BOUNDS ON MULTIGRADED REGULARITY

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Abstract. We explore the asymptotic behavior of the multigraded Castelnuovo–Mumford regularity of powers of ideals. Specifically, if $I$ is an ideal in the total coordinate ring $S$ of a smooth projective toric variety $X$, we bound the region $\text{reg}(I^n) \subseteq \text{Pic}(X)$ by proving that it contains a translate of $\text{reg}(S)$ and is contained in a translate of $\text{Nef}(X)$. Each bound translates by a fixed vector as $n$ increases. Along the way we prove that the multigraded regularity of a finitely generated torsion-free module is contained in a translate of $\text{Nef}(X)$ determined by the degrees of the generators of $M$, and thus contains only finitely many minimal elements.

1. Introduction

Building on the work of Swanson in [Swa97], Cutkosky–Herzog–Trung in [CHT99] and Kodiyalam in [Kod00] described the surprisingly predictable asymptotic behavior of Castelnuovo–Mumford regularity for powers of ideals on a projective space $\mathbb{P}^r$: given an ideal $I \subset \mathbb{K}[x_0, \ldots, x_r]$, there exist $d, e \in \mathbb{Z}$ such that for $n \gg 0$ the regularity of $I^n$ satisfies

$$\text{reg}(I^n) = dn + e.$$  

Due to the importance of regularity as a measure of complexity for syzygies and its geometric interpretation in terms of the cohomology of coherent sheaves [BEL91, CEL01], this phenomenon has received substantial attention [GGP95, Cha97, SS97, Röm01, TW05, BCH13], focused mostly on projective spaces. See [Cha13] for a survey.

Motivated by toric geometry, we turn our focus toward ideals in the multigraded total coordinate ring $S$ of a smooth projective toric variety $X$, for which a generalized notion of regularity was introduced by Maclagan and Smith [MS04]. In this setting the regularity of a $\text{Pic}(X)$-graded module is a subset of $\text{Pic}(X)$ that is closed under the addition of nef divisors. A natural question is thus whether there is an analogous description for the asymptotic shape of $\text{reg}(I^n) \subseteq \text{Pic}(X)$.

In Theorem 4.1 we bound multigraded regularity by establishing regions “inside” and “outside” of $\text{reg}(I^n)$ which translate linearly by a fixed vector as $n$ increases (see the figure in Example 4.2). The inner bound depends on the Betti numbers of the Rees ring $S[It]$, while the outer bound depends only on the degrees of the generators of $I$.

Theorem 4.1. There exists a degree $a \in \text{Pic}(X)$, depending only on $I$, such that for each integer $n > 0$ and each pair of degrees $\mathbf{q}_1, \mathbf{q}_2 \in \text{Pic}(X)$ satisfying $\mathbf{q}_1 \geq \deg f_i \geq \mathbf{q}_2$ for all generators $f_i$ of $I$, we have

$$n\mathbf{q}_1 + a + \text{reg}(S) \subseteq \text{reg}(I^n) \subseteq n\mathbf{q}_2 + \text{Nef}(X).$$

It is worth emphasizing that our result holds over smooth projective toric varieties with arbitrary Picard rank. Indeed, toric varieties of higher Picard rank introduce a wrinkle that is not present in existing asymptotic results on Castelnuovo–Mumford regularity: in general there are infinitely many possible regularity regions compatible with two given bounds. (In contrast, when $\text{Pic}(X) = \mathbb{Z}$, inner and outer bounds correspond to upper and lower bounds, respectively, with only finitely many integers between each pair.) Nevertheless, since multigraded regularity

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is invariant under positive translation by $\text{Nef} \, X$, an outer bound in the shape of the nef cone cannot contain an infinite expanding chain of regularity regions.

Surprisingly, we will see in Example 3.2 that even on a Hirzebruch surface $X$ the regularity of a finitely generated module may not be contained in the union of finitely many translates of $\text{Nef} \, X$. In the case of powers of ideals, however, the absence of torsion over $S$ implies that the regularity has finitely many minimal elements. More generally, in Theorem 3.11 we construct a nef-shaped outer bound determined by the degrees of generators of a torsion-free module (see the figure in Example 3.13). We use the idea that if the truncation $M_{\geq d}$ is not generated in a single degree $d$ then $M$ is not $d$-regular (see Theorem 3.3 for a simpler case).

**Theorem 3.11.** Let $M$ be a finitely generated graded torsion-free $S$-module with $\tilde{M} \neq 0$. Then $\text{reg} \, M$ is contained in a translate of $\text{Nef} \, X$. In particular, $\text{reg} \, M$ has finitely many minimal elements.

It remains an interesting problem to characterize modules with torsion whose regularity is contained in a translate of $\text{Nef} \, X$. Note that the regularity of a finitely generated module is always contained in a translate of $\text{Eff} \, X$ (see Proposition 3.7). In fact, the existence of a module whose regularity contains infinitely many minimal elements is a consequence of the difference between the effective and nef cones of $X$. This possibility highlights a theme from [BCHS21, BKLY22] that algebraic properties which coincide over projective spaces can diverge in higher Picard rank.

**Outline.** The organization of the paper is as follows: Section 2 introduces background results and our notation. Section 3 shows that the multigraded regularity of $S$ lies inside $\text{Nef} \, X$, in Theorem 3.3, and that the multigraded regularity of a finitely generated torsion-free $S$-module is contained in an appropriate translate of $\text{Nef} \, X$, in Theorem 3.11. Section 4 gives explicit inner and outer bounds for the multigraded regularity of powers of an ideal, in Theorem 4.1.

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2. Notation and Definitions

Throughout we work over a base field $\mathbb{K}$ and denote by $\mathbb{N}$ the set of non-negative integers. Let $X$ be a smooth projective toric variety determined by a fan. The total coordinate ring of $X$ is a $\text{Pic}(X)$-graded polynomial ring $S$ over $\mathbb{K}$ with an irrelevant ideal $B \subset S$. Write $\text{Eff} \, X$ for the monoid in $\text{Pic} \, X$ generated by the degrees of the variables in $S$.

Fix minimal generators $C = (c_1, \ldots, c_r)$ for the monoid $\text{Nef} \, X$ of classes in $\text{Pic} \, X$ represented by numerically effective divisors. For $\lambda \in \mathbb{Z}^r$, write $\lambda \cdot C$ to represent the linear combination $\lambda_1 c_1 + \cdots + \lambda_r c_r \in \text{Pic} \, X$, and similarly for other tuples in $\text{Pic} \, X$. Write $|\lambda|$ for the sum $\lambda_1 + \cdots + \lambda_r$.

We use a partial order on $\text{Pic} \, X$ induced by $\text{Nef} \, X$: given $a, b \in \text{Pic} \, X$, we write $a \leq b$ when $b - a \in \text{Nef} \, X$.

**Example 2.1.** The Hirzebruch surface $\mathcal{H}_t = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(t))$ is a smooth projective toric variety whose associated fan, shown left in Figure 1, has rays $(1, 0), (0, 1), (-1, t)$, and $(0, -1)$. For each ray there is a corresponding prime torus-invariant divisor. In particular, the total coordinate ring
of \( \mathcal{H}_t \) is the polynomial ring \( S = \mathbb{K}[x_0, x_1, x_2, x_3] \) and its irrelevant ideal is \( B = \langle x_0, x_2 \rangle \cap \langle x_1, x_3 \rangle \).

![Diagram showing fans and cones]

**Figure 1.** Left: fan of \( \mathcal{H}_2 \). Right: the cones Nef \( \mathcal{H}_2 \) (dark blue) and Eff \( \mathcal{H}_2 \) (blue).

Choosing a basis for \( \text{Pic} \mathcal{H}_t \cong \mathbb{Z}^2 \), the grading on \( S \) can be given as \( \deg x_0 = \deg x_2 = (1,0) \), \( \deg x_1 = (-1,1) \), and \( \deg x_3 = (0,1) \). The effective and nef cones are illustrated on the right.

For a Pic\((X)\)-graded \( S \)-module \( M \) and \( d \in \text{Pic} X \), denote by \( M \geq d \) the submodule of \( M \) generated by all elements of degrees \( d' \) satisfying \( d' \geq d \) (c.f. [MS04, Def. 5.1]). Denote by \( M \) the quasi-coherent sheaf on \( X \) associated to \( M \), as in [Cox95, §3].

We now recall the notion of multigraded Castelnuovo–Mumford regularity introduced by Maclagan and Smith.

**Definition 2.2** (c.f. [MS04, Def. 1.1]). Let \( M \) be a graded \( S \)-module. For \( d \in \text{Pic} X \), we say \( M \) is \( d \)-regular if the following hold:

1. \( H^i_B(M)_b = 0 \) for all \( i > 0 \) and all \( b \in \bigcup_{|\lambda|=i-1} (d - \lambda \cdot C + \text{Nef} \ X) \) where \( \lambda \in \mathbb{N}^r \).
2. \( H^i_B(M)_b = 0 \) for all \( b \in \bigcup_j (d + c_j + \text{Nef} \ X) \).

We write \( \text{reg} M \) for the set of \( d \) such that \( M \) is \( d \)-regular.

### 3. Finite Generation of Multigraded Regularity

We begin by constructing an outer bound for the regularity of \( I^n \)—a subset of Pic \( X \) that contains \( \text{reg} (I^n) \). In [Kod00], Kodiyalam constructs this from a bound on the degrees of the generators of \( I^n \). However, more nuanced behavior can occur in the multigraded setting. The following example shows that the degree of a minimal generator of an ideal does not bound its regularity on an arbitrary toric variety.

**Example 3.1.** Let \( I = \langle x_0 x_3, x_0 x_2, x_1 x_2 \rangle \) be an ideal in the total coordinate ring of the Hirzebruch surface \( \mathcal{H}_t \), with notation as in Example 2.1. A local cohomology computation verifies that \( I \) is \((1,1)\)-regular. However \( x_0 x_2 \) is a minimal generator with \( \deg(x_0 x_2) = (2,0) \not\leq (1,1) \).

The existence of a similar example with \( H^i_B(M) \neq 0 \) was noted by Maclagan and Smith, who asked whether \( B \)-torsion was necessary in [MS04, §5]. Example 3.1 shows that it is not.

Perhaps more unexpectedly, it is also possible for the regularity of a finitely generated module to have infinitely many minimal elements with respect to \( \text{Nef} \ X \), as is the case in the following simple example pointed out by Daniel Erman.

**Example 3.2.** Let \( M = S/\langle x_2, x_3 \rangle \) be the coordinate ring of a single point on \( \mathcal{H}_t \) (see Example 2.1). Since \( \langle x_2, x_3 \rangle \) is saturated we have \( H^0_B(M) = 0 \). Furthermore, since the support of \( \widetilde{M} \) has dimension 0 we must have \( H^i_B(M) = 0 \) for \( i \geq 2 \). Thus \( \text{reg} M \) is determined entirely by \( H^1_B(M) \), which vanishes exactly where the Hilbert function of \( M \) agrees with its Hilbert polynomial.
The Hilbert function of \( M \) is equal to 1 inside \( \text{Eff} \mathcal{H}_t \) and 0 outside of it. Hence \( \text{reg} M = \text{Eff} \mathcal{H}_t \). When \( t > 0 \) this cone does not contain finitely many minimal elements with respect to \( \text{Nef} X \), as illustrated in Figure 2.

**Figure 2.** The multigraded regularity of \( M \) (green) is an infinite staircase contained in a translate of the effective cone of \( \mathcal{H}_2 \) (blue).

The regularity of the module in Example 3.2 is contained in a translate of \( \text{Eff} X \), which does give an outer bound. We will see in Proposition 3.7 that this is true for all \( M \). At the same time many modules, for instance \( S/\langle x_0, x_1 \rangle \), do have regularity regions contained in translates of \( \text{Nef} X \). Thus an outer bound in the shape of \( \text{Eff} X \) would not be tight in general. In particular, we will see in Corollary 3.12 that an outer bound in the shape of \( \text{Nef} X \) exists for an ideal \( I \subseteq S \) and thus \( \text{reg} I \) has finitely many minimal elements. We begin with the case \( I = S \).

### 3.1. Regularity of the Coordinate Ring

In this section we show that the pathology seen in Example 3.2—a regularity region contained in no translate of \( \text{Nef} X \)—does not occur for the total coordinate ring of a smooth projective toric variety. In particular we show that \( \text{reg} S \subseteq \text{Nef} X \).

In [MS04, Prob. 6.12], Maclagan and Smith asked for a combinatorial characterization of toric varieties \( X \) such that \( \text{Nef} X \subseteq \text{reg} S \). Theorem 3.3 below shows that when \( X \) is smooth and projective, \( \text{Nef} X \subseteq \text{reg} S \) is in fact equivalent to the a priori stronger condition that \( \text{reg} S = \text{Nef} X \). It still remains an interesting question to characterize such toric varieties. For instance, the only Hirzebruch surface with this property is \( \mathcal{H}_1 \).

**Theorem 3.3.** Using the notation from Section 2, we have \( \text{reg} S \subseteq \text{Nef} X \). In particular, \( \text{reg} S \) contains finitely many minimal elements.

**Proof.** Take \( d \in \text{reg} S \). By [MS04, Thm. 5.4] the truncation \( S_{\geq d} \) is generated by the monomials of \( S_d \), so there is a surjection \( S_d \otimes \mathbb{K} S \to S_{\geq d}(d) \) which sheafifies to a surjection \( S_d \otimes \mathcal{O} \to \mathcal{O}(d) \). Hence \( \mathcal{O}(d) \) is generated by global sections, so by [CLS11, Thm. 6.3.11] \( d \) is nef.

An application of Dickson’s lemma (e.g. [CLO15, §2.4 Thm. 5]), suggested by Will Sawin [Saw], shows that \( \text{reg} S \) has finitely many minimal elements, finishing the proof.

**Lemma 3.4.** A subset \( V \subseteq \text{Nef} X \) contains finitely many minimal elements with respect to \( \leq \) on \( \text{Pic} X \).

Elements of \( V \) can be written as linear combinations \( \lambda \cdot C \) of the monoid generators of \( \text{Nef} X \). The minimal elements of \( V \) must have coefficients \( \lambda \in \mathbb{N}^r \) that are minimal in the component-wise partial order on \( \mathbb{N}^r \). By Dickson’s lemma only finitely many possible coefficients exist. \( \square \)

**Example 3.5.** The multigraded regularity of the coordinate ring of the Hirzebruch surface \( \mathcal{H}_2 \) is contained in the nef cone of \( \mathcal{H}_2 \), as illustrated in Figure 3.

Though we do not directly use Theorem 3.3 in the next section, we do rely on the idea of the proof. For an arbitrary module \( M \), if \( d \in \text{reg} M \) then the truncation \( M_{\geq d} \) is generated in a single degree \( d \), meaning that \( \tilde{M}(d) \) is globally generated. This no longer immediately implies that \( d \)
is nef, but Lemma 3.6 below connects the difference between $d$ and the degrees of the generators of $M$ to monomials in truncations of $S$ itself.

We also use the chamber complex of the rays of Eff $X$, which is described in [MS04, §2]. By definition, this chamber complex is the coarsest fan with support Eff $X$ which refines all triangulations of the degrees of the variables of $S$. It partitions Eff $X$ into cones that govern many geometric properties of Spec $S$, including its GIT quotients, birational geometry, and Hilbert polynomials (c.f. [CLS11, Ch. 14-15], [HKP06, §5]).

For our purposes we need only the existence of a strongly convex rational polyhedral fan that covers Eff $X$ and contains Nef $X$ as a cone. We will refer to the maximal cones as chambers and the codimension one cones as walls. In particular, Nef $X$ is a chamber.

**Lemma 3.6.** Let $\Gamma$ be a chamber of Eff $X$ other than Nef $X$, and let $a_1, \ldots, a_n \in \text{Pic } X$. If $a_i \in \Gamma \setminus \text{Nef } X$ for all $i$, then there exist monomials $m_i \in S_{\geq a_i}$ such that $\prod_i m_i$ is not generated by the monomials of $S_{\sum a_i}$.

**Proof.** Since $\Gamma$ and Nef $X$ intersect at most in a wall of $\Gamma$ and no $a_i$ lies in $\Gamma \cap \text{Nef } X$, their sum $b = \sum a_i$ must also be in $\Gamma \setminus \text{Nef } X$. Consider the multiplication maps

$$S_b \otimes \mathbb{R} S \xrightarrow{\varphi} S(b) \quad \text{and} \quad \bigotimes \mathbb{R} S_{\geq a_i}(a_i).$$

Suppose the proposition is false. Then the image of $\psi$ must be contained in the image of $\varphi$, else we could choose $(m_i) \in \bigotimes \mathbb{R} S_{\geq a_i}(a_i)$ with image not generated by the monomials of $S_b$. Note that each $S_{\geq a_i}(a_i)$ sheafifies to $\mathcal{O}(a_i)$, so sheafifying the entire diagram gives

$$S_b \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{O}(b) \quad \text{and} \quad \mathcal{O}(b).$$

In particular, the image of $\psi$ is still contained in the image of $\varphi$. Since $\psi$ sheafifies to an isomorphism, $\varphi$ sheafifies to a surjection. This implies $b \in \text{Nef } X$, which is a contradiction. $\square$

### 3.2. Regularity of Torsion-Free Modules

The goal of this section is to prove that the multi-graded regularity of an ideal $I \subseteq S$ has only finitely many minimal elements. We will prove this more generally for finitely generated torsion-free $S$-modules.

Proposition 3.7 shows that the regularity of an arbitrary finitely generated module is contained in some translate of Eff $X$. Under the stronger assumption that $M$ is torsion-free, Proposition 3.8 shows that we can also eliminate degrees that are in a translate of Eff $X$ but not Nef $X$.

**Proposition 3.7.** Let $M$ be a finitely generated graded $S$-module with $\tilde{M} \neq 0$. Suppose the degrees of all minimal generators of $M$ are contained in Eff $X$. Then $\text{reg } M \subseteq \text{Eff } X$. 

Figure 3. The regularity of $S$ (dark green) is contained in Nef $\mathcal{H}_2$ (dark blue).
Let $\mathbf{d} \in \text{reg } M$ and suppose for contradiction that $\mathbf{d} \not\in \text{Eff } X$. The degree $\mathbf{d}$ part $M_\mathbf{d}$ generates $M_{\geq \mathbf{d}}$ by [MS04, Thm. 5.4]. By hypothesis all elements of $M$ have degrees inside $\text{Eff } X$, so $M_\mathbf{d} = 0$ and thus $M_{\geq \mathbf{d}} = 0$. The modules $M$ and $M_{\geq \mathbf{d}}$ define the same sheaf by [MS04, Lem. 6.8], so $M_{\geq \mathbf{d}} = 0$ contradicts $\widetilde{M} \neq 0$. \hfill $\square$

**Proposition 3.8.** Let $M$ be a finitely generated graded torsion-free $S$-module with $\widetilde{M} \neq 0$. Suppose $\Gamma$ is a chamber of $\text{Eff } X \setminus \text{Nef } X$. If $\mathbf{d} = \deg f_i \in \Gamma \setminus \text{Nef } X$ for all generators $f_i$ of $M$, then $M$ is not $\mathbf{d}$-regular.

**Proof.** Assume on the contrary that $M$ is $\mathbf{d}$-regular. Let $a_i = \mathbf{d} - \deg f_i$ for each $i$. By choice of $\mathbf{d}$ we have $a_i \in \Gamma \setminus \text{Nef } X$. Hence by Lemma 3.6 there exist monomials $m_i \in S_{\geq a_i}$ such that $\prod_i m_i$ is not generated by the monomials of $S_{\sum a_i}$. Consider the elements $m_i f_i \in \widetilde{M}_\mathbf{d}$.

Since $M$ is $\mathbf{d}$-regular, the degree $\mathbf{d}$ part $M_\mathbf{d}$ generates $M_{\geq \mathbf{d}}$ by [MS04, Thm. 5.4]. Let $g_1, \ldots, g_s$ with $\deg g_j = \mathbf{d}$ be generators for $M_{\geq \mathbf{d}}$. Thus we must have relations

$$m_i f_i = \sum_j b_{i,j} g_j = \sum_j b_{i,j} \left( \sum_k a_{j,k} f_k \right) = \sum_k c_{i,k} f_k$$

for some $b_{i,j}, a_{j,k}, c_{i,k} \in S$ with $\deg b_{i,j} = \deg m_i - a_i$ and $\deg a_{j,k} = a_k$. These relations form a partial presentation matrix

$$A = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} - \begin{bmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & \cdots & c_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,n} & c_{2,n} & \cdots & c_{n,n} \end{bmatrix}$$

for $M$. In particular, $\det(A) \in \text{Fitt}_0 M \subseteq \text{ann } M$ by [Eis95, Prop. 20.7], so $\det(A) M = 0$.

Since there are no zerodivisors on a torsion-free $S$-module, we must have $\det(A) = 0$, but this is impossible: note that $\det(A)$ contains the monomial $m = \prod_i m_i$ and that $\det(A) \in \mathbf{d} + I$ for $I = \prod_k (c_{1,k}, c_{2,k}, \ldots, c_{n,k})$, then observe that $I \subseteq \prod_k (a_{1,k}, a_{2,k}, \ldots, a_{n,k}) \subseteq S \otimes_S S_{\sum a_k}$ since $\deg a_{j,k} = a_k$. Hence $\det(A) = 0$ implies $m \in I \subseteq S \otimes_S S_{\sum a_k}$ and contradicts our choice of $m_i$. \hfill $\square$

**Remark 3.9.** Example 3.2 shows that Theorem 3.11 is not true without the torsion-free hypothesis. In practice, however, we only need that the element $\det A$ from (1) is a nonzerodivisor on $M$ for some choice of $m_i$ as in Lemma 3.6. Given a specific toric variety, this may be possible to verify directly in some cases where $M$ is not torsion-free.

We will use the following technical lemma about the walls of $\text{Nef } X$ to find a vector satisfying the hypotheses of Proposition 3.8.

**Lemma 3.10.** Given $a_1, \ldots, a_n \in \text{Nef } X$ and $\mathbf{d} \in \text{Eff } X \setminus \text{Nef } X$, there exists a chamber $\Gamma$ sharing a wall $W$ with $\text{Nef } X$ and $\mathbf{w}$ in the relative interior of $W$ such that $\mathbf{d} + \mathbf{w} \in \Gamma$ and $\mathbf{d} + \mathbf{w} \in a_i + \Gamma$ for all $i$.

**Proof.** Consider the cone $P$ defined by all rays of $\text{Nef } X$ in addition to a primitive element along $\mathbf{d}$. Since $\text{Nef } X \subseteq P$, at least one wall $W$ of $\text{Nef } X$ must be in the interior of $P \subseteq \text{Eff } X$. Let $\Gamma$ be the chamber across $W$ from $\text{Nef } X$. Since $\mathbf{d} \not\in \text{Nef } X$, for each $\mathbf{w} \in W$ we have $\mathbf{d} + \mathbf{w} \not\in \text{Nef } X$.

Now consider the cone $Q$ defined by all supporting hyperplanes of $\text{Nef } X$ and $\Gamma$ except the hyperplane containing $W$. Since $W$ is in the intersection of the open half-spaces defining $Q$, it lies in the interior of $Q$. Therefore we can find $\mathbf{w}$ in the relative interior of $W \subset Q$ so that $\mathbf{d} + \mathbf{w} \in a_i + Q \subseteq a_i + (\Gamma \cup \text{Nef } X)$ for all $i$. By hypothesis $a_i + \text{Nef } X \subseteq \text{Nef } X$ so $\mathbf{d} + \mathbf{w} \not\in a_i + \text{Nef } X$. Hence $\mathbf{d} + \mathbf{w} \in a_i + \Gamma$ for all $i$. \hfill $\square$

**Theorem 3.11.** Let $M$ be a finitely generated graded torsion-free $S$-module with $\widetilde{M} \neq 0$. Suppose the degrees of all minimal generators of $M$ are contained in $\text{Nef } X$. Then $\text{reg } M \subseteq \text{Nef } X$. In particular, $\text{reg } M$ has finitely many minimal elements.
Proof. Suppose there exists $d \in \text{reg} M \setminus \text{Nef} X$. Since $M$ satisfies the hypothesis of Proposition 3.7, we can assume that $d \in \text{Eff} X$. Using Lemma 3.10, we can find $w$ in the relative interior of a wall separating $\text{Nef} X$ and an adjacent chamber $\Gamma$ such that $d + w \in \Gamma$ and $d + w \notin \text{deg} f_i + \Gamma$ for all $i$. It follows from Proposition 3.8 that $d + w \notin \text{reg} M$, which is a contradiction because $w \in \text{Nef} X$ and $\text{reg} M$ is invariant under positive translation by $\text{Nef} X$.

The conclusion that $\text{reg} M$ has finitely many minimal elements follows from Lemma 3.4. □

Corollary 3.12. Let $M$ be a finitely-generated torsion-free $S$-module. If $\text{deg} f_i \in b + \text{Nef} X$ for all generators $f_i$ of $M$ then $\text{reg} M \subseteq b + \text{Nef} X$.

Example 3.13. Consider the Hirzebruch surface $\mathcal{H}_2$, with notation from Example 2.1, and let $M$ be the torsion-free module with presentation

$$S(3,-3) \oplus S(2,-2) \oplus S(1,-2) \left\langle \begin{array}{ccc}
    x_0^2 x_1 & x_1^2 x_2 & x_2^2 x_3
\end{array} \right\rangle S(0,-4).$$

Since the degrees of the generators are contained in $(-3,2) + \text{Nef} \mathcal{H}_2$, by Corollary 3.12 the multigraded regularity of $M$ is contained in a translate of the nef cone, illustrated in Figure 5.

![Figure 5](image.png)

Figure 5. The multigraded regularity (dark green) of the module $M$ is contained in a translate $(-3,2) + \text{Nef} \mathcal{H}_2$ (light green) of the nef cone of $\mathcal{H}_2$ (dark blue).

### 4. Powers of Ideals and Multigraded Regularity

Throughout this section let $I = \langle f_1, \ldots, f_s \rangle \subseteq S$ be an ideal and let $P$ be the vector with coordinates $p_i = \text{deg} f_i \in \text{Pic} X$. We are interested in the asymptotic behavior of the multigraded regularity of $I^n$ as $n$ increases. In particular, we prove the following theorem:

**Theorem 4.1.** There exists a degree $a \in \text{Pic} X$, depending only on $I$, such that for each integer $n > 0$ and each pair of degrees $q_1, q_2 \in \text{Pic} X$ satisfying $q_1 \geq p_i \geq q_2$ for all $i$, we have

$$nq_1 + a + \text{reg} S \subseteq \text{reg}(I^n) \subseteq nq_2 + \text{Nef} X.$$
**Proof.** The inner bound will follow from Proposition 4.8. The outer bound follows from Corollary 3.12 by noting that \( \deg \prod_{j=1}^{n} f_{ij} = \sum_{j=1}^{n} p_{ij} \in nq_2 + \text{Nef} X \) for all products of \( n \) choices of generators of \( I \), and such products generate \( I^n \). \( \square \)

**Example 4.2.** Let \( I = \langle x_0 x_3, x_1^3 x_4 \rangle \) and \( J = \langle x_3, x_0^3 x_1 \rangle \) be two ideals in the total coordinate ring of the Hirzebruch surface \( H_2 \), with notation as in Example 2.1. Figure 6 shows the multigraded regularity of powers of \( I \) and \( J \) along with the bounds from Theorem 4.1.

![Figure 6](image)

**Figure 6.** The inner (dark green) and outer (light green) bounds for powers of \( I \) and \( J \). The circles correspond to the degrees of the generators of each power.

**Remark 4.3.** If \( q_2 \) is not nef, then the bounds in Theorem 4.1 will not increase with \( n \) in the partial order on \( \text{Pic} X \). We can see that this behavior is necessary by taking \( I \) to be a principal ideal generated outside of \( \text{Nef} X \).

### 4.1. The Rees Ring

One way to find a subset of the regularity of a module is by using its multigraded Betti numbers. In order to describe \( \text{reg}(I^n) \), we would thus like a uniform description of the Betti numbers of \( I^n \) for all \( n \). For this purpose, consider the multigraded Rees ring of \( I \):

\[
S[I t] := \bigoplus_{n \geq 0} I^n t^n \subseteq S[t],
\]

which is a \( \text{Pic}(X) \times \mathbb{Z} \)-graded noetherian ring with \( \deg ft^k = (\deg f, k) \) for \( f \in S \). Let \( R = S[T_1, \ldots, T_s] \) be the \( \text{Pic}(X) \times \mathbb{Z} \)-graded ring with \( \deg(T_i) = (\deg f_i, 1) = (p_i, 1) \). Notice that there is a surjective map of graded \( S \)-algebras:

\[
\begin{align*}
R & \longrightarrow S[I t] \\
T_i & \longrightarrow f_i t
\end{align*}
\]

Since \( R \) is a finitely generated standard graded algebra over \( S \), taking a single degree of a finitely generated \( R \)-module in the auxiliary \( \mathbb{Z} \) grading yields a finitely generated \( S \)-module.

**Definition 4.4.** For a \( \text{Pic}(X) \times \mathbb{Z} \)-graded \( R \)-module \( M \), define \( M^{(n)} \) to be the \( \text{Pic}(X) \)-graded \( S \)-module

\[
M^{(n)} := \bigoplus_{a \in \text{Pic} X} M_{(a,n)}.
\]
Following [Kod00], we record three important properties of this operation.

**Lemma 4.5.** Consider the functor $-^{(n)} : M \mapsto M^{(n)}$ from the category of Pic($X$) × $\mathbb{Z}$-graded $R$-modules to the category of Pic($X$)-graded $S$-modules.

(i) $-^{(n)}$ is an exact functor.

(ii) $S[I]\{t\}^{(n)} \cong I^n$.

(iii) $R(-a, -b)^{(n)} \cong R^{(n-b)}(-a) \cong \bigoplus_{|\nu|=n-b} S(-\nu \cdot P - a)$ where $\nu \in \mathbb{N}^n$.

Since $S[I]$ is a finitely generated module over the polynomial ring $R$, it has a finite free resolution. Applying $-^{(n)}$ gives a resolution by (i), which has cokernel $I^n$ by (ii) and whose terms are finitely generated free $S$-modules by (iii). Thus we can constrain the Betti numbers of $I^n$ in terms of those of $S[I]$.

### 4.2. Regularity of Powers of Ideals.

Given a description of the Betti numbers of $I^n$ in terms of $n$, we obtain an inner bound on $\text{reg}(I^n)$ using the following lemma.

**Lemma 4.6.** If $F_*$ is a finite free resolution for $M$ with $F_j = \bigoplus_i S(-a_{i,j})$ and $H^0_B(M) = 0$ then

$$\bigcap_i \bigcup_{\lambda = j} (a_{i,j} - \lambda \cdot C + \text{reg} S) \subseteq \text{reg} M$$

(2)

where $C = (c_1, \ldots, c_r)$ is the sequence of nef generators for $X$ and the union is over $\lambda \in \mathbb{N}^r$.

**Remark 4.7.** This result amounts to switching the union and intersection in the statement of [MS04, Cor. 7.3] for modules with $H^0_B(M) = 0$, which increases the size of the subset by allowing a different choice of $\lambda$ for each $i, j$.

**Proof.** Fix $d$ in the left hand side of (2) and consider the hypercohomology spectral sequence for $F_*$ (see [BCHS21, Thm. 4.14] for a description of this spectral sequence). We must show that $M$ is $d$-regular, meaning that $H^k_B(M)_d - \mu \cdot C = 0$ for all $d$ and all $\mu$ with $|\mu| = k - 1$. Since $F_*$ is a resolution for $M$, a diagonal of our spectral sequence converges to $H^k_B(M)$. Thus it is sufficient to prove that this entire diagonal vanishes in degree $d - \mu \cdot C$, i.e. that

$$H^k_B(F_j)_d - \mu \cdot C = \bigoplus_i H^k_B(S(-a_{i,j}))_d - \mu \cdot C = 0$$

(3)

for all $j$. This is satisfied for $k = 0$ by hypothesis. Now fix $k > 0$, $\mu$, $j$, and $i$. By choice of $d$ we have $d \in a_{i,j} - \lambda \cdot C + \text{reg} S$ for some $\lambda$ with $|\lambda| = j$, so that $d - a_{i,j} + \lambda \cdot C \in \text{reg} S$. Call this degree $d'$, and let $d' = (\lambda + \mu) \cdot C$, where $|\lambda + \mu| = k + j - 1$. Then by the definition of the regularity of $S$ we have $H^k_B(S)_d' - c = 0$ where

$$d' - c = d - a_{i,j} + \lambda \cdot C - (\lambda + \mu) \cdot C = d - \mu \cdot C.$$

Hence each summand in (3) is zero for $k > 0$, as desired. \qed

**Proposition 4.8.** There exists a degree $a \in \text{Pic} X$, depending only on the Rees ring of $I$, such that for each integer $n > 0$ and degree $q \in \text{Pic} X$ satisfying $q \geq \deg f_i$ for all homogeneous generators $f_i$ of $I$, we have

$$nq + a + \text{reg} S \subseteq \text{reg}(I^n).$$

**Proof.** Let $F_*$ be a minimal Pic($X$) × $\mathbb{Z}$-graded free resolution of $S[I]$ as an $R$-module, and write $F_j = \bigoplus_i R(-a_{i,j} - b_{i,j})$ for $a_{i,j} \in \text{Pic} X$ and $b_{i,j} \in \mathbb{Z}$. By Lemma 4.5, applying the $-^{(n)}$ functor to $F_*$ yields a (potentially non-minimal) resolution of $S[I]^{(n)} \cong I^n$ consisting of free $S$-modules

$$F_j^{(n)} \cong \bigoplus_i R(-a_{i,j} - b_{i,j})^{(n)} \cong \bigoplus_i \bigoplus_{|\nu|=n-b_{i,j}} S(-\nu \cdot P - a_{i,j})$$

(4)
where \( P = (\deg f_1, \ldots, \deg f_s) \) is the sequence of degrees of the homogeneous generators \( f_i \) of \( I \). From this Lemma 4.6 gives the following bound on the regularity of \( I^n \):
\[
\bigcap_{i,j} \bigcup_{|\lambda|=j} [\nu \cdot P + a_{i,j} - \lambda \cdot C + \reg S] \subseteq \reg(I^n).
\]
(4)

Note that \( b_{0,0} = 0 \), as \( S[I^t] \) is a quotient of \( R \), and thus \( b_{i,j} \geq 0 \) for all \( i, j \), as \( R \) is positively graded in the \( \mathbb{Z} \) coordinate.

Take \( a \in \Pic X \) so that \( a \geq a_{i,j} \) for all \( i, j \). There are only finitely many \( a_{i,j} \) because \( S[I^t] \) is a finitely generated \( R \)-module and \( R \) is noetherian. We may now simplify the left hand side of (4) by noting three things: (i) for all \( |\lambda| = j \) and all \( j \) we have \( \reg S \subseteq -\lambda \cdot C + \reg S \), (ii) if \( |\nu| = n - b_{i,j} \) then \( (n - b_{i,j})q \in \nu \cdot P + \reg S \), and (iii) for all \( i \) and all \( j \) we have \( nq + a \in (n - b_{i,j})q + a_{i,j} + \reg S \). Combining these facts gives that
\[
\reg(I^n) \supseteq \bigcap_{i,j} \bigcup_{|\lambda|=j} [\nu \cdot P + a_{i,j} - \lambda \cdot C + \reg S]
\]
\[
\supseteq \bigcap_{i,j} [\nu \cdot P + a_{i,j} + \reg S]
\]
\[
\supseteq \bigcap_{i,j} [(n - b_{i,j})q + a_{i,j} + \reg S]
\]
\[
\supseteq nq + a + \reg S.
\]
\( \square \)

A similar problem is to characterize the asymptotic behavior of regularity for symbolic powers of \( I \). Note that the symbolic Rees ring of \( I \) is not necessarily noetherian (see [GS21], for instance), so our argument for the existence of the degree \( a \) in the proof of Proposition 4.8 does not work in this case. More generally, if \( \mathcal{I} = \{I_n\} \) is a filtration of ideals, then one may ask for sufficient conditions so that \( \reg(I_n) \) is uniformly bounded.

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