Nuclear Schiff moment in nuclei with soft octupole and quadrupole vibrations

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Abstract

Nuclear forces violating parity and time reversal invariance (P,T-odd) produce P,T-odd nuclear moments, for example, the nuclear Schiff moment. In turn, this moment can induce the electric dipole moment in the atom. The nuclear Schiff moment is predicted to be enhanced in nuclei with static quadrupole and octupole deformation. The analogous suggestion of the enhanced contribution to the Schiff moment from the soft collective quadrupole and octupole vibrations in spherical nuclei is tested in this article in the framework of the quasiparticle random phase approximation with separable quadrupole and octupole forces applied to the odd $^{217-221}$Ra and $^{217-221}$Rn isotopes. We confirm the existence of the enhancement effect due to the soft modes. However, in the standard approximation the enhancement is strongly reduced by a small weight of the corresponding “particle + phonon” component in a complicated wave function of a soft nucleus. The perspectives of a better description of the structure of heavy soft nuclei are discussed.

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I. INTRODUCTION

The search for interactions violating time reversal ($T$-) invariance is an important part of studies of fundamental symmetries in nature. The manifestations of $CP$-violation (and therefore, through the $CPT$-theorem, of $T$-invariance) in systems of neutral $K$- and $B$-mesons [1] set limits on physical effects beyond the standard model. The main hopes for the extraction of nucleon-nucleon and quark-quark interactions violating fundamental symmetries emerge from the experiments with atoms and atomic nuclei, see the recent review [2] and references therein. For example, the best limits on $P, T$-odd forces have been obtained from the measurements of the atomic electric dipole moment (EDM) in the $^{199}$Hg [3] and $^{129}$Xe nuclei. As we know from past experience with $P$-odd forces, see the review article [4], there are powerful many-body mechanisms in heavy atoms and nuclei which allow one to expect a significant amplification of effects generated on the level of elementary interactions. There are also suggestions for using possible molecular and solid state enhancement mechanisms [6, 7, 8].

Theoretical calculations of atomic EDM proceed through the nuclear Schiff moment $S$ since the nuclear EDM is shielded by atomic electrons [9]. The Schiff moment produces the $P, T$-odd electrostatic potential that, in turn, induces the atomic EDM. The expectation value of the vector operator $S$ in a stationary nuclear state characterized by certain quantum numbers of angular momentum, $JM$, is possible only for $J \neq 0$ owing to the requirements of rotational invariance. Since all even-even nuclei have zero ground state spin, we need to consider an odd-$A$ nucleus. Furthermore, the non-zero expectation value of a polar vector $S$ requires parity non-conservation in a nucleus; in addition, being proportional to the $T$-odd pseudoscalar $\langle (S \cdot J) \rangle$, this expectation value reveals the violation of $T$-invariance.

A reliable evaluation of the nuclear Schiff moment should include the estimates of renormalization effects due to “normal” strong interactions inside the nucleus. The core polarization by the odd nucleon is important, especially in the case of the odd neutron, as $^{199}$Hg and $^{129}$Xe. Calculations [10, 11, 12, 13] show that the resulting configuration mixing, depending on details of the method, may change the result of the independent particle model by a factor of about 2. A possibility of using accidental proximity of nuclear levels with the same spin and opposite parity was pointed out in Refs. [9, 14]. In such approaches, possible coherent enhancement mechanisms are usually not considered.
The statistical many-body enhancement of parity non-conservation in the region of the high level density of neutron resonances was predicted theoretically (see e.g. reviews \cite{5, 15} and references therein). The existence of such enhancement is now well documented experimentally \cite{16}. The simultaneous violation of parity and time reversal invariance can be enhanced by the firmly established $\mathcal{P}$-violation due to $\mathcal{P}$-odd $\mathcal{T}$-even weak interactions.

The idea of a possible role of static octupole deformation \cite{15, 17, 18, 19} exploited the parity doublets which appear in the presence of pear-shaped intrinsic deformation of the mean field. The doublet partners have similar structure and relatively close energies so that they can be more effectively mixed by $\mathcal{P}$-odd forces. The Schiff moment in the body-fixed frame is enhanced being in fact proportional to the collective octupole moment. The microscopic calculations \cite{20, 21} predict a resulting Schiff moment by two-three orders of magnitude greater than in spherical nuclei; this enhancement was confirmed in Refs. \cite{22, 23}. The uncertainties related to the specific assumptions on $\mathcal{P}, \mathcal{T}$-odd forces and different approximations for nuclear structure are on the order of a factor 2 for the resulting Schiff moment.

It was suggested in Ref. \cite{24} that soft octupole vibrations observed in some regions of the nuclear chart more frequently than static octupole deformation may produce a similar enhancement of the Schiff moment. This would make heavy atoms containing nuclei with large collective Schiff moments attractive for future experiments in search for $\mathcal{P}, \mathcal{T}$-violation; experiments of this type are currently under progress or in preparation in several laboratories. Recently we performed \cite{25} the estimate of the Schiff moment generated in nuclei with the quadrupole deformation and soft octupole mode and showed that the result is nearly the same as in the case of the static octupole deformation.

A related idea is explored in the present paper. It is known that some nuclei are soft with respect to both quadrupole and octupole modes, see for example recent predictions for radioactive nuclei along the $N = Z$ line \cite{26}. The light isotopes of Rn and Ra are spherical but with a soft quadrupole mode and therefore large amplitude of quadrupole vibrations. The spectra of these nuclei display long quasivibrational bands \cite{27} based on the ground state and on the octupole phonon, with positive and negative parity, respectively. These bands are connected via low-energy electric dipole transitions. This situation seems a-priori to be favorable for the enhancement of $\mathcal{P}, \mathcal{T}$-odd effects.

Below we examine the question whether the enhancement indeed exists in some spherical
nuclei which have both collective quadrupole and octupole modes. The main weak interaction mixing can be expected to occur between the levels of the same spin and opposite parity that carry a significant admixture of the \((\text{particle} + 2^+ \text{ phonon})\) and \((\text{particle} + 3^- \text{ phonon})\) states with the odd nucleon in the same single-particle orbit. In the odd-neutron nuclei the Schiff moment is originated by the proton contribution to the collective phonon. A number of such nuclei have an appropriate opposite parity level with the same angular momentum close to the ground state. The mixing between the states of above mentioned nature can imitate the evolution of the spherical nucleus to the deformed pear-shaped state. In analogy with Refs. [20, 21], this might lead to enhancement of Schiff moments.

We will show that the effect does exist but, in the framework of the standard quasiparticle random phase approximation (QRPA), it is essentially compensated by the strong spreading of the single-particle strength that significantly reduces the weight of the desired configuration in the odd-nucleus ground state. We concentrate here only on physics connected to the soft vibrational modes. At the next stage the task of theory should be to go beyond the RPA and to combine all effects generated by \(\mathcal{P}, \mathcal{T}\)-violating interactions, including the influence of the core polarization and weak interaction admixtures to the phonon structure which are not discussed on the present work. Such effects should be included for the quantitative answer; however they are not expected to lead to a qualitative enhancement.

II. COLLECTIVE SCHIFF MOMENT IN SPHERICAL NUCLEI

Consider an odd-\(A\) nucleus with two close levels of the same spin \(J\) and opposite parity, ground state \(|\text{g.s.}\rangle\) and excited state \(|x\rangle\). The energies of these states are \(E_{\text{g.s.}}\) and \(E_x\), respectively. Let \(W\) be a \((\mathcal{P}, \mathcal{T})\)-odd interaction mixing nuclear states. Assuming that the mixing matrix elements of \(S\) and \(W\) are real, we can write down the Schiff moment emerging in the actual mixed ground state as

\[
S = 2 \frac{\langle \text{g.s.}|W|x\rangle \langle x|S|\text{g.s.}\rangle}{E_{\text{g.s.}} - E_x}.
\]

However, as it was explained in Ref. [1], in the case of mixing of close single-particle states one should not expect necessarily a large enhancement. For example, in a simple approximate model, where the strong nuclear potential is proportional to nuclear density and the spin-orbit interaction is neglected, the matrix element \(\langle \text{g.s.}|W|x\rangle\) contains the single-
particle momentum operator. This matrix element is proportional to \((E_{g.s.} - E_x)\), so that the small energy denominator cancels out. As mentioned above, the collective Schiff moments in nuclei with static octupole deformation may be by 2-3 orders of magnitude stronger than single-particle moments in spherical nuclei.

Consider the following example of a mechanism generating a collective Schiff moment in spherical nuclei \[25\]. Let an odd-\(A\) nucleus have the unperturbed ground state of spin \(J\) built as a zero spin core plus an unpaired nucleon in the spherical mean field orbit with angular momentum \(j = J\). The interaction between the odd particle and vibrations of the core causes an admixture of a quadrupole phonon to the ground state if the nuclear spin \(J > 1/2\):

\[
|\text{g.s.}\rangle = a_0 |j = J\rangle + a_2 |j \otimes 2^+\rangle. \tag{2}
\]

An opposite parity state with the same spin \(J\) can be formed by coupling an octupole phonon to the ground state particle,

\[
|x\rangle = |j \otimes 3^-\rangle. \tag{3}
\]

A \((\mathcal{P}, \mathcal{T})\)-odd Schiff moment \(\mathbb{I}\) can mix the states \(|\text{g.s.}\rangle\) and \(|x\rangle\). To first order in the non-relativistic nucleon velocity \(p/m\), the \((\mathcal{P}, \mathcal{T})\)-odd interaction can be presented as \(\mathbb{II}\)

\[
W_{ab} = \frac{G}{\sqrt{2}m} \left( \eta_{ab} \sigma_a - \eta_{ba} \sigma_b \right) \cdot \nabla_a \delta(r_a - r_b) + \eta'_{ab} [\sigma_a \times \sigma_b] \cdot \{(p_a - p_b), \delta(r_a - r_b)\}, \tag{4}
\]

where \(\{ , \}\) is an anticommutator, \(G\) is the Fermi constant of the weak interaction, \(m\) is the nucleon mass, and \(\sigma_{a,b}, r_{a,b},\) and \(p_{a,b}\) are the spins, coordinates, and momenta, respectively, of the interacting nucleons \(a\) and \(b\). The dimensionless constants \(\eta_{ab}\) and \(\eta'_{ab}\) characterize the strength of the \((\mathcal{P}, \mathcal{T})\)-odd nuclear forces; in fact, experiments on measurement of the EDMs are aimed at extracting the values of these constants.

In the case of static deformation, in the “frozen” body-fixed frame the intrinsic collective Schiff moment \(S_{\text{intr}}\) of the deformed nucleus can exist without any \((\mathcal{P}, \mathcal{T})\)-violation. The estimate found in Refs. \([20, 21]\) gives

\[
S_{\text{intr}} \approx \frac{9}{20\pi \sqrt{35}} e Z R^3 \beta_2 \beta_3 = \frac{3}{5 \sqrt{35}} O_{\text{intr}} \beta_2, \tag{5}
\]

where \(\beta_2\) and \(\beta_3\) are the static quadrupole and octupole deformation parameters, respectively, and \(O_{\text{intr}}\) is the static octupole moment. Of course, in the space-fixed laboratory frame, the nucleus has definite angular momentum rather than fixed orientation, and this makes the expectation value of the Schiff moment to vanish in the case of no \((\mathcal{P}, \mathcal{T})\)-violation.
The relation (5) is expected to hold \cite{25} for the *dynamic* quadrupole and octupole deformations in systems with spherical equilibrium shape if the effective dynamic deformation parameters $\beta_2$ and $\beta_3$ (measured by the multipole transition strength for the phonon states in the adjacent even-even nucleus) have a magnitude similar to that in deformed nuclei. This can be the case under an assumption that the ground state of the considered odd-$A$ nucleus can be well approximated by an unpaired particle with an admixture of a single phonon of a corresponding multipolarity. It is therefore important to try to perform a detailed calculation and learn more about the validity of such estimates. Below we present such calculations in the framework of the conventional QRPA for quadrupole and octupole phonons.

III. QRPA PHONONS AND PARTICLE-PHONON COUPLING

We start with the calculation of quadrupole ($J^\pi = 2^+$) and octupole ($J^\pi = 3^-$) phonon states in heavy even-even nuclei based on a conventional model Hamiltonian,

$$ H = H_{s-p} + H_{\text{pair}} + H_{\text{phon}}. $$

The first term, $H_{s-p}$, describes the single-particle mean field. The following Woods-Saxon parameterization \cite{28} was used in calculations:

$$ V_{c}^{p(n)} = -49.6 \left( 1 + (-) 0.86 \frac{N - Z}{A} \right) \text{MeV} $$

and

$$ V_{ls} = 18.8 \left( \frac{A}{A - 1} \right)^2 \text{MeV} $$

are the strength parameters for the central and spin-orbital potentials, respectively; $R = 1.3 A^{1/3}$ fm and $a = 0.7$ fm are the radius and diffuseness parameters. The second term of eq. (6), $H_{\text{pair}}$, is the pairing interaction with the strength

$$ G_p = \frac{17.9 + 0.176 (N - Z)}{A} $$

for protons and

$$ G_n = \frac{18.95 - 0.078 (N - Z)}{A} $$
for neutrons. The BCS formalism was used that yields a quasiparticle basis and corresponding Bogoliubov transformation coefficients \(u_j\) and \(v_j\). The last term,

\[
H_{\text{phon}} = \sum_{\lambda\mu} \omega_{\lambda} Q_{\lambda\mu}^\dagger Q_{\lambda\mu},
\]

presents RPA phonons obtained with a separable multipole-multipole interaction. The building blocks of the model are the two-quasiparticle RPA phonons,

\[
Q_{\lambda\mu}^\dagger = \frac{1}{2} \sum_{j_1,j_2} \left[ A_{j_1,j_2}^\lambda [\alpha_{j_1}^\dagger \alpha_{j_2}^\dagger]_{\lambda\mu} - (-)^{\lambda-\mu} B_{j_1,j_2}^\lambda [\alpha_{j_1}^{\dagger} \alpha_{j_2}^{\dagger}]_{\lambda-\mu} \right],
\]

where \(A_{j_1,j_2}\) and \(B_{j_1,j_2}\) are the forward and backward phonon amplitudes,

\[
\begin{align*}
A_{j_1,j_2}^\lambda &= \frac{1}{\sqrt{2Z(\lambda)}} f(\lambda, j_1, j_2) \xi_{12}^{(+)} - (\lambda) ;
B_{j_1,j_2}^\lambda &= \frac{1}{\sqrt{2Z(\lambda)}} f(\lambda, j_1, j_2) \xi_{12}^{(+)} + (\lambda),
\end{align*}
\]

where \(Z(\lambda)\) is a normalization factor, \(f(\lambda, j_1, j_2) = \langle j_1 | r^{\lambda} Y_\lambda | j_2 \rangle\), and the coherence factors of the Bogoliubov canonical transformation are \(\xi_{12}^{(\pm)} = u_1 v_2 \pm v_1 u_2\) and \(\eta_{12}^{(\pm)} = u_1 u_2 \pm v_1 v_2\).

The phonon frequencies \(\omega_{\lambda}\) are the roots of the characteristic RPA equation for each multipolarity \(\lambda\),

\[
X(\lambda) \equiv \frac{1}{2\lambda + 1} \sum_{j_1,j_2} \frac{[f(\lambda, j_1, j_2) \xi_{12}^{(+)}]^2 \varepsilon(\lambda, j_1, j_2)}{\varepsilon^2(\lambda, j_1, j_2) - \omega_{\lambda}^2} = \frac{1}{\chi(\lambda)},
\]

where \(\chi(\lambda)\) is the strength parameter and \(\varepsilon(\lambda, j_1, j_2) = \varepsilon(j_1) + \varepsilon(j_2)\) is the unperturbed two-quasiparticle energy. Solving these equations one obtains the energies of the phonons and internal structure of the phonon operator hidden in the amplitudes of different two-quasiparticle components. The values of the strength parameters, \(\chi(2)\) and \(\chi(3)\), were chosen to reproduce the excitation energies of the \(2^+_1\) and \(3^-_1\) states in the neighboring even-even nuclei.

We are interested in the evaluation of the Schiff moment for the odd-neutron nuclei. Therefore the second step of calculations is the solution for the odd-A nucleus with the even-even core excitations described above. If we neglect the (quasiparticle+two-phonon) components in the wave function of an excited state of an odd-A nucleus, then the corresponding wave functions have the following form:

\[
\Psi_n(J^z) = C_n \left( \alpha_{JM}^\dagger \delta_{jJ} + \sum_{\lambda,j} D_{jj}^\lambda \alpha_{j}^\dagger \otimes Q_{\lambda}^+(\lambda) \right) \Psi_0,
\]
TABLE I: Phonon energies in even-even nuclei and the corresponding parameters of the multipole-multipole interaction.

| Nucleus   | $\omega_2$, MeV | $\omega_3$, MeV | $\chi(2)$, $10^{-4}$ | $\chi(3)$, $10^{-5}$ |
|-----------|-----------------|-----------------|----------------------|----------------------|
| $^{216}_{88}$Ra$_{128}$ | 0.69            | 1.40            | 8.1                  | 1.2                  |
| $^{216}_{86}$Rn$_{130}$ | 0.46            | 1.11            | 6.9                  | 1.2                  |
| $^{218}_{88}$Ra$_{130}$ | 0.39            | 0.79            | 6.9                  | 1.2                  |
| $^{218}_{86}$Rn$_{132}$ | 0.32            | 0.84            | 5.9                  | 1.1                  |
| $^{220}_{88}$Ra$_{132}$ | 0.18            | 0.47            | 5.9                  | 1.1                  |
| $^{220}_{86}$Rn$_{134}$ | 0.24            | 0.66            | 5.3                  | 1.1                  |

where the operator $\alpha^\dagger$ creates a quasiparticle with respect to the ground state wave function $\Psi_0$ of the even-even nucleus. The energy $E_n(J)$ of the $n^{th}$ state with angular momentum $J$ in the odd-mass nucleus, the amplitudes of the quasiparticle-phonon components,

$$D^{\lambda,n}_{jj} = \sqrt{\frac{2\lambda + 1}{(2J + 1)2Z(\lambda)}} \frac{f^{\lambda}(J)\eta^{(-)}_{jj}}{\varepsilon(j) + \omega_\lambda - E_n},$$

(12)

and the amplitude of the single-quasiparticle component,

$$C^{-2}_{jn} = 1 + \sum_{\lambda,j} (D^{\lambda,n}_{jj})^2,$$

(13)

are obtained from the solution of the secular equation $^{30}$:

$$\varepsilon(J) - E_n = \sum_{\lambda,j} \left(D^{\lambda,n}_{jj}\right)^2 \left[\varepsilon(j) + \omega_\lambda - E_n\right].$$

(14)

We performed the calculations for several isotopes of radium and radon. The lighter even-even isotopes, $A = 216 - 218$, are considered to be still spherical; the ones with $A = 220$ are transitional and the heavier isotopes $A = 222 - 224$ acquire more pronounced deformed structure. In the six even-even nuclei treated here, see Table 1, the $2^+$ and $3^-$ states were experimentally observed with excitation energies $\omega_2$ and $\omega_3$ listed in Table 1. The values of the coupling constants $\chi(2)$ and $\chi(3)$, also shown in this Table, were found by fitting the lowest RPA $2^+$ and $3^-$ states to the corresponding experimental excitation energies.

The single-particle states were calculated in a spherical Woods-Saxon potential with the parameters indicated above. For heavier isotopes this might not be an appropriate approx-
imation; the purpose of our calculations for these nuclei was to examine the systematics of the results in the cases when the phonons are lowered in energy.

In Table II we show the composition of the ground state wave functions and their suitable parity partners in the six odd-neutron nuclei considered in this work. One can see from the wave functions that while the dominant component for negative parity is the $3^{-} \otimes g_{9/2}$ configuration, the corresponding component $2^{+} \otimes g_{9/2}$ is of the order of 10-30%. The spacings between the ground states and their parity partners turn out to be of order 1-2 MeV, Table I. The mixing of these two components will contribute the most to the Schiff moment. We should note here that the calculated spectra of the odd-neutron nuclei do not agree well with experiment even in the cases of $A = 217, 219$. This indicates that the model used for the description of coupling of the odd particle with the core vibrations is too restrictive.

IV. THE SCHIFF MOMENT

Having determined the wave functions one may proceed with the calculation of the Schiff moment according to Eq. (1), where we will use the notations $|J_{g}^{\pi}\rangle$ and $|J_{-g}^{\pi}\rangle$ for the ground state of the odd-neutron nucleus and for the first excited state of opposite parity, respectively,

$$S(J_{g}^{\pi}) = 2 \frac{\langle J_{g}^{\pi}|W|J_{-g}^{\pi}\rangle \langle J_{-g}^{\pi}J_{g}\rangle S_{z}}{E(J_{g}^{\pi}) - E(J_{-g}^{\pi})}.\quad (15)$$

Here $W$ is the $(P, T)$-violating nucleon-nucleon interaction given by Eq. (4) and $\langle J_{-g}^{\pi}J_{g}|S|J_{g}^{\pi}J_{g}\rangle$ is the matrix element of the Schiff operator $S$,

$$S_{\mu} = \frac{1}{10} \sqrt{\frac{4\pi}{3}} \sum_{i} Y_{1\mu}(\theta_{i}, \varphi_{i}) e_{i} \left[ r_{i}^{2} - \frac{5}{3} r_{ch}^{2} r_{i} \right],\quad (16)$$

for the maximum projection $M = J_{g}$ of angular momentum, and $r_{ch}^{2}$ is the mean square charge radius.

Since only the proton components of the phonons contribute to the matrix element of the Schiff moment, this matrix element for the case of the odd neutron can be written as

$$\langle J_{g}^{\pm}M = J_{g}|S_{z}|J_{g}^{\mp}M = J_{g}\rangle$$

$$= \sqrt{\frac{J_{g}(2J_{g} + 1)}{(J_{g} + 1)^{2}}} \sum_{j,\lambda,\lambda'} (-1)^{j+\lambda'+J_{g} + 1} \tilde{D}_{J_{g}^{\pm}j}^{\lambda} \tilde{D}_{J_{g}}^{\lambda'} J_{g}^{\pm} j \left\{ \frac{\lambda \lambda'}{J_{g} J_{g}} 1 \right\} \langle\lambda|S||\lambda'\rangle,\quad (17)$$
TABLE II: Structure of the ground \((J^\pi)\) and excited state with the same spin and opposite parity \((J^{-\pi})\) in odd-neutron nuclei (probabilities of various components in the wave functions).

|                  | \(9/2^+\) | \(9/2^-\) | \(1h_{9/2}\) | \(10^{-4}\) |
|------------------|------------|------------|--------------|-------------|
| \(^{217}\text{Ra}_{129}\) | \(2g_{9/2}\) | 0.71       | \(2g_{9/2} \otimes 2^+\) | 0.14        |
|                  | \(1j_{15/2} \otimes 3^-\) | 0.08       | \(3d_{5/2} \otimes 2^+\) | 0.05        |
| \(^{217}\text{Rn}_{131}\) | \(2g_{9/2}\) | 0.73       | \(2g_{9/2} \otimes 2^+\) | 0.11        |
|                  | \(1j_{15/2} \otimes 3^-\) | 0.07       | \(3d_{5/2} \otimes 2^+\) | 0.07        |
| \(^{219}\text{Ra}_{131}\) | \(2g_{7/2}\) | 0.43       | \(2f_{7/2}\) | 0.02        |
|                  | \(1i_{11/2} \otimes 2^+\) | 0.27       | \(2g_{9/2} \otimes 3^-\) | 0.98        |
|                  | \(2g_{7/2} \otimes 2^+\) | 0.11       |                        |             |
|                  | \(2g_{9/2} \otimes 2^+\) | 0.08       |                        |             |
|                  | \(3d_{3/2} \otimes 2^+\) | 0.09       |                        |             |
| \(^{219}\text{Rn}_{133}\) | \(3d_{5/2}\) | 0.48       | \(2f_{7/2}\) | 0.01        |
|                  | \(2g_{9/2} \otimes 2^+\) | 0.27       | \(2g_{9/2} \otimes 3^-\) | 0.98        |
|                  | \(3d_{5/2} \otimes 2^+\) | 0.12       |                        |             |
|                  | \(4s_{1/2} \otimes 2^+\) | 0.09       |                        |             |
| \(^{221}\text{Ra}_{133}\) | \(3d_{5/2}\) | 0.48       | \(2f_{5/2}\) | 0.04        |
|                  | \(2g_{9/2} \otimes 2^+\) | 0.25       | \(2g_{9/2} \otimes 3^-\) | 0.86        |
|                  | \(3d_{5/2} \otimes 2^+\) | 0.13       | \(1i_{11/2} \otimes 3^-\) | 0.09        |
|                  | \(4s_{1/2} \otimes 2^+\) | 0.10       |                        |             |
|                  | \(3d_{3/2} \otimes 2^+\) | 0.03       |                        |             |
| \(^{221}\text{Rn}_{135}\) | \(2g_{7/2}\) | 0.44       | \(2f_{7/2}\) | 0.02        |
|                  | \(1i_{11/2} \otimes 2^+\) | 0.26       | \(2g_{9/2} \otimes 3^-\) | 0.98        |
|                  | \(2g_{7/2} \otimes 2^+\) | 0.12       |                        |             |
|                  | \(3d_{3/2} \otimes 2^+\) | 0.12       |                        |             |
|                  | \(2g_{9/2} \otimes 2^+\) | 0.04       |                        |             |
TABLE III: Mixing matrix elements of weak interaction (m.e. $W$), of the Schiff moment (m.e. $S$), and final value for the Schiff moment, $S$, in the ground state of the odd-neutron nucleus.

| Nucleus | $E_+$, MeV | $E_-$, MeV | m.e. $W$, $\eta \cdot 10^{-2}$ eV | m.e. $S$, e·fm$^3$ | $S$, $\eta \cdot 10^{-8}$ e·fm$^3$ |
|---------|-------------|-------------|-----------------|-----------------|-----------------|
| $^{217}_{88}$Ra$_{129}$ | 0.09 | 2.44 | 0.3 | -0.1 | -0.03 |
| $^{217}_{86}$Rn$_{131}$ | -0.12 | 1.80 | 0.1 | -0.1 | -0.01 |
| $^{219}_{88}$Ra$_{131}$ | 0.49 | 1.47 | -1.3 | -0.1 | 0.30 |
| $^{219}_{86}$Rn$_{133}$ | -0.09 | 1.42 | 0.2 | -0.1 | -0.03 |
| $^{221}_{88}$Ra$_{133}$ | -0.77 | 1.05 | 0.2 | -0.2 | -0.07 |
| $^{221}_{88}$Rn$_{135}$ | 0.24 | 1.16 | -0.5 | -0.1 | 0.06 |

where $\tilde{D} = C \cdot D$ - is the amplitude of the (quasiparticle plus phonon) component, and the sum over $\lambda$ and $\lambda'$ is in reality reduced to one term with $\lambda^\pi = 2^+$ and $\lambda'^\pi = 3^-$. The matrix element of the Schiff operator between the quadrupole and octupole phonon states in the even nucleus has the following form in terms of the RPA amplitudes (9):

$$\langle 2^+||S||3^- \rangle = \sqrt{35} \sum_{j_1,j_2,j_3} \eta_{j_1,j_2,j_3}^{(-)} \langle j_1||S||j_2 \rangle \left\{ \begin{array}{ccc} 2 & 3 & 1 \\ j_1 & j_2 & j_3 \end{array} \right\} (A_{j_2j_3}^{(2+)} A_{j_3j_1}^{(3-)} + B_{j_2j_3}^{(2+)} B_{j_3j_1}^{(3-)}) \right) . \quad (18)$$

The results of the calculations are shown in Table III.

V. MATRIX ELEMENT OF WEAK INTERACTION

The calculation of the weak interaction matrix element appearing in Eqs. (1) and (15) is elaborate and contains a number of contributions. We collect these calculations more in detail in Appendix. Here we present only the final expressions for the important contributions and numerical results, Table III.

In this work we account only for the direct contribution of the weak interaction, Eq. (4), to the nuclear mean field for a particle $b = p, n$ at a point $r = r_n$,

$$\overrightarrow{W}_b(r) = \frac{G}{\sqrt{2} \cdot 2m} \sum_c \eta_{bc}(\vec{\sigma}n) \frac{d^2 J_c}{dr^2} + \frac{1}{4\pi} \nu_c^2 R_c^2(r) \frac{G}{\sqrt{2} \cdot 2m} \eta(\vec{\sigma}n) \frac{1}{4\pi} \frac{d\rho(r)}{dr} , \quad (19)$$

where $\nu_c^2$ is the occupancy of the single-particle state $c$ in the core. We expect this term to be the largest since all particles in the core add coherently in Eq. (19). The weak
matrix elements are typically of the order $10^{-2} \eta$ eV. The main contribution comes from the mixing between the (quasiparticle plus phonon) component of the excited state and the pure quasiparticle component of the ground state.

The reduced matrix elements $\langle 2^+|S||3^- \rangle$ in even nuclei are of the order $(1-2) e \cdot \text{fm}^3$, and the matrix elements of the Schiff operator between the ground state and its parity partner turn out to be around $0.1 - 0.2$ in units of $e \cdot \text{fm}^3$. The results for the Schiff moment in this model are within the limits $(0.04-0.6) \eta \cdot 10^{-8} e \cdot \text{fm}^3$, i.e. of the same order as in pure single-particle models in even nuclei. The single-particle contribution unrelated to the soft modes [11, 12, 29] is of the same order of magnitude and it should be added.

VI. DISCUSSION

How can one reconcile this result with the large enhancement found in the calculations for the nuclei with static quadrupole+octupole deformation? We need to stress that in those calculations the deformations are introduced explicitly from the beginning (in some calculations, only the quadrupole deformation is taken at a starting point while the octupole part is introduced via coupling of a particle to the $3^-$ phonon of the deformed core). Apparently, it is essential to start from a particle moving in a quadrupole deformed field in order to find large enhancement factors for the Schiff moment. The attempt to reproduce this effect by having in the ground state of the odd-\textit{A} nucleus an admixture of a particle plus a single $2^+$ phonon can succeed only in limiting cases.

In the case of static quadrupole, $\beta_2$, and octupole, $\beta_3$, deformations, the weak matrix element is $\propto \beta_3$, the Schiff moment operator $\propto \beta_2 \beta_3 / \Delta E_{\pm}$ as discussed in [20, 21]. Although the small denominator $\Delta E_{\pm} \sim 50$ keV was the main source of enhancement, it was also important to have noticeable values of the deformation parameters. Since in dynamical models with low phonon frequency $\omega$ the dynamic deformation parameters $\beta \propto 1/\omega$, one may expect a considerable enhancement at $\omega \to 0$, close to the phase transitions to static deformation. Such limiting cases will be discussed below.

The effects of quadrupole deformation can be mimicked by a high degree of coherence in a quadrupole mode. Similarly to that, the transition of a spherical nucleus to an octupole deformed state was discussed in the RPA framework in the past [31]. It was found that even for realistic interactions the energies of the lowest $3^-$ RPA root collapse in certain areas
of the periodic table. The stability of the nucleus against variation of a certain collective
variable \( Q \) can be expressed in terms of the static polarizability defined for instance through
the dependence of constrained Hartree-Fock energy \( E/A \) on the constraining term \( \alpha Q \),
\[
P = \frac{1}{2} \left[ \frac{\partial^2 (E/A)}{\partial \alpha^2} \right]_{\alpha=0}.
\] (20)
In the RPA, this quantity can be calculated as
\[
P = 2 \frac{m_{-1}}{A},
\] (21)
by relating it to the \( m_{-1} \) moment (the inverse energy weighted sum rule),
\[
m_{-1} = \sum_n \frac{|\langle n|Q|0 \rangle|^2}{\omega_n},
\] (22)
with \( \omega_n \) being the RPA energies corresponding to excitation from the ground state \( |0 \rangle \) to
the excited state \( |n \rangle \) with quantum numbers dictated by the symmetry of the operator \( Q \).
As polarizability increases the system becomes less stable against deformation characterized
by the collective variable \( Q \); a negative value of \( P \) indicates a situation in which a collapse
has occurred (in the RPA framework).
In order to take advantage of such properties and examine the behavior of the Schiff
moment in situations close to the RPA collapse that signals a transition from a spherical to
a pear-shaped nucleus (quadrupole + octupole deformation), we can consider the limit of
the RPA frequencies \( \omega_2 \) and \( \omega_3 \) going to zero. Introducing formally a small scaling factor \( y \),
\[
\omega_{2,3} \rightarrow y \omega_{2,3}, \quad y \ll 1,
\] (23)
we separate the singular in this limit part of the secular equation (10) as
\[
X(\lambda) = \frac{1}{2\lambda + 1} \sum_{12} \left| F_{12}^{(+) \ast} \right|^2 \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 + \epsilon_2} \right) + O(y^2).
\] (24)
The normalization factor \( Z(\lambda) \) goes to zero linearly in \( y \), and the phonon amplitudes \( \xi \)
grow \( \propto 1/\sqrt{y} \). As a result of this overcritical coupling, the energy of the ground state in
the odd nucleus collapses as well (goes to \( -\infty \) as \( 1/\sqrt{y} \)). The wave function components
(11) have a constant limit with the single-particle amplitude \( C^2 \rightarrow 1/2 \); the same effect was
observed in [32] for the strong coupling of a quasiparticle with phonons.
In Table IV the results of the calculation for \(^{219,221}\text{Ra}\) are shown as a function of \( y \)-
scaling. As the collective frequencies go down, the reduced matrix element \( \langle 2||S||3 \rangle \) in
TABLE IV: Scaling of RPA solutions with the low-lying frequency of collective oscillations in the even core: normalization factors $Z(\lambda)$; the reduced matrix element $\langle 2||S||3 \rangle$ of the Schiff moment in the even nucleus; energies $E_+$ and $E_-$ of the ground and excited state in the odd nucleus, in MeV; the matrix element (m.e. $W$) of the weak interaction between these states; the matrix element (m.e. $S$) of the Schiff moment in the odd nucleus; and the final result for the expectation value of the Schiff moment; the matrix elements of $W$ and $S$ are given in the same units as in Table III.

| Nucleus | $y$ | $Z(2)$ | $Z(3)$ | $\langle 2||S||3 \rangle$ | $E_+$ | $E_-$ | m.e. $W$ | m.e. $S$ | $S$ |
|---------|-----|--------|--------|---------------------|--------|--------|-----------|-----------|-----|
| $^{219}$Ra | 1.0 | 479 | 21696 | 1.7 | 0.5 | 1.5 | -1.3 | -0.1 | 0.3 |
| | 0.1 | 43 | 1825 | 20 | -3.8 | -1.9 | 1.1 | -0.2 | -0.2 |
| | 0.01 | 4 | 182 | 195 | -17 | -19 | 53 | -0.2 | 6.2 |
| $^{221}$Ra | 1.0 | 324 | 18100 | 2.2 | -0.8 | 1.0 | 0.2 | -0.2 | -0.1 |
| | 0.1 | 31 | 1650 | 23 | -6.1 | -2.9 | -19 | -0.5 | 6 |
| | 0.01 | 3 | 165 | 235 | -23 | -21 | -253 | -2.7 | 560 |

the even nucleus, the matrix element of the weak interaction in the odd nucleus and the resulting Schiff moment become large. This is especially pronounced in $^{221}$Ra, where the Schiff moment is strongly enhanced at the collapse threshold.

In the RPA framework the matrix element of the Schiff moment between the states of opposite parity in the odd nucleus can be presented in terms of the amplitudes $\tilde{D}^{\lambda}_{jj'}$, eq. (11), as

$$\langle +|S| - \rangle \propto \sum_{\lambda_1\lambda_2} \tilde{D}_{jj'}^{\lambda_1(+)} \tilde{D}_{jj'}^{\lambda_2(-)} \langle \lambda_1||S||\lambda_2 \rangle. \quad (25)$$

In the case of $^{219}$Ra, at the experimental values of the phonon frequencies, $y = 1$, there exists only one large combination of amplitudes,

$$|\tilde{D}_{g9/2, g9/2}^{2(+)}|^2 = 0.08, \quad |\tilde{D}_{g9/2, g9/2}^{3(-)}|^2 = 0.98. \quad (26)$$

Here the excited negative parity state is an almost pure combination $2g9/2 \otimes 3^{-}$. With reduction of frequencies, the amplitudes \([26]\) rapidly decrease going to limiting values of $|\tilde{D}^{2(+)}|^2 \approx 0.01$ and $|\tilde{D}^{3(-)}|^2 \approx 0.02$. The wave function (quasiparticle + phonon) turns out to be spread over many states, and the enhancement of the Schiff moment due to the growth of the matrix element $\langle 2||S||3 \rangle$ starts with further diminishing frequencies only after
the spreading saturates in a given single-particle space. The natural conclusion is that the enhancement of the Schiff moment in the presence of soft quadrupole and octupole modes is a real physical effect which is however hindered by the complexity of the mixed states in the limit of strong coupling between quasiparticles and soft mode phonons.

However, the ansatz for the wave function of the odd nucleus becomes invalid in this limit, and many-phonon components carry a large fraction of the total normalization. Leaving the full solution of this problem for the future, we just show that the enhancement occurs beyond the RPA framework.

VII. SIMPLE MODEL BEYOND RPA

The strong quasiparticle-phonon coupling leads to stationary states which are close to the coherent states, or condensates, of the phonon field. This limit is different from that used in (11), where only a one-phonon admixture was taken into account.

We consider the Hamiltonian (for simplicity with one single-particle j-level but the extension is straightforward):

\[
H = \epsilon \sum_m a_m^\dagger a_m + \sum_{\lambda \mu} \omega_\lambda Q_\lambda^\dagger Q_\mu, \\
+ \sum_{\lambda \mu ; m m'} x_\lambda \left( \begin{array}{ccc} \lambda & j & j \\ \mu & -m & m' \end{array} \right) (-)^{j+m} \hat{\rho}_{m'm} \left( Q_{\lambda\mu} + (-)^{\lambda-\mu} Q_{\lambda=\mu}^\dagger \right),
\]

(27)

where \(x_\lambda\) are the parameters of the particle-phonon coupling, \(Q_{\lambda\mu}\) phonon operators, and \(\hat{\rho}_{m'm} = a_m^\dagger a_{m'}\) is the operator of the single-particle density matrix that satisfies the operator equation of motion:

\[
[\hat{\rho}_{m'm}, H] = \sum_{\lambda \mu ; m''} x_\lambda \left( \begin{array}{ccc} \lambda & j & j \\ \mu & -m' & m'' \end{array} \right) (-)^{j+m'} \hat{\rho}_{m''m} \left( Q_{\lambda\mu} + (-)^{\lambda-\mu} Q_{\lambda=\mu}^\dagger \right) - \sum_{\lambda \mu ; m''} x_\lambda \left( \begin{array}{ccc} \lambda & j & j \\ \mu & -m & m'' \end{array} \right) (-)^{j+m} \hat{\rho}_{m'm''} \left( Q_{\lambda\mu}^\dagger + (-)^{\lambda-\mu} Q_{\lambda=\mu} \right).
\]

(28)

The phonon operators satisfy

\[
[Q_{\lambda\mu}, H] = \omega_\lambda Q_{\lambda\mu} + x_\lambda \sum_{m m'} (-)^{\lambda-\mu} \left( \begin{array}{ccc} \lambda & j & j \\ -\mu & -m & m' \end{array} \right) (-)^{j+m} \hat{\rho}_{m'm}.
\]

(29)
For monopole phonons, \( \lambda = 0 \), the exact solution \([33]\) leads to the coherent cloud of phonons with the Poisson distribution of the phonon numbers. For \( \lambda \neq 0 \) the analytical solution does not exist; however, we can use the method of Ref. \([34]\) that allows to separate the main effects in the adiabatic limit of slow collective motion. We assume that the even core is spherical with low-frequency collective modes, \( \omega_2 \) and \( \omega_3 \). Then in the odd nucleus we have a phonon condensate \( \langle Q_{\lambda \mu} \rangle \) that can be found from \([29]\),

\[
\langle Q_{\lambda \mu} \rangle = -\frac{x_{\lambda}}{\omega_{\lambda}} \sum_{mm'} (-)^{\lambda-\mu} \left( \begin{array}{ccc} \lambda & j & j \\ -\mu & -m & m' \end{array} \right) (-)^{j+m} \langle \rho_{m'm} \rangle.
\]

This condensate is created by the average odd-particle density matrix \( \langle \rho_{m'm} \rangle \) that determines self-consistently a direction of the intrinsic quantization axis. This direction is arbitrary in agreement with full rotational invariance, and we can expand this density matrix over multipoles,

\[
\langle \rho_{m'm} \rangle = \sum_{L\Lambda} (-)^{L+j-m} \left( \begin{array}{ccc} L & j & j \\ \Lambda & -m & m' \end{array} \right) Y_{L\Lambda} \rho_L,
\]

where the spherical function shows the “rotational behavior” of the condensate in the laboratory frame of the odd nucleus. Then we find the relation between the condensate and the density matrix multipoles through effective deformation parameters \( \beta_{\lambda} \),

\[
\langle Q_{\lambda \mu} \rangle = \beta_{\lambda} Y_{\lambda \mu}^*, \quad \beta_{\lambda} = \frac{1}{\omega_{\lambda}} \frac{x_{\lambda}}{2\lambda + 1} \rho_{\lambda}.
\]

As mentioned earlier, the effective deformation parameters induced in the odd nucleus by soft modes of the even core are inversely proportional to the corresponding frequencies. (In our notations we have \( Y_{\lambda \mu}^* \) because this function lowers the projection by \( \mu \), in accordance with the definition of \( Q_{\lambda \mu} \).)

The relation between the states of the odd and even nuclei is established through the equation of motion for single-particle operators linearized after the substitution of the phonon dynamics by the condensate,

\[
[a_m, H] = \epsilon a_m + 2 \sum_{\lambda \mu; m'} \lambda \left( \begin{array}{ccc} \lambda & j & j \\ \mu & -m & m' \end{array} \right) (-)^{j+m} a_{m'} \beta_{\lambda} Y_{\lambda \mu}^*.
\]

In order to diagonalize this set of equations, we are looking for the particle operator in the form

\[
a_m = \sum_k D_{mk}^j c_k,
\]
where \( c_k \) are particle annihilation operators with projection \( k \) onto the axis defined by the phonons, and \( D^j_{mk} \) are Wigner rotational functions.

After simple algebra, this procedure results in the occupation numbers \( n_k \) of the particle in the self-consistent phonon field,

\[
\langle c_k^\dagger c_k \rangle = n_k \delta_{kk'},
\]

and the “Nilsson-type” energies of the odd particle in the body-fixed frame,

\[
\mathcal{E}_k = \epsilon + \sum_\lambda \frac{2}{\sqrt{4\pi(2\lambda + 1)}} \frac{x_\lambda^2}{\omega_\lambda} (-)^{j-k+\lambda} \binom{\lambda}{j}{j}{k}{k'} \rho_\lambda,
\]

similar to how it would appear as a result of a self-consistent multipole-multipole interaction.

The original lab-frame density matrix (31) is diagonalized in the multipole form,

\[
\rho_\lambda = (-)^\lambda \sqrt{\frac{4\pi}{2\lambda + 1}} \sum_k (-)^{j-k} \binom{j}{j}{\lambda}{-k}{k} n_k.
\]

In particular,

\[
\rho_0 = - \sqrt{\frac{4\pi}{2j + 1}} \sum_k n_k,
\]

where we get the total number of unpaired particles (equal to 1 in our simplified case).

The model can be advanced further taking into account the non-adiabatic corrections to the condensate, analogously to the frequency expansion in (24).

Now we can show that there is a strong enhancement of the Schiff matrix element compared to the value \( S^0 \equiv \langle 2||S_1||3 \rangle \) in the even nucleus. The collective contribution to this operator is given by

\[
S_{1\nu} = S^0 \sum_{\mu\mu'} (-)^{\nu+\mu'} \binom{1}{-\nu}{2}{\mu}{3}{-\mu'} (Q_3^\dagger Q_2 + (-)^{\mu+\mu'} Q_2^\dagger Q_3)\rho_{\mu\mu'}. \tag{39}
\]

Using the deformation parameters (32) for the odd nucleus, we obtain

\[
\frac{S_{1\nu}}{S^0} = 2\beta_2 \beta_3 \sum_{\mu\mu'} (-)^{\nu+\mu'} \binom{1}{-\nu}{2}{\mu}{3}{-\mu'} Y_{3\mu'} Y_{2-\mu} = - \frac{1}{\sqrt{\pi}} \beta_2 \beta_3 Y^*_{1\nu}. \tag{40}
\]

With the self-consistent values of deformation parameters (32),

\[
\frac{S_{1\nu}}{S^0} = - \frac{1}{35\sqrt{\pi}} \frac{x_2 x_3}{\omega_2 \omega_3} \rho_2 \rho_3 Y^*_{1\nu}. \tag{41}
\]
Consequently, in the odd nucleus we reduced the problem to that for intrinsic deformation and corresponding intrinsic Schiff moment, compare eq. (5). In contrast to the RPA, the wave function of the odd particle is not fragmented being a self-consistent mean field solution in the presence of the condensate. The enhancement due to small frequencies is clearly present in (41) with no reduction that appeared in the RPA calculation.

VIII. CONCLUSION

We presented the calculations of the Schiff moment in odd-neutron nuclei with soft quadrupole and octupole vibrational modes present in the even-even spherical core. The conventional approach used above is based on the QRPA which allows one to define microscopically the structure of collective modes and coupling of the odd particle to phonons of those modes. We confirmed the effect of enhancement of the Schiff moment for small quadrupole and octupole frequencies. The even-even system close to the onset of deformation acts essentially similarly to the statically deformed system, where the enhancement was established earlier.

However, the QRPA supplemented by the standard ansatz for the (particle + one phonon) states in the odd system, turns out to be inadequate. The enhancement requires strong particle-phonon coupling. In this case, the single-particle strength is widely fragmented, and a small weight of mixing configurations (particle + quadrupole phonon) and (particle + octupole phonon) with the same spherical orbit for the particle practically cancels the enhancement of the Schiff matrix element between $2^+$ and $3^-$ phonons. The outcome for the Schiff moment in this type of calculation is on the same level of magnitude as for purely single-particle states. The QRPA calculations do not predict small energy denominators $\Delta_\pm$ which would provide a strong enhancement of the effect. In general, the spectra of odd-neutron nuclei in the case of soft transitional even core cannot be reliably reproduced by the standard version of the QRPA. The experimental (poorly known) spectra of $^{219,221}$Ra show the states, sometimes with unknown spin, of parity opposite to that of the ground state at $\Delta E \leq 100$ keV.

We demonstrated the presence of the effect by artificially diminishing vibrational frequency to the limit when the single-particle spread is saturated. We also sketched the more general formalism beyond the RPA that is capable of describing the strong coupling limit
for a particle in the field of soft phonons. The self-consistently emerging coherent phonon
state acts analogously to static deformation, in agreement with the original idea. We hope
to present corresponding advanced calculations elsewhere.

We have considered only the lowest symmetric quadrupole phonon in the present paper.
In the presence of few active protons and neutrons, there is a possibility of low-lying “mixed-
symmetry” quadrupole phonon in vibrational nuclei that is known to be connected to the
cubic $3^{-}$ phonon by a considerably stronger E1 strength than the symmetric lowest
quadrupole phonon [35]. Therefore one may expect additional enhancement if the coupling
to the “mixed-symmetry” phonon is taken into consideration.

The conventional core polarization effects are also important for getting a certain quan-
titative prediction, although they are not expected to lead to a significant amplification of
the effect. In the future the full calculations taking into account on equal footing collective
and single-particle effects, including the $\mathcal{P}, \mathcal{T}$-violating corrections to the vibrational modes
and core polarization, are to be carried out. It is desirable to extend the studies to lighter
nuclei $A \approx 100$ with pronounced octupole phonon states and more experimental data
available. In particular, there are indications for the intrinsic coupling between quadrupole
and octupole degrees of freedom, see global systematics [36] and recent experimental data
[37]. This may help to understand better the possibility for the enhancement of the Schiff
moment and to explore physical correlations between the calculated values of the Schiff
moment and complex nuclear structure.

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Michigan State University.

Appendix

Our wave function of the odd nucleus includes two types of states, pure quasiparticle (QP)
and quasiparticle+phonon. Then, any matrix element is a sum of four terms. One term
mixes pure QP components of the wave functions, the other two mix QP and QP+phonon
components, and the last one mixes two QP+phonon components. It is convenient to
illustrate graphically different contributions to the weak matrix element.

1. QP and QP Mixing

Here we calculate the weak matrix element between pure QP components in the mean field approximation \(19\):

\[
ME_1 = \langle QP|\hat{W}_{\text{mean}}|QP \rangle = \langle \psi_0|\alpha J_0^\dagger \alpha \alpha J_0^\dagger \alpha |\psi_0 \rangle \eta^{(+)}_{\mu \nu} W_{\mu \nu} =
\]

\[
= \left( \delta J_0 \delta J_0^\dagger - \langle \psi_0|\alpha J_0^\dagger \alpha \alpha J_0 \alpha |\psi_0 \rangle \right) \eta^{(+)}_{\mu \nu} W_{\mu \nu}.
\]

Using Eq. \(8\) we can express the products of \(\alpha J_0\) via phonon creation and annihilation operators \(Q_\lambda^\dagger\) and \(Q_\lambda\):

\[
ME_1 = \eta^{(+)}_{\mu \nu} W_{\mu \nu} \sum_\lambda (-1)^{J_0 + J_0^\dagger} B_{\mu J_\lambda}^\lambda B_{\nu J_\lambda}^\lambda \frac{2J_\lambda + 1}{2J_0 + 1} \eta^{(+)}_{\mu \nu} W_{\mu \nu}.
\]

The first term corresponds to Fig. \(\text{a}\), while the second term corresponds to Fig. \(\text{b}\). The sum over \(\lambda\) includes all RPA solutions of Eq. \(10\). It is interesting to recalculate this matrix element in a different way:

\[
\langle \psi_0|\alpha J_0^\dagger \alpha \alpha J_0^\dagger \alpha |\psi_0 \rangle = \langle \psi_0|\alpha \alpha J_0^\dagger \alpha J_0^\dagger \alpha |\psi_0 \rangle.
\]

In this case the matrix element will be expressed as

\[
ME_1 = \sum_\lambda (-1)^{J_0 + J_0^\dagger} A_{\mu J_\lambda}^\lambda A_{\nu J_\lambda}^\lambda \frac{2J_\lambda + 1}{2J_0 + 1} \eta^{(+)}_{\mu \nu} W_{\mu \nu}
\]

that corresponds to Fig. \(\text{c}\). Both calculations should give the same result due to the completeness of RPA solutions. If we leave only the most collective phonon in the sum, the
result will be different. In this case we prefer the first way of calculation where the pure QP contribution appears explicitly. In fact, this ambiguity is of minor importance since the weight of the QP component in the wave function of the excited state is rather small. The final result is:

\[ ME_1(J_0, J_0') = \eta^{(+)}_{J_0, J_0'} \hat{W} J_0 J_0' + \sum_{\lambda} (-1)^J_0 + J_\lambda - J_0 \hat{B}_J^\lambda J_0 \hat{B}_J^\lambda J_0' 2J_\lambda + 1 \eta^{(+)}_{\mu\nu} \hat{W}_{\mu\nu}. \]

2. QP and QP + Phonon Mixing

Now we calculate the weak matrix element between pure QP and QP+phonon components:

\[ ME_2 = \langle QP|\hat{W}_{\text{mean}}|QP + \text{phonon}\rangle = \]
\[ = \langle \psi_0|\alpha_{J_0}(\alpha_\mu \alpha_\nu - \alpha_\mu^{+} \alpha_\nu^{+})\alpha_{J_2}^{+} Q_2^{+}\xi_{J_2}^{(-)}\hat{W}_{\mu\nu} C_{J_2 M_2 \lambda_2 \mu_2}. \]

After simple transformations ME2 can be presented as follows:

\[ ME_2(J_0, J_2 \lambda_2) = \frac{\sqrt{2J_2 + 1}}{\sqrt{2J_0 + 1}} \sum_{J_2} \left( \xi_{J_0, J_2}^{(-)} \hat{W}_{J_0, J_2} B_{J_2}^{\lambda_2} - A_{J_0, J_2}^{\lambda_2} \xi_{J_0, J_2}^{(-)} \hat{W}_{J_0, J_2} \right). \]

Here the first term corresponds to Fig. 2a and the second term corresponds to Fig. 2b. Note the different Bogolubov factors that appear here compared to those in the previous
3. QP+Phonon and QP Mixing

Here we calculate the similar weak matrix element between QP+phonon and QP components:

$$ME_3 = \langle QP + \text{phonon}\vert \hat{W}_{\text{mean}}\vert QP \rangle =$$

$$= \langle \psi_0\vert Q_{\lambda_1}^0 \alpha_{J_1}^\dagger (\alpha_\mu^\dagger \alpha_\nu - \alpha_\mu^\dagger \alpha_\nu) \alpha_{J_0}^\dagger \vert \psi_0 \rangle \frac{1}{2} \xi_{\mu\nu} (-) W_{\mu
u} C_{J_0 J_1 \lambda_1 \lambda_1}.$$

The resulting expression is:

$$ME_3(J_0', J_1 \lambda_1) = (-1)^{J_1+J_0} \sqrt{\frac{2A_1+1}{2J_0+1}} \sum_{J'} \left( B_{J_1 J_1'}^\lambda \xi_{J_1 J_0'} (-) W_{J_1 J_0'} - \xi_{J_1 J_1'} (-) W_{J_1 J_1'} A_{J_1 J_0'}. \right).$$

Again, the first term corresponds to Fig. 2c, and the second term corresponds to Fig. 2d.

4. QP+Phonon and QP+Phonon Mixing

There are several terms in this type of the weak matrix element. Some of them are presented in Fig. 3, there are also additional contributions with $\lambda_1 \leftrightarrow \lambda_2$ and $\lambda_1 = \lambda_2$. Corresponding diagrams for $\lambda_1 = \lambda_2$ are similar to those in Fig. 1a,b.

$$ME_4 = \langle QP + \text{phonon}\vert \hat{W}_{\text{mean}}\vert QP + \text{phonon} \rangle =$$
\[ \eta^{(+)}_{\mu \nu} W_{\mu \nu} \langle \psi_0 | Q_{\lambda_1} \alpha_{J_1} \alpha_{\nu}^+ \alpha_{J_2} Q_{\lambda_2}^\dagger | \psi_0 \rangle C^{J_0 M_0}_{J_1 M_1 \lambda_1 \mu_1} C^{J_0 M_0}_{J_2 M_2 \lambda_2 \mu_2}. \]

It is convenient to split this matrix element into several terms,

\[ \langle \psi_0 | Q_{\lambda_1} \alpha_{J_1} \alpha_{\nu}^+ \alpha_{J_2} Q_{\lambda_2}^\dagger | \psi_0 \rangle = \delta_{J_1 \mu} \delta_{J_2 \nu} \langle \psi_0 | Q_{\lambda_1} Q_{\lambda_2}^\dagger | \psi_0 \rangle + \delta_{J_1 J_2} \langle \psi_0 | Q_{\lambda_1} \alpha_{\nu}^+ Q_{\lambda_2}^\dagger | \psi_0 \rangle - \delta_{J_1 \mu} \langle \psi_0 | Q_{\lambda_1} \alpha_{J_2}^+ \alpha_{J_\nu} Q_{\lambda_2}^\dagger | \psi_0 \rangle - \delta_{J_2 \nu} \langle \psi_0 | Q_{\lambda_1} \alpha_{J_\nu}^+ \alpha_{J_1} Q_{\lambda_2}^\dagger | \psi_0 \rangle + \langle \psi_0 | Q_{\lambda_1} \alpha_{\nu}^+ \alpha_{J_1} \alpha_{J_2} Q_{\lambda_2}^\dagger | \psi_0 \rangle. \]

The second term in this equation corresponds to Fig. 3b. Here, the weak interaction mixes 2\(^+\) and 2\(^-\) or 3\(^-\) and 3\(^+\) phonon states. Since we do not include 2\(^-\) and 3\(^+\) phonons in our consideration, we omit this term.

**Part 1:**

\[ \delta_{J_1 \mu} \delta_{J_2 \nu} \langle \psi_0 | Q_{\lambda_1} Q_{\lambda_2}^\dagger | \psi_0 \rangle \rightarrow \delta_{\lambda_1 \lambda_2} \eta^{(+)}_{J_1 J_2} W_{J_1 J_2}. \]

The result here is given by

\[ ME4.1(J_1 \lambda_1, J_2 \lambda_2) = \delta_{\lambda_1 \lambda_2} \eta^{(+)}_{J_1 J_2} W_{J_1 J_2}. \]

The corresponding graph is similar to that of Fig. 4b, where one should add a parallel phonon line and set \( J_1, J_2 \) instead of \( J_0, J_0' \) for the QP lines.

**Part 2:**

\[ -\delta_{J_1 \mu} \langle \psi_0 | Q_{\lambda_1} \alpha_{J_2}^+ \alpha_{J_\nu} Q_{\lambda_2}^\dagger | \psi_0 \rangle \approx -\delta_{J_1 \mu} \langle \psi_0 | Q_{\lambda_1} [\alpha_{J_2}^+ \alpha_{J_\nu}, Q_{\lambda_2}^\dagger] | \psi_0 \rangle. \]

After simple calculation one obtains

\[ ME4.2(J_1 \lambda_1, J_2 \lambda_2) = \sqrt{(2\lambda_1 + 1)(2\lambda_2 + 1)} \sum_{J_a J_b} \eta^{(+)}_{J_1 J_a} W_{J_1 J_a} \]

\[ \left( (-1)^{\lambda_1}(-1)^{J_1 - J_b} A_{J_a J_b}^{\lambda_2} A_{J_b J_1}^{\lambda_1} \left\{ \lambda_1 \ J_b \ J_2 \ \lambda_2 \ J_0 \ J_1 \right\} + (-1)^{\lambda_1}(-1)^{J_1 + J_b} B_{J_a J_b}^{\lambda_1} B_{J_b J_2}^{\lambda_2} \frac{\delta_{J_b J_0}}{2J_0 + 1} \right). \]

The second term here corresponds to Fig. 3a, and the first term corresponds to Fig. 3b with \( \lambda_1 \leftrightarrow \lambda_2 \).

**Part 3:**

\[ -\delta_{J_2 \nu} \langle \psi_0 | Q_{\lambda_1} \alpha_{J_\nu}^+ \alpha_{J_1} Q_{\lambda_2}^\dagger | \psi_0 \rangle \approx -\delta_{J_2 \nu} \langle \psi_0 | Q_{\lambda_1} [\alpha_{J_\nu}^+ \alpha_{J_1}, Q_{\lambda_2}^\dagger] | \psi_0 \rangle. \]

The result is

\[ ME4.3(J_1 \lambda_1, J_2 \lambda_2) = \sqrt{(2\lambda_1 + 1)(2\lambda_2 + 1)} \sum_{ab} \eta^{(+)}_{J_a J_2} W_{J_a J_2} \]

\[ \left( (-1)^{\lambda_1}(-1)^{J_1 - J_b} A_{J_a J_b}^{\lambda_2} A_{J_b J_1}^{\lambda_1} \left\{ \lambda_1 \ J_b \ J_2 \ \lambda_2 \ J_0 \ J_1 \right\} + (-1)^{\lambda_1}(-1)^{J_1 + J_b} B_{J_a J_b}^{\lambda_1} B_{J_b J_2}^{\lambda_2} \frac{\delta_{J_b J_0}}{2J_0 + 1} \right). \]
\[
\left( (-1)^{\lambda_2} (-1)^{J_1-J_0} A_{J_1,J_0}^\lambda A_{J_0,J_1}^\mu \begin{pmatrix}
\lambda_1 & J_b & J_2 \\
\lambda_2 & J_0 & J_1
\end{pmatrix} + (-1)^{\lambda_1} (-1)^{J_1+J_b} B_{J_1,J_b}^\lambda B_{J_b,J_1}^\mu \frac{\delta_{J_0,J_b}}{2 J_0+1} \right).
\]

The second term here corresponds to Fig. 3b, and the first term corresponds to Fig. 3b with \(\lambda_1 \leftrightarrow \lambda_2\). These terms are similar to those calculated in Part 2.

Part 4: This is the most complicated part of the calculation.

\[
\langle \psi_0 | Q_{\lambda_1} \alpha_\mu^\dagger \alpha_{J_2}^\dagger \alpha_{J_1} Q_{\lambda_2}^\dagger | \psi_0 \rangle = \langle \psi_0 | \left( [Q_{\lambda_1}, \alpha_\mu^\dagger \alpha_{J_2}^\dagger] + \alpha_\mu^\dagger \alpha_{J_2}^\dagger Q_{\lambda_1} \right) \left( [\alpha_\nu \alpha_{J_1}, Q_{\lambda_2}^\dagger] + \alpha_{\nu}^\dagger \alpha_{\nu} Q_{\lambda_2} \right) | \psi_0 \rangle =
\]

\[
= \langle \psi_0 | \left( A_{J_1,J_2}^{\lambda_1} C_{J_2,J_1}^{\lambda_2 \mu_1} J_2 M_2 + \alpha_\mu^\dagger \alpha_{J_2}^\dagger Q_{\lambda_1} \right) \left( A_{J_1,J_2}^{\lambda_2} C_{J_2,J_1}^{\lambda_2 \mu_2} J_1 M_1 J_2 M_2 + \alpha_{\nu}^\dagger \alpha_{\nu} Q_{\lambda_2} \right) | \psi_0 \rangle =
\]

\[
= A_{J_1,J_2}^{\lambda_1} A_{J_1,J_2}^{\lambda_2} C_{J_2,J_1}^{\lambda_1 \mu_1} J_2 M_2 + C_{J_2,J_2}^{\lambda_2 \mu_2} J_1 M_1 J_2 M_2 + \zeta C_{J_2,J_1}^{\lambda_1 \mu_1} J_2 J_1 M_2 + C_{J_2,J_2}^{\lambda_2 \mu_2} J_1 J_2 M_1 \langle \psi_0 | Q_{\lambda_3} Q_{\lambda_1} Q_{\lambda_2}^\dagger Q_{\lambda_4}^\dagger | \psi_0 \rangle.
\]

Here

\[
\zeta = (-1)^{\lambda_3+\mu_3+\lambda_4+\mu_4},
\]

and we use the RPA approximations

\[
[Q_{\lambda_1}, \alpha_\mu^\dagger \alpha_{J_2}^\dagger] = A_{J_1,J_2}^{\lambda_1} C_{J_2,J_1}^{\lambda_1 \mu_1} J_2 M_2 + O(\alpha^\dagger \alpha) \approx A_{J_1,J_2}^{\lambda_1} C_{J_2,J_1}^{\lambda_1 \mu_1} J_2 M_2,
\]

\[
[\alpha_\nu \alpha_{J_1}, Q_{\lambda_2}^\dagger] = A_{J_1,J_2}^{\lambda_2} C_{J_2,J_1}^{\lambda_2 \mu_2} J_1 M_1 J_2 M_2 + O(\alpha^\dagger \alpha) \approx A_{J_1,J_2}^{\lambda_2} C_{J_2,J_1}^{\lambda_2 \mu_2} J_1 M_1 J_2 M_2.
\]

After some calculations we obtain

\[
ME.4.4(J_1 \lambda_1, J_2 \lambda_2) = \sqrt{(2 \lambda_1 + 1)(2 \lambda_2 + 1)} \sum_{ab} \eta_{J_0,J_b}^{(+) \dagger} W_{J_a,J_b}
\]

\[
\left( (-1)^{\lambda_1} (-1)^{J_1-J_0} A_{J_1,J_0}^\lambda A_{J_0,J_1}^\mu \begin{pmatrix}
\lambda_1 & J_b & J_2 \\
\lambda_2 & J_0 & J_1
\end{pmatrix} + (-1)^{\lambda_1} (-1)^{J_1+J_b} B_{J_1,J_b}^\lambda B_{J_b,J_1}^\mu \frac{\delta_{J_0,J_b}}{2 J_0+1} \right) +
\]

\[
+ \delta_{\lambda_1,\lambda_2} \sum_{\lambda,\mu} \zeta (-1)^{J_1-J_b} B_{J_1,J_b}^{\lambda} B_{J_1,J_2}^{\mu} B_{J_2,J_1}^{\lambda} \frac{2 J_0 + 1}{2 J_1 + 1} \eta_{\mu}^{(+) \dagger} W_{\mu \nu}.
\]

The second term corresponds to Fig. 3b, and the first term corresponds to Fig. 3b with \(\lambda_1 \leftrightarrow \lambda_2\). The last term corresponds to Fig. 1b with the additional parallel phonon line and \(J_1, J_2\) instead of \(J_0, J_0\) for the QP lines.

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[1] R. Fleischer, Phys. Rep. 370, 537 (2002).
[2] J.S.M. Ginges and V.V. Flambaum, Phys. Rep. 397, 63 (2004).
[3] M.V. Romalis, W.C. Griffith, J.P. Jacobs, and E.N. Fortson, Phys. Rev. Lett. 86, 2505 (2001).
[4] J.P. Jacobs, W.M. Klipstein, S.K. Lamoreaux, B.R. Heckel, and E.N. Fortson, Phys. Rev. A 52, 3521 (1995).
[5] V.V. Flambaum and G.F. Gribakin, Prog. Part. Nucl. Phys. 35 423 (1995).
[6] J.J. Hudson, B.E. Sauer, M.R. Tarbutt, and E.A. Hinds, Phys. Rev. Lett. 89, 023003 (2002).
[7] S.K. Lamoreaux, Phys. Rev. A 66, 022109 (2002).
[8] T.N. Mukhamedjanov and O.P. Sushkov, Phys. Rev. A 72, 034501 (2005).
[9] V.V. Flambaum, I.B. Khriplovich, and O.P. Sushkov, J. Exptl. Theor. Phys. 87, 1521 (1984) [JETP 60, 873 (1984)].
[10] V.V. Flambaum and O.K. Vorov, Phys. Rev. C 49, R1827 (1994).
[11] V.F. Dmitriev and R.A. Sen’kov, Yad. Fiz. 66, 1988 (2003) [Phys. At. Nucl. 66, 1940 (2003)].
[12] V.F. Dmitriev, R.A. Sen’kov, and N. Auerbach, Phys. Rev. C 71, 035501 (2005).
[13] J.H. de Jesus and J. Engel, Phys. Rev. C 72, 045503 (2005).
[14] W.C. Haxton and E.M. Henley, Phys. Rev. Lett. 51, 1937 (1983).
[15] O.P. Sushkov and V.V. Flambaum, Usp. Fiz. Nauk 136, 3 (1982) [Sov. Phys. Usp. 25, 1 (1982)].
[16] G.E. Mitchell, J.D. Bowman, S.I. Pentila, and E.I. Sharapov, Phys. Rep. 354, 157 (2001).
[17] O.P. Sushkov and V.V. Flambaum, Yad. Fiz. 31, 55 (1980) [Phys. At. Nucl. 31, 55 (1980)].
[18] V.V. Flambaum and V.G. Zelevinsky, Phys. Lett. B 350, 8 (1995).
[19] V. Spevak and N. Auerbach, Phys. Lett. B 359, 254 (1995).
[20] N. Auerbach, V.V. Flambaum, and V. Spevak, Phys. Rev. Lett. 76, 4316 (1996).
[21] V. Spevak, N. Auerbach, and V.V. Flambaum, Phys. Rev. C 56, 1357 (1997).
[22] J. Engel, M. Bender, J. Dobaczewski, J.H. de Jesus, and P. Olbratowski, Phys. Rev. 68, 025501 (2003).
[23] J. Dobaczewski and J. Engel, Phys. Rev. Lett. 94, 232502 (2005).
[24] J. Engel, J.L. Friar, and A.C. Hayes, Phys. Rev. C 61, 035502 (2000).
[25] V.V. Flambaum and V.G. Zelevinsky, Phys. Rev. C 68, 035502 (2003).
[26] K. Kaneko, M. Hasegawa, and T. Mizusaki, Phys. Rev. C 66, 051306(R) (2002).
[27] J. Armstrong and V. Zelevinsky, Preprint MSUCL-1125, April 1999; BAPS 44, 397 (1999).
[28] J. Dudek, Z. Szymanski, and T. Werner, Phys. Rev. C 23, 920 (1981).
[29] V.V. Flambaum, I.B. Khriplovich and O.P. Sushkov, Nucl. Phys. A449, 750 (1986).

[30] V.G. Soloviev, *Theory of Atomic Nuclei: Quasiparticles and Phonons* (IOP Publishing, Bristol, 1992).

[31] A. Abbas, N. Auerbach, N. Van Giai, and L. Zamick, Nucl. Phys. A367, 189 (1981).

[32] Ch. Stoyanov and V. Zelevinsky, Phys. Rev. C 70, 014302 (2004).

[33] S.T. Belyaev and V.G. Zelevinsky, Yad. Fiz. 2, 615 (1965) [Sov. J. Nucl. Phys. 2, 442 (1966)].

[34] S.T. Belyaev and V.G. Zelevinsky, Yad. Phys. 11, 741 (1970) [Sov. J. Nucl. Phys. 11, 416 (1970)].

[35] N. Pietralla, C. Fransen, A. Gade, N. A. Smirnova, P. von Brentano, V. Werner, and S. W. Yates, Phys. Rev. C 68, 031305 (2003).

[36] M.P. Metlay *et al.*, Phys. Rev. C 52, 1801 (1995).

[37] W.F. Mueller *et al.*, Phys. Rev. C 73, 014316(R) (2006).