A two-step transition description of underdamped phase diffusion

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Abstract. A two-step transition model describing phase diffusion and switching process in underdamped Josephson junctions is discussed. The model takes into account the phase particle’s escape rate out of the potential well and transition rate from phase diffusion to the running state. Using as examples the experimental switching current distributions of two Nb/AlO\textsubscript{x}/Nb junctions of different sizes fabricated on the same chip, we extract the transition rate, which turns out to follow the predicted Arrhenius law in the thermal regime but is greatly enhanced when macroscopic quantum tunneling becomes the dominant escape mechanism. Our results show the validity of applying the model to obtain the transition rate out of the phase diffusion state into the running state from measured switching current distribution of underdamped Josephson junctions. The temperature and bias-current dependent data provide strong evidence for phase diffusion in both thermal and quantum regimes.

1. Introduction
Diffusion of Brownian particles in tilted washboard potentials has been studied for many years and is receiving increasing interest in various branches of physics [1] and chemistry [2]. For the phase particles in current-biased Josephson junction (JJ) systems, a number of studies have been focused on the classical regime where thermal activation (TA) is the dominant escape mechanism [3, 4, 5, 6, 7, 8, 9, 10, 11]. Recently there have also been experimental [12] and theoretical [13, 14, 15] works that extend to the quantum regime where macroscopic quantum tunneling (MQT) prevails.

Previous experiments using JJs have identified three distinctive dynamical states, namely the trapped, diffusive, and running states as shown schematically in Fig. 1. In this paper, we describe a two-step transition model for quantitatively investigating the phase diffusion (PD) process [12], which takes into account the particle’s escape rate out of the potential well \( \Gamma_1 \) (=\( \Gamma_{\text{TA}} \) or \( \Gamma_{\text{MQT}} \)) and the transition rate \( \Gamma_2 \) from PD to the running state. The model is applied to underdamped Nb/AlO\textsubscript{x}/Nb trilayer JJs in which PD occurs in both TA and MQT regimes.
The experimentally measurable quantity is junction’s switching current distribution 

\[ \Gamma_{\text{MQT}} = \frac{1}{\pi h} \int df \rho(I') \exp\left(\frac{-\pi I' dI'}{\rho(I')^2}\right) \] 

In Fig. 1, phase particles in the trapped, diffusive, and running states are denoted by state \( n \) with occupation probability \( \rho_n \) in a tilted washboard potential.

2. The two-step transition model

In Fig. 1, phase particles in the trapped, diffusive, and running states are denoted by state \( n \) (\( n = 1, 2, 3 \), respectively) with occupation probability \( \rho_n \). In this way, it is clear that \( \rho_n \) should satisfy the following master equation:

\[
\begin{align*}
\frac{d\rho_1}{dt} &= -\Gamma_1 \rho_1 \\
\frac{d\rho_2}{dt} &= \Gamma_1 \rho_1 - \Gamma_2 \rho_2 \\
\frac{d\rho_3}{dt} &= \Gamma_2 \rho_2.
\end{align*}
\]

The experimentally measurable quantity is junction’s switching current distribution \( P(I) \). Since \( P(I) = d\rho_3/dI \), it follows straightforwardly that

\[ \Gamma_2(I) = \frac{(dI/dI')P(I)}{1 - \int_0^1 P(I')dI' - e^{-\pi I^2 B/I^2} \int_0^I \Gamma_1(I'/dI')} \]  

Eq. (2) shows that \( \Gamma_2 \) can be extracted from measured \( P(I) \) since in the classical and quantum regimes \( \Gamma_1 \) is given by the well-known formulae \( \Gamma_{\text{MQT}} = (\omega_p/2\pi) a_t \exp(-\Delta U/k_B T) \), where \( a_t = 4/(\sqrt{1 + Qk_B T/1.8\Delta U} + 1)^2 \) and \( \Delta U = (2\sqrt{2}/3)E_J (1 - i^2)^{3/2} \) [16], and \( \Gamma_{\text{MQT}} = (\omega_p/2\pi)a_q \exp(-(7.2\Delta U/h\omega_p)(1+0.87/Q)) \), where \( a_q \simeq [120\pi(7.2\Delta U/h\omega_p)]^{1/2} \) [17], respectively. In the above expressions \( i = I/I_c \) is the junction’s bias current normalized to its critical current, \( E_J = hI_c/2e \) is the Josephson coupling energy, \( \omega_p = \omega_0 (1 - i^2) \) \( 1/4 \) is the plasma frequency with \( \omega_0 = (2\pi I_c/\Phi_0 C)^{1/2} \), and \( Q = \omega_p RC \) is the quality factor (\( C \) and \( R \) being junction’s capacitance and damping resistance).

Notice that in the limit of \( \Gamma_2 \to \infty \), Eq. (2) leads directly to \( \Gamma_1(I) = (dI/dI')P(I)/[1 - \int_0^1 P(I')dI'] \) which is identical to the result of Fulton and Dunkleberger [18] in which PD is absent. In the opposite limit of \( \Gamma_2 \ll \Gamma_1 \), the same expression is obtained with \( \Gamma_1 \) replaced by \( \Gamma_2 \): \( \Gamma_2(I) = (dI/dI')P(I)/[1 - \int_0^1 P(I')dI'] \). These results mean that the much slower process plays the major role in determining \( P(I) \), as expected. In the more interesting situation of \( \Gamma_2 \sim \Gamma_1 \), Eq. (2) enables one to separate the effect of \( \Gamma_2 \) on switching current distributions from that of \( \Gamma_1 \). The inverse procedure of computing \( P(I) \) from \( \Gamma_1 \) and \( \Gamma_2 \) is given by:

\[ P(I) = \frac{\Gamma_2}{(dI/dI')^2} e^{-\pi I^2 B/I^2} \int_0^I \Gamma_1 e^{-\pi I^2 B/I^2} \int_0^I (\Gamma_1 - \Gamma_2) dI'' dI'. \]
Similarly, in the quantum regime the boundary can be obtained from $\Gamma_{MQT}(I_m) \sim 1/\tau$, or equivalently by replacing $T_0$ with $T_{cr}$ in Eq. (4) since $E_J$ is in general independent of $T$ for $T < T_{cr}$. In Fig. 2, the solid lines are the boundaries calculated using a typical value of $\tau = 100 \mu s$ and the experimentally determined value of $I_m \sim 255$ nA for junction L, up to which $\sigma$ narrows so PD exists [see Fig. 4(b) below]. Since $T_{cr} = \hbar \omega_p[(1 + 1/4Q^2)^{1/2} - 1/2Q]/2\pi k_B \sim \hbar \omega_0/2\pi k_B$, it is almost constant in our experiment when $E_J$ is reduced by decreasing the junction size while keeping the ratio $I_c/C$ constant. If dc-SQUIDs were used, $T_{cr}$ would be proportional to $\sqrt{E_J} \sim \sqrt{T_c}$ due to the constant junction area and thus the capacitance $C$. The phase boundary between quantum and thermal PD for the two cases are depicted in Fig. 2 as red and blue dashed lines, respectively. It can be seen that the area of quantum PD becomes much larger in the present case (hatched area to the left of the vertical red dashed line) as compared to that if dc-SQUIDs were used (cross-hatched area).

The two dotted lines in Fig. 2 represent the $E_J$ of junctions S (lower) and L (upper), respectively. It can be seen clearly that PD occurs for junction S at all temperatures while junction L will proceed from MQT to TA and finally to PD as temperature increases. In Fig. 3, experimental $P(I)$ measured by the time-of-flight technique [12, 19] for junctions S and L are plotted. The measured temperature dependence of the width $\sigma$ and mean $I_s$ are shown in Fig. 4.

**Table 1.** Parameters of two Nb/AlO$_x$/Nb junctions S and L denoted by their smaller and larger area. $R_N$ is normal-state resistance obtained from I-V curves. $I_c$, $C$, and $R$ for L are determined from fits to experiment using TA and MQT theories below 450 mK and Monte Carlo simulations above it. Those for S are obtained considering its $R_N$ ratio to L. Area for L is estimated from fitted $C$ and a specific capacitance of $50 \mu F/\mu m^2$, and that for S is obtained via its $R_N$ ratio to L. Nominal areas for junctions S and L were 0.52 and 1.61 $\mu m^2$, respectively.

| Junction | Area(\( \mu m^2 \)) | $R_N$(k\( \Omega \)) | $I_c$(nA) | $C$(\( \mu F \)) | $R$(\( \Omega \)) | $T_{cr}$(mK) | $T_0$(mK) |
|----------|-----------------|-----------------|------------|--------------------|----------------|-------------|-------------|
| S        | 0.39            | 15.1            | 122        | 19.6               | 1800           | 140         | < 25        |
| L        | 1.54            | 3.84            | 480        | 77                 | 315            | 125         | \~450       |

Eqs. (2) and (3) thus allow us to quantitatively investigate the dependencies of PD on the bias current and temperature, and its influence on the switching current distribution $P(I)$.

3. Experiment

In order to contrast PD in TA and MQT regimes, we use two Nb-AlO$_x$-Nb trilayer junctions of different sizes fabricated on the same chip (see Table 1) with $T_0 \ll T_{cr}$ and $T_0 \gg T_{cr}$, respectively. Here, $T_0$ is the temperature above which PD occurs and $T_{cr}$ is the classical-to-quantum crossover temperature below which MQT is the dominant escape mechanism. PD ($T_0$) is confirmed (determined) from its key signature, namely the narrowing of the width $\sigma$ of switching current distribution $P(I)$ as temperature increases [7, 8, 9, 10]. The present experiment is advantageous for study quantum PD, as shown in Fig. 2, when compared to those using dc-SQUIDs [7, 8]. In the discussion of the boundary separating TA and PD Kivioja et al. [8] applied the condition $\Gamma_{TA}(I_m) \sim 1/\tau$, where $\tau$ is the time scale of the measurement and $I_m = 4I_c/\pi Q_0 \ (Q_0 = \omega_0 R_C)$ is the bias current below which the phase particle has a finite probability of being retrapped after each escape. This argument leads to the following phase boundary line in the ($T$, $E_J$) plane below which PD occurs:

$$E_J^{PD} = \frac{3}{4\sqrt{2}} \ln \left( \frac{\omega_0 \tau}{2\pi} \right) \frac{k_B T_0}{(1 - I_m/I_c)^{3/2}} \ .$$  (4)

Similarly, in the quantum regime the boundary can be obtained from $\Gamma_{MQT}(I_m) \sim 1/\tau$, or equivalently by replacing $T_0$ with $T_{cr}$ in Eq. (4) since $E_J$ is in general independent of $T$ for $T < T_{cr}$. In Fig. 2, the solid lines are the boundaries calculated using a typical value of $\tau = 100 \mu s$ and the experimentally determined value of $I_m \sim 255$ nA for junction L, up to which $\sigma$ narrows so PD exists [see Fig. 4(b) below]. Since $T_{cr} = \hbar \omega_p[(1 + 1/4Q^2)^{1/2} - 1/2Q]/2\pi k_B \sim \hbar \omega_0/2\pi k_B$, it is almost constant in our experiment when $E_J$ is reduced by decreasing the junction size while keeping the ratio $I_c/C$ constant. If dc-SQUIDs were used, $T_{cr}$ would be proportional to $\sqrt{E_J} \sim \sqrt{T_c}$ due to the constant junction area and thus the capacitance $C$. The phase boundary between quantum and thermal PD for the two cases are depicted in Fig. 2 as red and blue dashed lines, respectively. It can be seen that the area of quantum PD becomes much larger in the present case (hatched area to the left of the vertical red dashed line) as compared to that if dc-SQUIDs were used (cross-hatched area).

The two dotted lines in Fig. 2 represent the $E_J$ of junctions S (lower) and L (upper), respectively. It can be seen clearly that PD occurs for junction S at all temperatures while junction L will proceed from MQT to TA and finally to PD as temperature increases. In Fig. 3, experimental $P(I)$ measured by the time-of-flight technique [12, 19] for junctions S and L are plotted. The measured temperature dependence of the width $\sigma$ and mean $I_s$ are shown in Fig. 4.
Figure 3. Experimentally measured $P(I)$ of junctions S (a) and L (b) at some temperatures indicated. Inset shows the $I$-$V$ trace of junction S at 30 mK.

Figure 4. (a) $\sigma$ and $I_s$ of junctions S (a) and L (b) (symbols). Lines in (b) are calculated results. Inset shows $\sigma$ of junction S plotted in semi-logarithmic scale displaying a slope turning near $T_{cr}^S = 140$ mK.

The lines in Fig. 4(b) are calculated using MQT, TA, and Monte Carlo simulations in respective temperature ranges, from which parameters of junctions L and S are determined (see Table I). When $\sigma$ of junction S is plotted semi-logarithmically (see inset), it displays a clear change of slope at $T_{cr}^S = 140$ mK, indicating that the particle escape mechanism switched between TA and MQT. The results are consistent with those presented in Fig. 2.

4. Phase diffusion: Classical regime

In Figs. 5(a) and 6(a), we plot $\Gamma_1$ (solid lines) at several temperatures calculated using the parameters in Table I for junctions S and L, respectively. The corresponding $\Gamma_2$ (symbols) extracted from the measured $P(I)$ using Eq. (2) is shown in Figs. 5(b) and 6(b).

For junction S, it can be seen from Fig. 5(a) that at $T = 800$ mK, $\Gamma_1$ is several orders of magnitude greater than $\Gamma_2$. The measured $P(I)$ is therefore entirely determined by $\Gamma_2$. As temperature decreases, $\Gamma_1$ is seen to progressively approach $\Gamma_2$. To further understand the behavior of the PD state, we examine the data in Fig. 5(a) by looking at the temperature dependence of $\Gamma_2$ for fixed bias currents (thus fixed potentials). In Fig. 5(b), we plot $\Gamma_2$ versus $1/T$ at three bias currents of 48, 52, and 56 nA, which shows distinct features below and above $T_{cr}^S = 140$ mK. While the data for $T > T_{cr}^S$ follow the straight lines, indicating that $\Gamma_2$ in the classical regime obeys the Arrhenius law, $\Gamma_2$ displays a much weaker $1/T$ dependence at $T \ll T_{cr}^S$.

We note that similar behavior in the classical regime was discussed previously by Vion et al. [5] for overdamped junctions where the diffusive particle is considered to overcome an effective dissipation barrier. In that case, the transition rate from PD to the running state, which retains the familiar Kramers form, was derived. Fitting the data above $T_{cr}^S$ using $\Gamma_2 = a \exp(-b/T)$, we obtain $a = 5.2 \times 10^7$ sec$^{-1}$, $b = 2.3$ K for $I = 48$ nA (dashed line) and $a = 3.3 \times 10^8$ sec$^{-1}$, $b = 1.7$ K for $I = 52$ nA (dotted line). The effective barrier $b$ appears smaller than the corresponding.
calculated barrier height $\Delta U$ of 2.68 K and 2.46 K due to the net forward motion of the diffusive particles as intuitively expected. In Fig. 6(b), $\Gamma_2$ versus $1/T$ for junction $L$ at bias current of 220 nA is shown. Corresponding values of $a$, $b$, and $\Delta U$ are found to be $2.7 \times 10^9$ sec$^{-1}$, 8.7 K, and 8.9 K, respectively.

The above results indicate that in the thermal regime a dissipation-barrier description is valid for PD in underdamped JJs and there is a striking similarity between particle’s escape described by $\Gamma_1$ and transition from PD to the running states described by $\Gamma_2$. It can be seen from Figs. 5(a) and 6(a) that when PD is present $\Gamma_2$ is usually smaller than $\Gamma_1$ except in a narrow temperature range around $T_{cr}$, which is therefore more influential to $P(I)$. Considering that the attempt frequency of the measured JJs is on the order of the plasma frequency of $\omega_p/2\pi \sim 10^{10}$ sec$^{-1}$, comparing the values of the above data of $a$ vs. $\omega_p/2\pi$ and $b$ vs. $\Delta U$ for both junctions $S$ and $L$ suggests that $\Gamma_2$ becomes smaller as a result of the reduction of both the attempt frequency and barrier height.

5. Phase diffusion: Quantum regime

An interesting result of junction $S$ shown in Fig. 5(b) is that below $T_{cr}^S$, where MQT and quantum fluctuations dominate, $\Gamma_2$ clearly deviates from the Arrhenius law and displays a much weaker dependence on $1/T$. Though theoretical treatment of quantum PD in underdamped systems is not yet available we can gain some qualitative understanding of the problem from that of overdamped systems. For example, Machura et al. recently investigated the diffusion problem of overdamped particles in tilted washboard potentials using the Smoluchowski equation incorporating quantum fluctuations [14]. They found that the particle’s average velocity $\langle v \rangle$
increases with increasing temperature and quantum effects always assist the particle to overcome barriers and to pass longer distances, leading to a larger $\langle v \rangle$ than that the particle would have in absence of quantum fluctuations. Because in our underdamped junctions the dc voltage, which is proportional to $\langle v \rangle$, produced by PD is too low to be detected directly [20], it can nevertheless be expected that a larger $\langle v \rangle$ would result in a larger $\Gamma_2$ since the increased kinetic energy makes transitions to the running state easier. For this reason, the data in Fig. 5(b) are consistent with the theoretical prediction since extrapolating $\Gamma_2$ from the classical regime to the quantum regime would lead to rates that are much lower than the experimental data. Therefore, the much weaker $1/T$ dependence of $\Gamma_2$ below $T_{cr}^S$, in a stark contrast to the Arrhenius behavior above $T_{cr}^S$, manifests the quantum nature of the diffusion process at $T < T_{cr}^S$.

6. Summary
We described a two-step transition model for underdamped phase diffusion with which the effects of escape rate $\Gamma_1$ (from the trapped state) and the transition rate $\Gamma_2$ (from PD to the running state) on switching current distributions can be separated and $\Gamma_2$ could be determined from the measured $P(I)$ directly. We applied the model to underdamped Nb Josephson junctions in which phase diffusion occurs in both classical and quantum regimes. Using junctions of different sizes fabricated from the same trilayer we were able to calibrate the relevant parameters of the small junction and at the same time extended PD to temperatures well below the classical-to-quantum crossover temperature. It was found that $\Gamma_2$ vs. $T$ at fixed bias current, and thus fixed potential landscape, follows the Arrhenius law when thermal activation is the dominant escape mechanism. The more interesting finding was that when escape is caused by MQT and quantum fluctuations play an important role, $\Gamma_2$ is exponentially higher than that expected for the classical PD and has a much weaker $1/T$ dependence. The similarities between the temperature dependence of $\Gamma_1$ and $\Gamma_2$ in underdamped Josephson junctions going from classical to quantum regimes were striking. Our results demonstrated that the two-step transition model is valid and effective in analyzing the data involving classical and quantum PD processes in underdamped systems with tilted periodic potentials.

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