Quantizing Geometry or Geometrizing the Quantum?

Benjamin Koch
Pontificia Universidad Católica de Chile,
Av. Vicuña Mackenna 4860,
Santiago, Chile

(Dated: April 20, 2010)

The unsatisfactory status of the search for a consistent and predictive quantization of gravity is taken as motivation to study the question whether geometrical laws could be more fundamental than quantization procedures. In such an approach the quantum mechanical laws should emerge from the geometrical theory. A toy model that incorporates the idea is presented and its necessary formulation in configuration space is emphasized.

PACS numbers: 04.62.+v, 03.65.Ta

I. QUANTIZING GEOMETRY

The dream of finding a unified description of all physical phenomena is facing a profound problem: “The deep incompatibility between the indefinite nature of quantum mechanics and the rigid geometrical formulation of general relativity.” A common assumption is that quantum mechanics, as it is usually formulated, is a fact of “nature” and thus it is more fundamental than general relativity. Consequently most approaches try to apply one of the well defined quantization procedures to the physical degrees of freedom of space-time (or to some deeper theory that gives rise to space-time). Some of the most popular approaches along this line are:

String theory is for many the most famous candidate for a unified theory of nature [1]. It also lead to interesting conjectures about the relation between certain tree level string theories and quantum field theory [2]. But until today it could not live up to its promises concerning the uniqueness of what this theory actually predicts and explains.

Loop quantum gravity is a canonical approach for the quantization of space-time. In earlier stages of its development it lead to the development of geometrodynamics [3] and [4] supergravity that has very nice features at the Planck scale [5]. However, up to now it was not possible to show that it really contains general relativity in some classical limit [6].

Causal dynamical triangulation and causal sets are disciplines that earn more and more attention [7]. They show the emergence of four dimensional space-time by starting from a discrete causal structure. Until now those approaches are limited to asking very basic questions on such as the dimensionality of space-time but they don’t allow to derive an effective gravitational action.

Induced gravity theories try to show the emergence of curved space-time in a mean field approximation of some underlying microscopic degrees of freedom [8]. It is assumed that this mechanism is similar to the mechanism that allows to get fluid dynamics from Bose-Einstein condensation. Up to now those models manage to mimic some possible features of (quantized) general relativity but a complete picture is still missing. An other alternative for emergence of gravity is based on the idea that the existence of curved space-time emerges from non-geometric statistical laws [9, 10].

Renormalization group approaches are working in the imaginary time formalism. Given an ultra violet (UV) completion and the existence of a non-trivial fixed point in the running couplings of the completed gravitational action this approach might present a renormalizable version of gravity [11, 12]. Until now the strict applicability of the imaginary time formalism and the form of the UV completion are open issues.

Anisotropic models postulate a different scaling behavior of space and time in the UV regime, which allows to construct a power counting renormalizable theory [13] in the UV. However, recent studies claim that the infrared limit of the theory is not identical to massless gravity [14].

Further research has been done on asymptotic quantization [15], twistors [16], non-commutative [17, 18] and discretized [19] geometry.

Despite of impressive progress in some directions, the original task remains unsolved in all those approaches.

II. GEOMETRIZING THE QUANTUM

Given the problems in applying the laws of quantum mechanics to the geometry of space-time we want to ask the following question:

“Could it be that (classical) geometry is more fundamental than the rules of quantization?”

A. Conceptual problems

Necessarily, answering this question with “yes” would mean that the undeniable observable effects of quantization have to emerge from the deeper theory (in this case a classical geometrical theory). Such an approach faces immediately two mayor problems

• Determinism
  is, in contrast to quantum mechanics, part of most
geometric theories (such as general relativity). This means for example that in causal geometrical theories uncertainties are just a result of unknown initial conditions, whereas in standard quantum mechanics they are an irremucible concept.

- **Non-locality:**
In principle it is possible to construct deterministic (hidden variable) theories that are in agreement with the predictions of quantum mechanics. However, those theories have to pay a price in order to evade “no go” theorems such as the Bell inequalities [20]. They have to contain non-local interactions.

**B. A conceptual bridge**

There exists a self consistent deterministic formulation of quantum mechanics, which also reproduces all typical experimental results [43]. It was first suggested by de Broglie, then shown to be consistent by Bohm [21,22] and later further developed by several authors [20,23,24]. It will be referred to as dBB (de Broglie-Bohm) theory. In this proposal, the dBB theory will be an essential piece when building the bridge from classical geometry to a quantum theory. We will now shortly present its formulation for the case of a relativistic system of n-bosonic quantum theory. We will now shortly present its formulation for the case of a relativistic system of n-bosonic particles as given by [24]: Let $|0\rangle$ be state vector of the vacuum and $|n\rangle$ be an arbitrary n-particle state. The corresponding n-particle wave function is [24]

$$\psi(x_1; \ldots ; x_n) = \frac{P_n}{\sqrt{n!}} \langle 0 | \hat{\Phi}(x_1) \ldots \hat{\Phi}(x_n) | n \rangle \ ,$$

where the $\hat{\Phi}(x_j)$ are scalar Klein-Gordon field operators. The symbol $P_n$ denotes symmetrization over all positions $x_j$ which we will keep in mind but not write explicitly any more. For free fields, the wave function (1) satisfies the equation

$$\left( \sum_j^n \partial_j^2 - M^2 \right) \psi(x_1; \ldots ; x_n) = 0 \ .$$

Including this definition one has three coupled real differential equations. For further convenience the coordinates for the n particles can be labeled as

$$x^L = (t, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \ ,$$

which also implies the $1+3n$ dimensional co- and contravariant derivatives $\partial_L$ and $\partial^L$. Now the three real equations of the dBB theory read

$$2MQ \equiv (\partial^L S)(\partial_L S) - nM^2 \ \text{with} \ (4)$$

$$Q \equiv \frac{\hbar^2}{2M} \partial^L \partial_L P \ ,$$

$$0 \equiv \partial_L \left( P^2 (\partial^L S) \right) \ ,$$

(5)

$$p^L \equiv M \frac{dx^L}{ds} \equiv -\partial^L S \ .$$

(6)

Applying the total derivative $d/ds \equiv dx^L/ds\partial_L$ to eq. gives a Newtonian type of equation of motion

$$\frac{d^2 x^L}{ds^2} = \frac{(\partial^N S)(\partial^L \partial_N S)}{M^2} \ .$$

(7)

It is crucial to note that this theory addresses the two previously mentioned conceptual issues and thus makes the dBB theory a good framework for the program of geometrizing the quantum.

First, it is deterministic in the sense that given initial positions and given initial field configurations for $S$ and $P$ determine the final state of the system.

Second, it is deeply non-local, because the functions $S$ and $P$ simultaneously depend on the positions of all the n-particles.

A further remark: the dBB theory contextural and therefore not affected by the Kochen-Specker theorem [25].

**III. EMERGENT QUANTUM MECHANICS**

The idea that quantum mechanics might not be fundamental but rather emerge from an underlying classical system has been proposed in various ways.

**A. Various appearances of the idea**

Although the focus of this paper is on the possible geometric origin of quantum mechanics it is instructive to give a list of proposals that point into a similar direction.

**Statistical emergence of quantum mechanics:**

In [26,27] it was shown that quantum mechanical correlations arise when considering finite subsystems of classical statistical systems with originally infinite degrees of freedom. An application of this observation to quantum gravity is perceivable but was not attempted yet.

**Gauge emergence of quantum mechanics:**

Based on a new kind of local gauge transformation a non-linear field theory has been proposed that contains
quantum field theory as an infrared limit. Also a
special classical supersymmetric model was suggested to
give rise to a quantum mechanical system. A possible
unification with general relativity was not explored yet.

Dissipative emergence of both, quantum and gravity:
Dissipative deterministic systems can give rise to quan-
tum operators and symmetries that are not present in the
original theory at the microscopic scale. Further con-
jecturing that those symmetries are the ones of dif-
feomorphism invariance (general relativity) might give an
identikit picture of a future theory of quantum gravity.

Geometrical emergence of quantum mechanics:
The similarity between Weyl geometry and the struc-
ture of quantum mechanical equations was first noticed
in. Other studies in this direction focused on the
Ricci flow or on a geometric reduction of the
dimensionality of space-time. Using local confor-
mal transformations (Weyl geometry) it was even pos-
sible to formulate a geometrical theory that contains in
certain limits both general relativity and the equations
of Bohmian mechanics. The impressive success
of those (Weyl geometry) models is limited to the single
particle case because the dBB theory is only consistent
if it also contains the non-local interactions due to multi
particle dynamics.

B. Geometry of configuration space

It was shown that existing models for the geometrical
emergence of quantum mechanics are incomplete, since
they can’t explain the non-local interactions in the multi
particle dBB theory. Continuing previous work in this
direction a possible way to fill this gap will be
presented.

The $1+3n$ dimensional configuration space of $n$-
particles with a common time coordinate will be consid-
ered. Following the notation in eq. the coordinates in
this (possibly curved) space-time will be denoted as

$$\hat{x}^A = (\hat{t}, \hat{x}_1, \ldots, \hat{x}_n) \quad .$$

The toy model for the curvature of this space will be
a single scalar equation which is a $1+3n$ dimensional
analog to the Nordström theory

$$\hat{R} = \kappa \hat{T}_M \quad .$$

The left hand side contains the Ricci scalar (correspond-
ing to a metric $\hat{g}_{\Lambda \Sigma}$). The right hand side contains some
coupling constant $\kappa$ and the trace of the energy momen-
tum tensor $\hat{T}$. The symbol $|S$ indicates complete sym-
metrization of the terms with respect to the interchange
of two configuration coordinates $\hat{x}_i \leftrightarrow \hat{x}_j, \ldots$. Just like
in the case of the bosonic Klein-Gordon equation we will
keep this in mind without explicitly writing it into the
following equations. The symmetrization further fixes
the coordinate system for the four dimensional subspaces
and forces all block diagonal submetrics to be identical
$\hat{g}_{\mu \nu} \leftrightarrow \hat{g}^{\mu \nu}$. In order to describe the local conformal part
of this theory separately and for simplification one as-
sumes the metric $\hat{g}$ to split up into a conformal function
$\phi(x)$ and a flat part $\eta$

$$\hat{g}_{\Lambda \Gamma} = \phi^{3/4} \eta_{LG} \quad .$$

The inverse of the metric is

$$\hat{g}^{\Lambda \Gamma} = \phi^{-3/4} \eta^{LG} \quad .$$

Indices with a lower Greek and a lower Roman index can
be identified $\hat{\partial}_\Lambda \equiv \partial_L$. From this follows for example that
the adjoint derivatives are not identical, in both notations

$$\hat{\partial}^\Lambda = \hat{\partial}^{\Lambda \Sigma} \hat{\partial}_\Sigma = \phi^{-3/4} \eta^{LS} \partial_S = \phi^{-3} \partial^L \quad .$$

The geometrical dual to the first dBB equation:
An Extension of the Hamilton Jacobi stress energy tensor
$\hat{T}_M$ can be defined by subtracting a mass term $M^2$
for every particle

$$\hat{T}_M = \hat{p}^\Lambda \hat{p}_\Lambda - n \hat{M}_G^2$$

$$= (\hat{\partial}^\Lambda S_H)(\hat{\partial}_\Lambda S_H) - n \hat{M}_G^2$$

$$= \phi^{-3}((\partial^L S_H)(\partial_L S_H) - n \hat{M}_G^2)$$

$$= \phi^{3/4} \hat{T}_M \quad .$$

The Hamilton principle function $S_H$ defines the local
momentum $\hat{\partial}^\Lambda = \hat{M}_G \partial \hat{x}^\Lambda / \partial \hat{s} = -\hat{\partial}^\Lambda S_H$. Combining (13),
(12), (10), and (9) gives

$$\frac{12n}{\kappa (1-3n)} \frac{\partial^L \partial_L \phi}{\phi} = (\partial^L S_H)(\partial_L S_H) - n \hat{M}_G^2 \quad .$$

This is exactly the first dBB equation if one identifies

$$\phi(x) = P(x) \quad ,$$

$$S_H(x) = S(x) \quad ,$$

$$\kappa = \frac{12n}{(1-3n)} / \hbar^2 \quad ,$$

$$M^2 = \hat{M}_G^2 \quad .$$

Note that the matching conditions demand a negative
coupling $\kappa$.

The geometrical dual to the second dBB equation:
In order to find the dual to the second Bohmian equa-
tion one can exploit that the stress-energy tensor
is covariantly conserved

$$\nabla_\Lambda \hat{T}^{\Lambda \Delta} = 0 \quad .$$

This is true if the following relations are fulfilled

$$\nabla_\Lambda (\hat{\partial}^\Lambda S_H) = 0 \quad ,$$

$$\hat{\partial}^\Lambda S_H \nabla_\Lambda (\hat{\partial}_\Lambda S_H) = 0 \quad ,$$

$$\hat{\partial}^\Lambda S_H \nabla_\Lambda (\hat{\partial}^\Lambda S_H) = 0 \quad .$$
In addition to the covariant conservation of momentum \( \Gamma_\Lambda^\Sigma \) and the conservation of squared momentum \( \Gamma_{\Lambda|\Sigma} \), the tensor nature of \( \Gamma_\Lambda^\Sigma \) also demands \( \Gamma_{\Lambda|\Sigma} \). In order to calculate the covariant derivatives in (17-19), one needs to know the Levi Civita connection

\[
\Gamma^\Sigma_{\Lambda\Delta} = \frac{1}{2} \phi^{-\frac{1}{2}n\rho} \left( \partial_\Delta g_{\Sigma\rho} + \partial_\Sigma g_{\Delta\rho} - \partial_\rho g_{\Delta\Sigma} \right) = \frac{1}{2} \phi^{-\frac{1}{2}n\rho} \left[ (\partial_\Delta \phi) \phi^{-\frac{1}{2}n} \right] \delta^\Sigma_L + (\partial_D \phi) \phi^{-\frac{1}{2}n \rho} \right] \delta^\Sigma_L
\]

\[
-\left( \partial^\Sigma \phi^{-\frac{1}{2}n \rho} \right) n_L D
\]

It is this form of the connection that gives rise to the nonmetricity in Weyl geometry. Using eq. (20), the condition \( \Gamma^\Sigma_{\Lambda\Delta} \) reads

\[
\tilde{\nabla}_\Lambda (\phi^\Lambda S_H) = \phi^{-\frac{1}{2}n \rho} \partial_\Lambda \left[ \phi^2 \left( \partial^\rho S_H \right) \right] = 0 \quad \text{. (21)}
\]

With the matching conditions \( \text{[15]} \), the above equation is identical to the second Bohmian equation \( \text{[15]} \).

The geometrical dual to the third dBB equation: According to the Hamilton-Jacobi formalism the derivatives of the Hamilton principle function \( (S_H) \) define the momenta

\[
\dot{p}_\Lambda = -\left( \partial_\Lambda S_H \right) \quad \text{. (22)}
\]

Therefore, with the prescription \( \text{[12]} \) and the matching condition \( \text{[15]} \) one sees that the third Bohmian equation \( \text{[6]} \) is fulfilled.

The geometrical dual to the dBB equation of motion: From differential geometry it is known that the validity of the geodesics equation of motion results in the conservation of the stress energy tensor. Nevertheless, it is a good consistency check \( \text{[41]} \) to explicitly calculate the geodesic equation

\[
\frac{d^2 \hat{x}^\Lambda}{ds^2} + \hat{\Gamma}^\Lambda_{\Delta\Sigma} \frac{dx^\Delta}{ds} \frac{dx^\Sigma}{ds} = \hat{p}^\Lambda \cdot \hat{f}(\hat{x}) \quad \text{. (23)}
\]

Inserting eq. (20) into eq. (23) and using the matching conditions eq. \( \text{[15]} \) the dBB equation of motion \( \text{[17]} \) is obtained.

IV. SUMMARY

It is advocated that “geometrizing the quantum” might be a viable alternative to the standard approaches to quantum gravity. The main conceptual problems of the new approach are discussed. Using the example of a scalar geometrical toy model (incorporating gravity is beyond the scope of this proposal) and mapping this model to the dBB interpretation of the multi particle Klein-Gordon equation, it is shown how those problems can be evaded. It is argued that such a mechanism only can work consistently if the geometrical theory is formulated in the \((1+n)\) dimensional configuration space of the system.

The author wants to thank J.M. Isidro and D. Dolce for their remarks.

\[\text{[1]} \] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, Nucl. Phys. B 256, 253 (1985); E. Witten, Phys. Rev. D 44, 314 (1991).

\[\text{[2]} \] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

\[\text{[3]} \] J. A. Wheeler, Annals Phys. 2, 604 (1957).

\[\text{[4]} \] S. Deser and B. Zumino, Phys. Lett. B 62, 335 (1976).

\[\text{[5]} \] C. Rovelli, Living Rev. Rel. 1, 1 (1998) [arXiv:gr-qc/9710006].

\[\text{[6]} \] C. Rovelli, Living Rev. Rel. 1, 1 (1998) [arXiv:gr-qc/9710006].

\[\text{[7]} \] A. Ashtekar, New J. Phys. 7, 198 (2005) [arXiv:gr-qc/0401054].

\[\text{[8]} \] L. Bombelli, J. H. Lee, D. Meyer and R. Sorkin, Phys. Rev. Lett. 59, 521 (1987); J. Ambjorn, J.Jurkiewicz and R. Loll, Phys. Rev. Lett. 93, 131301 (2004) [arXiv:hep-th/0404156].

\[\text{[9]} \] A. Ashtekar, New J. Phys. 7, 198 (2005) [arXiv:gr-qc/0401054].

\[\text{[10]} \] A. D. Sakharov, Sov. Phys. Dokl. 12, 1040 (1968) [Dokl. Akad. Nauk Ser. Fiz. 177, 70 (1967 SOPUA,34.394.1991 GRGVA,32.365-367.2000)]; C. Barcelo, S. Liberati and M. Visser, Living Rev. Rel. 8, 12 (2005) [arXiv:gr-qc/0505065].

\[\text{[11]} \] C. Rovelli, Living Rev. Rel. 5, 75 (2002) [arXiv:gr-qc/9911028].

\[\text{[12]} \] M. Reuter, Phys. Rev. D 57, 971 (1998) [arXiv:hep-th/9605030]; M. Reuter and F. Saueressig, Phys. Rev. D 65, 065016 (2002) [arXiv:hep-th/0110054].

\[\text{[13]} \] D. F. Litim, Phys. Rev. Lett. 92, 201301 (2004) [arXiv:hep-th/0312114].

\[\text{[14]} \] A. Ashtekar, New J. Phys. 7, 198 (2005) [arXiv:gr-qc/0401054].

\[\text{[15]} \] P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]]; P. Horava, Phys. Rev. Lett. 102, 161301 (2009) [arXiv:0902.3657 [hep-th]]; R. G. Cai, L. M. Cao and N. Ohta, Phys. Rev. D 80, 024003 (2009) [arXiv:0904.3670 [hep-th]]; T. P. Sotiriou, M. Visser and S. Weinfurtner, Phys. Rev. Lett. 102, 251601 (2009) [arXiv:0904.4364 [hep-th]].

\[\text{[16]} \] C. Charmousis, G. Niz, A. Padilla and P. M. Saffin, arXiv:0905.2379 [hep-th]; M. Sakamoto, Phys. Rev. D 79, 124038 (2009) [arXiv:0905.4326 [hep-th]].

\[\text{[17]} \] A. Ashtekar, Phys. Rev. Lett. 46, 573 (1981).

\[\text{[18]} \] R. Penrose and W. Rindler, Cambridge, Uk: Univ. Pr. (1986) 501p.

\[\text{[19]} \] A. Connes, Commun. Math. Phys. 182, 155 (1996) [arXiv:hep-th/9603053].
[18] P. Nicolini, Int. J. Mod. Phys. A 24, 1229 (2009) [arXiv:0807.1939 [hep-th]].
[19] R. Gambini and J. Pullin, Phys. Rev. Lett. 94, 101302 (2005) [arXiv:gr-qc/0409057].
[20] J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press; ISBN 0521368693, 1988).
[21] D. Bohm, Phys. Rev. 85, 166 (1951).
[22] D. Bohm, Phys. Rev. 85, 172 (1951).
[23] P. R. Holland and J. P. Vigier, Nuovo Cim. B 88, 20 (1985); P. R. Holland, Phys. Lett. A 128, 9 (1988); P. R. Holland, Phys. Rept. 224, 95 (1993); W. Struyve and H. Westman, AIP Conf. Proc. 844, 321 (2006) [arXiv:quant-ph/0602229].
[24] H. Nikolic, Found. Phys. Lett. 17, 363 (2004) [arXiv:quant-ph/0208185]; H. Nikolic, Found. Phys. Lett. 18, 123 (2005) [arXiv:quant-ph/0302152]; H. Nikolic, Found. Phys. Lett. 18, 549 (2005) [arXiv:quant-ph/0406173]; H. Nikolic, Found. Phys. 37, 1563 (2007) [arXiv:quant-ph/0609163].
[25] S. Kochen and E. P. Specker, Journal of Mathematics and Mechanics 17, 59 (1976).
[26] C. Wetterich, arXiv:quant-ph/0212031.
[27] C. Wetterich, arXiv:hep-th/0104074.
[28] H. T. Elze, arXiv:quant-ph/0604142.
[29] G. 't Hooft, Class. Quant. Grav. 16, 3263 (1999) [arXiv:gr-qc/9903084]; G. 't Hooft, arXiv:hep-th/0003005.
[30] E. Santamato, J. Math. Phys. 25, 2477 (1984); E. Santamato, Phys. Rev. D 32, 2615 (1985).
[31] R. Carroll, arXiv:math-ph/0703065.
[32] J. M. Isidro, J. L. G. Santander and P. F. de Cordoba, arXiv:0808.2351 [hep-th]; S. Abraham, P. F. de Cordoba, J. M. Isidro and J. L. G. Santander, arXiv:0810.2236 [hep-th]; S. Abraham, P. F. de Cordoba, J. M. Isidro and J. L. G. Santander, arXiv:0810.2356 [hep-th]; J. M. Isidro, J. L. G. Santander and P. F. de Cordoba, arXiv:0912.1535 [hep-th].
[33] R. Bonal, I. Quiros and R. Cardenas, arXiv:gr-qc/0010010.
[34] R. Carroll, arXiv:quant-ph/0406004; R. Carroll, arXiv:quant-ph/0406203; R. Carroll, arXiv:math-ph/0703007; R. Carroll, arXiv:0705.3924 [gr-qc].
[35] B. Koch, arXiv:0810.2865 [quant-ph]; B. Koch, arXiv:0810.2768 [hep-th].
[36] B. Koch, arXiv:0901.4106 [gr-qc].
[37] F. Shojai and A. Shojai, Int. J. Mod. Phys. A 15, 1859 (2000) [arXiv:gr-qc/0010012]; A. Shojai, Int. J. Mod. Phys. A 15, 1757 (2000) [arXiv:gr-qc/0010013]; F. Shojai and A. Shojai, Pramana 58, 13 (2002) [arXiv:gr-qc/0109052]; F. Shojai and A. Shojai, arXiv:gr-qc/0404102; A. Shojai, F. Shojai and N. Dadhich, Int. J. Mod. Phys. A 20, 2773 (2005) [arXiv:gr-qc/0504137]; F. Shojai and A. Shirinifard, Int. J. Mod. Phys. D 14, 1333 (2005) [arXiv:gr-qc/0504138].
[38] R. Bonal, I. Quiros and R. Cardenas, arXiv:gr-qc/0010010.
[39] R. Carroll, arXiv:gr-qc/0406004; R. Carroll, arXiv:quant-ph/0406203; R. Carroll, arXiv:math-ph/0703007; R. Carroll, arXiv:0705.3924 [gr-qc].
[40] B. Koch, arXiv:0810.2765 [hep-th]; B. Koch, arXiv:0810.2786 [hep-th].
[41] B. Koch, arXiv:0901.4106 [gr-qc].
[42] G. Nordstrom, Ann. d. Phys. 40, 872 (1913); G. Nordstrom, Ann. d. Phys. 42, 533 (1913).
[43] Since it can not be distinguished experimentally from the standard formulation of quantum mechanics it’s probably better to call it an interpretation.