Evolving Loss Functions with Multivariate Taylor Polynomial Parameterizations

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Abstract—Loss function optimization for neural networks has recently emerged as a new direction for metalearning, with Genetic Loss Optimization (GLO) providing a general approach for the discovery and optimization of such functions. GLO represents loss functions as trees that are evolved and further optimized using evolutionary strategies. However, searching in this space is difficult because most candidates are not valid loss functions. In this paper, a new technique, Multivariate Taylor expansion-based genetic loss-function optimization (TaylorGLO), is introduced to solve this problem. It represents functions using a novel parameterization based on Taylor expansions, making the search more effective. TaylorGLO is able to find new loss functions that outperform those found by GLO in many fewer generations, demonstrating that loss function optimization is a productive avenue for metalearning.

Index Terms—loss function, optimization, neural networks, metalearning, evolution, evolutionary strategies

I. INTRODUCTION

As deep learning systems have become more complex, their architectures and hyperparameters have become increasingly difficult and time-consuming to optimize by hand. In fact, many good designs may be overlooked by humans with prior biases. Therefore, automating this process, known as metalearning, has become an essential part of the modern machine learning toolbox. Metalearning approaches this problem from numerous angles, both by optimizing different aspects of the architecture from hyperparameters \cite{1} to topologies \cite{2}, and by using different approaches from Bayesian optimization to evolutionary computation \cite{3}.

Recently, loss-function discovery and optimization has emerged as a new type of metalearning. It aims to tackle neural network's root training goal, by discovering better ways to define what is being optimized. In doing so, it makes it possible to regularize the solutions automatically. Genetic Loss Optimization (GLO) \cite{4} provided an initial implementation of this idea using a combination of genetic programming and evolutionary strategies.

However, loss functions can be challenging to represent in an optimization system because they have a discrete nested structure as well as continuous coefficients. GLO tackled this problem by discovering and optimizing loss functions in two separate steps: evolution of structure and optimization of coefficients. Such separate processes make it challenging to find a mutually optimal structure and coefficients. Furthermore, the structured search space is difficult since small changes to the genotype do not always result in small changes in the phenotype and can easily make a function invalid.

In an ideal case, loss functions would be smoothly mapped into arbitrarily long, fixed-length vectors in a Hilbert space. This mapping should be smooth, well-behaved, well-defined, incorporate both a function's structure and coefficients, and should by its very nature make large classes of infeasible loss functions a mathematical impossibility.

This paper introduces such an approach: Multivariate Taylor expansion-based genetic loss-function optimization (TaylorGLO). By using a novel parameterization for loss functions, the key pieces of information that affect a loss function’s behavior can be compactly represented in a vector. Such vectors can then be optimized for a specific task using a Covariance-Matrix Adaptation Evolutionary Strategy (CMA-ES). Select techniques can be used to narrow down the search space and speed up evolution.

Loss functions discovered by TaylorGLO outperform the standard cross-entropy loss (or log loss) on the MNIST dataset as well as the Baikal loss, discovered by the original GLO technique, with significantly fewer evaluations than GLO. Like Baikal, TaylorGLO loss functions have good performance characteristics on reduced training datasets.

The following section reviews literature in metalearning and loss function optimization, including the original GLO technique, and provides a background on multivariate Taylor approximations. A description of how such Taylor approximations are leveraged in a new loss function parameterization follows, and the TaylorGLO approach is described. TaylorGLO’s efficacy is then evaluated on the MNIST image classification task.

II. RELATED WORK

Designing performant deep neural networks involve a high-level of manual tuning of both the architecture and hyperparameters. The field of metalearning was developed to tackle this issue algorithmically \cite{1}. Significant work exists in neural network metalearning using a variety of approaches \cite{5}, with recent work tackling, for example, the evolution of entire architectures \cite{2}. It is therefore compelling to apply metalearning to the design of loss functions as well.

A. Loss Function Metalearning

Deep neural networks are traditionally trained iteratively, whereby model parameters (i.e., weights and biases) are updated
using gradients propagated backwards through the network, starting from an error given by a loss function \[\mathcal{L}\]. Loss functions represent the primary training objective of a neural network. In many tasks, such as classification and language modeling, the cross-entropy loss (also known as the log loss) is used almost exclusively. While in some approaches a regularization term (e.g. \(L^2\) weight regularization) is added to the the loss function definition, the core component is still the cross-entropy loss. The cross-entropy loss is motivated by information theory; it aims to minimize the number of bits needed to identify a message from the true distribution, using a code from the predicted distribution.

Other types of tasks that do not fit neatly into a single-label classification framework often have different task-specific loss functions. This observation suggests that it is useful to change the loss function appropriately. Indeed, in statistics it is known that different regression loss functions have unique properties, such as the Huber Loss which is resilient to outliers compared to other loss functions. Every instance where a loss function is chosen for a task without a specific justification is an opportunity to find a more optimal loss function automatically.

Genetic Loss Optimization (GLO) provided an initial study into metalearning of loss functions. GLO is based on a two-phase approach that (1) evolves a function structure using a tree representation, and (2) optimizes a structure’s coefficients using an evolutionary strategy. GLO was able to discover Baikal, a new loss function that outperformed the cross-entropy loss on tested image classification tasks, and made it possible to learn with smaller datasets. Baikal appeared to have these characteristics due to an implicit regularization effect that GLO discovered.

Tree-based representations, as used in GLO, have been dominant in genetic programming because they are flexible and can be applied to a variety of function evolution domains. For example, this genetic programming succeeded in discovering nontrivial mathematical formulas that underly physical systems using noisy experimental data.

However, due to the two-step approach that GLO takes, GLO is unable to discover a function with mutually beneficial structure and coefficients. Additionally, the majority of loss function candidates in GLO are not useful, since, for example, many of them have discontinuities. Further mutations to tree-based representations can have disproportionate effects on the functions they represent. GLO’s search is thus inefficient, requiring large populations that are evolved for many generations.

The technique presented in this paper, TaylorGLO, aims to solve the representation problems by using evolutionary strategies and a novel loss function parameterization based on multivariate Taylor expansions. Furthermore, since such representations are continuous, the approach can take advantage of CMA-ES as the search method, resulting in faster search.

B. Multivariate Taylor Expansions

Taylor expansions are a well-known function approximator that can represent differentiable functions within the neighborhood of a point using a polynomial series. Below, the common univariate Taylor expansion formulation is presented, followed by a natural extension to arbitrarily-multivariate functions.

Given a \(C^{k_{\text{max}}}\) smooth (i.e., first through \(k_{\text{max}}\) derivatives are continuous), real-valued function, \(f(x) : \mathbb{R} \to \mathbb{R}\), a \(k\)-th order Taylor approximation at point \(a \in \mathbb{R}\), \(\hat{f}_k(x, a)\), where \(0 \leq k \leq k_{\text{max}}\), can be constructed as,

\[
\hat{f}_k(x, a) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x - a)^n. \tag{1}
\]

Conventional, univariate Taylor expansions have a natural extension to arbitrarily high-dimensional inputs of \(f\). Given a \(C^{k_{\text{max}}+1}\) smooth, real-valued function, \(f(x) : \mathbb{R}^n \to \mathbb{R}\), a \(k\)-th order Taylor approximation at point \(a \in \mathbb{R}^n\), \(\hat{f}_k(x, a)\), where \(0 \leq k \leq k_{\text{max}}\), can be constructed. The stricter smoothness constraint compared to the univariate case allows for the application of Schwarz’s theorem on equality of mixed partials, obviating the need to take the order of partial differentiation into account.

Let us define an \(n\)-th degree multi-index, \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\), where \(\alpha_i \in \mathbb{N}_0\), \(|\alpha| = \sum_{i=1}^{n} \alpha_i\), \(\alpha! = \prod_{i=1}^{n} \alpha_i!\), \(x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}\), and \(x \in \mathbb{R}^n\). Multivariate partial derivatives can be concisely written using a multi-index \(\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}\). \tag{2}

Thus, discounting the remainder term, the multivariate Taylor expansion for \(f(x)\) at \(a\) is

\[
\hat{f}_k(x, a) = \sum_{\forall \alpha, |\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha f(a)(x - a)^\alpha. \tag{3}
\]

The unique partial derivatives in \(\hat{f}_k\) and \(a\) are parameters for a \(k\)th order Taylor expansion. Thus, a \(k\)th order Taylor expansion of a function in \(n\) variables requires \(n\) parameters to define the center, \(a\), and one parameter for each unique multi-index \(\alpha\), where \(|\alpha| \leq k\). That is:

\[
\#\text{parameters}(n, k) = n + \binom{n + k}{k} = n + \frac{(n + k)!}{n!k!}. \tag{4}
\]

The multivariate Taylor expansion can be leveraged for a novel loss-function parameterization. It enables TaylorGLO, a way to efficiently optimize loss functions, as will be described in subsequent sections.

III. LOSS FUNCTIONS AS MULTIVARIATE TAYLOR EXPANSIONS

Let an \(n\)-class classification loss function be defined as

\[
\mathcal{L}_f(x, y) = -\frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i). \tag{5}
\]
The function \( f(x, y) \) can be replaced by its \( k \)th-order, bivariate Taylor expansion, \( \hat{f}_k(x, y, a_x, a_y) \). More sophisticated loss functions can be supported by having more input variables, beyond \( x \) and \( y \), such as a time variable or unscaled logits. This approach can be useful, for example, to evolve loss functions that change as training progresses.

For example, a loss function in \( x \) and \( y \) has the following 3rd-order parameterization with parameters \( \theta \) (where \( \alpha = (\theta_0, \theta_1) \)):

\[
\mathcal{L}(x, y) = -\frac{1}{n} \sum_{i=1}^{n} \left[ \theta_2 + \theta_3(y_i - \theta_1) + \frac{1}{2} \theta_4(y_i - \theta_1)^2 + \frac{1}{6} \theta_5(y_i - \theta_1)^3 + \theta_6(x_i - \theta_0) + \theta_7(x_i - \theta_0)(y_i - \theta_1) + \frac{1}{2} \theta_8(x_i - \theta_0)^2 + \frac{1}{2} \theta_9(x_i - \theta_0)(y_i - \theta_1) + \frac{1}{6} \theta_{10}(x_i - \theta_0)^3 \right]
\]

Notably, the reciprocal-factorial coefficients can be integrated to be a part of the parameter set by direct multiplication if desired.

As will be shown in this paper, the technique makes it possible to train neural networks that are more accurate and learn faster, than those with tree-based loss function representations. Representing loss functions in this manner confers several useful properties:

- Guarantees smooth functions;
- Functions do not have poles (i.e., discontinuities going to infinity or negative infinity) within their relevant domain;
- Can be implemented purely as compositions of addition and multiplication operations;
- Can be trivially differentiated;
- Nearby points in the search space yield similar results (i.e., the search space is locally smooth), making the fitness landscape easier to search;
- Valid loss functions can be found in fewer generations and with higher frequency;
- Loss function discovery is consistent and does not depend as much on a specific initial population; and
- The search space has a tunable complexity parameter (i.e., the order of the expansion).

These properties are not necessarily held by alternative function approximators. For instance:

**Fourier series** are well suited for approximating periodic functions [16], therefore, they are not as well suited for loss functions, whose local behavior within a narrow domain is important. Being a composition of waves, Fourier series tend to have many critical points within the domain of interest. Gradients fluctuate around such points, making gradient descent infeasible. Additionally, close approximations require a large number of terms, which in itself can be injurious, causing large, high-frequency fluctuations, known as “ringing”, due to Gibb’s phenomenon [17].

**Padé approximants** can be more accurate approximations than Taylor expansions; indeed, Taylor expansions are a special case of Padé approximants where \( M = 0 \) [18]. However, unfortunately, Padé approximants can model functions with one or more poles, which valid loss functions typically should not have. These problems still exist, and are exacerbated, for Chisholm approximants [19] (a bivariate extension) and Canterbury approximants [20] (a multivariate generalization).

**Laurent polynomials** can represent functions with discontinuities, the simplest being \( x^{-1} \). While Laurent polynomials provide a generalization of Taylor expansions into negative exponents, the extension is not useful because it results in the same issues as Padé approximants.

**Polyharmonic splines** seem like an excellent fit since they can represent continuous functions within a finite domain. However, the number of parameters is prohibitive in multivariate cases.

The multivariate Taylor expansion is therefore a better choice than the alternatives. It makes it possible to optimize loss functions efficiently in TaylorGLO, as will be described next.

**IV. THE TAYLOR-GLO APPROACH**

TaylorGLO (Figure 1) aims to find the optimal parameters for a loss function parameterized as a multivariate Taylor expansion, as described in Section III. The parameters for a Taylor approximation (i.e., the center point and partial derivatives) are referred to as \( \theta_j, \theta_j \in \Theta \), \( \Theta = \mathbb{R}^{\# \text{params}} \). TaylorGLO strives to find the vector \( \theta_j \in \Theta \) that parameterizes the optimal loss function for a task. Because the values are continuous, as opposed to discrete graphs of the original GLO, it is possible to use continuous optimization methods.

In particular, Covariance Matrix Adaptation Evolutionary Strategy (CMA-ES) [14] is a popular population-based, black-box optimization technique for rugged, continuous spaces. CMA-ES functions by maintaining a covariance matrix around...
a mean point that represents a distribution of solutions. At each generation, CMA-ES adapts the distribution to better fit evaluated objective values from sampled individuals. In this manner, the area in the search space which is being sampled at each step dynamically grows, shrinks, and moves as needed to maximize sampled candidates’ fitnesses. The specific variant of CMA-ES that TaylorGLO uses is $(\mu/\mu, \lambda)$-CMA-ES [21], which incorporates weighted rank-$\mu$ updates [22] to reduce the number of objective function evaluations that are needed.

In TaylorGLO, CMA-ES is used to find try to find $\theta^*_f$. At each generation, CMA-ES samples points in $\Theta$ whose fitness is determined; this is accomplished by training a model with the corresponding loss function and evaluating the model on a validation dataset. Fitness evaluations may be distributed across multiple machines in parallel. An initial vector of $\theta^*_f = 0$ is chosen as a starting point in the search space to avoid bias.

Note that fully training a model can prove to be prohibitively expensive for all but the simplest problems. Fundamentally, there is a positive correlation between performance near the beginning of training and at the end of training. In order to identify the most promising candidates, it is enough to train the models only partially. This type of approximate evaluation is widely done in the field [23], [24]. An additional positive effect is that evaluation then favors loss functions that learn more quickly.

For a loss function to be useful, it must have a derivative that depends on the prediction. Therefore, internal terms that do not contribute to $\frac{\partial}{\partial \theta} L_f(x, y)$ can be trimmed away. This implies that any term, $t$ within $f(x_i, y_i)$, where $\frac{\partial}{\partial \theta_i} t = 0$, can be replaced with 0.

For example, this refinement simplifies Equation (5) to:

$$L(x, y) = -\frac{1}{n} \sum_{i=1}^{n} \left[ \theta_2(y_i - \theta_1)^2 + \frac{1}{2} \theta_3(y_i - \theta_1)^2 + \frac{1}{6} \theta_4(y_i - \theta_1)^3 + \theta_5(x_i - \theta_0)(y_i - \theta_1) \right] + \frac{1}{2} \theta_6(x_i - \theta_0)(y_i - \theta_1)^2 + \frac{1}{2} \theta_7(x_i - \theta_0)(y_i - \theta_1),$$

(7)

providing a reduction in the number of parameters from twelve to eight.

V. EXPERIMENTAL SETUP

This section presents the experimental setup that was used to evaluate the TaylorGLO technique. The MNIST image classification task was used as to measure the technique’s efficacy, and provide a point of comparison against the original GLO technique and the standard cross-entropy loss function.

A. Domain

The domain used for evaluation was the MNIST Handwritten Digits [25], a widely used dataset where the goal is to classify $28 \times 28$ pixel images as one of ten digits. MNIST has 55,000 training samples, 5,000 validation samples, and 10,000 testing samples. The dataset is well understood and relatively quick to train. Being a classification problem, MNIST training is traditionally framed with the standard cross-entropy loss:

$$L_{\text{Cross-Entropy}} = -\frac{1}{n} \sum_{i=0}^{n} x_i \log(y_i),$$

(8)

where $x$ is sampled from the true distribution, $y$ is from the predicted distribution, and $n$ is the number of classes. The cross-entropy loss is used as a baseline in this paper’s experiments.

The same CNN architecture presented in the GLO study [4] is used to provide a direct point of comparison. The model architecture is standard, consisting of convolution and pooling layers. Notably, this architecture contains a dropout layer [26] to explicitly provide regularization. Additionally, as in GLO’s experimentation methodology, training is based on stochastic gradient descent (SGD) with a batch size of 100, a learning rate of 0.01, and, unless otherwise specified, occurred over 20,000 steps.

B. TaylorGLO

A CMA-ES was set to have a population size $\lambda = 28$ and an initial step size $\sigma = 1.2$. These values were found to work well experimentally. The candidates were third-order (i.e., $k = 3$) TaylorGLO loss functions (Equation 7). Such functions were found experimentally to have a better trade-off between evolution time and performance compared to second- and fourth-order TaylorGLO loss functions (although the differences were relatively).

During candidate evaluation, models were trained for 10% of a full training run. On MNIST, this equates to 2,000 steps (i.e., 4 epochs).

C. Implementation Details

Due to the number of partial training sessions that are needed to evaluate TaylorGLO loss function candidates, training was distributed across the network to a cluster, composed of dedicated machines with NVIDIA GeForce GTX 1080Ti GPUs. Training itself was implemented with TensorFlow [27] in Python. The primary components of TaylorGLO (i.e., the genetic algorithm and CMA-ES) were implemented in the Swift programming language which allows for easy parallelization. These components run centrally on one machine and asynchronously dispatch work to the cluster.

VI. RESULTS

The subsequent subsections present experimental results that show how the TaylorGLO evolution process functions, and how the loss functions from TaylorGLO can be used as high-performance, drop-in replacements for the cross-entropy loss function.

A. Baseline

On MNIST, the average testing accuracy for models trained with the cross-entropy loss was 0.9903, with a standard deviation of 0.0005 ($n = 10$). At 10% through the training process, corresponding to the amount of training undergone to evaluate TaylorGLO candidates, the average testing accuracy was 0.9656 with a standard deviation of 0.0015 ($n = 10$).
B. TaylorGLO Evolution

Figure 2 shows an overview of the evolution process over 60 generations on the MNIST dataset, more than sufficient to reach convergence. TaylorGLO is able to quickly discover highly-performing loss functions, with all improvements being discovered within 20 generations. Generations’ average validation accuracy approaches generations’ best accuracy as evolution progresses, indicating that populations as a whole are improving. Notably, every single partial training session completed successfully without any instability, a stark contrast to GLO, whose unbounded search space often results in pathological functions.

Figure 3 shows the shapes and parameters of each generation’s highest-scoring loss function. They are plotted as if they were being used for binary classification, using the procedure described in the GLO study [4]. The functions are colored according to their generation. Additionally, plotted loss function curves are vertically shifted such that their loss at \( y_0 = 1 \) is zero; the raw value of a loss function is not relevant, the derivative, however, is. The functions have a distinct pattern throughout the evolution process, where early generations show a wider variety of shapes that converge in later generations towards curves with a shallow minimum around \( y_0 = 0.8 \) (the best loss function found on MNIST—described below—specifically had a minimum at \( y_0 = 0.8238 \)). Most notably, these well performing loss functions are not monotonically decreasing as the cross-entropy loss is. They each have a local minima near, but not at, \( y_0 = 1 \), after which the loss increases again as \( y_0 \) approaches 1. This shape provides an implicit regularization effect in the same manner that the Baikal loss function does [4]: it discourages overfitting by preventing the model from being too confident in its predictions.

This structure becomes quite evident when dimensionality reduction using t-SNE [28] is performed on every candidate loss function within a run of TaylorGLO evolution, as illustrated in Figure 4. A visualization of all TaylorGLO loss function candidates using t-SNE [28] on MNIST. Colors map to each candidate’s generation. Loss function populations show a clear evolutionary path over time.

Fig. 2. TaylorGLO evolution of loss functions for MNIST. Red dots mark generations with a new best loss function. TaylorGLO discovers good loss functions in very few generations. The best loss function discovered had a 2k-step validation accuracy of 0.9948. For comparison, the average 2k-step validation accuracy for the cross-entropy loss across ten runs is 0.9903.

Fig. 3. The best loss functions (top) and their respective parameters (bottom) from each generation of TaylorGLO on MNIST. Loss functions are plotted in a binary classification modality, and are colored according to their generation (i.e., earlier loss functions are on the blue side of the spectrum, while mature loss functions are red). TaylorGLO explores varying shapes of solutions before narrowing down on functions in the orange-red band. This result can also be seen on the bottom plot, where parameters become less chaotic over time.

Fig. 4. A visualization of all TaylorGLO loss function candidates using t-SNE [28] on MNIST. Colors map to each candidate’s generation. Loss function populations show a clear evolutionary path over time.
Testing Accuracy

0.88 0.90 0.92 0.94 0.96 0.98 1.00

TaylorGLO is a practical replacement for GLO which can significantly outperform the cross-entropy loss. The TaylorGLO loss function was discovered with 96% fewer evaluations than the BaikalCMA loss function from GLO.

The TaylorGLO loss function was discovered with 96% fewer evaluations (that is, 448 evaluations) compared to the top TaylorGLO loss function subsequent to the discovery of BaikalCMA, while the BaikalCMA required 11,120 evaluations.

The TaylorGLO loss function was a modest improvement over the previous best loss function from generation 16, which had a valid testing accuracy of 0.9958.

Over 10 fully-trained models, the best TaylorGLO loss function achieved a mean testing accuracy of 0.9951, with a standard deviation of 0.0005, while the cross-entropy loss only reached 0.9899, and the BaikalCMA loss function from the original GLO technique reached 0.9947. These figures are shown in Figure 5. The testing accuracy improvement the TaylorGLO loss function confers over the cross-entropy loss is highly statistically significant, with a p-value of $5.9 \times 10^{-15}$, in a heteroscedastic, two-tailed T-test, with 10 samples from each distribution. Using the same type of T-test, the TaylorGLO loss function’s improvement over BaikalCMA has a p-value of 0.0625.

Notably, the top loss function from TaylorGLO on MNIST slightly outperforms the top loss function from the original GLO technique, while requiring significantly fewer generations and partial training sessions. BaikalCMA required 11,120 partial evaluations (i.e., 100 individuals over 100 GP generations plus 32 individuals over 35 CMA-ES generations, ignoring evaluations subsequent to the discovery of BaikalCMA), while the top TaylorGLO loss function only required 448 partial evaluations (that is, 4.03% as many in this case). Thus, TaylorGLO is a practical replacement for GLO which can achieve comparable results with significantly fewer evaluations.

C. Performance on Reduced Datasets

The best TaylorGLO loss function on MNIST has significantly better performance characteristics than the cross-entropy loss function when applied on reduced datasets. This property appears to be a characteristic of optimized loss functions.

Figure 6 presents a comparison of accuracies for models trained on different portions of the MNIST dataset. Overall, TaylorGLO significantly outperforms the cross-entropy loss. All models were trained for the same number of steps (i.e., 20,000).

D. Evaluation Length Sensitivity

a) 200-step: TaylorGLO is surprisingly resilient when evaluations during evolution are shortened to 200 steps (i.e., 0.4 epochs) of training. With so little training, returned accuracies are noisy and dependent on each individual network’s particular random initialization. On a 60-generation run with 200-step evaluations, the best evolved loss function had a mean testing accuracy of 0.9946 across ten samples, with a standard deviation of 0.0016. While slightly lower, and significantly more variable, than the accuracy for the best loss function that was found on the main 2,000-step run, the accuracy is still significantly higher than that of the cross-entropy baseline, with a p-value of $6.3 \times 10^{-6}$. This loss function was discovered in generation 31, requiring 1,388.8 2,000-step-equivalent partial evaluations. That is, evolution with 200-step partial evaluations is over three-times less sample sample efficient than evolution with 2,000-step partial evaluations.

b) 20,000-step: On the other extreme, where evaluations consist of the same number of steps as a full training session, one would expect better loss functions to be discovered, and more reliably, because the fitness estimates are less noisy. Surprisingly, that is not the case: The best loss function had a mean testing accuracy of 0.9945 across ten samples, with a standard deviation of 0.0015. While also slightly lower, and also significantly more variable, than the accuracy for the best loss function that was found on the main 2,000-step run, the accuracy is significantly higher than the cross-entropy baseline, with a p-value of $5.1 \times 10^{-6}$. This loss function was discovered in generation 45, requiring 12,600 2,000-step-equivalent partial evaluations.
evaluations; over 28-times less sample sample efficient as evolution with 2,000-step partial evaluations.

These results thus suggest that there is an optimal way to evaluate candidates during evolution, resulting in lower computational cost and better loss functions. Notably, the best evolved loss functions from all three runs (i.e., 200-, 2,000-, and 20,000-step) have similar shapes, reinforcing the idea that partial-evaluations can provide useful performance estimates.

VII. DISCUSSION AND FUTURE WORK

TaylorGLO’s sample efficiency compared to GLO allows it to scale to larger networks and more complex datasets. In preliminary experiments TaylorGLO was applied to the more challenging CIFAR-10 dataset on a ResNet model with five residual blocks, commonly known as ResNet-32, using standard hyperparameters. This setup has been heavily engineered and manually tuned by the research community, and the architecture is also narrow and deep. TaylorGLO was able to find a loss function on generation 12 with very similar performance characteristics to the cross-entropy loss, without needing any prior human knowledge. Over ten runs each, the TaylorGLO loss function reached a mean testing accuracy of 0.9115, compared to 0.9105 for cross-entropy, and a standard deviation of 0.0032, compared to 0.0033. While it may be possible to improve upon this result, it is also possible that loss function optimization is less effective on architectures like ResNet that are narrow and deep. An important direction of future work is therefore to evolve both loss functions and architectures together, taking advantage of possible synergies between them.

Function evolution is a wide field. Genetic programming (GP) approaches are a dominant technique in it, as they have been in prior loss function metalearning work. GP systems are often able to discover solutions to problems that outperform human-built solutions. Early work aimed at developing pattern recognition systems using rulesets represented as trees and has grown to be a flexible framework that can be applied to different domains. However, while GP is very flexible, it can be overly so. Thoughtfully-designed search spaces for continuous optimization can lead to better results. TaylorGLO was developed by working backwards from the desiderata of a loss function evolution system, leading to loss functions that can outperform those found by GLO in a more sample efficient manner.

Future work will involve leveraging additional input variables in TaylorGLO loss functions, such as the percentage of training steps completed. TaylorGLO may then find loss functions that are best suited for different points in training, where, for example, different kinds of regularization work best. Unintuitive changes to the training process, such as cycling learning rates, have been able to improve model performance; evolution could be a useful way to discover similar techniques.

The proper choice of loss function may depend on other types of state as well. For example, batch statistics could help evolve loss functions that are more well-tuned to each batch; intermediate network activations could expose information that may help tune the function for deeper networks like ResNet; deeper information about the characteristics of a model’s weights and gradients, such as that from spectral decomposition of the Hessian matrix, could assist the evolution of loss functions that are able to adapt to the current fitness landscape.

VIII. CONCLUSION

This paper proposes multivariate Taylor expansion-based genetic loss-function optimization (TaylorGLO), a new technique for loss-function metalearning. TaylorGLO leverages a novel parameterization for loss functions that is designed to have desirable properties and allows for a continuous optimization approach to be used, instead of genetic programming. TaylorGLO loss functions are able to significantly outperform the cross-entropy loss on MNIST, and provide a slight improvement over the original GLO technique while requiring many fewer candidates to be evaluated. Analyses suggest that TaylorGLO is a mature technique that can provide customized loss functions that result in higher testing accuracies and better data utilization.

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