Algebro-Geometric Solutions for 
Kadomtsev-Petviashvili Hierarchy

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Abstract

Based on the idea of symmetric constraint, we apply the Gesztesy-Holden’s method to derive explicit representations of the Baker-Akhiezer function $\psi_1$ of the KP hierarchy, from which we provide theta function representations of algebro-geometric solutions for the whole Kadomtsev-Petviashvili (KP) hierarchy. This provides a approach to obtain some special subclasses of algebro-geometric solutions for the KP hierarchy and other high dimensional hierarchy of equations.

Key words: KP hierarchy, symmetric constraint, Baker-Akhiezer function, Gesztesy-Holden’s method, algebro-geometric solutions.

1 Introduction

In the past few decades there have been many remarkable developments in the theory of integrable nonlinear partial differential equations [1], [4], [13], [58]. One of the most widely studied integrable equation in 2+1 dimensions is the Kadomtsev-Petviashvili (KP) equation [11], [29],

$$u_t = \frac{1}{4} u_{xxx} + 3 u u_x + \frac{3}{4} \partial_x^{-1} u_{yy},$$

(1.1)

which also has been generalized in various forms [26], [42], [58]. There are also different approaches to the description of the algebraical and geometrical aspects of the KP equation. One is the theory of bilocal recursion operators [21], [34], [54], where the KP appears as a member of a hierarchy of commuting flows. Another is the Sato’s theory [10], [11], [29], [55], which

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is based on the treatment of partial differential KP equations as dynamical systems on the infinite-dimensional algebra of pseudo-differential operators [2]. By introducing an infinite set of time variables one can also treat the integrable equations as flows on infinite-dimensional Grassmannian manifolds [29]. Moreover, this theory reveals deep interrelations between the Hamiltonian structures of the KP hierarchy and two-dimensional conformal field theory as well as $W_{1+\infty}$ algebras [57]. In the scheme of the Sato’s theory, the KP equation is considered as the simplest member of a hierarchy of equations which can be brought to bilinear form and solved by the $\tau$-function approach.

The problem of constructing the algebro-geometric solutions is one of the most challenging problems of the theory of integrable systems, and many mathematicians and physicists spent much efforts to obtain the algebro-geometric solutions for almost all equations that are known to be integrable. This kinds of solutions were originally studied on the KdV equation based on the inverse spectral theory and algebro-geometric method developed by pioneers such as Novikov [14], Dubrovin et al. [15], [16], Its & Matveev [27], [28], Lax [21], and McKean & van Moerbeke [46] for $1+1$ systems, and extended by Krichever in 1976 for $2+1$ systems like KP [38], [39] in the late 1970s. Later this theory has been developed to the whole hierarchies of nonlinear integrable equations by Gesztesy, Holden et al. using polynomial recursion method [12], [23], [24], [25]. Another breakthrough in this area was made by Mumford in early 1980s, who observed that integrable equations like KdV, KP or sine-Gordon, are hidden in Fay’s trisecant formula. Mumford’s approach are based on degenerated versions of Fay’s identity, which reveals the relations between algebro-geometric solutions of integrable equations and a purely algebro-geometric identity [30], [32], [33]. Owing to their work, the theory of theta functions and Abelian varieties developed in the domain of complex analysis and algebraic geometry can be directly linked with the theory of integrable systems. A detailed introduction to this aspect can be found in the survey article [45].

Before turning to the main text, it seems appropriate to review some related literature about KP equation in detail as usual. An important development of the algebro-geometric method was the passage from $1+1$ systems to the integration of $2+1$ KP-like systems, realized by Krichever in 1976 [38], who constructed algebro-geometric solutions of the KP equation on a basis of a purely algebraic formulation of algebro-geometric approach. A complete description of smooth, real, algebro-geometric solutions of the KP equation was obtained in [17] by separate nontrivial work. The algebro-geometric spectral theory of the stationary periodic 2D Schrödinger operator was de-
veloped in the works [16], [49]. In ref. [47], Miller found 2+1-dimensional dynamics of the equations of the KP hierarchy can be approximated by the simpler 1+1-dimensional dynamics of the equations of a sequence of larger and larger vector nonlinear Schrödinger hierarchies, which could be used to develop powerful numerical methods for solving the KP equations. A explicit theta function solution of the KP equation was derived with the help of separation technique by Cao et al. [6]. Matveev & Salle found larger families of solutions to the KP equation, also expressed by means of the Riemann theta functions, by applying the dressing formulae [43], [44].

The purpose of this paper is to construct the algebro-geometrical solutions for the whole KP hierarchy by extending the Gesztesy-Holden’s method of [25] to 2+1 dimensional case based on the idea of symmetry constraint. The KP hierarchy can be consistently constrained in many different ways to yield hierarchies of equations in 1 + 1 independent variables [7], [9], [20], [37], [47], [51], [53], [54], [56]. The idea of studying these constraints comes from the reduction of 1+1-dimensional integrable soliton equations to finite-dimensional integrable equations [3], [19], [22]. A well-known example is the restriction of the KdV flow to the pure multisoliton submanifold [22], where we impose the constraint $u = \sum_{i=1}^{N} c_i \Psi_i^2$ on the KdV potential $u$ and eigenfunctions $\Psi_i$. This leads to the finite-dimensional integrable system

$$\Psi_{i,xx} = \lambda^2 \Psi_i + \sum_{k=1}^{N} c_k \Psi_k^2 \Psi_i, i = 1, \ldots, N.$$ 

For the KP equations, there are two kinds of constraints. The one is the so-called $k$-reductions for which the operator $L^k$ is forced to become a purely differential operator leads to hierarchies of 1 + 1-dimensional equations [11], [29]. These reductions are associated with constraints imposed separately on the potentials $u$, and eigenfunctions $\psi$ and $\psi^\ast$. Meanwhile, it has been shown that the KP hierarchy also admits another type of constraints relating the potentials $u$, with the eigenfunctions $\psi$ and $\psi^\ast$. One of them is the symmetry constraints for 2+1 dimensional soliton equations, which have been discussed the first time in the papers [7], [35]. In ref. [35], Konopelchenko et al. proved that the auxiliary linear problems

$$\psi_{t_2} = \psi_{xx} + 2u\psi,$$

$$\psi_{t_3} = \psi_{xxx} + 3u\psi_x + \frac{3}{2} u_x \psi + \frac{3}{2} \partial_x^{-1} u_{t_2} \psi, \quad (1.2)$$
and its adjoint
\[ \psi^{*}_{t_2} = -\psi^{*}_{xx} - 2u\psi^{*}, \]
\[ \psi^{*}_{t_3} = \psi^{*}_{xxx} + 3u\psi^{*}_x + \frac{3}{2}u_x \psi^{*} - \frac{3}{2}\partial_x^{-1}u_{t_2} \psi^{*} \]  
(1.3)

which arise from the third flow and second flow of KP hierarchy are constrained to the first two nontrivial flows of the AKNS hierarchy by identifying the potential \( u \) of the KP equation to \( \psi \psi^{*} \). Further, it has been shown that imposing the symmetry constraint

\[ u_x = (\psi \psi^{*})_x \]
on the auxiliary linear problems of the KP hierarchy

\[ \psi_{t_m} = L_m^+ \psi, \quad \psi^{*}_{t_m} = -(L_m^+)^* \psi, \]  
(1.4)

leads to standard AKNS hierarchy [36]. Here \( L \) is the usual first-order formal pseudo-differential Lax operator defined in [22] and \((L_m^+)^*\) is the formal adjoint to the operator \( L_m^+ \), i.e. if \( L_m^+ = \sum v_j \partial_x^j \), then \((L_m^+)^* = \sum (-\partial_x)^j v_j\).

Further motivation for the new constraints lies in the methods of solving integrable equations in 1+1- or 2+1-dimensions by the nonlinearization of linear problems [5], [8], [59]. Thus, imposing some constraint on a 2+1 dimensional equation is a possible way to obtain a submanifold of solutions of the KP equations by solving two equations in 1+1 dimensions.

Since any non-singular algebraic curve can be used as spectral curve of KP equation, it is essential to consider some constraints about KP hierarchy if we really want to get a explicit form of Baker-Akhiezer function from this point of view. However, this way of consideration produces mixed results. On the one hand, the study of KP equations is restricted to only certain specific types of spectral curves, which narrows the classes of possible solutions. On the other hand, it makes problems more concrete, which can increase the expressions of solutions corresponding to this specific curve and allows us to have a unified way to study solutions of the whole KP hierarchy. Thus, considering the constraints of KP equations enables us to get larger families of algebro-geometric solutions in this sense, which is also our start point.

This paper is organized as follows. In section 2, we formulate some fundamental knowledge about Sato’s KP hierarchy, AKNS hierarchy and the relations between them in literature. Then a basic initial problem is introduced as solutions of a linear system which is defined by classical squared
basis functions. In section 3, we shall introduce hyperelliptic curves associated with KP hierarchy and study the dynamics of auxiliary spectral points \( \{\mu_j\}_{j=1}^n, \{\nu_j\}_{j=1}^n \) with respect to \( x, y, t_{r+1} \) and corresponding trace formula. In section 4, we present explicit representation for Baker-Akhiezer function \( \psi_1, \psi_2 \) and consider their analytic properties. In section 5, it will be shown the function \( \psi_1 \) derived in section 4 is not only connected to the Baker-Akhiezer function of the KP equation, but a quantity to define new 2+1 dimension systems possessing close relations with the KP equation. Moreover, we shall derive theta function representation for \( \psi_1, \psi_2, q, p \), and algebro-geometric solutions of the whole KP hierarchy.

The whole approach discussed in the present paper is a general one and gives similar results for other 2+1 dimensional and higher dimensional soliton equations, for instance, for the modified KP hierarchy and Davey-Stewartson equation, etc.

2 Sato KP hierarchy, AKNS hierarchy and basic initial value problem

In the Sato approach [III], the KP hierarchy is described by the isospectral deformations of the eigenvalue problem

\[
L\psi = \lambda \psi, \quad \lambda \in \mathbb{C},
\]

where the pseudodifferential operator \( L \) is given by

\[
L = \partial + \sum_{j=1}^{\infty} u_{j+1} \partial^{-j}, \quad \partial = \partial_x,
\]

and \( u_j \) are functions in infinitely many variables \( (t_1, t_2, \ldots) \) with \( t_1 = x, t_2 = y \). We denote by \( B_m \) the differential part of \( L^m \):

\[
B_m = (L^m)_+ = \sum_{j=0}^{m} b_{m,j} \partial^j.
\]
The coefficients $b_{m,j}$ in (2.3) can be uniquely determined by the coordinates $u_j$, and their $x$ derivatives. Explicitly,

\[ B_1 = \partial, \]
\[ B_2 = \partial^2 + 2u_2, \]
\[ B_3 = \partial^3 + 3u_2\partial + 3u_3 + 3u_{2,x}, \]
\[ B_4 = \partial^4 + 4u_2\partial^2 + (4u_3 + 6u_{2,x})\partial + 4u_4 + 6u_{3,x} \]
\[ + 4u_{2,xx} + 6u_2^2, \text{ etc.} \]

From the compatibility conditions of (2.1) and

\[ \phi_{t_m} = B_m \phi, \quad (2.4) \]

we have

\[ L_{t_m} = [B_m, L], \quad (2.5) \]

or equivalently,

\[ (B_m)_{t_n} - (B_n)_{t_m} = [B_n, B_m]. \quad (2.6) \]

The KP hierarchy is obtained from (2.5) or (2.6) for infinite coordinates \{\(u_j\)\}_{j=2}^{\infty}. For example, from (2.5), we derive

\[ u_{2,t_2} = 2u_{3,x} + u_{2,xx}, \quad (2.7) \]
\[ u_{3,t_2} = 2u_{4,x} + u_{3,xx} + 2u_2u_{2,x}, \quad (2.8) \]
\[ u_{4,t_2} = 2u_{5,x} + u_{4,xx} + 4u_{2,x}u_3 - 2u_2u_{2,xx}, \quad (2.9) \]
\[ \ldots \ldots \]
\[ u_{2,t_3} = 3u_{4,x} + 3u_{3,xx} + u_{2,xxx} + 6u_2u_{2,x}, \quad (2.10) \]
\[ u_{3,t_3} = 3u_{5,x} + 3u_{4,xx} + u_{3,xxx} + 6(u_2u_3)_x, \quad (2.11) \]
\[ \ldots \ldots \]
\[ u_{2,t_4} = 4u_{5,x} + 6u_{4,xx} + 4u_{3,xxx} + u_{2,xxxx} + 12(u_2u_3)_x \]
\[ + 6(u_2u_2)_x, \quad (2.12) \]
\[ \ldots \ldots \]

Eliminating $u_3$, $u_4$, $u_5$ from (2.7), (2.8), (2.9) and taking into account (2.10), (2.12), one obtains

\[ u_{t_3} = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}\partial^{-1}u_{yy} \quad (2.13) \]
\[ u_{t_4} = \frac{1}{2}u_{xxy} + 4uu_y + 2u_x\partial^{-1}u_y + \frac{1}{2}\partial^{-2}u_{yyy}. \quad (2.14) \]
where we denote by $u = u_2$. Equation (2.13) is just the KP equation and equation (2.14) is the first higher order flows of the KP hierarchy. Similarly, the KP and higher-order KP equation can also be derived from (2.6) with $n = 2, m = 3$ and $n = 2, m > 3$, respectively.

Now we recall some basic results about AKNS hierarchy. The ref. [25] provides two complexified versions of AKNS hierarchy, zero-curvature and matrix differential operator formalisms. In the following, we shall construct the standard AKNS hierarchy in a similar manner. Generally, the AKNS hierarchy is introduced by developing its zero-curvature formalism. To this end one defines the sequences of differential polynomials \{\hat{f}_\ell\}_{\ell \in \mathbb{N}_0}, \{\hat{g}_\ell\}_{\ell \in \mathbb{N}_0}, \text{ and } \{\hat{h}_\ell\}_{\ell \in \mathbb{N}_0}, recursively by

\begin{align*}
\hat{f}_0 &= -q, \quad \hat{g}_0 = -\frac{1}{2}, \quad \hat{h}_0 = p, \\
\hat{f}_{\ell+1} &= -\hat{f}_{\ell,x} + 2q\hat{g}_{\ell+1}, \quad \ell \in \mathbb{N}_0, \\
\hat{g}_{\ell+1,x} &= pf_{\ell} + qh_{\ell}, \quad \ell \in \mathbb{N}_0, \\
\hat{h}_{\ell+1} &= h_{\ell,x} - 2pg_{\ell+1}, \quad \ell \in \mathbb{N}_0.
\end{align*}

Here we emphasize that $q, p$ should be regarded as functions of infinitely many variables $(t_1, t_2, t_3, \ldots)$. Explicitly, one computes

\begin{align*}
f_0 &= -q, \\
f_1 &= q_x + c_1(-q), \\
f_2 &= -q_{xx} + 2pq^2 + c_1q_x + c_2(-q), \\
&\quad \ldots, \\
g_0 &= -\frac{1}{2}, \\
g_1 &= c_1(-\frac{1}{2}), \\
g_2 &= -pq + c_2(-\frac{1}{2}), \\
g_3 &= pq_x - pxq + c_1(-pq) + c_3(-\frac{1}{2}), \\
&\quad \ldots, \\
h_0 &= p, \\
h_1 &= px + c_1p, \\
h_2 &= px - 2p^2q + c_1px + c_2p, \quad \text{etc.},
\end{align*}

where \{c_\ell\}_{\ell=0}^\infty \subset \mathbb{C} are integration constants. By introducing the homogeneous coefficients \hat{f}_\ell, \hat{g}_\ell, \hat{h}_\ell by vanishing of the integration constants $c_k, k =
\[ f_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{f}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{g}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{h}_k, \quad c_0 = 1. \] (2.22)

Then one introduces
\[ U(z) = \begin{pmatrix} -z^2 & q \\ p & -z^2 \end{pmatrix}, \quad z \in \mathbb{C}, \]
\[ V_{n+1}(z) = \begin{pmatrix} -G_{n+1}(z) & F_n(z) \\ -H_n(z) & G_{n+1}(z) \end{pmatrix}, \quad n \in \mathbb{N}_0, \] (2.24)

where \( G_{n+1}, F_n, H_n \) are polynomials with respect to \( z \):
\[ G_{n+1}(z) = \sum_{\ell=0}^{n+1} g_{n+1-\ell} z^\ell, \]
\[ F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell, \]
\[ H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell. \] (2.25-2.27)

and corresponding homogeneous polynomials are defined by
\[ \hat{F}_0(z) = F_0(z) = -q, \]
\[ \hat{F}_\ell(z) = F_\ell(z)|_{c_k=0,k=1,\ldots,\ell} = \sum_{k=1}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell \in \mathbb{N}, \]
\[ \hat{G}_0(z) = G_0(z) = -1/2, \]
\[ \hat{G}_{\ell+1}(z) = G_{\ell+1}(z)|_{c_k=0,k=1,\ldots,\ell+1} = \sum_{k=1}^{\ell+1} \hat{f}_{\ell+1-k} z^k, \quad \ell \in \mathbb{N}, \]
\[ \hat{H}_0(z) = H_0(z) = p, \]
\[ \hat{H}_\ell(z) = H_\ell(z)|_{c_k=0,k=1,\ldots,\ell} = \sum_{k=1}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell \in \mathbb{N}. \] (2.28-2.30)
For fixed $n \in \mathbb{N}_0$, the stationary and time-dependent AKNS hierarchy are defined by demanding zero curvature equation

$$V_{n+1,x}(z) = [U(z), V_{n+1}(z)]$$  (2.31)

and

$$U_{t_{n+1}} - V_{n+1,x}(z) = [U(z), V_{n+1}(z)],$$  (2.32)

respectively. Equation (2.31) is equivalent to

$$0 = -V_{n+1,x} + [U(z), V_{n+1}(z)] = \begin{pmatrix} G_{n+1,x} - pF_n - qH_n & -F_n,x - zF_n + 2qG_{n+1} \\ H_n,x - zH_n - 2pG_{n+1} & -G_{n+1,x} + pF_n + qH_n \end{pmatrix}$$

and hence stationary AKNS hierarchy can be introduced as follows

$$s$-AKNS_{n+1}(q,p) = -\begin{pmatrix} h_{n+1}(q,p) \\ f_{n+1}(q,p) \end{pmatrix} = 0, \quad n \in \mathbb{N}_0.$$  (2.33)

Explicitly,

$$s$-AKNS_1(q,p) = \begin{pmatrix} -p_x + c_1(-p) \\ -q_x + c_1q \end{pmatrix} = 0,$$

$$s$-AKNS_2(q,p) = \begin{pmatrix} -p_{xx} + 2p^2q + c_1(-p_x) + c_2(-p) \\ q_{xx} - 2pq^2 + c_1(-q_x) + c_2q \end{pmatrix} = 0,$$

$$s$-AKNS_3(q,p) = \begin{pmatrix} -p_{xxx} + 6pp_xq + c_1(-p_{xx} + 2p^2q) + c_2(-p_x) + c_3(-p) \\ -q_{xxx} + 6pqq_x + c_1(q_{xx} - 2pq^2) + c_2(-q_x) + c_3q \end{pmatrix} = 0, \quad \text{etc.}$$

Similarly, from (2.32) it follows

$$0 = U_{t_{n+1}} - V_{n+1,x} + [U(z), V_{n+1}(z)] = \begin{pmatrix} 0 & q_{t_{n+1}} - F_n,x - zF_n + 2qG_{n+1} \\ p_{t_{n+1}} + H_n,x - zH_n - 2pG_{n+1} & 0 \end{pmatrix}$$

and hence time-dependent AKNS hierarchy

$$AKNS_{n+1}(q,p) = \begin{pmatrix} p_{t_{n+1}} - h_{n+1}(q,p) \\ q_{t_{n+1}} - f_{n+1}(q,p) \end{pmatrix} = 0, \quad n \in \mathbb{N}_0.$$  (2.36)
Explicitly,

\[
\text{AKNS}_1(q,p) = \left( \begin{array}{c} pt_1 - p_x + c_1(-p) \\ qt_1 - q_x + c_1q \end{array} \right) = 0,
\]

\[
\text{AKNS}_2(q,p) = \left( \begin{array}{c} pt_2 - p_{xx} + 2p^2q + c_1(-p_x) + c_2(-p) \\ qt_2 + q_{xx} - 2pq^2 + c_1(-q_x) + c_2q \end{array} \right) = 0,
\]

\[
\text{AKNS}_3(q,p) = \left( \begin{array}{c} pt_3 - p_{xxx} + 6pp_xq + c_1(-p_{xx} + 2p^2q) + c_2(-p_x) + c_3(-p) \\ qt_3 - q_{xxx} + 6pqq_x + c_1(q_{xx} - 2pq^2) + c_2(-q_x) + c_3q \end{array} \right) = 0, \quad \text{etc.}
\]

Next we turn to discuss the relations between KP and AKNS hierarchy. To this end we denote by

\[
\tilde{\text{AKNS}}_{n+1}(q,p) = \text{AKNS}_{n+1}(q,p) |_{c_\ell=0, \ell=1,...,n+1}, \quad n \in \mathbb{N}_0,
\]

the corresponding homogeneous AKNS equations.

**Theorem 2.1** (see \[53\] or \[9\]). Assume \(p, q\) is a compatible solution of the system

\[
\tilde{\text{AKNS}}_2(q,p) = 0, \quad \text{AKNS}_{r+1}(q,p) = 0, \quad r \geq 2.
\]

Then

\[
u(x,y,t_{r+1}) = -q(x, -y, (-1)^r t_{r+1}) p(x, -y, (-1)^r t_{r+1})
\]
gives a solution of the nth KP equation.

Given these preparations, we turn to study the KP equations. Now \(q, p\) are considered as functions of variables \(t_1, t_2, t_{r+1}\) with \(t_1 = x, t_2 = y\) and we shall start from the following auxiliary linear problem

\[
\psi_x(z) = U(z) \psi(z), \quad \psi_y(z) = \tilde{V}_2(z) \psi(z),
\]

\[
\psi_{t_{r+1}}(z) = \tilde{V}_{r+1}(z) \psi(z), \quad z \in \mathbb{C}, \quad r \geq 2,
\]

where \(\tilde{V}_k(z) = V_k(z)|_{c_\ell=0, \ell=1,...,k},\) and \(\psi(z) = (\psi_1(z, x, y, t_{r+1}), \psi_2(z, x, y, t_{r+1}))^T.\)

Let

\[
\psi^{\pm}(z) = (\psi_1^{\pm}(z, x, y, t_{r+1}), \psi_2^{\pm}(z, x, y, t_{r+1}))
\]

\footnote{One can also start from the following linear problems

\[
\psi_x(z) = U(z) \psi(z), \quad \psi_{t_{m+1}}(z) = \tilde{V}_{m+1}(z) \psi(z), \quad z \in \mathbb{C}, \quad m = 2, 3, \ldots
\]

where \(q, p\) are considered as functions of \(x, t_2, t_3, \ldots\) and obtain similar results.}
be two fundamental solutions of linear system (2.43). Then we can define
three squared basis functions \( G, F, H \) by
\[
G(z) = \frac{1}{2} (\psi^+(z)\psi^-(z) + \psi^-(z)\psi^+(z)),
\]
\[
F(z) = \psi^+(z)\psi^-(z),
\]
\[
H(z) = \psi^+(z)\psi^-(z).
\]
(4.44)

Using (2.23), (2.24) and (2.43), one finds \( G, F, \) and \( H \), satisfy the following linear system
\[
G_x = pF + qH,
\]
\[
G_y = \hat{F}_1H - \hat{H}_1F,
\]
(2.45)
\[
G_{tr+1} = \hat{F}_rH - \hat{H}_rF,
\]
(2.46)
\[
F_x = 2qG - zF,
\]
(2.47)
\[
F_y = 2\hat{F}_1G - 2\hat{G}_2F,
\]
(2.48)
\[
F_{tr+1} = 2\hat{F}_rG - 2\hat{G}_{r+1}F,
\]
(2.49)
\[
H_x = 2pG + zH,
\]
(2.50)
\[
H_y = 2\hat{G}_2H - 2\hat{H}_1G,
\]
(2.51)
\[
H_{tr+1} = 2\hat{G}_{r+1}H - 2\hat{H}_rG.
\]
(2.52)

For fixed \( r \), solutions of linear system (2.45)-(2.53) are connected with the following basic initial value problem of AKNS system (2.55).

**Theorem 2.1.** Assume \( q, p \in C^\infty(\mathbb{R}^{2+1}) \). Moreover, suppose \( q, p \) is a solution satisfying (2.40), (2.41). Then the collection of polynomials
\[
\{(G, F, H) | (G_{j+1}, F_j, H_j), j \in \mathbb{N}_0\}
\]
gives a sequence of special solutions for (2.45)-(2.53). In particular, \((G_{n+1}, F_n, H_n), n \in \mathbb{N}_0\), corresponds to solutions of the following initial problem
\[
\begin{aligned}
q_y - \hat{f}_2 = 0, \\
p_y - \hat{h}_2 = 0, \\
q_{tr+1} - \hat{f}_{r+1} = 0, \\
p_{tr+1} - \hat{h}_{r+1} = 0,
\end{aligned}
\]
\[
-f_{n+1} = 0, \\
-h_{n+1} = 0, \quad r \geq 2.
\]
(2.55)
Proof. From (2.31) and (2.32) we know (2.55) is equivalent to the following zero curvature representation
\[
\frac{\partial}{\partial x} V_j(z) - \frac{\partial}{\partial t} U(z) = [U(z), V_j(z)], \quad j = 2, r + 1, n + 1,
\] (2.56)
introducing \(\frac{\partial}{\partial t_{n+1}} = 0\). Thus, it suffices to show
\[
[V_{j+1}(z) - \frac{\partial}{\partial t_j}, V_{k+1}(z) - \frac{\partial}{\partial t_k}] = 0, \quad j, k = 2, r + 1, n + 1.
\] (2.57)
and we proceed to prove this as follows. Define 2 × 2 matrix-valued differential expression
\[
L = \frac{1}{2} \left( \begin{array}{cc} \frac{d}{dx} - q & q \\ p & \frac{d}{dx} \end{array} \right),
\] (2.58)
\[
P_{n+1} = \sum_{j=0}^{n+1} \left( \begin{array}{cc} -g_{n+1-\ell} & f_{n-\ell} \\ -h_{n-\ell} & g_{n+1-\ell} \end{array} \right) L^{\ell}, \quad n \in \mathbb{N}_0, \quad f_{-1} = h_{-1} = 0.
\] (2.59)
Then it is not difficult to verify that (2.56) is equivalent to
\[
[P_{j+1} - \frac{\partial}{\partial t_j}, L] = 0, \quad j = 2, r + 1, n + 1.
\]
By Corollary 2 of Theorem 4.2 in [40] it follows
\[
[P_{j+1} - \frac{\partial}{\partial t_j}, P_{k+1} - \frac{\partial}{\partial t_k}] = 0, \quad j, k = 2, r + 1, n + 1.
\]
and hence we obtain
\[
[V_{j+1}(z) - \frac{\partial}{\partial t_j}, V_{k+1}(z) - \frac{\partial}{\partial t_k}] = [P_{j+1} - \frac{\partial}{\partial t_j}, P_{k+1} - \frac{\partial}{\partial t_k}]|_{\text{ker}(M - z)} = 0, \quad j, k = 1, \ldots, m,
\] (2.60)
where
\[
\text{ker}(M - z) = \{ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \mathbb{R}^{s+1} \rightarrow \mathbb{C}^2 | (M - z)\Psi = 0 \}, \quad z \in \mathbb{C}.
\]
This completes the proof. \(\square\)

Remark 2.2. This proposition can also be proved by introducing a fundamental meromorphic function \(\phi\) on Riemann surface \(X\), where \(q, p\) should be considered as functions of \(x, y, t_{r+1}\) (see [22]).
3 Spectral curve, Dubrovin-type equations and trace formula

In what follows, we shall introduce the spectral curve associated with KP hierarchy. To emphasize the difference between different solutions in (2.54), we add a subscript in each of $G, F, H$, that is, $(G_{n+1}, F_n, H_n), \ n \in \mathbb{N}$.

Then using (2.45)-(2.53) and theorem 2.1, we have

$$(G_{n+1} - F_n H_n)^t_j = 0, \ n \in \mathbb{N}, \ j = 1, 2, t_{r+1},$$

and hence $G_{n+1} - F_n H_n$ is $x, y, t_{r+1}$-independent implying

$$(G_{n+1} - F_n H_n) = R_{2n+2}, \ n \in \mathbb{N},$$

where integration constant $R_{2n+2}$ is a polynomial of degree $2n + 2$. Let

$$R_{2n+2}(z) = \sum_{j=0}^{2n+2-j} (-1)^j s_j z^{2n+2-j},$$

$s_0 = 1/4, \ s_j \in \mathbb{C}, j = 1, \ldots, 2n$,

and hyperelliptic curve associated with the $r$th KP equation is then introduced as follows:

$$X : \mathcal{P}(z, y) = y^2 - 4R_{2n+2}(z) = y^2 - 4G_{n+1}^2 + 4F_n H_n = 0.$$

The curve $X$ is compactified by joining two points $P_{\infty\pm}$ at infinity but for notational simplicity the compactification is also denoted by $X$. Points $P$ on $X \setminus \{P_{\infty\pm}\}$ are denote by pairs $(z, y)$, where $y(\cdot)$ is the meromorphic function on $X$ satisfying $\mathcal{P}(z, y) = 0$. The complex structure on $X$ is then defined in the usual way. Hence, $X$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n$ in a standard manner. Moreover, we denote the upper and lower sheets $\Pi_\pm$ by

$$\Pi_\pm = \{(z, \pm 2\sqrt{R_{2n+2}(z)}) \in X| z \in \Pi\},$$

where $\Pi$ denotes the cut plane $\mathbb{C} \setminus \mathcal{C}$ and $\mathcal{C}$ is the union of $n$ nonintersecting cuts joining two different branches of $\sqrt{R_{2n+2}(z)}$. The holomorphic sheet exchange map on $X$ is defined by

$$* : \ X \to X,$$

$$P = (z, 2\sqrt{R_{2n+2}(z)}) \mapsto P^* = (z, -2\sqrt{R_{2n+2}(z)}),$$

$$P_{\infty\pm} \mapsto P_{\infty\mp}^* = P_{\infty\mp}.$$
Moreover, positive divisors on $X$ of degree $n$ are denoted by

$$
\mathcal{D}_{P_1, \ldots, P_r} : \begin{cases} 
X \to \mathbb{N}_0, \\
P \to \mathcal{D}_{P_1, \ldots, P_n}(P) = \begin{cases} 
k & \text{if } P \text{ occurs } k \text{ times in } \{P_1, \ldots, P_n\}, \\
0 & \text{if } P \notin \{P_1, \ldots, P_n\}.
\end{cases}
\end{cases}
$$

(3.5)

In particular, the divisor $(\phi(\cdot))$ of a meromorphic function $\phi(\cdot)$ on $X$ is defined by

$$
(\phi(\cdot)) : X \to \mathbb{Z}, \quad P \mapsto \omega_{\phi}(P),
$$

(3.6)

where $\omega_{\phi}(P) = m_0 \in \mathbb{Z}$ if $(\phi \circ \zeta_P^{-1})(\zeta) = \sum_{n=m_0}^{\infty} c_n(P)\zeta^n$ for some $m_0 \in \mathbb{Z}$ by using a chart $(U_P, \zeta_P)$ near $P \in X$. Finally, we introduce symmetric functions of $x_1, \ldots, x_n$

$$
\begin{align*}
\Psi_0(x) &= 1, \quad \Psi_1(x) = \sum_{j=1}^{n} x_j, \quad \Psi_2(x) = \sum_{j,k=1, j<k}^{n} x_j x_k, \\
\Psi_3(x) &= \sum_{j,k,l=1, j<k<l}^{n} x_j x_k x_l, \quad \text{etc.}
\end{align*}
$$

(3.7)

where

$$
x = (x_1, \ldots, x_n).
$$

Now we turn to the parameter representations of $G_{n+1}, F_n, H_n$, which are described by the evolution of auxiliary spectrum points $\mu_j(x, y, t_{r+1}), \nu_j(x, y, t_{r+1}), j = 1, \ldots, n$. This procedure is standard, which is similar with 1+1 dimensional case.

**Theorem 3.1.** Solutions of (2.45)-(2.53) can also be expressed as

$$
\begin{align*}
G_{n+1} &= \sum_{j=0}^{n+1} (-1)^j \hat{g}_j(x, y, t_{r+1}) z^{n+1-j}, \\
F_n &= -q(x, y, t_{r+1}) \prod_{j=1}^{n} (z - \mu_j(x, y, t_{r+1})), \\
H_n &= p(x, y, t_{r+1}) \prod_{j=1}^{n} (z - \nu_j(x, y, t_{r+1})).
\end{align*}
$$

(3.8)
where
\[ \dot{g}_0 = 1/2, \dot{g}_1 = s_1, \]
\[ \dot{g}_2 = -pq - s_1^2 + s_2, \]
\[ \dot{g}_j = -\sum_{\nu=1}^{j-1} \dot{g}_j \dot{g}_{j-\nu} - pq \sum_{\alpha+\beta=j} \Psi_{\alpha}(\mu)\Psi_{\beta}(\mu) + s_j, \quad s = 3, \ldots, n+1, \]
and \( \{\mu_j(x,y,t_{r+1})\}_{j=1}^n \), \( \{\nu_j(x,y,t_{r+1})\}_{j=1}^n \) are \( n \) roots of \( \mathcal{F}_n, \mathcal{H}_n \), respectively. In particular, if \( \{\mu_j(x,y,t_{r+1})\}_{j=1}^n \) are mutually distinct and finite, then they satisfy the Dubrovin-type equations
\[ \mu_{j,x} = -2\sqrt{\mathcal{H}_2n+2(\mu_j)} \frac{1}{\prod_{k\neq j}(\mu_j - \mu_k)}, \]
\[ \mu_{j,y} = -2\tilde{F}_1(\mu_j)\sqrt{\mathcal{H}_2n+2(\mu_j)} \frac{1}{q \prod_{k\neq j}(\mu_j - \mu_k)}, \]
\[ \mu_{j,t_{r+1}} = -2\tilde{F}_2(\mu_j)\sqrt{\mathcal{H}_2n+2(\mu_j)} \frac{1}{p \prod_{k\neq j}(\mu_j - \mu_k)}, \]
and similar statement is also true for \( \{\nu_j(x,y,t_{r+1})\}_{j=1}^n \), where (3.10)-(3.12) change to
\[ \nu_{j,x} = -2\sqrt{\mathcal{H}_2n+2(\nu_j)} \frac{1}{\prod_{k\neq j}(\nu_j - \nu_k)}, \]
\[ \nu_{j,y} = -2\tilde{H}_1(\mu_j)\sqrt{\mathcal{H}_2n+2(\mu_j)} \frac{1}{p \prod_{k\neq j}(\mu_j - \mu_k)}, \]
\[ \nu_{j,t_{r+1}} = -2\tilde{H}_2(\mu_j)\sqrt{\mathcal{H}_2n+2(\mu_j)} \frac{1}{p \prod_{k\neq j}(\mu_j - \mu_k)}. \]
Finally, \( q, p \) and \( \mu_j, \nu_j \) are connected by the following trace formula
\[ \sum_{j=1}^n \mu_j = \frac{q_x}{q} - 2s_1, \quad \sum_{j=1}^n \nu_j = -\frac{p_x}{p} + 2s_1. \]
Proof. First, insertion of (3.8) into (3.2), (3.3) and a comparison powers of \( z \) yields (3.9). Then by (2.48), (2.49), (2.50), (2.51), (2.52) and (2.53), taking into account (3.4), (3.8), one derives (3.10)-(3.15). Moreover, combining (2.25) with (3.9) yields relations
\[ \dot{g}_j = (-1)^j g_j, \quad c_1 = -2s_1, \quad c_2 = 2(s_2 - s_1^2), \]
\[ c_3 = 4s_1(s_2 - s_1^2) - 2s_3, \quad \text{etc.} \]
and formula (3.16) is the direct result of (3.8), (3.17) and theorem 2.1.

4 Baker-Akhiezer function

Baker-Akhiezer function plays a very important role in finite gap integration of soliton equations and it permits us to obtain the Riemann theta function representation for solutions of a given equation and there are numerous articles have been devoted to this subject \[4\], \[31\], \[12\], \[24\]. In this section, we shall construct the explicit form of "Baker-Akhiezer function" associated with the \(r\)th KP equation, which consists of spectral parameter \(z\), potentials \(q,p\), and then study its analytic properties. Moreover, we will find the conservation relations of soliton equations play a key role in the construction of Baker-Akhiezer function, which reflects symmetry is the intrinsic character of classical integrable system.

To find explicit form of Baker-Akhiezer function

\[
\psi(P) = (\psi_1(P,x,y,t_{r+1}),\psi_2(P,x,y,t_{r+1})), \ P \in X,
\]

which satisfies

\[
\begin{align*}
\psi_x(P) &= U(z)\psi(P), \ 
\psi_y(P) &= \hat{V}_2(z)\psi(P), \ 
\psi_{t_{r+1}}(P) &= \hat{V}_{r+1}(z)\psi(P), \\
\psi_1(P,x_0,y_0,t_{r+1},0) &= 1, \ (P,x_0,y_0,t_{r+1},0) \in X \times \mathbb{R}^3,
\end{align*}
\]

we need some preparations.

Lemma 4.1. Suppose \(q,p \in C^\infty(\mathbb{R}^{2+1})\) and \(z \in \mathbb{C}\). Then \(G_{n+1}, F_n, H_n\) and basic fundamental solutions \((\psi_1^\pm, \psi_2^\pm)\) of linear system (4.1) have the following algebraic relation:

\[
(\psi_1^+(z)\psi_2^-(z) - \psi_1^-(z)\psi_2^+(z))^2 = 4R_{2n+2}^2(z). \tag{4.2}
\]

If we take \(\psi_1^+(z)\psi_2^-(z) - \psi_1^-(z)\psi_2^+(z) = 2\sqrt{R_{2n+2}(z)},\) then

\[
\begin{align*}
\psi_1^+(z)\psi_2^-(z) &= G_{n+1}(z) + \sqrt{R_{2n+2}(z)}, \tag{4.3} \\
\psi_1^-(z)\psi_2^+(z) &= G_{n+1}(z) - \sqrt{R_{2n+2}(z)}, \tag{4.4} \\
\frac{\psi_2^+(z)}{\psi_1^+(z)} &= \frac{H_n(z)}{G_{n+1}(z) \pm \sqrt{R_{2n+2}(z)}} = \frac{G_{n+1} \mp \sqrt{R_{2n+2}(z)}}{F_n}. \tag{4.5}
\end{align*}
\]

Proof. Expressions (4.2) - (4.5) can easily be verified by (2.44), (3.2). \(\square\)
Lemma 4.2. Suppose $q, p \in C^\infty(\mathbb{R}^{2+1})$. Then we have the following relations

$$\left[ \frac{q(x,y,t_{r+1})}{\mathcal{F}_n(z,x,y,t_{r+1})} \right]_{t_{j+1}} = \left[ \frac{\hat{F}_j(z,x,y,t_{r+1})}{\mathcal{F}_n(z,x,y,t_{r+1})} \right]_{x^j}, \quad j = 1, r, \quad (4.6)$$

$$\left[ \frac{p(x,y,t_{r+1})}{\mathcal{H}_n(z,x,y,t_{r+1})} \right]_{t_{j+1}} = -\left[ \frac{\hat{H}_j(z,x,y,t_{r+1})}{\mathcal{F}_n(z,x,y,t_{r+1})} \right]_{x^j}, \quad j = 1, r, \quad (4.7)$$

$$\left[ \frac{\hat{F}_1(z,x,y,t_{r+1})}{\mathcal{F}_n(z,x,y,t_{r+1})} \right]_{t_{r+1}} = \hat{F}_2(z,x,y,t_{r+1}), \quad (4.8)$$

$$\left[ \frac{\hat{H}_1(z,x,y,t_{r+1})}{\mathcal{F}_n(z,x,y,t_{r+1})} \right]_{t_{r+1}} = \hat{H}_2(z,x,y,t_{r+1}), \quad (4.9)$$

Proof. The proof is straightforward. By (2.49), (2.50) and

$$q_{t_{j+1}} - \hat{F}_j, x - z\hat{F}_j + 2q\hat{G}_{j+1} = 0, \quad j = 1, r, \quad (4.10)$$

we have

$$\left[ q, \frac{\mathcal{F}_n}{\mathcal{F}_n^2} \right]_{t_{j+1}} = \frac{q_t}{\mathcal{F}_n} - \frac{q_{t_{n,t_j}}}{\mathcal{F}_n^2} = \frac{\hat{F}_j, x + z\hat{F}_j - 2q\hat{G}_{j+1}}{\mathcal{F}_n} - \frac{q(2\hat{F}_j, G_{n+1} - 2\hat{G}_{j+1}, \mathcal{F}_n)}{\mathcal{F}_n^2} = \frac{\hat{F}_j, x - \hat{F}_j, \mathcal{F}_n, x}{\mathcal{F}_n^2} = \left[ \frac{\hat{F}_j}{\mathcal{F}_n} \right]_{x^j}, \quad j = 1, r.$$

and (4.7)-(4.9) follows in a similar manner. \qed

Now we lift the fundamental solution $(\psi^\pm_1(z), \psi^\pm_2(z))$, $z \in \mathbb{C}$, to Riemann surface $X$. Define the Baker-Akhiezer function by

$$\psi_j(P, x, y, t_{r+1}) = \psi^+_j(z, x, y, t_{r+1}), \quad j = 1, 2, \quad y(P) = 2\sqrt{\mathcal{R}_{2n+2}(z)}, \quad \text{for } P \in \Pi_+,$$

$$\psi_j(P, x, y, t_{r+1}) = \psi^-_j(z, x, y, t_{r+1}), \quad j = 1, 2, \quad y(P) = -2\sqrt{\mathcal{R}_{2n+2}(z)}, \quad \text{for } P \in \Pi_-,$$

$$\lim_{P \rightarrow P_0} \psi_j(P, x, t_{r+1}) = \lim_{P \rightarrow P_0} \psi_j(P^*, x, t_{r+1}), \quad \text{for } j = 1, 2, \quad P \in \Pi_\pm, \quad P_0 \in \mathbb{C},$$

where we choose the branches of $\sqrt{\mathcal{R}_{2n+2}(z)}$ satisfying

$$\lim_{z \rightarrow \infty} \frac{\mathcal{R}_{n+1}}{\sqrt{\mathcal{R}_{2n+2}(z)}} = 1.$$
Moreover, let
\[ \hat{\mu}_j(x, y, t) = \left( \mu_j(x, y, t_{r+1}), -2 \sqrt{\mathcal{R}_j(x, y, t_{r+1})} \right) \in \Pi_- \]
\[ \hat{\nu}_j(x, y, t) = \left( \nu_j(x, y, t_{r+1}), 2 \sqrt{\mathcal{R}_j(x, y, t_{r+1})} \right) \in \Pi_+ \]
\[ j = 1, \ldots, n. \]

Based on above preparations, we shall study explicit forms of Baker-Akhiezer function \( \psi_j(P) \), \( j = 1, 2 \).

**Theorem 4.3.** Suppose \( q, p \in C^\infty(\mathbb{R}^{2+1}) \). Then Baker-Akhiezer function satisfying condition (4.1) can be expressed as

\[
\psi_1(P, x, y, t_{r+1}) = \sqrt{\mathcal{F}_n(z, x, y, t_{r+1})} \exp \left( -\frac{1}{2} \int_{x_0}^x q(x', y, t_{r+1})y(P) \frac{dx'}{\mathcal{F}_n(z, x', y, t_{r+1})} \right.
\]

\[ - \left. \frac{y(P)}{2} \int_{y_0}^y \frac{\hat{F}_1(z, x_0, y', t_{r+1})}{\mathcal{F}_n(z, x_0, y', t_{r+1})} dy' - \frac{y(P)}{2} \int_{t_{r+1,0}}^{t_{r+1}} \frac{\hat{F}_r(z, x_0, y_0, t')}{\mathcal{F}_n(z, x_0, y_0, t')} dt' \right), \tag{4.11} \]

\[
\psi_2(P, x, y, t_{r+1}) = \sqrt{\mathcal{H}_n(z, x, y, t_{r+1})} \exp \left( \frac{1}{2} \int_{x_0}^x p(x', y, t_{r+1})y(P) \frac{dx'}{\mathcal{F}_n(z, x', y, t)} \right.
\]

\[ - \left. \frac{y(P)}{2} \int_{y_0}^y \frac{\hat{H}_1(z, x_0, y', t)}{\mathcal{F}_n(z, x_0, y', t)} dy' - \frac{y(P)}{2} \int_{t_{r+1,0}}^{t_{r+1}} \frac{\hat{H}_r(z, x_0, y_0, t')}{\mathcal{F}_n(z, x_0, y_0, t')} dt' \right). \tag{4.12} \]
Proof. By (2.43), (2.48)-(2.53), (4.5), one obtains

\[
\psi_{1,x}^\pm(z) = -\frac{z}{2}\psi_1^\pm(z) + q\psi_2^\pm(z)
= \left[\left( -q\mathcal{G}_{n+1}(z) + \frac{1}{2}\mathcal{F}_{n}(z) \right) + q\frac{\mathcal{H}(z)}{\mathcal{H}_{n+1}(z)} \right] \psi_1^\pm(z)
= \left[\left( -q\mathcal{G}_{n+1}(z) + \frac{1}{2}\mathcal{F}_{n}(z) \right) + q\frac{\mathcal{H}(z)}{\mathcal{H}_{n+1}(z) \pm \sqrt{\mathcal{R}_{2n+2}(z)}} \right] \psi_1^\pm(z)
= \pm q\sqrt{\mathcal{R}_{2n+2}(z) + \frac{1}{2}\mathcal{F}_{n}(z)} \psi_1^\pm(z), \quad (4.13)
\]

\[
\psi_{1,t,j+1}^\pm(z) = -\hat{G}_{j+1}(z)\psi_{1}^\pm(z) + \hat{F}_{j}(z)\psi_{2}^\pm(z)
= \left[\left( -\hat{F}_{j}(z)\mathcal{G}_{n+1}(z) + \frac{1}{2}\mathcal{F}_{n}(z) \right) + \hat{F}_{j}(z)\frac{\mathcal{H}(z)}{\mathcal{H}_{n+1}(z) \pm \sqrt{\mathcal{R}_{2n+2}(z)}} \right] \psi_1^\pm(z)
= \left[\left( -\hat{F}_{j}(z)\mathcal{G}_{n+1}(z) + \frac{1}{2}\mathcal{F}_{n}(z) \right) + \hat{F}_{j}(z)\frac{\mathcal{H}(z)}{\mathcal{H}_{n+1}(z) \pm \sqrt{\mathcal{R}_{2n+2}(z)}} \right] \psi_1^\pm(z)
= \pm \hat{F}_{j}(z)\sqrt{\mathcal{R}_{2n+2}(z) + \frac{1}{2}\mathcal{F}_{n}(z)} \psi_1^\pm(z), \quad j = 1, r, \quad (4.14)
\]

and similarly

\[
\psi_{2,x}^\pm(z) = p\psi_1^\pm(z) + \frac{z}{2}\psi_2^\pm(z)
= \left[ p\psi_1^\pm(z) + \left( \frac{2\mathcal{F}_{n}(z) - p\mathcal{G}_{n+1}(z)}{\mathcal{H}_{n}(z)} \right) \psi_2^\pm(z) \right] \psi_1^\pm(z)
= \left[ p\mathcal{G}_{n+1}(z) + \frac{2\mathcal{F}_{n}(z)}{\mathcal{H}_{n}(z)} + \left( \frac{2\mathcal{F}_{n}(z) - p\mathcal{G}_{n+1}(z)}{\mathcal{H}_{n}(z)} \right) \psi_2^\pm(z) \right] \psi_1^\pm(z)
= \pm p\sqrt{\mathcal{R}_{2n+2}(z) + \frac{1}{2}\mathcal{F}_{n}(z)} \psi_2^\pm(z), \quad (4.15)
\]

\[
\psi_{2,t,j+1}^\pm(z) = -\hat{H}_{j}(z)\psi_1^\pm(z) + \hat{G}_{j+1}(z)\psi_{2}^\pm(z)
= \left[ -\hat{H}_{j}(z)\psi_1^\pm(z) + \left( \frac{2\mathcal{F}_{n}(z)}{\mathcal{H}_{n}(z)} + \hat{H}_{j}(z)\frac{\mathcal{G}_{n+1}(z)}{\mathcal{H}_{n}(z)} \right) \psi_2^\pm(z) \right] \psi_1^\pm(z)
= \left[ -\hat{H}_{j}(z)\mathcal{G}_{n+1}(z) + \frac{2\mathcal{F}_{n}(z)}{\mathcal{H}_{n}(z)} + \hat{H}_{j}(z)\frac{\mathcal{G}_{n+1}(z)}{\mathcal{H}_{n}(z)} \right] \psi_2^\pm(z)
= \pm \hat{H}_{j}(z)\sqrt{\mathcal{R}_{2n+2}(z) + \frac{1}{2}\mathcal{F}_{n}(z)} \psi_2^\pm(z), \quad j = 1, r. \quad (4.16)
\]
Thus, we have

\[
\begin{align*}
    d\ln(\psi_1^{\pm}(z, x, y, t_{r+1})) &= \left( \frac{\mp q \sqrt{\mathcal{R}_{2n+2}(z)} + \frac{1}{2} \mathcal{F}_{n,x}(z)}{\mathcal{F}_n(z)} \right) dx \\
    &+ \left( \frac{\mp \tilde{F}_1(z) \sqrt{\mathcal{R}_{2n+2}(z)} + \frac{1}{2} \mathcal{F}_{n,y}(z)}{\mathcal{F}_n(z)} \right) dy \\
    &+ \left( \frac{\mp \hat{F}_2(z) \sqrt{\mathcal{R}_{2n+2}(z)} + \frac{1}{2} \mathcal{F}_{n,t_{r+1}}(z)}{\mathcal{F}_n(z)} \right) dt_{r+1}
\end{align*}
\]

and

\[
\begin{align*}
    d\ln(\psi_2^{\pm}(z, x, y, t_{r+1})) &= \left( \frac{\pm p \sqrt{\mathcal{R}_{2n+2}(z)} + \frac{1}{2} \mathcal{H}_{n,x}(z)}{\mathcal{H}_n(z)} \right) dx \\
    &+ \left( \frac{\pm \tilde{H}_1(z) \sqrt{\mathcal{R}_{2n+2}(z)} + \frac{1}{2} \mathcal{H}_{n,y}(z)}{\mathcal{H}_n(z)} \right) dy \\
    &+ \left( \frac{\pm \hat{H}_2(z) \sqrt{\mathcal{R}_{2n+2}(z)} + \frac{1}{2} \mathcal{H}_{n,t_{r+1}}(z)}{\mathcal{H}_n(z)} \right) dt_{r+1}
\end{align*}
\]

(4.17)

According to relations (4.7)-(4.9), it follows that the integrals

\[
\int_{(x_0, y_0, t_{r+1,0})}^{(x, y, t_{r+1})} d\ln\left( \psi_j^{\pm}(z, x, y, t_{r+1}) \sqrt{\mathcal{F}_n(z, x, y, t_{r+1})} \right), \quad j = 1, 2
\]

(4.19)

is independent of the path. Therefore taking into account the normalization condition \(\psi_1(P, x_0, y_0, t_{r+1,0}) = 1\), and choosing a special path \((x_0, y_0, t_{r+1,0}) \rightarrow (x_0, y_0, t_{r+1}) \rightarrow (x_0, y, t_{r+1}) \rightarrow (x, y, t_{r+1})\) in (4.19), we finally obtain (4.11) and (4.12).

Basic properties of Baker-Akhiezer functions \(\psi_j(P), j = 1, 2\) are summarized in the following result.

**Lemma 4.4.** Suppose \(q, p \in C^\infty(\mathbb{R}^{2+1})\) and \(P \in X \backslash \{P_{\infty}\}\). Then the
Baker-Akhiezer function derived in \([4.17]\), \([4.18]\) satisfy

\[
\psi_1(P, x, y, t_{r+1}) \psi_1(P^*, x, y, t_{r+1}) = \frac{\mathcal{F}_n(z, x, y, t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}, \tag{4.20}
\]

\[
\psi_2(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1}) = \frac{\mathcal{H}_n(z, x, y, t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}, \tag{4.21}
\]

\[
\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1}) = \frac{\mathcal{G}_{n+1}(z, x, y, t) + \sqrt{\mathcal{R}_{2n+2}(z)}}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}, \tag{4.22}
\]

\[
\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1}) + \psi_1(P^*, x, y, t_{r+1}) \psi_2(P, x, y, t_{r+1}) = \frac{2 \mathcal{G}_{n+1}(z, x, y, t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}, \tag{4.23}
\]

\[
\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1}) - \psi_1(P^*, x, y, t_{r+1}) \psi_2(P, x, y, t_{r+1}) = \frac{2 \sqrt{\mathcal{R}_{2n+2}(z)}}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}. \tag{4.24}
\]

**Proof.** It can be easily seen that \((4.20)\) and \((4.21)\) hold by \((4.11)\) and \((4.12)\), respectively. Then from \((4.20)\), \((4.21)\) and the fact that

\[
\left(\frac{1}{2}(\psi_1(P) \psi_2(P^*) + \psi_1(P^*) \psi_2(P)), \psi_1(P) \psi_1(P^*), \psi_2(P) \psi_2(P^*)\right)
\]

and

\[
\left(\frac{\mathcal{G}_{n+1}(z, x, y, t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}, \frac{\mathcal{F}(z, x, y, t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}, \frac{\mathcal{H}_n(z, x, y, t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}\right)
\]

are both solutions of linear system \((2.45)-(2.53)\) satisfying the same initial condition, one gets \((4.23)\). Using \((4.20)\), \((4.21)\) and \((4.21)\), we have

\[
\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1}) - \psi_1(P^*, x, y, t_{r+1}) \psi_2(P, x, y, t_{r+1}) = \sqrt{(\psi_1(P) \psi_2(P^*) + \psi_1(P^*) \psi_2(P))^2 - 4 \psi_1(P) \psi_1(P^*) \psi_2(P) \psi_2(P^*)}
\]

\[
= \frac{2 \sqrt{\mathcal{R}_{2n+2}(z)}}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})}. \tag{4.25}
\]

Finally, \((4.22)\) is the direct result of \((4.23)\), \((4.24)\). \(\square\)

Next we consider the analytic property and asymptotic behavior of \(\psi_1(P) \psi_2(P^*)\), and \(\psi_j(P), j = 1, 2\).
Lemma 4.5. The function $\psi_1(P)\psi_2(P^*)$ is a meromorphic function on $X$ with divisor

$$
(\psi_1(P)\psi_2(P^*)) = \mathcal{D}_{\hat{\mu}^*(x,y,t_1)} - \mathcal{D}_{\hat{\mu}^*(x_0,y_0,t_0)P_{\infty}}, \quad (4.26)
$$

where we abbreviate $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)$, $\hat{\nu} = (\hat{\nu}_1, \ldots, \hat{\nu}_n)$. Moreover, using the local coordinate $\zeta = z^{-1}$ near $P_{\infty}$, we have

$$
\lim_{P \to P_{\infty}^+} \psi_1(P, x, y, t)\psi_2(P^*, x, y, t) = \begin{cases} \\
1 + O(1), & \text{as } P \to P_{\infty}^+,
\end{cases}
$$

Proof. Noticing (3.2), (3.4), (4.22) and

$$
\psi_1(P, x, y, t)\psi_2(P^*, x, y, t) = \mathcal{H}_{n+1}(z, x, y, t_{r+1}) + \mathcal{F}_{n}(z, x_0, y_0, t_{r+1,0})
$$

one easily proves (4.26) and (4.27). □

Now we turn to study the analytic structure of $\psi_j(P, x, y, t_{r+1})$ on $X\backslash \{P_{\infty}\}$.

Theorem 4.6. Assume auxiliary spectrum points $\mu_j(x, y, t_{r+1}), \nu_j(x, y, t_{r+1}), j = 1, \ldots, n$, are mutually distinct and finite for all $(x, y, t_{r+1}) \in \Omega$, where $\Omega \in \mathbb{R}^3$ is an open interval. Moreover, let $P \in X\backslash \{P_{\infty}\}$. Then

I. $\psi_1(P, x, y, t_{r+1})$ and $\psi_2(P, x, y, t_{r+1})$ are meromorphic on $P \in X\backslash \{P_{\infty}\}$. Their divisor of poles coincides with $\mathcal{D}_{\hat{\mu}(x_0,y_0,t_{r+1,0})}$.

II. The divisor of zeros for $\psi_1(P, x, y, t_{r+1})$ and $\psi_2(P, x, y, t_{r+1})$ coincides with $\mathcal{D}_{\hat{\mu}(x,y,t_{r+1})}$ and $\mathcal{D}_{\hat{\mu}(x,y,t_{r+1})}$, respectively.

III. As $P \to P_{\infty}$, the asymptotic behavior of $(\psi_1(P, x, y, t_{r+1}), \psi_2(P, x, y, t_{r+1}))^T$
is given by the equations,

\[
\begin{pmatrix}
\psi_1(P,x,y,t_{r+1}) \\
\psi_2(P,x,y,t_{r+1})
\end{pmatrix} = \left[ \begin{pmatrix} 0 & 1 \\ \frac{q(x,y,t_{r+1})}{q(x_0,y_0,t_{r+1})} O(1) \end{pmatrix} \right] \zeta^{-1} + \left[ \begin{pmatrix} q(x_0,y_0,t_{r+1}) \\ O(1) \end{pmatrix} \right] + O(\zeta) + \exp \left( \frac{1}{2} (x - x_0) \zeta^{-1} - \frac{1}{2} (y - y_0) \zeta^{-2} - \frac{1}{2} (t_{r+1} - t_{r+1}) \zeta^{-(r+1)} \right), \text{ at } P \to P_\infty^-,
\]

(4.28)

and

\[
\begin{pmatrix}
\psi_1(P,x,y,t_{r+1}) \\
\psi_2(P,x,y,t_{r+1})
\end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 \\ 0 & -p(x,y,t_{r+1}) \end{pmatrix} \right] \zeta + O(\zeta^2) + \exp \left( -\frac{1}{2} (x - x_0) \zeta^{-1} + \frac{1}{2} (y - y_0) \zeta^{-2} + \frac{1}{2} (t_{r+1} - t_{r+1}) \zeta^{-(r+1)} \right), \text{ at } P \to P_\infty^+.
\]

(4.29)

**Proof.** First we study the function \( \psi_1 \). By (3.10), (3.11), (3.12) it follows

\[
-\frac{1}{2} q(x',y,t_{r+1}) y(P) \xrightarrow{P \to \hat{\mu}_j(x',y,t_{r+1})} \frac{\partial x'}{\partial x} \ln \sqrt{z - \mu_j(x',y,t_{r+1})},
\]

(4.30)

\[
-\frac{1}{2} y(P) \xrightarrow{P \to \hat{\mu}_j(x_0,y',t_{r+1})} \frac{\partial y'}{\partial y} \ln \sqrt{z - \mu_j(x_0,y',t_{r+1})},
\]

(4.31)

\[
-\frac{1}{2} y(P) \xrightarrow{P \to \hat{\mu}_j(x_0,y_0,t')} \frac{\partial t'}{\partial t} \ln \sqrt{z - \mu_j(x_0,y_0,t')},
\]

(4.32)
and hence one obtains

\[
\exp \left( -\frac{1}{2} \int_{x_0}^{x} \frac{q(x',y,t_{r+1})y(P)}{\mathcal{F}_n(z,x',y,t_{r+1})} dx' - \frac{y(P)}{2} \int_{y_0}^{y} \frac{\hat{F}_1(z,x_0,y',t_{r+1})}{\mathcal{F}_n(z,x_0,y',t_{r+1})} dy' \right.
\]
\[\left. - \frac{y(P)}{2} \int_{t_{r+1,0}}^{t_{r+1}} \frac{\hat{F}_r(z,x_0,y_0,t')}{\mathcal{F}_n(z,x_0,y_0,t')} dt' \right) = \exp \left( \int_{x_0}^{x} \partial_{x'} \ln \sqrt{z - \mu_j(x',y,t_{r+1})} dx' + \int_{y_0}^{y} \partial_{y'} \ln \sqrt{z - \mu_j(x_0,y',t_{r+1})} dy' \right.
\]
\[\left. + \int_{t_{r+1,0}}^{t_{r+1}} \partial_{t'} \ln \sqrt{z - \mu_j(x_0,y_0,t')} dt' + O(1) \right)
\]

\[
= \begin{cases} 
\sqrt{z - \mu_j(x_0,y_0,t_{r+1})}^{-1}, & P \to \hat{\mu}_j(x_0,y_0,t_{r+1,0}), \\
\sqrt{z - \mu_j(x,y,t_{r+1})}, & P \to \hat{\mu}_j(x,y,t_{r+1}) \neq \hat{\mu}_j(x_0,y_0,t_{r+1,0}), \\
O(1), & P \to \hat{\mu}_j(x,y,t_{r+1}) = \hat{\mu}_j(x_0,y_0,t_{r+1,0}), \\
O(1), & P \to \text{other points} \neq \hat{\mu}_j(x,y,t_{r+1}), \hat{\mu}_j(x_0,y_0,t_{r+1,0}).
\end{cases}
\]

Then taking into account (4.11), one proves I. and II. for \( \psi_1 \). Next we study the asymptotic behaviour of \( \psi_1 \) near \( P_{\infty \pm} \). Again using (4.11) and
local coordinate $\zeta = z^{-1}$ near $P_{\infty \pm}$, one infers

$$
\psi_1(P, x, y, t_{r+1}) = \exp \left( \frac{1}{2} \int_{x_0}^{x} q(x', y, t_{r+1}) y(P) \, dx' \right)
$$

$$
- \frac{y(P)}{2} \int_{y_0}^{y} \frac{F_1(z, x_0, y', t_{r+1})}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})} \, dy' - \frac{y(P)}{2} \int_{t_{r+1,0}}^{t_{r+1}} \frac{F_1(z, x_0, y_0, t')}{\mathcal{F}_n(z, x_0, y_0, t_{r+1})} \, dt'
$$

$$
= \exp \left( \frac{1}{2} \int_{x_0}^{x} \left( -\frac{q(x', y, t_{r+1})}{\sum_{j=0}^{\infty} f_j(x', y, t_{r+1}) \zeta^{j+1}} + q_j(x', y, t_{r+1}) \right) \, dx' \right)
$$

$$
+ \frac{1}{2} \int_{y_0}^{y} \left( -\frac{F_1(z, x_0, y', t_{r+1})}{\sum_{j=0}^{\infty} f_j(x_0, y', t_{r+1}) \zeta^{j+1}} + q_j(x_0, y', t_{r+1}) \right) \, dy'
$$

$$
+ \frac{1}{2} \int_{t_{r+1,0}}^{t_{r+1}} \left( -\frac{F_1(x_0, y_0, t')}{\sum_{j=0}^{\infty} f_j(x_0, y_0, t') \zeta^{j+1}} + q_j(x_0, y_0, t') \right) \, dt', \quad \text{as } P \to P_{\infty -},
$$

$$
= \exp \left( \frac{1}{2} \int_{x_0}^{x} \frac{q(x', y, t_{r+1})}{\sum_{j=0}^{\infty} f_j(x', y, t_{r+1}) \zeta^{j+1}} \, dx' \right)
$$

$$
+ \frac{1}{2} \int_{y_0}^{y} \frac{F_1(x_0, y', t_{r+1})}{\sum_{j=0}^{\infty} f_j(x_0, y', t_{r+1}) \zeta^{j+1}} \, dy'
$$

$$
+ \frac{1}{2} \int_{t_{r+1,0}}^{t_{r+1}} \frac{F_1(x_0, y_0, t')}{\sum_{j=0}^{\infty} f_j(x_0, y_0, t') \zeta^{j+1}} \, dt', \quad \text{as } P \to P_{\infty +},
$$

$$
= \exp \left( \frac{1}{2} (x - x_0) \zeta^{-1} - \frac{1}{2} (y - y_0) \zeta^{-2} \right)
$$

$$
- \frac{1}{2} (t - t_{r+1,0}) \zeta^{-(r+1)} + O(\zeta), \quad \text{as } P \to P_{\infty -},
$$

$$
\exp \left( -\frac{1}{2} (x - x_0) \zeta^{-1} + \frac{1}{2} (y - y_0) \zeta^{-2} + \frac{1}{2} (t - t_{r+1,0}) \zeta^{-(r+1)} + O(\zeta) \right), \quad \text{as } P \to P_{\infty +}.
$$

(4.34)

Here we have used asymptotic spectral expansion

$$
\frac{\mathcal{F}_n(z, x, y, t_{r+1})}{y(P)} = \sum_{j=0}^{\infty} f_j(x, y, t_{r+1}) \zeta^{j+1}, \quad \text{as } P \to P_{\infty \pm}, \quad (4.35)
$$

which can be derived by induction as in [25]. Applying Lemma 4.5 and (4.34), we may derive related results for $\psi_2$. ❑
5 Algebro-geometric solutions

In this section, we shall first give a detailed description of the function $\psi_1$ and then study the theta function representation of Baker-Akhiezer functions $\psi_1(P), \psi_2(P)$ and algebro-geometric solutions of the KP hierarachy.

The function $\psi_1(P, x, y, t_{r+1})$ derived in last section plays very important roles. Let us consider the case $r = 2$ for example. By introducing

$$\phi(P, x, y, t_3) = \psi_1(P, x, y, t_3) \exp\left(-\frac{1}{2}(x-x_0)z-\frac{1}{2}(y-y_0)z^2-\frac{1}{2}(t_3-t_{3,0})z^3\right),$$

and using theorem 4.6, one infers the function $\phi$ possess the same properties \textbf{I}, \textbf{II} as $\psi_1$, and the following expansions

$$\phi(P, x, y, t_3) \zeta \rightarrow 0 = \begin{cases} 
(1 + O(\zeta)) \exp(-(x-x_0)z - (y-y_0)z^2) & \text{as } P \rightarrow P_{\infty+}, \\
-(t_3-t_{3,0})z^3) & \text{as } P \rightarrow P_{\infty-}, \\
O(1) & \text{elsewhere} 
\end{cases}$$

where we use local coordinates $\zeta = z^{-1}$ near $P_{\infty \pm}$. Thus, the function $\phi$ gives one explicit form of Baker-Akhiezer function for the KP equation and one can use it to obtain the algebro-geometric solutions of the KP equation following the way in [40].

Another important fact is the function $\psi_1(P, x, y, t_3)$ describes a new (2+1) system, which is closely related with the KP equation. Let

$$\Phi(P, x, y, t_3) = \psi_1(P, x, y, t_3) \exp\left(\frac{1}{2}(x-x_0)z - \frac{1}{2}(y-y_0)z^2 - \frac{1}{2}(t_3-t_{3,0})z^3\right),$$

and then we have the following results.

\textbf{Theorem 5.1.} The function $\Phi(P, x, y, t_3)$ satisfies the following auxiliary linear problem:

$$L_2\Phi = 0, \quad L_3\Phi = 0,$$

where the operators $L_2$ and $L_3$ are given by the formulas

$$L_2 = \partial_y + \partial_x^2 + u_0 \partial_x + u_1,$$
$$L_3 = \partial_{t_3} + \partial^3 + v_0 \partial_x^2 + v_1 \partial_x + v_2,$$

and $u_0, u_1, v_0, v_1, v_2$ are coefficients independent of $P$ which are determined
by the following conditions
\[
(\partial_y - L_2)\Phi = O(\zeta) \exp \left( (x - x_0)\zeta^{-1} - (y - y_0)\zeta^{-2} - (t_3 - t_3,0)\zeta^{-3} \right),
\]
as \( P \to P_{\infty-} \), \( \zeta = z^{-1} \). (5.6)

\[
(\partial_{t_3} - L_3)\Phi = O(\zeta) \exp \left( (x - x_0)\zeta^{-1} - (y - y_0)\zeta^{-2} - (t_3 - t_3,0)\zeta^{-3} \right),
\]
as \( P \to P_{\infty-} \), \( \zeta = z^{-1} \). (5.7)

Proof. For convenience we denote by
\[
\Delta(x,y,t_3) = (x - x_0)\zeta^{-1} - (y - y_0)\zeta^{-2} - (t_3 - t_3,0)\zeta^{-3}
\]
and suppose
\[
\psi_1(P,x,y,t_3) \to 0 = \left[ \sum_{j=0}^{\infty} \Theta_j(x,y,t_3)\zeta^j \right] e^{\Delta(x,y,t_3)}, \text{ as } P \to P_{\infty-}. \quad (5.9)
\]
Then it follows from (5.9) that
\[
\psi_{1,x}(P,x,y,t_3) \to 0 = \left[ \Theta_0 + \sum_{j=0}^{\infty} (\Theta_j + \Theta_{j+1})\zeta^j \right] e^{\Delta(x,y,t_3)}, \quad (5.10)
\]
\[
\psi_{1,xx}(P,x,y,t_3) \to 0 = \left[ \Theta_0 - 2\Theta_1 + \sum_{j=0}^{\infty} (\Theta_j + 2\Theta_{j+1})\zeta^j \right] e^{\Delta(x,y,t_3)}, \quad (5.11)
\]
\[
\psi_{1,xxx}(P,x,y,t_3) \to 0 = \left[ \Theta_0 - 3\Theta_1 + \sum_{j=0}^{\infty} (3\Theta_j + \Theta_{j+2})\zeta^j \right] e^{\Delta(x,y,t_3)}, \quad (5.12)
\]
and
\[
\psi_{1,y}(P,x,y,t_3) \to 0 = \left[ -\Theta_0 - \Theta_1 + \sum_{j=0}^{\infty} (\Theta_j - \Theta_{j+2})\zeta^j \right] e^{\Delta(x,y,t_3)}, \quad (5.13)
\]
\[
\psi_{1,t_3}(P,x,y,t_3) \to 0 = \left[ -\Theta_0 - 3\Theta_1 - \Theta_0 + \sum_{j=0}^{\infty} (\Theta_j - \Theta_{j+3})\zeta^j \right] e^{\Delta(x,y,t_3)}. \quad (5.14)
\]
Inserting (5.14) into (5.4), (5.5), and taking into account (5.6), (5.7), one obtains

\[ 2\Theta_{0,x} + u_{0}\Theta_{0} = 0, \quad (5.15) \]
\[ \Theta_{0,y} + 2\Theta_{1,x} + \Theta_{0,xx} + u_{0}\Theta_{1} + u_{1}\Theta_{0} = 0, \quad (5.16) \]
\[ 3\Theta_{0,x} + v_{0}\Theta_{0} = 0, \quad (5.17) \]
\[ 3\Theta_{1,x} + 3\Theta_{0,xx} + v_{0}(\Theta_{1} + 2\Theta_{0,x}) + v_{1}\Theta_{0} = 0, \quad (5.18) \]
\[ 3\Theta_{2,x} + 3\Theta_{1,xx} + \Theta_{0,xxx} + v_{0}(\Theta_{2} + 2\Theta_{1,x} + \Theta_{0,xx}) + v_{1}(\Theta_{1} + \Theta_{0,x}) + v_{2}\Theta_{0} = 0. \quad (5.19) \]

Here \( \Theta_{j}, j = 0, 1, 2, \ldots \), possess explicit representations which arise from (4.11) or (5.34). Thus solving the system (5.15)-(5.19), we may derive \( u_{0}, u_{1}, v_{0}, v_{1}, v_{2} \) which are expressed by \( \Theta_{j} \).

Since analytic properties of \( L_{2}\psi_{1} \) and \( L_{3}\psi_{1} \) are identical with \( \psi_{1} \), except for possible different asymptotic behavior at \( P_{\infty} \), one concludes

\[ L_{2}\psi_{1} = 0, \quad L_{3}\psi_{1} = 0 \quad (5.20) \]

hold by (5.6), (5.7).

**Theorem 5.2.** Assume the condition of Theorem 5.1 holds. Then the compatibility condition for (5.4), (5.5) is equivalent to

\[ 2v_{0,x} - 3u_{0,x} = 0, \quad (5.21) \]
\[ v_{0,y} + v_{0,xx} + 2v_{1,x} - 3u_{0,xx} - 3u_{1,x} + u_{0}v_{0,x} - 2v_{0}u_{0,x} = 0, \quad (5.22) \]
\[ -u_{0,t_{3}} + v_{1,y} + v_{1,xx} + 2v_{2,x} - u_{0,xxx} - 3u_{1,xx} - 2v_{0}u_{1,x} + u_{0}v_{1,x} - v_{1}u_{0,x} - v_{0}u_{0,xx} = 0, \quad (5.23) \]
\[ -u_{1,t_{3}} + v_{2,y} + v_{2,xx} + u_{0}v_{2,x} - u_{1,xxx} - 2v_{0}u_{1,xx} - v_{1}u_{1,x} = 0. \quad (5.24) \]

**Proof.** Imposing on \( \psi_{1} \) the requirement \( \psi_{1,t_{3}y} = \psi_{1,gt_{3}} \), one easily get (5.21)-(5.24).

**Remark 5.3.** The function \( \psi_{1}(x, y, t_{3}) \) is connected with three equations, i.e. the second and third flows of ANKS hierarchy, KP equation, and the new system (5.21)-(5.24). Thus, it is possible to get the 'Bäcklund transformation' of algebro-geometric solutions between KP equation and the new 2+1 system. **Similar statement is true for** \( r > 2 \).

Now we turn to study the theta function representations for \( \psi_{1}(P), \psi_{2}(P) \) and algebro-geometric solutions of the whole KP hierarchy.

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First, choosing a convenient base point \( Q_0 \in X \setminus \{ P_{\infty} \} \), the Abel maps \( A_{Q_0}(\cdot) \) and \( \alpha_{Q_0}(\cdot) \) are defined by

\[
A_{Q_0} : X \to J(X) = \mathbb{C}^n / L_n,
\]

\[
P \mapsto A_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,n}(P)) \quad = \left( \int_{Q_0}^P \omega_1, \ldots, \int_{Q_0}^P \omega_n \right) \pmod{L_n},
\]

and

\[
\alpha_{Q_0} : \text{Div}(X) \to J(X),
\]

\[
D \mapsto \alpha_{Q_0}(D) = \sum_{P \in \mathcal{K}_n} D(P) A_{Q_0}(P),
\]

where \( L_n = \{ z \in \mathbb{C}^n | z = N + \Gamma M, \, N, \, M \in \mathbb{Z}^n \} \), and \( \Gamma, \Xi_{Q_0} \) are the Riemann matrix and the vector of Riemann constants, respectively. Moreover, we choose a homology basis \( \{ a_j, b_j \}_{j=1}^n \) on \( X \) in such a way that the intersection matrix of the cycles satisfies

\[
a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, \, k = 1, \ldots, n. \quad (5.25)
\]

For brevity, define the function \( \bar{z} : X \times \sigma^n X \to \mathbb{C}^n \) by

\[
\bar{z}(P, Q) = \Xi_{Q_0} - A_{Q_0}(P) + \alpha_{Q_0}(D_Q), \quad P \in \mathcal{K}_n, \quad Q = (Q_1, \ldots, Q_n) \in \sigma^n \mathcal{K}_n,
\]

here \( \bar{z}(\cdot, Q) \) is independent of the choice of base point \( Q_0 \). The Riemann theta function \( \theta(z) \) associated with \( X \) and the homology is defined by

\[
\theta(z) = \sum_{n \in \mathbb{Z}} \exp(2\pi i < n, \bar{z} > + \pi i < n, n \Gamma >), \quad z \in \mathbb{C}^n,
\]

where \( < \bar{B}, \bar{C} > = \bar{B} \cdot \bar{C}' = \sum_{j=1}^N \bar{B}_j C_j \) denotes the scalar product in \( \mathbb{C}^{n-1} \).

Let \( \omega^{(2)}_{P_{\infty}, q} \) be the normalized differentials of the second kind with a unique pole at \( P_{\infty} \), respectively, and principal parts

\[
\omega^{(2)}_{P_{\infty}, q} = \left( \zeta^{-2-q} + O(1) \right) d\zeta, \quad P \to P_{\infty}, \quad \zeta = z^{-1}, \quad q \in \mathbb{N}_0 \quad (5.27)
\]

\[2\sigma^n X = \frac{X \times \ldots \times X}{\sigma^n}.\]
Moreover, we introduce the notation $\omega_{P_{\infty+}P_{\infty-}}$ be normalized differential of third kind satisfying
\[
\int_{Q_0}^{P} \omega_{P_{\infty-}P_{\infty+}} = \zeta^{-1} + c_{\infty+} + O(\zeta), \quad P \to P_{\infty+}, \zeta = z^{-1}, c_{\infty+} \in \mathbb{C},
\]
and
\[
\int_{a_j}^{\infty} \omega_{P_{\infty+}P_{\infty-}} = 0 \quad j = 1, \ldots, n.
\]
Moreover, let $q, p$ satisfy (2.55). Moreover, let $\psi\in X\setminus\{P_{\infty+}\}$ and suppose the divisors $D_{\mu}(x,y,t_{r+1}), D_{\mu}(x,y,t_{r+1})$ are nonspecial. Then

**Lemma 5.4.** Assume spectral curve defined in (3.4) is nonsingular and $q, p$ satisfy (2.55). Moreover, let $P \in X\setminus\{P_{\infty\pm}\}$ and suppose the divisors $D_{\hat{\mu}}(x,y,t_{r+1})$, $D_{\hat{\mu}}(x,y,t_{r+1})$ are nonspecial. Then

\[
\psi_1(P, x, y, t_{r+1})\psi_2(P^*, x, y, t_{r+1}) = \left(\frac{1}{q(x_0, y_0, t_{r+1})}\right) \theta(\hat{z}(P_{\infty+}, \hat{\mu}(x_0, y_0, t_{r+1}))) \theta(\hat{z}(P_{\infty-}, \hat{\mu}(x_0, y_0, t_{r+1})))
\]

\[
\times \left(\frac{1}{q(x_0, y_0, t_{r+1})}\right) \theta(\hat{z}(P_{\infty+}, \hat{\mu}(x_0, y_0, t_{r+1}))) \theta(\hat{z}(P_{\infty-}, \hat{\mu}(x_0, y_0, t_{r+1})))
\]

\[
\times \exp \left(\int_{Q_0}^{P} \omega_{P_{\infty-}P_{\infty+}} - c_{\infty+}\right)
\]

(5.30)
Proof. We define

$$\mathbb{R}^1(P, x, t_{r+1}) = \frac{1}{q(x_0, y_0, t_{r+1})} \frac{\theta(z(P_{\infty+}, \mu^*(x_0, y_0, t_{r+1}))) \theta(z(P_{\infty-}, \mu(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty+}, \mu^*(x, y, t_{r+1}))) \theta(z(P_{\infty-}, \mu(x, y, t_{r+1})))} \times \exp \left( \int_{Q_0} \omega_{P_{\infty+}} - c_{\infty+} \right)$$

(5.32)

$$\mathbb{R}^2(P, x, t_{r+1}) = \frac{1}{q(x_0, y_0, t_{r+1})} \frac{\theta(z(P_{\infty+}, \mu^*(x_0, y_0, t_{r+1}))) \theta(z(P_{\infty-}, \mu(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty+}, \mu^*(x, y, t_{r+1}))) \theta(z(P_{\infty-}, \mu(x, y, t_{r+1})))} \times \exp \left( \int_{Q_0} \omega_{P_{\infty+}} - c_{\infty+} \right).$$

(5.33)

Then it is not difficult to know divisors of $\psi_1(P, x, y, t_{r+1})$ coincide with that of $\mathbb{R}^j(P, x, t_{r+1}), j = 1, 2,$ on $X$. Therefore, the quantity

$$\frac{\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1})}{\mathbb{R}^j(P, x, y, t_{r+1}), j = 1, 2,}$$

is a constant which is independent on $P$ and we denote it by $C(x, y, t_{r+1})$. Then taking $P \to P_{\infty+}$ and using Lemma 4.5, one obtains

$$C(x, y, t_{r+1}) = \lim_{P \to P_{\infty+}} \frac{\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1})}{\mathbb{R}^j(P, x, y, t_{r+1})} = 1, \quad j = 1, 2,$$

and hence equality $\psi_1(P, x, y, t_{r+1}) \psi_2(P^*, x, y, t_{r+1}) = \mathbb{R}^j(P, x, y, t_{r+1}), j = 1, 2,$ hold for any $P \in X$. We complete the proof. \(\Box\)

Given these preparations, one finally derives the following theta function representation for Baker-Akhiezer functions $\psi_j(P, x, y, t_{r+1}), j = 1, 2,$ and solutions $q(x, y, t_{r+1}), p(x, y, t_{r+1})$.

**Theorem 5.5.** Assume spectral curve defined in (3.4) is nonsingular and $q, p$ satisfy (2.25). Moreover, let $P \in X \setminus \{P_{\infty+}\}$ and suppose the divisors $D_q(x, y, t_{r+1}), D_p(x, y, t_{r+1})$ are nonspecial. Then functions $\psi_1, \psi_2, q, p$ have the
following theta function representations

\[
\psi_1(P, x, y, t_{r+1}) = C_1(x, y, t_{r+1}) \frac{\theta(z(P, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P, \hat{\mu}(x_0, y_0, t_{r+1})))} \exp \left( (x - x_0) \int_{Q_0}^P \Omega^{(2)}_0 \right) \\
- (y - y_0) \int_{Q_0}^P \Omega^{(2)}_1 - (t - t_{r+1,0}) \int_{Q_0}^P \Omega^{(2)}_r, \quad (5.34)
\]

\[
\psi_2(P, x, y, t_{r+1}) = C_2(x, y, t_{r+1}) \frac{\theta(z(P, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P, \hat{\mu}(x_0, y_0, t_{r+1})))} \exp \left( (x - x_0) \int_{Q_0}^P \Omega^{(2)}_0 \right) \\
- (y - y_0) \int_{Q_0}^P \Omega^{(2)}_1 - (t - t_{r+1,0}) \int_{Q_0}^P \Omega^{(2)}_r + \int_{Q_0}^P \omega_{P_{\infty+P_{\infty-}}} \right), \quad (5.35)
\]

\[
= \tilde{C}_2(x, y, t_{r+1}) \frac{\theta(z(P, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P, \hat{\mu}(x_0, y_0, t_{r+1})))} \exp \left( (x - x_0) \int_{Q_0}^P \Omega^{(2)}_0 \right) \\
+ (y - y_0) \int_{Q_0}^P \Omega^{(2)}_1 + (t - t_{r+1,0}) \int_{Q_0}^P \Omega^{(2)}_r + \int_{Q_0}^P \omega_{P_{\infty+P_{\infty-}}} \right), \quad (5.36)
\]

and

\[
q(x, y, t_{r+1}) = q(x_0, y_0, t_{r+1,0}) \frac{\theta(z(P_{\infty+}, \hat{\mu}(x_0, y_0, t_{r+1,0})))}{\theta(z(P_{\infty+}, \hat{\mu}(x, y, t_{r+1})))} \\
\times \frac{\theta(z(P_{\infty-}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty-}, \hat{\mu}(x_0, y_0, t_{r+1,0})))} \times \exp \left( -2(x - x_0)\omega^{(2)}_0 \right) \\
+ 2(y - y_0)\omega^{(2)}_1 + 2(t - t_{r+1,0})\omega^{(2)}_r, \quad (5.37)
\]

\[
p(x, y, t_{r+1}) = -e^{-2\omega_{\infty-}} \frac{\theta(z(P_{\infty-}, \hat{\mu}(x_0, y_0, t_{r+1,0})))}{q(x_0, y_0, t_{r+1,0})} \frac{\theta(z(P_{\infty-}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty-}, \hat{\mu}(x_0, y_0, t_{r+1,0})))} \\
\times \frac{\theta(z(P_{\infty+}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty+}, \hat{\mu}(x_0, y_0, t_{r+1,0})))} \times \exp \left( 2(x - x_0)\omega^{(2)}_0 \right) \\
- 2(y - y_0)\omega^{(2)}_1 - 2(t - t_{r+1,0})\omega^{(2)}_r, \quad (5.38)
\]
where

\[ C_1(x, y, t_{r+1}) = \frac{\theta(z(P_{\infty+}, \tilde{\mu}(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty+}, \tilde{\mu}(x, y, t_{r+1})))} \times \exp \left( - (x - x_0)\omega_0^{(2)} \right) \]
\[ + (y - y_0)\omega_1^{(2)} + (t - t_{r+1})\omega_r^{(2)}, \]
\[ C_2(x, y, t_{r+1}) = \frac{e^{-c_{\infty}} \theta(z(P_{\infty+}, \tilde{\mu}(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty+}, \tilde{\mu}(x, y, t_{r+1})))} \times \exp \left( (x - x_0)\omega_0^{(2)} \right) \]
\[ - (y - y_0)\omega_1^{(2)} - (t - t_{r+1})\omega_r^{(2)}, \]
\[ \tilde{C}_2(x, y, t_{r+1}) = \frac{e^{-c_{\infty}} \theta(z(P_{\infty+}, \tilde{\mu}^*(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty+}, \tilde{\mu}^*(x, y, t_{r+1})))} \times \exp \left( (x - x_0)\omega_0^{(2)} \right) \]
\[ - (y - y_0)\omega_1^{(2)} - (t - t_{r+1})\omega_r^{(2)}, \]

and the initial values \( q(x_0, y_0, t_{r+1}) \) and \( p(x_0, y_0, t_{r+1}) \) are constrained by

\[ q(x_0, y_0, t_{r+1})p(x_0, y_0, t_{r+1}) = - \frac{\theta(z(P_{\infty+}, \tilde{\mu}(x_0, y_0, t_{r+1}))) \theta(z(P_{\infty+}, \tilde{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty+}, \tilde{\mu}(x_0, y_0, t_{r+1}))) \theta(z(P_{\infty+}, \tilde{\mu}(x, y, t_{r+1})))} e^{-2c_{\infty}}. \]

Moreover, the Abel map linearizes the auxiliary divisors \( \mathcal{D}_{\tilde{\mu}(x,y,t_{r+1})}, \mathcal{D}_{\tilde{\mu}(x,y,t_{r+1})} \) in the sense that

\[ \alpha Q_0(\mathcal{D}_{\tilde{\mu}(x,y,t_{r+1})}) = \alpha Q_0(\mathcal{D}_{\tilde{\mu}(x_0,y_0,t_{r+1})}) - U_0^{(2)}(x - x_0) + U_1^{(2)}(y - y_0) \]
\[ + U_r^{(2)}(t_{r+1} - t_{r+1}), \]
\[ \alpha Q_0(\mathcal{D}_{\tilde{\mu}(x,y,t_{r+1})}) = \alpha Q_0(\mathcal{D}_{\tilde{\mu}(x_0,y_0,t_{r+1})}) - U_0^{(2)}(x - x_0) + U_1^{(2)}(y - y_0) \]
\[ + U_r^{(2)}(t_{r+1} - t_{r+1}) + A_{Q_0}(P_{\infty-}) - A_{Q_0}(P_{\infty+}). \]

**Proof.** First, we prove expression (5.34). Denote the right hand of (5.34) by \( \tilde{\psi}_1 \), and then one finds \( \psi_1 \) is meromorphic on \( X \setminus \{ P_{\infty+} \} \) with simple zeros at \( \tilde{\mu}_j(x, y, t_{r+1}), j = 1, \ldots, n \), and simple poles at \( \tilde{\mu}_j(x_0, y_0, t_{r+1}), j = 1, \ldots, n \), by Riemann vanishing theorem. A comparison of [4.28], [4.29], [5.34] for \( \tilde{\psi}_1 \), taking into [5.27], [5.28] shows \( \psi_1 \) and \( \tilde{\psi}_1 \) have identical exponential behavior up to order \( O(1) \) near \( P_{\infty+} \). Thus, \( \psi_1 \) and \( \tilde{\psi}_1 \) share the same singularities and zeros and the Riemann-Roch-type uniqueness result then proves that \( \psi_1 \) and \( \tilde{\psi}_1 \) coincide up to normalization and we denote this normalization constant by \( C_1(x, y, t_{r+1}) \). Hence \( \psi_1(P, x, y, t_{r+1}) \) has the
form of (5.34). Comparing (4.29) with (5.34) and taking into account the asymptotic behavior near $P_{\infty}$, we have

$$C_1(x, y, t_{r+1}) = \frac{\theta(z(P_{\infty}, \hat{\mu}(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty}, \hat{\mu}(x, y, t_{r+1})))} \exp \left( - (x - x_0)\omega_0^{(2)} ight. 
+ \left. (y - y_0)\omega_1^{(2)} + (t - t_{r+1})\omega_2^{(2)} \right).$$

(5.45)

A comparison of (4.22), (5.34), (5.35) and (5.36) then yields

$$C_2(x, y, t_{r+1}) = \frac{e^{-c_\infty} - \theta(z(P_{\infty}, \hat{\mu}(x_0, y_0, t_{r+1})))}{q(x_0, y_0, t_{r+1})} \frac{\theta(z(P_{\infty}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty}, \hat{\mu}(x_0, y_0, t_{r+1})))} \times \exp \left( (x - x_0)\omega_0^{(2)} ight. 
- \left. (y - y_0)\omega_1^{(2)} - (t - t_{r+1})\omega_2^{(2)} \right)$$

(5.46)

and

$$\tilde{C}_2(x, y, t_{r+1}) = \frac{e^{-c_\infty} - \theta(z(P_{\infty}, \hat{\mu}^*(x_0, y_0, t_{r+1})))}{q(x_0, y_0, t_{r+1})} \frac{\theta(z(P_{\infty}, \hat{\mu}^*(x, y, t_{r+1})))}{\theta(z(P_{\infty}, \hat{\mu}^*(x_0, y_0, t_{r+1})))} \times \exp \left( (x - x_0)\omega_0^{(2)} ight. 
- \left. (y - y_0)\omega_1^{(2)} - (t - t_{r+1})\omega_2^{(2)} \right).$$

(5.47)

On the other hand, from the asymptotic behavior of both sides in (4.20) and
for $P \to P_{\infty^+}$, one infers
\[
q(x, y, t_{r+1}) = q(x_0, y_0, t_{r+1}) C_1(x, y, t_{r+1})^2 \frac{\theta(z(P_{\infty^-}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty^-}, \hat{\mu}(x_0, y_0, t_{r+1})))} \\
\times \frac{\theta(z(P_{\infty^+}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty^+}, \hat{\mu}(x_0, y_0, t_{r+1})))} \\
= q(x_0, y_0, t_{r+1}) \frac{\theta(z(P_{\infty^+}, \hat{\mu}(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty^+}, \hat{\mu}(x, y, t_{r+1})))} \\
\times \frac{\theta(z(P_{\infty^-}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty^-}, \hat{\mu}(x_0, y_0, t_{r+1})))} \times \exp \left( -2(x - x_0)\omega_0^{(2)} \\
+ 2(y - y_0)\omega_1^{(2)} + 2(t - t_{r+1})\omega_r^{(2)} \right), \tag{5.48}
\]
and
\[
p(x, y, t_{r+1}) = -q(x_0, y_0, t_{r+1}) C_2(x, y, t_{r+1})^2 \frac{\theta(z(P_{\infty^-}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty^-}, \hat{\mu}(x_0, y_0, t_{r+1})))} \\
\times \frac{\theta(z(P_{\infty^+}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty^+}, \hat{\mu}(x_0, y_0, t_{r+1})))} \\
= -e^{-2c_{\infty^-}} \frac{\theta(z(P_{\infty^-}, \hat{\mu}(x_0, y_0, t_{r+1})))}{\theta(z(P_{\infty^-}, \hat{\mu}(x, y, t_{r+1})))} \\
\times \frac{\theta(z(P_{\infty^+}, \hat{\mu}(x, y, t_{r+1})))}{\theta(z(P_{\infty^+}, \hat{\mu}(x_0, y_0, t_{r+1})))} \times \exp \left( 2(x - x_0)\omega_0^{(2)} \\
- 2(y - y_0)\omega_1^{(2)} - 2(t - t_{r+1})\omega_r^{(2)} \right). \tag{5.49}
\]
Moreover, \[(5.42)\] follows from \[(5.48)\] and \[(5.49)\] by taking $(x, y, t_{r+1}) = (x_0, y_0, t_{r+1})$. Finally, the linearization property of the Abel map in \[(5.43)\] and \[(5.44)\] is a standard investigation of the differentials
\[
\Omega_i(x, y, t_{r+1}) = d\ln(\psi_i(\cdot, x, y, t_{r+1})), \quad i = 1, 2,
\]
or standard Langrange interpolation procedure (see \[25, 50\]). \qed

**Theorem 5.6.** Assume spectral curve defined in \[(3.3)\] is nonsingular and $q, p$ satisfy \[(2.55)\]. Moreover, let $P \in X \setminus \{P_{\infty^+}\}$ and the divisors $D_{\hat{\mu}(x, y, t_{r+1})}$, $D_{\hat{\mu}(x, y, t_{r+1})}$ are nonspecial. The algho-geomertic solutions of $r$th KP equation ($r \geq 2$) possess the following two kinds of theta function representations
\[
u(x, y, t_{r+1}) = \partial^2_y \ln \left( \theta(z(P_{\infty^+}, \hat{\mu}(x, -y, (-1)^rt_{r+1}))) \right) - \frac{\lambda_0}{2}, \tag{5.50}
\]
and
\[
    u(x, y, t_{r+1}) = \frac{\theta(z(P_{\infty+}, \mu(x, -y, (-1)^r t_{r+1})) \theta(z(P_{\infty-}, \mu(x, -y, (-1)^r t_{r+1}))}{\theta(z(P_{\infty-}, \mu(x, -y, (-1)^r t_{r+1}))} \theta(z(P_{\infty+}, \mu(x, -y, (-1)^r t_{r+1}))} 	imes e^{-2c_{\infty-}},
\]
where \( \lambda_0 \in \mathbb{C} \).

Proof. Firstly let us compute (5.50). Using (4.4), one infers
\[
    \psi_{1,xx} = \left( \frac{z^2}{4} + \frac{z q_x}{2 q} + q p \right) \psi_1 + \frac{q_x}{q} \psi_{1,x}.
\]
Suppose \( \psi_1 \) has the following expansions near \( P \to P_{\infty+} \)
\[
    \psi_1(P, x, y, t_{r+1}) \overset{\zeta \to 0}{=} \left( 1 + \sum_{j=1}^{\infty} \gamma_j(x, y, t_{r+1}) \zeta^j \right) \exp \left( -\frac{1}{2}(x - x_0)(\zeta^{-1} - 1) \right.
\]
\[
    + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1} + \frac{1}{2}(y - y_0)(\zeta^{-2} + O(1))
\]
\[
    + \frac{1}{2}(t_{r+1} - t_{r+1,0})(\zeta^{-(r+1)} + O(1)), \quad \zeta = z^{-1},
\]
where the constants \( \lambda_j, j = 1, 2, \ldots \), arise from Abel differentials of the
second kind. Then we have

\[
\psi_{1,x}(P,x,y,t_{r+1}) \xrightarrow{\zeta \to 0} \left( \sum_{j=1}^{\infty} \Upsilon_j(x,y,t_{r+1}) \zeta^j \right) \exp \left( -\frac{1}{2}(x-x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1}) \right) \\
+ \frac{1}{2}(y-y_0)(\zeta^{-2} + O(1)) + \frac{1}{2}(t_{r+1} - t_{r+1,0})(\zeta^{-(r+1)} + O(1)) \right) \\
+ \left( 1 + \sum_{j=1}^{\infty} \Upsilon_j(x,y,t_{r+1}) \right) \exp \left( -\frac{1}{2}(x-x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1}) \right) \\
+ \frac{1}{2}(y-y_0)(\zeta^{-2} + O(1)) + \frac{1}{2}(t_{r+1} - t_{r+1,0})(\zeta^{-(r+1)} + O(1)) \right) \\
\times \left( -\frac{1}{2} \right) \times \left( \zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1} \right), \quad \text{as } P \to P_{\infty+}, \quad (5.54)
\]

\[
\psi_{1,xx}(P,x,y,t_{r+1}) \xrightarrow{\zeta \to 0} \left( \sum_{j=1}^{\infty} \Upsilon_j(x,x,y,t_{r+1}) \zeta^j \right) \exp \left( -\frac{1}{2}(x-x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1}) \right) \\
+ \frac{1}{2}(y-y_0)(\zeta^{-2} + O(1)) + \frac{1}{2}(t_{r+1} - t_{r+1,0})(\zeta^{-(r+1)} + O(1)) \right) \\
+ 2 \left( \sum_{j=1}^{\infty} \Upsilon_{j,x}(x,y,t_{r+1}) \zeta^j \right) \exp \left( -\frac{1}{2}(x-x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1}) \right) \\
+ \frac{1}{2}(y-y_0)(\zeta^{-2} + O(1)) + \frac{1}{2}(t_{r+1} - t_{r+1,0})(\zeta^{-(r+1)} + O(1)) \right) \\
\times \left( -\frac{1}{2} \right) \times \left( \zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1} \right) + \left( 1 + \sum_{j=1}^{\infty} \Upsilon_j(x,y,t_{r+1}) \right) \zeta^j \\
\times \exp \left( -\frac{1}{2}(x-x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1}) + \frac{1}{2}(y-y_0)(\zeta^{-2} + O(1)) \\
+ \frac{1}{2}(t_{r+1} - t_{r+1,0})(\zeta^{-(r+1)} + O(1)) \right) \times \left( -\frac{1}{2} \right)^2 \times \left( \zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1} \right)^2, \\
\quad \text{as } P \to P_{\infty+}. \quad (5.55)
\]

Inserting (5.53)-(5.55) into (5.52), we get

\[
pq = \frac{\lambda_0}{2} - \Upsilon_{1,x}. \quad (5.56)
\]
Next one has to determine $\Upsilon_{1,x}$. Using (5.43), (5.44), one derives

$$
\psi_1(P, x, y, t_{r+1}) = -\Theta_{\zeta} \left( \frac{\Theta_{\zeta}(P_{r+1}, \hat{\mu}(x, y, t_{r+1}))}{\Theta_{\zeta}(P_{r+1}, \hat{\mu}(x, y, t_{r+1}))} \right) \times \exp \left( -\frac{1}{2}(x - x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^j) + \frac{1}{2}(y - y_0)(\zeta^{-2} + O(1)) \right)
+ \frac{1}{2}(t_{r+1} - t_{r+1})(\zeta^{-(r+1)} + O(1)),
\zeta \to 0
\left( 1 + \frac{\partial_{L_{\zeta}^{(2)}} \Theta_{\zeta}(P_{r+1}, \hat{\mu}(x, y, t_{r+1}))}{\Theta_{\zeta}(P_{r+1}, \hat{\mu}(x, y, t_{r+1}))} \right)
\times \exp \left( -\frac{1}{2}(x - x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^j) + \frac{1}{2}(y - y_0)(\zeta^{-2} + O(1)) \right)
+ \frac{1}{2}(t_{r+1} - t_{r+1})(\zeta^{-(r+1)} + O(1)),
\zeta \to 0
\right)^{-1}
+ \frac{1}{2}(t_{r+1} - t_{r+1})(\zeta^{-(r+1)} + O(1)) 
\times \exp \left( -\frac{1}{2}(x - x_0)(\zeta^{-1} + \sum_{j=0}^{\infty} \lambda_j \zeta^j) + \frac{1}{2}(y - y_0)(\zeta^{-2} + O(1)) \right)
+ \frac{1}{2}(t_{r+1} - t_{r+1})(\zeta^{-(r+1)} + O(1)),
(5.57)
$$

where $\partial_{L_{\zeta}^{(2)}}$ denotes the direction derivative of $\Theta$ function along vector $L_{\zeta}^{(2)} \in \mathbb{C}^n$ at the point $\zeta(P_{r+1}, \hat{\mu}(x, y, t_{r+1}))$. Thus, comparing (5.53) with (5.54), we have

$$
\Upsilon_{1,x} = -\partial^2_x \ln \left( \Theta_{\zeta}(P_{r+1}, \hat{\mu}(x, y, t_{r+1})) \right),
(5.58)
$$

Employing (5.48), (5.49), (5.50), (5.58) and Theorem 2.1, one obtains (5.50). Finally, (5.51) is the direct result of (5.37), (5.38) and Theorem 2.1. \(\square\)

**Remark 5.7.** Here we give a few remarks on the special case $r = 2$.

(i) In case $r = 2$, expressions (5.50) and (5.51) give rise to the algebro-geometric solutions of KP equation and real algebro-geometric solution of KP equation can easily derived from (5.51) by studying reduction condition $q = \pm \bar{p}$. Moreover, The transformation $x \to ix, y \to iy, t_3 \to it_3$ transforms (5.50) and (5.51) to algebro-geometric solutions of unstable version of KP
equation (KP1 equation).

(ii) The spectral curve of the KP equation (1.1) is hyperelliptic which compactified by two different points \( P_{\infty}^{\pm} \) at infinity. Similarly we can discuss algebro-geometric solution of the KP hierarchy on some specific tringular curves following this way. Moreover, if (5.50) and (5.51) are independent of \( y \), then we can derive theta function representations for algebro-geometric solutions of KdV equation. A similar remark applies also to the generation of the solutions of the KP equation from the solutions of the Boussinesq equation, given the assumption that \( u = u(x, y) \) and the dependence of \( \psi_j(x, y, t_3, P) \), \( j = 1, 2 \), on \( t_3 \) is purely exponential.

(iii) It is not difficult to verify the expression

\[
u(x, y, t_3) = -2q(x, y, t_3)p(x, y, t_3) = 2\partial_x^2 \ln (\theta(\tilde{z}(P_{\infty}^{+}, \tilde{\mu}(x, y, t_3)))) - \lambda_0 \quad (5.59)\]

gives rise to the algebro-geometric solutions of standard KP equation [40]

\[
u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} uu_x + \frac{3}{4} \partial_x^{-1} u_{yy} \quad (5.60)\]

6 Outlook

An important feature of this construction is that the Riemann surface \( X \) is not completely arbitrary. In fact, it has been proved that \( N \)-component vector nonlinear Schrödinger hierarchies (VNLS) are contained within the KP hierarchy and general algebro-geometric solutions of the KP hierarchy can be approximated by solutions of \( N \)-component VNLS hierarchies, in the limit of large \( N \) [47]. Therefore, how to derive the algebro-geometric solutions of \( N \)-component vector nonlinear Schrödinger hierarchies by extending the method of Gesztesy, \textit{et al.} remains to be a nontrivial work, which gives a complete answer to algebro-geometric solutions of the whole KP hierarchy.

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