LAwson HOMology FOR PROJECTive VARIETIES WITH 
C*-action

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Abstract. The Lawson homology of a smooth projective variety with a C*-action is given in terms of that of the fixed point set of this action. We also consider such a decomposition for the Lawson homology of certain singular projective varieties with a C*-action. As applications, we calculate the Lawson homology and higher Chow groups for several examples.

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1. INTRODUCTION

Let $X$ be a smooth complex projective variety of dimension $n$ with a holomorphic C*-action. Let $\varphi_t : X \to X$ be the flow corresponding to the C*-action.

Note that for a given C*-action, the flow $\varphi_t$ can be decomposed into an angular $S^1$ and a radial flow. Averaging a Kähler metric over $S^1$ and applying an argument of Frankel [Fra], we find a function $f : X \to \mathbb{R}$ of Bott-Morse type whose gradient generates the radial action. Recall that a Bott-Morse is a real value smooth function whose critical point set is a closed (real) submanifold and whose Hessian is non-degenerate in the normal direction.

The fixed point set $F$ of the action is assumed to be nontrivial. Let $F_1, \ldots, F_{\nu}$ be the connected components of $F$ and set $\lambda_j := n - \frac{1}{2} I_j - \dim C_{F_j}$, where $I_j$ denote the index of $f$ on $F_j$. Note that $I_j$ is always an even number.

Let $Z_r(X)$ be the space of all algebraic $r$-cycles on $X$. The Lawson homology $L_r H_k(X)$ of $r$-cycles is defined by

$$L_r H_k(X) := \pi_{k-2r}(Z_r(X)) \text{ for } k \geq 2r \geq 0,$$

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where $Z_r(x)$ is provided with a natural topology so that it is an abelian topological group (cf. [F], [L1] and [L2]). For general background, the reader is referred to [L2]. For $r < 0$, let $L_rH_k(X) := H_k(X, \mathbb{Z})$ and $L_rH_k(X)_0 := L_rH_k(X) \otimes \mathbb{Q}$.

The first result in this note is the following Lawson Homology Basis Formula:

**Theorem 1.1.** Let $X$ be a smooth complex projective variety of dimension $n$ with a holomorphic $\mathbb{C}^*$-action and let $F_j$ and $\lambda_j$ be as above. There are isomorphisms of Lawson homology groups

$$\bigoplus_{j=1}^\nu L_{r-\lambda_j}H_{k-2\lambda_j}(F_j) \cong L_rH_k(X)$$

for all $k \geq 2r \geq 0$.

**Remark 1.2.** For $r = 0$, by Dold-Thom theorem, $L_0H_k(Y) \cong H_k(Y, \mathbb{Z})$ for any $k \in \mathbb{Z}$ and any projective varieties $Y$. In this case, the isomorphism in Theorem 1.1 is called the Homology Basis Formula, which holds for $X$ is a compact Kähler manifold (cf. [CS]), by using the Bialynicki-Birula decomposition (cf. [B-B]). Such a result has been shown to hold even on Riemannian manifold admitting a generalized Morse-Stokes flow [HaLa].

Now we consider certain singular projective varieties. A **filtration** on a projective variety $X$ is a nest family $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ of closed algebraic subsets $X_i - X_{i-1}$ is locally closed in $X$.

**Theorem 1.3.** Let $X$ be a (not necessarily smooth) projective variety which admits a filtration $\{X_i\}_{i=0}^\nu$ such that $p_i : X_i - X_{i-1} \rightarrow F_i$ is a locally trivial fibration over a projective variety $F_i$ with fibers $\mathbb{C}^{\lambda_i}$. Then there is isomorphism of Lawson homology groups

$$\bigoplus_{j=1}^\nu L_{r-\lambda_j}H_{k-2\lambda_j}(F_j) \cong L_rH_k(X)$$

for all $k \geq 2r \geq 0$.

**Remark 1.4.** This is the Lawson homology analog of the homology basis theorem for a space with a filtration (cf. [CG]).

The next result is on Lawson homology for certain singular complex varieties with $\mathbb{C}^*$-actions. Recall that a $\mathbb{C}^*$-action $\mathbb{C}^* \times X \rightarrow X$ on a projective variety $X$ is called **singularity preserving** as $t \rightarrow 0$ if there exists an equivariant Whitney stratification of $X$ such that for every stratum $A$, and for every point $x \in A$, the limit $x_0 := \lim_{t \rightarrow 0} t \cdot x$ is also in $A$. In this situation, $X_j^+ \rightarrow F_j$ is an affine space bundle of fiber dimension which is denoted by $\lambda_j$, where $F_1, ..., \nu$ are the fixed point components and $X_j^+ = \{ x \in X | \lim_{t \rightarrow 0} t \cdot x \in F_j \}$ (cf. [CG]).

**Corollary 1.5.** If $X$ admits a singularity preserving $\mathbb{C}^*$-action as $t \rightarrow 0$ with fixed point components $F_1, ..., \nu$ such that for any stratum $A$ of $X$ and for any fixed point component $F_j$, either $A \cap F_j = \emptyset$ or $(A \cap F_j)^+ = A \cap X_j^+$, then we have

$$\bigoplus_{j=1}^\nu L_{r-\lambda_j}H_{k-2\lambda_j}(F_j) \cong L_rH_k(X)$$

for all $k \geq 2r \geq 0$.

The proof of these results is given in section 2. Applications and examples are given in section 3.
2. Proof of main results

In this section we first review Bialynicki-Birula decomposition’s decomposition and Carrell–Sommejé’s construction in proving the Homology Basis Formula and then give a proof of Theorem [10].

Let \( X \) be a smooth complex projective variety with a nontrivial \( \mathbb{C}^* \)-action \( \mathbb{C}^* \times X \rightarrow X, (\lambda, x) \mapsto \lambda \cdot x \). Denote by \( F_j, j = 1, 2, ..., \nu \) the connected components of \( F \). There are two \( \mathbb{C}^* \)-invariant decompositions of \( X \), the plus and minus decompositions (cf. [B-B]):

\[
X = \bigcup_{j=1}^\nu X_j^+, \quad \text{where} \quad X_j^+ = \{ x \in X | \lim_{\lambda \rightarrow 0} \lambda \cdot x \in F_j \}
\]

and

\[
X = \bigcup_{j=1}^\nu X_j^-, \quad \text{where} \quad X_j^- = \{ x \in X | \lim_{\lambda \rightarrow \infty} \lambda \cdot x \in F_j \}.
\]

For each \( j \), let \( n_j = \dim F_j \) and set \( \lambda_j = \dim X_j^+ - n_j \), which coincides with the one defined in the introduction by using the index of the associated Morse function (cf. [CS, HaLa]). Then \( \dim X_j^- = n - \lambda_j \).

Assume that \( X_j \) is one of \( X_j^+ \) or \( X_j^- \). Each \( X_j \) is a smooth quasi-projective variety of \( X \) Zariski open in its closure; the natural map \( p_j : X_j \rightarrow F_j \) can be given the structure of an algebraic \( \mathbb{C}^* \)-equivariant vector bundle of rank \( \lambda_j \); and the normal bundle of \( F_j \) in \( X_j \) is a subbundle of the normal bundle of \( F_j \) in \( X \).

**Theorem 2.1** ([B-B]). Let \( X \) be a smooth projective variety over \( \mathbb{C} \) admitting a nontrivial \( \mathbb{C}^* \)-action. Then

1. \( F \) is a finite disjoint union \( F = \bigsqcup_{j=1}^{\nu} F_j \) of smooth projective varieties.
2. After a suitable numbering of the component of the fixed point set \( F = \bigsqcup_{j=1}^{\nu} F_j \), the union \( X_i = \bigcup_{j=1}^\nu X_j^+ \) is a closed subvariety of \( X \) for all \( i = 1, 2, ..., \nu \).

Now we briefly review Chow motives, Lawson homology and their relations. Let \( X \) and \( Y \) be smooth projective varieties. A **correspondence** \( \Gamma \) from \( X \) to \( Y \) is a cycle (or an equivalent class of cycles depending on the context) on \( X \times Y \). We denote the group of correspondences of rational equivalence classes between varieties \( X \) and \( Y \) by \( \text{Corr}_d(X, Y) := CH_{d \cdot X + d}(X \times Y) \).

For \( \Gamma \in \text{Corr}_d(X, Y) \) and \( \alpha \in L_p H_k(X) \), the push-forward morphism is defined by

\[
\Gamma_* : L_p H_k(X) \rightarrow L_{p+d} H_{k+2d}(Y), \quad \Gamma_*(\alpha) = p_{2*}(p_1^* \alpha \cdot \Gamma),
\]

where \( \cdot \) is the intersection in Lawson homology (cf. [FG]). Moreover, from the construction, the action \( \Gamma_* \) depends only on the algebraic equivalent class \( \Gamma \).

Let \( \mathcal{V} \) denote the category of (not necessarily connected) complex smooth projective varieties. Let \( X \) and \( Y \) be smooth projective varieties. Suppose \( X = \bigsqcup X_\alpha \) is the decomposition of \( X \) into connected components. The group of correspondences of degree \( r \) from \( X \) to \( Y \) is defined as

\[
\text{Corr}^r(X, Y) := \oplus CH^{\dim X_\alpha + r}(X_\alpha \times Y).
\]

The composition of correspondences \( f \in \text{Corr}^r(X, Y) \) and \( g \in \text{Corr}^s(Y, Z) \) gives a correspondence in \( \text{Corr}^{r+s}(X, Z) \). A correspondence \( p \in \text{Corr}^0(X, X) \) is called a projector of \( X \) if \( p^2 = p \). The category of **Chow motives** \( CHM \) is given as
follows. Objects in $\text{CHM}$ are triples $(X, p, r)$, also denoted by $h(X, p)(-r)$, where $X \in \mathcal{V}$, $p$ is a projector of $X$, and $r \in \mathbb{Z}$. In particular, the motive $h(X, \text{id}_X)(-r)$ is simply denoted by $h(X)(-r)$. Morphisms in $\text{CHM}$ are defined as

$$\text{Hom}_{\text{CHM}}((X, p, r), (Y, q, s)) := q \circ \text{Corr}^{s-r}(X, Y) \circ p.$$ 

The composition of morphisms is defined using the composition of correspondences.

The following result proved by N. A. Karpenko in [K]:

**Theorem 2.2 (Karpenko).** Let $X$ be a smooth projective variety. Assume $X$ admits a filtration by closed subvarieties $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_\nu = X$ such that there exist flat morphisms $f_i : X_i - X_{i-1} \rightarrow F_i$, of relative dimension $\lambda_i$ over smooth projective varieties $F_i$ ($1 \leq i \leq \nu$), such that the fiber of every $f_i$ over every point $y$ of $F_i$ is isomorphic to the affine space $\mathbb{C}^{\lambda_i}$. Then there exists an isomorphism in $\text{CHM}$:

$$h(X) \simeq \bigoplus_{i=0}^\nu h(F_i)(\lambda_i).$$

This implies the following corollary.

**Corollary 2.3 (HuLi).** Using the notations in Theorem 2.2 we have

$$L_r H_k(X) \simeq \bigoplus_{i=0}^\nu L_{r-\lambda_i} H_{k-2\lambda_i}(F_i).$$

Therefore, we get the isomorphism in Theorem 1.1.

**Remark 2.4.** Equation (2) implies the decomposition of any oriented cohomology theory (cf. [NZ]). The Lawson homology is such a theory. The combination of Theorem 2.1 and 2.2 implies a motivic decomposition for smooth complex projective variety with $\mathbb{C}^\ast$-action (cf. [BF]).

The proof of Theorem 1.1

Set $W_i := X_i - X_{i-1}$ and let $\Gamma_i$ be the topological closure of $\{(x, p_i(x)) | x \in W_i\} \subset X \times F_i$. This is a closed algebraic subvariety of $X \times F_i$. Since $W_i \subset X$ is locally closed, the graph of $p_i$ is Zariski open in $\Gamma_i$. Let $g_i : \Gamma_i \rightarrow F_i$ be the projection onto $X_i$ and $g_i^\ast : L_{r-\lambda_i} H_{k-2\lambda_i}(F_i) \rightarrow L_r H_k(\Gamma_i)$ the induced map on Lawson homology. The existence of such a “wrong way” homomorphism $g_i^\ast$ in this case follows essentially from the constructions in [CG], since for each variety $V \subset F_i$ of pure dimension $q$, $g_i^{-1}(V) = \{(p_i^{-1}(y), y) | y \in V\}$ is a variety of pure dimension $q + \lambda_i$ in $\Gamma_i$. This induces a continuous map $g_i^\ast : Z_q(F_i) \rightarrow Z_{q+\lambda_i}(\Gamma_i)$ in the equidimensional topology for cycle spaces. This topology coincides with the natural Chow topology we used in defining Lawson homology (cf. [LF3]). By composing the homomorphism $L_r H_k(\Gamma_i) \rightarrow L_r H_k(X)$ induced by the projection $\Gamma_i \rightarrow X$, we get a homomorphism $\mu_i : L_{r-\lambda_i} H_{k-2\lambda_i}(F_i) \rightarrow L_r H_k(X)$ for each $i = 1, 2, \ldots, \nu$. Note that the image of $\mu_i$ is in $L_r H_k(X_i)$ so it factors through $\beta_i : L_{r-\lambda_i} H_{k-2\lambda_i}(F_i) \rightarrow L_r H_k(X_i)$.

Now we consider the following exact sequences (Comparing to that in the proof of Theorem 1 in [CG])

$$\begin{array}{cccccc}
L_r H_{k+1}(W_i) & \xrightarrow{\partial} & L_r H_k(X_{i-1}) & \xrightarrow{\alpha_{i-1}} & L_r H_k(X_i) & \xrightarrow{\alpha_i} & L_r H_k(W_i) \\
0 & \xrightarrow{\delta_i} & \oplus_{j<i} A_{r,k,j} & \oplus_{j\leq i} A_{r,k,j} & A_{r,k,i} & \rightarrow & 0
\end{array}$$
where \( A_{r,k,j} := L_{r-\lambda_j} H_{k-2\lambda_j} (F_j) \), \( \alpha_i = \oplus_{j,s} \beta_j \), \( \beta_i \) are given above, and the map \( \delta_i \) is the composed map of \( \beta \) and the restriction \( L_r H_k (X_i) \to L_r H_k (W_i) \). From the construction, we see that the map \( \delta_i \) is induced by the fiber bundle map \( p_i : W_i \to F_i \). Since the induced map \( p_i^* : L_q H_k (F_i) \to L_{q+\lambda_i} H_{k+2\lambda_i} (W_i) \) is an isomorphism for all \( k \geq 2q \geq 0 \) (cf. [FG], Prop. 2.3), we see that \( \delta_i \) is an isomorphism. By induction on \( i \), \( \alpha_{i-1} \) is an isomorphism. Since the diagram commutes, we have \( \partial = 0 \). Now we complete the proof of Theorem 1.3 by the Five lemma and induction on \( i \).

The proof of Corollary 1.3. It follows from Theorem 4 in [CG] and Theorem 1.3 above.

5. Applications and Examples

3.1. Projective Cone. Let \( \mathbb{P}^n \subset \mathbb{P}^{n+1} \) be a linear hyperplane defined by \( z_{n+1} = 0 \), where \([z_0 : z_1 : \cdots : z_{n+1}]\) is the homogeneous coordinates of \( \mathbb{P}^{n+1} \). Set \( P = [0 : 0 : \cdots : 1] \) and note that \( P \in \mathbb{P}^{n+1} \) is a point out of the hyperplane \( \mathbb{P}^n \). Let \( X \subset \mathbb{P}^n \) be a projective algebraic variety and let \( \Sigma^P (X) \) be the suspension of \( X \) (or Complex cone on \( X \)) with vertex \( P \).

Under this setting, one has the following result.

**Corollary 3.1.** If \( k \geq 2r \geq 0 \), then

\[
L_r H_k (\Sigma^P (X)) \cong \begin{cases} 
L_{r-1} H_{k-2} (X), & \text{if } r > 0, \\
H_{k-2} (X, \mathbb{Z}), & \text{if } r = 0 \text{ and } k > 0, \\
\mathbb{Z}, & \text{if } r = 0 \text{ and } k = 0.
\end{cases}
\]

**Remark 3.2.** The Lawson homology theory was originated from this fundamental result, first proved by Blaine Lawson (cf. [L1]). It is now called the Algebraic Suspension Theorem. Since the proof of our main theorems (Theorem 1.3 and 1.4) depends implicitly on this result. The “proof” below is a rephrase of Lawson’s theorem in our words.

**Proof.** Consider the action of \( \mathbb{C}^* \) on \( \mathbb{P}^{n+1} \) by

\[
\Phi_t : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}, \quad [z_0 : \cdots : z_n : z_{n+1}] \mapsto [z_0 : \cdots : z_n : tz_{n+1}].
\]

Restricted to \( \Sigma^P (X) \), we get an induced \( \mathbb{C}^* \)-action from \( \Phi_t \). The fixed point set of the induced action is \( F = F_1 \cup F_2 \), where \( F_1 \cong X \) and \( F_2 = P \). Using notations in Theorem 1.3, we see from the definition that \( X^+_1 \to F_1 \) is a fiber bundle with fibers \( \mathbb{C} \) and \( X^+_2 = P \). By Theorem 1.3 we have

\[
L_r H_k (\Sigma^P (X)) \cong L_{r-1} H_{k-2} (X) \oplus L_r H_k (P).
\]

If \( r > 0 \), then \( L_r H_k (P) = 0 \). If \( r = 0 \) and \( k > 0 \), then from definition \( L_r H_k (P) = 0 \) and \( L_{r-1} H_{k-2} (X) \cong H_{k-2} (X, \mathbb{Z}) \). In particular, since \( H_{k-2} (X, \mathbb{Z}) = 0 \) for \( k = 1 \), \( L_0 H_1 (\Sigma^P (X)) = 0 \) for any \( X \). If \( r = k = 0 \), then \( L_0 H_0 (\Sigma^P (X)) = L_0 H_0 (P) = \mathbb{Z} \).

3.2. Projective varieties admitting a Cell decomposition. A projective variety \( X \) is said to admit a cell decomposition if there exists a filtration \( \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_N = X \) such that \( X_i - X_{i-1} \) is isomorphic to \( \mathbb{C}^{\lambda_i} \) for all \( i \) (where \( 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \)). Sometimes we also call such an \( X \) is cellular. The Lawson homology for those \( X \) was proved to isomorphic to the singular homology with integral coefficients ([LF2], §5).
Corollary 3.3. Let $X$ admit a cell decomposition defined as above. Then we have
\[ L_r H_k(X) \cong \mathbb{Z}^{n_i} \]
for all $k \geq 2r \geq 0$, where $n_i$ is the number of $i$ such that $k = 2\lambda_i$. In particular, $L_r H_k(X) = 0$ if $k$ is odd.

Proof. By applying Theorem 1.3 to the special case that all $F_i$ are points, we have
\[ L_r H_k(X) \cong \bigoplus_{i=0}^{n} L_{r-\lambda_i} H_{k-2\lambda_i}(pt). \]

Note that $L_r H_k(X) \cong \mathbb{Z}^{n_i}$.

Example 3.4. Let $X \subset \mathbb{P}^{2d+1}$ be a split projective quadric of dimension $2d$, defined as the hypersurface in $\mathbb{P}^{2d+1}$ by
\[ X = \left\{ [x_0 : \cdots : x_d : y_0 \cdots : y_d] \in \mathbb{P}^{2d+1} \mid \sum_{i=0}^{d} x_i y_i = 0 \right\}. \]

Then for $k \geq 2r \geq 0$, we have
\[ L_r H_k(X) \cong L_r H_k(\mathbb{P}^d) \oplus L_{r-d} H_{k-2d}(\mathbb{P}^d) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 2d, \\ \mathbb{Z} & \text{if } 0 \leq k \leq 4d \text{ even but } k \neq 2d, \\ 0 & \text{otherwise.} \end{cases} \]

Proof. The quadric $X$ admits a $(\mathbb{C}^*)^d = (\mathbb{C}^*)^{d+1}/\mathbb{C}^*$-action given by
\[ (\mathbb{C}^*)^d \times X \to X, [x_0 : \cdots : x_d : y_0 \cdots : y_d] \mapsto [t_0 x_0 : \cdots : t_d x_d : y_0 \cdots : y_d]. \]

For simplicity, we denote $[x_0 : \cdots : x_d : y_0 \cdots : y_d]$ by $[x : y]$ and $[t_0 x_0 : \cdots : t_d x_d : y_0 \cdots : y_d]$ by $[tx : y]$.

The fixed point set $F$ of this action is $F_1 \cup F_2$, where
\[ F_1 = \{ [x_0 : \cdots : x_d : y_0 \cdots : y_d] | x_0 = \cdots = x_d = 0 \} \]
and
\[ F_2 = \{ [x_0 : \cdots : x_d : y_0 \cdots : y_d] | y_0 = \cdots = y_d = 0 \}. \]

Both $F_1$ and $F_2$ are isomorphic to $\mathbb{P}^d$. There is a decomposition of $X = X_1^+ \cup X_2^+$ coming from this action, where $X_1^+ := \{ [x : y] \mid \lim_{t \to 0} [tx : y] \in F_1 \}$ and $X_2^+ := \{ [x : y] \mid \lim_{t \to 0} [tx : y] \in F_2 \}$. A direct calculation shows that $X_1^+ = F_1$ and $X_2^+$ is a locally trivial fibration over $F_2$ with fiber $\mathbb{C}^d$. By Theorem 1.3 we have
\[ L_r H_k(X) \cong L_r H_k(\mathbb{P}^d) \oplus L_{r-d} H_{k-2d}(\mathbb{P}^d). \]

Since $L_r H_k(\mathbb{P}^d) \cong H_k(\mathbb{P}^d, \mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq k \leq 2d$ even and 0 otherwise, we get Equation (4).
The following example relates to the above one but the projective variety is singular.

**Example 3.5.** Let $X \subset \mathbb{P}^{2d+1}$ be a projective variety of dimension $2d$, defined as the hypersurface in $\mathbb{P}^{2d+1}$ by

$$X = \left\{ [x_0 : \cdots : x_d : y_0 : \cdots : y_d] \in \mathbb{P}^{2d+1} \mid \sum_{i=0}^{d} x_i^m y_i = 0 \right\}$$

for integers $m > 1$. Then for $k \geq 2r \geq 0$, we have

$$(6)$$

$$L_r H_k(X) \cong L_r H_k(\mathbb{P}^d) \oplus L_{r-d} H_{k-2d}(\mathbb{P}^d) = \begin{cases} 
\mathbb{Z}^2 & \text{if } k = 2d, \\
\mathbb{Z} & \text{if } 0 \leq k \leq 4d \text{ even but } k \neq 2d, \\
0 & \text{otherwise.}
\end{cases}$$

**Proof.** We use the notations in the proof of Example 3.4. The hypersurface $X \subset \mathbb{P}^{2d+1}$ admits a $(\mathbb{C}^*)^d = (\mathbb{C}^*)^{d+1}/\mathbb{C}^*$-action given by

$$(\mathbb{C}^*)^d \times X \to X, (t, [x : y]) \mapsto [tx : y].$$

The fixed point set $F$ of this action is $F_1 \cup F_2$, where

$$F_1 = \{ [x : y] | x = 0 \} \text{ and } F_2 = \{ [x : y] | y = 0 \}.$$ 

A simple calculation shows that the set of singular points of $X$ is $F_1$. Let $X_1^+$ and $X_2^+$ be the same meaning as in the proof of Example 3.4. Then $X = X_1^+ \cup X_2^+$, where $X_2^+$ is a locally trivial fibration over $F_2$ with fibers $\mathbb{C}^d$. Although $X$ is singular, both $X_1^+$ and $X_2^+$ are smooth. By Theorem 1.3 we have

$$L_r H_k(X) \cong L_r H_k(F_1) \oplus L_{r-d} H_{k-2d}(F_2) \cong L_r H_k(\mathbb{P}^d) \oplus L_{r-d} H_{k-2d}(\mathbb{P}^d).$$

This completes the calculation in this example. \qed

The following is another example of a singular projective variety which admits a cell decomposition.

**Example 3.6.** Let $G$ be a connected reductive group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $B$ be the variety of all Borel subalgebras of $\mathfrak{g}$. For each nilponent element $N \in \mathfrak{g}$, we set $B_N := \{ b \in B | N \in b \}$. Then we have

$$L_r H_k(B_N) \cong H_k(B_N, \mathbb{Z})$$

for all $k \geq 2r \geq 0$.

**Proof.** It was shown that $B_N$ admits a cell-decomposition (cf. [DL]). \qed

We end this subsection by the following example.

**Example 3.7.** Let $S$ be a smooth rational projective surface and let $Hilb^d(S)$ be the Hilbert scheme of $d$ points. Then we have

$$L_r H_k(Hilb^d(S)) \cong \mathbb{Z}^{c_k,d(S)},$$

where $c_k,d(S)$ is the coefficient of $z^k t^d$ in the power series expansion

$$\prod_{k=1}^{\infty} \frac{1}{(1 - z^{2k+2} t^k)(1 - z^{2k} t^k)}.$$
Proof. There is a \((\mathbb{C}^*)^2\)-action on \(S\) whose fixed point set \(F_S\) is finite. A result of Fogarty says that \(\text{Hilb}^d(S)\) is a smooth projective variety. Since \(S\) admits a \((\mathbb{C}^*)^2\)-action with isolated fixed points, this action induces a natural \((\mathbb{C}^*)^2\)-action on \(\text{Hilb}^d(S)\), which also has finite isolated fixed points. To see this, we note first that a fixed point \(P\) in \(\text{Hilb}^d(S)\) whose support must be in \(F_S\). We need to show that the number of ideals \(I_P \subset \mathbb{C}[S]\), invariant under \((\mathbb{C}^*)^2\), is finite. Since \(F_S\) is finite, we only need to show the case that the number of invariant ideals \(I \subset \mathbb{C}[S]\) is finite, where the reduced subscheme structure on \(\text{Spec}(\mathbb{C}[S]/I)\) is a point \(s\) in \(F_S\). Note that the affine coordinate ring at \(s\) is \(\mathbb{C}[T_1, T_2]\). The \((\mathbb{C}^*)^2\)-action on \(\mathbb{C}[T_1, T_2]\) is given by \((T_1, T_2) \mapsto (\lambda T_1, \mu T_2)\). An invariant ideal \(I\) is generated by monomials \(T_1^p T_2^q\) for some \(p, q \geq 0\). Let \(m\) be the maximal ideal such that \(\text{Spec}(\mathbb{C}[T_1, T_2]/m)\) is \(s\). One sees that \(m = (T_1, T_2)\).

If the dimension of \(\mathbb{C}[T_1, T_2]/I\) is less than or equal to \(N\), then \(I \subset (T_1, T_2)^N\). There are only finite number of such ideals corresponding to the pairs \((p, q)\) such that \(p + q \leq N\). Now \(N = d\) is given, so the number of invariant ideals \(I\) is finite. Therefore there are finite fixed point in \(\text{Hilb}^d(S)\) under the induced \((\mathbb{C}^*)^2\)-action. So there is no torsion elements in \(L_r H_k(\text{Hilb}^d(S))\) and its rank coincides to that of \(H_k(\text{Hilb}^d(S))\) for all \(k \geq 2r \geq 0\).

To identify \(L_r H_k(\text{Hilb}^d(S))\), it is enough to compute \(b_k(\text{Hilb}^d(S))\) for \(k\) even since \(b_{\text{odd}}(\text{Hilb}^d(S)) = 0\). These numbers has been calculated in [Ch]. That is, \(b_k(\text{Hilb}^d(S)) = c_{k, d}(S)\). From the defining generating function for \(c_{k, d}(S)\), one observes that \(c_{k, d}(S) = 0\) for odd \(k\). Hence

\[
L_r H_k(\text{Hilb}^d(S)) \cong \mathbb{Z}^{c_{k, d}(S)}. 
\]

\(\square\)

Remark 3.8. The Lawson homology of \(\text{Hilb}^d(S)\) with rational coefficients was computed implicitly in [HuL4]. The integral homology of the Hilbert scheme of \(d\) points on \(\mathbb{P}^2\) was computed in [ES], where the idea in this example is from.

3.3. Fiber bundles with cellular fibers. Let \(E\) be a fiber bundle over a quasi-projective variety \(X\) with cellular fibers \(Y\), where a filtration \(\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_N = X\) such that \(X_i - X_{i-1}\) is isomorphic to \(\mathbb{C}^{\lambda_i}\) for all \(i\).

Then we have isomorphisms

\[
(7) \quad L_r H_k(E) \cong \bigoplus_{i=0}^{N} L_{r-i} H_{k-2\lambda_i}(X), \quad \forall k \geq 2r \geq 0.
\]

Proof of Equation (7). By induction and long localization sequences for Lawson homology, we reduced ourselves to the case that \(E\) is trivial. We can further reduce to the case that \(X\) is projective. In this case, Equation (7) follows from Theorem 1.33.\(\square\)

In particular, this includes the Projective Bundle Formula (cf. [FG Prop. 2.5]), the product of any projective variety with a cellular variety, and the \(d\)-fold symmetric product of a smooth projective algebraic curve for \(d\) large.

3.4. Toric varieties. In this subsection, we compute Lawson homology groups for toric varieties \(X\) of \(\dim X = n\). For background on toric varieties, the reader is referred to Fulton’s book [Fu].
If $X$ is a smooth projective toric variety, then $X$ admits a cell decomposition. So the calculation of Lawson homology for $r$-cycles coincides with that of singular homology with the corresponding integral coefficients for all $r \geq 0$, as pointed out above. Explicitly,

**Example 3.9.** Let $X = X(\Delta)$ be a smooth projective toric variety associated to the fan $\Delta$ and denoted by $d_k$ the number of $k$-dimensional cones in $\Delta$. Then

$$L_rH_k(X) \cong \begin{cases} \mathbb{Z}^{b_{2m}(X)}, & \text{for } k = 2m \text{ even,} \\ 0, & \text{for } k \text{ odd,} \end{cases}$$

where $b_{2m}(X) = \sum_{i=m}^{n} \binom{i}{m} d_{n-i}$ is the $2m$-th Betti number of $X$.

For general projective toric variety $X$, we can not expect that there is an isomorphism between Lawson homology and the singular homology in $\mathbb{Z}$-coefficient. However, as hinted in [FM], we have the following result:

**Proposition 3.10.** The Lawson homology $L_rH_k(X)\mathbb{Q}$ with rational coefficients of a simplicial toric variety $X$ is isomorphic to the rational homology $H_k(X,\mathbb{Q})$ for $k \geq 2r \geq 0$.

**Proof.** First we note that for any $X = X(\Delta)$ where $\Delta$ is simplicial, there is a filtration for $X$, i.e., $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_N = X$ such that $Y_i := X_i - X_{i-1}$ is a quotient $\mathbb{C}^{n-k}/G_i$ of an affine space by a finite group (cf. [FM] §5.2). For such a quotient, the Lawson homology and Borel-Moore homology are the spaces invariant by the group:

$$L_*H_*(\mathbb{C}^m/G) \cong \{L_*H_*(\mathbb{C}^m)\}^G \text{ (cf. [HuLi] Prop. 3.1) }$$

and

$$H_*^{BM}(\mathbb{C}^m/G,\mathbb{Q}) \cong \{H_*^{BM}(\mathbb{C}^m,\mathbb{Q})\}^G,$$

where $H_*^{BM}(\cdot,\mathbb{Q})$ denotes the Borel-Moore homology with rational coefficient. Since we only consider the Borel-Moore homology for non-compact algebraic set, we simply denote it by $H_*(\cdot,\mathbb{Q})$.

Note that we have the following commutative diagram of long exact sequences:

$$\cdots \rightarrow L_rH_{k+1}(Y_i)_{\mathbb{Q}} \rightarrow L_rH_k(X_{i-1})_{\mathbb{Q}} \rightarrow L_rH_k(X_i)_{\mathbb{Q}} \rightarrow L_rH_{k-1}(X_{i-1})_{\mathbb{Q}} \rightarrow \cdots \rightarrow H_{k+1}(Y_i,\mathbb{Q}) \rightarrow H_k(X_{i-1},\mathbb{Q}) \rightarrow H_k(X_i,\mathbb{Q}) \rightarrow H_{k-1}(X_{i-1},\mathbb{Q}) \rightarrow \cdots$$

where the first and fourth vertical isomorphisms follows from above arguments, the second and fifth vertical isomorphisms follows from the induction.

Now the proposition follows from the Five lemma and the induction on $i$. \[\square\]

The next result is about an isomorphism between Lawson homology and higher Chow groups for toric varieties. Recall that (cf. [BI]) for each $m \geq 0$, let

$$\Delta[d] := \{t \in \mathbb{C}^{d+1} \mid \sum_{i=0}^{m} t_i = 1\} \cong \mathbb{C}^d.$$ 

and let $z^l(X,F)$ denote the free abelian group generated by irreducible subvarieties of codimension-$l$ on $X \times \Delta[d]$ which meets $X \times F$ in proper dimension for each face $F$ of $\Delta[d]$. Using intersection and pull-back of algebraic cycles, we can define face and degeneracy relations and obtain a simplicial abelian group structure for
Let $|z_l(X, *)|$ be the geometric realization of $z_l(X, *)$. Then the higher Chow group is defined as

$$\text{Ch}_l(X, k) := \pi_k(|z_l(X, *)|)$$

and set $\text{Ch}_l(X, k) := \text{Ch}^{n-l}(X, k)$. It was shown by Friedlander-Gabber \[FG\] that there is a natural map from the higher Chow groups to Lawson homology groups

$$FM : \text{Ch}_r(X, m) \to L_rH_{2r+k}(X)$$

for all $r, m \geq 0$.

**Theorem 3.11.** Let $X$ be an arbitrary toric variety. The higher Chow group $\text{Ch}_r(X, m)$ of $X$ is isomorphic to the Lawson homology group $L_rH_{2r+m}(X)$ of $X$. In particular, the algebraic equivalence coincide with the rational equivalence for projective toric varieties.

**Proof.** First we show that

$$\text{Ch}_r((\mathbb{C}^*)^N, k - 2r) \cong L_rH_{k}(((\mathbb{C}^*)^N))$$

for all $k \geq 2r \geq 0$ and $N \geq 0$. This statement follows from the induction and the fact that $L_rH_{k}(X \times \mathbb{C}^*) \cong \text{Ch}_r(X \times \mathbb{C}^*, k - 2r)$ for all $k \geq 2r \geq 0$ if $L_rH_{k}(X) \cong \text{Ch}_r(X, k - 2r)$ for all $k \geq 2r \geq 0$. To see the fact, one notes that there exists a commutative diagram

$$
\begin{array}{cccccc}
\text{Ch}_r(X, m) & \cong & \text{Ch}_r(X \times \mathbb{C}, m) & \cong & \text{Ch}_r(X \times \mathbb{C}^*, m) & \cong & \text{Ch}_r(X, m - 1) \\
L_rH_{k}(X) & \cong & L_rH_{k}(X \times \mathbb{C}) & \cong & L_rH_{k}(X \times \mathbb{C}^*) & \cong & L_rH_{k-1}(X) \\
\end{array}
$$

of long exact sequences from higher Chow groups to Lawson homology groups. By assumption, the first and fourth vertical maps are isomorphisms, where $m = k - 2r$. By the homotopy invariant property of higher Chow groups (cf. \[BI\] and Lawson homology groups (cf. \[LI\] or \[FG\] Prop. 2.3)) and the assumption, we get isomorphisms for the second and fifth vertical maps. Hence by the Five lemma, the middle vertical map is an isomorphism. This completes the proof of the statement in Equation \[S\] for all $k \geq 2r \geq 0$ and $N \geq 0$.

Note that for any toric variety $X = X(\Delta)$, there is a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_1 = \emptyset$ by closed algebraic subsets such that $U_i := X_i - X_{i-1}$ is the disjoint union of orbits $O_{\sigma_i}$, where $\sigma$ runs over the cones of dimension $n - i$. Since an orbit $O_{\sigma_i}$ is isomorphic to $(\mathbb{C}^*)^{n_{\sigma_i}}$ for some integer $n_{\sigma_i}$. Therefore the proposition follows from induction on $i$ and the commutative diagrams of long exact sequences

$$
\begin{array}{cccccc}
\text{Ch}_r(X_i, m + 1) & \cong & \text{Ch}_r(X_{i-1}, m) & \cong & \text{Ch}_r(U_i, m) & \cong & \text{Ch}_r(X_{i-1}, m - 1) \\
L_rH_{k+1}(U_i) & \cong & L_rH_{k}(X_{i-1}) & \cong & L_rH_{k}(U_i) & \cong & L_rH_{k-1}(X_{i-1}) \\
\end{array}
$$

where $m = k - 2r$. The first and fourth vertical isomorphisms follows from Equation \[S\], while the second and fifth vertical isomorphisms are the inductive assumptions. By the Five lemma, we get the isomorphism of the middle one. \[\square\]

**Proposition 3.12.** For any integer $k \geq 2r \geq 0$ and $n \neq 0$, we have the following formula

$$L_rH_{k}((\mathbb{C}^*)^n) = \mathbb{Z}^{a_{r,k,n}},$$

where $a_{r,k,n} := \binom{n}{k-n}$ if $k \geq r + n$ and 0 otherwise.
Proof. First we show that for any projective variety $X$, we have the following isomorphism:

$$L_r H_k(X \times \mathbb{C}^*) \cong L_{r-1} H_{k-2}(X) \oplus L_r H_{k-1}(X). \tag{11}$$

To see this, note that the pair $(X \times \mathbb{C}, X \times \{0\})$, we have the long exact sequence of Lawson homology:

$$
... \xrightarrow{\partial} L_r H_k(X) \xrightarrow{i_*} L_r H_k(X \times \mathbb{C}) \xrightarrow{Res} L_r H_k(X \times \mathbb{C}^*) \xrightarrow{\partial} L_r H_{k-1}(X) \rightarrow ...
$$

where $i : X = X \times \{0\} \rightarrow X \times \mathbb{C}$ is the inclusion, $Res$ is restriction map and $\partial$ is the boundary map.

The long exact sequence of Lawson homology for the pair $(X \times \mathbb{P}^1, X \times \{0\})$ is

$$
... \xrightarrow{\partial} L_r H_k(X) \xrightarrow{i_*} L_r H_k(X \times \mathbb{P}^1) \xrightarrow{Res} L_r H_k(X \times \mathbb{C}) \xrightarrow{\partial} L_r H_{k-1}(X) \rightarrow ...
$$

where $i_\infty : X = X \times \{\infty\} \rightarrow X \times \mathbb{P}^1$ is the inclusion.

From the $C^1$-homotopy invariance of Lawson homology, one gets $i_{0*} = i_{\infty*} : L_p H_k(X) \rightarrow L_p H_k(X \times \mathbb{P}^1)$, where $i_0 : X = X \times \{0\} \rightarrow X \times \mathbb{P}^1$ is the inclusion. From the definition of $i$ and $i_0$, we have $i_* = Res \circ i_{0*}$, where $Res : L_r H_k(X \times \mathbb{P}^1) \rightarrow L_r H_k(X \times \mathbb{C})$ is the restriction map. Hence we obtain

$$i_* = Res \circ i_{0*} = Res \circ i_{\infty*} = 0.$$

Therefore, Equation (12) is broken into short exact sequences

$$0 \rightarrow L_r H_k(X \times \mathbb{C}) \xrightarrow{Res} L_r H_k(X \times \mathbb{C}^*) \xrightarrow{\partial} L_r H_{k-1}(X) \rightarrow 0.$$

This sequence splits since the map $Z_r(X \times \mathbb{C}^*) = Z_r(X \times \mathbb{C})/Z_r(X \times \{0\}) \rightarrow Z_{r-1}(X) \simeq Z_r(X \times \mathbb{C})$ given by $c \mapsto c \cap (X \times \{0\})$ gives a section of the projection $Z_r(X \times \mathbb{C}) \rightarrow Z_r(X \times \mathbb{C})/Z_r(X \times \{0\})$. So we get Equation (11). The proof of the proposition is completed by induction on $n$. 

We set $\chi_p(X) := \sum_{i \geq 2p} (-1)^i \text{rank}(L_p H_k(X))$ whenever $L_p H_k(X)$ are finitely generated and vanishes for $k$ large. In this situation, $\chi_p(X)$ is well-defined. The proposition has the following corollaries.

**Corollary 3.13.** $\chi_p((\mathbb{C}^*)^n)$ is well-defined and $\chi_p((\mathbb{C}^*)^n) = \sum_{i=p}^n (-1)^{n+i}(i)$. In particular, $\chi_p((\mathbb{C}^*)^n) \neq 0$ for $1 \leq p \leq n$.

**Corollary 3.14.** Let $X = X(\Delta)$ be an arbitrary toric variety of dimension $n$. Then

$$\chi_p(X) = \sum_{i=0}^{n-p} \sum_{j=p}^{n-i} (-1)^{n-i+j} d_i \cdot \binom{n-i}{j},$$

where $d_i$ is the number of $i$-dimensional cones in $\Delta$.

**Proof.** Note that there is a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_1 = \emptyset$ by closed algebraic subsets such that $U_i := X_i - X_{i-1}$ is the disjoint union of orbits $O_{\sigma}$, where $\sigma$ runs over the cones of dimension $n-i$. By the long exact sequence of Lawson homology for $(X_i, X_{i-1})$, we get

$$\chi_p(X_i) = \chi(X_{i-1}) + \chi_p(U_i). \tag{13}$$
Since $U_i$ is the disjoint union of orbits of $O_\sigma$, each $O_\sigma$ is isomorphic to $(\mathbb{C}^*)^{n-i}$. Hence $\chi_p(U_i) = d_i \cdot \chi_p((\mathbb{C}^*)^{n-i})$. By taking the sum of Equation (13) for $i$ from 1 to $n$, we get

$$\chi_p(X) = \sum_{i=0}^{n-p} d_i \cdot \chi_p((\mathbb{C}^*)^{n-i}).$$

Now we complete the proof of the corollary by applying Corollary 3.13.

Note that if $p = 0$, $\chi_p(X)$ coincides with the Euler number of $X$. In this case, $\chi_0(X) = d_n$.

3.5. Symmetric products of Homogeneous varieties. Let $X$ be a projective variety and denoted by $SP^d X$ the $d$-th symmetric product of $X$.

**Proposition 3.15.** If $X$ is a complex projective variety which admits a cell decomposition, then we have isomorphisms

$$L_r H_k(SP^d X) \cong H_k(SP^d X, \mathbb{Q})$$

for any $k \geq 2r \geq 0$.

**Proof.** Since $X$ admits a cell decomposition, $X \times X$ also admits a cell decomposition (cf. [LF2, §5]), where $X \times X$ is a $d$-fold self-product of $X$. So we get $L_r H_k(X \times X) \cong H_k(X \times X)$.

Since both Lawson homology and the singular homology are the subspaces invariant by the symmetric group $\Sigma_d$:

$$L_r H_k(SP^d X) \cong \{L_r H_k(X \times X)\}^{\Sigma_d} \quad \text{(cf. [HuLi, Prop. 3.1])}$$

and

$$H_k(SP^d X, \mathbb{Q}) \cong \{H_k(X \times X, \mathbb{Q})\}^{\Sigma_d},$$

we obtain isomorphisms in the proposition.

In the case of Proposition 3.15 the dimension of $\mathbb{Q}$-vector space $L_r H_k(SP^d X) \mathbb{Q}$ is given by MacDonald formula [M].

**Remark 3.16.** The method of the computation for Lawson homology groups of the examples above also works for the higher Chow groups.

**References**

[BB] A. Biaynicki-Birula, Some theorems on actions of algebraic groups. Ann. of Math. (2) 98 (1973), 480–497.

[B] S. Bloch, Algebraic cycles and higher K-theory. Adv. in Math. 61 (1986), no. 3, 267–304.

[Br] P. Brosnan, On motivic decompositions arising from the method of Biaynicki-Birula. (English summary) Invent. Math. 161 (2005), no. 1, 91–111.

[CG] J. B. Carrell and R. M. Goresky, A decomposition theorem for the integral homology of a variety. Invent. Math. 73 (1983), no. 3, 367–381.

[CS] J. B. Carrell and A. J. Sommese, Some topological aspects of $\mathbb{C}^*$ actions on compact Kähler manifolds. Comment. Math. Helv. 54 (1979), no. 4, 567–582.

[Ch] J. Cheah, On the cohomology of Hilbert schemes of points. J. Algebraic Geom. 5 (1996), no. 3, 479–511.

[DLP] C. De Concini; G. Lusztig and C. Procesi Homology of the zero-set of a nilpotent vector field on a flag manifold. J. Amer. Math. Soc. 1 (1988), no. 1, 15–34.

[ES] G. Ellingsrud and S. Strømme, On the homology of the Hilbert scheme of points in the plane. Invent. Math. 87 (1987), no. 2, 343–352.

[Fra] T. Frankel, Fixed points and torsion on Kähler manifolds Ann. of Math. (2) 70 1959 1–8.
LA WRSON HOMOLOGY FOR PROJECTIVE V ARIETIES WITH C∗-ACTION

[13] E. Friedlander, Algebraic cycles, Chow varieties, and Lawson homology. Compositio Math. 77 (1991), no. 1, 55–93.

[FG] E. Friedlander and O. Gabber, Cycle spaces and intersection theory. Topological methods in modern mathematics. (Stony Brook, NY, 1991), 325–370, Publish or Perish, Houston, TX, 1993.

[Fu] W. Fulton, Introduction to toric varieties. Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993. xii+157 pp. ISBN: 0-691-00049-2

[HaLa] F. R. Harvey and H. B. Lawson, Jr. Finite volume flows and Morse theory. Ann. of Math. (2) 153 (2001), no. 1, 1–25.

[HuLi] W. Hu and L. Li, Lawson homology, morphic cohomology and Chow motives. arXiv:0711.0383. Accepted for publication on Nach. Math.

[K] N. A. Karpenko, Cohomology of relative cellular spaces and of isotropic flag varieties. (Russian. Russian summary) Algebra i Analiz 12 (2000), no. 1, 3–69; translation in St. Petersburg Math. J. 12 (2001), no. 1, 1–50

[Li] H. B. Lawson, Jr. Algebraic cycles and homotopy theory. Ann. of Math. 129 (1989), 253–291.

[L2] H. B. Lawson, Jr. Spaces of algebraic cycles. pp. 137-213 in Surveys in Differential Geometry, 1995 vol.2, International Press, 1995.

[LF1] P. Lima-Filho, Lawson homology for quasiprojective varieties. Compositio Math. 84(1992), no. 1, 1–23.

[LF2] P. Lima-Filho, On the generalized cycle map. (English summary) J. Differential Geom. 38 (1993), no. 1, 105–129.

[LF3] P. Lima-Filho, The topological group structure of algebraic cycles. Duke Math. J. 75 (1994), no. 2, 467–491.

[M] I. G. Macdonald, The Poincaré polynomial of a symmetric product. Proc. Cambridge Philos. Soc. 58 1962 563–568.

[NZ] A. Nenashev and K. Zainoulline, Oriented cohomology and motivic decompositions of relative cellular spaces. (English summary) J. Pure Appl. Algebra 205 (2006), no. 2, 323–340.

[Pe] Peters, C., Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups. Math. Z. 234 (2000), no. 2, 209–223.

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