UNIFORM REGULARITY AND VANISHING VISCOSITY LIMIT FOR THE FREE SURFACE NAVIER-STOKES EQUATIONS

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ABSTRACT. We study the inviscid limit of the free boundary Navier-Stokes equations. We prove the existence of solutions on a uniform time interval by using a suitable functional framework based on Sobolev conormal spaces. This allows us to use a strong compactness argument to justify the inviscid limit. Our approach does not rely on the justification of asymptotic expansions. In particular, we get a new existence result for the Euler equations with free surface from the one for Navier-Stokes.

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1. Introduction

The study of fluid motions with free interfaces has attracted a lot of attention during the last thirty years. When the fluid is viscous, that is to say for the Navier-Stokes equation, the local well-posedness for the Cauchy problem was shown for example in the classical works by Beale [7] and Tani [52], we also refer to the work by Solonnikov [50] and also to [28] for example. These results rely heavily on the smoothing effect due to the presence of viscosity. When viscosity is neglected, it is much more difficult to control the regularity of the free surface. The first attempts to get local well-posedness for the free-surface Euler equation go back to the works by Nalimov [42] and H. Yoshihara [60] for small data in two dimensions. A major breakthrough was achieved in the nineties by S. Wu [55, 56] who solved the irrotational problem in dimension two and three (see also the works by Craig [14], Beale, Hou, and Lowengrub [8], Ogawa and Tani [43], Schneider and Wayne [47], Lannes [33], Ambrose and Masmoudi [3, 4]). In this irrotational case, where dispersive effects are predominant, an almost global existence result was achieved by S. Wu [57] in the two-dimensional case and global existence results in three dimensions were proved recently by Germain, Masmoudi and Shatah [18, 19] and by S. Wu [58]. We also refer to [1] for a local well-posedness result below the standard regularity threshold. For the full system (without the irrotational assumption) there are well posedness results starting with the works by Christodoulou and Lindblad [12] and Lindblad [34]. Alternative approach were proposed by Coutand and Shkoller [13], Shatah and Zeng [49] and P. Zhang and Z. Zhang [61]. Note that the well-posedness of the Euler equation requires a Taylor sign condition for the pressure on the boundary.

A classical problem in fluid mechanics, since Reynolds numbers for real flows are very high, is to study the behaviour of the solutions of the Navier-Stokes equation at high Reynolds number which corresponds to small viscosity. A natural conjecture is of course to expect that the limit is given by a solution of the Euler equation. Nevertheless in the presence of boundaries the situation can be more complicated as we shall describe below. The main result of this work is the justification of the inviscid limit when we start from the free surface Navier-Stokes equation.

The study of the vanishing viscosity limit for the Navier-Stokes equation has a long history. We shall distinguish three different approach to the problem. The first one consists in proving that strong solutions to the Navier-Stokes equation exist on an interval of time independent of the viscosity parameter and then passing to the limit by strong compactness arguments. The second approach is based on weak compactness arguments starting from Leray weak solutions of the Navier-Stokes equation. Note that in these two approach the well posedness of the Euler equation is not used. On the contrary, the third approach relies on the a priori knowledge that sufficiently smooth solutions of the Euler equation exist: it consists either in comparing directly by a modulated type energy argument a Leray weak solution of the Navier-Stokes equation with an approximate solution whose leading order is given by a smooth solution the Euler equation or in directly justifying asymptotic expansions by studying a modified Navier-Stokes equation for the remainder.
When there is no boundary, the vanishing viscosity limit can be justified by using the three types of methods: we refer to Swann [51] and Kato [31] (see also [39]) for results with the first approach, to DiPerna and Majda [16, 15] for the second approach in the two-dimensional case and to the book [36] for example for the third approach. Note that the second approach yields the existence of global weak solution for the Euler equation and thus in three dimensions only results with the first and third approach are known.

In a fixed domain with boundary, the situation is more complicated, the situation depends on the type of boundary conditions that we impose on the boundary.

Let us first describe the situation in the case of the homogeneous Dirichlet boundary condition. For the first approach, the main difficulty is that for data of size of order one, all the known existence results of strong solutions for the Navier-Stokes equation yield uniform estimates that are valid only on a time interval which shrinks to zero when the viscosity goes to zero. This has a physical explanation: there is formation of boundary layers in a small vicinity of the boundary. Indeed, in a small vicinity of the boundary, the solution \( u_\varepsilon \), \( \varepsilon \) being the viscosity parameter, of the Navier-Stokes equation is expected to behave like 
\[
\lim_{\varepsilon \to 0} u_\varepsilon \sim U(t, y, z/\varepsilon) ,
\]
if the boundary is locally given by \( z = 0 \), where \( U \) is some profile which is not described by the Euler equation. Because of this small scale behaviour, one can see that there is no hope to get uniform estimates in all the spaces for which the 3-D Euler equation is known to be well-posed (\( H^s \), \( s > 5/2 \) for example).

Most of the works on the topic are thus concentrated on the justification of the third approach: namely the construction of an high order approximate solution (involving two spatial scales thus of matched asymptotics type) and the control of the remainder. Nevertheless, the equation that governs the boundary layers that appear in the formal expansion, the Prandtl equation, can be ill-posed in Sobolev spaces \([17]\), thus it is not always possible to even construct an approximate solution. Moreover, even when this is possible, some instabilities make impossible the control of the remainder on a suitable interval of time \([22], [27]\). Consequently, in a domain with fixed boundary with homogeneous Dirichlet boundary condition, the only full result is the work by Sammartino and Caflisch [46] where the asymptotic expansion is justified in the analytic framework. There is also a result by Kato [32] where some necessary condition for the convergence to hold is exhibited and the work [38] where the vertical viscosity is assumed much smaller than the horizontal one. The case of a non-homogeneous Dirichlet boundary condition which happens for example when there is injection or suction on the boundary is also studied. In this case the size of the boundary layer is \( \varepsilon \) (against \( \sqrt{\varepsilon} \) in the previous case) which makes the situation very different. The construction and justification of the asymptotic expansion, even for general hyperbolic-parabolic systems have been widely studied recently, we refer to \([21, 23, 24, 41, 26, 45, 54]\).

Let us finally describe the situation for the Navier-Stokes equation with a free surface. At first we note that there is a minimal amount of regularity which is needed in order to define the motion of the free surface, therefore there is up to our knowledge no suitable notion of weak solution for this equation. Consequently, it seems that to justify the inviscid limit one can only use either the first approach or the justification of matched asymptotic expansions. The main drawbacks of this last approach is first that a lot of regularity is needed in order to construct the expansion and second that the a priori knowledge of the well-posedness of the free surface Euler equation is needed. From a mathematical point of view, this is not very satisfactory since the local theory for the Euler equation is much more difficult to establish than the one for the Navier-Stokes equation. Moreover, because of the parabolic behaviour of the equation one do not expect to need more regularity on the data in order to solve the viscous equation than the inviscid one. In this work, we shall thus avoid to deal with the justification of asymptotic expansions as it is usually done and deal with the first approach. In this approach, one of the main difficulty is to prove that the local smooth solution of the Navier-Stokes equation (which comes from the classical works \([7, 52]\)) can be continued on an interval of time independent of the viscosity. In the vicinity of the
free surface the expected behaviour of the solution is \( u^\varepsilon \sim u(t,x) + \varepsilon \sqrt{U}(t,y,z/\varepsilon) \). Again, we observe that the \( H^s \) norm \( s > 5/2 \) still cannot be bounded on an interval of time independent of \( \varepsilon \). Nevertheless, the situation seems more favorable since one can expect that the Lipschitz norm which is the crucial quantity for the inviscid problem remains bounded. We shall consequently use a functional framework based on conormal Sobolev spaces which minimizes the needed amount of normal regularity but which gives a control of the Lipschitz norm of the solution in order to get that the solution of the Navier-Stokes equation exists and is uniformly bounded on an interval of time independent of \( \varepsilon \). Such an approach to the inviscid limit was recently used in our work [40] in the much simpler case of the Navier-Stokes equation with the Navier boundary condition on a fixed boundary. As a consequence of our estimates, we shall justify the inviscid limit by strong compactness arguments. Note that we shall also get as a corollary the existence of a solution to the free surface Euler equation. In particular this gives an approach by a physical regularization to the construction of solutions of the Euler equation which does not use the Nash-Moser iteration scheme. Of course, in order to get our result, we shall need to assume the Taylor sign condition for the pressure on the boundary which is needed to have local well-posedness for the limit Euler system.

1.1. The free surface Navier-Stokes equations. Let us now write down the system that we shall study. We consider the motion of an incompressible viscous fluid at high Reynolds number submitted to the influence of gravity. The equation for motion is thus the incompressible Navier-Stokes equation

\[
\partial_t u + u \cdot \nabla u + \nabla p = \varepsilon \Delta u, \quad \nabla \cdot u = 0, \quad x \in \Omega_t, \ t > 0
\]

where \( u(t,x) \in \mathbb{R}^3 \) is the velocity of the fluid and \( p \in \mathbb{R} \) is the pressure. We shall assume that the fluid domain \( \Omega_t \) is given by

\[
\Omega_t = \{ x \in \mathbb{R}^3, \ x_3 < h(t,x_1,x_2) \}
\]

thus the surface is given by the graph \( \Sigma_t = \{ x \in \mathbb{R}^3, \ x_3 = h(t,x_1,x_2) \} \) where \( h(t,x_1,x_2) \) which defines the free surface is also an unknown of the problem. The parameter \( \varepsilon > 0 \) which is the inverse of a Reynolds number is small. In this work, we shall focus on the 3-D equation set in the simplest domain. Note that this framework is relevant to describe for example atmospheric or oceanic motions. All our results are also valid in the two-dimensional case and can be extended to more general domains.

On the boundary, we impose two boundary conditions. The first one is of kinematic nature and states that fluid particles do not cross the surface:

\[
\partial_t h = u(t,x_1,x_2,h(t,x_1,x_2)) \cdot N, \quad (x_1,x_2) \in \mathbb{R}^2
\]

where \( N \) is the outward normal given by

\[
N = \begin{pmatrix} -\partial_1 h \\ -\partial_2 h \\ 1 \end{pmatrix}.
\]

The second one expresses the continuity of the stress tensor on the surface. Assuming that there is vacuum above the fluid, we get

\[
p n - 2 \varepsilon S u n = g h n, \quad x \in \Sigma_t
\]

where \( S \) is the symmetric part of the gradient

\[
S u = \frac{1}{2} (\nabla u + \nabla u^t),
\]
\( \mathbf{n} \) is the outward unit normal (\( \mathbf{n} = \mathbf{N}/|\mathbf{N}| \)) and \( g \) is a positive number which measures the influence of the gravity on the motion of the fluid (in the dimensional form of the equation this would be the acceleration of gravity constant). Note that the term involving the gravity force in (1.1) has been incorporated in the pressure term, this is why it appears in the right-hand side of (1.5). We neglect surface tension effects.

At \( t = 0 \), we supplement the system with the initial condition

\[
(1.6) \quad h_{/t=0} = h_0, \quad u_{/t=0} = u_0.
\]

Note that this means in particular that the initial domain \( \Omega_0 \) is given.

As usual in free boundary problems, we shall need to fix the fluid domain. A convenient choice in our case is to consider a family of diffeomorphism \( \Phi(t, \cdot) \) under the form

\[
(1.7) \quad \Phi(t, \cdot) : \mathcal{S} = \mathbb{R}^2 \times (-\infty, 0) \to \Omega_t
\]

\[
\begin{align*}
x &= (y, z) \mapsto (y, \varphi(t, y, z))
\end{align*}
\]

and we have to choose \( \varphi(t, \cdot) \). Note that we have to ensure that \( \partial_z \varphi > 0 \) in order to get that \( \Phi(t, \cdot) \) is a diffeomorphism.

Once the choice of \( \varphi \) is made, we reduce the problem into the fixed domain \( \mathcal{S} \) by setting

\[
(1.8) \quad v(t, x) = u(t, \Phi(t, x)), \quad q(t, x) = p(t, \Phi(t, x)) \quad x \in \mathcal{S}.
\]

Many choices are possible. A basic choice is to take \( \varphi(t, y, z) = z + \eta(t, y) \). Such a choice would fit in the inviscid case i.e. for the Euler equation when \( \varepsilon = 0 \). Indeed, the energy estimate and the physical condition yield that \( h \) and \( u \) have the same regularity and hence this choice yields that \( \Phi \) has in \( \mathcal{S} \) the same regularity as \( h \). In the Navier-Stokes case, the dissipation term in the energy estimate yields that \( \sqrt{\varepsilon} u \) is one derivative smoother than \( u \) and \( h \) and hence it is natural to choose a diffeomorphism such that \( \Phi \) has the same regularity as \( u \) in \( \mathcal{S} \). This is not given by our previous choice since by using the boundary condition (1.3), we shall get that \( \sqrt{\varepsilon} h \) and hence \( \sqrt{\varepsilon} \varphi \) gains only \( 1/2 \) derivative. The easiest way to fix this difficulty is to take a smoothing diffeomorphism as in [48], [33]. We thus choose \( \varphi \) such that

\[
(1.9) \quad \varphi(t, y, z) = Az + \eta(t, y, z)
\]

where \( \eta \) is given by the extension of \( h \) defined by

\[
(1.10) \quad \hat{\eta} (\xi, z) = \chi(z\xi) \hat{h}(\xi)
\]

where \( \hat{\cdot} \) stands for the Fourier transform with respect to the \( y \) variable and \( \chi \) is a smooth, even, compactly supported function such that \( \chi = 1 \) on \( B(0, 1) \). The number \( A > 0 \) is chosen so that

\[
(1.11) \quad \partial_z \varphi(0, y, z) \geq 1, \quad \forall x \in \mathcal{S}
\]

which ensures that \( \Phi(0, \cdot) \) is a diffeomorphism.

Another way to reduce the problem to a fixed domain would be to use standard Lagrangian coordinates. This coordinate system also has the advantage that \( \Phi \) in \( \mathcal{S} \) has the same regularity as the velocity in \( \mathcal{S} \). Nevertheless, here since the velocity will only have conormal regularity in \( \mathcal{S} \), the Lagrangian map will also have only conormal regularity. Consequently one main advantage of the choice (1.10) compared to Lagrangian coordinates is that we still have for \( \eta \) in \( \mathcal{S} \) a standard Sobolev regularity even if \( v \) only has a conormal one.

To transform the equations by using (1.8), we introduce the operators \( \partial_i^\varphi, i = t, 1, 2, 3 \) such that

\[
\begin{align*}
\partial_i u \circ \Phi(t, \cdot) &= \partial_i^\varphi v, \quad i = t, 1, 2, 3
\end{align*}
\]

we shall give the precise expression of these operators later. We thus get that by using the change of variable (1.7), the equation (1.1) for \( u, p \) becomes an equation for \( v, q \) in the fixed domain \( \mathcal{S} \)
which reads:

\[(1.12) \quad \partial_t \varphi + (v \cdot \nabla \varphi) v + \nabla \varphi q = \varepsilon \Delta \varphi v = 2\varepsilon \nabla \varphi \cdot S^\varphi v, \quad \nabla \varphi \cdot v = 0, \quad x \in S\]

where we have naturally set

\[\nabla \varphi q = \begin{pmatrix} \partial_1 \varphi q \\ \partial_2 \varphi q \\ \partial_3 \varphi q \end{pmatrix}, \quad \Delta \varphi = \sum_{i=1}^{3} (\partial_i \varphi)^2, \quad \nabla \varphi \cdot v = \sum_{i=1}^{3} \partial_i \varphi v_i, \quad v \cdot \nabla \varphi = \sum_{i=1}^{3} v_i \partial_i \varphi\]

and \(S^\varphi v = \frac{1}{2}(\nabla \varphi v + (\nabla \varphi v)^t)\). The two boundary conditions \((1.13), (1.15)\) become on \(z = 0\)

\[(1.13) \quad \partial_t h = v \cdot N = -v_y(t, y, 0) \cdot \nabla h(t, y) + v_3(t, y, 0), \quad q n - 2\varepsilon S^\varphi v \cdot n = gh n\]

where we use the notation \(v = (v_y, v_3)^t\). We shall also use set \(\nabla = (\nabla_y, \partial_z)^t\) From now on, we shall thus study in \(S\) the system \((1.12), (1.13), (1.14)\) with \(\varphi\) given by \((1.9), (1.10)\). We add to this system the initial conditions

\[(1.15) \quad h/t=0 = h_0, \quad v/t=0 = v_0.\]

To measure the regularity of functions defined in \(S\), we shall use Sobolev conormal spaces. Let us introduce the vector fields

\[Z_i = \partial_i, \quad i = 1, 2, \quad Z_3 = \frac{z}{1-z} \partial_z.\]

We define the Sobolev conormal space \(H^m_{co}(S)\) as

\[H^m_{co}(S) = \{ f \in L^2(S), \quad Z^\alpha f \in L^2(S), \quad |\alpha| \leq m \}\]

where \(Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}\) and we set

\[\|f\|_{m}^2 = \sum_{|\alpha| \leq m} \|Z^\alpha f\|_{L^2}^2, \quad \|f\| = \|f\|_0 = \|f\|_{L^2}.\]

In a similar way, we set

\[W^m_{co}(S) = \{ f \in L^\infty(S), \quad Z^\alpha f \in L^\infty(S), \quad |\alpha| \leq m \}\]

and

\[\|f\|_{m, \infty} = \sum_{|\alpha| \leq k} \|Z^\alpha f\|_{L^\infty}.\]

For vector fields, we also take the sums over its components. Note that the use of these spaces is classical in (hyperbolic) boundary value problems, see \([25, 29, 44, 53]\) for example.

Finally for functions defined on \(\mathbb{R}^2\) (like \(h\) in our problem), we use the notation \(|\cdot|_m\) for the standard Sobolev norm.

1.2. Main results. Our aim is to get a local well-posedness result for strong solutions of \((1.12), (1.13), (1.14)\) which is valid on an interval of time independent of \(\varepsilon\) for \(\varepsilon \in (0, 1]\). Note that such a result will also imply the local existence of strong solutions for the Euler equation. As it is well-known, \([56, 3, 13, 49]\), when there is no surface tension, a Taylor sign condition is needed to get local well-posedness for the Euler equation. For the Euler equation in a domain of the form \((1.2)\), the Taylor sign condition reads

\[(1.16) \quad - \partial_N p + g \geq c_0 > 0, \quad x \in \Sigma_t.\]
Before stating our main result, we need to understand what kind of Taylor sign condition (necessary in order to get a uniform with respect to \( \varepsilon \) local existence result) we have to impose for the Navier-Stokes equation. By using the divergence free condition, we get as usual that the pressure \( q \) solves in \( S \) the elliptic equation
\[
\Delta^\varepsilon q = -\nabla^\varepsilon \cdot (v \cdot \nabla^\varepsilon v).
\]
Moreover, by using the boundary condition (1.25), we get that on the boundary
\[
q/\varepsilon_{z=0} = 2\varepsilon S^\varepsilon v \cdot n + gh.
\]
We shall thus decompose the pressure into an "Euler" part and a "Navier-Stokes" part by setting \( q = q^E + q^{NS} \) with
\[
\Delta^\varepsilon q^E = -\nabla^\varepsilon \cdot (v \cdot \nabla^\varepsilon v), \quad q^E/\varepsilon_{z=0} = gh
\]
and
\[
\Delta^\varepsilon q^{NS} = 0, \quad q^{NS}/\varepsilon_{z=0} = 2\varepsilon S^\varepsilon v \cdot n.
\]
The main idea is that the part \( q^{NS} \) is small when \( \varepsilon \) is small while \( q^E \) which is of order one is the part which should converge to the pressure of the Euler equation when \( \varepsilon \) goes to zero. Consequently, the Taylor sign condition has to be imposed on \( q^E \). After the change of coordinates, this becomes
(1.17)
\[
g - \partial^\varepsilon q^E/\varepsilon_{z=0} \geq c_0 > 0.
\]
Note that since we shall indeed prove that the part \( q^{NS} \) of the pressure is small when \( \varepsilon \) is small, we shall actually obtain that for \( \varepsilon \) sufficiently small, the total pressure verifies the Taylor condition.

Finally, let us introduce a last notation, we denote by \( \Gamma = Id - n \otimes n \) the projection on the tangent space of the boundary.

Our main result reads:

**Theorem 1.1.** For \( m \geq 6 \), consider initial data \((h^0_0, v^0_0)\) such that
(1.18)
\[
\sup_{\varepsilon \in (0,1)} \left( |h^0_0|_m + \varepsilon^{\frac{1}{2}} |h^0_0|_{m+\frac{1}{2}} + |v^0_0|_m + \|\partial_z v^0_0\|_{m-1} + \|\partial_z v^0_0\|_{1,\infty} + \varepsilon^{\frac{1}{2}} \|\partial^2_{zz} v^0_0\|_{L^\infty} \right) \leq R_0,
\]
and assume that the Taylor sign condition (1.17) is verified and that the compatibility condition \( \Pi S^\varepsilon h^0 n/\varepsilon_{z=0} = 0 \) holds. Then, there exists \( T > 0 \) and \( C > 0 \) such that for every \( \varepsilon \in (0,1] \), there exists a unique solution \((v^\varepsilon, h^\varepsilon)\) of (1.12), (1.13), (1.14), (1.9), (1.10) which is defined on \([0,T]\) and satisfies the estimate:
(1.19)
\[
\sup_{[0,T]} \left( |v^\varepsilon|_m^2 + |h^\varepsilon|_m^2 + \|\partial_z v^\varepsilon\|_{m-2}^2 + \|\partial_z v^\varepsilon\|_{1,\infty}^2 + \|\partial_z v^\varepsilon\|_{L^4([0,T],H^m_{\varepsilon 0})}^2 \right) \leq C.
\]
Moreover, we also have the estimate
(1.20)
\[
\sup_{[0,T]} \left( \varepsilon |h^\varepsilon|_{m+\frac{1}{2}}^2 + \varepsilon \|\partial_z v^\varepsilon\|_{L^\infty}^2 \right) + \varepsilon \int_0^T \left( \|\nabla v^\varepsilon\|_m^2 + \|\nabla \partial_z v^\varepsilon\|_{m-2}^2 \right) \leq C.
\]

Note that in the above result, we have separated the estimates (1.19) that are independent of \( \varepsilon \) from the ones in (1.20) that depend on \( \varepsilon \) and are useful to get a closed estimate for the Navier-Stokes case. This is why we have stated an estimate of \( \|\partial_z v^\varepsilon\|_{m-1} \) which is not pointwise in time but only \( L^4 \). The reason for which we do not expect \( \sup_{[0,T]} \|\partial_z v^\varepsilon\|_{m-1} \) to be uniformly bounded with respect to \( \varepsilon \) will be explained below. It is related to the boundary control of the vorticity for the Navier-Stokes system. Nevertheless we point out that there is no loss of regularity, namely we have that \( \partial_z v^\varepsilon(t, \cdot) \in H^m_{\varepsilon 0} \) for every \( t \in [0,T] \) with bounds that may blow up with \( \varepsilon \). We shall also provide estimates for the pressure during the proof.

Also, note that the uniform existence time \( T \) is a priori also limited by the validity of the Taylor sign condition (1.17). For the Euler equation, it was pointed out by S. Wu [55] that in our infinite
depth framework with zero vorticity in the bulk, a maximum principle applied to the equation for the pressure yields that the Taylor sign condition always holds. Nevertheless, this argument breaks down when vorticity is not zero. Finally, let us point out that the compatibility condition \( \Pi S^\varphi v_0 \mathbf{n} \big|_{z=0} = 0 \) that we assume at the initial time is exactly the same as (1.8) in [7].

The main part of the paper will be devoted to the proof of Theorem 1.1. We shall explain the main steps and the main difficulties slightly below. We immediately see that by using the compactness provided by the uniform estimates of Theorem 1.1, we shall easily get as a corollary, the justification of the inviscid limit and the existence of a solution to the limit Euler system:

**Theorem 1.2.** Under the assumptions of Theorem 1.1, if we assume in addition that there exists \((h_0, v_0)\) such that

\[
\lim_{\epsilon \to 0} \|v_0^\epsilon - v_0\|_{L^2(S)} + |h_0^\epsilon - h_0|_{L^2(\mathbb{R}^2)} = 0.
\]

Then there exists \((h(t, x), v(t, x))\) with \(Z^\alpha \nabla v \in L^\infty([0, T] \times S)\), \(|\alpha| \leq 1\) and

\[
v \in L^\infty([0, T], H^m(S)), \quad \partial_z v \in L^\infty([0, T], H^{m-2}(S)), \quad h \in L^\infty([0, T], H^m(\mathbb{R}^2))
\]

such that

\[
\lim_{\epsilon \to 0} \sup_{[0, T]} \left( \|v^\epsilon - v\|_{L^2(S)} + \|v^\epsilon - v\|_{L^\infty(S)} + |h^\epsilon - h|_{L^2(\mathbb{R}^2)} + |h^\epsilon - h|_{W^{1, \infty}(\mathbb{R}^2)} \right) = 0
\]

and which is the unique solution to the free surface Euler equation in the sense that

\[
\partial_t^\epsilon v + (v \cdot \nabla v^\epsilon) v + \nabla^\epsilon q = 0, \quad \nabla^\epsilon : v = 0, \quad x \in S
\]

with the boundary conditions

\[
\partial_t^\epsilon h = v \cdot N \quad \text{and} \quad q = gh
\]

at \(z = 0\), \(\varphi\) being still defined by (1.9), (1.10).

We thus provide the justification of the inviscid limit in \(L^2\) and \(L^\infty\) norms. The convergence in higher norms (conormal for \(v\), standard Sobolev for \(h\)) follows by interpolation and the uniform estimate (1.19). Note that we do not obtain by compactness the convergence of \(v^\epsilon\) in Lipschitz norm. This is expected since in that norm, the boundary layer profile cannot be neglected when we pass to the limit. The above result also provides an existence and uniqueness result of solutions (with minimal normal regularity of the velocity) to the free surface Euler system which is new. Note that by using the equation for the vorticity, one can easily propagate higher normal regularity and thus recover the results of [34, 49, 61]. In particular, we get that \(\partial_z v \in L^\infty([0, T], H^{m-1}_c(S))\).

1.3. Sketch of the proof and organization of the paper. In order to prove Theorem 1.1 since classical local existence results of smooth solutions are available in the literature [6, 52], the main difficulty is to get a priori estimates on a time interval small but independent of \(\epsilon\) of the quantities that appear in (1.19), (1.20) in terms of the initial data only for a sufficiently smooth solution of the equation. We shall suppress the subscript \(\epsilon\) in the proof of the a priori estimates for notational convenience, we shall simply denote \((v^\epsilon, h^\epsilon)\) by \((v, h)\). Let us define:

\[
\mathcal{N}_m(T, v, h) = \sup_{[0, T]} \left( \|v(t)\|_m^2 + \|\partial_z v\|_{m-2}^2 + \|h(t)\|_m^2 + \epsilon \|h(t)\|_{m+1}^2 + \epsilon \|\partial_{zz} v(t)\|_{L^\infty}^2 + \|\partial_z v(t)\|_{1, \infty}^2 \right)
\]

\[
+ \|\partial_z v\|_{L^2([0, T], H^{m-1}_c)}^2 + \epsilon \int_0^T \|\nabla v\|_m^2 + \epsilon \int_0^T \|\nabla \partial_z v\|_{m-2}^2 < +\infty.
\]

The crucial point is thus to get a closed control of the above quantity on a sufficiently small time interval which is independent of \(\epsilon\). The main part of the paper will be devoted to these a priori estimates. Many steps will be needed in order to get the result.
Step 1: Estimates of $v$ and $h$. The first step will be to estimate $Z^\alpha v$ and $Z^\alpha h$ for $0 \leq |\alpha| \leq m$. When $\alpha = 0$, the estimate is a consequence of the energy identity for free surface Navier-Stokes equation which reads

$$\frac{d}{dt} \left( \int_S |v|^2 \, d\mathcal{V}_t + g \int_{z=0} |h|^2 \, dy \right) + 4\varepsilon \int_S |S^\varepsilon v|^2 \, d\mathcal{V}_t = 0.$$ 

Here $d\mathcal{V}_t$ stands for the natural volume element induced by the change of variable (1.9): $d\mathcal{V}_t = \partial_z \varphi(t, y, z) \, dydz$.

The difficulty to get estimates for higher order derivatives is that the coefficients (which depend on $h$) in the equation (1.12) are not smooth enough (even with the use of the smoothing diffeomorphism that we have taken) to control the commutators in the usual way. For example, for the transport term which reads:

$$\partial_t^\varepsilon + v \cdot \nabla \varphi = \partial_t + v_y \partial_y + \frac{1}{\partial_z \varphi} (v \cdot N - \partial_t \eta) \partial_z, \quad N = (-\partial_1 \varphi, -\partial_2 \varphi, 1)^t,$$

the commutator between $Z^\alpha$ and this term in the equation involves in particular the term $(v \cdot Z^\alpha N) \partial_z v$ which can be estimated only with the help of $\|Z^\alpha N\|_{L^2} \sim |h|_{m+\frac{1}{2}}$. This yields a loss of $1/2$ derivative. We also get similar problems when we compute for the pressure term the commutator between $Z^\alpha$ and $\nabla^\varepsilon q$. This difficulty was solved by Alinhac in [2]. The main idea is that some cancellation occurs when we use the good unknown $V^\alpha = Z^\alpha v - \partial_z^\varepsilon v Z^\alpha \eta$. Indeed, let us write our equation under the abstract form

$$\mathcal{N}(v, q, \varphi) = \partial_t^\varepsilon v + (v \cdot \nabla^\varepsilon) v + \nabla^\varepsilon q - 2\varepsilon \nabla^\varepsilon \cdot (S^\varepsilon v).$$

Then, if $\mathcal{N}(v, q, \varphi) = 0$, the linearized equation can be written as

$$D\mathcal{N}(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = \left( \partial_t^\varepsilon + (v \cdot \nabla^\varepsilon) - 2\varepsilon \nabla^\varepsilon \cdot (S^\varepsilon \cdot) \right) (\dot{v} - \partial_z^\varepsilon v \dot{\varphi}) + \nabla^\varepsilon (\dot{q} - \partial_z^\varepsilon q \varphi)
+ (\dot{v} \cdot \nabla^\varepsilon) v - \varphi (\partial_z^\varepsilon v \cdot \nabla^\varepsilon) v.$$

This means that the fully linearized equation has the same structure as the equation linearized with respect to the $v$ variable only thanks to the introduction of the good unknown. The justification of this identity will be recalled in section 2.

By using this crucial remark, we get that the equation for $(Z^\alpha v, Z^\alpha q, Z^\alpha \eta)$ can be written as

$$\partial_t^\varepsilon V^\alpha + v \cdot \nabla^\varepsilon V^\alpha + \nabla^\varepsilon Q^\alpha - 2\varepsilon \nabla^\varepsilon \cdot S^\varepsilon V^\alpha = l.o.t.$$

with $V^\alpha = Z^\alpha v - \partial_z^\varepsilon v Z^\alpha \eta$, $Q^\alpha = Z^\alpha q - \partial_z^\varepsilon q Z^\alpha \eta$ and the $l.o.t$ means lower order terms with respect to $v$ and $h$ that can be controlled by $\mathcal{N}_{m,T}(v, h)$. Consequently, we can perform an $L^2$ type energy estimate for this equation. By using energy estimates, we shall get an identity under the form

$$\frac{d}{dt} \frac{1}{2} \int_S |V^\alpha|^2 \, d\mathcal{V}_t + \frac{d}{dt} \frac{1}{2} \int_{z=0} (g - \partial_z^\varepsilon q^E) |Z^\alpha h|^2 = \cdots$$

which yields a good control of the regularity of the surface only if the sign condition $g - \partial_z^\varepsilon q^E \geq c_0 > 0$ is matched.

The main conclusion of this step will be that

$$(1.24) \quad \| (Z^m v - \partial_z^\varepsilon v Z^\alpha \eta)(t) \|^2 + |h(t)|^2 \leq C_0 + t \Lambda(R) + \int_0^t \| \partial_z v \|^2 \| m_{-1}$$

where $C_0$ depends only on the initial data and $\Lambda$ is some continuous increasing function in all its arguments (independent of $\varepsilon$) as soon as

$$Q_m(t) = \| v \|^2 m + |h|^2 m + \| \partial_z v \|^2 m_{-2} + \| v \|^2 2 \infty + \| \partial_z v \|^2 1 \infty + \varepsilon \| \partial_z v \|^2 1 \infty \leq R$$

for $t \in [0, T^\varepsilon]$. 

9
Step 2: Normal derivative estimates I. In order to close the argument, we need estimates for $\partial_z v$. We shall first estimate $\|\partial_z v\|_{L^\infty_t (H^{m-2}_{co})}$. This is not sufficient to control the right-hand side in (1.24), but this will be important in order to get the $L^\infty$ estimates. The main idea is to use the equivalent quantity

$$S_N = \Pi S^\varphi v N$$

which vanishes on the boundary. This allows to perform conormal estimates on the convection-diffusion type equation with homogeneous Dirichlet boundary condition satisfied by $S_N$. This yields again an estimate under the form

$$\|\partial_z v(t)\|_{m-2}^2 \leq C_0 + t\Lambda(R) + \int_0^t \|\partial_z v\|_{m-1}^2.$$

Step 3: $L^\infty$ estimates. We also have to estimate the $L^\infty$ norms that occur in the definition of $Q_m$. Note that we can not use Sobolev embedding (as it is classically done) since the conormal spaces do not control normal derivatives. The estimate of $\|v\|_{2,\infty}$ is a consequence of an anisotropic Sobolev estimate and thus, the difficult part is to estimate $\|\partial_z v\|_{1,\infty}$. Again, it is more convenient to estimate the equivalent quantity $\|S_N\|_{1,\infty}$ since $S_N$ solves a convection diffusion equation with homogeneous boundary condition. The estimate of $\|S_N\|_{L^\infty}$ is a consequence of the maximum principle for this equation. The control of $\|Z_i S_N\|_{L^\infty}$ is more difficult. The main reason is that a crude estimate of the commutator between $Z_i$ and the variable coefficient operator $\Delta^\varphi$ involves terms with more normal derivatives: two normal derivatives of $S_N$ and hence three normal derivatives of $v$. To overcome this difficulty, we note that at this step, the regularity of the surface is not really a problem: we want to estimate a fixed number of derivatives of $v$ in $L^\infty$ while $m$ can be considered as large as we need. Consequently, the idea is to change the coordinate system into a normal geodesic one in order to get the simplest possible expression for the Laplacian. By neglecting all the terms that can be estimated by the previous steps, we get a simple equation under the form

$$\partial_t \tilde{S}_N + z\partial_z w_3(t,y,0)\partial_z \tilde{S}_N + w_y(t,y,0) \cdot \nabla_y \tilde{S}_N - \varepsilon \partial_{zz} \tilde{S}_N = l.o.t$$

where $\tilde{S}_N$ stands for $S_N$ expressed in the new coordinate system and $w$ is the vector field that we obtain from $v$ by the change of variable. This is a one-dimensional Fokker Planck type equation (with an additional drift term in the tangential direction that can be eliminated by using lagrangian coordinates in this direction) for which the Green function is explicit and hence, we can use it to estimate $\|Z_i \tilde{S}_N\|_{L^\infty}$. Again the conclusion of this step is an estimate of the form

$$\|\partial_z v\|_{1,\infty}^2 + \varepsilon \|\partial_{zz} v\|_{2,\infty}^2 \leq C_0 + t\Lambda(R) + \int_0^t \|\partial_z v\|_{m-1}^2.$$

Step 4: Normal derivative estimate II. In order to close our estimate, we still need to estimate $\|\partial_z v\|_{m-1}$. For this estimate it does not seem a good idea to use $S_N$ as an equivalent quantity for $\partial_z v$. Indeed, the equation for $Z^{m-1}S_N$ involves $Z^{m-1}D^2 p$ as a source term and we note that since the Euler part of the pressure involves a harmonic function that verifies $p^E = gh$ on the boundary, we have that

$$Z^{m-1}D^2 p^E \sim Z^{m-1}D^2 h \sim |h|_{m+\frac{1}{2}}$$

and hence we do not have enough regularity of the surface. To get a better estimate, it is natural to try to use the vorticity $\omega = \nabla^\varphi \times v$ in order to eliminate the pressure. We shall get that indeed $Z^{m-1}\omega$ solves an equation under the form

$$\partial_t Z^{m-1}\omega + V \cdot \nabla Z^{m-1}\omega - \varepsilon \Delta^\varphi Z^{m-1}\omega = l.o.t$$

Nevertheless, while for the Euler equation the vorticity which solves a transport equation with a characteristic boundary is easy to estimate, for the Navier-Stokes system in a domain with
boundaries it is much more difficult to estimate. The difficulty in the case of the Navier-Stokes system is that we need an estimate of the value of the vorticity at the boundary to estimate it in the interior. Since on the boundary we have roughly \( Z^{m-1} \omega \sim Z^m v + Z^m h \), we only have by using a trace estimate a (uniform) control by known quantities (and in particular the energy dissipation of the Navier-Stokes equation) of

\[
\sqrt{\varepsilon} \int_0^t \left| Z^{m-1} \omega \right|^2_{L^2(\mathbb{R}^2)} dt.
\]

In this case, by a simple computation on the heat equation which is given in section 10.2 we see that the best estimate that we can expect, when we study the problem with zero initial datum

\[
\partial_t f - \varepsilon \Delta f = 0, \quad z < 0, \quad f_{/z=0} = f^b
\]

and with the boundary data \( f^b \) that satisfies an estimate as above is

\[
\int_0^{+\infty} e^{-2\gamma t} \left\| (\gamma + |\partial_t|)^{\frac{1}{2}} f \right\|_{L^2(S)}^2 dt \leq \sqrt{\varepsilon} \int_0^{+\infty} e^{-2\gamma t} \left\| f^b \right\|_{L^2(\mathbb{R}^2)}^2 dt \leq C.
\]

Consequently, we see that we get a control of \( f \) in \( H^1((0, T), L^2) \) which gives by Sobolev embedding an estimate of \( f \) in \( L^4([0, T], L^2(\Omega)) \) only.

Motivated by this result on the heat equation, we shall get an estimate of \( \|Z^{m-1} \omega\|_{L^4((0, T), L^2)} \). Note that the transport term in the equation has an important effect. Indeed, in the previous example of the heat equation, if we add a constant drift \( c \cdot \nabla f \) in the equation, we obtain a smoothing effect under the form

\[
\int_0^{+\infty} e^{-2\gamma t} \left\| (\gamma + |\partial_t + c \cdot \nabla|)^{\frac{1}{2}} f \right\|_{L^2}^2 dt.
\]

In order to estimate \( \|Z^{m-1} \omega\|_{L^4((0, T), L^2)} \), we shall consequently first switch into Lagrangian coordinates in order to eliminate the transport term and hence look for an estimate of \( \|(Z^{m-1} \omega) \circ X\|_{H^1((0, T), L^2)} \) since \( (Z^{m-1} \omega) \circ X \) solves a purely parabolic equation. Note that the price to pay is that the parabolic operator that we get after the change of variable has coefficients with limited uniform regularity (in particular with respect to the time and normal variables). To get this estimate, we use a microlocal symmetrizer based on a "partially" semiclassical paradifferential calculus i.e. based on the weight \( (\gamma^2 + |\tau|^2 + |\sqrt{\varepsilon} \xi|^{\frac{4}{3}})^{\frac{1}{2}} \). The use of paradifferential calculus in place of pseudo-differential one is needed because of the only low regularity uniform estimates of the coefficients that are known. The main properties of this calculus can be seen as a consequence of the general quasi-homogeneous calculus studied in [41] and is recalled at the end of the paper. By Sobolev embedding this yields an estimate of \( \|(Z^{m-1} \omega) \circ X\|_{L^4([0, T], L^2)} \) and thus of \( \|Z^{m-1} \omega\|_{L^4([0, T], L^2)} \) by a change of variable. This finally allows to get an estimate of \( \|Z^{m-1} \partial_x v\|_{L^4((0, T), L^2)} \).

The estimate of \( N_{m,T}(v, h) \) on some uniform time follows by combining the estimates of the four steps. Note that at the end, we also have to check that the Taylor sign condition and the condition that \( \Phi(t, \cdot) \) is a diffeomorphism remain true.

**Organisation of the paper.** The paper is organized as follows. In section 2 we recall the main properties of Sobolev conormal spaces, in particular, the product laws, embeddings and trace estimates that we shall use. We also state some geometric identities that are linked to the change of variable (2.4) and the Korn inequality that we shall use to control the energy dissipation term in (1.12). In section 3 we study the regularity properties of \( \eta \) given by (1.10). Next, the main part of the paper is devoted to the proof of the a priori estimates leading to the proof of Theorem 1.1. We first recall the energy identity for the Navier-Stokes equation (1.12) in section 4. The aim of the next three sections is to get the estimates of higher order conormal derivatives. In section 5...
where \( \Lambda \) and we define the norms: notation. We set for \( m \) that we shall use for Sobolev conormal spaces. It will be convenient to also use the following

Some properties of Sobolev conormal spaces.

2.1. \textbf{Some properties of Sobolev conormal spaces.} We have already recalled the notations for Sobolev conormal spaces. It will be convenient to also use the following notation. We set for \( m \geq 1 \):

\[
E^m = \{ f \in H^m_{co}, \quad \partial_z f \in H^{m-1}_{co} \}, \quad E^{m, \infty} = \{ f \in W^{m, \infty}_{co}, \quad \partial_z f \in W^{m-1, \infty}_{co} \}
\]

and we define the norms:

\[
\| f \|_{E^m}^2 = \| f \|_m^2 + \| \partial_z f \|_{m-1}^2, \quad \| f \|_{E^{m, \infty}} = \| f \|_{m, \infty} + \| \partial_z f \|_{m-1, \infty}.
\]

We shall also use Sobolev tangential spaces defined for \( s \in \mathbb{R} \) by

\[
H^s_{tan}(\Sigma) = \{ f \in L^2(\Sigma), \quad \Lambda^s f \in L^2(\Sigma) \}
\]

where \( \Lambda^s \) is the tangential Fourier multiplier by \((1+|\xi|^2)^{\frac{s}{2}} \) i.e. \( \mathcal{F}_y(\Lambda^s f)(\xi, z) = (1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}_y(f)(\xi, z) \)

where \( \mathcal{F}_y \) is the partial Fourier transform in the \( y \) variable. We set

\[
\| f \|_{H^s_{tan}} = \| \Lambda^s f \|_{L^2}.
\]

Note that we have

\[
\| f \|_{H^s_{tan}} \lesssim \| f \|_m
\]

for \( m \in \mathbb{N}, \; m \geq s \).

In this paper, we shall use repeatedly the following Gagliardo-Nirenberg-Moser type properties of the Sobolev conormal spaces:

\textbf{Proposition 2.1.} \textit{We have the following products, and commutator estimates:}

- For \( u, \; v \in L^\infty \cap H^k_{co}, \; k \geq 0 \):
  \[
  \| Z^{\alpha_1} u Z^{\alpha_2} v \| \lesssim \| u \|_{L^\infty} \| v \|_k + \| v \|_{L^\infty} \| u \|_k, \quad |\alpha_1| + |\alpha_2| = k.
  \]

- For \( 1 \leq |\alpha| \leq k, \; g \in H^{k-1}_{co} \cap L^\infty, \; f \in H^k_{co}, \) such that \( Z f \in L^\infty \), we have
  \[
  \| [Z^{\alpha}, f] g \| \lesssim \| Z f \|_{k-1} \| g \|_{L^\infty} + \| Z f \|_{L^\infty} \| g \|_{k-1}.
  \]

- For \( |\alpha| = k \geq 2, \) we define the symmetric commutator \([Z^{\alpha}, f, g] = Z^{\alpha}(fg) - Z^{\alpha} f g - f Z^{\alpha} g\). Then, we have the estimate
  \[
  \| [Z^{\alpha}, f, g] \| \lesssim \| Z f \|_{L^\infty} \| Z g \|_{k-2} + \| Z g \|_{L^\infty} \| Z f \|_{k-2}.
  \]
Proof. The proof of \eqref{2.1} is classical and can be found for example in \cite{25}. The commutator estimates \eqref{2.2}, \eqref{2.3} follow from \eqref{2.1} and the Leibnitz formula. Indeed, we have

\[
[Z^\alpha, f]g = \sum_{\beta + \gamma = \alpha, \beta \neq 0} C_{\beta, \gamma} Z^\beta f Z^\gamma g
\]

and hence since $\beta \neq 0$, we can write $Z^\beta = Z^{\tilde{\beta}} Z_i$, with $|\tilde{\beta}| = |\beta| - 1$, to get that

\[
\| Z^{\tilde{\beta}} Z_i f Z^\gamma g \| \lesssim \| Z f \| \| g \|_{k-1} + \| g \| \| Z f \|_{k-1}
\]

thanks to \eqref{2.1}. The proof of \eqref{2.3} can be obtained by a similar argument. \hfill \Box

We shall also need embedding and trace estimates for these spaces:

**Proposition 2.2.**

- For $s_1 \geq 0$, $s_2 \geq 0$ such that $s_1 + s_2 > 2$ and $f$ such that $f \in H^{s_1}_{\tan}$, $\partial_z f \in H^{s_2}_{\tan}$, we have the anisotropic Sobolev embedding:

\[
\| f \|_{L^\infty} \lesssim \| \partial_z f \|_{H^{s_2}_{\tan}} \| f \|_{H^{s_1}_{\tan}}.
\]

- For $f \in H^1(S)$, we have the trace estimates:

\[
|f(\cdot, 0)|_{H^s(\mathbb{R}^2)} \leq C \| \partial_z f \|_{H^{s_2}_{\tan}} \| f \|_{H^{s_1}_{\tan}},
\]

with $s_1 + s_2 = 2s \geq 0$.

As a consequence of \eqref{2.1}, we shall use very often that:

**Remark 2.3.** For $k \geq 5$, we have:

\[
\| f \|_{2, \infty}^2 \lesssim \| \partial_z f \|_{k-2} \| f \|_k, \quad k \geq 5.
\]

**Proof.** To get \eqref{2.3}, we first note that thanks to the one-dimensional Sobolev estimate that we have

\[
|\hat{f}(\xi, z)| \leq \left( \int_{-\infty}^0 |\partial_z \hat{f}(\xi, z)| \, dz \right)^{\frac{1}{2}}
\]

and hence, we obtain from Cauchy-Schwarz and the fact that $s_1 + s_2 > 2$ that

\[
\| f \|_{L^\infty} \lesssim \int_{\xi} |f(\xi, z)| \, dz \lesssim \left( \int_{\mathbb{R}^2} (1 + |\xi|)^{s_1 + s_2} \int_{-\infty}^0 |\partial_z \hat{f}(\xi, z)| \, dz \, d\xi \right)^{\frac{1}{2}} \lesssim \| \partial_z \Lambda^{s_1} f \|_{2} \| \Lambda^{s_2} f \|_{2}.
\]

The trace estimates \eqref{2.5} are also elementary in $S$. To get the estimate, it suffices to write that

\[
|f(\cdot, 0)|_{H^s(\mathbb{R}^2)}^2 = 2 \int_{\mathbb{R}^2} \int_{-\infty}^0 \partial_z \Lambda^s f(z, y) \Lambda^s f(z, y) \, dz \, dy
\]

and the result follows from Cauchy-Schwarz and the fact that

\[
\int_{\mathbb{R}^2} \partial_z \Lambda^s f(z, y) \Lambda^s f(z, y) \, dy = \int_{\mathbb{R}^2} \partial_z \Lambda^{s_1} f(z, y) \Lambda^{2s-s_1} f(z, y) \, dy.
\]

For later use, we also recall the classical tame Sobolev-Gagliardo-Nirenberg-Moser and commutator estimates in $\mathbb{R}^2$:

**Proposition 2.4.** For $s \in \mathbb{R}$, $s \geq 0$, we have

\[
|\Lambda^s(fg)|_{L^2(\mathbb{R}^2)} \leq C_s \left( |f|_{L^\infty(\mathbb{R}^2)} |g|_{H^s(\mathbb{R}^2)} + |g|_{L^\infty(\mathbb{R}^2)} |f|_{H^s(\mathbb{R}^2)} \right),
\]

\[
\||\Lambda^s f|Vg||_{L^2(\mathbb{R}^2)} \leq C_s \left( |\nabla f|_{L^\infty(\mathbb{R}^2)} |g|_{H^s(\mathbb{R}^2)} + |\nabla g|_{L^\infty(\mathbb{R}^2)} |f|_{H^s(\mathbb{R}^2)} \right)
\]

and

\[
|u^2|^\frac{1}{2} \lesssim |u|_{1, \infty} |\nabla u|_{\frac{1}{2}}.
\]
where $\Lambda^s$ is the Fourier multiplier by $(1 + |\xi|^2)^s$.

These estimates are also classical, we refer for example to [11]. Note that the last estimate can be obtained as a consequence of (1) and (5) in Theorem 15.2.

We shall also need to use results about semiclassical paradifferential calculus in section 10. They are described in section 15.

2.2. The equations in the fixed domain. By using the change of variables (1.8) and the definition (1.9), we obtain that

$$(\partial_i u)(t, y, \varphi) = (\partial_i v - \partial_i \varphi \partial_z v)(t, y, z), \quad i = 0, 1, 2$$

$$(\partial_3 u)(t, y, \varphi) = \left(\frac{1}{\partial_z \varphi} \partial_z v\right)(t, y, z)$$

where we set $\partial_0 = \partial_t$. We thus introduce the following operators

$$\partial_i^\varphi = \partial_i - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z, \quad i = 0, 1, 2, \quad \partial_3^\varphi = \frac{1}{\partial_z \varphi} \partial_z$$

in order to have

$$(2.9) \quad \partial_i u \circ \Phi = \partial_i^\varphi v, \quad i = 0, 1, 2, 3.$$ 

This yields by using the change of variable (1.7) that $(v, q)$ solves in the fixed domain $S$ the system (1.12), (1.13), (1.14) that was introduced previously. Note that thanks to this definition, since the operators $\partial_i$ commute, we immediately get that

$$(2.10) \quad [\partial_i^\varphi, \partial_j^\varphi] = 0, \quad \forall i, j.$$ 

Since the Jacobian of the change of variable (1.7) is $\partial_z \varphi$, it is natural to use on $S$ when performing energy estimates the following weighted $L^2$ scalar products:

$$\int_S fg \, d\nu_t, \quad d\nu_t = \partial_z \varphi(t, y, z) \, dydz$$

and for vector fields on $S$.

$$\int_S v \cdot w \, d\nu_t$$

where $\cdot$ stands for the standard scalar product of $\mathbb{R}^3$.

With this notation, we have the following integration by parts identities for the operators $\partial_i^\varphi$ and the above weighted scalar products:

**Lemma 2.5.**

$$(2.11) \quad \int_S \partial_i^\varphi f \, g \, d\nu_t = - \int_S f \partial_i^\varphi g \, d\nu_t + \int_{z=0} f g \, d\mathbf{N}_t \, dy, \quad i = 1, 2, 3,$$

$$(2.12) \quad \int_S \partial_i^\varphi f \, g \, d\nu_t = \partial_i \int_S f \, g \, d\nu_t - \int_S f \partial_i^\varphi g \, d\nu_t - \int_{z=0} f g \, \partial_i h$$

where the outward normal $\mathbf{N}$ is given by (1.4).

Note that in the above formulas, though it is not always explicitly mentioned, in each occurrence of $\varphi$, this function has to be taken at the time $t$.

We recall that by the choice (1.9), we have that on $z = 0, \varphi = h$. These formulas are straightforward consequences of the standard integration by parts formulas. Also, as a straightforward corollary, we get:
Corollary 2.6. Assume that \( v(t, \cdot) \) is a vector field on \( S \) such that \( \nabla^\varphi \cdot v = 0 \), then for every smooth functions \( f, g \) and smooth vector fields \( u, w \), we have the identities:

\[
\int_S (\partial_t^2 f + v \cdot \nabla^\varphi f) f dV_t = \frac{1}{2} \partial_t \int_S |f|^2 dV_t - \frac{1}{2} \int_{z=0} |f|^2 (\partial_t h - v \cdot N) dy,
\]

\[
\int_S \Delta^\varphi f g dV_t = - \int_S \nabla^\varphi f \cdot \nabla^\varphi g dV_t + \int_{z=0} \nabla^\varphi f \cdot N f dy,
\]

\[
\int_S \nabla^\varphi \cdot (S^\varphi u) w dV_t = - \int_S S^\varphi u \cdot S^\varphi w dV_t + \int_{z=0} (S^\varphi u N) \cdot w dy.
\]

2.3. Alinhac good unknown. In order to perform high order energy estimates, we shall need to study the equation that we get after applying conormal derivatives to the equation \( 1.12 \). The standard way to proceed is to say that the obtained equation has the structure

\[
\partial_t^2 Z^\alpha v + (v \cdot \nabla^\varphi) Z^\alpha v + \nabla^\varphi Z^\alpha q = \varepsilon \Delta^\varphi Z^\alpha v + l.o.t
\]

where \textit{l.o.t} stands for lower order terms and then to perform the natural energy estimate for this equation. The difficulty in free boundary problems is that very often \( \varphi \) is not smooth enough to consider all the standard commutators as lower order terms. Nevertheless there is a crucial cancellation pointed out by Alinhac \[2\] which allows to perform high order energy estimates. Since applying a derivative to the equation is like linearizing the equation, this cancellation can be explained in terms of a link between full an partial linearization. Let us set

\[
\mathcal{N}(v, q, \varphi) = \partial_t^2 v + (v \cdot \nabla^\varphi) v + \nabla^\varphi q - 2\varepsilon \nabla^\varphi \cdot (S^\varphi v),
\]

\[
d(v, \varphi) = \nabla^\varphi \cdot v,
\]

\[
\mathcal{B}(v, q, \varphi) = 2\varepsilon S^\varphi v N - (q - gh) N.
\]

It is more convenient to use the above form of equation \( 1.12 \) in view of the boundary condition \( 1.3 \). Note that by formal differentiation with respect to all the unknowns, we obtain the linearized operators:

\[
DN(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = \partial_t^2 \dot{v} + (v \cdot \nabla^\varphi) \dot{v} + \nabla^\varphi \dot{q} - 2\varepsilon \nabla^\varphi \cdot (S^\varphi \dot{v}) + (\dot{v} \cdot \nabla^\varphi) v - \partial_t^2 v (\partial_t \varphi + v \cdot \nabla^\varphi \varphi) + \partial_t^2 \dot{\varphi} \nabla^\varphi \varphi + 2\varepsilon \nabla^\varphi (\partial_t^2 \varphi \otimes \nabla^\varphi \varphi + \nabla^\varphi \varphi \otimes \partial_t^2 \varphi) + 2\varepsilon \partial_t^2 (S^\varphi v) \nabla^\varphi \varphi,
\]

\[
Dd(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \nabla^\varphi \cdot \dot{v} - \nabla^\varphi \varphi \cdot \partial_t^2 \varphi v,
\]

\[
DB(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = 2\varepsilon S^\varphi \dot{v} N - \partial_t^2 \dot{\varphi} \varphi \otimes \nabla^\varphi \varphi N - \nabla^\varphi \varphi \otimes \partial_t^2 \varphi v N - (\dot{q} - gh) N + (2\varepsilon S^\varphi v - (q - gh)) \dot{N},
\]

where \( \dot{N} = (-\partial_t \varphi, -\partial_2 \varphi, 0)^t \).

We have the following crucial identity first observed by Alinhac \[2\] which relates full and partial linearization.

Lemma 2.7. We have the following identities:

\[
DN(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = \partial_t^2 v + (v \cdot \nabla^\varphi) \dot{v} - 2\varepsilon \nabla^\varphi \cdot (S^\varphi \cdot ) \dot{v} - \partial_t^2 v \varphi + \nabla^\varphi \dot{q} - \partial_t^2 q \varphi
\]

\[
Dd(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \nabla^\varphi \cdot \dot{v} - \nabla^\varphi \varphi \cdot \partial_t^2 \varphi v + \varphi \partial_t^2 (N(v, q, \varphi)) - (\partial_t^2 \varphi v - \dot{\varphi}) v
\]

\[
DB(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = 2\varepsilon S^\varphi \dot{v} N + 2\varepsilon \varphi \partial_t^2 (S^\varphi v) N - (\dot{q} - gh) N + (2\varepsilon S^\varphi v - (q - gh)) \dot{N},
\]
As a consequence of this lemma, if \((v, q, \varphi)\) solves (1.12) we get that
\[
(2.20) 
DN(v, q, \varphi) \cdot (\dot{v}, \dot{q}, \dot{\varphi}) = \left( \partial_x^\alpha + (v \cdot \nabla \varphi) - \varepsilon \Delta \varphi \right) (\dot{v} - \partial_x^\alpha v \dot{\varphi}) + \nabla \varphi (\dot{q} - \partial_x^\alpha v \dot{\varphi}) + (\dot{v} \cdot \nabla \varphi)v - \varphi (\partial_x^\alpha v \cdot \nabla \varphi)v
\]
and
\[
(2.21) 
\nabla \varphi \cdot (\dot{v} - \partial_x^\alpha v \dot{\varphi}) = 0.
\]

The main consequence of these identities is that even if the naive form (2.16) of the equation for high order derivatives cannot be used, we can almost use it in the sense that if we replace \(Z^\alpha v\) and \(Z^\alpha q\) by the corresponding “good unknowns” \(Z^\alpha v - \partial_x^\alpha v Z^\alpha \eta\), \(Z^\alpha q - \partial_x^\alpha v Z^\alpha \eta\) in the left hand side, then we indeed get lower order terms in the right hand side of (2.16).

**Proof.** The proof follows from simple algebraic manipulations. There are many ways to explain this cancellation. It can be seen as a consequence of (2.10).

Let us set
\[
A_i(v, \varphi) = \partial_x^\alpha v, \quad F_{ij}(v, \varphi) = \partial_x^\alpha \partial_x^\beta v
\]
for \(i = 0, 1, 2, 3\). We note that for \(i = 0, 1, 2\)
\[
DA_i(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \partial_x^\alpha \dot{v} - \partial_x^\alpha v \partial_x^\alpha \dot{\varphi} = \partial_x^\alpha (\dot{v} - \partial_x^\alpha v \dot{\varphi}) + \partial_x^\alpha \partial_x^\alpha v \dot{\varphi}.
\]
Next, since \(\partial_x^\alpha\) and \(\partial_x^\beta\) commute, we also have
\[
\partial_x^\alpha \partial_x^\beta v = \partial_x^\beta \partial_x^\alpha v = \partial_x^\alpha (A_i(v, \varphi))
\]
and consequently, we find that
\[
(2.22) 
DA_i(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \partial_x^\alpha (\dot{v} - \partial_x^\alpha v \dot{\varphi}) + \dot{\varphi} \partial_x^\alpha (A_i(v, \varphi)).
\]
A similar computation shows that this relation is also true for \(i = 3\). In a similar way, we have that for \(i = 1, 2, j = 1, 2\)
\[
DF_{ij}(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \partial_x^\alpha (\partial_x^\gamma v - \partial_x^\gamma v \partial_x^\alpha \dot{\varphi}) - \partial_x^\alpha \dot{\varphi} \partial_x^\alpha (\partial_x^\alpha v)
\]
\[
= \partial_x^\alpha (\dot{v} - \partial_x^\alpha v \dot{\varphi}) + \dot{\varphi} \partial_x^\alpha \partial_x^\alpha v
\]
and hence we find by using again (2.10) that
\[
(2.23) 
DF_{ij}(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \partial_x^\alpha (\dot{v} - \partial_x^\alpha v \dot{\varphi}) + \dot{\varphi} \partial_x^\alpha (F_{ij}(v, \varphi)).
\]
A similar computation shows that this relation also holds true when \(i = 3\) or \(j = 3\).

The proof of Lemma 2.7 easily follows by using the two relations (2.22), (2.23).

\[\square\]

2.4. **Control from the dissipation term.** In view of the integration by parts formula (2.14), (2.16) we need to prove that the control of quantities like \(\int_S |\nabla \varphi|^2 dV_t\) yield a control of the standard \(H^1\) norm of \(f\). We also need Korn type inequalities to control the energy dissipation term. This is the aim of this final part of this preliminary section.

**Lemma 2.8.** Assume that \(\partial_2 \varphi \geq c_0\) and \(\|\nabla \varphi\|_{L^\infty} \leq 1/c_0\) for some \(c_0 > 0\), then there exists \(\Lambda_0 = \Lambda(1/c_0)\) such that
\[
\|\nabla f\|_{L^2(S)}^2 \leq \Lambda_0 \int_S |\nabla \varphi|^2 dV_t.
\]
Proof. By using the definition of the operators $\partial_i^c$, we first note that
\[ |\partial_i f| \leq |\partial_i \varphi| |\partial_i^c f|. \]

Hence we find
\[ \|\partial_i f\|_{L^2(S)} \leq \Lambda_0 \|\partial_i^c f\|_{L^2(S)} \]
and since $d\mathcal{V}_t = \partial_i \varphi dx \geq c_0 dx$ by assumption, this yields
\[ \|\partial_i f\|^2_{L^2(S)} \leq \Lambda_0 \int_S |\partial_i^c f|^2 d\mathcal{V}_t. \]

Next, since for $i = 1, 2$, we have
\[ |\partial_i f| \leq |\partial_i^c f| + |\partial_i \varphi| |\partial_i^c f| \leq \Lambda_0 |\nabla^c f|, \]
we also obtain that
\[ \|\partial_i f\|^2_{L^2(S)} \leq \Lambda_0 \int_S |\nabla^c f|^2 d\mathcal{V}_t, \quad i = 1, 2. \]

This ends the proof of Lemma 2.8. \qed

In the next proposition we state an adapted version of the classical Korn inequality in $\mathcal{S}$:

**Proposition 2.9.** If $\partial_i \varphi \geq c_0$, $\|\nabla \varphi\|_{L^\infty} + \|\nabla^2 \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then there exists $\Lambda_0 = \Lambda(1/c_0) > 0$, such that for every $v \in H^1(S)$, we have
\[ \|\nabla v\|^2_{L^2(S)} \leq \Lambda_0 \left( \int_S |\nabla^c v|^2 d\mathcal{V}_t + \|v\|^2_{L^2(S)} \right) \]
where
\[ S^c v = \frac{1}{2} (\nabla^c v + \nabla^c v^t). \]

For the sake of completeness, we shall give a proof of this estimate in subsection 14.3.

### 3. Preliminary estimates of $\eta$

In this section, we shall begin our a priori estimates for a sufficiently smooth solution of (1.9), (1.10). We assume that $A$ is chosen such that $\partial_z \varphi_0(y, z) \geq 1$ at the initial time.

We shall work on an interval of time $[0, T^c]$ such that
\[ \partial_z \varphi(t, y, z) \geq c_0, \forall t \in [0, T^c] \]
for some $c_0 > 0$. This ensures that for every $t \in [0, T^c]$, $\Phi(t, \cdot)$ is a diffeomorphism. We shall first estimate $\eta$ given by (1.10) in terms of $h$ and $v$.

Our first result is:

**Proposition 3.1.** We have the following estimates for $\eta$ defined by (1.10)
\[ \forall s \geq 0, \quad \|\nabla \eta(t)\|_{H^s(S)} \leq C_s |h(t)|_{s+\frac{1}{2}}, \]
\[ \forall s \in \mathbb{N}, \quad \|\nabla \partial t \eta\|_{H^s(S)} \leq C_s (1 + \|v\|_{L^\infty} + |h|_{L^\infty}) (\|v\|_{E^{-\frac{1}{2}}(S)} + \|\partial_t h\|_{s+\frac{1}{2}}) \]
and moreover, we also have the $L^\infty$ estimates
\[ \forall s \in \mathbb{N}, \quad \|\eta\|_{W^{s, \infty}} \leq C_s |h|_{s, \infty}, \]
\[ \forall s \in \mathbb{N}, \quad \|\partial t \eta\|_{W^{s, \infty}} \leq C_s (1 + |\nabla h|_{s, \infty}) \|v\|_{s, \infty} \]
where $C_s$ depends only on $s$.

Note that the above estimate ensures that $\eta$ has a standard Sobolev regularity in $\mathcal{S}$ and not only a conormal one. This is one of the main advantage in the choice of the diffeomorphism given by (1.7) and (1.9), (1.10).
Proof. From the explicit expression (1.10), we get that
\[
\int_{-\infty}^{0} (|\xi|^2 |\hat{\eta}(\xi,z)|^2 + |\partial_z \hat{\eta}(\xi,z)|^2) \, dz \lesssim |\xi| |\hat{h}(\xi)|^2
\]
and hence (3.2) follows by using the Bessel identity.

By using (3.2), we get that
\[
\|\nabla \partial_t \eta\|_{s} \lesssim |\partial_t h|_{s+\frac{1}{2}}.
\]
By using (1.13) and (2.6), we get by setting \(v^b(t,y) = v(t,y,0)\) that
\[
|\partial_t h|_{s+\frac{1}{2}} \leq |v^b \cdot N|_{s+\frac{1}{2}} \lesssim \|v\|_{L^\infty} |\nabla h|_{s+\frac{1}{2}} + (1 + |\nabla h|_{L^\infty}) |v^b|_{s+\frac{1}{2}}
\]
and hence (3.3) follows by using the trace inequality (2.5).

For the \(L^\infty\) estimates, we observe that we can write
\[
\eta(y,z) = \frac{1}{z^2} \psi(\cdot z) \ast_y h := \psi_z \ast_y h
\]
where \(\ast_y\) stands for a convolution in the \(y\) variable and \(\psi\) is in \(L^1(\mathbb{R}^2)\). Consequently, from the Young inequality for convolutions, we get that
\[
\|\eta\|_{L^\infty} \lesssim |h|_{L^\infty}, \quad \|\partial_t \eta\|_{L^\infty} \lesssim |\nabla h|_{L^\infty}, \quad i = 1, 2.
\]
For \(\partial_z \eta\), we note that
\[
\partial_z \hat{\eta} = (\xi_1 \partial_1 \chi + \xi_2 \partial_2 \chi)(z\xi) \hat{h}(\xi) = \nabla \chi(z\xi) \cdot F_y(\nabla h)(\xi).
\]
This yields that
\[
\partial_z \eta = \frac{1}{z^2} \psi^{(1)}(\cdot z) \ast_y \nabla h
\]
where \(\psi^{(1)}\) is again an \(L^1\) function and hence we obtain that
\[
\|\partial_z \eta\|_{L^\infty} \lesssim |\nabla h|_{L^\infty}.
\]
To get (3.4), the estimates of higher derivatives follow by induction.

To prove (3.5), we write thanks to (3.4) and (1.3) that
\[
\|\partial_t \eta\|_{W^{s,\infty}} \lesssim |v^b \cdot N|_{s,\infty} \lesssim \|v\|_{s,\infty} (1 + |\nabla h|_{s,\infty}).
\]

Next, we shall study how the smoothing effect of the Navier-Stokes equation on the velocity can be used to improve the regularity of the surface.

Since in the following we need to estimate very often expressions like \(f / \partial_z \varphi\) where \(f \in H^s(S)\) or \(H^s_{co}(S)\), we shall first state a general Lemma

**Lemma 3.2.** For every \(m \in \mathbb{N}\), we have
\[
\|f / \partial_z \varphi\|_m \leq \Lambda \left( \frac{1}{c_0}, |h|_{1,\infty} + \|f\|_{L^\infty} \right) (|h|_{m+\frac{1}{2}} + \|f\|_m)
\]

**Remark 3.3.** Of course, the above estimate is also valid in standard Sobolev spaces:
\[
\|f / \partial_z \varphi\|_{H^s} \leq \Lambda \left( \frac{1}{c_0}, |h|_{1,\infty} + \|f\|_{L^\infty} \right) (|h|_{s+\frac{1}{2}} + \|f\|_{H^s})
\]
for \(s \in \mathbb{R}, s \geq \frac{1}{2} \).
From a standard energy estimate for this transport equation, we thus obtain

\[
\frac{f}{\partial_z \varphi} = \frac{f}{A} - \frac{f}{A} \frac{\partial_z \eta}{A + \partial_z \eta} = \frac{f}{A} - \frac{f}{A} F(\partial_z \eta)
\]

where \( F(x) = x/(A + x) \) is a smooth function which is bounded together with all its derivatives on \( A + x \geq c_0 > 0 \) and such that \( F(0) = 0 \). Consequently, by using (2.1), we get that

\[
\| F(\partial_z \eta) \|_m \lesssim \Lambda \left( \frac{1}{c_0}, \| \nabla \eta \|_{L^\infty} \right) \| \partial_z \eta \|_m
\]

and further that

\[
\| \frac{f}{\partial_z \varphi} \|_m \lesssim \| f/A \|_m + \Lambda \left( \frac{1}{c_0}, \| \nabla \eta \|_{L^\infty} + \| f/A \|_{L^\infty} \right) \left( \| \partial_z \eta \|_m + \| f/A \|_m \right).
\]

The result follows by using (3.3) and (3.2).

Proof.

Next, we study the gain in the regularity of the surface which is induced by the gain of regularity on the velocity for the Navier-Stokes equation.

**Proposition 3.4.** For every \( m \in \mathbb{N}, \varepsilon \in (0, 1) \), we have the estimate

\[
\varepsilon \| h(t) \|^2_{m + \frac{1}{2}} \leq \varepsilon \| h_0 \|^2_{m + \frac{1}{2}} + \varepsilon \int_0^t \| v^b \|_{m + \frac{1}{2}}^2 + \int_0^t \Lambda_{1, \infty}(\| \nabla \|_{m + \frac{1}{2}}^2 + \varepsilon \| h \|^2_{m + \frac{1}{2}}) \, dt
\]

where

\[
\Lambda_{1, \infty} = \Lambda(\| \nabla h \|_{L^\infty(\mathbb{R}^2)} + \| v \|_{1, \infty})
\]

and \( v^b = v_{/z=0} \).

Note that by using the trace inequality (2.5) (with \( s_1 = 0, s_2 = 1, s = 1/2 \)), we can write that

\[
\varepsilon \int_0^t \| v^b \|_{m + \frac{1}{2}}^2 \leq \varepsilon \int_0^t \| \nabla \|_{m}^2 + \varepsilon \int_0^t \| v \|_{m}^2
\]

hence the right hand side in the estimate of Proposition 3.4 can indeed be absorbed by an energy dissipation term. Nevertheless, since the exact form of the energy dissipation term for our high order estimates will be more complicated, we shall give the exact way to control this term later.

**Proof.** By using (1.3), we get that

\[
\partial_t \Lambda^{m + \frac{1}{2}} h + v_y(t, y, 0) \cdot \nabla \Lambda^{m + \frac{1}{2}} h - \Lambda^{m + \frac{1}{2}} v_3(t, y, 0) + [\Lambda^{m + \frac{1}{2}}, v_y(t, y, 0)] \cdot \nabla h = 0.
\]

From a standard energy estimate for this transport equation, we thus obtain

\[
\frac{d}{dt} \frac{1}{2} \varepsilon \| h \|^2_{m + \frac{1}{2}} \lesssim \varepsilon \| \nabla v^b \|_{L^\infty(\mathbb{R}^2)} \| h \|^2_{m + \frac{1}{2}} + \varepsilon (\| v^b \|_{m + \frac{1}{2}} + \| [\Lambda^{m + \frac{1}{2}}, v_y(t, y, 0)] \cdot \nabla h \|_{L^2(\mathbb{R}^2)}) \| h \|_{m + \frac{1}{2}}
\]

where \( v^b = v(t, y, 0) \). By using the commutator estimate (2.7), we have

\[
\| [\Lambda^{m + \frac{1}{2}}, v_y(t, y, 0)] \cdot \nabla h \|_{L^2(\mathbb{R}^2)} \lesssim \| \nabla v^b \|_{L^\infty(\mathbb{R}^2)} \| h \|_{m + \frac{1}{2}} + \| \nabla h \|_{L^\infty(\mathbb{R}^2)} \| v^b \|_{m + \frac{1}{2}}
\]

and hence, we obtain

\[
\frac{d}{dt} \frac{1}{2} \varepsilon \| h \|^2_{m + \frac{1}{2}} \lesssim \varepsilon \| \nabla v^b \|_{L^\infty(\mathbb{R}^2)} \| h \|^2_{m + \frac{1}{2}} + (1 + \| \nabla h \|_{L^\infty(\mathbb{R}^2)}) \varepsilon \| v^b \|_{m + \frac{1}{2}} \| h \|_{m + \frac{1}{2}}.
\]
The result follows from the Young inequality
\begin{equation}
abla \leq \delta a^p + Cb^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad a, b \geq 0,
\end{equation}
with \( p = q = 2 \) and an integration in time. This ends the proof of Proposition 3.3.

By combining Proposition 3.1 and Proposition 3.4, we also obtain that:

**Corollary 3.5.** For every \( m \in \mathbb{N}, \varepsilon \in (0, 1) \), we have
\[\varepsilon \| \nabla \eta(t) \|_{H^m(S)}^2 \leq \varepsilon C_m \| h_0 \|_{m+\frac{1}{2}}^2 + \varepsilon \int_0^t \| v \|_{m+\frac{1}{2}}^2 + \int_0^t \Lambda_{1, \infty} (\| v \|_{m}^2 + \varepsilon \| h \|_{m+\frac{1}{2}}^2) \, dt\]
where \( \Lambda_{1, \infty} \) is defined in (3.3).

4. **Basic \( L^2 \) estimate**

We now start the first main part of our a priori estimates, namely estimates for \( Z^m v \) and \( Z^m h \). The easiest case is when \( m = 0 \) as corresponds to the physical energy.

**Proposition 4.1.** For any smooth solution of (1.12), we have the energy identity:
\[
\frac{d}{dt} \left( \int_S |v|^2 \, dV_t + g \int_{z=0} |h|^2 \, dy \right) + 4\varepsilon \int_S |S^\varphi v|^2 \, dV_t = 0.
\]

**Proof.** By using (1.12) and the boundary condition (1.3), we get that
\[
\frac{d}{dt} \int_S |v|^2 \, dV_t = 2 \int_S \nabla v \cdot (2\varepsilon S^\varphi v - q \text{Id}) \cdot v \, dV_t
\]
and hence by using the integration by parts formula (2.12), we find that
\[
\frac{d}{dt} \int_S |v|^2 \, dV_t + 4\varepsilon \int_S |S^\varphi v|^2 \, dV_t = 2 \int_S q \nabla v \cdot v \, dV_t + 2 \int_{z=0} (2\varepsilon S^\varphi v - q \text{Id}) N \cdot v \, dy.
\]
Next, by using successively (1.5) and (1.3), we observe that
\[
2 \int_{z=0} (2\varepsilon S^\varphi v - q \text{Id}) N \cdot v \, dy = -2 \int_{z=0} gh v \cdot N \, dy = - \int_{z=0} g \frac{d}{dt} |h|^2 \, dy
\]
and the result follows.

**Corollary 4.2.** If \( \partial_2 \varphi \geq c_0 > 0, |h|_{2, \infty} \leq \frac{1}{c_0} \) for \( t \in [0, T^\varepsilon] \), then we have
\[
\| v(t) \|^2 + \varepsilon \int_0^t \| \nabla v \|^2 \leq \Lambda \left( \frac{1}{c_0} \right) \left( \| v_0 \|^2 + \int_0^t \| v \|^2 \right), \quad \forall t \in [0, T^\varepsilon].
\]

**Proof.** It suffices to combine Proposition 4.1 and Propositions 2.9, 2.8.

5. **Equations satisfied by \( (Z^\alpha v, Z^\alpha h, Z^\alpha q) \)**

5.1. **A commutator estimate.** The next step in order to perform higher order conormal estimates is to compute the equation satisfied by \( Z^\alpha v \). We thus need to commute \( Z^\alpha \) with each term in the equation (1.12). It is thus useful to establish the following general expressions and estimates for commutators that we shall use many times.

We first notice that for \( i = 1, 2, 3 \), we have for any smooth function \( f \)
\begin{equation}
Z^\alpha \partial^\alpha_i f = \partial^\alpha_i Z^\alpha f - \partial^\alpha_i f \partial^\alpha_i Z^\alpha \eta + C^\alpha_i(f)
\end{equation}
where the commutator \( C^\alpha_i(f) \) is given for \( \alpha \neq 0 \) and \( i \neq 3 \) by
\begin{equation}
C^\alpha_i(f) = C^\alpha_{i,1}(f) + C^\alpha_{i,2}(f) + C^\alpha_{i,3}(f)
\end{equation}
We give the proof for Lemma 5.1.

To estimate the first term in $C_{i,1}$, we use the same kind of arguments: by using (3.2) and (1.9), we first get that

$$
\|\partial_z f [Z^\alpha, \partial_i \varphi, \frac{1}{\partial_z \varphi}] \| \leq \Lambda \left( \frac{1}{c_0}, \| \nabla \varphi \|_{1, \infty} + |h|_{2, \infty} \right) \left( \| \nabla \eta \|_{m-1} + |h|_{m-1, \frac{1}{2}} \right).
$$

And hence by using (3.7) and (3.2), (3.4), we find

$$
\|\partial_z f [Z^\alpha, \partial_i \varphi, \frac{1}{\partial_z \varphi}] \| \leq \Lambda \left( \frac{1}{c_0}, \| \nabla \varphi \|_{1, \infty} + |h|_{2, \infty} \right) |h|_{m-1, \frac{1}{2}}.
$$

To estimate the second type of terms in $C_{i,2}$, we note that for $|\alpha| \geq 1$, we can write

$$
Z^\alpha \left( \frac{1}{\partial_z \varphi} \right) = -Z^{\tilde{\alpha}} \left( \frac{Z_j \partial_j \eta}{\partial_z \varphi} \right)^2, \quad |\tilde{\alpha}| = |\alpha| - 1,
$$
hence, we obtain for $|\alpha| \geq 2$ that
\[
\partial_1 \varphi \left( Z^\alpha \left( \frac{1}{\partial_2 \varphi} \right) + \frac{Z^\alpha}{\partial_2 \varphi} \right) \partial_2 f = -\partial_1 \varphi \partial_2 f \left[ Z^\alpha, \frac{1}{\partial_2 \varphi} \right] Z_j \partial_j \eta
\]
and by using again (2.2), (5.7) and (3.2), (3.4), we also obtain that
\[
\| C_{i,2}^\alpha(f) \| \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + \| \nabla f \|_{L^\infty} \right) |h|_{m-\frac{1}{2}}.
\]

It remains to estimate $C_{\alpha}$. Since
\[
[Z_3, \partial_2] = -\partial_2 \left( \frac{z}{1+z} \right) \partial_2,
\]
we can prove by induction that
\[
[Z^\alpha, \partial_2] h = \sum_{|\beta| \leq m-1} c_\beta \partial_2 (Z^\beta h)
\]
for some harmless smooth bounded functions $c_\beta$. This yields
\[
\left\| \frac{\partial_1 \varphi}{\partial_2 \varphi} [Z^\alpha, \partial_2] f \right\|_{L^2} + \left\| \frac{\partial_1 \varphi}{\partial_2 \varphi} f \right\|_{L^2} \leq \Lambda \left( \frac{1}{c_0} \| \partial_1 \varphi \|_{L^\infty} + \| \partial_2 f \|_{L^\infty} \right) \left( \left\| \partial_2 f \right\|_{m-1} + \| \nabla \eta \|_{m-1} \right)
\]
Indeed, the last term comes from the fact that thanks to (1.9), the commutator $[Z^\alpha, \partial_2] f$ can be decomposed into an harmless bounded term and the commutator $[Z^\alpha, \partial_2] \eta$ which is in $L^2$. This yields by using again (3.2)
\[
\| C_{i,2}^\alpha \| \leq \Lambda \left( \frac{1}{c_0} \| \partial_1 \varphi \|_{L^\infty} + \| \partial_2 f \|_{L^\infty} \right) \left( \left\| \partial_2 f \right\|_{m-1} + \| \nabla \eta \|_{m-1} \right)
\]
(5.9)
\[
\leq \Lambda \left( \frac{1}{c_0} \| h \|_{1,\infty} + \| \nabla f \|_{L^\infty} \right) \left( \left\| \partial_2 f \right\|_{m-1} + \| h \|_{m-\frac{1}{2}} \right).
\]

To end the proof of Lemma 5.1, it suffices to collect (5.5), (5.7), (5.9).

The estimate for $C^\alpha_3$ can be obtained through very similar arguments. This ends the proof of Lemma 5.1.

5.2. Interior equation satisfied by $(Z^\alpha v, Z^\alpha q, Z^\alpha \varphi)$. We shall prove the following:

**Lemma 5.2.** For $1 \leq |\alpha| \leq m$, let us set $V^\alpha = Z^\alpha v - \partial_2^\alpha v Z^\alpha \eta$, $Q^\alpha = Z^\alpha q - \partial_2^\alpha q Z^\alpha \eta$, then we get the system
\[
\partial_2^\alpha V^\alpha + v \cdot \nabla \varphi V^\alpha + \nabla \varphi Q^\alpha - 2\varepsilon \nabla \varphi \cdot S^\varphi V^\alpha + C^\alpha(q) + C^\alpha(T) = \varepsilon D^\alpha(S^\varphi v) + \varepsilon \nabla \varphi \cdot (E^\alpha(v)) + (\partial_2^\alpha v \cdot \nabla \varphi) Z^\alpha \eta,
\]
(5.10)
\[
\nabla \varphi \cdot V^\alpha + C^\alpha(d) = 0.
\]
(5.11)

where the commutators $C^\alpha(q)$, $C^\alpha(d)$ and $E^\alpha(v)$ satisfy the estimates:
\[
\| C^\alpha(q) \| \leq \Lambda \left( \frac{1}{c_0} \| h \|_{2,\infty} + \| \nabla q \|_{1,\infty} \right) \left( \| \nabla q \|_{m-1} + \| h \|_{m-\frac{1}{2}} \right),
\]
(5.12)
\[
\| C^\alpha(d) \| \leq \Lambda \left( \frac{1}{c_0} \| h \|_{2,\infty} + \| \nabla v \|_{1,\infty} \right) \left( \| \nabla v \|_{m-1} + \| h \|_{m-\frac{1}{2}} \right),
\]
(5.13)
\[
\| C^\alpha(T) \| + \| E^\alpha(v) \| \leq \Lambda \left( \frac{1}{c_0} \| h \|_{2,\infty} + \| v \|_{E^2,\infty} \right) \left( \| v \|_{E^m} + \| h \|_{m-\frac{1}{2}} \right)
\]
(5.14)

and $D^\alpha(S^\varphi v)$ is given by
\[
D^\alpha(S^\varphi v)_{ij} = 2 C^\alpha_{ij}(S^\varphi v)_{ij}.
\]
Note that we will control the commutator $D^\alpha(S^\varphi v)$ later by using integration by parts.

**Proof.** We need to compute the equation solved by $Z^\alpha v$.

By using the previous notations, we first note that

\begin{equation}
Z^\alpha \nabla^\varphi q = \nabla^\varphi Z^\alpha q - \partial_z^\varphi q \nabla^\varphi Z^\alpha \varphi + C^\alpha(q)
\end{equation}

where

\[ C^\alpha(q) = \begin{pmatrix} C^\alpha_1(q) \\ C^\alpha_2(q) \\ C^\alpha_3(q) \end{pmatrix} \]

and thus thanks to lemma $5.1$, we have the estimate

\begin{equation}
||C^\alpha(q)|| \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + ||\nabla q||_{1,\infty} (||\nabla q||_{m-1} + |h|_{m-\frac{1}{2}}) \right).
\end{equation}

In a similar way, we get that

\begin{equation}
Z^\alpha \left( \nabla^\varphi \cdot v \right) = \nabla^\varphi \cdot Z^\alpha v - \nabla^\varphi Z^\alpha \varphi \cdot \partial_z^\varphi v + C^\alpha(d)
\end{equation}

where

\[ C^\alpha(d) = \sum_{i=1}^{3} C^\alpha_i(v) \]

and hence, thanks to Lemma $5.1$, we also have that

\begin{equation}
||C^\alpha(d)|| \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + ||\nabla v||_{1,\infty} (||\nabla v||_{m-1} + |h|_{m-\frac{1}{2}}) \right).
\end{equation}

Next, we shall expand the transport part of $(1.12)$.

\[ Z^\alpha \left( \partial_t^\varphi + v \cdot \nabla^\varphi \right) v. \]

We first note that

\begin{equation}
\partial_t^\varphi + v \cdot \nabla^\varphi = \partial_t + v_y \cdot \nabla_y v + V_z \partial_z
\end{equation}

where $V_z$ is defined by

\begin{equation}
V_z = \frac{1}{\partial_z^\varphi} v_z = \frac{1}{\partial_z^\varphi} (v \cdot N - \partial_t \varphi) = \frac{1}{\partial_z^\varphi} (v \cdot N - \partial_t \eta).
\end{equation}

and where $N(t, y, z)$ is defined as

\[ N(t, y, z) = \left( - \partial_1 \varphi(t, y, z), -\partial_2 \varphi(t, y, z), 1 \right)^t = \left( - \partial_1 \eta(t, y, z), -\partial_2 \eta(t, y, z), 1 \right)^t. \]

Note that $N$ is defined in the whole $S$ and that $N(t, y, 0)$ is indeed the outward normal to the boundary that we have used before. For the vector fields $v_z$ and $V_z$, we have the estimates:

**Lemma 5.3.** We have for $m \geq 2$, the estimates

\begin{equation}
||v_z||_{1,\infty} + ||V_z||_{1,\infty} \leq \Lambda \left( \frac{1}{c_0}, ||v||_{1,\infty} + |h|_{2,\infty} \right),
\end{equation}

\begin{equation}
||Zv_z||_{m-2} + ||ZV_z||_{m-2} \leq \Lambda \left( \frac{1}{c_0}, ||v||_{1,\infty} + |h|_{2,\infty} \right) (||v||_{E^{m-1}} + |h|_{m-\frac{1}{2}}).
\end{equation}

**Proof of Lemma 5.3.**

For the first estimate, we use that

\[ ||v_z||_{L^\infty} + ||V_z||_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, ||v||_{L^\infty} + ||\nabla \eta||_{L^\infty} + ||\partial_t \eta||_{L^\infty} \right) \leq \Lambda \left( \frac{1}{c_0}, ||v||_{L^\infty} + |h|_{1,\infty} \right) \]
thanks to (3.4), (3.5). In the same way, we get that
\[ \|Zv_z\|_{L^\infty} + \|ZV_z\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, \|v\|_{1,\infty} + \|\nabla \eta\|_{W^{1,\infty}} + \|\partial_t \eta\|_{W^{1,\infty}} \right) \]
by using also (3.5).

By using (2.1) and Proposition 3.1, we also obtain that
\[ \|Zv_z\|_{m-2} \lesssim \|\nabla \partial_t \eta\|_{m-2} + \|v\|_{L^\infty} (1 + \|\nabla \eta\|_{m-1}) + \|\nabla \eta\|_{L^\infty} \|v\|_{m-1} \]
(5.23)
\[ \lesssim \Lambda \left( \|v\|_{L^\infty} + |h|_{1,\infty} \right) \left( \|v\|_{E^{m-1}} + |h|_{m-\frac{1}{2}} \right). \]

Note that \( \partial_t \eta \) is not in \( L^2 \), this why we are obliged to apply one derivative to \( v_z \) before estimating it in \( L^2 \) in order to use Proposition 3.1. Finally, we note that we have
\[ \|ZV_z\|_{m-2} \lesssim \|Z\left( \frac{1}{\partial_z \varphi} \right) v_z\|_{m-2} + \|\frac{1}{\partial_z \varphi} Zv_z\|_{m-2} \]
and hence, by using (2.1), (3.7), we obtain that
\[ \|ZV_z\|_{m-2} \leq \Lambda \left( \frac{1}{c_0}, \|\nabla \eta\|_{1,\infty} + \|v_z\|_{1,\infty} \right) \left( \|Zv_z\|_{m-2} + \|\partial_z \eta\|_{m-2} \right). \]
Finally, (5.22) follows from (5.24) by using (5.23), (5.21) and Proposition 3.1.

This ends the proof of Lemma 5.3.

Remark 5.4. By using similar arguments, we also get that for \( m \geq 3 \), we have
\[ \|\partial_z Zv_z\|_{m-3} \leq \Lambda \left( \|v\|_{E^{1,\infty}} + |h|_{2,\infty} \right) \left( \|v\|_{E^{m-1}} + |h|_{m-\frac{1}{2}} \right) \]
and
\[ \|\partial_z V_z\|_{m-3} \leq \Lambda \left( \|v\|_{E^{1,\infty}} + |h|_{2,\infty} \right) \left( \|v\|_{E^{m-2}} + |h|_{m-\frac{1}{2}} \right). \]

By using the identity (5.19), we thus get that
\[ Z^\alpha (\partial^\varphi_t + v \cdot \nabla \varphi) v = (\partial_t + v_y \cdot \nabla_y + v_z \partial_z) Z^\alpha v + (v \cdot Z^\alpha N - \partial_t Z^\alpha \eta) \partial^\varphi_t v - \partial^\varphi_t Z^\alpha \eta (v \cdot N - \partial_t \eta) \partial^\varphi_t v + C^\alpha(T) \]
(5.25)
\[ = (\partial^\varphi_t + v \cdot \nabla \varphi) Z^\alpha v - \partial^\varphi_t v \left( \partial^\varphi_t + v \cdot \nabla \varphi \right) Z^\alpha \eta + C^\alpha(T) \]
where the commutator \( C^\alpha(T) \) is defined by
\[ C^\alpha(T) = \sum_{i=1}^{5} T_i^\alpha, \]
\[ T_1^\alpha = [Z^\alpha, v_y] \partial_y v, \quad T_2^\alpha = [Z^\alpha, v_z] \partial_z v, \quad T_3^\alpha = \frac{1}{\partial^\varphi_t} [Z^\alpha, v_z] \partial_z v, \]
\[ T_4^\alpha = \left( Z^\alpha \left( \frac{1}{\partial^\varphi_t} \right) \right) v_z \partial_z v, \quad T_5^\alpha = v_z \partial_z v \left( \frac{Z^\alpha, \partial_t \eta}{(\partial^\varphi_t)^2} \right) + v_z [Z^\alpha, \partial_z] v, \]
\[ T_6^\alpha = [Z^\alpha, v_z, \frac{1}{\partial^\varphi_t}] \partial_z v. \]
To estimate these commutators, we use similar arguments as in the proof of Lemma 5.1. In particular, we use the estimates (2.2), (2.3) and (3.7) combined with Lemma 5.3 and also again the estimates (3.2), (3.4). This yields
\[ \|C^\alpha(T)\| \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^{2,\infty}} \right) \left( \|v\|_{E^{m}} + |h|_{m-\frac{1}{2}} \right) \]
and hence the estimate (5.14) for \( C^\alpha(T) \) is proven.
It remains to compute \( \varepsilon \Delta \varphi v \). Since \( \nabla \varphi \cdot v = 0 \), it is more convenient to use that \( \Delta \varphi v = 2 Z^a \nabla \varphi \cdot (S^a v) \). By using (5.11), we thus use the following expansion

\[
2Z^a \nabla \varphi \cdot (S^a v) = 2 \nabla \varphi \cdot (Z^a S^a v) - 2 \partial_z^a S^a v \nabla \varphi (Z^a \varphi) + D^a (S^a v)
\]

where

\[
(5.27) \quad D^a (S^a v) = 2 C_{ij}^a (S^a v)_{ij}
\]

by using the summation convention over repeated indices. Next, since we can again expand

\[
(5.28) \quad 2Z^a (S^a v) = 2S^a (Z^a v) - \partial_z^a v \otimes \nabla \varphi Z^a \varphi - \nabla \varphi Z^a \varphi \otimes \partial_z^a v + E^a (v)
\]

with

\[
(\mathcal{E}^a v)_{ij} = C_i^a (v_j) + C_j^a (v_i),
\]

we obtain that

\[
(5.29) \quad \varepsilon \Delta \varphi v = 2 \varepsilon \nabla \varphi \cdot S^a (Z^a v) - 2 \varepsilon \nabla \varphi \cdot \left( \partial_z^a v \otimes \nabla \varphi Z^a \varphi - \nabla \varphi Z^a \varphi \otimes \partial_z^a v \right) - 2 \varepsilon \left( \partial_z^a S^a v \right) \nabla \varphi (Z^a \varphi)
\]

\[
+ \varepsilon D^a (S^a v) + \varepsilon \nabla \varphi \cdot (\mathcal{E}^a v).
\]

Thanks to Lemma 5.1, we also have the estimate

\[
(5.30) \quad \| \mathcal{E}^a (v) \| \leq \Lambda \left( \frac{1}{\varepsilon_0}, |h|_{2,\infty} + \| \nabla v \|_{1,\infty} \right) \left( \| v \|_m + \| \partial_z v \|_{m-1} + |h|_{m-\frac{5}{2}} \right).
\]

Consequently, by collecting (5.15), (5.17), (5.25) and (5.29), we get in view of (2.17), (2.18) that \( (Z^a v, \partial Z^a q, \partial Z^a \varphi) \) solves the equation

\[
D N (v, q, \varphi) \cdot (Z^a v, \partial Z^a q, \partial Z^a \varphi) - (Z^a v \cdot \nabla \varphi) v + C^a (q) + C^a (T) = \varepsilon D^a (S^a v) + \varepsilon \nabla \varphi \cdot (\mathcal{E}^a v)
\]

and the constraint

\[
D d(v, \varphi) \cdot (Z^a v, \partial Z^a \varphi) + C^a (d) = 0.
\]

Consequently, by using Lemma 2.7 and (2.20), (2.21), we get that \( (V^a, Q^a) \) with \( V^a = Z^a v - \partial_z^a Z^a \eta \), \( Q^a = Z^a q - \partial_z^a q Z^a \eta \) solves

\[
\partial_t^a V^a + v \cdot \nabla \varphi V^a + \nabla \varphi Q^a - 2 \varepsilon \nabla \varphi \cdot S^a v \varphi + C^a (q) + C^a (T)
\]

\[
= \varepsilon D^a (S^a v) + \varepsilon \nabla \varphi \cdot (\mathcal{E}^a v) + (\partial_z^a v \cdot \nabla \varphi) Z^a \eta
\]

with

\[
\nabla \varphi \cdot V^a + C^a (d) = 0.
\]

Since we have proven the estimates (5.30), (5.16), (5.18), this ends the proof of Lemma 5.2.

\[
\Box
\]

5.3. **Estimates of the boundary values.** We shall also need to compute the boundary condition satisfied by \( Z^a v \) when \( \alpha_3 = 0 \) (for \( \alpha_3 \neq 0 \), we have \( Z^a v = 0 \) on the boundary). As a preliminary, in order to control the commutators, we first establish the following

**Lemma 5.5.** For every \( s \geq 0, s \in \mathbb{R} \), we have the following estimates at \( z = 0 \):

\[
(5.31) \quad | \nabla v (\cdot, 0) |_s \leq \Lambda \left( \| v \|_{1,\infty} + |h|_{2,\infty} \right) \left( \| v (\cdot, 0) \|_{s+1} + |h|_{s+1} \right).
\]

**Proof.** Note that the estimate is obvious for \( | \partial_z v (\cdot, 0) |_s, i = 1, 2 \). Consequently, the only difficulty is to estimate \( | \partial_z v (\cdot, 0) |_s \). Since \( \nabla \varphi \cdot v = 0 \), we get that

\[
\partial_z v \cdot \mathbf{N} = \partial_z \varphi \left( \partial_1 v_1 + \partial_2 v_2 \right) = \left( A + \partial_z \eta \right) \left( \partial_1 v_1 + \partial_2 v_2 \right)
\]

and hence that

\[
(5.32) \quad \partial_z v \cdot \mathbf{n} = \frac{1}{|\mathbf{N}|} \left( A + \partial_z \eta \right) \left( \partial_1 v_1 + \partial_2 v_2 \right), \quad |\mathbf{N}| = \left( 1 + (\partial_1 \eta)^2 + (\partial_2 \eta)^2 \right)^{\frac{1}{2}}.
\]
Consequently, by using (2.6) and (3.7) (or actually its version on the boundary), we get that
\[ |\partial_z v \cdot n|_s \leq \Lambda \left( \|v\|_{1,\infty} + |h|_{2,\infty}(\mathbb{R}^2) + \|\partial_z \eta\|_{L^\infty} \right) \left( |v(\cdot, 0)|_{s+1} + |H N|_{s-1} + |\partial_z \eta(\cdot, 0)|_s \right) \]
and hence, by using Proposition 3.1 and the trace estimate (2.5) which yields
\[ |\partial_z \eta(\cdot, 0)|_s \lesssim \|\partial_z \eta\|_{H^{s+\frac{1}{2}}(S)} \lesssim |h|_{s+1}, \]
we get
\[ |\partial_z v \cdot n|_s \leq \Lambda \left( \|v\|_{1,\infty} + |h|_{2,\infty} \right) \left( |v(\cdot, 0)|_{s+1} + |h|_{s+1} \right). \tag{5.33} \]
It remains to control \( \Pi(\partial_z v) \) where \( \Pi = I d - n \otimes n \) is the orthogonal projection on \( (n)^\perp \) which is the tangent space to the boundary. We shall use the boundary condition (1.5) which yields
\[ \Pi(S^\varphi v n) = 0. \tag{5.34} \]
To expand \( \Pi(S^\varphi v n) \), we shall use the local basis in \( \Omega_t \) induced by (1.7) that we denote by \( (\partial_{y^1}, \partial_{y^2}, \partial_{y^3}) \). Note that by definition, we have by using (1.5) that \( (\partial_{y^i} u)(t, \Phi(t, \cdot)) = \partial_{y^i} v \). We also consider the induced riemannian metric defined by \( g_{ij} = \partial_{y^i} \cdot \partial_{y^j} \) and we define as usual the metric \( g^j{}^i \) as the inverse of \( g_{ij} \). Then we obtain that
\[ 2S u n = \partial_u u + g^j{}^i (\partial_{y^i} u \cdot n) \partial_{y^j}. \tag{5.35} \]
Consequently, since
\[ \partial_u u = -\partial_1 \varphi \partial_1 u - \partial_2 \varphi \partial_2 u + \partial_3 u = -\partial_1 \varphi \partial_1^u v - \partial_2 \varphi \partial_2^u v + \partial_3^u v, \]
we get from (5.34) that
\[ \Pi(\partial_z v) = \frac{\partial_z \varphi}{1 + |\nabla h|^2} \left( \Pi(\partial_1 h \partial_1 v + \partial_2 h \partial_2 v) - (g^i{}^j (\partial_{y^i} v \cdot n)\Pi\partial_{y^j}) \right). \tag{5.36} \]
Consequently, by using the same product estimates as above, we find that
\[ |\Pi(\partial_z v)|_s \leq \Lambda \left( \|v\|_{1,\infty} + \|\partial_z v \cdot n\|_{L^\infty} + |h|_{2,\infty} \right) \left( |v(\cdot, 0)|_{s+1} + |h|_{s+1} + |\partial_z v \cdot n|_s \right) \]
\[ \leq \Lambda \left( \|v\|_{1,\infty} + |h|_{2,\infty} \right) \left( |v(\cdot, 0)|_{s+1} + |h|_{s+1} \right), \]
where the last estimate comes from (5.33) and the fact that from (5.32) we obviously have
\[ \|\partial_z v \cdot n\|_{L^\infty} \leq \Lambda \left( \|\nabla \eta\|_{L^\infty} + \|v\|_{1,\infty} \right). \]
To conclude, it suffices to combine the last estimate and (5.33) since
\[ |(\partial_z v)(\cdot, 0)|_s \leq |\Pi(\partial_z v)|_s + |\partial_z v \cdot n|_s. \]
\[ \square \]

Next, we shall obtain the boundary condition for \( Z^\alpha v \). Again, note that the only interesting case occurs when \( \alpha_3 = 0 \). Indeed, otherwise \( Z^\alpha v = 0, Z^\alpha \eta = 0 \) on the boundary. We start with the study of the boundary condition (1.14).

**Lemma 5.6.** For every \( \alpha, 1 \leq |\alpha| \leq m \) such that \( \alpha_3 = 0 \) we have that on \( \{z = 0\} \)
\[ 2\varepsilon S^\varphi V^\alpha \mathbf{N} - (Z^\alpha q - g Z^\alpha h) \mathbf{N} + (2\varepsilon S^\varphi v - (q - gh)) Z^\alpha \mathbf{N} = C^\alpha(B) - 2\varepsilon Z^\alpha h \partial_\varepsilon (S^\varphi v) \mathbf{N} \]
where \( V^\alpha = Z^\alpha v - \partial_\varepsilon^\varphi v Z^\alpha \eta, Q^\alpha = Z^\alpha Q - \partial_\varepsilon^\varphi q Z^\alpha \eta \) and the commutator \( C^\alpha(B) \) enjoys the estimate:
\[ |C^\alpha(B)|_{L^2(\mathbb{R}^2)} \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |v|_{E^{2,\infty}} \right) \left( \varepsilon |v|^b_m + \varepsilon |h|^b_m \right) \tag{5.38} \]
Proof. By applying $Z^\alpha$ to the boundary condition (1.5) and by using the expansion (5.28), we get that
\begin{equation}
\varepsilon (2 S^\varphi (Z^\alpha v) - \partial^\varphi Z^\alpha \varphi - \nabla^\varphi Z^\alpha v \otimes \partial^\varphi v ) \mathbf{N} - (Z^\alpha q - g Z^\alpha h) \mathbf{N} + (2 \varepsilon S^\varphi v - (q - gh)) Z^\alpha \mathbf{N} = C^\alpha (\mathcal{B})
\end{equation}
where
\[ C^\alpha (\mathcal{B}) = - \varepsilon \mathcal{E}^\alpha (v) - C^\alpha (\mathcal{B})_1 + C^\alpha (\mathcal{B})_2 \]
with $\mathcal{E}$ defined after (5.28) and
\begin{align*}
C^\alpha (\mathcal{B})_1 &= \sum_{\beta + \gamma = \alpha, \ 0 < |\beta| < |\alpha|} \varepsilon Z^\beta (S^\varphi v) Z^\gamma \mathbf{N}, \\
C^\alpha (\mathcal{B})_2 &= \sum_{\beta + \gamma = \alpha, \ 0 < |\beta| < |\alpha|} Z^\beta (q - gh) Z^\gamma \mathbf{N}.
\end{align*}
Next, by using the product and commutator estimates of Proposition 2.4 on the boundary, we obtain that
\begin{align*}
|C^\alpha (\mathcal{B})_1|_{L^2(\mathbb{R}^2)} &\lesssim \varepsilon \| S^\varphi v \|_{1, \infty} |Z \mathbf{N}|_{m-2} + \varepsilon |S^\varphi v|_{m-1} |Z \mathbf{N}|_{L^\infty(\mathbb{R}^2)} \\
&\leq \Lambda \left( \frac{1}{c_0}, \| \nabla \eta \|_{1, \infty} + \| \nabla v \|_{1, \infty} \right) (\varepsilon |h|_m + \varepsilon |\nabla v|_{m-1})
\end{align*}
and hence, thanks to (3.4) and (5.31), we get that
\begin{align*}
|C^\alpha (\mathcal{B})_1|_{L^2(\mathbb{R}^2)} &\lesssim \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + \| v \|_{E^2, \infty} \right) (\varepsilon |h|_m + \varepsilon |v^b|_m).
\end{align*}
To estimate $C^\alpha (\mathcal{B})_2$, we first note that thanks to (1.13), we have $Z^\beta (q - gh) = Z^\beta (\varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n})$ and we get in a similar way that
\begin{align*}
|C^\alpha (\mathcal{B})_2|_{L^2(\mathbb{R}^2)} &\lesssim \left( |Z q^{NS}|_{L^\infty} + |h|_{L^\infty} \right) |\nabla h|_{m-1} + |\nabla h|_{1, \infty} (|Z q^{NS}|_{m-2}) \\
&\leq \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + \| v \|_{E^2, \infty} \right) (\varepsilon |v^b|_m + \varepsilon |h|_m)
\end{align*}
where we have set $q^{NS} = \varepsilon S^\varphi v \mathbf{n} \cdot \mathbf{n}$ and the last estimate comes from (5.31).

It remains to estimate $\mathcal{E}^\alpha (v)$ and hence $|C^\alpha_i (v_j)|_{L^2(\mathbb{R}^2)}$. At first, we note that $C^\alpha_i (v_j) = 0$ since we only consider the case that $\alpha_3 = 0$. Then, we get as in the proof of Lemma 5.1 that
\begin{align*}
|C^\alpha_i (v_j)|_{L^2(\mathbb{R}^2)} &\leq \Lambda \left( \frac{1}{c_0}, \| \nabla \eta \|_{1, \infty} + \| \nabla v \|_{1, \infty} \right) (|\nabla v|_{m-1} + |\nabla \eta|_{m-1}).
\end{align*}
Finally, we can use Lemma 5.5 and (3.4), (3.2) to get that
\begin{equation}
|C^\alpha_i (v_j)|_{L^2(\mathbb{R}^2)} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + \| v \|_{E^2, \infty} \right) (|v^b|_m + |h|_m).
\end{equation}
This yields
\begin{align*}
|\varepsilon \mathcal{E}^\alpha (v)|_{L^2(\mathbb{R}^2)} &\leq \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + \| v \|_{E^2, \infty} \right) (\varepsilon |v^b|_m + \varepsilon |h|_m)
\end{align*}
and the estimate (5.38) follows.

To get (5.37) from (5.39), it suffices to use Lemma 2.7.

It remains to study the kinematic boundary condition (1.13). 

\[ \square \]
Lemma 5.7. For every $\alpha$, $1 \leq |\alpha| \leq m$ such that $\alpha_3 = 0$ we have that on $\{z = 0\}$

\[(5.40)\]
\[\partial_t Z^\alpha h - v^b \cdot Z^\alpha N - V^\alpha \cdot N = C^\alpha(h)\]

where

\[(5.41)\]
\[|C^\alpha(h)|_{L^2(\mathbb{R}^2)} \leq \Lambda \left( \frac{1}{c_0} |v|_{E^1,\infty} + |h|_{2,\infty} \right) \left( |h|_m + \|v\|_{E^m} \right).\]

Proof. We immediately get from (1.13) that

\[\partial_t Z^\alpha h + v^b \cdot Z^\alpha N + V^\alpha \cdot N = -[Z^\alpha, v^b, \nabla_y h] - \frac{\partial_z v^b}{\partial_z \varphi} \cdot N Z^\alpha h := C^\alpha(h)\]

where we recall that we use the notation $f^b = f/_{z=0}$ for the trace on $z = 0$ Consequently, by using the formula (2.3) on the boundary, we get that

\[\|C^\alpha(h)\| \leq \Lambda \left( \frac{1}{c_0} |Zv|_{L^\infty} + |h|_{2,\infty} + |\partial_z v|_{L^\infty} \right) \left( |h|_m + |Zv^b|_{m-2} \right)\]

and the result follows by using the trace estimate (2.5).

\[\square\]

6. Pressure estimates

In view of the equation (5.10), in order to estimate the right hand side, we need to control the pressure. This is the aim of this section. By applying $\nabla \varphi \cdot$ to the equation (1.12), we get that the pressure $q$ solves in $S$ the system

\[(6.1)\]
\[\Delta \varphi q = -\nabla \varphi (v \cdot \nabla \varphi v), \quad q/_{z=0} = 2\varepsilon S^\varphi n \cdot n + gh.\]

We shall split the pressure into an "Euler" and a "Navier-Stokes" part with the following decomposition:

\[(6.2)\]
\[q = q^E + q^{NS}\]

where $q^E$ solves

\[(6.3)\]
\[\Delta \varphi q^E = -\nabla \varphi (v \cdot \nabla \varphi v), \quad q^E/_{z=0} = gh,\]

and $q^{NS}$ solves

\[(6.4)\]
\[\Delta \varphi q^{NS} = 0, \quad q^{NS}/_{z=0} = 2\varepsilon S^\varphi v n \cdot n.\]

The idea behind this decomposition is that we shall need more regularity on $v$ to estimate $q^{NS}$ but thanks to the gain of the $\varepsilon$ factor, this term will be controlled by the viscous regularization. The Euler pressure solves the same equation with the same boundary condition as in the free boundary Euler equation. Thanks to the explicit expressions of the operators $\partial_i^\varphi$, we get that the operator $\Delta^\varphi$ can be expressed as

\[(6.5)\]
\[\Delta^\varphi f = \frac{1}{\partial_z \varphi} \nabla \cdot (E \nabla f),\]

with the matrix $E$ defined by

\[
E = \begin{pmatrix}
\partial_z \varphi & 0 & -\partial_1 \varphi \\
0 & \partial_z \varphi & -\partial_2 \varphi \\
-\partial_1 \varphi & -\partial_2 \varphi & \frac{1+(\partial_1 \varphi)^2+(\partial_2 \varphi)^2}{\partial_z \varphi}
\end{pmatrix} = \frac{1}{\partial_z \varphi} PP^* \]
Proof. We can construct by standard variational arguments a solution of (6.10) which satisfies the
\begin{align}
(6.13)\quad \nabla \cdot (P \nabla \varphi) = \frac{1}{\partial_z \varphi} \nabla \cdot (P v), \quad \nabla \varphi f = \frac{1}{\partial_z \varphi} P^* \nabla f, \quad P = \begin{pmatrix}
\partial_z \varphi & 0 & 0 \\
0 & \partial_z \varphi & 0 \\
-\partial_1 \varphi & -\partial_2 \varphi & 1
\end{pmatrix}.
\end{align}

Note that \( E \) is symmetric positive and that if \( \| \nabla \varphi \|_{L^\infty} \leq \frac{1}{c_0} \) and \( \partial_z \varphi \geq c_0 > 0 \) then there exists \( \delta(c_0) > 0 \) such that
\begin{align}
(6.14)\quad EX \cdot X \geq \delta |X|^2, \quad \forall X \in \mathbb{R}^3.
\end{align}
Moreover, we note that
\begin{align}
(6.15)\quad E \subset W^{k,\infty} \leq \Lambda \left( \frac{1}{c_0} |h|_{k+1,\infty} \right)
\end{align}
and that by using the decomposition
\begin{align}
E = \text{Id}_A + \tilde{E}, \quad \tilde{E} = \begin{pmatrix}
\partial_z \eta & 0 & -\partial_1 \eta \\
0 & \partial_z \eta & -\partial_2 \eta \\
-\partial_1 \eta & -\partial_2 \eta & A(\partial_1 \eta)^2 + (\partial_2 \eta)^2 - \partial_z \eta
\end{pmatrix}, \quad \text{Id}_A = \text{diag}(1, 1, 1/A),
\end{align}
we have the estimate
\begin{align}
(6.16)\quad \| \tilde{E} \|_{H^s} \leq \Lambda \left( \frac{1}{c_0} |h|_{1,\infty} \right) |h|_{s+\frac{1}{2}}.
\end{align}
Before proving the estimates that we need for \( q^E \) and \( q^{NS} \), we shall establish general elliptic estimates for the Dirichlet problem of the operator \( \nabla \cdot (E \nabla \cdot) \).

Lemma 6.1. Consider the elliptic equation in \( S \)
\begin{align}
(6.17)\quad -\nabla \cdot (E \nabla \varphi) = \nabla \cdot F, \quad \rho_{/z=0} = 0.
\end{align}
Then we have the estimates:
\begin{align}
(6.18)\quad \| \nabla \varphi \| \leq \Lambda \left( \frac{1}{c_0} |h|_{1,\infty} \right) \| F \|_{L^2}, \quad \| \nabla^2 \varphi \| \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} \right) (\| \nabla \cdot F \| + \| F \|_1),
\end{align}
\begin{align}
(6.19)\quad \| \nabla \varphi \|_k \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_3 + \| F \|_{H^2_{\text{tan}}} + \| \nabla \cdot F \|_{H^1_{\text{tan}}} \right) (|h|_{k+\frac{1}{2}} + \| F \|_k), \quad k \geq 1,
\end{align}
\begin{align}
(6.20)\quad \| \partial_{zz} \varphi \|_{k-1} \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_3 + \| F \|_{H^2_{\text{tan}}} + \| \nabla \cdot F \|_{H^1_{\text{tan}}} \right) (|h|_{k+\frac{1}{2}} + \| F \|_k + \| \nabla \cdot F \|_{k-1}), \quad k \geq 2.
\end{align}

Note that the estimate \( (6.19) \) is tame in the sense that it is linear in the highest norm with respect to the source term and \( h \). Nevertheless, in the statement, the function \( \Lambda \) depends on \( k \).

Proof. We can construct by standard variational arguments a solution of \( (6.10) \) which satisfies the homogeneous \( H^1 \) estimate
\begin{align}
(6.21)\quad \| \nabla \varphi \| \leq \Lambda \left( \frac{1}{c_0} \right) \| F \|.
\end{align}
Next, the classical elliptic regularity result for \( (6.10) \) (see \[20\], \[33\] for example) yields
\begin{align}
(6.22)\quad \| \nabla \varphi \|_{H^1} \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} \right) (\| \nabla \cdot F \| + \| F \|_{H^1_{\text{tan}}}).
\end{align}
Before proving \( (6.19) \), we shall need another auxiliary estimate for low order derivatives in order to control \( \| \nabla \varphi \|_{L^\infty} \). Indeed, from \( (2.4) \), we get that
\begin{align}
(6.23)\quad \| \nabla \varphi \|_{L^\infty} \lesssim \| \partial_{zz} \nabla \varphi \|_{H^1_{\text{tan}}} \| \nabla \varphi \|_{H^2_{\text{tan}}}.
\end{align}
By applying the tangential derivation $\partial_y^\alpha$ to \ref{6.10} for $|\alpha| = 2$, we get that
\[-\nabla \cdot (E\nabla\partial_y^\alpha\nabla \rho) = \nabla \cdot \left( (\partial_y^\alpha E) + \nabla \cdot ((\partial_y^\alpha, E)\nabla \rho) \right), \quad \partial_y^\alpha \rho_{/z=0} = 0\]
and the standard energy estimate gives that
\[\|\nabla \partial_y^\alpha \rho\| \leq \Lambda \left( \frac{1}{c_0} (\|\partial_y^\alpha F\| + \|\partial_y^\alpha, E\| \nabla \rho\|) \right).\]
To control the commutator, we can expand it to get that
\[\|\partial_y^\alpha, E\| \nabla \rho\| \leq \|E\|_{\infty,1} \|\nabla \rho\|_{\mathcal{H}_{1,z}} + \|\partial_y^\alpha, \tilde{E}\| \nabla \rho\| \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} \|\nabla \rho\|_{\mathcal{H}_{1,z}} + \|\partial_y^\alpha, \tilde{E}\| \nabla \rho\| \right),\]
where we have used \ref{6.8} for the second estimate. To control the above last term, we use successively the Holder inequality and the Sobolev inequality to get that
\[\|\partial_y^\alpha, \tilde{E}\| \nabla \rho\| \leq \|\partial_y^\alpha, \tilde{E}\| L_{\infty,\alpha} \|\nabla \rho\| L_{\infty,\alpha} \| \|\partial_y^\alpha, \tilde{E}\| \|\nabla \rho\| H^2_{1,z}, \quad |\alpha| = 2.\]
Consequently, by using \ref{6.9} and \ref{6.15}, we obtain
\[\|\nabla \rho\| H^2_{1,z} \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_3 \right) \|\nabla \cdot F\| + \|F\| H^2_{1,z}.\]
Since the equation \ref{6.10} gives
\[\partial_{zz} \rho = \frac{1}{E_{33}} \left( \nabla \cdot F - \partial_z \left( \sum_{j < 3} E_{3,j} \partial_j \rho - \sum_{i < 3, j} \partial_i (E_{ij} \partial_j \rho) \right) \right)\]
we also obtain by using \ref{6.13}, \ref{6.15} and \ref{6.17} that
\[\|\partial_{zz} \rho\| H^1_{1,z} \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_3 \right) \left( \|\nabla \cdot F\| H^1_{1,z} + \|F\| H^2_{1,z} \right).\]
By using the anisotropic Sobolev embedding \ref{6.16} and \ref{6.18}, \ref{6.20}, we finally get that
\[\|\nabla q\| L_{\infty} \leq \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_3 \right) \left( \|\nabla \cdot F\| H^1_{1,z} + \|F\| H^2_{1,z} \right).\]
We can now establish the estimate \ref{6.12} by induction. Note that the case $k = 1$ is just a consequence of \ref{6.11}. Next, we want to apply derivatives to the equation, use the induction assumption to control commutators and use the energy estimate \ref{6.14} for the obtained equation. Note that for the last part of the argument we need the source term to be in divergence form. Consequently, a difficulty arises since $Z_3$ does not commute with $\partial_z$. To solve this difficulty, we can introduce a modified field such that
\[\tilde{Z}_3 \partial_z = \partial_z Z_3.\]
This is achieved by setting
\[\tilde{Z}_3 f = Z_3 f + \frac{1}{(1 - z)^2} f.\]
Let us also set $\tilde{Z}^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$. Note that
\[\|\tilde{Z}^\alpha f - Z^\alpha f\| \lesssim \|f\|_{|\alpha|-1}.\]
By applying $\tilde{Z}^\alpha$ to \ref{6.10}, we first get
\[\nabla \cdot (Z^\alpha (E\nabla \rho)) = \nabla \cdot \left( Z^\alpha F + (\tilde{Z}^\alpha - Z^\alpha) F_h - (\tilde{Z}^\alpha - Z^\alpha) (E\nabla \rho)_h \right)\]
where for a vector $X \in \mathbb{R}^3$, we set $X_h = (X_1, X_2, 0)^t$. Next, this yields
\[-\nabla \cdot (E \cdot \nabla Z_3^\alpha \rho) = \nabla \cdot \left( Z^\alpha F + (\tilde{Z}^\alpha - Z^\alpha) F_h - (\tilde{Z}^\alpha - Z^\alpha) (E\nabla \rho)_h \right) + \nabla \cdot C\]
\[\text{for } 0 \leq\]
where the commutator $C$ is given by
\[
C = -(E[Z^\alpha, \nabla]\rho) - \sum_{\beta+\gamma=\alpha, \beta\neq 0} c_{\beta, \gamma} Z^\beta E \cdot Z^\gamma \nabla \rho).
\]

From the standard energy estimate since $Z^\alpha \rho$ vanishes on the boundary, we get that
\[
(6.24) \quad \|\nabla \rho\| \leq \Lambda \left( \frac{1}{c_0} \right) \left( \|F\| + \|E \nabla \rho\| \right) + \|E[Z^\alpha, \nabla]\rho\| + \| \sum_{\beta+\gamma=\alpha, \beta\neq 0} c_{\beta, \gamma} Z^\beta E Z^\gamma \nabla \rho\|.
\]

To estimate the right hand-side, we first use (2.1) and (6.9) to get
\[
\|E \nabla \rho\| \lesssim (1 + \|\tilde{E}\|_{L^\infty}) \|\nabla \rho\| + \|\nabla \rho\|_{L^\infty} \|\tilde{E}\|_{L^1} + \Lambda \left( \frac{1}{c_0}, |h|_{1, \infty} \right) |\nabla \rho\| + \Lambda \left( \frac{1}{c_0}, |h|_{1, \infty} \right) |h|_{k-\frac{1}{2}} \|\nabla \rho\|_{L^\infty}.
\]

Next, $\|\nabla \rho\|_{k-1}$ is controlled by the induction assumption and we use (6.21) to get
\[
\|E \nabla \rho\|_{k-1} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + |h|_{3} + \|F\|_{H^3_{\text{tan}}} + \|\nabla \cdot F\|_{H^2_{\text{tan}}} \right) \left( |h|_{k-\frac{1}{2}} + \|F\|_{k-1} \right).
\]

In a similar way, we can use (5.8) to get that
\[
[Z^\alpha, \partial_z] = \sum_{|\beta|\leq|\alpha|-1} \varphi_{\alpha, \beta} \partial_z Z^\beta.
\]

for some harmless bounded functions $\varphi_{\alpha, \beta}$ and hence
\[
\|E[Z^\alpha, \nabla]\rho\| \leq \Lambda \left( \frac{1}{c_0}, |h|_{1, \infty} \right) \|\nabla \rho\|_{k-1}
\]

and we get by using (2.1) and (6.9) that
\[
\| \sum_{\beta+\gamma=\alpha, \beta\neq 0} c_{\beta, \gamma} Z^\beta E Z^\gamma \nabla \rho\| \lesssim \Lambda \left( |h|_{2, \infty} + \|\nabla \rho\|_{L^\infty} \right) \left( |h|_{k+\frac{1}{2}} + \|\nabla \rho\|_{k-1} \right).
\]

To conclude, we plug the two last estimates in (6.24) and we use the induction assumption to estimate $\|\nabla \rho\|_{k-1}$ and (6.21). This proves (6.12).

Finally, to get (6.13), it suffices to use the equation (6.10) in the form (6.19) and to use again the product estimates (2.1), and (6.8), (6.9), (6.21) together with the previous estimate (6.12).

Also, it will be convenient to have at our disposal the same kind of estimates as in Lemma 6.1 but with nonhomogeneous Dirichlet condition.

**Lemma 6.2.** Consider the elliptic equation in $S$
\[
(6.25) \quad -\nabla \cdot (E \nabla \rho) = 0, \quad \rho_{/z=0} = f^b.
\]

Then we have the estimates:
\[
(6.26) \quad \|\nabla \rho\|_{H^k(S)} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + |h|_{3} + |f^b|_{1, \infty} + |f^h|_{\frac{3}{2}} \right) \left( |h|_{k+\frac{1}{2}} + |f^h|_{k+\frac{1}{2}} \right).
\]

The regularity involving $|h|_{3}$ in the estimate (6.26) is not optimal, nevertheless, since at the end we shall use the Sobolev embedding to control $|h|_{2, \infty}$, this is sufficient.

**Proof.** We write $\rho$ under the form $\rho = \rho^H + \rho^r$ where $\rho^H$ will absorb the boundary data and $\rho^r$ solves
\[
(6.27) \quad -\nabla \cdot (E \nabla \rho^r) = \nabla \cdot (E \nabla \rho^H), \quad \rho^r_{/z=0} = 0.
\]
For $\rho^H$, we choose as in (1.10)
\[
\tilde{\rho}^H(\xi, z) = \chi(z \xi) \tilde{f}^b.
\]
Consequently, the estimates of Proposition 3.1 are also valid for $\rho^H$ in particular, we have for every $k \geq 0$:
\[
\|\nabla \rho^H\|_{H^k} \lesssim |f^b|_{k+\frac{1}{2}}, \quad \|\rho^H\|_{k, \infty} \lesssim \|f^b\|_{k, \infty}.
\]
Next, we can estimate the solution of (6.27). In Lemma 6.1 we have studied the elliptic problem (6.10) in Sobolev conormal spaces. Nevertheless, by using the same approach one can also get the estimate corresponding to (6.12) in standard Sobolev spaces. This yields
\[
\|\nabla \rho^H\|_{H^k} \leq \Lambda \left(\frac{1}{c_0}, |h|_{2, \infty} + |h|_3 + \|E \nabla \rho^H\|_{H^2}\right) \left(|h|_{k+\frac{1}{2}} + \|E \nabla \rho^H\|_{H^k}\right).
\]
To estimate the right hand side, since $\tilde{E}$ and $q^H$ have a standard Sobolev regularity in $S$, we get thanks to (6.28), (6.9) and (6.8) that
\[
\|E \nabla q^H\|_{H^k} \lesssim \Lambda \left(|h|_{1, \infty} + |f^b|_{1, \infty}\right) \left(|h|_{k+\frac{1}{2}} + |f^b|_{k+\frac{1}{2}}\right).
\]
This yields
\[
\|\nabla \rho^H\|_{H^k} \lesssim \Lambda \left(\frac{1}{c_0}, |h|_{2, \infty} + |f^b|_{1, \infty} + |h|_3 + |f^b|_{\frac{1}{2}}\right) \left(|h|_{k+\frac{1}{2}} + |f^b|_{k+\frac{1}{2}}\right).
\]
Consequently, (6.26) is proven.

We shall first establish regularity estimates for $q^{NS}$

**Proposition 6.3.** For $q^{NS}$, we have for $m \geq 1$, the estimate:
\[
\|\nabla q^{NS}\|_{H^{m-1}} \leq \Lambda \left(\frac{1}{c_0}, |h|_{2, \infty} + \|v\|_{E^{2, \infty}} + |h|_4 + \|v\|_{E^4}\right) \left(|\epsilon v^b|_{m+\frac{1}{2}} + \epsilon |h|_{m+\frac{1}{2}}\right),
\]
and the $L^\infty$ estimate
\[
\|\nabla q^{NS}\|_{L^\infty} \leq \epsilon \Lambda \left(\frac{1}{c_0}, |h|_{2, \infty} + \|v\|_{E^{2, \infty}} + |h|_4 + \|v\|_{E^4}\right).
\]

**Proof.** We need to estimate the solution of
\[
- \nabla \cdot (E \nabla q^{NS}) = 0, \quad q^{NS}_{/z = 0} = 2\epsilon S^\varphi v n \cdot n,
\]
therefore, it suffices to use Lemma 6.2 with $f^b = 2\epsilon S^\varphi v n \cdot n$. Since, we have
\[
|f^b|_{1, \infty} \leq \Lambda \left(\frac{1}{c_0}, |v|_{E^{1, \infty}} + |h|_{2, \infty}\right)
\]
and hence by using (5.32) and (5.36), we find
\[
|f^b|_{1, \infty} \leq \Lambda \left(\frac{1}{c_0}, |v|_{E^{1, \infty}} + |h|_{2, \infty} + \epsilon \|v\|_{2, \infty}\right)
\]
and
\[
|f^b|_{m+\frac{1}{2}} \leq \Lambda \left(\frac{1}{c_0}, |v|_{E^{1, \infty}} + |h|_{2, \infty}\right) \left(\epsilon \|\nabla v(\cdot, 0)\|_{m-\frac{1}{2}} + \epsilon |h|_{m+\frac{1}{2}}\right)
\]
\[
\leq \Lambda \left(\frac{1}{c_0}, |v|_{E^{1, \infty}} + |h|_{2, \infty}\right) \left(\epsilon \|v(\cdot, 0)\|_{m+\frac{1}{2}} + \epsilon |h|_{m+\frac{1}{2}}\right)
\]
where the last estimate comes from (5.34) In particular, since the above estimate holds for every $m$, we also have
\[
|f^b|_{\frac{m}{2}} \leq \Lambda \left(\frac{1}{c_0}, |v|_{E^{1, \infty}} + |h|_{2, \infty} + \epsilon |h|_{\frac{1}{2}} + \epsilon |v^b|_{\frac{1}{2}}\right).
\]
To conclude and get (6.30), we use the trace estimate (2.5) which yields
\[ \|v^b\|_2 \lesssim \|v\|_3 + \|v\|_4. \]
Finally, to obtain (6.31), it suffices to use the standard Sobolev embedding in $S$ to write
\[ \|\nabla q^{NS}\|_{L^\infty} \lesssim \|\nabla q^{NS}\|_2 \]
combined with (6.30) and the trace estimate (6.33). This ends the proof of Proposition 6.3.

It remains to estimate $q^E$.

**Proposition 6.4.** For $q^E$, we have the estimate:
\[ \|\nabla q^E\|_{m-1} + \|\partial_{zz} q^E\|_{m-2} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^1,\infty} + |h|_3 + \|v\|_{E^3} \right) \left( |h|_{m-\frac{1}{2}} + \|v\|_{E^m} \right), \]
\[ \|\nabla q^E\|_{1,\infty} + \|\partial_z^2 q^E\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^1,\infty} + |h|_4 + \|v\|_{E^3} \right), \]
\[ \|\nabla q^E\|_{2,\infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^1,\infty} + |h|_5 + \|v\|_{E^3} \right) \]

**Proof.** By using (6.33) and (6.6), we see that we have to solve the elliptic problem
\[ -\nabla \cdot (E \nabla q^b) = \nabla \cdot (P(v \cdot \nabla^2 v) = \partial_z \varphi \nabla^2 v \cdot \nabla^2 v, (y, z) \in S, \quad q^b_{/z=0} = gh. \]
We can split this equation in two parts by setting $q^E = q^b + q^i$ where $q^b$ solves the homogeneous equation
\[ -\nabla \cdot (E \nabla q^b) = 0 \]
in $S$ with nonhomogeneous boundary condition $q^b_{/z=0} = gh$ and $q^i$ solves
\[ -\nabla \cdot (E \nabla q^i) = \nabla \cdot (P(v \cdot \nabla^2 v) = \partial_z \varphi \nabla^2 v \cdot \nabla^2 v, (y, z) \in S, \quad q^i_{/z=0} = 0. \]
We get the estimate of $q^b$ as a consequence of Lemma 6.2 with $f^b = gh$. We find
\[ \|\nabla q^b\|_{m-1} + \|\partial_{zz} q^b\|_{m-2} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + |h|_3 \right) |h|_{m-\frac{1}{2}}. \]
To estimate $q^i$ we can use Lemma 6.1. This yields
\[ \|\nabla q^i\|_{m-1} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + \|P(v \cdot \nabla^2 v)\|_2 + \|\partial_z \varphi \nabla^2 v \cdot \nabla^2 v\|_1 \right) \left( |h|_{m-\frac{1}{2}} + \|P(v \cdot \nabla^2 v)\|_{m-1} \right). \]
By using again product estimates we thus find
\[ \|\nabla q^i\|_{m-1} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^1,\infty} + |h|_3 + \|v\|_{E^3} \right) \left( |h|_{m-\frac{1}{2}} + \|v\|_{E^m} \right). \]
In a similar way, using Lemma 6.1 we get the estimate for $\partial_{zz} q^i$.
To get (6.35), we use (2.4) to write that
\[ \|\nabla q^E\|_{k,\infty} \lesssim \|\partial_z \nabla q^E\|_{k+1} \|\nabla q^E\|_{k+2}, \quad k = 1, 2 \]
and hence we get the estimate for $\|\nabla q^E\|_{k,\infty}, k = 1, 2$ by using (6.34) with $m = 4$ or $m = 5$. Finally, to estimate $\|\partial_{zz} q^E\|_{L^\infty}$, it suffices to rewrite the equation (6.37) under the form (6.19) and to use the estimate for $\|\nabla q^E\|_{1,\infty}$ just established.

\[ \square \]
This ends the proof of Proposition \[6.4\]. It remains a last estimate for \( q^E \) that will be used for the control of the Taylor sign condition.

**Proposition 6.5.** For \( T \in (0,T^*) \), we have the estimate
\[
\int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} \leq \int_0^T \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} \right) \left( \|v\|_6 + \|\partial_z v\|_4 + \|v\|_{E^2,\infty} + \|h\|_6 + |h|_{3,\infty} \right) \cdot (1 + \varepsilon\|\partial_z v\|_{L^\infty} + \varepsilon\|\partial_z v\|_3).
\]

The proof of proposition \[6.5\] is a consequence of the following general lemma:

**Lemma 6.6.** Consider \( \rho \) the solution of the equation
\[
(6.38) \quad - \nabla \cdot (E \nabla \rho) = \nabla \cdot F, \quad \rho|_{z=0} = 0
\]
then for every \( k > 1 \), \( \rho \) satisfies the estimate
\[
\|(\partial_z \rho)^b\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{k+2,\infty} \right) \left( \|F\|_{H^k_{\text{tan}}} + |(F)^b|_{L^\infty(\mathbb{R}^2)} \right).
\]

We recall that we denote the trace of a function \( f \) on the boundary \( z = 0 \) by \( f|_{z=0} \) or \( f^b \).

**Proof.** The first step is to establish that \( \rho \) satisfies for every \( m \geq 0 \) the estimate:
\[
(6.39) \quad \|\nabla \rho\|_{H^m_{\text{tan}}} \leq \Lambda \left( \frac{1}{c_0}, |h|_{m+1,\infty} \right) \|F\|_{H^m_{\text{tan}}}.
\]
To get this estimate, it suffices to apply \( \partial_0^\alpha \) with \( |\alpha| \leq m \) to \[6.38\] and to use the standard energy estimate in a classical way. Next, we can rewrite \[6.38\] as
\[
(6.40) \quad - E_{33} \partial_z \rho = \partial_z F + R
\]
with \( R \) given by
\[
(6.41) \quad R = \sum_{i<3} \partial_i F_i + \partial_z E_{33} \partial_z \rho + \sum_{i<3} \left( \partial_i (E \nabla \rho)_i + \partial_z (E_{33} \partial_z \rho) \right).
\]
For the moment, we see \[6.40\] as an ordinary differential equation in \( z \), the variable \( y \) being only a parameter. We multiply \[6.40\] by \( \partial_z \rho \) and we integrate in \( z \) to get after integration by parts:
\[
(6.42) \quad |\partial_z \rho(0)|^2 \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} \right) \left( |\partial_z \rho|^2_{L^2} + |R|^2_{L^2} + |F(0)| \cdot |\partial_z \rho(0)| + \left| \int_{-\infty}^0 F \partial_z \rho \, dz \right| \right).
\]
Note that here \( |\cdot| \) stands for the absolute value. To estimate the last term, we use again the equation \[6.40\] to obtain
\[
\left| \int_{-\infty}^0 F \partial_z \rho \, dz \right| \leq \left| \int_{-\infty}^0 F \frac{\partial_z F}{E_{33}} \, dz \right| + \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} \right) |F|_{L^2} |R|_{L^2}
\]
and hence from an integration by parts we find
\[
\left| \int_{-\infty}^0 F \partial_z \rho \, dz \right| \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} \right) \left( |F(0)| + |F|^2_{L^2} + |R|^2_{L^2} \right).
\]
Consequently, we get from \[6.40\] that
\[
|\partial_z \rho(0)| \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} \right) \left( |\partial_z \rho|_{L^2} + |R|_{L^2} + |F(0)| \right).
\]
Now, by taking the supremum in \( y \) and by using the two-dimensional Sobolev embedding for the right hand side (except for the last term), we find that
\[
\|(\partial_z \rho)^b\|_{L^\infty(\mathbb{R}^2)} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} \right) \left( \|\partial_z \rho\|_{H^k_{\text{tan}}} + |R|_{H^k_{\text{tan}}} + |F|^b\|_{L^\infty(\mathbb{R}^2)} \right)
\]
for $k > 1$. To conclude, we see from the definition of $R$ that
\[
\|R\|_{H^k_{tan}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{k+2, \infty} \right) \|\nabla \phi\|_{H^{k+1}_{tan}} + \|F\|_{H^{k+1}_{tan}}
\]
and hence, we get from (6.39) that
\[
| (\partial_t \rho)^b |_{L^\infty(\mathbb{R}^2)} \leq \Lambda \left( \frac{1}{c_0}, |h|_{k+2, \infty} \right) (\|F\|_{H^{k+1}_{tan}} + |F^b|_{L^\infty(\mathbb{R}^2)}).
\]
This ends the proof of Lemma 6.6. \hfill \square

We are now in position to give the proof of Proposition 6.5:

**Proof of proposition 6.5.**

We note that $\partial_t q^E$ solves the elliptic equation
\[
\nabla \cdot (E \nabla \partial_t q^E) = \nabla \cdot (\partial_t (P(v \cdot \nabla^\varphi v))) - \nabla \cdot (\partial_t E (\nabla q^E)), \quad \partial_t q^E_{j=0} = g \partial_t h
\]
consequently, we can again split $\partial_t q^E = q^i + q^B$ where $q^B$ absorbs the boundary term:
\[
\nabla \cdot (E \nabla q^B) = 0, \quad q^B_{j=0} = g \partial_t h
\]
and $q^i$ solves
\[
\nabla \cdot (E \nabla \partial_t q^i) = \nabla \cdot (\partial_t (P(v \cdot \nabla^\varphi v))) - \nabla \cdot (\partial_t E (\nabla q^E)), \quad q^i_{j=0} = 0.
\]

The estimate of $\|\nabla q^B\|_{L^\infty}$ is a consequence of Lemma 6.2. Indeed, for $k > 3/2$, we have
\[
\|\nabla q^B\|_{L^\infty} \lesssim \|\nabla q^B\|_{H^k}
\]
and from the estimate of Lemma 6.2, we find that
\[
\|\nabla q^B\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + |h|_3 + |\partial_t h|_3 \right).
\]

From the boundary condition (1.13), we obtain
\[
|\partial_t h|_3 \leq \Lambda (\|v\|_{L^\infty} + \|h\|_{L^\infty}) (|h|_4 + |v^b|_3) \leq \Lambda (\|v\|_{L^\infty} + \|h\|_{L^\infty} + |h|_4 + \|v\|_{E^4})
\]
where the last estimate comes from the trace inequality (2.5). We thus finally get for $q^B$ that
\[
\|\nabla q^B\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2, \infty} + \|v\|_{L^\infty} + |h|_4 + \|v\|_{E^4} \right).
\]

It remains to estimate $q^i$. We shall use Lemma 6.6 but we need to be careful with the structure of the right hand side in (6.43). We first note thanks to the identities (6.6) that
\[
\nabla \cdot (\partial_t (P(v \cdot \nabla^\varphi v))) = \nabla \cdot ( (\partial_t P) v \cdot \nabla^\varphi v ) + \nabla \cdot ( P (\partial_t v \cdot \nabla^\varphi v) ) + \partial_t \varphi \nabla^\varphi \cdot (v \cdot \partial_t (\nabla^\varphi v) )
\]
For the last term, by using again (6.6) and the summation convention on repeated indices, we can write
\[
\partial_t \varphi \nabla^\varphi \cdot (v \cdot \partial_t (\nabla^\varphi v)) = \partial_t \varphi \partial_t^i v^j (v \cdot \partial_t \left( \frac{1}{\partial_t \varphi} P^* \nabla v_i \right) = \partial_t \varphi \partial_t^j v^i (v \cdot \partial_t \left( \frac{1}{\partial_t \varphi} P^* \nabla v_j \right) + \partial_t \varphi \partial_t^i v^j (v \cdot \nabla^\varphi \partial_t v_i)
\]
and the crucial observation is that since $\nabla^\varphi \cdot v = 0$, we have
\[
\partial_t^j (v \cdot \nabla^\varphi \partial_t v_i) = \partial_t^j (v_j \partial_t^j \partial_t v_i) = \partial_t^j (\partial_t v_i \partial_t^j v_j) + \partial_t^j (\partial_t^j \partial_t v_i v_j).
\]
Again since $\nabla^\varphi \cdot v = 0$, we can write
\[
\partial_t^j \partial_t v_i = - \frac{1}{\partial_t \varphi} \nabla \cdot (\partial_t P v)
\]
and hence, we obtain
\[
\partial_t^i (v \cdot \nabla^\varphi \partial_t v_i) = \partial_t^j (\partial_t v_i \partial_t^j v_j) - \partial_t^j \left( v_j \frac{1}{\partial_t \varphi} \nabla \cdot (\partial_t P v) \right).
\]
Consequently, we have proven that
\[
\nabla \cdot (\partial_t (P(v \cdot \nabla \varphi))) = \nabla \cdot ((\partial_t P) v \cdot \nabla \varphi) + 2 \nabla \cdot (P(\partial_t v \cdot \nabla \varphi)) + \nabla \cdot (v \cdot \partial_t (\frac{1}{\partial_z \varphi} P^*) \nabla v) - \nabla \cdot \left( \frac{1}{\partial_z \varphi} \nabla \cdot (\partial_t P v) \right).
\]

This allows to observe that (6.43) is under the form (6.38) with
\[
(6.45) \quad F = (\partial_t P) v \cdot \nabla \varphi + 2P(\partial_t v \cdot \nabla \varphi) + v \cdot \partial_t \left( \frac{1}{\partial_z \varphi} P^* \right) \nabla v - \frac{1}{\partial_z \varphi} \nabla \cdot (\partial_t P v) + \partial_t E \nabla q^E.
\]

Consequently, by using Lemma 6.6 for \( k = 2 \), we get that
\[
|\partial_z q^i_{/z=0}|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0} |h|_{4, \infty} \right) (\|F\|_3 + |F^b|_{L^\infty})
\]
and hence that
\[
(6.46) \quad \int_0^T |\partial_z q^i_{/z=0}|_{L^\infty} \leq \int_0^T \Lambda \left( \frac{1}{c_0} |h(t)|_{4, \infty} \right) (\|F\|_3 + |F^b|_{L^\infty}).
\]

Next, by using Proposition 3.1 and (2.1), we get that
\[
\|F\|_3 \leq \Lambda \left( \frac{1}{c_0} \|v\|_{E^4} + \|v\|_{E^1, \infty} + |h|_4 + |h|_{1, \infty} + |\partial_t h|_5 + |\partial_t h|_{2, \infty} + \|\nabla \varphi\|_3 \right) (1 + \|\partial_t v\|_{L^\infty} + \|\partial_t v\|_3).
\]

Therefore, by using again the boundary condition (1.13) and Proposition 6.4 we find
\[
\|F\|_3 \leq \Lambda \left( \frac{1}{c_0} \|v\|_{E^5} + \|v\|_{E^2, \infty} + \|v\|_6 + |h|_6 + |h|_{3, \infty} \right) (1 + \|\partial_t v\|_{L^\infty} + \|\partial_t v\|_3).
\]

By using again the expression (6.45), we also find
\[
|F^b|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0} |h|_{3, \infty} + \|v\|_{E^2, \infty} + \|\nabla \varphi\|_{L^\infty} \right) (1 + \|\partial_t v\|_{L^\infty})
\leq \Lambda \left( \frac{1}{c_0} |h|_{3, \infty} + \|v\|_{E^2, \infty} + |h|_4 + \|v\|_{E^4} \right) (1 + \|\partial_t v\|_{L^\infty})
\]
where the last estimate comes from Proposition 6.4. Consequently, by plugging these last two estimates in (6.46), we find
\[
\int_0^T |\partial_z q^i_{/z=0}|_{L^\infty} \leq \int_0^T \Lambda \left( \frac{1}{c_0} \|v\|_{E^5} + \|v\|_{E^2, \infty} + \|v\|_6 + |h|_6 + |h|_{3, \infty} \right) (1 + \|\partial_t v\|_{L^\infty} + \|\partial_t v\|_3).
\]

To conclude, we can use the equation (1.12) to get
\[
\|\partial_t v\|_3 + \|\partial_t v\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0} \|v\|_{E^5} + \|v\|_{E^2, \infty} + |h|_5 + |h|_{2, \infty} \right)
\cdot \left( 1 + \varepsilon \|\partial_z v\|_{L^\infty} + \varepsilon \|\partial_z v\|_3 + \|\nabla \varphi\|_{L^\infty} + \|\nabla \varphi\|_3 \right)
\]
and then Proposition 6.3 and Proposition 6.4 combined with the trace estimate (2.5) to find
\[
\|\partial_t v\|_3 + \|\partial_t v\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0} \|v\|_{E^5} + \|v\|_{E^2, \infty} + |h|_5 + |h|_{2, \infty} \right) \cdot \left( 1 + \varepsilon \|\partial_z v\|_{L^\infty} + \varepsilon \|\partial_z v\|_3 \right).
\]

This ends the proof of Proposition 6.5.
7. Conormal estimates for $v$ and $h$

7.1. Control given by the good unknown. To perform our higher order energy estimates, we shall use the good unknown $V^\alpha = Z^\alpha v - \partial_z^\alpha Z^\alpha \varphi$. A crucial point is therefore to establish that the control of this type of quantity and $Z^\alpha h$ yield a control of all the needed quantities.

We shall perform a priori estimates on an interval of time $[0,T^c]$ for which we assume that

$$\partial_z \varphi \geq c_0, \quad |h|_{L^\infty} \leq \frac{1}{c_0}, \quad g - (\partial_z^\alpha q^E)/z = 0, \quad \forall t \in [0,T^c]$$

for some $c_0 > 0$. In the following, we shall use the notation $\Lambda_0 = \Lambda(1/c_0)$. Note in particular that this will allow us to use the Korn inequality recalled in Proposition 2.9.

Let us introduce a few notations. As we have already used, we set

$$V^\alpha = Z^\alpha v - \partial_z^\alpha v Z^\alpha h, \quad \alpha \neq 0, \quad V^0 = v$$

(and $V^0 = v$), and we shall use the norms:

$$\|V^m(t)\|^2 = \sum_{|\alpha| \leq m} \|V^\alpha(t)\|^2, \quad \|S^\alpha V^m(t)\|^2 = \sum_{|\alpha| \leq m} \|S^\alpha V^\alpha(t)\|^2.$$

By using (7.1), we get that for every $t \in [0,T^c]$, we have the equivalence

$$\|v\|_m \lesssim \|V^m\| + \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) |h|_{m_{-\frac{1}{2}}}, \quad \|V^m\| \lesssim \|v\|_m + \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) |h|_{m_{-\frac{1}{2}}}.$$

7.2. Main estimate.

Proposition 7.1. Let us define for $t \in [0,T^c]$

$$\Lambda_\infty(t) = \Lambda \left( \frac{1}{c_0}, \|v(t)\|_{E_{2,\infty}} + \|\partial_z v\|_{L^\infty} + |h(t)|_{4} + |v(t)|_{E^4} \right),$$

then, for every $t \in [0,T]$, a sufficiently smooth solution of (1.12), (1.13), (1.14) satisfies for every $m \geq 0$ the estimate

$$\|V^m(t)\|^2 + |h(t)|^2_{m} + \epsilon |h(t)|^2_{m+\frac{1}{2}} + \epsilon \int_0^t \|\nabla V^m\|^2$$

$$\leq \Lambda_0 \left( \|V^m(0)\|^2 + |h(0)|^2_{m} + \sqrt{\epsilon} |h(0)|^2_{m+\frac{1}{2}} \right) + \int_0^t \Lambda_\infty(t) \|\partial_z v\|^2_{m-1}$$

where as before $\Lambda_0 = \Lambda(1/c_0)$.

Proof. By using the equations (5.10), (5.11), and the boundary condition (5.37), the same integrations by parts as in the proof of Lemma 4.1 yield that

$$\frac{d}{dt} \int_S |V^\alpha|^2 dV_t + 4\epsilon \int_S |S^\alpha V^\alpha|^2 dV_t = \mathcal{R}_S + \mathcal{R}_C + 2 \int_{z=0} (2\epsilon S^\alpha V^\alpha - Q^\alpha Id) \cdot V^\alpha dy$$

where we set

$$\mathcal{R}_S = 2 \int_S \left( \epsilon D^\alpha(S^\alpha v) + \epsilon \nabla^\alpha \cdot (\mathcal{E}^\alpha(v)) \right) \cdot V^\alpha dV_t,$$

$$\mathcal{R}_C = -2 \int_S \left( (C^\alpha(T) + C^\alpha(q)) \cdot V^\alpha - C^\alpha(d)Q^\alpha \right) dV_t.$$

Let us start with the analysis of the last term in (7.4). Note that this is the crucial term for which we shall need to use the physical condition. We first notice that if $\alpha_3 \neq 0$ since $V^\alpha_{/z=0} = 0,$
this term vanishes. Consequently, we only need to study the case $\alpha_3 = 0$. By using \eqref{5.37}, we get that
\begin{equation}
\int_{z=0} \left(2\varepsilon S^\varphi V^\alpha - Q^\alpha \text{Id}\right) N \cdot V^\alpha = \int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q Z^\alpha h\right) N \cdot V^\alpha - \int_{z=0} \left(2\varepsilon S^\varphi v - (q - gh)\text{Id}\right) Z^\alpha N \cdot V^\alpha \, dy + \mathcal{R}_B
\end{equation}
where
\begin{equation}
\mathcal{R}_B = \int_{z=0} (C^\alpha(B) - 2\varepsilon Z^\alpha h \partial_z^\varphi \left(S^\varphi v\right) N) \cdot V^\alpha \, dy.
\end{equation}
\end{proof}

To estimate $\mathcal{R}_B$, we can use \eqref{5.38} to get that
\begin{align*}
|\mathcal{R}_B| &\leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^{2,\infty}}\right) \left(\varepsilon(1 + \|\partial_z v\|_{L^{\infty}}) |h|_m + \varepsilon |v^b|_m\right) |(V^\alpha)^b|_{L^2} \\
&\leq \Lambda \varepsilon |h|_{m + \frac{1}{2}} |(V^\alpha)^b|_{L^2}.
\end{align*}

Note that in the proof, when we do not specify the time variable, this means that all the quantities that appear are evaluated at time $t$. To estimate the second term in the right hand side of \eqref{7.7}, we first note that
\begin{align*}
\int_{z=0} \left(2\varepsilon S^\varphi v - (q - gh)\text{Id}\right) Z^\alpha N \cdot V^\alpha \, dy &= \int_{z=0} \left(2\varepsilon S^\varphi v - q^{NS}\text{Id}\right) Z^\alpha N \cdot V^\alpha \, dy \\
\end{align*}
since $q = q^E + q^{NS}$ and $q^E = gh$ on the boundary. This yields
\begin{equation}
\left|\int_{z=0} \left(2\varepsilon S^\varphi v - q^{NS}\text{Id}\right) Z^\alpha N \cdot V^\alpha \, dy\right| \leq |Z^\alpha \nabla h|_{L^\infty} \left|\int_{z=0} \left(2\varepsilon S^\varphi v - q^{NS}\text{Id}\right)(V^\alpha)^b \right|_{L^1}.
\end{equation}

Since $q^{NS} = S^\varphi v n \cdot n$ on the boundary, we obtain by using \eqref{2.8} that
\begin{equation}
\left|\int_{z=0} \left(2\varepsilon S^\varphi v - q^{NS}\text{Id}\right) Z^\alpha N \cdot V^\alpha \, dy\right| \leq \Lambda \varepsilon |h|_{m + \frac{1}{2}} |(V^\alpha)^b|_{L^2}.
\end{equation}

To express the first term in the right hand side of \eqref{7.7}, we use first use the decomposition $q = q^E + q^{NS}$ and we write
\begin{align*}
\int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q Z^\alpha h\right) N \cdot V^\alpha &= \int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q^E Z^\alpha h\right) N \cdot V^\alpha \, dy + \int_{z=0} \partial_z^\varphi q^{NS} Z^\alpha h V^\alpha \cdot N.
\end{align*}

For the second term, we get thanks to \eqref{6.31} that
\begin{equation}
\left|\int_{z=0} \partial_z^\varphi q^{NS} Z^\alpha h V^\alpha \cdot N\right| \leq \|\partial_z^\varphi q^{NS}\|_{L^{\infty}} |h|_m |(V^\alpha)^b| \leq \Lambda \varepsilon |h|_m |(V^\alpha)^b|.
\end{equation}

For the first term, we use the kinematic boundary condition under the form given by Lemma \ref{5.7}
\begin{align*}
\int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q^E Z^\alpha h\right) N \cdot V^\alpha \, dy &= \int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q^E Z^\alpha h\right) \partial_t Z^\alpha h \\
&\quad - \int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q^E Z^\alpha h\right) v^b \cdot (Z^\alpha \nabla y h - C^\alpha(h)) \, dy.
\end{align*}

Thanks to an integration by parts, Proposition \ref{6.4} and \eqref{6.41}, we obtain
\begin{align*}
\left|\int_{z=0} \left(- g Z^\alpha h + \partial_z^\varphi q^E Z^\alpha h\right) v^b \cdot (Z^\alpha \nabla y h - C^\alpha(h)) \, dy\right| &\lesssim \|v\|_{1,\infty} (1 + \|\partial_z^\varphi q^E\|_{1,\infty}) (|Z^\alpha h| + |C^\alpha(h)|) \\
&\leq \Lambda |h|_m + \|v\|_{E^{1,\alpha}} |h|_m.
\end{align*}
and we finally write
\[
\int_{z=0} \left( -g Z^\alpha h + \partial_\varepsilon q^E \partial_\varepsilon Z^\alpha h \right) \partial_t Z^\alpha h = -\frac{1}{2} \cdot \frac{d}{dt} \int_{z=0} (g - \partial_\varepsilon q^E) |Z^\alpha h|^2 - \int_{z=0} \partial_t (\partial_\varepsilon q^E) |Z^\alpha h|^2.
\]
Consequently, gathering all the previous estimates, we have proven from (7.7) that
\[
(7.9) \quad \int_{z=0} (2\varepsilon S^\varepsilon V^\alpha - Q^\alpha Id)N \cdot V^\alpha = -\frac{1}{2} \cdot \frac{d}{dt} \int_{z=0} (g - \partial_\varepsilon q^E) |Z^\alpha h|^2 + \tilde{\mathcal{R}}_B
\]
where
\[
|\tilde{\mathcal{R}}_B| \leq \Lambda_{\infty} (\varepsilon (1 + \|\partial_\varepsilon v\|_{L^\infty}) |h|_m + \varepsilon |b|^b_m) |(V^\alpha)^b|
\]
\[
+ \varepsilon |h|_{m+\frac{1}{2}} |(V^\alpha)^b|_{\frac{1}{2}} + (1 + |\partial_\varepsilon q^E|_{L^\infty}) |h|^2_m + \|v\|_{E^m} |h|_m
\]
and we can use successively, (7.2), that
\[
(7.10) \quad |b|^b_m \leq \Lambda_{\infty} ((V^m)_{\cdot,0} + |h|_m),
\]
and the trace inequality (2.5) which yields
\[
|(V^\alpha)^b|^2_{L^2} \lesssim \|\partial_\varepsilon V^\alpha\| \|V^\alpha\|, \quad \|V^\alpha\|_{\frac{1}{2}} \lesssim \|\nabla V^\alpha\| + \|V^\alpha\|
\]
to get that
\[
(7.11) \quad |\tilde{\mathcal{R}}_B| \leq \varepsilon \|\nabla V^m\| \|V^m\| + \Lambda_{\infty} \|\partial_\varepsilon v\|_{m-1} |h|_m
\]
\[
+ \Lambda_{\infty} (1 + |(\partial_\varepsilon \partial_t q^E)^b|_{L^\infty}) (|h|^2_m + \varepsilon |h|^2_{m+\frac{1}{2}} + \|v^m\|^2)
\]
By plugging (7.9) into (7.4), we get that
\[
(7.12) \quad \frac{d}{dt} \int_{S} |V|^2 d\mathcal{V} + \int_{z=0} (g - \partial_\varepsilon q^E) |Z^\alpha h|^2 dy + 4\varepsilon \int_{S} S^\varepsilon |V^\alpha|^2 d\mathcal{V} = \mathcal{R}_S + \mathcal{R}_C + \tilde{\mathcal{R}}_B
\]
and it remains to estimate the commutators \(\mathcal{R}_S, \mathcal{R}_C\).

Let us begin with the estimate of \(\mathcal{R}_C\). By using (5.13), (5.14) and the Sobolev embedding, we immediately get that
\[
(7.13) \quad |\mathcal{R}_C| \leq \Lambda_{\infty} (\|v\|_m + \|\partial_\varepsilon v\|_{m-1} + |h|_m) \|v^m\| + \int \mathcal{C}^\alpha(q) \cdot v^m d\mathcal{V}_t.
\]
To estimate the last term, we could use directly (5.12). Neverthess, the estimate (5.12) implies that we need to control \(\|\nabla q\|_{1,\infty}\) and the control of \(\|\nabla q^{NS}\|_{1,\infty}\) through Sobolev embedding and (6.30) would involve a dependence in \(\varepsilon |\phi|^s\) with \(s > 4\) in the definition of \(\Lambda_{\infty}\). We can actually easily get a better estimate by using that \(\mathcal{C}^\alpha(q) = \mathcal{C}^\alpha(q^E) + \mathcal{C}^\alpha(q^{NS})\) and by handling the terms in two different ways. For the Euler pressure, we can use directly (5.12) and Proposition 6.4 to get that
\[
(7.14) \quad \|\mathcal{C}^\alpha(q^E)\| \|v^m\| \lesssim \Lambda_{\infty} \|v\|_{E^{m-1}} \|v\|_{E^m} \|v^m\|.
\]
To estimate \(\mathcal{C}^\alpha(q^{NS})\) we can have a closer look at the structure of the commutator. By using the decomposition (5.2), (5.7), (5.9) combined with Proposition 6.3, we have that
\[
\|\mathcal{C}_{i,2}^\alpha(q^{NS})\| + \|\mathcal{C}_{i,3}^\alpha(q^{NS})\| \leq \Lambda_{\infty} \|v\|_{E^{m-1}} \|h|_m\|
\]
Consequently, it only remains to study \(\mathcal{C}_{i,1}^\alpha(q^{NS}) = -[Z^\alpha \partial_\varepsilon \psi, \partial_\varepsilon q^{NS}]\)(the case \(i = 3\) can be handled by similar arguments). For this term, it is actually better to write it under the form
\[
\mathcal{C}_{i,1}^\alpha(q^{NS}) = -\left( [Z^\alpha \partial_\varepsilon \psi, \partial_\varepsilon q^{NS} - Z^\alpha \partial_\varepsilon \psi] \partial_\varepsilon q^{NS} \right).
\]
We can use the commutator estimate \( [2.2] \) for the first term and Proposition \([3.1]\) and Proposition \([6.3]\) to get that
\[
\| [Z^\alpha, \frac{\partial}{\partial \varphi}] \partial_\varphi q^{NS} \| + \| Z^\alpha \left( \frac{\partial}{\partial \varphi} \right) \partial_\varphi q^{NS} \| \leq \Lambda \left( \frac{1}{c_0} \varepsilon^{-1} \| \partial_\varphi q^{NS} \|_{L^\infty} \right) \left( \| \partial_\varphi q^{NS} \|_{m-1} + \varepsilon |h|_{m+\frac{1}{2}} \right)
\leq \Lambda_{\infty}(\varepsilon |h|_{m+\frac{1}{2}} + \varepsilon |v^b|_{m+\frac{1}{2}}).
\]
This yields
\[
|R_C| \leq \Lambda_{\infty}(\|v\|_{m} + \|\partial_\varphi v\|_{m-1} + |h|_{m} + \varepsilon |h|_{m+\frac{1}{2}} + \varepsilon |v^b|_{m+\frac{1}{2}})\|V^m\|.
\]
To estimate the last term \( \varepsilon |v^b|_{m+\frac{1}{2}}\|V^m\| \) in the above estimate, we can write that
\[
|v^b|_{m+\frac{1}{2}} \leq \sum_{|\alpha| \leq m} |V^\alpha|_{\frac{1}{2}} + |\partial_\varphi v^\alpha h|_{\frac{1}{2}}
\]
and we use \([2.3]\) and the trace estimate \([2.5]\) to get that
\[
|v^b|_{m+\frac{1}{2}} \leq \|\nabla V^m\| + \|V\|_{m} + \Lambda_{\infty}|h|_{m+\frac{1}{2}}.
\]
Hence, we find
\[
|R_C| \leq \varepsilon \|\nabla V^m\| \|V^m\| + \Lambda_{\infty} \|\partial_\varphi v\|_{m-1} \|V\|_{m} + \Lambda_{\infty} \left( \|V^m\|^2 + |h|_{m}^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right).
\]
It remains to estimate \( R_S \) which is given by \([7.5]\). For the second term, we use an integration by parts:
\[
\int_S \varepsilon \nabla^2 (E^\alpha(v)) \cdot V^\alpha d\vartheta = -\int_S \varepsilon E^\alpha(v) \cdot \nabla V^\alpha d\vartheta + \int_{\vartheta = 0} \varepsilon E^\alpha(v) N \cdot V^\alpha dy.
\]
The estimate of the first term comes directly from \([5.14]\), we get
\[
\left| \int_S \varepsilon E^\alpha(v) \cdot \nabla V^\alpha d\vartheta \right| \leq \Lambda_{\infty}(\|v\|_{E^m} + |h|_{m}) \varepsilon \|\nabla V^\alpha\|.
\]
To estimate the boundary term, we need an estimate of \( E^\alpha(v) \) on the boundary. The same arguments yielding the proof of \([5.14]\) can be used by using the commutator estimates on the boundary to get that
\[
|E^\alpha(v)(\cdot,0)|_{L^2} \leq \Lambda_{\infty}(\|h\|_{m} + |v^b|_{m} + |\partial_\varphi v^b|_{m-1})
\]
and hence by using \([5.31]\), we find
\[
|E^\alpha(v)(\cdot,0)|_{L^2} \leq \Lambda_{\infty}(\|h\|_{m} + |v^b|_{m}).
\]
Consequently, we have proven that
\[
\left| \int_S \varepsilon \nabla^2 (E^\alpha(v)) \cdot V^\alpha d\vartheta \right| \leq \Lambda_{\infty} \left( \|v\|_{E^m} + |h|_{m} \varepsilon \|\nabla V^\alpha\| + (\|h\|_{m} + |v^b|_{m}) \varepsilon \|(V^\alpha)^b\| \right).
\]
and by using again the trace inequality, we find that
\[
(7.17) \quad \left| \int_S \varepsilon \nabla^2 (E^\alpha(v)) \cdot V^\alpha d\vartheta \right| \leq \Lambda_{\infty} \varepsilon \|\nabla V^m\| \left( \|V^m\| + \|\partial_\varphi v\|_{m-1} + |h|_{m} \right)
\]
It remains to estimate \( \varepsilon \int_S D^\alpha(S^\varphi v) \cdot V^\alpha d\vartheta \). This is given in the following lemma:

**Lemma 7.2.** We have the estimate:
\[
\varepsilon \left| \int_S D^\alpha(S^\varphi v) \cdot V^\alpha d\vartheta \right| \leq \Lambda_{\infty} \left( \varepsilon \|\nabla V^m\| \left( \|V^m\| + \|\partial_\varphi v\|_{m-1} + |h|_{m+\frac{1}{2}} \right) + \varepsilon \|v\|^2_{E^m} + |h|_{m+\frac{1}{2}}^2 \right)
\leq \varepsilon \|\partial_\varphi v\|_{L^\infty} \left( \|h\|_{m}^2 + \|V^m\|^2 \right).
\]
We shall first finish the proof of Proposition \[7.1\] and then prove Lemma \[7.2\].

From (7.12), we can sum over \(\alpha\) (for \(\alpha = 0\), we use the basic estimate (7.2), integrate in time, use our a priori assumption (7.1) on the Taylor condition and use (7.16), (7.17) and Lemma (7.2) to get that

\[
\begin{align*}
(7.18) & \quad \|V^m(t)\|^2 + |h(t)|^2_m + \int_0^t \|S^p V^m\|^2 \leq \Lambda_0 (\|V^m(0)\|^2 + |h(0)|^2_m) \\
& \quad + \int_0^t \left( \varepsilon \Lambda_\infty \|\nabla V^m\| (\|V^m\| + \|\partial_z v\|_{m-1} + |h|_{m+\frac{1}{2}}) \\
& \quad + \Lambda_\infty (1 + |(\partial_z \partial_q F)|_b \|\partial_v\|_{L\infty}) \left[ |h|^2_m + \|V^m\|^2 + \varepsilon |h|^2_{m+\frac{1}{2}} + \Lambda_\infty \|\partial_v\|^2_{m-1} \right] \right). 
\end{align*}
\]

Now, we can use Proposition \[2.8\] and the Korn inequality of Proposition \[2.9\] to get that

\[
\|\nabla V^m\| \leq \Lambda \left( \frac{1}{c_0} \right) (\|S^p V^m\| + \|v\|_m)
\]

and the estimate of Proposition \[7.1\] follows from (7.18) and the Young inequality \[3.10\]. This ends the proof of Proposition \[7.1\].

It remains the:

**Proof of Lemma \[7.2\].** By using (5.27), we actually have to estimate

\[
\mathcal{R}_{Si} = \varepsilon \int_S C_{j}^\alpha(S^p v)_{ij} V_j^\alpha d\mathcal{V}_t
\]

\[
(7.19) = \varepsilon \int_S C_{j,1}^\alpha(S^p v)_{ij} V_j^\alpha d\mathcal{V}_t + \varepsilon \int_S C_{j,2}^\alpha(S^p v)_{ij} V_j^\alpha d\mathcal{V}_t + \varepsilon \int_S C_{j,3}^\alpha(S^p v)_{ij} V_j^\alpha d\mathcal{V}_t
\]

\[
(7.20) := \mathcal{R}_{Si}^1 + \mathcal{R}_{Si}^2 + \mathcal{R}_{Si}^3
\]

where we have used the decomposition 5.2. For the first term, by using the definition of the symmetric commutator we see that we need to estimate terms like

\[
\varepsilon \int_S Z^\beta (\frac{\partial_{ij} \varphi}{\partial z \varphi}) (Z^\gamma \partial_z (S^p v)_{ij}) V_j^\alpha d\mathcal{V}_t
\]

where \(\beta\) and \(\gamma\) are such that \(\beta \neq 0, \gamma \neq 0\) and \(|\beta| + |\gamma| = m\). By using (5.8), we can reduce the problem to the estimate of

\[
\varepsilon \int_S c_\gamma Z^\beta (\frac{\partial_{ij} \varphi}{\partial z \varphi}) \partial_z (Z^\gamma (S^p v)_{ij}) V_j^\alpha d\mathcal{V}_t
\]

with \(\beta\) as before (thus \(|\beta| \leq m-1\) and \(|\gamma| \leq |\gamma| \leq m-1\). By using an integration by parts, we are lead to the estimate of three types of terms:

\[
I_1 = \varepsilon \int_S Z^\beta (\frac{\partial_{ij} \varphi}{\partial z \varphi}) Z^\gamma (S^p v)_{ij} \partial_z V_j^\alpha d\mathcal{V}_t, \quad I_2 = \varepsilon \int_S \left( \partial_z Z^\beta (\frac{\partial_{ij} \varphi}{\partial z \varphi}) \right) Z^\gamma (S^p v)_{ij} V_j^\alpha d\mathcal{V}_t,
\]

and the term on the boundary

\[
I_3 = \varepsilon \int_{y=0} Z^\beta (\frac{\partial_{ij} \varphi}{\partial z \varphi}) Z^\gamma (S^p v)_{ij} V_j^\alpha dy.
\]

For the first term, since \(\beta \neq 0\), we get by using again (2.11), (3.7) and Proposition 3.1 that

\[
|I_1| \leq \varepsilon \|\nabla V^m\| \Lambda_\infty \left( \|v\|_m + \|\partial_z v\|_{m-1} + |h|_{m+\frac{1}{2}} \right).
\]
To estimate $I_2$, we can first use (6.8) to expand it as a sum of terms under the form

$$\tilde{I}_2 = \varepsilon \int_S \left( c_\beta Z_\beta \partial_z \left( \frac{\partial \varphi}{\partial z} \right) \right) Z^\gamma (S^\varphi v)_v V_j^\alpha d\mathcal{V}_t$$

with $|\tilde{\beta}| \leq \beta$. If $\gamma = 0$, since $|\tilde{\beta}| \leq m - 1$, we just write

$$|\tilde{I}_2| \leq \varepsilon \| \partial_z \left( \frac{\partial \varphi}{\partial z} \right) \|_{m-1} \| S^\varphi \|_{L^\infty} \| V_j^\alpha \|$$

while for $\gamma \neq 0$, we use (2.1) to get

$$|\tilde{I}_2| \leq \varepsilon \| \partial_z \left( \frac{\partial \varphi}{\partial z} \right) \|_{L^\infty} \| S^\varphi v \|_m \| V_j^\alpha \| + \varepsilon \| S^\varphi v \|_{1,\infty} \| \partial_z \left( \frac{\partial \varphi}{\partial z} \right) \|_{m-1}.$$  

Consequently, by using (3.7) and Proposition 3.1, we obtain

$$\|I_2\| \leq \Lambda_\infty \| V^m \| \left( \varepsilon \| S^\varphi v \|_m + \varepsilon \| h \|_{m+\frac{1}{2}} \right).$$

To conclude, we need to relate $\| S^\varphi v \|_m$ to the energy dissipation term. By using (5.1) and Lemma 5.1 combined with the identity (7.22), we get that

$$\| S^\varphi v \|_m \leq \| S^\varphi V^m \| + \Lambda_\infty \left( 1 + \| \partial_z v \|_{L^\infty} \right) \| h \|_m + \Lambda_\infty \left( \| \nabla v \|_{m-1} + \| h \|_{m-\frac{1}{2}} \right)$$

and hence we finally obtain that

$$|I_3| \leq \Lambda_\infty \varepsilon \left( |h|_{m+\frac{1}{2}} + \| (S^\varphi v)^b \|_{m-1} \right) \| (V^\alpha)^h \| \leq \Lambda_\infty \varepsilon \left( |h|_{m+\frac{1}{2}} + \| \varphi \|_{m} \right) \| (V^\alpha)^h \|$$

where we have used (5.31) in the second estimate. By using again (7.10) and the trace inequality, this yields

$$|I_3| \leq \Lambda_\infty \left( \varepsilon \| \nabla V^m \| \| V^m \| + \varepsilon \| h \|_{m+\frac{1}{2}}^2 + \varepsilon \| v \|_{E^m}^2 \right).$$

Consequently, we get from the previous estimates that

$$\|\mathcal{R}_{Si}^1\| \leq \Lambda_\infty \left( \varepsilon \| \nabla V^m \| \| V^m \| + \| h \|_{m+\frac{1}{2}}^2 + \varepsilon \| v \|_{E^m}^2 \right) + \varepsilon \| \partial_z v \|_{L^\infty} \left( \| h \|_{m+\frac{1}{2}}^2 + \| V^m \|_{E^m}^2 \right).$$

The estimate of $\mathcal{R}_{Si}^2$ is straightforward, from the definition after (5.2), we immediately get that

$$\|\mathcal{R}_{Si}^2\| \leq \varepsilon \Lambda_\infty \left( 1 + \| \partial z v \|_{L^\infty} \right) \| V^m \| \| h \|_{m-\frac{1}{2}}.$$  

It remains to estimate $\mathcal{R}_{Si}^3$. By using the definition of $C_{i,3}^\alpha (S^\varphi v)$, we get by using again (5.8) that

$$\varepsilon \left| \int_S \frac{\partial \varphi}{\partial z} \partial_z \left( S^\varphi v \right) [Z^\alpha, \partial_z] (S^\varphi v) d\mathcal{V}_t \right| \leq \varepsilon \Lambda_\infty \left( 1 + \| \partial_z v \|_{L^\infty} \right) \| h \|_{m-\frac{1}{2}} \| V^m \|.$$  

For the term

$$\varepsilon \left| \int_S \frac{\partial \varphi}{\partial z} V^\alpha [Z^\alpha, \partial_z] (S^\varphi v) d\mathcal{V}_t \right|,$$

we perform an integration by parts as in the estimate of $\mathcal{R}_{Si}^1$. We obtain by similar arguments that

$$\varepsilon \left| \int_S \frac{\partial \varphi}{\partial z} V^\alpha [Z^\alpha, \partial_z] (S^\varphi v) d\mathcal{V}_t \right| \leq \Lambda_\infty \varepsilon \left( \| V^m \| + \| \nabla V^m \| \right) \| v \|_{E^m}.$$
From the two last estimates and (7.22), (7.22), we finally obtain that
\[ |R_S| \leq \Lambda_\infty \left( \varepsilon \| \nabla v \| \left( \| V^m \| + \| \partial_v v \|_{m-1} + |h|_{m+\frac{1}{2}} \right) + \varepsilon \| \nabla v \|_{m-1} \right) \]
+ \varepsilon \| \partial_z v \|_{L^\infty} \left( |h|_{m} + \| V^m \|^2 \right).

This end the proof of Lemma 7.2.

8. Normal derivative estimates part I

In order to close the estimate of Proposition 7.1, we need to estimate \( \| \partial_v v \|_{m-1} \). For the normal component of \( v \), this is given for free, we have

**Lemma 8.1.** For every \( m \geq 1 \), we have
\[
(8.1) \quad \| \partial_v v \cdot n \|_{m-1} \leq \Lambda \left( \frac{1}{c_0} \| \nabla v \|_{L^\infty} \right) \left( \| V^m \| + |h|_{m-\frac{1}{2}} \right)
\]
where \( N = (-\partial_1 \varphi, -\partial_2 \varphi, 1)^t \), \( n = N / |N| \).

Note that in the above definition \( N \) is defined in the whole \( S \) via \( \varphi \).

**Proof.** From (1.12), the divergence free condition yields
\[
(8.2) \quad \partial_z v \cdot N = \partial_z \varphi \left( \partial_1 v_1 + \partial_2 v_2 \right)
\]
and hence the estimate follows from (2.1) and Proposition 3.1. \( \square \)

The next step is to estimate the tangential components of \( \partial_z v \). We shall proceed in two steps. As a first step, we shall estimate \( \sup_{[0,T]} \| \partial_z v \|_{m-2} \). This estimate will be important in order to control the \( L^\infty \) norms that occur in the definition of \( \Lambda_\infty \) in in Proposition 7.1 in terms of known quantities via Sobolev embeddings and \( L^\infty \) estimates. We shall prove in the end of the paper the more delicate estimate which allows to control \( \int_0^T \| \partial_z v \|_{m-1} \).

We begin with the following remark:

**Lemma 8.2.** For every \( k \geq 0 \), we have
\[
(8.3) \quad \| \partial_v v \|_k \leq \Lambda \left( \frac{1}{c_0}, \| \nabla v \|_{L^\infty} \right) \left( \| S_n \|_k + |h|_{k+\frac{1}{2}} + \| v \|_{k+1} \right),
\]
\[
(8.4) \quad \| \partial_{zz} v \|_k \leq \Lambda \left( \frac{1}{c_0}, |v|_{E^8,\infty} \right) \left( \| \nabla S_n \|_k + |h|_{k+\frac{1}{2}} + \| v \|_{k+2} \right)
\]
where
\[
(8.5) \quad S_n = \Pi S^x v n, \quad \Pi = Id - n \otimes n
\]
and \( n \) is defined by \( n = N / |N| \).

The main consequence of (8.3) is that for \( k \leq m-1 \), since \( \| v \|_{k+1} \) is estimated in Proposition 7.1, we can look for an estimate of \( \| S_n \|_k \) instead of \( \| \partial_v v \|_k \).

**Proof.** For the proof of Lemma 8.2, we can combine the identity (5.35) which can be written in the equivalent form
\[
(8.6) \quad 2S^x v n = \partial_n u + g^{ij} (\partial_j v \cdot n) \partial_y^i
\]
and the fact that
\[
(8.7) \quad \partial_N u = \frac{1 + |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2}{\partial_z \varphi} \partial_z v - \partial_1 \varphi \partial_1 v - \partial_2 \varphi \partial_2 v
\]
to obtain
\[ \| \partial_z v \|_k \leq \Lambda \left( \frac{1}{c_0}, \|v\|_{L^\infty} \right) (\|V^{k+1}\| + |h|_{k+\frac{1}{2}} + \| \partial_z v \cdot n \|_k + \| S^\varphi v \|_k). \]

Recall that we have the estimate \( |h|_{2, \infty} \leq 1/c_0 \) thanks to (7.1). To estimate \( \| \partial_z v \cdot n \|_k \), we use Lemma 8.1 and to estimate the last term, we note that

(8.8) \[ S^\varphi v \cdot n = S_n + (S^\varphi v \cdot n) n = S_n + D^\varphi v \cdot n n \]

and we use Lemma 8.1. This yields (8.3).

To obtain (8.4), we first observe that by using again (8.2), we obtain that

\[ \| \partial_{zz} v \cdot n \|_k \leq \Lambda \left( \frac{1}{c_0}, \|v\|_{E^2, \infty} \right) (|h|_{k+\frac{3}{2}} + \| v \|_{k+2} + \| S_n \|_{k+1}). \]

and hence, we find by using (8.3) that

(8.9) \[ \| \partial_{zz} v \cdot n \|_k \leq \Lambda \left( \frac{1}{c_0}, \|v\|_{E^2, \infty} \right) (|h|_{k+\frac{3}{2}} + \| v \|_{k+2} + \| S_n \|_{k+1}). \]

To get the estimate of the other components of \( \partial_{zz} v \), it suffices to use again (8.6), (8.7) combined with the previous estimates. This ends the proof of Lemma 8.2.

The aim of the following Proposition is to prove the estimate on \( S_n \). Let us recall that \( \Lambda_\infty \) is defined in (7.3) in Proposition 7.1 and that \( \Lambda_0 = \Lambda(1/c_0) \).

**Proposition 8.3.** For every \( m \geq 2 \), the solution of (1.12), (1.13), (1.14) satisfies, for every \( t \in [0, T_\epsilon] \), the estimate

\[
\| S_n \|_{m-2}^2 + \varepsilon \int_0^t \| \nabla S_n \|_{m-2}^2 \leq \Lambda_0 \| S_n(0) \|_{m-2}^2 + \int_0^t \Lambda_\infty (\| V^m \|^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 + \| S_n \|_{m-2}^2) \]
\[
+ \int_0^t \Lambda_\infty \| \partial_z v \|_{m-1}^2 + \varepsilon \int_0^t \| \nabla V^m \|^2.
\]

Note that the last term on the right-hand side of (8.10) can be estimated by using Proposition 7.1. We also point out that in the above estimate, because of the dependence in \( |h|_m \) in the right-hand side, we cannot change \( m - 2 \) into \( m - 1 \). The occurrence of this term is mainly due to the Euler part of the pressure when we study the equation for \( S_n \). Indeed, when we perform energy estimates on the equation for \( S_n \), the Hessian \( D^2 p^E \) of the pressure is handled as a source term. Since \( p^E = gh \) on the boundary, we get that the estimate of \( \| D^2 p^E \|_k \) necessarily involves \( |h|_{k+\frac{1}{2}} \). This is why we are restricted to \( k \leq m - 3/2 < m - 1 \).

**Proof.** The first step is to find an equation for \( S_n \). From (1.12), we find the equation

\[
\partial_t S^\varphi v + (v \cdot \nabla^\varphi) \nabla^\varphi v + (\nabla^\varphi v)^2 + (D^\varphi)^2 q - \varepsilon \Delta^\varphi \nabla^\varphi v = 0
\]

where \((D^\varphi)^2 q \) stands for the Hessian matrix of the pressure, \((D^\varphi)^2 q)_{ij} = \partial_i^\varphi \partial_j^\varphi q \) and hence by taking the symmetric part of the equation, we get

(8.11) \[ \partial_t S^\varphi v + (v \cdot \nabla^\varphi) S^\varphi v + \frac{1}{2} ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2 + (D^\varphi)^2 q - \varepsilon \Delta^\varphi (S^\varphi v) = 0. \]

This allows to get an evolution equation for \( S_n = \Pi(S^\varphi v n) \):

(8.12) \[ \partial_t S_n + (v \cdot \nabla^\varphi) S_n - \varepsilon \Delta^\varphi (S_n) = F_S \]
where the source term $F_S$ is given by

\begin{equation}
F_S = F_S^1 + F_S^2,
\end{equation}

with

\begin{equation}
F_S^1 = -\frac{1}{2} \Pi ((\nabla^2 v)^2 + ((\nabla^2 v)^2)^2) \mathbf{n} + (\partial_i \Pi + v \cdot \nabla^2 \Pi) S^2 v \mathbf{n} + \Pi S^2 v (\partial_t \mathbf{n} + v \cdot \nabla^2 \mathbf{n})
\end{equation}

\begin{equation}
F_S^2 = -2\varepsilon \partial_i \Pi \partial_i (S^2 v \mathbf{n}) - 2\varepsilon \Pi (\partial_i (S^2 v) \partial_i \mathbf{n}) - \varepsilon (\Delta^2 \Pi) S^2 v \mathbf{n} - \varepsilon \Pi S^2 v \Delta^2 \mathbf{n} - \Pi ((D^2 q) q) \mathbf{n}.
\end{equation}

Note that we have used the summation convention for the last term. By using Proposition 3.1 and Propositions 6.4, 6.3, and the product estimates (2.1), we have that the source term $F_S^2$ is bounded by

\begin{equation}
\|F_S^1\|_{m-2} \leq \Lambda_\infty (\|\nabla^2 v\|_{m-2} + |h|_{m-\frac{1}{4}} + \|v\|_{m-2}).
\end{equation}

We recall that the above notation implicitly means that all the quantities are evaluated at time $t$; Note that by using Lemma (8.2), we can rewrite this estimate as

\begin{equation}
\|F_S^1\|_{m-2} \leq \Lambda_\infty (\|S_n\|_{m-2} + |h|_{m-\frac{1}{2}} + \|v\|_{m-2}).
\end{equation}

In a similar way, we get for $F_S^2$ that

\begin{equation}
\|F_S^2\|_{m-2} \leq \Lambda_\infty \varepsilon (\|\partial_z v\|_{m-2} + \|\partial_x v\|_{m-1} + \|v\|_{m-1} + |h|_{m-\frac{1}{2}}) + \Lambda_\infty \|\nabla q\|_{L^{m-1}}.
\end{equation}

Indeed, let us give more details on one example:

\begin{equation}
\|\varepsilon \partial_i \Pi \partial_i (S^2 v \mathbf{n})\|_{m-2} \lesssim \varepsilon \|\partial_i \Pi\|_{L^\infty} \|\partial_i (S^2 v \mathbf{n})\|_{m-2} + \varepsilon \|\partial_i (S^2 v \mathbf{n})\|_{L^\infty} \|\partial_i \Pi\|_{m-2}
\end{equation}

and the result follows by using (2.1) again (note that $\Lambda_\infty$ involves $\varepsilon \frac{\|\partial_z v\|_{L^\infty}}{2}$, but this improvement does not seem useful. Next, we can use Proposition 6.3 and Proposition 6.4 to estimate the pressure. This yields

\begin{equation}
\|F_S^2\|_{m-2} \leq \Lambda_\infty \varepsilon (\|\partial_z v\|_{m-2} + |h|_{m-\frac{1}{2}} + |v|_{m+\frac{1}{2}}) + \Lambda_\infty (\|v\|_{m} + |h|_{m-\frac{1}{2}})
\end{equation}

and by using Lemma 8.2 and Lemma 2.8, we can rewrite this last estimate in the alternative form

\begin{equation}
\|F_S^2\|_{m-2} \leq \Lambda_\infty \varepsilon (\|\nabla^2 S_n\|_{m-2} + |h|_{m+\frac{1}{2}} + |v|_{m+\frac{1}{2}}) + \Lambda_\infty (\|v\|_{m} + |h|_{m-\frac{1}{2}}).
\end{equation}

Note that thanks to the boundary condition (1.14), we have that $S_n$ verifies an homogeneous Dirichlet boundary condition on $z = 0$,

\begin{equation}
(S_n)_{z=0} = 0
\end{equation}

hence we shall be able to estimate $S_n$ through standard energy estimates. We shall first prove for $m \geq 2$ the following estimate by induction:

\begin{equation}
\|S_n(t)\|_{m-2}^2 + \varepsilon \int_0^t \|\nabla^2 S_n\|_{m-2}^2 \leq \Lambda_0 \|S_n(0)\|_{m-2}^2 + \int_0^t \Lambda_\infty (\|v\|_{L^m}^2 + |h|_{m+\frac{1}{2}}^2 + \|S_n\|_{m-2}^2) + \varepsilon \int_0^t |v|_{m+\frac{1}{2}} \|S_n\|_{m-2}^2.
\end{equation}

For the $L^2$ estimate (which corresponds to $m = 2$), by using the boundary condition (8.18) and Lemma 2.5, we obtain

\begin{equation}
\frac{d}{dt} \frac{1}{2} \int_S |S_n|^2 \, dv + \varepsilon \int_S |\nabla^2 S_n|^2 \, dv = \int_S F_S \cdot S_n \, dv.
\end{equation}
To conclude, we integrate in time and we use (8.16), (8.17) and the Young inequality to absorb the term \( \|\nabla^{\varphi} S_n\| \) in the left hand side. Note that thanks to (7.1), we can again use that

\[
\|\nabla^{\varphi} S_n(t)\|^2 \leq \Lambda_0 \int_\mathcal{S} |\nabla^{\varphi} S_n|^2 \, dv. \tag{8.20}
\]

Assuming that the estimate (8.19) is proven for \( k \leq m - 3 \), we shall prove that it is verified for \( m - 2 \). We first need to compute the equation satisfied by \( Z^{\alpha} S_n \) for \( |\alpha| \leq m - 2 \). By using the expression (5.19) for the transport part of the equation, we get that

\[
\partial_t^\varphi Z^{\alpha} S_n + (v \cdot \nabla^{\varphi}) Z^{\alpha} S_n - \varepsilon \Delta^{\varphi} Z^{\alpha}(S_n) = F_S + C_S \tag{8.21}
\]

where the commutator is given by

\[
\begin{aligned}
&\mathcal{C}_S = C_1^1 + C_2^2 \\
&\text{with}
&\quad C_1^1 = [Z^{\alpha} v_y] \cdot \nabla_y S_n + [Z^{\alpha}, V_z] \partial^{\alpha}_Z S_n := C_{S_y} + C_{S_z}, \quad C_2^2 = -\epsilon [Z^{\alpha}, \Delta^{\varphi}] S_n.
\end{aligned}
\]

Since \( Z^{\alpha} S_n \) vanishes on the boundary, we get by using Corollary 2.6 and a standard energy estimate that

\[
\begin{aligned}
&\frac{d}{dt} \frac{1}{2} \int_\mathcal{S} |Z^{\alpha} S_n|^2 \, dv + \varepsilon \int_\mathcal{S} |\nabla^{\varphi} Z^{\alpha} S_n|^2 \, dv = \int_\mathcal{S} (F_S + C_S) \cdot Z^{\alpha} S_n \, dv \\
&\text{and we need to estimate the right-hand-side. We shall begin with the part involving } C_S. \quad \text{Thanks to the decomposition (8.22), we first observe that thanks to (2.2)}
&\quad \|C_{S_y}\| \leq \Lambda_{\infty}(\|S_n\|_{m-2} + \|v\|_{E^{m-2}}).
\end{aligned} \tag{8.23}
\]

To estimate \( C_{S_z} \), we need to study more carefully the commutator in order to avoid the appearance of \( \|\partial^{\alpha}_Z S_n\|_k \) which is not controlled (and cannot be controlled uniformly in \( \epsilon \) due to the presence of boundary layers). By expanding the commutator and by using (5.8), we see that we have to estimate terms like

\[
\|Z^{\beta} V_z \partial^{\gamma}_Z S_n\|
\]

with \( |\beta| + |\gamma| \leq m - 2, |\gamma| \leq m - 3 \). Next, we shall use that

\[
Z^{\beta} V_z \partial^{\gamma}_Z S_n = \frac{1 - z}{z} Z^{\beta} V_z Z_3 Z^{\gamma} S_n
\]

and we can finally rewrite the above expression as a sum of terms under the form:

\[
\begin{aligned}
&c_\beta Z^{\beta} \left( \frac{1 - z}{z} V_z \right) Z_3 Z^{\gamma} S_n
\end{aligned} \tag{8.25}
\]

where \( c_{\beta} \) are harmless bounded functions and \( |\tilde{\beta}| \leq |\beta| \). Indeed, this comes from the fact that \( Z_3((1 - z)/z) = \tilde{c}(1 - z)/z \) for \( \tilde{c} \) a smooth bounded function.

If \( \tilde{\beta} = 0 \), we get that

\[
\begin{aligned}
\left| c_\beta Z^{\beta} \left( \frac{1 - z}{z} V_z \right) Z_3 Z^{\gamma} S_n \right| \lesssim \|S_n\|_{m-2}
\end{aligned}
\]

When \( \tilde{\beta} \neq 0 \) we can use (2.1) to obtain that

\[
\begin{aligned}
\left| c_\beta Z^{\beta} \left( \frac{1 - z}{z} V_z \right) Z_3 Z^{\gamma} S_n \right| \lesssim \|Z\left( \frac{1 - z}{z} V_z \right)\|_{L^n} \|S_n\|_{m-2} + \|S_n\|_{L^n} \|Z\left( \frac{1 - z}{z} V_z \right)\|_{m-3}
\end{aligned}
\]

Next, we observe that

\[
\|Z\left( \frac{1 - z}{z} V_z \right)\|_{L^n} \lesssim \|V_z\|_{W^{1,\infty}} + \|Z V_z\|_{W^{1,\infty}} \lesssim \|V_z\|_{E^{2,\infty}}
\]

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since thanks to \((5.20)\) and \((1.13)\), we have that \(V_z\) vanishes on the boundary. Moreover, again since \(V_z\) vanishes on the boundary, we shall prove that

\[
\|Z(\frac{1-z}{z}V_z)\|_{m-3} \lesssim \|ZV_z\|_{m-3} + \|\partial_z Z V_z\|_{m-3} + \|\partial_z V_z\|_{m-3}.
\]

(8.26)

Since

\[
\|Z(\frac{1-z}{z}V_z)\|_{m-3} \lesssim \|\frac{1-z}{z}ZV_z\|_{m-3} + \|\frac{1}{z(1-z)}V_z\|_{m-3},
\]

we have to estimate

\[
\|\frac{1-z}{z} Z^\beta Z V_z\|, \quad \|\frac{1}{z(1-z)} Z^\beta V_z\|, \quad |\beta| \leq m-3.
\]

By using the following variants of the Hardy inequality:

**Lemma 8.4.** If \(f(0) = 0\), we have the inequalities,

\[
\int_{-\infty}^{0} \frac{1}{z^2(1-z)^2} |f(z)|^2 \, dz \lesssim \int_{-\infty}^{0} |\partial_z f(z)|^2 \, dz,
\]

\[
\int_{-\infty}^{0} \frac{(1-z)^2}{z} |f(z)|^2 \lesssim \int_{-\infty}^{0} \left( |f(z)|^2 + |\partial_z f(z)|^2 \right) \, dz.
\]

The estimate (8.26) follows. Note that we had to be careful in these estimates since \(V_z\) is not in \(L^2\) while its derivatives are. The proof of this Lemma which only relies on an integration by parts is given in section \([14]\). Looking at the previous estimates, we have thus proven that

\[
\|Z^\alpha, V_z \partial_z S_n\| \leq \|V_z\|_{E^{2,\infty}} \|S_n\|_{m-2} + \|S_n\|_{L^\infty} \left( \|ZV_z\|_{E^{m-2}} + \|\partial_z V_z\|_{m-3} \right).
\]

Thanks to Proposition \((3.1)\), we have that \(\|V_z\|_{E^{2,\infty}} \leq \Lambda_{\infty}\). Moreover, by using Lemma \((5.3)\) and Remark \((5.4)\) we also find that

\[
\|ZV_z\|_{E^{m-2}} \leq \Lambda_{\infty} \left( \|v\|_{E^{m-1}} + |h|_{m-\frac{1}{2}} \right).
\]

We have thus proven the commutator estimate

\[
\|Z^\alpha, V_z \partial_z S_n\| \leq \Lambda_{\infty} \left( \|v\|_{E^{m-1}} + |h|_{m-\frac{1}{2}} + \|S_n\|_{m-2} \right)
\]

(8.27)

and hence, in view of \((8.28)\), this yields

\[
\|C_{S_{1}}^{2}\| \leq \Lambda_{\infty} \left( \|S_n\|_{m-2} + \|v\|_{E^{m-1}} + |h|_{m-\frac{1}{2}} \right).
\]

(8.28)

Now, we shall estimate the term involving \(C_{S_{2}}^{2}\). To expand this commutator, we can use the expression \((6.5)\) of \(\Delta^\varphi\). This yields

\[
C_{S_{2}}^{2} = C_{S_{1}}^{2} + C_{S_{2}}^{2} + C_{S_{3}}^{2}
\]

where

\[
C_{S_{1}}^{2} = \varepsilon [Z^\beta, \frac{1}{\partial_z \varphi}] \nabla \cdot (E \nabla S_n), \quad C_{S_{2}}^{2} = \varepsilon \frac{1}{\partial_z \varphi} [Z^\alpha, \nabla] \cdot (E \nabla S_n), \quad C_{S_{3}}^{2} = \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot ([Z^\alpha, E \nabla] S_n).
\]

By expanding the first commutator \(C_{S_{1}}^{2}\), we get that we need to estimate terms under the form

\[
\varepsilon \int_{S} Z^\beta \left( \frac{1}{\partial_z \varphi} \right) Z^\gamma (\nabla \cdot (E \nabla S_n)) \cdot Z^\alpha S_n \, dV_i
\]

with \(\beta + \tilde{\gamma} = \alpha, \beta \neq 0\) and next we can commute \(Z^\gamma\) and \(\nabla\) (we recall that \(Z_3\) and \(\partial_z\) does not commute) to reduce the problem to the estimate of

\[
\varepsilon \int_{S} Z^\beta \left( \frac{1}{\partial_z \varphi} \right) \partial_i Z^\gamma ((E \nabla S_n)_i) \cdot Z^\alpha S_n \, dV_i
\]
with $|\gamma| \leq |\tilde{\gamma}|$. Since $S_n$ vanishes on the boundary, we can integrate by parts to get

$$
\left| \varepsilon \int_S Z^\beta \frac{1}{\partial_x \varphi} \partial_x Z^\gamma ((E \nabla S_n)_j) \cdot Z^\alpha S_n \, dV_t \right| \leq \\
\Lambda_0 \left( \varepsilon \| Z^\beta \frac{1}{\partial_x \varphi} \partial_x Z^\gamma ((E \nabla S_n)_j) \|_\infty \| \nabla^\varphi Z^\alpha S_n \| + \| S_n \|_{m-2} \right) + \varepsilon \| \partial_t Z^\beta \frac{1}{\partial_x \varphi} \partial_x Z^\gamma ((E \nabla S_n)_j) \| \| S_n \|_{m-2} \right) \\
A_0 \left( \varepsilon \| Z^\beta \frac{1}{\partial_x \varphi} \partial_x Z^\gamma ((E \nabla S_n)_j) \|_\infty \| \nabla^\varphi Z^\alpha S_n \| + \| S_n \|_{m-2} \right) + \varepsilon \| \partial_t Z^\beta \frac{1}{\partial_x \varphi} \partial_x Z^\gamma ((E \nabla S_n)_j) \| \| S_n \|_{m-2} \right).
$$

Next, we can use (2.1), Proposition 3.1 and (6.8), (6.9) to obtain

$$
\varepsilon^{\frac{3}{2}} \left| Z^\beta \frac{1}{\partial_x \varphi} Z^\gamma ((E \nabla S_n)_j) \right| \leq \varepsilon^{\frac{3}{2}} \left( \| Z \frac{1}{\partial_x \varphi} \|_\infty \| E \nabla S_n \|_{m-3} + \| E \nabla S_n \|_\infty \left| Z \frac{1}{\partial_x \varphi} \right|_{m-3} \right)
$$

$$
\leq \varepsilon^{\frac{3}{2}} \Lambda_0 \| \nabla S_n \|_{m-3} + \Lambda_\infty \varepsilon \| S_n \|_{m-2}^{\frac{1}{2}}.
$$

Note that $\Lambda_\infty$ contains a control of $\sqrt{\varepsilon} \| \partial_x \varphi \|_\infty$ that we have used to get the last line. From the same arguments, we also get that

$$
\varepsilon \left| \partial_t Z^\beta \frac{1}{\partial_x \varphi} \partial_x Z^\gamma ((E \nabla S_n)_j) \right| \leq \varepsilon \Lambda_0 \| \nabla S_n \|_{m-3} + \Lambda_\infty \varepsilon \| S_n \|_{m-2}^{\frac{1}{2}}
$$

We have thus proven the estimate

$$
(8.29) \quad \left| \int_S C_{S_1}^2 \cdot Z^\alpha S_n \, dV_t \right| \leq \Lambda_0 \left( \varepsilon^{\frac{3}{2}} \| \nabla^\varphi Z^\alpha S_n \| + \| S_n \|_{m-2} \right)
$$

$$
(\varepsilon^{\frac{3}{2}} \| \nabla S_n \|_{m-3} + \| S_n \|_{m-2} + \Lambda_\infty (\| S_n \|_{m-2} + \varepsilon \| S_n \|_{m-2}^{\frac{1}{2}})).
$$

By using (5.8) we see that to handle the term involving $C_{S_2}^2$, we have to estimate

$$
\varepsilon \int_S \partial_t Z^\beta (E \nabla S_n) \cdot Z^\alpha S_n \, dydz
$$

with $|\beta| \leq m-3$, hence we perform an integration by parts as above to get that

$$
(8.30) \quad \left| \int_S C_{S_2}^2 \cdot Z^\alpha S_n \, dV_t \right| \leq \Lambda_0 \varepsilon \frac{1}{2} \| \nabla^\varphi Z^\alpha S_n \| (\varepsilon^{\frac{3}{2}} \| \nabla S_n \|_{m-3} + \Lambda_\infty (\| S_n \|_{m-2} + \varepsilon \| S_n \|_{m-2}^{\frac{1}{2}})).
$$

In a similar way, by integrating by parts, we get that

$$
(8.31) \quad \left| \int_S C_{S_3}^2 \cdot Z^\alpha S_n \, dV_t \right| \leq \varepsilon \| [Z^\alpha, E \nabla] S_n \| \| \nabla Z^\alpha S_n \|
$$

$$
\leq \Lambda_0 \varepsilon \frac{1}{2} \| \nabla^\varphi Z^\alpha S_n \| (\varepsilon^{\frac{3}{2}} \| \nabla S_n \|_{m-3} + \Lambda_\infty (\| S_n \|_{m-2} + \varepsilon \| S_n \|_{m-2}^{\frac{1}{2}})).
$$

In view of (8.29), (8.30), (8.31), we have actually proven that

$$
(8.32) \quad \left| \int_S C_{S_2}^2 \cdot Z^\alpha S_n \, dV_t \right| \leq \Lambda_0 \left( \varepsilon^{\frac{3}{2}} \| \nabla Z^\alpha S_n \| + \| S_n \|_{m-2} \right)
$$

$$
(\varepsilon^{\frac{3}{2}} \| \nabla S_n \|_{m-3} + \| S_n \|_{m-2} + \Lambda_\infty (\| S_n \|_{m-2} + \varepsilon \| S_n \|_{m-2}^{\frac{1}{2}})).
$$

To conclude our energy estimate, we can use the identity (8.23), the estimates (8.14), (8.15) and (8.28), (8.32), to obtain

$$
\frac{d}{dt} \frac{1}{2} \int_S |Z^\alpha S_n|^2 \, dV_t + \frac{\varepsilon}{2} \int_S |\nabla Z^\alpha S_n|^2 \, dV_t
$$

$$
\leq \Lambda_\infty (\| v \|_{E^m} + \| h \|_{m-2}^{\frac{1}{2}} + \varepsilon \| h \|_{m-2}^{\frac{1}{2}}) \left( \| S_n \|_{m-2} + \| h \|_{m-2}^{\frac{1}{2}} + \varepsilon \| S_n \|_{m-2} + \Lambda_0 \| \nabla S_n \|_{m-2} \right).
$$
Note that we have used the Young inequality and (8.20) for \( Z^\alpha S_n \) to absorb the term \( \| \nabla Z^\alpha S_n \|_{m-2} \) by the energy dissipation term in the left hand side. Next, we can integrate in time to obtain

\[
\| S_n \|_{m-2}^2 + \varepsilon \int_0^t | \nabla^\varepsilon S_n |_{m-2}^2 \\
\leq \Lambda_0 \| S_n(0) \|_{m-2}^2 + \int_0^t \left( \Lambda_{\infty}(\| v \|_{E^m} + | h |_{m-\frac{1}{2}} + \varepsilon \frac{1}{2}| h |_{m+\frac{1}{2}}) (\| S_n \|_{m-2} + \| h \|_{m-\frac{1}{2}}) \\
+ \varepsilon \int_0^t | v^\beta |_{m+\frac{1}{2}} | S_n |_{m-2} + \Lambda_0 \varepsilon \int_0^t \| \nabla S_n \|_{m-3}^2 
\]

and we finally get (8.19) by using the induction assumption to control \( \Lambda_0 \varepsilon \int_0^t \| \nabla S_n \|_{m-3}^2 \).

To end the proof of Proposition 8.3, we can use again (7.15) to get

\[
\frac{d}{dt} \| S_n \|_{m-2} \leq \int_0^t \| \nabla V^m \|_{m-2}^2 + \int_0^t \Lambda_{\infty}(\| V^m \|_{m-2} + \| S_n \|_{m-2} + \varepsilon | h |_{m+\frac{1}{2}}^2). 
\]

and we can use Lemma 2.8 to replace \( \| \nabla^\varepsilon S_n \|_{m-2} \) by \( \| \nabla S_n \|_{m-2} \) in the left hand side. This ends the proof of Proposition 8.3.

\[\Box\]

9. \( L^\infty \) ESTIMATES

In view of the estimate of Proposition 8.3 to close the argument, we need to estimate the \( L^\infty \) norms contained in \( \Lambda_{\infty} \) and \( \int_0^t \| \partial_z v \|_{m-1}^2 \). We shall first estimate the \( L^\infty \) norms in terms of the quantities in the left hand side of the estimate of Proposition 8.3 and Proposition 7.1.

We shall begin with the estimates which can be easily obtained through Sobolev embeddings.

**Proposition 9.1.** We have the following estimates:

- for \( k \in \mathbb{N} \),
  \[
  | h |_{k,\infty} + \sqrt{\varepsilon} | h |_{k+1,\infty} \lesssim | h |_{2+k} + \varepsilon \frac{1}{2}| h |_{2+k+\frac{1}{2}},
  \]
- for the velocity, we have
  \[
  \| v(t) \|_{2,\infty} \leq \Lambda \left( \frac{1}{c_0}, | h |_{4,\infty} + \| V^4 \| + \| S_n \|_{3} \right)
  \]
- and also
  \[
  \| \partial_z v \|_{1,\infty} \leq \Lambda_0 \left( \| S_n \|_{1,\infty} + \| v \|_{2,\infty} \right), \quad \sqrt{\varepsilon} \| \partial_z v \|_{L^\infty} \leq \Lambda_0 \left( \sqrt{\varepsilon} \| \partial_z S_n \|_{L^\infty} + \| S_n \|_{1,\infty} + \| v \|_{2,\infty} \right).
  \]

**Proof.** The estimate (9.1) is a direct consequence of the Sobolev embedding in dimension 2.

To get the estimate (9.2), we first use the anisotropic Sobolev embedding (2.4) to obtain that

\[
\| Z^\alpha v \|_{\infty} \lesssim \| \partial_z v \|_{3} \| v \|_{4}, \quad | \alpha | \leq 2
\]

and next we use (8.2), (8.6) and (8.7) to estimate \( \| \partial_z v \|_{3} \) in terms of \( \| S_n \|_{3} \). Note that here we estimate the product by putting always \( v \) in \( L^2 \) and the various coefficients in \( L^\infty \) (and thus, we use Proposition 3.1 to control them in \( L^\infty \)). This yields

\[
\| \partial_z v \|_{3} \leq \Lambda \left( \frac{1}{c_0}, | h |_{4,\infty} + \| S_n \|_{3} + \| v \|_{4} \right).
\]

Finally, by using the definition of \( V^\alpha = Z^\alpha v - \partial_z^\alpha v Z^\alpha h \), we also get that

\[
\| v \|_{4} \lesssim \| V^4 \| + \Lambda \left( \frac{1}{c_0}, | h |_{4,\infty} \right) \| \partial_z v \| \leq \Lambda \left( \frac{1}{c_0}, | h |_{4,\infty} \right) \left( \| V^4 \| + \| S_n \| + \| v \|_{1} \right).
\]

Next, by crude interpolation, we can write

\[
\| v \|_{1} \lesssim \| v \|_{\frac{3}{2}} \| v \|_{\frac{1}{2}}.
\]
Proposition 9.4. For \( v \) and since \( V \), we have
\[
\|v\|_4 \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{4,\infty} + \|V^4\| + \|S_n\| + \|v\| \right)
\]
and since \( V^0 = v \), we have actually proven that
\[
\|v\|_4 \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{4,\infty} + \|V^4\| + \|S_n\| \right).
\]
By plugging this estimate in (9.5) and (9.2), we finally obtain (9.2).

The estimates in (9.3) are also a consequence of (8.2) and (8.6), (8.7).

As a consequence of Proposition 9.1, we see that the only \( L^\infty \) norms that appear in the estimates of Proposition 7.1 and Proposition 8.3, which remain to be estimated are \( \|S_n\|_{1,\infty} \) and \( \varepsilon \frac{1}{2} \|\partial_z S_n\|_{L^\infty} \).

Moreover, by using the following Lemma, we see that actually we only have to prove estimates for \( \|\chi S_n\|_{1,\infty} \) and \( \varepsilon \frac{1}{2} \|\partial_z S_n\|_{L^\infty} \) for \( \chi(z) \) compactly supported and such that \( \chi = 1 \) in a vicinity of \( z = 0 \).

**Lemma 9.2.** For any smooth cut-off function \( \chi \) such that \( \chi = 0 \) in a vicinity of \( z = 0 \), we have for \( m > k + 3/2 \):
\[
(9.6) \quad \| \chi f \|_{W^{k,\infty}} \lesssim \| f \|_m.
\]

To get this Lemma, it suffices to use the Sobolev embedding of \( H^k(S) \) in \( L^\infty(S) \) for \( k > 3/2 \) and then to use the fact that away from the boundary the conormal Sobolev norm \( \| \cdot \|_k \) is equivalent to the standard \( H^k \) norm.

To summarize the estimates of Proposition 9.1 and the last remark, it is convenient to introduce the notation
\[
(9.7) \quad Q_m(t) = \|h(t)\|_{m}^2 + \varepsilon |h(t)|_{\frac{m+1}{2}}^2 + \|V^m\|_2^2 + \|S_n(t)\|_{m-2}^2 + \|S_n(t)\|_{1,\infty}^2 + \varepsilon \|\partial_z S_n\|_{L^\infty}^2.
\]

and use it to state

**Corollary 9.3.** For \( m \geq 6 \), we have the following estimates for the \( L^\infty \) norms:
\[
\|h(t)\|_{4,\infty} + \|v(t)\|_{2,\infty} + \|\partial_z v(t)\|_{1,\infty} + \sqrt{\varepsilon} \|\partial_z v(t)\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, Q_m(t) \right).
\]

The next step will be to estimate \( \|S_n\|_{1,\infty} \) and \( \varepsilon \frac{1}{2} \|\partial_z S_n\|_{L^\infty} \). Let us set
\[
(9.8) \quad \Lambda_{\infty,m}(t) = \Lambda \left( \frac{1}{c_0}, |h|_{4,\infty} + \|v\|_{L^2,\infty} + \varepsilon \frac{1}{2} \|\partial_z v\|_{L^\infty} + |h|_m + \|v\|_m + \|\partial_z v\|_{m-2} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right).
\]

Note that thanks to Corollary 9.3 and (8.3), we have
\[
(9.9) \quad \Lambda_{\infty,m}(t) \leq \Lambda \left( \frac{1}{c_0}, Q_m(t) \right), \quad m \geq 6.
\]

**Proposition 9.4.** For \( t \in [0,T^e] \), and \( m \geq 6 \), we have the estimate
\[
\|S_n(t)\|_{1,\infty} \leq \Lambda_{0} \left( \frac{1}{c_0}, |h(t)|_m + \|V^m(t)\| + \|S_n(t)\|_{m-2} \right)
\]
\[
+ \int_0^t \varepsilon \|\nabla V^m\|_2^2 + \varepsilon \|\nabla S_n\|_{m-2}^2 \right) + (1 + t) \int_0^t \Lambda_{\infty,m}(t).\]

Note that in the above estimate, the terms \( \Lambda \left( \frac{1}{c_0}, |h(t)|_m + \|V^m(t)\| + \|S_n(t)\|_{m-2} \right) \) and \( \int_0^t \varepsilon \|\nabla V^m\|_2^2 + \varepsilon \|\nabla S_n\|_{m-2}^2 \right) \) can be estimated by using the estimates of Proposition 7.1 and Proposition 8.3.
Proof. To estimate \( \| S_n \|_{1,\infty} \), we shall perform directly \( L^\infty \) estimates on the convection diffusion equation (8.12) solved by \( S_n \). As before, the main interest in the use of \( S_n \) is the fact that it verifies the homogeneous Dirichlet boundary condition (8.18) on the boundary.

The estimate of \( \| S_n \|_{L^\infty} \) is a direct consequence of the maximum principle for the convection diffusion equation (8.12) with homogeneous Dirichlet boundary. We find

\[
\| S_n \|_{L^\infty} \leq \| S_n(0) \|_{L^\infty} + \int_0^t \| F_S \|_{L^\infty}
\]

and from a brutal estimate, we get in view of the expressions (8.14), (8.15) that

\[
\| F_S \|_{L^\infty} \leq \Lambda_{\infty,m} + \Lambda_0 \| \nabla q \|_{E^{1,\infty}}.
\]

To estimate the pressure, we use (6.35) to get

\[
\|\nabla q^E\|_{E^{1,\infty}} \leq \Lambda_{\infty,m}
\]

and (6.30) and the Sobolev embedding in \( S \) to get

\[
\|\nabla q^NS\|_{E^{1,\infty}} \lesssim \|\nabla q^NS\|_{H^{3/2}} \leq \Lambda_\infty (\|\nabla v\|_2 + \|\varepsilon h\|_2).
\]

To control \( \|\varepsilon v\|_2 \), we use again (7.15) to get

\[
\|\nabla q^NS\|_{E^{1,\infty}} \lesssim \Lambda_{\infty,m} \varepsilon (\|\nabla V^k\| + \|V^k\| + |h|_{k+1}), \quad k \geq 4.
\]

Consequently, we obtain by using the Cauchy-Schwarz and Young inequalities that

\[
\| S_n(t) \|_{L^\infty} \leq \| S_n(0) \|_{L^\infty} + \varepsilon \int_0^t \| S^\varepsilon V^m \|^2 + (1 + t) \int_0^t \Lambda_{\infty,m}\|
\]

The next step is to estimate \( \| \chi ZS_n \|_{L^\infty} \). Indeed, from the definition of Sobolev conormal spaces, we get that

\[
\| ZS_n \|_{L^\infty} \lesssim \| \chi ZS_n \|_{L^\infty} + \| v \|_{2,\infty}
\]

and hence thanks to (9.2), we find

\[
\| ZS_n \|_{L^\infty} \lesssim \| \chi ZS_n \|_{L^\infty} + \Lambda (1 + h| m + \| V^m \| + \| S_n \|_{m-2}), \quad m \geq 6.
\]

The main difficulty is to handle the commutator between the fields \( Z_i \) and the Laplacian \( \Delta^\varepsilon \). As in our previous work [40], it is convenient for this estimate to use a coordinate system where the Laplacian has the simplest expression. We shall thus use a normal geodesic coordinate system in the vicinity of the boundary. Note that we have not used this coordinate system before because it requires more regularity for the boundary: to get an \( H^m \) (or \( C^m \)) coordinate system, we need the boundary to be \( H^{m+1} \) (or \( C^{m+1} \)). Nevertheless, at this step, this is not a problem since we want to estimate a fixed low number of derivatives of the velocity while we can assume that the boundary is \( H^m \) for \( m \) as large as we need. We shall choose the cut-off function \( \chi \) in order to get a well defined coordinate system in the vicinity of the boundary.

We define a different parametrization of the vicinity of the boundary of \( \Omega_t \) by

\[
\Psi(t,\cdot) : S = \mathbb{R}^2 \times (-\infty,0) \rightarrow \Omega_t
\]

\[
x = (y,z) \mapsto \left( \begin{array}{c}
y \\
h(t,y)
\end{array} \right) + zn^b(t,y)
\]

where \( n^b \) is the unit exterior normal \( n^b(t,y) = (-\partial_1 h, -\partial_2 h, 1)/|N| \). Note that \( D\Psi(t,\cdot) \) is under the form \( M_0 + R \) where \( |R|_{\infty} \lesssim \varepsilon |h|_{2,\infty} \) and

\[
M_0 = \begin{pmatrix}
1 & 0 & -\partial_1 h \\
0 & 1 & -\partial_2 h \\
\partial_1 h & \partial_2 h & 1
\end{pmatrix}
\]

with
is invertible. This yields that \( \Psi(t, \cdot) \) is a diffeomorphism from \( \mathbb{R}^2 \times (-\delta, 0) \) to a vicinity of \( \partial \Omega_t \) for some \( \delta \) which depends only on \( c_0 \), and for every \( t \in [0, T^\epsilon] \) thanks to (7.1). By this parametrization, the scalar product in \( \Omega_t \) induces a Riemannian metric on \( T \mathcal{S} \) which has the block structure

\[
(9.15) \quad g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}
\]

and hence, the Laplacian in this coordinate system reads:

\[
(9.16) \quad \Delta_y f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\tilde{g}} f
\]

where \( |g| \) denotes the determinant of the matrix \( g \) and \( \Delta_{\tilde{g}} \) which is defined by

\[
\Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1 \leq i, j \leq 2} \partial_{y'} (\tilde{g}^{ij} \frac{\partial}{\partial y'} f)
\]

involves only tangential derivatives. Note that the drawback of this system is that it depends on \( h \) and \( \nabla h \) via \( n \) thus it loses one degree of regularity.

To use this coordinate system, we shall first localize the equation for \( S^\varphi v \) in a vicinity of the boundary. Let us set

\[
(9.17) \quad S^\chi = \chi(z) S^\varphi v
\]

where \( \chi(z) = \kappa(\frac{z}{\delta(c_0)}) \in [0, 1] \) and \( \kappa \) is smooth compactly supported and takes the value 1 in a vicinity of \( z = 0 \). Note that this choice implies that

\[
(9.18) \quad |\chi^{(k)}(z)| \leq \Lambda_0
\]

for every \( k \geq 1 \).

Thank to (8.11), we get for \( S^\chi \) the equation

\[
(9.19) \quad \partial^\varphi S^\chi + (v \cdot \nabla^\varphi) S^\chi - \varepsilon \Delta^\varphi S^\chi = F_{S^\chi}
\]

where the source term \( F_{S^\chi} \) can be split into

\[
(9.20) \quad F_{S^\chi} = F^\chi + F_v
\]

with

\[
F^\chi = (V_z \partial_z \chi) S^\varphi v - \varepsilon \nabla^\varphi \chi \cdot \nabla^\varphi S^\varphi v - \varepsilon \Delta^\varphi \chi S^\varphi v,
\]

\[
F_v = -\chi (D^\varphi)^2 q - \frac{\chi}{2} ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2).
\]

Note that thanks to Lemma [5.3] and [9.6], since all the terms in \( F^\chi \) are supported away from the boundary, we get that

\[
(9.21) \quad \|F^\chi\|_{1, \infty} \leq \Lambda \left( \frac{1}{c_0}, \|v\|_{1, \infty} + \|h\|_{2, \infty} \right) \|v\|_5 \leq \Lambda_{\infty, m}
\]

Next, we define implicitly in \( \Omega_t \), \( \tilde{S} \) by \( \tilde{S}(t, \Phi(t, y, z)) = S^\chi(t, y, z) \) and then \( S^\Psi \) in \( \mathcal{S} \) by \( S^\Psi(t, y, z) = \tilde{S}(t, \Psi(t, y, z)) = S^\chi(t, \Phi(t, y, z)^{-1} \circ \Psi) \). The change of variable is well-defined since we can choose \( \tilde{S} \) to be supported in the domain where \( \Psi^{-1} \) is well defined by taking \( \delta \) sufficiently small. Since \( S^\chi \) solves (9.19), we get that \( \tilde{S} \) solves

\[
\partial_t \tilde{S} + u \cdot \nabla \tilde{S} - \varepsilon \Delta \tilde{S} = F_{S^\chi}(t, \Phi(t, \cdot)^{-1})
\]

in \( \Omega_t \) and hence by using (9.16), we get that \( S^\Psi \) solves in \( \mathcal{S} \) the convection diffusion equation

\[
(9.22) \quad \partial_t S^\Psi + w \cdot \nabla S^\Psi - \varepsilon (\partial_{zz} S^\Psi + \frac{1}{2} \partial_z (\ln |g|) \partial_z S^\Psi + \Delta_{\tilde{g}} S^\Psi) = F_{S^\chi}(t, \Phi^{-1} \circ \Psi)
\]
where the vector field \( w \) is given by
\[
(9.23) \quad w = \nabla(D\Psi)^{-1}(v(t, \Phi^{-1}\circ \Psi) - \partial_t \Psi)
\]
with \( D\Psi \) the jacobian matrix of \( \Psi \) (with respect to the space variables). Note that \( S^\Psi \) is compactly supported in \( z \) in a vicinity of \( z = 0 \). The function \( \chi(z) \) is a function with a slightly larger support such that \( \nabla S^\Psi = S^\Psi \). The introduction of this function allows to have \( w \) also supported in a vicinity of the boundary. We finally set
\[
(9.24) \quad S^\Psi_n(t, y, z) = \Pi^b(t, y)S^\Psi_n^b(t, y)
\]
with \( \Pi^b = \text{Id} - n^b \otimes n^b \). Note that \( \Pi^b \) and \( n^b \) are independent of \( z \). This yields that \( S^\Psi_n \) solves
\[
(9.25) \quad \partial_t S^\Psi_n + w \cdot \nabla S^\Psi_n - \varepsilon(\partial_{zz} + \frac{1}{2}\partial_z(\ln |g|)\partial_z)S^\Psi_n = F^\Psi_n
\]
where
\[
(9.26) \quad F^\Psi_n = \left(\Pi^b F_{\nabla n}^b + F^\Psi_n^{1,1} + F^\Psi_n^{2,1}\right).
\]
with
\[
(9.27) \quad F^\Psi_n^{1,1} = (\partial_t + w_y \cdot \nabla_y)S^\Psi_n^b + \Pi^b S^\Psi_n^b \partial_t + w_y \cdot \nabla_y)n^b,
\]
\[
(9.28) \quad F^\Psi_n^{2,2} = -\varepsilon\Pi^b(\Delta g S^\Psi_n^b)n^b.
\]

We shall thus estimate \( S^\Psi_n \) which solves the equation \((1.10) \) in \( S \) with the homogeneous Dirichlet boundary condition \((S^\Psi_n)/_{z=0} = 0 \) (indeed, on the boundary \( z = 0 \), we have that \( S^\Psi_n = \Pi S^\sharp \nu n = S_n \)).

In the following, we shall use very often the following observation:

**Lemma 9.5.** Consider \( T : S \rightarrow S \) such that \( T(y, 0) = y, \forall y \in \mathbb{R}^2 \) and let \( g(x) = f(Tx) \). Then, we have for every \( k \geq 1 \), the estimate:
\[
\|g\|_{k,\infty} \leq \Lambda(\|\nabla T\|_{k-1,\infty})\|f\|_{k,\infty}.
\]

This is just a statement of the fact that Sobolev conormal spaces are invariant by diffeomorphisms which preserve the boundary. The proof is just a consequence of the chain rule and the fact that the family \((Z_1, Z_2, Z_3)\) generates the set of vector fields tangent to the set of \( \partial S \). A similar statement holds for the \( \|\cdot\|_n \) norm. Note that it is equivalent to estimate \( S^\Psi_n \) or \( S_n \). Indeed, since \( S^\Psi_n = \Pi^b S^\Psi(t, \Phi^{-1}\circ \Psi)n^b \), the above lemma yields
\[
\|S^\Psi_n\|_{1,\infty} \leq \Lambda(\|h\|_{2,\infty})\|\Pi^b S^\Psi \nu n^b\|_{1,\infty}
\]
and we observe that \( \|\Pi - \Pi^b\| + |n - n^b| = \mathcal{O}(z) \) in the vicinity of the boundary to get
\[
(9.29) \quad \|S^\Psi_n\|_{1,\infty} \leq \Lambda_0(\|S_n\|_{1,\infty} + \|v\|_{2,\infty}).
\]

From the same arguments, we also obtain that
\[
\|S_n\|_{1,\infty} \leq \Lambda_0(\|S^\Psi_n\|_{1,\infty} + \|v\|_{2,\infty})
\]
\[
(9.30) \quad \leq \Lambda_0\left(\|S^\Psi_n\|_{1,\infty} + \Lambda\left(\frac{1}{c_0}|h|_m + \|V^m\| + \|S_n\|_{m-2}\right)\right), \quad m \geq 6
\]
where the last estimate comes from \((9.2) \). By using repeatedly these arguments, we also obtain
\[
(9.31) \quad \|w\|_{1,\infty} \leq \Lambda\left(\frac{1}{c_0}|h|_{3,\infty} + \|v\|_{1,\infty} + \|\partial_t \Psi\|_{1,\infty}\right) \leq \Lambda\left(\frac{1}{c_0}|h|_{3,\infty} + \|v\|_{2,\infty} + \|\partial_t h\|_{2,\infty}\right) \leq \Lambda_{\infty, m}
\]
where \( \Lambda_{\infty, m} \) is defined by \((9.8) \). Note that we have used the boundary condition \((1.3) \) to get the last part of the estimate.
To estimate $Z_i S_n^\Psi = \partial_i S_n^\Psi$, $i = 1, 2$, we can proceed as in the proof of estimate (9.12). We first apply $\partial_i$, $i = 1, 2$ to (9.25) to get

$\tag{9.32} \partial_i \partial_j S_n^\Psi + w \cdot \nabla \partial_i S_n^\Psi - \varepsilon (\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z) \partial_i S_n^\Psi = \partial_i F_n^\Psi - \partial_i w \cdot \nabla S_n^\Psi - \frac{\varepsilon}{2} \partial_z S_n^\Psi \partial^2_{zz} (\ln |g|).$

From the maximum principle, we thus get that

$\| \partial_i S_n^\Psi (t) \|_{L^\infty} \leq \| \partial_i S_n^\Psi (0) \|_{L^\infty} + \int_{t_0}^t \left( \| \partial_i F_n^\Psi \|_{L^\infty} + \| \partial_i w \cdot \nabla S_n^\Psi \|_{L^\infty} + \Lambda \left( \frac{1}{c_0}, |h|_{3, \infty} + \varepsilon \| \partial_3 S_n^\Psi \| \right) \right) dt$.

$\tag{9.33}$

Note that for the last term, we have used that $\Psi$ involves two derivatives of $h$ but is $C^\infty$ in $z$ in view of the definition (9.14) and hence that $\partial^2_{zz} |g|$ has the regularity of $\partial_3 |g|$.

It remains to estimate the right-hand side in this last estimate. We first note that

$\| \partial_i w \cdot \nabla S_n^\Psi \|_{L^\infty} \leq \| w \|_{1, \infty} \| S_n^\Psi \|_{1, \infty} + \| \partial_i w_3 \partial_3 S_n^\Psi \|_{L^\infty} \leq \Lambda_{m, m} + \| \partial_i w_3 \partial_3 S_n^\Psi \|_{L^\infty}.$

Next, thanks to the definition (9.28) of $w$, we note that on the boundary, we have

$\tag{9.34} w^b = (D\Psi(t, y, 0))^{-1} \left( v^b - \left( 0 \atop \partial_3 h \right) \right).$

Since on the boundary, we have

$D\Psi(t, y, 0) = \begin{pmatrix} 1 & 0 & n_1^b \\ 0 & 1 & n_2^b \\ \partial_3 h & \partial_3 h & n_3^b \end{pmatrix},$

we get that

$\tag{9.35} ((D\Psi(t, y, 0))^{-1} Y)_3 = Y \cdot n^b$

for every $Y \in \mathbb{R}^3$ and hence in particular that on the boundary

$\tag{9.36} w^b_3 = v^b \cdot n - \partial_3 h = \frac{1}{|N|} (v^b \cdot N - \partial_3 h) = 0$

thanks to the boundary condition (1.13). Consequently, since $\partial_3 w_3$ also vanishes on the boundary we get that

$\tag{9.37} \| \partial_i w_3 \partial_3 S_n^\Psi \|_{L^\infty} \leq \| \partial_3 \partial_3 S_n^\Psi \|_{L^\infty} \| S_n^\Psi \|_{1, \infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{3, \infty} + \| v \|_{E^2, \infty} \right) \leq \Lambda_{m, m}.$

Note that for the last estimate, we have used again that $\Psi$ is $C^\infty$ in $z$. It remains to estimate $\| F_n^\Psi \|_{1, \infty}$ in the right-hand side of (9.33). By using (9.26) and the expressions (9.27), (9.28), we get

$\| F_n^\Psi,1 \|_{1, \infty} \leq \Lambda \left( \frac{1}{c_0}, |\partial_3 h|_{2, \infty} + |h|_{3, \infty} + \| v \|_{1, \infty} \right) \| v \|_{E^2, \infty} + \varepsilon \| \partial_3 v \|_{2, \infty} \right) \leq \Lambda_{m, m}$

and by using again the fact that $|\Pi - \Pi^b| + |n - n^b| = O(z)$, we get that

$\| F_n^\Psi,2 \|_{1, \infty} \leq \Lambda_{m, m} (\varepsilon \| S_n \|_{3, \infty} + \varepsilon \| v \|_{4, \infty}).$

Next, from the definitions (9.26), (9.20), we get thanks to (9.21) that

$\| F_n^\Psi \|_{1, \infty} \leq \Lambda_{m, m} \left( 1 + \varepsilon \| S_n \|_{3, \infty} + \varepsilon \| v \|_{4, \infty} + \| \Pi^b ((D^e)^2 q) n^b \|_{1, \infty} \right).$

To estimate the pressure term, we use that $\Pi \nabla^e$ involves only conormal derivatives and the decomposition (9.2) to obtain

$\| \Pi^b ((D^e)^2 q) n^b \|_{1, \infty} \leq \Lambda_0 \left( \| \nabla q^E \|_{2, \infty} + \| \nabla q^{NS} \|_{2, \infty} \right).$
For the Euler part \( q^E \) of the pressure, we use (6.35) to get \( \| \nabla q^E \|_{2,\infty} \leq \Lambda_\infty \) and for \( q^NS \), we can use again Proposition 6.3 to get that an estimate analogous to (9.11) holds for \( \| \nabla q^NS \|_{2,\infty} \) but with the restriction \( k \geq 5 \) for the right hand side. This yields

\[
\| \Pi^0 ((D^r)^2 q)^n \|_{1,\infty} \leq \Lambda_0 (1 + \varepsilon \| S^r V^m \|).
\]

We have thus proven that

\[
(9.38) \quad \| F_n \|_{1,\infty} \leq \Lambda_{\infty,m}(1 + \varepsilon \| \nabla V^m \| + \varepsilon \| S_n \|_{3,\infty} + \varepsilon \| v \|_{4,\infty}).
\]

Consequently, by combining (9.33), (9.37) and (9.38), we obtain

\[
(9.39) \quad \| \partial_t S_n^\psi(t) \|_{L^\infty} \leq \| \partial_t S_n^\psi(0) \|_{L^\infty} + \int_0^t \Lambda_{\infty,m}(1 + \varepsilon \| S^r V^m \| + \varepsilon \| S_n \|_{3,\infty} + \varepsilon \| v \|_{4,\infty}).
\]

The next step is to estimate \( \| Z_3 S_n^\psi \|_{L^\infty} \). This is more delicate due to the bad commutation between \( Z_3 \) and \( \varepsilon \partial_{zz} \). We shall use a more precise description of the solution of (9.25). First, it is convenient to eliminate the term \( \varepsilon \partial_t \ln |g| \partial_\nu \) in the equation (9.25). We set

\[
(9.40) \quad \rho(t, y, z) = |g|^{1/4} S_n^\psi = |g|^{1/4} \Pi^0(t, y) S_n^\psi n^1(t, y)
\]

This yields that \( \rho \) solves

\[
(9.41) \quad \partial_t \rho + w \cdot \nabla \rho - \varepsilon \partial_{zz} \rho = |g|^{1/4} \left( F_n^\psi + F_g \right) := \mathcal{H}
\]

where

\[
(9.42) \quad F_g = \frac{\rho}{|g|^{1/2}} \left( \partial_t + w \cdot \nabla - \varepsilon \partial_{zz} \right) |g|^{1/4}.
\]

Since

\[
(9.43) \quad \| Z_3 S_n^\psi \|_{L^\infty} \leq \Lambda_0 \| \rho \|_{1,\infty}, \quad \| \rho \|_{1,\infty} \leq \Lambda_0 \| S_n^\psi \|_{1,\infty}
\]

it is equivalent to estimate \( \| Z_3 S_n^\psi \|_{1,\infty} \) or \( \| \rho \|_{1,\infty} \). We shall thus estimate \( \rho \) which solves the equation (9.41) in \( \mathcal{S} \) with the homogeneous Dirichlet boundary condition \( \rho_{/z=0} = 0 \).

This estimate will be a consequence of the following Lemma:

**Lemma 9.6.** Consider \( \rho \) a smooth solution of

\[
(9.44) \quad \partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{H}, \quad z < 0, \quad \rho(t, y, 0) = 0, \quad \rho(t = 0) = \rho_0
\]

for some smooth vector field \( w \) such that \( w_3 \) vanishes on the boundary. Assume that \( \rho \) and \( \mathcal{H} \) are compactly supported in \( z \). Then, we have the estimate:

\[
\| Z_i \rho(t) \|_{\infty} \lesssim \| Z_i \rho_0 \|_{\infty} + \| \rho_0 \|_{\infty} + \int_0^t \left( (\| w \|_{E^2,\infty} + \| \partial_{zz} w_3 \|_{L^\infty}) \left( \| \rho \|_{1,\infty} + \| \rho \|_{4} \right) + \| \mathcal{H} \|_{1,\infty} \right).
\]

We have already used an estimate of the same type in our previous work [40]. The proof of the Lemma is given in section 14.1. From Lemma 9.6 we get that

\[
(9.45) \quad \| Z_3 \rho(t) \|_{\infty} \lesssim \| Z_3 \rho_0 \|_{\infty} + \| \rho_0 \|_{\infty} + \int_0^t \left( (\| w \|_{E^2,\infty} + \| \partial_{zz} w_3 \|_{L^\infty}) \left( \| \rho \|_{1,\infty} + \| \rho \|_{4} \right) + \| \mathcal{H} \|_{1,\infty} \right).
\]

Next, by using (9.38) and (9.42), we obtain

\[
\| \mathcal{H} \|_{1,\infty} \leq \Lambda_{\infty,m}(1 + \varepsilon \| S^r V^m \| + \varepsilon \| S_n \|_{3,\infty} + \varepsilon \| v \|_{4,\infty}) + \Lambda \left( \frac{1}{c_0}, \| h \|_{4,\infty} + \| v \|_{1,\infty} \right)
\]

\[
(9.46) \quad \leq \Lambda_{\infty,m}(1 + \varepsilon \| S^r V^m \| + \varepsilon \| S_n \|_{3,\infty} + \varepsilon \| v \|_{4,\infty}).
\]

Hence, from (9.40), we get that

\[
\| \rho \|_{4} \leq \Lambda \left( \frac{1}{c_0}, \| h \|_{6} + \| S_n^\psi \|_{4} \right) \leq \Lambda_{\infty,m}
\]
since we assume that \( m \geq 6 \) and from the definition \( (9.23) \), we also have \( \|w\|_{E^2,\infty} \leq \Lambda_{\infty,m} \). We only need to be more careful with the term \( \|\partial_z w_3\|_{L^\infty} \) in the right hand side of \( (9.45) \) since it contains two normal derivatives. Looking at the expression \( (9.23) \) of \( w \), we first note that since \( \Psi \in C^\infty \) in \( z \), we have
\[
\left\| \partial_z (\nabla (D\Psi^{-1} \partial_t \Psi)) \right\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, |h|_{2,\infty} + |\partial_t h|_{1,\infty} \right) \leq \Lambda_{\infty,m}
\]
therefore, we only need to estimate \( \partial_z (\nabla D\Psi^{-1} v(t, \Phi^{-1} \circ \Psi)) \). We can first use that
\[
\left\| \partial_z (\nabla D\Psi^{-1} v(t, \Phi^{-1} \circ \Psi))_3 \right\|_{L^\infty} \leq \left\| \nabla (\nabla (D\Psi(t, y, \Phi^{-1} \circ \Psi) \circ \Psi) \circ \Psi) \right\|_{L^\infty} + \Lambda_{\infty,m} \|v\|_{E^2,\infty}
\]
and then use the observation \( (9.35) \) to get that
\[
\left\| \partial_z (\nabla D\Psi^{-1} v(t, \Phi^{-1} \circ \Psi))_3 \right\|_{L^\infty} \leq \left\| \nabla \partial_z \left( v(t, \Phi^{-1} \circ \Psi) \cdot n^b \right) \right\|_{L^\infty} + \Lambda_{\infty,m}.
\]
We thus need to compute
\[
\nabla \partial_z \left( v(t, \Phi^{-1} \circ \Psi) \cdot n^b \right).
\]
Note that \( v(t, \Phi^{-1} \circ \Psi) = u(t, \Psi) \) where \( u \) is defined in \( \Omega_t \) and let us set \( u^\Psi(t, y, z) = u(t, \Psi) \). Since \( u \) is divergence free in \( \Omega_t \), the expression in local coordinates of the divergence yields
\[
\partial_z u^\Psi \cdot n^b = -\frac{1}{2} \partial_z (\ln |g|) u^\Psi \cdot n^b - \nabla g \cdot u^\Psi
\]
where \( u^\Psi_y = \Pi^b u^\Psi \) and \( \nabla g \cdot \cdot \) is the divergence operator associated to the metric \( \tilde{g} \) in the definition \( (9.15) \) and hence involves only tangential derivatives of \( u^\Psi \). This means as before that for \( u^\Psi \cdot n^b = v(t, \Phi^{-1} \circ \Psi) \cdot n^b \), we can replace one normal derivative by tangential derivatives. Consequently, we get from \( (9.23) \) that
\[
\left\| \partial_z w_3 \right\|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0}, \left\| u^\Psi \right\|_{E^1,\infty} + \|h\|_{3,\infty} + |\partial_t h|_{L^\infty} \right) \leq \Lambda_{\infty,m}.
\]
Note that we have again used the fact that \( \Psi \in C^\infty \) in \( z \). Finally, we note that
\[
\|\rho\|_4 \leq \Lambda \left( \frac{1}{c_0}, |h|_6 + \|S_n\|_4 + \|v\|_5 \right) \leq \Lambda_{\infty,m}, \quad \|Z_n \eta_n\|_\infty + \|\eta_0\|_\infty \leq \Lambda_0 \|v(0)\|_{E^2,\infty},
\]
and hence we obtain from \( (9.45) \) that
\[
\|\rho(t)\|_{1,\infty} \leq \Lambda_0 \|v\|_{E^2,\infty} + \int_0^t \Lambda_0 \left( 1 + \varepsilon \|S^\rho V^m\| + \varepsilon \|S_n\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} \right), \quad m \geq 5.
\]
Consequently, by combining \( (9.48) \), \( (9.39) \) and \( (9.43) \), \( (9.30) \), we get that
\[
\|S_n\|_{1,\infty} \leq \Lambda_0 \|v(0)\|_{E^2,\infty} + \Lambda \left( \frac{1}{c_0}, \|V^m(t)\| + \|S_n(t)\|_{m-2} + |h(t)|_m \right)
\]
\[
+ \int_0^t \Lambda_{\infty,m} \left( 1 + \varepsilon \|\nabla V^m\| + \varepsilon \|S_n\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} \right).
\]
To conclude, we use Sobolev estimates to control the last two terms. At first, thanks to \( (2.4) \), we get that
\[
\sqrt{\varepsilon} \|S_n\|_{3,\infty} \leq \left( \sqrt{\varepsilon} \|\partial_z S_n\|_4 \right)^{\frac{1}{2}} \left( \sqrt{\varepsilon} \|S_n\|_5 \right)^{\frac{1}{2}} \leq \Lambda_{\infty,m} \left( \sqrt{\varepsilon} \|\nabla S_n\|_4 \right)^{\frac{1}{2}} \left( \sqrt{\varepsilon} \|\nabla v\|_4 \right)^{\frac{1}{2}}
\]
and we use that for \( \alpha \neq 0, \ |\alpha| \leq 4, \)
\[
\sqrt{\varepsilon} \|\nabla Z^\alpha v\| \leq \sqrt{\varepsilon} \|\nabla V^\alpha\| + \Lambda_{\infty,m}.
\]
Indeed, note that in particular, the term \( \sqrt{\varepsilon} \|\partial_z v\|_{L^\infty} \) is in \( \Lambda_{\infty,m} \). Hence, we have proven that
\[
\sqrt{\varepsilon} \|S_n\|_{3,\infty} \leq \Lambda_{\infty,m} + \sqrt{\varepsilon} \|\nabla V^m\| + \sqrt{\varepsilon} \|\nabla S_n\|_{m-2}.
\]
From the same arguments, we also obtain
\begin{equation}
\sqrt{\varepsilon} \| v \|_{4, \infty} \leq \sqrt{\varepsilon} (\| \nabla v \|_5 + \| v \|_6) \leq \Lambda_{\infty, m} + \sqrt{\varepsilon} \| \nabla V^m \|.
\end{equation}
for \( m \geq 6 \). Consequently, we get from (9.49) and the Cauchy-Schwarz inequality that for \( m \geq 6 \)
\begin{equation}
\| S_n \|_{1, \infty} \leq \Lambda_0 \| v(0) \|_{E^{2, \infty}} + \Lambda \left( \frac{1}{c_0}, \| V^m(t) \| + \| S_n(t) \|_{m-2} + |h(t)|_m \right) + (1 + t) \int_0^t \Lambda_{\infty, m} + \int_0^t (\varepsilon \| \nabla V^m \|^2 + \varepsilon \| \nabla S_n \|^2_m).
\end{equation}
This ends the proof of Proposition 9.4.

It will be useful in the future to remember the estimates (9.50), (9.51) that we have just establish:

**Lemma 9.7.** For \( m \geq 6 \), we have the estimate
\begin{equation}
\sqrt{\varepsilon} \int_0^t \| \nabla v \|_{3, \infty} \leq \int_0^t \Lambda_{\infty, m} + \int_0^t (\varepsilon \| \nabla V^m \|^2 + \varepsilon \| \nabla S_n \|^2_m).
\end{equation}
We recall that \( \Lambda_{\infty, m} \) was defined in (9.8). To get the proof of this lemma, it suffices to use that
\[ \| \nabla v \|_{3, \infty} \leq \Lambda_{\infty, m} (\| S_n \|_{3, \infty} + \| v \|_{4, \infty}) \]
thanks to (8.6), (8.7) and (8.2) and then to use (9.50), (9.51) and the Young inequality.

Our next \( L^\infty \) estimate will be the one of \( \sqrt{\varepsilon} \| \partial_z S_n \|_{L^\infty} \) which is still missing.

**Proposition 9.8.** Assuming that the initial data satisfy the boundary condition \( \partial_z \eta \), we have for \( m \geq 6 \) the estimate
\begin{equation}
\varepsilon \| \partial_z v(t) \|_{L^\infty} \leq \Lambda_0 (\| v(0) \|_{E^{2, \infty}} + \varepsilon \| \partial_z v_0 \|_2^2) + \Lambda \left( \frac{1}{c_0}, |h(t)|_m + \| V^m(t) \| + \| S_n(t) \|_{m-2} \right) + \Lambda_0 \int_0^t \varepsilon (\| \nabla S_n \|^2_{m-2} + \| \nabla V^m \|^2) + (1 + t) \int_0^t (1 + \frac{1}{\sqrt{1 - \tau}}) \Lambda_{\infty, m} d\tau
\end{equation}
for \( t \in [0, T^\varepsilon] \).

Note that in our estimates, this is the only place where we use the compatibility condition on the initial data. We also again point out that the terms \( \Lambda (\frac{1}{c_0}, |h(t)|_m + \| V^m(t) \| + \| S_n(t) \|_{m-2}) \) and \( \int_0^t \varepsilon (\| \nabla S_n \|^2_{m-2} + \| \nabla V^m \|^2) \) can be estimated by using Proposition 9.1 and Proposition 8.3.

**Proof.** As in the proof of Proposition 9.4, we shall first reduce the problem to the estimate of \( \sqrt{\varepsilon} \| \partial_z \rho \|_{L^\infty} \). By using (9.21), and (9.30), we obtain
\[ \sqrt{\varepsilon} \| \partial_z S_n \|_{L^\infty} \leq \Lambda_0 (\| v \|_{E^{2, \infty}} + \sqrt{\varepsilon} \| \partial_z \rho \|_{L^\infty}) \]
and hence, by using again Proposition 9.1 and in particular (9.3) and (9.2), we obtain that
\begin{align}
\sqrt{\varepsilon} \| \partial_z S_n \|_{L^\infty} & \leq \Lambda_0 (\| S_n \|_{1, \infty} + \| v \|_{2, \infty} + \sqrt{\varepsilon} \| \partial_z \rho \|_{L^\infty}) \\
& \leq \Lambda_0 \left( \| S_n \|_{1, \infty} + \Lambda \left( \frac{1}{c_0}, |h|_m + \| V^m \| + \| S_n \|_{m-2} \right) + \sqrt{\varepsilon} \| \partial_z \rho \|_{L^\infty} \right)
\end{align}
(9.53) so that it only remains to estimate \( \sqrt{\varepsilon} \| \partial_z \rho \|_{L^\infty} \). Again note that we use the fact that \( g \) is \( C^\infty \) in \( z \) to write the first estimate. Since \( \eta \) solves the convection-diffusion equation (9.41) in \( z > 0 \) with zero Dirichlet boundary condition on the boundary, we can use the one-dimensional heat kernel of \( z > 0 \)
\[ G(t, z, z') = \frac{1}{\sqrt{4\pi t}} (e^{-\frac{(z-z')^2}{4t}} - e^{-\frac{(z+z')^2}{4t}}) \]
Consequently, by combining the two last estimates and (9.55), we get that 

\[ \varepsilon \frac{1}{2} \partial_v \rho(t, y, z) = \sqrt{\varepsilon} \int_{0}^{+\infty} \partial_z G(t, z, z') \rho_0(y, z') dz' + \int_{0}^{t} \sqrt{\varepsilon} \partial_z G(t - \tau, z, z') (H(\tau, y, z') - w \cdot \nabla \rho) dz' d\tau. \]

Since \( \rho_0 \) vanishes on the boundary thanks to the compatibility condition, we can integrate by parts the first term to obtain 

\[ \varepsilon \frac{1}{2} \| \partial_v \rho(t) \|_{L^\infty} \leq \varepsilon \frac{1}{2} \| \partial_z \rho_0 \|_{L^\infty} + \int_{0}^{t} \frac{1}{\sqrt{t - \tau}} (\| H \|_{L^\infty} + \| w \cdot \nabla \rho \|_{L^\infty}). \]  

Next, we use the definition of the source term \( H \) in (9.41) to get 

\[ \| H \|_{L^\infty} \leq \Lambda_{\infty, m} (1 + \| \nabla q \|_{E^1, \infty} + \varepsilon(\| S_n \|_{2, \infty} + \| v \|_{3, \infty})) \]

and hence, thanks to Proposition 9.4 and 9.10, we get 

\[ \| H \|_{L^\infty} \leq \Lambda_{\infty, m} (1 + \varepsilon \| v \|_{2} + \varepsilon(\| S_n \|_{2, \infty} + \| v \|_{3, \infty})). \]

By using the trace inequality (2.3) and (2.4), we find for \( m \geq 6 \)

\[ \varepsilon \| v \|_{2} \leq \varepsilon \| \partial_z v \|_{4} \| v \|_{0}^{\frac{1}{2}} \leq \Lambda_{\infty, m} \varepsilon \| \nabla v \|_{2}^{\frac{1}{8}} \leq \Lambda_{\infty, m} (1 + \varepsilon \| \nabla v \|_{2}^{\frac{1}{8}}) \]

and 

\[ \varepsilon \| S_n \|_{2, \infty} \leq \varepsilon \| \nabla S_n \|_{\frac{1}{2}} \| S_n \|_{4} \leq \Lambda_{\infty, m} \varepsilon \| \nabla S_n \|_{\frac{1}{2}} \| S_n \|_{m-2}, \]

\[ \varepsilon \| v \|_{3, \infty} \leq \varepsilon \| \nabla v \|_{2}^{\frac{1}{2}} \| v \|_{0}^{\frac{1}{2}} \leq \Lambda_{\infty, m} (1 + \varepsilon \| \nabla v \|_{2}). \]

Consequently, we have proven that 

\[ \| H \|_{L^\infty} \leq \Lambda_{\infty, m} (1 + \| \nabla v \|_{2}^{\frac{1}{8}} + \| \nabla S_n \|_{\frac{1}{2}} \| S_n \|_{m-2}) \]

Finally, by using (9.36), we can write as we have used to get (9.37) that 

\[ \| w \cdot \nabla \rho \|_{L^\infty} \leq \| w \|_{E^1, \infty} \| \rho \|_{1, \infty} \leq \Lambda_{\infty, m}. \]

Consequently, by combining the two last estimates and (9.55), we get that 

\[ \varepsilon \frac{1}{2} \| \partial_v \rho(t) \|_{L^\infty} \leq \Lambda_0 (\varepsilon \frac{1}{2} \| \partial_z v(0) \|_{L^\infty} + \| v(0) \|_{E^2, \infty}) + \int_{0}^{t} \frac{\Lambda_0 \varepsilon \| \nabla S_n \|_{m-2} + \varepsilon \| \nabla v \|_{2}^{\frac{1}{8}}}{(t - \tau)^{\frac{1}{2}}} d\tau. \]

To conclude, we use the Hölder inequality and in particular that 

\[ \left( \int_{0}^{t} \frac{\Lambda_0}{(t - \tau)^{\frac{1}{2}}} \| \nabla S_n \|_{m-2} \right)^{\frac{1}{2}} \left( \int_{0}^{t} \frac{1}{(t - \tau)^{\frac{1}{2}}} d\tau \right)^{\frac{1}{2}} \]

and a similar estimate for the term involving \( \| \nabla v \|_{2}^{\frac{1}{8}} \) to get the estimate 

\[ (\varepsilon \frac{1}{2} \| \partial_v \rho(t) \|_{L^\infty})^2 \leq \Lambda_0 \left( (\varepsilon \frac{1}{2} \| \partial_z v(0) \|_{L^\infty})^2 + \| v(0) \|_{E^2, \infty}^2 \right) + \Lambda_0 \| v(t) \|_{E^2, \infty}^2 \]

\[ + \Lambda_0 \int_{0}^{t} (\varepsilon \| \nabla S_n \|_{m-2} + \varepsilon \| \nabla v \|_{2})^2 + (1 + t) \int_{(t - \tau)^{\frac{1}{2}}}^{t} \frac{\Lambda_0}{(t - \tau)^{\frac{1}{2}}} d\tau. \]

To conclude the proof of Proposition 9.8 we can combine the last estimate and (9.55) with the estimate of Proposition 9.4. □

By using arguments close to the ones we have just used, we can also establish that
Lemma 9.9. For $m \geq 6$, assume that $\sup_{[0,T]} \Lambda_{\infty,m}(t) \leq M$. Then, there exists $\Lambda(M)$ such that we have the estimate

$$\int_0^t \sqrt{\varepsilon} \| \partial_z v \|_{1,\infty} \leq \Lambda(M) \left( (1 + t)^2 \left( 1 + \int_0^t \varepsilon \left( \| \nabla V^m \|^2 + \| \nabla S_n \|_{m-2}^2 \right) \right) \right).$$

Note that, by combining the previous lemma and Lemma 9.7, we obtain

Corollary 9.10. For $m \geq 6$, assume that $\sup_{[0,T]} \Lambda_{\infty,m}(t) \leq M$. Then, there exists $\Lambda(M)$ such that, we have the estimate

$$\int_0^t \sqrt{\varepsilon} \| \partial_z v \|_{1,\infty} \leq \Lambda(M) \left( (1 + t)^2 \left( 1 + \int_0^t \varepsilon \left( \| \nabla V^m \|^2 + \| \nabla S_n \|_{m-2}^2 \right) \right) \right).$$

Proof. We first note by using (9.24), (9.30) and again (8.2), (8.6) (8.7) that

$$\int_0^t \sqrt{\varepsilon} \| \partial_z v \|_{1,\infty} \leq \Lambda(M) \int_0^t \sqrt{\varepsilon} \left( \| \partial_z \rho \|_{1,\infty} + \sqrt{\varepsilon} \| \nabla v \|_{2,\infty} \right)$$

and hence, thanks to Lemma 9.7, we obtain

$$\int_0^t \sqrt{\varepsilon} \| \partial_z v \|_{1,\infty} \leq \Lambda(M)(1 + t) \left( \int_0^t \sqrt{\varepsilon} \| \partial_z \rho \|_{1,\infty} + \int_0^t \varepsilon \left( \| \nabla V^m \|^2 + \| \nabla S_n \|_{m-2}^2 \right) \right).$$

To estimate $\| \partial_z \rho \|_{1,\infty}$, note that a brutal use of the Duhamel formula (9.54) is not sufficient. Indeed, even for the fields $Z_i$, $i = 1, 2$ which commute with $G$, we get that

$$\sqrt{\varepsilon} \| \partial_z Z \|_{L^\infty} \leq \frac{1}{\sqrt{t}} \| \rho_0 \|_{1,\infty} + \int_0^t \frac{1}{\sqrt{t - \tau}} \left( \| \mathcal{H} \|_{1,\infty} + \| \cdot \|_{1,\infty} \right)$$

and the problem is that

$$\| \cdot \|_{1,\infty}$$

cannot be estimated in terms of controlled quantities ($\| \cdot \|_{2,\infty} \sim \| \partial_z \|_{2,\infty}$ is not controlled). Consequently, we need to be more precise and incorporate the transport term in the argument. We shall use the following lemma

Lemma 9.11. Consider $\rho$ a smooth solution of

$$\partial_t \rho + \rho \cdot \nabla \rho = \varepsilon \partial_z \rho + \mathcal{H}, \quad z < 0, \quad \rho(t, y, 0) = 0$$

for some smooth vector field $w$ such that $w_3$ vanishes on the boundary. Assume that $\rho$ and $\mathcal{H}$ are compactly supported in $z$ and that $\sup_{[0,T]} \left( \| w \|_{E^2,\infty} + \| \partial_z w_3 \| \right) \leq M$. Then, there exists $\Lambda(M)$ such that we have the estimate:

$$\sqrt{\varepsilon} \| \partial_z \rho(t) \|_{1,\infty} \leq \Lambda(M) \left( \frac{1}{\sqrt{t}} \| \rho(0) \|_{1,\infty} + \int_0^t \frac{1}{\sqrt{t - \tau}} \left( \| \rho \|_{1,\infty} + \| \rho \|_4 + \| \mathcal{H} \|_{1,\infty} \right) \right), \quad \forall t \in [0, T].$$

The proof of this Lemma is given in section (14.2). By using the previous Lemma, for the equation (9.44) and the estimates (9.46), (9.47), we find that

$$\sqrt{\varepsilon} \| \partial_z \rho(t) \|_{1,\infty} \leq \Lambda(M) \left( \frac{1}{\sqrt{t}} \| \rho(0) \|_{1,\infty} + \int_0^t \frac{1}{\sqrt{t - \tau}} \left( 1 + \varepsilon \| \nabla V^m \| + \varepsilon \| S_n \|_{3,\infty} + \varepsilon \| V \|_{4,\infty} \right) \right)$$

and hence by using (9.50) and (9.51), we obtain

$$\sqrt{\varepsilon} \| \partial_z \rho(t) \|_{1,\infty} \leq \Lambda(M) \left( \frac{1}{\sqrt{t}} \| \rho(0) \|_{1,\infty} + \int_0^t \frac{1}{\sqrt{t - \tau}} \left( 1 + \varepsilon \| \nabla V^m \| + \varepsilon \| \nabla S_n \|_{m-2} \right) \right)$$
for \( m \geq 6 \). From integration in time, we obtain
\[
\int_0^t \sqrt{\varepsilon} \| \partial_z \rho(t) \|_{1, \infty} \leq \Lambda(M) \left( \sqrt{t} \| \rho(0) \|_{1, \infty} + \sqrt{t} \int_0^t \left( 1 + \varepsilon \| \nabla v^m \| + \varepsilon \| \nabla S_n \|_{m-2} \right) \right).
\]
We finally get (9.50) by combining the last estimate and (9.57). Note that the terms involving the initial data can be estimated by \( \Lambda(M) \).

\[\square\]

10. Normal derivative estimates part II

In the estimate of Proposition 8.3, we see that the left hand side is still insufficient to control the term \( \int_0^t \| \partial_z v \|_{m-1}^2 \). It does not seem possible to estimate it by estimating \( \| S_n \|_{m-1} \). Indeed, in the right hand side of (8.21), the term involving the pressure in \( F^2 \) does not enjoy any additional regularity. The main obstruction comes from the Euler part \( q^E \) of the pressure. Since \( q^E = gh \) on the boundary the estimate given by Proposition 6.34 is optimal in terms of the regularity of \( h \): the estimate of \( \| D^2 q^E \|_{m-1} \) necessarily involves \( |h|_{m+\frac{1}{2}} \) which cannot be controlled uniformly in \( \varepsilon \). The solution will be to use the vorticity instead of \( S_n \) to perform this estimate. The main difficulty will be that the vorticity does not vanish on the boundary. Note that this difficulty is not a problem in the inviscid case since there is always a good energy estimate for the transport equation solved by the vorticity even if it does not vanish on the boundary.

Let us set \( \omega = \nabla \varphi \times v \) (note that we have in an equivalent way that \( \omega = (\nabla \times u)(t, \Phi) \)). Since
\[
\omega \times n = \frac{1}{2} (D^\varphi v \cdot n - (D^\varphi v)^t n) = S^\varphi v \cdot n - (D^\varphi v)^t n,
\]
we get as in (8.6) that
\[
\omega \times n = \frac{1}{2} \partial_n u - g^{ij} (\partial_j v \cdot n) \partial_{y^i}.
\]
Consequently, we get by using also (8.7) that
\[
\| Z^{m-1} \partial_z v \| \leq \Lambda_{\infty, 6} (\| v \|_m + |h|_{m-\frac{1}{2}} + \| \omega \|_{m-1})
\]
and hence we shall estimate \( \| \omega \|_{m-1} \) in place of \( \| \partial_z v \|_{m-1} \). By applying \( \nabla \varphi \times \cdot \) to the equation (1.12), we get the vorticity equation in \( S \)
\[
\partial_t Z^\omega + v \cdot \nabla \varphi \omega - \omega \cdot \nabla \varphi v = \varepsilon \Delta Z^\varphi \omega.
\]
By using (10.1) and the boundary condition (1.14), we note that on the boundary, we have
\[
\omega \times n = \Pi (\omega \times n) = -\Pi (g^{ij} (\partial_j v \cdot n) \partial_{y^i})
\]
and thus \( \omega \times n \) does not vanish on the boundary. Consequently, there is no gain to consider \( \omega \times n \) in place of \( \omega \) since the equation for \( \omega \times n \) is more complicated. In this subsection, we shall thus estimate \( Z^{m-1} \omega \).

Thanks to (10.3), we get that \( Z^\alpha \omega \) for \( |\alpha| \leq m - 1 \) solves in \( S \) the equation
\[
\partial_t Z^\alpha \omega + v \cdot \nabla \varphi Z^\alpha \omega - \varepsilon \Delta Z^\alpha \omega = F
\]
where the source term \( F \) is given by
\[
F = Z^\alpha (\omega \cdot \nabla \varphi v) + C_S
\]
where \( C_S \) is given as in (8.22) by
\[
C_S = C_S^1 + C_S^2
\]
with
\[
C_S^1 = [Z^\alpha v_y] \cdot \nabla y \omega + [Z^\alpha, V_z] \partial_z \omega : = C_{Sy} + C_{Sz}, \quad C_S^2 = -\varepsilon [Z^\alpha, \Delta \varphi] \omega.
\]
In addition, by using Lemma 5.5, we get that on the boundary
\[(10.7) \quad |(Z^{\alpha} \omega)^{b}| \leq \Lambda_{\infty,6}(\|v^{b}|_{m} + |h|_{m}).\]
Note that by using the trace Theorem, we get that
\[(\mathbf{Z}^{\alpha} \omega)^{b} \leq \Lambda_{\infty,6}(\|\nabla V^{m}\|^{\frac{1}{2}}\|V^{m}\|^{\frac{1}{2}} + |V^{m}| + |h|_{m}).\]
Consequently, the only way we can control this boundary value is through the estimate
\[\varepsilon^{\frac{1}{2}} \int_{0}^{t} |(Z^{\alpha} \omega)^{b}|^{2} \leq \varepsilon \int_{0}^{t} \|\nabla V^{m}\|^{2} + \int_{0}^{t} \Lambda_{\infty,6}(\|V^{m}\|^{2} + |h|_{m}^{2})\]
and to see that the left hand side can be estimated by using Proposition 7.1. Nevertheless, it will be useful to keep the slightly sharper form of the above inequality which reads
\[(10.8) \quad \varepsilon^{\frac{1}{2}} \int_{0}^{t} |(Z^{\alpha} \omega)^{b}|^{2} \leq \Lambda_{\infty,6}(\int_{0}^{t} \varepsilon \|\nabla V^{m}\|\|V^{m}\| + \|V^{m}\|^{2} + |h|_{m}^{2}).\]
The main difficulty will be to handle this non-homogeneous boundary value problem for the convection-diffusion equation (10.4) since because of (10.8) the boundary value is at a low level of regularity (it is $L_{t,y}^{2}$ and no more).

To split the difficulty, we set
\[(10.9) \quad Z^{\alpha} \omega = \omega^{\alpha}_{h} + \omega^{\alpha}_{nh}\]
where $\omega_{nh}$ solves in $\mathcal{S}$ the equation (10.4), that is to say
\[(10.10) \quad \partial_{t}^{\varepsilon} \omega^{\alpha}_{nh} + v \cdot \nabla^{\varphi} \omega^{\alpha}_{nh} - \varepsilon \Delta^{\varphi} \omega^{\alpha}_{nh} = F\]
with the initial and boundary conditions
\[(10.11) \quad (\omega^{\alpha}_{nh})^{b} = 0, \quad (\omega^{\alpha}_{nh})_{t=0} = \omega_{0}.\]
while $\omega_{h}$ will solve in $\mathcal{S}$ the homogeneous equation
\[(10.12) \quad \partial_{t}^{\varepsilon} \omega^{\alpha}_{h} + v \cdot \nabla^{\varphi} \omega^{\alpha}_{h} - \varepsilon \Delta^{\varphi} \omega^{\alpha}_{h} = 0\]
with the initial and boundary conditions
\[(10.13) \quad (\omega^{\alpha}_{h})^{b} = (Z^{\alpha} \omega)^{b}, \quad (\omega^{\alpha}_{h})_{t=0} = 0.\]

The main result of this section is the following proposition:

**Proposition 10.1.** For $T \in [0, T^{\varepsilon}], T^{\varepsilon} \leq 1$, assume that for $M > 0$, the estimate
\[(10.14) \quad \sup_{[0,T]} \Lambda_{\infty,6}(t) + \varepsilon \int_{0}^{T} (\varepsilon \|\nabla V^{6}\|^{2} + \varepsilon \|\nabla S_{n}\|_{4}^{2}) \leq M\]
holds. Then, there exists $\Lambda(M)$ such that
\[\|\omega\|_{L^{4}([0,T],H_{0}^{m-1}(S))}^{2} \leq \Lambda_{0}\|\omega(0)\|^{2}_{m-1} + \Lambda(M) \int_{0}^{t} (\|V^{m}\|^{2} + \|\omega\|_{m-1}^{2} + |h|_{m}^{2} + \varepsilon |h|_{m+rac{1}{2}}^{2})
+ \Lambda_{0} \int_{0}^{t} \varepsilon \|\nabla S_{n}\|_{m-2}^{2} + \varepsilon \|\nabla V^{m}\|^{2}).\]

The proof follows from the splitting (10.9) and the estimates of Proposition 10.2 and 10.4.
10.1. Estimate of $\omega_{nh}^\alpha$. Let us first estimate $\omega_{nh}^\alpha$. We shall use the notation

$$\|\omega_{nh}^{m-1}(t)\|^2 = \sum_{|\alpha| \leq m-1} \|\omega_{nh}^\alpha\|^2, \quad \int_0^t \|\nabla \omega_{nh}^{m-1}\|^2 = \sum_{|\alpha| \leq m-1} \|\nabla \omega_{nh}^\alpha\|^2.$$  

**Proposition 10.2.** For $t \in [0,T]$ the solution $\omega_{nh}^\alpha$ of (10.10), (10.11) satisfies the estimate:

$$\|\omega_{nh}^{m-1}(t)\|^2 + \epsilon \int_0^t \|\nabla \omega_{nh}^{m-1}\|^2 \leq \Lambda_0 \|\omega(0)\|_{m-1} + \int_0^t \Lambda_{\infty,6} (\|\omega\|^2_{m-1} + h\|_{m+\frac{1}{2}}^2) + \Lambda_0 \epsilon \|\nabla S_n\|^2_{m-2}.$$  

Note that the term $\Lambda_0 \int_0^t \|\nabla \varphi S_n\|^2_{m-2}$ in the right-hand side of the above estimate can be estimated by using Proposition 8.5.

**Proof.** Since we have for $\omega_{nh}^\alpha$ an homogeneous Dirichlet boundary condition, we deduce from (10.10) and a standard energy estimate that

$$\frac{d}{dt} \frac{1}{2} \int_S |\omega_{nh}^\alpha|^2 dV_t + \epsilon \int_S |\nabla \omega_{nh}^\alpha|^2 dV_t = \int_S F \cdot \omega_{nh}^\alpha dV_t$$  

and we need to estimate the right hand side with $F$ given by (10.5). By using (2.1), we get that

$$\|Z^\alpha (\omega \cdot \nabla \varphi)\| \leq \Lambda_{\infty,6} (\|\omega\|_{m-1} + \|\nabla \varphi\|_{m-1}) \leq \Lambda_{\infty,6} (\|\omega\|_{m-1} + \|v\|_{m} + |h|_{m-\frac{1}{2}}).$$  

Next, to estimate $C_S$, we observe that we can use (8.32) to estimate the part involving $C^2_S$. Indeed, it suffices, to change $S_n$ into $\omega$ and $m$ into $m + 1$ to obtain

$$\int_S C^2_S \cdot \omega_{nh}^\alpha dV_t \leq \Lambda_0 (\epsilon^\frac{1}{2} \|\nabla \omega_{nh}^\alpha\|^2 + \|\omega\|_{m-1})$$

$$(\epsilon^\frac{1}{2} \|\nabla \omega\|_{m-2} + \|\omega\|_{m-1} + \Lambda_{\infty,6} (|h|_{m-\frac{1}{2}}^2 + \epsilon^\frac{1}{2} |h|_{m+\frac{1}{2}})).$$

To estimate $C^1_S$, we can use the same decomposition of $C^1_S$ as the one given after (8.22). For $C^2_S$, we can also use (8.22) with $S_n$ changed into $\omega$ and $m$ into $m + 1$. This yields

$$\|C_{S_1}\| \leq \Lambda_{\infty,6} (\|\omega\|_{m-1} + \|v\|_{E_m}) \leq \Lambda_{\infty,6} (\|\omega\|_{m-1} + \|v\|_{m} + |h|_{m})$$

The commutator $C_{S_2}$ requires more care. Indeed, in (8.27), we remark that we cannot change $m$ into $m + 1$ since $h$ is not smooth enough uniformly in $\epsilon$: such an estimate would involve $|h|_{m+\frac{1}{2}}$ (and without a gain of $\epsilon^\frac{1}{2}$). By using as in (8.25) that this commutator can be expanded into a sum of terms under the form

$$c_{\beta} Z^{\beta} \left(1 - \frac{z}{\bar{z}} V_\eta\right) Z_3 Z^\gamma \omega$$

where the constraints are now $|\beta| + |\gamma| \leq m - 1$, $|\gamma| \leq m - 2$. Since $V_\zeta = v_\zeta / \partial_\zeta \varphi = (v \cdot N - \partial_\zeta \eta) / \partial_\zeta \varphi$, we use as before (2.1) and that $V_\zeta$ vanishes on the boundary to write

$$\|c_{\beta} Z^{\beta} \left(1 - \frac{z}{\bar{z}} V_\zeta\right) Z_3 Z^\gamma \omega\| \leq \Lambda_{\infty,6} \|\omega\|_{m-1} + \|\omega\|_{L^\infty} + |Z\left(1 - \frac{z}{\partial_\zeta \varphi} v_\zeta\right)|_{m-2}$$

$$\leq \Lambda_{\infty,6} (\|\omega\|_{m-1} + |h|_{m-\frac{1}{2}} + \frac{1}{|z|} |Z v_\zeta|_{m-2}).$$

Next, by using Lemma 8.4, we obtain that

$$\left\|\frac{1}{|z|} Z v_\zeta\right\|_{m-2} \leq \|Zv_\zeta\|_{m-2} + \|\partial_\zeta Z v_\zeta\|_{m-2} \leq \Lambda_{\infty,6} (\|v\|_{E_m} + \sum_{|\alpha|=m-1} \|\partial_\zeta Z^\alpha N - \partial_\zeta Z^\alpha \partial_\eta\|)$$

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and the last term requires some care. Since $\eta$ is given by (10.10), we first note that

$$|Z_3\eta| \lesssim |\tilde{\chi}(\xi|z)\tilde{h}|$$

where $\tilde{\chi}$ has a slightly bigger support than $\chi$ and thus we get that $Z_3$ acts as a zero order operator:

$$\|\nabla Z_3\eta\| \lesssim |h|^{\frac{1}{2}}.$$  (10.18)

This yields that if $\alpha_3 \neq 0$, we gain at least one derivative and thus, have

$$\|v \cdot \partial_z Z^\alpha N - \partial_z Z^\alpha \partial_t \eta\| \leq \Lambda_{\infty,6}(|h|_{m-\frac{1}{2}} + |\partial h|_{m-\frac{1}{2}}) \leq \Lambda_{\infty,6}(|h|_{m-\frac{1}{2}} + |v|^b_{m-\frac{1}{2}})$$

where the last estimate comes from the boundary condition (1.5). Hence by using (2.5), we get

$$\|v \cdot \partial_z Z^\alpha N - \partial_z Z^\alpha \partial_t \eta\| \leq \Lambda_{\infty,6}(|h|_{m-\frac{1}{2}} + \|v\|_{E^{m-1}}).$$

Consequently, we only need to estimate

$$\|v \cdot \partial_z Z^\alpha N - \partial_z Z^\alpha \partial_t \eta\|$$

when $|\alpha| = m - 1$, $\alpha_3 = 0$. By using the expression (3.6), we note that

$$v \cdot \partial_z Z^\alpha N - \partial_z Z^\alpha \partial_t \eta = v_1 \partial_z (\psi_z \ast_y \partial_1 Z^\alpha h) + v_2 \partial_z (\psi_z \ast_y \partial_2 Z^\alpha h) - \partial_z (\psi_z \ast_y \partial_4 Z^\alpha h) := T_\alpha.$$  

For $z \leq -1$, we can use the smoothing effect of the convolution to get that

$$\|T_\alpha\|_{L^2(S\cap |z| \geq 1)} \leq \Lambda_{\infty,6} \left(\|\partial_z \left(\frac{1}{2m-1}\psi_z \ast_y \nabla h\right)\|_{L^2(S\cap |z| \geq 1)} + \|\partial_z \left(\frac{1}{2m-1}\psi_z \ast_y \partial_4 h\right)\|_{L^2(S\cap |z| \geq 1)}\right)$$

where $\tilde{\psi}$ has the same properties as $\psi_z$. This yields

$$\|T_\alpha\|_{L^2(S\cap |z| \geq 1)} \leq \Lambda_{\infty,6}(|h|_{\frac{1}{2}} + \|v\|_{E^1}).$$  (10.19)

For $|z| \leq 1$, We shall rewrite $T_\alpha$ as

$$T_\alpha = v_1^b \partial_z (\psi_z \ast_y \partial_1 Z^\alpha h) + v_2^b \partial_z (\psi_z \ast_y \partial_2 Z^\alpha h) - \partial_z (\psi_z \ast_y \partial_4 Z^\alpha h) + \mathcal{R}$$

where

$$|\mathcal{R}| \leq \Lambda_{\infty,6} |z| |\partial_z (\psi_z \ast Z^\alpha \nabla h)| \leq |Z_3 (\psi_z \ast Z^\alpha \nabla h)|.$$  

Consequently, by using again the observation (10.18), we get that

$$\|\mathcal{R}\|_{L^2(S\cap |z| \leq 1)} \leq \Lambda_{\infty,6} |h|_{m-\frac{1}{2}}.$$  (10.21)

To estimate the first term in (10.20), we write

$$v_1^b \partial_z (\psi_z \ast_y \partial_1 Z^\alpha h) + v_2^b \partial_z (\psi_z \ast_y \partial_2 Z^\alpha h) = \partial_z (\psi_z \ast_y (v_1^b \partial_1 Z^\alpha h + v_2^b \partial_2 Z^\alpha h - \partial_4 Z^\alpha h))$$

$$+ \partial_z \left(\int_{\mathbb{R}^2} \left( (v_1(t,y,0) - v_1(t,y',0)) \psi_z(y-y') \partial_1 Z^\alpha h(t,y') \right) \right.\left. + (v_2(t,y,0) - v_2(t,y',0)) \psi_z(y-y') \partial_2 Z^\alpha h(t,y') \right).$$

For the second term, we can use the Taylor formula for $v_1$ to get that

$$|v_1(t,y,0) - v_1(t,y',0)||\partial_z \psi_z(y-y')| \leq \Lambda_{\infty,6} \frac{1}{z^2} \tilde{\psi}(\frac{y-y'}{z})$$

where $\tilde{\psi}$ is still an $L^1$ function. This yields that

$$\sup_{z \in (-1,0)} \left\|\partial_z \left(\int_{\mathbb{R}^2} \left( (v_1(t,y,0) - v_1(t,y',0)) \psi_z(y-y') \partial_1 Z^\alpha h(t,y') \right) \right.\right.$$

$$\left. + (v_2(t,y,0) - v_2(t,y',0)) \psi_z(y-y') \partial_2 Z^\alpha h(t,y') \right\|_{L^2(\mathbb{R}^2)} \leq \Lambda_{\infty,6} |h|_{m}.$$  (10.22)
For the first term, we shall use Lemma 5.7. For $|\alpha| = m - 1$, we get from Lemma 5.7 that
\[
v^b_1 \partial_1 Z^\alpha h + v^b_2 \partial_2 Z^\alpha h - \partial_t Z^\alpha h = -C^\alpha(h) - (V^\alpha)^b \cdot N - v^b_3
\]
and hence also that
\[
\|v^b_1 \partial_1 Z^\alpha h + v^b_2 \partial_2 Z^\alpha h - \partial_t Z^\alpha h\|_{H^{1/2}(\mathbb{R}^2)} \leq \Lambda_{\infty,6}(|v^b|_{m - \frac{1}{2}} + |h|_{m - \frac{1}{2}}).
\]
Consequently, we obtain that
\[
\|\partial_z (\psi \ast_y (v^b_1 \partial_1 Z^\alpha h + v^b_2 \partial_2 Z^\alpha h - \partial_t Z^\alpha h))\|_{L^2(S)} \leq |v^b_1 \partial_1 Z^\alpha h + v^b_2 \partial_2 Z^\alpha h - \partial_t Z^\alpha h|_{\frac{1}{2}}
\]
\[
\leq \Lambda_{\infty,6}(|v^b|_{m - \frac{1}{2}} + |h|_{m - \frac{1}{2}}).
\]
By combining (10.20), (10.21), (10.22) and the last estimate with the trace Theorem, we finally obtain that
\[
\|T_\alpha\|_{L^2(S;|z| \leq 1)} \leq \Lambda_{\infty,6}(\|v\|_{E^m} + |h|_m).
\]
Since we had already proven (10.19), we finally obtain that
\[
(10.23) \quad \|C_{S_z}\| \leq \Lambda_{\infty,6}(\|v\|_{E^m} + |h|_m).
\]
To conclude the proof of Proposition 10.2, we use the energy estimate (10.14), the commutator estimates (10.16), (10.17), (10.23), Lemma 2.8 and the Young inequality to obtain
\[
\|\omega_{m-1}^{n-1}(t)\|^2 + \varepsilon \int_0^t \|\nabla \omega_{m-1}^{n-1}\|^2 \leq \Lambda_0\|\omega(0)\|^2_{m-1} + \int_0^t \Lambda_{\infty,6}(\|v\|^2_{E^m} + |\omega|^2_{m-1} + |h|^2_{m-\frac{1}{2}} + \varepsilon |h|^2_{m+\frac{1}{2}}) + \Lambda_0 \int_0^t \varepsilon \|\nabla \omega\|^2_{m-2}.
\]
To end the proof, we note that thanks to (2.1), we have
\[
\sqrt{\varepsilon} \|\nabla \omega\|_{m-2} \leq \Lambda_0\left(\sqrt{\varepsilon} \|\partial_z v\|_{m-2} + \sqrt{\varepsilon} \|\partial_z v\|_{m-1}\right) + \Lambda_\infty |h|_{m-\frac{1}{2}}
\]
and we use again Lemma S.2 and (10.2) to obtain that
\[
\|\omega_{m-1}^{n-1}(t)\|^2 + \varepsilon \int_0^t \|\nabla \omega_{m-1}^{n-1}\|^2 \leq \Lambda_0\|\omega(0)\|^2_{m-1} + \int_0^t \Lambda_{\infty,6}(\|V_m\|^2 + |\omega|^2_{m-1} + |h|^2_{m} + \varepsilon |h|^2_{m+\frac{1}{2}}) + \Lambda_0 \int_0^t \varepsilon \|\nabla S_n\|^2_{m-2}.
\]
This ends the proof of Proposition 10.2. □

10.2. Estimate of $\omega_h^{\alpha}$. It remains to get an estimate for $\int_0^t |\omega_{h}^{\alpha} m-1|^2$ where $\omega_h^{\alpha}$ solves the homogeneous equation (10.12), with the non-homogeneous boundary condition (10.13). Note that the only estimate on $(Z^\alpha)^b$ that we have at our disposal is the $L^2$ in time estimate (10.8). This creates two difficulties, the first one is that we cannot easily lift this boundary condition and perform a standard energy estimate. The second one is that due to this poor estimate on the boundary value, we cannot hope an estimate of $\sup_{[0, T]} \|\omega_{h}^{\alpha} m-1\|$ independent of $\varepsilon$. 

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10.2.1. A simple computation on the heat equation. To understand the difficulty, let us consider the heat equation

\[ \partial_t f - \varepsilon \Delta f = 0, \quad x = (y, z) \in S \]

with zero initial data and the boundary condition \( f(t, y, 0) = f^b(t, y) \)

**Lemma 10.3.** The solution of the above equation satisfies the estimate:

\[
\int_0^{+\infty} e^{-2\gamma t} \| (\gamma + |\partial_t|)^{\frac{\alpha}{2}} f \|_2^2 dt \leq \sqrt{\varepsilon} \int_0^{+\infty} e^{-2\gamma t} |f^b|_2^2 \, dt.
\]

Consequently, we see that we get a control of \( f \) which is \( H^1_t (L^2(S)) \). Note that by Sobolev embedding, this implies that we can expect a control of \( \| f \|_{L^2_t (L^2(S))} \) (and not of \( \| f \|_{L^{r \infty}_t (L^2(S))} \)).

**Proof.** We can use a Laplace Fourier transform in time and \( y \) to get the equation

\[-\varepsilon \partial_{zz} \hat{f} + (\gamma + i\tau + \varepsilon |\xi|^2) \hat{f} = 0, \quad \hat{f}_{/z=0} = \hat{f}^b\]

where

\[ \hat{f}(\gamma, \tau, \xi, z) = \int_0^{+\infty} \int_{\mathbb{R}^2} e^{-\gamma t - i\tau y - i\xi y} f(t, y, z) \, dt \, dy \]

with \( \gamma > 0 \).

We can solve explicitly this equation to get

\[ \hat{f} = e^{(\gamma + i\tau + \varepsilon |\xi|^2)^{\frac{1}{2}} x \cdot \xi} \hat{f}^b, \quad z < 0 \]

and hence, we find that

\[ |\hat{f}(\gamma, \tau, \xi, \cdot)|_{L^2_\gamma}^2 \leq \frac{\sqrt{\varepsilon}}{(\gamma + |\tau| + \varepsilon |\xi|^2)^{\frac{1}{2}}} |\hat{f}^b|_{L^2_\gamma}^2. \]

This yields

\[ (\gamma + |\tau|)^{\frac{1}{2}} |\hat{f}(\gamma, \tau, \xi, \cdot)|_{L^2_\gamma}^2 \leq \sqrt{\varepsilon} |\hat{f}^b|_{L^2_\gamma}^2 \]

and the result follows from the Bessel-Parseval identity. \( \square \)

10.2.2. Statement of the estimate of \( \omega^\alpha_h \).

**Proposition 10.4.** For \( T \in [0, T^c] \), \( T^c \leq 1 \), assume that for \( M > 0 \), the estimate

\[ \sup_{[0,T]} \Lambda_{\infty,6}(t) + \int_0^T (\varepsilon \| \nabla V^6 \|^2 + \varepsilon \| \nabla S_h \|^2) \leq M \]

holds. Then, there exists \( \Lambda(M) \) such that for \( |\alpha| \leq m - 1 \), we have

\[ \| \omega^\alpha_h \|^2_{L^2([0,T], L^2(S))} \leq \Lambda(M) \int_0^T (\| V^m \|^2 + \| h \|^2_m) + \varepsilon \int_0^T \| \nabla V^m \|^2. \]

Again, note that the last term in the right hand side of the above inequality can be estimated by using Proposition 7.11.

We will prove this estimate by using a microlocal symmetrizer. In the convection diffusion equation (10.12), the convection term creates some difficulties. Indeed, the convection operator \( \partial_t + v \cdot \nabla \) dominates in the low frequency regime. For this operator, the boundary is characteristic since \( V_z \) vanishes on the boundary and the fact that \( V_z \) is not uniformly zero in a vicinity of the boundary is not convenient when performing microlocal energy estimate (see 37). More importantly, if we add a convection term to (10.24), even constant, that is to say that we study

\[ \partial_t f + c \cdot \nabla g f - \varepsilon \Delta f = 0 \]
for \( c \in \mathbb{R}^2 \), then the result of Lemma \[10.3\] becomes
\[
\int_0^{+\infty} e^{-2\gamma t} \left\| (\gamma + |\partial_t + c \cdot \nabla_y|^{1/2}) f \right\|^2 dt \leq \sqrt{\varepsilon} \int_0^{+\infty} e^{-2\gamma t} |f|^2_{L^2(\mathbb{R}^2)} dt.
\]
In other words, the smoothing effect does not occur in the direction \( \partial_t \) but in the direction \( \partial_t + c \cdot \nabla_y \).

This will be much more difficult to detect when \( c \) has variable coefficients. Consequently, in order to fix this difficulty, we shall use Lagrangian coordinates in order to eliminate the convection term and study a purely parabolic problem.

### 10.2.3. Lagrangian coordinates

Let us define a parametrization of \( \Omega_t \) by
\[
(10.26) \quad \partial_t X(t, x) = u(t, X(t, x)) = v(t, \Phi(t, \cdot)^{-1} \circ X), \quad X(0, x) = \Phi(0, x)
\]
where \( \Phi(t, \cdot)^{-1} \) stands for the inverse of the map \( \Phi(t, \cdot) \) defined by \[1.7\]. Note that the meaning of the initial condition is that we choose the parametrization of \( \Omega_0 \) which is given by \[1.7\]. Let us also define \( J(t, x) = |\det \nabla X(t, x)| \) the Jacobian of the change of variable.

We have the following estimates for \( X \):

**Lemma 10.5.** Under the assumption of Proposition \[10.4\] we have for \( t \in [0, T] \) the estimates
\[
\begin{align*}
(10.27) & \quad |J(t, x)|_{W^{1, \infty}} + |1/J(t, x)|_{W^{1, \infty}} \leq \Lambda_0, \\
(10.28) & \quad \|\nabla X(t)\|_{L^\infty} + \|\partial_t \nabla X\|_{L^\infty} \leq \Lambda_0 e^{\Lambda(M) t}, \\
(10.29) & \quad \|\nabla X\|_{1, \infty} + \|\partial_t \nabla X\|_{1, \infty} \leq \Lambda(M) e^{\Lambda(M) t}, \\
(10.30) & \quad \sqrt{\varepsilon} \|\nabla^2 X\|_{1, \infty} + \sqrt{\varepsilon} \|\partial_t \nabla^2 X\|_{L^\infty} \leq \Lambda(M)(1 + t^2) e^{\Lambda(M) t}.
\end{align*}
\]

**Proof.** Since \( u \) is divergence free we get that \( J(t, x) = J_0(x) \) and the first estimate follows from Proposition \[3.1\]. Next, by using the ordinary differential equation \[10.26\], we get that
\[
\partial_t DX = DvD\Phi^{-1} DX
\]
where \( D \) stands for the differential with respect to the space variable and hence we find
\[
|\nabla X(t, \cdot)|_{L^\infty} \leq \Lambda_0 + \Lambda_0 \int_0^t \|v\|_{E^{1, \infty}} \|\nabla X(t)\|_{L^\infty} \leq \Lambda_0 + \Lambda(M) \int_0^t |\nabla X(t)|_{L^\infty}.
\]
This yields the first part of \[10.25\] by using the Gronwall inequality. Moreover, by using again the equation \[10.31\], we get \[10.29\]. Next, by applying one conormal derivative to \[10.31\], we also get that
\[
\|\nabla X\|_{1, \infty} \leq \Lambda_0 + \Lambda_0 \int_0^t \|v\|_{E^{2, \infty}} \|\nabla X\|_{1, \infty}
\]
and hence we get \[10.29\] from the Gronwall inequality. The estimate for the time derivative follows again from the equation. The estimate \[10.30\] can be obtained in the same way. By applying \( \sqrt{\varepsilon} \nabla \) to \[10.31\], we find
\[
\sqrt{\varepsilon} \|\nabla^2 X\|_{1, \infty} \leq \Lambda(M) + \Lambda(M) \int_0^t \sqrt{\varepsilon} \|\nabla^2 X\|_{1, \infty} + \Lambda(M) e^{\Lambda(M) t} \int_0^t \sqrt{\varepsilon} \|\nabla^2 v\|_{1, \infty}
\]
and hence, by using Corollary \[9.10\] and the assumption \[10.25\], we find
\[
\sqrt{\varepsilon} \|\nabla^2 X\|_{1, \infty} \leq \Lambda(M)(1 + t^2) e^{\Lambda(M) t} + \Lambda(M) \int_0^t \sqrt{\varepsilon} \|\nabla^2 X\|_{1, \infty}
\]
and the first part of \[10.30\] follows by using the Gronwall inequality. For the second part of \[10.30\], it suffices to apply \( \sqrt{\varepsilon} \nabla \) to \[10.31\] and to use \[10.25\] and that \( \sqrt{\varepsilon} \|\nabla^2 v\|_{L^\infty} \leq \Lambda_{\infty, 0} \). □
10.2.4. Equation in Lagrangian coordinates. Let us set
\begin{equation}
\Omega^\alpha = e^{-\gamma t} \omega_h^\alpha(t, \Phi^{-1} \circ X)
\end{equation}
where \(\gamma > 0\) is a large parameter to be chosen. Then \(\Omega^\alpha\) solves in \(S\) the equation
\begin{equation}
a_0(\partial_t \Omega^\alpha + \gamma \Omega^\alpha) - \varepsilon \partial_t (a_{ij} \partial_j \Omega^\alpha) = 0
\end{equation}
where we have used the summation convention. Note that \(a_0 = |J_0|^\frac{1}{2}\) and that the matrix \((a_{ij})_{i,j}\) is defined by
\[(a_{ij})_{ij} = |J_0|^\frac{1}{2} P^{-1}, \quad P_{ij} = \partial_i X \cdot \partial_j X.
\]
Note that the coefficients of the matrix \((a_{ij})\) will therefore match the estimates of Lemma 10.5. Moreover, we also obtain thanks to Lemma 10.5 that
\begin{equation}
a_0(\partial_t \Omega^\alpha + \gamma \Omega^\alpha) - \varepsilon \partial_t (a_{ij} \partial_j \Omega^\alpha) = 0
\end{equation}
and thanks to (10.35) and (10.32), we obtain by change of variable (let us recall that \(J\) satisfies the estimate (10.27)) that
\begin{equation}
\Omega^\alpha_{/z=0} = (\Omega^\alpha)^b := e^{-\gamma t} \omega_h^\alpha(t, \Phi^{-1} \circ X(t, y, 0)).
\end{equation}
We shall prove that:

**Theorem 10.6.** There exists \(\gamma_0\) which depends only on \(M\) defined by (10.25) such that for \(\gamma \geq \gamma_0\), the solution of (10.33) with the boundary condition (10.35) satisfies the estimate
\[\|\Omega^{m-1}\|_{H^4(0,T,L^2)}^2 \leq \Lambda(M) \sqrt{\varepsilon} \int_0^T \|\Omega^{m-1}\|_{L^2(\mathbb{R}^2)}^2 \]
where \(\Lambda(M)\) is uniformly bounded for \(T \in [0,1]\).

Note that we define the norm \(H^4((0,T), L^2)\) as
\begin{equation}
\|f\|_{H^4((0,T), L^2)}^2 = \inf \{ \|Pf\|_{H^4(\mathbb{R}, L^2(S))} : Pf = f \text{ on } [0,T] \times S \}
\end{equation}
and we define the norm on the whole space by Fourier transform in time. Let us first explain how to deduce the proof of Proposition 10.4 from the estimate of Theorem 10.6. We first observe that
\[\|\Omega^\alpha\|_{L^2([0,T],L^2(S))} \leq C \|\Omega^\alpha\|_{L^2(S,L^4(0,T))} \leq C \|\Omega^{m-1}\|_{H^4(0,T,L^2)},\]
where the last estimate comes from the one-dimensional Sobolev embedding \(H^4 \subset L^4\). Note that with the definition (10.35), \(C\) does not depend on \(T\) since \(C\) is given by the Sobolev embedding in the whole space. Next, we use (10.6) to get
\[\|\Omega^\alpha\|_{L^4([0,T],L^2(S))} \leq \Lambda(M) \sqrt{\varepsilon} \int_0^T e^{-\gamma t} \|\Omega^{m-1}\|_{L^2(\mathbb{R}^2)}^2\]
and thanks to (10.35) and (10.32), we obtain by change of variable (let us recall that \(J\) satisfies the estimate (10.27)) that
\[\|\omega_h^\alpha\|_{L^4([0,T],L^2(S))} \leq \Lambda(M) \sqrt{\varepsilon} \int_0^T \|\omega_h\|_{m-1}^2\]
We finally end the proof of Proposition 10.4 by using (10.33) and the Young inequality.

It remains to prove Theorem 10.6. In the next subsection, we shall study a constant coefficient model. In the analysis of this model, we will construct symbols which will allow us to study the original problem with the help of the paradifferential calculus described in section 15.
10.2.5. **Symbolic Analysis.** In this section, we shall perform a symbolic analysis. This will allow to get our energy estimate for the equation (10.33), by using the semi-classical paradifferential calculus with parabolic homogeneity described in section [15].

Let us consider \( a = (a_0(z),a_{ij}(z)) \) smooth in \( z \) where \( a_{ij} \) is a symmetric matrix and \( a_0 \) a function with values in the compact set \( \mathcal{K} \) defined by

\[
(10.37) \quad a_0 \geq m, \quad a_{3,3} \geq m, \quad |a| + \sqrt{\varepsilon |\partial_z a|} \leq M, \quad (a_{ij}) \geq c_0 \text{Id}.
\]

where \( m, c_0 \) and \( M \) are positive numbers. This means that we neglect the dependence in \( t, y \) in the coefficients of (10.33). The symbolic version of (10.33) becomes

\[
(10.38) \quad \left( \frac{a_0}{a_{3,3}}(\gamma + i\tau) + A_y(a,\sqrt{\varepsilon} \xi) \right) \Omega + A_z(a,\sqrt{\varepsilon} \xi) \sqrt{\varepsilon} \partial_z \Omega = \varepsilon \partial_{zz} \Omega + R + F, \quad z < 0, \quad \Omega(0) = \Omega^b
\]

where \( A_y \) is homogeneous of degree one in \( \xi \), \( A_z \) is homogeneous of degree one in \( \xi \) and

\[
(10.39) \quad A_y(a,\xi) = \sum_{1 \leq i,j \leq 2} \frac{a_{ij}}{a_{3,3}} \xi_i \xi_j, \quad A_z(a,\xi) = -2 \sum_{1 \leq k \leq 2} \frac{a_{k3}}{a_{3,3}} i \xi_k,
\]

and

\[
R = i \sum_{1 \leq k \leq 2} \sqrt{\varepsilon} \frac{\partial_x a_{3k}}{a_{3,3}} \sqrt{\varepsilon} \xi_k \Omega + \sqrt{\varepsilon} \frac{\partial_x a_{33}}{a_{3,3}} \sqrt{\varepsilon} \partial_z \Omega.
\]

Note that we have incorporated \( F \) in (10.38) which is a given source term. Since in the system (10.33), the equations for the components of \( \Omega \) are not coupled, we can study separately each component and hence we shall assume that \( \Omega \in \mathbb{R} \) in this subsection.

Thanks to (10.37), we have that

\[
(10.40) \quad |R| \lesssim \sqrt{\varepsilon} |\xi| |\Omega| + \sqrt{\varepsilon} |\partial_z \Omega|.
\]

Let us set \( \xi^e = (\gamma, \tau, \varepsilon \xi) \) and \( \langle \xi^e \rangle = (\gamma^2 + \tau^2 + |\sqrt{\varepsilon} \xi|^4)^{\frac{1}{2}} \). We rewrite (10.38) as a system by setting

\[
(10.41) \quad U = (\Omega, \sqrt{\varepsilon} \partial_z \Omega / \langle \xi^e \rangle)^t.
\]

This yields the system

\[
(10.42) \quad \sqrt{\varepsilon} \partial_z U = \langle \xi^e \rangle A(a, \tilde{\xi}^e) U + F
\]

where

\[
(10.43) \quad A(a, \tilde{\xi}) = \begin{pmatrix} 0 & 1 \\ \frac{a_0}{a_{3,3}} (\tilde{\gamma} + i \tilde{\tau}) + A_y(a, \tilde{\xi}) & A_z(a, \tilde{\xi}) \end{pmatrix}
\]

and where we set \( \tilde{\xi} = (\tilde{\gamma}, \tilde{\tau}, \tilde{\xi}) \) with

\[
\tilde{\gamma} = \gamma / \langle \xi \rangle^2, \quad \tilde{\tau} = \tau / \langle \xi \rangle^2, \quad \tilde{\xi} = \xi / \langle \xi \rangle
\]

and for \( \tilde{\xi}^e \), we replace \((\gamma, \tau, \varepsilon \xi)\) by \((\gamma, \tau, \varepsilon \xi)\). Moreover, the source term \( F \) is defined by

\[
(10.44) \quad F = \frac{1}{\langle \xi^e \rangle}(0, R + F).
\]

The boundary condition at \( z = 0 \) becomes

\[
(10.45) \quad \Gamma U = \Omega^b, \quad \Gamma(U_1, U_2)^t = U_1
\]

where the writing \( U = (U_1, U_2)^t \) is related to the block structure of (10.43).

Note that \( \tilde{\xi} \) is in the compact set \( \tilde{\gamma}^2 + \tilde{\tau}^2 + |\tilde{\xi}|^4 = 1, \tilde{\gamma} \geq 0. \)
Proposition 10.7. There exists $\gamma_0 > 0$ such that for every $\gamma \geq \gamma_0$, we have for the solution of \eqref{10.42}, \eqref{10.45} the estimate
\[
\langle \xi^2 \rangle|U|^2_{L^2_x} + \sqrt{\varepsilon}|U(0)|^2 \leq C\sqrt{\varepsilon} |\Omega|^2 + \left(\frac{|F|_{L^2_x}}{\langle \xi^2 \rangle} \right)^2.
\]

Note that $\gamma_0$ only depends on the estimates \eqref{10.37}. In terms of $\Omega$, the previous proposition gives the estimate
\[
(10.46) \quad \langle \xi^2 \rangle|U|^2_{L^2_x} = \langle \xi^2 \rangle|\Omega|^2_{L^2_x} + \frac{\varepsilon}{\langle \xi^2 \rangle} |\partial_x \Omega|^2_{L^2_x} \leq C\sqrt{\varepsilon} |\Omega|^2 + \left(\frac{|F|_{L^2_x}}{\langle \xi^2 \rangle} \right)^2.
\]

We shall prove this Proposition by using the symmetrizers method.

Lemma 10.8. There exists $S(a, \tilde{\xi})$ symmetric, smooth in its argument and $\kappa > 0$ such that
\[
SA + (SA)^* \geq \kappa Id, \quad S + \Gamma^* \Gamma \geq \kappa Id
\]
for every $(a, \tilde{\xi}) \in K \times S_+$ where $K$ is the compact set defined by \eqref{10.37} and
\[
S_+ = \{\tilde{\xi}, \langle \xi, \tilde{\xi} \rangle = 1, \tilde{\gamma} \geq 0\}.
\]

The proof of the Proposition can be easily obtained from the Lemma. We multiply the equation by $S(a(z), \tilde{\xi})U$ and we integrate in $z$. This yields by using Lemma 10.8
\[
\kappa \langle \xi^2 \rangle|U|^2_{L^2_x} + \sqrt{\varepsilon}|U(0)|^2 \leq C \left( |F|_{L^2_x} |U|_{L^2_x} + \sqrt{\varepsilon} |\partial_x S|_{L^\infty} |U|_{L^2_x} + \sqrt{\varepsilon} |\Gamma U(0)|^2 \right).
\]

Note that $\sqrt{\varepsilon} |\partial_x S|_{L^\infty}$ is uniformly bounded thanks to \eqref{10.37} and hence we get from Cauchy-Schwarz that
\[
\langle \xi^2 \rangle|U|^2_{L^2_x} + \sqrt{\varepsilon}|U(0)|^2 \leq C \left( \frac{|F|_{L^2_x}}{\langle \xi^2 \rangle} + |U|^2_{L^2_x} \right) + \sqrt{\varepsilon} |\Gamma U(0)|^2.
\]

By using \eqref{10.44}, we get that $|F| \lesssim |U|$ and hence the result follows by choosing $\gamma$ sufficiently large.

It remains the proof of Lemma 10.8. It can be obtained from classical arguments.

We first prove that the eigenvalues of $A$ has nonzero real part. If $X = (X_1, X_2)$ is an eigenvector of $A(a, \tilde{\xi})$ associated to the eigenvalue $\mu$, we get that $X_2 = \mu X_1$ and that
\[
a_{33}\mu^2 X_1 = \left( a_0(\tilde{\gamma} + i\tilde{\tau}) + a_{3,3}A_y(\tilde{\xi}) + \mu a_{3,3}A_3(\tilde{\xi}) \right) X_1.
\]

If we assume that $\mu = i\lambda$, this yields
\[
a_0(\tilde{\gamma} + i\tilde{\tau}) + \eta \eta \bar{\eta} a_{ij} = 0
\]
where $\eta = (\xi_1, \xi_2, \lambda)$. By using \eqref{10.37}, we find by taking the real part that $\eta = 0$ and $\tilde{\gamma} = 0$ and thus also $\tilde{\tau} = 0$. This is impossible for $\langle \xi^2 \rangle = 1$. Consequently, there is no eigenvalue on the imaginary axis. Moreover, we easily see that there is one eigenvalue with positive real part $\mu_+$ and one with negative real part $\mu_-$. This yields that we can diagonalize $A$ in a smooth way: there exists a smooth invertible matrix $P(a, \tilde{\xi})$ such that
\[
A = P \begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix} P^{-1}.
\]

We thus choose $S$ in a classical way under the form
\[
S = (P^{-1})^* \begin{pmatrix} 1 & 0 \\ 0 & -\delta \end{pmatrix} P^{-1}
\]
with $\delta > 0$ to be chosen. The first property on $S$ in Theorem 10.8 is therefore met. Note that smooth projections on the subspace associated to the positive and respectively negative eigenvalue of $A$ are given by

$$
\Pi_+ = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}, \quad \Pi_- = P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}
$$

To get the second one, we note that if $X$ is an eigenvector of $A(a, \zeta)$ associated with the eigenvalue of positive real part (we recall that we study the equation for $z < 0$) we necessarily have $\Gamma X \neq 0$. In other words, we have $\text{Ker} \Gamma \cap \text{Ker} \Pi_-$. This yields that the map $X \in \mathbb{R}^2 \mapsto (\Gamma X, \Pi_- X)$ is invertible and hence by compactness, there exists $\alpha > 0$ such that for every $X \in \mathbb{R}^2$, every $a$ in the compact set (10.37) and $\tilde{\zeta} = 1, \tilde{\gamma} \geq 0$, we have

$$
|\Gamma X|^2 + |\Pi_- (a(0), \zeta) X|^2 \geq \alpha |X|^2.
$$

This allows to choose $\delta$ sufficiently small in order to get the second property in Lemma 10.8.

10.3. Energy estimate via microlocal symmetrizer. We shall now give the proof of Theorem 10.6. Let us take $T \in [0, T^*]$. We consider the solution of (10.33), (10.35) on $[0, T]$. Since the initial value is zero, we can assume that $\Omega^\alpha$ is zero for $t \leq 0$. Thanks to Lemma 10.5, we can assume that $a_0$ and $a_{ij}$ verify the estimates (10.37) on $[0, T] \times S$ by a suitable choice of the numbers $m, M$ and $c_0$ (thus these numbers depend on $\Lambda_{\infty, T}$) Note that thanks to Lemma 10.5 we can also assume that

$$
(10.47) \quad \|\partial_{x,y} (a_0, a_{ij})\|_{L^\infty} + \sqrt{\varepsilon} \|\partial_{x,y} \nabla a_{ij}\|_{L^\infty} \leq M.
$$

We can choose extensions of these coefficients on $\mathbb{R} \times S$ such that the new coefficients still satisfy (10.37) and the above estimate. We shall not use a different notation for the extensions. We also extend the boundary value $(\Omega^\alpha)^ b$ in (10.35) by 0 for $t \geq T$ and for $t \leq 0$ and denote by $g$ this new function. We shall consider the solution of

$$
(10.48) \quad a_0 (\partial_t + \gamma) \rho - \varepsilon \partial_i (a_{ij} \partial_j \rho) = 0, \quad (t, x) \in \mathbb{R} \times S, \quad \rho (t, y, 0) = g
$$

which vanishes for $t \leq 0$. By using a standard uniqueness result for this parabolic equation, we get that $\rho = \Omega^\alpha$ on $[0, T] \times S$ consequently, by using the notations of section 13, it suffices to prove the estimate:

**Proposition 10.9.** The solution of (10.48) satisfies the estimate

$$
(10.49) \quad \|\rho\|_{H^{\frac{1}{2}} (\mathbb{R}, L^2 (S))}^2 \leq C \sqrt{\varepsilon} \|g\|_{L^2 (\mathbb{R}^2)}^2.
$$

for $\gamma$ sufficiently large, where $C$ depending only of the parameters in (10.37), (10.47) (in particular it is independent of $\varepsilon$)

Indeed, let us assume that this last proposition is proven, then since

$$
\|\rho\|_{H^{\frac{1}{2}} (\mathbb{R}, L^2 (S))} \leq \|\rho\|_{H^{\frac{1}{2}} \gamma, \varepsilon}
$$

we find that

$$
(10.50) \quad \|\Omega^\alpha\|_{H^{\frac{1}{2}} ([0, T], L^2 (S))}^2 \leq C \sqrt{\varepsilon} \|\Omega^\alpha\|_{L^2 ([0, T] \times \mathbb{R}^2)}^2.
$$

We shall thus focus on the proof of (10.49). Again, we shall consider that $\rho$ is a scalar function. By using the notations of the previous subsection, we can define two symbols (with $z$ as parameter) $a_y$ and $a_z$ by

$$
da_y (X, \zeta, z) = A_y (a(t, y, z), \xi), \quad a_z (X, \zeta, z) = A_z (a(t, y, z), \xi),
$$

in such a way that we can rewrite (10.48) under the form

$$
\varepsilon \partial_{zz} \rho = \frac{a_0}{a_{33}} (\partial_t + \gamma) \rho + a_y (X, \sqrt{\varepsilon} \partial_y, z) \rho + a_z (X, \sqrt{\varepsilon} \partial_y, z) \sqrt{\varepsilon} \partial_z \rho + R^1
$$

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where
\[ R^1 = \frac{\sqrt{\varepsilon} \partial_{aij}}{a_{3,3}} \sqrt{\varepsilon} \partial_j \rho. \]

In view of the model estimate of Proposition \ref{10.7} we need to control \[ \|R^1\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon}. \]
Let us set
\[ b_{ij} = \frac{\sqrt{\varepsilon} \partial_{aij}}{a_{3,3}}. \]

Let us start with the estimates of the terms where \( j \neq 3 \). We first note that
\[ \|b_{ij} \sqrt{\varepsilon} \partial_j \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \|\sqrt{\varepsilon} \partial_j (b_{ij} \rho)\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} + \|\sqrt{\varepsilon} \partial_j b_{ij} \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon}. \]

For the second term, thanks to the uniform estimate (10.37) and (10.47), we can write
\[ \|\sqrt{\varepsilon} \partial_j b_{ij} \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \frac{1}{\gamma^2} \|\sqrt{\varepsilon} \partial_j b_{ij} \rho\|_{L^2} \leq \frac{C}{\gamma^2} \|\rho\|_{L^2}. \]

For the first term thanks to the definition of the weighted norm, we get
\[ \|\sqrt{\varepsilon} \partial_j (b_{ij} \rho)\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \|b_{ij} \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \frac{C}{\gamma^2} \|\rho\|_{L^2}. \]

Consequently, we have proven that
\[ \|b_{ij} \sqrt{\varepsilon} \partial_j \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \frac{C}{\gamma^2} \|\rho\|_{L^2}. \]

for \( j \neq 3 \). It remains the case that \( j = 3 \) which is more complicated. We shall use a duality argument and the paradifferential calculus of section 15. We first write
\[ \|b_{ij} \sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \frac{1}{\gamma^2} \|b_{ij} \sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon}. \]

Next for any test function \( f \) in the Schwartz class, we write
\[ |(b_{ij} \sqrt{\varepsilon} \partial_x \rho, f)_{L^2}| \leq |(\sqrt{\varepsilon} \partial_x \rho, b_{ij} f)_{L^2}| \leq \|\sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \|b_{ij} f\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon}. \]

Next, thanks to the estimate (10.37), (10.47), we get in particular that \( \|b_{ij}\|_{L^\infty} \) and \( \|\nabla_{t,y} b_{ij}\|_{L^\infty} \) are uniformly bounded, thus we find
\[ \|b_{ij} f\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \|T_{b_{ij}} f\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} + \|b_{ij} - T_{b_{ij}}\|_{H^1,\gamma,\varepsilon} \leq C \|f\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \]
by using in Theorem 15.3 the estimate (1) and the first estimate in (5). This proves by duality that
\[ \|b_{ij} \sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq C \|\sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \]
and hence that
\[ \|b_{ij} \sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \frac{C}{\gamma^2} \|\sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon}. \]

Consequently, we have proven that
\[ \|R^1\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \leq \frac{C}{\gamma^2} \left( \|\rho\|_{L^2} + \|\sqrt{\varepsilon} \partial_x \rho\|_{\mathcal{H}^\frac{1}{2},\gamma,\varepsilon} \right). \]

Note that in view of the left hand side in the model estimate of Proposition \ref{10.7}, the above estimate is a good estimate since for \( \gamma \) sufficiently large, it will be possible to absorb this term by the principal term of our estimate.

Next, we can replace products by paraproducts in the equation (10.52) to rewrite it under the form
\[ \varepsilon \partial_{zz} \rho = T_{a_0/a_{33}}^{\varepsilon,\gamma} (\partial_t + \gamma) \rho + T_{a_y}^{\varepsilon,\gamma} \rho + T_{a_z}^{\varepsilon,\gamma} \sqrt{\varepsilon} \partial_x \rho + R \]
where
\begin{align}
R &= R^1 + R^2, \\
R^2 &= \left(\frac{a_0}{a_{33}} - T^{\varepsilon,\gamma}_{a_0/a_{33}}\right)(\partial_t + \gamma) - \sum_{1 \leq i,j \leq 2} \left(\frac{a_{ij}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{ij}/a_{33}}\right)\varepsilon \partial_i \partial_j \rho \\
&\quad - 2 \sum_{1 \leq k \leq 2} \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\varepsilon \partial_k \partial_z \rho.
\end{align}

As before, we need to estimate \(\|R^2\|_{H^{-\frac{1}{2},\gamma,\varepsilon}}\). For the first term in the definition of \(R^2\), we can use (10.37) and (10.47) to write
\[
\left\| \left(\frac{a_0}{a_{33}} - T^{\varepsilon,\gamma}_{a_0/a_{33}}\right)(\partial_t + \gamma)\rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} \\
\leq \left\| (\partial_t + \gamma)\left(\frac{a_0}{a_{33}} - T^{\varepsilon,\gamma}_{a_0/a_{33}}\right)\rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} + \|\partial_t\left(\frac{a_0}{a_{33}}\right)\rho\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} + \|T^{\varepsilon,\gamma}\left(\frac{a_0}{a_{33}}\right)\rho\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} \\
\leq \left\| \left(\frac{a_0}{a_{33}} - T^{\varepsilon,\gamma}_{a_0/a_{33}}\right)\rho \right\|_{H^{\frac{1}{2},\gamma,\varepsilon}} + \frac{C}{\gamma^4} \|\rho\|_{L^2} \\
\leq \frac{C}{\gamma^4} \|\rho\|_{L^2}
\]

where we have used the \(L^2\) continuity of paraproducts (i.e. (1) with \(\mu = 0\) in Theorem 15.3) and the first estimate in (5) of Theorem 15.3 to get the two last lines. For the second type of terms in \(R^2\), we can proceed in the same way (note that \(\varepsilon \partial_{ij}\) is an operator of order 2 and hence has the same order as \(\partial_i\) in our calculi) to obtain that
\[
\left\| \left(\frac{a_{ij}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{ij}/a_{33}}\right)\varepsilon \partial_i \partial_j \rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} \leq \frac{C}{\gamma^4} \|\rho\|_{L^2}.
\]

For the last type of terms in \(R^2\), we first proceed in the same way to write
\[
\left\| \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\varepsilon \partial_k \partial_z \rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} \\
\leq \left\| \sqrt{\varepsilon} \partial_k \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} + \left\| \sqrt{\varepsilon} \partial_k \left(\frac{a_{k3}}{a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} + \left\| T^{\varepsilon,\gamma}\left(\frac{a_{k3}}{a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}} \\
\leq \frac{1}{\gamma^4} \left\| \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\varepsilon \partial_z \rho \right\|_{L^2} + \frac{C}{\gamma^2} \left\| \sqrt{\varepsilon} \partial_z \rho \right\|_{H^{-\frac{1}{2},\gamma,\varepsilon}}.
\]

Indeed, to get the last line, we have used the continuity of the paraproduct and that the term \(\|\sqrt{\varepsilon} \partial_k \left(\frac{a_{k3}}{a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho \|_{H^{-\frac{1}{2},\gamma,\varepsilon}}\) can be estimated in a similar way as it was done to get (10.32). To estimate the first term in the right hand side of the above inequality, we can again use a duality argument. For any test function \(f\), we have
\[
\left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho, f\right)_{L^2} = \left(\sqrt{\varepsilon} \partial_z \rho, \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)f\right) - \left(\sqrt{\varepsilon} \partial_z \rho, (T^{\varepsilon,\gamma}_{a_{k3}/a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}) f\right)
\]
where we have used that \(a_{k3}/a_{33}\) is a scalar real valued function. Consequently, by using again Theorem 15.3 we obtain that
\[
\left| \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho, f\right|_{L^2} \leq C\|\sqrt{\varepsilon} \partial_z \rho\|_{H^{-1,\gamma,\varepsilon}} \|f\|_{L^2}
\]
and hence that
\[
\left\| \left(\frac{a_{k3}}{a_{33}} - T^{\varepsilon,\gamma}_{a_{k3}/a_{33}}\right)\sqrt{\varepsilon} \partial_z \rho \right\|_{L^2} \leq C\|\sqrt{\varepsilon} \partial_z \rho\|_{H^{-1,\gamma,\varepsilon}}.
\]
Consequently, we finally obtain that
\[
\left\| \left( \frac{a_{ij}}{a_{33}} - T_{a_{ij}/a_{33}}^{\varepsilon, \gamma} \right) \varepsilon \partial_i \partial_j \rho \right\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}} \leq \frac{C}{\gamma^2} \left\| \sqrt{\varepsilon} \partial_z \rho \right\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}}
\]
and hence by collecting the previous estimates and (10.53), we get that
\[
(10.57) \quad \|R\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}} \leq \frac{C}{\gamma^4} \left( \|\rho\|_{L^2} + \|\sqrt{\varepsilon} \partial_z \rho\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}} \right).
\]
Next, we can rewrite (10.54) as a first order system by setting
\[
(10.58) \quad U = (\rho, T_{1/(\langle \xi \rangle)}^{\varepsilon, \gamma} \sqrt{\varepsilon} \partial_z \rho)^t.
\]
Note that \(T_{1/(\langle \xi \rangle)}^{\varepsilon, \gamma}\) is just the Fourier multiplier by \(1/\langle \xi \rangle\). This yields the system
\[
(10.59) \quad \sqrt{\varepsilon} \partial_z U = T_M^{\varepsilon, \gamma} U + F, \quad z < 0
\]
where the symbol \(M \in \Gamma_1\) is given by
\[
M(X, \zeta, z) = \langle \xi \rangle A(a(X, z), \tilde{\xi}), \quad \tilde{\xi} = (\tilde{\gamma}, \tilde{\tau}, \tilde{\xi}) = (\gamma/\langle \xi \rangle^2, \tau/\langle \xi \rangle^2, \xi/\langle \xi \rangle)
\]
where \(A\) is defined in (10.43) and the source term \(F\) is defined by
\[
F = (0, T_{1/(\langle \xi \rangle)}^{\varepsilon, \gamma} R + C)^t
\]
where \(C\) is a lower order commutator. By using (10.57) and (2) in Theorem 15.3 to estimate \(C\), we get that
\[
(10.60) \quad \|F\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}} \leq \frac{C}{\gamma^4} \|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}.
\]
The boundary condition for (10.59) becomes
\[
(10.61) \quad \Gamma U|_{z=0} = g
\]
where \(\Gamma\) is still defined by (10.46). We can then perform an energy estimate for the system (10.59), (10.61) by using the symmetrizer \(S\) constructed in Lemma 10.8. Indeed let us define a symbol \(S(X, \zeta, z) \in \Gamma_1\) by
\[
S(X, \zeta, z) = S(a(X, z), \tilde{\zeta}, z).
\]
From Lemma 10.8 we get that at the level of symbols, we have
\[
\text{Re } S M \geq \kappa(\zeta), \quad \text{Re } S|_{z=0} + \Gamma^* \Gamma \geq \kappa.
\]
Consequently, we can perform an energy estimate in a standard way by taking the scalar product of (10.59) with \(T_S^{\varepsilon, \gamma} U\) and by using Theorem 15.3 (in particular the estimates (2), (3) and the Garding inequality (4)). This yields
\[
\|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}^2 + \sqrt{\varepsilon}\|U(0)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{C}{\gamma^4} \left( \|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}^2 + \sqrt{\varepsilon}\|U(0)\|_{L^2(\mathbb{R}^3)}^2 \right) + \sqrt{\varepsilon} C \|g\|_{L^2(\mathbb{R}^3)}^2 + \|F, T_{1/(\langle \xi \rangle)}^{\varepsilon, \gamma} U\|_{L^2}.\]
Since by using (10.60), we get
\[
\|F, T_{1/(\langle \xi \rangle)}^{\varepsilon, \gamma} U\|_{L^2} \leq \|F\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}} \|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}} \leq \frac{C}{\gamma^4} \|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}^2.
\]
Consequently for \(\gamma\) sufficiently large (with respect to \(C\)), we obtain the estimate
\[
\|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}^2 \leq \sqrt{\varepsilon} C \|g\|_{L^2(\mathbb{R}^3)}^2.
\]
To conclude, we note that
\[
\|U\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}^2 = \|\rho\|_{H^{\frac{1}{2}, \gamma, \varepsilon}}^2 + \|\sqrt{\varepsilon} \partial_z \rho\|_{H^{-\frac{1}{2}, \gamma, \varepsilon}}^2.
\]
This ends the proof of Proposition 10.9

11. Proof of Theorem 1.1: uniform existence

In this section, we shall prove how we can combine all our energy estimates to get our uniform existence result. Let us fix \( m \geq 6 \). We consider initial data such that

\[
\mathcal{I}_m(0) = \|v_0\|_{E^m} + |h_0|_m + \sqrt{\varepsilon}|h_0|_{m+4} + \|v_0\|_{E^{2,\infty}} + \varepsilon\|\partial_z v(0)\|_{L^{\infty}} < +\infty.
\]

For such data, we are not aware of a local existence result. We could prove it by using our energy estimates and a classical iteration scheme. Nevertheless, we can also avoid this by using the available classical existence results in Sobolev spaces (for example [6, 52]). Indeed, we can first smooth the initial velocity and consider a sequence \( \alpha^\delta \) that meets the assumption of [6, 52]. This allows to get a positive time \( T^{\varepsilon, \delta} \) for which a solution \( v \) associated to this initial data exists in the space \( K^\varepsilon([0, T^{\varepsilon, \delta}] \times S) = H^\varepsilon([0, T], L^2) \cap L^2([0, T], H^r) \). Next, we can get by standard parabolic energy estimates that additional regularity propagates from the initial data, that is to say that on \([0, T^{\varepsilon, \delta}]\), we have

\[
\mathcal{K}(T) = \sup_{[0,T]} (\|v(t)\|_{E^m} + \|\partial_z v\|_{E^m-2} + |h(t)|_m + \varepsilon|h(t)|_{m+4} + \varepsilon\|\partial_z v(t)\|_{E^{2,\infty}} + \varepsilon\|\partial_z v(t)\|_{L^{\infty}} < +\infty.
\]

Moreover, we can also get from the initial condition that (7.1) is valid on \([0, T^{\varepsilon, \delta}]\) (possibly by taking \( T^{\varepsilon, \delta} \) smaller). Note that since we do not propagate any additional normal regularity, we do not need additional compatibility conditions. We shall not detail this step since it can be done by classical energy estimates (much simpler than the ones we have proven since in this stage we are not interested in estimates independent of \( \varepsilon \).)

An important remark is that if \( \mathcal{K}(T_0) < +\infty \), then the solution can be continued on \([0, T_1]\), \( T_1 > T_0 \) with \( \mathcal{K}(T_1) < +\infty \). Indeed if \( \mathcal{K}(T_0) < +\infty \), we can use the parabolic regularity for the Stokes problem on \([T_0/2, T_0]\) to get that the solution actually enjoys much more standard Sobolev regularity on \([T_0/2, T_0]\) (note that we assume that the surface \( h \) is \( H^3 \)) and in particular, we find that \( u(T_0) \in H^{r-1}, \ 3 < r < 7/2 \). This allows to use again the result of Beale [6] to continue the solution and the previous argument about the propagation of additional regularity to get our claim.

Next we want to use this remark to prove that the solution can be continued on an interval of time independent of \( \varepsilon \) and \( \delta \). Towards this, we first note that it is equivalent to control \( \mathcal{K}(T) \) and \( \mathcal{E}_m(T) \) where

\[
\mathcal{E}_m(T) = \sup_{[0,T]} Q_m(t) + D_m(T) + \|\omega\|_{L^4([0,T], H^{m-1/2})}^2
\]

with \( Q_m \) defined by (9.7) and

\[
D_m(T) = \varepsilon \int_0^T (\varepsilon\|\nabla v_m\|^2 + \varepsilon\|\nabla S_{n}\|_{m-2}^2).
\]

Indeed, the fact that \( \mathcal{K}(T) \leq \Lambda(\frac{1}{c_0}, \mathcal{E}_m(T)) \) is a consequence of (7.2), (8.1), (10.2), Lemma 8.2 and Corollary 9.3 while the reverse inequality is just a consequence of product estimates.

For two parameters \( R \) and \( c_0 \) to be chosen \( 1/c_0 << R \), we can thus define a time \( T^{\varepsilon, \delta} \)

\[
T^{\varepsilon, \delta} = \sup \{ T \in [0,1], \ \mathcal{E}_m(t) \leq R, \ |h(t)|_{2,\infty} \leq 1/c_0, \ \partial_z \varphi(t) \geq c_0, \ g - \partial_z^2 q^E(t) \geq \frac{c_0}{2}, \ \forall t \in [0,T].\}
\]
At first, let us notice that thanks to Corollary 9.3, we have that for $T \leq T_{*,\delta}^c$,
\[ \Lambda_{\infty,\delta}(T) \leq \Lambda(R) \]
where $\Lambda_{\infty,m}$ is defined by (9.8). Thanks to Corollary 9.10, we also have
\[ \int_0^T \sqrt{\varepsilon} \| \nabla^2 v \|_{1,\infty} \leq \Lambda(R). \]
This allows to use Proposition 7.1, Proposition 9.4, Proposition 9.8 and Proposition 10.1 to get that
\[ \mathcal{E}_m(T) \leq \Lambda \left( \frac{1}{c_0}, \mathcal{I}_m(0) \right) + \Lambda(R) \left( T^\frac{1}{2} + \Lambda(R) \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} + \Lambda(R) \int_0^T |\omega|_{m-1}^2 \right). \]
Consequently, from the Cauchy-Schwarz inequality, we find that
\[ \mathcal{E}_m(T) \leq \Lambda \left( \frac{1}{c_0}, \mathcal{I}_m(0) \right) + \Lambda(R) \left( T^\frac{1}{2} + \Lambda(R) \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} \right). \]
To estimate the last term in the right hand side, we can use Proposition 6.5 to find
\[ \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} \leq \Lambda(R) (T + \int_0^T (\varepsilon \| \partial_z z_v \|_{L^\infty} + \varepsilon \| \partial_z z_v \|_3)) \]
and hence thanks to (8.4), we find
\[ \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} \leq \Lambda(R) (T + \int_0^T \varepsilon \| \partial_z S_n \|_3) \leq \Lambda(R) \sqrt{T} \]
where the last estimate comes again from the Cauchy-Schwarz inequality. Consequently, we obtain from (11.2) that
\[ \mathcal{E}_m(T) \leq \Lambda \left( \frac{1}{c_0}, \mathcal{I}_m(0) \right) + \Lambda(R) T^{\frac{1}{2}}. \]
Moreover, thanks to the equation (1.11), we get that
\[ |h(t)|_{2,\infty} \leq |h(0)|_{2,\infty} + \Lambda(R) T, \quad \forall t \in [0, T] \]
and also
\[ \partial_z \varphi(t) \geq 1 - \int_0^t \| \partial_t \nabla \eta \|_{L^\infty} \geq 1 - \Lambda(R) T, \quad \forall t \in [0, T] \]
since we have chosen $A$ so that (1.11) is verified. Finally, since
\[ g - (\partial^2_z q^E)^b(t) = (g - (\partial^2_z q^E)^b)_{t=0} - \Lambda(R) \int_0^t (1 + |(\partial_t \partial_z q^E)^b|_{L^\infty}), \]
we get from (11.3) that
\[ g - (\partial^2_z q^E)^b \geq (g - (\partial^2_z q^E)^b)_{t=0} - \sqrt{t} \Lambda(R), \quad \forall t \in [0, T]. \]
In view of (11.7), (11.5), (11.6) and (11.4), we can take $R = 2\Lambda(|h(0)|_{2,\infty}, \mathcal{I}_m(0))$ to get that there exists $T_*$ which depends only on $\mathcal{I}_m(0)$ (and hence does not depend on $\varepsilon$ and $\delta$) so that for $T \leq \min \{ T_*, T_{*,\delta}^c \}$, we have
\[ \mathcal{E}_m(T) \leq R/2, \quad |h(t)|_{2,\infty} \leq \frac{1}{2c_0}, \quad \partial_z \varphi(t) \geq 2c_0, \quad g - \partial^2_z q^E(t) \geq \frac{3}{4} c_0, \quad \forall t \in [0, T]. \]
This yields $T_{*,\delta}^c \geq T_*$. Indeed otherwise, our criterion about the continuation of the solution would contradict the definition of $T_{*,\delta}^c$.  

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We have thus proven that for the smoothed initial data \(v_0^\delta\), there exists an interval of time \([0, T_\ast]\) independent of \(\varepsilon\) and \(\delta\) for which the solution exists and such that \(\mathcal{N}_m(T_\ast) < +\infty\). To get the existence of solution without the additional regularity for the initial data, it suffices to pass to the limit. The fact that \(\mathcal{N}_m(T_\ast)\) is uniformly bounded in \(\delta\) allows to pass to the limit easily by using strong compactness arguments. We shall not give more details since the arguments are very close to the ones that allow to get the inviscid limit that we shall detail below.

12. Uniqueness

12.1. Uniqueness for Navier-Stokes. In this section, we shall prove the uniqueness part of Theorem 1.1. We consider two solutions \((v^1, \varphi^1, q^1)\) with the same initial data of \((1.12), (1.13), (1.14)\) which satisfy on \([0, T]\),

\[
\mathcal{N}_m^i(T) \leq R, \quad i = 1, 2
\]

where \(\mathcal{N}_m\) is defined above in \((1.1)\) and the superscript \(i\) refers to one of the two solutions. We also assume that for each solution \((7.1)\) is verified. We set \(v = v^1 - v^2\), \(h = h^1 - h^2\), \(q = q^1 - q^2\).

At first, as in the proof of Proposition 3.4 (see \((3.9)\)), we can show from \((12.5)\) that

\[
(\partial_t + v^1_y \cdot \nabla_y + V^1_z \partial_z) v + \nabla^\varphi q - \varepsilon \Delta^\varphi v = \mathcal{F}
\]

with \(\mathcal{F}\) given by

\[
\mathcal{F} = (v^1_y - v^2_y) \cdot \nabla_y v^2 + (V^1_z - V^2_z) \partial_z v^2 - \varepsilon \left(\frac{1}{\partial_{\varphi^1}} - \frac{1}{\partial_{\varphi^2}}\right) ((P^1)^* \nabla q^2) + \varepsilon \frac{1}{\partial_{\varphi^1}} \nabla \cdot (E^1 \nabla v^2) + \varepsilon \frac{1}{\partial_{\varphi^2}} \nabla \cdot ((E^1 - E^2) \nabla v^2).
\]

In a similar way, thanks to \((6.6)\), we can write the divergence free condition under the form

\[
\nabla^\varphi \cdot v = -\varepsilon \left(\frac{1}{\partial_{\varphi^1}} - \frac{1}{\partial_{\varphi^2}}\right) \nabla \cdot (P^1 v^2) - \varepsilon \frac{1}{\partial_{\varphi^2}} \nabla \cdot ((P^1 - P^2) v^2).
\]

On the boundary, we obtain from \((1.13), (1.14)\) that

\[
\partial_t h + (v^1_y b^1_y) \cdot \nabla h - ((v^1)^b - (v^2)^b) = -((v^1)^b - (v^2)^b) \cdot \nabla h^2
\]

and

\[
q \mathbf{n}^1 - 2\varepsilon S^\varphi v \mathbf{n}^1 = gh \mathbf{n}^1 + 2\varepsilon (S^{\varphi^1} - S^{\varphi^2}) v^1 \mathbf{n}^1 + 2\varepsilon S v^2 v^2 (\mathbf{n}^1 - \mathbf{n}^2).
\]

At first, as in the proof of Proposition 3.4 (see \((3.9)\)), we can show from \((12.5)\) that

\[
\varepsilon |h(t)|^2 \leq \varepsilon \int_0^t \|\nabla v\|_{L^2(S)}^2 + \Lambda(R) \int_0^t (\|v\|_{L^2(S)}^2 + \varepsilon |h|^2) d\tau.
\]

Next, we can use again a standard energy estimate for \((12.2)\), by using Proposition 3.1 and Proposition 6.3 from a lengthy but easy computation, we obtain for \(v = v^1 - v^2\), \(h = h^1 - h^2\) that

\[
\|v(t)\|_{L^2(S)}^2 + \|h(t)\|_{L^2(S)}^2 + \varepsilon \int_0^t \|\nabla v\|_{L^2(S)}^2 \leq \Lambda(R) \int_0^t (|h|^2_{H^2_0(\mathbb{R}^2)} + \|v\|_{L^2(S)}^2 + \|\nabla(q^1 - q^2)\|_{L^2(S)} + \|v\|_{L^2(S)}).
\]
Note that we have used the estimates of Proposition 6.4 to estimate the pressure. We first apply (12.9) and by using the definition of $S^i v^i$, $i = 1, 2$, we obtain integrals like
\[ \varepsilon \int_{z=0} \partial_z v_j^1 \partial_j h v_j dy \]
where $1 \leq i \leq 2$. We can estimate it by
\[ \left| \varepsilon \int_{z=0} \partial_z v_j^1 \partial_j h v_j dy \right| \leq \varepsilon \left| \partial_z v_j^1 v_j \right|_{H^\frac{1}{2}(\mathbb{R}^2)} + \left| \partial_j h \right|_{H^\frac{1}{2}(\mathbb{R}^2)} \leq \Lambda(R) \varepsilon \left\| v \right\|_{H^1(S)} \left\| h \right\|_{H^\frac{1}{2}(\mathbb{R}^2)} \]
where the last estimate follows from (12.8) and the trace Theorem. We can then use the Young inequality.

From the equation for the pressure, we can also thanks to the estimates of section 6 obtain that
\[ \left\| \nabla (q^1 - q^2) \right\|_{L^2(S)} \leq \Lambda(R) \left( \left\| h \right\|_{H^\frac{1}{2}} + \left\| v \right\|_{H^1(S)} \right). \]
Note that there is no $\varepsilon$ in front of $\left\| h \right\|_{H^\frac{1}{2}}$ in this estimate because of the Euler part of the pressure. This yields
\[ \left\| v(t) \right\|_{L^2(S)}^2 + \left\| h(t) \right\|_{L^2(S)}^2 + \varepsilon \int_0^t \left\| \nabla v \right\|_{L^2(S)}^2 \leq \Lambda(R) \int_0^t \left( \left\| h \right\|_{H^\frac{1}{2}(\mathbb{R}^2)}^2 + \left\| v \right\|_{L^2(S)}^2 \right). \]
Consequently, we get the uniqueness for Navier-Stokes by combining the last estimate and (12.8). Note that the above estimate is not uniform in $\varepsilon$ and thus does not allow to recover the uniqueness for Euler.

12.2. Uniqueness for Euler.

Proposition 12.1. Consider $(v^i, h^i)$, $i = 1, 2$ two solutions of (1.22), (1.23) defined on $[0, T]$ with the same initial data and the regularity stated in Theorem 1.2. Then $v^1 = v^2$ and $h^1 = h^2$.

Proof. We assume that
\[ \sup_{i, [0, T]} \left( \left\| v^i \right\|_m + \left\| \partial_z v^i \right\|_{m-2} + \left\| h^i \right\|_m + \left\| \partial_z v^i \right\|_{1, \infty} \right) \leq R. \]
The proof relies almost only on arguments that have been used previously, we shall consequently only give the main steps. Let us also set $v = v^1 - v^2$, $h = h^1 - h^2$. By using at first the same crude estimate as we have used above, we first obtain that
\[ \left\| v(t) \right\|_{L^2(S)}^2 + \left\| h(t) \right\|_{L^2(S)}^2 \leq \Lambda(R) \int_0^t \left( \left\| h \right\|_{H^\frac{1}{2}}^2 + \left\| v \right\|_1^2 + \left\| \partial_z v \right\|_{L^2}^2 \right). \]
Note that we have used the estimates of Proposition 6.4 to estimate the pressure.

Next, we shall estimate $\left\| v \right\|_1$ and $\left\| h \right\|_1$. Let us set
\[ \mathcal{E}(v, q, \varphi) = (\partial_t^q + v \cdot \nabla^q) v + \nabla^q q. \]
We first apply $Z_j$ for $j = 1, 2, 3$ to obtain:
\[ D\mathcal{E}(v^i, q^i, \varphi^i) \cdot (Z_j v^i, Z_j q^i, Z_j \varphi^i) = 0 \]
for $i = 1, 2$. We thus obtain that
\[ D\mathcal{E}(v^2, q^2, \varphi^2) \cdot (Z_j v^2, Z_j q^2, Z_j \varphi^2) + (D\mathcal{E}(v^1, q^1, \varphi^1) - D\mathcal{E}(v^2, q^2, \varphi^2)) \cdot (Z_j v^1, Z_j q^1, Z_j \varphi^1) = 0 \]
where we also set \( \varphi = \varphi^1 - \varphi^2 \), \( q = q^1 - q^2 \). Consequently, by using Lemma 2.7, we can introduce the good unknowns
\[
V_j = Z_j v - \partial_x^2 v^2 Z_j \varphi, \quad Q_j = Z_j q - \partial_x^2 q^2 Z_j \varphi
\]
to obtain that
\[
(\partial_t^2 + v^2 \cdot \nabla^2) V_j + \nabla^2 Q_j = \mathcal{R}
\]
where
\[
\mathcal{R} = -(V_j \cdot \nabla^2 v^2 + Z_j \varphi (\partial_x^2 v^2 \cdot \nabla^2) v^2 - (D\mathcal{E}(v^1, q^1, \varphi^1) - D\mathcal{E}(v^2, q^2, \varphi^2)) \cdot (Z_j v^1, Z_j q^1, Z_j \varphi^1).
\]
Consequently, by using that \( q^2 \) verifies the Taylor sign condition on \([0, T]\), we can proceed as in the proof of Proposition 7.1 to get the estimate:
\[
\|V_j(t)\|_{L^2(S)}^2 + \|Z_j h\|_{L^2(\mathbb{R}^2)}^2 \leq \Lambda(R) \int_0^t \left( |h|^2 + \|v\|_1^2 + \|\partial_x v\|_{L^2}^2 \right).
\]
In view of the estimates (12.9), (12.10), we still need to estimate \( \|\partial_x v\| \) in order to conclude from the Gronwall Lemma. Let us set \( \omega^i = \nabla v^i \times v^i, i = 1, 2 \) and \( \omega = \omega^1 - \omega^2 \). By using an estimate like (10.2), we first obtain that
\[
\|\partial_x v\|_{L^2} \leq \Lambda(R) \left( \|\omega\|_{L^2(S)} + |h|_1 + \|v\|_1 \right)
\]
and hence we see that it only remains to estimate \( \|\omega\|_{L^2(S)} \). Since \( \omega^i \) solves the equation
\[
(\partial_t^2 + v^i \cdot \nabla^2) \omega^i - \omega^i \cdot \nabla^2 v^i = 0,
\]
a standard estimate on the difference yields
\[
\|\omega(t)\|_{L^2(S)}^2 \leq \Lambda(R) \int_0^t \left( |h|^2 + \|v\|_1^2 + \|\partial_x v\|_{L^2}^2 + \|\omega(t)\|_{L^2(S)}^2 \right).
\]
It suffices to combine (12.9), (12.10), (12.11), (12.12) to end the proof of Proposition 12.1.

13. PROOF OF THEOREM 12.2 INIVCILD LIMIT

From the uniform estimates of Theorem 14.1 we have that for \( \varepsilon \in (0, 1] \)
\[
\mathcal{N}_m(T) = \sup_{[0,T]} \left( \|v^\varepsilon(t)\|^2 + \|\partial_x v^\varepsilon\|^2 + |h(t)|^2 + \varepsilon \|h(t)\|_{L^2}^2 + \varepsilon \|\partial_x v(t)\|^2 + \|v(t)\|_{E^{2,\infty}}^2 \right)
\]
+ \( \|\partial_x v^\varepsilon\|_{L^4([0,T], H^{m-1}_{co,loc})}^2 + \varepsilon \int_0^T \|\nabla v^\varepsilon\|_m^2 + \varepsilon \int_0^T \|\nabla^2 v^\varepsilon\|_{m-2}^2 \leq R \).

In particular, we have that \( h^\varepsilon \) is bounded in \( L^\infty([0,T], H^m) \), that \( v^\varepsilon \) is bounded in \( L^\infty([0,T], H^m_{co,loc}) \) and \( \nabla v^\varepsilon \) is bounded in \( L^\infty([0,T], H^m_{co,loc}) \). Thus for every \( t \), \( v^\varepsilon(t) \) is compact in \( H^{m-1}_{co,loc} \). Next, by using the equation, we also get that \( \partial_t v^\varepsilon \) is bounded in \( L^2([0,T], H^m_{co,loc}) \) and \( \partial t h^\varepsilon \) is bounded in \( L^2([0,T], L^2) \). Moreover, from Proposition 6.3 and Proposition 6.4 we also get that \( \nabla q^\varepsilon \) is bounded in \( L^2([0,T], L^2) \). By classical arguments, we deduce that there exists a sequence \( \varepsilon_n \) which tends to zero and \( (v, h, q) \) such that \( v^\varepsilon_n \) tends to \( v \) strongly in \( C([0,T], H^{m-1}_{co,loc}) \), \( h^\varepsilon_n \) tends to \( h \) strongly in \( C([0,T], H^{m-1}_{loc}) \), \( \nabla q^\varepsilon \) tends to \( \nabla q \) weakly in \( L^2([0,T] \times S) \), and
\[
\sup_{[0,T]} \left( \|v\|_m^2 + \|\partial_x v\|_{m-2}^2 + |h(t)|^2 + \|v(t)\|_{E^{2,\infty}}^2 \right) + \|\partial_x v^\varepsilon\|_{L^4([0,T], H^{m-1}_{co,loc})}^2 \leq R.
\]
These convergences allow to pass to the limit in the equations by classical arguments and hence, we find that \( (v, h, \nabla q) \) solves the Euler equation (12.22) in \( S \). Since, we can also assume
that the trace $v_j^{\varepsilon}_{j=0}$ converges weakly in $L^2((0, T] \times \mathbb{R}^2)$, we also get that the boundary condition $\partial_t h = v \cdot N$ is verified in the weak sense. To pass to the limit, in the boundary condition (11.13), we note that since the Lipschitz norm of $v^\varepsilon$ is uniformly bounded, it only remains in the limit $q = gh$.

For the pressure, we note that because of Proposition 6.3 we have that $\|\nabla (q^{\varepsilon}) \|^2_{L^2(S)} \leq \varepsilon \Lambda(R)$ and hence tends to zero strongly.

We have thus proven that $(v, h)$ is a solution of the free surface Euler equation that satisfies the estimate (13.12). To prove the strong convergence of $h^\varepsilon(t)$ to $(v, h)$, we can use the conservation of energy for the solution $(v, h)$ of the free surface Euler equation (that we formally get by taking $\varepsilon = 0$ in Proposition 4.1) to get that the traceline $h(t)$ to $(v, h)$ converges strongly to $(v, h)$. Since the strong convergence of $h$ gives the one of $J^\varepsilon$, we finally obtain by combining with the uniform bounds (13.1) that $(v^\varepsilon(t), h^\varepsilon(t))$ converge strongly towards $(v, h)$ in $L^2(S) \times L^2(\mathbb{R}^2)$. The $L^\infty$ convergence can be finally obtained thanks to the $L^2$ convergence, the uniform bounds (13.1) and the inequality (2.4) (with $s_2 = 0$). This ends the proof of Theorem 11.2.

14. Proof of the technical lemmas

14.1. Proof of Lemma 9.6. The estimate of $\|\rho\|_{L^\infty}$ and $\|\partial_i \rho\|_{L^\infty} = \|Z_i \rho\|_{L^\infty}$, $i = 1, 2$ can be easily obtained from the maximum principle as in [9.12]. Indeed, we get that $\partial_t \rho$ solves the equation

$$\partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz} \partial_t \rho + \partial_t H - \partial_t w \cdot \nabla \rho$$

still with an homogeneous Dirichlet boundary condition. Consequently, by using again the maximum principle, we find

$$\|\nabla y \rho\|_{L^\infty} \leq \|\rho_0\|_{1, \infty} + \int_0^t \left(\|H\|_{1, \infty} + \|\partial_t w \cdot \nabla \rho\|_{L^\infty}\right).$$

To estimate the last term in the above expression, we use that $u_3$ vanishes on the boundary to get

$$\|\partial_t w \cdot \nabla \rho\|_{L^\infty} \lesssim \|w\|_{1, \infty} \rho_{1, \infty} + \|\partial_x \partial_t w_3\|_{L^\infty} \|Z_3 \rho\|_{L^\infty} \lesssim \|w\|_{E^2, \infty} \|\rho\|_{1, \infty}.$$

It remains to estimate $\|Z_3 \rho\|_{L^\infty}$ which is the most difficult term. We cannot use the same method as previously due to the bad commutator between $Z_3$ and the Laplacian. We shall use a more precise description of the solution of (9.44). We shall first rewrite the equation (9.44) as

$$\partial_t \rho + z \partial_z w_3(t, y, 0) \partial_z \rho + w_y(t, y, 0) \cdot \nabla y \rho - \varepsilon \partial_{zz} \rho = H - R := G$$

where

$$R = (w_y(t, x) - w_y(t, y, 0)) \cdot \nabla y \rho + (w_3(t, x) - z \partial_z w_3(t, y, 0)) \partial_z \rho.$$

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The idea will be to use an exact representation of the Green’s function of the operator in the left-hand side to perform the estimate.

Let $S(t, \tau)$ be the $C^0$ evolution operator generated by the left hand side of the above equation. This means that $f(t, y, z) = S(t, \tau)f_0(y, z)$ solves the equation

$$\partial_t f + z\partial_z w_3(t, y, 0)\partial_z f + w_9(t, y, 0) \cdot \nabla_y f - \varepsilon \partial_{zz} f = 0, \quad z > 0, \quad t > \tau, \quad f(t, y, 0) = 0.$$ 

with the initial condition $f(\tau, y, z) = f_0(y, z)$. Then we have the following estimates:

**Lemma 14.1.** There exists $C > 0$ independent of $\varepsilon$ such that

$$\|z\partial_z S(t, \tau)f_0\|_{L^\infty} \leq C(\|f_0\|_{L^\infty} + \|z\partial_z f_0\|_{L^\infty}), \quad \forall t \geq \tau \geq 0. \tag{14.3}$$

We shall postpone the proof of the Lemma until the end of the section.

By using Duhamel formula, we deduce that

$$\rho(t) = S(t, \tau)\rho_0 + \int_0^t S(t, \tau)G(\tau)\,d\tau. \tag{14.4}$$

Consequently, by using (14.3) in Lemma 14.1 we obtain

$$\|Z_3\rho\|_{L^\infty} \lesssim (\|\rho_0\|_{L^\infty} + \|z\partial_z \rho_0\|_{L^\infty} + \int_0^t (\|G\|_{L^\infty} + \|z\partial_z G\|_{L^\infty})) \tag{14.5}$$

Since $\rho$ and $G$ are compactly supported, we obtain

$$\|Z_3\rho\|_{L^\infty} \lesssim (\|\rho_0\|_{1, \infty} + \int_0^t \|G\|_{1, \infty}).$$

It remains to estimate the right hand side. First, let us estimate the term involving $R$. Since $w_3(t, y, 0) = 0$, we have

$$\|R\|_{L^\infty} \lesssim \|w_9\|_{L^\infty} \|\nabla_y \rho\|_{L^\infty} + \|\partial_z w_3\|_{L^\infty} \|Z_3 \rho\|_{L^\infty} \lesssim \|w\|_{E^1, \infty} \|\rho\|_{1, \infty}.$$ 

Next, in a similar way, we get that

$$\|ZR\|_{L^\infty} \lesssim \|w\|_{2, \infty} \|\rho\|_{1, \infty} + \|(w_9(t, x) - w_9(t, y, 0)) \cdot Z\nabla_y \rho\|_{L^\infty} + \|(w_3(t, x) - z\partial_z w_3(t, y, 0)) Z\partial_z \rho\|_{L^\infty}$$

By using the Taylor formula and the fact that $\rho$ is compactly supported in $z$, this yields

$$\|ZR\|_{L^\infty} \lesssim \|w\|_{2, \infty} \|\rho\|_{1, \infty} + \|\partial_z w_9\|_{L^\infty} \|\varphi(z) Z\nabla_y \rho\|_{L^\infty} + \|\partial_z w_3\|_{L^\infty} \|\varphi^2(z) Z\partial_z \rho\|_{L^\infty}$$

with $\varphi(z) = z/(1 - z)$ and hence, we obtain

$$\|R\|_{1, \infty} \lesssim (\|w\|_{E^{2, \infty}} + \|\partial_z w_3\|_{1, \infty}) (\|\rho\|_{1, \infty} + \|\varphi(z)\|_{2, \infty}).$$

The additional factor $\varphi$ in the last term is crucial to close our estimate. Indeed, by the Sobolev embedding (2.7), we have that for $|\alpha| = 2$

$$\|\varphi Z^\alpha \eta\|_{L^\infty} \lesssim \|Z^\alpha \eta\|_{s_2} + \|\partial_z (\varphi Z^\alpha \eta)\|_{s_1}$$

with $s_1 + s_2 > 2$, thus, with $s_1 = 1$, $s_2 = 2$, we obtain from definition of $Z_3$ that

$$\|\varphi Z^\alpha \eta\|_{L^\infty} \lesssim \|\eta\|_{4}, \quad |\alpha| = 2. \tag{14.6}$$

Consequently, we get that

$$\|R(t)\|_{1, \infty} \lesssim (\|w\|_{E^{2, \infty}} + \|\partial_z w_3\|_{L^\infty}) (\|\rho\|_{1, \infty} + \|\rho\|_{4}). \tag{14.7}$$

Finally, the proof of Proposition 3.3 follows from the last estimate and (14.5).

It remains to prove Lemma 14.1.
Proof of Lemma 14.1. Let us set \( f(t, y, z) = S(t, \tau) f_0(y, z) \), then \( f \) solves the equation

\[
\partial_t f + z \partial_z w_3(t, y, 0) \partial_z f + w_y(t, y, 0) \cdot \nabla_y f - \varepsilon \partial_{zz} f = 0, \quad z > 0, \quad f(t, y, 0) = 0.
\]

We can first transform the problem into a problem in the whole space. Let us define \( \tilde{f} \) by

\[
\tilde{f}(t, y, z) = f(t, y, z), \quad z > 0, \quad \tilde{f}(t, y, z) = -f(t, y, -z), \quad z < 0
\]

then \( \tilde{f} \) solves

\[
\partial_t \tilde{f} + z \partial_z w_3(t, y, 0) \partial_z \tilde{f} + w_y(t, y, 0) \cdot \nabla_y \tilde{f} - \varepsilon \partial_{zz} \tilde{f} = 0, \quad z \in \mathbb{R}
\]

with the initial condition \( \tilde{f}(\tau, y, z) = \tilde{f}_0(y, z) \).

We shall get the estimate by using an exact representation of the solution.

To solve (14.9), we can first define

\[
g(t, y, z) = f(t, \Phi(t, \tau, y), z)
\]

where \( \Phi \) is the solution of

\[
\partial_t \Phi = w_y(t, \Phi, 0), \quad \Phi(t, \tau, y) = y.
\]

Then, \( g \) solves the equation

\[
\partial_t g + z \gamma(t, y) \partial_z g - \varepsilon \partial_{zz} g = 0, \quad z \in \mathbb{R}, \quad g(\tau, y, z) = \tilde{f}_0(y, z)
\]

where

\[
(14.11) \quad \gamma(t, y) = \partial_z w_3(t, \Phi(t, \tau, y), 0)
\]

which is a one-dimensional Fokker-Planck type equation (note that now \( y \) is only a parameter in the problem). By a simple computation in Fourier space, we find the explicit representation

\[
g(t, x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi \varepsilon \int_\tau^t e^{2z(\Gamma(s)-\Gamma(t))} ds}} \exp \left( -\frac{(z-z')^2}{4\varepsilon \int_\tau^t e^{2z(\Gamma(s)-\Gamma(t))} ds} \right) \tilde{f}_0(y, e^{-\Gamma(t)} z') dz'
\]

(14.12)

\[
= \int_{\mathbb{R}} k(t, \tau, y, z - z') \tilde{f}_0(y, e^{-\Gamma(t)} z') dz'
\]

where \( \Gamma(t) = \int_\tau^t \gamma(s, y) ds \) (note that \( \Gamma \) depends on \( y \) and \( \tau \), we do not write down explicitly this dependence for notational convenience).

Note that \( k \) is non-negative and that \( \int_{\mathbb{R}} k(t, \tau, y, z) dz = 1 \), thus, we immediately recover that

\[
\|g\|_{L^\infty} \leq \|\tilde{f}_0\|_{L^\infty}.
\]

Next, we observe that we can write

\[
(14.13) \quad z \partial_z k(t, \tau, y, z - z') = (z - z') \partial_z k - z' \partial_z l(t, \tau, y, z - z')
\]

with

\[
\int_{\mathbb{R}} \left| (z - z') \partial_z k \right| dz' \lesssim 1
\]

and thus by using an integration by parts, we find

\[
\|z \partial_z g\|_{L^\infty} \lesssim \|\tilde{f}\|_{L^\infty} + \left\| e^{-\Gamma(t)} \int_{\mathbb{R}} k(t, \tau, y, z') z' \partial_z \tilde{f}_0(y, e^{-\Gamma(t)} z') dz' \right\|_{L^\infty}.
\]

By using (14.11), this yields

\[
\|z \partial_z g\|_{L^\infty} \lesssim \|\tilde{f}_0\|_{L^\infty} + \|z \partial_z \tilde{f}_0\|_{L^\infty}.
\]

By using (14.8) and (14.10), we obtain

\[
\|z \partial_z f\|_{L^\infty} \lesssim \|z \partial_z \tilde{f}\|_{L^\infty} \lesssim \|\tilde{f}_0\|_{L^\infty} + \|z \partial_z \tilde{f}_0\|_{L^\infty} \lesssim \|f_0\|_{L^\infty} + \|z \partial_z f_0\|_{L^\infty}.
\]
This ends the proof of Lemma 14.1.

14.2. **Proof of Lemma 9.11.** We use the same idea as in the proof of Lemma 9.6. We first estimate \( \sqrt{\varepsilon} \| \partial_z Z_i \rho \|_{L^\infty}, \ i = 1, 2 \). We get for \( \partial_i \rho \) the equation
\[
\partial_t \partial_i \rho + w \cdot \nabla \partial_i \rho = \varepsilon \partial_z \partial_i \rho + \partial_i \mathcal{H} - \partial_i w \cdot \nabla \rho \]
that we rewrite as
\[
\partial_t \partial_i \rho + z \partial_z w_3(t, y, 0) \partial_z \partial_i \rho + w_y(t, y, 0) \cdot \nabla_y \partial_i \rho - \varepsilon \partial_z \partial_i \rho = \partial_i \mathcal{H} - \partial_i w \cdot \nabla \rho - R := G
\]
where
\[
R^1 = (w_y(t, x) - w_y(t, y, 0)) \cdot \nabla_y \partial_i \rho + (w_3(t, x) - z \partial_z w_3(t, y, 0)) \partial_z \partial_i \rho.
\]
By using the notations before Lemma 14.1, we obtain
\[
\partial_i \rho = S(t, \tau) \partial_i \rho_0 + \int_0^t S(t, \tau) G(\tau) \, d\tau
\]
and we shall use the following semigroup estimates for \( S \):

**Lemma 14.2.** Under the assumption of Lemma 9.11 on \( w \), we have that for \( 0 \leq \tau \leq t \leq T \):
\[
\sqrt{\varepsilon} \| \partial_z S(t, \tau) f_0 \|_{L^\infty} \leq \frac{\Lambda(M)}{\sqrt{t}} \| f_0 \|_{L^\infty}, \quad \sqrt{\varepsilon} \| \partial_z (z \partial_z S(t, \tau) f_0) \|_{L^\infty} \leq \frac{\Lambda(M)}{\sqrt{t}} (\| f_0 \|_{L^\infty} + \| z \partial_z f_0 \|_{L^\infty})
\]
where \( \Lambda(M) \) does not depend on \( \varepsilon \).

Let us postpone the proof of this Lemma until the end of the section. By using Lemma 14.3, we thus get that
\[
\sqrt{\varepsilon} \| \partial_z \partial_i \rho(t) \|_{L^\infty} \leq \Lambda(M) \left( \frac{1}{\sqrt{t}} \| \partial_t \rho(0) \|_{L^\infty} + \int_0^t \frac{1}{\sqrt{t - \tau}} \left( \| \mathcal{H} \|_{1, \infty} + \| \partial_i w \cdot \nabla \rho \|_{L^\infty} + \| R^1 \|_{L^\infty} \right) \right).
\]
Next, we can use (14.2) and (14.7) to get that
\[
\| R^1 \|_{L^\infty} \leq \Lambda(M) (\| \rho \|_{1, \infty} + \| \rho \|_{4}).
\]
This yields
\[
\sqrt{\varepsilon} \| \partial_z \partial_i \rho(t) \|_{L^\infty} \leq \Lambda(M) \left( \frac{1}{\sqrt{t}} \| \partial_t \rho(0) \|_{L^\infty} + \int_0^t \frac{1}{\sqrt{t - \tau}} \left( \| \mathcal{H} \|_{1, \infty} + \| \rho \|_{1, \infty} + \| \rho \|_{4} \right) \right).
\]
The estimate of \( \| \partial_z (Z_3 \rho) \|_{L^\infty} \), we directly use the Duhamel formula (14.4) and we use the second estimate given in Lemma 14.3. This ends the proof of Lemma 14.2. It only remains to prove Lemma 14.3.

**Proof of Lemma 14.3.** We can use again that the solution of the equation is given by the representation (14.10), (14.12) and it suffices to notice that the kernel \( k \) has the property
\[
| \partial_z k | \leq \frac{\Lambda(M)}{\sqrt{\varepsilon (t - \tau)}} | k |
\]
and the result follows from standard convolution estimates.
14.3. **Proof of Proposition 2.9** Thanks to Lemma 2.8, the estimate (2.24) for \( v \) is actually equivalent to the standard Korn inequality in \( \Omega_t \) for \( u = v \circ \Phi^{-1} \). For the sake of completeness, we shall sketch the argument.

We first note that

\[
\int_S |S^\varepsilon v|^2 \, d\mathcal{V}_t \geq c_0 \|S^\varepsilon v\|^2_{L^2(S)}.
\]

Next, we shall reduce the problem to the classical Korn inequality in \( S \) for an auxiliary vector field. Let us set

\[
w_i = v_i + \partial_i \varphi v_3, \quad i = 1, 2, \quad w_3 = \partial_3 \varphi v_3.
\]

We note that for \( 1 \leq i, j \leq 2 \), we have

\[
2(Sw)_{ij} = \partial_i^2 v_j + \partial_j^2 v_i + \partial_j \varphi (\partial_i^2 v_i + \partial_i^3 v_3) + \partial_i \varphi (\partial_j^2 v_j + \partial_j^3 v_3) + 2 \partial_i \varphi \partial_j \varphi \partial_3^2 v_3 + 2 \partial_i^2 \varphi v_3
\]

and hence that

\[
(Sw)_{ij} = (S^\varepsilon v)_{ij} + \partial_j \varphi (S^\varepsilon v)_{i,3} + \partial_i \varphi (S^\varepsilon v)_{j,3} + \partial_i \varphi \partial_j \varphi S^\varepsilon (v)_{3,3} + \partial_3^2 \varphi v_3.
\]

In a similar way, we have that

\[
(Sw)_{i,3} = \partial_3 \varphi (S^\varepsilon v)_{i,3} + \partial_i \varphi \partial_3 \varphi (S^\varepsilon v)_{3,3} + \partial_3^2 \varphi v_3, \quad i = 1, 2,
\]

\[
(Sw)_{3,3} = (\partial_3 \varphi)^2 (S^\varepsilon v)_{3,3} + \partial_3^2 \varphi v_3.
\]

This yields

\[
(14.14) \quad \|Sw\|_{L^2(S)} \leq \Lambda_0 (\|S^\varepsilon v\|_{L^2(S)} + \|v\|_{L^2(S)}).
\]

Next, the Korn inequality in \( S \) for \( w \), yields for some \( C > 0 \):

\[
\|\nabla w\|_{L^2(S)} \leq C (\|Sw\|_{L^2(S)} + \|w\|_{L^2(S)}).
\]

Consequently, we obtain that

\[
\|\nabla w\|_{L^2(S)} \leq \Lambda_0 (\|S^\varepsilon v\|_{L^2(S)} + \|v\|_{L^2(S)}) + C \|w\|_{L^2(S)}.
\]

Moreover, since \( \partial_3 \varphi \geq \eta \), we have from the definition of \( w \) that

\[
\|w\|_{L^2(S)} \leq \Lambda_0 \|v\|_{L^2(S)}, \quad \|\nabla v\|_{L^2(S)} \leq \Lambda_0 (\|\nabla w\|_{L^2(S)} + \|v\|_{L^2(S)}),
\]

the result follows by combining these inequalities.

Finally, let us recall the proof of (14.14). We can define an extension of \( w \) in \( \mathbb{R}^3 \) by \( \tilde{w} = w, \ z < 0 \) and

\[
\tilde{w}_1(y, z) = 2w_1(y, z) - w_1(y, -3z), \quad \tilde{w}_3(y, z) = -2w_3(y, z) + 3u_3(y, -z), \quad z > 0.
\]

Since,

\[
\|Sw\|_{L^2(\mathbb{R}^3)} \leq 5 \|Sw\|_{L^2(S)},
\]

In \( \mathbb{R}^3 \), we obviously have that

\[
\|Sw\|^2_{L^2(\mathbb{R}^3)} = \frac{1}{2} \|\nabla \tilde{w}\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2} \|\nabla \cdot \tilde{w}\|^2_{L^2(\mathbb{R}^3)} \geq \frac{1}{2} \|\nabla \tilde{w}\|^2_{L^2(\mathbb{R}^3)}
\]

and the conclusion follows from the remark that

\[
\|\nabla w\|_{L^2(S)} \leq \|\nabla \tilde{w}\|_{L^2(\mathbb{R}^3)}.
\]
14.4. **Proof of Lemma 8.4.** Let us start with the first inequality. For any test function \( f \), we get by an integration by parts
\[
\int_{-\infty}^{0} \frac{1}{z(1-z)} f \partial_z f \, dz = \frac{1}{2} \int_{-\infty}^{0} \frac{1-2z}{z^2(1-z)^2} |f|^2 \, dz \geq \frac{1}{2} \int_{-\infty}^{0} \frac{1}{z^2(1-z)^2} |f|^2 \, dz
\]
and we get the result from the Cauchy-Schwarz inequality.

For the second inequality, we just write
\[
\int_{-\infty}^{0} \left( \frac{1-z}{z} \right)^2 f(z)^2 \, dz \leq 4 \int_{-1}^{0} \frac{1}{z^2} f(z)^2 \, dz + 4 \int_{-\infty}^{-1} |f(z)|^2 \, dz
\]
and we use the standard Hardy inequality on \((-1,0)\) to estimate the first term.

15. **Results on paradifferential calculus**

In this section, we shall state the results on paradifferential calculus with parabolic homogeneity that we need for our estimates of section 11. We shall first define a calculus as in the Appendix B of [11] without the semiclassical parameter \( \varepsilon \). In a second step, we shall deduce the result for the "partially" semiclassical version (that is to say with weight \( \langle \xi^\alpha \rangle = (\gamma^2 + \tau^2 + \varepsilon^2 |\xi|^4) \frac{4}{n} \)) that we need.

We first define operators acting on functions defined on \( \mathbb{R}^3_{(t,y)} \). The calculus for functions defined on \( \mathbb{R}_t \times S \) will immediately follow since the \( z \) variable will only be a parameter.

For notational convenience, we use in this section the notation \( X = (t,y) = (x_0, x_1, x_2) \) for the "space" variable and \( (\tau, \xi) \) for the corresponding Fourier variables, we also use the notation \( \zeta = (\gamma, \tau, \xi) = (\gamma, \eta) \in \mathbb{R}^4_+ = [1, +\infty[ \times \mathbb{R}^3 \) where \( \gamma \geq 1 \) will be a parameter. We define the weight:
\[
\langle \zeta \rangle = (\gamma^2 + \tau^2 + |\xi|^4)^\frac{4}{n}
\]
with |·| the Euclidian norm of \( \mathbb{R}^2 \). Note that this weight corresponds to the quasihomogeneous weight with \( p = 2, p_0 = 1, p_1 = p_2 = 2 \) in the general framework of [11]. We also point out that since we shall only consider \( \gamma \geq 1 \), we do not need to make any difference between \( \langle \zeta \rangle \) and \( \Lambda(\zeta) = (1 + \langle \zeta \rangle)^{2p} \frac{4}{n} \) in the notation of [11]. By using this weight, we define a scale of modified Sobolev type spaces by
\[
\mathcal{H}^{s,\gamma}(\mathbb{R}^3) = \{ u \in S'(\mathbb{R}^3), \| u \|^2_{\mathcal{H}^{s,\gamma}} < +\infty \}
\]
where
\[
\| u \|^2_{\mathcal{H}^{s,\gamma}} = \int_{\mathbb{R}^3} \langle \zeta \rangle^{2s} |\hat{u}(\tau, \xi)|^2 \, d\tau d\xi,
\]
\( \hat{u} \) being the Fourier transform of \( u \).

Next, we define our class of symbols:

**Definition 15.1.** For \( \mu \in \mathbb{R} \)
\begin{itemize}
  \item the class \( \Gamma_0^\mu \) denotes the space of locally bounded matrices \( a(X, \zeta) \) on \( \mathbb{R}^3 \times \mathbb{R}^4_+ \) which are \( C^\infty \) with respect to \( (\tau, \xi) = \eta \) and such that for all \( \alpha \in \mathbb{N}^4 \), there is a constant \( C_\alpha \) such that
    \[
    \forall (X, \zeta), \quad |\partial_\eta^{\alpha} a(X, \zeta)| = C_\alpha \langle \zeta \rangle^{\mu - \langle \alpha \rangle}
    \]
    where \( \langle \alpha \rangle = 2\alpha_0 + \alpha_1 + \alpha_2 \).
  \item \( \Gamma_1^\mu \) denotes the set of symbols \( a \in \Gamma_0^\mu \) such that \( \partial_i a \in \Gamma_0^\mu \), for \( i = 0, 1, 2 \).
\end{itemize}

The spaces \( \Gamma_0^\mu \) are equipped with the seminorms
\[
\| a \|_{(\mu, N)} = \sup_{\langle \alpha \rangle \leq N} \sup_{\mathbb{R}^3 \times \mathbb{R}^4_+} \langle \zeta \rangle^{\langle \alpha \rangle - \mu} |\partial_\eta^{\alpha} a(X, \zeta)|.
\]
Paradifferential operators are defined as pseudodifferential operators associated to a suitable regularization of the above symbols. For a symbol $a$ in the above class, we define a smooth symbol $\sigma_a(X, \zeta)$ defined by

$$\mathcal{F}_X \sigma_a(\hat{X}, \zeta) = \psi(\hat{X}, \zeta) \mathcal{F}_X a(\hat{X}, \zeta)$$

where $\mathcal{F}_X$ stands for the partial Fourier transform with respect to the $X$ variable and $\psi$ is an admissible cut-off function. We say that a smooth function $\psi$ is admissible if there exists $0 < \delta_1 < \delta_2 < 1$ such that

$$\psi(\hat{X}, \zeta) = 1, \text{ for } \langle \hat{X} \rangle \leq \delta_1 \langle \zeta \rangle, \quad \psi(\hat{X}, \zeta) = 0, \text{ for } \langle \hat{X} \rangle \geq \delta_2 \langle \zeta \rangle$$

where $\langle \hat{X} \rangle = (\gamma^2 + \hat{X}_0^2 + |(\hat{X}_1, \hat{X}_2)|^2)^{1/2}$ and that for every $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha, \beta}$ such that

$$\forall \hat{X}, \zeta, \quad |\partial^\alpha_{\eta} \partial^\beta_X \psi(\hat{X}, \zeta)| \leq C_{\alpha, \beta} \langle \zeta \rangle^{-\langle \alpha \rangle - \langle \beta \rangle}.$$ 

For a given admissible cut-off function, we then define an operator associated to the symbol $a$, $T_a^\gamma$ by

$$T_a^\gamma u = Op(\sigma_a)u$$

where $Op(\sigma_a)$ is the pseudodifferential operator defined by

$$(15.1) \quad Op(\sigma_a)u(X) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{iX\eta} \sigma_a(X, \zeta) \hat{u}(\eta) \, d\eta.$$

In this definition, the operator $T_a^\gamma$ depends on the choice of the admissible cut-off function. Nevertheless, it can be shown that the difference between two operators defined with the same symbol but with two different admissible cut-off function is a lower order operator. We refer to [11] for more details. Note that viewing $a(X) \in L^\infty$ as a symbol in $\Gamma^0$, the operator $T_a^\gamma$ can be related to Bony’s paraproduct ([10]).

The main interest of this class of operators is that it enjoys a nice symbolic calculus.

**Theorem 15.2.** We have the following results:

1. **Boundedness** For $a \in \Gamma^\mu_0$, and every $s \in \mathbb{R}$, there exists $C > 0$ (which depends only on semi norms of $a$) such that for every $\gamma \geq 1$ and $u \in \mathcal{H}^{s+\mu, \gamma}$, we have

$$\|T_a^\gamma u\|_{\mathcal{H}^{s, \gamma}} \leq C\|u\|_{\mathcal{H}^{s+\mu, \gamma}}.$$

2. **Product** For $a \in \Gamma^\mu_1$, $b \in \Gamma^\mu_1$ and $s \in \mathbb{R}$, we have

$$\|T_a^\gamma T_b^\gamma u - T_{ab}^\gamma u\|_{\mathcal{H}^{s, \gamma}} \leq C \left( \|a\|_{(\mu, N)} \|\nabla_X b\|_{(\mu', N)} + \|\nabla_X a\|_{(\mu, N)} \|b\|_{(\mu', N)} \right) \|u\|_{\mathcal{H}^{s+\mu+\mu'-1, \gamma}}$$

where $C$, $\mu$ and $N$ depend only on $s$, $\mu$, $\mu'$.

3. **Adjoint** For $a \in \Gamma^\mu_1$ (recall that we allow $a$ to be matrix valued), denote by $(T_a^\gamma)^*$ the adjoint of $a$ and $a^*(X, \zeta)$ the adjoint matrix of $a$, then we have

$$\|(T_a^\gamma)^* - T_{a^*a}^\gamma\|_{\mathcal{H}^{s, \gamma}} \leq C \|\nabla_X a\|_{(\mu, N)} \|u\|_{\mathcal{H}^{s+\mu+\mu'-1, \gamma}}$$

where $C$ and $N$ only depend on $\mu$ and $s$.

4. **Garding inequality** For $a \in \Gamma^\mu_1$, assume that there exists $c > 0$ such that

$$\forall X, \zeta, \quad \text{Re } a(X, \zeta) \geq c(\zeta)^\mu.$$

Then there exists $C > 0$ (depending only on $\|a, \nabla_X a\|_{(\mu, N)}$) such that for $\gamma \geq C$, we have

$$\frac{C}{2} \|u\|^2_{\mathcal{H}^{\frac{\gamma}{2}, \gamma}} \leq \text{Re } (T_a^\gamma u, u)_{L^2}$$
(5) **Paraprod**. If \( a(X) \in L^\infty \), \( \nabla_X a \in L^\infty \), there exists \( C > 0 \) such that for every \( u \), we have the estimates
\[
\|au - T^\gamma_au\|_{H^{\epsilon, \gamma}} \leq C\|\nabla_X a\|_{L^\infty}\|u\|_{H^{\epsilon, \gamma}},
\]
\[
\gamma\|au - T^\gamma_au\|_{H^{\epsilon, \gamma}} \leq C\|\nabla_X a\|_{L^\infty}\|u\|_{H^{\epsilon, \gamma}},
\]
\[
\|a\partial_j u - T^\gamma_au\partial_j u\|_{H^{\epsilon, \gamma}} \leq C\|\nabla_X a\|_{L^\infty}\|u\|_{H^{\epsilon, \gamma}},
\]
where \( \partial_0 = \partial_t \), \( p_0 = 1 \) and \( p_j = 2 \), \( j = 1, 2 \).

For the proof of these results, we refer to [41] Appendix B. The meaning of the assumption in the Garding inequality is that Re \( a := \frac{1}{2}(a + a^*) \) for a matrix is bounded from below on the support of \( w \).

When dealing with paraprod\( ucts \), it is useful to recall that thanks to the definition of the operators, we have when \( a = a(X) \) that
\[
\partial_j (T^\gamma_au) = T^\gamma_au\partial_j u + T^\gamma_{\partial_j a}u.
\]

Another useful remark about products is that if \( a = a(X, \zeta) \) but \( b = b(\zeta) \) does not depend on \( X \), then we have
\[
T^\gamma_{ab} = T^\gamma_a T^\gamma_b.
\]

To conclude the description of the calculus, let us notice that if \( a(X, z, \zeta) \) and \( u(X, z) \) depends also on \( z \in (-\infty, 0) \) which plays the role of a parameter, all the results of Theorem 15.2 remain true by adding a supremum in \( z \) in the definition of the \( L^\infty \) norms and by defining the \( \| \cdot \|_{H^{s, \gamma}} \) norm of a function defined on \( \mathbb{R} \times \mathcal{S} \) by
\[
\|u\|_{H^{s, \gamma}}^2 = \int_{-\infty}^0 \int_{\mathbb{R}^3} \langle \zeta \rangle^{2s} |\mathcal{F}_{t,y}u(\tau, \xi, z)|^2 \, d\tau d\xi dz.
\]

Let us turn to the semiclassical version of the calculus that we need. We shall use the semiclassical type weight \( \langle \zeta \rangle = (\gamma^2 + \tau^2 + |\sqrt{\varepsilon}\xi|^4)^{\frac{1}{4}} \) where \( \varepsilon \in (0, 1) \). We thus define a new weighted norm as
\[
\|u\|_{H^{s, \gamma, \varepsilon}}^2 = \int_{\mathbb{R}^3} \langle \zeta \rangle^{2s} |\hat{u}(\tau, \xi)|^2 \, d\tau d\xi,
\]

For pseudodifferential operators, the semiclassical quantization corresponding to \( \| \cdot \|_{H^{s, \gamma, \varepsilon}} \) for a symbol \( \sigma(X, \zeta) \) is
\[
Op^\varepsilon(\sigma)u(X) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{iX\cdot\eta} \sigma(X, \gamma, \tau, \sqrt{\varepsilon}\xi) \hat{u}(\eta) \, d\eta, \quad \eta = (\tau, \xi)
\]

Note that this definition is different from the quantization used in [41], this is due to the fact that we shall study a problem of purely parabolic nature. Let us define the "scaling map" \( H^\varepsilon \) by
\[
H^\varepsilon u = \sqrt{\varepsilon} u(t, \sqrt{\varepsilon}y).
\]

Then, we note that
\[
Op^\varepsilon(\sigma) = H^{-1}_\varepsilon Op(\sigma^\varepsilon)H^\varepsilon
\]
where \( \sigma^\varepsilon \) is given by \( \sigma^\varepsilon(X, \zeta) = \sigma(t, \sqrt{\varepsilon}y, \zeta) \). This motivates the following definition: for a symbol \( a(X, \zeta) \in T^\mu_0 \), we define a semiclassical paradifferential calculus by
\[
T^\varepsilon_{a, \gamma}u = H^{-1}_\varepsilon T^\gamma_{a, \gamma}H^\varepsilon u
\]
with \( a^\varepsilon(X, \zeta) = a(t, \sqrt{\varepsilon}y, \zeta) \).

Next, we observe that
\[
\|H^\varepsilon u\|_{H^{s, \gamma}} = \|u\|_{H^{s, \gamma, \varepsilon}}
\]
and that if \( a \) is in \( \Gamma_0^\mu \), then the family \( (a^\varepsilon)_{\varepsilon \in (0,1)} \) is uniformly bounded in \( \Gamma_0^\mu \). Moreover, if \( a \in \Gamma_1^\mu \), then \( a^\varepsilon \) and \( \nabla_X a^\varepsilon \) are uniformly bounded in \( \Gamma_1^\mu \). This allows to get from Theorem \( 15.2 \) the following properties of the semiclassical calculus:

**Theorem 15.3.** We have the following results:

1. **Boundedness** For \( a \in \Gamma_0^\mu \), and every \( s \in \mathbb{R} \), we have
   \[
   \|T_a^{\varepsilon,\gamma} u\|_{\mathcal{H}^s,\gamma,\varepsilon} \leq C\|u\|_{\mathcal{H}^{s+\mu,\gamma,\varepsilon}}.
   \]

2. **Product** For \( a \in \Gamma_1^\mu \), \( b \in \Gamma_1^\mu \) and \( s \in \mathbb{R} \), we have
   \[
   \|T_a^{\varepsilon,\gamma} T_b^{\varepsilon,\gamma} u - T_{ab}^{\varepsilon,\gamma} u\|_{\mathcal{H}^s,\gamma,\varepsilon} \leq C\|u\|_{\mathcal{H}^{s+\mu+\mu',\gamma,\varepsilon}}.
   \]

3. **Adjoint** For \( a \in \Gamma_1^\mu \), we have
   \[
   \|(T_a^{\varepsilon,\gamma})^* - T_{a^*}^{\varepsilon,\gamma}\|_{\mathcal{H}^s,\gamma,\varepsilon} \leq C\|u\|_{\mathcal{H}^{s+\mu-1,\gamma,\varepsilon}}.
   \]

4. **Garding inequality** For \( a \in \Gamma_1^\mu \), assume that there exists \( c > 0 \) such that
   \[
   \forall X, \xi, \quad Re a(X, \xi) \geq c(\xi)^\mu.
   \]
   Then there exists \( C > 0 \) such that for \( \gamma \geq C \), we have
   \[
   \frac{c}{2} \|T_{w}^{\varepsilon,\gamma} u\|^2_{\mathcal{H}^{s,\gamma,\varepsilon}} \leq Re (T_a^{\varepsilon,\gamma} u, u)_{L^2}.
   \]

5. **Paraproduct** If \( a(X) \in L^\infty \), \( \nabla_X a \in L^\infty \), then we have
   \[
   \|au - T_a^{\varepsilon,\gamma} u\|_{\mathcal{H}^{1,\gamma,\varepsilon}} \leq C\|u\|_{\mathcal{H}^{0,\gamma,\varepsilon}},
   \]
   \[
   \|\gamma au - T_a^{\varepsilon,\gamma} u\|_{\mathcal{H}^{0,\gamma,\varepsilon}} \leq C\|u\|_{\mathcal{H}^{1,\gamma,\varepsilon}},
   \]
   \[
   \|a\sqrt{\varepsilon}\alpha \partial_j u - T_a^{\varepsilon,\gamma} \sqrt{\varepsilon}\alpha \partial_j u\|_{\mathcal{H}^{0,\gamma,\varepsilon}} \leq C\|u\|_{\mathcal{H}^{2,\gamma,\varepsilon}}.
   \]

where \( \partial_0 = \partial_t \), \( p_0 = 1 \) and \( p_j = 2 \), \( j = 1, 2 \) and \( \alpha_0 = 0 \), \( \alpha_j = 1 \), \( j \geq 1 \).

As before, in all the above estimates \( C \) only depends on seminorms of the symbols and in particular, it is independent of \( \varepsilon \) for \( \varepsilon \in (0,1) \).

Note that in the above result, due to the anisotropy in our definition of the semiclassical calculus we do not gain powers of \( \varepsilon \) in the product and adjoint estimates.

Finally, if the symbols and \( u \) depend on the parameter \( z \) all the above results remain true as before with the definition
\[
\|u\|^2_{\mathcal{H}^{s,\gamma,\varepsilon}} = \int_0^0 \int_{\mathbb{R}^3} \langle \xi \rangle^{2s}|\mathcal{F}_{\tau y}u(\tau, \xi, z)|^2 d\tau d\xi dz.
\]

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