Analysis of the $[56, 2^+]$ Baryon Masses in the $1/N_c$ Expansion

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Abstract

The mass spectrum of the positive parity $[56, 2^+]$ baryons is studied in the $1/N_c$ expansion up to and including $O(1/N_c)$ effects with $SU(3)$ symmetry breaking implemented to first order. A total of eighteen mass relations result, several of which are tested with the available data. The breaking of spin-flavor symmetry is dominated by the hyperfine interactions, while spin-orbit effects are found to be small.

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In the mass range from 1600 to 2100 MeV there exists a set of positive parity baryons which might be assigned to an irreducible representation [56, 2+] of $SU(6) \otimes O(3)$, where $SU(6)$ is the spin-flavor group and $O(3)$ classifies the orbital excitations. Among the candidate states in that set, all non-strange states are known as well as seven strangeness $S = -1$ states. Some of the strange states are, however, established with low certainty (two stars or less in the Particle Data Listings [1]). In this letter the available empirical information is used to implement an analysis of the masses based on the $1/N_c$ expansion of QCD [2, 3], an approach that has turned out to be very successful in baryon phenomenology.

The $1/N_c$ expansion has been applied to the ground state baryons [4, 5, 6, 7, 8, 9, 10], and to excited baryons, where the masses and decays of the negative parity spin-flavor 70-plet [11, 12, 13, 14, 15] and the positive parity Roper 56-plet [16] have been analyzed. Two frameworks have been used in implementing the $1/N_c$ expansion for baryons. One framework is based on the contracted spin-flavor $SU(2N_f)_c$ symmetry, $N_f$ being the number of light flavors, which is a symmetry of QCD in the $N_c \to \infty$ limit [4, 12, 17]. In this framework commutation relations of operators like axial currents and hadron masses are constrained by consistency relations. The observed baryons at $N_c = 3$ are identified with the low lying spin states of an infinite representation of the contracted symmetry. The second framework makes use of the spin-flavor $SU(2N_f)$ algebra, with an explicit representation of operators that act on a space of states constructed as tensor products of $N_c$ valence quarks [7]. Both approaches are consistent and deliver equivalent results order by order in the $1/N_c$ expansion. From the practical point of view, however, the second one is easier to work with, especially at subleading orders in $1/N_c$, and for this reason it has been chosen in most analyses. Another advantage in this approach, is the possibility of using the language of the constituent quark model, as applied to the spin-flavor degrees of freedom, without any loss of generality.

The study of excited baryons is not free of difficulties. Although a significant amount of symmetry in the form of a contracted $SU(2N_f)_c$ is always present in the $N_c \to \infty$ limit [12, 18], there is no strict spin-flavor symmetry in that limit. Indeed, as it was shown in [11], spin-orbit interactions break spin-flavor symmetry at $O(N_c^0)$ in states belonging to mixed symmetric spin-flavor representations, and configuration mixing, i.e., mixing of states belonging to different spin-flavor multiplets in general occurs at $O(N_c^0)$ as well. The use of spin-flavor symmetry as a zeroth order approximation is therefore not warranted for excited baryons. However, a phenomenological fact is that spin-orbit interactions are very small (in
the real world with \( N_c = 3 \) they have a magnitude expected for \( \mathcal{O}(N_c^{-2}) \) effects), and since all sources of \( \mathcal{O}(N_c^0) \) spin-flavor breaking, including the configuration mixing, requires such orbital interactions, it is justified to treat them in practice as subleading. Thanks to this observation, the usage of spin-flavor \( SU(2N_f) \) as the zeroth order symmetry is justified. A second problem is posed by the fact that excited baryons have finite widths. The impact of this on the analyses of the masses is not fully clarified yet. One likely possibility is that their effects are included in the effective parameters that determine the masses’ \( 1/N_c \) expansion. This is an issue that has been recently considered in Ref. \([19]\).

The analysis of the \([56, 2^+]\) masses is made along the lines established in previous investigations of the \([70, 1^-]\) baryons \([11, 13, 15]\). The \([56, 2^+]\) multiplet contains two \( SU(3) \) octets with total angular momentum \( J = 3/2 \) and \( 5/2 \), and four decuplets with \( J = 1/2, 3/2, 5/2 \) and \( 7/2 \), as listed in Table \( \Box \). Note that the octets have spin \( S = 1/2 \) while the decuplets have spin \( S = 3/2 \) as in the ground state baryons. For non-strange states this is the \( I = S \) rule. The states are obtained by coupling the orbital part with \( \ell = 2 \) to the spin-flavor symmetric states, namely,

\[
|J J_z; (p = 2S, q), Y, I I_z\rangle_{\text{sym}} = \sum_{m, S_z} \left( \begin{array}{c} J \\ S_z \\ m \\ J_z \end{array} \right) |S S_z; (p, q), Y, I I_z\rangle_{\text{sym}} |\ell = 2 m\rangle (1)
\]

where \( (p, q) \) label the \( SU(3) \) representation and \( Y \) stands for the hypercharge. Note that, unlike the states in mixed-symmetric representations where excited and core quarks have to be distinguished for the purpose of building a basis of mass operators, such a distinction is unnecessary for the symmetric representation.

The mass operator can be expressed as a string of terms expanded in \( 1/N_c \):

\[
H_{\text{mass}} = \sum c_i O_i + \sum b_i \bar{B}_i
\] (2)

where the operators \( O_i \) are \( SU(3) \) singlets and the operators \( \bar{B}_i \) provide \( SU(3) \) breaking and are defined to have vanishing matrix elements between non-strange states. The effective coefficients \( c_i \) and \( b_i \) are reduced matrix elements that encode the QCD dynamics and they are determined by a fit to the empirically known masses.

The operators \( O_i \) and \( \bar{B}_i \) can be expressed as positive parity and rotationally invariant products of generators of \( SU(6) \otimes O(3) \) as it has been explained elsewhere \([11]\). A generic \( n \)-body operator has the structure

\[
o^{(n)} = \frac{1}{N_{n-1}} O_{\ell} O_{SF}, \tag{3}
\]
where the factors $O_\ell$ and $O_{SF}$ can be expressed in terms of products of generators of the orbital group $O(3)$ ($\ell_i$), and of the spin-flavor group $SU(6)$ ($S_i$, $T_a$ and $G_{ia}$), respectively. The explicit $1/N_c$ factors originate in the $n-1$ gluon exchanges required to give rise to an $n$-body operator. The matrix elements of operators may also carry a nontrivial $N_c$ dependence due to coherence effects [4]: for the states considered, $G_{ia}$ ($a = 1, 2, 3$) and $T_8$ have matrix elements of $O(N_c)$, while the rest of the generators have matrix elements of zeroth order.

At each order in $1/N_c$ and $\epsilon$, where the latter parameter measures $SU(3)$ breaking, there is a basis of operators. The construction of these bases is straightforward, and the operators are listed in Table I. The corresponding matrix elements between the states belonging to the $[56, 2^+]$ multiplet are given in Tables II and III. Note that the operators used in the analysis of the $[70, 1^-]$ masses are reduced to the operators given here in Table I. This can be shown using reductions, valid for the symmetric representation, of matrix elements involving excited quark and/or core operators, such as:

$$\langle \text{Sym} \mid s_i \mid \text{Sym} \rangle = \frac{1}{N_c} \langle \text{Sym} \mid S_i \mid \text{Sym} \rangle$$

$$\langle \text{Sym} \mid S^c_i \mid \text{Sym} \rangle = \frac{N_c - 1}{N_c} \langle \text{Sym} \mid S_i \mid \text{Sym} \rangle,$$

etc.,

where $S = s + S^c$, $s$ being the spin operator acting only on one quark (the excited one for instance), and $S^c$ acts on the remaining ($N_c - 1$) core quarks. Similarly, relations for two-body operators can also be derived, e.g.:

$$\langle \text{Sym} \mid s_i G^c_{ja} \mid \text{Sym} \rangle = \langle \text{Sym} \mid s_i (G_{ja} - g_{ja}) \mid \text{Sym} \rangle$$

$$= \frac{1}{N_c} \langle \text{Sym} \mid S_i G_{ja} - \frac{i}{4} \delta_{ij} T_a - \frac{i}{2} \epsilon_{ijk} G_{ka} \mid \text{Sym} \rangle$$

An important observation is that in the present case there is no $SU(3)$ singlet operator breaking spin-flavor symmetry at $O(N_c^0)$. In particular, operators involving the $O(3)$ generators, that in the mixed-symmetric spin-flavor representations can be $O(N_c^0)$, are demoted to $O(1/N_c)$ in the spin-flavor symmetric representation. At $O(N_c^{-1})$ only two singlet operators appear, the spin-orbit operator $O_2$ and the hyperfine operator $O_3$, both being two-body operators. At order $\epsilon$ there is one operator $O(N_c^0)$, namely $\bar{B}_1$ and two operators $O(N_c^{-1})$, namely $\bar{B}_{2,3}$.

Note that in the $56$-plet there are no state mixings in the $SU(3)$ symmetric limit. Only the operator $\bar{B}_2$ induces mixings. The mixings affect the octet and decuplet $\Sigma^{(8),(10)}$ and $\Xi^{(8),(10)}$ states, and in the limit of isospin symmetry there are four mixing angles, namely $\theta^\Sigma, \Sigma'$ and

...
The physical states are given by \( \Sigma_J = \Sigma_J^{(8)} \cos \theta_J^{\Sigma,\Sigma'} + \Sigma_J^{(10)} \sin \theta_J^{\Sigma,\Sigma'} \) and \( \Sigma'_J = -\Sigma_J^{(8)} \sin \theta_J^{\Sigma,\Sigma'} + \Sigma_J^{(10)} \cos \theta_J^{\Sigma,\Sigma'} \) and in a similar way for the cascades. The mixing angles are determined by the ratio of the matrix elements of the operator \( \bar{B}_2 \) to the spin-flavor mass splitting induced by the \( O(N_c^{-1}) \) singlet operators. This implies that the mixing angles are \( O(\epsilon N_c^0) \). The mixings affect the mass eigenvalues at \( O(\epsilon^2/N_c) \), which is beyond the accuracy of the present analysis.

Since there are twenty four independent masses in the isospin symmetric limit, and the basis consists of six operators, there are eighteen mass relations that hold independently of the values of the coefficients \( c_i \) and \( b_i \). These relations are depicted in Table IV. In addition to the Gell-Mann Okubo (GMO) relations for each octet (two such relations) and the equal spacing relations (EQS) for each decuplet (eight such relations), there are eight relations that involve states belonging to different \( SU(3) \) multiplets as well as different values of \( J \): the first three in the Table IV involve only the masses of non-strange states, while the remaining five relations have been chosen in such a way that several of them can be tested directly with the available data. These latter eight relations provide a useful test of the validity of the \( 1/N_c \) expansion as implemented in this analysis. The GMO and EQS relations cannot be tested due to the scarcity of information on strange baryons. If the one and two star states are excluded, there are four relations that can be tested, namely, the three non-strange ones (1, 2 and 3) and relation (4). If the one and two star states are included (three such states), there are three additional relations that can be tested, namely (5, 6 and 7). In all cases they are found to be satisfied within the experimental errors. Following [9], in order to compare to what extent the empirical accuracies of the mass relations match the theoretical expectations, each of the mass relations in Table IV is cast in the form LHS = RHS with the left hand side (LHS) and right hand side (RHS) possessing only terms with positive coefficients. The accuracy of the mass relations is then defined as \( |LHS - RHS|/[(LHS + RHS)/2] \). These ratios are \( O(\epsilon^2 N_c^{-2}) \) for the GMO and EQS relations, and \( O(N_c^{-3}) \), \( O(\epsilon^2 N_c^{-2}) \) and/or \( O(\epsilon N_c^{-3}) \) for the others. For \( N_c = 3 \), and \( \epsilon \sim 1/3 \), the ratios associated with the relations (1) to (8) in Table IV are estimated to be of the order of 4%. The ratios obtained with the physical masses are listed in the last column of Table IV and they are within that estimated theoretical range. It is important to emphasize that all these empirically verified relations represent a genuine test of spin-flavor symmetry and its breaking according to the \( 1/N_c \) expansion, as pointed out above. The fact
that they are all verified to the expected accuracy is remarkable and gives strong support to the analysis based on the premises of this work.

The fit to the available data, where states with three or more stars in the Particle Data Listings are included, leads to the effective constants $c_i$ and $b_i$ shown in Table II and the results for the masses shown in Table IV, where fourteen of them are predictions. The $\chi^2_{\text{dof}}$ of the fit is 0.7, where the number of degrees of freedom (dof) is equal to four. The errors shown for the predictions in Table IV are obtained propagating the errors of the coefficients in Table II. There is also a systematic error $\mathcal{O}(N c^{-2})$, resulting from having included only operators up to $\mathcal{O}(N c^{-1})$ in the analysis, which can be roughly estimated to be around 30 MeV.

In Table IV the partial contributions from each operator to the mass of the different members of the multiplet are also shown. The operator $O_1$ provides the spin-flavor singlet mass of about 1625 MeV. The breaking of spin-flavor symmetry by the $SU(3)$ singlet operators is essentially given in its entirety by the hyperfine interaction $O_3$, that produces a splitting between octet and decuplet states of approximately 240 MeV, while the spin-orbit operator $O_2$ is rather irrelevant inducing spin-flavor breaking mass shifts of less than 30 MeV. Note that $O_2$ is the sole source of the splittings between the two $N$ states and also between the $\Delta$ states. The weakness of $O_2$ is thus very convincingly established.

The breaking of $SU(3)$ is dominated by the operator $\bar{B}_1$, which gives a shift of about 200 MeV per unit of strangeness. The main role of the subleading operators $\bar{B}_{2,3}$ is to provide the observed $\Lambda - \Sigma$ splittings in the octets, and the different splittings between the $N$ and the average $\Lambda - \Sigma$ masses in the two octets, and the $\Sigma - \Delta$ splitting in the $J = 7/2$ decuplet. Finally, $\bar{B}_2$ gives the only contributions to the state mixings. The mixing angles that result from the fit are: $\theta_{3/2}^{\Sigma,\Sigma'} = -0.16, \theta_{5/2}^{\Sigma,\Sigma'} = -0.26, \theta_{3/2}^{\Xi,\Xi'} = -0.21$ and $\theta_{5/2}^{\Xi,\Xi'} = -0.19$ (in radians).

The better established $\Lambda - \Sigma$ splitting in the $J = 5/2$ octet is almost 100 MeV, while the other splitting in the $J = 1/2$ octet is small and slightly negative. The latter one involves however the one star state $\Sigma(1840)$, which might also be assigned to the radially excited $56'$. The large $N_c$ analysis implies that these splittings are $\mathcal{O}(\epsilon/N_c)$, and are produced only by the operators $\bar{B}_2$ and $\bar{B}_3$. The result from the fit indicates that the $\Lambda_{5/2} - \Sigma_{5/2}$ receives a contribution of 63 MeV from $\bar{B}_3$ and 40 MeV from $\bar{B}_2$. It is interesting to observe that several mass splitting differences receive only contributions from $\bar{B}_2$ as it is obvious from Table III. These involve the splittings in the octets ($\Lambda_{5/2} - N_{5/2}) - (\Lambda_{3/2} - N_{3/2}$),
\((\Sigma_{5/2} - N_{5/2}) - (\Sigma_{3/2} - N_{3/2})\) and \((\Xi_{5/2} - N_{5/2}) - (\Xi_{3/2} - N_{3/2})\), and the decuplet splittings
\((\Sigma_J - \Delta_J) - (\Sigma_{J'} - \Delta_{J'})\), \((\Xi_J - \Delta_J) - (\Xi_{J'} - \Delta_{J'})\) and \((\Omega_J - \Delta_J) - (\Omega_{J'} - \Delta_{J'})\). Further
information on these splittings would allow to pin down with better confidence the relevance
of \(\bar{B}_2\). The fit implies for instance that the contribution of \(\bar{B}_2\) to the \(\Omega_{1/2} - \Omega_{7/2}\) splitting
is about \(225 \pm 100\) MeV, a rather large effect. The operator \(\bar{B}_2\) involves the orbital angular
momentum operator, and since in all other known cases where orbital couplings occur their
effects are suppressed, the same would be expected here. The naive expectation is that the
coefficient of \(\bar{B}_2\) would be of order \(2\sqrt{3} \epsilon\) times the coefficient of \(O_2\). It is in fact substantially
larger. However, this result is not very conclusive, because \(b_2\) is largely determined by only
a few inputs resulting in a rather large relative error for this parameter. Related to this, the
sign of the coefficient of \(\bar{B}_2\) determines the ascending or descending ordering of the masses
of strange states in the decuplet as \(J\) increases. In the present analysis the higher \(J\) states
are lighter. However, the structure of \(SU(3)\) breaking splittings cannot be established better
because of the rather small number of strange states available for the fit. This is perhaps
the most important motivation for further experimental and lattice QCD study of the still
non-observed states.

It is of interest to draw some comparisons among the analyses carried out in previous
works, that include the ground state baryons [8, 9], the \([70, 1^-]\) baryons [13, 15], the Roper
multiplet \([56', 0^+]\) [16], and the present analysis. At the level of \(SU(3)\) singlet operators
the hyperfine interaction is \(O(1/N_c)\) in all cases. It is interesting to compare the strength of the
hyperfine interaction in the different multiplets estimating the strength of the quark pairwise
hyperfine interaction. In the large \(N_c\) limit that strength should be the same for different
low lying excited states. For the ground state baryons the hyperfine operator, up to terms
proportional to the identity operator, is given by: \(\sum_{i \neq j} s_i \cdot s_j\) where the indices \(i, j\) run
from 1 to \(N_c\). The ground state \(\Delta - N\) splitting then gives a strength of about 100 MeV for
this operator. In excited states with \(\ell = 1\), the results from the \([70, 1^-]\) plet depend in general
on the mixing angles used as an input [18]. For the particular choice of the angles used in
the analyses [13, 14, 15] the hyperfine interaction involving the excited quark and quarks
in the core (the operator \(O_7\) in [15]) is suppressed, indicating that the hyperfine interaction
is predominantly short range. The relevant hyperfine interaction is in this case the one
involving the \(N_c - 1\) quarks in the core, i.e., the indices \(i\) and \(j\) run only over the quarks
in the core. This leads in the \([70, 1^-]\) to a strength of about 160 MeV. In the \([56, 2^+], a
reasonable assumption is that a single quark is excited with $\ell = 2$. From the result obtained in the $[70, 1^-]$, it is expected that the excited quark will also have negligible participation in the hyperfine interaction. This will be given essentially by the hyperfine interaction of the core quarks. Using the relation $\left(1 - \frac{2}{N_c}\right) S^2 = (S^c)^2 - \frac{3}{4}$ valid in states belonging to the symmetric spin-flavor representation, the result of the fit implies a strength of 240 MeV. In the Roper $[56', 0^+]$ multiplet the situation is less clear, as all quarks may participate of the hyperfine interaction. The average strength in the core turns out to be about 160 MeV. These results indicate an increase in the strength of the hyperfine interaction in going from the ground state baryons to excited baryons. This suggests the presence of an underlying dynamical mechanism that might be possible to identify in specific models. The other $SU(3)$ singlet interaction common to all multiplets, the spin-orbit interaction, is weak in the two known cases, namely $[70, 1^-]$ and $[56, 2^+]$.

The $SU(3)$ breaking operator $\bar{B}_1 = -S$ gives a mass shift per unit of strangeness of about 200 MeV in all multiplets considered, which is in line with the the value of the strange quark mass. The operator $l_i g_{i8}$ which contributes at $O(\epsilon N_c^0)$ in the $56$-plet and at $O(\epsilon/N_c)$ in the $70$-plet, carries coefficients of similar size but different sign. This issue can be further clarified when the role of $\bar{B}_2$ is better established.

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| Operator                  | Fitted coef. (MeV) |
|--------------------------|--------------------|
| $O_1 = N_c \, 1$         | $c_1 = 541 \pm 4$  |
| $O_2 = \frac{1}{N_c} l_i S_i$ | $c_2 = 18 \pm 16$  |
| $O_3 = \frac{1}{N_c} S_i S_i$ | $c_3 = 241 \pm 14$ |

| $B_1 = -S$               | $b_1 = 206 \pm 18$  |
| $B_2 = \frac{1}{N_c} l_i G_{i8} - \frac{1}{2\sqrt{3}} O_2$ | $b_2 = 104 \pm 64$  |
| $B_3 = \frac{1}{N_c} S_i G_{i8} - \frac{1}{2\sqrt{3}} O_3$ | $b_3 = 223 \pm 68$  |

**TABLE I:** List of operators and the coefficients resulting from the fit with $\chi^2_{dof} = 0.7$

| Operator                  | $O_1$ | $O_2$ | $O_3$ |
|--------------------------|-------|-------|-------|
| $^{2}S_{3/2}$ $N_c$      | $-\frac{3}{2N_c}$ | $\frac{3}{4N_c}$ |
| $^{2}S_{5/2}$ $N_c$      | $\frac{1}{N_c}$ | $\frac{3}{4N_c}$ |
| $^{4}T_{1/2}$ $N_c$      | $-\frac{3}{2N_c}$ | $\frac{15}{4N_c}$ |
| $^{4}T_{3/2}$ $N_c$      | $-\frac{3}{N_c}$ | $\frac{15}{4N_c}$ |
| $^{4}T_{5/2}$ $N_c$      | $-\frac{1}{2N_c}$ | $\frac{15}{4N_c}$ |
| $^{4}T_{7/2}$ $N_c$      | $\frac{3}{N_c}$ | $\frac{15}{4N_c}$ |

**TABLE II:** Matrix elements of $SU(3)$ singlet operators.
| $B_1$  | $B_2$  | $B_3$  |
|-------|-------|-------|
| $N_J$ | 0     | 0     | 0     |
| $\Lambda_J$ | 1     | $\frac{3\sqrt{3} a_J}{4N_c}$ | $\frac{-3\sqrt{3}}{8N_c}$ |
| $\Sigma_J$ | 1     | $\frac{-\sqrt{3} a_J}{4N_c}$ | $\frac{\sqrt{3}}{8N_c}$ |
| $\Xi_J$ | 2     | $\frac{\sqrt{3} a_J}{N_c}$ | $\frac{-\sqrt{3}}{2N_c}$ |
| $\Delta_J$ | 0     | 0     | 0     |
| $\Sigma_J$ | 1     | $\frac{3\sqrt{3} b_J}{4N_c}$ | $\frac{-5\sqrt{3}}{8N_c}$ |
| $\Xi_J$ | 2     | $\frac{3\sqrt{3} b_J}{2N_c}$ | $\frac{-5\sqrt{3}}{4N_c}$ |
| $\Omega_J$ | 3     | $\frac{9\sqrt{3} b_J}{4N_c}$ | $\frac{-15\sqrt{3}}{8N_c}$ |

TABLE III: Matrix elements of $SU(3)$ breaking operators. Here, $a_J = 1, -2/3$ for $J = 3/2, 5/2$, respectively and $b_J = 1, 2/3, 1/9, -2/3$ for $J = 1/2, 3/2, 5/2, 7/2$, respectively.
\[
\begin{align*}
(1) \quad & \Delta_{5/2} - \Delta_{3/2} = N_{5/2} - N_{3/2} & \text{Accuracy: 0.6\%} \\
(2) \quad & 5(\Delta_{7/2} - \Delta_{5/2}) = 7(N_{5/2} - N_{3/2}) & \text{Accuracy: 1.8\%} \\
(3) \quad & \Delta_{7/2} - \Delta_{1/2} = 3(N_{5/2} - N_{3/2}) & \text{Accuracy: 1.5\%} \\
(4) \quad & 8(\Lambda_{3/2} - N_{3/2}) + 22(\Lambda_{5/2} - N_{5/2}) = 15(\Sigma_{5/2} - \Lambda_{5/2}) + 30(\Sigma_{7/2} - \Delta_{7/2}) & \text{Accuracy: 0.4\%} \\
(5) \quad & \Lambda_{5/2} - \Lambda_{3/2} + 3(\Sigma_{5/2} - \Sigma_{3/2}) = 4(N_{5/2} - N_{3/2}) & \text{Accuracy: 1.7\%} \\
(6) \quad & \Lambda_{5/2} - \Lambda_{3/2} + \Sigma_{5/2} - \Sigma_{3/2} = 2(\Sigma'_{5/2} - \Sigma'_{3/2}) & \text{Accuracy: 0.5\%} \\
(7) \quad & 7 \Sigma'_{3/2} + 5 \Sigma_{7/2} = 12 \Sigma'_{5/2} & \text{Accuracy: 0.5\%} \\
(8) \quad & 4 \Sigma'_{1/2} + \Sigma_{7/2} = 5 \Sigma'_{3/2} & \\
\end{align*}
\]

\begin{align*}
\text{(GMO)} \quad & 2(N + \Xi) = 3 \Lambda + \Sigma \\
\text{(EQS)} \quad & \Sigma - \Delta = \Xi - \Sigma = \Omega - \Xi
\end{align*}

| TABLE IV | The 18 independent mass relations include the GMO relations for the two octets and the two EQS for each of the four decuplets. The accuracy is calculated as explained in the text. |
| 1/$N_c$ expansion results | Partial results | Total | Empirical |
|---------------------------|----------------|-------|-----------|
|                           | $O_1$ | $O_2$ | $O_3$ | $\bar{B}_1$ | $\bar{B}_2$ | $\bar{B}_3$ |       |           |
| $N_{3/2}$                 | 1623  | -9    | 60    | 0          | 0          | 0          | 1674 ± 15 | 1700 ± 50 |
| $\Lambda_{3/2}$          | 206   | 45    | -48   |            |            |            | 1876 ± 39 | 1880 ± 30 |
| $\Sigma_{3/2}$           | 206   | -15   | 16    |            |            |            | 1881 ± 25 | (1840)    |
| $\Xi_{3/2}$              | 412   | 60    | -64   |            |            |            | 2081 ± 57 |           |
| $N_{5/2}$                | 1623  | 6     | 60    | 0          | 0          | 0          | 1689 ± 14 | 1683 ± 8  |
| $\Lambda_{5/2}$         | 206   | -30   | -48   |            |            |            | 1816 ± 33 | 1820 ± 5  |
| $\Sigma_{5/2}$          | 206   | 10    | 16    |            |            |            | 1920 ± 24 | 1918 ± 18 |
| $\Xi_{5/2}$             | 412   | -40   | -64   |            |            |            | 1997 ± 49 |           |
| $\Delta_{1/2}$          | 1623  | -27   | 301   | 0          | 0          | 0          | 1897 ± 32 | 1895 ± 25 |
| $\Sigma_{1/2}$         | 206   | 45    | -80   |            |            |            | 2068 ± 52 |           |
| $\Xi_{1/2}$             | 412   | 90    | -161  |            |            |            | 2237 ± 88 |           |
| $\Omega_{1/2}$          | 618   | 135   | -241  |            |            |            | 2408 ± 127|           |
| $\Delta_{3/2}$          | 1623  | -18   | 301   | 0          | 0          | 0          | 1906 ± 27 | 1935 ± 35 |
| $\Sigma'_{3/2}$        | 206   | 30    | -80   |            |            |            | 2061 ± 44 | (2080)    |
| $\Xi'_{3/2}$            | 412   | 60    | -161  |            |            |            | 2216 ± 76 |           |
| $\Omega_{3/2}$          | 618   | 90    | -241  |            |            |            | 2373 ± 110|           |
| $\Delta_{5/2}$          | 1623  | -3    | 301   | 0          | 0          | 0          | 1921 ± 21 | 1895 ± 25 |
| $\Sigma'_{5/2}$        | 206   | 5     | -80   |            |            |            | 2051 ± 37 | (2070)    |
| $\Xi'_{5/2}$            | 412   | 10    | -161  |            |            |            | 2181 ± 64 |           |
| $\Omega_{5/2}$          | 618   | 15    | -241  |            |            |            | 2313 ± 94 |           |
| $\Delta_{7/2}$          | 1623  | 18    | 301   | 0          | 0          | 0          | 1942 ± 27 | 1950 ± 10 |
| $\Sigma_{7/2}$         | 206   | -30   | -80   |            |            |            | 2036 ± 44 | 2033 ± 8  |
| $\Xi_{7/2}$             | 412   | -60   | -161  |            |            |            | 2131 ± 76 |           |
| $\Omega_{7/2}$          | 618   | -90   | -241  |            |            |            | 2229 ± 110|           |

TABLE V: Masses (in MeV) predicted by the 1/$N_c$ expansion as compared with the empirically known masses. The partial contributions to each mass by the operators in the basis are shown. Those partial contributions in blank are equal to the one above in the same column.