CONSTRUCTING THE DEMAND FUNCTION OF A STRICTLY CONVEX PREFERENCE RELATION

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Abstract. We give conditions under which the demand function of a strictly convex preference relation can be constructed.

Introduction

This paper gives conditions under which the demand function of a strictly convex preference relation can be constructed, and should be seen as a continuation of the work of Douglas Bridges [4, 5, 6, 8] to examine aspects of mathematical economics in a rigorously constructive manner, see also [12]. In particular, Bridges considered the problem that we consider here in [6]. Corollary 12 is a generalisation of the main result of [6] and our proof, although less elegant, is also somewhat simpler.

Following Bridges we take, as our starting point, the standard configuration in microeconomics consisting of a consumer whose consumption set $X$ is a compact, convex subset of $\mathbb{R}^n$ ordered by a strictly ordered preference relation $\succ$. For a given price vector $p \in \mathbb{R}^n$ and a given initial endowment $w$, the consumer’s budget set

$$\beta(p, w) = \{x \in X : p \cdot x \leq w\}$$

is the collection of all consumption bundles available to the consumer.

As detailed in [6], it is easy to show that classically, if $\beta(p, w) \neq \emptyset$, then there exists a unique $\succ$-maximal point $\xi_{p,w} \in \beta(p, w)$: $\xi_{p,w} \succ x$ for all $x \in \beta(p, w)$. Let $T$ be the set of pairs consisting of a price vector $p$ and an initial endowment $w$ for which $\beta(p, w)$ is inhabited. If the preference relation $\succ$ is continuous, then a sequential compactness argument gives the sequential, and hence pointwise, continuity of the demand function $F$ on $T$ which sends $(p, w)$ to the maximal element $\xi_{p,w}$ of $\beta(p, w)$ (see, for example, chapter 2, section D of [16]).

Bridges asked under what conditions can we

1. Compute the demand function $F$;
2. Compute a modulus of uniform continuity for $F$: given $\varepsilon > 0$, can we produce $\delta > 0$ such that if $(p, w), (p', w') \in T$ with $\|(p, w) - (p', w')\| < \delta$, then $\|F(p, w) - F(p', w')\| < \varepsilon$.

In [6] Bridges introduced the notion of a uniformly rotund preference relation and showed that if $\succ$ is uniformly rotund and you restrict $F$ to a compact subset of $T$ on which the consumer cannot be satiated, then $F$ is uniformly continuous. Theorem 12 shows that we do not need the hypothesis that our consumer is nonsatiated. Theorems 1 and 9 encapsulate what we can say about strictly convex preference relations, which is more than one might think.

We work in Bishop’s style constructive mathematics. Any proof in this framework embodies an algorithm, so when we show that there exists $x$ such that $P(x)$, our proof gives an explicit construction of an object $x$ together with a proof that $P(x)$ holds. Formally we take Bishop’s constructive mathematics to be Aczel’s constructive Zermelo-Fraenkel set theory (CZF) with intuitionistic logic and the axiom of dependent choice [2]. By interpreting CZF in Martin-Löf type theory [1], the
A preference relation $\succ$ on a set $X$ is a binary relation which is

- asymmetric: if $x \succ y$, then $\neg(y \succ x)$;
- negatively transitive: if $x \succ y$, then for all $z$ either $x \succ z$ or $z \succ y$.

If $x \succ y$, we say that $x$ is preferable to $y$. We write $x \succeq y$, $x$ is preferable or indifferent to $y$, for $\neg(y \succ x)$. We note that $x \succ x$ is contradictory, that $\succ$ and $\succeq$ are transitive, and that if either $x \succeq y \succ z$ or $x \succ y \succ z$, then $x \succ z$.

Let $\succ$ be a preference relation on a subset $X$ of $\mathbb{R}^N$.

- $\succ$ is a continuous preference relation if the graph
  \[
  \{(x, x') : x \succ x'\}
  \]
  of $\succ$ is open.
- $\succ$ is strictly convex if $X$ is convex and $tx + (1-t)x' \succ x$ or $tx + (1-t)x' \succ x'$ whenever $x, x' \in X$ with $x \neq x'$ and $t \in (0, 1)$.
- $X$ is uniformly rotund if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$, if $\|x - x'\| \geq \varepsilon$, then
  \[
  \left\{ \frac{1}{2} (x + x') + z : z \in B(0, \delta) \right\} \subset X,
  \]
  where $B(x, r)$ is the open ball of radius $r$ centred on $x$. The preference relation $\succ$ is uniformly rotund if $X$ is uniformly rotund and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - x'\| \geq \varepsilon$ ($x, x' \in X$), then for each $z \in B(0, \delta)$ either $\frac{1}{2} (x + x') + z \succ x$ or $\frac{1}{2} (x + x') + z \succ x'$.

A uniformly rotund preference relation is strictly convex.

A set $S$ is said to be inhabited if there exists $x$ such that $x \in S$. An inhabited subset $S$ of a metric space $X$ is located if for each $x \in X$ the distance

\[
\rho(x, S) = \inf \{\rho(x, s) : s \in S\}
\]

from $x$ to $S$ exists. If $X$ is located and its metric complement

\[-X = \{x \in \mathbb{R}^n : \rho(x, X) > 0\}\]

is also located, then $X$ is said to be bilocated. An $\varepsilon$-approximation to $S$ is a subset $T$ of $S$ such that for each $s \in S$, there exists $t \in T$ such that $\rho(s, t) < \varepsilon$. We say that $S$ is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable $\varepsilon$-approximation to $S$; totally bounded sets are located. A metric space $X$ is compact if it is complete and totally bounded. We will use $\| \cdot \|$ to represent the Euclidean norm, $\| \cdot \|_1$ for the norm $x \mapsto \sum_{i=1}^n x_i$ on $\mathbb{R}^n$, and we write $\rho, \rho_1$ for the respective induced metrics.

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1A set is finitely enumerable if it is the image of $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$, and a set is finite if it is in bijection with $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$; constructively these notions are distinct.
Constructing maxima

In this section we focus on the construction of maximally preferred elements of a consumption set \( X \). Our main result is

**Theorem 1.** Let \( \succ \) be a continuous, strictly convex preference relation on an inhabited, compact subset \( X \) of Euclidean space. Then there exists a unique \( \xi \in X \) such that \( \xi \succ x \) for all \( x \in X \).

Our proof proceeds by induction. The following lemma provides the key to proving the one dimensional case.

**Lemma 2.** Let \( \succ \) be a strictly convex preference relation on \([0, 1]\). Then either \( 1/2 \succ x \) for all \( x \in [0,1/4) \) or \( 1/2 \succ x \) for all \( x \in (3/4, 1] \).

**Proof.** Applying the strict convexity of \( \succ \) to \( 1/4 \in (0, 3/4), 1/2 \in (1/4, 3/4), 3/4 \in (1/2, 1) \) yields

\[
\frac{1}{4} > 0 \quad \text{or} \quad \frac{1}{4} > \frac{3}{4}; \\
\frac{1}{2} > \frac{1}{4} \quad \text{or} \quad \frac{1}{2} > \frac{3}{4}; \\
\frac{3}{4} > \frac{1}{4} \quad \text{or} \quad \frac{3}{4} > 1.
\]

It follows that either \( 1/2 > 1/4 > 0 \) or \( 1/2 > 3/4 > 1 \). In the first case suppose that \( x \succ 1/2 \) for some \( x \in [0, 1/4) \). Then, by the strict convexity and transitivity of \( \succ \), \( 1/4 \succ x \); this contradiction ensures that \( 1/2 \nleq x \) for all \( x \in [0, 1/4) \). Similarly, in the second case \( 1/2 \nleq x \) for all \( x \in (3/4, 1] \). \( \square \)

**Lemma 3.** If \( \succ \) is a strictly convex, continuous preference relation on \([0, 1]\), then there exists \( \xi \in [0, 1] \) such that \( \xi \succ x \) for all \( x \in [0, 1] \).

**Proof.** We inductively construct intervals \([\xi_n, \bar{\xi}_n]\) such that, for each \( n \),

1. \( [\xi_n, \bar{\xi}_n] \subset [\xi_{n-1}, \bar{\xi}_{n-1}] \);
2. \( \bar{\xi}_n - \xi_n = (4/5)^n \);
3. for each \( x \in [0, 1] \setminus [\xi_n, \bar{\xi}_n] \), there exists \( y \in [\xi_n, \bar{\xi}_n] \) such that \( y \nleq x \).

To begin the construction set \( \xi_0 = 0 \) and \( \bar{\xi}_0 = 1 \). At stage \( n \), rescaling for \( n > 1 \), we apply Lemma 2 if the first case obtains, then we set \( \xi_n = (3\xi_{n-1} + \bar{\xi}_{n-1})/4 \) and \( \bar{\xi}_n = \xi_{n-1} \). In the second case we set \( \xi_n = \xi_{n-1} \) and \( \bar{\xi}_0 = (\xi_{n-1} + 3\bar{\xi}_{n-1})/4 \). By the transitivity of \( \succ \), we need only check condition 3. for \( [\xi_{n-1}, \bar{\xi}_{n-1}] \setminus [\xi_n, \bar{\xi}_n] \), and by Lemma 2 \( y = (\xi_{n-1} + \bar{\xi}_{n-1})/2 \) suffices for each such point.

Let \( \xi \) be the unique intersection of the \([\xi_n, \bar{\xi}_n] \). Since \( \succ \) is continuous, the maximality of \( \xi \) follows from 3. \( \square \)

**Lemma 4.** If \( \succ \) is a strictly convex, continuous preference relation on \([a, b]\), where \( a \leq b \), then there exists \( \xi \in [a, b] \) such that \( \xi \succ x \) for all \( x \in [a, b] \).

**Proof.** Construct an increasing binary sequence \((\lambda_n)_{n \geq 1}\) such that

\[
\lambda_n = 0 \quad \Rightarrow \quad b - a < 1/n; \\
\lambda_n = 1 \quad \Rightarrow \quad b - a > 1/(n + 1).
\]

Without loss of generality, we may assume that \( \lambda_1 = 0 \). If \( \lambda_n = 0 \), set \( x_n = a \) and if \( \lambda_n = 1 - \lambda_{n-1} \), then we apply Lemma 3 after some scaling, to construct a \( \succ \)-maximal element \( x \) in \([a, b]\), and set \( x_k = x \) for all \( k \geq n \). Then for \( m > n \), \(|x_m - x_n| < 2/(m - 1)\), so \((x_n)_{n \geq 1}\) converges to some element \( \xi \in [a, b] \). If there exists \( x \neq \xi \) such that \( x \succ \xi \), then \( b - a > 0 \) and we get a contradiction to Lemma 3. The result now follows from continuity. \( \square \)
We use $\pi_i$ to denote the $i$-th projection function, and we write $[x, y]$ for
\[
\{tx + (1-t)y : t \in [0, 1]\}.
\]
Here is the proof of Theorem 1.

**Proof.** We proceed by induction on the dimension $n$ of the space containing $X$. Lemma 4 is just the case $n = 1$. Now suppose we have proved the result for $n$ and consider a strictly convex preference relation $\succ$ on a compact, convex subset $X$ of $\mathbb{R}^n$. Define a preference relation $\succ'$ on $\pi_1(X) = [a, b]$ by
\[
s \succ_i t \iff \exists x \in X \forall y \in X \ (\pi_1(x) = s \text{ and if } \pi_1(y) = t, \text{ then } x \succ y).
\]
Then $\succ'$ is strictly convex and sequentially continuous: let $s_1, s_2, t \in [a, b]$ with $s_1 < t < s_2$. By the induction hypothesis there exist $\xi_1, \xi_2$ such that $\pi_1(\xi_i) = s_i$ and $\xi_i \succ x$ for all $x \in X$ with $\pi_1(x) = s_i$ ($i = 1, 2$). Let $z$ be the unique element of $[\xi_1, \xi_2]$ such that $\pi_1(z) = t$. Then, by the strict convexity of $\succ$, either $z \succ \xi_1$ or $z \succ \xi_2$. In the first case $t \succ' s_1$ and in the second $t \succ' s_2$. Hence $\succ$ is strictly convex. That $\succ'$ is continuous is straightforward.

We can now apply Lemma 4 to construct a maximal element $\xi_1$ of $(\pi_1(X), \succ')$, and then the induction hypothesis to construct a maximal element of
\[
S = \{x \in X : \pi_1(x) = \xi_1\}
\]
with $\succ | S$. Clearly $\xi = \xi_1 \times \xi_2$ is a $\succ$-maximal element of $X$. The uniqueness of maximal elements follows directly from the strict convexity of $\succ$.

We shall have need for the following simple corollary, which is of independent interest.

**Corollary 5.** Under the conditions of Theorem 4, if $x \in X$ and $x \not\succ \xi$, then $\xi \succ x$.

**Proof.** Let $y = (x + \xi)/2$. Then either $y \succ x$ or $y \succ \xi$. Since $\xi \succ y$ the former must attain, so $\xi \succ y \succ x$. \qed

If we are not interested in uniqueness of maxima, then we might suppose that $\succ$ only satisfies the weaker condition of being *convex*: for all $x, y \in X$ and each $t \in [0, 1]$, either $(x + y)/2 \succ x$ or $(x + y)/2 \succ y$. We give a Browuerian counterexample\(^2\) to show that this condition is not strong enough to allow the construction of a maximal point. Let $x \in (-1/4, 1/4)$ and let $f : [0, 1] \to \mathbb{R}$ be the function given by
\[
f(t) = \begin{cases} 
sign(x)(t - x \lor 0) & t \in [0, x \lor 0] \\
0 & t \in [x \lor 0, 1 - x \lor 0] \\
-sign(x)(t - x \lor 0) & t \in [1 - x \lor 0, 1],
\end{cases}
\]
where $\text{sign}$\(^3\)
\[
\text{sign}(x) = \begin{cases} 
-1 & x < 0 \\
0 & x = 0 \\
1 & x > 0.
\end{cases}
\]
Define a preference realtion $\succ$ on $[0, 1]$ by
\[
t \succ s \iff f(t) > f(s).
\]

\(^2\)A Browuerian counterexample is a weak counterexample: it is not an example contradicting a proposition, but an example showing a proposition to imply a principle which is unacceptable in constructive mathematics. Generally these can be considered as unprovability results.

\(^3\)This is just convenient notation: formally sign is not a constructively well defined function, but the function $f$ does exist constructively.
It is easy to see that $\succ$ is continuous and convex. Further, if $x > 0$, then 0 is the unique maximal element, and if $x < 0$, then 1 is the unique maximal element. Now suppose that we can construct $\xi \in [0, 1]$ such that $\xi \succ t$ for all $t \in [0, 1]$: either $\xi > 0$ or $\xi < 1$. In the first case we have $\neg(x > 0)$ and in the second $\neg(x < 0)$, so the statement

‘Every continuous, convex preference relation on $[0, 1]$ has a maximal element’

implies $\forall x \in \mathbb{R} (x \leq 0 \lor x \geq 0)$, which is equivalent to the constructively unacceptable lesser limite principle of omniscience [9].

Continuous demand functions

We now consider a consumer whose consumption set $X$ is a closed convex subset of $\mathbb{R}^n$ ordered by a strictly convex preference relation $\succ$, and who has an initial endowment $w \in \mathbb{R}$. For a given price vector $p \in \mathbb{R}$, a consumers budget set

$$\beta(p, w) = \{x \in X : p \cdot x \leq w\}$$

is the collection of commodity bundles the consumer can afford. The collection of maximal elements of $\beta(p, w)$ form the consumers demand set for price $p$ and initial endowment $w$.

**Lemma 6.** If $p > 0$ and there exists $x \in X$ such that $p \cdot x \leq w$, then $\beta(p, w)$ is compact and convex.

**Proof.** Convexity is clear. See [6] for a proof that $\beta(p, w)$ is compact. $\square$

We use $\partial S$ to denote the boundary of a subset $S$ of some metric space.

**Lemma 7.** The boundary of $\beta(p, w)$ is compact.

**Proof.** If $X$ is colocated, then $\rho(x, \partial X) = \max\{\rho(x, X), \rho(x, -X)\}$ and hence the boundary of $X$ is located. Therefore it suffices to show that $-\beta(p, w)$ is located. This is similar to the proof of Lemma [6] $\square$

It now follows from Theorem [1] that the function $F$, the consumers demand function, that maps $(p, w)$, where $p$ is a price vector and $w$ an initial endowment, to the unique maximal element of $\beta(p, w)$, is well defined. By logical considerations we have that any function which can be proven to exist within Bishop’s constructive mathematics alone is classically continuous, so the consumers demand function is continuous in the classical setting.

We seek conditions under which $F$ is constructively continuous. A function on a locally compact space is said to be Bishop continuous if it is pointwise continuous, and is further uniformly continuous on every compact space. Since the uniform continuity theorem—every continuous function on a compact space is uniformly continuous—is not provable in Bishop’s constructive mathematics, this is the natural notion of continuity for us to consider. We study the continuity of $F$ by looking at the map $\Gamma$, on the set $T$ of all inhabited $\beta(p, w)$, taking $\beta(p, w)$ to $F(p, w)$. We give $T$ the Hausdorff metric: for located subsets $A, B$ of a metric space $Y$

$$\rho_H(A, B) = \max \{\sup \{\rho(a, B) : a \in A\}, \sup \{\rho(b, A) : b \in B\}\}.$$

Our next lemma shows how studying $\Gamma$ allows us to show the continuity of $F$.

**Lemma 8.** If $\Gamma$ is continuous, then $F$ is continuous. If $\Gamma$ is uniformly continuous, then for each $p \in \mathbb{R}^n$, $w \mapsto F(p, w)$ is uniformly continuous, and for each $w \in \mathbb{R}$, $p \mapsto F(p, w)$ is Bishop continuous.
In constructive mathematics, the uniform continuity theorem—every pointwise continuous function with compact domain is uniformly continuous—is closely related to the ‘semi-constructive’ fan theorem isolated by Brouwer. In the appendix we introduce Brouwer’s fan theorem (FT) and the notion of a weakly (uniformly) continuous predicate, and we give a version of the uniform continuity theorem for these predicates. Our next result says that adopting Brouwer’s fan theorem is sufficient to prove the classical result that $F$ is continuous when $\succ$ is continuous and strictly convex. We observe that if $\beta(p, w)$ is inhabited and every component of $p$ is positive, then

$$\beta(p, w) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} p_i x_i \leq w \right\}$$

is a diamond, and if the diameter

$$\sup\{\rho(x, y) : x, y \in \beta(p, w)\}$$

of $\beta(p, w)$ is positive, then $\beta(p, w)$ has inhabited interior.

**Theorem 9.** Suppose Brouwer’s full fan theorem holds. If $\succ$ is continuous and strictly convex, then $F$ is Bishop continuous.

**Proof.** Since FT implies that every continuous function on a compact space is uniformly continuous, it suffices, by Lemma 8, to show that $\Gamma$ is continuous. Fix $\varepsilon > 0$, and $(p, w) \in \mathbb{R}^{n+1}$ such that $\beta(p, w)$ is inhabited; we write $S = \beta(p, w)$ and $\xi = F(p, w)$.

Either $\rho(\xi, \partial S) > 0$ or $\rho(\xi, \partial S) < \varepsilon/2$. In the first case, let $\varphi$ be the natural bijection of $[0, 1]^n$ with $T \equiv \partial \beta(p, w) \setminus B_{\rho_1}(x, \varepsilon/2)$; without loss of generality, $\varphi$ is nonexpansive. We define a predicate on $[0, 1]$ by

$$P(x, \alpha, \delta) \iff \forall y \in B(\varphi(x), \delta) \xi \succ y.$$  

Then $P$ is a weakly continuous predicate: condition (i) follows from Corollary 5 and the lower pointwise continuity of $\succ$; condition (ii) follows from elementary geometry, given that $\varphi$ is nonexpansive. By Theorem 17, $P$ is weakly uniformly continuous and hence there exists $\delta > 0$ such that every $y \in B(x, \delta)$ is strictly less preferable than $\xi$ for all $x \in T$. If $\rho(x, S) < \min\{\delta, \varepsilon\}/2$, then $\rho(x, T) < \delta$, $x \in S$, or $x \in B(\xi, \varepsilon)$. In the first two cases $\xi \succ x$; it follows that $F(p', w') \in B(\xi, \varepsilon)$ whenever $\rho_H(\beta(p, w), \beta(p', w')) < \min\{\delta, \varepsilon\}/2$. \hfill \square

It may seem a little odd that we choose to work in Bishop’s constructive mathematics because we are interested in producing results with computational meaning, but that we then add an extra principle FT to our framework. In particular, the inconsistency of Brouwer’s fan theorem with recursive analysis 2 may cause some consternation. The constructive nature of the fan theorem can be intuitively justified as follows: in order to assert that $B$ is a bar we must have a proof that $B$ is a bar, and a proof is a finite object; therefore an examination of the finite information used in the proof that $B$ is a bar should reveal the uniform bound that the fan theorem gives us. Although this argument does not hold up in Bishop’s constructive mathematics, if your objects are presented in the right way (and indeed a very nature way from a computational point of view), then the fan theorem can be proved 11 17.

We pause here to give a consequence of Theorem 9. Consider a system with $N$ commodities, $n$ producers, and $m$ consumers. To each producer we associate a production set $Y_i \subset \mathbb{R}^N$; and to each consumer a consumption set $X_i \subset \mathbb{R}^N$ endowed with a preference relation $\succ_i$. Further we assume that each consumer has no initial endowment. A competitive equilibrium of an economy consists of a price vector $p \in \mathbb{R}^N$, points $\xi_1, \ldots, \xi_t \in \mathbb{R}^N$, and a vector $\eta$ in the aggregate production set

$$Y = Y_1 + \cdots + Y_n,$$
satisfying
\[ E1 \ \xi_i \in D_i(p) \text{ for each } 1 \leq i \leq m. \]
\[ E2 \ \mathbf{p} \cdot y \leq \mathbf{p} \cdot \eta = 0 \text{ for all } y \in Y. \]
\[ E3 \ \sum_{i=1}^{m} \xi_i = \eta. \]

An economy is said to have approximate competitive equilibria if for all \( \varepsilon > 0 \) there exist a price vector \( \mathbf{p} \in \mathbb{R}^N \), points \( \xi_1, \ldots, \xi_i \in \mathbb{R}^N \), and a vector \( \eta \) satisfying \( E1, E3 \), and
\[ AE \ \mathbf{p} \cdot \eta > -\varepsilon. \]

The work in [12] together with Theorem [9] gives the next result, which is an approximate version of McKenzie’s theorem on the existence of competitive equilibria [14].

**Theorem 10.** Assume that Brouwer’s fan theorem holds. Suppose that

(i) each \( X_i \) is compact and convex;
(ii) each \( \succ_i \) is continuous and strictly convex;
(iii) \( (X_i \cap Y)^\circ \) is inhabited for each \( i \);
(iv) \( Y \) is a located closed convex cone;
(v) \( Y \cap \{(x_1, \ldots, x_N) : x_i \geq 0 \text{ for each } i\} = \{0\} \); and
(vi) for each \( \mathbf{p} \in \mathbb{R}^N \) and each \( i \), if \( \sum_{i=1}^{m} F_i(p) \in Y \), then there exists \( x_i \in X_i \) such that \( x_i \succ_i F_i(p) \).

Then there are approximate competitive equilibria.

**Uniformly rotund preference relations.** In order to prove Theorem [9] we effectively strengthened our theory, and therefore weakened our notion of computable. The other natural approach toward proving the existence of a Bishop continuous demand function is to strengthen the conditions on \( \succ \). We follow the lead of Bridges in [6] and focus on uniformly rotund preference relations.

Hereafter, we extend the domain of \( \Gamma \) to all inhabited, compact, convex subsets of \( X \). Theorem [11] still ensures that \( \Gamma \) is well defined.

**Theorem 11.** If \( \succ \) is a uniformly rotund preference relation, then \( \Gamma \) is uniformly continuous.

**Proof.** Let \( S, S' \) be compact, convex subsets of \( X \) and let \( \xi, \xi' \) be their \( \succ \)-maximal points. Fix \( \varepsilon > 0 \) and let \( \delta' > 0 \) be such that if \( \|x - x'\| \geq \varepsilon \) (\( x, x' \in X \)), then for each \( z \in B(0, \delta') \) either \( \frac{1}{2}(x + x') + z \succ x \) or \( \frac{1}{2}(x + x') + z \succ x' \), and set \( \delta = \min\{\varepsilon, \delta'\}/2 \).

If \( \rho_H(S, S') < \delta \), then \( \|\xi - \xi'\| \). Let \( S, S' \) be such that \( \rho_H(S, S') < \delta \) and suppose that \( \|\xi - \xi'\| \geq \varepsilon \). Since \( S, S' \) are convex
\[ S \cap B((\xi + \xi')/2, \delta) \text{ and } S' \cap B((\xi + \xi')/2, \delta) \]
are both inhabited; let \( z \) be an element of the former set and let \( z' \) be an element of the latter. By the maximality of \( \xi \in S \) and our choice of \( \delta, z \succ \xi' \); similarly, \( z' \succ \xi \). Therefore
\[ \xi \succ z \succ \xi' \succ z' \succ \xi, \]
which is absurd. Hence \( \|\xi - \xi'\| \leq \varepsilon. \]

As a corollary we have the following improvement on the main result of [6].

**Corollary 12.** Let \( \succ \) be a uniformly rotund preference relation on a compact, uniformly rotund subset \( X \) of \( \mathbb{R}^n \), and let \( S \) be a subset of \( \mathbb{R}^n \times \mathbb{R} \) such that \( \beta(p, w) \) is inhabited for each \( (p, w) \in S \). Then for each \( p \in \mathbb{R}^n \), the function \( w \mapsto F(p, w) \) is uniformly continuous, and for each \( w \in \mathbb{R} \), the function \( p \mapsto F(p, w) \) is Bishop continuous. In particular, \( F \) is Bishop Continuous.

**Proof.** The result follows directly from Lemma [8] and Theorem [11].
Not surprisingly, a less uniform version of rotundness is enough to give us the pointwise continuity of $\Gamma$. A subset $X$ of $\mathbb{R}^n$ is rotund if for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x' \in X$, if $\|x - x'\| \geq \varepsilon$, then
\[
\left\{ \frac{1}{2} (x + x') + z : z \in B(0, \delta) \right\} \subset X.
\]
A preference relation $\succ$ is rotund if $X$ is rotund and for each $x \in X, \varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - x'\| \geq \varepsilon (x' \in X)$, then for each $z \in B(0, \delta)$ either $\frac{1}{2} (x + x') + z \succ x$ or $\frac{1}{2} (x + x') + z \succ x'$.

**Theorem 13.** If $\succ$ is a rotund preference relation, then $\Gamma$ is continuous.

*Proof.* The proof is, of course very similar to the proof of Theorem 11. Let $S$ be a compact, convex subset of $X$ and let $\xi$ be the unique maximal element of $S$. Fix $\varepsilon > 0$. Pick $\delta > 0$ such that if $\|\xi - x\| \geq \varepsilon (x \in X)$, then for each $z \in B(0, \delta)$ either $\frac{1}{2} (\xi + x) + z \succ x$ or $\frac{1}{2} (\xi + x) + z \succ x'$. If $S'$ is a compact, convex subset of $X$, with maxima $\xi'$, such that $\rho_H(S, S') < \delta$, then the assumption that $\|\xi - \xi'\| > \varepsilon$ leads to a contradiction as in the proof of Theorem 11. $\square$

By the next result, Theorem 11 can be used to improve on Theorem 9.

**Proposition 14.** Assume Brouwer’s fan theorem. If $\succ$ is continuous and strictly convex, then $\succ$ is uniformly rotund.

*Proof.* Without loss of generality,
\[
C = \{ (x, y) \in X^2 : \|x - y\| \geq \varepsilon \}
\]
is compact; moreover
\[
P((x, y), \varepsilon, \delta) \equiv \|x - y\| < \varepsilon \lor \forall z \in B((x + y)/2, \delta) (z \succ x \lor z \succ y)
\]
defines a continuous predicate on $C$. Hence $P$ is uniformly continuous by Theorem 17, but the uniformity of $P$ says precisely that $\succ$ is uniformly rotund. $\square$

**Corollary 15.** Suppose Brouwer’s full fan theorem holds. If $\succ$ is continuous and strictly convex, then $\Gamma$ is uniformly continuous.

**Appendix: The fan theorem and continuous predicates.** Let $2^\mathbb{N}$ denote the space of binary sequences, Cantor’s space, and let $2^*$ be the set of finite binary sequences. A subset $S$ of $2^*$ is decidable if for each $a \in 2^*$ either $a \in S$ or $a \notin S$. For two elements $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_n) \in 2^*$ we denote by $u \concat v$ the concatenation
\[
(u_1, \ldots, u_m, v_1, \ldots, v_n)
\]
of $u$ and $v$. For each $\alpha \in 2^\mathbb{N}$ and each $N \in \mathbb{N}$ we denote by $\overline{\alpha}(N)$ the finite binary sequence consisting of the first $N$ terms of $\alpha$. A set $B$ of finite binary sequences is called a bar if for each $\alpha \in 2^\mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\overline{\alpha}(N) \in B$. A bar $B$ is said to be uniform if there exists $N \in \mathbb{N}$ such that for each $\alpha \in 2^\mathbb{N}$ there is $n \leq N$ with $\alpha(n) \in B$. The strongest form of Brouwer’s fan theorem is:

FT: Every bar is uniform.

Brouwer introduced the fan theorem as a constructive principle and gave a philosophical justification for its use; it is no longer considered a valid principle of constructive mathematics, but is still used freely by some schools (see [9]).

A predicate $P$ on $S \times \mathbb{R}^+ \times \mathbb{R}^+$ is said to be a continuous predicate on $S$ if

(i) for each $x \in S$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that $P(x, \varepsilon, \delta)$;
(ii) if \(P(x, \varepsilon, \delta)\) and \(|x - y| < \delta' < \delta\), then \(P(y, \varepsilon, \delta - \delta')\).

If in addition, for each \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(P(x, \varepsilon, \delta)\) for all \(x \in S\), then \(P\) is a uniformly continuous predicate on \(S\).

**Theorem 16.** The statement

Every continuous predicate on \([0,1]\) is uniformly continuous.

**Proof.** Let \(P\) be a continuous predicate on \([0,1]\) and fix \(\varepsilon > 0\). Define a uniformly continuous function \(f\) from \(2^\mathbb{N}\) onto \([0,1]\) by

\[
f(\alpha) = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \left(\frac{(-1)^{a_n} + 1}{2}\right),
\]

where \(\alpha = (a_n)_{n \geq 1}\), and let

\[
B = \left\{a \in 2^\ast : \forall x \in (f(a - 0), f(a - 1)) P(x, \varepsilon, (2/3)^{|a|})\right\},
\]

where \(\sim\) denotes concatenation, \(0 = (0,\ldots)\), and \(i_1 = (1,0,\ldots)\). We show that \(B\) is a bar. Let \(\alpha \in 2^\mathbb{N}\), and, using (i), pick \(\delta > 0\) such that \(P(f(\alpha), \varepsilon, \delta)\). Pick \(n\) such that \((2/3)^{n-1} < 2\delta\). Then

\[
(f(\overline{\alpha}(n) \sim 0), f(\overline{\alpha}(n) \sim i_1))(2/3)^n \subset (f(\alpha) - \delta, f(\alpha + \delta)).
\]

It follows from condition (ii) that \(\alpha(n) \in B\); whence \(B\) is a bar.

By Brouwer's fan theorem, there exists \(N > 0\) such that for all \(\alpha \in 2^\mathbb{N}\) there is \(n < N\) with \(\alpha(n) \in B\). Then, by condition (ii), \(P(x, \varepsilon, (2/3)^N)\) for all \(x \in [0,1]\).

Conversely, let \(B\) be a bar that is closed under extension and define a predicate \(P\) by

\[
P(x, \varepsilon, \delta) \equiv \forall x \left(f(x) = \alpha \rightarrow \exists N > 0 (2^{-N} > \delta \land \overline{\alpha}(N) \in B)\right).
\]

It is easy to show that \(P\) is a pointwise continuous predicate. Hence \(P\) is uniformly continuous; in particular, there exists \(\delta > 0\) such that \(P(x, 1, \delta)\) holds for all \(x \in [0,1]\). Pick \(N > 0\) such that \(2^{-N} < \delta\). Then for all \(\alpha \in 2^\mathbb{N}\), \(\overline{\alpha}(N) \in B\). \(\square\)

Here is the result we need for the proof of Theorem 9.

**Theorem 17.** Assume the fan theorem and let \(P\) be a pointwise continuous predicate on \([0,1]^n\). Then \(P\) is a uniformly continuous predicate on \([0,1]^n\).

**Proof.** We proceed by induction on \(n\). The case \(n = 1\) is just one direction of Theorem 16. Suppose that the result holds for predicates on \([0,1]^{n-1}\), and let \(P\) be a predicate on \([0,1]^n\). For each \(x\) in \([0,1]\) let \(P_x\) be the predicate on \([0,1]\) given by

\[
P_x(z, \varepsilon, \delta) \iff P((z, x), \varepsilon, \delta).
\]

Then, since \(P\) is continuous, \(P_x\) is a continuous predicate for each \(x \in [0,1]\). It follows from our induction hypothesis that each \(P_x\) is uniformly continuous. Define a predicate \(P'\) on \([0,1]\) by

\[
P'(x, \varepsilon, \delta) \iff \forall y \in [0,1]^{n-1} P_x(y, \varepsilon, \delta).
\]

It is easily shown that \(P'\) is a continuous predicate and that \(P'(x, \varepsilon, \delta)\) holds for all \(x \in [0,1]\) if and only if \(P(x, \varepsilon, \delta)\) holds for all \(x \in [0,1]^n\). By Lemma 10, \(P'\) is uniformly continuous; whence \(P\) is uniformly continuous. \(\square\)
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