QCD Radiative Correction to the Hadronic Annihilation Rate of $1^{+-}$ Heavy Quarkonium

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Abstract

Hadronic annihilation rate of $1^{+-}$ heavy quarkonium is given to next-to-leading order in $\alpha_s$ and leading order in $v^2$ using a recently developed factorization formalism which is based on NRQCD. The result includes both the annihilation of P-wave color-singlet $Q\bar{Q}$ component, and the annihilation of S-wave color-octet $Q\bar{Q}$ component of the quarkonium. The notorious infrared divergences due to soft gluons, i.e., the Logarithms associated with the binding energy, encountered in previous perturbative calculations of $1^{+-}$ quarkonium decays are found to be explicitly cancelled, and a finite result for the decay width to order $\alpha_s^3$ is then obtained.

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The study of heavy quarkonium plays an important role in the understanding of quantum chromodynamics (QCD). In recent years, since the E760 collaboration at Fermilab [1, 2] observed charmonium states via resonant $p\bar{p}$-annihilation and measured their masses, widths, and branching fractions with unprecedented precision, problems about decay and production of heavy quarkonium have aroused much interest of people. In particular, with the observation of $1^{+−}$ charmonium state $h_c$ in experiment [1], the study of its properties such as mass and decay width becomes very interesting both theoretically and experimentally. In the earlier treatment of heavy quarkonium annihilation, quarkonium is only taken as a color-singlet bound state of a heavy quark $Q$ and its antiquark $\bar{Q}$. Calculation of decay rate is based on the assumption that the annihilation of $Q$ and $\bar{Q}$ is a short-distance process which, because of asymptotic freedom of QCD, can be computed in perturbation theory, and all nonperturbative effects could be factored into a constant: the wavefunction at the origin or its derivative at the origin. Using this factorization assumption to calculate the hadronic decay width of P-wave quarkonium $1^{+−}$ state, the infrared divergence appears in the limit of small binding energy [3] and this divergence cannot be factored into the derivative of the wavefunction. It is interpreted as a signal that the decay rate is sensitive to the nonperturbative effects which cannot be contained in the wavefunction. This implies that the earlier factorization assumption fails because it is not rigorous within the framework of QCD.

Recently, Bodwin, Braaten and Lepage [4] have developed a factorization formalism that allows systematic calculations of inclusive decay rates and production cross section of heavy quarkonium to any order in QCD coupling constant $\alpha_s$ and to any order in $v^2$, where $v$ is the typical relative velocity of the heavy quark. The factorization formula is based on the use of the effective field theory NRQCD (Nonrelativistic QCD). Using this rigorous factorization formalism, the decay rate can be written as a sum of a set of long-distance nonperturbative matrix elements of which each is multiplied by a short-distance coefficient which can be calculated in perturbative QCD. This approach is successful in the study of many processes about decay and production of heavy quarkonium [5]. In this paper, we apply NRQCD to the hadronic decay of $1^{+−}$ quarkonium and compute the QCD radiative corrections to its annihilation rate from both the color-singlet and color-octet $Q\bar{Q}$ Fock states. We will show the explicit cancellation of the infrared divergence between the color-singlet and color-octet $Q\bar{Q}$ components, and give a complete result at next to leading order in $\alpha_s$ and leading order in $v^2$.

In NRQCD, the effect of annihilation can be taken into account by adding 4-fermion operators to NRQCD Lagrangian:

$$\delta\mathcal{L}_{4-\text{fermion}} = \sum_n \frac{f_n(\alpha_s)}{m_{d_n}^4} \mathcal{O}_n,$$

where the sum is over all possible local 4-fermion operators $\mathcal{O}_n$ that annihilate and create a $Q\bar{Q}$ pair, and
$d_n$ is the scaling dimension of $O_n$. The short distance coefficients $f_n(\alpha_s)$ can be computed by matching perturbative amplitudes for $Q\bar{Q}$ scattering in NRQCD with the corresponding amplitudes in full QCD. The annihilation rate of a quarkonium state $H$ to light hadrons (LH) can be written as

$$\Gamma(H \rightarrow LH) = 2Im < H|\delta\mathcal{L}_{4-\text{fermion}}|H >$$

(2)

At a given order in $v^2$, the number of matrix elements can be reduced to a finite number by using velocity scaling rules for the matrix elements[4, 6]. These scaling rules consist of that for the operators and for the probabilities of the Fock states that give the leading contributions to the matrix elements.

At leading order in $v^2$, the decay width of $1^{+-}$ quarkonium state can be written as

$$\Gamma(1^{+-} \rightarrow LH) = 2Im f_1(1^1P_1)H_1 + 2Im f_8(1^1S_0)H_8 + O(v^2\Gamma),$$

(3)

where two nonperturbative parameters $H_1$ and $H_8$ can be defined rigorously in terms of matrix elements of a color-singlet and a color-octet 4-fermion operator in NRQCD

$$H_1 = \frac{<1^{+-}|O_1(1^1P_1)|1^{+-}>}{m^4},$$

$$H_8 = \frac{<1^{+-}|O_8(1^1S_0)|1^{+-}>}{m^2},$$

where

$$O_1(1^1P_1) = \psi^+(\frac{i}{2})\bar{D})\chi \cdot \chi^+(-\frac{i}{2}\bar{D})\psi,$$

$$O_8(1^1S_0) = \psi^+(T^a\chi \cdot \chi^+T^a\psi,$$

where $D$ is the space part of the covariant derivative $D^\mu$ and $T^a(a = 1, \cdots, N_c^2-1)$ is the $SU(N_c)$ color matrix, and $\psi$ and $\chi^+$ are the fields with two components for quark $Q$ and antiquark $\bar{Q}$ in NRQCD. Here including $H_8$ is due to the fact that for decays of $1^{+-}$ heavy quarkonium, an S-wave color-octet $Q\bar{Q}$ component in the wavefunction will contribute at the same order in $v$ as the P-wave color-singlet $Q\bar{Q}$ component, because the probability for annihilating an S-wave color-octet $Q\bar{Q}$ state is proportional to $v^0$ while this state has a probability at order of $v^2$ to be transmitted into a P-wave color-singlet $Q\bar{Q}$ state through the emission of soft gluon, whereas the dominate Fock state of $1^{+-}$ quarkonium is $|Q\bar{Q}>$ with $Q\bar{Q}$ pair in a color-singlet $1^1P_1$ state, and the probability for annihilation through P-wave color singlet $Q\bar{Q}$ is proportional to $v^2$.

In the following we first calculate the coefficients of non-perturbative matrix elements $H_1$ and $H_8$ to order $\alpha_s^3$ by matching the imaginary part of perturbative scattering amplitude of $Q\bar{Q} \rightarrow Q\bar{Q}$ in full QCD.
with that in NRQCD, and then derive the formula of $1^{+-}$ quarkonium decay width to an accuracy of next-to-leading order in $\alpha_s$. Using phenomenological parameters $H_1$ and $H_8$ determined from other processes, we finally give an approximate numerical estimate of the decay width.

We first calculate the imaginary part of $Q\bar{Q}$ forward scattering amplitude $\text{Im} \mathcal{M}$ in full QCD. For convenience, we consider $Q\bar{Q}$ scattering in the center of momentum frame with the momenta of the heavy quarks and antiquarks small compared to the heavy quark mass. We take the incoming $Q$ and $\bar{Q}$ to have momenta $\vec{p}$ and $-\vec{p}$, while the outgoing $Q$ and $\bar{Q}$ have momenta $\vec{p}'$ and $-\vec{p}'$. By the conservation of energy, we have $|\vec{p}'| = |\vec{p}| = p$. In order to compare with the result in NRQCD, following [4], in the expression of $\text{Im} \mathcal{M}$ to be calculated in full perturbative QCD, we write the 4-component Dirac spinors in the Dirac representation in terms of 2-component Pauli spinors via the substitutions,

$$u(\vec{p}) = \sqrt{E + m} \left( \frac{\xi}{E + m} \right),$$  \hspace{1cm} (4)

$$v(-\vec{p}) = \sqrt{E + m} \left( \frac{-\vec{p} \cdot \vec{\sigma} \eta}{E + m} \right),$$  \hspace{1cm} (5)

where $E = \sqrt{m^2 + p^2}$, $\xi$ and $\eta$ are 2-component spinors with color indices suppressed. The Dirac spinors $u(\vec{p}')$ and $v(-\vec{p}')$ have similar expressions in terms of Pauli spinors $\xi'$ and $\eta'$. The spinors (4) and (5) represent fermion states with standard nonrelativistic normalization.

![Feynman diagrams](image)

(a) \hspace{2cm} (b)

**Fig.1** Feynman diagrams contributing to $C^{full QCD}_{(T^a \otimes T^a)(1 \otimes 1)}$ at order $\alpha_s^2$

It is known from [5] that to leading order in $\alpha_s$, only the coefficient $\text{Im} f_8(1S_0)$ in (5) does not vanish and therefore only the color-octet matrix element $H_8$ contributes to the decay width. In the S-wave case, we expand the annihilation amplitude $\text{Im} \mathcal{M}$ in terms of velocities $\vec{v} = \vec{p}/E$ and $\vec{v}' = \vec{p}/m$ only to leading order, and reduce $\text{Im} \mathcal{M}$ to four terms

$$\text{Im} \mathcal{M}$$

$$= C_{(1\otimes 1)(1\otimes 1)} \xi^+ \eta^+ \xi + C_{(T^a \otimes T^a)(1\otimes 1)} \xi^+ T^a \eta^+ T^a \xi$$

$$+ C_{(1\otimes 1)(\sigma^i \otimes \sigma^j)} \xi^+ \sigma^i \eta^+ \sigma^j \xi + C_{(T^a \otimes T^a)(\sigma^i \otimes \sigma^j)} \xi^+ \sigma T^a \eta^+ \sigma T^a \xi.$$  \hspace{1cm} (6)
In order to determine \((Imf_8(1S_0))_0\), we only consider the coefficient \(C_{(T^a\otimes T^a)(1\otimes 1)}\) of the term \(\xi^{++}T^a\eta'^+T^a\xi\).

At order \(\alpha^2_s\), only two diagrams shown in Fig.1(a) and Fig.1(b) contribute to this coefficient. After decomposing the spinors and expanding them to leading order in \(\vec{v}\) and \(\vec{v}'\), we obtain the coefficient of the term \(\xi^{++}T^a\eta'^+T^a\xi\) in \(Im\mathcal{M}\)

\[
C^{full\ QCD}_{(T^a\otimes T^a)(1\otimes 1)} = \frac{(N_c^2 - 4)g^4}{16N_cm^2}(d - 2)(d - 3)\Phi(2),
\]

where the two massless particle phase space \(\Phi(2)\) in \(d = 4 - 2\epsilon\) dimension is integrated to give

\[
\Phi(2) = \frac{1}{8\pi}\left(\frac{4\pi}{4m^2}\right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(2 - 2\epsilon)}.
\]

A simple calculation leads to the final expression

\[
C^{full\ QCD}_{(T^a\otimes T^a)(1\otimes 1)} = \frac{\pi(N_c^2 - 4)\alpha_s^2}{4N_cm^2}\left(\frac{4\pi\mu^4}{4m^2}\right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(2 - 2\epsilon)}(1 - \epsilon)(1 - 2\epsilon),
\]

where the dimensionless coupling constant is defined as

\[
\alpha_s = \left(\frac{g^2}{4\pi}\right)\mu^{-2\epsilon}.
\]

While in NRQCD the \(\bar{Q}Q\) forward scattering amplitude can be reproduced by 4-fermion operators in the effective lagrangian, and the corresponding term \(\xi^{++}T^a\eta'^+T^a\xi\) in \(Im\mathcal{M}\) comes from operator \(O_8(1S_0)\), of which the coefficient is \(\frac{Imf_8(1S_0)}{m^2}\). By comparing it with (7), we get

\[
(Imf_8(1S_0))_0 = \frac{\pi(N_c^2 - 4)\alpha_s^2}{4N_c}\left(\frac{4\pi\mu^4}{4m^2}\right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(2 - 2\epsilon)}(1 - \epsilon)(1 - 2\epsilon).
\]

We keep \(\epsilon\) in the above expression for convenience of later calculations. In the limit \(\epsilon \to 0\), we get

\[
(Imf_8(1S_0))_0 = \frac{\pi(N_c^2 - 4)}{4N_c}\alpha_s^2,
\]

which has been given in (7). Here the subscript “0” in the coefficient means only the result at leading order in \(\alpha_s\) is taken, and the width can be written as

\[
\Gamma(1^{+-} \to LH) = \frac{\pi(N_c^2 - 4)}{2N_c}\alpha_s^2H_8 + O(\alpha_s \Gamma).
\]

In order to obtain the next-to-leading order result, we must take account of the effects coming from both color-octet component and color-singlet component of the quarkonium. In the following, we calculate
the coefficients \( \text{Im} f_1(1P_1) \) to leading order in \( \alpha_s \) and \( \text{Im} f_8(1S_0) \) to next-to-leading order in \( \alpha_s \), and then give the complete formula for the hadronic decay width of \( 1P_1 \) to order of \( \alpha_s^3 \) at leading order of \( v^2 \).

![Feynman diagrams contributing to \( C_f^{\text{full QCD}} \) at leading order in \( \alpha_s \)](image)

Via the same procedure as above, we consider the imaginary part of \( Q \) scattering amplitude, and calculate the coefficient \( \text{Im} f_1(1P_1) \) by matching a perturbative calculation in full QCD with the corresponding perturbative calculation in NRQCD. The P-wave case requires an expansion of the annihilation amplitude \( \text{Im} \mathcal{M} \) up to the first power of relative momenta \( \vec{p} \) and \( \vec{p}' \). At this order \( \text{Im} \mathcal{M} \) can be written as

\[
\text{Im} \mathcal{M} = C_{\bar{q}q(1\otimes 1)(1\otimes 1)} \vec{v}' \cdot \vec{v} \xi' + \vec{v} \xi' + \vec{v} \xi' + \vec{v} \xi'
\]

\[
+ C_{\bar{q}q(1\otimes 1)(\sigma\otimes \sigma)} \xi' + \vec{v} \cdot \vec{v} \xi' + \vec{v} \cdot \vec{v} \xi' + \vec{v} \cdot \vec{v} \xi'.
\]

The determination of \( \text{Im} f_1(1P_1) \) only requires calculating the coefficient \( C_{\bar{q}q(1\otimes 1)(1\otimes 1)} \) of the term \( \vec{v}' \cdot \vec{v} \xi' \). In full QCD to leading order in \( \alpha_s \), only the diagrams in Fig.2 contribute to this coefficient. After making a nonrelativistic expansion for \( \text{Im} \mathcal{M} \) to first order in \( v \) and \( v' \), we get

\[
C_{\bar{q}q(1\otimes 1)(1\otimes 1)}^{\text{full QCD}} = \int \frac{(N_c^2 - 4)C_F g_s^6}{8N_c^2 m^4} \frac{d-3}{48(d-1)} \left[ \left( -320 + 96d \right) \frac{1}{x_1^3 x_2^3} + \left( 768 - 240d + 4d^2 \right) \left( \frac{1}{x_1^4 x_2^4} + \frac{1}{x_1^4 x_2^3} \right) \right.
\]

\[
+ \left( -512 + 176d - 4d^2 \right) \left( \frac{1}{x_1^3 x_2^2} + \frac{1}{x_1^2 x_2^3} \right) + \left( 64 - 32d \right) \left( \frac{1}{x_1} + \frac{1}{x_2} \right)
\]

\[
+ \left( -1344 + 404d - 2d^2 \right) \frac{1}{x_1^3 x_2} + \left( 624 - 208d + 2d^2 \right) \left( \frac{1}{x_1^4 x_2} + \frac{1}{x_1^3 x_2^2} \right)
\]

\[
+ \left( -64 + 32d \right) \left( \frac{1}{x_1} + \frac{1}{x_2} \right) + \left( -168 + 68d - 3d^2 \right) \left( \frac{1}{x_1} + \frac{1}{x_2} \right)
\]

\[
+ \left( \text{two other permutations} \right) d\Phi(3).
\]
Here $x_i = k_i/m$ ($i = 1, 2, 3$) and $k_i$ denote the energies of the final-state gluons. The massless three-body phase space can be written as

$$d\Phi(3) = \frac{4m^2}{2(4\pi)^3} \left( \frac{4\pi}{4m^2} \right)^{2\epsilon} \frac{1}{\Gamma(2-2\epsilon)} [(1-x_1)(1-x_2)(1-x_3)]^{-\epsilon} dx_1 dx_2.$$  

After performing the integration for two invariant $x_1$ and $x_2$, we obtain

$$C_{\text{full } QCD}^{\vec{v}' \cdot \vec{v} (1 \otimes 1)(1 \otimes 1)} = \left( \frac{N_c^2 - 4}{3N_c^2m^2} \right) \frac{4\pi}{4m^2} (4\pi)^{2\epsilon} \left( \frac{\mu^{2\epsilon}}{\Gamma(2-2\epsilon)} \right) \left( \frac{1}{2\epsilon_{IR}} + \frac{7\pi^2 - 94}{48} \right)$$

$$= \left( \frac{Imf_{s}(1S_0)_{0}}{m^2} \right) \frac{4C_F\alpha_s}{3N_c\pi} \left( \frac{1}{2\epsilon_{IR}} - \gamma_E + \ln \left( \frac{4\pi\mu_{IR}^2}{4m^2} \right) + \frac{7\pi^2 - 118}{48} \right),$$

where $\gamma_E = 0.577$ is the Euler constant. Comparing (14) with the result obtained in ref.[3] which is regularized by the binding energy of $Q\bar{Q}$ pair, we find that if making the substitution $\ln \frac{m^2}{\epsilon_{IR}} \rightarrow -\frac{1}{2\epsilon_{IR}}$, the two results have the same divergent terms, but their finite terms are different due to different regularization schemes. Here $\epsilon$ is the binding energy of $Q\bar{Q}$ pair, which is defined as

$$\frac{\epsilon}{m} = \frac{4m^2 - M^2}{4m^2},$$

where $M$ is the mass of $Q\bar{Q}$ bound state. Here we control the infrared divergence using on-shell dimensional regularization, because off-shell binding energy regularization scheme will break manifest gauge invariance and conventional treatment of NRQCD is exact only for on-shell amplitudes. However, after taking account of the contribution from color-octet $Q\bar{Q}$ component we will find that the coefficient $Imf_{1}(1P_1)$ is infrared finite and the final result is independent of infrared regularization scheme.

**Fig.3** Feynman diagram contributing to $C_{\vec{v}' \cdot \vec{v} (1 \otimes 1)(1 \otimes 1)}^{NRQCD}$ through the operator $O_{1}(1P_1)$
In NRQCD, the $Q\bar{Q}$ forward scattering amplitude can be reproduced by operators in $\delta \mathcal{L}_{4-\text{fermion}}$.

When working at order $\alpha_s^3$, there are two 4-fermion operators which contribute to the coefficient $C_{\vec{v} \cdot \vec{v}' (1 \otimes 1)(1 \otimes 1)}$ of the term $\vec{v} \cdot \vec{v}' \eta^+ \eta^+ \xi$ in $\text{Im} \mathcal{M}$, which are

$$\delta \mathcal{L}_{4-\text{fermion}} = \frac{f_1(1P_1)}{m^2} \mathcal{O}_1(1P_1) + \frac{f_8(1S_0)}{m^2} \mathcal{O}_8(1S_0).$$

The color singlet operator $\mathcal{O}_1(1P_1)$ contributes through the tree diagram in Fig.3 which contains a 4-fermion vertex corresponding to $\mathcal{O}_1(1P_1)$, and the result is

$$\text{Im} \mathcal{M}_{\text{Fig.3}} = \frac{Im f_1(1P_1)}{m^2} \vec{v} \cdot \vec{v}' \eta^+ \eta^+ \xi. \quad (15)$$

Since $\text{Im} f_8(1S_0)$ is already known to be of order $\alpha_s^2$, it is necessary to compute the contribution of the operator $\mathcal{O}_8(1S_0)$ to an accuracy of $\alpha_s$. It is obvious that this contribution only comes from one-loop diagrams in Fig.4(a)–(d) which contain a 4-fermion vertex corresponding to $\mathcal{O}_8(1S_0)$, and these one-loop figures cause the transition from a color octet $Q\bar{Q}$ into a color singlet $Q\bar{Q}$. The overall contribution of diagrams in Fig.4 is

$$\text{Im} \mathcal{M}_{\text{Fig.4}} = \frac{Im f_8(1S_0)}{m^2} \frac{4C_F a_s}{3N_c \pi} \left[ -\frac{1}{2} \left( \frac{1}{\epsilon_{IR}} - \gamma_E + \ln \frac{4\pi \mu_{IR}^2}{4m^2} \right) \vec{v} \cdot \vec{v}' \eta^+ \eta^+ \xi \right] + \frac{1}{4 \left( \frac{1}{\epsilon_{UV}} - \gamma_E + \ln \frac{4\pi \mu_{UV}^2}{4m^2} \right)} \vec{v} \cdot \vec{v}' \eta^+ \eta^+ \xi, \quad (16)$$

where $\frac{1}{\epsilon_{IR}}$ is the IR (infrared) divergence and $\mu_{IR}$ is the corresponding scale, while $\frac{1}{\epsilon_{UV}}$ is the UV (ultraviolet) divergence and $\mu_{UV}$ is the corresponding scale. After the renormalization of operator $\mathcal{O}_8(1S_0)$ in the $\overline{MS}$ scheme the result is free from UV divergence, but the IR divergence still remains and it represents the nonperturbative nature of the annihilation amplitude. To order $\alpha_s^3$, $\text{Im} f_8(1S_0)$ on the right hand side of (16)
must be taken as \((Imf_8(1S_0))_0\), and then we obtain

\[
C_{NRQCD}^{\vec{v} \cdot \vec{v}(1\otimes 1)(1\otimes 1)} = \frac{Imf_1(1P_1)}{m^2} + \frac{(Imf_8(1S_0))_0}{m^2} 4C_F\alpha_s \frac{1}{3N_c \pi} \left[ -\frac{1}{2} \left( \frac{1}{\epsilon_{IR}} - \gamma_E + \ln \frac{4\pi \mu^2}{m^2} \right) + \ln \frac{\mu_{UV}}{2m} \right].
\] (17)

From (14) and (17), we find that the coefficients of IR divergence are the same. It is clear that the IR divergence appearing in (14) is proportional to the probability of transition between a color-singlet \(Q\bar{Q}\) pair and a color-octet \(Q\bar{Q}\) pair by the emission of soft gluon. This is the nonperturbative effect and must be factored into the long-distance matrix elements which have been defined explicitly in NRQCD. Comparing (14) with (17) and using (9), the finite coefficient \(Imf_1(1P_1)\) is found to be

\[
Imf_1(1P_1) = \frac{(N_c^2 - 4)C_F \alpha_s^3}{3N_c^2} \left( \frac{7\pi^2 - 118}{48} - \ln \frac{\mu}{2m} \right).
\] (18)

Obviously the previously encountered IR divergence has been canceled and factored into the nonperturbative matrix element. The operator \(\mathcal{O}_8(1S_0)\) satisfies the evolution equation

\[
\mu \frac{\partial O_8(1S_0)}{\partial \mu} = \alpha_s(\mu) \frac{4C_F}{3\pi N_c m^2} \mathcal{O}_1(1P_1),
\] (19)

which has been derived in [4]. We have neglected the subscript “UV” in \(\mu\) and we will keep this notation in our work.
We have derived the coefficient $\text{Im} f_8(1 S_0)$ to leading order in $\alpha_s$. In order to get the result to next-to-leading order, we must consider the imaginary part of scattering amplitude of $Q \bar{Q}$ pair to order in $\alpha_s^2$ in full QCD. The diagrams which contribute to the coefficient of the term $\xi^{\prime +} T^a \eta^{\prime } \eta^{+} T^a \xi$ in $\text{Im} \mathcal{M}$ to next-to-leading order in $\alpha_s$ are shown in Fig.5. We only give the representative diagrams and neglect the diagrams which give the same result as some of those in Fig.5. The contribution from each diagram in terms of the unrenormalized coupling constant has in general the following form:

$$\text{(Im} f_8(1S_0)\text{)}_0 \alpha_s \frac{\alpha_s}{\pi} f(\epsilon) A("\text{diagram}")$$

(20)
Table 1: $C_{\text{full QCD}}^{(T^a\otimes T^a)(1\otimes 1)}$ from individual diagrams shown in Fig.5

| Diagram  | Contribution from two-particle cut | Contribution from three-particle cut |
|----------|-----------------------------------|-------------------------------------|
| (a)      | $(C_F - \frac{N_c}{2})\left(\frac{1}{\pi} + \frac{1}{\epsilon_{IR}} + 2\ln 2 - 2\right)$ | 0                                   |
| (b)      | $C_F(-\frac{1}{\pi} + 3\ln 2 - 1)$ | 0                                   |
| (c)      | $C_F(-\frac{1}{2\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} - 3\ln 2 - 2)$ | 0                                   |
| (d)      | $(C_F - \frac{C_A}{2})\left(\frac{1}{\pi} - 2\ln 2 + \frac{\pi^2}{4}\right)$ | 0                                   |
| (e)      | $C_A\left[\frac{3}{2} - \frac{1}{2}\left(1 + \frac{1}{\epsilon_{IR}}\right) + 2 - \ln 2 + \frac{\pi^2}{4}\right]$ | $C_A\left[\frac{1}{2} - \frac{1}{2}\left(1 + \frac{1}{\epsilon_{IR}}\right) + 2 - \ln 2 + \frac{\pi^2}{4}\right]$ |
| (f)      | $C_A\left[\frac{N_c}{2\epsilon_{UV}} - \frac{N_c}{\epsilon_{IR}}\right]$ | $C_A\left[\frac{N_c}{2\epsilon_{UV}} + \frac{N_c}{\epsilon_{IR}}\right]$ |
| (g)      | $-\frac{N_f}{2\epsilon_{UV}} + \frac{N_f}{\epsilon_{IR}} \sum_{i} \ln \frac{m_i^2}{m^2}$ | $-\frac{N_f}{2\epsilon_{UV}} + \frac{N_f}{\epsilon_{IR}} \sum_{i} \ln \frac{m_i^2}{m^2}$ |
| (h)      | 0 | 0 |
| (i)      | 0 | 0 |
| (j)      | $C_F\left[-\left(\frac{1}{\pi} + \frac{1}{\epsilon_{IR}}\right) - 2 + 2\ln 2 + \frac{4\pi^2}{3}\right]$ | $C_F\left[-\left(\frac{1}{\pi} + \frac{1}{\epsilon_{IR}}\right) + 9 - \frac{4\pi^2}{3}\right]$ |
| (k) + (l) | 0 | 0 |
| (m) + (n) | 0 | 0 |

The values are normalized to $\frac{(\text{Im}f_{\text{Gluon}}(S_0))\mu}{m^2} f(\epsilon)\frac{\alpha_s}{\pi}$.

Here $\epsilon = (4 - d)/2$, $f(\epsilon) = (\frac{4\pi \mu^2}{4m^2})^\epsilon \Gamma(1 + \epsilon)$, $C_F = \frac{N_c^2 - 1}{2N_c}$, $C_A = N_c$, $N_f$ stands for the number of light flavors.

with

$$f(\epsilon) = (\frac{4\pi \mu^2}{4m^2})^\epsilon \Gamma(1 + \epsilon).$$

The imaginary part of these diagrams receives contributions from two-gluon cut, three-gluon cut and a “light” quark-antiquark pair plus one-gluon cut. The contribution of each individual diagram is calculated in Feynman gauge. Hence we have to add a ghost contribution both to two-gluon cut and to three-gluon cut in the diagram of Fig.5f. Our results for the contributions from individual diagrams are listed in table 1.

Divergences show up in the intermediate steps of the calculation, the dimensional regularization procedure is used by going to $d$ dimensions and introducing a scale $\mu$ through the standard replacement of the bare coupling constant $g \rightarrow g\mu^{(d-4)/2}$. Manifest gauge invariance and massless particle kinematics greatly simplify the calculations. The origin of the $\epsilon = 0$ poles is specified in the table by the subscripts UV and IR. In the table we give the regularized und unrenormalized results for these diagrams, which show a $1/(d - 4)$ divergence and a finite part.

The overall result for the unrenormalized first-order radiative correction to the coefficient $C_{(T^a\otimes T^a)(1\otimes 1)}$ in full QCD can be obtained by summing up all different individual contributions, and reads

\[ C_{\text{full QCD}}^{(T^a\otimes T^a)(1\otimes 1)} \]
\[
(Im f_s(I S_0))_0 = \frac{1}{m^2} \left\{ 1 + \frac{\alpha_s}{\pi} f(\epsilon) \left[ 2b_0 \frac{1}{\epsilon} + (C_F - C_A) \frac{\pi^2}{2} + A \right] \right\},
\]

where

\[
f(\epsilon) = \left( \frac{4\pi\mu^2}{4m^2} \right)^\epsilon \Gamma(1 + \epsilon),
\]

\[
b_0 = \frac{1}{12} (11C_A - 2n_f),
\]

\[
A = C_F \left( \frac{\pi^2}{4} - 5 \right) + C_A \left( \frac{122\pi^2}{9} - \frac{17\pi^2}{24} - \frac{8}{9} n_f \right),
\]

with \( C_F = \frac{N_c^2 - 1}{2N_c}, \ C_A = N_c \). The term with \( \frac{1}{\epsilon} \) indicates the Coulomb singularity which arises from the Coulomb exchange of the gluon between quark and antiquark in Fig.5(a). It is sensitive to the long-distance nonperturbative effect.

We find that the cancellation of the infrared divergences occurs in the overall result in spite of the fact that the individual cuts do not. This is the same as the corresponding color singlet coefficient \( C_{full \ QCD}^{T^a \otimes T^a}(1 \otimes 1) \). As a matter of fact it may be interesting to know that the infrared divergences of Fig.5a,c,k,l cancel each other, which is different from the color singlet coefficient, where the cancellation occurs between Fig.5a and Fig.5c, as well as between Fig.5k, Fig.5l, Fig.5m and Fig.5n respectively \[8, 9\].

Now we renormalize the coupling constant in the \( \overline{MS} \) scheme with

\[
\frac{\alpha_s}{\pi} = \frac{\alpha_s^{\overline{MS}}}{\pi} \left( 1 - \frac{\alpha_s^{\overline{MS}}}{\pi} b_0 \left( \frac{1}{\epsilon} + \ln 4\pi - \gamma_E \right) \right),
\]

and find

\[
C_{full \ QCD}^{T^a \otimes T^a}(1 \otimes 1) = \frac{\pi(N_c^2 - 4)}{4N_c m^2} \alpha_s^2 \left\{ 1 + \frac{\alpha_s}{\pi} \left[ \left( C_F - \frac{C_A}{2} \right) \frac{\pi^2}{2} + 4b_0 \ln \frac{\mu}{2m} + A \right] \right\},
\]

where we have suppressed the superscript \( \overline{MS} \) in \( \alpha_s \).

\[\text{I would like to thank Dr. Petrelli et al. to point out the problem of this term for our old version in their recent paper hep-ph/9707223. Now in this new version it is important to note that the two independent calculations are found to give a identical result.} \]
In order to determine $\text{Im} f_8(1S_0)$, we must calculate the corresponding contribution of $\delta L_{4-\text{fermion}}^{\text{full QCD}}$ to $C_{(T^a\otimes T^a)(1\otimes 1)}^{\text{NRQCD}}$ to next-to-leading order in $\alpha_s$. The relevant Feynman diagrams are shown in Fig.6. They contain a four-fermion vertex that corresponds to the term $\bar{\psi} T_a \chi \chi T_a \psi$ in the effective Lagrangian. In the limit $v \to 0$, only Fig.6(b) and Fig.6(c), which include Coulomb exchange of the gluon, contribute at next-to-leading order. Fig.6(a) gives leading order result

$$\text{Im} M_6(a) = \frac{\text{Im} f_8(1S_0)}{m^2} \xi^{T^a, T^a, \eta^+} \eta^+ T^a \xi. \quad (23)$$

The contribution from Fig.6(b) is

$$\text{Im} M_6(b) = \frac{\text{Im} f_8(1S_0)}{m^2} (C_F - \frac{C_A}{2}) \frac{\pi \alpha_s}{4v} \left[1 - \frac{i}{\pi} \left(\frac{1}{\epsilon_{IR}} - \gamma_E + \ln 4\pi - 2\ln \frac{m v}{\mu}\right)\right] \xi^{T^a, T^a, \eta^+} \eta^+ T^a \xi, \quad (24)$$

where the imaginary part arises because the incoming quark and antiquark can scatter on shell before being annihilated by the 4-fermion operator. The contribution from Fig.6(c) is

$$\text{Im} M_6(c) = \frac{\text{Im} f_8(1S_0)}{m^2} (C_F - \frac{C_A}{2}) \frac{\pi \alpha_s}{4v} \left[1 + \frac{i}{\pi} \left(\frac{1}{\epsilon_{IR}} - \gamma_E + \ln 4\pi - 2\ln \frac{m v}{\mu}\right)\right] \xi^{T^a, T^a, \eta^+} \eta^+ T^a \xi. \quad (25)$$

Add the contributions from Fig.6(a),(b),(c) together, we obtain the complete result for $C_{(T^a\otimes T^a)(1\otimes 1)}^{\text{NRQCD}}$ to order of $\alpha_s^3$ in NRQCD

$$C_{(T^a\otimes T^a)(1\otimes 1)}^{\text{NRQCD}} = \frac{\text{Im} f_8(1S_0)}{m^2} \left[1 + \frac{\alpha_s}{\pi} (C_F - \frac{C_A}{2}) \frac{\pi^2}{2v}\right]. \quad (26)$$

Comparing (22) and (26), we can read off the imaginary part of $f_8(1S_0)$ to next-to-leading order in $\alpha_s$:

$$\text{Im} f_8(1S_0) = \frac{(N_c^2 - 4) \pi \alpha_s^2}{4N_c} \left[1 + \frac{\alpha_s}{\pi} (4b_0\ln \frac{\mu}{2m} + A)\right]. \quad (27)$$
Note that the factorization approach reproduces the standard prescription of simply dropping the $1/v$ terms in the perturbatively calculated annihilation rate. It is clear that the Coulomb singularity can be factored into the nonperturbative part trivially in this factorization formula.

Having derived the coefficients $Im f_1(^1P_1)$ and $Im f_8(^1S_0)$, we finally come to the overall result for hadronic decay of $1^-$ quarkonium state to next-to-leading order in $\alpha_s$ at leading order of $v^2$. Substituting (18) and (27) into (3), we get

$$\Gamma(1^+ \rightarrow LH) = \frac{2(N_c^2 - 4)C_F\alpha_s^3}{3N_c^2} \left( \frac{7\pi^2 - 118}{48} - \ln \left( \frac{\mu}{2m} \right) \right) H_1$$

$$+ \frac{(N_c^2 - 4)\pi\alpha_s^2(\mu)}{2N_c} \left( 1 + \frac{\alpha_s}{\pi} \left( 4b_0\ln \left( \frac{\mu}{2m} \right) + A \right) \right) H_8(\mu).$$

(28)

Working to all orders in $\alpha_s(\mu)$, the final result is independent of $\mu$, since the coefficients depend on $\mu$ in such a way that they will cancel the $\mu$ dependence of matrix elements.

Now we apply our above result to the charmonium system to study the decay width of $h_c$. In (28) making a choice of $\mu = m_c$ and taking $N_c = 3, \; n_f = 3$ we obtain

$$\Gamma(h_c \rightarrow LH) = -0.16\alpha_s^3(m_c)H_1 + 2.62\alpha_s^2(m_c)(1 + 7.10\frac{\alpha_s(m_c)}{\pi})H_8$$

(29)

It is interesting to note that the contribution of color singlet component is negative and the QCD radiative correction from color octet component is very large. The matrix elements $H_1$ and $H_8$ have been defined explicitly in NRQCD and they are difficult to derive from first principles of QCD. People have tried to compute them using lattice simulations [10]. In practice they can be determined phenomenologically. Heavy quark spin symmetry provides approximate relations between them and the corresponding two parameters in the expressions of decay widths of P-wave triplet $\chi_{cJ}(J = 0, 1, 2)$ states. At leading order of $v^2$, they are equal respectively. A rough estimate of $H_1$ and $H_8$ have been given in [12] by comparing the theoretical result of $\chi_{cJ}$ decay to order $\alpha_s^3$ with experimental data. There they don’t give the coefficients of $H_8$ at order $\alpha_s^3$, because in $\chi_{cJ}$ decays, the contributions of color-octet component are the same for $J=0,1,2$, and can be treated as just one parameter. However it is not $H_8$ mentioned above. Here, as an approximation, using the estimated value for $H_1$ and $H_8$ to the order of $\alpha_s^2$ in [1],

$$H_1 = 15.3 \pm 3.7 MeV, \quad H_8 = 3.26 \pm 0.73 MeV,$$

$$\alpha_s(m_c) = 0.25 \pm 0.02,$$

we roughly get $\Gamma = 0.80 \pm 0.20 MeV$. A more reliable estimate will be obtained with a complete theoretical result for the $\chi_{cJ}$ decay width to order $\alpha_s^3$.  

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In this work we use a general factorization formula which is based on NRQCD to calculate the annihilation rate of $1^{+-}$ quarkonium, we see that the infrared divergence appearing in previous calculations can be factored into the long-distance perturbative matrix element rigorously. Our result is also free from the Coulomb singularity. The corresponding case of $\chi_{cJ}$ production through gluon fragmentation has been studied in [11]. It is clear from our calculation that the failure of previous factorization assumption is due to the fact that only color singlet component was considered and all contributions from color octet were neglected. In that sense the previous result is incomplete and therefore the infrared divergence may appear in some cases such as the annihilation and production of P-wave quarkonium even at leading order in $v^2$. Our calculation shows that the rigorous factorization formula can separate short-distance perturbative effects from long-distance nonperturbative effects correctly and can therefore provide a systematical calculation for quarkonium decay and production to any order in $\alpha_s$ and in $v^2$, because it is based on a solid theoretical foundation.
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