ON PROPERTIES OF SIMILARITY BOUNDARY OF
ATTRACTIONS IN PRODUCT DYNAMICAL SYSTEMS

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Abstract. Fractals in higher dimensional dynamical systems have significant roles in physics and other applied sciences. In this paper, one of the key property of fractals, called self similarity in product systems, is studied using the concept of similarity boundary. The relationship between similarity boundary of an attractor in a product space to one of its projection spaces is discussed. The impact of inverse invariance of similarity boundary on its coordinate iterated function system is analyzed. Fractals satisfying the strong open set condition, restricted to attractors in product spaces, are characterized. The relationship between similarity boundary of attractors in product spaces and their overlapping sets is also obtained. The equivalency of the restricted open set condition and the strong open set condition in product spaces, is proved. Self similarity of an attractor in a product system is characterized using the Hausdorff measure of its similarity boundary. Also, the Hausdorff dimensions of the overlapping set and similarity boundary of attractors for different types of iterated function systems are obtained.

1. Introduction. The concept of dynamical systems and fractals are abundantly used in many areas of physics and mathematics. However, this concept was conceptualized mathematically by Mandelbrot [27] in 1975. He is often called the father of fractal geometry. Later, one of the strong property of fractals, called self similarity, was introduced and studied by Hutchinson [22]. He also developed a mathematical tool known as iterated function system (IFS) for constructing self similar fractals in discrete dynamical systems. Many other mathematicians like Barnsley and Falconer contributed a lot to the theory [6], [7], [10]-[17]. Felix Hausdorff [21] introduced one of the important idea, called Hausdorff dimension, which plays a crucial role in the study of dynamical systems and attractors. Aside from Hausdorff dimension, similarity dimension was also introduced by Hutchinson [22] to quantify the complexity of fractal patterns. He studied separation properties of IFSs using open set conditions (OSC), given by Moran [29] in 1946, to see whether the Hausdorff dimension of an attractor coincides with its similarity dimension. Later Lalley [26] intensified it by introducing the strong open set condition (SOSC), since an attractor is almost insignificant in OSC. In 1999, Moran [28] derived another open set condition known as the restricted open set condition (ROSC) from SOSC and proved that

2020 Mathematics Subject Classification. 28A80, 31E05.

Key words and phrases. Fractal, iterated function system, dynamical system, self similarity, similarity boundary.

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ROSC is equivalent to SOSC under certain conditions on the open set that satisfies OSC. In the same year Jeesling and Krishnamurthi [23] introduced a weaker version of SOSC by defining a strong open set condition restricted to the attractor ($SOSC_K$). They also introduced an extended idea of topological boundary called similarity boundary to study the self similarity property. More related works can be seen in [24], [1]-[5].

Fractal objects in higher dimensional dynamical systems are incapable of being visualized and hence their study is highly complicated. Thus, to study topological and measure theoretical properties of an attractor of a product IFS using the similarity boundary has great importance. The product of two IFSs was introduced and studied by Duvall and Husch [9]. In [4] and [5], Rinju and Mathew introduced $(n, m) - IFS$, that iterates fractals in product metric spaces. In this paper, the concept of similarity boundary is extended to product spaces and their properties are studied. Elementary definitions and results are given in section 2. Section 3 deals with the properties of similarity boundary in product IFS. The relationships between similarity boundaries of attractors in product IFS and its coordinate IFSs are given as Theorem 3.1 and Theorem 3.2. One of the important properties of similarity boundary, called inverse invariance in product space, is also discussed. $SOSC_K$ in product space is also characterized in this section. In section 4, self similar measure, Hausdorff measure and Hausdorff dimension of similarity boundary of an attractor in a product space are discussed. It is also proved that Hausdorff dimension of similarity boundary of an attractor in product space is the same as the Hausdorff dimension of its overlapping set.

2. Preliminaries. Fundamental definitions and results that are necessary for the development of this paper are presented in this section. These definitions and results are from [6] and [9]. Throughout this paper, $N_\phi = \{1, 2, \cdots, n_\phi\}$, and $N_\Psi = \{1, 2, \cdots, n_\Psi\}$ where $n_\phi, n_\Psi \in \mathbb{N}$.

Let $(X, d)$ be a complete metric space and $H(X)$ be the collection of all nonempty compact subsets of $X$. Then $(H(X), h)$ is a complete metric space where $h(A, B) = \max\{d(A, B), d(B, A)\}$, where $d(A, B) = \max\{\min\{d(a, b) : b \in B\} : a \in A\}$. A function $\phi : X \to X$ is called a contraction mapping if, for all $(x, y) \in X$, $d(\phi(x), \phi(y)) \leq rd(x, y)$ for some constant $r \in [0, 1)$ and this $r$ is called as the contractivity factor of $\phi$. A contraction mapping is said to be a contraction similitude in the case where equality holds. Any contraction mapping $\phi$ on a complete metric space $X$ has exactly one fixed point, say $x_\phi$, according to Banach’s fixed point theorem and the point is given by $\lim_{n \to \infty} \phi^n(x) = x_\phi$ for any arbitrary point $x \in X$. An iterated function system (IFS) is a complete metric space $X$ together with a finite collection of contraction mappings $\phi_1, \phi_2, \phi_3, \cdots, \phi_n$, denoted as $\{X; \phi_1, \phi_2, \cdots, \phi_n\}$. The set function $w$ on $H(X)$, called the Hutchinson mapping, is defined as $w(K) = \bigcup_{i=1}^n \phi_i(K)$, for all $K \in H(X)$. This mapping possesses a unique fixed point called the attractor of the IFS, $w$ being a contraction mapping on $H(X)$, with respect to the Hausdorff metric.

Let $B$ be a subset of $X$. Then, for a non-negative real number $\alpha$ and $\delta > 0$, the outer measure is given by

$$H_\delta^n(B) = \inf \left\{ \sum_{i=1}^\infty (diam(U_i))^\alpha : B \subset \bigcup_{i=1}^\infty U_i, diam(U_i) \leq \delta, U_i \text{ is open} \right\}.$$
Then, the \( \alpha \)-dimensional Hausdorff measure is defined by \( H^\alpha(B) = \lim_{\delta \to 0} H_\delta^\alpha(B) \) if \( B \) measurable with respect to \( H_\delta^\alpha \). The Hausdorff dimension of \( B \subseteq X \), denoted by \( \dim_H(B) \), is given by the value \( \alpha = \inf\{ \alpha : H^\alpha(B) = 0 \} = \sup\{ \alpha : H^\alpha(B) = \infty \} \) such that

\[
H^\alpha(B) = \begin{cases} 
\infty & \text{if } \alpha < \dim_H(B) \\
0 & \text{if } \alpha > \dim_H(B) 
\end{cases}
\]

We say an IFS \( \Phi = \{ X; \phi_1, \phi_2, \cdots, \phi_n \Phi \} \) with attractor \( K_\Phi \) satisfies,

(1) **open set condition (OSC)** if there exists a non-empty open set \( V \subseteq X \), such that \( \phi_i(V) \subseteq V \) for all \( i = 1, 2, \cdots, n_\Phi \) and \( \phi_i(V) \cap \phi_j(V) = \emptyset \) for all \( i \neq j \), where \( i, j \in N_\Phi \).

(2) **strong open set condition (SOSC)** if there exists a non-empty open set \( V \subseteq X \) such that \( V \cap K_\Phi \neq \emptyset \), that satisfies OSC.

(3) **restricted open set condition (ROSC)** if there exists a non-empty open subset \( V \subseteq K_\Phi \) such that \( \phi_i(V) \subseteq V \) and \( \phi_i(V) \cap \phi_j(V) = \emptyset \) for all \( i = 1, 2, \cdots, n_\Phi \) where \( O_\Phi \) is the overlapping set, defined as \( O_\Phi = \bigcup_{i \neq j} (\phi_i(K_\Phi) \cap \phi_j(K_\Phi)) \).

(4) **strong open condition restricted to \( K_\Phi \)** if there exists an open set \( V \subseteq K_\Phi \) such that \( \phi_i(V) \subseteq V \) for all \( i \in N_\Phi \) and \( \phi_i(V) \cap \phi_j(K_\Phi) = \emptyset \) for all \( i, j \in N_\Phi \).

Consider IFS \( \Phi = \{ X; \phi_1, \phi_2, \cdots, \phi_n \Phi \} \) with attractor \( K_\Phi \). Then, in [24] a generalization of topological boundary of \( K_\Phi \) is introduced, which is called the similarity boundary \( B_\Phi \). It is defined as

\[
B_\Phi = \bigcup_{i \neq j} \phi_i^{-1}((\phi_i(K_\Phi) \cap \phi_j(K_\Phi))
\]

The product \( X \times Y \) of two metric spaces \( (X, d') \) and \( (Y, d'') \) is given by \( (X \times Y, d) \), where \( d((x, y), (\hat{x}, \hat{y})) = \sqrt{d(x, \hat{x})^2 + d''(y, \hat{y})^2} \). The product \( \Phi \times \Psi \) of two IFSs \( \Phi = \{ X; \phi_1, \phi_2, \cdots, \phi_n \Phi \} \) and \( \Psi = \{ Y; \psi_1, \psi_2, \cdots, \psi_n \Psi \} \) is defined as \( \Phi \times \Psi = \{ X \times Y; \zeta_j \} \), where \( \zeta_j = \phi_i \times \psi_j \) and \( (i, j) \in N_\Phi \times N_\Psi \). Moreover, the attractor \( K \) of \( \Phi \times \Psi \) coincides with \( K_\Phi \times K_\Psi \), where \( K_\Phi \) and \( K_\Psi \) are the attractors of the IFSs \( \Phi \) and \( \Psi \) respectively.

3. **Similarity boundary of an attractor in product IFS**. In this section, similarity boundary of an attractor in a product metric space is introduced and its properties are studied using coordinate IFS. The relationship between the similarity boundaries of attractors in a product space and corresponding coordinate spaces are studied.

Consider the product IFS \( \Phi \times \Psi \) where \( \Phi = \{ \mathbb{R}^2; (\frac{x}{2}, \frac{y}{2}), (\frac{x+1}{2}, \frac{y}{2}), (\frac{2x+1}{4}, \frac{y+1}{2}) \} \) and \( \Psi = \{ \mathbb{R}; \frac{z}{3}, \frac{z+1}{3}, \frac{z+2}{3} \} \). The attractor of \( \Phi \times \Psi \) is shown in Figure 1. Since the attractor of \( \Phi \) is the Sierpinski triangle, the similarity boundary \( B_\Phi \) of \( \Phi \) is the vertices \( \{(0,0), (\frac{1}{2},1), (1,0)\} \) in \( \mathbb{R}^2 \). The similarity boundary \( B_\Psi \) of \( \Psi \) is the end points of the unit interval. Now, the similarity boundary \( B_{\Phi \times \Psi} \) of the product IFS \( \Phi \times \Psi \) is the set \( \{(0,0,z), (\frac{1}{2},1,z), (1,0,z); 0 \leq z \leq 1 \} \). Clearly, \( B_\Phi \times B_\Psi \subseteq B_{\Phi \times \Psi} \), which is proved in the next theorem.

**Theorem 3.1.** Let \( \Phi = \{ X; \phi_1, \phi_2, \cdots, \phi_n \Phi \} \) and \( \Psi = \{ Y; \psi_1, \psi_2, \cdots, \psi_n \Psi \} \) be two IFSs and \( \Phi \times \Psi = \{ X \times Y; \xi_j, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi \} \) be their product IFS. Then

\[
B_\Phi \times B_\Psi \subseteq B_{\Phi \times \Psi}
\]
where $B_\Phi$, $B_\Psi$, and $B_{\Phi \times \Psi}$ are the similarity boundaries of attractors of the IFSs $\Phi$, $\Psi$ and $\Phi \times \Psi$ respectively.

**Proof.** Let $(x, y) \in B_\Phi \times B_\Psi$. That is, $x \in B_\Phi$ and $y \in B_\Psi$. So there exist $i, k \in N_\Phi$ with $i \neq k$ and $j, l \in N_\Psi$ with $j \neq l$ such that $x \in \phi_i^{-1}(\phi_j(K_\Phi) \cap \phi_k(K_\Phi))$ and $y \in \psi_j^{-1}(\psi_j(K_\Psi) \cap \psi_l(K_\Psi))$. That follows $\phi_i(x) \in \phi_i(K_\Phi) \cap \phi_k(K_\Phi)$ and $\psi_j(y) \in \psi_j(K_\Psi) \cap \psi_l(K_\Psi)$. Thus,

$$
\zeta_{ij}(x, y) \in [\phi_i(K_\Phi) \cap \phi_k(K_\Phi)] \times [\psi_j(K_\Psi) \cap \psi_l(K_\Psi)]
$$

$$
= \left[\phi_i(K_\Phi) \times \psi_j(K_\Psi)\right] \cap \left[\phi_k(K_\Phi) \times \psi_l(K_\Psi)\right]
$$

$$
= \zeta_{ij}(K) \cap \zeta_{kl}(K)
$$

and hence $(x, y) \in \zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K))$. Since $i \neq k$ and $j \neq l$, $ij \neq kl$. Therefore, $(x, y) \in B_{\Phi \times \Psi}$.

The containment becomes equality if the product IFS $\Phi \times \Psi$ is totally disconnected. However, the reverse containment is not true in general. For example consider two IFSs, $\Phi = \{\mathbb{R}, \frac{2}{3}, \frac{2}{3}^2\}$ and $\Psi = \{\mathbb{R}, \frac{2}{3}, \frac{2}{3}^2\}$. The attractor $K_\Phi$ of $\Phi$ is the unit interval $[0, 1]$ and the attractor $K_\Psi$ of $\Psi$ is the Cantor set. Here the similarity boundary of $K_\Phi$ is $B_\Phi = \{\frac{1}{2}\}$ and the similarity boundary of $K_\Psi$ is $B_\Psi = \emptyset$. Clearly, $B_\Phi \times B_\Psi = \emptyset$, but $B_{\Phi \times \Psi} = \{0, 1\} \times K_\Psi$. But this example gives an idea how to express the similarity boundary $B_{\Phi \times \Psi}$ of the attractor of a product IFS in terms of $B_\Phi$ and $B_\Psi$, which is given as the next theorem.

**Theorem 3.2.** Suppose $\Phi = \{X; \phi_1, \phi_2, \ldots, \phi_n\}$ and $\Psi = \{Y; \psi_1, \psi_2, \ldots, \psi_n\}$ are two IFSs with attractors $K_\Phi$ and $K_\Psi$ respectively. Let $B_\Phi$ and $B_\Psi$ be their similarity boundaries. Then the similarity boundary $B_{\Phi \times \Psi}$ of the attractor $K$ of the product IFS $\Phi \times \Psi$ is given by

$$
B_{\Phi \times \Psi} = (B_\Phi \times K_\Psi) \cup (K_\Phi \times B_\Psi).
$$

**Proof.** Suppose $(x, y) \in B_{\Phi \times \Psi}$. That is $(x, y) \in \zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K))$ for some $ij \neq kl$ where $i, k \in N_\Phi$ and $j, l \in N_\Psi$. That follows $\zeta_{ij}(x, y) \in \zeta_{ij}(K) \cap \zeta_{kl}(K)$ which implies $\zeta_{ij}(x, y) \in \zeta_{ij}(K)$ and $\zeta_{ij}(x, y) \in \zeta_{kl}(K)$. That is, $\phi_i(x) \times \psi_j(y) \in \phi_i(K_\Phi) \times \psi_j(K_\Psi)$ and $\phi_k(x) \times \psi_l(y) \in \phi_k(K_\Phi) \times \psi_l(K_\Psi)$. Thus, $\phi_i(x) \in \phi_i(K_\Phi) \cap \phi_k(K_\Phi)$ and $\psi_j(y) \in \psi_j(K_\Psi) \cap \psi_l(K_\Psi)$. That is, $x \in \phi_i^{-1}(\phi_i(K_\Phi) \cap \phi_k(K_\Phi))$ and $y \in \psi_j^{-1}(\psi_j(K_\Psi) \cap \psi_l(K_\Psi))$. Since $ij \neq kl$, either $i \neq k$ or $j \neq l$. If $i \neq k$, then
One of the main properties of similarity boundary is inverse invariance. The following result discusses the relationship between the inverse invariance of similarity boundaries of the attractors in product space and coordinate IFSs.

**Theorem 3.3.** Suppose \( \Phi = \{ X; \phi_1, \phi_2, \ldots, \phi_{n_\Phi} \} \) and \( \Psi = \{ Y; \psi_1, \psi_2, \ldots, \psi_{n_\Psi} \} \) are two IFSs with attractors \( K_\Phi \) and \( K_\Psi \) respectively. Let \( \Phi \times \Psi = \{ X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi \} \) be their product IFS with attractor \( K \) and \( B_{\Phi \times \Psi} \) and \( B_\Phi \) be the similarity boundaries of \( K, K_\Phi \) and \( K_\Psi \) respectively. If \( B_{\Phi \times \Psi} \) is inverse invariant, then either \( B_\Phi \) or \( B_\Psi \) is inverse invariant.

**Proof.** Assume that \( B_{\Phi \times \Psi} \) is inverse invariant. That is \( \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K \subseteq B_{\Phi \times \Psi} \) for all \((i, j) \in N_\Phi \times N_\Psi\). Let \( x \in \phi_i^{-1}(B_\Phi) \cap K_\Phi \) and \( y \in \psi_j^{-1}(B_\Psi) \cap K_\Psi \) for some \( i \in N_\Phi \) and \( j \in N_\Psi \). Then, \( \phi_i(x) \in B_\Phi \) and \( \psi_j(y) \in B_\Psi \) for some \( i \in N_\Phi \) and \( j \in N_\Psi \). That implies, \( \zeta_{ij}(x, y) = \phi_i(x) \times \psi_j(y) \in B_{\Phi \times \Psi} \subseteq B_{\Phi \times \Psi} \). Thus, \( (x, y) \in \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K \). Since \( x \in K_\Phi \) and \( y \in K_\Psi \), \( (x, y) \in K \times K_\Psi \). Therefore, \((x, y) \in \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K \). By assumption, \( B_{\Phi \times \Psi} \) is inverse invariant and hence \( (x, y) \in B_{\Phi \times \Psi} \). So, \( (x, y) \in (B_\Phi \times K_\Psi) \cup (K_\Phi \times B_\Psi) \) by Theorem 3.2. That is, either \((x, y) \in B_\Phi \times K_\Psi \) or \((x, y) \in K_\Phi \times B_\Psi \). If \((x, y) \in B_\Phi \times K_\Psi \), then \( x \in B_\Phi \) and if \((x, y) \in K_\Phi \times B_\Psi \), then \( y \in B_\Psi \). Since \( i \) and \( j \) are arbitrary, the result is true for all \( i \in N_\Phi \) and \( j \in N_\Psi \). Thus either \( B_\Phi \) or \( B_\Psi \) is inverse invariant.

The converse part of the theorem is not true. For example, consider two IFSs \( \Phi = \{ \mathbb{R}; \phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x+1}{2} \} \) and \( \Psi = \{ \mathbb{R}; \psi_1(x) = \frac{x}{2}, \psi_2(x) = \frac{x+1}{2} \} \). The attractor of the product IFS \( \Phi \times \Psi \) with similitudes \( \zeta_{ij} \)'s where \( i, j = 1, 2 \) is shown as Figure 2. Suppose \( B_\Phi, B_\Psi \) and \( B_{\Phi \times \Psi} \) denote the similarity boundaries of the attractors in \( \Phi, \Psi \) and \( \Phi \times \Psi \) respectively. Then, \( B_\Phi = B_\Psi = \{ 0, 1 \} \). Clearly, both \( B_\Phi \) and \( B_\Psi \) are inverse invariant. From Figure 2, it is easy to see that \( B_{\Phi \times \Psi} \) is the boundary of the unit square. That is, \( B_{\Phi \times \Psi} = \{ (x, 0), (x, 1), (0, y), (1, y); x, y \in [0, 1] \} \). Consider, \( \zeta_{11}(1, 1) = (\phi_1(1), \psi_1(1)) = \left( \frac{1}{2}, \frac{1}{2} \right) \). Since \( (1, 1) \in B_{\Phi \times \Psi} \), \( \left( \frac{1}{2}, \frac{1}{2} \right) \in \zeta_{11}(B_{\Phi \times \Psi}) \). But, \( \left( \frac{1}{2}, \frac{1}{2} \right) \notin B_{\Phi \times \Psi} \), which proves that \( B_{\Phi \times \Psi} \) is not inverse invariant.
The following theorem is a characterization of inverse invariance of the similarity boundary of an attractor in product IFS.

**Theorem 3.4.** Let \( \Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_{\Phi}, 1 \leq j \leq n_{\Psi}\} \) be a product IFS with attractor \( K \). Let \( B_{\Phi \times \Psi} \) be the similarity boundary of \( K \). Then, \( \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K \subseteq B_{\Phi \times \Psi} \) if and only if \( \zeta_{ij}(U_{\Phi \times \Psi}) \subseteq U_{\Phi \times \Psi} \) where \( U_{\Phi \times \Psi} = K \setminus B_{\Phi \times \Psi} \).

**Proof.** First assume that \( \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K \subseteq B_{\Phi \times \Psi} \). Suppose \( (x, y) \in \zeta_{ij}(U_{\Phi \times \Psi}) \). That is, \( (x, y) \in \zeta_{ij}(K \setminus B_{\Phi \times \Psi}) \subseteq \zeta_{ij}(K \setminus \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K) = \zeta_{ij}(K \setminus \zeta_{ij}^{-1}(B_{\Phi \times \Psi})). \) That implies, \( \zeta_{ij}^{-1}(x, y) \in K \) and \( \zeta_{ij}^{-1}(x, y) \notin \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \). Thus, \( (x, y) \in \zeta_{ij}(K) \subseteq K \) and \( (x, y) \notin B_{\Phi \times \Psi} \) and hence \( (x, y) \in K \setminus B_{\Phi \times \Psi} = U_{\Phi \times \Psi} \).

Conversely, assume \( \zeta_{ij}(U_{\Phi \times \Psi}) \subseteq U_{\Phi \times \Psi} \). Let \( (x, y) \in \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \cap K \). That is, \( \zeta_{ij}(x, y) \in B_{\Phi \times \Psi} \) and \( \zeta_{ij}(x, y) \in K \). Thus, \( \zeta_{ij}(x, y) \notin U_{\Phi \times \Psi} \) which follows \( \zeta_{ij}(x, y) \notin \zeta_{ij}(U_{\Phi \times \Psi}) \) by assumption. That is, \( \zeta_{ij}(x, y) \in \zeta_{ij}(K) \setminus \zeta_{ij}(U_{\Phi \times \Psi}) = \zeta_{ij}(K \setminus U_{\Phi \times \Psi}) \). Hence, \( (x, y) \in K \setminus U_{\Phi \times \Psi} = B_{\Phi \times \Psi} \). \( \square \)

In [24], a weaker version of SOSC of an IFS, called the strong open set condition restricted to its attractor (SOSC\(_{K}\)) where \( K \) is the attractor of IFS, is established. The significance of (SOSC\(_{K}\)) is greater than SOSC is some sense, because of the insignificance of ambient space. The next theorem gives a characterization for SOSC\(_{K}\) in product IFS.

**Theorem 3.5.** Suppose \( \Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_{\Phi}, 1 \leq j \leq n_{\Psi}\} \) is a product IFS with attractor \( K \). Then, \( \Phi \times \Psi \) satisfies SOSC\(_{K}\) if and only if \( \Phi \) and \( \Psi \) satisfy SOSC\(_{K_{\Phi}}\) and SOSC\(_{K_{\Psi}}\) respectively, where \( K_{\Phi} \) and \( K_{\Psi} \) are the attractors of the corrdinate IFSs \( \Phi \) and \( \Psi \).

**Proof.** Suppose \( \Phi \) satisfies SOSC\(_{K_{\Phi}}\) and \( \Psi \) satisfies SOSC\(_{K_{\Psi}}\). That is, there exist open sets \( V_{\Phi} \subseteq K_{\Phi} \) and \( V_{\Psi} \subseteq K_{\Psi} \) such that

1. \( \phi_i(V_{\Phi}) \subseteq V_{\Phi} \) for all \( i \in N_{\Phi} \) and \( \phi_i(V_{\Phi}) \cap \phi_k(K_{\Phi}) = \emptyset \) for all \( i, k \in N_{\Phi} \) with \( i \neq k \).
2. \( \psi_j(V_{\Psi}) \subseteq V_{\Psi} \) for all \( j \in N_{\Psi} \) and \( \psi_j(V_{\Psi}) \cap \psi_l(K_{\Psi}) = \emptyset \) for all \( j, l \in N_{\Psi} \) with \( j \neq l \).
Let \( V_\Phi \times \Psi = V_\Phi \times V_\Psi \). Then, \( V_\Phi \times \Psi \) is open in \( K \) and \( \zeta_{ij}(V_\Phi \times \Psi) = \zeta_{ij}(V_\Phi \times V_\Psi) = \phi_i(V_\Phi) \times \psi_j(V_\Psi) \subseteq V_\Phi \times V_\Psi = V_\Phi \times \Psi \) for all \((i, j) \in N_\Phi \times N_\Psi\). Moreover,

\[
\zeta_{ij}(V_\Phi \times \Psi) \cap \zeta_{kl}(K) = \zeta_{ij}(V_\Phi \times \Psi) \cap \zeta_{kl}(K \times \Psi)
\]

\[
= [\phi_i(V_\Phi) \times \psi_j(V_\Psi)] \cap [\phi_k(K_\Psi) \times \psi_l(K_\Psi)]
\]

\[
= [\phi_i(V_\Phi) \cap \phi_k(K_\Psi)] \times [\psi_j(V_\Psi) \cap \psi_l(K_\Psi)]
\]

\[
= \emptyset
\]

for all \( ij \neq kl \), which proves that \( \Phi \times \Psi \) satisfies \( \text{SOSC}_K \).

Conversely assume that \( \Phi \times \Psi \) satisfies \( \text{SOSC}_K \). That is, there exists an open set \( V \) in \( K \) such that \( \zeta_{ij}(V) \subseteq V \) for all \((i, j) \in N_\Phi \times N_\Psi\) and \( \zeta_{ij}(V) \cap \zeta_{kl}(K) = \emptyset \) for all \((i, j), (k, l) \in N_\Phi \times N_\Psi\) with \( ij \neq kl \). Let \( \Pi_1 \) be the first coordinate projection and \( \Pi_2 \) be the second coordinate projection. Clearly \( \Pi_1(V) \) and \( \Pi_2(V) \) are open in \( K_\Psi \) and \( K_\Psi \) respectively. Moreover,

1. \( \phi_i(\Pi_1(V)) = \Pi_1(\zeta_{ij}(V)) \subseteq \Pi_1(V) \) for all \( i \in N_\Phi \). Similarly, \( \psi_j(\Pi_2(V)) \subseteq \Pi_2(V) \) for all \( j \in N_\Psi \).

2. \( \phi_i(\Pi_1(V)) \cap \phi_k(K_\Psi) = \emptyset \) for all \( i, k \in N_\Phi \) with \( i \neq k \), as if there exists an element \( x \in X \) such that \( x \in \phi_i(\Pi_1(V)) \cap \phi_k(K_\Psi) \) for some \( i, k \in N_\Phi \) with \( i \neq k \). Then, \( x \in \phi_i(\Pi_1(V)) \) and \( x \in \phi_k(K_\Psi) \). That is, \( x \in \Pi_1(\zeta_{ij}(V)) \) for some \( j \in N_\Psi \) and \( x \in \Pi_1(\zeta_{kl}(K)) \) for some \( l \in N_\Psi \) with \( ij \neq kl \), which implies \( x \in \Pi_1(\zeta_{ij}(V) \cap \zeta_{kl}(K)) \). Hence, \( x \in \zeta_{ij}(V) \cap \zeta_{kl}(K) \), a contradiction. Similarly, \( \psi_j(\Pi_2(V)) \cap \psi_l(K_\Psi) = \emptyset \) for all \( j, l \in N_\Psi \) with \( j \neq l \).

\[
\square
\]

\textbf{Figure 3.} Attractor of the product IFS \( \Phi \times \Psi \).

Consider an example of a product IFS \( \Phi \times \Psi \) where \( \Phi = \Psi = \{ \mathbb{R}; \frac{\pi}{3}, \frac{\pi + 2}{3} \} \). The attractor \( K \) of \( \Phi \times \Psi \) is given in Figure 3. Since both \( \Phi \) and \( \Psi \) are totally disconnected, the similarity boundaries \( B_\Phi \) and \( B_\Psi \) are the empty set. Therefore, \( U_\Phi = K_\Psi \backslash B_\Phi = K = U_\Psi = K_\Psi \backslash B_\Psi \), where \( K_\Psi = K_\Psi \) is the attractor of \( \Phi \). From the figure, it is easy to see that \( \Phi \times \Psi \) is also totally disconnected. That implies \( B_\Phi \times \Psi \) is also empty. Thus, \( U_\Phi \times \Psi = K \backslash B_\Phi \times \Psi \). Therefore, \( U_\Phi \times \Psi = K = K_\Psi \times K_\Psi = U_\Psi \times U_\Psi \), in this particular example. The following theorem discusses the relationship between \( U_\Phi \times \Psi, U_\Phi \) and \( U_\Psi \) in general.

\textbf{Theorem 3.6.} Suppose \( \Phi \times \Psi = \{ X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi \} \) is a product IFS with coordinate IFSs \( \Phi = \{ X; \phi_1, \phi_2, \ldots, \phi_{n_\Phi} \} \) and \( \Psi = \{ Y; \psi_1, \psi_2, \ldots, \psi_{n_\Psi} \} \).
Let $K$, $K_\Phi$, and $K_\Psi$ be their respective attractors with similarity boundaries $B_{\Phi \times \Psi}$, $B_\Phi$, and $B_\Psi$. Then,

$$U_{\Phi \times \Psi} = (U_\Phi \times K_\Psi) \cap (K_\Phi \times U_\Psi)$$

where, $U_{\Phi \times \Psi} = K \setminus B_{\Phi \times \Psi}$, $U_\Phi = K \setminus U_\Psi$, and $U_\Psi = K_\Phi \setminus B_\Psi$.

**Proof.** Let $(x, y) \in U_{\Phi \times \Psi}$. That is, $(x, y) \in K \setminus B_{\Phi \times \Psi}$ which implies $(x, y) \in K$ and $(x, y) \notin B_{\Phi \times \Psi}$. Thus, $(x, y) \in K$ and $(x, y) \notin (B_\Phi \times K_\Psi) \cup (K_\Phi \times B_\Psi)$ which gives, $(x, y) \in K$, $(x, y) \notin (B_\Phi \times K_\Psi)$ and $(x, y) \notin (K_\Phi \times B_\Psi)$. That is, $x \in K_\Phi$ and $x \notin B_\Psi$. Also, $y \in K_\Psi$ and $y \notin B_\Phi$ and hence, $(x, y) \in (K_\Phi \times U_\Psi) \cap (U_\Phi \times K_\Psi)$.

Conversely assume that $(x, y) \in (K_\Phi \times U_\Psi) \cap (U_\Phi \times K_\Psi)$. That is, $(x, y) \in K_\Phi \times U_\Psi$ and $(x, y) \in U_\Phi \times K_\Psi$. That implies, $x \in K_\Phi \cap U_\Psi$ and $y \in U_\Phi \cap K_\Psi$ and hence, $x \notin B_\Phi$ and $y \notin B_\Psi$. Therefore, $(x, y) \in B_\Phi \times B_\Psi \subseteq B_{\Phi \times \Psi}$ and thus $(x, y) \in U_{\Phi \times \Psi}$. □

Consider a product IFS $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$ with attractor $K$. Let $B_{\Phi \times \Psi}$ be its similarity boundary. The next theorem discusses a property of $U_{\Phi \times \Psi} = K \setminus B_{\Phi \times \Psi}$ which is useful to see that, under certain condition on $B_{\Phi \times \Psi}$, $U_{\Phi \times \Psi}$ satisfies the conditions for $SOSC_K$.

**Theorem 3.7.** Suppose $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$ is a product IFS with attractor $K$. Let $B_{\Phi \times \Psi}$ be the similarity boundary of $K$ and $U_{\Phi \times \Psi} = K \setminus B_{\Phi \times \Psi}$. Then, $\zeta_{ij}(U_{\Phi \times \Psi}) \cap \zeta_{kl}(K) = \emptyset$ for all $ij \neq kl$, where $(i, j), (k, l) \in N_\Phi \times N_\Psi$.

**Proof.** Suppose there exists some $(x, y) \in X \times Y$ such that $(x, y) \in \zeta_{ij}(U_{\Phi \times \Psi}) \cap \zeta_{kl}(K)$ for some $ij \neq kl$. That is, $(x, y) \in \zeta_{ij}(U_{\Phi \times \Psi})$ and $(x, y) \in \zeta_{kl}(K)$ for some $ij \neq kl$, which follows that $(x, y) \in \zeta_{ij}((U_\Phi \times K_\Psi) \cap (K_\Phi \times U_\Psi)) = \zeta_{ij}(U_\Phi \times K_\Psi) \cap \zeta_{ij}(K_\Phi \times U_\Psi) = (\phi_i(U_\Phi) \times \psi_j(K_\Psi)) \cap (\phi_j(K_\Phi) \times \psi_i(U_\Psi))$ and $(x, y) \in \phi_k(K_\Phi) \times \psi_l(K_\Psi)$ for some $ij \neq kl$. Hence, $x \in \phi_i(U_\Phi) \cap \phi_k(K_\Phi)$ and $y \in \psi_j(U_\Psi) \times \psi_l(K_\Psi)$ for some $ij \neq kl$. Since, $ij \neq kl$, either $i \neq k$ or $j \neq l$.

**Case 1.** $i \neq k$ where $i, k \in N_\Phi$.

Since $x \in \phi_i(U_\Phi) \cap \phi_k(K_\Phi)$, $x \in \phi_i(U_\Phi)$ and $x \in \phi_k(K_\Phi)$. That is, $x \in \phi_i(K_\Phi \setminus B_\Phi) = \phi_i(K_\Phi) \setminus \phi_i(B_\Phi)$ and $x \in \phi_k(K_\Phi)$ and $x \notin \phi_i(B_\Phi)$. Since $x \notin \phi_i(B_\Phi) = \phi_i(\bigcup_{r \neq i} \phi^{-1}_r(\phi_r(K_\Phi) \cap \phi_r(K_\Phi)))$, $x \notin \phi_i(K_\Phi) \cap \phi_k(K_\Phi)$, a contradiction.

**Case 2.** $j \neq l$ where $j, l \in N_\Psi$.

Similar to case 1, $y \in \psi_j(U_\Psi) \times \psi_l(K_\Psi)$ leads to a contradiction. □

That is, by the above theorem, $U_{\Phi \times \Psi}$ satisfies the second condition of $SOSC_K$. By Theorem 3.4, $U_{\Phi \times \Psi}$ satisfies the first condition of $SOSC_K$ if $B_{\Phi \times \Psi}$ is inverse invariant. Thus, we can conclude that if $B_{\Phi \times \Psi}$ is inverse invariant, then $\Phi \times \Psi$ satisfies $SOSC_K$.

The overlapping set of an attractor in a product IFS $\Phi \times \Psi$ with similitudes $\zeta_{ij}$ is given by

$$O_{\Phi \times \Psi} = \bigcup_{ij \neq kl} (\zeta_{ij}(K) \cap \zeta_{kl}(K))$$

where $K$ is the attractor of $\Phi \times \Psi$.

Consider an example of a product IFS $\Phi \times \Psi$, where $\Phi = \{R; \frac{r}{2}, \frac{r+1}{2}\}$ and $\Psi = \{R; \frac{r}{2}, \frac{r+1}{2}\}$, with attractor $K$ (Figure 4).
Here the similarity boundary of $\Phi \times \Psi$ is $\{0, 1\} \times K_\Psi$, where $K_\Psi$ is the attractor of $\Psi$. Also, the overlapping set $O_{\Phi \times \Psi}$ of $K$ is the set $\{\frac{1}{2}\} \times K_\Psi$. Clearly, the inverse images of $O_{\Phi \times \Psi}$ under $\zeta_{ij}$s contain points of $B_{\Phi \times \Psi}$. This leads to the following theorem.

**Theorem 3.8.** Let $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$ be a product IFS with attractor $K$. Let $B_{\Phi \times \Psi}$ and $O_{\Phi \times \Psi}$ be the similarity boundary and overlapping set of $K$ respectively. Then,

$$B_{\Phi \times \Psi} \subseteq \bigcup_{i=1}^{n_\Phi} \bigcup_{j=1}^{n_\Psi} \zeta_{ij}^{-1}(O_{\Phi \times \Psi}).$$

**Proof.** Let $(x, y) \in B_{\Phi \times \Psi}$. That is, there exist $ij \neq kl$ where $(i, j), (k, l) \in N_\Phi \times N_\Psi$, such that $(x, y) \in \zeta_{ij}^{-1}((\zeta_{ij}(K) \cap \zeta_{kl}(K)))$, which gives $\zeta_{ij}(x, y) \in \zeta_{ij}(K) \cap \zeta_{kl}(K)$. Since $ij \neq kl$, $\zeta_{ij}(x, y) \in O_{\Phi \times \Psi}$ and, hence $(x, y) \in \zeta_{ij}^{-1}(O_{\Phi \times \Psi})$. Thus, $B_{\Phi \times \Psi} \subseteq \bigcup_{i=1}^{n_\Phi} \bigcup_{j=1}^{n_\Psi} \zeta_{ij}^{-1}(O_{\Phi \times \Psi})$. \hfill $\Box$

The next theorem says that the similarity boundary of an attractor of a product IFS does not intersect with any nonempty open set that satisfies the conditions for RO SC.

**Theorem 3.9.** Let $K$ be the attractor of the product IFS $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$. Suppose $V$ is a nonempty open subset of $K$. Then $\zeta_{ij}(V) \cap O_{\Phi \times \Psi} = \emptyset$ for all $(i, j) \in N_\Phi \times N_\Psi$ implies $V \cap B_{\Phi \times \Psi} = \emptyset$.

**Proof.** Assume $\zeta_{ij}(V) \cap O_{\Phi \times \Psi} = \emptyset$ for all $(i, j) \in N_\Phi \times N_\Psi$. Suppose there exists an element $(x, y) \in X \times Y$ such that $(x, y) \in V \cap B_{\Phi \times \Psi}$. By Theorem 3.8, $(x, y) \in V \cap \zeta_{ij}^{-1}(O_{\Phi \times \Psi})$, for some $(i, j) \in N_\Phi \times N_\Psi$. That is, $\zeta_{ij}(x, y) \in \zeta_{ij}(V) \cap O_{\Phi \times \Psi}$, which is a contradiction to the assumption. \hfill $\Box$

Consider a product IFS $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$. The attractor $K$ of $\Phi \times \Psi$ is the image of the product of the space of codes $N_\Phi \times N_\Psi = N_\Phi^\times \times N_\Psi^\times = N_\Phi \times N_\Psi \times \cdots \times N_\Psi \times N_\Psi \times \cdots$, where $N_\Phi = \{1, 2, \cdots, n_\Phi\}$ and
The similarity boundary $\partial K$ is invariant under the geometric shift map $\Phi \times \Psi$ if and only if $\zeta_{ij}(x, y) \in A$ for some $(i, j) \in N_\Phi \times N_\Psi$, where $A \subseteq K$. The next theorem is an important property of $\Phi \times \Psi$.

**Theorem 3.10.** Suppose $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, i \in N_\Phi, j \in N_\Psi\}$ is a product IFS with attractor $K$. Let $T$ be the geometric shift map on $K$. Then $(x, y) \in T(A)$ if and only if $\zeta_{ij}(x, y) \in A$ for some $(i, j) \in N_\Phi \times N_\Psi$, where $A \subseteq K$.

**Proof.** Let $(x, y) \in T(A)$. That is, $(x, y) = \pi \circ \sigma(ij)$ for some $ij = (i_1i_2i_3 \cdots j_1j_2j_3 \cdots) \in N_\Phi \times N_\Psi$. This implies $(x, y) = \pi(i_1i_2i_3 \cdots j_1j_2j_3 \cdots) = \bigcap_{m_i=2}^{\infty} \bigcap_{m_j=2}^{\infty} \zeta_i(i_1i_2i_3 \cdots j_1j_2j_3 \cdots)(K)$. Then, $\zeta_{ij}(x, y) = \pi(ij)$ and $\pi(ij) \in A$. Conversely, suppose that $\zeta_{ij}(x, y) \in A$ with $(x, y) \in K$ and $(i, j) \in N_\Phi \times N_\Psi$. Since $\pi$ is a projection map onto $K$, there exists $i \in N_\Phi$ and $j \in N_\Psi$ such that $(x, y) = \pi(ij)$. Then, $(x, y) = \pi \circ \sigma(i_1i_2 \cdots j_1j_2 \cdots) \in \pi \circ \sigma^{-1}(\zeta_{ij}(x, y)) \subseteq T(A)$. 

The following theorem states that the image of overlapping set in a product IFS under the geometric shift map is the same as the similarity boundary of its attractor.

**Theorem 3.11.** Let $\Phi \times \Psi$ be a product IFS with attractor $K$ and $T$ be the geometric shift map. Then

$$T(O_{\Phi \times \Psi}) = B_{\Phi \times \Psi},$$

where $O_{\Phi \times \Psi}$ is the overlapping set and $B_{\Phi \times \Psi}$ is the similarity boundary of $K$.

**Proof.** Let $(x, y) \in T(O_{\Phi \times \Psi})$. Then $\zeta_{ij}(x, y) \in O_{\Phi \times \Psi}$ for some $(i, j) \in N_\Phi \times N_\Psi$. That is $\zeta_{ij}(x, y) \in \zeta_{ij}(K \cap \zeta_{kl}(K))$ for some $(k, l) \in N_\Phi \times N_\Psi$ with $ij \neq kl$, which implies $(x, y) \in \zeta_{ij}^{-1}(\zeta_{ij}(K \cap \zeta_{kl}(K)))$ that implies $(x, y) \in B_{\Phi \times \Psi}$.

Conversely, assume $(x, y) \in B_{\Phi \times \Psi}$. That is, there exist some $(i, j), (k, l) \in N_\Phi \times N_\Psi$ with $ij \neq kl$, such that $(x, y) \in \zeta_{ij}^{-1}(\zeta_{ij}(K \cap \zeta_{kl}(K)))$. Thus, $\zeta_{ij}(x, y) \in \zeta_{ij}(K \cap \zeta_{kl}(K)) \subseteq O_{\Phi \times \Psi}$, since $ij \neq kl$. By the property of geometrical shift map, $(x, y) \in T(O_{\Phi \times \Psi})$.

Equivalency of the two separation conditions ROSC and SOSC in a product IFS $\Phi \times \Psi$ with attractor $K$ are proved in the next theorem using a certain condition on the similarity boundary of $K$.

**Theorem 3.12.** Let $\Phi \times \Psi$ be the product IFS of two IFSs with attractor $K$. Suppose the similarity boundary $B_{\Phi \times \Psi}$ of $K$ is inverse invariant. Then the following are equivalent.

1. $\Phi \times \Psi$ satisfies ROSC.
2. $U_{\Phi \times \Psi} = K \setminus B_{\Phi \times \Psi} \neq \emptyset$.
3. $\Phi \times \Psi$ satisfies SOSC.
Proof. To prove (1) $\implies$ (2), assume that $\Phi \times \Psi$ satisfies ROSC. That is, there exists an open set $V \subseteq K$ such that $\zeta_{ij}(V) \subseteq V$ for all $(i,j) \in N_\Phi \times N_\Psi$ and $\zeta_{ij}(V) \cap O_{\Phi \times \Psi} = \emptyset$ for all $(i,j) \in N_\Phi \times N_\Psi$. That implies $V \cap \zeta_{ij}^{-1}(O_{\Phi \times \Psi}) = \emptyset$, for all $(i,j) \in N_\Phi \times N_\Psi$. Suppose $U_{\Phi \times \Psi} = \emptyset$. That is, $K \subseteq B_{\Phi \times \Psi} \subseteq \bigcup_{i=1}^{n_\Psi} \bigcup_{j=1}^{n_\Phi} \zeta_{ij}^{-1}(O_{\Phi \times \Psi}) = \emptyset$, a contradiction to the assumption. Therefore, $U_{\Phi \times \Psi} = \emptyset$.

(2) $\implies$ (3). Suppose that $U_{\Phi \times \Psi} \neq \emptyset$. Clearly, $U_{\Phi \times \Psi}$ is an open subset of $K$ by definition. Since $B_{\Phi \times \Psi}$ is inverse invariant, $\zeta_{ij}(U_{\Phi \times \Psi}) \subseteq U_{\Phi \times \Psi}$, for all $(i,j) \in N_\Phi \times N_\Psi$, by Theorem 3.4. Also, $\zeta_{ij}(U_{\Phi \times \Psi}) \cap B_{\Phi \times \Psi} \subseteq \zeta_{ij}(U_{\Phi \times \Psi}) \cap B_{\Phi \times \Psi} = \emptyset$, by Theorem 3.7. Since $U_{\Phi \times \Psi}$ satisfies all the conditions for SOSC, $\Phi \times \Psi$ satisfies SOSC.

(3) $\implies$ (1). Assume $\Phi \times \Psi$ satisfies SOSC. So there exists an open set $V$ in $X \times Y$ with the condition $V \cap K \neq \emptyset$, such that $\bigcup_{i=1}^{n_\Phi} \bigcup_{j=1}^{n_\Psi} \zeta_{ij}(V) \subseteq V$ and $\zeta_{ij}(V) \cap B_{\Phi \times \Psi} = \emptyset$, for all $ij \neq kl$, where $i,k \in N_\Phi$ and $j,l \in N_\Psi$. Let $W = V \cap K$. Clearly, $W$ is a nonempty open subset of $K$. Moreover, $\zeta_{ij}(W) = \zeta_{ij}(V \cap K) = \zeta_{ij}(V) \cap \zeta_{ij}(K) \subseteq V \cap K = W$. Also, $\zeta_{ij}(W) \cap O_{\Phi \times \Psi} \subseteq \zeta_{ij}(V) \cap \emptyset$. \hfill \square

Using the next theorem we can characterize self similarity of an attractor in a product IFS using the Hausdorff measure of its similarity boundary.

**Theorem 3.13.** Let $\Phi \times \Psi$ be a product IFS and $K$ be its attractor. Then $K$ is self similar if and only if $H^\alpha(B_{\Phi \times \Psi}) = 0$, where $H^\alpha(B_{\Phi \times \Psi})$ is the $\alpha$-dimensional Hausdorff measure of the similarity boundary $B_{\Phi \times \Psi}$ of $K$.

**Proof.** Assume that $K$ is self similar. That is, $H^\alpha(\zeta_{ij}(K) \cap \zeta_{kl}(K)) = 0$, for all $ij \neq kl$. Let $A = \zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K)).$ Then $\zeta_{ij}(A) = \zeta_{ij}(K) \cap \zeta_{ij}(K)$ and hence, $H^\alpha(A) = H^\alpha(\zeta_{ij}(A)) = H^\alpha(\zeta_{ij}(K) \cap \zeta_{ij}(K)) = 0$. Being the finite union of such $A$s, $H^\alpha(B_{\Phi \times \Psi}) = 0$.

Conversely suppose that $H^\alpha(B_{\Phi \times \Psi}) = 0$. Since, $B_{\Phi \times \Psi} \subset K$, $H^\alpha(K) > 0$. Also, $\zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \subseteq B$ implies $\zeta_{ij}(K) \cap \zeta_{kl}(K) \subseteq \zeta_{ij}(B)$. Thus, $H^\alpha(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \leq H^\alpha(\zeta_{ij}(B_{\Phi \times \Psi})) = H^\alpha(B_{\Phi \times \Psi}) = 0$. Therefore, $K$ is self similar. \hfill \square

4. **Self similar measure of similarity boundary in product space.** The code space $\Omega$ in product space is defined by, $\Omega = \Pi \times \{\{1,2,\ldots,N_\Phi\} \times \Pi \times \{\{1,2,\ldots,N_\Psi\} \times \Pi \times \{\{1,2,\ldots,N_\Phi\} \times \Pi \times \{\{1,2,\ldots,N_\Psi\} \times \Pi \}$ Define the cylinder set $C_{(i_1,j_1),\ldots,(i_n,j_n)} = (i_1,j_1),\ldots,(i_n,j_n) \times [\Pi_{n+1} \{\{1,2,\ldots,N_\Phi\} \times \Pi \times \{\{1,2,\ldots,N_\Psi\} \times \Pi \times \{\{1,2,\ldots,N_\Phi\} \times \Pi \times \{\{1,2,\ldots,N_\Psi\} \times \Pi \}$. Now define a metric $\rho$ on $\Omega$ by,

$\rho((i_n,j_n),(k_n,l_n)) = \begin{cases} 1 & \text{if } (i_1,j_1) \neq (k_1,l_1) \\ s_{i_1,j_1} \cdots s_{i_m,j_m} & \text{if } (i_r,j_r) = (k_r,l_r) \text{ for all } 1 \leq r \leq m \\ 0 & \text{if } (i_r,j_r) \neq (k_r,l_r) \text{ for all } r \end{cases}
$

where $s_{i,j}$'s are contractivity factors of contractions $\zeta_{i,j}$ of the product IFS $\Phi \times \Psi = \{X \times Y; \zeta_{i,j}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$. The cylinder sets are both open and closed in $\Omega$ with respect to this metric $\rho$. Define a measure $\nu$ on $\Omega$ having the property, $\nu(C_{(i_1,j_1),\ldots,(i_n,j_n)}) = (s_{i_1,j_1} \cdots s_{i_n,j_n})^\alpha$, where $\alpha$ is the similarity dimension of $K$. Also, for this metric, Hausdorff measure, $H^\alpha(\Omega) = \nu(\Omega) = 1$.

Corresponding to the contractions $\zeta_{ij}$ on $K$, we define $\overline{\zeta}_{ij}$ be the contractions on $\Omega$ defined by, $\overline{\zeta}_{ij}(i_1,j_1),\overline{\zeta}_{ij}(i_2,j_2)) = ((i,j),(i_1,j_1),(i_2,j_2))$. Clearly, the contracting factor of $\overline{\zeta}_{ij}$ is the same as that of $\zeta_{ij}$ for all $(i,j) \in N_\Phi \times N_\Psi$. Let the address map be defined by $h : \Omega \to K$ be defined by, $h((i_1,j_1),\cdots) = \lim_{n \to \infty} \zeta_{i_1,j_1} \circ$
\( \zeta_{i,j} \circ \cdots \circ \zeta_{i,n, j_n}(K) \). Then \( h \) is well defined and continuous. Define the image measure \( \mu \) on \( K \) by \( \mu(L) = \nu(h^{-1}(L)) \), for \( L \subseteq K \). The next theorem helps to determine the image measure of a measurable set under similitude.

**Theorem 4.1.** Suppose \( K \) is the attractor of a product IFS \( \Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi \} \). Let \( \mu \) be the image measure on \( K \). Suppose, \( \mu(\zeta_{ij}(K)) = s_{ij}^o \). Then \( \mu(\zeta_{ij}(L)) = s_{ij}^o \mu(L) \), where \( L \) is \( \mu \)-measurable set in \( K \).

**Proof.** Let \( L \subseteq K \) be a \( \mu \)-measurable set. Then \( \mu(\zeta_{ij}(L)) = \nu(h^{-1}(\zeta_{ij}(L))) \). But \( h^{-1}(\zeta_{ij}(L)) \subseteq \overline{\zeta_{ij}(h^{-1}(L))} \) and \( \nu \) is a measure, \( \nu(\overline{\zeta_{ij}(h^{-1}(L))}) \leq \nu(h^{-1}(\zeta_{ij}(L))) \).

Thus,
\[
\nu(\overline{\zeta_{ij}(h^{-1}(L))}) = s_{ij}^o \nu(h^{-1}(L)) = s_{ij}^o \mu(L),
\]

since \( \nu \) is the Hausdorff measure on \( \Omega \). Therefore, \( \mu(\zeta_{ij}(L)) \geq s_{ij}^o \mu(L) \). Since \( L \subseteq K \), \( \zeta_{ij}(K) = (\zeta_{ij}(K) \setminus \zeta_{ij}(L)) \cup \zeta_{ij}(L) = \zeta_{ij}(K \setminus L) \cup \zeta_{ij}(L) \).

Therefore,
\[
\mu(\zeta_{ij}(K)) = \mu(\zeta_{ij}(K \setminus L)) + \mu(\zeta_{ij}(L))
\]

since the union is disjoint. Thus,
\[
s_{ij}^o = \mu(\zeta_{ij}(K \setminus L)) + \mu(\zeta_{ij}(L)) \leq s_{ij}^o \mu(K \setminus L) + s_{ij}^o \mu(L) = s_{ij}^o.
\]

Hence, \( \mu(\zeta_{ij}(L)) = s_{ij}^o \mu(L) \).

In the following theorem, an equivalent condition for the image measure of the similarity boundary to be zero is given.

**Theorem 4.2.** Suppose \( B_{\Phi \times \Psi} \) is the similarity boundary of \( K \), where \( K \) is the attractor of the product IFS \( \Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi \} \). Then the following are equivalent.

1. \( \mu(\zeta_{ij}(K)) \cap \zeta_{kl}(K) = 0 \) for \( ij \neq kl \).
2. \( \mu(B_{\Phi \times \Psi}) = 0 \).

**Proof.** Assume (1). Let \( L = \zeta_{i,j}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \) for some \( ij \neq kl \). That is, \( \zeta_{ij}(K) \cap \zeta_{kl}(K) \) and hence \( \mu(\zeta_{ij}(L)) = s_{ij}^o \mu(L) \). By assumption, \( \mu(\zeta_{ij}(K) \cap \zeta_{kl}(K)) = 0 \).

That implies, \( \mu(\zeta_{ij}(L)) = 0 \), and thus \( \mu(L) = 0 \). Being the finite union of such \( L \)'s, \( \mu(B_{\Phi \times \Psi}) = 0 \).

Now assume (2). Let \( L = \zeta_{i,j}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \), for some \( ij \neq kl \). Since, being finite intersection of compact sets, \( \zeta_{ij}(K) \cap \zeta_{kl}(K) \) is compact. Therefore, \( \zeta_{ij}(K) \cap \zeta_{kl}(K) \) is a Borel set in \( K \) and thus measurable. Therefore by Theorem 4.1, \( \mu(\zeta_{ij}(K) \cap \zeta_{kl}(K)) = s_{ij}^o \mu(L) \). But, \( L \subseteq B_{\Phi \times \Psi} \). Thus, \( \mu(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \leq s_{ij}^o \mu(B_{\Phi \times \Psi}) \). By assumption, \( \mu(B_{\Phi \times \Psi}) = 0 \) and \( \mu \) is a positive measure. Therefore, \( \mu(\zeta_{ij}(K) \cap \zeta_{kl}(K)) = 0 \).

Suppose \( \Phi \times \Psi \) satisfies SOSC. Then, the self similar measure of the similarity boundary \( B_{\Phi \times \Psi} \) of its attractor is zero, since \( \Phi \times \Psi \) satisfies SOSC, \( \mu(\zeta_{ij}(K) \cap \zeta_{kl}(K)) = 0 \). By Theorem 4.2, \( \mu(B_{\Phi \times \Psi}) = 0 \). Hausdorff dimension of the similarity boundary of an attractor of a product IFS which is totally disconnected or just touching is determined in the following theorem.

**Theorem 4.3.** Suppose the product IFS \( \Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi \} \) with attractor \( K \) is totally disconnected or just touching. Then, the Hausdorff dimension of the similarity boundary \( B_{\Phi \times \Psi} \) of \( K \) is zero.
Proof. Assume that $\Phi \times \Psi$ is totally disconnected. That is, $\zeta_{ij}(K) \cap \zeta_{kl}(K) = \emptyset$ for all $(i, j), (k, l) \in N_\Phi \times N_\Psi$ with $ij \neq kl$. Thus, $B_{\Phi \times \Psi} = \emptyset$ and hence $\dim_H(B_{\Phi \times \Psi}) = 0$.

Now assume $\Phi \times \Psi$ is just touching. Then, the overlapping set $O_{\Phi \times \Psi}$ is a finite set. Since $\zeta_{ij}$'s are bijective, $\zeta_{ij}^{-1}(O_{\Phi \times \Psi})$ is a finite set for all $(i, j) \in N_\Phi \times \Psi$. But, by Theorem 3.8, $B_{\Phi \times \Psi} \subseteq \bigcup_{i,j=1}^{n_\Phi} \bigcup_{n=1}^{n_\Psi} \zeta_{ij}^{-1}(O_{\Phi \times \Psi})$. Thus $B_{\Phi \times \Psi}$ is finite, being the finite union of finite sets. Since Hausdorff dimension of a finite set is zero, $\dim_H(B_{\Phi \times \Psi}) = 0$. \hfill \Box

In the case of an overlapping product IFS, Hausdorff dimension of its similarity boundary is non-zero. The next theorem gives a lower bound for the Hausdorff dimension of the similarity boundary of an attractor in an overlapping product IFS.

**Theorem 4.4.** Suppose $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$ is an overlapping IFS with attractor $K$. Then, the Hausdorff dimension of the similarity boundary $B_{\Phi \times \Psi}$ of $K$ is at least one. That is, $\dim_H(B_{\Phi \times \Psi}) \geq 1$.

*Proof.* Since $\Phi \times \Psi$ is an overlapping IFS, $\zeta_{ij}(K) \cap \zeta_{kl}(K)$ is a connected set for all $(i, j), (k, l) \in N_\Phi \times N_\Psi$ with $ij \neq kl$. Since each $\zeta_{ij}$ is a continuous function, $\zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K))$ is also a connected set containing more than one point. Thus, $\dim_H(\zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K))) \geq 1$. Hence,

$$\dim_H(B_{\Phi \times \Psi}) = \dim_H(\bigcup_{ij \neq kl} \zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K))) \geq 1.$$ \hfill \Box

The following theorem proves that the Hausdorff dimension of the similarity boundary of an attractor of a product IFS is the same as the Hausdorff dimension of its overlapping set.

**Theorem 4.5.** Let $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$ be a product IFS with attractor $K$. Let $B_{\Phi \times \Psi}$ and $O_{\Phi \times \Psi}$ be the similarity boundary and overlapping set of $K$ respectively. Then $\dim_H(B_{\Phi \times \Psi}) = \dim_H(O_{\Phi \times \Psi})$.

*Proof.* By Theorem 3.8,

$$B_{\Phi \times \Psi} \subseteq \bigcup_{i=1}^{n_\Phi} \bigcup_{j=1}^{n_\Psi} \zeta_{ij}^{-1}(O_{\Phi \times \Psi}).$$

Then

$$\dim_H(B_{\Phi \times \Psi}) \leq \dim_H \left( \bigcup_{i=1}^{n_\Phi} \bigcup_{j=1}^{n_\Psi} \zeta_{ij}^{-1}(O_{\Phi \times \Psi}) \right) = \max_{1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi} (\dim_H(\zeta_{ij}^{-1}(O_{\Phi \times \Psi}))).$$

Consider the overlapping set $O_{\Phi \times \Psi} = \bigcup_{ij \neq kl} (\zeta_{ij}(K) \cap \zeta_{kl}(K))$ where $(i, j), (k, l) \in N_\Phi \times N_\Psi$. Since for all $(i, j) \in N_\Phi \times N_\Psi$, $\zeta_{ij}$ is continuous and $K$ is compact, $\zeta_{ij}(K)$ is compact being the continuous image of a compact set. Every compact set is a Borel set and being finite union of Borel set, $O_{\Phi \times \Psi}$ is a Borel set. Since the inverse image of a Borel set under a continuous map is Borel, $\zeta_{ij}^{-1}(O_{\Phi \times \Psi})$ is a Borel set. Therefore,

$$\dim_H(O_{\Phi \times \Psi}) = \dim_H(\zeta_{ij}^{-1}(O_{\Phi \times \Psi})) = \dim_H(\zeta_{ij}^{-1}(O_{\Phi \times \Psi}))$$
for all $(i, j) \in N_\Phi \times N_\Psi$. That implies,

$$\dim_H(B_{\Phi \times \Psi}) \leq \max_{1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi} (\dim_H(O_{\Phi \times \Psi})) = \dim_H(O).$$

Thus, $\dim_H(O_{\Phi \times \Psi}) = \dim_H\left(\bigcup_{ij \neq kl} \zeta_{ij}(K) \cap \zeta_{kl}(K)\right)$

$$\leq \max_{1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi} \{\dim_H(\zeta_{ij}(K) \cap \zeta_{kl}(K))\}
= \max_{1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi} \{\dim_H(\zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K)))\}
= \dim_H(B_{\Phi \times \Psi}).$$

Hence, $\dim_H(O_{\Phi \times \Psi}) \leq \dim_H(B_{\Phi \times \Psi}). \quad \square$

The Hausdorff dimension of the similarity boundary of an attractor of a product
IFS is always less than the similarity dimension of the attractor, which is proved in
the next theorem.

**Theorem 4.6.** Let the product IFS $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$
with attractor $K$ satisfying $SOSC$ and let $\text{int}(K) \neq \emptyset$. If $B_{\Phi \times \Psi}$ is the similarity
boundary of $K$, then $\dim_H(B_{\Phi \times \Psi}) < \alpha_1 + \alpha_2$, where $\alpha_1$ and $\alpha_2$ are the similarity
dimensions of $\Phi$ and $\Psi$.

**Proof.** By Theorem 6.1, if the product IFS satisfies $SOSC$, then the upper box
counting dimension of the overlapping set, $\overline{\dim_B}(O_{\Phi \times \Psi})$, is strictly less than $\alpha_1 + \alpha_2$.

**Claim 1.** $\dim_H(\partial(K)) < \alpha_1 + \alpha_2$, where $\partial(K)$ is the topological boundary of $K$.

Let $(I, J)_k = \{(i_1, j_1), \ldots, (i_k, j_k) : i_r \in \{1, 2, \ldots, n_\Phi\}, j_r \in \{1, 2, \ldots, n_\Psi\}\}$. Then it is easy to see that, $K$ is the invariant set for the IFS
$\{X \times Y; \zeta_{ij} = \zeta_{i_1j_1} \circ \zeta_{i_2j_2} \circ \cdots \circ \zeta_{i_kj_k} : IJ = ((i_1, j_1), \ldots, (i_k, j_k)) \in (I, J)_k\}$.

**Claim 2.** There exists an element $k \in \mathbb{N}$ such that for some $IJ \in (I, J)_k$, $\zeta_{IJ}(K) \subseteq \text{int}(K)$.

Let $(x, y) \in \text{int}(K)$. Then, there is an $\epsilon > 0$, such that open ball $B_\epsilon(x, y) \subseteq \text{int}(K)$. Choose $k$ large enough so that $\max\{|s_{IJ} : IJ \in (I, J)_k\} < \frac{\epsilon}{\text{diam}(K)}$.

Again choose $IJ \in (I, J)_k$ such that $(x, y) \in \zeta_{IJ}(K)$. This is possible since $K = \bigcup_{IJ \in (I, J)_k} \zeta_{IJ}(K)$. Thus,

$$\text{diam}(\zeta_{IJ}(K)) = s_{IJ}\text{diam}K < \frac{\epsilon\text{diam}(K)}{\text{diam}(K)} = \epsilon,$$

with $(x, y) \in \zeta_{IJ}(K)$. Hence, $\zeta_{IJ}(K) \subseteq B_\epsilon(x, y) \subseteq \text{int}(K)$, which proves Claim 2.

Let $E = \{IJ \in (I, J)_k : \zeta_{IJ}(K) \cap \partial K \neq \emptyset\}$ and $F = \{IJ \in (I, J)_k : \zeta_{IJ}(K) \cap \text{int}(K)\} = (I, J)_k \setminus E$. Clearly, $E$ and $F$ are disjoint sets with $E \cup F = (I, J)_k$. Also, $F \neq \emptyset$ by Claim 2.

Let $L = \bigcup_{IJ \in E} \zeta_{IJ}(K)$. Since, $\partial K \subseteq K = \bigcup_{IJ \in I, J} \zeta_{IJ}(K)$, and $E$ contains all the
$IJ$ such that $\zeta_{IJ}(K) \cap \partial K \neq \emptyset$, $\partial K \subseteq L$. Also, $\zeta_{IJ}(\text{int}(K)) \subseteq \text{int}(K)$, since each $\zeta_{IJ}$ is a contraction. Therefore, $\zeta_{IJ}(\text{int}(K)) \cap \partial K = \emptyset$ for all $IJ \in (I, J)_k$. That implies, $\partial K \subseteq \bigcup_{IJ \in E} \zeta_{IJ}(\partial K)$ and, hence $\partial K \subseteq L$.

**Claim 3.** $\dim_H(L) < \alpha_1 + \alpha_2$.

$\alpha_1 + \alpha_2$ is the unique number that satisfies the equation $\sum_{IJ \in (I, J)_k} s_{IJ}^{\alpha_1 + \alpha_2} = 1$.
Let $\beta$ be the number that satisfies the equation $\sum_{IJ \in E} s_{IJ}^{\beta} = 1$. Since $E \subseteq (I, J)_k$,
The growth of a tumor can be evaluated by finding its fractal dimension. Many the concept of fractal dimension can use to measure the irregularity of a tumor. because of these properties, Fractal dimension provides a statistical indication of complexity and the dimensions and hence we can examine the irregular boundaries of tumors by fractal geometry.

**Theorem 4.7.**

Let $V = \text{int}(K)$. By assumption, $V \neq \emptyset$. Since in $\mathbb{R}^n \times \mathbb{R}^m$, the similitudes are open maps, $\text{int}(\zeta_{ij}(K)) = \zeta_{ij}(\text{int}(K)) = \zeta_{ij}(V)$. Suppose for some $ij \neq kl$, where $(i,j), (k,l) \in N_\Phi \times \Psi$, $\zeta_{ij}(V) \cap \zeta_{kl}(V) = \emptyset$. Then, $\zeta_{ij}(V) \cap \zeta_{kl}(V)$ is an open set and hence its Lebesgue measure, $\lambda(\zeta_{ij}(V) \cap \zeta_{kl}(V)) > 0$. But by assumption, $0 = \lambda(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \geq \lambda(\zeta_{ij}(V) \cap \zeta_{kl}(V)) > 0$, a contradiction. Therefore, $\text{int}(\zeta_{ij}(K)) \cap \text{int}(\zeta_{kl}(V)) = \emptyset$ for all $ij \neq kl$. That implies, $\zeta_{ij}(K) \cap \zeta_{kl}(K) = \partial(\zeta_{ij}(K)) \cap \partial(\zeta_{kl}(K))$. Since each $\zeta_{ij}$ is a homeomorphism on $\mathbb{R}^n \times \mathbb{R}^m$, $\partial(\zeta_{ij}(K)) = \zeta_{ij}(\partial K)$. Thus, $\zeta_{ij}(K) \cap \zeta_{kl}(K) = \zeta_{ij}(\partial K) \cap \zeta_{kl}(\partial K)$. That is, for all $ij \neq kl$, $\zeta_{ij}(K) \cap \zeta_{kl}(K) \subseteq \zeta_{ij}(\partial K)$, and hence $\zeta_{ij}^{-1}(\zeta_{ij}(K) \cap \zeta_{kl}(K)) \subseteq \partial K$. Thus, $B_{\Phi \times \Psi} \subseteq \partial K$ and hence $\dim_H(B_{\Phi \times \Psi}) < \alpha_1 + \alpha_2$. 

We conclude this section by proving that the similarity boundary of an attractor of a product IFS is an extension of its topological boundary.

**Theorem 4.7.** Suppose the product IFS $\Phi \times \Psi = \{X \times Y; \zeta_{ij}, 1 \leq i \leq n_\Phi, 1 \leq j \leq n_\Psi\}$ with attractor $K$ satisfies SOSC. If the similarity boundary $B_{\Phi \times \Psi}$ of $K$ is inverse invariant, then $B_{\Phi \times \Psi} = \partial K$, where $\partial K$ is the topological boundary of $K$.

**Proof.** By Claim 4 in Theorem 4.7, $B_{\Phi \times \Psi} \subseteq \partial K$. Now it is enough to prove the converse. Clearly, $B_{\Phi \times \Psi}$ is a closed set. Assume that, there exists an element $(x, y) \in \partial K \setminus B_{\Phi \times \Psi}$. Then, $\zeta_{ij}(x, y) \in \partial K$ for all $(i, j) \in N_\Phi \times N_\Psi$. Suppose there exists some $(i, j) \in N_\Phi \times N_\Psi$, such that $(x, y) \in \text{int}(K)$. Then, since $\zeta_{ij}(x, y) \in \zeta_{ij}(\partial K) = \partial(\zeta_{ij}(K))$, $\zeta_{ij}(x, y) \in \zeta_{kl}(K)$ for some $ij \neq kl$. That is, $\zeta_{ij}(x, y) \in \zeta_{ij}(K) \cap \zeta_{kl}(K)$ for some $ij \neq kl$. Thus, $(x, y) \in B_{\Phi \times \Psi}$, a contradiction. Therefore, if $(x, y) \in \partial K \setminus B_{\Phi \times \Psi}$, then $\zeta_{ij}(x, y) \in \partial K$ for all $(i, j) \in N_\Phi \times N_\Psi$. Also if $\zeta_{ij}(x, y) \in B_{\Phi \times \Psi}$, then $(x, y) \in \zeta_{ij}^{-1}(B_{\Phi \times \Psi}) \subseteq B_{\Phi \times \Psi}$, since $B_{\Phi \times \Psi}$ is inverse invariant. But this leads to a contradiction to the choice of $(x, y)$. Thus, $x \in \partial K \setminus B_{\Phi \times \Psi}$ implies $\zeta_{ij}(x, y) \in \partial K \setminus B_{\Phi \times \Psi}$, for all $(i, j) \in N_\Phi \times N_\Psi$. Since the choice of $x$ is arbitrary, $\zeta_{ij}(\partial K \setminus B_{\Phi \times \Psi}) \subseteq \partial K \setminus B_{\Phi \times \Psi}$, for all $(i, j) \in N_\Phi \times N_\Psi$. Let $A = \partial K \setminus B_{\Phi \times \Psi}$. Then, $\zeta_{ij}(A) \subseteq A$, for all $(i, j) \in N_\Phi \times N_\Psi$. Thus, $K \subseteq A$, which is a contradiction and, hence $\partial K \subseteq B_{\Phi \times \Psi}$. 

**Application.** Fractal theory has extensive applications in higher dimensional spaces fractal modeling allow us to examine many 3D objects that are complex and irregular in nature. One of the main application of fractal geometry lies in the area of image analysis ([18], [19], [20], [25]) and its significance has been increased in many areas of biology and medicine such as, pathology, molecular biology etc. Even though the characteristics of most of the objects in pathology such as branching of vessels and tumors have complex structures, they have a repeated patterns called self similarity.

The topological dimensions of many tumors are larger than their fractal dimensions and hence we can examine the irregular boundaries of tumors by fractal geometry. Fractal dimension provides a statistical indication of complexity and the magnitude of irregularity of objects and it helps to reduce many geometric parameters of an irregular structure to a single parameter. Because of these properties, the concept of fractal dimension can use to measure the irregularity of a tumor. The growth of a tumor can be evaluated by finding its fractal dimension. Many
studies ([3], [8]) show that the fractal dimension of benign tumor is low and that of a malignant tumor is high. So fractal dimension can be used as a mathematical tool to diagnose cancerous tumors.

In this paper we proposed the concept of the similarity boundary of an attractor in product spaces to study self similar fractals in higher dimensional space. For a given 3D tumor image, it is possible to find its product IFS whose attractor is same as the real image. By using the concepts and results that discussed in this paper, one can find similarity boundary and its dimension and hence Theorem 4.3 and Theorem 4.4 can be used to evaluate the fractal dimension of a tumor. Thus fractal modeling in higher dimensional spaces using the idea of similarity boundary can be used to diagnose cancer tumors.

5. Conclusion. Dynamical systems generate fractal objects by iteration. The iterative process creating attractors very often produce self similar sets. Discrete and random systems also have self similarity in different variations. The relationship between the boundaries of attractors in discrete dynamical systems and their coordinate projections are studied in this paper. Several properties of self similar sets in higher dimensional systems are analyzed using this concept. Open set conditions and different types of dimensions are also discussed from a product space perspective. More related applications will be studied in the future.

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Received April 2020; revised October 2020.

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