The Hopf modules category and the Hopf equation

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Abstract

Let \((A, \Delta)\) be a Hopf-von Neumann algebra and \(R\) be the unitary fundamental operator on \(A\) defined by Takesaki in [28]: \(R(a \otimes b) = \Delta(b)(a \otimes 1)\). Then \(R^{12}R^{23} = R^{23}R^{13}R^{12}\) (see lemma 4.9 of [28]). This operator \(R\) plays a vital role in the theory of duality for von Neumann algebras (see [28] or [2]). If \(V\) is a vector space over an arbitrary field \(k\), we shall study what we have called the Hopf equation: \(R^{12}R^{23} = R^{23}R^{13}R^{12}\) in \(\text{End}_k(V \otimes V \otimes V)\). Taking \(W := \tau R\tau\), the Hopf equation is equivalent with the pentagonal equation: \(W^{12}W^{13}W^{23} = W^{23}W^{12}\) from the theory of operator algebras (see [2]), where \(W\) are viewed as map in \(L(K \otimes K)\), for a Hilbert space \(K\). For a bialgebra \(H\), we shall prove that the classic category of Hopf modules \(HM^H\) plays a decisive role in describing all solutions of the Hopf equation. More precisely, if \(H\) is a bialgebra over \(k\) and \((M, \cdot, \rho) \in HM^H\) is an \(H\)-Hopf module, then the natural map \(R = R_{(M, \cdot, \rho)}\) is a solution for the Hopf equation. Conversely, the main result of this paper is a FRT type theorem: if \(M\) is a finite dimensional vector space and \(R \in \text{End}_k(M \otimes M)\) is a solution for the Hopf equation, then there exists a bialgebra \(B(R)\) such that \((M, \cdot, \rho) \in B(R)M^{B(R)}\) and \(R = R_{(M, \cdot, \rho)}\). By applying this result, we construct new examples of noncommutative and noncocommutative bialgebras which are different from the ones arising from quantum group theory. In particular, over a field of characteristic two, an example of five dimensional noncommutative and noncocommutative bialgebra is given.

0 Introduction

Let \(H\) be a bialgebra over a field \(k\). There are two fundamental categories in the theory of Hopf algebras and quantum groups: \(HM^H\), the category of \(H\)-Hopf modules and \(HYD^H\), the category of quantum Yetter-Drinfel’d modules. The objects in these categories are \(k\)-vector
spaces $M$ which are left $H$-modules $(M, \cdot)$, right $H$-comodules $(M, \rho)$, such that the following quite distinct compatibility relations hold:

$$\rho(h \cdot m) = \sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>}\quad (1)$$

in the case $\mathcal{H}M^H$, and respectively

$$\sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>} = \sum (h_{(2)} \cdot m)_{<0>} \otimes (h_{(2)} \cdot m)_{<1>} h_{(1)} \quad (2)$$

for the Yetter-Drinfel’d categories.

Traditionally, these two categories have been studied for completely different reasons: the classical category $\mathcal{H}M^H$ (or immediate generalisations of it: $\mathcal{A}M^H$, $\mathcal{A}M(H)^C$) is involved in the theory of integrals for a Hopf algebra (see [1], [27] or the more recent [19]), Clifford theory of representations ([17], [23], [24], [26]) and Hopf-Galois theory ([19], [24], etc.). The category $\mathcal{HYD}^H$, introduced in [31], plays an important role in the quantum Yang-Baxter equation, quantum groups, low dimensional topology and knot theory (see [13], [14], [20], [21], or [29]).

However, there are two connections between these categories. The first one was given by P. Schauenburg in [22]: it was proven that the category $\mathcal{HYD}^H$ is equivalent to the category $\mathcal{H}M^H$ of two-sided, two-cosided Hopf modules. The second was given recently in [4]. For $A$ an $H$-comodule algebra and $C$ an $H$-module coalgebra, Doi (see [7]) and independently Koppinen (see [12]) defined $\mathcal{A}M(H)^C$, the category of Doi-Koppinen Hopf modules, whose objects are left $A$-modules and right $C$-comodules and satisfy a compatibility relation which generalises ([1]). In [4] it was proven that $\mathcal{HYD}^H$ is isomorphic to $\mathcal{H}M(H^{\text{op}} \otimes H)^H$, where $H$ can be viewed as an $H^{\text{op}} \otimes H$-module (comodule) coalgebra (algebra). The isomorphism is the identity functor $\mathcal{M} \to \mathcal{H}M$. We hereby obtain a strong link between the categories $\mathcal{H}M^H$ and $\mathcal{HYD}^H$: both are particular cases of the same general category $\mathcal{A}M(H)^C$. This led us in [5], [6] to study the implications of the category $\mathcal{HYD}^H$ in the classic, non-quantic part of Hopf algebra theory. In [3] we start with the following classic theorem (see [19]): any finite dimensional Hopf algebra is Frobenius. In the language of categories, this result is interpreted as follows: the forgetful functor $\mathcal{H}M^H \to \mathcal{H}M$ is Frobenius (i.e., cf. [5], by definition has the same left and right adjoint) if and only if $H$ is finite dimensional. The next step is easy to take: we must generalize this result for the forgetful functor $\mathcal{HYD}^H \to \mathcal{HYD}^H$ and then apply it in the case of Yetter-Drinfel’d modules for the forgetful functor $\mathcal{HYD}^H \to \mathcal{HYD}^H$. We thus obtain the fact that the forgetful functor $\mathcal{HYD}^H \to \mathcal{HYD}^H$ is Frobenius if and only if $H$ is finite dimensional and unimodular (see theorem 4.2 of [5]). The same treatment was applied in [3] for the classic Maschke theorem. One of the major obstacles was to correctly define the notion of integral for the Doi-Hopf datum $(H, A, C)$, such as to be connected to the classic integral on a Hopf algebra (corresponding to the case $C=A=H$), as well as to the notion of total integral (corresponding to the case $C=A$) defined by Doi in [5]. This technique can be looked upon as a "quantisation" of the theorems from the classic theory of Hopf algebras. There are two steps to it: first, we seek to generalize a result for the category $\mathcal{A}M(H)^C$, then to apply it to the particular $\mathcal{HYD}^H$ case. There is also another approach to this "quantisation" technique, recently evidenced in [5] for the same Frobenius type theorem.
was first proven, by generalizing the classic result, that any finite dimensional Hopf algebras extensions is a $\beta$-Frobenius extension (or a Frobenius extension of second kind). Then, this theorem was "quantised" to the case of Hopf algebras extensions in $\mathcal{H}\mathcal{Y}\mathcal{D}^H$. The result includes the case of enveloping algebras of Lie coloralgebras.

Beginning with this paper, we shall tackle the reverse problem: we shall try to involve the category $\mathcal{H}\mathcal{M}^H$ in fields dominated until now by $\mathcal{H}\mathcal{Y}\mathcal{D}^H$, i.e. try a "dequantisation". For the beginning, it is enough to remind that the category $\mathcal{H}\mathcal{Y}\mathcal{D}^H$ is deeply involved in the quantum Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where $R \in \text{End}_k(M \otimes M)$, $M$ being a $k$-vector space. The starting point of this paper is the following question:

"Can the category $\mathcal{H}\mathcal{M}^H$ be studied in connection with a certain non-linear equation?"

The answer is affirmative and, surprisingly, the equation in which the category $\mathcal{H}\mathcal{M}^H$ is involved (which we shall call Hopf equation) is very close to the quantum Yang-Baxter equation. More precisely, it is

$$R_{12}R_{23} = R_{23}R_{13}R_{12}$$

The simple way of obtaining it from the quantum Yang-Baxter equation by just deleting the term $R_{13}$ on the left hand side has nevertheless unpleasant effects: first of all, if the Yang-Baxter equation is reduced to the solution of a homogenous system, in the case of the Hopf equation the system is not homogenous any more; secondly, if $R$ is a solution of the Hopf equation, $W := \tau R \tau$ (or $W := R^{-1}$, if $R$ is bijective) is not a solution for the Hopf equation, but for the pentagonal equation:

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$ 

An interesting connection between the pentagonal and the quantum Yang-Baxter equations is given in [30]. The pentagonal equation plays a fundamental role in the duality theory for operator algebras (see [2] and the references indicated here). If $H$ is a Hopf algebra, then

$$R : H \otimes H \rightarrow H \otimes H, \quad R(g \otimes h) = \sum h_{(1)}g \otimes h_{(2)}$$

is a bijective solution of the Hopf equation. Moreover, the comultiplication $\Delta$ can be rebuilt from $R$ by means of

$$\Delta(h) = R((1 \otimes h)z),$$

where $z \in H \otimes H$ such that $R(z) = 1 \otimes 1$. This operator was defined first by Takesaki in [28] for a Hopf-von Neumann algebra. The operator $W := \tau R \tau$ is called in [13] the evolution operator for a Hopf algebra and plays an important role in the description of the Markov transition operator for the quantum random walks (see [13] or [14]).

The Hopf equation can be viewed as a natural generalisation of the idempotent endomorphisms of a vector space: more precisely, if $f \in \text{End}_k(M)$, then $f \otimes I$ (or $I \otimes f$) is a solution.
of the Hopf equation if and only if \( f^2 = f \). We shall prove that if \( (M, \cdot, \rho) \in H_\mathcal{M}^H \) then the natural map 
\[
R_{(M,\cdot,\rho)}(m \otimes n) = \sum n_{<1>} \cdot m \otimes n_{<0>}
\]
is a solution of the Hopf equation. Conversely, the main result of this paper is a FRT type theorem which shows that in the finite dimensional case, any solution \( R \) of the Hopf equation has this form, i.e. there exists a bialgebra \( B(R) \) such that \( (M, \cdot, \rho) \in B(R)_\mathcal{M}^B(R) \). Similarly to [2], a solution \( R \) of the Hopf equation is called commutative if \( R_{12} R_{13} = R_{13} R_{12} \). In the finite dimensional case, any commutative solution of the Hopf equation has the form \( R = R(M,\cdot,\rho) \), where \( (M, \cdot, \rho) \) is a Hopf module over a commutative bialgebra \( B(R) \). This result can be viewed as the algebraic version of the theorem 2.2 of [2], which classifies all multiplicative, unitary and commutative operators which can be defined on a Hilbert space. In the last part we shall apply our theorem to constructing new examples of noncommutative, noncocommutative bialgebras which differ from those arising from the FRT theorem for the quantum Yang-Baxter equation. These bialgebras arise from the elementary maps of plane euclidian geometry: the projections of \( k^2 \) on the Ox and Oy coordinate axis. Surprisingly, over a field of characteristic 2, our FRT type construction supplies us an example of noncommutative and noncocommutative bialgebra of dimension 5.

Obviously, substituting the key map \( R_{(M,\cdot,\rho)} \) with \( \tau R_{(M,\cdot,\rho)} \), all results of this paper remain valid if we replace the Hopf equation with the pentagonal equation. We have preferred however to work with the Hopf equation, for historical reasons: this is how the issue has been raised for the first time in lemma 4.9 of [28].

This study was continued in [18], where new classes of bialgebras arising from the Hopf equation are introduced and analyzed.

## 1 Preliminaries

Throughout this paper, \( k \) will be a field. All vector spaces, algebras, coalgebras and bialgebras that we consider are over \( k \). \( \otimes \) and \( \text{Hom} \) will mean \( \otimes_k \) and \( \text{Hom}_k \). For a coalgebra \( C \), we will use Sweedler’s \( \Sigma \)-notation, that is, \( \Delta(c) = \sum_{c(1)} \otimes c(2) \), \( (I \otimes \Delta) \Delta(c) = \sum c(1) \otimes^2 c(2) \otimes c(3) \), etc. We will also use Sweedler’s notation for right \( C \)-comodules: \( \rho^*_M(m) = \sum m_{<0>} \otimes m_{<1>} \), for any \( m \in M \) if \( (M, \rho_M) \) is a right \( C \)-comodule. \( \mathcal{M}^C \) will be the category of right \( C \)-comodules and \( A_\mathcal{M} \) will be the category of left \( A \)-modules and \( A \)-linear maps, if \( A \) is a \( k \)-algebra.

Recall the following well known lemmas:

**Lemma 1.1** Let \( M \) be a finite dimensional vector space with \( \{m_1, \cdots, m_n\} \) a basis for \( M \) and let \( C \) be a coalgebra. We define the \( k \)-linear map \( \rho : M \rightarrow M \otimes C \), \( \rho(m_i) = \sum_{v=1}^n m_v \otimes c_{v,i} \), for all \( l = 1, \cdots, n \), where \( (c_{v,i})_{v,1} \) is a family of elements of \( C \). The following statements are equivalent:

**Lemma 1.2** Let \( M \) be a finite dimensional vector space with \( \{m_1, \cdots, m_n\} \) a basis for \( M \) and let \( C \) be a coalgebra. We define the \( k \)-linear map \( \rho : M \rightarrow M \otimes C \), \( \rho(m_i) = \sum_{v=1}^n m_v \otimes c_{v,i} \), for all \( l = 1, \cdots, n \), where \( (c_{v,i})_{v,1} \) is a family of elements of \( C \). The following statements are equivalent:
1. \((M, \rho)\) is a right \(C\)-comodule.

2. The matrix \((c_{vl})_{v,l}\) is comultiplicative, i.e.
   \[
   \Delta(c_{jk}) = \sum_{u=1}^{n} c_{ju} \otimes c_{uk}, \quad \varepsilon(c_{jk}) = \delta_{jk}
   \]
   (5)
   for all \(j, k = 1, \ldots, n\)

If we denote \(B = (c_{vl})_{v,l}\), then, as usual, the relations (5) can formally be written: \(\Delta(B) = B \otimes B\), \(\varepsilon(B) = I_n\).

**Lemma 1.2** Let \((C, \Delta, \varepsilon)\) be a coalgebra. Then, on the tensor algebra \((T(C), M, u)\), there exists a unique bialgebra structure \((T(C), M, u, \Delta, \varepsilon)\) such that \(\Delta(c) = \Delta(c)\) and \(\varepsilon(c) = \varepsilon(c)\) for all \(c \in C\). In addition, the inclusion map \(i : C \to T(C)\) is a coalgebra map.

Furthermore, if \(M\) is a vector space and \(\mu : C \otimes M \to M\), \(\mu(c \otimes m) = c \cdot m\) is a linear map, then there exists a unique left \(T(C)\)-module structure on \(M\), \(\overline{\mu} : T(C) \otimes M \to M\), such that \(\overline{\mu}(c \otimes m) = c \cdot m\), for all \(c \in C\), \(m \in M\).

Let \(H\) be a bialgebra. Recall that an (left-right) \(H\)-Hopf module is a left \(H\)-module \((M, \cdot)\) which is also a right \(H\)-comodule \((M, \rho)\) such that

\[
\rho(h \cdot m) = \sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>}
\]
(6)
for all \(h \in H\), \(m \in M\). \(H\mathcal{M}^H\) will be the category of \(H\)-Hopf modules and \(H\)-linear \(H\)-colinear homomorphisms.

**Lemma 1.3** Let \(H\) be a bialgebra, \((M, \cdot)\) a left \(H\)-module and \((M, \rho)\) a right \(H\)-comodule. Then the set

\[
\{ h \in H \mid \rho(h \cdot m) = \sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>}, \forall m \in M \}
\]

is a subalgebra of \(H\).

**Proof** Straightforward. \(\square\)

We obtain from this lemma that if a left \(H\)-module and right \(H\)-comodule \(M\) satisfies the condition of compatibility (5) for a set of generators as an algebra of \(H\) and for a basis of \(M\), then \(M\) is an \(H\)-Hopf module. If \((M, \cdot)\) is a left \(H\)-module and \((M, \rho)\) is a right \(H\)-comodule, the special map

\[
R_{(M, \cdot, \rho)} : M \otimes M \to M \otimes M, \quad R_{(M, \cdot, \rho)}(m \otimes n) = \sum n_{<1>} \cdot m \otimes n_{<0>}
\]
(7)
will play an important role in the present paper. It is useful to point out the following lemma. The proof is left to the reader.
Lemma 1.4 Let $H$ be a bialgebra, $(M, \cdot)$ a left $H$-module and $(M, \rho)$ a right $H$-comodule. If $I$ is a biideal of $H$ such that $I \cdot M = 0$, then, with the natural structures, $(M, \cdot')$ is a left $H/I$-module and $(M, \rho')$ a right $H/I$-comodule and $R_{(M, \cdot', \rho')} = R_{(M, \cdot, \rho)}$.

For a vector space $V$, $\tau : V \otimes V \to V \otimes V$ will denote the switch map, that is, $\tau(v \otimes w) = w \otimes v$ for all $v, w \in V$. If $R : V \otimes V \to V \otimes V$ is a linear map we denote by $R^{12}, R^{13}, R^{23}$ the maps of $\text{End}_k(V \otimes V \otimes V)$ given by

$$R^{12} = R \otimes I, \quad R^{23} = I \otimes R, \quad R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau).$$

Using the notation $R(u \otimes v) = \sum u_1 \otimes v_1$ then

$$R^{12}(u \otimes v \otimes w) = \sum u_1 \otimes v_1 \otimes w_0$$

where the subscript $(0)$ means that $w$ is not affected by the application of $R^{12}$.

Let $H$ be a bialgebra and $(M, \cdot)$ a left $H$-module which is also a right $H$-comodule $(M, \rho)$. Recall that $(M, \cdot, \rho)$ is a Yetter-Drinfel’d module if the following compatibility relation holds:

$$\sum h(1) \cdot m_{<0>} \otimes h(2)m_{<1>} = \sum (h(2) \cdot m)_{<0>} \otimes (h(2) \cdot m)_{<1>} h(1)$$

for all $h \in H$, $m \in M$. $\mathcal{YD}^H$ will be the category of Yetter-Drinfel’d modules and $H$-linear $H$-colinear homomorphism. If $(M, \cdot, \rho)$ is a Yetter-Drinfel’d module then the special map $R = R_{(M, \cdot, \rho)}$ given by the equation (7) is a solution of the quantum Yang-Baxter equation

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

If $M$ is a finite dimensional vector space and $R$ is a solution of the quantum Yang-Baxter equation, then there exists a bialgebra $A(R)$ such that $(M, \cdot, \rho) \in A(R) \mathcal{M}^A(R)$ and $R = R_{(M, \cdot, \rho)}$ (see [20]). For a further study of the Yetter-Drinfel’d category we refer to [14], [20], [21], [22], or to the more recent [4], [5], [6], [9].

2 The Hopf equation

We will start with the following

Definition 2.1 Let $V$ be a vector space and $R \in \text{End}_k(V \otimes V)$.

1. We shall say that $R$ is a solution for the Hopf equation if

$$R^{23} R^{12} = R^{12} R^{23} \quad (8)$$

2. We shall say that $R$ is a solution for the pentagonal equation if

$$R^{12} R^{13} R^{23} = R^{23} R^{12} \quad (9)$$
Remarks 2.2 1. The Hopf equation is obtained from the quantum Yang-Baxter equation
\[ R^{23} R^{13} R^{12} = R^{12} R^{13} R^{23} \]
by deleting the midle term from the right hand side.

2. Let \( \{m_i\}_{i \in I} \) be a basis of the vector space \( V \). Then an endomorphism \( R \) of \( V \otimes V \) is given by a family of scalars \((x^{kl}_{ij})_{i,j,k,l \in I}\) of \( k \) such that
\[
R(m_v \otimes m_u) = \sum_{i,j} x^{ji}_{uv} m_i \otimes m_j
\]
for all \( v, u \in I \). A direct computation shows us that \( R \) is a solution of the Hopf equation if and only if \((x^{kl}_{ij})_{i,j,k,l \in I}\) is a solution of the nonlinear equation
\[
\sum_{\alpha,\beta,\gamma} x^{k\gamma}_{j\alpha} x^{\alpha\beta}_{u\gamma} x^{i\beta}_{w\alpha} = \sum_{i} x^{ji}_{wu} x^{kl}_{iv}
\]
for all \( j, k, l, u, v, w \in I \). It follows that solving the system (11) is really a non-trivial problem.

3. Using the notation \( R(x \otimes y) = \sum x_1 \otimes y_1 \), for \( x, y \in V \), then \( R \) is a solution of the Hopf equation if and only if
\[
\sum x_{110} \otimes y_{101} \otimes z_{011} = \sum x_{01} \otimes y_{11} \otimes z_{10}
\]
for all \( x, y, z \in V \).

4. Suppose that \( R \in \text{End}_k(V \otimes V) \) is bijective. Then, \( R \) is a solution of the Hopf equation if and only if \( R^{-1} \) is a solution of the pentagonal equation.

5. Let \( A \) be an algebra and \( R \in A \otimes A \) be an invertible element such that the Hopf equation \( R^{23} R^{13} R^{12} = R^{12} R^{23} \) holds in \( A \otimes A \otimes A \). Then, the comultiplication
\[
\Delta : A \rightarrow A \otimes A, \quad \Delta(a) := R(1 \otimes a)R^{-1}
\]
for all \( a \in A \) is coassociative and an algebra map.

Indeed,
\[
(I \otimes \Delta) \Delta(a) = R^{23} R^{13} (1 \otimes 1 \otimes a)(R^{23} R^{13})^{-1}
\]
and
\[
(\Delta \otimes I) \Delta(a) = R^{12} R^{23} (1 \otimes 1 \otimes a)(R^{12} R^{23})^{-1}
\]
Let \( W := (R^{13})^{-1}(R^{23})^{-1} R^{12} R^{23} \). Then \( \Delta \) is coassociative if and only if
\[
(1 \otimes 1 \otimes a)W = W(1 \otimes 1 \otimes a)
\]
for all \( a \in A \). But, as \( R \) satisfies the Hopf equation, we have that \( W = R^{12} \), i.e. equation (12) holds.

In the next proposition we shall evidence a few equations which are equivalent to the Hopf equation.
Proposition 2.3 Let $V$ be a vector space and $R \in \text{End}_k(V \otimes V)$. The following statements are equivalent:

1. $R$ is a solution of the Hopf equation.

2. $T := \tau R$ is a solution of the equation: $T^{12} T^{23} T^{12} = T^{23} \tau^{12} T^{23}$.

3. $T := R \tau$ is a solution of the equation: $T^{23} T^{12} T^{23} = T^{12} T^{13} \tau^{23}$.

4. $W := \tau R \tau$ is a solution of the pentagonal equation.

Proof $1 \iff 2$ The proof will follow from the formulas:

$$T^{12} T^{23} T^{12} = \tau^{13} R^{23} R^{13} R^{12}, \quad T^{23} \tau^{12} T^{23} = \tau^{13} R^{12} R^{23}$$

and from the fact that $\tau^{13}$ is an automorphism of $V \otimes V \otimes V$. Let $x, y, z \in V$. Then $T(x \otimes y) = \sum y_1 \otimes x_1$. We have

$$T^{12} T^{23} T^{12} (x \otimes y \otimes z) = \sum z_{011} \otimes y_{101} \otimes x_{110} = \tau^{13} R^{23} R^{13} R^{12} (x \otimes y \otimes z)$$

and

$$T^{23} \tau^{12} T^{23} (x \otimes y \otimes z) = \sum z_{10} \otimes y_{11} \otimes x_{01} = \tau^{13} R^{12} R^{23} (x \otimes y \otimes z)$$

$1 \iff 3$ follows from the formulas:

$$T^{23} T^{12} T^{23} = R^{23} R^{13} R^{12} \tau^{13}, \quad T^{12} T^{13} T^{23} = R^{12} R^{23} \tau^{13}.$$

$1 \iff 4$ follows from the formulas:

$$W^{12} W^{13} W^{23} = R^{23} R^{13} R^{12} \tau^{13}, \quad W^{23} W^{12} = R^{12} R^{23} \tau^{13}.$$

From now on we shall study only the Hopf equation. In 2 numerous examples of operators $W$ which are solutions for the pentagonal equation are given. All these operators come from the theory of operator algebras. We shall present only one of them, which plays a key role in classifying the multiplicative and commutative operators (see theorem 2.2 from the above cited paper). Let $G$ be a locally compact group and $dg$ a right Haar measure on $G$. Then, $V_G(\xi)(s, t) = \xi(st, t)$ is a solution for the pentagonal equation. It follows that $\tau V_G \tau$ is a solution for the Hopf equation. Next, we shall present other, purely algebraic, examples of solutions for the Hopf equation.

Examples 2.4 1. The identity map $I_{V \otimes V}$ is a solution of the Hopf equation.

2. Let $V$ be a finite dimensional vector space and $u$ an automorphism of $V$. If $R$ is a solution of the Hopf equation then $^u R := (u \otimes u) R(u \otimes u)^{-1}$ is also a solution of the Hopf equation.
Indeed, as $\text{End}_k(V \otimes V) \cong \text{End}_k(V) \otimes \text{End}_k(V)$, we can view $R = \sum f_i \otimes g_i$, where $f_i, g_i \in \text{End}_k(V)$. Then $u R = \sum u f_i u^{-1} \otimes u g_i u^{-1}$ and

\[
(u R)^{12} (u R)^{23} = (u \otimes u \otimes u) R^{12} R^{23} (u \otimes u \otimes u)^{-1},
\]

\[
(u R)^{23} (u R)^{13} = (u \otimes u \otimes u) R^{23} R^{13} R^{12} (u \otimes u \otimes u)^{-1},
\]

hence $u R$ is also a solution of the Hopf equation.

3. Let $f, g \in \text{End}_k(V)$ such that $f^2 = f$, $g^2 = g$ and $fg = gf$. Then, $R := f \otimes g$ is a solution of the Hopf equation.

A direct computation shows that

\[
R^{23} R^{13} R^{12} = f^2 \otimes fg \otimes g^2, \quad R^{12} R^{23} = f \otimes gf \otimes g
\]

so the above conclusion follows. With this example in mind we can view the Hopf equation as a natural generalization of the idempotent endomorphism. That because $R = f \otimes I$ (or $R = I \otimes f$) is a solution of the Hopf equation if and only if $f^2 = f$.

We suppose now that $V$ is a two dimensional vector space with $\{v_1, v_2\}$ a basis of $V$. Let $f_q \in \text{End}_k(V)$ such that with respect to the given basis is

\[
f_q = \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}
\]

where $q$ is a scalar of $k$. Then $f_q^2 = f_q$.

Let $g_q \in \text{End}_k(V)$, $g_q = Id_V - f_q$. Then $g_q$ is also an idempotent endomorphism of $V$ and $g_q f_q = f_q g_q$. Thus, we obtain that $R_q = f_q \otimes g_q$ with respect to the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$ is

\[
R_q = \begin{pmatrix} 0 & -q & 0 & -q^2 \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

and $R_q$ is a solution for the Hopf equation.

Now let $g = Id_V$ and $R'_q = f_q \otimes Id_V$. Then with respect to the same ordonate basis of $V \otimes V$, $R'_q$ is given by

\[
R'_q = \begin{pmatrix} 1 & 0 & q & 0 \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

and $R'_q$ is also a solution of the Hopf equation.

4. The above $R_q$ and $R'_q$ are also solutions of the quantum Yang-Baxter equation, because each of them has the form $f \otimes g$ with $fg = gf$. In this example we will construct a solution for the Hopf equation which is not a solution of the quantum Yang-Baxter equation.
Let $G$ be a group and $V$ be a $G$-graded representation on $G$, that is $V$ is a left $k[G]$-module and there exists $\{V_\sigma \mid \sigma \in G\}$ a family of subspaces of $V$ such that

\[ V = \oplus_{\sigma \in G} V_\sigma \quad \text{and} \quad g \cdot V_\sigma \subseteq V_{g\sigma} \]

for all $g, \sigma \in G$. If $v_\sigma \in V_\sigma$ we shall write $\deg(v_\sigma) = \sigma$; if $v \in V$, then $v$ is a finite sum of homogenous elements $v = \sum v_\sigma$. The map

\[ R : V \otimes V \to V \otimes V, \quad R(u \otimes v) = \sum_{\sigma} \sigma \cdot u \otimes v_\sigma \tag{14} \]

is a solution of the Hopf equation and is not a solution of the quantum Yang-Baxter equation. Indeed, it is enough to prove that (8) holds only for homogenous elements. Let $u_\sigma \in V_\sigma$, $u_\tau \in V_\tau$ and $u_\theta \in V_\theta$. Then,

\[
R^{23} R^{13} R^{12} (u_\sigma \otimes u_\tau \otimes u_\theta) = R^{23} R^{13} (\tau \cdot u_\sigma \otimes u_\tau \otimes u_\theta) \\
= R^{23} (\theta \tau \cdot u_\sigma \otimes u_\tau \otimes u_\theta) \\
= \theta \tau \cdot u_\sigma \otimes \theta \cdot u_\tau \otimes u_\theta
\]

and

\[
R^{12} R^{23} (u_\sigma \otimes u_\tau \otimes u_\theta) = R^{12} (u_\sigma \otimes \theta \cdot u_\tau \otimes u_\theta) \\
(\deg(\theta \cdot u_\tau) = \theta \tau) = \theta \tau \cdot u_\sigma \otimes \theta \cdot u_\tau \otimes u_\theta
\]

Hence $R$ is a solution of the Hopf equation. On the other hand, by a direct computation we get

\[ R^{12} R^{13} R^{23} (u_\sigma \otimes u_\tau \otimes u_\theta) = \theta \tau \theta \cdot u_\sigma \otimes \theta \cdot u_\tau \otimes u_\theta \]

i.e. $R$ is not a solution of the quantum Yang-Baxter equation.

5. Let $G$ be a group and $V$ be a $G$-crossed module, that is $V$ is a left $k[G]$-module and there exists $\{V_\sigma \mid \sigma \in G\}$ a family of subspaces of $V$ such that

\[ V = \oplus_{\sigma \in G} V_\sigma \quad \text{and} \quad g \cdot V_\sigma \subseteq V_{g\sigma^{-1}} \]

for all $g, \sigma \in G$. Then $R$ given by (14) is a solution of the quantum Yang-Baxter equation and is not a solution of the Hopf equation.

6. Let $q$ be a scalar of $k$, $q \neq 0$, $q \neq 1$. Then the classical two dimensional Yang-Baxter operator

\[ R = \begin{pmatrix}
    q & 0 & 0 & 0 \\
    0 & 1 & q - q^{-1} & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & q
\end{pmatrix} \]

is a solution of the Yang-Baxter equation and is not a solution for the Hopf equation.

Indeed, the element in the $(1, 1)$-position of $R^{23} R^{13} R^{12}$ is $q^3$, while the element in the $(1, 1)$-position of $R^{12} R^{23}$ is $q^2$, i.e. $R$ is not a solution of the Hopf equation.
7. Let $H$ be a bialgebra. Then
\[ R : H \otimes H \to H \otimes H, \quad R(g \otimes h) = \sum h_{(1)}g \otimes h_{(2)} \]
for all $g, h \in H$, is a solution of the Hopf equation. This operator was defined by Takesaki in [28] for a Hopf-von Neumann algebra $(\mathcal{A}, \Delta)$.

8. Let $H$ be a Hopf algebra with an antipode $S$. Then $H/k$ is a Hopf-Galois extension (see [19]), i.e. the canonical map
\[ \beta : H \otimes H \to H \otimes H, \quad \beta(g \otimes h) = \sum gh_{(1)} \otimes h_{(2)} \]
is bijective. Then $\beta$ is a solution of the Hopf equation. Furthermore,
\[ R' : H \otimes H \to H \otimes H, \quad R'(g \otimes h) = \sum g_{(1)} \otimes S(g_{(2)})h \]
is also a solution of the Hopf equation.

In [4], the concept of multiplicative and commutative (respectively cocommutative) operator is introduced: that is, a unitary operator $W \in \mathcal{L}(K \otimes K)$, where $K$ is a Hilbert space, $W$ satisfies the pentagonal equation and $W_{12}W_{23} = W_{23}W_{13}$ (respectively $W_{12}W_{13} = W_{13}W_{12}$). We shall now introduce the corresponding concept for the Hopf equation.

**Definition 2.5** Let $V$ be a vector space and $R \in \text{End}_k(V \otimes V)$ be a solution of the Hopf equation. Then

1. $R$ is called commutative if $R_{12}R_{13} = R_{13}R_{12}$.

2. $R$ is called cocommutative if $R_{13}R_{23} = R_{23}R_{13}$.

**Remarks 2.6** 1. Let $R \in \text{End}_k(V \otimes V)$. Then $R$ is a commutative solution of the Hopf equation if and only if $W := \tau R\tau$ is a commutative solution of the pentagonal equation.

Indeed, $R_{12}R_{13} = R_{13}R_{12}$ if and only if
\[ \tau_{12}W_{12}\tau_{12}\tau_{13}W_{13}\tau_{13} = \tau_{13}W_{13}\tau_{13}\tau_{12}W_{12}\tau_{12}. \]  
(15)

Using the formulas
\[ \tau_{12}\tau_{13} = \tau_{23}\tau_{12}, \quad \tau_{13}\tau_{12} = \tau_{12}\tau_{23}, \]
\[ W_{12}\tau_{23} = \tau_{23}W_{13}, \quad \tau_{12}W_{13} = W_{23}\tau_{12}, \]
\[ W_{13}\tau_{12} = \tau_{12}W_{23}, \quad \tau_{23}W_{12} = W_{13}\tau_{23} \]
we get that the equation (15) is equivalent to
\[ \tau_{12}W_{13}\tau_{23}W_{12}\tau_{13} = \tau_{13}\tau_{12}W_{23}W_{13}\tau_{23}\tau_{12}. \]
The conclusion follows as
\[ \tau^{12} \tau^{13} \tau^{12} \tau^{23} = \tau^{23} \tau^{12} \tau^{13} \tau^{12} = \text{Id}. \]

2. Suppose that \( R \in \text{End}_k(V \otimes V) \) is bijective. Then, \( R \) is a cocommutative solution of the Hopf equation if and only if \( \tau R^{-1} \tau \) is a commutative solution of the Hopf equation.

Our example (4) can be generalized to arbitrary Hopf modules and evidences the role which can be played by the \( H \)-Hopf modules in solving the Hopf equation.

**Proposition 2.7** Let \( H \) be a bialgebra and \((M, \cdot, \rho)\) an \( H \)-Hopf module. Then:

1. the natural map
\[
R_{(M, \cdot, \rho)}(m \otimes n) = \sum n_{<1>} \cdot m \otimes n_{<0>}
\]
is a solution of the Hopf equation.

2. if \( H \) is commutative then \( R_{(M, \cdot, \rho)} \) is a commutative solution of the Hopf equation.

**Proof**

1. Let \( R = R_{(M, \cdot, \rho)} \). For \( l, m, n \in M \) we have
\[
R^{12} R^{23} (l \otimes m \otimes n) = R^{12} \left( \sum l \otimes n_{<1>} \cdot m \otimes n_{<0>} \right)
= \sum (n_{<1>} \cdot m)_{<1>} \cdot l \otimes (n_{<1>} \cdot m)_{<0>} \otimes n_{<0>}
\]
and
\[
R^{23} R^{13} R^{12} (l \otimes m \otimes n) = R^{23} R^{13} \left( \sum m_{<1>} \cdot l \otimes m_{<0>} \otimes n \right)
= R^{23} \left( \sum n_{<1>} m_{<1>} \cdot l \otimes m_{<0>} \otimes n_{<0>} \right)
= \sum n_{<2>} m_{<1>} \cdot l \otimes n_{<1>} \cdot m_{<0>} \otimes n_{<0>}
= \sum n_{<1>(2)} m_{<1>} \cdot l \otimes n_{<1>(1)} \cdot m_{<0>} \otimes n_{<0>}
\]
(\text{using (3)})
\[
= \sum (n_{<1>} \cdot m)_{<1>} \cdot l \otimes (n_{<1>} \cdot m)_{<0>} \otimes n_{<0>}
\]
i.e. \( R \) is a solution of the Hopf equation.

2. We have
\[
R^{12} R^{13} (l \otimes m \otimes n) = \sum m_{<1>} n_{<1>} \cdot l \otimes m_{<0>} \otimes n_{<0>}
\]
and
\[
R^{13} R^{12} (l \otimes m \otimes n) = \sum n_{<1>} m_{<1>} \cdot l \otimes m_{<0>} \otimes n_{<0>}
\]
As \( H \) is commutative, we obtain that \( R^{12} R^{13} = R^{13} R^{12} \). \( \square \)

**Remark 2.8** If \((M, \cdot, \rho)\) is an \( H \)-Hopf module then the map
\[
R'_{(M, \cdot, \rho)} : M \otimes M \to M \otimes M, \quad R'_{(M, \cdot, \rho)}(m \otimes n) = \sum m_{<0>} \otimes m_{<1>} \cdot n
\]
is a solution of the pentagonal equation, as \( R'_{(M, \cdot, \rho)} = \tau R_{(M, \cdot, \rho)} \tau \).
3 A FRT type construction for Hopf modules

In this section we shall prove the main result of the paper, which shows us that in the finite dimensional case any solution of the Hopf equation has the form $R_{(M,\cdot,\rho)}$.

**Theorem 3.1** Let $M$ be a finite dimensional vector space and $R \in \text{End}_k(M \otimes M)$ be a solution of the Hopf equation. Then

1. There exists a bialgebra $B(R)$ such that $M$ has a structure of $B(R)$-Hopf module $(M,\cdot,\rho)$ and $R = R_{(M,\cdot,\rho)}$.

2. The bialgebra $B(R)$ is a universal object with this property: if $H$ is a bialgebra such that $(M,\cdot',\rho') \in H\mathcal{M}^H$ and $R = R_{(M,\cdot',\rho')}$ then there exists a unique bialgebra map $f : B(R) \rightarrow H$ such that $\rho' = (I \otimes f)\rho$. Furthermore, $a \cdot m = f(a) \cdot' m$, for all $a \in B(R)$, $m \in M$.

3. If $R$ is commutative, then there exists a commutative bialgebra $\overline{B}(R)$ such that $M$ has a structure of $\overline{B}(R)$-Hopf module $(M,\cdot',\rho')$ and $R = R_{(M,\cdot',\rho')}$. 

**Proof** 1. The proof will be given is several steps. Let $\{m_1, \ldots, m_n\}$ be a basis for $M$ and $(x_{uv}^{ji})_{i,j,u,v}$ a family of scalars of $k$ such that

$$R(m_v \otimes m_u) = \sum_{i,j} x_{uv}^{ji} m_i \otimes m_j$$

(16)

for all $u, v = 1, \cdots, n$.

Let $(C, \Delta, \varepsilon) = \mathcal{M}^n(k)$, be the comatrix coalgebra of order $n$, i.e. $C$ is the coalgebra with the basis $\{c_{ij} \mid i,j = 1, \cdots, n\}$ such that

$$\Delta(c_{jk}) = \sum_{u=1}^n c_{ju} \otimes c_{uk}, \quad \varepsilon(c_{jk}) = \delta_{jk}$$

(17)

for all $j,k = 1, \cdots, n$. Let $\rho : M \rightarrow M \otimes C$ given by

$$\rho(m_l) = \sum_{v=1}^n m_v \otimes c_{vl}$$

(18)

for all $l = 1, \cdots, n$. Then, by lemma [1.1], $M$ is a right $C$-comodule. Let $T(C)$ be the bialgebra structure on the tensor algebra $T(C)$ which extends $\Delta$ and $\varepsilon$ (from lemma [1.2]). As the inclusion $i : C \rightarrow T(C)$ is a coalgebra map, $M$ has a right $T(C)$-comodule structure via

$$M \xrightarrow{\rho} M \otimes C \xrightarrow{I \otimes i} M \otimes T(C)$$

There will be no confusion if we also denote the right $T(C)$-comodule structure on $M$ with $\rho$. 

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Now, we will put a left $T(C)$-module structure on $M$ in such a way that $R = R_{(M, \cdot, \rho)}$. First we define
\[
\mu : C \otimes M \to M, \quad \mu(c_{ju} \otimes m_v) := \sum_i x_{uv}^{ji} m_i
\]
for all $j, u, v = 1, \ldots, n$. From lemma 1.2, there exists a unique left $T(C)$-module structure on $(M, \cdot)$ such that
\[
c_{ju} \cdot m_v = \mu(c_{ju} \otimes m_v) = \sum_i x_{uv}^{ji} m_i
\]
for all $j, u, v = 1, \ldots, n$. For $m_v, m_u$ the elements of the given basis, we have:
\[
R_{(M, \cdot, \rho)}(m_v \otimes m_u) = \sum_j c_{ju} \cdot m_v \otimes m_j = \sum_{i,j} x_{uv}^{ji} m_i \otimes m_j = R(m_v \otimes m_u)
\]
Hence, $(M, \cdot, \rho)$ has a structure of left $T(C)$-module and right $T(C)$-comodule such that $R = R_{(M, \cdot, \rho)}$.

Now, we define the obstructions $\chi(i, j, k, l)$ which measure how far away $M$ is from a $T(C)$-Hopf module. Keeping in mind that $T(C)$ is generated as an algebra by $(c_{ij})$ and using lemma 1.3 we compute
\[
\sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>} - \rho(h \cdot m)
\]
only for $h = c_{jk}$, and $m = m_l$, for $j, k, l = 1, \ldots, n$. We have:
\[
\sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>} = \sum_{u,v} c_{ju} \cdot m_v \otimes c_{uk} c_{vl} = \sum_i m_i \otimes \left( \sum_{u,v} x_{uv}^{ji} c_{uk} c_{vl} \right)
\]
and
\[
\rho(h \cdot m) = \rho(c_{jk} \cdot m_l) = \sum_{\alpha} x_{kl}^{j\alpha} (m_{\alpha})_{<0>} \otimes (m_{\alpha})_{<1>} = \sum_{i,\alpha} x_{kl}^{j\alpha} m_i \otimes c_{i\alpha} = \sum_i m_i \otimes \left( \sum_{\alpha} x_{kl}^{j\alpha} c_{i\alpha} \right)
\]
Let
\[
\chi(i, j, k, l) := \sum_{u,v} x_{uv}^{ji} c_{uk} c_{vl} - \sum_{\alpha} x_{kl}^{j\alpha} c_{i\alpha}
\]
for all $i, j, k, l = 1, \ldots, n$. Then
\[
\sum h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>} - \rho(h \cdot m) = \sum_i m_i \otimes \chi(i, j, k, l)
\]
Let $I$ be the two-sided ideal of $T(C)$ generated by all $\chi(i, j, k, l), i, j, k, l = 1, \ldots, n$. The key point of the proof follows:

$I$ is a bi-ideal of $T(C)$ and $I \cdot M = 0$.

We first prove that $I$ is also a coideal and this will result from the following formula:

$$\Delta(\chi(i, j, k, l)) = \sum_{a,b} \chi(i, j, a, b) \otimes c_{ak}c_{bl} + \sum_p c_{ip} \otimes \chi(p, j, k, l)$$

(21)

Indeed, we have:

$$\Delta(\chi(i, j, k, l)) = \sum_{u,v} x^{ji}_{uv} \Delta(c_{uk}) \Delta(c_{vl}) - \sum_{\alpha} x^{ji}_{\alpha} \Delta(c_{i\alpha})$$

$$= \sum_{a,b,u,v} x^{ji}_{uv} c_{ua}c_{vb} \otimes c_{ak}c_{bl} - \sum_{a,b} x^{ji}_{ab} c_{ip} \otimes c_{p\alpha}$$

$$= \sum_{a,b} \left( \chi(i, j, a, b) + \sum_{\gamma} x^{j\gamma}_{ab} c_{i\gamma} \right) \otimes c_{ak}c_{bl}$$

$$- \sum_{p} c_{ip} \otimes \left( -\chi(p, j, k, l) + \sum_{r,s} x^{jp}_{rs} c_{r\alpha}c_{s\beta} \right)$$

$$= \sum_{a,b} \chi(i, j, a, b) \otimes c_{ak}c_{bl} + \sum_{p} c_{ip} \otimes \chi(p, j, k, l)$$

where in the last equality we use the fact that

$$\sum_{a,b,\gamma} x^{j\gamma}_{ab} c_{i\gamma} \otimes c_{ak}c_{bl} = \sum_{p,r,s} x^{jp}_{rs} c_{ip} \otimes c_{r\alpha}c_{s\beta}$$

Hence, the formula (21) holds. On the other hand

$$\varepsilon(\chi(i, j, k, l)) = x^{ji}_{kl} - x^{ji}_{kl} = 0$$

so we proved that $I$ is a coideal of $T(C)$.

Now, in order to show that $I \cdot M = 0$, we shall use the fact that $R$ is a solution of the Hopf equation. For $z \in M, j, k = 1, \ldots, n$, we have the following formula:

$$\left( R^{23} R^{13} R^{12} - R^{12} R^{23} \right) (z \otimes m_k \otimes m_j) = \sum_{r,s} \chi(r, s, j, k) \cdot z \otimes m_r \otimes m_s$$

(22)

Let us compute

$$\left( R^{23} R^{13} R^{12} \right) (z \otimes m_k \otimes m_j) = \left( R^{23} R^{13} \right) \left( \sum_{\alpha} c_{ak} \cdot z \otimes m_\alpha \otimes m_j \right)$$

$$= R^{23} \left( \sum_{\alpha, \beta} c_{\beta j} c_{ak} \cdot z \otimes m_\alpha \otimes m_\beta \right)$$

$$= \sum_{a,\beta, r,s} x^{jr}_{\beta \alpha} c_{\beta j} c_{ak} \cdot z \otimes m_r \otimes m_s$$
On the other hand
\[
(R^{12}R^{23})(z \otimes m_k \otimes m_j) = R^{12}(\sum_s z \otimes c_{sj} \cdot m_k \otimes m_s)
\]
\[
= R^{12}(\sum_{s,\alpha} x^{s\alpha}_{jk} m_{\alpha} \otimes m_s)
\]
\[
= \sum_{r,s,\alpha} x^{s\alpha}_{rk} c_{\alpha} \cdot z \otimes m_r \otimes m_s
\]
It follows that
\[
(R^{23}R^{13}R^{12} - R^{12}R^{23})(z \otimes m_k \otimes m_j) = \sum_{r,s}(\sum_{\alpha,\beta} x^{s\beta}_{ja} c_{\alpha j} - \sum_{\alpha} x^{s\alpha}_{jk} c_{\alpha r}) \cdot z \otimes m_r \otimes m_s
\]
\[
= \sum_{r,s} \chi(r, s, j, k) \cdot z \otimes m_r \otimes m_s
\]
i.e. the formula (22) holds. But \(R\) is a solution of the Hopf equation, hence \(\chi(r, s, j, k) \cdot z = 0\), for all \(z \in M, j, k, r, s = 1, \ldots, n\). We conclude that \(I\) is a bi-ideal of \(T(C)\) and \(I \cdot M = 0\).

Define now
\[
B(R) = T(C)/I.
\]
\(M\) has a right \(B(R)\)-comodule structure via the canonical projection \(T(C) \to B(R)\) and a left \(B(R)\)-module structure as \(I \cdot M = 0\). As \((c_{ij})\) generate \(B(R)\) and in \(B(R)\), \(\chi(i, j, k, l) = 0\), for all \(i, j, k, l = 1, \ldots, n\), using (20) we get that \((M, \cdot, \rho) \in B(R)\mathcal{M}^{B(R)}\) and, by lemma (1.4), \(R = R_{(M, \cdot, \rho)}\).

2. Let \(H\) be a bialgebra and suppose that \((M, \cdot', \rho') \in \mathcal{H}\mathcal{M}^H\) and \(R = R_{(M, \cdot', \rho')}\). Let \((c'_{ij})_{i,j=1,\ldots,n}\) be a family of elements of \(H\) such that
\[
\rho'(m_i) = \sum_v m_v \otimes c'_{vl}
\]
Then
\[
R(m_v \otimes m_u) = \sum_j c'_{ju} \cdot' m_v \otimes m_j
\]
and
\[
c'_{ju} \cdot' m_v = \sum_i x^{ji}_{uv} m_i = c_{ju} \cdot m_v.
\]
Let
\[
\chi'(i, j, k, l) = \sum_{u,v} x^{ji}_{uv} c'_{ak} c'_{vl} - \sum_{\alpha} x^{j\alpha}_{kl} c'_{i\alpha}
\]
From the universal property of the tensor algebra \(T(C)\), there exists a unique algebra map \(f_1 : T(C) \to H\) such that \(f_1(c_{ij}) = c'_{ij}\), for all \(i, j = 1, \ldots, n\). As \((M, \cdot', \rho') \in \mathcal{H}\mathcal{M}^H\) we get that \(\chi'(i, j, k, l) = 0\), and hence \(f_1(\chi(i, j, k, l)) = 0\), for all \(i, j, k, l = 1, \ldots, n\). So the map \(f_1\) factorizes to the map
\[
f : B(R) \to H, \quad f(c_{ij}) = c'_{ij}
\]
Of course, for \(m_i\) an arbitrary element of the given basis of \(M\), we have
\[
(I \otimes f)\rho(m_i) = \sum_v m_v \otimes f(c_{vl}) = \sum_v m_v \otimes c'_{vl} = \rho'(m_i)
\]
Conversely, the relation \((I \otimes f)\rho = \rho'\) necessarily implies \(f(c_{ij}) = c'_{ij}\), which proves the uniqueness of \(f\). This completes the proof of the theorem.

3. For \(z \in M\) and \(j, k = 1, \cdots, n\) we have the formula:
\[
\left( R_{12}^2 R_{13} - R_{13}^2 R_{12} \right) (z \otimes m_k \otimes m_j) = \sum_{r,s} \left( c_{rk} c_{sj} - c_{sj} c_{rk} \right) \cdot z \otimes m_r \otimes m_s \tag{23}
\]

Let \(\mathcal{T}\) be the two-sided ideal of \(T(C)\) generated by \(I\) and all \([c_{rk}, c_{sj}]\). It follows from the formula
\[
\Delta ([c_{rk}, c_{sj}]) = \sum_{a,b} \left( [c_{ra}, c_{sb}] \otimes c_{bj} c_{ak} + c_{ra} c_{sb} \otimes [c_{ak}, c_{bj}] \right)
\]
that \(\mathcal{T}\) is also a coideal of \(T(C)\) and from equation (23) we get that \(\mathcal{T} \cdot M = 0\). Define now
\[
\mathcal{B}(R) = T(C)/\mathcal{T}.
\]

Then \(\mathcal{B}(R)\) is a commutative bialgebra, \(M\) has a structure of \(\mathcal{B}(R)\)-Hopf module \((M, ', \rho')\) and \(R = R_{(M, ', \rho')}\).

**Remark 3.2**
1. Our obstruction elements \(\chi(i, j, k, l)\) play the same role as the homogenous elements \(d(i, j, k, l)\) defined in [20] which correspond to the quantum Yang-Baxter equation: the two-sided ideal generated by them is also a coideal which annihilates \(M\). This was the key point of the proof.

2. The last point of our theorem can be viewed as an algebraic version of theorem 2.2. from [2]. All commutative bialgebras \(\mathcal{B}(R)\) are quotients for various bialgebra structures which can be given on \(k[Y_1, \cdots, Y_n]\).

## 4 Applications

In this section we shall construct new examples of noncommutative noncocommutative bialgebras arising from our FRT type theorem. As the relations through which we factor are not all homogenous, all our examples are different from the ones which appear in quantum group theory. A completely different method for constructing such objects uses Ore extensions and was recently evidenced in [3].

In the next propositions of this section, the relations \(\chi(i, j, k, l) = 0\) will be written in the lexicographical order according to \((i, j, k, l)\) starting with \((1, 1, 1, 1)\).

### 4.1 Back to euclidian geometry

There exists an intimate link between the quantum Yang-Baxter equation and the quantum plane \(k_q < x, y \mid xy = qyx >\) (see [13]).
We shall now show that the term "dequantisation" used in the introduction is not an abuse. More specifically, our FRT type construction for the Hopf equation supplies us with a way to construct noncommutative and noncocommutative bialgebras starting from the elementary maps of plane euclidian geometry: projections of \( k^2 \) on the \( Ox \) and \( Oy \) axis. The map \( f_0 \), corresponding to \( q = 0 \) in the equation (13), is in fact the projection of the plane \( k^2 \) on the \( Ox \) axis. We can now associate three bialgebras to this projection: the first one corresponds to the solution of the Hopf equation \( f_0 \otimes g_0 \), where \( g_0 = \text{Id}_{k^2} - f_0 \), i.e. is the projection of the plane \( k^2 \) on the \( Oy \) axis, the second corresponds to \( f_0 \otimes \text{Id}_{k^2} \), and the third corresponds to the \( f_0 \otimes f_0 \). In this way, we obtain the bialgebras denoted below by \( B_{0}^2(k) \), \( D_{0}^2(k) \) and \( E_{0}^2(k) \). If \( q \neq 0 \) and \( k = \mathbb{R} \), the map \( f_q \) given in (13) sends all the points of the \( \mathbb{R}^2 \) plane on the \( Ox \) axis under an angle \( \arctg(q) \) with respect to the \( Oy \) axis. Correspondingly, the bialgebras \( B_{q}^2(k) \), \( D_{q}^2(k) \) and \( E_{q}^2(k) \) are constructed.

**Proposition 4.1** Let \( q \) be a scalar of the field \( k \) and \( R_q \) be the solution of the Hopf equation given by

\[
R_q = \begin{pmatrix}
0 & -q & 0 & -q^2 \\
0 & 1 & 0 & q \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Let \( B_{q}^2(k) \) be the bialgebra \( B(R_q) \). Then:

1. If \( q = 0 \), the bialgebra \( B_{0}^2(k) \) is the free algebra generated by \( x, y, z \) with the relations:

\[
yx = x, \quad yz = 0.
\]

The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by:

\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = x \otimes z + z \otimes y
\]

\[
\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0.
\]

2. If \( q \neq 0 \), the bialgebra \( B_{q}^2(k) \) is the free algebra generated by \( A, B \) with the relation:

\[
A^2B = AB.
\]

The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by:

\[
\Delta(A) = A \otimes A, \quad \Delta(B) = q^{-1}AB \otimes B + (B - AB) \otimes A,
\]

\[
\varepsilon(A) = 1, \quad \varepsilon(B) = q.
\]

**Proof** Let \( M \) be a two dimensional vector space with \( \{m_1, m_2\} \) a basis. Put \( R = R_q \). With respect to the ordonate basis \( \{m_1 \otimes m_1, m_1 \otimes m_2, m_2 \otimes m_1, m_2 \otimes m_2\} \), \( R \) is given by:

\[
R(m_1 \otimes m_1) = R(m_2 \otimes m_1) = 0,
\]
\begin{align*}
R(m_1 \otimes m_2) &= -qm_1 \otimes m_1 + m_1 \otimes m_2, \\
R(m_2 \otimes m_2) &= -q^2 m_1 \otimes m_1 + q m_1 \otimes m_2
\end{align*}

Now, if we write

\[R(m_v \otimes m_u) = \sum_{i,j=1}^{2} x_{uv}^{ji} m_i \otimes m_j\]

we get that among the elements \((x_{uv}^{ji})\), the only nonzero elements are:

\[x_{21}^{11} = -q, \quad x_{21}^{21} = 1, \quad x_{22}^{11} = -q^2, \quad x_{22}^{21} = q.\]

The sixteen relations \(\chi(i, j, k, l) = 0\) are:

\[
\begin{align*}
-q c_{21} c_{11} - q^2 c_{21} c_{21} &= 0, \\
-q c_{21} c_{12} - q^2 c_{21} c_{22} &= 0, \\
-q c_{22} c_{11} - q^2 c_{22} c_{21} &= -q c_{11}, \\
-q c_{22} c_{12} - q^2 c_{22} c_{22} &= -q^2 c_{11}, \\
c_{21} c_{11} + q c_{21} c_{21} &= 0, \\
c_{21} c_{12} + q c_{21} c_{22} &= 0, \\
c_{22} c_{11} + q c_{22} c_{21} &= c_{11}, \\
c_{22} c_{12} + q c_{22} c_{22} &= q c_{11},
\end{align*}
\]

\[
\begin{align*}
0 &= 0, \\
0 &= 0, \\
0 &= -q c_{21}, \\
0 &= -q^2 c_{21}, \\
0 &= 0, \\
0 &= 0, \\
0 &= c_{21}, \\
0 &= q c_{21}.
\end{align*}
\]

Hence, \(c_{21} = 0\). Now, if we denote \(c_{11} = x, c_{22} = y, c_{12} = z\), there are only two linear independent relations:

\[yx = x, \quad yz + qy^2 = qx.\]

If \(q = 0\), then follows 1. If \(q \neq 0\), then \(x\) is an element in the free algebra generated by \(y\) and \(z\). Let \(A = y\) and \(B = z + qy = z + qA\). Then

\[x = q^{-1} AB\]

and by substituting in the first relation we get \(A^2 B = AB\). The formulas for \(\Delta\) and \(\varepsilon\) follow as the original \((c_{ij})\) was a comultiplicative matrix. \(\square\)

**Remark 4.2** The bialgebra \(B_0^2(k)\) is not a Hopf algebra. We can localize it to obtain a Hopf algebra. As \(\Delta(x) = x \otimes x, \Delta(y) = y \otimes y\) and \(\varepsilon(x) = \varepsilon(y) = 1\) we should add new generators which make \(x\) and \(y\) invertible. But then \(y = 1\) and \(z = 0\). It follows that if we localize the bialgebra \(B_0^2(k)\), we get the usual Hopf algebra \(k[X, X^{-1}]\), with \(\Delta(X) = X \otimes X\), \(\varepsilon(X) = X \otimes X\), and with the antipode \(S(X) = X^{-1}\).

**Proposition 4.3** Let \(q\) be a scalar of the field \(k\) and \(R'_q\) be the solution of the Hopf equation given by

\[
R'_q = \begin{pmatrix}
1 & 0 & q & 0 \\
0 & 1 & 0 & q \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

We denote by \(D_0^2(k)\) the bialgebra \(B(R'_q)\). Then:
1. If \( q = 0 \), the bialgebra \( D^2_0(k) \) is the free algebra generated by \( x, y, z \) with the relations:

\[
x^2 = x = yz, \quad zx = xz = z^2 = yz = 0.
\]

The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by:

\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = x \otimes z + z \otimes y
\]

\[
\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0.
\]

2. If \( q \neq 0 \), the bialgebra \( D^2_q(k) \) is the free algebra generated by \( A, B \) with the relations:

\[
A^2 = A^2, \quad BA = 0.
\]

The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by:

\[
\Delta(A) = A \otimes A + q^{-1}(A^2 - A) \otimes B, \quad \Delta(B) = A^2 \otimes B + B \otimes A - q^{-1}B \otimes B,
\]

\[
\varepsilon(A) = 1, \quad \varepsilon(B) = 0.
\]

**Proof** We start exactly as in the above proposition. We get that among the scalars \( (x^{ij}_{uv}) \), which define \( R \), the only nonzero elements are:

\[
x^{11}_{11} = x^{21}_{21} = 1, \quad x^{12}_{12} = x^{22}_{22} = q.
\]

Now the relations \( \chi(i, j, k, l) = 0 \) are:

\[
c_{11}c_{11} + qc_{11}c_{21} = c_{11}, \quad c_{11}c_{12} + qc_{11}c_{22} = qc_{11},
\]

\[
c_{12}c_{11} + qc_{12}c_{21} = 0, \quad c_{12}c_{12} + qc_{12}c_{22} = 0,
\]

\[
c_{21}c_{11} + qc_{21}c_{21} = 0, \quad c_{21}c_{12} + qc_{21}c_{22} = 0,
\]

\[
c_{22}c_{11} + qc_{22}c_{21} = c_{11}, \quad c_{22}c_{12} + qc_{22}c_{22} = qc_{11},
\]

\[
0 = c_{21}, \quad 0 = qc_{21}, \quad 0 = 0, \quad 0 = 0,
\]

\[
0 = 0, \quad 0 = 0, \quad 0 = c_{21}, \quad 0 = qc_{21}.
\]

Hence \( c_{21} = 0 \). If we denote \( c_{11} = x, c_{22} = y, c_{12} = z \) then we get the following six relations:

\[
x^2 = x = yz, \quad zx = 0, \quad z^2 + qzy = 0,
\]

\[
xyz + qxy = yz + qy^2 = qx.
\]

As \( c_{21} = 0 \), the comultiplication and the counity take the following form:

\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = x \otimes z + z \otimes y
\]

\[
\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0.
\]
Hence, if \( q = 0 \) we get exactly the relation of \( D_0^2(k) \). Suppose now that \( q \neq 0 \). Then \( x \) is in the free algebra generated by \( y, z \) and

\[
x = y^2 + q^{-1}yz = y(y + q^{-1}z)
\]

If we substitute \( x \) in the other five relations we get, after we multiply with \( q \) or \( q^2 \):

\[
y(z + qy)(y(z + qy) - q) = 0,
\]

\[
(y - 1)y(z + qx) = 0,
\]

\[
zy(z + qy) = 0,
\]

\[
z(z + qy) = 0,
\]

\[
y(z + qy)(z + qy - q) = 0.
\]

Using the fact that \( y(y - 1) = (y - 1)y \), the fifth relation follows from the second and the fourth. On the other hand, as \( y(y^2 - 1) = (y + 1)(y - 1)y \), the first relation follows from the second and the third. We have thus reduced the above five relations to only three:

\[
(y - 1)y(z + qy) = 0, \quad zy(z + qy) = 0, \quad z(z + qy) = 0.
\]

Further, let \( B = z \) and \( A = q^{-1}(z + qy) = q^{-1}(B + qy) \). It follows that \( y = A - q^{-1}B \). The third relation takes the form

\[
BA = 0
\]

and this implies the second one. Using \( BA = 0 \), the first relation becomes \( A^3 = A^2 \). The formulas for \( \Delta \) and \( \varepsilon \) follow from equation (24). We note that the element \( A - q^{-1}B \) is a groupal element of \( D_q^2(k) \).

\[\square\]

**Remark 4.4** \( R_q' \) is also a solution of the quantum Yang-Baxter equation. The bialgebra \( A(R_q') \) which we obtain applying the usual FRT construction is the free algebra generated by \( x, y, z, t \) with the relations:

\[
zx = xz = zy = z^2 = zt = 0, \quad xy - yx = qyz, \quad xt - tx = qtz,
\]

\[
xy + qxt = qx^2, \quad y^2 + qyt = qxy, \quad ty + qt^2 = qxt.
\]

The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given in such way that the matrix

\[
\begin{pmatrix}
x & y \\
z & t
\end{pmatrix}
\]

is comultiplicative.

In the next proposition we shall prove that the bialgebra \( E_q^2(k) \), with \( q \neq 0 \), which can be associated to the solution \( R_q'' = f_q \otimes f_q \) is not dependent of \( q \), i.e. \( E_q^2(k) \cong E_{q'}^2(k) \), for all \( q, q' \in k \setminus \{0\} \).
Proposition 4.5 Let $q$ be a scalar of the field $k$ and $R''_q$ be the solution of the Hopf equation given by

$$R''_q = \begin{pmatrix}
1 & q & q^2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$$

Let $E^2_q(k)$ be the bialgebra $B(R''_q)$. Then:

1. If $q = 0$, the bialgebra $E^2_0(k)$ is the free algebra generated by $x, y, z$ with the relations:

$$x^2 = x, \quad xz = z = z^2 = 0.$$ 

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by:

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = x \otimes z + z \otimes y$$

$$\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0.$$ 

2. If $q \neq 0$, the bialgebra $E^2_q(k)$ is the free algebra generated by $A, B$ with the relations:

$$B^3 = B^2.$$ 

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by:

$$\Delta(A) = A \otimes A, \quad \Delta(B) = B \otimes A + B^2 \otimes (B - A)$$

$$\varepsilon(A) = \varepsilon(B) = 1.$$ 

Proof We start exactly as in the above propositions. We get that among the scalars $(x^i_{jk})$, which define $R$, the only nonzero elements are:

$$x^1_{11} = 1, \quad x^1_{21} = x^1_{12} = q, \quad x^1_{22} = q^2.$$ 

Now the relations $\chi(i, j, k, l) = 0$ are:

\begin{align*}
c_{11}c_{11} + qc_{21}c_{11} + qc_{11}c_{21} + q^2c_{21}c_{21} &= c_{11} \\
c_{11}c_{12} + qc_{21}c_{12} + qc_{11}c_{22} + q^2c_{21}c_{22} &= qc_{11} \\
c_{12}c_{11} + qc_{22}c_{11} + qc_{12}c_{21} + q^2c_{22}c_{21} &= qc_{11} \\
c_{12}c_{12} + qc_{22}c_{12} + qc_{12}c_{22} + q^2c_{22}c_{22} &= q^2c_{11} \\
0 &= 0, & 0 &= 0, & 0 &= 0, & 0 &= 0, \\
0 &= c_{21}, & 0 &= qc_{21}, & 0 &= qc_{21}, & 0 &= q^2c_{21}, \\
0 &= 0, & 0 &= 0, & 0 &= 0, & 0 &= 0,
\end{align*}
Hence $c_{21} = 0$. If we denote $c_{11} = x$, $c_{22} = y$, $c_{12} = z$ then we get the following four relations:

\[ x^2 = x, \quad xz + qxy = qx, \quad zx + qyx = qx, \]

\[ z^2 + qyz + qzy + q^2y^2 = q^2x. \]

So, if $q = 0$, we obtain the relations of $E_0^2(k)$. If $q \neq 0$, then $x$ is in the free algebra generated by $y$ and $z$ and

\[ x = y^2 + q^{-1}zy + q^{-1}yz + q^{-2}z^2 = (y + q^{-1}z)^2. \]

If we substitute $x$ in the other three relations, only

\[ (y + q^{-1}z)^3 = (y + q^{-1}z)^2. \]

remains, the other two being linear dependent from this one. Now, if we denote $A = y$ and $B = y + q^{-1}z$, we obtain the description of $E_0^2(k)$. \hfill \Box

**Remarks 4.6**

1. If $q \neq 0$, it is interesting to denote that $B^2$ is a groupal element of $E_0^2(k)$. Indeed, we have

\[
\Delta(B^2) = B^2 \otimes A^2 + B^2 \otimes (AB - A^2) + B^2 \otimes (BA - A^2) + B^4 \otimes (B^2 - BA - AB + A^2)
\]

\[
= B^2 \otimes A^2 + B^2 \otimes (AB - A^2) + B^2 \otimes (BA - A^2) + B^4 \otimes (B^2 - BA - AB + A^2)
\]

\[
= B^2 \otimes B^2
\]

2. The bialgebra $D_0^2(k)$ is the quotient of $B_0^2(k)$ by the two-sided ideal (which is also a coideal) generated by

\[ x^2 - x, \quad zx, \quad xz, \quad z^2. \]

$D_0^2(k)$ is also a quotient of $E_0^2(k)$ by the two-sided ideal generated by

\[ yx - x, \quad yz. \]

3. Let $n \geq 2$ be a natural number. The bialgebras $B_n^0(k)$, $D_n^0(k)$ and $E_n^0(k)$ constructed in the previous propositions can be generalised to $B_n^0(k)$, $D_n^0(k)$ and respectively $E_n^0(k)$. We have chosen to construct them for the case $n = 2$ in order to better sense the flavour of plane euclidian geometry. For clarity reasons we shall describe $B_n^0(k)$.

Let $\pi_1 : k^n \to k^n$ be the projection of $k^n$ on the $Ox_1$ axis, i.e. $\pi_1((x_1, x_2, \ldots, x_n)) = (x_1, 0, \ldots, 0)$ for all $(x_1, x_2, \ldots, x_n) \in k^n$ and $\pi^1 := Id_{k^n} - \pi_1$, the projection of $k^n$ on the hiperplane $x_1 = 0$, that is $\pi^1((x_1, x_2, \ldots, x_n)) = (0, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n) \in k^n$. Then $\pi_1 \otimes \pi^1$ is a solution of the Hopf equation and the bialgebra $B_n^0(k) := B(\pi_1 \otimes \pi^1)$ can be described as follows:

- $B_n^0(k)$ is the free algebra generated by $(c_{ij})_{i,j=1,\ldots,n}$ with the relation

\[ c_{i1} = 0, \quad c_{jk}c_{il} = \delta_{kj}\delta_{il}c_{11} \]

for all $i, j \geq 2$ and $k, l \geq 1$, where $\delta_{uv}$ is the Kronecker simbol.
The comultiplication $\Delta$ and the counit $\varepsilon$ are given in such a way that the matrix $(c_{ij})_{i,j}$ is comultiplicative.

The proof is similar to the one of proposition [4.1]. Among the elements $(x_{ji}^{uv})$, which define $\pi_1 \otimes \pi_1$, the only nonzero elements are

$$x_{ti}^{11} = 1, \quad \forall t \geq 2.$$

If $i \neq 1$, all the relations $\chi(i, j, k, l) = 0$ are $0 = 0$, with the exception of the relations $\chi(i, j, j, 1) = 0$ for all $j \geq 2$, which give us $0 = c_{i1}$ for all $i \geq 2$. If $i = 1$ the relations $\chi(1, j, k, l) = 0$ give us $c_{jk}c_{1l} = \delta_{kj}\delta_{l1}c_{11}$ for all $j \geq 2$ and $k, l \geq 1$.

New types of bialgebras can be constructed starting from projections of $k^n$ on different intersections of hyperplanes.

\subsection{4.2 A five dimensional noncommutative noncocommutative bialgebra}

Let $p$ be a prime number. The classification of $p$ dimensional Hopf algebras over a field of positive characteristic is still an open problem (we remind that in [32] Zhu proved that, over an algebraically closed field of characteristic zero, any $p$ dimensional Hopf algebra is isomorphic to the groupal algebra $k[\mathbb{Z}_p]$). Classifying the $p$ dimensional bialgebras seems to be a much more complicated problem.

\textbf{Remark 4.7} We notice that $y$ does not appear in the relations of $E_2^0(k)$. As $\Delta(y - 1) = (y - 1) \otimes y + 1 \otimes (y - 1)$ and $\varepsilon(y - 1) = 1$, we get that the two-sided ideal generated by $y - 1$ is also a coideal. We can add the new relation $y = 1$ in the definition of $E_2^0(k)$ and we obtain a three dimensional noncocommutative bialgebra. We denote this bialgebra with $T(k)$. Then:

\begin{itemize}
  \item As a vector space, $T(k)$ is three dimensional with $\{1, x, z\}$ a $k$-basis.
  \item The multiplication rule is given by:
        $$x^2 = x, \quad xz = zx = z^2 = 0.$$
  \item The comultiplication $\Delta$ and the counity $\varepsilon$ are given by
        $$\Delta(x) = x \otimes x, \quad \Delta(z) = x \otimes z + z \otimes 1, \quad \varepsilon(x) = 1, \quad \varepsilon(z) = 0.$$
\end{itemize}

In [11], over a field of characteristic two, two examples of three dimensional bialgebras are given. Both of them are commutative and cocommutative. Our $T(k)$ differs from one of them only by the relation $\Delta(z) = x \otimes z + z \otimes 1$ (respectively $\Delta(z) = 1 \otimes z + z \otimes 1$ in [11]). This minor change of $\Delta$ (in our case $k$ being a field of arbitrary characteristic) makes $T(k)$ noncocommutative.

$T(k)^*$ is a three dimensional noncommutative bialgebra. It follows that $T(k) \otimes T(k)^*$ is a nine dimensional noncommutative and noncocommutative bialgebra.
If in the preceding remark we have constructed a noncocommutative but commutative three dimensional bialgebra, now we shall construct a special $R$ such that $B(R)$ is a five dimensional noncommutative noncocommutative bialgebra.

**Proposition 4.8** Let $k$ be a field and $R$ be the matrix of $M_4(k)$ given by

$$R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

Then:

1. $R$ is a solution of the Hopf equation if and only if $k$ has the characteristic two. In this case $R$ is commutative.

2. If $\text{char}(k) = 2$, then the bialgebra $B(R)$ is the free algebra generated by $x$, $y$, $z$ with the relations:

$$x^2 = x, \quad y^2 = z^2 = yx = yz = 0, \quad xy = y, \quad xz = zx = 0.$$  

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by:

$$\Delta(x) = x \otimes x + y \otimes z, \quad \Delta(y) = x \otimes y + y \otimes x + y \otimes y, \quad \Delta(z) = z \otimes x + x \otimes z + zy \otimes z,$$

$$\varepsilon(x) = 1, \quad \varepsilon(y) = \varepsilon(z) = 0.$$

Furthermore, $\dim_k(B(R)) = 5$.

**Proof** 1. By a direct computation we obtain:

$$R^{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$R^{23} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$R^{13} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
It follows that:

\[
R^{12}R^{23} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

and

\[
R^{23}R^{13}R^{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

where \( \alpha = 1 + 1 \). Hence, \( R \) is a solution of the Hopf equation if and only if \( \text{char}(k) = 2 \). In this case we also have that

\[
R^{12}R^{13} = R^{13}R^{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

i.e. \( R \) is commutative.

2. Suppose that \( \text{char}(k) = 2 \). Starting as in the proof of the above propositions, with a basis in a two dimensional vector space, we obtain that among \( (x^{kl}_{ij}) \) the only nonzero elements are

\[
x^{11}_{11} = x^{22}_{22} = x^{21}_{21} = x^{12}_{12} = x^{21}_{12} = 1.
\]

Now, the relations \( \chi(i, j, k, l) = 0 \), written in the lexicografical order are:

\[
c_{11}c_{11} = c_{11}, \quad c_{11}c_{12} = c_{12}, \quad c_{12}c_{11} = 0, \quad c_{12}c_{12} = 0, \\
c_{21}c_{11} + c_{11}c_{21} = 0, \quad c_{21}c_{12} + c_{11}c_{22} = c_{11}, \\
c_{22}c_{11} + c_{12}c_{21} = c_{11}, \quad c_{22}c_{12} + c_{12}c_{22} = c_{12}, \\
c_{11}c_{21} = c_{21}, \quad c_{11}c_{22} = c_{22}, \quad c_{12}c_{21} = 0, \quad c_{12}c_{22} = 0, \\
c_{21}c_{21} = 0, \quad c_{21}c_{22} = c_{21}, \quad c_{22}c_{21} = c_{21}, \quad c_{22}c_{22} = c_{22}.
\]

Now, if we denote \( c_{11} = x, \quad c_{12} = y, \quad c_{21} = z, \quad c_{22} = t \) and using that \( \text{char}(k) = 2 \) we get the following relations:

\[
x^2 = x, \quad y^2 = z^2 = yx = yz = 0, \quad xy = y, \quad xz = zx = z, \\
zy = x + t, \quad t^2 = t, \quad xt = t, \quad tx = x, \\
yt = 0, \quad ty = y, \quad zt = tz = z.
\]
So, \( t \) is in the free algebra generated by \( x, y, z \) and
\[
    t = zy - x.
\]
If we substitute \( t \) in all the relations in which \( t \) is involved, then these become identities. The relations given in the statement of the proposition remain. The formula for \( \Delta \) follows, as the matrix
\[
    \begin{pmatrix}
        x & y \\
        z & t
    \end{pmatrix}
\]
was comultiplicative.

We shall prove now that \( \dim_k(B(R)) = 5 \), more exactly we will show in an elementary way (without the diamond lemma) that \( \{1, x, y, z, zy\} \) is a \( k \)-basis for \( B(R) \). From the relations which define \( B(R) \) we obtain:
\[
    x(zy) = zy, \quad (zy)^2 = (zy)x = y(zy) = (zy)y = z(zy) = (zy)z = 0.
\]
All these relations give us that \( \{1, x, y, z, zy\} \) is a sistem of generators of \( B(R) \) as a vector space over \( k \). It remain to check that they are linear independent over \( k \). Let \( a, b, c, d, e \in k \) such that
\[
    a + bx + cy + dz + e(zy) = 0.
\]
First, we multiply to the left with \( y \) and we get \( a = 0 \). Then we multiply to the right with \( z \) and we obtain that \( b = 0 \). Now we multiply to the right with \( x \) and we get \( d = 0 \). If we multiply now to the left with \( z \) we get \( c = 0 \), and \( e = 0 \) follows. Hence \( \{1, x, y, z, zy\} \) is a \( k \)-basis for \( B(R) \).

\[\square\]

**Remarks 4.9** 1. The bialgebra \( B(R) \) constructed in the above proposition is not a Hopf algebra. We can localize \( B(R) \) and we get a Hopf algebra which is isomorphic to the grupal Hopf algebra \( k[G] \), where \( G \) is a group with two elements. Indeed, let \( S \) be a potential antipode. Then:
\[
    S(x)x + S(y)z = 1, \quad S(x)y + S(y)t = 0.
\]
If we multiply the second equation to the right with \( z \) we get \( S(y)z = 0 \), so \( S(x)x = 1 \). But \( x^2 = x \), so \( x = 1 \) and then \( y = 0 \), \( t = 1 \). We obtain the Hopf algebra \( k < z \mid z^2 = 0 >, \Delta(z) = z \otimes 1 + 1 \otimes z, \varepsilon(z) = 0 \). If we denote \( g = z + 1 \) then \( g^2 = 1 \), \( \Delta(g) = g \otimes g, \varepsilon(g) = 1 \), hence \( B(R) \) is isomorphic to the Hopf algebra \( k[G] \), where \( G = \{1, g\} \) is a group with two elements.

2. Directly from the proof, we obtain an elementary definition for the bialgebra \( B(R) \):
- As a vector space \( B(R) \) is five dimensional with \( \{1, x, y, z, t\} \) a \( k \)-basis.
- The multiplication rule is given by:
\[
    x^2 = x, \quad y^2 = z^2 = 0, \quad t^2 = t, \quad xy = y, \quad yx = 0, \quad xz = zx = z, \quad xt = t, \quad tx = x,
\]
yz = 0, \quad zy = x + t, \quad yt = 0, \quad ty = y, \quad zt = tz = z.

• The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given in such way that the matrix

\[
\begin{pmatrix}
x & y \\
z & t
\end{pmatrix}
\]

is comultiplicative.

3. As \( R \) is commutative we can construct the bialgebra \( B(R) \): it is the quotient \( k[X, Z]/(X^2 - X, Z^2, XZ - Z) \) of the polynomial bialgebra \( k[X, Z] \) with the coalgebra structure given by

\[
\Delta(X) = X \otimes X, \quad \Delta(Z) = X \otimes Z + Z \otimes X
\]

\[
\varepsilon(X) = 1, \quad \varepsilon(Z) = 0.
\]

4. Recently, in [10], considering quotients of the bialgebras \( B(R) \) for various solutions \( R \) of the Hopf-equation, numerous examples of finite dimensional noncommutative and nonco-commutative bialgebras are constructed. We shall present one of them, which is a quotient of \( E_0^2(k) \).

Let \( n \geq 2 \) be a natural number. Then, the two-sided ideal \( I \) of \( E_0^2(k) \) generated by \( y^n - y, zy, xy - x \) and \( yx - x \) is a biideal and \( B_{2n+1}(k) := E_0^2(k)/I \) is a \( 2n + 1 \)-dimensional noncommutative nonco-commutative bialgebra (see [10] for the proof). The bialgebra \( B_{2n+1}(k) \) can be described as follows:

• \( B_{2n+1}(k) \) is the free algebra generated by \( x, y, z \) with the relations:

\[
x^2 = x, \quad xz = zx = z^2 = zy = 0, \quad y^n = y, \quad xy = yx = x.
\]

• The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by:

\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = x \otimes z + z \otimes y
\]

\[
\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0.
\]

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