Zero Sound in Effective Holographic Theories

Bum-Hoon Lee∗†, Da-Wei Pang† and Chanyong Park†

∗ Department of Physics, Sogang University
Seoul 121-742, Korea
† Center for Quantum Spacetime, Sogang University
Seoul 121-742, Korea
bhl@sogang.ac.kr, pangdw@sogang.ac.kr, cyong21@sogang.ac.kr

Abstract

We investigate zero sound in $D$-dimensional effective holographic theories, whose action is given by Einstein-Maxwell-Dilaton terms. The bulk spacetimes include both zero temperature backgrounds with anisotropic scaling symmetry and their near-extremal counterparts obtained in 1006.2124 [hep-th], while the massless charge carriers are described by probe D-branes. We discuss thermodynamics of the probe D-branes analytically. In particular, we clarify the conditions under which the specific heat is linear in the temperature, which is a characteristic feature of Fermi liquids. We also compute the retarded Green’s functions in the limit of low frequency and low momentum and find quasi-particle excitations in certain regime of the parameters. The retarded Green’s functions are plotted at specific values of parameters in $D = 4$, where the specific heat is linear in the temperature and the quasi-particle excitation exists. We also calculate the AC conductivity in $D$-dimensions as a by-product.
1 Introduction

The AdS/CFT correspondence [1, 2] has revealed the deep relations between gauge theories and string theories and has provided powerful tools for understanding the dynamics of strongly coupled field theories in the dual gravity side. In recent years, this paradigm has been applied to investigate the properties of certain condensed matter systems [3]. The correspondence between gravity theories and condensed matter physics (sometimes is also named as AdS/CMT correspondence) has shed light on studying physics in the real world in the context of holography.
It is well known that in realistic condensed matter systems, the presence of a finite density of charge carriers is of great importance. According to the AdS/CFT correspondence, the dual bulk gravitational background should be charged black holes in asymptotically AdS spacetimes. The simplest example of such charged AdS black holes is Reissner-Nordström-AdS(RN-AdS) black hole, which has proven to be an efficient laboratory for studying the AdS/CMT correspondence. For instance, investigations of the fermionic two-point functions in this background indicated the existence of fermionic quasi-particles with non-Fermi liquid behavior [4, 5, 6], while the \( AdS_2 \) symmetry of the extremal RN-AdS black hole is crucial to the emergent scaling symmetry at zero temperature [7]. Moreover, adding a charged scalar in such background leads to superconductivity [8, 9, 10].

A further step towards a holographic model-building of strongly-coupled systems at finite charge density is to consider the leading relevant (scalar) operator in the field theory side, whose bulk gravity theory is an Einstein-Maxwell-Dilaton system with a scalar potential. Such theories at zero charge density were analyzed in detail in recent years as they mimic certain essential properties of QCD [11, 12, 13, 14, 15]. Solutions at finite charge density have been considered in [16, 17, 18, 19, 20, 21] in the context of AdS/CMT correspondence. Recently a general framework for the discussion of the holographic dynamics of Einstein-Maxwell-Dilaton systems with a scalar potential was proposed in [22], which was a phenomenological approach based on the concept of Effective Holographic Theory (EHT). The minimal set of bulk fields contains the metric \( g_{\mu\nu} \), the gauge field \( A_\mu \) and the scalar \( \phi \) (dual to the relevant operator). \( \phi \) appears in two scalar functions that enter the effective action: the scalar potential and the non-minimal Maxwell coupling. They studied thermodynamics of certain exact solutions and computed the DC and AC conductivity. The main advantage of this EHT approach is that it permits a parametrization of large classes of IR dynamics and allows investigations on important observables. However, it is not clear whether concrete EHTs can be embedded into string theories. For subsequent generalizations see [23, 24, 25, 26, 27, 28, 29, 30].

On the other hand, strongly coupled quantum liquids play an important role in condensed matter physics, where quantum liquids mean translationally invariant systems at zero (or low) temperature and at finite density. By now there are two successful phenomenological theories of quantum liquids: Landau’s Fermi-liquid theory and the theory of quantum Bose liquids, describing two different behaviors of a quantum liquid at low momenta and
temperatures. In particular, the specific heat of a Bose liquid at low temperature is proportional to $T^q$ in $q$ spatial dimensions, while the specific heat of a Fermi liquid scales as $T$ at low $T$, irrespective of the spatial dimensions.

One may wonder if the newly developed techniques in AdS/CFT correspondence can help us understand the behavior of quantum liquids. In [31] the authors considered a class of gauge theories with fundamental fields whose holographic dual in the appropriate limit was given in terms of the Dirac-Born-Infeld (DBI) action in AdS space. They found that the specific heat $\sim T^{2p}$ in $p$ spatial dimensions at low temperature and the system supported a sound mode at zero temperature, which was called “zero-temperature sound”. One interesting feature was that the “holographic zero sound” mode was almost identical to the zero sound in Fermi liquids: the real part of the dispersion relation was linear in momentum ($\omega = qv$) and the imaginary part had the same $q^2$ dependence predicted by Landau. The crucial difference was that the zero-temperature sound velocity coincided with the first-sound velocity, while generically the two velocities are not equal for a Fermi liquid. Such analysis was performed in the case of massive charge carriers in [32] and in the case of Sakai-Sugimoto model in [33]. The specific heat of general $Dp/Dq$ systems was calculated in [34] and the specific heat of Lifshitz black holes was discussed in [35] and [36], while the zero sound was also investigated in [36].

In this paper we will study the low-temperature specific heat and the holographic zero sound in effective holographic theories. Here the bulk effective theory is $D$-dimensional Einstein gravity coupled to a Maxwell term with non-minimal coupling and a scalar. It was found in [37] that the theory admitted both extremal and near-extremal solutions with anisotropic scaling symmetry. We consider dynamics of probe D-branes in the above mentioned backgrounds and find that by appropriately fixing the parameters in the effective theory, the specific heat can be proportional to $T$, resembling a Fermi liquid. We also compute the current-current retarded Green functions at low frequency and low momentum, and clarify the conditions when a quasi-particle excitation exists. Moreover, we also explore the possibility of observing the existence of Fermi surfaces in such a system by numerical methods. We find that although the system possesses some features of Fermi liquids, such as linear specific heat and zero sound excitation, we do not observe any characteristic structure in the wide range of $k$. In addition, the AC conductivity is also obtained as a by-product.
The rest of the paper is organized as follows: the exact solutions of the effective bulk theory will be reviewed in section 2 and the thermodynamics of massless charge carriers will be discussed in section 3. We shall calculate the correlation functions in section 4 and identify the quasi-particle behavior, while the existence of Fermi surfaces will be explored in section 5 via numerics. We will calculate the AC conductivity in section 6, including the zero density limit. Finally we will give a summary and discuss future directions.

2 The solution

In this section we will review the solutions obtained in [37], which can be seen as generalizations of the four-dimensional near-extremal scaling solution discussed in [22]. In the beginning we consider the following action in $D$-dimensions, without any reference to string theory or M-theory origin nor specifying the forms of the gauge coupling $f(\phi)$ and the scalar potential $V(\phi)$ explicitly,

$$ S = -\frac{1}{16\pi G_D} \int d^D x \sqrt{-g} [R + f(\phi) \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \frac{1}{2} (\partial \phi)^2 + V(\phi)]. \quad (2.1) $$

The resulting solutions are charged dilaton black holes, which have been investigated in the literature for a long period [38, 39, 40, 41]. Let us focus on solutions carrying electric charge only. The general configuration with planar symmetry can be written as follows

$$ ds^2 = -U(r) dt^2 + \frac{dr^2}{U(r)} + V(r) \sum_{i=1}^{D-2} dx_i^2, \quad \phi = \phi(r), \quad A_t = A_t(r), \quad A_r = A_t = 0. \quad (2.2) $$

After plugging in the scaling ansatz

$$ U(r) \sim r^\beta, \quad V(r) \sim r^\gamma $$

into the equations of motion, we can arrive at several constraints on the parameters and the scalar functions:

- We require that $\beta > 1$, so that the extremal solutions have smooth connections to the finite temperature solutions;
- The field equations indicate that $\beta \leq 2$ and $0 \leq \gamma \leq 2$ and $\beta = 2$ when the scale invariance is restored.
• The equations of motion determine that the scalar field must take the form

\[ \phi(r) = C_2 \log r + \phi_0, \] (2.4)

and both \( f(\phi) \) and \( V(\phi) \) are constrained to be exponential in \( \phi \), power law in \( r \).

• Once we have fixed \( \beta > 1 \), the metric must have \( \beta \geq \gamma \), where saturation occurs for vanishing flux, e.g. in \( AdS_D \) with \( \beta = \gamma = 2 \).

Subsequently, according to the constraints discussed above, we take the following forms of \( f(\phi) \) and \( V(\phi) \).

\[ f(\phi) = e^{\alpha \phi}, \quad V(\phi) = -V_0 e^{\eta \phi}. \] (2.5)

Then we will consider charged dilaton black holes with a Liouville potential [42]. Now the scaling ansatz turns out to be

\[ ds^2 = -C_1 r^\beta dt^2 + \frac{dr^2}{C_1 r^\beta} + C_3 r^\gamma \sum_{i=1}^{D-2} dx_i^2, \]

\[ \phi(r) = C_2 \log r + \phi_0, \quad A'(r) = \frac{Q}{r^{\alpha C_2 + \gamma \frac{D-2}{2}}}. \] (2.6)

Since \( \phi_0 \) and \( C_3 \) can be eliminated by rescaling \( r \) and \( x_i \), we shall set \( \phi_0 = 0 \) and \( C_3 = 1 \). The remaining parameters can be explicitly given in terms of \( \{V_0, \eta, \alpha\} \),

\[ \beta = 2 - \frac{2(D-2)(\alpha + \eta)}{(\alpha + \eta)^2 + 2(D-2)} \eta, \quad \gamma = \frac{2(\alpha + \eta)^2}{(\alpha + \eta)^2 + 2(D-2)}, \]

\[ C_2 = \frac{2}{\alpha + \eta}, \quad Q^2 = \frac{V_0^2 - \eta^2 - \alpha \eta}{2 + \alpha^2 + \alpha \eta}, \]

\[ C_1 = \frac{V_0[(\alpha + \eta)^2 + 2(D-2)]^2}{(D-2)(2 + \alpha^2 + \alpha \eta)[2(D-2) + (D-1)\alpha^2 - \eta^2(D-3) + 2\alpha \eta]}. \] (2.7)

It can be easily seen that there is no scale invariance in such backgrounds. We will restrict to \( \eta > 0 \) and \( V_0 > 0 \) without loss of generality, which implies that \( \phi \) must diverge to positive infinity at the horizon.

In the specific limit \( \eta = 0, \beta = 2 \), the scalar potential becomes constant and the scale invariance of the solutions can be restored:

• When we set \( \alpha \to +\infty, \gamma = 2 \) with vanishing flux, the scalar \( \phi \) becomes constant and the resulting solution is \( AdS_D \);
• When we set $\alpha \to 0$, $\gamma = 0$ with flux through $R^{D-2}$, $\phi$ becomes constant and the resulting solution is $AdS_2 \times R^{D-2}$;

• When we set $\alpha$ arbitrary, $0 \leq \gamma \leq 2$ with flux through $R^{D-2}$, the scalar $\phi \sim \log r$ and the resulting solution is the modified Lifshitz solution, whose dynamical exponent $z = 2/\gamma$.

Before coming to practical calculations we should determine the range of parameters. Firstly, by requiring that $\phi(r) \to +\infty$ for small $r$ and the flux to be real, one can impose the bound on $\alpha$ in terms of fixed $\eta$,

$$-\eta < \alpha < \frac{2 \eta}{\eta} - \eta.$$

(2.8)

Secondly, when the flux is zero,

$$\alpha = \frac{2}{\eta} - \eta, \quad \beta = \gamma = \frac{4}{2 + (D - 2)\eta^2} \equiv \gamma'.$$

we should require $\gamma' > 1$ to ensure a well-defined boundary in the sense of AdS/CFT,

$$\eta^2 < \frac{2}{D - 2}.$$

Combining constraint derived in the general background in the beginning of this section, we can obtain the following complete restrictions

$$1 < \beta \leq 2, \quad \gamma \leq \beta \leq 2, \quad -\eta < \alpha \leq \frac{2}{\eta} - \eta, \quad 0 \leq \eta < \sqrt{\frac{2}{D - 2}}.$$

(2.9)

We will impose such constraints in the subsequent calculations.

The near extremal solution can be obtained in a similar way,

$$ds^2 = -C_1 r^\beta f(r) dt^2 + \frac{dr^2}{C_1 r^\beta f(r)} + C_3 r^\gamma \sum_{i=1}^{D-2} dx_i^2,$$

(2.10)

where

$$f(r) = 1 - \left(\frac{r_w}{r}\right)^{w}, \quad w = \beta - 1 + \frac{D - 2}{2} \gamma.$$

(2.11)

\footnote{The reason why such a solution is called “modified Lifshitz” is that the scalar field must be constant in Lifshitz background, which is required by scaling symmetry \cite{43}. Properties of the modified Lifshitz solutions have been studied in \cite{44} and \cite{20}.}

\footnote{For details see \cite{37}.}
and the other parameters and fields remain the same as the extremal solutions. One can easily get the temperature

$$T = \frac{1}{4\pi} C_1 w r_+^{\beta-1}, \quad (2.12)$$

and the entropy density

$$s \equiv \frac{S_{BH}}{V_{R^{D-2}}} = \frac{1}{4G_D} r_+^{\gamma-2}. \quad (2.13)$$

### 3 Thermodynamics of probe D-branes

In this section we will investigate thermodynamics of massless charge carriers in the backgrounds reviewed in previous section. According to AdS/CFT, $N_f$ probe D-branes correspond to $N_f$ fields in the fundamental representation of the gauge group in the probe limit $N_f \ll N$ \cite{45, 46}. An efficient method for evaluating the DC conductivity and DC Hall conductivity of probe D-branes was proposed in \cite{47} and \cite{48}. Moreover, a holographic model building approach to “strange metallic” phenomenology was initiated in \cite{49}, where the bulk spacetime was a Lifshitz black hole and the charge carriers were described by D-branes. Here we will consider probe D-branes as massless charge carriers and explore the thermodynamics in the near-extremal background.

The dynamics of probe D-branes is described by the Dirac-Born-Infeld (DBI) action

$$S_{DBI} = -N_f T_D \text{Vol}(\Sigma) \int dt d\sigma^a x e^{-\phi} \sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab})},$$

$$= -\tau_{\text{eff}} \int dt d\sigma^a x e^{-\phi} \sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab})}, \quad (3.1)$$

where $T_D$ denotes the tension of D-branes, $g_{ab}$ is the induced metric and $F_{ab}$ is the $U(1)$ field strength on the worldvolume. In the second line we set $\tau_{\text{eff}} = N_f T_D \text{Vol}(\Sigma)$, where $\text{Vol}(\Sigma)$ denotes volume of the internal space that the D-branes may be wrapping. Furthermore, we assume that the D-branes are extended along $q \leq D - 2$ spatial dimensions of the black hole solution. If $q < D - 2$, the fundamental fields are propagating along certain $q$-dimensional defect. We will introduce a nontrivial worldvolume gauge field $A_t(r)$ and absorb the factor $2\pi\alpha'$ into $F_{ab}$. Since we are not studying realistic string theories, the Wess-Zumino terms will be omitted in the following discussions.

Before proceeding we should make sure that the backreaction of the probe branes onto the background can be neglected. Our discussion is along the line of \cite{49}. Expanding the
DBI action to quadric order of $F_{rt}$ in the background, we can obtain

$$S_{\text{DBI}} = -\tau_{\text{eff}} \int dt dr d^q x e^{-\phi} \sqrt{-g} \sqrt{1 + g^{tt} g^{rr} F_{rt}^2}, \quad (3.2)$$

To avoid backreaction of the probes on the background, the stress energy of the probes must be smaller than that generating the bulk spacetime. It can be easily seen that the stress energy of the original background $\sim \ell_P^{D-2} |\Lambda|$, where $\ell_P$ denotes the Planck length in $D$-dimensional spacetime and $\Lambda$ is the corresponding cosmological constant. Therefore by varying the quadric action of the probes with respect to $g_{tt}$, we can arrive at the following condition

$$\frac{e^{-\phi}}{\sqrt{1 + g^{tt} g^{rr} F_{rt}^2}} \ll \frac{\ell_P^{D-2} |\Lambda|}{\tau_{\text{eff}}}. \quad (3.3)$$

One can see that as long as the effective tension $\tau_{\text{eff}}$ is sufficiently small, the backreaction can be neglected.

In this background configuration, after performing the trivial integrations on $dt d^q x$ and dividing out the infinite volume of $\mathbb{R}^{1,q}$, we can obtain the action density.

$$S = -\tau_{\text{eff}} \int dr r^m \sqrt{1 - A_t'^2}, \quad (3.4)$$

where

$$m = \frac{1}{2} \gamma q - C_2 = \frac{(\alpha + \eta)[q(\alpha + \eta) + 2(D - 2)]}{(\alpha + \eta)^2 + 2(D - 2)}. \quad (3.5)$$

and the prime denotes derivative with respect to $r$. Then the charge density is given by

$$\rho \equiv \frac{\delta \mathcal{L}}{\delta A'_t} = \tau_{\text{eff}} \frac{r^m A'_t}{\sqrt{1 - A_t'^2}}. \quad (3.6)$$

We can also solve for $A'_t(r)$

$$A'_t = \frac{d}{\sqrt{r^{2m} + d^2}}, \quad d \equiv \frac{\rho}{\tau_{\text{eff}}}. \quad (3.7)$$

By plugging in the solution for $A'_t(r)$, we can find the on-shell action density

$$S_{\text{on-shell}} = -\tau_{\text{eff}} \int dr \frac{r^{2m}}{\sqrt{r^{2m} + d^2}}. \quad (3.8)$$

Following the methods used in [50], some interesting physical quantities like the chemical potential and the free energy, can be evaluated analytically. We will see that this is still
the case for our background. The chemical potential is given by

\[ \mu = \int_{r_+}^{\infty} A'_d \, dr, \]

\[ = \mu_0 - r_+ 2 F_1 \left( \frac{1}{2m}, \frac{1}{2}; 1 + \frac{1}{2m}, \frac{2m^2}{d^2} \right), \]  

(3.9)

where

\[ \mu_0 = d \frac{1}{m^2} B_0(m), \quad B_0(m) = \frac{1}{2} B(1 + \frac{1}{2m}, \frac{1}{2} - \frac{1}{2m}). \]  

(3.10)

Notice that in order to obtain the above results, we have made use of the following useful formulae for Beta function and incomplete Beta function

\[ B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} = \int_{0}^{\infty} \, du \, (1 + u)^{-(a+b)} u^{a-1}, \]

\[ B(x; a, b) = \int_{0}^{x/(1-x)} \, du \, (1 + u)^{-(a+b)} u^{a-1}. \]

as well as for Hypergeometric function

\[ B(x; a, b) = a^{-1} x \frac{a}{2} F_1 (a, 1 - b; a + 1, x), \]

\[ 2 F_1 (a; b, c, x) = (1 - x)^{-a} 2 F_1 (a, c - b; c, \frac{x}{x - 1}). \]

After choosing the grand-canonical ensemble, the free energy density is given by

\[ \Omega = -S_{\text{on-shell}} = \tau_{\text{eff}} \int_{r_+}^{\infty} \, dr \, \frac{t_{2m}}{\sqrt{r_{2m}^2 + d^2}}, \]

\[ \Omega_0 = -\frac{\tau_{\text{eff}}}{2(m + 1)d} \frac{t_{2m+1}}{2} 2 F_1 (1 + \frac{1}{2m}, \frac{1}{2}; 2 + \frac{1}{2m}, \frac{r_{2m}^2}{d^2}), \]

(3.11)

where

\[ \Omega_0 = -\frac{\tau_{\text{eff}}}{2(m + 1)d} \frac{t_{1+m}}{m} B(1 + \frac{1}{2m}, \frac{1}{2} - \frac{1}{2m}) \]

\[ = -\frac{\tau_{\text{eff}}}{(m + 1)B_0(m)^m} \mu_0^{m+1}. \]  

(3.12)

Moreover, other thermodynamic quantities can also be calculated from the thermodynamic relations. The charge density can be written as

\[ \rho = -\partial \Omega / \partial \mu = \tau_{\text{eff}} (\frac{\mu_0}{B_0(m)})^m = \tau_{\text{eff}} d, \]

(3.13)

which is consistent with previous result. The entropy density is given by

\[ s = -\partial \Omega / \partial T = \frac{\rho}{\beta - 1} \left( \frac{4\pi}{C_1 w} \right)^{\frac{1}{2m+1}} T^{2m+1} + \tau_{\text{eff}} \frac{4\pi}{2(\beta - 1)d} \left( \frac{4\pi}{C_1 w} \right)^{\frac{1}{2m+1}} T^{2(m+1)-1}, \]

(3.14)
Notice that when $\beta = 2$, there exists a nontrivial contribution to the entropy density at $T = 0$ like those observed in [35] and [36]. On the other hand, the entropy density is vanishing at extremality as long as $\beta \neq 2$. The specific heat is

$$c_V = T \frac{\partial s}{\partial T} = \frac{\rho(2 - \beta)}{(\beta - 1)^2} \left( \frac{4\pi}{C_1 w} \right)^{1/\beta} T^{3-\beta} + \frac{\tau_{\text{eff}}^2 (2m + 2 - \beta)}{2(\beta - 1)^2 \rho} \left( \frac{4\pi}{C_1 w} \right)^{2m+1/\beta} T^{2m+2-\beta}. \quad (3.15)$$

It is well known that for a gas of free bosons in $q$ spatial dimensions, the specific heat at low temperature is proportional to $T^p$, while for a gas of fermions the low temperature specific heat is proportional to $T$, irrespective of $p$. When $\beta \neq 2$ and $\tau_{\text{eff}}$ is sufficiently small, the first term dominates. One can easily obtain $\beta = 3/2$ when the specific heat is proportional to $T$. Then the parameter $\alpha$ can be expressed in terms of $\eta$

$$\alpha_{1\pm} = (2D - 5)\eta \pm \sqrt{2(D - 2)[2(D - 2)\eta^2 - 1]}, \quad (3.16)$$

Combining with (2.9), we can arrive at the following conclusions:

- When

$$\sqrt{\frac{1}{2(D - 2)}} \leq \eta < \sqrt{\frac{2}{3(D - 2)}},$$

both $\alpha_{1+}$ and $\alpha_{1-}$ are permitted solutions;

- When

$$\sqrt{\frac{2}{3(D - 2)}} \leq \eta < \sqrt{\frac{2}{D - 2}},$$

only $\alpha_{1-}$ is a permitted solution;

- When

$$0 < \eta < \sqrt{\frac{1}{2(D - 2)}},$$

there is no solution, which means that we cannot realize $c_V \propto T$ in this regime.

When $\beta = 2, \eta = 0$, the second term provides the only contribution. The linear dependence on $T$ fixes $m = 1/2$. Note that in this limit, $\gamma = 2\alpha^2/(\alpha^2 + 2(D - 2)) \equiv 2/z$. So

$$\alpha_{2\pm} = \frac{-4(D - 2) \pm \sqrt{16(D - 2)^2 + 4(D - 2)(2q - 1)}}{2(2q - 1)}, \quad (3.17)$$

By taking into account of (2.9), we can find that only $\alpha_{2+}$ is permitted.
We can also formally evaluate the “speed of sound”. In grand-canonical ensemble, the pressure is given by

\[ P = -\Omega_0 = \frac{\tau_{\text{eff}}}{(m+1)B_0(m)}\mu_0^{m+1}, \quad (3.18) \]

while the energy density is

\[ \varepsilon = \Omega_0 + \mu_0 \rho = \frac{m\tau_{\text{eff}}}{(m+1)B_0(m)}\mu_0^{m+1}. \quad (3.19) \]

Therefore,

\[ \varepsilon = mP, \quad \Rightarrow \quad c_s^2 = \frac{\partial P}{\partial \varepsilon} = \frac{1}{m}. \quad (3.20) \]

However, it was emphasized in [36] that this quantity is only the speed of normal/first sound in the relativistic case \( z = 1 \). Actually the speed of normal sound is dimensionful in a system with \( z > 1 \). We will calculate the holographic zero sound in the next section.

### 4 The holographic zero sound

In this section we will calculate the holographic zero sound in the anisotropic background at extremality. The basic strategy is to consider fluctuations of the worldvolume gauge field on the probe D-branes in the background with nontrivial \( A_t \). Such analysis was performed for \( AdS_{p+2} \) background in [31] and for Lifshitz background in [36]. We will calculate the holographic zero sound in a similar way and classify the behavior of the zero sound in different parameter ranges.

#### 4.1 The retarded Green’s functions

Zero sound should appear as a pole in the density-density retarded two-point function \( G^R_{tt}(\omega, k) \) at extremality [31]. In [36] the authors provided a general framework for evaluating the corresponding retarded Green’s functions with background metric

\[ ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{xx} \sum_{i=1}^{D-2} dx_i^2. \quad (4.1) \]

Here we will take the nontrivial dilaton into account. The symmetries in the spatial directions allow us to consider fluctuations of the gauge fields with the following form

\[ A_\mu(r) \to A_\mu(r) + a_\mu(t, r, x), \]

11
where \( x \) denotes one of the spatial directions. The quadratic action for the fluctuations is given by

\[
S_a^2 = \frac{\tau_{\text{eff}}^2}{2} \int dt dr dq e^{-\phi} \frac{g_{tt} f_{tx}^2 - |g_{tt}| a_x^2}{g_{xx}(|g_{tt}| - A_t^2)^{1/2}} + \frac{|g_{tt}| g_{rr} a_t^2}{(|g_{tt}| - A_t^2)^{3/2}},
\]

(4.2)

where \( f_{tx} = \partial_t a_x - \partial_x a_t \). Note that we are working in the gauge of \( a_r = 0 \). After performing the Fourier transform

\[
a_\mu(t, r, x) = \int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega t + i k x} a_\mu(\omega, r, k),
\]

the linearized equations of motion can be written as

\[
\partial_r \left[ e^{-\phi} \frac{g_{xx}^{3/2} |g_{tt}| a_t'}{|g_{tt}| - A_t^2} \right] - e^{-\phi} \frac{g_{xx}^{3/2-1} g_{rr}}{\sqrt{|g_{tt}| - A_t^2}} (k^2 a_t + \omega k a_x) = 0,
\]

(4.3)

\[
\partial_r \left[ e^{-\phi} \frac{g_{xx}^{3/2-1} |g_{tt}| a_x'}{|g_{tt}| - A_t^2} \right] + e^{-\phi} \frac{g_{xx}^{3/2-1} g_{rr}}{\sqrt{|g_{tt}| - A_t^2}} (\omega^2 a_x + \omega k a_t) = 0.
\]

(4.4)

In addition, the following constraint can be obtained by writing \( a_r \)'s equation of motion in \( a_r = 0 \) gauge

\[
g_{rr} g_{xx} \omega a_t' + (|g_{tt}| - A_t^2) k a_x' = 0.
\]

(4.5)

The above equations are not independent, as we can obtain the equation of motion for \( a_t \) by combining the constraint equation and the equation of motion for \( a_x \). Therefore it is sufficient to solve the constraint and the equation for \( a_t \) only. By introducing the gauge-invariant electric field

\[
E(r, \omega, k) = \omega a_x + k a_t,
\]

we can obtain the equation of motion for \( E \)

\[
E'' + \partial_r \ln \left( \frac{\frac{\omega^2}{u} |g_{tt}|^{1/2}}{u(2u^2 - \omega^2)} \right) E' - \frac{g_{rr}}{|g_{tt}|} (u^2 k^2 - \omega^2) E = 0,
\]

(4.6)

where

\[
u(r) = \sqrt{\frac{|g_{tt}| g_{rr} - A_t^2}{g_{rr} g_{xx}}}
\]

(4.7)

Moreover, the quadratic action can also be expressed in terms of \( E \),

\[
S_a^2 = \frac{\tau_{\text{eff}}^2}{2} \int dr d\omega dk e^{-\phi} \frac{g_{xx}^{3/2} g_{rr}^{1/2}}{u} \left[ E^2 + \frac{|g_{tt}|}{g_{rr}(u^2 k^2 - \omega^2)} E'^2 \right],
\]

(4.8)
For our specific background, the metric can be rewritten in terms of the new radial coordinate $z = 1/r$ as follows

$$ds^2 = -\frac{C_1}{z^\beta} dt^2 + \frac{z^{\beta-4}}{C_1} dz^2 + \frac{1}{z^\gamma} \sum_{i=1}^{D-2} dx_i^2. \quad (4.9)$$

In the new coordinate system, the solution of the worldvolume gauge field and the function $u(z)$ are given by

$$\dot{A}_t^2 = \frac{d^2 z^{2m}}{z^4 (1 + d^2 z^{2m})}, \quad u^2 = \frac{C_1}{z^{\beta-\gamma} (1 + d^2 z^{2m})}, \quad (4.10)$$

where dot denotes derivative with respect to $z$. Integrating the quadratic action by parts

$$S_{a^2} = \frac{\tau_{\text{eff}}}{2} \int d\omega dk \frac{\epsilon^2}{u(k^2 - \omega^2)} \dot{E}E, \quad (4.11)$$

introducing a cutoff at $z = \epsilon$ and taking the limit $\epsilon \to 0$, the quadratic action turns out to be

$$S_{a^2} = -\frac{\tau_{\text{eff}}}{2} \int d\omega dk \frac{\epsilon^{2-m}}{k^2} \dot{E}E. \quad (4.12)$$

After imposing the incoming boundary condition at the “horizon” $z \to 0$ and plugging in the solutions of $a_\mu$, the retarded correlation function reads [51]

$$G_{tt}^R(\omega, k) = \frac{\delta^2}{\delta a_t(\epsilon)^2} S_{a^2} = \left( \frac{\delta E(\epsilon)}{\delta a_t(\epsilon)} \right)^2 \frac{\delta^2}{\delta E(\epsilon)^2} S_{a^2}, \quad (4.13)$$

By defining

$$\Pi(\omega, k) \equiv \frac{\delta^2}{\delta E(\epsilon)^2} S_{a^2}, \quad (4.14)$$

the retarded correlation functions can be written in terms of $\Pi(\omega, k)$

$$G_{tt}^R(\omega, k) = k^2 \Pi(\omega, k), \quad G_{tx}^R(\omega, k) = \omega k \Pi(\omega, k), \quad G_{xx}^R(\omega, k) = \omega^2 \Pi(\omega, k). \quad (4.15)$$

### 4.2 Matching the solutions

In order to evaluate the retarded correlation functions, we should try to solve (4.6), whose analytic solutions are always difficult to find. We will leave the numerical work to section 5, while here we will obtain the low-frequency behavior of $\Pi(\omega, k)$ by solving (4.6) in different limits and matching the two solutions in an overlapped regime, following the
spirit of [31] and [36]. To be concrete, we will solve (4.6) in the limit of large $z$ and then expand the solution in the small frequency and momentum limit. Next we will take the small frequency and momentum limit first and then perform the large $z$ expansion. The integration constants can be fixed by matching the two solutions.

First let us take $z \to \infty$, which leads to the following equation for $E$

$$\ddot{E} + \frac{2 - \beta + \gamma}{z} \dot{E} + \frac{\omega^2 z^{2\beta - 4}}{C_1^2} E = 0. \quad (4.16)$$

The solution can be given in terms of a Hankel function of the first kind,

$$E = D_0 (\frac{x}{2})^\nu H_\nu^{(1)}(x), \quad x = \frac{\omega}{C_1 (\beta - 1)} z^{\beta - 1}, \quad \nu = \frac{1}{2} - \frac{\gamma}{2(\beta - 1)}. \quad (4.17)$$

In the limit of small frequency with $\nu \neq 0$, the asymptotic expansion reads

$$E = D_0 \Gamma \left( \frac{1}{2} + \frac{\gamma}{2(\beta - 1)} \right)^{-1} (1 - i \tan \frac{\pi \gamma}{2(\beta - 1)}) \left[ -\frac{i D_0}{\pi} \Gamma \left( \frac{\gamma}{2(\beta - 1)} - \frac{1}{2} \right) \left( \frac{\omega}{2C_1 (\beta - 1)} \right)^{1-\frac{\nu}{\beta-\gamma}} z^{\beta-\gamma-1} \right]. \quad (4.18)$$

It should be pointed out that the case of $\nu = 0$ must be treated separately. In this case the corresponding parameter is given by

$$\alpha_{3+} = -(D - 1) \eta + \sqrt{(D - 2)^2 \eta^2 + 2(D - 2)}. \quad (4.19)$$

Now the expansion contains a logarithmic term

$$E \approx D_0 + \frac{2i}{\pi} D_0 (\log(\omega z^{\beta - 1}) - \log(2C_1(\beta - 1)) + \gamma_E), \quad (4.20)$$

where $\gamma_E$ is the Euler constant.

Next we take $\omega z^{\beta/2} \ll 1, kz^{\gamma/2} \ll 1$ with $\omega k^{-\beta/\gamma}$ being fixed. Then the last term in (4.6) can be neglected and the equation of $E$ becomes

$$\ddot{E} + \left[ \partial_z \ln \left( e^{\gamma_E} \frac{g_{xx} + g_{rr}^{-1/2}}{u(k^2 u^2 - \omega^2)} \right) \right] \dot{E} = 0. \quad (4.21)$$

When $m \neq 1$ and $m \neq \gamma - \beta + 1$, the solution is given by

$$E = D_1 + D_2 \left[ C_1 k^2 z^{m-1} \left( \frac{1}{m\sqrt{1 + d_2 z^{2m}}} + \frac{1}{m - 1} F_1 \left( \frac{1}{2}, 1, -\frac{1}{2m}; \frac{3}{2} - \frac{1}{2m}, -d_2 z^{2m} \right) \right) \right. \left. -\omega^2 \frac{z^{m+n}}{m+n} F_1 \left( \frac{1}{2}, 1, \frac{n}{2m}; \frac{3}{2}, \frac{n}{2m}, -d_2 z^{2m} \right) \right], \quad (4.22)$$
where \( n \equiv \beta - \gamma - 1 \). For either \( m \neq 1 \) or \( m \neq \gamma - \beta + 1 \), the powers of \( z \) do not match, which will be displayed in Appendix A. We will make use of the following useful formulae for the asymptotic expansion

\[
\begin{align*}
2F_1(a, b; c, x) &= (1 - x)^{-a} \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}2F_1(a, c - b; a - b + 1, \frac{1}{1 - x}) \\
&\quad +(1 - x)^{-b} \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}2F_1(b, c - a; b - a + 1, \frac{1}{1 - x}), \\
2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, -x^2\right) &= x^{-1} \log(x + \sqrt{1 + x^2}).
\end{align*}
\]

Therefore the large \( z \) limit is given as follows when \( n \neq 0 \),

\[
E \simeq D_1 + D_2\left[\frac{C_1\mu_0 k^2}{md} - \frac{\omega^2 z^n}{nd} - \frac{\omega^2 d - \frac{m}{2md}}{2md}B\left(\frac{1}{2} + \frac{n}{2m} - \frac{n}{2m}\right)\right]. \quad (4.23)
\]

When \( n = 0 \) the expansion also contains a logarithmic term

\[
E \simeq D_1 + D_2\left[\frac{C_1\mu_0 k^2}{md} - \frac{\omega^2}{md} \log(2dz^m)\right]
\]

\[
\simeq D_1 + D_2\left[\frac{C_1\mu_0 k^2}{md} - \frac{\omega^2}{md} \log 2d + \frac{\omega^2}{(\beta - 1)d} \log \omega - \frac{\omega^2}{(\beta - 1)d} \log(\omega z^{\beta - 1})\right]. \quad (4.24)
\]

To evaluate the retarded correlation functions, we need small \( z \) expansion of the solution. It can be seen that the second term in the expansion always tends to zero more rapidly, being irrespective of \( n = 0 \) or not. Therefore we have

\[
E \simeq D_1 + D_2 \frac{C_1 k^2}{m - 1} z^{m - 1}, \quad (4.25)
\]

Assuming that \( m > \frac{1}{2} \), the leading order behavior of \( E \) reads \( E(\epsilon) \simeq D_1, \dot{E}(\epsilon) \simeq D_2 C_1 k^2 \epsilon^{m - 2} \), so the quadratic action turns out to be

\[
S_{a^2} = -\frac{\tau_{\text{eff}}}{2} \int d\omega dk \frac{\epsilon^{2 - m}}{k^2} E \dot{E}
\]

\[
= -\frac{\tau_{\text{eff}} C_1}{2} \int d\omega dk D_1 D_2,
\]

Thus

\[
\Pi(\omega, k) = \lim_{\epsilon \to 0} \frac{\delta^2}{\delta E(\epsilon)^2} S_{a^2} = \frac{\delta^2}{\delta D_1^2} S_{a^2} \big|_{\epsilon \to 0}, \quad (4.27)
\]

\[\text{the } m < 1 \text{ can be dealt with in a similar fashion, see footnote 7 of } [36].\]
The relation between the integration constants $D_1$ and $D_2$ can be obtained by matching the expansions of the solutions in different limits and eliminating the other integration constant $D_0$.

Finally we summarize our result for $\Pi(\omega, k)$

$$\Pi(\omega, k) \propto \frac{\tau_\text{eff} C_1}{\delta_1 k^2 - \delta_2 \omega^2 - \delta_3 G_0(\omega)}. \quad (4.28)$$

When $n \neq 0$ the parameters are given by

$$\delta_1 = -\frac{C_1 \mu_0 n}{\pi m (2C_1(\beta - 1))^{\frac{\pi}{\beta - 1}}} \Gamma(\frac{\gamma}{2(\beta - 1)} - \frac{1}{2}) \Gamma(\frac{\gamma}{2(\beta - 1)} + \frac{1}{2}),$$

$$\delta_2 = -\frac{nd \frac{\pi}{\beta - 1}}{2\pi m (2C_1(\beta - 1))^{\frac{\pi}{\beta - 1}}} \Gamma(\frac{\gamma}{2(\beta - 1)} - \frac{1}{2}) \Gamma(\frac{\gamma}{2(\beta - 1)} + \frac{1}{2}) B\left(\frac{1}{2} + \frac{n}{2m^2}, -\frac{n}{2m}\right),$$

$$\delta_3 = i + \tan\left(\frac{\pi \gamma}{2(\beta - 1)}\right), \quad G_0(\omega) = \omega^{2 - \frac{n}{\beta - 1}}. \quad (4.29)$$

When $n = 0$ the parameters are given by

$$\delta_1 = \frac{2C_1(\beta - 1) \mu_0}{\pi m}, \quad \delta_2 = i, \quad \delta_3 = -\frac{1}{\pi},$$

$$G_0(\omega) = \omega^2 \log(\delta \omega^2), \quad \delta = \frac{de^{2\gamma E}}{2C_1^2(\beta - 1)^2}. \quad (4.30)$$

### 4.3 Zero sound

The dispersion relation of the holographic sound mode is given by setting the denominator of $\Pi(\omega, k)$ to vanish. Similar to the situations discussed in [36], the value of $n$ determines which term in the denominator dominates.

It can be seen that when $n < 0$, the $\omega^2$ term dominates. Then we can expand $k(\omega)$

$$k = \pm \omega \sqrt{\frac{\delta_2}{\delta_1}} [1 + \frac{\delta_3}{2\delta_2} \omega^{-1 + \frac{\gamma}{\beta - 1}} + \mathcal{O}(\omega^{-2 + \frac{2\gamma}{\beta - 1}})], \quad (4.31)$$

and then invert to find

$$\omega(k) = \pm k \sqrt{\frac{\delta_1}{\delta_2}} - \frac{\delta_3}{2\delta_2} \frac{\delta_1}{\delta_2} \frac{\pi}{\pi - 1} k^{\frac{\gamma}{\beta - 1}} + \mathcal{O}(k^{-1 + \frac{2\gamma}{\beta - 1}}). \quad (4.32)$$

Notice that when $n < 0$, $\beta - 1 < \gamma$, so at low momentum $k^{\frac{\gamma}{\beta - 1}} < k$, that is, the real part is bigger than the imaginary part. Therefore this mode describes a quasi-particle excitation.
As a check of consistency, we can take the specific limit \( \beta = \gamma = 2, C_1 = 1, \lambda = 0, m = q \), which just gives the \( AdS_D \) background. One can obtain
\[
\omega(k) = \pm \frac{k}{\sqrt{q}} - i \frac{1}{\Gamma\left(\frac{1}{2} - \frac{1}{2q}\right)\Gamma\left(\frac{1}{2}\right)} k^2 + \mathcal{O}(k^3),
\]
(4.33)
which agrees with [31]. The speed of the holographic zero sound is given by
\[
v_0^2 = \frac{C_1 d^{\frac{n+1}{m}}}{m} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{2m}\right)\Gamma\left(\frac{1}{2}\right) - \frac{1}{2m}}{\Gamma\left(\frac{1}{2} + \frac{n}{2m}\right)\Gamma\left(-\frac{n}{2m}\right)}.
\]
(4.34)
It can be seen that in the relativistic case \( \beta = \gamma = 2 \), the speed of zero sound coincides with the speed of normal/first sound. One special case is \( m + n = 1 \), where all the \( \Gamma \) functions cancel and the speed of zero sound reads
\[
v_0^2 = \frac{C_1 m d^{n+1}}{m},
\]
(4.35)
which turns out to be finite as \( n \to 0 \). When \( m + n > 1 \) as well as \( n < 0, \Gamma\left(-\frac{n}{2m}\right) \) has a pole as \( n \to 0 \), so \( v_0^2 \) goes to zero from above.

When \( n > 0, G_0(\omega) \) dominates, then
\[
k(\omega) = \pm \left(\frac{\delta_1}{i\delta_3}\right)^{\frac{1}{2}} \omega^{1 - \frac{n}{2(\beta - 1)}} (1 + \frac{\delta_2}{2\delta_3} \omega^\frac{1}{\sqrt{q}} + \mathcal{O}(\omega^\frac{2}{\sqrt{q}})).
\]
(4.36)
Inverting the relation above,
\[
\omega = \left(\frac{\delta_1}{i\delta_3}\right)\frac{\beta^{\frac{1}{2}}}{2(\beta - 1) - n} k^{\frac{2(\beta - 1) - n}{2(\beta - 1) - n}} + \frac{\beta - 1}{2(\beta - 1) - n} \frac{\delta_2}{\delta_3} \left(\frac{\delta_1}{\delta_3}\right)^{\frac{\beta - n}{2(\beta - 1) - n}} k^{\frac{2(\beta - 1) - n}{2(\beta - 1) - n}} + \mathcal{O}(k^{\frac{2(\beta - 1) + 4n}{2(\beta - 1) - n}}),
\]
(4.37)
Notice that since \( \delta_1 \) is real and \( \delta_3 \) is complex, the leading term has a complex coefficient. Furthermore, the real and imaginary parts are of the same order, hence this mode is not a quasi-particle.

Finally when \( n = 0 \),
\[
k = \pm \frac{\omega}{\sqrt{\delta_1}} \sqrt{\delta_2 + \log(\delta_2^2)}.
\]
(4.38)
Expanding for small \( \omega \),
\[
k(\omega) = \pm \frac{\omega}{\sqrt{\delta_1}} \sqrt{\delta_3 \log(\delta_2^2)} - \frac{\omega \delta_2}{2\sqrt{\delta_1}} (\delta_3 \log(\delta_2^2))^{\frac{1}{2}} + \mathcal{O}(\omega \log^{-3/2}(\delta_2^2)).
\]
(4.39)
It can be seen that the dispersion relation differs from the holographic zero sound mode by logarithmic terms.
5 A numerical survey of Fermi surface

In previous section we observed a sound-like excitation in the regime of $n < 0$. It is known that such zero sound mode is associated with the deformation of the Fermi surface away from the spherical shape. The theory of normal Fermi liquids tells us that the jump in the distribution function can be observed as a singularity in the retarded current-current Green’s function in the $\omega = 0$ limit. To see if we can observe such Fermi surface we need the complete solution to the equation for the gauge fluctuations (4.6) with $\omega = 0$, at least numerically.

We investigated thermodynamics of probe D-branes in section 3, where we found that when $\beta \neq 2$, the specific heat was proportional to the temperature $T$ under certain conditions, which is just the behavior of Fermi liquids. As we have many groups of parameters $(\alpha, \eta)$ which lead to specific heat linear in $T$, we choose the following parameters $\alpha = 3/2, \eta = 1/2$ in $D = 4$ dimensional spacetime as an example. For simplicity we also fix $q = 1$ and $V_0 = 1$. Then the equation for the gauge fluctuations (4.6) in the $\omega = 0$ limit is reduced to

$$E'' + \frac{3(r^3 - 2d^2)}{2r(r^3 + d^2)}E' - \frac{15k^2 \sqrt{r}}{8(r^3 + d^2)} E = 0.$$  \hspace{1cm} (5.1)

In the near horizon region $r \to 0$, the perturbative solution of the above equation is given by

$$E = c_1 + c_2 r^4.$$ \hspace{1cm} (5.2)

Here, we choose the boundary condition $E = 0$ at the horizon, which is compatible to impose the incoming boundary condition for $\omega \neq 0$. So we set $c_1 = 0$. At the boundary $r = \infty$, the asymptotic solution of (5.1) becomes

$$E = A + Br^{-1/2}.$$ \hspace{1cm} (5.3)

Notice that since the first term is finite at the boundary, it corresponds to the source term and the coefficient of the second term implies the vev of the dual gauge operator. Then, the Green function is proportional to $-B/A$. In Figure 1, we plot the dependence of the Green’s function on the momentum $k$ for $d = 1$.

From the figure it can be easily seen that the characteristic structure does not appear in a wide range of $k$, while the specific heat and zero sound exhibit typical features of Fermi liquids. Such phenomenon was also observed in [33]. It was pointed out in [33] that
one difficulty in applying Landau’s theory of Fermi liquids was the assumption that the particle number should be conserved as the strength of the interaction was varied, which was not obvious for the case they discussed. Their conclusion seems to be still applicable to the present case.

6 AC conductivity

In this section we calculate the AC conductivity by making use of the correlation functions, which can be seen as a by-product of section 4. Furthermore, we will take the limit of zero density and compare the results with those appeared in previous literatures.

It can be easily seen that the current-current correlation function is given by

$$G^R_{xx}(\omega, k = 0) = \omega^2 \Pi(\omega, k = 0) \propto \frac{-\tau_{\text{eff}} C_1 \omega^2}{\delta_2 \omega^2 + \delta_3 G_0(\omega)}. \quad (6.1)$$

Therefore we can obtain the following expressions in the small frequency limit,

$$G^R_{xx}(\omega \to 0, k = 0) \propto -\tau_{\text{eff}} C_1 \delta_3^{-1} \omega^{\frac{n}{\tau}} \quad n > 0, \quad (6.2)$$

$$G^R_{xx}(\omega \to 0, k = 0) \propto \frac{\tau_{\text{eff}} C_1 \omega^2}{\delta_3 \log(\delta \omega^2)} \quad n = 0. \quad (6.3)$$
Recalling that the definition of AC conductivity is given by

$$\sigma(\omega) = -\frac{i}{\omega} G_{xx}^R(\omega \to 0, k = 0),$$  \hspace{1cm} (6.5)$$

we can arrive at the following results

$$\sigma(\omega) \propto i\tau_{\text{eff}} C_1 \delta_2^{-1} \omega^{\frac{n}{3}+1}, \quad n > 0,$$  \hspace{1cm} (6.6)$$

$$\sigma(\omega) \propto i\tau_{\text{eff}} C_1 \delta_3^{-1} \omega^{-1}(\log(\delta\omega^2))^{-1}, \quad n = 0,$$  \hspace{1cm} (6.7)$$

$$\sigma(\omega) \propto i\tau_{\text{eff}} C_1 \delta_2^{-1} \omega^{-1}, \quad n < 0.$$  \hspace{1cm} (6.8)$$

As a check of consistency, we consider the specific limit, $\beta = 2, \gamma = 2/z, n = 1-2/z$. When $n > 0, z > 2$, $\sigma(\omega) \propto i\tau_{\text{eff}} C_1 \delta_2^{-1} \omega^{-2/z}$, which agrees with the result obtained in [36]. The behavior of the AC conductivity is qualitatively similar to that investigated in [36]. To be concrete, when $n \neq 0$ $\delta_2$ is real while $\delta_3$ is complex. The conductivity is purely imaginary when $n < 0$ and has a simple pole at zero frequency. Then according to Kramers-Kronig relation, one can conclude that the real part of the conductivity, and hence the spectral function, consists of a delta function at zero frequency. When $n > 0$ the conductivity, and hence the spectral function has a power-law dependence, and no explicit quasi-particle excitation exists.

On the other hand, it was observed in [52] that after transforming the equation of $a_x$ into a Schrödinger-like form, the AC conductivity was directly related to the reflection amplitude for scattering off the potential. For our system, it can be observed that the equation for $a_x$ at $k = 0$ becomes

$$\partial_r [\frac{e^{-\phi} g_{2x}^{-2/3} |g_{tt}| a_x'}{|g_{tt}| g_{rr} - A_t^2}^{1/2}] + \frac{e^{-\phi} g_{2x}^{-1}}{\sqrt{|g_{tt}| g_{rr} - A_t^2}} \omega^2 a_x = 0$$  \hspace{1cm} (6.9)$$

in the original coordinate,

$$ds^2 = -C_1 r^\beta dt^2 + \frac{dr^2}{C_1 r^\beta} + C_2 r^\gamma \sum_{i=1}^{D-2} dx_i^2.$$
The asymptotic behavior of \( a_x \) is given by
\[
a_x(\omega) = \frac{E_x(\omega)}{i\omega} + \frac{J_x(\omega)}{\tau_{\text{eff}}(\gamma - \beta - m + 1)} r^{\gamma - \beta - m + 1},
\]
where we have introduced a background electric field \( E_x(t) \equiv \text{Re}E_x(\omega)e^{-i\omega t} \). The above equation can be put in a Schrödinger-like form,
\[
-\frac{d^2\Psi}{ds^2} + U(s)\Psi = \Omega^2\Psi,
\]
where
\[
a_x = \frac{r^{\gamma/2}}{\lambda^{1/2}}\Psi, \quad \lambda = \sqrt{r^{2m} + d^2}, \quad \frac{d}{dr} = r^{-\beta} \frac{d}{ds}, \quad \Omega = \frac{\omega}{C_1},
\]
and
\[
U(s) = r^{\gamma/2} \lambda^{-1/2} \left[ \frac{1}{2} \frac{d}{ds} (r^{-\gamma/2} \lambda^{-1/2} \frac{d\lambda}{ds}) - \frac{d}{ds} \left( \frac{\gamma}{2} r^{-\gamma/2 - 1} \lambda^{1/2} \frac{dr}{ds} \right) \right].
\]
Generally speaking, the AC conductivity can be obtained by numerical methods. However, in the zero density limit
\[
d = 0, \quad \lambda = r^m,
\]
one can find
\[
U(s) = \frac{U_0}{s^2}, \quad U_0 = \frac{(m - \gamma)(m - \gamma + 2\beta - 2)}{4(\beta - 1)^2},
\]
so the potential possesses the same form as those investigated in \cite{17} and \cite{20}. Assuming that the solution asymptotes to \( AdS_D \), we can obtain
\[
\text{Re}(\sigma) \sim \omega^\kappa, \quad \kappa = 2\nu_0 - 1, \quad \nu_0^2 = U_0 + \frac{1}{4}
\]
following their approach. In particular, when \( q = 2, C_2 = 0 \), the potential vanishes and the conductivity is constant at all temperatures
\[
\sigma(\omega) = \tau_{\text{eff}} \equiv \sigma_0,
\]
which agrees with the analysis performed in \cite{53}.

7 Summary and discussion

In this paper we explore the zero sound in \( D \)-dimensional effective holographic theories, whose bulk fields include the graviton \( g_{\mu\nu} \), the \( U(1) \) gauge field \( A_\mu \), and the scalar field
The solutions possess anisotropic scaling symmetry and they reduce to previously known examples, such as $AdS_D$, $AdS_2 \times R^{D-2}$ and “modified” Lifshitz solutions under certain conditions. We consider thermodynamics of massless probe D-branes in the near-extremal background and clarify the conditions under which the specific heat is linear in the temperature, which is a characteristic feature of Fermi liquids. Subsequently we study the zero sound mode by considering the fluctuations of the worldvolume gauge fields on the probe D-branes. Rather than analytically solving the equations of motion, we obtain the low-frequency behavior by solving the equation in two different limits and then matching the two solutions in a regime where the limits overlap, following [31] and [36]. The resulting behavior of the zero sound looks similar to that investigated in [36], that is, when the parameter $n \equiv \beta - \gamma - 1 < 0$, the dispersion relation reveals a quasi-particle excitation; while the zero sound is not a well-defined quasi-particle when $n \geq 0$. Furthermore, we plot the correlation function in $D = 4$ at $\omega = 0$ with fixed parameters which lead to linear specific heat. The result is that we cannot observe any characteristic structure of Fermi liquids in a wide range of $k$, which is similar to what was found in [33].

As a by-product, we also evaluate the AC conductivity via the current-current correlation function, which reduces to previously known results at specific limits.

By now there are mainly two approaches for studying condensed matter physics in the context of holography. One approach can be thought of as “top-down”, that is, we consider certain exact solutions or brane configurations in string/M theory which possess the desired properties of condensed matter systems. The main advantage is that we have clear understanding about the dual field theories, while it is difficult to find such solutions in string/M theory. A complementary approach can be seen as “bottom-up”, which means that we consider certain toy models of gravity which possess solutions with the desired properties. It allows a parametrization of large classes of IR dynamics and provides useful information in the dual field theory side. However, the main disadvantage is that the embeddings of such toy models into string/M theory are not obvious, thus many things in the field theory side remain unknown. However, the “bottom-up” approach is an efficient tool for investigating the AdS/CMT correspondence.

The background solutions we studied in this paper are exact solutions of theories with domain wall vacua. In [37] the author also constructed interpolates between two exact solutions of the single-exponential, domain wall gravity theory, which lead to the argu-
ment that the domain wall/QFT correspondence [54] can be considered as an effective holographic tool which is applicable in settings beyond the regime of domain wall supergravities. Therefore the domain wall/QFT correspondence can also be taken as one specific class of effective holographic theories. Moreover, the author of [37] argued that even when the UV completion of some bulk theory was unknown, if the theory admitted an approximate domain wall solution at some intermediate value of \( r \) then one could use domain wall/QFT correspondence to develop a holographic map. Thus it is interesting to develop the AdS/CMT holography by making use of this domain wall/QFT correspondence. In particular, we can establish precision holography in this anisotropic background along the line of [55] and study the fermionic correlation functions following [56, 57, 58]. We leave such fascinating projects in the future.

**Acknowledgments**

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) through the Center for Quantum Spacetime(CQUeST) of Sogang University with grant number 2005-0049409.

**A Asymptotic expansions of \( E(z) \) at specific values**

In this appendix we show that when the parameter \( m \) takes some specific values, the large \( z \) expansions of the solutions \( E(z) \) to equation (4.21) cannot match that of (4.17). Firstly, when \( m = 1 \) but \( m \neq -n \), the solution to (4.21) is given by

\[
E(z) = D_1 + D_2[C_1 k^2 \left( \frac{1}{\sqrt{1 + d^2 z^2}} + \log z - \log 2(1 + \sqrt{1 + d^2 z^2}) \right) + \omega^2 \left( \frac{z^{\beta - \gamma}}{\beta - \gamma^2} F_1 \left( \frac{1}{2}, \frac{\beta - \gamma}{2}; 1 + \frac{\beta - \gamma}{2}, -d^2 z^2 \right) - \frac{d^2 z^{2 + \beta - \gamma}}{2 + \beta - \gamma^2} F_1 \left( \frac{1}{2}, 1 + \frac{\beta - \gamma}{2}; 2 + \frac{\beta - \gamma}{2}, -d^2 z^2 \right) \right)].
\]  

(A.1)

When performing the expansion, we just focus on the powers of \( z \). The Hypergeometric function gives

\[
\sim \text{const} + z^{\beta - \gamma + 1} + z^{-\gamma},
\]  

(A.2)

which does not match that of (4.17).
Secondly, when \( m \neq 1 \) but \( m = -n \neq 0 \), the solution is
\[
E(z) = D_1 + D_2 \left[ C_1 k^2 \frac{z^{m-1}}{m} \left( \frac{1}{1 + d^2 z^{2m}} + \frac{1}{m - 1} F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{2m}, -d^2 z^{2m} \right) \right) \right. \\
\left. + \omega^2 \left( \frac{1}{1 - \beta + \gamma} + (1 - \beta + \gamma) \log z - \log(2 + 2 \sqrt{1 + d^2 z^{2m} - 2\beta}) \right) \right].
\] (A.3)

The \( \omega^2 \) term gives
\[
\sim \frac{1}{1 - \beta + \gamma} - \log 2d,
\] (A.4)

which does not match that of (4.21) either. Therefore we just considered the cases with \( m \neq 1 \) and \( m \neq \gamma - \beta + 1 \) in the main text.

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].
E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[3] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” Class. Quant. Grav. 26, 224002 (2009) [arXiv:0903.3246 [hep-th]].
C. P. Herzog, “Lectures on Holographic Superfluidity and Superconductivity,” J. Phys. A 42, 343001 (2009) [arXiv:0904.1975 [hep-th]].
J. McGreevy, “Holographic duality with a view toward many-body physics,” arXiv:0909.0518 [hep-th].
G. T. Horowitz, “Introduction to Holographic Superconductors,” arXiv:1002.1722 [hep-th].
S. Sachdev, “Condensed matter and AdS/CFT,” arXiv:1002.2947 [hep-th].
[4] S. S. Lee, “A Non-Fermi Liquid from a Charged Black Hole: A Critical Fermi Ball,” Phys. Rev. D 79, 086006 (2009) [arXiv:0809.3402 [hep-th]].

[5] H. Liu, J. McGreevy and D. Vegh, “Non-Fermi liquids from holography,” arXiv:0903.2477 [hep-th].

[6] M. Cubrovic, J. Zaanen and K. Schalm, “String Theory, Quantum Phase Transitions and the Emergent Fermi-Liquid,” Science 325, 439 (2009) [arXiv:0904.1993 [hep-th]].

[7] T. Faulkner, H. Liu, J. McGreevy and D. Vegh, “Emergent quantum criticality, Fermi surfaces, and AdS2,” arXiv:0907.2694 [hep-th].

[8] S. S. Gubser, “Breaking an Abelian gauge symmetry near a black hole horizon,” Phys. Rev. D 78, 065034 (2008) [arXiv:0801.2977 [hep-th]].

[9] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, “Building a Holographic Superconductor,” Phys. Rev. Lett. 101, 031601 (2008) [arXiv:0803.3295 [hep-th]].

[10] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, “Holographic Superconductors,” JHEP 0812, 015 (2008) [arXiv:0810.1563 [hep-th]].

[11] U. Gursoy and E. Kiritsis, “Exploring improved holographic theories for QCD: Part I,” JHEP 0802, 032 (2008) [arXiv:0707.1324 [hep-th]].

[12] U. Gursoy, E. Kiritsis and F. Nitti, “Exploring improved holographic theories for QCD: Part II,” JHEP 0802, 019 (2008) [arXiv:0707.1349 [hep-th]].

[13] S. S. Gubser, A. Nellore, S. S. Pufu and F. D. Rocha, “Thermodynamics and bulk viscosity of approximate black hole duals to finite temperature quantum chromodynamics,” Phys. Rev. Lett. 101, 131601 (2008) [arXiv:0804.1950 [hep-th]].

[14] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Deconfinement and Gluon Plasma Dynamics in Improved Holographic QCD,” Phys. Rev. Lett. 101, 181601 (2008) [arXiv:0804.0899 [hep-th]].

[15] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Holography and Thermodynamics of 5D Dilaton-gravity,” JHEP 0905, 033 (2009) [arXiv:0812.0792 [hep-th]].
[16] S. S. Gubser and F. D. Rocha, “Peculiar properties of a charged dilatonic black hole in $AdS_5$,” Phys. Rev. D 81, 046001 (2010) [arXiv:0911.2898 [hep-th]].

[17] K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, “Holography of Charged Dilaton Black Holes,” JHEP 1008, 078 (2010) arXiv:0911.3586 [hep-th].

[18] J. Gauntlett, J. Sonner and T. Wiseman, “Quantum Criticality and Holographic Superconductors in M-theory,” JHEP 1002, 060 (2010) [arXiv:0912.0512 [hep-th]].

[19] M. Cadoni, G. D’Appollonio and P. Pani, “Phase transitions between Reissner-Nordstrom and dilatonic black holes in 4D AdS spacetime,” JHEP 1003, 100 (2010) arXiv:0912.3520 [hep-th].

[20] C. M. Chen and D. W. Pang, “Holography of Charged Dilaton Black Holes in General Dimensions,” JHEP 1006, 093 (2010) arXiv:1003.5064 [hep-th].

[21] K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi and A. Westphal, “Holography of Dyonic Dilaton Black Branes,” arXiv:1007.2490 [hep-th].

[22] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, “Effective Holographic Theories for low-temperature condensed matter systems,” arXiv:1005.4690 [hep-th].

[23] B. H. Lee, S. Nam, D. W. Pang and C. Park, “Conductivity in the anisotropic background,” arXiv: 1006.0779 [hep-th].

[24] B. H. Lee, D. W. Pang and C. Park, “Strange Metallic Behavior in Anisotropic Background,” JHEP 1007, 057 (2010) [arXiv:1006.1719 [hep-th]].

[25] Y. Liu and Y. W. Sun, “Holographic Superconductors from Einstein-Maxwell-Dilaton Gravity,” JHEP 1007, 099 (2010) [arXiv:1006.2726 [hep-th]].

[26] U. Gursoy, “Gravity/Spin-model correspondence and holographic superfluids,” arXiv:1007.4854 [hep-th].

[27] T. Faulkner, G. T. Horowitz and M. M. Roberts, “Holographic quantum criticality from multi-trace deformations,” arXiv:1008.1581 [hep-th].
[28] A. Bayntun, C. P. Burgess, B. P. Dolan and S. S. Lee, “AdS/QHE: Towards a Holographic Description of Quantum Hall Experiments,” arXiv:1008.1917 [hep-th].

[29] M. Ali-Akbari and K. B. Fadafan, “Conductivity at finite ’t Hooft coupling from AdS/CFT,” arXiv:1008.2430 [hep-th].

[30] D. Astefanesei, N. Banerjee and S. Dutta, “Moduli and electromagnetic black brane holography,” arXiv:1008.3852 [hep-th].

[31] A. Karch, D. T. Son and A. O. Starinets, “Zero Sound from Holography,” arXiv:0806.3796 [hep-th].

[32] M. Kulaxizi and A. Parnachev, “Comments on Fermi Liquid from Holography,” Phys. Rev. D 78, 086004 (2008) [arXiv:0808.3953 [hep-th]].

[33] M. Kulaxizi and A. Parnachev, “Holographic Responses of Fermion Matter,” Nucl. Phys. B 815, 125 (2009) [arXiv:0811.2262 [hep-th]].

[34] A. Karch, M. Kulaxizi and A. Parnachev, “Notes on Properties of Holographic Matter,” JHEP 0911, 017 (2009) [arXiv:0908.3493 [hep-th]].

[35] B. H. Lee and D. W. Pang, “Notes on Properties of Holographic Strange Metals,” arXiv:1006.4915 [hep-th].

[36] C. Hoyos-Badajoz, A. O’Bannon and J. M. S. Wu, “Zero Sound in Strange Metallic Holography,” arXiv:1007.0590 [hep-th].

[37] E. Perlmutter, “Domain Wall Holography for Finite Temperature Scaling Solutions,” arXiv:1006.2124 [hep-th].

[38] G. W. Gibbons and K. i. Maeda, “Black Holes And Membranes In Higher Dimensional Theories With Dilaton Fields,” Nucl. Phys. B 298, 741 (1988).

[39] J. Preskill, P. Schwarz, A. D. Shapere, S. Trivedi and F. Wilczek, “Limitations on the statistical description of black holes,” Mod. Phys. Lett. A 6, 2353 (1991).

[40] D. Garfinkle, G. T. Horowitz and A. Strominger, “Charged black holes in string theory,” Phys. Rev. D 43, 3140 (1991) [Erratum-ibid. D 45, 3888 (1992)].
[41] C. F. E. Holzhey and F. Wilczek, “Black holes as elementary particles,” Nucl. Phys. B 380, 447 (1992) [arXiv:hep-th/9202014].

[42] K. C. C. Chan, J. H. Horne and R. B. Mann, “Charged Dilaton Black Holes with Unusual Asymptotics,” Nucl. Phys. B 447, 441 (1995) [arXiv:gr-qc/9502042].
R. G. Cai and Y. Z. Zhang, “Black plane solutions in four-dimensional spacetimes,” Phys. Rev. D 54, 4891 (1996) [arXiv:gr-qc/9609065].
R. G. Cai, J. Y. Ji and K. S. Soh, “Topological dilaton black holes,” Phys. Rev. D 57, 6547 (1998) [arXiv:gr-qc/9708063].
C. Charmousis, B. Gouteraux and J. Soda, “Einstein-Maxwell-Dilaton theories with a Liouville potential,” Phys. Rev. D 80, 024028 (2009) [arXiv:0905.3337 [gr-qc]].

[43] S. Kachru, X. Liu and M. Mulligan, “Gravity Duals of Lifshitz-like Fixed Points,” Phys. Rev. D 78, 106005 (2008) [arXiv:0808.1725 [hep-th]].

[44] M. Taylor, “Non-relativistic holography,” arXiv:0812.0530 [hep-th].

[45] A. Karch and E. Katz, “Adding flavor to AdS/CFT,” JHEP 0206, 043 (2002) [arXiv:hep-th/0205236].

[46] S. Kobayashi, D. Mateos, S. Matsuura, R. C. Myers and R. M. Thomson, “Holographic phase transitions at finite baryon density,” JHEP 0702, 016 (2007) [arXiv:hep-th/0611099].

[47] A. Karch and A. O’Bannon, “Metallic AdS/CFT,” JHEP 0709, 024 (2007) [arXiv:0705.3870 [hep-th]].

[48] A. O’Bannon, “Hall Conductivity of Flavor Fields from AdS/CFT,” Phys. Rev. D 76, 086007 (2007) [arXiv:0708.1994 [hep-th]].

[49] S. A. Hartnoll, J. Polchinski, E. Silverstein and D. Tong, “Towards strange metallic holography,” JHEP 1004, 120 (2010) [arXiv:0912.1061 [hep-th]].

[50] A. Karch and A. O’Bannon, “Holographic Thermodynamics at Finite Baryon Density: Some Exact Results,” JHEP 0711, 074 (2007) [arXiv:0709.0570 [hep-th]].

[51] D. T. Son and A. O. Starinets, “Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications,” JHEP 0209, 042 (2002) [arXiv:hep-th/0205051].
[52] G. T. Horowitz and M. M. Roberts, “Zero Temperature Limit of Holographic Superconductors,” JHEP 0911, 015 (2009) [arXiv:0908.3677 [hep-th]].

[53] C. P. Herzog, P. Kovtun, S. Sachdev and D. T. Son, “Quantum critical transport, duality, and M-theory,” Phys. Rev. D 75, 085020 (2007) [arXiv:hep-th/0701036].

[54] H. J. Boonstra, K. Skenderis and P. K. Townsend, “The domain wall/QFT correspondence,” JHEP 9901, 003 (1999) [arXiv:hep-th/9807137].

[55] I. Kanitscheider, K. Skenderis and M. Taylor, “Precision holography for non-conformal branes,” JHEP 0809, 094 (2008) [arXiv:0807.3324 [hep-th]].

[56] S. S. Gubser, F. D. Rocha and P. Talavera, “Normalizable fermion modes in a holographic superconductor,” arXiv:0911.3632 [hep-th].

[57] T. Faulkner, G. T. Horowitz, J. McGreevy, M. M. Roberts and D. Vegh, “Photoemission 'experiments' on holographic superconductors,” JHEP 1003, 121 (2010) [arXiv:0911.3402 [hep-th]].

[58] T. Faulkner, N. Iqbal, H. Liu, J. McGreevy and D. Vegh, “From black holes to strange metals,” arXiv:1003.1728 [hep-th].