Topological Zero-Thickness Cosmic Strings

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Abstract

In this paper, based on the gauge potential decomposition and the $\phi-$mapping theories, we study the topological structures and properties of the cosmic strings that arise in the Abelian-Higgs gauge theory in the zero-thickness limit. After a detailed discussion, we conclude that the topological tensor current introduced in our model is a better and more basic starting point than the generally used Nambu-Goto effective action for studying cosmic strings.

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I. INTRODUCTION

Cosmic strings, the idea of which was very popular in the 1980’s and much of 90’s, are linear topological defects that may be formed at phase transitions in the early universe \([1, 2]\). Though the recent observations of the cosmic microwave background by WMAP \([3]\) showed beyond doubt that cosmic strings could not provide an adequate explanation for the primordial density fluctuations, but could only contribute at the level of a few percent, they still have very fruitful implications for cosmology, for instance, they can generate gravitational lensing effects, gravitational waves, and so on. Meanwhile, the recent developments in superstring theory suggest that cosmic superstrings (macroscopic fundamental strings or one-dimensional Dirichlet branes) can play a role very similar to that of cosmic strings. Also, on the observational side, though the galaxy image pair CSL-1 \([4, 5]\) has been excluded as a candidate for a cosmic string lens \([6]\), it now still gives us the impression that, cosmic strings might provide a best observational window to the very early universe, the extremely high energy physics, and possibly to superstring theory, and thus have been of particular interest recently.

Though the cosmological evolution of the cosmic strings has been studied extensively, they were mostly focused on their dynamical properties. We note that, their topological properties are also very important and worth investigating. In this paper, by making use of the gauge potential decomposition and the \(\phi\)–mapping theories \([7, 8, 9, 10]\), we study the topological properties of the cosmic strings that arise in the Abelian-Higgs model with a broken U(1) gauge symmetry in the zero-thickness limit in detail. Since almost all of the ideas can carry over to cosmic superstrings, our results are also useful in superstring theory.

The paper is organized as follows: In Sec. III we give a slightly detailed review of the topological structures of the cosmic strings, showing that they can be totally decided by the distribution of some complex scalar field. Also we show that as we can deduce the Nambu-Goto effective action of the strings, all the cosmic strings in our model have a zero thickness. In Sec. III we study the topological properties of the zero-thickness cosmic strings, showing that cosmic strings not only represent trapped energy, but also represent trapped flux. We point out that, the winding numbers of the strings are the products of their Hopf indices and Brouwer degrees, and the flux carried by a string is its winding multiple of the flux quantum \(2\pi e\). For application, we show a nature picture of the evolution of the early universe. The
conclusion is given in Sec. IV, in which we suggest a better and more basic starting point for studying cosmic strings than the generally used Nambu-Goto effective action.

II. TOPOLOGICAL STRUCTURE OF COSMIC STRINGS

There are various kinds of cosmic strings in the literature \[1, 2\], in this paper, we mainly focus our attention on the cosmic strings that generally appear in the Abelian-Higgs model with a broken U(1) gauge symmetry, though we note that our results can be generalized to other cases easily. The Lagrangian density is given by

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - V(\phi),
\]

where \(\phi(x)\) is a complex scalar field, which can also be thought of as a pair of real fields \(\phi^1, \phi^2\), with \(\phi = \phi^1 + i\phi^2\), \(A_\mu\) is a gauge field, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the corresponding field strength tensor, and the covariant derivative \(D_\mu = \partial_\mu - i e A_\mu\). Usually the potential \(V(\phi)\) is a function only of \(|\phi|\), and here is taken to be the form

\[
V(\phi) = \frac{\lambda}{4} (\phi^* \phi - \eta^2)^2,
\]

where \(\eta\) is the vacuum expectation value (VEV) of \(|\phi|\). Also there are two dimensionless parameters, the gauge coupling constant \(e\) and the Higgs self-coupling \(\lambda\). Meanwhile, the metric is taken to be that of the spatially flat expanding universe

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2).
\]

For cosmic string formation, there are several mechanisms, such as the Kibble mechanism, the thermal fluctuations of the magnetic field, etc. Here, we assume the cosmic strings have already formed, and discuss their topological structures.

As the gauge potential \(A_\mu\) is initially introduced in the covariant derivative of \(\phi(x)\), it is natural to think that there is some relationship between the distributions of \(A_\mu\) and \(\phi(x)\). According to the gauge potential decomposition theory \[7, 8\] proposed by one of the authors (Duan), it can be seen clearly that, besides a gauge transformation term, \(A_\mu\) is totally decided by \(\phi(x)\)

\[
A_\mu = \frac{1}{e} \epsilon_{ab} \frac{\phi^a}{||\phi||} \partial_\mu \frac{\phi^b}{||\phi||} + \frac{1}{e} \partial_\mu \theta,
\]

where \(||\phi|| = \sqrt{\phi^a \phi^a}\) and \(\theta\) is only a phase factor.
In order to study the topological structures of the strings, we define a second order
topological tensor current as follow
\[ j^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\lambda\rho} (\partial_{\lambda} A_{\rho} - \partial_{\rho} A_{\lambda}), \] (5)
where \( g \) is the determinant of the metric \( g = \text{det}(g_{\mu\nu}) \). Substituting (4) into (5), it yields
\[ j^{\mu\nu} = \frac{1}{e \sqrt{-g}} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \partial_{\lambda} \phi^a \partial_{\rho} \phi^b \| \phi \| \| \phi \|, \] (6)
which shows that \( j^{\mu\nu} \) is antisymmetric and identically conserved
\[ \nabla_{\mu} j^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} j^{\mu\nu}) = 0. \] (7)
Following the \( \phi \)-mapping theory \[9, 10\] which was also proposed by author Duan, this
topological tensor current can be expressed in a compact \( \delta \)-function form
\[ j^{\mu\nu} = \frac{2\pi}{e} \delta(\phi) J^{\mu\nu}(\frac{\phi}{x}), \] (8)
where \( J^{\mu\nu}(\frac{\phi}{x}) \) is the general Jacobian tensor defined as
\[ \epsilon^{ab} J^{\mu\nu}(\frac{\phi}{x}) = \frac{\epsilon^{\mu\nu\lambda\rho}}{\sqrt{-g}} \partial_{\lambda} \phi^a \partial_{\rho} \phi^b. \] (9)
From (8) one sees clearly that \( j^{\mu\nu} \) is nontrivial only when \( \phi = 0 \), or equivalently
\[ \begin{cases} \phi^1(x) = 0 \\ \phi^2(x) = 0 \end{cases}. \] (10)

Suppose that for the equations (10), there are \( K \) different solutions. According to the
implicit function theorem, when the regular condition of \( \phi(x) \) at which the rank of the
Jacobian matrix \( (\partial_{\mu} \phi^a) \) is 2 satisfies, these solutions can be expressed as
\[ x^\mu = x_i^\mu(u^1, u^2), \quad i = 1, \cdots, K, \] (11)
where the subscript \( i \) represents the \( i \)th solution which is a two dimensional submanifold
spanned by the parameters \( u^c(c=1,2) \) with the metric tensor \( \gamma_{cd} = g_{\mu\nu}(\partial x^\mu / \partial u^c)(\partial x^\nu / \partial u^d) \) and called the \( i \)th singular submanifold \( N_i \) in the spacetime. For each \( N_i \) it can be proved
that there exists a local two dimensional normal submanifold \( M_i \) in the spacetime spanned
by the parameters \( v^A(A=1,2) \) with the metric tensor \( g_{AB} = g_{\mu\nu}(\partial x^\mu / \partial v^A)(\partial x^\nu / \partial v^B) \), which
is transversal to \( N_i \) at the intersection point \( p_i \). Then at the regular point \( p_i \), the regular condition can be expressed explicitly as

\[
J(\frac{\phi}{v}) = \frac{\partial(\phi^1, \phi^2)}{\partial(v^1, v^2)} \neq 0. (12)
\]

By using the \( \delta \)–function theory, one can prove that

\[
\delta(\phi) = \sum_i \frac{\beta_i \zeta_i}{J(\frac{\phi}{v})|_{p_i}} \delta(N_i), (13)
\]

where \( \beta_i \) is a positive integer called the Hopf index of \( \phi \)–mapping and \( \zeta_i = \text{sign} J(\phi/v)|_{p_i} = \pm 1 \) is the Brouwer degree. \( \delta(N_i) \) is the \( \delta \)–function on the singular submanifold \( N_i \) with the expression

\[
\delta(N_i) = \int_{N_i} \delta(x - x_i(u^1, u^2)) \sqrt{-\gamma}d^2u. (14)
\]

Substituting (13) into (8), one gets the final expansion form of \( j^{\mu\nu} \) on the \( K \) singular submanifolds

\[
j^{\mu\nu} = \frac{2\pi}{e} \sum_i \beta_i \zeta_i \delta(N_i) \frac{J^{\mu\nu}(\frac{\phi}{v})}{J(\frac{\phi}{v})|_{p_i}}, (15)
\]

or, in terms of parameters \( y^A = (v^1, v^2, u^1, u^2) \),

\[
j^{AB} = \frac{2\pi}{e} \sum_i \beta_i \zeta_i \delta(N_i) \frac{J^{AB}(\frac{\phi}{v})}{J(\frac{\phi}{v})|_{p_i}}. (16)
\]

Now, we see that, if taking \( u^1 \) and \( u^2 \) to be timelike evolution parameter \( t \) and spacelike string parameter \( \sigma \) (i.e. \( u^1 = t, u^2 = \sigma \)), respectively, as it can be proved that \( J^{\mu\nu}(\frac{\phi}{v})/J(\frac{\phi}{v})|_{p_i} \) or \( J^{AB}(\frac{\phi}{v})/J(\frac{\phi}{v})|_{p_i} \), has the dimension of velocity, the inner topological structures of \( j^{\mu\nu} \) or \( j^{AB} \) just represent \( K \) isolated singular strings moving in the universe. These singular strings are just the cosmic strings, and the two dimensional singular submanifolds \( N_i (i = 1, \cdots, K) \) are their world sheets. Meanwhile, we can classify these cosmic strings in terms of their Brouwer degrees \( \zeta_i \): a string is called a cosmic string if its \( \zeta > 0 \) and an anti cosmic string if its \( \zeta < 0 \).

Further, if we define the Lagrangian density of the cosmic strings as the generalization of Nielsen’s Lagrangian [11, 12]

\[
\mathcal{L}_{\text{str}} = -\mu \sqrt{\frac{1}{2} g_{\mu\lambda} g_{\nu\rho} j^{\mu\nu} j^{\lambda\rho}} = -\mu \frac{2\pi}{e} \delta(\phi) J(\frac{\phi}{v}), (17)
\]
where $\mu$ is the string tension, which is defined as the energy per unit length, then by using (13) and (14) one can get the action of the strings

$$S_{\text{str}} \equiv \int d^4 x \sqrt{-g} \mathcal{L}_{\text{str}} = -\mu \int d^4 x \sqrt{-g} \frac{2\pi}{e} \delta(\phi) J(\dot{\phi})$$

$$= \frac{2\pi}{e} \sum_i \beta_i \zeta_i S_i,$$

(18)

where

$$S_i = -\mu \int_{N_i} \sqrt{-\gamma} d^2 u$$

(19)

is exactly the Nambu-Goto effective action of the $i$th string, which shows that the thickness of the string is zero. So that the cosmic strings in our model all have a zero thickness. Actually, we can get an understanding of this in eq. (10) topologically, which shows that the field $\phi(x)$ is renormalized to a unit vector field $n(x) = \frac{\phi(x)}{||\phi||}$ except at its zero points. Though the real cosmic strings if there existed should not be zero thickness, we can still get many important properties of them by using this zero-thickness model, especially the topological properties which have no relations with their thickness. And many of the discussions in the literature are also based on the action (19) and by adding further correction terms.

III. PROPERTIES OF COSMIC STRINGS

Now we study the topological properties of the cosmic strings. Firstly, according to (10), we see that, from the mathematic point of view, the cosmic strings are just the intersection lines of two infinite sheets, so that topologically there are only two types of cosmic strings: the closed loops and the infinite long strings.

Secondly, from the definition of $j^{\mu\nu}$ (5), one knows that it’s the Hodge dual tensor of $F_{\mu\nu}$. According to (8), in the spacetime where $\phi(x) \neq 0$,

$$j^{\mu\nu} = 0 \rightarrow F_{\mu\nu} = 0.$$  

(20)

We can understand this in two ways, one is that $A_\mu$ is a pure gauge which has no independent physical meanings and can be chosen to be zero, and the other is that the coupling constant $e \rightarrow 0$. Both these two cases lead to the same result that the symmetry of $\phi(x)$ is global. This holds true before the symmetry-breaking phase transition when all the values of $\phi(x)$ are large. At that time, the Lagrangian density of $\phi(x)$ corresponds to

$$\mathcal{L}_\phi = \partial_\mu \phi^* \partial^\mu \phi - V(\phi).$$  

(21)
This is an interesting result, for the Lagrangian has the same form as that of inflation. Though in most inflationary scenarios, the inflaton field is a scalar one, our complex scalar field can also drive inflation very well. Actually, the inflationary model driven by $L$ scalar fields $\phi_i (i=1, \cdots, L)$ has also been discussed in [13].

After the symmetry is broken, $\phi(x)$ degenerates to its VEV, during which, according to the Kibble mechanism and other mechanisms, cosmic strings form, at the core of which $\phi(x)$ is zero and the symmetry can be regarded as unbroken. If there are some physical ‘magnetic fields’ initially in the universe which cannot exist in the broken phase, they must get confined into the cosmic strings. This suggests that $A_\mu$ gets physical meanings or the ‘magnetic fields’ get coupled to $\phi(x)(e \neq 0)$, just as [14, 15] stated, “the known cosmological phase transitions are not simple symmetry breaking transitions, but they involve a breakdown of a local gauge symmetry.” And from this, we see clearly that cosmic strings represent trapped ‘magnetic’ flux. From [15] or [16], it’s easy to see that the flux carried by the $i$th cosmic string is

$$\Phi_i = \Phi_0 W_i,$$

where $\Phi_0 = 2\pi/e$ is the flux quantum, and $W_i$ is the winding number of the $i$th string, determined by

$$W_i = \beta_i \zeta_i,$$

($i$ is not summed). This is also an important result. We have to mention that, although the Kibble mechanism only gives a contribution of $\pm 1$ to the winding number of the cosmic strings, some other mechanisms can indeed form strings with high windings, as discussed in [14, 13, 16].

Now, combining with the inflationary scenarios and GUTs, we could construct a brief picture of the evolution of the early universe: After the big bang, the universe was in a very hot and dense state filled with a hot ‘soup’ of some kinds of scalar fields and maybe some other massless particles. As the universe temperature $T$ is very high, $\phi(x)$ has a very large value, $|\phi|^2 \gg \eta^2$, so the center hump term $\eta^2$ in the potential $V(\phi)$ (2) is unimportant and can be ignored$^1$. This shows that the symmetry of the field $\phi(x)$ is complete and global, and fluctuations in any directions are equally likely. Then as the universe expands and cools,

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$^1$ Generally, $\eta^2$ is a function of the temperature $T$, but here we take it to be a constant for simplicity, as detailed in [17].
\( \phi(x) \) rolls down. If the rolling of \( \phi(x) \) is very slow, which satisfies the slow-roll conditions of inflation, the universe is in the stage of inflation \([18, 19]\), namely the expansion of the space continues in a quasi-exponential way (\( \sim e^{Ht} \)) for a period of time (\( \Delta t \sim 70H^{-1} \)). Then it comes to the final phase of inflation. The term \( \eta^2 \) in the potential \( (2) \) becomes important, and the fluctuations of \( \phi(x) \) over the hump are not permitted. Thus the field tends to settle towards one of its VEV \( \phi = \eta e^{i\alpha} \) and couple to the gauge field, during which, the Kibble mechanism and other mechanisms work. Therefore the symmetry of \( \phi(x) \) breaks down and the cosmic strings form, whose structures are described by \( (15) \) or \( (16) \) and which eventually form a random tangled network. As Jeannerot et al \([20]\) showed in an interesting recent study that topological or embedded cosmic strings formed at the end of inflation seem almost unavoidable, here we give quite a nature way to achieve this in our topological zero-thickness cosmic string model. And one thing that is worth noting is that, according to \( (7) \), as there is no strings originally, after their formation, the total windings of all the strings must be zero.

As an aside, from \( (2) \) and \( (8) \), it’s obvious that, in the core of the strings, \( \phi(x) \) vanishes, but the potential does not. Thus we see that, cosmic strings not only represent trapped flux, but also represent trapped potential energy (as well as gradient energy) \([17]\). (actually we can also see this from \( (18) \)). And the important thing is that, the density of this trapped energy which acts as a remnant of the earlier high-temperature universe may be similar to what it was before the symmetry-breaking phase transition, which suggests that cosmic strings might provide a best observational window to the very early universe, and to the extremely high energy physics. Meanwhile, to say the density of the string energy, we note that though for the zero-thickness cosmic strings, all their cores reach zero represented by a delta function \( \delta(\phi) \), the actual cosmic strings can in general have more than one thickness scale: the thickness of the field energy core and the thickness of the gauge core.

IV. CONCLUSION

In this paper, we mainly studied the topological structures and properties of the cosmic strings that appear in the Abelian-Higgs model with a broken U(1) gauge symmetry in the limit of zero-thickness. By using the gauge potential decomposition and the \( \phi \)–mapping theories, and discussing the properties of the zero points of the complex scalar field, we
obtained the topological structures of these strings, showing that under the regular condition 
\( J(\phi/v) \neq 0 \), the cosmic strings are isolated ones moving in the universe, their winding 
numbers are determined by the products of their Hopf indices and Brouwer degrees, they 
represent not only trapped energy but also trapped flux and the flux they carried are their 
windings multiple of the flux quantum. Meanwhile, as we have got the Nambu-Goto effective 
action \((19)\) of the cosmic strings from their topological structures \((15)\) or \((16)\), varying it 
with respect to \( x^\mu(u^c) \) will give the Euler-Lagrange equations of the strings

\[
x^\mu_{,c;c} + \Gamma^\mu_{\nu\lambda} \gamma^{cd} x^\nu_{,c} x^\lambda_{,d} = 0,
\]

where the coma and the semicolon represent the partial derivative and the covariant deriva-
tive, respectively, \( \Gamma^\mu_{\nu\lambda} \) is the four-dimensional Christoffel symbol, and the covariant Laplacian 
is

\[
x^\mu_{,c;c} = \frac{1}{\sqrt{-\gamma}} \partial_c (\sqrt{-\gamma} \gamma^{cd} x^\mu_d).
\]

This suggests that though we mainly discussed the topological properties of the cosmic 
strings based on their topological structures \((15)\) or \((16)\), these structures can also be used 
to study their dynamical properties. Therefore we conclude that the topological current 
\((15)\) or \((16)\) or the original definition form \((5)\) is very important for the cosmic strings, for it 
almost describes all their properties, and so that it can be used as a better and more basic
starting point than the generally used effective action \((19)\) for the studying of the cosmic 
strings.

Finally, we see that all the above discussions are based on the regular condition \((12)\), 
what will happen if this condition fails? Also, as cosmic superstrings or branes are in 
essence defects of various dimensions (though not necessarily topological), can these obtained 
topological properties be generalized to them? How to do it? They are our further works.

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[1] M. B. Hindmarsh and T. W. B. Kibble, Rept. Prog. Phys. 58, 477 (1995).

[2] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge 
University Press, Cambridge, England 1994).
[3] C. L. Bennett et al., Astrophys. J. Suppl. 148, 1 (2003).
[4] M. Sazhin et al., Mon. Not. Roy. Astro. Soc. 343, 353 (2003).
[5] M. V. Sazhin et al., Astrophys. J. 636, 5 (2005).
[6] E. Agol, C. J. Hogan and R. M. Plotkin, Phys.Rev. D 73, 087302 (2006).
[7] Y. S. Duan, S. L. Zhang and S. S. Feng, J. Math. Phys. 35, 4463 (1994).
[8] Y. S. Duan, G. H. Yang and Y. Jiang, Gen. Relativ. Gravit. 29, 715 (1997).
[9] Y. Jiang and Y. Duan, J. Math. Phys. 41, 6463 (2000).
[10] Y. Duan, L. Fu and G. Jia, J. Math. Phys. 41, 4379 (2000).
[11] H. B. Nielsen and P. Olesen, Nucl. Phys. B 57, 367 (1973).
[12] H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45 (1973).
[13] R. Easther and J. T. Giblin, Phys. Rev. D 72, 103505 (2005).
[14] A. Rajantie, Contemp. Phys. 44, 485 (2003).
[15] A. Rajantie, hep-ph/0311262.
[16] M. Donaire and A. Rajantie, Phys. Rev. D 73, 063517 (2006).
[17] A. C. Davis and T. W. B. Kibble, Contemp. Phys. 46, 313 (2005).
[18] A. D. Linde, JETP Lett. 38, 149 (1983).
[19] A. D. Linde, Phys. Lett. B 129, 177 (1983).
[20] R. Jeannerot, J. Rocher and M. Sakellariadou, Phys. Rev. D 68, 103514 (2003).