BRST Cohomology in Quantum Affine Algebra $U_q(\widehat{sl}_2)$

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ABSTRACT

Using free field representation of quantum affine algebra $U_q(\widehat{sl}_2)$, we investigate the structure of the Fock modules over $U_q(\widehat{sl}_2)$. The analysis is based on a $q$-analog of the BRST formalism given by Bernard and Felder in the affine Kac-Moody algebra $\widehat{sl}_2$. We give an explicit construction of the singular vectors using the BRST charge. By the same cohomology analysis as the classical case ($q = 1$), we obtain the irreducible highest weight representation space as a nontrivial cohomology group. This enables us to calculate a trace of the $q$-vertex operators over this space.
1. Introduction

Recently, a $q$-analog of the Knizhnik-Zamolodchikov ($q$-KZ) equation was discussed by Frenkel and Reshetikhin.\cite{1} They argued that the connection matrix of the solution of $q$-KZ equation provides the elliptic solution of quantum Yang-Baxter equation of the face type. In order to construct the solutions of $q$-KZ in the analogous way to the Feigin-Fuchs construction in conformal field theory, some free field representations of $U_q(\widehat{sl}_2)$ were considered by several people for an arbitrary representation level $k$.\cite{2,3,4}

At the same time, some of the exactly solvable lattice models in two dimension were reformulated based upon $U_q(\widehat{sl}_2)$.\cite{5} This formulation allows explicit evaluation of form factors of local operators, as a trace of certain $q$-vertex operators over the irreducible highest weight representation (IHWR) space of $U_q(\widehat{sl}_2)$.

In this letter, we discuss a $q$-analog of the bosonized version of the Bernard-Felder’s BRST formalism given for $\widehat{sl}_2$\cite{6,7,8} and investigate the structure of the Fock modules over $U_q(\widehat{sl}_2)$. We give a construction of the singular vectors and a picture fixing procedure, which restricts the Fock module to the one isomorphic to the expected $q$-analog of the Wakimoto module. We then obtain IHWR of $U_q(\widehat{sl}_2)$ as the BRST cohomology group in a complex of the Fock modules. As an application, we discuss a general formula which allows the calculation of traces of the $q$-vertex operators over IHWR. This formula is useful for the construction of the solutions of $q$-KZ equation (i.e. form factors) in the exactly solvable models.

2. Preliminaries

The quantum affine algebra $U_q(\widehat{sl}_2)$ is realized by the three bosonic fields $\Phi, \phi$ and $\chi$. These fields are given by\cite{2}

$$X(L; M, N|z; \alpha) = -\sum_{n \neq 0} \frac{[Ln]\alpha X,n}{[Mn][Nn]} z^{-n} q^{n|\alpha} + \frac{L \alpha X,0}{MN} \log z + \frac{LQX}{MN}$$

(1)

for $X = \Phi, \phi$ and $\chi$ and $L, M, N \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{R}$. Here as usual, $|q| < 1$ and

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad \text{for } m \in \mathbb{Z},$$

(2)
for $m \in \mathbb{Z}$. We also use the notation

$$X(N|z; \alpha) = X(L; L, N|z; \alpha).$$

The above three fields are quantized by imposing the following commutation relations:

$$[a_{\Phi,n}, a_{\Phi,m}] = \delta_{n+m,0} \frac{2n}{n}[(k+2)n], \quad [\tilde{a}_{\Phi,0}, Q_{\Phi}] = 2(k+2),$$

$$[a_{\phi,n}, a_{\phi,m}] = -\delta_{n+m,0} \frac{2n}{n}[2n], \quad [\tilde{a}_{\phi,0}, Q_{\phi}] = -4,$$

$$[a_{\chi,n}, a_{\chi,m}] = \delta_{n+m,0} \frac{2n}{n}[2n], \quad [\tilde{a}_{\chi,0}, Q_{\chi}] = 4,$$

with $\tilde{a}_{X,0} = \frac{q-q^{-1}}{2 \log q} a_{X,0}$ for $X = \Phi, \phi$ and $\chi$, and others commute.

Let us define the currents $J^3(z), J^\pm(z)$ by

$$J^3(z) = \partial_z \Phi(k+2 |q^{-2}z; -1) + \partial_z \phi(2 |q^{-k-2}z; -\frac{k}{2} - 1),$$

$$J^+(z) = -\left[ 1 \partial_z \exp \left\{ -\chi(k+2 |q^{-2}z; 0) \right\} \right. \exp \left\{ -\phi(2 |q^{-k-2}z; 1) \right\},$$

$$J^-(z) = \left[ k+2 \partial_z \exp \left\{ \Phi(k+2 |q^{-2}z; -\frac{k}{2} - 1) + \phi(2 |q^{-k-2}z; -1) + \chi(1; 2, k+2 |q^{-k-2}z; 0) \right\} \right. \exp \left\{ -\Phi(k+2 |q^{-2}z; \frac{k}{2} + 1) + \chi(1; 2, k+2 |q^{-k-2}z; 0) \right\}. $$

Here we use the $q$-difference operator with parameter $n \in \mathbb{Z}_{\geq 0}$

$$n \partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n}z)}{(q - q^{-1})z}. $$

The currents in (7) as well as the auxiliary fields $\psi(z)$ and $\varphi(z)$ defined by

$$
\psi(z) = K \exp \left\{ (q - q^{-1}) \sum_{k=1}^\infty J^3_k z^{-k} \right\},
$$

$$
\varphi(z) = K \exp \left\{ (q - q^{-1}) \sum_{k=1}^\infty J^+ k z^{-k} \right\},
$$

$$
\chi(z) = K \exp \left\{ (q - q^{-1}) \sum_{k=1}^\infty J^- k z^{-k} \right\}.
$$
\[ \varphi(z) = K^{-1} \exp \left\{ -(q - q^{-1}) \sum_{k=1}^{\infty} J_{-k}^3 z^k \right\}. \]  \hspace{1cm} (10)

with \( K = q^{\tilde{a}_0} \tilde{a}_0 \) satisfy the relations of \( U_q(\hat{sl}_2) \) obtained by Drinfeld:

\[ [J_n^3, J_m^3] = \delta_{n+m,0} \frac{1}{n} [2n] \gamma^{-n} - \gamma^{-n}, \quad [J_n^3, K] = 0, \]

\[ K J_n^\pm K^{-1} = q^{\pm 2} J_n^\pm, \]

\[ [J_n^3, J_m^\pm] = \pm \frac{1}{n} [2n] \gamma^{\mp |n|/2} J_{n+m}^\pm, \]

\[ J_{n+1}^\pm J_m^\pm - q^{\pm 2} J_m^\pm J_{n+1}^\pm = q^{\pm 2} J_n^\pm J_{m+1}^\pm - J^\pm J_{m+1}^\pm J_n^\pm, \]

\[ [J_n^+, J_m^-] = \frac{1}{q - q^{-1}} (\gamma^{n-m}/2 \psi_{n+m} - \gamma^{(m-n)/2} \varphi_{n+m}), \]  \hspace{1cm} (11)

where \( \gamma^{1/2} \equiv q^{1/2} \) is the center of the algebra. Here we gave the mode expansions as follows.

\[ \sum_{n \in \mathbb{Z}} J_n^3 z^{-n-1} = J^3(z), \quad \sum_{n \in \mathbb{Z}} J_n^\pm z^{-n-1} = J^\pm(z), \]

\[ \sum_{n \in \mathbb{Z}} \psi_n z^{-n} = \psi(z), \quad \sum_{n \in \mathbb{Z}} \varphi_n z^{-n} = \varphi(z). \]  \hspace{1cm} (12)

The standard Chevalley generators \( \{ e_i, f_i, t_i \} \) are given by the identification

\[ t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = J_0^+, \quad f_1 = J_0^-, \quad e_0 t_1 = J_1^-, \quad t_1^{-1} f_0 = J_{-1}^+. \]  \hspace{1cm} (13)

3. Fock Space

The highest weight state (HWS) \(|l> \rangle = |0 \rangle \) for the level \( k \) representation of \( U_q(\hat{sl}_2) \) is generated by \(|l >= \lim_{z \to 0} \Phi_l(z)|0 \rangle \). Here the \( q \)-analog of the primary field \( \Phi_l(z) \) is defined by\(^{[2,3,4]}\)

\[ \Phi_l(z) = \exp \{ \Phi(l; 2k + 2 |q^k z; k - \frac{k}{2} + 1) \} \]  \hspace{1cm} (14)

and \(|0 \rangle \) is the vacuum state, which is annihilated by \( a_{X,n}, n \geq 0 \) for \( X = \Phi, \phi \) and \( \chi \). The HWS \(|l> \rangle \) satisfies the relations \( t_1|l >= q^l|l>, \quad t_0|l >= q^{k-l}|l> \)
\[ c_i|l> = 0 \text{ for } i = 0, 1 \text{ and } p^L_0|l> = p^h_0|l> \text{ with } p \equiv q^{2(k+2)}, \quad h = \frac{l(l+2)}{4(k+2)} \]

Here the grading operator \( L_0 = L_0^\Phi + L_0^\phi + L_0^\chi \) is given by

\[ L_0^\Phi = \frac{\tilde{a}_\Phi,0(\tilde{a}_\Phi,0 + 2)}{4(k+2)} + \sum_{n \geq 1} \frac{n^2}{[2n][(k+2)n]} a_{\Phi,-n}a_{\Phi,n}, \quad (15) \]
\[ L_0^\phi = -\frac{\tilde{a}_\phi,0(\tilde{a}_\phi,0 - 2)}{8} - \sum_{n \geq 1} \frac{n^2}{[2n]^2} a_{\phi,-n}a_{\phi,n}, \quad (16) \]
\[ L_0^\chi = \frac{\tilde{a}_\chi,0(\tilde{a}_\chi,0 + 2)}{8} + \sum_{n \geq 1} \frac{n^2}{[2n]^2} a_{\chi,-n}a_{\chi,n}. \quad (17) \]

Let \( F_{l,s,t} \) be the Fock space on the vector \( |l; s, t> \equiv e^{\frac{s}{2}Q_\phi + \frac{t}{2}Q_\chi}|l> \)

\[ F_{l,s,t} = \left\{ \prod a_{\Phi,-n} \prod a_{\phi,-n'} \prod a_{\chi,-n''}|l; s, t> \right\}, \quad (18) \]

where \( n, n' \) and \( n'' \) are positive integers, and define \( F_l = \oplus_{s,t \in \mathbb{Z}} F_{l,s,t} \). Then the highest weight \( U_q(\hat{sI}_2) \) module

\[ V(\lambda_l) = U_q(\hat{sI}_2)|l> \quad (19) \]

of the highest weight \( \lambda_l = (k-l)\Lambda_0 + l\Lambda_1 \) with \( \Lambda_0, \Lambda_1 \) being the fundamental weight is embedded in the Fock space \( F_l \).

We are interested in obtaining the \( q \)-analog of the Wakimoto module, i.e. the Fock space which coincides with the Wakimoto module for \( \hat{sI}_2 \) in the limit \( q \to 1 \).

As is in the classical case, our Fock space \( F_l \) contains some redundancies comparing with the expected \( q \)-Wakimoto module. The origin of them is in the bosonization of the \( q \) counterpart of the first order fields \( \beta \) and \( \gamma \) by \( \phi \) and \( \chi \). That is, roughly speaking, in (7), we use the fields \( \phi \) and \( \chi \) only in the combination

\[ \beta(z) = -[1 \partial_z \xi(z)] : \exp\{-\phi(2|q^{-k-2}z; -1)\} :, \quad (20) \]
\[ \gamma(z) = \eta(z) : \exp\{\phi(2|q^{-k-2}z; 0)\} :, \quad (21) \]
where
\[
\xi(z) =: \exp \{-\chi(2q^{-k-2}z; 0) \} :,
\]
\[
\eta(z) =: \exp \{\chi(2q^{-k-2}z; 0) \} :.
\] (22) (23)

Therefore the zero-mode of \(\xi(z)\), \(\xi_0 = \oint \frac{dz}{z} \xi(z)\), is not contained in the module \(V(\lambda_l)\).

Note that, as in the classical case, we have the following operator product expansion (OPE).

\[
\xi(z)\eta(w) = -\eta(w)\xi(z) \sim \frac{1}{z-w}
\] (24)

and especially, \(\{\xi_0, \eta_0\} = 1\) for \(\eta_0 = \oint dz \eta(z)\). Furthermore, we have

\[
J^3(z)\eta(w) = \eta(w)J^3(z) \sim 0, \\
J^-(z)\eta(w) = -\eta(w)J^-(z) \sim 0, \\
J^+(z)\eta(w) = -\eta(w)J^+(z) \sim 1\partial_w \left[ \frac{1}{z-w} : \exp \{-\phi(2q^{-k-2}w; 1) \} : \right].
\] (25)

These OPEs allow \(\eta_0\) to commute with all the currents. We can therefore restrict the Fock space \(F_l\) on the kernel of \(\eta_0\).

We also have a redundancy of infinite pictures, i.e. the redundancy in the choice of the vacuum state for \(\beta\) and \(\gamma\). As in the classical case, we can fix picture by setting the constraint \(\tilde{a}_{\phi,0} + \tilde{a}_{\chi,0} = 0\) on the Fock space. This implies the constraint \(s = t\) in \(F_{l,s,t}\). Noting the fact that all the currents in (7) commute with \(\tilde{a}_{\phi,0} + \tilde{a}_{\chi,0}^{[2]}\), this constraint does not introduce any inconsistencies in our Fock space representation.

Because the above two procedure is the straightforward extension of those used in the classical theory\(^{[7,8,11]}\) we claim that our restricted Fock space \(\tilde{F}_l =\)
⊕_{s \in \mathbb{Z}} \text{Ker} \eta_0(F_{l,s,s}) \text{ is the } q\text{-Wakimoto module.}^* \text{ In the later section, we will evaluate the character for the irreducible highest weight representation of } U_q(\widehat{sl}_2) \text{ based on this restricted Fock space and will justify this point.}

4. Screening Charges and \(q\)-Vertex Operator

The \(q\)-analog of the screening operator is given by \([2,3]\)

\[ S(t) = \beta(t) : \exp\left\{ -\Phi(k + 2|q^{-2}t; -\frac{k}{2} - 1) \right\} : \quad (26) \]

This satisfies the following OPEs.

\[
\begin{align*}
J^3(z)S(t) &= S(t)J^3(z) \sim 0, \\
J^+(z)S(t) &= S(t)J^+(z) \sim 0, \\
J^-(z)S(t) &= S(t)J^-(z) \sim k+2\partial_t \left[ \frac{1}{z-t} : \exp\{ -\Phi(k + 2|q^{-2}z; \frac{k}{2} + 1) \} : \right]. \\
S(t)\eta(w) &= \eta(w)S(t) \\
&\sim 1\partial_t \left[ \frac{1}{t-w} : \exp\{ -\Phi(k + 2|q^{-2}w; -\frac{k}{2} - 1) - \phi(2|q^{-k-2}w; -1) \} : \right].
\end{align*}
\]

These OPEs allows the screening charge \(\int_0^c dt S(t)\) to commute with all the currents and the screening operator \(S(t)\) to commute with \(\eta_0\). Here the Jackson integral is defined by

\[
\begin{align*}
\int_0^c dt f(t) &= c(1-p) \sum_{m \in \mathbb{Z}} f(cp^m)p^m, \quad (28) \\
\int_0^c dt f(t) &= c(1-p) \sum_{m \in \mathbb{Z}_{\geq 0}} f(cp^m)p^m. \quad (29)
\end{align*}
\]

Now let us discuss a \(q\)-analog of the screened vertex operator.\([6]\) The \(q\)-screened

\[ \text{In Ref.}[3], \text{ Matsuo discussed the equivalent restriction of Fock space in the different bosonization scheme.} \]
vertex operator $\Phi^{(r)}_{l,m}(z) : \tilde{F}_{k = 1} \rightarrow \tilde{F}_{k = 3}$ with $2r = l_1 + l - l_3$ is defined by

$$
\Phi^{(r)}_{l,m}(z) = \int_0^{q^{-1}z} dp_1 \int_0^{q^{-1}z} dp_2 \cdots \int_0^{q^{-1}z} dp_r \Phi^{(l)}_{m}(z) S(t_1) \cdots S(t_r),
$$

(30)

where

$$
\Phi^{(l)}_{m-1}(z) = \frac{[m-1]!}{[l]!} \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \cdots \oint \frac{dw_{l-m+1}}{2\pi i}
$$

$$
\times \left[ \cdots \left[ \Phi^{(l)}_{l}(z), J^{-}(w_{1}) \right]_{q^1}, J^{-}(w_{2}) \right]_{q^2} \cdots J^{-}(w_{l-m+1}) \right]_{q^{2m}} (31)
$$

for $m = 1, 2, \ldots, l$. Here the integration regions in (30) are chosen such that the Jackson integral of any matrix elements of operators containing

$$
\Phi^{(l)}_{m}(z)S(t_{1})S(t_{2}) \cdots S(t_{j-1})_{k+2} \partial_{t_{j}} \left[ \frac{1}{w - t_{j}} : \exp \left\{ -\Phi(k + 2|q^{-2}t_{j}; \frac{k}{2} + 1) \right\} : S(t_{j+1}) \cdots S(t_{r}) \right], (32)
$$

for $j = 1, 2, \ldots, r$ and arbitrary $w$, vanishes.

As the classical integral, one can relate the above Jackson integral to the following ordered Jackson integral

$$
J_{r}(z) = \int_0^{q^{-1}z} dp_1 \int_0^{q^{-1}z} dp_2 \cdots \int_0^{q^{-1}z} dp_r \Phi^{(l)}_{m}(z) S(t_{1})S(t_{2}) \cdots S(t_{r}).
$$

(33)

Note that this ordered integral makes the matrix element of the operators containing (32) vanish, too. In (33), if the variable $t_i$ passes $t_j$, one gets the factor $A_s(t_i/t_j)$ defined by

$$
S(t_i)S(t_j) = A_s(t_i/t_j)S(t_j)S(t_i).
$$

(34)

From (26), one gets

$$
A_s(t_i/t_j) = \left( \frac{t_i}{t_j} \right)^{\frac{2}{t_j}} \frac{\vartheta_1 (\frac{1}{2\pi i} \ln(q^2 t_i/t_j)|\tau)}{\vartheta_1 (\frac{1}{2\pi i} \ln(q^{-2} t_i/t_j)|\tau)}, (35)
$$

8
with \( \tau \equiv \frac{\ln p}{2\pi i} \). In the classical limit \( q \to 1 \), \( \mathcal{A}_s \) goes to \( e^{2\pi i} \). Noting the fact that the function \( \mathcal{A}_s(z) \) is a pseudo constant, i.e. \( \mathcal{A}_s(p^m z) = \mathcal{A}_s(z) \) for \( m \in \mathbb{Z} \) and the definition of the Jackson integral, one gets

\[
\int_{0}^{c} d\rho \tau \mathcal{A}_s(\rho) f(\rho) = \mathcal{A}_s(c) \int_{0}^{c} d\rho f(\rho).
\]

Using this repeatedly, we obtain

\[
\Phi^{(r)}_{l,m}(z) = \prod_{j=1}^{r} \frac{1 - \mathcal{A}_s^j}{1 - \mathcal{A}_s} \mathcal{J}_r(z),
\]

where \( \mathcal{A}_s \equiv \mathcal{A}_s(q^{2+\varepsilon}) \) with \( \varepsilon > 0 \) being a regularization parameter.

Let us finally make a connection of our \( q \)-screened vertex operator \( \Phi^{(r)}_{l,m}(z) \) with the (type I) \( q \)-vertex operator \( \Phi^{\lambda_{l_3} V^{(l)}}_{\lambda_{l_1}}(z) : \tilde{F}_{l_1} \longrightarrow \tilde{F}_{l_3} \otimes V^{(l)}_{\lambda_{l_1}} \) in Ref.[5]. Here \( V^{(l)} \) denotes the \((l+1)\)-dimensional \( U_q(\hat{sl}_2) \) module with basis \( v^{(l)}_m, m = 0, 1, \ldots, l \).

The \( q \)-vertex operator \( \Phi^{\lambda_{l_3} V^{(l)}}_{\lambda_{l_1}}(z) \) is given by the \( q \)-screened vertex as follows.

\[
\Phi^{\lambda_{l_3} V^{(l)}}_{\lambda_{l_1}}(z) = z^{h_{l_3}} - h_{l_1} \Phi^{\lambda_{l_3} V^{(l)}}_{\lambda_{l_1}}(z),
\]

\[
\tilde{\Phi}^{\lambda_{l_3} V^{(l)}}_{\lambda_{l_1}}(z) = \sum_{m=0}^{l} \Phi^{(r)}_{l,m}(z) \otimes v^{(l)}_m.
\]

See Ref.[5] for the detailed properties of \( \tilde{\Phi}^{\lambda_{l_3} V^{(l)}}_{\lambda_{l_1}}(z) \).

5. BRST Cohomology

In this section, we show that our Fock modules (18) are reducible for the special value of the spin \( l \) due to the existence of singular vectors. Constructing these singular vectors explicitly, we discuss their resolution in the Fock spaces.

Let us define the BRST charge \( Q_n, n \in \mathbb{Z}_{>0} \) as follows.

\[
Q_n = \frac{1 - \mathcal{A}_s^n}{1 - \mathcal{A}_s} \int_{0}^{c} d\rho \tau \int_{0}^{c} d\rho t S(t) S(t_2) \cdots S(t_n) dt t_2 \cdots S(t_n).
\]
The BRST current

\[ J_n(t) = \frac{1 - A^n_s}{1 - A^s} \int_0^{q^2 t} \cdots \int_0^{q^2 t} d_p t_2 \cdots d_p t_n S(t) S(t_2) \cdots S(t_n) \]  

(41)

is a single valued function on \( \tilde{F}_{n,n'} \equiv \tilde{F}_{l_{n,n'}} \). Here \( l_{n,n'} = n - n' \frac{P}{P'} - 1 \) with non-negative integers \( n, n' \) and coprime positive integers \( P, P' \) satisfying \( \frac{P}{P'} \equiv k + 2 \).

It has been proved by Malikov \[15\] that the Verma module \( V(\lambda_l) \) over \( U_q(\hat{sl}_2) \) are reducible if and only if its classical counterpart is reducible, i.e. \( l \) satisfies the Kac-Kazhdan equation, \( l = n - n' \frac{P}{P'} - 1 \) with certain non-negative integers \( n, n' \).

In the following paragraph, we are concentrate on this case.

We have

**Proposition**

(i) \( Q_n Q_{P-n} = Q_{P-n} Q_n = 0 \).

(ii) The following infinite sequence

\[ \cdots \rightarrow Q_n \tilde{F}_{-n+2P,n'} \rightarrow Q_{P-n} \tilde{F}_{n,n'} \rightarrow Q_n \tilde{F}_{-n-n'} \rightarrow Q_{P-n} \tilde{F}_{n-2P,n'} \rightarrow Q_n \tilde{F}_{-n-2P,n'} \rightarrow Q_{P-n} \rightarrow \cdots \]  

(42)

is a complex.

The first statement follows from \( Q_n Q_{P-n} = Q_{P-n} Q_n = Q_P \) and \( A^P_s = 1 \).

Here the latter equality is proved for the cases \( k = 0, 1, 2 \) by using Riemann’s theta identity.\[17\] For example, in the case \( k = 2 \), the following identity is used.

\[ 2\vartheta_1(x|\tau)^4 = \sum_{\text{all spin structures}} e_\alpha \vartheta[\alpha] (2x|\tau) \vartheta[\alpha](0|\tau)^3, \]  

(43)

where \( e_{00} = e_{10} = 1 = -e_{01} = -e_{11} \). The remaining cases \( k \neq 0, 1, 2 \) are conjectured. The second statement follows from the fact that the BRST charge \( Q_n \) commutes with \( \eta_0 \) and the constraint \( \tilde{a}_{\phi,0} + \tilde{a}_{\chi,0} = 0 \).

We also have
Lemma

The singular vector $Q_n|n, -n' > (\neq 0)$ exist for positive integers $n'$ and $n$ with $1 \leq n \leq P - 1$. In addition, for the same integers $n'$ and $n$, there exists a state (cosingular vector) $|w \rangle \in \hat{F}_{n,n'}$ such that $Q_n|w \rangle = | -n, n' >$. Here $|n, n' >= |l_{n,n'} >$.

The proof follows from the analogous calculation to the classical case.\[^6\]

\[
Q_n|n, -n' > = \frac{1 - A^n_n}{1 - A^n_s} \int_0^\infty \cdots \int_0^\infty d_p v_1 d_p v_2 \cdots d_p v_n \prod_{j=2}^n v_j^{-\frac{1}{2}} \frac{(q^{-2}v_j; p)_\infty}{(q^2v_j; p)_\infty} 
\times \prod_{2 \leq i < j \leq n} v_j^{\frac{1}{2} - \frac{1}{2}} \frac{(q^{-2}v_j/v_i; p)_\infty}{(q^2v_j/v_i; p)_\infty} \beta(t) \prod_{j=2}^n \beta(tv_j) 
\times \exp \left\{ \sum_{m \geq 1} \frac{a_{\Phi,m}}{[ (k + 2)m ]} q^{-(\frac{k}{2} + 4)m} \Gamma_m (1 + \sum_{j=2}^n v_j^m) \right\} | -n, -n' > ,
\]

where we made a change of the variables $t_j = tv_j$, $j = 2, 3, ..., n$. Note $\beta_m|l > = 0$ with $\beta_m = \int \frac{dz}{2\pi i} z^m \beta(z)$ for $m \geq 0$ and the formula $\int_0^\infty d_p tt^l = (1 - p)^{l+1} \delta(p^{l+1})$, where $\delta(p^{l+1})$ is a delta function vanishing unless $l \neq -1$.\[^13\] We hence have a non-vanishing condition $n' > 0$ for the $t$ integration. Furthermore, we can evaluate the inner product of $Q_n|n, -n' >$ with a covector $< -n, -n'| (\gamma_n')^n$, where $< -n, -n'| -n, -n' > = 1$ and $\gamma_n = \int \frac{dz}{2\pi i} z^n \gamma(z)$. This is possible due to the formula established by Askey, Kadell and Aomoto.\[^14\]

\[
\int_0^\infty d_p t_1 \int_0^\infty d_p t_2 \cdots \int_0^\infty d_p t_r \prod_{j=1}^r t_j^{x-1} \frac{(p t_j; p)_\infty}{(p^{1-K} t_j; p)_\infty} \prod_{1 \leq i < j \leq r} t_i^{2K-1} \frac{(p^{1-K} t_j/t_i; p)_\infty}{(p^K t_j/t_i; p)_\infty} (t_i - t_j)
\]

for $A = \sum_{j=1}^r (x + 2K(r - 1) + 1 - j)(1 + (j - 1)K)$. Here, $(a; p)_\infty = \prod_{s=0}^\infty (1 - a p^s)$.
and

\[ \Gamma_p(x) = \frac{(p; p)_{\infty}}{(p^x; p)_{\infty}} (1 - p)^{1-x} \]  

(46)

is a \( q \)-gamma function.\textsuperscript{[14]} The result is given by

\[ < -n, -n' | (\gamma_n)' n Q_n | n, -n' > \sim \prod_{j=1}^{n} \frac{1 - A^j}{1 - A^j} \prod_{j=1}^{n-1} \frac{\Gamma_p(-j) \Gamma_p(j)}{\Gamma_p(j+2)}. \]

This implies the non-vanishing condition \( 1 \leq n \leq P - 1 \). The second part of the statement follows from the same calculation.\textsuperscript{[6]}

In the case \( 1 \leq n \leq P - 1 \) and \( n' = 1 \), we obtained consistent expressions of the singular vectors with those obtained by Malikov\textsuperscript{[15]} in the Chevalley basis. To show this coincidence in the other cases may be an interesting problem.

This lemma as well as the proposition indicates that the same \( U_q(\hat{sl}_2) \) submodule structure generated by the singular and cosingular vectors as the classical case may exist in our Fock space \( \hat{F}_l \). We thus claim that the IHWR \( H_{n,n'} \) of \( U_q(\hat{sl}_2) \) with the highest weight \( \lambda_{n,n'} \) is given by the BRST cohomology group as follows.

\[ \text{Ker} Q_n[2^s] \big/ \text{Im} Q_n[2^s-1] = \begin{cases} 0 & \text{for } s \neq 0 \\ H_{n,n'} & \text{for } s = 0 \end{cases}, \]  

(47)

where \( Q_n[2a] = Q_n \) and \( Q_n[2a-1] = Q_{P-n} \) with \( a \in \mathbb{Z} \). The proof may be carried out in the same way as in the classical case by applying Jantzen filtrations.\textsuperscript{[6,16]}

As a corollary, we have a formula for the trace over the IHWR of \( U_q(\hat{sl}_2) \)

\[ \text{Tr}_{H_{n,n'}} \mathcal{O} = \sum_{s \in \mathbb{Z}} (-)^s \text{Tr}_{\hat{F}_{n,n'}} \mathcal{O}^{[s]}, \]  

(48)

where the graded physical operator \( \mathcal{O}^{[s]} \) is defined recursively by the relations

\[ Q_n^{[s]} \mathcal{O}^{[s]} = \mathcal{O}^{[s+1]} Q_n^{[s]}, \quad \mathcal{O}^{[0]} = \mathcal{O}. \]  

(49)
The $q$-screened vertex operator $\Phi_{l,m}^{(r)}(z)$ is a typical physical operator. The BRST relations for this vertex are obtained from (30) and (40) as follows.

$$Q_{n_3}\Phi_{l,m}^{(r)}(z) = A_{l}^{n_3}\Phi_{l,m}^{(r+n_3-n_1)}(z)Q_{n_1},$$

$$QP_{-n_3}\Phi_{l,m}^{(r+n_3-n_1)}(z) = A_{l}^{P-n_3}\Phi_{l,m}^{(r)}(z)QP_{-n_1},$$

where $A_{l} \equiv A_{l}(q^{-l+\varepsilon})$ is defined by the relation $S(t)\Phi_{l}(z) = A_{l}(t/z)\Phi_{l}(z)S(t)$. By the explicit calculation, we have

$$A_{l}(z) = z^{-\frac{l}{k+2}}\vartheta_1\left(\frac{1}{2\pi i}\ln(q^{-l}z)|\tau\right)\vartheta_1\left(\frac{1}{2\pi i}\ln(q^{l}z)|\tau\right).$$

This is again a pseudo constant.

6. Trace over IHWR

We here discuss an application of the trace formula (48). We give an explicit picture fixing procedure mentioned previously.

The goal is to calculate the following trace

$$\text{Tr}_{\mathcal{H}_{n,n'}}\left(\zeta^{L_0-c'}\prod_{i=1}^{N}\Phi_{l_i,m_i}^{(r_i)}(z_i)\right)$$

where $c = \frac{3k}{k+2}$. From the charge conservation for the fields $\Phi, \phi$ and $\chi$, we obtain the selection rules

$$\sum_{i=1}^{N} l_i = 2\sum_{i=1}^{N} r_i, \quad \sum_{i=1}^{N} m_i = \sum_{i=1}^{N} r_i.$$

These are the same conditions as in the classical case.

We claim that the trace over the restricted Fock space is evaluated as

$$\text{Tr}_{\tilde{F}_{n,n'}^{[s]}}O^{[s]} = \text{Tr}_{\tilde{F}_{n,n'}^{[s]}}\left(\xi(w_0)\int\frac{dw}{2\pi i}\eta(w)O^{[s]}\right)|_{\tilde{a}_0=0,\tilde{a}_\chi=0}.$$
This is a direct extension of the classical one in Ref.[8]. By using this, we obtain a general formula for the trace of operators in the $\phi \chi$ sector over the restricted Fock space $\tilde{F}^{\phi \chi}$, where $\tilde{F}_{n,n'} = F^{\Phi}_{n,n'} \otimes \tilde{F}^{\phi \chi}$. The result is

$$\text{Tr}_{\tilde{F}^{\phi \chi}} \left( \zeta^{L_0^+ + L_0^+} z^{-\frac{\alpha}{2\pi} \phi \cdot 0} \prod_{i=1}^{N} \sum : e^{-\chi(2|q^a|x_i;0)} : \prod_{j=1}^{N} \sum : e^{-\chi(2|q^b|y_j;0)} : \prod_{r=1}^{M} : e^{\phi(2|q^c|z_r;d_r)} : \prod_{s=1}^{M} : e^{-\phi(2|q^d|w_s;h_s)} : \right)$$

$$= \frac{z^{-1} \prod_{r} Z_r}{\prod_{0 \leq i < i' \leq N} E(X_i, X_{i'}) \prod_{j < j'} E(Y_j, Y_{j'}) \prod_{r,s} E(\tau_r, \tau_s; d_r + h_s)} \times \frac{\prod_{0 \leq i \leq N} \vartheta \left( \sum_0^N X_i - \sum_1^N Y_j + \sum_1^M Z_r - \sum_1^M W_s - z^2 | \tau \right)}{\prod_{i=0}^{N} \vartheta \left( \sum_0^N X_i - \sum_1^N Y_j + \sum_1^M Z_r - \sum_1^M W_s - z^2 | \tau \right)}$$

$$= \frac{z^{-1} \prod_{r} Z_r}{\prod_{0 \leq i < i' \leq N} E(X_i, X_{i'}) \prod_{j < j'} E(Y_j, Y_{j'}) \prod_{r,s} E(\tau_r, \tau_s; d_r + h_s)} \times \frac{\prod_{0 \leq i \leq N} \vartheta \left( \sum_0^N X_i - \sum_1^N Y_j + \sum_1^M Z_r - \sum_1^M W_s - z^2 | \tau \right)}{\prod_{i=0}^{N} \vartheta \left( \sum_0^N X_i - \sum_1^N Y_j + \sum_1^M Z_r - \sum_1^M W_s - z^2 | \tau \right)}$$

where $X_i = q^{a_i} x_i, Y_j = q^{b_j} y_j, Z_r = q^{c_r} z_r, W_s = q^{d_s} w_s, X_0 = q^{-(k+2)} w_0, \zeta \equiv e^{2\pi i \tau}$. The functions $\eta(\tau)$ and $E(x, y)$ are Dedekind’s function and the prime form, respectively. In (56), we omitted an irrelevant factor depending only on $q$ and used the abridged notations $x_i$ to express $\frac{\ln x_i}{2\pi i}$ etc. in the theta functions. We also introduced the notation

$$E_q(w, z; \alpha) = \frac{(1 - q^\alpha z/w)}{\sqrt{z/w}} \prod_{s=1}^{\infty} (1 - q^\alpha \zeta^s z/w)(1 - q^\alpha \zeta^s w/z).$$

The formula (56) is a $q$-analog of the formula in Ref.[18] for genus one.

Using this formula, the character $\chi_{n,n'}(\tau)$ of the IHWR $\mathcal{H}_{n,n'}$ is evaluated as follows.

$$\chi_{n,n'}(\tau) = \text{Tr}_{\mathcal{H}_{n,n'}} \left( \zeta^{L_0^+} \tau^{J_0^+} \right)$$

$$= \frac{1}{\vartheta(\tau^2)} \sum_{s \in \mathbb{Z}} \left[ \zeta^{PP'(s-n\frac{\alpha'}{\alpha}\frac{\beta'}{\beta})^2 z^2P(s-n\frac{\alpha'}{\alpha}\frac{\beta'}{\beta})} - \zeta^{-PP'(s+n\frac{\alpha'}{\alpha}\frac{\beta'}{\beta})^2 z^2P(s+n\frac{\alpha'}{\alpha}\frac{\beta'}{\beta})} \right],$$

with $\tilde{z} = \frac{2\ln q}{\tau + \frac{\pi i}{\alpha \beta}}$. This coincides with the Weyl-Kac character formula for $\mathcal{H}_{1/2}$ being consistent with the result in the $q$–deformed Verma module.\(^{(19)}\)

* Do not confuse $\tau$ in (56) with the one in (36) and (53).
Conclusion

We discussed a $q$-analog of Bernard-Felder’s BRST formalism for $u_q(\widehat{sl}_2)$. We defined the nilpotent BRST operator $Q_n$ and constructed the singular vectors. Combining this with the picture fixing procedure, we obtained the IHWR of the quantum affine algebra $U_q(\widehat{sl}_2)$ as the BRST cohomology group. Hence a trace of the $q$-vertex operators over IHWR was made a calculable form.

This formula allows us to construct the solutions of the $q$-KZ equation. In addition, according to Ref.[5], the evaluation of the spin-spin correlation functions as well as the form factors in the higher spin XXZ model is now possible. We will discuss these subjects elsewhere.

Acknowledgement

The author would like to thank S.Helmke, K.Kimura, A.Matsuo, A.Nakayashiki, I.Ojima and J.Shiraishi for valuable discussions.

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