AN EXPLICIT ESTIMATE OF THE BERGMAN KERNEL FOR COMPACT RIEHMANN SURFACES

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Dedicated to Bo Berndtsson on occasion of his 70th birthday

ABSTRACT. We shall give an explicit version of Tian’s partial $C^0$-estimate for compact Riemann surfaces.

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1. INTRODUCTION

Let $(L, e^{-\phi})$ be a positive line bundle over an $n$-dimensional compact complex manifold $X$. Let $m$ be a positive integer. Let $K_X$ be the canonical line bundle over $X$. We call

$$K_{m\phi}(x) := \sup_{u \in H^0(X, K_X + mL)} \frac{i^n u(x) \wedge \bar{u}(x) e^{-m\phi(x)}}{\int_X i^n u \wedge \bar{u} e^{-m\phi}}$$

the Bergman kernel forms and

$$B_{m\phi}(x) := \sup_{u \in H^0(X, mL)} \frac{|u(x)|^2 e^{-m\phi(x)}}{\int_X |u|^2 e^{-m\phi} MA_{m\phi}}, \quad MA_{m\phi} := \frac{(i\partial \bar{\partial} (m\phi))^n}{n!},$$

the Bergman kernel functions. In [18] Tian proved that

$$\lim_{m \to \infty} \frac{K_{m\phi}}{MA_{m\phi}} = \lim_{m \to \infty} B_{m\phi} = \frac{1}{(2\pi)^n}.$$ 

Effective lower bound estimate for $B_{m\phi}$ is known as Tian’s partial $C^0$-estimate [19]. The first general result is obtained by Donaldson–Sun [8] using proof by contradiction, the Gromov–Hausdorff convergence theory and the Hörmander $\bar{\partial}$-estimate. The main result in this paper is the following explicit version of Tian’s partial $C^0$-estimate for compact Riemann surfaces.

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Our proof consists of three parts — an Ohsawa–Takegoshi type theorem, a variant of the Blocki–Zwonek estimate [4] and the isoperimetric inequality.

**Main Theorem:** Let \((L, e^{-\phi})\) be a positive line bundle over a compact Riemann surface \(X\). Put \(\omega := \text{MA}_{\phi} = i\partial\bar{\partial}\phi\). Assume that

\[
\deg(L) := \int_X c_1(L) = \frac{1}{2\pi} \int_X \omega \geq 4, \quad \sup_X \text{Ric} \omega \leq 1, \quad L_0 \geq 2\pi,
\]

where \(L_0\) denotes the infimum of the length of simple closed geodesics in \(X\), then \(K_{\phi}/\text{MA}_{\phi} \geq \frac{1}{8\pi}\).

Apply our main theorem to \(L - K_X\), we get

**Corollary:** Let \((L, e^{-\phi})\) be a positive line bundle over a compact Riemann surface \(X\). Put \(\omega := \text{MA}_{\phi} = i\partial\bar{\partial}\phi\). Assume that

\[
\deg(L) := \int_X c_1(L) = \frac{1}{2\pi} \int_X \omega \geq 8, \quad \sup_X \text{Ric} \omega \leq \frac{1}{2}, \quad L_0 \geq 4\pi,
\]

where \(L_0\) denotes the infimum of the length of simple closed geodesics in \(X\), then \(B_{\phi} \geq \frac{1}{16\pi}\).

It is known that \(L_0\) has a precise lower bound [11, Corollary 2.3.2] in terms of curvature, diameter and volume of \((X, \omega)\), hence the above corollary gives an explicit version of Tian’s partial \(C^0\)-estimate for compact Riemann surfaces (for recent works on Tian’s partial \(C^0\)-estimate, see [5, 6, 12, 13, 14, 17, 20, 21, 22], etc). At the end of this paper we shall give a remark to possible higher dimensional generalizations of our main theorem.

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2. **An Ohsawa–Takegoshi Type Theorem**

We shall use the following Ohsawa–Takegoshi type theorem [16], which is a special case of the main theorem in [10].

**Theorem 2.1.** Let \((L, e^{-\phi})\) be a positive line bundle on an \(n\)-dimensional compact complex manifold. Fix \(x \in X\). Assume that there is a non-positive function \(G\) smooth outside \(x\) such that \(G(z) - \log |z - x|^{2n}\) is smooth near \(x\) and

\[
i\partial\bar{\partial}\phi + \lambda i\partial\bar{\partial}G \geq 0
\]

on \(X\) for some constant \(\lambda > 1\). Then \(K_{\phi}\) in (1.1) satisfies

\[
K_{\phi}(x) \geq \frac{\lambda - 1}{\lambda} \lim_{t \to -\infty} e^{-t} \int_{G < t} \text{MA}_{\phi}(x) \text{MA}_{\phi}(x),
\]

where \(\text{MA}_{\phi}\) is defined in (1.2).

**Proof.** Let us rephrase the proof in [16]. Our curvature assumption implies that

\[
\phi^{t+\delta} := \phi + \lambda \max\{G - t, 0\}
\]
defines a singular metric on $\mathbb{C}^{t+i\alpha} \times L$ with non-negative curvature current. Hence Berndtsson’s theorem [11] implies that $\log K_{\phi}(x)$ is a convex function of $t$. By a direct computation (see the appendix in [16] or Theorem 3.8 in [2]) we find that

$$\lim_{t \to -\infty} e^{t}K_{\phi}(x) = \frac{\lambda - 1}{\lambda} \lim_{t \to -\infty} e^{-t} \int_{G < t} \mathcal{M}_{\phi}(x)$$

is finite since $G(z) - \log |z - x|^{2n}$ is smooth near $x$. Hence $e^{t}K_{\phi}(x) = e^{t + \log K_{\phi}(x)}$, as a convex function of $t$ bounded near $-\infty$, must be increasing. Thus

$$K_{\phi}(x) = e^{0}K_{\phi}(x) \geq \lim_{t \to -\infty} e^{t}K_{\phi}(x) = \frac{\lambda - 1}{\lambda} \lim_{t \to -\infty} e^{-t} \int_{G < t} \mathcal{M}_{\phi}$$

gives our estimate. \(\square\)

3. The Blocki–Zwonek estimate

We shall study the right hand side of (2.1) using a variant of Blocki–Zwonek’s estimate [4, Proof of Theorem 3] (see also [3, section 10] for related results).

**Lemma 3.1.** With the notation in Theorem 2.1. Let $\omega$ be an arbitrary Kähler form on $X$. Then

$$\frac{d}{dt} \int_{G < t} \omega_n \geq \frac{\sigma(G = t)^2}{2 \int_{G < t} i\partial \bar{\partial} G \wedge \omega_n}, \quad \omega_q := \omega^q/q!,$$

where

$$\sigma(G = t) := \int_{G = t} d\sigma, \quad d\sigma := \sqrt{2} \sum \frac{G_{\alpha} \omega^{\alpha \beta}}{|\partial G|_{\omega}} \frac{\partial}{\partial z_{\beta}} \omega_n,$$

is the measure of the hypersurface $\{G = t\}$ with respect to $\omega$.

**Proof.** Notice that

$$V := \frac{\partial}{\partial t} + \sum \frac{G_{\alpha} \omega^{\alpha \beta}}{|\partial G|_{\omega}} \frac{\partial}{\partial z_{\beta}}$$

satisfies $V(G - t) = 0$, hence it can be used to compute $\frac{d}{dt} \int_{G < t}$, in particular, we have

$$\frac{d}{dt} \int_{G < t} \omega_n = \int_{G < t} L_V \omega_n = \int_{G = t} V \omega_n = \frac{1}{\sqrt{2}} \int_{G = t} \frac{d\sigma}{|\partial G|_{\omega}}.$$

Hence the Cauchy–Schwarz inequality gives

$$\frac{d}{dt} \int_{G < t} \omega_n \geq \frac{\sigma(G = t)^2}{\sqrt{2} \int_{G = t} |\partial G|_{\omega} d\sigma} = \frac{\sigma(G = t)^2}{2 \int_{G = t} i\partial \bar{\partial} G \wedge \omega_n} = \frac{\sigma(G = t)^2}{2 \int_{G < t} i\partial \bar{\partial} G \wedge \omega_n},$$

where we use the Stokes theorem in the last equality. \(\square\)

Since

$$G_1 \leq G_2 \Rightarrow \int_{G_2 < t} \omega_n \leq \int_{G_1 < t} \omega_n.$$
in order to get the best estimate from (2.1), one should choose \( G \) to be the following *envelope*, say \( g_{\phi,x,\lambda} \), defined by

\[
\text{sup}\{ G \leq 0 : G(z) - \log |z - x|^{2n} \text{ is smooth near } x \text{ and } i\partial \overline{\partial} \phi + \lambda i\partial \overline{\partial} G \geq 0 \}.
\]

It is known that (see [7])

\[
\epsilon_x(L) := \text{sup}\{ \lambda \geq 0 : \text{there exists } G \leq 0 \text{ on } X \text{ such that the blue part in (3.1) holds} \}
\]

is equal to the *Seshadri constant* up to a constant factor \( n \). If \( 0 < \lambda < \epsilon_x(L) \) then \( g_{\phi,x,\lambda} \), as an envelope, must satisfy

\[
(i\partial \overline{\partial} \phi + \lambda i\partial \overline{\partial} g_{\phi,x,\lambda})^n = (2\pi n\lambda)^n \delta_x
\]
on \( \{g_{\phi,x,\lambda} < 0\} \), where \( \delta_x \) is the measure defined by \( \int_X f \delta_x = f(x) \). Thus if we choose \( G = g_{\phi,x,\lambda} \) and \( \omega = i\partial \overline{\partial} \phi \) then

\[
\int_{G_t} i\partial \overline{\partial} G \wedge \omega_{n-1} = \int_{G_t} (i\partial \overline{\partial} G + \omega/\lambda - \omega/\lambda) \wedge \omega_{n-1}
\]

\[
= \int_{G_t} (i\partial \overline{\partial} G + \omega/\lambda) \wedge \omega_{n-1} - \frac{n}{\lambda} \int_{G_t} \omega_n
\]

\[
\leq \int_{G < 0} (i\partial \overline{\partial} G + \omega/\lambda) \wedge \omega_{n-1} - \frac{n}{\lambda} \int_{G_t} \omega_n
\]

\[
= \frac{n}{\lambda} \left( \int_{G < 0} \omega_n - \int_{G_t} \omega_n \right),
\]

where we use \( \int_{G < 0} i\partial \overline{\partial} G \wedge \omega_{n-1} = 0 \) (since \( G = 0 \) outside \( \{G < 0\} \)) in (3.7). In case \( n = 1 \), (3.3) directly gives

\[
\int_{G_t} i\partial \overline{\partial} G = \int_{G_t} i\partial \overline{\partial} G + \omega/\lambda - \omega/\lambda = 2\pi - \int_{G_t} \omega/\lambda.
\]

Hence Lemma 3.1 implies

\[
\frac{d}{dt} \int_{G_t} \omega \geq \frac{\sigma(G = t)^2}{4\pi - \frac{2}{\lambda} \int_{G_t} \omega}.
\]

4. ISOPERIMETRIC INEQUALITY AND THE FINAL PROOF

We shall use the following result (see inequality (5.4) in [15, Proposition 5.2]).

**Lemma 4.1** (Isoperimetric inequality). Let \( U \) be an open subset of a compact Riemann surface \((X, \omega)\). Assume that

\[
A := \int_U \omega \leq \frac{1}{2} \int_X \omega.
\]

Then

\[
\sigma(\partial U)^2 \geq \min\{ L_0^2, A(4\pi - kA) \}, \quad k := \sup \text{Ric} \omega,
\]

where \( L_0 \) denotes the infimum of the length of simple closed geodesics in \( X \).
By the definition of the Seshadri constant in (3.2), we know that in case \( n = 1 \), the Hodge decomposition gives

\[
\epsilon_x = \deg(L) := \int_X c_1(L) = \frac{1}{2\pi} \int_X \omega.
\]

Hence if \( \int_X \omega \geq 8\pi \) then \( \epsilon_x \geq 4 \). Hence we can take \( \lambda = 2 \) in (3.9), which gives

\[
\left(4.3\right) \quad \frac{d}{dt} \int_{G<t} \omega \geq \frac{\sigma(G = t)^2}{4\pi - \int_{G<t} \omega}.
\]

Now, since \( \sup_X \text{Ric} \omega \leq 1 \), \( L_0^2 \geq 4\pi^2 \) and

\[
\int_{G<t} \omega \leq \int_{G<0} \omega = 2\pi \lambda = 4\pi \leq \frac{1}{2} \int_X \omega,
\]

apply the above lemma to \( U = \{ G < t \} \), then by (4.1), we have

\[
\sigma(G = t)^2 \geq \min\{4\pi^2, A(4\pi - A)\} = A(4\pi - A), \quad A := \int_{G<t} \omega.
\]

Thus (4.3) gives

\[
\frac{d}{dt} \int_{G<t} \omega \geq \int_{G<t} \omega,
\]

which implies that \( e^{-t} \int_{G<t} \omega \) is increasing with respect to \( t < 0 \). Hence

\[
\lim_{t \to -\infty} e^{-t} \int_{G<t} \omega \leq e^{-0} \int_{G<0} \omega = 2\pi \lambda = 4\pi.
\]

Hence (2.1) gives \( K_{\phi} \geq \frac{\omega}{8\pi} \). The proof of main theorem is now complete.

**Proof of the corollary.** Since \( \text{Ric} \omega \geq -1/2 \) we have

\[
\omega + (\text{Ric} \omega) \omega \geq \frac{1}{2} \omega := \omega'.
\]

Our assumption implies that

\[
\frac{1}{2\pi} \int_X \omega' \geq 4, \quad \sup_X \text{Ric} \omega' \leq 1, \quad L_0' \geq 2\pi,
\]

where \( L_0' \) denotes the infimum of the length of simple closed geodesics in \( (X, \omega') \). Hence our main theorem implies that

\[
\frac{B_{\phi} \omega}{\omega'} \geq \frac{1}{8\pi},
\]

from which our corollary follows. \( \square \)
5. A REMARK TO THE HIGHER DIMENSIONAL CASE

In the higher dimensional case, there is no simple formula like (4.2) for the Seshadri constant. Thus we need to estimate the Seshadri constant first, one possible approach is to use the distance function. The idea is the following. Let \((L, e^{-\phi})\) be a positive line bundle over an \(n\)-dimensional compact complex manifold \(X\). Put
\[
\omega := i\partial\overline{\partial}\phi,
\]
using Taylor expansion of the distance function
\[
d_x(z) := \text{the distance between } x \text{ and } z \text{ with respect to the Kähler metric } \omega,
\]
one may prove that
\[
(5.1) \quad i\partial\overline{\partial}\log d_x^{2n} + C\omega \geq 0, \quad n := \dim X,
\]
near \(x\) for some positive constant \(C\) (depending on \(\omega\)). Assume that the above identity holds on a small geodesic ball \(B_x(r)\) of radius \(r\) around \(x\). Put
\[
G(z) := \begin{cases} 
n \left(1 - \frac{d_x(z)^2}{r^2} + \log \frac{d_x(z)^2}{r^2}\right) & d_x(z) \leq r, \\
0 & d_x(z) > r,
\end{cases}
\]
then (5.1) implies
\[
i\partial\overline{\partial}G + \left(C + \frac{2n}{r^2}\right)\omega \geq 0
\]
on \(X\), which gives
\[
\epsilon_x(L) \geq \lambda := \left(C + \frac{2n}{r^2}\right)^{-1}.
\]
Hence by Theorem 2.1 and
\[
\lim_{t \to -\infty} e^{-t} \int_{G < t} \frac{\omega^n}{n!} = \frac{\pi^{n/2} n!}{e^{n!}}.
\]
we have:

**Theorem 5.1.** With the notation above, assume that
\[
\lambda := \left(C + \frac{2n}{r^2}\right)^{-1} > 1
\]
then we have
\[
K_\phi(x) \geq \frac{\lambda - 1}{\lambda} \frac{e^{n!}}{\pi^{n/2} n!}.
\]

Denote by \(r_{\text{inj}}\) the *injectivity radius* of \((X, \omega)\). The above estimate suggests to define the following *Greene-Wu constant*
\[
C(r) := \inf\{C \geq 0 : dd^c \log d_x^{2n} + C\omega \geq 0 \text{ on } B_x(r)\}, \quad 0 < r < r_{\text{inj}}.
\]
In case \((X, \omega)\) has non-positive sectional curvature, Greene-Wu [9] proved that \(C(r)\) is always zero for \(0 < r < r_{\text{inj}}\). But for \(\mathbb{P}^n\) with the Fubini-Study metric (positive curvature case), by a direct computation, we know that \(\lim_{r \to 0} C(r) > 0\) and \(C(r)\) goes to infinity when \(r\) goes to \(r_{\text{inj}}\).
This example suggests us to estimate upper bound of $C(r_{inj}/2)$. In case all sectional curvatures of $\omega$ are bounded above by one, Greene-Wu’s comparison theorem for $i\partial\bar{\partial}\log d_x$ could give an upper bound of $C(r_{inj}/2)$. But we do not know how to estimate $C(r_{inj}/2)$ using only the Ricci curvature. On the other hand, it might be possible to develop the higher dimensional Blocki-Zwonek estimate [3]. The main difficulty is that (3.6) is not an identity in the higher dimensional case and there is no isoperimetric inequality associated to (3.6) even for the $\mathbb{P}^2$ case. Hence one has to develop the isoperimetric theory for (3.5) instead.

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