Criticality theory for Schrödinger operators with singular potential

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Abstract

In a seminal work, B. Simon provided a classification of nonnegative Schrödinger operators \(-\Delta + V\) into subcritical and critical operators based on the long-term behaviour of the associated heat kernel. Later works by others developed an alternative subcritical/critical classification based on whether or not the operator admits a weighted spectral gap. All these works dealt only with potentials that ensured the validity of Harnack’s inequality, typically potentials in \(L^p_{\text{loc}}\) for some \(p > \frac{N}{2}\).

This paper extends such a weighted spectral gap classification to a large class of locally integrable potentials \(V\) called balanced potentials here. Harnack’s inequality will not hold in general for such potentials. In addition to the standard potentials in the space \(L^p_{\text{loc}}\) for some \(p > \frac{N}{2}\), this class contains potentials which are locally bounded above or more generally, those for which the positive and negative singularities are separated as well as those for which the interaction of the positive and negative singularities is not “too strong”.

We also establish for such potentials, under the additional assumption that \(V^+ \in L^p_{\text{loc}}\) for some \(p > \frac{N}{2}\), versions of the Agmon-Allegretto-Piepenbrink Principle characterising the subcritical/critical operators in terms of the cone of positive distributional supersolutions (or solutions).

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1 Introduction

1.1 A brief review of the literature

Understanding the nature of critical points of functionals has been one of the principal aims in the calculus of variations. Jacobi [15] was probably the first one to realise that the non-negativity of the second variation of a functional is closely related to the existence of nontrivial nonnegative solutions to the associated linear equation. This led to his well-known “non-conjugacy” condition valid for one-dimensional functionals of the type

$$J(u) = \int_0^1 f(t, u, u') dt.$$ 

Since then, analogous results in higher dimensions for linear second order symmetric elliptic operators with a potential term have been investigated by many authors. When the potential is well behaved, it has been shown that the oscillatory properties of solutions to such operators are closely related to the Morse index of the associated quadratic form (see for instance [3], [20] and [21]).

In this work, we consider Schrödinger operators of the type

$$-\Delta + V$$

in a domain \( \Omega \) of \( \mathbb{R}^N \), \( N \geq 1 \) with a potential \( V \in L^1_{\text{loc}}(\Omega) \). The properties of this operator will be related to the following associated quadratic form:

$$Q_V(\xi) = \int_\Omega \left\{ |\nabla \xi|^2 + V\xi^2 \right\}, \quad \xi \in W^{1,\infty}_c(\Omega),$$

where \( W^{1,\infty}_c(\Omega) \) stands for the space of Lipschitz continuous functions with compact support in \( \Omega \). To avoid any confusions, we introduce some basic definitions.

**Definition 1.1.** We shall call \( -\Delta + V \) a nonnegative operator in \( \Omega \) if \( Q_V(\xi) \geq 0 \) for all \( \xi \in W^{1,\infty}_c(\Omega) \).

**Definition 1.2.** We say that \( u \in L^1_{\text{loc}}(\Omega) \) is a distributional supersolution to \( -\Delta + V \) in \( \Omega \) if

$$Vu \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \int_\Omega u \left\{ -\Delta \xi + V\xi \right\} dx \geq 0, \quad \forall 0 \leq \xi \in C^2_c(\Omega).$$

A distributional solution is a function \( u \in L^1_{\text{loc}}(\Omega) \) such that (1.1) holds for all \( \xi \in C^2_c(\Omega) \).

We shall say that \( u \in L^1_{\text{loc}}(\Omega) \) is a strict distributional supersolution to \( -\Delta + V \) in \( \Omega \) if it is a distributional supersolution but not a distributional solution.

We define the following cone of all non-negative super-solutions to \( -\Delta + V \):

**Definition 1.3.** Let

$$\mathcal{C}_V(\Omega) := \{ u \in L^1_{\text{loc}}(\Omega) : u \neq 0, u \geq 0 \text{ and } u \text{ a distributional supersolution to } -\Delta + V \text{ in } \Omega \}.$$ 

It is easy to see that if \( u \in \mathcal{C}_V(\Omega) \), then \( -\Delta u + Vu \) is a non-negative Radon measure \( \mu \).

**Definition 1.4.** Given \( u \in \mathcal{C}_V(\Omega) \), the measure \( \mu := -\Delta u + V u \) will be called the Riesz measure associated to \( u \). We let \( \mathcal{M}_V(\Omega) := \{ \mu : \mu \text{ is the Riesz measure of some } u \in \mathcal{C}_V(\Omega) \} \).

In this work, we shall label as the “AAP Principle” (“Agmon-Allegretto-Piepenbrink Principle”) any statement asserting the equivalence between the strength of “non-negativity” of the Schrödinger operator \( -\Delta + V \) and the cone \( \mathcal{C}_V(\Omega) \) (or the Riesz measure space \( \mathcal{M}_V(\Omega) \)).
The contributions of Agmon [1], Allegretto [3], Gesztesy-Zhao [12], Piepenbrink [21], Pinchover-Tintaraev [25], and the regularity relaxations on \( V \) brought by them have led to the following version of the “AAP principle” (see Simon [29, Thm. C.8.1]):

**Assume that**

\[
V^+ \text{ is in the local Kato class and } V^- \text{ in Kato class in } \Omega.
\]

Then, the quadratic form \( Q_V \) is non-negative in \( \Omega \) if and only if the equation \((-\Delta + V)u = 0\) admits a positive distributional solution in \( \Omega \) i.e., iff \( 0 \in M_V(\Omega) \). Using the well-known Picone’s identity, one can easily show that the existence of a positive distributional solution implies that \( Q_V \) must be non-negative. The converse is slightly more difficult, and relies on the Harnack’s principle which holds if \( V \) satisfies (1.2). As pointed out by Fischer-Colbrie and Schoen, this principle extends to a complete manifold (see [11, Thm. 1]), and they used it to classify stable minimal surfaces in 3-manifolds of non-negative scalar curvature. For analytical applications, AAP type principles can be used to show various integral inequalities in Sobolev spaces.

Simons [28] also introduced a classification of non-negative operators \(-\Delta + V\) into subcritical and critical operators according to the long-time behavior of the associated heat kernel. In [19], Murata rephrased Simon’s idea from a different perspective. He considered potentials

\[
V \in L^p_{\text{loc}}(\Omega) \quad \text{with} \quad \begin{cases} \ p > N/2 & \text{for } N \geq 2, \\ \ p = 1 & \text{for } N = 1, \end{cases}
\]

and classified non-negative operators \(-\Delta + V\) by the sign of their fundamental solutions. He defined any such operator to be subcritical in \( \Omega \) if its fundamental solution at any point in \( \Omega \) is positive in \( \Omega \), and defined it to be critical in \( \Omega \) otherwise. The Laplacian in \( \mathbb{R}^N \) provides a very simple example to illustrate this classification: it is critical for \( N = 1, 2 \) (all fundamental solutions are of the type \(-|x|\) and \(-\log |x|\) which are not positive), whereas it is subcritical for \( N \geq 3 \) (since it has a positive fundamental solution of the type \(|x - y|^{2-N}\)). On Riemannian manifolds this classification has led to the notions of parabolic manifold (versus non-parabolic manifold), depending on whether the Laplace-Beltrami operator admits a positive fundamental solution at every point or not (see [13]).

A key feature of the classification of Murata is that subcritical/critical operator can be characterized in terms of the quadratic form \( Q_V \) as follows (see [19, Thm 2.4]):

A nonnegative operator \(-\Delta + V\) is subcritical in \( \Omega \) if and only if for any nonnegative function \( w \neq 0 \) with compact support satisfying (1.3), there exists \( \varepsilon > 0 \) such that \( Q_V(\xi) \geq \varepsilon \int_{\Omega} w \xi^2 \) for all \( \xi \in C^2_c(\Omega) \).

Furthermore, Murata also showed that this dichotomy is reflected in a very interesting way in the structure of positive solutions:

A nonnegative operator \(-\Delta + V\) is critical in \( \Omega \) if and only if the set of all positive solutions to \((-\Delta + V)u = 0 \) in \( \Omega \) is given by positive scalar multiples of a fixed positive solution.

Namely, in this case the equation \((-\Delta + V)u = 0\) admits up to scalar multiplication a unique positive solution which is usually called the “ground state”. In other words, for subcritical operators the associated quadratic form satisfies a weighted Poincaré inequality, or possesses
a “spectral gap”, whereas critical operators are the ones for which the non-negativity of the quadratic form cannot be improved at all. In particular, to show that \( Q_V \) can be improved, it is enough to find two positive linearly independent solutions to \((-\Delta + V)u = 0\) (see for instance lemma 10.19). This yields a strategy to see how far some well-known inequalities, for example classical “Hardy inequality” on a bounded domain \( \Omega^* \) of \( \mathbb{R}^N \), \( N \geq 3 \),

\[
\int_{\Omega^*} |\nabla \xi|^2 - H_N|x|^{-2}\xi^2 \geq 0 \quad \forall \xi \in C^2_c(\Omega^*),
\]

where \( H_N = \left( \frac{N-2}{2} \right)^2 \), can be improved (a question raised by Brezis and Vazquez).

Murata’s classification has been extended to non-symmetric elliptic operators with Hölder continuous coefficients by Pinchover [22]. Extension to the \( p \)-Laplace operators have been studied by Pinchover-Tintarev [26] and [24].

We remark that in one dimension, a rather exhaustive analysis of nonnegative operators \(-d^2/dx^2 + V\) has been done by Gesztesy and Zhao [12] under the only assumption that \( V \) is locally integrable; see theorems 3.1 and 3.6 there. Therefore, in this work we will mainly restrict our attention to dimensions bigger than one.

All the above works have one common feature: they impose regularity assumptions on the potential \( V \) (and the coefficients of the operator) that allow the application of Harnack’s inequality, which in particular guarantees that the solutions to \((-\Delta + V)u = 0\) are positive and continuous.

One maybe interested in frameworks that allow a study of operators which fail to satisfy the Harnack’s inequality or do not admit a fundamental solution at every point. More specifically, it will be interesting to extend the subcritical/critical classification and obtain the corresponding AAP Principle to the operator \(-\Delta + V\) for a larger class of locally integrable functions than previously considered.

In this context we summarise a part of recent work of Jaye, Mazya and Verbitsky [16] which treats potentials that are in the space \( H^{-1}_{loc} \). Given a distribution \( \sigma \) on \( \Omega \), we can define the following quadratic form

\[
Q_\sigma(\xi) = \int_{\Omega} |\nabla \xi|^2 + \langle \sigma, \xi^2 \rangle \quad \forall \xi \in C^\infty_c(\Omega).
\]

**Definition 1.5.** We say that a distribution \( \sigma \in \mathcal{D}'(\Omega) \) is

(i) form-bounded if there exist constants \( \beta_1, \beta_2 > 0 \) such that

\[
-\beta_1 \int_{\Omega} |\nabla \xi|^2 \leq \langle \sigma, \xi^2 \rangle \leq \beta_2 \int_{\Omega} |\nabla \xi|^2 \quad \forall \xi \in C^\infty_c(\Omega);
\]

(ii) upper (lower) bounded if the upper (lower) bound above holds.

We straightaway remark that if \( \sigma \) is form bounded, then in fact \( \sigma \in H^{-1}_{loc}(\Omega) \) (see [16] Lemma 2.6)). We note that the quadratic form \( Q_\sigma \) is non-negative if and only if \( \sigma \) is lower bounded with the constant \( \beta_1 \leq 1 \). The work [16] deals with the problem of finding equivalent and easier to verify conditions for the form-boundedness of a distributional potential \( \sigma \). They prove the following AAP-Principle :
Theorem 1.6. If $\sigma \in D'(\Omega)$ is form-bounded with $\beta_1 < 1$, then there exists an a.e. positive solution $u \in H_{loc}^1(\Omega)$ to the following problem

$$-\Delta u + \sigma u = 0 \text{ in } \mathcal{D}'(\Omega).$$

If $\sigma$ is a negative Borel measure, then $\sigma$ is lower bounded (and hence form-bounded) with $\beta_1 = 1$ iff there exists a positive superharmonic supersolution to the above equation.

More generally, they prove the equivalence of form boundedness of $\sigma$ to the existence of a solution to the Riccati-type equation $-\Delta u = |\nabla u|^2 - \sigma$ in $\mathcal{D}'(\Omega)$ which additionally satisfies a Caccioppoli-type inequality (see [16 Thm. 3.14]). But the subcritical/critical classification of nonnegative operators $-\Delta + \sigma$ is not treated in this work.

We remark that in theorem 1.6 the assumption $\beta_1 < 1$ implies a strong form of positivity, namely, that the pre-Hilbert space $(C_c^\infty(\Omega), \sqrt{Q_\sigma})$ is continuously imbedded in $H_{loc}^1(\Omega)$.

In this work, as we said before, our aim is to provide a subcritical/critical classification for nonnegative operators with locally integrable potentials $V$ as well as show the corresponding AAP-Principle. Using the above terminology, our main assumption is that $V$ is a “balanced” potential in $\Omega$ (see definition 6.3, 6.11) which in particular requires that $V$ is lower bounded with $\beta_1 \leq 1$ but not necessarily that it is upper bounded. In the latter half of the work we assume additionally that $V^+$ satisfies the stronger local integrability property: $V^+ \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$; that is, $V$ is “locally” upper bounded in $\Omega$. In fact, in our setting $V$ will be form-bounded in dimensions bigger than 2, and hence will belong to $H_{loc}^{1,1}(\Omega)$, if for instance $V^+ \in L^{\frac{N}{2}}_{loc}(\Omega)$. More importantly, in contrast to [16], for subcritical operators we only demand that the pre-Hilbert space $(W_c^{1,\infty}(\Omega), \sqrt{Q_V})$ imbeds continuously into $L_{loc}^1(\Omega)$.

A typical example is the critical Hardy operator

$$-\Delta - H_N|x|^{-2}$$

in a ball $B \subset \mathbb{R}^N$, $N \geq 3$, containing the origin. In fact, we base all our development on this weak assumption about the pre-Hilbert space which necessitates a restriction to balanced potentials as mentioned above. Nevertheless, we will find that almost all the well studied potentials fall into this category.

Another motivation to consider potentials $V$ which are at least locally integrable (and not more irregular) is the fact that in dimensions $N \geq 3$, the study of form bounded distributional potentials can be reduced to that of nonnegative quadratic forms $Q_V$ with $V$ nonpositive and locally integrable; see theorem 4.1 in [19].

Notations:

(i) For a set $A$, we write $A \Subset \Omega$ if $\bar{A}$ is a compact subset of $\Omega$.

(ii) The space of distributions on $\Omega$ will be denoted as $\mathcal{D}'(\Omega)$.

(iii) For a function $w \in L_{loc}^1(\Omega)$, we will write “$Q_V \geq w$ in $\Omega$” to express the condition:

$$Q_V(\xi) \geq \int_{\Omega} w\xi^2, \forall \xi \in W_c^{1,\infty}(\Omega).$$
(iv) For a function \( f \) its support \( \{ f \neq 0 \} \) will be denoted as \( \text{supp}(f) \), and given a space \( X \) of functions on \( \Omega \), we let \( X_c \) denote its subspace consisting of those with compact support in \( \Omega \).

(v) “\( \inf \)”, “\( \sup \)” will denote respectively the essential infimum and supremum.

(vi) \( \| \cdot \|_{p,A} \) will denote the \( L^p \)-norm on a set \( A \).

(vii) For \( N \geq 3 \) and \( 1 \leq p < \infty \), the space \( D^{1,p}_0(\Omega) \) is the completion of \( C^\infty_c(\Omega) \) with respect to the norm: \( \left( \int_\Omega |\nabla \cdot |^p \right)^{\frac{1}{p}} \).

(viii) Given two topological spaces \( X,Y \), we write \( X \hookrightarrow Y \) to mean that \( X \subset Y \) and the inclusion map is continuous.

1.2 Contents of the paper

In §2 we recall some preliminary results required in the rest of the paper.

In §3 we show how the energy space \( H^V(\Omega) \) can be constructed from a given nonnegative quadratic form \( Q_V \) on \( \Omega \) under the only assumption that \( V \in L^1_{\text{loc}}(\Omega) \). Some results relating the singularities of \( V \) to the “size” of the energy space are proved, and we also introduce the concept of an “energy solution in \( H^V(\Omega) \)” to the equation \( -\Delta u + Vu = f \).

We introduce in §4 two types of subcritical/critical classification for nonnegative operators \( -\Delta + V \) on \( \Omega \) based on whether the energy space imbeds into \( L^1_{\text{loc}} \) or \( L^2_{\text{loc}} \) space. The former imbedding leads to the \( L^1 \)-subcritical/critical set of operators and the latter into feebly and globally \( L^2 \)-subcritical/critical set of operators. Given that we work with \( V \) whose negative part \( V^- \) is especially unrestricted, it is nontrivial to show that these three notions of criticalities coincide (see §9).

In §5 we explore the various imbeddings of the energy space \( H^V(\Omega) \) when \( -\Delta + V \) is \( L^1 \)-subcritical in \( \Omega \). In this a special reference can be made to §5.1 which resembles a statement of continuity for the map \( \xi \mapsto \int_K V\xi, K \subset \Omega \) on the energy space \( H^V(\Omega) \) with a correction term.

We introduce the notion of “balanced” and “tame” potentials \( V \) in §6 which are central to the paper. We show that the concept of balanced potentials covers well-known ones as well as many that are not usually considered. We show that for balanced potentials \( V \) that are also \( L^1 \)-subcritical, the concept of energy solutions is equivalent to that of distributional solutions.

§7 deals with AAP principle for a general nonnegative operator \( -\Delta + V \). Assuming \( V \in L^1_{\text{loc}}(\Omega) \), we show that existence of a nonnegative (distributional) super solution to this operator implies that the associated quadratic form \( Q_V \) is nonnegative. The converse is proved under the additional condition that \( V^+ \in L^p_{\text{loc}}(\Omega) \) for some \( p > \frac{N}{N-2} \) (\( N \geq 2 \)), and relies on the \( L^1 \)-theory developed by Stampacchia for the operator \( -\Delta + V^+ \).

We exploit the result on equivalence of energy and distribution solutions to develop an AAP principle in §8 for \( L^1 \)-subcritical operators with balanced potentials.

Using the results from the previous sections, in subsection §9.1 we show that the notions of \( L^1 \)- and globally \( L^2 \)-criticalities (introduced in §4) coincide for nonnegative operators with balanced potentials. In subsection §9.2 we show that notions of \( L^1 \)- and feebly \( L^2 \)-criticalities too coincide for the class of tame potentials.
In §10, we exploit this equivalence between the two notions of criticality for tame potentials and provide characterisations (in the spirit of the AAP principle) of nonnegative subcritical operators in terms of their Riesz measures (see theorems 10.5 and 10.7) as well as the dimension of the cone of nonnegative (distributional) super solutions (see theorem 10.22).

Finally in §11 we give applications and examples of our results.

2 Some Preliminaries

We first recall the concept of harmonic capacity (called here as “Cap”):

**Definition 2.1.** For a compact set $K \subset \Omega$, the harmonic capacity of $K$ with respect to $\Omega$ is defined as:

$$\text{Cap}(K; \Omega) := \inf \left\{ \int_{\Omega} |\nabla \xi|^2 : \xi \in C_\infty^c(\Omega), \xi \geq 1 \text{ on } K \right\}.$$ 

The above definition can be used to define the capacity $\text{Cap}(A; \Omega)$ of any subset $A$ of $\Omega$. In particular, the following fact is well known:

If $\text{Cap}(A; \Omega) = 0$ for any bounded subset $A$ of $\Omega$ then $\text{Cap}(A \cap U; U) = 0$ for any open set $U \subset \mathbb{R}^N$. In such a situation, we will simply say $\text{Cap}(A) = 0$.

We have then the following characterisation for the validity of $Q_V \succeq 0$ when $V$ is non-positive (see chapter 2 in [18]):

**Theorem 2.2.** Let $V \in L^1_{\text{loc}}(\Omega)$ with $V^+ \equiv 0$. Then $Q_V \succeq 0$ in $\Omega$ if and only if the following inequality holds:

$$\frac{1}{4} \leq \sup_{K \in \mathcal{N}} \left( \int_K V^- \right) \leq 1$$

where $\mathcal{N} := \{K : K \text{ is a compact subset of } \Omega \text{ with } \text{Cap}(K; \Omega) > 0\}$.

The main trouble in using the above characterisation is that the calculation of capacity of sets is hard. In this context, we cite the following more concrete characterisation:

**Theorem 2.3.** [10, theorem 5.1] 

$$\left\{ V \in L^1_{\text{loc}}(\Omega) : Q_V \succeq 0 \text{ in } \Omega \right\} = \left\{ V \in L^1_{\text{loc}}(\Omega) : V \geq \text{div}\tilde{f} + |\tilde{f}|^2 \text{ for some } \tilde{f} \in (L^2_{\text{loc}}(\Omega))^N \right\}.$$ 

The inequality above should be understood in the sense of distributions.

**Definition 2.4.** Let $V \in L^1_{\text{loc}}(\Omega)$. Given $f \in L^1_{\text{loc}}(\Omega)$, we say that a function $u \in L^1_{\text{loc}}(\Omega)$ is a distribution solution of $-\Delta u + Vu = f$ in $\Omega$ if

$$Vu \in L^1_{\text{loc}}(\Omega) \text{ and } \int_{\Omega} u (-\Delta \xi + V \xi) = \int_{\Omega} f \xi, \text{ for all } \xi \in C_\infty(\Omega).$$

In the following lemma we recall results on the solvability for a Schrodinger operator with a non-negative potential and the right-hand side data given in $H^{-1}$. 


Lemma 2.5. Let $\Omega_*$ be a bounded open set, $W \in L^1_{\text{loc}}(\Omega_*)$, $W \geq 0$, and $f \in H^{-1}(\Omega_*)$. Then the problem
\begin{equation}
-\Delta u + Wu = f
\end{equation}
admits a unique distribution solution $u \in H^1_0(\Omega_*)$. Furthermore, for any compact set $K \subset \Omega_*$,
\begin{equation}
\|Wu\|_{L^1(K)} \leq 2\|W\|_{L^1(K)}^{1/2}\|f\|_{H^{-1}(\Omega_*)}.
\end{equation}
If in addition $f \geq 0$, the solution is nonnegative.

Proof. Let $(\cdot, \cdot)$ denote the duality pairing between $H^{-1}(\Omega_*)$ and $H^1_0(\Omega_*)$. Consider the functional
\[ J : H^1_0(\Omega_*) \to \mathbb{R} \cup \{\infty\}, \quad J(w) = \frac{1}{2} \int_{\Omega_*} \left\{ |\nabla w|^2 + W(x)w^2 \right\} dx - (f, w). \]
Since this functional is coercive and strictly convex, it admits a unique minimizer $u$ in the subspace $X := \{ u \in H^1_0(\Omega_*): J(u) < \infty \}$. For each $\xi \in C^1_c(\Omega_*)$, and $t \in \mathbb{R}$ we have $u + t\xi \in X$ and hence
\[ \frac{d}{dt} J(u + t\xi) \bigg|_{t=0} = 0. \]
This implies
\[ \int_{\Omega_*} \{ \nabla u \cdot \nabla \xi + Wu\xi \} = (f, \xi), \quad \forall \xi \in C^1_c(\Omega_*). \]
Furthermore, since $J(u) \leq J(0) = 0$ we deduce
\[ \int_{\Omega_*} \left\{ |\nabla u|^2 + W(x)u^2 \right\} dx \leq 2\|f\|_{H^{-1}(\Omega_*)}\|u\|_{H^1_0(\Omega_*)}. \]
Therefore, $\|u\|_{H^1_0(\Omega_*)} \leq 2\|f\|_{H^{-1}(\Omega_*)}$ and hence, $\|Wu^2\|_{L^1(K)} \leq 4\|f\|_{H^{-1}(\Omega_*)}^2$. So, on each $K \subset \Omega_*$ we get,
\[ \int_K |u| \leq \left( \int_K W \right)^{1/2} \left( \int_K |u|^2 \right)^{1/2} \leq 2\|W\|_{L^1(K)}^{1/2}\|f\|_{H^{-1}(\Omega_*)}. \]
Under the further assumption that $(f, \xi) \geq 0$ for all nonnegative $\xi \in H^1_0(\Omega_*)$, we have $(f, |u| - u) \geq 0$. In this case we deduce $J(u) \geq J(|u|)$, which proves that the unique minimizer is a nonnegative function.

Using the imbedding $L^\infty(\Omega_*) \hookrightarrow H^{-1}(\Omega_*)$, as a particular case of Lemma 2.5 we have

Corollary 2.6. Let $\Omega_*$ be a bounded open set, $W \in L^1_{\text{loc}}(\Omega_*)$, $W \geq 0$, and $f \in L^\infty(\Omega_*)$. Then the problem (2.1) admits a unique distribution solution $u \in H^1_0(\Omega_*)$. Furthermore, for any compact set $K \subset \Omega_*$,
\begin{equation}
\|Wu\|_{L^1(K)} \leq 2\|W\|_{L^1(K)}^{1/2}\|f\|_{L^\infty(\Omega_*)}.
\end{equation}
If in addition $f \geq 0$, the solution is nonnegative.
In order to solve with a general measure as the right-hand side data, we need to go out of the space $H^1_0(\Omega)$. Indeed, we require the following concept of “very weak” solutions:

**Definition 2.7.** Let $V \in L^1_{\text{loc}}(\Omega)$ and $\mu$ be a finite Borel measure on $\Omega$. Then, $u \in L^1(\Omega)$ is said to be a very weak solution of the problem $-\Delta u +Vu = \mu$ in $\Omega$, $u = 0$ on $\partial \Omega$ if $Vu \in L^1(\Omega)$ and

$$
\int_{\Omega} u(-\Delta \xi + V\xi) = \int_{\Omega} \xi d\mu \quad \text{for all } \xi \in C^2(\Omega) \cap C_0(\Omega).
$$

**Remark 2.8.** This notion of solution is equivalent to asking that $u \in W^{1,1}_0(\Omega)$ solve the above equation in the sense of distributions (see [27, Corollary 4.5]).

We then have the following fundamental result due to Stampacchia based on the duality method (see [30, théorème 9.1]):

**Proposition 2.9.** Let $\mu$ be a finite Borel measure on a bounded smooth domain $\Omega^*$ and $W \in L^p(\Omega^*)$ for some $p > N/2$ be a nonnegative function. Then there exists a very weak solution $u \in L^1(\Omega^*)$ of the problem $-\Delta u + Wu = \mu$ in $\Omega^*$, $u = 0$ on $\partial \Omega^*$. Furthermore, $u \in W^{1,p}_0(\Omega^*)$ for any $p \in [1, \frac{N}{N-1})$ and the following estimate holds:

$$
\|u\|_{W^{1,p}_0(\Omega^*)} \leq C(p,\Omega^*)|\mu|(\Omega^*), \quad \forall 1 \leq p < \frac{N}{N-1}.
$$

A very weak solution satisfying the above properties will be identical to $u$.

The following is a local version of regularity for solutions with measure data:

**Proposition 2.10.** Let $u$ be a distributional solution of $-\Delta u = \mu$ for some Radon measure $\mu$ in $\Omega$. Then $u \in W^{1,q}_{\text{loc}}(\Omega)$ for any $q \in [1, \frac{N}{N-1})$.

**Proof.** Fix any ball $B \subset \Omega$. Let $B_1, B_2$ be balls with $B \subset B_1 \subset B_2 \subset \Omega$. Define $\mu_1 := \mu \cdot B_1$. Consider the problem

$$
-\Delta v = \mu_1 \quad \text{in } B_2, \quad v = 0 \quad \text{on } \partial B_2.
$$

Then by proposition 2.9 above, there exists a unique very weak solution $v$ of the above problem with $v \in W^{1,q}_{\text{loc}}(B_2)$ for all $1 \leq q < \frac{N}{N-1}$. Let $\psi := u - v$. Then $\psi$ is harmonic in $B_1$ and hence is locally smooth there. Thus $u \in W^{1,q}(B)$ for any $q \in [1, \frac{N}{N-1})$.

The following weak maximum principle for distributional supersolutions is well known (see [8, proposition B.1]):

**Lemma 2.11.** Let $\Omega^*$ be a bounded smooth open set. If $u \in W^{1,1}_{\text{loc}}(\Omega^*)$ satisfies $-\Delta u \geq 0$ in $\mathcal{D}'(\Omega^*)$, then $u \geq 0$ a.e. in $\Omega^*$.

The following is a version of the strong maximum principle due to Ancona [4] and Brezis, Ponce [6] (see theorem 1), but in a distributional setting:

**Theorem 2.12.** (Strong Maximum Principle)

Let $U \subset \mathbb{R}^N$ be a non-empty open connected bounded set and $W \in L^1_{\text{loc}}(U)$ with $W \geq 0$. 


Therefore, the following Euler-Lagrange equation holds:
\[ -\Delta u + Wu \geq 0 \text{ in } D'(U), \]

then either \( u \equiv 0 \) a.e., or else \( \text{Cap}\{u = 0\} = 0. \) If the latter holds, we say \( u > 0 \) q.a.e. (quasi almost everywhere).

We note that, since \( Wu \in L^1_{loc}(U) \), from Remark 3 of [8] indeed we have \( \Delta u \leq Wu \) in the sense of measures and theorem 1 of [8] applies.

3 Energy space \( \mathcal{H}_V(\Omega) \)

Throughout this section we only assume \( V \in L^1_{loc}(\Omega) \).

**Definition 3.1.** Let \( W^{1,\infty}_c(\Omega) \) be the vector space of Lipschitz continuous functions with compact support in \( \Omega \). We define the quadratic form
\[ Q_V(u) := \int_\Omega \left\{ |\nabla u|^2 + Vu^2 \right\}, \quad u \in W^{1,\infty}_c(\Omega). \]

**Lemma 3.2.**
(i) \( Q_V \) is non-negative in \( W^{1,\infty}_c(\Omega) \) whenever \( Q_V \) is non-negative in \( C^1_c(\Omega) \).

(ii) Given \( u \in W^{1,\infty}_c(\Omega) \), we have \( |u| \in W^{1,\infty}_c(\Omega) \) and \( Q_V(u) = Q_V(|u|) \).

(iii) Assume \( Q_V \) is non-negative in \( C^1_c(\Omega) \). Then, \( Q_V \) is positive definite on \( W^{1,\infty}_c(\Omega) \) and in particular on \( C^1_c(\Omega) \).

**Proof.** (i) Let \( u \in W^{1,\infty}_c(\Omega) \). Let \( \rho_\varepsilon \in C^\infty(\Omega) \) be an approximation of the unity and define \( u_\varepsilon := u \ast \rho_\varepsilon \). Since \( u \) has compact support, we have \( u_\varepsilon \in C^\infty(\Omega) \) with all their supports contained in a fixed compact set in \( \Omega \) for all \( \varepsilon > 0 \) sufficiently small. Furthermore, \( u_\varepsilon \) converges uniformly to \( u \) in \( \Omega \) and \( \nabla u_\varepsilon \to \nabla u \) in \( (L^2(\Omega))^N \). We therefore conclude that \( Q_V(u_\varepsilon) \to Q_V(u) \). Non-negativity of \( Q_V \) in \( W^{1,\infty}_c(\Omega) \) follows from its nonnegativity in \( C^1_c(\Omega) \).

From the identity,
\[ Q_V(u - w) = Q_V(u) + Q_V(w) - 2 \int_\Omega \left\{ \nabla u \cdot \nabla w + Vuw \right\} \]
we in fact obtain that \( Q_V(u - u_\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

(ii) Well known.

(iii) Assume that \( Q_V(u_0) = 0 \) for some \( u_0 \in W^{1,\infty}_c(\Omega) \). Then, \( u_0 \) is a minimizer for \( Q_V \) on \( W^{1,\infty}_c(\Omega) \). Since \( |u_0| \in W^{1,\infty}_c(\Omega) \) and \( Q_V(u_0) = Q_V(|u_0|) \), we may assume that \( u_0 \geq 0 \) and the following Euler-Lagrange equation holds:
\[ \int_\Omega \nabla u_0 \cdot \nabla \xi + Vu_0 \xi = 0, \quad \forall \xi \in C^1_c(\Omega). \]

Therefore,
\[ \int_\Omega \nabla u_0 \cdot \nabla \xi + V^+ u_0 \xi \geq 0, \quad \forall \xi \in C^1_c(\Omega), \xi \geq 0. \]
By applying the strong maximum principle as stated in Theorem 2.12, we have the alternative: either \( u_0 > 0 \) a.e. or \( u_0 \equiv 0 \). Since \( u_0 \) has compact support, we conclude that indeed \( u_0 \equiv 0 \). □

**Definition 3.3.** Suppose \( Q_V \geq 0 \) in \( \Omega \), i.e., \( Q_V \) is non-negative on \( W^{1,\infty}_c(\Omega) \). From the above lemma we deduce that

\[
x_V(\xi, \phi) := \int_{\Omega} \nabla \xi \cdot \nabla \phi + V \xi \phi
\]

defines an inner product on \( W^{1,\infty}_c(\Omega) \times W^{1,\infty}_c(\Omega) \) with the associated norm \( \sqrt{Q_V} \).

The pre-Hilbert space \( (W^{1,\infty}_c(\Omega), \sqrt{Q_V}) \) will be denoted \( \mathcal{L}_V(\Omega) \) and its closure under the \( \sqrt{Q_V} \) norm will be called the energy space \( \mathcal{H}_V(\Omega) \) associated to the Schrödinger operator \( -\Delta + V \).

**Lemma 3.4.** \( C^\infty_c(\Omega) \) is a dense subspace of \( \mathcal{H}_V(\Omega) \).

**Proof.** From (i) in the proof of lemma 3.2 we obtain that \( C^\infty_c(\Omega) \) is dense in \( L^V(\Omega) \). Density in \( \mathcal{H}_V(\Omega) \) now follows from the fact that \( L^V(\Omega) \) is a dense subspace of \( \mathcal{H}_V(\Omega) \). □

In the following proposition, we state some relations between the singularities of \( V \) and the “size” of the corresponding \( \mathcal{H}_V(\Omega) \) space.

**Proposition 3.5.** Let \( U \) be an open set in \( \mathbb{R}^N \), \( N \geq 2 \) which is restricted to be bounded if \( N = 2 \). Assume \( V \in L^1_{loc}(U) \) to be such that \( Q_V \geq 0 \) in \( U \).

(i) Let \( V^+ \in L^\frac{N}{N-2}(U) \) if \( N \geq 3 \) and \( V^+ \in L^q(U) \) for some \( q > 1 \) when \( N = 2 \). Then

\[
(3.1) \quad (W^{1,\infty}_c(U), \| \cdot \|_{D^{1,2}_0(U)}) \hookrightarrow \mathcal{L}_V(U).
\]

(ii) If for some \( 1 \leq p < 2 \),

\[
(3.2) \quad (W^{1,\infty}_c(U), \| \cdot \|_{D^{1,p}_0(U)}) \hookrightarrow \mathcal{L}_V(U)
\]

then \( V^- \not\in L^\frac{N}{N-2}(B) \) for any ball \( B \Subset U \). In particular, if \( N = 2 \), the above imbedding is impossible.

**Proof.** (i) Let \( N \geq 3 \). Using Sobolev imbedding we note that for any \( \xi \in W^{1,\infty}_c(U) \),

\[
0 \leq Q_V(\xi) \leq \int_U \left\{ |\nabla \xi|^2 + V^+ \xi^2 \right\} \leq C \left( 1 + \| V^+ \|_{L^\frac{N}{N-2}(U)} \right) \| \xi \|_{W^{1,2}_c(U)}^2.
\]

A similar proof holds when \( N = 2 \).

(ii) We prove the contrapositive statement. Assume \( V^- \in L^\frac{N}{N-2}(B) \) for some ball \( B \Subset U \) with center \( x_0 \). Choose the sequence \( \{ \xi_k \} \):

\[
\xi_k(x) := (1 - k|x - x_0|) \chi_{\{|x-x_0| \leq \frac{1}{k}\}},
\]

where
and note that \( \{\xi_k\} \subset W^{1,\infty}_c(B) \) for \( k \) large. A straightforward computation and Hölder’s inequality give
\[
\int_U |\nabla \xi_k|^p \sim k^{p-N} \quad \int_U V^\frac{}{}^{-\xi_k^2} \leq O_k(1) k^{2-N} \|V\|_{L^{N/2}(\{|x-x_0| \leq \frac{1}{k}\})}.
\]
Hence, for some constant \( c > 0 \),
\[
\frac{Q_V(\xi_k)}{\|\xi_k\|_{D_0^\infty,U}^2} \geq \frac{c}{k^{2(1-N/p)}} \int_U \left\{ |\nabla \xi_k|^2 - V^{-\xi_k^2} \right\} \geq c k^{N(2-p)/p}.
\]
Since \( p \in [1,2) \), it follows that the imbedding (3.2) cannot hold.

**Corollary 3.6.** Let \( V^+ \in L^\infty_{\text{loc}}(\Omega) \) if \( N \geq 3 \) and \( V^+ \in L^2_{\text{loc}}(\Omega) \) for some \( q > 1 \) when \( N = 2 \). Then, for each open set \( U \Subset \Omega \) the imbedding (3.1) holds and denoting by \( T : \mathbb{D}_0^{1,2}(U) \to \mathcal{H}_V(U) \) the continuous linear extension of this imbedding, we have
\[
\|T(u)\|_{V,U}^2 = \int_U \left\{ |\nabla u|^2 + Vu^2 \right\} \quad \forall u \in H^1_c(U) \cap C_c(U).
\]

**Proof.** Consider the approximating sequence \( \{u_\epsilon := u * \rho_\epsilon\} \) considered in (i) of lemma 3.2 above. Clearly, \( u_\epsilon \to u \) in \( H^1_c(U) \). Hence, we have that \( \{u_\epsilon\} \) is a Cauchy sequence in \( \mathcal{L}_V(U) \). Necessarily, \( u_\epsilon \to T(u) \) in \( \mathcal{H}_V(U) \); that is, \( Q_V(u_\epsilon) \to \|T(u)\|_V,U \). Since \( u_\epsilon \to u \) in \( C(U) \) and \( \nabla u_\epsilon \to \nabla u \) in \( (L^2(U))^N \), we get
\[
Q_V(u_\epsilon) \to \int_U \left\{ |\nabla u|^2 + Vu^2 \right\}.
\]

In the energy space, we can consider the notions of solution and supersolutions in the following sense:

**Definition 3.7.** Let \( V \in L^1_{\text{loc}}(\Omega) \) and \( f \in L^1_{\text{loc}}(\Omega) \). We say that \( u_* \in \mathcal{H}_V(\Omega) \) is an “energy supersolution” to \(-\Delta u + Vu = f\) if
\[
a_{V,\Omega}(u_*, \xi) \geq \int_\Omega f \xi, \quad \forall \xi \in W^{1,\infty}_c(\Omega), \xi \geq 0.
\]

\((u_* \in \mathcal{H}_V(\Omega) \) will be called an “energy subsolution” if the inequality above is reversed.)

We say that \( u_* \in \mathcal{H}_V(\Omega) \) is an “energy solution” to \(-\Delta u + Vu = f\) if
\[
a_{V,\Omega}(u_*, \xi) = \int_\Omega f \xi, \quad \forall \xi \in W^{1,\infty}_c(\Omega).
\]

**4 Classification of Schrödinger operators : Two notions of Criticality**

**Definition 4.1.** We call the quadratic form \( Q_V \) to be supercritical in \( \Omega \), if \( Q_V \not\leq 0 \) in \( \Omega \); that is, \( Q_V(\xi) < 0 \) for some \( \xi \in W^{1,\infty}_c(\Omega) \).
We introduce the following classification of criticality to Schrödinger operators with only locally integrable potential $V$. We can use the pre-energy space $L^1_V(\Omega)$ to define the following $L^1$-version of criticality:

**Definition 4.2** ($L^1$-criticality). Let $V \in L^1_{\text{loc}}(\Omega)$. We say that the operator $-\Delta + V$ is:

(i) $L^1$-subcritical in $\Omega$, if $Q_V \succeq 0$ in $\Omega$ and $L^1_V(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega)$;

(ii) $L^1$-critical in $\Omega$, if $Q_V \succeq 0$ in $\Omega$ but $-\Delta + V$ is not $L^1$-subcritical in $\Omega$;

**Definition 4.3.** If $L^1_V(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega)$, we can uniquely extend this inclusion map to a continuous linear map $J : H_V(\Omega) \rightarrow L^1_{\text{loc}}(\Omega)$.

In particular, for any compact set $K \subset \Omega$, we have

$$\int_K |J(u)| \leq C_K \|u\|_{V,\Omega} \quad \text{for all } u \in H_V(\Omega).$$

The map $J$ may not be injective in general. For a large class of potentials called balanced potentials (see definition 6.3) we can show the injectivity; see corollary 6.8.

We now define two notions of criticality based on weighted $L^2$-spaces, one notion global in $\Omega$ and the other local.

**Definition 4.4** (Global $L^2$-criticality). Let $V \in L^1_{\text{loc}}(\Omega)$. The operator $-\Delta + V$ is said:

(i) **globally** $L^2$-subcritical in $\Omega$, if $Q_V \succeq 0$ in $\Omega$ and $L^1_V(\Omega) \hookrightarrow L^2(\Omega;wdx)$ for some non-negative weight $w \in L^1_{\text{loc}}(\Omega)$ satisfying $1/w \in L^1_{\text{loc}}(\Omega)$.

(ii) **globally** $L^2$-critical in $\Omega$ if $Q_V \succeq 0$ in $\Omega$, but it is not globally $L^2$-subcritical in $\Omega$.

**Definition 4.5** (Feeble $L^2$-criticality). Let $V \in L^1_{\text{loc}}(\Omega)$. The operator $-\Delta + V$ is said:

(i) **feebly** $L^2$-subcritical in $\Omega$, if $Q_V \succeq 0$ in $\Omega$ and $L^1_V(\Omega) \hookrightarrow L^2(K)$ for some compact set $K \subset \Omega$ with positive measure;

(ii) **feebly** $L^2$-critical in $\Omega$ if $Q_V \succeq 0$ in $\Omega$, but it is not feebly $L^2$-subcritical in $\Omega$.

The following implications are easy to show:

- global $L^2$-subcriticality $\implies$ feeble $L^2$-subcriticality
- global $L^2$-subcriticality $\implies$ $L^1$-subcriticality.

We refer to proposition 9.7 (see definition 6.3) for a large class of potentials $V$ for which all the three definitions of criticality are equivalent.

Note that our definitions differ from the ones given in the literature for more “regular” potentials $V$ (see for instance, [19] and [25]) in two respects. Firstly, in our setting the operator $-\Delta + V$ does not necessarily admit a Green’s function. Secondly, in the global $L^2$-subcriticality definition, we allow weights $w$ that may vanish in a null set and in the feeble $L^2$-criticality definition, the quadratic form is assumed to be “coercive” only on a compact set with positive measure.

The following proposition justifies the use of weights like $w$ and $\chi_K$ in our definitions of $L^2$-subcriticality.
\textbf{Proposition 4.6.} Let $N \geq 3$ and $V \in L^1_{\text{loc}}(\Omega)$. Assume that $Q_V \geq 0$ in $\Omega$ and
\[ (W^{1,\infty}_c(\Omega), \| \cdot \|_{W^{1,p}_0(\Omega)}) \hookrightarrow L_V(\Omega) \text{ for some } 1 \leq p < \frac{2N}{N+2}. \]
Then any nonnegative weight $w \in L^1_{\text{loc}}(\Omega)$ such that
\[ L_V(\Omega) \hookrightarrow L^2(\Omega; w \, dx) \]
must satisfy $\inf_B w = 0$ on any ball $B \Subset \Omega$. In particular, if $w \in C(\Omega)$ then $w \equiv 0$.

\textit{Proof.} Let $B \Subset \Omega$ be a ball. With our assumption on $p$, we note that $W^{1,p}_0(B) \not\subset L^2(B)$. Choose a sequence $\{\xi_n\} \subset C^\infty_c(B)$ such that
\[ \sup_n \|\xi_n\|_{W^{1,p}_0(B)} < \infty \quad \text{and} \quad \|\xi_n\|_{L^2(B)} \to \infty \text{ as } n \to \infty. \]
By the assumed imbedding, we note that $\{\xi_n\}$ is a bounded sequence in $L_V(\Omega)$. Therefore,
\[ \int_\Omega w|\xi_n|^2 \leq Q_V(\xi_n) \leq C < \infty. \]
This shows that $\inf_B w = 0$. \hfill \Box

\section{Various embeddings of the space $L_V(\Omega)$}

Recall from definition 3.3 that $L_V(\Omega) := (W^{1,\infty}_c(\Omega), \sqrt{Q_V})$.

\textbf{Proposition 5.1.} Let $V_i \in L^1_{\text{loc}}(\Omega)$ be such that $Q_{V_i} \geq 0$ in $\Omega$, $i = 1, 2$ and $V_1 \geq V_2$.

(i) Then, the identity map $\text{id} : L^1_{V_1}(\Omega) \to L^1_{V_2}(\Omega)$ is continuous and it admits a unique continuous linear extension $\text{id}^* : H_{V_1}(\Omega) \to H_{V_2}(\Omega)$;

(ii) If furthermore $-\Delta + V_i$ is an $L^1$-subcritical operator in $\Omega$ ($i = 1, 2$) with the corresponding extensions $J_i$, then $J_1 = J_2 \circ \text{id}^*$ and $\text{id}^*$ is injective whenever $J_1$ is injective.

\textit{Proof.} (i) Follows from the fact : $0 \leq Q_{V_2}(\xi) \leq Q_{V_1}(\xi)$ for all $\xi \in W^{1,\infty}_c(\Omega)$.

(ii) Let $u \in H_{V_1}(\Omega)$ and $\{\xi_n\} \subset C^\infty_c(\Omega)$ be such that $\xi_n \to u$ in $H_{V_1}(\Omega)$. Then, $\xi_n \to J_1(u)$ in $L^1_{\text{loc}}(\Omega)$. Note that $\{\xi_n\}$ is a Cauchy sequence in $H_{V_2}(\Omega)$. Let $\tilde{u}$ be the limit of $\{\xi_n\}$ in $H_{V_2}(\Omega)$. This means that $\text{id}^*(u) = \tilde{u}$ and $\xi_n \to J_2(\tilde{u})$ in $L^1_{\text{loc}}(\Omega)$. Therefore, $J_2(\text{id}^*(u)) = J_1(u)$. Injectivity of $\text{id}^*$ follows easily. \hfill \Box

\textbf{Lemma 5.2.} Assume $Q_V \geq 0$ in $\Omega$. Then, the following properties hold:

(i) $L^+_{V_+}(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega)$, $L^+_{V_+}(\Omega) \hookrightarrow L^2(\Omega, V^+ \, dx)$ and $L^+_{V_+}(\Omega) \hookrightarrow L^2(\Omega, V^+ \, dx)$.

(ii) Fix $\varphi \in C^2_c(\Omega)$. Then, there exists a constant $C := \sqrt{Q_V(\varphi)} + \|\Delta \varphi\|_{L^\infty(\text{supp(}\varphi))}$ such that
\[ \left| \int_\Omega V \varphi \xi \right| \leq C \left( \sqrt{Q_V(\xi)} + \int_{\text{supp}(\varphi)} |\xi| \right), \quad \forall \xi \in W^{1,\infty}_c(\Omega). \]
(iii) Suppose $\mathcal{L}_V(\Omega) \hookrightarrow L_{1\text{loc}}^1(\Omega)$. Fix $\varphi \in C_c^2(\Omega), \varphi \geq 0$. Consider the linear maps
\[(5.2) \quad \Phi^\pm : \mathcal{L}_V(\Omega) \rightarrow L^1(\Omega), \quad \text{given by } \Phi^\pm(\xi) = V^\pm \varphi \xi.\]

Then $\Phi^+$ is continuous iff $\Phi^-$ is continuous.

(iv) Suppose $\mathcal{L}_V(\Omega) \hookrightarrow L_{1\text{loc}}^1(\Omega)$. Then,
\[\mathcal{L}_V(\Omega) \hookrightarrow L_{1\text{loc}}^1(\Omega, V^+dx) \quad \text{iff} \quad \mathcal{L}_V(\Omega) \hookrightarrow L_{1\text{loc}}^1(\Omega, V^-dx).\]

Proof. (i) Note that $Q_{V^+} \geq 0$ in $\Omega$. Then, the embedding $\mathcal{L}_V^+(\Omega) \hookrightarrow \mathcal{L}_V(\Omega)$ is a special case of proposition 5.1. The other two imbeddings follow from the inequalities
\[\int_\Omega V^- \xi^2 \leq Q_{V^+}(\xi), \quad \int_\Omega V^+ \xi^2 \leq \int_\Omega \{ |\nabla \xi|^2 + V^+ \xi^2 \} \text{ for all } \xi \in W^{1,\infty}_c(\Omega).\]

(ii) Noting that $a_{V,\Omega}(\varphi, \xi) \leq \sqrt{Q_V(\varphi)}\sqrt{Q_V(\xi)}$ we deduce that
\[\int_\Omega V \varphi \xi = a_{V,\Omega}(\varphi, \xi) - \int_\Omega \nabla \varphi \cdot \nabla \xi \leq \sqrt{Q_V(\varphi)}\sqrt{Q_V(\xi)} + \int_\Omega (\Delta \varphi) \xi \leq C(V, \varphi) \left( \sqrt{Q_V(\xi)} + \int_{\text{supp}(\varphi)} |\xi| \right), \forall \xi \in W^{1,\infty}_c(\Omega),\]
where $C(V, \varphi) := \sqrt{Q_V(\varphi)} + \| \Delta \varphi \|_{L^\infty(\text{supp}(\varphi))}$. Replacing $\xi$ by $-\xi$ in the above calculation, we obtain (5.1).

(iii) Using $\mathcal{L}_V \hookrightarrow L_{1\text{loc}}^1(\Omega)$, we rewrite (5.1) as the following (replacing $\xi$ by $|\xi|$):
\[\left| \int_\Omega V^+ \varphi |\xi| - \int_\Omega V^- \varphi |\xi| \right| \leq C_\varphi \sqrt{Q_V(\xi)}, \forall \xi \in W^{1,\infty}_c(\Omega).\]

We conclude that
\[\int_\Omega V^\pm \varphi |\xi| \leq C_\varphi \sqrt{Q_V(\xi)} + \int_\Omega V^\pm \varphi |\xi|.\]
It follows now that $\Phi^+$ is continuous iff $\Phi^-$ is continuous.

(iv) follows from (iii) by noting that $\Phi^\pm$ is continuous for any $\varphi \in C_c^2(\Omega), \varphi \geq 0$, iff $\mathcal{L}_V(\Omega) \hookrightarrow L_{1\text{loc}}^1(\Omega, V^\pm dx)$.

Lemma 5.3. Assume $-\Delta + V$ is an $L^1$ subcritical operator in $\Omega$. Then
(i) $Q_V(\xi^\pm) \leq Q_V(\xi)$ for each $\xi \in W^{1,\infty}_c(\Omega)$;
(ii) Given $u \in \mathcal{H}_V(\Omega)$, we can find $u_\pm \in \mathcal{H}_V(\Omega)$ such that
\[u = u_+ - u_- \quad \text{and} \quad J(u_\pm) = (J(u))^\pm \geq 0.\]
(iii) Given $u \in \mathcal{H}_V(\Omega)$, choose $u_\pm \in \mathcal{H}_V(\Omega)$ as in (ii) above. Then,
\[\| u_\pm \|_{V, \Omega} \leq \| u \|_{V, \Omega}, \quad \| u_+ + u_- \|_{V, \Omega} \leq \| u \|_{V, \Omega}.\]
(iv) Given \( u \in \mathcal{H}_V(\Omega) \), choose \( u_\pm \in \mathcal{H}_V(\Omega) \) as in (ii) above. Then,
\[
a_{V,\Omega}(u_+, u_-) \leq 0.
\]

Proof. (i) For each \( \xi \in W^{1,\infty}_c(\Omega) \) we have
\[
\xi_+, \xi_- \in W^{1,\infty}_c(\Omega) \quad \text{and} \quad Q_V(\xi) = Q_V(\xi^+) + Q_V(\xi^-) \geq Q_V(\xi^\pm).
\]

(ii) Let \( \{ \xi_n \} \subset W^{1,\infty}_c(\Omega) \) be such that \( \xi_n \to u \) in \( \mathcal{H}_V(\Omega) \). From part (i) we get that \( Q_V(\xi^\pm_n) \) is also bounded. Therefore, there exists \( u_\pm \in \mathcal{H}_V(\Omega) \) such that (up to a subsequence)
\[
(5.3) \quad \xi^\pm_n \rightharpoonup u_\pm \quad \text{weakly in} \ \mathcal{H}_V(\Omega).
\]
Clearly, \( u = u_+ - u_- \). Recalling from definition 4.3 that \( J : \mathcal{H}_V(\Omega) \to L^1_{\text{loc}}(\Omega) \) is a continuous linear map, we have
\[
\begin{align*}
(a) & \quad \xi^\pm_n \overset{L^1_{\text{loc}}(\Omega)}{\to} J(u), \text{ which implies (up to subsequence) } \xi_n \overset{a.e.}{\rightharpoonup} J(u), \text{ and hence } \xi^\pm_n \overset{a.e.}{\rightharpoonup} J(u)^\pm; \\
(b) & \quad \xi^\pm_n \rightharpoonup J(u_\pm) \quad \text{in the weak topology on} \ L^1_{\text{loc}}(\Omega) \quad \text{(recall that} \ L^\infty_c(\Omega) \text{ is the dual of} \ L^1_{\text{loc}}(\Omega)); \\
& \quad \text{refer Thm. 4, p.182 of Horvath [14]).}
\end{align*}
\]
Let \( B \subset \Omega \) be a ball. By property (b) above, we have
\[
(5.4) \quad \xi^\pm_n \rightharpoonup J(u_\pm) \quad \text{weakly in} \ L^1(B).
\]
This is equivalent to \( \{ \xi^\pm_n \} \) being an equi-integrable family in \( L^1(B) \) (see lemma 12.3 in the appendix). By Vitali’s convergence Theorem \( \{ \xi^\pm_n \} \overset{a.e.}{\rightharpoonup} J(u)^\pm \) and is equi-integrable) we conclude that \( \xi^\pm_n \overset{L^1(B)}{\rightharpoonup} J(u)^\pm \). Hence in (5.4) we have strong convergence, and furthermore \( J(u_\pm) = J(u)^\pm \geq 0 \).

(iii) Let \( u \in \mathcal{H}_V(\Omega) \), and as in part (ii), consider a sequence \( \{ \xi_n \} \subset W^{1,\infty}_c(\Omega) \) such that \( \xi_n \to u \) in \( \mathcal{H}_V(\Omega) \) as well as
\[
\xi^+_n \to u_+ \quad \text{and} \quad \xi^-_n \to u_- \quad \text{weakly in} \ \mathcal{H}_V(\Omega).
\]
Hence, \( |\xi_n| \to u_+ + u_- \quad \text{weakly in} \ \mathcal{H}_V(\Omega) \). Since a norm is sequentially weakly lower semi-continuous and \( Q_V(|\xi|) = Q_V(\xi) \) for any \( \xi \in W^{1,\infty}_c(\Omega) \) we deduce that
\[
\| u_+ + u_- \|_{V,\Omega} \leq \liminf_{n \to \infty} Q_V(|\xi_n|) = \liminf_{n \to \infty} Q_V(\xi_n) = \| u \|_{V,\Omega}.
\]
In a similar manner, using (i), we can show that \( \| u_\pm \|_{V,\Omega} \leq \| u \|_{V,\Omega} \).

(iv) The parallelogram law and part (iii) imply the following
\[
a_{V,\Omega}(u_+, u_-) = \| u_+ + u_- \|^2_{V,\Omega} - \| u_+ - u_- \|^2_{V,\Omega} \leq 0.
\]
[\qed]
6 Balanced potentials, Energy and Distribution Solutions

6.1 Balanced potentials: definition and properties

Let \( V \in L^1_{\text{loc}}(\Omega) \) and \( Q_V \geq 0 \) in \( \Omega \). For \( U \Subset \Omega \) a non-empty open set, we consider the space

\[
\mathcal{L}_V(\Omega; U) := \left( W^{1,\infty}_{c}(\Omega), \| \cdot \|_{V, U} \right)
\]

with the norm

\[
\| \xi \|_{V, U} := \sqrt{Q_V(\xi)} + \int_U |\xi|, \quad \xi \in W^{1,\infty}_c(\Omega),
\]

and let \( \mathcal{B}_V(\Omega; U) \) denote the completion of this space.

**Remark 6.1.** (i) In general, the norms \( \| \cdot \|_{V, U} \) are not equivalent as \( U \) varies over \( \Omega \). For a general nonnegative operator equivalence of these norms seems to require a further restriction on \( V^+ \) (see corollary 9.6 and also [25, Theorem 1.4]).

(ii) When \( -\Delta + V \) is an \( L^1 \)-subcritical operator in \( \Omega \) or if \( L^V(\Omega; U) \hookrightarrow L^1_{\text{loc}}(\Omega) \) for any \( U \), it is easy to see that the norms \( \| \cdot \|_{V, U} \) are equivalent as \( U \) varies over \( \Omega \). In the former case all the spaces \( \mathcal{B}_V(\Omega; U) \) are equivalent to \( \mathcal{H}_V(\Omega) \).

The inclusion map \( \mathcal{L}_V(\Omega; U) \to L^1(U) \) being continuous, it admits a unique continuous extension to \( \mathcal{B}_V(\Omega; U) \) that will be denoted as \( J_U \). If \( -\Delta + V \) is an \( L^1 \)-subcritical operator in \( \Omega \), we can identify \( \mathcal{B}_V(\Omega; U) \) with \( \mathcal{H}_V(\Omega) \), and we will let \( J_U(u) := J(u)|_U \).

Given a closed ball \( B \subset \Omega \), let \( X_B \) denote the completion of the space \( C^2_c(B) \) under the \( C^2(B) \) norm and \( X^*_B \) denote its dual space. Based on (5.1), we have the following result.

**Proposition 6.2.** Let \( V \in L^1_{\text{loc}}(\Omega) \) be such that \( Q_V \geq 0 \) in \( \Omega \). Then for any closed ball \( B \subset \Omega \), and any open set \( U \Subset \Omega \) containing \( B \), the linear map

\[
(6.1) \quad \mathcal{L}_V(\Omega; U) \ni \xi 
\]

is continuous. In particular, if \( -\Delta + V \) is an \( L^1 \)-subcritical operator in \( \Omega \), then the map in (6.1) is continuous from \( \mathcal{L}_V(\Omega) \) to \( X^*_B \) for any such ball \( B \).

**Proof.** From (5.1), we easily obtain that

\[
\|V\xi\|_{X^*_B} := \sup_{\|\varphi\|_{X_B} \leq 1} \left| \int_{\Omega} V\xi \varphi \right| \leq C \|\xi\|_{V, U}, \quad \forall \xi \in W^{1,\infty}_c(\Omega).
\]

Denote by \( \mathcal{B}_{B,U} \) the unique continuous linear extension of the map given in (6.1) to \( \mathcal{B}_V(\Omega; U) \). Consider the following “multiplication by \( V \)” map

\[
\mathcal{M}_B : \{ w \in L^1(B) : Vw \in L^1(B) \} \to L^1(B) \quad \text{given by} \quad \mathcal{M}_B(w) := Vw.
\]

We consider situations where we can recover \( \mathcal{B}_{B,U} \) as the composition of \( J_U \) and the map \( \mathcal{M}_B \).
Definition 6.3. A potential \( V \in L^1_{loc}(\Omega) \) is called balanced in \( \Omega \) if

(i) \( Q_V \succeq 0 \) in \( \Omega \),

and for any closed ball \( B \subset \Omega \) there exists an open set \( U \Subset \Omega \) containing \( B \) such that

(ii) \( VJ_U(u) \in L^1(B) \) for any \( u \in B_V(\Omega; U) \),

(iii) \( \nabla_{B,U} = M_B \circ J_U \) on \( B_V(\Omega; U) \).

That is, for a balanced potential, the following diagram commutes:

\[
\begin{array}{c}
L^1(B) \\
\downarrow M_B \\
B_V(\Omega; U) \\
\downarrow V_{B,U} \\
L^1(B) \\
\downarrow M_B \circ J_U \\
X_B^* \\
\end{array}
\]

That is, for a balanced potential, the following diagram commutes:

\[
\begin{array}{c}
L^1(B) \\
\downarrow M_B \\
B_V(\Omega; U) \\
\downarrow V_{B,U} \\
L^1(B) \\
\downarrow M_B \circ J_U \\
X_B^* \\
\end{array}
\]

(6.2)

In the subsections 6.5 and 6.6 we will give examples of both balanced and non-balanced potentials.

Proposition 6.4. Let (i) and (ii) of definition 6.3 be satisfied. Then the following are equivalent:

(i) \( V \) is balanced in \( \Omega \),

(ii) the map \( M_B \circ J_U : B_V(\Omega; U) \to L^1(B) \subset X_B^* \) is continuous (in the \( X_B^* \) norm),

(iii) for any sequence \( \{\xi_n\} \subset W^{1,\infty}_c(\Omega) \) such that \( \xi_n \to u \) in \( B_V(\Omega; U) \), we have

\[
\int_B V\xi_n \varphi \to \int_B VJ_U(u) \varphi, \quad \forall \varphi \in C^2_c(\Omega).
\]

Proof. (i) \( \implies \) (ii): follows from the continuity of \( \nabla_{B,U} \).

(ii) \( \implies \) (iii): is trivial.

(iii) \( \implies \) (i): Note that \( \{V\xi_n\} \) is a sequence converging to \( \nabla_{B,U}(u) \) in \( X_B^* \) by proposition 6.2.

Since \( C^2_c(B) \) is dense in \( X_B^* \), assumption (iii) implies that \( \nabla_{B,U}(u) = VJ_U(u) = (M_B \circ J_U)(u) \). Thus, condition (iii) of definition 6.3 holds.

Remark 6.5. (i) If \( V \) is balanced in \( \Omega \), then \( \text{Range}(\nabla_{B,U}) \subset L^1(B) \).

(ii) One can easily check that the graph of the map \( M_B \) is always closed in \( L^1(B) \times L^1(B) \); but not necessarily so in \( L^1(B) \times X_B^* \). But, if definition 6.3 (i)-(ii) are satisfied and the graph of the map \( M_B \) is closed in \( L^1(B) \times X_B^* \), then \( V \) is balanced in \( \Omega \). This follows from proposition 6.2.

(iii) When \( L_V(\Omega) \hookrightarrow L^1_{loc}(\Omega) \) (i.e. \( -\Delta + V \) is an \( L^1 \)-subcritical operator in \( \Omega \)), we can identify \( B_V(\Omega; U) \) with \( J_V(\Omega) \). In this case, for each \( \varphi \in C^2_c(\Omega) \) consider the linear form

\[
L_V(\Omega) \ni \xi \mapsto \int_\Omega V\xi \varphi
\]

(6.3)
which is continuous by (5.1), and let \( J \) be as given in definition 6.3. Hence, in the subcritical case, the potential \( V \) is balanced iff \( VJ(u) \in L^1_{\text{loc}}(\Omega) \) for all \( u \in \mathcal{H}_V(\Omega) \) and the unique continuous extension of (6.3) to \( \mathcal{H}_V(\Omega) \) is given by

\[
\mathcal{H}_V(\Omega) \ni u \mapsto \int_\Omega VJ(u)\varphi.
\]

**Proposition 6.6.** (i) The class of balanced potentials form a convex set in \( L^1_{\text{loc}}(\Omega) \).

(ii) Let \( V \in L^1_{\text{loc}}(\Omega) \) be balanced in \( \Omega \) and \( W \in L^1_{\text{loc}}(\Omega) \) be nonnegative. Then \( V + W \) is balanced in \( \Omega \).

**Proof.** (i) Let \( V_1, V_2 \) be balanced in \( \Omega \) and let \( V := tV_1 + (1-t)V_2 \), \( t \in (0, 1) \). It is easy to see that \( Q_V \geq 0 \) in \( \Omega \). Fix a ball \( B \Subset \Omega \) and choose \( U_1, U_2 \) corresponding to \( V_1, V_2 \) from definition 6.3. Let \( U := U_1 \cup U_2 \). To avoid confusion we denote the maps \( J_{U_i} \) associated to the potentials \( V_i \) and the open set \( U_i \) by \( J_{U_i} \) for \( i = 1, 2 \).

Using the concavity of the square root function, we get for \( t \in (0, 1) \), \( \xi \in W^{1,\infty}(\Omega) \),

\[
t(\sqrt{Q_{V_1}(\xi)} + \int_{U_1} |\xi|) + (1-t)(\sqrt{Q_{V_2}(\xi)} + \int_{U_2} |\xi|) \leq \sqrt{Q_V(\xi)} + \int_{\Omega} |\xi|.
\]

From the above inequality, it follows that if \( \xi_n \to u \) in \( \mathcal{B}_V(\Omega;U) \), then \( \xi_n \to u_i \) (for some \( u_i \)) in \( \mathcal{B}_{V_i}(\Omega;U_i) \) for \( i = 1, 2 \). We note that for such a sequence \( \{\xi_n\} \), we obtain also that \( \xi_n \to J_{U}(u) \) in \( L^1(U) \) as well as \( \xi_n \to J_{U_i}(u_i) \) in \( L^1(U_i) \) for \( i = 1, 2 \). Therefore,

\[
J_{U}(u) = J_{U_1}(u_1) = J_{U_2}(u_2) \text{ on } B \text{ for any } u \in \mathcal{B}_V(\Omega;U).
\]

It follows that (ii) of definition 6.3 holds for this \( U \). Similarly, it also follows that (iii) of definition 6.3 holds since

\[
V \xi_n \to tV_1J_{U_1}(u_1) + (1-t)V_2J_{U_2}(u_2) = VJ_{U}(u) \text{ in } X_B^*.
\]

(ii) Clearly \( Q_{V+W} \geq 0 \) in \( \Omega \). Fix a ball \( B \Subset \Omega \) and choose \( U \) corresponding to \( B \) from definition 6.3 applied to \( V \). We note that

\[
\frac{1}{2}\left(\sqrt{Q_V(\xi)} + \int_{\Omega} |\xi|\right) + \frac{1}{2}\left(\int_{\Omega} |W|^2\right)^{\frac{1}{2}} \leq \sqrt{Q_{V+W}(\xi)} + \int_{\Omega} |\xi|.
\]

We denote the maps associated to the potentials \( V, V + W \) and the open set \( U \) by \( J_V \) and \( J_U \) respectively. As above, if \( \xi_n \to u \) in \( \mathcal{B}_{V+W}(\Omega;U) \), then \( \xi_n \to v \) for some \( v \in \mathcal{B}_V(\Omega;U) \), \( \xi_n \to J_U(u) \) in \( L^1(U) \) and \( \xi_n \to w \) in \( L^1(U; Wdx) \) for some \( w \). Arguing as before, we can verify (ii) and (iii) of definition 6.3 hold for \( V + W \). \( \square \)

### 6.2 Distribution Vs Energy solution

Distributional solutions to \(-\Delta u + Vu = h \) (see definition 2.4) do not necessarily belong to the energy space \( \mathcal{H}_V(\Omega) \) associated to the Schrödinger operator \(-\Delta + V\). For instance, non-zero constants are distributional solutions to \( \Delta u = 0 \) in \( \Omega \); however, non-zero constants do not lie in \( \mathcal{H}_0(\Omega) = D^{1,2}_0(\Omega) \) for dimensions larger than two.
When \( \mathcal{L}_V(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega) \), it is natural to ask if for an element in \( \mathcal{H}_V(\Omega) \), the notions of distributional solutions and energy solutions (see definition 3.7) are equivalent. This equivalence is not obvious since the energy solution \( u \) may not have the property that \( Vu \in L^1_{\text{loc}}(\Omega) \). But this equivalence is true under the restriction that \( V \) is balanced as shown by the following proposition.

**Proposition 6.7.** Assume \( V \in L^1_{\text{loc}}(\Omega) \) is balanced in \( \Omega \) and \( -\Delta + V \) is an \( L^1 \)-subcritical operator in \( \Omega \). Given \( f \in L^1_{\text{loc}}(\Omega) \), an element \( u \in H^V(\Omega) \) is an energy solution (see definition 3.7) to \( -\Delta u + Vu = f \) iff \( J(u) \) is a distributional solution.

**Proof.** Assume first \( u \in H^V(\Omega) \) is an energy solution and consider a sequence \( \{\xi_n\} \subset W^{1,\infty}(\Omega) \) such that \( \xi_n \to u \) in \( H^V(\Omega) \). Since

\[
\int_{\Omega} a_V(\xi_n, \varphi) = \int_{\Omega} f \varphi + a_V(\xi_n - u, \varphi) \quad \forall \varphi \in W^{1,\infty}(\Omega),
\]

we get

\[
\int_{\Omega} \xi_n(-\Delta \varphi) + \int_{\Omega} V \xi_n \varphi = \int_{\Omega} f \varphi + o_n(1), \quad \forall \varphi \in C^2_c(\Omega).
\]

Remark 5.5 (iii) implies,

\[
\int_{\Omega} J(u)(-\Delta \varphi) + \int_{\Omega} V J(u) \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in C^2_c(\Omega).
\]

Conversely, let \( u \in \mathcal{H}_V(\Omega) \) be such that \( J(u) \) is a distributional solution; in particular, \( V J(u) \in L^1_{\text{loc}}(\Omega) \). Tracing back the steps given above, we conclude that \( u \) is an energy solution as well. \( \Box \)

**Corollary 6.8.** Let \( V \in L^1_{\text{loc}}(\Omega) \) be balanced in \( \Omega \) and \( -\Delta + V \) an \( L^1 \)-subcritical operator in \( \Omega \). Then \( J \) is injective.

**Proof.** Let \( u_* \in \mathcal{H}_V(\Omega) \) be such that \( J(u_*) = 0 \). We note that, since 0 is a distributional solution to \( -\Delta u + Vu = 0 \), by the above proposition \( u_* \) is an energy solution to this same equation. Since an energy solution is unique in \( \mathcal{H}_V(\Omega) \), we get \( u_* = 0 \). \( \Box \)

**Lemma 6.9** (Comparison principle for energy solutions). Let \( V \in L^1_{\text{loc}}(\Omega) \) be such that \( -\Delta + V \) is an \( L^1 \)-subcritical operator in \( \Omega \). Let \( f \in L^1_{\text{loc}}(\Omega) \) and \( u \in \mathcal{H}_V(\Omega) \) be an energy super solution:

\[
a_V(\xi, u) \geq \int_{\Omega} f \xi, \quad \forall 0 \leq \xi \in W^{1,\infty}_c(\Omega).
\]

If \( f \geq 0 \), then \( J(u) \geq 0 \).

**Proof.** Using the equation satisfied by \( u \) we get

\[
a_V(\xi, u) \geq 0, \quad \forall 0 \leq \xi \in W^{1,\infty}_c(\Omega).
\]

By Lemma 5.3 (ii), we can find a sequence \( \{\xi_n\} \subset W^{1,\infty}_c(\Omega) \) such that

\[
(6.4) \quad \xi_n \to u \text{ in } \mathcal{H}_V(\Omega), \quad \xi_n^\pm \to u^\pm \text{ weakly in } \mathcal{H}_V(\Omega), \quad \xi_n^\pm \to J(u)^\pm \text{ in } L^1_{\text{loc}}(\Omega).
\]
Recalling \( u = u_+ - u_- \), we can now argue as follows:

\[
0 \leq \lim_{n \to \infty} a_{V,\Omega}(u, \xi_n) = a_{V,\Omega}(u, u_-) = a_{V,\Omega}(u_+, u_-) - a_{V,\Omega}(u_-, u_-) \leq -a_{V,\Omega}(u_-, u_-). \tag{by (6.4)}
\]

Hence \( u_- = 0 \) and thus, \( J(u) = J(u_+) = J(u)^+ \geq 0 \).

**Proposition 6.10** (Strong Maximum principle for energy solution). Assume \( V \in L^1_{loc}(\Omega) \) is balanced in \( \Omega \) and \(-\Delta + V \) is an \( L^1 \) subcritical operator in \( \Omega \). Given a non-negative function \( f \in L^1_{loc}(\Omega) \), \( f \neq 0 \), let \( u \in \mathcal{H}_V(\Omega) \) be the energy solution of \(-\Delta u + Vu = f \). That is,

\[
a_{V,\Omega}(u, \xi) = \int_{\Omega} f \xi, \quad \forall \xi \in W^{1,\infty}(\Omega).
\]

Then, \( J(u) \) (a distributional solution of \(-\Delta u + Vu = f \)) has a quasi-continuous representative \( \tilde{u} \) which satisfies

\[
\text{Cap}\{\{\tilde{u} = 0\}\} = 0.
\]

**Proof.** By corollary 6.8 and lemma 6.9, we have \( J(u) \neq 0 \) and \( J(u) \geq 0 \). Furthermore, \( J(u) \) is also a distributional solution of the corresponding equation from proposition 6.7. From proposition 2.10 we conclude that \( J(u) \in W^{1,1}(\Omega) \) and hence admits a quasicontinuous representative \( \tilde{u} \) in this space. Now the strong maximum principle in theorem 2.12 can be applied to \( \tilde{u} \) showing that \( \text{Cap}\{\{\tilde{u} = 0\}\} = 0 \).

6.3 Strongly Balanced Potentials

We note that any non-negative operator \(-\Delta + V \) leads to an estimate like (5.1) but not to similar estimates involving only the positive or negative parts of \( V \). We are therefore led to introduce the following more concrete formulation:

**Definition 6.11.** A function \( V \in L^1_{loc}(\Omega) \) is called strongly balanced in \( \Omega \) if \( Q_V \geq 0 \) in \( \Omega \) and for any compact \( K \subset \Omega \) there exists an open set \( U \supseteq \Omega \) containing \( K \) for which the linear map:

\[
\mathcal{L}_V(\Omega; U) \ni \xi \mapsto V^+ \xi \in L^1(K)
\]

is continuous. That is, there exists a constant \( c > 0 \) (depending on \( K, U \)) satisfying

\[
c \int_K V^+ |\xi| \leq \|\xi\|_{V,\Omega}^U := \sqrt{Q_V(\xi)} + \int_U |\xi|, \quad \forall \xi \in W^{1,\infty}(\Omega).
\]

**Remark 6.12.** (i) If all the norms \( \| \cdot \|_{V,\Omega}^U \) are equivalent as \( U \) varies over \( \Omega \), we can easily see that the definition 6.11 is the same as requiring that (6.6) holds for any pair \( K, U \).

(ii) Indeed, for the class of strongly balanced potentials in \( V(\Omega) \) (see definition 6.28 and proof of proposition 6.31) given the compact set \( K \) we cannot ensure that (6.6) holds for an arbitrary open set \( U \) containing \( K \).
Proposition 6.13. Let $V \in L^1_{\text{loc}}(\Omega)$ be such that $Q_V \succeq 0$ in $\Omega$. Then, the following statements are equivalent:

(i) $V$ is strongly balanced in $\Omega$;

(ii) for any compact set $K \subset \Omega$, there exists an open set $U \Subset \Omega$ containing $K$ for which the following linear map is continuous

$$L_V(\Omega; U) \ni \xi \mapsto V^\pm \xi \in L^1(K);$$

(iii) for any compact set $K \subset \Omega$, there exists an open set $U \Subset \Omega$ containing $K$ for which the following linear map is continuous

$$L_V(\Omega; U) \ni \xi \mapsto V\xi \in L^1(K).$$

Proof. (i) $\iff$ (ii): Let $V$ be strongly balanced in $\Omega$ and a compact $K$ be given. Corresponding to $K$ choose an open set $U_1$ from definition 6.11. Following a similar argument using (5.1) as in lemma 5.2 (iii) there exists an open set $U \Subset \Omega$ containing $K$ such that (6.7) is continuous. Converse is similar.

(i) $\iff$ (iii): Follows from the equivalence (i) and (ii).

Proposition 6.14. Let $V \in L^1_{\text{loc}}(\Omega)$ be a strongly balanced potential in $\Omega$. Then,

(i) for any compact $K \subset \Omega$ we may find an open set $U \Subset \Omega$ containing $K$ such that

$$VJ_U(u) \in L^1(K) \quad \forall u \in B_V(\Omega; U),$$

(ii) for $K, U$ as in (i), it holds

$$\xi_n \to u \quad \text{in} \quad B_V(\Omega; U) \implies V\xi_n \to VJ_U(u) \quad \text{in} \quad L^1(K).$$

Proof. (i) Given compact $K$ choose open set $U$ as in (i) above. Let $u \in B_V(\Omega; U)$. Choose a sequence $\{\xi_n\} \subset L_V(\Omega; U)$ converging to $u$. Then, it follows from (6.7) that $\{V^\pm \xi_n\}$ is Cauchy in $L^1(K)$ and hence converges there. Noting that $\xi_n \to J_U(u)$ in $L^1(U)$ and hence pointwise a.e. in $U$ for a subsequence, we get that $V^\pm J_U(u) \in L^1(K)$.

(ii) From (i) above, we obtain that $\{V\xi_n\}$ is Cauchy in $L^1(K)$. Noting that (up to a subsequence) $\xi_n \to J_U(u)$ pointwise a.e. in $K$, the result follows.

Finally, we can justify our terminology:

Corollary 6.15. $V \in L^1_{\text{loc}}(\Omega)$ is a strongly balanced potential in $\Omega$ iff it is balanced in $\Omega$ and the map

$$V_{B,U} : B_V(\Omega; U) \to L^1(B)$$

is continuous in the norm.

Proof. Follows from both statements proposition 6.14 (i) and (ii) along with a simple covering argument.

Remark 6.16. Let $V \in L^1_{\text{loc}}(\Omega)$ such that $Q_V \succeq 0$ in $\Omega$. Then, the proofs given in proposition above shows that (6.6) holds
(i) for any pair \((K,U)\) iff the linear maps
\[ \mathcal{L}_V(\Omega; U) \ni \xi \mapsto V^\pm \xi \in L^1_{\text{loc}}(\Omega) \]
are continuous for any \(U\).

(ii) for any pair \((K,U)\) such that \(K \subset U\) iff the linear maps
\[ \mathcal{L}_V(\Omega; U) \ni \xi \mapsto V^\pm \xi \in L^1_{\text{loc}}(U) \]
are continuous for any \(U\).

Next result provides an abstract result for ensuring strong balancedness.

**Proposition 6.17.** Let \(V \in L^1_{\text{loc}}(\Omega)\) be such that \(Q_V \succeq 0\) in \(\Omega\). Suppose for any ball \(B \subset \Omega\) there exists an open set \(U \subset \Omega\) containing \(B\) and a reflexive space \((Z_B, \| \cdot \|)\) consisting of \(L^1(B)\) functions closed under the \(| \cdot |\) operation such that

(i) \(\| |z| \|_B \leq \|z\|_B\) for all \(z \in Z_B\),

(ii) \(\mathcal{L}_V(\Omega; U) \hookrightarrow Z_B \hookrightarrow L^1(B)\),

(iii) \(Vz \in L^1(B)\) for all \(z \in Z_B\).

Then \(V\) is strongly balanced in \(\Omega\).

The above proposition uses the following basic result, whose proof is recalled in the appendix.

**Lemma 6.18.** Let \(K \subset \mathbb{R}^N\) be a compact set (with positive measure) and \((Z, \| \cdot \|)\) a reflexive space consisting of \(L^1(K)\) functions closed under the \(| \cdot |\) operation such that
\[ Z \hookrightarrow L^1(K) \quad \text{and} \quad \| |z| \| \leq \|z\| \quad \forall z \in Z. \]

Then given any sequence \(\{z_n\} \subset Z\) converging to some \(z \in Z\), we may find a subsequence \(\{z_{n_k}\}\) and \(z^* \in Z\) such that
\[ |z_{n_k}| \leq z^* \quad \text{in} \ K \ \forall k, \quad \text{and} \quad z_{n_k} \text{ converges pointwise to } z. \]

**Proof of Proposition 6.17.** Take any ball \(B \subset \Omega\) along with the corresponding \(U\) and a sequence \(\xi_n \to 0\) in \(\mathcal{L}_V(\Omega; U)\). It follows that \(\xi_n \to 0\) in \(Z_B\) (and hence in \(L^1(B)\)). We claim that \(V\xi_n \to 0\) in \(L^1(B)\). Consider any subsequence \(\{V\xi_{n_k}\}\). From Lemma 6.18, we may find a further subsequence \(\{\xi_{n_{k_j}}\}\) and a function \(z^* \in Z_B\) such that
\[ |\xi_{n_{k_j}}| \leq z^* \quad \text{in} \ B \ \forall j, \quad \text{and} \quad \xi_{n_{k_j}} \text{ converges pointwise to } 0. \]

Since \(Vz^* \in L^1(B)\) (by assumption), we have \(V\xi_{n_{k_j}} \xrightarrow{L^1(B)} 0\) (by Lebesgue dominated convergence theorem) and hence the claim follows. Proposition 6.13 (iii) shows \(V\) is strongly balanced in \(\Omega\). \(\square\)
Proposition 6.19.  
(i) The class of strongly balanced potentials form a convex set in $L^1_{loc}(\Omega)$.  

(ii) Let $V \in L^1_{loc}(\Omega)$ be a strongly balanced potential in $\Omega$ and $W \in L^1_{loc}(\Omega)$ be nonnegative. Then $V + W$ is strongly balanced in $\Omega$.

Proof. (i) Let $V_1, V_2$ be strongly balanced potentials in $\Omega$. Fix a compact set $K \subset \Omega$ and choose $c_1, c_2$ and $U_1, U_2$ corresponding to $V_1, V_2$ from definition 6.11. Let $c := \min\{c_1, c_2\}$ and $U := U_1 \cup U_2$. Using the convexity of the map $x \mapsto x^+$ and the concavity of the square root function, we get for $t \in (0, 1)$,

$$c \int_K (tV_1 + (1-t)V_2)^+ |\xi| \leq c_1 t \int_K V_1^+ |\xi| + c_2 (1-t) \int_K V_2^+ |\xi| \leq t \sqrt{Q_{V_1}(\xi)} + (1-t) \sqrt{Q_{V_2}(\xi)} + \int_U |\xi| \leq \sqrt{Q_{U_1+(1-t)V_2}(\xi)} + \int_U |\xi|.$$  

(ii) Fix a compact $K \subset \Omega$ and choose $c, U$ corresponding to $K$ from definition 6.11. Using the convexity of the map $x \mapsto x^+$ again,

$$\frac{c}{2} \int_K (V + W)^+ |\xi| \leq \frac{1}{2} \sqrt{Q_V(\xi)} + \frac{1}{2} \int_U |\xi| + c_K \left( \int_{\Omega} W |\xi|^2 \right)^{\frac{1}{2}} \leq c_K \left( \sqrt{Q_V(\xi)} + \sqrt{Q_{V+W}(\xi)} \right) + \frac{1}{2} \int_U |\xi| \leq 2c_K \sqrt{Q_{V+W}(\xi)} + \frac{1}{2} \int_U |\xi|.$$  

\[\square\]

6.4 Balanced condition for $L^1$-subcritical operators

In this subsection we show that for $L^1$-subcritical operators the balanced condition in definition 6.6 can be simplified and that the concepts of balanced and strongly balanced potentials coincide. We first restate some of the previous results for an $L^1$-subcritical operator. Recall that for these operators the space $\mathcal{L}_V(\Omega;U)$ is equivalent to $\mathcal{L}_V(\Omega)$ for any open set $U \subset \Omega$.

Proposition 6.20. Let $V \in L^1_{loc}(\Omega)$ be such that $\mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ (i.e. $-\Delta + V$ is an $L^1$-subcritical operator in $\Omega$). Then,

(i) $\mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega, V^+ dx)$.

(ii) Let $J$ be as in definition 6.3 and denote by $J^\pm$, the unique continuous extensions to $\mathcal{H}_V(\Omega)$ of the two imbeddings given in (i). Then, $J^\pm(u) = J(u)$ a.e. in $\text{supp}(V^\pm)$.

Proof. (i) follows from proposition 6.13 (i).

(ii) Let $\{\xi_n\} \subset W^{1,\infty}_c(\Omega)$ be a sequence converging to $u \in \mathcal{H}_V(\Omega)$. Then, $\{\xi_n\}$ converges to $J(u)$ in $L^1_{loc}(\Omega)$ and to $J^\pm(u)$ respectively in $L^1_{loc}(\Omega, V^\pm dx)$ as $n \to \infty$. The result follows from a.e. pointwise convergence (up to a subsequence) of $L^1$-convergent sequences.  

\[\square\]
For $L^1$-subcritical potentials of the form $V_0 := \text{div} \tilde{f} + |\tilde{f}|^2$ we show injectivity of the associated $J_0$ map by connecting it to the closability of the “magnetic operator”.

**Lemma 6.21.** Let $\tilde{f} \in (L^2_{\text{loc}}(\Omega))^N$ and denote $V_0 := \text{div} \tilde{f} + |\tilde{f}|^2$. Assume $V_0 \in L^1_{\text{loc}}(\Omega)$ is such that $-\Delta + V_0$ is an $L^1$-subcritical operator in $\Omega$ with the associated map $J_0$.

Then $J_0$ is injective iff the magnetic operator

$$T_\tilde{f}(\xi) := \nabla \xi - \tilde{f} \xi, \quad \xi \in C^1_c(\Omega)$$

is closable with domain $(C^1_c(\Omega), \| \cdot \|_1)$ to $(L^2(\Omega))^N$.

**Proof.** By a straightforward integration by parts, we get

$$Q_{V_0}(\xi) := \int_\Omega |\nabla \xi|^2 + V_0 \xi^2 = \int_\Omega |\nabla \xi - \tilde{f} \xi|^2 \forall \xi \in C^1_c(\Omega).$$

Let $T_\tilde{f}$ be closable. Take $\{\xi_n\} \subset C^1_c(\Omega)$ such that $\{\xi_n\}$ is a Cauchy sequence in $\mathcal{H}_V(\Omega)$ and $\xi_n \to 0$ in $L^1_{\text{loc}}(\Omega)$. Then, it follows from (6.12) that $\{T_\tilde{f}(\xi_n)\}$ is a Cauchy sequence in $(L^2(\Omega))^N$ and hence converges to 0 by the closability property. From (6.12) again, $\xi_n \to 0$ in $\mathcal{H}_{V_0}(\Omega)$ and hence $J_0$ is injective.

For the converse, let $\{\xi_n\} \subset C^1_c(\Omega)$ be such that

$$\xi_n \to 0 \text{ in } L^1_{\text{loc}}(\Omega) \quad \text{and} \quad T_\tilde{f}(\xi_n) \to \tilde{g} \text{ in } (L^2(\Omega))^N.$$ 

Then from (6.12) we get $\{\xi_n\}$ is a Cauchy sequence in $\mathcal{H}_{V_0}(\Omega)$ and hence by injectivity of $J_0$, $\xi_n \to 0$ in $\mathcal{H}_{V_0}(\Omega)$. By (6.12) again, we get $\tilde{g} = 0$. □

**Lemma 6.22.** $T_\tilde{f}$ is a closable operator from $L^1_{\text{loc}}(\Omega)$ to $(L^1(\Omega))^N$ for any $\tilde{f} \in (L^1_{\text{loc}}(\Omega))^N$.

**Proof.** Let $\{\xi_n\} \subset C^1_c(\Omega)$ be such that

$$\xi_n \to 0 \text{ in } L^1_{\text{loc}}(\Omega) \quad \text{and} \quad T_\tilde{f}(\xi_n) := \tilde{g}_n \to \tilde{g} \text{ in } (L^1(\Omega))^N.$$ 

Let us denote $\tilde{f} = (f_1, \cdots, f_N), \tilde{g} = (g_1, \cdots, g_N)$ and $\tilde{g}_n = (g^n_1, \cdots, g^n_N)$. Fix a cube $E := \Pi_{i=1}^N(a_i, b_i)$ and choose any $\varphi \in C^1_c(E)$. Consider the modified sequence $\{\tilde{\xi}_n := \varphi \xi_n\}$. Then, from (6.13),

$$\tilde{\xi}_n \to 0 \text{ in } L^1_{\text{loc}}(\Omega) \quad \text{and} \quad T_\tilde{f}(\tilde{\xi}_n) = \varphi T_\tilde{f}(\xi_n) + \nabla \varphi \xi_n := \tilde{g}_n \to \varphi \tilde{g} \text{ in } (L^1(\Omega))^N.$$ 

We note that $f_i$ is locally integrable on a.e. line parallel to the $i$-th co-ordinate axis. We can hence define for a.e. $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \Pi_{j \neq i}(a_j, b_j)$,

$$F_i(x) := -\int_{a_i}^{x_i} f_i(x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_N) \, ds.$$ 

We note that $F_i$ is absolutely continuous on $[a_i, b_i]$ for a.e. $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \Pi_{j \neq i}(a_j, b_j)$. Denoting $\tilde{g}_n = (\tilde{g}^n_1, \cdots, \tilde{g}^n_N)$, we can then write (6.14) as

$$\partial_i(\tilde{\xi}_n e^{F_i}) = \tilde{g}^n_i e^{F_i} \text{ a.e. in } (a_i, b_i) \text{ for } i = 1, \cdots, N.$$
In the above equation $\partial_i$ stands for pointwise $i$-th partial derivative. For any $i$ we note that $(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \cdots, x_N) \notin \text{supp}(\varphi)$. For any $x \in E$, integrating along $i$-th coordinate direction in (6.15),

$$
(e^{F_i} \tilde{\xi}_n)(x) = (e^{F_i} \tilde{\xi}_n)(x) - (e^{F_i} \tilde{\xi}_n)(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_N)
$$

$$
= \int_{a_i}^{x_i} (\tilde{g}_i^n e^{F_i})(x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_N) \, ds.
$$

(6.16)

Since $\tilde{\xi}_n \to 0$ and $\tilde{g}_i^n \to g_i$ in $L^1_{loc}(\Omega)$, we obtain for a.e. $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \Pi_{j \neq i}(a_j, b_j)$:

$$
\tilde{\xi}_n \to 0 \quad \text{and} \quad \tilde{g}_i^n \to \varphi g_i \quad \text{in} \quad L^1((a_i, b_i)).
$$

Taking subsequential limit in (6.16), for a.e. $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \Pi_{j \neq i}(a_j, b_j)$ and a.e. $x_i \in (a_i, b_i)$ we get,

$$
\int_{a_i}^{x_i} (\varphi g_i e^{F_i})(x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_N) \, ds = 0, \quad i = 1, 2, \cdots, N.
$$

Hence $\varphi g \equiv 0$ in $E$.

**Corollary 6.23.** Let $\tilde{f} \in (L^2_{loc}(\Omega))^N$ and assume $V_0 := \text{div} \tilde{f} + |\tilde{f}|^2 \in L^1_{loc}(\Omega)$ is such that $-\Delta + V_0$ is an $L^1$-subcritical operator in $\Omega$. Then the associated map $J_0$ is injective.

**Proof.** Follows from lemmas 6.21 and 6.22

Finally, we characterise balanced potentials in the $L^1$-subcritical operator context.

**Lemma 6.24.** Let $V \in L^1_{loc}(\Omega)$ be such that $-\Delta + V$ is an $L^1$-subcritical operator in $\Omega$. Then the following are equivalent:

(i) $V$ is balanced in $\Omega$,

(ii) $J$ is injective and $VJ(u) \in L^1_{loc}(\Omega)$ for any $u \in \mathcal{H}_V(\Omega)$,

(iii) $V$ is strongly balanced in $\Omega$.

**Proof.** (i) $\implies$ (ii): follows from definition 6.3 and corollary 6.8

(ii) $\implies$ (iii): Let $Z := J(\mathcal{H}_V(\Omega)) \subset L^1_{loc}(\Omega)$ with the norm

$$
\|z\| := \|u\|_{V, \Omega} \quad \text{where} \quad z = J(u), u \in \mathcal{H}_V(\Omega).
$$

Therefore, $Z$ and $\mathcal{H}_V(\Omega)$ are isometric. Given any $z = J(u)$, we write $u = u_+ - u_-$ as in lemma 5.3 Then, again as in that lemma,

$$
J(u_{\pm}) = (J(u))^\pm.
$$

Therefore, $|z| = J(u_+ + u_-) \in Z$ and

$$
\|z\| := \|u_+ + u_-\|_{V, \Omega} \leq \|u\|_{V, \Omega} := \|z\|.
$$

Thus, $V, Z$ satisfy assumptions of proposition 6.17 and hence $V$ is strongly balanced in $\Omega$.

(iii) $\implies$ (i): follows from corollary 6.15.

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Remark 6.25. If \( V := V_0 \) is as in corollary 6.23, then we have that the statements (i) and (iii) of lemma 6.24 are equivalent to the following

\[(ii)' : \quad V_0 J_0(u) \in L^1_{\text{loc}}(\Omega) \quad \text{for any} \quad u \in \mathcal{H}_{V_0}(\Omega).\]

6.5 Examples of Strongly Balanced Potentials

We now give examples of large class of strongly balanced potentials.

6.5.1 Some very general examples

Remark 6.26. Assume \( Q_V \succeq 0 \) in \( \Omega \).

(i) If either \( V^- \in L^\infty_{\text{loc}}(\Omega) \), or \( V^+ \in L^\infty_{\text{loc}}(\Omega) \), then \( V \) is strongly balanced in \( \Omega \). This follows from proposition 6.13.

(ii) Let \( V^+ \in L^q_{\text{loc}}(\Omega) \) for some \( q > 1 \). If \( L^V(\Omega) \hookrightarrow L^{q'}_{\text{loc}}(\Omega) \) (where \( q' = \frac{q}{q-1} \)), then \( V \) is strongly balanced in \( \Omega \).

Lemma 6.27. Let \( V \in L^1_{\text{loc}}(\Omega) \) be such that \( Q_V \succeq 0 \) in \( \Omega \).

(i) \( V_\varepsilon := (1 + \varepsilon)V^+ - V^- \) is a strongly balanced potential for any \( \varepsilon > 0 \).

(ii) \( \tilde{V}_\varepsilon := V^+ - (1 - \varepsilon)V^- \) is a strongly balanced potential for any \( \varepsilon > 0 \).

(iii) Define \( V_n := \min\{V, n\} \). Then \( V \) is strongly balanced in \( \Omega \) if \( Q_{V_n} \succeq 0 \) in \( \Omega \) for some \( n \).

Proof. (i) Clearly \( Q_{V_\varepsilon} \succeq 0 \) in \( \Omega \). Now, for any compact set \( K \subset \Omega \) and \( \xi \in W^{1,\infty}_{\text{c}}(\Omega) \),

\[
\frac{1}{1 + \varepsilon} \int_K V_\varepsilon^+ |\xi| = \int_K V^+ |\xi| \leq c_K \left( \int_K V^+ |\xi|^2 \right)^{\frac{1}{2}} \leq \frac{c_K}{\sqrt{\varepsilon}} \left( \int_K eV^+ |\xi|^2 \right)^{\frac{1}{2}} \leq \frac{c_K}{\sqrt{\varepsilon}} \sqrt{Q_{V_\varepsilon}(\xi)}.
\]

(ii) We note that

\[
\varepsilon \int_\Omega V^- |\xi|^2 \leq Q_{\tilde{V}_\varepsilon}(\xi), \quad \forall \xi \in W^{1,\infty}_{\text{c}}(\Omega).
\]

We may assume \( \varepsilon < 1 \) as otherwise the result is obvious by remark 6.26(i). As above,

\[
\frac{1}{1 - \varepsilon} \int_K \tilde{V}_\varepsilon^- |\xi| = \int_K V^- |\xi| \leq \frac{c_K}{\sqrt{\varepsilon}} \left( \int_K eV^- |\xi|^2 \right)^{\frac{1}{2}} \leq \frac{c_K}{\sqrt{\varepsilon}} \sqrt{Q_{\tilde{V}_\varepsilon}(\xi)}.
\]

That \( \tilde{V}_\varepsilon \) is strongly balanced in \( \Omega \) follows from proposition 6.13.

(iii) Noting that \( V_n \) is a strongly balanced potential (since it is bounded above), the result follows from the identity

\[
V = \min\{V, n\} + (V - n)\chi_{\{V \geq n\}}
\]

and proposition 6.19(ii).
6.5.2 Balanced potentials whose positive and negative singularities are separated

We now define a class of potentials whose positive and negative singularities are separated.

Definition 6.28. Let $\mathcal{V}(\Omega)$ denote the class of functions $V \in L^1_{loc}(\Omega)$ for which we can find a cover of $\Omega$ by open sets $\{U_j\}$, $U_j \Subset \Omega$, such that either $V^+$ is bounded on $U_j$ or else there exists an open set $O_j$ satisfying $U_j \Subset O_j \Subset \Omega$ and $V^-$ is bounded on $O_j$.

Proposition 6.29. Let $V \in \mathcal{V}(\Omega)$.

(i) If $h_1, h_2 \in L^\infty_{loc}(\Omega)$ and $h_1$ nonnegative, then $h_1 V + h_2 \in \mathcal{V}(\Omega)$.

(ii) $tV^+ - sV^- \in \mathcal{V}(\Omega)$ for all $t, s > 0$.

Proof. We take the cover $\{U_j\}$ for $V$. The same cover works for $h_1 V + h_2$ and $tV^+ - sV^-$, $t, s > 0$.

Example 6.30. (i) If $V^+ \in L^\infty_{loc}(\Omega)$, then clearly $V \in \mathcal{V}(\Omega)$ (we simply take $\{U_j\}$ to be any countable collection of open balls covering $\Omega$).

(ii) If there exists a compact set $K$ and an open set $O \Subset \Omega$ such that $K \subset O$, $\sup_{\Omega \setminus K} V < +\infty$ and $\inf_O V > -\infty$, then $V \in \mathcal{V}(\Omega)$. In this case we take any open set $U$ such that $K \subset U \Subset O$ to be a member of the open cover and note that $V$ is bounded from above outside $K$.

(iii) Let $S := \{x_i\}$ be a discrete set of points in $\Omega$ contained in an open set $U \subset \mathbb{R}^N$ such that $\overline{U} \subset \Omega$. Associated to each $x_i$ we choose $\alpha_i \in (-N, 0)$ and $a_i \geq 0$ such that $\sum_i a_i < \infty$. Then, any $V \in L^1_{loc}(\Omega)$ such that

$$0 \leq V \leq \sum_i a_i |x - x_i|^\alpha \text{ a.e. in } U \text{ and } V \leq 0 \text{ a.e. in } \Omega \setminus U$$

is in $\mathcal{V}(\Omega)$. Indeed, we choose open balls $B_i \Subset U$ centered at the singularity $x_i$ and excluding other singularities. Noting that $V$ is locally bounded above outside the union of these balls, it is easy to construct the cover $\{U_j\}$ as required by the definition 6.28 by enlarging the collection $\{B_j\}$. We can easily generalise this example by replacing the point singularities $\{x_i\}$ by a disjoint sequence of compact sets $\{K_i\}$ containing the positive singularities of $V$.

(iv) Let $U$ and $\{a_i\}$ be as in (iii). Let $S := \{x_i\}$ be any countable set such that $\overline{S} \Subset U$ ($S$ maybe even dense in a portion of $U$). Choose $\alpha \in (-N, 0)$. Then any $V \in L^1_{loc}(\Omega)$ satisfying

$$0 \leq V \leq \sum_i a_i |x - x_i|^\alpha \text{ a.e. in } U \text{ and } V \leq 0 \text{ a.e. in } \Omega \setminus U$$

will be in $\mathcal{V}(\Omega)$ as it satisfies conditions in example (ii).
Suppose there exist \( x_0 \in \Omega \) and a sequence of open balls \( B_{\epsilon_n}(x_0), \epsilon_n > 0 \), contained in \( \Omega \) such that

\[
\sup_{B_{\epsilon_n}} V = +\infty \quad \text{and} \quad \inf_{B_{\epsilon_n}} V = -\infty \quad \text{for all } n.
\]

Then \( V \) is not in \( \mathcal{V}(\Omega) \).

**Proposition 6.31.** Let \( V \in \mathcal{V}(\Omega) \) be such that \( Q_V \geq 0 \) in \( \Omega \). Then \( V \) is strongly balanced in \( \Omega \).

**Proof.** Let \( \xi \in W^{1,\infty}_c(\Omega) \). Let the cover \( \{ U_j \} \) be as given in the definition 6.28. If \( V^+ \) is bounded on \( U_j \) then clearly

\[
(6.17) \quad \int_{U_j} V^+|\xi| \leq (\sup_{U_j} V^+) \int_{U_j} |\xi|.
\]

If not, we can obtain open set \( O_j \) as in definition 6.28 on which \( V^- \) is bounded. Choose \( \varphi \in C^2_c(O_j), 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) on \( U_j \). Then, by (5.1),

\[
(6.18) \quad \int_{U_j} V^+|\xi| \leq C_j \left( \int_{O_j} V^-|\xi|\varphi + \sqrt{Q_V(\xi)} \right) \leq C_j \left( \int_{O_j} |\xi| + \sqrt{Q_V(\xi)} \right).
\]

From (6.17) and (6.18) we get that for any \( U_j \),

\[
\int_{U_j} V^+|\xi| \leq C_j \left( \int_{O_j} |\xi| + \sqrt{Q_V(\xi)} \right).
\]

Given a compact set \( K \subset \Omega \) we can cover \( K \) by finitely many \( U_j \)'s and use the above inequality to verify the inequality in (6.6) holds.

**Example 6.32.** Let \( N \geq 3, a_i \in \Omega, \alpha_i \in (1, N/2) \) and \( c_i \in \mathbb{R} \) for \( i = 1, \ldots, N \). We choose \( |c_i| \geq 1 \) if \( \alpha_i = 1 \). Define

\[
\vec{\Gamma} := (c_1|x - a_1|^{-\alpha_1}, \ldots, c_N|x - a_N|^{-\alpha_N}), \quad V_0 := \text{div } \vec{\Gamma} + |\vec{\Gamma}|^2.
\]

Using the proposition 6.31, we can check that \( V_0 \) (and hence any locally integrable \( V \geq V_0 \)) is strongly balanced in \( \Omega \).

For more examples of potentials satisfying the conditions in the proposition 6.31 see corollary 11.6 and remark 11.7.

### 6.5.3 Balanced potentials whose energy space imbeds in \( H^1_{loc} \)

The following result shows that a large class of locally integrable functions that fall in the framework of \([16]\) are balanced potentials. We also remark that this result allows a balanced potential to oscillate infinitely often near a singularity, but not in a “wild” manner; see remark 6.45 in this context. Example 6.44 (see again remark 6.45) shows that the assumption on \( V^+ \) made in the following lemma is sharp.

**Lemma 6.33.** Let \( V \in L^1_{loc}(\Omega) \) be such that \( V^+ \in L^{2N}_{loc}(\Omega) \) when \( N \geq 3 \) and \( V^+ \in L^p_{loc}(\Omega) \) for some \( p > 1 \) when \( N = 2 \). If \( Q_V \geq 0 \) in \( \Omega \) and \( \mathcal{L}_V(\Omega) \hookrightarrow H^1_{loc}(\Omega) \), then \( V \) is strongly balanced in \( \Omega \). Same conclusion holds if above conditions are imposed on \( V^- \) instead of \( V^+ \).
Proof. Let $N \geq 3$. We estimate for any ball $B \subset \Omega$ and $\xi \in W^{1,\infty}_c(\Omega)$,
\[
\int_B V^+|\xi| \leq \left( \int_B (V^+)^{\frac{N}{N+2}} \right)^{\frac{N+2}{N}} \left( \int_B |\xi|^{\frac{N}{N+2}} \right)^{\frac{N-2}{N}}
\leq \tilde{c}_B \|\xi\|_{H^1(B)}
\leq c_B \sqrt{Q_V(\xi)}.
\]
A simple covering argument shows that $V$ is strongly balanced in $\Omega$. A similar proof holds for $N = 1, 2$. The case of $V^-$ can be shown using proposition 6.13(i).

Next, we make the following definition which plays a main role in showing the equivalence of $L^1$ and feeble $L^2$ criticalities (see proposition 9.7):

Definition 6.34. We call $V \in L^1_{loc}(\Omega)$ a tame potential in $\Omega$ if $V$ is balanced in $\Omega$ and $V^+ \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$ ($N \geq 2$).

Lemma 6.35. Let $V \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$ when $N \geq 2$ and $V \in L^1_{loc}(\Omega)$ when $N = 1$. If $Q_V \geq 0$ in $\Omega$, then $V$ is strongly balanced (in fact, tame) in $\Omega$.

Proof. We shall use some results from [25] to show the lemma. If $-\Delta + V$ is $L^1$-critical in $\Omega$, let $\Phi > 0$ denote the ground state solution of this operator. Fix a ball $B \subset \Omega$ and a $\psi \in C^\infty_B(\Omega)$ such that $\int_\Omega \psi \Phi \neq 0$. Consider the quadratic form
\[
\tilde{Q}(\xi) := Q_V(\xi) + \left| \int_\Omega \psi \xi \right|^2, \quad \xi \in W^{1,\infty}_c(\Omega).
\]
Let $\tilde{F}(\Omega)$ denote the Hilbert space obtained by completing the space $\tilde{L}(\Omega) := (W^{1,\infty}_c(\Omega), (\tilde{Q})^{\frac{1}{2}})$. It can be shown (see theorem 1.4 in [25]) that the above quadratic forms generate equivalent norms on $\tilde{F}(\Omega)$ as such $\psi \in C^\infty_B(\Omega)$ is varied. Then from proposition 3.1 in [25] we obtain that if $-\Delta + V$ is $L^1$-subcritical in $\Omega$, then $L_V(\Omega) \hookrightarrow H^1_{loc}(\Omega)$ and if $-\Delta + V$ is $L^1$-critical in $\Omega$, then $L_V(\Omega) \hookrightarrow H^1_{loc}(\Omega)$. We can proceed as in the proof of the last lemma to conclude noting that
\[
\left| \int_\Omega \psi \xi \right| \leq c_B \int_B |\xi|.
\]

Remark 6.36. Since all our results beginning from section 8 except those in subsection 10.3 are valid for any tame potential, it follows that they generalise (in view of the above lemma) the corresponding results in Murata [19] and Pinchover-Tintaraev [25].

We provide examples of potentials for which a strengthened version of assumption (ii) of definition 6.3 is enough to ensure that it is strongly balanced.

Proposition 6.37. Let $N \geq 3$ and $V_0 := div \tilde{f} + |\tilde{f}|^2$ where $\tilde{f} := (f_1, \cdots, f_N)$ satisfies the following conditions (for $i = 1, \cdots, N$):

(i) $f_i \in L^{\frac{4N}{N+2}}_{loc}(\Omega)$

(ii) $(\partial_i f_i) w \in L^1_{loc}(\Omega)$ for any $w \in H^1_{loc}(\Omega)$.

If $V \in L^1_{loc}(\Omega)$ is such that $V \geq tV_0$ in $\Omega$ for some $t \in (0, 1)$, then $V$ is strongly balanced in $\Omega$. 

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Proof. From theorem 2.3 it follows that $Q_{V_0} \geq 0$ in $\Omega$. Fix any $t \in (0, 1)$. Then, $\mathcal{L}_{tv_0}(\Omega) \hookrightarrow H^1_{\text{loc}}(\Omega)$. We note that for any $w \in H^1_{\text{loc}}(\Omega)$ and any ball $B \subseteq \Omega$,
\[
\int_B |f|^2 w \leq \|f\|^2_{L^2(B)} \|w\|_{L^\infty(B)} < +\infty.
\]
Therefore, from Proposition 6.17 we obtain that $tv_0$ is strongly balanced in $\Omega$. That $V$ is also strongly balanced follows from proposition 6.19 (ii).

Let $\partial_i$ and $\partial_i^p$ stand respectively for the $i$-th distributional and pointwise derivatives. In what follows we fix a generic cube $E := \prod_{i=1}^N [a_i, b_i] \subseteq \Omega$. First we state the following standard result:

**Proposition 6.38.** Let $f \in L^1(E)$. If $\partial_i f \in L^1(E)$, then for a.e. $\tilde{x} \in \prod_{i=2}^N [a_i, b_i]$ the function $f(\cdot, \tilde{x})$ is absolutely continuous on $[a_1, b_1]$ and $\partial_i f(\cdot, \tilde{x}) = \partial_i^p f(\cdot, \tilde{x})$ a.e. in $[a_1, b_1]$. Corresponding statement holds for any $i = 2, \ldots, N$.

We now provide easier to verify sufficient conditions ensuring assumption (ii) of proposition 6.37 holds.

**Proposition 6.39.** (i) Let $f_i \in L^2_{\text{loc}}(\Omega)$. Suppose that for any cube $E \subseteq \Omega$, we can write $\partial_i f_i$ as
\[
(6.19) \quad \partial_i f_i = \partial_i g_+ - \partial_i g_- \quad \text{where } g_+ \in L^2(E), \partial_i g_+ \in L^1(E) \quad \text{and} \quad \partial_i g_- \geq 0.
\]
Then $\partial_i f_i$ satisfies condition (ii) in proposition 6.37.

(ii) Let $f_i \in L^2_{\text{loc}}(\Omega)$ and $\partial_i f_i \in L^1_{\text{loc}}(\Omega)$. Suppose that for any cube $E \subseteq \Omega$,
\[
\int_{\prod_{i=2}^N [a_i, b_i]} \left( \int_{a_1}^{b_1} |\partial_1 f_i| dx_1 \right)^2 d\tilde{x} < \infty.
\]
Then (6.19) holds for $f_i$. A corresponding statement is true for any $i = 2, \ldots, N$.

(iii) If $f_i \in L^2_{\text{loc}}(\Omega)$, $\partial_i f_i \in L^1_{\text{loc}}(\Omega)$ and $\partial_i f_i$ is locally bounded above (or below), then $f_i$ satisfies condition (ii) in proposition 6.37.

**Proof.** (i) For convenience, we may take $i = 1$. For $w \in H^1_{\text{loc}}(\Omega)$ and any $\varphi \in C^1_c(E)$ nonnegative, by Tonelli’s theorem and proposition 6.38
\[
\int_E (\partial_i g_\pm) |w| \varphi = \int_{\prod_{i=2}^N [a_i, b_i]} dx_2 \cdots dx_N \int_{a_1}^{b_1} (\partial_i g_\pm) |w| \varphi dx_1
\]
\[
= - \int_{\prod_{i=2}^N [a_i, b_i]} dx_2 \cdots dx_N \int_{a_1}^{b_1} g_\pm \partial_1 (|w| \varphi) dx_1
\]
\[
= - \int_E g_\pm \partial_1 (|w| \varphi) < +\infty.
\]

(ii) We define
\[
g_\pm(x_1, \tilde{x}) := \int_{a_1}^{x_1} (\partial_1 f_i) \pm(t, \tilde{x}) dt, \quad x_1 \in [a_1, b_1] \quad \text{for a.e. } \tilde{x} \in \prod_{i=2}^N [a_i, b_i].
\]
We also have \( g_\pm \in L^2(E) \), \( \partial_1 g_\pm = (\partial_1 f_1) \pm \in L^1(E) \) and hence \( \partial_1 f_1 = \partial_1 g_+ - \partial_1 g_- \).

(iii) We note that on any cube \( E \in \Omega \), we have \( \partial_1(Cx_i \pm f_i) \geq 0 \) for some constant \( C_E > 0 \) and can apply Tonelli’s theorem as above.

\[ \square \]

**Corollary 6.40.** If \( f_1(x) = \rho(x_1)\eta(x_2, x_3, \ldots, x_N) \) for some \( \rho \in W^{1,1}_{loc}(\mathbb{R}) \) and \( \eta \in L^2_{loc}(\mathbb{R}^{N-1}) \), then \( f_1 \) satisfies the assumptions of proposition 6.39 (ii) and hence \( (6.19) \) holds.

See also example 1.13 and theorem 4.1 in [16] in the context of the above two results.

**Remark 6.41.** From the construction of \( g_\pm \) in proposition 6.39 (ii) it is clear that \( (6.19) \) always holds if we relax the requirement \( g_\pm \in L^2_{loc}(\Omega) \) to \( g_\pm \in L^1_{loc}(\Omega) \).

### 6.5.4 Some well-known potentials are balanced

We provide some examples of strongly balanced potentials not in \( L^p_{loc} \) for some \( p > N/2 \) and as well as not in the class \( V \).

**Example 6.42.** (i) Let \( N \geq 3 \) and \( 1/S_N \) denote the norm of the imbedding \( \mathcal{D}_0^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2N/3}_w(\mathbb{R}^N) \). Assume \( V \in L^2_{loc}(\Omega) \) satisfies \( \|V\|_{L^2_w(\Omega)} \leq S_N^2 \). Then \( Q_{-V} \geq 0 \) in \( \Omega \) and hence \( -V^- \) is strongly balanced in \( \Omega \) by remark 6.26 (i). Thus \( V \) is strongly balanced in \( \Omega \) by proposition 6.19 (ii). Using example 6.30 (v), we can choose \( V^+, V^- \) such that \( V \) is not in \( V(\Omega) \).

(ii) Let \( V, W \) be as in proposition 6.19 (ii). Suppose that

\[ \tilde{c}_B := \int_B \frac{1}{W} < +\infty \text{ for any ball } B \Subset \Omega. \]

Then

\[ \frac{1}{\sqrt{\tilde{c}_B}} \int_B |\xi| \leq \frac{1}{\sqrt{\tilde{c}_B}} \left( \int_B \frac{1}{W} \right)^{\frac{1}{2}} \left( \int_B W|\xi|^2 \right)^{\frac{1}{2}} \leq \sqrt{Q_{V+W}(\xi)}. \]

That is, we additionally have that \( -\Delta + V + W \) is an \( L^1 \) subcritical operator in \( \Omega \).

We now give some examples that involve Hardy type potentials.

**Example 6.43.** Let \( N \geq 3 \) and consider the Hardy constant \( H_N := \frac{(N-2)^2}{4} \).

(i) Let \( \{x_i\}_{i=1}^k \subset \mathbb{R}^N \) and choose \( \{a_i\}_{i=1}^k \subset \mathbb{R} \setminus \{0\} \) such that

\[ \alpha := \sum_{i=1}^k \max\{a_i, 0\} \leq H_N. \]

Define

\[ V(x) := -\sum_{i=1}^k a_{i}|x - x_i|^{-2}, x \notin \{x_1, x_2, \ldots, x_k\}. \]

Then, we can write \( V = V_0 + V_1 \) by summing all the poles with \( a_i < 0 \) to get \( V_0 \) and those with \( a_i > 0 \) to get \( V_1 \). It can be shown that \( Q_{V_1} \geq 0 \) in \( \mathbb{R}^N \) (and \( L^p_{-N}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_0^{1,2}(\mathbb{R}^N) \) if \( \alpha < H_N \)); see proposition 1.2 in [10]. Since \( V_1 \leq 0 \), it is strongly balanced in \( \Omega \). Thus, by proposition 6.19 (ii), \( V \) is a strongly balanced potential in \( \mathbb{R}^N \).
(ii) If \( \{a_i\}_{i=1}^k \) in the previous example are chosen so that
\[
a_i < H_N \quad \text{and} \quad \sum_{i=1}^k a_i < H_N,
\]
then we can find at least one configuration of poles \( \{x_i\}_{i=1}^k \subset \mathbb{R}^N \) so that \( Q_V \geq 0 \) in \( \mathbb{R}^N \) and \( \mathcal{L}_V(\mathbb{R}^N) \hookrightarrow \mathcal{D}^{1,2}_0(\mathbb{R}^N) \) (see theorem 1.1 in [9]). We see that \( V \in \mathcal{V}(\mathbb{R}^N) \) from example 6.30 (iii) and hence is strongly balanced in \( \mathbb{R}^N \).

(iii) We take \( S := \{x_i\} \subset \mathbb{R}^N \) to be a countable dense set and \( \{a_i\} \) be a sequence of positive numbers whose sum is not bigger than \( H_N \). Consider the Hardy potential with poles on the dense set \( S \):
\[
V(x) := -\sum_i a_i |x - x_i|^{-2}.
\]
Recalling that \( -\Delta - H_N|x|^{-2} \) is a non-negative operator in \( \mathbb{R}^N \), we obtain for any \( \xi \in C_c^\infty(\mathbb{R}^N) \),
\[
\int_{\mathbb{R}^N} |\nabla \xi|^2 \geq \frac{1}{H_N} \sum_i a_i \int_{\mathbb{R}^N} |\nabla \xi(\cdot + x_i)|^2 \geq \sum_i a_i \int_{\mathbb{R}^N} |x|^{-2}\xi^2(\cdot + x_i) = -\int_{\mathbb{R}^N} V\xi^2.
\]
Thus \( Q_V \geq 0 \) in \( \mathbb{R}^N \) and since \( V \leq 0 \) it is a strongly balanced (in fact a tame) potential in \( \mathbb{R}^N \). Indeed, \( V \notin L^{1/2}(B) \) for any ball \( B \).

(iv) Consider the critical Hardy operator \( -\Delta + V, V := -H_N|x|^{-2} \), on any bounded domain \( U \subset \mathbb{R}^N, N \geq 3 \). Then it is well known that the operator is \( L^1 \)-subcritical in \( U \) and \( \mathcal{L}_V(U) \not\hookrightarrow H^1_0(U) \). Clearly, \( V \) is tame in \( U \).

For more examples of strongly balanced potentials see corollaries 11.5 and 11.6 in §11.2. These examples require some further results than presented so far.

### 6.6 Examples of non balanced potentials

The previous discussion shows that if the negative and positive parts of the potential \( V \) do not interact too much, then the potential is balanced. We describe below a non balanced potential that oscillates infinitely often near its point singularity. We also point out that these oscillations are “wild” in the sense of remark 6.45 below.

**Example 6.44.** Denote \( B := B_1(0) \subset \mathbb{R}^N, N \geq 3 \) and fix any \( t \in (0,1) \). Let \( H_N := (N-2)^2 \) denote the critical Hardy constant. We take any \( c \in (0, H_N/4] \), \( \alpha < 0 \) and consider the potential
\[
V_\alpha(x) := -c \sin^2(|x|^\alpha)|x|^{-2}, \quad x \in B.
\]
Define then the vectorfield \( \overrightarrow{\Gamma}_\alpha := (\sqrt{-V_\alpha}, 0, \cdots, 0) \). Thanks to Hardy’s inequality we have
\[
Q_{-4|\overrightarrow{\Gamma}_\alpha|^2} = Q_{4V_\alpha} \geq 0 \quad \text{in} \quad B.
\]
By setting \( \sigma_\alpha := \text{div} \overrightarrow{\Gamma}_\alpha \), an explicit computation gives,
\[
\sigma_\alpha(x) = \sqrt{c} x_1 \left( \alpha|x|^{\alpha-3} \cos(|x|^\alpha) - \sin(|x|^\alpha) |x|^{-3} \right).
\]
We choose $\alpha > 2 - N$ and check that $\sigma_\alpha \in L^1(B)$. Now, by theorem 4.1 (iii) in [16],

$$\left| \int_B \sigma_\alpha \xi^2 \right| \leq \int_B |\nabla \xi|^2 \quad \forall \xi \in W^{1,\infty}_c(B).$$

In particular, $\sigma_\alpha \in H^{-1}_{\text{loc}}(B)$. It is easy to see from the above inequality that we can identify $\mathcal{K}_{t\sigma_\alpha}(B)$ with $H^1_0(B)$. If we choose $\beta \in (\frac{2-N}{2}, 0)$, then

$$w_\beta(x) := |x|^\beta - 1 \in H^1_0(B).$$

A straightforward consideration involving radial-angular coordinates shows that

$$\sigma_\alpha w_\beta \notin L^1(B_\epsilon(0)) \quad \text{for any } \epsilon \in (0, 1) \quad \text{if } \alpha + \beta \leq 2 - N.$$  \hspace{1cm} (6.20)

Therefore, for any $\alpha \in (2 - N, \frac{2-N}{2})$ we may find $\beta \in (\frac{2-N}{2}, 0)$ such that (6.20) holds. Hence $t\sigma_\alpha$ is not balanced in $B$ for any $\alpha \in (2 - N, \frac{2-N}{2})$.

Remark 6.45. We note that $\sigma_\alpha^+ \notin L^{2N+2}_{\text{loc}}(B)$ if $\alpha \leq \frac{2-N}{2}$ and hence for such $\alpha$ the potential $t\sigma_\alpha$ for $t \in (0, 1)$ does not satisfy the assumptions of lemma 6.33. Conversely, if $\alpha > \frac{2-N}{2}$, then $\sigma_\alpha \in L^{2N+2}_{\text{loc}}(B)$ and lemma 6.33 ensures that $t\sigma_\alpha$ is balanced in $B$ for $t \in (0, 1)$. Thus, balanced potentials are allowed to oscillate infinitely often near a singularity as long as these oscillations are not “wild”.

7 AAP principle in the space of Distributions for general non-negative operators

In theorems 7.2 and 7.7 below, we will show that given a non-negative quadratic form $Q_V$ in $\Omega$ with $V \in L^p_{\text{loc}}(\Omega)$ for some $p > \frac{N}{2}$ ($N \geq 2$), the set $\mathcal{E}_V(\Omega)$ is non-empty and the converse for any $V \in L^1_{\text{loc}}(\Omega)$.

7.1 $\mathcal{E}_V(\Omega) \neq \emptyset$ implies $Q_V \geq 0$ in $\Omega$

We recall the following well-known result (whose proof is given in the appendix 12.2):

**Proposition 7.1.** Let $\Omega_*$ be a bounded open set, $W \in L^1_{\text{loc}}(\Omega_*)$, $W \geq 0$, $f \in L^\infty(\Omega_*)$, $f \geq 0$. If $u \in H^1_0(\Omega_*)$ is a non-negative nonzero weak solution of $-\Delta u + Wu = f$ in $\Omega_*$, then

$$\int_{\Omega_*} |
abla \xi|^2 \, dx + \int_{\Omega_*} W \xi^2 \geq \int_{\Omega_*} \left( \frac{f}{u} \right) \xi^2, \quad \forall \xi \in C^\infty(\Omega_*).$$

In particular, $f/u \in L^1_{\text{loc}}(\Omega_*)$.

We extend the above result to the distributional framework for any $V \in L^1_{\text{loc}}(\Omega)$.

**Theorem 7.2.** Let $V \in L^1_{\text{loc}}(\Omega)$ and $\Phi \in \mathcal{E}_V(\Omega)$ (i.e., $\Phi \neq 0$, $\Phi \geq 0$ satisfies $-\Delta \Phi + V\Phi \geq h$ in $\mathcal{D}'(\Omega)$ for some non-negative $h \in L^1_{\text{loc}}(\Omega)$). Then,

$$Q_V(\xi) \geq \int_\Omega \left( \frac{h}{\Phi} \right) \xi^2 \, dx, \quad \forall \xi \in W^{1,\infty}_c(\Omega).$$  \hspace{1cm} (7.1)

In particular, $h/\Phi \in L^1_{\text{loc}}(\Omega)$.  

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By similar arguments as in the claim above, but applied to $w \geq 0$, further, since $\Phi < 2.11$, we get $(7.3)$.

From (7.2) and proposition 7.1 we conclude that for all large $w^*$ in $\Omega$.

Let $w^* \rightarrow w_\infty$ a.e. Note that $0 \leq w_\infty \leq \Phi$ and $w_\infty \neq 0$. By dominated convergence theorem, $w_k \rightarrow w_\infty$ in $L^1_{loc}(\Omega_*)$ and hence $w_\infty$ is a distributional solution of $-\Delta w_\infty + V^+ w_\infty = h + V^- \Phi$ in $\Omega_*$. From theorem 2.12 we conclude that $w_\infty > 0$ a.e. in $\Omega_*$. From (7.2) and proposition 7.1 we conclude that for all large $k$,

$$\int_{\Omega_*} |\nabla \xi|^2 \, dx + \int_{\Omega_*} V^+ \xi^2 \geq \int_{\Omega_*} \left( \frac{H_k}{w_k} \right) \xi^2.$$  

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Noting that pointwise we have
\[ 0 \leq \frac{H_k}{w_k} = \frac{T_k(h + V\Phi)}{w_k} \to \frac{h + V\Phi}{w_\infty} \quad \text{a.e. in } \Omega, \]
and applying Fatou Lemma, we conclude
\[ \liminf_{k \to \infty} \int_{\Omega^*} \left( \frac{H_k}{w_k} \right) \xi^2 \geq \int_{\Omega^*} \left( \frac{h + V\Phi}{w_\infty} \right) \xi^2. \]
Finally, since \( w_\infty \leq \Phi \), from (7.3) and the above inequality we obtain,
\[ \int_{\Omega^*} |\nabla \xi|^2 + \int_{\Omega^*} V \xi^2 \geq \int_{\Omega^*} |\nabla \xi|^2 + \int_{\Omega^*} V^+ \xi^2 - \int_{\Omega^*} V^- \left( \frac{\Phi}{w_\infty} \right) \xi^2 \geq \int_{\Omega^*} \left( \frac{h}{\Phi} \right) \xi^2. \]
Since \( \xi \) was any test function with support in \( \Omega \), the theorem follows. \( \square \)

**Remark 7.3.** If in the above theorem
\[ \inf_B \frac{h}{\Phi} > 0 \quad \text{for any } B \subset \Omega \]
then in fact we obtain that \( \mathcal{L}_V(\Omega) \hookrightarrow L^2_{loc}(\Omega) \). This is a situation one encounters when \( V \) is regular (say in \( L^p_{loc}(\Omega) \) for some \( p > \frac{N}{2} \)).

We can now state a version of the above theorem for the energy supersolution.

**Proposition 7.4.** Assume \( V \in L^1_{loc}(\Omega), V \) is balanced and \( -\Delta + V \) is \( L^1 \)-subcritical (i.e., \( \mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega) \)). Given a non-negative non-zero function \( f \in L^1_{loc}(\Omega) \), let \( u \in \mathcal{H}_V(\Omega) \) be an energy supersolution of \( -\Delta u + Vu = f \). That is,
\[ a_{V,\Omega}(u, \xi) \geq \int_{\Omega} f \xi, \quad \forall 0 \leq \xi \in W^{1,\infty}_{c}(\Omega). \]

Then,
\[ Q_V(\xi) \geq \int_{\Omega} \left( \frac{f}{J(u)} \right) \xi^2 dx \quad \forall \xi \in W^{1,\infty}_{c}(\Omega). \]

**Proof.** It follows from lemma [6.9] and proposition [6.7] that \( J(u) \) is a non-negative non-zero distributional supersolution. In particular, from theorem [2.12] \( J(u) > 0 \) a.e. in \( \Omega \). The result now follows from theorem [7.2]. \( \square \)

### 7.2 \( V^+ \in L^p_{loc}(\Omega) \) for some \( p > \frac{N}{2} \) and \( Q_V \geq 0 \) in \( \Omega \) imply \( \mathcal{C}_V(\Omega) \neq \emptyset \)

From standard existence results for Green’s function (see, for instance, [32]), we have:

**Proposition 7.5.** Let \( V^+ \in L^p_{loc}(\Omega) \) for some \( p > \frac{N}{2} \) (\( N \geq 2 \)) and \( \Omega_* \subset \Omega \) be a domain with smooth boundary. Then, there exists a Green’s function \( G \) for \( -\Delta + V^+ \) in \( \Omega_* \) satisfying the following properties:

(i) \( G : \Omega_* \times \Omega_* \to \mathbb{R}, G > 0 \text{ in } \Omega_*, G(x,y) = G(y,x), \)
(ii) $G(x, y) \sim \delta(x)$ (the distance to the boundary function) uniformly for $x$ near $\partial \Omega$, and

(iii) for a given $f \in L^\infty(\Omega)$, the very weak solution of the boundary value problem

$$(-\Delta + V^+)u = f \text{ in } \Omega, \; u = 0 \text{ on } \partial \Omega$$

is given by the Green’s representation formula $u(x) = \int_{\Omega} G(x, y)f(y)dy$.

**Remark 7.6.** The existence of Green’s function, Green’s representation formula as well as Harnack inequality can be ensured under weaker assumptions on $V^+$, for example, membership in the local Morrey spaces or the local Kato class. Consequently, the analysis done here can be modified to handle these more general situations without changing the final results.

**Theorem 7.7.** Let $N \geq 2$. Assume $V \in L^1_{loc}(\Omega)$, $V^+ \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$ ($N \geq 2$) and $Q_V \geq 0$ in $\Omega$. Then there exists a non-negative Radon measure $\mu$ and an a.e. positive function $u \in W^{1,q}_{loc}(\Omega)$ for any $1 \leq q < \frac{N}{N-2}$ which is a distributional solution of $-\Delta u + V^+u = \mu$ in $\Omega$. In particular, $C_V(\Omega) \neq \emptyset$.

**Proof.** Consider an exhaustion $\{\Omega_m\}_{m=1}^\infty$ of $\Omega$ by open bounded sets $\Omega_m$ with smooth boundary such that $\Omega_m \Subset \Omega_{m+1} \Subset \Omega$. Let $V_n^- = \min\{V^-, n\}$, $n \in \mathbb{N}$. Define

$$\lambda_{n,m} := \inf_{\phi \in H_0^1(\Omega_m) \backslash \{0\}} \frac{\int_{\Omega_m} |\nabla \phi|^2 + (V^+ - V_n^-)\phi^2}{\int_{\Omega_m} \phi^2}.$$

Using the assumption $Q_V \geq 0$ in $\Omega$, we obtain that $\lambda_{n,m} \geq 0$. Since $V^+ \in L^\frac{N}{2}(\Omega_m)$, there exists a non-negative function $u_{n,m} \in H^1_0(\Omega_m)$ that achieves the above infimum. Clearly $u_{n,m}$ solves the following equation in the $H^1-$ weak sense:

$$\begin{cases}
-\Delta u_{n,m} + V^+u_{n,m} = (V_n^- + \lambda_{n,m})u_{n,m} \text{ in } \Omega_m, \\
u_{n,m} \geq 0 \text{ in } \Omega_m; \\
u_{n,m} = 0 \text{ on } \partial \Omega_m.
\end{cases}$$

Again by the assumption on $V^+$, we note that $u_{n,m}$ is continuous in $\overline{\Omega_m}$. Fix a ball $B_0 \Subset \Omega$. Let $x_{n,m} \in \overline{B_0}$ denote a point at which $u_{n,m}$ achieves its minimum in $\overline{B_0}$, that is:

$$u_{n,m}(x_{n,m}) \leq u_{n,m}(x) \text{ for all } x \in \overline{B_0}.$$

Let $G_m$ be the (positive) Green’s function for $-\Delta + V^+$ on $\Omega_m$. We now normalise $u_{n,m}$ as:

$$w_{n,m} := u_{n,m}/\rho_{n,m} \quad \text{where}$$

$$\rho_{n,m} := \int_{\Omega_m} (V_n^- + \lambda_{n,m})u_{n,m}(y)G_m(x_{n,m}, y)dy > 0.$$

Define

$$f_{n,m} := (V_n^- + \lambda_{n,m})w_{n,m}.$$

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Clearly \( w_{n,m} \in H^1_0(\Omega_m) \) satisfies in the \( H^1 \)–weak sense:

\[
\begin{cases}
-\Delta w_{n,m} + V^+ w_{n,m} = f_{n,m} & \text{in } \Omega_m, \ w_{n,m} \geq 0 \text{ in } \Omega_m, \\
\int_{\Omega_m} f_{n,m}(y) G_m(x_{n,m}, y) dy = 1.
\end{cases}
\]

From Green’s function representation,

\[
w_{n,m}(x) = \int_{\Omega_m} G_m(x, y) f_{n,m}(y) dy.
\]

We note from the above expression and the normalisation in (7.10) that \( w_{n,m}(x_{n,m}) = 1 \) for all \( n, m \). Clearly, from (7.7) and the above normalisation,

\[
(7.11) \ 1 = w_{n,m}(x_{n,m}) \leq w_{n,m}(x) \text{ for all } x \in B_0.
\]

Fix \( m_0 \geq 1 \). We note that \( G_m \geq G_{m_0} \) on \( \Omega_{m_0} \) for all \( m \geq m_0 \) (proof similar to that of the claim in theorem 7.2) and hence,

\[
\gamma(m_0) := \min_{x \in B_0, y \in \Omega_{m_0}} G_{m_0+1}(x, y) > 0.
\]

Therefore,

\[
G_m(x_{n,m}, y) \geq \gamma(m_0) \ \forall n \geq 1, m \geq m_0 + 1, y \in \Omega_{m_0}.
\]

Consequently,

\[
(7.12) \ \forall n \geq 1, m \geq m_0 + 1 : \ \gamma(m_0) \int_{\Omega_{m_0}} f_{n,m} \leq \int_{\Omega_{m_0}} f_{n,m}(y) G_m(x_{n,m}, y) dy \leq 1.
\]

Thus, we get that,

\[
(7.13) \ \|f_{n,m}\|_{L^1(\Omega_{m_0})} \leq \frac{1}{\gamma(m_0)} \text{ for all } n \geq 1, m \geq m_0 + 1 \text{ and } m_0 \geq 1.
\]

Consider the very weak solution \( \psi_{n,m} \) of the following problem for \( n \geq 1, m \geq m_0 + 2 \):

\[
\begin{cases}
-\Delta \psi_{n,m} + V^+ \psi_{n,m} = f_{n,m} & \text{in } \Omega_{m_0+2}, \\
\psi_{n,m} \geq 0 & \text{in } \Omega_{m_0+2}, \\
\psi_{n,m} = 0 & \text{on } \partial \Omega_{m_0+2}.
\end{cases}
\]

In what follows \( q \) will denote any number in the interval \([1, \frac{N}{N-1}]\). By the estimate in (2.4) in proposition 2.9 and (7.12),

\[
(7.14) \ \|\psi_{n,m}\|_{W^{1,q}(\Omega_{m_0+2})} \leq C(q, m_0) \|f_{n,m}\|_{L^1(\Omega_{m_0+2})} \leq \frac{C(q, m_0)}{\gamma(m_0+2)} \text{, } \forall n \geq 1, m \geq m_0 + 3.
\]

We note that \( h_{n,m} := w_{n,m} - \psi_{n,m} \) solves the following equation for all \( n \geq 1, m \geq m_0 + 2 \):

\[
\begin{cases}
-\Delta h_{n,m} + V^+ h_{n,m} = 0 & \text{in } \Omega_{m_0+2}, \\
h_{n,m} \geq 0 & \text{on } \partial \Omega_{m_0+2}.
\end{cases}
\]
By the weak comparison principle, \( h_{n,m} \geq 0 \) in \( \Omega_{m_0+2} \). By using Harnack’s inequality we obtain that

\[
\sup_{\Omega_{m_0+1}} h_{n,m} \leq C(m_0) \inf_{\Omega_{m_0+1}} h_{n,m} \leq C(m_0) \inf_{\mathbb{N}_0} h_{n,m} \leq C(m_0) h_{n,m}(x_{n,m})
\]

(7.15)

\[
\leq C(m_0) w_{n,m}(x_{n,m}) = C(m_0), \ \forall n \geq 1, m \geq m_0 + 2.
\]

Therefore, \( \{-\Delta h_{n,m}\} \) is a bounded sequence in \( L^p(\Omega_{m_0+1}) \). Therefore, by standard elliptic regularity and (7.15) we obtain that (for different constants \( C(m_0) \)),

\[
\|h_{n,m}\|_{W^{2,p}(\Omega_{m_0})} \leq C(m_0) \left( \|h_{n,m}\|_{L^p(\Omega_{m_0+1})} + \|V^+ h_{n,m}\|_{L^p(\Omega_{m_0+1})} \right)
\]

\[
\leq C(m_0) \left( 1 + \|V^+\|_{L^p(\Omega_{m_0+1})} \right) \|h_{n,m}\|_{L^\infty(\Omega_{m_0+1})}
\]

(7.16)

\[
\leq C(m_0) \left( 1 + \|V^+\|_{L^p(\Omega_{m_0+1})} \right) \forall n \geq 1, m \geq m_0 + 2.
\]

From the above estimates (7.14) and (7.16) we obtain

\[
\|w_{n,m}\|_{W^{1,q}(\Omega_{m_0})} \leq C(q, m_0), \ \forall n \geq 1, m \geq m_0 + 3.
\]

In what follows, we fix \( r \) to be the Hölder conjugate of \( p \), and we note that \( r \in [1, \frac{N}{N-2}) \). The last inequality together with Sobolev imbedding means that

\[
\{w_{n,m} : n \geq 1, m \geq m_0 + 3\} \text{ is relatively compact in } L^r(\Omega_{m_0}), \ \forall m_0 \geq 1.
\]

(7.17)

Consider the diagonal sequence \( \{w_{n,n}\} \) which we will denote by \( \{v_n\} \). By (7.17), there exists a subsequence of \( \{v_n\} \), which we will denote by \( \{v_{n_j}(1)\} \) and a non-negative function \( u_1 \in L^r(\Omega_1) \) such that

\[
v_{n_j}(1) \to u_1 \text{ in } L^r(\Omega_1) \text{ as } j \to \infty.
\]

Considering now the sequence \( \{v_{n_j}(1)\} \) and again applying (7.17) to it, we obtain a subsequence of \( \{v_{n_j}(1)\} \), which we will denote by \( \{v_{n_j}(2)\} \), and a non-negative function \( u_2 \in L^r(\Omega_2) \) such that

\[
v_{n_j}(2) \to u_2 \text{ in } L^r(\Omega_2) \text{ as } j \to \infty.
\]

It is easy to see that \( u_1 \equiv u_2 \) in \( \Omega_1 \). Proceeding inductively, we obtain subsequences \( \{v_{n_j}(m)\} \) and non-negative functions \( u_m \in L^r(\Omega_m) \) such that

\[
\begin{cases}
\{v_{n_j(m+1)}\}_{j} \subset \{v_{n_j(m)}\}_{j} \text{ for all } m; \\
v_{n_j(m)} \to u_m \text{ in } L^r(\Omega_m) \text{ as } j \to \infty, \text{ for each } m \in \mathbb{N}; \\
u_m \equiv u_{m+1} \text{ in } \Omega_{m+1}.
\end{cases}
\]

(7.18)

Define

\[
u \in L^r_{loc}(\Omega) \text{ as: } u \equiv u_m \text{ on } \Omega_m, m = 1, 2, 3, \cdots.
\]

(7.19)

We note that each \( v_{n_j}(m) \) solves:

\[
\int_{\Omega} v_{n_j}(m)(-\Delta \xi + V^+ \xi) = \int_{\Omega} (V^-_{n_j(m)} + \lambda_{n_j(m), n_j(m)}) v_{n_j}(m) \xi, \ \forall \xi \in C_c^\infty(\Omega_m).
\]

(7.20)
Take any $\xi \in C_c^\infty(\Omega)$, $\xi \geq 0$. Choose $m$ such that support of $\xi$ is contained in $\Omega_m$. Letting $j \to \infty$ in the above equation with such a $\xi$, using (7.18)-(7.19), the assumption $V^+ \in L^p_{loc}(\Omega)$ and Fatou’s lemma on the right hand side, we get,

$$\int_\Omega u(-\Delta \xi + V^+ \xi) \geq \int_\Omega V^{-u\xi}.$$  

Since $\xi \geq 0$ was arbitrary, we obtain that $-\Delta u + Vu \geq 0$ in the sense of distributions. Define the non-negative Radon measure $\mu := -\Delta u + Vu$. Then, we get that $u$ is a distribution solution of $-\Delta u + Vu = \mu$ in $\Omega$. From (7.19) and (7.11) we get that $u$ is non-negative in $\Omega$ and $\inf_{\Omega} u \geq 1$. Appealing to theorem 2.12 we obtain that $u>0$ a.e. in $\Omega$. That $u \in W^{1,q}_{loc}(\Omega)$ for $q \in [1, \frac{N}{N-1})$ follows from proposition 2.10.

**Theorem 7.8** (AAP Principle in $\mathcal{D}'(\Omega)$). Assume $V \in L^1_{loc}(\Omega)$.

(i) If $\mathcal{C}_V(\Omega) \neq \emptyset$ then $Q_V \succeq 0$ in $\Omega$.

(ii) Conversely, if $N \geq 2$, $V^+ \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$ and $Q_V \succeq 0$ in $\Omega$, then $\mathcal{C}_V(\Omega) \neq \emptyset$.

(iii) When $N = 1$, we have $Q_V \succeq 0$ in $\Omega$ iff $\mathcal{C}_V(\Omega) \neq \emptyset$.

**Proof.** The case $N \geq 2$ follows from theorems 7.2 and 7.7. When $N = 1$ we refer to theorem 3.1 in [12].

8 AAP principle for $L^1$-subcritical operators with balanced potential

Our assumptions on $V$ are rather weak and may prevent the existence of a Green’s function for the operator $-\Delta + V$ in $\Omega$. Nevertheless, if the operator is $L^1$-subcritical, within the energy space $\mathcal{H}_V(\Omega)$ it continues to retain the unique solvability and monotonicity properties that are usually implied by the existence of a nonnegative Green’s function.

**Lemma 8.1.** Let $V \in L^1_{loc}(\Omega)$ and $Q_V \succeq 0$ in $\Omega$. Then, $-\Delta + V$ is $L^1$-subcritical in $\Omega$ if and only if there exists a (unique) energy solution (see definition 3.7) $u \in \mathcal{H}_V(\Omega)$ solving $-\Delta u + Vu = f$ for any $f \in L^\infty_{c}(\Omega)$. Indeed, if $f$ is nonnegative, then so is $J(u)$.

**Proof.** Let $-\Delta + V$ be $L^1$-subcritical, i.e., $\mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega)$. Take any $f \in L^\infty_c(\Omega)$. Define the continuous linear map

$$\Phi_f : L^1_{loc}(\Omega) \to \mathbb{R} \quad \text{by} \quad \Phi_f(h) := \int_\Omega fh.$$  

Then $\Phi_f$ is continuous on $\mathcal{L}_V(\Omega)$, and can therefore be uniquely extended to a continuous linear map on $\mathcal{H}_V(\Omega)$. Using Riesz representation theorem, we can find a unique $u \in \mathcal{H}_V(\Omega)$ satisfying

$$a_{V,\Omega}(u,h) = \Phi_f(h), \quad \forall h \in W^{1,\infty}_c(\Omega).$$  

(8.1)

Conversely, assume that the operator equation is solvable in $\mathcal{H}_V(\Omega)$ (in the energy sense) for each given $f \in L^\infty_c(\Omega)$. By density of $W^{1,\infty}_c(\Omega)$ in $\mathcal{H}_V(\Omega)$ such a solution is unique. Suppose
The assertion that $\mathcal{L}_V(\Omega) \not\hookrightarrow L^1_{\text{loc}}(\Omega)$. Then, we may find a ball $B \subset \Omega$ and a sequence $\{\xi_n\} \subset W^{1,\infty}_c(\Omega)$ such that $\xi_n \to 0$ in $\mathfrak{H}_V(\Omega)$, but $\inf_B \int_B |\xi_n| > 0$. By considering $|\xi_n|$ in place of $\xi_n$ and noting that $Q_V(\xi_n) = Q_V(|\xi_n|)$, we may assume without loss of generality that $\xi_n \geq 0$. Let $u \in \mathfrak{H}_V(\Omega)$ be the energy solution of $-\Delta u + Vu = \chi_B$. It then follows that $\int_B \xi_n = a_{V,\Omega}(u, \xi_n) \to 0$ as $n \to \infty$. This gives a contradiction, thereby showing the imbedding $\mathcal{L}_V(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega).

The assertion that $J(u) \geq 0$ whenever $f \geq 0$ follows from lemma 6.9.

The following corollary shows that $L^1$—subcritical operators with balanced potentials possess a hidden nonnegative Green’s function.

**Corollary 8.2.** Let $V \in L^1_{\text{loc}}(\Omega)$ be such that $Q_V \geq 0$ in $\Omega$. Then,

(i) If $V$ is balanced in $\Omega$ and $-\Delta + V$ is $L^1$—subcritical in $\Omega$, then given any $f \in L_\infty(\Omega)$, there exists a unique distributional solution $u_* \in L^1_{\text{loc}}(\Omega)$ obtained as $u_* = J(u)$ for some $u \in \mathfrak{H}_V(\Omega)$, solving $-\Delta u_* + Vu_* = f$. Indeed, if $f$ is nonnegative, then so is the corresponding solution $u_*$. 

(ii) Conversely, if for any compact subset $K$ of $\Omega$ with positive measure, the problem $-\Delta u + Vu = \chi_K$ admits a nonnegative distributional solution $u$, then $-\Delta + V$ is $L^1$—subcritical in $\Omega$.

**Proof.** (i) Follows from the above lemma and proposition 6.7.

(ii) We take any such compact $K$ and the corresponding solution $u$. From theorem 2.12 we have $u > 0$ a.e. in $\Omega$ and from theorem 7.2 we obtain that $Q_V \succeq w := \chi_K/u$ in $\Omega$. Therefore,

$$
\left( \int_K \frac{1}{w} \right)^{\frac{1}{2}} \sqrt{Q_V(\xi)} \geq \left( \int_K \frac{1}{w} \right)^{\frac{1}{2}} \left( \int_K w|\xi|^2 \right)^{\frac{1}{2}} \geq \int_K |\xi|, \quad \forall \xi \in W^{1,\infty}_c(\Omega).
$$

**Remark 8.3.** From the above results, it is clear that if $V$ is balanced and $-\Delta + V$ is $L^1$—subcritical in $\Omega$, there exists no nonzero $u \in \mathfrak{H}_V(\Omega)$ such that $u_* = J(u)$ solves in the distributional sense $-\Delta u_* + Vu_* = 0$ in $\Omega$. In other words, in this situation, the only distributional solution $u_* \in \mathfrak{H}_V(\Omega)$ to $-\Delta u + Vu = 0$ in $\Omega$ is the trivial solution.

**Corollary 8.4.** Let $V \in L^1_{\text{loc}}(\Omega)$ and $Q_V \succeq 0$ in $\Omega$. Suppose for some $\epsilon_0$ chosen to be positive if $-\Delta + V$ is $L^1$—critical in $\Omega$ and zero otherwise, there exists $f_0 \in L_\infty(\Omega)$ such that the problem $-\Delta u + (V + \epsilon_0)u = f_0$ has no distribution solution in $\Omega$. Then $V$ is not a balanced potential in $\Omega$.

**Proof.** Suppose $V$ is balanced in $\Omega$. Then by proposition 6.7(ii), $V + \epsilon_0$ is balanced in $\Omega$. We note that the operator $-\Delta + V + \epsilon_0$ is $L^1$—subcritical in $\Omega$ and apply corollary 8.2 to conclude that for any $f \in L_\infty(\Omega)$ we obtain a distributional solution of $-\Delta u + (V + \epsilon_0)u = f$ in $\Omega$.

**Corollary 8.5.** Let $V \in L^1_{\text{loc}}(\Omega)$ and $Q_V \succeq 0$ in $\Omega$. Suppose for some $\epsilon_0$ chosen to be positive if $-\Delta + V$ is $L^1$—critical in $\Omega$ and zero otherwise, we have $\mathcal{C}_{V + \epsilon_0}(\Omega) = \emptyset$ (see definition 7.3). Then, $V$ is not balanced in $\Omega$.


Proof. Follows by the contrapositive argument as above and noting that if \( f \in L^\infty_c(\Omega) \), \( f \not\equiv 0 \) is chosen nonnegative, the corresponding distributional solution will be positive a.e. in \( \Omega \) from corollary 8.2 and theorem 2.12.

But, we can show that the above results imply the existence of a positive distributional solution to the equation \(-\Delta u + Vu = 0\) on any “strict” sub-domain of \( \Omega \).

**Proposition 8.6.** Assume \( V \in L^1_{loc}(\Omega) \), \( V \) is balanced in \( \Omega \) and \(-\Delta + V \) is \( L^1 \)-subcritical in \( \Omega \). Then, given any compact set \( K \subset \Omega \) with positive measure, there exists an a.e. positive distribution solution \( u \) satisfying \((-\Delta + V)u = 0 \) in \( \Omega \setminus K \).

**Proof.** Since \(-\Delta + V \) is \( L^1 \)-subcritical, by corollary 8.2 we can solve \((-\Delta + V)u = \chi_K \) in \( D'(\Omega) \) with \( u \geq 0 \). From proposition 2.10 we can find a quasicontinuous representative of \( u \), which solves the same PDE as \( u \) in the distributional sense, and from theorem 2.12, we get \( u > 0 \) a.e. in \( \Omega \).

**Corollary 8.7.** Assume \( V \in L^1_{loc}(\Omega) \), \( V \) is balanced in \( \Omega \) and \(-\Delta + V \) is \( L^1 \)-subcritical in \( \Omega \). Then, \( \mathcal{C}_V(\Omega) \) contains an infinite set of linearly independent functions.

**Proposition 8.8.** Assume that \( V \in L^1_{loc}(\Omega) \) is a balanced potential and \(-\Delta + V \) is \( L^1 \)-subcritical in \( \Omega \). Suppose additionally that for some compact submanifold \( \Sigma \) contained in \( \Omega \), the restriction map

\[ \mathcal{L}_V(\Omega) \to L^1(\Sigma), \quad \xi \mapsto \xi|_{\Sigma} \]

is continuous. Then there exists an a.e. positive distribution solution \( u \in L^1_{loc}(\Omega) \) solving \(-\Delta u + Vu = 0 \) in \( \Omega \setminus \Sigma \).

**Proof.** By Riesz representation argument, for any \( h \in L^\infty(\Sigma) \), \( h \geq 0, h \not\equiv 0 \), we obtain an Energy solution \( u_* \in \mathcal{H}_V(\Omega) \) such that

\[ a_V(u_*, \xi) = \int_{\Sigma} h\xi d\nu, \quad \forall \xi \in W^{1,\infty}_c(\Omega). \]

For simplicity we let \( h \equiv 1 \) on \( \Sigma \). We can then proceed as in proposition 6.7 to show that \( u := J(u_*) \) is a nonnegative distribution solution of \(-\Delta u + Vu = \nu \) in \( \Omega \). Therefore, \( u \not\equiv 0 \) and \(-\Delta u + Vu = 0 \) in \( \Omega \setminus \Sigma \).

**Remark 8.9.** The main point of the above results is that Harnack inequality may fail for the operator \(-\Delta + V \). It seems natural to push the arguments in proposition 8.6 further by taking a sequence \( \{K_n\} \) of compact sets approaching the boundary (or point at infinity) of \( \Omega \) and show that the corresponding sequence of solutions \( \{u_n\} \) converges (up to a subsequence) locally to an a.e. positive function \( u \) solving \((-\Delta + V)u = 0 \) in the distributional sense in the full domain \( \Omega \). Such an argument seems to require Harnack-type inequality for its success (see [2], and [26] for further generalisations).

This motivates the following

**Open problem:** Assume \( V \in L^1_{loc}(\Omega) \), \( V \) is balanced and \(-\Delta + V \) is \( L^1 \)-subcritical in \( \Omega \). Does \( 0 \in \mathcal{M}_V(\Omega) \) ?

**Corollary 8.10.** Assume \( V \in L^1_{loc}(\Omega) \), \( V \) is balanced and \(-\Delta + V \) is \( L^1 \)-subcritical in \( \Omega \). Then, there exists an open dense subset \( U \) of \( \Omega \) and an a.e. positive distribution solution \( u \) satisfying \((-\Delta + V)u = 0 \) in \( U \).
Proof. Take a compact $K \subset \Omega$ to be any “Cantor-like” set with positive measure and empty interior. Then $U := \Omega \setminus K$ is an open dense subset of $\Omega$ and then we apply proposition 8.6.

**Proposition 8.11** (AAP Principle in $\mathcal{H}_V(\Omega)$ for $L^1$-subcritical operator). Let $V \in L^1_{\text{loc}}(\Omega)$ be such that $Q_V \geq 0$ in $\Omega$. Then $-\Delta + V$ is $L^1$-subcritical in $\Omega$ iff there exists $u \in \mathcal{H}_V(\Omega)$ and $f \in L^1_{\text{loc}}(\Omega)$ with $\inf_B f > 0$ for any ball $B \Subset \Omega$ such that $-\Delta u + Vu = f$ in the energy sense in $\Omega$.

Proof. Let $-\Delta + V$ be $L^1$-subcritical. Consider an exhaustion $\{\Omega_n\}$ of $\Omega$ by open sets such that $\Omega_n \Subset \Omega$, $\Omega_n \subset \Omega_{n+1}$.

Set $K_1 := \overline{\Omega}$ and $K_{n+1} := \overline{\Omega_{n+1}} \setminus \Omega_n$ (for $n \geq 1$). Note that given a compact set $K \subset \Omega$,

$$\{n \in \mathbb{N}: K \cap K_n \neq \emptyset\}$$

is a finite set. By lemma 8.1, there exists $u_n \in \mathcal{H}_V(\Omega), u_n \neq 0$ such that

$$a_{V,\Omega}(u_n, \xi) = \int_{\Omega} \chi_{K_n} \xi, \quad \forall \xi \in W^{1,\infty}_c(\Omega).$$

Setting

$$\rho_n := \sum_{j=1}^{n} \frac{1}{j^2} \frac{u_j}{\|u_j\|_{V,\Omega}}, \quad f_n := \sum_{j=1}^{n} \frac{\chi_{K_j}}{j^2} \frac{\|u_j\|_{V,\Omega}},$$

we have also

$$a_{V,\Omega}(\rho_n, \xi) = \int_{\Omega} f_n \xi, \quad \forall \xi \in W^{1,\infty}_c(\Omega).$$

We easily see that

$$\rho_n \to u := \sum_{j=1}^{\infty} \frac{1}{j^2} \frac{u_j}{\|u_j\|_{V,\Omega}} \quad \text{in } \mathcal{H}_V(\Omega).$$

Concerning the sequence $\{f_n\}$, from the observation (8.2) we see that

$$\sum_{j=1}^{n} \frac{1}{j^2} \frac{\chi_{K_j}}{\|u_j\|_V} \to f := \sum_{j=1}^{\infty} \frac{1}{j^2} \frac{\chi_{K_j}}{\|u_j\|_V} \quad \text{in } L^1_{\text{loc}}(\Omega).$$

Note that $\inf_B f > 0$ for any ball $B \Subset \Omega$. Therefore, for each $\xi \in W^{1,\infty}_c(\Omega)$ we have $a_{V,\Omega}(\rho_n, \xi) \to a_{V,\Omega}(u, \xi)$ and $\int_{\Omega} f_n \xi \to \int_{\Omega} f \xi$. Hence, we get in the limit:

$$a_{V,\Omega}(u, \xi) = \int_{\Omega} f \xi, \quad \forall \xi \in W^{1,\infty}_c(\Omega).$$

Conversely, let $u \in \mathcal{H}_V(\Omega)$ be an energy solution of $(-\Delta + V)u = f$ for some $f \in L^1_{\text{loc}}(\Omega)$ with $\inf_B f > 0$ for any ball $B \Subset \Omega$. Suppose that $L_V(\Omega) \not\hookrightarrow L^1_{\text{loc}}(\Omega)$. Then, we may find a ball $B \Subset \Omega$ and a sequence $\{\xi_n\} \subset W^{1,\infty}_c(\Omega)$ such that $\xi_n \to 0$ in $\mathcal{H}_V(\Omega)$ but $\inf_n \int_B |\xi_n| > 0$. By
considering $|\xi_n|$ in place of $\xi_n$ and noting that $Q_V(\xi_n) = Q_V(|\xi_n|)$, we may assume without loss of generality that $\xi_n \geq 0$. It then follows that
\[
\inf_n \int_\Omega f \xi_n \geq (\inf_B f) \inf_n \int_B \xi_n > 0 \quad \text{and} \quad a_{V,\Omega}(u, \xi_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
This gives a contradiction to the assumption that $u$ is the energy solution of $-\Delta u + Vu = f$, thereby showing the imbedding $\mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega)$. \hfill \Box

**Theorem 8.12** (AAP Principle in $\mathcal{D}'(\Omega)$ for $L^1$-subcritical operator). Let $V \in L^1_{loc}(\Omega)$ be such that $Q_V \geq 0$ in $\Omega$.

(i) Let $V$ be balanced in $\Omega$. If $-\Delta + V$ is $L^1$-subcritical in $\Omega$, then there exists an a.e. positive function $u_* \in L^1_{loc}(\Omega)$ (obtained as $u_* = J(u)$ for some $u \in \mathcal{H}_V(\Omega)$) and $f \in L^1_{loc}(\Omega)$ with $\inf_B f > 0$ for any ball $B \Subset \Omega$, solving $-\Delta u_* + Vu_* = f$ in the sense of distributions in $\Omega$.

(ii) Conversely, if there exists a nonnegative function $u \in L^1_{loc}(\Omega)$ and $f \in L^1_{loc}(\Omega)$ with $\inf_B f > 0$ for any ball $B \Subset \Omega$, solving $-\Delta u + Vu = f$ in the sense of distributions in $\Omega$, then $-\Delta + V$ is $L^1$-subcritical in $\Omega$.

**Proof.** (i) follows from proposition 8.11 proposition 6.7, lemma 6.9 and theorem 2.12 (ii) follows in the same way as the proof of corollary 8.2 (ii) by noting that $w := f/u$ satisfies $1/w \in L^1_{loc}(\Omega)$. \hfill \Box

### 9 Equivalence of $L^1$ and $L^2$-subcriticality notions

#### 9.1 Equivalence of $L^1$- and global $L^2$-subcriticality when $V$ is balanced

When $V$ is balanced, it is a rather straightforward affair to prove the equivalence of the notions of $L^1$- and global $L^2$-subcriticalities as the following result shows:

**Proposition 9.1.** Assume $V \in L^1_{loc}(\Omega)$ is such that $Q_V \geq 0$ in $\Omega$.

(i) Let $V$ be balanced in $\Omega$. Then $-\Delta + V$ is globally $L^2$-subcritical in $\Omega$ if it is $L^1$-subcritical in $\Omega$.

(ii) $-\Delta + V$ is $L^1$-subcritical in $\Omega$ if it is globally $L^2$-subcritical in $\Omega$.

**Proof.** (i) By theorem 8.12 there exists $u_* \in L^1_{loc}(\Omega)$ obtained as $u_* = J(u)$ for some $u \in \mathcal{H}_V(\Omega)$ and $f \in L^1_{loc}(\Omega)$ with $\inf_B f > 0$ for any ball $B \Subset \Omega$, solving $-\Delta u_* + Vu_* = f$ in the distribution sense. Indeed, $u_* > 0$ a.e.. By theorem 7.2 we get $Q_V \geq f/u_*$ in $\Omega$. This in particular shows that $w := f/u_* \in L^1_{loc}(\Omega)$. It is easy to see that $1/w \in L^1_{loc}(\Omega)$. Therefore, $-\Delta + V$ is globally $L^2$-subcritical.

(ii) Conversely, let $-\Delta + V$ be globally $L^2$-subcritical with the weight $w$. Then given any compact set $K \subset \Omega$, by an application of Hölder’s inequality,
\[
\left( \int_K 1/w \right)^{\frac{1}{2}} \sqrt{Q_V(\xi)} \geq \left( \int_K 1/w \right)^{\frac{1}{2}} \left( \int_K w|\xi|^2 \right)^{\frac{1}{2}} \geq \int_K |\xi|, \quad \forall \xi \in W^{1,\infty}_c(\Omega).
\]
This shows that $-\Delta + V$ is $L^1$-subcritical. \hfill \Box
9.2 Equivalence of $L^1$ and feeble $L^2$-subcriticality when $V$ is tame in $\Omega$

In this subsection we assume $V^+ \in L^p_{\text{loc}}(\Omega)$ for some $p > \frac{N}{2}$ $(N \geq 2)$. Mainly, we show that the validity of the imbedding $\mathcal{L}_V(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega)$ is determined (somewhat surprisingly) by whether or not $\mathcal{L}_V(\Omega) \hookrightarrow L^1(K)$ for some compact set $K \subset \Omega$ with positive measure. From this, we deduce the equivalence between the notions of $L^1$- and feeble $L^2$- subcriticality.

**Proposition 9.2.** Let $\Omega_*$ be a bounded domain with smooth boundary and let $K_1, K_2$ be two disjoint compact sets with positive measure contained in $\Omega_*$. Assume that $V^+ \in L^p(\Omega_*)$ for some $p > \frac{N}{2}$ $(N \geq 2)$. Then, the unique weak solution $\psi_\alpha$ to the problem

\begin{equation}
-\Delta u + V^+ u = \alpha \chi_{K_1} - \chi_{K_2}, \quad u \in H^1_0(\Omega_*), \quad \alpha \in \mathbb{R},
\end{equation}

is in the space $W^{2,p}(\Omega_*)$ for all $\alpha$ and is positive a.e. in $\Omega$ for all $\alpha > 0$ large enough. In particular, $\psi_\alpha \in C(\overline{\Omega}_*)$ and satisfies the equation \((9.1)\) pointwise a.e. in $\Omega_*$.

**Proof.** Let $G, G_0$ be the Green’s functions of the operators $-\Delta + V^+$ and $-\Delta$ respectively in $\Omega_*$. It is well known that there exists a constant $c > 0$ such that

\[ c^{-1} G_0(x, y) \leq G(x, y) \leq c G_0(x, y) \text{ for all } x, y \in \Omega_* \]

The solution $\psi_\alpha$ to \((9.1)\) can be represented as follows

\[ \psi_\alpha(x) = \alpha \int_{K_1} G(x, y) dy - \int_{K_2} G(x, y) dy. \]

Hence, we deduce

\[ \left( \frac{\alpha}{c} \right) \int_{K_1} G_0(x, y) dy - c \int_{K_2} G_0(x, y) dy \leq \psi_\alpha(x) \leq c \alpha \int_{K_1} G_0(x, y) dy. \]

Therefore,

\begin{equation}
\left( \frac{\alpha}{c} \right) u_1(x) - c u_2(x) \leq \psi_\alpha(x) \leq c \alpha u_1(x) \text{ in } \Omega_*,
\end{equation}

where $u_1, u_2 \in H^1_0(\Omega_*)$ solve the following problems:

\[ -\Delta u_1 = \chi_{K_1} \quad \text{and} \quad -\Delta u_2 = -\chi_{K_2}. \]

We note that $u_1, u_2 \in C^1(\overline{\Omega}_*)$ and that by Hopf lemma, $u_2 < 0$ on $\partial \Omega_*$. Therefore, we obtain from \((9.2)\) that $\psi_\alpha$ is bounded in $\Omega_*$ for all $\alpha$ and can be made positive by choosing $\alpha > 0$ large enough. By elliptic regularity, $\psi_\alpha \in W^{2,p}(\Omega_*) \subset C(\overline{\Omega}_*)$ for all $\alpha$. \hfill \Box

**Lemma 9.3.** Assume that $p > \frac{N}{2}$ $(N \geq 2)$ and $\Omega_* \Subset \Omega$ be a bounded domain with smooth boundary, $u \in W^{2,p}(\Omega_*) \cap H^1_0(\Omega_*) \cap C(\overline{\Omega}_*)$ be a nonnegative function. Consider the extension

\[ \tilde{u} = \begin{cases} u & \text{in } \Omega_*, \\ 0 & \text{in } \Omega \setminus \Omega_. \end{cases} \]

Then, $\gamma(\tilde{u}) \in W^{2,1}_c(\Omega)$ for any function $\gamma \in C^2(\mathbb{R})$ with $\gamma(t) = 0$ for all $t \leq 0$. 

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This shows that the second distributional derivative of $\gamma$ is compactly contained in $\Omega$. Define

$$
\gamma(\phi_n) = \gamma(u) \text{ in } H^1(\Omega) \cap C(\overline{\Omega}),
$$

(9.3)

$$
\gamma'(\phi_n) \partial_i \phi_n = \gamma'(u) \partial_i u \text{ in } W^{1,1}(\Omega).
$$

From the facts that $\phi_n \to 0$ uniformly on $\partial \Omega$ and $\partial_i \phi_n \to \partial_i u$ in $W^{1,p}(\Omega)$ we obtain using the Trace embedding that

$$
\int_{\partial \Omega} |\gamma'(\phi_n) \partial_i \phi_n| \leq \max_{\partial \Omega} |\gamma'(\phi_n)| \int_{\partial \Omega} |\partial_i \phi_n| = o_n(1).
$$

From (9.3), the above estimate and the Trace embedding, we obtain that $\gamma'(u) \partial_i u = 0$ on $\partial \Omega$. That is,

$$
\gamma(u) \in H^1_0(\Omega) \text{ and } \partial_i \gamma(u) = \gamma'(u) \partial_i u \in W_0^{1,1}(\Omega).
$$

Therefore, for a test function $\xi \in C_c^\infty(\Omega),$

$$
\int_{\Omega} \gamma(\bar{u}) \partial_j \xi = \int_{\Omega} \gamma(u) \partial_j \xi = - \int_{\Omega} \gamma'(u) \partial_i u \partial_j \xi = \int_{\Omega} \partial_j (\gamma'(u) \partial_i u) \xi.
$$

This shows that the second distributional derivative of $\gamma(\bar{u})$ is clearly an $L^1(\Omega)$ function with support in $\Omega$.

**Proposition 9.4.** Assume $V^+ \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$ $(N \geq 2)$. Let $K_1, K_2$ be two disjoint compact subsets of $\Omega$. Then, there exists a nonnegative function $\Psi \in W^{2,1}_c(\Omega) \cap H^1_0(\Omega) \cap C_c(\Omega)$ and a constant $C > 0$ such that the following inequality is satisfied pointwise a.e. in $\Omega$:

$$
-\Delta \Psi + V^+ \Psi \leq C\chi_{K_1} - \chi_{K_2}.
$$

**Proof.** Choose a bounded domain $\Omega_s \Subset \Omega$ with smooth boundary such that $K_1, K_2$ are compactly contained in $\Omega_s$. Applying proposition 9.2, we choose a positive function $\psi := \psi_\omega \in W^{2,p}(\Omega_s) \cap H^1_0(\Omega_s) \cap C_0(\overline{\Omega_s})$ solving problem (9.1), and extend it by zero on $\Omega \setminus \Omega_s$ (while still referring to this extension as $\psi$).

Let $\gamma(s) = \chi_{(0,\infty)}s^3$. Then $\gamma \in C^2(\mathbb{R})$ and

$$
\gamma''(s) \geq 0 \text{ for all } s \in \mathbb{R}; \quad \gamma'(s) > 0 \text{ and } \gamma(s) - \gamma'(s)s < 0 \text{ for } s > 0.
$$

Define $\bar{\psi} := \gamma(\psi)$. Clearly $\bar{\psi} \geq 0$ and by Lemma 9.3, $\bar{\psi} \in W^{2,1}_c(\Omega)$. Hence, for a.e. in $\Omega_s,$

$$
-\Delta \bar{\psi} + V^+ \bar{\psi} = -\gamma''(\psi)|\nabla \psi|^2 - \gamma'(\psi)\Delta \psi + V^+ \gamma(\psi)
$$

$$
= -\gamma''(\psi)|\nabla \psi|^2 + \gamma'(\psi)(-\Delta \psi + V^+ \psi) + V^+ (\gamma(\psi) - \gamma'(\psi)\psi)
$$

(9.4)

$$
\leq \gamma'(\psi)(-\Delta \psi + V^+ \psi).
$$
Since $ψ$ is continuous in $\overline{\Omega}_*$ and positive in $\Omega_*$, we have for some positive constant $c$,

$$
\gamma'(ψ) \leq c \text{ in } K_1 \quad \text{and} \quad \gamma'(ψ) \geq \frac{1}{c} \text{ in } K_2.
$$

Since $ψ$ solves (9.1), we obtain from (9.4),

$$
-\Delta \overline{ψ} + V^+ \overline{ψ} \leq αc \chi_{K_1} - \left(\frac{1}{c}\right) \chi_{K_2} \text{ a.e. in } \Omega_*,
$$

Hence, $Ψ := c \overline{ψ}$ is the required function. □

**Lemma 9.5** (Non-Vanishing of null sequences). Assume $V^+ \in L^p_{\text{loc}}(\Omega)$ for some $p > \frac{N}{2}$ ($N \geq 2$), $V^- \in L^1_{\text{loc}}(\Omega)$ and $Q_V(\Omega) \geq 0$. Then, for any sequence $\{ξ_n\} \subset W^{1,\infty}(\Omega)$ such that $Q_V(ξ_n) → 0$ we have the following alternative

(i) either $\int_K |ξ_n| → 0$ for any compact set $K$ with positive measure,

(ii) or, $\inf_n \int_K |ξ_n| > 0$ for any compact set $K$ with positive measure.

**Proof.** We argue by contradiction. Let $K_1, K_2$ be two disjoint compact subsets of $\Omega$ with positive measure such that we can find a sequence $\{ξ_n\}$ satisfying

$$
Q_V(ξ_n) → 0, \quad \int_{K_1} |ξ_n| → 0 \quad \text{and} \quad \inf_n \int_{K_2} |ξ_n| > 0.
$$

Since $Q_V(ξ_n) = Q_V(|ξ_n|)$, without loss of generality we may assume that $ξ_n$ is nonnegative for all $n$. Consider the function $Ψ$ given by the previous proposition for the pair $K_1, K_2$. Let $Ω_n ⊂ Ω$ contain $K_1, K_2$ and the supports of $Ψ$ and $ξ_n$. Then, by corollary 3.6, we can expand for any $t \in \mathbb{R}$,

$$
\|T(ξ_n + tΨ)\|_{V, Ω_n} = \int_Ω \left\{ |∇(ξ_n + tΨ)|^2 + V(ξ_n + tΨ)^2 \right\}
= Q_V(ξ_n) + t^2 \|T(Ψ)\|_{V, Ω_n} + 2t \int_Ω ξ_n(-ΔΨ + VΨ)
\leq o_n(1) + t^2 \|T(Ψ)\|_{V, Ω_n} + 2t \int_Ω ξ_n(-ΔΨ + V^+Ψ)
$$

(9.5)

By the property of $Ψ$, we note that:

$$
\int_{K_1} ξ_n(-ΔΨ + V^+Ψ) ≤ o_n(1), \quad \sup_n \int_{K_2} ξ_n(-ΔΨ + V^+Ψ) < 0,
$$

and

$$
\int_{Ω \setminus (K_1 \cup K_2)} ξ_n(-ΔΨ + V^+Ψ) ≤ 0.
$$

Hence, for $t > 0$ small enough and a fixed large $n$, we deduce that the right-hand side of (9.5) is strictly negative, a contradiction. □
The above proposition implies the following alternative.

**Corollary 9.6.** Let \( V^+ \in L^p_{loc}(\Omega) \) for some \( p > \frac{N}{2} \) (\( N \geq 2 \)), \( V^- \in L^1_{loc}(\Omega) \) and \( Q_V \geq 0 \) in \( \Omega \). Then,

(i) either \( \mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega) \),

(ii) or, the restriction map \( \mathcal{L}_V(\Omega) \to L^1(K) \) is discontinuous for any compact set \( K \) with positive measure.

Finally, we can show the equivalence of the two notions of criticality when \( V \) is tame:

**Proposition 9.7.** Assume \( V \in L^1_{loc}(\Omega) \) and \( Q_V \geq 0 \) in \( \Omega \).

(i) Let \( V^+ \in L^p_{loc}(\Omega) \) for some \( p > \frac{N}{2} \) (\( N \geq 2 \)). Then, \(-\Delta + V\) is \( L^1 \)-subcritical if it is feebly \( L^2 \)-subcritical.

(ii) Suppose \( V \) is tame in \( \Omega \). Then, \(-\Delta + V\) is globally \( L^2 \)-subcritical in \( \Omega \) if and only if the restriction map \( \mathcal{L}_V(\Omega) \to L^1(K) \) is continuous for some compact set \( K \) with positive measure.

(iii) Suppose \( V \) is tame in \( \Omega \). Then, \(-\Delta + V\) is globally \( L^2 \)-subcritical in \( \Omega \) if and only if \(-\Delta + V\) is feebly \( L^2 \)-subcritical in \( \Omega \).

**Proof.**

(i) Let \( Q_V \geq c\chi_K \) in \( \Omega \) for some constant \( c > 0 \) and some compact set \( K \subset \Omega \) with positive measure. By an application of Hölder’s inequality,

\[
c^{-\frac{1}{2}}|K|^\frac{1}{2} \sqrt{Q_V(\xi)} \geq |K|^{\frac{1}{2}} \left( \int_K \xi^2 \right)^{\frac{1}{2}} \geq \int_K |\xi|, \quad \forall \xi \in W^{1,\infty}(\Omega).
\]

Appealing to corollary 9.6 we obtain that \( \mathcal{L}_V(\Omega) \hookrightarrow L^1_{loc}(\Omega) \).

(ii) follows from corollary 9.6 and proposition 9.1

(iii) follows from (i) and (ii) above.

10 AAP Principle for subcritical and critical operators with tame potentials

**Remark 10.1.** In view of the proposition 9.7 we will simply use the terminology “subcritical” (“critical”) instead of \( L^1 \), globally \( L^2 \) or feebly \( L^2 \)-subcritical (critical) whenever \( V \) is tame.

We can make more precise the distributional AAP Principle given in theorem 7.8 by considering separately the two subclasses of non-negative operators, namely the class of sub-critical and the critical operators. Indeed, we formulate this in lemmas 10.3 and 10.6 in terms of the Riesz measures. For tame potentials, a neat version is given in theorems 10.5 and 10.7. Since we will use the equivalence between the two criticalities stated in proposition 9.7 we will often restrict \( V \) to be tame in \( \Omega \).
10.1 Distributional AAP Principle for subcritical operators in terms of \textit{Riesz measures}

In the light of proposition 9.7, we first restate the results in section 8 when \( V \) is tame in \( \Omega \):

**Theorem 10.2.** Assume \( V \in L^1_{\text{loc}}(\Omega) \), \( V \) is tame in \( \Omega \). Then \( -\Delta + V \) is subcritical in \( \Omega \) if and only if there exists a non-negative distribution solution \( u_\ast \) of the equation \( -\Delta u + Vu = f \) where \( f \) is some nonnegative locally integrable function satisfying \( \inf_K f > 0 \) for some compact set \( K \subset \Omega \) with positive measure.

**Proof.** The “only if” part follows from propositions 9.1, 9.7 and theorem 8.12. Conversely, if such \( u_\ast \) and \( f \) exist, by theorem 7.2, we get \( Q_V \gtrless f/u_\ast \) in \( \Omega \). In particular \( f/u_\ast \notin L^1_{\text{loc}}(\Omega) \). Since \( f/u_\ast \not\equiv 0 \) in \( K \), for some \( c > 0 \) we also have that \( \{ f/u_\ast \geq c \} \cap K \) has positive measure. Then, \( Q_V \gtrless c\chi_{\tilde{K}} \) in \( \Omega \) for any compact \( \tilde{K} \subset \{ f/u_\ast \geq c \} \cap K \) with positive measure. Therefore, \( -\Delta + V \) is subcritical by proposition 9.7. \( \square \)

Next, we show the following “solvability” property for a subcritical operator based on proposition 9.7.

**Lemma 10.3.** Let \( V \in L^1_{\text{loc}}(\Omega) \) and \( V \) tame in \( \Omega \). If the problem \( -\Delta u + Vu = \chi_K \) admits a nonnegative distribution solution for some compact \( K \subset \Omega \) with positive measure, then the problem \( -\Delta u + Vu = f \) admits a distribution solution for any \( f \in L^\infty_c(\Omega) \).

**Proof.** By theorem 10.2, \( -\Delta + V \) is subcritical in \( \Omega \) and the solvability for any \( f \in L^\infty_c(\Omega) \) follows from corollary 8.2. \( \square \)

Finally, we show an AAP Principle for subcritical operators in terms of the \textit{Riesz measures}. We note that, in contrast with results in section 8, the solutions need not lie in the space \( \mathcal{H}_V(\Omega) \).

**Lemma 10.4.** Let \( V \in L^1_{\text{loc}}(\Omega) \) and \( Q_V \gtrless 0 \) in \( \Omega \).

\[ \begin{align*} 
\text{(i)} & \quad \text{Assume } V^+ \in L^p_{\text{loc}}(\Omega) \text{ for some } p > \frac{N}{2} \quad (N \geq 2). \text{ Then, } -\Delta + V \text{ is feebly } L^2\text{-subcritical in } \Omega \text{ if } \mathcal{M}_V(\Omega) \neq \{0\} \text{ (see definition 1.3).} \\
\text{(ii)} & \quad \text{Conversely, assume } V \text{ is balanced. Then } \mathcal{M}_V(\Omega) \neq \{0\} \text{ if } -\Delta + V \text{ is } L^1\text{-subcritical in } \Omega. 
\end{align*} \]

**Proof.** (i) We recall that \( \mathcal{M}_V(\Omega) \neq 0 \) by theorem 7.7. Let \( \mu \in \mathcal{M}_V(\Omega) \), \( \mu \neq 0 \) and \( u \in L^1_{\text{loc}}(\Omega) \) be a nonnegative function solving \( -\Delta u + Vu = \mu \) in \( \mathcal{D}'(\Omega) \). From proposition 2.10 \( u \in W^1_{\text{loc}}(\Omega) \) and from theorem 2.12 \( u > 0 \) a.e. in \( \Omega \). Take an exhaustion \( \{ \Omega_n \}_{n=1}^\infty \) of \( \Omega \) as follows:

\[ \Omega_n \text{ an open bounded set, } \Omega_1 \subset \Omega_n \subset \Omega_{n+1} \subset \Omega \text{ and } \mu(\Omega_1) > 0. \]

Choose balls \( B \Subset B_1 \subset \Omega_1 \) such that \( \mu(\Omega_1 \setminus B_1) > 0 \). Let \( \tilde{\mu} = \mu_+(\Omega_1 \setminus B_1) \). Let \( \theta_n \in W^{1,1}_0(\Omega_n) \) be a very weak solution (refer proposition 2.9) of

\[ -\Delta \theta_n + (V^+ + \chi_B)\theta_n = \tilde{\mu} \text{ in } \Omega_n, \quad \theta_n = 0 \text{ on } \partial \Omega_n. \]
Then, by remark 2.8, \( \theta_n \) solves the above equation in the distributional sense. By using the distributional Kato’s inequality as in the claim of theorem 7.2, we obtain that \( \theta_n \geq 0 \) in \( \Omega_n \) for all \( n \) as well as

\[
\theta_n \leq \theta_{n+1} \leq u \quad \text{in} \quad \Omega_n \quad \text{for all} \quad n \geq 1.
\]

Since \( \theta_n \in W_0^{1,1}(\Omega_n) \) is nontrivial, we may find a quasi-continuous representative of \( \theta_n \) (which we denote again by \( \theta_n \)) that is positive q.a.e. by theorem 2.12. By elliptic regularity and Harnack principle, \( \theta_n \) is continuous and positive outside the support of \( \tilde{\mu} \) and in particular on \( B \). Define \( z_n := u - \theta_n \). Then,

\[
-\Delta z_n + V z_n \geq (V^- + \chi_B)\theta_n \quad \text{in} \quad D'({\Omega_n}), \quad 0 \leq z_{n+1} \leq z_n \quad \text{in} \quad \Omega_n.
\]

Clearly \( z_n \) is nontrivial and from theorem 2.12 again, we get \( z_n > 0 \) a.e. in \( \Omega_n, n \geq 1 \). Appealing to theorem 7.2, we get

\[
\text{(10.1)} \quad Q_V \geq (V^- + \chi_B)(\theta_n/z_n) \quad \text{in} \quad \Omega_n.
\]

We see that \( w := (V^- + \chi_B)(\theta_1/z_1) \) is an a.e. positive function on \( B \). Therefore, for some \( c > 0 \), we have that the set \( \{w \geq c\} \cap B \) has positive measure. We choose a compact set \( \tilde{K} \subset \{w \geq c\} \cap B \) with positive measure.

We note from the monotonicity of \( \theta_n \) and \( z_n \) that

\[
\text{(10.2)} \quad (V^- + \chi_B)(\theta_n/z_n) \geq (V^- + \chi_B)(\theta_1/z_1) \chi_{\Omega_1} \quad \text{in} \quad \Omega_n, \forall n \geq 1.
\]

Then, from (10.1), (10.2) and the choice of \( \tilde{K} \) we get that,

\[
Q_V \geq c\chi_{\tilde{K}} \quad \text{in} \quad \Omega_n, \forall n \geq 1.
\]

Therefore, \( -\Delta + V \) is feebly \( L^2 \)-subcritical in \( \Omega \).

(ii) follows from corollary 8.2.

As a consequence of the above lemma, we have the following result for tame potentials.

**Theorem 10.5** (AAP principle in \( D'({\Omega}) \) for subcritical operators). Suppose that \( V \in L^1_{\text{loc}}({\Omega}) \) is tame in \( {\Omega} \). Then \( -\Delta + V \) is subcritical in \( {\Omega} \) if and only if \( M_V({\Omega}) \neq \emptyset \).

### 10.2 Distributional AAP Principle for Critical operators in terms of Riesz measures

We state the following AAP Principle for critical operators, which follows directly from lemma 10.4 and using the fact \( M_V({\Omega}) \neq \emptyset \) (see theorem 7.1):

**Lemma 10.6** (AAP Principle in \( D'({\Omega}) \) for Critical Operators). Let \( V \in L^1_{\text{loc}}({\Omega}) \) such that \( Q_V \geq 0 \) in \( {\Omega} \).

(i) Assume \( V^+ \in L^p_{\text{loc}}({\Omega}) \) for some \( p > \frac{N}{2} \) (\( N \geq 2 \)). Then, \( M_V({\Omega}) = \{0\} \) if \( -\Delta + V \) is feebly \( L^2 \)-critical in \( {\Omega} \).

(ii) Conversely, assume \( V \) is balanced. Then \( -\Delta + V \) is \( L^1 \)-critical in \( {\Omega} \) if \( M_V({\Omega}) = \{0\} \).

The above lemma implies the following result for tame potentials.
Theorem 10.7 (AAP principle in $D'((\Omega)$ for critical operators). Suppose that $V \in L^1_{\text{loc}}(\Omega)$ is tame in $\Omega$. Then $-\Delta + V$ is critical in $\Omega$ if and only if $M_V(\Omega) = \{0\}$. That is, any nonnegative super solution of $-\Delta + V$ in $\Omega$ is in fact a solution.

By applying theorems 7.7, 10.7 and 2.12, we obtain

Corollary 10.8. Assume $V \in L^1_{\text{loc}}(\Omega)$ is tame in $\Omega$ and $-\Delta + V$ is critical in $\Omega$. Then there exists an a.e. positive distribution solution $u$ satisfying $(-\Delta + V)u = 0$ in $\Omega$.

The above corollary should be viewed together with the proposition 8.6. We have then the following result for those nonnegative operators which extend as nonnegative operators to a bigger domain:

Proposition 10.9. Assume that $\overline{\Omega} \neq \mathbb{R}^N$. Let $V \in L^1_{\text{loc}}(\Omega)$ be tame in $\Omega$. If $-\Delta + V$ is subcritical in $\Omega$ we assume additionally that there exist an open set $\tilde{\Omega}$ and $\tilde{V} \in L^1_{\text{loc}}(\tilde{\Omega})$ such that

(i) $\tilde{\Omega} \setminus \Omega$ contains a compact set $K$ with positive measure

(ii) $\tilde{V}$ extends $V$ and $\tilde{V}$ is tame in $\tilde{\Omega}$ (in particular $Q_{\tilde{V}} \geq 0$ in $\tilde{\Omega}$).

Then there exists an a.e. positive distribution solution $u$ satisfying $(-\Delta + V)u = 0$ in $\Omega$.

Proof. If $-\Delta + V$ is critical in $\Omega$, by the above corollary, we are done. Otherwise, we consider the extended operator $-\Delta + \tilde{V}$. If $-\Delta + \tilde{V}$ is subcritical in $\tilde{\Omega}$ we apply proposition 8.6 with the compact set $K$. If it is critical, then the result follows again from the above corollary.

Remark 10.10. There are well-known examples where the above assumptions don’t hold, for example the boundary Hardy operator on a ball can’t be extended to a bigger domain as any such extension will not be locally integrable in the bigger domain.

Remark 10.11. To obtain the above conclusion for any nonnegative operator under the assumptions that $V$ is locally bounded or $V \in L^p_{\text{loc}}(\Omega)$ for $p > \frac{N}{2}$ ($N \geq 2$), see for example theorem 2.3 in [26] and [25].

The following is a restatement of the results given in proposition 8.6, corollary 10.8 and theorem 7.2.

Corollary 10.12. Assume $V \in L^1_{\text{loc}}(\Omega)$ and $V$ tame in $\Omega$. Then $Q_V \geq 0$ in $\Omega$ iff for any compact $K \subset \Omega$ with positive measure, there exists an a.e. positive distribution solution $u$ to the problem $-\Delta u + Vu = 0$ in $\Omega \setminus K$.

Proof. The “only if” part follows from corollaries 8.2 and 10.8. To show the converse, let $\xi \in W^{1,\infty}_c(\Omega)$. Choose a compact set $K \subset \Omega$ with positive measure disjoint from the support of $\xi$. Then, by assumption, there exists an a.e. positive distribution solution $u$ to the problem $-\Delta u + Vu = 0$ in $\Omega \setminus K$. Hence by theorem 7.2 $Q_V \geq 0$ on $\Omega \setminus K$. In particular, $Q_V(\xi) \geq 0$.

We recall here the definition of a “null” sequence:

Definition 10.13. Let $Q_V \geq 0$ in $\Omega$. A sequence $\{\xi_n\} \subset W^{1,\infty}_c(\Omega)$ is called a null sequence for $Q_V$ if $Q_V(\xi_n) \to 0$ and there exists a compact set $K \subset \Omega$ with positive measure such that $\int_K |\xi_n| = 1$ for all $n$. 51
The following proposition just restates the result in proposition [9.7].

**Proposition 10.14.** Let $V \in L^1_{\text{loc}}(\Omega)$ be tame in $\Omega$ and $Q_V \succeq 0$ in $\Omega$. Then $-\Delta + V$ is critical in $\Omega$ iff $Q_V$ has a null sequence.

The following results show that being a critical operator is an “unstable” state:

**Lemma 10.15.** Let $U \subset \Omega \subset \tilde{\Omega}$ be open sets such that $\tilde{\Omega} \setminus \Omega$ and $\Omega \setminus U$ contain a compact set with positive measure. Let $V \in L^1_{\text{loc}}(\tilde{\Omega})$ and $V^+ \in L^p_{\text{loc}}(\tilde{\Omega})$ for some $p > \frac{N}{2} (N \geq 2)$. If $Q_V \succeq 0$ in $\Omega$, then $-\Delta + V$ is $L^1$-subcritical in $U$. If $-\Delta + V$ is $L^1$-critical in $\Omega$ then it is supercritical in $\Omega$.

**Proof.** Fix a compact set $K \subset \Omega \setminus U$ of positive measure. We obviously have $Q_V + \chi_K \succeq \chi_K$ in $\Omega$. By proposition [9.7(i)] we have that $-\Delta + V + \chi_K$ is $L^1$-subcritical in $\Omega$ and hence we obtain that $-\Delta + V$ is $L^1$-subcritical in $U$.

Suppose $Q_V \succeq 0$ in $\tilde{\Omega}$. If $-\Delta + V$ is $L^1$-critical in $\tilde{\Omega}$, by the same arguments as above we obtain that $-\Delta + V$ is $L^1$-subcritical in $\Omega$, a contradiction. If $-\Delta + V$ is $L^1$-critical in $\tilde{\Omega}$ so will it be in $\Omega$, again the same contradiction. Thus, $-\Delta + V$ is supercritical in $\Omega$. 

**Corollary 10.16.** Let $V, U, \Omega, \tilde{\Omega}$ be as in the above lemma. If $V$ is balanced in $\Omega$, then $-\Delta + V$ is subcritical in $U$. If additionally $-\Delta + V$ is critical in $\Omega$ then it is supercritical in $\tilde{\Omega}$.

**Proof.** Follows from the above lemma and proposition [9.7].

**Proposition 10.17.** Assume $V \in L^1_{\text{loc}}(\Omega)$ is such that $Q_V \succeq 0$ in $\Omega$.

(i) Take any nonnegative $w \in L^1_{\text{loc}}(\Omega)$ such that $w \not\equiv 0$. Then $-\Delta + V + w$ is a feebly $L^2$-subcritical operator in $\Omega$. If $-\Delta + V$ is a feebly $L^2$-critical operator in $\Omega$, then $-\Delta + V - w$ is supercritical in $\Omega$.

(ii) If $V$ is tame in $\Omega$ and $w \in L^\infty_{\text{loc}}(\Omega), w \not\equiv 0$ and is nonnegative, in the statement (i) above we can replace “feebly $L^2$-critical (subcritical)” by critical (subcritical).

**Proof.** (i) We note that we may find a compact set $K$ of positive measure and a $c > 0$ such that $K \subset \{w \geq c\}$. We then see that,

$$Q_{V+w}(\xi) = Q_V(\xi) + \int_\Omega w \xi^2 \geq c \int_\Omega \chi_K \xi^2 \quad \forall \xi \in W^{1,\infty}(\Omega).$$

Hence, $-\Delta + V$ is feebly $L^2$-subcritical.

If the operator $-\Delta + V$ is feebly $L^2$-critical, there exists $\xi_0 \in W^{1,\infty}(\Omega)$ such that

$$\int_\Omega |\nabla \xi_0|^2 + V \xi_0^2 < c \int_\Omega \chi_K \xi_0^2 \leq \int_\Omega w \xi_0^2.$$

That is, $Q_{V-w}(\xi_0) < 0$.

(ii) follows from the fact that $V + w$ is tame in $\Omega$ (by proposition [6.6(ii)]) and proposition [9.7].
10.3 Characterisation of subcritical/critical operators in terms of dimension of $C_V(\Omega)$ when $V^+ \in L^1_{loc}(\Omega)$

Proposition 10.18. Let $\phi_1, \phi_2 \in C^2(\Omega)$ be positive in $\Omega$, and set $\Phi := \sqrt{\phi_1\phi_2}$. Then, $\Phi \in C^2(\Omega)$ satisfies

\[(10.3) \quad -\Delta \Phi = \frac{1}{4\Phi^3} |\phi_2\nabla \phi_1 - \phi_1\nabla \phi_2|^2 - \frac{1}{2\Phi}(\phi_2\Delta \phi_1 + \phi_1\Delta \phi_2) \text{ in } \Omega.
\]

Proof. Follows by direct calculation.

Lemma 10.19. Let $V^+ \in L^1_{loc}(\Omega)$ and $u_1, u_2 \in C_V(\Omega)$. Then $Q_V \geq w$ in $\Omega$ where

\[w := \frac{1}{4} \left| \frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2} \right|^2 \in L^1_{loc}(\Omega).
\]

Proof. The main idea of the proof is adapted from theorem 3.1 of Pinchover [23]. In view of proposition 2.10 we also obtain that $u_i \in W^{1,1}_{loc}(\Omega), i = 1, 2$. Let $\rho_n \in C^\infty(\mathbb{R}^N)$ be a sequence of standard mollifiers such that $supp(\rho_n) = \{|x| \leq 1/n\}$. Define

$\Omega_n := \{x \in \Omega : d(x, \partial \Omega) > 1/n\}; \phi_{i,n} := u_i * \rho_n$ in $\Omega_n, i = 1, 2$.

Then $\{\phi_{i,n}\}$ is a sequence of functions in $C^\infty(\Omega_n)$ such that

\[(10.4) \quad \phi_{i,n} > 0 \text{ and } \phi_{i,n} \to u_i \text{ in } W^{1,1}_{loc}(\Omega).
\]

Define

$\Phi_n := \sqrt{\phi_{1,n}\phi_{2,n}}, \Phi := \sqrt{u_1u_2}$.

By (10.4) and Vitali’s convergence theorem we note that (upto a subsequence),

\[(10.5) \quad \Phi_n \to \Phi \text{ in } L^1_{loc}(\Omega).
\]

Applying proposition 10.18 we get

\[(10.6) -\Delta \Phi_n = \frac{1}{4\Phi_n^3} |\phi_{2,n}\nabla \phi_{1,n} - \phi_{1,n}\nabla \phi_{2,n}|^2 - \frac{1}{2\Phi_n}(\phi_{2,n}\Delta \phi_{1,n} + \phi_{1,n}\Delta \phi_{2,n}) \text{ in } \Omega_n.
\]

We consider the second term on the right of the last equation:

\[
-\frac{1}{2\Phi_n}(\phi_{2,n}\Delta \phi_{1,n} + \phi_{1,n}\Delta \phi_{2,n}) = -\frac{1}{2} \sqrt{\frac{\phi_{2,n}}{\phi_{1,n}}}(u_1 * \Delta \rho_n) - \frac{1}{2} \sqrt{\frac{\phi_{1,n}}{\phi_{2,n}}}(u_2 * \Delta \rho_n)
\]

\[
\geq -\frac{1}{2} \sqrt{\frac{\phi_{2,n}}{\phi_{1,n}}}(V_{u_1} * \rho_n) - \frac{1}{2} \sqrt{\frac{\phi_{1,n}}{\phi_{2,n}}}(V_{u_2} * \rho_n)
\]

\[
= -\frac{1}{2} \sqrt{\frac{\phi_{2,n}}{\phi_{1,n}}}(V^+ u_1) * \rho_n + \frac{\phi_{1,n}}{\phi_{2,n}}(V^+ u_2) * \rho_n
\]

\[
+ \frac{1}{2} \sqrt{\frac{\phi_{2,n}}{\phi_{1,n}}}(V^- u_1) * \rho_n + \frac{\phi_{1,n}}{\phi_{2,n}}(V^- u_2) * \rho_n
\]

\[(10.7) \quad := f_n + g_n.
\]
We now estimate $f_n$. Using the fact that $V^+$ is locally bounded, for any ball $B \subseteq \Omega$ there exists a $C_B > 0$ such that

$$\sup_B |f_n| \leq C_B \sqrt{\phi_{1,n} \phi_{2,n}} = C_B \Phi_n \text{ for all } n.$$  

As before, by Vitaly’s convergence theorem, (10.4) and (10.5), we conclude that (upto a subsequence),

$$(10.8) \quad f_n \to -V^+ \Phi \text{ in } L^1_{loc}(\Omega).$$

From (10.4) and (10.5), using the fact that $g_n \geq 0$ and Fatou’s lemma, we can pass to the limit along a subsequence to obtain that for any nonnegative $\xi \in C^\infty_c(\Omega)$,

$$(10.9) \quad \liminf_{n \to \infty} \int \Omega g_n \xi \geq \int \Omega V^- \Phi \xi.$$  

Thus, using (10.8)-(10.9) in (10.7), we obtain from (10.6),

$$(10.10) \quad -\Delta \Phi \geq \frac{1}{4\Phi^2} \left| u_2 \nabla u_1 - u_1 \nabla u_2 \right|^2 - V \Phi \text{ in } \mathcal{D}'(\Omega).$$

Therefore, from theorem 7.2 we conclude that $Q_V \succeq w$ in $\Omega$. 

**Remark 10.20.**  
(i) We required $V^+ \in L^\infty_{loc}(\Omega)$ only to converge in (10.8). A more careful estimate may relax this condition on $V^+$.  
(ii) If $V \in L^p_{loc}(\Omega)$ for some $p > \frac{N}{2}$ ($N \geq 2$) and $u_1, u_2$ are nonnegative nontrivial distributional solutions, then again the conclusion of the above lemma holds. We can pass to the limit in (10.8) by noting that $u_1, u_2$ are positive continuous functions in $\Omega$.  
(iii) In the literature, results like the above lemma are applied to a pair of linearly independent elements in $\mathcal{C}_V(\Omega)$ in order to improve the quadratic form, typically for the Hardy-type inequalities.

**Corollary 10.21.** If $u_1$ and $u_2$ as in the above lemma are linearly independent, then $w \neq 0$ in $\Omega$.

**Proof.** Assume $w \equiv 0$ in $\Omega$. We consider the quasi-continuous representatives of $u_1, u_2$ whose zero sets are known to have Capacity zero (see theorem 2.12). Then we can follow the arguments in [17] (see proof of theorem 1.3) to show that necessarily $u_1$ and $u_2$ are linearly dependent. 

As a consequence, we have the following characterisation for subcritical and critical operators:

**Theorem 10.22.** Let $V^+ \in L^\infty_{loc}(\Omega)$. Then

(i) $-\Delta + V$ is subcritical in $\Omega$ iff $\mathcal{C}_V(\Omega)$ contains an infinite set of linearly independent functions.
(ii) $-\Delta + V$ is critical in $\Omega$ iff

$$C_V(\Omega) = \{ tu_0 : t > 0 \}$$

for some nonnegative nontrivial $u_0 \in L^1_{\text{loc}}(\Omega)$ solving $-\Delta u_0 + V u_0 = 0$ in $\mathcal{D}'(\Omega)$.

Proof. (i) The “only if” part follows from corollary 8.7. For the converse, suppose $u_1, u_2$ are two linearly independent functions in $C_V(\Omega)$. Then, from corollary 10.21, we get $w \not\equiv 0$ in $\Omega$. Thus from lemma 10.19 we obtain that $-\Delta + V$ is feebly $L^2$-subcritical in $\Omega$ and hence subcritical in $\Omega$.

(ii) The “if” part follows from the above theorem. For the converse (that is assuming $-\Delta + V$ is critical in $\Omega$), we note that $C_V(\Omega) \neq \emptyset$ by theorem 7.7 and by theorem 10.7 any element of $C_V(\Omega)$ is a distributional solution of $-\Delta u + V u = 0$ in $\Omega$. Conclusion follows again from the above theorem.

11 Examples and applications

11.1 Non-negative operators

Lemma 11.1. Let $u \in W^{2,1}_{\text{loc}}(\Omega)$ be such that $\inf_B u > 0$ for any ball $B \subset \Omega$ and $f \in L^1_{\text{loc}}(\Omega)$ be a nonnegative function. Let $V := (\Delta u + f)/u$. Then, $-\Delta + V$ is a non-negative operator.

Proof. Clearly, $V$ is locally integrable in $\Omega$ and $u$ solves $-\Delta u + Vu = f$ in $\mathcal{D}'(\Omega)$. Thus, the associated quadratic form satisfies $Q_V \succeq 0$ in $\Omega$ by theorem 7.2.

In the following example we show that it is possible to choose in this manner $V \in L^1(\Omega)$ but $V \not\in L^p(\Omega)$ for any $p > 1$.

Example 11.2. Denote by $I$ the unit interval $[0,1]$ and by $I^N$ the cube $[0,1]^N$. Let $f \in L^\infty(I^N)$ be a nonnegative function. Choose a function $\rho$ on $I$ such that $\rho \in L^1(I)$. Define $w : I^N \to \mathbb{R}$ by

$$w(x) = w(x_1, x_2, \cdots, x_N) := \int_0^{x_1} \left( \int_0^{x_2} \rho(s) ds \right) dt.$$

Clearly $w \in W^{2,1}(I^N) \cap C(I^N)$ and

$$\Delta w(x) = \frac{\partial^2 w}{\partial x_1^2}(x) = \rho(x_1) \quad \text{a.e. in } I^N.$$

Choosing $k > 0$ large enough so that $w + k > 0$ in $I^N$ and letting $u := w + k$, we see that $u \in W^{2,1}(I^N)$ is a positive continuous function which solves $-\Delta u + Vu = f$ in $\mathcal{D}'(I^N)$ where

$$V(x_1, x_2, \cdots, x_N) := (\rho(x_1) + f(x))/(w(x) + k).$$

Clearly, if we further choose $\rho \not\in L^p(I)$ for any $p > 1$, we get that $V \in L^1(I^N)$ but $V \not\in L^p(I^N)$ for any $p > 1$. By the above lemma $Q_V \succeq 0$ in $\Omega$.

Example 11.3. Let $\Omega := I^N, N \geq 3$. Choose $f \equiv 1$ and $\rho \in W^{1,1}(I)$ such that

(i) $\rho + 1 < 0$, 55
(ii) $\rho' \notin L^p(I)$ for any $p > 1$.

Let $V$ be the negative singular potential as given in (11.1). Thus, $Q_V \geq 0$ in $\Omega$. We choose the vector field

$$\vec{\Gamma} := \frac{1}{2\sqrt{N}}(\sqrt{-V}, \sqrt{-V}, \ldots, \sqrt{-V}).$$

Let $\sigma := \text{div} \, \vec{\Gamma}$. We note that

$$\sigma(x_1, x_2, \ldots, x_N) = \frac{\sqrt{-V(x)}}{4\sqrt{N}(\rho(x_1) + 1)} \left( V \int_{0}^{x_1} \rho(t) dt - \rho'(x_1) \right).$$

From the above equation and the properties of $\rho$, we see that $\sigma$ has the same integrability as $\rho'$ i.e., $\sigma \in L^p(I^N)$ only for $p = 1$. We check that $Q_{-\Delta \Phi} = Q_V \geq 0$ in $\Omega$. Therefore, from theorem 4.1 (iii) in [16] we obtain that

$$\Phi = f - \Delta \Phi + V \Phi \in H^1(\Omega).$$

That is, $\sigma$ is form bounded, $Q_\sigma \geq 0$ in $\Omega$ and the identity map

$$\text{id} : \left( W^{1,\infty}(\Omega), \| \cdot \|_{H^1(\Omega)} \right) \to \mathcal{L}_\sigma(\Omega)$$

is continuous.

Fix $t \in (0, 1)$. If we consider the potential $t\sigma$, we obtain from (11.2) that the above map is a homeomorphism onto $\mathcal{L}_{t\sigma}(\Omega)$. That is, the corresponding energy space $\mathcal{E}_{t\sigma}(\Omega)$ can be identified with $H^1_0(\Omega)$. By corollary 6.40 we obtain that $\rho'w \in L^1_{\text{loc}}(\Omega)$ for any $w \in H^1_{\text{loc}}(\Omega)$ and hence from proposition 6.17 we obtain that $t\sigma$ is a balanced potential in $\Omega$.

### 11.2 Subcritical operators

**Lemma 11.4.** Let $\Phi \in W^{2,1}_{\text{loc}}(\Omega)$ be such that $\inf_B \Phi > 0$ for any ball $B \subset \Omega$ and $f \in L^1_{\text{loc}}(\Omega)$ be a nonnegative function such that $\inf_K f > 0$ for some compact $K \subset \Omega$ with positive measure. Let $V := (\Delta \Phi + f)/\Phi$. Then, $-\Delta + V$ is a feebly $L^2$- subcritical operator.

**Proof.** We note that $-\Delta \Phi + V \Phi = f$ in $\mathcal{D}'(\Omega)$. Since $\Phi, f$ are nonnegative, by theorem 7.2 we obtain that $Q_V \geq f/\Phi$ in $\Omega$. The conclusion follows by noting that $f/\Phi \not\equiv 0$ in $K$.  

**Corollary 11.5.** Let $\Phi, f$ be as in the above lemma with the additional assumptions: $(\Delta \Phi)^+$ and $f$ belong to $L^1_{\text{loc}}(\Omega)$. Take $V$ as in the above lemma. Then, $V$ is a tame potential in $\Omega$ and given any $g \in L^\infty_c(\Omega)$, there exists a unique $u \in \mathcal{H}_V(\Omega)$ solving $-\Delta u + Vu = g$ in $\mathcal{D}'(\Omega)$.

**Proof.** We note that $V$ is tame in $\Omega$ by the above lemma and remark 6.26 (i). The proof of rest of the assertions follows from the above lemma, proposition 9.7 and corollary 8.2.

**Corollary 11.6.** Take any $\Phi \in W^{2,1}_{\text{loc}}(\Omega)$ and $f \in L^\infty_{\text{loc}}(\Omega)$ satisfying

$$\inf_B \Phi, \inf_B f > 0 \text{ for any ball } B \subset \Omega \text{ and } \Delta \Phi \in \mathcal{V}(\Omega) \text{ (see definition 6.23).}$$

Define $V := (\Delta \Phi + f)/\Phi$. Then, $V$ is balanced in $\Omega$ and given any $g \in L^\infty_c(\Omega)$, there exists a unique $u \in \mathcal{H}_V(\Omega)$ solving $-\Delta u + Vu = g$ in $\mathcal{D}'(\Omega)$.  

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Proof. We note that $-\Delta + V$ is globally $L^2$-subcritical in $\Omega$ by following the proof of lemma 11.4. Thus, the operator is $L^1$-subcritical in $\Omega$. Since $\Delta \phi \in \mathcal{V}(\Omega)$ we get that $V \in \mathcal{V}(\Omega)$ by proposition 6.20(i). Thus $V$ is strongly balanced in $\Omega$ by proposition 6.31. The proof of solvability follows from corollary 8.2.

Remark 11.7. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $\Gamma_N$ denote the fundamental solution of $-\Delta$ in $\mathbb{R}^N$. Assume $g \in L^1(\Omega) \cap L^r_{\text{loc}}(\Omega) \cap \mathcal{V}(\Omega)$ for some $r > 1$ and satisfies

$$\Psi(x) := -\int_{\mathbb{R}^N} \Gamma_N(y-x)g(y)\chi_\Omega(y)dy$$

is bounded below in $\Omega$.

This can be achieved for example, by taking $g^+ \in L^p(\Omega)$ for some $p > \frac{N}{2}$. Then letting $\Phi = \Psi - \inf_{\Omega} \Psi + 1$ we can satisfy the conditions in the last corollary.

Lemma 11.8. Let $W \in L^1_{\text{loc}}(\mathbb{R}^N)$ be a nonnegative function.

(i) Let $K \subset \mathbb{R}^N$ be a compact set with positive measure. Then, $-\Delta + W$ is subcritical in $\mathbb{R}^N \setminus K$, $N \geq 1$.

(ii) $-\Delta + W$ is subcritical in $\mathbb{R}^N$ if $N \geq 3$.

Proof. We remark that $Q_W \succeq Q_0$ in any $\Omega \subset \mathbb{R}^N$. Therefore, both $L^1$ and $L^2$ subcriticality of $-\Delta + W$ in $\Omega$ follow from the subcriticality of $-\Delta$ there.

(i) Since $Q_0 \succeq 0$ in $\mathbb{R}^N$, from corollary 10.16 we obtain that $-\Delta$ is subcritical in $\mathbb{R}^N \setminus K$, $N \geq 1$.

(ii) It is enough to show $-\Delta$ is $L^2$-subcritical in $\mathbb{R}^N$ if $N \geq 3$. Choose

$$\Phi_N(x) = (N(N-2))^{\frac{N-2}{2}}(1 + |x|^2)^{\frac{2-N}{2}}.$$

Then $\Phi_N$ is a non-negative locally integrable function solving $-\Delta \Phi_N = (\Phi_N)^{\frac{N+2}{N-2}}$ in $\mathcal{D}'(\mathbb{R}^N)$. The assertion now follows from theorem 10.2.

Corollary 11.9. Let $N \geq 1$ and $W \in L^1_{\text{loc}}(\mathbb{R}^N)$ be a nonnegative function. Then, given any compact set $K \subset \mathbb{R}^N$ of positive measure, there exists an a.e. positive distribution solution $u$ of $-\Delta u + Wu = \chi_K$ in $\mathbb{R}^N$.

Proof. Follows from the above lemma, remark 6.26(i) and corollary 8.6. In the following proposition, we generalise the example 6.43(iii).

Proposition 11.10. Let $V_0$ be a potential in $\Omega$ such that

$$-C|x|^{-\beta} \leq V_0(x) \leq C$$

for some $C > 0, \beta \in (0, N)$.

Assume $Q_{V_0} \succeq 0$ in $\Omega$. Let $S := \{x_i\} \subset \Omega$ be a countable set and $\alpha_i$ be nonnegative numbers such that

$$0 < \alpha := \sum_i \alpha_i \leq 1.$$
Define
\[ V(x) = \sum_i \alpha_i V_0(x - x_i). \]

(i) \( V \) is tame in \( \Omega \) and there exists an a.e.
positive distributional super solution to \(-\Delta + V \)
in \( \Omega \).

(ii) Let \( V_1 \) be a tame potential in \( \Omega \) such that
\[ V \leq V_1, \quad V_1 \neq V. \]
Then, given any \( f \in L^\infty_c(\Omega) \), there exists a
distribution solution \( u \) of
\[ -\Delta u + V_1 u = f \text{ in } \Omega. \]
In fact, given any compact set \( K \) in \( \Omega \) with positive measure, there exists an a.e.
positive distribution solution \( u \) of
\[ -\Delta u + V_1 u = 0 \text{ in } \Omega \setminus K. \]

(iii) If \( \alpha < 1 \), both the assertions in (ii) above hold for \( V \).

(iv) If \( \alpha = 1 \), given any compact set \( K \) in \( \Omega \) of positive measure and any
\( f \in L^\infty_c(\Omega \setminus K) \), there exists a distribution solution \( u \) of
\[ -\Delta u + Vu = f \text{ in } \Omega \setminus K. \]
As before, there exists an a.e.
positive distribution solution \( u \) of \(-\Delta u + Vu = 0 \) in
\( \Omega \setminus K. \)

Proof. (i) We note that \( V \) is bounded above in \( \Omega \) and by the pointwise estimate on \( V_0 \), it
is also locally integrable in \( \Omega \). Let \( \{\beta_i\} \) be any sequence of nonnegative numbers such that
\( \sum_i \beta_i \leq 1 \). Since \( Q_{V_0} \geq 0 \) in \( \Omega \), we obtain for all \( \xi \in C^\infty_c(\Omega) \),
\[ \int_\Omega |\nabla \xi|^2 \geq \sum_i \beta_i \int_\Omega |\nabla \xi(\cdot + x_i)|^2 \geq \sum_i \beta_i \int_\Omega V_0 \xi^2(\cdot + x_i). \] (11.3)
Taking \( \beta_i = \alpha_i \) in the above inequality, we get that \( Q_V \geq 0 \) in \( \Omega \) and hence by remark 6.26(i),
\( V \) is tame in \( \Omega \). Existence of a positive super solution follows from theorem 7.7.

(ii) From proposition 10.17(i), we get that \(-\Delta + V_1 \) is feebly \( L^2 \)-subcritical in \( \Omega \) and since \( V_1 \)
is tame in \( \Omega \), it is subcritical in \( \Omega \). We can appeal to corollary 8.2 and proposition 8.6.

(iii) Let \( \alpha < 1 \). Now, by the equation (11.3) above,
\[ \int_\Omega |\nabla \xi|^2 + V \xi^2 = (1 - \alpha) \int_\Omega |\nabla \xi|^2 + \alpha \left( \int_\Omega |\nabla \xi|^2 + \frac{1}{\alpha} \sum_i \alpha_i V_0(\cdot - x_i) \xi^2 \right) \]
\[ \geq (1 - \alpha) \int_\Omega |\nabla \xi|^2. \]
Thus, \( L_V(\Omega) \hookrightarrow D_0^{1,2}(\Omega) \). By Sobolev imbedding, \(-\Delta + V \) is subcritical in \( \Omega \). The assertions
about the existence of distribution solutions follow again from corollary 8.2 and proposition 8.6.

(iv) follows from corollaries 10.16, 8.2 and proposition 8.6. \( \square \)
Remark 11.11. (i) By taking $N \geq 3$, $\Omega = \mathbb{R}^N$ and $V_0 = -\left(\frac{(N-2)^2}{4}\right)|x|^{-2}$ (or any of its radial improvements in bounded domains), we see that the above proposition gives corresponding statements about the multi-polar Hardy operator with poles in the set $S$. These poles are allowed to be dense in some portion or all of $\Omega$.

(ii) For the multi-polar Hardy potential having only finitely many poles but where $\alpha_i$ can be any real number less than $\left(\frac{N-2}{2}\right)^2$, the problem of whether $L(V(\Omega)) \hookrightarrow \mathcal{D}^{1,2}_0(\mathbb{R}^N)$ is more involved and is treated in [9].

(iii) A similar construction can be made when the “mother potential” $V_0$ has a higher dimensional singular set on a plane by a weighted sum of countable translations of $V_0$ perpendicular to the plane.

11.3 Critical operators

Let $\Omega$ be a bounded domain and $V \in L_{\text{loc}}^1(\Omega)$. Define the subspace and the functional

$$H(V, \Omega) := \{u \in H^1_0(\Omega) : Vu^2 \in L^1(\Omega)\};$$

$$I_V : H(V, \Omega) \to \mathbb{R} \text{ by } I_V(u) := \int_{\Omega} \left\{|\nabla u|^2 + Vu^2\right\}.$$ Consider the corresponding “first eigenvalue”

$$\lambda_1 := \inf \left\{I_V(u) : u \in H(V, \Omega) \text{ and } \int_{\Omega} u^2 = 1\right\}.$$ The following are well-known examples of critical Schrödinger operators:

**Lemma 11.12.** Let $V^+ \in L_+^N(\Omega)$ ($N \geq 3$). Assume that $\lambda_1 > -\infty$ and is achieved by some element in $H(V, \Omega)$. Then, $-\Delta + V - \lambda_1$ is both feebly $L^2$–critical and $L^1$–critical in $\Omega$.

**Proof.** Let $\Phi \in H(V, \Omega)$ attain the infimum for $\lambda_1$. Considering $|\Phi|$ instead of $\Phi$, we may assume $\Phi$ is nonnegative. By the assumptions, we obtain that

$$(11.4) \quad I_V(u) - \lambda_1 \int_{\Omega} u^2 \geq 0 \quad \forall u \in H(V, \Omega) \text{ and } I_V(\Phi) - \lambda_1 \int_{\Omega} \Phi^2 = 0.$$ Therefore, $\Phi$ solves

$$-\Delta \Phi + V \Phi = \lambda_1 \Phi \text{ in } \mathcal{D}'(\Omega).$$

From theorem 2.12 we obtain that $\Phi > 0$ a.e. in $\Omega$. Suppose there exists a compact set $K \subset \Omega$ with positive measure and $c_K > 0$ such that

$$\int_{\Omega} \left\{|\nabla \xi|^2 + (V - \lambda_1)\xi^2\right\} \geq c_K \int_K \xi^2, \quad \forall \xi \in C_c^1(\Omega).$$

We now choose a sequence $\{\xi_n\} \subset C_c^1(\Omega)$ such that $\xi_n \to \Phi$ in $H^1_0(\Omega)$. Then, from the above inequality, using Sobolev imbedding and Fatou’s lemma we get

$$\int_{\Omega} \left\{|\nabla \Phi|^2 + (V^+ - \lambda_1)\Phi^2\right\} \geq \int_{\Omega} V^+ \Phi^2 + c_K \int_K \Phi^2.$$
From (11.4) we obtain that \( \Phi \equiv 0 \) in \( K \) a contradiction to its positivity a.e. This shows that \( -\Delta + V - \lambda_1 \) is feebly \( L^2 \)-critical in \( \Omega \). A similar argument shows that the operator is \( L^1 \)-critical in \( \Omega \) as well. \( \square \)

**Lemma 11.13.** The operator \(-\Delta\) is critical in \( \mathbb{R}^N \) when \( N = 1, 2 \).

**Proof.** Let \( u \in L^1_{\text{loc}}(\Omega) \) be a non-negative distribution solution of \(-\Delta u = f\) for some nonnegative locally integrable function \( f \). If \( N = 2 \), by the Louville property of nonnegative superharmonic functions we get \( u \) is identically constant and hence \( f \equiv 0 \). If \( N = 1 \), we note that \( u \) becomes a nonnegative concave function in \( \mathbb{R} \) and hence must be constant. Therefore, again \( f \equiv 0 \). Criticality follows from theorem 10.2. \( \square \)

### 11.4 Supercritical operators

The following results provide a way of constructing supercritical operators using a wide class of singular potentials. To state the result in full generality, given a continuous function \( \phi : [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \), we introduce the Orlicz class

\[
L_\phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable : } \int_\Omega \phi(|u|) < \infty \right\}.
\]

We denote by \( 2^* \) the critical Sobolev exponent \( \frac{2N}{N-2} \).

**Proposition 11.14.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \), \( N \geq 3 \). Let \( \alpha : [0, \infty) \to [0, \infty) \) be a continuous function such that

(i) \( \beta(t) := t^{-2}\alpha(t) \) is a continuous function in \( [0, \infty) \) with \( \beta(0) = 0 \)

(ii) \( H^1_0(\Omega) \setminus L_\alpha(\Omega) \neq \emptyset \) and \( L^{2^*}(\Omega) \subseteq L_\beta(\Omega) \).

For any \( w \in H^1_0(\Omega) \setminus L_\alpha(\Omega) \) define the potential \( V_w := -\beta(w) \). Then, the operator \(-\Delta + V_w\) is supercritical.

**Proof.** Let \( w \in H^1_0(\Omega) \setminus L_\alpha(\Omega) \). Since \( w \in L^{2^*}(\Omega) \) and by assumption \( L^{2^*}(\Omega) \subseteq L_\beta(\Omega) \), we obtain that \( V_w := -\beta(w) \in L^1(\Omega) \). Therefore \( w \in H^1_0(\Omega) \), but

\[
\int_\Omega V_w w^2 = -\int_\Omega \alpha(w) = -\infty.
\]

Let now \( \{\phi_n\} \subset C^\infty(\Omega) \) be a sequence converging in \( H^1_0(\Omega) \) to \( w \). From Fatou’s lemma we obtain that \( \lim_{n \to \infty} Q_{V_w}(\phi_n) = -\infty \). \( \square \)

**Corollary 11.15.** Let \( \Omega \subset \mathbb{R}^N, N \geq 3 \) be a bounded domain and \( p \in (2^*, 2^* + 2) \). If for each \( w \in H^1_0(\Omega) \setminus L^p(\Omega) \), we define the potential \( V_w := -|w|^{p-2} \), then the operator \(-\Delta + V_w\) is supercritical. Note that \( V_w \not\in L^q(\Omega) \) for some \( 2(N-1)/N < q < N/2 \).

**Proof.** Consider \( \alpha(t) := |t|^p \) and \( \beta(t) := |t|^{p-2} \) which is continuous (note that \( p > 2 \)). Since \( H^1_0(\Omega) \not\subset L^p(\Omega) \) and \( L^{2^*}(\Omega) \subset L^{p-2}(\Omega) \), the conclusion follows from the above proposition. \( \square \)
Corollary 11.16. Let $V$ be any of the potentials considered in proposition 11.14. Then there exists no non-trivial non-negative distributional supersolution to the operator $-\Delta + V$.

Proof. For $V$ as in proposition 11.14, we have that $Q^V \npreceq 0$ in $\Omega$. Therefore, in view of theorem 7.2 there cannot exist a non-trivial non-negative distributional supersolution to the operator $-\Delta + V$ in $\Omega$.

12 Appendix

12.1 Weak compactness in $L^1_{\text{loc}}$

The space $L^1_{\text{loc}}(\Omega)$ is a locally convex topological space with a countable family of seminorms defined by $| \cdot |_n := \int_{\Omega_n} |u|$ where $\{\Omega_n\}$ is an exhaustion of $\Omega$ by bounded open sets. Its dual can be identified with $L^\infty(\Omega)$ (see [14], Thm. 4, p.182). Weakly compact sets in $L^1_{\text{loc}}(\Omega)$ are characterized as follows:

Lemma 12.1. Let $K \subset L^1_{\text{loc}}(\Omega)$ and define $K_B := \{f|_B : f \in K\}$. Then the following statements are equivalent:

(i) $K$ is compact in the weak topology $\sigma(L^1_{\text{loc}}(\Omega),L^\infty(\Omega))$;

(ii) for each ball $B \Subset \Omega$, $K_B \subset L^1(B)$ is compact in the weak topology $\sigma(L^1(B),L^\infty(B))$;

(iii) $K$ is weakly closed and for each ball $B \Subset \Omega$, the set $K_B$ is equi-integrable in $L^1(B)$;

(iv) $K$ is sequentially compact in the weak topology $\sigma(L^1_{\text{loc}}(\Omega),L^\infty(\Omega))$;

(v) for each ball $B \Subset \Omega$, $K_B$ is sequentially compact in the weak topology $\sigma(L^1(B),L^\infty(B))$.

Proof. (i) $\implies$ (iv) Since $L^1_{\text{loc}}(\Omega)$ with weak topology is a locally convex topological vector space with a countable family of semi-norms that separate points, its weak topology is metrisable. The equivalence follows immediately.

(ii) $\iff$ (iii) This equivalence is the Dunford-Pettis Theorem.

(iii) $\iff$ (v) This is the Eberlein-Smulian Theorem.

(iv) $\iff$ (v) Since the dual of $L^1_{\text{loc}}(\Omega)$ is $L^\infty(\Omega)$, the conclusion follows from the following equivalences:

$$f_n \to f \text{ weakly in } L^1_{\text{loc}}(\Omega) \iff \int_{\Omega} (f_n - f) \xi \to 0 \quad \forall \xi \in L^\infty(\Omega)$$

$$\iff \int_{\Omega} (f_n - f)(\chi_B \xi) \to 0 \quad \forall \xi \in L^\infty(\Omega), \forall B \Subset \Omega$$

$$\iff \int_{B} (f_n - f) \xi \to 0 \quad \forall \xi \in L^\infty(B), \forall B \Subset \Omega$$

$$\iff f_n \to f \text{ weakly in } L^1(B), \forall B \Subset \Omega.$$
12.2 Proof of proposition 7.1

Fix $\varepsilon > 0$. Let $\xi \in C^\infty_c(\Omega^*)$ and take $\frac{\xi^2}{u + \varepsilon}$ as test function in the weak formulation for $u$. We then get

$$\int_{\Omega^*} \nabla u \cdot \nabla \left( \frac{\xi^2}{u + \varepsilon} \right) + \int_{\Omega^*} W u \left( \frac{\xi^2}{u + \varepsilon} \right) = \int_{\Omega^*} f \left( \frac{\xi^2}{u + \varepsilon} \right).$$

So,

$$- \int_{\Omega^*} \left| \frac{\xi}{u + \varepsilon} - \nabla \xi \cdot \nabla u \right|^2 + \int_{\Omega^*} \left\{ |\nabla \xi|^2 + W u \left( \frac{\xi^2}{u + \varepsilon} \right) \right\} = \int_{\Omega^*} f \left( \frac{\xi^2}{u + \varepsilon} \right).$$

That is,

$$\int_{\Omega^*} \left\{ |\nabla \xi|^2 + W u \left( \frac{\xi^2}{u + \varepsilon} \right) \right\} \geq \int_{\Omega^*} f \left( \frac{\xi^2}{u + \varepsilon} \right) + \int_{\Omega^*} \left| \frac{\xi}{u + \varepsilon} - \nabla \xi \cdot \nabla u \right|^2.$$

Since the set $\{u = 0\}$ has measure zero, Lebesgue theorem and Fatou’s Lemma yield

$$\int_{\Omega^*} \left\{ |\nabla \xi|^2 + W \xi^2 \right\} \geq \int_{\Omega^*} f \left( \frac{\xi}{u + \varepsilon} \right) + \int_{\Omega^*} \left| \frac{\xi}{u + \varepsilon} - \nabla \xi \cdot \nabla u \right|^2.$$

12.3 Proof of Lemma 6.18

Choose the subsequence $\{z_{n_k}\}$ such that

$$\|z_{n_k+1} - z_{n_k}\| \leq \frac{1}{2^k}, \quad k = 1, 2, \ldots$$

Let

$$\hat{z}_m := \sum_{k=1}^{m} |z_{n_{k+1}} - z_{n_k}|.$$

Then $\|\hat{z}_m\| \leq 1$ and hence $\{\hat{z}_m\}$ converges weakly (up to a subsequence, again denoted by $\{\hat{z}_m\}$) to some $\hat{z} \in \mathbb{Z}$. Since $\mathcal{Z} \hookrightarrow L^1(K)$, we have that $\{\hat{z}_m\}$ is also bounded in $L^1(K)$. We also note that, by monotone convergence theorem, $\{\hat{z}_m\}$ converges in $L^1(K)$. Hence necessarily $\hat{z}_m \leq \hat{z}$. By a telescoping series,

$$|z_{n_{m+1}}| \leq \hat{z}_m + |z_{n_1}| \leq \hat{z} + |z_{n_1}| := z^* \quad \forall m.$$

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References

[1] S. Agmon, On positivity and decay of solutions of second order elliptic on Riemannian manifolds, Methods of functional analysis and theory of elliptic equations, Proc. of the International meeting dedicated to the memory of Prof. Carlo Miranda, Instituto di Matematica R. Caccioppoli, University di Napoli, 1982.
[2] M. Aizenman, B. Simon, Brownian Motion and Harnack Inequality for Schrödinger Operators, Comm. on Pure and Applied Math. 25 (1982), 209–273.

[3] W. Allegretto, On the equivalence of two types of oscillation for elliptic operators, Pacific J. Math. 55 (1974), 319–328.

[4] A. Ancona, Une propriété d’invariance des ensembles absorbants par perturbation d’un opérateur elliptique, Comm. PDE 4 (1979), 321–337.

[5] H. Brezis, M. Marcus, A. Ponce, Nonlinear elliptic equations with measures revisited, Annals of Math. Studies (2007).

[6] H. Brezis, A. Ponce, Remarks on the strong maximum principle, Differential Integral Equations 16 (2003), 1–12.

[7] H. Brezis, A. Ponce, Kato’s inequality when $\Delta u$ is a measure, C. R. Math. Acad. Sci. Paris, Ser. I 338 (2004), 599–604.

[8] H. Brezis, M. Marcus, A. Ponce, Nonlinear elliptic equations with measures revisited, Mathematical aspects of nonlinear dispersive equations (Annals of Mathematics Studies, 163), 2007, 55–110.

[9] V. Felli, E.M. Marchini, S. Terracini, On Schrödinger operators with multipolar inverse-square potentials, J. Funct. Anal. 250 (2007), 265–316.

[10] V. Felli, S. Terracini, Elliptic equations with multi-singular inverse-square potential and critical nonlinearity, Comm. Partial Differential Equations 31 (2006), 469–495.

[11] D. Fischer-Colbrie, R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure Appl. Math. 33 (1980), 199–211.

[12] F. Gesztesy, Z. Zhao, On critical and subcritical Sturm-Liouville operators, J. Funct. Anal. 98 (1991), 311–345.

[13] A.A. Grigoryan, On the existence of positive fundamental solutions of the Laplace equation on Riemannian manifolds, Math. USSR Sbornik 56 (1987), 349–358.

[14] J. Horváth, Espaces de fonctions localement intégrables et de fonctions intégrables à support compact, Rev. Colombiana Mat. 21 (1987), 167–186.

[15] C.G. Jacobi, Zur Theorie der Variations-Rechnung und der differential-Gleichungen, Crelle’s Journal für die Reine und angewandte Mathematik 17 (1837), 68–82.

[16] B. Jaye, V.G. Maz’ya, I.E. Verbitsky, Existence and regularity of positive solutions of elliptic equations of Schrödinger type, J. Anal. Math. 118 (2012), 577–621.

[17] B. Kawohl, M. Lucia, S. Prashanth, Simplicity of the principal eigenvalue for indefinite quasilinear problems, Adv. Differential Equations 12 (2007), 407–434.

[18] V. Maz’ya, Sobolev Spaces, Grundlehren der mathematischen Wissenschaften 342, 2nd edition, Springer-Verlag, 2011.
[19] M. Murata, *Structure of positive solutions to* $(−Δ + V)u = 0$ *in* $\mathbb{R}^n$, *Duke Math. J.* **53** (1986), 869–943.

[20] J. Piepenbrink, *Finiteness of the lower spectrum of Schrödinger operators*, *Math. Z.* **140** (1974), 29–40.

[21] J. Piepenbrink, *Nonoscillatory elliptic equations*, *J. Differential Equations* **15** (1974), 541–550.

[22] Y. Pinchover, *On positive solutions of second-order elliptic equations, stability results, and classification*, *Duke Math. J.* **57** (1988), 955–980.

[23] Y. Pinchover, *On Criticality and Ground State of second order elliptic equations II*, *J. Differential Equations* **87** (1990), 353-364.

[24] Y. Pinchover, G. Psaradakis, On positive solutions of the $(p,A)$-Laplacian with potential in Morrey space. *Anal. PDE* 9 (2016), no. 6, 1317-1358.

[25] Y. Pinchover, K. Tintarev, *A ground state alternative for singular Schrödinger operators*, *J. Funct. Anal.* **230** (2006), 65–77.

[26] Y. Pinchover, K. Tintarev, *Ground state alternative for p-Laplacian with potential term*, *Calc. Var.* **28** (2007), 179–201.

[27] A.C. Ponce, *Selected problems on elliptic equations involving measures*, [http://arxiv.org/pdf/1204.0668](http://arxiv.org/pdf/1204.0668), Winner of the Concours annuel 2012 in Mathematics of the Académie Royale de Belgique, (2012).

[28] B. Simon, *Large time behavior of the* $L^p$-*norm of Schrödinger semigroups*, *J. Funct. Anal.* **40** (1981), 66–83.

[29] B. Simon, *Schrödinger semigroups*, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 447–526.

[30] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189–258.

[31] G. Stampacchia, *Equations elliptiques du second ordre à coefficients discontinus*, *Les Presses de l’Université de Montréal, Montréal*, 1966.

[32] Z. Zhao, *Green Function for Schrödinger Operator and Conditioned Feynman-Kac Gauge*, *J. of Math. Anal. and Applications* **116**, 309–334, 1986.