The Eight Cayley–Dickson Doubling Products

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Abstract. The purpose of this paper is to identify all eight of the basic Cayley–Dickson doubling products. A Cayley–Dickson algebra $\mathbb{A}_{N+1}$ of dimension $2^{N+1}$ consists of all ordered pairs of elements of a Cayley–Dickson algebra $\mathbb{A}_N$ of dimension $2^N$ where the product $(a, b)(c, d)$ of elements of $\mathbb{A}_{N+1}$ is defined in terms of a pair of second degree binomials $(f(a, b, c, d), g(a, b, c, d))$ satisfying certain properties. The polynomial pair $(f, g)$ is called a ‘doubling product.’ While $\mathbb{A}_0$ may denote any ring, here it is taken to be the set $\mathbb{R}$ of real numbers. The binomials $f$ and $g$ should be devised such that $\mathbb{A}_1 = \mathbb{C}$ the complex numbers, $\mathbb{A}_2 = \mathbb{H}$ the quaternions, and $\mathbb{A}_3 = \mathbb{O}$ the octonions. Historically, various researchers have used different yet equivalent doubling products.

Keywords. Cayley–Dickson algebra, Doubling product, Twisted group algebra, Quaternions, Octonions, Fano plane.

1. Introduction

Although it is recognized that there are several possible Cayley–Dickson doubling products, past researchers have restricted themselves to only two of them. Furthermore, the development of the basis vectors has proceeded historically in a fashion which obscures the high periodicity of the structure constants associated with the products of the Cayley–Dickson basis vectors. The purpose of this paper is to catalog all possible variants of the Cayley–Dickson doubling product and to suggest an alternate way to number the basis vectors. The alternate numbering method has been used in the past, for example in Shafer’s 1954 paper [14], but has fallen out of favor. By identifying the ordered pair of two sequences as the ‘shuffling’ of the two sequences the author demonstrates how this leads naturally to the numbering used by Shafer and others and how it illustrates the periodicity of the structure constants and corresponding ‘twist’ on the group product underlying the product of the basis vectors. As a result of cataloging the eight different Cayley–Dickson doubling products the author has identified one in particular $(a, b)(c, d) = (ac - b^*d, da^* + bc)$ having an interesting twist map. To the
2. Defining Properties

We shall take the following as necessary properties of a Cayley–Dickson algebra $\mathbb{A}_N$:

1. For each $N$ and each $x \in \mathbb{A}_N$,
   \[ 1 \cdot x = x \cdot 1 = x \]  
   (2.1)

2. There shall be a norm and an involution $^*$ such that for each $N$ and each $x, y \in \mathbb{A}_N$,
   \[ x + x^* \in \mathbb{A}_0 = \mathbb{R} \]  
   (2.2)
   \[ xx^* = x^*x = \|x\|^2 \]  
   (2.3)
   \[ (xy)^* = y^*x^* \]  
   (2.4)

3. There shall be an infinite sequence $e_0, e_1, e_2, \ldots$ of unit vectors such that
   (a) For each $N \geq 0$ if $p < 2^N$ then $e_p$ is a basis vector for $\mathbb{A}_N$
   (b) The set $\{ \pm e_p \mid 0 \leq p < 2^N \}$ is a group under the product on $\mathbb{A}_N$.
   (c) $e_0 = 1$ and if $p > 0$ then
   \[ e_p^2 = -e_0 \]  
   (2.5)
   (d) If $0 \neq p \neq q \neq 0$ then the anti-symmetric property holds:
   \[ e_p e_q + e_q e_p = 0 \]  
   (2.6)
   (e) If $0 \neq p \neq q \neq 0$ then there is an $r \neq 0$ such that either $e_p e_q = e_r$
   or $e_p e_q = -e_r$ and the quaternion property holds:
   \[ e_p e_q = e_r \text{ implies } e_q e_r = e_p \]  
   (2.7)

3. The Basis Vectors

The entire approach taken in this paper is highly dependent upon the manner in which the basis vectors are chosen. They are deliberately chosen to be the basis vectors of the Hilbert space $\ell^2$ of square-summable sequences. That is,

\[ e_0 = 1, 0, 0, 0, \ldots \]
\[ e_1 = 0, 1, 0, 0, \ldots \]
\[ e_2 = 0, 0, 1, 0, \ldots \]
  

In order to relate these basis vectors to the idea of ordered pairs, we identify a real number $r$ with the sequence

\[ r = r, 0, 0, 0, \ldots \]  
   (3.1)

and the ordered pair of two sequences

\[ x = x_0, x_1, x_2, \ldots \]
\[ y = y_0, y_1, y_2, \ldots \]
with the ‘shuffle’ of those two sequences

\[(x, y) = x_0, y_0, x_1, y_1, x_2, y_2, \ldots\]

This ‘shuffle basis’ immediately leads to two results: an inductive development of the basis vectors and the proper definition of the involution.

These basis vectors defined inductively as ordered pairs are

\[e_0 = 1\]  
\[e_{2p} = (e_p, 0)\]  
\[e_{2p+1} = (0, e_p)\]

Furthermore, since \[x + x^* \in \mathbb{R}\] is a requirement it is sufficient that

\[x^* = x_0, -x_1, -x_2, \ldots\]

with the result that

\[(x, y)^* = x_0, -y_0, -x_1, -y_1, \ldots\]

which, expressed in terms of ordered pairs is

\[(x, y)^* = (x^*, -y)\]

which is the traditional involution for Cayley–Dickson algebras. This also guarantees by induction that for each \(A\)

\[x^{**} = x\]

4. Devising an Adequate Product

It is required that \(1 \cdot x = x \cdot 1 = x\).

Let \((a, b)(c, d) = (f(a, b, c, d), g(a, b, c, d))\).

For it to be the case that \((a, b)(1, 0) = (a, b)\) and \((1, 0)(c, d) = (c, d)\) it is sufficient that \(f\) contain one of the terms \(ac\) or \(ca\) plus or minus some product of \(b\) or \(b^*\) with \(d\) or \(d^*\).

Furthermore, it is sufficient that \(g\) contain some product of \(a\) or \(a^*\) with \(d\) plus a product of \(c\) or \(c^*\) with \(b\).

\[f_0(a, b, c, d) = ca - b^*d\]
\[f_1(a, b, c, d) = ca - db^*\]
\[f_2(a, b, c, d) = ac - b^*d\]
\[f_3(a, b, c, d) = ac - db^*\]
\[f_4(a, b, c, d) = ca - bd^*\]
\[f_5(a, b, c, d) = ca - d^*b\]
\[f_6(a, b, c, d) = ac - bd^*\]
\[ f_7(a, b, c, d) = ac - d^*b \]

We already know that \( g \) must contain one of the following terms: \( ad, da, a*d \) or \( da^* \). In order for \( g \) to be zero when an ordered pair is multiplied by its conjugate it must be the case that, for each of these four options \( g \) must contain, respectively, the terms \( c^*b, bc^*, cb \) and \( bc \).

Therefore \( g \) must be one of the following four binomials:

\[
\begin{align*}
g_0(a, b, c, d) &= da^* + bc \\
g_1(a, b, c, d) &= a^*d + cb \\
g_2(a, b, c, d) &= ad + c^*b \\
g_3(a, b, c, d) &= da + bc^*
\end{align*}
\]

This results in a combination of 32 possible Cayley–Dickson doubling products. And indeed all 32 can be shown to satisfy properties (2.1) through (2.6) above. It is the quaternion property (2.7) which reduces the number from thirty-two to eight.

5. Twisted Group Algebra

The set \( W = \{0, 1, 2, 3, \ldots \} \) of whole numbers is a group under the bit-wise ‘exclusive or’ (XOR) operation on their binary representations, with group identity 0. For example, \( 27 \oplus 14 = 11011_2 \oplus 01110_2 = 10101_2 = 21 \). Rather than representing the group operation as addition, we will use juxtaposition. Thus for \( p, q \in W \) we will denote \( p \oplus q \) as simply \( pq \). This group operation on \( W^2 \) is pertinent since for \( p, q \in W \) and each of the 32 doubling products \((f_i, g_j)\) there is an \( \omega_{ij}(p, q) \in \{-1, 1\} \) such that

\[
e_p e_q = \omega_{ij}(p, q)e_{pq} \quad (5.1)
\]

The map \( \omega_{ij}: W \times W \to \{-1, 1\} \) is called a twist on the group \( W \) making each of the algebras resulting from the 32 doubling products a twisted group algebra \([6,8,13]\).

6. Interior Points of \( W^2 \) and Their Successors

Property (2.7) is a property of the interior points of \( W^2 \). The interior points of \( W^2 \) are the points \((p, q) \in W^2 \) such that \( 0 \neq p \neq q \neq 0 \). Every point of \( W^2 \) has four successor points \((2p, 2q), (2p, 2q + 1), (2p + 1, 2q)(2p + 1, 2q + 1)\). Interior points of \( W^2 \) which are not a successor point of an interior point are initial interior points. To establish a general property of the interior points of \( W^2 \) by induction one must first establish that the property holds for the initial interior points (the basis step). The inductive step consists of showing that if the property holds for an interior point \((p, q)\) then the property holds for the four successors of \((p, q)\).

We will find that for the initial interior points property (2.7) holds for only half of the 32 possible doubling products. Then we will find that only
8 of those 16 doubling products will satisfy property (2.7) for the remaining interior points of \( W^2 \).

Before beginning with the induction, let us re-express property (2.7) with property (5.1) in mind.

If \( 0 \neq p \neq q \neq 0 \) and if \( e_p e_q = e_{pq} \) then \( e_q e_{pq} = e_p \) and \( e_{pq} e_p = e_q \). \( (6.1) \)

We will adopt the notation \((p, q, r)\) to mean that \( e_p e_q = e_r \). Since \( e_p e_q + e_q e_p = 0 \) for \( 0 \neq p \neq q \), one but not both of \((p, q, pq)\) or \((q, p, pq)\) must be the case.

7. The Basis Step of the Inductive Proof of Property \((6.1)\)

First we will determine which of the 32 doubling products satisfy property \((6.1)\) for basis vector products \( e_p e_q \) where \((p, q)\) is an initial interior point of \( W^2 \). The initial interior points are precisely those interior points \((p, q)\) for which either \( p = 1 \) or \( q = 1 \) or for which \( p \) and \( q \) differ by exactly 1. Put another way, the initial interior points of \( W^2 \) consist of the ordered pairs of the forms \((2s, 1), (1, 2s), (2s + 1, 1), (1, 2s + 1), (2s, 2s + 1), (2s + 1, 2s)\) where \( s > 0 \).

For each interior point of \( W^2 \) we must have either one or the other but not both of the following quaternion properties:

\[(2s, 1, 2s + 1) \text{ and } (1, 2s + 1, 2s) \] \( \text{ and } (2s + 1, 2s, 1) \) \( Q \)
\[(1, 2s, 2s + 1) \text{ and } (2s, 2s + 1, 1) \text{ and } (2s + 1, 1, 2s) \] \( , Q^T \)

Consider the following

\[ e_{2s} e_1 = (e_s, 0)(0, 1) = \begin{cases} (0, e_s) = e_{2s+1} & \text{for } g_2, g_3 \text{ with all } f_i \\ (0, -e_s) = -e_{2s+1} & \text{for } g_0, g_1 \text{ with all } f_i \end{cases} \] \( (7.1) \)

\[ e_1 e_{2s+1} = (0, 1)(0, e_s) = \begin{cases} (e_s, 0) = e_{2s} & \text{for } f_4, f_5, f_6, f_7 \text{ with all } g_j \\ (-e_s, 0) = -e_{2s} & \text{for } f_0, f_1, f_2, f_3 \text{ with all } g_j \end{cases} \] \( (7.2) \)

\[ e_{2s+1} e_{2s} = (0, e_s)(e_s, 0) = \begin{cases} (0, 1) = e_1 & \text{for } g_2, g_3 \text{ with all } f_i \\ (0, -1) = -e_1 & \text{for } g_0, g_1 \text{ with all } f_i \end{cases} \] \( (7.3) \)

To make sense of these results we summarize in Table 1 which of the 32 doubling products satisfy a condition from either option \( Q \) or \( Q^T \). Any doubling product (shown in gray) which satisfies a condition from both options fails property \((6.1)\) and therefore is not a valid Cayley–Dickson doubling product.

8. The Inductive Step of the Proof of Property \((6.1)\)

Suppose \((p, q)\) is an interior point of \( W^2 \) and that \((p, q, pq)\) implies that \((q, pq, p)\) and \((pq, p, q)\). For which of the sixteen remaining doubling products does property \((6.1)\) follow for all four successors of \((p, q)\)?
Table 1. First elimination of doubling products

|   | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|------|------|------|------|------|------|------|------|
| $g_0$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ |
|     | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q$   | $Q$   | $Q$   |      |
| $g_1$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ |
|     | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q$   | $Q$   | $Q$   |      |
| $g_2$ | $Q$   | $Q$   | $Q$   | $Q$   | $Q$   | $Q$   | $Q$   | $Q$   |
|     | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q$   | $Q$   | $Q$   |      |
| $g_3$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q^T$ | $Q$   | $Q$   | $Q$   | $Q$   |

For each of the following four pairs of triple conditions only conditions from one of each pair may follow from the conditions $(p, q, pq)$, $(q, pq, p)$, $(pq, p, q)$ and a valid doubling product.

I. Contradictory conditions $A$ and $\bar{A}$

$(2p, 2q, 2pq)$ and $\ (2q, 2pq, 2p)$ and $\ (2pq, 2p, 2q)$ $(A)$

or

$(2q, 2p, 2pq)$ and $\ (2p, 2pq, 2q)$ and $\ (2pq, 2q, 2p)$ $(\bar{A})$

II. Contradictory conditions $B$ and $\bar{B}$

$(2p, 2q + 1, 2pq + 1)$ and $\ (2q + 1, 2pq + 1, 2p)$ and $\ (2pq + 1, 2p, 2q + 1)$ $(B)$

or

$(2q + 1, 2p, 2pq + 1)$ and $\ (2p, 2pq + 1, 2q + 1)$ and $\ (2pq + 1, 2q + 1, 2p)$ $(\bar{B})$

III. Contradictory conditions $C$ and $\bar{C}$

$(2p + 1, 2q, 2pq + 1)$ and $\ (2q, 2pq + 1, 2p + 1)$ and $\ (2pq + 1, 2q, 2q)$ $(C)$

or

$(2q, 2p + 1, 2pq + 1)$ and $\ (2p + 1, 2pq + 1, 2q)$ and $\ (2pq + 1, 2q, 2p + 1)$ $(\bar{C})$

IV. Contradictory conditions $D$ and $\bar{D}$

$(2p + 1, 2q + 1, 2pq)$ and $\ (2q + 1, 2pq, 2p + 1)$ and $\ (2pq, 2p + 1, 2q + 1)$ $(D)$

or

$(2q + 1, 2p + 1, 2pq)$ and $\ (2p + 1, 2pq, 2q + 1)$ and $\ (2pq, 2q + 1, 2q + 1)$ $(\bar{D})$

Suppose $(p, q)$ is an interior point of $W^2$ and that $(p, q, pq)$, $(q, pq, p)$, $(pq, p, q)$.

1. $e_{2p}e_{2q} = (e_p, 0)(e_q, 0)$

   $$e_{2p}e_{2q} = \begin{cases} (e_p e_q, 0) = (e_{pq}, 0) = e_{2pq} & \text{for } f_2, f_3, f_6, f_7 \\ (-e_p e_q, 0) = (-e_{pq}, 0) = -e_{2pq} & \text{for } f_0, f_1, f_4, f_5 \end{cases}$$

   (8.2)

So $(2p, 2q, 2pq)$ follows from conditions $A$ for $f_2, f_3, f_6, f_7$ and $(2q, 2p, 2pq)$ follows from conditions $\bar{A}$ for $f_0, f_1, f_4, f_5$. 
2. \[ e_{2q}e_{2pq} = (e_q, 0)(e_{pq}, 0) \]
\[ = \begin{cases} (e_qe_{pq}, 0) = (e_p, 0) = e_{2p} & \text{for } f_2, f_3, f_6, f_7 \\ (-e_qe_{pq}, 0) = (-e_p, 0) = -e_{2p} & \text{for } f_0, f_1, f_4, f_5 \end{cases} \] (8.3)

So \((2q, 2pq, 2p)\) follows from conditions \(A\) for \(f_2, f_3, f_6, f_7\) and \((2pq, 2q, 2p)\) follows from conditions \(\tilde{A}\) for \(f_0, f_1, f_4, f_5\).

3. \[ e_{2pq}e_{2p} = (e_{pq}, 0)(e_p, 0) \]
\[ = \begin{cases} (e_{pq}e_p, 0) = (e_q, 0) = e_{2q} & \text{for } f_2, f_3, f_6, f_7 \\ (-e_{pq}e_p, 0) = (-e_q, 0) = -e_{2q} & \text{for } f_0, f_1, f_4, f_5 \end{cases} \] (8.4)

So \((2pq, 2p, 2q)\) follows from conditions \(A\) for \(f_2, f_3, f_6, f_7\) and \((2p, 2pq, 2q)\) follows from conditions \(\tilde{A}\) for \(f_0, f_1, f_4, f_5\).

4. \[ e_{2p}e_{2q+1} = (e_p, 0)(0, e_q) \]
\[ = \begin{cases} (0, e_pe_q) = (0, e_{pq}) = e_{2pq+1} & \text{for } g_0, g_2 \\ (0, -e_pe_q) = (0, -e_{pq}) = -e_{2pq+1} & \text{for } g_1, g_3 \end{cases} \] (8.5)

So \((2p, 2q + 1, 2pq + 1)\) follows from conditions \(B\) for \(g_0, g_2\) and \((2q + 1, 2p, 2pq + 1)\) follows from conditions \(\tilde{B}\) for \(g_1, g_3\).

5. \[ e_{2q+1}e_{2pq+1} = (0, e_q)(0, e_{pq}) \]
\[ = \begin{cases} (e_qe_{pq}, 0) = (e_p, 0) = e_{2p} & \text{for } f_0, f_2, f_4, f_6 \\ (-e_qe_{pq}, 0) = (-e_p, 0) = -e_{2p} & \text{for } f_1, f_3, f_5, f_7 \end{cases} \] (8.6)

So \((2q + 1, 2pq + 1, 2p)\) follows from conditions \(B\) for \(f_0, f_2, f_4, f_6\) and \((2pq + 1, 2q + 1, 2p)\) follows from conditions \(\tilde{B}\) for \(f_1, f_3, f_5, f_7\).

6. \[ e_{2pq+1}e_{2p} = (0, e_{pq})(e_p, 0) \]
\[ = \begin{cases} (0, e_pe_p) = (0, e_q) = e_{2q+1} & \text{for } g_0, g_2 \\ (0, -e_pe_p) = (0, -e_q) = -e_{2q+1} & \text{for } g_1, g_3 \end{cases} \] (8.7)

So \((2pq + 1, 2p, 2q + 1)\) follows from conditions \(B\) for \(g_0, g_2\) and \((2pq + 1, 2q + 1, 2p)\) follows from conditions \(\tilde{B}\) for \(g_1, g_3\).

7. \[ e_{2p+1}e_{2q} = (0, e_p)(e_q, 0) \]
\[ = \begin{cases} (0, e_pe_q) = (0, e_{pq}) = e_{2pq} & \text{for } g_0, g_2 \\ (0, -e_pe_q) = (0, -e_{pq}) = -e_{2pq} & \text{for } g_1, g_3 \end{cases} \] (8.8)

So \((2pq + 1, 2p, 2q + 1)\) follows from conditions \(C\) for \(g_0, g_2\) and \((2pq + 1, 2q + 1, 2p)\) follows from conditions \(\tilde{C}\) for \(g_1, g_3\).
8.  
\[ e_{2p}e_{2q+1} = (e_p,0)(0,e_q) = \begin{cases} 
    (0,e_pe_q) = (0,e_{pq}) = e_{2pq+1} & \text{for } g_0,g_2 \\
    (0,-e_pe_q) = (0,-e_{pq}) = -e_{2pq+1} & \text{for } g_1,g_3 
\end{cases} \]  
\quad (8.9)

So \((2p,2q+1,2pq+1)\) follows from conditions \(C\) for \(g_0,g_2\) and \((2q+1,2p,2pq+1)\) follows from conditions \(\tilde{C}\) for \(g_1,g_3\).

9.  
\[ e_{q+1}e_{2pq+1} = (0,e_q)(0,e_{pq}) = \begin{cases} 
    (e_qe_{pq},0) = (e_p,0) = e_{2p} & \text{for } f_0,f_2,f_4,f_6 \\
    (-e_qe_{pq},0) = (-e_p,0) = -e_{2p} & \text{for } f_1,f_3,f_5,f_7 
\end{cases} \]  
\quad (8.10)

So \((2q+1,2pq+1,2p)\) follows from conditions \(C\) for \(f_0,f_2,f_4,f_6\) and \((2pq+1,2q+1,2p)\) follows from conditions \(\tilde{C}\) for \(f_1,f_3,f_5,f_7\).

10.  
\[ e_{p+1}e_{2q+1} = (0,e_p)(0,e_q) = \begin{cases} 
    (e_p,e_{pq},0) = (e_q,0) = e_{2p} & \text{for } f_0,f_2,f_4,f_6 \\
    (-e_pe_{pq},0) = (-e_q,0) = -e_{2p} & \text{for } f_1,f_3,f_5,f_7 
\end{cases} \]  
\quad (8.11)

So \((2p+1,2q+1,2pq)\) follows from conditions \(D\) for \(f_0,f_2,f_4,f_6\) and \((2q+1,2p+1,2pq)\) follows from conditions \(\tilde{D}\) for \(f_1,f_3,f_5,f_7\).

11.  
\[ e_{q+1}e_{2pq} = (0,e_q)(e_{pq},0) = \begin{cases} 
    (0,e_pe_{pq}) = (0,e_p) = e_{2p+1} & \text{for } g_0,g_2 \\
    (0,-e_pe_{pq}) = (0,-e_p) = -e_{2p+1} & \text{for } g_1,g_3 
\end{cases} \]  
\quad (8.12)

So \((2q+1,2pq,2p+1)\) follows from conditions \(D\) for \(g_0,g_2\) and \((2pq,2q+1,2p+1)\) follows from conditions \(\tilde{D}\) for \(g_1,g_3\).

12.  
\[ e_{2pq}e_{2p+1} = (pq,0)(0,e_p) = \begin{cases} 
    (0,e_{pq}e_p) = (0,e_q) = e_{2q+1} & \text{for } g_0,g_2 \\
    (0,-e_{pq}e_p) = (0,-e_q) = -e_{2q+1} & \text{for } g_1,g_3 
\end{cases} \]  
\quad (8.13)

So \((2pq,2p+1,2q+1)\) follows from conditions \(D\) for \(g_0,g_2\) and \((2p+1,2pq,2q+1)\) follows from conditions \(\tilde{D}\) for \(g_1,g_3\).

These results are compiled in Table 2 on page 10. Eight more of the doubling products are eliminated for inconsistency with property 6.1. Thus the inductive step of the proof of property 6.1 only succeeds for the eight remaining products.

\[ F_0: (a,b)(c,d) = (ca - b^*d, da^* + bc) \]
\[ F_1: (a,b)(c,d) = (ca - db^*, a^*d + cb) \]
\[ F_2: (a,b)(c,d) = (ac - b^*d, da^* + bc) \]
\[ F_3: (a,b)(c,d) = (ac - db^*, a^*d + cb) \]
\[ F_0^\uparrow: (a,b)(c,d) = (ca - bd^*, ad + c^*b) \]
Table 2. Second elimination of doubling products

|    | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|----|-------|-------|-------|-------|-------|-------|-------|-------|
| $g_0$ | $A$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $\bar{A}$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $B$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $\bar{B}$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $C$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $\bar{C}$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $D$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
|    | $\bar{D}$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |

$P_{1}^\top: (a, b)(c, d) = (ca - d^*b, da + bc^*)$
$P_{2}^\top: (a, b)(c, d) = (ac - bd^*, ad + c^*b)$
$P_{3}^\top: (a, b)(c, d) = (ac - d^*b, da + bc^*)$

The ‘transpose’ symbol is used for the second set of four doubling products since, as will later become apparent, the corresponding product matrices of unit vectors are transposes of each other.

The product $P_{3}^\top$ is the one most commonly used, but its transpose $P_3$ is a close second. The author has found no instance of the use of the other six doubling products.

All eight of these products result in the complex $i = e_1 = 0, 1, 0, 0, \ldots$. For the first four products, the quaternion $j = e_2 = 0, 0, 1, 0, 0, \ldots$ and $k = e_3 = 0, 0, 1, 0, \ldots$. For their transposes $k = e_2$ and $j = e_3$.

**9. Recursive Definition of Structure Constants for the Cayley–Dickson Products**

For $P_0, P_1, P_2$ and $P_3$,

\[(1, 2n, 2n + 1) \text{ for all } n > 0 \]  

(9.1)
whereas for $P_0^\top, P_1^\top, P_2^\top$ and $P_3^\top$,
\[(1, 2n + 1, 2n) \text{ for all } n > 0 \quad (9.2)\]

The ‘transpositive’ nature of these two properties induces a transpose relationship between pairs of the eight doubling products.

For all eight of the Cayley–Dickson doubling products, the second quaternion property 6.1 holds.

For $0 \neq p \neq q \neq r \neq 0$ \((p, q, r) \rightarrow (q, r, p) \rightarrow (r, p, q)\) (9.3)

For $P_0$ and $P_0^\top$ if $0 \neq p \neq q \neq 0$ then
\[(p, q, r) \rightarrow (2r, 2q, 2p)\]
\[\rightarrow (2p, 2q + 1, 2r + 1)\]
\[\rightarrow (2p + 1, 2q, 2r + 1)\]
\[\rightarrow (2p + 1, 2q + 1, 2r)\] (9.4)

For $P_1$ and $P_1^\top$ if $0 \neq p \neq q \neq 0$ then
\[(p, q, r) \rightarrow (2r, 2q, 2p)\]
\[\rightarrow (2r + 1, 2q + 1, 2p)\]
\[\rightarrow (2r + 1, 2q, 2p + 1)\]
\[\rightarrow (2r + 1, 2q + 1, 2p + 1)\] (9.5)

For $P_2$ and $P_2^\top$ if $0 \neq p \neq q \neq 0$ then
\[(p, q, r) \rightarrow (2p, 2q, 2r)\]
\[\rightarrow (2p, 2q + 1, 2r + 1)\]
\[\rightarrow (2p + 1, 2q, 2r + 1)\]
\[\rightarrow (2p + 1, 2q + 1, 2r)\] (9.6)

For $P_3$ and $P_3^\top$ if $0 \neq p \neq q \neq 0$ then
\[(p, q, r) \rightarrow (2p, 2q, 2r)\]
\[\rightarrow (2r + 1, 2q + 1, 2p)\]
\[\rightarrow (2r + 1, 2q, 2p + 1)\]
\[\rightarrow (2r + 1, 2q + 1, 2p + 1)\] (9.7)

The most commonly used of these eight doubling products is $P_3^\top$ [5, 10, 14] but $P_3$ has also been used [2, 3].

10. Application to Octonion Basis Vectors

Using induction rules 9.1 through 9.7 one can construct all the $(p, q, r)$ for any $A_N$. First using either rule 9.1 or 9.2 for all $n$ such that $2n + 1 < 2^N$ construct all $(1, q, r)$. Then use whichever of rules 9.4 through 9.7 applies to compute the remainder.

For example, using $P_0$, let us compute all the quaternion triplets for the octonions $O = A_3$. Applying rule 9.2 for $n = 1, 2, 3$ gives the quaternion
triplets $\{1, 2, 3\}$, $\{1, 4, 5\}$ and $\{1, 6, 7\}$. The remaining four can be obtained from $\{1, 2, 3\}$ and rule 9.4: $\{2, 6, 4\}$, $\{2, 5, 7\}$, $\{6, 3, 5\}$ and $\{4, 7, 3\}$.

The following are the triplets for $P_0$ through $P_3$.

$P_0$: $(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 6, 4), (2, 5, 7), (5, 6, 3), (3, 4, 7)$

$P_1$: $(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 6, 4), (5, 2, 7), (7, 4, 3), (3, 6, 5)$

$P_2$: $(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (7, 2, 5), (5, 6, 3), (3, 4, 7)$

$P_3$: $(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (5, 2, 7), (7, 4, 3), (3, 6, 5)$

Reverse these to obtain the triplets for $P_0^\top$ through $P_3^\top$.

$P_0^\top$: $(1, 3, 2), (1, 5, 4), (1, 7, 6), (2, 4, 6), (7, 5, 2), (5, 3, 6), (3, 7, 4)$

$P_1^\top$: $(1, 3, 2), (1, 5, 4), (1, 7, 6), (2, 4, 6), (5, 7, 2), (7, 3, 4), (3, 5, 6)$

$P_2^\top$: $(1, 3, 2), (1, 5, 4), (1, 7, 6), (2, 6, 4), (7, 5, 2), (5, 3, 6), (3, 7, 4)$

$P_3^\top$: $(1, 3, 2), (1, 5, 4), (1, 7, 6), (2, 6, 4), (5, 7, 2), (7, 3, 4), (3, 5, 6)$

For $P_0$, $P_3$ there are permutations of integers 1 through 7 which take any triple $(p, q, r)$ through all the triples for that product as well as its transpose. For example:

$P_0$: $(1263457)|(1, 2, 3), (2, 6, 4), (6, 3, 5), (3, 4, 7), (4, 5, 1), (5, 7, 2), (7, 1, 6)$

$P_3$: $(1243675)|(1, 2, 3), (2, 4, 6), (4, 3, 7), (3, 6, 5), (6, 7, 1), (7, 5, 2), (5, 1, 4)$

11. The Oriented Fano Plane

The products of the basis vectors for the octonions are customarily represented in the Fano Plane. The shuffle basis produces an especially nice representation of the eight doubling products. (See Fig. 1) Fano planes are typically numbered in a haphazard fashion although a better oriented construction exists. To construct an oriented Fano plane draw an equilateral triangle inscribed with a circle and construct the three altitudes. Label the center $e_1$. Next label a midpoint of one of the sides $e_2$. Label the vertex opposite that midpoint $e_3$. Label one of the two remaining midpoints $e_4$ and the vertex opposite that midpoint $e_5$. Label the remaining midpoint $e_6$ and the remaining vertex $e_7$. This results in one of only two versions of an oriented Fano plane, each being a reflection of the other in one of the altitudes.

Each of the three sides of the triangle represents a triple $(p, q, r)$. Likewise each altitude from a vertex to the midpoint of the opposite side as well as the circle through the midpoints represents such a triple.

For each of the eight doubling products, the sense of the three sides is the same–either clockwise ($\rightarrow$) around the triangle, or counter-clockwise ($\leftarrow$). If clockwise, then the three sides of the triangle represent $(5, 2, 7)$, $(7, 4, 3)$ and $(3, 6, 5)$. If counter-clockwise then the three sides represent $(7, 2, 5)$, $(5, 6, 3)$ and $(3, 4, 7)$.

The circle through the midpoints of the sides represents either $(2, 4, 6)$ in the clockwise sense (°) or $(2, 6, 4)$ in the counter-clockwise sense ($\circ$).

The three altitudes may all be in an ‘up’ sense from base to vertex (↑) or may all be in a ‘down’ sense from vertex to base (↓). So the altitudes must either be $(1, 2, 3)$, $(1, 4, 5)$, and $(1, 6, 7)$ or $(1, 3, 2)$, $(1, 5, 4)$, and $(1, 7, 6)$. All altitudes must have the same sense. See Fig. 1 for a breakdown of all the modes of the eight Cayley–Dickson doubling products.
This common orientation with regards to directions of the three sides as well as directions of the altitudes, together with the fact that for each \((p, q, r)\), \(r\) is the ‘bit-wise exclusive or’ of \(p\) and \(q\), motivates calling this an ‘oriented’ Fano plane. The sides all have the same orientation and the altitudes all have the same orientation.

Since the sides may have two senses and the circle may have two senses and the altitudes may have two senses, all \(2^3 = 8\) versions of the Cayley–Dickson products may be accommodated in the one diagram.

For the octonions and for any of the eight Cayley–Dickson doubling variations, knowing the ‘sense’ of \((1, 2, 3)\), \((2, 4, 6)\) and \((2, 5, 7)\) is sufficient to recover the sense of all octonion triples \((p, q, r)\) for each product using an oriented Fano plane.

12. The Twists on W Corresponding to the Eight Doubling Products

For each of the eight Cayley–Dickson doubling products of finite sequences, it’s multiplication table is divided into distinct \(2 \times 2\) blocks of the form
Cayley–Dickson doubling products is used. Results are shown in Table 3 on page 15. For all Cayley–Dickson products, and since for \( e \) or \( q \), or \( e \), \( p \) and \( q \) are the identity, it follows that for \( p > 1 \), \( e_p e_p^* = ||e_p||^2 = 1 \) and since \( e_p e_p^* = -e_p e_p = -\omega(p, p) e_0 = 1 \) it follows that for \( p > 1 \)

\[
\omega(p, p) = -1
\]

(12.2) and that for all \( p \)

\[
e_p^* = \omega(p, p) e_p
\]

(12.3)

From Eq. 2.6 it follows that

\[
\omega(q, p) + \omega(p, q) = 0 \quad \text{for } 0 \neq p \neq q \neq 0
\]

(12.4)

12.1. The Product of \( e_{2r} e_{2s} \)

Since \( e_{2r} = (e_r, 0) \) then \( e_{2r} e_{2s} = (e_r, 0) (e_s, 0) \) this could be called the case of \( b = d = 0 \). For each of the 32 distinct products, either \( e_{2r} e_{2s} = (e_s e_r, 0) \) or \( e_{2r} e_{2s} = (e_r e_s, 0) \). We shall consider the effect each of these alternatives upon the twist \( \omega \).

| \( P_0 \) | \( e_{2s} \) | \( e_{2s+1} \) | \( P_0^\top \) | \( e_{2s} \) | \( e_{2s+1} \) |
|---|---|---|---|---|---|
| \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_s e_r^*) \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_r e_s^*) \) |
| \( e_{2r+1} \) | \( (0, e_r e_s) \) | \( (-e_r e_s^*, 0) \) | \( e_{2r+1} \) | \( (0, e_r e_s^*) \) | \( (-e_s e_r^*, 0) \) |
| \( P_1 \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_s e_r^*) \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_r e_s^*) \) |
| \( e_{2r+1} \) | \( (0, e_r e_s) \) | \( (-e_r e_s^*, 0) \) | \( e_{2r+1} \) | \( (0, e_r e_s^*) \) | \( (-e_s e_r^*, 0) \) |
| \( P_2 \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_s e_r^*) \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_r e_s^*) \) |
| \( e_{2r+1} \) | \( (0, e_r e_s) \) | \( (-e_r e_s^*, 0) \) | \( e_{2r+1} \) | \( (0, e_r e_s^*) \) | \( (-e_s e_r^*, 0) \) |
| \( P_3 \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_s e_r^*) \) | \( e_{2r} \) | \( (e_s e_r, 0) \) | \( (0, e_r e_s^*) \) |
| \( e_{2r+1} \) | \( (0, e_r e_s) \) | \( (-e_r e_s^*, 0) \) | \( e_{2r+1} \) | \( (0, e_r e_s^*) \) | \( (-e_s e_r^*, 0) \) |

\[
\begin{bmatrix}
  e_{2r} e_{2s} & e_{2r} e_{2s+1} \\
  e_{2r+1} e_{2s} & e_{2r+1} e_{2s+1}
\end{bmatrix}
 =
\begin{bmatrix}
  (e_r, 0)(e_s, 0) & (e_r, 0)(0, e_s) \\
  (0, e_r)(e_s, 0) & (0, e_r)(0, e_s)
\end{bmatrix}
\]

The results of the four products will vary according to which of the eight Cayley–Dickson doubling products is used. Results are shown in Table 3 on page 15.

The basis elements \( e_0, e_1, e_2, \ldots \) are indexed by \( W = \{0, 1, 2, \ldots \} \) which is a group under the bit-wise ‘exclusive or’ of their binary representations and for each of the 32 Cayley–Dickson products there is a function \( \omega \) from \( W \times W \) to \( \{-1, 1\} \) such that for \( p, q \in W \)

\[
e_p e_q = \omega(p, q) e_{pq}
\]

(12.1) where \( pq \) is the group product of \( p \) and \( q \). The function \( \omega \) is called a ‘twist’ on the group \( W \) and turns the set of all finite sequences into a twisted group algebra.

12.1. The Product of \( e_{2r} e_{2s} \)
12.1.1. $e_2r e_{2s} = (e_s e_r, 0)$. Since $e_2r e_{2s} = \omega(2r, 2s)e_{2rs}$ and since $e_2r e_{2s} = \omega(s, r)e_{2rs}$ we may conclude that Whenever $e_2r e_{2s} = (e_s e_r, 0)$,
\[ \omega(2r, 2s) = \omega(s, r) \]  
(12.5)

12.1.2. $e_2r e_{2s} = (e_r e_s, 0)$. Since $e_2r e_{2s} = \omega(2r, 2s)e_{2rs}$ and since $e_2r e_{2s} = \omega(r, s)e_{2rs}$ we may conclude that Whenever $e_2r e_{2s} = (e_r e_s, 0)$,
\[ \omega(2r, 2s) = \omega(r, s) \]  
(12.6)

12.2. The Products $e_2r e_{2s+1}$ and $e_{2r+1} e_{2s}$

Since $e_2r e_{2s+1} = (e_r, 0)(0, e_s)$ this could be called the case of $b = c = 0$. And since $e_{2r+1} e_{2s} = (0, e_r)(e_s, 0)$ that could be called the case of $a = d = 0$. An inspection of the 32 alternate products shows that there are only four distinct possibilities for the products of $e_2r e_{2s+1}$ and $e_{2r+1} e_{2s}$.

12.2.1. $e_2r e_{2s+1} = (0, e_s e_r)$ and $e_{2r+1} e_{2s} = (0, e_r e_s^* )$. These two conditions imply that whenever $e_2r e_{2s+1} = (0, e_s e_r)$,
\[ \omega(2r, 2s + 1) = \omega(s, r) \]  
(12.7)

and that whenever $e_{2r+1} e_{2s} = (e_r e_s^*, 0)$,
\[ \omega(2r + 1, 2s) = \omega(s, s)\omega(r, s) = \begin{cases} -\omega(r, s) & \text{if } s > 0 \\ 1 & \text{otherwise} \end{cases} \]  
(12.8)

12.2.2. $e_2r e_{2s+1} = (0, e_r e_s)$ and $e_{2r+1} e_{2s} = (0, e_s^* e_r )$. These two conditions imply that whenever $e_2r e_{2s+1} = (0, e_r e_s)$,
\[ \omega(2r, 2s + 1) = \omega(r, s) \]  
(12.9)

and that whenever $e_{2r+1} e_{2s} = (e_s^* e_r, 0)$,
\[ \omega(2r + 1, 2s) = \omega(s, s)\omega(s, r) = \begin{cases} -\omega(s, r) & \text{if } s > 0 \\ 1 & \text{otherwise} \end{cases} \]  
(12.10)

12.2.3. $e_2r e_{2s+1} = (0, e_s e_r^* )$ and $e_{2r+1} e_{2s} = (0, e_r e_s )$. These two conditions imply that whenever $e_2r e_{2s+1} = (0, e_s e_r^* )$
\[ \omega(2r, 2s + 1) = \omega(r, r)\omega(s, s) = \begin{cases} -\omega(s, r) & \text{if } r > 0 \\ 1 & \text{otherwise} \end{cases} \]  
(12.11)

and that whenever $e_{2r+1} e_{2s} = (e_r e_s, 0)$,
\[ \omega(2r + 1, 2s) = \omega(r, s) \]  
(12.12)

12.2.4. $e_2r e_{2s+1} = (0, e_r^* e_s )$ and $e_{2r+1} e_{2s} = (0, e_s e_r )$. These two conditions imply that whenever $e_2r e_{2s+1} = (0, e_r^* e_s )$
\[ \omega(2r, 2s + 1) = \omega(s, s)\omega(r, s) = \begin{cases} -\omega(r, s) & \text{if } s > 0 \\ 1 & \text{otherwise} \end{cases} \]  
(12.13)

and that whenever $e_{2r+1} e_{2s} = (e_s e_r, 0)$,
\[ \omega(2r + 1, 2s) = \omega(r, s) \]  
(12.14)
12.3. The Product $e_{2r+1}e_{2s+1}$

Since $e_{2r+1} = (0, e_r)$ the product $e_{2r+1}e_{2s+1}$ could be called the case of $a = c = 0$.

12.3.1. $e_{2r+1}e_{2s+1} = -(e^*_re, 0)$. This implies that whenever $e_{2r+1}e_{2s+1} = -(e^*_re, 0)$

$$\omega(2r + 1, 2s + 1) = -\omega(s, s)\omega(s, r) = \begin{cases} \omega(s, r) & \text{if } s > 0 \\ -1 & \text{otherwise} \end{cases}$$

(12.15)

12.3.2. $e_{2r+1}e_{2s+1} = -(e_re^*_s, 0)$. This implies that whenever $e_{2r+1}e_{2s+1} = -(e_re^*_s, 0)$

$$\omega(2r + 1, 2s + 1) = -\omega(s, s)\omega(r, s) = \begin{cases} \omega(r, s) & \text{if } s > 0 \\ -1 & \text{otherwise} \end{cases}$$

(12.16)

12.3.3. $e_{2r+1}e_{2s+1} = -(e^*_sre, 0)$. This implies that whenever $e_{2r+1}e_{2s+1} = -(e^*_sre, 0)$

$$\omega(2r + 1, 2s + 1) = -\omega(r, r)\omega(s, r) = \begin{cases} \omega(s, r) & \text{if } r > 0 \\ -1 & \text{otherwise} \end{cases}$$

(12.17)

12.3.4. $e_{2r+1}e_{2s+1} = -(e^*_re_s, 0)$. This implies that whenever $e_{2r+1}e_{2s+1} = -(e^*_re_s, 0)$

$$\omega(2r + 1, 2s + 1) = -\omega(r, r)\omega(r, s) = \begin{cases} \omega(r, s) & \text{if } r > 0 \\ -1 & \text{otherwise} \end{cases}$$

(12.18)

13. The Eight Variations

The preceding results are summarized in the following tables:

| $P_0 : (a, b)(c, d) = (ca - b^*d, da^* + bc)$ |
|---|---|---|
| $\omega_0$ | $2s$ | $2s + 1$ |
| $2r$ | $\omega(s, r)$ | $-\omega(s, r)$ if $r > 0$ |
| | | $1$ otherwise |
| $2r + 1$ | $\omega(r, s)$ | $\omega(r, s)$ if $r > 0$ |
| | | $-1$ otherwise |

| $P^*_0 : (a, b)(c, d) = (ca - bd^*, ad + c^*b)$ |
|---|---|---|
| $\omega_0^*$ | $2s$ | $2s + 1$ |
| $2r$ | $\omega(s, r)$ | $\omega(r, s)$ |
| $2r + 1$ | $-\omega(s, r)$ if $s > 0$ | $\omega(r, s)$ if $s > 0$ |
| | | $1$ otherwise | $-1$ otherwise |
\[
P_1 : (a, b)(c, d) = (ca - db^*, a^*d + cb)
\]

\[
\begin{aligned}
&\omega_1 \quad 2s \quad 2s + 1 \\
&2r \quad \omega(s, r) \begin{cases}
-\omega(r, s) & \text{if } r > 0 \\
1 & \text{otherwise}
\end{cases} \\
&2r + 1 \quad \omega(s, r) \begin{cases}
\omega(r, s) & \text{if } r > 0 \\
-1 & \text{otherwise}
\end{cases}
\end{aligned}
\]

\[
P_1^\top : (a, b)(c, d) = (ca - d^*b, da + bc^*)
\]

\[
\begin{aligned}
&\omega_1^* \quad 2s \quad 2s + 1 \\
&2r \quad \omega(s, r) \quad \omega(s, r) \\
&2r + 1 \quad \begin{cases}
-\omega(r, s) & \text{if } s > 0 \\
\omega(r, s) & \text{if } s > 0 \\
1 & \text{otherwise} \\
-1 & \text{otherwise}
\end{cases}
\end{aligned}
\]

\[
P_2 : (a, b)(c, d) = (ac - b^*d, da^* + bc)
\]

\[
\begin{aligned}
&\omega_2 \quad 2s \quad 2s + 1 \\
&2r \quad \omega(r, s) \begin{cases}
-\omega(s, r) & \text{if } r > 0 \\
1 & \text{otherwise}
\end{cases} \\
&2r + 1 \quad \omega(r, s) \begin{cases}
\omega(s, r) & \text{if } r > 0 \\
-1 & \text{otherwise}
\end{cases}
\end{aligned}
\]

\[
P_2^\top : (a, b)(c, d) = (ac - bd^*, ad + c^*b)
\]

\[
\begin{aligned}
&\omega_2^* \quad 2s \quad 2s + 1 \\
&2r \quad \omega(r, s) \quad \omega(s, r) \\
&2r + 1 \quad \begin{cases}
-\omega(s, r) & \text{if } s > 0 \\
\omega(r, s) & \text{if } s > 0 \\
1 & \text{otherwise} \\
-1 & \text{otherwise}
\end{cases}
\end{aligned}
\]

\[
P_3 : (a, b)(c, d) = (ac - db^*, a^*d + cb)
\]

\[
\begin{aligned}
&\omega_3 \quad 2s \quad 2s + 1 \\
&2r \quad \omega(r, s) \begin{cases}
-\omega(r, s) & \text{if } r > 0 \\
1 & \text{otherwise}
\end{cases} \\
&2r + 1 \quad \omega(s, r) \begin{cases}
\omega(s, r) & \text{if } r > 0 \\
-1 & \text{otherwise}
\end{cases}
\end{aligned}
\]

\[
P_3^\top : (a, b)(c, d) = (ac - d^*b, da + bc^*)
\]

\[
\begin{aligned}
&\omega_3^* \quad 2s \quad 2s + 1 \\
&2r \quad \omega(r, s) \quad \omega(s, r) \\
&2r + 1 \quad \begin{cases}
-\omega(r, s) & \text{if } s > 0 \\
\omega(s, r) & \text{if } s > 0 \\
1 & \text{otherwise} \\
-1 & \text{otherwise}
\end{cases}
\end{aligned}
\]
Table 4. Twist blocks

| Twist | C   | L   | T   | D   | N   |
|-------|-----|-----|-----|-----|-----|
|       | 0 = | 0 > | 0 > | 0 > | 0 ≠ |
| \(\omega_0\) | \[1 1\] | \[1 -1\] | \[1 1\] | \[1 -1\] | \[-1 1\] |
| \(\omega_1\) | \[1 1\] | \[1 -1\] | \[1 1\] | \[1 -1\] | \[-1 -1\] |
| \(\omega_2\) | \[1 1\] | \[1 -1\] | \[1 1\] | \[1 -1\] | \[1 1\] |
| \(\omega_3\) | \[1 1\] | \[1 -1\] | \[1 1\] | \[1 -1\] | \[1 1\] |
| \(\omega_0^*\) | \[1 1\] | \[1 1\] | \[-1 1\] | \[1 1\] | \[-1 1\] |
| \(\omega_1^*\) | \[1 1\] | \[-1 1\] | \[-1 1\] | \[1 1\] | \[-1 1\] |
| \(\omega_2^*\) | \[1 1\] | \[-1 1\] | \[-1 1\] | \[1 1\] | \[1 1\] |
| \(\omega_3^*\) | \[1 1\] | \[-1 1\] | \[-1 1\] | \[1 1\] | \[-1 1\] |

14. The Twist Blocks

The twist tables for each of the eight \(\omega_k\) twist functions can each be subdivided into \(2 \times 2\) blocks with entries +1 or -1. The upper left corner \(C\) of each twist table for \(A_N\), \(N > 0\), corresponds to \(r = s = 0\). For \(N > 1\) the left side blocks \(L\) correspond to \(r > s = 0\). The top side blocks \(T\) correspond to \(0 = r < s\). The diagonal blocks \(-D\) correspond to \(0 < r = s\). For \(N > 2\), the interior blocks \(N\) correspond to \(0 ≠ r ≠ s ≠ 0\). Note that \(D\) is shown in its positive form in Table 4 on page 20 but appears as \(-D\) in the actual twist tables. Also notice that \(C\) is the same for all eight doubling products. Doubling products \(P_0\) through \(P_3\) differ from each other only in their values for \(N\) and products \(P_0^T\) through \(P_3^T\) also differ from each other only in their values of \(N\). The twist tables for \(P_k^T\) is the transpose of the twist table for \(P_k\).

Since every pair of basis vectors \(e_p, e_q\) has a product \(e_pe_q = \omega_k(p, q)e_{pq}\) where \(pq = p \oplus q\), the bit-wise ‘exclusive or’ (XOR) of the binary representations of \(p\) and \(q\) one only needs a table of values of \(\omega_k(p, q)\) or \(\omega_k^*(p, q)\) to recover the multiplication table for any particular \(0 ≤ k < 4\) and any \(A_N\).

15. Geometry of the \(\omega\) Twist Tables

The \(\omega\) tables for the eight doubling products are highly periodic when using the shuffle basis. However, the two most common doubling products \(\omega_3\) and
\( \omega_0 \) have a somewhat bewildering pattern to the eye. This is not the case for \( \omega_1 \) and \( \omega_2 \) or their transposes. The pattern for \( \omega_1 \) ‘alternates’ whereas as \( N \to \infty \), the \( \omega_2 \) table for \( A_{N+1} \) has the same appearance as the table for \( A_N \). Note that for any \( \omega \) table the table for \( A_N \) always forms the upper left quadrant of the table for \( A_{N+1} \).

In Figs. 2, 3, 4 and 5 we see visual representations of \( \omega_0 \) through \( \omega_3 \) for \( A_4 \) and \( A_7 \) respectively, with dark gray representing \( \omega(p, q) = +1 \) and light gray representing \( \omega(p, q) = -1 \). The upper left square in all images has coordinates \((p, q) = (0, 0)\) and the lower right corner has coordinates \((p, q) = (2^N - 1, 2^N - 1)\).

Figure 2. \( \omega_0 \) twist map for \( A_4 \) and \( A_7 \)

Figure 3. \( \omega_1 \) twist map for \( A_4 \) and \( A_7 \)
16. The Twist Tree for $\omega_2$

In [3] it was demonstrated how to use a tree diagram to determine $\omega_3(p, q)$ for any two non-negative integers $p$ and $q$. Figure 6 is the twist tree for $\omega_2$.

To use the tree to find $\omega_2(p, q)$ for non-negative integers $p$ and $q$ they must first be represented in their binary form. To illustrate, let $p = 93 = 101101_2$ and let $q = 37 = 0100101_2$. Next ‘shuffle’ the two bit strings together to obtain 10, 01, 10, 10, 11, 00, 11. Beginning with the top of the tree $C$ which represents the top left quadrant of the $\omega_2$ twist table a binary 1 is an instruction to move to the right branch of the tree and a binary 0 is an instruction to move to the left branch of the tree. So the first instruction 10 moves from $C$ to $L$. The next instruction 01 moves from $L$ to $-1$. Therefore $\omega_2(93, 37) = -1$. There is no need to finish traversing the
sequence 10, 01, 10, 10, 11, 00, 11 further because of the stability of $\omega_2$. Thus $e_{93}e_{37} = -e_{120}$ and $(37, 93, 120)$ is a structure constant for the product $P_2$.

Note that any sequence of instructions which terminates in $C, T$ or $L$ gives a result of $\omega_2(p, q) = 1$ and any which terminates in $-D$ gives $\omega_2(p, q) = -1$.

This approach to finding the value of the twist function developed in [3] was found useful by Flaut and Shpakivskyi in their study [9] of holomorphic functions in generalized Cayley–Dickson algebras.

The two dimensional graphs of $\omega_2$ in Fig. 4 suggests that

\[ \omega_2(p, q) = \begin{cases} 1 & \text{if } 1 < \frac{q}{p} < \frac{3}{2} \text{ and } \\ -1 & \text{if } \frac{2}{3} < \frac{q}{p} \leq 1 \end{cases} \]  

(16.1)

The trees in Fig. 6 suggest that if $m > n \geq 0$ and $r, s < 2^n$ then

\[ \omega_2(2^n + r, 2^m + s) = 1 \]  

(16.2)

\[ \omega_2(2^m + 2^n + r, 2^n + s) = 1 \]  

(16.3)

\[ \omega_2(2^m + r, 2^m + 2^n + s) = 1 \]  

(16.4)

17. Conclusion

We have identified the eight variations of the doubling-products which produce the Cayley–Dickson algebras and presented an alternate conception of the order pair of two sequences as consisting of the ‘shuffling’ of the two sequences. We have demonstrated how the corresponding ‘shuffle basis’ for Cayley–Dickson algebras reveals the periodic and fractal nature of the twist underlying the product of the basis vectors of the algebra and identified one in particular \((a, b)(c, d) = (ac - b^*d, da^* + bc)\) which has a particularly appealing twist map yet has heretofore not been investigated. We have also presented another instance of the use of a ‘twist tree’ for determining the twist function of a Cayley–Dickson algebra. In regard to Octonion algebras in particular we have suggested an alternate way to label the Fano Plane in such a way that the structure constants have a consistent orientation in the plane. It is hoped that the alternate perspectives offered in this paper will prove useful to other researchers.
Appendix

The author wrote the following bc [11] program for generating the $\omega$ twist maps in Figs. 2 through 5. The code generates a $\LaTeX$ document containing a PSTricks figure.

For $A_7$ it was necessary to increase the LaTeX memory allotment in the user’s (not the system’s) texmf.cnf file to ‘main_memory = 7500000’.

```
#TeXTables.bc
define sgn(x,y){
    auto p,q,lp,lq;
    #Note: lp means p is odd, !lp means p is even
    scale=0;
    p=x;
    q=y;
    if(p*q==0) return 1;
    lp=p%2;
    lq=q%2;
    if(p==1 && lq) return -1;
    /*The following lines are for omega 0
    if(!lp && !lq) return sgn(q/2,p/2);
    if(!lp && lq) return -sgn(q/2,p/2);
    return sgn(p/2,q/2);
    End of omega 0 lines*/
    /*The following lines are for omega 1
    if(!lp && lq) return -sgn(p/2,q/2);
    return sgn(q/2,p/2);
    End of omega 1 lines*/
    /*The following lines are for omega 2
    if(!lp && lq) return -sgn(q/2,p/2);
    #End of omega 2 lines*/
    /*The following lines are for omega 3
    if(!lp && !lq) return sgn(p/2,q/2);
    if(!lp && lq) return -sgn(p/2,q/2);
    return sgn(q/2,p/2);
    End of omega 3 lines*/
}
k=7; #Table for Cayley-Dickson Algebra A_k (set desired value)
print \documentclass{minimal}\n"
print "\usepackage{pstricks-add}\n"
print "\usepackage{graphicx}\n"
print "\begin{document}\n"
scale=5;
dot=24/2^k; # Sets the size of each square
du=20/2^k; # Sets the distance between squares
```
\begin{pspicture}(\textwidth,\textheight)\psframe[fillcolor=yellow](-1,-1)(1,1)\end{pspicture}\end{document}

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