is isomorphic to an algebra \(B[y]/I\) where \(B = A/(z)\) and \(I\) is generated by “monomials in \(y\),” which are elements of the form \(\overline{a_i y^{m_i}}\). It turns out that the inequality of the proposition is essentially equivalent to

\[
\text{rank}(\times g_1) \leq \text{rank}(\times g_2)
\]

where \(g_1\) and \(g_2\) are general linear forms of \(\text{Gr}(x)(A)\) and \(A\) and respectively. This simplifies the computation of the rank of a general linear form as it is often the case that even the rank of \(\times g_1\) is sufficiently large.
Throughout this paper $K$ denotes a field of any characteristic unless otherwise specified. When we discuss the strong Lefschetz property we assume the characteristic of $K$ is zero. Except for this all results are valid for any characteristic. In Section 2.2 and an early part of Section 2.3 we do use the exponential of a nilpotent matrix. However, this is not essential. We use the exponential, because it facilitates the description of the matrices of the commutator algebra $\mathfrak{C}(J)$. This is easy to see, so there will be no confusion. In [8], we consider Artinian algebras with non-standard grading, but in this paper all algebras are assumed to have the standard grading so the generators of algebras have degree one.

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2 The commutator algebra of a nilpotent matrix

2.1 The Jordan canonical form and Young diagrams

Let $K$ be a field. Let $M(n)$ denote the set of $n \times n$ matrices with entries in $K$. We are concerned with nilpotent matrices. Since all eigenvalues of a nilpotent matrix are 0, its Jordan canonical form is expressed by telling how it decomposes into Jordan cells. Thus we are led to use a Young diagram to indicate the Jordan canonical form of a nilpotent matrix. When we say that \((n_1, n_2, \ldots, n_r)\) is a partition of an integer $n$, it means that all $n_i$ are positive integers such that $n = n_1 + n_2 + \cdots + n_r$. When the terms in a partition of $n$ are arranged in decreasing order we may associate to it a Young diagram of size $n$ in the well known manner. The notation $T = T(n_1, n_2, \ldots, n_r)$ denotes the Young diagram with rows of $n_i$ boxes, $i = 1, 2, \ldots, r$, where it is tacitly assumed that $n_1 \geq n_2 \geq \cdots \geq n_r > 0$. The same Young diagram is also denoted by $T = \hat{T}(\nu_1, \ldots, \nu_p)$, by which it is meant that the integer $\nu_i$ is the number of boxes in the $i$th column of $T$.

Let $T$ be a Young diagram of size $n$. Suppose we number the boxes of $T$ with numbers 1, 2, \ldots, $n$. Then we may define a matrix $M = (a_{ij}) \in M(n)$ as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } j \text{ is next to and on the right of } i, \\
0 & \text{otherwise}. \end{cases} \quad (1)$$

It is easy to see that the matrix $M$ is nilpotent and the matrices given by different numberings of the same $T$ are different only by a permutation of rows and columns. As is well known there is a bijection between the set of Young diagrams of size $n$ and the conjugacy classes of matrices with a single eigenvalue in $M(n)$.

In this note we use only two numberings of a Young diagram. One is the horizontal numbering starting with the first row and ending with the last row, and the other vertical.
Here are examples of such numberings for the Young diagram $T = T(5, 3, 1)$.

$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 &
\end{array}$,  
$\begin{array}{cccc}
1 & 4 & 6 & 8 \\
2 & 5 & 7 & 9 \\
3 &
\end{array}$

When $T$ is numbered vertically we call the matrix defined by (1) the Jordan second canonical form. To write the matrix explicitly, let $n = \nu_1 + \cdots + \nu_p$ be the dual partition of $n = n_1 + \cdots + n_r$. In other words $\nu_i$ is the number of boxes of the $i$th column of $T(n_1, \cdots, n_r)$. Let $I_i$ be the $\nu_i \times \nu_i+1$ matrix

$$I_i = \begin{array}{cc}
E & \{\nu_i+1 \} \\
O & \{\nu_i - \nu_i+1 \}
\end{array} \nu_i+1$$

where $E$ is the $\nu_i+1 \times \nu_i+1$ identity and $O$ the $(\nu_i - \nu_i+1) \times \nu_i+1$ zero matrix. Then the Jordan second canonical form for $T = T(n_1, n_2, \cdots, n_r)$ is the matrix

$$\begin{array}{cccccccc}
O & I_1 & O & \cdots & O & O & \{\nu_1 \} & \\
O & O & I_2 & \cdots & O & O & \{\nu_2 \} & \\
O & O & O & \cdots & O & O & \{\nu_3 \} & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
O & O & O & \cdots & O & I_{p-1} & \{\nu_{p-1} \} & \\
O & O & O & \cdots & O & O & \{\nu_p \}
\end{array} \nu_1 \nu_2 \nu_3 \cdots \nu_p$$

When $T$ is numbered horizontally the matrix is usually called the Jordan canonical form. If it is necessary to distinguish from the second, we call it the Jordan first canonical form. In our previous paper [6] we used the term “Jordan second canonical form” in a slightly different sense from the use here.

For our convenience we state some notational conventions.

- $T = T(n_1, n_2, \cdots, n_r)$ indicates that $T$ is a Young diagram with $r$ rows, where $n_i$ is the number of boxes of the $i$th row. It is assumed that $n_1 \geq n_2 \geq \cdots \geq n_r > 0$. $T = \hat{T}(\nu_1, \cdots, \nu_p)$ denotes the Young diagram with the integer $\nu_i$ as the number of boxes in the $i$th column. Thus, for example, $T(5, 3, 1) = \hat{T}(3, 2, 2, 1, 1)$.

- If $T = T(n_1, \cdots, n_r) = \hat{T}(\nu_1, \cdots, \nu_p)$, then $n = n_1 + \cdots + n_r$ and $n = \nu_1 + \cdots + \nu_p$ are dual partitions to each other.
• When we say that \( n = n_1 + \cdots + n_r \) is a partition of \( n \), it is not assumed that the terms are arranged in either decreasing or increasing order, but it is assumed that each term is positive.

• When we know that the sequence in a partition \( n = n_1 + \cdots + n_r \) is put in decreasing order, we use the term “dual partition” for \( \hat{T}(n_1, \ldots, n_r) \), identifying it with the Young diagram.

• When \( T = T(n_1, \ldots, n_r) \), sometimes \( T \) is referred to as a sequence or a partition in the obvious sense.

2.2 The linear hull of a generic exponential matrix

Until Proposition 1 begins, we assume that \( \text{char } K = 0 \). Let \( J \in M(n) \). As is well known, the exponential of \( J \) is the following:

\[
\exp(J) = E + J + \frac{1}{2!}J^2 + \frac{1}{3!}J^3 + \cdots
\]

We are interested in the linear hull of the set \( \{\exp(xJ) | x \in K \} \), where \( J \) is a nilpotent matrix. Suppose for the moment that \( J \) is a single \( n \times n \) Jordan block of a nilpotent matrix, so

\[
J = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& & \ddots & & \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Then

\[
\exp(xJ) = \begin{pmatrix}
1 & x & x^2/2! & \cdots & x^{n-1}/(n-1)! \\
0 & 1 & x & \cdots & \\
0 & 0 & 1 & \cdots & x^2/2! \\
0 & 0 & 0 & \cdots & x \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Thus the linear hull of \( \{\exp(xJ) | x \in K \} \) is the vector space spanned by \( E, J, J^2, J^3, \ldots, J^{n-1} \).

If \( x \) is an indeterminate, we say \( \exp(xJ) \) is a **generic exponential** of \( J \). By an **augmented exponential** we mean a matrix of type either \( (O|\exp(J)) \) or its vertical version,

\[
\begin{pmatrix}
\exp(J) \\
O
\end{pmatrix},
\]

where \( O \) is a zero block of an arbitrary size as long as it fits \( \exp(J) \). If \( J \) is a single Jordan cell of a nilpotent matrix, the linear hull of an augmented generic exponential of \( J \) is the vector space consisting of matrices which are one of the following types.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 \\
0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \\
0 & 0 & 0 & 0 & 0 & x_0 & x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_0
\end{pmatrix}
\]
2.3 The commutator algebra of a nilpotent matrix as a set

For \( J \in \mathbb{M}(n) \) we denote by \( \mathcal{C}(J) \) the commutator algebra of \( J \), namely

\[
\mathcal{C}(J) = \{ X \in \mathbb{M}(n) | XJ = JX \}.
\]

Note that \( \mathcal{C}(J) \) is an associative algebra with identity.

Let \( T = T(n_1, n_2, \ldots, n_r) \) be a Young diagram of size \( n \) numbered horizontally and let \( J \in \mathbb{M}(n) \) be the Jordan canonical matrix of partition \( (n_1, n_2, \ldots, n_r) \), so \( J \) is the matrix defined by the equation (1). Let \( J_i \) be the \( i \)th diagonal block of \( J \), namely, \( J_i \) is the Jordan cell of size \( n_i \). For the moment let us write \( \text{expo}(J_i) \) for \( \exp(J_i) \) augmented by \( O \). Recall that \( \text{expo}(J_i) \) is determined by its size. (For definition see Section 2.2.) Introduce letters \( x_{ij} \) as many as the number of pairs \((i, j)\) for \( 1 \leq i, j \leq r \), and define, for each pair \((i, j)\), the matrix \( X_{ij} \) of size \( n_i \times n_j \) as follows:

\[
X_{ij} = \begin{cases}
\text{expo}(x_{ij}J_j) & \text{if } i \leq j, \\
\text{expo}(x_{ij}J_i) & \text{if } i > j.
\end{cases}
\]

Put

\[
X = (X_{ij})
\]

by which we mean the \( n \times n \) block matrix with blocks \( X_{ij} \) defined above. The following lemma was proved by Turnbull and Aitken [11] and also by Gantmacher [4]. The proof here is due to Basili [1].

Lemma 1 (Gantmacher[4], Turnbull and Aitken[11]) The set \( \mathcal{C}(J) \) coincides with the linear hull of the set \( \{ X \in \mathbb{M}(n) | x_{ij} \in K \} \) as defined in (7) and (8).

Proof. Suppose \( M \) is an \( n \times n \) matrix. Let \( M = (M_{pq}) \) be the block decomposition shown in the picture below.

\[
M = \begin{array}{cccc}
M_{11} & M_{12} & \cdots & M_{1r} \\
M_{21} & M_{22} & \cdots & M_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
M_{r1} & M_{r2} & \cdots & M_{rr}
\end{array}
\begin{array}{c}
\{n_1 \\
\{n_2 \\
\vdots \\
\{n_r \\
\begin{array}{c}
n_1 \\
n_2 \\
\vdots \\
n_r
\end{array}
\end{array}
\end{array}
\]

Then one sees that the condition \( MJ = JM \) means that \( M_{pq}J_q = J_pM_{pq} \) for all \( 1 \leq p, q \leq r \). Thus the assertion follows from Lemma[2] below.
Lemma 2 Let \( Z = (z_{ij}) \) be a \( p \times q \) matrix. Let \( J_1 \) and \( J_2 \) be the Jordan cells of sizes \( p \) and \( q \) respectively.

(i) If \( p \geq q \), then \( J_1Z = ZJ_2 \) implies that \( z_{21} = z_{31} = \cdots = z_{p1} = 0 \) and \( z_{ij} = z_{(i+1)(j+1)} \) for all \( i, j \) such that \( 1 \leq i \leq p - 1 \) and \( 1 \leq j \leq q - 1 \).

(ii) If \( p < q \), then \( J_1Z = ZJ_2 \) implies that \( z_{p1} = z_{p2} = \cdots = z_{p(q-1)} = 0 \) and \( z_{ij} = z_{(i+1)(j+1)} \) for all \( i, j \) such that \( 1 \leq i \leq p - 1 \) and \( 1 \leq j \leq q - 1 \).

Proof is straightforward.

Remark 3 (i) A generic matrix \( X \) in \( \mathfrak{C}(J) \) decomposes as \( X = (X_{ij}) \), where each block \( X_{ij} \) is of the type shown in (5) for \( i \geq j \) and in (6) for \( i < j \). Entries in the last columns in the blocks \( X_{ij} \) for \( i \geq j \) and those in the first rows in \( X_{ij} \) for \( i < j \) are algebraically independent.

(ii) As before let \( (n_1, \ldots, n_r) \) be the partition of the nilpotent matrix \( J \) put in the Jordan canonical form. Put \( A = K[z]/(z^{n_1}) \) and \( V = \bigoplus_{r=1}^{r} K[z]/(z^{n_i}) \). Regard \( V \) as an \( A \)-module. Then we may use monomials as a basis of \( V \) so that \( J \) is the matrix for the multiplication map

\[ xz : V \rightarrow V. \]

It is not difficult to see that \( \mathfrak{C}(J) \) coincides with \( \text{End}_A(V) \). Details are left to the reader. In this paper this is used only to indicate another proof of Theorem [6]. (See the last paragraph preceding Theorem [6].)

Example 4 Let \( T = T(3, 3, 2) \). Then a general element of \( \mathfrak{C}(J) \) is of the form

\[
\begin{pmatrix}
  a & a' & a'' & b & b' & b'' & c & c' \\
  a & a' & 0 & b & b' & 0 & c & 0 \\
  0 & a & 0 & 0 & b & 0 & 0 & 0 \\
  d & d' & d'' & e & e' & e'' & f & f' \\
  0 & d & d' & 0 & e & e' & 0 & f \\
  0 & 0 & d & 0 & 0 & e & 0 & 0 \\
  0 & g & g' & 0 & h & h' & i & i' \\
  0 & 0 & g & 0 & 0 & h & 0 & i \\
\end{pmatrix}
\]  

(10)

Example 5 Let \( T = T(4, 2, 1) \). Then a general element of \( \mathfrak{C}(J) \) is of the form

\[
\begin{pmatrix}
  a & a' & a'' & a''' & b & b' & c \\
  0 & a & a' & a'' & 0 & b & 0 \\
  0 & 0 & a & a' & 0 & 0 & 0 \\
  0 & 0 & 0 & a & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & d & d' & e \\
  0 & 0 & 0 & d & 0 & e & 0 \\
  0 & 0 & g & 0 & 0 & h & i \\
\end{pmatrix}
\]  

(11)
3 The structure of the commutator algebra of a nilpotent matrix

To say anything about the structure of the algebra $C(J)$ we need to look at the Young diagram $T$ for $J$ more in detail. Throughout this section we fix $T$ and $J$ as follows:

1. $T = T(n_1, \ldots, n_r)$ is a Young diagram of size $n$.
2. $J$ is the Jordan canonical form of a nilpotent matrix of type $T$.

Let $(f_1, f_2, \ldots, f_s)$ be the finest subsequence of $(n_1, \ldots, n_r)$ such that $f_1 > f_2 > \cdots > f_s > 0$. Then we can rewrite the same sequence $(n_1, \ldots, n_r)$ as

$$(n_1, \ldots, n_r) = (f_1, \underbrace{f_2, \ldots, f_2}_{m_2}, \underbrace{f_3, \ldots, f_s}_{m_s}) = (f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_s)$$

(12)

The integer $m_j$ is the multiplicity of $f_j$. Let us call $m_1, \ldots, m_s$ the multiplicity sequence of $T(n_1, \ldots, n_r)$. Note that it gives us a partition of the number $r$ of rows of $T$, namely, $r = m_1 + m_2 + \cdots + m_s$.

Recall that the Jacobson radical of a ring is defined to be the intersection of

$$\text{ann}(M)$$

where $\text{ann}(M)$ denotes the annihilator of the module $M$ and $M$ runs over all simple (right) modules. The Jacobson radical is a two sided nilpotent ideal and if it is 0 then the ring is said to be semisimple. (See e.g., early pages of Herstein [9].) The following is a known result. E.g., this can be implied by [3 Theorem 3.5.2] with the identification of $C(J)$ with $\text{End}_A(V)$, where $A = K[z]/(z^{n_1})$ as described in Remark 3 (ii). We give a direct proof after Proposition 8 and Example 9 where we show a number of properties of a generic matrix of $C(J)$.

**Theorem 6** Let $C(J) \subset M(n)$ be the commutator algebra of $J$. Let $m_1, m_2, \ldots, m_s$ be the multiplicity sequence of $T$. Let $\rho$ be the Jacobson radical of $C(J)$. Then there is a surjective homomorphism

$$\Phi : C(J) \longrightarrow M(m_1) \times M(m_2) \times \cdots \times M(m_s)$$

with $\text{ker} \Phi = \rho$.

Proof is postponed to the end of Definition 10. The map $\Phi$ is defined in (19).

Let $M \in M(n)$. Using the partition $n = n_1 + \cdots + n_r$ we decompose $M$ into blocks as indicated in (9). When $M \in M(n)$ is considered as a block matrix in this way for a Young diagram $T = T(n_1, \ldots, n_r)$, we write $M = (x_{ij}^{(kl)})$, by which we mean that the element $x_{ij}^{(kl)}$
is the \((i, j)\)-entry of the \((k, l)\)-block of \(M\). Note that \(M\) has square diagonal blocks. For \(M = (x_{ij}^{(kl)})\) we define the matrix \(\hat{M}\) by

\[
\hat{M} = (x_{kl}^{(ij)}).
\]

The matrices \(M\) and \(\hat{M}\) differ only by a certain permutation of rows and columns. To name the permutation explicitly, let \(T_h\) and \(T_v\) be the same Young diagram with the horizontal and vertical numberings respectively. Let \(\pi : T_h \rightarrow T_v\) be the permutation of the integers \(\{1, 2, \ldots, n\}\) which, as in the picture below, claims that if a box is numbered \(i\) in \(T_h\) then the same box is numbered \(\pi(i)\) in \(T_v\).

Let \(P = (p_{ij})\) be the matrix defined by

\[
p_{ij} = \begin{cases} 
1 & \text{if } j = \pi(i), \\
0 & \text{otherwise.}
\end{cases}
\] (13)

Then one sees easily that \(\hat{M} = P^{-1}MP\) for \(M \in \mathfrak{C}(J)\).

Notice that the indices \(i\) and \(j\) of blocks of \(\hat{M} = (a_{kl}^{(ij)})\) run over 1 through \(n_1\), since \(n_1\) is the size of the biggest block of \(M = (a_{ij}^{(kl)})\). Let \(n = \nu_1 + \cdots + \nu_p\) be the dual partition of \(n = n_1 + \cdots + n_r\). (So \(\nu_i\) is the number of boxes of the \(i\)th column of \(T\).) Then one sees with a little contemplation that the size of the \((i, j)\)-block of \(\hat{M}\) is \(\nu_i \times \nu_j\). We are interested in the diagonal blocks of \(\hat{M}\). So we introduce a definition.

**Definition 7** For a Young diagram \(T = T(n_1, \ldots, n_r)\) and a block matrix \(M = (a_{ij}^{(kl)})\) as in (9) above, we define

\[
N_i
\]

to be the \(i\)th diagonal block of \(\hat{M}\) for \(i = 1, 2, \ldots, p\). Note that \(N_i\) is a square matrix of size \(\nu_i\), where \((\nu_1, \ldots, \nu_p)\) is the dual partition of \(T = T(n_1, \ldots, n_r)\).

Recall that \(J\) is the \(n \times n\) Jordan matrix with partition \(T = T(n_1, \ldots, n_r)\).

**Proposition 8** Using the notation above we have:

(i) For \(M \in \mathfrak{C}(J)\), the matrix \(\hat{M}\) is block upper triangular. Namely, if \(i > j\) the \((i, j)\)-block of \(\hat{M}\) is \(O\).

(ii) \(\hat{J}\) is the Jordan second canonical form of \(T\).
(iii) Let $\mathfrak{C}(\hat{J}) \subset M(n)$ be the commutator algebra of $\hat{J}$. Then the map $\mathfrak{C}(J) \to \mathfrak{C}(\hat{J})$ defined by

$$ M \mapsto \hat{M} $$

is a natural isomorphism of algebras.

(iv) Let $T' = T(n_1 - 1, n_2 - 1, \ldots, n_r - 1)$, and let $J'$ be the Jordan canonical matrix with partition $T'$. Then $\hat{M}$ decomposes as

$$ \hat{M} = \begin{bmatrix} N_1 & \ast \\ O & M' \end{bmatrix} $$

where $M' \in \mathfrak{C}(J')$.

(v) If $\nu_1 = \nu_2$, then $N_1 = N_2$ for every $M \in \mathfrak{C}(J)$. (For the definition of $N_i$ see Definition 7.)

(vi) If $\nu_1 > \nu_2$, then $N_1$ decomposes as follows:

$$ N_1 = \begin{bmatrix} N_2 & G' \\ O & G \end{bmatrix} \begin{bmatrix} r - m_s \\ \vdots \end{bmatrix} $$

(14)

Furthermore all entries of $G$ are algebraically independent of any other entry of $M$ if $M$ is generic in $\mathfrak{C}(J)$.

(vii) $N_1$ decomposes as:

$$ N_1 = \begin{bmatrix} G_1 & \ast & \ast & \ast & \ast \\ O & G_2 & \ast & \ast & \ast \\ O & O & G_3 & \ast & \ast \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & G_s \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_s \end{bmatrix} $$

(15)

Proof. (i) Suppose $j > i$. Let $x_{kl}^{(ij)}$ be the $(k,l)$-entry of the $(i,j)$-block of $\hat{M}$. Originally it is the $(i,j)$-entry of $(k,l)$-block of $M$. In Lemma 1 we showed that each block is “upper triangular.” Hence $x_{kl}^{(ij)} = 0$ for any $k,l$.

(ii) Left to the reader.

(iii) With $P$ as defined by (13), we have $\hat{M} = P^{-1}MP$. Hence the assertion follows.

(iv) This is not difficult to see. Details are left to the reader.

(v) That $\nu_1 = \nu_2$ means that each block of $M$ has at least two rows and two columns. Hence every block of $M$ has the $(2,2)$ entry and moreover $x_{11}^{(kl)} = x_{22}^{(kl)}$ for all $(k,l)$. This shows that $N_1 = N_2$.

(vi) That $\nu_1 > \nu_2$ means that $\nu_1 - \nu_2 = m_s$ and that $f_s = 1$. So there are $m_s^2$ blocks of size $1 \times 1$ in $M$. If we write $M$ with variable entries, then these blocks consist of algebraically independent entries.
independent elements by Remark 3. This implies that the entries of $G_s$ consist of algebraically independent elements and are algebraically independent of any other entries of $M$.

(vii) In view of (iv) the assertion follows inductively from (v) and (vi).

**Example 9** Let $T=T(3,2,2)$. Then a general element of $\mathfrak{C}(J)$ is:

\[
M = \begin{pmatrix}
  a & a' & a'' & b & b' & c & c' \\
  0 & a & a' & 0 & b & 0 & c \\
  0 & 0 & a & 0 & 0 & 0 & 0 \\
  0 & d & d' & e & e' & f & f' \\
  0 & 0 & d & 0 & e & 0 & f \\
  0 & g & g' & h & h' & i & i' \\
  0 & 0 & g & 0 & h & 0 & i \\
\end{pmatrix}
\] (16)

By a certain permutation it becomes $\hat{M}$ as follows:

\[
\hat{M} = \begin{pmatrix}
  a & b & c & a' & b' & c' & a'' \\
  0 & e & f & d' & e' & f' & d'' \\
  0 & h & i & g & h' & i' & g' \\
  0 & 0 & 0 & a & b & c & d' \\
  0 & 0 & 0 & e & f & d & d'' \\
  0 & 0 & 0 & h & i & g & g' \\
  0 & 0 & 0 & 0 & 0 & 0 & a \\
\end{pmatrix}
\] (17)

Note that the block decomposition of $\hat{M}$ gives us the dual partition $7 = 3 + 3 + 1$. The first diagonal block of $\hat{M}$ is the matrix $N_1 = \begin{pmatrix}
  a & b & c \\
  0 & e & f \\
  0 & h & i \\
\end{pmatrix}$. The second $N_2$ is identical with $N_1$. The third $N_3 = (a)$.

**Definition 10** With the notation made in Definition 7 we call the sequence of matrices $(N_1, N_2, \cdots, N_p)$ the **coarse diagonal blocks** of $\hat{M}$. We call the sequence $(G_1, \cdots, G_s)$ the diagonal blocks of $N_1$. We apply the term analogously to all $N_i$. Hence the diagonal blocks of $N_i$ are $(G_1, G_2, \cdots, G_i)$ with a certain $t$ depending on $i$. By the **fine diagonal blocks** of $\hat{M}$ we mean the totality of the diagonal blocks:

\[
(\text{diag}(N_1), \text{diag}(N_2), \cdots, \text{diag}(N_p))
\] (18)

**Proof of Theorem** Let $M \in \mathfrak{C}(J)$. Let $N_1, N_2, \cdots, N_p$ be the coarse diagonal blocks of $\hat{M}$, and $G_1, \cdots, G_s$ be the diagonal blocks of $N_1$. Define the map

\[
\Phi : \mathfrak{C}(J) \longrightarrow \mathbf{M}(m_1) \times \cdots \times \mathbf{M}(m_s)
\] (19)

by $\Phi(M) = (G_1, \cdots, G_s)$. By Proposition 8 (vi) and (vii) it is easy to see that $\Phi$ is surjective and also that the kernel of $\Phi$ is nilpotent. Since $\mathbf{M}(m_1) \times \cdots \times \mathbf{M}(m_s)$ is semisimple, the proof of Theorem 6 is complete.
Remark 11 The matrices $G_i$ defined above are precisely the same as $\tilde{A}_{\alpha\beta}$ with $\alpha = \beta = i$ in Basili [1, p.60]. Basili shows that $M$ is nilpotent if and only if each $\bar{A}_{ii}$ is nilpotent. This is particularly obvious after the proof of Theorem 6. Thus we recover Basili’s identification of the nilpotent commutator with the inverse image under $\Phi$ of the locus where the $A_{ii}$ or $G_i$ are nilpotent.

In the next proposition we would like to redefine $\Phi$ in a coordinate free manner. Let $V$ be a finite dimensional vector space over $K$. We use the same letter $J$ as before to denote a nilpotent element of $\text{End}(V)$. The notation $\mathcal{C}(J)$ is used in the same meaning as for the matrix. Namely $\mathcal{C}(J) = \{M \in \text{End}(V) | MJ = JM\}$. Note that the subspaces $\ker J^i$ and $\text{im} J^i$ of $V$ are $\mathcal{C}(J)$-modules for every integer $i$, and so are their sums and intersections. Let $p$ be the least integer such that $J^p = 0$. (To avoid the trivial case we assume $p > 1$.) We have a descending chain of subspaces:

$$V = \ker J^p + \text{im} J \supset \ker J^{p-1} + \text{im} J \supset \ker J^{p-2} + \text{im} J \supset \cdots \supset \ker J^0 + \text{im} J = \text{im} J$$

From among the sequence of successive quotients

$$(\ker J^{p-i} + \text{im} J)/(\ker J^{p-i-1} + \text{im} J)$$

for $i = 0, 1, \cdots, p - 1$, pick the non-zero vector spaces and rename them

$$U_1, U_2, \cdots, U_s$$

(20)

Note that $U_1 = V/(\ker J^{p-1} + \text{im} J)$. In fact everything may be regarded as a module over $K[J]$, which is a commutative local ring. So we may use Nakayama’s Lemma to see $V \neq (\ker J^{p-1} + \text{im} J)$.

Likewise consider the ascending chain of subspaces:

$$0 = \text{im} J^p \cap \ker J \subset \text{im} J^{p-1} \cap \ker J \subset \text{im} J^{p-2} \cap \ker J \subset \cdots \subset \text{im} J \cap \ker J \subset \text{im} J^0 \cap \ker J = \ker J$$

Let

$$W_1, W_2, \cdots, W_{s'}$$

be the subsequence of non-zero terms of the successive quotients

$$(\text{im} J^{p-i-1} \cap \ker J)/(\text{im} J^{p-i} \cap \ker J)$$

for $i = 0, 1, \cdots, p - 1$.

(We note $W_1 = \text{im} J^{p-1} \cap \ker J$.) The spaces $U_i$ and $W_i$ are $\mathcal{C}(J)$-modules. Hence the module structure induces the algebra homomorphisms $\phi_i : \mathcal{C}(J) \longrightarrow \text{End}(U_i)$ for $i = 1, 2, \cdots, s$. Define the algebra homomorphism

$$\phi : \mathcal{C}(J) \longrightarrow \text{End}(U_1) \times \text{End}(U_2) \times \cdots \times \text{End}(U_s)$$

(22)

by the concatenation $\phi = (\phi_1, \cdots, \phi_s)$ of all $\phi_i : \mathcal{C}(J) \longrightarrow \text{End}(U_i)$. Similarly define the algebra homomorphism

$$\phi' : \mathcal{C}(J) \longrightarrow \text{End}(W_1) \times \text{End}(W_2) \times \cdots \times \text{End}(W_{s'})$$

(23)

by the concatenation of $\phi_i' : \mathcal{C}(J) \longrightarrow \text{End}(W_i)$.

Now we have
Proposition 12  
(i)  $s = s'$

(ii) $\dim U_i = \dim W_i$ for $i = 1, 2, \ldots, s$

(iii) If we identify $\text{End}(U_i)$ and $\text{End}(W_i)$ with a full matrix ring using suitable bases, then the homomorphism $\Phi$ defined in (19) coincides with $\phi$ in (22) and with $\phi'$ in (23).

Proof. Let $n = \dim V$. Let $B$ be a basis of $V$ in which $J$ is put in the Jordan first canonical form. Once and for all we fix such a basis and we identify $\text{End}(V)$ and $M(n)$. Suppose that the matrix for $J$ decomposes into Jordan cells as denoted by the Young diagram $T = T(n_1, n_2, \ldots, n_r)$. We index the boxes of $T$ by the basis elements of $V$ themselves in such a way that it satisfies the following condition:

$e, e' \in B$ and $e' = Je \iff$ the box $e'$ is next to and on the right of $e$.

(cf. Equation (11) of Section 2.1.) Here the “box $e$” means that the box is indexed by $e$. Henceforth we use the words “box in $T$” and a “basis element” in $B$ interchangeably.

Let $(m_1, m_2, \ldots, m_r)$ be the multiplicity sequence of $T$ so we have, by (12) in Section 3,

$$(n_1, \ldots, n_r) = (f_1, \ldots, f_{1\cdot m_1}, f_2, \ldots, f_{2\cdot m_2}, \ldots, f_{r\cdot m_r}).$$

Let us call the set of boxes in $T$ a “rectangle” if it consists of all rows of the same length $f_i$ for some $i$. Let $T_i$ be the $i$th rectangle. (So $T_i$ consists of $m_i \times f_i$ boxes.) We may write $T = T_1 \cup T_2 \cup \cdots \cup T_{s'}$ as a disjoint union of Young subdiagrams aligned left. Let $e$ be a basis element of $V$ in the first column of $T$ and let $e'$ be a basis element at the end of a row of $T$.

We have that

- the box $e$ belongs to $T_j \iff \begin{cases} J^ke = 0 & \text{if } k = f_j, \\ J^ke \neq 0 & \text{if } k < f_j, \end{cases}$

and

- the box $e'$ belongs to $T_j \iff \begin{cases} e' \in \text{im} J^k & \text{if } k = f_j - 1, \\ e' \not\in \text{im} J^k & \text{if } k > f_j - 1. \end{cases}$

Thus the basis elements in the first column of $T_i$ span the space $U_i$ and those in the last column span $W_i$. This shows (i) and (ii). To prove (iii), we have to show that for $M \in \mathcal{C}(J)$, both $\phi(M)$ and $\phi'(M)$, for each $i$, are represented by the matrix $G_i$ defined in Proposition (S) with a suitable basis for $V$. Number the boxes of $T$ vertically. Then one sees that $\phi_i(M)$, for $M \in \mathcal{C}(J)$, corresponds to the $i$th diagonal block $G_i$ of $N_1$ as was shown in Proposition (S) (vii).

Also it is not difficult to see that $\phi(M) \in \text{End}(W_i)$ corresponds to the last fine diagonal block in $N_{f_{i+1} - 1}$, which is also $G_i$. This completes the proof.

Remark 13 Let $J \in \text{End}(V)$ be nilpotent and let $U_1, \ldots, U_s$ and $W_1, \ldots, W_s$ be the vector spaces defined above. Suppose that $T(n_1, \ldots, n_r)$ is the Jordan decomposition for $J$. Let $(f_1, f_2, \ldots, f_s)$ be the finest subsequence of $(n_1, \ldots, n_r)$ such that $f_1 > f_2 > \cdots > f_s > 0$. Then we have:

(i) $U_i = (\ker J^{f_i} + \text{im} J) / (\ker J^{f_i - 1} + \text{im} J)$ for $i = 1, 2, \ldots, s$. 

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(ii) \( W_i = (\ker f_i \cap \text{im} f_i) / (\ker f_i \cap \text{im} J_i) \) for \( i = 1, 2, \ldots, s \).

These could have been the definitions of the vector spaces \( U_i \) and \( W_i \). The statement of Proposition 12 \((iii)\) means that \( \phi_i \) and \( \phi'_i \) are equivalent in the sense that there are bijective linear maps \( \psi_i : U_i \to W_i \), which make the following diagrams commutative

\[
\begin{array}{ccc}
U_i & \phi_i(X) & U_i \\
\downarrow \psi_i & & \downarrow \psi_i \\
W_i & \phi'_i(X) & W_i
\end{array}
\]

for every \( X \in \mathcal{C}(J) \). This shows that \( U_i \) and \( W_i \) are isomorphic as modules over \( \mathcal{C}(J) \). If we take grading into account, we have the isomorphism

\[
U_i[1 - f_i] \cong W_i,
\]

since the degrees of the boxes of the first column of \( T_i \) are different from those in the last column of \( T_i \) by \( f_i - 1 \) row-wise. This isomorphism can be proved more directly as follows.

Consider the diagram. (We write \( 0 : J \) for \( \ker f_i \).)

\[
\begin{array}{ccccccc}
0 & \to & 0 : J & \to & 0 : J & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 : J^i & \to & 0 : J^{i+1} & \to & 0 : J \cap \text{im} J^i \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 : J^{i-1} & \to & 0 : J^i & \to & 0 : J \cap \text{im} J^{i-1} \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 : J^{i-1} / J(0 : J^i) & \to & 0 : J^i / J(0 : J^{i+1}) & \to & X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0
\end{array}
\]

The definition of the maps should be self-explanatory. Also note the isomorphisms

\[
\frac{0 : J^i}{J(0 : J^{i+1})} \cong \frac{0 : J^i + \text{im} J}{\text{im} J}
\]

and the same for \( i - 1 \) instead of \( i \). Then the last horizontal exact sequence shows that \( X \) is isomorphic to \( U_j \) if \( i = f_j \). But the last vertical exact sequence shows that \( X \cong W_j \). Thus we have \( U_j \cong W_j \) for all \( j \).

Remark 14 1. \( U_1, \ldots, U_s \) are simple \( \mathcal{C}(J) \)-modules of different isomorphism types, and these exhaust all isomorphism types of simple \( \mathcal{C}(J) \)-modules.
2. When $J$ is given as the multiplication map $xz \in \text{End}(A)$, $A$ an Artinian $K$-algebra, we term the modules $U_1, \cdots, U_s$ the “central simple modules” of the pair $(A, z)$, and study them further in [7] and [8]. We say “central simple” in the sense it has no proper submodules over the centralizer of $J$.

3. Let $A$ be an Artinian Gorenstein $K$-algebra, not necessarily graded, and let $z \in A$ be any non-unit element. Let $(f_1^{m_1}, \cdots, f_s^{m_s})$ be the partition for the Jordan decomposition of the nilpotent element $xz \in \text{End}(A)$. Then it is possible to define $U_1, \cdots, U_s, W_1, \cdots, W_s$ as in Remark [9]. Since $A$ is Artinian Gorenstein, $\text{Hom}_A(A, A)$ is an exact functor. Hence we have the isomorphism

$$\text{Hom}_A(U_i, A) \cong W_i, \text{ and } \text{Hom}_A(W_i, A) \cong U_i.$$ 

So

$$\text{Hom}_A(U_i, A) \cong U_i, \text{ and } \text{Hom}_A(W_i, A) \cong W_i.$$ 

This explains the symmetry of the Hilbert function of $U_i$ shown in [7, Proposition 4.6] and [8, Proposition 5.3].

**Proposition 15** Let $T$, $J$ and $\mathfrak{C}(J)$ be the same as above. Given $M \in \mathfrak{C}(J)$, it is possible to put the fine diagonal blocks of $\hat{M}$ in the Jordan canonical form by conjugation without affecting the shape of $\hat{J}$.

**Proof.** We have to find an invertible matrix $H$ such that $H^{-1}\hat{J}H = \hat{J}$ and the fine diagonal blocks of $(H^{-1}\hat{M}H)$ are Jordan first canonical forms. Let $N_1, \cdots, N_p$ be the diagonal blocks as in the proof of Theorem [5]. Let $G_1, \cdots, G_s$ be the diagonal blocks of $N_1$. Let $F_i \in \text{M}(m_i)$ be an invertible matrix such that $F_i^{-1}G_iF_i$ is the Jordan canonical form of $G_i$. Let $H_1 \in \text{M}(r)$ be the matrix which has $F_i$ as the $i$th diagonal block and $0$ off diagonal. Then $H_1$ puts the diagonal blocks of $N_1$ into the Jordan canonical forms. In the same way define $H_i$ for $N_i$ for each $1 \leq i \leq p$. Finally define $H \in \text{M}(n)$ so that it has $H_i$ as the $i$th diagonal block and $0$ off diagonal. One sees easily that $H$ does not change the shape of $\hat{J}$, so it has the desired property.

**Proposition 16** Let $T = (n_1, n_2, \cdots, n_r)$, $J$ and $\mathfrak{C}(J)$ be the same as before. Let $m_1, \cdots, m_s$ be the multiplicity sequence of $T$ so that we have

$$T(n_1, \cdots, n_r) = T(f_1, \cdots, f_1, f_2, \cdots, f_2, \cdots, f_s, \cdots, f_s)$$

where $f_1, \cdots, f_s$ is the descending subsequence of $n_1, \cdots, n_r$. Let $M \in \mathfrak{C}(J)$ and let $N_1$ be the first coarse diagonal block of $\hat{M}$. Suppose that the diagonal blocks of $N_1$ are $(G_1, \cdots, G_s)$ and moreover that all entries of $\hat{M}$ are $0$ except in the diagonals $N_1, \cdots, N_p$. Then we have:

$$\text{rank } (\hat{M} + \hat{J}) \geq \text{rank } G_1^{l_1} + \text{rank } G_2^{l_2} + \cdots + \text{rank } G_s^{l_s} + \text{rank } J.$$ 

Assume furthermore that all entries of $N_1$ are zero outside of the diagonal blocks $G_1, \cdots, G_s$. Then we have:

$$\text{rank } (\hat{M} + \hat{J}) = \text{rank } G_1^{l_1} + \text{rank } G_2^{l_2} + \cdots + \text{rank } G_s^{l_s} + \text{rank } J.$$
Proof. First we prove the second assertion. Assume \( s = 1 \). It means that \( T = T(n_1, \ldots, n_1) \).

In this case the matrix \( \hat{M} + \hat{J} \) has the form:

\[
\begin{array}{cccccc}
N_1 & E & O & \cdots & O & O \\
O & N_2 & E & \cdots & O & O \\
O & O & N_3 & \cdots & O & O \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & N_{p-1} & E \\
O & O & O & \cdots & O & N_p \\
\end{array}
\]

\[m \quad m \quad m \quad \cdots \quad m\]

(24)

Here \( G_1 = N_1 = N_2 = \cdots = N_p \), and \( p = n_1 = f_1 \) and \( m = r \). We are going to apply basic row and column operations to this matrix so that the rank can be computed. Use row operations by block so the matrix becomes:

\[
\begin{array}{cccccc}
G & E & O & \cdots & O & O \\
-G^2 & O & E & \cdots & O & O \\
G^3 & O & O & \cdots & O & O \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\pm G^{p-1} & O & O & \cdots & O & E \\
\pm G^p & O & O & \cdots & O & O \\
\end{array}
\]

\[m \quad m \quad m \quad \cdots \quad m\]

(25)

(We have put \( G = G_1 \).) Now use column operations to kill all the matrices in the blocks of the first column, except the block at the bottom. Thus the rank is equal to \( \text{rank} G^p + m(f_1 - 1) \) and the assertion of the proposition is proved for the case \( s = 1 \). Now assume \( s > 1 \). Write \( T \) as a disjoint union of rectangles, \( T = T_1 \sqcup T_2 \sqcup \cdots \sqcup T_s \). (It means that \( T_i = T(f_i, \ldots, f_i) \) for each \( i \).) We proceed to the general case by induction on \( s \). As before let \( N_1, N_2, \cdots, N_p \) be the coarse diagonal blocks of \( \hat{M} \) and let \( I_1, I_2, \cdots, I_{p-1} \) be the above diagonal blocks for \( \hat{J} \).

(These are described as matrices (3) and (4) in Section 2.2.) Recall that \( I_i \) is a matrix of size \( \nu_i \times \nu_{i+1} \). If \( \nu_i = \nu_{i+1} \), then \( I_i \) is the identity matrix of that size and if \( \nu_i > \nu_{i+1} \) then it is the identity of size \( \nu_{i+1} \) augmented by a zero block from below. Also recall that the diagonal blocks of \( N_1 \) are \( G_1, \cdots, G_s \) as shown in the figure (15). By Proposition 8 \( G_1 \) is contained in \( N_i \) for every \( i \) as the first diagonal block. Let us write \( m := m_1 \) for the size of \( G_1 \). Let \( E^{(i)} \) be the submatrix of \( I_i \) consisting of the first \( m \) rows and \( m \) columns. \( (E^{(i)} \) is nothing but the identity matrix of size \( m \).) Notice that the \( G_1 \) appears \( p \) times as a fine diagonal block of \( \hat{M} \) (one time in every \( N_i \)), and that except the first there is \( E^{(i)} \) somewhere above \( G_1 \) in the same column block. Now making row operations using \( E^{(i)} \) it is possible to kill all \( G_1 \) of the same column block. As in the case \( s = 1 \) we are left with consecutive powers of \( G_1 \) in the first
column block. Now make column operations, using various $E^{(i)}$ to annihilate everything on the same row block. Then as a result all $G_1$ disappear except the single $G^p$ at the bottom of the first column block. Moreover from every $N_i$ the first row block disappears. Now for the rows and columns that were not involved by the above procedure we may apply the induction hypothesis for $T(f_{m_2}, \cdots, f_{m_i}, \cdots, f_s)$. This completes the proof for the second statement.

The proof for the first inequality is proved similarly.

4 Application to the theory of Artinian $K$-algebras

Before we state our main theorem we introduce some definitions. By dimension we will mean dimension as a $K$-vector space.

**Definition 17** Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian $K$-algebra, where $A_0 = K$ is a field and $A_c \neq 0$.

- The algebra $A$ has the weak Lefschetz property (WLP) if there is a linear element $y \in A_1$ such that the multiplication $\times y : A_i \to A_{i+1}$ is either injective or surjective for all $i = 0, 1, \cdots, c - 1$. A weak Lefschetz element is a linear element $y$ with this property.

- The algebra $A$ has the strong Lefschetz property (SLP) if there is a linear element $y \in A_1$ such that the multiplication $\times y^{c-2i} : A_i \to A_{c-i}$ is bijective for all $i = 0, 1, \cdots, [c/2]$. We call a linear element $y$ with this property a strong Lefschetz element.

- An element $y \in A_1$ is a general linear form if there is a non-empty Zariski open set $U \subset A_1$, such that $y$ has the same Jordan canonical form as every element $y' \in U$. (A general linear form exists if $K$ is infinite.)

- The Sperner number of $A$ is $\text{Max}\{\dim A_i | i = 0, 1, 2 \cdots, c\}$.

- The CoSperner number of $A$ is $\sum_{i=0}^{c-1} \text{Min}\{\dim A_i, \dim A_{i+1}\}$.

**Remark 18** It is easy to see the following.

(i) For any $y \in A$, the rank of $\times y$ does not exceed the CoSperner number of $A$.

(ii) For any $y \in A$, the dimension of $A/yA$ is no less than the Sperner number of $A$.

(iii) A linear form $y \in A$ is a weak Lefschetz element for $A$ if and only if the rank of $\times y$ is equal to the CoSperner number of $A$.

(iv) If the Hilbert function of $A$ is unimodal, then we have

$$\text{Sperner}A + \text{CoSperner}A = \dim A.$$
(v) If \( A \) has the strong/weak Lefschetz property, then a general linear form is a strong/weak Lefschetz element.

(vi) Suppose that \( y \in A \) is a linear form and \( T = T(n_1, \cdots, n_r) \) is the partition for the nilpotent endomorphism \( xy \in \text{End}(A) \). Then \( y \) is a weak Lefschetz element if and only if \( r \), the number of the Jordan blocks of \( xy \), is equal to the Sperner number of \( A \). Also \( y \) is a strong Lefschetz element if and only if the dual partition \( \hat{T}(\nu_1, \cdots, \nu_p) \) of \( T \) is the one obtained from the unimodal Hilbert function of \( A \). (cf. [8] Lemma 3.7.)

We are going to apply Proposition 16 to Artinian \( K \)-algebras to evaluate the rank of a general linear form. First we review some basic facts on the associated form ring of an Artinian algebra with respect to the principal ideal generated by a linear form. This was motivated by the necessity to prove Proposition 20.

Let \( A \) be a graded Artinian \( K \)-algebra and let \( z \in A_1 \) be any linear form of \( A \). Put

\[
\text{Gr}(z)(A) = A/(z) \oplus (z)/(z^2) \oplus (z^2)/(z^3) \oplus \cdots \oplus (z^{p-1})/(z^p).
\]

Here \( p \) is the least integer such that \( z^p = 0 \). As is well known \( \text{Gr}(z)(A) \) is endowed with a commutative ring structure. The multiplication in \( \text{Gr}(z)(A) \) is given by

\[
(a + (z^{i+1}))(b + (z^{j+1})) = ab + (z^{i+j+1}),
\]

where \( a \in (z^i) \) and \( b \in (z^j) \). Note that \( \text{Gr}(A) \) inherits a grading from \( A \) and in this sense \( \text{Gr}(A) \) and \( A \) have the same Hilbert function. For a non-zero element \( a \in A \) there is \( i \) such that \( a \in (z^i) \setminus (z^{i+1}) \). In this case we write \( a^* \in \text{Gr}(A) \) for the natural image of \( a \) in \( (z^i)/(z^{i+1}) \).

Let \( \times : A \to \text{End}(A) \) be the regular representation of the algebra \( A \). (So \( \times a \) is the endomorphism of \( A \) defined by \( \times a(b) = ab \) for \( a, b \in A \).) Let \( z \) be a linear form of \( A \). Since \( A \) is Artinian, \( \times z \) is nilpotent. Let \( T = T(n_1, \cdots, n_r) \) be the Young diagram for the Jordan canonical form of \( \times z \).

One sees easily that the number \( r \) of parts of \( T \) is equal to \( \dim \ker[\times z : A \to A] = \dim A/(z) \), since each Jordan cell of \( \times z \) contributes 1 to the dimension of the kernel. The Young diagram \( T(n_1 - 1, n_2 - 1, \cdots, n_r - 1) \), with zero’s deleted, corresponds to the Jordan canonical form of the induced map \( \times \P \in \text{End}(A/(0 : z)) \). Thus, inductively, it follows that the dual of the partition \( T(n_1, \cdots, n_r) \) is \( T(\nu_1, \nu_2, \cdots, \nu_p) \), with the integers \( \nu_i = \dim(z^{i-1})/(z^i) \).

Let \( B \subset A \) be a \( K \)-basis of \( A \) in which \( \times z \) is written as a Jordan canonical form. We identify the boxes of \( T = T(n_1, \cdots, n_r) \) and the elements of \( B \). With this identification a row of \( T \) is a basis for a Jordan cell of \( \times z \).

Let \( B_i = B \cap ((z^{i-1}) \setminus (z^i)) \). It is easy to see that \( B_i \sqcup B_{i+1} \sqcup \cdots \sqcup B_p \) is a \( K \)-basis for the ideal \( (z^{i-1}) \). With the identification of \( B \) and the boxes of \( T \), the set \( B_i \) corresponds to the boxes in the \( i \)th column of \( T \).

Now let \( B^* \) be the natural image of \( B \) in \( \text{Gr}(A) \), i.e., \( B^* = \{ b^* \in \text{Gr}(A) \mid b \in B \} \). Similarly let \( B_i^* = \{ b^* \mid b \in B_i \} \). One sees immediately that \( B^* \) is a basis of \( \text{Gr}(A) \) in which the map \( \times z^* \)
is represented by a Jordan canonical form. It is also immediate to see that $B^*_i$ is a $K$-basis for \((z^i-1)/(z^i)\).

Now we would like to prove

**Proposition 19** We use the notation above.

(i) The linear maps $\times z \in \text{End}(A)$ and $\times z^* \in \text{End}(\text{Gr}(z)(A))$ have the same Jordan canonical form.

(ii) Let $y \in A$ be any element. Then $\times y \in \mathcal{E}(\times z)$ and $\times y^* \in \mathcal{E}(\times z^*)$.

(iii) Let $y \in A$ be any linear form. Let $P$ be the matrix for $\times y$ with the basis $B$ and similarly $Q$ the matrix for $\times y^*$ with the basis $B^*$. Then the coarse diagonal blocks of $\hat{P}$ and those of $\hat{Q}$ coincide.

(iv) The kernel of the multiplication map $z^* : \text{Gr}(z)(A) \rightarrow \text{Gr}(z)(A)$ is given by

$$\bigoplus_{\alpha=1}^{p} \left( (z^{i-1}) \cap (0 : z) + (z^i) \right)/(z^i)$$

*Proof.* (i) Consider the ideal of $\text{Gr}(A)$ generated by a power of $z^*$. First note that $(z^*)^\alpha = (z^\alpha)^*$. Now it is easy to see that $(z^*)^\alpha \text{Gr}(z)(A) \cong (z^\alpha)/(z^{\alpha+1}) \oplus \cdots$, which implies that $\text{rank}(\times z)^\alpha = \text{rank}(\times z^*)^\alpha$ for all $\alpha = 1, 2, 3, \ldots$. This shows that they have the same Jordan canonical form.

(ii) Trivial.

(iii) A coarse diagonal block of $\hat{P}$ is the matrix for the induced map $\times y : A \in \text{End}(\times z^\alpha)/(z^{\alpha+1})$ with the basis $B^*_\alpha$. Thus the assertion follows immediately.

(iv) Left to the reader.

Let $R = K[x_1, x_2, \cdots, x_d]$ be the polynomial ring and let $I \subset R$ be a homogeneous ideal such that $A = R/I$ is an Artinian $K$-algebra. Put $Z = x_d$. For any homogeneous element $f \in R$ it is possible to write uniquely

$$f = f_0 + f_1Z + f_2Z^2 + \cdots + f_kZ^k$$

where $f_i$ is a homogeneous polynomial in $K[x_1, \cdots, x_{n-1}]$. Let $i$ be the least integer such that $f_i \neq 0$. In this case we will write $\text{In}'(f) = f_iZ^i$. Furthermore we define $\text{In}'(I)$ to be the ideal of $R$ generated by the set $\{\text{In}'(f)\}$, where $f$ runs over homogeneous elements of $I$. It is well known that $R/\text{In}'(I) \cong \text{Gr}(z)(A)$. (Here we have set $z = Z \mod I$) Suppose that $d = 2$. Then one notices that $\text{In}'(I)$ coincides with the ideal generated by the initial terms of $I$ with respect to the reverse lexicographic order with $x_1 > x_2$. The same notation, $\text{In}'(I)$, and $\text{Gr}(z)(I)$ will be applied for a graded submodule $I$ of a finite colength in a free $R$-module.
Proposition 20 Let $K$ be an infinite field, and let $V$ be a finite vector space over $K$. Let $J \in \text{End}(V)$ be nilpotent. Choose a basis of $V$ so that we may identify $\text{End}(V) = M(n)$, where $n = \dim V$ and $J$ is put in the Jordan first canonical form. Let $\mathfrak{C}(J) \subset \text{End}(V)$ be the commutator algebra of $J$. Let $M \in \text{End}(V) = M(n)$ be nilpotent such that $M \in \mathfrak{C}(J)$. Let $N_1, \ldots, N_p$ be the coarse diagonal blocks of $\hat{M}$. Let $M^\dagger$ be the matrix such that $\hat{M}^\dagger = \text{diag}(N_1, \cdots, N_p)$. Then $M^\dagger \in \mathfrak{C}(J)$. Moreover we have
\[
\text{rank}(M^\dagger + \lambda J) \leq \text{rank}(M + \lambda J)
\]
for most $\lambda \in K$.

Proof. It is easy to see that $M^\dagger \in \mathfrak{C}(J)$ so we omit the proof. To prove the second assertion let $R = K[y, z]$ be the polynomial ring in two variables. Define an algebra homomorphism $R \rightarrow \text{End}(V)$ by $y \mapsto M$ and $z \mapsto J$. Then we may regard $V$ as an $R$ module with support in the maximal ideal $(y, z)$. (Note that $V$ is not necessarily graded.) Now $M$ is the matrix for the multiplication map $\times y : V \rightarrow V$ and $J$ for $\times z : V \rightarrow V$. Let
\[
\text{Gr}_{(z)}(V) = V/zV \oplus zV/z^2V \oplus \cdots \oplus z^{p-1}V/z^pV
\]
The module $\text{Gr}_{(z)}(V)$ has naturally the structure of $R$-module. One notices that $\times z \in \text{End}(\text{Gr}_{(z)}(V))$ has the same Jordan canonical form as $J$. Moreover the matrix for $\times y \in \text{End}(\text{Gr}_{(z)}(V))$ is $M^\dagger$. Let $g \in R$ be a general linear form. Now by [7, Proposition 3.3] we have
\[
\dim V/gV \leq \dim \text{Gr}_{(z)}(V)/g\text{Gr}_{(z)}(V)
\]
This proves the assertion as we may assume that $g = y + \lambda z$ for a sufficiently general $\lambda \in K$.

Theorem 21 Let $K$ be an infinite field and let $A = \bigoplus A_i$ be a graded Artinian $K$-algebra and let $z \in A$ be any linear form. Suppose that the Jordan decomposition of the nilpotent element
\[
\times z \in \text{End}(A)
\]
is given by
\[
T = T(n_1, \cdots, n_r) = T(f_1, \cdots, f_{m_1}, f_2, \cdots, f_{m_2}, \cdots, f_s, \cdots, f_{m_s}).
\]
Let $y \in A$ be a linear form linearly independent of $z$. Let $J, M \in \text{End}(A)$ be the matrices for $z, y$ with a basis of $A$ so that $J$ is in the Jordan second canonical form. Let $N_1, \cdots, N_r$ be the coarse diagonal blocks of $\hat{M} \in \mathfrak{C}(J)$ (as defined in Definition 10) and let $G_1, \cdots, G_s$ be the diagonal blocks of $N_1$. Then we have:

(i) $\text{rank } G_{1}^{f_1} + \text{rank } G_{2}^{f_2} + \cdots + \text{rank } G_{s}^{f_s} + \text{rank } J \leq \text{rank}(M + \lambda J)$ for most $\lambda \in K$. 

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(ii) The equality
\[ \text{rank} \ G^f_1 + \text{rank} \ G^f_2 + \cdots + \text{rank} \ G^f_s + \text{rank} \ J = \text{CoSperner}(A), \]
implies that \( y + \lambda z \) is a weak Lefschetz element of \( A \) for most \( \lambda \in K \).

Proof. (i) Put \( G = \text{Gr}(z)(A) \) and let \( z^*, y^* \in G \) be the initial forms of \( z, y \) respectively. Recall that \( \times z \in \text{End}(A) \) and \( \times z^* \in \text{End}(G) \) have the same Jordan canonical form. We may choose a basis \( B \subset A \) such that \( J := \times z \) is in Jordan canonical form as well as \( \times z^* \) with \( B^* \). Let \( M^\dagger \) be the matrix for \( \times y^* \) with \( B^* \). Since \( y \notin (z) \), the matrix \( \hat{M}^\dagger \in \mathcal{C}(\hat{J}) \) consists of only diagonal blocks by the way the multiplication is defined in \( G \). Moreover they are the same as those of \( \hat{M} \). Thus we have
\[ \text{rank}(M^\dagger + \lambda J) \leq \text{rank}(M + \lambda J) \]
for most of \( \lambda \in K \) by Proposition 20. Now the first inequality immediately follows from Proposition 16. (ii) This follows from Proposition 18 (iii).

The following Theorem was proved in [7, Theorem 1.2]. The proof is essentially the same as that of Theorem 21 (ii) above.

**Theorem 22** Let \( A \) be an Artinian Gorenstein \( K \)-algebra and \( z \in A \) a linear form. Let
\[ U_1, \ldots, U_s \]
be the central simple modules defined in (20) for the nilpotent endomorphism \( \times z \in \text{End}(A) \). Suppose that all \( U_i \) have the strong Lefschetz property as \( A \)-modules. Then \( A \) has the strong Lefschetz property.

**Remark 23** Let \( A \) be an Artinian \( K \)-algebra and \( z \in A \) a linear form. The central simple modules of \( (A, z) \) are defined to be the non-zero modules of the form \( (0 : z^f + (z))/0 : z^{f-1} + (z) \). They are modules over the algebra \( A/(z) \) and are determined by the Jordan canonical form of \( \times z \). Let \( G = \text{Gr}(z)(A) \) be the associated form ring. Then the endomorphisms \( \times z^* \in \text{End}(G) \) and \( \times z \in \text{End}(A) \) have the same Jordan canonical form. Thus the central simple modules of \( (G, z^*) \) can be regarded as the same modules over of \( A/(z) \) with the identification \( G/(z^*) = A/(z) \). Suppose that \( A \) is Gorenstein. Then, even though \( G \) may not be Gorenstein, the strong Lefschetz property of the central simple modules of \( (A, z) \) implies that \( G \) has the strong Lefschetz property. (See [8, Theorem 5.2].)

5 Examples

In the following examples we show how Theorem 21 can be used to compute the rank of a general linear form for \( A \) and to prove the weak Lefschetz property of \( A \). We proved in [5] that every Artinian complete intersection in codimension three over a field of characteristic zero has the WLP. The method to prove it in Examples 24 and 25 are different from the one used in [5].
Example 24 Assume that $K$ is a field of char $K \neq 2$. Let $R = K[x, y, z], I = (x^2 + y^2 + z^2, x^4 + y^4 + z^4, xyz)$, and $A = R/I$. (We use the same letter $z$ for the image of $z$ in $A$.) Then $x \cdot z \in \text{End}(A)$ is represented by the partition

$$24 = \underbrace{5 + 5 + 5 + 5}_4 + \underbrace{1 + 1 + 1 + 1}_4$$

The dual partition is

$$24 = 8 + 4 + 4 + 4 + 4$$

The Young diagram is as follows:

```
+---+---+---+---+---+---+
|   |   |   |   |   |   |
+---+---+---+---+---+---+
|   |   |   |   |   |   |
+---+---+---+---+---+---+
|   |   |   |   |   |   |
|   |   |   |   |   |   |
+---+---+---+---+---+---+
|   |   |   |   |   |   |
+---+---+---+---+---+---+
|   |   |   |   |   |   |
|   |   |   |   |   |   |
+---+---+---+---+---+---+
|   |   |   |   |   |   |
```

(26)

The rank of a general linear form of $A$ is $18$. $A$ has the strong Lefschetz property, but $z$ itself is not a strong Lefschetz element, since the rank of $x \cdot z$ is $14$.

Proof. Since char $K \neq 2$, we have that $A$ is Artinian. Consider the exact sequence

$$0 \rightarrow A/0 : z \rightarrow A \rightarrow A/(z) \rightarrow 0$$

The first column of the Young diagram corresponds to $A/(z)$, with the Hilbert function $1 + 2t + 2t^2 + 2t^3 + t^4$. So the dimension is $8$. Put $B = K[x, y, z]/(x^2 + y^2 + z^2, xy, x^4 + y^4 + z^4)$. Then it is easy to see that $zA \cong A/(0 : z) \cong K[x, y, z]/(x^2 + y^2 + z^2, xy, x^4 + y^4 + z^4)$. The ideal $zA$ corresponds to the diagram with the first column deleted. Now it is easy to compute $\dim B/(0 : z^i) = 4(4 - i)$, for $i = 0, 1, 2, 3$. Thus we have verified the partition for $x \cdot z$ is $T(8, 4, 4, 4, 4) = T(5, 5, 5, 1, 1, 1, 1, 1)$. Put $J = x \cdot z$. Then a general member of $\mathfrak{C}(J)$ has coarse diagonal blocks $N_1, \ldots, N_5$ whose sizes are $(8, 4, 4, 4, 4)$ respectively and $N_1$ has two diagonal blocks $G_1$ and $G_2$ of size $4$. Let $U_1$ and $U_2$ be the central simple modules as defined in Remark 15, and let $g$ be a general linear form of $A$. Let $G_1$ be a matrix for the induced map $x \cdot g \in \text{End}(U_1)$ and $G_2$ for $x \cdot g \in \text{End}(U_2)$. Notice that we have the exact sequence

$$0 \rightarrow U_2 \rightarrow A/(z) \rightarrow U_1 \rightarrow 0,$$

where the first map sends $1$ to $xy$. Thus we have

$$\begin{cases}
U_1 \cong K[x, y]/(x^2 + y^2, xy), \\
U_2 \cong K[x, y]/(x^2 + y^2, xy)[-2].
\end{cases}$$

In the notation of Proposition 16, $s = 2, f_1 = 5, f_2 = 1$ and $\text{rank} G_1^{f_1} + \text{rank} G_2^{f_2} + \text{rank} J = 0 + 2 + 16 = 18$. Since the CoSperner number of $A$ is $18$, this shows that $A$ has the weak Lefschetz property by Theorem 21. By direct computation or using [10] Theorem 2.9 or [5] Proposition 4.4, it follows that $U_1$ and $U_2$ have the SLP. Hence by Theorem 22, $A$ has the SLP.

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Example 25 Assume that $K$ is an infinite field of char $K \neq 2$. Let $R = K[x, y, z]$ and let $A = R/(x^4 + y^4 + z^4, xy^3 + x^2z^2, y^3z)$. Then, as is easily calculated, $z \in \text{End}(A)$ is represented by

$$T = T(7, 7, 7, 7, 7, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1) = \hat{T}(16, 12, 12, 6, 6, 6, 6).$$

Thus using the notation of Proposition 16, $f_1 = 7, f_2 = 3, f_3 = 1$. Put $J = xz$. Let $M$ be a general member of $\mathcal{C}(J)$. The first coarse diagonal block $N_1$ of $M$ is of size 16 and it consists of fine diagonal blocks $G_1, G_2, G_3$ of sizes 6, 6, 4 respectively. Let $U_1, U_2, U_3$ be the central simple modules defined in Remark 13. Then, as with the previous example, it is not difficult to see that $A$ is Artinian and that

$$U_1 \cong K[x, y]/(x^2, y^3), \quad U_2 \cong K[x, y]/(x^2, y^3)[-2], \quad U_3 \cong K[y]/(y^3)[-3].$$

Let $g \in A$ be a general linear form and let $G_1$ be the matrix for the induced map $g \in \text{End}(U_1)$. Thus one sees that $G_1$ and $G_2$ are the nilpotent matrix with Jordan decomposition $T(4, 2)$ and $G_3$ with $T(4)$. Thus

$$\text{rank } G_1 + \text{rank } G_2 + \text{rank } G_3 + \text{rank } J = 0 + 1 + 3 + 48 = 52.$$  

This is equal to the CoSperner number of $A$. By Theorem 21, this shows that $A$ has the weak Lefschetz property. As in the previous Example, $z$ is not a weak Lefschetz element, but by Theorem 22 $A$ has the strong Lefschetz property.

Example 26 Let $R = K[w, x, y, z]$ be the polynomial ring, and put

$$A = R/(w^2, wx, x^3, xy, y^3, yz, z^3).$$

(We use the same letters $w, x, \cdots$ to denote their images in $A$.) The Jordan decomposition of $J := xz \in \text{End}(A)$ is represented by the partition $T = T(3, 3, 3, 1, 1, 1, 1) = \hat{T}(8, 4, 4)$. For a general linear form $g \in A$, the matrix $xg \in \mathcal{C}(J)$ has three coarse diagonal blocks of sizes 8, 4, 4. The first block $N_1$ has two diagonal blocks of size four each. One sees that

$$U_1 = \text{A}/(0 : z^2 + (z)) \cong K[w, x, y, z]/(w^2, wx, x^3, y, z), \quad U_2 = ((0 : z) + (z))/(z) \cong K[w, x, y]/(w^2, x, y^3)[−1].$$

(To see this, notice that $(0 : z) + (z))/(z)$ is a principal ideal of $A/(z)$ generated by $y$.) Both $U_1$ and $U_2[1]$ have the Hilbert function $(1, 2, 1)$. Let $g \in A$ be a general linear form let $G_1$ be the matrix for the induced maps $xg \in \text{End}(U_1)$. Then rank$(G_1)^3 + \text{rank } G_2 + \text{rank } J = 0 + 2 + 8 = 10$, which is equal to the CoSperner number of $A$. Hence $A$ has the WLP. In fact $A$ has the SLP, but Theorem 22 does not apply since $A$ is not Gorenstein. However, [8, Theorem 5.2] does apply.

Alternatively we may let $w$ do the role of $z$. The Jordan decomposition for $xw$ is given by $T = T(2, 2, 2, 2, 1, 1, 1, 1, 1, 1) = \hat{T}(11, 5)$. We have

$$U_1 = \text{A}/((0 : w) + (w)) \cong K[w, x, y, z]/(w, x, y^3, yz, z^3), \quad U_2 = ((0 : w) + (w))/(w) \cong K[x, y, z]/(x^2, y, z^3)[−1].$$

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$U_1$ has the Hilbert function $(1, 2, 2)$, and $U_2 (0, 1, 2, 2, 1)$. (Notice that $(1, 2, 2, 0, 0) + (0, 1, 2, 2, 1) = (1, 3, 4, 2, 1)$ is the Hilbert function of $A/(z)$.) Let $g \in A$ be a general linear form and let $G_1$ and $G_2$ be matrices for $\times g \in \text{End}(U_1)$ and $\times g \in \text{End}(U_2)$ respectively. Then $\text{rank}(G_1)^2 + \text{rank}(G_2) + \text{rank}(J) = 1 + 4 + 5 = 10$. This also shows that $A$ has the WLP. Note that we can use neither Theorem 22 nor [8, Theorem 5.2] to prove $A$ has the SLP, since $A$ is not Gorenstein and since $U_1$ does not have a symmetric Hilbert function.

**Example 27** Let $K$ be a field of characteristic 0. Let $R = K[x_1, x_2, \cdots, x_n]$ be the polynomial ring with $n \geq 2$. Let

\[
I = (x_1^2, x_1x_2, x_2^3, x_2x_3, \cdots, x_{n-1}x_n, x_n^3)
\]

Then $R/I$ has the SLP.

To prove this we first consider a similar but simpler example as follows.

**Example 28** Let $K$ be a field of characteristic 0. Let $n > 1$ be an integer.

(i) $K[x_1, x_2, \cdots, x_n]/(x_1^2, x_1x_2, \cdots, x_{n-1}x_n, x_n^3)$ has the SLP.

(ii) $A = K[x_0, x_1, \cdots, x_n]/(x_0^2, x_1^2, x_2^3, x_2x_3, \cdots, x_{n-1}x_n, x_n^3)$ has the WLP for any positive integer $\alpha$.

By [6, Proposition 18], (i) follows from (ii). To prove (ii) we would like to use Theorem 21 so that the same proof works for Example 27 also. Put $z = x_n$. Note that $\dim A = 2^n\alpha$. Furthermore note that $\dim A/(z) = 2^{n-1}\alpha$ and $\dim(z)/(z^2) = \dim(z^2)/(z^3) = 2^{n-2}\alpha$. This shows that the Jordan canonical form for $\times z \in \text{End}(A)$ is given by the partition:

\[
T = T(3, 3, \cdots, 3, 1, 1, \cdots, 1) = \tilde{T}(2^{n-1}\alpha, 2^{n-2}\alpha, 2^{n-2}\alpha)
\]

The Young diagram is shown in the picture below:

```
  \vdots  \vdots  \vdots
  \vdots  \vdots  \vdots
  1    2
```

(27)

Now we see that the first coarse diagonal block $N_1$ of the matrix $\times g \in C(\widehat{\times z})$, where $g$ is a general linear form, is of size $2^{n-1}\alpha$ and it consists of two fine diagonal blocks of size $2^{n-2}\alpha$ each. Note that

\[
\begin{aligned}
U_1 &:= A/(0 : z^2) + (z) \cong K[x_0, x_1, \cdots, x_{n-2}]/(x_0^2, x_1^2, \cdots, x_{n-2}^2) \\
U_2 &:= (0 : z) + (z)/(z) \cong K[x_0, x_1, \cdots, x_{n-2}]/(x_0^2, x_1^2, \cdots, x_{n-2}^2)[-1]
\end{aligned}
\]
Since $U_1$ and $U_2$ have the SLP, if $g \in A$ is a general linear form, then the rank of $g^3 \in \text{End}(U_i)$ can be computed from the Hilbert series of $U_i$. Now let $G_i$ be a matrix for $g \in \text{End}(U_i)$. Then we have $\text{rank}(G_1)^3 = \dim U_1/(0 : g^3)$ and $\text{rank} G_2 = \dim U_2/0 : g$. Thus, using Lemma 29 below,

$$\text{rank}(G_1)^3 + \text{rank} G_2 + \text{rank}(xz) =$$

$$(\dim U_1 - s(n - 2) - s'(n - 2) - s''(n - 2)) + (\dim U_2 - s(n - 2)) + 2^{n-1} \alpha$$

$$= 2^n \alpha - s(n) = \text{CoSperner number of } A$$

This completes the proof. (The algebra $A$ in fact has the SLP. This can be proved using [8, Theorem 5.2].)

Verification of the following lemma is left to the reader.

**Lemma 29** Fix a positive integer $\alpha$. Let $A$ be as above. Define the polynomial $h_n(q)$ by

$$h_n(q) = (q^{\alpha-1} + q^{\alpha-2} + \cdots + q + 1)(q + 1)^n$$

Define the integers $s(n), s'(n), s''(n)$ to be the first three terms of the coefficients of the polynomial $h_n(q)$ put in the decreasing order. Then we have

1. $h_n(q)$ is the Hilbert series of $A$.
2. $h_{n-2}(q)$ is the Hilbert series of $U_1$ and $U_2[1]$.
3. $s(n)$ is the Sperner number of $A$ and $s(n - 1)$ is the Sperner number of $U_1$ and $U_2$.
4. $2^n \alpha - s(n)$ is the CoSperner number of $A$ and $2^{n-1} \alpha - s(n - 1)$ is the CoSperner number of $U_1$ and $U_2$.
5. $s(n) = s(n - 1) + s'(n - 1)$, for $n \geq 1$.
6. $s(n) = 2s(n - 2) + s'(n - 2) + s''(n - 2)$, for $n \geq 2$.

Now we prove Example 27. As in the previous example it suffices to prove the WLP for

$$A := K[x_0, x_1, \cdots, x_n]/(x_0^\alpha, x_1^2, x_1 x_2, x_2^3, x_2 x_3, \cdots, x_{n-1} x_n, x_n^3)$$

for any positive integer $\alpha$.

Put $A^{(n)} = A, B^{(n-1)} = A^{(n-2)}[z]/(z^2)$. Then we have the exact sequence:

$$0 \rightarrow (z) \rightarrow A^{(n)} \rightarrow A^{(n-1)} \rightarrow 0.$$

But

$$(z)[1] \cong A/(0 : z) \cong K[x_0, x_1, \cdots, x_n]/(x_0^\alpha, x_1^2, x_1 x_2, x_2^3, x_2 x_3, \cdots, x_{n-1} x_n, x_n^2) \cong B^{(n-1)}.$$

Note that $A^{(n)}$ and $B^{(n)}$ have Hilbert series

$$(1 + q + \cdots + a^{\alpha-1})(1 + q)^n.$$

We are going to induct on $n$ so we assume the SLP for $A^{(n-2)}$ and $B^{(n-2)}$. Put $z = x_n$. Then one sees easily that $xz$ has the same Jordan decomposition as the one treated in Example 28. Thus the same proof as Example 28 works verbatim in this case.

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