Hypergraphs and hypermatrices with symmetric spectrum

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Abstract

It is well known that a graph is bipartite if and only if the spectrum of its adjacency matrix is symmetric. In the present paper, this assertion is dissected into three separate matrix results of wider scope, which are extended also to hypermatrices. To this end the concept of bipartiteness is generalized by a new monotone property of cubical hypermatrices, called odd-colorable matrices. It is shown that a nonnegative symmetric $r$-matrix $A$ has a symmetric spectrum if and only if $r$ is even and $A$ is odd-colorable. This result also solves a problem of Pearson and Zhang about hypergraphs with symmetric spectrum and disproves a conjecture of Zhou, Sun, Wang, and Bu.

Separately, similar results are obtained for the $H$-spectram of hypermatrices.

Keywords: hypergraphs; hypermatrices; eigenvalues; $H$-eigenvalues; symmetric spectrum; odd transversal.

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1 Introduction

The purpose of this paper is to extend the following well-known result in spectral graph theory:

**Theorem B** A graph is bipartite if and only if its adjacency matrix has a symmetric spectrum.

Recall that the spectrum of a complex square matrix $A$ is called symmetric if it is the same as the spectrum of $-A$.

Notwithstanding the fame of Theorem B, we find that it is a certain mismatch, obtained by forcing together several more general statements, with no regard to their key differences. To clarify this point we shall distill a few one-sided implications from the mix of Theorem B.

Thus, for any $n \times n$ complex matrix $A = [a_{ij}]$ and nonempty sets $I \subset [n], J \subset [n]$, write $A[I,J]$ for the submatrix of all $a_{ij}$ with $i \in I$ and $j \in J$. Now, call $A$ bipolar if there is a partition $[n] = U \cup W$ such that $A[U,U] = 0$ and $A[W,W] = 0$. Clearly, the adjacency matrix of a bipartite graph is bipartite, but the above definition extends to any square matrix.

As with graphs, by negating eigenvectors over one of the partition sets, we get:

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Proposition 1 If a matrix is bipartite, then its spectrum is symmetric.

Clearly, Proposition 1 immediately implies half of Theorem B, but is much more general, and besides has nothing to do with graphs. Aiming at the other half of Theorem B, note that the existence of a general converse of Proposition 1 is highly unlikely. Indeed, the matrix

\[ H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]

dashes hopes for a converse of Proposition 1 even within the class of real symmetric matrices; additionally, the Kronecker powers of \( H_2 \) provide infinitely many examples to the same effect.

Furthermore, letting \( I_n \) be the identity matrix of order \( n \), we see that the matrices

\[ \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \]

are cospectral; yet the first one is bipartite, whereas the second one is not. Therefore, in general, bipartiteness cannot be inferred from the spectra of real symmetric matrices. Obviously, pairing Proposition 1 with a converse in the spirit of Theorem B will badly squash its scope, so it is better left as is. Hints for possible development are given in Proposition 6 and Question 7 below.

Nonetheless, it is interesting to find for which matrices spectral conditions may imply bipartiteness. With this goal in mind, we narrow the focus to nonnegative matrices, and arrive at the following statement:

Theorem 2 Let \( A \) be an irreducible nonnegative square matrix with spectral radius \( \rho \). If \( -\rho \) is an eigenvalue of \( A \), then \( A \) is bipartite.

Theorem 2 was proved by Cvetković, Doob, and Sachs ([3], p. 83) for strongly connected digraphs, but their proof extends with no change to any irreducible nonnegative square matrix. Let us emphasize that Theorem 2 also has nothing to do with graphs—it’s crux is the irreducibility of \( A \), and the proof rests entirely on the Perron-Frobenius theory.

It should be noted that in [7], Esser and Harary stated similar results for digraphs, but omitted the requirement for strong connectivity, thereby compromising their theorems. For example, under the spell of Theorem B, they claim on p. 18 of [7] that "A digraph is bipartite if and only if its adjacency matrix has a symmetric spectrum." This is easily disproved; e.g., the digraph with adjacency matrix

\[ A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]
has spectrum \( \{1, 1, -1, -1\} \), but is not bipartite. Moreover, since \( A_1 \) is cospectral with the bipartite matrix

\[
A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

it follows that bipartite digraphs cannot be characterized by their spectra in general. For additional examples to the same effect the construction of \( A_1 \) can be generalized as follows: take two disjoint bipartite graphs \( G \) and \( H \), and add all arcs from \( G \) to \( H \). The resulting digraph is non-bipartite, as it contains transitive 3-cycles, but its spectrum is symmetric, as it is the multiset union of the spectra of \( G \) and \( H \).

On the positive side, Theorem 2 implies the following corollary:

**Corollary 3** If \( A \) is a symmetric nonnegative matrix with symmetric spectrum, then \( A \) is bipartite.

In the light of Proposition 1, Theorem 2, and Corollary 3, we see that Theorem B is just a modest corollary of more general results. Thus, the main goal of this paper is to generalize Proposition 1, Theorem 2, and Corollary 3 to hypergraphs and hypermatrices.

Let us recall that in [16], Pearson and Zhang raised a similar problem, which can be stated as: characterize all connected \( r \)-graphs with symmetric spectrum. Some incomplete solutions to this problem were given in [12], [19], and [22]. In this paper we find necessary and sufficient conditions for the symmetry of the spectrum of a nonnegative symmetric hypermatrices and obtain a complete solution of the problem of Pearson and Zhang. We also disprove a conjecture stated by Zhou, Sun, Wang, and Bu in [22].

The structure of the remaining part of the paper is as follows: In Section 2 we give some basic definitions and results for hypermatrices and hypergraphs. Section 3 is dedicated to properties of hypermatrices and hypergraphs that generalize bipartiteness. In that section we construct some families of hypergraphs that disprove the conjecture of Zhou et al. mentioned above. The main results of the paper are in Section 4, where we characterize nonnegative symmetric hypermatrices with symmetric spectrum. Finally, in Section 5 we discuss a few general questions of spectral hypergraph theory.

## 2 Hypermatrices and their eigenvalues

Let \( r \geq 2 \), and let \( n_1, \ldots, n_r \) be positive integers. An \( r \)-matrix of order \( n_1 \times \cdots \times n_r \) is a function defined on the Cartesian product \([n_1] \times \cdots \times [n_r]\). In this note we consider only the case \( n_1 = \ldots, n_r = n \) and call such an \( r \)-matrix a cubical \( r \)-matrix of order \( n \).\(^1\)

\(^1\)In graph theory the order of a (hyper)graph is the number of its vertices, and in much of matrix theory the order of a square matrix means the number of its rows. We keep these meanings.
Hereafter, “matrix” will stand for “r-matrix” with unspecified r; thus, ordinary matrices will be referred to as “2-matrices”. We denote matrices by capital letters, whereas their values are denoted by the corresponding lowercase letter with the variables listed as subscripts. For example, if A is an r-matrix of order n, we let \( a_{i_1,\ldots,i_r} := A(i_1,\ldots,i_r) \) for all \( i_1,\ldots,i_r \in [n] \).

In analogy to 2-matrices, given a cubical r-matrix A of order n and a set \( X \subseteq [n] \), we write \( A[X] \) for the cubical matrix \( a_{i_1,\ldots,i_r} \) of all \( \{i_1,\ldots,i_r\} \in X^r \), and call \( A[X] \) a principal submatrix of A induced by X.

Let A be a cubical r-matrix of order n. Following the general setup of [14], define the eigenvalues of A as in [4]: an eigenvalue of A is a complex number \( \lambda \) that satisfies the equations
\[
\lambda x_k^{r-1} = \sum_{i_2,\ldots,i_r} a_{k,i_2,\ldots,i_r} x_{i_2} \cdots x_{i_r} \quad k = 1,\ldots,n,
\]
for some nonzero complex vector \((x_1,\ldots,x_n)\), called an eigenvector to \( \lambda \). The eigenvalues of A are the roots of its characteristic polynomial \( \phi_A(x) \) (see [10] for details on \( \phi_A(x) \)) and the multiset of all roots of \( \phi_A(x) \) is called the spectrum of A. In particular, the multiplicity of an eigenvalue \( \lambda \) as a root of \( \phi_A(x) \) is called the algebraic multiplicity of \( \lambda \). The spectral radius \( \rho(A) \) of A is the largest modulus of its eigenvalues.

A useful subset of eigenvalues was introduced by Qi in [14]: an H-eigenvalue of A is a real number \( \lambda \), which satisfies the equations (1) for some nonzero real vector \((x_1,\ldots,x_n)\), called an H-eigenvector to \( \lambda \). The H-spectrum of A is defined as the set of all H-eigenvalues and the H-spectral radius \( \rho_H(A) \) of A is the largest modulus of its H-eigenvalues.

Defining symmetry of the spectrum of a cubical hypermatrix is a problem of its own. Following the familiar path, we say that the spectrum of a cubical matrix A is symmetric if it is the same as the spectrum of \( -A \). It can be shown that the spectrum of A is symmetric if and only if for every eigenvalue \( \lambda \) of A, \( -\lambda \) is also an eigenvalue of A with the same algebraic multiplicity as \( \lambda \). In contrast, we say that the H-spectrum of a cubical matrix A is symmetric if for every H-eigenvalue \( \lambda \) of A, \( -\lambda \) is also an H-eigenvalue of A. Thus, eigenvalue multiplicity is irrelevant for the symmetry of the H-spectrum.

We also suggest a geometric spectral symmetry:

**Definition 4** The spectrum of a cubical matrix A of order n is called geosymmetric if there exists a unitary diagonal operator \( \Phi : \mathbb{C}^n \to \mathbb{C}^n \) such that if \( \lambda \) is an eigenvalue of A with eigenvector \( x \), then \( -\lambda \) is also an eigenvalue of A with eigenvector \( \Phi(x) \).

**Definition 5** The H-spectrum of a cubical matrix A of order n is called geosymmetric if there exists an orthogonal diagonal operator \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) such that if \( \lambda \) is an eigenvalue of A with eigenvector \( x \), then \( -\lambda \) is also an eigenvalue of A with eigenvector \( \Phi(x) \).

Geosymmetric spectrum is a new concept even for 2-matrices. For example, the spectrum of the 2-matrix
\[
\begin{bmatrix}
I_n & 0 \\
0 & -I_n
\end{bmatrix}
\]
is symmetric, but not geosymmetric. Hence, Proposition 1 may be put in a stronger form:
Proposition 6 If a 2-matrix is bipartite, then its spectrum is geosymmetric.

The new detail prompts a new search for a reasonable converse:

Question 7 Which square 2-matrices with geosymmetric spectrum are bipartite?

It is not hard to see that for 2-matrices “geosymmetric spectrum” implies “symmetric spectrum”, and for any matrix “geosymmetric H-spectrum” implies “symmetric H-spectrum”, but the general relation is not so clear:

Question 8 Let $r \geq 3$ and $A$ be an $r$-matrix with geosymmetric spectrum. Is it always true that the spectrum of $A$ is symmetric?

We shall also need a Perron-Frobenius type theorem, so we give some definitions next: The digraph $\mathcal{D}(A)$ of a cubical $r$-matrix of order $n$ is defined by setting $V(\mathcal{D}(A)) := [n]$ and letting $\{k, j\} \in E(\mathcal{D}(A))$ whenever there is a nonzero entry $a_{k,i_2,\ldots,i_r}$ such that $j \in \{i_2, \ldots, i_r\}$. Following [8], a cubical matrix is called weakly irreducible if its digraph is strongly connected; if a cubical matrix is not weakly irreducible, it is called weakly reducible.

The combined work of Chang, Pearson, and Zhang [4], Yang and Yang [20], and Friedland, Gaubert, and Han [8] laid the ground for a Perron-Frobenius theory of nonnegative hypermatrices. Of this large body of work we shall need the following theorem:

Theorem 9 If $A$ is a nonnegative cubical matrix, then $\rho(A)$ is an eigenvalue of $A$. If $A$ is also weakly irreducible and $x$ is a nonnegative eigenvector to $\rho(A)$, then $x$ is positive.

2.1 Eigenvalues of real symmetric matrices

A cubical $r$-matrix is called symmetric if $a_{i_1,\ldots,i_r} = a_{p(i_1,\ldots,i_r)}$ for every $(i_1,\ldots,i_r) \in [n]^r$ and every permutation $p(i_1,\ldots,i_r)$ of $(i_1,\ldots,i_r)$.

For a real symmetric 2-matrix eigenvalues can be alternatively defined by taking the Lagrange multipliers at the critical points of the matrix quadratic form over the Euclidean sphere. In [14] and [15], Qi showed that some of these relations carry over to $r$-matrices.

Let $A$ be a real symmetric $r$-matrix of order $n$, and for any real vector $x := (x_1, \ldots, x_n)$, define the polynomial form $P_A(x)$ of $A$ as

$$P_A(x) := \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r} x_{i_1} \cdots x_{i_r}.$$ 

Note that polynomial forms generalize quadratic forms to $r$-matrices. In particular, note the crucial identity

$$\frac{dP_A(x)}{dx_k} = r \sum_{i_2,\ldots,i_r} a_{k,i_2,\ldots,i_r} x_{i_2} \cdots x_{i_r}.$$ 

5
Further, write $S_r^{n-1}$ for the set of all real $n$-vectors $(x_1, \ldots, x_n)$ with $|x_1|^r + \cdots + |x_n|^r = 1$, and define the parameter $\eta(A)$ of $A$ as

$$\eta(A) := \max\{P_A(x) : x \in S_r^{n-1}\}.$$

In [15], Qi showed that if $A$ is a symmetric nonnegative matrix, then $\rho(A) = \eta(A)^2$. We shall need this result with an extra detail, so we reproduce the proof of Qi.

**Proposition 10** If $A$ is a nonnegative symmetric matrix, then $\rho(A) = \eta(A)$. If $x \in S_r^{n-1}$ and $\eta(A) = P_A(x)$, then $x$ is an $H$-eigenvector to $\rho(A)$.

**Proof** Suppose that $(x_1, \ldots, x_n)$ is a nonnegative eigenvector to $\rho(A)$. Since equations (1) are homogeneous, we may force $x_1^r + \cdots + x_n^r = 1$, and so

$$\rho(A) = \rho(A) \sum_{k \in [n]} x_k^r = \sum_{k \in [n]} \sum_{i_2, \ldots, i_r} a_{k,i_2, \ldots, i_r} x_k x_{i_2} \cdots x_{i_r} = P_A(x) \leq \eta(A).$$

Next, let $\eta(A) = P_A(x_1, \ldots, x_n)$ for some $(x_1, \ldots, x_n) \in S_r^{n-1}$. The function $|y_1|^r + \cdots + |y_n|^r$ has continuous derivatives in each variable; hence the Lagrange multiplier method implies that there is some $\lambda$ such that

$$\lambda x_k^{r-1} = \sum_{i_2, \ldots, i_r} a_{k,i_2, \ldots, i_r} x_{i_2} \cdots x_{i_r}, \quad k = 1, \ldots, n.$$

Thus, $\lambda$ is an $H$-eigenvalue of $A$. Now we find that

$$\rho(A) \geq \lambda \geq \lambda \sum_{k \in [n]} x_k^r = \sum_{k \in [n]} \sum_{i_2, \ldots, i_r} a_{k,i_2, \ldots, i_r} x_k x_{i_2} \cdots x_{i_r} = \eta(A) \geq \rho(A).$$

Hence, equality holds throughout, and so $(x_1, \ldots, x_n)$ is an eigenvector to $\rho(A)$.

Note that the digraph $D(A)$ of a symmetric matrix $A$ is an undirected 2-graph. If $A$ is a weakly reducible symmetric matrix, then $D(A)$ is disconnected and the vertices of each component of $D(A)$ induce a weakly irreducible principal submatrix of $A$, called a component of $A$. Clearly, $A$ is a block diagonal matrix of its components. It is not hard to see that every eigenvalue of $A$ is an eigenvalue of one or more of its components, and vice versa. In [2], Cooper and Dutle proved a result about the characteristic polynomial of disconnected graphs, which was extended by Hu, Huang, Ling, and Qi [10] to weakly reducible matrices, and in [18], Shao, Shan, and Zhang deduced an explicit relation between the characteristic polynomials of an $r$-matrix $A$ and of its components:

Let $A$ be a symmetric weakly reducible $r$-matrix of order $n$. If $A_1, \ldots, A_k$ are the components of $A$ and $n_1, \ldots, n_k$ are their orders, then

$$\phi_A(x) = \prod_{i \in [k]} (\phi_{A_i}(x))^{(r-1)^{n_i}}. \quad (2)$$

\footnote{For hypergraphs this equality has been proved by Cooper and Dutle in [2].}
2.2 Hypergraphs

An \( r \)-graph consists of a set of vertices \( V(G) \) and a set of edges \( E(G) \), which are subsets of \( V(G) \) with exactly \( r \) elements. Hereafter, “graph” will stand for “\( r \)-graph” with unspecified \( r \); thus, ordinary graphs will be referred to as “2-graphs”. The order of a graph is the number of its vertices. If \( G \) is of order \( n \) and \( V(G) \) is not defined explicitly, it is assumed that \( V(G) = [n] \).

Given an \( r \)-graph \( G \) with vertex set \( V(G) = [n] \), the adjacency matrix \( A(G) \) of \( G \) is the \( r \)-matrix of order \( n \), whose entries are defined by

\[
a_{i_1, \ldots, i_r} := \begin{cases} 
1, & \text{if } \{i_1, \ldots, i_r\} \in E(G); \\
0, & \text{otherwise}. 
\end{cases}
\]

The eigenvalues of \( G \) are the eigenvalues of \( A(G) \), the \( H \)-eigenvalues of \( G \) are the \( H \)-eigenvalues of \( A(G) \), and \( \rho(G) := \rho(A(G)) \).

Since the adjacency matrix \( A(G) \) of any graph \( G \) is symmetric, the digraph of \( A(G) \) is a 2-graph, which is just the 2-section of \( G \) (see, e.g., [12], p. 533). Hence, \( A(G) \) is weakly irreducible if and only if \( G \) is connected.

A graph \( G \) is called \( k \)-chromatic if its vertices can be partitioned into \( k \) sets so that each edge intersects at least two sets. The chromatic number \( \chi(G) \) of \( G \) is the smallest \( k \) for which \( G \) is \( k \)-chromatic. Similarly, a graph \( G \) is called \( k \)-partite if its vertices can be partitioned into \( k \) sets so that no edge has two vertices from the same set.

3 Odd-colorings and odd transversals

Let \( r \geq 2 \) and \( r \) be even. A cubical \( r \)-matrix \( A \) of order \( n \) is called odd-colorable if there exists a map \( \varphi : [n] \rightarrow [r] \) such that if \( a_{i_1, \ldots, i_r} \neq 0 \), then

\[
\varphi(i_1) + \cdots + \varphi(i_r) = r/2 \pmod{r}.
\]

The function \( \varphi \) is called an odd-coloring of \( A \).

Accordingly, we say that a graph is odd-colorable if its adjacency matrix is odd-colorable. Note that if a graph is odd colorable, so are its subgraphs; hence, “being odd-colorable” is a monotone graph property.

Next, write \( I_X \) for the indicator function of a set \( X \subseteq [n] \), and let \( A \) be a cubical \( r \)-matrix of order \( n \). A set \( X \subseteq [n] \) is called an odd transversal of \( A \) if \( a_{i_1, \ldots, i_r} \neq 0 \) implies that

\[
I_X(i_1) + \cdots + I_X(i_r) = 1 \pmod{2}.
\]

A matrix \( A \) with an odd transversal is called an odd transversal matrix.\(^3\)

Accordingly, we say that a graph is an odd transversal graph, if its adjacency matrix has an odd transversal; that is to say, an odd transversal of a graph is a vertex set that intersects each

\(^3\)Odd transversal matrices were introduced by Chen and Qi in [5] under the name “weakly odd-bipartite tensors”.
edge in an odd number of vertices. Note that “having an odd transversal” also is a monotone property of graphs.

The purpose of this section is to investigate odd-colorings and odd transversals of graphs and matrices. To begin with, note that if \( A \) is a square 2-matrix, then the following three properties are equivalent:
- \( A \) is odd-colorable;
- \( A \) has an odd transversal;
- \( A \) is bipartite.

However, for larger \( r \) the situation is more complicated. Let us stress the fact that odd-colorable \( r \)-matrices are defined only if \( r \) is even, whereas \( r \)-matrices with odd transversals may exist for any \( r \); e.g., the adjacency matrices of \( r \)-partite \( r \)-graphs have odd transversals.

First, we show that if \( r \) is even, “having an odd transversal” always implies “odd-colorable.”

**Proposition 11** If \( r \) is even and \( A \) is an \( r \)-matrix with an odd transversal, then \( A \) is odd-colorable.

**Proof** Let \( X \) be an odd transversal of \( A \). For every \( i \in [n] \), let \( \varphi (i) := (r/2) I_X (i) \). If \( a_{i_1, \ldots, i_r} \neq 0 \), then
\[
I_X (i_1) + \cdots + I_X (i_r) = 1 \pmod{2};
\]
hence,
\[
\varphi (i_1) + \cdots + \varphi (i_r) = (r/2) I_X (i_1) + \cdots + (r/2) I_X (i_r) = r/2 \pmod{r}.
\]
Therefore, \( \varphi (i) \) is an odd-coloring of \( A \) and so \( A \) is odd-colorable. \( \square \)

As it turns out, if \( r = 2 \pmod{4} \), then Proposition 1 can be inverted, that is to say, “having an odd transversal” and “odd-colorable” are equivalent properties if \( r = 2 \pmod{4} \).

**Proposition 12** Let \( r = 2 \pmod{4} \). An \( r \)-matrix \( A \) is odd-colorable if and only if it has an odd transversal.

**Proof** Set \( r = 4k + 2 \). In view of Proposition 1, we only need to show that if \( A \) is odd-colorable, then it has an odd transversal. Let \( \varphi : [n] \to [4k + 2] \) be an odd-coloring of \( A \). Write \( X \) for the set of all \( i \in [n] \) such that \( \varphi (i) \) is odd. We shall show that \( X \) is an odd transversal of \( A \). Indeed, if \( a_{i_1, \ldots, i_r} \neq 0 \), then
\[
\varphi (i_1) + \cdots + \varphi (i_r) = 2k + 1 \pmod{4k + 2}.
\]
Therefore, among the numbers \( \varphi (i_1), \ldots, \varphi (i_r) \), the number of the odd ones is odd, which implies that
\[
I_X (i_1) + \cdots + I_X (i_r) = 1 \pmod{2},
\]
and so, \( X \) is an odd transversal of \( A \). \( \square \)

Since Proposition 12 does not cover the case \( r = 0 \pmod{4} \), Zhou, Sun, Wang, and Bu stated a conjecture on p. 9 of [22], which would imply that Proposition 12 holds for any even

\[\text{Let us note that this argument has been used before, e.g., in the proof of Theorem 11 of [22].}\]
are also 2-chromatic, but odd-colorable graphs in transversal. Therefore, $G$ is even and $X$ contradicting that

Proof Let $n \geq 8k$. Partition $[n]$ into two sets $A$ and $B$ so that $|A| \geq 4k$ and $|B| \geq 4k$. Define the $4k$-graph $G$ by setting $V(G) := [n]$ and letting

$$E(G) := \{e : e \subset [n], \ |e \cap A| = 2k \text{ and } |e \cap B| = 2k\}.$$ 

To see that $G$ is odd-colorable, define a map $\varphi : [n] \to [4k]$ by letting

$$\varphi(i) := \begin{cases} 4k, & \text{if } i \in A; \\ 1, & \text{if } i \in B. \end{cases}$$

For every edge $\{i_1, \ldots, i_{4k}\} \in E(G)$, we see that

$$\varphi(i_1) + \cdots + \varphi(i_{4k}) = 2k \pmod{4k};$$

thus, $G$ is odd-colorable.

Assume for a contradiction that $X \subset [n]$ is an odd transversal. Then either $|X \cap A| > 2k$ or $|X \cap B| > 2k$, for otherwise there are $e_1 \subset A \setminus X$ and $e_2 \subset B \setminus X$ such that $|e_1| = |e_2| = 2k$; hence, $e_1 \cup e_2 \in E(G)$ and $(e_1 \cup e_2) \cap X = \emptyset$, contradicting that $X$ is a transversal.

Assume by symmetry that $|X \cap A| > 2k$. Then $|X \cap B| < 2k$, for otherwise there are $e_1 \subset X \cap A$ and $e_2 \subset X \cap B$ such that $|e_1| = |e_2| = 2k$; hence, $e_1 \cup e_2 \in E(G)$ and $(e_1 \cup e_2) \cap X = 4k$, contradicting that $X$ is an odd transversal.

Since $|X \cap A| > 2k$ and $|X \cap B| < 2k$, there are $e_1 \subset X \cap A$ and $e_2 \subset B \setminus X$ such that $|e_1| = |e_2| = 2k$; hence, $e_1 \cup e_2 \in E(G)$ and $(e_1 \cup e_2) \cap X = 2k$, contradicting that $X$ is an odd transversal. Therefore, $G$ has no odd transversals. \qed

To construct another family of odd-colorable but not odd-transversal graphs, note that if $r$ is even and $G$ is odd-colorable, then $\chi(G) \leq r$, whereas if $G$ is odd-transversal, then $\chi(G) = 2$. The graphs constructed in Proposition 13 are also 2-chromatic, but odd-colorable graphs in general may have higher chromatic number, as shown below:

Proposition 14 Let $k$ be a positive integer. If $n \geq 16k$, then there exists a family of 3-chromatic odd-colorable $4k$-graphs of order $n$.

Proof Let $n \geq 16k$ and let partition $[n]$ into three sets $A$, $B$, and $C$ so that $|A| \geq 6k$, $|B| \geq 6k$ and $|C| \geq 4k$. First, define four families of $4k$-subsets of $[n]$:

$$E_1 := \{e : e \subset [n], \ |e \cap A| = 2k \text{ and } |e \cap C| = 2k\},$$

$$E_2 := \{e : e \subset [n], \ |e \cap B| = 2k \text{ and } |e \cap C| = 2k\},$$

$$E_3 := \{e : e \subset [n], \ |e \cap A| = k \text{ and } |e \cap B| = 3k\},$$

$$E_4 := \{e : e \subset [n], \ |e \cap A| = 3k \text{ and } |e \cap B| = k\}.$$
Now, define a $4k$-graph $G$ by setting $V(G) := [n]$ and letting $E(G) := E_1 \cup E_2 \cup E_3 \cup E_4$.

To see that $G$ is odd-colorable, define the map $\varphi : [n] \rightarrow [4k]$ by letting

$$\varphi(i) := \begin{cases} 
1 & \text{if } i \in A; \\
4k - 1 & \text{if } i \in B; \\
4k & \text{if } i \in C.
\end{cases}$$

Let $e \in E(G)$. We shall check that $\varphi(i)$ is an odd-coloring. Indeed, if $e \in E_1$, then

$$\sum_{i \in e} \varphi(i) = 4k \cdot 2k + 2k = 2k \pmod{4k},$$

and if $e \in E_2$, then

$$\sum_{i \in e} \varphi(i) = 4k \cdot 2k + 2k(4k - 1) = 2k \pmod{4k}.$$

If $e \in E_3$, then

$$\sum_{i \in e} \varphi(i) = k + 3k(4k - 1) = 2k \pmod{4k},$$

and finally, if $e \in E_4$, then

$$\sum_{i \in e} \varphi(i) = 3k + k(4k - 1) = 2k \pmod{4k}.$$

Hence if $\{i_1, \ldots, i_{4k}\} \in E(G)$, then $\varphi(i_1) + \cdots + \varphi(i_{4k}) = 2k \pmod{4k}$; thus, $G$ is odd-colorable.

Since the classes $A$, $B$, and $C$ do not span edges, $\chi(G) \leq 3$. Assume for a contradiction that $\chi(G) = 2$, and let $X$ and $Y$ be the two color classes of $G$. Clearly, either $|A \cap X| \geq 3k$ or $|A \cap Y| \geq 3k$. By symmetry, assume that $|A \cap X| \geq 3k$. Therefore, $|Y \cap B| > 5k$, for otherwise $X$ would contain an edge from $E_4$. Clearly, either $|C \cap X| \geq 2k$ or $|C \cap Y| \geq 2k$. If $|A \cap X| \geq 2k$, then $X$ contains an edge from $E_1$; if $|A \cap Y| \geq 2k$, then $Y$ contains an edge from $E_2$. This contradiction completes the proof.

It is not clear how large may the chromatic number of odd-colorable graphs be, so we would like to raise a question:

**Question 15** Let $r = 0 \pmod{4}$. What is the maximum chromatic number of an odd-colorable $r$-graph of order $n$.

### 4 Odd-colorings and eigenvalues

In this section we discuss matrices and graphs with symmetric spectrum. The first results for $r$-graphs were given in [12], Theorem 8.9 and Proposition 8.10 that read as:

If $G$ is an $r$-graph and $-\rho(G)$ is an eigenvalue of $G$, then $r$ is even.

If $r$ is even and $G$ is an $r$-graph with an odd transversal, then $G$ has symmetric spectrum.

These facts extend to symmetric nonnegative $r$-matrices as well. Here is our generalization of Proposition 1 from the Introduction.
Theorem 16 Let \( r \geq 2 \) and \( r \) be even. If an \( r \)-matrix \( A \) is odd-colorable, then the spectrum of \( A \) is symmetric and geosymmetric.

Proof Let \( n \) be the order of \( A \) and \( \varphi : [n] \to [r] \) be an odd coloring of \( A \), i.e., if \( a_{j_1 \ldots j_r} \neq 0 \), then

\[
\varphi(j_1) + \cdots + \varphi(j_r) = r/2 \pmod{r}.
\]

First, we show that the spectrum of \( A \) is geosymmetric. Define a map \( \Phi : \mathbb{C}^n \to \mathbb{C}^n \) by letting \( \Phi(x_1, \ldots, x_n) := (y_1, \ldots, y_n) \), where for each \( k \in [n] \), we set \( y_k := e^{2\varphi(k)\pi i/r}x_k \). Clearly, \( \Phi \) is a unitary diagonal operator. Let \( \lambda \) be an eigenvalue of \( A \) with eigenvector \( (x_1, \ldots, x_n) \). We see that

\[
\sum_{j_2, \ldots, j_r} a_{k_j} x_{j_2} \cdots x_{j_r} = \sum_{j_2, \ldots, j_r} a_{k_j} \prod_{i=2}^r x_{j_i} e^{\pi i - 2\varphi(k)\pi i/r} =
\]

\[
= - \sum_{j_2, \ldots, j_r} a_{k_j} \prod_{i=2}^r x_{j_i} e^{2\varphi(k)(r-1)\pi i/r}
\]

\[
= -\lambda x_k^{-1} e^{2\varphi(k)(r-1)\pi i/r} = -\lambda y_k^{-1}.
\]

Therefore, \( -\lambda \) is an eigenvalue of \( A \) with eigenvector \( (y_1, \ldots, y_n) \), so the spectrum of \( A \) is geosymmetric.

To finish the proof, note that in [17], Shao showed that if \( (z_1, \ldots, z_n) \) is a vector with nonzero entries and a matrix \( B \) is defined as

\[
b_{j_1, \ldots, j_r} = z_{j_1}^{-r} a_{j_1, \ldots, j_r} z_{j_1} \cdots z_{j_r}, \quad (j_1, \ldots, j_r) \in [n]^r,
\]

then \( A \) and \( B \) have the same spectrum. Setting \( z_k := e^{2\varphi(k)\pi i/r} \) for each \( k \in [n] \), in view of (4), we find that \( B = -A \); thus the spectrum of \( A \) is symmetric, completing the proof.

Our next goal is to generalize Theorem 2 to symmetric hypermatrices. The result that we shall state comes as a consequence of several results of Yang and Yang [21], which for convenience we combine into one theorem. Recall that in [21], Yang and Yang gave numerous results about weakly irreducible nonnegative matrices with more than one eigenvalue of modulus equal to the spectral radius. For the case of symmetric matrices, their Theorems 3.9, 3.10, and 3.11 imply the following statement:

Theorem 17 Let \( A \) be a weakly irreducible, nonnegative, symmetric \( r \)-matrix of order \( n \). If \( \rho(A) e^{i\theta} \) is an eigenvalue of \( A \), then there is a function \( \varphi : [n] \to [r] \) such that if \( a_{j_1 \ldots j_r} \neq 0 \), then

\[
e^{2\varphi(j_1)\pi i/r} \cdots e^{2\varphi(j_r)\pi i/r} = e^{i\theta} e^{2\varphi(j_1)\pi i} = \cdots = e^{i\theta} e^{2\varphi(j_r)\pi i}.
\]

Using this result, we encounter no difficulty in generalizing Theorem 2:

Theorem 18 Let \( A \) be a weakly irreducible, nonnegative, symmetric \( r \)-matrix. If \( -\rho(A) \) is an eigenvalue of \( A \), then \( r \) is even and \( A \) is odd-colorable.
Proof Let $n$ be the order of $A$. Theorem 17 implies that there exists a function $\varphi : [n] \to [r]$ such that if $a_{j_1, \ldots, j_r} \neq 0$, then
\[
e^{2\varphi(j_1)\pi i/r} \cdots e^{2\varphi(j_r)\pi i/r} = e^{i\pi e^{2\varphi(j_1)\pi i}}.
\]
Taking the $r$th power of both sides, we find that
\[
(\varphi(j_1) + \cdots + \varphi(j_r))2\pi = r\pi \pmod{2\pi}.
\]
Hence, $r$ is even and
\[
\varphi(j_1) + \cdots + \varphi(j_r) = r/2 \pmod{r}.
\]
Therefore, $\varphi$ is an odd-coloring of $A$, and so $A$ is odd-colorable. \hfill \Box

Corollary 19 If $G$ is a connected graph, then the spectrum of $G$ is symmetric if and only if $r$ is even and $G$ is odd-colorable.

Corollary 19 completely solves the problem of Pearson and Zhang mentioned in the introduction. However, it is possible to further strengthen this assertion by dropping the premise for connectivity. Thus, using Theorem 18, we generalize Corollary 3 as follows:

Theorem 20 If $A$ is a symmetric nonnegative $r$-matrix with symmetric spectrum, then $A$ is odd-colorable.

Proof Assume that $A$ is weakly reducible, for Theorem 18 takes care of the other case. Let $A_1, \ldots, A_k$ be the components of $A$ and let $\rho = \rho(A)$. Since $\rho$ is an eigenvalue of $A$, we see that $-\rho$ also is an eigenvalue of $A$; hence, $-\rho$ is eigenvalue of a component of $A$. Without loss of generality we assume that $-\rho$ is an eigenvalue of $A_1$. Thus, $\rho(A_1) = \rho$, and Theorem 18, together with Theorem 16, implies that the spectrum of $A_1$ is symmetric.

Further, zero all entries of $A_1$ and write $A'$ for the resulting $r$-matrix. Using equation (2), it is not hard to see that $A'$ also has a symmetric spectrum. Clearly, $A_2, \ldots, A_k$ are components of $A'$, and all other components of $A'$ are zero diagonal entries. Iterating this argument, we end up with a zero matrix, after finding that all components of $A$ have symmetric spectrum. Hence, all components of $A$ are odd-colorable, and so is $A$. \hfill \Box

A similar theorem holds also for matrices with geosymmetric spectrum; we omit its proof.

Theorem 21 If $A$ is a symmetric nonnegative $r$-matrix with geosymmetric spectrum, then $A$ is odd-colorable.

Note that the conclusion of Theorem 18 may fail for non-symmetric matrices. For instance, let $A$ be the 3-matrix of order six such that
\[
a_{1,2,3} = a_{2,3,4} = a_{3,4,5} = a_{4,5,6} = a_{5,6,1} = a_{6,1,2} = 1,
\]
and all other entries are zero. The six eigenequations of \( A \) are

\[
\lambda x_1^2 = x_2x_3, \quad \lambda x_2^2 = x_3x_4, \quad \lambda x_3^2 = x_4x_5, \quad \lambda x_4^2 = x_5x_6, \quad \lambda x_5^2 = x_6x_1, \quad \lambda x_6^2 = x_1x_2.
\]

It is not hard to see that if \( \lambda \neq 0 \), then \( \lambda^6 = 1 \), and so the spectral radius of \( A \) is 1. Define the vector \( (y_1, \ldots, y_6) \), by setting \( y_k := e^{2k\pi i/6} \) for each \( k \in \{1, \ldots, 6\} \). We see that

\[
y_2y_3 = e^{4\pi i/6}e^{6\pi i/6} = -y_1^2, \quad y_3y_4 = e^{6\pi i/6}e^{8\pi i/6} = -y_2^2, \quad y_4y_5 = e^{8\pi i/6}e^{10\pi i/6} = -y_3^2,
\]

\[
y_5y_6 = e^{10\pi i/6}e^{12\pi i/6} = -y_4^2, \quad y_6y_1 = e^{12\pi i/6}e^{2\pi i/6} = -y_5^2, \quad y_1y_2 = e^{2\pi i/6}e^{4\pi i/6} = -y_6^2.
\]

Hence, \( (y_1, \ldots, y_6) \) is an eigenvector to the eigenvalue \(-1\).

Since Theorem 18 extends Theorem 2 only to symmetric \( r \)-matrices, we conclude with a corresponding question:

**Question 22** Which weakly irreducible nonnegative cubical \( r \)-matrices \( A \) have \(-\rho(A)\) as an eigenvalue?

### 4.1 Odd transversals and \( H \)-eigenvalues

The symmetry of the \( H \)-spectrum of \( r \)-graphs is somewhat simpler. In [12], Theorem 8.7, we proved the following statement:

*If \( G \) is a connected graph and \(-\rho(G)\) is an \( H \)-eigenvalue of \( G \), then \( G \) has an odd transversal.*

The short proof of this assertion extends with minor changes to nonnegative matrices:

**Theorem 23** If \( A \) is a weakly irreducible, nonnegative, symmetric \( r \)-matrix and \(-\rho(A)\) is an \( H \)-eigenvalue of \( A \), then \( r \) is even and \( A \) has an odd transversal.

**Proof** Let \( A \) be of order \( n \) and set \( \rho := \rho(A) \). Suppose that \(-\rho\) is an \( H \)-eigenvalue of \( A \) and let \( x := (x_1, \ldots, x_n) \) be an \( H \)-eigenvector to \(-\rho\) such that \( x \in S_r^{n-1} \). Clearly, the vector \( y := (|x_1|, \ldots, |x_n|) \) also belongs to \( S_r^{n-1} \). We have

\[
-\rho x_k^{r-1} = \sum_{i_2, \ldots, i_r} a_{k,i_2,\ldots,i_r} x_i x_{i_2} \cdots x_{i_r}, \quad k = 1, \ldots, n.
\]

Hence, for each \( k \in [n] \), we find that

\[
\rho |x_k|^r = \left| \sum_{i_2, \ldots, i_r} a_{k,i_2,\ldots,i_r} x_i x_{i_2} \cdots x_{i_r} \right| \leq \sum_{i_2, \ldots, i_r} a_{k,i_2,\ldots,i_r} |x_k| |x_{i_2}| \cdots |x_{i_r}|.
\]  \hspace{1cm} (5)

Adding these inequalities, we get

\[
\rho = \rho \sum_{k \in [n]} |x_k|^r \leq \sum_{k \in [n]} \sum_{i_2, \ldots, i_r} a_{k,i_2,\ldots,i_r} |x_k| |x_{i_2}| \cdots |x_{i_r}| = P_G(y) \leq \rho.
\]
Therefore $P_G(y) = \rho$ and Proposition 10 implies that $y$ is a nonnegative eigenvector to $\rho$, which by Theorem 9 must be positive. In addition, equality holds in (5) for every $k \in [n]$; thus,

$$- \text{sign}(x_k^r) = \text{sign}(x_kx_{i_2} \cdots x_{i_r})$$

whenever $a_{k,i_2,\ldots,i_r} \neq 0$. Since $A$ is symmetric, we get

$$(-1)^r \text{sign}(x_{i_1}^r) \cdots \text{sign}(x_{i_r}^r) = \text{sign}(x_{i_1} \cdots x_{i_r})^r,$$

whenever $a_{i_1,\ldots,i_r} \neq 0$, and thus $r$ is even. Therefore, (6) implies that $x_{i_1} \cdots x_{i_r} < 0$ whenever $a_{i_1,\ldots,i_r} \neq 0$, and so the set of indices of the negative entries of $(x_1,\ldots,x_n)$ is an odd transversal of $A$. □

As in Theorem 20, we obtain the following generalization of Corollary 3:

**Corollary 24** If $A$ is a symmetric nonnegative $r$-matrix with geosymmetric $H$-spectrum, then $A$ has an odd-transversal.

Finally, here is a converse of Corollary 24, which completes the picture for the $H$-spectrum.

**Theorem 25** Let $r \geq 2$ and $r$ be even. If an $r$-matrix $A$ has an odd transversal, then its $H$-spectrum is symmetric and geosymmetric.

**Proof** Let $A$ be of order $n$ and let $X$ be an odd transversal of $A$. Hence, if $a_{j_1,\ldots,j_r} \neq 0$, then

$$I_X(j_1) + \cdots + I_X(j_r) = 1 \pmod{2}.$$

We shall prove that the $H$-spectrum of $A$ is geosymmetric; the symmetry follows immediately, as the $H$-spectrum is a simple set.

Define a map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by letting $\Phi(x_1,\ldots,x_n) := (y_1,\ldots,y_n)$, where for each $k \in [n]$, we set $y_k := (2I_X(k) - 1)x_k$. Clearly $\Phi$ is an orthogonal diagonal operator. Let $\lambda$ be an $H$-eigenvalue of $A$ with $H$-eigenvector $(x_1,\ldots,x_n)$. Since $2I_X(k) - 1 = -1$ if $I_X(k) = 1$, and $2I_X(k) - 1 = 1$ if $I_X(k) = 0$, we see that

$$\sum_{j_2,\ldots,j_r} a_{k,j_2,\ldots,j_r}y_{j_2} \cdots y_{j_r} = -\sum_{j_2,\ldots,j_r} a_{k,j_2,\ldots,j_r}x_{j_2} \cdots x_{j_r} = -\lambda y_k^r - 1.$$

Hence, $-\lambda$ is an eigenvalue with eigenvector $(y_1,\ldots,y_n)$, and so the $H$-spectrum of $A$ is geosymmetric. □

The conclusion of this subsection is that if we considered only real eigenvectors, the existence of odd-colorable graphs that are not odd-transversal would not have been made clear. This distinction shows that complex eigenvectors and eigenvalues may play structural role in spectral hypergraph theory.
5 Concluding remarks

In this short section we briefly address three topics in spectral hypergraph theory: definition of adjacency matrix, relevance of algebraic spectra, and “odd-bipartiteness”.

**Adjacency matrix.** The traditional definition of the adjacency (hyper)matrix (see, e.g., [9] and [11]) represents edges by 1. This tradition was challenged by Cooper and Dutle in [2], who chose to represent the edges of an $r$-graph by the value $1/(r-1)!$, thereby scaling down all eigenvalues and simplifying a number of expressions. While this novelty has been widely accepted, it has drawbacks. For example, in the new setup the $k$th slice of the adjacency matrix is not the adjacency matrix of the link graph of the vertex $k$. We believe that the correct definition of the adjacency matrix is given by (3), and its scaled version should be called “scaled adjacency matrix”.

**Relevance of the algebraic spectrum.** Given the adjacency matrix, it is possible to build a spectral theory for hypergraphs, after adopting a particular spectral theory of hypermatrices. The theory of the determinants developed in [10] gives a solid ground for such endeavor, but is hardly acceptable in full generality for hypergraphs. The main problem comes from the fact that there are overwhelmingly many algebraic eigenvalues: indeed, a result of Qi in [14] implies that an $r$-graph of order $n$ has $n(r-1)^{n-1}$ eigenvalues. Certainly not all of those are combinatorially relevant. For instance, any vector with at most $r-2$ nonzero entries is an eigenvector to the eigenvalue 0.

Here is an example showing that algebraic multiplicity may be combinatorially irrelevant: Let $r \geq 3$ and $G$ be an $r$-graph of order $n$. Let $m_0$ be the multiplicity of the eigenvalue 0 and $m_1, \ldots, m_k$ be the multiplicities of the remaining eigenvalues of $G$. Now, add an isolated vertex to $G$ and write $H$ for the resulting graph. Using (2), we see that $H$ has the same eigenvalues as $G$, but the multiplicity of 0 increases to $m_0^{r-1} + (r-1)^n$, and the multiplicities of the remaining eigenvalues become $m_1^{r-1}, \ldots, m_k^{r-1}$. Given how simple the operation of adding an isolated vertex is, it is difficult to accept that it may affect the multiplicities of the nonzero eigenvalues.

**Odd transversals.** The study of transversals is one the oldest and most important topics in hypergraph theory (see, Ch. 2 of [1]). The concept “odd transversal” seems to have been used first by Cowan et al. in [6] and later by Rautenbach and Szigeti in [13]. The connection of odd transversals to spectral symmetry was studied first in [12]. In recent literature, odd transversal graphs have been called “odd bipartite hypergraphs.” This combination of words is contradictory, since hypergraphs cannot be bipartite. Besides, there is no need for a new term.

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