RELATIVE CHOW STABILITY AND OPTIMAL WEIGHTS

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ABSTRACT. For a polarized Kähler manifold \((X, L)\), we show the equivalence between relative balanced embeddings introduced by Mabuchi and \(\sigma\)-balanced embeddings introduced by Sano, answering a question of Hashimoto. We give a GIT characterization of the existence of a \(\sigma\)-balanced embedding, and relate the optimal weight \(\sigma\) to the action of \(\text{Aut}_0(X, L)\) on the Chow line of \((X, L)\).

1. INTRODUCTION

By definition [2], an extremal Kähler metric on a polarized Kähler manifold \((X, L)\) is a critical point of the Calabi functional, which assigns to each Kähler metric in \(c_1(L)\) the \(L^2\)-norm of its scalar curvature. Constant scalar curvature Kähler metrics (cscK for short) are special examples of extremal metrics. Initially stated for cscK metrics, the Yau-Tian-Donaldson conjecture, refined by Székelyhidi [23], predicts that the existence of an extremal Kähler metric on a given polarized Kähler manifold \((X, L)\) should be equivalent to a relative GIT stability notion of \((X, L)\). This conjecture should be seen as an infinite dimensional Kempf-Ness correspondence.

In [4], Donaldson introduced a finite dimensional approximation to this Kempf-Ness correspondence, via quantization. Let \(\text{Aut}_0(X, L)\) be the automorphism group of \((X, L)\) modulo the \(\mathbb{C}^*\)-action by rotation on the fibers of \(L\). If this group is discrete, cscK metrics can be approximated by a sequence of balanced metrics. Balanced metrics are particular Kähler metrics induced by projective embeddings \(X \hookrightarrow \mathbb{P}(H^0(X, L^k)^*)\). They appear as zeros of a finite dimensional moment map, and their existence correspond to the Chow stability of \((X, L^k)\) [26, 12, 18, 19]. From the general theory of moment maps, balanced metrics satisfy a unicity property. As a corollary, one obtains uniqueness of a cscK metric in \(c_1(L)\) under the assumption on \(\text{Aut}(X, L)\) [4]. The quantization method is a powerful tool in the study of cscK metrics [4, 5], and it is natural to extend it to the situation \(\text{Aut}_0(X, L) \neq 0\).

Generalisation of this approximation process to extremal metrics has been pioneered by Mabuchi [13, 14, 15, 16]. His approach can be understood in the framework of relative stability [23, 1], which naturally appears in GIT in the presence of non-discrete stabilizers. In the quantization setting, elements of \(\text{Aut}_0(X, L)\) acting on the Chow line of \((X, L)\) are an obstruction to Chow stability [14, 16, 17]. This obstruction is a source of examples of cscK manifolds that are not asymptotically Chow stable [3, 17]. It is then natural to consider a stability notion relatively to a maximal torus of symmetries \(T \subset \text{Aut}_0(X, L)\), so-called Chow polystability relative
Since then, other notions of quantization of extremal metrics have been introduced. Extremal metrics can be seen as self-similar solutions to the Calabi flow. As noticed by Sano, it is also natural to consider Donaldson’s iteration process \([4]\) as a discretisation of the Calabi flow. The notion of \(\sigma\)-balanced metrics, for \(\sigma \in \text{Aut}_0(X, L)\), provides a quantization of extremal metrics by self-similar solutions to Donaldson’s iteration process \([20]\). The \(\sigma\)-balanced metrics appear as zeros of a finite dimensional moment map, and can be used to recover minimization properties, unicity or splitting results for extremal metrics \([21, 22]\). Another point of view is given by Hashimoto, whose defined a quantization of the extremal vector field \([8]\). We refer to \([9]\) for a good reference on these different notions of quantization.

In this note, we show that Mabuchi’s relatively balanced metrics and Sano’s \(\sigma\)-balanced metrics are equivalent, for a suitable choice of \(\sigma\). In fact, for a given maximal compact torus \(T \subset \text{Aut}_0(X, L)\), there is a unique \(\sigma \in T^c\) (modulo \(T\)) allowing the existence of a \(\sigma\)-balanced metric. Such an automorphism \(\sigma\) will be called an optimal weight.

**Theorem 1.1.** Let \((X, L)\) be a polarized compact Kähler manifold and let \(T \subset \text{Aut}_0(X, L)\) be a maximal compact torus. Then \((X, L)\) is Chow polystable relative to \(T\) if and only if it admits a \(\sigma\)-balanced embedding with \(\sigma \in T^c\) optimal weight.

The paper is organized as follows. Section 2 is a brief review on Chow stability, in relation to balanced embeddings. We introduce necessary notations, definitions and results. In section 3, we define the relative Chow scheme \(\text{CHOW}^T_n(d)\). Restricting to the smooth case, \(\text{CHOW}^T_n(d)\) parametrizes \(T^c\)-invariant subvarieties of \(\mathbb{CP}^N\) of degree \(d\) and dimension \(n\), for \(T^c\) a given subtorus of \(\text{Aut}(\mathbb{CP}^N)\). It provides the natural framework for relative Chow stability. We then prove Theorem 1.1 in Section 4. Along the way, we explain how an optimal weight \(\sigma\) twists the torus of symmetries \(T\) to another torus \(T^\sigma\) so that the Chow line of \((X, L)\) becomes \(T^\sigma\)-invariant. Finally, we give a closed formula relating the optimal weight \(\sigma\) to the \(T\)-character acting on the Chow line of \((X, L)\) in Section 4.3.

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2. **Chow stability and balanced embeddings**

Let \((X, L)\) be a polarized Kähler manifold of complex dimension \(n\). By Kodaira’s embedding theorem, replacing \(L\) by a sufficiently large tensor power, the following map defines an embedding:

\[
\iota: X \rightarrow \mathbb{P}(V^*) \\
x \mapsto [ev_x],
\]
where $ev_x$ denotes the evaluation map at $x \in X$ and $V := H^0(X, L)$. Let $\mathcal{B}(V)$ be the space of basis of $V$. For any basis $s = \{s_{\alpha}\} \in \mathcal{B}(V)$ we define an isomorphism
\begin{equation}
\Phi_s : \mathbb{P}(V^*) \rightarrow \mathbb{C}\mathbb{P}^N
\end{equation}
and thus an embedding $f_s := \Phi_s \circ \iota$ of $X$ in $\mathbb{C}\mathbb{P}^N$, where $N + 1 = \dim(V)$. In this section, we will consider a moduli problem for embedded submanifolds of $\mathbb{C}\mathbb{P}^N$ with same degree and dimension as $X_s := f_s(X)$.

2.1. Chow stability. The idea behind Chow points and schemes is to replace the embedded submanifold $X$ by a hypersurface in a higher dimensional projective space. Consider $n + 1$ copies of the dual projective space $\mathbb{C}\mathbb{P}^{N^*} \times \cdots \times \mathbb{C}\mathbb{P}^{N^*}$ and the divisor:
\[ D_{X_s} := \{(H_0, \cdots, H_n) \in \mathbb{C}\mathbb{P}^{N*} \times \cdots \times \mathbb{C}\mathbb{P}^{N*} \mid H_0 \cap \cdots \cap H_n \cap X_s \neq \emptyset\}. \]

Set
\[ W := (\text{Sym}^d(\mathbb{C}^{N+1}))^{\otimes n+1}. \]

The divisor $D_{X_s}$ is defined by an element $\text{Chow}(X_s)$ in $W$, up to a constant. The corresponding point $\text{Chow}(X_s) \in \mathbb{P}(W)$ is called the Chow point of $X_s$. A remarkable fact is that $X_s$ is entirely characterized by $\text{Chow}(X_s)$. Set $d = \int_X c_i(L)$ the degree of $X_s$ in $\mathbb{C}\mathbb{P}^N$. Then, consider the space of $n$-dimensional subvarieties of $\mathbb{C}\mathbb{P}^N$ of degree $d$:
\[ \mathcal{CHOW}_{\mathbb{P}^N}(n, d) := \{Y \hookrightarrow \mathbb{C}\mathbb{P}^N \mid \dim(Y) = n, \deg(Y) = d\}. \]

We then have an injective map:
\begin{equation}
\text{Chow} : \mathcal{CHOW}_{\mathbb{P}^N}(n, d) \rightarrow \mathbb{P}(W)
\end{equation}
\[ (Y \hookrightarrow \mathbb{C}\mathbb{P}^N) \mapsto \text{Chow}(Y), \]

and the Chow scheme $\mathcal{CHOW}_N(n, d)$ is by definition the image of the Chow map:
\[ \mathcal{CHOW}_N(n, d) := \text{Chow}(\mathcal{CHOW}_{\mathbb{P}^N}(n, d)). \]

By construction, Chow is a $1 : 1$ correspondence between $\mathcal{CHOW}_{\mathbb{C}\mathbb{P}^N}(n, d)$ and the Chow scheme $\mathcal{CHOW}_N(n, d)$, and the identification is often implicit in the literature.

Remark 2.1. In this paper, we restrict ourselves to smooth subvarieties of $\mathbb{C}\mathbb{P}^N$. For moduli considerations, one may consider subschemes as well.

The $\text{SL}_{N+1}(\mathbb{C})$-action on $\mathbb{C}^{N+1}$ induces an action on $W$, and we are interested in the GIT quotient of the Chow scheme under this action. To simplify notation, set $G^c = \text{SL}_{N+1}(\mathbb{C})$. We will restrict ourselves to a single $G^c$-orbit:
\[ \mathcal{Z} := G^c \cdot \text{Chow}(X_s) \]

Recall the definition:

**Definition 2.2.** The polarized manifold $(X, L)$ is Chow polystable if the $G^c$-orbit of $\text{Chow}(X_s)$ is closed in $W$ for any $s \in \mathcal{B}(V)$.

Note that this definition is independent on the choice of $s \in \mathcal{B}(V)$, as seen below. We can assume, up to replacing $L$ by a high tensor power, that $\text{Aut}_0(X, L)$ acts on $L$ and thus on $V$ (see e.g. [10]). We then consider the induced group representation
\[ \rho : \text{Aut}_0(X, L) \rightarrow \text{SL}(V). \]
The natural right action of $G^c$ on $\mathcal{B}(V)$ commutes with the left action of $\mathbb{C}^* \times \text{Aut}_0(X)$ on $\mathcal{B}(V)$, where $\mathbb{C}^*$ acts by scalar multiplication. The space $\mathcal{Z}$ is then identified with the quotient space

$$\mathcal{Z} \simeq \mathcal{B}(V)/(\mathbb{C}^* \times \text{Aut}_0(X))$$

via the Chow map. The bundle $\mathcal{O}_{\mathbb{P}(W)}(1)$ restricts to an ample line bundle $\mathcal{L}$ on $\mathcal{Z}$ and $(\mathcal{Z}, \mathcal{L})$ is a smooth polarized Kähler manifold. In the next section, we recall the moment map condition for Chow polystability.

2.2. Balanced embeddings. We briefly recall the symplectic aspects related to Chow stability. For more details and proofs we refer to [26, 12, 18, 19]. Denote by $G = SU(N+1)$ the compact form of $G_c$. The line bundle $\mathcal{L}$ carries a hermitian metric $h$, called Chow metric, whose curvature $\Omega$ is a $G$-invariant Kähler form on $\mathcal{Z}$ and such that the $G$-action on $(\mathcal{Z}, \Omega)$ is hamiltonian. Denote by $\mathfrak{g} = \text{Lie}(G)$ the Lie algebra of $G$, identified to its dual with the invariant non-degenerate pairing

$$<\xi, \eta> = \text{trace}(\xi \cdot \eta^*) .$$

Denote also by $m_0$ the moment map for the $G$-action on $(\mathbb{CP}^N, \omega_{FS})$:

$$m_0 : \mathbb{CP}^N \rightarrow \mathfrak{g} \quad z \mapsto i\left(\frac{z}{|z|^2}\right)_0$$

where the subscript 0 stands for the trace-free part. Then the moment map $\mu_0$ for the $G$-action on $(\mathcal{Z}, \Omega_{\mathcal{Z}})$ reads:

$$\mu_0 : \mathcal{Z} \rightarrow \mathfrak{g} \quad s \mapsto \int_{X_s} m_0 \omega_{FS}.$$

Definition 2.3. A zero of $\mu_0$ is called a balanced embedding.

From [25, 12], the associated GIT notion is Chow polystability:

Theorem 2.4. The manifold $(X, L)$ is Chow polystable if and only if it admits a balanced embedding.

If $\text{Aut}_0(X, L)$ is not discrete, there is a caracter on its Lie algebra whose vanishing is a necessary condition to Chow polystability of $(X, L)$. To introduce this caracter, we first express the $G^c$-action on $\mathcal{L}$ by mean of the moment map. For any $\xi \in \mathfrak{g}^c$, denote by $X_\xi$ the infinitesimal action of $\xi$ on $\mathcal{Z}$. Then the lift of $X_\xi$ is given by

$$\tilde{X}_\xi = X_\xi^h + 2\pi <\mu_0, \xi> X_1,$$

where $X_\xi^h$ is the horizontal lift with respect to the Chern connection of $(\mathcal{L}, h)$ and where $X_1$ denotes the vector field generating the $U(1)$ action on the fibers of $(\mathcal{L}, h)$. Let $s \in \mathcal{Z}$, $G_s$ be its stabiliser in $G$ and $\mathfrak{g}_s = \text{Lie}(G_s)$. Define the map $\mathcal{F}$:

$$\mathcal{F} : \mathfrak{g}_s^c \rightarrow \mathbb{C} \quad \xi \mapsto <\mu_0(s), \xi> .$$

We refer to [25] for a proof of the following:

Proposition 2.5. For any $\xi \in \mathfrak{g}_s$, the function

$$G^c \ni g \mapsto <\mu_0(g \cdot s), \text{Ad}_g \xi>$$

is constant. Moreover $\mathcal{F}$ defines a character on $\mathfrak{g}_s^c \simeq \text{Lie}(\text{Aut}_0(X, L))$. 

Remark 2.6. The character $F$ is a quantization of the celebrated Futaki character \[6\].

By definition, the existence of a balanced embedding imposes the vanishing of $F$. For any $\xi \in \mathfrak{g}$, $X_\xi$ and its lift $X^h_\xi$ vanish at $s$. However, if $\langle \mu_0(s), \xi \rangle$ is not zero, the orbit of a lift $x \in \mathcal{L}_s$ under the $G_c^s$-action will contain zero, and the $G^c$-orbit will not be closed. To overcome this issue, following Székelyhidi [23], one considers a relative notion of stability. In the next section, we introduce a relative Chow scheme and the associated notion of stability.

3. Relative Chow stability

In the following, we will restrict ourselves to embedded submanifolds of $\mathbb{C}P^N$ that are invariant under a specified torus of symmetries.

3.1. Relative embeddings. Fix a maximal torus $T$ in $\text{Aut}_0(X, L)$, and denote its complexification $T^c$. Recall that we have a representation $\rho : \text{Aut}_0(X, L) \to \text{SL}(V)$.

To simplify notations, we also denote the image of $T^c$ under $\rho$ by $T^c$. The action of the complexified torus on $V$ induces a weight decomposition $V = \bigoplus_{\chi \in w_V(T)} V(\chi)$ where $w_V(T)$ is the space of weights for this action. Let $N^V(\chi)$ be the dimension of $V(\chi)$. We consider the space of basis adapted to this decomposition:

$$B^T(V) := \left\{ (s^\chi_i)_{\chi \in w_V(T); i=1..N^\chi} \in (V)^{N^\chi +1} \mid \det(s^\chi_i) \neq 0 \text{ and } \forall(\chi, i), \ s^\chi_i \in V(\chi) \right\}.$$

Assume now that the embedding $f_s = \Phi_s \circ \imath$ is given by some $s \in B^T(V)$. The representation $\rho$ induces via the choice of basis $s$ a representation of $T^c$ on $\mathbb{C}P^N$:

$$\rho : T^c \to \text{SL}(\mathbb{C}P^N).$$

and a weight decomposition

$$\mathbb{C}_{N^\chi} := \bigoplus_{\chi \in w_V(T)} \mathbb{C}^{N^\chi}$$

where $T^c$ acts with weight $\chi$ on $\mathbb{C}^{N^\chi}$. Note that these representations and weight decompositions do not depend on $s \in B^T(V)$, and by construction $X_s$ is invariant under the $\rho(T^c)$-action. Consider now the set of subvarieties of $\mathbb{C}P^N$ of dimension $n$ and degree $d$ that are invariant under the $T^c$-action:

$$\mathcal{C}_N W^T_{\mathbb{P}^N}(n, d) := \{ Y \to \mathbb{C}P^N \mid \dim(Y) = n, \ degree(Y) = d, \ \rho(T^c) \cdot Y = Y \}.$$

Definition 3.1. The relative Chow scheme $\mathcal{C}_N W^T_{\mathbb{P}^N}(n, d)$ is defined by:

$$\mathcal{C}_N W^T_{\mathbb{P}^N}(n, d) := \text{Chow}(\mathcal{C}_N W^T_{\mathbb{P}^N}(n, d)).$$

Alternatively, $\rho$ induces a natural representation of $T^c$ on $\text{Sym}^d(\mathbb{C}^{N^\chi})$, and thus on $W = (\text{Sym}^d(\mathbb{C}^{N^\chi}))^{\otimes n+1}$. The relative Chow scheme is then given by the invariant Chow points in $\mathbb{P}(W)$. Denote the weight decomposition on $W$ by

$$W = \bigoplus_{\chi \in w_V(T)} W(\chi).$$
By construction, for any $Y \in \mathcal{H}O W^T_{P^N}(n,d)$, the associated divisor $D_Y$ is $T^c$-invariant for the diagonal action of $T^c$ on $\mathbb{P}^{N_s} \times \ldots \times \mathbb{P}^{N_s}$, and thus there is a unique weight $\chi_Y \in w_T(T)$ such that $\text{Chow}(Y) \in W(\chi_Y)$. Thus the relative Chow scheme lies in the disjoint union

$$\mathcal{H}O W^T_{X}(n,d) \subset \bigcup_{\chi \in w_W(T)} \mathbb{P}(W(\chi)),$$

where we identify $\mathbb{P}(W(\chi))$ with its image in $\mathbb{P}(W)$. Consider now the centralizer of $\rho(T^c)$ in $SL_N(\mathbb{C})$:

$$G^c_T = S(\Pi_N GL_{N_s}(\mathbb{C}))$$

which is the complexification of

$$G_T := S(\Pi_N U(N^c)).$$

The $G^c_T$-action on $\mathbb{CP}^N$ induces a $G_T^c$-action on $\mathcal{H}O W^T_{P^N}(n,d)$. We also have an induced action on $W$, under which the Chow map is $G^c_T$-equivariant. The orbit of Chow($X_s$) under this action corresponds to the set of Chow points:

$$\text{Chow}(\{ \Phi_s \circ \iota(X), g \in G^c_T \}) = \text{Chow}(\{ \Phi_s \circ \iota(X), s \in B^T(V) \})$$

for the natural right action of $G^c_T$ on $B^T(V)$. In the next section, we consider the GIT notions associated to the $G^c_T$-action on $\mathcal{H}O W^T_{P^N}(n,d)$.

3.2. Relative stability. We come back to the symplectic picture initiated in Section 2.2 following the abstract setting of [23]. The obstruction [6] motivates the following: instead of looking for zeros of $\mu_0$, try to minimize the function

$$G^c \ni g \mapsto ||\mu_0(g \cdot s)||^2.$$

A critical point for the square norm of the moment map satisfies the Euler-Lagrange condition $\mu_0(s) \in \mathfrak{g}_s$. Equivalently, $\mu_0(s)$ is equal to its orthogonal projection on the stabiliser of $s$. To have a common stabiliser for any $s$, we consider $T$ a maximal compact subgroup of $G_s$ as in Section 3.1 and restrict ourselves to the $G^c_T$ orbit of $s$:

$$Z^T := G^c_T \cdot s.$$

For any $s \in Z^T$, the stabilizer of $s$ in $G_T$ is $T$. Denote by $t$ the Lie algebra of $T$. We introduce the extremal vector $\mu_T$ to be the orthogonal projection of $\mu_0(s)$ onto $t$. One obtains the relative notions of balanced embeddings and Chow stability [13][14]:

**Definition 3.2.** An embedding $f_s$ is balanced relative to $T$ if $\mu_0(s) = \mu_T$.

Consider at the level of Lie algebras the orthogonal decompositions

$$\mathfrak{g}_T = t \oplus \mathfrak{g}_{T^\perp} \quad \text{and} \quad \mathfrak{g}^c_T = t^c \oplus \mathfrak{g}^c_{T^\perp}.$$

Denote by $G_{T^\perp}$ and $G^c_{T^\perp}$ the connected Lie subgroups of $G^c_T$ associated to $\mathfrak{g}_{T^\perp}$ and $\mathfrak{g}^c_{T^\perp}$.

**Definition 3.3.** The polarized manifold $(X, L)$ is Chow polystable relative to $T$ if the $G^c_{T^\perp}$-orbit of $\text{Chow}(X_s)$ is closed in $W$ for any $s \in B^T(V)$.

Mabuchi proved the following theorem [13]:

**Theorem 3.4.** The polarized manifold $(X, L)$ is Chow polystable relative to $T$ if and only if it admits a balanced embedding relative to $T$. 


Note that the restriction to the $G_{\tau^{-1}}$-orbit is crucial. Indeed, as noticed in Section 3.1, and by $T^c$-equivariance, there is a unique $\chi_X \in w_W(T)$ such that the set $\text{Chow}(\{X_s, s \in B^T(V)\})$ lies in $\mathbb{P}(W(\chi_X))$. By construction, this character corresponds to the exponential of (a multiple) of the character $F$ defined in (6). Thus, if $F \neq 0$, the $T^c$ orbit of $\hat{\text{Chow}}(X_s)$ will contain zero for any $s \in B^T(V)$ and its $G_{\tau}$-orbit will not be closed.

**Definition 3.5.** The destabilizing character of $(X, L)$ is the unique $\chi_X \in w_W(T)$ such that $\text{Chow}(X_s) \in \mathbb{P}(W(\chi_X))$ for any $s \in Z^T$. In the following section, we explain how $\sigma$-balanced embeddings overcome the issue caused by the destabilizing character, while being equivalent to relatively balanced embeddings.

### 4. GIT of $\sigma$-balanced metrics

An alternative notion of relative balanced embeddings has been introduced by Sano [20]. A $\sigma$-balanced metric is a self-similar solution to Donaldson’s dynamical system, and provides a quantization of extremal metrics [21].

#### 4.1. Definition of $\sigma$-balanced embeddings

We refer to [22] for a detailed treatment of this section. Let $\sigma \in T^c$. We also denote by $\sigma$ its image in $G_{\tau}$ under the representation $\rho$. From the symplectic point of view, the idea behind $\sigma$-balanced embeddings is to twist the invariant pairing $<\cdot, \cdot>$ by $\sigma \in T^c$ to obtain a moment map orthogonal to the stabiliser $t$. That way, the obstruction $F$ from (6) vanishes. Equivalently, one can twist the moment map $\mu_0$ and the symplectic form $\Omega$. On the space $Z^T$, consider the map:

\begin{equation}
(\sigma, \xi) \mapsto \int_{X_s} m_{\sigma} \omega_F \xi,
\end{equation}

where

\begin{equation}
m_{\sigma} = i(\frac{(\sigma \cdot z)(\sigma \cdot z)}{|z|^2})_0.
\end{equation}

There is a $G_T$ invariant Kähler form $\Omega^\sigma$ on $Z^T$ so that $\mu_0^\sigma$ is a moment map for the $G_T$ action on $(Z^T, \Omega^\sigma)$.

**Definition 4.1.** The embedding $f_s : X \to \mathbb{C}P^N$ is called $\sigma$-balanced if $\mu_0^\sigma(s) = 0$.

The following provides a character on $t^c$, independent on $s \in Z^T$, whose vanishing is a necessary condition for the existence of a $\sigma$-balanced embedding:

\begin{equation}
(\sigma, \xi) \mapsto \int_{X_s} m_{\sigma} \omega_F \xi,
\end{equation}

where

\begin{equation}
m_{\sigma} = i(\frac{|\sigma \cdot z|^2}{|z|^2})_0.
\end{equation}

This character is actually the linearisation of a convex functional $\mathcal{J}$ on $T^c$:

\begin{equation}
\mathcal{J} : T^c \to \mathbb{R}
\end{equation}

\begin{equation}
\sigma \mapsto \int_{X_s} |\sigma \cdot z|^2 \omega_F^2.
\end{equation}

As a corollary, there is a unique $\sigma \in T^c$, up to $T$, so that $\mathcal{J} = 0$. This motivates the following definition:

**Definition 4.2.** An element $\sigma \in T^c$ such that $\mathcal{J} = 0$ is called optimal weight.

In the next section, we investigate the relation between optimal weight an extremal vector $\mu_T$, as well as the relation between $\sigma$-balanced embeddings and Chow stability.
4.2. **Twisting the torus.** To relate $\mu$ and $\mu^\sigma$, we need to consider a bigger group action, and include the homothetic translation on $L$, or $\mathbb{C}^{N+1}$, in the picture. Consider the $\mathbb{C}^*$ action on $L$ given by rotations on the fibers. It induces an action on $B(V)$ that correspond to the action by homotheties on $\mathbb{C}^{N+1}$. Set

$$\tilde{G}^c = \mathbb{C}^* \times G^c,$$

with Lie algebra isomorphic to $\mathfrak{gl}(N+1)$. Then the moment map for the action of $G = S^1 \times G$ on $(\mathbb{CP}^N, \omega_{FS})$ is

$$m : \mathbb{CP}^N \to \mathfrak{u}(N+1)$$

$$z \mapsto i(\frac{\omega_{FS}}{z}).$$

The $\tilde{G}^c$ actions descend to $(\mathbb{Z}, \mathcal{L})$, with moment map given at $s$ by integration of $m$ over $X_s$. In the relative situation, set

- $\tilde{T} = S^1 \times T$, maximal compact subgroup of symmetries in Aut($L$),
- $\tilde{T}^c$ its complexification,
- $\tilde{G}_T^c = \mathbb{C}^* \times G_T^c$, centraliser of $\tilde{T}$ in $\tilde{G}^c$,
- $\tilde{G}_T = S^1 \times G_T$, its compact form,
- $\mathfrak{t}, \mathfrak{t}^c, \mathfrak{g}_t^c, \tilde{\mathfrak{g}}_T^c$ the corresponding Lie algebras.

The action of these groups descends to an action on $\mathbb{S}^T_s$. Fix now $\sigma \in T^c$. Note that for any $s \in Z^T$, its stabiliser is now $T^c$. The $\tilde{G}_T$-action is hamiltonian for $\Omega$ with moment map:

$$\mu : Z^T / s \to \tilde{\mathfrak{g}}_T$$

$$\frac{\mathfrak{g}}{s} \mapsto \int_{X_s} m \omega_{FS}.$$

We fix now an optimal weight $\sigma \in T^c$. Note that we still have the orthogonal decomposition

(12) $$\tilde{\mathfrak{g}}_T = \mathfrak{t}^c \oplus \mathfrak{g}_{T^c} = \mathfrak{t} \oplus \mathfrak{g}_{T^c}.$$ 

Setting $t^c := \sigma \cdot t \cdot \sigma^*$, the decomposition (12) induces the decomposition into orthogonal subspaces (actually Lie subalgebras, see below):

$$\tilde{\mathfrak{g}}_T = (\sigma \mathfrak{o}^c \oplus t^c) \oplus (\sigma^{-1} \cdot \mathfrak{g}_{T^c} \cdot (\sigma^{-1})^*).$$

Fix now $\xi = \sigma \eta \sigma^* \in (\sigma \mathfrak{o}^c \oplus t^c)$. The quantity $< \mu, \xi >$ is constant on $Z^T$. Indeed, if $\xi \in t^c$, $< \mu, \xi > = \mu(\eta)$ as $\eta$ is trace-free. Then $< \mu, \xi > = \mathfrak{F}_s(\eta) = 0$, as $\sigma$ is chosen optimal. For $\xi = \sigma \mathfrak{o}^c$, $< \mu, \xi > = \mathcal{S}(\sigma)$, also independent on $s$. Thus the differential of $< \mu, \xi >$ vanishes on $Z^T$, and $i_{\mathcal{X}_s} \Omega = 0$ by definition of a moment map. The non-degeneracy of $\Omega$ forces $\mathcal{X}_s$ to be zero, and $\xi$ belongs to the Lie algebra of the stabiliser of $s \in Z^T$ for any $s$, that is $t$. We have shown that

(14) $$(\sigma \mathfrak{o}^c \oplus t^c) = \hat{t}.$$ 

From (12) and (13) we deduce for any optimal weight $\sigma \in T^c$ the decomposition:

$$\tilde{\mathfrak{g}}_T = (\sigma \mathfrak{o}^c \oplus t^c) \oplus \mathfrak{g}_{T^c}.$$ 

From this we have:

**Theorem 4.3.** Let $\sigma$ be an optimal weight and let $s \in Z^T$. Then $X_s$ is balanced relatively to $T$ if and only if it is $\sigma$-balanced.
It is clear that

Proof. The $\sigma$-balanced condition is equivalent to $\mu_0^\sigma = 0$, that is $\sigma \mu \sigma^* = c_\sigma 1$ for some real constant $c_\sigma$. By choice of optimal weight, $\forall \xi \in \mathfrak{t}$,

$$\mathcal{F}'(\xi) = \langle \mu^\sigma, \xi \rangle = \langle \sigma \mu \sigma^*, \xi \rangle = 0,$$

where we use that $\xi$ is trace-free. Thus, $s$ is $\sigma$-balanced if and only if $\sigma \mu(s) \sigma^* \in (1 \oplus \mathfrak{t})$. From [14], this is equivalent to $\mu(s) \in (1 \oplus \mathfrak{t})$, that is $s$ being balanced relative to $T$. □

As a corollary, we recover Theorem 1.1

Corollary 4.4. $(X, L)$ admits a $\sigma$-balanced embedding if and only if it is Chow polystable relative to $T$.

We now relate the choice of optimal weight $\sigma \in T^c$ to the relative Chow scheme. From now on, we shall assume that the restriction of the pairing $< \cdot, \cdot >$ to $\mathfrak{t}$ is normalized to be rational on the kernel of the exponential map.

Lemma 4.5. For $\sigma$ an optimal weight, $(\mathfrak{t}^\sigma)^c$ is the Lie algebra of an algebraic torus in $\widetilde{G}_T^c$.

Proof. It is clear that $\mathfrak{t}^\sigma$ is an abelian Lie algebra, and generates a compact analytic subgroup of $\widetilde{G}_T$. Remains to show that it generates an algebraic group. The projection of $\mu$ to $\mathfrak{t}$ doesn’t depend on $s$, and equals $c_1 + \mu_T$ for a constant $c$. By [23] Lemma 3.3], $c_1 + \mu_T$ generates an algebraic subgroup of $\widetilde{G}_T$. Note that the proof of [23] Lemma 3.3] can be adapted directly to our context, without assuming the existence of a solution to $\mu(s) = c_1 + \mu_T$, because $T$ stabilizes all elements $s \in \mathbb{Z}^T$. But then $\sigma^{-1} c_\sigma 1(\sigma^{-1})^* = c_1 + \mu_T$ (see proof of Theorem 4.3), so that $\sigma^{-1} c_\sigma 1(\sigma^{-1})^*$ also generates an algebraic subgroup. We conclude that $\sigma(c^{-1}_\sigma) 1 \sigma^*$ satisfies the rationality condition that ensures that $\mathfrak{t}^\sigma (= c^{-1}_\sigma \cdot \mathfrak{t}^\sigma)$ generates an algebraic torus in $\widetilde{G}_T^c$. □

Let’s denote by $T^\sigma$ the algebraic torus in $\widetilde{G}_T^c$ generated by $\mathfrak{t}^\sigma$. Note that any $X_s$, $s \in \mathbb{Z}^T$, is $T^\sigma$-invariant, so that $\mathbb{Z}^T$ embeds into the relative Chow scheme $\mathcal{O}(\mathcal{O} W^\sigma_N (n, d))$ via the Chow map. As in Section 5 there is a weight decomposition under the $T^\sigma$-action:

$$W = (\text{Sym}^d(\mathbb{C}^{N+1}))^{\otimes n+1} = \bigoplus_{\chi \in w_W(T^\sigma)} W(\chi)$$

and $\mathbb{Z}^T$ is embedded in $\mathbb{P}(W(\chi'))$ for a unique $\chi' \in w_W(T^\sigma)$.

Lemma 4.6. The character $\chi'$ is trivial.

Proof. As $\sigma$ is an optimal weight, $\mu$ is orthogonal to the Lie algebra $\mathfrak{t}^\sigma$. From [5], we see that the linearisation of the action of $T^\sigma$ is trivial, that is $\chi' = 0$. □

Let $G_\sigma^c$ be the connected Lie subgroup of $\widetilde{G}_T^c$ generated by $(\mathfrak{t}^\sigma)^c \oplus \mathfrak{g}_T^{c}$). We then have shown:

Theorem 4.7. The manifold $(X, L)$ admits a $\sigma$-balanced embedding if and only if the $G_\sigma^c$-orbit of $CHOW(X_s)$ is closed for any $s \in \mathbb{B}^T(V)$.

The optimal weight provide a relative setting where the only destabilizing symmetry is generated by $\sigma^{-1} 1(\sigma^{-1})^*$. 
4.3. Optimal weight and destabilizing character. Recall from Section 4.1 that there is a unique $\chi_X \in w_W(T)$ such that the set Chow($\{X_s, s \in B^T(V)\}$ lies in $\mathbb{P}(W(\chi_X))$. In this short section, we give the precise relation between $\chi_X$ and optimal weight $\sigma$. For this, we will restrict our attention to the $\tilde{T}^c$ action on $W$.

First notice that there is a $1-1$ correspondence between the weights $w_W(T)$ of the $\tilde{T}^c$-action and the weights of the $T^c$-action on $W$ given by

$$w_W(T) \to w_W(\tilde{T})$$

$$\chi \to \chi' : (\lambda, t) \to \chi_d(n+1) \cdot \chi(t)$$

where $(\lambda, t) \in \mathbb{C}^* \times T^c$. In particular, the space $W$ decomposes into the same direct sum of invariant subspaces under the action of these two tori, and Chow($X_s) \in \mathbb{P}(W(\chi'_X))$ for any $s \in B^T(V)$.

Let $\sigma$ be an optimal weight and set

$$\alpha^\sigma := \sigma^{-1} \cdot \jmath(\sigma) \text{Id} \cdot (\sigma^{-1})^*.$$

**Proposition 4.8.** The relation between the optimal weight $\sigma$ and the destabilizing character of $(X, L)$ is given by

$$\alpha^\sigma = \chi'_X.$$

**Proof.** From the proof of Lemma 4.3 $\alpha^\sigma$ is rational, and thus can be written

$$\alpha^\sigma = \chi_X^\sigma,$$

for $\chi^\sigma$ a character in $\text{Hom}(\tilde{T}^c, \mathbb{C}^*)$ (by using derivation, we identify the characters and elements in $(\tilde{F})^*$). Let $\mathbb{C}^\sigma$ be the trivial line bundle over $\mathbb{Z}^T$, with $\tilde{T}^c$-linearisation given by $(\chi^\sigma)^{-1}$. Recall from the discussion of Section 4.2 that $\alpha^\sigma$ is the orthogonal projection $\pi_{\mu_1} \mu$ of $\mu$ to $\mathfrak{i}$, independently on $s \in \mathbb{Z}^T$. Then by construction, the moment map for the $\tilde{T}^c$-action on $(\mathbb{Z}^T, \mathcal{L}^{n^\sigma} \otimes \mathbb{C}^\sigma)$ is nothing but

$$n^\sigma \pi_{\mu_1} - \chi^\sigma = 0.$$

Thus the points in $\mathbb{Z}^T$ are semi-stable with respect to the $\tilde{T}^c$-action on $\mathcal{L}^{n^\sigma} \otimes \mathbb{C}^\sigma$. This action must then be trivial. But $\tilde{T}^c$ acts on $\mathcal{L}^{n^\sigma} \otimes \mathbb{C}^\sigma$ with weight

$$(\chi'_X)^n^\sigma \cdot (\chi^\sigma)^{-1}.$$

We conclude that $\alpha^\sigma = \chi'_X$. \hfill \Box

**Remark 4.9.** Note that from Proposition 4.3 the extremal $\alpha^\sigma$ only depends on the connected component of $\text{Chow}^T_N(n, d)$ where the Chow line of $(X, L)$ lies. In particular, it is invariant under embedded $T$-equivariant complex deformations of $(X, L)$. Considering tensor powers $L^k$ of $L$, and letting $k$ go to infinity, the extremal vectors $\alpha^\sigma_k$ converge to the extremal vector field of $(X, L)$ with respect to $T$ (see [22]). One recovers the fact that the extremal vector field is invariant under polarized $T$-equivariant complex deformations [11].

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