An Example of a Right Loop Admitting Only Discrete Topolization

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Abstract

Being motivated by [4] and [5], an example of a right loop admitting only discrete topolization is given.

Key words: Right loop, Normalized Right Transversal.

1 Introduction

Definition 1.1. A groupoid \((S, \circ)\) is called a right loop (resp. left loop) if

1. for all \(x, y \in S\), the equation \(X \circ x = y\) (resp. \(x \circ X = y\)), where \(X\) is unknown in the equation, has a unique solution in \(S\). In notation we write it as \(X = y/x\) (resp. \(X = x/y\))

2. if there exists \(1 \in S\) such that \(1 \circ x = x \circ 1 = x\) for all \(x \in S\).

Let \((S, \circ)\) be a right loop (resp. left loop). For \(u \in S\) the map \(R_u^o : S \to S\) (resp. \(L_u^o : S \to S\)) defined by \(R_u^o(x) = x \circ u\) (resp. \(L_u^o(x) = u \circ x\)) is a bijection on \(S\). We will drop the superscript, if the binary operation is clear.

Definition 1.2. A groupoid \((S, \circ)\) is called a loop if it is right loop as well as left loop. A loop \(S\) is said to be commutative if \(x \circ y = y \circ x\) for all \(x, y \in S\).
Definition 1.3. A loop $S$ is said to be **inverse property loop (I.P. loop)** if for each $x \in S$ there exists $x^{-1} \in S$ such that $x^{-1} \circ (x \circ y) = y$ and $(y \circ x) \circ x^{-1} = y$ for all $y \in S$.

Let $G$ be a group and $H$ be a subgroup of $G$. A **normalized right transversal (NRT)** $S$ of $H$ in $G$ is a subset of $G$ obtained by choosing one and only one element from each right coset of $H$ in $G$ and $1 \in S$. Then $S$ has a induced binary operation $\circ$ given by $\{x \circ y\} = Hxy \cap S$, with respect to which $S$ is a right loop with identity 1, (see [3, Proposition 4.3.3, p.102],[2]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [2, Theorem 3.4, p.76]).

Let $T(G,H)$ denote the set of all NRTs of $H$ in $G$. Let $S \in T(G,H)$ and $\circ$ be the induced binary operation on $S$. Let $x,y \in S$ and $h \in H$. Then $x.y = f(x,y)(x \circ y)$ for some $f(x,y) \in H$ and $x \circ y \in S$. Also $x.h = \sigma_x(h)x\theta h$ for some $\sigma_x(h) \in H$ and $x\theta h \in S$. This gives us a map $f : S \times S \to H$ and a map $\sigma : S \to H^H$ defined by $f((x,y)) = f(x,y)$ and $\sigma(x)(h) = \sigma_x(h)$. Also $\theta$ is a right action of $H$ on $S$. The quadruple $(S,H,\sigma,f)$ is a $c$-groupoid (see [2, Definition 2.1, p. 71]). In fact, every $c$-groupoid comes in this way (see [2, Theorem 2.2, p.72]). The same is observed in [1] but with different notations (see [1, Section 3, p. 289]).

**Definition 1.4.** A right loop $(S, \circ)$ is said to be **topological right loop** if $S$ is a topological space and the operations $\circ$ and $/$ are continuous.

Let $x,y \in S$. Then the map $t \mapsto (y/x) \circ t$ from $S$ to $S$ takes $x$ to $y$.

**Observation 1.5.** Let $(S, \circ)$ be a topological right loop. Then

(i) the topological right loop $S$ is homogeneous. For any $x,y \in S$, the map $t \mapsto (y/x) \circ t$ is a homeomorphism sending $x$ to $y$.

(ii) every $T_1$ topological right loop is Hausdorff. Note that the map $f : S \times S \to S$ defined by $f(x,y) = x/y$ is countinous, where $x,y \in S$. Then $\Delta = \{(x,x) | x \in S\} = f^{-1}(1)$ is a closed subset of $S \times S$.

**Remark 1.6.** From now onwards, we will assume that our topological right loop is $T_1$.\[\]
2 An Example of a Right Loop Admitting Only Discrete Topolization

Let \((U, \circ)\) be a loop. Let \(e\) denote the identity of the loop \(U\). Let \(B \subseteq U \setminus \{e\}\) and \(\eta \in \text{Sym}(U)\) such that \(\eta(e) = e\). Define an operation \(\circ'\) on the set \(U\) as

\[
x \circ' y = \begin{cases} 
  x \circ y & \text{if } y \notin B \\
  y \circ \eta(x) & \text{if } y \in B
\end{cases}
\]  

(1)

It can be checked that \((U, \circ')\) is a right loop. Let us denote this right loop as \(U_B^\eta\). If \(B = \emptyset\), then \(U_B^\eta\) is the loop \(U\) itself. If \(\eta\) is fixed, then we will drop the subscript \(\eta\). It can be checked that if \(y \notin B\), then \(R_y \circ' y = R_y \circ y\) and if \(y \in B\), then \(R_y \circ' y = L_y \circ \eta\).

In following example, we will observe that right loop structure on each NRTs in infinite dihedral group of non-normal subgroup of order 2 can be obtained in the manner defined in the equation (1).

Example 2.1. Let \(U = \mathbb{Z}\), the infinite cyclic group. Define a map \(\eta : \mathbb{Z} \to \mathbb{Z}\) by \(\eta(i) = -i\), where \(i \in \mathbb{Z}\). Note that \(\eta\) is a bijection on \(\mathbb{Z}\). Let \(\emptyset \neq B \subseteq \mathbb{Z} \setminus \{0\}\). We denote \(\mathbb{Z}_B^\eta\) by \(\mathbb{Z}_B^\eta\).

Let \(G = D_\infty = \langle x, y | x^2 = 1, xyx = y^{-1} \rangle\) and \(H = \{1, x\}\). Let \(N = \langle y \rangle\). Let \(\epsilon : N \to H\) be a function with \(\epsilon(1) = 1\). Then \(T_\epsilon = \{ \epsilon(y^i)y^i | 1 \leq i \leq n \} \subseteq \mathcal{T}(G, H)\) and all NRTs \(T \in \mathcal{T}(G, H)\) are of this form. Let \(B = \{ i \in \mathbb{Z} | \epsilon(y^i) = x \}\). Since \(\epsilon\) is completely determined by the subset \(B\), we shall denote \(T_\epsilon\) by \(T_B\). Clearly, the map \(\epsilon(y^i)y^i \mapsto i\) from \(T_\epsilon\) to \(\mathbb{Z}^B\) is an isomorphism of right loops. So we may identify the right loop \(T_B\) with the right loop \(\mathbb{Z}^B\) by means of the above isomorphism. We observe that \(T_0 = N \cong \mathbb{Z}\).

Lemma 2.2. Let \(G\) be a group and \(H\) a subgroup of \(G\). Let \(S \in \mathcal{T}(G, H)\) such that \(S\) is an I.P. loop with respect to the induced binary operation on it. Then \(L_{1/a}^{-1} = L_a\).

Proof. By [1, Proposition 3.3, p. 290; Proposition 3.6, p. 293], \(1/a = a'\), where \(a'\) denote the left inverse of \(a \in S\). Let \(x \in S\) and \(y = L_{1/a}^{-1}(x) = L_a^{-1}(x) \Rightarrow x = a' \circ y \Rightarrow (a')' \circ x = (a')' \circ (a' \circ y)\).

By [1, Proposition 3.6, p.293], \((a')' = a\). Also, since \(S\) is I.P. loop, \((a')' \circ (a' \circ y) = y\). Then above equation becomes \(a \circ x = y \Rightarrow L_a(x) = y\). This shows that \(L_{1/a}^{-1} = L_a\).
Following, we will give an example of a right loop where only discrete topolization is possible:

**Example 2.3.** Let $G_1$ be a countable group with relation $x^2 = e$, where $e$ is the identity of $G_1$ and $G_2$ be a finite group with identity $1$ on which two distinct involutive automorphism $\phi$ and $\psi$ exists. Fix two distinct elements $a, b \in G_1$. Let $(L = G_1 \times G_2, \ast)$ be commutative I.P. loop as in [5].

Let $B = \{(e, 1)/(a, 1), (b, 1)\}$. Let $\eta \in \text{Sym}(L)$ such that $\eta(e, 1) = (e, 1)$. Define a binary operation $\circ'$ on $L$ as defined in the equation (1). Consider a topology on $L$ so that $L$ is a topological right loop. Then the map $\alpha = R^{\circ'}_b (R^{\circ'}_e (R^{\circ'}_a (R^{\circ'}_e (R^{\circ'}_a L^{\ast}_e ) )^{-1} )^{-1} )^{-1}$ is a homeomorphism. One can note that $\alpha = L^{\ast}_b (L^{\ast}_e )^{-1} L^{\ast}_a (L^{\ast}_e )^{-1} L^{\ast}_a L^{\ast}_b = L^{\ast}_b L^{\ast}_a L^{\ast}_a L^{\ast}_b$ (by Lemma 2.2).

Also note that $\alpha$ is same as given in [5]. This $\alpha$ moves only finitely many points. Therefore, the right loop $(L, \circ')$ can not be topolized in non-discrete manner.

Let $S$ be a right loop. Then the group generated by $R_a$ for all $a \in S$ is called as the right multiplication group of $S$. At the end, we ask following question:

**Question:** Let $S$ be a right loop admitting only discrete topology. What topology can the right multiplication group of $S$ can have?

**References**

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