Exact wave functions of bound $\mu^-$ for calculating ordinary muon capture rates

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Abstract. The goal of the present contribution is twofold: (i) To compute exact wave functions for a muon bound in the extended Coulomb potential of a muonic atom by solving the Dirac equation within the context of genetic algorithms and neural network techniques using experimental finite-size charge-densities for the attracting nucleus. (ii) To calculate partial and total rates of the ordinary muon capture in various muonic atoms. In contrast to the majority of previous realistic calculations for $\mu^-$-capture rates, in our present work we utilize the above mentioned exact wave functions for a muon orbiting at the 1s and 2p atomic orbits. The required many-body nuclear wave functions are obtained by diagonalizing the eigenvalue problem of the quasi-particle random phase approximation (QRPA).

1. Introduction

As is well known, when negative muons, produced by the decay of slow pions (e.g. in a meson factory at Fermilab, USA, or at JPARC, Japan), slow down in matter, it is possible for them to be captured in atomic orbits. Afterwards, fast electromagnetic cascades bring the muon of the muonic atom down to the innermost (1s or 2p) orbits [1, 2]. The bound in such an orbit muon may disappear, either by decay, known as muon decay in orbit and represented by the reaction

$$\mu_b^- \to e^- + \bar{\nu}_e + \nu_\mu,$$

or by capture by the nucleus the main channel of which is the ordinary muon capture represented by [3, 4]

$$\mu_b^- + (A, Z) \to (A, Z - 1)^* + \nu_\mu.$$

We mention that, some of the most sensitive recent experiments (at Fermilab and JPARC) search extensively for additional channels that involve muon number non-conservation predicted to occur in modern gauge theories [5]. The most important exotic possibilities of this kind are: (i) The neutrinoless muon to electron conversion

$$\mu_b^- + (A, Z) \to (A, Z) + e^-,$$

which violates the muon number and is allowed by lepton conservation (this is searched by the mu2e experiment at Fermilab and the COMMET experiment at JPARC), and (ii) the muon to positron conversion

$$\mu_b^- + (A, Z) \to (A, Z - 2) + e^+$$
which violates both the muon number and the lepton number as well.

In the present paper we focus our attention on process (2) which has been the subject of extensive experimental and theoretical investigations started on the early 50’s. Recently, however, the interest for this process has been revived due to its important role in astrophysics and especially in supernova dynamics [3]. For terrestrial experiments studying the muon capture by a nuclear target \((A,Z)\), a competitive channel is the radiative muon capture [1, 2]

\[ \mu^- + (A, Z) \rightarrow (A, Z-1)^* + \nu_\mu + \gamma \]  

(5)

It should be stressed that the ordinary muon capture Eq. (2) is one of the most useful channels for testing the nuclear structure models, as the low-energy electron scattering on nuclei.

2. Theoretical Framework - Brief description of the Formalism

2.1. The interaction Hamiltonian of the muon capture process

In the context of the current-current interaction hypothesis, the effective interaction Hamiltonian for the ordinary muon capture takes the form [3, 4]

\[ \hat{H}_{\text{eff}} = \int d^3x \hat{H}_{\text{eff}}(x) = \frac{G}{\sqrt{2}} \int d^3x \hat{J}_\lambda^\mu(x)(j^\dagger)^\lambda(x) \]  

(6)

where \( j^\dagger \) denotes the lepton charge-changing current

\[ j^\dagger = \bar{\psi}_{\nu_\mu} \gamma_\lambda(1 - \gamma_5) \tau_+ \psi_\mu \]  

(7)

(\( \bar{\psi}_{\nu_\mu} \), \( \psi_\mu \)) are the lepton spinors, and \( J_\lambda^- \) represents the hadronic charge-changing current written as

\[ J_\lambda^- = \bar{u}_n(P')[(F_V \gamma_\lambda + F_M \sigma_{\lambda\gamma} q_\mu + F_A \gamma_\lambda \gamma_5 + F_P \gamma_5 q_M)\tau_- u_p(P) \]  

(8)

In the latter equation, \( u_{p,n}(P) \) represent the proton or neutron spinor. The quantities \( F_V, F_M, F_A, F_P \) are the well known nucleon form factors for the polar-vector, magnetic, axial-vector and pseudoscalar components of the hadronic current, respectively. All these form factors are functions of the four-momentum transfer \( q^2 \), given by the kinematics of the reaction. In our convention [6, 7] we write

\[ q^2 = q_0^2 - q^2 \]  

(9)

The magnitude of the three momentum transfer is \( q = |q| \), which enters the nuclear matrix elements, can be approximated by [3]

\[ q \equiv \nu \simeq m_\mu - \varepsilon_b + E_i - E_f \]  

(10)

In the latter equation, \( m_\mu \) is the muon rest mass, \( \varepsilon_b \) is the muon binding energy at the muonic atom, and \( E_i, E_f \) represent the energies of the initial (ground) and final nuclear states, respectively. In Eq. (10) we neglect the nuclear recoil and assume that the initial momentum of the bound muon is negligible.

2.2. Partial and total muon capture rates

In our nuclear structure calculations, the even-even nuclear systems are supposed to be spherically symmetric with an initial (ground) state \(|J_i^{\pi_i}\rangle = |0^+\rangle\). After the \( \mu^- \) capture, the studied nucleus is left in a final excited state \(|J_f^{\pi_f}\rangle \) having angular momentum \( J_f \) and parity \( \pi_f \).

Using natural units (\( \hbar = c = 1 \)), the partial capture rate for a transition from the ground state to an excited state is written as [3, 4]

\[ \sigma_{i\rightarrow f} = \frac{\Omega(\varepsilon_f)^2}{2\pi} \sum_{s_f,s_i} \frac{1}{(2J_i + 1)} \sum_{M_f,M_i} |\langle f | \hat{H}_{\text{eff}} | i \rangle|^2 \]  

(11)
where $\varepsilon_f$ is the energy of the outgoing $\nu_\mu$ and $\Omega$ stands for the quantization volume. After replacing $\hat{H}_{\text{eff}}$ in Eq. (11) with its expression from Eq. (6) and elaborating on the result, the partial muon capture rate $\sigma_{i \rightarrow f}$ reads \[3, 4\]

$$
\sigma_{if} = \frac{G^2(\varepsilon_f)^2}{2\pi} \left\{ \frac{1}{2J_i + 1} \sum_{J=0}^{\infty} |\langle J_f || \Phi^\mu(\hat{M}_J - \hat{L}_J) || J_i \rangle|^2 + \sum_{J=1}^{\infty} |\langle J_f || \Phi^\mu(\hat{T}^{el}_J - \hat{T}^{mag}_J) || J_i \rangle|^2 \right\} \tag{12}
$$

The operators $\hat{M}_J$ (Coulomb or charge), $\hat{L}_J$ (longitudinal), $\hat{T}^{el}_J$ (transverse electric) and $\hat{T}^{mag}_J$ (transverse magnetic) contain polar-vector and axial-vector components (see e.g. Refs. \[6, 7\]). The nuclear wave functions for the initial $|J_i\rangle$ and the final $|J_f\rangle$ states, in the present work are constructed within the context of the quasi-particle random phase approximation (QRPA) \[6, 7\]. In Eq. (12) $\Phi^\mu$ stands for the muon wave function. Usually, in the allowed muon capture process, an average value for the 1s state, i.e. $\langle \Phi^\mu_{1s} \rangle^2$, is taken out of the integrals of Eq. (12) \[1, 3\]. This is given in terms of the effective nuclear charge $Z_{\text{eff}}$ which sees the bound muon \[5\]. In the present work, however, exact wave functions for the bound $\mu^-$ orbiting at the innermost atomic orbits of a muonic atom are obtained by solving the Schrödinger and Dirac equations as is described below \[5\].

**3. Computation of the exact wave function for a bound $\mu^-$**

3.1. **Solving the Schrödinger equation**

For non-relativistic bound muons we can use only the large component of the muon spinor (the small one is assumed to be zero) in the calculations of the ordinary muon capture rates. In such cases, one solves the Schrödinger equation which for the reduced radial muon wave function $u(r) = r\Phi^\mu(r)$ is written as \[5\]

$$
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2}u(r) + V(r)u(r) = Eu(r). \tag{13}
$$

where $V(r)$ represents the extended Coulomb potential originating from the finite-size nuclear charge-density ($E$ is the energy eigenvalue) and $\mu$ denotes the reduced mass of the muon-nucleus system. The latter equation is solved by assuming that $u(r)$ can be written as a superposition of appropriately parametrized sigmoid functions and applying the same techniques as in solving the Dirac equation (see below) \[5, 8\].

3.2. **Solving the Dirac equation**

In the present contribution we consider both radial components of the bound muon wave function, the large, $f(r)$, and the small, $g(r)$. They are given from the solution of the Dirac equations \[3, 9\]

$$
\frac{d}{dr}f(r) + \frac{1}{r}f(r) = \frac{1}{\hbar}[\mu c^2 - E + V(r)]g(r) \tag{14}
$$

$$
\frac{d}{dr}g(r) - \frac{1}{r}g(r) = \frac{1}{\hbar}[\mu c^2 + E - V(r)]f(r), \tag{15}
$$

The solution of these equations is obtained by utilizing neural network techniques and genetic algorithms in order to optimize the parametric expressions discussed in Sect. 3.1. In Fig. 1, the large and small components of the Dirac spinor for a muon orbiting in the $\Phi_{1s}$ (left) and $\Phi_{2p}$ (right) orbits of a $^{28}\text{Si}$ muonic atom, are shown. For comparison, the wave function obtained by solving the Schrödinger equation (13) is also shown in the case of the $\Phi_{1s}$ wave function.
4. Construction of the many body nuclear wave functions

The second basic ingredient in our present study is the construction of the initial and final nuclear states entering the capture rates of Eqs. (11) and (12). We deduce them within the QRPA, the main features of which are outlined in Refs. [6, 7].

The nuclear ground state of even-even isotopes is reliably calculated from the solution of the BCS equations obtained with the single-particle energies of the effective field of a Coulomb-corrected Woods-Saxon potential plus the pairing interaction (monopole) part of a one-meson exchange Bonn CD two-body potential. Similarly, the final (excited) nuclear states of an even-even isotope are computed by solving the QRPA equations as described e.g. in Refs. [6, 7]. Results for partial and total muon capture rates in a set of nuclei (\(^{28}\)Si, \(^{64}\)Zn, \(^{98}\)Mo, \(^{124}\)Sn, \(^{156}\)Gd, \(^{208}\)Pb, etc.), obtained as discussed above, will be published elsewhere [9].

5. Summary

The ordinary \(\mu^-\) capture by atomic nuclei is studied in currently interesting nuclei (\(^{28}\)Si, \(^{64}\)Zn, \(^{98}\)Mo, \(^{124}\)Sn, \(^{156}\)Gd, \(^{208}\)Pb, etc.) [9]. The nuclear wave functions are computed within the context of the QRPA by employing the Bonn CD residual interaction that is a realistic description of the two-body nuclear forces. For the reduced matrix elements of the relevant tensor multipole operators, coming out of the Donnelly-Walecka projection method, we employ the computational approach developed by our group recently [8]. The required muon wave functions are obtained by solving the Dirac equation using neural network techniques and genetic algorithms. As a special case in our present work, we illustrate the \(\Phi_{1s}\) and \(\Phi_{2p}\) wave functions for a muon bound in the \(^{28}\)Si nucleus. Results for the capture rates are expected to be published soon.

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