Remarks on Wolff’s inequality for hypersurfaces

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Abstract

We run an iteration argument due to Pramanik and Seeger, to provide a proof of sharp decoupling inequalities for conical surfaces and for $k$-cones. These are extensions of results of Łaba and Pramanik to sharp exponents.

1. Statements of the results

For $n \geq 2$, let $L_0 \subset \mathbb{R}^{n+1}$ be an affine subspace of dimension $n$ that does not pass through the origin. Let $S_0 \subset L_0$ be a smooth compact surface (possibly with boundary) of dimension $n-1$ whose Gaussian curvature does not vanish at every point. The surface $S$ given by

$$S = \{tx \in \mathbb{R}^{n+1} : x \in S_0; t \in [C_1, C_2]\}$$

for some $0 < C_1 < C_2$ is called a conical surface induced by $S_0$. For each $a \in S$, there exists a unique $b \in S_0$ such that $a = tb$ for some $t \in [C_1, C_2]$. We denote by $\eta(a)$ the convex hull $[C_1b, C_2b]$ in $\mathbb{R}^{n+1}$, and call $\eta(a)$ the 1-plane at $a$.

Now we follow the approach of Łaba and Pramanik [5] to introduce the notion of conical surfaces of higher co-dimensions.

Let $L_0$ be an $n$-dimensional linear subspace of $\mathbb{R}^{n+k}$. Let $v_1, v_2, \ldots, v_k$ be linearly independent vectors such that

$$\{x + c_1v_1 + c_2v_2 + \cdots + c_kv_k \in \mathbb{R}^{n+k} : x \in L_0, (c_1, c_2, \ldots, c_k) \in \mathbb{R}^k\} = \mathbb{R}^{n+k}.$$

For simplicity, let $v_0 = 0 \in \mathbb{R}^{n+k}$. For each $i = 0, 1, 2, \ldots, k$, denote

$$L_i = L_0 + v_i.$$

For each $L_i$, we fix a bounded and convex solid $F_i$ such that $E_i := \partial F_i$ is a $C^\infty$ surface and has non-vanishing Gaussian curvature at every point on it. We fix some rotation $R$ satisfying $R(L_i) = \mathbb{R}^{n} \times c_i$ for some $c_i \in \mathbb{R}^k$ for all $i$. By discarding the last $k$ coordinates we can identify $L_i$ with $\mathbb{R}^{n}$. Thus for each unit normal vector $x \in S^{n-1}$ in $\mathbb{R}^{n}$, each $E_i$ contains exactly one point $a_i$ such that $x$ is the outward normal vector to $E_i \subset L_i$ at $a_i$. We say that
a $(k + 1)$-tuple of points $(x_0, \ldots, x_k)$ is *good* if $x_i \in E_i$ for every $0 \leq i \leq k$, and if the outward unit normal vectors to $E_i$ at $x_i$ are the same. The *$k$-cone* $S$ in $\mathbb{R}^{n+k}$ induced by the collection $\{E_i\}_{i=0}^k$ is defined by

$$S = \bigcup_{(x_0, \ldots, x_k) \text{ good}} \eta(x_0, \ldots, x_k),$$

where $\eta(x_0, \ldots, x_k)$ denotes the convex hull generated by $x_0, \ldots, x_k$ in $\mathbb{R}^{n+k}$. By the discussion of [5, section 7], $S$ is a smooth surface of co-dimension 1 in $\mathbb{R}^{n+k}$. Also, each $a \in S$ belongs to $\eta(x_0, \ldots, x_k)$ for exactly one good $(k + 1)$-tuple $(x_0, \ldots, x_k)$. We will call $\eta(x_0, \ldots, x_k)$ the $k$-plane at $a$, and denote it by $\eta(a)$.

Let $S$ be a conical surface induced by $S_0$ or a $k$-cone induced by $\{E_i\}_{i=0}^k$. For each $a \in S$, denote by $n_a$ the unit normal vector to $S$ at $a$. For a small number $\delta > 0$, we denote by $\mathcal{M}_\delta$ a $\delta^{1/2}$-separated subset of $S_0$ (resp. $E_0$) when $S$ is a conical surface (resp. a $k$-cone). Moreover, denote by $N_\delta S$ the $\delta$-neighborhood of $S$. Throughout the paper, we are interested in a covering of $N_\delta S$ satisfying the following assumption.

**Assumption (A)** For each small $\delta > 0$ and each $a \in S$, let $\Pi_{a,\delta}$ be a rectangular box centered at $a$, of dimensions $C\delta \times C\delta^\frac{1}{2} \times \cdots \times C\delta^\frac{1}{2} \times C \times \cdots \times C$, where the short direction is normal to $S$ at $a$, the long directions are parallel to the $k$-plane $\eta(a)$ at $a$, and the mid-length directions are tangent to $S$ at $a$ but perpendicular to the $k$-plane $\eta(a)$. Then:

$A_1$: $C_1 \Pi_{a,\delta} \subset N_\delta S \cap \{x \in \mathbb{R}^{n+k} : (x - a) \cdot n_a \leq \delta\}$ for some small constant $C_1 > 0$;

$A_2$: $\{\Pi_{a,\delta}\}_{a \in \mathcal{M}_\delta}$ forms a finitely overlapping covering of $N_\delta S$;

$A_3$: for every $a \in \mathcal{M}_\delta$, there are at most $O(1)$ distinct $b \in \mathcal{M}_\delta$ such that $|n_a - n_b| \leq C_2 \delta^{1/2}$;

$A_4$: if $0 < \delta \leq \sigma$ and if $\Pi_{a,\delta} \cap \Pi_{b,\sigma} \neq \emptyset$ for some $a, b \in S$, then $\Pi_{a,\delta} \subset C_3 \Pi_{b,\sigma}$.

This group of assumptions is identical to that in [5]. The constants $C, C_1, C_2, C_3$ are independent of the parameter $\delta$ and the choice of $\mathcal{M}_\delta$.

Let $\Xi_a$ be a smooth function in $\mathbb{R}^{n+k}$ with $c \leq \|\Xi_a\|_{L^1(\mathbb{R}^{n+k})} \leq c^{-1}$ for some small $c$ such that $\text{supp}(\Xi_a) \subset \Pi_{a,\delta}$ and $\{\Xi_a\}_{a \in \mathcal{M}_\delta}$ forms a smooth partition of unity of $N_\delta S$. We can always take such partition of unity (depending on $\delta$). We first take functions $\Xi_a$ satisfying $\Xi_a \sim 1$ on $c\Pi_{a,\delta}$ and $\text{supp}(\Xi_a) \subset \Pi_{a,\delta}$ for some small constant $c > 0$. We then modify them to be a partition of unity. The $L^1$ estimate follows from the integration by parts.

Our first result is the following:

**Theorem 1.1.** Let $n \geq 2$ and $k \geq 1$. Let $S$ be a $k$-cone in $\mathbb{R}^{n+k}$. Under Assumption (A), if $\text{supp}(\hat{f}) \subset N_\delta S$, then for $p \geq 2 + 4/(n - 1)$, we have

$$\|f\|_{L^p(\mathbb{R}^{n+k})} \leq C_{p,\epsilon} \delta^{-\frac{n-1}{4} + \frac{n+1}{2p} - \epsilon} \left(\sum_{a \in \mathcal{M}_\delta} \|\Xi_a \ast f\|_{L^p(\mathbb{R}^{n+k})}^2\right)^{\frac{1}{2}},$$

for every $\epsilon > 0$. 

By a standard interpolation, the above estimate (1.1) further implies
\[ \|f\|_{L^p} \leq C_{p, \epsilon} \delta^{-\epsilon} \left( \sum_{a \in \mathcal{M}_\delta} \| \Xi_a \ast f \|_{L^p}^2 \right)^{1/2} \] (1.2)
for every \( 2 \leq p \leq 2 + 4/(n - 1) \) and every \( \epsilon > 0 \). Up to the arbitrarily small factor \( \epsilon > 0 \), both (1.1) and (1.2) are sharp. For the sharpness we refer to the discussion in the introduction of the paper [5].

Theorem 1.1 involves \( k \)-cones. Recall that \( k \)-cones are generated by the boundaries \( E_i \) of bounded and strictly convex bodies \( F_i \subset L_i \) with \( 0 \leq i \leq k \), whose Gaussian curvature does not vanish at every point. That means, for each \( i \), if \( L_i \) is identified with \( \mathbb{R}^n \) in a canonical manner, then at every point on \( E_i \), all the principle curvatures are positive. However, in the definition of a conical surface \( S \) induced by \( S_0 \), we only assumed \( S_0 \) to have non-vanishing Gaussian curvatures. That means principle curvatures might have different signs. Therefore, the \( l^2 \) decoupling for conical surfaces does not necessarily imply the \( l^p \) decoupling for conical surfaces. We refer to [2] for more discussion. For conical surfaces, we prove

**Theorem 1.2.** Let \( n \geq 2 \). Let \( S \) be a conical surface in \( \mathbb{R}^{n+1} \). Under Assumption (A), if \( \text{supp}(\hat{f}) \subset \mathcal{N}_\delta S \), then for \( p \geq 2 + 4/(n - 1) \), we have
\[ \|f\|_{L^p(\mathbb{R}^{n+1})} \leq C_{p, \epsilon} \delta^{-\frac{n-1}{2}} \left( \sum_{a \in \mathcal{M}_\delta} \| \Xi_a \ast f \|_{L^p(\mathbb{R}^{n+1})}^p \right)^{\frac{1}{p}} , \]
for every \( \epsilon > 0 \).

Theorem 1.1 and Theorem 1.2 are extensions of results in Łaba and Pramanik [5] to sharp exponents. Our proof relies on an iteration argument and on results of Bourgain and Demeter [1, 2]. This iteration argument was first used by Pramanik and Seeger [7], and was later used by Bourgain and Demeter [1] to obtain sharp decoupling estimates for the cone. For the prior developments on Wolff’s inequalities, we refer to Wolff [8], Łaba and Wolff [6], Garrigós and Seeger [3, 4].

For some examples of \( k \)-cones and conical surfaces and applications of the decouplings for those surfaces, we refer to the introduction of the paper [5].

For \( (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n \), we use the notation \( \zeta = (\zeta_1, \ldots, \zeta_n) \) and \( \zeta' = (\zeta_1, \ldots, \zeta_{n-1}) \). Throughout the paper, we write \( A \lesssim B \) if \( A \leq cB \) for some constant \( c > 0 \), and \( A \sim B \) if \( c^{-1}A \leq B \leq cB \). The constant \( c \) will in general depend on fixed parameters such as \( p, n \) and sometimes on the variable parameter \( \epsilon \) but not the parameter \( \delta \).

2. Proof of Theorem 1.1

A truncated hyperbolic paraboloid \( H_{\nu}^{n-1} \) in \( \mathbb{R}^n \) is defined for \( \nu = (v_1, \ldots, v_{n-1}) \in (\mathbb{R} \setminus \{0\})^{n-1} \) as
\[ H_{\nu}^{n-1} = \{ (\zeta_1, \ldots, \zeta_{n-1}, v_1\zeta_1^2 + \cdots + v_{n-1}\zeta_{n-1}^2) : |\zeta_i| \leq 1 \} . \]
When \( v_i = 1 \) for all \( i \), we use \( P^{n-1} \) instead of \( H_{\nu}^{n-1} \). We denote by \( \mathcal{N}_\delta H_{\nu}^{n-1} \) the \( \delta \)-neighbourhood of \( H_{\nu}^{n-1} \). Let \( \mathcal{P}_\delta \) be a finitely overlapping cover of \( \mathcal{N}_\delta H_{\nu}^{n-1} \) with \( \delta \times \delta^{1/2} \times \cdots \times \delta^{1/2} \) rectangular boxes \( \Pi_{a, \delta} \) centered at \( a \). Moreover, denote \( \mathcal{M}_\delta = \{ a : \Pi_{a, \delta} \in \mathcal{P}_\delta \} \). For each \( a \in \mathcal{M}_\delta \), let \( \Psi_a \) be a smooth function in \( \mathbb{R}^n \) with \( \| \Psi_a \|_{L^1(\mathbb{R}^n)} \sim 1 \) and \( \text{supp}(\Psi_a) \subset \Pi_{a, \delta} \) such that \( \{ \Psi_a \}_{a \in \mathcal{M}_\delta} \) forms a smooth partition of unity of \( \mathcal{N}_\delta H_{\nu}^{n-1} \).

To prove Theorem 1.1, we will use the following theorem due to Bourgain and Demeter.
THEOREM 2.1 ([I, theorem 1.1]). Denote \( p_0 = 2(n+1)/(n-1) \). If \( \text{supp}(\hat{f}) \subset N_\delta P^{n-1} \), then

\[
\|f\|_{L^p(\mathbb{R}^n)} \lesssim_\delta \delta^{-\varepsilon} \left( \sum_{a \in M_\delta} \|\hat{\Psi}_a \ast f\|_{L^p(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}},
\]

for every \( \varepsilon > 0 \).

In the forthcoming proof of Theorem 1.1, we consider only the endpoint \( p_0 = 2(n+1)/(n-1) \). The estimate for the general range follows from the interpolation with the trivial estimate at \( p = \infty \).

2.1. In the first step of the proof, we will slice our surface into small pieces so that we can exploit local properties of a \( k \)-cone. Let \( \{e_i\}_{i=1}^{n+k} \) be a collection of standard orthonormal bases in \( \mathbb{R}^{n+k} \). By a linear transformation, we may assume that \( L_0 = \text{span}(e_1, \ldots, e_n) \) and \( L_i = L_0 + e_{n+i} \) for each \( 1 \leq i \leq k \).

Fix a small parameter \( \varepsilon > 0 \). This \( \varepsilon \) is essentially the same as the one in the statement of Theorem 1.1. We may also assume that \( \varepsilon^{-1} \) is a natural number. We define a sliced surface \( \tilde{S} \) by

\[
\tilde{S} = S \cap (\mathbb{R}^n \times \{(\tau_1, \ldots, \tau_k) : c_i \leq \tau_i \leq c_i + 4\delta^{k/2}\}),
\]

for some \( c_i \) with \( 1 \leq i \leq k \). We will prove the decoupling for the sliced surface \( \tilde{S} \) first.

PROPOSITION 2.2. If \( \text{supp}(\hat{f}) \subset N_\delta \tilde{S} \), then

\[
\|f\|_{L^p(\mathbb{R}^{n+i})} \lesssim_\delta \delta^{-\varepsilon} \left( \sum_{a \in M_\delta} \|\hat{\Psi}_a \ast f\|_{L^p(\mathbb{R}^{n+i})}^2 \right)^{\frac{1}{2}}.
\]

The desired decoupling inequalities for the surface \( S \) can be deduced from Proposition 2.2. To see this, let \( \{\hat{\psi}_j\}_{j \in \mathbb{Z}} \) be a partition of unity of \( \mathbb{R} \) such that

\[
\|\hat{\psi}_j\|_{L^1(\mathbb{R})} \sim 1 \quad \text{and} \quad \text{supp}(\hat{\psi}_j) \subset [(j-2)\delta^{k/2}, (j+2)\delta^{k/2}].
\]

For each \( J = (j_1, \ldots, j_k) \in \mathbb{Z}^k \), we define

\[
f_J(x,t) = \int_{\mathbb{R}^n \times \mathbb{R}^2} \prod_{i=1}^k \hat{\psi}_{j_i}(\tau_i) \hat{f}(\xi,\tau) e^{2\pi i (x \cdot \xi + t \cdot \tau)} \, d\xi \, d\tau.
\]

Here \( \tau = (\tau_1, \ldots, \tau_k) \). Note that \(|\{J \in \mathbb{Z}^k : f_J \not= 0\}| = O(\delta^{-k/2}) \). Hence, by the triangle inequality

\[
\|f\|_{L^p(\mathbb{R}^{n+i})} \lesssim_\delta \delta^{-k/2} \max_{J \in \mathbb{Z}^k} \|f_J\|_{L^p(\mathbb{R}^{n+i})}.
\]

By Proposition 2.2 and Young’s inequality, the last expression can be further bounded by

\[
\delta^{-2k} \max_{J \in \mathbb{Z}^k} \left( \sum_{a \in M_\delta} \|\hat{\Psi}_a \ast f\|_{L^p(\mathbb{R}^{n+i})}^2 \right)^{\frac{1}{2}} \lesssim_\delta \delta^{-2k} \left( \sum_{a \in M_\delta} \|\hat{\Psi}_a \ast f\|_{L^p(\mathbb{R}^{n+i})}^2 \right)^{\frac{1}{2}}.
\]

Hence, what remains is to show Proposition 2.2.

2.2. Our argument relies on an iteration. This iteration argument first appeared in Pramanik and Seeger [7]. We will deduce Proposition 2.2 from the following proposition.
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**Proposition 2.3.** Fix $\mu$ such that $2\mu + \epsilon/2 \leq 1$ and $\mu \geq \epsilon/2$. Let $a \in \mathcal{M}_{\beta^+}$. If $\text{supp}(\hat{f}) \subset N_0\delta$, then

$$\|\Xi_a \ast f\|_{L^{p_0}(\mathbb{R}^{n+1})} \lesssim \delta^{-\epsilon^3} \left( \sum_{b \in \mathcal{M}_{\beta^+}} \|\Xi_a \ast \Xi_b \ast f\|_{L^{p_0}(\mathbb{R}^{n+1})}^2 \right)^{\frac{1}{2}}.$$ 

We postpone the proof of Proposition 2.3 to the next subsection, and continue by

**Proof of Proposition 2.2.** First of all, by the triangle inequality and Hölder’s inequality, we obtain

$$\|f\|_{p_0} \lesssim \delta^{-C\epsilon} \left( \sum_{a \in \mathcal{M}_{\beta^+}} \|\Xi_a \ast f\|_{p_0}^2 \right)^{\frac{1}{2}},$$

for some large constant $C > 0$. Next, by applying Proposition 2.3 with $\mu = \epsilon/2$, the last expression can be further bounded by

$$\delta^{-\epsilon^3 - C\epsilon} \left( \sum_{a \in \mathcal{M}_{\beta^+}} \sum_{b \in \mathcal{M}_{\beta^+}} \|\Xi_a \ast \Xi_b \ast f\|_{p_0}^2 \right)^{\frac{1}{2}} \lesssim \epsilon \delta^{-\epsilon^3 - C\epsilon} \left( \sum_{b \in \mathcal{M}_{\beta^+}} \|\Xi_b \ast f\|_{p_0}^2 \right)^{\frac{1}{2}}.$$ 

The last inequality follows from

$$\|a \in \mathcal{M}_{\beta^+} : \Pi_{a,\beta^+} \cap \Pi_{b,\beta^+} \cap N_0\delta \neq \emptyset\| = O(1),$$

which further follows from Assumption $A_3$ and Assumption $A_4$. Repeatedly apply Proposition 2.3 with $\mu = \mu_l = \epsilon/4$ starting with $l = 3$ until $l = 2/\epsilon - 1$. In the end we have

$$\|f\|_{p_0} \lesssim \epsilon \delta^{-2C\epsilon} \left( \sum_{a \in \mathcal{M}_{\beta^+}} \|\Xi_a \ast f\|_{p_0}^2 \right)^{\frac{1}{2}}.$$ 

This finishes the proof of Proposition 2.2.

2.3. In this subsection, we will prove Proposition 2.3 by using Theorem 2.1. Let $B \subset \mathbb{R}^{n+k}$ be a ball of radius $r_B := \delta^{-2(\mu+\frac{1}{2})}$, centered at $c_B$. Let $C$ be a large constant. Define a weight $\omega_B$ associated with the ball $B$ by $(1 + \cdots - C_B/r_B)^{-C}$. To prove Proposition 2.3, by a simple localisation argument, it suffices to prove

$$\|\Xi_a \ast f\|_{L^{p_0}(\omega_B)} \lesssim \delta^{-\epsilon^3} \left( \sum_{b \in \mathcal{M}_{\beta^+}} \|\Xi_a \ast \Xi_b \ast f\|_{L^{p_0}(\omega_B)}^2 \right)^{\frac{1}{2}}.$$ 

Let $a \in \eta(y_0, \ldots, y_k)$ for some good $(k + 1)$-tuple $(y_0, \ldots, y_k)$. Under certain affine transformations, we may assume that $y_0$ lies in the origin and

$$y_i = (0, \ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0) \text{ for each } 1 \leq i \leq k.$$ 

Moreover, we assume that the normal vector to the surface $S$ at the point $y_i$ is given by $e_n$ for every $0 \leq i \leq k$. By using a partition of unity, we may assume, that each $E_i$, viewed as a hypersurface in $L_i$, can be represented as the graph of a smooth function $G_i : (-\epsilon_0, \epsilon_0)^{n-1} \rightarrow \mathbb{R}$ for some fixed small constant $\epsilon_0 > 0$. Under these assumptions,
we observe that $\nabla G_i(0) = (0, \ldots, 0) \in \mathbb{R}^{n-1}$ for each $i$. Moreover, for those points that are different from the origin, we have

**Claim 2.1.** For each $1 \leq i \leq k$, there exists a smooth function $h_i : (-\epsilon_0, \epsilon_0)^{n-1} \to \mathbb{R}^{n-1}$ such that

$$\nabla G_i(h_i(\xi')) = \nabla G_0(\xi') \text{ for all } \xi' \in (-\epsilon_0, \epsilon_0)^{n-1}.$$  
Moreover, $h_i(\xi') = J_i \cdot \xi' + O(|\xi'|^2)$ for some positive definite matrix $J_i$.

**Proof.** For each $\xi'$, let us consider the level set $\{\eta' \in \mathbb{R}^{n-1} : \nabla G_i(\eta') = \nabla G_0(\xi')\}$. Note that this set is not empty. Recall that $G_i$ is a strictly convex smooth function. Hence the existence and smoothness of $h_i$ can be guaranteed by the implicit function theorem.

To obtain an asymptotic of the function $h_i$ near the origin, we differentiate both sides of the equation $\nabla G_i(h_i(\xi')) = \nabla G_0(\xi')$, and obtain $(HG_i)(\nabla h_i) = HG_0$. Here $HG_i$ is the Hessian matrix of the function $G_i$. Since $E_i$ is strictly convex, $HG_i$ is a positive definite matrix. Thus, $\nabla h_i$ is also a positive definite matrix. The identity $h_i(\xi') = J_i \cdot \xi' + O(|\xi'|^2)$, with some positive definite matrix $J_i$, immediately follows from Taylor’s theorem. This completes the proof of the claim.

Denote $h_0(\xi') = \xi'$. By Claim 2.1, if $\epsilon_0$ is chosen small enough, then a good $(k+1)$-tuple containing $(\xi', G_0(\xi'), 0, \ldots, 0) := P_0(\xi')$ also contains

$$(h_1(\xi'), G_1(h_1(\xi'))), 0, \ldots, 0, 1, 0, \ldots, 0) := P_i(\xi').$$

for each $1 \leq i \leq k$. Hence, w.l.o.g. we may assume that the $k$-cone $\tilde{S} \cap \Pi_{a,\delta^\alpha}$ is given by

$$\left\{ \left( 1 - \sum_{j=1}^{k} \theta_j \right) P_0(\xi') + \sum_{j=1}^{k} \theta_j P_j(\xi') : |\xi'| \lesssim \delta^\alpha, \ 0 \leq \theta_j \leq \delta^{\beta/2} \right\}. \tag{2.2}$$

We claim that the $k$-cone given by (2.2) is contained in the $\delta^{2\mu + \xi}$ neighbourhood of a cylinder. To be precise, we will use the cylinder

$$\left\{ P_0(\xi') + \sum_{j=1}^{k} \theta_j e_{n+j} : |\xi'| \lesssim \delta^\mu, \ 0 \leq \theta_j \leq \delta^{\beta/2} \right\}. \tag{2.3}$$

That is, we will show that, given an arbitrary point on the $k$-cone (2.2), its distance with the cylinder (2.3) is smaller than $O(\delta^{2\mu + \delta^\xi})$. Given a point in (2.2), we write it as

$$\left( 1 - \sum_{j=1}^{k} \theta_j \right) \xi' + \sum_{j=1}^{k} \theta_j h_j(\xi'), \left( 1 - \sum_{j=1}^{k} \theta_j \right) G_0(\xi') + \sum_{j=1}^{k} \theta_j G_j(h_j(\xi')), \theta_1, \ldots, \theta_k \right).$$

We calculate its distance with the point

$$\left( 1 - \sum_{j=1}^{k} \theta_j \right) \xi' + \sum_{j=1}^{k} \theta_j h_j(\xi'), G_0 \left( 1 - \sum_{j=1}^{k} \theta_j \right) \xi' + \sum_{j=1}^{k} \theta_j h_j(\xi'), \theta_1, \ldots, \theta_k \right)$$. 

from the cylinder \((2 \cdot 3)\). This amounts to proving

\[
\left| G_0 \left( \left( 1 - \sum_{j=1}^{k} \theta_j \right) \xi' + \sum_{j=1}^{k} \theta_j h_j(\xi') \right) - \left( 1 - \sum_{j=1}^{k} \theta_j \right) G_0(\xi') \right| + \sum_{j=1}^{k} \theta_j G_j(h_j(\xi')) \lesssim \delta^{2\mu + \frac{\epsilon}{2}}.
\]

By the triangle inequality, it suffices to show

\[
\left| G_0 \left( \left( 1 - \sum_{j=1}^{k} \theta_j \right) \xi' + \sum_{j=1}^{k} \theta_j h_j(\xi') \right) - G_0(\xi') \right| \lesssim \delta^{2\mu + \frac{\epsilon}{2}}
\]

and

\[
|G_0(\xi') - G_j(h_j(\xi'))| \lesssim \delta^{2\mu}.
\]

The latter follows directly from Taylor’s formula. To prove the former estimate, we write

\[
G_0(\xi') = (\xi')^T [HG_0(0)](\xi') + O(|\xi'|^3) \text{ with } HG_0(0) \text{ the Hessian matrix of the function } G_0 \text{ at the origin. Moreover, we know that } HG_0(0) \text{ is positive definite. Using this formula, we just need to show that}
\]

\[
\left| \left( 1 - \sum_{j=1}^{k} \theta_j \right) \xi' + \sum_{j=1}^{k} \theta_j h_j(\xi') - \xi' \right| \lesssim \delta^{\mu + \frac{\epsilon}{2}},
\]

which follows via a direct calculation.

So far we have verified that the \(k\)-cones \((2 \cdot 2)\) lies in a \(\delta^{2\mu + \frac{\epsilon}{2}}\)-neighbourhood of the cylinder \((2 \cdot 3)\). Hence to prove the localised decoupling inequality \((2 \cdot 1)\), by the uncertainly principle, it is the same as proving a corresponding decoupling inequality associated with the cylinder \((2 \cdot 3)\), which further follows from Theorem 2-1 and Fubini’s theorem. This finishes the proof of Proposition 2-3.

### 3. Proof of Theorem 1-2

In this section, we use the notations defined at the beginning of Section 2. The proof of Theorem 1-2 essentially follows via the same argument as that of Theorem 1-1.

Let \(H_0^{n-1} \approx \) be the hyperbolic paraboloid defined in the beginning of Section 2. To prove Theorem 1-2, we will use the following theorem due to Bourgain and Demeter.

**Theorem 3-1** ([2, theorem 1-1]). Denote \(p_0 = 2(n + 1)/(n - 1)\). Fix \(\nu \in (\mathbb{R} \setminus \{0\})^{n-1}\).

If \(\text{supp}(\hat{f}) \subset N_\delta H_0^{n-1}\), then

\[
\|f\|_{L^p(\mathbb{R}^n)} \lesssim \epsilon \delta^{\frac{n}{m} - \frac{n-1}{m} - \epsilon} \left( \sum_{a \in \mathcal{M}_a} \|\Psi_a \ast f\|_{L^{p_0}(\mathbb{R}^n)} \right)^{\frac{1}{m}},
\]

for every \(\epsilon > 0\).

The role of Theorem 3-1 in the proof of Theorem 1-2 is similar to that of Theorem 2-1 in the proof of Theorem 1-1. However, in contrast with the proof of Theorem 1-1, we need a rescaled version of Theorem 3-1. This is because the exponent of \(\delta\) in (3-1) is not arbitrarily small, which requires us to carefully deal with the exponent of \(\delta\) there.
By performing simple rescaling argument to Theorem 3-1, we have the following proposition.

**Proposition 3-2.** Denote $p_0 = 2(n + 1)/(n - 1)$. Fix $v \in (\mathbb{R} \setminus 0)^n$ and $a, \alpha > 0$. If $\text{supp}(\hat{f}) \subset \mathcal{N}_{a^2 + a} H_v^{n-1} \cap \{ (\xi_1, \ldots, \xi_n) : |\xi_i| \leq \delta^\alpha \times \mathbb{R} \}$, then

$$
\| f \|_{L^{p_0}(\mathbb{R}^n)} \lesssim \epsilon^{\frac{1}{p_0}} \delta_a(\frac{1}{p_0} - \frac{n}{2})^{-\epsilon} \left( \sum_{a \in \mathcal{M}_{a^2 + a}} \| \Psi_a * f \|_{L^{p_0}(\mathbb{R}^n)} \right)^{\frac{1}{p_0}},
$$

for every $\epsilon > 0$.

**Proof.** We define the linear transform $L$ by

$$
L(\xi_1, \ldots, \xi_{n-1}, \xi_n) = (\delta^\alpha \xi_1, \ldots, \delta^\alpha \xi_{n-1}, \delta^{2\alpha} \xi_n).
$$

Applying Theorem 3-1 to a function $g$ satisfying $\hat{g} = \hat{f} \circ L$ gives

$$
\| g \|_{L^{p_0}(\mathbb{R}^n)} \lesssim \epsilon^{\frac{1}{p_0}} \delta_a(\frac{1}{p_0} - \frac{n}{2})^{-\epsilon} \left( \sum_{a \in \mathcal{M}_{a^2 + a}} \| \Psi_a * g \|_{L^{p_0}(\mathbb{R}^n)} \right)^{\frac{1}{p_0}}.
$$

Changing back to the original variables completes the proof.

The forthcoming proof of Theorem 1-2 is similar to the one in Section 2. As we did in Section 2, by interpolation, it suffices to consider only the endpoint $p_0 = 2(n + 1)/(n - 1)$.

3-1. In the first step of the proof, we will slice our surface into small pieces so that we can exploit local properties of the conical surface. By a linear transformation, we may assume that $L_0 = \mathbb{R}^d \times \{1\}$ and $C_1 = 1$.

Fix a small parameter $\epsilon > 0$. This $\epsilon$ is essentially the same as the one in the statement of Theorem 1-2. We may also assume that $\epsilon^{-1}$ is a natural number. We define a sliced surface $\tilde{S}$ by

$$
\tilde{S} = S \cap (\mathbb{R}^n \times \{ \tau_1 : d \leq \tau_1 \leq d + 4\delta^{\epsilon/2} \})
$$

for some $d$. We will prove the decoupling for the sliced surface $\tilde{S}$ first.

**Proposition 3-3.** If $\text{supp}(\hat{f}) \subset \mathcal{N}_{a} \tilde{S}$, then

$$
\| f \|_{L^{p_0}(\mathbb{R}^{n+1})} \lesssim \epsilon^{\frac{1}{p_0}} \delta_a^{\frac{2}{p_0} - \frac{n}{2} - \epsilon} \left( \sum_{a \in \mathcal{M}_{a}} \| \Psi_a * f \|_{L^{p_0}(\mathbb{R}^{n+1})} \right)^{\frac{1}{p_0}}.
$$

The desired decoupling inequalities for the surface $S$ can be deduced from Proposition 3-3. This can be shown by using arguments in Subsection 2-1. This is very similar to the argument in Subsection 2-1. As before, let $\{ \hat{\psi}_j \}_{j \in \mathbb{Z}}$ be a partition of unity of $\mathbb{R}$ such that

$$
\| \psi_j \|_{L^1(\mathbb{R})} \sim 1 \text{ and supp}(\hat{\psi}_j) \subset [(j - 2)\delta^{\epsilon/2}, (j + 2)\delta^{\epsilon/2}].
$$

For each $j_1 \in \mathbb{Z}$, we define

$$
f_{j_1}(x, t) = \int_{\mathbb{R}^n \times \mathbb{R}} \hat{\psi}_{j_1}(\tau_1) \hat{f}(\zeta, \tau_1) e^{2\pi i (x \cdot \xi + t \tau_1)} d\xi d\tau_1.
$$

Since $|\{j_1 \in \mathbb{Z} : f_{j_1} \neq 0\}| = O(\delta^{-\epsilon/2})$. Hence, by the triangle inequality

$$
\| f \|_{L^{p_0}(\mathbb{R}^{n+1})} \lesssim \delta^{\epsilon/2} \max_{j_1 \in \mathbb{Z}} \| f_{j_1} \|_{L^{p_0}(\mathbb{R}^{n+1})}.
$$
By Proposition 3-3 and Young’s inequality, the last expression can be further bounded by

$$
\delta^{-2\epsilon} \max_{j_1 \in \mathbb{R}} \left( \sum_{a \in \mathcal{M}_3} \| \Xi_a \ast f j_1 \|_{L^p_0(\mathbb{R}^{n+1})}^2 \right)^{\frac{1}{4}} \lesssim \delta^{-2\epsilon} \left( \sum_{a \in \mathcal{M}_3} \| \Xi_a \ast f \|_{L^p_0(\mathbb{R}^{n+1})}^2 \right)^{\frac{1}{4}}.
$$

Hence, what remains is to show Proposition 3-3.

### 3.2. We will deduce Proposition 3-3 from the following proposition.

**Proposition 3-4.** Fix $\mu > 0$ such that $2\mu + \epsilon/2 \leq 1$ and $\mu \geq \epsilon/2$. Let $a \in \mathcal{M}_{\delta^p}$. If $\text{supp}(\hat{f}) \subset \mathcal{N}_0S$, then

$$
\| \Xi_a \ast f \|_{L^p_0(\mathbb{R}^{n+1})} \lesssim_{\epsilon} \delta^{\frac{\mu}{\mu_0} - \frac{\mu_1}{2} - \epsilon} \left( \sum_{b \in \mathcal{M}_{2\mu + \epsilon/2}} \| \Xi_b \ast f \|_{L^p_0(\mathbb{R}^{n+1})}^2 \right)^{\frac{1}{2}}.
$$

Proposition 3-3 can be deduced from Proposition 3-4 by arguments in Subsection 2.2. As we pointed out before, this argument traces back to Pramanik and Seeger [7]. We postpone the proof of Proposition 3-4 to the next subsection, and continue by

**Proof of Proposition 3-3.** To apply Proposition 3-4, we slice the frequency part by Hölder’s inequality:

$$
\| f \|_{p_0} \lesssim \delta^{-C\epsilon} \left( \sum_{a \in \mathcal{M}_{\delta^p}} \| \Xi_a \ast f \|_{p_0}^2 \right)^{\frac{1}{2}},
$$

for some large constant $C > 0$. Next, by applying Proposition 3-4 with $\mu = \epsilon/2$, the last expression can be further bounded by

$$
\delta^{-C\epsilon} \delta^{\frac{\mu}{\mu_0} - \frac{\mu_1}{2} - \epsilon} \left( \sum_{a \in \mathcal{M}_{\delta^p}} \sum_{b \in \mathcal{M}_{\delta^p/2}} \| \Xi_a \ast \Xi_b \ast f \|_{p_0}^2 \right)^{\frac{1}{2}} \lesssim \delta^{-C\epsilon} \delta^{\frac{\mu}{\mu_0} - \frac{\mu_1}{2} - \epsilon} \left( \sum_{b \in \mathcal{M}_{\delta^p/2}} \| \Xi_b \ast f \|_{p_0}^2 \right)^{\frac{1}{2}}.
$$

The last inequality follows from

$$
|\{a \in \mathcal{M}_{\delta^p} : \Pi_{a,\delta^p} \cap \Pi_{\delta^p/2} \cap \mathcal{N}_0S \neq \emptyset\}| = O(1),
$$

which further follows from Assumption $A_3$ and Assumption $A_4$. Repeatedly apply Proposition 3-4 with $\mu = \mu_l = \epsilon/4$ starting with $l = 3$ until $l = 2/\epsilon - 1$. In the end we have

$$
\| f \|_{p_0} \lesssim_{\epsilon} \delta^{-C\epsilon} \delta^{\frac{\mu}{\mu_0} - \frac{\mu_1}{2} - \epsilon} \left( \sum_{a \in \mathcal{M}_{\delta^p}} \| \Xi_a \ast f \|_{p_0}^2 \right)^{\frac{1}{2}}.
$$

This finishes the proof of Proposition 3-3.

### 3.3. In this subsection, we will deduce Proposition 3-4 from Proposition 3-2. To do this, we will just follow the arguments used in Subsection 2.3. Let $B \subset \mathbb{R}^{n+1}$ be a ball of radius...
\( r_B := \delta^{-(2\mu+\frac{1}{2})} \), centered at \( c_B \). Let \( C \) be a large constant. Define a weight \( w_B \) associated with the ball \( B \) by \((1 + \cdots - c_B/r_B)^{-C} \). To prove Proposition 3.4, by a simple localisation argument, it suffices to prove

\[
\| \Xi_a * f \|_{L^p_0(w_B)} \lesssim \epsilon \delta^{-\frac{2\mu}{p_0} + \frac{\epsilon}{2}} \left( \sum_{b \in M_{\rho \delta^{p_0} + \epsilon/2}} \| \Xi_a * \Xi_b * f \|_{L^p_0(w_B)} \right)^{\frac{1}{p_0}}. \tag{3.2}
\]

Under certain linear transformation, we may assume that \( a = (0, \ldots, 0, 1) \) and \( S_0 \) is represented as the graph of a smooth function \( G \) with \( G(0) = 0 \) and \( \nabla G(0) = (0, \ldots, 0) \in \mathbb{R}^{n-1} \). Hence, w.l.o.g. we may assume that the conical surface \( \tilde{S} \cap \Pi_{a, \rho} \) is given by

\[
\{(1 + \theta)(\xi', G(\xi'), 1) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : |\xi'| \lesssim \delta^{\mu}, 0 \leq \theta \leq \delta^{\frac{\epsilon}{2}} \}. \tag{3.3}
\]

We claim that this surface is contained in the \( \delta^{2\mu+\epsilon/2} \)-neighbourhood of the following cylinder

\[
\{(\xi', G(\xi')) : |\xi'| \lesssim \delta^\mu \} \times \mathbb{R}. \tag{3.4}
\]

To see this, we take any point in (3.3), and we write it as

\[
((1 + \theta) \xi', (1 + \theta) G(\xi'), 1 + \theta).
\]

We calculate its distance with the point from the cylinder (3.4). This amounts to proving

\[
|(1 + \theta) G(\xi') - G((1 + \theta) \xi')| \lesssim \delta^{2\mu+\frac{\epsilon}{2}},
\]

which follows directly from Taylor’s formula.

So far we have verified that the conical surfaces (3.3) lies in a \( \delta^{2\mu+\epsilon/2} \)-neighbourhood of the cylinder (3.4). Hence to prove the localised decoupling inequality (3.2), by the uncertainty principle, it is the same as proving a corresponding decoupling inequality associated with the cylinder (3.4), which further follows from Proposition 3.2 and Fubini’s theorem. This finishes the proof of Proposition 3.4.

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