In this paper we study the $H^2$ global regularity for solutions of the $p(x)$-Laplacian in two-dimensional convex domains with Dirichlet boundary conditions. Here $p : \Omega \to [p_1, \infty)$ with $p \in \text{Lip}(\Omega)$ and $p_1 > 1$.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ and let $p : \Omega \to (1, +\infty)$ be a measurable function. In this work, we study the $H^2$ global regularity of the weak solution of the following problem

\[
\begin{cases}
-\Delta_{p(x)}u = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]

where $\Delta_{p(x)}u = \text{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the $p(x)$-Laplacian. The hypothesis over $p$, $f$ and $g$ will be specified later.

Note that, the $p(x)$-Laplacian extends the classical Laplacian ($p(x) \equiv 2$) and the $p$-Laplacian ($p(x) \equiv p$ with $1 < p < +\infty$).

This operator has been recently used in image processing and in the modeling of electrorheological fluids, see [3,5,24].

Motivated by the applications to image processing problems, in [8], the authors study two numerical methods to approximate solutions of the type of (1.1). In Theorem 7.2, the authors prove the convergence in $W^{1, p(\cdot)}(\Omega)$ of the conformal Galerkin finite element method. It is of our interest to study, in a future work, the rate of this convergence. In general, all the error bounds depend on the global regularity of the second derivatives of the solutions, see for example [6,22]. However, there appear to be no existing regularity results in the literature that can be applied here, since all the results have either a first order or local character.

The $H^2$ global regularity for solutions of the $p$-Laplacian is studied in [22]. There the authors prove the following: Let $1 < p \leq 2$, $g \in H^2(\Omega)$, $f \in L^q(\Omega)$ ($q > 2$) and $u$ be the unique weak solution of (1.1). Then:

- If $\partial \Omega \in C^2$ then $u \in H^2(\Omega)$;
• If $\Omega$ is convex and $g = 0$ then $u \in H^2(\Omega)$;
• If $\Omega$ is convex with a polygonal boundary and $g \equiv 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Regarding the regularity of the weak solution of (1.1) when $f = 0$, in [1,7], the authors prove the $C^{1,\alpha}_{loc}$ regularity (in the scalar case and also in the vectorial case). Then, in the paper [15] the authors study the case where the functional has the so-called $(p,q)$-growth conditions. Following these ideas, in [17], the author proves that the solutions of (1.1) are in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ if $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with $C^{1,\gamma}$ boundary, $p(x)$ is a H"{o}lder function, $f \in L^\infty(\Omega)$ and $g \in C^{1,\gamma}(\overline{\Omega})$; while in [4], the authors prove that the solutions are in $H^2_{loc}(\{x \in \Omega: p(x) \leq 2\})$ if $p(x)$ is uniformly Lipschitz (Lip(\Omega)) and $f \in W^{2,1,1}_{loc}(\Omega) \cap L^\infty(\Omega)$.

Our aim, it is to generalize the results of [22] in the case where $p(x)$ is a measurable function. To this end, we will need some hypothesis over the regularity of $p(x)$. Moreover, in all our result we can avoid the restriction $g = 0$, assuming some regularity of $g(x)$.

On the other hand, to prove our results, we can assume weaker conditions over the function $f$ than the ones on [4]. Since, we only assume that $f \in L^{q(\cdot)}(\Omega)$, we do not have a priori that the solutions are in $C^{1,\alpha}(\Omega)$. Then we cannot use it to prove the $H^2$ global regularity. Nevertheless, we can prove that the solutions are in $C^{1,\alpha}(\overline{\Omega})$, after proving the $H^2$ global regularity.

The main results of this paper are:

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^2$ boundary, $p \in \text{Lip}(\overline{\Omega})$ with $p(x) \geq p_1 > 1$, $g \in H^2(\Omega)$ and $u$ be the weak solution of (1.1). If

\begin{align*}
\text{(F1)} & \quad f \in L^{q(\cdot)}(\Omega) \text{ with } q(x) \geq q_1 > 2 \text{ in the set } \{x \in \Omega: p(x) \leq 2\}; \\
\text{(F2)} & \quad f \equiv 0 \text{ in the set } \{x \in \Omega: p(x) > 2\},
\end{align*}

then $u \in H^2(\Omega)$.

**Theorem 1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with convex boundary, $p \in \text{Lip}(\overline{\Omega})$ with $p(x) \geq p_1 > 1$, $g \in H^2(\Omega)$ and $u$ be the weak solution of (1.1). If $f$ satisfies (F1) and (F2) then $u \in H^2(\Omega)$.

Using the above theorem we can prove the following:

**Corollary 1.3.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^2$ with polygonal boundary, $p$ and $f$ as in the previous theorem, $g \in W^{2,q(\cdot)}(\Omega)$ and $u$ be the weak solution of (1.1) then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.

Observe that this result extends the one in [17] in the case where $\Omega$ is a polygonal domain in $\mathbb{R}^2$.

**Organization of the paper.** The rest of the paper is organized as follows. After a short Section 2 where we collect some preliminary results, in Section 3, we study the $H^2$-regularity for the non-degenerated problem. In Section 4 we prove Theorem 1.1. Then, in Section 5, we study the regularity of the solution $u$ of (1.1) if $\Omega$ is convex. In Section 6, we make some comments on the dependence of the $H^2$-norm of $u$ on $p_1$. Lastly, in Appendices A and B we give some results related to elliptic linear equation with bounded coefficients and Lipschitz functions, respectively.

2. Preliminaries

We now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ and $p : \Omega \to [1, +\infty)$ be a measurable bounded function, called a variable exponent on $\Omega$ and denote $p_1 := \text{essinf } p(x)$ and $p_2 := \text{esssup } p(x)$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} \, dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\{k > 0; \varrho_{p(\cdot)}(u/k) \leq 1\}.$$ 

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

For the proofs of the following theorems, we refer the reader to [12].
Theorem 2.1 (Hölder’s inequality). Let $p, q, s : \Omega \to [1, +\infty]$ be measurable functions such that
\[
\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)} \quad \text{in } \Omega.
\]
Then the inequality
\[
\|f g\|_{L^s(\Omega)} \leq 2 \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}
\]
holds for all $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$.

Let $W^{1,p}(\Omega)$ denote the space of measurable functions $u$ such that $u$ and the distributional derivative $\nabla u$ are in $L^p(\Omega)$. The norm
\[
\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}
\]
makes $W^{1,p}(\Omega)$ a Banach space.

Theorem 2.2. Let $p'(x)$ be such that $1/p(x) + 1/p'(x) = 1$. Then $L^{p'}(\Omega)$ is the dual of $L^p(\Omega)$. Moreover, if $p_1 > 1$, $L^{p_1}(\Omega)$ and $W^{1,p_1}(\Omega)$ are reflexive.

We define the space $W_0^{1,p}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Then we have the following version of Poincaré’s inequity (see Theorem 3.10 in [21]).

Lemma 2.3 (Poincaré’s inequality). If $p : \Omega \to [1, +\infty)$ is continuous in $\overline{\Omega}$, there exists a constant $C$ such that for every $u \in W_0^{1,p}(\Omega)$,
\[
\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.
\]

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that $p$ is log-Hölder continuous in $\Omega$ if there exists a constant $C_{\log}$ such that
\[
|p(x) - p(y)| \leq \frac{C_{\log}}{\log(1 + \frac{1}{|x-y|})} \quad \forall x, y \in \Omega.
\]

It was proved in [10, Theorem 3.7], that if one assumes that $p$ is log-Hölder continuous then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ (see also [9,12,13,21,25]).

We now state the Sobolev embedding theorem (for the proofs see [12]). Let
\[
p^*(x) := \begin{cases} \frac{p(x)N}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N \end{cases}
\]
be the Sobolev critical exponent. Then we have the following:

Theorem 2.4. Let $\Omega$ be a Lipschitz domain. Let $p : \Omega \to [1, \infty)$ and $p$ be log-Hölder continuous. Then the imbedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*(\Omega)}(\Omega)$ is continuous.

3. $H^2$-regularity for the non-degenerated problem for any dimension

In this section we assume that $\Omega$ is a bounded domain in $\mathbb{R}^N$, with $N \geq 2$.

We want to study higher regularity of the weak solution of the regularized equation,
\[
\begin{cases}
-\text{div}(\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]
where $0 < \varepsilon \leq 1$, and $f \in \text{Lip}(\Omega)$ and $g \in W^{1,p}(\Omega)$.

The existence of a weak solution of (3.2) holds by Theorem 13.3.3 in [12].

Remark 3.1. Given $\varepsilon \geq 0$, $p \in C^0(\overline{\Omega})$ for some $\alpha_0 > 0$, and $g \in L^\infty(\Omega)$ we have the following results:

(1) Since $f, g \in L^\infty(\Omega)$, by Theorem 4.1 in [18], we have that $u \in L^\infty(\Omega)$.

(2) By Theorem 1.1 in [17], $u \in C^{1,\alpha}(\Omega)$ for some $\alpha$ depending on $p_1, p_2$, $\|u\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$. Moreover, given $\Omega_0 \subset \subset \Omega$, $\|u\|_{C^{1,\alpha}(\Omega_0)}$ depends on the same constants and $\text{dist}(\Omega_0, \partial \Omega)$. 

Finally, by Theorem 1.2 in [17], if \( \partial \Omega \in C^{1, \gamma} \) and \( g \in C^{1, \gamma}(\partial \Omega) \) for some \( \gamma > 0 \) then \( u \in C^{1, \alpha}(\Omega) \), where \( \alpha \) and \( \|u\|_{C^{1, \alpha}(\Omega)} \) depend on \( p_1, p_2, N, \|u\|_{L^\infty(\Omega)}, \|p\|_{C^\gamma(\Omega)}, \alpha_0 \) and \( \gamma \).

We will first prove the \( H^2 \)-local regularity assuming only that \( p(x) \) is Lipschitz. Then, we will prove the global regularity under the stronger condition that \( \nabla p(x) \) is Hölder.

3.1. \( H^2 \)-local regularity

While we were finishing this paper, we found the work [4], where the authors give a different proof of the \( H^2 \)-local regularity of the solutions of \((3.2)\). Anyhow, we leave the proof for the completeness of this paper.

**Theorem 3.2.** Let \( p, f \in \text{Lip}(\Omega) \) with \( p_1 > 1 \) and \( u \) be a weak solution of \((3.2)\), then \( u \in H^2_{\text{loc}}(\Omega) \).

**Proof.** First, let us define for any function \( F \) and \( h > 0 \),
\[
\Delta^h F(x) = \frac{F(x + h) - F(x)}{h},
\]
where \( h = he_k \) and \( e_k \) is a vector of the canonical base of \( \mathbb{R}^N \).

Let \( \eta(x) = \xi(x)^2 \Delta^h u(x) \) where \( \xi \) is a regular function with compact support. Therefore, if we take \( v_e = (|u|^2 + \varepsilon)^{1/2} \) and \( \eta < \text{dist}(\text{supp}(\xi), \partial \Omega) \), we have
\[
\int_\Omega \left( v_e(x)^{p(x)-2} \nabla u(x), \nabla \eta(x) \right) dx = \int_\Omega f(x) \eta(x) dx,
\]
\[
\int_\Omega \left( v_e(x) + h \right)^{p(x)+h-2} \nabla u(x + h), \nabla \eta(x) \right) dx = \int_\Omega f(x + h) \eta(x) dx.
\]

Subtracting, using that \( \nabla \eta = 2\xi \nabla \xi \Delta^h u + \xi^2 \Delta^h (\nabla u) \) and dividing by \( h \) we obtain
\[
I = \int_\Omega \left( \Delta^h \left( v_e(x)^{p(x)-2} \nabla u \right), \Delta^h (\nabla u) \right) \xi^2 dx
= -2 \int_\Omega \left( \Delta^h \left( v_e(x)^{p(x)-2} \nabla u \right), \xi \nabla \xi \Delta^h u \right) dx + \int_\Omega \xi^2 \Delta^h f \Delta^h u dx
= 2 \int_\Omega \left( \int_0^1 \left( v_e(x + ht)^{p(x)+ht-2} \nabla u(x + ht) \right) dt \right) \frac{\partial}{\partial x_k} (\xi \nabla \xi \Delta^h u) dx
+ \int_\Omega \xi^2 \Delta^h f \Delta^h u dx
= II + III.
\]

Now, let us fix a ball \( B_R \) such that \( B_{3R} \subset \subset \Omega \) and take \( \xi \in C^\infty_0(\Omega) \) supported in \( B_{2R} \) such that \( 0 \leq \xi \leq 1 \), \( \xi = 1 \) in \( B_R \), \( |

\nabla \xi | \leq 1/R \) and \( |D^2 \xi | \leq CR^{-2} \).

By Remark 3.1, there exists a constant \( C_1 > 0 \) such that \( |\nabla u| \leq C_1 \) in \( B_{3R} \), therefore we get
\[
II \leq 2 \int_{B_{3R}} \frac{C}{R} |\Delta^h u_{3R} | \xi dx + 2 \int_{B_{3R}} \frac{C}{R^2} |\Delta^h u | dx
\leq \frac{C}{R} \int_{B_{2R}} |\Delta^h (\nabla u) | \xi dx + CR^{N-2}.
\]

On the other hand, since \( f \) is Lipschitz we have that
\[
|f(x + h) - f(x)| \leq C_2 h
\]
for some constant \( C_2 > 0 \). This implies that
\[
III \leq C_2 R^N.
\]
Therefore, summing II and III, and using Young’s inequality, we have that for any $\delta > 0$

$$I \leq \delta \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx + C,$$

(3.3)

for some constant $C$ depending on $R$ and $\delta$.

On the other hand observe that $I = I_1 + I_2$ where

$$I_1 = \frac{1}{h} \int_{B_{2R}} \left( (v_\varepsilon(x+h)^{p(x+h)-2} \nabla u(x+h) - v_\varepsilon(x)^{p(x+h)-2} \nabla u(x)), \Delta^h(\nabla u) \right) \xi^2 \, dx,$$

and

$$I_2 = \frac{1}{h} \int_{B_{2R}} \left( v_\varepsilon(x)^{p(x+h)} - v_\varepsilon(x)^{p(x)} \right) \frac{\nabla u(x)}{v_\varepsilon(x)^2}, \Delta^h(\nabla u) \right) \xi^2 \, dx.$$

Using that $p(x)$ is Lipschitz and the fact that $|\nabla u(x)| \leq C_1$ we have that, for some $b$ between $p(x+h)$ and $p(x)$,

$$\frac{1}{h} |v_\varepsilon(x)^{p(x+h)} - v_\varepsilon(x)^{p(x)}| = \left| v_\varepsilon(x)^b \log(v_\varepsilon(x)) \right| \frac{p(x+h) - p(x)}{h} \leq C,$$

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

Therefore, we have that

$$-I_2 \leq CC_1 \varepsilon^{-1} \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx.$$

By (3.3), the last inequality and using again Young’s inequality we have that, for any $\delta > 0$,

$$I_1 \leq \delta \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx + C,$$

(3.4)

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

To finish the proof, we have to find a lower bound for $I_1$. By the well-known inequality, we have that

$$\left( (v_\varepsilon(x+h)^{p(x+h)-2} \nabla u(x+h) - v_\varepsilon(x)^{p(x+h)-2} \nabla u(x)), (\nabla u(x+h) - \nabla u(x)) \right) \geq C_\xi |\nabla u(x+h) - \nabla u(x)|^2,$$

where

$$C_\xi = \begin{cases} \varepsilon^{(p(x+h)-2)/2} & \text{if } p(x+h) \geq 2, \\ (p(x+h) - 1) \varepsilon^{(p(x+h)-2)/2} & \text{if } p(x+h) \leq 2. \end{cases}$$

Therefore, using that $p_1 > 1$, we arrive at

$$I_1 \geq \int_{B_{2R}} C h^{-2} |\nabla u(x+h) - \nabla u(x)|^2 \xi^2 \, dx = C \int_{B_{2R}} |\Delta^h(\nabla u(x))|^2 \xi^2 \, dx.$$

Finally combining the last inequality with (3.4) we have that

$$\int_{B_R} |\Delta^h(\nabla u(x))|^2 \, dx \leq C(N, p, f, \varepsilon).$$

This proves that $u \in H^2_{\text{loc}}(\Omega)$.

$\square$

3.2. $H^2$-global regularity

Now we want to prove that if $f \in \text{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$, the regularized equation (3.2) has a weak solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for an $\alpha \in (0, 1)$. We already know, by Remark 3.1, that $u \in C^{1,\alpha}(\overline{\Omega})$. Then, we only need to prove that $u \in C^2(\Omega)$.

Lemma 3.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^{1,\gamma}$, $p \in C^{1,\beta}(\Omega) \cap C^{\alpha}(\overline{\Omega})$, $f \in \text{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$. Then, the Dirichlet Problem (3.2) has a solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. 
Proof. Observe that by Theorem 3.2, we know that the solution is in $H^2_{\text{loc}}(\Omega)$. Then for any $\Omega' \subset \subset \Omega$ we can derive the equation and look at the solution of (3.2) as the solution of the following equation,

$$
\begin{aligned}
L_{\varepsilon}u &= a(x) \quad \text{in } \Omega', \\
u &= u \quad \text{on } \partial \Omega'.
\end{aligned}
$$

(3.5)

Here,

$$
L_{\varepsilon}u = a_{ij}^{\varepsilon}(x)u_{x_i x_j}
$$

with

$$
a_{ij}^{\varepsilon}(x) = \delta_{ij} + (p(x) - 2) \frac{u_{x_i} u_{x_j}}{v_2^2}, \quad v_2 = (\varepsilon + |\nabla u|^2)^{\frac{1}{2}} \quad \text{and}
$$

$$
a_{\varepsilon}(x) = \ln(v_2) \langle \nabla u, \nabla p \rangle + f v_2^{2-p}. \quad (3.6)
$$

The operator $L_{\varepsilon}$ is uniformly elliptic in $\Omega$, since for any $\xi \in \mathbb{R}^N$

$$\min\{ (p_1 - 1), 1 \} |\xi|^2 \leq a_{ij}^{\varepsilon} \xi_i \xi_j \leq \max\{ (p_2 - 1), 1 \} |\xi|^2. \quad (3.7)
$$

On the other hand, by Remark 3.1, $u \in C^{1,\alpha}(\overline{\Omega})$. Then, $a_{ij}^{\varepsilon} \in C^{\alpha}(\overline{\Omega})$, since $\varepsilon > 0$. Using that $f \in \text{Lip}(\Omega)$, we have that $a \in C^{\rho}(\Omega)$ where $\rho = \min(\alpha, \beta)$. If $\partial \Omega' \in C^2$, as $u$ is the unique solution of (3.5), by Theorem 6.13 in [19], we have that $u \in C^{2,\rho}(\Omega')$. This ends the proof. $\square$

Remark 3.4. By the $H^2$ global estimate for linear elliptic equations with $L^\infty(\Omega)$ coefficients in two variables (see Lemma A.1 and (3.7)) we have that

$$
\|u\|_{H^2(\Omega)} \leq C\left( \|a\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)} \right)
$$

where $u$ is the solution of (3.2) and $C$ is a constant independent of $\varepsilon$.

4. Proof of Theorem 1.1

Before proving the theorem, we will need a global bound for the derivatives of the solutions of (3.2).

Lemma 4.1. Let $f \in L^{q'(1)}(\Omega)$ with $q'(x) \leq p^*(x)$, $g \in W^{1,p^*(1)}(\Omega)$, $\varepsilon > 0$ and $u_{\varepsilon}$ be the weak solution of (3.2) then

$$
\|\nabla u_{\varepsilon}\|_{L^{q'(1)}(\Omega)} \leq C
$$

where $C$ is a constant depending on $\|f\|_{L^{q'(1)}(\Omega)}, \|g\|_{W^{1,p^*(1)}(\Omega)}$ but not on $\varepsilon$.

Proof. Let

$$
J(v) := \int_\Omega \frac{1}{p(x)} (|\nabla v|^2 + \varepsilon)^{p(x)/2} \, dx.
$$

By the convexity of $J$ and using (3.2) we have that

$$
J(u_{\varepsilon}) \leq J(g) - \int_\Omega (|\nabla u_{\varepsilon}|^2 + \varepsilon)^{(p-2)/2} \nabla u_{\varepsilon}(\nabla g - \nabla u_{\varepsilon}) \, dx
$$

$$
\leq C \left( 1 + \int_\Omega f(u_{\varepsilon} - g) \, dx \right)
$$

$$
\leq C \left( 1 + \|f\|_{L^{q'(1)}(\Omega)} \|u_{\varepsilon} - g\|_{L^{p^*(1)}(\Omega)} \right)
$$

$$
\leq C \left( 1 + \|f\|_{L^{q'(1)}(\Omega)} \|\nabla u_{\varepsilon} - \nabla g\|_{L^{p^*(1)(\Omega)}} \right),
$$

where in the last inequality we are using that $W^{1,p^*(1)}(\Omega) \hookrightarrow L^{p^*(1)}(\Omega)$ continuously and Poincaré’s inequality.

Thus we have that there exists a constant independent of $\varepsilon$ such that

$$
\int_\Omega |\nabla u_{\varepsilon}|^{p(x)} \, dx \leq C\left( 1 + \|u_{\varepsilon}\|_{L^{p^*(1)}(\Omega)} \right),
$$
and using the properties of the $L^p(\Omega)$-norms this means that
$$\|\nabla u_\epsilon\|_{L^p(\Omega)}^m \leq C(1 + \|\nabla u_\epsilon\|_{L^p(\Omega)}),$$
for some $m > 1$. Therefore $\|\nabla u_\epsilon\|_{L^p(\Omega)}$ is bounded independent of $\epsilon$. \(\square\)

To prove Theorem 1.1, we will use the results of Section 3. Therefore, we will first need to assume that $p \in C^{1,\beta}(\Omega) \cap C(\overline{\Omega})$.

**Theorem 4.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^2$ boundary, $p \in C^{1,\beta}(\Omega) \cap C^{\alpha}(\overline{\Omega})$ with $p(x) \equiv p_1 > 1$, $g \in H^2(\Omega)$ and $u$ be the weak solution of (1.1), if $f$ satisfies (F1) and (F2) then $u \in H^2(\Omega)$.

**Proof.** Let $f_\epsilon \in \text{Lip}(\Omega)$ and $g_\epsilon \in C^{2,\alpha}(\overline{\Omega})$ such that
$$\begin{align*}
f_\epsilon \to f & \quad \text{strongly in } L^q(\Omega), \\
g_\epsilon \to g & \quad \text{strongly in } H^2(\Omega),
\end{align*}$$
as $\epsilon \to 0$. Observe that, since $f(x) = 0$ if $p(x) > 2$, we can take $f_\epsilon \equiv 0$ in $\{x \in \Omega: \ p(x) > 2\}$.

Now, let us consider the solution of (3.2) as the solution of
$$\begin{align*}
a_{11}^\epsilon(\cdot) \frac{\partial^2 u_\epsilon}{\partial x_1^2} + 2a_{12}^\epsilon(\cdot) \frac{\partial^2 u_\epsilon}{\partial x_1 \partial x_2} + a_{22}^\epsilon(\cdot) \frac{\partial^2 u_\epsilon}{\partial x_2^2} = a_\epsilon(\cdot) & \quad \text{in } \Omega, \\
u_\epsilon = g_\epsilon & \quad \text{on } \partial \Omega,
\end{align*}$$
where $a_{11}^\epsilon, a_{12}^\epsilon, a_{22}^\epsilon, a_\epsilon$ are defined as in Lemma 3.3, substituting $f$ and $g$ by $f_\epsilon$ and $g_\epsilon$ respectively. By Lemma 3.3 we know that $u_\epsilon \in C^{2}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

First we will prove that $\{u_\epsilon\}_{\epsilon \in (0,1]}$ is bounded in $H^2(\Omega)$. By Remark 3.4, we have that
$$\|u_\epsilon\|_{H^2(\Omega)} \leq C(\|a_\epsilon(\cdot)\|_{L^2(\Omega)} + \|g_\epsilon\|_{H^2(\Omega)}) \leq C(\|\ln(v_\epsilon)\|_{L^2(\Omega)} + \|f_\epsilon\|_{L^2(\Omega)} + \|g_\epsilon\|_{H^2(\Omega)}).
$$
Taking $\Omega_1 = \{x \in \Omega: \ |\nabla u_\epsilon(x)| > 1\}$, using that $p(x)$ is Lipschitz and Hölder’s inequality, we have
$$\|\ln(v_\epsilon)\|_{L^2(\Omega)} \leq C(\|\ln^2(v_\epsilon)\|_{L^p(\Omega)}^{1/2} + \|v_\epsilon\|_{L^p(\Omega)}^{1/2} + \|u_\epsilon\|_{L^p(\Omega)} + 1).
$$
On the other hand, since $q(x) \geq q_1 = 0$, we have that $q(\cdot) \leq p^*(\cdot)$. Then, as $\|f_\epsilon\|_{L^p(\Omega)}$ and $\|g_\epsilon\|_{H^2(\Omega)}$ are bounded independent of $\epsilon$, using Lemma 4.1 we conclude that $\|\nabla u_\epsilon\|_{L^p(\Omega)}$ is uniformly bounded.

Observe that, for all $s > 0$ there exists a constant $C > 0$ such that
$$\ln(v_\epsilon) \leq C v_\epsilon^{s/2} \leq C |\nabla u_\epsilon|^{s/2} \quad \text{in } \Omega_1,$$
thus
$$\|\ln^2(v_\epsilon)\|_{L^p(\Omega)} \leq C \|\nabla u_\epsilon\|_{L^p(\Omega)}^{1+s} \leq C \|\nabla u_\epsilon\|_{L^p(\Omega)}^{1+s} \leq C \|u_\epsilon\|_{H^2(\Omega)}^{1+s}.$$
In the last line, we are using that $2^* = \infty$, since $N = 2$.
Then, by the last inequality, (4.8) and (4.9), we get
$$\|u_\epsilon\|_{H^2(\Omega)} \leq C(\|u_\epsilon\|_{H^2(\Omega)}^{1+s} + \|f_\epsilon\|^2 \|v_\epsilon^{-p}\|_{L^2(\Omega)} + 1).
$$
(4.10)
Taking
$$A_1 = \{x \in \Omega: \ p(x) = 2\} \quad \text{and} \quad A_2 = \{x \in \Omega: \ p(x) < 2\}$$
and using that $f_\epsilon \equiv 0$ in $\{x \in \Omega: \ p(x) > 2\}$, we have that
$$\|f_\epsilon\|^2 \|v_\epsilon^{-p}\|_{L^2(\Omega)} \leq \|f_\epsilon\|_{L^2(A_1)}^2 + \|f_\epsilon\|^2 \|v_\epsilon^{-p}\|_{L^2(A_2)}.$$
Since $\|f_\epsilon\|_{L^2(A_1)}$ is bounded, to prove that $\{u_\epsilon\}_{\epsilon \in (0,1]}$ is bounded in $H^2(\Omega)$, we only have to find a bound of $\|f_\epsilon\|^2 \|v_\epsilon^{-p}\|_{L^2(A_2)}$. 

Let us define in $A_2$ the function
\[
\tilde{q}(x) = \begin{cases} 
\frac{1}{2p(x) - 3} + 1 & \text{if } \frac{1}{q(x)} + \frac{2}{3} < p(x) < 2, \\
\frac{q(x)}{2} + 1 & \text{if } p(x) < \frac{1}{q(x)} + \frac{3}{2}.
\end{cases}
\]

It is easy to see that $2 < \tilde{q}(x) \leq q(x)$ for any $x \in A_2$.

On the other hand, let us denote $\mu(x) = \frac{2q(x)}{q(x) - 2}$ and $\gamma(x) = \mu(x)(2 - p(x))$ then
\[
1 < 1 + \frac{2}{q_2} \leq \gamma(x) \leq \max \left\{ 2, 2 + \frac{8}{q_1 - 2} \right\} \quad \forall x \in A_2.
\]

Now, using Hölder's inequality with exponent $\tilde{q}(x)/2$, we have
\[
\left\| f_\epsilon v_\epsilon^{2-p} \right\|_{L^2(A_2)} \leq C \left\| f_\epsilon \right\|_{L^{\tilde{q}()}(A_2)} \left\| v_\epsilon^{2-p} \right\|_{L^{\mu()}(A_2)}.
\]

Then, if $\gamma(x) \leq 1$ we have $\left\| v_\epsilon^{2-p} \right\|_{L^{\mu()}(A_2)} \leq 1$ and since $\tilde{q}(x) \leq q(x)$ we get
\[
\left\| f_\epsilon v_\epsilon^{2-p} \right\|_{L^2(A_2)} \leq C.
\]

If $\gamma(x) > 1$, we have
\[
\left\| v_\epsilon^{2-p} \right\|_{L^{\mu()}(A_2)} \leq \left\| v_\epsilon \right\|_{L^{\gamma()}(A_2)}^{2-p} \leq C(1 + \left\| \nabla u_\epsilon \right\|_{L^{\gamma()}(A_2)}^{2-p}),
\]

where in the last inequality we are using that $\epsilon \leq 1$.

Since $2^* = \infty$ and $1 < \gamma(x) \leq \gamma(y) \leq 2 < \infty$, by the Sobolev embedding inequality, we have that
\[
\left\| \nabla u_\epsilon \right\|_{L^{2^*}(A_2)}^{2-p_1} \leq C \left\| u_\epsilon \right\|_{H^{2^*(2)}(A_2)}^{2-p_1} \leq C \left\| u_\epsilon \right\|_{H^2(\Omega)}^{2-p_1}.
\]

Combining this last inequality with inequalities (4.12), (4.11), (4.10) and the fact that $\tilde{q}(x) \leq q(x)$, we get
\[
\left\| u_\epsilon \right\|_{H^2(\Omega)} \leq C \left( \left\| u_\epsilon \right\|_{H^{2^*(2)}(A_2)}^{(1+s)/2} + \left\| u_\epsilon \right\|_{H^2(\Omega)}^{2-p_1} \right) + 1.
\]

Finally, we get that for any $0 < s < 1$ there exists a constant $C = C(p, q, f, s)$ such that
\[
\left\| u_\epsilon \right\|_{H^2(\Omega)} \leq C.
\]

Then, there exists a subsequence still denoted by $\{u_\epsilon\}_{\epsilon \in (0, 1)}$ and $u \in H^1(\Omega)$ such that
\[
u_\epsilon \rightarrow u \quad \text{strongly in } H^1(\Omega),
\]
\[
u_\epsilon \rightharpoonup u \quad \text{weakly in } H^2(\Omega).
\]

It is clear that $u$ satisfies the boundary condition.

Lastly, by Proposition 3.2 in [2], there exists a constant $M > 0$ independent of $\epsilon$ such that
\[
\left| \left( \epsilon + |\nabla u_\epsilon|^2 \right)^{\frac{p(x)-2}{2}} \nabla u_\epsilon - \left( \epsilon + |\nabla u|^2 \right)^{\frac{p(x)-2}{2}} \nabla u \right| \leq M \left| \nabla (u_\epsilon - u) \right|^{p(x)-1}
\]

for all $x \in \Omega$. Then, passing to the limit in the weak formulation of (3.2) and using the above inequality, we have that
\[
\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \ dx = \int_\Omega f \varphi \ dx
\]

for any $\varphi \in C_0^\infty(\Omega)$. Therefore $u \in H^2(\Omega)$ and solves (1.1).

Now, we are able to prove the theorem.

**Proof of Theorem 1.1.** First, we consider the case $p \in C^1(\Omega)$. Let $p_\epsilon \in C_0^\infty(\Omega)$ such that $p_\epsilon \rightarrow p$ in $C^1(\Omega)$. Now, we define
\[
f_\epsilon(x) = \begin{cases} 
f(x) & \text{if } p_\epsilon(x) \leq 2, \\
0 & \text{if } p_\epsilon(x) > 2.
\end{cases}
\]

Observe that $f_\epsilon \rightarrow f$ in $L^{q(\Omega)}(\Omega)$ as $\epsilon \rightarrow 0$. 

Then, by Theorem 4.2, the solution \( u_\epsilon \) of (1.1) (with \( p_\epsilon \) and \( f_\epsilon \) instead of \( p \) and \( f \)) is bounded in \( H^2(\Omega) \) by a constant independent of \( \epsilon \). Therefore, there exists a subsequence still denoted \( \{u_\epsilon\}_{\epsilon \in (0,1]} \) and \( u \in H^2(\Omega) \) such that
\[
\begin{align*}
u_\epsilon &\rightharpoonup u \quad \text{in} \ H^1(\Omega), \\
u_\epsilon &\rightharpoonup u \quad \text{weakly in} \ H^2(\Omega).
\end{align*}
\] (4.15)

It remains to prove that \( u \) is a solution of (1.1). Let \( \varphi \in C_0^\infty(\Omega) \), then
\[
\int_\Omega f_\epsilon \varphi \mathrm{d}x = \int_\Omega |\nabla u_\epsilon|^{p_\epsilon(x)-2} \nabla u_\epsilon \nabla \varphi \mathrm{d}x \\
= \int_\Omega |\nabla u_\epsilon|^{p(x)-2} \nabla u_\epsilon \nabla \varphi \mathrm{d}x + \int_\Omega (|\nabla u_\epsilon|^{p_\epsilon(x)-2} - |\nabla u_\epsilon|^{p(x)-2}) \nabla u_\epsilon \nabla \varphi \mathrm{d}x. \tag{4.16}
\]

Therefore, using that \( H^2(\Omega) \rightharpoonup W^{1,p(\cdot)}(\Omega) \) compactly, we have that
\[
\int_\Omega |\nabla u_\epsilon|^{p(x)-2} \nabla u_\epsilon \nabla \varphi \mathrm{d}x \rightarrow \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \mathrm{d}x. \tag{4.17}
\]

On the other hand, we have
\[
|\nabla u_\epsilon(x)|^{p_\epsilon(x)-1} - |\nabla u_\epsilon(x)|^{p(x)-1} = |\nabla u_\epsilon(x)|^{b_\epsilon(x)} \log(|\nabla u_\epsilon(x)|) (p_\epsilon(x) - p(x)),
\]
where \( b_\epsilon(x) = p_\epsilon(x)\theta + (1 - \theta)p(x) - 1 \) for some \( 0 < \theta < 1 \). Therefore, using that \( 2^* = \infty \) and that \( p_\epsilon \rightarrow p \) uniformly, we obtain
\[
\int_\Omega (|\nabla u_\epsilon|^{p_\epsilon(x)-2} - |\nabla u_\epsilon|^{p(x)-2}) \nabla u_\epsilon \nabla \varphi \mathrm{d}x \rightarrow 0. \tag{4.18}
\]

Then, using that \( f_\epsilon \rightarrow f \) in \( L^0(\Omega) \), (4.16), (4.17) and (4.18), we conclude that \( u \) is a solution of (1.1).

Now, we consider the case \( p \in \text{Lip}(\Omega) \). By Lemmas B.1 and B.2 there exists \( p_\epsilon \in C^1(\Omega) \) such that \( |\Omega \setminus \Omega_0| < \epsilon \) where
\[
\Omega_0 = \{x \in \Omega: p_\epsilon(x) = p(x) \text{ and } \nabla p_\epsilon(x) = \nabla p(x)\}.
\]

We define \( f_\epsilon \) as in (4.14). Then, the solution \( u_\epsilon \) of (1.1) with \( p_\epsilon \) and \( f_\epsilon \) instead of \( p \) and \( f \) is bounded in \( H^2(\Omega) \) by a constant independent of \( \epsilon \). Therefore there exists a subsequence still denoted \( \{u_\epsilon\}_{\epsilon \in (0,1]} \) and \( u \in H^2(\Omega) \) satisfying (4.15).

Lastly, we prove that \( u \) is a solution of (1.1). Let \( \varphi \in C_0^\infty(\Omega) \). By Hölder’s inequality, since \( 2^* = \infty \) and by (3) of Lemma B.2 we have
\[
\int_{\Omega \setminus \Omega_0} (|\nabla u_\epsilon|^{p_\epsilon(x)-2} - |\nabla u_\epsilon|^{p(x)-2}) \nabla u_\epsilon \nabla \varphi \mathrm{d}x \\
\leq C \left( \|\nabla u_\epsilon\|_{L^p(\Omega)} \|1\|_{L^p(\Omega)} + \|\nabla u_\epsilon\|_{L^p(\Omega)} \|1\|_{L^p(\Omega)} \right) \\
\leq C \|\nabla u_\epsilon\|_{H^2(\Omega)} \left( \|1\|_{L^p(\Omega)} \|1\|_{L^p(\Omega)} \right).
\]

Then, since \( \|u_\epsilon\|_{H^2(\Omega)} \) is bounded independent of \( \epsilon \) and \( |\Omega \setminus \Omega_0| < \epsilon \) we obtain that
\[
\int_{\Omega \setminus \Omega_0} (|\nabla u_\epsilon|^{p_\epsilon(x)-2} - |\nabla u_\epsilon|^{p(x)-2}) \nabla u_\epsilon \nabla \varphi \mathrm{d}x \rightarrow 0.
\]

Therefore, since (4.16), (4.17) again hold, using that \( f_\epsilon \rightarrow f \) in \( L^0(\Omega) \), and the above equation, we conclude that \( u \) is a solution of (1.1). \( \square \)

5. The convex case

Lastly, we want to prove that the solution is in \( H^2(\Omega) \) if we only assume that \( \partial \Omega \) is convex. We want to remark here that this result generalizes the one in Theorem 2.2 in [22] in two ways. In that paper the authors consider the case \( p = \text{constant} \) and \( g = 0 \). Instead, we are allowed to cover the case where \( g \) is any function in \( H^2(\Omega) \) and \( p(x) \in \text{Lip}(\Omega) \).

Remark 5.1. Let \( \Omega \) be a convex set and \( p : \Omega \rightarrow [1, \infty) \) be log-continuous in \( \Omega \). Then, there exists a sequence \( \{\Omega_m\}_{m \in \mathbb{N}} \) of convex subset of \( \Omega \) with \( C^2 \) boundary such that \( \Omega_m \subset \Omega_{m+1} \) for any \( m \in \mathbb{N} \) and \( |\Omega \setminus \Omega_m| \rightarrow 0 \).
(1) Then, there exists a constant \( C \) depending on \( p(x) \), \(|\Omega|\) such that
\[
\|v\|_{L^p(\Omega_m)} \leq C\|\nabla v\|_{L^p(\Omega_m)} \quad \forall v \in W^{1,p}(\Omega_m),
\]
for any \( m \in \mathbb{N} \). This follows by Theorem 3.3 in [21], using that \( \Omega_m \subset \Omega_{m+1} \) for any \( m \in \mathbb{N} \).

(2) The Lipschitz constants of \( \Omega_m \) (\( m \in \mathbb{N} \)) are uniformly bounded (see Remark 2.3 in [22]). Therefore, the extension operators
\[
E_{1,m} : W^{1,p}(\Omega_m) \to W^{1,p}(\Omega) \quad \text{and} \quad E_{2,m} : H^2(\Omega_m) \to H^2(\Omega)
\]
define as Theorem 4.2 in [11] satisfy that \( \|E_{1,m}\| \) and \( \|E_{2,m}\| \) are uniformly bounded.

(3) By (2) and Corollary 8.3.2 in [12], there exists a constant \( C \) independent of \( m \) such that
\[
\|v\|_{L^{p^*}(\Omega_m)} \leq C\|v\|_{W^{1,p}(\Omega_m)} \quad \forall v \in W^{1,p}(\Omega_m),
\]
for any \( m \in \mathbb{N} \).

We want to remark that all the constants of the above inequalities are independent of \( p_1 \) (see Section 6 for the applications).

**Proof of Theorem 1.2.** We begin taking \( \{\Omega_m\}_{m \in \mathbb{N}} \) as in Remark 5.1 and \( u_m \) the solution of
\[
\begin{cases}
-\Delta_p u_m = f & \text{in } \Omega_m, \\
u_m = g & \text{on } \partial \Omega_m.
\end{cases}
\]

By Theorem 1.1, \( u_m \in H^2(\Omega_m) \) for any \( m \in \mathbb{N} \). Moreover, \( u_m \) solves
\[
\begin{cases}
mu_m = \alpha_{ij}^m(x)u_{m,i}x_j = \alpha^m(x) & \text{in } \Omega_m, \\
u_m = g & \text{on } \partial \Omega_m,
\end{cases}
\]
with
\[
\alpha_{ij}^m(x) = \delta_{ij} + \left( p(x) - 2 \right) \frac{u_{m,i}x_j(x)}{|\nabla u_m(x)|^2},
\]
\[
\alpha^m(x) = \ln \left( |\nabla u_m(x)| \right) \left( |\nabla u_m(x), \nabla p(x)| + f(x) |\nabla u_m(x)| \right)^{2-p(x)}.
\]

Then \( v_m = u_m - g \) solves
\[
\begin{cases}
L^m v_m = -L^m g + \alpha^m(x) & \text{in } \Omega_m, \\
v_m = 0 & \text{on } \partial \Omega_m.
\end{cases}
\]

Thus, using that \( v_m \in H^2(\Omega_m) \cap H^1_0(\Omega_m) \) and since the coefficients \( \alpha_{ij}^m(x) \) are bounded independent of \( m \), we can argue as in Theorem 2.2 in [22] and obtain
\[
\|v_m\|_{H^2(\Omega_m)} \leq C\|L^m g - f |\nabla u_m|^{2-p(x)} + \ln(|\nabla u_m|) |\nabla u_m| \|_{L^2(\Omega_m)}
\leq C \left( \|\nabla u_m\|_{L^{2-p}(\Omega_m)} + \ln(|\nabla u_m|) \|\nabla u_m\|_{L^2(\Omega_m)} + 1 \right) \tag{5.19}
\]
where the constant \( C \) is independent of \( m \).

As in Lemma 4.1 we can prove, using Remark 5.1(1) and (3), that the norms \( \|\nabla u_m\|_{L^{p^*}(\Omega_m)} \) are uniformly bounded. Therefore, proceeding as in Theorem 4.2, we obtain
\[
\|\ln(|\nabla u_m|) |\nabla u_m| \|_{L^2(\Omega_m)} + \|f |\nabla u_m|^{2-p} \|_{L^2(\Omega_m)} \leq C \left( \|\nabla u_m\|_{L^{p(x)+1}^2(\Omega_{2,m})}^{(1+1)/(1+1)} + \|\nabla u_m\|_{L^{p^*}(\Omega_{2,m})}^{2-p} \right) + 1), \tag{5.20}
\]
with \( C \) independent of \( m \), where
\[
\Omega_{1,m} = \{x \in \Omega_m: |\nabla u_m(x)| > 1\} \quad \text{and} \quad A_{2,m} = \{x \in \Omega_m: p(x) < 2\}.
\]

Now, using Remark 5.1(3) and (2), we have that for any \( r > 1 \)
\[
\|v_m\|_{W^{1,r}(\Omega_m)} \leq \|E_{2,m}v_m\|_{W^{1,1}(\Omega)} \leq C \|E_{2,m}v_m\|_{H^2(\Omega)} \leq C \|v_m\|_{H^2(\Omega_m)} \tag{5.21}
\]
where \( C \) is independent of \( m \).
Therefore, using (5.19), (5.20) and (5.21), we get
\[
\|v_m\|_{H^2(\Omega_m)} \leq C \left( \|v_m\|_{H^2(\Omega_m)}^{(1+s)/2} + \|v_m\|_{H^2(\Omega_m)}^{-p_1} + \|g\|_{H^2(\Omega_m)}^{(1+s)/2} + \|g\|_{H^2(\Omega_m)}^{-p_1} + 1 \right)
\leq C \left( \|v_m\|_{H^2(\Omega_m)}^{(1+s)/2} + \|v_m\|_{H^2(\Omega_m)}^{-p_1} + 1 \right),
\]
where the constant $C$ is independent of $m$. This proves that $\{\|v_m\|_{H^2(\Omega_m)}\}_{m \in N}$ is bounded.

Now we have, as in the proof of Theorem 2.2 in [22], that there exist a subsequence still denote $\{v_m\}_{m \in N}$ and a function $v \in H^2(\Omega) \cap H^1_0(\Omega)$ such that
\[
v_m \to v \quad \text{strongly in } H^1(\Omega')
\]
for any $\Omega' \subset \subset \Omega$. Then $u = v + g \in H^2(\Omega)$ and
\[
u_m \to u \quad \text{strongly in } H^1(\Omega')
\]
for any $\Omega' \subset \subset \Omega$. Thus, using (4.13), we have
\[
|\nabla u_m|^{p(x)-2} \nabla u_m \to |\nabla u|^{p(x)-2} \nabla u \quad \text{strongly in } L^{p(\cdot)}(\Omega')
\]
for any $\Omega' \subset \subset \Omega$.

On the other hand, for any $\varphi \in C^\infty_0(\Omega)$ there exists $m_0$ such that for all $m \geq m_0$
\[
\int_{\Omega_m} |\nabla u_m|^{p(x)-2} \nabla u_m \nabla \varphi \, dx = \int_{\Omega_m} f \varphi \, dx.
\]
Therefore, using (5.22) we have that $u$ is a weak solution of (1.1). □

**Proof of Corollary 1.3.** By the previous theorem we have that $u \in H^2(\Omega)$, then we can derive Eq. (1.1) and obtain
\[
\begin{align*}
-\alpha_{ij}(x) u_{ij} &= \alpha(x) \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega,
\end{align*}
\]
where
\[
\alpha_{ij}(x) = \delta_{ij} + (p(x) - 2) \frac{u_i(x)u_j(x)}{|\nabla u(x)|^2},
\]
\[
\alpha(x) = \ln\left(|\nabla u(x)|\right) |\nabla u(x), \nabla p(x)| + f(x) |\nabla u(x)|^{2-p(x)}.
\]

Using that $f \in L^q(\Omega)$ with $q > q_1 > 2$ and following the lines in the proof of Theorem 4.2, we have that $\alpha(x) \in L^s(\Omega)$ with $s > 2$. Therefore, by Remark A.3, we have that $u \in C^{1,\alpha}(\Omega)$. □

**6. Comments**

In the image processing problem it is of interest the case where $p_1$ is close to 1. By this reason, we are also interested in the dependence of the $H^2$-norm on $p_1$.

If $N = 2$, $g \in H^2(\Omega)$ and $u_\varepsilon$ is the solution of (3.2), we have by Lemma A.1, (3.6) and (3.7), that there exists a constant $C$ independent of $p_1$ and $\varepsilon$ such that
\[
\|u_\varepsilon\|_{H^2(\Omega)} \leq \frac{C}{(p_1 - 1)^s} \left( \|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)} \right),
\]
where $\kappa = 1$ if $\Omega$ is convex and $\kappa = 2$ if $\partial \Omega \in C^2$. Therefore, using that the Poincaré inequality and the embedding $W^{1, p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ hold in the case $p_1 = 1$ and following the lines of Theorem 1.1 and Theorem 1.2 we have that
\[
\|u\|_{H^2(\Omega)} \leq \frac{C}{(p_1 - 1)^\kappa},
\]
where the constant $C$ is independent of $p_1$. 
Appendix A. Regularity results for elliptic linear equations with coefficients in $L^\infty$

Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ and

$$M u = a_{ij}(x) u_{x_i x_j},$$

such that $a_{ij} = a_{ji}$ and for any $\xi \in \mathbb{R}^N$

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2,$$

(A.1)

and

$$M_1 \leq a_{11}(x) + a_{22}(x) \leq M_2 \quad \text{in} \ \Omega$$

(A.2)

where $\lambda$, $\Lambda$, $M_1$ and $M_2$ are positive constant.

In the next lemma, we will give an $H^2$-bound for solutions of

$$\begin{cases}
M u = f & \text{in} \ \Omega, \\
u = g & \text{on} \ \partial \Omega,
\end{cases}$$

(A.3)

In fact, the following result is proved in Theorem 37, III in [23], but the dependence of the bounds on the ellipticity and the $L^\infty$-norm of $(a_{ij}(x))$ are not explicit. Then, following the proof of the mentioned theorem we can prove

Lemma A.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$. Then, if $u$ is a solution of (A.3) and $u \in H^2(\Omega)$ we have that

$$\|u\|_{H^2(\Omega)} \leq C \kappa \left( \|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)} \right),$$

where $\kappa = 1$ if $\Omega$ is convex and $\kappa = 2$ if $\partial \Omega \in C^2$ and $C$ is a constant independent of $\lambda$.

Proof. In this proof, we denote $u_{ij} = u_{x_i x_j}$ for all $i, j = 1, 2$ and $C$ is a constant independent of $\lambda$.

First, we consider the case $g \equiv 0$. Using (A.1), we have that

$$(a_{11}(x) + a_{22}(x)) (u_{12}^2 - u_{11} u_{22}) = \sum_{i,j,k=1}^2 a_{ij} u_{ik} u_{kj} - \Delta u \sum_{i,j=1}^2 a_{ij} u_{ij} \geq \lambda \sum_{i,k=1}^2 u_{ki}^2 - \Delta u f(x).$$

Then, using Young’s inequality, we get

$$\frac{\lambda}{2(a_{11}(x) + a_{22}(x))} \sum_{i,k=1}^2 u_{ki}^2 \leq \frac{4}{\lambda(a_{11}(x) + a_{22}(x))} f(x)^2 + u_{12}^2 - u_{11} u_{22},$$

and by (A.2), we have that

$$\sum_{i,k=1}^2 u_{ki}^2 \leq \frac{C}{\lambda^2} f(x)^2 + \frac{C}{\lambda} (u_{12}^2 - u_{11} u_{22}).$$

(A.4)

Now, using (37.4) and (37.6) in [23], we have that for any $u \in H^2(\Omega)$

$$\int_{\Omega} (u_{12}^2 - u_{11} u_{22}) \, dx = - \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 \frac{H}{2} \, ds$$

(A.5)

where $H$ is the curvature of $\partial \Omega$. If $\Omega$ is convex, then $H \geq 0$ and therefore, using (A.4) and (A.5), we have that

$$\|D^2 u\|_{L^2(\Omega)} \leq C \frac{\lambda}{\lambda} \|f\|_{L^2(\Omega)}.$$  

(A.6)

In the general case, we can use the following inequality

$$\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 \, ds \leq C \left( 1 + \delta^{-1} \right) \int_{\Omega} |\nabla u|^2 \, dx + \delta \int_{\Omega} \sum_{i,k=1}^2 u_{ki}^2 \, dx$$

(A.7)

for any $\delta > 0$. See Eq. (37.6) of [23].
Then, by (A.4), (A.5), using that \( H \) is bounded and (A.7) (choosing \( \delta \) properly) we arrive at

\[
\int \sum_{k=1}^{2} u_{ik}^{2} \, dx \leq \frac{C}{\lambda^{2}} \left( \int \| f(x)^{2} \, dx + \int \| \nabla u \|^{2} \, dx \right). \tag{A.8}
\]

On the other hand, using that \( Lu = f \) in \( \Omega \), (A.1) and the Poincaré inequality, we have

\[
\| \nabla u \|_{L^{2}(\Omega)} \leq \frac{C}{\lambda} \| f \|_{L^{2}(\Omega)}. \tag{A.9}
\]

Therefore, by (A.8) and (A.9), we get

\[
\| D^{2}u \|_{L^{2}(\Omega)} \leq \frac{C}{\lambda^{2}} \| f \|_{L^{2}(\Omega)}.
\]

Thus, by the last inequality, (A.9) and (A.6) the lemma is proved in the case \( g = 0 \).

When \( g \) is any function in \( H^{2}(\Omega) \) the lemma follows taking \( v = u - g \). \( \Box \)

The following theorem is proved in Corollary 8.1.6 in [20].

**Theorem A.2.** Let \( \Omega \) be a convex polygonal domain in \( \mathbb{R}^{2} \), \( \mathcal{M} \) satisfying (A.1) and \( u \in H^{2}(\Omega) \cap H^{1}_{0}(\Omega) \) be a solution of (A.3) with \( g = 0 \) and \( f \in L^{p}(\Omega) \) with \( p > 2 \). Then \( \nabla u \in C^{\mu}(\partial \Omega) \) for some \( 0 < \mu < 1 \).

**Remark A.3.** Observe that the above theorem holds also if we consider any \( g \in W^{2,p}(\Omega) \), since we can take \( v = u - g \) in (A.3) and use that \( W^{2,p}(\Omega) \hookrightarrow C^{1,1-2/p}(\partial \Omega) \).

**Appendix B. Lipschitz functions**

Using the linear extension operator defined in [14], we have the following lemma.

**Lemma B.1.** Let \( \Omega \) be a bounded open domain with Lipschitz boundary and \( f \in \text{Lip}(\partial \Omega) \). Then, there exists a function \( \overline{f} : \mathbb{R}^{N} \rightarrow \mathbb{R} \) such that \( \overline{f} \) is a Lipschitz function, \( \sup_{\mathbb{R}^{N}} \overline{f} = \inf_{\mathbb{R}^{N}} \overline{f} = \max_{\mathbb{R}^{N}} f \).

**Lemma B.2.** Let \( f : \mathbb{R}^{N} \rightarrow \mathbb{R} \) be a Lipschitz function. Then for each \( \varepsilon > 0 \), there exists a \( C^{1} \) function \( f_{\varepsilon} : \mathbb{R}^{N} \rightarrow \mathbb{R} \) such that

1. \( |x \in \mathbb{R}^{N} : f_{\varepsilon}(x) \neq f(x) \text{ or } Df_{\varepsilon}(x) \neq Df(x)| \leq \varepsilon. \)
2. There exists a constant \( C \) depending only on \( N \) such that

\[
\| Df_{\varepsilon} \|_{L^{\infty}(\mathbb{R}^{N})} \leq C \text{Lip}(f).
\]
3. If \( 1 < f_{1} \leq f(x) \leq f_{2} \) in \( \mathbb{R}^{N} \), we have

\[
1 < f_{\varepsilon}(x) \leq f_{2} + CE^{\frac{1}{N}} \quad \text{in } \mathbb{R}^{N}
\]

with \( C \) a constant depending only on \( N \).

**Proof.** Items (1) and (2) follow by Theorem 1, p. 251 in [16].

To prove (3), let us define

\[
\Omega_{0} = \{ x \in \mathbb{R}^{N} : f_{\varepsilon}(x) = f(x) \text{ and } Df_{\varepsilon}(x) = Df(x) \}
\]

and let us suppose that there exists \( x \in \mathbb{R}^{N} \setminus \Omega_{0} \) such that \( f_{\varepsilon}(x) = f_{2} + \delta \) with \( \delta > 0 \). If \( x_{0} \in \Omega_{0} \), by (2), we have

\[
C \text{Lip}(f)|x - x_{0}| \geq f_{\varepsilon}(x) - f_{\varepsilon}(x_{0}) = f_{2} + \delta - f(x_{0}) \geq \delta.
\]

Then \( B_{\rho}(x) \subset \mathbb{R}^{N} \setminus \Omega_{0} \) where \( \rho = \delta(C \text{Lip}(f))^{-1} \) and using (1) we get \( \delta \leq C\varepsilon^{1/N} \), for some constant \( C \) independent of \( \varepsilon \). Analogously we can prove the other inequality. \( \Box \)
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