An even better Density Increment Theorem and its application to Hadwiger’s Conjecture

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Dedicated to the memory of Robin Thomas

Abstract

In 1943, Hadwiger conjectured that every graph with no $K_t$ minor is $(t-1)$-colorable for every $t \geq 1$. In the 1980s, Kostochka and Thomason independently proved that every graph with no $K_t$ minor has average degree $O(t\sqrt{\log t})$ and hence is $O(t\sqrt{\log t})$-colorable. Recently, Norin, Song and the author showed that every graph with no $K_t$ minor is $O(t(\log t)^{\beta})$-colorable for every $\beta > 1/4$, making the first improvement on the order of magnitude of the $O(t\sqrt{\log t})$ bound. More recently, the author showed that every graph with no $K_t$ minor is $O(t(\log t)^{\beta})$-colorable for every $\beta > 0$; more specifically, they are $t \cdot 2^{O((\log \log t)^{2/3})}$-colorable. Building on that work, we show in this paper that every graph with no $K_t$ minor is $O(t(\log \log t)^{24})$-colorable.

1 Introduction

All graphs in this paper are finite and simple. Given graphs $H$ and $G$, we say that $G$ has an $H$ minor if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. We denote the complete graph on $t$ vertices by $K_t$.

In 1943 Hadwiger made the following famous conjecture.

Conjecture 1.1 (Hadwiger’s conjecture \[Had43\]). For every integer $t \geq 1$, every graph with no $K_t$ minor is $(t - 1)$-colorable.

Hadwiger’s conjecture is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. For an overview of major progress on Hadwiger’s conjecture, we refer the reader to \[NPS19\], and to the recent survey by Seymour \[Sey16\] for further background.

The following is a natural weakening of Hadwiger’s conjecture, which has been considered by several researchers.

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Conjecture 1.2 (Linear Hadwiger’s conjecture \[RS98\], \[Kaw07\], \[KM07\]). There exists a constant \(C > 0\) such that for every integer \(t \geq 1\), every graph with no \(K_t\) minor is \(Ct\)-colorable.

For many decades, the best general bound on the number of colors needed to properly color every graph with no \(K_t\) minor had been \(O(t\sqrt{\log t})\), a result obtained independently by Kostochka \[Kos82\], \[Kos84\] and Thomason \[Tho84\] in the 1980s. The results of \[Kos82\], \[Kos84\], \[Tho84\] bound the “degeneracy” of graphs with no \(K_t\) minor. Recall that a graph \(G\) is \(d\)-degenerate if every non-empty subgraph of \(G\) contains a vertex of degree at most \(d\). A standard inductive argument shows that every \(d\)-degenerate graph is \((d+1)\)-colorable. Thus the following bound on the degeneracy of graphs with no \(K_t\) minor gives a corresponding bound on their chromatic number and even their list chromatic number.

Theorem 1.3 (\[Kos82\], \[Kos84\], \[Tho84\]). Every graph with no \(K_t\) minor is \(O(t\sqrt{\log t})\)-degenerate.

Very recently, Norin, Song and the author \[NPS19\] improved this with the following theorem.

Theorem 1.4 (\[NPS19\]). For every \(\beta > \frac{1}{4}\), every graph with no \(K_t\) minor is \(O(t(\log t)^\beta)\)-colorable.

In \[NS19b\], Norin and Song extended Theorem 1.4 to odd minors. In \[NP20\], Norin and the author extended Theorem 1.4 to list coloring. Even more recently, the author in \[Pos20a\] further improved the bound in Theorem 1.4 as follows.

Theorem 1.5. Every graph with no \(K_t\) minor is \(t \cdot 2^{O((\log \log t)^{2/3})}\)-colorable. Hence for every \(\beta > 0\), every graph with no \(K_t\) minor is \(O(t(\log t)^\beta)\)-colorable.

The main result of this paper is the following.

Theorem 1.6. Every graph with no \(K_t\) minor is \(O(t(\log \log t)^{24})\)-colorable.

1.1 A better density increment theorem

The key to the improvement is a nearly optimal density increment theorem as follows.

Theorem 1.7. There exists a constant \(C = C_1 > 0\) such that the following holds. Let \(G\) be a graph with \(d(G) \geq C\), and let \(D > 0\) be a constant. Let \(s = D/d(G)\) and let \(q_{\ref{thm:1.7}}(s) := C(1 + \log s)^{24}\). Then \(G\) contains at least one of the following:

(i) a minor \(J\) with \(d(J) \geq D\), or

(ii) a subgraph \(H\) with \(v(H) \leq q_{\ref{thm:1.7}}(s) \cdot \frac{D^2}{d(G)}\) and \(d(H) \geq \frac{d(G)}{q_{\ref{thm:1.7}}(s)}\).

In \[NS19a\], Norin and Song had proved Theorem 1.7 with \(g(s) = s^\alpha\) for any \(\alpha > \frac{1}{\log(3/2)}\) and \(1 \approx 0.7095\). Using that result, they showed that every graph with no \(K_t\) minor is \(O(t(\log t)^{0.354})\)-colorable. Shortly thereafter, in \[Pos19\], the author improved this to \(g(s) = s^{o(1)}\). That result was then combined in \[NPS19\] with the earlier work to yield Theorem 1.4. The function \(g(s)\) in \[NPS19\] was not explicitly found. It is not hard to derive an
explicit function of $g(s) = 2^{O((\log s)^{2/3}+1)}$ from Lemma 2.5 in [NPS19]. A slight modification to the proof was presented in [Pos20b], yielding $g(s) = 2^{O(\sqrt{\log s}+1)}$. The main result of this paper is the new bound listed above.

This implies the following improvement to Theorem 2.2 in [Pos20b] where the function was $f(t) := 2^{C\sqrt{\log \log t}}$.

**Theorem 1.8.** There exists an integer $C = C_{1.8} > 0$ such that the following holds: Let $t \geq 3$ be an integer and define $f_{1.8}(t) := C(\log \log t)^{24}$. For every integer $k \geq t$, if $G$ is a graph with $d(G) \geq k \cdot f_{1.8}(t)$ and $G$ contains no $K_t$ minor, then $G$ contains a $k$-connected subgraph $H$ with $\nu(H) \leq t \cdot f_{1.8}(t) \cdot \log t$.

We prove Theorem 1.8 in Section 2.

### 1.2 Proof of Theorem 1.6

We recall the proof outline of Theorem 1.5, which split according to two main cases as determined by the following key definition.

**Definition 1.9.** Let $s$ be a nonnegative integer. We say that a graph $G$ is $s$-chromatic-separable if there exist two vertex-disjoint subgraphs $H_1, H_2$ of $G$ such that $\chi(H_i) \geq \chi(G) - s$ for each $i \in \{1, 2\}$ and that $G$ is $s$-chromatic-inseparable otherwise.

To prove Theorem 1.6, we need the following two main lemmas used for the proof of Theorem 1.5. Their proofs can be found in [Pos20a].

**Lemma 1.10** (Lemma 2.3 in [Pos20a]). There exists an integer $C = C_{1.10} > 0$ such that the following holds: Let $t \geq 3$ be an integer. If $G$ is a $Ct \log \log t$-chromatic-inseparable graph with $\chi(G) \geq Ct \cdot f_{1.8}(t) + \log \log t$, then $G$ contains a $K_t$ minor.

**Lemma 1.11** (Lemma 2.4 in [Pos20a]). There exists an integer $C = C_{1.11} > 0$ such that the following holds: Let $t \geq 3$ be an integer and let $m$ be a constant such that $m \geq Ct$. If $G$ is a graph with $\chi(G) \geq Cm \log \log t$ and every subgraph $H$ of $G$ with $\chi(H) \geq \chi(G)/2$ is $m$-chromatic-separable, then $G$ contains a $K_t$ minor.

We are now ready to prove our main result Theorem 1.6 assuming Lemmas 1.10 and 1.11.

**Proof of Theorem 1.6.** We prove the contrapositive. Let $C_{1.6} = C_{1.8} \cdot C_{1.10} \cdot C_{1.11}^2$. Let $G$ be a graph with $\chi(G) \geq C_{1.6}(\log \log t)^{24}$. Let $m := \max\{C_{1.10} C_{1.11} \cdot t \log \log t\}$. Let $k_1 := C_{1.10} \cdot t \cdot f_{1.8}(t) + \log \log t$ and let $k_2 := C_{1.11} \cdot m \cdot \log \log t$. Note that by choice of $C_{1.6}$, we have that $\chi(G) \geq \max\{2k_1, k_2\}$.

By Lemma 1.11 as $\chi(G) \geq k_2$, we have that $G$ contains a $K_t$ minor as desired or that $G$ contains an $m$-chromatic-inseparable subgraph $H$ with $\chi(H) \geq \chi(G)/2 \geq k_1$. We may assume the latter case or we are done. But then by Lemma 1.10 as $\chi(H) \geq k_1$, we have that $H$ contains a $K_t$ minor and hence that $G$ contains a $K_t$ minor as desired.
1.3 Outline of Paper

In Section 2, we prove Theorem 1.8. In Section 3, we introduce our more technical main theorem, Theorem 3.1, and derive Theorem 1.7 from it. In Section 4, we outline the proof of Theorem 3.1 while reviewing some preliminary definitions; namely, Theorem 3.1 follows from two other main theorems, Theorems 4.7 and 4.8. In Section 5, we prove Theorem 4.7 which shows that a very unbalanced bipartite graph of high minimum degree has either a small, dense subgraph or an \((\ell + 1)\)-bounded minor with density almost \(\ell\) times the original. In Section 6, we prove Theorem 4.8 which shows that a graph of high density has either a small, dense subgraph, or a very unbalanced bipartite graph of high density, or a \(k\)-bounded minor with density almost \(k\). As mentioned above, we combine in Section 4 these results to prove Theorem 3.1 by choosing \(k\) and \(\ell = k^{1/3}\). Finally in Section 7, we discuss impediments to improving the bound in Theorem 1.6.

1.4 Notation

We use largely standard graph-theoretical notation. We denote by \(v(G)\) and \(e(G)\) the number of vertices and edges of a graph \(G\), respectively, and denote by \(d(G) = e(G)/v(G)\) the density of a non-empty graph \(G\). We use \(\chi(G)\) to denote the chromatic number of \(G\), and \(\kappa(G)\) to denote the (vertex) connectivity of \(G\). The degree of a vertex \(v\) in a graph \(G\) is denoted by \(\deg_G(v)\) or simply by \(\deg(v)\) if there is no danger of confusion. We denote by \(G[X]\) the subgraph of \(G\) induced by a set \(X \subseteq V(G)\). If \(A\) and \(B\) are disjoint subsets of \(V(G)\), then we let \(G(A, B)\) denote the bipartite subgraph with \(V(G(A, B)) = A \cup B\) and \(E(G(A, B)) = \{uv \in E(G) : u \in A, v \in B\}\).

2 Proof of Theorem 1.8

In this section, we prove Theorem 1.8. We need an explicit form of Theorem 1.3 as follows.

Theorem 2.1 ([Kos82]). Let \(t \geq 2\) be an integer. Then every graph \(G\) with \(d(G) \geq 3.2t \sqrt{\log t}\) has a \(K_t\) minor.

We also require the following classical result of Mader [Mad72] which ensures that every dense graph contains a highly-connected subgraph.

Lemma 2.2 ([Mad72]). Every graph \(G\) contains a subgraph \(G'\) such that \(\kappa(G') \geq d(G)/2\).

We are now ready to prove Theorem 1.8 which we restate for convenience.

Theorem 1.8. There exists an integer \(C = C_{1.8} > 0\) such that the following holds: Let \(t \geq 3\) be an integer and define \(f_{1.8}(t) := C(\log \log t)^{24}\). For every integer \(k \geq t\), if \(G\) is a graph with \(d(G) \geq k \cdot f_{1.8}(t)\) and \(G\) contains no \(K_t\) minor, then \(G\) contains a \(k\)-connected subgraph \(H\) with \(v(H) \leq t \cdot f_{1.8}(t) \cdot \log t\).

Proof of Theorem 1.8 Let \(C_{1.8} = \lfloor 11 \cdot (25)^{24} \cdot C_{1.7} \rfloor\). Let \(G\) be a graph with \(d(G) \geq k \cdot f_{1.8}(t)\). Let \(D = 3.2t \sqrt{\log t}\) and \(s = D/d(G)\). Now Theorem 1.7 applies to \(G\). However, Theorem 1.7(i) does not hold as otherwise \(G\) has a \(K_t\) minor by Theorem 2.1.
So we may assume that Theorem 1.7(ii) holds. That is, there exists a subgraph \( H \) of \( G \) with \( d(H) \geq d(G)/g(1.7) \) and \( v(H) \leq g(1.7) \cdot D^2/d(G) \leq g(1.7) \cdot 11t \log t \). Note that \( s \leq 3.2 \sqrt{\log t} \) as \( k \geq t \). Hence

\[
g(1.7)(s) \leq g(1.7)(3.2 \sqrt{\log t}) \\
\leq C(1.7)(1 + \log(3.2 \sqrt{\log t}))^{24} \\
\leq C(1.7)(3 + \log t)^{24} \\
\leq C(1.7)(25 \log \log t)^{24} \\
\leq (25)^{24} \cdot C(1.7)(\log \log t)^{24},
\]

since \( t \geq 3 \). As \( t \geq 3 \) and \( C(1.8) \geq 11 \cdot (25)^{24} \cdot C(1.7) \) it follows that

\[
11 \cdot g(1.7)(s) \leq f(1.8)(t),
\]

and hence

\[
d(H) \geq \frac{d(G)}{g(1.7)(s)} \geq \frac{k \cdot f(1.8)(t)}{g(1.7)(s)} \geq 11k \geq 2k.
\]

By Lemma 2.2, \( H \) contains a subgraph \( H' \) such that \( \kappa(H') \geq d(H)/2 \geq k \). Since \( v(H') \leq v(H) \leq t \cdot f(1.8)(t) \cdot \log t \), we have that \( H' \) is as desired. \( \square \)

3 Outline of Proof of Density Increment Theorem

Recall that our goal in this paper is to prove Theorem 1.7. In fact, we prove the following more technical theorem which is an improvement over similar theorems in [NPS19, Pos20a].

**Theorem 3.1.** Let \( k \geq 100 \) be an integer and let \( \ell = \left\lceil k^{1/3} \right\rceil \). Let \( \epsilon \in (0, \frac{1}{32k^{1/3}}] \) and let \( G \) be a graph with \( d = d(G) \geq 1/\epsilon \). Then \( G \) contains at least one of the following:

1. a subgraph \( H \) with \( v(H) \leq 12 \cdot k^{10/3} \cdot d \) and \( e(H) \geq \epsilon^2 d^2/2 \), or
2. an \((\ell + 1)\)-bounded minor \( G' \) with \( d(G') \geq \ell \cdot (1 - \frac{2}{7}) \cdot d \), or
3. a \(k\)-bounded minor \( G' \) with \( d(G') \geq k \cdot (1 - \frac{7}{k}) \cdot d \).

The proof of Theorem 3.1 occupies Sections 4, 5 and 6. The proof is similar to the one by the author in [Pos19] with some important tweaks. We have the following immediate corollary of Theorem 3.1.

**Corollary 3.2.** Let \( k \geq 100 \) be an integer. Let \( \epsilon \in (0, \frac{1}{32k^{1/3}}] \) and let \( G \) be a graph with \( d = d(G) \geq 1/\epsilon \). Then \( G \) contains at least one of the following:

1. a subgraph \( H \) with \( v(H) \leq 12 \cdot k^{10/3} \cdot d \) and \( e(H) \geq \epsilon^2 d^2/2 \), or
2. an \( \ell \)-bounded minor \( G' \) with \( d(G') \geq \ell \cdot (1 - \frac{2}{k\ell}) \cdot d \) for some integer \( \ell \) with \( k^{1/3} < \ell \leq k \).

Now we are ready to derive Theorem 1.7 from Corollary 3.2. We restate Theorem 1.7 for convenience.
Theorem 1.7. There exists a constant \( C = C_{1.7} > 0 \) such that the following holds. Let \( G \) be a graph with \( d(G) \geq C \), and let \( D > 0 \) be a constant. Let \( s = D/d(G) \) and let \( q_{1.7}(s) := C(1 + \log s)^{24} \). Then \( G \) contains at least one of the following:

(i) a minor \( J \) with \( d(J) \geq D \), or

(ii) a subgraph \( H \) with \( v(H) \leq g_{1.7}(s) \cdot D^2 \) and \( d(H) \geq \frac{d(G)}{q_{1.7}(s)} \).

Proof of Theorem 1.7. Let \( C_{1.7} = 2^{8-15} = 2^{120} \). We proceed by induction on \( s \). If \( s \leq 1 \), then \( J = G \) is a minor of \( G \) with \( d(J) = d(G) \geq s \cdot d(G) = D \) and (i) holds as desired. So we may assume that \( s > 1 \).

Let \( k = \frac{1}{4} \cdot (C_{1.7})^{1/2} \cdot (1 + \log s)^3 = 2^{13}(1 + \log s)^3 \). Since \( \log s \geq 0 \), we have that \( k \geq 2^{13} = 8192 > e^9 > 100 \).

We apply Corollary 3.2 to \( G \) with this \( k \) and \( \varepsilon = \frac{1}{32k^{1/3}} \).

First suppose that Corollary 3.2(i) holds. That is, there exists a subgraph \( H \) with \( v(H) \leq 12 \cdot k^{10/3} \cdot d \) and \( e(H) \geq \varepsilon^2 \cdot d^2 / 2 \). Now
\[
\frac{d(H)}{v(H)} = \frac{e(H)}{v(H)} \geq \frac{\varepsilon^2}{24 \cdot k^{10/3} \cdot d} = \frac{1}{3 \cdot 2^{13} \cdot k^8 \cdot d}.
\]

Note that
\[
12 \cdot k^{10/3} \leq 3 \cdot 2^{13} \cdot k^8 \leq (4k)^8 \leq q_{1.7}(1 + \log s)^{24} = q_{1.7}(s).
\]
Hence
\[
v(H) \leq 12 \cdot k^{10/3} \cdot d \leq q_{1.7}(s) \cdot d \leq q_{1.7}(s) \cdot \frac{d^2}{d(G)}
\]
since \( s \geq 1 \) and furthermore
\[
d(H) \geq \frac{1}{3 \cdot 2^{13} \cdot k^8 \cdot d} = \frac{d}{q_{1.7}(s)}.
\]
But then (ii) holds as desired.

So we may assume that Corollary 3.2(i) holds. That is, there exists an \( \ell \)-bounded minor \( G' \) of \( G \) with
\[
d(G') \geq \ell \cdot \left(1 - \frac{3}{k^{1/3}}\right) \cdot d
\]
for some integer \( \ell \) with \( k^{1/3} < \ell \leq k \). Let \( d' = d(G') \) and \( s' = D/d' \). Note that since \( k \geq 27 = 3^3 \), we have that \( \ell \geq k^{1/3} \geq 3 \). Hence
\[
d' \geq \ell \cdot \left(1 - \frac{2}{k^{1/3}}\right) \cdot d \geq \frac{\ell}{2} \cdot d > d,
\]
and reciprocally
\[
s' \leq \frac{s}{\ell \cdot \left(1 - \frac{3}{k^{1/3}}\right)} \leq \frac{2s}{\ell} < s.
\]
Since \( s' < s \), we have by induction that at least one of (i) or (ii) holds for \( G' \).
First suppose that (i) holds for $G'$. That is, there exists a minor $J$ of $G'$ with $d(J) \geq D$.
But then $J$ is also a minor of $G$ and (i) holds for $G$ as desired.

So we may assume that (ii) holds for $G'$. That is, there exists a subgraph $H'$ of $G'$ with
\[ v(H') \leq q_{1.7}(s') \cdot \frac{D^2}{d'} \]
and
\[ d(H') \geq \frac{d'}{q_{1.7}(s')} . \]

But then $H'$ corresponds to a subgraph $H$ of $G$ with $v(H) \leq \ell \cdot v(H')$ and $e(H) \geq e(H')$. Now
\[ v(H) \leq \ell \cdot v(H') \leq \ell \cdot q_{1.7}(s') \cdot \frac{D^2}{d'} \leq \left( \frac{q_{1.7}(s')}{1 - \frac{3}{k^{1/3}}} \right) \cdot \frac{D^2}{d} . \]

Similarly
\[ d(H) = \frac{e(H)}{v(H)} \leq \frac{e(H')}{\ell \cdot v(H')} = d(H') \cdot \frac{\ell}{\ell \cdot q_{1.7}(s')} \geq \left( \frac{1 - \frac{3}{k^{1/3}}}{q_{1.7}(s')} \right) \cdot d . \]

Note that
\[ k^{1/3} \geq 3(1 + \log s) , \]

since $C_{1.7} \geq 2^{48} \geq 3^{3.8}$. Hence
\[ \frac{1}{1 - \frac{3}{k^{1/3}}} \leq 1 + \frac{6}{k^{1/3}} \leq 1 + \frac{2}{1 + \log s} , \]

where the first inequality follows since $\frac{3}{k^{1/3}} \leq \frac{1}{2}$ as $k^{1/3} \geq 6$. On the other hand,
\[ \log(\ell) \geq \log(k^{1/3}) \geq \log(e^3) \geq 3 , \]

since $k^{1/3} \geq e^3$ as $k \geq e^9$. Hence
\[ \log s' \leq \log \left( \frac{2s}{\ell} \right) \leq \log(s) + 1 - \log(\ell) \leq \log(s) - 2 . \]

Thus
\[ \frac{q_{1.7}(s')}{q_{1.7}(s)} \leq \frac{(1 + \log s')^{24}}{(1 + \log s)^{24}} \leq \frac{1 + \log s'}{1 + \log s} \leq \frac{1 + \log(s) - 2}{1 + \log s} = 1 - \frac{2}{1 + \log s} . \]

We now have that
\[ \frac{q_{1.7}(s')}{1 - \frac{2}{k^{1/3}}} \leq \left( 1 - \frac{2}{1 + \log s} \right) \left( 1 + \frac{2}{1 + \log s} \right) q_{1.7}(s) \leq q_{1.7}(s) . \]

Hence
\[ v(H) \leq q_{1.7}(s) \cdot \frac{D^2}{d} , \]
and
\[ d(H) \geq \frac{d}{q_{1.7}(s)} , \]

and (ii) holds as desired. \qed
4 Outline of the Proof of Theorem \[\text{3.1}\]

In this section we introduce additional definitions used in the proof of Theorem \[\text{3.1}\] and outline its proof.

**Definition 4.1.** Let \( G \) be a graph, and let \( K, d \geq 1, \varepsilon \in (0, 1) \) be real. We say that

- a vertex of \( G \) is \((K, d)\)-small in \( G \) if \( \deg_G(v) \leq Kd \) and \((K, d)\)-big otherwise;
- two vertices of \( G \) are \((\varepsilon, d)\)-mates if they have at least \( \varepsilon d \) common neighbors;
- \( G \) is \((K, \varepsilon, d)\)-unmated if every \((K, d)\)-small vertex in \( G \) have strictly fewer than \( \varepsilon d \) \((\varepsilon, d)\)-mates.

Here is a useful proposition and corollary.

**Proposition 4.2.** For all \( K, d \geq 1, \varepsilon \in (0, 1) \) and every graph \( G \) at least one of the following holds:

(i) there exists a subgraph \( H \) of \( G \) with \( v(H) \leq 3Kd \) and \( e(H) \geq \varepsilon^2 d^2 / 2 \), or

(ii) \( G \) is \((K, \varepsilon, d)\)-unmated.

**Proof.** Assume that \( G \) is not \((K, \varepsilon, d)\)-unmated. Then there exists \( v \in V(G) \) with at least \( \varepsilon d \) \((\varepsilon, d)\)-mates. Let \( v_1, \ldots, v_{\lceil \varepsilon d \rceil} \) be distinct \((\varepsilon, d)\)-mates of \( v \). Let \( H = G[N(v) \cup \{v, v_1, \ldots, v_{\lceil \varepsilon d \rceil}\}] \). Then \( v(H) \leq 1 + Kd + \lceil \varepsilon d \rceil \leq 3Kd \) and \( e(H) \geq \varepsilon^2 d^2 / 2 \). Thus (i) holds, as desired. \( \square \)

**Corollary 4.3.** Let \( K, k, d \geq 1, \varepsilon \in (0, 1) \), and let \( G' \) be a \( k \)-bounded minor of a graph \( G \). Then at least one of the following holds:

(i) there exists a subgraph \( H \) of \( G \) with \( v(H) \leq 3kKd \) and \( e(H) \geq \varepsilon^2 d^2 / 2 \), or

(ii) \( G' \) is \((K, \varepsilon, d)\)-unmated.

**Proof.** Assume that \( G' \) is not \((K, \varepsilon, d)\)-unmated. By Proposition 4.2 applied to \( G' \), there exists a subgraph \( H' \) of \( G' \) with \( v(H') \leq 3Kd \) and \( e(H') \geq \varepsilon^2 d^2 / 2 \). Since \( H' \) is a \( k \)-bounded minor of \( G \), it corresponds to a subgraph \( H \) of \( G \) with \( v(H) \leq k \cdot v(H') \leq 3kKd \) and \( e(H) \geq e(H') \geq \varepsilon^2 d^2 / 2 \). \( \square \)

**Definition 4.4.** Let \( F \) be a non-empty forest in a graph \( G \). Let \( k, d, s \geq 1 \) be real and let \( \varepsilon, c \in (0, 1) \). We say \( F \) is

- \((k, d)\)-small if every vertex in \( V(F) \) is \((k, d)\)-small in \( G \),
- \((\varepsilon, d)\)-mate-free if no two distinct vertices in any component of \( F \) are \((\varepsilon, d)\)-mates in \( G \),
- \((c, d)\)-clean if \( e(G) - e(G/F) \leq c \cdot d \cdot v(F) \),
- \( k \)-bounded if \( v(T) \leq k \) for every component \( T \) of \( F \), and
• a \((k, p)\)-shrubbery if \(k - p < \nu(T) \leq k\) for every component \(T\) of \(F\).

**Definition 4.5.** Let \(\ell \geq 1\) be an integer. An \(\ell\)-star is a star with \(\ell\) leaves. An \(\ell^-\text{-star}\) is a star with at least one but at most \(\ell\) leaves. Let \(G\) be a graph and let \((A, B)\) be a partition of \(V(G)\). Let \(\ell \geq 1\) be an integer. We say a forest \(F\) is an \(\ell^-\text{-star-matching}\) from \(B\) to \(A\) if every component \(T\) of \(F\), then \(T\) is an \(\ell^-\text{-star}\), the center of \(T\) is in \(B\) and the leaves of \(T\) are in \(A\). Similarly we define \(\ell\text{-star-matching}, \ell\text{-claw-matching} and \ell^-\text{-claw-matching}\) from \(B\) to \(A\) as above if every component of \(F\) is an \(\ell\)-star (resp. \(\ell\)-claw and \(\ell^-\text{-claw}\)) instead of an \(\ell^-\text{-star}\).

Here is a simple but useful proposition whose proof we omit.

**Proposition 4.6.** Let \(G\) be a graph. If \(uv \in E(G)\), then
\[
e(G) - e(G/uv) = 1 + |N(u) \cap N(v)|.
\]

The proof of Theorem 3.1 is based on the following two theorems.

**Theorem 4.7.** Let \(K, \ell \geq 2\) be integers with \(K \geq \ell^2(\ell + 1)\), and let \(\varepsilon_0 > 0\) and \(d_0 \geq 1/\varepsilon_0\) be real. Let \(G = (A, B)\) be a bipartite graph such that \(|A| \geq \ell|B|\) and every vertex in \(A\) has at least \(d_0\) neighbors in \(B\). Then \(G\) contains at least one of the following:

(i) a subgraph \(H\) with \(\nu(H) \leq 4\ell Kd_0\) and \(e(H) \geq \varepsilon_0^2d_0^2/2\).

(ii) an \((\ell + 1)\)-bounded minor \(H\) with \(d(H) \geq \frac{\ell^2}{\ell + 1}(1 - 8\ell^3\varepsilon_0)d_0\).

**Theorem 4.8.** Let \(K \geq k \geq 27\) be integers with \(K \geq 4\cdot k^{4/3}\). Let \(\ell = k^{1/3}\). Let \(\varepsilon \in (0, \frac{1}{32k^{2/3}}]\). Let \(G\) be a graph with \(d = d(G) \geq 2/\varepsilon\). Then \(G\) contains at least one of the following:

(i) a subgraph \(H\) with \(\nu(H) \leq 3k^2Kd\) and \(e(H) \geq \varepsilon^2d^2/2\), or

(ii) a bipartite subgraph \(H = (X, Y)\) with \(|X| \geq \ell|Y|\) such that every vertex in \(X\) has at least \((1 - 8k^2\varepsilon)d\) neighbors in \(Y\), or

(iii) a \(k\)-bounded minor \(G'\) with \(d(G') \geq k \cdot \left(1 - \frac{2}{\ell}\right) \cdot d\).

We prove Theorem 4.7 in Section 5 and Theorem 4.8 in Section 6. We finish this section by deriving Theorems 3.1 which we restate for convenience, from Theorems 4.7 and 4.8

**Theorem 3.1.** Let \(k \geq 100\) be an integer and let \(\ell = \lfloor k^{1/3} \rfloor\). Let \(\varepsilon \in (0, \frac{1}{32k^{2/3}}]\) and let \(G\) be a graph with \(d = d(G) \geq 1/\varepsilon\). Then \(G\) contains at least one of the following:

(i) a subgraph \(H\) with \(\nu(H) \leq 12 \cdot k^{10/3} \cdot d\) and \(e(H) \geq \varepsilon^2d^2/2\), or

(ii) an \((\ell + 1)\)-bounded minor \(G'\) with \(d(G') \geq \ell \cdot \left(1 - \frac{2}{\ell}\right) \cdot d\), or

(iii) a \(k\)-bounded minor \(G'\) with \(d(G') \geq k \cdot \left(1 - \frac{2}{\ell}\right) \cdot d\).
Proof of Theorem 3.4. We apply Theorem 4.8 to $G$ with $K = 4 \cdot k^{4/3}$. If Theorem 4.8(i) holds then (i) holds as desired. If Theorem 4.8(iii) holds, then (iii) holds as desired.

So we may assume that Theorem 4.8(ii) holds, that is there exists a bipartite subgraph $H = (X, Y)$ with $|X| \geq \ell|Y|$ such that every vertex in $X$ has at least $(1 - 8k^2\varepsilon)d$ neighbors in $Y$. We next apply Theorem 4.7 with $d_0 = (1 - 8k^2\varepsilon)d$ and $\varepsilon_0 = 2\varepsilon$ to $H$.

First assume Theorem 4.7(i) holds. That is, there exists a subgraph $H_0$ of $H$ with $v(H_0) \leq 4\ell \cdot K \cdot d_0$ and $e(H_0) \geq \varepsilon_0^2d_0^2/2$. Note then that

$$v(H_0) \leq 16 \cdot k^{5/3} \cdot d \leq 12 \cdot k^{10/3} \cdot d,$$

since $k \geq 2$, and

$$e(H_0) = 4\varepsilon^2 \cdot (1 - 8k^2\varepsilon)^2 \cdot d^2/2 \geq \varepsilon^2d^2/2,$$

since $8k^2\varepsilon \leq 1/2$, as $\varepsilon \leq \frac{1}{16k^2}$. But then (i) holds as desired.

So we may assume that Theorem 4.7(ii) holds. That is, $H$ contains an $(\ell + 1)$-bounded minor $H_0$ with

$$d(H_0) \geq \frac{\ell^2}{\ell + 1} \cdot (1 - 8\ell^3\varepsilon_0) \cdot d_0$$

$$\geq \ell \cdot \left(1 - \frac{1}{\ell + 1}\right) (1 - 16k\varepsilon) \cdot (1 - 8k^2\varepsilon) \cdot d$$

$$\geq \ell \cdot \left(1 - \frac{1}{\ell}ight) (1 - 24k^2\varepsilon) \cdot d,$$

$$\geq \ell \cdot \left(1 - \frac{2}{\ell}ight) (1 - \frac{24k^2}{32k^{7/3}}) \cdot d,$$

$$\geq \ell \cdot \left(1 - \frac{2}{\ell}ight) \cdot d,$$

since $\ell \leq k^{1/3}$. Hence (ii) holds with $G' = H_0$, as desired. \hfill \Box

5 Dense Minors in Unbalanced Bipartite Graphs

In this section, we prove Theorem 4.7. The proof is nearly identical to that of the author in Theorem 3.4 as presented in [Pos19]. To prove Theorem 4.7 we need the following three lemmas, Lemmas 5.1, 5.2, and 5.3 from there. We include their proofs for completeness.

Lemma 5.1. Let $\ell \geq 1, d_B > d_A \geq 0$ be integers. Let $G$ be a graph and let $(A, B)$ be a partition of $V(G)$ with $|A| \geq \ell|B|$ and $B$ is independent. If every vertex in $A$ has at least $d_B$ neighbors in $B$ and at most $d_A$ neighbors in $A$, then $G$ contains an $\ell$-claw-matching $F$ from $B$ to $A$ such that every vertex in $V(F) \cap A$ has at most $d_A$ neighbors in $B \setminus V(F)$.

Proof. Let $F_0$ be an $\ell$-claw-matching from $B$ to $A$ such that $|V(F_0) \cap A|$ is maximized. Assume first that $V(F_0) \cap A = A$. Note that $|V(F_0) \cap A| \leq \ell \cdot |V(F_0) \cap B|$. Then $V(F_0) \cap B = B$ because $|V(F_0) \cap A| = |A| \geq \ell|B|$. Hence $V(F_0) = V(G)$ and $F = F_0$ is as desired. So we may assume that $A \setminus V(F_0) \neq \emptyset$. 

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Let \( u \in A \setminus V(F_0) \). By the maximality of \( F_0 \), \( N_G(u) \cap B \subseteq V(F_0) \cap B \). For each \( v \in V(G) \) with \( v \neq u \), we say that a path \( P \) from \( u \) to \( v \) is a \((u,v)\)- alternating path if

- \( P \) is a path in \( G(A,B) \), and
- every internal vertex of \( P \) has degree exactly one in \( F_0 \cap P \) (that is - informally - that every other edge of \( P \) is in \( F_0 \)), and
- there does not exist a triangle of \( G \) containing an edge of \( F_0 \) and an edge of \( P - E(F_0) \).

Let \( B_u \) be the set of all vertices \( v \in B \) such that there exists a \((u,v)\)- alternating path. Then \( B_u \neq \emptyset \) as \( d_B > d_A \).

**Claim 5.1.1.** For all \( v \in B_u \), we have \( v \in V(F_0) \) and the component of \( F_0 \) containing \( v \) has exactly \( \ell \) edges.

**Proof.** Let \( v \in B_u \). Then there exists a \((u,v)\)- alternating path \( P \). Let \( F'_0 = F_0 \cup P \). Since \( P \) is a path in \( G(A,B) \), we have \( E(F_0) \subseteq E(G(A,B)) \). It follows that \( v \in V(F_0) \) and the component of \( F_0 \) containing \( v \) has exactly \( \ell \) edges, else \( F'_0 \) is an \( \ell \)- claw-matching from \( B \) to \( A \) with \( |V(F'_0) \cap A| > |V(F_0) \cap A| \), contrary to the choice of \( F_0 \). \( \square \)

Let \( F \) be the subgraph of \( F_0 \) consisting of all the components \( T \) of \( F_0 \) such that \( T \) contains a vertex in \( B_u \). By Claim 5.1.1 \( F \) is an \( \ell \)- claw-matching from \( B \) to \( A \). It remains to show that every vertex in \( V(F) \cap A \) has at most \( d_A \) neighbors in \( B \setminus V(F) \). Let \( w \in V(F) \cap A \) and let \( x \) be a neighbor of \( w \) in \( B \setminus V(F) \). Then there exists \( v \in B_u \) such that \( vw \in E(F) \). By the definition of \( B_u \), there exists a \((u,v)\)- alternating path \( P \). Then \( w \notin V(P) \). Let \( P' = P + vw + wx \). Then \( P' \) is a path in \( G(A,B) \) from \( u \) to \( x \) such that every other edge is in \( F \). Note that \( P + vw \) is a \((u,w)\)- alternating path. By the maximality of \( F_0 \), \( x \in V(F_0) \). By the choice of \( F \), \( x \notin B_u \). Thus \( P' \) is not a \((u,x)\)- alternating path. It follows that \( x \) is the center of a star \( T \) in \( F_0 \setminus V(F) \) such that \( wx \) is contained in a triangle \( wxz \), where \( z \in A \cap V(T) \). Since \( w \) has at most \( d_A \) neighbors in \( A \), we see that \( w \) has at most \( d_A \) neighbors in \( B \setminus V(F) \), as desired. \( \square \)

We now apply Lemma 5.1 to obtain a mate-free \( \ell \)-claw-matching in a dense unbalanced bipartite graph assuming that the graph itself is unmated.

**Lemma 5.2.** Let \( K \geq \ell \geq 1 \) and \( d_0 \geq 1 \) be integers, and let \( \varepsilon_0 \in (0,1) \) be real. Let \( G = (A,B) \) be a bipartite graph such that \( |A| \geq \ell |B| \) and every vertex in \( A \) has at least \( d_0 \) neighbors in \( B \). If \( G \) is \((K,\varepsilon_0,d_0)\)- unmated, then \( G \) contains an \((\varepsilon_0,d_0)\)- mate-free \( \ell \)-claw-matching \( F \) from \( B \) to \( A \) such that every vertex in \( V(F) \cap A \) has at most \( \varepsilon_0 d_0 \) neighbors in \( B \setminus V(F) \).

**Proof.** Since \( K \geq 1 \) and \( G \) is \((K,\varepsilon_0,d_0)\)-unmated, we see that every vertex of \( A \) is \((K,d_0)\)- small, and has at most \( \varepsilon_0 d_0 \) many \((\varepsilon_0,d_0)\)- mates in \( G \). Let \( G' \) be obtained from \( G \) by adding all possible edges \( uv \), where \( u,v \in A \) are \((\varepsilon_0,d_0)\)- mates in \( G \). Then in \( G' \), every vertex of \( A \) has at least \( d_B = d_0 \) neighbors in \( B \) and at most \( d_A = \varepsilon_0 d_0 < d_B \) neighbors in \( A \). By Lemma 5.1 \( G' \) contains an \( \ell \)-claw-matching \( F \) from \( B \) to \( A \) such that every vertex in \( V(F) \cap A \) has at most \( d_A \) neighbors in \( B \setminus V(F) \). It remains to show that every component \( T \) of \( F \) is \((\varepsilon_0,d_0)\)- mate-free.
Let $x, y \in V(T)$ be distinct. We may assume that $x \in A$. Assume first that $y \in B$. Then $xy \in E(G)$, and so $x$ and $y$ are not $(\varepsilon_0, d_0)$-mates in $G$, because $G$ is bipartite. So we may assume that $y \in A$. Since $T$ is an $\ell$-claw in $G'$, we see that $xy \notin E(G')$. By the choice of $G'$, $x$ and $y$ are not $(\varepsilon_0, d_0)$-mates in $G$, as desired. \hfill \square

Next we clean the $\ell$-claw-matching obtained from Lemma 5.2. To do this, we have to remove components whose centers are big in $G[V(F)]$ and then switch edges as necessary.

**Lemma 5.3.** Let $K \geq \ell \geq 1$ be integers. Let $\varepsilon_1 \in (0, 1)$, and let $d_1 \geq \frac{1}{\varepsilon_1}$ be an integer. Let $G = (A, B)$ be a bipartite graph such that $|A| = |B|$ and every vertex in $A$ has exactly $d_1$ neighbors in $B$. Suppose $G$ is $(K, \varepsilon_1, d_1)$-unmated, and has an $(\varepsilon_1, d_1)$-mate-free $\ell$-claw-matching $F_1$ from $B$ to $A$ with $V(F_1) = V(G)$. Then $G$ contains at least one of the following:

(i) a subgraph $H$ of $G$ with $v(H) \leq (\ell + 1)(K + 1)d_1$ and $e(H) \geq \varepsilon_1^2d_1^2/2$, or

(ii) a $(K, d_1)$-small, $(\varepsilon_1, d_1)$-mate-free, $(\ell \cdot (\ell + 1)^2\varepsilon_1, d_1)$-clean $\ell$-claw-matching $F$ from $B$ to $A$ such that $v(F) \geq v(G) \left(1 - \frac{\ell(\ell + 1)}{K}\right)$.

**Proof.** Since $G$ is bipartite and $K \geq 1$, we see that every vertex in $A$ is $(K, d_1)$-small in $G$. Note that $e(G) = d_1|A| = d_1\ell \cdot |B|$. Hence the number of $(K, d_1)$-big vertices in $G$ is at most $\frac{\ell|B|}{K} \cdot \frac{\nu(G)}{\ell + 1}$. Let $F^*$ be the subgraph of $F_1$ consisting of all the components $T$ of $F_1$ such that each $T$ contains only $(K, d_1)$-small vertices of $G$. Then $F^*$ is a $(K, d_1)$-small, $(\varepsilon_1, d_1)$-mate-free $\ell$-claw-matching from $B$ to $A$ in $G$ and

$$v(F^*) \geq v(G) - \left(\frac{\ell}{K(\ell + 1)} \cdot v(G)\right) (\ell + 1) = v(G) \left(1 - \frac{\ell}{K}\right).$$

Given distinct components $T_1, T_2$ of an $\ell$-claw-matching $F$ from $B$ to $A$ and edges $u_1v_1 \in E(T_1)$ and $u_2v_2 \in E(T_2)$ with $v_1, v_2 \in B$, we say that $\{u_1v_1, u_2v_2\}$ is a bad pair of $F$ if $u_1v_2, u_2v_1 \in E(G)$. Now let $F$ be an $(\varepsilon_1, d_1)$-mate-free $\ell$-claw-matching from $B$ to $A$ with $V(F) = V(F^*)$ such that $F$ has the minimum number of bad pairs. Then $F$ is $(K, d_1)$-small. We now bound the maximum number of bad pairs of $F$ that an edge of $F$ can belong to.

Let $T$ be a component of $F$ with $uv \in E(T)$. Assume that $\{(uv, u_iv_i)\}_{i=1}^b$ are $b = \lceil (\ell + 1)(\varepsilon_1d_1 + 1) \rceil$ bad pairs of $F$, where $u_1v_1 \in E(T_1), u_2v_2 \in E(T_2), \ldots, u_bv_b \in E(T_b)$, and $T_1, T_2, \ldots, T_b$ are distinct components of $F \setminus V(T)$. We may further assume that $v, v_1, \ldots, v_b \in B$. Then there are at most $\ell\varepsilon_1d_1$ trees $T'$ in $\{T_1, T_2, \ldots, T_b\}$ such that a vertex in $T - v$ has an $(\varepsilon_1, d_1)$-mate in $T'$. We may assume that no vertex in $T - v$ has an $(\varepsilon_1, d_1)$-mate in $T_i$ for all $i$, where $1 \leq i < b - \ell\varepsilon_1d_1$. For each such $i$, let $T'_i$ be obtained from $T$ by deleting $u$ and adding the edge $v_iu_i$, and let $T''_i$ be obtained from $T_i$ by deleting $u_i$ and adding the edge $v_iu_i$. Let $F'_i$ be obtained from $F \setminus V(T \cup T_i)$ by adding $T'_i$ and $T''_i$. Then $F'_i$ is an $(\varepsilon_1, d_1)$-mate-free $\ell$-claw-matching from $A$ to $B$ with $V(F'_i) = V(F) = V(F^*)$ for all $i \in [b]$. Let $X$ be the union of vertex set of all components of $F \setminus V(T)$ containing a neighbor of $u$ or $v$. Then $|X| \leq (\ell + 1)(K + 1)d_1$. Let $H = G[X]$. It follows from the choice of $F$ that for every $i < b - \ell\varepsilon_1d_1$ there are at least $b$ bad pairs of $F'_i$, which are not bad pairs of $F$. Each such pair must contain one of the edges $vu_i$ and $v_iu$. It follows that $\deg_H(v_i) + \deg_H(u_i) \geq b$, and consequently $e(H) \geq b(b - \ell\varepsilon_1d_1 - 1)/2 \geq \varepsilon_1^2d_1^2/2$. Hence (i) holds.
We may now assume that every edge of $F$ belongs to at most $\ell(\ell+1)(\varepsilon_1d_1+1)$ bad pairs of $F$. Then there are at most $\ell(\ell+1)(\varepsilon_1d_1+1) \cdot e(F)/2$ bad pairs of $F$ in total. Note that every pair of edges of $G$ that become parallel in $G/E(F)$ corresponds to a bad pair or a common neighbor of two leaves of some component in $F$. Note that $e(F) \leq v(F)$. Since $F$ is $(\varepsilon_1, d_1)$-mate-free, it follows that

$$e(G) - e(G/E(F)) \leq e(F) + \left(\frac{\ell}{2}\right)\varepsilon_1d_1 \cdot \frac{v(F)}{\ell+1} + \ell(\ell+1)(\varepsilon_1d_1+1) \cdot \frac{e(F)}{2}$$

$$\leq \ell \cdot v(F) + \frac{\ell}{2}\varepsilon_1d_1 \cdot v(F) + \ell(\ell+1)\varepsilon_1d_1v(F)$$

$$\leq (\ell+1)^2\varepsilon_1d_1 \cdot v(F),$$

since $\ell \geq 1$, $\varepsilon_1d_1 \geq 1$ and $e(F) \leq v(F)$. Hence $F$ is $((\ell+1)^2\varepsilon_1, d_1)$-clean and (ii) holds. □

We finish this section by proving Theorem 4.7, which we restate below for convenience. The bound in outcome (ii) has been improved over the previous version in [Pos19] namely by a factor of 2 to be nearly optimal. This is accomplished by adding the assumption that $K \geq \ell^2(\ell+1)$.

**Theorem 4.7.** Let $K, \ell \geq 2$ be integers with $K \geq \ell^2(\ell+1)$, and let $\varepsilon_0 > 0$ and $d_0 \geq 1/\varepsilon_0$ be real. Let $G = (A, B)$ be a bipartite graph such that $|A| \geq \ell|B|$ and every vertex in $A$ has at least $d_0$ neighbors in $B$. Then $G$ contains at least one of the following:

(i) a subgraph $H$ with $v(H) \leq 4\ell Kd_0$ and $e(H) \geq \varepsilon_0^2d_0^2/2$.

(ii) an $(\ell+1)$-bounded minor $H$ with $d(H) \geq \frac{\ell^2}{\ell+1}(1 - 8\ell^3\varepsilon_0)d_0$.

**Proof of Theorem 4.7.** Assume first that $G$ is not $(K, \varepsilon_0, d_0)$-unmated. By Proposition 4.2(i), there exists a subgraph $H$ of $G$ with $v(H) \leq 3Kd_0$ and $e(H) \geq \varepsilon_0^2d_0^2/2$. Hence (i) holds because $\ell \geq 1$.

Assume next that $G$ is $(K, \varepsilon_0, d_0)$-unmated. By Lemma 5.2, $G$ contains an $(\varepsilon_0, d_0)$-mate-free $\ell$-claw-matching $F_1$ from $B$ to $A$ such that every vertex in $V(F_1) \cap A$ has at most $\varepsilon_0d_0$ neighbors in $B \setminus V(F_1)$. Let $d_1 = [d_0(1-\varepsilon_0)]$ and $\varepsilon_1 = \frac{\varepsilon_0d_0}{d_1}$. Then $\varepsilon_1 \in (0, 1)$ and $d_1\varepsilon_1 = d_0\varepsilon_0 \geq 1$. Let $G'$ be obtained from $G$ with $V(G') = V(F_1)$ and $E(F_1) \subseteq E(G')$ such that every vertex in $V(F_1) \cap A$ has exactly $d_1$ neighbors in $V(F_1) \cap B$ in $G'$. Since $G$ is $(K, \varepsilon_0, d_0)$-unmated, we see that $G'$ is $(K, \varepsilon_1, d_1)$-unmated. Furthermore, $F_1$ is an $(\varepsilon_1, d_1)$-mate-free $\ell$-claw-matching from $V(F_1) \cap B$ to $V(F_1) \cap A$ with $V(F_1) = V(G')$ in $G'$. By Lemma 5.3 applied to $G'$ with parameters $K, \ell, \varepsilon_1, d_1$, at least one of Lemma 5.3(i) or (ii) holds for $G'$.

First suppose that Lemma 5.3(i) holds for $G'$. That is, $G'$ has a subgraph $H$ with $v(H) \leq (\ell+1)(K+1)d_1$ and $e(H) \geq \varepsilon_0^2d_1^2/2$. But then (i) holds for $G$ because $K \geq \ell \geq 1$, $d_1 \leq d_0$ and $\varepsilon_1d_1 \geq \varepsilon_0d_0$.

So we may assume that Lemma 5.3(ii) holds for $G'$. That is, there exists a $(K, d_1)$-small, $(\varepsilon_1, d_1)$-mate-free $((\ell+1)^2\varepsilon_1, d_1)$-clean $\ell$-claw-matching $F$ from $V(F_1) \cap B$ to $V(F_1) \cap A$ in $G'$ such that $v(F) \geq v(G')\left(1 - \frac{\ell(\ell+1)}{K}\right)$. Now let $H = G'/E(F)$. Then $H$ is an $(\ell+1)$-bounded minor of $G$ with

$$v(H) \leq \left(\frac{1}{\ell+1} + \frac{\ell}{K}\right) \cdot v(G') \leq \frac{1}{\ell} \cdot v(G').$$

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because $K \geq \ell^2(\ell + 1)$. Since $F$ is $((\ell + 1)^2\epsilon_1, d_1)$-clean, we have
\[
e(H) \geq e(G') - (\ell + 1)^2\epsilon_1 d_1 \cdot v(F) \\
\geq d_1 \cdot \frac{\ell}{\ell + 1} \cdot v(G') - (\ell + 1)^2\epsilon_1 d_1 \cdot v(G') \\
\geq \left( d_0 \cdot (1 - \epsilon_0) \cdot \frac{\ell}{\ell + 1} - (\ell + 1)^2 \cdot 2\epsilon_0 d_0 \right) \cdot v(G') \\
\geq \frac{\ell}{\ell + 1} \cdot (1 - 8\ell^3\epsilon_0)d_0 \cdot v(G'),
\]
where we use the fact that $1 + 2(\ell + 1)^3 \leq 8\ell^3$ since $\ell \geq 2$. Hence
\[
d(H) = \frac{e(H)}{v(H)} \geq \frac{\ell^2}{\ell + 1} \cdot (1 - 8\ell^3\epsilon_0) \cdot d_0.
\]
and (ii) holds, as desired. \qed

6 Dense Minors in General Graphs

In this section we prove Theorem 4.8. First, we need the following definitions.

**Definition 6.1.** A *rooted tree* $T$ is a tree $T$ with a specified vertex $v$ called its *root* which we denote by $\text{root}(T)$. If $v \in V(T) \setminus \{\text{root}(T)\}$ and $e \in E(T)$ is an edge incident with a vertex $v$ such that the component $H$ of $T - e$ containing $v$ does not contain $\text{root}(T)$, then we say that $e$ is a *central edge for* $v$ in $T$ and that $H$ is a *peripheral piece for* $v$. We say a vertex $v \in T \setminus \{\text{root}(T)\}$ is *p-peripheral* if the peripheral piece for $v$ contains at most $p$ vertices, and *p-central* otherwise.

We say a vertex $v \in V(T) \setminus \{\text{root}(T)\}$ is a *terminal* if the peripheral piece for $v$ consists only of $v$ and leaves of $T$.

Note that a vertex is 1-peripheral in $T$ if and only if it is a leaf. Note also that the peripheral pieces of 1-central terminal vertices of a rooted tree are vertex-disjoint.

**Definition 6.2.** A *rooted forest* $F$ is a forest $F$ where every component has a specified root vertex, or equivalently, where each component of $F$ is a rooted tree. Let $C(F)$ denote the components of $F$. For $C$ a component of $F$, we let $\text{root}(C)$ denote the root of $C$.

**Definition 6.3.** Let $F$ be a rooted forest. We define the *farness of $F$* as
\[
\text{far}(F) = \sum_{C \in C(F)} \sum_{v \in C} \text{dist}(\text{root}(C), v),
\]
where $\text{dist}(u, v)$ denotes the *distance* between $u$ and $v$ (i.e. the length of the shortest path between $u$ and $v$). We say a rooted forest $F$ in a graph $G$ is $(c, d, k)$-shiny if
\[
e(G) - e(G/F) \leq c \cdot d \cdot (2k \cdot v(F) - \text{far}(F)).
\]
We are now ready to prove Theorem 4.8 which we restate for convenience.

**Theorem 4.8.** Let \( K \geq k \geq 27 \) be integers with \( K \geq 4 \cdot k^{4/3} \). Let \( \ell = k^{1/3} \). Let \( \varepsilon \in (0, \frac{1}{32k^{1/3}}] \). Let \( G \) be a graph with \( d = d(G) \geq 2/\varepsilon \). Then \( G \) contains at least one of the following:

1. a subgraph \( H \) with \( v(H) \leq 3k^2 K d \) and \( e(H) \geq \varepsilon^2 d^2 / 2 \), or
2. a bipartite subgraph \( H = (X,Y) \) with \( |X| \geq \ell |Y| \) such that every vertex in \( X \) has at least \( (1 - 8k^2 \varepsilon) d \) neighbors in \( Y \), or
3. a \( k \)-bounded minor \( G' \) with \( d(G') \geq k \cdot (1 - \frac{2}{k}) \cdot d \).

**Proof of Theorem 4.8.** Suppose not. Let \( G \) be a counterexample with \( v(G) \) minimized. Thus we may assume that \( d(H) < d(G) \) for every proper subgraph \( H \) of \( G \), and hence \( \delta(G) \geq d \). Since (i) does not hold for \( G \), we have by Proposition 4.2 that \( G \) is \((K, \varepsilon, d)\)-unmated.

Let \( A \) denote the set of all \((K, d)\)-small vertices of \( G \), and let \( B = V(G) \setminus A \) be the set of \((K, d)\)-big vertices. Then \( K \cdot d \cdot |B| \leq 2e(G) = 2d \cdot v(G) \). Hence \( |B| \leq \frac{2}{K} \cdot v(G) \leq \frac{1}{2k^{4/3}} \cdot v(G) \) since \( K \geq 4 \cdot k^{4/3} \). Then

\[
|A| \geq \left( 1 - \frac{1}{2 \cdot k^{4/3}} \right) \cdot v(G) \geq \frac{4\ell}{K} \cdot v(G) \geq 2\ell |B|,
\]

as \( k \geq 27 \).

Let \( p = k^{2/3} \) and \( c = 2k\varepsilon \). Let \( F \) be a \((K, d)\)-small, \((\varepsilon, d)\)-mate-free, \((c, d, k)\)-shiny rooted \((k, p)\)-shrubbery in \( G \) so that \( v(F) \) is maximized and subject to that \( \text{far}(F) \) is minimized.

**Claim 6.3.1.** \( F \) is nonempty.

**Proof.** First suppose that every vertex in \( A \) has at least \( (1 - 2k\varepsilon) d \) neighbors in \( B \). Since \( |A| \geq 2\ell |B| > \ell |B| \), we have that (ii) holds with \( H = G(A, B) \), a contradiction.

So we may assume that some vertex \( w \in A \) has at least \( 2k\varepsilon d \) neighbors in \( A \). Note that \( 2k\varepsilon d \geq k\varepsilon d + k \) as \( \varepsilon d \leq 1 \). Since \( G \) is \((K, \varepsilon, d)\)-unmated, every vertex in \( A \) has strictly fewer than \( \varepsilon d (\varepsilon, d) \)-mates in \( G \). It follows that there exists \( w_1, w_2, \ldots, w_{k-1} \in A \) such that no two vertices of \( \{v\} \cup \{w_i : i \in [k-1]\} \) are \((\varepsilon, d)\)-mates in \( G \).

Let \( T \) be a rooted star with the edge set \( \{vw_i : i \in [k-1]\} \) with root \( w \). Since \( G \) is \((K, \varepsilon, d)\)-unmated, we see that \( T \) is a \((K, d)\)-small, \((\varepsilon, d)\)-mate-free rooted \((k, p)\)-shrubbery in \( G \). Note that

\[
e(G) - e(G/E(T)) \leq e(G[V(T)]) + \left( \frac{k}{2} \right) \varepsilon d \leq \left( \frac{k}{2} \right) (\varepsilon d + 1) = k^2 \varepsilon d.
\]

Since \( v(T) = k \) and \( \text{far}(T) = k - 1 \), we find that

\[
e(G) - e(G/E(T)) \leq k^2 \varepsilon d \leq c \cdot d(2k \cdot v(T) - \text{far}(F)),
\]

since \( c \geq \varepsilon \). Hence \( T \) is \((c, d, k)\)-shiny. But now \( T \) satisfies the conditions for \( F \) as desired. \( \square \)
By Claim 6.3.1 $F$ is non-empty. Note that $V(F) \subseteq A$ as $F$ is $(K, d)$-small. Let $A' = A \setminus V(F)$.

Let $C$ be the set of roots and $p$-central terminals of components $T$ of $F$ with $v(T) = k$. Since the peripheral pieces of 1-central terminals of a rooted tree are vertex-disjoint, we have that the number of $p$-central terminals in a component $T$ of $F$ is at most $k/p$. Thus, we find that

$$|C| \leq \left( \frac{1}{k} + \frac{1}{p} \right) \cdot v(G) \leq \frac{2}{p} \cdot v(G),$$

since $p \leq k$.

Since $G$ is $(K, \varepsilon, d)$-unmated, we see that every vertex in $A$ has fewer than $\varepsilon d$ $(\varepsilon, d)$-mates in $G$. We next prove several claims.

**Claim 6.3.2.** Every $k$-bounded minor of $G$ is $(Kk, \varepsilon, d)$-unmated.

**Proof.** Let $G'$ be a $k$-bounded minor of $G$. By Corollary 4.3 applied to $G$ and $G'$ with parameters $Kk, k, d, \varepsilon$, it follows that either $G$ has a subgraph $H$ with $v(H) \leq 3k^2Kd$ and $e(H) \geq \varepsilon d^2/2$, or $G'$ is $(Kk, \varepsilon, d)$-unmated. In the first case, (i) holds, a contradiction. Hence $G'$ is $(Kk, \varepsilon, d)$-unmated. 

Let $G' = G/E(F)$. Then $G'$ is a $k$-bounded minor of $G$. By Claim 6.3.2 $G'$ is $(Kk, \varepsilon, d)$-unmated. Let $F_1$ be the set of components $T$ of $F$ such that $v(F) = k$. Let $W_1$ be the set of terminals of components in $F_1$ that are not in $C$. Let $W_2 = V(F_1) \setminus (C \cup W_1)$ and let $F_2 = F \setminus V(F_1)$. Then every component of $F_2$ has at most $k - 1$ vertices.

**Claim 6.3.3.** If $v \in A'$, then $v$ has at most $2(k - 1)\varepsilon d$ neighbors in $V(F_2) \cup W_2$ in $G$.

**Proof.** Suppose not. Let $F_3$ be the set of all components $T$ of $F$ such that no vertex in $T$ is an $(\varepsilon, d)$-mate of $v$ in $G$. Since $v$ has fewer than $\varepsilon d$ many $(\varepsilon, d)$-mates in $G$, and each component of $F_3$ has at most $k - 1$ vertices, we see that $F_3$ has more than $\varepsilon d$ distinct components each containing a neighbor of $v$ that is in $V(F_2) \cup W_2$. Note that every vertex of $A \setminus V(F)$ and every vertex corresponding to a component of $F$ are $(Kk, d)$-small in $G'$. Since $G'$ is $(Kk, \varepsilon, d)$-unmated, we see that $v$ has fewer than $\varepsilon d$ many $(\varepsilon, d)$-mates in $G'$. Thus there exists a component $T$ in $F_3$ such that $v$ has a neighbor $w$ in $T \cap (V(F_2) \cup W_2)$ and the vertex $v_T$ corresponding to $T$ in $G'$ is not an $(\varepsilon, d)$-mate of $v$ in $G'$. Therefore $|N_{G'}(v) \cap N_{G'}(v_T)| \leq \varepsilon d$.

First suppose that $w \in V(F_2)$. Let $T' = T + vw$ and $F' = (F \setminus V(T)) \cup T'$. Then $F'$ is a $(K, d)$-small rooted $(k, p)$-shrubbery since $V(F') \subseteq A$ and $v(T') = v(T) + 1 \leq k$. By the choice of $F_3$, $T'$ is $(\varepsilon, d)$-mate-free. Thus $F'$ is $(\varepsilon, d)$-mate-free. Note that

$$v(F') = v(F) + 1.$$

Furthermore,

$$\text{far}(F') \leq \text{far}(F) + \text{dist}(w, \text{root}(T)) \leq \text{far}(F) + k.$$

Note that

$$\text{e}(G') - \text{e}(G/F') \leq |N_{G'}(v) \cap N_{G'}(v_T)| + 1 \leq \varepsilon d + 1.$$
Recall that \( e(G) - e(G') \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) \) since \( F \) is \((c, d, k)\)-shiny. Therefore

\[
e(G) - e(G/F') = (e(G) - e(G')) + (e(G') - e(G/F')) \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + \varepsilon d + 1 \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + 2\varepsilon d \leq cd \cdot (2k \cdot v(F') - \text{far}(F')) = cd \cdot (2k \cdot v(F') - \text{far}(F')),\]

because \( 1 \leq \varepsilon d \) and \( 2\varepsilon \leq c \). Thus \( F' \) is \((c, d, k)\)-shiny. But then \( v(F') > v(F) \), contrary to the maximality of \( v(F) \).

So we may assume that \( w \in W_2 \). Thus \( w \) is not a terminal of \( T \). Hence by definition of terminal, there exists a leaf \( v' \) of \( T \) in the peripheral piece for \( v \) such that \( \text{dist}(v', \text{root}(C)) \geq \text{dist}(w, \text{root}(C)) + 2 \).

Let \( T' = T \setminus \{v'\} + vw \) and \( F' = (F \setminus V(T)) \cup T' \). Then \( F' \) is a \((K, d)\)-small rooted \((k, p)\)-shrubbery since \( V(F') \subseteq A \) and \( v(T') = v(T) \leq k \). By the choice of \( F_3 \), \( T' \) is \((\varepsilon, d)\)-mate-free. Thus \( F' \) is \((\varepsilon, d)\)-mate-free. Note that

\[ v(F') = v(F). \]

Furthermore,

\[ \text{far}(F') \leq \text{far}(F) - 1, \]

since

\[ \text{dist}(v, \text{root}(T')) = \text{dist}(w, \text{root}(T')) + 1 \leq \text{dist}(v', \text{root}(T')) - 1. \]

Note that

\[ e(G') - e(G/F') \leq |N_{G'}(v) \cap N_{G'}(v_T)| + 1 \leq \varepsilon d + 1. \]

Recall that \( e(G) - e(G') \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) \) since \( F \) is \((c, d, k)\)-shiny. Therefore

\[
e(G) - e(G/F') = (e(G) - e(G')) + (e(G') - e(G/F')) \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + \varepsilon d + 1 \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + 2\varepsilon d \leq cd \cdot (2k \cdot v(F') - \text{far}(F')) = cd \cdot (2k \cdot v(F') + \text{far}(F')),\]

because \( 1 \leq \varepsilon d \) and \( 2\varepsilon \leq c \). Thus \( F' \) is \((c, d, k)\)-shiny. But then \( v(F') = v(F) \) and \( \text{far}(F') < \text{far}(F) \), contrary to the minimality of the farness of \( F \).

\[ \square \]

**Claim 6.3.4.** If \( v \in A' \), then \( v \) has at most \((2k + 1)\varepsilon d\) neighbors in \( A' \) in \( G \).

**Proof.** Suppose not. Hence \( v \) has at least \((2k + 1)\varepsilon d\) neighbors in \( A' \) in \( G \). Then \( v \) has at least \((2k + 1)\varepsilon d - \lfloor \varepsilon d \rfloor \geq 2k \varepsilon d\) neighbors in \( A' \) that are not \((\varepsilon, d)\)-mates of \( v \) in \( G \). Let \( r = \lceil 2k \varepsilon d \rceil \) and \( v_1, \ldots, v_r \in A' \) be neighbors of \( v \) such that \( v \) and \( v_i \) are not \((\varepsilon, d)\)-mates in \( G \) for all \( i \in [r] \). Then for each \( i \in [r], v_i \) has fewer than \( \varepsilon d \) many \((\varepsilon, d)\)-mates in \( G \), in particular, \( v_i \) has fewer than \( \varepsilon d \) many \((\varepsilon, d)\)-mates in \( \{v_1, v_2, \ldots, v_r\} \) in \( G \). Since \( r = \lceil 2k \varepsilon d \rceil \),
it follows that there exists a subset \( I \) of \([r]\) with \(|I| = k - 1\) such that no two vertices of \( \{v_i : i \in I\} \) are \((\varepsilon, d)\)-mates in \( G \). We may assume without loss of generality that \( I = [k - 1] \).

Let \( T^* \) denote the rooted star with edge set \( \{vv_i : i \in [k - 1]\} \) where \( v \) is the root. Let \( F^* = F \cup T^* \). Then \( F^* \) is a \((K, \varepsilon, d)\)-mate-free rooted \((k, p)\)-shrubbery in \( G \) because \( V(F^*) \subseteq A \) and \( T^* \) is \((\varepsilon, d)\)-mate-free in \( G \).

Note that
\[
v(F^*) = v(F) + k.
\]
Furthermore,
\[
\text{far}(F^*) = \text{far}(F) + \text{far}(T^*) \leq \text{far}(F) + k,
\]
since \( \text{far}(T^*) = k - 1 \leq k \). Since \( G' \) is \((Kk, \varepsilon, d)\)-unmated and each vertex of \( T^* \) is \((Kk, d)\)-small in \( G' \), we have
\[
e(G'') - e(G/F^*) \leq e(G[V(T^*)]) + \left( \frac{k}{2} \right) \varepsilon d \leq \left( \frac{k}{2} \right) (\varepsilon d + 1) \leq k^2 \varepsilon d,
\]
since \( \varepsilon d \geq 1 \) and \( k \geq 2 \). It follows that
\[
e(G) - e(G/F^*) = (e(G) - e(G')) + (e(G') - e(G/F^*))
\leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + k^2 \varepsilon d
\leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + cd k^2
= cd \cdot (2k \cdot v(F) - \text{far}(F^*) + k^2)
\leq cd \cdot (2k \cdot v(F^*) - \text{far}(F^*))
\]
because \( c \geq \varepsilon \), \( \text{far}(F^*) \leq \text{far}(F) + k \) and \( v(F^*) = v(F) + k \). Hence \( F^* \) is \((c, d, k)\)-shiny. But then \( v(F^*) > v(F) \), contrary to the maximality of \( v(F) \).

\[\Box\]

**Claim 6.3.5.** If \( v \in A' \), then \( v \) has at most \( 4k^2 \varepsilon d \) neighbors in \( W_1 \) in \( G \).

**Proof.** Suppose not. Thus there exist at least \( 4k \varepsilon d \) distinct components of \( F_1 \) each containing a neighbor of \( v \) that is in \( W_1 \). Let \( r = \lceil 3k \varepsilon d \rceil \). Then there exist distinct components \( T_1, T_2, \ldots, T_r \) of \( F_1 \) such that for each \( i \in [r] \), \( T_i \) does not contain an \((\varepsilon, d)\)-mate of \( v \) in \( G \), and there exists \( v_i \in V(T_i) \cap W_1 \) such that \( vv_i \in E(G) \). This is possible because \( v \) has fewer than \( \varepsilon d \) many \((\varepsilon, d)\)-mates in \( G \). For each \( i \in [r] \), let \( e_i \) be a central edge for \( v_i \) in \( T_i \), let \( H_i \) be the peripheral piece for \( v_i \) and \( e_i \) in \( T_i \), and let \( T_i' = T_i \setminus V(H_i) \). Note that for each \( i \in [r] \), we have that \( v(H_i) \leq p \) since \( v_i \) is \( p \)-peripheral in \( T_i \) as \( v_i \in W_1 \).

For each \( S \subseteq [r] \) with \( S \neq \emptyset \), let \( T_S \) denote the rooted tree with edge set \( \bigcup_{i \in S} (E(H_i) \cup \{vv_i\}) \) and root \( v \). Let \( F_S = (F \setminus \bigcup_{i \in S} V(T_i)) \cup (\bigcup_{i \in S} T_i') \cup T_S \). Then \( \{v\} = V(F_S) \setminus V(F) \) and \( v(F_S) = v(F) + 1 \). Let \( S \subseteq [r] \) be chosen such that

(i) \( 2 \leq v(T_S) \leq k \), and

(ii) \( T_S \) is \((\varepsilon, d)\)-mate-free in \( G \), and

(iii) \( e(G') - e(G/F_S) \leq 2 \varepsilon d |S| \), and
Furthermore, note that $S$ exists because $S = \{i\}$ for any $i \in [r]$ satisfies the above conditions (i-iii). Note that $F_S$ is $(K, d)$-small and $(\varepsilon, d)$-mate-free because $V(F_S) \subseteq A$ and $T_S$ is $(\varepsilon, d)$-mate-free. Note that

$$v(F_S) = v(F) + 1.$$ 

Furthermore,

$$\text{far}(F_S) \leq \text{far}(F) + \text{far}(T_S) \leq \text{far}(F) + 2(k - 1),$$

where the last inequality follows since every vertex in $T_S$ has distance at most 2 in $T_2$ from the root $v$. Since $|S| \leq k - 1$, by assumption (iii), we have

$$e(G') - e(G/F_S) \leq 2\varepsilon d(k - 1) \leq cd$$

because $c = 2k\varepsilon$. Hence

$$e(G) - e(G/F_S) = (e(G) - e(G')) + (e(G') - e(G/F_S)) \leq cd \cdot (2k \cdot v(F) - \text{far}(F)) + cd \leq cd \cdot (2k \cdot v(F) - \text{far}(F) + 1) \leq cd \cdot (2k \cdot v(F_S) - \text{far}(F_S)).$$

Hence $F_S$ is $(c, d, k)$-shiny. Recall that $v(F_S) > v(F)$. By the choice of $F$, it follows that $F_S$ is not a $(k, p)$-shrubbery. Thus $v(T_S) \leq k - p$ and so $|S| \leq k - p$. Let

$$R = \{i \in [r] \setminus S : \text{ no vertex in } H_i \text{ has an } (\varepsilon, d)\text{-mate in } \cup_{j \in S} V(H_j) \text{ in } G\}.$$ 

Since $G$ is $(K, \varepsilon, d)$-unmate and $|\cup_{j \in S} V(H_j)| < k$, we have

$$|R| \geq r - |S| - |\cup_{j \in S} V(H_j)| \cdot \varepsilon d \geq 3k\varepsilon d - |S| - k\varepsilon d \geq 2k\varepsilon d - (k - 1) > k\varepsilon d.$$ 

Note that $G/F_S$ is a $k$-bounded minor of $G$. By Claim 6.3.2, $G/F_S$ is $(Kk, \varepsilon, d)$-unmate. Clearly, every vertex of $A \setminus V(F_S)$ is $(Kk, d)$-small in $G/F_S$. Similarly, every vertex corresponding to a component of $F_S$ is $(Kk, d)$-small in $G/F_S$. Let $v_{T_S}$ denote the vertex corresponding to $T_S$ in $G/F_S$. Then $v_{T_S}$ has at most $\varepsilon d$ many $(\varepsilon, d)$-mates in $G/F_S$ because $G/F_S$ is $(Kk, \varepsilon, d)$-unmate. For each $i \in R$, let $v_{T_i}$ be the vertex of $G/F_S$ corresponding to $T_i$. Since $|R| > k\varepsilon d$, there exists $q \in R$ such that $v_{T_q}$ and $v_{T_S}$ are not $(\varepsilon, d)$-mates in $G/F_S$.

Let $S' = S \cup \{q\}$. Then $|S'| = |S| + 1 \leq k - 1$ since $|S| \leq k - p \leq k - 2$ as $p \geq 2$. Moreover, $F_{S'}$ is $(\varepsilon, d)$-mate-free since $q \in R$. Let $F_q = F_S - e_q$. Then $F_{S'} = F_q + vv_q$ and $F_q$ is a proper spanning subgraph of $F_S$. Note that $H_q$ is a component of $F_q$. Let $v_{H_q}$ and $v_{T_q}$ be the vertices of $G/F_q$ corresponding to $H_q$ and $T_{q'}$, respectively. Since $v_{T_q}$ and $v_{T_S}$ are not $(\varepsilon, d)$-mates in $G/F_S$, it follows that

$$|N_{G/F_S}(v_{T_q}) \cap N_{G/F_S}(v_{T_S})| \leq \varepsilon d.$$ 

Note that every common neighbor of $v_{H_q}$ and $v_{T_S}$ in $G/F_q$ is a common neighbor of $v_{T_q}$ and $v_{T_S}$ in $G/F_S$ except possibly for $v_{T_q}$. Hence

$$|N_{G/F_q}(v_{H_q}) \cap N_{G/F_q}(v_{T_S})| \leq \varepsilon d + 1.$$ 

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By Proposition 4.6, $e(G/F_q) \geq e(G/F_S) + 1$ and

$$e(G/F_q) - e(G/F_{S'}) \leq |N_{G/F_q}(v_{H_q}) \cap N_{G/F_q}(v_{T_S})| + 1 \leq \varepsilon d + 2.$$  

It follows that

$$e(G/F_S) - e(G/F_{S'}) = (e(G/F_S) - e(G/F_q)) + (e(G/F_q) - e(G/F_{S'}))$$

$$\leq -1 + (\varepsilon d + 2) \leq 2\varepsilon d.$$  

By assumption (iii), $e(G') - e(G/F_S) \leq 2\varepsilon d|S|$. Thus

$$e(G') - e(G/F_{S'}) = (e(G') - e(G/F_S)) + (e(G/F_S) - e(G/F_{S'}))$$

$$\leq 2\varepsilon d(|S| + 1)$$

$$= 2\varepsilon d|S'|.$$  

But then $S'$ satisfies conditions (i-iii) with $v(T_{S'}) > v(T_S)$, contrary to the maximality of $S$. This completes the proof of Claim 6.3.5. 

Back to the main proof. By Claims 6.3.3, 6.3.4, and 6.3.5 if $v \in A'$, then has at most

$$|N(v) \setminus (B \cup C)| \leq 2(k - 1)\varepsilon d + (2k + 1)\varepsilon d + 4k^2\varepsilon d \leq 8k^2\varepsilon d.$$  

Since $\delta(G) \geq d$, it follows every vertex in $A'$ has at least $(1 - 8k^2\varepsilon)d$ neighbors in $B \cup C$.  

Now first suppose that $|A'| > \ell|B \cup C|$. Then $H = G(X, Y)$ satisfies (ii) where $X = A'$ and $Y = B \cup C$, a contradiction.  

So we may assume that

$$|A'| \leq \ell|B \cup C| = \ell(|B| + |C|) \leq \ell \cdot \left(\frac{1}{k^{4/3}} + \frac{1}{p}\right) \cdot v(G) \leq \frac{2\ell}{p} \cdot v(G),$$

since $p = k^{2/3} \leq k^{4/3}$. Recall that $G' = G/E(F)$. Let $G_1 = G \setminus A$ and let $G'_1 = G_1/E(F) = G' \setminus A$. Thus

$$v(G'_1) \leq |B| + |C(F)| \leq \frac{1}{k^{4/3}} \cdot v(G) + \frac{1}{k - p} \cdot v(G) \leq \frac{1}{k - 2p} v(G),$$

as

$$\frac{1}{k - 2p} - \frac{1}{k - p} = \frac{p}{(k - p)(k - 2p)} \geq \frac{p}{k^2} = \frac{1}{k^{4/3}}.$$  

Since $F$ is $(c, d, k)$-shiny, we have by definition that

$$e(G) - e(G') \leq c \cdot d \cdot (2k \cdot v(F) - \text{far}(F)) \leq c \cdot d \cdot 2k \cdot v(G),$$

where we used the facts that $\text{far}(F) \geq 0$ and $v(F) \leq v(G)$. Yet every edge of $G_1$ that is not an edge of $G'_1$ is an edge of $G$ that is not an edge of $G'$. Thus

$$e(G_1) - e(G'_1) \leq e(G) - e(G'),$$

and hence

$$e(G'_1) \geq e(G_1) - c \cdot d \cdot 2k \cdot v(G).$$

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Let $G_2 = G(A', V(G) \setminus (B \cup C))$. By Claims 6.3.3, 6.3.4, and 6.3.5 we have that
\[ e(G_2) \leq 8k^2 \varepsilon |A'| \leq 8k^2 \varepsilon \cdot v(G). \]

Let $G_3 = G[A' \cup B \cup C]$. Since every proper subgraph of $G$ has density smaller than $G$, we find that $d(G_3) < d$ and hence
\[ e(G_3) < |A' \cup B \cup C| \cdot d = (|A'| + |B| + |C|) \cdot d \leq (\ell + 1)(|B| + |C|) \cdot d \leq \frac{2(\ell + 1)}{p} \cdot d \cdot v(G). \]

Thus
\[ e(G_1) \geq e(G) - e(G_2) - e(G_3) \geq \left( 1 - 8k^2 \varepsilon - \frac{2(\ell + 1)}{p} \right) \cdot d \cdot v(G). \]

But then
\[ e(G_1') \geq e(G_1) - c \cdot d \cdot v(G) \geq \left( 1 - c \cdot 2k - 8k^2 \varepsilon - \frac{2(\ell + 1)}{p} \right) \cdot d \cdot v(G), \]

and hence
\[ d(G_1') \geq (k - 2p) \left( 1 - c \cdot 2k - 8k^2 \varepsilon - \frac{2(\ell + 1)}{p} \right) \cdot d. \]
\[ \geq k \cdot \left( 1 - \frac{2p}{k} \right) \left( 1 - 4k^2 \varepsilon - 8k^2 \varepsilon - \frac{4\ell}{p} \right) \cdot d, \]

where we used the fact that $c = 2\varepsilon k$. Substituting $p = k^{2/3}$, $\ell = k^{1/3}$ and using the fact that $\varepsilon \leq \frac{1}{32k^{1/3}}$, we find that
\[ d(G_1') \geq k \cdot \left( 1 - \frac{2}{k^{1/3}} \right) \left( 1 - 12k^2 \cdot \frac{1}{32k^{7/3}} - \frac{4}{k^{1/3}} \right) \cdot d, \]
\[ \geq k \cdot \left( 1 - \frac{2}{k^{1/3}} \right) \left( 1 - \frac{5}{k^{1/3}} \right) \cdot d, \]
\[ \geq k \cdot \left( 1 - \frac{7}{k^{1/3}} \right) \cdot d. \]

Since $G_1'$ is a $k$-bounded minor of $G$, we have that (iii) holds for $G$, a contradiction. \qed

### 7 Concluding Remarks

The main obstacle now to improving the bound in Theorem 1.6 using this approach is remains improving the function $g(s) = O((1 + \log s)^{24})$ in Theorem 1.7. The author has made a reasonable attempt to optimize the value of 24. Perhaps a more careful interplay between the parameters could improve this value further. Further bottlenecks beyond improving $g(s)$ exists for certain better bounds. For a bound of $O(t(\log \log t)^2)$, there is a bottleneck caused by the division into the inseparable and separable cases. It is conceivable that the two cases can be combined to reduce this to a bound of $O(t \log \log t)$. Beyond that point, Hadwiger’s conjecture seems to become quite difficult as there would then be two new distinct bottlenecks: the $O(t \log \log t)$ factor from the separable case; the $O(t \log \log t)$ bound for small graphs which is used in the inseparable case.
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