Toric Ideals of Lattice Path Matroids and Polymatroids

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June 15, 2010

Abstract

We show that the toric ideal of a lattice path polymatroid is generated by quadrics corresponding to symmetric exchanges, and give a monomial order under which these quadrics form a Gröbner basis. We then obtain an analogous result for lattice path matroids.

1 Introduction

If \( B = \{i_1, i_2, \ldots, i_r\} \) is a basis of a matroid \( M \), the toric map of \( M \) sends the base ring variable \( Y_B \) to the monomial \( x_{i_1} x_{i_2} \cdots x_{i_r} \) and is naturally extended over all polynomials of variables indexed by bases of \( M \). White has conjectured [11] that the kernel of this map is generated by quadratic binomials corresponding to symmetric exchanges between pairs of bases of \( M \).

Lattice path matroids, introduced by Bonin, de Mier and Noy in [3] and studied further in [4], are an especially nice class of matroids whose bases are in correspondence with certain planar lattice paths. Subclasses of these matroids appeared in [8] and [1]. In [10], the study of enumerative properties of such matroids gave rise to a related class of discrete polymatroids, in the sense of Herzog and Hibi [7], known as lattice path polymatroids.

As in [7], toric maps can be defined for discrete polymatroids as well, inspiring a generalization of White’s conjecture. In Theorem 3.1 we show that White’s conjecture holds for lattice path polymatroids. We also provide a monomial order under which the generating set of symmetric exchange binomials forms a Gröbner basis. We then show how a lattice path matroid can be realized as the set of squarefree bases of a related lattice path polymatroid, allowing us to prove an analogue of Theorem 3.1 for such matroids.

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2 Preliminaries

We assume the reader has a basic knowledge of matroid theory (see [9]). All our monomials are in the variables \( \{x_0, x_1, x_2, \ldots \} \). When \( m \) is a monomial, we write \( d_i(m) \) to mean the degree of \( x_i \) in \( m \).

**Definition 2.1.** Let \( \Gamma \) be a finite collection of monomials. Then \( \Gamma \) is a discrete polymatroid if it satisfies the following two properties: 1) If \( m \in \Gamma \) and \( m' \) divides \( m \), then \( m' \in \Gamma \), and 2) If \( m, m' \in \Gamma \) and the degree of \( m \) is greater than that of \( m' \), there exists an index \( i \) such that \( d_i(m) > d_i(m') \) and \( x_im' \in \Gamma \).

Thus, a matroid can be viewed as a squarefree discrete polymatroid. It is easily seen that the maximal monomials of a discrete polymatroid \( \Gamma \) are all of the same degree. In keeping with standard matroid terminology, we refer to these maximal monomials as bases, and we say their degree is the rank of \( \Gamma \). Discrete polymatroids were introduced by Herzog and Hibi in [7], where the following polymatroidal analogue of the classical symmetric exchange property for matroids was proven.

**Proposition 2.2.** Let \( m \) and \( m' \) be bases of a discrete polymatroid \( \Gamma \), and choose \( i \) with \( d_i(m) > d_i(m') \). Then there exists an index \( j \) with \( d_j(m) < d_j(m') \), such that both \( \frac{x_i}{x_j}m \) and \( \frac{x_j}{x_i}m' \) are bases of \( \Gamma \).

In the case of the above proposition, we say that the bases \( \frac{x_j}{x_i}m \) and \( \frac{x_i}{x_j}m' \) are obtained from \( m \) and \( m' \) via a symmetric exchange.

2.1 Lattice paths

Fix two integers \( n, r > 0 \). For our purposes, a lattice path is a sequence of unit-length steps in the plane, each either due north or east, beginning at the origin and ending at the point \( (n, r) \). For a lattice path \( \sigma \), define a set \( N(\sigma) \subseteq [n + r] \) by the following rule: \( i \in N(\sigma) \iff \) the \( i \)th step of \( \sigma \) is north.

Let \( \sigma \) and \( \tau \) be lattice paths. We say that \( \sigma \) is above \( \tau \) if for all \( i \leq n \) the \( i \)th east step of \( \sigma \) lies on or above the \( i \)th east step of \( \tau \). In this case, we write \( \sigma \succeq \tau \).

Now fix two lattice paths \( \alpha \) and \( \omega \) with \( \alpha \succeq \omega \).

**Theorem 2.3.** [3] The collection \( \{N(\sigma) : \alpha \succeq \sigma \succeq \omega \} \) is the set of bases of a matroid.

We write \( M(\alpha, \omega) \) to denote the matroid determined by the paths \( \alpha \) and \( \omega \).

Matroids arising in this fashion are known as lattice path matroids. For any lattice path \( \sigma \), define a monomial \( m(\sigma) \) by the following rule: the degree of \( x_i \) in \( m(\sigma) \) is the number of north steps taken by \( \sigma \) along the vertical line \( x = i \).

**Theorem 2.4.** [10] The collection \( \{m(\sigma) : \alpha \succeq \sigma \succeq \omega \} \) is the set of bases of a discrete polymatroid.
Figure 1: The lattice path matroid $\mathcal{M}(\alpha, \omega)$ where $N(\alpha) = \{1, 2, 4, 6\}$ and $N(\omega) = \{3, 5, 7, 8\}$. If $\sigma$ is the bold path, $m(\sigma) = x_1^3 x_3$.

We call such discrete polymatroids \textit{lattice path polymatroids}, and write $\Gamma(\alpha, \omega)$ to denote the polymatroid determined by $\alpha$ and $\omega$. For a lattice path $\sigma$ whose last step is east, let $\sigma^+$ be the path obtained from $\sigma$ by removing its last east step, and adding an east step at the beginning. It is easily seen that the lattice path matroid $\mathcal{M}(\alpha, \omega)$ is coloop-free if and only if $\alpha$ and $\omega$ share no north steps (and thus the last step of $\alpha$ is east). That is, $\mathcal{M}(\alpha, \omega)$ is coloop-free if and only if $\alpha^+ \succeq \omega$. The following theorem motivated the introduction of lattice path polymatroids.

\textbf{Theorem 2.5.} [10] Suppose the lattice path matroid $\mathcal{M}(\alpha, \omega)$ is coloop-free. Then its $h$-vector is the $f$-vector (or degree sequence) of $\Gamma(\alpha^+, \omega)$.

\textbf{Example 2.6.} If $\alpha$ is the path consisting of $r$ north steps followed by $n$ east steps and $\omega$ is any other path, then $\mathcal{M}(\alpha, \omega)$ is a shifted matroid. In [8], it is shown that every shifted matroid is of this form. In this case, bases of the polymatroid $\Gamma(\alpha, \omega)$ are generators of the smallest Borel-fixed ideal containing $m(\omega)$ (see [6]).

2.2 Toric ideals

The base ring of a polymatroid $\Gamma$ is the polynomial ring $\mathbb{C}[Y_m : m \text{ is a basis of } \Gamma]$. If $n$ and $n'$ are obtained from $m$ and $m'$ by a symmetric exchange, we call $Y_n Y_{n'} - Y_{n'} Y_n$ a symmetric exchange binomial. The toric ideal of $\Gamma$ is the kernel of the map $\phi : \mathbb{C}[Y_m : m \text{ is a basis of } \Gamma] \rightarrow \mathbb{C}[x_0, x_1, x_2, \ldots]$ defined by

\[ \phi(Y_{m_1} Y_{m_2} \cdots Y_{m_t}) = m_1 m_2 \cdots m_t \]

and extended by linearity. Clearly, any symmetric exchange binomial is in the toric ideal of $\Gamma$.

\textbf{Conjecture 2.7} (White’s conjecture, adapted for polymatroids). \textit{The toric ideal of } $\Gamma$ \textit{is generated by symmetric exchange binomials.}

For a set $V = \{m_1, m_2, \ldots, m_t\}$ of bases of $\Gamma$, write $M_V$ as short for the base ring monomial $Y_{m_1} Y_{m_2} \cdots Y_{m_t}$. Now for any monomial $\mu$ of degree $> r$, we define a simple graph $\mathcal{G}(\mu)$, known as a \textit{symmetric exchange graph}, as follows. The vertices of $\mathcal{G}(\mu)$ are all sets $V = \{m_1, m_2, \ldots, m_t\}$ of bases of $\Gamma$ with $\phi(M_V) = \mu$ (that is,
Suppose that $G(\mu)$ is connected for any monomial $\mu$ of degree $> r$. Then Conjecture 2.7 holds for $\Gamma$.

Proof. Any polynomial in the toric ideal of $\Gamma$ can be written as a sum of binomials of the form $M_V - M_W$, where $V$ and $W$ are vertices of some $G(\mu)$. Since $G(\mu)$ is connected, there exists a path $V = V_0, V_1, V_2, \ldots, V_k = W$ where each $V_i$ and $V_{i+1}$ are connected by an edge. Now write

$$M_V - M_W = (M_V - M_{V_1}) + (M_{V_1} - M_{V_2}) + (M_{V_2} - M_{V_3}) + \cdots + (M_{V_{k-1}} - M_W).$$

Since each parenthesized term in this sum is the product of a monomial with a symmetric exchange binomial, the result follows. \qed
where \( \ell \) to \((i,j)\) \(\Gamma\) so that we may apply Theorem 2.10. Therefore every \( \mathcal{G}(\mu) \) is connected, and Theorem 2.8 gives us that the toric ideal of \( \Gamma \) is generated by symmetric exchange binomials.

To finish the proof, we apply Buchberger’s algorithm (again, see [5]) to the set of symmetric exchange binomials. The S-pair of two symmetric exchange binomials consists either of replacing \( M_V \) with \( M_{V'} \), where \( V \to V' \) is a directed edge of \( \mathcal{G}^{\ell}(\mu) \) or of replacing \( M_W \) with \( M_{W'} \), where \( W \to W' \) is a directed edge of \( \mathcal{G}^{\ell}(\mu) \). Let \( V_0 \) be the unique sink of \( \mathcal{G}^{\ell}(\mu) \). By Lemma 2.9 this binomial reduces to \( M_{V_0} - M_{V_0} = 0 \).

\[ \square \]

3 LATTICE PATH POLYMATROIDS

For the remainder of this paper, fix \( n \) and \( r \) and let \( \alpha \) and \( \omega \) be two lattice paths to \((n,r)\) with \( \alpha \succeq \omega \). To eliminate excess notation we often identify a path \( \sigma \) with the associated monomial \( m(\sigma) \), writing, for example, \( Y_\sigma \) rather than \( Y_{m(\sigma)} \) and \( d_\sigma(\sigma) \) rather than \( d_\sigma(m(\sigma)) \). This section is devoted to proving the following theorem.

**Theorem 3.1.** Let \( \Gamma = \Gamma(\alpha,\omega) \) be a lattice path polymatroid. Then the toric ideal of \( \Gamma \) is generated by symmetric exchange binomials. Moreover, there exists a monomial order on the base ring of \( \Gamma \) under which the symmetric exchange binomials form a \( \text{Gröbner basis for the toric ideal.} \)

First, we build a monomial order on the base ring of a lattice path polymatroid \( \Gamma \) so that we may apply Theorem 2.10.

For any lattice path \( \sigma \), define \( \ell(\sigma) \) to be the following \( nr \)-tuple:

\[
(\ell_{0,r}, \ell_{0,r-1}, \ldots, \ell_{0,1}, \ell_{1,r}, \ell_{1,r-1}, \ldots, \ell_{1,1}, \ldots, \ell_{n-1,r}, \ell_{n-1,r-1}, \ldots, \ell_{n-1,1})
\]

where \( \ell_{i,j} = 1 \) if the topmost north step of \( \sigma \) along the line \( x = i \) goes from \((i,j-1)\) to \((i,j)\), and \( \ell_{i,j} = 0 \) otherwise. For a base ring monomial \( M = Y_{\sigma_1} Y_{\sigma_2} \cdots Y_{\sigma_t} \), let \( \ell(M) = \sum_{1 \leq i \leq t} \ell(\sigma_i) \), where the sum of vectors is taken componentwise.

Now let \( M \) and \( M' \) be base ring monomials, and write \( M >_\ell M' \) whenever \( \ell(M) \) lexicographically precedes \( \ell(M') \). Note that \( >_\ell \) is not yet a total order, as clearly there may be monomials \( M \neq M' \) with \( \ell(M) = \ell(M') \).

Indeed, let \( M = Y_{\sigma_1} Y_{\sigma_2} \cdots Y_{\sigma_t} \) and \( M' = Y_{\tau_1} Y_{\tau_2} \cdots Y_{\tau_t} \) be two distinct base ring monomials with \( \ell(M) = \ell(M') \), where the indexing paths of these monomials are ordered so that whenever \( i < j \), \( \ell(\sigma_i) \) lexicographically precedes \( \ell(\sigma_j) \) and \( \ell(\tau_i) \) precedes \( \ell(\tau_j) \). Extend the definition of \( >_\ell \) to say that \( M >_\ell M' \) if \( \ell(\tau_i) \) lexicographically precedes \( \ell(\sigma_i) \) for the least \( i \) such that \( \sigma_i \neq \tau_i \).
If \( n = r = 4 \) and \( \sigma_1 \) and \( \sigma_2 \) are the paths above, \( \ell(Y_{\sigma_1}Y_{\sigma_2}) = (0, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0) \).

Since a path \( \sigma \) is clearly determined by the vector \( \ell(\sigma) \), this completes \( >_\ell \) to a total order on all monomials in the base ring (once we set \( M >_\ell 1 \) for any monomial \( M \)). Moreover, if \( M >_\ell M' \), then \( MN >_\ell M'N \), since \( \ell(MN) = \ell(M) + \ell(N) \) and \( \ell(M'N) = \ell(M') + \ell(N) \). Thus \( >_\ell \) is a monomial order.

**Definition 3.2.** Let \( V = \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \) be a vertex of \( G(\mu) \) for some \( \mu \), where again we have written \( \sigma_i \) to mean \( m(\sigma_i) \). We call the vertex \( V \) thin if it has the following two properties:

1: For any two paths \( \sigma_i, \sigma_j \in V \), either \( \sigma_i \succeq \sigma_j \) or \( \sigma_j \succeq \sigma_i \).

2: For any \( i \), the \( i \)th east steps of any two paths in \( V \) are at most a unit length apart.

**Proposition 3.3.** Let \( V \) be a vertex of some \( G(\mu) \) that is not thin. Then there is a vertex \( V' \) of \( G(\mu) \) resulting from a symmetric exchange between two bases in \( V \) such that \( M_V >_\ell M_{V'} \). In other words, \( V \rightarrow V' \) is a directed edge of \( G^\ell(\mu) \).

**Proof.** Let \( V = \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \). We handle two cases, each corresponding to a way in which a vertex may fail to be thin.

First suppose that the \((i + 1)\)st east step of some path in \( V \) is more than a unit length above the \((i + 1)\)st east step of another path in \( V \), and let \( i \) be minimal with
this property. Let $\sigma_p$ be the path with the highest $(i+1)^{st}$ east step, and let $\sigma_q$ be the path with the lowest. By the minimality of $i$, $d_i(\sigma_p) > d_i(\sigma_q)$. Since the two paths eventually meet, there must be some $j > i$ such that $d_j(\sigma_p) < d_j(\sigma_q)$. Let $j$ be minimal with this property, let $\sigma'_p$ be the path obtained from $\sigma_p$ by adding a north step along $x = j$ and removing one along $x = i$, and let $\sigma'_q$ be the path obtained from $\sigma_q$ by adding a north step along $x = i$ and removing one along $x = j$. Note that $\sigma'_p$ and $\sigma'_q$ are the results of a symmetric exchange between $\sigma_p$ and $\sigma_q$, although we still need to show that both $\sigma'_p$ and $\sigma'_q$ are paths in $\Gamma(\alpha, \omega)$. To see this, note that the minimality of $j$ implies that every east step of $\sigma_p$ between $x = i$ and $x = j$ is strictly above the corresponding east step of $\sigma_q$. Thus $\sigma_p \succeq \sigma'_p \succeq \sigma_q$ and $\sigma_p \succeq \sigma'_q \succeq \sigma_q$, meaning both $\sigma'_p$ and $\sigma'_q$ are between $\alpha$ and $\omega$. Let $V'$ be the vertex resulting from this symmetric exchange. Then $V'$ is identical to $V$ to the left of the line $x = i$. Since neither $\sigma'_p$ nor $\sigma'_q$ attains the same height on the line $x = i$ as $\sigma_p$, it follows that $M_V >_\ell M_{V'}$.

Now suppose that no two paths in $V$ are ever more than a unit length apart, and let $i$ be the least index so that $V$ fails to be thin at the line $x = i$. Then there are paths $\sigma_p$ and $\sigma_q$ of $V$ such that every east step of $\sigma_p$ to the left of $x = i$ is on or above the corresponding east step of $\sigma_q$ (though the two do not always coincide), but the $(i+1)^{st}$ step of $\sigma_q$ is a unit length above that of $\sigma_p$. It is clear that $d_i(\sigma_p) > d_i(\sigma_q)$. Let $j$ be the least index greater than $i$ such that $d_j(\sigma_p) > d_j(\sigma_q)$, let $\sigma'_p$ be the path obtained from $\sigma_p$ by deleting a north step along $x = j$ and adding one along $x = i$, and let $\sigma'_q$ be the path obtained from $\sigma_q$ by deleting a north step along $x = i$ and adding one along $x = j$. The same argument from the first paragraph of this proof shows that both $\sigma'_p$ and $\sigma'_q$ are paths in $\Gamma(\alpha, \omega)$. Again, let $V'$ be the vertex resulting from this symmetric exchange. Since every east step of $\sigma_p$ in between $x = i$ and $x = j$ is exactly a unit length above the corresponding east step of $\sigma_q$, it follows that $\ell(M_V) = \ell(M_{V'})$. Writing $>_\text{lex}$ for lexicographic order, we have the following chain:

$$
\ell(\sigma'_p) >_\text{lex} \ell(\sigma_p) >_\text{lex} \ell(\sigma_q) >_\text{lex} \ell(\sigma'_q).
$$

Thus $M_V >_\ell M_{V'}$. 

\[\square\]

**Lemma 3.4.** Let $\mu$ be a monomial so that $\mathcal{G}(\mu)$ is nonempty. Then $\mathcal{G}(\mu)$ has exactly one thin vertex.

**Proof.** Existence follows from Proposition 3.3 and the easy fact that a finite acyclic directed graph has at least one sink.

To prove uniqueness, let $V = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ be a thin vertex, ordered so that $\sigma_1 \succeq \sigma_2 \succeq \cdots \succeq \sigma_t$, and suppose $V$ is uniquely determined to the left of the line $x = i$ (where we allow $i = 0$). Since $V$ is thin, there is an index $k$ and a number $p$ so that the $i^{th}$ east steps of the paths $\sigma_1, \sigma_2, \ldots, \sigma_k$ coincide and lie on the line $y = p$ and the $i^{th}$ east steps of the paths $\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_t$ coincide and lie on the line $y = p - 1$. Now write $d_i(\mu) = qt + r$, with $r < t$. 

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If \( r \leq t - k \), then the paths \( \sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_{k+r} \) must each have \( q + 1 \) steps along the line \( x = i \), while the rest must have \( q \) north steps along this line. If \( r > t - k \), then each of the paths \( \sigma_1, \sigma_2, \ldots, \sigma_{t-k} \) and \( \sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_t \) must have \( q + 1 \) steps along \( x = i \), and the rest must have \( q \) steps.

Thus, \( V \) is uniquely determined to the left of the line \( x = i + 1 \), and the result follows.

**Proof of Theorem 3.1.** Proposition 3.3 and Lemma 3.4 imply that any \( G^\ell(\mu) \) has a unique sink (namely its thin vertex), so Theorem 2.10 finishes the proof.

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**4 Lattice path matroids**

The goal of this section is to prove the following analogue of Theorem 3.1 for lattice path matroids.

**Theorem 4.1.** Let \( M = M(\alpha, \omega) \) be a lattice path matroid. Then the toric ideal of \( M \) is generated by symmetric exchange binomials. Moreover, there exists a monomial order on the base ring of \( M \) under which the symmetric exchange binomials form a Gröbner basis for the toric ideal.

**Proof.** Let \( \sigma \) be a lattice path to the point \((n, r)\) with \( N(\sigma) = \{a_1, a_2, \ldots, a_r\} \), where \( a_1 < a_2 < \cdots < a_r \). Define a lattice path \( \overline{\sigma} \) to the point \((n + r, r)\) by \( N(\overline{\sigma}) = \{a_1 + 1, a_2 + 2, \ldots, a_r + r\} \), and note that \( m(\overline{\sigma}) = x_{a_1}x_{a_2}\cdots x_{a_r} \).

![Figure 4: A lattice path matroid](image)

Figure 4: A lattice path matroid \( M(\alpha, \omega) \) and the associated polymatroid \( \Gamma(\alpha, \omega) \).

If \( \sigma \) is the bold path, note that \( N(\sigma) = \{2, 3, 4, 7\} \) and \( m(\overline{\sigma}) = x_2x_3x_4x_7 \).

Define a function from \( M = M(\alpha, \omega) \) to \( \Gamma = \Gamma(\alpha, \omega) \) by \( \sigma \to \overline{\sigma} \), and note that a lattice path \( \sigma \) in between \( \overline{\sigma} \) and \( \overline{\sigma} \) is in the image of this map if and only if it has no more than one north step along every line \( x = i \), which is equivalent to \( m(\sigma) \) being squarefree.

For a vertex \( V = \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \) of some \( G_M(\mu) \), let \( \overline{V} = \{\overline{\sigma_1}, \overline{\sigma_2}, \ldots, \overline{\sigma_t}\} \) denote the corresponding vertex of \( G_\Gamma(\mu') \) for some monomial \( \mu' \). We can now define a monomial order \( >_L \) on the base ring of \( M \) by

\[
M_V >_L M_{\overline{V}} \iff M_{\overline{\sigma}} >_L M_{\overline{\sigma'}}.
\]
Thus, the graph $\mathcal{G}_L^\mu(M)$ is a directed subgraph of $\mathcal{G}_L^\mu(\mu')$ for some monomial $\mu'$. Because a symmetric exchange between two squarefree monomials results in two squarefree monomials, Proposition 3.3 and Lemma 3.4 imply that each directed graph $\mathcal{G}_L^\mu(M)$ has a unique sink, and we can apply Theorem 2.10.

Acknowledgements. Thanks to Craig Huneke and Joe Bonin for many insightful conversations.

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