Variants of Jacobi polynomials in coding theory

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Abstract
In this paper, we introduce the notion of the complete joint Jacobi polynomial of two linear codes of length \( n \) over \( \mathbb{F}_q \) and \( \mathbb{Z}_k \). We give the MacWilliams type identity for the complete joint Jacobi polynomials of codes. We also introduce the concepts of the average Jacobi polynomial and the average complete joint Jacobi polynomial over \( \mathbb{F}_q \) and \( \mathbb{Z}_k \). We give a representation of the average of the complete joint Jacobi polynomials of two linear codes of length \( n \) over \( \mathbb{F}_q \) and \( \mathbb{Z}_k \) in terms of the compositions of \( n \) and its distribution in the codes. Further we present a generalization of the representation for the average of the \((g + 1)\)-fold complete joint Jacobi polynomials of codes over \( \mathbb{F}_q \) and \( \mathbb{Z}_k \). Finally, we give the notion of the average Jacobi intersection number of two codes.

Keywords Codes · Weight enumerators · Jacobi polynomials

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1 Introduction
In [7], MacWilliams, Mallows and Conway provided the notion of the joint weight enumerator of two \( \mathbb{F}_q \)-linear codes. The joint weight enumerators of codes were studied by Dougherty, Harada and Oura [5] over the finite ring \( \mathbb{Z}_k \) of integers modulo \( k (k \geq 2) \). Furthermore, the average of the joint weight enumerators of two binary codes were introduced by Yoshida [10].
Consecutively, Yoshida [11] defined the average intersection number of two binary codes. In [2], Chakraborty and Miezaki studied the average of the complete joint weight enumerators of codes over $\mathbb{F}_q$ and $\mathbb{Z}_k$ and defined the average complete joint weight enumerators as a generalization of the average joint weight enumerators in [10]. They also studied the average intersection number of two codes over $\mathbb{F}_q$ and $\mathbb{Z}_k$. The notion of the Jacobi polynomial of a code over $\mathbb{F}_q$ was introduced by Ozeki [9]. A successive study of the Jacobi polynomial of a binary code was carried out by Bonnecaze et al. [1] to construct various kinds of designs. The concept of the $g$-th Jacobi polynomial of a code over $\mathbb{F}_q$ was introduced by Honma et al. [6]. They also obtained the MacWilliams identity for the $g$-th Jacobi polynomials. In the present paper, we give the notion of the complete joint Jacobi polynomials of codes over $\mathbb{F}_q$ and $\mathbb{Z}_k$ and obtain the MacWilliams type identity for the polynomials. We also define the average Jacobi polynomial and the average complete joint Jacobi polynomial of codes over $\mathbb{F}_q$ and $\mathbb{Z}_k$, and give an analogue of the main theorem in [10] for each of the polynomial. Moreover, as a generalization of the complete joint Jacobi polynomials and the average complete joint Jacobi polynomials, we introduce the concept of the $g$-fold complete joint Jacobi polynomials and the average $(g+1)$-fold complete joint Jacobi polynomials, respectively of codes over $\mathbb{F}_q$ and $\mathbb{Z}_k$. Finally, we define the average Jacobi intersection number and give a formula to compute the numbers. We also give some numerical examples of the average Jacobi intersection number for Type II codes.

Throughout this paper, we assume that $\mathcal{R}$ denotes either the finite field $\mathbb{F}_q$ of order $q$, where $q$ is a prime power or the ring $\mathbb{Z}_k$ of integers modulo $k$ for some positive integer $k \geq 2$.

This paper is organized as follows. In Sect. 2, we give definitions and some basic properties of linear codes over $\mathcal{R}$. In Sect. 3, we give the MacWilliams type identity (Theorem 3.1) for the complete joint Jacobi polynomials of codes over $\mathcal{R}$. In Sect. 4, we give an analogue to the main theorem in [10] for the average Jacobi polynomials (Theorem 4.1) as well as for the average complete joint Jacobi polynomials (Theorem 4.2) over $\mathcal{R}$. In Sect. 5, we give a generalization of the MacWilliams identity (Theorem 5.1) for the $g$-fold complete joint Jacobi polynomials of codes over $\mathcal{R}$. We also give a generalization of Theorem 4.2 for the average $(g+1)$-fold complete joint Jacobi polynomials of codes over $\mathcal{R}$ (Theorem 5.2). In Sect. 6, we define the average Jacobi intersection number and give a formula (Theorem 6.1) to compute this number. We also give some numerical examples for some Type II codes over $\mathbb{F}_2$. From the observation of the numerical examples, we enclose the section with two conjectures (Conjecture 6.1, Conjecture 6.2).

## 2 Preliminaries

An $\mathbb{F}_q$-linear code of length $n$ is a vector subspace of $\mathbb{F}_q^n$, and a $\mathbb{Z}_k$-linear code of length $n$ is an additive group of $\mathbb{Z}_k^n$. The elements of an $\mathcal{R}$-linear code are called codewords. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be the elements of $\mathcal{R}^n$. Then the inner product of two elements $\mathbf{u}, \mathbf{v} \in \mathcal{R}^n$ is given by

$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$ 

The dual code of an $\mathcal{R}$-linear code $C$ of length $n$ is defined by

$$C^\perp := \{ \mathbf{v} \in \mathcal{R}^n \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in C \}.$$ 

An $\mathcal{R}$-linear code $C$ is called self-dual if $C = C^\perp$. The weight of $\mathbf{u} \in \mathcal{R}^n$ is denoted by $\text{wt}(\mathbf{u})$ and defined to be the number of $i$'s such that $u_i \neq 0$. A self-dual code over $\mathbb{F}_2$ of length $n \equiv 0 \pmod{8}$ is called Type II if the weight of each codeword of the code is a multiple of 4.
Let the elements of $\mathcal{R}$ be $0 = \omega_0, \omega_1, \ldots, \omega_{|\mathcal{R}|} - 1$ in some fixed order. Then the composition of an element $u \in \mathcal{R}^n$ is defined as

$$\text{comp}(u) := \ell(u) := (\ell_a(u) : a \in \mathcal{R}),$$

where $\ell_a(u)$ denotes the number of coordinates of $u$ that are equal to $a \in \mathcal{R}$. Obviously,

$$\sum_{a \in \mathcal{R}} \ell_a(u) = n.$$

In general, a composition $L$ of $n$ is a vector with non-negative integer components $L_a$ for $a \in \mathcal{R}$ such that

$$\sum_{a \in \mathcal{R}} L_a = n.$$

**Definition 2.1** Let $C$ be an $\mathcal{R}$-linear code of length $n$. We denote

$$T^C_L := \{ u \in C | \text{comp}(u) = L \}.$$

Then the complete weight enumerator for $C$ is defined as

$$cwe_C ((x_a)_{a \in \mathcal{R}}) := \sum_{u \in C} \prod_{a \in \mathcal{R}} x_{\ell_a(u)} = \sum_L A^C_L \prod_{a \in \mathcal{R}} x_{L_a},$$

where $A^C_L = |T^C_L|$. In more general, the complete weight enumerator of $C$ for genus $g$ is defined as

$$cwe^g_C ((x_a)_{a \in \mathcal{R}^g}) := \sum_{u_1, \ldots, u_g \in C} \prod_{a \in \mathcal{R}^g} x_{n_a(u_1, \ldots, u_g)}$$

where $n_a(u_1, \ldots, u_g)$ denotes the number of $i$ such that $a = (u_{1i}, \ldots, u_{gi})$.

Now fix $w \in \mathcal{R}^n$. We denote by $\text{comp}_w(u) := r(u; w)$ the Jacobi composition of $u \in \mathcal{R}^n$ with respect to $w$ having the components $r_a(u; w)$ for $a \in \mathcal{R}^2$ which are defined as

$$r_a(u; w) := \# \{ i | (u_i, w_i) = a \}.$$

Clearly

$$\sum_{a \in \mathcal{R}^2} r_a(u; w) = n.$$

In general, a Jacobi composition $R$ of $n$ is a vector with non-negative integer components $R_a$ for $a \in \mathcal{R}^2$ such that

$$\sum_{a \in \mathcal{R}^2} R_a = n.$$

**Definition 2.2** Let $C$ be an $\mathcal{R}$-linear code of length $n$ and $w \in \mathcal{R}^n$. We denote

$$T^C_{R,w} := \{ u \in C | \text{comp}_w(u) = R \}.$$

Then the complete Jacobi polynomial of $C$ with respect to $w \in \mathcal{R}^n$ is defined as

$$Jac(C, w; (x_a)_{a \in \mathcal{R}^2}) := \sum_{u \in C} \prod_{a \in \mathcal{R}^2} x_{r_a(u; w)} = \sum_R B^C_{R,w} \prod_{a \in \mathcal{R}^2} x_{R_a},$$

where $B^C_{R,w} = |T^C_{R,w}|$. 
Remark 2.1 \( \mathcal{J}_{C,D}(\{x_a\}_{a \in \mathbb{R}^2}) := \sum_{u,v \in D} \prod_{a \in \mathbb{R}^2} x_a^n(u,v), \)
where \( n_a(u,v) := \#\{i \mid (u_i, v_i) = a\}. \)

**3 Complete joint Jacobi polynomial and MacWilliams identity**

Let us fix \( \mathbf{w} \in \mathbb{R}^n. \) Then we denote by

\[
\text{comp}_w(u, v) := h(u, v; w) := (h_a(u, v; w) : a \in \mathbb{R}^3)
\]

the joint Jacobi composition of \( u, v \in \mathbb{R}^n \) with respect to \( w, \) where \( h_a(u, v; w) := \#\{i \mid (u_i, v_i, w_i) = a\}. \) Clearly

\[
\sum_{a \in \mathbb{R}^3} h_a(u, v; w) = n.
\]

In general, a joint Jacobi composition \( H \) of \( n \) denotes a vector with non-negative integer components \( H_a \) for \( a \in \mathbb{R}^3 \) such that

\[
\sum_{a \in \mathbb{R}^3} H_a = n.
\]

**Definition 3.1** Let \( C \) and \( D \) be two \( \mathbb{R} \)-linear codes of length \( n. \) Then the complete joint Jacobi polynomial of \( C \) and \( D \) with respect to \( w \in \mathbb{R}^n \) is denoted by \( \mathfrak{Jac}(C, D; w; \{x_a\}_{a \in \mathbb{R}^3}) \) and defined as

\[
\mathfrak{Jac}(C, D; w; \{x_a\}_{a \in \mathbb{R}^3}) := \sum_{u,v \in D} \prod_{a \in \mathbb{R}^3} x_a^{h_a(u,v,w)}
= \sum_H B_H^{C,D,w} \prod_{a \in \mathbb{R}^3} x_a^{H_a}.
\]

where \( B_H^{C,D,w} := \#\{(u, v) \in C \times D \mid \text{comp}_w(u, v) = H\}. \)

**Remark 3.1** Let \( C \) and \( D \) be two \( \mathbb{R} \)-linear code of length \( n, \) and \( w \in \mathbb{R}^n. \) Then we have

1. If \( C = \{(0, 0, \ldots, 0)\}, \) then \( \mathfrak{Jac}(C, D; w) = J\text{ac}(D; w). \)
2. If \( D = \{(0, 0, \ldots, 0)\}, \) then \( \mathfrak{Jac}(C, D; w) = J\text{ac}(C; w). \)
3. If \( C = D \) and \( w = (0, 0, \ldots, 0), \) then \( \mathfrak{Jac}(C, D; w) = c\text{we}_C^{(2)}. \)

In this section, we give the MacWilliams type identity for the complete joint Jacobi polynomial of \( \mathbb{R} \)-linear codes. We review \([4,5,7]\) to introduce some fixed characters over \( \mathbb{R}. \)

Let \( \mathbb{R} = \mathbb{F}_q, \) where \( q = p^f \) for some prime number \( p. \) A character \( \chi \) of \( \mathbb{F}_q \) is a homomorphism from the additive group \( \mathbb{F}_q \) to the multiplicative group of non-zero complex numbers. Now let \( F(x) \) be a primitive irreducible polynomial of degree \( f \) over \( \mathbb{F}_p \) and let \( \lambda \) be a root of \( F(x). \) Then any element \( \alpha \in \mathbb{F}_q \) has a unique representation as:

\[
\alpha = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_{f-1} \lambda^{f-1},
\]
where \( \alpha_i \in \mathbb{F}_p \). We define \( \chi(\alpha) := \zeta_p^{\alpha_0} \), where \( \zeta_p \) is the \( p \)-th primitive root of unity, and \( \alpha_0 \) is given by Equation (1).

Again if \( R = \mathbb{Z}_k \), then for \( \alpha \in \mathbb{Z}_k \), we define \( \chi(\alpha) := \zeta_k^{\alpha} \), where \( \zeta_k \) is the \( k \)-th primitive root of unity.

Now for any \( \alpha \in R \), we have the following property:

\[
|R|^{-1} \sum_{i=0}^{|R|-1} \chi(\alpha \omega^i) := \begin{cases} |R| & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha \neq 0. \end{cases}
\]

Now we give the MacWilliams relation for the complete joint Jacobi polynomial of codes over \( R \).

**Theorem 3.1** (MacWilliams Identity) Let \( C \) and \( D \) be two \( R \)-linear codes of length \( n \), and \( \text{Jac}(C, D, w; \{x_a\}_{a \in \mathbb{R}^3}) \) be a complete joint Jacobi polynomial for codes \( C \) and \( D \) with respect to \( w \in \mathbb{R}^n \). Then

\[
\text{Jac}(C, D^\perp, w; \{x_a\}_{a \in \mathbb{R}^3}) = \frac{1}{|D|} \text{Jac} \left( C, D, w; \left\{ \sum_{b \in R} \chi(a_2 b) x(a_1 b a_3) \right\}_{a \in \mathbb{R}^3} \right).
\]

**Proof** Let

\[
\delta_{D^\perp}(v) := \begin{cases} 1 & \text{if } v \in D^\perp, \\ 0 & \text{otherwise} \end{cases}.
\]

Then we have the following identity

\[
\delta_{D^\perp}(v) = \frac{1}{|D|} \sum_{d \in D} \chi(d \cdot v).
\]

Now

\[
\text{Jac}(C, D^\perp, w; \{x_a\}_{a \in \mathbb{R}^3})
\]

\[
= \sum_{c \in C} \sum_{d^\perp \in D^\perp} \prod_{a \in \mathbb{R}^3} x_{h_a(c, d^\perp ; w)}
\]

\[
= \sum_{c \in C} \sum_{v \in \mathbb{R}^n} \delta_{D^\perp}(v) \prod_{a \in \mathbb{R}^3} x_{h_a(c, v ; w)}
\]

\[
= \frac{1}{|D|} \sum_{c \in C} \sum_{v \in \mathbb{R}^n} \sum_{d \in D} \chi(d \cdot v) \prod_{a \in \mathbb{R}^3} x_{h_a(c, v ; w)}
\]

\[
= \frac{1}{|D|} \sum_{c \in C} \sum_{d \in D} \chi(d v_1 + \cdots + d_n v_n) \prod_{1 \leq i \leq n} x(c_i v_i w_i)
\]

\[
= \frac{1}{|D|} \sum_{c \in C} \prod_{1 \leq i \leq n} \sum_{v_i \in \mathbb{R}} \chi(d_i v_i) x(c_i v_i w_i)
\]

\[
= \frac{1}{|D|} \sum_{c \in C} \prod_{a = (a_1, a_2, a_3) \in \mathbb{R}^3} \left( \sum_{b \in \mathbb{R}} \chi(a_2 b) y(a_1 b a_3) \right)^{h_a(c, d ; w)}
\]

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= \frac{1}{|D|} \Jac (C, D, w; \left\{ \sum_{b \in \mathcal{R}} \chi(a_1 b x(a_2 b a_3)) \right\}_{a \in \mathcal{R}^3}).

Hence the proof is completed. \( \square \)

**Corollary 3.1** Let \( C \) and \( D \) be two \( \mathcal{R} \)-linear codes of length \( n \). Then

(i) \( \Jac (C^\perp, D, w; \left\{ x_a \right\}_{a \in \mathcal{R}^3}) = \frac{1}{|C|} \Jac (C, D, w; \left\{ \sum_{b \in \mathcal{R}} \chi(a_1 b x(ba_3)) \right\}_{a \in \mathcal{R}^3}). \)

(ii) \( \Jac (C^\perp, D^\perp, w; \left\{ x_a \right\}_{a \in \mathcal{R}^3}) = \frac{1}{|C||D|} \Jac (C, D, w; \left\{ \sum_{b_1, b_2 \in \mathcal{R}} \chi(a_1 b_1 + a_2 b_2 x(b_1 b_2 a_3)) \right\}_{a \in \mathcal{R}^3}). \)

Now by Remark 3.1 and by Theorem 3.1 we have the MacWilliams type identity for the complete Jacobi polynomial of codes over \( \mathcal{R} \) as follows:

\[
Jac(C^\perp, w; \left\{ x_a \right\}_{a \in \mathcal{R}^2}) = \frac{1}{|C|} Jac\left(C, w; \left\{ \sum_{b \in \mathcal{R}} \chi(a_1 b x(ba_2)) \right\}_{a \in \mathcal{R}^2}\right).
\]

### 4 Main results

We write \( S_n \) for the symmetric group acting on the set \( \{1, 2, \ldots, n\} \), equipped with the composition of permutations. For any \( \mathcal{R} \)-linear code \( C \), the code \( C^\sigma := \{ u^\sigma \mid u \in C \} \) for any permutation \( \sigma \in S_n \) is called **permutationally equivalent** to \( C \), where \( u^\sigma := (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \).

**Definition 4.1** Let \( C \) be an \( \mathcal{R} \)-linear code, and \( w \in \mathcal{R}^n \). Then the **average Jacobi polynomial** of \( C \) with respect to \( w \in \mathcal{R}^n \) is defined as follows:

\[
Jac^{av}(C, w; \left\{ x_a \right\}_{a \in \mathcal{R}^2}) := \frac{1}{n!} \sum_{\sigma \in S_n} Jac(C^\sigma, w; \left\{ x_a \right\}_{a \in \mathcal{R}^2}).
\]

Clearly, we have the MacWilliams identity for the average Jacobi polynomial as follows:

\[
Jac^{av}(C^\perp, w; \left\{ x_a \right\}_{a \in \mathcal{R}^2}) = \frac{1}{|C|} Jac^{av}(C, w; \left\{ \sum_{b \in \mathcal{R}} \chi(a_1 b x(ba_2)) \right\}_{a \in \mathcal{R}^2}).
\]

Now we have the following result. We will prove the following theorem in a more general setting in Theorem 4.2. Before stating the theorem we put

\[
{n \choose n_1, \ldots, n_k} := \frac{n!}{n_1! \cdots n_k!}.
\]
Theorem 4.1 Let $C$ be an $\mathcal{R}$-linear code of length $n$, and $w \in \mathcal{R}^n$. Again let $L$ be the composition of $n$ and $R$ be the Jacobi composition of $n$ such that

$$L = \left( \sum_{b \in \mathcal{R}} R_{(\omega b_0), \ldots, R_{(\omega b_{|R|-1} b)}}, \ldots, R_{(\omega b_{|R|-1} b)} \right),$$

$$\ell(w) = \left( \sum_{b \in \mathcal{R}} R_{(b_0 b_0), \ldots, R_{(b_{|R|-1} b_{|R|-1})}} \right).$$

Then

$$\text{Jac}^{av}(C, w; \{x_a\}_{a \in \mathcal{R}^3}) = \sum_{L, R} \lambda^C \prod_{b \in \mathcal{R}} \left( \frac{\ell_b(w)}{n \left( L_{(\omega b_0), \ldots, L_{(\omega b_{|R|-1} b)} R_a \right)} \right) \prod_{a \in \mathcal{R}^2} \chi_R^{Ra}.$$

Definition 4.2 Let $C$ and $D$ be two $\mathcal{R}$-linear codes of length $n$. Then the average complete joint Jacobi polynomial of codes $C$ and $D$ with respect to $w \in \mathcal{R}^n$ is defined as follows:

$$\text{Jac}^{av}(C, D, w; \{x_a\}_{a \in \mathcal{R}^3}) := \frac{1}{n!} \sum_{\sigma \in S_n} \text{Jac}(C^\sigma, D, w; \{x_a\}_{a \in \mathcal{R}^3}).$$

Remark 4.1 We have the following remarks.

1. $\text{Jac}^{av}(C, D, w; \{x_a\}_{a \in \mathcal{R}^3}) \neq \text{Jac}^{av}(D, C, w; \{x_a\}_{a \in \mathcal{R}^3}).$
2. There exists some $\sigma \in S_n$ such that

$$\text{Jac}^{av}(C^\sigma, D^\sigma, w; \{x_a\}_{a \in \mathcal{R}^3}) \neq \text{Jac}^{av}(C, D, w; \{x_a\}_{a \in \mathcal{R}^3}).$$

3. If $D = \{(0, 0, \ldots, 0)\}$, then

$$\text{Jac}^{av}(C, D, w; \{x_a\}_{a \in \mathcal{R}^3}) = \text{Jac}^{av}(C, w; \{x_a\}_{a \in \mathcal{R}^3}).$$

From Theorem 3.1, we have the MacWilliams identity for the average joint Jacobi polynomial as follows:

(i) $\text{Jac}^{av}(C, D^\perp, w; \{x_a\}_{a \in \mathcal{R}^3})$

$$= \frac{1}{|D|} \text{Jac}^{av} \left( C, D, w; \left\{ \chi(a_2 b) x_{(a_1 b a_3)} \right\}_{a \in \mathcal{R}^3} \right).$$

(ii) $\text{Jac}^{av}(C^\perp, D, w; \{x_a\}_{a \in \mathcal{R}^3})$

$$= \frac{1}{|C|} \text{Jac}^{av} \left( C, D, w; \left\{ \chi(a_1 b) x_{(b a_2 a_3)} \right\}_{a \in \mathcal{R}^3} \right).$$

(iii) $\text{Jac}^{av}(C^\perp, D^\perp, w; \{x_a\}_{a \in \mathcal{R}^3})$

$$= \frac{1}{|C||D|} \text{Jac}^{av} \left( C, D, w; \left\{ \sum_{a_1, b_2 \in \mathcal{R}} \chi(a_1 b_1 + a_2 b_2) x_{(b_1 b_2 a_3)} \right\}_{a \in \mathcal{R}^3} \right).$$

Now we have the following result analogue to the main theorem in [10] which represents the average of the complete joint Jacobi polynomials of two codes $C$ and $D$ of length $n$ with respect to $w \in \mathcal{R}^n$ by using the compositions of $n$ and its distribution in the codes.
Theorem 4.2 Let $C$ and $D$ be two $\mathcal{R}$-linear codes of length $n$, and $w \in \mathcal{R}^n$. Let $L$ be the composition of $n$ and $R$ be the Jacobi composition of $n$. Again let $H$ be the joint Jacobi composition of $n$ such that

$$L = \left( \sum_{a=(a_1,a_2) \in \mathcal{R}^2} H_{(ba_1a_2)} : b \in \mathcal{R} \right),$$

$$R = \left( \sum_{b \in \mathcal{R}} H_{(ba_1a_2)} : a = (a_1, a_2) \in \mathcal{R}^2 \right).$$

Then

$$\mathcal{J}ac^w(C, D, w; \{x_c\}_{c \in \mathcal{R}^3}) = \sum_{L, R, H} A_L B_R D_w \prod_{a \in \mathcal{R}^2} \left( \frac{R_a}{n} \right) \prod_{c \in \mathcal{R}^3} x_c H_c.$$

Proof Let $C$ and $D$ be two $\mathcal{R}$-linear codes of length $n$. Then the joint Jacobi polynomial of $C$ and $D$ with respect to $w \in \mathcal{R}^n$ is

$$\mathcal{J}ac(C, D, w; \{x_c\}_{c \in \mathcal{R}^3}) = \sum_{H} B_{H}^{C, D, w} \prod_{c \in \mathcal{R}^3} x_c H_c,$$

where $\sum_{c \in \mathcal{R}^3} H_c = n$. Now define

$$N_{L, R, H}^{C, D, w} := \#\{(u, v) \in C \times D \mid \text{comp}(u) = L, \text{comp}(v) = R, \text{comp}_{w}(u, v) = H\}.$$

Therefore, $B_{H}^{C, D, w} = N_{L, R, H}^{C, D, w}$, for

$$L = \left( \sum_{a=(a_1,a_2) \in \mathcal{R}^2} H_{(ba_1a_2)} : b \in \mathcal{R} \right),$$

$$R = \left( \sum_{b \in \mathcal{R}} H_{(ba_1a_2)} : a = (a_1, a_2) \in \mathcal{R}^2 \right).$$

Hence we can write from Equation (2)

$$\mathcal{J}ac(C, D, w; \{x_c\}_{c \in \mathcal{R}^3}) = \sum_{L, R, H} N_{L, R, H}^{C, D, w} \prod_{c \in \mathcal{R}^3} x_c H_c.$$

Now

$$\sum_{\sigma \in S_n} N_{L, R, H}^{C, D, w} = \#\{(u, v, \sigma) \in T_L^{C} \times T_R^{D, w} \times S_n \mid \text{comp}_{w}(u^\sigma, v) = H\} = \sum_{u \in T_L^{C}} \sum_{v \in T_R^{D, w}} \#\{\sigma \in S_n \mid \text{comp}_{w}(u^\sigma, v) = H\}.$$
We observe that the order of a subgroup of \( S_n \) which stabilizes \( u \in T_L^C \) is \( \prod_{b \in \mathfrak{R}} L_b! \). Therefore

\[
\sum_{\sigma \in S_n} N_{L,R,H}^{C^\sigma,D,w} = \sum_{u \in T_L^C} \sum_{v \in T_R^D} \prod_{b \in \mathfrak{R}} L_b! M_{L,H},
\]

where

\[
M_{L,H} := \# \{ u' \in \mathfrak{R}^n \mid \text{comp}(u') = L, \text{comp}_w(u', v) = H \}.
\]

Therefore

\[
\sum_{\sigma \in S_n} N_{L,R,H}^{C^\sigma,D,w} = A_{L} B_{R}^{D,w} \prod_{b \in \mathfrak{R}} L_b! \prod_{a \in \mathfrak{R}^2} \frac{R_a!}{\prod_{i=0}^{n-1} H(\omega^i a_1 a_2)!} \prod_{b \in \mathfrak{R}} L_b! \prod_{a \in \mathfrak{R}^2} H(\omega^i a_1 a_2)! \prod_{b \in \mathfrak{R}} \prod_{c \in \mathfrak{R}^3} x_{H^c}.
\]

Now we have

\[
\mathfrak{Jac}_{av}^\sigma(C, D, w; \{ x_c \}_{c \in \mathfrak{R}^3}) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathfrak{Jac}(C^\sigma, D, w; x_c : c \in \mathfrak{R}^3) = \frac{1}{n!} \sum_{L,R,H} \sum_{\sigma \in S_n} N_{L,R,H}^{C^\sigma,D} \prod_{c \in \mathfrak{R}^3} x_{c}^H.
\]

This completes the proof. \( \square \)

### 5 Average of \((g + 1)\)-fold complete joint Jacobi polynomials

In this section, we present a generalization of the average complete joint Jacobi polynomials of \( \mathfrak{R} \)-linear codes. We call these Jacobi polynomials the average \((g + 1)\)-fold complete joint Jacobi polynomials. We obtain a generalized MacWilliams identity for these Jacobi polynomials. We also discuss an analogy of Theorem 4.2 for the average \((g + 1)\)-fold complete joint Jacobi polynomials.
Let \( w \in \mathfrak{H}^n \). Then we denote the \( g \)-fold joint Jacobi composition of \( u_1, \ldots, u_g \in \mathfrak{H}^n \) with respect to \( w \) by

\[
\text{comp}_w(u_1, \ldots, u_g) := h(u_1, \ldots, u_g; w)
\]

\[
:= (h_\alpha(u_1, \ldots, u_g; w) : \alpha \in \mathfrak{H}^{g+1}),
\]

where \( h_\alpha(u_1, \ldots, u_g; w) \) denotes the number of coordinate position \( i \) such that \( \alpha = (u_{1i}, \ldots, u_{gi}, w_i) \). It is immediate that

\[
\sum_{\alpha \in \mathfrak{H}^{g+1}} h_\alpha(u_1, \ldots, u_g; w) = n.
\]

Now we define the \( g \)-fold joint Jacobi composition \( H^{(g)} \) of \( n \) by \( H^{(g)} = (H_\alpha^{(g)} : \alpha \in \mathfrak{H}^{g+1}) \), where the non-negative integer components \( H_\alpha^{(g)} \) satisfy the following condition:

\[
\sum_{\alpha \in \mathfrak{H}^{g+1}} H_\alpha^{(g)} = n.
\]

**Definition 5.1** Let \( C_1, C_2, \ldots, C_g \) be \( \mathfrak{H} \)-linear codes of length \( n \). Then the \( g \)-fold complete joint Jacobi polynomial of \( C_1, C_2, \ldots, C_g \) with respect to \( w \in \mathfrak{H}^n \) is defined as follows:

\[
\text{Jac}(C_1, \ldots, C_g, w; \{x_\alpha\}_{\alpha \in \mathfrak{H}^{g+1}}) := \sum_{u_1 \in C_1, \ldots, u_g \in C_g, \alpha \in \mathfrak{H}^{g+1}} x_\alpha h(u_1, \ldots, u_g; w)
\]

\[
= \sum_{H^{(g)}} B_{H^{(g)}}^{C_1, \ldots, C_g, w} \prod_{\alpha \in \mathfrak{H}^{g+1}} x_\alpha H_\alpha^{(g)},
\]

where \( B_{H^{(g)}}^{C_1, \ldots, C_g, w} \) is the number of \( g \)-tuple \( (u_1, \ldots, u_g) \in C_1 \times \cdots \times C_g \) such that \( \text{comp}_w(u_1, \ldots, u_g) = H^{(g)} \).

The MacWilliams identity for the \( g \)-fold joint weight enumerators of codes over \( \mathbb{Z}_k \) was given in [5]. Recently the MacWilliams identity for the \( g \)-th Jacobi polynomial of a binary code was given in [6]. Now we give a generalized MacWilliams identity for the \( g \)-fold complete joint Jacobi polynomials of \( \mathfrak{H} \)-linear codes. Let \( \tilde{C}_i \) denote either \( C_i \) or \( C_i^\perp \). Then

\[
\delta(C_i, \tilde{C}_i) := \begin{cases} 
0 & \text{if } \tilde{C}_i = C_i, \\
1 & \text{if } \tilde{C}_i = C_i^\perp.
\end{cases}
\]

We recall the character \( \chi \) from Sect. 3. Let \( T_R = (\chi(ab))_{a,b \in \mathfrak{H}} \) be an \( |\mathfrak{H}| \times |\mathfrak{H}| \) matrix.

**Theorem 5.1** (Generalized MacWilliams identity) Let \( C_1, \ldots, C_g \) be \( \mathfrak{H} \)-linear codes of length \( n \), and \( \text{Jac}(C_1, \ldots, C_g, w; \{x_\alpha\}_{\alpha \in \mathfrak{H}^{g+1}}) \) be the \( g \)-fold complete joint Jacobi polynomial for \( C_1, \ldots, C_g \) with respect to \( w \in \mathfrak{H}^n \). Then

\[
\text{Jac}(\tilde{C}_1, \ldots, \tilde{C}_g, w; \{x_\alpha\}_{\alpha \in \mathfrak{H}^{g+1}}) = \frac{1}{|C_1|^{\delta(C_1, \tilde{C}_1)} \cdots |C_g|^{\delta(C_g, \tilde{C}_g)}} T_R^{\delta(C_1, \tilde{C}_1)} \otimes \cdots \otimes T_R^{\delta(C_g, \tilde{C}_g)} \otimes T_R^0 \text{Jac}(C_1, \ldots, C_g, w; \{x_\alpha\}_{\alpha \in \mathfrak{H}^{g+1}}),
\]

where \( T_R^0 \) denotes the identity matrix \( I \) of order \( |\mathfrak{H}| \).
Proof Let $\tilde{C}_k$ be $C_k^\perp$ and $\tilde{C}_i$ be $C_i$ for $k \neq i$. Then it is sufficient to show that

$$\begin{align*}
|C_k| \text{Jac}(C_1, \ldots, C_{k-1}, C_k^\perp, C_{k+1}, \ldots, C_g, w; \{x_a\}_{a \in \mathcal{R}^{g+1}}) \\
= I \otimes \cdots \otimes I \otimes T_{\mathcal{R}} \otimes I \otimes \cdots \otimes I \otimes I^{(g+1)-\text{th}} \\
\tilde{\text{Jac}}(C_1, \ldots, C_{k-1}, C_k, C_{k+1}, \ldots, C_g, w; \{x_a\}_{a \in \mathcal{R}^{g+1}}).
\end{align*}$$

The proof is straightforward. So, we leave it for the readers. □

Definition 5.2 Let $C$, $D_1$, $D_2$, $D_g$ be $(g + 1)$ $\mathcal{R}$-linear codes of length $n$. Then the average $(g + 1)$-fold complete joint Jacobi polynomial of codes $C$, $D_1$, $D_2$, $D_g$ with respect to $w \in \mathcal{R}^n$ is defined as follows:

$$\begin{align*}
\tilde{\text{Jac}}^\text{av}(C, D_1, \ldots, D_g, w; \{x_a\}_{a \in \mathcal{R}^{g+2}}) \\
:= \frac{1}{n!} \sum_{\sigma \in S_n} \tilde{\text{Jac}}(C^\sigma, D_1, \ldots, D_g, w; \{x_a\}_{a \in \mathcal{R}^{g+2}}).
\end{align*}$$

It is immediate from Theorem 5.1 that the average $(g + 1)$-fold complete joint Jacobi polynomials of $\mathcal{R}$-linear codes $C$, $D_1$, $D_2$, $D_g$ satisfy the following MacWilliams type identity:

$$\begin{align*}
\tilde{\text{Jac}}^\text{av}(\tilde{C}, \tilde{D}_1, \ldots, \tilde{D}_g, w; \{x_a\}_{a \in \mathcal{R}^{g+2}}) \\
= \frac{1}{|C|^{\delta(C, \tilde{C})}|D_1|^{\delta(D_1, \tilde{D}_1)} \cdots |D_g|^{\delta(D_g, \tilde{D}_g)}} \\
T_{\mathcal{R}}^{\delta(C, \tilde{C})} \otimes T_{\mathcal{R}}^{\delta(D_1, \tilde{D}_1)} \otimes \cdots \otimes T_{\mathcal{R}}^{\delta(D_g, \tilde{D}_g)} \otimes T_{\mathcal{R}}^0 \\
\tilde{\text{Jac}}^\text{av}(C, D_1, \ldots, D_g, w; \{x_a\}_{a \in \mathcal{R}^{g+2}}),
\end{align*}$$

where notations carry the same meaning as in Theorem 5.1.

Now we have the following generalization of Theorem 4.2. The proof of the following theorem is not so difficult. So, we leave it for the readers.

Theorem 5.2 Let $C$, $D_1$, $D_2$, $D_g$ be $(g + 1)$ $\mathcal{R}$-linear codes of length $n$, and $w \in \mathcal{R}^n$. Let $L$ be the composition of $n$, where $L = (L_a : a \in \mathcal{R})$, and $R_1, \ldots, R_g$ be the Jacobi compositions of $n$, where $R_i = (R_{1b} : b \in \mathcal{R}^2)$ for $i = 1, \ldots, g$. Let $H^{(g)}$ be the $g$-fold joint Jacobi composition of $n$, where $H^{(g)} = (H_c^{(g)} : c = (c_1, \ldots, c_g, c_{g+1}) \in \mathcal{R}^{g+1})$ such that for $i = 1, \ldots, g$,

$$R_{ib} = \sum_{\tilde{c} = (c_1, \ldots, c_{i-1}, b_1, c_{i+1}, \ldots, c_{g+1})} H_{\tilde{c}}^{(g)}, \text{ where } b = (b_1, b_2) \in \mathcal{R}^2.$$

Again let $H^{(g+1)}$ be the $(g + 1)$-fold joint Jacobi composition of $n$, where $H^{(g+1)} = (H_d^{(g+1)} : d = (d_0, d_1, \ldots, d_g, d_{g+1}) \in \mathcal{R}^{g+2})$ such that

$$L_a = \sum_{\tilde{d} = (a, d_1, \ldots, d_g, d_{g+1})} H_{\tilde{d}}^{(g+1)} \text{ for } a \in \mathcal{R},$$

$$H_c^{(g)} = \sum_{\tilde{d} = (d_0, c_1, \ldots, c_g, c_{g+1})} H_{\tilde{d}}^{(g+1)} \text{ for } c = (c_1, \ldots, c_g, c_{g+1}) \in \mathcal{R}^{g+1}.$$
Then
\[ \text{jac}^{au}(C, D_1, \ldots, D_g, w; \{x_{e}\}_{e \in \mathcal{R}^{g+2}}) = \sum_{L, H^{(g)}, H^{(g+1)}} A_L B_{H^{(g)}} D_1, \ldots, D_g, w \]
\[ \prod_{c=(c_1, \ldots, c_{g+1}) \in \mathcal{R}^{g+1}} \left( H^{(g+1)}_{c_{(0)c_1 \ldots c_{g}c_{g+1})}, \ldots, H^{(g+1)}_{c_{(0)|c|-1}c_{g}c_{g+1})} \right) \prod_{d \in \mathcal{R}^{g+2}} \xi_d^{H^{(g+1)}} \]

6 Average Jacobi intersection number

The notion of the average intersection number was introduced in [10] for binary linear codes. In [2], the average intersection number for \( \mathcal{R} \)-linear codes was studied. In this section, we give the notion of the average Jacobi intersection number for \( \mathcal{R} \)-linear codes.

For \( u, w \in \mathcal{R}^n \), we define \( u^w = (u^w_1, \ldots, u^w_n) \) such that
\[ u^w_i = \begin{cases} u_i & \text{if } w_i = 0, \\ 0 & \text{otherwise.} \end{cases} \]

Let \( C \) and \( D \) be two \( \mathcal{R} \)-linear codes of length \( n \). Then the Jacobi intersection of the codes \( C \) and \( D \) with respect to \( w \in \mathcal{R}^n \) is defined as,
\[ C \cap w D := \{ (u, v) \in C \times D \mid u^w = v^w \}. \]

Now we defined the average Jacobi intersection number for \( C \) and \( D \) with respect to \( w \in \mathcal{R}^n \) as follows:
\[ \Delta^w(C, D) := \frac{1}{n!} \sum_{\sigma \in S_n} |C^\sigma \cap w D|. \]

Clearly, \( h_a(u, v; w) = 0 \) for \( a = (a_1, a_2, a_3 : a_1 \neq a_2, a_3 = 0) \in \mathcal{R}^3 \) if and only if \( u^w = v^w \). Thus we have the following remark.

**Remark 6.1** For \( a = (a_1, a_2, a_3) \in \mathcal{R}^3 \), we let
\[ y_a = \begin{cases} 0 & \text{if } a_1 \neq a_2 \text{ and } a_3 = 0, \\ 1 & \text{otherwise.} \end{cases} \]

Then \( \text{jac}^{au}(C, D, w; \{x_{a}\}_{a \in \mathcal{R}^3}) = \Delta^w(C, D) \).

Now we have the following result.

**Theorem 6.1** Let \( C \) and \( D \) be two \( \mathcal{R} \)-linear codes of length \( n \) and \( w \in \mathcal{R}^n \). Again let \( R' \) and \( R'' \) be two Jacobi compositions of \( n \) such that
\[ R'_{(a_00)} = R''_{(a_00)}, \text{ for } a \in \mathcal{R} \]
and
\[ \sum_{a \in \mathcal{R}} R'_{(ab)} \leq \ell_b(w), \text{ for } b \in \mathcal{R} \]
Let $L$ be the composition of $n$ such that

$$L = \left( \sum_{a \in \mathfrak{A}} R'_{(00a)}, \ldots, \sum_{a \in \mathfrak{A}} R'_{(0|R|-1a)} \right),$$

Then

$$\Delta^w(C, D) = \sum_{R', R'', L} A^C_L B^D_{R''} \frac{\prod_{a \in \mathfrak{A}} \ell_a(w)}{n} \left( R'_{(00a)} \cdots R'_{(0|R|-1a)} \right) \left( L_{w00}, \ldots, L_{w|R|-1} \right).$$

**Proof** Let $T^C_L$ be the set of all elements of $C$ with composition $L$ of $n$, and $T^{D,w}_L$ be the set of all elements of $D$ with Jacobi composition $R''$ of $n$ with respect to $w \in \mathfrak{A}^n$. Then we can write

$$n! \Delta^w(C, D) = \sum_{\sigma \in S_n} |C^\sigma \cap_{w} D|$$

$$= \# \{(u, v, \sigma) \in C \times D \times S_n \mid (u^w)^{\sigma} = v^w \}$$

$$= \sum_{R'', L} \sum_{u \in T^C_L} \sum_{v \in T^{D,w}_L} \# \{ \sigma \in S_n \mid (u^w)^{\sigma} = v^w \}$$

$$= \sum_{R', R'', L} A^C_L B^D_{R''} \prod_{b \in \mathfrak{A}} L_b ! \prod_{a \in \mathfrak{A}} \frac{\ell_a(w)}{R'_{(00a)} ! \cdots R'_{(0|R|-1a)} !}.$$

Hence this completes the proof. \qed

**Some numerical examples**

Here we give some examples of the average Jacobi intersection numbers for some Type II codes over $\mathbb{F}_2$.

1. Let $e_8$ be the extended Hamming code, and $w \in \mathbb{F}_2^8$.
   (i) If $wt(w) = 1$, then $\Delta^w(e_8, e_8) = 4.8$.
   (ii) If $wt(w) = 2$, then $\Delta^w(e_8, e_8) = 6.4$.
   (iii) If $wt(w) = 3$, then $\Delta^w(e_8, e_8) = 9.6$.

2. Let $w \in \mathbb{F}_2^{16}$.
   (i) If $wt(w) = 1$, then
      $$\Delta^w(e_8^2, e_8^2) \approx 5.90769230769,$$
      $$\Delta^w(d_{16}^+, d_{16}^+) \approx 5.90769230769,$$
      $$\Delta^w(d_{16}^+, e_8^2) \approx 5.90769230769.$$
   (ii) If $wt(w) = 2$, then
      $$\Delta^w(d_{16}^+, d_{16}^+) \approx 7.87692307692,$$
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\[ \Delta^w(e_8^2, e_8^2) \approx 7.87692307692, \]
\[ \Delta^w(d_{16}^+, e_8^2) \approx 7.87692307692. \]

(iii) If \( \text{wt}(w) = 3 \), then
\[ \Delta^w(d_{16}^+, d_{16}^+) \approx 11.8153846154, \]
\[ \Delta^w(e_8^2, e_8^2) \approx 11.8153846154, \]
\[ \Delta^w(d_{16}^+, e_8^2) \approx 11.8153846154. \]

(3) Let \( g_{24} \) be the extended Golay code, and \( w \in \mathbb{F}_2^{24} \).

(i) If \( \text{wt}(w) = 1 \), then
\[ \Delta^w(g_{24}, g_{24}) \approx 6.02048106692, \]
\[ \Delta^w(d_{24}^+, d_{24}^+) \approx 6.08859978358, \]
\[ \Delta^w(g_{24}, d_{24}^+) \approx 5.9447244582. \]

(ii) If \( \text{wt}(w) = 2 \), then
\[ \Delta^w(d_{24}^+, d_{24}^+) \approx 8.11813304477, \]
\[ \Delta^w(g_{24}, g_{24}) \approx 8.02730808923, \]
\[ \Delta^w(g_{24}, d_{24}^+) \approx 7.92569659443. \]

(ii) If \( \text{wt}(w) = 3 \), then
\[ \Delta^w(d_{24}^+, d_{24}^+) \approx 12.1771995672, \]
\[ \Delta^w(g_{24}, g_{24}) \approx 12.0409962134, \]
\[ \Delta^w(g_{24}, d_{24}^+) \approx 11.8885448916. \]

(iii) If \( \text{wt}(w) = 4 \), then
\[ \Delta^w(g_{24}, g_{24}) \approx 20.0581090736. \]

(iv) If \( \text{wt}(w) = 5 \), then
\[ \Delta^w(g_{24}, g_{24}) \approx 36.0720806541. \]

A combinatorial \( t-(n, k, \lambda) \) design (or a \( t \)-design for short) is a pair \( \mathcal{D} = (X, \mathcal{B}) \), where \( X \) is a set of \( n \) points, and \( \mathcal{B} \) a collection of \( k \)-element subsets of \( X \) called blocks, with the property that any \( t \) points are contained in precisely \( \lambda \) blocks.

Let \( X := \{1, 2, \ldots, n\} \), and \( u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n \). Then the support of \( u \) is the set of indices of its nonzero coordinates: \( \text{supp}(u) := \{i \mid u_i \neq 0\} \). For an \( \mathbb{F}_q \)-linear code \( C \) of length \( n \), let us define \( \mathcal{B}(C_w) := \{\text{supp}(u) \mid u \in C_w\} \), where \( C_w := \{u \in C \mid \text{wt}(u) = w\} \). In general, \( \mathcal{B}(C_w) \) is a multi-set. For a \( \mathbb{F}_q \)-linear code \( C \) of length \( n \), we say that \( C_w \) is a \( t \)-design if \( (X, \mathcal{B}(C_w)) \) is a \( t \)-design. We say that an \( \mathbb{F}_q \)-linear code \( C \) of length \( n \) is \( t \)-homogeneous if for every given nonzero weight \( w \), \( C_w \) is a \( t \)-design.

Observing the values of the average Jacobi intersection numbers of some binary Type II codes, we have the following conjectures.
Conjecture 6.1 Let $C$ and $D$ be two Type II codes of length $n$ over $\mathbb{F}_2$, and $w \in \mathbb{F}_2^n$. If $C$ and $D$ are $t$-homogeneous and $\text{wt}(w) \leq t$, then

$$
\lim_{n \to \infty} \Delta^w(C, D) = \begin{cases} 6 & \text{if } \text{wt}(w) \leq 1, \\ 8 & \text{if } \text{wt}(w) = 2, \\ 12 & \text{if } \text{wt}(w) = 3, \\ 20 & \text{if } \text{wt}(w) = 4, \\ 36 & \text{if } \text{wt}(w) = 5. \\
\end{cases}
$$

Conjecture 6.2 If $C$ and $D$ are $t$-homogeneous over $\mathbb{F}_q$, and $\text{wt}(w) \leq t$ for $w \in \mathbb{F}_q^n$, then the average Jacobi intersection number of $C$ and $D$ with respect to $w$ can be uniquely determined.

Dougherty defined the $g$-fold joint weight enumerators for codes over Frobenius rings, and gave a generalization of the MacWilliams relation for the weight enumerators in [4]. In future work, we will give a Frobenius-code analogue of the results in this paper. Motivated by the work done by Miezaki and Oura in [8], Chakraborty, Miezaki and Oura introduced the concept of the average complete joint cycle index and gave a relation between the average complete joint cycle index and the average complete joint weight enumerator of codes in [3]. We will discuss a relation between Jacobi polynomials of codes and cycle indices in the subsequent papers.

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References

1. Bonnecaze A., Mourrain B., Solé P.: Jacobi polynomials, Type II codes, and designs. Des. Codes Cryptogr. 16, 215–234 (1999).
2. Chakraborty H.S., Miezaki T.: Average complete joint weight enumerators and self-dual codes. Des. Codes Cryptogr. 89(6), 1241–1254 (2021).
3. Chakraborty H.S., Miezaki T., Oura M.: On the cycle index and the weight enumerator II, submitted.
4. Dougherty S.T.: Algebraic Coding Theory Over Finite Commutative Rings. SpringerBriefs in Mathematics. Springer, Cham (2017).
5. Dougherty S.T., Harada M., Oura M.: Note on the $g$-fold joint weight enumerators of self-dual codes over $\mathbb{Z}_k$. Appl. Algebra Eng. Commun. Comput. 11, 437–445 (2001).
6. Honma K., Okabe T., Oura M.: Weight enumerator, intersection enumerator and Jacobi polynomial. Discret. Math. 343(6), 111815 (2020).
7. MacWilliams F.J., Mallows C.L., Sloane N.J.A.: Generalizations of Gleason’s theorem on weight enumerators of self-dual codes. IEEE Trans. Inf. Theory 18, 794–805 (1972).
8. Miezaki T., Oura M.: On the cycle index and the weight enumerator. Des. Codes Cryptogr. 87(6), 1237–1242 (2019).
9. Ozeki M.: On the notion of Jacobi polynomials for codes. Math. Proc. Camb. Philos. Soc. 121(1), 15–30 (1997).
10. Yoshida T.: The average of joint weight enumerators. Hokkaido Math. J. 18, 217–222 (1989).
11. Yoshida T.: The average intersection number of a pair of self-dual codes. Hokkaido Math. J. 20, 539–548 (1991).

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