VERY FREE CURVES ON FANO COMPLETE INTERSECTIONS

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Abstract. In this paper, we show that general Fano complete intersections over an algebraically closed field of arbitrary characteristics are separably rationally connected. Our proof also implies that general log Fano complete intersections with smooth tame boundary divisors admit very free $\mathbb{A}^1$-curves.

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1. INTRODUCTION

Throughout the paper, we work over an algebraically closed field $k$ with characteristic $\text{char } k$.

1.1. The background and main results. The existence of rational curves in higher dimensional varieties greatly shapes the geometry. The following definitions describe the existence of large amount of rational curves:

Definition 1.1 ([Kol96] IV.3). Let $X$ be a smooth proper variety defined over $k$.

(1) A variety $X$ is rationally connected (RC) if there is a family of irreducible proper rational curves $g : U \to Y$ and an evaluation morphism $u : U \to X$ such that the morphism $u^{(2)} : U \times_Y U \to X \times X$ is dominant.

(2) A variety $X$ is rationally chain connected (RCC) if there is a family of two pointed chains of rational curves $g : V \to Y$ such that the evaluation morphism $ev : V \to X \times X$ is dominant.

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A variety $X$ is separably rationally connected (SRC) if there exists a proper rational curve $f : \mathbb{P}^1 \to X$ such that the image lies in the smooth locus of $X$ and the pullback of the tangent sheaf $f^* T_X$ is ample. Such rational curves are called very free curves.

We refer to Kollár’s book [Kol96] for the background. It is known that SRC implies RC, and RC implies RCC. All the notations of rational connectedness are equivalent for smooth varieties in characteristic zero. The converse statement is not known in positive characteristics.

The fundamental results of Campana [Cam92] and Kollár, Miyaoka and Mori [KMM92] show that Fano varieties, i.e., smooth varieties with ample anticanonical bundles are rationally chain connected. In particular, Fano varieties are all SRC in characteristic zero.

It has been pointed out by Kollár that separable rational connectedness is the right notion for rational connectedness in arbitrary characteristics. They have both nice geometric and arithmetic applications as follows. Geometrically, when the base field is algebraically closed,

1. Graber-Harris-Starr [GHS03] proves the famous theorem asserts that over characteristic zero, a proper family of varieties over an algebraic curve whose general fiber is smooth RC admits a section;
2. de Jong and Starr in [dJS03] generalize the result of [GHS03] by showing the existence of sections over positive characteristics when general fibers are smooth SRC.
3. The weak approximation for families of separably rationally connected varieties was studied by [HT06], [HT08a].
4. Tian and Zong [TZ12] show that the Chow group of 1-cycles on a smooth proper SRC variety is generated by rational curves.

Arithmetically, we have:

1. when the base field is a local field, Kollár [Kol99] shows that a smooth proper SRC variety admits a very free curve through any rational point;
2. when the base field is a finite field, Kollár and Szabo [KS03] show that a smooth proper SRC variety admits a very free curve through any zero dimensional cycle after a finite base change;
3. when the base field is a large field, Hu [Hu10] proves interesting results on the weak approximation conjecture for SRC varieties at places of good reductions.

Despite the nice behavior of SRC varieties, many important varieties which are RC over characteristic zero, but not known in the positive characteristic cases. The following question is the major motivation of the present paper:

**Question 1.2** (Kollár). In arbitrary characteristic, is every smooth Fano variety separably rationally connected?
The first testing example is Fano complete intersections in projective spaces. The difficulty is to prove separable rational connectedness in low characteristics [KS03, Conjecture 14].

The question is known for general Fano hypersurfaces by [Zhu11], where very free curves are constructed explicitly over degenerate Fano varieties. In this paper, we provide an answer in the complete intersection case:

**Theorem 1.3.** Over arbitrary characteristics, a general Fano complete intersection in \( \mathbb{P}^n \) is separably rationally connected.

Our theorem eliminates the SRC condition in [TZ12, Theorem 1.7].

**Corollary 1.4.** Let \( X \) be a general complete intersection of type \((d_1, \cdots, d_l)\) in \( \mathbb{P}^n \) such that \( d_1 + \cdots + d_l \leq n - 1 \). Then the Chow group of 1-cycles on \( X \) is generated by lines.

The search of the very free curves leads us to the logarithmic version of Theorem 1.3, which turns out to be the key:

**Theorem 1.5.** Over arbitrary characteristics, a general log Fano complete intersection with a general tame boundary in \( \mathbb{P}^n \) is separably \( \mathbb{A}^1 \)-connected.

We refer to Section 1.2, and 1.3 for more details of the definitions and the proof of the above results.

**Question 1.6.** Can we drop the tame condition on the boundary in Theorem 1.5?

Our construction of very free log maps provides much more general result over characteristic zero:

**Proposition 1.7.** Assume \( \text{char } k = 0 \). Let \( X \) be a log Fano smooth variety, i.e., \(- (K_X + D) \) is ample, with a smooth irreducible boundary divisor \( D \). Then \( X \) is separably \( \mathbb{A}^1 \)-connected if it is separably \( \mathbb{A}^1 \)-uniruled.

The proof of Proposition 1.7 will be given in the end of Section 4.2. Combining the above result with Lemma 3.5 and 4.1, we immediately have:

**Corollary 1.8.** With the same assumptions as above, when the normal bundle of \( D \) is effective, then \( (X, D) \) is \( \mathbb{A}^1 \)-uniruled. In particular, it is \( \mathbb{A}^1 \)-connected.

The condition in above corollary is first stated in the work of Hassett-Tschinkel [HT08b]. The \( \mathbb{A}^1 \)-uniruledness seems to be a more natural setting — it includes for example \((\mathbb{P}^1, \{\infty\})\), or Hirzebruch surfaces with the negative curve as the boundary, which are \( \mathbb{A}^1 \)-uniruled, but the divisor \( \mathcal{O}_D(D) \) is not effective.

Keel-McKernan [KM99] prove that any log Fano pair over complex numbers is either \( \mathbb{A}^1 \)-uniruled or uniruled. It is natural to ask:

**Question 1.9.** Let \( (X, D) \) be a log smooth log Fano variety with \( D \) irreducible. Is the pair \( (X, D) \) always \( \mathbb{A}^1 \)-uniruled?
Remark 1.10. It should be emphasized that our proof of Theorem 1.3 and 1.5 is constructive, which allows one to write down the exact degree of the very free curves in each case. We leave the details of this to interested readers.

On the other hand, it seems to us that $\mathbb{A}^1$-connectedness itself is a very useful conception for the study of quasi-projective varieties. The results of Theorem 1.5 and Proposition 1.7 provide a lot of interesting and concrete examples for $\mathbb{A}^1$-connectedness. In our subsequent paper [CZ13], we will study the properties of $\mathbb{A}^1$-connectedness for general log smooth varieties, and following the work of [HT08b], an application to Zariski density of integral points over function field of curves will be considered.

We next summarize the ideas used in the proof of Theorem 1.3 and 1.5.

1.2. Log Fano complete intersections. Let $(X, D)$ be a smooth pair consisting of a smooth variety $X$ and a smooth divisor $D \subset X$. This defines the divisorial log structure on $X$:

\[(1.2.1) \quad \mathcal{M}_X := \{ s \in \mathcal{O}_X \mid s|_{X \setminus D} \in \mathcal{O}^* \}.\]

Let $X^\dagger = (X, \mathcal{M}_X)$ be the log scheme defined by the smooth pair $(X, D)$. In this case, $X^\dagger$ is log smooth. We refer to [Kat89] for the basic terminologies of logarithmic geometry. When there is no danger of confusion, we may use $(X, D)$ for the log scheme $X^\dagger$ to specify the boundary $D$.

Consider the projective space $\mathbb{P}^n$ with homogeneous coordinates $\bar{x} = [x_0 : \cdots : x_n]$.

Throughout this paper, we fix a sequence of non-negative integers

\[(1.2.2) \quad d_1, \cdots, d_l, d_b\]

such that $d_i > 0$ for $i = 1, \cdots, l$, and $d_b + \sum_{i=1}^{l} d_i \leq n$. Choose a collection of general homogeneous polynomials in $\bar{x}$:

$F_1, F_2, \cdots, F_l, G$

with degrees $\deg G = d_b$ and $\deg F_i = d_i$ for all $i$.

Let $X \subset \mathbb{P}^n$ be the sub-scheme defined by $F_i$ for $i = 1, \cdots, l$, and $D \subset X$ be the locus cut out by $G$. Since $G$ and $F_i$ are general, we may assume that $(X, D)$ is a smooth pair. We call the corresponding log scheme $X^\dagger$ a log $(d_1, \cdots, d_l; d_b)$-complete intersection. We say $X^\dagger$ has a tame boundary if $\text{char } k \nmid d_b$. When $d_b = 0$, $X$ is a Fano $(d_1, \cdots, d_l)$-complete intersection in the usual sense.

1.3. Separable $\mathbb{A}^1$-connectedness. Let $X^\dagger \to B^\dagger$ be a morphism of log schemes. A stable log map over a log scheme $S^\dagger$ is a commutative diagram

\[(1.3.1) \quad \begin{array}{ccc}
C^\dagger & \xrightarrow{f} & X^\dagger \\
\downarrow & & \downarrow \\
S^\dagger & \xrightarrow{} & B^\dagger
\end{array}\]
such that $C^\dagger \to B^\dagger$ is a family of log curves over $S^\dagger$ as defined in [Kat96, Ols07], and the underlying map $f$ is a family of usual stable maps to the underlying family of targets $\mathcal{X}/B$.

The theory of stable log maps is developed by Gross-Siebert [GS13], and independently by Abramovich-Chen [Che10, AC]. The most important result about log maps we will need in this paper, is that they form an algebraic stack. Both [GS13] and [Che10] assume to work over field of characteristic zero for the purpose of Gromov-Witten theory, but the proof of algebraicity works in general. Olsson’s log cotangent complex [Ols05] provides a well-behaved deformation theory for studying stable log maps when $\mathcal{X}^\dagger \to B^\dagger$ is log smooth.

For a log smooth scheme $X^\dagger$ over Spec $k$, we write $\Omega_{X^\dagger}$ and $T_{X^\dagger} = \Omega_{X^\dagger}^\vee$ for the log cotangent and tangent bundles respectively. Generalizing Definition 1.1(3), we introduce the terminologies which are crucial to our construction:

**Definition 1.11.** A log scheme $X^\dagger$ given by a divisorial log smooth pair $(X, D)$ is called separably $\mathbb{A}^1$-connected (respectively separably $\mathbb{A}^1$-uniruled) if there is a single-marked, genus zero log map $f : C^\dagger/S^\dagger \to X^\dagger$ with $C \cong \mathbb{P}^1$ and $S$ a geometric point, such that $f^*T_{X^\dagger}$ is ample (respectively semi positive), and the tangency at the marking is non-trivial. We call such log stable map a very free (respectively free) $\mathbb{A}^1$-curve.

**Remark 1.12.** (1) Since the tangency at the marking is non-trivial, the image of the marked point has to lie on the boundary $D$. We refer to [ACGM] for the canonical evaluation spaces of the markings.

(2) The definition of log maps allow the image of components of the source curve lies in the boundary divisor. But when $f^*T_{X^\dagger}$ is semi positive, a general deformation yields a map with smooth source curve whose image meeting the boundary divisor only at the markings, see Lemma 3.8. Thus, the above definition of $\mathbb{A}^1$-uniruledness is compatible with the definition in [KM99].

1.4. **Proof of Theorem 1.3 and 1.5.** Same as in [Zhu11], the approach we will use here is by taking degenerations. However, this time we are able to chase the deformation theory with the help of logarithmic geometry. We summarize the steps in the proof, and refer to later sections for the technical details.

First, consider a general Fano $(d_1, \cdots, d_l)$-complete intersection $X$. We take a general simple degeneration of $X$ as in Section 3.1, and obtain a singular fiber by gluing a general log Fano $(d_1, \cdots, d_l-1; 1)$-complete intersection $(X_1, D)$ and a general log Fano $(d_1, \cdots, 1; d_l-1)$-complete intersection $(X_2, D)$ along the boundary divisor $D$.

**Observation 1.13** (See Proposition 3.9). The general fiber is SRC if

(1) $(X_1, D)$ is separably $\mathbb{A}^1$-connected;

(2) $(X_2, D)$ is separably $\mathbb{A}^1$-uniruled.
Second, note that $D$ is a general Fano $(d_1, \cdots, d_{l-1}, d_l - 1, 1)$-complete intersection in one-dimensional lower. We reduce both (1) and (2) in Observation 1.13 to the SRC property of the boundary $D$:

**Observation 1.14** (See Lemma 4.2 and Proposition 4.3).

(1) $(X_1, D)$ is separably $\mathbb{A}^1$-connected if $D$ is SRC.

(2) $(X_2, D)$ is separably $\mathbb{A}^1$-uniruled if $D$ is SRC (or separably uniruled).

Finally, the inductive process ends when $D$ is a projective space, which is of course separably rationally connected.

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2. **Free $\mathbb{A}^1$ lines on log Fano complete intersections**

The following is a variation of [Ang12, Theorem 4.2], which allows us to construct free $\mathbb{A}^1$ lines explicitly on log Fano complete intersections.

**Lemma 2.1.** Let $X$ be a smooth $(d_1, \cdots, d_l)$-complete intersection in $\mathbb{P}^n$ defined by

$$\{F_1 = \cdots = F_l = 0\}$$

and $D = \bigcup_{j=1}^k D_j$ be a simple normal crossings divisor on $X$ with each irreducible component $D_j$ defined by $\{G_j = 0\}$ for all $j$. Let $X^\dagger$ be the log variety associated to the pair $(X, D)$, and write $d'_j = \deg G_j$. Then the log tangent bundle $T_{X^\dagger}$ is the middle cohomology of the following complex:

$$\mathcal{O}_X \xrightarrow{A} \mathcal{O}_X(1)^{\oplus(n+1)} \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus d} \xrightarrow{B} \sum_{i=1}^l \mathcal{O}_{\mathbb{P}^n}(d_i) \oplus \sum_{j=1}^k \mathcal{O}_{\mathbb{P}^n}(d_j),$$

with the arrows defined by

$$A = (x_0, \cdots, x_n, d'_1, \cdots, d'_k)^T$$

and

$$B = \begin{pmatrix} \text{Jac } \tilde{F} & 0 \\ \text{Jac } \tilde{G} & \text{diag}(\tilde{G}) \end{pmatrix}$$

where $\tilde{F} = (F_1, \cdots, F_l)$, $\tilde{G} = (G_1, \cdots, G_k)$, Jac $\tilde{F}$ and Jac $\tilde{G}$ are the corresponding Jacobian matrices, and diag $\tilde{G}$ denotes the diagonal matrix.

Furthermore, when $\text{char } k \nmid d'_j$ for all $j$, the log tangent bundle $T_{X^\dagger}$ is given by the kernel of the following morphism

$$\mathcal{O}_X(1)^{\oplus(n+1)} \oplus \mathcal{O}_X^{\oplus(k-1)} \xrightarrow{B'} \sum_{i=1}^l \mathcal{O}_X(d_i) \oplus \sum_{j=1}^k \mathcal{O}_X(d_j),$$

where

$$B' = \begin{pmatrix} \text{Jac } \tilde{F} & 0 \\ \text{Jac } \tilde{G} & \text{diag}(G_1, \cdots, G_{k-1}) \end{pmatrix}$$
Proof. In case $\vec{F} = 0$, the result is proved in [Ang12, Theorem 4.2]. In case $\vec{G} = 0$ and $\vec{F}$ is non-trivial, the tangent bundle $T_X$ is given by the middle cohomology of the following:

$$\mathcal{O}_X \xrightarrow{A} \mathcal{O}_X(1)^{\oplus (n+1)} \xrightarrow{\text{Jac} \vec{F}} \sum_{i=1}^l \mathcal{O}_X(d_i).$$

Now the statement follows from combining the above sequence with [Ang12, Theorem 4.2].

Proposition 2.2. Let $(X, D)$ be a general log Fano $(d_1, \ldots, d_i; d_b)$-complete intersection in $\mathbb{P}^n$ with $e = \sum_{i=1}^l d_i + d_b \leq n$. If $\text{char } k \nmid d_b$, then the pair $(X, D)$ is separably log-uniruled by lines. Furthermore, the restriction of the log tangent bundle to such a log free line has splitting type

$$\mathcal{O}(1)^{\oplus (n+1-e)} \oplus \mathcal{O}^{\oplus (e-l-1)}.$$

Proof. By the log deformation theory, it suffices to produce a pair $(X, D)$ log-smooth along a line such that the restriction of the log tangent bundle is semi positive.

Let $L$ be the line defined by

$$\{x_2 = \cdots = x_n = 0\}.$$ 

For simplicity, we introduce $m_j = \sum_{i=1}^j d_i$, and set $m_0 = 0$ and $m_{l+1} = e$. Choose the following homogeneous polynomials:

\begin{align*}
F_i &= x_{m_{i-1}+2} \cdot x_{0}^{d_{i-1}} + x_{m_{i-1}+3} \cdot x_{1} \cdot x_{0}^{d_{i-2}} + \cdots + x_{m_{i}+1} \cdot x_{1}^{d_{i-1}} \\
G &= x_{1}^{d_{b}} + x_{m_{j}+2} \cdot x_{1}^{d_{b-2}} \cdot x_{0} + \cdots + x_{m_{j+1}} \cdot x_{0}^{d_{b-1}}.
\end{align*}

for $i \in \{1, \ldots, c\}$. Note that when $d = 1$, we get

\begin{equation}
G = x_1.
\end{equation}

and all $F_i$ remain the same. We then check that

1. $L$ lies in the smooth locus of $X$;
2. $L$ intersects $D$ only at the point $[x_0 : x_1] = [1 : 0]$, which is a smooth point of $D$.

By Lemma 2.1, we have a morphism of sheaves over $L$:

$$H^0(\mathcal{O}^{\oplus (n+1)}) \xrightarrow{B'} H^0 \left( \sum_{i=1}^c \mathcal{O}(d_i - 1) \oplus \mathcal{O}(d-1) \right),$$

which is surjective by the assumption $\text{char } k \nmid d_b$, and the choice of polynomials $F_i$ and $G$. Furthermore, the splitting type of the log tangent bundle on the line is as desired. This finishes the proof. \hfill \spadesuit
3. Reduction to log Fano varieties via degeneration

3.1. Simple degeneration. Consider a log smooth morphism of fine and saturated log schemes \( \pi : X^\dagger \to p^\dagger \) with \( p^\dagger \) the standard log points, i.e. \( \overline{\mathcal{M}}_{p^\dagger} = \mathcal{M}_{p^\dagger}/k^* \cong \mathbb{N} \).

**Definition 3.1.** We call such log smooth morphism \( \pi : X^\dagger \to p^\dagger \) a simple degeneration if the underlying space \( X_0 \) is given by two smooth varieties \( Y_1 \) and \( Y_2 \) intersecting transversally along a connected smooth divisor \( D \).

**Remark 3.2.** By [Ols03b], any log smooth morphism \( \pi \) as above which is a simple degeneration, admits a canonical log structure \( \tilde{\pi} : \tilde{X}^\dagger \to p^\dagger \) and a morphism \( g : p^\dagger \to p^\dagger \) such that \( \pi \) is the pull-back of \( \tilde{\pi} \) along \( g \).

Assume we are in the situation of Section 1.2. We fix a smooth \((d_1, \cdots, d_{l-1})\)-complete intersection \( W \) of codimension \( l + 1 \) in \( \mathbb{P}^n \) cut out by \( F_1, \cdots, F_{l-1} \). Let \( G_l \) be the product \( G_1 G_2 \) of two homogeneous polynomials of degree \( a \) and \( d_l - a \) and let \( F_l \) be a homogeneous polynomial of degree \( d_l \).

Consider the pencil of divisors in \( Z \subset W \times \mathbb{A}^1 \) defined by \( \{ t \cdot F_l + G_l = 0 \} \).

Let \( \pi : Z \to \mathbb{A}^1 \) be the projection to the second factor.

For a general choice of \( F_1, \cdots, F_l, G_1 \), and \( G_2 \), there exists an open neighbourhood \( U \subset \mathbb{A}^1 \) of 0 satisfying the following properties:

1. \( \pi : \mathcal{X} := \pi^{-1}U \to U \) is a flat family of \((d_1, \cdots, d_l)\)-complete intersections in \( \mathbb{P}^n \);
2. the general fibers \( X_t \) are smooth;
3. the special fiber \( X_0 \) is a union of a smooth \((d_1, \cdots, d_{l-1}, a)\)-complete intersection \( X_1 \) and a smooth \((d_1, \cdots, d_{l-1}, d_l - a)\)-complete intersection \( X_2 \);
4. the intersection \( D \) of \( X_1 \) and \( X_2 \) is a smooth \((d_1, \cdots, d_{l-1}, a, d_l - a)\)-complete intersection;
5. the singular locus of the total space is given by the base locus \( \{ F_l = 0 \} \cap D \), and is of codimension one in \( D \).

Let \( \mathcal{X}^\circ \) be the complement of \( \{ F_l = 0 \} \cap D \) in \( \mathcal{X} \). \( \mathcal{X}^\circ \) is the smooth locus of the total space \( \mathcal{X} \) in the usual sense. Consider the canonical divisorial log structure \( \mathcal{M}_{\mathcal{X}^\circ} \) associate to the pair \( (\mathcal{X}^\circ, \partial \mathcal{X}^\circ := \pi^{-1}(0)) \), and the log structure \( \mathcal{M}_{\mathbb{A}^1} \) associated to \( (\mathbb{A}^1, 0) \). Then we have a morphism of log schemes

\[
\pi^\dagger : (\mathcal{X}^\circ, \mathcal{M}_{\mathcal{X}^\circ}) \to (\mathbb{A}^1, \mathcal{M}_{\mathbb{A}^1}).
\]

Write

\[
\pi^\dagger_0 : Y^\dagger \to 0^\dagger
\]

for the fiber over \( 0 \in \mathbb{A}^1 \). The closure of the underlying scheme \( Y \) of \( Y^\dagger \) is given by \( X_1 \cup X_2 \).
Lemma 3.3. The log map $\pi^\dagger$ is a log smooth morphism of fine and saturated log schemes and the central fiber $Y^\dagger$ is a simple degeneration. In particular, the general point of $D$ lies in the log smooth locus of $\pi^\dagger$.

Proof. This follows since the pair $(X^\circ, \partial X^\circ)$ over $\mathbb{A}^1$ is a simple normal crossings degeneration. 

3.2. The gluing construction.

Lemma 3.4. Let $\pi : Y^\dagger \to p^\dagger$ be a simple degeneration with its canonical log structures as in Definition 3.1. Let $f : C \to Y$ be a usual genus zero stable map with $C$ given by two irreducible components $C_1$ and $C_2$ glued along a node $x \in C$. Further assume that $f^{-1}(D) = x$ with the same contact orders $c$ on each components. Then there is a stable log map given by the following diagram:

$$
\begin{array}{ccc}
C^\dagger & \xrightarrow{f} & Y^\dagger \\
p^\dagger & \downarrow{u} & p^\dagger
\end{array}
$$

over the underlying stable map $f$, such that on the level of characteristics $\tilde{u}^\circ : \mathbb{N} \to \mathbb{N}$ is multiplication by $c$.

Proof. This is constructed in [Kim10, Section 5.2.3].

We next consider a log smooth variety $X^\dagger$ given by a smooth variety $X$ and a smooth divisor $D \subset X$. Consider the $\mathbb{P}^1$-bundle $\mathbb{P} := \mathbb{P}(N_{D/X} \oplus \mathcal{O}_D)$ with two disjoint divisors $D_0 \cong D_\infty \cong D$ such that $N_{D_0/\mathbb{P}} \cong N_{D/X}^\vee$ and $N_{D_\infty/\mathbb{P}} \cong N_{D/X}$. Gluing $\mathbb{P}$ and $X$ by identifying $D_0$ with $D$, we obtain a scheme $Y$. By [Kat96, Theorem 11.2], there is simple degeneration with the central fiber $\pi : Y^\dagger \to p^\dagger$ a log smooth simple degeneration as in Lemma 3.4. By [GS13, Proposition 6.1], there is a log map $g : Y^\dagger \to X^\dagger$ contracting the $\mathbb{P}^1$-bundle $\mathbb{P}$ to the divisor $D$.

Lemma 3.5. Consider a genus zero stable map

$$f : C \to D$$

such that $C \cong \mathbb{P}^1$ and $\deg(N_{D/X})|_C = c \geq 0$. Then there is a log map $f^\dagger : C^\dagger \to X^\dagger$ over $f$ with a unique marking $\sigma \in C^\dagger$ of contact order $c$.

Proof. Pick an arbitrary point $\sigma \in C$, and fix an isomorphism

$$(N_{D/X})|_C \cong \mathcal{O}_C(c \cdot \sigma).$$

Choose a section $s \in H^0(\mathcal{O}_C(c \cdot \sigma))$ with a zero of order $c$ at $\sigma$. Thus, the section $s$ defines a map

$$f' : C \to \mathbb{P}$$

which is tangent to $D_\infty$ at $\sigma$ only of contact order $c$, and does not meet $D_0$.

By [Kim10, Section 5.2.3], there is a log map

$$
\begin{array}{c}
C^\dagger \overset{f}{\longrightarrow} Y^\dagger \\
\downarrow \hspace{1cm} \downarrow \\
p^\dagger \overset{}{\longrightarrow} p^\dagger.
\end{array}
$$

Now the composition $f := g \circ f'$ defines the log map we want. ♠

**Lemma 3.6.** Notations as above, consider a genus zero stable map

$$f : C \rightarrow X$$

such that

1. $C$ has two irreducible components $C_1$ and $C_2$ meeting at the node $p$;
2. $f|_{C_1}$ only meets $D$ at the node $x$ of contact order $c_1$;
3. $f(C_2) \subset D$, and $\deg(f^*(N_{D/X}))(c_2) = c_2$.

Assume that $c_1 + c_2 \geq 0$. Then there is a stable log map $f : C^\dagger/p^\dagger \rightarrow X^\dagger$ over $f$ with a single marked point $\sigma \in C_2$ of contact order $c_1 + c_2$.

**Proof.** We define a morphism of sheaves over $C_2$:

$$
\mathcal{O} \oplus \mathcal{O}(-c_2) \rightarrow \mathcal{O}(c_1)
$$

where the arrow $\mathcal{O} \rightarrow \mathcal{O}(c_1)$ is defined by the effective divisor $c \cdot x$, and $\mathcal{O}(-c_2) \rightarrow \mathcal{O}(c_1)$ is defined by the effective divisor $(c_1 + c_2) \cdot \sigma$. This defines a morphism $C_2 \rightarrow \mathbb{P}$ tangent to $D_\infty$ and $D_0$ at $\sigma$ and $x$ with contact orders $c_1 + c_2$ and $c_1$ respectively. We are in the situation of Lemma 3.4. Thus, there is a stable log map $f'$ as in (3.2.1) over the underlying map $f$. The composition of $f := f' \circ g$ yields the stable log map as in the statement. ♠

**Remark 3.7.** In Lemma 3.6, the marking $\sigma$ can be removed if $c_1 + c_2 = 0$.

For the reader’s convenience, we include the following result, which is known to experts:

**Lemma 3.8.** Consider a genus zero log map

$$
\begin{array}{c}
C^\dagger \overset{f}{\longrightarrow} X^\dagger \\
\downarrow \hspace{1cm} \downarrow \\
S^\dagger \overset{}{\longrightarrow} B^\dagger
\end{array}
$$

where the underlying $S$ is a geometric point, $X^\dagger \rightarrow B^\dagger$ is a log smooth family, and the log structure of $B^\dagger$ over the generic point is trivial. Assume $f^*T_{X^\dagger/B^\dagger}$ is semi-positive. Let $f'$ be a general smoothing of $f$. Then

1. the source curve of $f'$ is irreducible;
2. the map $f'$ only meets the locus $\partial X^\dagger$ with non-trivial log structure at the marked points.
**Proof.** Let $\mathcal{R}$ be the moduli space of stable log maps, and $\mathfrak{M}$ be the moduli space of genus zero pre-stable curves with its canonical log structure. Then the semi-positivity of $f^*TX^\dagger/B^\dagger$ implies the morphism of usual algebraic stacks

$$\mathcal{R} \to \mathcal{Log}_{\mathfrak{M} \times B^\dagger}$$

is smooth at the point $[f] \in \mathcal{R}$, see [Che10, Section 2.5]. Here $\mathcal{Log}_{\mathfrak{M} \times B^\dagger}$ is Olsson’s log stack parameterizing log structures over $\mathfrak{M} \times B^\dagger$, see [Ols03a]. By assumption $\mathcal{Log}_{\mathfrak{M} \times B^\dagger}$ contains an open dense sub-stack with trivial log structures. Thus, a general deformation $f'$ satisfies the conditions in the statement, see for example [Che10, Section 3.2].

3.3. From Fano to log Fano via a simple degeneration.

**Proposition 3.9.** Notations as in Lemma 3.3. Consider the two log Fano varieties $X^\dagger_i$ associated to $(X_i, D)$ for $i = 1, 2$ as in Section 3.1. If $X^\dagger_1$ is separably $\mathbb{A}^1$-connected, and $X^\dagger_2$ is separably $\mathbb{A}^1$-uniruled, then general fibers of (3.1.1) are separably rationally connected.

**Proof.** By assumption, we may take a very free $\mathbb{A}^1$ curve $f_1 : C^\dagger_1 \to X^\dagger_1$, and a free $\mathbb{A}^1$ curve $f_2 : C^\dagger_2 \to X^\dagger_2$. Write $\sigma_i$ for the unique marking on $C^\dagger_i$ for $i = 1, 2$. Note that the marking of the free log map sweep out general points on the boundary divisor [KM99, Corollary 5.5(3)]. We may assume that $f_1(\sigma_1) = f_2(\sigma_2)$, and $f_i(C^\dagger_i \setminus \{\sigma_i\}) \cap D_i = \emptyset$ for $i = 1, 2$.

After composing $f_i$ with some generically étale multiple cover by rational curves ramified at $\sigma_i$, we may assume that $f_1$ and $f_2$ have the same contact orders along the common boundary. By Lemma 3.4, we may glue $f_1$ and $f_2$ along the markings, and obtain a stable log map $f : C^\dagger \to X^\dagger$ where the underlying curve $C$ is a rational curve with one node obtained by gluing $C_1$ and $C_2$ along the markings.

Since the pullback of the log tangent bundles $f_1^*T_{X^\dagger_1}$ and $f_2^*T_{X^\dagger_2}$ are at least semi positive, there exists a smoothing $f'$ of $f$ to the general fiber of the one parameter degeneration by Lemma 3.8. Since $f_1^*T_{X^\dagger_1}$ is ample by assumption, a general smoothing $f'$ is very free.

4. Reduction to the Fano boundary

4.1. Separably $\mathbb{A}^1$-uniruledness. The following can be found in [KM99, 5.2]. For completeness, we include the proof here.

**Lemma 4.1.** Let $X^\dagger$ be a log smooth scheme given by a normal crossing pair $(X, \sum_{i=1}^k D_i)$. Then we have an exact sequence

$$0 \to \mathcal{O}_{D_i} \to T_{X^\dagger}|_D \to T_{D^\dagger_i} \to 0,$$

where $D^\dagger_i$ is given by the pair $(D_i, \sum_{j \neq i} D_j|_{D_i})$. 
Proof. Write $Z^\dagger$ to be the log scheme given by $(X, \sum_{j \neq i} D_j)$. Consider the exact sequence over $X$:

$$0 \to \Omega_{Z^\dagger} \to \Omega_{X^\dagger} \to \mathcal{O}_{D_i} \to 0$$

Applying $\otimes \mathcal{O}_{D_i}$ to the above sequence, we have

$$0 \to \text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{D_i}, \mathcal{O}_{D_i}) \to \Omega_{Z^\dagger}|_{D_i} \to \Omega_{X^\dagger}|_{D_i} \to \mathcal{O}_{D_i} \to 0.$$  

Note that $\text{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{D_i}, \mathcal{O}_{D_i}) = N_{D_i/X}^\dagger$. Now the statement follows from taking the dual of the above exact sequence.

Combining Lemma 3.5 and 4.1, we get:

**Lemma 4.2.** Notations as in Lemma 4.1, assume $D_i^\dagger$ is separably $\mathbb{A}^1$-uniruled, and the restriction of the normal bundle $N_D$ to a log free curve in $D_i^\dagger$ is semi positive, then $X^\dagger$ is separably $\mathbb{A}^1$-uniruled.

In particular, when $D$ is smooth, $X^\dagger$ is separably $\mathbb{A}^1$-uniruled if $D$ is separably uniruled and nef.

### 4.2. Separably $\mathbb{A}^1$-connectedness.

The goal of this section is to prove the following:

**Proposition 4.3.** Let $X^\dagger$ be a general log Fano $(d_1, \cdots, d_l; d_b)$-complete intersection given by the pair $(X, D)$ as in Section 1.2. If $D$ is separably rationally connected and $\text{char } k \nmid d_b$, then $X^\dagger$ is separably $\mathbb{A}^1$-connected.

**Proof of Proposition 4.3.** Let $\sigma, \sigma_1$ be two general points on $C_1$. Since $\text{char } k \nmid d_b$, we may choose a log free line $f_2 : C_2^\dagger \to X^\dagger$ constructed in Proposition 2.2 with the unique marking $\sigma_2$ having image $f_1^\prime(\sigma_1)$. By Lemma 3.6, we may glue $f_1$ and $f_2$ by identifying $\sigma_1$ and $\sigma_2$, and obtain a stable log map $f : C^\dagger \to X^\dagger$ with one marking $\sigma$ and one node $p$.

If we restrict (4.1.1) to $C_1$, there are two possibilities:

1. $T_{X^\dagger}|_{C_1}$ is ample.
2. $T_{X^\dagger}|_{C_1}$ is a trivial extension of $T_D|_{C_1}$ by $\mathcal{O}_{C_1}$.

In the first case, a general smoothing $f$ is very free by Lemma 3.8. In the second case, $T_{X^\dagger}|_{C_1}$ is only semi positive.

Consider the composition

$$(4.2.1) T_{C_2^\dagger}|_p \xrightarrow{df_2} T_{X^\dagger}|_p \xrightarrow{\delta} T_D|_p.$$

**Lemma 4.4.** The push-forward morphism $df_2$ is injective when $\text{char } k \nmid d_b$.

**Proof.** It suffices to show that pullback morphism $(df_2)^\vee : \Omega_{X^\dagger}|_p \to \Omega_{C^\dagger}|_p$ is surjective. We check this using a local computation. Locally at $p$, there is a log 1-form $dg/g$ where $g$ is the defining equation of the boundary. Since the image of $C_2$ is a log free line, $(df_2)^\vee(dg/g) = d_b \cdot dt/t \neq 0$.

**Lemma 4.5.** The composite morphism (4.2.1) is a zero morphism.
Proof. Applying Lemma 4.1 to both $C^\dagger_2$ and $X^\dagger$, and restricting to $\sigma$, we have the commutative diagram:

\[
\begin{array}{c}
0 \rightarrow k_{\sigma_2} \xrightarrow{\cong} T_{C^\dagger_2|\sigma_2} \rightarrow 0 \\
0 \rightarrow \mathcal{O}_{D|\sigma_2} \rightarrow T_{X^\dagger|\sigma_2} \rightarrow T_{D|\sigma_2} \rightarrow 0
\end{array}
\]

The statement then follows. ♠

Let $E$ be the codimension one vector subspace in $T_{X^\dagger|p}$ which corresponds to $T_{D|p}$. To make a log very free curve, it suffices to increase the positivity outside $E$. By Proposition 2.2, the splitting type of $T_{X^\dagger|C^2}$ is $O(1)^{\oplus(n+1-e)} \oplus O^{\oplus(e-1-l)}$. Let $E'$ be the canonical subspace of $T_{X^\dagger|p}$ which corresponds to the factor $O(1)^{\oplus(n+1-e)}$. By Lemma 4.4, $E'$ contains the log tangent direction $T_{C^\dagger_2|p}$.

By Lemma 4.5, $E'$ as a vector subspace in $f^*T_{X^\dagger|p}$ is contained in the kernel of $T_{X^\dagger|p} \rightarrow T_{D|p}$. Since $E$ is of codimension one, the two vector subspaces $E'$ and $E$ span $T_{X^\dagger|p}$. Now Proposition 4.3 follows by applying Proposition 4.9 below. ♠

Proof of Proposition 1.7. By adjunction, $D$ is Fano, and hence rationally connected. We may then choose a very free rational curve $f : C \rightarrow D$ through general points of $D$. Now the theorem is proved by gluing a $f$ with a free $A^1$-curve in $(X, D)$, and applying the same argument as for Proposition 4.3. ♠

4.3. A result from elementary transform.

Construction 4.6. Let $C$ be the union of $C_1$ and $C_2$ glued at a node $p$. Let $q : C \rightarrow T$ be a smoothing of $C$ with $C$ the fiber over $0 \in T$. Let $s_1$ and $s_2$ be two sections of $q$ both of which specialize to two distinct points $y_1, y_2$ on $C_1$. Consider a locally free sheaf $\mathcal{E}$ of rank $r$ on $Y$, satisfying the following property:

1. $\mathcal{E}|_{C_1}$ is isomorphic to $\mathcal{O} \oplus \mathcal{F}$, where $\mathcal{F}$ is a positive sub-bundle. Let $E$ be the canonical codimension one subspace of $\mathcal{E}|_p$ which corresponds to $\mathcal{F}$.
2. $\mathcal{E}|_{C_2} \cong \mathcal{T} \oplus \mathcal{O}^{\oplus r-k}$, where $1 \leq k \leq r$ and $\mathcal{T}$ is positive. Let $E'$ be the canonical subspace of $\mathcal{E}|_p$ which corresponds to $\mathcal{T}$.
3. $E'$ and $E$ span $\mathcal{E}|_p$.

Consider the following composition:

\[
\begin{array}{c}
r : \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee|_{C_2} \rightarrow \mathcal{T}^\vee
\end{array}
\]

Clearly $r$ is surjective. Let $K^\vee$ be the kernel of $r$, i.e., the elementary transform of $\mathcal{E}^\vee$ along $\mathcal{T}^\vee$.

\[
\begin{array}{c}
0 \rightarrow K^\vee \rightarrow \mathcal{E}^\vee \xrightarrow{r} \mathcal{T}^\vee \rightarrow 0
\end{array}
\]
Dualizing the above short exact sequence over $\mathcal{C}$, we get a long exact sequence

$$0 \longrightarrow \text{Hom}_\mathcal{O}(T^\vee, \mathcal{O}_\mathcal{C}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{K} \longrightarrow \text{Ext}^1_\mathcal{O}(T^\vee, \mathcal{O}_\mathcal{C}) \longrightarrow 0.$$ 

The first term vanishes because it is the dual of a torsion sheaf. The last term is isomorphic to $\mathcal{T} \otimes \mathcal{O}_{\mathcal{C}^2} \mathcal{O}_{\mathcal{C}^2}(\mathcal{C}^2)$ by [Eis95, A3.46 b] and

$$\text{Ext}^1_\mathcal{O}(\mathcal{O}_{\mathcal{C}^2}, \mathcal{O}_\mathcal{C}) \cong \mathcal{O}_{\mathcal{C}^2}(\mathcal{C}^2).$$

Thus we obtained a short exact sequence

$$(4.3.3) \quad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \otimes \mathcal{O}_{\mathcal{C}^2} \mathcal{O}_{\mathcal{C}^2}(\mathcal{C}^2) \longrightarrow 0.$$ 

Lemma 4.7. $h^1(C_2, \mathcal{K}|_{C_2}(-p)) = 0$.

Proof. Restricting the short exact sequence (4.3.2) to $C_2$, applying the functor $\text{Hom}_\mathcal{O}_{\mathcal{C}^2}(\ast, \mathcal{O}_{\mathcal{C}^2})$, and combining with (4.3.3), we obtain

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{E}|_{C_2} \longrightarrow \mathcal{K}|_{C_2} \longrightarrow \mathcal{T} \otimes \mathcal{O}_{\mathcal{C}^2} \mathcal{O}_{\mathcal{C}^2}(\mathcal{C}^2) \longrightarrow 0.$$ 

The quotient bundle $\mathcal{E}|_{C_1}/\mathcal{T}$ is a trivial vector bundle and the last term of the exact sequence is isomorphic to $\mathcal{T}(-p_i)$. In particular, we have

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}^2}(-p)^{\oplus(r-k)} \longrightarrow \mathcal{K}|_{C_2}(-p) \longrightarrow \mathcal{T}(-2p) \longrightarrow 0.$$ 

The lemma follows from the vanishing of $H^1$ of the first and the third term of the above sequence. ♠

Lemma 4.8. $h^1(C_1, \mathcal{K}|_{C_1}(-y_1 - y_2)) = 0$.

Proof. Restricting the short exact sequence (4.3.2) to $C_1$, we get

$$\mathcal{K}^\vee|_{C_1} \longrightarrow \mathcal{E}^\vee|_{C_1} \longrightarrow \mathcal{T}^\vee|_{C_1} \longrightarrow 0.$$ 

The above sequence is also left exact. Indeed, since $\mathcal{T} \otimes \mathcal{O}_{\mathcal{C}^2} \mathcal{O}_{\mathcal{C}^2}(\mathcal{C}^2)|_{C_1}$ is torsion, by restricting (4.3.3) to $C_1$ and taking the dual over $C_1$, we have the injection from $\mathcal{K}^\vee|_{C_1}$ to $\mathcal{E}^\vee|_{C_1}$.

In other words, the vector bundle $\mathcal{K}|_{C_1}$ is the elementary transform of $\mathcal{E}|_{C_1}$ along $p$ with the specific subspace $E'$. By condition (3) of the construction, $E'$ does not lie in $\mathcal{F}$ at $p$. This implies that $\mathcal{K}$ is ample on $C_1$. The statement follows.

Proposition 4.9. With the same notations and constructions as above, the restriction of $\mathcal{E}$ to a general fiber $\mathcal{C}_t$ is positive.

Proof. By the construction, we know $\mathcal{K}|_{\mathcal{C}_t}$ is isomorphic to $\mathcal{E}|_{\mathcal{C}_t}$. Since $\mathcal{K}$ is locally free on $\mathcal{C}_t$, it is flat over $\mathcal{T}$. By upper semicontinuity, it suffices to show that $h^1(C, \mathcal{K}(\mathcal{C}^2(-y_1 - y_2))) = 0$. This follows from the above two lemmas and the restriction short exact sequence. ♠
References

[AC] Dan Abramovich and Qile Chen, *Stable logarithmic maps to Deligne-Faltings pairs II*, The Asian Journal of Mathematics, accepted.

[ACGM] D. Abramovich, Q. Chen, D. Gillam, and S. Marcus, *The evaluation space of logarithmic stable maps*, arXiv:1012.5416v1, preprint.

[Ang12] Elena Angelini, *The torelli problem for logarithmic bundles of hypersurface arrangements in the projective space*, Ph.D thesis (2012).

[Cam92] F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 539–545. MR 1191735 (93k:14050)

[Che10] Q. Chen, *Stable logarithmic maps to Deligne-Faltings pairs I*, August 2010, Annals of Math, to appear. arXiv:1008.3090.

[CZ13] Qile Chen and Yi Zhu, *On the $\mathbb{A}^1$-connected varieties*, manuscript in preparation, 2013.

[dJS03] A. J. de Jong and J. Starr, *Every rationally connected variety over the function field of a curve has a rational point*, Amer. J. Math. 125 (2003), no. 3, 567–580. MR 1981034 (2004h:14018)

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)

[GS03] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

[GS13] Mark Gross and Bernd Siebert, *Logarithmic Gromov-Witten invariants*, J. Amer. Math. Soc. 26 (2013), no. 2, 451–510. MR 3011419

[HT06] Brendan Hassett and Yuri Tschinkel, *Weak approximation over function fields*, Invent. Math. 163 (2006), no. 1, 171–190. MR 2208420 (2003m:14081)

[HT08a] ________, *Approximation at places of bad reduction for rationally connected varieties*, Pure Appl. Math. Q. 4 (2008), no. 3, Special Issue: In honor of Fedor Bogomolov. Part 2, 743–766. MR 2435843 (2010h:14081)

[HT08b] ________, *Log Fano varieties over function fields of curves*, Invent. Math. 173 (2008), no. 1, 7–21. MR 2403393 (2009c:14080)

[Hu10] Yong Hu, *Weak approximation over function fields of curves over large or finite fields*, Math. Ann. 348 (2010), no. 2, 357–377. MR 2672306 (2011h:14072)

[Kat89] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224. MR 1040550 (91d:14020)

[Kat96] Fumiharu Kato, *Log smooth deformation theory*, Tohoku Math. J. (2) 48 (1996), no. 3, 317–354. MR 1404507 (99a:14012)

[Kim10] Bumsig Kim, *Logarithmic stable maps*, New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008), Adv. Stud. Pure Math., vol. 59, Math. Soc. Japan, Tokyo, 2010, pp. 167–200. MR 2683209 (2011m:14019)

[KM99] Seán Keel and James McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. 140 (1999), no. 669, viii+153. MR 1610249 (99m:14068)

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rationally connected varieties*, J. Algebraic Geom. 1 (1992), no. 3, 429–448. MR 1158625 (93i:14014)

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)

[Kol99] ________, *Rationally connected varieties over local fields*, Ann. of Math. (2) 150 (1999), no. 1, 357–367. MR 1715330 (2000h:14019)
[KS03] János Kollár and Endre Szabó, *Rationally connected varieties over finite fields*, Duke Math. J. **120** (2003), no. 2, 251–267. MR 2019976 (2005h:14090)

[Ols03a] Martin C. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 5, 747–791. MR MR2032986 (2004k:14018)

[Ols03b] , *Universal log structures on semi-stable varieties*, Tohoku Math. J. (2) **55** (2003), no. 3, 397–438. MR MR1993863 (2004f:14025)

[Ols05] , *The logarithmic cotangent complex*, Math. Ann. **333** (2005), no. 4, 859–931. MR MR2195148 (2006j:14017)

[Ols07] , *(Log) twisted curves*, Compos. Math. **143** (2007), no. 2, 476–494. MR MR2309994 (2008d:14021)

[TZ12] Zhiyu Tian and Runpu Zong, *One cycles on rationally connected varieties*, arXiv:1209.4342.

[Zhu11] Yi Zhu, *Fano hypersurfaces in positive characteristic*, arXiv:1111.2964v1 (2011).

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