A Mathematical Model of the Strain and Stress Kinetics during Welding of Thin-Walled Products

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Abstract. The paper dwells on the mathematical model of the strain and stress of the elements of the thin-walled systems. A version of the sophisticated theory of shells with the use of several base surfaces has been developed at the Kutaisi Technical University [3,8]. The theory is based on a kinematic hypothesis thereby facilitating the construction of a three-dimensional field of deformation of shell by deformation of two or more surfaces. The use of several base surfaces allows not only for accounting the transverse shears and crimping, but also, with account for the shell thickness, for modeling the mechanical and thermal phenomena on the front surfaces of the layers. In doing so, the geometrical and mechanistic interpretation of generalized displacements and generalized internal forces is clear enough, and the basic equations are simple. The model is based on a geometrically linear version of the theory of shells with the use of several base surfaces and the theory of non-isothermal plastic flow [4]. The developed mathematical model of the strain and stress kinetics allows for evaluating the temperature and strain-stress states of thin-walled products during welding.

1 Introduction

Theoretical studies of welding deformations and stresses give rise to considerable difficulties due to a mixed picture of the interrelated physimechanical and chemical phenomena occurring in the material during welding [1, 2]. It is necessary to mention that the welding production is distinguished by the variety of both the products themselves (materials, geometrical dimensions, shapes, etc.) and external effects (methods for the supply and capacity of energy being supplied, methods of spatial registration of products, etc.). In such conditions, one cannot use the simplified approaches because of inadequacies of the appropriate mathematical model.

The mechanical properties of weldable thin-walled structures are studied theoretically by a theory of the elastoplastic plates and shells. However, an analysis of the published works shows that the elastoplastic shells are little-investigated from the perspectives of the sophisticated theories, while the results of a number of works highlight the importance of including the transverse shears and crimping in plasticity problems. A version of the sophisticated theory of shells with the use of several base surfaces has been developed at the Kutaisi Technical University [3]. The theory is based on a kinematic hypothesis thereby facilitating the construction of a three-dimensional field of deformation of shell by deformation of two or more surfaces. The use of several base surfaces allows not only for accounting the transverse shears and crimping, but also, with account for the shell thickness, for modeling the mechanical and thermal phenomena on the front surfaces of the layers. In doing so, the geometrical and mechanistic interpretation of generalized displacements and generalized internal forces is clear enough, and the basic equations are simple.

2 Methodology

The model is based on a geometrically linear version of the theory of shells with the use of several base surfaces and the theory of non-isothermal plastic flow [4]. The geometrical linearity denotes similarity of the metrics of the deformed and undeformed states of shell, and the possibility of using the linear tensor of deformations [5]:

$$
\varepsilon_{ij} = \frac{1}{2} \left( \nabla_i U_j + \nabla_j U_i \right)
$$

where $U_{j}$—the components of the displacement vector; $\nabla_j$—operator of covariant differentiation.

Since the kinematic hypothesis uses not one, but two base surfaces, there have been introduced specific parameters of the shell space associated with both surfaces (Fig. 1):

$$
\tilde{R}(\chi^1, \chi^2, \chi^3) = \tilde{R}(\chi^1, \chi^2) + \chi^3 \left( \tilde{R}(\chi^1, \chi^2) - \tilde{R}(\chi^1, \chi^2) \right)
$$

The vector ratios
set the parameters of the shell space base surfaces. Let us
call conditionally the first surface $S_1$ the lower base
surface, but the second one $S_2$ – the upper surface. The vector
\[
\vec{H}(X^1, X^2) = \vec{R}(X^1, X^2) - \vec{R}(X^1, X^2) 
\]  
(4)

Which is perpendicular to the surface $S$, assigns
the points of the base surfaces with the same Gaussian
coordinates $(X^1, X^2)$.

\textbf{Figure 1.} The parameters of the shell space with two base
surfaces.

There have been accepted the following designations: the Latin indexes 1, 2, 3, but the Greek ones – 1, 2; the
magnitudes with a negative sign (-) are set on the lower
base surface, and the magnitudes with a positive sign (+) –
on the upper base surface; the index with a negative
sign (-) means that tensor component by this index is
referred to a basis vector on the lower base surface, and
with a positive sign (+) – on the upper base surface.

The main ratios of the theory of non-isothermal
plastic flow, linking the increments of the components of
deformation tensor $\delta \sigma^{ij}$ with the increments of the
components of deformation tensor $\delta \epsilon_{km}$ can be written as the
defining relationships for an anisotropic elastic body
[4]:
\[
\delta \sigma^{ij} = \Lambda^{ijk m} (\delta \epsilon_{km} - \alpha_{km} \delta t T),
\]  
(5)
where, $\Lambda^{ijk m}$ – the components of tensor of the
elastoplastic properties depending on the stress state,
temperature and the achieved degree of hardening in a
given point of the shell; $\alpha_{km}$ – the components of tensor
of the temperature linear expansion coefficients.

The formula (5) necessitates representation of the
equations of the theory of shells with the use of several
base surfaces in the increments as well. According to the
main kinematic hypothesis, the points of the shell receive
the additional displacements:
\[
\delta \vec{U}(X^1, X^2, X^3) = (1 - X^3) \delta \vec{U}(X^1, X^2) + X^3 \delta \bar{U}(X^1, X^2), 
\]  
(6)
where $\delta \bar{U}(X^1, X^2)$, $\delta \bar{U}(X^1, X^2)$ – the increments of the
displacements of the appropriate pints of the base
surfaces – the increments of the generalized displacements
of the mechanical systems.

The temperature increments in the shell points are
approximated similarly:
\[
\delta T(X^1, X^2, X^3) = (1 - X^3) \delta T(X^1, X^2) + X^3 \bar{T}(X^1, X^2) 
\]  
(7)
where $\delta T(X^1, X^2)$, $\bar{T}(X^1, X^2)$ set the distributions of the
temperature increments on the lower and upper base
surfaces, accordingly.

The basic variational equation of the theory in the
increments derived from the principle of possible
displacements, has the form as follows:
\[
\int \left\{ \left[ (\delta \bar{T})^{(-)} \beta_i N_{\beta}^{(+) -} - (\delta T)^{(-)} \right]^{(-)} \delta U_i \right\} d \Gamma 
\]  
\[
+ \int \left\{ (\delta \bar{T})^{(+)} \beta_i N_{\beta}^{(-) (+)} - (\delta T)^{(+) (+)} \right\}^{(+) (+)} \delta U_i \right\} d \Gamma 
\]  
\[ 
- \int \delta U_i \left[ \left\{ \frac{\partial \bar{S}}{\partial \beta_i} (\delta \bar{T})^{(-)} + \delta \bar{U}_i^{(-)} + \delta \bar{U}_i^{(-)} \right\} + \delta U_i^{(+) (+)} \right\}^{(+) (+)} d S = 0. 
\]  
(8)
where $\bar{S}$ – the increments of the generalized
internal forces, corresponding with the increments of
stresses acting on the transverse shell areas; $N$ – the
outer normal unit vector to a border $\Gamma$ placed on the
tangential plane to a surface $S$ and equal to $N_{\beta}^{(-)} \bar{R}$;
$\delta T, \bar{T}$ – the increments of the generalized forces,
conditioned by the increments of the external surface
forces acting on the transverse boundary surfaces; $\Gamma$ –
the border of the internal base surface; $\delta \bar{U}_i^{(-)}$ and $\delta \bar{U}_i^{(+) (+)}$ – the
increments of the internal generalized forces,
corresponding with the increments of the external
volumetric and external surface forces applied to the
appropriate surfaces.

It follows from the equation (8) that, by reason of
the arbitrariness of $\delta U_1$ and $\delta U_i$ for the internal points
of the base surface $S$:
\[
\delta \bar{S}_i^{(-)} = \frac{\partial \bar{S}}{\partial \beta_i} (\delta \bar{T})^{(-)} + \delta \bar{U}_i^{(-)} + \delta \bar{U}_i^{(-)} = 0
\]  
\[ 
\delta \bar{S}_i^{(-)} = \frac{\partial \bar{S}}{\partial \beta_i} (\delta \bar{T})^{(-)} + \delta \bar{U}_i^{(-)} + \delta \bar{U}_i^{(-)} = 0. 
\]  
(9)
Based on the formulas (1) and (5), for the
increments of the internal generalized forces, we can
obtain the decompositions:
\[
\delta T^{(+) (-)} = \bar{T} \beta m \delta \bar{U}_m^{(-)} + \bar{T} \beta m \delta \bar{U}_m^{(-)} 
\]  
\[ + \delta \bar{U}_m^{(-)} + \delta \bar{U}_m^{(-)} + \delta \bar{U}_m^{(-)} 
\]  
\[ + \bar{T} \beta m \delta \bar{U}_m^{(-)} - B \bar{B} \beta m \bar{T} = B \beta m \bar{T} ;
\]  
\[ \]
\[
\delta \dot{t} \beta^S = T \beta^S \delta m \nabla \delta U_m + \nabla \beta^S \delta m \nabla \delta U_m + T \beta^S \delta m \delta U_m,
\]
\[
\delta \dot{\gamma} = \Theta \beta^S \delta m \nabla \delta U_m + \Theta \delta m \nabla \delta U_m + \Theta \beta^S \delta m \delta U_m,
\]
\[
\delta \delta^S = \Theta \beta^S \delta m \nabla \delta U_m + \Theta \delta m \nabla \delta U_m + \Theta \beta^S \delta m \delta U_m,
\]
whose coefficients:
\[
\begin{align*}
\frac{\delta}{\delta \beta^S} &= \frac{1}{2} \left[ A^1_{ij} \delta^m + A^1_{ij} \delta^m \right] \nabla \beta^S \delta m \\
+ &X^3 \Gamma^m_{ij} \beta^S \delta m \nabla \delta U_m,
\end{align*}
\]
\[
\frac{\delta}{\delta \beta^S} = \frac{1}{2} \left[ A^1_{ij} \delta^m + A^1_{ij} \delta^m \right] \nabla \beta^S \delta m \\
+ &X^3 \Gamma^m_{ij} \beta^S \delta m \nabla \delta U_m,
\]
where,
\[
\nabla = \nabla \beta^S \delta m = \nabla \beta^S \delta m \nabla \delta U_m 
\]

The extracted equations are complex, and the only way to solve them is to develop the numerical algorithms, particularly on the basis of the finite elements method [6].

Let us decompose the domain of integration \( \int \) into the finite elements and consider one finite element \( S_e \). Let us use the displacement fields of the points of the base surfaces on the element [6]:
\[
\delta U_i = \delta U_i^k \Psi_k(x_1, x_2) \quad \delta U_i = \delta U_i^k \Psi_k(x_1, x_2),
\]
where \( \Psi_k(x_1, x_2) \) – the local interpolating (basis) functions; \( N \) is equal to 1, 2, ..., \( N_e \) (\( N_e \) – the number of the double points in the finite element); \( \xi^m \) – the arbitrary curvilinear coordinate system.

The covariant derivatives are approximated:
\[
\nabla \delta U_i = \frac{\partial \delta U_i}{\partial x^k} = \delta U_k \frac{\partial \Psi_i}{\partial x^k},
\]

The variational equation for the finite element has the form as follows
\[
\int_{S_e} \left[ \left( \frac{\partial \Psi_i}{\partial x^k} \right) \delta U_k + \Psi_i \delta \Psi_k \right] \delta U_i dx = 0,
\]

(14)
\[ P_N = \left( 2 Q^M + Q^H \right) \int \frac{h}{\delta} \Psi_N \Psi_M dS - q \int q \Psi_N \Psi_M dS \]

\[ + T_H \int \alpha \Psi_N \Psi_M dS \]

\[ + \frac{1}{2} \left( S_{N(i-1)} + S_{N,i} \right) \]

By this means, in order to find the node temperatures \( T^M \), \( T^H \), at the end of the i-th time step, it is necessary to solve the system of linear algebraic equations (19).

The developed mathematical model was used when studying the temperature and the strain-stress states of the plate during welding [9]. The plate consisting of steel 12X18H10T with the size of 0,4 x 0,4 m and the thickness of 0,01 m, was smelting in the middle, by nonconsumable tungsten electrode. The arc speed \( v = 0,0028 \) mps, heat effective capacity - \( q = 3,670 \) kW.

### 3 Conclusions

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Using the theory of shells having several base surfaces, there has been obtained the system of differential equations describing the elastoplastic and variational temperature problems. In addition, using the method of finite elements, there has been developed the method for solving these problems. Also, the numerical study of the obtained mathematical model has been carried out.

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