Toric extremal Kähler-Ricci solitons are Kähler-Einstein

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Abstract. In this short note, we prove that a Calabi extremal Kähler-Ricci soliton on a compact toric Kähler manifold is Einstein. This solves for the class of toric manifolds a general problem stated by the authors that they solved only under some curvature assumptions.

Introduction

Let $M^{2n}$ be a compact Kähler manifold and let $\Omega \in H^{1,1}(M)$ be a Kähler class. In the attempt to identify “special” representatives of $\Omega$, several notions of “canonical” Kähler metrics have been introduced. A natural choice are of course Kähler-Einstein metrics, generalized by extremal metrics and Kähler-Ricci solitons (KRS). Extremal metrics are defined to be critical points of the Calabi functional

$$\omega \mapsto \int_M s_\omega^2 \omega^n$$

that maps the Kähler metric $\omega$ to the $L^2$-norm of its scalar curvature. The Euler-Lagrange equation of the Calabi functional is

(1) \quad \text{grad}_\omega (s_\omega) \text{ is holomorphic.}

Kähler-Ricci solitons are Kähler metrics that satisfy the relation

(2) \quad \rho + c_\omega = L_X \omega

with their Ricci form $\rho$, for some vector field $X$ that is holomorphic and, in the compact case, is the gradient of a smooth function $f : M \to \mathbb{R}$. The KRS equation forces $\omega$ to lie in the class $2\pi c_1(M)$.

In [2] we addressed the problem whether the same $\omega \in 2\pi c_1(M)$ can be extremal and a KRS without being Einstein and we proved the following.

Theorem 1 ([2]). A compact extremal KRS with positive holomorphic sectional curvature is Kähler-Einstein.

Toric manifolds are compact Kähler $2n$-manifolds admitting an effective Hamiltonian action of an $n$-torus $T$ by Kähler automorphism. Although in an algebraic geometric context, Fulton calls them a “remarkably fertile testing ground for general theories” and, also from the Kähler geometric point of view, their richness of symmetries makes them a large park of examples.

As compact symplectic manifolds, they are characterized by the image of their moment map, that is a Delzant polytope, i.e. a convex polytope $\Delta \subset \mathbb{R}^n$ with

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certain combinatoric properties. Given a compact symplectic toric manifold with moment image \( \Delta \), all possible compatible complex structure are described by a single function, as we explain below.

The \( \mathbb{T} \)-invariant Kähler geometry on a dense subset is well described in the coordinates given by the moment map itself. In these coordinates, the extremal condition has a particularly simple description, see e.g. [1].

Separately, it is known that every toric Fano manifolds admits a KRS, see e.g. [4] and references therein where, in addition, Donaldson explains also the relation between the soliton field \( X \) and the Delzant polytope. The existence of extremal metrics in the toric setting is discussed in [1].

The purpose of this note is to prove the following result.

**Theorem 2.** A compact toric Calabi-extremal Kähler-Ricci soliton is Kähler-Einstein.

This solves the problem stated in [2] for the class of toric Kähler metrics, that can have holomorphic sectional curvature of any sign and so are not included in Theorem [1].

The proof of Theorem 2 is based on the combinatoric properties of Delzant polytopes and the boundary behavior of the Abreu potential. The problem in its full generality remains open.

**Problem.** Prove that every extremal Kähler-Ricci soliton is Einstein or find a counterexample.

Another class of manifold related to toric Kähler manifolds is given by toric bundles, where the existence of KRS has been studied in [5]. It would be interesting to apply the techniques of toric geometry from [1, 4] to study the existence of extremal or constant scalar curvature Kähler metrics in this class of manifolds and establish an analogue of Theorem 2.

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**1. Proof of Theorem 2**

Let \( (M, g, \omega) \) be a toric Kähler manifold, with moment map \( \mu: M \to \Delta = \mu(M) \subset \mathbb{R}^n \). The moment image can be written as

\[
\Delta = \{ x \in \mathbb{R}^n : \ell_k(x) \geq b_k, 1 \leq k \leq d \}
\]

as intersection of the \( d \) half-spaces \( \{ x \in \mathbb{R}^n : \ell_k(x) - b_k \geq 0 \} \).

The linear functions \( \ell_k \) are defined by \( \ell_k(x) = \langle u_k, x \rangle \), where \( u_k \) is the normal to the facet \( \{ \ell_k(x) = 0 \} \cap \Delta \). The combinatoric property of being Delzant implies the following.

**Lemma 1.1.** Let \( \Delta \) be a Delzant polytope in \( \mathbb{R}^n \). Then the vertices of \( \Delta \) cannot lie on the any affine hyperplane.
Proof. Let \( P \) be a vertex of \( \Delta \). By definition of Delzant polytope, the exactly \( n \) edges meeting at \( P \) are of the form \( tv_i \) for \( t \in [0, a_i] \) and the \( v_i \) can be taken to be a basis of \( \mathbb{Z}^n \). Further \( n \) vertices are of the form \( P_i = a_i v_i \) and they cannot lie on the same affine hyperplane of \( \mathbb{R}^n \) as the \( v_i \) are linearly independent over \( \mathbb{R} \). \( \square \)

Given a compact toric symplectic manifold \((M, \omega)\) with Delzant polytope \( \Delta \), consider the dense subset \( M^0 = \{ p \in M : \text{the } T\text{-action is free at } p \} \simeq \Delta^0 \times \mathbb{T} \), where \( \Delta^0 \) is the interior of \( \Delta \) and \((x, y) \in \Delta^0 \times \mathbb{T}\) are the symplectic coordinates. In these coordinates, the \( T \)-action is just the group multiplication on the second component. In particular, \( T \)-invariant tensor fields on \( M^0 \) depend only on \( x \in \Delta^0 \).

All \( T \)-invariant complex structures compatible with \( \omega \) are determined by the Abreu potential, a function \( g : \Delta^0 \rightarrow \mathbb{R} \) given by

\[
2g(x) = \sum \ell_k(x) \log \ell_k(x) + h(x),
\]
on the interior of \( \Delta \), where the \( \ell_k \) are from (3) and \( h \) is a smooth function on \( \Delta \).

In the \((x, y)\)-coordinates, the symplectic form is the canonical \( \omega = dx_i \wedge dy_i \) and the Kähler metric corresponding \( g \) as in (4) is \( g_{ij}(x) dx_i \cdot dx_j \), where \( G = (g_{ij}) \) is the (Euclidean) Hessian of \( g \). The matrix \( G \) has to be singular on the boundary of \( \Delta \) in order for the metric to extend smoothly on the whole \( M \). However, it is possible to describe the behavior of \( G \) on the vertices of \( \Delta \).

Lemma 1.2. The inverse of the Hessian matrix \( G \) vanishes at the vertices of \( \Delta \).

Proof. Without loss of generality, up to translations and to a transformation of \( \text{SL}(n, \mathbb{Z}) \), we can assume that 0 is a vertex and that the edges meeting there are the coordinate axes \( x_1, \ldots, x_n \).

The transformed polytope is then given by

\[
\Delta = \bigcap_{i=1}^n \{ x \in \mathbb{R}^n : x_i \geq 0 \} \cap \bigcap_{i=n+1}^d \{ x \in \mathbb{R}^n : \ell_k(x) \geq 0 \}
\]
and the linear functions \( \ell_k \) do not vanish at zero.

The Abreu potential \( g \) is given by

\[
2g(x) = \sum_{i=1}^n x_i \log x_i + \sum_{i=n+1}^d \ell_i(x) \log \ell_i(x) + h(x)
\]
and its Hessian matrix is

\[
G_{ij} = \frac{\delta_{ij}}{x_j} + \tilde{h}_{ij}(x)
\]

where the function \( \tilde{h}_{ij} \) is given by

\[
\tilde{h}_{ij} = \sum_{k=n+1}^d \frac{\ell_k(x) \ell_{k,j}(x)}{\ell_k(x)} + h_{ij}.
\]

From [1 Thm. 2.8], the determinant of \( G \) is given by

\[
\frac{1}{\det G} = \delta(x)x_1 \cdots x_n \cdot \ell_{n+1}(x) \cdots \ell_d(x)
\]
for some function $\delta$ strictly positive and smooth on the whole $\Delta$.

The entry $g^{ij}$ of $G^{-1}$ is given by

$$g^{ij}(x) = \frac{1}{\det G} \text{cof}(G)_{ij}$$

where $\text{cof}(G)$ is the cofactor matrix of $G$.

The conclusion follows from the claim that

$$\text{cof}(G)_{ij} = o\left(\frac{1}{x_1 \cdots x_n}\right).$$

From (5) one can see that, after eliminating the $i$-th row and the $j$-th column, the variables $x_i$ and $x_j$ can appear at the denominator only in the derivatives of $\tilde{h}$, but from (6) we see that their limit for $x \to 0$ is finite, so the claim is true. $\square$

Abreu’s characterization [1] of toric extremal metrics relies on the fact that a $T$-invariant function has a holomorphic gradient if, and only if, it is an affine function in the symplectic coordinates. We use this on the scalar curvature and on the potential $f$ of $X$.

Proof of Theorem 2. From the preserved quantity (in our notation)

$$s + |\nabla f|^2 + 2f = \text{const}$$

that holds for every Ricci soliton, see e.g. [3], plus the extremal assumption, it follows that both $f$ and $|\nabla f|^2$ are affine functions in the interior of $\Delta$.

If $f = a \cdot x$, then one has that

$$|\nabla f|^2 = a^T G^{-1}(x) a$$

is an affine function as well. If we consider its extension to the whole $\mathbb{R}^n$, it is zero in all the vertices of $\Delta$ by the Lemma 1.2. On the other hand, the zeros of a nonzero affine function is a proper affine hyperplane, so by Lemma 1.1 we can conclude that the length of $X$ must be the zero function. So $X = 0$ and the metric is Einstein. $\square$

References

[1] M. Abreu, $Kähler$ geometry of toric manifolds in symplectic coordinates, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 1–24.

[2] Simone Calamai and David Petrecca, On Calabi extremal $Kähler$-Ricci solitons, Proc. Amer. Math. Soc. 144 (2016), no. 2, 813–821. MR 3430856

[3] B. Chow et al., The Ricci Flow: Techniques and Applications: Geometric Aspects, Mathematical surveys and monographs, vol. 135, American Mathematical Society, 2007.

[4] S. K. Donaldson, $Kähler$ geometry on toric manifolds, and some other manifolds with large symmetry, arXiv:0803.0985.

[5] F. Podestà and A. Spiro, $Kähler$-Ricci solitons on homogeneous toric bundles, J. Reine Angew. Math. 642 (2010), 109–127. MR 2658183

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