GAPS IN SUMSETS OF \( s \) PSEUDO \( s \)-TH POWER SEQUENCES

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Abstract. We study the length of the gaps between consecutive members in the sumset \( sA \) when \( A \) is a pseudo \( s \)-th power sequence, with \( s \geq 2 \). We show that, almost surely, \( \lim \sup (b_{n+1} - b_n)/\log(b_n) = s^s s!/\Gamma^s(1/s) \), where \( b_n \) are the elements of \( sA \).

1. Introduction

Erdős and Rényi \cite{erdos1960} proposed in 1960 a probabilistic model for sequences \( A \) growing like the \( s \)-th powers: they build a probability space \( (U, T, P) \) and a sequence of independent random variables \( (\xi_n)_{n \in \mathbb{N}} \) with values in \( \{0, 1\} \) and \( P(\xi_n = 1) = s^n/n^{1+s}; \) to any \( u \in U \) they associate the sequence of integers \( A = A_n \) such that \( n \in A_u \) if and only if \( \xi_n(u) = 1 \). In short, the events \( \{n \in A\} \) are independent and \( P(n \in A) = 1/n^{1+s}; \) the counting function of these random sequences \( A \) satisfies almost surely the asymptotic relation \( |A \cap [1, x]| \sim x^{1/s} \), whence the terminology pseudo \( s \)-th powers. Erdős and Rényi studied the random variable \( r_s(A, n) \) which counts the number of representations of \( n \) in the form \( n = a_1 + \cdots + a_s \), \( a_1 \leq \cdots \leq a_s \), \( a_i \in A \). For the simplest case \( s = 2 \) they proved that \( r_2(A, n) \) converges to a Poisson distribution with parameter \( \pi/8 \), when \( n \to \infty \). They also claimed the analogous result for \( s > 2 \) but their analysis did not take into account the dependence of some events. J. H. Goguel \cite{goguel1940} proved indeed that for each integer \( d \), the sequence of the integers \( n \) such that \( r_s(A, n) = d \) has almost surely the density \( \lambda^d_s e^{-\lambda_s}/d! \), where \( \lambda_s = \Gamma^s(1/s)/(s^s s!) \). B. Landreau \cite{landreau1991} gave a proof of this result based on correlation inequalities and also showed that the sequence of random variables \( (r_s(A, n))_n \) converges in law towards the Poisson distribution with parameter \( \lambda_s \).

In particular, both the sets of the integers belonging, or not belonging, to \( sA = \{a_1 + \cdots + a_s : a_i \in A\} \) have almost surely a positive density and it makes sense to study the length of the gaps in \( sA \). The aim of the paper is to obtain a precise estimate for the maximal length of such gaps.

Theorem 1. For any \( s \geq 2 \) the sequence \( sA = (b_n)_n \), sum of \( s \) copies of a pseudo \( s \)-th power sequence \( A \), satisfies almost surely

\[
\lim \sup_{n \to \infty} \frac{b_{n+1} - b_n}{\log b_n} = \frac{s^s s!}{\Gamma^s(1/s)}.
\]

We remark that this result is heuristically consistent with the easier fact that for a random sequence \( S \) with \( P(n \in S) = 1 - e^{-\lambda} \), we have \( \lim \sup (s_{m+1} - s_m)/\log s_m = 1/\lambda \) almost surely.

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2. Notation and general lemmas

2.1. Notation. We retain the notation of the introduction, for the probability space $(\mathcal{U}, T, \mathbb{P})$ and the definition of the random sequences $A = A_n$, where the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s} n^{-1+1/s}$. We further use the following notation.

i) We write $\omega$ to denote a set of distinct integers and we denote by $E_\omega$ and $E^c_\omega$ the events

$$E_\omega := \{\omega \subset A\} \quad \text{and} \quad E^c_\omega := \{\omega \not\subset A\}$$

respectively. We write $\omega \sim \omega'$ to mean that $\omega \cap \omega' \neq \emptyset$ but $\omega \neq \omega'$: e remark that $\omega \sim \omega'$ if and only if the events $E_\omega$ and $E_{\omega'}$ are distinct and dependent.

If $\omega = \{x_1, \ldots, x_r\}$ we write

$$\sigma(\omega) = \{a_1 x_1 + \cdots + a_r x_r : a_1 + \cdots + a_r = s, \ a_i \geq 1\}$$

for the set of all integers that can be written as a sum of $s$ integers using all the integers $x_1, \ldots, x_r$. We denote by $\Omega_s$ the family of sets

$$\Omega_s = \{\omega : \sigma(\omega)\}.$$

ii) Given $\alpha > 0$, we denote by $I_i$ the interval $[i, i + \alpha \log i]$ and we denote by $F_i$ the event

$$F_i := \{sA \cap I_i = \emptyset\}.$$

We denote by $\Omega_{I_i}$ the family of sets

$$\Omega_{I_i} = \{\omega : \sigma(\omega) \cap I_i \neq \emptyset\}.$$

iii) We finally let $\lambda_s = \frac{\Gamma(s)}{s!} s^{-s}$.

2.2. Probabilistic lemmas. We use the following generalization of the Borel-Cantelli Lemma, proved indeed by P. Erdős and A. Rényi in 1959 [2].

**Theorem 2** (Borel-Cantelli Lemma). Let $(F_i)_{i \in \mathbb{N}}$ be a sequence of events and let

$$Z_n = \sum_{i \leq n} P(F_i).$$

If the sequence $(Z_n)_n$ is bounded, then, with probability 1, only finitely many of the events $F_i$ occur.

If the sequence $(Z_n)_n$ tends to infinity and

$$\lim_{n \to \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

then, with probability 1, infinitely many of the events $F_i$ occur.

**Theorem 3** (Janson’s Correlation Inequality [1]). Let $(E_\omega)_{\omega \in \Omega}$ be a finite collection of events which are intersections of elementary independent events and assume that $P(E_\omega) \leq 1/2$ for any $\omega \in \Omega$. Then

$$\prod_{\omega \in \Omega} P(E^c_\omega) \leq P\left( \bigcap_{\omega \in \Omega} E^c_\omega \right) \leq \prod_{\omega \in \Omega} P(E^c_\omega) \times \exp\left(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})\right),$$

where $\omega \sim \omega'$ means that the events $E_\omega$ and $E_{\omega'}$ are distinct and dependent.
2.3. A technical lemma.

**Lemma 1.** Given $1 \leq t \leq s-1$ and positive integers $a_1, \ldots, a_t$ we have, as $z$ tends to infinity:

i) \[ \sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+t/s}. \]

ii) \[ \sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t < z}} (x_1 \cdots x_t)^{-1+1/s} (z - (a_1 x_1 + \cdots + a_t x_t))^{-2t/s} \ll z^{-1/s} \log z. \]

iii) \[ \sum_{1 \leq x_1 < \cdots < x_s \leq z} (x_1 \cdots x_s)^{-1+1/s} \sim s^s \lambda_s. \]

**Proof.** i) We have

\[
\sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} = (a_1 \cdots a_t)^{-1/s} \sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t = z}} (a_1 x_1 \cdots a_t x_t)^{-1+1/s}
\]

\[
\leq (a_1 \cdots a_t)^{-1/s} \sum_{y_1, \ldots, y_t \geq 1} (y_1 \cdots y_t)^{-1+1/s}.
\]

If $y_1 + \cdots + y_t = z$ then at least one of them, say $y_t$, is greater than $z/t$ and is determined by $y_1, \ldots, y_{t-1}$. Thus,

\[
\sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+1/s} \sum_{y_1, \ldots, y_t \geq 1} (y_1 \cdots y_{t-1})^{-1+1/s}
\]

\[
\ll z^{-1+1/s} \left( \sum_{y \leq z} y^{-1+1/s} \right)^{t-1}
\]

\[
\ll z^{-1+1/s} (z^{1/s})^{t-1} \ll z^{-1+t/s}.
\]

ii) We have

\[
\sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t < z}} (x_1 \cdots x_t)^{-1+1/s} (z - (a_1 x_1 + \cdots + a_t x_t))^{-2t/s}
\]

\[
= \sum_{m \leq z/2} (z - m)^{-2t/s} \sum_{\substack{x_1, \ldots, x_t \geq 1 \atop a_1 x_1 + \cdots + a_t x_t = m}} (x_1 \cdots x_t)^{-1+1/s}
\]

(by i) \[
\ll \sum_{m \leq z/2} (z - m)^{-2t/s} \ll z^{-2t/s} + \sum_{m \leq z/2} (z - m)^{-2t/s} m^{-1+t/s}
\]

\[
\ll z^{-2t/s} z^{t/s} + z^{-1+t/s} \sum_{m \leq z/2} (z - m)^{-2t/s}
\]

\[
\ll z^{-t/s} + z^{-1+t/s} \left( 1 + \log z + z^{-1-2t/s} \right)
\]

\[
\ll z^{-t/s} + z^{-1+t/s} \log z
\]

\[
\ll z^{-1/s} \log z.
\]
Remark 1. Except in the case when \( s = 2 \) and \( t = 1 \), the upper bound in ii) may be replaced by \( z^{-1/s} \).

iii) It follows from Lemma 3 of [5].

\[ \square \]

3. Proof of Theorem 1

3.1. Combinatorial lemmas.

Lemma 2. We have

\[ \sum_{\omega \in \Omega_s} P(E_\omega) \sim \lambda_s \]  
as \( z \to \infty \).

Proof.

\[ \sum_{\omega \in \Omega_s} P(E_\omega) = \sum_{\omega \in \Omega_s, |\omega| = s} P(E_\omega) + \sum_{\omega \in \Omega_s, |\omega| \leq s-1} P(E_\omega). \]

The main contribution comes from the first sum.

\[ \sum_{\omega \in \Omega_s, |\omega| = s} P(E_\omega) = \frac{1}{s^s} \sum_{1 \leq x_1 < \cdots < x_s \leq z} (x_1 \cdots x_s)^{-1+1/s} \sim \lambda_s \]  
as \( z \to \infty \), by Lemma 1 iii). For the second sum we have

\[ \left( \text{Lemma 1 ii) } \right) \sum_{\omega \in \Omega_s, |\omega| \leq s-1} P(E_\omega) \ll \sum_{r \leq s-1} z^{r-1} \ll z^{-1/s}. \]

Lemma 3. For any \( z \leq z' \) we have

\[ \sum_{\omega \in \Omega_z, \omega' \in \Omega_{z'}} P(E_\omega \cap E_{\omega'}) \ll z^{-1/s} \log z. \]

Proof. If \( \omega \in \Omega_z \) then there exist some \( r \leq s \) and some positive integers \( a_1, \ldots, a_r \) with \( a_1 + \cdots + a_r = s \) such that \( a_1 x_1 + \cdots + a_r x_r = z \). Thus, any pair of sets \( \omega \sim \omega' \) with \( \omega \in \Omega_z, \omega' \in \Omega_{z'} \), \( z \leq z' \) is of the form

\[ \omega = \{x_1, \ldots, x_1, u_{t+1}, \ldots, u_r\} \]

\[ \omega' = \{x_1, \ldots, x_1, v_{t+1}, \ldots, v_{r'}\} \]

with \( 1 \leq t \leq r, r' \leq s \) and positive integers \( a_1, \ldots, a_r \) and \( b_1, \ldots, b_{r'} \) with

\[ a_1 x_1 + \cdots + a_r x_r + a_{t+1} u_{t+1} + \cdots + a_r u_r = z \]

\[ b_1 x_1 + \cdots + b_r x_r + b_{t+1} v_{t+1} + \cdots + b_{r'} v_{r'} = z'. \]

Of course if \( r = t \) then \( \omega = \{x_1, \ldots, x_r\} \) and \( r' \geq t + 1 \). Otherwise \( \omega = \omega' \). And similarly, when \( r' = t \), we have \( r \geq t + 1 \).

Given \( z, z', t, r, r', a_1, \ldots, a_r, b_1, \ldots, b_{r'} \) we estimate the sum

\[ \sum_{\omega \sim \omega' \atop \omega \in \Omega_z, \omega' \in \Omega_{z'}} P(E_\omega \cap E_{\omega'}) \]
where the sum is extended to the pairs $\omega \sim \omega'$ satisfying the above conditions. We distinguish several cases according to the values of $r$ and $r'$.

- If $r \geq t + 1$ and $r' \geq t + 1$, we have

$$
\sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) 
\leq \sum_{x_1, \ldots, x_t \in I_{\omega'}} (x_1 \cdots x_t)^{-1+1/s} \times \left( \sum_{u_{t+1}, \ldots, u_r \in I_{\omega'}} (u_{t+1} \cdots u_r)^{-1+1/s} \right) \times \left( \sum_{v_{t+1}, \ldots, v_{r'} \in I_{\omega'}} (v_{t+1} \cdots v_{r'})^{-1+1/s} \right)
$$

By Lemma 1(i) we have

$$
\sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) 
\ll \sum_{x_1, \ldots, x_t \in I_{\omega'}} (x_1 \cdots x_t)^{-1+1/t} (z - (a_1 x_1 + \cdots + a_t x_t))^{-1} (z' - (b_1 x_1 + \cdots + b_t x_t))^{-1/s} 
\ll \sum_{x_1, \ldots, x_t \in I_{\omega'}} (x_1 \cdots x_t)^{-1+1/t} (z - (a_1 x_1 + \cdots + a_t x_t))^{-1} (z' - (b_1 x_1 + \cdots + b_t x_t))^{-1/s}
$$

Using the inequality $AB \leq A^2 + B^2$, we get

$$
\sum_{\omega \sim \omega'}^* P(E_\omega \cap E_{\omega'}) 
\leq \sum_{x_1, \ldots, x_t \in I_{\omega'}} (x_1 \cdots x_t)^{-1+1/t} (z - (a_1 x_1 + \cdots + a_t x_t))^{-2} 
+ \sum_{x_1, \ldots, x_t \in I_{\omega'}} (x_1 \cdots x_t)^{-1+1/t} (z' - (b_1 x_1 + \cdots + b_t x_t))^{-2} 
\ll z^{-1/s} \log z.
$$
Lemma 4. Let \( \alpha > 0 \) and the interval \( I_i = [i, i + \alpha \log i] \). For any \( i \leq j \) we have
\[
\sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s}(\log i)^2(\log j).
\]

Proof.\[
\sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'}) \leq \sum_{z \in I_i} \sum_{\omega \in \Omega_{\omega_z}} \sum_{\omega' \in \Omega_{\omega_z'}} P(E_\omega \cap E_{\omega'}) \ll \sum_{z \in I_i} z^{-1/s} \log z \ll (\log i)^2(\log j)i^{-1/s}.
\]

Lemma 5. We have
\[
\prod_{\omega \in \Omega_{I_i}} P(E_{\omega}) = i^{-\alpha \log i + o(1)}.
\]

Proof. We observe that
\[
\prod_{z \in I_i} \prod_{\omega \in \Omega_{z}} P(E_{\omega}) \leq \prod_{\omega \in \Omega_{I_i}} P(E_{\omega}) \leq \prod_{\omega \in \Omega_{I_i}} P(E_{\omega}) = \prod_{z \in I_i} \prod_{\omega \in \Omega_{z}} P(E_{\omega}).
\]

Writing \( P(E_{\omega}) = 1 - P(E_{\omega}) \) and taking logarithms we have
\[
\log \left( \prod_{z \in I_i} \prod_{\omega \in \Omega_{z}} P(E_{\omega}) \right) = \sum_{z \in I_i} \sum_{\omega \in \Omega_{z}} \log (1 - P(E_{\omega})) \\
\sim -\sum_{z \in I_i} \sum_{\omega \in \Omega_{z}} P(E_{\omega}) \\
\sim -\sum_{z \in I_i} \lambda_z \\
\sim -\alpha \lambda_i \log i.
\]
On the other hand,
\[
\log \left( \prod_{z \in I_i} \prod_{\omega \in \Omega_{i|z|=s}} P(E^c_\omega) \right) = \sum_{z \in I_i} \sum_{\omega \in \Omega_{i|z|=s}} \log(1 - P(E_\omega)) \\
\sim - \sum_{z \in I_i} \sum_{\omega \in \Omega_{i|z|=s}} P(E_\omega) \\
= - \sum_{x_1 < \cdots < x_s} \sum_{x_1 + \cdots + x_s = z} \frac{1}{s^s} (x_1 \cdots x_s)^{-1+1/s}
\]
(Lemma 1 iii) \sim - \lambda_s \alpha \log i.
\]

Lemma 6. We have
\[
P(F_i) = i^{-\alpha \lambda_s + o(1)}.
\]

Proof. We observe that
\[
F_i = \bigcap_{\omega \in \Omega_{I_i}} E^c_\omega.
\]
Since \(P(E_\omega) \leq 1/2\) for any \(\omega\), Theorem 3 applies and we have
\[
\prod_{\omega \in \Omega_{I_i}} P(E^c_\omega) \leq P(F_i) \leq \prod_{\omega \in \Omega_{I_i}} P(E^c_\omega) \times \exp \left( 2 \sum_{\omega, \omega' \in \Omega_{I_i}} P(E_\omega \cap E_{\omega'}) \right).
\]
After Lemma 5 we only need to prove
\[
\sum_{\omega, \omega' \in \Omega_{I_i}} P(E_\omega \cap E_{\omega'}) = o(1).
\]
But it is a consequence of Lemma 4 with \(j = i\).
\[
\sum_{\omega, \omega' \in \Omega_{I_i}} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s + o(1)}.
\]

Lemma 7. If \(i < j\) and \(I_i \cap I_j = \emptyset\) then
\[
\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E^c_\omega) \leq P(F_i) P(F_j) \left( 1 + O(j^{-1/s} \log j) \right).
\]

Proof. It is clear that
\[
\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E^c_\omega) = \left( \prod_{\omega \in \Omega_{I_i}} P(E^c_\omega) \right) \left( \prod_{\omega \in \Omega_{I_j}} P(E^c_\omega) \right) \left( \prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E^c_\omega) \right)^{-1}.
\]
The lower bound of the Janson’s inequality, applied to the first two products, gives
\[
\prod_{\omega \in \Omega_{I_i} \cup \Omega_{I_j}} P(E^c_\omega) \leq P(F_i) P(F_j) \left( \prod_{\omega \in \Omega_{I_i} \cap \Omega_{I_j}} P(E^c_\omega) \right)^{-1}.
\]
The logarithm of the last factor is

$$- \sum_{\omega \in \Omega_i \cap \Omega_j} \log (1 - P(E_\omega)) \sim \sum_{\omega \in \Omega_i \cap \Omega_j} P(E_\omega)$$

Since $I_i \cap I_j = \emptyset$, if $\omega \in \Omega_i \cap \Omega_j$ then $|\omega| \leq s - 1$. Thus

$$\sum_{\omega \in \Omega_i \cap \Omega_j} P(E_\omega) \leq \sum_{|\omega| \leq s - 1} P(E_\omega) \leq \sum_{z \in I_j} \sum_{r \leq s - 1} \sum_{a_1, \ldots, a_r = 1} x_1 \cdots x_r - 1 + 1/s$$

(Lemma [II.9]) \(\ll j^{-1/s} \log j\).

Thus

$$\left( \prod_{\omega \in \Omega_i \cap \Omega_j} P(E_{\omega}^c) \right)^{-1} \leq 1 + O(j^{-1/s} \log j)$$

which ends the proof of the Lemma. \(\square\)

### 3.2. End of the proof.

After these Lemmas we are ready to finish the proof of Theorem 1.

If $\alpha > 1/\lambda_s$ then

$$\sum_i P(F_i) = \sum_i i^{-\alpha \lambda_s + o(1)} < \infty$$

and Theorem 2 implies that with probability 1 only finite many events $F_i$ occur. This proves that

$$\limsup_{k \to \infty} \frac{b_{k+1} - b_k}{\log b_k} \leq 1/\lambda_s.$$  

If $\alpha < 1/\lambda_s$ then

$$Z_n = \sum_{i \leq n} P(F_i) = \sum_{i \leq n} i^{-\alpha \lambda_s + o(1)} = n^{1-\alpha \lambda_s + o(1)} \to \infty.$$  

If in addition

$$\lim_{n \to \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

Theorem 2 implies that with probability 1 infinitely many events $F_i$ occur and

$$\limsup_{k \to \infty} \frac{b_{k+1} - b_k}{\log b_k} \geq 1/\lambda_s.$$  

We next prove (3). We observe that

$$F_i \cap F_j = \bigcap_{\omega \in \Omega_i \cup \Omega_j} E_{\omega}^c,$$

so we can use Janson inequality to get

$$P(F_i \cap F_j) \leq \prod_{\omega \in \Omega_i \cup \Omega_j} P(E_{\omega}^c) \times \exp \left( 2 \sum_{\omega, \omega' \in \Omega_i \cup \Omega_j} P(E_\omega \cap E_{\omega'}) \right).$$
Observe that
\[
\sum_{\omega, \omega' \in \Omega_i \cup \Omega_j} P(E_\omega \cap E_{\omega'}) \leq \sum_{\omega, \omega' \in \Omega_i} P(E_\omega \cap E_{\omega'}) + \sum_{\omega, \omega' \in \Omega_j} P(E_\omega \cap E_{\omega'}) + \sum_{\omega \in \Omega_i, \omega' \in \Omega_j} P(E_\omega \cap E_{\omega'}). \]

Applying Lemma 4 to the three sums we have

Thus,
\[
\text{Since } \alpha_1 \leq \lambda \text{, the number } \beta = (1 - \alpha_1)/2 \text{ is positive. Now we split the sum in (3) into three sums:}
\]

\[
\Delta_1n = \sum_{1 \leq i < j \leq n \atop n^2 < i < j - \alpha \log j} P(F_i \cap F_j) - P(F_i)P(F_j)
\]

\[
\Delta_2n = \sum_{1 \leq i < j \leq n \atop i < n^2} P(F_i \cap F_j) - P(F_i)P(F_j)
\]

\[
\Delta_3n = \sum_{1 \leq i < j \leq n \atop j - \log j < i \leq j} P(F_i \cap F_j) - P(F_i)P(F_j)
\]

i) Estimate of \(\Delta_1n\). Since in this case we have \(I_i \cap I_j = \emptyset\), we can apply Lemma 7 to (5) to get

\[
\prod_{\omega \in \Omega_i \cup \Omega_j} P(E_\omega^c) \leq P(F_i)P(F_j)(1 + O(j^{-1/6}\log j)).
\]

This inequality and (5) gives

\[
P(F_i \cap F_j) \leq P(F_i)P(F_j)(1 + O(j^{-1/6}\log j)),
\]

so

\[
P(F_i \cap F_j) - P(F_i)P(F_j) \ll P(F_i)P(F_j)i^{-1/6}(\log i)^2(\log j) \ll n^{-\beta/2 + o(1)}P(F_i)P(F_j).
\]
Thus

\[ \Delta_{1n} \ll n^{-\beta/s + o(1)} \sum_{i,j \leq n} P(F_i)P(F_j) \ll n^{-\beta/s + o(1)}Z_n^2. \]  

ii) Estimate of \( \Delta_{2n} \). In this case we use the crude estimate

\[ P(F_i \cap F_j) - P(F_i)P(F_j) \leq P(F_i \cap F_j) \leq P(F_j). \]

We have

\[ \Delta_{2n} \leq \sum_{j \leq n} P(F_j) \leq n^\beta \sum_{j \leq n} P(F_j) \leq n^{\beta + o(1)}Z_n^2, \]

since \( Z_n = n^{1-\alpha} + o(1) = n^{2\beta + o(1)} \).

iii) Estimate of \( \Delta_{3n} \). Again we use (7) and we have

\[ \Delta_{3n} \leq \sum_{j \leq n} \sum_{\log j \leq i \leq j} P(F_j) \leq \alpha \log n \sum_{j \leq n} P(F_j) \leq n^{-2\beta + o(1)}Z_n^2. \]

Finally, using the estimates in (6), (8) and (9) we have

\[ \sum_{1 \leq i < j \leq n} \frac{P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} \ll n^{-\beta/s + o(1)} + n^{-\beta + o(1)} + n^{-2\beta + o(1)} \to 0. \]

This ends the proof of (3) and hence that of Theorem 1.

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