EQUIANGULAR SPIRAL MODES OF POWER LAW DISKS
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The stability of self-similar power-law disks to non-axially symmetrical disturbances of zero frequency ($\omega = 0$) is discussed. It is established that marginally unstable modes either have $\omega = 0$ or consist of continua with modes of all frequencies becoming unstable together. A careful study of all modes close to $\omega = 0$ yields a remarkable conclusion.
1. Introduction.

Following early developments by Brodetsky & Snow, Kalnajs has given a pretty development of the theory of gravitational potential of equiangular spiral waves. Donner, in a fine but unpublished Cambridge thesis (supervised by DLB), gave an account of viscous spiral waves in both the linear and non-linear régimes. The relationship of the vanishing viscosity limit of these waves to the stability problem of spiral structure was not completely elucidated. Zang and Toomre have explored the stability of the $V = \text{constant}$ Mestel disk in stellar dynamics. Zang found $m = 1$ more unstable than the $m = 2$ modes. Toomre used the stable modes to illustrate very beautifully the effects of the swing amplifier.

Spruit in a more recent paper that parallels Donner’s analysis has discussed the non-linear equiangular spiral viscous accretion disk.

We were led to take a new look at the inviscid problem because the dimensional argument of section 3 showed that there is no characteristic frequency of a ‘power-law’ disk. It was then unclear how such a disk could pick out a characteristic frequency for its marginally stable mode. This paper finds all the $\omega = 0$ modes and resolves this question.
2. Self-Similarity.

A flat disk is said to be self-similar if any physical quantity \( q \) defining the configuration at the point \( R, \phi \) is related to that quantity at other points by

\[
q(\lambda R, \phi + f(\lambda)) = A_1(\lambda)q(R, \phi). \tag{2.1}
\]

So the physical quantity at a point \( \lambda \) times further out and rotated by \( f(\lambda) \) is a multiple of that quantity at the original point. When \( \lambda = 1 \) we ask that the transformed point be coincident with the original one so we take \( f(1) = 0 \) and \( A_1(1) = 1 \). Furthermore, while relationship (1) shall be true of all the physical quantities defining the configuration and for all values of \( \lambda \), nevertheless the function \( A_1(\lambda) \) will depend on the physical dimensions mass, length, time, etc. of \( q \) and be such that \( A_1(\lambda) \equiv 1 \) if \( q \) is dimensionless. \( f(\lambda) \) does not depend on \( q \).

Differentiating (2.1) with respect to \( \lambda \) and then putting \( \lambda = 1 \), we deduce

\[
R \frac{\partial q}{\partial R} + \alpha \frac{\partial q}{\partial \phi} = bq, \tag{2.2}
\]

where \( \alpha = f'(1) \) and \( b = A'_1(1) \).

Equation (2.2) may be rewritten

\[
R \frac{\partial Q}{\partial R} + \alpha \frac{\partial Q}{\partial \phi} = 0, \tag{2.3}
\]

where

\[
Q = \ln q - b \ln R = \ln(q R^{-b}). \tag{2.4}
\]

The general solution of (2.3) is

\[
Q = S(\phi - \alpha \ln R), \tag{2.5}
\]

where \( S \) is an arbitrary function so the general solution of (2.2) is

\[
q = R^b F(\phi - \alpha \ln R). \tag{2.6}
\]

Hence in all self-similar disks the pressure, density, velocity of rotation, etc. are all of the form (2.6). Notice that \( S \) and \( F \) are constant on equiangular spirals.

3. Axially-symmetrical self-similar disks in Equilibrium
When we impose axial symmetry the functions $F$ in equation (2.6) all become constants. Our equilibrium disks will have a fluid velocity $V(R)\hat{\phi}$ and centrifugal force together with the pressure gradient in the disk will balance the gravity of the disk. It is useful to define the fictitious circular velocity $V_c(R)\hat{\phi}$ as that velocity which would by itself balance the gravity

$$\frac{V_c^2}{R} = g(R),$$

while

$$\frac{V_0^2}{R} = g(R) + \frac{1}{\Sigma_0} \frac{\partial p_0}{\partial R},$$

where $\Sigma_0(R)$ is the surface density of the disk and $p_0(R)$ is the integral of the pressure through the (negligible) thickness of the disk. In accordance with equation (2.6) we take $V_c$ of the form

$$V_c = B_1 R^{-\beta}$$

and note that because $V_0$ and $V_c$ have the same dimensions, $V_0$ is also proportional to $R^{-\beta}$. Now $p_0/\Sigma_0$ has the dimensions of $V_c^2/R$ so it must vary as $R^{-(2\beta+1)}$. The dimensions of $G \Sigma$ are those of $[GM/R^2]$ and $g$, so $G \Sigma$ varies as $V_c^2/R \propto R^{-(2\beta+1)}$. 

Putting these results together

$$V_0^2/R \propto V_c^2/R = g \propto G \Sigma \propto R^{-(2\beta+1)},$$

$$p_0 = \Sigma_0 \sigma^2 \propto R^{-(4\beta+1)}.$$ 

Inserting these results into (3.2), we find

$$V_0^2 = V_c^2 [1 - (4\beta + 1)\sigma^2/V_c^2].$$

If the pressure decreases outwards $\sigma^2/V_c^2 \leq (4\beta + 1)^{-1}$.

Now the gravitational fields of power-law disks are readily calculated e.g. by Mestel’s method, which gives

$$g = 2\pi G \Sigma_0/L(\beta),$$

where

$$L(\beta) = \frac{\Gamma(1-\beta)\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\beta)\Gamma(\frac{1}{2}-\beta)} \quad \text{and} \quad -\frac{1}{2} < \beta < \frac{1}{2}.$$ 

We have found the approximate formula

$$L(\beta) = (\frac{1}{2} - \beta)(1.88\beta^3 + 1.78\beta^2 + 2.44\beta + 2)/(2\beta + 1)$$

gives a better than 1.2% fit to this function in the required range $-\frac{1}{2} < \beta \leq \frac{1}{2}$ (Table 1).
Table 1

| $\beta$ | $L(\beta)$ | Approximation |
|---------|------------|---------------|
| .5      | 0          | 0             |
| .4      | .188      | .188          |
| .3      | .367      | .368          |
| .2      | .550      | .552          |
| .1      | .754      | .755          |
| 0       | 1.000     | 1.000         |
| -.1     | 1.326     | 1.329         |
| -.2     | 1.818     | 1.829         |
| -.3     | 2.726     | 2.755         |
| -.4     | 5.304     | 5.348         |
| -.5     | $\infty$  | $\infty$     |

Hence

$$2\pi G \Sigma_0 = L(\beta) B_1^2 R^{-(2\beta+1)} = L(\beta) V_c^2 / R$$ \hspace{1cm} (3.10)

$$p_0 = \sigma^2 \Sigma_0 = \left( \frac{\sigma^2}{V_c^2} \right) \left( \frac{L B_1^4}{2\pi G} \right) R^{-(4\beta+1)}.$$ \hspace{1cm} (3.11)

For given values of $B_1$ and $|\beta| < \frac{1}{2}$ and for a (constant) chosen ratio of $\sigma^2/V_c^2 \leq \frac{1}{(4\beta+1)}$, equation (3.3), (3.6), (3.10) and (3.11) define an equilibrium self-similar rotating disk.

The particular interest of the problem

Notice that the only dimensionful constants defining the equilibrium disks are $G = [M^{-1} L^3 T^{-2}]$ and $B_1 = [L^{\beta+1} T^{-1}]$. The other constants involved $\beta$ and $\sigma/V_c$ are both dimensionless. When we are interested in perturbations about the equilibrium we shall take perturbed surface pressures and perturbed surface densities to be related by

$$\Delta p/p_0 = \gamma \Delta \Sigma / \Sigma_0$$

where $\gamma$ is another dimensionless constant. For $\beta \neq 1$ it is not possible to make a characteristic frequency from the properties of the equilibrium because nothing of dimensions $[T^{-1}]$ can be made from $G$ and $B_1$ which characterize the equilibrium.

Now consider an imaginary linear series of equilibria with the same values of $G$, $B_1$ and $\beta$ but with gradually decreasing values of $\sigma^2/V_c^2$ starting from the value $(4\beta+1)^{-1}$ which is entirely pressure supported. We shall take $\gamma > 3/2$ so that the system is then stable. As we decrease the pressure support and increase
the rotation speed to compensate, we know we must encounter a system that is marginally unstable since the system with \( \sigma = 0 \) is unstable to Jeans’s instability at very short wavelengths. Furthermore, we could perform the experiment of exciting modes of only one chosen \( e^{im\phi} \) symmetry. For each \( m \) value there will be a different marginally unstable mode. Each of these must have a natural frequency \( \omega_m \) which might or might not be zero. However, \( \omega_m \) must be a property of the equilibrium configuration and that has no way of making a characteristic frequency since no constants with dimensions \( T^{-1} \) can be made from \( G \) and \( B_1 \). We deduce that the characteristic frequencies of isolated marginally unstable modes, whatever the \( m \) may be, must be zero. This implies that the ‘pattern’ speeds \( \Omega_p = -\omega/m \) of isolated modes must also be zero. As we shall see, this greatly simplifies the calculations. However, there is another strange possibility originally envisaged by Birkhoff in his book. If a mode is not isolated but part of a continuum then it can be that the system first becomes unstable with a whole continuum of modes, each with a different frequency all becoming unstable together. Such modes would be self-similar copies of each other, that of frequency \( \omega_m \) being related to that of frequency \( \omega'_m = \lambda^{-(\beta+1)}\omega_m \) by having a scale in space \( \lambda \) times as large. In this case the system would first go unstable at all scales simultaneously as a result of its self similarity. Which of these possibilities occurs?

For axially-symmetrical modes instability sets in through \( \omega = 0 \) for all inviscid systems with no meridional motions (see e.g. Lynden-Bell & Ostriker). Because of this first, Schmitz (1986, 1988 & 1989) and Schmitz & Ebert (1987), and later we (Lemos et al. 1991) studied just those modes and found that the stability criterion was remarkably close to Toomre’s local one. Furthermore, the radial displacements of the marginally stable modes were of the form

\[
\xi_R \propto R^{(4\beta+1)/2} \cos(\alpha \ln R + \text{const})
\]

or in complex form, \( \xi_R \propto R^Z \) where \( Z = 2\beta + \frac{1}{2} + i\alpha \). For non-axially symmetrical modes there is no such nutcracker theorem to indicate that instability should set in at \( \omega = 0 \) but the above arguments indicate that either \( \omega = 0 \) or the mode structure is truly remarkable with modes of all real frequencies going unstable simultaneously. To spiral structure theorists it would be odd if marginal stability took place at \( \omega = 0 \) for modes of every non-zero \( m \), since this would imply a zero pattern speed for the spiral structure; however, the pattern speed is often associated with a corotation point at or near the edge of the disk and the power law disks are infinite so this could imply a zero pattern speed for them even if this were not generally the case.

These considerations greatly aroused our interest in studying modes with frequencies close \( \omega = 0 \) for the power law disks. As we shall show, the perturbation
equations at $\omega = 0$ are self-similar under a radial scaling, so the $\omega = 0$ modes can be found exactly for every $m$. Having found all these modes we then embark on a more delicate discussion to discover whether such modes are marginally unstable or whether they are ordinary stable modes that happen to have zero frequency in non-rotating axes. If the former were true, we would find the marginally unstable modes so we could deduce the criterion for stability and the form of the instability at each value of $m$. If the latter were true, we would have proved the remarkable result that modes at all frequencies become marginally unstable together so these could be sought at any chosen frequency. We shall adopt the Agatha Christie approach, not giving the answer until we find it. Those spoil-sports who merely want the culprit named can read the end of the paper first.

4. Disturbances with $\omega = 0$

Many years ago now, Lynden-Bell & Ostriker created a powerful formalism for discussing stability problems of this type. They showed that the eigenvalue problem for $\omega$ would be reduced to the solution of the equation

$$-\omega^2 A \cdot \xi + \omega B \cdot \xi + C \cdot \xi = 0$$

where $A$, $B$, and $C$ were Hermitian operators and $\xi e^{i\omega t}$ was the displacement vector involved in the perturbation. $A$ is positive definite. For the particular case in which the undisturbed equilibrium is axially symmetric (i.e. no meridional circulation), $\xi$ may be taken to vary as $e^{im\phi}$ and the different $m$ components each obey a similar equation with simpler operators $A_m$, $B_m$, and $C_m$ replacing $A$, $B$, $C$. Lynden-Bell and Ostriker’s formalism was developed for a 3-dimensional system of density $\rho_0(r)$ so we must integrate in the vertical direction over all $z$ to obtain equations for a flat disk in which case our $\Sigma_0(R)$ replaces $\rho_0(r)$ and all $z$ components vanish. The operator equation then takes the simple form

$$\left(-\omega^2 A_m + \omega B_m + C_m\right) \cdot \begin{pmatrix} \xi_R + i\xi_\phi \\ \xi_R - i\xi_\phi \end{pmatrix} = 0$$

where

$$A_m = \Sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B_m = -2\Sigma_0 \Omega_0 \begin{pmatrix} m + 1 & 0 \\ 0 & m - 1 \end{pmatrix}$$

where $\Omega_0(R) = V_0(R)/R$,

$$C_m = T_m + V_m + P_m$$

7
Here
\[ T_m = -\Sigma_0 \Omega_0^2 \begin{pmatrix} (m+1)^2 & 0 \\ 0 & (m-1)^2 \end{pmatrix} \] (4.6)
while \( V_m \) and \( P_m \) are the operators obtained from \( V \cdot \xi \) and \( P \cdot \xi \) by taking \( \xi \) to behave as \( e^{im\phi} \) and making them operate on the vector \( \zeta = \begin{pmatrix} \xi_R + i\xi_\phi \\ \xi_R - i\xi_\phi \end{pmatrix} \) instead of \( \begin{pmatrix} \xi_R \\ \xi_\phi \end{pmatrix} \).

\( V \cdot \xi \) and \( P \cdot \xi \) are defined by
\[ -V \cdot \xi = \Sigma_0 \Delta(\nabla \psi) \] (4.7)
\[ P \cdot \xi = \Sigma_0 \Delta \left( \frac{1}{\Sigma} \nabla p \right) \] (4.8)
where \( \psi \) is the gravitational potential and \( \Delta \) is the Lagrangian displacement operator
\[ \Delta [Q(r, t)] = Q(r + \xi, t) - Q_0(r, t). \] (4.9)
Here \( Q \) is any quantity evaluated in the perturbed flow, \( Q_0 \) is the same quantity evaluated in the unperturbed flow and \( \xi = \xi(r, t) \). To first order in \( \xi \), \( \Delta \) is related to the Eulerian change \( \delta \) evaluated at the same point in both flows by
\[ \Delta = \delta + \xi \cdot \nabla. \] (4.10)

Lynden-Bell & Ostriker (equation 25) show
\[ P \cdot \xi = \nabla \left[ (1 - \gamma)p_0 \text{div}\xi \right] - p_0 \nabla(\text{div}\xi) - \nabla \left[ (\xi \cdot \nabla)p_0 \right] + (\xi \cdot \nabla)\nabla p_0 \] (4.11)
and from (4.7) and (4.10)
\[ -V \cdot \xi = \Sigma_0 (\xi \cdot \nabla)\nabla \psi_0 + \Sigma_0 \nabla \delta \psi. \] (4.12)
\( \delta \psi \) has to be evaluated in terms of \( \xi \) by use of Poisson’s integral remembering that \( \delta \Sigma = -\text{div}(\Sigma_0 \xi) \). However, although Lynden-Bell & Ostriker used that method to demonstrate that \( V \) was Hermitian, there is a simpler method for the \( \omega = 0 \) modes based on their self-similarity.

The scalars involved in the variational principle of Lynden-Bell & Ostriker are quantities such as \( \int \int \zeta^* T_m \cdot \zeta R d\phi dR \). The operator \( T_m \) itself is proportional
to $\Sigma_0 \Omega_0^2 \propto R^{-(4\beta+3)}$ while the $RdR$ integrations raise this to $R^{-(4\gamma+1)}$. This suggests that $\zeta$ will vary as $R^{+(4\beta+1)/2}$, so we shall hereafter write

$$\zeta = R^{(4\beta+1)/2} \eta.$$  \hspace{1cm} (4.13)

This agrees with the behaviour found for the axially symmetrical modes by Lemos et al. who discuss the interesting behaviour at large $R$. We write $D = R\partial/\partial R$ and after some lengthy calculations done independently by each of us, we obtain

$$P_n \cdot \zeta = R^{2\beta-3/2} \Sigma_0 V_c^2 P_{\Xi_m} \cdot \eta$$  \hspace{1cm} (4.14)

where $P_n \cdot \eta = -\frac{1}{2} \sigma_0^2 M \cdot \left( \begin{array}{c} \eta_+ \\ \eta_- \end{array} \right)$ and $M$ is the matrix

$$\left( \begin{array}{cc} \gamma[(D^2-(\beta_2+m)^2)] + 2\beta_1(1+m) & \gamma[(D-m)^2-\beta_2^2] \\ \gamma[(D+m)^2-\beta_2^2] & \gamma[D^2-(\beta_2-m)^2] + 2\beta_1(1-m) \end{array} \right).$$  \hspace{1cm} (4.15)

Here

$$\sigma_0^2 = \sigma^2/V_c^2, \quad \beta_1 = 4\beta + 1 \text{ and } \beta_2 = 2\beta + 3/2.$$  \hspace{1cm} (4.16)

Similarly

$$T_n \cdot \zeta = R^{2\beta-3/2} \Sigma_0 V_0^2 T_{\Xi_m} \cdot \eta$$  \hspace{1cm} (4.17)

where using (3.6) we find

$$T_{\Xi_m} \cdot \eta = -(1-\beta_1 \sigma_0^2) \left( \begin{array}{c} (m+1)^2 \\ 0 \\ (m-1)^2 \end{array} \right) \cdot \left( \begin{array}{c} \eta_+ \\ \eta_- \end{array} \right).$$  \hspace{1cm} (4.18)

To calculate the gravity term we see from (4.12) that we need an expression for $\delta \psi$ which arises from $\delta \Sigma$.

$$\delta \Sigma = -\text{div}(\Sigma_0 \xi) = \Sigma_0 R^{-1}(2\beta \xi_R - D \xi_R - im\xi_R \phi)$$

$$= -\Sigma_0 R^{2\beta-1/2} \left[ (D + \frac{1}{2}) \eta_R + im\eta_\phi \right]$$  \hspace{1cm} (4.19)

Now $\Sigma_0 \propto R^{-2\beta-1}$, so

$$R^{3/2} \delta \Sigma \propto (D + \frac{1}{2}) \eta_R + im\eta_\phi.$$  \hspace{1cm} (4.20)

Kalnajs has shown that for flat disks the Fourier transform with respect to $u = \ln R$ of $R^{3/2} \delta \Sigma$ is simply related to the Fourier transform in $\ln R$ of $R^{1/2} \delta \psi$. In particular if $R^{3/2} \delta \Sigma \propto e^{iku} e^{im\phi}$, then

$$R^{1/2} \delta \psi = 2\pi G K(k, m) R^{3/2} \delta \Sigma$$  \hspace{1cm} (4.21)
where
\[
K(k, m) = \frac{1}{2} \frac{\Gamma(m + \frac{1}{2} + ik)/2 \Gamma(m + \frac{1}{2} - ik)/2}{\Gamma((m + \frac{3}{2} + ik)/2) \Gamma((m + \frac{3}{2} - ik)/2)} \Gamma\left(\frac{m + \frac{3}{2} + ik}{2}\right) / \Gamma\left(\frac{m + \frac{3}{2} - ik}{2}\right) \Gamma\left(\frac{m + \frac{3}{2} + 1}{2}\right) / \Gamma\left(\frac{m + \frac{3}{2} - 1}{2}\right).
\]

(4.22)

For \( m \geq 2 \) Donner has shown this to be well approximated by
\[
K(k, m) \simeq 1/s
\]
where
\[
s^2 = (k^2 + m^2 + \frac{1}{4}).
\]

(4.24)

To extend Donner’s approximation to \( m = 0, 1 \) we use the exact recurrence relation
\[
K(k, m - 1)K(k, m) = \frac{1}{(m - \frac{1}{2})^2 + k^2}
\]
which follows from the properties of the \( \Gamma \) function. We thus obtain
\[
K(k, 1) = [K(k, 2) \left(\frac{9}{4} + k^2\right)]^{-1} \simeq \sqrt{k^2 + \frac{17}{4}} \left(\frac{k^2 + \frac{9}{4}}{k^2 + \frac{1}{4}}\right)^{-1}
\]

(4.26)

\[
K(k, 0) = [K(k, 1) \left(\frac{1}{4} + k^2\right)]^{-1} \simeq \left[\sqrt{k^2 + \frac{17}{4}}\right]^{-1} \left(\frac{k^2 + \frac{9}{4}}{k^2 + \frac{1}{4}}\right).
\]

(4.27)

These formulae are accurate to better than 1% as Table 2 shows. Thus when \( \eta \propto e^{iku + im\phi} \), we find using (4.20) and (4.21)
\[
R^\beta \delta \psi = 2\pi G K R^\beta \delta \Sigma = -2\pi G K \Sigma_0 R^{2\beta+1}[(ik + \frac{1}{2}\eta_R + im\eta_\phi].
\]

(4.28)

(More generally a superposition of components of different ks may be needed).

Returning to (4.12) and using (3.10), we finally obtain
\[
V_m \cdot \xi - \Sigma_0 V^2 c R^{2\beta=3/2} \eta_m \cdot \eta
\]

(4.29)

where (when only one \( k \) component is present)
\[
\eta_m \cdot \eta = \left[-\frac{1}{2} L K \left(\frac{1}{4} - (ik - m)^2 \right)^2 \left(\frac{1}{4} - (ik + m)^2 \right)^2 \right] \cdot \left(\begin{array}{c}
\beta + 1 \\
\beta + 1
\end{array}\right) \left(\begin{array}{c}
\eta_+ \\
\eta_-
\end{array}\right)
\]

(4.30)

Our operator equation (4.2) can now be written in dimensionless form
\[
\left(-\frac{\omega^2}{\Omega_c^2} a + \frac{\omega b}{\Omega_c} + \xi \right) \cdot \eta = 0,
\]

(4.31)
where \( \Omega_c = V_c/R \),

\[
a_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
b_m = -\frac{2\Omega_0}{\Omega_c} \begin{pmatrix} m+1 & 0 \\ 0 & m-1 \end{pmatrix},
\]

and

\[
c_m(t_m + p_m + v_m).
\]

(4.32)

(4.33)

(4.34)

Notice that when \( \omega = 0 \) only the \( c_m \) term survives. No term in \( t_m, p_m \) or \( v_m \) involves \( u = \ln R \) except through the operator \( D = \partial/\partial u \). Hence these modes have an \( e^{iku} \) behaviour and we may replace the operator \( D \) by \( ik \) where it occurs in \( p_m \).

This justifies our use of this form for \( v_m \) for these modes and reduces the problem of finding them to algebra. To find such \( \omega = 0 \) modes we need the determinant of \( c_m \) to be zero. We find them by calculating the determinant numerically as a function of \( k \). Figure 1 plots the \( k \) so found against \( \beta \). Figure 2 shows the spirals that result. Toomre’s \( Q \) parameter gives the stability of the axially-symmetric modes, both in the local approximation (and also quite accurately, \(< 4\%\), for the global modes of Lemos et al.). To compare with the Local analysis of a gas disk we recall that for such modes

\[
\omega^2 = k_1^2 c^2 - 2\pi G \Sigma_0 |k_1| + \kappa^2
\]

(4.35)

where we use the suffix 1 to distinguish the wave number \( k_1 \) in \( R \) from the wave number \( k \) in \( \ln R \) used earlier. \( c \) is the velocity of sound \( (\gamma p_0/\Sigma_0)^{\frac{1}{2}} \), \( \kappa \) is related to \( \Omega_c \) via

\[
\kappa^2 = 4\Omega_c \left[ \frac{1}{2R} \frac{d}{dR}(\Omega_c R^2) \right] = 2(1 - \beta)\Omega_c^2.
\]

(4.36)

The expression for \( \omega^2 \) may be rewritten

\[
\omega^2 = c^2 \left( k_1 - \frac{\pi G \Sigma_0}{c^2} \right)^2 + (\kappa^2 - \pi^2 G^2 \Sigma_0^2/c^2)
\]

(4.37)

which demonstrates that the least stable wave number is given by \( k_1 = \pi G \Sigma_0/c^2 \) and the condition for stability to axially symmetrical modes is \( Q > 1 \), where

\[
Q = \frac{\kappa c}{\pi G \Sigma_0} = \frac{\sqrt{8(1 - \beta)}}{L(\beta)} \frac{c}{V_c} = \frac{\sqrt{8\gamma(1 - \beta)}}{L(\beta)} \sigma_0.
\]

(4.38)

It is well known that Toomre’s original criterion for stars only differs from this in that 3.36 replaces the value of \( \pi \) in the denominator. Here the gaseous \( Q \)
defined above is the relevant one. It is also interesting to compare the least stable wavenumber \( k'_0 = k_0 R \) with the value of \( k \). From (4.35)

\[
\begin{align*}
  k_0 R &= \frac{\pi G \Sigma_0 R}{c^2} = \frac{1}{2} L(\beta) V^2_c / c^2 = \frac{1}{2} L(\beta) / (\gamma \sigma_0^2) \\
  k_0 R &= \frac{4(1 - \beta)}{Q^2 L(\beta)}
\end{align*}
\]  

(4.39)

When the modes are non-axially symmetrical, the approximation commonly used for tightly wound modes is found from (4.35) or (4.37) by replacing \( \omega \) by \( \omega + \Omega_0(R) \). Equation (4.35) then takes the form

\[
-\frac{\omega^2}{\Omega_c^2} - \frac{\omega}{\Omega_c} 2m \frac{\Omega_0}{\Omega_c} + \left\{ m^2 (1 - \beta_1 \sigma_0^2) - [k_1^2 c^2 - 2\pi G \Sigma_0 |k| + \kappa^2 \frac{1}{\Omega_c^2}] \right\} = 0
\]

apart from the matrix form of equation (4.31) we see there is a close similarity with this approximate dispersion relationship. We therefore expect that \( \omega = 0 \) will occur near where the large curly bracket is zero and that the value of \( k \) for which \( \omega = 0 \) will be approximately

\[
k'_1 = k_1 R = \left[ m^2 (1 - (4\beta + 1) \sigma_0^2) - \frac{\kappa^2}{\Omega_c^2} + \frac{\pi^2 G^2 \Sigma_0^2}{\Omega_c^2 c^2} \right] \frac{1}{2} V_c / c^2 + \frac{\pi G \Sigma_0 R}{c^2}
\]

Re-writing this expression in terms of our dimensionless variables we find

\[
k'_1 = \left[ m^2 (1 - (4\beta + 1) \sigma_0^2) - 2(1 - \beta) + \frac{L(\beta)^2}{4\gamma \sigma_0^2} \right] \frac{1}{\gamma \frac{1}{2} \sigma_0} + \frac{L(\beta)}{2\gamma \sigma_0^2}
\]  

(4.40)

which is compared with the exactly calculated results in Figure 3. \( k'_1 \) given by (4.40) is good approximation for the \( \omega = 0 \) modes in all but the very open spirals. \( k_0 R \) shows no agreement.

5. Marginal Instability

The existence of stationary modes does not imply marginal stability. To ensure the latter we need the limit of growing modes as the growth rate tends to zero. Returning to (4.1) we see that unless \( \mathbf{B} : \xi = 0 \), modes with \( \omega \) close to zero will obey

\[
\omega \mathbf{B} : \xi = -\mathbf{C} : \xi
\]
However $B$ and $C$ are Hermitian and we can see from (4.33) that at least for all modes with $m > 1$, $B$ is negative definite. Hence the eigenvalues $\omega$ are necessarily real and for any eigen $\xi$

$$\omega = -\frac{\langle \xi^* \cdot C \cdot \xi \rangle}{\langle \xi^* \cdot B \cdot \xi \rangle}$$

Thus, provided $\langle \xi^* \cdot B \cdot \xi \rangle \neq 0$, all the eigen frequencies close to $\omega = 0$ are real. Now we may evaluate $\langle \xi^* \cdot B \cdot \xi \rangle$ for our $\omega = 0$ mode it can only be slightly different for the infinitesimally different mode with $\omega \to 0$. Hence, provided $\langle \xi^* \cdot B \cdot \xi \rangle \neq 0$ for our $\omega = 0$ modes, there can be no marginally unstable modes close to $\omega = 0$.

Now in the form (4.33) $b_m$ is diagonal, so all we need to ensure is that for our $\omega = 0$ mode

$$\eta_+^* b_{11} \eta_+ + \eta_-^* b_{22} \eta_- \neq 0$$

multiplying by $c_{22}$ the LHS becomes

$$\eta_+^* \eta_+ (b_{11} c_{22} + b_{22} c_{11})$$

where we have used the fact that $c_{mn} \cdot \eta = 0$ to change the $\eta_-$ terms to $\eta_+$ ones by writing

$$c_{22} \eta_-^* = -c_{12} \eta_+^* \text{ and } c_{12} \eta_- = -c_{11} \eta_+.$$

Thus, provided $\eta_+$ is non zero, the condition $b_{11} c_{22} + b_{22} c_{11} \neq 0$ implies that $\langle \xi \cdot B \cdot \xi \rangle \neq 0$ for the marginal mode. (Even if $\eta_+$ is zero, a similar argument multiplying by $c_{11}$ to start with yields the same result provided $\eta_-$ is non zero and they can not both be zero as then $\eta$ vanishes). Now

$$b_{11} c_{22} + b_{22} c_{11} = \frac{2 \Omega_0 m}{\Omega_c} \left[ -2(m^2 - 1 + \beta) - L K (m^2 + k^2 - \frac{3}{4}) + \sigma_0^2 \{ 2 \beta_1 (m^2 - 1) + \gamma [(k - \beta_2)^2 + m^2] \} \right]$$

To find out whether this can ever be zero for a marginally stable mode we solved the equation $b_{11} c_{22} + b_{22} c_{11} = 0$ for $\sigma_0^2$ and inserted the result into $||c_m||$ for each $k$, the result was negative for every $k$ and every $m \neq 0$.

We deduce that any marginally stable mode with $m \neq 0$ has a non-zero frequency $\omega_m$ and an associated corotation resonance (provided $\omega_m$ is positive) at $R^{\beta+1} = B_1/\omega_m$. The re-scalings of this mode by factors $\lambda$ yield an infinity of marginally stable modes with frequencies $\lambda^{-(\beta+1)} \omega_m$. Thus, marginally stable modes occur simultaneously at all frequencies. We find this result to be unexpected and remarkable. One may now search for a marginally unstable mode by first choosing a frequency; any one is as good as any other; it is unnecessary to search through all possible frequency as one would have to do for a non-self-similar system.
Axially Symmetric Modes

It is of interest to see how the $\omega = 0$ modes studied earlier fit into the present analysis. Setting $m = 0$, $c_m$ becomes a multiple of the matrix \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]
and $b_{11}c_{22} + b_{22}c_{11}$ vanishes. Thus the determinant $||c_m||$ and $<\xi^* \cdot B \cdot \xi>$ both vanish. This is associated with the existence of the trivial neutral displacement with $\xi_\phi$ independent of $\phi$. To find the marginally unstable modes one must expand the determinant

$$\left| \left| \begin{array}{c} a_m \\
\frac{\omega}{\Omega_c} b \\
\end{array} \right| \right| = \left( \frac{\omega}{\Omega_c} \right)^2 (a_{11}b_{22} + a_{22}b_{11}) - \left( \frac{\omega}{\Omega_c} \right)^2 (a_{11}c_{22} + a_{22}c_{11}) = 0$$

but for $m = 0$, $a_{11}b_{22} + a_{22}b_{11} = 0$, so the equation for the non-trivial $\omega = 0$ axially symmetric modes is

$$\frac{\omega^4}{\Omega_c^2} ||a_m|| = (a_{11}c_{22} + a_{22}c_{11}) = 0$$

For $m = 0$, $a_{11}c_{22} + a_{22}c_{11} - b_{11}b_{22} = 0$ yields

$$\sigma_0^2 \left[ \gamma (k^2 + \beta_2^2) - 4\beta_1 \right] + 2 (1 - \beta) - LK (k^2 + \frac{1}{4}) = 0$$

which is the condition of marginal stability for $m = 0$ modes found by Lemos et al. in a slightly different notation.

Conclusion.

In line with the anti-spiral theorem of Lynden-Bell and Ostriker the condition $||c_m|| = 0$ involves $k^2$, $m^2$ and $\Omega_c^2$ but not the signs of $k$, $m$, $\Omega_c$. This implies that to every trailing spiral mode with $\omega = 0$ there corresponds a leading spiral mode with $\omega = 0$ and that we could take pairs of linear combinations of them which are not spiral at all! Only for growing or decaying modes can this symmetry be broken in a dissipationless system.

The non-axially symmetric $\omega = 0$ modes are not the key to the stability problem, rather they are examples of stable steady modes which can be found exactly. However, that result leads to the remarkable conclusion that modes of all frequencies become marginally unstable together, so that marginally stable modes can be sought at any frequency and others can then be found by self-similar scaling.
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| k  | K(k,0)  | Approx | K(k,1) | Approx | K(k,2) | Approx |
|----|---------|--------|--------|--------|--------|--------|
| 0  | 4.377   | 4.366  | 0.914  | 0.904  | 0.486  | 0.485  |
| 0.2| 3.834   | 3.812  | 0.9018 | 0.904  | 0.484  | 0.483  |
| 0.4| 2.810   | 2.799  | 0.868  | 0.871  | 0.478  | 0.476  |
| 0.6| 2.004   | 1.993  | 0.818  | 0.823  | 0.468  | 0.466  |
| 0.8| 1.479   | 1.468  | 0.760  | 0.765  | 0.456  | 0.452  |
| 1  | 1.145   | 1.135  | 0.699  | 0.705  | 0.440  | 0.436  |
| 2  | 0.519   | 0.512  | 0.453  | 0.460  | 0.353  | 0.348  |
| 4  | 0.252   | 0.250  | 0.244  | 0.247  | 0.224  | 0.222  |
| 6  | 0.167   | 0.166  | 0.165  | 0.166  | 0.159  | 0.158  |
| 8  | 0.125   | 0.125  | 0.124  | 0.125  | 0.121  | 0.121  |
| 10 | 0.1001  | 0.0999 | 0.0996 | 0.0999 | 0.0982 | 0.0979 |
Figure Captions

**Figure 1.** $k$ the dimensionless wavenumber in $lnR$ space for $\omega = 0$ waves in power law disks with rotations $V \propto R^{-\beta}$. The ‘ring’ modes $m = 0$ are marginally unstable. The $m \neq 0$, $\omega = 0$ modes are stable.

**Figure 2a.** $\omega = 0$ modes of $V \propto R^{-\beta}$ gas disks: On the left $\beta = 0$, $Q = 1$, $\gamma = 2$; bottom $m = 1$, $k = 7$; second $m = 2$, $k = 9.45$; third $m = 3$, $k = 11.9$; top $m = 4$, $k = 14.5$. On the right $\beta = 0$, $Q = 1.5$, $\gamma = 2$; bottom $m = 1$, $k = 2.6$; second $m = 2$, $k = 4.6$; third $m = 3$, $k = 6.1$; top $m = 4$, $k = 7.7$. Notice that the larger $Q$ gives more open spirals.

**Figure 2b.** $\beta = -0.2$. For equal $Q$ and $m$ the less sheared rotation law gives more open spirals $Q = 1$, $\gamma = 2$, $m = 2$, $k = 5.8$.

**Figure 3.** The dimensionless wavenumber of the exact $\omega = 0$ mode is plotted against $k'_0$, the value of $k_0R$ for *marginal stability* in the local tightly-wound approximation. $k$ and $k'_0$ give disparate curves $k'_1$, the $k_1R$ of the $\omega = 0$ *stable* mode in the tightly wound approximation is a reasonably good approximation for large $k$. 
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