Time Variant Distribution of Sugi Log Prices based on Geometric Mean-Reverting Model for Risk Valuation

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Abstract: Time variant distribution of stochastic price dynamics models, plays an important role in evaluating the risk of forest management at any given point in time. In this paper, we present demonstrative results on the use of time variant distribution for risk evaluation, using a geometric mean-reverting stochastic model and log price data. The data used for this study comes from national monthly statistical data of 4m long sugi (Cryptomeria japonica) log prices, and the parameters of the model were derived from a pseudo-likelihood approach, using discretization by the Euler method. The time variant distribution for the stochastic model was numerically computed by applying the method of lines to the Fokker–Planck equation. The results of this study showed that when the management risk is defined by the probability that a price falls below a given threshold price to sustain forest management, the risk increase over time. This was true for all scenarios with different reverted mean values. It was also revealed through this study that the management risk under a higher threshold price tends to reach the risk neutral point of 50% for sustaining forest management, earlier than those scenarios with a lower threshold price.

Keywords: log price, mean-reverting process, risk valuation, stochastic differential equation, time variant distribution

1. Introduction

The stochastic nature of timber product markets has a big impact on forest management decisions, which could ultimately have a negative effect on sustainable forest management. The importance of the stochastic nature of the timber price dynamics, as a subject area, is revealed in the work done by several researchers (Reed 1984; Brazee and Mendelsohn 1988; Clarke and Reed 1989; Clarke and Reed 1990; Teeter and Caulfield 1991; Zinkhan 1991; Thomson 1992; Reed and Ye 1994; Yin and Newman 1995; Yin and Newman 1996; Yoshimoto and Shoji 1998; Willassen 1998; Plantinga 1998; Morck et al. 1989; Thorsen 1999; Brazee and Bulte 2000; Fina et al. 2000; Hughes 2000; Yoshimoto 2002; Sdal 2003; Insley 2002; Insley and Rollins 2005; Penttinen 2006). When constructing a stochastic model for timber price dynamics, several candidates can be proposed depending upon model assumptions. In recent literature on stochastic modeling for price dynamics, stochastic differential equations are often utilized where the drift and volatility terms are a function of time and state. Changing these functional forms, different models can be created (see Duffie 1992).

An extensive review of literature shows that a geometric Brownian motion has been widely used as a stochastic model for price dynamics, due to the ease of its application on these models. A geometric Brownian motion has been used by Clarke and Reed (1989), Thomson (1992), Yoshimoto and Shoji (1998), and Yoshimoto (2002). Other researchers such as Haight and Holmes (1991) have employed the non-stationary random walk for the log-transformed price process. For microeconomic reasons of long-run marginal cost of production reflecting a mean price, researchers such as Insley (2002), and Insley and Rollins (2005), have used a geometric mean-reverting process. Yoshimoto and Shoji (2002) tested 13 different state dependent volatility models for 13 time series data of log price. They revealed the importance of preparing candidate models for finalizing an appropriate model for the target price dynamics. Yoshimoto (2009) used a geometric mean-reverting process to seek the minimum harvest age and threshold price for sustaining forest management. Searching for such a threshold price often relies on discretization of the continuous process by using a binomial approximation within the framework of stochastic dynamic programming.

In addition, in order to capture the price dynamics by stochastic models, it is also important to seek price distribution at any given point of time. This makes it possible to estimate more precisely
probability of the price that falls below or rises above a given threshold price to sustain forest management. In this paper, we present the numerical results for the time variant distribution based on a geometric mean-reverting stochastic model for log price data, and a preliminary study on the use of the derived distribution for risk evaluation. The data used for this study comes from national monthly statistical data of 4m long sugi (Cryptomeria japonica) log prices, and the parameters of the model are set using estimated parameters in Yoshimoto (2009), which assumes that the reverted mean price reflects the induced costs of timber production as well as the derived threshold price to sustain forest management using stochastic dynamic programming. The time variant distribution for the stochastic model is numerically computed by applying the method of lines to the Fokker–Planck equation.

The rest of the paper is organized as follows. In the section that follows, the equation for the time variant probability density function for a general type of a continuous-time stochastic volatility model is presented as well as a numerical computation method to derive the probability density. In the third section, earlier results of the parameter values of a geometric mean-reverting stochastic model for the sugi log price are presented from the work by Yoshimoto (2009). This is then followed by the results obtained from the numerical computation of the time variant distribution. The paper ends with concluding remarks in the last section.

2. Time Variant Distribution and Its Numerical Computation

Let $x$ be a diffusion process defined by a stochastic differential equation (SDE) for all values of $t \geq t_0$:

\begin{align*}
\frac{dx}{dt} &= f(x)dt + g(x)d\omega_t \\
x(t_0) &= x_0.
\end{align*}

The time variant probability density function of the process $x$ satisfies the following Kolmogorov forward (Fokker–Planck) equation:

\begin{align*}
\frac{\partial}{\partial t} p(t; x; x_0) &= -\frac{\partial}{\partial x} (f(x)p(t; x; x_0)) + \frac{\partial^2}{\partial x^2} (g^2(x)p(t; x; x_0)) \\
p(t_0; x; x_0) &= \delta(x - x_0),
\end{align*}

where $\delta(\cdot)$ is the Dirac delta function (see Ozaki 1985). The stationary distribution of the diffusion process $x$ (when $t \to \infty$) then follows the following:

\begin{equation}
\frac{d}{dx} p_{\infty}(x) = 2 \frac{(f(x) - g(x)g_x(x))}{g^2(x)} p_{\infty}(x).
\end{equation}

For computational purposes, the Fokker–Planck equation can be rewritten as

\begin{equation}
p_1(t, x) = a(x)p(t, x) + b(x)p_x(t, x) + c(x)p_{xx}(t, x),
\end{equation}

where

\begin{align*}
a(x) &= \frac{1}{2}(g^2(x))_{xx} - f_x(x) \\
b(x) &= (g^2(x))_x - f(x) \\
c(x) &= \frac{1}{2}g^2(x).
\end{align*}

Here, the initial point $x_0$ is omitted on purpose.

With the use of the method of lines, the above partial differential equation (PDE) can be approximated using a system of ordinary differential equations (ODEs) (see Larsson and Thomée 2009). Let us denote $U_k(t) = p(t, y_k; x_0)$, where $y_k = a + kh$ is a discretization of the spatial interval $[a, b] \subset \mathbb{R}$ of probable values of $x$. Here, $h = (b - a)/(N + 1)$ for all values of $k = 0, \ldots, (N + 1)$. It is assumed that the following boundary conditions hold at $y_k = a$ and $y_k = b$:

\begin{align*}
p(t, a; x_0) &= 0 \\
p(t, b; x_0) &= 0,
\end{align*}

\[2\]
that is,
\[ U_0(t) = 0 \]
\[ U_{N+1}(t) = 0. \]

The second order approximation (by finite difference) for the spatial derivative of \( p \) becomes
\[ p_x(t, y_k) = \frac{U_{k+1}(t) - U_{k-1}(t)}{2h} \]
\[ p_{xx}(t, y_k) = \frac{U_{k+1}(t) - 2U_k(t) + U_{k-1}(t)}{h^2}, \]
and yields the following system of ODEs for all values of \( k = 1, \ldots, N \):
\[ \frac{d}{dt} U_k(t) = \mu(y_k)U_{k-1}(t) + \lambda(y_k)U_k(t) + \rho(y_k)U_{k+1}(t), \]
where
\[ \mu(x_k) = \frac{c(x_k)}{h^2} - \frac{b(x_k)}{2h} \]
\[ \lambda(x_k) = \frac{a(x_k)}{h^2} - \frac{2c(x_k)}{h^2} \]
\[ \rho(x_k) = \frac{b(x_k)}{2h} - \frac{c(x_k)}{h^2}. \]

In the matrix form, the above linear system of \( N \) equations can be written as
\[ \begin{bmatrix} dU_1/dt \\ dU_2/dt \\ \vdots \\ dU_{N-1}/dt \\ dU_N/dt \end{bmatrix} = \begin{bmatrix} \lambda(x_1) & \rho(x_1) & 0 & \ldots & 0 & 0 & 0 \\ \mu(x_2) & \lambda(x_2) & \rho(x_2) & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mu(x_{N-1}) & \lambda(x_{N-1}) & \rho(x_{N-1}) \\ 0 & 0 & 0 & \ldots & 0 & \mu(x_N) & \lambda(x_N) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix} \]
with the initial condition:
\[ U_k(t_0) = \begin{cases} 1/h & y_k \equiv x_0 \\ 0 & \text{otherwise} \end{cases} \]

Eq.[21] is an approximation to the Dirac delta function, \( \delta(\cdot) \), which satisfies two basic properties of the Dirac function: 1) \( \delta(x - x_0) = 0 \) for all \( x \neq x_0 \), and 2) \( \int_{-\infty}^{\infty} \delta(x - x_0)dx = 1. \)

Finally, an approximation of \( \tilde{p}(t, y_k; x_0) \) to \( p(t, y_k; x_0) \) is given by
\[ \tilde{p}(t, y_k; x_0) = \frac{U_k(t)}{h \sum_{l=1}^{N} U_l(t)}. \]

It must be noted that in order to implement the above numerical method, the following have to be considered. The values of \( a \) and \( b \) are set based on the previous knowledge of the problem or by the method of “try-and-error”. Since the above linear system of \( N \) ODEs is “stiff”, it can numerically be solved using the backward differential formulas (Matlab code ode15s in MATLAB 2012; see Shampine and Reichelt 1997). A Matlab script for computing the time variant probability density of the diffusion process as described here is given in Appendix.

3. Geometric Mean-Reverting Model for Sugi Log Price

3.1. Parameter estimation of a geometric mean-reverting model

The following volatility model or geometric mean-reverting model was applied to 4m long sugi log price:
\[ dx = (\alpha - \beta x)dt + \sigma xdw \]
\[ x(t_0) = x_0. \]
its reverted mean $\hat{x}$ is calculated by $\hat{x} = 2\alpha/(\sigma^2 + 2\beta)$. As shown in Yoshimoto (2009), the induced costs for harvest related activities was assumed to reflect the cost considered in the price dynamics. This was implemented by assuming the value for the reverted mean price. This was done because the direct estimation of the parameters resulted in a zero reverted mean, which is unacceptable in economics. By using different sets of parameters with the assumed reverted mean, we can also investigate how the derived distribution changes. We used monthly time series data of sugi (*Cryptomeria japonica*) log price from January 1975 to September 2006. This information came from the Japanese Forestry Agency’s annual report on timber supply and demand (Rinyacho 1975–2006) (see Figure 1). Sugi log dimensions included log diameters from 14–22 cm and log lengths from 3.65–4.0 meters.

Figure 1. National average monthly price of sugi log from 1975–2006.

Parameter estimation for the geometric mean-reverting model, Eq.[23], was carried out by the pseudo-likelihood approach, based on discretization by the Euler method (see Prakasa Rao 1999) as follows. First, the log-transformed variable, $y_t = \ln(x_t)$ was introduced for $x_t$ or $x(t)$. Under Ito Lemma (Gardiner 1985), we have

$$
    dy_t = \frac{d\ln(x_t)}{dx_t} dx_t + \frac{1}{2} \frac{d^2\ln(x_t)}{dx_t^2} (dx_t)^2 = (\frac{\alpha}{x_t} - \beta - \frac{1}{2}\sigma^2) dt + \sigma dB_t
$$

A new variable, $y_t$, has a drift term equal to $(\frac{\alpha}{x_t} - \beta - \frac{1}{2}\sigma^2)dt$ and a constant volatility term. Eq.[25] is then discretized under the assumption that the drift and diffusion term are constant over the small interval of time.

$$
    y_{n+1} - y_n = (\frac{\alpha}{x_t} - \beta - \frac{1}{2}\sigma^2)(t_{n+1} - t_n) + \sigma(B_{t_{n+1}} - B_{t_n})
$$

where $t_n$ is the time of the $n$-th observation and $y_{t_n}$ is the corresponding log-transformed price data. The last term on the right hand side, $\sigma(B_{t_{n+1}} - B_{t_n})$ is normally distributed with mean 0 and variance $\sigma^2(t_{n+1} - t_n)$. Thus, for the series of the log-transformed observation, $(y_{t_0}, y_{t_1}, y_{t_2}, \ldots, y_{t_N})$ with unknown parameters, $(\alpha, \beta, \sigma)$, the likelihood function $p(y_{t_0}, y_{t_1}, y_{t_2}, \ldots, y_{t_N})$ can be defined by

$$
    p(y_{t_0}, y_{t_1}, y_{t_2}, \ldots, y_{t_N}) = p(y_{t_0}) \prod_{n=0}^{N-1} p(y_{n+1}|y_n)
$$

Under the normality of $\sigma(B_{t_{n+1}} - B_{t_n})$, the conditional probability is given by

$$
    p(y_{n+1}|y_n) = \frac{1}{\sqrt{2\pi\sigma^2(t_{n+1} - t_n)}} \exp\left[ -\frac{1}{2} \frac{(y_{n+1} - y_n - (\frac{\alpha}{x_t} - \beta - \frac{1}{2}\sigma^2)(t_{n+1} - t_n))^2}{\sigma^2(t_{n+1} - t_n)} \right]
$$

Yoshimoto and Jimenez
From the likelihood function for the log-transformed observation, \((y_0, y_1, y_2, \ldots, y_N)\), the likelihood function for the original observation, \((x_0, x_1, x_2, \ldots, x_N)\) for log prices is expressed as

\[
p(x_0, x_1, x_2, \ldots, x_N) = p(y_0, y_1, y_2, \ldots, y_N) \left| \frac{\partial}{\partial(x_0, x_1, x_2, \ldots, x_N)} \right|
\]

where \(\left| \frac{\partial}{\partial}\right|\) denotes the absolute value of the determinant of the Jacobian matrix. The maximum likelihood estimates are obtained by maximizing Eq.[29] with respect to a set of unknown parameters.

For the computational purpose, the following log-likelihood maximization is usually applied:

\[
\log p(x_0, x_1, x_2, \ldots, x_N) = -\frac{1}{2} \sum_{n=0}^{N-1} \log(2\pi \sigma^2(t_{n+1} - t_n)) - \frac{(y_{n+1} - y_n - (\mu - \beta \cdot \sigma^2)(t_{n+1} - t_n))}{\sigma^2(t_{n+1} - t_n)} + \log p(y_n) - \sum_{n=0}^{N-1} \log(x_{2n})
\]

Under the Euler method with time interval equal to 1/12 for the monthly data set, we obtained \((\alpha, \beta, \sigma) = (0.0, 0.027749, 0.0588299)\). Applying this to \(\hat{x} = 2\alpha/(\sigma^2 + 2\beta)\), we had a 0 value for the reverted mean price. Thus, we manually re-estimated the parameters by assuming the long-term expectation of the reverted mean with five values, 5,000, 8,000, 10,000, 12,000, and 15,000 Yen/m³.

The estimated parameters \(\alpha, \beta\) and \(\sigma\) of Eq.[23] under five different reverted mean values are presented in Table 1. These were obtained from Yoshimoto (2009). Table 2, on the other hand, shows the derived threshold price with a different reverted mean to sustain forest management using a stochastic dynamic programming model (Yoshimoto 2009). Note that the reverted mean was also assumed to be the induced unit cost for the management. Thus, below the reverted mean, the management would result in negative profit.

### Table 1. Estimated parameters under five different reverted mean values (Yoshimoto 2009).

| Reverted Mean \((1000 \times \text{Yen/m}^3)\) | \(\alpha\)     | \(\beta\)   | \(\sigma\)     |
|--------------------------------------------|---------------|-------------|---------------|
| 5                                         | 0.1786551     | 0.0339952   | 0.0589202     |
| 8                                         | 0.3198313     | 0.0382381   | 0.0590600     |
| 10                                        | 0.4244126     | 0.0406958   | 0.0590837     |
| 12                                        | 0.5299780     | 0.0417486   | 0.0591819     |
| 15                                        | 0.5555251     | 0.0352734   | 0.0593572     |

### 3.2. Numerically computed distribution for the probability density function

Given an initial price of \(x_0 = 32,000\) Yen/m³, we applied the numerical method presented in Section 2 for the computation of the time variant distribution of the model, Eqs.[23]–[24], using the parameters presented in Table 1. Figures 2 to 6 show the estimated probability density \(\hat{p}(t_n, x; x_0)\) for the price at a given time of \(t_n\) with the set of estimated parameters under five different reverted mean values in Table 1, respectively. Here, we focused on a set of ten different time instants as in \((t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = (5, 10, 15, 20, 25, 30, 35, 40, 45, 50)\) in years. The corresponding threshold price in Table 2 was also plotted by the red dotted vertical line in figures. In addition, the probability that a price falls below a given threshold price was estimated as a risk to sustainable forest management. That is, when a price falls below the threshold price level, forest management activities would cease. Hence the probability of this event can be regarded as being “management risky”. The calculated probabilities of this event were expressed by “\(P_b\)” in figures. For example, given the unit price of 17,400 Yen/m³ in Figure 2, “\(P_d\)” becomes 0.00017968 at \(t_1 = 5\) (years).

### Table 2. The reverted mean price vs. threshold price (Yoshimoto 2009).

| Reverted Mean \((1000 \times \text{Yen/m}^3)\) | Threshold Price \((1000 \times \text{Yen/m}^3)\) |
|--------------------------------------------|--------------------------------------------|
| 5                                         | 17.4                                      |
| 8                                         | 20.7                                      |
| 10                                        | 20.0                                      |
| 12                                        | 16.5                                      |
| 15                                        | 20.6                                      |
Figure 2. Estimated probability density $\tilde{p}(t_n, x; x_0)$ under the reverted mean of 5,000 Yen/m$^3$.

Figure 3. Estimated probability density $\tilde{p}(t_n, x; x_0)$ under the reverted mean of 8,000 Yen/m$^3$. 
Time Variant Distribution

Figure 4. Estimated probability density $\tilde{p}(t_n, x; x_0)$ under the reverted mean of 10,000 Yen/m$^3$.

Figure 5. Estimated probability density $\tilde{p}(t_n, x; x_0)$ under the reverted mean of 12,000 Yen/m$^3$. 
Figure 6. Estimated probability density $\tilde{p}(t_n, x; x_0)$ under the reverted mean of 15,000 Yen/m$^3$.

Table 3 shows the management risk as defined in the above in all scenarios. Figure 7 expresses changes of the management risk over time. When the threshold price was equal to 20,700 Yen/m$^3$, the parameter set showed the highest degree of curvature with the sigmoid form, followed by the scenario with a threshold price of 20,000 Yen/m$^3$, and then the scenario with a threshold price of 17,400 Yen/m$^3$. The other two scenarios with a threshold price of 20,600 Yen/m$^3$ and 16,500 Yen/m$^3$ showed a slightly slow increase in the probability over time. These findings reflect the differences in time for a probability to reach the risk neutral line of 50%. The higher the degree of curvature, the shorter the time is required for the probability or management risk to reach the 50% neutral line. In other words, when the level of a threshold price is higher, the management would become risky at an earlier time more likely than a scenario with a lower level of a threshold price.

Table 3. The estimated management risk with five different threshold prices.

| Time | Threshold Price (1000 x Yen/m$^3$) |
|------|-----------------------------------|
|      | 17.4 | 20.7 | 20 | 16.5 | 20.6 |
| 5    | 0.00018 | 0.01165 | 0.00469 | 0.00000 | 0.00292 |
| 10   | 0.03810 | 0.19156 | 0.12303 | 0.00634 | 0.00307 |
| 15   | 0.20150 | 0.45343 | 0.34504 | 0.05305 | 0.17771 |
| 20   | 0.42235 | 0.65956 | 0.54970 | 0.14812 | 0.29143 |
| 25   | 0.61637 | 0.79482 | 0.70097 | 0.26590 | 0.39967 |
| 30   | 0.75768 | 0.87778 | 0.80375 | 0.38301 | 0.49145 |
| 35   | 0.85138 | 0.92729 | 0.87115 | 0.48732 | 0.56719 |
| 40   | 0.91040 | 0.95654 | 0.91476 | 0.57487 | 0.62895 |
| 45   | 0.94649 | 0.97380 | 0.94291 | 0.64593 | 0.67909 |
| 50   | 0.96819 | 0.98403 | 0.96114 | 0.70254 | 0.71975 |
4. Concluding Remarks

In this paper, a preliminary study of the time variant distribution of the geometric mean-reverting model for a 4m sugi log price was carried out. The time variant distribution was numerically computed by applying the method of lines to the Fokker–Planck equation, which corresponds to the geometric mean-reverting model, using the parameters estimated in Yoshimoto (2009) for the national monthly data. In a sustainable forest management practice that consider risk, not only is it extremely important to know how the target price would change over time, but it is also important to know how the time variant distribution of the target price would change over time. This is because the time variant distribution provides risk information on the price to fall below or rise above the threshold price level to sustain forest management over time. If we know that a forest management activity becomes risky in the near future, it would be possible to treat or apply some policy measures to avoid a wrong management decision.

Our results show that the higher the threshold price to sustain forest management, the shorter the time is required for the management to be risky. Thus, the management would become risky at an earlier time than cases with a lower threshold price. This is suggested by the estimated probability that the price falls below the threshold price. Clearly, this illustrates the important role that the time variant distribution of the stochastic price dynamics models might play in evaluating the risk of sustainable forest management at any given point in time.

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Appendix (scripts of Matlab and R)

The following is a Matlab script to compute the time variant probability density of the diffusion process, \(dy = (\alpha - \beta \cdot y)dt + \sigma \cdot y \cdot dw\), based on the method of lines:

```matlab
%--------------------------------------------------------------------------------
% Sugi
% Data divided by 1000 yen for the estimated parameters
% Reverted Mean = 5 000 yen
% Threshold Price = 17 400 yen
% Inputs:
% P0: initial price
% ThP: the threshold price
% y0: initial value for y
% yf: initial value for y
% ya: lowest value for y
% yb: highest value for y
% My: the total number of points of the spatial discretization of [ya,yb]
% FileName: the name to specify the diffusion model (here lsde.nn)
% Outputs:
% time: time instants where the probability density is computed (a vector).
% yp: spatial discretization of [ya,yb] where the density is computed (a vector).
% Dist: the probability density at each instant of time and space (a matrix).
% x0: spatial discretization of [ya,yb] where the density is computed (a vector).
% Pb: the risk of forest management (a vector).

close all
clear all
P0=32000/1000; ThP=17400/1000;
alpha=0.1786551; beta=0.0339952; sigma=0.0589202; rho=1;
T=[0 5 10 15 20 25 30 35 40 45 50];
y0=100/1000;
yf=80000/1000;
My=1501;
[time,Pric,Dist,P00]=distKPE(P0,y0,yf,My,T,‘lsde.nn’,alpha,beta,sigma,rho);
NT=length(T);
Pb=zeros(1,NT-1);
PbDen=Dist;
```
for i=2:NT
    Area=sum(Dist(2:end,i).*diff(Pric));
    PbDen(:,i)=Dist(:,i)./Area;
    Density=PbDen(:,i);
    index=find(Pric<=ThP);
    Pb(i-1)=sum(Density(index(2:end)).*diff(Pric(index)));
end
save('RM5','P0','ThP','alpha','beta','sigma','T','Pric','PbDen','Pb');

% Definition of Function distKPE to compute the probability density.

function [time,yp,Dist,x0] = distKPE(y0,ya,yb,My,T,FileName,alpha,beta,sigma,rho)
    dy=(yb-ya)/(My-1);
    yp=ya:dy:yb;
    yp=yp(2:end-1)';
    ix0=find(yp>=y0,1);
    x0=yp(ix0);
    yt0=zeros(length(yp),1);
    yt0(ix0)=1/dy;
    M=KPE([],yp,FileName,alpha,beta,sigma,rho);
    options = odeset('RelTol',1.0e-6,'AbsTol',1.0e-9);
    tic
    [time Dist] = ode15s(@Fname,T,yt0,options,M);
    toc
    time=time';
    Dist=Dist';
end

% Definition of Function M=KPE.

function M=KPE(t,y,FileName,alpha,beta,sigma,rho)
    N=length(y);
    h=y(2)-y(1);
    h2=h.*h;
    [mu,muy,s2,s2y,s2yy]=feval(FileName,t,y,alpha,beta,sigma,rho);
    aa=s2yy./2-muy;
    bb=s2y-mu;
    cc=s2./2;
    Ukm1=cc./h2-0.5.*bb./h;
    Uk =aa-2.*cc./h2;
    Ukp1=0.5.*bb./h+cc./h2;
    zv=zeros(1,N-1);
    M=diag(Uk) + [zv 0; [diag(Ukm1(2:N)) zv'] + [[zv' diag(Ukp1(1:N-1))]; zv 0];
end

% Definition of Function Fname.

function f=Fname(t,y,M)
    f=M*y;
end
% Definition of Function lsde.nn.

function [mu,muy,s2,s2y,s2yy] = lsde.nn(t,y,alpha,beta,sigma,rho)
    M=length(y);
    mu=alpha-beta.*y; % drift
    muy=-beta.*ones(M,1); % derivative of the drift
    g2=sigma.*sigma;
    rho2=2.*rho;
    s2=g2.*y.^rho2; % square of the diffusion
    s2y=g2.*rho2.*y.^(rho2-1); % derivative of the square
    s2yy=g2.*rho2.*(rho2-1).*y.^(rho2-2); % 2nd derivative of the square
end

The following is R-script to draw Figure 2.

% RM5.mat is created by the above Matlab-script.
install.packages("R.matlab")
require("R.matlab")
library(lattice)
ID <- c(5,8,10,12,15) % Reverted Mean
ThdPrice <- c(17.4,20.7,20,16.5,20.6) %Threshold Price
par(ask=TRUE)
for(i in 1:1){
    filename <- paste("RM",ID[i],".mat",sep="")
    filename1 <- paste("FigRM",ID[i],".pdf",sep="")
    title1 <- paste("Reverted Mean=",ID[i]," (x10^3 Yen/m^3)"")
    a<-readMat(filename)
    b<-data.frame(a)
    t<-rbind(
        cbind(b$T.2,b$Pric,b$PbDen.2,b$Pb.1),
        cbind(b$T.3,b$Pric,b$PbDen.3,b$Pb.2),
        cbind(b$T.4,b$Pric,b$PbDen.4,b$Pb.3),
        cbind(b$T.5,b$Pric,b$PbDen.5,b$Pb.4),
        cbind(b$T.6,b$Pric,b$PbDen.6,b$Pb.5),
        cbind(b$T.7,b$Pric,b$PbDen.7,b$Pb.6),
        cbind(b$T.8,b$Pric,b$PbDen.8,b$Pb.7),
        cbind(b$T.9,b$Pric,b$PbDen.9,b$Pb.8),
        cbind(b$T.10,b$Pric,b$PbDen.10,b$Pb.9),
        cbind(b$T.11,b$Pric,b$PbDen.11,b$Pb.10))
    t<-data.frame(t)
    t$X4<-round(t$X4,7)
    t$X1<-as.factor(paste("At",t$X1," years Pb="))
    y <- ThdPrice[i]
    xyplot(X3 X2|X1,data=t,
        panel = function(...){
            panel.abline(v=y, lty = "dotted", col = "red")
            panel.xlab(...),
            par.settings = list(strip.background=list(col="lightgrey")),
            type="l",col="black",layout=c(1,10),
            index.cond=list(c(10,8,7,6,5,4,3,2,1,9)),
            xlab= expression(paste("Price (x10^-3 Yen/m^3","\)")),ylab="Probability",
            main=title1,ylim=c(0,0.20))
    }
}