Constraints on Non-Holomorphic Correction in $\mathcal{N}=2$ Superspace

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Abstract

We study quantum corrections on four derivative term of vector multiplets in $\mathcal{N}=2$ supersymmetric Yang-Mills theory. We first show splitting of quantum correction on gauge neutral hypermultiplets from $U(1)$ vector multiplets at four derivative order. We then revisit the non-renormalization theorem given by N. Seiberg and M. Dine and show the non-renormalization theorem in mixed (Coulomb plus Higgs) branch even though gauge neutral hypermultiplet develops the vacuum expectation value.

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1 Introduction

In the last decade supersymmetric field theories have been much explored and exact solutions are well-often obtained not only perturbatively but also non-perturbatively. The key ingredient is holomorphic argument (See [1] for reviews of $\mathcal{N} = 1$ supersymmetric field theories). For $\mathcal{N} = 2$ SYM theory in four dimensions, the low energy effective action at leading order is completely determined by $\mathcal{N} = 2$ superspace chiral integral of holomorphic function [2]. Notice that Kähler potential is written as non-holomorphic term from $\mathcal{N} = 1$ supersymmetric point of view. More supersymmetry, more controls of quantum corrections! For $\mathcal{N} = 4$ SYM theory we expect it is strongly constrained. However, analysis is hampered because of the lack of manifest $\mathcal{N} = 4$ superspace. If such a formalism is known, we would directly obtain the non-renormalization theorem of four-derivative term, which is expected to be chiral integral in $\mathcal{N} = 4$ superspace. Despite this obstacle, M. Dine and N. Seiberg have shown the non-renormalization theorem [3] of four derivative operators of purely chiral multiplets by taking advantage of $\mathcal{N} = 2$ superspace formulation and symmetry arguments in Coulomb branch of SU(2) gauge theory without turning on the vacuum expectation value of gauge neutral hypermultiplet. This result has been extended to SU(N) gauge theory in several papers [4, 5, 6] and these results are important in the context of recent studies of Matrix theory, and in the ADS/CFT correspondence of string theory. Although extra symmetries are assigned appropriately to be equivalent to $\mathcal{N} = 4$ supersymmetry, all of these analysis are still based on the $\mathcal{N} = 2$ superspace. Studies of the next-to-leading correction in various branch are also important, however, crossing quantum corrections among chiral multiplets and hypermultiplets at four derivative order have not been studied in detail because of the lack of simple $\mathcal{N} = 2$ superspace formulation as far as we know.

On the other hand, splitting of quantum correction among $U(1)$ chiral multiplets and gauge neutral hypermultiplets at two derivative order has been already observed in [7, 8] since $\mathcal{N} = 2$ supersymmetry implies that the kinetic term of chiral multiplet is constrained to be Kähler geometry while that of hypermultiplet is to be hyper-Kähler geometry. With the aid of superspace Feynman rule, perturbative study of $\mathcal{N} = 2$ non-holomorphic correction indicates the Kähler-like-geometry for four derivative terms involving only vector multiplets [9] and we then expect that this splitting structure is held even at four derivative order.

In this letter, we first show absence of crossing quantum corrections among gauge neutral hypermultiplets and $U(1)$ vector multiplets on $\mathcal{N} = 2$ non-holomorphic term at four derivative order. We then revisit the non-renormalization theorem of four derivative operator of gauge field for $\mathcal{N} = 2$ SYM theory [3] and clarify that it is true even though
in mixed branch.

2 Splitting of Quantum Corrections at Four Derivative Order

Let us begin with $\mathcal{N} = 2$ SYM theory based on a group $G$ of rank $r$ which is spontaneously broken by the vacuum expectation values of scalar fields either of chiral multiplets or of hypermultiplets. In the following we focus on such a mixed branch that is Abelian theory with $v(< r)$ Abelian chiral multiplets and $h$ gauge neutral hypermultiplets. If the theory is asymptotically-free, regions of large vacuum expectation value of scalar fields in chiral multiplet correspond to the weak coupling regime. Taking all the vacuum expectation values large enough, all massive fields become very heavy and almost decouple from the theory. It is then reasonable that the low energy “Wilsonian” effective action is given in terms of $v$ Abelian chiral multiplets and $h$ gauge neutral hypermultiplet. According to supersymmetric derivative expansion, four derivative operators would arise as non-holomorphic corrections on $\mathcal{N} = 2$ supersymmetric effective action in the following form

$$\int d^4 \theta d^4 \bar{\theta} \mathcal{H}(\Psi^a, \Psi^{a\dagger}; S^i, S^{i\dagger}),$$

where we denote gauge neutral $\mathcal{N} = 2$ hypermultiplets by $S^i$, $i = 1, 2, ..., h$ and $\mathcal{N} = 2$ Abelian chiral multiplets by $\Psi^a$, $a = 1, 2, ..., v$, respectively. To analyze the structure of some scalar function $\mathcal{H}$, it is convenient to replace $\mathcal{N} = 2$ superfields in terms of $\mathcal{N} = 1$ superfield and because of $\mathcal{N} = 2$ supersymmetry, it is enough to consider the following function

$$\int d^4 \theta d^4 \bar{\theta} \mathcal{H}(\Phi^a, \Phi^{a\dagger}; Q^i, Q^{i\dagger}, \tilde{Q}^{i\dagger}),$$

where we denote gauge neutral hypermultiplet by $\mathcal{N} = 1$ chiral superfields $Q^i$, $\tilde{Q}^{i\dagger}$, $i = 1, 2, ..., h$ and $\mathcal{N} = 2$ Abelian chiral superfield by $\mathcal{N} = 1$ superfields $\Phi^a$, $W^a_\alpha$, $a = 1, 2, ..., v$ respectively.

Manipulating super-covariant derivatives, this includes a crossing four derivative term, $\partial_\phi \partial_{\tilde{q}^i} \mathcal{H} \cdot \partial^2 \phi \partial^2 \tilde{q}^{i\dagger}$ where $\phi$ and $\tilde{q}^{i\dagger}$ are the scalar component of chiral multiplet and that of hypermultiplet, respectively. $\mathcal{N} = 1$ supersymmetry implies that $\partial_\phi \partial_{\tilde{q}^i} \mathcal{H} \cdot (-i \bar{\zeta} \bar{\sigma}^\mu \partial_\mu \partial^2 \psi)$ must be accompanied in order to cancel $\mathcal{N} = 1$ supersymmetric variation, where we denote hypermultiplet fermion and chiral multiplet fermion, associated with $\mathcal{N} = 1$ supersymmetry transformation, by $\bar{\zeta}$ and $\psi$, respectively. However, such a term is not allowed since to

\[\text{in the scale-invariant case we take the bare coupling to be small.}\]

\[\text{we follow the conventions and the reduction from } \mathcal{N} = 2 \text{ superfield to } \mathcal{N} = 1 \text{ superfield of } [3].\]

\[\text{Due to } \mathcal{N} = 2 \text{ supersymmetry, we do not consider explicit dependence of } \mathcal{N} = 1 \text{ superfield strength } W_\alpha \text{ on } \mathcal{H} \text{ while for vector superfields } V \text{ we can show that the only possible dependence upon this function is Fayet-Iliopoulos D-term by supersymmetric gauge invariance.}\]
cancel the other supersymmetric variation within $\mathcal{N} = 2$ supersymmetry, there must be a term involving four derivatives, a gauge field and a scalar field, out of which no Lorentz invariant combination can be formed. We then obtain $\partial_\phi \partial_\bar{q} \mathcal{H} = 0$. This implies
\[ \mathcal{H}(\Psi^a, \Psi^{a\dagger}; S^i, S^{i\dagger}) = \mathcal{H}_V(\Psi^a, \Psi^{a\dagger}) + \mathcal{H}_H(S^i, S^{i\dagger}), \]
and we find the vacuum expectation value of hypermultiplet never appears in the Kähler metric of chiral multiplet even at four derivative order. Further constraints will be given by thinking of coupling as a background fields as we will see in the next section.

3 Revisiting the Non-renormalization Theorem of Four Derivative Operators in $\mathcal{N} = 2$ finite SYM theory

We would like to revisit the non-renormalization theorem of four derivative operator of $\mathcal{N} = 2$ finite SYM theory in mixed branch as an immediate consequence of previous section. Consider $SU(N)$ SYM theory, breaking to single $U(1)$ case. With $\mathcal{N} = 2$ superspace formulation, the low energy effective action is described by a $U(1)$ chiral multiplet $\Psi$ and some gauge neutral massless hypermultiplets $S_i$. The kinetic term for chiral superfield is given by the chiral integral
\[ \int d^2 \theta d^2 \bar{\theta} \tau \Psi^2, \]
where $\tau$ is usual holomorphic gauge coupling [2]. Using the scale invariance and holomorphy, this term remains quadratic after quantum corrections are included. It is important to note that $\tau$ is chiral. On the other hand, terms with four derivative arises from
\[ \int d^4 \theta d^4 \bar{\theta} \mathcal{H}(\Psi, \bar{\Psi}^\dagger; S_i, S_i^{\dagger}). \]
As we have shown in the previous section, no crossing term among $U(1)$ chiral multiplet and gauge neutral hypermultiplets is effectively generated, so four derivative terms of gauge field arise from purely chiral part. We then restrict our attention to terms given by purely chiral multiplet
\[ \int d^4 \theta d^4 \bar{\theta} \mathcal{H}_V(\Psi, \bar{\Psi}^\dagger, \tau, \tau^{\dagger}). \]
Both the scale-invariance and the $U(1)_R$ invariance provide a unique form
\[ \mathcal{H} = c \ln \Psi \ln \bar{\Psi}^\dagger, \]
with a constant $c$. To determine the $\tau$ dependence, we promote $\tau$ as a background chiral superfield. This expression spoils both the scale-invariance and the $U(1)_R$ invariance unless it is independent of $\tau$. Then we find that such a correction arises only from one-loop level.

\[ ^5 \text{Instanton calculation in [11] further confirms this result and determine the constant } c = 1/(8\pi)^2. \]
4 Remarks

In this letter we have first presented absence of crossing quantum corrections among \( U(1) \) chiral multiplets and gauge neutral hypermultiplets at four derivative order. From the viewpoint of symmetries of Lagrangian, we do not know how they control the crossing quantum corrections among \( U(1) \) chiral multiplets and gauge neutral hypermultiplets but \( \mathcal{N} = 2 \) supersymmetry controls them since a variation of Lagrangian under supersymmetry is always total divergence and it is a symmetry of action. The similar aspects can be seen in the Chern-Simons Theory of three dimensional gauge theory \([13]\).

We then revisit the non-renormalization theorem given by M. Dine and N. Seiberg. Our proof of the non-renormalization theorem is given in terms of Wilsonian effective action. The Wilsonian effective action is different from ordinary 1PI effective action if interacting massless particles exist. As reported in \([13]\), there is holomorphic anomaly related to IR issues and resulting the violation of non-renormalization theorem. Since they are absent in Wilsonian effective action, it is well-often favored in the study of supersymmetric theories. However, as reported in \([7]\), there are violating terms of special Kähler geometry even in a naive Wilsonian effective action and authors of \([14]\) proposed the Wilsonian effective action with field dependent cut-off. Further discussion about issues of regularization is beyond the scope of this letter.

The final remark is as follows: It is interesting to know the non-renormalization theorem when the low energy effective action is described by more than two \( U(1) \) chiral superfields. In our proof of non-renormalization theorem in mixed branch we have simply restricted to \( SU(N) \) breaking to single \( U(1) \) case. Since renormalization is UV nature, it is independent of IR structure and there might be non-renormalization theorem even if more than two \( U(1) \) gauge symmetries survive. Unfortunately, we do not know how to obtain the exact solution by utilizing symmetries of Lagrangian. For example, Such a term

\[
f(\tau, \tau^\dagger) \frac{\Psi^a \bar{\Psi}^c}{\Psi^b \bar{\Psi}^d},
\]

do not conflict with both the scale-invariance and the \( U(1)_R \) invariance. In perturbative study of \( \mathcal{N} = 2 \) SYM theory\([4]\), the terms like \((8)\) is absent but how it goes non-perturbatively? Direct test by instanton calculation is of great interest.

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