Implications of a running spectral index for slow roll inflation

Richard Easther\(^1\) and Hiranya V Peiris\(^2,3\)

\(^1\) Department of Physics, Yale University, New Haven, CT 06520, USA
\(^2\) Kavli Institute for Cosmological Physics and Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA
E-mail: richard.easther@yale.edu and hiranya@cfcp.uchicago.edu

Received 5 July 2006
Accepted 24 August 2006
Published 18 September 2006

Online at stacks.iop.org/JCAP/2006/i=09/a=010
doi:10.1088/1475-7516/2006/09/010

Abstract. We analyse the weak (2\(\sigma\)) evidence for a running spectral index seen in the three year WMAP data set and its implications for single field, slow roll inflation. We assume that the running is comparable to the central value found from the WMAP data analysis, and use the Hubble slow roll formalism to follow the evolution of the slow roll parameters. For all parameter choices consistent with a large, negative running, single field, slow roll inflation lasts less than 30 e-folds after CMB scales leave the horizon. Thus, a definitive observation of a large negative running would imply that any inflationary phase requires multiple fields or the breakdown of slow roll. Alternatively, if single field, slow roll inflation sources the primordial fluctuations, we can expect the observed running to move much closer to zero as the CMB is measured more accurately at small angular scales.

Keywords: CMBR experiments, CMBR theory, inflation, power spectrum

\(^3\) Hubble Fellow.
One of the most perplexing features of the three year data set\textsuperscript{4} from the Wilkinson Microwave Anisotropy Probe (WMAP) \cite{1}--\cite{3} is that it hints at a significant \textquoteleft running\textquoteright{} in the scalar spectral index. This feature was seen, with roughly the same level of significance, in the WMAPI analysis \cite{4,5}. The evidence for a running index persists when large scale structure data are added to the analysis \cite{3}, but is diluted by the addition of Lyman-\alpha forest data, both with WMAPI \cite{6} and WMAPII \cite{7}--\cite{9}. Joint analyses require a melange of data sets to increase their coverage in $k$-space and thus harbour the possibility of systematic normalization uncertainties or a tension in relative normalization between data sets, which could manifest itself as a spurious running. This source of error is eliminated when WMAPII is considered on its own, but only at the cost of reduced $k$-coverage.

We stress that WMAPII does not demand a non-vanishing running, as zero is $\sim 2\sigma$ from the central value. Rather, we spell out the consequences of a large running—particularly with a negative sign—and the likely consequences for inflation. We find that a running similar to the WMAPII centroid rules out all simple models of inflation: those driven by a single, minimally coupled, scalar field whose evolution is well described by the slow roll formalism. Inflation, if it happens at all, would thus be non-minimal. Alternatively, future analyses of data sets with a larger $k$-coverage will yield a running much closer to zero than the value extracted from WMAPII.

We proceed by using the Hubble slow roll (HSR) hierarchy \cite{10}--\cite{14}, which connects astrophysical determinations of the primordial power spectra (both tensor and scalar) to the inflationary potential. In \cite{14} we showed how a Monte Carlo Markov chain (MCMC) analysis of cosmological data can constrain the HSR parameters. These parameters are associated with flow equations that determine their scale dependence. We use this system of equations to compute the number of e-folds of inflation which occur after a mode with wavenumber $k_0$ leaves the horizon. Choosing $\xi$ large enough to explain the central value found for the running in Spergel \textit{et al} guarantees that any single field model of slow roll inflation will run for less than 30 e-folds after CMB scales leave the horizon.

At a minimum, fitting the parameters in the HSR hierarchy to data provides an optimal way of imposing an \textquoteleft inflationary prior\textquoteright{} on the cosmological parameter estimation

\textsuperscript{4} We denote this as WMAPII, and the one year data set as WMAPI.
process. The most optimistic outcome of this process is that we can constrain these parameters well enough to effectively ‘reconstruct’ a portion of the inflaton’s effective potential from astrophysical observations. As we explain below, one can increase the number of e-folds by adding a fourth slow roll parameter. This term would be third order in the slow roll expansion\(^5\), and at least four parameters would be needed to specify the inflationary dynamics. In this case, the slow roll approximation may apply in the sense that the inflaton energy density is dominated by the potential. However, the first four terms of the slow roll expansion would be on a roughly equal footing, putting severe constraints on the inflationary parameter space.

2. The slow roll approximation

As summarized in [14], the dynamics of single field inflation can be written in the Hamilton–Jacobi form, where overdots correspond to time derivatives and primes denote derivatives with respect to \(\phi\). The HSR parameters \(\ell \lambda_H\) obey the infinite hierarchy of differential equations

\[
\epsilon(\phi) \equiv \frac{m_{Pl}^2}{4\pi} \left[ \frac{H'(\phi)}{H(\phi)} \right]^2;
\]

\[
\ell \lambda_H \equiv \left( \frac{m_{Pl}^2}{4\pi} \right) \ell \left( \frac{H'}{H} \right)^{\ell-1} \frac{d(\ell+1)H}{d\phi(\ell+1)}, \quad \ell \geq 1.
\]

The usual slow roll parameters are \(\eta = \lambda_H^1\) and \(\xi = \lambda_H^2\). If we set the higher order terms to zero at some fiducial point, these differential equations ensure they vanish at all other times. The potential is given by

\[
V(\phi) = \frac{3m_{Pl}^4}{8\pi} H^2(\phi) \left[ 1 - \frac{1}{3} \epsilon(\phi) \right].
\]

Liddle showed that the hierarchy can be solved exactly when truncated at order \(M\) [13], so

\[
\frac{H(\phi)}{H_0} = 1 + B_1 \left( \frac{\phi}{m_{Pl}} \right) + \cdots + B_{M+1} \left( \frac{\phi}{m_{Pl}} \right)^{M+1}.
\]

The \(B_i\) are specified by the initial values of the HSR parameters,

\[
B_1 = \sqrt{4\pi \epsilon_0}, \quad B_{\ell+1} = \frac{(4\pi)^\ell}{(\ell+1)!} B_{1}^{\ell+1} \lambda_{H,0}^{\ell}.
\]

where the subscript 0 refers to their values at the moment the fiducial mode \(k_0\) leaves the horizon, and \(\phi = \phi_0 = 0\). The number of e-folds, \(N\), is given by

\[
\frac{dN}{d\phi} = \frac{4\pi}{m_{Pl}^2} \frac{H}{H'};
\]

Finally, \(\phi\) and \(k\) are related by

\[
\frac{d\phi}{d\ln k} = -\frac{m_{Pl}}{2\sqrt{\pi}} \frac{\sqrt{\epsilon}}{1 - \epsilon}.
\]

\(^5\) The usual counting is that \(\epsilon\) and \(\eta\) are the lowest order terms, \(\xi\) second order, and the next term is thus third order.
We now turn to the inflationary observables,

\[ n_s = 1 + 2\eta - 4\epsilon - 2(1 + C)\epsilon^2 - \frac{1}{2}(3 - 5C)\epsilon\eta + \frac{1}{2}(3 - C)\xi, \]

\[ r = 16\epsilon [1 + 2C(\epsilon - \eta)], \]

\[ \alpha = -\frac{1}{1 - \epsilon} \left\{ 2\xi + 8\epsilon^2 - 10\epsilon\eta + \frac{7C - 9}{2}\epsilon\xi + \frac{3 - C}{2}\xi\eta \right\}, \]

where \( C = 4(\ln 2 + \gamma) - 5 \), \( C' = -2 + \ln 2 + \gamma \), and we have introduced the customary notation \( \alpha = \frac{dn_s}{d\ln k} \) and \( r \) is tensor:scalar ratio.\(^6\) We retain all terms in \( \alpha \) up to quadratic order in the slow roll parameters, anticipating that \( \xi \) may be as large as \( \epsilon \) or \( \eta \).

### 3. Cosmological constraints

Consider the following very weak constraints:

\[ 0.8 < n_s < 1.2, \]

\[ r < 1.2. \]

These bounds are very generous, and correspond to the 2σ error on a fit to WMAP alone with \( r \) and \( \frac{dn_s}{d\ln k} \) in the parameter set. The allowed region shrinks considerably when large scale structure data are added. With a pivot \( k_0 = 0.002 \text{ Mpc}^{-1} \) the central value of \( n_s \) is 1.16.\(^7\) However, this result depends sensitively on the choice of the pivot. We can choose a \( k_0 \) inside the range of scales that contribute to the CMB so that the central value of \( n_s \) is unity, and the allowed range falls inside the bounds specified by equation (11). Making this adjustment will have no impact on the analysis that follows.

The running is given by equation (10), where \( \xi \) is the only term linear in the slow roll parameters and it dominates the expression when \( |\alpha| \) is large. In particular, for large negative \( \alpha \), we need \( \xi > 0 \). While \( \epsilon \) and \( \eta \) can become large for fixed \( n_s \), they make a positive contribution if one moves out along the degeneracy direction \( 2\eta \sim 4\epsilon \), thanks to the coefficients on \( \epsilon\eta \) and \( \epsilon^2 \) in equation (10). Figure 1 shows the cuts these constraints put on the \((\epsilon, \eta)\) plane for \( \xi = 0 \) and \( \xi = 0.01 \). Finally, we show the region where \( \alpha > 0 \).

Again we remind the reader that WMAP II does not require that \( \alpha < 0 \) and the purpose of this paper is to explore the consequences of measuring a value of \( \alpha \) near the WMAP centroid. This number depends on whether one allows for a contribution from primordial tensors. For the pure scalar case, the distribution peaks at \( \alpha \sim -0.05 \). Including tensors actually makes the central value larger, but this is not necessary for us to establish the conclusions we reach below.

The minimal required amount of inflation is not well defined. If inflation is to happen before the electroweak phase transition, the number of e-folds \( N \) must be greater than 30, and \( N \sim 55 \) for GUT scale inflation. These numbers are mildly dependent on the reheating mechanism, but are sufficient for our purposes. Any model with \( N < 30 \) is

---

\(^6\) Beware of the distinction between \( C \) and \( C' \)—these quantities are sometimes confused in the literature.

\(^7\) [http://lambda.gsfc.nasa.gov/product/map/current/params/lcdm_run_tens_wmap.cfm](http://lambda.gsfc.nasa.gov/product/map/current/params/lcdm_run_tens_wmap.cfm)
Implications of a running spectral index for slow roll inflation

Figure 1. We show the regions of the $(\epsilon, \eta)$ plane excluded by the assumed bounds on the spectral parameters. As $\xi$ increases, the portion of the plane for which $\alpha > 0$ shrinks.

unlikely to provide a workable explanation for the large scale homogeneity and isotropy of the universe. Because we can compute the running of the HSR parameters as a function of $\phi$ (or, equivalently, $k$), we can obtain the remaining number of e-folds for any choice of $\{\epsilon_0, \eta_0, \xi_0\}$. The end of inflation is signified by the instant when $\epsilon = 1$, and we find $N$ from equation (6) alongside the flow equations (1) and (2). The result of this calculation is displayed in figure 2 for four values of $\xi$. We truncate the slow roll hierarchy at third order and choose $\xi$ consistent with a large negative running. We see that the fraction of the parameter space which satisfies the observed constraints on $n_s$ and $r$ while yielding $\alpha \lesssim -0.03$ and $N > 30$ is essentially of measure zero.

4. Discussion

We have shown that a large, negative running of the scalar spectral index cannot be produced by a single field inflationary model which is well described by the truncated slow roll expansion. Specifically, the large running leads to an unacceptably small value of $N$, the number of e-folds of inflation after the largest observable scales leave the horizon. The low values of $N$ can be ameliorated by adding a further slow roll
Implications of a running spectral index for slow roll inflation

Figure 2. We show the regions of the \((\epsilon, \eta)\) plane excluded by the assumed bounds on the spectral parameters for the same cuts displayed in figure 1. We include contours (for \(N = 15, 30, 45, 60\)) showing the number of e-foldings occurring after the fiducial mode \(k_0 = 0.002 \text{ Mpc}^{-1}\) leaves the horizon. For \(\xi = 0.03\) the only regions of the \((\epsilon, \eta)\) plane with an acceptable value of \(n_s\) have \(N < 15\). The running is a weak function of \(\epsilon\) and \(\eta\)—the values of \(\alpha\) above are taken from \(\xi\) alone.

parameter \((\lambda)\) to the analysis which, if carefully chosen, moves \(\xi\) closer to zero as inflation continues. However, in this case the inflationary dynamics are described by four numbers, \(\{\epsilon_0, \eta_0, \xi_0, 3\lambda_0\}\), all of equal importance. Such a model may be ‘slowly rolling’ in the sense that \(\epsilon \ll 1\), but the slow roll expansion is not trustworthy when its first four terms are of equal weight. Nor is there is any guarantee the sequence can be safely truncated after \(3\lambda_0\).

The difficulty of reconciling a large running with a sufficient number of e-folds of inflation has been recognized previously (e.g. [15, 16]), as the topic received considerable attention in the wake of the WMAP data release. Chung et al [15] show how to construct explicit models with a large running, while Malquarti et al [16] use a modified version of Monte Carlo reconstruction to illuminate the tension between a large running and providing the required number of e-folds. Further, a specific class of slowly rolling models with significant running is probed using Monte Carlo Reconstruction techniques in [17]. The specific contribution made in this paper is to fully spell out the range of \(\alpha\) that yields acceptable values of \(n_s\) and \(N\), without introducing higher order terms into the slow
roll expansion\(^8\). While we stress that the WMAPII data set has not made a definitive detection of any running, the current central value of \(\alpha\) derived from the WMAPII data set lies outside the permitted range.

A corollary of our result is that if the inflationary power spectrum does have significant scale dependence at CMB scales then this running must be transient. As noted above, the literature contains a number of inflationary models with a large running and \(N > 30\) (e.g. \([15, 16, 18, 19]\)). Without exception these models have a ‘running of the running’, so that \(d\alpha/d\ln k \neq 0\), and the running is only large over a finite range of \(k\) values. When the spectrum is characterized in terms of \(n_s\) and \(\alpha\), the running is constant by construction. Using the HSR formalism, \(\alpha\) is explicitly scale dependent, since \(\xi\) is a function of \(\phi\) and thus of \(N\). When we truncate the HSR hierarchy at \(\xi\), \(\xi(N)\) cannot change rapidly enough to produce a strongly scale dependent \(\alpha\): this requires the inclusion of a non-zero \(^3\lambda\). An explicit example of this type of model is given by \([18, 19]\), where the running is driven by a \(\phi^{22}\) term in the potential, and \(\xi\) is strongly scale dependent. One can compute \(d\xi/dN\) for this potential and it is dominated by the \(V^{(4)}\) term, which is correlated with \(^3\lambda\).

This precisely in accord with our analysis: a large running is not ruled out \textit{a priori} but obtaining such a running in combination with an acceptable number of e-folds necessitates non-trivial terms at high order in slow roll. Needless to say, there is no guarantee that inflation is well described by slow roll, or that the potential can be fully described by just one or two parameters. Models with ‘features’ in the potential yielding a local violation of the slow roll conditions have been considered in the past (e.g. \([20]–[22]\)). Further, models of inflation with two or more interacting fields can easily yield complicated spectra \([23]\), or one could turn to scenarios with two or more bursts of inflation \([24]–[28]\). However, invoking any of these options would dramatically complicate the theoretical understanding of inflation.

This analysis depends on the Hubble slow roll hierarchy, and its ability to incorporate constraints arising from the duration of inflation alongside those from the perturbation spectra \([14]\). The role of the \(\xi\) (or ‘jerk’) term in the dynamics of the flow equations is discussed in detail in \([29]\). This analysis focuses on ‘attractors’ in the slow roll parameter space in the presence of non-zero high order terms in the slow roll hierarchy. Likewise, inflationary models generated via the flow equations \([11, 22]\) or \textit{Monte Carlo reconstruction} \([12]\) can provide solutions with a large, negative running and \(N \gtrsim 55\), but these again rely on contributions from higher order terms. Cline and Hoi \([30]\) argue that the support for running in the WMAP II data set comes from the multipoles with \(30 \lessapprox \ell \lessapprox 40\) (which closely resemble their counterparts in the WMAPII data set), whereas the analysis here assumes that the running applies to the full range of \(k\) in the primordial spectrum. This is not unreasonable, since engineering a spectrum with a localized running would again require a significant contribution from higher order slow roll parameters.

Of course, the most prosaic resolution of this conundrum is that the weak evidence for a significant running in the WMAPII data set is a statistical artefact that will evaporate as more data become available. Indeed, adding Lyman-\(\alpha\) forest data is known to reduce the need for a negative value of \(\alpha\) \([6]–[9]\). In particular, we anticipate that the Planck mission will significantly enhance our understanding of \(\alpha\) by providing high quality measurements of the fundamental power spectrum over a larger wavelength range than WMAP. This will

\(^8\) Moreover, by avoiding the use a Monte Carlo sampling of the parameter space, we can be sure that we have not overlooked very small ‘corners’ of parameter space which might yield an acceptable spectrum.
allow a measurement of the running without the need to combine several heterogeneous data sets, a process that carries the risk of systematic biases in their relative normalization.

Acknowledgments

We are grateful to Jose Espinosa, Will Kinney, Eugene Lim, and members of the WMAP science team for useful discussions. RE is supported in part by the United States Department of Energy, grant DE-FG02-92ER-40704. HVP is supported by NASA through Hubble Fellowship grant No HF-01177.01-A awarded by the Space Telescope Science Institute, which is operated by the Association of Universities for Research in Astronomy, Inc., for NASA, under contract NAS 5-26555.

References

[1] Hinshaw G et al., 2006 Preprint astro-ph/0603451
[2] Page L et al., 2006 Preprint astro-ph/0603450
[3] Spergel D N et al., 2006 Preprint astro-ph/0603449
[4] Spergel D N et al. (WMAP Collaboration), 2003 Astrophys. J. Suppl. 148 175 [astro-ph/0302209]
[5] Bennett C L et al., 2003 Astrophys. J. Suppl. 148 1 [astro-ph/0302207]
[6] Seljak U et al. (SDSS Collaboration), 2005 Phys. Rev. D 71 103515 [SPIRES] [astro-ph/0407372]
[7] Lewis A., 2006 Preprint astro-ph/0603753
[8] Viel M, Haehnelt M G and Lewis A., 2006 Mon. Not. R. Astron. Soc. Lett. 370 L51 [astro-ph/0603753]
[9] Spergel D N et al. (WMAP Collaboration), 2003 Astrophys. J. Suppl. 148 175 [astro-ph/0302209]
[10] Seljak U, Slosar A and McDonald P., 2006 Preprint astro-ph/0603587
[11] Adams J A, Cresswell B and Easther R, 2001 Phys. Rev. D 64 083508 [SPIRES] [astro-ph/0102236]
[12] Peiris H V et al., 2003 Astrophys. J. Suppl. 148 213 [SPIRES] [astro-ph/0302225]
[13] Sasaki M and Stewart E D, 1996 Prog. Theor. Phys. 95 71 [SPIRES] [astro-ph/9507001]
[14] Silk J and Turner M S, 1987 Phys. Rev. D 35 419 [SPIRES]
[15] Holman R, Kolb E W, Vadas S L and Wang Y, 1991 Phys. Lett. B 269 252 [SPIRES]
[16] Polarski D and Starobinsky A A, 1992 Nucl. Phys. B 385 623 [SPIRES]
[17] Lyth D H and Stewart E D, 1996 Phys. Rev. D 53 1784 [SPIRES] [hep-ph/9510204]
[18] Burgess C P, Easther R, Mazumdar A, Mota D F and Multamaki T, 2005 J. High Energy Phys. JHEP05(2005)067 [SPIRES] [hep-th/0501125]
[19] Chongchitnan S and Elstathiou G., 2005 Phys. Rev. D 72 083520 [SPIRES] [astro-ph/0508355]
[20] Cline J M and Hoi L, 2006 Preprint astro-ph/0603403