ON THE LINEAR COMPLEXITY OF GENERALIZED CYCLOTOMIC QUATERNARY SEQUENCES WITH LENGTH $2pq$

MINGLONG QI, SHENGWU XIONG, JINGLING YUAN, WENBI RAO, AND LUO ZHONG

Abstract. In this paper, the linear complexity over $\text{GF}(r)$ of generalized cyclotomic quaternary sequences with period $2pq$ is determined, where $r$ is an odd prime such that $r \geq 5$ and $r \notin \{p, q\}$. The minimal value of the linear complexity is equal to $\frac{4pq + p + q + 1}{2}$ which is greater than the half of the period $2pq$. According to the Berlekamp-Massey algorithm, these sequences are viewed as enough good for the use in cryptography. We show also that if the character of the extension field $\text{GF}(r^m)$, $r$, is chosen so that $(\frac{2}{r}) = (\frac{2}{r}) = \frac{1}{r}$, $r \mid 3pq - 1$, and $r \mid 2pq - 4$, then the linear complexity can reach the maximal value equal to the length of the sequences.

1. Introduction

Pseudo-random sequences play important roles in fields such as communication systems, simulation, and cryptography [1, 2]. In cryptography, sequences with good balance and high linear complexity are preferable [2, 3, 4]. According to the Berlekamp-Massey algorithm [5], if the linear complexity is greater than the half of the period, then the sequences are viewed as enough good for cryptographic uses. Because of their good algebraic structure [6, 8], cyclotomic sequences of different periods and orders find many applications in cryptography and communication [4, 7, 8, 9, 10].

A family of generalized cyclotomic quaternary sequences of length $2p$ was constructed in [11], of which the linear complexity and the autocorrelation were studied in [10] and in [12], respectively. In [13], Chang et al considered the linear complexity of quaternary cyclotomic sequences with period $2pq$. Using the technique of Fourier spectrums of sequences, the authors of [14] determined the linear complexity of generalized cyclotomic sequences with period $pq$. The linear complexity of Whiteman’s generalized cyclotomic binary sequences with the period $p^{m+1}q^{n+1}$ was computed out in [20]. Very recently in [21, June 2015], Li et al studied the linear complexity of generalized cyclotomic binary sequences with the length $2p^{m+1}q^{n+1}$. Because of easy and efficient hardware implementation, researches on linear complexity and autocorrelation of generalized cyclotomic binary and quaternary sequences are intensive [7, 8, 9, 13, 14, 17].

In this paper, we consider the linear complexity over $GF(r)$ of the generalized cyclotomic quaternary sequences with period $2pq$ constructed in [13]. The rest of the paper is organized as follows: in Section 2, basic definitions, notations and

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related lemmas of previous works needed to prove the main results are given. In Section 3, we give main theorems of this paper and their proofs. In Section 4, we give a brief conclusion and remarks.

2. Preliminaries

Let \( p, q \) be two distinct odd primes, \( r \) be an odd prime such that \( r \geq 5 \) and \( r \neq p, q \), and \( m \) be the order of \( r \) modulo \( pq \). Then, by the Chinese Remainder Theorem, there is a \( 2pq^{th} \) root of unity in the extension field \( \mathbf{GF}(r^m) \). Throughout the paper, we keep the meanings of \( r, m, \) and \( \mathbf{GF}(r^m) \), unchanged. From here and hereafter, the meanings of the following symbols are kept unchanged: \( p, q \) are not only two distinct odd primes, but also \( \gcd(p - 1, q - 1) = 2 \), which implicates that, the generalized cyclotomic quaternary sequences with the period \( 2pq \) are the order of \( r \). Let \( T = pq \), and \( \beta \) be a \( 2pq^{th} \) root of unity in \( \mathbf{GF}(r^m) \). We use \( \beta_T, \beta_p, \beta_q \), and \( \beta_2 \), to denote a \( pq^{th}, \) \( p^{th}, \) a \( q^{th} \) and a second root of unity in \( \mathbf{GF}(r^m) \), respectively. We define \( M_p = q \cdot (q^{-1} \mod p) \mod pq), \quad M_2 = u \cdot (u^{-1} \mod 2) \mod 2u \) where \( u \in \{p, q, T\}, \quad M_p = p \cdot (p^{-1} \mod q) \mod pq), \quad 2_p^{-1} = 2^{-1} \mod p), \quad 2_q^{-1} = 2^{-1} \mod q), \quad 2_T^{-1} = 2^{-1} \mod T). \) It is clear that, \( M_2 \equiv u \mod 2u \), with \( u \in \{p, q, T\}. \) In addition, let \( g \) be a common odd primitive root of \( p \) and \( q \), and \( x \) be an integer which is computed by the following congruent system:

\[
(2.1) \quad \begin{cases} 
  x \equiv g \quad \pmod{2p}, \\
  x \equiv 1 \quad \pmod{2q}.
\end{cases}
\]

If \( sq + tp = 1 \), then by the Generalized Chinese Remainder Theorem,

\[
(2.2) \quad x \equiv g \cdot s \cdot q + t \cdot p \quad \pmod{2pq}.
\]

We define the following sets:

\[
(2.3) \quad \begin{cases} 
  D_0^{(2T)} =< g > \quad \pmod{2T}, \\
  D_1^{(2T)} =< g > x \quad \pmod{2T}, \\
  D_0^{(T)} =< g > \quad \pmod{T}, \\
  D_1^{(T)} =< g > x \quad \pmod{T}, \\
  D_0^{(2p)} =< g^2 > \quad \pmod{2p}, \\
  D_1^{(2p)} =< g^2 > g \quad \pmod{2p}, \\
  D_0^{(2q)} =< g^2 > \quad \pmod{2q}, \\
  D_1^{(2q)} =< g^2 > g \quad \pmod{2q}, \\
  D_0^{(p)} =< g^2 > \quad \pmod{p}, \\
  D_1^{(p)} =< g^2 > g \quad \pmod{p}, \\
  D_0^{(q)} =< g^2 > \quad \pmod{q}, \\
  D_1^{(q)} =< g^2 > g \quad \pmod{q}.
\end{cases}
\]
From (2.3), the following partitions are straightforward:

\[
\begin{align*}
Z_{2T}^* &= D_0^{(2T)} \cup D_1^{(2T)}, \\
Z_T^* &= D_0^{(T)} \cup D_1^{(T)}, \\
Z_{2p}^* &= D_0^{(2p)} \cup D_1^{(2p)}, \\
Z_q^* &= D_0^{(2q)} \cup D_1^{(2q)}, \\
Z_{2T}^* &= D_0^{(p)} \cup D_1^{(p)}, \\
Z_q^* &= D_0^{(q)} \cup D_1^{(q)}, \\
Z_{2T}^* &= D_0^{(2T)} \cup 2Z_T^* \cup qZ_{2p}^* \cup pZ_{2q}^* \cup 2qZ_q^* \cup 2pZ_q^* \cup \{pq\} \cup \{0\}.
\end{align*}
\]  

(2.4)

Define four parameters as follows:

\[
\begin{align*}
A_0 &= \sum_{i \in D_0^{(p)}} \beta_i^p, & A_1 &= \sum_{i \in D_1^{(p)}} \beta_i^p, \\
B_0 &= \sum_{i \in D_0^{(q)}} \beta_i^q, & B_1 &= \sum_{i \in D_1^{(q)}} \beta_i^q.
\end{align*}
\]  

(2.5)

Below is the definition of the generalized cyclotomic quaternary sequences of length $2T$ constructed by Chang et al in [13], \{s(t) : t \mod 2T = 0, 1, \ldots , 2T - 1\},

\[
s(t) = \begin{cases}
0 & \text{if } t \in \{0\}, \\
e & \text{if } t \in \{T\}, \\
0 & \text{if } t \in D_0^{(2T)} \cup qD_0^{(2p)} \cup pD_0^{(2q)}, \\
a & \text{if } t \in D_1^{(2T)} \cup qD_1^{(2p)} \cup pD_1^{(2q)}, \\
b & \text{if } t \in 2D_0^{(T)} \cup 2qD_0^{(p)} \cup 2pD_0^{(q)}, \\
c & \text{if } t \in 2D_1^{(T)} \cup 2qD_1^{(p)} \cup 2pD_1^{(q)}, \\
d & \text{if } t \in 2D_0^{(2T)} \cup 2qD_0^{(2p)} \cup 2pD_0^{(2q)}.
\end{cases}
\]  

(2.6)

In (2.6), $a, b, c$ and $d$ belong to $\mathbb{F}_4$, but $e$ belongs to $\mathbb{F}_2^*$. Let $S = \{0, 1, 2, 3\} \subset \text{GF}(r)$. Then, there exists always a mapping from $\mathbb{F}_4$ to $S$, $\psi$, such that $\psi(0) = 0, \psi(1) = \psi(2) = 2$ and $\psi(3) = 3$. We thus obtain a particular instance of (2.6):

\[
s(t) = \begin{cases}
0 & \text{if } t \in \{0\}, \\
2 & \text{if } t \in \{T\}, \\
0 & \text{if } t \in D_0^{(2T)} \cup qD_0^{(2p)} \cup pD_0^{(2q)}, \\
1 & \text{if } t \in D_1^{(2T)} \cup qD_1^{(2p)} \cup pD_1^{(2q)}, \\
2 & \text{if } t \in 2D_0^{(T)} \cup 2qD_0^{(p)} \cup 2pD_0^{(q)}, \\
3 & \text{if } t \in 2D_1^{(T)} \cup 2qD_1^{(p)} \cup 2pD_1^{(q)}.
\end{cases}
\]  

(2.7)

In this paper, we study the linear complexity of the sequences defined in (2.7) over $\text{GF}(r^m)$.

Let \{s(t) : t = 0, 1, \ldots , N - 1\} be a sequence over $\text{GF}(r^m)$ with period $N$. Then, the linear complexity of $s(t)$ over $\text{GF}(r^m)$, denoted by $\text{LC}(s)$, is the least integer $L$ which satisfies next recurrent relation:

\[
s(t + L) = c_{L-1}s(t + L - 1) + \cdots + c_1s(t + 1) + c_0s(t)
\]

for $t \geq 0$, where $c_0, c_1, \ldots , c_{L-1} \in \text{GF}(r^m)$. The minimal polynomial and the generating polynomial over $\text{GF}(r^m)[x]$ related to $s(t)$ are given below respectively

\[
M_s(x) = x^L - \sum_{i=0}^{L-1} c_i x^i,
\]

(2.8)

\[
G_s(x) = \sum_{t=0}^{N-1} s(t)x^t.
\]
The following equation relates both the minimal polynomial and the generating polynomial of the sequence \( s(t) \):

\[
M_s(x) = \frac{x^N - 1}{\gcd(x^N - 1, G_s(x))}.
\]

From (2.9), the linear complexity can be calculated by

\[
LC(s) = \deg(M_s(x)) = N - \deg(\gcd(x^N - 1, G_s(x))).
\]

In this paper, we focus on how to compute the degree of \( \gcd \) in (2.10) over the extension field \( \text{GF}(r^m) \), for the sequence defined in (2.7). We will compute out the number of zeros of the generating polynomial \( G_s(x) \) under the form \( \beta^k \) for \( k \in \mathbb{Z}_T \), from which the degree of \( \gcd(x^{2T} - 1, G_s(x)) \) can be deduced.

We write down a series of lemmas of previous works related to and necessary for proof of the main theorems.

**Lemma 2.1.** Let \( A_0, A_1, B_0, \) and \( B_1 \) be the parameters defined in (2.5).

1. If \( p \equiv 1 \pmod{4} \), and \( q \equiv 1 \pmod{4} \), then \( A_0(1 + A_0) \equiv \frac{p-1}{4} \pmod{r} \), and \( B_0(1 + B_0) \equiv \frac{q-1}{4} \pmod{r} \).
2. If \( p \equiv 3 \pmod{4} \), and \( q \equiv 3 \pmod{4} \), then \( A_1(1 + A_1) \equiv -\frac{p+1}{4} \pmod{r} \), and \( B_1(1 + B_1) \equiv -\frac{q+1}{4} \pmod{r} \).

**Proof.** Since \( r \) is an odd prime such that \( r \geq 5 \) and \( r \neq p,q \), and \( m \) is the order of \( r \) modulo \( T \), hence \( r^m \equiv 1 \pmod{pq} \). In other words, \( \gcd(r^m,p) = 1 \) and \( \gcd(r^m,q) = 1 \). In addition, \( \beta_p \) and \( \beta_q \) are the \( p^\text{th} \) root of unity and the \( q^\text{th} \) root of unity over the extension field \( \text{GF}(r^m) \), respectively. From (2.5), \( A_0 = \sum_{i \in D_0^{(p)}} \beta_p^i \), and \( B_0 = \sum_{i \in D_0^{(q)}} \beta_q^i \). Lemma 2.1 (1) follows from (13) in [4]. The proof of Lemma 2.1 (2) is similar to that of Lemma 2.1 (1), and mentioned at first time in [18].

**Lemma 2.2.** \( [10, 13] \)

1. if \( a \in D_i^{(u)} \), then \( aD_j^{(u)} \equiv D_{i+j}^{(u)} \pmod{2} \).
2. \( D_2^{(2u)} \equiv D_1^{(u)} \pmod{2} \).

Where \( u \in \{p,q,T\} \).

**Lemma 2.3.** \( [19] \)

1. \( 2 \in D_0^{(u)} \) if only if \( u \equiv \pm 1 \pmod{8} \), and \( 2 \in D_1^{(u)} \) if only if \( u \equiv \pm 3 \pmod{8} \), where \( u \in \{p,q\} \).
2. (Quadratic Reciprocity).

\[
\left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \left( \frac{p}{q} \right).
\]

**Lemma 2.4.** \( [14] \) \( A_0 \) and \( A_1 \in \text{GF}(r) \) if only if \( r \in D_0^{(p)} \).
3. Computing the Linear Complexity According to Various Cases

For easy expression of some results, we give the definition of the sets which classify the pairs of \((p, q)\). Let \(s = p \mod 8\) and \(t = q \mod 8\), then
\[
(\pm 1, \pm 1) = \{(p, q) : s = \pm 1 \text{ and } t = \pm 1\}, \\
(\pm 3, \pm 3) = \{(p, q) : s = \pm 3 \text{ and } t = \pm 3\}, \\
(\pm 1, \pm 3) = \{(p, q) : s = \pm 1 \text{ and } t = \pm 3\}, \\
(\pm 3, 1) = \{(p, q) : s = \pm 3 \text{ and } t = \pm 1\}.
\]
As we require that \(\gcd(p - 1, q - 1) = 2\), it is clear that \(p, q \notin (1, 1) \cup (-3, -3) \cup (1, -3) \cup (-3, 1)\).

Let \(\left(\frac{r}{p}\right)\) denote the Legendre symbol of \(r\) modulo \(p\). Next, we give a series of lemmas necessary to prove the main theorems.

**Lemma 3.1.** Define a mapping \(f\), such that for \(k \in D_0^{(pq)} \cup D_1^{(pq)}\), \(k \mapsto f(k) = (k \mod p, k \mod q)\). Then, the following mappings deduced from the mapping \(f\) are bijective.

\[
(1) \quad f : D_0^{(pq)} \to D_0^{(p)} \times D_0^{(q)} \cup D_1^{(p)} \times D_1^{(q)}, \\
(2) \quad f : D_1^{(pq)} \to D_1^{(p)} \times D_1^{(q)} \cup D_1^{(p)} \times D_1^{(q)}.
\]

**Proof.** We only prove Lemma 3.1 (1) since the proof for Lemma 3.1 (2) is similar. At first, we prove the range of \(f\) is such that \(\textrm{Ran}(f) \subseteq D_0^{(p)} \times D_0^{(q)} \cup D_1^{(p)} \times D_1^{(q)}\). Let \(k_1 = k \mod p\), and \(k_2 = k \mod q\). Without loss of generality, suppose \((k_1, k_2) \in D_0^{(p)} \times D_1^{(q)}\), and we go to prove a contradiction. Since \(k \in D_0^{(pq)}\), there is a \(n\) such that \(k \equiv g^n \mod pq\), where \(0 \leq n < \frac{(p - 1)(q - 1)}{2}\). There must be a pair of integers \(i\) and \(j\), with \(0 \leq i < \frac{(p - 1)}{2}\) and \(0 \leq j < \frac{(q - 1)}{2}\), such that \(k_1 \equiv g^{2i} \pmod p\), and \(k_2 \equiv g^{2j + 1} \pmod q\). From above discussion, we have the following congruent system:

\[
(3.1) \quad \begin{cases} 
g^n \equiv g^{2i} \pmod p, \\
g^n \equiv g^{2j + 1} \pmod q.
\end{cases}
\]

From (3.1), it follows

\[
(3.2) \quad \begin{cases} 
n \equiv 2i \pmod {p - 1}, \\
n \equiv 2j + 1 \pmod {q - 1}.
\end{cases}
\]

Remark that (3.2) implies that \(\gcd(p - 1, q - 1) = 1\), which is contradictory to the predefined condition \(\gcd(p - 1, q - 1) = 2\).

By the Chinese Remainder Theorem, it is obvious that the mapping \(f\) is injective. We go to prove that \(f\) is surjective as well. Recall the meaning of the symbols used in the above proof. Let \(s(q - 1) + t(p - 1) = 2\), and \((k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}\). Hence, \(k_1 \equiv g^{2i} \pmod p\), and \(k_2 \equiv g^{2j} \pmod q\). Resolve the following congruent system:

\[
(3.3) \quad \begin{cases} 
n \equiv 2i \pmod {p - 1}, \\
n \equiv 2j \pmod {q - 1}.
\end{cases}
\]
By the Generalized Chinese Remainder Theorem, the solution of (3.3) is equal to
\begin{equation}
(3.4) \quad n \equiv s(q-1)i + t(p-1)j \pmod{\frac{(p-1)(q-1)}{2}}.
\end{equation}

It is clear that \(g^n \pmod{pq} \in D_0^{(pq)}\).

**Lemma 3.2.** Let \(f\) be a mapping, such that for \(k \in D_0^{(2pq)} \cup D_1^{(2pq)}\), \(k \mapsto f(k) = (k \pmod{2}, k \pmod{pq}) = (1, k \pmod{pq})\). Then, the following mappings related to \(f\) are bijective.

(1)
\[f : D_0^{(2pq)} \to Z_2^* \times D_0^{(pq)} = \{1\} \times D_0^{(pq)},\]

(2)
\[f : D_1^{(2pq)} \to Z_2^* \times D_1^{(pq)} = \{1\} \times D_1^{(pq)}.
\]

**Proof.** The proof is simple and similar to that of Lemma 3.1. \(\square\)

**Lemma 3.3.** Let \(f\) be a mapping, such that for \(k \in D_0^{(2p)} \cup D_1^{(2p)}\), \(k \mapsto f(k) = (k \pmod{2}, k \pmod{p}) = (1, k \pmod{p})\). Then, the following mappings related to \(f\) are bijective.

(1)
\[f : D_0^{(2p)} \to Z_2^* \times D_0^{(p)} = \{1\} \times D_0^{(p)},\]

(2)
\[f : D_1^{(2p)} \to Z_2^* \times D_1^{(p)} = \{1\} \times D_1^{(p)}.
\]

**Proof.** It is obvious. \(\square\)

**Lemma 3.4.**
\begin{equation}
(3.5) \quad 2_T^{-1} \in \begin{cases}
D_0^{(T)}(2,2) \in D_0^{(p)} \times D_0^{(q)} \cup D_0^{(p)} \times D_1^{(q)}, \\
D_1^{(T)}(2,2) \in D_1^{(p)} \times D_1^{(q)} \cup D_1^{(p)} \times D_0^{(q)}.
\end{cases}
\end{equation}

Recall that \(T = pq\), and \(2_T^{-1} = 2^{-1} \pmod{T}\).

**Proof.** It is clear that \(2_T^{-1} = 2_p^{-1} \pmod{p}\), and \(2_T^{-1} = 2_q^{-1} \pmod{q}\). The conclusion follows from Lemma 2.3 and Lemma 3.1. \(\square\)

We define six integer functions in order to simplify notation.
\begin{equation}
(3.6) \quad \begin{cases}
A_0(k) = \sum_{i \in D_0^{(p)}} \beta^k_i, \quad A_1(k) = \sum_{i \in D_1^{(p)}} \beta^k_i, \\
B_0(k) = \sum_{i \in D_0^{(q)}} \beta^k_i, \quad B_1(k) = \sum_{i \in D_1^{(q)}} \beta^k_i, \\
Z^{(p)}(k) = A_0(k) + A_1(k), \\
Z^{(q)}(k) = B_0(k) + B_1(k).
\end{cases}
\end{equation}

Where \(k \in Z_2T\).

**Lemma 3.5.**
\begin{equation}
(1) \quad \sum_{n \in D_0^{(2pq)}} \beta^{kn} = \beta_2^k (A_0(2_p^{-1}q_p^{-1}k)B_0(2_q^{-1}p_q^{-1}k)+A_1(2_p^{-1}q_p^{-1}k)B_1(2_q^{-1}p_q^{-1}k)).
\end{equation}
(2) \[
\sum_{n \in D_0^{1(2pq)}} \beta^{kn} = \beta_2^k \left( A_0 (2^{-1}_p q^{-1}_p k) B_1 (2^{-1}_q p^{-1}_q k) + A_1 (2^{-1}_p q^{-1}_p k) B_0 (2^{-1}_q p^{-1}_q k) \right).
\]

Where \( k \in \mathbb{Z}_{2T} \).

**Proof.** We give only the proof for case (1). By Lemma 3.2 and Lemma 3.1 we have \( n = pq + 2 \cdot 2^{-1}_T (M_p k_1 + M_q k_2) \mod 2T \), with \( n \in D_0^{1(2pq)} \), \( k_1 = n \mod p \), \( k_2 = n \mod q \), and \((k_1, k_2) \in D_0^{(p)} \times D_0^{(q)} \cup D_1^{(p)} \times D_1^{(q)} \).

\[
\sum_{n \in D_0^{1(2pq)}} \beta^{kn} = \beta_2^k \left( \sum_{(k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}} \beta_{T}^{k_1} (2^{-1}_p q^{-1}_q k_1) \sum_{k_2 \in D_0^{(q)}} \beta_{T}^{k_2} (M_p k_1 + M_q k_2) \right) + \sum_{(k_1, k_2) \in D_1^{(p)} \times D_1^{(q)}} \beta_{T}^{k_1} (2^{-1}_p q^{-1}_q k_1) \sum_{k_2 \in D_1^{(q)}} \beta_{T}^{k_2} (M_p k_1 + M_q k_2).
\]

**Lemma 3.6.**

(1) \[
\sum_{n \in qD_0^{(p)}} \beta^{kn} = \beta_2^k A_0 (2^{-1}_p k),
\]
\[
\sum_{n \in qD_1^{(p)}} \beta^{kn} = \beta_2^k A_1 (2^{-1}_p k).
\]

(2) \[
\sum_{n \in pD_0^{(q)}} \beta^{kn} = \beta_2^k B_0 (2^{-1}_q k),
\]
\[
\sum_{n \in pD_1^{(q)}} \beta^{kn} = \beta_2^k B_1 (2^{-1}_q k).
\]

Where \( k \in \mathbb{Z}_{2T} \).

**Proof.** Using Lemma 3.3 both for \( p \) and \( q \), and the same proof method as that for Lemma 3.5 we can obtain the conclusion of the actual lemma. \( \square \)

**Lemma 3.7.**

(1) If \( \left( \frac{a}{p} \right) = \left( \frac{a}{q} \right) \), then

\[
\sum_{i \in D_0^{(T)}} \beta^{ki} = A_0(k) B_0(k) + A_1(k) B_1(k),
\]
\[
\sum_{i \in D_1^{(T)}} \beta^{ki} = A_0(k) B_1(k) + A_1(k) B_0(k).
\]

(2) If \( \left( \frac{a}{p} \right) = -\left( \frac{a}{q} \right) \), then

\[
\sum_{i \in D_0^{(T)}} \beta^{ki} = A_0(k) B_1(k) + A_1(k) B_0(k),
\]
\[
\sum_{i \in D_1^{(T)}} \beta^{ki} = A_0(k) B_0(k) + A_1(k) B_1(k).
\]
Proof. We only prove the expansion formula for $\sum_{i \in D_1^{(T)}} \beta_i^k$ in case (1). Let $i \equiv M_p m + M_q n \pmod{pq}$ where $m \equiv i \pmod{p}$, $n \equiv i \pmod{q}$, and $i \in D_1^{(T)}$. We distinguish two cases $(\frac{i}{q}) = (\frac{i}{p}) = 1$ and $(\frac{i}{q}) = (\frac{i}{p}) = -1$. It is clear that $q_p^{-1} = q_f^{-1}$ (mod $p$) $\in D_0^{(p)}$ if $(\frac{i}{p}) = 1$, and $q_p^{-1} \in D_1^{(p)}$ if $(\frac{i}{p}) = -1$, according to the Legendre Symbol definition. Same situation for $p_q^{-1}$. We use Lemma 3.1 to decompose the summation over $D_1^{(T)}$ into the summations over $D_0^{(p)} \times D_1^{(q)} \cup D_1^{(p)} \times D_0^{(q)}$.

\begin{equation}
(1) \quad (\frac{\bar{i}}{p}) = (\frac{\bar{i}}{q}) = 1.
\end{equation}

\[\sum_{i \in D_1^{(T)}} \beta_i^k = \sum_{m \in D_0^{(p)}} \beta_m^k \cdot \sum_{n \in D_1^{(q)}} \beta_n^k + \sum_{m \in D_1^{(p)}} \beta_m^k \cdot \sum_{n \in D_0^{(q)}} \beta_n^k \]

\[= \sum_{m \in D_0^{(p)}} \sum_{n \in D_1^{(q)}} (\beta_m^k \cdot \beta_n^k) + \sum_{m \in D_1^{(p)}} \sum_{n \in D_0^{(q)}} (\beta_m^k \cdot \beta_n^k) \]

Note that $\beta_p^k = \beta_p$ and $\beta_q^k = \beta_q$. Using Lemma 2.2 we eliminate the exponents $p_q^{-1}$ and $q_p^{-1}$ from above formula:

\[\sum_{i \in D_1^{(T)}} \beta_i^k = \sum_{m \in D_0^{(p)}} (\beta_m^k + \sum_{n \in D_1^{(q)}} (\beta_n^k) = \sum_{m \in D_1^{(p)}} (\beta_m^k + \sum_{n \in D_0^{(q)}} (\beta_n^k) \]

\[= \sum_{m \in D_0^{(p)}} (\beta_m^k + \sum_{n \in D_1^{(q)}} (\beta_n^k) + \sum_{m \in D_1^{(p)}} (\beta_m^k + \sum_{n \in D_0^{(q)}} (\beta_n^k) \]

\[= A_0(k)B_1(k) + A_1(k)B_0(k). \]

\begin{equation}
(2) \quad (\frac{\bar{i}}{q}) = (\frac{\bar{i}}{p}) = -1. \text{ Note that in this case, } q_p^{-1} \in D_1^{(p)} \text{ and } p_q^{-1} \in D_1^{(q)}. \text{ The rest of the proof is same as for } (\frac{\bar{i}}{q}) = (\frac{\bar{i}}{p}) = 1.
\end{equation}

By the definition of generating polynomial in (2.8), we can write down the one of the sequence defined in (2.7):

\[G_s(t) = 2t^T + \left( \sum_{i \in D_1^{(2T)}} + \sum_{i \in D_1^{(2q)}} + \sum_{i \in D_1^{(2p)}} \right) t^i + \]

\[2 \left( \sum_{i \in D_0^{(2T)}} + \sum_{i \in D_0^{(2q)}} + \sum_{i \in D_0^{(2p)}} \right) t^i + \]

\[3 \left( \sum_{i \in D_1^{(2T)}} + \sum_{i \in D_1^{(2q)}} + \sum_{i \in D_1^{(2p)}} \right) t^i. \]

Since $\beta$ is the $2T$th root of unity in the extension field $\mathbb{GF}(r^m)$, all the zeros of the polynomial $x^{2T} - 1$ over $\mathbb{GF}(r^m)[x]$ are under the form $\beta^k$, with $k \in \mathbb{Z}_{2T}$. If the same $\beta^k$ is also the zero of the generating polynomial in (3.7), it simply means that $x - \beta^k$ is a common factor of $G_s(x)$ and $x^{2T} - 1$. By explicitly compute out all the zeros of the generating polynomial $G_s(x)$ under the form $\beta^k$, we can determine the degree of the corresponding minimal polynomial, from which the linear complexity of the sequence in (2.7) is straightforward.
Lemma 3.8. If (2, 2) be determined as follows:

\[ G_s(\beta^k) = 2\beta_2^k + \left( \sum_{i \in D_1^{(T)}} \beta_2^{2^i \cdot k_i} + \sum_{i \in D_1^{(T)}} \beta_2^{k_i} \right) + \left( \sum_{i \in D_1^{(p)}} \beta_p^{2^i \cdot k_i} + \sum_{i \in D_1^{(p)}} \beta_p^{k_i} \right) + \left( \sum_{i \in D_1^{(q)}} \beta_q^{-1 \cdot k_i} + \sum_{i \in D_1^{(q)}} \beta_q^{k_i} \right) + 2 \left( Z^{(p)}(k) \cdot Z^{(q)}(k) + Z^{(p)}(k) + Z^{(q)}(k) \right) \]  

(3.8)

Where \( k \in \mathbb{Z}_{2T} \).

3.1. 2 is a quadratic residue both modulo \( p \) and modulo \( q \). Let \( k_1 = k \mod p \) and \( k_2 = k \mod q \). Next lemma gives the values of \( G_s(\beta^k) \) for \( k \in \mathbb{Z}_{2T} \).

Lemma 3.8. If \((2, 2) \in D_0^{(p)} \times D_0^{(q)}\), then the values of \( G_s(\beta^k) \) where \( k \in \mathbb{Z}_{2T} \) can be determined as follows:

1. \( k \in \{0, pq\} \cup Z_{2T}^\ast \cup qZ_{2p}^\ast \cup pZ_{2q}^\ast \),

\[ G_s(\beta^k) = \begin{cases} 
3pq - 1 & k \in \{0\}, \\
2pq - 4 & k \in \{pq\}, \\
-4 & k \in Z_{2T}^\ast \cup qZ_{2p}^\ast \cup pZ_{2q}^\ast. 
\end{cases} \]

2. \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \).
   (a) If \( k \in 2D_0^{(T)} \) and \((k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}\) and \( \left( \frac{2}{p} \right) = -\left( \frac{2}{q} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_0 + A_1B_1 + A_1 + B_1). \]

(b) If \( k \in 2D_0^{(T)} \) and \((k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}\) and \( \left( \frac{2}{q} \right) = \left( \frac{2}{p} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_1 + A_1B_0 + A_1 + B_1). \]

(c) If \( k \in 2D_0^{(T)} \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)}\) and \( \left( \frac{2}{q} \right) = -\left( \frac{2}{p} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_0 + A_1B_1 + A_0 + B_0). \]

(d) If \( k \in 2D_0^{(T)} \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)}\) and \( \left( \frac{2}{q} \right) = \left( \frac{2}{p} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_1 + A_1B_0 + A_0 + B_0). \]

(e) If \( k \in 2D_1^{(T)} \) and \((k_1, k_2) \in D_0^{(p)} \times D_1^{(q)}\) and \( \left( \frac{2}{q} \right) = -\left( \frac{2}{p} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_1 + A_1B_0 + A_1 + B_0). \]

(f) If \( k \in 2D_1^{(T)} \) and \((k_1, k_2) \in D_0^{(p)} \times D_1^{(q)}\) and \( \left( \frac{2}{q} \right) = \left( \frac{2}{p} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_0 + A_1B_1 + A_1 + B_0). \]

(g) If \( k \in 2D_1^{(T)} \) and \((k_1, k_2) \in D_1^{(p)} \times D_0^{(q)}\) and \( \left( \frac{2}{q} \right) = -\left( \frac{2}{p} \right) \), then

\[ G_s(\beta^k) = 2(A_0B_1 + A_1B_0 + A_0 + B_1). \]
\( (h) \) If \( k \in 2D_1^{(T)} \) and \( (k_1, k_2) \in \text{D}_1^{(p)} \times \text{D}_1^{(q)} \) and \( (\frac{k_1}{k_2}) = (\frac{2}{3}) \), then
\[
G_s(\beta^k) = 2(A_0B_0 + A_1B_1 + A_0 + B_1).
\]
\( (3) \) \( k \in 2qD_0^{(p)} \cup 2qD_1^{(p)} \).
(a) If \( k \in 2qD_0^{(p)} \) and \( (\frac{k}{p}) = -1 \) or \( k \in 2qD_1^{(p)} \) and \( (\frac{k}{p}) = 1 \), then
\[
G_s(\beta^k) = 2A_0.
\]
(b) If \( k \in 2qD_0^{(p)} \) and \( (\frac{k}{p}) = 1 \) or \( k \in 2qD_1^{(p)} \) and \( (\frac{k}{p}) = -1 \), then
\[
G_s(\beta^k) = 2A_1.
\]
\( (4) \) \( k \in 2pD_0^{(q)} \cup 2pD_1^{(q)} \).
(a) If \( k \in 2pD_0^{(q)} \) and \( (\frac{k}{q}) = -1 \) or \( k \in 2pD_1^{(q)} \) and \( (\frac{k}{q}) = 1 \), then
\[
G_s(\beta^k) = 2B_0.
\]
(b) If \( k \in 2pD_0^{(q)} \) and \( (\frac{k}{q}) = 1 \) or \( k \in 2pD_1^{(q)} \) and \( (\frac{k}{q}) = -1 \), then
\[
G_s(\beta^k) = 2B_1.
\]

Proof. Using Lemma 3.3 and Lemma 2.2 to eliminate the exponents \( 2^{-1}_T, 2^{-1}_p \), and \( 2^{-1}_q \) from (3.8), we get a new form of (3.8):
\[
G_s(\beta^k) = 2\beta^k_2 + \left( \beta^k_2 + 1 \right) \left( \sum_{i \in \text{D}_1^{(p)}} \beta^k_i + \sum_{i \in \text{D}_1^{(q)}} \beta^k_i + \sum_{i \in \text{D}_1^{(q)}} \beta^k_i \right) + 2 \left( Z^{(p)}(k) \cdot Z^{(q)}(k) + Z^{(p)}(k) + Z^{(q)}(k) \right).
\]
(3.9)

Where \( k \in Z_{2T} \).

In (3.9), let \( s_1 = \sum_{i \in \text{D}_1^{(p)}} \beta^k_i + \sum_{i \in \text{D}_1^{(q)}} \beta^k_i + \sum_{i \in \text{D}_1^{(q)}} \beta^k_i \), \( s_2 = Z^{(p)}(k) \), \( s_3 = Z^{(q)}(k) \), \( s_4 = Z^{(p)}(k) \cdot Z^{(q)}(k) \), and \( s_5 = \beta^k_2 \). It is easy to compute the values of \( s_2 \) and \( s_3 \), for all \( k \in Z_{2T} \), see below:
\[
s_2 = \begin{cases} 
-1 & k \in Z^{*}_2 \cup 2Z^{*}_2 \cup qZ^{*}_2 \cup 2qZ^{*}_p, \\
p - 1 & k \in \{0, T\} \cup pZ^{*}_q \cup 2pZ^{*}_q.
\end{cases}
\]
(3.10)
\[
s_3 = \begin{cases} 
-1 & k \in Z^{*}_2 \cup 2Z^{*}_2 \cup pZ^{*}_2 \cup 2pZ^{*}_q, \\
q - 1 & k \in \{0, T\} \cup qZ^{*}_2 \cup 2qZ^{*}_p.
\end{cases}
\]
(3.11)

After having computed \( s_2 \) and \( s_3 \), it is straightforward to find the values of \( s_4 \). On the other hand, computation of \( s_5 \) is trivial. Due to limited space and similarity of proof process, we consider only the case (3) of Lemma 3.3 that is, \( k \in 2qZ^{*}_p \).

By Lemma 2.2, \( A_1(k) = A_1 \) if \( k \in D_0^{(p)} \), and \( A_1(k) = A_0 \) if \( k \in D_1^{(p)} \).

Remark that if \( (\frac{k}{p}) = 1 \), then \( q \) \( (\text{mod} \ p) \) \( D_0^{(p)} \) \( \text{mod} \ p \) \( D_1^{(p)} \). Next, we compute \( s_1 \) for \( k \in 2qZ^{*}_p \). Let \( k = 2qk_1 \), where \( k_1 = k \) \( (\text{mod} \ p) \). It is obvious that \( B_0(k) = B_1(k) = \frac{k - 1}{2q} \), \( A_0(k) + A_1(k) = s_2 = Z^{(p)}(k) = -1 \), \( B_0(k) + B_1(k) = s_3 = Z^{(q)}(k) = q - 1 \), and \( s_5 = \frac{\beta^k_2}{2} = 1 \). By Lemma 3.3 we distinguish two cases:
Let (2/3) that correspond to the case (3) in Lemma 3.8.

\[ s_1 = \sum_{i \in D_1^{(p)}} \beta_2^{ki} + \sum_{i \in D_1^{(q)}} \beta_p^{ki} + \sum_{i \in D_1^{(s)}} \beta_q^{ki} \]

\[ = A_0(k)B_1(k) + A_1(k)B_0(k) + A_1(k) + B_1(k) \]

\[ = (A_0(k) + A_1(k)) \cdot \frac{q-1}{2} + A_1(k) + \frac{q-1}{2} \]

\[ = (-1) \cdot \frac{q-1}{2} + A_1(k) + \frac{q-1}{2} = A_1(k). \]

Note that \( s_1 = A_1(k) = A_1(2qk_1) = A_1(qk_1) \) since \( 2 \in D_0^{(p)} \). Using Lemma 2.2, possible combination for \( k_1 \) and \( q \) (mod \( p \) \( \in \{D_0^{(p)}, D_1^{(p)}\} \) gives the following values to \( s_1 \):

\[
(3.12) \quad s_1 = \begin{cases} 
A_0 & k_1 \in D_0^{(p)} \text{ and } \left( \frac{2}{p} \right) = -1 \text{ or } \\
A_1 & k_1 \in D_1^{(p)} \text{ and } \left( \frac{2}{p} \right) = 1, \\
A_1 & k_1 \in D_0^{(p)} \text{ and } \left( \frac{2}{p} \right) = 1 \text{ or } \\
k_1 \in D_1^{(p)} \text{ and } \left( \frac{2}{p} \right) = -1. 
\end{cases}
\]

(2) \( \left( \frac{4}{p} \right) = -\left( \frac{2}{q} \right) \). After rather similar computation, we find that the final values of \( s_1 \) in this case are the same as that listed in (3.12).

From (3.9) - (3.12), we obtain

\[ G_s(\beta^k) = 2\beta_2^k + \left( \beta_2^k + 1 \right) \left( \sum_{i \in D_1^{(p)}} \beta_i^{ki} + \sum_{i \in D_1^{(q)}} \beta_p^{ki} + \sum_{i \in D_1^{(s)}} \beta_q^{ki} \right) \]

\[ + 2 \left( Z^{(p)}(k) \cdot Z^{(q)}(k) + Z^{(p)}(k) + Z^{(q)}(k) \right) \]

\[ = 2 \cdot s_5 + (1 + s_3) \cdot s_1 + 2(s_2 \cdot s_3 + s_2 + s_3) \]

\[ = 2 \cdot 1 + (1 + 1) \cdot s_1 + 2((-1) \cdot (q-1) + (-1) + (q - 1)) \]

\[ = 2s_1. \]

Hence, for \( k \in 2qZ_p^* \) we get

\[ G_s(\beta^k) = 2s_1 = \begin{cases} 
2A_0 & k_1 \in D_0^{(p)} \text{ and } \left( \frac{2}{p} \right) = -1 \text{ or } \\
2A_1 & k_1 \in D_1^{(p)} \text{ and } \left( \frac{2}{p} \right) = 1, \\
2A_1 & k_1 \in D_0^{(p)} \text{ and } \left( \frac{2}{p} \right) = 1 \text{ or } \\
k_1 \in D_1^{(p)} \text{ and } \left( \frac{2}{p} \right) = -1, 
\end{cases} \]

that correspond to the case (3) in Lemma 3.8.

We below list main theorems of this subsection:

**Theorem 3.1.** Let \( (2, 2) \in D_0^{(p)} \times D_0^{(q)}, \left( \frac{p}{q} \right) = 1 \text{ and } \left( \frac{q}{p} \right) = 1 \). Then, the linear complexity over \( GF(r^m) \) of the generalized cyclotomic quaternary sequences with period \( 2T \) specified in (2.7), \( LC(s) \), can be determined according to next two cases.
(1) \((p, q) \in (1, -1)\) and \(r \mid p - 1\) and \(r \mid q + 1\) OR \((p, q) \in (-1, 1)\) and \(r \mid p + 1\) and \(r \mid q - 1\):

\[
LC(s) = \frac{5pq + p + q + 1}{4}.
\]

(2) \((p, q) \in (-1, -1)\) and \(r \mid p + 1\) and \(r \mid q + 1\):

\[
LC(s) = \frac{4pq - p - q + 2}{2}.
\]

Proof. Since \((\frac{p}{q})_1 = 1\) and \((\frac{p}{q})_0 = 1\), it means that \(r \in D_0^{(p)}\) and \(r \in D_0^{(q)}\). Hence, by Lemma \(2.3\) \(A_0, A_1, B_0, B_1 \in \text{GF}(r)\). For Theorem \(3.1\) (1), we only prove the part where \((p, q) \in (1, -1)\) and \(r \mid p - 1\) and \(r \mid q + 1\), since proof for another part is symmetric.

Since \((p, q) \in (1, -1), p \equiv 1 \pmod{4}\) and \(q \equiv 3 \pmod{4}\). Because \(r \mid p - 1\) and \(r \mid q + 1\), by Lemma \(2.3\) \(A_0, A_1, B_0, B_1 \in \{0, -1\}\).

If \(A_0 \in \{0, -1\}\), by Lemma \(3.8\) (3), \(G_s(\beta^k) = 0\) for \(k \in 2qD_0^{(p)}\) or \(k \in 2qD_1^{(p)}\). It means that \(\prod_{k \in D} (x - \beta^k)\) are common factors of both \(x^{2pq} - 1\) and \(G_s(x)\), where \(D = 2qD_0^{(p)}\) or \(D = 2qD_1^{(p)}\). The number of the common factors is equal to \(\frac{2 - 1}{2}\).

On the other hand, if \(B_0 \in \{0, -1\}\), by Lemma \(3.8\) (4), \(G_s(\beta^k) = 0\) for \(k \in 2pD_0^{(q)}\) or \(k \in 2pD_1^{(q)}\). Hence, there are \(\frac{2 - 1}{2}\)'s common factors between \(x^{2pq} - 1\) and \(G_s(x)\), that are under the form \(\prod_{k \in D} (x - \beta^k)\) for \(D = 2qD_0^{(q)}\) or \(D = 2qD_1^{(q)}\).

In order to find other zeros of \(G_s(\beta^k)\) with \(k \in \mathbb{Z}_{2T}\), we define eight quantities that correspond to the eight sub-cases in Lemma \(3.8\) (2):

\[
\begin{align*}
a &= 2(A_0B_0 + A_1B_1 + A_1 + B_1), \\
b &= 2(A_0B_1 + A_1B_0 + A_1 + B_1), \\
c &= 2(A_0B_0 + A_1B_1 + A_0 + B_0), \\
d &= 2(A_0B_1 + A_1B_0 + A_0 + B_0), \\
e &= 2(A_0B_1 + A_1B_0 + A_1 + B_0), \\
f &= 2(A_0B_0 + A_1B_1 + A_1 + B_0), \\
g &= 2(A_0B_1 + A_1B_0 + A_0 + B_1), \\
h &= 2(A_0B_0 + A_1B_1 + A_0 + B_1).
\end{align*}
\]

Note that for each \((A_0, B_0) \in \{0, -1\} \times \{0, -1\}\) from Table \(\mathbf{1}\) there are three sub-cases in Lemma \(3.8\) (2) where \(G_s(\beta^k) = 0\) for \(k \in \mathbb{Z}_{T_0}^{(p)} \cup \mathbb{Z}_{T_1}^{(q)}\), that contribute in total \(3(p - 1)(q - 1)/4\)'s zeros to \(G_s(\beta^k)\) for \(k \in \mathbb{Z}_{2T}\). If we examine closely those three sub-cases, we could remark that it is required that \((\frac{p}{q})_2 = (\frac{q}{p})_2\) which is equivalent to \((p, q) \in (1, -1) \cup (-1, 1)\) according to the Quadratic Reciprocity Law (See Lemma \(2.3\) (2)).

From Lemma \(3.8\) (1), it is obvious that \(G_s(\beta^k) \neq 0\) for \(k \in \{0, \ pq\} \cup \mathbb{Z}_{2T}^{(q)} \cup pqZ_{2p}^{(p)} \cup pZ_{2q}^{(q)}\). By \(2.10\),

\[
LC(s) = 2pq - \frac{3(p - 1)(q - 1)}{4} - \frac{p - 1}{2} - \frac{q - 1}{2}
= \frac{5pq + p + q + 1}{4};
\]
Table 1. Values of $G_s(\beta^k)$ in Lemma 3.8(2)

| $A_0$ | $A_1$ | $B_0$ | $B_1$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|-------|-------|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | -1    | 0     | -1    | -2  | -4  | 2   | 0   | -2  | 0   | -2  | 0   |
| 0     | -1    | -1    | 0     | -2  | 0   | 2   | 0   | -2  | -4  | 2   | 0   |
| -1    | 0     | 0     | -1    | -2  | 0   | 2   | 0   | -2  | 0   | -2  | 0   |
| -1    | 0     | -1    | 0     | 2   | 0   | -2  | 0   | -2  | 0   | -2  | 0   |

$^1 (\frac{\zeta}{p}) = 1$ and $(\frac{\zeta}{q}) = 1$.

and for Theorem 3.1 (2),

$$LC(s) = 2pq - \frac{p - 1}{2} - \frac{q - 1}{2}$$

$$= \frac{4pq - p - q + 2}{2}.$$ 

\[\square\]

Algorithm 1 Algorithm to compute linear complexity

1: procedure $LC(p, q, r)$
2: Construct the set $D_1 = D_1(2T) \cup qD_1(2p) \cup pD_1(2q)$.
3: Construct the set $D_2 = 2D_0(T) \cup 2qD_0(p) \cup 2pD_0(q)$.
4: Construct the set $D_3 = 2D_1(T) \cup 2qD_1(p) \cup 2pD_1(q)$.
5: Construct $G_s(x)$ by (3.7).
6: Construct the polynomial $P(x) = x^{2T} - 1$.
7: Compute $\gcd(G_s(x), P(x))$.
8: return $2pq - \deg(\gcd(G_s(x), P(x)))$.
9: end procedure

Example 3.1. We implement Algorithm 1 using the computer algebra system Maple, in particular, using the Maple built-in function Gcdex to compute the gcd of $P(x) = x^{2T} - 1$ and the generating polynomial $G_s(x)$ defined in (3.7). Given a pair of primes $p$ and $q$, and $r$ such that the conditions in Theorem 3.1 are satisfied, let $LC_{Alg}$ denote the linear complexity computed by Algorithm 1 and $LC_{Th}$ denote the linear complexity calculated using the formula in Theorem 3.1. In Table 2 some numerical examples are listed. We observe that the numerical result by running the procedure in Algorithm 1 coincide the ones predicted by Theorem 3.1.

Theorem 3.2. Let $(2, 2) \in D_0(p) \times D_0(q)$, $(\frac{\zeta}{p}) = 1$ and $(\frac{\zeta}{q}) = -1$. Then, the linear complexity over $\mathbf{GF}(r^m)$ of the generalized cyclotomic quaternary sequences with period $2T$ specified in (2.7), $LC(s)$, can be determined according to following cases.

1. If $(p, q) \in (1, -1)$ and $r \mid p - 1$ and $r \nmid 3q - 1$ and $r \nmid 2q - 4$ OR $(p, q) \in (-1, 1)$ and $r \mid p + 1$ and $r \nmid 3q + 1$ and $r \nmid 2q + 4$, then

$$LC(s) = \frac{3pq + q}{2}.$$
Table 2. Linear Complexity Calculated By Theorem 3.1

| p   | q   | r | $LC_{Algo}$ | $LC_{Th}$ | Condition |
|-----|-----|---|-------------|-----------|-----------|
| 41  | 79  | 5 | 4079        | 4079      | $C_1^a$   |
| 113 | 167 | 7 | 23659       | 23659     | $C_1$     |
| 89  | 263 | 11| 29347       | 29347     | $C_1$     |
| 79  | 41  | 5 | 4079        | 4079      | $C_2^b$   |
| 167 | 113 | 7 | 23659       | 23659     | $C_2$     |
| 263 | 89  | 11| 29347       | 29347     | $C_2$     |
| 311 | 313 | 13| 121835      | 121835    | $C_2$     |
| 79  | 239 | 5 | 37604       | 37604     | $C_3^c$   |
| 167 | 223 | 7 | 74288       | 74288     | $C_3$     |
| 103 | 311 | 13| 63860       | 63860     | $C_3$     |

$\text{a} (p, q) \in \{(1, -1) \text{ and } r \mid p - 1 \text{ and } r \mid q + 1,}$

$\text{b} (p, q) \in \{(-1, 1) \text{ and } r \mid p + 1 \text{ and } r \mid q - 1,}$

$\text{c} (p, q) \in \{(-1, -1) \text{ and } r \mid p + 1 \text{ and } r \mid q + 1.$

(2) If $(p, q) \in (1, -1)$ and $r \mid p - 1$ and $(r \mid 3q - 1$ or $r \mid 2q - 4)$ and $r \neq 5$ OR

$(p, q) \in (-1, 1)$ and $r \mid p + 1$ and $(r \mid 3q + 1$ or $r \mid 2q + 4)$ and $r \neq 5$, then

$$LC(s) = \frac{3pq + q - 2}{2}.$$

(3) If $(p, q) \in (1, -1)$ and $r \mid p - 1$ and $(r \mid 3q - 1$ or $r \mid 2q - 4)$ and $r = 5$ OR

$(p, q) \in (-1, 1)$ and $r \mid p + 1$ and $(r \mid 3q + 1$ or $r \mid 2q + 4)$ and $r = 5$, then

$$LC(s) = \frac{3pq + q - 4}{2}.$$

Remark 3.1. In Theorem 3.2 if $p \equiv 1 \pmod{8}$ and $r \mid p - 1$, by Lemma 3.8 (1),

$r \mid 3q - 1$ and $r \mid 2q - 4$ imply that $\beta^0 = 1$ and $\beta^p$ are zeros of $G_s(\beta^k)$ for $k \in \mathbb{Z}_{2T}$, respectively. For $p \equiv -1 \pmod{8}$ and $r \mid p + 1$, again by Lemma 3.8 (1), $\beta^0$ is zero of $G_s(\beta^k)$ if only if $r \mid 3q + 1$ and, $\beta^p$ is zero of $G_s(\beta^k)$ if only if $r \mid 2q + 4$. It is obvious that $r = 5$ if only if both $r \mid 3q - 1$ and $r \mid 2q - 4$ hold for the case $p \equiv 1 \pmod{8}$ and $r \mid p - 1$, or if only if both $r \mid 3q + 1$ and $r \mid 2q + 4$ hold for the case $p \equiv -1 \pmod{8}$ and $r \mid p + 1$.

Remark 3.2. In Table 3, the values of $G_s(\beta^k)$ for $k \in 2D^T_q \cup 2D^T_1$ that correspond to the eight sub-cases in Lemma 3.8 (2) are listed, where $A_0, A_1 \in \{0, -1\}$ and, $B_0$ and $B_1$ take arbitrary values but $B_0 + B_1 = -1$. If $A_0 = 0, G_s(\beta^k) = 0$ for the sub-cases $d$ and $h$ in Lemma 3.8 (2); if $A_0 = -1, G_s(\beta^k) = 0$ for the sub-cases $b$ and $h$ in Lemma 3.8 (2). For the above two cases $A_0 = 0$ and $A_0 = -1$, from Lemma 3.8 (2), we can observe that the condition $\left(\frac{2}{q}\right) = \left(\frac{2}{7}\right)$ is required, that is equivalent to the condition $p \equiv 1 \pmod{8}$ and $q \equiv -1 \pmod{8}$ or $p \equiv -1 \pmod{8}$ and $q \equiv 1 \pmod{8}$ by the Quadratic Reciprocity law (see Lemma 2.3 (2)). In other words, for $(p, q) \in (1, -1)$ or $(p, q) \in (-1, 1)$, if $A_0, A_1 \in \{0, -1\}$, then $G_s(\beta^k)$
Proof of Theorem 3.2. Since \( \left( \frac{c}{p} \right) = 1 \) and \( \left( \frac{c}{q} \right) = -1 \), by Lemma 2.4, \( A_0, A_1 \in \mathbf{GF}(r) \) and \( B_0, B_1 \in \mathbf{GF}(r^m) \setminus \mathbf{GF}(r) \). If \( p \equiv 1 \) (mod 8) and \( r \mid p - 1 \) or \( p \equiv -1 \) (mod 8) and \( r \mid p + 1 \), then by Lemma 2.1, \( A_0, A_1 \in \{ 0, -1 \} \). From Lemma 3.3, \( G_s(\beta^k) \) for \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \) has no zeros.

for \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \), has \( \frac{(p-1)(q-1)}{2} \) zeros. But for \( (p, q) \in (-1, -1) \), \( G_s(\beta^k) \) for \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \) has no zeros.

Proof of Theorem 3.2. Since \( \left( \frac{c}{p} \right) = 1 \) and \( \left( \frac{c}{q} \right) = -1 \), by Lemma 2.4, \( A_0, A_1 \in \mathbf{GF}(r) \) and \( B_0, B_1 \in \mathbf{GF}(r^m) \setminus \mathbf{GF}(r) \). If \( p \equiv 1 \) (mod 8) and \( r \mid p - 1 \) or \( p \equiv -1 \) (mod 8) and \( r \mid p + 1 \), then by Lemma 2.1, \( A_0, A_1 \in \{ 0, -1 \} \). From Lemma 3.3, \( G_s(\beta^k) \) for \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \) has no zeros. We next prove each case in Theorem 3.2.

1. Case \( (p, q) \in (1, -1) \) and \( r \mid p - 1 \) and \( r \nmid 3q - 1 \) and \( r \nmid 2q - 4 \) OR \( (p, q) \in (-1, 1) \) and \( r \mid p + 1 \) and \( r \nmid 3q + 1 \) and \( r \nmid 2q + 4 \). From Remark 3.2, neither \( \beta^0 \) nor \( \beta^{pq} \) is a zero of \( G_s(\beta^k) \) for \( k \in \{ 0, pq \} \cup Z_{2T} \cup qZ_{2T} \cup pZ_{2T} \). From Remark 3.2, \( G_s(\beta^k) \) for \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \) has \( \frac{(p-1)(q-1)}{2} \) zeros. In addition that there are \( \frac{p-1}{2} \) zeros for \( k \in 2qD_0^{(p)} \cup 2qD_1^{(p)} \), the linear complexity could be directly computed out by

\[
LC(s) = 2pq - \frac{(p-1)(q-1)}{2} - \frac{p-1}{2} = \frac{3pq+q}{2}.
\]

2. Case \( (p, q) \in (1, -1) \) and \( r \mid p - 1 \) and \( r \mid 3q - 1 \) or \( r \mid 2q - 4 \) and \( r \neq 5 \) OR \( (p, q) \in (-1, 1) \) and \( r \mid p + 1 \) and \( r \mid 3q + 1 \) or \( r \mid 2q + 4 \) and \( r \neq 5 \). By above discussion, these conditions imply that \( G_s(\beta^k) \) has an extra zero that is either \( \beta^0 = 1 \) or \( \beta^{pq} \), so the linear complexity equals

\[
LC(s) = 2pq - \frac{(p-1)(q-1)}{2} - \frac{p-1}{2} - 1 = \frac{3pq+q-2}{2}.
\]

3. Case \( (p, q) \in (1, -1) \) and \( r \mid p - 1 \) and \( r \mid 3q - 1 \) or \( r \mid 2q - 4 \) and \( r = 5 \) OR \( (p, q) \in (-1, 1) \) and \( r \mid p + 1 \) and \( r \mid 3q + 1 \) or \( r \mid 2q + 4 \) and \( r = 5 \). These conditions imply that \( G_s(\beta^k) \) has two extra zeros that are \( \beta^0 = 1 \) and \( \beta^{pq} \), so the linear complexity equals

\[
LC(s) = 2pq - \frac{(p-1)(q-1)}{2} - \frac{p-1}{2} - 2 = \frac{3pq+q-4}{2}.
\]

\[\square\]

Theorem 3.3. Let \((2, 2) \in D_0^{(p)} \times D_0^{(q)}, \left( \frac{c}{p} \right) = 1, \left( \frac{c}{q} \right) = -1, \) and \((p, q) \in (-1, -1)\). Then, the linear complexity over \( \mathbf{GF}(r^m) \) of the generalized cyclotomic quaternary sequences with period \( 2T \) given in (27), \( LC(s) \) can be determined according to following cases.

| \( A_0 \) | \( A_1 \) | \( B_0 \) | \( B_1 \) | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( g \) | \( h \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | -1 | *2 | * | * | 0 | * | * | * | * | 0 |
| -1 | 0 | * | * | * | 0 | * | * | * | * | 0 |
(1) If $r \mid p + 1$ and $r \mid 3q + 1$ and $r \mid 2q + 4$, then
\[ LC(s) = \frac{4pq + p - 1}{2}. \]
(2) If $r \mid p + 1$ and $(r \mid 3q + 1$ or $r \mid 2q + 4$) and $r \neq 5$, then
\[ LC(s) = \frac{4pq + p - 3}{2}. \]
(3) If $r \mid p + 1$ and $(r \mid 3q + 1$ or $r \mid 2q + 4$) and $r = 5$, then
\[ LC(s) = \frac{4pq + p - 5}{2}. \]

Proof. Since $(p, q) \in (-1, -1)$, by Remark 3.2, $G_s(\beta^k)$ for $k \in 2D_0^{(T)} \cup 2D_1^{(T)}$ has no zeros. The rest of the proof is similar to that for Theorem 3.2. □

3.2. 2 is a quadratic non-residue both modulo $p$ and modulo $q$.

**Lemma 3.9.** Let $(2, 2) \in D_1^{(p)} \times D_1^{(q)}$. Then the values of $G_s(\beta^k)$ for $k \in \mathbb{Z}_{2T}$ can be determined as follows:

1) $k \in \{0, pq\} \cup 2qZ_p^* \cup 2pZ_q^*$.

\[ G_s(\beta^k) = \begin{cases} 
3pq - 1 & k \in \{0\}, \\
2pq - 4 & k \in \{pq\}, \\
-1 & k \in 2qZ_p^* \cup 2pZ_q^*.
\end{cases} \]

2) $k \in D_0^{(2T)} \cup D_1^{(2T)}$. Let $k_1 \equiv k \pmod{p}$ and $k_2 \equiv k \pmod{q}$.
   (a) If $k \in D_0^{(2T)}$ and $(k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}$, then
   \[ G_s(\beta^k) = A_0 - A_1 + B_0 - B_1 - 4. \]
   (b) If $k \in D_0^{(2T)}$ and $(k_1, k_2) \in D_1^{(p)} \times D_0^{(q)}$, then
   \[ G_s(\beta^k) = A_1 - A_0 + B_1 - B_0 - 4. \]
   (c) If $k \in D_1^{(2T)}$ and $(k_1, k_2) \in D_0^{(p)} \times D_1^{(q)}$, then
   \[ G_s(\beta^k) = A_0 - A_1 + B_0 - B_1 - 4. \]
   (d) If $k \in D_1^{(2T)}$ and $(k_1, k_2) \in D_1^{(p)} \times D_0^{(q)}$, then
   \[ G_s(\beta^k) = A_1 - A_0 + B_0 - B_1 - 4. \]

(3) $k \in 2D_0^{(T)} \cup 2D_1^{(T)}$.
   (a) If $k \in 2D_0^{(T)}$ and \( k \equiv \left(\begin{smallmatrix} 2 \\ p \end{smallmatrix}\right) \) or $k \in 2D_1^{(T)}$ and \( k \equiv \left(\begin{smallmatrix} 2 \\ q \end{smallmatrix}\right) \), then
   \[ G_s(\beta^k) = 2(A_0B_0 + A_1B_1 - 1). \]
   (b) If $k \in 2D_0^{(T)}$ and \( k \equiv \left(\begin{smallmatrix} 2 \\ q \end{smallmatrix}\right) \) or $k \in 2D_1^{(T)}$ and \( k \equiv \left(\begin{smallmatrix} 2 \\ p \end{smallmatrix}\right) \), then
   \[ G_s(\beta^k) = 2(A_0B_1 + A_1B_0 - 1). \]

(4) $k \in qD_0^{(2p)} \cup qD_1^{(2p)}$.
   (a) If $k \in qD_0^{(2p)}$ and \( k \equiv \left(\begin{smallmatrix} 1 \\ p \end{smallmatrix}\right) \) or $k \in qD_1^{(2p)}$ and \( k \equiv \left(\begin{smallmatrix} 2 \\ p \end{smallmatrix}\right) \), then
   \[ G_s(\beta^k) = A_1 - A_0 - 4. \]
(b) If \( k \in qD_0^{(2p)} \) and \( \left( \frac{q}{p} \right) = -1 \) or \( k \in qD_1^{(2p)} \) and \( \left( \frac{q}{p} \right) = 1 \), then

\[ G_s(\beta^k) = A_0 - A_1 - 4. \]

(5) \( k \in pD_0^{(2q)} \cup pD_1^{(2q)}. \)

(a) If \( k \in pD_0^{(2q)} \) and \( \left( \frac{p}{q} \right) = 1 \) or \( k \in pD_1^{(2q)} \) and \( \left( \frac{p}{q} \right) = -1 \), then

\[ G_s(\beta^k) = B_1 - B_0 - 4. \]

(b) If \( k \in pD_0^{(2q)} \) and \( \left( \frac{p}{q} \right) = -1 \) or \( k \in pD_1^{(2q)} \) and \( \left( \frac{p}{q} \right) = 1 \), then

\[ G_s(\beta^k) = B_0 - B_1 - 4. \]

Proof. It is clear that \( 2_p^{-1} \in D_1^{(p)} \) and \( 2_q^{-1} \in D_1^{(q)} \). By Lemma 3.1 and Lemma 3.4, \( 2_p^{-1} \in D_0^{(T)} \). By Lemma 2.2, it can be deduced the following equations

\[
\sum_{i \in D_1^{(p)}} \beta_{1-i} = \sum_{i \in 2^{-1}D_1^{(T)}} \beta_{-i} = \sum_{i \in D_1^{(p)}} \beta_{i},
\]

\[
\sum_{i \in D_1^{(p)}} \beta_{p^{-1}i} = \sum_{i \in 2^{-1}D_1^{(p)}} \beta_{pi} = \sum_{i \in D_1^{(p)}} \beta_{pi},
\]

\[
\sum_{i \in D_1^{(q)}} \beta_{q^{-1}i} = \sum_{i \in 2^{-1}D_1^{(q)}} \beta_{qi} = \sum_{i \in D_1^{(q)}} \beta_{qi}.
\]

Substitute the group of above equations into (3.8), it leads to a new form for \( G_s(\beta^k) \) with \( k \in \mathbb{Z}_{2T}: \)

\[
G_s(\beta^k) = 2\beta_2^k + \left( \beta_2^k + 1 \right) \sum_{i \in D_1^{(p)}} \beta_{i}^k + \left( \beta_2^k \sum_{i \in D_0^{(p)}} \beta_{i}^k + \sum_{i \in D_1^{(p)}} \beta_{i}^k \right) + \left( \beta_2^k \sum_{i \in D_0^{(q)}} \beta_{i}^k + \sum_{i \in D_1^{(q)}} \beta_{i}^k \right) + 2 \left( Z^{(p)}(k) \cdot Z^{(q)}(k) + Z^{(p)}(k) + Z^{(q)}(k) \right).
\]

(3.13)

The rest of the proof is similar to that of Lemma 3.8.

\[ \square \]

Corollary 3.1. Let \( 2 \in D_1^{(p)}, 2 \in D_1^{(q)}, \left( \frac{p}{q} \right) = 1 \) and \( \left( \frac{q}{p} \right) = 1 \). In addition, let \( r \mid p+16 \) and \( r \mid q+16 \) if \( p \equiv 3 \pmod{8} \) and \( q \equiv 3 \pmod{8} \), \( r \mid p-16 \) and \( r \mid q+16 \) if \( p \equiv 5 \pmod{8} \) and \( q \equiv 3 \pmod{8} \) and, \( r \mid p+16 \) and \( r \mid q-16 \) if \( p \equiv 3 \pmod{8} \) and \( q \equiv 5 \pmod{8} \). Then, \( G_s(\beta^k) \), where \( k \in qD_0^{(2p)} \cup qD_1^{(2p)} \cup pD_0^{(2q)} \cup pD_1^{(2q)} \), has exactly \( \frac{p-1}{2} + \frac{q-1}{2} \)'s zeros.
Lemma 2.1, for the case where $r | p + 16$ and $r | q + 16$ with $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$, resolve next congruent system:

\[
\begin{align*}
A_0 + A_1 &= -1, \\
B_0 + B_1 &= -1, \\
p + 16 &\equiv 0 \pmod{r}, \\
q + 16 &\equiv 0 \pmod{r}, \\
A_1(A_1 + 1) &= -\frac{r + 1}{4} \pmod{r}, \\
B_1(B_1 + 1) &= -\frac{q + 1}{4} \pmod{r}.
\end{align*}
\]

(3.14)

Now consider the case where $r | p - 16$ and $r | q + 16$ with $p \equiv -3 \pmod{8}$ and $q \equiv 3 \pmod{8}$ and next congruent system:

\[
\begin{align*}
A_0 + A_1 &= -1, \\
B_0 + B_1 &= -1, \\
p - 16 &\equiv 0 \pmod{r}, \\
q + 16 &\equiv 0 \pmod{r}, \\
A_0(A_0 + 1) &= -\frac{r + 1}{4} \pmod{r}, \\
B_1(B_1 + 1) &= -\frac{q + 1}{4} \pmod{r}.
\end{align*}
\]

(3.15)

And finally for the case where $r | p + 16$ and $r | q - 16$ with $p \equiv 3 \pmod{8}$ and $q \equiv -3 \pmod{8}$, consider the following congruent system:

\[
\begin{align*}
A_0 + A_1 &= -1, \\
B_0 + B_1 &= -1, \\
p + 16 &\equiv 0 \pmod{r}, \\
q - 16 &\equiv 0 \pmod{r}, \\
A_1(A_1 + 1) &= -\frac{r + 1}{4} \pmod{r}, \\
B_0(B_0 + 1) &= -\frac{q + 1}{4} \pmod{r}.
\end{align*}
\]

(3.16)

By a simple modular arithmetic computation, the congruent systems (3.14), (3.15) and (3.16) give rise to the same solutions:

\[
\pm (A_0 - A_1) - 4 \equiv 0 \pmod{r},
\]

(3.17)

and

\[
\pm (B_0 - B_1) - 4 \equiv 0 \pmod{r}.
\]

(3.18)

If $A_0 - A_1 - 4 \equiv 0 \pmod{r}$ or $A_1 - A_0 - 4 \equiv 0 \pmod{r}$, then by Lemma 3.9 (4), $G_s(\beta^k)$, where $k \in qD_0^{(2p)} \cup qD_1^{(2p)}$, has $\frac{p - 1}{2}$ zeros. On the other hand, if $B_0 - B_1 - 4 \equiv 0 \pmod{r}$ or $B_1 - B_0 - 4 \equiv 0 \pmod{r}$, then by Lemma 3.9 (5), $G_s(\beta^k)$, where $k \in pD_0^{(2q)} \cup pD_1^{(2q)}$, has $\frac{q - 1}{2}$ zeros. From above discussion, (3.17) and (3.18), $G_s(\beta^k)$ has exactly $\frac{p - 1}{2} + \frac{q - 1}{2}$’s zeros, where $k \in qD_0^{(2p)} \cup qD_1^{(2p)} \cup pD_0^{(2q)} \cup pD_1^{(2q)}$. \[\square\]

**Corollary 3.2.** Let $2 \in D_1^{(p)}$, $2 \in D_1^{(q)}$, $(\frac{p}{r}) = 1$ and $(\frac{q}{r}) = 1$. In addition, let $r | p + 1$ and $r | q + 1$ if $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$, $r | p - 1$ and $r | q + 1$ if $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$ and, $r | p + 1$ and $r | q - 1$ if $p \equiv 3$.
Theorem 3.4. Let $2 \mid k$ if $r \equiv q \equiv \pm 1 \pmod{4}$. By Lemma 2.4, $|G_5(\beta^k)|$ is determined according to following cases: 

**Proof.** By Lemma 2.4, $A_0, A_1, B_0, B_1 \in \mathbf{GF}(r)$. By Lemma 2.4, $A_0, A_1, B_0, B_1 \in \{0, -1\}$, since $p, q \equiv \pm 1 \pmod{4}$. By Lemma 3.9 (3), $A_0B_0 + A_1B_1 = 1$ or $A_0B_1 + A_1B_0 = 1$. In other words, $G_5(\beta^k)$ has $\frac{(p-1)(q-1)}{2}$'s zeros, where $k \in 2D_0^{(T)} \cup 2D_1^{(T)}$. □

**Corollary 3.3.** Let $2 \in D_1^{(p)}, 2 \in D_2^{(q)}, (\frac{q}{p}) = 1 \text{ and } (\frac{p}{q}) = 1$. In addition, let $r \mid p + 16$ and $r \mid q + 16$ if $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$, $r \mid p - 16$ and $r \mid q + 16$ if $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$ and, $r \mid p + 16$ and $r \mid q - 16$ if $p \equiv 3 \pmod{8}$ and $q \equiv 5 \pmod{8}$. If $r = 5$, then $G_5(\beta^k)$ has at least $\frac{(p-1)(q-1)}{2} - 1$ plus $s$'s zeros, where $k \in \mathbf{Z}_{2T}$.

**Proof.** Remark that $5 \mid p + 16$ and $5 \mid q + 16$ imply $5 \mid p \pm 1$ and $5 \mid q \pm 1$. By Corollary 3.1 and 3.2, it follows the result of Corollary 3.3. □

**Corollary 3.4.** Let $2 \in D_1^{(p)}, 2 \in D_2^{(q)}, (\frac{q}{p}) = 1 \text{ and } (\frac{p}{q}) = 1$. In addition, let $r \mid p + 4$ and $r \mid q + 4$ if $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$, $r \mid p - 4$ and $r \mid q + 4$ if $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$ and, $r \mid p + 4$ and $r \mid q - 4$ if $p \equiv 3 \pmod{8}$ and $q \equiv 5 \pmod{8}$. Then $G_5(\beta^k)$ has $\frac{(p-1)(q-1)}{4}$'s zeros, where $k \in D_0^{(2T)} \cup D_1^{(2T)}$. 

**Proof.** By Lemma 2.4, $A_0, A_1, B_0, B_1 \in \mathbf{GF}(r)$. By Lemma 2.4 and using the same proof method as for Corollary 3.1, it can be deduced the following equations:

$$
\begin{cases}
\pm(A_0 - A_1) \equiv 2 \pmod{r}, \\
\pm(B_0 - B_1) \equiv 2 \pmod{r}.
\end{cases}
$$

Substitute the above equations into Lemma 3.9 (2), it leads to that in one of four sub-cases, $G_5(\beta^k) = 0$. In other words, there are $\frac{(p-1)(q-1)}{4}$'s zeros for $G_5(\beta^k)$ where $k \in D_0^{(2T)} \cup D_1^{(2T)}$. □

**Theorem 3.4.** Let $2 \in D_1^{(p)}, 2 \in D_2^{(q)}, (\frac{q}{p}) = 1 \text{ and } (\frac{p}{q}) = 1$. In addition, let $r \mid p + 16$ and $r \mid q + 16$ if $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$, $r \mid p - 16$ and $r \mid q + 16$ if $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$ and, $r \mid p + 16$ and $r \mid q - 16$ if $p \equiv 3 \pmod{8}$ and $q \equiv 5 \pmod{8}$. Then, the linear complexity over $\mathbf{GF}(r^m)$ of the generalized cyclotomic quaternary sequences with period $2T$ specified in (2.7) can be determined according to following cases:

1. if $(p, q) \in (3, 3)$ and $r \notin \{5, 13, 59, 127\}$ OR $(p, q) \in (3, -3) \cup (-3, 3)$ and $r \notin \{5, 43, 769\}$,

$$
\text{LC}(s) = 2pq - \frac{p + q}{2} + 1
$$

2. if $(p, q) \in (3, 3)$ and $r \in \{13, 59, 127\}$ OR $(p, q) \in (3, -3) \cup (-3, 3)$ and $r \in \{43, 769\}$,

$$
\text{LC}(s) = 2pq - \frac{p + q}{2}
$$

3. if $r = 5$,

$$
\text{LC}(s) = \frac{3pq + 1}{2}
$$
Remark 3.3. Follow the context in Theorem 3.4 and let $r \mid p + 16$ and $r \mid q + 16$ for the case where $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$. It is easy to check that if $r \in \{13, 59\}$ then $r \mid 3pq - 1$, and if $r = 127$ then $r \mid 2pq - 4$. By Lemma 3.9 (1), $r \mid 3pq - 1$ implies that $\beta^0 = 1$ is a zero of $G_s(\beta^k)$, and $r \mid 2pq - 4$ leads to that $\beta^pq$ is a zero of $G_s(\beta^k)$ as well, where $k \in Z_{2T}$.

Remark 3.4. Consider the context in Theorem 3.4. Let $r \mid p - 16$ and $r \mid q + 16$ if $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$, and $r \mid p + 16$ and $r \mid q - 16$ if $p \equiv 3 \pmod{8}$ and $q \equiv 5 \pmod{8}$. It is straightforward to verify that if $r = 769$ then $r \mid 3pq - 1$, and if $r = 43$ then $r \mid 2pq - 4$. Combining Remark 3.3 and the condition $r \not\in \{5, 13, 59, 127\}$ or $r \not\in \{5, 43, 769\}$, it prevents that $\beta^0 = 1$ and $\beta^pq$ are zeros to $G_s(\beta^k)$, where $k \in Z_{2T}$, and the case where $r = 5$.

Proof of Theorem 3.4. We prove each case of Theorem 3.4.

(1) By Remark 3.3 and Corollary 3.1, $G_s(\beta^k)$ has exactly $\frac{p-1}{2} + \frac{q-1}{2}$'s zeros, where $k \in Z_{2T}$. By (2.10),

$$LC(s) = 2pq - \frac{p-1}{2} - \frac{q-1}{2} = 2pq - \frac{p+q}{2} + 1.$$ 

(2) By Remark 3.3, either $\beta^0 = 1$ or $\beta^pq$ but not both is a zero. Hence, again by Corollary 3.1, $G_s(\beta^k)$ has exactly $\frac{p-1}{2} + \frac{q-1}{2} + 1$'s zeros, where $k \in Z_{2T}$. Hence,

$$LC(s) = 2pq - \frac{p-1}{2} - \frac{q-1}{2} - 1 = 2pq - \frac{p+q}{2}.$$ 

(3) By Corollary 3.3, $G_s(\beta^k)$ has exactly $\frac{p-1}{2} + \frac{q-1}{2} + \frac{(p-1)(q-1)}{2}$'s zeros, where $k \in Z_{2T}$. By (2.10),

$$LC(s) = 2pq - \frac{p-1}{2} - \frac{q-1}{2} - \frac{(p-1)(q-1)}{2} = \frac{3pq + 1}{2}.$$ 

\[ \square \]

Theorem 3.5. Let $2 \in D_1^{(p)}$, $2 \in D_1^{(q)}$, $\left(\frac{p}{q}\right) = 1$, and $\left(\frac{q}{p}\right) = 1$. In addition, let $r \mid p + 4$ and $r \mid q + 4$ if $p \equiv 3 \pmod{8}$ and $q \equiv 3 \pmod{8}$, $r \mid p - 4$ and $r \mid q + 4$ if $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$, and $r \mid p + 4$ and $r \mid q - 4$ if $p \equiv 3 \pmod{8}$ and $q \equiv 5 \pmod{8}$. Then, the linear complexity over $GF(r^m)$ of the generalized cyclotomic quaternary sequences with period $2T$ specified in (2.7) can be determined according to following cases:

(1) if $(p, q) \in \{(3, 3) \text{ OR } (3, -3) \cup (-3, 3) \text{ and } r \neq 7$,

$$LC(s) = \frac{7pq + p + q - 1}{4}.$$ 

(2) If $(p, q) \in \{(3, 3) \text{ OR } (3, -3) \cup (-3, 3) \text{ and } r = 7$,

$$LC(s) = \frac{7pq + p + q - 5}{4}.$$ 

Remark 3.5. In Theorem 3.3 for the case where $(p, q) \in \{(3, 3), (3, -3) \cup (-3, 3)$ and $r 
eq 7$, then

$$LC(s) = \frac{7pq + p + q - 1}{4}.$$ 

For the case
(p, q) ∈ (3, −3) ∪ (−3, 3), if r = 7, then r | 3pq − 1. That is, β₀ = 1 is a zero of \(G_s(\beta^k)\). Where \(k \in Z_{2T}\).

**Proof of Theorem 3.7.** Using Remark 3.5 and Corollary 3.4. □

**Theorem 3.6.** Let \(2 \in D_1^{(p)}, 2 \in D_1^{(q)}, \left(\frac{2}{p}\right) = 1\) and \(\left(\frac{2}{q}\right) = 1\). In addition, let \(r | p + 1\) and \(r | q + 1\) if \(p \equiv 3 \pmod{8}\) and \(q \equiv 3 \pmod{8}\), \(r | p − 1\) and \(r | q + 1\) if \(p \equiv 5 \pmod{8}\) and \(q \equiv 3 \pmod{8}\) and, \(r | p + 1\) and \(r | q − 1\) if \(p \equiv 3 \pmod{8}\) and \(q \equiv 5 \pmod{8}\). Then, the linear complexity over \(GF(r^m)\) of the generalized cyclotomic quaternary sequences with period \(2T\) specified in (2.7) can be determined according to following cases:

1. if \((p, q) \in (3, 3)\) and \(r \not= 5\) OR \((p, q) \in (3, −3) \cup (−3, 3)\) and \(r \not= 5\), then

\[
LC(s) = \frac{3pq + p + q − 1}{2}
\]

2. If \(r = 5\), then

\[
LC(s) = \frac{3pq + 1}{2}
\]

**Proof.** Using Corollary 3.2 and 3.3. □

**Theorem 3.7.** Let \(2 \in D_1^{(p)}, 2 \in D_1^{(q)}, \left(\frac{2}{p}\right) = 1\) and \(\left(\frac{2}{q}\right) = −1\). In addition, let \(r | p + 16\) if \(p \equiv 3 \pmod{8}\), \(r | p − 16\) if \(p \equiv 5 \pmod{8}\). Then, the linear complexity over \(GF(r^m)\) of the generalized cyclotomic quaternary sequences with period \(2T\) specified in (2.7) can be determined according to following cases:

1. If \((p, q) \in (3, ±3)\) and \(r | 48q + 1\) and \(r | 8q + 1\) OR \((p, q) \in (−3, 3)\) and \(r | 48q − 1\) and \(r | 8q − 1\), then

\[
LC(s) = 2pq − \frac{p − 1}{2}
\]

2. If \((p, q) \in (3, ±3)\) and \((r | 48q + 1) OR (r | 8q + 1)\) and \(r \not= 5\) OR \((p, q) \in (−3, 3)\) and \((r | 48q − 1) OR (r | 8q − 1)\) and \(r \not= 5\), then

\[
LC(s) = 2pq − \frac{p − 1}{2} − 1.
\]

3. If \((p, q) \in (3, ±3)\) and \((r | 48q + 1) OR (r | 8q + 1)\) and \(r = 5\) OR \((p, q) \in (−3, 3)\) and \((r | 48q − 1) OR (r | 8q − 1)\) and \(r = 5\), then

\[
LC(s) = 2pq − \frac{p − 1}{2} − 2.
\]

**Remark 3.6.** In Theorem 3.7, the condition \(r | 48q + 1\) or \(r | 8q + 1\) for the case where \(p \equiv 3 \pmod{8}\) and \(r | p + 16\), and the one \(r | 48q − 1\) or \(r | 8q − 1\) for the case where \(p \equiv 5 \pmod{8}\) and \(r | p − 16\) imply that by Lemma 3.1 (1), \(β^0 = 1\) or \(β^pψ\) is a zero of \(G_s(β^k)\) for \(k \in Z_{2T}\), respectively. It is obvious that if \(r = 5\), then \(r | 48q + 1\) implies \(r | 8q + 1\), vice versa, and \(r | 48q − 1\) implies \(r | 8q − 1\), vice versa.

**Proof of Theorem 3.7.** Using Corollary 3.1 and Remark 3.6. □

**Example 3.2.** We give some pairs of \((p, q)\) which match the condition that is \((p, q) \in (3, 3)\) and \((r | 48q + 1) OR (r | 8q + 1)\) and \(r = 5\), and compute the linear
complexity both by Algorithm [1] and the formula in Theorem 3.7 (3). We found
that numerical and theoretical results are identical.

\[(p, q) = \]

\[
\{(19, 83), (19, 443), (59, 43), (59, 83), \\
(59, 163), (59, 283), (59, 443), (139, 83), \\
(139, 443), (179, 43), (179, 83), (179, 163), \\
(179, 283), (179, 443), (379, 83), (379, 443), \\
(419, 43), (419, 83), (419, 163), (419, 283), \cdots \}.
\]

For example, for all

\[(p, q) \in \{(59, 43), (59, 83), (59, 163), (59, 283), (59, 443)\},
\]

\(G_s(\beta^k)\) has same number of zeros which equals \(\frac{p-1}{2}+2 = 31\), where \(k \in \mathbb{Z}_{2T}\), but for
each pair of \((p, q)\), the linear complexity is different, that is equal to \(2pq - \frac{p-1}{2} = 2pq - 31\). For instance, if \(p = 59\) and \(q = 43\), then \(LC(s) = 5043\).

### 3.3. 2 is a quadratic non-residue modulo \(p\) and a quadratic residue modulo \(q\).

**Lemma 3.10.** Let \(2 \in D_0^{(p)}\) and \(2 \in D_0^{(q)}\). For \(k \in \mathbb{Z}_{2T}\), let \(k_1 \equiv k \) (mod \(p\))
and \(k_2 \equiv k \) (mod \(q\)). Then, the values of \(G_s(\beta^k)\) can be determined according to
various range of \(k\).

1. \(k \in \{0, pq\} \cup 2qZ_p^* \cup pZ_2^*\).

\[
G_s(\beta^k) = \begin{cases} 
3pq - 1 & k \in \{0\}, \\
2pq - 4 & k \in \{pq\}, \\
-1 & k \in 2qZ_p^*, \\
-4 & k \in pZ_2^*.
\end{cases}
\]

2. \(k \in D_0^{(2T)} \cup D_1^{(2T)}\).

(a) If \(k \in D_0^{(2T)}\) and \((k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}\) and \(\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)\) or \(k \in D_1^{(2T)}\)
and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)}\) and \(\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)\), then

\[G_s(\beta^k) = (A_0 - A_1)(B_0 - B_1 + 1) - 4.\]

(b) If \(k \in D_0^{(2T)}\) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)}\) and \(\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)\) or \(k \in D_1^{(2T)}\)
and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)}\) and \(\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)\), then

\[G_s(\beta^k) = (A_0 - A_1)(B_0 - B_1 - 1) - 4.\]

(c) If \(k \in D_1^{(2T)}\) and \((k_1, k_2) \in D_0^{(p)} \times D_1^{(q)}\) and \(\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)\) or \(k \in D_0^{(2T)}\)
and \((k_1, k_2) \in D_0^{(p)} \times D_0^{(q)}\) and \(\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)\), then

\[G_s(\beta^k) = (A_1 - A_0)(B_0 - B_1 - 1) - 4.\]
(d) If \( k \in D_1^{(2T)} \) and \((k_1, k_2) \in D_1^{(p)} \times D_0^{(q)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{p}{q} \right) \) or \( k \in D_0^{(2T)} \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)} \) and \( \left( \frac{a}{p} \right) = -\left( \frac{p}{q} \right) \), then

\[
G_s(\beta^k) = (A_1 - A_0)(B_0 - B_1 + 1) - 4.
\]

(3) \( k \in 2D_0^{(T)} \cup 2D_1^{(T)} \).

(a) If \( k \in 2Z_1^* \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)} \cup D_0^{(p)} \times D_1^{(q)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{p}{q} \right) \) or \( k \in 2Z_1^* \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)} \cup D_0^{(p)} \times D_1^{(q)} \) and \( \left( \frac{a}{p} \right) = -\left( \frac{p}{q} \right) \), then

\[
G_s(\beta^k) = 2B_0.
\]

(b) If \( k \in 2Z_1^* \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)} \cup D_0^{(p)} \times D_0^{(q)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{p}{q} \right) \) or \( k \in 2Z_1^* \) and \((k_1, k_2) \in D_1^{(p)} \times D_1^{(q)} \cup D_0^{(p)} \times D_0^{(q)} \) and \( \left( \frac{a}{p} \right) = -\left( \frac{p}{q} \right) \), then

\[
G_s(\beta^k) = 2B_1.
\]

(4) \( k \in qZ_{2p}^* \).

(a) If \( k_1 \in D_1^{(p)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = 1 \) or \( k_1 \in D_0^{(p)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = -1 \), then \( k_1 \in D_1^{(p)} \) and \( \left( \frac{a}{p} \right) = -\left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = 1 \) or \( k_1 \in D_0^{(p)} \) and \( \left( \frac{a}{p} \right) = -\left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = -1 \), then

\[
G_s(\beta^k) = A_0 - A_1 - 4.
\]

(b) If \( k_1 \in D_0^{(p)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = 1 \) or \( k_1 \in D_0^{(p)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = -1 \), then \( k_1 \in D_0^{(p)} \) and \( \left( \frac{a}{p} \right) = -\left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = -1 \), then

\[
G_s(\beta^k) = A_0 - A_0 - 4.
\]

(5) \( k \in 2pZ_{q}^* \).

(a) If \( k_2 \in D_1^{(q)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = 1 \) or \( k_2 \in D_0^{(q)} \) and \( \left( \frac{a}{p} \right) = \left( \frac{q}{p} \right) \) and \( \left( \frac{a}{p} \right) = -1 \), then

\[
G_s(\beta^k) = 2B_0.
\]
(b) If $k_2 \in D_0^{(q)}$ and \( \left( \frac{q}{p} \right) = \left( \frac{p}{q} \right) = 1 \) or $k_2 \in D_0^{(q)}$ and \( \left( \frac{q}{p} \right) = -\left( \frac{p}{q} \right) = -1 \) or $k_2 \in D_1^{(q)}$ and \( \left( \frac{q}{p} \right) = -\left( \frac{p}{q} \right) = 1 \) or $k_2 \in D_1^{(q)}$ and \( \left( \frac{q}{p} \right) = -\left( \frac{p}{q} \right) = -1 \), then

\[
G_s(\beta^k) = 2B_1.
\]

**Proof.** Similar to the proof of Lemma 3.8.

**Corollary 3.5.** Let $2 \in D_1^{(q)}$, $2 \in D_0^{(q)}$, \( \left( \frac{q}{p} \right) = 1 \) and \( \left( \frac{p}{q} \right) = 1 \). In addition, let $r \mid p \pm 4$ if $p = \pm 3 \pmod{8}$, and $r \mid q \pm 1$ if $q = \mp 1 \pmod{8}$. Then, $G_s(\beta^k)$ has exactly $\frac{3(p-1)(q-1)}{4} + \frac{q-1}{2}$ zeros, where $k \in Z_{2T}^* \cup 2Z_T^* \cup 2pZ_q^*$.

**Proof.** As for the proof of Corollary 3.1 from the conditions given in the actual Lemma, we can obtain $A_0 - A_1 = \pm 2$ and $B_0, B_1 \in \{0, -1\}$ or $B_0 - B_1 = \pm 1$. From Lemma 3.10 (2), $G_s(\beta^k) = 0$ for $k \in Z_{2T}^*$ in one of the four sub-cases, that contributes $\frac{3(p-1)(q-1)}{4}$ zeros. By the same way, we get $\frac{q-1}{2}$ zeros for $k \in Z_{2T}^*$ from Lemma 3.11 (3), and $\frac{q-1}{2}$ zeros for $k \in 2pZ_q^*$ from Lemma 3.11 (5), of $G_s(\beta^k)$. Hence, the number of zeros of $G_s(\beta^k)$ where $k \in Z_{2T}^* \cup 2Z_T^* \cup 2pZ_q^*$ equals

\[
\frac{3(p-1)(q-1)}{4} + \frac{q-1}{2}.
\]

**Corollary 3.6.** Let $2 \in D_1^{(q)}$, $2 \in D_0^{(q)}$, \( \left( \frac{q}{p} \right) = 1 \) and \( \left( \frac{p}{q} \right) = 1 \). In addition, let $r \mid p \pm 16$ if $p = \pm 3 \pmod{8}$, and $r \mid q \pm 1$ if $q = \mp 1 \pmod{8}$. Then, $G_s(\beta^k)$ has exactly $\frac{3(p-1)(q-1)}{4} + \frac{q-1}{2}$ zeros, where $k \in qZ_{2p}^* \cup 2Z_T^* \cup 2Z_q^* \cup 2pZ_q^*$.

**Proof.** By Lemma 2.4, $A_0, A_1, B_0, B_1 \in \text{GF}(r)$. By Lemma 2.1 and referring to the proof of Corollary 3.1, we obtain $A_0 - A_1 = \pm 4$ and $B_0, B_1 \in \{0, -1\}$ or $B_0 - B_1 = \pm 1$. Looking at Lemma 3.11 (4) with $A_0 - A_1 = \pm 4$, we remark that $G_s(\beta^k) = 0$ for $k \in qZ_{2p}^*$ in one of the two sub-cases, that contributes $\frac{q-1}{2}$ zeros. On the other hand, the condition $B_0, B_1 \in \{0, -1\}$ leads to $\frac{3(p-1)(q-1)}{4} + \frac{q-1}{2}$ zeros as proved in Corollary 3.5. Hence, the number of zeros of $G_s(\beta^k)$ where $k \in qZ_{2p}^* \cup 2Z_T^* \cup 2pZ_q^*$ equals

\[
\frac{3(p-1)(q-1)}{4} + \frac{p-1}{2} + \frac{q-1}{2}.
\]

**Corollary 3.7.** Let $2 \in D_1^{(q)}$, $2 \in D_0^{(q)}$, \( \left( \frac{q}{p} \right) = 1 \) and \( \left( \frac{p}{q} \right) = -1 \). In addition, let $r \mid p \pm 16$ if $p = \pm 3 \pmod{8}$. Then, $G_s(\beta^k)$ has exactly $\frac{q-1}{2}$ zeros, where $k \in qZ_{2p}^*$.

**Proof.** Since \( \left( \frac{q}{p} \right) = 1 \) and \( \left( \frac{p}{q} \right) = -1 \), by Lemma 2.4, $A_0, A_1 \in \text{GF}(r)$ and $B_0, B_1 \in \text{GF}(r^m) \setminus \text{GF}(r)$. The rest of the proof is similar to that of Corollary 3.6.
Corollary 3.8. Let $2 \in D_{1}^{(p)}, 2 \in D_{0}^{(q)}, (\frac{q}{p}) = -1$ and $(\frac{p}{q}) = 1$. In addition, let $r \mid q \pm 1$ if $q \equiv \mp 1 \pmod{8}$. Then, $G_{s}(\beta^{k})$ has exactly $(p-1)(q-1) + \frac{q-1}{2}$’s zeros, where $k \in 2Z_{T}^{*} \cup 2pZ_{q}^{*}$.

Proof. See the proof for Corollaries 3.5, 3.7. □

Corollary 3.9. Let $(\frac{p}{q}) = -1$ and $(\frac{q}{p}) = -1$. Then, there are no $r$ with $r \geq 5$ such that $r \mid 3pq - 1$ and $r \mid 2pq - 4$.

Proof. The conditions $r \mid 3pq - 1$ and $r \mid 2pq - 4$ imply that $r = 5$. But it is impossible that $r = 5$ occurs. If $r = 5$, by the Quadratic Reciprocity Law in Lemma 2.3 (2), $(\frac{p}{5}) = (\frac{q}{5}) = (\frac{q}{q}) = -(\frac{q}{q}) = -1$. With $r \mid 3pq - 1$, it leads to $3pq \equiv 1 \pmod{5}$. Hence, $(\frac{3pq}{5}) = (\frac{3}{5}) \cdot (\frac{p}{5}) \cdot (\frac{q}{5}) = -1 = (\frac{q}{q}) = 1$, that is a contradiction. □

Theorem 3.8. Let $2 \in D_{1}^{(p)}, 2 \in D_{0}^{(q)}, (\frac{p}{q}) = 1$ and $(\frac{q}{p}) = 1$. In addition, let $r \mid p \pm 4$ if $p \equiv \pm 3 \pmod{8}$, and $r \mid q \pm 1$ if $q \equiv \mp 1 \pmod{8}$. Then, the linear complexity over $\mathbb{GF}(r^{m})$ of the generalized cyclotomic quaternary sequences with period $2T$ specified in (2.7) can be determined according to following cases:

1. if $(p, q) \in (3, 1) \cup (-3, -1)$ and $r \neq 13$ OR $(p, q) \in (3, -1)$ and $r \neq 11$ , then
   \[
   LC(s) = \frac{5pq + 3p + q - 1}{4}
   \]

2. If $(p, q) \in (3, 1) \cup (-3, -1)$ and $r = 13$ OR $(p, q) \in (3, -1)$ and $r = 11$, then
   \[
   LC(s) = \frac{5pq + 3p + q - 5}{4}
   \]

Remark 3.7. We consider two particular conditions that are $r = 13$ for the case where $(p, q) \in (3, 1) \cup (-3, -1)$, and $r = 11$ for the case where $(p, q) \in (3, -1)$ in Theorem 3.8, respectively. It is easy to check that, for the two cases if $r = 13$ and $r = 11$ respectively, then $r \mid 3pq - 1$. In other words, by Lemma 3.9 (1), $\beta^{0} = 1$ is a zero for $G_{s}(\beta^{k})$ where $k \in Z_{2T}$.

Proof of Theorem 3.8. Using Corollary 3.6 and (2.10) to prove Theorem 3.8 (1), and Corollary 3.3, (2.10), and Remark 3.7 to prove Theorem 3.8 (2). □

Theorem 3.9. Let $2 \in D_{1}^{(p)}, 2 \in D_{0}^{(q)}, (\frac{p}{q}) = 1$ and $(\frac{q}{p}) = 1$. In addition, let $r \mid p \pm 16$ if $p \equiv \pm 3 \pmod{8}$, and $r \mid q \pm 1$ if $q \equiv \mp 1 \pmod{8}$. Then, the linear complexity over $\mathbb{GF}(r^{m})$ of the generalized cyclotomic quaternary sequences with period $2T$ specified in (2.7) can be determined according to following cases:

1. if $(p, q) \in (3, 1) \cup (-3, -1)$ and $r \neq 7$ OR $(p, q) \in (3, -1)$ and $r \notin \{7, 47\}$ , then
   \[
   LC(s) = \frac{3pq + 1}{2}
   \]

2. If $(p, q) \in (3, 1) \cup (-3, -1)$ and $r = 7$ OR $(p, q) \in (3, -1)$ and $r \in \{7, 47\}$ , then
   \[
   LC(s) = \frac{3pq - 1}{2}
   \]

Remark 3.8. In Theorem 3.9, $r = 7$ implies that $\beta^{0} = 1$ is a zero of $G_{s}(\beta^{k})$ where $k \in Z_{2T}$ for $(p, q) \in (3, 1) \cup (-3, -1)$. While for $(p, q) \in (3, -1)$, $r = 47$ implicates that $\beta^{0} = 1$ is a zero, and $r = 7$ implies that $\beta^{pq}$ is a zero, of $G_{s}(\beta^{k})$ where $k \in Z_{2T}$.
Proof of Theorem 3.9. Using Corollary 3.6 and (2.10) to prove Theorem 3.9 (1), and Corollary 3.6 (2.10), and Remark 3.8 to prove Theorem 3.9 (2). □

Theorem 3.10. Let $2 \in D_1^{(p)}$, $2 \in D_0^{(q)}$, $(\frac{r}{p}) = 1$ and $(\frac{r}{q}) = -1$. In addition, let $r \mid p \pm 16$ if $p \equiv \pm 3 \pmod{8}$. Then, the linear complexity over $\mathbf{GF}(r^m)$ of the generalized cyclotomic quaternary sequences with period $2T$ given in (2.7) can be determined according to following cases:

1. If $(p, q) \in (3, \pm 1)$ and $r \nmid 48q + 1$ and $r \nmid 8q + 1$ OR $(p, q) \in (-3, -1)$ and $r \nmid 48q - 1$ and $r \nmid 8q - 1$, then
   \[
   LC(s) = \frac{4pq - p + 1}{2}.
   \]

2. If $(p, q) \in (3, \pm 1)$ and $(r \mid 48q + 1$ or $r \mid 8q + 1)$ and $r \not= 5$ OR $(p, q) \in (-3, -1)$ and $(r \mid 48q - 1$ or $r \mid 8q - 1)$ and $r \not= 5$, then
   \[
   LC(s) = \frac{4pq - p - 1}{2}.
   \]

3. If $(p, q) \in (3, \pm 1)$ and $(r \mid 48q + 1$ or $r \mid 8q + 1)$ and $r = 5$ OR $(p, q) \in (-3, -1)$ and $(r \mid 48q - 1$ or $r \mid 8q - 1)$ and $r = 5$, then
   \[
   LC(s) = \frac{4pq - p - 3}{2}.
   \]

Remark 3.9. In Theorem 3.10 for $(p, q) \in (3, \pm 1)$, $r \mid 48q + 1$ and $r \mid 8q + 1$ imply that $\beta^0 = 1$ and $\beta^{pq}$ are zeros of $G_s(\beta^k)$ where $k \in \mathbb{Z}_{2T}$, respectively. While for $(p, q) \in (-3, -1)$, $\beta^0 = 1$ is a zero of $G_s(\beta^k)$ if only if $r \mid 48q - 1$, and $\beta^{pq}$ is a zero of $G_s(\beta^k)$ if only if $r \mid 8q - 1$. It is obvious that if $r = 5$, then $r \mid 48q \pm 1$ implicate $r \mid 8q \pm 1$, vice versa.

Proof of Theorem 3.10. Using Corollary 3.7 and (2.10) to prove Theorem 3.9 (1), and Corollary 3.7 (2.10), and Remark 3.9 to prove Theorem 3.10 (2), and (3). □

Theorem 3.11. Let $2 \in D_1^{(p)}$, $2 \in D_0^{(q)}$, $(\frac{r}{p}) = -1$ and $(\frac{r}{q}) = 1$. In addition, let $r \mid q \pm 1$ if $q \equiv \mp 1 \pmod{8}$. Then, the linear complexity over $\mathbf{GF}(r^m)$ of the generalized cyclotomic quaternary sequences with period $2T$ given in (2.7) can be determined according to following cases:

1. If $(p, q) \in (3, 1)$ and $r \nmid 3p - 1$ and $r \nmid 2p - 4$ OR $(p, q) \in (\pm 3, -1)$ and $r \nmid 3p + 1$ and $r \nmid 2p + 4$, then
   \[
   LC(s) = \frac{3pq + p}{2}.
   \]

2. If $(p, q) \in (3, 1)$ and $(r \mid 3p - 1$ or $r \mid 2p - 4)$ and $r \not= 5$ OR $(p, q) \in (\pm 3, -1)$ and $(r \mid 3p + 1$ or $r \mid 2p + 4)$ and $r \not= 5$, then
   \[
   LC(s) = \frac{3pq + p - 2}{2}.
   \]

3. If $(p, q) \in (3, 1)$ and $(r \mid 3p - 1$ or $r \mid 2p - 4)$ and $r = 5$ OR $(p, q) \in (\pm 3, -1)$ and $(r \mid 3p + 1$ or $r \mid 2p + 4)$ and $r = 5$, then
   \[
   LC(s) = \frac{3pq + p - 4}{2}.
   \]
Remark 3.10. In Theorem 3.11, the conditions $r \mid 3p+1$ and $r \mid 2p+4$ signify that \( \beta^0 = 1 \) and \( \beta^{pq} \) are zeros of \( G_s(\beta^k) \) for \( k \in \mathbb{Z}_{2T} \) if \( (p,q) \in \{(\pm 3,-1)\} \), respectively. If \( (p,q) \in \{(3,1)\} \), the same conditions change into $r \mid 3p-1$ and $r \mid 2p-4$, respectively. It is clear that if \( r = 5 \), then $r \mid 3p \pm 1$ implicate $r \mid 2p \pm 4$, vice versa.

Proof of Theorem 3.11. Using Corollary 3.8 and (2.10) to prove Theorem 3.11 (1), and Corollary 3.8, (2.10), and Remark 3.10 to prove Theorem 3.11 (2) and (3). □

Theorem 3.12. Let $\left(\frac{r}{p}\right) = -1$ and $\left(\frac{r}{q}\right) = -1$. Then, the linear complexity over $\mathbf{GF}(r^m)$ of the generalized cyclotomic quaternary sequences with period $2T$ specified in (2.7) can be determined according to following cases:

1. if $r \nmid 3pq - 1$ and $r \nmid 2pq - 4$, then $LC(s) = 2pq$.
2. If $r \mid 3pq - 1$ or $r \mid 2pq - 4$, then $LC(s) = 2pq - 1$.

Proof. Since $\left(\frac{r}{p}\right) = -1$ and $\left(\frac{r}{q}\right) = -1$, by Lemma 2.4, \( A_0, A_1, B_0, B_1 \in \mathbf{GF}(r^m) \setminus \mathbf{GF}(r) \). From Lemma 3.9, \( G_s(\beta^k) \) has no zeros for \( k \in \mathbb{Z}_{2T} \), except for two possible zeros which are \( \beta^0 = 1 \) and \( \beta^{pq} \), respectively. By Corollary 3.9 no $r$ such that both $r \mid 3pq - 1$ and $r \mid 2pq - 4$ hold together. □

4. Conclusion

In this paper, we computed out the linear complexity over $\mathbf{GF}(r)$ of the generalized cyclotomic quaternary sequences of length $2T$ specified in (2.7). The results show that the minimal value of the linear complexity is equal to $\frac{5pq+p+q+1}{4}$ which is greater than $T$, the half of the period of the sequences. According to the Berlekamp-Massey algorithm [5], these sequences are viewed as enough good for the use in cryptography. Remark that if the character of the extension field $\mathbf{GF}(r^m)$, $r$, is chosen so that \( r \geq 5 \), \( r \neq p,q \), (\( \frac{r}{p}\) = (\( \frac{r}{q}\)) = -1, \( r \mid 3pq - 1 \), and \( r \mid 2pq - 4 \), then by Theorem 3.12, the linear complexity can reach the maximal value which is the length of the sequences, $2T$.

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