Breather-like pulses in a medium with the permanent dipole moment

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Abstract
The solutions of the reduced Maxwell–Bloch equations for an anisotropic two-level medium, which describe the propagation of electromagnetic pulses having a duration from a few field oscillations, are studied. An influence of the permanent dipole moment of the quantum transition on dynamics of the pulses and their spectrum is considered.

Keywords: nonlinear coherent processes, optical solitons, resonant media, optical anisotropy, higher harmonic generation

1 Introduction

The significant attention was paid during the last years on investigation of the nonlinear coherent phenomena in the anisotropic media. Distinctive feature of such the media is that the stationary states of quantum particles being contained in them possess no the parity. For this reason, the diagonal elements of the matrix of the dipole moment operator and their difference commonly referred to as permanent dipole moment (PDM) of the transition are distinct from zero.

The role of PDM in the second harmonic generation in asymmetric semiconductor quantum wells was recognized and extensively studied (see, e.g., and references therein). Its relative influence on the conversion efficiency increases for larger pump/harmonic wavelengths, especially at higher pump powers. Also, propagation in the resonant optically uniaxial media of the two-component electromagnetic pulses consisting of the short-wavelength ordinary and long-wavelength extraordinary components was investigated in. It was revealed that, in the conditions of strong interaction between the components, pulses can pass through the anisotropic medium in the modes distinct from the self-induced transparency.

The propagation of one-component extremely short electromagnetic pulses of femtosecond durations through the media having PDM was considered in works. As opposite to the ultrashort pulses, a high-frequency carrier wave is absent in the Fourier spectrum of such the pulses. It was noted in that nonzero PDM may have a strong effect on the medium response for very short pulses, while an application of the usual scheme of the rotating-wave approximation eliminates its influence in the longer pulses case. Complete integrability of the reduced Maxwell–Bloch equations for the two-level medium with PDM in the frameworks of the inverse

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scattering transformation method\textsuperscript{14–15} was established in paper\textsuperscript{3}. The pulse solutions of these equations on constant (but not arbitrary) background were obtained. The background was chosen in such a manner that the problem considered was reduced to the isotropic medium case. Dynamics of these pulses at the pump presence was considered in\textsuperscript{4}. The system of the full Maxwell–Bloch equations was studied in work\textsuperscript{5}, where stationary extremely short pulses decaying exponentially and algebraically were found. An asymmetry caused by PDM on the polarity of these pulses was revealed. Also, the effects of passing the extremely short pulses with a duration up to several oscillations of electromagnetic field through a medium possessing PDM were investigated numerically\textsuperscript{10}. It was shown that there exists solitary stable bipolar signal with a nonzero time area (nonzero breather). The recent results summarized above make valuable to clarify the role of PDM in the formation the one-component pulses of the self-induced transparency (including, the extremely short pulses and ultrashort ones as the limiting cases). In present report, we shall study the exact solutions of the breather type of system of the reduced Maxwell–Bloch equations for an anisotropic medium.

2 The model and basic equations

Let’s consider optically uniaxial medium, whose anisotropy is created by the electric field. It splits the energy levels owing to the Stark effect, retaining the degeneration of electronic levels on the absolute value of projection $M$ of the total angular momentum. Thus, $\pi$-transition ($\Delta M = 0$) and doubly degenerate $\sigma$-transitions ($|\Delta M| = 1$) are formed in an electronic subsystem.

Let the electromagnetic pulse propagate in the positive direction of $y$ axis perpendicular to the optical axis $z$ of the medium. Suppose that only the extraordinary component $E_e$ of its electric field, which is parallel to $z$ axis, is distinct from zero. One can show that such the pulse will interact with the $\pi$-transition only. The system of reduced Maxwell–Bloch equations describing this process in the unidirectional propagation approximation\textsuperscript{16} has next form:

$$\frac{\partial \sigma_3}{\partial t} = i \frac{d}{\hbar} E_e (\sigma - \sigma^*)$$

(1)

$$\frac{\partial \sigma}{\partial t} = i \left( \omega_0 + \frac{D}{\hbar} E_e \right) \sigma + 2i \frac{d}{\hbar} E_e \sigma_3,$$

(2)

$$\frac{\partial E_e}{\partial y} + \frac{n_e}{c} \frac{\partial E_e}{\partial t} = -2\pi i N d \omega_0 \frac{n_e c}{n_e c} (\sigma - \sigma^*),$$

(3)

where $\sigma_3 = (\rho_{22} - \rho_{11})/2$ is the population inversion of the quantum level; $\sigma = \rho_{12}$; $\rho_{jk}$ ($k = 1, 2$) are the elements of the density matrix; $d$, $D$ and $\omega_0$ are the dipole moment, PDM and the frequency of $\pi$-transition, respectively; $n_e$ is the extraordinary refractive index; $N$ is the concentration of the $\pi$-transitions. In the equations given, we ignore the relaxation and inhomogeneous broadening of the spectral line of resonant absorption.

System (1–3) coincides with the reduced Maxwell–Bloch equations for the isotropic media\textsuperscript{16} if $D = 0$. One can see that the extraordinary component of electric field fulfills here two functions: it causes the quantum transitions and, simultaneously, shifts dynamically their frequency. In the case of the two-component pulses\textsuperscript{6–9}, these functions were executed by its short-wavelength ordinary and long-wavelength extraordinary components accordingly. In so doing, the ordinary component generated extraordinary one due to PDM of the resonant $\sigma$-transitions.

It is convenient for subsequent consideration to introduce new variables

$$u = \frac{d E_e}{\hbar \omega_0}, \quad \tau = \omega_0 \left( t - \frac{n_e}{c} y \right), \quad \eta = 2 \frac{\pi N d^2}{n_e c \hbar} y.$$  

(4)

Then, equations (1–3) are rewritten as

$$\frac{\partial \sigma_3}{\partial \tau} = i u (\sigma - \sigma^*),$$

(5)

$$\frac{\partial \sigma}{\partial \tau} = i (1 + 2 k u) \sigma + 2 i u \sigma_3,$$

(6)
\[ \frac{\partial u}{\partial \eta} = i(\sigma^* - \sigma), \quad (7) \]

where
\[
k = \frac{D}{2a}.
\]

An integrability of equations (5)–(7) with arbitrary parameter \( k \) in the frameworks of the inverse scattering transformation method\(^{13-15} \) was revealed in\(^3 \). The corresponding overdetermined system of linear equations (Lax pair) has the next form
\[
\begin{cases}
\frac{\partial \psi}{\partial \tau} = L(\lambda) \psi(\lambda), \\
\frac{\partial \psi}{\partial \eta} = A(\lambda) \psi(\lambda),
\end{cases} \quad (8)
\]

where \( \psi = \psi(\tau, \eta, \lambda) = (\psi_1, \psi_2)^T \) is the vector solution of the Lax pair, \( \lambda \) is so-called spectral parameter, matrices \( L(\lambda) \) and \( A(\lambda) \) are defined as follows
\[
L(\lambda) = \begin{pmatrix}
\frac{ik}{2\sqrt{1+k^2}} + i\sqrt{1+k^2}u & \frac{i\lambda}{2} \\
\frac{i\lambda}{2} & -\frac{ik}{2\sqrt{1+k^2}} - i\sqrt{1+k^2}u
\end{pmatrix},
\]
\[
A(\lambda) = \frac{i(1+k^2)\lambda}{1-(1+k^2)\lambda^2} \begin{pmatrix}
\frac{2k\sigma_3 - \sigma - \sigma^*}{\sqrt{1+k^2}\lambda} & 2\sigma_3 + k(\sigma + \sigma^*) + \sqrt{1+k^2}(\sigma - \sigma^*) \\
2\sigma_3 + k(\sigma + \sigma^*) + \sqrt{1+k^2}(\sigma - \sigma^*) & -2k\sigma_3 + \sigma + \sigma^*
\end{pmatrix}.
\]

One can check by direct calculation that the compatibility condition of both the equations of Lax pair (8):
\[
\frac{\partial L(\lambda)}{\partial \eta} - \frac{\partial A(\lambda)}{\partial \tau} + [L(\lambda), A(\lambda)] = 0, \quad (9)
\]
is equivalent to system (5)–(7).

3 **Exponentially and rationally decreasing breather-like pulses**

Being integrable by the inverse scattering transformation method, equations (5)–(7) have the multisoliton solutions. The one-soliton pulse and its algebraic limit can be obtained by direct integration of these equations in stationary case as it was done, for example, for the system of the full Maxwell–Bloch equations\(^5 \). The formulas arising at it differ from ones presented in\(^5 \) by the definition of the pulse velocity only. Applying the Darboux transformation technique\(^17 \) to find the two-soliton breather-like pulse solution of system (5)–(7) on the zero background, we come to the following expression for variable \( u \):
\[
u = -\frac{2}{\sqrt{1+k^2}} \frac{\partial}{\partial \tau} \arctan \left( \frac{4k\alpha_1 \Omega \sinh A_R + Z_+ T^{-1} \cos A_I}{Z_- \cosh A_R - 4k\alpha_T \Omega \sin A_I} \right), \quad (10)
\]

where
\[
A_R = \frac{T^{-1}}{\sqrt{1+k^2}} \left( \tau + 16(1+k^2)(Z^2_+ + 4Z + 16k^2a^2_R) \frac{Z\sigma_0}{A} \eta \right) + c_1,
\]
\[
A_I = \frac{\Omega}{\sqrt{1+k^2}} \left( \tau - 16(1+k^2)(Z^2_- - 4Z - 16k^2a^2_I) \frac{Z\sigma_0}{A} \eta \right) + c_2,
\]
\[
A = Z_+^4 + 8Z_+^2(Z - 4k^2a^2_R) + 16(1 - 4a^2_R)(Z^2 - 4k^4a^2_R),
\]
\[
\Omega = a_R Z_+/Z, \quad T^{-1} = a_I Z_- /Z, \quad Z = 4(a^2_R + a^2_I), \quad Z_\pm = Z \pm k^2. \quad (11)
\]
Real constants $a_R$, $a_I$, $c_1$ and $c_2$ are the free parameters of the pulse. Constant $\sigma_0$ is an initial population of the quantum level ($|\sigma_0| \leq 1/2$). The formulas for variables $\sigma_3$ and $\sigma$ are cumbersome and omitted.

The time area of this two-soliton solution is distinct from zero:

$$\int_{-\infty}^{\infty} u \, dt = -\frac{4 \text{sign}(k)}{\sqrt{1 + k^2}} \arctan \left[ |Tk| \frac{a_I^2}{a_R^2 + a_I^2} \right].$$

For this reason, we refer to it as the breather-like pulse to distinguish from the well-known breather pulse in isotropic media, whose time area is equal to zero. An existence of the breather type solutions with nonzero area was established numerically for system (5)–(7) in 10.

Quantities $\Omega$ and $T$, which characterize the frequency and duration of the pulse, can be regarded as its free parameters instead of $a_R$ and $a_I$. Then, one finds from equations (11)

$$a_R = \frac{\Omega}{2} \left( 1 + \frac{R_+}{\Omega T} \right),$$

$$a_I = 1 + \frac{R_-}{2T},$$

where

$$R_\pm = \frac{1}{\sqrt{2}} \sqrt{r \pm ((\Omega^2 - k^2)T^2 - 1)},$$

$$r = \sqrt{((\Omega^2 - k^2)T^2 + 1)^2 + 4k^2T^2}.$$ 

Here the arithmetic root is taken in the definition of $r$, and the signs of $R_+$ and $R_-$ are chosen in such a manner that $R_+ R_- = \Omega T$. Without loss of a generality, we shall suppose in the sequel that parameters $a_R$ and $a_I$ (or $\Omega$ and $T$) are positive. If $\Omega T \gg 1$, then solution (11) is nothing but the pulse with higher-frequency filling (ultrashort pulse), while at $\Omega T < 1$ variable $u$ may no change the polarity as it happens for the extremely short pulses.

Expanding the right-hand side of equation (10) in the Taylor series at a neighborhood of point $k = 0$ and retaining the first two terms, we come to formulas

$$u = u_0 + ku_1,$$

$$u_0 = 4\Omega \frac{\Omega T \cosh B_R \sin B_I + \sinh B_R \cos B_I}{\Omega^2 T^2 (\cosh 2B_R + 1) + \cos 2B_I + 1},$$

$$u_1 = -\frac{4\Omega^2 T^2 (\cosh 2B_R + 1)(\cos 2B_I + 1)}{(\Omega^2 T^2 (\cosh 2B_R + 1) + \cos 2B_I + 1)^2},$$

where

$$B_R = \frac{\tau}{T} + \frac{4T((\Omega^2 + 1)T^2 + 1)\sigma_0 \eta}{((\Omega + 1)^2 T^2 + 1)((\Omega - 1)^2 T^2 + 1)},$$

$$B_I = \Omega \tau - \frac{4\Omega T^2 ((\Omega^2 - 1)T^2 + 1)\sigma_0 \eta}{((\Omega + 1)^2 T^2 + 1)((\Omega - 1)^2 T^2 + 1)}.$$ 

It follows from the expressions presented that the modulus of the Fourier transform of $u_0$ achieves the maxima on odd harmonics of the basic frequency $\Omega$, while the modulus of the Fourier transform of $u_1$ has them on even harmonics, including zeroth one. Since the generation of the secondary harmonics due to PDM is nonlinear effect, they are more localized at the center of the pulse. In the case of breather-like pulses, an asymmetry induced by PDM manifests itself in that the signs of $k$ and the zero harmonic are opposite. Similar asymmetry on the polarity of a signal for the extremely short pulses was revealed under analytical and numerical investigations in 5, 10.
The conclusions made above are most obvious for the pulses with slowly varying envelope. Indeed, if condition $\Omega T \gg 1$ holds, then formulas (12) and (13) become simpler:

$$u_0 = \frac{2 \sin B_I}{T \cosh B_R},$$  \hspace{1cm} (14)$$

$$u_1 = -\left(\frac{2 \cos B_I}{\Omega T \cosh B_R}\right)^2.$$  \hspace{1cm} (15)

Carrying out the Fourier transform with expressions (14) and (15):

$$F(\nu, u_{0,1}) = \int_{-\infty}^{\infty} e^{i\nu \tau} u_{0,1} d\tau,$$

one obtains

$$F(\nu, u_0) = -i\pi \exp(-i\theta_R \nu) \left( \frac{\exp(i(\theta_I - \Omega T \theta_R))}{\cosh \pi T(\nu + \Omega)/2} - \frac{\exp(i(\Omega T \theta_R - \theta_I))}{\cosh \pi T(\nu - \Omega)/2} \right),$$

$$F(\nu, u_1) = -\frac{\exp(-i\theta_R \nu)}{\Omega^2} \left( \frac{2\nu}{\sinh \pi T \nu/2} + (\nu + 2\Omega) \frac{\exp(2i(\theta_I - \Omega T \theta_R))}{\sinh \pi T(\nu + 2\Omega)/2} + \frac{(\nu - 2\Omega) \exp(2i(\Omega T \theta_R - \theta_I))}{\sinh \pi T(\nu - 2\Omega)/2} \right),$$

where $\theta_R$ and $\theta_I$ are the values of $B_R$ and $B_I$ at $\tau = 0$.

It is seen that the maxima of the moduli of $F(\nu, u_0)$ and $F(\nu, u_1)$ are reached at $\nu = \Omega$ and at $\nu = 0, 2\Omega$, respectively. The width of the spectral lines is equal to $T^{-1}$, while the maximum values of $|F(\nu, u_1)|$ are proportional to $\Omega^{-2} T^{-1}$. Thus, an efficiency of the secondary harmonic generation grows with reducing the carrier frequency of the pulse. This agrees with the results of the previous studies (see, e.g.,1). An influence of PDM on the pulses having the high-frequency filling is weak. This is due to the fact that an average value of the frequency detuning $2k u$ (see formula (6)) on the duration of such the pulse tends to zero.

Let us return to expression (10) again. The time dependence of $u$ and $\sigma_3$ is plotted below. Parameters

![Profiles of $u$ and $\sigma_3$](image)

Fig. 1. Profiles of $u$ and $\sigma_3$ with $k = 1$, $\sigma_0 = -0.5$, $\bar{\Omega} = 1$ and $\bar{T} = 10$.

$\bar{\Omega} = \Omega/\sqrt{1 + k^2}$ and $\bar{T} = \sqrt{1 + k^2} T$ of the pulse are selected so that it excites the medium strongly (see Fig. 1b). The curve of the modulus of the Fourier transform of $F(\nu, u)$ is given on Fig. 2. Note, that the zeroth harmonic exists in the Fourier spectrum in accordance with that the time area of the breather-like pulses differs from zero. For the pulses with $\Omega T \approx 1$, this fact was established in$^{10}$ (see Fig. 3 also). Since the principal maximum on
Fig. 2. Modulus of Fourier transform $F(\nu, u)$ of the pulse presented on Fig. 1.

Fig. 3. Profiles of $u$ with $k = 1$, $\sigma_0 = -0.5$, $\tilde{\Omega} = 1$, $\tilde{T} = 1$ (solid curve) and with $k = 0$ (dotted line).

Fig. 2 is arranged to the left from $\tilde{\Omega}$, the secondary harmonics are generated most effectively by the pulses, whose basic carrier frequency on an input of the anisotropic medium is lesser than the resonant frequency. The same effect takes place under reducing a duration of incident pulse due to broadening of its spectral width. Also, it is possible to show that the position of the principal maximum on $\nu$ axis in the Fourier spectrum is displaced to a red region with growing $|k|$. The curve on Fig. 4 shows a difference of the spectral structure of $u$’s with identical $\tilde{\Omega}$ and $\tilde{T}$ in the cases of anisotropic and isotropic media. The lapse to the right of $\tilde{\Omega}$ is a consequence of increasing an asymmetry of the principal peak of the Fourier spectrum in the medium with PDM.

In limiting case $2(a_R + ia_I) \rightarrow k \exp i\varphi$ (or $T \rightarrow \infty$) formula (10) yields

$$u = -2 \frac{\Omega \cot \varphi \left( \zeta_1 \sin^2 \varphi \cos \zeta_2 + (\cos^2 \varphi - 2) \sin \zeta_2 + 2 \sin \varphi \right)}{\sqrt{1 + k^2} \left( \zeta_1 \sin \varphi + \cos \zeta_2 \right)^2 + \cot^2 \varphi (1 - \sin \varphi \sin \zeta_2)^2},$$

(16)

where

$$\zeta_1 = \Omega \tau + \frac{4\Omega(1 + \Omega^2)\sigma_0\eta}{(1 - \Omega^2)^2} + c_1, \quad \zeta_2 = \Omega \tau + \frac{4\Omega\sigma_0\eta}{1 - \Omega^2} + c_2,$$

$$\Omega = \frac{k \cos \varphi}{\sqrt{1 + k^2}}.$$
Here variable $u$ decreases rationally. An arbitrary parameter of the pulse is real constant $\varphi$ determining its carrier frequency, which is always less resonant one.

Existence of rationally decreasing pulses is the distinctive feature of the anisotropic media. So, algebraic one-component extremely short pulse was found in⁵, and two-component one-parameter pulses were constructed in⁹. Curves of $u$, $\sigma_3$ and $|F(\nu,u)|$ for the pulse (16) are represented in Figs 5 and 6. Comparing curves on Figs 2 and 6, one can see that the position on $\nu$ axis of maxima of the secondary harmonics is more shifted in the red region for the rationally decreasing pulses. Besides, the expressed asymmetry has not only the principal peak, but the secondary peaks of the Fourier spectrum also.

Performing limiting procedures $\varphi \to \pm \pi/2$ and $-c_1 \to \pm \cos c_2$, we obtain from equation (16):

$$u = \frac{-2k}{1 + k^2 + k^2(\tau + 4\sigma_0\eta)^2}.$$  

(17)

If we impose additional condition $c_2 \to \pm \pi/2$ in so doing, then, instead of (17), formula (16) turns into following

$$u = -6k \frac{w_1}{w_2}.$$  

(18)
Fig. 6. Modulus of Fourier transform \( F(\nu, u) \) of the rationally decreasing breather-like pulse. Values of the parameters correspond to the pulse on Fig. 5.

where

\[
 w_1 = k^4(\tau + 4\sigma_0\eta)^4 + 6k^2(\tau + 4\sigma_0\eta)^2(1 + k^2)\tau + 4(1 - 3k^2)\sigma_0\eta) - 3(1 + k^2)^2, \\
 w_2 = k^6(\tau + 4\sigma_0\eta)^6 + 3k^4(\tau + 4\sigma_0\eta)^3(1 + k^2)\tau + 4(1 + 9k^2)\sigma_0\eta) + \\
 + 9k^2(3(1 + k^2)^2\tau^2 + 8(1 + k^2)(3 - k^2)\sigma_0\tau\eta + 16(3 - 2k^2 + 11k^4)\sigma_0^2\eta^2) + 9(1 + k^2)^3.
\]

At the end of this section, we discuss briefly the spectral data\(^{13-15}\) corresponding to the solutions studied. The discrete spectrum of breather-like pulse \((10)\) includes four simple points. They are located on the plane of the Lax pair \((8)\) spectral parameter at

\[
 \lambda = \pm \frac{4(a_R + ia_I)^2 - k^2}{2(a_R + ia_I)\sqrt{1 + k^2}}, \quad \lambda = \pm \frac{4(a_R - ia_I)^2 - k^2}{2(a_R - ia_I)\sqrt{1 + k^2}}.
\]

The points of the discrete spectrum of rationally decreasing pulse \((16)\) are doubly degenerate and located at

\[
 \lambda = \pm \frac{ik\sin \varphi}{\sqrt{1 + k^2}}.
\]

Solution \((17)\) has two simple points of the discrete spectrum at \(\lambda = \pm ik/\sqrt{1 + k^2}\). This rational pulse is a particular case of the one-soliton solution of system \((4) - (7)\), which is stationary. The points of the discrete spectrum of solution \((18)\) are the same as for previous one, but they are doubly degenerate.

4 Conclusion

In the present report, we have considered the propagation through the two-level medium possessing PDM of the electromagnetic pulses with duration from a several oscillations of the field. The explicit breather-like solutions of corresponding system of the reduced Maxwell–Bloch equations are studied. Unlike to the breathers in an isotropic medium, these ones have the time area distinct from zero. An existence of the solutions of such a kind (namely, the nonzero breather) was revealed under numerical analysis in\(^{10}\). As in the case of the extremely short electromagnetic pulses\(^{5,10}\), it is shown here that an asymmetry on the polarity of a signal takes place for the breather-like pulses of arbitrary duration: the signs of the zero harmonic and PDM are opposite. It follows from the Fourier analysis of the solutions under discussion that an influence of PDM on an efficiency of
the secondary harmonics generation grows with a reduction of the basic carrier frequency of the pulse and its duration shortening. As a result, the pulses, whose carrier frequency on an input of the medium is less than resonant one, will generate the secondary harmonics most effectively. This concern, in particular, the rationally decreasing breather-like pulses that exist only in the case of the anisotropic media.

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