Cylindrical contact Homology and Topological Entropy

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Abstract

The objective of this paper is to study the relation between contact homology and topological entropy. We define a version of growth rate for cylindrical contact homology that we call exponential homotopical growth rate and which is well adapted to study the topological entropy of Reeb vector fields. We prove that, if for a contact manifold there exists a contact form for which the cylindrical contact homology has exponential homotopical growth rate, then for all other contact forms associated to this contact manifold the topological entropy of their Reeb flow is positive. Using this result, we provide numerous new examples of contact 3-manifolds for which every associated Reeb flow has positive topological entropy.

1 Introduction

The objective of this paper is to investigate the connections between certain topological invariants of a contact 3-manifold and the topological entropy of Reeb flows associated to this contact manifold. The topological entropy is an important dynamical invariant that encodes in a single non-negative number the chaotic behavior of a dynamical system. Despite its central importance in the theory of dynamical systems it has not received much attention in the study of the dynamics of Reeb vector fields, with the important exception of the work Macarini and Schlenk [26] on positivity of topological entropy of Reeb flows on the unit tangent bundle of energy hyperbolic manifolds with the contact structure associated to the geodesic flow. Inspired by their work and that of Colin and Honda, and Vaugon on the exponential growth rate of contact homology, we construct many new examples of contact 3-manifolds for which every Reeb flow has positive topological entropy. By the work of Katok [23] [24], this implies that every such Reeb flow has a compact invariant set on which the dynamics of the flow is conjugated to a shift.

We first recall some definitions from contact geometry. A 1-form $\alpha$ on a $(2n + 1)$-dimensional manifold $Y$ is called a contact form if $\alpha \wedge (d\alpha)^n$ is a volume form on $Y$. The hyperplane $\xi = \ker(\alpha)$ is called the contact structure. For us a contact manifold will be a pair $(Y, \xi)$ where $\xi$ is the kernel of some contact form $\alpha$ in $Y$ (we point out that in the literature these are sometimes called co-oriented contact manifolds). When $\alpha$ satisfies $\xi = \ker(\alpha)$, we will say that $\alpha$ is a contact form associated to $(Y, \xi)$. Notice that for
a contact manifold there exists infinitely many different contact forms associated to it. Given a contact form \( \alpha \), its Reeb vector field is the unique vector field \( X_\alpha \) such that \( \alpha(X_\alpha) = 1 \) and \( i_{X_\alpha}d\alpha = 0 \); the Reeb flow of \( \alpha \) is the flow of the vector field \( X_\lambda \). We will refer to the periodic orbits of the Reeb flow as Reeb orbits. A contact form \( \alpha \) is called hypertight if it doesn’t have any contractible Reeb orbits. For a Reeb orbit \( \gamma \) of the Reeb flow of \( \alpha \) its action \( A(\gamma) \) is the period of \( \gamma \), which coincides with \( \int_{S^1} \gamma^*\alpha \). Lastly, a Reeb orbit \( \gamma \) is said to be non-degenerate when 1 is not an eigenvalue of the linearisation \( D\phi^{A(c)}_{X_\alpha}|_\xi \) of the Poincare return map associated to the \( \gamma \).

Our results are inspired by the philosophy that complicated topological structure can imply chaotic behavior for dynamical systems associated to such a structure. Two important examples of this phenomena are: the result that for manifolds with complicated loop space the geodesic flow always has positive topological entropy (see [28] for an exposition of this topic); and the result that every diffeomorphism of a surface which is isotopic to a pseudo-Anosov diffeomorphism has positive topological entropy. We proceed now to explain our main results.

Let \( M \) be a manifold and \( X \) be a \( C^k \) \( (k \geq 1) \) vector field. Our first result relates the growth, relative to the period \( T \), of the number of distinct homotopy classes in a manifold \( M \) containing periodic orbits of the flow \( \phi_X \) with period smaller than \( T \), and the topological entropy of \( \phi_X \). More precisely, denote by \( \Lambda^T_X \) the set of free homotopy classes of \( M \) containing periodic orbits of the flow \( \phi_X \) with period smaller than \( T \), and the topological entropy of \( \phi_X \). More precisely, denote by \( \Lambda^T_X \) the set of free homotopy classes of \( M \) which contain a periodic orbit of \( \phi_X \) with period smaller than \( T \). We denote by \( N_X(T) \) the cardinality of \( \Lambda^T_X \). We then have the following result:

**Theorem 1.** If for real numbers \( a > 0; b \) we have \( N_X(T) \geq e^{aT+b} \) then \( h_{top}(X) \geq a \).

Theorem 1 might be a folklore result in the theory of dynamical systems; however as we have not found it proved (or even stated) anywhere in the literature we provide a complete proof of it in section 2. It contains as a special case Ivanov’s inequality for surface diffeomorphisms (see [22]). Our motivation for this result is to apply it for Reeb flows. Contact homology allows one to carry information about the dynamical behavior of one special Reeb flow associated to a contact manifold, to all other Reeb flows associated to the same contact manifold. In section 4 we introduce the notion of exponential homotopical growth rate of contact homology; this growth rate differs from the ones already studied in the literature and is specially designed to allow one to apply Theorem 1 to obtain results about the topological entropy of Reeb flows. This is made via the following theorem which builds on theorem 1:

**Theorem 2.** Let \( (M, \ker(\lambda_0)) \) be a contact manifold such that the cylindrical contact homology \( CH_{cyl}(\lambda_0) \) of \( (M, \lambda_0) \) has exponential homotopical growth rate with exponential weight \( a \). Then for every \( C^k \) \( (k \geq 2) \) contact form \( \lambda \) associated to \( (M, \ker(\lambda_0)) \) the Reeb flow of \( X_\lambda \) has positive topological entropy. More precisely, if \( \lambda = f\lambda_0 \) for a function \( f > 0 \), then \( h_{top}(X_\lambda) \geq \frac{a}{\max f} \).

For more precise information on the theorem and the definition of exponential hom-
topical growth rate of contact homology we refer to section 4. In section 3, we present, for the sake of completeness, the definition of cylindrical contact homology and its properties. We point the attention of the reader that Theorem 2 above allows one to obtain estimates for the topological entropy for Reeb flows when the contact form is $C^k$ for $k \geq 2$. In contrast with that, the techniques used in [26], produce estimates for the topological entropy only for $C^\infty$ contact forms as they depend on the use of Yomdin’s theorem (see [28]), which fails for finite regularity.

Our other results are concerned with the existence of examples of contact manifolds which have a contact form with exponential homotopical growth rate of cylindrical contact homology. We show that in dimension 3 they exist in abundance, and it follows from Theorem 2 that every Reeb flow on these contact manifolds has chaotic behavior. In section 4 we construct such examples for manifolds with a non-trivial JSJ decomposition and with a hyperbolic component that fibers over the circle. This is the content of:

**Theorem 3.** Let $M$ be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary such that the component $M_0$ satisfies:

- $M_0$ is a suspension of a surface with boundary $S$ by a diffeomorphism $h : S \to S$ with pseudo-Anosov monodromy

Then, $M$ can be given infinitely many different contact structures $\xi_k$, such that for each $\xi_k$ there exists a contact form $\lambda_k$ associated to $(M, \xi_k)$ having exponential homotopical growth of cylindrical contact homology.

In section 6, we study the contact homology of contact 3-manifolds $(M, \lambda_F)$ obtained via a special integral Dehn surgery on the unit tangent bundle $T_1S$ of a hyperbolic surface $(S, g)$. This Dehn surgery is performed on the neighbourhood of a Legendrian curve $L_c$ which is the Legendrian lift of a separating geodesic. The surgery we perform is the contact version of Handel-Thurston surgery introduced in [13] to produce non-algebraic Anosov Reeb flows in 3-manifolds: we refer to it as the Foulon-Hasselblatt surgery. In the paper [13] the authors restrict their attention to integer surgeries with positive surgery coefficient in order to produce an Anosov Reeb flow; our methods work for both negative and positive and negative coefficients as the Anosov condition does not play a role in our results.

**Theorem 4.** Let $(M, \lambda_F)$ be the contact manifold endowed with the contact form obtained via the integral Foulon-Hasselblatt surgery on the Legendrian $L_c \subset T_1S$. Then $(M, \lambda_F)$ has exponential homotopical growth rate of contact homology.

In section 7, we discuss the dynamical consequences of our results, other related results and questions we consider for future research.
Remark: it is important to point that all the results above do not depend on the Polyfolds technology which is being developed Hofer, Wysocki and Zehnder. This is the case because the versions of contact homology used for proving the results above, involve only somewhere injective pseudoholomorphic curves; in this situation transversality can be achieved by “classical” perturbation methods as is done in [9]. Therefore this versions of contact homology can be defined in complete rigour without the use of Polyfolds.

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2 Homotopic growth of periodic orbits and topological entropy

Throughout this section $M$ will denote a compact manifold. We endow $M$ with an auxiliary Riemannian metric $g$, which induces a distance function $d_g$ on $M$, whose injective radius we denote by $\epsilon_g$. Let $\tilde{M}$ be the universal cover of $M$, $\tilde{g}$ be the Riemannian metric that makes the covering map $\pi: \tilde{M} \to M$ an isometry, and $d_{\tilde{g}}$ be the distance generated by the metric $\tilde{g}$.

Let $X$ be a vector field in $M$ with no singularities and $\phi^t_X$ the flow generated by $X$. We call $P^X(T)$ the number of periodic orbits of $\phi^t$ with period in $[0,T]$. For us a periodic orbit of $X$ is a pair $(\gamma,c,T)$ where $\gamma$ is the set of parametrizations of a given immersed curve $c: S^1 \to M$, and $T$ is a positive real number (called the period of the orbit) such that:

- $\gamma \in [\gamma]_c \iff \gamma: \mathbb{R} \to M$ parametrizes $c$ and $\dot{\gamma}(t) = X(\gamma(t))$
- for all $\gamma \in [\gamma]_c$ we have $\gamma(T + t) = \gamma(t)$ and $\gamma([0,T]) = c$

We say that a periodic orbit $(\gamma,c,T)$ is in a free homotopy class $l$ of $M$ when $c \in l$.

By a parametrized periodic orbit $(\gamma,T)$ we mean a periodic orbit $(\gamma,c,T)$ with a fixed choice of parametrization $\gamma \in [\gamma]_c$. A parametrized periodic orbit $(\gamma,T)$ is said to be in a free homotopy class $l$ when the underlying periodic orbit $(\gamma,c,T)$ is in $l$.

We now recall a definition of topological entropy due to Bowen [6] which will be very useful for us. Let $T$ and $\delta$ be positive real numbers. A set $S$ is said to be $T,\delta$-separated if for all $q_1 \neq q_2 \in S$ we have:

$$\max_{t \in [0,T]} d_g(\phi^t_X(q_1),\phi^t_X(q_2)) > \delta.$$ (1)
We denote by $n^{T,\delta}$ the maximal cardinality of a $T,\delta$-separated set for the flow $\phi_X$. Then we define the $\delta$-entropy $h_\delta(\phi_X)$ to be:

$$h_\delta(\phi_X) = \limsup_{T \to +\infty} \frac{\log(n^{T,\delta})}{T}$$  \hspace{1cm} (2)

The topological entropy $h_{\text{top}}$ is then defined by the formula:

$$h_{\text{top}}(\phi_X) = \lim_{\delta \to 0} h_\delta(\phi_X)$$ \hspace{1cm} (3)

One can prove that the topological entropy doesn’t depend on the metric $d_g$ but only on the topology determined by the metric. For these and other structural results about topological entropy we refer the reader to any standard textbook in dynamics such as [17] and [29].

From the work of Kaloshin and others it is well known that the exponential growth rate of periodic orbits $\limsup_{T \to +\infty} \frac{\log(P_X(T))}{T}$ can be much bigger than the topological entropy. However, if the orbits appear sufficiently quickly in different free homotopy classes more can be said.

Let $\Lambda$ denote the set of free homotopy classes of loops in $M$, and $\Lambda_0 \subset \Lambda$ be the subset of primitive free homotopy classes. Denote by $\Lambda_X^T \subset \Lambda$ the set of free homotopy classes $\varphi$ such that there exists a periodic orbit of $\phi^t_X$ with period smaller or equal to $T$ which is homotopic to $\varphi$. We denote by $N_X(T)$ the cardinality of $\Lambda_X^T$.

Let $\{((\gamma_i,T_i); 1 \leq i \leq n\}$ be a set of parametrized periodic orbits of $X$. For a number $T$ satisfying $T \geq T_i$ for all $i \in \{1,\ldots,n\}$ and a constant $\delta > 0$, we denote by $\Lambda_X^{T,\delta}(((\gamma_1,T_1),\ldots,(\gamma_n,T_n)))$ the subset of $\Lambda$ such that:

- $l \in \Lambda_X^{T,\delta}(((\gamma_1,T_1),\ldots,(\gamma_n,T_n)))$ if, and only if, there exist a parametrized periodic orbit $(\hat{\gamma}, \hat{T})$ with period $\hat{T} \leq T$ in the free homotopy class $l$ and a number $i_l \in \{1,\ldots,n\}$ for which $\max_{t \in [0,T]}(d_g(\gamma_{\hat{i}}(t), \hat{\gamma}(t))) \leq \delta$.

Notice that:

$$\Lambda_X^{T,\delta}(((\gamma_1,T_1),\ldots,(\gamma_n,T_n))) = \bigcup_{i \in \{1,\ldots,n\}} \Lambda_X^{T,\delta}(((\gamma_i,T_i)))$$ \hspace{1cm} (4)

We are ready to prove the main result in this section. Theorem 1 below is well known to be true in the particular cases where $\phi_X$ is a geodesic flow, where it follows from Manning’s inequality (see [24] and [28]); and where $\phi_X$ is the suspension of surface diffeomorphism with pseudo-Anosov monodromy, where it follows from Ivanov’s theorem (see [22]). It can be seen as a generalization of these results in the sense that it includes them as particular cases and that it applies to many other situations. Our argument is inspired by the remarkable proof of Ivanov’s inequality given by Jiang in [22].

**Theorem 1.** If for real numbers $a > 0$, $b$ we have $N_X(T) \geq e^{aT+b}$ then $h_{\text{top}}(\phi_X) \geq a$.  

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**Proof:** The theorem will follow if we prove that for all $\delta < \frac{\epsilon_0}{10r}$ we have $h_\delta(\phi_X) \geq a$.

From now on fix $0 < \delta < \frac{\epsilon_0}{10r}$.

**Step 1:** for any point $p \in M$ let $V_{4\delta}(p)$ be the $4\delta$-neighbourhood of $\pi^{-1}(p)$. Because $\delta < \frac{\epsilon_0}{10r}$, it is clear that $V_{4\delta}(p)$ is the disjoint union:

$$V_{4\delta}(p) = \bigcup_{\tilde{p} \in \pi^{-1}(p)} B_{4\delta}(\tilde{p})$$

where the ball $B_{4\delta}(\tilde{p})$ is taken with respect to the metric $\tilde{g}$.

Because of compactness of $M$, there exists a constant $0 < k_1$ which does not depend on $p$, such that if $B$ and $B'$ are two distinct connected components of $V_{4\delta}(p)$ we have $d_{\tilde{g}}(B, B') > k_1$.

Again because of compactness, we know that the vector field $\tilde{X} := \pi^*X$ is bounded in the norm given by the metric $\tilde{g}$. Combining this with the inequality in the last paragraph, one obtains the existence of a constant $0 < k_2$, which again doesn’t depend on $p$ such that, if $\tilde{v} : [0, R] \to \tilde{M}$ is a parametrized trajectory of $\phi_{\tilde{X}}$ such that $\tilde{v}(0) \in B$ and $\tilde{v}(R) \in B'$ for $B \neq B'$ are connected components of $V_{4\delta}(p)$ then $R > k_2$.

From the last assertion we deduce the existence of a constant $\tilde{K}$, depending only $\tilde{g}$ and $X$, such that for every $p \in M$ and every a parametrized trajectory $\tilde{v} : [0, T] \to \tilde{M}$ of $\phi_{\tilde{X}}$, the number $L^T(p, \tilde{v})$ of distinct connected components of $V_{4\delta}(p)$, intersected by the curve $\tilde{v}([0, T])$ satisfies:

$$L^T(p, \tilde{v}) < \tilde{K}T$$

**Step 2:** we claim that for every parametrized periodic orbit $(\gamma', T')$ of $X$ we have:

$$\sharp(\Lambda_X^{T,\delta}((\gamma', T'))) < \tilde{K}T$$

for all $T > T'$.

To see that this is true take $\tilde{\gamma}'$ be a lift of $\gamma'$ and let $p' = \gamma'(0)$ and $\tilde{p}' = \tilde{\gamma}'(0)$. We consider the set $\{B_j : 1 \leq j \leq m^T(\gamma')\}$ of connected components of $V_{4\delta}(p')$ satisfying:

- $B_j \neq B_k$ if $j \neq k$,
- if $B$ is a connected component of $V_{4\delta}(p')$ which intersects $\tilde{\gamma}'([0, T])$ then $B = B_j$ for some $j \in \{1, \ldots, m^T((\gamma', T'))\}$,
- if $j < i$ then $B_j$ is visited by the trajectory $\tilde{\gamma}'$ before $B_i$.

From step 1, we know that $m^T((\gamma', T')) < \tilde{K}T$.

For each $l \in \Lambda_X^{T,\delta}((\gamma', T'))$ pick $(\chi_l, T_l)$ in $l$ to be a parametrized periodic orbit which satisfies $d_{\tilde{g}}(\chi_l(t), \gamma'(t)) < \delta$ for all $t \in [0, T]$. There exists a lift $\tilde{\chi}_l$ of $\chi_l$ satisfying $d_{\tilde{g}}(\tilde{\chi}_l(t), \tilde{\gamma}'(t)) < \delta$.

From the triangle inequality it is clear that the point $q_l = \tilde{\chi}_l(0)$ is in the connected component $B_1$ which contains $\tilde{p}'$. We will show that $\tilde{\chi}_l(T_l)$ is contained in $B_j$ for some $j \in \{1, \ldots, m^T(\gamma')\}$. Because $\pi(\tilde{\chi}_l(0)) = \pi(\tilde{\chi}_l(T_l))$, we have:
Then, if $m < \frac{N_X(T)}{KT}$, we denote by $\tilde{p}_j'$ the unique element $\pi^{-1}(p')$ for which we have $d_3(\tilde{\chi}_l(T), \tilde{p}_j') < \delta$. Using the triangle inequality we now obtain:

$$d_3(\tilde{\gamma}(T), \tilde{p}_j') \leq d_3(\tilde{\gamma}(T), \tilde{\chi}_l(T)) + d_3(\tilde{\chi}_l(T), \tilde{p}_j') < \delta + \delta$$

which already implies that $\tilde{\chi}_l(T) \in V_{4\delta}(p')$. From the inequalities above we conclude that $\tilde{\gamma}(T)$ and $\tilde{\chi}_l(T)$ are in the connected component of $V_{4\delta}(p')$ that contains $\tilde{p}_j'$. Because this connected component contains $\tilde{\gamma}(T)$, it is therefore one of the $B_j$ for which we have $d_3(\tilde{\gamma}(T), \tilde{\chi}_l(T)) < \delta$. Therefore it is clear, because $l \geq s + 1$, that if $\tilde{\chi}_l(T)$ and $\tilde{\chi}_l(T')$ are in a same component of $V_{4\delta}(p')$, then the closed curves $\chi([0, T_l])$ and $\chi([0, T'])$ are be freely homotopic. This contradicts our choice of $(\chi_l, T_l)$ and $(\chi_l, T_l')$ and the fact that $l \neq l'$.

We have thus concluded that the map $\gamma_{T, \delta} : \Lambda^{T, \delta}_X(\gamma') \to [0, \gamma(T')]$ is injective, which implies that $\tilde{\gamma}(\Lambda^{T, \delta}_X(\gamma', T')) \leq T(\gamma', T') < K T$.

**Step 3:** inductive step.

As an immediate consequence of step 2 we have that if $\{\gamma_i, T_i; 1 \leq i \leq m\}$ is a set of parametrized periodic orbits of $X$ we have $\tilde{\gamma}(\Lambda^{T, \delta}_X(\gamma_1, T_1), ..., \gamma_m, T_m)) \leq m K T$.

**Inductive claim:** fix $T > 0$ and suppose that $S_m^T = \{\gamma_i, T_i; 1 \leq i \leq m\}$ is a set of parametrized periodic orbits such that $T \geq T_i$ for all $i \in \{1, ..., m\}$ satisfying:

- (a) the free homotopy classes $l_i$ of $(\gamma_i, T_i)$ and $l_j$ of $(\gamma_j, T_j)$ satisfy $l_i \neq l_j$ if $i \neq j$,
- (b) for every $i \neq j$ we have $\max_{t \in [0, T]} \frac{d}{g}(\gamma_i(t), \gamma_j(t)) > \delta$.

Then, if $m < \frac{N_X(T)}{KT}$, there exists a parametrized periodic orbit $(\gamma_{m+1}, T_{m+1} \leq T)$ such that its homotopy class $l_{m+1}$ satisfy $l_{m+1} \neq l_i$ and:

$$\max_{t \in [0, T]} \frac{d}{g}(\gamma_{m+1}(t), \gamma_i(t)) > \delta$$

for all $i \in \{1, ..., n\}$.

**Proof of the claim:** notice that $\tilde{\gamma}(\Lambda^{T, \delta}_X(\gamma_1, T_1), ..., \gamma_m, T_m)) \leq m K T$. Therefore, because $m < \frac{N_X(T)}{KT}$, there exists a free homotopy $l_{m+1} \in \Lambda^{T, \delta}_X(\gamma_1, T_1), ..., \gamma_m, T_m))$. Choose a parametrized periodic orbit $(\gamma_{m+1}, T_{m+1})$ with $T_{m+1} \leq T$ in the homotopy class $l_{m+1}$.

As $l_{m+1} \notin \Lambda^{T, \delta}_X(\gamma_1, T_1), ..., \gamma_m, T_m))$, we must have $\max_{t \in [0, T]} \frac{d}{g}(\gamma_{m+1}(t), \gamma_i(t)) > \delta$ for all $i \in \{1, ..., m\}$; thus completing the proof of the claim.
Step 4: obtaining a $T, \delta$ separated set.

The strategy is now to use the inductive step to obtain a set $S_X^T = \{(\gamma_i, T_i); 1 \leq i \leq \lfloor \frac{N_X(T)}{KT} \rfloor \}$, satisfying conditions (a) and (b) above with the maximum possible cardinality. We start with a set $S_X^T = \{(\gamma_1, T_1)\}$, which clearly satisfies conditions (a) and (b), and if $1 < \lfloor \frac{N_X(T)}{KT} \rfloor$ we apply the inductive step to obtain a parametrized periodic orbit $(\gamma_2, T_2 \leq T)$ such that $S_X^T = \{(\gamma_1, T_1), (\gamma_2, T_2 \leq T)\}$ satisfies (a) and (b). We can go on applying the inductive step to produce sets $S_X^{m} = \{(\gamma_i, T_i); 1 \leq i \leq m\}$ satisfying the desired conditions (a) and (b) as long as $m - 1$ is smaller than $\lfloor \frac{N_X(T)}{KT} \rfloor$. By this process we can construct a set $S_X^T = \{(\gamma_i, T_i); 1 \leq i \leq \lfloor \frac{N_X(T)}{KT} \rfloor \}$ such that for all $i, j \in \{1, \ldots, \lfloor \frac{N_X(T)}{KT} \rfloor \}$:

- (a) the free homotopy classes $l_i$ of $(\gamma_i, T_i)$ and $l_j$ of $(\gamma_j, T_j)$ satisfy $l_i \neq l_j$ if $i \neq j$,
- (b) for every $i \neq j$ we have $\max_{t \in [0, T]} d_g(\gamma_i(t), \gamma_j(t)) > \delta$.

For each $i \in \{1, \ldots, \lfloor \frac{N_X(T)}{KT} \rfloor \}$ let $q_i = \gamma_i(0)$. We define the set $P_X^T := \{q_i; 1 \leq i \leq \lfloor \frac{N_X(T)}{KT} \rfloor \}$. The condition (b) satisfied by $S_X^T$ implies that $P_X^T$ is a $T, \delta$-separated set.

From the definition we used of the topological entropy we have:

$$h_\delta(\phi_X) \geq \limsup_{T \to +\infty} \frac{\log(\lfloor \frac{N_X(T)}{KT} \rfloor)}{T} \quad (11)$$

Step 5: suppose now that for constants $a > 0$ and $b$ we have $N_X(T) \geq e^{aT+b}$. For every $\epsilon > 0$ we know that for $T$ big enough we have $e^T > KT$. This implies that:

$$\limsup_{T \to +\infty} \frac{\log(\lfloor \frac{N_X(T)}{KT} \rfloor)}{T} \geq \limsup_{T \to +\infty} \frac{\log(\lfloor e^{aT+b} \rfloor)}{T} = \limsup_{T \to +\infty} \frac{\log(\lfloor e^{(a-\epsilon)T+b} \rfloor)}{T} \quad (12)$$

It is clear that $\limsup_{T \to +\infty} \frac{\log(\lfloor e^{(a-\epsilon)T+b} \rfloor)}{T} = (a - \epsilon)$. We have thus proven that if for constants $a > 0$ and $b$ we have $N_X(T) \geq e^{aT+b}$ then $h_\delta(\phi_X) \geq (a - \epsilon)$. Because $\epsilon$ can be taken arbitrarily small we obtain:

$$h_\delta(\phi_X) \geq a \quad (13)$$

Step 6: we have so far concluded that for all $\delta < \frac{\epsilon_0}{10}$ we have $h_\delta(\phi_X) \geq a$. We then have:

$$h_{top}(\phi_X) = \lim_{\delta \to 0} h_\delta(\phi_X) \geq a \quad (14)$$

finishing the proof of the theorem. \hfill \Box

Remark: one could naively believe that there exists a constant $\delta_g > 0$ depending only on the metric $g$, such that if two parametrized periodic orbits $(\gamma_1, T_1)$ and $(\gamma_2, T_2)$ satisfy $\sup_{t \in [0, \max\{T_1, T_2\}]} \{d_g(\gamma_1(t), \gamma_2(t))\} < \delta_g$ then $(\gamma_1, T_1)$ and $(\gamma_2, T_2)$ are freely homotopic.
This would make the proof of Theorem 1 much shorter, however such a constant does not exist; one can easily find for any $\delta > 0$, flows in the 3-torus with two periodic orbits $(\gamma_1, T_1)$ and $(\gamma_2, T_2)$ which are in different primitive free homotopy classes and satisfy $\sup_{t \in [0, \max\{T_1, T_2\}]} \{d_g(\gamma_1(t), \gamma_2(t))\} < \delta$.

## 3 Contact homology

### 3.1 Pseudo-holomorphic curves in symplectic cobordisms

To define the contact homology theories used in this paper we use pseudoholomorphic curves in symplectizations of contact manifolds and symplectic cobordisms. Pseudo-holomorphic curves were introduced in symplectic manifolds by Gromov in [15] and adapted to symplectizations and symplectic cobordisms by Hofer [18]; see also [4] as a general reference for pseudoholomorphic curves in symplectic cobordisms.

#### 3.1.1 Cylindrical almost complex structures

Let $(Y, \xi)$ be a contact manifold and $\lambda$ an associated contact form. The symplectization of $(Y, \xi)$ is the product $\mathbb{R} \times Y$ with the symplectic form $d(e^s \lambda)$ (where $s$ denotes the $\mathbb{R}$ coordinate in $\mathbb{R} \times Y$). $d\lambda$ restricts to a symplectic form on the vector bundle $\xi$ and it is well known that the set $j(\lambda)$ of $d\lambda$-compatible almost complex structures on the symplectic vector bundle $\xi$ is non-empty and contractible. Notice, that if $Y$ is 3-dimensional the set $j(\lambda)$ doesn’t depend on the contact form $\lambda$ associated to $(Y, \xi)$.

For $j \in j(\lambda)$ we can define an $\mathbb{R}$-invariant almost complex structure $J$ on $\mathbb{R} \times Y$ by demanding that:

$$J \partial_s = X_\lambda, \quad J |_{\xi} = j \quad (15)$$

We will denote by $J(\lambda)$ the set of almost complex structures in $\mathbb{R} \times Y$ that are $\mathbb{R}$-invariant, $d(e^s \lambda)$-compatible and satisfy the equation (15) for some $j \in j(\lambda)$.

#### 3.1.2 Exact symplectic cobordisms with cylindrical ends

An exact symplectic cobordism is, naively speaking, an exact symplectic manifold $(W, \omega)$ that outside a compact subset is like a cylindrical end of a symplectization. We restrict our attention to exact symplectic cobordisms having only one positive end and one negative end.

Let $(W, \omega = d\kappa)$ be an exact symplectic manifold with no boundary, and let $(Y, \lambda^+)$ and $(Y^-, \lambda^-)$ be contact manifolds. We will say that $(W, \omega = d\kappa)$ is an exact symplectic cobordism from $(Y, \lambda^+)$ to $(Y^-, \lambda^-)$ when: $W = W^+ \cup \widehat{W} \cup W^-$, with $W^+ \cap W^-$ empty and $\widehat{W}$ is compact; and there exist numbers $R^+ > 0$ and $R^- > 0$, and diffeomorphisms $\Psi^+: W^+ \to [R^+, +\infty) \times Y^+$ and $\Psi^-: W^- \to (-\infty, R^-] \times Y^-$, such that:
\[ \Psi^+: W^+ \to [R^+, +\infty) \times Y^+ \text{ satisfies } (\Psi^+)^*(e^{s-R^+}\lambda^+) = \kappa; \quad (16) \]
\[ \Psi^-: W^- \to (-\infty, R^-) \times Y^- \text{ satisfies } (\Psi^-)^*(e^{s-R^-}\lambda^-) = \kappa. \quad (17) \]

In such a cobordism, we say that an almost complex structure \( J \) is cylindrical if:

\[ J \text{ coincides with } J^+ \in \mathcal{J}(C^+\lambda^+) \text{ in the region } W^+ \quad (18) \]
\[ J \text{ coincides with } J^- \in \mathcal{J}(C^-\lambda^-) \text{ in the region } W^- \quad (19) \]
\[ \mathcal{J} \text{ is compatible with } \varpi \text{ in } \tilde{W} \quad (20) \]

where \( C^+ > 0 \) and \( C^- > 0 \) are constants.

The set of cylindrical almost complex structures in \((\mathbb{R} \times Y, \varpi)\) coinciding with \( J^+ \) on \( W^+ \) and \( J^- \) on \( W^- \) is denoted by \( \mathcal{J}(J^-, J^+) \), and it is well known to be non-empty and contractible. We will write \( \lambda^+ \succex \lambda^- \) when there exists an exact symplectic cobordism from \( \lambda^+ \) to \( \lambda^- \) as above. We remind the reader that \( \lambda^+ \succex \lambda \) and \( \lambda \succex \lambda^- \) implies \( \lambda^+ \succex \lambda^- \); or in other words that existence of exact symplectic cobordisms is transitive; see [4] for a detailed discussion on symplectic cobordisms with cylindrical ends. Notice, that a symplectization is a particular case of an exact symplectic cobordism.

**Remark:** we point out to the reader that in many references in the literature, a slightly different definition of cylindrical almost complex structures is used: instead of demanding that \( J \) satisfies equations (18) and (19), the stronger condition that \( J \) coincides with \( J^\pm \in \mathcal{J}(\lambda^\pm) \) in the region \( W^\pm \) is demanded. We need to consider this more relaxed definition of cylindrical almost complex structures when we study the cobordism maps of cylindrical contact homologies in section 3.3.

### 3.1.3 Splitting symplectic cobordisms

Let \( \lambda^+, \lambda \) and \( \lambda^- \) be contact forms associated to \((Y, \xi)\) such that \( \lambda^+ \succex \lambda, \lambda \succex \lambda^- \). For \( \epsilon > 0 \) sufficiently small, it is easy to see that one also has \( \lambda^+ \succex (1+\epsilon)\lambda \) and \( (1-\epsilon)\lambda \succex \lambda^- \). Then, for each \( R > 0 \), it is possible to construct an exact symplectic form \( \varpi_R = d\kappa_R \) on \( W = \mathbb{R} \times Y \) satisfying:

\[
\begin{align*}
\kappa_R &= e^{s-R-2}\lambda^+ \text{ in } [R+2, +\infty) \times Y \quad (21) \\
\kappa_R &= f(s)\lambda \text{ in } [-R, R] \times Y \quad (22) \\
\kappa_R &= e^{s+R+2}\lambda^- \text{ in } (-\infty, -R-2] \times Y \quad (23)
\end{align*}
\]

where \( f: [-R, R] \to [1-\epsilon, 1+\epsilon], f(-R) = 1-\epsilon, f(R) = 1+\epsilon \) and \( f' > 0 \). In \((\mathbb{R} \times Y, \varpi_R)\) we consider a compatible cylindrical almost complex structure \( \tilde{J}_R \); but we demand an extra condition on \( \tilde{J}_R \):
Again we divide $W$ in regions: $W^+ = [R + 2, +\infty) \times Y$, $W^{(\lambda^+, \lambda)} = [R, R + 2] \times Y$, $W(\lambda) = [-R, R] \times Y$, $W^{(\lambda^-, \lambda)} = [-R - 2, -R] \times Y$ and $W^- = (-\infty, -R - 2] \times Y$. The family of exact symplectic cobordisms with cylindrical almost complex structures $(\mathbb{R} \times Y, \varpi_R, \tilde{J}_R)$ is called a splitting family from $\lambda^+$ to $\lambda^-$ along $\lambda$.

### 3.1.4 Pseudoholomorphic curves

Let $(S, i)$ be a closed Riemann surface without boundary, $\Gamma \subset S$ be a finite set. Let $\lambda$ be a contact form in $Y$ and $J \in J(\lambda)$. A finite energy pseudoholomorphic curve in the symplectization $(\mathbb{R} \times Y, J)$ is a map $\tilde{w} : S \setminus \Gamma \to \mathbb{R} \times Y$ that satisfies:

$$\partial J(\tilde{w}) = d\tilde{w} \circ i - J \circ d\tilde{w} = 0$$

and

$$0 < E(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{S \setminus \Gamma} \tilde{w}^* d(q\lambda)$$

where $\mathcal{E} = \{q : \mathbb{R} \to [0, 1]; q' \geq 0\}$. The quantity $E(\tilde{w})$ is called the Hofer energy, and was introduced in [18]. We write $\tilde{w} = (s, w) \in \mathbb{R} \times Y$. The operator $\partial J$ above is called the Cauchy-Riemann operator for the almost complex structure $J$.

For an exact symplectic cobordism $(W, \varpi)$ from $\lambda^+$ to $\lambda^-$ as considered above, and $\overline{J} \in J(J^-, J^+)$ a finite energy pseudoholomorphic curve is again a map $\tilde{w} : (S \setminus \Gamma \to W$ satisfying:

$$d\tilde{w} \circ i = \overline{J} \circ d\tilde{w},$$

and

$$0 < E_{\lambda^-}(\tilde{w}) + E_c(\tilde{w}) + E_{\lambda^+}(\tilde{w}) < +\infty,$$

where:

$E_{\lambda^-}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\tilde{w}^{-1}(W^-)} \tilde{w}^* d(q\lambda^-)$,

$E_{\lambda^+}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\tilde{w}^{-1}(W^+)} \tilde{w}^* d(q\lambda^+)$,

$E_c(\tilde{w}) = \int_{\tilde{w}^{-1}W(\lambda^-, \lambda^+)} \tilde{w}^* \varpi$.

These energies were also introduced in [18].

In splitting symplectic cobordisms we use a slightly modified version of energy. Instead of demanding $0 < E_{\lambda^-}(\tilde{w}) + E_c(\tilde{w}) + E_{\lambda^+}(\tilde{w}) < +\infty$ we demand:

$$0 < E_{\lambda^-}(\tilde{w}) + E_{\lambda^-, \lambda}(\tilde{w}) + E_{\lambda}(\tilde{w}) + E_{\lambda, \lambda^+}(\tilde{w}) + E_{\lambda^+}(\tilde{w}) < +\infty$$

where:

$E_{\lambda}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\tilde{w}^{-1}W(\lambda)} \tilde{w}^* d(q\lambda)$,
$$E_{\lambda,\lambda}(\tilde{w}) = \int_{\tilde{w}^{-1}(W(\lambda,\lambda))} \tilde{w}^\ast \omega,$$
$$E_{\lambda,\lambda}(\bar{w}) = \int_{\bar{w}^{-1}(W(\lambda,\lambda))} \bar{w}^\ast \omega,$$
and $E_{\lambda,\lambda}(\tilde{w})$ and $E_{\lambda,\lambda}(\bar{w})$ are as above.

The elements of the set $\Gamma \subset S$ are called punctures of the pseudoholomorphic $\tilde{w}$. The work of Hofer et al. [18] [8] allows us do classify the punctures in two types: positive punctures and negative punctures. This classification is done according to the behaviour of a $\tilde{w}$ in the neighbourhood of the puncture. Before presenting this classification we introduce some notation: we let $B_\delta(z)$ be the ball of radius $\delta$ centered at the puncture $z$, and denote by $\partial(B_\delta(z))$ its boundary. With this in hand, we can describe the types of punctures as follows:

- $z \in \Gamma$ is called positive interior puncture when $z \in \Gamma \setminus \Gamma_\theta$ and $\lim_{z' \to z} s(z') = +\infty$, and in this case there exists a sequence $\delta_n \to 0$ and Reeb orbit $\lambda^+$ of $X_{\lambda^+}$, such that $w(\partial(B_\delta_n(z)))$ converges in $C^\infty$ to $\lambda^+$ as $n \to +\infty$.
- $z \in \Gamma$ is called negative interior puncture when $z \in \Gamma \setminus \Gamma_\theta$ and $\lim_{z' \to z} s(z') = -\infty$, and in this case there exists a sequence $\delta_n \to 0$ and Reeb orbit $\lambda^-$ of $X_{\lambda^-}$, such that $w(\partial(B_\delta_n(z)))$ converges in $C^\infty$ to $\lambda^-$ as $n \to +\infty$.

The results in [18] and [8] imply that these are indeed the only real possibilities we need to consider for the behaviour of the $\tilde{w}$ near punctures. Intuitively, we have that at the punctures, the pseudoholomorphic curve $\tilde{w}$ detects Reeb orbits. When for a puncture $z$, there is a subsequence $\delta_n$ such that $w(\partial(B_\delta_n(z)))$ converges to a given Reeb chord $c$ (orbit $\gamma$), we will say that $\tilde{w}$ is asymptotic to this Reeb orbit $\gamma$ at the puncture $z$.

If a pseudoholomorphic curve is asymptotic to a non-degenerate Reeb orbit at a puncture, more can be said about its asymptotic behaviour in neighbourhoods of this puncture. In order to describe the behaviour of $\tilde{w}$ near a puncture $z$, we take a neighbourhood $U \subset S$ of $z$ that admits a holomorphic chart $\psi_U : (U, z) \to (\mathbb{D}, 0)$. Using polar coordinates $(r, t) \in (0, +\infty) \times S^1$ we can write $x \in (\mathbb{D} \setminus 0)$ as $x = e^{-r}t$. With this notation, it is shown in [18] [8], that if $z$ is a positive interior puncture on which $\tilde{w}$ is asymptotic to a non-degenerate Reeb orbit $\lambda^+$ of $X_{\lambda^+}$, then $\tilde{w} \circ \psi_u^{-1}(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^\ast(t) = w(r, t)$ converges uniformly in $C^\infty$ to a Reeb orbit $\gamma$ of $X_{\lambda^+}$, and the convergence rate is exponential.

Similarly, if $z$ is a negative interior puncture on which $\tilde{w}$ is asymptotic to a non-degenerate Reeb orbit $\lambda^-$ of $X_{\lambda^-}$, then $\tilde{w} \circ \psi_u^{-1}(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^\ast(t) = w(r, t)$ converges uniformly in $C^\infty$ to a Reeb orbit $\gamma$ of $-X_{\lambda^-}$ as $r \to +\infty$, and the convergence rate is exponential.

Remark: the fact that the convergence of pseudoholomorphic curves near punctures to Reeb orbits is of exponential nature is a consequence of the asymptotic formula obtained in [18]. Such formulas are necessary for the Fredholm theory that gives the dimension of the space of pseudoholomorphic curves with fixed asymptotic data.
The discussion above can be summarised by saying that near punctures the finite pseudoholomorphic curves detect Reeb orbits. It is exactly this behavior that makes these objects useful for the study of dynamics of Reeb vector fields. When a pseudoholomorphic curve approaches a Reeb orbit near a puncture we say that the pseudoholomorphic is asymptotic to the Reeb orbit at that puncture.

For us it will be important to consider the moduli spaces $M(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ of genus 0 pseudoholomorphic curves, modulo reparametrisation, with one positive puncture asymptotic to a non-degenerate Reeb orbit $\gamma$ and negative punctures asymptotic to non-degenerate orbits $\gamma_1', \ldots, \gamma_m'$. The work of Dragnev [9] shows that the linearization $D\overline{\nu}J$ at any element $M(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ is a Fredholm map (we remark that this property is valid for more general moduli spaces of curves with prescribed asymptotic behaviour). One would like to conclude that the dimension of a connected component of $M(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ is given by the Fredholm index of an element of $M(\gamma, \gamma_1', \ldots, \gamma_m'; J)$; however this is not always the case as problems might appear when multiply covered pseudoholomorphic curves appear.

**Fact:** as a consequence of the exactness of the symplectic cobordisms considered above we obtain that the energy $E(\tilde{w})$ of $\tilde{w}$ satisfies $E(\tilde{w}) \leq 5A(\tilde{w})$ where $A(\tilde{w})$ is the sum of the action of the Reeb orbits detected by the punctures of $\tilde{w}$ counted with multiplicity.

### 3.2 Full contact homology

Full contact homology was introduced in [10] as an important invariant of contact structures. We refer the reader to [10] and [3] for detailed presentations of the material contained in this subsection.

Let $(Y^{2n+1}, \ker(\lambda))$ be a contact manifold with $\lambda$ a non-degenerate contact form. We denote by $\mathcal{P}(\lambda)$ the set of good periodic orbits of the Reeb vector field $X_\lambda$. To each orbit $\gamma \in \mathcal{P}(\lambda)$, we define a $\mathbb{Z}_2$-grading $|\gamma| = (\mu_{\mathbb{C}Z}(\gamma) + (n - 2)) \mod 2$. An orbit $\gamma$ is called good if it is either simple, or if $\gamma = (\gamma')^i$ for a simple orbit $\gamma'$ with the same grading of $\gamma$.

$\mathfrak{A}(Y, \lambda)$ is defined to be the supercommutative, $\mathbb{Z}_2$ graded, $\mathbb{Q}$ algebra with unit, generated by $\mathcal{P}(\lambda)$ (an algebra with this properties is sometimes referred in the literature as a commutative super-algebra or a super-ring). The $\mathbb{Z}_2$-grading on the elements of the algebra is obtained by considering on the generators the grading mentioned above and extending it to $\mathfrak{A}(Y, \lambda)$.

$\mathfrak{A}(Y, \lambda)$ can be equipped with a differential $d_J$. Denote by $\mathcal{M}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ to be the moduli space of finite energy pseudoholomorphic curves of genus 0 and Fredholm index $k$ modulo reparametization, with one positive puncture asymptotic to $\gamma$ and negative punctures asymptotic to $\gamma_1', \ldots, \gamma_m'$ in the symplectization $(\mathbb{R} \times Y, J)$. As the almost complex structure $J$ is $\mathbb{R}$-invariant in $\mathbb{R} \times Y$, we have an $\mathbb{R}$-action on $\mathcal{M}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ and we denote by $\overline{\mathcal{M}}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J) = \mathcal{M}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J)/\mathbb{R}$. Lastly we denote by $\overline{\overline{\mathcal{M}}}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ the compactification of $\overline{\mathcal{M}}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ as presented in [4]. The moduli space $\overline{\overline{\mathcal{M}}}^k(\gamma, \gamma_1', \ldots, \gamma_m'; J)$ also involves pseudoholomorphic buildings that appear.

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as limits of a sequence of curves in $\tilde{M}^k(\gamma, \gamma'_1, ..., \gamma'_m; J)$ that “break”; we refer the reader to [4] for a more detailed description of these moduli spaces. To define our differential we need the following hypothesis:

**Hypothesis H**: there exists an abstract perturbation of the Cauchy-Riemann operator $\partial_J$ such that the compactified moduli spaces $\tilde{M}(\gamma, \gamma'_1, ..., \gamma'_m; J)$ of solutions of the perturbed equation are unions of branched manifolds with corners and rational weights whose dimension is given by the Conley-Zehnder index of the asymptotic orbits and the relative homology class of the solution.

The proof that Hypothesis H is true is still not written and is one of the main objectives behind the development of the Polyfold technology by Hofer, Wysocki and Zehnder. We define:

$$d_J\gamma = m(\gamma) \sum_{\gamma'_1, ..., \gamma'_m} \frac{C(\gamma'_1, ..., \gamma'_m)}{m!} \gamma'_1 \gamma'_2 ... \gamma'_m$$

where $C(\gamma'_1, ..., \gamma'_m)$ is the algebraic count of points in the 0-dimensional manifold

$$\tilde{M}^0(\gamma, \gamma'_1, ..., \gamma'_2; J)/\mathbb{R}$$

and $m(\gamma)$ is the multiplicity of $\gamma$. $d_J$ is extended to the whole algebra by the Leibnitz rule. Under hypothesis H it was proved in [10] that $(d_J)^2 = 0$. We have therefore that $(\mathfrak{A}(Y, \lambda), d_J)$ is a differential $\mathbb{Z}_2$ graded super-commutative algebra. We define:

**Definition 1.** The **full contact homology** $C^{\text{full}}(\lambda, J)$ of $(Y, \lambda)$ is the homology of the complex $(\mathfrak{A}, d_J)$.

Under Hypothesis H, it was also proved in [10] that the full contact homology does not depend on the contact form $\lambda$ adapted to $(Y, \xi)$ and on the almost complex structure $J$.

### 3.3 Cylindrical contact homology

Suppose now that $(Y, \lambda)$ is a contact manifold, and $\lambda$ is a non-degenerate hypertight contact form. For hypertight contact manifolds we can define a simpler version of contact homology called cylindrical contact homology. We denote by $CH^{\text{cyl}}(\lambda)$ the $\mathbb{Z}_2$-graded $\mathbb{Q}$-vector space generated by the elements of $\mathcal{P}(\lambda)$. The differential $d_J^{\text{cyl}} : CH^{\text{cyl}}(\lambda) \to CH^{\text{cyl}}(\lambda)$ will count elements in the moduli space $\tilde{M}^0(\gamma, \gamma'; J)/\mathbb{R}$. For the generators $\gamma \in \mathcal{P}(\lambda)$ we define:

$$d_J^{\text{cyl}}(\gamma) = \text{cov}(\gamma) \sum_{\gamma' \in \mathcal{P}(\lambda)} C(\gamma, \gamma'; J)\gamma'$$

for $C(\gamma, \gamma'; J)$ as defined above and where $\text{cov}(\gamma)$ is the covering number of $\gamma$. For $\lambda$ hypertight and assuming Hypothesis H is true, Eliashberg, Givental and Hofer proved in [10] that $(d_J^{\text{cyl}})^2 = 0$.

**Definition 2.** The **cylindrical contact homology** $CH^{\text{cyl}}(\lambda)$ of $(Y, \lambda)$ is the homology of the complex $(CH^{\text{cyl}}(\lambda), d_J^{\text{cyl}})$.
Also under Hypothesis H, this homology doesn’t depend on the hypertight contact form \( \alpha \) associated to \((Y, \xi)\) or on the almost complex structure \( J \).

Denote by \( \Lambda \) the set of free homotopy classes of \( Y \). It is clear that for each \( \rho \in \Lambda \) the subspace \( \text{CH}^0_{\text{cyl}}(\lambda) \subset \text{CH}_{\text{cyl}}(\lambda) \) generated by the set \( \mathcal{P}_\rho(\lambda) \) of good periodic orbits in \( \rho \) is a subcomplex of \((\text{CH}_{\text{cyl}}(\lambda), d^\text{cyl}_J)\) because the differential \( C(\gamma, \gamma'; J) \) can be non-zero only for orbits \( \gamma' \) which are freely homotopic to \( \gamma \). Moreover the restriction \( d^\text{cyl}_J |_{\text{CH}^0_{\text{cyl}}(\lambda)} \) has image in \( \text{CH}^0_{\text{cyl}}(\lambda) \). From now on we will denote the restriction \( d^\text{cyl}_J : \text{CH}^0_{\text{cyl}}(\lambda) \to \text{CH}^0_{\text{cyl}}(\lambda) \) by \( d^0_J \). Denoting by \( \text{CH}^0_{\text{cyl}} \) the homology of \((\text{CH}_{\text{cyl}}^0(\lambda), d^0_J)\) we thus have the following direct sum:

\[
\text{CH}^0_{\text{cyl}}(\lambda) = \bigoplus_{\rho \in \Lambda} \text{CH}^0_{\text{cyl}}(\lambda)
\]

The fact that we can define partial versions of cylindrical contact homology restricted to certain free homotopy classes will be of crucial importance for us. It will allow us to obtain results in our examples without resorting to Hypothesis H. We explain now how to do this.

### 3.3.1 Cylindrical contact homology in special homotopy classes

Let \( \Lambda_0 \) denote the set of primitive free homotopy classes of \( Y \). Let \((Y, \lambda)\) be a contact manifold, and \( \rho \in \Lambda \) be either in \( \Lambda_0 \) or a free homotopy class which contains only simple closed Reeb orbits. From the work of Dragnev [9], we have that there exists a generic subset \( \mathcal{J}^\text{reg}(\lambda) \) of \( \mathcal{J}(\lambda) \) such that for all \( J \in \mathcal{J}^\text{reg}(\lambda) \):

- for all Reeb orbits \( \gamma_1, \gamma_2 \in \rho \), the moduli space of pseudoholomorphic cylinders \( \mathcal{M}(\gamma_1, \gamma_2; J) \) is transverse, i.e. the linearized Cauchy-Riemann operator \( D\partial_J(\tilde{w}) \) is surjective for all \( \tilde{w} \in \mathcal{M}(\gamma_1, \gamma_2; J) \);

- for all Reeb orbits \( \gamma_1, \gamma_2 \in \rho \), each connected component \( \mathcal{L} \) of the moduli space \( \mathcal{M}(\gamma_1, \gamma_2; J) \) is a manifold whose dimension is given by the Fredholm index of any \( \tilde{w} \in \mathcal{L} \).

In this case, for \( J \in \mathcal{J}^\text{reg}(\lambda) \), we define:

\[
d^\text{cyl}_J(\gamma) = \text{cov}(\gamma) \sum_{\gamma' \in \mathcal{P}_\rho(\lambda)} C^\rho(\gamma, \gamma'; J) \gamma' = \sum_{\gamma' \in \mathcal{P}_\rho(\lambda)} C^\rho(\gamma, \gamma'; J) \gamma'
\]

where \( C^\rho(\gamma, \gamma'; J) \) is the number of points the moduli space \( \widehat{\mathcal{M}}^1(\gamma, \gamma'; J)/\mathbb{R} \). The second equality follows from the fact that all Reeb orbits in \( \rho \) are simple, which implies \( \text{cov}(\gamma) = 1 \).

Because of Dragnev’s transversality result, in the present situation the differential \( d^\text{cyl}_J : \text{CH}_{\text{cyl}}(\lambda) \to \text{CH}_{\text{cyl}}(\lambda) \) is well-defined and satisfies \((d^\text{cyl}_J)^2 = 0\). Therefore, in this situation, we can define the cylindrical contact homology \( \text{CH}^\rho_{\text{cyl}}(\lambda) \) without the need of Hypothesis H. Once the transversality for \( J \) has been achieved, and using coherent orientations constructed in [5], the proof that \( d^\text{cyl}_J \) is well-defined and that \((d^\text{cyl}_J)^2 = 0\), is a
For \( \rho \) as in the theorem above, \( d^p_J(\gamma) : CH^p_{cy}(\lambda) \to CH^p_{cy}(\lambda) \) is well-defined and is finite sum: the moduli space \( \tilde{M}^1(\gamma, \gamma'; J) \) can be non-empty only if \( A(\gamma') \leq A(\gamma) \). It then follows from the non-degeneracy of \( \lambda \) the numbers \( C_{cy}(\gamma, \gamma'; J) \) can be different of zero only for finitely many \( \gamma' \). To see that \( C_{cy}(\gamma, \gamma'; J) \) is finite for every \( \gamma' \in \rho \) suppose by contradiction that there is a sequence \( \tilde{w}_i \) of distinct elements of \( M^1(\gamma, \gamma'; J)/\mathbb{R} \). By the SFT compactness theorem \([4]\) such a sequence has a convergent subsequence that converges to a pseudoholomorphic building \( \tilde{w} \) which has Fredholm index 1. Because of the hypertightness of \( \lambda \), no bubbling can occur and all the levels \( \tilde{w}_1, ..., \tilde{w}_k \) of the building \( \tilde{w} \) are pseudoholomorphic cylinders. From the hypothesis on \( \rho \) all these cylinders are somewhere injective pseudoholomorphic curves, and because of the regularity of \( J \) have to have Fredholm index \( \geq 1 \). Combining this we have \( 1 = I_F(\tilde{w}) = \sum(I_F(\tilde{w}_i)) \geq k \) which implies \( k = 1 \). Thus \( \tilde{w} \in \tilde{M}^1(\gamma, \gamma'; J)/\mathbb{R} \) and is the limit of a sequence of distinct elements of \( M^1(\gamma, \gamma'; J)/\mathbb{R} \). This is absurd because \( \tilde{M}^1(\gamma, \gamma'; J)/\mathbb{R} \) is a 0-dimensional manifold. \( \square \)

For \( \rho \) as in the theorem above, \( (d^p_J)^2 = 0 \): if we write

\[
d^p_j \circ d^p_j(\gamma) = \sum_{\gamma'' \in \mathcal{P}_\rho(\lambda)} m_{\gamma, \gamma''} \gamma''
\]

we know that \( m_{\gamma, \gamma''} \) is the number of two-level pseudoholomorphic buildings \( \tilde{w} = (\tilde{w}_1, \tilde{w}_2) \) such that \( \tilde{w}_1 \in \tilde{M}^1(\gamma, \gamma'; J) \) and \( \tilde{w}_2 \in \tilde{M}^1(\gamma', \gamma''; J) \), for some \( \gamma' \in \mathcal{P}_\rho(\lambda) \). Because of transversality of \( \tilde{w}_1 \) and \( \tilde{w}_2 \) we can perform gluing and this implies that \( \tilde{w} \) is in the boundary of the moduli space \( \tilde{M}^2(\gamma, \gamma''; J) \). Taking a sequence \( \tilde{w}_i \) of elements in \( \tilde{M}^2(\gamma, \gamma''; J) \) converging to the boundary of \( \tilde{M}^2(\gamma, \gamma''; J) \) and arguing similarly as above, we have that this sequence converges to a pseudoholomorphic building \( \tilde{w} \), whose levels are somewhere injective pseudoholomorphic cylinders. Using that \( I_F(\tilde{w}) = 2 \) we again obtain that \( \tilde{w} \) can have at most 2 levels. As \( \tilde{w} \) is in the boundary of \( \tilde{M}^2(\gamma, \gamma''; J) \) it cannot have 1 level, and is therefore a two-level pseudoholomorphic building whose levels have Fredholm index 1. Summing up, \( \tilde{w} = (w^1, w^2) \), where \( w^1 \in \tilde{M}^1(\gamma, \gamma'; J) \) and \( w^2 \in \tilde{M}^1(\gamma', \gamma''; J) \), for some \( \gamma' \in \mathcal{P}_\rho(\lambda) \).

The discussion above implies that \( m_{\gamma, \gamma''} \) is actually the count with signs of boundary components of the compactified moduli space \( \tilde{M}^2(\gamma, \gamma''; J)/\mathbb{R} \); because the signs of this count are determined by coherent orientations of \( \tilde{M}^2(\gamma, \gamma''; J)/\mathbb{R} \) \([5]\), we have just proved that \( m_{\gamma, \gamma''} = 0 \). \( \square \)

Exact symplectic cobordism induce homology maps for the SFT-invariants. We describe now, how this is done for cylindrical contact homology. Let \( (Y^+, \lambda^+) \) and \( (Y^-, \lambda^-) \) be contact manifolds, with \( \lambda^+ \) and \( \lambda^- \) hypertight contact forms and \( (W, \xi) \) an exact symplectic cobordism from \( (Y^+, \lambda^+) \) to \( (Y^-, \lambda^-) \). Suppose that \( \rho \) is either a primitive free homotopy class or that all the closed Reeb orbits of both \( \lambda^+ \) and \( \lambda^- \) which belong to \( \rho \) are simple. Choose almost complex structures \( J^+ \in J_{reg}(\lambda^+) \) and \( J^- \in J_{reg}(\lambda^-) \). From the work of Dragnev \([9]\) (see also section 2.3 in \([27]\)) we know that there is a generic
subset $\mathcal{J}_{reg}^\rho(J^-, J^+) \in \mathcal{J}(J^-, J^+)$ such that for $\hat{J} \in \mathcal{J}_{reg}^\rho(J^-, J^+)$, $\gamma^+ \in P_\rho(\lambda^+)$ and $\gamma^- \in P_\rho(\lambda^-)$:

- all the curves $\tilde{w}$ in the moduli spaces $\mathcal{M}(\gamma^+, \gamma^-; \hat{J})$ are regular
- the connected components $\mathcal{V}$ of $\mathcal{M}(\gamma^+, \gamma^-; \hat{J})$ have dimension equal to the Fredholm index of any pseudoholomorphic curve in $\mathcal{V}$

In this case we can define a map $\Phi^{\hat{J}} : CH^\rho_{cyl}(\lambda^+) \to CH^\rho_{cyl}(\lambda^-).$

$$\Phi^{\hat{J}}(\gamma^+) = \sum_{\gamma^- \in P_\rho(\lambda^-)} n_{\gamma^+, \gamma^-} \gamma^-$$

(36)

where $n_{\gamma^+, \gamma^-}$ is the number pseudoholomorphic cylinders with Fredholm index 0, positively asymptotic to $\gamma^+$ and negatively asymptotic to $\gamma^-$. Using a combination of compactness and gluing (see [3]) one proves that $\Phi^{\hat{J}} \circ d^\rho_{J^+} = d^\rho_{J^-} \circ \Phi^{\hat{J}}$, and thus we have a map $\Phi^{\hat{J}} : CH^\rho_{cyl}(\lambda^+) \to CH^\rho_{cyl}(\lambda^-)$ on the homology level.

We study the cobordism map in the following situation: let $(V = \mathbb{R} \times Y, \varpi)$ be an exact symplectic cobordism from $(Y, M\lambda)$ to $(Y, m\lambda)$ where $M > m > 0$, for a hypertight contact form $\lambda$. Suppose that one can make an isotopy of exact symplectic cobordisms $(\mathbb{R} \times Y, \varpi_t)$ from $(Y, M\lambda)$ to $(Y, m\lambda)$, with $\varpi_t$ satisfying $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\lambda_0)$. We consider the space $\tilde{\mathcal{J}}(J, J)$ of smooth homotopies:

$$t \in [0, 1]; J_t \in \mathcal{J}(J, J)$$

(37)

with $J_0 = J_V$ and $J_1 \in \mathcal{J}_{reg}(\lambda)$, and $J_t$ is compatible with $\varpi_t$. $J^t$ is a deformation of $J_0$ to $J_1$, through asymptotically cylindrical almost complex structure in for the cobordisms $(\mathbb{R} \times Y, \varpi_t)$. For Reeb orbits $\gamma, \gamma' \in P^\rho(\lambda)$ we consider the moduli space:

$$\tilde{\mathcal{M}}^1(\gamma, \gamma'; J_t) = \{(t, \tilde{w}) \mid t \in [0, 1] \text{ and } \tilde{w} \in \tilde{\mathcal{M}}^1(\gamma, \gamma'; J_t)\}$$

(38)

By using the techniques presented in [9], we know that there is a generic subset $\tilde{\mathcal{J}}_{reg}(J, J) = \tilde{\mathcal{J}}(J, J)$ such that $\tilde{\mathcal{M}}^1(\gamma, \gamma'; J_t)$ is a 1-dimensional smooth manifold with boundary, where $(t, \tilde{w}) \in \tilde{\mathcal{M}}^1(\gamma, \gamma'; J_t)$. The crucial condition that makes this valid is again the fact that the all the pseudoholomorphic curves that make part of this moduli space are somewhere injective.

We have the following proposition which is a consequence of the combination of work of Eliashberg, Givental and Hofer [10] and Dragnev [9].

**Proposition 1.** Let $(Y, \lambda)$ be a hypertight contact manifold and $\rho$ choose $\lambda^+ = M\lambda$ and $\lambda^- = m\lambda$ where $M > m > 0$ are constants, and $\rho$ be either a primitive free homotopy class or one such all the closed Reeb orbits $\lambda$ in $\rho$ are simple. Choose an almost complex structure $J \in \mathcal{J}_{reg}^\rho(\lambda)$, and set $J^+ = J^- = J$. Let $(W = \mathbb{R} \times Y, \varpi)$ is an exact symplectic cobordism from $M\lambda$ to $m\lambda$, and choose a regular almost complex structure $\tilde{J} \in \mathcal{J}_{reg}^\rho(J^-, J^+)$. Then, if there is an homotopy $(\mathbb{R} \times Y, \varpi_t)$ through exact symplectic cobordisms from $M\lambda$ to $m\lambda$, with $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\lambda)$, we have that the map $\Phi^{\tilde{J}} : CH^\rho_{cyl}(\lambda) \to CH^\rho_{cyl}(\lambda)$ is chain homotopic to the identity.
The proof is again a combination of compactness and gluing, which we sketch below and refer the reader to [3] and [10] for more details.

**Sketch of the proof:** we define initially the following map

\[ K : CH^p_{cyl}(\lambda) \to CH^p_{cyl}(\lambda) \]  

(39)

that counts finite energy, Fredholm index \(-1\) pseudoholomorphic cylinders in the cobordisms \((\mathbb{R} \times Y, \omega_t)\) for \(t \in [0,1]\). Because of the regularity of our homotopy, the moduli space of index \(-1\) cylinders whose positive puncture detects a fixed Reeb orbit \(\gamma\) is finite, and therefore the map \(K\) is well defined.

Notice that for \(t = 1\) the cobordism map \(\Phi^J_1\) is the identity, and the pseudoholomorphic curves that define it are just trivial cylinders over Reeb orbits. For \(t = 0\), \(\Phi^J_0\) counts index 0 cylinders in the cobordisms \((\mathbb{R} \times Y, \omega)\). From the regularity of \(J_0, J_1\) and the homotopy \(J_t\), we have that the pseudoholomorphic cylinders involved in these two maps belong to the 1-dimensional moduli spaces \(\tilde{M}^1(\gamma, \gamma'; J_t)\).

By using a combination of compactness and gluing we can show that the boundary \(\mathcal{W}\) of the moduli spaces \(\tilde{M}^1(\gamma, \gamma'; J_t)\) is exactly the set of pseudoholomorphic buildings \(\tilde{w}\) with two levels \(\tilde{w}_c\) and \(\tilde{w}_s\) such that: \(\tilde{w}_c\) is an index \(-1\) cylinder in a cobordism \((\mathbb{R} \times Y, \omega_t)\) and \(\tilde{w}_s\) is index 1 pseudoholomorphic cylinder in the symplectization of \(\lambda\) above or below \(\tilde{w}_c\). Such two level buildings are exactly the ones involved in the maps \(K \circ d^J_{cyl} + d^J_{cyl} \circ K\).

As a consequence one has that the difference between the maps \(\Phi^J_1 = Id\) and \(\Phi^J\) is equal to \(K \circ d^J_{cyl} + d^J_{cyl} \circ K\); implying that \(\Phi^J\) is chain homotopic to the identity. \(\Box\)

The result above can be used to show that \(CH^p_{cyl}(\lambda_0)\) does not depend on the regular almost complex structure \(J\) used to define the differential \(d_J\).

### 4 Exponential homotopical growth rate and estimates for \(h_{top}\)

In this section we define the exponential homotopical growth rate of contact homology and relate it to the topological entropy of Reeb vector fields. The basic idea is to use non-vanishing of cylindrical contact homology of \((M, \xi)\) in a free homotopy class to obtain existence of Reeb orbits in such an homotopy class for any contact form associated to \((M, \xi)\); this idea is present in the works [21] and [27]. It is straightforward to see that the period and action of a Reeb orbit are equal and in the sequel we will use the same notation to refer period and action of Reeb orbits.

Let \((M, \ker(\lambda_0))\) be a contact manifold with \(\lambda_0\) a hypertight contact form. For \(T > 0\) we define \(\tilde{\Lambda}^T(\lambda_0)\) to be the set of free homotopy classes of \(M\) such that \(\rho \in \tilde{\Lambda}^T(\lambda_0)\) if, and only if, all Reeb orbits of \(X_{\lambda_0}\) in \(\rho\) are simply covered, have action/period smaller than \(T\) and \(CH^p_{cyl}(\lambda_0) \neq 0\). We define \(N^cyl_T(\lambda_0)\) to be the cardinality \(|\tilde{\Lambda}^T(\lambda_0)|\).

**Definition:** we say that the cylindrical contact homology \(CH^p_{cyl}(\lambda_0)\) of \((M, \lambda_0)\) has exponential homotopical growth rate with exponential weight \(a > 0\) if there exists a real
number \( b \), such that \( N_{\tilde{T}}^{cyl}(\lambda_0) = \tilde{\mathcal{H}}_{\tilde{T}}(\lambda_0) \geq e^{aT+b} \).

The main result of this section is the following theorem:

**Theorem 2.** Let \((M, \ker(\lambda_0))\) be a contact manifold such that the cylindrical contact homology \( CH_{cyl}^{R_0}(\lambda_0) \) of \((M, \lambda_0)\) has exponential homotopical growth rate with exponential weight \( a \). Then for every \( C_k \) (\( k \geq 2 \)) contact form \( \lambda \) associated to \((M, \ker(\lambda_0))\) the Reeb flow of \( X_\lambda \) has positive topological entropy. More precisely, if \( \lambda = f\lambda_0 \) for a function \( C_k \) function \( f \) > 0, then \( h_{\text{top}}(X_\lambda) \geq \frac{a}{\max f} \).

**Proof:** If \( \lambda = c\lambda_0 \) for a constant \( c > 0 \) the theorem follows easily from Theorem 1. We can assume thus that \( \lambda = f\lambda_0 \) for a non-constant function \( f \) > 0 and write \( E = \max f \).

**Step 1:** We assume initially that \( \lambda \) is non-degenerate and \( C^\infty \). For every \( \epsilon > 0 \) it is possible to construct an exact symplectic cobordism from \((M, \lambda_0)\) to the symplectization of \((M, \lambda_0)\) and \( J \) and \( J_0 \) of the cobordism, and with \( J \) on \([-R,R] \times M \).

Using these cobordisms, it is possible to construct a splitting family \((\mathbb{R} \times M, \omega_R, J_R)\) from \((E + \epsilon)\lambda_0 \) to \( e\lambda_0 \), such that for every \( R > 0 \) \((\mathbb{R} \times M, \omega_R, J_R)\) is homotopical to the symplectization of \((M, \lambda_0)\). We fix a regular almost complex structure \( J_0 \in J_{cyl}^{\epsilon}(\lambda_0) \) and \( J \in J(\lambda) \), and demand that \( J_R \) coincides with \( J_0 \) in the positive and negative ends of the cobordism, and with \( J \) on \([-R,R] \times M \).

Let \( \rho \in \mathcal{F}_\lambda \). We claim that for every \( R \) there exists a finite \( \rho \)-pseudoholomorphic cylinder \((\tilde{w} \in \mathbb{R} \times M, J_R)\) positively asymptotic to a Reeb orbit in \( P_{\rho}(\lambda_0) \) and negatively asymptotic to an orbit in \( P_{\rho}(\lambda_0) \).

If this was not true for a certain \( R > 0 \), then because of the absence of \( \rho \)-pseudoholomorphic cylinders asymptotic to Reeb orbits in \( P_{\rho}(\lambda_0) \) we have \( J_R \in J_{cyl}^{\epsilon}(J_0, J_0) \). Therefore, the map \( \Phi^{J_R} : CH_{cyl}^{R_0}(\lambda_0) \to CH_{cyl}^{J_0}(\lambda_0) \) induced by \((\mathbb{R} \times M, \omega_R, J_R)\) is well-defined. But because there are no \( \rho \)-pseudoholomorphic cylinders asymptotic to Reeb orbits in \( P_{\rho}(\lambda_0) \), we have that the map \( \Phi^{J_R} : CH_{cyl}^{R_0}(\lambda_0) \to CH_{cyl}^{J_0}(\lambda_0) \) vanishes. On the other hand, from section 3.3 we know that \( \Phi^{J_R} \) is the identity. As \( \Phi^{J_R} \) vanishes and is the identity we conclude that \( CH_{cyl}^{J_0}(\lambda_0) = 0 \), contradicting that \( \rho \in \mathcal{F}_\lambda \).

**Step 2:**

Let \( \rho \in \mathcal{F}_\lambda \) and \( R_n \to +\infty \) be a strictly increasing sequence and \( \tilde{w}_n : \mathbb{R} \times (S^1 \times \mathbb{R}, i) \to (\mathbb{R} \times M, J_{R_n}) \) be a sequence of \( \rho \)-pseudoholomorphic cylinders with one positive puncture asymptotic to an orbit in \( P_{\rho}(\lambda_0) \) and one negative puncture asymptotic to an orbit in \( P_{\rho}(\lambda_0) \). Notice that, because of the properties of \( \rho \) the energy of \( \tilde{w}_n \) is uniformly bounded.

Therefore we can apply the SFT compactness theorem to obtain a subsequence of \( \tilde{w}_n \) which converges to a pseudoholomorphic building \( \tilde{w} \); notice that for this we need to use the non-degeneracy of \( \lambda \). Moreover we can give a very precise description of the building.

Let \( \tilde{w}_k \) for \( k \in \{1, \ldots, m\} \) be the levels of the pseudoholomorphic building \( \tilde{w} \). Because the topology of our curve doesn’t change on the breaking we have the following picture:
• the upper level $\tilde{w}^1$ has one pseudoholomorphic curve, which has one positive puncture, which is asymptotic to an orbit $\gamma_0 \in P_{\rho}(\lambda_0)$, and several negative punctures. All of the negative punctures detect contractible orbits, except one that detect a Reeb orbit $\gamma_1$ which is also in $\rho$

• on every other level $\tilde{w}^k$ there is a special curve which has one positive puncture, which is asymptotic to a Reeb orbit $\gamma_{k-1}$ in $\rho$ and possibly several negative. Of the negative punctures there is one that is asymptotic to an orbit $\gamma_k$ in $\rho$ and all the others are contractible.

Because of the splitting behavior of the cobordisms $\mathbb{R} \times (S^1 \times \mathbb{R}, i) \to (\mathbb{R} \times M, J_{R^\omega})$ it is clear that there exists a $k_0$, such that the level $\tilde{w}^{k_0}$ is in an exact symplectic cobordism from $(E + \epsilon)\lambda_0$ to $\lambda$. This implies that the special orbit $\gamma_{k_0}$ is a Reeb orbit of $X_{\lambda}$ in the homotopy class $\rho$.

Notice that $A(\gamma_{k_0}) \leq (E + \epsilon)T$. This implies that all the other orbits appearing as punctures of the building $\tilde{w}$ have action smaller than $(E + \epsilon)T$. In particular this implies that $\gamma_{k_0}$ has action smaller than $(E + \epsilon)T$.

As we can do the construction above for any $\epsilon > 0$ we can obtain a sequence of Reeb orbits $\gamma_{K_\alpha(j)}$ which are all in $\rho$ and such that $A(\gamma_{K_\alpha(j)}) \leq (E + \epsilon)T$. Using Arzela-Ascoli’s Theorem one can extract a convergent subsequence of $\gamma_{i_\epsilon}^j$; its limit $\gamma_\rho$ is clearly a Reeb orbit of $\lambda$ in the free homotopy class $\rho$ and with action smaller or equal to $ET$.

**Step 3:** estimating $N_{X_{\lambda}}(T)$.

From step 2, we know that if $\rho \in \tilde{\Lambda}_T(\lambda_0)$ then there is a Reeb orbit $\gamma_\rho$ of the Reeb flow of $X_\lambda$ with $A(\gamma_\rho) \leq ET$. Recalling that the period and the action of a Reeb orbit coincide, we obtain that $N_{X_{\lambda}}(T) \geq \sharp(\tilde{\Lambda}_T^{\geq \epsilon}(\lambda_0))$. Under the hypothesis of the theorem we have:

$$N_{X_{\lambda}}(T) \geq \epsilon^{\frac{a}{\#(\mathbb{R})} + b} \quad (40)$$

Applying Theorem 1 we then obtain $h_{\text{top}}(X_{\lambda}) \geq \frac{\#}{\mathbb{R}}$. This proves the theorem in the case $\lambda$ is $C^\infty$ and non-degenerate.

**Step 4:** passing to the case of a general $C^k \geq 2$ contact form $\lambda$ (the case where $\lambda$ is degenerate is included here).

Let $\lambda_i$ be a sequence of non-degenerate contact forms converging in the $C^k$-topology to a contact form $\lambda$ which is $C^k$ ($k \geq 2$) and possibly degenerate. For every $\epsilon > 0$ there is $i_0$ such that for $i > i_0$; there exists an exact symplectic cobordism from $(E + \epsilon)\lambda_0$ to $\lambda_i$.

Fixing then an homotopy class $\rho \in \tilde{\Lambda}_T(\lambda_0)$ we know, by the previous steps, that there exists a Reeb orbit $\gamma_\rho(n)$ of $\lambda_n$ in the homotopy class $\rho$ with action smaller than $(E + \epsilon)T$. By taking the sequence $\gamma_\rho(n)$ and applying Arzela-Ascoli’s theorem we obtain a subsequence which converge to a Reeb orbit $\gamma_{\epsilon, \rho}$ of $X_{\lambda}$ with $A(\gamma_{\epsilon, \rho}) \leq (E + \epsilon)T$; notice that here we use that $\lambda$ is at least $C^2$ so that $X_{\lambda}$ is at least $C^1$, in order to be able to use Arzela-Ascoli’s theorem.
Because $\epsilon > 0$ above can be taken arbitrarily close to 0 we can actually obtain a sequence $\gamma_{j,\rho}$ of Reeb orbits of $X_\lambda$ whose homotopy class is $\rho$ such that the actions $A(\gamma_{j,\rho})$ converges to $ET$. Again applying Arzela-Ascoli’s theorem, we obtain that the sequence $\gamma_{j,\rho}$ has a convergent subsequent, which converges to an orbit $\gamma_\rho$ satisfying $A(\gamma_{j,\rho}) \leq ET$.

Reasoning as in step 3 above, we obtain that $N_{X_\lambda}(T) \geq e^{\frac{aT}{2} + b}$ and applying theorem 1 we obtain the desired estimate for the topological entropy. This finishes the proof of the theorem.

\section{Contact 3-manifolds with a hyperbolic component}

In this section we will prove the following theorem:

\textbf{Theorem 3.} Let $M$ be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary such that the component $M_0$ satisfies:

- $M_0$ is a suspension of a surface with boundary $S$ by a diffeomorphism $h : S \rightarrow S$ with pseudo-Anosov monodromy

Then, $M$ can be given infinitely many different contact structures $\xi_k$ such that there exists a hypertight contact form $\lambda_k$ associated to $(M, \xi_k)$ with exponential homotopical growth of contact homology.

We denote by $S$ a surface with boundary and $\omega$ a symplectic form on $S$. Let $h$ be a symplectomorphism of $(S, \omega)$ to itself, with pseudo-Anosov monodromy and which is the identity on a neighbourhood of $\partial S$. We follow a well known recipe to construct a suitable contact form in the mapping torus $\Sigma(S, h)$.

We choose a primitive $\beta$ for $\omega$ such that for coordinates $(r, \theta) \in [-\epsilon, 0] \times S^1$ in a neighbourhood $V$ of $\partial S$ we have $\beta = f(r)d\theta$, where $f > 0$ and $f' > 0$. On $\mathbb{R} \times S$ we consider the contact form $\tilde{\alpha}$:

$$\tilde{\alpha} = dt + \epsilon(1 - F_i(t))\beta + \epsilon F_i(t)h^*\beta$$

when $t \in [i, i + 1]$ (41)

where $F_0 : \mathbb{R} \rightarrow [0, 1]$ is a smooth non-decreasing function which satisfies $F_0(t) = 0$ for $t \in (-\infty, \frac{1}{100})$ and $F_0(t) = 1$ for $t \in (\frac{1}{100}, +\infty)$; $F_i(t) = F_0(t - i)$ and $\epsilon > 0$. It is a simple computation to show that for $\epsilon$ small enough $\tilde{\alpha}$ is a contact form. For $t \in [0, 1]$, we have that Reeb vector field $X_{\tilde{\alpha}}$ is equal to $\partial_t + v(p, t)$, where $v(p, t)$ is the unique vector tangent to $S$ that satisfies $\omega(v(p, t), \cdot) = F_0(t)\beta - F_0^*h^*\beta$.

Considering on the diffeomorphism $H : \mathbb{R} \times S \rightarrow \mathbb{R} \times S$ defined by $H(t, p) = (t - 1, h(p))$. The mapping torus $\Sigma(S, h)$ is defined by:

$$\Sigma(S, h) := (\mathbb{R} \times S)/_{(t, p) \sim H(t, p)}$$

and we denote by $\pi : \mathbb{R} \times S \rightarrow \Sigma(S, h)$ the associated covering map.
Because $H^*\tilde{\alpha} = \tilde{\alpha}$, there exists a unique contact form $\alpha$ on $\Sigma(S,h)$ such that $\pi^*\alpha = \tilde{\alpha}$. Notice that for the neighbourhood $S^1 \times V$ of $\partial\Sigma(S,h)$, $\alpha = dt + \epsilon f(r)d\theta$, and thus $X_\alpha$ is tangent to $\partial\Sigma(S,h)$.

The Reeb vector field $X_\alpha$ on $\Sigma(S,h)$ is transverse to the surfaces $\{t\} \times S$ for $t \in \mathbb{R}/\mathbb{Z}$. This implies that $\{0\} \times S$ is a global surface of section for the Reeb flow of $\alpha$, and by our expression of $X_{\tilde{\alpha}}$ the first return map of the Reeb flow of $\alpha$ is isotopic to $h$.

5.1 Contact 3-manifolds containing $(\Sigma(S,h), \alpha)$ as a component

Let $W$ be a compact oriented irreducible 3-manifold such that $\partial(W)$ is a union of incompressible tori. The following theorem of Colin and Honda [8] tells us that $W$ admits a hypertight contact form tangent to the boundary:

**Theorem:** (Colin-Honda [8]) Let $W$ be a compact, oriented, irreducible 3-manifold such that $\partial(W)$ is a union of incompressible tori. Then there exists a hypertight contact form $\zeta$ such that, in a neighbourhood $(\mathbb{R}/K\mathbb{Z}) \times S^1 \times I$ with oriented coordinates $((t, \vartheta), \tilde{r})$ of each component of $\partial(W)$, $\zeta = \cos(\tilde{r})d\tilde{t} \pm \sin(\tilde{r})d\vartheta$.

Now suppose we are given a finite collection compact oriented irreducible 3-manifolds $\{M_i, 0 \leq i \leq l\}$ with $M_0 = \Sigma(S,h)$, and that can be suitably glued along their boundaries to give an oriented 3-manifold $M$. This means that $\{M_i, 0 \leq i \leq l\}$ is the JSJ decomposition of the 3-manifold $M$. By using the above theorem of Colin and Honda we can endow each $M_i$ with a hypertight contact form $\alpha_i$ (compatible with the orientation of $M_i$) tangent to the boundary of $M_i$; on the special piece $M_0$ we consider the the contact form $\alpha_0$ equal to $\alpha$ constructed above (which is also tangent to the boundary).

We sketch how to obtain a contact form on $M$ from the $\alpha_i$ above and refer the reader to [8] for a complete presentation. From the description above of $\alpha_i$, we know that in a neighbourhood of $\partial(\Sigma(S,h))$ diffeomorphic to $S^1 \times V$, with coordinates $(t, r, \vartheta)$, we have $\alpha = dt + f(r)d\vartheta$ where $f > 0$ and $f' > 0$.

For a natural number $n > 3$ we consider a neck $T$ of the form $\mathbb{R}/K\mathbb{Z} \times [0,1] \times S^1$ with coordinates $(t', r', \vartheta')$. Let $g_1$ and $g_2$ be functions from $\times[0,n]$ to $\mathbb{R}$ satisfying:

- $g_1(0) = 1$ and $g_1^{(j)}(0) = 0$ for all positive integer $j$
- $g_2^{(j)}(0) = f^{(j)}(0)$ for all non-negative integer $j$
- $(g_1 g_2' - g_1' g_2)(r') > 0$ for all $r' \in [0, n]$
- $g_1(r') = \cos(2\pi r')$ and $g_2(r') = \sin(2\pi r')$ if $r' \in [1, n]$

Then $\nu_n = g_1(r) dt + g_2(r) d\vartheta$ is a contact form in $T_n$. $\nu_n$ interpolates between the contact form on a neighbourhood of the boundary of $\partial(\Sigma(S,h))$ to the contact form $\cos(\tilde{r})d\tilde{t} - \sin(\tilde{r})d\tilde{\vartheta}$ in the boundary of $M_i$ for $i \geq 1$ by the above theorem of Colin and Honda. By introducing the necks $T_n$ we can interpolate the contact forms in the boundaries of the components $M_i$ to obtain a contact form $\lambda_0$ on $M$. Notice that the orientations induced by the contact forms on the components have to match when we make
the gluing in order to be able to glue the contact forms on the pieces. The hypertightness of \( \lambda_0 \) is a consequence of the hypertightness of the \( \alpha_i \), combined with the fact that all the periodic orbits in the neck \( T_n \) represent non-trivial homology classes in the incompressible tori; because of the incompressibility of the boundary tori of \( M_i \) no Reeb orbits becomes contractible after gluing if they were not contractible in the component.

Because we can introduce the necks \( T_n \) as above, we can obtain for our given 3-manifold \( M \), contact structures with arbitrarily large Giroux torsion. As Giroux torsion distinguishes contact structures we obtain infinitely many different contact structures on \( M \). We will denote these different contact forms obtained by adding necks by \( \lambda_k \) and their contact structures in \( M \) by \( \xi_k = \ker(\lambda_k) \).

By doing an arbitrarily small perturbation of \( \alpha_0 \) supported in the interior of \( \Sigma(S,h) \) we can obtain a contact form \( \hat{\alpha} \) such that all Reeb orbits which are not freely homotopic to curves in \( \partial \Sigma(S,h) \) are non-degenerate, and \( \{0\} \times S \) is a global surface of section for the flow of \( X_{\hat{\alpha}} \). We will consider in \( M \) the contact-form \( \lambda^k \) associated to \( \xi_k \) which restricts to \( \alpha_i \) in the components \( M_i \) for \( 1 \leq i \leq l \), and to \( \hat{\alpha} \) in \( M_0 \).

### 5.2 Proof of Theorem 3

It is clear that theorem 3 will follow from the exponential homotopical growth of cylindrical contact homology of \( (M,\lambda_k) \), and therefore from the following proposition which is essentially due to Vaugon in \[30\]:

**Proposition 2.** \( \lambda_k \) has exponential homotopical growth of cylindrical contact homology.

Before proving the proposition we introduce some necessary ideas and notation.

The first return map of \( X_{\hat{\alpha}} \) is a diffeomorphism \( \hat{h} : S \to S \) which is homotopic \( h \) and therefore to a pseudo-Anosov map. The Reeb orbits of \( X_{\hat{\alpha}} \) are in one-to-one correspondence with periodic orbits of \( \hat{h} \). Moreover we have that two Reeb orbits \( \gamma_1 \) and \( \gamma_2 \) of \( X_{\hat{\alpha}} \) are freely homotopic if and only if their associated periodic orbits are in the same Nielsen class. Thus there is an injective map \( \Xi \) from the set \( N \) of Nielsen classes to the set \( \bigwedge \) of free homotopy classes of Reeb orbits in \( \Sigma(S,h) \).

Denote by \( N_k \) the set of distinct Nielsen classes which contain only periodic orbits of \( \hat{h} \) of period smaller or equal to \( k \). Because of the pseudo-Anosov monodromy of \( \hat{h} \) we know that there are constants \( a > 0 \) and \( b \in \mathbb{R} \), such that \( \sharp(\hat{N}_k) \geq e^{ak+b} \). Analogously define the subset \( \bigwedge_T(\Sigma(S,h)) \) of free homotopy classes of \( \Sigma(S,h) \) which contains at least one Reeb orbit of \( X_{\hat{\alpha}} \) and contains only Reeb orbits with action smaller than \( T \).

Because \( \hat{h} \) is the first return map for a global surface of section of the flow \( X_{\alpha} \), there exists a constant \( \eta > 0 \) such that if \( \bar{\nu} \in N_k \to \Xi(\bar{\nu}) \in \bigwedge_{\eta k}(\Sigma(S,h)) \). This implies that the \( \sharp(\bigwedge_T(\Sigma(S,h))) \geq e^{\frac{aT}{2}+b} \). Notice that by doing the perturbation sufficiently small \( \eta \) can be taken to be as close as desired to 1.

Let \( \bigwedge_T^0(\Sigma(S,h)) \) be the subset of \( \bigwedge_T(\Sigma(S,h)) \) which contains free homotopy classes in \( \Sigma(S,h) \) which are primitive and different from the ones generated by curves in \( \partial \Sigma(S,h) \)
(we denote by $\Lambda^0_0(\Sigma(S,h))$ the set $\Lambda^0_{+\infty}(\Sigma(S,h))$). Because the fundamental group of $\partial\Sigma(S,h)$ grows quadratically we know that $\sharp\Lambda^0_0(\Sigma(S,h)) \geq e^{\frac{T}{\eta} + b}$.

We are now ready for the proof of the proposition which, as we mentioned, is essentially the argument of Vaugon in [30]. Proof of Proposition 2:

Step 1:
Let $i : \Sigma(S,h) \to M$ be the injection we obtain from looking at $\Sigma(S,h)$ as a component of $M$. Because of the incompressibility of $\partial\Sigma(S,h)$ in $M$, the associated map $i_* : \Lambda^0_0(\Sigma(S,h)) \to \Lambda(M)$ is injective for any $T > 0$ (where $\Lambda(M)$ denotes the free loop space of $M$). Using the Reeb flow of $\lambda_k$ is tangent to $\partial\Sigma(S,h)$, we know that for every $\rho \in i_*(\Lambda^0_0(\Sigma(S,h)))$ all the Reeb orbits of $X_{\lambda_k}$ in $\rho$ are in the interior of $\Sigma(S,h)$. Therefore the map $i_* : \Lambda^0_T(\Sigma(S,h)) \to \Lambda_T(M)$ is well defined (where by $\Lambda_T(M)$ we denote the set of free homotopy classes of $M$ which only contain Reeb orbits with action smaller than $T$).

Step 2: for every $\varrho \in i_*(\Lambda^0_0(\Sigma(S,h)))$ we have $\text{CH}_\varrho^cyl(\lambda_k) \neq 0$.

Vaugon showed (see the proofs of Lemma 7.11 and Theorems 1.3 and 1.2 on pages 27 and 28 in [30]; see also [11]) that the numbers of even and odd Reeb orbits in $\varrho$ differ. For Euler characteristic reasons this implies that $\text{CH}_\varrho^cyl(\lambda_k) \neq 0$.

Step 3:
Remember that in section 4 we defined $N^cyl^0_T(\lambda_k)$ to be the number of different free homotopy classes $\varrho$ in $\Lambda_T(M)$ which contained only simple Reeb orbits with action smaller then $T$ and such that $\text{CH}_\varrho^cyl(\lambda_k) \neq 0$.

Combining the last two steps we have:

$$N^cyl^0_T(\lambda_k) \geq \sharp(i_*(\bigwedge^0_T(\Sigma(S,h)))) = \sharp(\bigwedge^0_T(\Sigma(S,h))) \geq e^{\frac{T}{\eta} + b}. \quad (43)$$

As mentioned earlier, Theorem 3 now follows from Theorem 2.

6 Graph manifolds and Handel-Thurston surgery

In [16] Handel and Thurston used Dehn surgery to obtain non-algebraic Anosov flows in 3-manifolds. Their surgery was adapted to the contact setting by Foulon and Hasselblatt in [13], who interpreted it as a Legendrian surgery and used it to produce non-algebraic Anosov Reeb flows on 3-manifolds. We consider here a surgery that includes the Foulon-Hasselblatt one as a particular case: they restrict their attention to Dehn surgeries with positive integer coefficients while we consider the case of any integer coefficient.

6.1 The surgery

We start by fixing some notation. Let $(S,g)$ be an oriented hyperbolic surface and $c : S^1 \to S$ an embedded oriented separating geodesic of $g$. We denote by $\pi : (D,g) \to (S,g)$
the locally isometric universal covering of \((S, g)\) by the the hyperbolic disc \((\mathbb{D}, g)\) such that \((-1, 1) \times \{0\} \subset \pi^{-1}(c(S^1));\) this is always possible since the segment \((-1, 1) \times \{0\}\) of the real axis is a geodesic in \((\mathbb{D}, g)\). We denote by \(v(\theta)\) the unique unitary vector field over \(c(\theta)\) satisfying \(\angle(c'(\theta), v(\theta)) = -\frac{\pi}{2}\). For coordinates \(z = x + iy \in \mathbb{D}\), the lift of \(v(\theta)\) to \((-1, 1) \times \{0\}\) is the vector field \(-\partial_y\) over \((-1, 1) \times \{0\}\). Also, let \(\Pi : T_1S \to S\) denote the base point projection.

Because \(c\) is a separating geodesic, we can cut \(S\) along \(c\) to obtain two oriented hyperbolic surfaces with boundary which we denote by \(S_1\) and \(S_2\); the vector field \(v(\theta)\) points inside \(S_2\) and outside \(S_1\). This decomposition of \(S\) induces a decomposition of \(T_1S\) in \(T_1S_1\) and \(T_1S_2\). Both \(T_1S_1\) and \(T_1S_2\) are 3-manifolds whose boundary is the torus formed by the the unit fibers over \(c\).

Denote by \(V_{c,\delta}\) the closed \(\delta\)-neighbourhood of the \(c\) the geodesic \(c\) for the hyperbolic metric \(g\). For \(\delta > 0\) sufficiently small we have that \(V_{c,\delta}\) is an annulus such that the only closed geodesics contained in \(V_{c,\delta}\) are the covers of \(c\), and that satisfies the following convexity property: if \(\tilde{V}\) is the connected component of \(\pi^{-1}(V_{c,\delta})\) containing \((-1, 1) \times \{0\}\), then every segment of a hyperbolic geodesic starting and ending in \(\tilde{V}\) is completely contained in \(\tilde{V}\). Also it follows from the conventions adopted above, that if we denote by \(U^+\) the upper hemisphere of the \(\mathbb{D}\) composed of points with positive imaginary component and by \(U^-\) the lower hemisphere of the \(\mathbb{D}\) composed of points with negative imaginary component, we have:

\[
\tilde{V} \cap U^+ \subset \pi^{-1}(S_1) \text{ and } \tilde{V} \cap U^- \subset \pi^{-1}(S_2).
\]

The fact has the following important consequence: if \(\nu([0, K])\) is a hyperbolic geodesic segment starting and ending at \(V_{c,\delta}\) and contained in one of the \(S_i\), then \([\nu]\) is a non-trivial homotopy class in the relative fundamental group \(\pi_1(S_i, V_{c,\delta})\).

On the unit tangent bundle \(T_1S\) we consider consider the contact form \(\lambda_g\) whose Reeb vector field is the geodesic vector field for the hyperbolic metric \(g\). It is well known that the lifted curve \((c(\theta), v(\theta))\) in \(T_1S\) is Legendrian on the contact manifold \((T_1S, \ker(\lambda_g))\). The geodesic vector field \(X_{\lambda_g}\) over the Legendrian curve the geodesic vector field coincides with the horizontal lift of \(v\) (see section 1.3 of [28]), and therefore points inward \(T_1S_2\) and outwards \(T_1S_1\), and is normal to \(\partial(T_1S_2)\) for the Sasaki metric.

Moreover if \(\delta > 0\) is small enough we know that for every \(\vartheta \in L_c\) there exists numbers \(t_1 < 0\) and \(t_2 > 0\) such that:

\[
\phi^{t_1}_{\lambda_g}(\vartheta) \in T_1S_1 \setminus \Pi^{-1}(V_{c,\delta}) \quad (45)
\]

\[
\phi^{t_2}_{\lambda_g}(\vartheta) \in T_1S_2 \setminus \Pi^{-1}(V_{c,\delta}) \quad (46)
\]

Following [13], we know that there exists a neighbourhood \(B_{3\eta}^{3\eta}\) of \(L_c\) on which we can find coordinates \((t, s, w) \in (-3\eta, 3\eta) \times S^1 \times (-2\epsilon, 2\epsilon)\) such that:

25
\[ \lambda_g = dt + wds, \]  
\[ L_c = \{0\} \times S^1 \times \{0\}, \]  
\[ \{0\} \times \{\theta\} \times (-2\epsilon, 2\epsilon) \] is a local parametrization of the unitary fiber over \( \theta \in L_c \), and \( \epsilon < \frac{n}{4q|\pi|} \), where \( q \) is a fixed integer. Let \( W^- = \{-3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon) \) and \( W^+ = \{+3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon) \). It is clear that \( \Pi(W^-) \subset S_1 \) and \( \Pi(W^+) \subset S_2 \). Because on \( B_{2\epsilon}^{3\eta} \) the Reeb vector field \( X_{\lambda_g} \) is given by \( \partial_t \), it is clear that for every point \( p \in B_{2\epsilon}^{3\eta} \) there are \( p^- \in W^-, p^+ \in W^+, t^- \in (-6\eta, 0) \) and \( t^+ \in (0, 6\eta) \) for which:

\[ \phi^{t^-}_{X_{\lambda_g}}(p) = p^- \quad \text{and} \quad \phi^{t^+}_{X_{\lambda_g}}(p) = p^+ \]  

This means that trajectories of the flow of \( X_{\lambda_g} \) that enter the box \( B_{2\epsilon}^{3\eta} \) enter through \( W^- \) and exit through \( W^+ \); they cannot stay inside \( B_{2\epsilon}^{3\eta} \) for positive or negative time. We can say even more about these trajectories.

For \( \sigma = (p, \hat{p}) \in (S \times T_pS) \) in \( W^+ \cup W^- \) let \( \tilde{\sigma} = (\tilde{p}, \tilde{\hat{p}}) \) be a lift of \( \sigma \) to the unit tangent bundle \( T_1 \mathbb{D} \) such that \( \tilde{p} \in \tilde{V} \). The geodesic vector field \( X_{\lambda_g} \) in \( \tilde{\sigma} \) coincides with the horizontal lift of \( \hat{\sigma} \) \((26)\) (section 1.3). For \( \delta, \eta > 0 \) and \( \epsilon < \frac{n}{4|\pi|} \) sufficiently small we can guarantee that:

- \( B_{2\epsilon}^{3\eta} \) is contained in \( V_{c,\delta} \)
- for the lifts \( \tilde{\sigma} = (\tilde{p}, \tilde{\hat{p}}) \) of points in \( W^+ \cup W^- \) as above, the vector \( \tilde{\hat{p}} \) (which is the projection of the geodesic vector field \( X_{\lambda_g}(\tilde{\sigma}) \)) satisfies \( \angle(\tilde{\hat{p}}, -\partial_y) < \delta \)

Shrinking \( \delta > 0, \eta > 0 \) and \( 0 < \epsilon < \frac{n}{4|\pi|} \) if necessary this implies that for every \( \sigma^+ \in W^+ \) there exists \( t_{\sigma^+} > 0 \) and for every \( \sigma^- \in W^- \) there exists \( t_{\sigma^-} < 0 \) such that:

\[ \phi^{t_{\sigma^+}}_{X_{\lambda_g}}(\sigma^+) \in (T_1S_2) \setminus V_{c,\delta} \quad \text{and} \quad \forall t \in [0, t_{\sigma^+}] \quad \phi^{t}_{X_{\lambda_g}}(\sigma^+) \notin B_{2\epsilon}^{3\eta} \]
\[ \phi^{t_{\sigma^-}}_{X_{\lambda_g}}(\sigma^-) \in (T_1S_1) \setminus V_{c,\delta} \quad \text{and} \quad \forall t \in [t_{\sigma^-}, 0] \quad \phi^{t}_{X_{\lambda_g}}(\sigma^-) \notin B_{2\epsilon}^{3\eta} \]

To prove this last condition above one uses the fact that \( \angle(\tilde{\hat{p}}, -\partial_y) < \delta \) is small and studies the behavior of geodesics in \( (\mathbb{D}, g) \) starting at points close to the real axis and with initial velocity close to \( -\partial_y \). It is easy to see that such geodesics have to cut through the region \( V_{c,\delta} \) and visit the interior of both \( S_1 \setminus V_{c,\delta} \) and \( S_2 \setminus V_{c,\delta} \). From now on we will assume that \( \delta > 0, \eta > 0 \) and \( 0 < \epsilon < \frac{n}{4|\pi|} \) are such that the all the above mentioned properties described for them being sufficiently small, hold simultaneously.

Consider the following map \( F : B_{2\epsilon}^{2q} \setminus \overline{B_{\epsilon}^{q}} \to B_{2\epsilon}^{2\eta} \setminus \overline{B_{\epsilon}^{3\eta}} \):

\[ F(t, s, w) = (t, s + f(w), w) \quad \text{for} \quad (t, s, w) \in (\eta, 2\eta) \times S^1 \times (-2\epsilon, 2\epsilon) \]

where \( f(w) = -q\mathcal{R}(\frac{w}{\pi}) \) for our previously chosen integer \( q \) and \( \mathcal{R} : [-1, 1] \to [0, 2\pi] \) satisfies \( \mathcal{R} = 0 \) on a neighbourhood of \(-1\), \( \mathcal{R} = 2\pi \) on a neighbourhood of \( 1 \), \( 0 \leq \mathcal{R}' \leq 4 \) and \( \mathcal{R}' \) is an even function; and \( F \) is the identity otherwise.
Our new 3-manifold $M$ is obtained by gluing $T_1S \setminus B^3_{2\epsilon}$ and $B_{2\epsilon}^{2n}$ using the map $F$: \[
 M = (T_1S \setminus B^3_{\epsilon}) \cup B_{2\epsilon}^{2n} \big/ (x \in B_{2\epsilon}^{2n} \setminus B^3_{\epsilon}) \sim (F(x) \in T_1S \setminus B^3_{\epsilon}) \tag{53} \]

Notice that $T_1S = (T_1S \setminus B^3_{\epsilon}) \cup B_{2\epsilon}^{2n} \big/ (x \in B_{2\epsilon}^{2n} \setminus B^3_{\epsilon}) \sim (x \in T_1S \setminus B^3_{\epsilon})$. This clarifies our construction of $M$ and shows that $M$ is obtained from $T_1S$ via a Dehn surgery on $L_\epsilon$.

We follow [13] to endow $M$ with a contact form which coincides $\lambda_g$ outside of $B_{2\epsilon}^{2n}$. As a preparation we define the function $\beta : (-3\eta, 3\eta) \to \mathbb{R}$:

- $\beta$ is equal to 1 in an open neighbourhood of $[-2\eta, 2\eta]$,
- $|\beta'| \leq \frac{\eta}{\gamma}$ and $\text{supp} \beta$ is contained in $[-3\eta, 3\eta]$.

Using $\beta$ we define:

$$r(t, w) = \beta(t) \int_{-2\epsilon}^w x f'(x) dx \tag{54}$$

We point out to the reader that $\text{supp}(r)$ is contained in $B_{2\epsilon}^{3\eta}$ and therefore so is $\text{supp}(dr)$. Notice also, that in $B_{2\epsilon}^{2n} \setminus B^3_{\epsilon}$ one has $dr = \frac{\eta}{\gamma} f'(w) dw$.

Again following [13] we define in $T_1S \setminus B^3_{\epsilon}$ the 1-form:

$$A_r = dt + wds + dr \text{ for } (-3\eta, -\eta), \tag{55}$$
$$A_r = dt + wds - dr \text{ for } (\eta, 3\eta), \tag{56}$$
$$A_r = \lambda_g \text{ otherwise.} \tag{57}$$

Notice that because $\text{supp}(dr)$ is contained in $B_{2\epsilon}^{3n}$ the 1-form $A_r$ is well-defined.

On the box $B_{2\epsilon}^{2n}$ we define:

$$\tilde{A} = dt + wds + dr \tag{58}$$

Computing we obtain $F^*(A_r) = \tilde{A}$ which means that the gluing map $F$ allows us to glue the 1-forms $A_r$ and $\tilde{A}$. We denote by $\lambda_F$ the 1-form in $M$ obtained by gluing $\tilde{A}$ and $A_r$. We will denote by $\tilde{B}$ the following region:

$$\tilde{B} = ((B_{2\epsilon}^{3n} \setminus B^3_{\epsilon}) \subset M) \cup B_{2\epsilon}^{2n} \big/ (x \in B_{2\epsilon}^{2n} \setminus B^3_{\epsilon}) \sim (F(x) \in (B_{2\epsilon}^{3n} \setminus B^3_{\epsilon})) \tag{59}$$

The importance of this region lies in the fact that in $M \setminus \tilde{B} = T_1S \setminus B_{3\epsilon}^{3n}$, the contact form $\lambda_F$ coincides with $\lambda_g$.

Following [FH] one shows through a direct computation that $(dt + wds \pm dr) \wedge (dw \wedge ds) = (1 \pm \frac{\eta}{2\epsilon}) dt \wedge dw \wedge ds$. Using the fact that $\epsilon < \frac{\eta}{3|\eta|}$ one gets that $|\frac{\eta}{2\epsilon}| < 1$, thus obtaining that $(dt + wds \pm dr)$ is a contact form. It follows from this that $A_r$ and $\tilde{A}$ are contact forms in their respective domains and therefore $\lambda_F$ is a contact form in $M$. More strongly, Foulon and Hasselblatt proceed to show that if $q$ is non-negative the Reeb flow of $\lambda_F$ is an Anosov Reeb flow.
6.2 Hypertightness and exponential homotopical growth for $\lambda_F$

For $q \in \mathbb{N}$ the hypertightness of $\lambda_F$ follows from the fact that its Reeb flow is Anosov [12]. In this subsection we give an independent and completely geometrical proof of hypertightness of $\lambda_F$, which is valid for every $q \in \mathbb{Z}$.

To understand the topology of Reeb orbits of $\lambda_F$ we will study trajectories that enter the surgery region $\tilde{B}$. We start by studying trajectories in $B_{2\varepsilon}^{2n}$. On this region we have:

$$X_{\lambda_F} = \frac{\partial_t}{1 + \partial_t \rho}$$  \hfill (60)

This implies, similarly to the case of $\lambda_g$, that for points $p \in B_{2\varepsilon}^{2n}$ trajectory $\phi_{X_{\lambda_F}}^t (p)$ leaves the box $B_{2\varepsilon}^{2n}$ in forward and backward time. More precisely, there exists a constant $\tilde{a} > 0$ depending only on $\lambda_F$, such that for $p \in B_{2\varepsilon}^{2n}$ there are $\tilde{p}^- \in \tilde{W}^- = \{-2\eta\} \times S^1 \times [-2\varepsilon, 2\varepsilon]$, $\tilde{p}^+ \in \tilde{W}^+ = \{+2\eta\} \times S^1 \times [-2\varepsilon, 2\varepsilon]$, $\tilde{t}^- \in (-\tilde{a}, 0]$ and $\tilde{t}^+ \in [0, \tilde{a})$ such that:

$$\phi_{X_{\lambda_F}}^{\tilde{t}^-} (\tilde{p}) = \tilde{p}^- \quad \text{and} \quad \phi_{X_{\lambda_F}}^{\tilde{t}^+} (\tilde{p}) = \tilde{p}^+$$  \hfill (61)

We now analyse the trajectories of points $\tilde{p}^- \in \tilde{W}^-$ and $\tilde{p}^+ \in \tilde{W}^+$. For this, we first notice that on $\tilde{B} \setminus B^n_{\eta}$ the contact form $\lambda_F$ is given by: $dt + wds \pm dr$ and therefore we have in this region:

$$X_{\lambda_F} = \frac{\partial_t}{1 \pm \partial_t \rho}$$  \hfill (62)

which is still a positive multiple of $\partial_t$.

This implies that for every $\tilde{p}^- \in \tilde{W}^-$ and $\tilde{p}^+ \in \tilde{W}^+$ there exist $\tilde{t}^0 < 0$ and $\tilde{t}^0 < 0$ such that

$$\phi_{X_{\lambda_F}}^{\tilde{t}^-} (\tilde{p}^-) \in \mathcal{W}^- \quad \text{and} \quad \phi_{X_{\lambda_F}}^{\tilde{t}^+} (\tilde{p}^+) \in \mathcal{W}^+$$  \hfill (63)

Again using that $X_{\lambda_F}$ is a positive multiple of $\partial_t$ on $\tilde{B} \setminus B_{2\varepsilon}^{2n}$ we have that for every point $p$ in $\tilde{B} \setminus B_{2\varepsilon}^{2n}$ whose $t$ coordinate is in $[2\eta, 3\eta]$ the trajectory of the flow $\phi_{X_{\lambda_F}}^t$ going through $p$ is a straight line with fixed coordinates $s$ and $w$, that goes from $\tilde{W}^+$ to $\mathcal{W}^+$. Analogously, for every point $p$ in $\tilde{B} \setminus B_{2\varepsilon}^{2n}$ whose $t$ coordinate is in $[-3\eta, -2\eta]$ the trajectory of the backward flow of $\phi_{X_{\lambda_F}}^t$ going through $p$ is a straight line $\tilde{W}^-$ to $\mathcal{W}^-$. Summing up, with all the cases considered above we have showed that for every point $p \in \tilde{B}$ the trajectory of the flow $\phi_{X_{\lambda_F}}^t$ going through $p$ for $t = 0$ intersects $\mathcal{W}^-$ for non-positive time and $\mathcal{W}^+$ for for non-negative time. Therefore, the trajectory must enter in $\tilde{B}$ through $\tilde{W}^-$ and leave through $\mathcal{W}^+$. Now, because on $M \setminus \tilde{B} = T_1 S \setminus B_{2\varepsilon}^{2n}$ the contact form $\lambda_F$ coincides with $\lambda_g$ we have that trajectories of $X_{\lambda_F}$ starting at $\mathcal{W}^-$ at the time $t = 0$ have to leave $M \setminus N$ for negative time before reentering on $\tilde{B}$; similarly the trajectories starting at $\mathcal{W}^+$ have to leave $M \setminus N$ for positive time before reentering on $\tilde{B}$. More precisely, one can use equations (50) and (51) (on page 22) to see that for $p^- \in \mathcal{W}^-$
and \( p^+ \in \mathcal{W}^+ \) there exist \( t_{p^-} < 0 \) and \( t_{p^+} > 0 \) such that:

\[
\begin{align*}
\phi_{X_{\lambda_F}}^t(p^+) & \in M_2 \setminus N \text{ and } \forall t \in [0, t_{p^+}] \ \phi_{X_{\lambda_F}}^t(p^+) \notin \tilde{B} \\
\phi_{X_{\lambda_F}}^t(p^-) & \in M_1 \setminus N \text{ and } \forall t \in [t_{p^-}, 0] \ \phi_{X_{\lambda_F}}^t(p^-) \notin \tilde{B}
\end{align*}
\] (64) (65)

where

\[
\begin{align*}
M_1 & = (T_1S_1 \setminus B_1^0) \cup B_2^{2\eta}(-) \backslash (x \in B_2^{2\eta}(-) \setminus \overline{B}_1^0) \sim (F(x) \in ((B_2^{2\eta} \cap T_1S_1) \setminus \overline{B}_1^0) \\
M_2 & = (T_1S_2 \setminus B_1^0) \cup B_2^{2\eta}(+) \backslash (x \in B_2^{2\eta}(+) \setminus \overline{B}_1^0) \sim (F(x) \in ((B_2^{2\eta} \cap T_1S_2) \setminus \overline{B}_1^0) \\
N & = \Pi^{-1}(V_{c,\delta}) \cup B_2^{2\eta}(-) \backslash (x \in B_2^{2\eta}(-) \setminus \overline{B}_1^0) \sim (F(x) \in ((B_2^{2\eta} \cap T_1S_1) \setminus \overline{B}_1^0)
\end{align*}
\] (66) (67) (68)

for \( B_2^{2\eta}(-) = [-2\eta, 0] \times S^1 \times (-2\varepsilon, 2\varepsilon) \) and \( B_2^{2\eta}(+) = [0, 2\eta] \times S^1 \times (-2\varepsilon, 2\varepsilon) \).

**Remark:** it is not hard to see that \( M = M_1 \cup M_2 \sim (\tilde{F}(x) \in \partial M_2) \). Here \( \tilde{F} \) is a Dehn twist which coincides with \((s + f(w), w)\) for \( w \in [-2\varepsilon, 2\varepsilon] \) and is the identity elsewhere. This picture of \( M \) is closer to the one in the paper [10] and shows that \( M \) is a graph manifold (a graph manifold is one whose JSJ decomposition consists of Seifert \( S^1 \) bundles). By using this description of \( M \) and applying Van-Kampen’s to analyse the fundamental group of \( M \), Handel and Thurston show that, for \( q \) not belonging to a finite subset of \( \mathbb{Z} \), no finite cover of \( M \) is a Seifert manifold thus obtaining that \( M \) is an “exotic” graph manifold.

From their definition one sees that as manifolds, \( M_1 \cong T_1S_1 \) and \( M_2 \cong T_1S_2 \). This implies \( \partial M_1 \) and \( \partial M_2 \) are incompressible tori. If we look at \( M_1 \) and \( M_2 \) as submanifolds of \( M \), their boundary \( T \) coincide and is also incompressible in \( M \). We remark that \( M_1 \setminus N \) is diffeomorphic to \( T_1S_i \setminus \Pi^{-1}(V_{c,\delta}) \) which is diffeomorphic to \( T_1S_i \) for \( i = 1, 2 \).

In a similar way we can describe the topology of \( N \). Let \( N_i = M_i \cap N \); reasoning identically as one does to show that \( M_i \) is diffeomorphic to \( T_1S_i \) one shows that \( N_i \) is diffeomorphic to a thickened two torus \( \mathcal{T}^2 \times [-1, 1] \). As \( N \) is obtained from \( N_1 \) and \( N_2 \) by gluing them along \( T \) (which is a boundary component of both of them) we have that \( N \) is also diffeomorphic to the product \( \mathcal{T}^2 \times [-1, 1] \).

The discussion above proves the following:

**Lemma 1.** For all \( \tilde{p} \in \tilde{B} \) the trajectory \( \{ \phi_{X_{\lambda_F}}^t(\tilde{p}); t \in \mathbb{R} \} \) intersects \( M_1 \setminus N \) and \( M_2 \setminus N \).

**Proof:** we have already established that for \( \tilde{p} \in \tilde{B} \setminus B_2^{2\eta} \) its trajectory intersects \( \mathcal{W}^- \) and \( \mathcal{W}^+ \). The lemma follows in this case from equations (64) and (65).

For the others \( \tilde{p} \in \tilde{B} \) (in the complement of \( \tilde{B} \setminus B_2^{2\eta} \)) we saw that their trajectory intersects \( \mathcal{W}^- \) and \( \mathcal{W}^+ \). But trajectories of points in \( \mathcal{W}^- \) intersect \( \mathcal{W}^- \) and trajectories of points in \( \mathcal{W}^+ \) intersect \( \mathcal{W}^+ \). This implies that also for such \( \tilde{p} \) its trajectory intersects \( \mathcal{W}^- \) and \( \mathcal{W}^+ \). Thus, also in this case, the lemma follows in this case from equations (64) and (65). \( \square \)
Notice that trajectories can only enter in $\tilde{B}$ through the wall $W^-$ which is contained in $M_1$ and can only exit $\tilde{B}$ through the wall $W^+$ which is contained in $M_2$. We also point out that all trajectories of the flow $\phi^{t}_{X_{\lambda_{F}}}$ are transversal to $\mathbb{T}$, with the exception of the two Reeb orbits which correspond to the hyperbolic geodesic $c$ (they continue to exist as periodic orbits after the surgery because they are distant from the surgery region).

We will deduce from the previous discussion the following important lemma:

**Lemma 2.** Let $\gamma([0,T])$ be a trajectory of $X_{\lambda_{F}}$ such that $\gamma(0) \in \mathbb{T}$, $\gamma(T') \in \mathbb{T}$ and for all $t \in (0,T')$ we have $\gamma(t) \notin \mathbb{T}$; notice that in such a situation $\gamma([0,T]) \subset M_i$ for some $i$ equals to 1 or 2. Then $\gamma([0,T]) \cap (M_i \setminus N)$ is non-empty.

**Proof:**

**First case:** suppose that $\gamma([0,T]) \cap \tilde{B}$ is empty. In this case $\gamma([0,T])$ also exists as a hyperbolic geodesic with endpoints in the closed geodesic $c$. It follows from the convexity of the hyperbolic metric that $[\gamma([0,T])] \in \pi_1(T_1 S_i, \mathbb{T})$ is non-trivial. This implies that $[\gamma([0,T])] \in \pi_1(M_i, \mathbb{T})$ is non-trivial which can be true only if $\gamma([0,T]) \cap (M_i \setminus N)$ is non-empty since $N$ is a tubular neighbourhood of $\mathbb{T}$.

**Second case:** suppose that $\gamma([0,T]) \cap \tilde{B}$ is non-empty and $\gamma([0,T]) \subset M_2$. Take $\hat{t} \in [0,T']$ such that $\gamma(\hat{t}) \in \tilde{B}$. We know from our previous discussion that there are $\hat{t}_1 \leq \hat{t} \leq \hat{t}_2$ such that $\gamma([\hat{t}_1, \hat{t}_2]) \subset \mathbb{T}$, $\gamma(\hat{t}_1) \in (T \cap \tilde{B})$ and $\gamma(\hat{t}_1) \in W^+$; notice that in coordinates $(t,s,w)$, $T \cap \tilde{B}$ is the annulus $\{0\} \times S^1 \times (-2\epsilon, 2\epsilon)$. From this picture it is clear that for $t$ smaller that $\hat{t}_1$ the trajectory enters in $M_1$; therefore we have that $\hat{t}_1 = 0$ and $\gamma([0,\hat{t}_2]) \subset \tilde{B}$. Notice also that for all $t$ slightly bigger than $\hat{t}_2$ the trajectory is outside $\tilde{B}$. Because trajectories of $X_{\lambda_{F}}$ can only enter $\tilde{B}$ in $M_1$ we obtain that $\gamma([\hat{t}_2, T'])$ does not intersect the interior of $\tilde{B}$ and therefore exists as a hyperbolic geodesic in $T_1 S_2$. Now, using equations (50) and (51) we obtain that, because $\gamma(\hat{t}_2) \in W^+$, the trajectory $\gamma : [\hat{t}_2, T'] \rightarrow M_2$ has to intersect $M_2 \setminus N$ before hitting $\mathbb{T}$ at $t = T'$. Thus there is some $t \in (\hat{t}_2, T')$ for which $\gamma(t) \in M_2 \setminus N$.

**Third case:** the proof in the case where $\gamma([0,T]) \cap \tilde{B}$ is non-empty and $\gamma([0,T]) \subset M_1$ is analogous to the one of the Second case.

This three cases exhaust all possibilities and therefore prove the lemma.  

Our reason for introducing the above decomposition of $M$ into $M_1$ and $M_2$ and for proving the lemmas above is to introduce the following representation of Reeb orbits of $\lambda_{F}$. Let $(\gamma, T)$ be a Reeb orbit of $\lambda_{g}$ which intersects both $M_1 \setminus N$ and $M_2 \setminus N$. We can assume that the chosen parametrization of the Reeb orbit satisfies: $\gamma(0) \in \partial N$, and that there are $t_+ > 0$ and $t_- < 0$ such that:

$$\gamma(t+) \in M_1 \setminus N \text{ and } \gamma([0, t_+)) \in M_1 \cup N$$  \hspace{1cm} (69)

$$\gamma(t-) \in M_2 \setminus N \text{ and } \gamma([t-, 0]) \in M_2 \cup N$$  \hspace{1cm} (70)

This means that in an interval of the origin $\gamma$ is coming from $M_2 \setminus N$ and going to $M_1 \setminus N$. It follows from Lemma 2 that there exists a unique sequence $0 = t_0 < t_\frac{1}{2} < t_1 < t_\frac{1}{2} < ... < t_n = T$ such that $\forall k \in \{0, ..., n - 1\}$:
\( \gamma((t_k, t_{k+\frac{1}{2}})) \subset M_i \) for \( i \) equals to 1 or 2

- \( \gamma((t_k, t_{k+\frac{1}{2}})) \subset M_i \) for \( i \) equals to 1 or 2

- \( \gamma((t_{k+\frac{1}{2}}, t_{k+1})) \in N \) and there is a unique \( \tilde{t}_k \in [t_{k+\frac{1}{2}}, t_{k+1}] \) such that \( \gamma(\tilde{t}_k) \in \mathbb{T} \)

- if \( \gamma((t_k, t_{k+\frac{1}{2}})) \subset M_i \) then \( \gamma((t_{k+1}, t_{k+\frac{1}{2}})) \subset M_j \) for \( j \neq i \)

Notice that \( \gamma([t_0, t_1]) \subset M_1 \) and \( \gamma([t_{n-1}, t_{n-\frac{1}{2}}]) \subset M_2 \). This implies that \( n = 2n' \) is even, and that \( \gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_1 \) for \( k \) even, and \( \gamma([t_{k'}, t_{k'+\frac{1}{2}}]) \subset M_2 \) for \( k' \) odd. For each \( k \in \{0, \ldots, 2n' - 1\} \) the existence of the unique \( \tilde{t}_k \) in the interval \( [t_{k+\frac{1}{2}}, t_{k+1}] \) for which \( \gamma(\tilde{t}_k) \in \mathbb{T} \) is guaranteed from Lemma 2 and the fact that \( \mathbb{T} \) is the hypersurface that separates \( M_1 \) and \( M_2 \).

In order to obtain information on the free homotopy class of \( (\gamma, \mathbb{T}) \) we observe that for \( \gamma((t_k, t_{k+\frac{1}{2}})) \) coincides with a hyperbolic geodesic segment in \( T_1 S_i \) starting and ending \( V_{c,d} \).

Therefore, as we have previously seen the homotopy class \( [\gamma((t_k, t_{k+\frac{1}{2}}))] \) in \( \pi_1(T_1 S_i, V_{c,d}) \) is non-trivial which implies that \( \gamma((t_k, t_{k+\frac{1}{2}})) \) is a non-trivial relative homotopy class in \( \pi_1(M_i, N) \). We consider now the curve \( \gamma((\tilde{t}_k, \tilde{t}_{k+1})) \): it is the concatenation of 3 curves, the first and the third ones being completely contained in \( N \) and the middle one being \( \gamma((t_k, t_{k+\frac{1}{2}})) \): from this description and the fact that \( \gamma((t_k, t_{k+\frac{1}{2}})) \) is a non-trivial relative homotopy class in \( \pi_1(M_i, N) \) it is clear that \( \gamma((\tilde{t}_k, \tilde{t}_{k+1})) \) is also non-trivial in \( \pi_1(M_i, N) \) (and also non-trivial in \( \pi_1(M_i, \mathbb{T}) \)).

We now denote by \( \tilde{M} \) the universal cover of \( M \) and and \( \tilde{\gamma} : \tilde{M} \to M \) a covering map.

From the incompressibility of \( \mathbb{T} \) it follows that every lift of \( \mathbb{T} \) is an embedded plane in \( \tilde{M} \).

We denote by \( \tilde{N}^0 \) a lift of \( N \): because \( N \) is a thickened neighbourhood of an incompressible torus it follows that \( \tilde{N}^0 \) is diffeomorphic to \( \mathbb{R}^2 \times [-1, 1] \), i.e. it is a thickened neighbourhood of an embedded plane in \( \tilde{M} \).

Because \( N \) separates \( M \) in two components, it follows that \( \tilde{N}^0 \) separates \( \tilde{M} \) is two connected components, \( \partial(\tilde{N}^0) \) is the union of two embedded planes \( \tilde{P}_0^+ \) and \( \tilde{P}_0^- \) which are characterized by the fact that there are neighbourhoods \( V_+ \) and \( V_- \) of, respectively, \( \tilde{P}_0^+ \) and \( \tilde{P}_0^- \) such that \( \tilde{\gamma}(V_+) \subset M_1 \) and \( \tilde{\gamma}(V_-) \subset M_2 \). We will denote by \( C_0^+ \) the connected component of \( \tilde{M} \setminus \tilde{N}^0 \) which intersects \( V_+ \), and by \( C_0^- \) the connected component of \( \tilde{M} \setminus \tilde{N}^0 \) which intersects \( V_- \).

As we saw earlier, the trajectory \( [\gamma((t_k, t_{k+\frac{1}{2}}))] \) is a non-trivial relative homotopy class in \( \pi_1(M_i, N) \). It is not hard to see that it remains as such in \( \pi_1(M, N) \). Let \( T_i = \partial(N) \cap M_i \); because \( N \) is a tubular neighbourhood of \( \mathbb{T}_i \) it is clear that \( [\gamma((t_k, t_{k+\frac{1}{2}}))] \) would be trivial in \( \pi_1(M_i, \mathbb{T}_i) \) if and only if it is trivial in \( \pi_1(M_i, N) \), and we know it is not. As \( T_i \) is isotopic to \( \mathbb{T} \) it is also an incompressible torus that divide \( M \) in 2 components. Now, \( [\gamma((t_k, t_{k+\frac{1}{2}}))] \) would be trivial in \( \pi_1(M_i \setminus \text{int}(N), \mathbb{T}_i) \) if, and only if, there is a curve \( c \) in \( T_i \) which endpoints \( \gamma(t_k) \) and \( \gamma(t_{k+\frac{1}{2}}) \) such that the concatenation \( \gamma \ast c \) is contractible in \( (M_i \setminus \text{int}(N)) \). Because of the incompressibility of \( \mathbb{T}_i \), such a curve \( \gamma \ast c \) is contractible in \( (M_i \setminus \text{int}(N)) \) if, and only if, it is contractible in \( M \). This implies that \( [\gamma((t_k, t_{k+\frac{1}{2}}))] \) would be trivial in \( \pi_1(M, T_i) \) if, and only if, it was trivial in \( \pi_1((M_i \setminus \text{int}(N), T_i)) \) which we know it is not the case. Lastly, because \( N \) is a tubular neighbourhood of \( \mathbb{T}_i \) it is clear that as \( [\gamma((t_k, t_{k+\frac{1}{2}}))] \) is not trivial \( \pi_1(M, T_i) \) it cannot be trivial in \( \pi_1(M, N) \), as we wished to show.
Let now $\tilde{\gamma}$ be a lift of $\gamma$ such that $\tilde{\gamma}(0) \in \tilde{N}^0$. We know that $\tilde{\gamma}([t_{2n'-\frac{1}{2}} - T, t_{\frac{1}{2}}]) \subset \tilde{N}^0$. It will be useful to us to define the following sequence:

\[ \tilde{t}_i = q_i T + t_{r_i}, \]  

(71)

where $q_i$ and $r_i < 2n'$ are the unique integers such that $i = q_i(2n') + r_i$. Associated to $\tilde{t}_i$ we associate the lift $\tilde{N}^i$ of $N$, which is determined by the property that $\tilde{\gamma}(\tilde{t}_i) \in \tilde{N}^i$. It is clear that the sequence $\tilde{N}^i$ contains all lifts of $N$ which are intersected by the curve $\tilde{\gamma}(\mathbb{R})$. For the lifts $\tilde{N}^i$ we define the connected components $C^i_+$ and $C^i_-$ of $\tilde{M} \setminus \tilde{N}^i$, and the planes $P^i_+$ and $P^i_-$ analogously as how we defined them for $\tilde{N}^0$. A priori it could be that for $i \neq j$ we had $\tilde{N}^i = \tilde{N}^j$. We will show however, that this cannot happen.

Firstly, $\tilde{N}^0 \neq \tilde{N}^1$ because $\gamma([t_0, t_1])$ is non-trivial in $\pi_1(M, N)$. Also, we have that $\tilde{N}^1 \subset C^0_+$ because $\gamma([t_0, t_{\frac{1}{2}}]) \subset M_1$. An identical reasoning shows that $\tilde{N}^2 \neq \tilde{N}^1$ and:

\[ \tilde{N}^2 \subset C^1_+. \]  

(72)

On the other hand we have that $\tilde{N}^0 \subset C^1_+$, because $\tilde{\gamma}([\tilde{t}_0, t_{\frac{1}{2}}])$ gives a path totally contained in $\tilde{M} \setminus \tilde{N}^1$ connecting $\tilde{N}^0$ and $P^i_1$. As $\tilde{N}^2 \subset C^1_+$ and $\tilde{N}^0 \subset C^1_+$, we must have $\tilde{N}^2 \neq \tilde{N}^0$. In an identical way, one shows that $\tilde{N}^3 \neq \tilde{N}^1$, and more generally that $\tilde{N}^{i+2} \neq \tilde{N}^i$ and $\tilde{N}^{i+1} \neq \tilde{N}^i$.

Now for $\tilde{N}^3$, we have that $\tilde{N}^3 \subset C^2_+$. As $\tilde{\gamma}([\tilde{t}_0, t_{\frac{1}{2}}])$ is a path completely contained in $\tilde{M} \setminus \tilde{N}^2$ connecting $\tilde{N}^0$ and $P^2_+$ we obtain that $\tilde{N}^0 \subset C^2_+$, and therefore $\tilde{N}^3 \neq \tilde{N}^0$.

Proceeding inductively along this line one obtains that $\tilde{N}^i \neq \tilde{N}^0$ for all $i \neq 0$, and more generally, $\tilde{N}^i \neq \tilde{N}^j$ for all $i \neq j$. As a consequence of this, we obtain that the curve $\tilde{\gamma}(\mathbb{R})$ cannot be homeomorphic to a circle and therefore $\gamma(\mathbb{R})$ cannot be contractible.

We are ready for the main result of this subsection:

**Proposition 3.** $\lambda_F$ is hypertight.

*Proof:* there are two possibilities for Reeb orbits.

**Possibility 1:** the Reeb orbit $\gamma$ visits both $M_1 \setminus N$ and $M_2 \setminus N$.

In this case, we have just showed above that $\gamma$ is not contractible.

**Possibility 2:** the Reeb orbit $\gamma$ is completely contained in $M_i$ for $i$ equal to 1 or 2.

In this case, the Reeb orbit does not visit the surgery region $\tilde{B}$. Therefore it existed also before the surgery as a closed hyperbolic geodesic in $M_i \setminus \tilde{B} = T_1 S_i \setminus B^{3n}_{2\epsilon}$. Such a closed geodesic is non-contractible in $T_1 S_i$ which is diffeomorphic to $M_i$. We have thus obtained that $\gamma \subset M_i$ is non-contractible in $M_i$.

Looking now at $M_i$ as a submanifold with boundary of $M$, we remind the reader that $\partial M_i$ is an incompressible torus in $M$. This implies that every non-contractible closed curve in $M_i$ remains non-contractible in $M$; therefore $\gamma$ is also a non-contractible Reeb orbit for this case. \qed
6.2.1 Exponential homotopical growth of cylindrical contact homology for \( \lambda_F \)

We proceed now to obtain more information on the properties of periodic orbits of \( X_{\lambda_F} \). We state the following important fact:

**Fact:** for a periodic orbit \((\gamma, T)\) which visits both \( M_1 \setminus N \) and \( M_2 \setminus N \) we have that any curve freely homotopic to \((\gamma, T)\) must always intersect \( T \).

**Proof of fact:** as we saw earlier the lift \( \tilde{\gamma} \) intersects all the elements of the sequence \( \tilde{N}_i \) (of lifts of \( N \)) which satisfy \( \tilde{N}_i \neq \tilde{N}_j \) for all \( i \neq j \).

Introducing an auxiliary distance \( d \) on the compact manifold \( M \) (coming from a Riemannian metric) we obtain an auxiliary distance \( \tilde{d} \) on \( \tilde{M} \) by pulling \( d \) back by the covering map. It is clear that for \( i \) sufficiently big the \( \tilde{d} \) distance between \( \tilde{N}_i \) and \( \tilde{N}_0 \) becomes arbitrarily large. As a consequence, one obtains that for each \( K > 0 \) there exists \( t_K > 0 \) such that \( \tilde{d}(\tilde{\gamma}(\pm t_K), \tilde{N}_0) > K \).

Let now \( \zeta : [0, T] \rightarrow M \) be closed curve freely homotopic to \( \gamma([0, T]) \). An homotopy \( H : [0, T] \times [0, 1] \rightarrow M \) generates an homotopy \( \tilde{H} : \mathbb{R} \times [0, 1] \rightarrow \tilde{M} \) from a lift \( \tilde{\gamma} \) and a lift \( \tilde{\zeta} \). Using the fact that \( H \) is uniformly continuous one proves that there exists a constant \( O > 0 \) such that \( \tilde{d}(\tilde{H}([t] \times [0, 1]), \tilde{\gamma}(t)) < O \) for all \( t \in \mathbb{R} \).

Take now \( K > 2O \). Using the triangle inequality and the facts that \( \tilde{d}(\tilde{H}([t] \times [0, 1]), \tilde{\gamma}(t)) < O \) and \( \tilde{d}(\tilde{\gamma}(\pm t_K), \tilde{N}_0) > K \) we obtain \( H([t_K] \times [0, 1]) \) is always in the same connected component of \( \tilde{M} \setminus \tilde{N}_0 \) and must thus intersect \( \tilde{N}_0 \). Even more, because \( \tilde{\zeta}(\mathbb{R}) \) intersects both components of \( \partial(\tilde{N}_0) \) we have that \( \zeta \) visits both components of \( M \setminus N \) and therefore has to intersect \( T \); this completes the proof of the fact.

\( \square \)

We are now ready for the most important result of this section:

**Theorem 4.** The cylindrical contact homology \( \text{CH}_c^{\text{cy}d}(M, \lambda_F) \) has exponential homotopical growth rate with respect to the action.

**Proof:**

**Step 1:** a special class of Reeb orbits.

We will obtain our estimate by looking at Reeb orbits which are completely contained in the component \( M_1 \). As we saw previously such orbits never cross the surgery region \( \tilde{B} \). Thus they are in a region where \( \lambda_F \) coincides with \( \lambda_g \), and such Reeb orbits exist also as closed geodesics in \( (S_1, g) \). Conversely, every closed geodesic in \( (S_1, g) \) does not cross the region \( B^{3n}_{2\epsilon} \) and thus also exist as Reeb orbit of \( \lambda_F \). This gives a bijective correspondence between closed geodesics of \( (S_1, g) \) which are not homotopic to a multiple of \( \partial S_1 \) and Reeb orbits of \( \lambda_F \) which are completely contained in \( M_1 \).

Let \( \Lambda(S_1) \) denote the set of free homotopy classes in \( S_1 \) which are not multiple cover of \( \partial S_1 \). We know that each \( \rho \in \Lambda(S_1) \) contains exactly one closed geodesic \( c_\rho \). Let \( \gamma_\rho \) be the lift of \( c_\rho \) to \( T_1S_1 \) which is a Reeb orbit of \( \lambda_g \); as we saw above each \( \gamma_\rho \) can also be seen as a
Reeb orbit of $\lambda_F$. We will denote by $\Lambda((S_1) \leq T$ the set primitive of free homotopy classes in $S_1$ whose unique closed geodesic has period smaller or equal to $T$. Because $g$ is hyperbolic it is a classic fact that there exist constants $a > 0$, $b$ such that: $\pi((\Lambda(S_1)) \leq T) \geq e^{aT+b}$.

Let $\Theta : \Lambda((S_1)) \to \Lambda(T_1,S_1)$ (where $\Lambda(T_1,S_1)$ is the free loop space of $T_1S_1$), be the map which associates $c_\rho$ to $\gamma_\rho$ in $T_1S_1$. $\Theta : \Lambda((S_1)) \to \Lambda(T_1,S_1)$ is easily seen to be injective. Because $T_1S_1$ is diffeomorphic to $M_1$ we can also view $\Theta((\Lambda(S_1)))$ as a subset of the free loop space $\Lambda((M_1))$ of $M_1$.

Step 2:
Let $i : M_1 \to M$ be the injection obtained by looking at $M_1$ as a component of $M$. As remarked above the boundary $\partial(i(M_1))$ is an incompressible torus in $M$. We consider the induced map of free loop spaces $i_* : \Lambda(M_1) \to \Lambda(M)$. As a consequence of the incompressibility of $\partial(i(M_1))$, the restriction of $i_*$ to $\Theta((\Lambda(S_1)))$ is injective.

To see that, it suffices to show the following claim: if $\zeta$ and $\zeta'$ are curves in $M_1$ which cannot be isotoped to a curve in $\partial M_1$ and which are in the same free homotopy class in $M$, then $\zeta$ and $\zeta'$ are freely homotopic in $M_1$. For $\zeta$ and $\zeta'$ satisfying the hypothesis of our claim there is a cylinder $G$ in $M$ whose boundary components are $\zeta$ and $\zeta'$ which intersects $\partial M_1$ transversely. In such a case, $G$ intersects $\partial M_1$ in a finite collection of curves $\{w_n\}$ which are all contractible in $M$ (this contractibility is because $\zeta$ and $\zeta'$ cannot be isotoped to a curve in $\partial M_1$). The incompressibility of $\partial M_1$ implies that these $\{w_n\}$ are all contractible already in $\partial M_1$. Now, we cut the discs in $G$ whose boundary are the curves $c_n$ and substitute them by discs contained in $\partial M_1$; this produces a cylinder $G'$ completely contained in $M_1$ whose boundaries are $\zeta$ and $\zeta'$. This implies that $\zeta$ and $\zeta'$ were already in the same free homotopy class in $M_1$.

From step one, we know that for each $\rho \in i_*(\Theta((\Lambda(S_1))))$ there is a Reeb orbit $\gamma_\rho$ in $\rho$.

Step 3: for each $\rho \in i_*(\Theta((\Lambda(S_1))))$, we have that $\gamma_\rho$ is the unique Reeb orbit in $\rho$.

Let $\gamma$ be a Reeb orbit in $\rho$. If it is contained in $M_1$, we know that $\gamma$ exists also as a closed geodesic in $(S_1,g)$. Using an argument as in step 2 above, it is easy to show that $\gamma$ and $\gamma_\rho$ are freely homotopic in $M_1$, and therefore also in $T_1S_1$. Projecting to $S_1$ we obtain that $\gamma$ and $\gamma_\rho$ are lifts of geodesics of $(S_1,g)$ in a same free homotopy class of $S_1$. But for each free homotopy class of $S_1$ there is a unique closed geodesic of $(S_1,g)$; this implies that $\gamma = \gamma_\rho$.

Step 3 will now follow if we prove the following claim: every Reeb orbit of $\lambda_F$ in $\rho$ is completely contained in $M_1$.

Proof of the claim: if $\gamma$ was contained in $M_2$ then it would be possible to isotopy $\gamma_\rho$ to a curve completely contained in $\partial M_1$; which by definition of $\Lambda(S_1)$ is impossible.

The only remaining possibility is that $\gamma$ visit both $M_1 \setminus N$ and $M_2 \setminus N$ (the reason for that is that if $\gamma$ is completely contained in $M_1 \cup N$ convexity of the hyperbolic metric implies that $\gamma$ is in $M_1$). In the current situation, as we have just seen every curve which is freely homotopic to $\gamma$ has to intersect the torus $T$; as $\gamma_\rho$ does not intersect $T$ it cannot be freely homotopic to $\gamma$ which implies
that $\gamma \notin \rho$ finishing the proof of step 3.

**Step 4:** end of the proof.

From the previous steps we know that for each $\rho \in i_*(\Theta(\bigwedge(S_1)))$, there exists a unique Reeb orbit $\gamma_\rho \in \rho$. This implies that for each $\rho \in i_*(\Theta(\bigwedge(S_1)))$, the cylindrical contact homology $CH_{cyl}^\rho(M,\lambda_F) \neq 0$.

Let $\rho \in i_*(\Theta(\bigwedge(S_1)\leq T))$; then as we showed all Reeb orbits in $\rho$ have action smaller or equal than $T$ and $CH_{cyl}^\rho(M,\lambda_F) \neq 0$. This implies that:

$$N_T^{cyl}(\lambda_F) \geq i_*(\Theta(\bigwedge(S_1)\leq T)))$$  \hspace{1cm} (73)

As $i_*$ restricted to $\Theta(\bigwedge(S_1)\leq T))$ is injective, and $\Theta$ is injective we obtain that:

$$\sharp(i_*(\Theta(\bigwedge(S_1)\leq T))) = \sharp(\bigwedge(S_1)\leq T) \geq e^{aT+b}. \hspace{1cm} (74)$$

Combined with the last equation, it gives:

$$N_T^{cyl}(\lambda_F) \geq e^{aT+b} \hspace{1cm} (75)$$

\[\blacksquare\]

### 7 Conclusion

By the work of Katok [23] [24] we know that for a flow in a 3-manifold positivity of topological entropy is equivalent to the existence of a Smale horseshoe in the flow. For a flow a “horseshoe” is a compact invariant set where the dynamics of the flow are conjugate to that of a shift map. In particular the number of hyperbolic periodic orbits on a “horseshoe” of a 3-dimensional flow grows exponentially with respect to the period; see the recent paper [25] for a refined estimate of this growth. As a consequence for the contact 3-manifolds $(M,\xi)$ considered in theorems 3 and 4, we have that for any Reeb flow associated to $(M,\xi)$ the number of hyperbolic Reeb orbits grows exponentially with the action. An interesting property of the entropy estimate used in this paper, [1] and [26] is that it gives estimates on the growth of the number of hyperbolic Reeb orbits also for degenerate contact forms; this is not possible when making estimates using just contact homology.

We first mention that there is another way to study topological entropy for Reeb flows by studying growth rate of Reeb chords; this is the path followed by Macarini and Schlenk in [26]. In the paper [1] we study the growth rate of Reeb chords in the manifolds considered in theorems 4 and 5 for special Legendrian curves in these manifolds by means of a version of Legendrian contact homology. We also obtain there the effect this growth rate has on the topological entropy.

We know that the consequences of topological entropy in higher dimensions are not so strong the ones in low dimensions, in particular positive topological entropy does not implies the existence of a horseshoe in higher dimensions. However it is natural to ask the following question:
**Question 1:** does exponential homotopical growth of periodic orbits for a Reeb flow implies the existence of a compact invariant set where the dynamics are conjugated to a shift in higher dimensions?

We would like to consider is if we can obtain more dynamical information about the examples study in theorems 3 and 4. As an example, one can ask:

**Question 2:** let $(M, \xi)$ be a manifold satisfying the hypothesis of either theorem 3 or 4, and $\lambda$ an associated contact form. Is it true that for the Reeb flow $\phi_{X_\lambda}$ there exists an invariant region of positive measure (with respect to the measure $\lambda \wedge d\lambda$) where the dynamics of the Reeb flow is ergodic?

One important property of many of the contact 3-manifolds considered in Theorem 3 (section 5) is that they have positive Giroux torsion. By a theorem of Gay [14] (see also [21]) manifolds with positive Giroux torsion are not strongly fillable. This implies that many of the contact manifolds satisfying the claims of Theorem 3 are not strongly fillable and are therefore quite different from the examples studied in [26] which are unit tangent bundles and thus strongly fillable. Notice also that in theorem 3 we showed the existence of 3-manifolds with hyperbolic components which can be given infinitely many different contact structures whose Reeb flows always have positive topological entropy property. This motivates the following questions:

**Question 3:** are there examples of hypertight contact structures on closed hyperbolic 3-manifolds for which there is a contact form with exponential homotopical growth of cylindrical contact homology? For examples of hypertight contact structures on some hyperbolic 3-manifolds see [13]

**Question 4:** are there examples in higher dimensions of non-symplectically fillable contact manifolds for which every associated Reeb flow has positive topological entropy? Are there examples in higher dimensions of manifolds which have infinitely many different contact structures such that for all of them, every associated Reeb flow has positive topological entropy?

Lastly we mention that the techniques used in this paper and in [1] can also be used in combination with the ideas of Momin [27] to study chaotic behavior of Reeb flows associated to $S^3$ with its unique tight contact structure, provided this Reeb flow has a complicated knot as a Reeb orbit. This is done in [2].
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