AUTOMORPHIC FORMS AND LORENTZIAN KAC–MOODY ALGEBRAS. PART II

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Abstract. We give variants of lifting construction, which define new classes of modular forms on the Siegel upper half-space of complex dimension 3 with respect to the full paramodular groups (defining moduli of Abelian surfaces with arbitrary polarization). The data for these liftings are Jacobi forms of integral and half-integral indices. In particular, we get modular forms which are generalizations of the Dedekind eta-function. Some of these forms define automorphic corrections of Lorentzian Kac–Moody algebras with hyperbolic generalized Cartan matrices of rank three classified in Part I of this paper. We also construct many automorphic forms which give discriminants of moduli of K3 surfaces with conditions on Picard lattice. These results are important for Mirror Symmetry and theory of Lorentzian Kac–Moody algebras.

§0. Introduction

In Part I we developed the general theory of reflective automorphic forms and their particular case of Lie reflective automorphic forms on Hermitean symmetric domains of type IV. These automorphic forms are very important in Mirror Symmetry (for K3’s and Calabi–Yau’s) and for Lorentzian Kac–Moody algebras. In Part I, in particular, we showed that this theory is similar to the theory of hyperbolic root systems (it is its mirror symmetric variant). We explained that reflective automorphic forms are very exceptional. Conjecturally their number is finite similarly to finiteness results for corresponding hyperbolic root systems with some condition of finiteness of volume for fundamental polyhedron (i.e., of elliptic or parabolic type). We believe and hope to show in further publications that classification of reflective automorphic forms and corresponding hyperbolic root systems is the key step in classification of some important class of Calabi–Yau’s (see [GN6]). For example, finiteness results for hyperbolic root systems of elliptic and parabolic type and for reflective automorphic forms are related with finiteness of families of these Calabi–Yau’s.

We demonstrated in Part I a general method of classification of the hyperbolic root systems on the example of symmetric (and twisted to symmetric) hyperbolic generalized Cartan matrices of elliptic type of rank 3 and with a lattice Weyl vector. Let us denote the set of these matrices by \( \mathfrak{A} \). It contains 60 matrices. In this Part II we consider methods of construction of reflective automorphic forms. In particular, for many generalized Cartan…

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matrices \( A \in \mathfrak{A} \) we find their mirror symmetric objects — automorphic forms \( F \) defining automorphic corrections of Kac–Moody algebras \( g(A) \) corresponding to \( A \). These forms \( F \) define so-called automorphic Lorentzian Kac–Moody algebras \( g_F \) containing \( g(A) \) and having good automorphic properties. In this paper we consider mainly 3-dimensional automorphic forms \( F \) with respect to the full paramodular groups, i.e. automorphic forms on the Siegel upper half-space \( \mathbb{H}_2 \) of complex dimension 3 with respect to the paramodular groups \( \Gamma_t \subset \text{Sp}_4(\mathbb{Q}) \) (the threefold \( A_t = \Gamma_t \setminus \mathbb{H}_2 \) is the moduli space of Abelian surfaces with polarization of type \((1,t)\)). We remark that the results and methods of this paper can be generalized to automorphic forms with respect to \( O(n,2) \) (see [G4]–[G5]). We do not consider in this paper the generalized Cartan matrices \( A \in \mathfrak{A} \) related with automorphic forms \( F \) with respect to some congruence subgroups of the paramodular groups. We will treat them later.

First we define the advance version of the Maass lifting (see [M1]–[M2]) in the form proposed in [G1]–[G5]. This new variant (we call it arithmetic lifting, with polarization of type \((1,t)\)) provides modular forms with respect to the full paramodular group \( \Gamma_t \) with a character \( \chi : \Gamma_t \to \mathbb{C}^* \) of order \( Q \) where \( Q \) is necessarily a divisor of 12, because \( \chi \) is trivial on the commutator subgroup of \( \Gamma_t \). The datum for the lifting is a holomorphic Jacobi form \( \phi_{k,t}(\tau,z) \) of weight \( k \) of integral or half-integral index \( t \) with a character \( \upsilon_{SL} \times \upsilon_H \) of the Jacobi group \( \Gamma^J = \text{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z}) \) induced by a character \( \upsilon_{SL} : \text{SL}_2(\mathbb{Z}) \to \{ \pm 1 \} \) of order \( Q \) and by a binary character \( \upsilon_H : H(\mathbb{Z}) \to \{ \pm 1 \} \) of the Heisenberg group. In Theorem 1.12 we define several variants of the arithmetic lifting numerated by \( \mu \in (\mathbb{Z}/Q\mathbb{Z})^* \):

\[
\text{Lift}_\mu(\phi)(Z) = \sum_{m \equiv \mu \text{ mod } Q} \sum_{m > 0} m^{2-k} (\tilde{\phi}_{k,t} | T^{(Q)}_\mu(m))(Z) \tag{0.1}
\]

where \( \tilde{\phi}_{k,t}(Z) = \phi_{k,t}(\tau, z) \exp(2\pi i t \omega) \) is the modular form with respect to the maximal parabolic subgroup \( \Gamma_{\infty,t} \) associated with \( \phi_{k,t}(\tau, z) \) (this parabolic subgroup defines a 1-dimensional boundary component of the threefold \( A_t \)) and where the operators \( T^{(Q)}_\mu(m) \) are the “minus”–embedding of the standard elements \( T^{(Q)}(m) \) of the Hecke ring of the principal congruence subgroup \( \Gamma_1(Q) \subset \text{SL}_2(\mathbb{Z}) \) in the Hecke ring of the parabolic subgroup \( \Gamma_{\infty,t} \). The Fourier expansion of the lifting (0.1) can be written in terms of Fourier coefficients of the Jacobi form used for the lifting.

In §1 we construct several series of Jacobi forms which one can use as data for the arithmetic lifting (0.1). For example, the Jacobi theta-series

\[
\vartheta(\tau, z) = \vartheta_{11}(\tau, z) = \sum_{m \in \mathbb{Z}} \left( \frac{-4}{m} \right) q^{m^2/8} r^{m/2}
\]

is a Jacobi form of weight \( \frac{1}{2} \) and index \( \frac{1}{2} \). (For matrix \((\tau \ z \ z \ \omega) \in \mathbb{H}_2 \) we use three formal variables \( q = \exp(2\pi i \tau) \), \( r = \exp(2\pi i z) \) and \( s = \exp(2\pi i \omega) \).) A typical modular form which we construct is the cusp form

\[
\Delta_1(Z) = \text{Lift}_1(\eta(\tau) \vartheta(\tau, z)) = \sum_{M \geq 1} \sum_{n, m > 0, l \in \mathbb{Z}} \sum_{n, m \equiv 1 \text{ mod } 6} \sum_{4nm - 3l^2 = M^2} \sum_{a | (n,l,m)} \left( \frac{6}{a} \right) q^{n/6} r^{l/2} s^{m/2}
\]

(0.2)
of weight one with respect to the paramodular group $\Gamma_3$ with a character of order 6. This is the lifting of $\eta(\tau)\vartheta(\tau, z)$ where $\eta(\tau)$ is the Dedekind $\eta$-function. For a primitive matrix $N = \begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix}$ (i.e. $(n, l, m) = 1$) the Fourier coefficient $f(N)$ of $\Delta_1(Z)$ is equal to $\pm 1$ or 0. One can consider $\Delta_1(Z)$ as a variant of Dedekind $\eta$-function in three variables. Below we give some evidence for this point of view.

The Dedekind $\eta$-function is a character for an affine Kac–Moody algebra. The cusp form $\Delta_1(Z)$ defines the automorphic correction $g_{\Delta_1}$ for the symmetric generalized Cartan matrix $A_{3,II} \in \mathfrak{A}$ (see Theorem 1.3.1 in Part I). It follows that $\Delta_1(Z)$ is the denominator function of the generalized Kac–Moody superalgebra $g_{\Delta_1}$ and it has an infinite product expansion due to the Weyl–Kac–Borcherds denominator identity (see [GN1, §6] and [R]). We find this expansion in §2. It gives multiplicities of roots of $g_{\Delta_1}$.

To describe this infinite product expansion, let us define a weak Jacobi form $\phi_{03}(\tau, z) = \sum_{n \geq 0, l} f_3(n, l)q^n r^l$ of weight 0 and index 3 with integral Fourier coefficients:

$$\phi_{03}(\tau, z) = \left(\frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)}\right)^2 = r^{-1} \left(\prod_{n \geq 1} (1 + q^{n-1}r)(1 + q^n r^{-1})(1 - q^{2n-1}r^2)(1 - q^{2n-1}r^{-2})\right)^2.$$  

We have the identity

$$\Delta_1(Z) = q^{3/4} r^{1/2} s^{1/2} \prod_{n, l, m \in \mathbb{Z}, (n, l, m) > 0} (1 - q^{n} r^l s^m)^{f_3(nm, l)}. \quad (0.3)$$

This formula is an example of the exponential or Borcherds lifting. In general, one can define the exponential lifting of a Jacobi form $\phi_{0, t}(\tau, z) = \sum_{n, l} f(n, l)q^n r^l$ of weight zero as follows (compare with (0.1)):

$$\text{Exp-Lift}(\phi_{0, t})(Z) = \psi(Z) \exp \left(-\sum_{m \geq 1} m^{-1} \phi_{0, t}|_0 T_- (m)(Z)\right) \quad (0.4)$$

where

$$\psi(Z) = \eta(\tau)^{f(0, 0)} \prod_{l > 0} \left(\frac{\vartheta(\tau, lz)}{\eta(\tau)} \exp(\pi i l^2 \omega)\right)^{f(0, l)}.$$  

The $T_- (m)$ are the operators from (0.1) if we put there $Q = 1$. In Theorem 2.1 we describe properties of the lifting (0.4). This theorem is a modified version of the Borcherds construction given in [Bo6, Theorem 10.1].

The arithmetic and exponential liftings have similar behavior with respect to the action of Hecke operators, if we modify, in appropriate way, the action of Hecke operators on (0.4). Let $X = \sum_i \Gamma_t g_i \in H(\Gamma_t)$ be an element of the Hecke ring of the group $\Gamma_t$. For a modular form $F(Z)$ of weight $k$ with respect to $\Gamma_t$ we define the Hecke product

$$[F]_X(Z) = \prod_i (F|_{kg_i})(Z). \quad (0.5)$$
If $X$ has a good reduction modulo $t$, then we proved in [GN4, Theorem A.7] that

$[\text{Exp-Lift}(\phi_{0,t})]_X = c \cdot \text{Exp-Lift}(\phi_{0,t}|_0 J^{(t)}_0(X))$

where $J^{(t)}_0(X)$ is a natural projection of the Hecke ring $\mathcal{H}(\Gamma_t)$ on the Hecke-Jacobi ring of the parabolic subgroup $\Gamma_\infty$ and $c$ is a constant.

Another type of Hecke product is the multiplicative symmetrisation

$$\text{Ms}_p : F(Z) \mapsto \prod_{M_i \in \Gamma_t \cap \Gamma_{tp} \setminus \Gamma_{tp}} (F|_k M_i)(Z)$$

which maps modular forms with respect to $\Gamma_t$ to modular forms with respect to $\Gamma_{pt}$ for a prime $p$. Theorem 3.3 says that

$$\text{Ms}_p(\text{Exp-Lift}(\phi_{0,t})) = c \cdot \text{Exp-Lift}(\phi_{0,t}|_{T^-(p)})$$

where $T^-(p)$ is the Hecke operator used in (0.1) and (0.4). To prove the relations of the exponential lifting with the defined Hecke correspondences we use the methods developed in [G2], [G6]–[G8] (see also [G11]) in the case of the lifting of type (0.1).

In §2–§4 we find many cases when the arithmetic lifting (or a finite Hecke product of type (0.5) or (0.6) of lifted modular forms) is equal to the exponential lifting. This is true for the most fundamental Siegel modular forms $\Delta_5(Z)$ (the product of even theta-constants), $\Delta_2(Z)$, $\Delta_1(Z)$ and $\Delta_{1/2}(Z)$ (the index denotes the weight) which are the modular forms with respect to the groups $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$ respectively. The first three functions are cusp forms. The last one is a Siegel theta-constant (the “most odd” even theta-constant) which we consider as a “trivial” lifting of the Jacobi theta-series. This interpretation leads to the fact that this theta-constant is a modular form with respect to the full paramodular group $\Gamma_4$ and has infinite product expansion of type (0.3) (see Theorem 1.11 where also another singular modular form with respect to $\Gamma_{36}$ is constructed).

The modular forms $\Delta_k(Z)$ ($k = 1/2, 1, 2, 5$) have very simple divisors. They coincide with the irreducible Humbert surfaces $H_1$ of discriminant 1 in the corresponding modular threefolds. The $\Delta_5(Z)$ is associated with the hyperbolic root system defined by a triangle with vertices at infinity on the hyperbolic plane (see Fig.1 in [GN5]), the $\Delta_2(Z)$ and $\Delta_1(Z)$ are associated with similar right quadrangle and right hexagon (see Fig.2 and Fig.3 in [GN5] respectively). The singular modular form $\Delta_{1/2}(Z)$ is connected with the right $\infty$-gon with vertices at infinity and defines the generalized Kac–Moody superalgebra with infinitely many simple real roots (i.e. of parabolic type). In §5.1 we describe relations between the modular forms constructed in §1–§4 and generalized Cartan matrices $A \in \mathfrak{A}$.

Many modular forms constructed in §1–§4 (in particular, all modular forms of $\Delta$-type) give discriminant automorphic forms for moduli spaces of algebraic K3 surfaces with some special Picard lattices and give Mirror Symmetry for K3 surfaces in the variant we considered in [GN3] (see also [GN6]). We describe these cases in Sect. 5.2.

Below we summarize properties of the cusp form $\Delta_1(Z)$ (and other $\Delta$-functions as well) as the 3-dimensional generalization of the Dedekind $\eta$-function:

(a) $\Delta_1(Z)$ is the unique (up to a constant) cusp form of weight one with respect to the full paramodular group $\Gamma_3$. (We remark that the weight $\frac{1}{2}$ is the singular weight for $Sp_4(\mathbb{Q})$, thus one is the minimal weight for which a cusp form may exist.)
(b) \( \Delta_1(Z) \) is a root of order six from the first cusp form (with trivial character) with respect to \( \Gamma_3 \).

(c) \( \Delta_1(Z) \) satisfies the Euler type identity (the infinite sum in (0.2) is equal to the product in (0.3)) and it is the denominator function of a generalized Kac–Moody superalgebra.

(d) The divisor of \( \Delta_1(Z) \) is discriminant of the moduli space of \( K3 \) surfaces with the Picard lattice described in Sect. 5.2.

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1. The lifting of Jacobi forms of half-integral indices

In this chapter we consider a new variant of the arithmetic lifting (or the Maass or Saito–Kurokawa lifting) of Jacobi forms to the Siegel modular forms (see [M1]–[M2] and [G1]–[G5]). The Siegel modular forms under construction are modular forms with respect to a full paramodular group (integral symplectic groups of a skew-symmetric form) with a character (or a multiplier system).

In what follows we consider three types of automorphic forms of integral and half-integral weight: modular forms for \( SL_2(\mathbb{Z}) \), Siegel modular forms with respect to a paramodular group and Jacobi forms of integral and half-integral index.

Let
\[
\mathbb{H}_n = \{ Z = tZ \in M_n(\mathbb{C}), \; Z = X + iY, \; Y > 0 \}
\]
be the Siegel upper-half space of genus \( n \). We denote by \( |_k \) \( (k \in \mathbb{Z}/2) \) the standard slash operator on the space of functions on \( \mathbb{H}_n \):
\[
(F|_k M)(Z) := \det(CZ + D)^{-k}F(M < Z >) \tag{1.1}
\]
where
\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{R}) \quad \text{and} \quad M < Z > = (AZ + B)(CZ + D)^{-1}.
\]

For a half-integral \( k \) we choose one of the holomorphic square roots by the condition \( \sqrt{\det(Z/i)} > 0 \) for \( Z = iY \in \mathbb{H}_n \).

**Definition 1.1.** Let \( k \) be integral or half-integral. Let \( \Gamma \subset Sp_n(\mathbb{R}) \) be a subgroup which contains a principal congruence subgroup. A modular form of weight \( k \) for \( \Gamma \) with a multiplier system (or a character) \( v : \Gamma \to \mathbb{C}^\times \) is a holomorphic function on \( \mathbb{H}_n \) which satisfies the functional equation
\[
(F|_k M)(Z) = v(M)F(Z) \quad \text{for any } M \in \Gamma.
\]

For \( n = 1 \) we have to add the standard growth condition (the holomorphicity) at the cusps of the group \( \Gamma \). We denote by \( \mathcal{M}_k(\Gamma, v) \) (resp. \( \mathcal{N}_k(\Gamma, v) \)) the space of all modular (resp. cusps) forms of weight \( k \).
1. Jacobi modular forms of half-integral index.

We shall consider Jacobi forms of integral and half-integral weight. To define the corresponding multiplier system we use the multiplier system of the \( \eta \)-function. The Dedekind eta-function

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) = \sum_{n \in \mathbb{N}} \left( \frac{12}{n} \right) q^{n^2/24},
\]

where \( \tau \in \mathbb{H}_1 \), \( q = \exp(2\pi i \tau) \) and

\[
\left( \frac{12}{n} \right) = \begin{cases} 
1 & \text{if } n \equiv \pm 1 \text{ mod } 12 \\
-1 & \text{if } n \equiv \pm 5 \text{ mod } 12 \\
0 & \text{if } (n,12) \neq 1,
\end{cases}
\]

satisfies the functional equation

\[
\eta(a\tau + b) = v_\eta(M)(c\tau + d)^{\frac{1}{2}} \eta(\tau) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})
\]

where \( v_\eta(M) \) is a 24th root of unity. The Dedekind eta-function is a modular form of weight \( 1/2 \). (One can consider it as a modular form with respect to the double cover \( \widetilde{SL}_2(\mathbb{Z}) \) of \( SL_2(\mathbb{Z}) \). The commutator subgroup of \( \widetilde{SL}_2(\mathbb{Z}) \) has index 24.) For arbitrary even \( D \) the \( \eta \)-multiplier system defines the character \( v^D_\eta \) of \( SL_2(\mathbb{Z}) \).

**Lemma 1.2.** Let \( D \) be even integral. We set \( Q = \frac{24}{(24,D)} \) where \( (24,D) = \text{g.c.d.} \). Then \( v^D_\eta \) is a character of \( SL_2(\mathbb{Z}) \) and

\[
\text{Ker}(v^D_\eta) \supset \Gamma_1(Q) = \{ M \in SL_2(\mathbb{Z}) \mid M \equiv E_2 \text{ mod } Q \}. \tag{1.3}
\]

**Proof.** The lemma follows from the fact that \( SL_2(\mathbb{Z})/[SL_2(\mathbb{Z}), SL_2(\mathbb{Z})] \cong \mathbb{Z}/12\mathbb{Z} \) or from the exact formula for the multiplier system \( v_\eta(M) \) in terms of the entries of the matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). For even \( D \) it is given by

\[
v^D_\eta(M) = \begin{cases} 
\exp \left( \frac{2\pi i D}{24}((a + d)c - bd(c^2 - 1) - 3c) \right) & c \equiv 1 \text{ mod } 2 \\
\exp \left( \frac{2\pi i D}{24}((a + d)c - bd(c^2 - 1) + 3(d - cd - 1)) \right) & d \equiv 1 \text{ mod } 2.
\end{cases}
\]

This formula gives us a better result for special \( D \): if \( Q = 3, 6 \)

\[
\text{Ker}(v^D_\eta) \supset \Gamma_0^0(Q) = \{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b, c \equiv 0 \text{ mod } Q, \ a, d \equiv \pm 1 \text{ mod } Q \},
\]

if \( Q = 12 \)

\[
\text{Ker}(v^D_\eta) \supset \Gamma_{1,5}(12) = \{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b, c \equiv 0 \text{ mod } 12, \ a, d \equiv 1, 5 \text{ mod } 12 \}. 
\]
Jacobi forms of half-integral indices appear naturally as Fourier-Jacobi coefficients of the theta-constants or well known $Sp_4(\mathbb{Z})$-cusp form $\Delta_5(Z)$ of weight 5 (see (1.24) and Example 1.14 below). Let us introduce a maximal parabolic subgroup of the integral symplectic group

$$\Gamma_{\infty} = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}. \quad (1.4)$$

The Jacobi group is defined by $\Gamma_J = \Gamma_{\infty}/\{\pm E_4\} \cong SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, where $H(\mathbb{Z})$ is the integral Heisenberg group, which is the central extension

$$0 \to \mathbb{Z} \to H(\mathbb{Z}) \to \mathbb{Z} \times \mathbb{Z} \to 0.$$

We use the following embeddings "tilde" of $SL_2(\mathbb{Z})$ and $\ldots$ of $H(\mathbb{Z})$ in $\Gamma_J$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H(\mathbb{Z}) \cong \left\{ \lambda, \mu, \kappa \in \mathbb{Z} \mid [\lambda, \mu; \kappa] = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & \mu \\ 0 & 0 & 1 - \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}. \quad (1.5)$$

There exists the unique binary character $v_H$ of $H(\mathbb{Z})$

$$v_H([\lambda, \mu; \kappa]) := (-1)^{\lambda+\mu+\lambda\mu+\kappa} \quad (1.6)$$

which can be extended to the Jacobi group.

**Lemma 1.3.** Let $\chi$ be a character of $SL_2(\mathbb{Z})$. Then

$$(\chi \times v_H)(\gamma \cdot h) := \chi(\gamma) \cdot v_H(h), \quad (\chi \times id_H)(\gamma \cdot h) := \chi(\gamma)$$

are correctly defined characters of the Jacobi group where $\gamma \in SL_2(\mathbb{Z})$ and $h \in H(\mathbb{Z})$.

**Proof.** It is easy to see that

$$v_H(\gamma^{-1}h\gamma) = v_H(h)$$

for any $\gamma \in SL_2(\mathbb{Z})$.

$\square$

In what follows we fix the notations

$$Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2, \quad e(z) = \exp(2\pi iz), \quad q = e(\tau), \quad r = e(z), \quad s = e(\omega). \quad (1.7)$$
Definition 1.4. Let $t$ be an integral or half-integral positive number. A holomorphic function $\phi(\tau, z)$ on $\mathbb{H}_1 \times \mathbb{C}$ is called a Jacobi form of weight $k$ and index $t$ with a multiplier system (or a character) $v : \Gamma^J \rightarrow \mathbb{C}^\times$ if the function

$$\tilde{\phi}(Z) := \phi(\tau, z) \exp(2\pi it\omega), \quad Z \in \mathbb{H}_2,$$

is a $\Gamma_\infty$-modular form of weight $k$ with the multiplier system $v$, i.e if it satisfies the functional equation

$$(\tilde{\phi}|_k M)(Z) = v(M)\tilde{\phi}(Z) \quad \text{for any } M \in \Gamma^J$$

and has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, l} f(n, l) \exp(2\pi i(n\tau +lz))$$

where the summation is taken over $n$ and $l$ from some free $\mathbb{Z}$-modules. The condition $f(n, l) = 0$ unless $4tn - l^2 \geq 0$ is equivalent to the holomorphicity of $\phi$ at infinity. The form $\phi(\tau, z)$ is called a Jacobi cusp form if $f(n, l) = 0$ unless $4tn - l^2 > 0$. We call a holomorphic function $\phi(\tau, z)$ a nearly holomorphic Jacobi form of weight $k$ and index $t$ if it satisfies the functional equation of the definition and there exists $n \in \mathbb{N}$ such that $\Delta(\tau)^n \phi(\tau, z)$ is a Jacobi form. We call the form $\phi(\tau, z)$ a weak Jacobi form if $f(n, l) \neq 0$ only for $n \geq 0$ in its Fourier expansion.

We denote the space of all Jacobi forms (resp. cusp forms, weak forms or nearly holomorphic Jacobi forms) with the multiplier system $v$ by $J_{k,t}^J(v)$ (resp. $J_{k,t}^{\text{cusp}}(v)$, $J_{k,t}^{\text{weak}}(v)$ or $J_{k,t}^{\text{nh}}(v)$). We remark that $J_{k,t}(v)$ contains only zero if $k \leq 0$, but there exist weak Jacobi forms of non-positive weights. One can give a similar definition for any congruence-subgroup of the Jacobi group, but with the definition given above we would like to emphasize that there exists a large class of Jacobi forms of half-integral indices with respect to the full Jacobi group. In what follows the two classical examples are very important.

Example 1.5. Jacobi theta-series, the Jacobi triple product and the quintuple product.

The Jacobi theta-series is defined as

$$\vartheta(\tau, z) = \sum_{n \equiv 1 \mod 2} (-1)^{n-1} \exp(\frac{\pi in^2}{4}\tau + niz) = \sum_{m \in \mathbb{Z}} \left(-\frac{4}{m}\right) q^{m^2/8}r^{m/2}$$

where

$$\left(-\frac{4}{m}\right) = \begin{cases} 1 & \text{if } m \equiv \pm 1 \mod 4 \\ 0 & \text{if } m \equiv 0 \mod 2. \end{cases}$$

It is known that $\vartheta(\tau, z)$ satisfies the following transformations formulae with respect to the standard generators of $SL_2(\mathbb{Z})$ and $H(\mathbb{Z})$:

$$\vartheta(\tau, z + \lambda \tau + \mu) = (-1)^{\lambda+\mu} \exp(-\pi i (\lambda^2 \tau + 2\lambda z)) \vartheta(\tau, z)$$

$$\vartheta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \exp\left(-\frac{3\pi i}{4}\right) \sqrt{\tau} \exp\left(\pi i \frac{z^2}{\tau}\right) \vartheta(\tau, z)$$

$$\vartheta(\tau + 1, z) = \exp\left(\frac{\pi i}{4}\right) \vartheta(\tau, z).$$
This is a Jacobi form of weight 1/2 and index 1/2 with a multiplier system \( v_\vartheta \). By definition, \( v_\vartheta \) is the multiplier system of the function \( \vartheta(\tau, z) \exp(\pi i \omega) \). The first functional equation for \( \vartheta(\tau, z) \) is equivalent to

\[
\vartheta(\tau, z) \exp(\pi i \omega)|_{\lambda, \mu; \kappa} = (-1)^{\lambda+\mu+\lambda\mu+\kappa} \vartheta(\tau, z) \exp(\pi i \omega).
\]

Thus the restriction of \( v_\vartheta \) to \( H(\mathbb{Z}) \) is the character \( v_H \) defined in (1.6). Moreover we have

\[
\frac{\partial \vartheta(\tau, z)}{\partial z} \bigg|_{z=0} = 2\pi i \sum_{n \equiv 1 \mod 4} \left( -\frac{4}{n} \right) q^{n^2/8} = 2\pi i \eta(\tau)^3.
\]

Thus

\[
v_\vartheta(\tilde{M}) = v_\eta(M)^3 \quad \text{for} \quad M \in SL_2(\mathbb{Z}).
\]

The formula for \( \eta(\tau)^3 \) is due to Jacobi and follows from his famous triple formula

\[
\prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{3}{8} m(m-1)} r^m
\]

or equivalently

\[
\vartheta(\tau, z) = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n).
\] (1.9)

We recall the quintuple product formula. We shall use it in the form

\[
\sum_{n \in \mathbb{Z}} \left( \frac{12}{n} \right) q^{n^2/24} r^{n/2} = q^{\frac{1}{24} r^{-1}} \prod_{n \geq 1} \left( 1 + q^{n-1}r \right) \left( 1 + q^n r^{-1} \right) \left( 1 - q^{2n-1} r^2 \right) \left( 1 - q^{2n-1} r^{-2} \right) \left( 1 - q^n \right)
\]

where the generalized Kronecker symbol was defined in (1.2). The left hand side of the last identity is a Jacobi modular form.

**Lemma 1.6.** The function

\[
\vartheta_{3/2}(\tau, z) = \sum_{n \in \mathbb{Z}} \left( \frac{12}{n} \right) q^{n^2/24} r^{n/2} \in J_{1/2, 3/2}(v_\eta \times v_H)
\]

is a Jacobi modular form of weight 1/2 and index 3/2 with the multiplier system \( v_\eta \times v_H \).

**Proof.** The statement follows from the identity

\[
\vartheta_{3/2}(\tau, z) = \frac{\eta(\tau) \vartheta(\tau, 2z)}{\vartheta(\tau, z)}
\]

and \( \vartheta(\tau, 2z) \in J_{1/2, 2}(v_\eta^2 \times \text{id}_H) \). One can prove this directly or using the Hecke operator \( \Lambda_2 \) which will be defined in the next section.

\( \square \)
1.2. Hecke operators.

We shall use some Hecke operators on the space of Jacobi modular forms. Let us define the Hecke ring of the Jacobi group $\Gamma^J$ (equivalently, the parabolic subgroup $\Gamma_\infty$ defined in (1.4)) and its congruence subgroup

$$\Gamma^J(Q) = \Gamma_1(Q) \times H(\mathbb{Z})$$

where $\Gamma_1(Q)$ is the principal congruence subgroup of $SL_2(\mathbb{Z})$ (see (1.3)). We denote the corresponding subgroup of $\Gamma_\infty$ by $\Gamma_\infty(Q)$. An element of the Hecke ring is a formal finite sum of double left cosets with respect to $\Gamma_\infty$ (about the Hecke rings of this type see [G6]–[G8], [G11]). If

$$X = \sum_i a_i \Gamma_\infty N_i \Gamma_\infty = \sum_j b_j \Gamma_\infty M_j \in H(\Gamma_\infty),$$

we define in a standard way its action on the space of all $\Gamma_\infty$-modular forms

$$(F|kX)(Z) := \begin{cases} \sum_j \nu(M_j)^{2k-3} b_j (F|kM_j)(Z) & \text{if } k \neq 0 \\ \sum_j b_j (F|0M_j)(Z) & \text{if } k = 0, \end{cases}$$

where $\nu(M_j)$ is the degree of the simplectic similitude $M_j$. We use the normalizing factor $\nu(M)^{2k-3}$ connected with $Sp_4$ since the corresponding Hecke operators are also used in the construction of some $L$-functions (e.g. see [G3], [G11] for the $Spin-L$-function of Siegel modular forms and [G8]–[G9] for the skew-symmetric square of the standard $L$-function of $SU(2,2)$-modular forms).

Let us recall the definition of the Hecke element $T^{(Q)}(m)$ of the ring $H(\Gamma_1(Q), M_2^+(Q))$

where

$$M_2^+(Q) = \{(a \ b \ c \ d) \in M_2^+(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod Q \}.$$ 

For $(m, Q) = 1$ we have

$$T^{(Q)}(m) = \sum_{ad = m \atop b \mod d} \Gamma_1(Q) \sigma_a \begin{pmatrix} a & Qb \\ 0 & d \end{pmatrix},$$

where $a > 0$ and $\sigma_a \in SL_2(\mathbb{Z})$ such that $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \mod Q$.

We shall use two different types of Hecke operator on the space of Jacobi forms $J_{k,t}(\chi)$:

$$\Lambda_n = \Gamma_\infty(Q) \text{diag}(1, n, 1, n^{-1}) \Gamma_\infty(Q) = \Gamma_\infty(Q) \text{diag}(1, n, 1, n^{-1}) \in H(\Gamma_\infty(Q)) \quad (1.10)$$

and

$$T^{(Q)}_{-}(m) = \sum_{ad = m \atop b \mod d} \Gamma_\infty(Q) \tilde{\sigma}_a \begin{pmatrix} a & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} Qb & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in H(\Gamma_\infty(Q)) \quad (1.11)$$
where \((m, Q) = 1\). These elements realize the embeddings of the semigroup \(\mathbb{N}^\times\) (equivalently, the Hecke ring \(H(\{1\}, \mathbb{N}^\times)\) of the trivial group) and the Hecke ring of \(\Gamma_1(Q)\) into the Hecke ring of the parabolic subgroups \(\Gamma_\infty(Q) \subset \Gamma_\infty\). One can consider \(H(\Gamma_\infty)\) as a non-commutative extension of the usual commutative Hecke ring \(H(Sp_4(\mathbb{Z}))\) (see the papers of the first author mentioned above for a general case of the Hecke rings of parabolic subgroups of the classical groups over local fields and \([G2]\) for the case of the paramodular groups).

Let us suppose that the \(SL_2(\mathbb{Z})\)-part of the character \(\chi\) of Jacobi group has the conductor \(Q\). Then element (1.11) defines the following operator on the spaces of Jacobi forms of integral weight \(k\) and index \(t\) \((t \in \mathbb{Z}/2)\) with the character \(\chi\):

\[
\tilde{\phi}|_k T_\infty^Q(m)(Z) = m^{2k-3} \sum_{\substack{a d = m \\ b \mod{\frac{1}{d}}}} d^{-k} \chi(a) \phi\left(\frac{a \tau + b Q}{d}, az\right) \exp(2\pi i m t \omega).
\] (1.12)

The element \(\Lambda_n\) defines a Hecke operator for an arbitrary (integral or half-integral) weight \(k\)

\[
(\phi|_k \Lambda_n)(Z) = n^{-k} \phi(\tau, nz) \exp(2\pi i n^2 t \omega).
\] (1.13)

Usually we shall omit the variable \(\omega\) in the action of Hecke operators.

**Lemma 1.7.** Let \(\phi(\tau, z) \in J_{k,t}(\chi \times \nu^\varepsilon_H)\), where \(\chi\) is a character of \(SL_2(\mathbb{Z})\) and \(\varepsilon = 0\) or \(= 1\). Then

\[
\phi|_k \Lambda_n \in J_{k,tn^2}(\chi \times \nu^{n \varepsilon}_H).
\]

We assume that \(k\) is integral, \(\Gamma_1(Q) \subset \text{Ker}(\chi)\) and \((m, 2^s Q) = 1\). Then

\[
(\phi|_k T_\infty^Q(m))(\tau, z) \in J_{k,mt}(\chi_m \times \nu^\varepsilon_H)
\]

where \(\chi_m\) is a character of \(SL_2(\mathbb{Z})\) defined by

\[
\chi_m(\alpha) := \chi(\alpha_m)
\]

with \(\alpha_m \in SL_2(\mathbb{Z})\) such that \(\alpha_m \equiv \begin{pmatrix} 1 & 0 \\ 0 & m^{-1} \end{pmatrix} \mod{Q}\).

**Remark.** It is clear from the definition that the space \(J_{k,t}(\chi \times \nu^\varepsilon_H)\) is not empty only if \(2t \equiv \varepsilon \mod{2}\).

**Proof.** It is easy to check that

\[
(\phi|_k \Lambda_n)|_k[\lambda, \mu; \kappa] = (-1)^{m\lambda + m\mu + m^2\lambda \mu + m^2 \kappa} \phi|_k \Lambda_n.
\]

This proves the first statement. Any representative from the system indicated in (1.11) acts on \(\omega\) by multiplication on \(m\). Thus the form

\[
\tilde{\psi}(Z) = \psi(\tau, z)e^{2\pi it \omega} = (\tilde{\phi}|_k T_\infty^Q(m))(Z)
\]

is a Jacobi form of index \(mt\). Let us find its character. For any \(\alpha \in SL_2(\mathbb{Z})\)

\[
\tilde{\psi}|_k \tilde{\alpha} = \sum_{M \in T_\infty^Q(m)} \tilde{\phi}|_k(\tilde{\alpha}_M \cdot M)
\]
(see (1.5)). By definition of the system of representatives \( \{ M \} \) given in (1.11),

\[
\alpha_M \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \alpha \mod Q.
\]

This gives the formula for the character \( \chi_m(\alpha) \). If index \( t \) is half-integral, then for arbitrary \( [\lambda, \mu; \kappa] \in H(\mathbb{Z}) \) we have

\[
\psi|_k \widetilde{\sigma}_a \left( \begin{array}{cccc} a & 0 & Qb & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \left[ \begin{array}{c} \lambda \\ \mu \\ \kappa \end{array} \right] = (-1)^{d\lambda+am+\mu b}\kappa - Qb\lambda(d\lambda+1)+m\kappa \phi|_k \widetilde{\sigma}_a \left( \begin{array}{cccc} a & 0 & Qb & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).
\]

This finishes the proof. \( \Box \)

Using the exact formula for \( v^D_\eta \) from Lemma 1.2, we obtain

**Lemma 1.8.** Let \( \chi \) be a character of \( \Gamma_\infty \) of type \( v^D_\eta \times v^E_H \), where \( D \) is even. Let \( Q = 24/(D, 24) \) and \( m \equiv -1 \mod Q \). Then

\[ \chi_m = \overline{\chi} \]

where bar denotes complex conjugation.

### 1.3. Paramodular groups and Humbert modular surfaces.

It is well known that the symplectic group of rank four over a field is isomorphic to an orthogonal group of signature \( (3, 2) \). We recall the corresponding construction over \( \mathbb{Z} \). We fix a \( \mathbb{Z} \)-module \( L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4 \). Arbitrary \( \mathbb{Z} \)-linear map \( g : L \to L \) induces the linear map \( \wedge^2 g : \wedge L \wedge L \to \wedge L \wedge L \) of the \( \mathbb{Z} \)-module \( \wedge L \wedge \wedge L \) of integral bivectors. \( \wedge L \wedge \wedge L \) is isomorphic to the module of integral skew-symmetric matrices \( (e_i \wedge e_j) \) corresponding to the elementary skew-symmetric matrix \( E_{ij} \) having only two non-zero elements \( e_{ij} = 1 \) and \( e_{ji} = -1 \). The scalar product \( (u, v) \) on \( \wedge L \wedge \wedge L \) is defined by \( u \wedge v = (u, v)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4 L \). This is an even unimodular integral symmetric bilinear form of signature \( (3, 3) \) on \( \wedge L \wedge \wedge L \) which is invariant with respect to the action of \( SL(L) \) on \( \wedge L \wedge \wedge L \). We recall that if the matrix \( X \) corresponds to the bivector \( x = \sum_{i<j} x_{ij}e_i \wedge e_j \), then \( (x, x) = 2 \text{Pf}(X) \), where \( \text{Pf}(X) \) is the Pfaffian of \( X \) and \( \text{Pf}(MX^tM) = (\text{det}(M) \text{Pf}(X)) \).

We fix a skew-symmetric form \( J_t \) on \( L \) by the property:

\[ J_t(x, y)e_1 \wedge e_2 \wedge e_3 \wedge e_4 = -x \wedge y \wedge w_t, \quad w_t = te_1 \wedge e_3 + e_2 \wedge e_4. \]

The group

\[ \tilde{\Gamma}_t = \{ g : L \to L \mid J_t(gx, gy) = J_t(x, y) \} \]

is called the integral paramodular group of level \( t \). The lattice \( L_t = w_t^{-1} \) consisting of all elements of \( \wedge L \wedge L \) orthogonal to \( w_t \) has the basis

\[ f_1 = e_1 \wedge e_2, \quad f_2 = e_2 \wedge e_3, \quad f_3 = te_1 \wedge e_3 - e_2 \wedge e_4, \quad f_{-2} = e_4 \wedge e_1, \quad f_{-1} = e_4 \wedge e_3. \quad (1.14) \]
The Pfaffian defines a quadratic form $S_t$ of signature $(3, 2)$ on the lattice $L_t$ which has the following matrix

$$
S_t = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 2t & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

in the basis $\{f_i\}$. It gives a homomorphism from the integral paramodular group to the orthogonal group of the lattice $L_t$

$$\Lambda^2 : \tilde{\Gamma}_t \to O(L_t).$$

The paramodular group $\tilde{\Gamma}_t$ is conjugate to a subgroup of the usual rational symplectic group $Sp_4(Q)$:

$$\Gamma_t := I_t \tilde{\Gamma}_t I_t^{-1} = \left\{ \begin{pmatrix}
* & t* & * & * \\
* & * & * & t^{-1}
\end{pmatrix} \in Sp_4(Q) \mid \text{all } * \text{ are integral} \right\},$$

where $I_t = \text{diag}(1, 1, 1, t)$. Therefore we get a homomorphism

$$\Phi : \Gamma_t \to O(L_t), \quad \Phi(g) = \Lambda^2(I_t^{-1} g I_t). \quad (1.16)$$

One can check, that

$$\Phi(\Gamma_t) \subset \tilde{\text{SO}}(L_t) = \tilde{O}(L_t) \cap SO(L_t), \quad \text{Ker } \Phi = \{ \pm E_4 \},$$

where

$$\tilde{O}(L_t) = \{ g \in O(L_t) \mid \forall \ell \in \hat{L_t} \quad g\ell - \ell \in L_t \}$$

($\hat{L_t}$ denotes the dual lattice) is the subgroup of the orthogonal group consisting of elements acting identically on the discriminant group

$$A_{L_t} := \hat{L_t}/L_t = (2t)^{-1}Z/Z \cong Z/2tZ$$

equipped with the corresponding finite quadratic form $q_{L_t}$ (see [N1]). The next lemma is well known (see, for example, [GH1])

**Lemma 1.9.** The homomorphism $\Phi$ defines an isomorphism between $\Gamma_t$ and $\tilde{\text{SO}}^+(L_t) = SO^+(L_t) \cap \tilde{\text{SO}}(L_t)$, where $SO^+(L_t)$ denote the subgroup of elements with real spinor norm equals one.

The paramodular group has normal extensions generated by the $\Phi$-preimages of the elements of $SO^+(L_t)$. From the result of [N1], it follows that $SO^+(L_t)/\tilde{\text{SO}}^+(L_t) \cong O(q_{L_t})$ where $O(q_{L_t})$ is the finite orthogonal group of the discriminant group $(A_{L_t}, q_{L_t})$. For the case of the lattice $L_t$

$$O(q_{L_t}) = \{ b \mod 2t \mid b^2 \equiv 1 \mod 4t \} \cong (Z/2Z)^{\nu(t)} \quad (1.17)$$
where \( \nu(t) \) is the number of prime divisors of \( t \). Using (1.16), one can define a normal extension \( \Gamma_t^* \) of the paramodular group

\[
\Gamma_t^* = \langle \Gamma_t, V_d \rangle \quad \text{where} \quad d||t > \subset Sp_4(\mathbb{R}),
\]

where for a strict divisor \( d \) of \( t \) (\( d||t \) means that \( d|t \) and \( (d, t/d) = 1 \)) we set

\[
V_d = (\sqrt{t})^{-1} \tilde{V}_d, \quad \tilde{V}_d = \begin{pmatrix}
\frac{dx - t}{d} & 0 & 0 \\
-\frac{y}{d} & 0 & 0 \\
0 & 0 & \frac{d}{t}
\end{pmatrix} \in GSp_4(\mathbb{Z}) \quad (xd - yt = 1).
\]

(1.18)

Then we have \( \Gamma_t^*/\Gamma_t \cong O(q_{Lt}) \). For a square free \( t \) the group \( \Gamma_t^* \) is a maximal discrete group acting on \( \mathbb{H}_2 \). The homomorphism \( \bigwedge^2 \) defines the isomorphism

\[
\Phi : P\Gamma_t^* \rightarrow PO^+(Lt).
\]

We shall use also a double normal extension

\[
\Gamma_t^+ = \Gamma_t \cup \Gamma_t V_t, \quad V_t = \frac{1}{\sqrt{t}} \begin{pmatrix}
0 & t & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & t \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(1.19)

One can prove (see [G1, Lemma 2.2])

**Lemma 1.10.** The group \( \Gamma_t^+ \) is generated by the maximal parabolic subgroup \( \Gamma_{\infty, t} = \Gamma_t \cap \Gamma_{\infty}(\mathbb{Q}) \) and \( V_t \).

We shall use the exact images of the generators of \( \Gamma_t^+ \) with respect to \( \Phi \):

\[
\Phi \left( \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \right) = \begin{pmatrix}
a & -b & 0 & 0 \\
-c & d & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & a & b
\end{pmatrix}, \quad \Phi \left( \begin{pmatrix}
\lambda & \mu \\
\kappa & t
\end{pmatrix} \right) = \begin{pmatrix}
1 & 0 & 2\mu & t\lambda - \kappa \\
0 & 1 & 2\lambda & t\mu^2 \\
0 & 0 & 0 & \lambda \\
0 & 0 & 0 & \mu
\end{pmatrix},
\]

\[
\Phi(V_t) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \Phi(J_t) = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The involution \( \Phi(V_t) \) acts on the discriminant group \( A_{Lt} \) as multiplication on \(-1\), thus we have the isomorphism

\[
\Phi : P\Gamma_t^+ \rightarrow P\tilde{O}^+(Lt).
\]

We remark that for the case of a perfect square \( t = d^2 \) the group \( \Gamma_{d^2} \) is conjugate (by the matrix \( \text{diag}(1, d, 1, d^{-1}) \)) to a subgroup of \( \Gamma_1 = Sp_4(\mathbb{Z}) \). In this realization we have

\[
\Gamma_{d^2} \cong \Gamma_{d_1} = \{ M \in Sp_4(\mathbb{Z}) \mid M \equiv \begin{pmatrix}
* & 0 & * & 0 \\
0 & * & 0 & * \\
* & 0 & * & 0 \\
0 & 0 & * & 0
\end{pmatrix} \mod d \}, \quad V_{d^2} \cong V_{d, 1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(1.20)
The real orthogonal group $O^+(L_t \otimes \mathbb{R})$ acts on the homogeneous domain $\Omega^+_t$ of type IV corresponding to the lattice $L_t$, which is a subdomain of a quadric in the projective space $\mathbb{P}^4$

$$\Omega^+_t = \{ Z \in \mathbb{P}(L_t \otimes \mathbb{C}) \mid (Z, Z) = 0, (Z, \overline{Z}) < 0 \}^+$$

$$= \{ Z = i((tz_2^2 - z_1z_3), z_3, z_2, z_1, 1) \cdot z_0 \mid \text{Im}(z_1z_3 - tz_2^2) > 0, \text{Im}(z_1) > 0 \}$$

$$\cong \{ \tilde{z} = (z_3, z_2, z_1) \in \mathbb{C}^3 \mid \text{Im}(z_1z_3 - tz_2^2) > 0, \text{Im}(z_1) > 0 \} = \mathbb{H}_t^+,$$

where $\mathbb{H}_t^+$ is the 3-dimensional tube domain of type IV (see the part I of this paper for a general definition). Let us define an isomorphism $\phi_t$ of the Siegel upper half-plane $\mathbb{H}_2$ with $\mathbb{H}_t^+$ as

$$\psi_t \left( \begin{array}{c} \tau \\ z \\ \omega \end{array} \right) = i(t\omega, z, \tau) \in \mathbb{H}_t^+.$$ 

Using the formulae for $\Phi$-images of generators of the symplectic group given above we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}_2 & \xrightarrow{g} & \mathbb{H}_2 \\
\downarrow \psi_t & & \downarrow \psi_t \\
\mathbb{H}_t^+ & \xrightarrow{\Phi(g)} & \mathbb{H}_t^+ \\
\end{array}$$

(1.21)

where $g \in Sp_4(\mathbb{R})$.

Let us compare the Fourier expansions of modular forms with respect to symplectic and orthogonal groups. For $F(Z) \in \mathcal{M}_k(\Gamma_t, \chi)$ we have

$$F(Z) = \sum_{N=(n \frac{l}{2} \frac{m}{2}) \geq 0} a(N) \exp(2\pi i \text{tr}(NZ)) = \sum_{N=(n \frac{l}{2} \frac{m}{2}) \geq 0} a(N) q^n r^l s^m$$

where $n, l$ and $m$ run over some free $\mathbb{Z}$-modules depending on the character $\chi$. If we consider $F(\phi_t(Z))$ as a modular form with respect to the orthogonal group $SO^+(L_t)$ then we can rewrite the Fourier expansion above as

$$F(\tilde{z}) = F(z_1f_2 + z_2f_3 + z_3f_3) = \sum_{\ell=nf_2-lf_3+mf_{-2}, -\ell, l \geq 0} a(\ell) \exp(-2\pi i (\ell, z) L_t) = \sum\ell a(\ell) q^n r^l s^m$$

where the summation is taken over the same $n, l, m$ like in the symplectic Fourier expansion above, and $\tilde{f}_3 = \frac{1}{2}f_3$, $q = \exp(2\pi i z_1)$, $r = \exp(2\pi i z_2)$, $s = \exp(2\pi i z_3)$. Thus

$$a\left( \begin{array}{c} n \\ l/2 \\ m/2 \end{array} \right) = a(nf_2-lf_3+mf_{-2}).$$

A primitive $\ell = (e, a, -\frac{b}{2}, c, f) \in \widehat{L}_t$ (here primitive means $(e, a, b, c, f) = 1$) determines the quadratic rational divisor

$$\mathcal{H}_\ell = \{ Z \in \Omega^+_t \mid (\ell, Z) = 0 \} \cong \{ \left( \begin{array}{c} \tau \\ z \\ \omega \end{array} \right) \in \mathbb{H}_2 \mid t\ell f(z^2 - \tau \omega) + t\omega + bz + a\tau + c = 0 \} \quad (1.22)$$
of the discriminant $D(\ell) = 2t(\ell, \ell) = b^2 - 4tef - 4tac$. Due to the isomorphism $\Phi$, the groups $\Gamma_t$, $\Gamma_t^+$ and $\Gamma_t^*$ act on the set of all rational quadratic divisors with a fixed discriminant. We define a Humbert modular surface $H_\ell$ in the Siegel modular threefold $\mathcal{A}_\ell = \Gamma_t \backslash \mathbb{H}_2$ ($\mathcal{A}_\ell^+ = \Gamma_t^+ \backslash \mathbb{H}_2$ or $\mathcal{A}_\ell^* = \Gamma_t^* \backslash \mathbb{H}_2$ respectively) by

$$H_\ell = \pi_t \left( \bigcup_{g \in \Gamma_t} g^*(\mathcal{H}_\ell) \right),$$

where $\pi_t : \mathbb{H}_2 \to \mathcal{A}_t$ is the natural projection. We remark that according to our definition a Humbert modular surface is irreducible. The $SO(L_t)$-orbit of a primitive vector $\ell \in \mathcal{L}_t$ depends only on the norm of $\ell$ and its image in the discriminant group. Thus any Humbert surface in $\mathcal{A}_t$ of discriminant $D$ can be represented in the form

$$H_D(b) = \pi_t(\{Z \in \mathbb{H}_2 | a\tau + bz + t\omega = 0\})$$  \hspace{1cm} (1.23)

where $a, b \in \mathbb{Z}$, $D = b^2 - 4ta$ and $b \text{ mod } 2t$. From that follows that the number of Humbert surfaces in $\mathcal{A}_t$ of discriminant $D$ is equal to $\# \{ b \text{ mod } 2t | b^2 \equiv D \text{ mod } 4t \}$. The involution $V_t$ acts on the discriminant group by multiplication on $-1$. Therefore the number of the Humbert surfaces of discriminant $D$ in $\mathcal{A}_t^+$ is equal to $\# \{ \pm b \text{ mod } 2t | b^2 \equiv D \text{ mod } 4t \}$. About the theory of Humbert surfaces for the case of a non-principle polarization see [vdG] and [GH1].

1.4. The “trivial” lifting of the Jacobi theta-function.

We define the arithmetic lifting of Jacobi forms as an action of a formal operator $L$-function on a given Jacobi modular form. We may illustrate this approach with a classical example. Let us consider the even theta-series

$$\vartheta(\tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi in^2\tau).$$

The periodic function $\exp(2\pi i \tau)$ is invariant with respect to the parabolic subgroup $\Gamma_0 = \{( \pm 1_n \pm m \mid m \in \mathbb{Z} \} \subset SL_2(\mathbb{Z})$. The element $[n^{-1}] = \Gamma_0(n^{-1}_0 0)\Gamma_0$ of the Hecke ring $H(\Gamma_0)$ acts on the function $\exp(2\pi i \tau)$ in a very simple way: $\exp(2\pi i \tau)[[n^{-1}] = \exp(2\pi i n^2\tau)$. Thus

$$\vartheta(\tau) = 1 + 2 \sum_{n \in \mathbb{N}} \exp(2\pi i \tau)[[n^{-1}] = 1 + 2 \exp(2\pi i \tau)\left( \sum_{n \in \mathbb{N}} [n^{-1}] \right)$$

where we consider $[\zeta(1)] = \sum_{n \geq 1} [n^{-1}]$ as a formal Dirichlet series over the Hecke ring $H(\Gamma_0)$ which has an expansion in the formal infinite product $[\zeta(1)] = \prod_p (1 - [p^{-1}])^{-1}$ over all primes. We can interpret the last representation of the theta-series as a lifting of $\Gamma_0$-modular form $\exp(2\pi i \tau)$ defined by the formal zeta-function $[\zeta(1)]$. (See [G6] for some applications of this representation of $\vartheta(\tau)$.)

In this section we use this kind of a “trivial” lifting to define two special singular modular forms with respect to the full paramodular group $\Gamma_4$ and $\Gamma_{36}$. The first form is one of the classical Siegel theta-constants. The behavior of the theta-constants with respect to the
of order 4 is a modular form of weight 1. The function

\[ \Theta_{1,1}(Z) = \frac{1}{2} \sum_{l_1, l_2 \in \mathbb{Z}} \exp \left( \pi i \left( \frac{Z[l_1 + 1]}{l_2 + 1} \right) + l_1 + l_2 \right) = \frac{1}{2} \sum_{n, m \in \mathbb{Z}} \left( \frac{-4}{n} \right) \left( \frac{-4}{m} \right) q^{n^2/8} r^{nm/4} s^{m^2/8}, \]

where \( Z[M] = t M \), is a modular form with respect to the subgroup \( \Gamma_{2,1} \subset \text{Sp}_4(\mathbb{Z}) \) conjugated to the paramodular group \( \Gamma_4 \) (see (1.20)). Our second example is a modular form of weight 1/2 with respect to the full paramodular group \( \Gamma_{36} \).

**Theorem 1.11.** 1. The function

\[ \Delta_{1/2}(Z) = \frac{1}{2} \sum_{n, m \in \mathbb{Z}} \left( \frac{-4}{n} \right) \left( \frac{-4}{m} \right) q^{n^2/8} r^{nm/2} s^{m^2/2} \]

is a modular form of weight 1/2 with respect to the double extension \( \Gamma_{4}^+ \) of the paramodular group \( \Gamma_{4}(\text{see (1.19)}) \) with a multiplier system \( v_8 : \Gamma_{4}^+ \rightarrow \{ \sqrt[3]{\mathbb{Q}} \} \) which is induced by \( v_3 \times v_H \). It means, that

\[ v_8 |_{SL_2(\mathbb{Z})} = v_3, \quad v_8 |_{H(\mathbb{Z})} = v_H, \quad v_8([0, 0; \frac{3}{4}) = \exp \left( \frac{\pi i k}{4} \right) \quad (k \in \mathbb{Z}) \]

and \( \Delta_{1/2}(V_4(Z)) = \Delta_{1/2}(Z) \). Moreover the divisor of \( \Delta_{1/2}(Z) \) on \( \mathcal{A}_{4}^+ = \Gamma_{4}^+ \setminus \mathbb{H}_2 \) is exactly equal to the Humbert modular surface of discriminant 1

\[ \text{Div}_{\mathcal{A}_{4}^+}(\Delta_{1/2}(Z)) = H_1 = \pi_{1/4}^+(\{Z \in \mathbb{H}_2 \mid z = 0\}). \]

2. The function

\[ D_{1/2}(Z) = \frac{1}{2} \sum_{m, n \in \mathbb{Z}} \left( \frac{12}{n} \right) \left( \frac{12}{m} \right) q^{n^2/24} r^{nm/2} s^{m^2/2} \]

is a modular form of weight 1/2 with respect to the normal extension \( \Gamma_{36}^* = [\Gamma_{36}, V_4, V_9, V_{36}] \) of order 4 of \( \Gamma_{36} \) with a multiplier system \( v_{24} : \Gamma_{36}^* \rightarrow \{ \sqrt[3]{\mathbb{Q}} \} \) induced by \( v_\eta \times v_H \), i.e.

\[ v_{24} |_{SL_2(\mathbb{Z})} = v_\eta, \quad v_{24} |_{H(\mathbb{Z})} = v_H, \quad v_{24}([0, 0; \frac{\kappa}{36}) = \exp \left( \frac{\pi i k}{12} \right) \quad (k \in \mathbb{Z}) \]

and

\[ D_{1/2}(V_{36}(Z)) = D_{1/2}(V_9(Z)) = D_{1/2}(V_4(Z)) = D_{1/2}(Z). \]

Moreover the divisor of \( D_{1/2}(Z) \) on the threefold \( \mathcal{A}_{36}^* = \Gamma_{36}^* \setminus \mathbb{H}_2 \) consists of the three irreducible components with multiplicity one

\[ \text{Div}_{\mathcal{A}_{36}^*}(D_{1/2}) = H_{4}^* + H_{9}^*(27) + H_{16}^*(32), \]

where \( H_{4}^* = \pi_{36}^*(\{Z \in \mathbb{Z}_2 \mid 2z - 1 = 0\}) \) and

\[ H_{9}^*(27) = \pi_{36}^*(\{5\tau + 27z + 36\omega = 0\}), \quad H_{16}^*(32) = \pi_{36}^*(\{7\tau + 32z + 36\omega = 0\}). \]
Proof. Let us define a lifting of the Jacobi theta-series \( \tilde{\theta}(Z) = \theta(\tau, z) \exp(\pi i \omega) \) using a formal Dirichlet \( L \)-series over the Hecke ring \( H(\Gamma_{\infty}) \)

\[
L_{\pm} \left( \frac{1}{2}, \left( \frac{-4}{m} \right) \right) = \sum_{m \geq 1} m^{-\frac{1}{2}} \left( \frac{-4}{m} \right) \Lambda_m
\]

(see (1.10), (1.13)). After action of the formal operator Dirichlet series on \( \tilde{\theta}(Z) \), we get

\[
\Delta_{1/2}(Z) = \sum_{m > 0} m^{-\frac{1}{2}} \left( \frac{-4}{m} \right) \tilde{\theta} \left( \frac{Z}{4} \right) \Lambda_m(Z) = \sum_{n \in \mathbb{Z}, m \in \mathbb{N}} \left( \frac{-4}{m} \right) q^{n^2/8} r^{nm/2} s^{m^2/2}.
\]

Thus \( \Delta_{1/2}(Z) \) is invariant with weight \( k \) and with the multiplier system mentioned in the theorem with respect to the parabolic subgroup \( \Gamma_{\infty,4} = \Gamma_4 \cap \Gamma_{\infty}(\mathbb{Q}) \) of \( \Gamma_4 \). Moreover we see from this representation that \( \Delta_{1/2}(Z) \) is invariant with respect to the transformation \( q \to s^4, s \to q^{\frac{1}{4}} \) (i.e. \( \tau \to 4\omega, \omega \to \tau/4 \)) defined by \( V_4 \) (see (1.19)). According to Lemma 1.10 the \( \Delta_{1/2}(Z) \) is a modular form with respect to \( \Gamma_4^+ \). The last group is conjugate to \( \Gamma_{2,1}^+ \) (see (1.20)). Thus the function

\[
\Theta_{1,1}(Z) = \Delta_{\frac{1}{2}} \left( \left( \frac{\tau}{z/2}, \frac{z/2}{\omega/4} \right) \right)
\]

is a modular form with respect to the subgroup \( \Gamma_{2,1} \) of \( Sp_4(\mathbb{Z}) \). From the construction of \( \Delta_{1/2}(Z) \) as a lifting of the triple product, it follows that its divisor contains the Humbert surfaces \( H_1 \). To prove that this is the full divisor of \( \Delta_{1/2}(Z) \), let us consider a \( Sp_4(\mathbb{Z}) \)-modular form

\[
F_5(Z) = \prod_{g \in \Gamma_{2,1}^+ \setminus \Gamma_1} \left( \Theta_{1,1} \left( \frac{Z}{4} g \right) (Z) \right).
\]

Since \([\Gamma_1 : \Gamma_{2,1}^+] = \frac{p^4 + p^2}{2}\), the modular form \( F_5(Z) \) has weight 5. This is a cusp form, because \( F_5 \left( \left( \frac{Z}{z}, \omega \right) \right) \equiv 0 \) for \( z = 0 \). Up to a constant there exists only one \( Sp_4(\mathbb{Z}) \)-cusp form of weight 5 whose divisor is equal to \( H_1 \) (see [Fr2]). From this fact, it easily follows that the divisor of \( \Delta_{1/2}(Z) \) on \( A_4^+ \) is exactly \( H_1 \).

The construction of \( D_{1/2}(Z) \) is similar. The form

\[
D_{1/2}(Z) = \sum_{m > 0} m^{-\frac{1}{2}} \left( \frac{12}{m} \right) \tilde{\vartheta}_{3/2} \left( \frac{Z}{4} \right) \Lambda_m(Z) = \sum_{n \in \mathbb{Z}, m \in \mathbb{N}} \left( \frac{12}{m} \right) \left( \frac{12}{n} \right) q^{n^2/24} r^{nm/2} s^{m^2/2}
\]

is modular with respect to the parabolic subgroup \( \Gamma_{\infty,36} = \Gamma_{36} \cap \Gamma_{\infty}(\mathbb{Q}) \) with a multiplier system \( v_n \times v_H \) and is invariant with respect to the action of \( V_{36} \). The involutions \( V_9 \) and \( V_4 \) satisfy the relation \( \Gamma_{36}(V_9V_4) = \Gamma_{36}V_{36} \). Taking \( V_9 \) in the form (1.18) we see that the involution \( V_9 \) does not change the Fourier expansion. Thus \( D_{1/2}(Z) \) is a modular form with respect to \( \Gamma_{36}^* \).

The threefold \( A_{36}^+ \) contains the only Humbert surface \( H_4^+ \) of discriminant 4 which is a part of the divisor of \( D_{1/2}(Z) \) since \( \vartheta_{3/2}(\tau, z) \equiv 0 \) if \( z = \frac{1}{2} \). We remark that \( A_{36}^+ = \Gamma_{36}^+ \setminus \mathbb{H}_2 \) contains two Humbert surfaces of discriminant 4:

\[
H_4^+(2) = \pi_{36}^+(\{2z - 1 = 0\}) \quad \text{and} \quad H_4^+(34) = \pi_{36}^+(\{8\tau + 34z + 36\omega = 0\}).
\]
Let us prove that $D_{1/2}(Z)$ is anti-invariant with respect to the involutions which define the surfaces $H^*_9(27)$ and $H^*_{16}(32)$. For that we consider this function as a modular form with respect to the orthogonal group $PO(L_{36})$ (see Sect. 1.3)

$$D_{1/2}(z) = \frac{1}{2} \sum_{n,m \in \mathbb{Z}} \left( \frac{12}{n} \right) \left( \frac{12}{m} \right) \exp \left( 2\pi i \left( \frac{n^2}{24} z_1 + \frac{nm}{2} z_2 + \frac{m^2}{24} z_3 \right) \right).$$

The Humbert surface $H^*_9(27)$ is defined by the element $l_9 = (0, 5, \frac{27}{72}, 1, 0) \in \hat{L}_{36}$ (see (1.21) and (1.23)). In the basis $(f_2, f_3, f_{-2})$ defined in (1.14), the involution $\sigma_{l_9}$ has the matrix

$$\left( \sigma_{l_9} \right) = \begin{pmatrix} 81 & -24 \cdot 90 & 400 \\ 6 & -161 & 30 \\ 16 & -24 \cdot 18 & 81 \end{pmatrix}.$$

Thus

$$D_{1/2}(\sigma_{l_9}(z)) = \sum_{n \in \mathbb{Z}, m \in \mathbb{N}} \left( \frac{12}{n} \right) \left( \frac{12}{m} \right) \exp \left( 2\pi i \left( \frac{(20m-9n)^2}{24} z_1 + \frac{(20m-9n)(9m-4n)}{2} z_2 + \frac{(9m-4n)^2}{24} z_3 \right) \right) = -D_{1/2}(z)$$

and $D_{1/2}(Z)$ has zero along the Humbert surface $H^*_9(27)$. For the surface $H^*_{16}(32)$ the proof is similar.

Using the “projection” of type (1.24) of the modular form $D_{1/2}(\left( \begin{smallmatrix} \tau \\ z/6 \\ \omega/36 \end{smallmatrix} \right))$ with respect to the group $\Gamma_{6,1}$ to the space of modular forms with respect to $Sp_4(\mathbb{Z})$, we can prove that its divisor consists of three Humbert surfaces $H^*_4$, $H^*_9(27)$ and $H^*_{16}(32)$ with multiplicities one, because the weight of the $Sp_4(\mathbb{Z})$-modular form, whose divisor is exactly $H_4$, is known (see [GN4] for the construction of such functions). In §2 we shall prove the statement about the divisor of $D_{1/2}(Z)$ using another method, which also gives us infinite product expansions of $\Delta_{1/2}(Z)$ and $D_{1/2}(Z)$ (see (2.11) and (2.14)).

\[\square\]

### 1.5. The arithmetic lifting of Jacobi forms of half-integral index.

The next theorem is a generalization of a construction of cusp forms with respect to the paramodular groups proposed in [G1] and [G3]. This is an advanced version of Maass lifting (see [M1]–[M2]). The main construction is valid for $SO(n, 2)$-group (see [G4]–[G5]).

**Theorem 1.12.** Let $k$ be integral, $t$ be integral or half-integral, $D$ be an even divisor of 24. We take the conductor $Q = 24/D$ and $\mu \in (\mathbb{Z}/Q\mathbb{Z})^*$. Let us consider a Jacobi cusp form $\phi \in J_{k,t}^{\text{cusp}}(v^D_\eta \times v^*_{H})$, where $\varepsilon = 0$ or $\varepsilon = 1$. Then the function

$$F_\phi(Z) = \text{Lift}_\mu(\phi)(Z) = \sum_{m \equiv \mu \mod Q, m > 0} m^{2-k}(\widetilde{\phi}|_k T^Q(m))(Z)$$
(see (1.12)) is a cusp form with respect to a paramodular group, and

\[ \text{Lift}_\mu(\phi)(Z) \in \mathfrak{M}_k(\Gamma_{Qt}^+, \chi_{D, \varepsilon, \mu}) \quad (\text{if } tQ \in \mathbb{Z}) \]
\[ \text{Lift}_\mu(\phi)(Z) \in \mathfrak{M}_k(\Gamma'_{4Qt}, \chi_{1, \mu}) \quad (\text{if } tQ \text{ is half-integral}) \]

where \( \Gamma'_{4Qt} = \delta_2 \Gamma_{4Qt}^+ \delta_2^{-1} \) (\( \delta_2 = \text{diag}(1, 2, 1, 2^{-1}) \)). If \( \text{Lift}_\mu(\phi) \neq 0 \), then the \( \chi_{D, \varepsilon, \mu} \) is a character of order \( Q \) or \( 2Q \) of the group \( \Gamma_{Qt}^+ \) or \( \Gamma'_{4Qt} \) respectively. This character is induced by \( v_{\eta, \mu}^D \times v_H^\varepsilon \), where \( v_{\eta, \mu}^D \) is a character of \( SL_2(\mathbb{Z}) \)-conjugated to \( v_{\eta}^D \) (see Lemma 1.7), and by the relations

\[ \chi_{D, \varepsilon, \mu}(V_{Qt}) = (-1)^k, \quad \chi_{D, \varepsilon, \mu}([0, 0; \frac{K}{Qt}]) = \exp\left(2\pi i \frac{\mu K}{Q}\right) \quad (\kappa \in \mathbb{Z}). \]

If \( \mu = 1 \), then \( \text{Lift}_1(\phi)(Z) \neq 0 \) for \( \phi \neq 0 \), i.e. we have an embedding of the space \( J_{k, t}^{\text{cusp}}(v_{\eta}^D \times v_H^\varepsilon) \) into the space of Siegel modular forms.

**Remark 1.** We denote \( \text{Lift}_1 \) by \( \text{Lift} \). If \( \mu \neq 1 \), the lifting can be zero for some non-zero Jacobi forms.

**Remark 2.** It is easy to give a variant of this theorem for non-cusp Jacobi forms. One should only add a natural condition on weights and characters of \( SL_2 \)-Eisenstein series in order to get convergence of the lifting (compare with [G4]). It gives interesting examples of Eisenstein series with respect to the paramodular groups. We hope to describe them somewhere else.

**Remark 3.** In the notation \( \mathfrak{M}_k(\Gamma_{t}, \chi_{D, \varepsilon, \mu}) \) we will put the corresponding character \( v_{\eta, \mu}^D \times v_H^\varepsilon \) of the Jacobi group instead of the induced character of \( \Gamma_{Qt}^+ \).

**Proof.** The convergence of the series defining \( \text{Lift}_\mu(\phi) \) follows from the upper bound of Jacobi cusp forms of weight \( k \) and index \( t \) on \( \mathbb{H}_1 \times \mathbb{C} \):

\[ |\phi(\tau, z)| < Cv^{-\frac{\kappa}{2}} \exp(2\pi ty^2/v), \]

where \( v = \text{Im } \tau > 0 \), \( y = \text{Im } z \) and the constant \( C \) does not depend on \( \tau \) and \( z \). To prove the last inequality, we take the function

\[ \phi^*(\tau, z) = v^\frac{k}{2} \exp(-2\pi ty^2/v)|\phi(\tau, z)| \]

which is \( \Gamma^J \)-invariant and is bounded on any compact subset in \( \mathbb{H}_1 \times \mathbb{C} \). If we take the following realization of the fundamental domain \( D \) of \( \Gamma^J \) on \( \mathbb{H}_1 \times \mathbb{C} \)

\[ D = \{(\tau, \alpha \tau + \beta) \mid -1 \leq \alpha, \beta \leq 1, \tau \in SL_2(\mathbb{Z}) \setminus \mathbb{H}_1\}, \]

then the function \( \phi^* \) is bounded on the set \( \{(\tau, z) \in D, \text{Im } \tau > C\} \) because \( \phi^*(\tau, z) \to 0 \) as \( v \to \infty \) for any cusp form \( \phi(\tau, z) \).
Let us consider the Fourier expansion of the Jacobi form $\phi(\tau, z)$ with the character $v^D_\eta \times v^\varepsilon_H$

$$\phi(\tau, z) = \sum_{\substack{n \equiv D \mod 24 \atop l \equiv \varepsilon \mod 2 \atop n > 0, 4nt > l^2}} f(n, l) \exp \left(2\pi i \left(\frac{n}{24} \tau + \frac{l}{2} z\right)\right)$$

$$= \sum_{\substack{N \equiv 1 \mod Q \atop l \equiv \varepsilon \mod 2 \atop N > 0, 4NDt > l^2}} f(ND, l) \exp \left(2\pi i \left(\frac{N}{Q} \tau + \frac{l}{2} z\right)\right).$$

By the definition (1.12) of the Hecke operators, we have

$$m^{2-k} (\tilde{\phi}|_k T^{(Q)}_{-}(m))(Z) =$$

$$m^{k-1} \sum_{ad = m} d^{-k} v^D_\eta(\sigma_a) \sum_{\substack{N \equiv 1 \mod Q \atop l \equiv \varepsilon \mod 2 \atop b \equiv \varepsilon \mod d}} f(DN, l) \exp \left(2\pi i \left(\frac{N(a\tau + bQ)}{Qd} + \frac{al}{2} z + mt\omega\right)\right)$$

$$= \sum_{ad = m} a^{k-1} v^D_\eta(\sigma_a) \sum_{dN_1 \equiv 1 \mod Q} f(dN_1 D, l) \exp \left(2\pi i \left(\frac{aN_1}{Q} \tau + \frac{al}{2} z + mt\omega\right)\right).$$

If $m \equiv \mu \mod Q$, then the condition $dN_1 \equiv 1 \mod Q$ ($N = dN_1$) is equivalent to $aN_1 \equiv \mu \mod Q$, because $ad \equiv \mu \mod Q$, $Q$ is a divisor of 24 and for arbitrary $d$ with $(d, 24) = 1$ it is true that $d^2 \equiv 1 \mod 24$. Hence taking the summation over all $m \equiv \mu \mod Q$, we get

$$\sum_{m \equiv \mu \mod Q} m^{2-k} (\tilde{\phi}|_k T^{(Q)}_{-}(m))(Z) =$$

$$\sum_{a > 0, d > 0} \sum_{ad \equiv \mu \mod Q} a^{k-1} v^D_\eta(\sigma_a) \sum_{aN_1 \equiv \mu \mod Q, \atop l \equiv \varepsilon \mod 2} f(dN_1 D, l) \exp \left(2\pi i \left(\frac{aN_1}{Q} \tau + \frac{al}{2} z + ad\varepsilon\right)\right)$$

$$= \sum_{\substack{N, M > 0 \atop N, M \equiv \mu \mod Q}} \left( \sum_{a|N, L, M} a^{k-1} v^D_\eta(\sigma_a) f\left(\frac{NMD}{a^2}, \frac{L}{a}\right) \right) \exp \left(2\pi i \left(\frac{N}{Q} \tau + \frac{L}{2} z + M\varepsilon\omega\right)\right).$$

(1.26)

Let us suppose that $Qt$ is integral. Then either $\varepsilon = 0$ or $\varepsilon = 1$ and $Q$ is even. In both cases $\tilde{\phi}|_k T^{(Q)}_{-}(m)$ is a $\Gamma_\infty$-modular form with character $v^D_{\eta, \mu} \times v^\varepsilon_H$ for all $m \equiv \mu \mod Q$ (see Lemma 1.8). Moreover

$$\tilde{\phi}|_k T^{(Q)}_{-}(m)|_{k[0, 0; \frac{K}{Qt}] = \exp \left(2\pi i \frac{K\mu}{Q}\right)} \tilde{\phi}|_k T^{(Q)}_{-}(m).$$

Thus the action of the center of the parabolic subgroup $\Gamma_\infty, Qt = \Gamma_{Qt} \cap \Gamma_\infty(Q)$ on $F_\phi(Z)$ is given by the character

$$\nu_Q([0, 0; \frac{K}{Qt}]) = \exp \left(2\pi i \frac{\mu K}{Q}\right)$$
of order $Q$. Therefore the lifting $F_\phi(Z)$ is a $\Gamma_{\infty,Qt}$-modular form with character $v^{\phi}_{\eta,\mu} \times v^\omega_H \times \nu_Q$. The Fourier expansion calculated above shows that $F_\phi(Z)$ is invariant under the change of the variables \{\(\tau \to Qt\omega, \ z \to z, \ \omega \to (Qt)^{-1}\tau\}\}. This transformation is made by the element $V_{Qt}$ (see (1.19)). Thus

$$\left( F_\phi|_k V_{Qt} \right)(Z) = (-1)^k F_\phi(Z).$$

Moreover we have $F_\phi|_k J_{Qt} = F_\phi$, where

$$J_{Qt} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (Qt)^{-1} \\ -1 & 0 & 0 & 0 \\ 0 & -Qt & 0 & 0 \end{pmatrix}$$

is the standard element from the group $\Gamma_{Qt}$, since $V_{Qt} I V_{Qt} I = J_{Qt}$ where $I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_{\infty}$.

This proves that for arbitrary $M \in \Gamma_{Qt}^+ = \Gamma_{Qt} \cup \Gamma_{Qt} V_{Qt}$

$$\left( F_\phi|_k M \right)(Z) = \chi(M) F(Z)$$

where $\chi(M)^Q = 1$ for $M \in \Gamma_t$. If $F_\phi(Z) \not\equiv 0$, then we have a character

$$\chi : \Gamma_{tQ}^+ \to \mathbb{C}^*$$

of the paramodular group extending the character $v^{\phi}_{\eta,\mu} \times v^\omega_H$ because for $k \in \mathbb{Z}$ the operator $|_k$ defines the action of the group on the space of functions. We denote this character $\chi_{D,\varepsilon,\mu}$. Thus $F_\phi(Z)$ is a $\Gamma_{Qt}^+$-modular form with this character. If $\mu = 1$, then the lifting is an injective embedding of the space of Jacobi forms into the corresponding space of $\Gamma_t$-modular forms, because the first Fourier-Jacobi coefficient of $F_\phi(Z)$ is $\phi(\tau, z) \not\equiv 0$. This finishes the proof in the case $tQ \in \mathbb{Z}$.

Let us suppose that $tQ$ is half-integral. Then $\varepsilon = 1$ and $Q$ is odd, i.e. $Q = 3$. Let us consider the Jacobi form

$$\psi(\tau, z) = \phi(\tau, 2z) = 2^{-k}(\phi|_k \Lambda_2)(\tau, z) \in J_k,4t(v^D_q \times 1_H).$$

Its lifting $F_\psi(Z)$ is a modular form of weight $k$ with respect to the group $\Gamma_{4Qt}$ with a character induced by the character $v^{\phi}_{\eta,\mu} \times \text{id}_H$. The operators $T_{-}(Q)(m)$ and $\Lambda_2$ commute. Thus

$$F_\psi(Z) = 2^k \sum_{m \equiv \mu \ mod \ Q} m^{2-k}(\tilde{\phi}|_k \Lambda_2)(Z) = (F_\phi|_k \delta_2)(Z)$$

where $\delta_2 = \text{diag}(1, 2, 1, 2^{-1})$. It proves that $F_\phi$ is a modular form with respect to $\delta_2^{-1} \Gamma_{4Qt} \delta_2$ with a character of order 3 if $Qt$ is half-integral.

The proof that the lifting is a cusp form is the same as for $Q = 1$ (see [G1]).
1.6. Examples of the arithmetic lifting.

Below we give series of examples of liftings of Jacobi forms. It provides us with many important Siegel modular forms with respect to the paramodular groups. These modular forms have interesting applications to the algebraic geometry and to the theory of Lorentzian Kac–Moody algebras. It was proved in [G1] that the field of rational functions with respect to $\Gamma_t$ (or, equivalently, the moduli space of Abelian surfaces with polarization of type $(1, t)$) might be rational only for the twenty exceptional polarizations $t = 1, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36$. This list partly explains our interest to the modular forms of small weight with respect to $\Gamma_t$. Another starting point for the construction of these modular forms is the classification of hyperbolic generalized Cartan matrices in the part I of this paper (see also [GN4]). The Siegel modular forms obtained as the arithmetic lifting of holomorphic Jacobi forms will define automorphic Lorentzian Kac–Moody algebras corresponding to these generalized Cartan matrices. One of the advantages of the arithmetic lifting is that we have formula (1.26) for all Fourier coefficients of the lifted form in terms of Fourier coefficients of the corresponding Jacobi form. The only factor we have to calculate is $v_D^D(\sigma_a) = \pm 1$. According to Lemma 1.2, we have

$$v_D^D(\sigma_a) = \begin{cases} 1 & \text{if } Q_D = 1, 2, 3, 6 \\ \left(\frac{-4}{a}\right) & \text{if } Q_D = 4, 12 \end{cases} \tag{1.27}$$

for $Q_D = 24/(24, D)$.

**Example 1.13.** The case of trivial character. If $\chi = \text{id}_{SL_2} \times \text{id}_H$, we get the lifting constructed in [G1] and [G4]

$$\text{Lift} : \mathcal{J}_c^{\text{cusp}}(k, t) \rightarrow \mathfrak{M}_k(\Gamma_t).$$

The case of $t = 1$ is the original Maass lifting defined in [M1]. It is known that $\dim(J_{1, t}) = 0$ (see [Sk]). Thus the arithmetic lifting with trivial character gives us modular forms of weight $k \geq 2$. For $k = 3$ we obtain canonical differential forms on the moduli spaces of Abelian surfaces with $(1, t)$-polarization (see [G1], [G2]) or on some finite quotients of such threefolds (see [GH1]). In the examples below we shall see that if we admit Jacobi forms with commutator characters (characters of type $v_{\eta}^D \times v_{H}^f$), we can construct roots of order $d$ ($d$ is a divisor of 12) from some $\Gamma_t$-modular forms.

**Example 1.14.** Three dimensional variants of Dedekind $\eta$-function: $\Delta_1(Z)$, $\Delta_2(Z)$ and $\Delta_5(Z)$. We construct a series of examples of cusp forms using the Jacobi forms of the type $\eta^d(\tau) \vartheta(\tau, z) \in \mathcal{J}_c^{\text{cusp}}(v_{\eta}^{d+3} \times v_{H})$ where $d = 1, 3, 9, 23$. For any $d$ mentioned above we obtain the cusp forms

$$\text{Lift}_1(\eta^d(\tau) \vartheta(\tau, z)) \in \mathfrak{M}_{d+1}(\Gamma_{t_d}, v_{\eta}^{d+3} \times v_{H})$$

for the paramodular groups with $t_1 = 3, t_3 = 2, t_9 = 1$ and $t_{23} = 6$. Let us put

$$\eta(\tau)^d = \sum_{n \in \mathbb{N}} \tau_d(n)q^{n/24}.$$
Then according to (1.26) we get an exact formula for the Fourier coefficients of the lifting \( \text{Lift}_1(\eta^d(\tau)\vartheta(\tau,z))(Z) \) in terms of \( \tau_d(n) \). For all cases when we know an elementary expression for \( \tau_d(n) \), we get an elementary formula for the Fourier coefficients of the lifting. In particular for \( d = 1 \) and \( d = 3 \) we obtain two very nice cusp forms of weight 1 and 2 with respect to the group \( \Gamma_3 \) and \( \Gamma_2 \) respectively with the following Fourier expansions

\[
\Delta_1(Z) = \sum_{M \geq 1} \sum_{n,m>0,l \in \mathbb{Z}} \sum_{n,m \equiv 1 \mod 6} \sum_{4nm-3l^2=M^2} \left( \frac{-4}{l} \right) \left( \frac{12}{M} \right) \frac{6}{a} q^{n/6} r^{l/2} s^{m/2} \in \mathcal{M}_1(\Gamma_3, v_4^4 \times v_H)
\]

(where we use notation (1.7)) and

\[
\Delta_2(Z) = \sum_{N \geq 1} \sum_{n,m>0,l \in \mathbb{Z}} N \left( \frac{-4}{N} \right) \sum_{a \mid (n,l,m)} \left( \frac{-4}{a} \right) q^{n/4} r^{l/2} s^{m/2} \in \mathcal{M}_2(\Gamma_2, v_6^6 \times v_H).
\]

Hence all Fourier coefficients \( a(N) \) of the *cusp form of weight one* \( \Delta_1(Z) \) corresponding to the primitive matrices \( N \) are equal to \( \pm 1 \) or 0! We may say that this function is the simplest Siegel cusp form. \( \Delta_1(Z) \) and \( \Delta_2(Z) \) have properties similar to the Dedekind \( \eta \)-function. For example, \( \Delta_1(Z) \) is a root of order 6 of the unique up to a constant cusp form (with trivial character) of weight 6 for \( \Gamma_3 \) and \( \Delta_2(Z) \) is a root of order 4 from the unique cusp form of weight 8 for \( \Gamma_2 \). Both functions have infinite product expansions and they are the discriminants of moduli spaces of \( K3 \) surfaces of special types. (See \S 5 below and [GN2]–[GN3] where \( \Delta_2 \) was used.)

The modular forms \( \Delta_1(Z) \) and \( \Delta_2(Z) \) are also connected with the theory of moduli spaces \( A_t = \Gamma_t \setminus \mathbb{H}_2 \) of Abelian surfaces with polarization of type \((1,t)\) (without level structure). Using \( \Delta_1(Z) \) and \( \Delta_2(Z) \) we can prove the rationality of the moduli spaces \( A_3 \) and \( A_2 \) respectively.

We would like to remark that \( \Delta_1(Z)^3 dZ = \Delta_1(Z)^3 d\tau \wedge dz \wedge d\omega \) defines a canonical differential form on the double cover

\[
^{2}A_3 = ^{2}\Gamma_3 \setminus \mathbb{H}_2 \xrightarrow{2:1} A_3.
\]

where \( ^{2}\Gamma_3 = \text{Ker}(\chi_6^3) \) is a subgroup of index 2 of \( \Gamma_3 \) (\( \chi_6 \) is the character of \( \Delta_1(Z) \) of order 6). For some other applications of constructed modular forms to the moduli spaces of Abelian and Kummer surfaces see [G2], [GH1]–[GH2].

For \( d = 9 \) the lifting of \( \eta(\tau)^9 \vartheta(\tau,z) \) is equal to the well known Siegel cusp form \( \Delta_5(Z) \) of weight 5 with respect to the full Siegel modular group \( \Gamma_1 = \text{Sp}_4(Z) \) with the non-trivial binary character

\[
\Delta_5(Z) = \text{Lift}(\eta(\tau)^9 \vartheta(\tau,z)) = \frac{1}{64} \prod_{(a,b)} \vartheta_{a,b}(Z)
\]

where the product is taken over all even Siegel theta-constants (compare this definition with construction (1.24)). This function was constructed as a lifting by Maass in [M2]. It
the second Jacobi form, thus
\[ \Gamma_F \]

(We denote by has the Fourier expansion
\[ \Delta \]

It was proved in [GN1]–[GN3] that the functions \( \Delta \) and \( \Delta_2 \) are Weyl–Kac denominator functions for some generalized Kac–Moody superalgebras.

**Example 1.15.** The function \( \Delta_{11}(Z) \) and the \( \mu \)-Lifting for \( \mu = -1 \). The functions \( \Delta_1(Z) \), \( \Delta_2(Z) \) and \( \Delta_5(Z) \) are connected with the first case of Theorem 1.12 when \( Qt \) is even. Now we consider the second one when \( Qt \) is odd. Let us take Jacobi cusp forms
\[ \eta^2(\tau)\vartheta(\tau,2z) \in J_{3,2}(v_6^8 \times \text{id}_H), \quad \eta^2(\tau,2z) \in J_{11,2} \]

where the last Jacobi form has trivial character. Then \( Q = 3 \) for the first and \( Q = 1 \) for the second Jacobi form, thus
\[ \Delta_{11}(Z) = \text{Lift}(\eta(\tau)^{21}\vartheta(\tau,2z)) \in \mathcal{N}_{11}(\Gamma_2), \]
\[ F_3^{(6)}(Z) = \text{Lift}(\eta(\tau)^5\vartheta(\tau,2z)) \in \mathcal{N}_3(\Gamma_6, v_6^8 \times \text{id}_H). \]

(We denote by \( F_k^{(t)}(Z) \) a modular form of weight \( k \) with respect to the paramodular group \( \Gamma_t \).) According to Theorem 1.12 and Lemma 1.8 for \( Q = 3 \) (\( \mu \equiv 2 \equiv -1 \mod 3 \))
\[ \text{Lift}_2(\eta^5(\tau)\vartheta(\tau,2z)) \in \mathcal{N}_3(\Gamma_6, v_6^{16} \times \text{id}_H), \]
if it does not vanish identically. To prove this, we calculate
\[ (\eta^5(\tau)\vartheta(\tau,2z)|_{3} T_{-}^{(3)}(2))(Z) = q^{\frac{2}{3}}r^{-3}(1 - r^2)(1 - r)^4 + \cdots \neq 0. \]

In fact we can prove that
\[ (\eta^5(\tau)\vartheta(\tau,2z)|_{3} T_{-}^{(3)}(2))(Z) = \eta(\tau)\vartheta^4(\tau,z)\vartheta(\tau,2z) \in J_{3,4}(v_6^{16} \times \text{id}_H). \]
The product of two “conjugated” liftings is a cusp form with trivial character
\[ F_6^{(6)}(Z) = \text{Lift}_1(\eta(\tau)^5\vartheta(\tau,2z)) \cdot \text{Lift}_2(\eta(\tau)^5\vartheta(\tau,2z)) \in \mathcal{N}_6(\Gamma_6). \]

The construction of the arithmetic lifting provides us with some information about divisor of lifted forms. Using Theorem 1.12 and (1.12) one can prove

**Lemma 1.16.** Let \( \text{Lift}_\mu(\phi) \) be the lifting defined in Theorem 1.12. Let us assume that \( \phi(\tau,z)|_{z=0} \equiv 0 \), and this zero has order \( m \). Then \( \text{Lift}_\mu(\phi) \) has zero of order at least \( m \) along the Humbert surface \( H_1(0) \), where \( H_1(0) = \pi_{Qt}^{+}((\{Z | z = 0\}) \subset A_{Qt}^{+} \) (see Sect. 1.3).

More generally, let \( M \subset \mathbb{Q}/\mathbb{Z} \) be a subset which is invariant with respect to multiplication on arbitrary \( a \in \mathbb{Z} \) such that \( (a,Q) = 1 \). Let us assume that \( \phi(\tau,z)|_{z=\alpha} \equiv 0 \) of order \( m \) for all \( \alpha \in M \). Then \( \text{Lift}_\mu(\phi) \) has zero of order at least \( m \) along the Humbert surface \( H(\alpha) = \pi_{Qt}^{+}((\{Z | z = \alpha\}) \subset A_{Qt}^{+} \) for any \( \alpha \in M \).

In particular, for the form \( F_6^{(6)}(Z) \) given above
\[ \text{Div}_{A_6^{+}}(F_6(Z)) \supset 2H_1 + 2H_4. \]
Examples 1.17. The quintuple product. Let us consider the quintuple product

$$\vartheta_{3/2}(\tau, z) = \sum_{n \in \mathbb{Z}} \left( \frac{12}{n} \right) q^{n^2/24} r^{n/2} \in J^{\text{cusp}}_{1/2}(\mathbb{H})$$

(see Lemma 1.6). Then for $d = 1, 3, 5, 11$ we can construct the following Siegel cusp forms

$$\text{Lift}(\eta(\tau)^d \vartheta_{3/2}(\tau, z)) \in \mathcal{M}_{d+1}(\Gamma_{t_d}, v_{\eta}^{d+1} \times v_H)$$

for the paramodular groups of level $t_1 = 18, t_3 = 9, t_5 = 6$ and $t_{11} = 3$ respectively. For $d = 7$ and $23$ we define

$$\text{Lift}(\eta(\tau)^7 \vartheta_{3/2}(\tau, 2z)) \in \mathcal{M}_4(\Gamma_{18}, v_\eta^8 \times \text{id}_H),$$

$$\text{Lift}(\eta(\tau)^{23} \vartheta_{3/2}(\tau, 2z)) \in \mathcal{M}_{12}(\Gamma_6).$$

Like in Example 1.14 we may calculate the exact form of the Fourier expansion of such forms. The case of $d = 1$ and $d = 3$, when we get the cusp forms of weight $1$ and $2$, are of the special interest. They are

$$D_1(Z) = \sum_{M \geq 1} \sum_{n,m \geq 0, l \in \mathbb{Z}} \frac{(12/M)}{a(n,l,m)} \sum_{a \mid (n,l,m)} \left( -\frac{4}{a} \right) q^{n/12} r^{l/2} s^{3m/2} \in \mathcal{M}_1(\Gamma_{18}, v_\eta^2 \times v_H), \quad (1.29)$$

$$D_2(Z) = \sum_{N \geq 1} \sum_{n,m \geq 0, l \in \mathbb{Z}} N \left( -\frac{4}{N} \right) \left( \frac{12}{l} \right) \sum_{a \mid (n,l,m)} \left( \frac{6}{a} \right) q^{n/6} r^{l/2} s^{3m/2} \in \mathcal{M}_2(\Gamma_9, v_\eta^4 \times v_H). \quad (1.30)$$

Next two examples show how one can use the Hecke operators $\Lambda_n$ to construct new Jacobi cusp forms of half-integral index.

Lemma 1.18. Let $a, b \in \mathbb{N}$ be such that $(a, b) = 1$. Then

$$\vartheta(\tau, az)\vartheta(\tau, bz) \in J^{\text{cusp}}_{1/2}(a^2 + b^2)(v_\eta^6 \times v_H), \quad \text{if } ab \text{ is even},$$

$$\vartheta_{3/2}(\tau, az)\vartheta_{3/2}(\tau, bz) \in J^{\text{cusp}}_{1/2}(a^2 + b^2)(v_\eta^2 \times v_H^{a+b}), \quad \text{if } (ab, 6) \neq 1,$$

$$\vartheta(\tau, az)\vartheta_{3/2}(\tau, bz) \in J^{\text{cusp}}_{1/2}(a^2 + 3b^2)(v_\eta^4 \times v_H^{a+b}), \quad \text{if } (a, 3) = 1 \lor (a, 2) = 2 \lor (b, 6) \neq 1.$$

Proof. Let us consider the Fourier expansion

$$\vartheta(\tau, az)\vartheta(\tau, bz) = \sum_{n,m \in \mathbb{Z}} \left( \frac{-4}{n} \right) \left( \frac{-4}{m} \right) q^{n^2 + m^2} r^{am + bm}.$$  

For an arbitrary Fourier coefficient $f(N, L)$ the norm of its index

$$2(a^2 + b^2)N - L^2 = \frac{1}{4}(am - bm)^2 \geq 0$$

can be zero only if $n$ or $m$ is even, because $ab$ is even. Since $\left( \frac{-4}{2n} \right) = 0$, we get the first statement of the lemma. The proof of the second and third statement is similar.
Example 1.19. Cusp forms of weight one. The Jacobi cusp forms of weight 1 from the last lemma generate a series of Siegel cusp forms of weight one:

\[
\begin{align*}
\text{Lift}(\vartheta(\tau, az)\vartheta(\tau, bz)) &\in \mathcal{M}_1(\Gamma_2(a^2+b^2), v_\eta^6 \times v_{H}), \\
\text{Lift}(\vartheta_{3/2}(\tau, az)\vartheta_{3/2}(\tau, bz)) &\in \mathcal{M}_1(\Gamma_{18}(a^2+b^2), v_\eta^2 \times v_{H}^{\alpha+b}), \\
\text{Lift}(\vartheta(\tau, az)\vartheta_{3/2}(\tau, bz)) &\in \mathcal{M}_1(\Gamma_{3'(a^2+3b^2)}, v_\eta^4 \times v_{H}^{\alpha+b}).
\end{align*}
\]

Example 1.20. \(\vartheta(\tau, z)\vartheta(\tau, 2z)-\)lifting series. We can use Jacobi cusp forms of weight one to produce Siegel modular forms of small weights with respect to \(\Gamma_t\) of small level \(t\). We give some examples for the groups \(\Gamma_4 - \Gamma_{10}\) in the case of \(\vartheta(\tau, z)\vartheta(\tau, 2z)\). First we have

\[
\text{Lift}(\vartheta(\tau, z)\vartheta(\tau, 2z)) \in \mathcal{M}_1(\Gamma_{10}, v_\eta^6 \times v_{H}).
\]

We may combine the last Jacobi cusp form together with \(\eta(\tau)\) or \(\vartheta(\tau, z)\). For example we can get the cusp form of minimal weight and with trivial character for \(\Gamma_5\)

\[
F_5^{(5)}(Z) = \text{Lift}(\eta(\tau)^3\vartheta(\tau, z)^6\vartheta(\tau, 2z)) \in \mathcal{M}_5(\Gamma_5).
\]

According to Lemma 1.16

\[
\text{Div}_{\mathcal{A}_5}(F_5(Z)) \supset 7H_1 + H_4
\]

where \(H_d\) are the corresponding Humbert surfaces in the threefold \(\mathcal{A}_5 = \Gamma_5^+ \setminus \mathbb{H}_2\). We also obtain

\[
\begin{align*}
\text{Lift}(\eta^6\vartheta(\tau, z)\vartheta(\tau, 2z)) &\in \mathcal{M}_4(\Gamma_5, v_\eta^{12} \times v_{H}), & \text{Lift}(\eta^3\vartheta^2 \cdot \vartheta(\tau, 2z)) &\in \mathcal{M}_3(\Gamma_6, v_\eta^{12} \times \text{id}_H), \\
\text{Lift}(\eta^3 \cdot \vartheta(\tau, 2z)) &\in \mathcal{M}_2(\Gamma_7, v_\eta^{12} \times v_{H}), & \text{Lift}(\eta^3 \cdot \vartheta(\tau, 2z)^2) &\in \mathcal{M}_3(\Gamma_9, v_\eta^{12} \times v_{H}), \\
\text{Lift}(\eta^9 \cdot \vartheta(\tau, 2z)) &\in \mathcal{M}_7(\Gamma_4), & \text{Lift}(\vartheta \cdot \vartheta(\tau, 2z)^3) &\in \mathcal{M}_2(\Gamma_{13}, v_\eta^{12} \times v_{H}).
\end{align*}
\]

In the lemma below we consider a construction which is similar to the quintuple product (see Lemma 1.6).

Lemma 1.21. The functions

\[
\phi_{1,4}(\tau, z) = \eta(\tau)^2 \frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)} \in J_{1,4}^{\text{cusp}}(v_\eta^2 \times \text{id}_H),
\]

\[
\psi(\tau, z) = \eta(\tau)^3 \left(\frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)}\right)^2 \in J_{2,4}^{\text{cusp}}(v_\eta^3 \times \text{id}_H)
\]

are holomorphic Jacobi forms.

Proof. Let us consider a weak Jacobi form of weight 0 and index 4

\[
\phi_{0,4}(\tau, z) = \frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)} = r + 1 + r^{-1} + q(-r^4 - r^3 + r + 2 + r^{-1} - r^{-3} - r^{-4}) + \ldots
\]

The Fourier coefficient \(f(n, l)\) of \(\phi_{0,4}\) depends only on the “norm” \(16n - l^2\) and \(\pm l\) mod 8. Moreover \(16n - l^2 \geq -16\) if \(f(n, l) \neq 0\). From the exact form of coefficients with \(q^0\) it follows that \(16n - l^2 \geq -1\) if \(f(n, l) \neq 0\), and \(f(n, l) = 1\) if \(16n - l^2 = -1\). Thus \(\phi(\tau, z)\) is a cusp form. The condition \(16n - l^2 \geq -1\) implies holomorphicity of \(\psi(\tau, z)\).
Example 1.22. The lifting of the Jacobi cusp form \( \phi_{1,4} \) considered in the last lemma gives us Siegel cusp forms of weight 1, 2 and 4

\[
\text{Lift} (\phi_{1,4}) \in \mathcal{N}_1(\Gamma_{48}, v_7^2 \times \text{id}_H), \quad \text{Lift} (\eta^2 \phi_{1,4}) \in \mathcal{N}_2(\Gamma_{24}, v_7^4 \times \text{id}_H),
\]

\[
\text{Lift} (\eta^6 \phi_{1,4}) \in \mathcal{N}_4(\Gamma_{12}, v_7^6 \times \text{id}_H).
\]

All these modular forms have zero along the Humbert surface \( H_9 \) in the corresponding moduli spaces \( \mathcal{A}^+_t \).

One can use some differential operators to construct new Jacobi forms with a commutator character. The following lemma is a direct reformulation of Eichler–Zagier construction.

Lemma 1.23. (See [EZ, Theorem 9.5].) Let

\[
\phi_1 \in J_{k_1,m_1}(v_7^{d_1} \times v_H^{\varepsilon_1}), \quad \phi_2 \in J_{k_2,m_2}(v_7^{d_2} \times v_H^{\varepsilon_2})
\]

be two Jacobi forms of integral or half-integral indices, where \( \varepsilon_i = 0 \) or 1. Then one can define the Jacobi form

\[
[\phi_1, \phi_2] = \frac{1}{2\pi i} (m_2 \phi_1' \phi_2 - m_1 \phi_1 \phi_2') \in J_{k_1+k_2+1,m_1+m_2}(v_7^{d_1+d_2} \times v_H^{\varepsilon_1+\varepsilon_2}),
\]

where

\[
\phi'(\tau, z) = \frac{\partial \phi(\tau, z)}{\partial z}.
\]

Proof. One should consider the Jacobi function \( \psi = \phi_1^{m_2} / \phi_2^{m_1} \) of index zero. Then

\[
\left( \frac{\phi_2^{m_1+1}}{\phi_1^{m_2-1}} \right) \frac{\partial \psi(\tau, z)}{\partial z} = (m_2 \phi_1' \phi_2 - m_1 \phi_1 \phi_2') \in J_{k_1+k_2+1,m_1+m_2}(v_7^{d_1+d_2} \times v_H^{\varepsilon(m_1+m_2)}).
\]

\[ \square \]

We can combine two Jacobi theta-functions \( \vartheta(\tau, z) \) and \( \vartheta_{3/2}(\tau, z) \) using the differential operator of Lemma 1.23.

Lemma 1.24. The Jacobi form

\[
\phi_{2,2}(\tau, z) = 2[\vartheta(\tau, z), \vartheta_{3/2}(\tau, z)] \in J_{2,2}(v_7^4 \times \text{id}_H)
\]

is a cusp form with integral Fourier coefficients and

\[
\frac{\phi_{2,2}(\tau, z)}{\eta(\tau)} \in J_{4,2}(v_7^3 \times \text{id}_H).
\]

Proof. We can calculate its Fourier expansion

\[
\phi_{2,2}(\tau, z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (3m-n) \left( \frac{4}{m} \right) \left( \frac{12}{n} \right) q^{3m^2+16n^2} r^{m+n} \quad (1.34)
\]
For all Fourier coefficients \( f(N, L) \) of this Jacobi form we have \( 8N - L^2 = \frac{1}{12}(3m-n)^2 \geq \frac{1}{3} \). The Fourier expansion shows that \( \phi_{2,2}/\eta \) is still holomorphic.

**Remark.** One can check that \([\phi_1, \phi_2]\) in Lemma 1.23 is a Jacobi cusp form (maybe, identical to zero in some cases).

The Jacobi form constructed in the last lemma, produces liftings

\[
\text{Lift}(\phi_{2,2}) \in \mathfrak{M}_2(\Gamma_1, v^4_\eta \times \text{id}_H),
\text{Lift}(\eta^2 \phi_{2,2}) \in \mathfrak{M}_3(\Gamma_8, v^6_\eta \times \text{id}_H),
\text{Lift}(\eta^4 \phi_{2,2}) \in \mathfrak{M}_4(\Gamma_6, v^8_\eta \times \text{id}_H),
\text{Lift}(\eta^8 \phi_{2,2}) \in \mathfrak{M}_6(\Gamma_4, v^{12}_\eta \times \text{id}_H),
\text{Lift}(\eta^{20} \phi_{2,2}) \in \mathfrak{M}_{12}(\Gamma_2).
\]

One can rewrite the definition of \( \phi_{2,2} \) using only the Jacobi theta-series \( \vartheta(\tau, z) \)

\[
\phi_{2,2}(\tau, z) = 2\eta(\tau) \frac{[\vartheta(\tau, z), \vartheta(\tau, 2z)]}{\vartheta(\tau, z)}.
\]

It gives another formula for the non-cusp form \( \phi_{2,2}(\tau, z)/\eta(\tau) \).

**Example 1.25.** The Jacobi form \( \phi_{3,1}(\tau, z) \). One can construct Jacobi forms of half-integral indices using also Jacobi forms of integral indices. For example, let us consider the quotient

\[
\phi_{3,1}(\tau, z) = \frac{\phi_{12,1}(\tau, z)}{\eta(\tau)^{18}} \in \mathfrak{M}_3(\Gamma_8, v^{12}_\eta \times \text{id}_H)
\]

where

\[
\phi_{12,1}(\tau, z) = (r + 10 + r^{-1})q + (10r^2 - 88r - 132 - 88r^{-1} + 10r^{-2})q^2 + \ldots
\]

is the unique Jacobi cusp form of weight 12 and index 1 with integral coprime Fourier coefficients (see [EZ]). This quotient is again a holomorphic Jacobi form since \( 4n - l^2 \geq 3 \) for all Fourier coefficients of \( \phi_{12,1} \). The Jacobi form \( \phi_{3,1}(\tau, z) \) is very useful in many questions. We have the following lifting for \( d = 2 \) and \( d = 6 \)

\[
\text{Lift}(\eta^2 \phi_{3,1}) \in \mathfrak{M}_4(\Gamma_3, v^6_\eta \times \text{id}_H),
\text{Lift}(\eta^6 \phi_{3,1}) \in \mathfrak{M}_6(\Gamma_2, v^{12}_\eta \times \text{id}_H).
\]

One can use these modular forms and the modular forms from the first example to construct all generators of the graded rings of symmetric modular forms for \( \Gamma_2 \) and \( \Gamma_3 \). We are planning to consider these type of questions in a publication which follows.

### § 2. Infinite Product Expansion

In this section we consider an exponential or Borcherds lifting of Jacobi forms of weight 0 in the space of meromorphic Siegel modular forms with respect to a paramodular group. It will give us infinite product expansion of the modular forms constructed in § 1 and description of their divisors.
2.1 Borcherds lifting. We prove below a variant of the Borcherds construction of automorphic forms as infinite products (see [Bo6, Theorem 10.1]). We remark that Theorem 10.1 in [Bo6] was formulated only for an unimodular lattice. The statement and the main part of the proof are valid for an arbitrary lattice \( L \) of signature \((n + 2, 2)\) with two orthogonal unimodular isotropic planes if one replaces in the theorem a nearly holomorphic modular form \( f(\tau) \) with a nearly holomorphic Jacobi form \( \phi(\tau, z) \) of weight 0 and index 1 with respect to an anisotropic lattice of rank \( n \), and if one uses results about generators of the orthogonal group \( \hat{O}(L) \) proved in [G4]. In this section we consider the case of the full paramodular group \( \Gamma_t \).

Let

\[
\phi_{0,t}(\tau, z) = \sum_{n, l \in \mathbb{Z}} f(n, l) q^n r^l \in J_{0,t}^{nh} \quad (2.1)
\]

be a nearly holomorphic Jacobi form of weight 0 and index \( t \) (i.e. \( n \) might be negative in the Fourier expansion). We recall the notations \( q = \exp(2\pi i \tau) \), \( r = \exp(2\pi i z) \), \( s = \exp(2\pi i \omega) \) and \( \phi_{0,t}(Z) = \phi_{0,t}(\tau, z) \exp(2\pi it \omega) \). Let

\[
\phi_{0,t}^{(0)}(z) = \sum_{l \in \mathbb{Z}} f(0, l) r^l \quad (2.2)
\]

be the \( q^0 \)-part of \( \phi_{0,t}(\tau, z) \). The Fourier coefficient \( f(n, l) \) of \( \phi_{0,t} \) depends only on the norm \( 4tn - l^2 \) of \((n, l)\) and \( l \mod 2t \). From the definition of nearly holomorphic forms, it follows that the norm of indices of non-zero Fourier coefficients are bounded from below.

**Theorem 2.1.** Assume that the Fourier coefficients of Jacobi form \( \phi_{0,t} \) from (2.1) are integral. Then the product

\[
\text{Exp-Lift}(\phi_{0,t})(Z) = B_{\phi}(Z) = q^A r^B s^C \prod_{n, l, m \in \mathbb{Z}} (1 - q^n r^l s^m)^{f(nm, l)}, \quad (2.3)
\]

where

\[
A = \frac{1}{24} \sum_l f(0, l), \quad B = \frac{1}{2} \sum_{l > 0} lf(0, l), \quad C = \frac{1}{4} \sum_l l^2 f(0, l),
\]

and \((n, l, m) > 0\) means that if \( m > 0 \), then \( l \) and \( n \) are arbitrary integers, if \( m = 0 \), then \( n > 0 \) and \( l \in \mathbb{Z} \) or \( l < 0 \) if \( n = m = 0 \), defines a meromorphic modular form of weight \( f^{(0,0)} \) with respect to \( \Gamma_t^+ \) with a character (or a multiplier system if the weight is half-integral) induced by \( v^{2A}_n \times v^{2B}_H \). All divisors of \( \text{Exp-Lift}(\phi_{0,t})(Z) \) on \( \mathcal{A}_t^+ \) are the Humbert modular surfaces \( H_D(b) \) of discriminant \( D = b^2 - 4ta \) (see (1.23)) with multiplicities

\[
m_{D,b} = \sum_{n > 0} f(n^2 a, nb). \quad (2.4)
\]

Moreover

\[
B_{\phi}(V_t(Z)) = (-1)^D B_{\phi}(Z) \quad \text{with} \quad D = \sum_{n \leq 0} \sigma_1(-n) f(n, l) \quad (2.5)
\]
where \( \sigma_1(n) = \sum_{d|n} d \).

**Remark.** One obtains the same statement for any Jacobi form \( \phi_{0,t}(\tau, z_1, \ldots, z_n) \) of weight 0 and index \( t \) where \( S \) is an even anisotropic quadratic form of rank \( n \). Then the function \( \text{Exp-Lift}(\phi_{0,t}^{(S)}) \) is a meromorphic automorphic form with respect to the orthogonal group \( \tilde{SO}^+(U^2 \oplus S(t)) \) of signature \((n + 2, 2)\) where \( U \) is the unimodular isotropic plane and \( S(t) \) is the lattice of rank \( n \) with the quadratic form \( tS \).

**Proof.** The product (2.3) is a particular case of automorphic products considered in [Bo6, Theorem 5.1]. It converges for \( \det(\text{Im}(\tau)) > c \), where \( c \) is sufficiently large, and it can be extended to a multi-valued meromorphic function on \( \mathbb{H}_2 \) whose singularities, including zeros, lie on rational quadratic divisors \( \mathcal{H}_t \) of type (1.22).

Let us decompose the product of Theorem 2.1 in two factors

\[
B_\phi(Z) = q^{A_r B_s C} \prod_{(n,l,m) > 0} (1 - q^{n r^l s^m} f^{(n,l)}) \times \prod_{n,l,m \in \mathbb{Z}, m > 0} (1 - q^{n r^l s^m} f^{(nm,l)})
\]

and let us calculate the Fourier expansion of the logarithm of the second factor:

\[
\log \left( \prod_{n,l,m \in \mathbb{Z}, m > 0} (1 - q^{n r^l s^m} f^{(nm,l)}) \right) = - \sum_{n,l \in \mathbb{Z}, m > 0} f^{(nm,l)} \sum_{c \geq 1} \frac{1}{c} q^{en r^c s^m t^c}.
\]

The last sum can be written as the action of the formal Dirichlet series \( \sum_{m \geq 1} m^{-1} T(m) \) on the Jacobi form \( \phi_{0,t} \) (see Theorem 1.12). Thus we obtain

\[
\log \left( \prod_{n,l,m \in \mathbb{Z}, m > 0} \ldots \right) = - \sum_{m \geq 1} m^{-1} \left( \phi_{0,t} | T(m) \right)(Z).
\]

This expansion shows us that the second factor in (2.6) is invariant with respect to the action of the parabolic subgroup \( \Gamma_{\infty,t} \) whenever the product converges.

It is easy to see that the first factor in (2.6) is equal to a product of Jacobi theta-series and Dedekind eta-functions

\[
q^{A_r B_s C} \prod_{(n,l,m) > 0} (1 - q^{n r^l s^m} f^{(n,l)}) = \eta(\tau) f^{(0,0)} \prod_{l > 0} \left( \frac{\partial(\tau, l z) e^{\pi i l^2 \omega \eta(\tau)}}{\eta(\tau)} \right)^{f^{(0,l)}}.
\]

The last identity explains the form of the factor \( q^{A_r B_s C} \) in the definition of the function of the theorem. Thus we proved that

\[
q^{A_r B_s C} \prod_{n,l,m \in \mathbb{Z}, (n,l,m) > 0} (1 - q^{n r^l s^m} f^{(nm,l)})
\]

\[
= \eta(\tau)^{f^{(0,0)}} \prod_{l > 0} \left( \frac{\partial(\tau, l z) e^{\pi i l^2 \omega \eta(\tau)}}{\eta(\tau)} \right)^{f^{(0,l)}} \exp \left( - \sum_{m \geq 1} m^{-1} \phi_{0,t} | T_- (m)(Z) \right)
\]

(2.7)
whenever the product converges. Thus $B_\phi(Z)$ transforms like a $\Gamma_{\infty,t}$-modular form of weight $\frac{1}{2}(1,0)$ with the multiplier system of the theorem. It is useful to write down the whole product $B_\phi(Z)$ in terms of Hecke operators $T_-(m)$. We can get such expression using the involution $V_t$ (see (1.19))

$$\text{Exp-Lift}(\phi_{0,t})(Z) = B_\phi(Z) =$$

$$= q^{A_t B_s C} \exp \left( - \sum_{m \geq 1} m^{-1} \tilde{\phi}_{0,t}[T_-(m)](Z) \right) \exp \left( - \sum_{m \geq 1} m^{-1}(\tilde{\phi}_{0,t}^{(0)} + \phi_{0,t}^{(0)})(0)\right) V_t(Z).$$

(2.8)

The functions $\phi_{0,t}^{(0)}(z) = \sum_{l} f(0, l)r^l$ and $\tilde{\phi}_{0,t}^{(0)}(Z) = \sum_{l} f(0, l)r^l s^l$ are not Jacobi forms, and we fix the standard system of representatives (1.11) in $T_-(m)$ to define the corresponding formal action. The exponent of the function $\tilde{\phi}_{0,t}^{(0)}$ in (2.8) defines the subproduct over all $(n, l, 0)$ with $n > 0$ in (2.3). The exponent with the function $\phi_{0,t}^{(0)}$, which does not depend on $\tau$ and $\omega$, defines the finite subproduct over $(0, l, 0)$ with $l > 0$. The representation (2.8) shows us analogy between the exponential lifting and the lifting of holomorphic forms defined in Theorem 1.12. In §3 we shall use (2.8) to prove that the exponential lifting commutes with some Hecke correspondence. (In fact one can consider the factor before the exponent in (2.7) as the Hecke operator $T(0)$.)

Let us check the behavior of $B_\phi(Z)$ with respect to the involution $V_t : (q, r, s) \mapsto (s^t, r^t, q^{1/t})$. In the product only the terms $(1 - q^{n_r l}s^{tm})f^{(n, m, l)}$ with $n < 0$ do not have $V_t$-pairing terms. Thus

$$\frac{B_\phi(V_t(Z))}{B_\phi(Z)} = (sq^{1/t})^{tA-tD-C} \prod_{(n, l, m) > 0} \prod_{n < 0} \frac{(s^{-tn} - r^tq^m)f^{(n, m, l)}}{(q^{-n_r l} - s^{tm})f^{(n, m, l)}} = (-1)^D(sq^{1/t})^{tA-tD-C}$$

where $D$ was defined in (2.5). The finite product in the last formula is equal to $(-1)^D$. The first factor is equal to one according to

Lemma 2.2. For arbitrary nearly holomorphic Jacobi form $\phi_{0,t} = \sum_{n, l} f(n, l)q^n r^l$ the identity

$$24(tA - tD - C) = t \sum_l f(0, l) - 24t \sum_{n < 0, l} \sigma_1(n)f(n, l) - 6 \sum_l l^2 f(0, l) = 0$$

is valid.

Proof of the lemma. We use the differential operator

$$L_k = 8\pi i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} - \left( \frac{2k - 1}{z} \right) \frac{\partial}{\partial z}$$

defined in [EZ, §3]. If the Jacobi form $\phi_{12,t}(\tau, z) = \Delta(\tau)\phi_{0,t}(\tau, z)$ is weak holomorphic (i.e. when $f(n, l) \neq 0$ implies $n \geq -1$), then $t \sum_l f(0, l) - 24t f(-1, l) - 6 \sum_l l^2 f(0, l)$ is the constant term of $(D_2 \phi_{12,t})(\tau)$ (see [EZ, Theorem 3.1]) which is a cusp form of weight 14 for $SL_2(\mathbb{Z})$. Thus it is equal to zero. For arbitrary $\phi_{0,t}$ let us consider

$$f_2(\tau) = \Delta(\tau)^{-1} L_{12}(\Delta(\tau) \phi_{0,t}(\tau, z))|_{z=0}$$
which is a nearly holomorphic $SL_2(\mathbb{Z})$-form of weight 2. Let $\phi_{0,\ell}(\tau, z) = \sum_{\nu \geq 0} \chi_\nu(\tau)z^\nu$ is a Taylor expansion around $z = 0$ where $\chi_\nu(\tau) = \frac{(2\pi i)^\nu}{\nu!} \sum_n (\sum_f \nu f(n, \ell))q^n$. Thus

$$f_2(\tau) = 8\pi i\left(\frac{\Delta'(\tau)}{\Delta(\tau)} - \chi_0'(\tau)\right) - 48\chi_2(\tau)$$

where $(2\pi i)^{-1} \frac{\Delta'(\tau)}{\Delta(\tau)} = E_2(\tau) = 1 - 24\sum_{n \geq 1} \sigma_1(n)q^n$. It shows that the sum in the right hand side of the identity of the lemma is (up to the constant $(4\pi i)^2$) the constant term of $f_2(\tau)$. But the constant term of any nearly holomorphic modular form of weight 2 is equal to zero (see [Bo6, Lemma 9.2]). The lemma is proved.

The involution $V_\ell$ and the group $\Gamma_{\infty, \ell}$ generate the double extension $\Gamma^+_\ell$ of the paramodular group. Hence we have proved that $B_\phi(Z)$ transforms like a Siegel modular form of weight $f(0,0)/2$ with a character (or a multiplier system) $v : \Gamma^+_\ell \to \mathbb{C}^*$ induced by the character of the product of the Jacobi forms and the Dedekind $\eta$-functions in (2.7) together with the relation (2.5).

By [Bo6] (see the proof of Theorem 5.1 and Theorem 10.1 there) any two branches of the analytical continuation of the modular product $B_\phi$ differ by multiplication on a non-zero constant. Any Humbert surface in $A^+_\ell$ can be represented in the form $H_\ell$ with a primitive $\ell = (0, a, -b, 2t, 1, 0) \in L^*_t$ (see Sect. 1.3). For such $\ell$ with $D = 2t(\ell, \ell) = b^2 - 4at > 0$, the divisor $H_\ell$ has non-trivial intersection with a neighborhood of $\mathbb{H}_2$ at infinity where the product $B_\phi(Z)$ converges if it does not coincide with zero of a factor of type $(1 - q^{na}r^{nb}s^{tn})$ with $f(n^2a, nb) \neq 0$ of the product. Therefore $B_\phi(Z)$ is holomorphic univalent along any quadratic divisor $H_D(b)$ or it has zero (or pole) along this divisor of order $m_{D,b} = \sum_{n>0} f(n^2a, nb)$. The theorem is proved.

2.2 The basic Siegel modular forms. In the rest of the paper we show that many of modular forms of small weights constructed as the arithmetic lifting of Jacobi forms of half-integral indices have infinite product expansion, i.e. they can be represented as an exponential lifting. In this subsection we consider the most fundamental examples of Siegel modular forms $\Delta_1/2(Z)$, $\Delta_1(Z)$, $\Delta_2(Z)$ and $\Delta_5(Z)$ with the divisor equals to Humbert modular surface $H_1$ in the corresponding threefold $A^+_1$ (for $t = 4, 3, 2, 1$ respectively). These modular forms define the automorphic corrections of the most important Lorentzian Kac–Moody algebras of signature $(2, 1)$. An automorphic correction of a Lorentzian Kac–Moody algebra is defined by the Fourier coefficients of a modular form with appropriate character with respect to the Weyl group of a reflective hyperbolic lattice. According to the Weyl–Kac–Borcherds denominator formula for generalized Kac–Moody superalgebras, the multiplicities of the infinite product expansion define the multiplicities of positive roots of the corresponding algebra.

We start with some general remarks. If one has an automorphic product of type considered above, then it is an important and difficult problem to calculate an exact form of the Fourier coefficients of this product. For example, this coefficients defines the set of imaginary simple roots of automorphic generalized Kac–Moody algebra (see [Bo1]–[Bo5] and [GN1]–[GN4]). Our approach to this problem is to find identities between arithmetic and exponential liftings. If an arithmetic lifting $Lift(\phi_{k,d})$ would coincide with exponential
one, then there exists a natural formula for the Jacobi form of weight zero \( \phi_{0,t} \) which is
the datum for the exponential lifting

\[
\phi_{0,t} = \frac{m^{2-k} \phi_{k,d}|kT_-|m_2}{\phi_{k,d}}
\]

(2.9)

where \( T_-|m_2 \) is the second Hecke operator of the sequence of operators used to produce
the arithmetic lifting of \( \phi_{k,d} \). Thus one needs to check Fourier coefficients with negative
norm of \( \phi_{k,d}|kT_-|m_2/\phi_{k,d} \). If it is good enough, then using information about divisors or
arguments with dimension of the space of modular forms one can prove an identity between
liftings. We really do this in many cases below. To illustrate this method, we start with an
infinite product expansion of modular forms of the singular weight constructed in Theorem
1.11.

Example 2.3. Infinite product expansion of theta-functions. Let us analyze the Jacobi
form (2.9) for the case of two liftings of singular Jacobi forms (see Theorem 1.11). For the
case of the theta-function \( \Delta_{1/2}(Z) \) we get the weak Jacobi form defined in (1.33)

\[
\phi_{0,4}(\tau, z) = \frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)} = \sum_{n \geq 0, l \in \mathbb{Z}} f_4(n, l)q^n r^l
\]

\[
= r^{-1} \prod_{m \geq 1} (1 + q^{m-1}r + q^{2m-2}r^2)(1 + q^m r^{-1} + q^{2m} r^{-2}) \prod_{n \equiv 1, 2 \mod 3} (1 - q^n r^3)(1 - q^n r^{-3})
\]

\[
= (r + 1 + r^{-1}) - q(r^4 + r^3 - r + 2 - r^{-1} + r^{-3} + r^{-4}) + q^2(\ldots)
\]

(2.10)

where all Fourier coefficients \( f_4(n, l) \) of the weak Jacobi form are integral (in fact they are
Fourier coefficients of automorphic forms of weight \(-1/2\)). Thus according to Theorem 2.1,
\( \text{Exp-Lift}(\phi_{0,4}) \) is a modular form of weight 1/2 with respect to the paramodular group \( \Gamma_4^+ \n
\)having irreducible Humbert modular surface \( H_1 \) as its divisor. It implies that the quotient
\( \Delta_{1/2}(Z)/\text{Exp-Lift}(\phi_{0,4})(Z) \) is a holomorphic automorphic function invariant with respect
to \( \Gamma_4^+ \), thus it is a constant. Moreover we get the following infinite product expansion of
\( \Delta_{1/2}(Z) \):

\[
\frac{1}{2} \sum_{n,m \in \mathbb{Z}} \left( \frac{-4}{n} \right) \left( \frac{-4}{m} \right) q^{n^2/8} r^{nm/2} s^{m^2/2} = q^{1/8} r^{1/2} s^{1/2} \prod_{(n,l,m) > 0} (1 - q^n r^l s^{4m}) f_4(n,m,l).
\]

(2.11)

Let us consider the weak Jacobi form (2.9) connected with \( D_{1/2}(Z) \)

\[
\phi_{0,36}(\tau, z) = \frac{\vartheta_{3/2}(\tau, 5z)}{\vartheta_{3/2}(\tau, z)} = \frac{\vartheta(\tau, 10z)\vartheta(\tau, z)}{\vartheta(\tau, 5z)\vartheta(\tau, 2z)} = \sum_{n \geq 0, l \in \mathbb{Z}} f_{36}(n, l) q^n r^l
\]

\[
= r^{-2} \prod_{n \geq 1} \frac{(1 + q^{n-1}r^5)(1 + q^n r^{-5})(1 - q^{2n-1}r^{10})(1 - q^{2n-1}r^{-10})}{(1 + q^{n-1}r)(1 + q^n r^{-1})(1 - q^{2n-1}r^2)(1 - q^{2n-1}r^{-2})}
\]

\[
= (r^2 - r^1 + 1 - r^{-1} + r^{-2}) + q^2(-r^{17} + \ldots) + q^5(r^{27} + \ldots)
\]

\[
+ q^7(r^{32} + \ldots) + q^8(r^{34} + \ldots) + \ldots,
\]

(2.12)
where we include in the last formula only summands \( q^n r^l \) with the negative norm \( 144n - l^2 \):
\[
144 \cdot 2 - 17^2 = -1, \quad 144 \cdot 5 - 27^2 = -9, \quad 144 \cdot 7 - 32^2 = -16, \quad 144 \cdot 8 - 34^2 = -4.
\]

In calculations with weak Jacobi forms we shall often use the following simple considerations. Let
\[
\phi_{0,t}(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} f_t(n, l) q^n r^l \in J_{0,t}^{\text{weak}}.
\]
Then \( f_t(n, l) \) depends only on the norm \( 4nt - l^2 \) and \( \pm l \text{ mod } 2t \); \( f_t(n, l) = 0 \) if \( 4nt - l^2 < -t^2 \). Moreover, to calculate all Fourier coefficients with negative norm, it is enough to find \( f_t(n, l) \) for
\[
n \leq \left\lfloor \frac{t}{4} \right\rfloor (t \not\equiv 0 \text{ mod } 4) \quad \text{or} \quad n \leq \frac{t}{4} - 1 (t \equiv 0 \text{ mod } 4).
\]
These arguments imply the fact that the last part of (2.12) contains all orbits of possible Fourier coefficients with negative norm. (See (4.5) below for another formula for \( \phi_{0,36}(\tau, z) \).) Using Theorem 2.1, we get
\[
\operatorname{Div}_{A^+_n} \left( \text{Exp-Lift}(\phi_{0,36}) \right) = H_4(2) + H_4(34) + H_9(27) + H_{16}(32).
\]
(See the notation (1.23)). The surfaces \( H_4(2) \) and \( H_4(34) \) are equivalent with respect to the group \( \Gamma_{36}^* \) (see Theorem 1.11). This finishes the proof of the statement of Theorem 1.11 about the divisor of \( D_{1/2}(Z) \) and gives us the infinite product expansion of \( D_{1/2}(Z) \):
\[
\frac{1}{2} \sum_{m, n \in \mathbb{Z}} \left( \frac{12}{n} \right) \left( \frac{12}{m} \right) q^{n^2/24} r^{nm/2} s^{3m^2/2} = q^{1/24} r^{1/2} s^{3/2} \prod_{(n, l, m) > 0} (1 - q^{n r l s^{36 m}}) f_{36}(nm, l).
\]

**Example 2.4.** The modular form \( \Delta_5(Z) \). We recall that there exists a weak Jacobi form of weight zero and index one
\[
\phi_{0,1}(\tau, z) = \frac{\phi_{12,1}(\tau, z)}{\eta(\tau)^{24}} = (r + 10 + r^{-1}) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) + q^2(\ldots)
\]
which is one of the standard generators of the graded ring of weak Jacobi forms. There is a formula for Fourier coefficients of \( \phi_{12,1}(\tau, z) \) in terms of H. Cohen’s numbers (see [EZ], §9). The function \( \phi_{0,1} \) gives the following result from [GN1] about the product of even theta-constants:
\[
\Delta_5(Z) = \sum_{n, l, m \equiv 1 \text{ mod } 2 \atop n, m > 0} a_{(n, l, m)} (-1)^{l+q+2} a^4 \tau_9 \left( \frac{4nm - l^2}{a^2} \right) q^{n/2} r^{l/2} s^{m/2}
\]
\[
= (qrs)^{1/2} \prod_{n, l, m \in \mathbb{Z} \atop (n, l, m) > 0} (1 - q^{r l s^{m}}) f_{1}(nm, l) \in \mathfrak{M}_5(\Gamma_1, \chi_2)
\]
where \( f_1(n, l) \) are the Fourier coefficients of \( \phi_{0,1}(\tau, z) \). We note that (2.9) provides us with another formula for \( \phi_{0,1}(\tau, z) \):
\[
\phi_{0,1}(\tau, z) = -\frac{\langle \eta(\tau)^9 \theta(\tau, z) \rangle_5 T_-(3)}{\eta(\tau)^9 \theta(\tau, z)}
\]
which represents \( \phi_{0,1}(\tau, z) \) as a sum of four infinite products.

Our next purpose is to construct the infinite product expansion of the modular forms \( \Delta_1 \) and \( \Delta_2 \) defined in Example 1.14. For a large \( m \) \((m = 5 \text{ and } 7 \text{ in the case under consideration})\) the calculation of \( \phi_{k,d}|T_m(\Delta_1)/\phi_{k,d} \) from (2.9) might be rather long, thus we prefer to construct the same function in a non-direct way. In the next lemma we define two weak Jacobi forms of weight 0 using the quintuple product and the Jacobi forms constructed in Lemma 1.24.

**Lemma 2.5.** The functions

\[
\phi_{0,2}(\tau, z) = \frac{\phi_{2,2}(\tau, z)}{\eta(\tau)^4} = \frac{1}{2} \eta(\tau)^{-4} \sum_{m,n \in \mathbb{Z}} (3m-n) \left( \frac{-4}{m} \right) \frac{1}{n} q^{\frac{3m^2+n^2}{2} + \frac{m+n}{2}}
\]

\[
= \sum_{n \geq 0, l \in \mathbb{Z}} f_2(n, l) q^n r^l = (r + 4 + r^{-1}) + q(r^{\pm 3} - 8r^{\pm 2} - r^{\pm 1} + 16) + q^2(\ldots) \quad (2.18)
\]

(\( r^{\pm l} \) means that we have two summands with \( r^l \) and with \( r^{-l} \)) and

\[
\phi_{0,3}(\tau, z) = \left( \frac{\vartheta_{3/2}(\tau, z)}{\eta(\tau)} \right)^2 = \left( \frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)} \right)^2 = \sum_{n \geq 0, l} f_3(n, l) q^n r^l
\]

\[
= r^{-1} \left( \prod_{n \geq 1} (1 + q^{n-1}r)(1 + q^n r^{-1})(1 - q^{2n-1} r^2)(1 - q^{2n-1} r^{-2}) \right)^2
\]

\[
= (r + 2 + r^{-1}) + q(-4r^{\pm 3} - 4r^{\pm 2} + 2r^{\pm 1} + 4) + q^2(\ldots) \quad (2.19)
\]

are weak Jacobi forms of weight 0 and index 2 and 3 respectively. Moreover, their Fourier coefficients satisfy the property: if \( 4tn - l^2 < 0 \) and \( f_t(n, l) \neq 0 \), then \( 4tn - l^2 = -1 \) and \( f_t(n, l) = 1. \)

**Proof.** According to (2.13) one has to calculate the \( q^0 \)-part of the Fourier expansions of the given Jacobi forms. We also remark that for a prime \( p \) Fourier coefficient \( f_p(n, l) \) of a Jacobi form of index \( p \) depends only on the norm \( 4np - l^2 \). Thus (2.15) gives the Fourier coefficients \( f_2(n, l) \) with \( 8n - l^2 = -1, 4, 7, 8 \) and (2.17) gives \( f_3(n, l) \) with \( 12n - l^2 = -1, 3, 8, 11, 12. \)

\[\square\]

The modular form \( \Delta_1 \) with respect to \( \Gamma_3 \) and \( \Delta_2 \) with respect to \( \Gamma_2 \) (see Example 1.14) are analogous to \( \Delta_5 \) and they also satisfy a generalized Euler-type identity (for \( \Delta_2 \) it was given in [GN1]). Using the modular forms \( \Delta_1, \Delta_2 \) and \( \Delta_5 \), we can define the automorphic corrections of symmetric generalized Cartan matrices of elliptic type of rank three with a lattice Weyl vector of Theorem 1.3.1 in Part I. Comparing the divisors of \( \Delta_2(Z) \) and \( \Delta_1(Z) \) with the divisors of \( \text{Exp-Lift}(\phi_{0,t})(Z) \) for \( t = 2 \) and 3 respectively we obtain
Theorem 2.6. The following identities are valid:

\[
\Delta_1(Z) = \sum_{M \geq 1} \sum_{m>0, l \in \mathbb{Z}} \left( \frac{-4}{l} \right) \left( \frac{12}{M} \right) \sum_{a \mid (n,l,m)} \left( \frac{6}{a} \right) q^{n/6} r^l/2 s^{m/2} \\
= q^{\frac{1}{8}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{n,l,m \in \mathbb{Z}} (1 - q^n r^l s^{3m})^{f_3(nm,l)} \in \mathcal{M}_1(\Gamma^+_3, \chi_6) \tag{2.20}
\]

where the character \( \chi_6 : \Gamma^+_3 \to \sqrt{1} \) is induced by \( v_\eta^4 \times v_H \) and

\[
\Delta_2(Z) = \sum_{N \geq 1} \sum_{m>0, l \in \mathbb{Z}} N \left( \frac{-4}{N} \right) \sum_{a \mid (n,l,m)} \left( \frac{-4}{a} \right) q^{n/4} r^l/2 s^{m/2} \\
= q^{\frac{1}{16}} r^{\frac{1}{4}} s^{\frac{1}{4}} \prod_{n,l,m \in \mathbb{Z}} (1 - q^n r^l s^{2m})^{f_2(nm,l)} \in \mathcal{M}_2(\Gamma^+_2, \chi_4) \tag{2.21}
\]

where \( \chi_4 : \Gamma^+_2 \to \sqrt{1} \) is induced by \( v_\eta^6 \times v_H \). Moreover the divisor of these modular forms is the irreducible Humbert surface \( H_1 \)

\[
\text{Div}_{\mathcal{A}_4^+}(\Delta_1(Z)) = H_1, \quad \text{Div}_{\mathcal{A}_2^+}(\Delta_2(Z)) = H_1.
\]

One can check (one only needs to calculate the \( q^0 \)-part of the corresponding Fourier expansion) that the weak Jacobi forms used above as the data for the exponential lifting of all \( \Delta_k \)-forms are related by simple relations

\[
\phi_{0,1}(\tau, 2z) = \phi_{0,2}^2(\tau, z) - 8\phi_{0,4}(\tau, z) = \phi_{0,1}(\tau, z) \cdot \phi_{0,3}(\tau, z) - 12\phi_{0,4}(\tau, z). \tag{2.22}
\]

The form \( \phi_{0,3} \) is the square of a weak Jacobi form of index \( \frac{3}{2} \) with character \( \text{id}_{\text{SL}_2} \times v_H \)

\[
\xi_{0,\frac{3}{2}}(\tau, z) = \frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)} = r^{-\frac{1}{2}} \prod_{n \geq 1} \left( 1 + q^{n-1} r^1 \right) \left( 1 + q^n r^{-1} \right) \left( 1 - q^{2n-1} r^2 \right) \left( 1 - q^{2n-1} r^{-2} \right)
\]

(or, equivalently, \( \xi_{0,\frac{3}{2}}(\tau, z) = \eta(\tau)^{-1} \vartheta_{3/2}(\tau, z) \)). We set

\[
\xi_{0,6}(\tau, z) = \xi_{0,\frac{3}{2}}(\tau, 2z) = (r + r^{-1}) - q(r^5 + r^{-1} + r^{-5}) + q^2(\ldots) \in J_{0,6}^{\text{weak}}. \tag{2.23}
\]

The weak Jacobi form \( \xi_{0,6} \) has only two orbits of the Fourier coefficients with negative norm of their indices. They are \( r \) and \( -qr^5 \). It is clear that \( \eta(\tau)\xi_{0,6}(\tau, z) \) is holomorphic. The weak Jacobi \( \phi_{0,2} \) also has an expression in terms of Jacobi theta-series

\[
\phi_{0,2} \cdot \phi_{0,4} - \phi_{0,3}^2 = \xi_{0,6} \quad \text{or} \quad \phi_{0,2}(\tau, z) = \frac{\vartheta(\tau, 2z)^2}{\vartheta(\tau, z) \vartheta(\tau, 3z)} + \frac{\vartheta(\tau, 4z) \vartheta(\tau, z)}{\vartheta(\tau, 2z) \vartheta(\tau, 3z)}. \tag{2.24}
\]

We obtain similar formulae for \( \phi_{0,1}(\tau, z) \) below (see (3.16), (3.34) and (3.35)).
§3. LIFTINGS AND HECKE CORRESPONDENCE

In this section we shall prove that the exponential lifting commutes with some Hecke correspondence. We consider the multiplicative Hecke operators used in [GN4] and the multiplicative analogue of the operator of symmetrisation studied in [G2].

3.1. Multiplicative symmetrisation.

In [G2] we defined a Hecke operator which transforms modular forms with respect to $\Gamma_t$ into modular forms with respect to $\Gamma_{tp}$

$$\text{Sym}_{t,p} : \mathcal{M}_k(\Gamma_t) \to \mathcal{M}_k(\Gamma_{tp}), \quad \text{Sym}_{t,p} : F \mapsto \sum_{M \in (\Gamma_t \cap \Gamma_{tp}) \backslash \Gamma_{tp}} F|_{k,M}. \quad (3.1)$$

We call this operator the operator of $p$-symmetrisation. It can be represented as an action of an element from the Hecke ring $H(\Gamma_{\infty,t})$ of the parabolic subgroup $\Gamma_{\infty,t} \subset \Gamma_t$. To see this, we take a system of representatives

$$(\Gamma_t \cap \Gamma_{tp}) \backslash \Gamma_{tp} = \{ J_{tp}, \nabla(\frac{b}{tp}), b \mod p \} \quad (3.2)$$

where

$$J_t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \\ -1 & 0 & 1 & 0 \\ 0 & -t & 0 & 0 \end{pmatrix}, \quad \nabla(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

It is valid $J_t J_{tp} = \text{diag}(1, p, 1, p^{-1})$, thus for any $F \in \mathcal{M}_k(\Gamma_t)$ we have

$$\text{Sym}_{t,p}(F) = F|_{k}(\Lambda_p + \sum_{b \mod p} \nabla(\frac{b}{tp})).$$

The last operator is defined by the following element in the parabolic Hecke ring

$$\text{Sym}_p = \Lambda_p + \nabla_{t,p} \in H(\Gamma_{\infty,t}), \quad \text{where} \quad \nabla_{t,p} = \sum_{b \mod p} \Gamma_{\infty,t} \nabla(\frac{b}{tp}) \quad (3.3)$$

and $\Lambda_p$ is the element (1.10). The operator of $p$-symmetrisation is injective if $(t,p) = 1$ and it commutes with the arithmetic lifting. If $\phi \in J_{k,t}$, then

$$\text{Sym}_{t,p}(\text{Lift}(\phi)) = p^{3-k} \text{Lift}(\phi|_{k,T_-(p)}). \quad (3.4)$$

(see Satz 2.10 and Satz 3.1 in [G2]). Let us define the multiplicative analogue of the $p$-symmetrisation.

**Definition 3.1.** Let $F \in \mathcal{M}_k(\Gamma_t, \chi)$ where $k \in \mathbb{Z}/2$. Then for a prime $p$ we define the operator of multiplicative symmetrisation

$$\text{Ms}_p : F \mapsto p^{-k} \chi(J_t) \prod_{M_i \in (\Gamma_t \cap \Gamma_{tp}) \backslash \Gamma_{tp}} F|_{k,M_i}. \quad (3.5)$$

(The additional constant $p^{-k}\chi(J_t)$ makes formulae simpler.)
Lemma 3.2. Let $p$ be an arbitrary prime and $F \in \mathcal{M}_k(\Gamma_t, \chi)$. Then

$$\text{Ms}_p(F)(Z) \in \mathcal{M}_{k(p+1)}(\Gamma_{tp}, \chi^{(p)})$$

where $\chi^{(p)}$ is a character of $\Gamma_{tp}$. Moreover if the modular form $F$ is zero along a Humbert surface $H_t \subset A_t$ of discriminant $D$, then $\text{Ms}_p(F)$ is zero along $\text{Ms}_p(H_t)$ which is a sum (with some multiplicities) of Humbert surfaces with discriminant $D$ and $p^2D$ in $A_{tp}$.

Proof. Using (3.2) and interpretation (3.3) of the $p$-symmetrisation as a Hecke operator, we get

$$\text{Ms}_p(F)(Z) = F\left(\frac{\tau}{pz}, \frac{pz}{p^2\omega}\right) \prod_{m \equiv \beta \mod{p}} F\left(\frac{\tau}{z}, \frac{z}{\omega + \frac{\beta}{tp}}\right) \in M_{k(p+1)}(\Gamma_{tp}, \chi^{(p)}) \quad (3.6)$$

where $\chi^{(p)}$ is the character with the properties $\chi^{(p)}|_{SL_2(\mathbb{Z})} = (\chi|_{SL_2(\mathbb{Z})})^{p+1}$ and $\chi^{(p)}|_{H(\mathbb{Z})} = (\chi|_{H(\mathbb{Z})})^p$. The formula (3.6) explains also the action of $\text{Ms}_p$ on the rational quadratic divisors.

Similar to the arithmetic lifting and the $p$-symmetrisation, the exponential lifting commutes with the multiplicative symmetrisation. We have the following analog of (3.4).

Theorem 3.3. Let $\phi \in J^{4th}_{0, t}$ be like in Theorem 2.1. Then for an arbitrary prime $p$ we have

$$\text{Ms}_p(\text{Exp-Lift}(\phi_{0, t})) = \text{Exp-Lift}(\phi_{0, t}|_{T_-}(p)).$$

Proof. To prove the theorem, we use the representation (2.8) of the exponential lifting. The formula (3.6) shows that $\text{Ms}_p$ can be written as the action of $\text{Sym}_p$ on the function under the exponent in (2.8). Let us consider the product of the formal Dirichlet series $\sum_{m=1}^{\infty} T_-(m)m^{-1}$ over the Hecke ring $H(\Gamma_{\infty, t})$ with $\text{Sym}_p = \Delta_p + \nabla_{t,p}$. According to our definition of the normalizing factor of Hecke operators in the case of weight zero (see Sect. 1.2), we can consider the Hecke ring $H(\Gamma_{\infty, t})$ modulo its central element $\Delta(p) = \Gamma_{\infty, t}(pE_4)$. I.e. for any $X \in H(\Gamma_{\infty, t})$ the Hecke operators $X$ and $\Delta(p)X$ are identical. We recall that $\Delta(p)\Lambda_p = T_-(p, p)$ where $T_-(p, p)$ is the embedding of $T(p, p) = SL_2(\mathbb{Z})(pE_2)SL_2(\mathbb{Z}) \in H(SL_2(\mathbb{Z}))$ into the Hecke ring $H(\Gamma_{\infty, t})$. Using the definition one can check that

$$T_-(m)\nabla_{t,p} = \begin{cases} pT_-(m) & \text{if } m \equiv 0 \mod p \\ \nabla_{t,p}T_-(m) & \text{if } m \not\equiv 0 \mod p. \end{cases}$$

Thus

$$\left(\sum_{m=1}^{\infty} T_-(m)m^{-1}\right)(T_-(p, p) + \nabla_{t,p})$$

$$= \sum_{m \geq 1} \left(T_-(mp) + pT_\left(\frac{m}{p}\right)T_-(p, p)\right)m^{-1} + \sum_{(m, p) = 1} \nabla_{t,p}T_-(m)m^{-1}$$

$$= T_-(p) \sum_{m \geq 1} T_-(m)m^{-1} + \nabla_{t,p} \cdot \sum_{(m, p) = 1} T_-(m)m^{-1}, \quad (3.7)$$
since \( T(m)T(p) = T(mp) + pT(p, p)T(m/p) \) in \( H(SL_2(\mathbb{Z})) \).

Let us consider the representation (2.8) for the exponential lifting of \( \phi_{0,t} \). According to (3.7) we have the following identity for the main factor in (2.8)

\[
\exp \left( - \sum_{m \geq 1} (\tilde{\phi}_{0,t} | T_{-}(m)) | \text{Sym}_p(Z) \right) = \exp \left( - \sum_{m \geq 1} (\tilde{\phi}_{0,t} | T_{-}(p)) | T_{-}(m)(Z) \right),
\]

since \( \nabla_{t,p} \) defines zero operator on the space of Jacobi functions of weight 0 and index \( t \):

\[
(\tilde{\phi}_{0,t} | \nabla_{t,p})(Z) = \tilde{\phi}_{0,t}(Z) \cdot \sum_{b \mod p} \exp \left( \frac{2\pi ib}{p} \right) = 0.
\]

Let us consider the second exponent in (2.8). We have the formal identity

\[
V_t \left( \Lambda_p + \sum_{b \mod p} \nabla \left( \frac{b}{tp} \right) \right) = T_{-}(p)V tp \left( \sqrt{pE_4} \right)^{-1}
\]

where we consider \( T_{-}(p) \) as the formal sum of the left cosets fixed in (1.11) and \( V_t \) is involution (1.19). The second exponent in (2.8) is not \( \Gamma_{\infty,t} \)-invariant, but it is invariant with respect to the minimal parabolic subgroup \( \Gamma_{00} \), which is the intersection of \( \Gamma_{\infty,t} \) with the subgroup of the upper-triangular matrices in \( \Gamma_t \). This parabolic subgroup is the semidirect product of the subgroup of all upper-triangular matrices in \( SL_2(\mathbb{Z}) \) with the Heisenberg group. Thus we still can consider \( T_{-}(m) \) as an element of the Hecke ring \( H(\Gamma_{00}) \) of this minimal parabolic subgroup if we take \( T_{-}(m) \) in the standard form (1.11). (See [G10], where Hecke rings of parabolic subgroups of this type were considered for \( GL_n \) over a local field.) Thus for the second exponent in (2.8) we get

\[
\exp \left( - \sum_{m \geq 1} m^{-1}(\tilde{\phi}_{0,t}^0 + \phi_{0,t}^0) | T_{-}(m)) | V_t | \text{Sym}_p(Z) \right)
\]

\[
= \exp \left( - \sum_{m \geq 1} m^{-1}(\tilde{\phi}_{0,t}^0 + \phi_{0,t}^0) | T_{-}(p)) | T_{-}(m)) | V tp (Z) \right).
\]

This finishes the proof.

\[ \square \]

The multiplicative symmetrisation will give us some modular forms of small weight with simple divisor. We consider below the multiplicative \( p \)-symmetrisation of \( \Delta_k \)-functions (for \( k = 1/2, 1, 2, 5 \)) for different primes to produce an infinite product expansion of some functions constructed in §1 as the arithmetic lifting, i.e. we get new identities between arithmetic and exponential liftings. We recall that for an arbitrary Jacobi form \( \phi_{0,t}(\tau, z) = \sum_{n,l} f(n, l)q^n r^l \) we have the following formula for the Fourier coefficient of of the Jacobi form \( (\phi_{0,t} | T_{-}(m))(\tau, z) = \sum_{n,l} f_m(n, l)q^n r^l \)

\[
f_m(N, L) = m \sum_{a | (N, L, m)} a^{-1} f \left( \frac{Nm}{a^2}, \frac{L}{a} \right).
\]

(3.8)
In particular, \( f_m(0,0) = \sigma_1(m) f(0,0) \). We use the identities (see (2.10), (2.15) (2.18), (2.19))

\[
\begin{align*}
\phi_{0,1}|T_-(2)(\tau, z) &= r^2 + 2r^1 + 30 + 2r^{-1} + r^{-2} + q(\ldots), \\
\phi_{0,1}|T_-(3)(\tau, z) &= r^3 + 3r^1 + 40 + 3r^{-1} + r^{-3} + q(\ldots), \\
\phi_{0,2}|T_-(2)(\tau, z) &= r^2 + 2r^1 + 12 + 2r^{-1} + r^{-2} + q(\ldots), \\
\phi_{0,2}|T_-(3)(\tau, z) &= r^3 + 3r^1 + 16 + 3r^{-1} + r^{-3} + q(\ldots), \\
\phi_{0,3}|T_-(2)(\tau, z) &= r^2 + 2r^1 + 6 + 2r^{-1} + r^{-2} + q(\ldots), \\
\phi_{0,3}|T_-(3)(\tau, z) &= r^3 + 3r^1 + 8 + 3r^{-1} + r^{-3} + q(\ldots), \\
\phi_{0,4}|T_-(2)(\tau, z) &= r^2 + 2r^1 + 3 + 2r^{-1} + r^{-2} + q(\ldots).
\end{align*}
\]

(3.9)

**Example 3.4.** The modular form \( \Delta_{11} \). Let us consider \( \text{Ms}_2(\Delta_5) \). According to the first identity in (3.9), we get

\[
\Delta_{11}(Z) = \frac{\text{Ms}_2(\Delta_5(Z))}{\Delta_2(Z)^2} \in \mathcal{N}_{11}(\Gamma_2)
\]

with trivial character and with \( \text{Div}_{\Lambda_2^+} (\Delta_{11}) = H_1 + H_4 \). The \( \Gamma_2 \)-modular form of weight 11 \( \text{Lift}(\eta^{21}(\tau) \vartheta(\tau, 2z)) \) defined in Example 1.15 has zero along \( H_1 + H_4 \) (see Lemma 1.16). Thus we have proved the identity

\[
\Delta_{11}(Z) = \text{Lift}(\eta^{21}(\tau) \vartheta(\tau, 2z)) = \text{Exp-Lift}\left(\phi_{0,1}|T_-(2) - 2\phi_{0,2}\right)
\]

\[
= qrs^2 \prod_{n,l,m \in \mathbb{Z}} (1 - q^n r^l s^{2m}) f_{2}^{(11)}(n,m) \in \mathcal{N}_{11}(\Gamma_2)
\]

(3.11)

where

\[
\phi_{0,2}^{(11)}(\tau, z) = \phi_{0,1}|T_-(2)(\tau, z) - 2\phi_{0,2}(\tau, z) = \sum_{n \geq 0, l \in \mathbb{Z}} f_{2}^{(11)}(n,l) q^n r^l.
\]

Below we give another formula for the form \( \Delta_{11}(Z) \) and for the function \( \phi_{0,2}^{(11)} \) using a different multiplicative Hecke correspondence (see (3.14) and (3.33)).

Theorem 3.2 is valid for all prime \( p \), including the divisors of \( t \). Let us consider \( p = t = 2 \). The Hecke operator \( \Lambda_2 \) (see (1.10)) transforms the \( \Gamma_1 \)-modular form \( \Delta_5 \) to the \( \Gamma_4 \)-modular form \( \Delta_5^{(4)} \)

\[
\Delta_5^{(4)}\left(\begin{array}{c}
\tau \\
z \\
\omega
\end{array}\right) = \Delta_5\left(\begin{array}{c}
\tau \\
2z \\
4\omega
\end{array}\right) \in \mathcal{N}_5(\Gamma_4, v_\eta^{12} \times \text{id}_H)
\]

(3.12)

with divisor \( H_4 + H_1 \) in \( \Lambda_4^+ \). Almost the same function we get taking the multiplicative 2-symmetrisation of \( \Delta_2(Z) \). Comparing divisors of modular forms, we get the relation

\[
\text{Ms}_2(\Delta_2(Z)) = \Delta_5^{(4)}(Z) \Delta_{1/2}(Z)^2 \in \mathcal{N}_6(\Gamma_4, v_\eta^{18} \times \text{id}_H).
\]

(3.13)

(The modular form \( \text{Ms}_2(\Delta_2(Z)) \) has the divisor \( H_4 + 3H_1 \) in \( \Lambda_4^+ \).) The last identity shows that we can define \( \Delta_2(Z) \) in terms of Siegel theta-constants.

In the next lemma we prove some identities between the main Jacobi forms \( \phi_{0,t} \) for \( t = 1, 2, 3, 4 \) which have been used in the exponential liftings (2.16), (2.20), (2.21) and the Jacobi form defined in (3.11).
Lemma 3.5. The identities

\[ \phi_{0,1}^{(11)} = \phi_{0,1}|T_- (2) - 2\phi_{0,2} = \phi_{0,1}^2 - 20\phi_{0,2}, \]
\[ \phi_{0,2}|T_- (2)(\tau, z) = \phi_{0,1}(\tau, 2z) + 2\phi_{0,4}(\tau, z), \]
\[ 4\phi_{0,1} = \phi_{0,2}|T_+ (2) \]

are valid where \( T_+ (2) = \Gamma_\infty \text{diag}(1, 1, 1, 2)\Gamma_\infty \in H(\Gamma_\infty). \)

Proof. To prove (3.14) and (3.15), we have to calculate the \( q^0\)-part of the corresponding Jacobi forms using (3.8). To prove (3.16), we apply the Hecke operator

\[ \Lambda_{2}^{\ast} = \Gamma_{\infty} \text{diag}(1, p^{-1}, 1, p)\Gamma_{\infty} = \sum_{\lambda, \mu \mod p} \sum_{\kappa \mod p^2} \left( \begin{array}{cccc}
1 & 0 & 0 & \mu \\
p^{-1}\lambda & p^{-1} & p^{-1}\mu & p^{-1}\kappa \\
0 & 0 & 1 & -\lambda \\
0 & 0 & 0 & p
\end{array} \right) \in H(\Gamma_{\infty}) \]

(3.17)

to (3.15):

\[ \phi_{0,2}|(T_- (2) \cdot \Lambda_{2}^{\ast})(\tau, z) = \phi_{0,1}|(\Lambda_{2} \cdot \Lambda_{2}^{\ast})(\tau, z) + 2\phi_{0,4}|\Lambda_{2}^{\ast}(\tau, z). \]

We know that \( T_- (p)\Lambda_{2}^{\ast} = p^2 T_+ (p) \) and \( \Lambda_{p}\Lambda_{p}^{\ast} = p^4 \) (see [G2], [G3] about properties of these operators and for more details). To finish the proof of (3.16), we show that \( \phi_{0,4}|\Lambda_{2}^{\ast} = 0 \) (it means that \( \phi_{0,4} \) is a “new” Jacobi form). We have

\[ \phi_{0,4}|\Lambda_{2}^{\ast}(\tau, z) = 4 \sum_{\mu, \lambda \mod 2} \phi_{0,4}(\tau, \frac{\lambda z + \mu}{2}) = \sum_{\lambda \mod 2} \sum_{n, l \in \mathbb{Z}} f_4(n, 2l) q^{n + \lambda + \lambda^2} r^{2 + \lambda - l}. \]

This Fourier expansion shows that the Jacobi form \( \phi_{0,4}|\Lambda_{2}^{\ast} \) of weight 0 and index 1 is holomorphic, thus \( \phi_{0,4}|\Lambda_{2}^{\ast} \equiv 0. \)

Remark 3.6. Fourier coefficients of the function \( \phi_{0,1}(\tau, z) \), which is one of the canonical generators of the graded ring of weak Jacobi forms, are equal to the multiplicities of roots of the automorphic Kac–Moody Lie algebra defined by \( \Delta_5 (Z) \). The identity (3.16) gives us a new expression for this function which is simpler than the expression

\[ \phi_{0,1}(\tau, z) = (144\Delta(\tau))^{-1} (E_{3}^{2}(\tau)E_{4,1}(\tau, z) - E_{6}(\tau)E_{6,1}(\tau, z)) \]

from [EZ]. To calculate the Fourier coefficients of \( \phi_{0,2} \), one has to use only \( \eta(\tau)^{-\frac{1}{2}} \) and the theta-function \( \phi_{2,2} \) (see Lemma 1.24). Let us calculate the action of \( T_+ (2) \) on the Fourier coefficients of \( \phi_{0,2}(\tau, z) \). By definition

\[ T_+ (2) = \sum_{a, b, c \mod 2} \Gamma_{\infty} \left( \begin{array}{ccc}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1
\end{array} \right) + \sum_{\lambda \mod 2} \Gamma_{\infty} \left( \begin{array}{ccc}
2 & 0 & 0 \\
\lambda & 1 & 0 \\
0 & 0 & -\lambda
\end{array} \right). \]

Thus

\[ \phi_{0,2}|T_+ (2)(\tau, z) \]

\[ = 2 \sum_{a, b \mod 2} \phi_{0,2}(\tau + \frac{a}{2}, z + \frac{b}{2}) + 2 \sum_{\lambda \mod 2} \phi_{0,2}(2\tau, \lambda \tau + z) \exp \left( 2\pi i (\lambda^2 \tau + 2\lambda z) \right) \]

\[ = 8 \sum_{n, l} f_2(2n, 2l) q^n r^l + 2 \sum_{\lambda \mod 2} f_2(\frac{n + \lambda + \lambda^2}{2}, l + 2\lambda) q^n r^l. \]
It gives us the following formula for the Fourier coefficients \( f_1(N) = f_1(n, l) \) \((N = 4n - l^2)\) of the Jacobi form \( \phi_{0,1}(\tau, z) \) in terms of Fourier coefficients \( f_2(M) = f_2(n, l) \) \((M = 8n - l^2)\) of \( \phi_{0,2}(\tau, z) \)

\[
f_1(N) = 2f_2(4N) + 2^{-1}\left(\frac{-N}{2}\right) + 1) f_2(N)
\]

where

\[
\left(\frac{-N}{2}\right) = \left(\frac{-N}{8}\right) = \begin{cases} 
1 & \text{if } -N \equiv 1 \mod 8 \\
-1 & \text{if } -N \equiv 5 \mod 8 \\
0 & \text{if } -N \equiv 0 \mod 4.
\end{cases}
\]

We continue the list of Euler type identities between infinite sums and infinite products.

**Example 3.7.** Weight 3 modular form with respect to \( \Gamma_6 \). For the case of the multiplicative 2-symmetrisation of \( \Delta_1(Z) \) we obtain a cusp form constructed in Example 1.20

\[
\text{Ms}_2(\Delta_1) = \text{Exp-Lift}(\phi_{0.3}|T_-(2)) = \text{Lift}(\vartheta(\tau, 2z) \vartheta(\tau, z)^2 \eta(\tau)^3) \in \mathcal{M}_3(\Gamma_6, v_7^{12} \times \text{id}_H)
\]

with \( \text{Div}_{A_6^+}(F_3^{(6)}) = H_4 + 3H_1 \). Therefore \( \text{Ms}_2(\Delta_1) \) defines a canonical differential form with known divisor on the double covering of the moduli space of Abelian surfaces with polarization of type \((1,6)\). For more information on this subject see [GH2].

We may construct an interesting modular form using \( \Delta_{1/2}(Z) \). Its 2-symmetrisation is a modular form of half-integral weight

\[
F_{3/2}^{(8)} = \text{Ms}_2(\Delta_{1/2}) = \text{Exp-Lift}(\phi_{0.4}|T_-(2)) \in \mathcal{M}_3/2(\Gamma_8, v_7^9 \times \text{id}_H)
\]

with \( \text{Div}_{A_8^+}(F_{3/2}^{(8)}) = H_4(2) + 3H_1 \), where \( H_4(2) = \pi_8(\{2z = 1\}) \subset A_8^+ \). The square of \( F_{3/2}^{(8)}(Z) \) is a modular form of weight 3.

The multiplicative 3-symmetrisation of \( \Delta_1(Z) \), \( \Delta_2(Z) \) and \( \Delta_5(Z) \) produces modular forms with divisor \( H_9 \). We obtain the following three functions

\[
F_{16}^{(3)}(Z) = \frac{\text{Ms}_3(\Delta_5(Z))}{\Delta_1(Z)^4} = \text{Exp-Lift}(\phi_{0.1}|T_-(3) - 4\phi_{0.3}) \in \mathcal{M}_{16}(\Gamma_3, v_7^8 \times \text{id}_H)
\]

\[
F_8^{(6)} = \text{Ms}_3(\Delta_2) = \text{Exp-Lift}(\phi_{0.2}|T_-(3)) \in \mathcal{N}_8(\Gamma_6)
\]

\[
F_4^{(9)} = \text{Ms}_3(\Delta_1) = \text{Exp-Lift}(\phi_{0.3}|T_-(3)) \in \mathcal{N}_4(\Gamma_9, v_7^{16} \times \text{id}_H)
\]

with the divisors

\[
\text{Div}_{A_4^+}(F_{16}^{(3)}) = H_9, \quad \text{Div}_{A_6^+}(F_8^{(6)}) = H_9 + 4H_1, \quad \text{Div}_{A_9^+}(F_4^{(9)}) = H_9(3) + 4H_1
\]

where \( H_9(3) = \pi_9(\{3z = 1\}) \subset A_9^+ \). (We remark that there are two non-equivalent irreducible Humbert surfaces of discriminant 9 in \( A_9^+ \).)
3.2. Multiplicative Hecke operators.

Let $F \in \mathcal{M}_k(\Gamma_t, \chi)$ be a modular form of integral weight and

$$X = \Gamma_t M \Gamma_t = \sum_i \Gamma_t g_i \in \mathcal{H}_* (\Gamma_t) = \bigotimes_{(p,t)=1} \mathcal{H}_p (\Gamma_t) \cong \bigotimes_{(p,t)=1} \mathcal{H}_p (\Gamma_1)$$

be a Hecke element with a good reduction modulo all primes dividing $t$. Then we can define a multiplicative Hecke operator

$$[F] X := \prod_i F|_{k g_i}. \quad (3.24)$$

This is again a $\Gamma_t$-modular form. We call it the Hecke product of $F$ defined by $X$. In Theorem A.7 of [GN4, Appendix A] we proved that the exponential lifting commutes with multiplicative Hecke operators. More exactly, for arbitrary $\phi \in J_{0h}^{nh}$ and $X \in \mathcal{H}_* (\Gamma_t)$ the identity

$$[\text{Exp-Lift}(\phi_{0,t})] X = c \cdot \text{Exp-Lift} (\phi | J_0^{(t)} (X)) \quad (3.25)$$

is valid, where $J_0^{(t)}$ is a natural projection of the Hecke ring $\mathcal{H}_* (\Gamma_t)$ into the Hecke-Jacobi ring of the parabolic subgroup of $\Gamma_t$ (see [G2], [G6], [G8]) and $c$ is a constant. The operator $J_0^{(t)}$ is the same one which appears in the commutative relation between the arithmetic lifting and Hecke operators (see [G8]).

We consider below (3.25) for the Hecke operator $T(p) = \Gamma_t \text{diag}(1,1,p,p) \Gamma_t$ in the case of good and bad reduction. If $T(p) \in H(\Gamma_t)$ ($(p,t) = 1$) we have

$$T(p) = \Gamma_t \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sum_{a_1,a_2,a_3 \mod p} \Gamma_t \begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & a_2 & a_3 \end{pmatrix \frac{t}{p} \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sum_{a \mod p} \Gamma_t \begin{pmatrix} 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sum_{b \mod p} \Gamma_t \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (3.26)$$

and

$$J_0(T(p)) = T_0(p) + p^2 + p \quad (3.27)$$

where $T_0(p)$ is the Hecke-Jacobi operator

$$T_0(p) = \sum_{b \mod p} \sum_{c \mod p^2} \Gamma_{\infty} \begin{pmatrix} 1 & 0 & c & b \\ 0 & p & 0 & p b \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} + \sum_{a,b \mod p} \sum_{a \neq 0} \Gamma_{\infty} \begin{pmatrix} p & 0 & a & ab \\ 0 & p & 0 & ab^2 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & p \end{pmatrix} + \sum_{\lambda \mod p} \Gamma_{\infty} \begin{pmatrix} p^2 & 0 & 0 & 0 \\ p \lambda & p & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.28)$$

The element $T_0(p)$ defines an operator on the space of Jacobi forms which does not change the index

$$T_0(p) : J_{k,t} \rightarrow J_{k,t}.$$

It coincides (up to a constant) with the Hecke operator $T_p$ defined in [EZ, \S 4].
Therefore we prove exists only one \( \lambda \) where \( N^\lambda \) for the Fourier coefficients depending on \( p \) of \( T_0(p) \) on the Jacobi forms of index \( t \) for a divisor \( p \) of \( t \). Let \( \phi_{0,t}(\tau, z) \) be an arbitrary Jacobi form of weight zero and index \( t \) with Fourier expansion

\[
\tilde{\phi}_{0,t}(Z) = \sum_{n,l \in \mathbb{Z}} g(n, l) q^n r^l s^t = \sum_N g(N) \exp(2\pi i \text{tr}(NZ))
\]

where \( N = \left( \begin{array}{c} n \\ \frac{1}{2} \end{array} \right) \in M_2(\mathbb{Z}) \) (\( t \) is fixed). In accordance with (3.28) we get

\[
\tilde{\phi}_{0,t}|T_0(p)(Z) = p^3 \sum_{l \equiv 0 \mod p} g\left( \left( \begin{array}{c} n \\ \frac{1}{2} \end{array} \right) \right) \exp\left( 2\pi i \text{tr}(p^{-2}N \left( \begin{array}{c} 1 \\ 0 \end{array} \right) ) \right) + \sum_{n,l \in \mathbb{Z}} G_p(N) g(N) \sum_{\lambda \mod p} \exp\left( 2\pi i \text{tr}(N \left( \begin{array}{c} p \\ \lambda \end{array} \right) ) \right)
\]

where \( N[M] = {}^tM NM \) and we denote by \( G_p(N) \) the Gauss sum

\[
G_p(n, l, t) = -p + \sum_{a,b \mod p} \exp\left( 2\pi i \frac{na + lab + tab^2}{p} \right).
\]

Changing \( N \) to \( N[M^{-1}] \) in the first and third sum, we get the formula for the Fourier coefficient \( g_p(n, l) \) of Jacobi form \( \phi_{0,t}|T_0(p)(\tau, z) \) of index \( t \)

\[
g_p(n, l) = p^3 g(p^2n, pl) + G_p(n, l, t) g(n, l) + \sum_{\lambda \mod p} g\left( \frac{n + \lambda l + \lambda^2 t}{p^2}, \frac{l + 2\lambda}{p} \right) \tag{3.29}
\]

where we put \( g(n, l) = 0 \) if \( n \notin \mathbb{Z} \) or \( l \notin \mathbb{Z} \). In the case of a good reduction, when \( (p, t) = 1 \), the \( G_p(N) \) is given by the formula

\[
G_p(n, l, t) = p \left( \frac{-(4nt - l^2)}{p} \right), \quad (p, t) = 1
\]

(see (3.19) about definition of the Kronecker symbol for \( p = 2 \)). If \( p|t \), we can represent the formula for \( G_p \) in the form useful for exact calculations

\[
G_p(n, l, t) = p \begin{cases} 
0 & \text{if } l \not\equiv 0 \mod p \\
p - 1 & \text{if } (n, l) \equiv (0, 0) \mod p \\
-1 & \text{if } n \not\equiv 0 \mod p, \text{ and } l \equiv 0 \mod p.
\end{cases}
\]

For the Fourier coefficients depending on \( \lambda \) in (3.29) we have \( \text{det}(N \left( \begin{array}{c} p \\ \lambda \end{array} \right)) = p^2 \text{det}(N) \). If \( (t, p) = 1 \), then \( n + \lambda l + \lambda^2 t \) is a full square \( \mod p \) for \( 4nt - l^2 \equiv 0 \mod p^2 \). Thus there exists only one \( \lambda \mod t \) which gives us a non-trivial term in the third summand in (3.29). Therefore we prove
Lemma 3.8. Let us suppose that the Fourier coefficient \( g(n, l) \) of the Jacobi form \( \phi_{0, t} \) of weight zero and index \( t \) depends only on the norm \( n = 4nt - l^2 \in \mathbb{Z} \). We denote \( g(N) = g(n, l) \). For any prime \( p \) such that \( (t, p) = 1 \), the Fourier coefficients of \( \phi_{0, t}|T_0(p) \) are given by the formula

\[
g_p(N) = p^3 g(p^2 N) + p \left( \frac{-N}{p} \right) g(N) + g \left( \frac{N}{p^2} \right)
\]

where we set \( g \left( \frac{N}{p} \right) = 0 \) if \( \frac{N}{p} \notin \mathbb{Z} \).

Example 3.9. The Igusa modular form \( \Delta_{35}(Z) \). The Igusa modular form \( \Delta_{35}(Z) \) is the first Siegel modular form of odd weight with respect to \( \Gamma_1 = Sp_4(\mathbb{Z}) \). It has weight 35 (see [Ig1]). We defined this modular form in [GN4] as a Hecke product of \( \Delta_5(Z) \). In this example we recall this construction. Let us take the modular form \( \Delta_5(Z) \) which has the divisor \( H_1 \) in \( \mathcal{A}_1 \). Using the system of representatives \( T(p) \), we then get

\[
[\Delta_5(Z)]_{T_2} = \prod_{a, b, c \mod 2} \Delta_5 \left( \frac{z_1 + a}{2}, \frac{z_2 + b}{2}, \frac{z_3 + c}{2} \right) \prod_{a \mod 2} \Delta_5 \left( \frac{z_1 + a}{2}, z_2, 2z_3 \right) \Delta_5 \left( 2z_1, 2z_2, \frac{z_1 + a}{2} \right) \times \Delta_5 \left( 2z_1, 2z_2, 2z_3 \right) \prod_{b \mod 2} \Delta_5 \left( 2z_1, -z_1 + z_2, \frac{z_1 - 2z_2 + z_3 + b}{2} \right). \tag{3.30}
\]

One can check that \( \text{div}_{\mathcal{A}_1} ([\Delta_5(Z)]_{T(p)}) = (p + 1)^2 H_1 + H_{p^2} \). Thus

\[
\Delta_{35}(Z) = \frac{[\Delta_5(Z)]_{T_2}}{\Delta_5(Z)^8} = \text{Exp-Lift} \left( \phi_{0, 1} | (T_0(2) - 2) \right) \in \mathfrak{M}_{35}(\Gamma_1)
\]

and

\[
\Delta_{35}(Z) = q^{2rs^2} (q - s) \prod_{n, l, m \in \mathbb{Z}, (n, l, m) > 0} (1 - q^n r^l s^m) f_1^{(2)}(4nm - l^2) \tag{3.31}
\]

where \( f_1(4n - l^2) = f_1(n, l) \) are the Fourier coefficients of \( \phi_{0, 1}(\tau, z) \) and

\[
f_1^{(2)}(N) = 8f_1(4N) + 2 \left( \frac{-N}{2} \right) f_1(N) + f_1 \left( \frac{N}{4} \right)
\]

according to Lemma 3.8. Remark that we cannot construct \( \Delta_{35}(Z) \) as an arithmetic lifting of a holomorphic Jacobi form. Nevertheless (3.29) gives us \( \Delta_{35}(Z) \) as a finite Hecke product of the lifted form \( \Delta_5(Z) \).

Example 3.10. The modular form \( D_6(Z) \). Using (3.25)–(3.26) and Lemma 3.8, we get modular forms with divisor \( H_{p^2} \) in \( \mathcal{A}_1^2 \) and \( \mathcal{A}_1^3 \) respectively

\[
F_p^{(2)}(Z) = c_2 \frac{[\Delta_2(Z)]_{T_2}}{\Delta_2(Z)^{(p+1)^2}} = \text{Exp-Lift} \left( \phi_{0, 2} | (T_0(p) - p - 1) \right) \in \mathfrak{M}_{2p(p^2 - 1)}(\Gamma_2),
\]

\[
F_p^{(3)}(Z) = c_3 \frac{[\Delta_1(Z)]_{T_2}}{\Delta_1(Z)^{(p+1)^2}} = \text{Exp-Lift} \left( \phi_{0, 3} | (T_0(p) - p - 1) \right) \in \mathfrak{M}_{p(p^2 - 1)}(\Gamma_3, \chi_3(p))
\]

where \( c_2, c_3 \) are integers.
where \( p \neq 2 \) for \( \Delta_2 \) and \( p \neq 3 \) for \( \Delta_1 \). The character \( \chi_2^{(p)} \) is trivial or has order two. (The \( c_p \) are constants which can be easily calculated.) In particular we get the modular form

\[
D_6(Z) = \frac{2^{22}[\Delta_1(Z)]^{T(2)}}{\Delta_1(Z)^9} = \text{Exp-Lift}(\phi_{0,3}|(T_0(2) - 3))
\]

which we have constructed as arithmetic lifting of the Jacobi form \( \eta^{11} \partial_{3/2} \) in Example 1.17. Thus according to (3.9) we get

\[
D_6(Z) = \text{Lift}(\eta^{11} \partial_{3/2}) = q^{\frac{1}{2}}r^{\frac{1}{2}}s^{\frac{3}{2}} \prod_{n, l, m \in \mathbb{Z}} (1-q^n r^l s^{3m})^{g_3(nm, l)} \in \mathcal{H}_6(\Gamma_3^{+}, v_\eta^{12} \times v_H)
\]

where

\[
\phi_{0,3}^{(6)}(\tau, z) = \phi_{0,3}|(T_0(2) - 3)(\tau, z) = \sum_{n, l} g_3(n, l)q^n r^l = r^2 - r + 12 - r^{-1} + r^{-2} + q(\ldots).
\]

**Example 3.11.** Hecke product for \( p = t = 2 \) and \( p = t = 3 \). Using (3.29) we can consider the cases of bad reduction \( p = t = 3 \) and \( p = t = 3 \). The Hecke operator \( T^+(2) = \Gamma_2^{+} \text{diag}(1, 1, 2) \Gamma_2^{+} \) from the Hecke ring \( \mathcal{H}(\Gamma_2^{+}) \) of the maximal normal extension \( \Gamma_2^{+} \) contains 18 left cosets: the 15 cosets from (3.26) and

\[
\sum_{a, b \text{mod } 2, (a, b) \neq (0, 0)} \Gamma_2^{+} \left( \begin{array}{ccc} -a & 2 & b \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right).
\]

Therefore the modular form \([\Delta_2]|_{T^+(2)} \) of weight 36 has divisor \( 2H_4 + 9H_1 \) and we obtain the identity

\[
\Delta_{11}(Z)^2 = \left( \text{Lift}(\eta^{21}(\tau) \partial(\tau, 2z)) \right)^2 = \left( \text{Exp-Lift}(\phi_{0,2}^{(11)}(Z)) \right)^2
\]

because

\[
\phi_{0,2}|T_0(2)(\tau, z) = 2r^2 + 44 + 2r^{-2} + q(\ldots).
\]

As a corollary we can continue the identity (3.14) between the basic Jacobi forms (see Lemma 3.5)

\[
\phi_{0,2}^{(11)} = \phi_{0,1}|T_-(2) - 2\phi_{0,2} = \phi_{0,1}^2 - 20\phi_{0,2} = \frac{1}{2} \phi_{0,2}|T_0(2).
\]

Similarly to the case \( p = 2 \) we obtain for \( p = t = 3 \) the Jacobi form \( \phi_{0,3}|T_0(3)(\tau, z) = 2r^3 + 68 + 2r^{-3} + q(\ldots) \). Therefore the Hecke product for \( p = 3 \) of \( \Gamma_3 \)-modular form \( \Delta_1 \) is related with

\[
\mathcal{F}_{16}^{(3)}(Z) = \text{Exp-Lift}\left( \frac{1}{2} \phi_{0,3}|(T_0(3) - 2) \right) \in \mathcal{H}_{16}(\Gamma_3, v_\eta^{8} \times \text{id}_H)
\]

with divisor \( H_0 \). We have constructed this modular form in (3.22) using multiplicative 3-symmetrisation of \( \Delta_5 \). In terms of Jacobi forms it is equivalent to the relation

\[
\phi_{0,3}|T_0(3) = 2\phi_{0,1}|T_-(3) - 6\phi_{0,3}.
\]

Recall that the Jacobi form \( \phi_{0,3}(\tau, z) \) is the square of an infinite product (see (2.19)).
Example 3.12. $\Gamma_4$-modular forms with $H_{p^2}$-divisors. Our next examples are connected with the Jacobi form $\phi_{0,4}$. In the case of good reduction ($p \neq 2$), we can define the same function as in Example 3.10. Since
\[
\phi_{0,4}|T_0(p)(\tau, z) = r^p + pr + (p^3 + 1) + r^{-p} + pr^{-1} + q(\ldots) \quad (p \neq 2)
\]
we obtain a modular form
\[
F_p^{(4)}(Z) = \text{Exp-Lift}(\phi_{0,4}|(T_0(p) - p - 1)) \in \mathfrak{H}_{p(p^2-1)/2}(\Gamma_4, v_{q}^{p(p^2-1)} \times \text{id}_H)
\]
with divisor $H_{p^2}$. For instance for $p = 3$ we get a modular form of weight 12 with divisor $H_9$ in $\mathbb{A}_4^+$. Using (3.29) for $p = 2$, we obtain a very nice identity
\[
\phi_{0,4}|T_0(2)(\tau, z) = \phi_{0,1}(\tau, 2z). \quad (3.34)
\]
It gives us the second new formula for the generator $\phi_{0,1}(\tau, z)$ (see [3.16]). We can represent it in an equivalent form using the operator $\Lambda_2^*$ (see (3.17)). One can check (see [G2]) that
\[
p^{-1}T_0(p) \cdot \Lambda_2^* = T_+(1, p^2) = \sum_{a,b,c \equiv 0, 0 \mod p^2} \Gamma_\infty \frac{1 \ 0 \ a \ b \ c}{0 \ 1 \ b \ 0 \ 0 \ 0 \ p^2} + \sum_{a,b \equiv 0, 0 \mod p^2} \frac{1 \ 0 \ a \ b \ c}{0 \ 1 \ b \ 0 \ 0 \ 0 \ p^2} + \sum_{a \equiv 0, 0 \mod p^2} \frac{1 \ 0 \ a \ b \ c}{0 \ 1 \ b \ 0 \ 0 \ 0 \ p^2}
\]
Thus (3.34) is equivalent to
\[
8\phi_{0,1}(\tau, z) = \phi_{0,4}|T_+(1, 4)(\tau, z). \quad (3.35)
\]
We recall that $\phi_{0,4}(\tau, z)$ is given by an infinite product (see (2.10)). One can find a formula for the action of $T_+(1, 4)$ on Fourier coefficients of Jacobi forms similar to the formula (3.19) for $T_+(2)$.

§ 4. Reflective modular forms

In this section we construct some other modular forms with known divisors and prove new identities between arithmetic and exponential liftings. Our main aim is to construct reflective modular forms, i.e. forms with divisors defined by reflections in the corresponding orthogonal group. We are especially interesting in modular forms of divisors with multiplicity one.

4.1. The modular form $D_2(Z)$. In what follows we shall use new weak Jacobi forms in order to prove that some modular forms constructed in §1 as the arithmetic lifting have infinite product expansion. Using the basic weak Jacobi form from §2 (see (2.15), (2.19), (2.23)), we define a weak Jacobi form of weight 0 and index 9
\[
\phi_{0,9}(\tau, z) = \phi_{0,1}(\tau, 3z) + 7\phi_{0,3}(\tau, z)\xi_{0,6}(\tau, z) - \phi_{0,3}(\tau, z)^3
\]
\[
= (r^2 - r + 4 - r^{-1} + r^{-2}) + \cdots + q^2(r^9 + \ldots). \quad (4.1)
\]
Since $\eta^6\phi_{0,9}$ is holomorphic, the part of the Fourier expansion written above contains all Fourier coefficients of negative norm. Using this Jacobi form we can construct the function $D_2(Z)$ (see (1.30)) from Example 1.17 as exponential lifting.
Theorem 4.1. The following identity is valid:

\[ D_2(Z) = \sum_{N \geq 1} \sum_{\substack{m > 0, l \in \mathbb{Z} \\ n, m \equiv 1 \mod 6 \\ 4nm - l^2 = N^2}} N \left( \frac{-4}{N} \right) \left( \frac{12}{l} \right) \sum_{a | (n, l, m)} \left( \frac{6}{a} \right) q^{n/6} r^{l/2} s^{3m/2} \]

\[ = \frac{1}{\sqrt{2\pi}} \frac{1}{s^{1/2}} \prod_{n, l, m \in \mathbb{Z}} (1 - q^n r^l s^9m)^{f_9(nm, l)} \in \mathcal{M}_2(\Gamma_9, v_9^4 \times v_H). \] (4.2)

Moreover \( D_2(Z) \) has the divisor

\[ \text{Div}_{A_9^+}(D_2(Z)) = H_4(2) + H_9(9) \]

(see (1.23)).

Proof. The lifted form \( D_2(Z) \) has zero along \( H_4(2) \) according to Lemma 1.16. To prove the identity we have to check that \( D_2(Z) \) has zero along \( H_9(9) \). This Humbert surface is defined by the element \( \ell_9 = 2f_2 + \frac{1}{2} f_3 + f_2 \in \hat{L}_9 \) (see (1.14) and (1.22)). In the basis (1.14) the matrix of the reflection \( \sigma_{\ell_9} \) has the form

\[ (\sigma_{\ell_9}) = \begin{pmatrix} 9 & -72 & 16 \\ 2 & -17 & 8 \\ 4 & -36 & 9 \end{pmatrix}. \]

Thus, if we consider \( D_2 \) as a modular form with respect to the orthogonal group of lattice \( L_9 \), we get

\[ D_2(\sigma_{\ell_9}(\mathfrak{z})) = \sum_{n, l, m} a(n, l, m) \exp \left( 2\pi i \left( \frac{9n-24l+16m}{6} z_1 + \frac{12n-17l+24m}{2} z_2 + \frac{2n-6l+9m}{6} z_3 \right) \right) = -D_2(\mathfrak{z}), \]

because the Fourier coefficient \( a(n, l, m) \) has the factor \( \left( \frac{12}{l} \right) \).

\[ \square \]

Similarly to \( \phi_{0,36}(\tau, z) \) (see (2.12)) we introduce a weak Jacobi form of index 12

\[ \xi_{0,12}(\tau, z) = \frac{\partial_{3/2}(\tau, 3z)}{\partial_{3/2}(\tau, z)} = \frac{\partial(\tau, 6z) \partial(\tau, z)}{\partial(\tau, 3z) \partial(\tau, 2z)}. \] (4.3)

Using the same arguments as in the case of the weak Jacobi form \( \phi_{0,36} \), we see that one needs to calculate the Fourier coefficients of \( \phi_{0,12}(\tau, z) \) for \( q^n \) with \( n \leq 3 \). It gives us:
Lemma 4.2. The weak Jacobi form

\[ \xi_{0,12}(\tau, z) = (r - 1 + r^{-1}) \prod_{n \geq 1} (1 - q^{n}r^{\pm 1} + q^{2n}r^{\pm 2})(1 + q^{2n-1}r^{\pm 2} + q^{4n-2}r^{\pm 4}) \]

\[ (1 + q^{3n-2}r^{\pm 3})(1 + q^{3n-1}r^{\pm 3})(1 - q^{6n-1}r^{\pm 6})(1 - q^{6n-5}r^{\pm 6}) \]

(\pm m means that there are two factors in the product: with \(-m\) and with \(+m\) has a non-zero Fourier coefficient \(f_{12}(n, l)\) of negative norm only for \(48n - l^2 = -1\) (\(r\) and \(-qr^7\)) and \(48n - l^2 = -4\) \((-q^2r^{10})\) and its first Fourier coefficients are

\[ \xi_{0,12}(\tau, z) = (r - 1 + r^{-1}) - q(r^7 + \ldots) - q^2(r^{10} + \ldots) - q^3(r^{12} + \ldots) + \ldots. \]

In particular, \(\eta^2 \xi_{0,12}\) is holomorphic.

Using \(\xi_{0,12}\) we define

\[ \phi_{0,18}(\tau, z) = \xi_{0,12}(\tau, z) \cdot \xi_{0,6}(\tau, z) = \frac{\vartheta(\tau, 6z) \vartheta(\tau, 4z) \vartheta(\tau, z)}{\vartheta(\tau, 3z) \vartheta(\tau, 2z)^2} \]

\[ = (r^2 - r^2 + r^{-1} + r^{-2}) + \ldots + q^3(r^{15} + \ldots) + q^4(2r^{17} + \ldots) + \ldots. \quad (4.4) \]

Since \(\eta^3 \phi_{0,18}\) is holomorphic, one needs to know only the Fourier coefficients with norm \(\geq -9\). All Fourier coefficients of this type are written in the Fourier expansion above. We give, without proof, the identity for the modular form \(D_1(Z)\) (see (1.29)) from Example 1.17:

\[ D_1(Z) = \sum_{M \geq 1} \sum_{m > 0, l \in \mathbb{Z}} \left( \frac{12}{Ml} \right) \sum_{n, m \equiv 1 \text{ mod } 12, 4mn - l^2 = M^2} \left( -\frac{4}{a} \right) q^{n/12}r^{l/2}s^{3m/2} \]

\[ = q^{\frac{1}{12}}r^{\frac{1}{12}}s^{\frac{1}{2}} \prod_{n, l, m \in \mathbb{Z}, (n,l,m) > 0} (1 - q^n r^l s^{18m}) f_{18}(nm, l) \in \mathfrak{N}_1(\Gamma_{18}, v_N^2 \times v_H). \quad (4.5) \]

The weak Jacobi forms introduced above provide another formula for the Jacobi form \(\phi_{0,36}\) (see (2.12)) used in the exponential lifting of the singular modular form \(D_{1/2}(Z)\)

\[ \phi_{0,36}(\tau, z) = \phi_{0,4}(\tau, 3z) - \xi_{0,6}(\tau, 2z) \xi_{0,12}(\tau, z). \quad (4.6) \]

4.2. Anti-symmetric modular forms.

The arithmetic lifting provides us with modular forms which are invariant with respect to the main exterior involution \(V_t\) of the group \(\Gamma_t\) (see (1.19)). Using the exponential lifting, one can construct anti-invariant modular forms, i.e. forms satisfying \(F(V_t(Z)) = -F(Z)\). For example for \(t = 1\) we have the anti-invariant form \(F(Z) = \Delta_{35}(Z)\). In this subsection we construct anti-invariant modular forms for arbitrary \(\Gamma_t\) \((t > 1)\). For \(\Gamma_2, \Gamma_3\) and \(\Gamma_4\) these modular forms have only the Humbert surface \(H_{4t}(0) = \pi_t\{\tau - tw = 0\}\) as their divisor. (We remark that for \(t = 4\) there are two irreducible Humbert surfaces with discriminant 16.)
Let us consider the function \( \psi_{0,t}(\tau, z) = \Delta(\tau)^{-1}E_{12,t}(\tau, z) \) where

\[
\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + 253q^3 + \ldots
\]

and \( E_{12,t}(\tau, z) \) is a Jacobi–Eisenstein series of weight 12 and index \( t \).

There exists a formula for Fourier coefficients of \( E_{k,1} \) in terms of H. Cohen’s numbers (see [EZ, §2]). One can find the table of the values of Fourier coefficients of \( E_{4,1}(\tau, z) \) and \( E_{6,1}(\tau, z) \) in [EZ, §1]. Using the basic Jacobi forms \( \phi_{0,2}(\tau, z) = r^{\pm 1} + 4 + \ldots \), \( \phi_{0,3}(\tau, z) = r^{\pm 1} + 2 + \ldots \) and the forms \( \phi_{11}(\tau, z) = r^{\pm 2} + 22 + \ldots \) and \( \phi_{6}(\tau, z) = r^{\pm 2} - r^{\pm 1} + 12 + \ldots \), which are the data for the exponential liftings (3.11) and (3.32), we define

\[
\psi_{0,2}(\tau, z) = \Delta(\tau)^{-1}E_{6,1}(\tau, z)^2 - 2\phi_{0,2}^{(11)}(\tau, z) + 176\phi_{0,2}(\tau, z)
= \sum_{n \geq 0, l \in \mathbb{Z}} c_2(n, l)q^n r^l = q^{-1} + 24 + q(\ldots),
\]

\[
\psi_{0,3}(\tau, z) = \Delta(\tau)^{-1}E_{4,1}(\tau, z)^3 - 3\phi_{0,3}^{(6)}(\tau, z) - 171\phi_{0,3}(\tau, z)
= \sum_{n \geq 0, l \in \mathbb{Z}} c_3(n, l)q^n r^l = q^{-1} + 24 + q(\ldots).
\]

The Jacobi forms \( \psi_{0,p} \) (\( p = 2, 3 \)) contain the only type of Fourier coefficients with indices of negative norm. This is \( q^{-1} \) of norm \(-4p\). Thus we can use both functions to produce the exponential liftings

\[
\Psi^{(2)}_{12}(Z) = \text{Exp-Lift}(\psi_{0,2}) = q \prod_{n, l, m \in \mathbb{Z}} (1 - q^n r^l s^{2m})^{c_2(n, l, m)} \in \mathcal{M}_{12}(\Gamma_2),
\]

\[
\Psi^{(3)}_{12}(Z) = \text{Exp-Lift}(\psi_{0,3}) = q \prod_{n, l, m \in \mathbb{Z}} (1 - q^n r^l s^{3m})^{c_3(n, l, m)} \in \mathcal{M}_{12}(\Gamma_3).
\]

According to Theorem 2.1

\[
\Psi^{(p)}_{12}(V_p < Z >) = -\Psi^{(p)}_{12}(Z) \quad (p = 2, 3) \quad \text{and} \quad \text{Div}_{A_p}(\Psi^{(p)}_{12}) = \begin{cases} H_8 & \text{for } p = 2 \\ H_{12} & \text{for } p = 3 \end{cases}.
\]

The Fourier-Jacobi expansion of \( \Psi^{(p)}_{12} \) starts with coefficients

\[
\Psi^{(p)}_{12}(Z) = \Delta_{12}(\tau) - \Delta_{12}(\tau) \psi_{0,p}(\tau, z) \exp(2\pi i p \omega) + \ldots.
\]

Therefore the constructed modular forms \( \Psi^{(p)}_{12}(Z) \) (\( p = 2, 3 \)) are not cusp forms.

If we do the same for \( t = 4 \), we get a Jacobi form we used to construct \( \Delta_{35}(Z) \). Let us take the Jacobi form

\[
\phi_{0,1}((T_0(2) - 2)(\tau, 2z) = q^{-1} + (r^4 + 70 + r^{-4}) + q(\ldots).
\]
Its exponential lifting is zero along two Humbert surfaces with discriminant 16. To delete
the second component, we consider the additional Jacobi–Eisenstein series which has the
constant term equals zero (such a series exists if the index contains a perfect square). For
\( t = 4 \) this Jacobi–Eisenstein series is the eight power of the Jacobi theta-series \( \vartheta(\tau, z) \).

Remark 4.3. It is easy to modify the calculation of Fourier coefficients in [EZ] to find an
exact formula for the Fourier expansion of the additional Eisenstein series with a perfect
square index. It gives us, for example, a formula for the eight power of the triple product
in terms of H.Cohen numbers. We hope to present such formulae later.

Using \( \vartheta(\tau, z)^8 \), we define

\[
\psi_{0,4}(\tau, z) = (\phi_{0,1}((T_0(2) + 26))(\tau, 2z) - \Delta(\tau)^{-1}E_4(\tau)\vartheta(\tau, z)^8 - 8(\phi_{0,4}((T_0(3) + 4))(\tau, z)
\]

\[
= \sum_{n \geq 0, l \in \mathbb{Z}} c_4(n, l)q^{nr^4} = q^{-1} + 24 + q(\ldots).
\]

Similarly to \( \psi_{0,2} \) and \( \psi_{0,3} \) the Jacobi form \( \psi_{0,4} \) contains only the Fourier coefficients of type
\( q^{-1} \) with index of negative norm. Taking its exponential lifting we obtain the \( \Gamma_4 \)-modular
form of weight 12

\[
\Psi_{12}^{(4)}(Z) = \exp\text{-Lift}(\psi_{0,4})(Z) = q \prod_{n, l, m \in \mathbb{Z}} \left(1 - q^n r^l s^4 m\right)c_4(nm, l) \in \mathcal{M}_{12}(\Gamma_4).
\]

According to Theorem 2.1, \( \Psi_{12}^{(4)}(Z) \) is anti-invariant and \( \Div_{\mathcal{A}_4}(\Psi_{12}^{(4)}) = H_8(0) \).

Remark 4.4. The fact that all three modular forms constructed above have weight 12
has the following explanation using the Borcherds singular modular form \( \Phi_{12}(\tau, z, \omega) \) with
respect to the orthogonal group of the lattice \( L \) of signature (26, 2), which is the orthogonal
sum of two unimodular hyperbolic plains \( U \) and the Leech lattice \( L \). See [Bo5] and also
[Bo4], [Bo2]. The modular form \( \Phi_{12}(\tau, z, \omega) \) is anti-invariant, i.e.

\[
\Phi_{12}(\tau, z, \omega) = -\Phi_{12}(\omega, z, \tau).
\]

Let us take a primitive \( \ell \in L \) such that \( \ell^2 = 2t \) (\( t > 1 \)). Then the orthogonal group of
the lattice \( 2U \oplus < 2t > \) (see the notation in Sect. 1.3) acts on the tree
dimensional subdomain \( \mathbb{H}^+ \) of the tube homogeneous domain of dimension 26. The \( \mathbb{H}^+ \)
is isomorphic to the Siegel upper half plane and the restriction \( \Phi|_{\mathbb{H}^+} \) is a Siegel modular form
of weight 12 with respect to the group \( \Gamma_t \) (see (1.21) and Lemma 1.9). Thus for arbitrary \( t > 1 \) there exists an anti-symmetric modular form of weight 12 with respect to \( \Gamma_t \). Since
\( \Psi_{12}^{(t)}(Z) \) (\( t = 2, 3, 4 \)) have weight 12 and the divisor \( H_{4t} \), they coincide with \( c\Phi|_{\mathbb{H}^+} \). It also
proves that the divisor of \( \Phi|_{\mathbb{H}^+} \) is exactly equal to \( H_{4t} \) for \( t = 2, 3, 4 \). For arbitrary \( t \) the
restriction \( \Phi|_{\mathbb{H}^+} \) may have additional divisors.

4.3. The reflective modular forms with divisors \( H_1 \) and \( H_4 \).
Example 4.5. The first cusp form for $\Gamma_5$. In (1.32) we defined a cusp form of weight 5 with respect to $\Gamma_5$ with trivial character. Let us construct this function as the exponential lifting. We set
\[ \phi_{0,5}(\tau, z) = \phi_{0,2}(\tau, z)\phi_{0,3}(\tau, z) = (r^2 + 6r^{-1} + 10 + 6r^{-1} + r^{-2}) + q(-3r^4 + \ldots) + \ldots. \]
Its Fourier expansion does not contain the summand $qr^5$ with $N = 20n^2 - l^2 = -5$. Thus
\[ \Div_{A_5^+}(\ExpLift(\phi_{0,5})) = 7H_1 + H_4. \]
According to Lemma 1.16
\[ F_5^{(5)}(Z) = \ExpLift(\phi_{0,5}) \in \mathfrak{M}_5(\Gamma_5). \quad (4.10) \]
We can define another weak Jacobi form of index 5
\[ \phi_{0,5}^{(2)}(\tau, z) = 2\phi_{0,2}(\tau, z)\phi_{0,3}(\tau, z) - \phi_{0,1}(\tau, z)\phi_{0,4}(\tau, z) = (r^{\pm 2} + r^{\pm 1} + 8) + q(r^5 + \ldots) + \ldots. \]
According to Theorem 2.1
\[ \Div_{A_5^+}(\ExpLift(\phi_{0,5}^{(2)})) = 2H_1 + H_4 + H_5. \]
One can prove that
\[ G_4^{(5)}(Z) = \ExpLift(\phi_{0,5}^{(2)}) \in \mathfrak{M}_4(\Gamma_5, v_{\eta}^{12} \times v_H). \quad (4.11) \]
Example 4.6. The cusp forms for $\Gamma_6$. If the polarization $t$ contains more than one of primes, the structure of the graded ring of modular forms reflects the properties of rings of modular forms for different polarizations. In [G2] it was proved injectivity of the $p$-symmetrisation $\text{Sym}_{t,p} : \mathfrak{M}_k(\Gamma_t) \to \mathfrak{M}_k(\Gamma_{tp})$ if $(t, p) = 1$ (see (3.1)). In this example we present infinite product expansions of the $\Gamma_6$-cusp forms of weight 3 constructed in Example 1.15 and 1.20.

According to the dimension formula of the space of Jacobi forms (see [EZ], [SkZ]) we have $\dim(J_{6,6}^{cusp}) = 1$. A square root of a generator of this space was defined in Example 1.20: $\phi_{6,6}(\tau, z) = (\eta(\tau)^3\vartheta(\tau, z)\vartheta(\tau, 2z))^2$. Its arithmetic lifting is
\[ F_3^{(6)}(Z) = \ExpLift(\phi_{6,6}) \in \mathfrak{M}_3(\Gamma_6^+, v_{\eta}^{12} \times \id_H). \]
For (1,6)-polarization there are two Humbert surfaces with discriminant 1 in $A_6^+$:
\[ H_1(1) = \pi_6(\{Z \in \mathbb{H}_2 \mid z = 0\}), \quad H_1(5) = \pi_6(\{Z \in \mathbb{H}_2 \mid \tau + 5z + 6\omega = 0\}), \]
and one Humbert surface $H_4$ of discriminant 4. According to Lemma 1.16 $\Div_{A_6^+}(F_3^{(6)})$ contains $3H_1(1) + H_4$. One can prove that
\[ F_3^{(6)}(Z) = \ExpLift(3\phi_{0,3}^2 - 2\phi_{0,2}\phi_{0,4}) \quad (4.12) \]
and
\[ \Div_{A_6^+}(F_3^{(6)}(Z)) = 3H_1(1) + 2H_1(5) + H_4. \]
In Example 1.15 we defined $(-1)$-Lift of the Jacobi form $\eta(\tau)^5\vartheta(\tau, 2z)$. One can prove that
\[ \ExpLift(\phi_{0,3}^2) \in \mathfrak{M}_3(\Gamma_6^+, v_{\eta}^{16} \times \id_H), \quad (4.13) \]
and their divisors are equal to $2H_1(1) + 4H_1(5) + H_4$ and $5H_1(1) + H_4$ respectively.
Example 4.7. The first cusp form for $\Gamma_7$. For the $\Gamma_7$-modular form of weight 2 from Example 1.20 we have

$$F_2^{(7)}(Z) = \text{Lift}(\vartheta(\tau, z)^3 \vartheta(\tau, 2z)) = \text{Exp-Lift}(\phi_{0,3} \cdot \phi_{0,4})(Z) \in \mathcal{M}_2(\Gamma_7, v_{12}^1 \times v_H)$$

(4.15)

with

$$\text{Div}_{A_7^1}(F_2^{(7)}) = 4H_1 + H_4.$$ 

This divisor is a part of the divisor of the arithmetic lifting (see Lemma 1.16). Since $\eta^4 \phi_{0,3} \phi_{0,4}$ is holomorphic, we have to check in the Fourier expansion of $\phi_{0,3} \phi_{0,4}$ only Fourier coefficients with norm $\geq -4$. We see that

$$\phi_{0,3}(\tau, z) \phi_{0,4}(\tau, z) = (r^2 + 3r + 4 + 3r^{-1} + r^{-2}) + \ldots,$$

thus its exponential lifting has the divisor mentioned above. This finishes the proof of (4.15).

The first non-zero cusp form for $\Gamma_7$ has weight 4 (see [G2, §3]). The Jacobi form $\phi_{4,7}(\tau, z) = \vartheta(\tau, z)^6 \vartheta(\tau, 2z)^2$ is the cusp form of weight 4 and index 7, and there exists the only one such form. Moreover,

$$F_4^{(7)} = \text{Lift}(\phi_{4,7}) = (F_2^{(7)})^2$$

is the first cusp form with respect to $\Gamma_7$ with trivial character.

Example 4.8. The cusp form of weight 1 for $\Gamma_{10}$. We set

$$\phi_{0,10}(\tau, z) = \phi_{0,4}(\tau, z) \xi_{0,6}(\tau, z) = \frac{\vartheta(\tau, 4z) \vartheta(\tau, 3z)}{\vartheta(\tau, 2z) \vartheta(\tau, z)}$$

$$= (r^2 + r + 2 + r^{-1} + r^{-2}) + \ldots + q^2(r^9 + \ldots).$$

(4.16)

The Jacobi forms $\eta^2 \phi_{0,4}$ and $\eta \xi_{0,6}$ are holomorphic at infinity. Thus in (4.16) we have all types of Fourier coefficients of $\phi_{0,10}(\tau, z)$ of negative norm. One can prove that the exponential lifting of $\phi_{0,10}$ coincides with the modular form of weight 1 for $\Gamma_{10}$ constructed in Example 1.20

$$F_1^{(10)}(Z) = \text{Lift}(\vartheta(\tau, z) \vartheta(\tau, 2z)) = \text{Exp-Lift}(\phi_{0,10}) \in \mathcal{M}_1(\Gamma_{10}, v_6^6 \times v_H)$$

(4.17)

and $\text{Div}_{A_{10}^1}(F_1^{(10)}) = 2H_1(1) + H_1(9) + H_4$.

5. Some applications

In this section we shortly describe some applications of automorphic forms constructed in §1–§4. We hope to give more detailed and advanced applications in further papers.
5.1. Denominator identities for automorphic Lorentzian Kac–Moody algebras.

Almost all identities between arithmetic and exponential liftings proved above give denominator formulae for Lorentzian Kac–Moody algebras. This means that they give Fourier expansions (or the corresponding q-expansions) of Lie type. See Sect. 2.5 of Part I and considerations below.

We denote by \( M_t = U(4t) \oplus \langle 2 \rangle \) \((t \in \mathbb{N}/4)\) a hyperbolic lattice with the bases \( f_2, \hat{f}_3, f_{-2} \) and with the Gram matrix

\[
\begin{pmatrix}
0 & 0 & -4t \\
0 & 2 & 0 \\
-4t & 0 & 0
\end{pmatrix}.
\]

We denote an element \( nf_2 + lf_3 + mf_{-2} \) of this lattice (or of \( M_t \otimes \mathbb{Q} \)) by its coordinates \((n, l, m)\).

All our formulae have the form:

\[
\Phi = \sum_{(n, l, m) \in M_t} N(n, l, m)q^{\rho_1+n\rho_2+l\rho_3} \prod_{(n, l, m) > 0} (1 - q^{n\rho_1+l\rho_2+m\rho_3})^{f(n, l, m)}
\]

for some \( W, \mathcal{M}, \rho = (\rho_1, \rho_2, \rho_3) \) which we define below. The “exponent” \( e^{(n, l, m)} := q^{n\rho_1+l\rho_2+m\rho_3} \). The Weyl group \( W \) is a reflection subgroup \( W \subset W(M_t) \subset O(M_t) \), and \( \mathcal{M} \) is a fundamental polyhedron of \( W \) in the hyperbolic space \( \mathcal{L}(M_t) \) defined by the hyperbolic lattice \( M_t \). The \( W \) and \( \mathcal{M} \) are defined by an acceptable set \( P(\mathcal{M}) \subset M_t \) of elements orthogonal to faces of \( \mathcal{M} \) and directed outward. The set \( P(\mathcal{M}) \) is also called the set of simple real roots. The main property of this set is that \((\alpha, \alpha) > 0 \) and \((\alpha, \alpha) | 2(M_t, \alpha)\) for any \( \alpha \in P(\mathcal{M}) \). Moreover, it defines a generalized Cartan matrix

\[
A = \left( \frac{2(\alpha, \alpha')}{(\alpha, \alpha)} \right), \quad \alpha, \alpha' \in P(\mathcal{M}),
\]

which means that all diagonal elements of \( A \) are equal to 2 and all non-diagonal elements are non-positive integers. The generalized Cartan matrix \( A \) is the main invariant of the Fourier expansion of Lie type. The Weyl group \( W \) is generated by the reflections \( s_\alpha \in O^+(M_t), \alpha \in P(\mathcal{M}) \). We recall that \( s_\alpha(\alpha) = -\alpha \) and \( s_\alpha \) is identical on the \( \alpha^\perp \). The \( \epsilon : W \to \{ \pm 1 \} \) is a character. It is defined by the set \( P(\mathcal{M})_{\pm} = \{ \alpha \in P(\mathcal{M}) | \epsilon(s_\alpha) = -1 \} \) which is called the set of even simple real roots. Respectively, the set \( P(\mathcal{M})_{\mp} = P(\mathcal{M}) \setminus P(\mathcal{M})_{\pm} = \{ \alpha \in P(\mathcal{M}) | \epsilon(s_\alpha) = 1 \} \) is called the set of odd simple real roots. By definition, \( a = (n, l, m) \in \mathbb{R}_{++}^+ \mathcal{M} \) if \((a, P(\mathcal{M})) \leq 0 \) and \( a \not= 0 \). The lattice Weyl vector \( \rho = (\rho_1, \rho_2, \rho_3) \in \mathbb{R}_{++}^+ \mathcal{M} \cap (M_t \otimes \mathbb{Q}) \) is defined by the property: \((\rho, \alpha) = -(\alpha, \alpha)/2\) for any \( \alpha \in P(\mathcal{M}) \). It is true that \((\rho, \rho) < 0 \). The case \((\rho, \rho) < 0 \) is called elliptic. The case \((\rho, \rho) = 0 \) is called parabolic. All Fourier coefficients \( N(n, l, m), N(a) \) and multiplicities \( f(n, l, m) \) are integers. The inequality \( a = (n, l, m) > 0 \) means that \( a \in M_t \) and either \((a, \rho) < 0 \) or \( a \in \mathbb{Q}_{++}^+ \rho \) (if \((\rho, \rho) = 0 \)). If the multiplicity \( f(n, l, m) \not= 0 \), then either
\( a \in W(P(\mathcal{M}))_\theta \) and \( f(n, l, m) = 1 \) (i.e. \( a \) is a positive even real root), or \( a \in W(P(\mathcal{M}))_\tau \) and \( f(n, l, m) = -1 \) (i.e. \( a \) is a positive odd real root), or \( a \in 2W(P(\mathcal{M}))_\tau \) (i.e. \( a \) is a positive even real root which is multiple to an odd real root) and \( f(n, l, m) = 1 \), or \((a, a) \leq 0\) (i.e. \( a \) is a positive imaginary root).

Below we describe sets \( P(\mathcal{M})_\theta \), \( P(\mathcal{M})_\tau \) and the generalized Cartan matrices \( \Phi \) for all our formulae. To describe some of them we also use the group of symmetries of \( P(\mathcal{M}) \) which is the group

\[
\text{Sym}(P(\mathcal{M})) = \{ g \in O^+(M_t) \mid g(P(\mathcal{M})) = P(\mathcal{M}), g(P(\mathcal{M}))_\tau = P(\mathcal{M})_\tau \}.
\]

We also use notation \( W(k_1, ..., k_r)(K) \) for the reflection subgroup of a lattice \( K \) generated by reflections in all primitive elements \( \delta \in K \) such that \((\delta, \delta) \in \{ k_1, ..., k_n \} \subset \mathbb{N}\). The group \( W(K) \) denote the full reflection group of the lattice \( K \) which is generated by reflections in all elements \( \delta \in K \) with positive squares \((\delta, \delta) > 0\). For all cases the semi-direct product \( W \rtimes \text{Sym}(P(\mathcal{M})) \) has finite index in \( O^+(M_t) \) (i.e. \( W \) has restricted arithmetic type) and for almost all of them \( W \rtimes \text{Sym}(P(\mathcal{M})) \) is equal to the full group \( O^+(M_t) \). Here \( O^+(M_t) \) denote the subgroup of \( O(M_t) \) which fixes the light cone of \( M_t \).

5.1.1. Cases \( \Phi = \Phi_{t,j,\mu} \) and \( \Phi = \tilde{\Phi}_{t,1,\mu} \) where \( t = 1, 2, 3, 4 \), \( j = 0, I, II \) and \( \mu = \overline{0}, \overline{1}, \overline{2}, \overline{3}. \) For these cases
\[
\begin{align*}
\Phi_{1,II,\overline{5}} &= \Delta_5 \text{ (formula (2.16) and [GN1], [GN2]);} \\
\Phi_{2,II,\overline{5}} &= \Delta_2 \text{ (formula (2.21) and [GN1]);} \\
\Phi_{3,II,\overline{5}} &= \Delta_1 \text{ (formula (2.20));} \\
\Phi_{4,II,\overline{5}} &= \Delta_{1/2} \text{ (formula (2.11));} \\
\Phi_{1,0,\overline{5}} &= \Delta_{35} \text{ (formula (3.31) and [GN4]);} \\
\Phi_{2,0,\overline{5}} &= \Delta_{11} \text{ (formula (3.11));} \\
\Phi_{3,0,\overline{5}} &= D_6 \cdot \Delta_1 \text{ (formula (3.32));} \\
\Phi_{4,0,\overline{5}} &= \Delta_5^{(4)} \text{ (formula (3.12));} \\
\Phi_{t,1,\overline{5}} &= \Phi_{t,0,\overline{5}}/\Phi_{t,II,\overline{5}}, \quad t = 1, 2, 3, 4 \text{ (see [GN4]);} \\
\Phi_{t,II,\overline{5}}(Z) &= \Phi_{t,0,\overline{5}}(Z)/\Phi_{t,II,\overline{5}}(2Z), \quad t = 1, 2, 3, 4 \text{ (see [GN4]);} \\
\Phi_{t,1,\overline{1}} &= 1 \text{ for } t = 1 \text{ and } \Phi_{t,1,\overline{1}} = \Psi_{t,1,\overline{1}} \text{ for } t = 2, 3, 4 \text{ (formulae (4.7), (4.8), (4.9));} \\
\Phi_{t,3,\overline{5}} &= \Phi_{t,3,\overline{5}} / \Phi_{t,1,\overline{5}}, \quad t = 1, 2, 3, 4; \\
\Phi_{t,1,\overline{5}} &= \Phi_{t,1,\overline{5}} / \Phi_{t,1,\overline{5}}, \quad t = 1, 2, 3, 4.
\end{align*}
\]

According to our definitions, \( \Phi_{1,j,\overline{1}} = \Phi_{1,j,\overline{1}} \) and \( \Phi_{1,1,\overline{1}} = \tilde{\Phi}_{1,1,\overline{1}} \). We denote the corresponding to these forms \( A, W, \mathcal{M} \) and \( \rho \) by the same indexes and “tilde”. We describe them below. For all of them we have

\[
W_{t,j,\mu} \rtimes \text{Sym}(P_{t,j,\mu}(\mathcal{M}_{t,j,\mu})) = O^+(M_t), \quad \tilde{W}_{t,1,\mu} \rtimes \text{Sym}(\tilde{P}_{t,1,\mu}(\tilde{\mathcal{M}}_{t,1,\mu})) = O^+(M_t).
\]

We have for \((t, I, \overline{1})\) and \((t, 0, \overline{1})\):

\[
W_{t,1,\overline{1}} = W_{t,0,\overline{1}} = W(M_t) = W^{(2,8,8t)}(M_t)
\]

with the same fundamental polyhedron \( \mathcal{M}_{t,1,\overline{1}} = \mathcal{M}_{t,0,\overline{1}} \).
**Cases** \((t, I, \overline{T})\), \(t = 1, 2, 3, 4\). We have

\[
P_{t, I, \overline{T}}(\mathcal{M}_{t, I, \overline{T}}) = \{\delta_1 = (0, -1, 0), \delta_2 = (1, 2, 0), \delta_3 = (-1, 0, 1)\}, \quad P_{t, I, \overline{T}}(\mathcal{M}_{t, I, \overline{T}}) = \{\delta_1\}
\]

with the Gram matrix

\[
G(P_{t, I, \overline{T}}(\mathcal{M}_{t, I, \overline{T}})) = (\delta_i, \delta_j) = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & -4t \\ 0 & -4t & 8t \end{pmatrix}
\]

and the generalized Cartan matrix

\[
A_{t, I, \overline{T}} = \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -t \\ 0 & -1 & 2 \end{pmatrix}
\]

with the set of odd indices \(\{1\}\). We have

\[
\rho_{t, I, \overline{T}} = \left(\frac{2t + 3}{2t}, \frac{1}{2}, \frac{3}{2t}\right).
\]

The matrix \(A_{1, I, \overline{T}} = A_{1, I, \overline{T}}\) is equal to the matrix \(r = -59/2\) of the Table 1 in Part I and is twisted to the symmetric generalized Cartan matrix \(A_{1,0}\) of Theorem 1.3.1 in Part I.

**Cases** \((t, 0, \overline{T})\), \(t = 1, 2, 3, 4\). We have:

\[
P_{t, 0, \overline{T}}(\mathcal{M}_{t, 0, \overline{T}}) = \{2\delta_1 = (0, -2, 0), \delta_2 = (1, 2, 0), \delta_3 = (-1, 0, 1)\} \quad \text{and} \quad P_{t, 0, \overline{T}}(\mathcal{M}_{t, 0, \overline{T}}) = \emptyset
\]

with the Gram matrix

\[
G(P_{t, 0, \overline{T}}(\mathcal{M}_{t, 0, \overline{T}})) = \begin{pmatrix} 8 & -8 & 0 \\ -8 & 8 & -4t \\ 0 & -4t & 8t \end{pmatrix}
\]

and the generalized Cartan matrix

\[
A_{t, 0, \overline{T}} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -t \\ 0 & -1 & 2 \end{pmatrix}
\]

with the empty set of odd indices. We have

\[
\rho_{t, 0, \overline{T}} = \left(\frac{t + 2}{t}, 1, \frac{2}{t}\right).
\]

The matrix \(A_{1,0, \overline{T}} = A_{1,0, \overline{T}}\) is equal to the symmetric generalized Cartan matrix \(A_{1,0}\) of Theorem 1.3.1 in Part I.
Cases \((t, I, T)\), \(t = 1, 2, 3, 4\). We have
\[
\tilde{W}_{t,I,T} = W^{(8,8t)}(M_t)
\]
with the fundamental polyhedron
\[
\tilde{M}_{t,I,T} = \text{Sym}(\tilde{P}_{t,I,T}(\tilde{M}_{t,I,T}))(\tilde{M}_{t,I,T})
\]
where \(\text{Sym}(\tilde{P}_{t,I,T}(\tilde{M}_{t,I,T}))\) is generated by the reflection in \(\delta_1 = (0, -1, 0)\). Respectively,
\[
\tilde{P}_{t,I,T}(\tilde{M}_{t,I,T}) = \text{Sym}(\tilde{P}_{t,I,T}(\tilde{M}_{t,I,T}))(\delta_2, \delta_3)
\]
\[
= \{\delta_2 = (1, 2, 0), \delta_3 = (-1, 0, 1), \delta_2' = (1, -2, 0)\},
\]
\[
\tilde{P}_{t,I,T}(\tilde{M}_{t,I,T}) = \emptyset.
\]
We have
\[
G(\tilde{P}_{t,I,T}(\tilde{M}_{t,I,T})) = \begin{pmatrix}
  8 & -4t & -8 \\
-4t & 8t & -4t \\
-8 & -4t & 8
\end{pmatrix},
\]
the generalized Cartan matrix
\[
\tilde{A}_{t,I,T} = \begin{pmatrix}
  2 & -t & -2 \\
-1 & 2 & -1 \\
-2 & -t & 2
\end{pmatrix},
\]
with the empty set of odd indices. The lattice Weyl vector
\[
\tilde{\rho}_{t,I,T} = \left(\frac{t+1}{t}, \frac{1}{t}\right).
\]
The matrix \(\tilde{A}_{1,I,T} = \tilde{A}_{1,II,0}\) gives the symmetric generalized Cartan matrix \(A_{1,I}\) of Theorem 1.3.1 in Part I. The matrix \(\tilde{A}_{4,I,T}\) gives the generalized Cartan matrix corresponding to \(r = -5/2\) of Table 1 in Part I. This matrix is twisted to the matrix \(A_{1,II}\).

Cases \((t, II, T)\), \(t = 2, 3, 4\). The case \((1, II, T)\) coincides with \((1, II, 0)\) and will be considered later together with all cases \((t, II, 0)\). We have
\[
W_{t,II,T} = W^{(2,8t)}(M_t), \quad t = 2, 3, 4,
\]
with the fundamental polyhedron
\[
\mathcal{M}_{t,II,T} = \text{Sym}(P_{t,II,T}(\mathcal{M}_{t,II,T}))(\mathcal{M}_{t,0,T}), \quad t = 2, 3, 4,
\]
where \(\text{Sym}(P_{t,II,T}(\mathcal{M}_{t,II,T}))\) is generated by the reflection in \(\delta_2 = (1, 2, 0)\) and
\[
P_{t,II,T}(\mathcal{M}_{t,II,T}) = \text{Sym}(P_{t,II,T}(\mathcal{M}_{t,II,T}))(2\delta_1, \delta_3) = \]
\[ = \{ 2\delta_1 = (0, -2, 0), \delta_3 = (-1, 0, 1), \delta_3' = (t-1, 2t, 1), 2\delta_1' = (2, 2, 0) \} \]

with the Gram matrix

\[
G(P_{t,II,T}(M_{t,II,T})) = \begin{pmatrix}
8 & 0 & -8t & -8 \\
0 & 8t & -4t^2 + 8t & -8t \\
-8t & -4t^2 + 8t & 8t & 0 \\
-8 & -8t & 0 & 8 \\
\end{pmatrix}
\]

and the generalized Cartan matrix

\[
A_{t,II,T} = \begin{pmatrix}
2 & 0 & -2t & -2 \\
0 & 2 & -t + 2 & -2 \\
-2 & -t + 2 & 2 & 0 \\
-2 & -2t & 0 & 2 \\
\end{pmatrix}
\]

with the empty set of odd indices. The lattice Weyl vector

\[
\rho_{t,II,T} = \left( \frac{t+1}{t}, 1, \frac{1}{t} \right).
\]

The matrix \( A_{4,II,T} \) gives the third generalized Cartan matrix for \( r = -2 \) of Table 1 in Part I. It is twisted to the generalized Cartan matrix \( A_{2,I} \) of Theorem 1.3.1 in Part I.

**Cases** \((t, T), \ t = 2, 3, 4\). For these cases

\[
W_{t,T} = W^{(st)}(M_t), \quad t = 2, 3, 4,
\]

with the fundamental polyhedron

\[
M_{t,T} = \text{Sym}(P_{t,T}(M_{t,T}))(M_{t,0,T}), \quad t = 2, 3, 4,
\]

where (we consider only \( t = 2, 3, 4 \)) the group \( \text{Sym}(P_{t,T}(M_{t,T})) \) is generated by the reflections in \( \delta_1 = (0, -1, 0) \) and \( \delta_2 = (1, 2, 0) \). We have

\[
\rho_{t,T} = (1, 0, 0)
\]

and for \( \delta_3 = (-1, 0, 1) \)

\[
P_{t,T}(M_{t,T}) = P_{t,T}(M_{t,T}) \eta = \text{Sym}(P_{t,T}(M_{t,T})) (\delta_3) = \{ \text{primitive } \delta \in M_t \mid (\delta, \delta) = 8t, \ 4t | (\delta, M_t), \ (\delta, \rho) = -4t \}.
\]

These case is parabolic, the group \( \text{Sym}(P_{t,T}(M_{t,T})) \) is the affine reflection group on a line (it is isomorphic to \( \mathbb{Z} \) up to index two). The Gram matrix is

\[
G(P_{t,T}(M_{t,T})) = (\alpha, \alpha'), \ \alpha, \alpha' \in P_{t,T}(M_{t,T}),
\]

and the generalized Cartan matrix is

\[
A_{t,T} = \begin{pmatrix}
(\alpha, \alpha') \\
4t
\end{pmatrix}, \ \alpha, \alpha' \in P_{t,T}(M_{t,T}),
\]

with the empty set of odd indices. It is symmetric of parabolic type and has a lattice Weyl vector. It is interesting that for \( t = 2, 3 \) this matrix does not have a parabolic submatrix of the type \( \tilde{A}_1 \).
Case \((t, 0, \overline{0}), \ t = 1, 2, 3, 4\). For this case
\[
W_{t, 0, \overline{0}} = W^{(2,8)}(M_t).
\]
and
\[
\mathcal{M}_{t, 0, \overline{0}} = \text{Sym}(P_{t, 0, \overline{0}}(\mathcal{M}_{t, 0, \overline{0}}))(\mathcal{M}_{t, 0, \overline{T}}).
\]
The group Sym\((P_{1,0,\overline{0}}(\mathcal{M}_{1,0,\overline{0}}))\) is trivial and the case \((1, 0, 0)\) coincides with the \((1, 0, \overline{T})\) which has been considered before. Below we assume that \(t = 2, 3, 4\). We have
\[
P_{t, 0, \overline{0}}(\mathcal{M}_{t, 0, \overline{0}}) = P_{t, 0, \overline{0}}(\mathcal{M}_{t, 0, \overline{0}}) = \text{Sym}(P_{t, 0, \overline{0}}(\mathcal{M}_{t, 0, \overline{0}}))(2\delta_1, \delta_2) = \{2\delta_1 = (0, -2, 0), \delta_2 = (1, 2, 0), \delta'_2 = (0, 2, 1)\}
\]
where Sym\((P_{t,0,\overline{0}}(\mathcal{M}_{t,0,\overline{0}}))\) is generated by the reflection in \(\delta_3 = (-1, 0, 1)\). We have
\[
\rho_{t, 0, \overline{0}} = \left(\frac{2}{t}, 1, \frac{2}{t}\right),
\]
\[
G(P_{t, 0, \overline{0}}(\mathcal{M}_{t, 0, \overline{0}})) = \begin{pmatrix}
8 & -8 & -8 \\
-8 & 8 & -4t + 8 \\
-8 & -4t + 8 & 8
\end{pmatrix}
\]
and
\[
A_{t, 0, \overline{0}} = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -t + 2 \\
-2 & -t + 2 & 2
\end{pmatrix}
\]
with the empty set of odd indices. The matrices \(A_{t,0,\overline{0}}, \ t = 1, 2, 3,\) coincide with the symmetric generalized Cartan matrices \(A_{t,0}\) of Theorem 1.3.1 of Part I. The matrix \(A_{4,0,\overline{0}}\) coincides with the matrix \(A_{1,1,0}\) of this theorem.

Case \((t, I, \overline{0}), \ t = 1, 2, 3, 4\). For this case
\[
W_{t, I, \overline{0}} = W^{(2,8)}(M_t)
\]
and
\[
\mathcal{M}_{t, I, \overline{0}} = \text{Sym}(P_{t, I, \overline{0}}(\mathcal{M}_{t, I, \overline{0}}))(\mathcal{M}_{t, I, \overline{T}}).
\]
The group Sym\((P_{1,1,\overline{0}}(\mathcal{M}_{1,1,\overline{0}}))\) is trivial and the case \((1, I, 0)\) coincides with the \((1, I, \overline{T})\) which has been considered before. Below we assume that \(t = 2, 3, 4\). We have
\[
P_{t, I, \overline{0}}(\mathcal{M}_{t, I, \overline{0}}) = \text{Sym}(P_{t, I, \overline{0}}(\mathcal{M}_{t, I, \overline{0}}))(\delta_1, \delta_2) = \{\delta_1 = (0, -1, 0), \delta_2 = (1, 2, 0), \delta'_2 = (0, 2, 1)\}
\]
and
\[
P_{t, I, \overline{0}}(\mathcal{M}_{t, I, \overline{0}})\overline{T} = \{\delta_1 = (0, -1, 0)\}.
where $\text{Sym}(P_{t,I,0}(\mathcal{M}_{t,I,0}))$ is generated by the reflection in $\delta_3 = (-1, 0, 1)$. We have

$$\rho_{t,I,0} = \left( \frac{3}{2t}, \frac{1}{2}, \frac{3}{2t} \right),$$

$$G(P_{t,I,0}(\mathcal{M}_{t,I,0})) = \begin{pmatrix} 2 & -4 & -4 \\ -4 & 8 & -4t + 8 \\ -4 & -4t + 8 & 8 \end{pmatrix}$$

and

$$A_{t,I,0} = \begin{pmatrix} 2 & -4 & -4 \\ 1 & 2 & -t + 2 \\ -1 & -t + 2 & 2 \end{pmatrix}$$

with the set $\{1\}$ of odd indices. In the Table 1 of Part I, the matrix $A_{2,I,0}$ corresponds to $r = -17/2$, the matrix $A_{3,I,0}$ corresponds to the second matrix with $r = -11/2$, the matrix $A_{4,I,0}$ corresponds to the third matrix with $r = -4$. These matrices are twisted to the matrices $A_{2,0}, A_{3,0}, A_{1,II}$ of Theorem 1.3.1 in Part I respectively.

**Cases** $(t, I, 0)$, $t = 1, 2, 3, 4$. For these cases

$$\tilde{W}_{t,I,0} = W^{(8)}(M_t)$$

with the fundamental polyhedron

$$\tilde{\mathcal{M}}_{t,I,0} = \text{Sym}(\tilde{P}_{t,I,0}(\tilde{\mathcal{M}}_{t,I,0}))(\mathcal{M}_{t,I,1}).$$

The case $(1, I, 0)$ coincides with $(1, I, 1)$ and has been considered. Below we suppose that $t = 2, 3, 4$. For $t = 2, 3, 4$, the group $\text{Sym}(\tilde{P}_{t,I,0}(\tilde{\mathcal{M}}_{t,I,0}))$ is generated by the reflections in $\delta_1 = (0, -1, 0)$ and $\delta_3 = (-1, 0, 1)$, and is isomorphic to $D_2$. We have

$$\tilde{P}_{t,I,0}(\tilde{\mathcal{M}}_{t,I,0}) = \tilde{P}_{t,I,0}(\tilde{\mathcal{M}}_{t,I,0}) = \text{Sym}(\tilde{P}_{t,I,0}(\tilde{\mathcal{M}}_{t,I,0}))(\delta_2) = \{\delta_2 = (1, 2, 0), \delta_2 = (0, 2, 1), \delta_2 = (0, -2, 1), \delta_2 = (1, -2, 0)\},$$

with the Gram matrix

$$G(\tilde{P}_{t,I,0}(\tilde{\mathcal{M}}_{t,I,0})) = \begin{pmatrix} 8 & -4t + 8 & -4t - 8 & -8 \\ -4t + 8 & 8 & -8 & -4t - 8 \\ -4t - 8 & -8 & 8 & -4t + 8 \\ -8 & -4t - 8 & -4t + 8 & 8 \end{pmatrix},$$

and the generalized Cartan matrix

$$\tilde{A}_{t,I,0} = \begin{pmatrix} 2 & -t + 2 & -t - 2 & -2 \\ -t + 2 & 2 & -t - 2 & 0 \\ -t - 2 & -2 & 2 & -t + 2 \\ -2 & -t - 2 & -t + 2 & 2 \end{pmatrix}. $$
with the empty set of odd indices. The lattice Weyl vector
\[ \tilde{\rho}_{t,\overline{I}} = \left( \frac{1}{t}, 0, \frac{1}{t} \right). \]

For \( t = 1, 2, 3 \), the generalized Cartan matrices \( \tilde{A}_{t,\overline{I}} \) give the symmetric generalized Cartan matrices \( A_{t,\overline{I}} \), and \( \tilde{A}_{4,\overline{I}} \) gives the symmetric generalized Cartan matrix \( A_{2,\overline{I}} \) of Theorem 1.3.1 in Part I.

**Cases** \( (t, II, 0) \), \( t = 1, 2, 3, 4 \). For these cases
\[ W_{t, II, 0} = W^{(2)}(M_t) \]
with
\[ P_{t, II, 0}(M_{t, II, 0}) = \text{Sym}(P_{t, II, 0}(M_{t, II, 0}))(\mathcal{M}_{t,0,\overline{I}}) \]
where the group \( \text{Sym}(P_{t, II, 0}(M_{t, II, 0})) \) is generated by the reflections in \( \delta_2 = (1, 2, 0) \) and \( \delta_3 = (-1, 0, 1) \). We can take
\[ \rho_{t, II, 0} = \left( \frac{1}{2t}, \frac{1}{2}, \frac{1}{2t} \right). \]

Then
\[ P_{t, II, 0}(M_{t, II, 0}) = P_{t, II, 0}(M_{t, II, 0}) = \text{Sym}(P_{t, II, 0}(M_{t, II, 0}))(\delta_1), \quad \delta_1 = (0, -1, 0). \]

We have
\[
\begin{align*}
P_{1, II, 0}(M_{1, II, 0}) &= \{ \delta_{11} = (0, -1, 0), \delta_{12} = (1, 1, 0), \delta_{13} = (0, 1, 1) \}, \\
P_{2, II, 0}(M_{2, II, 0}) &= \{ \delta_{11} = (0, -1, 0), \delta_{12} = (1, 1, 0), \delta_{13} = (1, 3, 1), \delta_{14} = (0, 1, 1) \}, \\
P_{3, II, 0}(M_{3, II, 0}) &= \{ \delta_{11} = (0, -1, 0), \delta_{12} = (1, 1, 0), \\
& \quad \delta_{13} = (2, 5, 1), \delta_{14} = (2, 7, 2), \delta_{15} = (1, 5, 2), \delta_{16} = (0, 1, 1) \} ,
\end{align*}
\]
and
\[
\begin{align*}
P_{4, II, 0}(M_{4, II, 0}) &= \text{Sym}(P_{4, II, 0}(M_{4, II, 0}))(0, -1, 0)) \\
&= \{ \delta \in M_4 | (\delta, \delta) = 2, (\delta, \rho) = -1 \}. 
\end{align*}
\]

It follows that \( G(P_{t, II, 0}(M_{t, II, 0})) = A_{t, II, 0} \) where for \( t = 1, 2, 3 \) the generalized Cartan matrix \( A_{t, II, 0} \) is equal to the generalized Cartan matrix \( A_{t, II} \) of Theorem 1.3.1 in Part I, and the \( A_{4, II, 0} \) is the symmetric parabolic generalized Cartan matrix (it is infinite)
\[ A_{4, II, 0} = ((\alpha, \alpha')) \quad \alpha, \alpha' \in P_{4, II, 0}(M_{4, II, 0}). \]
5.1.2. The case $D_2(Z)$. The automorphic form $D_2(Z)$ was considered in Theorem 4.1. For this case $t = 9$ and the Weyl group is

$$W = W^{(2,8,18)}(M_9).$$

The full group $W(M_9) = W^{(2,8,18,72)}(M_9)$ has the fundamental polyhedron $\mathcal{M}_0$ with the set of primitive orthogonal vectors

$$P(\mathcal{M}_0) = \{\alpha_1 = (0, -1, 0), \alpha_2 = (1, 2, 0), \alpha_3 = (2, 9, 1), \delta = (-1, 0, 1)\}$$

with the Gram matrix

$$G(P(\mathcal{M}_0)) = \begin{pmatrix}
2 & -4 & -18 & 0 \\
-4 & 8 & 0 & -36 \\
-18 & 0 & 18 & -36 \\
0 & -36 & -36 & 72
\end{pmatrix}.$$ 

We have

$$\rho = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{6}\right).$$

The group $\text{Sym}(P(\mathcal{M}))$ is generated by the reflection in $\delta = (-1, 0, 1)$, and

$$P(\mathcal{M}) = \text{Sym}(P(\mathcal{M}))(\alpha_1, \alpha_2, \alpha_3) = \{(0, -1, 0), (1, 2, 0), (2, 9, 1), (1, 9, 2), (0, 2, 1)\},$$

$$P(\mathcal{M})_\mathfrak{T} = \{(0, -1, 0)\}.$$ 

The Gram matrix

$$G(P(\mathcal{M})) = \begin{pmatrix}
2 & -4 & -18 & -18 & -4 \\
-4 & 8 & 0 & -36 & -28 \\
-18 & 0 & 18 & -36 & -36 \\
-18 & -36 & -18 & 18 & 0 \\
-4 & -28 & -36 & 0 & 8
\end{pmatrix}$$

and the generalized Cartan matrix

$$A(P(\mathcal{M})) = \begin{pmatrix}
2 & -4 & -18 & -18 & -4 \\
-1 & 2 & 0 & -9 & -7 \\
-2 & 0 & 2 & -2 & -4 \\
-2 & -4 & -2 & 2 & 0 \\
-1 & -7 & -9 & 0 & 2
\end{pmatrix}$$

with the set $\{1\}$ of odd indices. This is the generalized Cartan matrix of the case $r = -3/2$ of Table 1 in Part I. It is twisted to the symmetric generalized Cartan matrix $A_{1,\text{III}}$ of Theorem 1.3.1 in Part I.
5.1.3. The case $D_{1/2}(Z)$. The final formula for the automorphic form $D_{1/2}(Z)$ is given in (2.14). For this case $t = 36$ and the Weyl group is

$$W = W^{(2,8,18,32)}(M_{36}).$$

The group $W(M_{36}) = W^{(2,8,18,32,72,288)}(M_{36})$ has the fundamental polyhedron $\mathcal{M}_0$ with the set of primitive orthogonal vectors

$$P(\mathcal{M}_0) = \{\alpha_1 = (0, -1, 0), \alpha_2 = (1, 2, 0), \alpha_3 = (7, 32, 1), \alpha_4 = (5, 27, 1),
\delta_1 = (2, 18, 1), \delta_2 = (-1, 0, 1)\}$$

with the Gram matrix

$$G(P(\mathcal{M}_0)) = \begin{pmatrix} 2 & -4 & -64 & -54 & -36 & 0 \\ -4 & 8 & -16 & -36 & -72 & -144 \\ -64 & -16 & 32 & 0 & -144 & -864 \\ -54 & -36 & 0 & 18 & -36 & -576 \\ -36 & -72 & -144 & -36 & 72 & -144 \\ 0 & -144 & -864 & -576 & -144 & 288 \end{pmatrix}.$$ 

The fundamental polyhedron $\mathcal{M}$ for the Weyl group $W$ is

$$\mathcal{M} = \text{Sym}(P(\mathcal{M}))(\mathcal{M}_0), \quad P(\mathcal{M}) = \text{Sym}(P(\mathcal{M}))(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

$$P(\mathcal{M})_{\mathcal{T}} = \text{Sym}(P(\mathcal{M}))(\alpha_1).$$

The lattice Weyl vector is

$$\rho = \left(\frac{1}{24}, \frac{1}{2}, \frac{1}{24}\right).$$

The symmetry group $\text{Sym}(P(\mathcal{M}))$ is generated by the reflections in $\delta_1 = (2, 18, 1)$ and $\delta_2 = (-1, 0, 1)$. It is infinite, and this case is parabolic with the parabolic generalized Cartan matrix

$$A(P(\mathcal{M})) = \left(\frac{2(\alpha, \alpha')}{(\alpha, \alpha)}\right), \quad \alpha, \alpha' \in P(\mathcal{M}).$$

5.2. Discriminants of moduli of K3 surfaces and Mirror Symmetry.

A lattice $T$ with signature $(n, 2)$ is called 2-reflective if there exists an automorphic form $\Phi$ on $\Omega(T)$ with respect to a subgroup of $O(T)$ of finite index such that the divisor of $\Phi$ is union of quadratic divisors orthogonal to elements of $T$ with norm 2. The automorphic form $\Phi$ is called 2-reflective (for $T$). Obviously, any overlattice $T \subset T'$ of finite index of a 2-reflective lattice $T$ is also 2-reflective. Thus, one 2-reflective lattice $T$ gives many other 2-reflective lattices. A 2-reflective lattice $T$ is called strongly 2-reflective if there exists a 2-reflective automorphic form $\Phi$ for $T$ such that the divisor of $\Phi$ is union of all quadratic divisors orthogonal to elements of $T$ with norm 2. The automorphic form $\Phi$ is then called strongly 2-reflective. Conjecturally (see [N6] and Part I) number of 2-reflective lattices $T$ of rank $\geq 5$ is finite. 2-reflective lattices and strongly 2-reflective lattices are important for moduli of K3-surfaces. By the global Torelli Theorem (see [P-Š–Š]) and epimorphicity of period map (see [Ku]), moduli $\mathcal{M}_S$ of K3 surfaces with a condition
the paramodular group $\hat{O}(L_t)$ where $L_t = U^2 \oplus \langle 2t \rangle$, $t \in \mathbb{N}$ (see Sect. 1.3), and have the divisors which are unions of quadratic divisors $H_\delta$, where $\delta$ are some primitive roots of $L_t$ completely described by the construction of $\Phi$. Here $\delta \in L_t$ is a root of $L_t$ if the reflection in $\delta$ belongs to $O(L_t)$.

For each automorphic form $\Phi$ of the theorem, we replace $L_t$ by a canonical (with respect to $L_t$) sublattice $T \subset (L_t \otimes \mathbb{Q})(m)$, $m \in \mathbb{Q}$. Here canonical means that $O(T) = O(L_t)$ in $O(L_t \otimes \mathbb{Q})$. Then a primitive root $\delta$ in $L_t$ is replaced by a primitive root $\tilde{\delta} \in T$, $\tilde{\delta} \in \mathbb{Q}_{++}$. Obviously, then $\Phi$ will be automorphic with respect to the corresponding subgroup of finite
index of $O(T)$. Below, for each $\Phi$, we give a procedure which changes $L_t$ by $T$ in such a way that any quadratic divisor $H_\delta$, $\delta \in L_t$, of $\Phi$ will be replaced by $H_{\tilde{\delta}}$, $\tilde{\delta} \in T$ and $(\delta, \tilde{\delta}) = 2$ in the lattice $T$. Moreover, any element $\tilde{\delta} \in T$ with the norm 2 gives a rational quadratic divisor of $\Phi$. Thus, the lattice $T$ will be strongly 2-reflective with the strongly 2-reflective automorphic form $\Phi$.

For $\Delta_{35}$ ($t = 1$) and $\Psi_{12}^{(t)}$, $t = 2, 3, 4$, we put $T = L_t = U^2 \oplus \langle 2t \rangle$.

For $\Delta_5$, $\Delta_2$, $\Delta_1$, $\Delta_{1/2}$ which correspond to $t = 1, 2, 3, 4$ respectively, one should put $T = (L_t)^*\{4t\}$. Then $T = U(4t)^2 \oplus \langle 2 \rangle$.

For $D_6 \cdot \Delta_1$, $F_5^{(5)}$, $F_2^{(7)}$ which correspond to odd $t = 3, 5, 7$ respectively, we take $T \otimes \mathbb{Z}_p = (L_t \otimes \mathbb{Z}_p)^*\{(t)\}$ for $p \neq 2$, and $T \otimes \mathbb{Z}_2 = (L_t \otimes \mathbb{Z}_2)(t)$. Here $\mathbb{Z}_p$ denote $p$-adic integers. Then $T = U(t)^2 \oplus \langle 2 \rangle$.

For $F_3^{(6)}$ and $F_1^{(10)}$ which correspond to even $t = 6, 10$ respectively, with odd $t/2$, we first replace $L_t$ by its overlattice of index 2 (it is unique). We get an odd lattice $\tilde{L}_t = U^2 \oplus \langle t/2 \rangle$. The lattice $L_t$ is its maximal even sublattice. Then we consider $T = (\tilde{L}_t)^*\{(t) = U(t)^2 \oplus \langle 2 \rangle$.

Using properties of the paramodular orthogonal group $O(L_t)$ described in Sect. 1.3, one can check the properties of $T$ we claimed. It finishes the proof.

The automorphic forms $\Phi$ of Theorem 5.2.1 can be used to continue series of examples of the variant of Mirror Symmetry for K3 surfaces we suggested in [GN3]. We denote the automorphic form $\Phi$ corresponding to the lattice $T$ from Theorem 5.2.1 by $\Phi_T$. We remark that for the most part of cases the form $\Phi_T$ has zeros of multiplicity one and is the unique strongly reflective automorphic form for $T$ with this property (by Koecher principle, see [Ba], [F1] for example). All other forms have very small multiplicities of zeros and are automorphic with respect to natural groups of small indices in the full orthogonal groups. It follows that they are very exceptional.

The automorphic forms $\Phi_T$ may be considered as “algebraic functions” on moduli $\mathcal{M}_{T,L}$ of K3 surfaces with the condition $T^\perp := T(-1)_{\mathbb{Z}K3}$ on the Picard lattice (B-model). For any $T$, the hyperbolic lattices $T^\perp$ may be easily computed using the discriminant forms technique (see [N1]) and give very interesting moduli of K3 surfaces. We hope to study equations of the corresponding K3 surfaces in further publications.

Let us consider an isotropic element $c$ in the first summand $U(k)$ of the lattice $T$. We then get a hyperbolic lattice

$$S = (c_T^+/[c])(-1) = \begin{cases} U \oplus \langle -2t \rangle & \text{for } T = U^2 \oplus \langle 2t \rangle, \\ U(k) \oplus \langle -2 \rangle & \text{for } T = U(k)^2 \oplus \langle 2 \rangle, \end{cases}$$

and can consider moduli $\mathcal{M}_S$ of K3 surfaces with condition $S$ on the Picard lattice (A-model). A general K3 surface $X$ from this moduli has Picard lattice $S$. The Fourier expansion of $\Phi_T$ at the cusp $c$ which we used in this paper is then related with the geometry of non-singular rational curves on these K3 surfaces $X$ (see [GN3]).

More exactly it means (see details in [GN3] and Part I) that replacing $M_t$ by the lattice $S$, we have the similar to (5.1) Fourier expansion related with the group $W = W^{(-2)}(S)$, some its quadratic character $\epsilon$, a generalized lattice Weyl vector $\rho \in S \otimes \mathbb{Q}$, and a fundamental polyhedron $\mathcal{M}$ for $W$. For this case, we can interpret $\mathbb{R}_{++} \mathcal{M} \cap S$ as the set $\text{NEF}(S)$ of numerically effective (or irreducible with non-negative square) elements of the Picard lattice $S$ of $X$ and the set $P(\mathcal{M})$ as the set of irreducible non-singular rational
curves on $X$. We can interpret $\alpha > 0$, $\alpha \in S$ (for the product part of (5.1)), as the set $\text{EF}(S)$ of effective elements of the Picard lattice $S$ for the K3 surfaces $X$. We get the identity

$$\Phi_T(z) = \sum_{w \in W(\mathbb{Z}^2)} \varepsilon(w) \left( \exp(2\pi i (w(\rho) \cdot z)) - \sum_{a \in \text{NEF}(S)} N(a) \exp(2\pi i (w(\rho + a) \cdot z)) \right)$$

where $z \in \Omega(V^+(S)) = S \otimes \mathbb{R} + iV^+(S)$ and $V^+(S)$ is the light cone of effective elements of $S$. For the lattices $T = U(4)^2 \oplus \langle 2 \rangle$ and $U(8)^2 \oplus \langle 2 \rangle$ these formulae were given in [GN3].

Considering linear system $|k\rho|$, one gets very canonical and beautiful projective models of $X$. The case $T = U(12)^2 \oplus \langle 2 \rangle$ corresponding to $\Delta_1$ is especially nice. For this case the Picard lattice $S = U(12) \oplus \langle -2 \rangle$. The linear system $|6\rho|$ gives the embedding of $X$ in $\mathbb{P}^4$ as intersection of quadric and cubic. The surface $X$ has exactly 6 non-singular rational curves, all of them have degree 6. The set of their classes is $P_{3,11,\pi}(\mathcal{M}_{3,11,\pi}) \subset S$ for the $\rho$ from Sect. 5.1.1.

We recall (see Part I) that a lattice $T$ with two negative squares is called reflective if there exists an automorphic form $\Phi$ on $\Omega(T)$ with respect to a subgroup of finite index of $O(T)$ such that the zero divisor of $\Phi$ is union of quadratic divisors orthogonal to roots of $T$. The automorphic form $\Phi$ is called reflective for $T$. Conjecturally (see Part I) the set of reflective lattices $T$ is finite up to multiplication of the form of $T$. It is an interesting problem to find all reflective lattices $T$ and their reflective automorphic forms.

In this paper we found many reflective lattices $T$ and reflective automorphic forms for $\text{rk } T = 5$. They are lattices

$$T = U^2 \oplus \langle 2t \rangle, \quad t = 1, 2, \ldots, 9, 10, 12, 16, 18, 36,$$

related with the paramodular groups. Using canonical with respect to $T$ sublattices of $(T \otimes \mathbb{Q})(k)$ (defined in the proof of Theorem 5.2.1), one can construct many other reflective lattices. For example, we get lattices of Theorem 5.2.1. Without any doubt these lattices are also very important for the theory of Lorentzian Kac–Moody algebras, geometry of K3 surfaces and mirror symmetry (see [GN3] and [GN6]).

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