Scaling of Eliassen-Palm flux vectors

Martin Jucker

Climate Change Research Centre and Centre of Excellence for Climate Extremes, University of New South Wales, Sydney, New South Wales, Australia

Correspondence
Martin Jucker, Climate Change Research Centre, Mathews Building Level 4, University of New South Wales, Sydney, NSW 2052, Australia.
Email: publications@martinjucker.com

Funding information
Australian Research Council, Grant/Award Numbers: CE170100023, FL150100035

Abstract
Eliassen-Palm flux is one of the main diagnostics tools for wave propagation and wave-mean flow interaction in atmospheric dynamics and in particular stratosphere-troposphere coupling. Even though the theory has been derived in the 1960s, there is still no consensus about how to display the flux vectors in a plot. This is particularly true where both the troposphere and stratosphere are of importance. Some of the traditional methods are to scale the arrows by either pressure, the exponential of height, the square root of pressure, or even by an arbitrary factor. But the arguments for any of those methods are subjective, and they result in both different amplitudes and direction. Here, we propose an objective way of scaling EP flux vectors, in either linear or logarithmic pressure or height coordinates, which allows for a physically sound representation throughout the entire atmosphere.

KEYWORDS
analysis, atmosphere, climate, dynamics, Eliassen-Palm flux, stratosphere

1 | INTRODUCTION

Eliassen-Palm flux (EP flux, Eliassen and Palm, 1960) is omnipresent as a diagnostic tool for wave-mean flow interaction, and in particular stratosphere-troposphere coupling. It shows the direction of small amplitude atmospheric waves as vectors and at the same time acceleration (or deceleration) of zonal mean zonal wind via its divergence (Andrews and McIntyre, 1976, 1978). For finite amplitude theory see for example, Nakamura and Zhu (2010). While the divergence is a scalar and therefore easily represented in a figure, the flux itself is a vector field (here denoted \( \mathbf{F} \)) and great care must be taken in representing the magnitude and direction of the arrows. The main difficulties arise from the exponential decrease of mass with height, and the non-trivial aspect ratios resulting from plotting the components in degrees latitude—pressure/height space, as this is not a Cartesian coordinate system.

Unfortunately, there is no consensus in literature about how exactly this should be done. For instance, Andrews et al. (1983, henceforth ‘AMS83’) state that ‘in practice it is found that multiplication of \( \mathbf{F} \) by \( p^{-1} \) keeps magnitudes roughly comparable throughout the middle atmosphere. However, there appears to be no decisive theoretical justification for such a scaling’. They then scale the arrows at each grid point with an undisclosed number, and only consider the arrow direction in their analysis.

Probably the most important effort to use a physically consistent scaling was undertaken by Edmon et al. (1980, henceforth ‘EHM80’), who derived the appropriate expressions for the meridional and vertical components \( (F_\phi, F_p) \) for display in pressure coordinates. Their motivation was to find an expression which would assure that the apparent derivatives with respect to latitude and pressure in the figure correctly represent the EP flux divergence and therefore the acceleration of the zonal flow.
While EHM80 also briefly discuss the effect of using logarithmic pressure axes, they do not explicitly show the resulting components. Instead, they argue that given the unavoidable non-conservative dissipation of wave activity along the vertical path spanning many scale heights, efforts to rescale EP flux vectors in log-\(p\) plots would be futile in any case. This is an unfortunate conclusion, and given the number of publications showing EP flux vectors over multiple scale heights, we believe it is worth using a geometrically consistent scaling for more clarity. Dunkerton et al. (1981) expand on EMH80's discussion, and provide more detail on how to plot EP fluxes in \(z\)-space, but apply volume rather than mass weighting in the vertical, again leading to vanishingly small arrows at high altitudes.

Palmer (1981) give somewhat more details about how to include figure aspect ratio, with explicit values for an aspect ratio scaling constant \(c\). However, such a constant value is again only applicable for linear pressure axis (as we will show below). They work in log-\(p\) or \(z\)-coordinates, and manually set the density to a constant value of one, without any physical nor geometric reason. Later, Baldwin et al. (1985, henceforth ‘BEH85’) suggested multiplication with \(\exp(z/H)\) (they work in \(z\)-coordinates as well), which is the same as AMS83’s division by pressure, without compelling geometric or physical arguments. These authors also remove a multiplicative factor of cosine of latitude. Other authors use the inverse square root of pressure (e.g., Taguchi and Hartmann, 2006) and even such influential organisations as NOAA’s Physical Sciences Laboratory (former Earth System Research Laboratory)\(^1\) and the University of Reading\(^2\) recommend using the inverse square root of pressure plus an arbitrary constant multiplication above an arbitrary pressure level.

In this letter, we will derive a geometrically and physically consistent scaling for EP flux vectors, taking into account spherical geometry, the figure aspect ratio and the units of the vector components. It is a simple derivation, but the reasoning is more geometric than physical, which is probably why previous authors came to conclusions such as the one by AMS83 cited above. It is surprising how such arbitrary scaling has been accepted by the research community, when a correct way of displaying scientific data is so important. For instance, we will show that using the square root of pressure is ill-informed and should only be used with great caution.

This letter is organised as follows: Section 2 derives the scaling for EP flux arrow plots which conserve the direction and amplitude (to a constant factor) in any linear or logarithmic plot with arbitrary aspect ratio. Section 3 describes how to represent EP flux vectors in log-pressure or \(z\)-coordinates consistently. Section 4 then concludes by showing the differences between our scaling and the most important scalings used in literature as described above. Python code to compute EP fluxes and display them on an arbitrary figure is part of the Python package aostools (Jucker, 2020b).

## 2 Vector Scaling

We start from the expression of the EP flux components in pressure coordinates as in Equation (2.1) of AMS83

\[
\begin{align*}
\Phi &= -u\theta' + p\frac{\partial \phi}{\partial p} , \\
\Phi_p &= f\left(\phi, \frac{1}{\cos \phi} \frac{\partial (\Pi \cos \phi)}{\partial \phi} \frac{\sqrt{\theta'}}{\phi} - u\phi' \right)
\end{align*}
\]

and

\[
\mathbf{F} = (\Phi, \Phi_p) = a \cos \phi \left( \phi, \frac{\partial f}{\partial p} \right).
\]

All notations are standard, with the convention of overbars indicating zonal averages and their departures. Subscripts refer to partial derivatives and primes denoting zonal averages and their departures. Terms in parentheses correspond to partial derivatives and primes denoting zonal averages and their departures. The pressure in hPa, \(f\phi\) is in units of m\(^3\)/s\(^2\), and, assuming pressure in hPa, \(f\phi\) is in m\(^3\)/hPa/s\(^2\). Therefore \(f\phi\) and \(f\phi_p\) are in units of m\(^3\)/s\(^2\) and m\(^3\)hPa/s\(^2\) respectively. As described by EHM80, if one tries to plot the vector fields \((\Phi, \Phi_p)\) or \((\Phi, \Phi_p)\) in a latitude-vertical plot, the directions of the arrows will not visually represent the physical effects of the waves (such as divergence and convergence, and the direction of propagation). To make sure the derivative with respect to the coordinates used on the x- and y-axes corresponds to the divergence of \(\mathbf{F}\), EHM80 define (their Equation [3.13])

\[
\langle \hat{\mathbf{F}}_\phi, \hat{\mathbf{F}}_p \rangle = \frac{2\pi}{g} a^2 \cos^2 \phi \left( f\phi, a f_p \right),
\]

which are in units of m\(^3\)/rad and m\(^3\)/hPa respectively. With this scaling, the mass weighted divergence of EP flux is simply \(\partial_\phi \Phi + \partial_p \Phi_p\), and if one plots the vectors \(\mathbf{F} = (\hat{\mathbf{F}}_\phi, \hat{\mathbf{F}}_p)\) in a linear and equal aspect ratio plot with the latitude in radians, the arrows will show the physically correct picture. However, EP fluxes are very rarely plotted with these required specifications: Latitude is usually shown in degrees, and the aspect ratio is usually arbitrary, as even if the plot itself is squared, pressure spans over a 1000 hPa while latitude only extends over 2\(\pi\) radians or 180\(^\circ\) maximum. But of course, these numbers can be arbitrary when not plotting the entire atmosphere; if only part of the domain is shown, for instance above 250 hPa and the northern extratropics only, the
vectors have to be scaled accordingly. But the most important factor is the scale of the pressure axis, which is often logarithmic rather than linear. Many authors work in log-pressure coordinates $z \sim \log(p)$, which we will discuss in detail in Section 3. First, we need to derive the scaling factors $\alpha$ and $\beta$ to account for units and plot aspect ratio such that the vector

$$\mathbf{\hat{F}} = (\alpha \mathbf{\hat{F}}_\phi, \beta \mathbf{\hat{F}}_p)$$  \hspace{1cm} (5)

shows both physically correct direction and amplitude (to a constant factor) at any point on the plot. This corresponds to a change of variables, with the derivative giving the necessary scaling coefficients, for example,

$$\alpha = \frac{\partial X}{\partial x}, \beta = \frac{\partial Y}{\partial y},$$  \hspace{1cm} (6)

where $x'$, $y'$ are the physical coordinates (latitude, pressure, etc.) and $X$, $Y$ are the coordinates in the plot, that is, the length along the axis. We will use the convention that $X$, $Y$ denote the axis length in inches, and $x$, $y \in [0, 1]$ are the fraction along the axes (see Figure 1), and consider the two cases of linear and logarithmic plots. Note that we use inches here (with no loss of generality) as a standard of measuring image size and resolution (e.g., dots per inch) which most plotting software adheres to.

### 2.1 Linear latitude-pressure plots

For a linear axis, the coordinate transformation is straightforward, and takes the simple form

$$\alpha^{\text{lin}} = \frac{X}{x'(x = 1) - x'(x = 0)}, \beta^{\text{lin}} = \frac{Y}{y'(y = 1) - y'(y = 0)}.$$ \hspace{1cm} (7)

For instance, in a linear latitude-pressure plot, where latitude increases from left to right and pressure decreases from bottom to top, this becomes

$$\alpha^{\text{lin}} = \frac{X}{\Delta \phi \cdot \pi / 180},$$  \hspace{1cm} (8)

$$\beta^{\text{lin}} = \frac{Y}{-\Delta p},$$  \hspace{1cm} (9)

where $\Delta \phi = \max(\phi) - \min(\phi)$ and $\Delta p = \max(p) - \min(p)$. Now $\alpha^{\text{lin}}$ has units of inches per radian and $\beta^{\text{lin}}$ has units of inches per hPa, and the minus sign in $\beta^{\text{lin}}$ comes from the inverted pressure axis. Note that if latitude is inverted, as for example in BEH85, $\Delta \phi = \min(\phi) - \max(\phi) < 0$. The scaling of Equations (8) and (9) differs from that proposed in EHM80 only in that it explicitly includes the figure aspect ratio. However, this step is still essential for preserving arrow angles. As described by many subsequent authors, it will yield very small arrows in the stratosphere. But that is only a result of using a logarithmic axis rather than linear vertical axis, not of any physical shortcoming of the theory (nor physical decrease of wave activity flux), and we propose scaling for a logarithmic axis in the next section.

### 2.2 Logarithmic latitude-pressure or latitude-height plots

Most studies concerned with stratosphere-troposphere coupling plot EP flux vectors in both the troposphere and the stratosphere, and therefore utilise a logarithmic pressure axis, or equivalently log-pressure ($z$) coordinates. Using the same scaling as for linear axes will not work, and has led to arbitrary methods for re-scaling to make the vectors visible in the stratosphere. However, these arbitrary scalings are ill-informed, and it is relatively simple to derive a consistent scaling for logarithmic axes.

For the derivation, assume $y \in [0, 1]$ the position along the $y$-axis as in (7), and $p_0 = p(y = 0)$ and $p_1 = p(y = 1)$. For instance, $p_0 = 1,000$ hPa and $p_1 = 1$ hPa. Then, at any given pressure $p$ along the $(\log_{10})$ logarithmic $y$-axis,

$$y(p) = \frac{\log_{10}(p_0) - \log_{10}(p)}{\log_{10}(p_0) - \log_{10}(p_1)} = \frac{\log_{10}(p_0/p)}{\log_{10}(p_0/p_1)} = \frac{\ln(p_0/p)}{\ln(p_0/p_1)}.$$ \hspace{1cm} (10)

Now we can directly apply Equation (6):
that they visually represent wave propagation, but also that their divergence corresponds to zonal mean acceleration. For this, we need to find $\Delta_z$ such that

$$\int \frac{1}{\rho_0} \nabla \cdot \mathbf{F} \, dm = \int \Delta_z \rho_0 \, d\phi \, dz,$$  

where $dm$ is the mass element of a zonally symmetric portion of the atmosphere and takes the form

$$dm = \rho_0 2\pi a^2 \cos \phi \, d\phi \, dz.$$  

Equations (16) and (17) are the counterparts of Equations (3.11) and (3.10) in EHM80 in $z$-coordinates. The density $\rho_0$ appears explicitly on the right hand side of Equation (16) to make sure the units of $\Delta_z$ are m$^3$, just like $\Delta$ in EHM80. Similarly, $\rho_0$ must be included in the definition of $dm$. Further following EHM80 yields the vector components

$$\left( F_{\phi z}^x, F_{\phi z}^z \right) = \frac{2\pi}{\rho_0} a^2 \cos \phi \, \left( \frac{1}{a} F_{\phi}^x z F_{\phi}^z \right),$$  

for which the mass weighted divergence of EP flux is now $\partial_\phi F_{\phi z}^x + \partial_z F_{\phi z}^z$. We note that considering mass instead of volume introduces the factor of $1/\rho_0$ which is missing in Dunkerton et al. (1981), and assures that the vector components do not become exponentially smaller with height. Similarly, AMS83 convert $(F_{\phi}, F_p)$ to $(F_{\phi}^x = F_{\phi} p/ p_0, F_z = -F_p/p_0)$ with $z = -\ln(p/p_0)$. These factors come from replacing $\theta_p$ with $\theta_z$ with $H = 1$). However, such scaling still produces rapidly decaying vector sizes with height, and AMS83 still have to ‘multiply them by scalar normalizing factors, which differ from one grid point to another’.

The missing physical explanation from those arguments is that $d\phi dp$ includes mass weighting via the pressure element $dp$, whereas $d\phi dz$ does not, and the density has to be explicitly taken out of $\Delta_z$ in Equation (16). This in turn causes a division by $\rho_0$ to appear in Equation (18), which ascertains that the vectors do not become vanishingly small in the stratosphere. There is also no need to artificially set the density to unity throughout the domain as done by some authors.

Naturally, vertical propagation of waves throughout the atmospheric column does not happen purely conservatively, such that dissipation will decrease wave activity and we cannot expect EP flux arrows to remain of constant size over multiple scale heights. However, the exponential effect of mass dominates the effect of non-conservative dissipation and in practice EP flux arrows scaled with our method will be of comparable amplitude (Figure 2).

$$\frac{\partial y}{\partial p} = -\frac{1}{\ln(p_0/p_1)} \frac{\partial}{\partial p} \ln(p) \,(11)$$

Finally,

$$\rho \log \frac{\partial y}{\partial p} = Y = -\frac{Y}{\ln(p_0/p_1)} \cdot \rho.$$

Again, the units of $\rho \log \phi$ are inches per hPa. It is now obvious why AMS83 and others find that dividing the arrows by pressure is a good way to display EP flux arrows: It’s the required factor when changing variables from $p$ to $\log(p)$. While previous authors failed to find a physical reason, there is a geometric reason why this is true, and there is an additional scaling factor of $Y/\ln(p_0/p_1)$ which has to be included. It also becomes clear that scaling by the square root of pressure as in for example, Taguchi and Hartmann (2006) has neither a physical nor a geometric basis.

3 | $F_p$ Versus $F_z$

It is often convenient to work with the log-pressure coordinate $z = -H \ln(p/p_0)$, as $z$ has units of meters and is close to geometric height in the atmosphere. Using $z$ instead of $p$ has the consequence that the EP flux components (1)–(3) change form to (Andrews et al., 1987, Equation 3.5.3)

$$f_{\phi z}^x = -u' v' + \frac{v'}{\theta_z} = f_{\phi z},$$  

and

$$f_z = \left( f - \frac{1}{\cos \phi} \frac{\partial (\cos \phi)}{\partial \phi} \right) \frac{v'}{\theta_z} - u' w',$$  

and

$$\left( F_{\phi z}^x, F_{\phi z}^z \right) = \rho_0 a \cos \phi \left( f_{\phi z}^x, f_{\phi z}^z \right).$$  

$f_{\phi z}^x$ and $f_{\phi z}^z$ are the same as the change of variable from $p$ to $z$ cancels out in the second term, and we can drop the superscript. However, the units of $f_z$ are m$^2$/s$^2$ compared to hPa-m/s$^2$ for $f_p$. Most importantly, both terms in Equation (15) are now multiplied by density $\rho_0 = \rho \exp (-z/H)$. Closely following EHM80 and Dunkerton et al. (1981), we want to plot the vectors $\mathbf{F}$ in such a way that they visually represent wave propagation, but also
This letter shows that the most widely used scaling via dividing by pressure happens to be the correct scaling for a logarithmic axis (with additional constant factors, see Equation (12)). We will now compare the arrows resulting from our derivation to some of the most widely referenced methods. We note that it is very rare for authors to give details on whether or not they included figure aspect ratio for their plots, and we assume here that they do (the arrow direction depends on the ratio between $X$ and $Y$ in, e.g., Equation (7)). Many graphical packages also include automatic scaling algorithms, but these ’intelligent’ functions cannot guess the physical meaning of the axis dimensions, and often fail in representing both the angles and amplitudes of the arrows.

Figure 2 shows a selection of plots which we reconstructed from the PSL website (panels a,c,e), from
The methods of Palmer (1981) and AMS83 (both scaled by most important point to make from Figure 2 is that only physical analyses using those methods might lead to different interpretations. In contrast, all plots confirm that the newly derived method shows useful and accurate information throughout the entire plot domain, and is desirable over the other tested alternatives.

5 | CONCLUSIONS

In this letter, we show a derivation of the scaling of EP flux vectors for consistent plotting, and provide Python code for use (Jucker, 2019). In addition to presenting the derivation for plots with arbitrary aspect ratio, we also discuss the difference between working with \( F_z \) and \( F_p \) and how to scale these vectors on logarithmic pressure axes. Explicitly, the proposed scaling is summarised in Table 2. Given how simple the scaling is, we hope that this letter is useful as a reference for other authors struggling with the way to plot EP arrows for the analysis of atmospheric dynamics and in particular stratosphere-troposphere coupling.

ACKNOWLEDGMENTS

This research was supported by ARC grant FL150100035 and the ARC Centre of Excellence for Climate Extremes which is supported by the Australian Research Council via grant CE170100023. The data for plotting was downloaded from ERA5 (Hersbach et al., 2020) via the Copernicus Climate Data Store (Copernicus Climate Change Service Climate Data Store, 2017, last accessed Aug 2019). The analysis scripts are available online (Jucker, 2020a) and are part of the Python package aostools (Jucker, 2020b).

CONFLICT OF INTEREST

The author declares no conflict of interest.

ORCID

Martin Jucker © https://orcid.org/0000-0002-4227-315X

ENDNOTES

1 https://psl.noaa.gov/data/epflux/img/EP_Flux_Calculation_and_Display.pdf.
2 http://www.met.reading.ac.uk/~pn904784/snap/ep\_flux\_calculations.html.
3 https://psl.noaa.gov/data/epflux/.
4 daily instantaneous data at 00 UTC on 2.5° and 37-level grid. Data available from Jucker (2020a).
REFERENCES

Andrews, D.G., Holton, J.R. and Leovy, C.B. (1987) Middle Atmosphere Dynamics. International Geophysics Series. San Diego, California: Academic Press.

Andrews, D.G., Mahlman, J.D. and Sinclair, R.W. (1983) Eliassen-Palm diagnostics of wave-mean flow interaction in the GFDL "SKYHI" general circulation model. *Journal of the Atmospheric Sciences*, 40, 2768–2784. https://doi.org/10.1175/1520-0469%281983%29040%3C2768%3AETWATM%3E2.0.CO%3B2.

Andrews, D.G. and McIntyre, M.E. (1976) Planetary waves in horizontal and vertical shear: the generalized Eliassen-Palm relation and the mean zonal acceleration. *Journal of the Atmospheric Sciences*, 33, 2031–2048. https://doi.org/10.1175/1520-0469%281976%29033%3C2031%3APWIHAV%3E2.0.CO;2.

Andrews, D.G. and McIntyre, M.E. (1978) Generalized Eliassen-Palm and Charney-Drazin theorems for waves on axisymmetric mean flows in compressible atmospheres. *Journal of Atmospheric Sciences*, 35, 175. https://doi.org/10.1175/1520-0469(1978)035%3C0175:GEPACD%3E2.0.CO;2.

Baldwin, M.P., Edmon, H.J. and Holton, J.R. (1985) A diagnostic study of Eddy-mean flow interactions during FGGE SOP-1. *Journal of the Atmospheric Sciences*, 42, 1838–1845. https://doi.org/10.1175/1520-0469%281985%29042%3C1838%3ADSOEM%3E2.0.CO%3B2.

Nakamura, N. and Zhu, D. (2010) Finite-amplitude wave activity and diffusive flux of potential vorticity in Eddy-mean flow interaction. *Journal of the Atmospheric Sciences*, 67, 2701–2716. https://doi.org/10.1175/2010JAS3432.1.

Taguchi, M. and Hartmann, D.L. (2006) Increased occurrence of stratospheric sudden warmings during El Niño as simulated by WACCM. *Journal of Climate*, 19, 324–332. https://journals.ametsoc.org/jcli/article/19/3/324/31248/Increased-Occurrence-of-Stratospheric-Sudden.

How to cite this article: Jucker M. Scaling of Eliassen-Palm flux vectors. *Atmos Sci Lett*. 2021;22: e1020. https://doi.org/10.1002/asl.1020