A Note on Fluxes and Superpotentials
In $\mathcal{M}$-theory Compactifications
On Manifolds of $G_2$ Holonomy

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We consider the breaking of $\mathcal{N} = 1$ supersymmetry by non-zero $G$-flux when $\mathcal{M}$-theory is compactified on a smooth manifold $X$ of $G_2$ holonomy. Gukov has proposed a superpotential $W$ to describe this breaking in the low-energy effective theory. We check this proposal by comparing the bosonic potential implied by $W$ with the corresponding potential deduced from the eleven-dimensional supergravity action. One interesting aspect of this check is that, though $W$ depends explicitly only on $G$-flux supported on $X$, $W$ also describes the breaking of supersymmetry by $G$-flux transverse to $X$.
1. Introduction

One route to possible $\mathcal{M}$-theory phenomenology is to consider $\mathcal{M}$-theory compactifications on eleven-dimensional spaces of the form $M_4 \times X$, where $M_4$ denotes flat Minkowski space. When the seven-fold $X$ possesses a metric of $G_2$ holonomy, then $M_4 \times X$ is a vacuum solution of Einstein’s equation. Further, there exists one covariantly constant spinor on $X$, leading to an effective theory with $\mathcal{N} = 1$ supersymmetry in four dimensions. However, in contrast to $\mathcal{M}$-theory compactifications on Calabi-Yau four-folds \cite{1}, if we generalize this background ansatz to allow for non-zero $G$-flux and a warped product metric on $M_4 \times X$, then no supersymmetric vacua away from the trivial $G = 0$ background exist. This was demonstrated, for instance, in \cite{2}, \cite{3}.

The issue of supersymmetry-breaking by $G$-flux on $M_4 \times X$ is interesting, as this $G$-flux also generates a cosmological constant in the four-dimensional theory \cite{4}, \cite{5}. As in the case of compactifications on Calabi-Yau four-folds \cite{6}, the breaking of supersymmetry by $G$-flux can be effectively described in the four-dimensional theory by introducing a superpotential $W$ \cite{7} for the moduli of the compactification.

In the case of compactifications on Calabi-Yau four-folds, the superpotentials proposed in \cite{6} have been directly verified by a Kaluza-Klein reduction of the effective $\mathcal{M}$-theory action \cite{8}. One purpose of this note is to perform a similar check of the superpotential describing $G$-flux in compactifications on manifolds of $G_2$ holonomy. We compare the bosonic potential derived from $W$ with the corresponding bosonic potential obtained from the $\mathcal{M}$-theory effective action. In addition we want to explore a phenomenon that arises in $\mathcal{M}$-theory compactification to four dimensions but not in compactification to three dimensions on a Calabi-Yau four-fold. As in the Freund-Rubin solution, while preserving the four-dimensional symmetries, the $G$-field can have a component with all indices tangent to $M_4$, possibly triggering the breaking of supersymmetry. We will show that the minimal superpotential that describes the components of $G$ along $X$ actually also incorporates the component tangent to $M_4$. This observation has interesting implications for the structure of the parameter space of compactifications with $G$-flux.

The outline for this note is the following. In Section 2 we review the low-energy structure of Kaluza-Klein compactification of the $\mathcal{M}$-theory effective action on a smooth manifold $X$ of $G_2$ holonomy. We directly find the effective bosonic potential for $G$.

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1 Compactification on a smooth $X$ is not phenomenologically viable, since the low-energy theory will contain only abelian vector multiplets with no charged matter. When $X$ is allowed to have appropriate singularities, non-abelian gauge-groups and charged chiral matter \cite{6}, \cite{10} can be present, but of course the low-energy supergravity approximation is no longer valid.
In Section 3 we introduce and motivate the superpotential $W$. We then derive the bosonic potential for $G$ following from $W$. We find a potential which naively differs from the result of Section 2.

Finally in Section 4, we show how the two can be reconciled.

2. Kaluza-Klein Reduction of $\mathcal{M}$-Theory on Manifolds of $G_2$ Holonomy

In this section, we first review the structure of the massless $N = 1$ multiplets that arise when eleven-dimensional supergravity is compactified on $X$ (with $G = 0$), as has been discussed in [11], [12], [13]. The four-dimensional effective theory possesses $b_2(X)$ abelian vector superfields $V^j$ and $b_3(X)$ neutral chiral superfields $Z^i$. These superfields describe massless modes of the flat $C$-field and the metric on $X$.

To describe these modes explicitly, let us choose bases of harmonic forms $\{\omega_j\}$ for $\mathcal{H}^2(X)$ and $\{\phi_i\}$ for $\mathcal{H}^3(X)$. We then make a Kaluza-Klein ansatz for $C$,

$$C = c^i(x) \phi_i + A^j_{\mu}(x) dx^\mu \wedge \omega_j. \quad (2.1)$$

This ansatz is slightly oversimplified as it applies only when the $G$-flux is trivial. The scalars $c^i$ and the vectors $A^j_{\mu}$ describe the holonomies of a flat $C$-field. Because these holonomies take values in $U(1)$ and not $\mathbb{R}$, the fields appearing in (2.1) (in particular the scalars $c^i$) should also be regarded as taking values in $U(1)$ rather than $\mathbb{R}$. This observation deserves emphasis—it can also be understood by noting that under “large” gauge-transformations which add to $C$ a closed 3-form on $X$ of appropriately normalized periods, the $c^i$ undergo integral shifts.

We also note that when $X$ has $G_2$ holonomy, $b_1(X) = 0$, for the same reasons as in the case of Calabi-Yau three-folds. So no harmonic 1-forms on $X$ appear in the ansatz for $C$. Each vector $A^j_{\mu}(x)$ in (2.1) gives rise to one abelian vector superfield $V^j$, and each scalar $c^i(x)$ in (2.1) appears as the real component of the complex scalar $z^i$ in the chiral superfield $Z^i$.

The corresponding imaginary components of the $z^i$ describe massless fluctuations in the background metric on $X$. Recall that associated to any metric of $G_2$ holonomy on $X$ is a unique covariantly constant (hence closed and co-closed) 3-form $\Phi$. Given any such metric, we may associate to it the cohomology class $[\Phi]$ in $H^3(X; \mathbb{R})$, and this assignment is invariant under diffeomorphisms of $X$. In fact, as was shown by Joyce [14], the moduli space of $G_2$ holonomy metrics on $X$, modulo diffeomorphisms isotopic to the identity, is a smooth
manifold of dimension $b_3(X)$. Further, near a point in the moduli space corresponding to the equivalence class of metrics associated to $\Phi$, the moduli space is locally diffeomorphic to an open ball about $[\Phi]$ in $H^3(X; \mathbb{R})$. These results imply that massless modes of the metric on $X$ may be parametrized by introducing $b_3(X)$ scalars $s^i(x)$ defined by

$$\Phi = s^i(x) \phi_i .$$

(The $s^i$ are presumed to fluctuate around some point away from the origin.) Thus the $s^i$ in (2.2) naturally combine with the $c^i$ as $z^i = c^i + i s^i$. Note that this holomorphic combination $C + i\Phi$ on $X$ is analogous to the complexified Kähler class $B + iJ$ familiar from compactification on Calabi-Yau three-folds.

We now recall the bosonic action of eleven-dimensional supergravity [15],

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \left[ \sqrt{-g} R - \frac{1}{2} G \wedge \ast G - \frac{1}{6} C \wedge G \wedge G \right].$$

(2.3)

Higher derivative corrections in the $\mathcal{M}$-theory effective action, such as the $CI_8(R)$ term [16], will not be relevant for the following. We also find it useful to introduce $T_2$ and $T_5$, the $M2$-brane and $M5$-brane tensions, and to recall the relations between $\kappa_{11}^2$, $T_2$, and $T_5$ (as derived, for example, in [17])

$$\frac{1}{2\kappa_{11}^2} = \frac{1}{2\pi} T_2 T_5 ,$$

$$T_5 = \frac{1}{2\pi} T_2^2 .$$

(2.4)

Henceforth, we set $T_2 = 1$ to obtain the standard flux quantization conditions on $G$. In these units, $\kappa_{11}^2 = 2\pi^2$ and $T_5 = 1/2\pi$.

We wish to determine the potential induced for the moduli $Z^i$ when $G \neq 0$. We assume that $G$ respects the Lorentz symmetry of $M_4$ and so decomposes as

$$G = G_0 + G_X ,$$

(2.5)

where $G_0 = G|_{M_4}$ and $G_X = G|_X$. As explained in [18], Dirac quantization on $X$ generally requires that $\frac{1}{2\pi} G_X - \frac{1}{2} \lambda$ has integral periods, where $\lambda = p_1(X)/2$. So if $\lambda$ were not even in $H^4(X; \mathbb{Z})$, our earlier assumption that $G_X = 0$ would not have been consistent quantum mechanically. However, the following simple argument (see footnote 2 of [12]) implies that when $X$ is a spin seven-fold (one consequence of $G_2$ holonomy), then $\lambda$ is always even. Consider $S^1 \times X$. This is a spin eight-fold, and $p_1(S^1 \times X) = p_1(X)$, so it
suffices to consider \( \lambda \) on \( S^1 \times X \). But on any spin eight-fold, \( \lambda \) being even is equivalent to the intersection form on \( H^4 \) being even (for a proof of this standard fact via index theory, see [18]). Finally, the intersection form on \( S^1 \times X \) is even for trivial reasons. So we learn that \( \frac{1}{2\pi} G_X \) must have integral periods on \( X \), consistent with \( G_X = 0 \).

The quadratic \( GG \) term in \( S_{11} \) now descends directly to a pair of terms in the low-energy action for the metric moduli,

\[
\delta S^{(GG)}_4 = - \frac{1}{8\pi^2} \int d^4x \left[ \text{vol}(X) G_0 \wedge \ast G_0 + \sqrt{-g_4} \int_X G_X \wedge \ast G_X \right].
\] (2.6)

A term in the low-energy action for the moduli of the \( C \)-field is also induced from the \( CGG \) Chern-Simons term in \( S_{11} \),

\[
\delta S^{(CGG)}_4 = - \frac{1}{8\pi^2} \int d^4x \left[ G_0 \int_X C \wedge G_X \right].
\] (2.7)

(In computing this interaction, which will be converted to an ordinary potential in Section 4, we have dropped boundary terms from infinity on \( M_4 \) which can be absorbed in the background value of \( C \). A factor of three has arisen because the Chern-Simons term is cubic in \( C \).)

Let us now adopt a slightly more suggestive notation. We dualize the flux \( G_0 \) on \( M_4 \) by introducing a scalar \( f \) satisfying

\[
G_0 = f \, dx^0 \wedge \ldots \wedge dx^3, \quad \ast G_0 = -f,
\] (2.8)
in a coframe adapted to the metric. We also define

\[
\theta \equiv \frac{1}{4\pi} \int_X C \wedge G_X.
\] (2.9)

The expression \( \theta \) is a seven-dimensional Chern-Simons form on \( X \). It is not well-defined as a real number, due to the fact that \( C \) is only defined up to shifts \( C \to C + \phi_i \), where the harmonic forms \( \frac{1}{2\pi} \phi_i \) are normalized to have integral periods on \( X \). At first glance, one might have thought that \( \theta \) is consequently defined only modulo \( \pi \cdot \text{integer} \). In fact, because the class \( \lambda \) of \( X \) is even, a careful treatment of \( \theta \), as given in Section 3 of [13], shows that \( \theta \) is actually well-defined modulo \( 2\pi \cdot \text{integer} \). Hence our notation is correct in suggesting that \( \theta \) is an angle.

\[\text{We will not distinguish notationally between the four-, seven-, and eleven-dimensional Hodge } \ast, \text{ but the distinction should be clear from context.}\]
The four-dimensional effective potential for $C$ and $\Phi$ is then determined from (2.6) and (2.7) after we pass to Einstein frame, rescaling the four-dimensional metric $g_{\mu \nu} \rightarrow 2\pi^2 \text{vol}(X)^{-1} g_{\mu \nu}$. We find this potential to be

$$V(C, \Phi) = -\frac{1}{32\pi^6} \text{vol}(X)^3 f^2 + \frac{\pi^2}{2} \text{vol}(X)^{-2} \int_X G_X \wedge \star G_X + \frac{\theta}{2\pi^2} f.$$  

(2.10)

The additional factors of $\text{vol}(X)$ in the $f^2$ term arise from the explicit factor in (2.6) and the scaling of the four-dimensional Hodge $\star$. This term also has the “wrong” sign as it is really a kinetic energy, so we slightly abuse the terminology in referring to $V$ as a “potential”. The $\theta$ term, as it descends from the eleven-dimensional Chern-Simons term, remains independent of $\text{vol}(X)$ under the rescaling.

3. The Superpotential

3.1. Motivating the Superpotential

We can now introduce the superpotential $W(Z^i)$, essentially proposed in [7], which describes the breaking of supersymmetry by $G$-flux on $X$. We consider

$$W(Z^i) = \frac{1}{8\pi^2} \int_X \left( \frac{1}{2} C + i \Phi \right) \wedge G_X .$$  

(3.1)

The relative factor of $1/2$ between the two terms in $W$ is required by supersymmetry. For under a variation

$$C \rightarrow C + \delta C, \quad \Phi \rightarrow \Phi + \delta \Phi ,$$  

(3.2)

the superpotential varies as

$$W \rightarrow W + \frac{1}{8\pi^2} \int_X (\delta C + i \delta \Phi) \wedge G_X .$$  

(3.3)

Note that a relative factor of 2 has appeared in the variation of the first term due to its quadratic dependence on $C$ and the fact that, whereas $d\Phi = 0$, $dC = G$. $\delta W$ is linear in $\delta C + i \delta \Phi$ as required for holomorphy. The condition for unbroken supersymmetry and zero cosmological constant of a four-dimensional $N = 1$ supergravity theory with superpotential $W$ is that, in the vacuum,

$$W = dW = 0.$$  

(3.4)

The latter condition, for $W$ above, is sufficient to imply that $G_X$ must vanish in a supersymmetric vacuum with zero cosmological constant.
A discussion of the proper interpretation of the former condition is called for, since the term $\int_X C \wedge G_X/(4\pi)^2 = \theta/4\pi$ in $W$ is only well-defined modulo $1/2 \cdot$ integer. We have no way to pick a natural definition of this expression as a real number, so the best we can do is to say that all possibilities differing by $\theta \to \theta + 2\pi$ are allowed. Thus, the theory depends on an integer that is not fixed when the $C$-field on $X$ (and its curvature $G_X$) are given.

What is the physical interpretation of this integer? Heuristically, it corresponds to the value of the period $\int_X G_7/(2\pi)^2$ (= $T_5 \int_X G_7/2\pi$ in our conventions), where $G_7$ is the seven-form field dual to $G$. In fact, the classical equation of motion $dG_7 = -\frac{1}{2} G \wedge G + \ldots$ (the $\ldots$ being gravitational corrections that we do not consider here) shows that $\int_X G_7/(2\pi)^2$ is not constant because $G_7$ is not closed. As observed by Page [20], what is constant and should be quantized is rather $\int_X (G_7 + \frac{1}{2} C \wedge G)/(2\pi)^2$, and since $\frac{1}{2} \int_X C \wedge G/(2\pi)^2$ (= $\theta/2\pi$) is anyway only defined modulo an integer, we introduce no additional ambiguity if we take $\int_X (G_7 + \frac{1}{2} C \wedge G) = 0$. Thus, $\frac{1}{2} \int_X C \wedge G/(2\pi)^2$ can “stand in” for the $G_7$-flux, and the possibility of adding an integer to its value amounts to the possibility of shifting the $G_7$-flux by an integer number of quanta.

At this point, we can see that the parameter space of $M$-theory compactifications with $G$-flux is not, as one might have supposed, a product of the space of $C$-fields on $X$ with a copy of $\mathbb{Z}$ parametrized by the $G_7$-flux. Rather, the parameter space is fibered over the space of $C$-fields, with the fiber being a copy of $\mathbb{Z}$; but the fibration is non-trivial. Consider varying $C$ by $C \to C + C'$, where $dC' = 0$. If $C'$ has trivial periods, this change in the $C$-field is topologically trivial. But $\frac{1}{2} \int_X C \wedge G/(2\pi)^2$ changes in a non-trivial way. If $G/2\pi$, restricted to $X$, is divisible by an integer $m$, then the change in $\frac{1}{2} \int_X C \wedge G/(2\pi)^2$ will always be an integer multiple of $m$, and so the change in the $G_7$-flux is likewise a multiple of $m$. Thus, the only invariant under this process is the value of $G_7$ modulo $m$.

For example, if $m = 1$, $G_7$ can be varied arbitrarily, and the overall parameter space of $C$-fields plus $G_7$-flux is connected. Only when $G_X = 0$ is the parameter space what one would expect naively: a product of the space of $C$-fields with a copy of $\mathbb{Z}$ parameterizing the $G_7$-flux.

For precise computations, it is awkward to work with $G_7$, since (as $G$ is closed and $G_7$ is not) the theory is naturally formulated with a three-form field $C$ and not with a dual six-form field. In Section 4, by treating the $C$-field quantum mechanically, we will show how the picture we have just described is reproduced if one works with $G$, rather than (as in the last few paragraphs) the dual $G_7$. The need to work quantum mechanically when
one formulates the discussion in terms of $G$ should not come as a surprise; duality typically relates a classical description in one variable to a dual quantum mechanical description.

Previously, it was suggested \[3\] that a superpotential describing the effects of $G$-flux along $M_4$ and $X$ would be

$$ W = \int_X G_7 + \int_X (C + i\Phi) \wedge G_X. \quad (3.5) $$

One might have naively thought that the two terms in (3.5) involving $G_7$ and $G_X$ were describing independent effects due to $G$-flux along $M_4$ and $X$. As we have seen, this interpretation would be problematic because generically the parameter space of $G_7$-flux fibers non-trivially over the space of $C$-fields on $X$. A closely related observation is that neither the term involving $G_7$ nor the term involving $G_X$ individually respects holomorphy. So the relative normalization of the terms is not arbitrary but fixed by holomorphy, contrary to what the naive interpretation would suggest. In fact, taking $\int_X (G_7 + \frac{1}{2} C \wedge G) = 0$, we see that (3.5) corresponds, up to normalization, to the proposed superpotential in (3.1), whose holomorphy we verified and in which $G_7$ does not explicitly appear.

The superpotential (3.1) can also be motivated and its normalization fixed from an argument given in \[6\]. Consider an $M5$-brane having worldvolume $\mathbb{R}^{2,1} \times \Sigma$, where $\Sigma$ is a 3-cycle on $X$ which is calibrated by $\Phi$. Such a calibrated 3-cycle is a supersymmetric 3-cycle \[21\] and has minimal volume, $\text{vol}(\Sigma) = |\int_{\Sigma} \Phi|$, within its homology class. So the wrapped $M5$-brane appears as a BPS domain wall in the four-dimensional theory. The tension $\tau$ of such a BPS domain wall in the low-energy $d = 4, \mathcal{N} = 1$ Wess-Zumino model describing the $Z^i$ is \[22\]

$$ \tau = 2|\Delta W|, \quad (3.6) $$

where $\Delta W$ is the change in $W$ upon crossing the wall.

On the other hand, the $M5$-brane is a magnetic source for $G_X$, and the class of $G_X$ must change upon crossing the wall. This change is $\Delta G_X = 2\pi \delta_{\Sigma}$, where $\delta_{\Sigma}$ is the fundamental class of $\Sigma$. Hence (assuming $C = 0$) we have

$$ 2|\Delta W| = \frac{1}{(2\pi)^2} \left| \int_X \Phi \wedge \Delta G_X \right|, $$

$$ = \frac{1}{2\pi} \left| \int_{\Sigma} \Phi \wedge G_X \right| = T_5 \text{vol}(\Sigma), \quad (3.7) $$

as we expect for an M5-brane domain wall. The dependence on $C$ follows from supersymmetry.
We now derive the bosonic potential $U$ which follows from the superpotential $W$. Recall that, in terms of the Kähler potential $K$, the metric on moduli space $g_{i\bar{j}}$, and $W$, the potential $U$ is given by \[ U = \exp(K) \left[ g^{i\bar{j}} D_i W \overline{D_j W} - 3|W|^2 \right], \] (3.8)

where $D_i W = \partial_i W + \partial_i K \cdot W$ is the covariant derivative of $W$ as a section of a line bundle over the moduli space parametrized locally by the $Z^i$.

To evaluate $U$, we need to know $g_{i\bar{j}}$ and $K$. The metric $g_{i\bar{j}}$ can be determined most directly from the kinetic terms of the $c^i$ fields. These kinetic terms arise from reducing the term $-\frac{1}{8\pi^2} \int G \wedge \ast G$ in the action $S_{11}$. We find these kinetic terms to be (in Einstein frame)

\[ L_{\text{kin}} = -\frac{1}{4} \text{vol}(X)^{-1} \partial_{\mu} c^i \partial^{\mu} c^j \int_X \phi_i \wedge \ast \phi_j. \] (3.9)

On the other hand, the metric $g_{i\bar{j}}$ appears in the four-dimensional action in the term $L_{\text{kin}} = -g_{i\bar{j}} \partial_\mu z^i \partial^{\mu} \overline{z}^j$. So (3.9) determines the metric $g_{i\bar{j}}$ to be

\[ g_{i\bar{j}} = \overline{\partial_j} \partial_i K = \frac{1}{4} \text{vol}(X)^{-1} \int_X \phi_i \wedge \ast \phi_j. \] (3.10)

We now claim that the Kähler potential is given, up to shifts $K(Z^i, \overline{Z}^i) \rightarrow K + f(Z^i) + f^*(\overline{Z}^i)$, by

\[ K = -3 \log \left[ \frac{1}{2\pi^2} \frac{1}{7} \int_X \Phi \wedge \ast \Phi \right]. \] (3.11)

The general form for $K$ has appeared in \[12\], \[13\], but we must be careful to check the factor of $-3$ appearing in the normalization of $K$. We will make this check directly by computing $\overline{\partial_j} \partial_i K$.

### 3.2. Mathematical Preliminaries and Computing $\overline{\partial_j} \partial_i K$

In order that the following be self-contained, we must review a few facts about the group $G_2$ and metrics of $G_2$ holonomy (for which a good general reference is \[24\]). As very lucidly described in \[25\], the group $G_2$ can be defined as the subgroup of $GL(7, \mathbb{R})$ preserving a particular 3-form on $\mathbb{R}^7$. In terms of coordinates $(x^1, \ldots, x^7)$ on $\mathbb{R}^7$, this 3-form is

\[ \Phi = \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}, \] (3.12)
where we abbreviate $\theta^{i_1 \ldots i_n} = dx^{i_1} \wedge \ldots \wedge dx^{i_n}$. $G_2$ also preserves the Euclidean metric $ds^2 = (dx^1)^2 + \ldots + (dx^7)^2$ (as we expect, since $G_2$ occurs as the holonomy of $X$ and so must be a subgroup of $O(7)$) and hence the dual 4-form with respect to this metric,

$$\star \Phi = \theta^{4567} + \theta^{2367} + \theta^{2345} + \theta^{1357} - \theta^{1346} - \theta^{1256} - \theta^{1247}. \quad (3.13)$$

We note that

$$dx^1 \wedge \ldots \wedge dx^7 = \frac{1}{7} \Phi \wedge \star \Phi,$$ 

(3.14)

so that $G_2$ also preserves the orientation of $\mathbb{R}^7$.

When $X$ possesses a metric of $G_2$ holonomy, then at each point $p$ of $X$ there exists a local frame in which the covariantly constant 3-form $\Phi$ takes the form in (3.12) and the metric takes the Euclidean form. Hence the local relation (3.14) immediately implies that

$$\frac{1}{7} \int_X \Phi \wedge \star \Phi = \text{vol}(X). \quad (3.15)$$

This relation explains the factor of $\frac{1}{7}$ appearing in $K$, which, besides being very natural, will give the correct normalization of $U$.

Because we are primarily interested in 3-forms on $X$, our interest naturally lies in the $G_2$ representation $\Lambda^3(\mathbb{R}^7)^*$ consisting of rank-three anti-symmetric tensors. $\Lambda^3(\mathbb{R}^7)^*$ decomposes under $G_2$ into irreducible representations $1 \oplus 7 \oplus 27$. The trivial representation $1$ is generated by the invariant 3-form (3.12) which we used to define $G_2$. The 7 arises from the fundamental representation of $G_2$ as a subgroup of $GL(7, \mathbb{R})$ via the map $(\mathbb{R}^7)^* \hookrightarrow \Lambda^3(\mathbb{R}^7)^*$, sending $\alpha \mapsto \star(\alpha \wedge \Phi)$. Note that $\star(\cdot \wedge \Phi)$ defines a (non-zero) $G_2$-equivariant map, which must be an isomorphism onto its image by Schur’s lemma. The 27 can then be characterized as the set of those elements $\lambda$ in $\Lambda^3(\mathbb{R}^7)^*$ which satisfy $\lambda \wedge \Phi = \lambda \wedge \star \Phi = 0$. This identification follows again from the fact that $\cdot \wedge \star \Phi$ and $\cdot \wedge \Phi$ are $G_2$-equivariant maps.

Now, as is familiar from the case of Calabi-Yau three-folds, the decomposition of $\Lambda^3(\mathbb{R}^7)^*$ into irreducible representations of $G_2$ implies a corresponding decomposition of $\Lambda^3 T^* X$ under the holonomy. The Laplacian on $X$ respects this decomposition, implying a corresponding classification of the harmonic 3-forms on $X$, $\mathcal{H}^3(X) \cong \mathcal{H}^3_1(X) \oplus \mathcal{H}^3_7(X) \oplus \mathcal{H}^3_{27}(X)$. In fact, the Laplacian of $X$ as an operator on $p$-forms depends only on the representation of the holonomy, not on $p$. Hence the dimension of $\mathcal{H}^p_R(X)$ for some representation $R$ depends only on $R$, not $p$. Thus, since $\mathcal{H}^1(X) = \mathcal{H}^1_1(X) = 0$, we also have that $\mathcal{H}^3_7(X) = 0$. Further, from our characterization of the 27 above, we see that $\mathcal{H}^3_1(X)$ is orthogonal to $\mathcal{H}^3_{27}(X)$ in the usual inner product $(\alpha, \beta) = \int_X \alpha \wedge \star \beta$. 

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We are now prepared to evaluate $\partial_j \partial_i \mathcal{K}$. First we observe that $\int_X \Phi \wedge \ast \Phi$ is a homogeneous function of the $s^i$ of degree $\frac{7}{2}$. This observation follows from (3.13) and the fact that under a scaling of the local coframe $dx^i \rightarrow \lambda dx^i$, $\Phi$ scales as $\Phi \rightarrow \lambda^3 \Phi$, or equivalently the $s^i$ scale as $s^i \rightarrow \lambda^3 s^i$, and $\text{vol}(X)$ scales as $\text{vol}(X) \rightarrow \lambda^7 \text{vol}(X)$. Remembering that $s^i = \text{Im}(z^i)$, homogeneity of $\int_X \Phi \wedge \ast \Phi$ in the $s^i$ then implies that

$$\partial \mathcal{K} / \partial z^i = i \frac{7}{2} \int_X \Phi_i \wedge \ast \Phi.$$  (3.16)

To evaluate a second derivative of $\mathcal{K}$, we must evaluate $\frac{\partial}{\partial s^i} (\ast \Phi)$ arising from the numerator of (3.16). Here $\Phi : s^i \mapsto s^i \phi_i$ is a linear map of the (local) coordinates $s^i$ on the moduli space, and for a fixed metric on $X$, the Hodge $\ast$ is certainly a linear operator on $\mathcal{H}^3(X)$. However, since the operator $\ast$ depends on the metric, hence the moduli $s^i$, in general $\ast \Phi(s^i)$ depends nonlinearly on the $s^i$. Following Joyce [14] (where this derivative appears), we will denote $\Theta(s^i) \equiv \ast \Phi(s^i)$ to emphasize this nonlinearity.

Let us restrict to a local coordinate patch (diffeomorphic to $\mathbb{R}^7$) on $X$ and on this patch consider $\Theta$ as a map on a small open ball about the canonical 3-form $\Phi$ in $\Lambda^3 T^* \mathbb{R}^7$. $\Theta$ is well-defined, since for $\Xi$ sufficiently small, a local change of frame can always be found taking $\Phi + \Xi$ to the canonical form (3.12). The metric associated to $\Phi + \Xi$ in the frame for which $\Phi + \Xi$ is canonical is the Euclidean metric, and so $\Theta$ is very easy to evaluate in this frame.

We now consider the derivative $D\Theta$. $D\Theta$ is locally linear (i.e. linear over $C^\infty(\mathbb{R}^7)$), so it suffices to consider $D\Theta$ as a linear map on $\Lambda^3(\mathbb{R}^7)^*$. Observe that, since $\Phi$ is $G_2$-invariant, $D\Theta$ is actually a $G_2$-equivariant map. If we denote by $\pi^1$, $\pi^7$, and $\pi^{27}$ the projections on $\Lambda^3(\mathbb{R}^7)^* \cong 1 \oplus 7 \oplus 27$, then Schur’s lemma implies that $D\Theta$ decomposes as

$$D\Theta = D\Theta \circ \pi^1 + D\Theta \circ \pi^7 + D\Theta \circ \pi^{27} = a \ast \circ \pi^1 + b \ast \circ \pi^7 + c \ast \circ \pi^{27},$$  (3.17)

for some constants $a$, $b$, $c$.

The above expression for $D\Theta$ certainly holds when we consider evaluating $D\Theta$ on the restrictions of elements of $\mathcal{H}^3(X)$ to the local patch. But the expression is also a sensible global expression on $X$, so it must in fact hold globally. It remains only to evaluate the constants $a$ and $c$ (since $\mathcal{H}^3_7 = 0$, $b$ is irrelevant for us).  

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3 This statement follows from a simple dimension count, noting that $\dim GL(7, \mathbb{R}) = 49$, $\dim G_2 = 14$ and $\dim \Lambda^3(\mathbb{R}^7)^* = 35$, so that $\dim GL(7, \mathbb{R}) - \dim G_2 = \dim \Lambda^3(\mathbb{R}^7)^*$.

4 $b = 1$ for the curious.
We determine \( a \) and \( c \) by explicit computation in \( \Lambda^3(\mathbb{R}^7)^* \). We fix \( a \) by once more considering the scaling \( dx^i \rightarrow \lambda dx^i \), under which \( \Phi \rightarrow \lambda^3 \Phi \) and \( \Phi \rightarrow \lambda^4 \Phi \). So \( a = 4/3 \).

To fix \( c \), let us consider the 3-form \( \Xi = \theta^{123} - \theta^{145} \). Since \( \Xi \wedge \Phi = \Xi \wedge \Phi \wedge \Phi = 0 \), we see that \( \Xi \) transforms in the 27 of \( G_2 \). By a change in the frame \( dx^2, dx^3 \rightarrow (1 - \epsilon/2)dx^2, dx^3 \) and \( dx^4, dx^5 \rightarrow (1 + \epsilon/2)dx^4, dx^5 \), the sum \( \Phi + \epsilon \Xi \) can be brought to the canonical form, to linear order in \( \epsilon \). Dualizing and transforming back to the original frame, we easily see that \( \Theta(\Phi + \epsilon \Xi) = \star \Phi - \epsilon \star \Xi + O(\epsilon^2) \), which fixes \( c = -1 \). So we finally conclude that

\[
\frac{\partial}{\partial s^i}(\star \Phi) = \frac{4}{3} \star \pi^1(\phi_i) - \star \pi^{27}(\phi_i).
\]

(3.18)

This derivative in hand, we evaluate \( \partial \mathcal{K}/\partial z^i/j^z \) as

\[
\overline{\partial_j} \partial_i \mathcal{K} = \frac{7}{4} \frac{1}{\text{vol}(X)} \left\{ \int_X \phi_i \wedge \star \frac{4}{3} \pi^1(\phi_j) - \pi^{27}(\phi_j) \right\} - \frac{1}{\text{vol}(X)} \int_X \phi_i \wedge \star \Phi \cdot \frac{7}{3} \int_X \phi_j \wedge \star \Phi \right\},
\]

\[
= \frac{1}{4} \text{vol}(X)^{-1} \left\{ \int_X \phi_i \wedge \star \pi^1(\phi_j) - \int_X \phi_i \wedge \star \pi^{27}(\phi_j) \right\},
\]

\[
= \frac{1}{4} \text{vol}(X)^{-1} \int_X \phi_i \wedge \star \phi_j,
\]

(3.19)

where we have noted that \( \pi^1(\phi_i) = \Phi \cdot (\int_X \phi_i \wedge \star \Phi)/(\int_X \Phi \wedge \star \Phi) \). Comparing to (3.10), we see that \( \mathcal{K} \) is the properly normalized Kähler potential.

3.3. Computing the Potential

The rest of the calculation of \( U \) from (3.8) is now direct. We have

\[
D_i W = \frac{1}{8\pi^2} \int_X \phi_i \wedge G_X + \frac{7}{16\pi^2} \left[ \int_X \phi_i \wedge \Phi \cdot \Phi \right] \cdot \int_X \frac{1}{2} (C + i \Phi) \wedge G_X.
\]

(3.20)

So

\[
g^{i\bar{j}} D_i W D_{\bar{j}} W = \frac{1}{7(2\pi)^4} \int_X \Phi \wedge \Phi \cdot \left\{ \int_X G_X \wedge \Phi \wedge G_X + \frac{21}{4} \left( \int_X \Phi \wedge \Phi \right)^{-1} \cdot \left( \int_X \Phi \wedge G_X \right)^2 \right. \]

\[
+ \left( \frac{7}{2} \right)^2 \left( \int_X \Phi \wedge \Phi \right)^{-1} \cdot \left( \frac{1}{2} \right) \int_X C \wedge G_X \right\}.
\]

(3.21)

Here we have noted, for instance, that \( g^{i\bar{j}} \int_X \phi_i \wedge G_X \cdot \int_X \phi_j \wedge G_X = \frac{1}{4} \int_X \Phi \wedge \Phi \cdot \int_X G_X \wedge \Phi \wedge G_X \). This relation may be checked by expanding \( G_X = f^i(s) \cdot \phi_i \) for some functions \( f^i(s) \) (although \( G_X \) is independent of the metric, the basis \( \{ \phi_i \} \) is not). So we
find
\[
U = \frac{7^3 \pi^2}{2} \left( \int_X \Phi \wedge \ast \Phi \right)^{-2} \left[ \frac{1}{7} \int_X G_X \wedge \ast G_X + \left( \int_X \Phi \wedge \ast \Phi \right)^{-1} \left( \frac{1}{2} \int_X C \wedge G_X \right)^2 \right],
\]
\[
= \frac{\pi^2}{2} \text{vol}(X)^{-2} \left[ \int_X G_X \wedge \ast G_X + \text{vol}(X)^{-1} \left( \frac{1}{2} \int_X C \wedge G_X \right)^2 \right].
\]
(3.22)

Comparing \( V \) in (2.10) to \( U \) in (3.22), we see that the superpotential produces the correct \( G_X^2 \) term, but of course the terms in \( V \) involving \( f \) do not appear in \( U \). In terms of the angle \( \theta \), we can write
\[
U = \frac{\pi^2}{2} \text{vol}(X)^{-2} \left[ \int_X G_X \wedge \ast G_X + 8\pi^6 \text{vol}(X)^{-3} \left( \frac{\theta}{2\pi} \right)^2 \right].
\]
(3.23)

4. Comparing the Potentials

There seems to be an obvious disagreement between \( V \) and \( U \). They do not even depend on the same variables. The component \( G_0 = f dx^0 \wedge \ldots \wedge dx^3 \) of the \( G \)-field appears in \( V \) as
\[
V(f) = -\frac{1}{32\pi^6} \text{vol}(X)^3 f^2 + \frac{\theta}{2\pi} f.
\]
(4.1)
Hence \( V \) depends on \( G_0 \) as well as on the \( C \)-field along \( X \). By contrast, \( U \) depends only on the \( C \)-field along \( X \).

We will now see that \( V \) and \( U \) can be reconciled by treating \( G_0 \) quantum mechanically. In fact, quantum mechanically, there are only discrete allowed values for \( f \); upon setting it equal to an allowed value, \( V \) coincides with \( U \).

A four-form field in 3+1 dimensions is analogous to a two-form field – the curvature of an abelian gauge field – in 1+1 dimensions. We will simply interpret (4.1) as the action for a four-form field and treat it quantum mechanically. (The underlying eleven-dimensional supergravity action also contains couplings of \( G_0 \) to fermions, but these are inessential for our present purposes.) In the analogy with 1+1-dimensional abelian gauge theory, the term in (4.1) that is proportional to \( \theta \) is analogous to the theta-angle of abelian gauge theory.

Classically, if one were allowed to just minimize \( V \) with respect to \( f \), having nonzero \( \theta \) would induce a non-zero value for \( f \) and hence a non-zero energy density. This is roughly
what emerges from a proper quantum mechanical treatment \cite{20}. If one proceeds naively, the classical value for \( f \) in (4.1) is

\[
f = 8\pi^5 \text{vol}(X)^{-3} \theta,
\]

leading to the induced energy density

\[
E(\theta) = 8\pi^6 \text{vol}(X)^{-3} \left( \frac{\theta}{2\pi} \right)^2.
\]

Quantum mechanically, one really wants the energies of all of the states; one finds \cite{26} that the quantum states of this system are labeled by an integer \( n \), with the energy density of the \( n^{th} \) state being

\[
E_n(\theta) = 8\pi^6 \text{vol}(X)^{-3} \left( n + \frac{\theta}{2\pi} \right)^2.
\]

A different way to describe the above results is that (4.3) is, indeed, the correct formula for the energy, but in interpreting \( \theta/2\pi \) as a real number, one must include all possibilities, differing by the possible addition of an integer. Of course, one could, for any given \( \theta \) pick the \( \text{– generically unique – integer that minimizes the energy. However, we prefer to show that our two potentials } V \text{ and } U \text{ agree (when } V \text{ is treated quantum mechanically) without imposing any such restriction. For this, we simply use (4.3), but accepting all real lifts of } \theta \text{ differing by multiples of } 2\pi, \text{ as we have anyway done throughout this paper.}

Comparing to (3.23), we see that the vacuum energy density \( E(\theta) \) due to \( G_0 \) is exactly the second term in \( U \), so that the superpotential \( W \) indeed captures the effects of supersymmetry-breaking by both \( G_X \) and \( G_0 \).

Finally, we note that the potential \( U \) is positive-definite. In the regime of large \( \text{vol}(X) \), for which our effective \( M \)-theory action is valid, no vacuum exists and we see a runaway to infinite volume, a familiar situation in supergravity compactifications. We might also consider contributions to the potential from membrane instantons wrapping calibrated 3-cycles \( \Sigma \) on \( X \), but the leading instanton contribution to the potential \cite{12} in the large volume regime is of order \( e^{-\text{vol}(\Sigma)} \) and so will not help to stabilize the runaway.

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