PRINCIPAL TORUS BUNDLES OF LORENTZIAN 
S-MANIFOLDS AND THE ϕ-NULL OSSERMAN CONDITION

LETIZIA BRUNETTI AND ANGELO V. CALDARELLA

Abstract. The main result we give in this brief note relates, under suitable hypotheses, the ϕ-null Osserman, the null Osserman and the classical Osserman conditions to each other, via semi-Riemannian submersions as projection maps of principal torus bundles arising from a Lorentzian S-manifold.

1. Introduction

The Jacobi operator is, for several reasons, one of the most interesting objects induced by the curvature operator. On a (semi-)Riemannian manifold \((M, g)\), let us consider the unit spacelike \(S^+(M)\) (resp. timelike \(S^-(M)\)) sphere bundle with fiber
\[
S^+_p(M) = \{z \in T_pM | g_p(z, z) = \pm 1\},
\]
and put \(S(M) = \bigcup_{p \in M} S^+_p(M) \cup S^-_p(M)\).

For any \(z \in S_p(M)\), \(p \in M\), the Jacobi operator with respect to \(z\) is the endomorphism \(R_z : z^\perp \to z^\perp\) such that \(R_z(\cdot) = R_p(\cdot, z)z\) \((20)\), where \(R\) is the \((1, 3)\)-type curvature tensor on \((M, g)\).

The Jacobi operator is obviously self-adjoint, hence a great deal of study has been carried out about the behaviour of its eigenvalues in the Riemannian case since R. Osserman proposed his Conjecture in [33] (see also [32]). Indeed, one easily sees that Riemannian space-forms are characterized by having Jacobi operators with exactly one constant eigenvalue corresponding to the sectional curvature. Those Riemannian manifolds whose Jacobi operators have eigenvalues independent both of the vector \(z \in S^+_p(M)\) and of the point \(p \in M\) are the Osserman manifolds. Any locally flat or locally rank-one symmetric space is an Osserman manifold, whilst the converse statement is known as the Osserman Conjecture. Several authors have dealt with this Conjecture, providing positive answers in many cases \([12, 13, 14, 28, 29, 30]\).

One gets a different situation when considering the indefinite setting, where a semi-Riemannian manifold \((M, g)\) is said to be spacelike (resp. timelike) Osserman, if the characteristic polynomial of \(R_z\) is independent of both \(z \in S^+_p(M)\) (resp. \(z \in S^-_p(M)\)) and \(p \in M\). It is known that \((M, g)\) being spacelike Osserman is equivalent to \((M, g)\) being timelike Osserman \((19, 20)\), but several counterexamples to the Osserman Conjecture were found (see for example [5], [6], [21]) for non-Lorentzian semi-Riemannian manifolds.

Finally, in the Lorentzian setting a complete solution for the Osserman Conjecture was provided in a sequence of works by E. García-Río, D.N. Kupeli and M.E. Vázquez-Abal \((17, 18)\), together with N. Blažič, N. Bokan and P. Gilkey.

Date: May 2, 2014.
1991 Mathematics Subject Classification. Primary 53C50; Secondary 53B30, 53C25, 53C15.
Key words and phrases. Lorentz manifold. Semi-Riemannian manifold. Semi-Riemannian submersion. Jacobi operator. Osserman condition. Indefinite S-manifold.
They proved that a Lorentzian manifold is Osserman if and only if it has constant sectional curvature (see also [20]).

It was defined a very fruitful new Osserman-related condition for Lorentzian manifolds in [18]. There, the authors introduced the Jacobi operator $\bar{R}_u$ with respect to a null (lightlike) vector $u$, and then they studied the so-called null Osserman conditions with respect to a unit timelike vector (see also [20]).

Here, we are concerned with an Osserman-related condition derived by the null Osserman condition, which is known as the $\varphi$-null Osserman condition, introduced and studied by the first author in [7] for manifolds carrying Lorentzian globally framed $f$-structures. This condition appears to be a natural generalization of the null Osserman condition, to which it reduces when considering Lorentzian almost contact structures. This was motivated by the following considerations: although any Lorentzian Sasaki manifold $(M, \varphi, \xi, \eta, g)$ with constant $\varphi$-sectional curvature is globally null Osserman with respect to the timelike vector field $\xi$, there is no similar result when we consider Lorentzian $\mathcal{S}$-manifolds, which generalize Lorentzian Sasaki ones, and moreover, as we proved in [8], no Lorentzian $\mathcal{S}$-manifold can be neither null Osserman, nor Osserman. Further basic properties of such manifolds are studied in [7] and developed in [8]. We refer the reader to both works for more details about the $\varphi$-null Osserman condition, whilst the general reference for the whole Osserman framework is [20].

In this short note, we deal with the study of some relationships among the above three Osserman-related notions, providing a few results of equivalence, obtained by considering a natural structure of principal torus bundle arising from a Lorentzian $\mathcal{S}$-manifold, which involves semi-Riemannian submersions.

Indeed, from [3], a strong link between $f$-structures and Riemannian submersions is well-known. Namely, any compact and connected manifold endowed with a regular and normal $g.f.$-$f$-structure is the total space of a torus principal bundle over a complex manifold, which, under suitable hypotheses, can be a Kähler manifold. Moreover, as also proved in [3], a compact, connected and regular Riemannian $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta_\alpha, g)$, with each $\xi_\alpha$ regular, projects itself onto a compact Kähler manifold and onto a compact and regular Sasakian manifold. These results have been extended to the semi-Riemannian case by the first author, together with A.M. Pastore, who in [10] proved that a compact, connected and regular indefinite (in particular, Lorentzian) $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta_\alpha, g)$ projects itself onto a compact (indefinite) Kähler manifold and onto a compact and regular indefinite (Lorentzian) Sasaki manifold, via semi-Riemannian submersions.

Based on the above, after recalling, in Section 2, some basic features of (almost) $\mathcal{S}$-manifolds, in Section 4 we carry on an investigation on the possibilities of projectability of the $\varphi$-null Osserman conditions via semi-Riemannian submersions with a Lorentzian $\mathcal{S}$-manifold as total space, and either a Lorentzian Sasakian manifold or a Kähler manifold as base space. Using some properties established in [3], which we briefly recall in Section 3, together with a few properties of semi-Riemannian submersions, and under an additional assumption on the eigenvectors of the Jacobi operators, we obtain equivalence results relating the $\varphi$-null Osserman condition with the classical and the null Osserman condition in the framework of principal torus bundles constructed on a given Lorentzian $\mathcal{S}$-manifold.

In what follows, all smooth manifolds are supposed to be connected, and all tensor fields and maps are assumed to be smooth. Moreover, according to [24], for the Riemannian curvature tensor of a semi-Riemannian manifold $(M, g)$ we use the definition $R(X, Y, Z, W) = g(R(Z, W)Y, X) = g(\langle [\nabla_Z, \nabla_W] - \nabla_{[Z,W]} \rangle Y, X)$ for any vector fields $X, Y, Z, W$ on $M$. 
Finally, for any $p \in M$ and any linearly independent vectors $x, y \in T_pM$ spanning a non-degenerate plane $\pi = \operatorname{span}(x, y)$, that is $g_p(x, x)g_p(y, y) - g_p(x, y)^2 \neq 0$, the sectional curvature of $(M, g)$ at $p$ with respect to $\pi$ is, by definition, the real number
\[
k_p(\pi) = k_p(x, y) = \frac{R_p(x, y, x, y)}{\Delta(\pi)},\]
where $\Delta(\pi) = g_p(x, x)g_p(y, y) - g_p(x, y)^2$.

2. Preliminaries

Let us recall some basic definitions and facts about (almost) $S$-manifolds needed in the rest of the paper.

Framed $f$-manifolds were originally considered by H. Nakagawa in [26] and [27], based on the notion of $f$-structure, which was firstly introduced in 1963 by K. Yano (see, for example, [22], [23]) and, in the subsequent years, by several authors ([1], [3], [11], [25], [29]).

A globally framed $f$-structure (briefly g.f.f-structure) on a manifold $M$ is a non-vanishing $(1,1)$-type tensor field $\varphi$ on $M$ of constant rank satisfying the following conditions: $\varphi^2 + \varphi = 0$, and the subbundle $\ker(\varphi)$ is parallelizable. This is equivalent to the existence of $s$ linearly independent vector fields $\xi_\alpha$ and 1-forms $\eta^\alpha$ ($\alpha \in \{1, \ldots, s\}$), $s$ being the dimension of $\ker(\varphi)$) at any point $p \in M$, such that
\[
\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad \text{and} \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta.
\]
Each $\xi_\alpha$ is said to be a characteristic vector field of the structure, and a manifold $M$ carrying a g.f.f-structure is denoted by $(M, \varphi, \xi_\alpha, \eta^\alpha)$, and called a g.f.f-manifold. When $s = 1$ (resp.: $s = 0$), we have an almost contact (resp.: almost complex) structure. From (2.1) one easily has $\varphi\xi_\alpha = 0$ and $\eta^\alpha \circ \varphi = 0$, for any $\alpha \in \{1, \ldots, s\}$. Furthermore, $\operatorname{Im}(\varphi)$ is a distribution on $M$ of even rank $r = 2n$ on which $\varphi$ acts as an almost complex tensor field, and one has the splitting $TM = \operatorname{Im}(\varphi) \oplus \ker(\varphi)$, hence $\dim(M) = 2n + s$. A g.f.f-manifold is said to be normal if the $(1,2)$-type tensor field $N = [\varphi, \varphi] + 2d\eta^\alpha \otimes \xi_\alpha$ vanishes.

In [9], the authors study the properties of a g.f.f-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ endowed with a compatible indefinite metric, that is a semi-Riemannian metric $g$ verifying
\[
g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y),
\]
for all $X, Y \in \Gamma(TM)$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$. Such a manifold is said to be an indefinite metric g.f.f-manifold and denoted by $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$. From (2.2) one also has $g(X, \xi_\alpha) = \varepsilon_\alpha \eta^\alpha(X)$ and $g(\varphi X, \varphi Y) = -g(\varphi X, Y)$, for any $X, Y \in \Gamma(TM)$, and the splitting $TM = \operatorname{Im}(\varphi) \oplus \ker(\varphi)$ becomes orthogonal.

The fundamental 2-form $\Phi$ of an indefinite metric g.f.f-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is defined by $\Phi(X, Y) = g(\varphi X, \varphi Y)$. If $\Phi = d\eta^\alpha$, for any $\alpha \in \{1, \ldots, s\}$, the manifold is said to be an indefinite almost $S$-manifold. Finally, a normal indefinite almost $S$-manifold is, by definition, an indefinite $S$-manifold. Such a manifold is characterized by the identity $(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi X$, where $\xi = \sum_{\alpha=1}^{s} \xi_\alpha$ and $\eta = \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha$. It follows that $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$ and $\nabla_{\xi_\alpha} \xi_\beta = 0$, for any $\alpha, \beta \in \{1, \ldots, s\}$, and each $\xi_\alpha$ is a Killing vector field.

For more details on (almost) $S$-manifolds the reader is referred to [15] in the Riemannian case, and to [9] for the indefinite case.
3. Lorentzian $S$-manifolds and the $\varphi$-null Osserman condition.

The notion of $\varphi$-null Osserman condition is derived from that of null Osserman, which we briefly recall here, following [13] and [20].

Let $(M,g)$ be a Lorentzian manifold and $p \in M$. If $u \in T_pM$ is a lightlike (or null) vector, that is $u \neq 0$ and $g_p(u,u) = 0$, then $\text{span}(u) \subset u^\perp$. We can endow the quotient space $\bar{u}^\perp = u^\perp/\text{span}(u)$, whose canonical projection is $\pi: u^\perp \to \bar{u}^\perp$, with a positive definite inner product $\bar{g}$ defined by $\bar{g}(\bar{x},\bar{y}) = g_p(x,y)$, where $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$, obtaining the Euclidean vector space $(\bar{u}^\perp,\bar{g})$.

The Jacobi operator with respect to $\bar{u}$ is the endomorphism $\bar{R}_u: \bar{u}^\perp \to \bar{u}^\perp$ defined by $\bar{R}_u(\bar{x}) = \pi(\bar{R}_{\bar{p}}(x,u)u)$, for all $\bar{x} = \pi(x) \in \bar{u}^\perp$. It is easy to see that $\bar{R}_u$ is a self-adjoint endomorphism, hence it is diagonalizable.

If $z \in T_pM$ is a unit timelike vector, the null congruence set of $z$ is defined to be the set $\mathcal{N}(z) = \{ u \in T_pM | g_p(u,u) = 0, \ g_p(u,z) = -1 \}$. The elements of $\mathcal{N}(z)$ are in one-to-one correspondence to those of the set $S(z) = \{ x \in z^\perp | g_p(x,x) = 1 \}$, called the celestial sphere of $z$, via the map $\psi: \mathcal{N}(z) \to S(z)$ such that $\psi(u) = u - z$.

**Definition 3.1** ([13], [20]). A Lorentzian manifold $(M,g)$ is said to be null Osserman with respect to $z$, $z \in T_pM$ being a unit timelike vector, if the eigenvalues of $\bar{R}_u$ and their multiplicities are independent of $u \in \mathcal{N}(z)$.

Following [7] and [8], we recall the basic facts related with the definition of the $\varphi$-null Osserman condition.

Let $(M,\varphi,\xi,\eta^\alpha, g)$ be a Lorentzian $g,f,f$-manifold, with $\dim(M) = 2n + s$, and $\alpha \in \{1,\ldots,s\}$, $s \geq 1$. It is easy to see that one of the characteristic vector fields has to be timelike and, without loss of generality, we assume it is $\xi_1$. If $p \in M$, we define the $\varphi$-celestial sphere of $(\xi_1)_p$ to be the set $S_\varphi((\xi_1)_p) = S((\xi_1)_p) \cap \text{Im}(\varphi_p)$, and the $\varphi$-null congruence set of $(\xi_1)_p$ to be $\mathcal{N}_\varphi((\xi_1)_p) = \psi^{-1}(S_\varphi((\xi_1)_p))$.

**Definition 3.2** ([7], [8]). A Lorentzian $g,f,f$-manifold $(M,\varphi,\xi,\eta^\alpha, g)$ is said to be $\varphi$-null Osserman with respect to $(\xi_1)_p$, $p \in M$, if the eigenvalues of $\bar{R}_u$ and their multiplicities are independent of $u \in \mathcal{N}_\varphi((\xi_1)_p)$.

Fix $p \in M$ and consider $u \in \mathcal{N}_\varphi((\xi_1)_p)$. Since we can write $u = (\xi_1)_p + x$, with $x \in S_\varphi((\xi_1)_p)$, there is a natural one-to-one correspondence between the two kinds of Jacobi operator $R_z: x^\perp \to x^\perp$ and $\bar{R}_u: \bar{u}^\perp \to \bar{u}^\perp$. In [8] it is provided the relationship between these two operators with respect to the $\varphi$-null Osserman condition, which we summarize in the following proposition.

**Proposition 3.3** ([8]). Let $(M,\varphi,\xi,\eta^\alpha, g)$ be a Lorentzian $S$-manifold, $\dim(M) = 2n + s$, $s \geq 1$. For any $p \in M$, $M$ is $\varphi$-null Osserman with respect to $(\xi_1)_p$ if and only if the eigenvalues of $R_x$ with their multiplicities are independent of $x \in S_\varphi((\xi_1)_p)$.

The above result enables us to write the definition of the $\varphi$-null Osserman condition in terms of operator $R_x$, $x \in S_\varphi((\xi_1)_p)$, instead of $\bar{R}_u$, $u \in \mathcal{N}_\varphi((\xi_1)_p)$. It is clear that, in the case of a Lorentzian Sasakian manifold, the $\varphi$-null Osserman condition reduces to that of null Osserman one.

4. Principal torus bundles and the $\varphi$-null Osserman condition.

From [8] it is known that under an assumption of regularity it is possible to relate metric $g,f,f$-manifolds both to almost complex and to almost contact metric manifolds via Riemannian submersions. The semi-Riemannian version of the results of [8] is provided in [10], where it is possible to find the following result.
Theorem 4.1. Let \((M, \varphi, \xi_0, \eta^\alpha, g)\) be a compact, connected and regular indefinite \(S\)-manifold, with \(\dim(M) = 2n + s\), \(s \geq 2\). Then, there exists a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & M' \\
\downarrow \pi & & \downarrow \pi' \\
N & \xrightarrow{\tau'} & N
\end{array}
\]

where \(N\) is a \(2n\)-dimensional compact Kähler manifold, either indefinite or not, and \(M'\) is a \((2n + 1)\)-dimensional compact and regular Sasakian manifold, indefinite or not. All the maps are semi-Riemannian submersions with totally geodesic fibres, and more precisely:

- \(\tau\) is the projection of a principal \(\mathbb{T}^{s-1}\)-bundle over \(M'\);
- \(\pi'\) is the projection of a principal \(S^1\)-bundle over \(N\);
- \(\pi\) is the projection of a principal \(\mathbb{T}^k\)-bundle over \(N\).

where \(\mathbb{T}^k\) is the \(k\)-dimensional torus, for any \(k \in \mathbb{N}, k \geq 1\).

For the notion of regularity of a distribution and of a \(g.f.f\)-structure the reader is referred to [34] and [9]. The general idea of this result, as contained in [9], is to fibrate \(M\) by any \(s - r\) of the vector fields \(\xi_0\)'s, to obtain a principal \(\mathbb{T}^{s-1}\)-bundle over \(M'\). The remaining \(r\) characteristic vector fields are then projectable to \(M'\), inducing a \(g.f.f\)-structure on \(M'\) and preserving the regularity. Thus, \(M'\) can be fibrated again by its \(r\) characteristic vector fields, obtaining a principal \(\mathbb{T}^r\)-bundle over \(N\), which finally produces a commutative diagram. In particular, if we fibrate a Lorentzian \(S\)-manifold \(M\) by the \(s - 1\) spacelike characteristic vector fields, in Theorem 4.1 we obtain that \(N\) is a Kähler manifold and \(M'\) is a Lorentz Sasakian manifold.

We are going to find out some informations about the possibility of projecting the \(\varphi\)-null Osserman condition both onto the null Osserman condition and the classical Osserman condition, via the previous fibrations.

In general (see [16], [31]), given a \(C^\infty\)-submersion \(f : (M, g) \to (B, g')\) between semi-Riemannian manifolds, i.e. a map whose differential \((df)_p\) is surjective, for all \(p \in M\), then \(V = (\ker(df)_p)_{p \in M}\) and \(H = (\ker(df)_{p})_{p \in M}\), are, by definition, the \emph{vertical} and the \emph{horizontal} distributions of \(f\). Such a map is said to be a semi-Riemannian submersion if each fibre \(f^{-1}(p')\), \(p' \in B\), is a (semi-)Riemannian submanifold of \(M\) and the restriction of \(g_p\) to \(H_p\) is an isometry for all \(p \in M\). A vector field \(U\) (resp. \(X\)) on \(M\) such that \(U_p \in \mathcal{V}_p\) (resp. \(X_p \in \mathcal{H}_p\)) is called \emph{vertical} (resp. \emph{horizontal}). A vector field \(X\) on \(M\) such that there exists a vector field \(X'\) on \(B\) for which \(f_\ast X = X'\) is said to be \emph{projectable}, and any horizontal, projectable vector field on \(M\) is said to be \emph{basic}. The vertical distribution is always integrable, with the fibres of \(f\) as leaves. Denoting by \(v\) and \(h\) the projections of \(TM\) onto \(V\) and \(H\), respectively, the O'Neill tensors of \(f\) are the \((1,2)\)-type tensor fields \(T\) and \(A\) on \(M\) defined by:

\[
T(X, Y) = T_{X,Y} := v\nabla_{v,X}hY + h\nabla_{v,X}vY, \\
A(X, Y) = A_{X,Y} := v\nabla_{h,X}hY + h\nabla_{h,X}vY.
\]

They are both \(g\)-skew-symmetric tensors, and they satisfy the following fundamental properties:

\[
T_UW = T_WU, \quad U, W \in \mathcal{V} \\
A_{X,Y} = -A_{Y,X} = \frac{1}{2}v[X, Y], \quad X, Y \in \mathcal{H}
\]

It follows that the horizontal distribution is integrable if and only if \(A = 0\), and in this case the leaves are totally geodesic submanifolds of \(M\). Furthermore, the fibres of \(f\) are totally geodesic semi-Riemannian submanifolds of \(M\) if and only if \(T = 0\).
Lemma 4.2. Let \((M, \varphi, \xi_0, \eta^0, g)\) be a Lorentzian \(S\)-manifold, with \(\dim(M) = 2n + s, \ s \geq 1\). Let \(\pi : M \to N\) be a principal \(\mathbb{T}^s\)-bundle over a Kähler manifold, as in Theorem 4.1. We have:

\[
A_X Y = -g(X, \varphi Y)\bar{\xi}, \quad A_X \xi_0 = -\varepsilon_0 \varphi X,
\]

for any \(X, Y \in \text{Im}(\varphi)\) and any \(\alpha \in \{1, \ldots, s\}\), where \(\xi = \sum_{\alpha=1}^s \xi_\alpha\).

Proof. By construction of \(\pi\), we have \(\mathcal{H}_p = \text{Im}(\varphi_p)\) and \(\mathcal{V}_p = \text{span}\{s_1, \ldots, s_s\}\) for any \(p \in M\). Thus, since \(\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X\), by direct calculation we get:

\[
A_X Y = -g(X, \varphi Y)\bar{\xi}, \quad A_X \xi_\alpha = -\varepsilon_\alpha \varphi X,
\]

for all \(X, Y \in \mathcal{H}\). Analogously, we have \(A_X \xi_\alpha = h(\nabla_X \xi_\alpha) = -\varepsilon_\alpha \varphi X\) for all \(X \in \mathcal{H}\) and \(\alpha \in \{1, \ldots, s\}\).

For a semi-Riemannian submersion \(f : (M, g) \to (B, g')\), let us denote by \(R^*\) the \((1,3)\)-type \(\mathcal{H}\)-valued tensor field on \(M\) such that, if \(X, Y, Z \in \Gamma(TM)\) are basic vector fields \(f\)-related to \(X', Y', Z' \in \Gamma(TB)\), then \(R^*(X, Y, Z)\) is the unique basic vector field \(f\)-related to \(R^*(X', Y', Z')\). Thus, for any \(x \in \mathcal{H}_p\), one can consider the self-adjoint endomorphism \(R^*_x : x^+ \cap \mathcal{H}_p \to x^+ \cap \mathcal{H}_p\) such that \(R^*_x = R^*_x(y, x, x)\).

Lemma 4.3. Let \(f : (M, g) \to (B, g')\) be a semi-Riemannian submersion. For any orthogonal vectors \(x, y \in \mathcal{H}_p\) one has

\[
R^*_y(y) = h_p R_x(y) - 3A_x A_x(y).
\]

Proof. From standard formulas on the curvature tensors of a submersion (see [10], pag. 13), we have

\[
g_p(R^*_y(y, z), z) = R^*_p(x, y, x, z)
\]

\[
= R_p(x, y, x, z) + 2g_p(A_x(y), A_x(z)) - g_p(A_x(y), A_x(z))
\]

\[
= g_p(h_p R_x(y, z), z) - 3g_p(A_x A_x(y), z)
\]

for any \(z \in x^+ \cap \mathcal{H}_p\), which yields (4.2).

Proposition 4.4. Let \((M, \varphi, \xi_0, \eta^0, g)\) be a Lorentzian \(S\)-manifold, with \(\dim(M) = 2n + s, \ s \geq 1\). Let \(\pi : M \to N\) be a principal \(\mathbb{T}^s\)-bundle over a Kähler manifold \((N, J, g)\), as in Theorem 4.1. Let \(p \in M\), and suppose that, for any \(x \in S_\varphi(s_1)\), \(\varphi x\) is an eigenvector of \(R_x\). Then, \(M\) is \(\varphi\)-null Osserman with respect to \(s_1\) if and only if \(N\) is Osserman at \(p' = \pi(p)\).

Proof. Suppose first that \(s \geq 2\). Fix \(p' \in N\), with \(p' = \pi(p)\), \(p \in M\), and let \(x' \in T_{p'} N\) a unit vector, and \(y', z' \in x'^+\). Let \(x \in S_\varphi(s_1)\), \(V = x^+ \cap \text{Im}(\varphi_p)\) and \(y, z \in V\) such that \(x' = (d\pi)_p(x), \ y' = (d\pi)_p(y)\) and \(z' = (d\pi)_p(z)\). Then

\[
g_p(R^*_y(y, z), z) = G_p((d\pi)_p(R^*_p(y)), (d\pi)_p(z)) = G_p(R^*_y(y, z), z'),
\]

which implies that the Jacobian operators \(R^*_x : V \to V\) and \(R^*_y : x^+ \to x^+\) have the same characteristic polynomial. Using (4.4) one has \(A_x A_x(y) = -(s - 2)g_p(y, \varphi x)\varphi x\) and since \(R_x\) leaves the subspace \(V\) invariant, (4.2) gives

\[
R^*_x(y) = R_x(y) + 3(s - 2)g_p(y, \varphi x)\varphi x
\]

for any \(y \in V\). Observe that if \(\varphi x\) is an eigenvector of \(R_x\), we have

\[
g_p(R_x(y), \varphi x)\varphi x = g_p(y, R_x(\varphi x))\varphi x = R_y(g_p(y, \varphi x)\varphi x),
\]
that is the endomorphism of $V$ such that $y \mapsto g_p(y, \varphi x)\varphi x$ commutes with $R_x$.

This implies they are simultaneously diagonalizable, and if $\lambda_i, i \in \{1, \ldots, r\}$ are the eigenvalues of $R_x$, counted with multiplicities, with $\lambda_1$ relative to $\varphi x$, then $\lambda_1 + 3(s-2), \lambda_j, j \in \{2, \ldots, r\}$ are the eigenvalues of $R_x^*$. By Proposition 3.3 we obtain our statement.

If $s = 1$ then the proof goes through as above, except for the fact that one has $A_\alpha A_{\beta}(y) = g_p(y, \varphi x)\varphi x$.

\[ \text{□} \]

**Remark 4.5.** It is clear, from the previous proof, that in case $s = 2$ the hypothesis of $\varphi x$ being an eigenvector of $R_x$ can be dropped without affecting the result. Furthermore, in case $s = 1$, the statement is relative to the projection $\pi'$ of the commutative diagram in the Theorem 4.4.

**Lemma 4.6.** Let $(M, \varphi, \xi_\alpha, \eta^a, g)$ be a Lorentzian $S$-manifold, with $\dim(M) = 2n + s, s \geq 2$. Let $\alpha : M \to M'$ be a principal $\mathbb{T}^{s-1}$-bundle over a Lorentz Sasakian manifold, as in Theorem 4.4. We have:

\[ (4.3) \quad A_\alpha Y = -g(X, \varphi Y) \sum_{\alpha = 2}^{s} \xi_\alpha, \quad A_\alpha \xi_\alpha = -\varphi X, \]

for any $X, Y \in \text{Im}(\varphi) \oplus \text{span}(\xi_1)$ and any $\alpha \in \{2, \ldots, s\}$.

**Proof.** By construction of $\tau$, we have the splitting $\mathcal{H}_p = \text{Im}(\varphi_p) \oplus \text{span}(\xi_1)$ and $\mathcal{V}_p = \text{span}(\xi_2_p, \ldots, \xi_s_p)$ for any $p \in M$. Proceeding along the same lines as the proof of Lemma 4.2 we get (4.3). \[ \text{□} \]

**Proposition 4.7.** Let $(M, \varphi, \xi_\alpha, \eta^a, g)$ be a Lorentzian $S$-manifold, with $\dim(M) = 2n + s, s \geq 2$. Let $\alpha : M \to M'$ be a principal $\mathbb{T}^{s-1}$-bundle over a Lorentz Sasakian manifold $M'$ with structure $(\varphi', \xi', \eta', g')$ as in Theorem 4.4. Let $p \in M$, and suppose that, for any $x \in S_p((\xi_1)_p), \varphi x$ is an eigenvector of $R_x$. Then, $M$ is $\varphi$-null Osserman with respect to $(\xi_1)_p$ if and only if $M'$ is null Osserman with respect to $(\xi'_p, p') = \tau(p)$.

**Proof.** One can follow the same proof of Proposition 4.4 where, using (4.3), one has $A_\alpha A_{\beta}(y) = -(s-1)g_p(y, \varphi x)\varphi x$. \[ \text{□} \]

Propositions 4.4 and 4.7 can be summarized as follows.

**Theorem 4.8.** Let $(M, \varphi, \xi_\alpha, \eta^a, g)$ be a compact, connected and regular Lorentzian $S$-manifold, with $\dim(M) = 2n + s, s \geq 2$. Consider the commutative diagram of principal torus bundles

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & M' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
N & \xrightarrow{\tau} & N'
\end{array}
\]

where $N$ is a $2n$-dimensional compact Kähler manifold and $M'$ is a $(2n + 1)$-dimensional compact and regular Lorentz Sasakian manifold, with unit timelike characteristic vector field $\xi' = \tau_*(\xi_1)$. Let $p \in M$, and suppose that $\varphi x$ is an eigenvector of $R_x$ for any $x \in S_p((\xi_1)_p)$. The following three statements are equivalent.

(a) $M$ is $\varphi$-null Osserman with respect to $(\xi_1)_p$;
(b) $N$ is Osserman at $q = \pi(p)$;
(c) $M'$ is null Osserman with respect to $\xi'_p, p' = \tau(p)$.

**Remark 4.9.** It is clear that the three Osserman-type conditions in the above theorem can be also considered either pointwise or globally. Moreover, if we use
the pointwise conditions, from the equivalence $(a) \iff (b)$ it follows that $N$ is Einstein at each point and the connectedness implies that it is a Kähler-Einstein manifold.

**Remark 4.10.** In case $\tau : M \to M'$ is a principal $\mathbb{T}^{s-1}$-bundle from a Lorentzian $\mathcal{S}$-manifold $(M, \varphi, \xi_s, \eta_s, g)$ with $\dim(M) = 2n + s$, $s \geq 2$, over a Sasakian manifold $M'$ with structure $(\varphi', \xi', \eta', g')$ as in Theorem 4.1, we could ask about the Osserman condition on $M'$. Let us suppose $M'$ pointwise Osserman, since it is odd-dimensional, it has constant sectional curvature $c$ ([12, 20]). Being $k(X', \xi') = 1$, for any $X' \in \text{Im}(\varphi')$, then $c = 1$ and $M'$ is locally isometric to the sphere $S^{2n+1}$ with its standard Sasakian structure (see [2], p. 114). By construction of the bundle projection $\tau$, we can suppose that $\mathcal{H}_p = \text{Im}(\varphi_p) \oplus \text{span}(\xi_p)$ and $\mathcal{V}_p = \text{span}(\xi_1, \ldots, \xi_{s-1})$. Hence, with calculations similar to those of Lemma 4.2 one has $A_X Y = g(Y, \varphi X) \sum_{\alpha=1}^{s-1} \xi_\alpha$. By standard formulas on sectional curvatures of the total and the base spaces of a semi-Riemannian submersion (see [10], p. 14) we have

$$k(x, \varphi x) = k'(x', \varphi' x') - 3g(A_\varphi x, A_\varphi x) = 1 - 3(s - 3), \quad x \in \text{Im}(\varphi_p),$$

which gives a necessary condition on the $\varphi$-sectional curvature of $M$ for $M'$ to be an Osserman Sasakian manifold.

**Remark 4.11.** Analogously, in case $\tau : M \to M'$ is a principal $\mathbb{T}^{s-1}$-bundle from a Lorentzian $\mathcal{S}$-manifold $(M, \varphi, \xi_s, \eta_s, g)$, with $\dim(M) = 2n + s$, $s \geq 2$, over a Lorentzian Sasakian manifold $M'$ with structure $(\varphi', \xi', \eta', g')$ as in Theorem 4.1, we could ask again about the Osserman condition on $M'$. It is known that any connected Lorentzian Osserman manifold is a space-form ([20]), and since $k(X', \xi') = -1$, for any $X' \in \text{Im}(\varphi')$, $M'$ has constant sectional curvature $c = -1$. As in the previous calculations, using (4.3), we have

$$k(x, \varphi x) = k'(x', \varphi' x') - 3g(A_\varphi x, A_\varphi x) = -1 - 3(s - 1), \quad x \in \text{Im}(\varphi_p),$$

which is a necessary condition on the $\varphi$-sectional curvature of $M$ for $M'$ to be a Lorentzian Osserman manifold.

**References**

[1] D.E. Blair, Geometry of manifolds with structural group $U(n) \times O(s)$. J. Differential Geom. 4 (1970), 155–167.

[2] D.E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics 203, Birkhäuser, Boston, 2002.

[3] D.E. Blair, G. Ludden and K. Yano, Differential geometric structures on principal toroidal bundles. Trans. Amer. Math. Soc. 181 (1973), 175–184.

[4] N. Blažič, N. Bokan and P. Gilkey, A note on Osserman Lorentz manifolds. Bull. London Math. Soc. 29 (1997), 227–230.

[5] N. Blažič, N. Bokan and Z. Rakić, A note on Osserman conjecture and isotropic covariant derivative of curvature. Proc. Amer. Math. Soc. 128 (2000), 245–253.

[6] A. Bonome, R. Castro, E. García-Río, L. Hervella and R. Vázquez-Lorenzo, Nonsymmetric Osserman indefinite Kähler manifolds. Proc. Amer. Math. Soc. 126 (1998), 2763–2769.

[7] L. Brunetti, An Osserman-type condition on $g.f.f$-manifolds with Lorentz metric. Submitted for publication. Preprint: [http://arxiv.org/abs/1106.6317v1](http://arxiv.org/abs/1106.6317v1)

[8] L. Brunetti and A.V. Caldarella, Curvature properties of $\varphi$-null Osserman Lorentzian $\mathcal{S}$-manifolds. Submitted for publication. Preprint: [http://arxiv.org/abs/1112.1050v1](http://arxiv.org/abs/1112.1050v1)

[9] L. Brunetti and A.M. Pastore, Curvature of a class of indefinite globally framed $f$-manifolds. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 51 (99) (2008), no. 3, 183–204.

[10] L. Brunetti and A.M. Pastore, Examples of indefinite globally framed $f$-structures on compact Lie groups. Publ. Math. Debrecen 80/1-2 (2012), 215–234.

[11] J.L. Cabrerizo, L.M. Fernandez and M. Fernandez, The curvature tensor fields on $f$-manifolds with complemented frames. An. Științ. Univ. Al. I. Cuza Iași Secț. I a Mat. 36 no. 2 (1990), 151-161.

[12] Q.S. Chi, A curvature characterization of certain locally rank-one symmetric space. J. Differential Geom. 28 (1988), 187–202.
[13] Q.S. Chi, Quaternionic Kähler manifolds and a curvature characterization of two-point homogeneous space. Illinois J. Math. 35 (1991), 408–418.
[14] Q.S. Chi, Curvature characterization and classification of rank-one symmetric space. Pacific J. Math. 150 (1991), 31–42.
[15] K.L. Duggal, S. Ianuş and A.M. Pastore, Maps interchanging f-structures and their harmonicity. Acta Appl. Math. 67 (2001), 91–115.
[16] M. Falcitelli, S. Ianuş S and A.M. Pastore, Riemannian Submersions and Related Topics. World Sci. Publishing, River Edge, NJ (2004).
[17] E. García-Río and D.N. Kupeli, Four-Dimensional Osserman Lorentzian manifolds. New Develop. in Diff. Geom. (Delsucen, 1994), 201–211, Math. Appl. 350, Kluwer Acad. Publ., Dordrecht, 1996.
[18] E. García-Río, D.N. Kupeli and M.E. Vázquez-Abal, On a problem of Osserman in Lorentzian geometry. Differential Geom. Appl. 7 (1997), 85–100.
[19] E. García-Río, D.N. Kupeli, M.E. Vázquez-Abal and R. Vázquez-Lorenzo, Osserman affine connections and their Riemannian extensions. Differential Geom. Appl. 11 (1999), 145–153.
[20] E. García-Río, D.N. Kupeli and R. Vázquez-Lorenzo, Osserman Manifolds in Semi-Riemannian Geometry. Lecture Notes in Mathematics 1777, Springer-Verlag, Berlin, 2002.
[21] E. García-Río, M.E. Vázquez-Abal and R. Vázquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds. Proc. Amer. Math. Soc. 126 (1998), 2771–2778.
[22] S.I. Goldberg and K. Yano, On normal globally framed f-manifolds. Tôhoku Math. J. 22 (1970), 362–370.
[23] S.I. Goldberg and K. Yano, Globally framed f-manifolds. Illinois J. Math. 15 (1971), 456–474.
[24] S. Ishihara, Normal structure $f$ satisfying $f^3 + f = 0$. Kōdai Math. Sem. Rep. 18 (1966), 36–47.
[25] H. Nakagawa, f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian. Kōdai Math. Sem. Rep. 18 (1966), 161–183.
[26] H. Nakagawa, On framed f-manifolds. Kōdai Math. Sem. Rep. 18 (1966), 293–306.
[27] Y. Nikolayevsky, Osserman manifolds of dimension 8. Manuscripta Math. 115 (2004), 31–53.
[28] Y. Nikolayevsky, Osserman Conjecture in dimension $n \neq 8$. Ann. Math. 168 (2008), 505–522.
[29] Y. Nikolayevsky, Osserman manifolds of dimension 16. Contemporary Geometry and Related Topics, Proc. Conf. Belgrade, 2005 (2006), 379–398.
[30] B. O’Neill, The fundamental equations of a submersion. Michigan Math. J. 13 (1966), 459–469.
[31] R. Osserman, Curvature in the Eighties. Amer. Math. Monthly 97 (1990), 731–756.
[32] R. Osserman and P. Sarnak, A new curvature invariant and entropy of geodesic flow. Invent. Math. 77 (1984), 455–462.
[33] R. Palais, A global formulation of the Lie theory of transformation groups. Mem. Am. Math. Soc. 22 (1957), 1–123.
[34] J. Vanzura, Almost r-contact structures. Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. 26 (1972), 97–115.
[35] K. Yano, On a structure defined by a tensor field $f$ of type $(1, 1)$ satisfying $f^3 + f = 0$. Tensor N.S. 14 (1963), 99–109.

Department of Mathematics, University of Bari “Aldo Moro”
Via E. Orabona, 4 – 70125 Bari (ITALY)
E-mail address, L. Brunetti: brunetti@dm.uniba.it
E-mail address, A.V. Caldarella: caldarella@dm.uniba.it