Control Theoretical Approach to Quantum Control

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We derive the quantum stochastic master equation for bosonic systems without measurement theory but control theory. It is shown that the quantum effect of the measurement can be represented as the correlation between dynamical and measurement noise. The transfer function representation allows us to analyze a dynamical uncertainty relation which imposes strong constraints on the dynamics of the linear quantum systems. In particular, quantum systems preserving the minimum uncertainty are uniquely determined. For large spin systems, it is shown that local dynamics are equivalent to bosonic systems. Considering global behavior, we find quantum effects to which there is no classical counterparts. A control problem of producing maximal entanglement is discussed as the stabilization of a filtering process.

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I. INTRODUCTION

The quantum stochastic master equation has been derived by analyzing a measurement process carefully \cite{1,2} in combination with the notion of positive operator valued measure \cite{3}. Since it is based on the model that the measurement is made on the quantum system indirectly via an environment, the stochastic master equation is deeply involved in the input-output formulation \cite{4,5}, which was introduced as a response to the physical necessity for the formulation of quantum system interacting the traveling field. This formulation is compatible with control theory and extensively analyzed using the transfer function representation \cite{6}. Due to the control theoretical analysis, the full quantum treatment of feedback system and controller design was developed for quantum noise reduction problems \cite{7}.

The quantum stochastic master equation for bosonic fields has a decent property that the gaussianity of a density matrix is preserved in time. Thus, the full dynamics of the density matrix can be described by up to the second moments \cite{6,8,9}, and the covariance matrix of the canonical observables obeys the matrix analog of second order equation, the Riccati differential equation, which is widely known in control theory such as linear quadratic Gaussian control, estimation, $H_2$ and $H_\infty$ control \cite{10,11}. These formulations of the quantum stochastic master equation is, however, based on the Hamiltonian formulation and incorporated with physical treatments, and therefore sometimes misled engineers about the back-action of the quantum measurement.

In this paper, we derive the stochastic master equation from a general description of linear quantum systems via the transfer function representation \cite{6}. In this treatment, we do not assume any underlying physics and Hamiltonians. The only assumption imposed here is the noncommutativity of the input and the output signals. Although all parameters of the linear quantum system are left undetermined, the noncommutativity leads to strong constraints on the transfer function and allows us to obtain the equivalent model to that of physics. These constraints are characterized by the fact that the poles and zeros of the transfer function are distributed in the complex plane subject to a certain rule. This includes significant information when one designs a quantum system because the zero and pole of the transfer function completely determine the statistical properties of the output. Then, the stochastic master equation can be derived as a conditional density of the linear system with the constraints. The quantum mechanical constraints can also be characterized by the detectability condition via the Hamiltonian matrix associated with the algebraic Riccati equation.

These ideas can be extended to spin systems using the quasiprobability distribution function on the sphere \cite{13}. At first we will show the fermionic analog of the relation between the superoperators and the differential operators by means of the so-called star product. This relation allows us to establish the same formalism as the bosonic case for the local behavior of a large spin. The global behavior of the large spin is considered for quantum non-demolition (QND) measurement. This formulation of the spin system shows that the leading expansion with respect to the spin number has the corresponding classical system. Dealing with higher order expansion, we will see a quantum effect of measurement in the back-action process to which there are no classical counterparts.

For experimental and practical reasons, spin systems are thought of as an important basis for quantum information processing, and controlling spin systems is a indispensable technology for it. In particular, spin entanglement is an important subject of quantum control and a lot of experiments have been performed using QND measurement in recent years \cite{14,15,16}. This problem can be reduced to the stabilization of the spin system with a stochastic noise. We will consider the production of the maximal entanglement with the use of feedback based on the global description of the large spin system.
This paper starts with Sec. II which is a brief review of classical linear system theory for introducing control theoretical notions to quantum systems. In Sec. III and XIV, filtering theory is overviewed. Sec. V introduce the general formulation of linear quantum systems, and the uncertainty relation is stated in terms of the transfer function in Sec. VI. We formulate measurement on linear quantum systems in Sec. VII through which the uncertainty relation is stated again in terms of detectability in Sec. IX. The mathematical basis for the spin system is developed in Sec. XII. The local behavior and the global behavior of the spin system are considered in Sec. XIII and XIV, respectively. The production of the spin entanglement based on the global behavior is discussed in Sec. XV.

II. LINEAR SYSTEM THEORY

A linear system is described in the time domain as

\[ \dot{x}(t) = Ax(t) + Bu(t), \]
\[ m(t) = Cx(t) + Du(t), \]

where vectors \( u, m \) and \( x \) represent the input, the output, and the state of the system, respectively. The linearity of the system allows us to express it in the frequency domain as

\[ m(s) = G(s)u(s), \]

where \( u(s), m(s) \) are the Laplace transforms of the input and the output. A function \( G(s) \) relating \( u(s) \) to \( m(s) \) is called a transfer function, defined as

\[ G(s) = C(sI - A)^{-1}B + D. \]

Each element of the transfer function is referred to as \( A, B, C \) and \( D \) matrices, respectively.

The transfer function representation of the linear system shows that the state \( x \) is a hidden information carrier from the input to the output, and the choice of \( x \) is not important in the relation between them. The input-output relation is invariant under the similarity transformation of \( x \), i.e., for any nonsingular matrix \( T \),

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \]

For the same reason, the stability is characterized by an invariant quantity under the similarity transformation. The system is said to be stable if \( \text{Re}\lambda(A) < 0 \). \( \lambda(A) \) denotes the eigenvalues of \( A \), which are called the poles of \( G \).

\( B \) matrix determines the relationship between the state of the system and the input, and thereby involves in controllability of the system. The pair \( (A, B) \) is said to be controllable if, for any set of complex numbers \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \), there exist a matrix \( K \) such that \( \lambda(A + BK) = \Lambda \). This definition indicates that the poles of \( G \) are assignable anywhere in the complex plane \( \mathbb{C} \) by means of state feedback \( u = Kx \). The pair \( (A, B) \) is said to be stabilizable if there exist a matrix \( F \) such that \( A + BF \) is stable. This is the case if uncontrollable modes are stable. Likewise, \( C \) matrix determines which element of the state is visible for an observer. The pair \( (C, A) \) is said to be observable if \( (A^T, C^T) \) is controllable, and detectable if there exists \( L \) such that \( A + LC \) is stable.

In case of single-input-single-output (SISO) systems, due to \( D \) matrix, there exists an \( s \) on the complex plane for which \( G = 0 \). Such a point is called the zero of \( G \) and plays an important role in a wide class of control problems as a “norm” of the system is determined by both of the pole and zero of \( G \). For example, if the input is a noise and the output is a quantity to be not affected by the noise, then one would design the system to have a small norm by appropriately distributing the poles and zeros on \( C \).

III. FILTERING

The notion of observability is concerned with the determinacy of the initial state \( x(0) \) from the output segment \( \{m(s)|0 \leq s < t\} \) of arbitrary length for a deterministic measurement. For a noisy measurement, a filtering process is required to extract information about the system subject to a certain criterion.

A. uncorrelated noises

Let us consider a nonlinear system with two independent noises represented as

\[ dx = a(x)dt + b(x)dw \]
\[ dm = c(x)dt + Ddv \]

where \( w, v \) are independent normalized Wiener processes, and we have assumed that \( D \) is a constant matrix. The first equation describes the dynamics of \( x \) driven by the stochastic signal \( v \), and the second one is the noisy measurement process. This system has the joint density \( p(x, m) \) of two random variables \( x, m \), and only \( m \) is directly visible. A filtering problem is to find an optimal estimate of \( x, \hat{x} \), from the measurement outcome \( m \). If we define the optimality by minimization of the mean square error \( P = E[(x - \hat{x})(x - \hat{x})^T] \), then the estimate is given by a conditional expectation. For an arbitrary function \( \phi(x) \), the conditional expectation

\[ \pi_t(\phi) := E[\phi|m(s)\ (0 \leq s \leq t)] \]

satisfies

\[ d\pi_t(\phi) = \pi_t(\mathcal{L}\phi)dt + [\pi_t(\phi c^T) - \pi_t(\phi)\pi_t(c^T)](DD^T)^{-1}d\mathcal{W} \]
where
\[ L = \sum_i a_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{ij} b_{ij} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \]
and \( d\mathbf{\nu} := dm - \pi_i(c) dt \), which is called the innovation process, is another Wiener process. Note that if \( D D^T \) is singular, the inverse can be replaced by the pseudoinverse.

From the conditional expectation (14), it can be easily seen that the conditional density
\[ p(x, t) = \Pr[x(t)|m(s) \ (0 \leq s \leq t)] \]
satisfies
\[ dp = L^* p dt + (c - \tilde{c})^T (D D^T)^{-1} p \ d\mathbf{\nu}, \]
where \( L^* \) is the adjoint operator of \( L \), or the Fokker-Planck operator.

In the linear case, \( a(x) = A x \), \( b(x) = B \), \( c(x) = C x \), the mean \( \hat{x} \) and covariance matrix \( P \) with respect to the conditional density \( p \) are given by
\[ \dot{x} = A \hat{x} + B \tilde{c} \ dt + C \dot{\mathbf{\nu}}, \]
\[ \dot{P} = A P + P A^T + B B^T - C (D D^T)^{-1} C^T \]
where \( \dot{\mathbf{\nu}} = dm - C \dot{x} dt \) is the innovation process. This is known as the Kalman filter. The second equation is referred to as the Riccati differential equation.

### B. correlated noises

Let us consider a linear system with two normalized Wiener processes that are correlated as
\[ \langle d\mathbf{v}(s) d\mathbf{v}^T(t) \rangle = S dt \delta(t - s). \]  
In this case, the conditional expectation is given by the Kalman filter
\[ \dot{x} = A \hat{x} + (P C^T + B S) (D D^T)^{-1} \ d\mathbf{\nu}, \]
\[ \dot{P} = A P + P A^T + B B^T - (P C^T + B S) (D D^T)^{-1} (P C^T + B S)^T. \]
As in the uncorrelated case, the conditional density is derived as the dual expression of the Kalman filter, given by
\[ dp = L^* p dt + \left[ C(x - \hat{x}) - \nabla BS \right] (D D^T)^{-1} p \ d\mathbf{\nu}. \]

Note that the correlation of the two noises appears as a derivative of first order in the coefficient of the innovation process.

### IV. HAMILTONIAN MATRIX

It can be shown that under the measurement for an infinitely long time, the Kalman filter is in a stationary state if the stability is assured. (The convergence also holds even if the system is not stable, provided certain conditions are imposed.) Then, the Riccati differential equation for the covariance matrix is reduced to the algebraic Riccati equation, which is deeply related to the Hamiltonian matrix.

A matrix \( H \) is said to be Hamiltonian if
\[ \Sigma H + H^T \Sigma = 0, \]
where
\[ \Sigma = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \]
From this definition, it can be easily seen that the Hamiltonian matrix is of the form
\[ H = \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix}. \]
where \( W, Q \) are symmetric matrices. Corresponding to this matrix, let us define an algebraic Riccati equation as
\[ X A + X A^T + X R X - Q = 0. \]

Let us denote by \( \text{Ric}[H] \) a solution \( X \) satisfying \( \Re \lambda(A + RX) < 0 \).

From Eq. (15), it follows that
\[ \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} A + RX & R \\ 0 & -(A + RX)^T \end{bmatrix}, \]
which implies that \( \lambda(H) = \lambda(A + RX) \cup \lambda(-A - RX) \).

Using Eq. (15) again, we have
\[ H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (A + RX). \]
It turns out that \( \text{Ric}[H] \) is given by \( X \) satisfying Eq. (16) with \( \Re \lambda(A + RX) < 0 \). For \( H \) which has eigenvalues on the imaginary axis, it is basically possible to obtain a solution of the Riccati equation due to the same procedure. This is the case if a quantum system is under QND measurement, as will be seen later.

For an algebraic Riccati equation of the form
\[ A X + X A^T + B B^T - X C^T C P = 0, \]
the corresponding Hamiltonian matrix is written as
\[ H = \begin{bmatrix} A^T & -C^T C \\ -B B^T & -A \end{bmatrix}. \]
In this case, there exists \( \text{Ric}[H] \) iff \( (C, A) \) is detectable and \( (A, B) \) has no uncontrollable modes on the imaginary axis. Moreover, if these conditions hold, \( \text{Ric}[H] \geq 0 \).
V. LINEAR QUANTUM SYSTEM

In this section, we will introduce a basic formulation of linear quantum systems. Let \( u \) be a noncommutative variable such that
\[
[u(t), u^\dagger(t')] = \delta(t - t').
\]
The signal can be decomposed into the real and imaginary parts as
\[
u_x = u + u^\dagger, \quad u_y = -i(u - u^\dagger).
\]
Physically, these operators correspond to the amplitude of two phases of a signal constituted of the infinite number of independent bosonic modes in free space.

Let us consider a quantum system which converts the signal \( u \) into another quantum signal \( m(t) \) in the same space according to a completely positive map \( \Gamma \) given by \( m(t) = \Gamma(u(t)) = u(t) \ast g(t) \),
\[
(17)
\]
where \( g \) is a function determined by \( \Gamma \), and \( \ast \) represents the convolution of the operator \( u \) and the function \( g \). \( u, m \) can be thought of as the input and the output of the quantum system. In the linear case, an adequate rotation in the output complex amplitude plane can choose the form of the output operator \( m \) such that \( m_x, m_y \) respond to \( u_x, u_y \), respectively. Then, the two phases can be decoupled as
\[
\left[
\begin{array}{c}
m_x \\
m_y
\end{array}
\right] = G \left[
\begin{array}{c}
u_x \\
v_y
\end{array}
\right] = \left[
\begin{array}{cc}
x & 0 \\
0 & y
\end{array}
\right] \left[
\begin{array}{c}
u_x \\
v_y
\end{array}
\right],
\]
(18)

where \( G_x, G_y \) are the transfer functions for each phase given by
\[
G_x = \left[
\begin{array}{cc}
A_x & B_x \\
C_x & D_x
\end{array}
\right], \quad G_y = \left[
\begin{array}{cc}
A_y & B_y \\
C_y & D_y
\end{array}
\right].
\]
(19)

In the time domain, this system can be represented as
\[
\frac{dx}{dy} = \left[
\begin{array}{cc}
A_x & 0 \\
0 & A_y
\end{array}
\right] \left[
\begin{array}{c}
x \\
y
\end{array}
\right] dt + \left[
\begin{array}{cc}
B_x & 0 \\
0 & B_y
\end{array}
\right] \left[
\begin{array}{c}
u_x \\
u_y
\end{array}
\right],
\]
(20a)
\[
\frac{dm_x}{dm_y} = \left[
\begin{array}{cc}
x & 0 \\
0 & C_y
\end{array}
\right] \left[
\begin{array}{c}
x \\
y
\end{array}
\right] dt + \left[
\begin{array}{cc}
D_x & 0 \\
0 & D_y
\end{array}
\right] \left[
\begin{array}{c}
u_x \\
u_y
\end{array}
\right],
\]
(20b)

where \( x, y \) are the internal observables of the system. They are noncommutative in general, however the explicit relation between them is not necessary. Thus, although we denote them by \( x, y \), it does not necessarily mean that \( x, y \) correspond to position and momentum.

The advantage of using transfer functions is that we can simplify the expression of complex networks. For example, the cascade connection of two quantum systems represented by \( G_1, G_2 \), which constitutes the simplest quantum network, is described by the product of the two transfer functions
\[
G_1G_2 = \left[
\begin{array}{cccc}
A_1 & B_1C_2 & B_1D_2 \\
0 & A_2 & B_2 \\
C_1 & D_1C_2 & D_1D_2
\end{array}
\right].
\]
(22)

For any type of connections of systems, we can obtain a simple expression using transfer functions.

VI. CORRELATION AND NOISE REDUCTION

Let us consider a quantum system in a density matrix \( \rho \) on a Hilbert space \( \mathcal{H} \). To evaluate the correlation of noncommutative observables of the system with respect to \( \rho \), we introduce a pre-inner product of operators \( Q, R \) on \( \mathcal{H} \)
\[
\langle Q, R \rangle_\rho = \frac{1}{2} \text{Tr} \left[ \rho (QR^\dagger + R^\dagger Q) \right].
\]
(23)

Let \( q(t) \) and \( r(t) \) be self-adjoint operators on \( \mathcal{H} \), and \( q(\omega) \) and \( r(\omega) \) be the corresponding Fourier transforms, respectively. The power spectrum \( S_{qr} \) of \( q \) and \( r \) is defined as
\[
\langle q(\omega), r(\omega') \rangle_\rho = S_{qr}(\omega) \delta(\omega - \omega'),
\]
(24)

The correlation function \( R_{qr}(t) \) of operators \( q(t) \) and \( r(t) \) is then defined by inverse Fourier transform of the power spectrum \( S_{qr}(\omega) \).

Returning to the decoupled system \([19]\), it turns out that the absolute value of the transfer function determines the power spectrum of the output signals, i.e., the power spectrum of \( m_x \) is related to that of \( u_x \) via
\[
S_{m_x m_x}(\omega) = |G_x(\omega)|^2 S_{u_x u_x}(\omega).
\]
(25)

If the transfer function \( G_x \) is unitary, then the power spectrums of the input and output signals are equivalent. Such a system cannot change the statistical property of the input signal at all. However, if we design the system such that the absolute value of the transfer function \( G_x \) is less than unity on the imaginary axis of the complex plane \( C \), the fluctuation of the input is to be reduced through the system in \( x \)-phase. Requiring stability of \( G_x \), reducing the fluctuation of the input in \( x \)-phase can be stated as
\[
|G_x(s)| < 1
\]
(26)
in the left half plane of \( C \). The output of the system satisfying the condition \([20]\) is a squeezed state. If \( G_x \) has a zero on the origin in \( C \) then the system can produce the perfect squeezing asymptotically. This is the case if we use the quantum mechanical feedback and parametric amplifier, no matter how weak the performance of the amplifier is \([21]\).

At this point, it seems that each element of the transfer function can take an arbitrary value, and there seems to be no differences between classical and quantum systems. However, there is a very strong constraint on the transfer function in the quantum case due to the noncommutativity of the input and output signals, as will be seen in the next section.

VII. UNCERTAINTY AND TRANSFER FUNCTION

The specific feature of quantum signals is the existence of an additional skew-symmetric form. For arbitrary op-
Consider the SISO linear quantum system with an input signal $u$ satisfying Eq. (17). In the frequency domain, the input $u$ is characterized by

$$[u_x(\omega), u_y(\omega)]_\rho = \frac{1}{4\pi} \delta(\omega + \omega').$$  

(28)

It has been shown that the noncommutativity of the input and the output result in a following relation between the two phases:

$$G_x(s)G_y(-s) \geq 1.$$  

(29)

If the output $m$ also satisfies the relation

$$[m_x(\omega), m_y(\omega')]_\rho = \frac{1}{4\pi} \delta(\omega + \omega'),$$  

(30)

and the equality of Eq. (29) is achieved, i.e.,

$$G_x(s)G_y(-s) = 1.$$  

(31)

This is the case when the input and the output are in the same space.

The relation allows us to state the quantum theoretical limitations on linear quantum systems in terms of control theory. Suppose that we design a quantum system such that $G_x$ is stable and has a zero on the origin in $C$ to reduce the noise, as stated in the previous section. This system can produce the perfect squeezing asymptotically. Then, according to the relation (31), the transfer function of the other phase, $G_y$, has a pole at the origin, and consequently, $G_y$ is unavoidably unstable and the noise is amplified to infinity. That is to say, in the quantum case, the zeros of $G_x$ and the poles of $G_y$ are distributed symmetrically with respect to the origin in $C$. And also, the trade-off between the two phases can be easily seen by taking the norm of the both side of Eq. (20), i.e., $|G_x||G_y| \geq 1$.

It is worth noting that from Eqs. (29,31), the stochastic properties of the output is unconcerned with the noncommutativity of the internal observables. Once we obtain the transfer function of a quantum system, the characteristics of the internal observables is not significant for the input-output relation.

Eq. (31) allows us to simplify the transfer function of the quantum SISO linear system. One can choose $B_x = B_p = -C_x$ with an appropriate similar transformation. Since Eq. (31) holds for an arbitrary $s$, it follows that $D_xD_p = 1$. For a physical reason, it is natural to assume $[G_x] = [G_y]$ at infinite frequencies. Thus, with an appropriate phase rotation, we have $D_x = D_p = 1$. Furthermore, Eq. (31) leads to

$$s(1 - C_xC_p^{-1}) - (A_x + A_p + C_x^2) = 0,$$

which verifies that $C_x = C_p$ and $A_x + A_p = -C_x^2$. After all, we can express a general linear quantum system as

$$\begin{bmatrix}
\frac{1}{2}C_x^2 + F & 0 & -C_x \\
0 & 0 & 0 \\
C_x & 0 & 1
\end{bmatrix} \begin{bmatrix}
m_x \\
dm_x \\
dm_y
\end{bmatrix} = \begin{bmatrix}
-C_x & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & C_x
\end{bmatrix} \begin{bmatrix}
m_x \\
dm_x \\
dm_y
\end{bmatrix},$$  

(32)

where $F$ is a free parameter. Note that this form also holds for multi-input-multi-output systems if each matrix is nonsingular.

VIII. MEASUREMENTS ON LINEAR SYSTEMS

Let us consider a measurement of a single phase of the output, say $m_x$, of the linear quantum system. This is called the homodyne measurement. The measurement projects the signal $u_x, u_y$ onto a commutative counterparts $\xi, \eta$ with the same stochastic properties as the quantum ones. Since we are measuring only $x$-phase, no information on $y$-phase is valid. Thus, we can represent the measured system as

$$\begin{align}
d\begin{bmatrix} x \\ y \end{bmatrix} &= A \begin{bmatrix} x \\ y \end{bmatrix} dt + B \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}, \\
dm &= C \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix} d\xi \\ d\eta \end{bmatrix},
\end{align}$$  

(33a)

(33b)

where $m$ is the measurement outcome, $x, y$ are the states (we have used the same notations as Eq. (21) although they are commutative here), and each matrix is given by

$$A = \begin{bmatrix}
\frac{1}{2}C_x^2 + F & 0 \\
0 & -\frac{1}{2}C_x^2 - F
\end{bmatrix}, B = \begin{bmatrix}
-C_x & 0 \\
0 & 0
\end{bmatrix},$$

$$C = \begin{bmatrix}
C_x & 0 \\
0 & 0
\end{bmatrix}, D = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.$$  

This is alternatively represented as

$$dm = \begin{bmatrix}
\frac{1}{2}C_x^2 + F & 0 & -C_x \\
0 & -\frac{1}{2}C_x^2 - F & 0 \\
C_x & 0 & 1
\end{bmatrix} \begin{bmatrix}
d\xi \\
d\eta \\
0
\end{bmatrix}.$$  

(34)

From the expression above, it is obvious that the second element of $m$ is irrelevant.

The projection postulate of the quantum theory means that the quantum system under measurements obeys a density conditioned on the measurement outcomes. Since for linear systems with correlated noises the conditional density is given by Eq. (12), which is obtained via the Kalman filter, we have the conditional density of the

$$p(x, y|m) = \frac{\rho(x, y|m)}{Z(m)},$$  

(35)

where $Z(m)$ is the normalizing factor.
quantum system on the phase space
\[
\begin{align*}
dp &= \left[ -\frac{\partial}{\partial x}(\frac{1}{2}C_x^2 + F)x - \frac{\partial}{\partial y}(\frac{1}{2}C_y^2 - F)y \\
&+ \frac{1}{2}C_x^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] dt \\
&+ C_x \left[ x - \langle x \rangle \right] p \, dx,
\end{align*}
\]
where \( \xi \) is the innovation process. Using the correspondences between the density matrix \( \rho \) and the density on the phase space \( p \),
\[
\begin{align*}
ap &= \left( \alpha + \frac{1}{2} \frac{\partial}{\partial \alpha^*} \right) \rho, \\
ap^\dagger &= \left( \alpha^* - \frac{1}{2} \frac{\partial}{\partial \alpha}\right) \rho, \\
apa &= \left( \alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*} \right) \rho, \\
ap^\dagger a &= \left( \alpha^* + \frac{1}{2} \frac{\partial}{\partial \alpha^*} \right) \rho,
\end{align*}
\]
where \( a \) is the annihilation operator and \( \alpha := (x + iy)/2 \), we obtain the quantum stochastic master equation
\[
\begin{align*}
\dot{\rho} &= \frac{C_x^2}{2} (2apa^\dagger - a^\dagger ap - paa^\dagger) dt \\
&+ [F(a^\dagger a - aa), \rho] dt + C_x(a\rho + paa^\dagger - \langle x \rangle \rho) \, dx.
\end{align*}
\]
Note that the derivation of this stochastic master equation started from Eq. \( \text{(31)} \). The derivatives in RHS’s of Eq. \( \text{(35)} \) come from the non-commutativity of the operators \( a, a^\dagger \), and therefore the derivative in the innovation process of Eq. \( \text{(34)} \) is the consequence of the fact that we are measuring the operator-valued variable. On the other hand, as the Kalman filter, the derivative results from the correlation between the dynamical noise and the measurement noise, as stated earlier. Thus, the quantum effect of the measurement on linear quantum systems is expressed as the classical noise correlation.

Let us consider the first moments of noncommutative observables \( x, y \) subject to Eq. \( \text{(35)} \), defined as \( \hat{x} = \text{Tr} (a + a^\dagger) \rho, \hat{y} = i\text{Tr} (a^\dagger - a) \rho. \) They are calculated as
\[
\begin{align*}
\dot{\hat{x}} &= A \left[ \hat{x} \right] dt + (PC^T + B)(DD^T)^{-1} \left[ \frac{d\xi}{dt} \right], \quad (37)
\end{align*}
\]
where the covariance matrix \( P \) obeys a Riccati differential equation
\[
\begin{align*}
\dot{P} &= AP + PA^T + BB^T \\
&+ (PC^T + BS)(DD^T)^{-1} (PC^T + BS)^T.
\end{align*}
\]
Now it is obvious that this is equivalent to Eq. \( \text{(10)} \).

Assume that \( (C, A) \) is detectable and \( (A, B) \) is stabilizable. Then, the system is in the stationary state after the measurement for an infinitely long time \( [20] \), and Eq. \( \text{(35)} \) can be reduced to an algebraic Riccati equation associated with a Hamiltonian matrix
\[
H = \begin{bmatrix}
\frac{C_x^2}{2} + F & 0 & -C_y^2 \\
0 & -\frac{C_y^2}{2} - F & 0 \\
0 & 0 & -\frac{C_y^2}{2} - F
\end{bmatrix}.
\]
If \( F > -C_y^2/2 \), the eigenspace corresponding to the negative eigenvalues of \( H \) is spanned by
\[
\begin{bmatrix}
1 \\
0 \\
1 + \frac{2F}{C_y^2}
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
1 + \frac{2F}{C_y^2}
\end{bmatrix}.
\]
According to Eq. \( \text{(10)} \), one can obtain the stationary covariance matrix \( P \) given by
\[
P = \begin{bmatrix}
1 + \frac{2F}{C_y^2} & 0 \\
0 & (1 + \frac{2F}{C_y^2})^{-1}
\end{bmatrix}.
\]
This implies the trade-off between the variances of the two phases, as expected from the uncertainty relation. It should be noted that this trade-off results from Eq. \( \text{(31)} \). If \( F = 0 \), the system asymptotically goes to the vacuum state, which is also physically natural.

**IX. STATIONARY UNCERTAINTY**

So far, we have seen the constraints resulting from Eq. \( \text{(31)} \). In this section, we shall consider the general case of Eq. \( \text{(29)} \). For a quantum system the covariance matrix satisfies the following inequality \( \text{(19)} \):
\[
P + \Omega \geq 0,
\]
where
\[
\Omega := \begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix}.
\]
In the static case, \( P = \text{Ric}[H] \) with a Hamiltonian matrix
\[
H = \begin{bmatrix}
(A - BSC)^T & -C^T C \\
-B(I - S^2)B^T & -(A - BSC)
\end{bmatrix},
\]
where the transpose is generalized to the complex conjugate.

Let us consider the similarity transformation of \( H \) with a matrix
\[
T = \begin{bmatrix}
I & 0 \\
\Omega & I
\end{bmatrix}.
\]
From the definition \( \text{(13)} \), it turns out that \( THT^{-1} \) is also a Hamiltonian matrix. From the definition of \( \text{Ric}[H] \), we have
\[
H \left[ \frac{I}{\text{Ric}[H]} \right] = \left[ \frac{I}{\text{Ric}[H]} \right] N
\]
for a matrix \( N \) such that \( \text{Re}(\lambda(N)) < 0 \). Premultiplication of this equation by \( T \) verifies that

\[
THT^{-1} \left[ \begin{array}{c}
I \\
\text{Ric}[H] + \Omega
\end{array} \right] = \left[ \begin{array}{c}
I \\
\text{Ric}[H] + \Omega
\end{array} \right] N,
\]

which implies that \( \text{Ric}[THT^{-1}] = \text{Ric}[H] + \Omega \).

The stationary uncertainty relation \( \text{SIR} \) is now equivalent to the condition that \( \text{Ric}[THT^{-1}] \geq 0 \). Let us define \( A_T, B_T \) and \( C_T \) by

\[
THT^{-1} = \left[ \begin{array}{ccc}
A_T^T & -C_T^T C_T \\
-B_T B_T^T & -A_T
\end{array} \right].
\]

From the statement on the positivity of the solution to the algebraic Riccati equation in Sec.IV, all linear quantum systems are subject to the condition that \( (C_T, A_T) \) is detectable and \( (A_T, B_T) \) has no uncontrollable modes on the imaginary axis.

For example, in the case of the linear quantum system \( \text{LQ} \), or the corresponding Hamiltonian matrix \( \text{H} \), it can be easily shown that

\[
THT^{-1} = \left[ \begin{array}{cccc}
\frac{C^2}{2} + F & -iC^2 & -C^2 & 0 \\
0 & -\frac{C^2}{2} - F & 0 & 0 \\
0 & 0 & -\frac{C^2}{2} - F & 0 \\
0 & 0 & -iC^2 & \frac{C^2}{2} + F
\end{array} \right].
\]

For \( F > -C^2/2 \), this system satisfies the condition shown above. Moreover, in this case, \( B_T = 0 \) reflects the fact that the system achieves the equality of the uncertainty relation.

X. WHAT DO WE CONTROL?

In the classical case, the Kalman filter is used to estimate the state of a system, and we usually assume that the system is not influenced by the estimator, i.e., the evolution of the system does not depend on how we estimate the state. Then, the system \( \text{S} \) is described by the joint density \( p(x, m) \), and the Kalman filter \( \text{K} \) is used to obtain the conditional density \( p(x|m) \), which describes our knowledge about the system. Using the conditional expectation \( E(x|m) \), we control the marginal density \( p(x) \).

In the quantum case, the joint density \( p(x, y) \) describes the system before measurements, or pre-measurement system. If we make a measurement on the system \( \text{Q} \), it no longer obeys \( \text{Q} \) but the Kalman filter \( \text{K} \). This is what the back-action of measurements means and the consequence of the projection postulate. In other words, the quantum system reflects our knowledge about the system. Thus, the innovation process can be thought of as the back-action process of the measurement, and our knowledge and the quantum system itself are continuously updated by the measurement outcome. There are still uncertainties on the initial state, and the quantum system under the measurement is different from our knowledge. However, the stability of the Kalman filter guarantees that they agree with each other asymptotically. In this case, the target of our control using the conditional expectation \( E(x|m) \) is the Kalman filter or our knowledge itself, instead of the marginal density \( p(x) \), and consequently, quantum control with measurements is equivalent to controlling a nonlinear system with state feedback.

XI. EXAMPLES

In this section, we will see simple examples in which the stochastic master equation derived from a system theoretical point of view consists with a physical derivation.

A. single mode

In the case of a cavity, the input and the output is the traveling wave in free space and the state is a single mode inside the cavity. The infinitesimal evolution of the system is given by a unitary operator

\[
U(dt) = \exp[K(a \otimes du^\dagger - a^\dagger \otimes du)],
\]

where \( a \) is the annihilation operator for the cavity mode, \( u \) is the traveling field before the interaction with the cavity, and \( K \) is a coupling constant. The cavity system before measurements can be expressed as

\[
\begin{bmatrix}
dm_x \\
dm_y
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
du_x \\
du_y
\end{bmatrix},
\]

where \( m \) is the traveling field immediately after the interaction. Each element of the transfer function is given by

\[
A = \begin{bmatrix}
-\frac{K^2}{2} & 0 \\
0 & -\frac{K^2}{2}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
-K & 0 \\
0 & -K
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
K & 0 \\
0 & K
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

Thus, from Eq.\text{M}, the cavity under the homodyne measurement is described by a density matrix \( \rho \) satisfying

\[
d\rho = \frac{K^2}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a a^\dagger) dt + K(a\rho + \rho a^\dagger - (a + a^\dagger)\rho) du,
\]

where \( du \) is the innovation process. This is equivalent to the standard stochastic master equation for the optical system \( \text{S} \).
B. deterministic input

Let us consider an additional deterministic input to the system described by a Hamiltonian

\[ H = -h_y(a + a^\dagger) - ih_x(a - a^\dagger), \]

where \( h_x, h_y \) are control gains. The conditional expectation is then given by

\[
\begin{align*}
    d \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} & = A \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} dt + \begin{bmatrix} h_x \\ h_y \end{bmatrix} dt + (PC^T - B)dy, \\
    \dot{P} & = AP + PA + BB^T - (PC^T + B)(DD^T)^{-1}(PC^T + B)^T,
\end{align*}
\]

(46a)

(46b)

where each matrix is given by

\[
A = \begin{bmatrix} -K \delta^2 & 0 \\ 0 & -K \delta^2 \end{bmatrix}, \quad B = \begin{bmatrix} -K & 0 \\ 0 & -K \end{bmatrix},
\]

\[
C = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Since the control input \( h_x, h_y \) is deterministic and linear, the covariance matrix \( P \) is independent of the input. This fact simplifies the design of the input to stabilize the Kalman filter. However, this is not the case in general, as will be seen later, and it is difficult to show the stability.

C. additional noise

Assume that a classical noise is added to the system during signal processing before we obtain the measurement outcome \( m_t \). The whole system is described by

\[
d \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = A \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} dt + \begin{bmatrix} h_x \\ h_y \end{bmatrix} dt + (PC^T - B)dy,
\]

(47)

where \( \zeta \) is the additional Winer process independent of \( \xi, \eta \), and \( \delta \) is the strength of the noise. For this equation, \( DDT = 1 + \delta^2 \), so the stochastic master equation is given by

\[
d\rho = \frac{K^2}{2}(2apa^\dagger - aa^\dagger \rho - \rho a a^\dagger)dt + \frac{K}{1 + \delta^2}(a \rho + \rho a^\dagger - \langle a + a^\dagger \rangle \rho)dy.
\]

This is equivalent to the stochastic master equation for the photon detector with efficiency \( 1/(1 + \delta^2) \), in which \( \zeta \) is thought of as a noise induced by undesirable optical modes. In reality, we cannot discriminate between the classical and quantum noise in the measurement process. Eq. (45) has assumed an ideal signal amplification of the photon detector. If we can obtain the full dynamical model of the measurement process, including information and noise processing in the photon detector, the stochastic master equation would be of a different form. It would be described by the Kalman filter for a cascade system.

XII. SPIN WIGNER FUNCTION

The phase space formulation is compatible with the classical formulation and allows us to understand quantum systems on the classical language. We have seen that once the transfer function representation of a quantum system is obtained, the stochastic master equation naturally follows from the Kalman filter in a classical way. A spin system is described by a density on a sphere, instead of the phase space, and the spherical constraint leads to the nonlinearity of the dynamics. The calculation of a conditional density then leads to a general filtering problem. For spins, however, there does not necessarily exist the classical filter corresponding to the quantum stochastic master equation. In this section, we will examine the classical counterparts to spin systems under measurements.

For an operator \( X \) on a Hilbert space representing an \( S \)-spin system, let us define a function as

\[
W_X(\Omega) = \text{Tr} X w(\Omega),
\]

(48)

where

\[
w(\Omega) = \sqrt{\frac{4\pi}{2S + 1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} Y^*_{LM}(\Omega) T_{LM}.
\]

(49)

\( Y_{LM}(\Omega) \) are the spherical harmonics and \( T_{LM} \) are the irreducible tensors of rank \( L \), or the polarization operator, which are, in terms of the basis spin functions, defined as

\[
T_{LM} = \sqrt{\frac{2L + 1}{2S + 1}} \sum_{mm'} C_{Sm' LM}^S|Sm'\rangle\langle Sm|,
\]

(50)

where \( C_{Sm' LM}^S \) are Clebsch-Gordan coefficients. For a density matrix \( \rho \) of the spin system, the function \( W_\rho \) has the same properties as the optical Wigner function. For example, the expectation of an arbitrary operator \( A \) with respect to \( \rho \) is expressed as

\[
\text{Tr} \rho A = \frac{2S + 1}{4\pi} \int d\Omega W_\rho W_A,
\]

(51)

where \( d\Omega = \sin \theta d\theta d\phi \). From \( W_\rho, \rho \) can be reconstructed via the relation

\[
\rho = \frac{2S + 1}{4\pi} \int d\Omega w(\Omega) W_\rho(\Omega).
\]

(52)

Let us define an operator \( \mathcal{P} \) as

\[
\mathcal{P}(W_A W_B) = W_{AB}.
\]

(53)

For the \( S \)-spin system, \( \mathcal{P} \) is given by

\[
\mathcal{P} = \sum_j \frac{(-1)^j}{j!(2S + j + 1)!} \sqrt{2S + 1} \mathcal{F}^{-1}(\mathcal{L}^2) \times \{ \mathcal{R}^+ j \mathcal{F}(\mathcal{L}^2) \otimes \mathcal{R}^{-j} \mathcal{F}(\mathcal{L}^2) \},
\]

(54)
where \( \tilde{F} \) is defined by
\[
\tilde{F}(\mathcal{L}^2)Y_{LM} = \sqrt{(2S + L + 1)!(2S - L)!} Y_{LM},
\]
and
\[
\mathcal{L}^2 = -\left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right),
\]
\[
\mathcal{R}_{\pm}^{\pm} = \prod_{k=0}^{j} (k \cot \theta - \frac{\partial}{\partial \theta} \mp \frac{i}{\sin \theta} \mp \frac{\partial}{\partial \phi}).
\]

Note that \( \mathcal{L}^2 \) and \( \mathcal{R}_{\pm}^{\pm} \) are commutative. In the large spin limit \( S \gg 1 \), \( \mathcal{P} \) can be simplified as
\[
\mathcal{P} = 1 \otimes 1 + \frac{\epsilon}{2}(\mathcal{R}^{-1} \otimes \mathcal{R}^+ - \mathcal{R}^+ \otimes \mathcal{R}^{-1}),
\]
where \( \epsilon = (2S + 1)^{-1} \ll 1 \). Let us define complex numbers \( \alpha, \alpha^* \) as
\[
\alpha = e^{i\phi}\sin \theta, \quad \alpha^* = e^{-i\phi}\sin \theta.
\]
In the limit of \( \theta \ll 1 \), \( \mathcal{R}^\pm \) can be represented as
\[
\mathcal{R}^{-1} \sim -e^{i\phi} \frac{\partial}{\partial \alpha},
\]
\[
\mathcal{R}^+ \sim -e^{-i\phi} \frac{\partial}{\partial \alpha^*}.
\]

The spin angular momentum operator or briefly the spin operator, can be represented by a set of three \( (2S+1) \times (2S+1) \) matrices. Using the polarization operators, the spherical components of the spin operator are given by
\[
\begin{align*}
S_\pm &= \mp \frac{\sqrt{S(S+1)(2S+1)^2}}{3} T_{1\pm 1}, \\
S_0 &= \frac{\sqrt{S(S+1)(2S+1)}}{3} T_{10},
\end{align*}
\]
and the cartesian components are
\[
\begin{align*}
S_x &= (S_+ + S_-), \\
S_y &= -i(S_+ - S_-), \\
S_z &= S_0.
\end{align*}
\]

For the spherical components of the spin operator, the definition \( 13 \) yields
\[
\begin{align*}
W_{S_+} &= \sqrt{\frac{S(S+1)}{2}} e^{i\phi}\sin \theta = \sqrt{\frac{S(S+1)}{2}} \alpha, \\
W_{S_-} &= \sqrt{\frac{S(S+1)}{2}} e^{-i\phi}\sin \theta = \sqrt{\frac{S(S+1)}{2}} \alpha^*, \\
W_{S_0} &= \sqrt{S(S+1)} \cos \theta.
\end{align*}
\]
Let us consider the action of the spin operators on the density matrix in terms of the spin Wigner function. According to Eq.\( (55) \), we have
\[
S_+ \rho \leftrightarrow \sqrt{\frac{S(S+1)}{2}}(\alpha + \frac{\epsilon}{2} \frac{\partial}{\partial \alpha^*}) W_\rho, \quad (60)
\]
\[
S_- \rho \leftrightarrow \sqrt{\frac{S(S+1)}{2}}(\alpha^* - \frac{\epsilon}{2} \frac{\partial}{\partial \alpha}) W_\rho, \quad (61a)
\]
\[
\rho S_+ \leftrightarrow \sqrt{\frac{S(S+1)}{2}}(\alpha^* + \frac{\epsilon}{2} \frac{\partial}{\partial \alpha^*}) W_\rho, \quad (61c)
\]
\[
\rho S_- \leftrightarrow \sqrt{\frac{S(S+1)}{2}}(\alpha - \frac{\epsilon}{2} \frac{\partial}{\partial \alpha}) W_\rho. \quad (61d)
\]
These relations are similar to the bosonic case \( 33 \) except for the small parameter \( \epsilon \) in the second terms.

**XIII. LOCAL BEHAVIOR OF LARGE SPINS**

The correspondences shown above involve the complex parameters \( \alpha, \alpha^* \). This fact implies that in the large spin limit, the spin operator can be related to the creation and annihilation operators of the bosonic system. In fact, the relation between the spin operator and creation and annihilation operators can be directly seen from the commutation relation of the spin operator. Let us define operators \( A_\pm, A_z \) by
\[
A_\pm = \mu S_\mp, \\
A_z = -S_z + \frac{1}{2\mu^2},
\]
where \( \mu \) is a constant. They obey the commutation relations
\[
[A_-, A_+] = 1 - 2\lambda^2 A_z, \\
[A_\pm, A_\mp] = A_\pm.
\]
Thus, if \( \lambda \to 0 \), we have the correspondences
\[
A_+ \leftrightarrow a^+, \quad (62a)
A_- \leftrightarrow a, \quad (62b)
A_z \leftrightarrow a^+ a. \quad (62c)
\]
From Eq.\( (56) \), it turns out that the north pole of the sphere corresponds to the origin of the phase space so that the the spin function \( \langle SS \rangle \) is the ground state. Thus, we have
\[
\lim_{\epsilon \to 0} A_z \langle SS \rangle = 0,
\]
which implies \( S \sim (2\mu^2)^{-1} \), and consequently, \( S \gg 1 \) is required for the correspondences to hold.

Let us consider the measurement of \( x \) component of the spin operator with damping. The stochastic master equation is given by

\[
d\rho = \frac{K^2}{2} (2S_+ \rho S_- - \rho S_- S_+ - S_- S_+ \rho) dt \\
+ K(S_+ \rho + \rho S_- - \langle S_+ + S_\rho \rangle dt,
\]

where \( \chi \) is the innovation process resulting from the measurement of \( S_x \). Define two variables as

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = \sqrt{2S + 1} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \alpha.
\]

Eq. (61) leads to the stochastic master equation for the spin Wigner function written as

\[
dW = \frac{K^2}{2e} \left[ \frac{\partial}{\partial X} X + \frac{\partial}{\partial Y} Y + \frac{\partial^2}{\partial X^2} \right] W dt \\
+ \frac{K}{\sqrt{\epsilon}} \left[ X + \frac{\partial}{\partial X} - \langle X \rangle \right] W dt d\chi + \frac{K}{\sqrt{\epsilon}} \left[ Y + \frac{\partial}{\partial Y} - \langle Y \rangle \right] W dt d\chi.
\]

This is the same as the evolution of the conditional density for the bosonic system. Thus, the local behavior of the large spin is equivalent to the bosonic system, and the stochastic master equation can also be derived as the Kalman filter of the following pre-measurement system:

\[
d \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{K^2}{2e} \left[ \begin{array}{cc} 0 & -K^2 \\ -K^2 & 0 \end{array} \right] \begin{bmatrix} X \\ Y \end{bmatrix} dt + \left[ \begin{array}{cc} -K^2 \\ 0 \end{array} \right] \begin{bmatrix} d\xi \\ d\eta \end{bmatrix},
\]

\[
dm = \frac{K}{\sqrt{\epsilon}} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}.
\]

Each element of \( A, B \) and \( C \) matrices grows with the number of spins, and consequently the measurement noise relatively becomes small accordingly. In other words, the measurement of the large number spin is almost deterministic.

In the case of QND measurement of the \( x \) component of the spin, \( S_x \), the stochastic master equation is given by

\[
d\rho = \frac{K^2}{2} [S_x, [S_x, \rho]] dt + K(\{S_x, \rho\} - 2\langle S_x \rangle \rho) d\chi.
\]

To zeroth order of \( \epsilon \), Eq. (61) yields

\[
dW = \frac{K^2}{4e} \frac{\partial^2}{\partial Y^2} W dt + \frac{K}{\sqrt{2e}} (X - \langle X \rangle) W dt d\chi.
\]

From Eq. (61), it can be easily seen that this stochastic master equation is equivalent to the Kalman filter of the following pre-measurement system:

\[
d \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{K}{\sqrt{2e}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw
\]

\[
dm = \frac{K}{\sqrt{2e}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} dt + dw,
\]

where \( w, v \) are independent normalized Wiener processes. These equations exactly reflect what QND measurement means. Eq. (60) indicates that under QND measurement of \( S_x \), the corresponding variable \( X \) is not affected by the dynamical noise. However, there is the measurement noise \( v \) which comes from the interaction between the spin and the external field for extracting information from the system. Because of the measurement noise, the measured system must be described by the conditional density, or the Kalman filter of Eq. (65), and then \( X \) is subject to the back-action noise \( \chi \).

**XIV. GLOBAL BEHAVIOR OF SPINS**

In the previous section, we have seen that the local behavior of a large spin is well-described on the local \( XY \) plane at the north pole of the sphere. The \( x \) and \( y \) components of the spin is equivalent to the two phases of a bosonic system, and thereby described as the filter of a linear classical system. For the global behavior of the large spin under QND measurement, it is convenient to use the \( z \) component for a mathematical simplicity. We assume here that, in addition to the QND measurement, there is a control parameter which allows us to rotate the spin about \( y \) axis with a control gain \( h \). The stochastic master equation for the conditioned density matrix is given by

\[
d\rho = \left( -ih[S_y, \rho] - \frac{K^2}{2} [S_z, [S_z, \rho]] \right) dt \\
+ K \left( \{S_z, \rho\} - 2\langle S_z \rangle \rho \right) d\chi,
\]

where \( \chi \) is the innovation process.

Let us rewrite the stochastic master equation for the spin Wigner function \( W \). For details, see Appendix. It can be shown that the commutation relation with the cartesian components of the spin operator \( S_i (i = x, y, z) \) corresponds to the orbital angular momentums \( L_i \) as

\[
[S_i, \rho] \leftrightarrow -L_i W.
\]

Unlike the commutation relation, the anticommutation relation with \( S_z \) has a complex form. It can be given to order of \( \epsilon \) by

\[
\{S_z, \rho\} \leftrightarrow \left[ \cos \frac{\theta}{\epsilon} - \frac{\epsilon}{2} \left( \cos \theta + \sin \frac{\theta}{\epsilon} \partial_\theta + \cos \theta L_z^2 \right) \right] W.
\]

The stochastic master equation is then written for the spin Wigner function as

\[
dW = \left[ ihL_y - \frac{K^2}{2} L_z^2 \right] W dt \\
+ K \left[ \left( \frac{1}{\epsilon} - \frac{\epsilon}{2} \left( \cos \theta - \langle \cos \theta \rangle \right) \right) - \frac{\epsilon}{2} \left( \sin \frac{\theta}{\epsilon} \partial_\theta + \cos \theta L_z^2 \right) \right] W dt d\chi.
\]

From Eq. (61), it can be easily seen that this stochastic master equation is equivalent to the Kalman filter of the following pre-measurement system:
It is worth noting that the innovation process includes second order derivatives $L^2$. In the case of the classical nonlinear filtering, the innovation term is involved in derivatives of at most first order, which occurs if the dynamical and measurement noises are correlated. Unlike the bosonic case, the stochastic master equation for the spin system therefore has no classical counterpart in the first or higher order approximation with respect to $\epsilon$, and the derivative of second order represents a quantum effect of the measurement process.

To order of $1/\epsilon$, the stochastic master equation has a classical counterpart, and it would be insightful to see the corresponding before-measurement system. One can easily show that the stochastic master equation is equivalent to the filtering of the following pre-measurement system:

$$
\begin{align*}
\dot{x} &= Ax dt + Bx dw, \quad (68a) \\
 dm &= C\dot{x} dt + \sqrt{\epsilon} dv, \quad (68b)
\end{align*}
$$

where

$$
A = \begin{bmatrix} -\frac{K^2}{2} & 0 & h \\ 0 & -\frac{K^2}{2} & 0 \\ -h & 0 & 0 \end{bmatrix},
B = \begin{bmatrix} 0 & -K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
C = \begin{bmatrix} 0 & 0 & K \end{bmatrix},
$$

and $x = [x \ y \ z]^T$ corresponds to each cartesian components of the spin operator, and $w, v$ are independent Wiener processes. The double commutation relation, $[S_z, [S_z, \rho]]$, of Eq. (68) produces the exponential decay of $x, y$ (A matrix) and mixing of $x, y$ (B matrix) in Eq. (68a). Apart from the control input $h$, $z$ is completely static. The second equation shows that the measurement process would be deterministic in the large spin limit $\epsilon \to 0$.

The measured system is described by the filtering of Eq. (68), which is given by

$$
\begin{align*}
\dot{x} &= Ax dt + PC^T d\xi, \\
\dot{p} &= AP + PA^T + BVB^T - \frac{1}{\epsilon} PC^T CP, \\
\dot{v} &= AV + V A^T + BVB^T,
\end{align*}
$$

where $P$ is the covariance matrix and $V$ is the conditional second moment. Note that the control input $h$ is of the bilinear form in Eq. (68), so that the covariance matrix $P$ is dependent on the control input $h$, unlike the bosonic case of Eq. (68).

### XV. DISCUSSION

For quantum information theoretical purposes, we want to prepare entangled states with the use of feedback. Entanglement can be produced by quantum mechanical ambiguities. In the case of an ensemble of $N$ spin-$1/2$ particles, the coherent superposition of states in which half of the spins are up and the rest are down is maximally ambiguous, e.g.,

$$
\sum_{\sigma} |\uparrow(1) \cdots \uparrow(\frac{N}{2}) \downarrow(\frac{N}{2}+1) \cdots \downarrow(N)\rangle,
$$

where $\sigma$ represents permutation. This state can be characterized by the mean values of all components of the spin operator being zero. In particular, $z$ component is deterministic, i.e., the variance of $z$ component of this state is also zero. The direction of the spins in the $xy$ plane is, however, completely indeterminant and this ambiguity is the resource of entanglement.

At first, let us examine the asymptotic behavior of the system (69), especially the covariance matrix in case of $h = 0$. Assume that the initial state of the $S$-spin is a spin coherent state pointing $x$ axis. The initial second moment is given by

$$
V_0 = \begin{bmatrix} S^2 & 0 & 0 \\
0 & \frac{S}{2} & 0 \\
0 & 0 & \frac{S}{2} \end{bmatrix}.
$$

Eq. (59) yields the stationary second moment

$$
V_s = \begin{bmatrix} \frac{S}{2}(S + \frac{1}{2}) & 0 & 0 \\
0 & \frac{S}{2}(S + \frac{1}{2}) & 0 \\
0 & 0 & 0 \end{bmatrix}.
$$

Then, the covariance matrix $P$ is given by $\text{Ric}[H]$, where $H$ is defined as

$$
H = \begin{bmatrix}
-\frac{K^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{K^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{S}{2}(S + \frac{1}{2}) & 0 & 0 & K^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{S}{2}(S + \frac{1}{2}) & 0 & 0 & K^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
$$

Simple manipulation verifies

$$
P_s = \begin{bmatrix}
\frac{S}{2K^2}(S + \frac{1}{2}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{S}{2K^2}(S + \frac{1}{2}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
$$

As expected, the variance of $z$ goes to zero and the other two elements are equally distributed. Though $z$ is static in the stationary situation, it is not stable as in Eq. (69a) and fluctuated by the back action during the transitional phase. As a result, it is not necessarily possible to obtain the maximal entanglement by the QND measurement only.

It is expected that the maximal entanglement can be obtained by stabilizing $z$ (10). Both of the quantum system under measurements and our knowledge on the system are described by the conditional expectation with different initial states. In the case of the linear system with an additive control input as in Eq. (10), the covariance is independent of the input and the difference of the initial states converges under a certain condition so that
we can make use of linear state feedback. This is not the case in general, and the stabilization requires robustness for the difference of the initial state, nonlinear feedback with higher order correlations and so on.

For the system, stabilization would be simplified using switching control. In addition, the design freedom is increased by introducing dynamics into the controller such that \( dh = Q dt \), where \( Q \) is a function. The stability is examined due to a stochastic analog of Lyapunov theory. We focus on the stabilization of \( z \) and \( h \) so that it would be enough to consider a Lyapunov function introduced in Sec. XI. The infinitesimal operator is given by

\[
\mathcal{L} = -h \frac{\partial}{\partial z} + Q \frac{\partial}{\partial u} + \frac{P_z^2}{2} \frac{\partial^2}{\partial u^2}.
\]

For \( U = z^2 + h^2 \), \( 2U = h(-2xz + Q) + P_z^2 \). Since \( \epsilon \ll 1 \), Eq. (69) leads to \( P_z^2 \sim -P_z^2/\epsilon \). Hence, a controller \( Q = 2xz - kh \) with \( k \) such that \( k > (P_z/h)^2 \) can drive \( z \) close to zero. Then, the local description introduced in Sec. XIII is valid and we can utilize the linear control introduced in Sec. XI.

**XVI. CONCLUSION**

In this paper, we started the formulation of quantum systems from the general treatment of linear systems with the noncommutative input and output. The transfer function representation provides the model of the quantum system not dependent on the commutativity of the internal observables. We obtain the quantum mechanical constraints on the model from the commutation relation of the input and the output. This constraint can be characterized in two different ways. One is the fact that the zeros and poles of the transfer functions for the two phase are distributed symmetrically in the complex plane. This characterization provides important information for quantum system synthesis because the properties of the system are completely determined by the poles and zeros. The other is concerned with the positivity of the solution to the algebraic Riccati equation. According to this constraint, there does not exist a quantum system which does not satisfy a certain detectability condition.

We have shown that the quantum stochastic master equation naturally follows from the conditional density, or the Kalman filter, of the system satisfying the conditions shown above. Then, the innovation process of the Kalman filter can be thought of as the back action of the measurement. These are also true for the large spin system to zeroth order of the number of spins. If we incorporate with higher order approximations, the spin system reveals the quantum effect to which there are no classical counterparts.

**Appendix : Proof of Eq. (67)**

Let us consider the commutation relation \([S_y, w]\) first. From the definitions \((66)\), we have

\[
[S_y, w] = -i\sqrt{\frac{4\pi}{2S + 1}} \sum_{LM} Y^*_{LM} [S_+ - S_-, w],
\]

where

\[
[S_+ - S_-, w] = \sqrt{\frac{L(L+1)}{2}} \left( c^{LM+1}_{LM} T_{LM+1} - c^{LM-1}_{LM} T_{LM-1} \right).
\]

Each Clebsch-Gordan coefficient in this equation is calculated as

\[
c^{LM+1}_{LM} = -\frac{\sqrt{L^2 - M^2 + L - M}}{2L(L+1)},
\]

\[
c^{LM-1}_{LM} = \frac{\sqrt{L^2 - M^2 + L + M}}{2L(L+1)}.
\]

Thus, the commutation relation turns out to be

\[
[S_y, w] = i\sqrt{\frac{4\pi}{2S + 1}} \sum_{LM} T_{LM} \left( \sqrt{L(L+1)} - M(M - 1) Y^*_{LM-1} - \sqrt{L(L+1)} - M(M + 1) Y^*_{LM+1} \right)
\]

Using the differential relation of spherical harmonics

\[
\frac{\partial Y^*_{LM}}{\partial \theta} \pm M \cot \theta Y^*_{LM} = \pm \sqrt{L(L+1)} - M(M \pm 1) Y^*_{LM \mp 1} e^{\mp i\phi},
\]

the commutation relation can be rewritten as

\[
[S_y, w] = \frac{i}{2} \sqrt{\frac{4\pi}{2S + 1}} \sum_{LM} T_{LM} \left( e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \theta} \right) + e^{-i\phi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \theta} \right) \right) Y^*_{LM}
\]

\[
= - L_y w,
\]

where \( L_y \) is an orbital angular momentum operator in the cartesian components. That is to say, for the operator \( w \), the commutator with respect to the spin operator \( S_y \) can be replaced by the multiplication of the orbital angular momentum operators. For \([S_z, w]\), the commutation relation

\[
[S_z, T_{LM}] = MT_{LM},
\]

leads to

\[
[S_z, w] = \sqrt{\frac{4\pi}{2S + 1}} \sum_{LM} MY^*_{LM} T_{LM}
\]

\[
= -i \frac{\partial}{\partial \phi} w
\]

\[
= -L_z w.
\]

(73)
For the anticommutation relation \( \{ S_z, w \} \), let us start from the definitions again, i.e.,

\[
\{ S_z, w \} = \sqrt{4\pi S(S+1)} \sum_{LM} Y_{LM}^* \{ T_{10}, T_{LM} \}.
\]

The anticommutator of the irreducible tensors is given by

\[
\{ T_{10}, T_{LM} \} = \sqrt{3(2+1)} \sum_{L'} \left( (-1)^{L'} - (-1)^L \right)
\]

in which Clebsch-Gordan coefficients are given by

\[
C_{LM 10}^{L+1M} = \sqrt{(L+M+1)(L-M+1)} \sqrt{2L+1},
\]

\[
C_{LM 10}^{L-1M} = -\sqrt{(L+M)(L-M)} \sqrt{2L+1},
\]

and the 6j-symbols are

\[
\begin{align*}
\{ L & \quad 1 \quad L+1 \\
S & \quad S \quad S \}
\end{align*}
\]

\[
= (1)^{L-1} \frac{2}{2} \sqrt{(2S+L+2)(2S-L)(L+1)} \sqrt{(2L+3)(2L+1)S(S+1)(2S+1)},
\]

\[
\begin{align*}
\{ L & \quad 1 \quad L-1 \\
S & \quad S \quad S \}
\end{align*}
\]

\[
= (1)^{L} \frac{2}{2} \sqrt{(2S+L+2)(2S-L-1)L} \sqrt{(2L+1)(2L-1)S(S+1)(2S+1)},
\]

In addition, Using the differential relation of spherical harmonics

\[
\sin \theta \frac{\partial}{\partial \theta} Y_{LM} = \frac{L \cos \theta Y_{LM}}{\sqrt{2L + 1}} - \frac{\sqrt{2L + 1}(L^2 - M^2) Y_{L-1M}}{2L - 1} + \frac{\sqrt{2L + 1}(L + 1)^2 - M^2)}{2L + 3} Y_{L+1M},
\]

one can rewrite the anticommutator as

\[
\{ S_z, w \} = \sqrt{\frac{4\pi}{2S+1}} \sum_{LM} T_{LM} \frac{1}{2L+1} \left[ \left( \frac{\cos \theta - \frac{\epsilon}{2} \cos \theta (L(L+1)+1) - \frac{\epsilon}{2} \sin \theta \frac{\partial}{\partial \theta} \right) Y_{LM}^* \right.
\]

where we have used the property of the total angular momentum \( L^2 Y_{LM} = L(L+1) Y_{LM} \). Eqs. (72,73,74) in combination with the definition (69) verify Eq. (70).

[1] H.J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag, Berlin, 1993).
[2] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **47** 642 (1993).
[3] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, 1983).
[4] C.W. Gardiner and H.J. Collett, Phys. Rev. A **31** 3761 (1984).
[5] H.J. Collett and C.W. Gardiner, Phys. Rev. A **30** 1386 (1984).
[6] M. Yanagisawa and H. Kimura, IEEE Trans. Automat. Contr. **48** 2107 (2003).
[7] M. Yanagisawa and H. Kimura, IEEE Trans. Automat. Contr. **48** 2121 (2003).
[8] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **49** 1350 (1994).
[9] M. Yanagisawa and H. Kimura, in Learning, Control and Hybrid Systems *Lecture Notes in Control and Information Sciences* Vol.241, 294 (Springer-Verlag, 1998).
[10] A.C. Doherty and K. Jacobs, Phys. Rev. A **60** 1549 (1999).
[11] K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control* (Prentice Hall, New Jersey, 1995).
[12] H. Kimura, *Chain-Scattering Approach to H-Infinity Control* (Birkhauser, 1996).
[13] R.L. Stratonovich, Sov. Phys.JETP **31** 1012 (1956).
[14] A. Kuzmich, L. Mandel, and N. P. Bigelow, Phys. Rev. Lett. **85** 5643 (2000).
[15] JM Geremia, J.K. Stockton, and H. Mabuchi, quant-ph/0402137.
[16] J.K. Stockton, R. van Handel, and H. Mabuchi, quant-ph/0401107.
[17] C.M. Caves, Phys. Rev. D **26** 1817 (1982).
[18] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[19] A.C. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[20] T. Kailath and L. Ljung, IEEE Trans. Automat. Contr. **21** 385 (1976).
[21] A.B. Klimov and P. Espinosa, J. Phys. A: Math. Gen. **35** 8435 (2002).
[22] L.K. Thomsen, S. Mancini, H.M. Wiseman, Phys. Rev. D **74** 085006 (2006).
A 65 061801 (2002).

[23] H.J. Kushner, *Stochastic Stability and Control*. (Academic Press, 1967).