PARTIAL SUMS OF THE NORMALIZED RABOTNOV FUNCTIONS

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Abstract. Let \( R_\alpha,\beta(z) = z + \sum_{n=1}^\infty A_n z^{n+1} \) be the sequence of partial sums of the normalized Rabotnov functions \( R_\alpha,\beta(z) = z + \sum_{n=1}^\infty A_n z^{n+1} \) where \( A_n = \beta^n \Gamma(1+\alpha) \Gamma((1+\alpha)(n+1)) \).

The purpose of the present paper is to determine lower bounds for \( R\{ R_\alpha,\beta(z) \} \), \( R\{ (R_\alpha,\beta)_m(z) \} \), \( R\{ R'_\alpha,\beta(z) \} \), \( R\{ (R'_\alpha,\beta)_m(z) \} \), and \( R\{ (\|R_\alpha,\beta\|_m(z)) \} \) where \( \|R_\alpha,\beta\|_m \) is the Alexander transform of \( R_\alpha,\beta \). Several examples of the main results are also considered.

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1. Introduction and preliminaries

In 1948, Rabotnov [19] introduced a special function applied in viscoelasticity. This function, known today as the Rabotnov fractional exponential function or briefly Rabotnov function, is defined as follows

\[
R_\alpha,\beta(z) = z^\alpha \sum_{n=0}^\infty \frac{(\beta)^n z^{n(1+\alpha)}}{\Gamma((n+1)(1+\alpha))}, \quad (\alpha, \beta, z \in \mathbb{C}).
\] (1.1)

Rabotnov function is the particular case of the familiar Mittag-Leffler function [16] widely used in the solution of fractional order integral equations or fractional order differential equations. The relation between the Rabotnov function and Mittag-Leffler function can be written as follows

\[
R_\alpha,\beta(z) = z^\alpha E_{1+\alpha,1+\alpha}(\beta z^{1+\alpha}),
\]

where \( E \) is Mittag-Leffler function and \( \alpha, \beta, z \in \mathbb{C} \). Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found in [3, 4, 11, 12].

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^\infty a_n z^n
\] (1.2)

which are analytic in the open unit disc \( \mathcal{U} = \{ z : |z| < 1 \} \) and hold the normalization condition \( f(0) = f'(0) - 1 = 0 \). Further, by \( S \) we shall denote the class of all functions in \( A \) which are univalent in \( \mathcal{U} \).
The Alexander transform $\mathbb{I}[f]: \mathbb{U} \to \mathbb{C}$ of $f$ is defined by [2]

$$\mathbb{I}[f] = \int_0^z f(t)dt = z + \sum_{n=2}^\infty \frac{a_n}{n} z^n.$$ 

It is clear that the Rabotnov function $R_{\alpha,\beta}(z)$ does not belong to the family $A$. Thus, it is natural to consider the following normalization of Rabotnov function

$$R_{\alpha,\beta}(z) = z^{1+\frac{1}{1+\alpha}} \Gamma (1 + \alpha) R_{\alpha,\beta}(z^{1+\alpha})$$

$$= z + \sum_{n=1}^\infty \frac{\beta^n \Gamma (1 + \alpha)}{\Gamma ((1 + \alpha)(n + 1))} z^{n+1}, \ z \in \mathbb{U}, \quad (1.3)$$

Geometric properties including starlikeness, convexity and close-to-convexity for the normalized Rabotnov function $R_{\alpha,\beta}(z)$ were recently investigated by Eker and Ece in [7].

Throughout this paper, we shall restrict our attention to the case of real-valued $\alpha \geq 0$, $\beta > 0$ and $z \in \mathbb{U}$.

The concept of finding the lower bound of the real part of the ratio of the partial sum of analytic functions to its infinite series sum was introduced firstly by Silvia [22]. Silverman in [21] found the partial sums of convex and starlike functions by developed more useful techniques. After that, several researchers investigated such partial sums for different subclasses of analytic functions. For more work on partial sums, the interested readers are referred to [5, 8, 9, 10, 15, 17, 18, 20].

Recently, some researchers have studied on partial sums of special functions. For example, Orhan and Yagmur in [23] determined lower bounds for the normalized Struve functions to its sequence of partial sums. Some lower bounds for the quotients of normalized Dini functions and their partial sum, as well as for the quotients of the derivative of normalized Dini functions and their partial sums were obtained by Aktaş and Orhan in [1]. Din et al. [6] found the partial sums of two kinds normalized Wright functions and the partial sums of Alexander transform of these normalized Wright functions. While Kazımoğ in [14] studied the partial sums of the normalized Miller-Ross Function.

In this paper, we study the ratio of a function of the form (1.3) to its sequence of partial sums

$$(R_{\alpha,\beta}(z))_m(z) = z + \sum_{n=1}^m \frac{\beta^n \Gamma (1 + \alpha)}{\Gamma ((1 + \alpha)(n + 1))} z^{n+1}, \ m \in \mathbb{N}, \quad (1.4)$$

and for $m = 0$, we have $(R_{\alpha,\beta}(z))_0(z) = z$.

when the coefficients of $R_{\alpha,\beta}$ satisfy certain conditions. We determine the lower bounds of

$$\Re \left\{ \frac{R_{\alpha,\beta}(z)}{(R_{\alpha,\beta})_m(z)} \right\}, \Re \left\{ \frac{(R_{\alpha,\beta})_m(z)}{R_{\alpha,\beta}(z)} \right\}, \Re \left\{ \frac{R'_{\alpha,\beta}(z)}{(R_{\alpha,\beta})_m'(z)} \right\}, \Re \left\{ \frac{R'_{\alpha,\beta}}{R_{\alpha,\beta}(z)} \right\},$$

$$\Re \left\{ \frac{\mathbb{I}[R_{\alpha,\beta}](z)}{(\mathbb{I}[R_{\alpha,\beta}])_m(z)} \right\}, \Re \left\{ \frac{(\mathbb{I}[R_{\alpha,\beta}])_m(z)}{\mathbb{I}[R_{\alpha,\beta}](z)} \right\}.$$
where $I[R_{\alpha,\beta}]$ is the Alexander transform of $R_{\alpha,\beta}$.

In order to obtain our results we need the following lemmas.

**Lemma 1.1.** If $n \in \mathbb{N}$ and $\alpha \geq 0$, then

$$(1 + \alpha)^{n-1}(n - 1)!\Gamma (1 + \alpha) \leq \Gamma ((1 + \alpha) n).$$

(1.5)

**Lemma 1.2.** Let $\alpha \geq 0$ and $0 < \beta < \alpha + 1$. Then the function $R_{\alpha,\beta} : \mathfrak{U} \rightarrow \mathbb{C}$ defined by (1.3) satisfies the following inequalities:

(i) $|R_{\alpha,\beta}(z)| \leq e^{\beta/(\alpha + 1)}, \quad z \in \mathfrak{U}$.

(ii) $|R'_{\alpha,\beta}(z)| \leq \frac{n \beta}{\alpha + 1} e^{\beta/(\alpha + 1)}, \quad z \in \mathfrak{U}$.

(iii) $|I[R_{\alpha,\beta}](z)| \leq \frac{1}{2} \left( e^{\beta/(\alpha + 1)} + 1 \right), \quad z \in \mathfrak{U}$.

**Proof.** (i) By using the inequality (1.5) of Lemma 1.2, we obtain

$$|R_{\alpha,\beta}(z)| = \left| z + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma (1 + \alpha)}{\Gamma ((1 + \alpha) (n + 1))} z^n \right|$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma (1 + \alpha)}{\Gamma ((1 + \alpha) (n + 1))}$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{\beta^n}{(1 + \alpha)^n n!}$$

$$= e^{\beta/(\alpha + 1)}, \quad z \in \mathfrak{U}.$$

(ii) To prove (ii), using the inequality (1.5) of Lemma 1.1, we have

$$|R'_{\alpha,\beta}(z)| = \left| 1 + \sum_{n=1}^{\infty} \frac{(n + 1) \beta^n \Gamma (1 + \alpha)}{\Gamma ((1 + \alpha) (n + 1))} z^n \right|$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{(n + 1) \beta^n \Gamma (1 + \alpha)}{\Gamma ((1 + \alpha) (n + 1))}$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{(n + 1) \beta^n}{(1 + \alpha)^n n!}$$

$$= \frac{(\alpha + 1 + \beta) e^{\beta/(\alpha + 1)}}{\alpha + 1}, \quad z \in \mathfrak{U}.$$

(iii) Making the use of the inequality (1.5), we get

$$(n + 1)\Gamma ((1 + \alpha) (n + 1)) \geq 2(1 + \alpha)^{n-1}(n - 1)!\Gamma (1 + \alpha), \quad n \in \mathbb{N}.$$
We thus find

\[
|\mathbb{I}[\mathbb{R}_{\alpha,\beta}(z)]| = \left| z + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{(n+1)\Gamma((1+\alpha)(n+1))} z^{n+1} \right|
\]

\[
\leq 1 + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{(n+1)\Gamma((1+\alpha)(n+1))}
\]

\[
\leq 1 + \sum_{n=1}^{\infty} \frac{\beta^n \Gamma(1+\alpha)}{(n+1)\Gamma((1+\alpha)(n+1))}
\]

\[
\leq 1 + \sum_{n=1}^{\infty} \frac{\beta^n}{2(1+\alpha)^n n!}
\]

\[
= \frac{1}{2} e^{\frac{\beta}{\alpha} + 1} + \frac{1}{2}, \quad z \in \mathfrak{U}.
\]

□

Let \( w(z) \) be an analytic function in \( \mathfrak{U} \): In the sequel, we will use the following well-known result:

\[
\Re \left\{ \frac{1 + w(z)}{1 - w(z)} \right\} > 0, \quad z \in \mathfrak{U} \text{ if and only if } |w(z)| < 1, \quad z \in \mathfrak{U}.
\]

**Theorem 1.3.** Let \( \alpha \geq 0 \) and \( 0 < \beta < \alpha + 1 \). Then

\[
\Re \left\{ \frac{\mathbb{R}_{\alpha,\beta}(z)}{(\mathbb{R}_{\alpha,\beta})_m(z)} \right\} \geq 2 - e^{\frac{\beta}{\alpha+1}}, \quad z \in \mathfrak{U},
\]

(1.6)

and

\[
\Re \left\{ \frac{(\mathbb{R}_{\alpha,\beta})_m(z)}{\mathbb{R}_{\alpha,\beta}(z)} \right\} \geq \frac{1}{e^{\beta/(\alpha+1)}}, \quad z \in \mathfrak{U}.
\]

(1.7)

**Proof.** From inequality (i) of Lemma 1.2, we get

\[
1 + \sum_{n=1}^{\infty} |A_n| \leq e^{\frac{\beta}{\alpha+1}},
\]

or equivalently

\[
\left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=1}^{\infty} |A_n| \leq 1
\]

where \( A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \).

In order to prove the inequality (1.6), we consider the function \( w(z) \) defined by
\[
\frac{1 + w(z)}{1 - w(z)} = \frac{1}{e^{\beta/(\alpha+1)} - 1} \left[ \frac{\mathbb{R}_{\alpha, \beta}(z)}{\mathbb{R}_{\alpha, \beta}(m(z))} - (2 - e^{\beta/(\alpha+1)}) \right]
\]
\[
= 1 + \sum_{n=1}^{m} A_n z^n + \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} A_n z^n/n.
\]

(1.8)

Now, from (1.8) we can write
\[
w(z) = \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} A_n z^n
\]
\[
= 2 + 2 \sum_{n=1}^{m} A_n z^n + \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} A_n z^n/n.
\]

and
\[
|w(z)| \leq \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n| + \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n| - \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n|.
\]

This implies that \(|w(z)| \leq 1\) if and only if
\[
2 \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n| \leq 2 - \sum_{n=1}^{m} |A_n|.
\]

Which further implies that
\[
\sum_{n=1}^{m} |A_n| + \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n| \leq 1.
\]

(1.9)

It suffices to show that the left hand side of (1.9) is bounded above by \(\left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=1}^{\infty} |A_n|\), which is equivalent to
\[
\left( \frac{2 - e^{\beta/(\alpha+1)}}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=1}^{m} |A_n| \geq 0.
\]

To prove (1.7), we write
\[
\frac{1 + w(z)}{1 - w(z)} = \frac{e^{\beta/(\alpha+1)}}{e^{\beta/(\alpha+1)} - 1} \left[ \frac{\mathbb{R}_{\alpha, \beta}(z)}{\mathbb{R}_{\alpha, \beta}(m(z))} - \frac{1}{e^{\beta/(\alpha+1)}} \right]
\]
\[
= 1 + \sum_{n=1}^{m} A_n z^n - \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} A_n z^n/n.
\]

Therefore
\[
|w(z)| \leq \left( \frac{e^{\beta/(\alpha+1)}}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n| \leq 1.
\]

The last inequality is equivalent to
\[
\sum_{n=1}^{m} |A_n| + \left( \frac{1}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} |A_n| \leq 1.
\]

(1.10)
Since the left hand side of (1.10) is bounded above by \( \left( \frac{1}{-\left(\frac{\alpha+1}{\nu+1}\right)} \right) \sum_{n=1}^{\infty} |A_n| \), this completes the proof. \( \square \)

**Theorem 1.4.** Let \( \alpha \geq 0 \), \( 0 < \beta < \alpha + 1 \) and \( 2(\alpha + 1) \geq (\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) \). Then

\[
\mathfrak{R} \left\{ \frac{\mathbb{R}_{\alpha,\beta}'(z)}{(\mathbb{R}_{\alpha,\beta})_m'(z)} \right\} \geq \frac{2(\alpha + 1) - (\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right)}{\alpha + 1}, \quad z \in \mathcal{U}, \tag{1.11}
\]

and

\[
\mathfrak{R} \left\{ \frac{(\mathbb{R}_{\alpha,\beta})_m'(z)}{\mathbb{R}_{\alpha,\beta}'(z)} \right\} \geq \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right)}, \quad z \in \mathcal{U}. \tag{1.12}
\]

**Proof.** From part (ii) of Lemma 1.2, we observe that

\[
1 + \sum_{n=1}^{\infty} (n + 1) |A_n| \leq \left( \frac{\alpha + 1 + \beta}{\alpha + 1} \right) e^\left(\frac{\alpha}{\nu+1}\right),
\]

where \( A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \). This implies that

\[
\left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \sum_{n=1}^{\infty} (n + 1) |A_n| \leq 1.
\]

Consider

\[
\frac{1 + w(z)}{1 - w(z)} = \left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \left\{ \frac{\mathbb{R}_{\alpha,\beta}'(z)}{(\mathbb{R}_{\alpha,\beta})_m'(z)} \right\} - \left( \frac{2(\alpha + 1) - (\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right)}{\alpha + 1} \right)
\]

\[
1 + \sum_{n=1}^{m} (n + 1) A_n z^n + \left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \sum_{n=m+1}^{\infty} (n + 1) A_n z^n
\]

\[
= \frac{1 + \sum_{n=1}^{m} (n + 1) A_n z^n + \left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \sum_{n=m+1}^{\infty} (n + 1) A_n z^n}{1 + \sum_{n=1}^{m} (n + 1) A_n z^n}.
\]

Therefore

\[
|w(z)| \leq \left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \sum_{n=m+1}^{\infty} (n + 1) |A_n| \leq 1.
\]

The last inequality is equivalent to

\[
\sum_{n=1}^{m} (n + 1) |A_n| + \left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \sum_{n=m+1}^{\infty} (n + 1) |A_n| \leq 1. \tag{1.13}
\]

It suffices to show that the left hand side of (1.13) is bounded above by

\[
\left( \frac{\alpha + 1}{(\alpha + 1 + \beta) e^\left(\frac{\alpha}{\nu+1}\right) - (\alpha + 1)} \right) \sum_{n=1}^{\infty} (n + 1) |A_n|.
\]
which is equivalent to
\[
\left(\frac{2(\alpha + 1) - (\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}}}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right) \sum_{n=1}^{m} (n + 1) |A_n| \geq 0
\]
which holds true for \(2(\alpha + 1) - (\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} \geq 0\).

To prove the result (1.12), we write
\[
\frac{1 + w(z)}{1 - w(z)} = \left(\frac{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}}}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right) \left[\frac{R_{\alpha,\beta}'(z)}{R_{\alpha,\beta}(z)} - \frac{\alpha + 1}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right]
\]
where
\[
|w(z)| \leq \frac{\left(\frac{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}}}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right) \sum_{n=m+1}^{\infty} (n + 1) |A_n|}{2 - 2 \sum_{n=1}^{m} (n + 1) |A_n| - \left(\frac{(2\alpha + 2) - (\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}}}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right) \sum_{n=m+1}^{\infty} (n + 1) |A_n|} \leq 1.
\]
The last inequality is equivalent to
\[
\sum_{n=1}^{m} (n + 1) |A_n| + \left(\frac{\alpha + 1}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right) \sum_{n=m+1}^{\infty} (n + 1) |A_n| \leq 1. \quad (1.14)
\]
It suffices to show that the left hand side of (1.14) is bounded above by
\[
\left(\frac{\alpha + 1}{(\alpha + 1 + \beta) e^{\frac{\alpha}{\alpha + 1}} - (\alpha + 1)}\right) \sum_{n=1}^{\infty} (n + 1) |A_n|,
\]
the proof is complete. \(\square\)

**Theorem 1.5.** Let \(\alpha \geq 0\), \(0 < \beta < \alpha + 1\) and \(1 < e^{\frac{\alpha}{\alpha + 1}} < e\). Then
\[
\Re\left\{\frac{\mathbb{I}[R_{\alpha,\beta}](z)}{(\mathbb{I}[R_{\alpha,\beta}])_m(z)}\right\} \geq \frac{3 - e^{\beta/(\alpha + 1)}}{2}, \quad z \in \mathfrak{M}, \quad (1.15)
\]
and
\[
\Re\left\{\frac{(\mathbb{I}[R_{\alpha,\beta}])_m(z)}{\mathbb{I}[R_{\alpha,\beta}](z)}\right\} \geq \frac{2}{e^{\beta/(\alpha + 1)} - 1}, \quad z \in \mathfrak{M}. \quad (1.16)
\]
where \(\mathbb{I}[R_{\alpha,\beta}]\) is the Alexander transform of \(R_{\alpha,\beta}\).

**Proof.** To prove (1.15), we consider from part (iii) of Lemma 1.2 so that
\[
1 + \sum_{n=1}^{\infty} \frac{|A_n|}{n + 1} \leq \frac{1}{2} \left(e^{\frac{\alpha}{\alpha + 1}} + 1\right),
\]
or equivalently

\[
\left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=1}^{\infty} \frac{|A_n|}{n+1} \leq 1
\]

where \( A_n = \frac{\beta^n \Gamma(1+\alpha)}{\Gamma((1+\alpha)(n+1))} \). Now, we write

\[
\frac{1 + w(z)}{1 - w(z)} = \frac{\left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right)}{\left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \left[ 1 + \sum_{n=1}^{m} \frac{|A_n|}{n+1} z^n + \left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} \frac{|A_n|}{n+1} z^n \right]}
\]

\( \frac{1 + w(z)}{1 - w(z)} = 1 + \sum_{n=1}^{m} \frac{|A_n|}{n+1} z^n + \left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} \frac{|A_n|}{n+1} z^n. \) (1.17)

Now, from (1.17) we can write

\[
w(z) = \frac{\left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right)}{2 + 2 \sum_{n=1}^{m} \frac{|A_n|}{n+1} z^n + \left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} \frac{|A_n|}{n+1} z^n}
\]

Using the fact that \(|w(z)| \leq 1\), we get

\[
\left| 2 - 2 \sum_{n=1}^{m} \frac{|A_n|}{n+1} - \left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} \frac{|A_n|}{n+1} \right| \leq 1.
\]

The last inequality is equivalent to

\[
\sum_{n=1}^{m} \frac{|A_n|}{n+1} + \left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=m+1}^{\infty} \frac{|A_n|}{n+1} \leq 1. \quad (1.18)
\]

It suffices to show that the left hand side of (1.18) is bounded above by \( \left( \frac{2}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=1}^{m} \frac{|A_n|}{n+1} \), which is equivalent to

\[
\left( \frac{3 - e^{\beta/(\alpha+1)}}{e^{\beta/(\alpha+1)} - 1} \right) \sum_{n=1}^{m} \frac{|A_n|}{n+1} \geq 0.
\]

The proof of (1.16) is similar to the proof of Theorem 1.3. □

Observe that \( \Re_{1,1}(z) = \sqrt{z} \sinh \left( \sqrt{z} \right) \). Thus, taking \( m = 0 \) in Theorem 1.3, we immediately obtain the following result.

**Corollary 1.6.** The following inequalities hold true:

\[
\Re \left\{ \frac{\sinh \left( \sqrt{z} \right)}{\sqrt{z}} \right\} \geq 2 - \sqrt{e} \approx 0.35128, \quad z \in \mathfrak{U}, \quad (1.19)
\]

\[
\Re \left\{ \frac{\sqrt{z}}{\sinh \left( \sqrt{z} \right)} \right\} \geq \frac{1}{\sqrt{e}} \approx 0.60653, \quad z \in \mathfrak{U}. \quad (1.20)
\]

**Remark 1.7.** Putting \( m = 0 \) in inequality (1.11), we obtain \( \Re \left\{ \Re_{\alpha,\beta}^i(z) \right\} > 0 \). In view of Noshiro-Warschawski Theorem (see [13]), we have that the normalized Rabotnov function is univalent in \( \mathfrak{U} \) for \( 2(\alpha + 1) \geq (\alpha + 1 + \beta) e^{\frac{\beta}{(\alpha+1)}} \).
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