Bridging Distributional and Risk-sensitive Reinforcement Learning with Provable Regret Bounds

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Abstract
We study the regret guarantee for risk-sensitive reinforcement learning (RSRL) via distributional reinforcement learning (DRL) methods. In particular, we consider finite episodic Markov decision processes whose objective is the entropic risk measure (EntRM) of return. We identify a key property of the EntRM, the monotonicity-preserving property, which enables the risk-sensitive distributional dynamic programming framework. We then propose two novel DRL algorithms that implement optimism through two different schemes, including a model-free one and a model-based one.

We prove that both of them attain $O\left(\exp\left(\frac{\beta H}{6}\right)^{-1} H \sqrt{HS^2AT}\right)$ regret upper bound, where $S$ is the number of states, $A$ is the number of actions, $H$ is the time horizon, and $T$ is the total number of time steps. It matches RSVI2, proposed in Fei et al. (2021), with a much simpler regret analysis. To the best of our knowledge, this is the first regret analysis of DRL that bridges DRL and RSRL in terms of sample complexity. Finally, we improve the existing lower bound by proving a tighter bound of $\Omega\left(\exp\left(\frac{\beta H}{6}\right)^{-1} H \sqrt{SAT}\right)$ for the $\beta > 0$ case, which recovers the tight lower bound $\Omega(H \sqrt{SAT})$ in the risk-neutral setting.

Keywords: distributional reinforcement learning, risk-sensitive reinforcement learning, regret bounds, episodic MDP, entropic risk measure

1. Introduction
Standard reinforcement learning (RL) seeks to find an optimal policy that maximizes the expected return (Sutton and Barto, 2018). This approach is often referred to as risk-neutral RL, as it focuses on the mean functional of the return distribution. However, in high-stakes applications, such as finance (Davis and Lleo, 2008; Bielecki et al., 2000), medical treatment (Ernst et al., 2006), and operations research (Delage and Mannor, 2010), decision-makers are often risk-sensitive and aim to maximize a risk measure of the return distribution.

Since the pioneering work of Howard and Matheson (1972), risk-sensitive reinforcement learning (RSRL) based on the exponential risk measure (EntRM) has been applied to a wide range of domains (Shen et al., 2014; Nass et al., 2019; Hansen and Sargent, 2011). EntRM offers a trade-off between the expected return and its variance and allows for the adjustment of risk-sensitivity through a risk parameter (see Equation 1). However, existing approaches typically require complicated algorithmic designs to handle the non-linearity of EntRM.
Distributional reinforcement learning (DRL) has demonstrated superior performance over traditional methods in some challenging tasks under a risk-neutral setting (Bellemare et al., 2017; Dabney et al., 2018b,a). Unlike value-based approaches, DRL learns the entire return distribution instead of a real-valued value function. Given the distributional information, it is natural to leverage it to optimize a risk measure other than expectation (Dabney et al., 2018a; Singh et al., 2020; Ma et al., 2020). Despite the intrinsic connection between DRL and RSRL, existing works on RSRL via DRL approaches lack regret analysis (Dabney et al., 2018a; Ma et al., 2021; Achab and Neu, 2021). Consequently, it is challenging to evaluate and improve these DRL algorithms in terms of sample efficiency, leading to a reasonable question:

\textit{Can distributional reinforcement learning methods achieve near-optimal regret with a risk-sensitive purpose?}

In this work, we answer this question positively by providing two DRL algorithms with regret upper bounds. We devise two novel DRL algorithms with principled exploration schemes for risk-sensitive control in the tabular MDP setting. In particular, the proposed algorithms implement the principle of optimism in the face of uncertainty (OFU) at the distributional level to balance the exploration-exploitation trade-off. By providing the first regret analysis of DRL, we theoretically verify the efficacy of DRL for RSRL. Therefore, our work bridges the gap between DRL and RSRL with regard to sample complexity.

1.1 Related Work

Related work in DRL has rapidly grown since Bellemare et al. (2017), with numerous studies aiming to improve performance in the risk-neutral setting (see Rowland et al., 2018; Dabney et al., 2018b,a; Barth-Maron et al., 2018; Yang et al., 2019; Lyle et al., 2019; Zhang et al., 2021). However, only a few works have considered risk-sensitive behavior, including Dabney et al. (2018a); Ma et al. (2021); Achab and Neu (2021). None of these works have addressed sample complexity.

A large body of work has investigated RSRL using the EntRM in various settings (Borkar, 2001, 2002; Borkar and Meyn, 2002; Borkar, 2010; Bäuerle and Rieder, 2014; Di Masi et al., 2000; Di Masi and Stettner, 2007; Cavazos-Cadena and Hernández-Hernández, 2011; Jaśkiewicz, 2007; Ma et al., 2020; Mihatsch and Neumeier, 2002; Osogami, 2012; Patek, 2001; Shen et al., 2013, 2014). However, these works either assume known transition and reward or consider infinite-horizon settings without considering sample complexity.

Our work is related to two recent studies by Fei et al. (2020) and Fei et al. (2021) in the same setting. Fei et al. (2020) introduced the first regret-guaranteed algorithms for risk-sensitive episodic Markov decision processes (MDPs), but their regret upper bounds contain an unnecessary factor of \(\exp(|\beta|H^2)\) and their lower bound proof contains errors, leading to a weaker bound. Fei et al. (2021) improved on their algorithm by removing the \(\exp(|\beta|H^2)\) factor, but their regret analysis is complicated and their lower bound is not fixed. More recently, Achab and Neu (2021) proposed a risk-sensitive deep deterministic policy gradient framework, but their work is fundamentally different from ours as they consider the conditional value at risk (CVaR) and focus on infinite-horizon settings. For a detailed comparison with their work, please refer to Appendix A.
1.2 Contributions

These are the main contributions of the paper:

- The development of a risk-sensitive distributional dynamic programming (RS-DDP) framework, establishing a distributional Bellman optimality equation under risk-sensitive settings by utilizing the monotonicity-preserving property of the Entropy Risk Measure (EntRM).

- The proposal of two deep reinforcement learning (DRL) algorithms that enforce the Optimism in the Face of Uncertainty (OFU) principle in a distributional fashion through different schemes, along with regret upper bounds of $\tilde{O}(\frac{\exp(\beta H)}{\beta H}H\sqrt{SAK})$ for both algorithms. This is the first analysis of a DRL algorithm in a finite episodic Markov Decision Process (MDP) in the risk-sensitive setting. The proposed algorithms are simpler and more interpretable than previous work by Fei et al. (2021).

- The filling of gaps in the proof of the lower bound in Fei et al. (2020), resulting in a tight lower bound of $\Omega(\frac{\exp(\beta H)}{\beta H}H\sqrt{SAT})$ for $\beta > 0$. This lower bound is independent of $S$ and $A$ and recovers the tight lower bound in the risk-neutral setting ($\beta \to 0$).

2. Preliminaries

We provide the technical background in this section.

2.1 Notations

We write $[M : N] \triangleq \{M, M + 1, ..., N\}$ and $[N] \triangleq [1 : N]$ for any positive integers $M \leq N$. We adopt the convention that $\sum_{i=m}^{n} a_i \triangleq 0$ if $n < m$ and $\prod_{i=m}^{n} a_i \triangleq 1$ if $n < m$. We use $\mathbb{I}\{\cdot\}$ to denote the indicator function. For any $x \in \mathbb{R}$, we define $[x]^+ \triangleq \max\{x, 0\}$. We define the step function with parameter $c$ as $\psi_c(x) \triangleq \mathbb{I}\{x \geq c\}$. Note that $\psi_c$ represents the CDF corresponding to a deterministic variable taking value $c$. We denote by $\mathcal{D}([a, b])$, $\mathcal{D}_M$ and $\mathcal{D}$ the set of distributions supported on $[a, b]$, $[0, M]$ and the set of all distributions respectively. For a random variable (r.v.) $X$, we use $\mathbb{E}[X]$ and $\mathbb{V}[X]$ to denote its expectation and variance. We use $\mathcal{O}(\cdot)$ to denote $O(\cdot)$ omitting logarithmic factors.

2.2 Episodic MDP

An episodic MDP is identified by $\mathcal{M} \triangleq (\mathcal{S}, \mathcal{A}, (P_h)_{h \in [H]}, (r_h)_{h \in [H]}, H)$, where $\mathcal{S}$ is the state space, $\mathcal{A}$ the action space, $P_h : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$ the probability transition kernel at step $h$, $r_h : \mathcal{S} \times \mathcal{A} \to \mathcal{D}([0, 1])$ the collection of reward functions at step $h$ and $H$ the length of one episode. The agent interacts with the environment for $K$ episodes. At the beginning of episode $k$, Nature selects an initial state $s^k_1$ arbitrarily. In step $h$, the agent takes action $a^k_h$ and observes random reward $r^k_h(s^k_h, a^k_h)$ and reaches the next state $s^k_{h+1} \sim P_h(\cdot|s^k_h, a^k_h)$. The episode terminates at $H + 1$ with $r_{H+1} = 0$, then the agent proceeds to next episode.

For each $(k, h) \in [K] \times [H]$, we denote by $\mathcal{H}_h^k \triangleq (s^1_h, a^1_h, s^2_h, a^2_h, \ldots, s^1_h, a^1_h, \ldots, s^k_h, a^k_h)$ the (random) history up to step $h$ episode $k$. We define $\mathcal{F}_k \triangleq \mathcal{H}_h^{k-1}$ as the history up to episode $k - 1$. We describe the interaction between the algorithm and MDP in two levels.
In the level of episode, we define an algorithm as a sequence of functions $A = (A_k)_{k \in [K]}$, each mapping $F_k$ to a policy $A_k(F_k) \in \Pi$. We denote by $\pi^k = A_k(F_k)$ the policy at episode $k$. In the level of step, a (deterministic) policy $\pi$ is a sequence of functions $\pi = (\pi_h)_{h \in [H]}$ with $\pi_h : S \rightarrow A$.

2.3 Risk Measure

For any two random variables $X \sim F$ and $Y \sim G$, we say that $Y$ dominates $X$ as well as $G$ dominates $F$ if $\forall x \in \mathbb{R}, F(x) \geq G(x)$, and we write $Y \succeq X$ as well as $G \succeq F$. A risk measure $T$ is a functional mapping from a set of random variables $\mathcal{X}$ to the reals that satisfies certain properties of the following:

- (M) Monotonicity: $X \preceq Y \Rightarrow T(X) \leq T(Y)$, $\forall X, Y \in \mathcal{X}$,
- (T) Translation-invariance: $T(X + c) = T(X) + c$, $\forall X \in \mathcal{X}$, $\forall c \in \mathbb{R}$,
- (C) Convexity: $T(\theta X_1 + (1 - \theta) X_2) \leq \theta T(X_1) + (1 - \theta) T(X_2)$, $\forall X_1, X_2 \in \mathcal{X}$, $\forall \theta \in [0, 1]$,
- (D) Distribution-invariance: $F_{X_1} = F_{X_2} \Rightarrow T(X_1) = T(X_2)$.

A mapping $T : \mathcal{X} \rightarrow \mathbb{R}$ is called a risk measure if it satisfies the properties (M) and (T). A distribution-invariant risk measure is a risk measure satisfying the property (D). In this paper we only consider distribution-invariant risk measures. For this reason we overload notations and write $T(F_X) := T(X)$. Convex risk measure is a more general class of risk measures that satisfies property (M), (T) and (C).

In this paper, we are interested in the EntRM, which is a well-known risk measure in risk-sensitive decision-making, including mathematical finance (Föllmer and Schied, 2016), Markovian decision processes (Bäuerle and Rieder, 2014). The EntRM value of a r.v. $X \sim F$ with coefficient $\beta \neq 0$ is defined as

$$U_\beta(X) \triangleq \frac{1}{\beta} \log(\mathbb{E}_{X \sim F} [\exp(\beta X)]) = \frac{1}{\beta} \log \left( \int_{\mathbb{R}} \exp(\beta x) dF(x) \right).$$

Since EntRM is distribution-invariant, we write $U_\beta(F) = U_\beta(X)$ for $X \sim F$. For $\beta$ with small absolute value, using Taylor’s expansion we have

$$U_\beta(X) = \mathbb{E}[X] + \frac{\beta}{2} \mathbb{V}[X] + O(\beta^2). \quad (1)$$

Hence for a decision-maker with the goal of maximizing the EntRM value, she tends to be risk-seeking (favoring high uncertainty in $X$) if $\beta > 0$ and risk-averse (favoring low uncertainty in $X$) if $\beta < 0$. $|\beta|$ controls the risk-sensitivity. It reduces to the mean functional when $\beta \rightarrow 0$.

3. Risk-sensitive Distributional Dynamic Programming

Bellemare et al. (2017); Rowland et al. (2018) have discussed the infinite-horizon distributional dynamic programming in the risk-neutral setting, which will be referred to as the
classical DDP. There is a big gap between the risk-sensitive MDP and the risk-neutral one. In this section, we establish a novel DDP framework for risk-sensitive control.

We define the return for a policy $\pi$ starting from state-action pair $(s, a)$ at step $h$

$$Z^\pi_h(s, a) \triangleq \sum_{h'=h}^{H} r_{h'}(s_{h'}, a_{h'}), \quad s_h = s, a_h = \pi_h(s_h'), s_{h+1} \sim P_{h'}(\cdot|s_{h'}, a_{h'}).$$

Define $Y^\pi_h(s) \triangleq Z^\pi_h(s, \pi_h(s))$, then it is immediate that

$$Z^\pi_h(s, a) = r_h(s, a) + Y^\pi_h(s'), s' \sim P_h(\cdot|s, a).$$

There are two sources of randomness in $Z^\pi_h(s, a)$: the transition $P^\pi_h$ and the next-state return $Y^\pi_{h+1}$. Denote by $\nu^\pi_h(s)$ and $\eta^\pi_h(s, a)$ the cumulative distribution function (CDF) corresponding to $Y^\pi_h(s)$ and $Z^\pi_h(s, a)$ respectively. For the risk-sensitive purpose, we define the action-value function of a policy $\pi$ at step $h$ as $Q^\pi_h(s, a) \triangleq U_\beta(Z^\pi_h(s, a))$, which is the EntRM value of the return distribution, for each $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$. The value function is defined as $V^\pi_h(s) \triangleq Q^\pi_h(s, \pi_h(s)) = U_\beta(Y^\pi_h(s))$. We focus on the control setting, in which the goal is to find an optimal policy to maximize the value function, that is,

$$\pi^*(s) \triangleq \arg \max_{(\pi_1, \ldots, \pi_H) \in \Pi} V^\pi_1 \ldots \pi_H(s).$$

We write $\pi = (\pi_1, \ldots, \pi_H)$ to emphasize that it is a multi-stage maximization problem. Exhaustive search suffers exponential computational complexity. In the risk-neutral setting, the principle of optimality holds, in other words, the optimal policy of the tail sub-problem is the tail optimal policy (Bertsekas et al., 2000). Therein the multi-stage maximization problem can be reduced to multiple single-stage maximization problems. However, the principle does not always hold for general risk measures. For example, the optimal policy for CVaR may be non-Markovian/history-dependent (Shapiro et al., 2021).

We identify the monotonicity-preserving property of EntRM, upon which we retain the principle of optimality.

**Lemma 1** The EntRM satisfies the monotonicity-preserving property: $\forall F_1, F_2, G \in \mathcal{D}, \forall \theta \in [0, 1]$,

$$U_\beta(F_2) \leq U_\beta(F_1) \Rightarrow U_\beta((1-\theta)F_2 + \theta G) \leq U_\beta((1-\theta)F_1 + \theta G).$$

**Proof** We only prove the case that $\beta > 0$. The case that $\beta < 0$ follows analogously. For any two distributions $F_1$ and $F_2$ such that $U_\beta(F_1) > U_\beta(F_2)$, we have

$$U_\beta(F_1) = \frac{1}{\beta} \log \int_\mathbb{R} \exp(\beta x) dF_1(x) > \frac{1}{\beta} \log \int_\mathbb{R} \exp(\beta x) dF_2(x) = U_\beta(F_2),$$

5
Hence ERM satisfies the monotonicity-preserving property. Moreover, the property \((T)\) entails that the EntRM value of the current return \(Z_h^\pi(s,a)\) equals the sum of the immediate reward \(r_H(s,a)\) and the value of the future return \(Y_h^\pi(s')\)

\[
U_\beta(Z_h^\pi(s,a)) = U_\beta(r_H(s,a) + Y_h^\pi(s')) = r_h(s,a) + U_\beta(Y_h^\pi(s'))
\]

where we write \([P_h \nu_{h+1}^\pi](s,a) = \sum_{s'} P_h(s'|s,a)\nu_{h+1}^\pi(s')\). These observations imply the principle of optimality.

**Proposition 2 (Principle of optimality)** Let \(\pi^* = \{\pi_1^*, \pi_2^*, ..., \pi_H^*\}\) be an optimal policy and assume that we visit some state \(s\) using policy \(\pi\) at time-step \(h\) with positive probability. Consider the sub-problem defined by the following maximization problem

\[
\max_{\pi \in \Pi} V_h^\pi(s) = r_h(s,a) + U_\beta([P_h \nu_{h+1}^\pi](s,a)).
\]

Then the truncated optimal policy \(\{\pi_h^*, \pi_{h+1}^*, ..., \pi_H^*\}\) is optimal for this sub-problem.

For notational simplicity, we write \(\pi_{h_1:h_2} = \{\pi_{h_1}, \pi_{h_1+1}, ..., \pi_{h_2}\}\) for two positive integers \(h_1 < h_2 \leq H\).

**Proof** Notice that there exists some optimal policy for sub-problems at each step, which will be shown in Proposition 3. Suppose that the truncated policy \(\pi_{h:H}^*\) is not optimal for this subproblem, then there exists an optimal policy \(\tilde{\pi}_{h:H}\) such that

\[
\exists \tilde{s}_h \text{ occurring with positive probability, } V_{\tilde{h}_{h:H}}^\tilde{\pi}(\tilde{s}_h) > V_{\tilde{h}_{h:H}}^{\pi_{h:H}}(\tilde{s}_h).
\]

There exists a state \(\tilde{s}_{h-1}\) which occurs with positive probability and \(P_{h-1}(\tilde{s}_h|\tilde{s}_{h-1}, \pi_{h-1}^*|\tilde{s}_{h-1}) > 0\) such that

\[
U_\beta(\nu_{h-1}^{\pi_{h-1}^*}(\tilde{s}_{h-1})) = r_{h-1}(\tilde{s}_{h-1}, \pi_{h-1}^*|\tilde{s}_{h-1}) + U_\beta\left(P_{h-1}\nu_{h-1}^{\pi_{h-1}^*}(\tilde{s}_{h-1}, \pi_{h-1}^*|\tilde{s}_{h-1})\right)
\]

\[
> r_{h-1}(\tilde{s}_{h-1}, \pi_{h-1}^*|\tilde{s}_{h-1}) + U_\beta\left(P_{h-1}\nu_{h-1}^{\pi_{h-1}^*}(\tilde{s}_{h-1}, \pi_{h-1}^*|\tilde{s}_{h-1})\right)
\]

\[
= U_\beta(\nu_{h-1}^{\pi_{h-1}^*}(\tilde{s}_{h-1}))
\]

where the inequality is due to the strict monotonicity preserving property of \(U_\beta\). It follows that \(\{\pi_{h-1}^*, \tilde{\pi}_h, ..., \tilde{\pi}_H\}\) is a strictly better policy than \(\{\pi_{h-1}^*, \pi_{h}^*, ..., \pi_H^*\}\) for the subproblem
from \( h - 1 \) to \( H \). Suppose for \( h' + 1 \in [2, h - 1] \), \( \{\pi_{h' + 1}, \ldots, \pi_{h}, \ldots, \pi_{H}\} \) is a strictly better policy than \( \{\pi_{h' + 1}, \ldots, \pi_{h}, \ldots, \pi_{H}\} \) for the sub-problem from \( h' + 1 \) to \( H \). Similarly we can obtain that \( \{\pi_{h' + 1}, \ldots, \pi_{h}, \ldots, \pi_{H}\} \) is also a strictly better policy than \( \{\pi_{h' + 1}, \ldots, \pi_{h}, \ldots, \pi_{H}\} \). Repeating the above arguments finally yields that \( \{\pi_{1}, \pi_{2}, \ldots, \pi_{H}\} \) is a strictly better policy than \( \pi^* = \{\pi_{1}, \pi_{2}, \ldots, \pi_{H}\} \). This is contradicted to the assumption that \( \pi^* = \{\pi_{1}, \pi_{2}, \ldots, \pi_{H}\} \) is an optimal policy.

It further induces the distributional Bellman optimality equation.

**Proposition 3 (Distributional Bellman optimality equation)** For arbitrary initial state \( s_1 \), the optimal policy \( (\pi_h^*)_{h \in [H]} \) is given by the following backward recursions:

\[
\begin{align*}
\nu_h^*(s) &= \psi_0, \quad \eta_h(s, a) = [P_h \nu_{h+1}^*](s, a)(\cdot - r_h(\cdot|s, a)), \\
\pi_h^*(s) &= \arg \max_{a \in A} Q_h^*(s, a) = U^\beta(\eta_h(s, a), \nu_h^*(s)) = \eta_h(s, \pi_h^*(s)),
\end{align*}
\]

where \( F(\cdot-c) \) denotes the CDF obtained by shifting \( F \) right by \( c \). Furthermore, the sequence \( (\eta_h^*)_{h \in [H]} \) and \( (\nu_h^*)_{h \in [H]} \) are the sequence of distributions corresponding to the optimal returns at each step.

**Proof** Throughout the proof we drop the dependence on * for the ease of notation. The proof follows from induction. Notice that by distributional Bellman equation, \( \eta_h(s_h) \) and \( V_h(s_h) \) are the return distribution at state \( s_h \) at step \( h \) following policy \( \pi_h \) and value function respectively. At step \( H \), it is obvious that \( \pi_H \) is the optimal policy that maximizes the ERM value at the final step for each state \( S_H \in S \). Now fix \( h \in [H - 1] \), assume that \( \pi_{h+1} \) is the optimal policy for the subproblem

\[
V_{h+1}^{\pi_{h+1}}(s_{h+1}) = \max_{\pi_{h+1}} V_{h+1}^{\pi_{h+1}}(s_{h+1}), \forall s_{h+1}.
\]

In other words,

\[
U_\beta(\nu_{h+1}(s_{h+1})) = U_\beta(\nu_{h+1}^{\pi_{h+1}}(s_{h+1})) = \max_{\pi_{h+1}} U_\beta(\nu_{h+1}^{\pi_{h+1}}(s_{h+1}))
\geq U_\beta(\nu_{h+1}^{\pi_{h+1}}(s_{h+1})), \forall \pi_{h+1}, \forall s_{h+1}.
\]

It follows that \( \forall s_h \),

\[
V_h(s_h) = Q_h(s_h, \pi_h(s_h)) = U_\beta(\nu_{h}^{\pi_{h+1}}(s_h)) = \max_{a_h} U_\beta(\eta_h(s_h, a_h))
= \max_{a_h} \left\{ r_h(s_h, a_h) + \max_{\pi_{h+1}} U_\beta(\nu_{h+1}^{\pi_{h+1}}(s_h)) \right\}
\geq \max_{a_h} \left\{ r_h(s_h, a_h) + \max_{\pi_{h+1}} U_\beta(\nu_{h+1}^{\pi_{h+1}}(s_h)) \right\}
= \max_{\pi_h} \left\{ r_h(s_h, a_h) + \max_{\pi_{h+1}} U_\beta(\nu_{h+1}^{\pi_{h+1}}(s_h)) \right\}
\geq \max_{\pi_{h+1}} U_\beta(\nu_{h}^{\pi_{h+1}}(s_h)).
\]
Hence $V_h$ is the optimal value function at step $h$ and $\pi_{h:H}$ is the optimal policy for the sub-problem from $h$ to $H$. The induction is completed.

For simplicity, we define the distributional Bellman operator $B(P, r): \mathcal{G}^S \rightarrow \mathcal{G}^{S \times A}$ with associated model $(P, r) = (P(s, a), r(s, a))_{(s, a) \in S \times A}$ as

$$[B(P, r)\nu](s, a) \triangleq [P\nu](s, a)(\cdot - r_h(s, a)), \quad \forall (s, a) \in S \times A.$$ 

Hence we can rewrite Equation 2 in a compact form:

$$\nu^*_{h+1}(s) = \psi_0, \quad \eta^*_h(s, a) = [B(P, r_h)\nu^*_{h+1}](s, a),$$

$$\pi^*_h(s) = \arg\max_{a \in A} U_\beta(\eta^*_h(s, a)), \quad \nu^*_h(s) = \eta^*_h(s, \pi^*_h(s)), \forall (s, a, h) \in S \times A \times [H].$$

Finally, we define the regret of an algorithm $\mathcal{A}$ interacting with an MDP $\mathcal{M}$ for $K$ episodes as

$$\text{Regret}(\mathcal{A}, \mathcal{M}, K) \triangleq \sum_{k=1}^K V^*_1(s^k_1) - V^*_h(s^k_1).$$

Note that the regret is a random variable since $\pi^k$ is a random quantity. We denote by $\mathbb{E}[\text{Regret}(\mathcal{A}, \mathcal{M}, K)]$ the expected regret. We will omit $\pi$ and $\mathcal{M}$ if it is clear from the context.

4. RODI-MF

We introduce Algorithm 1: Model-Free Risk-sensitive Optimistic Distribution Iteration (RODI-MF) in this section.

For completeness, we introduce some additional notations here. For two CDFs $F$ and $G$ over reals, we define the supremum distance between them $\|F - G\|_\infty \triangleq \sup_x |F(x) - G(x)|$.

We define the $\ell_1$ distance between two probability mass functions (PMFs) with the same support $P$ and $Q$ as $\|P - Q\|_1 \triangleq \sum_i |P_i - Q_i|$. We denote by $B_{\infty}(F, c) := \{G \in \mathcal{G} \mid \|G - F\|_\infty \leq c\}$ the supremum norm ball consisting of CDFs centered at $F$ with radius $c$. With slight abuse of notations, we denote by $B_1(P, c)$ the $l_1$ norm ball consisting PMFs centered at $P$ with radius $c$.

Planning phase (Line 4-12). At a high level, the algorithm implements an optimistic version of approximate RS-DDP from step $H + 1$ to step 1 in each episode. In Line (5-7), it performs sample-based Bellman update. To make it clear, we introduce the superscript $k$ to the variables of Algorithm 1 in episode $k$. For example, $\eta^k_h$ denotes $\eta_h$ in episode $k$. Specifically, for those visited state-action pairs, we claim that Line 6 is equivalent to a model-based Bellman update. Denote by $\mathbb{I}_h^k(s, a) \triangleq \mathbb{I}\{(s^k_{h+1}, a^k_{h}) = (s, a)\}$ and $N^k_h(s, a) \triangleq \sum_{\tau \in [k-1]} \mathbb{I}_h^k(s, a)$. Fix a tuple $(s, a, k, h)$ such that $N^k_h(s, a) \geq 1$. We denote by $\hat{P}^k_h(\cdot | s, a)$ the empirical transition model

$$\hat{P}^k_h(s' | s, a) \triangleq \frac{1}{N^k_h(s, a)} \sum_{\tau \in [k-1]} \mathbb{I}_h^k(s, a) \cdot \mathbb{I}\{s^\tau_{h+1} = s'\}.$$ 

Observe that for any $\nu \in \mathcal{G}^S$, we have
\[
\left[\hat{P}_h^k \nu\right](s, a) = \sum_{s' \in S} \hat{P}_h^k(s'|s, a) \nu(s') = \frac{1}{N_h^k(s, a)} \sum_{\tau \in [k-1]} \sum_{s' \in S} \mathbb{I}_h^\tau(s, a) \cdot \mathbb{I}\{s_{\tau+1}^\tau = s'\} \nu(s') \\
= \frac{1}{N_h^k(s, a)} \sum_{\tau \in [k-1]} \sum_{s' \in S} \mathbb{I}_h^\tau(s, a) \cdot \mathbb{I}\{s_{\tau+1}^\tau = s'\} \nu(s_{\tau+1}^\tau) \\
= \frac{1}{N_h^k(s, a)} \sum_{\tau \in [k-1]} \sum_{s' \in S} \mathbb{I}_h^\tau(s, a) \nu(s_{\tau+1}^\tau).
\]

Hence the update formula in Line 6 of Algorithm 1 can be rewritten as

\[
\eta_h^k(s, a) = \left[\hat{P}_h^k \nu_{h+1}^k\right](s, a)(\cdot - r_h(s, a)) = \mathbb{B}(\hat{P}_h^k, r_h)\nu_{h+1}^k(s, a).
\]

Line 6 is equivalent to a model-based Bellman update with empirical model \( \hat{P}_h^k \). Alternatively, the unvisited \((s, a)\) remains to be the return distribution corresponding to the highest possible reward \( H + 1 - h \). The algorithm then computes the optimism constants \( c_h^k \) (Line 8) and enforces OFU through the distributional optimism operator \( O_{\infty}^{c_h^k} \) (Line 9) to obtain the optimistically plausible return distribution \( \eta_h^k \). The choice of \( c_h^k \) will be discussed later. The optimistic return distribution yields the optimistic value function, from which the algorithm generates the greedy policy \( \pi_h^k \). The policy \( \pi_h^k \) will be used in the interaction phase.

Interaction phase (Line 14-17). In Line (15-16), the agent interacts with the environment using policy \( \pi^k \) and updates the counts \( N_h^k \) based on new observations.

4.1 Exponential Entropic Risk Measure

In our analysis, we consider the exponential entropic risk measure (EERM)

\[
E_\beta(F) \triangleq \exp(\beta U_\beta(F)) = \int_\mathbb{R} \exp(\beta x) dF(x),
\]

which is the EntRM followed by an exponential transformation. The exponential transformation preserves the order in the sense that \( \forall \beta \neq 0, \)

\[
U_\beta(F) \geq U_\beta(G) \iff \text{sign}(\beta)E_\beta(F) \geq \text{sign}(\beta)E_\beta(G).
\]

Hence we can get the distributional Bellman optimality equation in terms of EERM

\[
\nu_h^*(s) = \psi_0, \quad \eta_h^k(s, a) = [P_h \nu_{h+1}^k](s, a)(\cdot - r_h(s, a)), \\
\pi_h^k(s) = \arg \max_{a \in \mathcal{A}} \text{sign}(\beta)E_\beta(\eta_h^k(s, a)), \quad \nu_h^k(s) = \eta_h^k(s, \pi_h^k(s)). \tag{4}
\]

**Proposition 4 (Equivalence between EntRM and EERM)** The policy \( \pi^* \) generated by Equation 4 is an optimal policy for ERM and EERM. In addition, the value distribution generated by Equation 4 is identical to the optimal value distribution for ERM.

**Proof** The proof follows from induction. Observe that the only difference between Equation 4 and Equation 2 is the way of generating \( \pi_h^k \). For notational convenience, denote the
quantity generated in Equation 2 and Equation 4 by $(\cdot)^*$ and $(\cdot)^{\ast}$ respectively. Consider the base case. It is obvious that $\eta_H^\ast = \tilde{\eta}_H^\ast$. It follows that $\pi_H^\ast(s) = \tilde{\pi}_H^\ast(s)$ for each $s$ since

$$U_\beta(F) \geq U_\beta(G) \iff \text{sign}(\beta)E_\beta(F) \geq \text{sign}(\beta)E_\beta(G).$$

Then we have $\nu_H^\ast = \tilde{\nu}_H^\ast$. Now suppose $\nu_{h+1}^\ast = \tilde{\nu}_{h+1}^\ast$ for $h + 1 \in [2 : H]$, we can prove that $\eta_h^\ast = \tilde{\eta}_h^\ast$, $\pi_h^\ast = \tilde{\pi}_h^\ast$ and $\nu_h^\ast = \tilde{\nu}_h^\ast$. The induction is completed. $\blacksquare$

We provide two important properties of EERM that will be used to establish the regret upper bounds, including the Lipschitz continuity and linearity. Denote by $L_M$ the Lipschitz constant of the EERM $E_\beta : \mathcal{D}([0, M]) \to \mathbb{R}$ with respect to the infinity norm $\|\cdot\|_\infty$, in other words,

$$E_\beta(F) - E_\beta(G) \leq L_M\|F - G\|_\infty, \forall F, G \in \mathcal{D}_M.$$

Lemma 5 provides a tight Lipschitz constant of EERM, which relates the distances between distributions to the difference measured by their EERM values.

**Lemma 5 (Lipschitz property of EERM)** $E_\beta$ is Lipschitz continuous with respect to the supremum norm over $\mathcal{D}_M$ with $L_M = |\exp(\beta M) - 1|$. Moreover, $L_M$ is tight in terms of both $|\beta|$ and $M$.

**Proof** We first provide the proof for the case $\beta > 0$. For any $F, G \in \mathcal{D}_M$, without loss of generality we assume $\int_0^M G(x) d\exp(\beta x) - \int_0^M F(x) d\exp(\beta x) \geq 0$, otherwise we switch the order.

$$|E_\beta(F) - E_\beta(G)| = \left| \int_0^M \exp(\beta x) dF(x) - \int_0^M \exp(\beta x) dG(x) \right|$$

$$= \left| \exp(\beta x)F(x)|_0^M - \int_0^M F(x) d\exp(\beta x) - \exp(\beta x)G(x)|_0^M + \int_0^M G(x) d\exp(\beta x) \right|$$

$$= \int_0^M (G(x) - F(x)) d\exp(\beta x) \leq \int_0^M |G(x) - F(x)| d\exp(\beta x)$$

$$\leq \|F - G\|_\infty \int_0^M 1 d\exp(\beta x) = (\exp(\beta M) - 1) \|F - G\|_\infty.$$ 

For the case $\beta < 0$, we assume $\int_0^M G(x) d\exp(\beta x) - \int_0^M F(x) d\exp(\beta x) \geq 0$.

$$|E_\beta(F) - E_\beta(G)| = \int_0^M (G(x) - F(x)) d\exp(\beta x) = \int_0^M (G(x) - F(x)) \beta \exp(\beta x) dx$$

$$\leq \int_0^M |G(x) - F(x)| |\beta| \exp(\beta x) dx$$

$$\leq \|F - G\|_\infty \int_0^M -1 d\exp(\beta x) = (1 - \exp(\beta M)) \|F - G\|_\infty$$

$$= |\exp(\beta M) - 1| \|F - G\|_\infty.$$ 

Thus $L_M = |\exp(\beta M) - 1|$ for EERM. To show the tightness of the constant, consider two scaled Bernoulli distributions $F = (1 - \mu_1)\psi_0 + \mu_1\psi_M$ and $G = (1 - \mu_2)\psi_0 + \mu_2\psi_M$, where
\( \mu_1, \mu_2 \in (0,1) \) are some constants. It holds that
\[
|E_\beta(F) - E_\beta(G)| = |\mu_1 \exp(\beta M) + 1 - \mu_1 - (\mu_2 \exp(\beta M) + 1 - \mu_2)|
\]
\[
= |\mu_1 - \mu_2| |\exp(\beta M) - 1| = \|F - G\|_\infty L_M,
\]
where the last equality holds since \( \|F - G\|_\infty = |F(0) - G(0)| = |\mu_1 - \mu_2| \) (independent of \( M \)). More formally, we have
\[
\inf_{M>0, \beta>0} \sup_{F,G \in \mathcal{D}_M} \frac{|E_\beta(F) - E_\beta(G)|}{\|F - G\|_\infty} = |\exp(\beta M) - 1| = L_M.
\]

Notice that \( \lim_{\beta \to 0} L_M = 0 \), which coincides with the fact that \( \lim_{\beta \to 0} E_\beta = 1 \).

Another key property of EERM is the linearity
\[
E_\beta(\theta F + (1 - \theta)G) = \theta E_\beta(F) + (1 - \theta)E_\beta(G),
\]
which sharpens the regret bounds. In contrast, EntRM is non-linear in the distribution, which could induce a factor of \( \exp(|\beta|H) \) when controlling the error propagation across time-steps. It would further lead to a compounding factor of \( \exp(|\beta|H^2) \) in the regret bound.

### 4.2 Distributional Optimism over the Return Distribution

For the ease of presentation, we only consider the case of \( \beta > 0 \) in the sequel. The case \( \beta < 0 \) follows from similar arguments. We first give a formal definition of optimism at distributional level.

**Definition 6** For two CDFs \( F \) and \( G \), we say that \( F \) is more optimistic than \( G \) if \( U_\beta(F) \geq U_\beta(G) \).

This reflects the intuition that the more optimistic distribution should own larger EntRM value. Since the exponential transformation preserves the order, \( F \) is more optimistic than \( G \) if \( E_\beta(F) \geq E_\beta(G) \). Following Keramati et al. (2020), we define the distributional optimism operator \( O^\infty_c : \mathcal{D}([a,b]) \mapsto \mathcal{D}([a,b]) \) with level \( c \in (0,1) \) as
\[
(O^\infty_c F)(x) \triangleq [F(x) - c\mathbb{I}_{[a,b]}(x)]^+.
\]
The optimism operator shifts the distribution \( F \) down by at most \( c \) over \([a,b]\), and retain the value 1 at \( b \). It ensures that \( O^\infty_c F \) remains in \( \mathcal{D}([a,b]) \) and dominates all the other CDFs in \( \mathcal{D}([a,b]) \).

**Lemma 7** Let \( T \) be a functional (not necessarily a risk measure) satisfying the monotonicity, i.e., \( T(F) \leq T(G) \) for any \( F \preceq G \). For any \( G \in \mathcal{D}([a,b]) \), it holds that if \( G \in B_\infty(F,c) \), then \( G \preceq O^\infty_c F \). Moreover, it holds that
\[
O^\infty_c F \in \operatorname{arg max}_{G \in B_\infty(F,c) \cap \mathcal{D}([a,b])} T(G).
\]
Proof Let $G \in \mathcal{G}([a, b]) \cap B_\infty(F, c)$. It follows from the definition of $B_\infty(F, c)$ that $\sup_{x \in [a, b]} |F(x) - G(x)| \leq c$, therefore for any $x \in [a, b]$, $G(x) \geq \max(F(x) - c, 0) = (Q_c^\infty F)(x)$. Since $Q_c^\infty F$ dominates any $G$ in $\mathcal{G}([a, b]) \cap B_\infty(F, c)$, the monotonicity of $T$ leads to the result.

We define the EERM value produced by the algorithm as $W^k_h(s) \triangleq E_\beta(\nu^k_h(s))$ and $J^k_h(s, a) \triangleq E_\beta(\eta^k_h(s, a))$ for all $(s, a, h)$. Similarly, we define $W^*_h(s) \triangleq E_\beta(\nu^*_h(s))$ and $J^*_h(s, a) \triangleq E_\beta(\eta^*_h(s, a))$ for all $(s, a, h)$.

Denote by $\iota = \log(SAT/\delta_1)$. For any $\delta \in (0, 1)$, we define the good event as

$$G_\delta \triangleq \left\{ \left\| \hat{P}_h^k(\cdot | s, a) - P_h(\cdot | s, a) \right\|_1 \leq \sqrt{\frac{2S}{N^k_h(s, a) \bigvee 1}} \iota, \forall(s, a, k, h) \in S \times A \times [K] \times [H] \right\},$$

under which the empirical distributions concentrates around the true distributions w.r.t. $\| \cdot \|_1$. Using Lemma 11, the monotonicity of EERM, and inductions, we arrive at Proposition 8, which guarantees the sequence $\{W^k_1(s_1^k)\}_{k \in [K]}$ produced by Algorithm 1 is indeed optimistic compared to the optimal value $\{W^*_1(s_1^k)\}_{k \in [K]}$.

**Proposition 8 (Optimism)** Conditioned on event $G_\delta$, the sequence $\{W^k_1(s_1^k)\}_{k \in [K]}$ produced by Algorithm 1 are all greater than or equal to $W^*_1(s_1^k)$, i.e.,

$$W^k_1(s_1^k) = E_\beta(\nu^k_1(s_1^k)) \geq E_\beta(\nu^*_1(s_1^k)) = W^*_1(s_1^k), \forall k \in [K].$$

**Lemma 9 (High probability good event)** For any $\delta \in (0, 1)$, the event $G_\delta$ is true with probability at least $1 - \delta$.

**Fact 1 (\ell_1 concentration bound, Weissman et al. (2003))** Let $P$ be a probability distribution over a finite discrete measurable space $(X, \Sigma)$. Let $P_n$ be the empirical distribution of $P$ estimated from $n$ samples. Then

$$\mathbb{P}(\left\| \hat{P}_n - P \right\|_1 \geq \epsilon) \leq \exp\left(-\frac{n\epsilon^2}{2|X|}\right).$$

In other words, with probability at least $1 - \delta$,

$$\left\| \hat{P}_n - P \right\|_1 \leq \sqrt{\frac{2|X|}{n} \log \frac{1}{\delta}}.$$

Lemma 9 does not directly follow from a union bound together with Fact 1 since the case $N^k_h(s, a) = 0$ need to be checked.

**Proof** Fix some $(s, a, k, h) \in S \times A \times [K] \times [H]$. If $N^k_h(s, a) = 0$, then we have $\hat{P}_h^k(\cdot | s, a) = \frac{1}{S} 1$. A simple calculation yields that for any $P_h(\cdot | s, a)$

$$\left\| \frac{1}{S} 1 - P_h(\cdot | s, a) \right\|_1 \leq 2 \leq \sqrt{2S \log(1/\delta)}.$$
It follows that

$$\mathbb{P}\left( \left\| \hat{P}_h^k(s,a) - P_h(s,a) \right\|_1 \leq \sqrt{\frac{2S}{N_h^k(s,a)} \log(1/\delta)} \mid N_h^k(s,a) = 0 \right) = 1 > 1 - \delta.$$  

The event is true for the unseen state-action pairs. Now we consider the case that $N_h^k(s,a) > 0$. By Fact 1, we have that for any integer $n \geq 1$

$$\mathbb{P}\left( \left\| \hat{P}_h^k(s,a) - P_h(s,a) \right\|_1 \leq \sqrt{\frac{2S}{N_h^k(s,a)} \log(1/\delta)} \mid N_h^k(s,a) = n \right) \leq 1 - \delta.$$  

Thus we have

$$\mathbb{P}\left( \left\| \hat{P}_h^k(s,a) - P_h(s,a) \right\|_1 \leq \sqrt{\frac{2S \log(1/\delta)}{N_h^k(s,a)}} \right) = \sum_{n=0,1,\ldots} \mathbb{P}\left( \left\| \hat{P}_h^k(s,a) - P_h(s,a) \right\|_1 \leq \sqrt{\frac{2S \log(1/\delta)}{N_h^k(s,a)} \mid N_h^k(s,a) = n} \right) \mathbb{P}(N_h^k(s,a) = n) \geq (1 - \delta) \sum_{n=0,1,\ldots} \mathbb{P}(N_h^k(s,a) = n) = 1 - \delta.$$  

Applying a union bound over all $(s,a,k,h) \in S \times A \times [K] \times [H]$ and rescaling $\delta$ leads to the result.

Lemma 9 suggests that $\mathcal{G}_\delta$ holds with probability $1 - \delta$. We will verify the distributional optimism conditioned on $\mathcal{G}_\delta$.

**Lemma 10** For any $F_i \in \mathcal{F}$ and any $\theta, \theta' \in \Delta_n$ with any $n \geq 2$, it holds that

$$\left\| \sum_{i=1}^n \theta_i F_i - \sum_{i=1}^n \theta'_i F_i \right\|_\infty \leq \left\| \theta - \theta' \right\|_1.$$  

**Proof**

$$\left\| \sum_{i=1}^n \theta_i F_i - \sum_{i=1}^n \theta'_i F_i \right\|_\infty = \sup_{x \in \mathbb{R}} \left| \sum_{i=1}^n (\theta_i F_i - \theta'_i F_i) \right| \leq \sup_{x \in \mathbb{R}} \sum_{i=1}^n |\theta_i - \theta'_i| F_i(x) \leq \sum_{i=1}^n |\theta_i - \theta'_i| = \left\| \theta - \theta' \right\|_1.$$  

**Lemma 11** (Bound on the optimistic constant) For any bounded distributions $\{F_i\}_{i \in [n]}$ and any $\theta, \theta' \in \Delta_n$ it holds that if $c \geq \left\| \theta - \theta' \right\|_1$, then

$$\sum_{i=1}^n \theta_i F_i \leq O_c^\infty \left( \sum_{i=1}^n \theta'_i F_i \right).$$
**Proof** Without loss of generality assume $F \in \mathcal{F}_{\lambda}^n$. By Lemma 10, for any $x$

$$\left| \sum_{i=1}^{n} (\theta_i' - \theta_i) F_i(x) \right| \leq \|\theta' - \theta\|_1.$$ 

For any $x \in [0, M + 1)$, 

$$O_c^+ \left( \sum_{i=1}^{n} \theta_i' x_i \right)(x) = \left[ \sum_{i=1}^{n} \theta_i' x_i - c \right] = \left[ \sum_{i=1}^{n} \theta_i' x - r \right] + \left[ \sum_{i=1}^{n} (\theta_i' - \theta_i) x_i - c \right] + \left[ \sum_{i=1}^{n} \theta_i' x_i - c \right] \leq \left[ \sum_{i=1}^{n} \theta_i' x_i - c \right] + \left[ \sum_{i=1}^{n} \theta_i' x - r \right] = \sum_{i=1}^{n} \theta_i' x_i - r.$$

Now we give the proof of Proposition 8.

**Proof** The proof follows from induction. Fix $k \in [K]$. For any $(s, a)$ such that $N^k_H(s, a) > 0$

$$J^k_H(s, a) = E_{\beta} (\eta^k_H(s, a)) = E_{\beta} (\psi_{r_H(s, a)}) = \exp (\beta r_H(s, a)) = J^k_H(s, a).$$

Consider the unvisited $(s, a)$ with $N^k_H(s, a) = 0$

$$J^k_H(s, a) = E_{\beta} (\eta^k_H(s, a)) = E_{\beta} (\psi_1) = \exp (\beta) \geq J^k_H(s, a).$$

Thus $W^k_H(s) = max_a J^k_H(s, a) \geq max_a J^k_H(s, a) = W^k_H(s), \forall s$, therefore the base case holds.

Now suppose for $h + 1 \in [2 : H]$, it holds that $W^k_{h+1}(s) \geq W^k_{h+1}(s), \forall s$. Consider arbitrary $(s, a)$ with $N^k_H(s, a) > 0$, we apply Lemma 11 with $	heta = P_h(s, a), \theta' = \hat{P}^k_h(s, a), F = \nu^k_{h+1}$ to obtain

$$[P_h \nu^k_{h+1}](s, a) \leq O^\infty_{\alpha}(\eta^k_{h+1}(s, a))$$

since $c^k_h(s, a) = \sqrt{\frac{2S}{N^k_H(s, a) + 1}} \geq \left\| P_h(s, a) - \hat{P}^k_h(s, a) \right\|_1$ for $h \in [H - 1]$. It follows that

$$J^k_h(s, a) = E_{\beta} (O^\infty_{\beta}(\eta^k_{h+1})(s, a)) = \exp (\beta r_H(s, a)) \geq \exp (\beta r_H(s, a)) \cdot [P_h \nu^k_{h+1}(s, a)] = \exp (\beta r_H(s, a)) \cdot [P_h W^*_{h+1}](s, a) = J^k_h(s, a), \forall (s, a),$$

where the first inequality is due to the monotonicity-preserving property, and the second inequality follows from the induction assumption. For the unvisited $(s, a)$ at step $h$,

$$J^k_h(s, a) = E_{\beta} (\psi_{H+1-h}) = \exp (\beta (H + 1 - h)) \geq J^k_h(s, a).$$
Finally it follows that for any \( s \),

\[
W^k_h(s) = \max_a J^k_h(s,a) \geq \max_a J^*_h(s,a) = W^*_h(s).
\]

The induction is completed.

**Algorithm 1 RODI-MF**

1: Input: \( T \) and \( \delta \)
2: Initialize \( N_h(\cdot,\cdot) \leftarrow 0; \eta_h(\cdot,\cdot), \nu_h(\cdot) \leftarrow \psi_{H+1-h} \forall h \in [H] \)
3: for \( k = 1 : K \) do
4:   for \( h = H : 1 \) do
5:     if \( N_h(\cdot,\cdot) > 0 \) then
6:       \( \eta_h(\cdot,\cdot) \leftarrow \frac{1}{N_h(\cdot,\cdot)} \sum_{\tau \in [k-1]} \Pi^k_h(\cdot,\cdot) \nu_{h+1}(s_{h+1})(\cdot - r_h(\cdot,\cdot)) \)
7:     end if
8:     \( c_h(\cdot,\cdot) \leftarrow \sqrt{\frac{2S}{N_h(\cdot,\cdot)} \max_l} \)
9:     \( \eta_h(\cdot,\cdot) \leftarrow O(c_h(\cdot,\cdot)) \eta_h(\cdot,\cdot) \)
10: \( \pi_h(\cdot) \leftarrow \arg \max_a U_\beta(\eta_h(\cdot, a)) \)
11: \( \nu_h(\cdot) \leftarrow \eta_h(\cdot, \pi_h(\cdot)) \)
end for
13: Receive \( s^k_1 \)
14: for \( h = 1 : H \) do
15:   \( a^k_h \leftarrow \pi_h(s^k_h) \) and transit to \( s^k_{h+1} \)
16: \( N_h(s^k_h, a^k_h) \leftarrow N_h(s^k_h, a^k_h) + 1 \)
end for
18: end for

4.3 Regret Upper Bound of RODI-MF

**Theorem 12 (Regret upper bound of RODI-MF)** For any \( \delta \in (0,1) \), with probability \( 1 - \delta \), the regret of Algorithm 1 is bounded as

\[
\text{Regret(RODI-MF,K)} \leq \mathcal{O}
\left( \frac{1}{|\beta|} L_H H \sqrt{S^2AK \log(4SAT/\delta)} \right) = \tilde{\mathcal{O}} \left( \frac{\exp(|\beta|H) - 1}{|\beta|} H \sqrt{S^2AK} \right).
\]

**Remark 13** The above results match the best-known results in Fei et al. (2021). In particular, our algorithms attain exponentially improved regret bounds than those of RSVI and RSQ in Fei et al. (2020) with a factor of \( \exp(|\beta|H^2) \).

**Remark 14** For \( \beta \) with small absolute value, using Taylor’s expansion

\[
U_\beta(Z^\pi) = \mathbb{E}[Z^\pi] + \frac{\beta}{2} \mathbb{V}[Z^\pi] + \mathcal{O}(\beta^2).
\]

Since \( r_h \in [0,1] \) by assumption, \( \mathbb{E}[Z^\pi] = \mathbb{E}[\sum_{h=1}^H r_h(s_h, a_h)] = \mathcal{O}(H) \) and \( \mathbb{V}[Z^\pi] = \mathbb{E}[(Z^\pi)^2] - \mathbb{E}^2[Z^\pi] = \mathcal{O}(H^2) \). The decision-maker does not know the environment. If she wants to balance the risk \( \frac{\beta}{2} \mathbb{V}[Z^\pi] \) and the expectation \( \mathbb{E}[Z^\pi] \), \( \beta \) should be set in the order of \( \mathcal{O}(1/H) \).
Remark 15 By choosing $|\beta| = O(1/H)$, we can eliminate the exponential dependency on $H$ and achieve polynomial regret bound akin to the risk-neutral setting. Therefore, DRL can achieve $O\left(H \sqrt{HS^2 AT}\right)$ regret bound for RSRL with reasonable risk-sensitivity.

Lemma 16 Let $0 < m \leq a < b$, it holds that

$$\log(b) - \log(a) \leq \frac{1}{m}(b - a).$$

Proof We first prove the case $\beta > 0$. The proof can be readily adapted to other cases. We define $\Delta_h^k \triangleq W_h^k - W_{\pi h}^k = E_\beta(\nu_h^k) - E_\beta(\nu_{\pi h}^k) \in D_h^S$ with

$$D_h \triangleq [1 - \exp(\beta(H + 1 - h)), \exp(\beta(H + 1 - h)) - 1]$$

and $\delta_h^k \triangleq \Delta_h^k(s_h^k)$. For any $(s, h)$ and any $\pi$, we let $P_{h}^\pi(\cdot | s) \triangleq P_{h}(\cdot | s, \pi_h(s))$. Observe that the regret can be bounded as

$$\text{Regret}(K) = \sum_{k=1}^{K} \frac{1}{\beta} \log\left(W_1^k(s_1^k)\right) - \frac{1}{\beta} \log\left(W_1^{\pi h}(s_1^k)\right)$$

$$= \sum_{k=1}^{K} \frac{1}{\beta} \log\left(W_1^k(s_1^k)\right) - \frac{1}{\beta} \log\left(V_1^k(s_1^k)\right) + \frac{1}{\beta} \log\left(W_1^k(s_1^k)\right) - \frac{1}{\beta} \log\left(V_1^{\pi h}(s_1^k)\right)$$

$$\leq \sum_{k=1}^{K} \frac{1}{\beta} \log\left(W_1^k(s_1^k)\right) - \frac{1}{\beta} \log\left(W_1^{\pi h}(s_1^k)\right)$$

$$\leq \frac{1}{\beta} \sum_{k=1}^{K} W_1^k(s_1^k) - W_1^{\pi h}(s_1^k) = \frac{1}{\beta} \sum_{k=1}^{K} \delta_h^k,$$

where the last inequality follows from Lemma 16 and that both $W_1^k(s_1^k)$ and $W_1^{\pi h}(s_1^k)$ are larger than 1. We can decompose $\delta_h^k$ as follows

$$\delta_h^k = E_\beta\left(\nu_h^k(s_h^k)\right) - E_\beta\left(\nu_{\pi h}^k(s_h^k)\right)$$

$$= E_\beta\left(O_{c_h}\left(\left[\hat{P}_h^{\pi h} \nu_{h+1}^k \right] (s_h^k)(- r_h^k)\right)\right) - E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)(- r_h^k)\right)$$

$$= \exp(\beta r_h^k) \left(E_\beta\left(O_{c_h}\left(\left[\hat{P}_h^{\pi h} \nu_{h+1}^k \right] (s_h^k)\right)\right) - E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right)\right)$$

$$+ \exp(\beta r_h^k) \left(E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right) - E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right)\right)$$

$$+ \exp(\beta r_h^k) \left(E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right) - E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right)\right)$$

$$+ \exp(\beta r_h^k) \left(E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right) - E_\beta\left([P_h^{\pi h} \nu_{h+1}^k] (s_h^k)\right)\right).$$
Using the Lipschitz property of EERM,
\[
(a) \leq \exp(\beta r_h^k) \cdot L_{H-h} \left\| \mathcal{O}_{c_h}^\infty \left( \left[ \hat{P}^{n^k}_{h} \nu_{h+1}^k \right] (s_h^k) \right) - \left[ \hat{P}^{n^k}_{h} \nu_{h+1}^k \right] (s_h^k) \right\| _\infty \\
\leq \exp(\beta r_h^k) \cdot L_{H-h} c_h^k \\
\leq \exp(\beta) (\exp(\beta(H-h)) - 1) c_h^k \\
\leq (\exp(\beta(H + 1 - h)) - 1) \sqrt{\frac{2S}{(N_h^k \lor 1)^t}}.
\]

We can bound \((b)\) as
\[
(b) = \exp(\beta r_h^k) \left( E_\beta \left( \left[ \hat{P}^{n^k}_{h} \nu_{h+1}^k \right] (s_h^k) \right) - E_\beta \left( \left[ P^{n^k}_{h} \nu_{h+1}^k \right] (s_h^k) \right) \right) \\
\leq \exp(\beta) L_{H-h} \left\| \left[ \hat{P}^{n^k}_{h} \nu_{h+1}^k \right] (s_h^k) - \left[ P^{n^k}_{h} \nu_{h+1}^k \right] (s_h^k) \right\| _\infty \\
\leq \exp(\beta) (\exp(\beta(H-h)) - 1) \left\| \hat{P}^{n^k}_{h} (s_h^k) - P^{n^k}_{h} (s_h^k) \right\| _1 \\
\leq (\exp(\beta(H + 1 - h)) - 1) \sqrt{\frac{2S}{(N_h^k \lor 1)^t}},
\]

where the second inequality is due to Lemma 10. Observe that
\[
(c) = \exp(\beta r_h^k) \left[ P^{n^k}_{h} (V_{h+1}^k - V_{h+1}^{n^k}) \right] (s_h^k) = \exp(\beta r_h^k) \left[ P^{n^k}_{h} \Delta_{h+1}^k \right] (s_h^k) = \exp(\beta r_h^k) (\epsilon_h^k + \delta_h^k).
\]

where \(\epsilon_h^k \triangleq [P^{n^k}_{h} \Delta_{h+1}^k](s_h^k) - \Delta_{h+1}^k(s_h^k)\) is a martingale difference sequence with \(\epsilon_h^k \in 2D_{h+1}\) a.s. for all \((k,h) \in [K] \times [H]\), and \(\epsilon_h^k \triangleq \left\| \hat{P}^k(s_h^k) - P^k(s_h^k) \right\|_1\). Since
\[
(a) + (b) \leq 2L_{H+1-h} c_h^k,
\]
we can bound \(\delta_h^k\) recursively as
\[
\delta_h^k \leq 2L_{H+1-h} c_h^k + \exp(\beta r_h^k) (\epsilon_h^k + \delta_h^k).
\]

Repeating the procedure, we can get
\[
\delta_h^k \leq 2 \sum_{H=1}^{H-1} L_{H+1-h} \prod_{i=1}^{h-1} \exp(\beta r_h^k) c_h^k + \sum_{H=1}^{H-1} \prod_{i=1}^{H-1} \exp(\beta r_h^k) \epsilon_h^k + \prod_{i=1}^{H-1} \exp(\beta r_h^k) \delta_h^k \\
\leq 2 \sum_{H=1}^{H-1} (\exp(\beta(H + 1 - h)) - 1) \exp(\beta(h - 1)) c_h^k + \sum_{H=1}^{H-1} \prod_{i=1}^{H-1} \exp(\beta r_h^k) \epsilon_h^k + \exp(\beta(H - 1)) \delta_h^k \\
\leq 2 \sum_{H=1}^{H-1} (\exp(\beta H) - 1) c_h^k + \sum_{H=1}^{H-1} \prod_{i=1}^{H-1} \exp(\beta r_h^k) \epsilon_h^k + \exp(\beta(H - 1)) \delta_h^k.
\]

It follows that
\[
\sum_{K=1}^{K} \delta_h^k \leq 2(\exp(\beta H) - 1) \sum_{K=1}^{K} \sum_{h=1}^{H-1} c_h^k + \sum_{K=1}^{K} \sum_{h=1}^{H-1} \prod_{i=1}^{H-1} \exp(\beta r_h^k) \epsilon_h^k + \sum_{K=1}^{K} \exp(\beta(H - 1)) \delta_h^k.
\]
Now we bound each term separably. The first term can be bounded as

$$2(\exp(\beta(H + 1)) - 1) \sum_{k=1}^{K} \sum_{h=1}^{H-1} c_k^h = 2(\exp(\beta(H + 1)) - 1) \sum_{h=1}^{H-1} \sum_{k=1}^{K} \sqrt{\frac{2S}{N_k^h \lor 1}} \leq 4(\exp(\beta(H + 1)) - 1) \sum_{h=1}^{H-1} \sqrt{2S^2 AK}.$$ 

$$= 4(\exp(\beta(H + 1)) - 1)(H - 1)\sqrt{2S^2 AK}.$$ 

Observe that

$$\prod_{i=1}^{h} \exp(\beta r_i^k) e_k^h \in \exp(\beta h)D_h = \exp(\beta h)[1 - \exp(\beta(H + 1 - h))], \exp(\beta(H + 1 - h)) - 1]$$

$$\subseteq [1 - \exp(\beta(H + 1)), \exp(\beta(H + 1)) - 1],$$

thus we can bound the second term by Azuma-Hoeffding inequality: with probability at least 1 - \delta', the following holds

$$\sum_{k=1}^{K} \sum_{h=1}^{H-1} \prod_{i=1}^{h} \exp(\beta r_i^k) e_k^h \leq (\exp(\beta(H + 1)) - 1)\sqrt{2KH \log(1/\delta')}.$$ 

The third term can be bounded as

$$\sum_{k=1}^{K} \exp(\beta(H - 1)) \delta_H^k = \exp(\beta(H - 1)) \sum_{k=1}^{K} W_H^k(s_H^k) - W_H^\pi(s_H^k)$$

$$= \exp(\beta(H - 1)) \sum_{k=1}^{K} [\mathbb{I}(N_H^k = 0) \exp(\beta) + \mathbb{I}(N_H^k > 0) \exp(\beta r_H(s_H^k)) - \exp(\beta r_H(s_H^k))]$$

$$\leq \exp(\beta(H - 1))(\exp(\beta) - 1) \sum_{k=1}^{K} \mathbb{I}(N_H^k = 0)$$

$$\leq (\exp(\beta H) - 1)SA < (\exp(\beta(H + 1)) - 1)SA$$

Using a union bound and let \(\delta = \delta' = \frac{\delta}{2}\), we have that with probability at least 1 - \delta,

$$\text{Regret}(K) \leq \frac{\exp(\beta(H + 1)) - 1}{\beta} \left(4(H - 1)\sqrt{2S^2 AK} + \sqrt{2KH} + SA\right)$$

$$= \tilde{O}\left(\frac{\exp(\beta H) - 1}{\beta H} H\sqrt{HS^2 AT}\right),$$

where \(\iota \equiv \log(2SAT/\delta)\).
For the $\beta < 0$ case. Using similar arguments, we arrive at

$$\text{Regret}(K) \leq \sum_{k=1}^{K} \frac{1}{\beta} \log \left( W_1^k(s_1^k) \right) - \frac{1}{\beta} \log \left( W_1^{\pi k}(s_1^k) \right)$$

$$= \sum_{k=1}^{K} \frac{1}{-\beta \exp(\beta H)} \sum_{k=1}^{K} W_1^{\pi k}(s_k^k) - W_1^k(s_1^k),$$

where the last inequality is due to that both $W_1^{\pi k}(s_k^k)$ and $W_1^k(s_k^k)$ is larger than or equal to $\exp(\beta H)$. We can finally get

$$\text{Regret}(K) \leq \tilde{O} \left( \frac{1}{-\beta \exp(\beta H)} \sqrt{HS^2 AT} \right) = \tilde{O} \left( \frac{\exp(|\beta| H) - 1}{|\beta|} \sqrt{HS^2 AT} \right).$$

5. RODI-MB

We introduce the second algorithm Model- Based Risk-sensitive Optimistic Distribution Iteration (RODI-MB). Algorithm 2 is a model-based algorithm because it explicitly maintains the empirical transition model in each episode. Interestingly, it is equivalent to a non-distributional reinforcement learning algorithm (Algorithm 3) that deals with the one-dimensional values instead of the distributions, which saves the computational complexity and space complexity.

Planning phase (Line 4-10). Analogous to Algorithm 1, the algorithm also performs approximate DDP together with the OFU principle. First, it applies the distributional optimistic operator to the empirical transition model $\hat{P}_h^k$ to get the optimistic transition model $\bar{P}_h^k$. Then the algorithm uses $\bar{P}_h^k$ to execute Bellman update to generate the optimistic return distribution $\eta_h^k$. The remaining steps are the same as Algorithm 1.

Interaction phase (Line 11-15). In Line (13-14), the agent interacts with the environment using policy $\pi^k$ and updates the counts $N_h^{k+1}$ and empirical transition model $\hat{P}_h^{k+1}$ based on the new observations.

5.1 Distributional Optimism over the Model

We consider the optimism among the space of PMFs rather than CDFs. We aim to obtain an optimistic transition model $\bar{P}_h^k(s, a)$ from the empirical one $\hat{P}_h^k(s, a)$. To be more specific, the return distribution $\nu_h^k$ computed from $\hat{P}_h^k(s, a)$ and $\nu_h^{k+1}$ is expected to be more optimistic than the optimal one $\eta_h^k$ with high probability. We define the distributional optimism operator $O_\alpha^L : \mathcal{D}(S) \rightarrow \mathcal{D}(S)$ over space of PMFs with level $\alpha$ and future return $\nu \in \mathcal{D}$ as

$$O_\alpha^L \left( \hat{P}(s, a), \nu \right) \triangleq \arg \max_{P \in B_\alpha(\hat{P}(s, a), \alpha)} U_\beta([P \nu]).$$
Proof

The proof follows from induction. Fix \( s, a, k, h \). For any PMF \( P \in B_1(\hat{P}^k_H(s, a), c^k_H(s, a)) \), we have

\[
E_\beta \left( \left[ \hat{P}^k_H \nu^k_{h+1} \right](s, a) \right) \geq E_\beta \left( \left[ P \nu^k_{h+1} \right](s, a) \right).
\]

**Proof** Use the definition of \( O^1_c \) and Proposition 4.

**Lemma 17 (Optimistic model)** Fix \( s, a, k, h \). For any PMF \( P \in B_1(\hat{P}^k_H(s, a), c^k_H(s, a)) \), we have

\[
E_\beta \left( \left[ \hat{P}^k_H \nu^k_{h+1} \right](s, a) \right) \geq E_\beta \left( \left[ P \nu^k_{h+1} \right](s, a) \right).
\]

**Proof** Use the definition of \( O^1_c \) and Proposition 4.

**Lemma 18 (Optimism)** Conditioned on event \( G_\delta \), the sequence \( \{W^k_1(s_1)\}_{k \in [K]} \) produced by Algorithm 2 are all greater than or equal to \( W^*_1(s_1) \), i.e.,

\[
W^k_1(s_1) = E_\beta(\nu^k_1(s_1)) \geq E_\beta(\nu^*_1(s_1)) = W^*_1(s_1), \forall k \in [K].
\]

**Proof** The proof follows from induction. Fix \( k \in [K] \). For \( h = H \), we have that for the visited \( (s, a) \)

\[
J^k_H(s, a) = E_\beta(\eta^k_H(s, a)) = \exp(\beta r_H(s, a)) = J^*_H(s, a).
\]

For the unvisited \( (s, a) \)

\[
J^k_H(s, a) = \exp(\beta) \geq J^*_H(s, a)
\]

Thus \( W^k_H(s) = \max_a J^k_H(s, a) \geq \max_a J^*_H(s, a) = W^*_H(s), \forall s \). Now suppose for \( h + 1 \in [2 : H] \), it holds that \( W^k_{h+1}(s) \geq W^*_H(s), \forall s \). It follows that for the \( (s, a) \) with \( N^k_h(s, a) > 0 \)

\[
J^k_h(s, a) = \exp(\beta r_h(s, a)) E_\beta \left( \left[ \hat{P}^k_h \nu^k_{h+1} \right](s, a) \right) \\
\geq \exp(\beta r_h(s, a)) E_\beta \left( \left[ P_h \nu^k_{h+1} \right](s, a) \right) \\
\geq \exp(\beta r_h(s, a)) E_\beta \left( \left[ P_h \nu^*_h \right](s, a) \right) \\
= J^*_h(s, a), \forall (s, a).
\]

The first inequality is due to Theorem 17. The second inequality follows from the induction assumption. For the unvisited \( (s, a) \), we have \( J^k_h(s, a) = \exp(\beta (H + 1 - h)) \geq J^*_h(s, a) \). Since

\[
W^k_h(s) = \max_a J^k_h(s, a) \geq \max_a J^*_h(s, a) = W^*_h(s), \forall s
\]

The induction is completed.

---

\( O^1_c \) takes also the future return distribution \( \nu \) as input, which is different from \( O^\infty_c \). Given a model \( \hat{P}(s, a) \) and return distribution \( \nu \), \( O^1_c \) returns a model that yields the largest ERM value among all possible models in \( B_1(\hat{P}(s, a), c) \). The ERM satisfy an interesting property that enables an efficient approach to perform \( O^1_c \) (see Appendix B).

Recall that Lemma 9 suggests that \( G_\delta \) holds with probability \( 1 - \delta \), therefore it suffices to consider the case conditioned on \( G_\delta \). Due to the equivalence between EntRM and EERM, we will verify the optimism in terms of EERM for \( \beta > 0 \).
Algorithm 2 RODI-MB
1: Input: $T$ and $\delta$
2: $N^1_h(\cdot, \cdot) \leftarrow 0$; $\hat{P}^1_h(\cdot, \cdot) \leftarrow \frac{1}{S} 1 \forall h \in [H]$ 
3: for $k = 1 : K$ do 
4: \hspace{1cm} $\nu^k_{H+1}(\cdot) \leftarrow \psi_0$ 
5: \hspace{1cm} for $h = H : 1$ do 
6: \hspace{2cm} if $N^k_h(\cdot, \cdot) > 0$ then 
7: \hspace{3cm} $\tilde{P}^k_h(\cdot, \cdot) \leftarrow O^1_{c^k_h(\cdot, \cdot)} \left( \hat{P}^k_h(\cdot, \cdot), \nu^k_{H+1} \right)$ 
8: \hspace{3cm} $\nu^k_h(\cdot, \cdot) \leftarrow \text{arg max}_a U^\beta(\nu^k_h(\cdot, \cdot), \tilde{P}^k_h(\cdot, \cdot))$ 
9: \hspace{2cm} else 
10: \hspace{3cm} $\nu^k_h(\cdot, \cdot) \leftarrow \psi_{H+1-h}$ 
11: \hspace{1cm} end if 
12: \hspace{1cm} $\pi^k_h(\cdot) \leftarrow \text{arg max}_a U^\beta(\nu^k_h(\cdot, \cdot), a)$ 
13: \hspace{1cm} $\nu^k_h(\cdot, \cdot) \leftarrow \nu^k_h(\cdot, \pi^k_h(\cdot))$ 
14: \hspace{1cm} end for 
15: \hspace{1cm} Receive $s^k_1$ 
16: \hspace{1cm} for $h = 1 : H$ do 
17: \hspace{2cm} $a^k_h \leftarrow \pi^k_h(s^k_h)$ and transit to $s^k_{h+1}$ 
18: \hspace{2cm} Compute $N^{k+1}_h(\cdot, \cdot)$ and $\hat{P}^{k+1}_h(\cdot, \cdot)$ 
19: \hspace{1cm} end for 
20: end for 

Algorithm 3 ROVI
1: Input: $T$ and $\delta$
2: $N^1_h(\cdot, \cdot) \leftarrow 0$; $\hat{P}^1_h(\cdot, \cdot) \leftarrow \frac{1}{S} 1 \forall h \in [H]$ 
3: for $k = 1 : K$ do 
4: \hspace{1cm} $W^{k}_{H+1}(\cdot) \leftarrow 1$ 
5: \hspace{1cm} for $h = H : 1$ do 
6: \hspace{2cm} if $N^k_h(\cdot, \cdot) > 0$ then 
7: \hspace{3cm} $\tilde{P}^k_h(\cdot, \cdot) \leftarrow O^1_{c^k_h(\cdot, \cdot)} \left( \hat{P}^k_h(\cdot, \cdot), W^k_{h+1} \right)$ 
8: \hspace{3cm} $J^k_h(\cdot, \cdot) \leftarrow \exp(\beta(\cdot + 1 - h))$ 
9: \hspace{2cm} else 
10: \hspace{3cm} $J^k_h(\cdot, \cdot) \leftarrow \text{sign}(\beta) J^k_h(s^k_h, a)$ 
11: \hspace{1cm} end if 
12: \hspace{1cm} $W^k_h(\cdot) \leftarrow \text{max}_a J^k_h(\cdot, a)$ 
13: \hspace{1cm} end for 
14: \hspace{1cm} Receive $s^k_1$ 
15: \hspace{1cm} for $h = 1 : H$ do 
16: \hspace{2cm} $a^k_h \leftarrow \text{arg max}_a \text{sign}(\beta) J^k_h(s^k_h, a)$ and transit to $s^k_{h+1}$ 
17: \hspace{2cm} Compute $N^{k+1}_h(\cdot, \cdot)$ and $\hat{P}^{k+1}_h(\cdot, \cdot)$ 
18: \hspace{1cm} end for 
19: end for
5.2 Equivalence to ROVI

Risk-sensitive Optimistic Value Iteration (ROVI) is a non-distributional algorithm that deals with the value function rather than the return distribution. ROVI-MB can be shown to be equivalent to ROVI in the sense that they generate the policy sequence in the same way, which implies that their trajectories \( \mathcal{H}_F \) follow the same distribution. The intuition behind the equivalence is the connection between EntRM and EERM and the linearity of \( k \).

**Definition 19** For two algorithms \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \), we say that \( \mathcal{A} \) is equivalent to \( \tilde{\mathcal{A}} \) (vice versa) if for any \( k \in [K] \), any \( \mathcal{F}_k \) it holds that \( \mathcal{A}(\mathcal{F}_k) = \tilde{\mathcal{A}}(\mathcal{F}_k) \).

It follows from the induction that the whole history/trajectory \( \mathcal{F}_{K+1} \) generated by the interaction between each of two equivalent algorithms and any MDP instance follows the same distribution. Thus, the two algorithms enjoy equal regret.

**Proposition 20 (Equivalence between ROVI and ROVI-MB)** Algorithm 2 is equivalent to Algorithm 3.

**Proof** We only prove the case that \( \beta > 0 \). The case that \( \beta < 0 \) follows analogously. Fix an arbitrary \( k \in [K] \) and \( \mathcal{F}_k = \{s_1^k, a^k_1, R_1^k, \cdots, s^k_H, a^k_H, R^k_H\} \). Denote by \( \mathcal{A} \) (\( \tilde{\mathcal{A}} \)) and \( \{\pi^k_h\} \) (\( \{\tilde{\pi}^k_h\} \)) Algorithm 3 (Algorithm 2) and the associated policy sequence. It suffices to prove that \( \pi^k \) coincides with \( \tilde{\pi}^k \) for the same history \( \mathcal{F}_k \). By the definition of the two algorithms

\[
\tilde{\pi}^k_h(s) = \arg \max_a Q^k_h(s, a) = U_\beta(\tilde{\eta}^k_h(s, a)), \quad \pi^k_h(s) = \arg \max_a J^k_h(s, a).
\]

If \( J^k_h(s, a) = E_\beta(\eta^k_h(s, a)) = \exp(\beta Q^k_h(s, a)) \) for any \( (s, a) \), then \( \pi^k_h = \tilde{\pi}^k_h \) due to the monotonicity of the exponential function. We will prove that \( J^k_h(s, a) = E_\beta(\eta^k_h(s, a)) \) for any \( (s, a) \) by the induction. Notice that \( J^k_h(s, a) = E_\beta(\eta^k_H(s, a)) \). Assume that \( J^k_h(s, a) = E_\beta(\eta^k_h(s, a)) \) for all \( (s, a) \) for some \( h \in [H] \). It follows that \( \pi^k_h = \tilde{\pi}^k_h \) and

\[
W^k_h(s) = \max_a J^k_h(s, a) = J^k_h(s, \pi^k_h(s)) = E_\beta(\eta^k_h(s, \pi^k_h(s))) = E_\beta(\eta^k_h(s, \tilde{\pi}^k_h(s))) = E_\beta(\eta^k_H(s, a))
\]

Given the same history \( \mathcal{F}_k \), the two algorithms share the empirical transition model \( \tilde{P}^k_{h-1} \), the count \( N^k_{h-1} \), and the optimism constants \( c^k_{h-1} \). Therefore they also share the optimistic transition model \( \tilde{P}^k_{h-1} \). According to the update formula of Algorithm 3, we have that for any \( (s, a) \) with \( N^k_h(s, a) > 0 \)

\[
J^k_{h-1}(s, a) = \exp(\beta r_{h-1}(s, a)) \left[ \tilde{P}^k_{h-1}W^k_h \right](s, a) = \exp(\beta r_{h-1}(s, a))E_\beta \left( \left[ \tilde{P}^k_{h-1}\nu^k_h \right](s, a) \right)
\]

Moreover, the unvisited \( (s, a) \) (\( N^k_h(s, a) = 0 \)) satisfies

\[
J^k_{h-1}(s, a) = \exp(\psi_{H+1-h}) = \exp(\beta(h + 1 - h)) = E_\beta \left( \eta^k_{h-1}(s, a) \right).
\]

Thus the proof is completed. \( \blacksquare \)
5.3 Regret Upper Bound of RODI-MB/ROVI

Theorem 21 (Regret upper bound of RODI-MB/ROVI) For any $\delta \in (0, 1)$, with probability $1 - \delta$, the regret of Algorithm 1 or Algorithm 3 is bounded as

$$
\text{Regret}(\text{RODI-MF}, K) = \text{Regret}(\text{ROVI}, K) \leq O\left( \frac{1}{|\beta|} L_H H \sqrt{S^2 A K \log(4SAT/\delta)} \right)
$$

$$
= \tilde{O} \left( \frac{\exp(|\beta| H) - 1}{|\beta| H \sqrt{S^2 A K}} \right).
$$

Since Algorithm 3 is a classical algorithm, it is natural to use the classical analysis to derive the regret bounds. That being said, we will show that the distributional analysis yields a tighter bound than the non-distributional analysis. In particular, the latter one yields a regret bound that explodes as $\beta$ approaches zero, but our analysis can recover the desired order when reduced to the risk-neutral setting.

**Proof** The regret can be bounded as

$$
\text{Regret}(K) \leq \frac{1}{\beta} \sum_{k=1}^{K} W_1^k(s^k_1) - W_1^\pi(s^k_1) = \frac{1}{\beta} \sum_{k=1}^{K} \delta^k.
$$

We can decompose $\delta^k_h$ as follows

$$
\delta^k_h = E_\beta(\nu_h^k(s^k_h)) - E_\beta(\nu^\pi_h(s^k_h))
$$

$$
= \exp(\beta r^k_h) E_\beta \left( \left[ \tilde{P}_h^k \nu_{h+1}^k \right](s^k_h) \right) - \exp(\beta r^k_h) E_\beta \left( \left[ P_h^\pi \nu_{h+1}^\pi \right](s^k_h) \right)
$$

$$
= \exp(\beta r^k_h) E_\beta \left( \left[ \tilde{P}_h^k \nu_{h+1}^k \right](s^k_h) \right) - \exp(\beta r^k_h) E_\beta \left( \left[ P_h^\pi \nu_{h+1}^\pi \right](s^k_h) \right)
$$

$$
+ \exp(\beta r^k_h) E_\beta \left( \left[ P_h^\pi \nu_{h+1}^\pi \right](s^k_h) \right) - \exp(\beta r^k_h) E_\beta \left( \left[ P_h^\pi \nu_{h+1}^\pi \right](s^k_h) \right)
$$

$$
= \exp(\beta r^k_h) \left[ \tilde{P}_h^k W_{h+1}^k \right](s^k_h) - \exp(\beta r^k_h) \left[ P_h^\pi W_{h+1}^\pi \right](s^k_h)
$$

$$
+ \exp(\beta r^k_h) \left[ P_h^\pi W_{h+1}^\pi \right](s^k_h) - \exp(\beta r^k_h) \left[ P_h^\pi W_{h+1}^\pi \right](s^k_h).
$$

Both distributional analysis and non-distributional analysis seem to be viable to deal with (a), but the non-distributional analysis turns out to yield an unsatisfactory bound.

**Non-distributional analysis.** Notice that $W_{h+1}^k(s) \leq \exp(\beta(H - h))$, $\forall s$. Thus the following holds

$$
(a) = \exp(\beta r^k_h) \left[ \tilde{P}_h^k W_{h+1}^k \right](s^k_h) - \left[ P_h^\pi W_{h+1}^\pi \right](s^k_h)
$$

$$
= \exp(\beta r^k_h) \left[ \left( \tilde{P}_h^k - P_h^\pi \right) W_{h+1}^k \right](s^k_h)
$$

$$
\leq \exp(\beta) \left\| \tilde{P}_h^k - P_h^\pi \right\|_1 \max_s W_{h+1}^k(s)
$$

$$
\leq \exp(\beta(H + 1 - h)) c_h.
$$
**Distributional analysis.** Using the Lipschitz property of EERM

\[
(a) \leq L_{H+1-h} \left\| \tilde{P}_h^k \nu_{h+1}^k \right\|_\infty \leq \left\| \tilde{P}_h^k \nu_{h+1}^k \right\|_\infty
= L_{H+1-h} \left\| \tilde{P}_h^k \nu_{h+1}^k - P_h^k \nu_{h+1}^k \right\|_\infty
\leq L_{H+1-h} \left\| \tilde{P}_h^k - P_h^k \right\|_1 \leq L_{H+1-h} c_h^k
= (\exp(\beta(H + 1 - h)) - 1)c_h^k,
\]

where the second inequality is due to Lemma 10. The two types of analysis lead to different coefficients. Consider the risk-neutral setting \( \beta \to 0 \). For the distributional analysis, the coefficient appears in the regret bound as

\[
\lim_{\beta \to 0} \frac{\exp(\beta(H + 1 - h)) - 1}{\beta} = H + 1 - h
\]

In contrast, the non-distributional analysis leads to that

\[
\lim_{\beta \to 0} \frac{\exp(\beta(H + 1 - h))}{\beta} = \infty.
\]

For small \( \beta \), the distributional analysis recovers the order of the corresponding risk-neutral algorithm. However, the non-distributional analysis yields a exploding factor as \( \beta \to 0 \). Therefore, it is not proper to use the classical analysis to obtain the regret bound of Algorithm 3. Term (b) is bounded as

\[
(b) = \exp(\beta r_h^k) \left[ P_h^k (W_{h+1}^k - W_{h+1}^k) \right] (s_h^k) = \exp(\beta r_h^k) \left[ P_h^k \Delta_{h+1}^k \right] (s_h^k)
= \exp(\beta r_h^k)(e_h^k + \delta_{h+1}^k),
\]

where \( e_h^k \triangleq [P_h^k \Delta_{h+1}^k](s_h^k) - \Delta_{h+1}^k(s_{h+1}^k) \) is a martingale difference sequence with \( e_h^k \in 2D_{h+1} \) a.s. for all \((k, h) \in [K] \times [H]\). In summary, we can bound \( \delta_h^k \) recursively as

\[
\delta_h^k \leq L_{H+1-h} c_h^k + \exp(\beta r_h^k)(e_h^k + \delta_{h+1}^k).
\]

Repeating the procedure, we can get

\[
\delta_h^k \leq \sum_{h=1}^{H-1} \sum_{i=1}^{h-1} \frac{H-1}{h-1} L_{H+1-h} \prod_{i=1}^{h-1} \exp(\beta r_i^k) c_i^k + \sum_{h=1}^{H-1} \prod_{i=1}^{h-1} \exp(\beta r_i^k) e_h^k + \prod_{i=1}^{H-1} \exp(\beta r_i^k) \delta_H^k
\leq \sum_{h=1}^{H-1} \frac{H-1}{h-1} (\exp(\beta(H + 1 - h)) - 1)\exp(\beta(h - 1)) e_h^k + \sum_{h=1}^{H-1} \prod_{i=1}^{h-1} \exp(\beta r_i^k) e_h^k + \exp(\beta(H - 1)) \delta_H^k
\leq \sum_{h=1}^{H-1} (\exp(\beta H) - 1)e_h^k + \sum_{h=1}^{H-1} \prod_{i=1}^{h-1} \exp(\beta r_i^k) e_h^k + \exp(\beta(H - 1)) \delta_H^k.
\]

It follows that

\[
\sum_{k=1}^{K} \delta_h^k \leq (\exp(\beta H) - 1) \sum_{h=1}^{H-1} e_h^k + \sum_{h=1}^{H-1} \sum_{i=1}^{K} \prod_{i=1}^{h-1} \exp(\beta r_i^k) e_h^k + \sum_{k=1}^{K} \exp(\beta(H - 1)) \delta_H^k.
\]
The following follows analogously: with probability at least $1 - \delta$,

$$\text{Regret}(K) \leq \frac{\exp(\beta(H + 1)) - 1}{\beta} \left( 2(H - 1)\sqrt{2SAT} + \sqrt{2KH} + SA \right)
= \tilde{O}\left( \frac{\exp(\beta H) - 1}{\beta H} H\sqrt{HS^2AT} \right),$$

where $\iota \triangleq \log(2SAT/\delta)$.

**Remark 22** If we use non-distributional analysis, we will arrive at

$$\text{Regret}(K) \leq \tilde{O}\left( \frac{\exp(\beta H)}{\beta} \sqrt{HS^2AT} \right),$$

which blows up as $\beta \to 0$.

**Remark 23** Compared to the traditional/non-distributional analysis dealing with scalars, our analysis is distribution-centered, and we call it the distributional analysis. The distributional analysis deals with the distributions of the return rather than the risk measure values of the return. In particular, it involves the operations of the distributions, the optimism between different distributions, the error caused by estimation of distribution, etc. These distributional aspects fundamentally differ from the traditional analysis that deals with the scalars (value functions).

6. Regret Lower Bound

The proof of Fei et al. (2020) reduces the regret lower bound to the two-armed bandit regret lower bound. Since the two-armed bandit is a special case of MDP with $S = 1$, $A = 2$ and $H = 1$, the reduction-based proof only leads to a lower bound independent of $S, A,$ and $H$. Instead, our tight lower bound follows a totally different roadmap motivated by Domingues et al. (2021). Domingues et al. (2021) proves the tight minimax lower bound $H\sqrt{SAT}$ for risk-neutral MDP. However, the generalization to risk-sensitive MDP is non-trivial. The main technical challenge is due to the non-linearity of EntRM. The proof in Fei et al. (2020) heavily relies on the linearity of expectation, allowing the exchange between taking the risk measure (expectation) and the summation. In the risk-sensitive setting, the non-linearity of EntRM requires new proof techniques.

**Assumption 1** Assume $S \geq 6, A \geq 2,$ and there exists an integer $d$ such that $S = 3 + \frac{A^d - 1}{A - 1}$. We further assume that $H \geq 3d$ and $\bar{H} \triangleq \frac{d}{3} \geq 1$.

**Theorem 24 (Tighter lower bound)** Assume Assumption 1 holds and $\beta > 0$. Let $L \triangleq (1 - \frac{1}{\beta})(S - 3) + \frac{1}{\beta}$. Then for any algorithm $\mathcal{A}$, there exists an MDP $\mathcal{M}_{\mathcal{A}}$ such that for $K \geq 2 \exp(\beta(H - \bar{H} - d))HLA$ we have

$$\mathbb{E}[\text{Regret}(\mathcal{A}, \mathcal{M}_{\mathcal{A}}, K)] \geq \frac{1}{72 \sqrt{6}} \frac{\exp(\beta H/6) - 1}{\beta H} H\sqrt{SAT}.$$
Remark 25 As $\beta \to 0$, it recovers the tight lower bound for risk-neutral episodic MDP $\Omega(H\sqrt{SAT})$ (see Domingues et al., 2021).

Before presenting the proof of Theorem 24, we first show the correction of the lower bound in Fei et al. (2020).

6.1 Correction of Lower Bound

Fei et al. (2020) presents the following lower bound.

**Proposition 26 (Theorem 3, Fei et al. (2020))** For sufficiently large $K$ and $H$, the regret of any algorithm obeys

$$\mathbb{E}[\text{Regret}(K)] \gtrsim \frac{e^{\beta H/2} - 1}{|\beta|} \sqrt{TK \log T}.$$  

However, the lower bound itself and the proof are incorrect. The major mistake appears at the second inequality of the following statements in their proof

$$\mathbb{E}[\text{Regret}(K)] \gtrsim \frac{\exp(\beta H/2) - 1}{\beta} \sqrt{K \log(K)} \quad \gtrsim \frac{\exp(\beta H/2) - 1}{\beta} \sqrt{KH \log(KH)}.$$

The authors establish the second inequality based on the following fact

**Fact 2 (Fact 5, Fei et al. (2020))** For any $\alpha > 0$, the function $f_\alpha \triangleq e^{\alpha x} - 1 - x$, $x > 0$ is increasing and satisfies $\lim_{x \to 0} f_\alpha = \alpha$.

In fact, we can only use Fact 2 to derive $\frac{\exp(\beta H/2) - 1}{\beta} \gtrsim H$, which combined with the first inequality yields

$$\mathbb{E}[\text{Regret}(K)] \gtrsim H \sqrt{KH \log(KH)}.$$  

It is a weaker lower bound and does not feature the dependence on $\beta$. The best result we can get based on the original proof is that

**Proposition 27 (Correction of Theorem 3, Fei et al. (2020))** For sufficiently large $K$ and $H$, the regret of any algorithm obeys

$$\mathbb{E}[\text{Regret}(K)] \gtrsim \frac{e^{\beta H/2} - 1}{|\beta|} \sqrt{K \log K}.$$  

6.2 Proof of Theorem 24

We define $\text{kl}(p, q) \triangleq p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q}$ as the KL divergence between two Bernoulli distributions with parameters $p$ and $q$. We define the probability measure induced by an algorithm $A$ and an MDP instance $\mathcal{M}$ as

$$\mathbb{P}_A^\mathcal{M}(F^{K+1}) \triangleq \prod_{k=1}^{K} \mathbb{P}_{A_k(F^k), \mathcal{M}}(T^{k}_{H} | s^{k}_{1}).$$
where \( P_{\pi, M} \) is the probability measure induced by a policy \( \pi \) and \( M \), which is defined as
\[
P_{\pi, M}(I_H|s_1) \triangleq \prod_{h=1}^{H} \pi_h(a_h|s_h)P_h^M(s_{h+1}|s_h, a_h).
\]

Note that the probability measure for the truncated history \( H^k_h \) can be obtained by marginalization
\[
P_{\mathcal{A}, M}(H^k_h) = P_{\mathcal{A}, M}(F^k)P_{\mathcal{A}, M}(I^k_h).
\]

We denote by \( P_{\mathcal{A}, M} \) and \( E_{\mathcal{A}, M} \) the probability measure and expectation induced by \( \mathcal{A} \) and \( M \). For the sake of simplicity, the dependency on \( \mathcal{A} \) and \( M \) may be dropped if it is clear in the context.

**Fact 3 (Lemma 1, Garivier et al. (2019))** Consider a measurable space \((\Omega, \mathcal{F})\) equipped with two distributions \( P_1 \) and \( P_2 \). For any \( \mathcal{F} \)-measurable function \( Z : \Omega \to [0, 1] \), we have
\[
KL(P_1, P_2) \geq kl(E_1[Z], E_2[Z]),
\]
where \( E_1 \) and \( E_2 \) are the expectations under \( P_1 \) and \( P_2 \) respectively.

**Fact 4 (Lemma 5, Domingues et al. (2021))** Let \( M \) and \( M' \) be two MDPs that are identical except for their transition probabilities, denoted by \( P_h \) and \( P'_h \), respectively. Assume that we have \( \forall (s, a), P_h(\cdot|s, a) \ll P'_h(\cdot|s, a) \). Then, for any stopping time \( \tau \) with respect to \((I_k)_{k \geq 1}\) that satisfies \( P_M[\tau < \infty] = 1 \)
\[
KL(P_M, P_{M'}) = \sum_{(s, a, h) \in S \times A \times [H-1]} E_M[N^r_h(s, a)] KL(P_h(\cdot|s, a), P'_h(\cdot|s, a)).
\]

**Lemma 28** If \( \epsilon \geq 0, p \geq 0 \) and \( p + \epsilon \in [0, \frac{1}{2}] \), then
\[
kl(p, p + \epsilon) \leq \frac{\epsilon^2}{2p(1-p)} \leq \frac{\epsilon^2}{p}.
\]

**Proof** Fix \( q \in [0, 1] \), let \( h(p) := kl(p, q) \). It is immediate that
\[
h'(p) = \log \frac{p}{q} - \log \frac{1-p}{1-q},
\]
\[
h''(p) = \frac{1}{p(1-p)} > 0.
\]

Therefore \( h(p) \) is strictly convex, increasing in \((q, 1)\) and decreasing in \((0, q)\). By Taylor’s expansion, we have that
\[
h(p) = h(q) + h'(q)(p - q) + \frac{1}{2} h''(r)(p - q)^2 = \frac{(p - q)^2}{2r(1-r)}
\]
for some \( r \in [p, q] \) \( (p < q) \) or \( r \in [q, p] \) \( (p > q) \). In particular, for any \( \epsilon \geq 0 \) such that \( q = p + \epsilon \leq \frac{1}{2} \) it follows that
\[
kl(p, p + \epsilon) = \frac{(p - q)^2}{2r(1-r)}|_{q=p+\epsilon} = \frac{\epsilon^2}{2r(1-r)} \leq \frac{\epsilon^2}{2p(1-p)} \leq \frac{\epsilon^2}{p},
\]
where the first inequality follows from the fact that \( r \mapsto r(1 - r) \) is increasing in \([p, p + \epsilon] \subset [0, \frac{1}{2}]\) and the second inequality is due to that \( 1 - p \geq \frac{1}{2} \).

The proof of Theorem 24 adopts the same construction of hard MDP class \( C \) as Domingues et al. (2021).

**Proof** We consider the case that \( \beta > 0 \). Fix an arbitrary algorithm \( \mathcal{A} \). We introduce three types of special states for the hard MDP class: a waiting state \( s_w \) where the agent starts and may stay until stage \( \bar{H} \), after that it has to leave; a good state \( s_g \) which is absorbing and is the only rewarding state; a bad state \( s_b \) that is absorbing and provides no reward. The rest \( S - 3 \) states are part of a \( A \)-ary tree of depth \( d - 1 \). The agent can only arrive \( s_w \) from the root node \( s_{\text{root}} \) and can only reach \( s_g \) and \( s_b \) from the leaves of the tree.

Let \( \bar{H} \in [H - d] \) be the first parameter of the MDP class. We define \( \bar{H} \triangleq \bar{H} + d + 1 \) and \( H' \triangleq H + 1 - \bar{H} \). We denote by \( \mathcal{L} \triangleq \{s_1, s_2, \ldots, s_L\} \) the set of \( L \) leaves of the tree. For each \( u^* \triangleq (h^*, \ell^*, a^*) \in \{d + 1 : \bar{H} + d\} \times \mathcal{L} \times \mathcal{A} \), we define an MDP \( \mathcal{M}_{u^*} \) as follows. The transitions in the tree are deterministic, hence taking action \( a \) in state \( s \) results in the \( a \)-th child of node \( s \). The transitions from \( s_w \) are defined as

\[
P_h(s_w | s_w, a) \triangleq \mathbb{I}\{a = a_w, h \leq \bar{H}\} \quad \text{and} \quad P_h(s_{\text{root}} | s_w, a) \triangleq 1 - P_h(s_w | s_w, a).
\]

The transitions from any leaf \( s_i \in \mathcal{L} \) are specified as

\[
P_h(s_g | s_i, a) \triangleq p + \Delta_{u^*}(h, s_i, a) \quad \text{and} \quad P_h(s_b | s_i, a) \triangleq 1 - p - \Delta_{u^*}(h, s_i, a),
\]

where \( \Delta_{u^*}(h, s_i, a) \triangleq \epsilon \mathbb{I}\{h, s_i, a) = (h^*, \ell^*, a^*)\} \) for some constants \( p \in [0, 1] \) and \( \epsilon \in [0, \min(1 - p, p)] \) to be determined later. \( p \) and \( \epsilon \) are the second and third parameters of the MDP class. Observe that \( s_g \) and \( s_b \) are absorbing, therefore we have \( \forall a, P_h(s_g | s_g, a) \triangleq P_h(s_b | s_b, a) \triangleq 1 \). The reward is a deterministic function of the state

\[
r_h(s, a) \triangleq \mathbb{I}\{s = s_g, h \geq \bar{H}\}.
\]

Finally we define a reference MDP \( \mathcal{M}_0 \) which differs from the previous MDP instances only in that \( \Delta_0(h, s_i, a) \triangleq 0 \) for all \( (h, s_i, a) \). For each \( \epsilon, p \) and \( \bar{H} \), we define the MDP class

\[
\mathcal{C}_{\bar{H}, p, \epsilon} \triangleq \mathcal{M}_0 \cup \{\mathcal{M}_{u^*}\}_{u^* \in \{d + 1 : \bar{H} + d\} \times \mathcal{L} \times \mathcal{A}}.
\]

The total expected ERM value of \( \mathcal{A} \) is given by

\[
\begin{align*}
E_{\mathcal{A}, \mathcal{M}_{u^*}} &\left[ \sum_{k=1}^{K} \beta^k \left( \sum_{h=1}^{\bar{H}} r_h(s_h^k, a_h^k) \pi^k \right) \right] \\
= E_{\mathcal{A}, \mathcal{M}_{u^*}} &\left[ \sum_{k=1}^{K} \frac{1}{\beta} \log E_{\mathcal{A}, \mathcal{M}_{u^*}} \left[ \exp \left( \beta \sum_{h=1}^{\bar{H}} r_h(s_h^k, a_h^k) \right) \right] \right] \\
= E_{\mathcal{A}, \mathcal{M}_{u^*}} &\left[ \sum_{k=1}^{K} \frac{1}{\beta} \log E_{\pi^*, \mathcal{M}_{u^*}} \left[ \exp \left( \beta \sum_{h=1}^{\bar{H}} \mathbb{I}\{s_h^k = s_g\} \right) \right] \right] \\
= E_{\mathcal{A}, \mathcal{M}_{u^*}} &\left[ \sum_{k=1}^{K} \frac{1}{\beta} \log E_{\pi^*, \mathcal{M}_{u^*}} \left[ \exp(\beta H') \mathbb{I}\{s_h^k = s_g\} \right] \right] \\
= E_{\mathcal{A}, \mathcal{M}_{u^*}} &\left[ \sum_{k=1}^{K} \frac{1}{\beta} \log(\exp(\beta H')\mathbb{P}_{\pi^*, \mathcal{M}_{u^*}}(s_H^k = s_g) + \mathbb{P}_{\pi^*, \mathcal{M}_{u^*}}(s_H^k = s_b)) \right],
\end{align*}
\]
where the second equality follows from the fact that the reward is non-zero only after step $H$, the third equality is due to that the agent gets into absorbing state when $h \geq H$. Define $x_h^k \triangleq (s_h^k, a_h^k)$ for each $(k, h)$ and $x^* \triangleq (s^*, a^*)$, then it is not hard to obtain that

$$
P_{\pi^*, u^*} \left[ s^k_H = s_g \right] = \sum_{h=1}^{H+d} \mathbb{P}_{\pi^*, u^*} \left( s^k_h \in \mathcal{L} \right) + \mathbb{I} \left( h = H^* \right) \mathbb{P}_{\pi^*, u^*}(x^k_h = x^*) \varepsilon
$$

$$= p + \varepsilon \mathbb{P}_{\pi^*, u^*}(x^k_{H^*} = x^*).$$

For an MDP $M_{u^*}$, the optimal policy $\pi^*, M_{u^*}$ starts to traverse the tree at step $h^* - d$ then chooses to reach the leaf $s_{h^*}$ and performs action $a^*$. The corresponding optimal value in any of the MDPs is $V^*, M_{u^*} = \frac{1}{\beta} \log(\exp(\beta H')(p + \varepsilon) + 1 - p - \varepsilon)$. Define $p_{u^*}^k \triangleq \mathbb{P}_{\pi^*, u^*}(x^k_{h^*} = x^*)$, then the expected regret of $\mathcal{A}$ in $M_{u^*}$ can be bounded below as

$$E_{\mathcal{A}, M_{u^*}} \left[ \text{Regret}(\mathcal{A}, M_{u^*}, K) \right] = E_{\mathcal{A}, M_{u^*}} \left[ \sum_{k=1}^{K} V^*, M_{u^*} - U_{\beta} \left( \sum_{h=1}^{H} \tau_h(x^k_{h}) | \pi^* \right) \right]$$

$$= E_{\mathcal{A}, M_{u^*}} \left[ \sum_{k=1}^{K} \frac{1}{\beta} \log \frac{\exp(\beta H')(p + \varepsilon) + 1 - p - \varepsilon}{\exp(\beta H')(p + \varepsilon p_{u^*}^k) + 1 - p - \varepsilon p_{u^*}^k} \right]$$

$$= E_{\mathcal{A}, M_{u^*}} \left[ \sum_{k=1}^{K} \frac{1}{\beta} \log \left( 1 + \frac{\varepsilon (1 - p_{u^*}^k)(\exp(\beta H') - 1)}{\exp(\beta H')(p + \varepsilon p_{u^*}^k) + 1 - p - \varepsilon p_{u^*}^k} \right) \right]$$

$$\geq E_{\mathcal{A}, M_{u^*}} \left[ \frac{\exp(\beta H') - 1}{4\beta} \varepsilon \sum_{k=1}^{K} (1 - p_{u^*}^k) \right]$$

$$= \frac{\exp(\beta H') - 1}{4\beta} K \varepsilon \left( 1 - K E_{\mathcal{A}, M_{u^*}} \left[ N_K(u^*) \right] \right)$$

The first inequality holds by setting $p + \varepsilon \leq \exp(-\beta H')$. The second inequality holds by letting $\varepsilon \leq 2 \exp(-\beta H')$ since $\log(1 + x) \geq \frac{x}{2}$ for $x \in [0, 1]$. The last equality follows from the fact that

$$E_{\mathcal{A}, M_{u^*}} \left[ p_{u^*}^k \right] = E_{\mathcal{A}, M_{u^*}} \left[ \mathbb{P}_{\pi^*, u^*}(x^k_{h^*} = x^*) \right] = \mathbb{P}_{\mathcal{A}, u^*}(x^k_{h^*} = x^*) = E_{\mathcal{A}, u^*} \left[ \mathbb{I} \left( x^k_{h^*} = x^* \right) \right]$$

and the definition of $N_K(u^*) \triangleq \sum_{k=1}^{K} \mathbb{I} \left( x^k_{h^*} = x^* \right)$.
The maximum of the regret can be bounded below by the mean over all instances as
\[
\max_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \text{Regret} (∅, \mathcal{M}_{u^*}, K) \geq \frac{1}{H L A} \sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \text{Regret} (∅, \mathcal{M}_{u^*}, K)
\]
\[
\geq \exp(\beta H') - 1 \frac{K \epsilon}{4 \beta} \left( 1 - \frac{1}{L A K H} \sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \mathbb{E}_{u^*} [N_K(u^*)] \right).
\]

Observe that it can be further bounded if we can obtain an upper bound on \(\sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \mathbb{E}_{u^*} [N_K(u^*)]\), which can be done by relating each expectation to the expectation under the reference MDP \(\mathcal{M}_0\).

By applying Fact 3 with \(Z = \frac{N_K(u^*)}{K} \in [0, 1]\), we have
\[
\text{kl} \left( \frac{1}{K} \mathbb{E}_0 [N_K(u^*)], \frac{1}{K} \mathbb{E}_{u^*} [N_K(u^*)] \right) \leq \text{KL}(\mathbb{P}_0, \mathbb{P}_{u^*}).
\]

By Pinsker’s inequality, it implies that
\[
\frac{1}{K} \mathbb{E}_{u^*} [N_K(u^*)] \leq \frac{1}{K} \mathbb{E}_0 [N_K(u^*)] + \sqrt{\frac{1}{2} \text{KL}(\mathbb{P}_0, \mathbb{P}_{u^*})}.
\]

Since \(\mathcal{M}_0\) and \(\mathcal{M}_{u^*}\) only differs at stage \(h^*\) when \((s, a) = x^*\), it follows from Fact 4 that
\[
\text{KL}(\mathbb{P}_0, \mathbb{P}_{u^*}) = \mathbb{E}_0 [N_K(u^*)] \text{kl}(p, p + \epsilon).
\]

By Lemma 28, we have \(\text{kl}(p, p + \epsilon) \leq \frac{2}{p^{\epsilon}}\) for \(\epsilon \geq 0\) and \(p + \epsilon \in [0, \frac{1}{2}]\). Consequently,
\[
\frac{1}{K} \sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \mathbb{E}_{u^*} [N_K(u^*)]
\leq \frac{1}{K} \mathbb{E}_0 \left[ \sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} N_K(u^*) \right] + \frac{\epsilon}{\sqrt{2p}} \sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \sqrt{\mathbb{E}_0 [N_K(u^*)]}
\leq 1 + \frac{\epsilon}{\sqrt{2p}} \sqrt{L A K H},
\]

where the second inequality is due to the Cauchy-Schwartz inequality and the fact that \(\sum_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} N_K(u^*) = K\).

It follows that
\[
\max_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \text{Regret} (∅, \mathcal{M}_{u^*}, K) \geq \frac{\exp(\beta H') - 1}{4 \beta} \frac{K \epsilon}{L A H} \left( 1 - \frac{1}{L A H} \frac{\sqrt{L A K H}}{K} \right).
\]

Choosing \(\epsilon = \frac{\sqrt{p}}{2} \left( 1 - \frac{1}{L A H} \right) \sqrt{\frac{L A H}{K}}\) maximizes the lower bound
\[
\max_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \text{Regret} (∅, \mathcal{M}_{u^*}, K) \geq \frac{\sqrt{p}}{8 \sqrt{2}} \frac{\exp(\beta H') - 1}{\beta} \left( 1 - \frac{1}{L A H} \right)^2 \sqrt{L A K H}.
\]
Since $S \geq 6$ and $A \geq 2$, we have $\bar{L} = (1 - \frac{1}{4})(S - 3) + \frac{1}{4} \geq \frac{S}{4}$ and $1 - \frac{1}{L \bar{A} H} \geq 1 - \frac{1}{\frac{3}{2} 2} = \frac{2}{3}$.

Choose $\bar{H} = \frac{H}{3}$ and use the assumption that $d \leq \frac{H}{3}$ to obtain that $H' = H - d - \bar{H} \geq \frac{H}{3}$. Now we choose $p = \frac{1}{4} \exp(-\beta H')$ and $\epsilon = \sqrt{\frac{2}{2} (1 - \frac{1}{L \bar{A} H}) \sqrt{L \bar{A} H}} \frac{1}{\epsilon} \exp(-\beta H'/2) \sqrt{L \bar{A} H} \leq \frac{1}{\epsilon} \exp(-\beta H')$ if $K \geq 2 \exp(\beta H') \bar{L} \bar{A} H$. Such choice of $p$ and $\epsilon$ guarantees the assumption of Lemma 28 and that $p + \epsilon \leq \exp(-\beta H')$, $\epsilon \leq 2 \exp(-\beta H')$. Finally we use the fact that $\sqrt{L \bar{A} H} \geq \frac{1}{2\sqrt{2}} \sqrt{3} \sqrt{S A K H}$ to obtain

$$\max_{u^* \in [d+1:H+d] \times \mathcal{L} \times \mathcal{A}} \text{Regret}(\mathcal{A}, \mathcal{M}_{u^*}, K) \geq \frac{1}{72\sqrt{6}} \frac{\exp(\beta H/6) - 1}{\beta} \sqrt{S A K H}.$$ 

Theorem 24 recovers the tight lower bound for standard episodic MDP, implying that the exponential dependence on $|\beta|$ and $H$ in the upper bounds is indispensable. Yet, it is not clear whether a similar lower bound holds for $\beta < 0$, which is left as a future direction.

7. Conclusion and Discussion

In this paper, we present a novel framework for risk-sensitive distributional dynamic programming. We introduce two DRL algorithms, a model-free and a model-based approach, which implement the OFU principle at the distributional level to strike a balance between exploration and exploitation under the risk-sensitive setting. We provide rigorous theoretical analysis demonstrating that both algorithms achieve near-optimal regret upper bounds compared to our improved lower bound.

Looking forward, there are several promising avenues for future research. Our current regret upper bound has an additional factor of $\sqrt{HS}$ compared to the lower bound, and it may be possible to eliminate this factor through further algorithmic improvements or refined analysis techniques. Additionally, extending our DRL algorithm from tabular MDP to linear function approximation settings would be an interesting and valuable direction for future investigation. Lastly, it would be worthwhile to explore whether our DDP framework can be applied to other risk measures beyond the ones considered in this paper.

A. Comparisons with Related Works

In this section, we compare our work with closely related works in details. First, we summarize the differences between our work and Achab and Neu (2021) as follows.

- Setting. Achab and Neu (2021) considers the discounted MDP with infinite horizon, but we consider the episodic MDP setting. Moreover, Achab and Neu (2021) assumes that the model is known, while we propose DRL algorithms when the model is unknown. Neither RL algorithms suitable for unknown model nor sample complexity guarantee is provided in their work.

- Risk measure. Achab and Neu (2021) establishes the RS-DDP framework using the risk measure Conditional Value at Risk, while our work considers the entropic risk measure.
Fei et al. (2020, 2021) solved the risk-sensitive MDP problem using valued-based RL, which estimates and constructs the optimistic version of the EntRM value function. Fei et al. (2021) proposed the RSVI2 algorithm that improved upon Fei et al. (2020) and achieved the best result with the regret upper bound of $\tilde{O}(\frac{\exp(|\beta|H^2)}{|\beta|}H\sqrt{S^2AK})$. The significance of the proposed algorithms is three-fold.

- Our algorithms are the first distributional reinforcement learning algorithms with provably regret guarantees, suggesting that DRL can work well and even matches the performance of the SOTA value-based RL algorithm for risk-sensitive control in terms of sample complexity. The idea of leveraging the distributional information for risk-sensitivity purposes is natural since the risk measure value is obtained by applying the risk measure/functional to the return distribution. However, existing works on risk-sensitive control via DRL approaches lack regret analysis. Thus, it is difficult to evaluate and improve their algorithms for sample efficiency. Therefore, our algorithms with near-optimal regret upper bounds bridge the gap between the DRL and risk-sensitive MDP in the theoretic RL community.

- Compared with Fei et al. (2021), our algorithms are simpler and easier to interpret, leading to clean regret analysis. [21] implements optimism by adding a bonus to the risk measure value function. It designed an exploration mechanism called doubly decaying bonus to remove the $\exp(|\beta|H^2)$ factor from Fei et al. (2020). The doubly decaying bonus decays across the episode and the horizon, which is complicated and not straightforward. Instead, our algorithms implement the distributional optimism by iteratively constructing the optimistic return distribution. The distributional optimism does not involve a complicated bonus design. It only requires a simple application of distributional optimism operator with a constant decaying across the episode. Moreover, the doubly decaying bonus obscures the regret analysis, while our distributional-based analysis is clean and easy to follow.

- Our algorithm may be generalized to risk-sensitive MDP with other risk measures. The analysis of Fei et al. (2020, 2021) is particularly suitable for the EntRM. It is unclear whether it is possible to extend to other risk measures. Under the distributional perspective, our algorithm maintains a sequence of optimistically plausible estimates of the return distribution. Since the distributional information suffices to deal with any risk measure, our algorithm may motivate the design of similar algorithms for other risk measures.

### B. Additional Property of EntRM

We state some lemmas about the monotonicity-preserving property and their proofs here. The results hold for general risk measures satisfying the monotonicity-preserving property.

**Lemma 29** Let $T$ be a risk measure satisfying the monotonicity-preserving property. For any $F$ and $G$ such that $T(F) < T(G)$ and $0 \leq \theta' < \theta \leq 1$,

$$T(\theta F + (1 - \theta)G) < T(\theta' F + (1 - \theta')G).$$
Proof Let $\tilde{\theta} = \frac{\theta'}{\theta + \theta'} \in [\theta', \theta]$ and $\tilde{\theta} = \theta - \theta' \in (0, 1]$. It holds that
\[
\theta F + (1 - \theta) G = \tilde{\theta} F + (1 - \tilde{\theta})(\tilde{\theta} F + (1 - \tilde{\theta})G)
\]
\[
\theta' F + (1 - \theta') G = \tilde{\theta} G + (1 - \tilde{\theta})(\tilde{\theta} F + (1 - \tilde{\theta})G).
\]
The result follows from the monotonicity-preserving property
\[
T(\tilde{\theta} F + (1 - \tilde{\theta})(\tilde{\theta} F + (1 - \tilde{\theta})G)) < T(\tilde{\theta} G + (1 - \tilde{\theta})(\tilde{\theta} F + (1 - \tilde{\theta})G)).
\]

Lemma 30 Let $T$ be a risk measure satisfying the monotonicity-preserving property and $n \geq 2$ be an arbitrary integer. If $T(F_i) \geq T(G_i), \forall i \in [n]$ (and $T(F_j) \neq T(G_j)$ for some $j \in [n]$) then $T\left(\sum_{i=1}^{n} \theta_i F_i\right) \geq (\sum_{i=1}^{n} \theta_i G_i)$ for any $\theta \in \Delta_n$ (and $\theta_j \neq 0$).

Proof The proof follows from induction. Note that \[\sum_{i=1}^{n} \theta_i F_i = \theta_1 F_1 + (1 - \theta_1) \sum_{i=2}^{n} \frac{\theta_i}{1 - \theta_1} F_i\] and \[\sum_{i=2}^{n} \frac{\theta_i}{1 - \theta_1} F_i \in \mathcal{D},\] therefore by Lemma 29 we have $T(\sum_{i=1}^{n} \theta_i F_i) \geq T(\tilde{\theta}_1 G_1 + \sum_{i=2}^{n} \theta_i F_i)$. Suppose that for some $k \in [n-1]$ it holds that $T(\sum_{i=1}^{n} \theta_i F_i) \geq T(\sum_{i=1}^{k} \theta_i F_i + \sum_{i=k+1}^{n} \theta_i F_i)$.

Since
\[
\sum_{i=1}^{k} \theta_i G_i + \sum_{i=k+1}^{n} \theta_i F_i = \theta_{k+1} F_{k+1} + \sum_{i=1}^{k} \theta_i G_i + \sum_{i=k+2}^{n} \theta_i F_i
\]
\[
= \theta_{k+1} F_{k+1} + (1 - \theta_{k+1}) \left[\sum_{i=1}^{k} \frac{\theta_i}{1 - \theta_{k+1}} G_i + \sum_{i=k+2}^{n} \frac{\theta_i}{1 - \theta_{k+1}} F_i\right]
\]
and \[\frac{1}{1 - \theta_{k+1}} \left[\sum_{i=1}^{k} \theta_i G_i + \sum_{i=k+2}^{n} \theta_i F_i\right] \in \mathcal{D},\] it follows that
\[
T\left(\sum_{i=1}^{n} \theta_i F_i\right) \geq T\left(\sum_{i=1}^{k} \theta_i G_i + \sum_{i=k+1}^{n} \theta_i F_i\right) \geq T\left(\sum_{i=1}^{k+1} \theta_i G_i + \sum_{i=k+2}^{n} \theta_i F_i\right).
\]
The induction is completed. If in addition $T(F_j) > T(G_j)$ for some $j \in [n]$, the proof follows analogously by replacing the inequality with the strict inequality and the fact that $\theta_j > 0$. □

Lemma 31 (Monotonicity-preserving under pairwise transport) Let $T$ be a risk measure satisfying the monotonicity-preserving property. Suppose $n \geq 2$ and $(F_i)_{i \in [n]}$ satisfies $T(F_1) \leq T(F_2) \ldots \leq T(F_n)$. For any $\theta, \theta' \in \Delta_n$ and any $1 \leq i < j \leq n$ such that
\[
\begin{align*}
\theta'_i &\leq \theta_i, \\
\theta'_j &\geq \theta_j, \\
\theta'_k &\equiv \theta_k, \quad k \neq i, j
\end{align*}
\]
It holds that $T(\sum_{i=1}^{n} \theta_i F_i) \leq T(\sum_{i=1}^{n} \theta'_i F_i)$. 33
**Proof** Observe that
\[
\sum_{k=1}^{n} \theta'_k F_k = \theta'_i F_i + \theta'_j F_j + \sum_{k \neq i,j} \theta'_k F_k = \theta'_i F_i + \theta'_j F_j + \sum_{k \neq i,j} \theta_k F_k \\
= (\theta'_i F_i + \theta'_j F_j) + (1 - \theta_i - \theta_j) \sum_{k \neq i,j} \theta_k F_k.
\]

By Lemma 29, it suffices to prove \( T(\frac{1}{\theta_i + \theta_j} (\theta'_i F_i + \theta'_j F_j)) \geq T(\frac{1}{\theta_i + \theta_j} (\theta_i F_i + \theta_j F_j)) \). The result follows from the definition and the fact that \( T(F_i) \leq T(F_j) \) and \( \theta'_i \leq \theta_i \).

**Lemma 32 (Monotonicity-preserving under block-wise transport)** Suppose \( n \geq 2 \) and \((F_i)_{i \in [n]}\) satisfies \( T(F_1) \leq T(F_2) \leq \cdots \leq T(F_n) \). It holds that \( T(\sum_{i=1}^{n} \theta_i F_i) \leq T(\sum_{i=1}^{n} \theta'_i F_i) \) for any \( \theta, \theta' \in \Delta_n \) satisfying \( \exists k \in [n], \theta'_i \leq \theta_i \) if \( i \leq k \) and \( \theta'_i \geq \theta_i \) otherwise.

**Proof** Fix \( k \in [n] \). We rewrite the assumption imposed to \( \theta' \) as \( \theta'_i = \theta_i - \delta_i \) for \( i \leq k \) and \( \theta'_i = \theta_i + \delta_i \) for \( i > k \), where each \( \delta_i \geq 0 \). It will be shown that there exists a sequence \( \{\theta^l\}_{l \in [k]} \) satisfying \( \theta^0 = \theta \) and \( \theta^k = \theta' \) such that \( T(\theta^l) \leq T(\theta^{l+1}) \), then the proof shall be completed.

The sequence is constructed as follows: at the \( l \)-th iteration, we transport probability mass \( \delta_l \) of \( \theta_l \) to the probability mass of \( k+1, \ldots, n \). Specifically, we start from moving to the least number \( i_l \geq i_{l-1} \) that satisfy \( \theta_{i_l}^{l-1} < \theta_{i_l}^l \) and sequentially move to the next one if there is remaining mass. The iteration stops until all the mass \( \delta_l \) are transported. Repeating the procedure for \( k \) times we obtain \( \theta^k = \theta' \). The inequality \( T(\theta^l) \leq T(\theta^{l+1}) \) for each iteration follows from Lemma 31.

Recall that the distributional optimism operator \( O^1_c : \mathcal{D}(S) \mapsto \mathcal{D}(S) \) over space of PMFs with level \( c \) and future return \( \nu \in \mathcal{D}^S \) as
\[
O^1_c \left( \tilde{P}, \nu \right) \triangleq \arg \max_{P \in B_1(\tilde{P}, c)} U_\beta([P \nu]).
\]

By Lemma 32, \( O^1_c \left( \tilde{P}, \nu \right) \) can be computed as follows
- sort \( \nu \) in the ascending order such that \( U_\beta(\nu^1) \leq U_\beta(\nu^2) \cdots \leq U_\beta(\nu^S) \)
- permute \( \tilde{P} \) in the order of \( \nu \)
- move probability mass \( \tilde{c} \) of the first \( S - 1 \) states sequentially to the \( S \)-th state

The computational complexity of the three steps are \( O(S \log(S)) \), \( O(S) \), and \( O(S) \). Therefore the computational complexity of applying \( O^1_c \) in Line 6 of Algorithm 2 is only \( O(S \log(S)) \).

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