Distributional conformal prediction*

Victor Chernozhukov†  Kaspar Wüthrich‡  Yinchu Zhu§

First version on arXiv: September, 17, 2019  This version: August 24, 2021

Abstract

We propose a robust method for constructing conditionally valid prediction intervals based on models for conditional distributions such as quantile and distribution regression. Our approach can be applied to important prediction problems including cross-sectional prediction, $k$-step-ahead forecasts, synthetic controls and counterfactual prediction, and individual treatment effects prediction. Our method exploits the probability integral transform and relies on permuting estimated ranks. Unlike regression residuals, ranks are independent of the predictors, allowing us to construct conditionally valid prediction intervals under heteroskedasticity. We establish approximate conditional validity under consistent estimation and provide approximate unconditional validity under model misspecification, overfitting, and with time series data. We also propose a simple “shape” adjustment of our baseline method that yields optimal prediction intervals.

Keywords: prediction intervals, quantile regression, distribution regression, conditional validity, model-free validity

*We are grateful to the Editor, two anonymous referees, Dimitris Politis, and Allan Timmermann for valuable comments. Wüthrich is also affiliated with CESifo and the Ifo Institute. Chernozhukov gratefully acknowledges funding by the National Science Foundation. The usual disclaimer applies.

†Massachusetts Institute of Technology; 50 Memorial Drive, E52-361B, Cambridge, MA 02142, USA; Email: vchern@mit.edu

‡Department of Economics, University of California San Diego, 9500 Gilman Dr., La Jolla, CA 92093, USA; Email: kwuthrich@ucsd.edu

§Brandeis University; 415 South Street, Waltham, MA 02453, USA; Email: yinchuzhu@brandeis.edu
1 Introduction

We develop a robust approach for constructing prediction intervals based on models for conditional distributions. The proposed method is generic and can be implemented using a great variety of flexible and powerful methods, including conventional quantile regression (QR) (Koenker and Bassett, 1978), distribution regression (DR) (e.g., Foresi and Peracchi, 1995; Chernozhukov et al., 2013), as well as non-parametric and high-dimensional machine learning methods such as quantile neural networks (e.g., Taylor, 2000) and quantile trees and random forests (e.g., Chaudhuri and Loh, 2002; Meinshausen, 2006).

We observe data \( \{(Y_t, X_t)\}_{t=1}^T \), where \( Y_t \) is a continuous outcome of interest and \( X_t \) is a \( p \times 1 \) vector of predictors. Our task is to predict \( Y_{T+1} \) given knowledge of \( X_{T+1} \). This setting encompasses many classical cross-sectional and time series prediction problems. Moreover, our approach can be applied to synthetic control settings where the goal is to predict counterfactuals in the absence of a policy intervention (e.g., Cattaneo et al., 2019; Chernozhukov et al., 2021) and to the problem of predicting individual treatment effects (e.g., Kivaranovic et al., 2020b; Lei and Candès, 2020).

With iid (or exchangeable data), standard conformal prediction methods, which are based on modeling the conditional mean, yield prediction intervals \( \hat{C}_{(1-\alpha)} \) that satisfy
\[
P \left( Y_{T+1} \in \hat{C}_{(1-\alpha)} (X_{T+1}) \right) \geq 1 - \alpha
\]
for a given miscoverage level \( \alpha \in (0, 1) \). A prediction interval satisfying this property is said to be unconditionally valid. Unconditionally valid prediction intervals guarantee accurate coverage on average, treating \( (Y_{T+1}, X_{T+1}) \) and \( \{(Y_t, X_t)\}_{t=1}^T \) as random.

However, in many applications, unconditional validity may be unsatisfactory. Let us consider three examples; see Romano et al. (2019a); Foygel Barber et al. (2021) for further examples and discussions. First, from a fairness perspective, data-driven recommendation systems should guarantee equalized coverage across protected groups, in which case the goal is to construct prediction intervals that are valid conditional on a protected attribute such as race or gender (Romano et al., 2019a). Second, as in Section 5.1, consider the problem of predicting stock returns given the realized volatility. Since the distribution of returns is more dispersed when the variance is higher, a natural prediction algorithm should yield wider prediction intervals for higher values of volatility. That is, the prediction interval should be valid conditional on the known value of realized volatility rather than on average. Third, as in Section 5.2, suppose our goal is to predict wages based on an individual’s education and experience. An unconditionally valid prediction interval exhibits coverage 90% on average across all individuals but may contain the true wage of high-school dropouts with no work
experience with probability zero. A more useful prediction interval should exhibit correct coverage conditional on an individual’s observed education and experience and contain the true wage with 90% probability for every single individual.

Motivated by this discussion, we develop a distributional conformal prediction (DCP) method for constructing prediction intervals that are approximately valid conditional on the full vector of predictors $X_{T+1}$, while treating $Y_{T+1}$ and $\{(Y_t, X_t)\}_{t=1}^T$ as random:

$$P \left( Y_{T+1} \in \widehat{C}_{(1-\alpha)}(X_{T+1}) \mid X_{T+1} \right) \geq 1 - \alpha + o_P(1). \quad (2)$$

A prediction interval satisfying property (2) as $T \to \infty$ is said to be approximately conditionally valid.\(^1\)

While the requirement in (2) is natural in many applications, there are also other notions of conditional validity. Instead of conditioning on $X_{T+1}$ (object conditional), one can also study the conditional coverage probability given the training sample $\{(Y_t, X_t)\}_{t=1}^T$ (training conditional) or given $Y_{T+1}$ (label conditional) or combinations of them; see Vovk (2012) for a detailed discussion. By Proposition 2 of Vovk (2012), inductive conformal predictions (also known as split-sample conformal predictions) automatically achieve training conditional validity as long as the training sample is large enough. In classification problems (the support of $Y_{T+1}$ is a finite set), label conditional validity is often of great interest as it is important to know the error rates for different categories and provides useful information on false positive and false negative rates (Vovk, 2012). In Vovk (2012), label conditional validity is achieved by forming the conformity score within each category. Both training and label conditional validity can be achieved in a distribution-free way, i.e., for a given procedure, the conditional validity holds for any distribution of the data.

However, object conditional validity in the sense of (2) cannot be achieved in a distribution-free way for non-trivial predictions. By Vovk (2012); Lei and Wasserman (2014); Foygel Barber et al. (2021), any prediction set satisfying (2) for every probability distribution of $(X_t, Y_t)$ has infinite Lebesgue measure with non-trivial probability. Therefore, we only aim to achieve (2) for a limited class of probability distributions. The construction of the proposed prediction set $\widehat{C}_{(1-\alpha)}$ relies on learning the conditional distribution $Y_t \mid X_t$ and we only hope for conditional validity in (2) in the class of distributions that can be learned well. In particular, this class of distributions are those satisfying our regularity conditions.

Our empirical results demonstrate the importance of using DCP instead of standard conformal prediction methods based on modeling the conditional mean. When predicting

\(^1\)See, for example, Lei and Wasserman (2014); Sesia and Candes (2020); Foygel Barber et al. (2021) for a further discussion of the difference between conditional and unconditional validity.
daily stock returns in Section 5.1, the coverage probability of the 90% mean-based conformal prediction interval can drop to around 50% when the realized volatility is high. By contrast, DCP provides a coverage probability close to 90% for all values of realized volatility. This finding is important since volatility tends to be high during periods of crisis when accurate risk assessments are most needed. When predicting wages in Section 5.2, we find that the DCP prediction intervals contain the true wage with probability close to 90% for most individuals, whereas standard mean-based conformal prediction intervals either substantially under- or overcover.

To motivate DCP, note that a conditionally valid prediction interval is given by

\[
\left[ Q\left(\frac{\alpha}{2}, x\right), Q\left(1 - \frac{\alpha}{2}, x\right) \right],
\]

where \( Q(\tau, x) \) is the \( \tau \)-quantile of \( Y_t \) given \( X_t = x \). To implement the prediction interval (3), a plug-in approach would replace \( Q \) with a consistent estimator \( \hat{Q} \)

\[
\left[ \hat{Q}\left(\frac{\alpha}{2}, x\right), \hat{Q}\left(1 - \frac{\alpha}{2}, x\right) \right].
\]

This approach exhibits two well-known drawbacks. First, it will often exhibit undercover- age in finite samples (e.g., Romano et al., 2019b). Second, it is neither conditionally nor unconditionally valid under misspecification.

We build upon conformal prediction (Vovk et al., 2005, 2009) and use the conditional ranking as a conformity score. This choice is particularly useful when working with regression models for conditional distributions such as QR and DR.\(^2\) Our method is conditionally valid under correct specification, while the construction of the procedure as a conformal prediction method guarantees the unconditional validity under misspecification. Let \( F(y, x) = P(Y_t \leq y \mid X_t = x) \) denote the conditional cumulative distribution function (CDF) of \( Y_t \) given \( X_t = x \). Throughout the paper, we assume that \( F(\cdot, X_t) \) is a continuous function almost surely. Our method is based on the probability integral transform, which states that the conditional rank,

\[
U_t := F(Y_t, X_t),
\]

has the uniform distribution on \((0, 1)\) and is independent of \( X_t \).

To construct the prediction interval, we test the plausibility of each \( y \in \mathbb{R} \). By the probability integral transform, conditional on \( X_{T+1} \), \( F(Y_{T+1}, X_{T+1}) \) belongs to \([\alpha/2, 1 - \alpha/2]\) with probability \( 1 - \alpha \). Thus, collecting all values \( y \in \mathbb{R} \) satisfying \( F(y, X_{T+1}) \in [\alpha/2, 1 - \alpha/2] \) yields a conditionally valid prediction interval in the sense of (2). We operationalize this idea by proposing a conformal prediction procedure based on the estimated ranks, \( \hat{U}_t^{(y)} := \hat{F}^{(y)}(Y_t, X_t) \). For each \( y \in \mathbb{R} \), \( \hat{F}^{(y)} \) is an estimator of \( F \) obtained based on the augmented

\(^2\)This transformation is also very useful in other prediction problems (e.g., Politis, 2015).
data, \( \{(Y_t, X_t)\}_{t=1}^{T+1} \), where \( Y_{T+1} = y \). Data augmentation is a key feature of conformal prediction. It implies the model-free unconditional exact finite-sample validity with iid (or exchangeable) data and, thus, guards against model misspecification and overfitting. Without data augmentation, the resulting prediction intervals are not exactly valid, not even with correct specification and iid data.

Our baseline method asymptotically coincides with the oracle interval in (3). This oracle interval may not be the shortest possible prediction interval in general. Therefore, we also develop a simple and easy-to-implement adjustment of our baseline method for improving efficiency, which we refer to as optimal DCP. In Section 5.2, we show empirically that optimal DCP yields shorter prediction intervals than baseline DCP when the conditional distribution is skewed.

We establish the following theoretical performance guarantees for the baseline and optimal DCP.

(i) Asymptotic conditional validity under consistent estimation of the conditional CDF

(ii) Unconditional validity under model misspecification:

(a) Finite-sample validity with iid (or exchangeable) data

(b) Asymptotic validity with time series data

(iii) For optimal DCP:

(a) Under weak conditions: asymptotic conditional validity and optimality (shortest length)

(b) Under strong conditions: asymptotic convergence to the optimal prediction interval

1.1 Motivating Example

We illustrate the advantages of DCP relative to mean-based conformal prediction (e.g., Lei et al., 2018) based on the following simple analytical example.

\[
Y_t = X_t + X_t \varepsilon_t, \quad X_t \sim \text{Uniform}(0, 1), \quad \varepsilon_t \sim N(0, 1).
\]

Our motivating example draws on Koenker and Bassett (1982); Koenker (2005a); Lei et al. (2018); Romano et al. (2019b). We focus on the population conformal prediction (or oracle) problem under correct specification and abstract from finite sample issues.

Mean-based conformal prediction is based on the residuals \( R_t = Y_t - E(Y_t \mid X_t) = Y_t - X_t = X_t \varepsilon_t \). The mean-based prediction interval is

\[
C_{(1-\alpha)}^{\text{reg}}(x) = [x - Q_{|R|}(1-\alpha), x + Q_{|R|}(1-\alpha)],
\]
where \( Q_{|R|}(1 - \alpha) \) is the \((1 - \alpha)\)-quantile of the distribution of \(|R_t|\). An important property and drawback of \( C_{(1-\alpha)}^{\text{reg}} \) is that its length, \( 2 \cdot Q_{|R|}(1 - \alpha) \), is fixed and does not depend on \( X_{T+1} = x \) (Lei et al., 2018; Romano et al., 2019b). This feature implies that \( C_{(1-\alpha)}^{\text{reg}} \) is not adaptive to the heteroskedasticity in the location-scale model (5) and not conditionally valid.

DCP is based on the ranks \( U_t = \Phi(\varepsilon_t) \), where \( \Phi(\cdot) \) is the CDF of \( N(0, 1) \). The DCP prediction interval is

\[
C_{(1-\alpha)}^{\text{dcp}}(x) = [x - x \cdot Q_{|\varepsilon|}(1 - \alpha), x + x \cdot Q_{|\varepsilon|}(1 - \alpha)] ,
\]

(7)

where \( Q_{|\varepsilon|}(1 - \alpha) = \Phi^{-1}(1 - \alpha/2) \) is the \((1 - \alpha)\)-quantile of \(|\varepsilon_t|\). Unlike \( C_{(1-\alpha)}^{\text{reg}} \), the length of \( C_{(1-\alpha)}^{\text{dcp}} \), \( 2x \cdot Q_{|\varepsilon|}(1 - \alpha) \), depends on \( X_{T+1} = x \). Our construction automatically adapts to the heteroskedasticity in model (5) and is conditionally valid.

Figure 2 provides an illustration. Panel (a) shows that the conditional length of \( C_{(0.9)}^{\text{reg}} \) is constant, whereas the length of \( C_{(0.9)}^{\text{dcp}} \) varies as a function of \( x \). \( C_{(0.9)}^{\text{dcp}} \) is shorter than \( C_{(0.9)}^{\text{reg}} \) for low values and wider for high values of \( x \). Panel (b) shows that \( C_{(0.9)}^{\text{dcp}} \) is valid for all \( x \), whereas \( C_{(0.9)}^{\text{reg}} \) overcovers for low values and undercovers for high values of \( x \). Figure 2 illustrates the advantage of our method. For predictor values where the conditional variance is low, it yields shorter prediction intervals, while ensuring conditional coverage for values where the conditional dispersion is large by suitably enlarging the prediction interval.

**Figure 1: Motivating example**

**1.2 Related Literature**

We build on and contribute to the literature on conformal prediction (e.g., Vovk et al., 2005; Vovk, 2012; Vovk et al., 2009; Lei et al., 2013; Lei and Wasserman, 2014; Lei et al., 2018;
Within the conformal prediction literature, our paper is most closely related to Lei and Wasserman (2014), Lei et al. (2018), and Romano et al. (2019b). Lei and Wasserman (2014) propose conditionally valid and asymptotically efficient conformal prediction intervals based on estimators of the conditional density. We take a different and complementary approach, allowing researchers to leverage powerful regression methods for modeling conditional distributions, including QR and DR approaches. Lei et al. (2018) develop conformal prediction methods based on regression models for conditional expectations. However, as discussed in Section 1.1, this approach is not conditionally valid under heteroskedasticity. They also propose a locally weighted conformal prediction approach, where the regression residuals are weighted by the inverse of a measure of their variability. This approach can alleviate some of the limitations of mean-based conformal prediction but is motivated by and based on restrictive locations-scale models. By contrast, our approach is generic and exploits flexible and substantially more general models for the whole conditional distribution.

Romano et al. (2019b) propose a split conformal approach based on QR models, which they call conformalized quantile regression (CQR). See also Sesia and Candes (2020); Kivaranovic et al. (2020a) for related approaches and Vovk et al. (2020) for a general approach to adaptive conformal prediction. Their approach is based on splitting the data into two subsets, \( T_1 \) and \( T_2 \). Based on \( T_1 \), they estimate two separate quantile functions \( \hat{Q}(\alpha/2, x) \) and \( \hat{Q}(1 - \alpha/2, x) \) and construct the prediction intervals as

\[
\left[ \hat{Q}(\alpha/2, x) - Q_E(1 - \alpha), \hat{Q}(1 - \alpha/2, x) + Q_E(1 - \alpha) \right],
\]

where \( Q_E(1 - \alpha) \) is the \((1 - \alpha)(1 + 1/|T_2|)\)-th empirical quantile of

\[
E_t = \max \left\{ \hat{Q}(\alpha/2, X_t) - Y_t, Y_t - \hat{Q}(1 - \alpha/2, x) \right\}
\]

in \( T_2 \). Constructing prediction intervals based on deviations from quantile estimates is similar to working with deviations from mean estimates, as the deviations are measured in absolute levels. By contrast, exploiting the probability integral transform, our approach is generic and relies on permuting ranks, which naturally have the same scaling on (0, 1). Note, however, that our paper was inspired by Romano et al. (2019b) and we view our proposal as a (fully quantile-rank based) refinement of Romano et al. (2019b).

Our adjustment for constructing efficient prediction intervals is related to and inspired by conformal prediction literature on minimum-volume prediction sets based on density estimators (e.g., Lei et al., 2013; Lei and Wasserman, 2014; Eck and Crawford, 2019; Izbicki...
et al., 2019, 2020) and nearest-neighbor estimators Gyorfi and Walk (2020). It is most closely related and can be viewed as an alternative to conformal histogram regression (Sesia and Romano, 2021). The main differences between our approach and conformal histogram regression are the following. First, our method is based on an optimization problem formulated in terms of estimated quantile functions and does not require estimating a conditional density or histogram. Second, we do not work with nested sets but instead use a simple adjustment of our baseline conformity score. Finally, our approach works for general outcome distributions and does not rely on assuming unimodal distributions.

Conceptually, our paper is further related to the transformation-based model-free prediction approach developed in Politis (2013) and Politis (2015) in that we rely on transformations of the original setup into one that is easier to work with (i.e., ranks which are uniformly distributed) and study the properties of our approach in a model-free setting. An important difference is the implementation of the resulting procedure. The transformation-based approach is based on the bootstrap, whereas our approach is based on permuting ranks. Permuting ranks estimated based on the augmented data guarantees the model-free finite sample validity of our method with exchangeable data. To our knowledge, no exact finite-sample validity results have been developed for the bootstrap-based approach.

2 Distributional Conformal Prediction

Here we introduce DCP. We present a full and a split sample version of our method.

2.1 Full Distributional Conformal Prediction

Let $y$ denote a test value for $Y_{T+1}$. We test plausibility of each value $y \in \mathbb{R}$, collect all plausible values, and report them as the prediction set. In practice, we consider a grid of test values $Y_{\text{trial}}$. Define the augmented data $Z(y) = \{Z_i(y)\}_{i=1}^{T+1}$, where

$$
Z_i(y) = \begin{cases} 
(Y_i, X_i) & \text{if } 1 \leq t \leq T \\
(y, X_i) & \text{if } t = T + 1
\end{cases}
$$

(8)

Based on the augmented dataset $Z(y)$, we estimate the conditional CDF using a suitable

\footnote{For example, we can choose $Y_{\text{trial}}$ to be a fine grid between $-\max_{1 \leq t \leq T} |Y_t|$ and $\max_{1 \leq t \leq T} |Y_t|$. This choice has a theoretical justification since, under exchangeability, $P(|Y_{T+1}| \geq \max_{1 \leq t \leq T} |Y_t|) \leq 1/(1 + T)$ (Chen et al., 2016); see also the discussion in the \texttt{conformalInference} R-package (https://github.com/ryantibs/conformal).}
method such as QR and DR, which are discussed in more detail in the SI Appendix. Let $\hat{F}(y)$ denote the estimator for $F$ based on the augmented sample. If the original estimate is not monotonic, we rearrange it (e.g., Chernozhukov et al., 2009, 2010) so that $\hat{F}(y)(\cdot, x)$ is always monotonic. To simplify the exposition, we keep these rearrangements implicit.

We compute the ranks $\{\hat{U}_t^{(y)}\}_{t=1}^{T+1}$, where

$$
\hat{U}_t^{(y)} = \begin{cases} 
\hat{F}(y)(Y_t, X_t) & \text{if } 1 \leq t \leq T \\
\hat{F}(y)(y, X_t) & \text{if } t = T + 1 
\end{cases}
$$

(9)

and obtain p-values as

$$
\hat{p}(y) = \frac{1}{T+1} \sum_{t=1}^{T+1} 1\{\hat{V}_t^{(y)} \geq \hat{V}_{T+1}^{(y)}\},
$$

(10)

where $\hat{V}_t^{(y)} := \psi(\hat{U}_t^{(y)})$, and $\psi(\cdot)$ is a deterministic function. For our baseline method, we use $\psi(x) = |x - 1/2|$. In Section 4, we show how to choose $\psi$ optimally to ensure efficiency. Prediction intervals are computed as $\hat{C}_{\alpha}^{\text{full}}(X_{T+1}) = \{y \in Y_{\text{trial}} : \hat{p}(y) > \alpha\}$.\(^4\)

We summarize our approach in Algorithm 1.

**Algorithm 1 (Full DCP).**

**Input:** Data $\{(Y_t, X_t)\}_{t=1}^{T}$, miscoverage level $\alpha \in (0, 1)$, a point $X_{T+1}$, test values $Y_{\text{trial}}$

**Process:** For $y \in Y_{\text{trial}},$

1. define the augmented data $Z^{(y)}$ as in (9)
2. compute $\hat{p}(y)$ as in (10)

**Output:** Return $(1 - \alpha)$ prediction set $\hat{C}_{\alpha}^{\text{full}}(X_{T+1}) = \{y \in Y_{\text{trial}} : \hat{p}(y) > \alpha\}$

2.2 Split Distributional Conformal Prediction

An important drawback of full DCP (Algorithm 1) is its computational burden due to the grid search. Since $\hat{F}(y)$ is obtained based on the augmented data, one has to choose $Y_{\text{trial}}$ and re-estimate the entire conditional distribution for all $y \in Y_{\text{trial}}$. Therefore, we propose a split conformal procedure that exploits sample splitting, avoids grid search, and only requires estimating $F$ once. Sample splitting is a popular approach for improving the computational performance of conformal prediction methods (e.g., Lei et al., 2018; Romano et al., 2019b).

**Algorithm 2 (Split DCP).**

\(^4\)Instead of $\hat{C}_{\alpha}^{\text{full}}(X_{T+1})$ we typically report the closed interval $\tilde{C}_{\alpha}^{\text{full}}(X_{T+1}) = \left[\min(\hat{C}_{\alpha}^{\text{full}}(X_{T+1})), \max(\hat{C}_{\alpha}^{\text{full}}(X_{T+1}))\right]$. 

9
**Input:** Data \(\{(Y_t, X_t)\}_{t=1}^{T}\), miscoverage level \(\alpha \in (0,1)\), point \(X_{T+1}\)

**Process:**

1. Split \(\{1, \ldots, T\}\) into \(T_1 := \{1, \ldots, T_0\}\) and \(T_2 := \{T_0 + 1, \ldots, T\}\).
2. Obtain \(\hat{F}\) based on \(\{Z_t\}_{t \in T_1}\).
3. Compute \(\{\hat{V}_t\}_{t \in T_2} = \{\psi(\hat{U}_t)\}_{t \in T_2}\), where \(\hat{U}_t = \hat{F}(Y_t, X_t)\).
4. Compute \(\hat{Q}_{T_2}\), the \((1 - \alpha)(1 + 1/|T_2|)\) empirical quantile of \(\{\hat{V}_t\}_{t \in T_2}\).

**Output:** Return \((1 - \alpha)\) prediction set \(\hat{C}_{split}^{(1-\alpha)}(X_{T+1}) = \{y : \psi\left(\hat{F}(y, X_{T+1})\right) \leq \hat{Q}_{T_2}\}\).

(Since \(\hat{F}(\cdot, X_{T+1})\) is monotonic, \(\hat{C}_{split}^{(1-\alpha)}(X_{T+1})\) is an interval.)

In Algorithm 2, we split \(\{1, \ldots, T\}\) into \(\{1, \ldots, T_0\}\) and \(\{T_0 + 1, \ldots, T\}\). With iid data, one can also consider random splits.

Split DCP lends itself naturally to simple in-sample validity checks with both cross-sectional and time series data as illustrated in Section 5.

### 3 Theoretical Performance Guarantees

In this section, we establish the theoretical properties of our procedure. We focus on full-sample DCP (Algorithm 1). For the split-sample approach (Algorithm 2), we provide a modified version (Algorithm S1) in the SI Appendix and present its theoretical properties in Section 4.

When the data are iid (or exchangeable), our method achieves finite-sample unconditional validity in a model-free manner, as a consequence of general results on conformal inference and permutation inference more generally (e.g., Vovk et al., 2005; Hoeffding, 1952).

**Theorem 1** (Finite sample unconditional validity). Suppose that the data are iid or exchangeable and that the estimator of the conditional distribution is invariant to permutations of the data. Then

\[
P\left( Y_{T+1} \in \hat{C}_{full}^{(1-\alpha)}(X_{T+1}) \right) \geq 1 - \alpha.
\]

The proof of Theorem 1 is standard and omitted. Theorem 1 highlights the strengths and drawbacks of conformal prediction methods. Most commonly-used estimators of the conditional CDF such as QR and DR are invariant to permutations of the data. As a result, Theorem 1 provides a model-free unconditional performance guarantee in finite samples, allowing for arbitrary misspecification of the model of the conditional CDF. On the other hand, it has a major theoretical drawback. Even with iid data, it provides no guarantee at all on conditional validity.
Our next theoretical results provide a remedy. We impose the following weak regularity conditions.

**Assumption 1.** Suppose that there exists a non-random function $F^*(\cdot, \cdot)$ such that the following conditions hold as $T \to \infty$. Define $V_t := \psi(F^*(Y_t, X_t))$ for $1 \leq t \leq T + 1$.

1. There exists a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$ and $(T + 1)^{-1} \sum_{t=1}^{T+1} \phi(\hat{V}_t - V_t) = o_P(1)$ and $\hat{V}_{T+1} = V_{T+1} + o_P(1)$, where $\hat{V}_t := \hat{V}_t(Y_{T+1}) = \psi(\hat{F}(Y_{T+1}))(Y_t, X_t))$ for $1 \leq t \leq T + 1$.
2. $\sup_{v \in \mathbb{R}} |\hat{G}(v) - G(v)| = o_P(1)$, where $\hat{G}(v) = (T + 1)^{-1} \sum_{t=1}^{T+1} 1\{V_t < v\}$ and $G(\cdot)$ is the distribution function of $V_{T+1}$.
3. $\sup_{x_1 \neq x_2} |G(x_1) - G(x_2)|/|x_1 - x_2|$ is bounded.

Assumption 1 allows for some flexibility with respect to the model estimator. Here, we only require $F^*$ to be a non-random function, which may or may not be $F$. The interpretation is straightforward when $F^* = F$ since this simply means that the estimator $\hat{F}$ is consistent for $F$. We discuss the case of $F^* \neq F$ after Theorem 2 below. Note that we can replace the consistency requirement in Assumption 1 with a stronger uniform consistency requirement, $\sup_{x,y} |\hat{F}(y, x) - F^*(y, x)| = o_P(1)$.

We also notice that the quantities $\hat{V}_t$ and $V_t$ are defined under the true $Y_{T+1}$. This means that $\hat{F}(y)$ uses $y = Y_{T+1}$. In other words, the estimator $\hat{F}$ based on the sample $\{(X_t, Y_t)\}_{t=1}^{T+1}$ would be consistent for some $F^*$ if $Y_{T+1}$ were observed. Since the goal of Assumption 1 is to guarantee the coverage probability for $Y_{T+1}$, the conditions in Assumption 1 only need to hold for $y = Y_{T+1}$.

Notice that $\hat{F}$ is consistent for $F^*$ under a very weak norm, and no rate condition is required. When $\psi(x) = |x - 1/2|$, a simple example of $\phi(\cdot)$ in Assumption 1 is $\phi(x) = x^q$ for some $q > 0$; in other words, a sufficient condition is $(T + 1)^{-1} \sum_{t=1}^{T+1} |\hat{F}(Y_t, X_t) - F^*(Y_t, X_t)|^q = o_P(1)$, which can be verified for many existing estimators with $q = 2$.

The following lemma gives the basic consistency result.

**Lemma 1.** Let Assumption 1 hold. Then $\hat{G}(\hat{V}_{T+1}) = G(V_{T+1}) + o_P(1)$, where $\hat{G}(v) = (T + 1)^{-1} \sum_{t=1}^{T+1} 1\{V_t < v\}$.

By Assumption 1, $G(\cdot)$ is uniformly continuous and thus continuous. Since $G(\cdot)$ is the distribution function of $V_{T+1}$, we have that $G(V_{T+1})$ has the uniform distribution on $(0, 1)$, i.e., $P(G(V_{T+1}) \leq \alpha) = \alpha$. This implies the unconditional asymptotic validity.

---

5This is not really much different from assuming that $\hat{F}$ based on the sample $\{(X_t, Y_t)\}_{t=1}^{T+1}$ is consistent for some $F^*$.
Theorem 2 (Asymptotic unconditional validity). Let Assumption 1 hold. Then
\[ P \left( Y_{T+1} \in \hat{C}_{\text{full}}^{(1-\alpha)} (X_{T+1}) \right) = 1 - \alpha + o(1). \]

Theorem 2 establishes the asymptotic unconditional validity of the procedure. Since Theorem 1 already establishes the unconditional validity in finite-samples for iid or exchangeable data without assuming any consistency of \( \hat{F} \), the main purpose of Theorem 2 is to address the case of non-exchangeable data (e.g., time series data with ergodicity), especially when the model is misspecified (i.e., if \( F^* \neq F \)).

To illustrate model misspecification, consider the popular linear QR model, which assumes \( Q(\tau, x) = x^T \beta(\tau) \) and thus \( F(y, x) = F(y, x; \beta) = \int_0^1 1\{x^T \beta(\tau) \leq y\} d\tau. \) This model is typically estimated by \( \hat{\beta}(\tau) = \arg\min_{\beta} \sum_{t=1}^{T+1} \rho_\tau(Y_t - X_t^T \beta) \) with \( \rho_\tau(a) = a(\tau - 1\{a < 0\}). \) Under misspecification \( (Q(\tau, x) \neq x^T \beta(\tau)) \), \( \hat{\beta}(\tau) \) is still estimating \( \beta^*(\tau) = \arg\min_{\beta} \sum_{t=1}^{T+1} E\rho_\tau(Y_t - X_t^T \beta) \) and \( F^* \) is defined using \( \beta^*(\cdot) \), e.g., \( F^*(y, x) = \int_0^1 1\{x^T \beta^*(\tau) \leq y\} d\tau. \) For parametric models, \( F^* \) is usually the probability limit of \( \hat{F} \). In general, we can consider a model \( F \) and minimize the empirical risk \( \hat{F} = \arg\min_{g \in F} \sum_{t=1}^{T+1} L(Y_t, X_t, g) \) for some loss function \( L \). Even if the model is misspecified \( (F \notin F) \), it is still possible to show that \( \hat{F} \) is close (in some norm) to \( F^* = \arg\min_{g \in F} \sum_{t=1}^{T+1} E[L(Y_t, X_t, g)] \). In the SI Appendix, we provide a more detailed discussion of this and some theoretical results verifying the consistency requirement in Assumption 1 for the time series case; see also Chernozhukov et al. (2018) for a general discussion of conformal prediction in time series settings.

The cost of allowing for misspecification is that one cannot guarantee conditional validity when \( F^* \neq F \). On the other hand, Lemma 1 implies that the prediction intervals are conditionally valid when \( F^* = F \).

Theorem 3 (Asymptotic conditional validity). Let Assumption 1 hold with \( F^* = F \). Then
\[ P \left( Y_{T+1} \in \hat{C}_{\text{full}}^{(1-\alpha)} (X_{T+1}) \mid X_{T+1} \right) = 1 - \alpha + o_P(1). \]

Theorems 2–3 establish the asymptotic validity of our procedure under weak and easy-to-verify conditions. They formalize the key intuition that conditional validity hinges on the quality of the estimator \( \hat{F} \) of the conditional CDF.\(^6\)

\(^6\)In Theorem 3, we assume \( F^* = F \). Since the first version of this paper was written, Candès et al. (2021) have provided more general results where \( F^* \approx F \).
4 Extension: Optimal DCP

In Section 3, we have seen that a generic conformity score \( \psi(y, x) = |F(y, x) - 1/2| \) leads to conditional validity if the conditional distribution \( F \) can be estimated consistently. We now characterize an optimal choice of conformity score that results in the shortest prediction interval. Detailed implementation algorithms, technical assumptions, and proofs are provided in the SI Appendix.

Let \( Z \) and \( X \) denote the support of \( Z_t = (Y_t, X_t) \) and \( X_t \), respectively. The optimal prediction interval is

\[
C_{(1-\alpha)}^{opt}(x) = [r_1(x, \alpha), r_2(x, \alpha)],
\]

where the functions \( r_1(\cdot, \cdot), r_2(\cdot, \cdot) \) satisfy that for any \( x \in X \),

\[
r_2(x, \alpha) - r_1(x, \alpha) = \min_{F(z_2, x) - F(z_1, x) \geq 1-\alpha} z_2 - z_1.
\]

The question is whether it is possible to design a conformity score that achieves the above optimal prediction interval. To answer this question formally, we consider a generic conformity score \( \psi(y, x) \), which might contain components that need to be estimated.

Permuting a large number of values of \( \psi(Y_t, X_t) \) in conformal predictions amounts to taking the sample \( (1-\alpha) \)-quantile of \( \psi(Y_t, X_t) \); for example, following Algorithm 2, one would \( (1-\alpha)(1+1/|T_2|) \) empirical quantile of \( \psi(Y_t, X_t) \). Assuming a law of large numbers, this empirical quantile would be close to the population \( (1-\alpha) \)-quantile of \( \psi(Y_t, X_t) \), leading to the asymptotic conformal prediction interval for \( Y_{T+1} \)

\[
C_{(1-\alpha)}^{conf}(X_{T+1}) = \{ y : \psi(y, X_{T+1}) \leq Q_\psi(1-\alpha) \},
\]

where \( Q_\psi(1-\alpha) \) is the \( (1-\alpha) \)-quantile of \( \psi(Y_t, X_t) \). The following result shows how to construct the optimal conformity score \( \psi \).

**Lemma 2.** Let \( \psi_*(y, x) = |F(y, x) - b(x, \alpha) - (1-\alpha)/2| \), where \( b(\cdot, \cdot) \) is a function satisfying that for any \( x \in X \),

\[
b(x, \alpha) \in \arg \min_{z \in [0, \alpha]} Q(z + 1 - \alpha, x) - Q(z, x).
\]

Let \( C_{(1-\alpha)}^{conf}(X_{T+1}) \) be defined as in (13) with the above conformity score \( \psi_* \). Assume that \( F(\cdot, x) \) is a continuous function for any \( x \in X \). Then \( Q_\psi(1-\alpha) = (1-\alpha)/2 \) and

\[
\mu(C_{(1-\alpha)}^{opt}(X_{T+1})) = \mu(C_{(1-\alpha)}^{conf}(X_{T+1})) \text{ almost surely,}
\]

\[
\mu(C_{(1-\alpha)}^{opt}(X_{T+1})) = \mu(C_{(1-\alpha)}^{conf}(X_{T+1})) \text{ almost surely,}
\]

\[
\mu(C_{(1-\alpha)}^{opt}(X_{T+1})) = \mu(C_{(1-\alpha)}^{conf}(X_{T+1})) \text{ almost surely,}
\]

\[
\mu(C_{(1-\alpha)}^{opt}(X_{T+1})) = \mu(C_{(1-\alpha)}^{conf}(X_{T+1})) \text{ almost surely,}
\]

\[
\mu(C_{(1-\alpha)}^{opt}(X_{T+1})) = \mu(C_{(1-\alpha)}^{conf}(X_{T+1})) \text{ almost surely,}
\]

\[
\mu(C_{(1-\alpha)}^{opt}(X_{T+1})) = \mu(C_{(1-\alpha)}^{conf}(X_{T+1})) \text{ almost surely,}
\]
where $\mu(\cdot)$ denotes the Lebesgue measure. If the optimization problem in (11) has a unique solution for any $x \in \mathcal{X}$, then

$$C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) = C_{(1-\alpha)}^{\text{conf}}(X_{T+1}) \text{ almost surely.}$$

Lemma 2 motivates conformity scores of the form

$$\psi^*(y, x) = |F(y, x) - [b(x, \alpha) + (1 - \alpha)/2]|,$$

where $b(\cdot, \cdot)$ solves (14). Compared to the choice of $\psi(y, x) = |F(y, x) - 1/2|$ mentioned in Section 3, we can view $\psi^*$ as having a “shape” adjustment $b(x, \alpha) - \alpha/2$. Since $F(Y_t, X_t)$ is independent of $X_t$, the optimal conformity score measures the distance between two independent components: $F(Y_t, X_t)$ and $1/2 + (b(X_t, \alpha) - \alpha/2)$. Hence, by Lemma 2, in order to take into account the shape of the conditional distribution $F(\cdot, x)$, it suffices to consider the scalar quantity

$$1/2 + (b(x, \alpha) - \alpha/2).$$

In some special cases, the “shape” adjustment can be shown to be zero, i.e., $b(x, \alpha) = \alpha/2$. One typical example is when $F(\cdot, x)$ is a symmetric uni-modal distribution with a well-defined conditional density.\footnote{In this case, $Q(1/2 + \delta, x) - Q(1/2, x) = Q(1/2, x) - Q(1/2 - \delta, x)$ and the conditional density is increasing on $(-\infty, Q(1/2, x))$ and decreasing on $(Q(1/2, x), \infty)$. One can show $b(x, \alpha) = \alpha/2$ by taking the first-order derivative for the optimization problem in (14) and setting it to zero.}

Therefore, the choice of $\psi(y, x) = |F(y, x) - 1/2|$ mentioned in Section 3 is optimal in these cases. However, Lemma 2 provides a construction that achieves optimality more generally. By the definition of $\psi^*$ and $Q_\psi(1 - \alpha) = (1 - \alpha)/2$, the prediction interval is

$$C_{(1-\alpha)}^{\text{conf}}(x) = [Q(b(x, \alpha), x), Q(b(x, \alpha) + 1 - \alpha, x)].$$

(15)

We illustrate this in Figure 2 with $\alpha = 0.1$. (15) implies that $b(x, \alpha)$ is the quantile-index of the lower bound of the interval. For the symmetric distribution in the left panel, we see $b(x, \alpha) = 0.05$, which is $\alpha/2$. For the asymmetric distribution in the right panel, we see that $b(x, \alpha) = 0.007$, which is far away from $\alpha/2 = 0.05$.\footnote{In this case, $Q(1/2 + \delta, x) - Q(1/2, x) = Q(1/2, x) - Q(1/2 - \delta, x)$ and the conditional density is increasing on $(-\infty, Q(1/2, x))$ and decreasing on $(Q(1/2, x), \infty)$. One can show $b(x, \alpha) = \alpha/2$ by taking the first-order derivative for the optimization problem in (14) and setting it to zero.}
Figure 2: Optimal prediction intervals

The first result in Lemma 2 is general and allows for the lack of uniqueness of the optimal prediction interval. For example, if $F$ is the uniform distribution on a certain interval, then all conditionally valid prediction intervals have the same length. Clearly, in this case, achieving the optimal length is the only goal one can hope for. When we can uniquely define the optimal prediction interval, Lemma 2 implies that the conformal procedure can recover the uniquely defined optimal interval, not just achieving the optimal length.

Lemma 2 also confirms the insight of Lei and Wasserman (2014): the optimal confidence set for $X_{T+1} = x$ should take the form $\{y : f(y, x) \geq c(x)\}$ for some $c(x) > 0$, where $f(y, x) = \partial F(y, x) / \partial y$. Assume that $F(\cdot, x)$ is a uni-modal distribution and $f(\cdot, x)$ is a continuous function for any $x \in X$. Then this confidence set is an interval. This means that $\{y : f(y, x) \geq c(x)\} = [c_1(x), c_2(x)]$ and $f(c_1(x), x) = f(c_2(x), x) = c(x)$. We notice that $c_1(x), c_2(x)$ are related to our results in that $c_1(x) = Q(b(x, \alpha), x)$ and $c_2(x) = Q(b(x, \alpha) + 1 - \alpha, x)$. To see this, simply observe that the first-order condition of the optimization problem in (14) is $1/f(Q(z + 1 - \alpha, x) - 1/f(Q(z, x)) = 0$, which implies that

\[
f(Q(b(x, \alpha) + 1 - \alpha, x)) = f(Q(b(x, \alpha), x)).\]

To make the procedure operational, we provide the conformal prediction interval $\hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1})$ defined in Algorithm S1 in the SI Appendix. We can provide the following guarantee.

**Theorem 4.** Let Assumption S1 in the SI Appendix hold. Then

\[
P\left(Y_{T+1} \in \hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1}) \mid X_{T+1}\right) = 1 - \alpha + o_P(1)
\]

and

\[
\mu \left(\hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1})\right) \leq \mu \left(C_{(1-\alpha)}^{\text{opt}}(X_{T+1})\right) + o_P(1).
\]
The main requirements in Assumption S1 in the SI Appendix are consistency of \( \hat{F} \) and that the density \( f \) bounded below on its support. This is quite mild in the sense that it does not imply that the optimal prediction interval in (11) is uniquely defined. For example, it allows \( f \) to be a uniform distribution. Therefore, as discussed above, the conformal prediction interval would have approximately the shortest length but might not converge to \( C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) \) in (11).

The following theorem provides a stronger result about \( \hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1}) \) based on stronger assumptions.

**Theorem 5.** Let Assumption S2 in the SI Appendix hold. Consider the conformal prediction interval \( \hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1}) \) defined in Algorithm S1 in the SI Appendix. Then

\[
\mu \left( \hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1}) \triangle C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) \right) = o_P(1),
\]

where \( \triangle \) denotes the symmetric difference of sets (i.e., \( A \triangle B = (A \setminus B) \cup (B \setminus A) \)), \( C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) \) is defined in (11).

The key component of Assumption S2 in the SI Appendix is consistent estimation of \( b \). Theorem 5 shows that \( \hat{C}_{(1-\alpha)}^{\text{conf}}(X_{T+1}) \) is close to \( C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) \) in the sense that the symmetric difference between these two sets has vanishing Lebesgue measure.

## 5 Empirical Applications

We illustrate the performance of DCP in two empirical applications and provide a comparison to alternative approaches. We consider eight different conformal prediction methods.

1. **DCP-QR**: DCP with QR (Algorithm 2)
2. **DCP-QR\(^*\)**: Optimal DCP with QR (Algorithm S1 in SI Appendix)
3. **DCP-DR**: DCP with DR (Algorithm 2)
4. **CQR**: CQR with QR (Romano et al., 2019b)
5. **CQR-m**: CQR variant (Sesia and Candes, 2020; Kivaranovic et al., 2020a) with QR
6. **CQR-r**: CQR variant (Sesia and Candes, 2020) with QR.
7. **CP-OLS**: Mean-based split conformal prediction with OLS
8. **CP-loc**: Locally-weighted conformal prediction (Lei et al., 2018) with OLS
All computations were carried out in R (R Core Team, 2021). Code and data for replicating the empirical results are deposited on Github (https://github.com/kwuthrich/Replication_DCP).

5.1 Predicting Stock Market Returns

Here we consider the problem of predicting stock market returns, which are known to exhibit substantial heteroskedasticity; see Chapter 13 in Elliott and Timmermann (2016) for a recent review and the references therein. We use data on daily returns of the market portfolio (CRSP value-weighted portfolio) from July 1, 1926, to June 30, 2021.\footnote{The CRSP data are constructed from the Fama/French 3 Factors data (Kenneth R. French, 2021) available from Kenneth R. French's data library (accessed August 17, 2021).} We use lagged realized volatility $X_t$ to predict the present return $Y_t$.\footnote{We compute realized volatility as the square root of the sum of squared returns over the last 22 days.} Daily returns are not iid and exhibit time series dependence. In the SI Appendix, we show that the key conditions underlying our theoretical results hold when the data are $\beta$-mixing. Several stochastic volatility models for asset returns, including the popular GARCH models, can be shown to be $\beta$-mixing (e.g., Boussama, 1998; Carrasco and Chen, 2002; Francq and Zakoïan, 2006).

We evaluate the performance of the different methods by splitting the data into a holdout and a test sample. To account for the dependence in the data, we present results averaged over five consecutive prediction exercises. In the first exercise, we apply split conformal prediction with an equal split ($|T_1| = |T_2|$) to the first 50% of observations and use the next 10% for testing. In the second exercise, we drop the first 10% of the observations, apply split conformal prediction to the next 50% of observations, and use the next 10% for testing and so on.

Figure 3 plots the empirical coverage probabilities for 20 bins obtained by dividing up the support of $X_t$ based on equally spaced quantiles. DCP-QR and DCP-QR$^*$ yield prediction intervals with coverage levels that are almost constant across all bins and close to the nominal level. They outperform DCP-DR, which undercovers in high-volatility regimes. The conditional coverage properties of DCP-QR and DCP-QR$^*$ are very similar to CQR, CQR-m, CQR-r, and CP-loc. This suggest that location-scale models, which are nested by QR, provide a good approximation of the conditional distribution. CP-OLS exhibits over-coverage under low-volatility regimes and substantial undercoverage under high-volatility regimes. This finding has important practical implications since the volatility tends to be high during periods of crisis, which is precisely when accurate risk assessments are most needed.
Figure 3: Conditional coverage 90% prediction intervals by realized volatility

Figure 4 shows the conditional length of the prediction intervals. DCP-QR, DCP-QR*, CQR, CQR-m, CQR-r, and CP-loc yield prediction intervals of similar length. The DCP-DR prediction intervals are somewhat shorter than those of the QR-based methods at the upper tail. Finally, CP-OLS yields prediction intervals that are almost constant across all values of realized volatility; they are longer at the lower tail and shorter at the upper tail.\textsuperscript{10}

Figure 4: Conditional coverage 90% prediction intervals by realized volatility

\textsuperscript{10}The CP-OLS prediction intervals are not exactly constant because we are reporting results averaged over five experiments.
5.2 Predicting Wages Using CPS Data

We consider the problem of predicting wages using individual characteristics. We use the 2012 CPS data provided in the R-package hdm (Chernozhukov et al., 2016), which contains information on \( N = 29217 \) observations. Here we use the index \( i \) instead of \( t \). To illustrate the impact of skewness on the performance of the different prediction methods, we use the hourly wage as our dependent variable \( Y_i \).\(^{11}\) Predictors \( X_i \) include indicators for gender, marital status, educational attainment, region, experience, experience squared, and all two-way interactions such that \( \text{dim}(X_i) = 100 \) after removing constant variables.

Following Romano et al. (2019b) and Sesia and Candes (2020), we evaluate the performance of the different methods by randomly holding out 20% of the data for testing, \( I_{\text{test}} \), and applying split conformal prediction with an equal split to the remaining 80% of the data. We repeat the whole experiment 20 times.

Panel (a) of Table 1 shows that all conformal prediction methods exhibit excellent unconditional coverage properties, confirming the theoretical finite sample guarantees. To assess and compare the conditional coverage properties, for each method, we compute conditional coverage probabilities as the predictions from logistic regressions of \( \{ Y_i \in C_{\text{split}}^{(1-\alpha)}(X_i) \}_{i \in I_{\text{test}}} \) on \( \{ X_i \}_{i \in I_{\text{test}}} \), where \( C_{\text{split}}^{(1-\alpha)} \) is the split conformal prediction interval obtained by the corresponding method. The less dispersed the predicted coverage probabilities are around the nominal level \( 1 - \alpha = 0.9 \), the better the overall conditional coverage properties of a method. Panel (b) of Table 1 plots the standard deviation of the predicted coverage probabilities.\(^{12}\) DCP-QR\(^*\) yields the lowest dispersion of all methods. The predicted coverage probabilities based on DCP-QR are less dispersed than those obtained from CQR, CQR-m, CQR-r. CP-loc yields a higher dispersion than the methods based on QR and DR, which demonstrates the value-added of using flexible models of the conditional distribution. Overall, DCP performs much better than CP-OLS for which the predicted coverage probabilities exhibit a very high dispersion. Figure 5 in the SI Appendix plots histograms of the predicted coverage probabilities.

Table 2 shows the average length of the prediction intervals. DCP-QR\(^*\) produces the shortest prediction intervals among of all methods. This demonstrates the practical advantage of the shape adjustment when the conditional distribution is skewed. The results also suggest a trade-off between conditional coverage accuracy and average length. For example,

\[ \text{We obtain the hourly wage by exponentiating the log hourly wage provided in the dataset.} \]

\[ \text{Using } \sqrt{1/|I_{\text{test}}| \sum_{i \in I_{\text{test}}} (\text{Coverage}_i - 0.9)^2}, \text{ where } \text{Coverage}_i \text{ is the predicted coverage probability, instead of the standard deviation yields very similar results.} \]
CP-OLS and CP-loc, which both exhibit poor conditional coverage properties, yield shorter prediction intervals than DCP-QR.

Table 1: Coverage 90% prediction intervals

|            | DCP-QR | DCP-QR* | DCP-DR | CQR   | CQR-m | CQR-r  | CP-OLS | CP-loc |
|------------|--------|---------|--------|-------|-------|--------|--------|--------|
| (a)        |        |         |        |       |       |        |        |        |
| Unconditional coverage | 0.90   | 0.90    | 0.90   | 0.90  | 0.90  | 0.90   | 0.90   | 0.90   |
| (b)        |        |         |        |       |       |        |        |        |
| Dispersion of predicted conditional coverage ($\times 100$) | 1.80    | 1.71    | 3.08   | 2.21  | 2.36  | 2.30   | 11.13  | 4.11   |

Table 2: Average length 90% prediction intervals

|            | DCP-QR | DCP-QR* | DCP-DR | CQR   | CQR-m | CQR-r  | CP-OLS | CP-loc |
|------------|--------|---------|--------|-------|-------|--------|--------|--------|
|            | 34.22  | 29.61   | 33.69  | 34.52 | 34.84 | 34.63  | 33.84  | 32.66  |

References

Angrist, J., Chernozhukov, V., and Fernández-Val, I. (2006). Quantile regression under misspecification, with an application to the us wage structure. *Econometrica*, 74(2):539–563.

Belloni, A. and Chernozhukov, V. (2011). L1-penalized quantile regression in high-dimensional sparse models. *The Annals of Statistics*, 39(1):82–130.

Boussama, F. (1998). *Ergodicité, mélange et estimation dans les modeles garch*. PhD Thesis, Paris 7.

Bradley, R. C. (2005). Basic properties of strong mixing conditions. a survey and some open questions. *Probability surveys*, 2:107–144.

Bradley, R. C. (2007). *Introduction to strong mixing conditions*, volume 1. Kendrick Press Heber City.

Candès, E. J., Lei, L., and Ren, Z. (2021). Conformalized survival analysis. arXiv preprint arXiv:2103.09763.
Carrasco, M. and Chen, X. (2002). Mixing and moment properties of various garch and stochastic volatility models. *Econometric Theory*, 18(1):17–39.

Cattaneo, M. D., Feng, Y., and Titunik, R. (2019). Prediction intervals for synthetic control methods. arXiv:1912.07120.

Chaudhuri, P. (1991). Global nonparametric estimation of conditional quantile functions and their derivatives. *Journal of Multivariate Analysis*, 39(2):246 – 269.

Chaudhuri, P. and Loh, W.-Y. (2002). Nonparametric estimation of conditional quantiles using quantile regression trees. *Bernoulli*, 8(5):561–576.

Chen, W., Wang, Z., Ha, W., and Barber, R. F. (2016). Trimmed conformal prediction for high-dimensional models. arXiv preprint arXiv:1611.09933.

Chernozhukov, V., Fernández-Val, I., and Galichon, A. (2009). Improving point and interval estimators of monotone functions by rearrangement. *Biometrika*, 96(3):559–575.

Chernozhukov, V., Fernández-Val, I., and Galichon, A. (2010). Quantile and probability curves without crossing. *Econometrica*, 78(3):1093–1125.

Chernozhukov, V., Fernández-Val, I., and Melly, B. (2013). Inference on counterfactual distributions. *Econometrica*, 81(6):2205–2268.

Chernozhukov, V., Fernández-Val, I., Melly, B., and Wüthrich, K. (2020). Generic inference on quantile and quantile effect functions for discrete outcomes. *Journal of the American Statistical Association*, 115(529):123–137.

Chernozhukov, V., Hansen, C., and Spindler, M. (2016). hdm: High-dimensional metrics. *R Journal*, 8(2):185–199.

Chernozhukov, V., Wüthrich, K., and Yinchu, Z. (2018). Exact and robust conformal inference methods for predictive machine learning with dependent data. In Bubeck, S., Perchet, V., and Rigollet, P., editors, *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 732–749. PMLR.

Chernozhukov, V., Wüthrich, K., and Zhu, Y. (2021). An exact and robust conformal inference method for counterfactual and synthetic controls. *Journal of the American Statistical Association*, 0(0):1–16.

Dedecker, J., Doukhan, P., Lang, G., Rafael, L. R. J., Louhichi, S., and Prieur, C. (2007). *Weak dependence*. Springer.
Dedecker, J. and Louhichi, S. (2002). Maximal inequalities and empirical central limit theorems. In Empirical process techniques for dependent data, pages 137–159. Springer.

Eck, D. J. and Crawford, F. W. (2019). Efficient and minimal length parametric conformal prediction regions. arXiv preprint arXiv:1905.03657.

Elliott, G. and Timmermann, A. (2016). Economic Forecasting. Princeton University Press.

Foresi, S. and Peracchi, F. (1995). The conditional distribution of excess returns: An empirical analysis. Journal of the American Statistical Association, 90(430):451–466.

Foygel Barber, R., Candès, E. J., Ramdas, A., and Tibshirani, R. J. (2021). The limits of distribution-free conditional predictive inference. Information and Inference: A Journal of the IMA, 10(2):455–482.

Francq, C. and Zakoïan, J.-M. (2006). Mixing properties of a general class of GARCH (1,1) models without moment assumptions on the observed process. Econometric Theory, 22(5):815–834.

Gyorfi, L. and Walk, H. (2020). Nearest neighbor based conformal prediction. Stuttgarter Mathematische Berichte 2020-002.

He, X., Ng, P., and Portnoy, S. (1998). Bivariate quantile smoothing splines. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 60(3):537–550.

Hoeffding, W. (1952). The Large-Sample Power of Tests Based on Permutations of Observations. The Annals of Mathematical Statistics, 23(2):169 – 192.

Izbicki, R., Shimizu, G., and Stern, R. B. (2020). Cd-split and hpd-split: efficient conformal regions in high dimensions. arXiv preprint arXiv:2007.12778.

Izbicki, R., Shimizu, G. T., and Stern, R. B. (2019). Flexible distribution-free conditional predictive bands using density estimators. arXiv preprint arXiv:1910.05575.

Kenneth R. French (2021). Kenneth French Data Library. Fama/French 3 Factors [Daily] data. URL: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. Accessed August 17, 2021.

Kivaranovic, D., Johnson, K. D., and Leeb, H. (2020a). Adaptive, distribution-free prediction intervals for deep networks. In Chiappa, S. and Calandra, R., editors, Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics, volume 108 of Proceedings of Machine Learning Research, pages 4346–4356. PMLR.
Kivaranovic, D., Ristl, R., Posch, M., and Leeb, H. L. (2020b). Conformal prediction intervals for the individual treatment effect. arXiv:2006.01474.

Koenker, R. (2004). Quantile regression for longitudinal data. *Journal of Multivariate Analysis*, 91(1):74 – 89. Special Issue on Semiparametric and Nonparametric Mixed Models.

Koenker, R. (2005a). *Quantile Regression*. Econometric Society Monographs. Cambridge University Press.

Koenker, R. (2005b). *Quantile regression*. Number 38. Cambridge university press.

Koenker, R. and Bassett, G. (1978). Regression quantiles. *Econometrica*, 46:33–50.

Koenker, R. and Bassett, G. (1982). Robust tests for heteroscedasticity based on regression quantiles. *Econometrica*, 50(1):43–61.

Koenker, R., Ng, P., and Portnoy, S. (1994). Quantile smoothing splines. *Biometrika*, 81(4):673–680.

Komunjer, I. (2013). Chapter 17 - quantile prediction. In Elliott, G. and Timmermann, A., editors, *Handbook of Economic Forecasting*, volume 2 of *Handbook of Economic Forecasting*, pages 961 – 994. Elsevier.

Lei, J., GSell, M., Rinaldo, A., Tibshirani, R. J., and Wasserman, L. (2018). Distribution-free predictive inference for regression. *Journal of the American Statistical Association*, 113(523):1094–1111.

Lei, J., Robins, J., and Wasserman, L. (2013). Distribution-free prediction sets. *Journal of the American Statistical Association*, 108(501):278–287.

Lei, J. and Wasserman, L. (2014). Distribution-free prediction bands for non-parametric regression. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(1):71–96.

Lei, L. and Candès, E. J. (2020). Conformal inference of counterfactuals and individual treatment effects. arXiv:2006.06138.

Li, Y. and Zhu, J. (2008). L1-norm quantile regression. *Journal of Computational and Graphical Statistics*, 17(1):163–185.

Meinshausen, N. (2006). Quantile regression forests. *Journal of Machine Learning Research*, 7:983–999.
Meyn, S. P. and Tweedie, R. L. (2012). *Markov chains and stochastic stability*. Springer Science & Business Media.

Politis, D. N. (2013). Model-free model-fitting and predictive distributions. *TEST*, 22(2):183–221.

Politis, D. N. (2015). *Model-free prediction and regression: a transformation-based approach to inference*. Springer, New York.

R Core Team (2021). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rio, E. (2017). *Asymptotic Theory of Weakly Dependent Random Processes*. Springer.

Romano, Y., Barber, R. F., Sabatti, C., and Candes, E. J. (2019a). With malice towards none: Assessing uncertainty via equalized coverage. arXiv:1908.05428.

Romano, Y., Patterson, E., and Candes, E. J. (2019b). Conformalized quantile regression. NeurIPS.

Sesia, M. and Candes, E. J. (2020). A comparison of some conformal quantile regression methods. *Stat*, 9(1):e261. e261 sta4.261.

Sesia, M. and Romano, Y. (2021). Conformal histogram regression. arXiv preprint arXiv:2105.08747.

Taylor, J. W. (2000). A quantile regression neural network approach to estimating the conditional density of multiperiod returns. *Journal of Forecasting*, 19(4):299–311.

van der Vaart, A. and Wellner, J. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer Science & Business Media.

Vershynin, R. (2018). *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press.

Vovk, V. (2012). Conditional validity of inductive conformal predictors. In Hoi, S. C. H. and Buntine, W., editors, *Proceedings of the Asian Conference on Machine Learning*, volume 25 of *Proceedings of Machine Learning Research*, pages 475–490, Singapore Management University, Singapore. PMLR.

Vovk, V., Gammerman, A., and Shafer, G. (2005). *Algorithmic Learning in a Random World*. Springer.
Vovk, V., Nouretdinov, I., and Gammerman, A. (2009). On-line predictive linear regression. *The Annals of Statistics*, 37(3):1566–1590.

Vovk, V., Petej, I., Tocaceli, P., Gammerman, A., Ahlberg, E., and Carlsson, L. (2020). Conformal calibrators. In Gammerman, A., Vovk, V., Luo, Z., Smirnov, E., and Cherubin, G., editors, *Proceedings of the Ninth Symposium on Conformal and Probabilistic Prediction and Applications*, volume 128 of *Proceedings of Machine Learning Research*, pages 84–99. PMLR.

Wu, Y. and Liu, Y. (2009). Variable selection in quantile regression. *Statistica Sinica*, 19(2):801–817.
A Details for Section 4

Here we describe in detail how to implement the optimal prediction intervals described in Section 4.

A.1 Implementation

We now consider the estimation of \( b(\cdot, \cdot) \). We assume that \( \hat{F}(y, x) \) is monotonic in \( y \); if not, we first rearrange it. We define \( \hat{Q}(\cdot, \cdot) \) by

\[
\hat{Q}(\tau, x) = \inf \left\{ y : \hat{F}(y, x) \geq \tau \right\}.
\]

Define

\[
L(x) = \min_{z \in [0, \alpha]} Q(z + 1 - \alpha, x) - Q(z, x)
\]

and

\[
\hat{L}(x) = \min_{z \in [0, \alpha]} \hat{Q}(z + 1 - \alpha, x) - \hat{Q}(z, x).
\]

Let \( \hat{b}(x, \alpha) \) be a function such that \( \hat{b}(x, \alpha) \in [0, \alpha] \) and

\[
\hat{Q}(\hat{b}(x, \alpha) + 1 - \alpha, x) - \hat{Q}(\hat{b}(x, \alpha), x) = \hat{L}(x).
\]

We propose the following algorithm.

Algorithm S1 (Optimal split DCP).

**Input:** Data \( \{(Y_t, X_t)\}_{t=1}^T \), miscoverage level \( \alpha \in (0, 1) \), and point \( X_{T+1} \)

**Process:**

1. Split \( \{1, \ldots, T + 1\} \) into \( T_1 := \{1, \ldots, T_0\} \) and \( T_2 := \{T_0 + 1, \ldots, T\} \).
2. Obtain \( \hat{F} \) and \( \hat{b} \) based on \( \{Z_t\}_{t \in T_1} \).
3. Compute \( \hat{V}_t^* \) with \( \hat{V}_t^* = \hat{F}(Y_t, X_t) - \hat{b}(X_t, \alpha) - \frac{1}{2}(1 - \alpha) \).
4. Compute \( \hat{Q}_{T_2}^* \), the \( (1 - \alpha)(1 + 1/|T_2|) \) empirical quantile of \( \{|\hat{V}_t^*|\}_{t \in T_2} \).

**Output:** Return the prediction set

\[
\hat{C}_{\alpha}^{\text{conf}}(X_{T+1}) = \left\{ y : \left| \hat{F}(y, X_t) - \hat{b}(X_{T+1}, \alpha) - \frac{1}{2}(1 - \alpha) \right| \leq \hat{Q}_{T_2}^* \right\}.
\]

(Since \( \hat{F}(\cdot, X_{T+1}) \) is monotonic, \( \hat{C}_{\alpha}^{\text{conf}}(X_{T+1}) \) is an interval.)
A.2 Regularity conditions

We now provide the regularity condition for Theorem 4. For simplicity we focus on the case of iid data. Recall that a legitimate cumulative distribution function $F(\cdot)$ on $\mathbb{R}$ is a non-decreasing right-continuous function such that $\lim_{z \to -\infty} F(z) = 0$ and $\lim_{z \to \infty} F(z) = 1$.

**Assumption S1.** Suppose that the following hold:

1. The data $\{(Y_t, X_t)\}_{t \in T_2}$ is iid and $|T_2| \to \infty$.
2. For any $x \in \mathcal{X}$, $\hat{F}(\cdot, x)$ is a legitimate cumulative distribution function on $\mathbb{R}$ with probability one.
3. $\sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}(x)} |\hat{F}(y, x) - F(y, x)| = o_P(1)$ as $|T_1| \to \infty$, where $\mathcal{Y}(x)$ is the support of conditional distribution $Y_t | X_t = x$.

4. There exist constants $C_1, C_2 > 0$ such that $\min_{y \in \mathcal{Y}(x)} f(y, x) \geq C_1$ and $\sup_{y \in \mathcal{Y}(x)} |y| \leq C_2$ for any $x \in \mathcal{X}$.

The following assumption is used to prove the results in Theorem 5.

**Assumption S2.** Suppose that the following hold.

1. $|T_2|^{-1} \sum_{t \in T_2} (\hat{F}(Y_t, X_t) - U_t)^2 = o_P(1)$ and $|T_2|^{-1} \sum_{t \in T_2} (\hat{b}(X_t, \alpha) - b(X_t, \alpha))^2 = o_P(1)$, where $b(\cdot, \cdot)$ is the unique function satisfying the requirement in Lemma 2.
2. $\sup_{v \in \mathbb{R}} |\hat{G}_*(v) - G_*(v)| = o_P(1)$, where $\hat{G}_*(v) = |T_2|^{-1} \sum_{t \in T_2} 1\{\hat{V}_t^* \leq v\}$ and $G_*(\cdot)$ is the distribution function of $V_t^* = U_t - b(X_t, \alpha) - \frac{1}{2}(1 - \alpha)$.
3. There exists a constant $C_1 > 0$ such that for any $x \in \mathcal{X}$, $\inf_{y \in s(x)} f(y, x) \geq C_1$, where $s(x) = [s_1(x), s_2(x)]$ is the support of the distribution $Y | X = x$.
4. $\sup_{y \in \mathbb{R}} |\hat{F}(y, X_{T+1}) - F(y, X_{T+1})| = o_P(1)$ and $\hat{b}(X_{T+1}, \alpha) = b(X_{T+1}, \alpha) + o_P(1)$.

5. There exists a constant $C_2 > 0$ such that for any $x \in \mathcal{X}$, $\max\{|s_1(x)|, |s_2(x)|\} \leq C_2$.

The key requirement in Assumption S2 is the consistency of $\hat{b}$. Since $\hat{b}$ is a solution to the optimization problem in (16), we can establish its consistency using the same argument for the consistency of an M-estimator. Under Assumption S1, we only need to impose the convexity of the mapping $z \mapsto Q(z + 1 - \alpha, x) - Q(z)$. A simple sufficient condition is that there exists constants $\kappa_1, \kappa_2 > 0$ such that for any $x \in \mathcal{X}$ and for any $z$ with $|b(x, \alpha) - z| \leq \kappa_1$, $\frac{\partial^2}{\partial z^2} (Q(z + 1 - \alpha, x) - Q(z)) \geq \kappa_2$. 

27
For uni-modal distributions, the above condition can be verified once we assume that the density \( f(\cdot, x) \) is not too flat around \( Q(b(x, \alpha), x) \) and \( Q(b(x, \alpha) + 1 - \alpha, x) \). A similar condition is imposed as Assumption 2 in Lei and Wasserman (2014).

## B Regression models for conditional distributions

An important advantage of the proposed approach is that it allows researchers to leverage powerful regression methods for estimating conditional CDFs. This section discusses semiparametric (and potentially penalized) QR and DR models, which are very popular in applied research. We emphasize that our method is generic and also works in conjunction with nonparametric estimators (e.g., Chaudhuri, 1991; Koenker et al., 1994; He et al., 1998) as well as high-dimensional methods based on trees and random forests (e.g., Chaudhuri and Loh, 2002; Meinshausen, 2006) and neural networks (e.g., Taylor, 2000).

### B.1 Quantile regression methods

QR methods impose a model for the conditional quantiles \( Q(\tau, x) \). The implied model for the conditional CDF is (Chernozhukov et al., 2013)

\[
F(y, x) = \int_0^1 1 \{Q(\tau, x) \leq y\} d\tau.
\]

A leading example is where \( Q(\tau, x) \) is assumed to be linear:

\[
Q(\tau, x) = x^\top \beta(\tau)
\]

If \( X_t \) is low dimensional, the parameter of interest \( \beta(\tau) \) can be estimated using linear QR (Koenker and Bassett, 1978) as the solution to a convex program

\[
\hat{\beta}(\tau) \in \arg \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T+1} \rho_\tau \left( Y_t - X_t^\top \beta \right),
\]

where \( \rho_\tau(u) := u(\tau - 1\{u < 0\}) \) is the check function. In problems where \( X_t \) is high-dimensional, it may be convenient to consider a penalized version of program (19):

\[
\hat{\beta}(\tau) \in \arg \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T+1} \rho_\tau \left( Y_t - X_t^\top \beta \right) + \mathcal{P}(\beta),
\]

where \( \mathcal{P}(\beta) \) is a penalty function. Examples of \( \mathcal{P}(\beta) \) include \( \ell_1 \)-penalties (e.g., Koenker, 2004; Li and Zhu, 2008; Belloni and Chernozhukov, 2011) and SCAD (e.g., Wu and Liu,
The conditional distribution can be estimated as
\[
\hat{F}(y, x) = \int_0^1 1 \left\{ x^\top \hat{\beta}(\tau) \leq y \right\} d\tau.
\]

### B.2 Distribution regression methods

Instead of modeling the conditional quantile function, one can directly model the conditional CDF using DR (e.g., Foresi and Peracchi, 1995; Chernozhukov et al., 2013, 2020). DR methods impose a generalized linear model for the CDF:
\[
F(y, x) = \Lambda \left( x^\top \beta(y) \right),
\]
where \( \beta(y) \) is the parameter of interest and \( \Lambda(\cdot) \) is a known link function, for example, the Probit or Logit link.

If \( X_t \) is low dimensional, the parameters \( \beta(y) \) can be estimated as
\[
\hat{\beta}(y) \in \text{arg max}_{\beta \in \mathbb{R}^p} \sum_{t=1}^{T+1} \left[ 1 \{ Y_t \leq y \} \log \left( \Lambda \left( X_t^\top \beta \right) \right) + 1 \{ Y_t > y \} \log \left( 1 - \Lambda \left( X_t^\top \beta \right) \right) \right].
\]

When \( \Lambda(\cdot) \) is the Probit (Logit) link, this is simply a Probit (Logit) regression of \( 1 \{ Y_t \leq y \} \) on \( X_t \). In high dimensional settings, one can use a penalized version of program (21) (e.g., with an \( \ell_1 \)-penalty or an elastic net penalty). The conditional distribution can be estimated as
\[
\hat{F}(y, x) = \Lambda \left( x^\top \hat{\beta}(y) \right).
\]

### C Proofs

#### C.1 Proof of Lemma 1

The argument is similar to Theorem 2 in Chernozhukov et al. (2018). Let \( \delta > 0 \) be a constant to be chosen later. Define quantities \( R_T = \sup_{v \in \mathbb{R}} |\hat{G}(v) - G(v)| \) and \( W = \sup_{x_1 \neq x_2} |G(x_1) - G(x_2)|/|x_1 - x_2| \).

Let \( A = \{ t : |\hat{V}_t - V_t| \geq \delta \} \). Fix \( x \in \mathbb{R} \). Then
\[
(T + 1) \left| \hat{G}(x) - G(x) \right|
\leq \left| \sum_{t \in A} \left( 1\{\hat{V}_t < x\} - 1\{V_t < x\} \right) \right| + \left| \sum_{t \in A^c} \left( 1\{\hat{V}_t < x\} - 1\{V_t < x\} \right) \right|
\leq |A| + \left| \left( \sum_{t \in A^c} 1\{\hat{V}_t < x\} \right) - \left( \sum_{t \in A^c} 1\{V_t < x\} \right) \right| \tag{22}
\]
where (i) follows by the fact that the difference of two indicators takes value in \{-1, 0, 1\}. We notice that for \(t \in A^c\), \(V_t - \delta < \hat{V}_t < V_t + \delta\). Therefore,
\[
\sum_{t \in A^c} 1\{V_t < x - \delta\} \leq \sum_{t \in A^c} 1\{\hat{V}_t < x\} \leq \sum_{t \in A^c} 1\{V_t < x + \delta\}.
\]
Since \(\sum_{t \in A^c} 1\{V_t \leq x\}\) is also between \(\sum_{t \in A^c} 1\{V_t \leq x - \delta\}\) and \(\sum_{t \in A^c} 1\{V_t \leq x + \delta\}\), it follows that
\[
\left| \left( \sum_{t \in A^c} 1\{\hat{V}_t < x\} \right) - \left( \sum_{t \in A^c} 1\{V_t < x\} \right) \right| 
\leq \left( \sum_{t \in A^c} 1\{V_t \leq x + \delta\} \right) - \left( \sum_{t \in A^c} 1\{V_t \leq x - \delta\} \right) 
\leq (T + 1) \left[ G(x + \delta) - G(x - \delta) \right] + \left( \sum_{t \in A} 1\{V_t \leq x + \delta\} - 1\{V_t \leq x - \delta\} \right) 
\leq (T + 1) \left[ G(x + \delta) - G(x - \delta) \right] + |A| 
\leq (T + 1) (2\delta W + 2R_T) + |A|,
\]
where (i) follows by the fact that the difference of two indicators takes value in \{-1, 0, 1\}. Combining the above display with (22), we obtain that
\[
(T + 1) \left| \hat{G}(x) - \hat{G}(x) \right| \leq 2|A| + (T + 1) (2\delta W + 2R_T).
\]
Since the right-hand side does not depend on \(x\), we have that
\[
\sup_{x \in \mathbb{R}} |\hat{G}(x) - \hat{G}(x)| \leq 2\frac{|A|}{T + 1} + 2\delta W + 2R_T.
\]
To bound \(|A|\), we notice that
\[
|A| \phi(\delta) \leq \sum_{t \in A} \phi(|\hat{V}_t - V_t|) \leq \sum_{t=1}^{T+1} \phi(|\hat{V}_t - V_t|) \leq o_p(T + 1).
\]
Hence, the above two displays imply that
\[
\sup_{x \in \mathbb{R}} |\hat{G}(x) - G(x)| \leq \sup_{x \in \mathbb{R}} |\hat{G}(x) - \hat{G}(x)| + R_T \leq o_p(1/\phi(\delta)) + 2\delta W + 3R_T. \tag{23}
\]
Now we fix an arbitrary $\eta \in (0,1)$. Choose $\delta = \eta/(6W)$. Since $1/\phi(\delta)$ is a constant and $R_T = o_P(1)$ by assumption, (23) implies that
\[
\limsup_{T \to \infty} P \left( \sup_{x \in \mathbb{R}} \left| \hat{G}(x) - \hat{G}(x) \right| > \eta \right) \\
\leq \limsup_{T \to \infty} P \left( |o_P(1/\phi(\delta))| > \eta/3 \right) + \limsup_{T \to \infty} P \left( |2\delta W| > \eta/3 \right) + \limsup_{T \to \infty} P \left( |R_T| > \eta/9 \right) = 0.
\]
Since $\eta > 0$ is arbitrary, we have
\[
\sup_{x \in \mathbb{R}} \left| \hat{G}(x) - G(x) \right| = o_P(1).
\]
Thus,
\[
\hat{G} (\hat{V}_{T+1}) = G(\hat{V}_{T+1}) + o_P(1) = G(V_{T+1}) + o_P(1),
\]where (i) follows by $|G(\hat{V}_{T+1}) - G(V_{T+1})| \leq W|\hat{V}_{T+1} - V_{T+1}|$. The proof is complete.

C.2 Proof of Theorem 2

Notice that
\[
P \left( \hat{Y}_{T+1} \in \hat{C}_{\text{full}}_{(1-\alpha)} (X_{T+1}) \right) = P \left( \frac{1}{T+1} \sum_{t=1}^{T+1} 1 \left\{ \hat{V}_{t} (\hat{Y}_{T+1}) \geq \hat{V}_{t} (Y_{T+1}) \right\} > \alpha \right) = P \left( 1 - \hat{G}(\hat{V}_{T+1}) > \alpha \right).
\]
By Lemma 1, $\hat{G}(\hat{V}_{T+1}) = G(V_{T+1}) + o_P(1)$. Since $G(\cdot)$ is continuous, $G(V_{T+1})$ has the uniform distribution on $(0,1)$. The desired result follows.

C.3 Proof of Theorem 3

Notice that $U_t = F(Y_t, X_t)$ is independent of $X_t$. Since $V_{T+1} = \psi(U_{T+1})$, $V_T$ is also independent of $X_{T+1}$. This means that
\[
P(G(V_{T+1}) \leq \alpha \mid X_{T+1}) = P(G(V_{T+1}) \leq \alpha).
\]
Since $G(\cdot)$ is the distribution function of $V_{T+1}$ and is a continuous function, we have that $P(G(V_{T+1}) \leq \alpha) = \alpha$. The desired result follows by Lemma 1 and
\[
P \left( Y_{T+1} \in \hat{C}_{\text{full}}_{(1-\alpha)} (X_{T+1}) \mid X_{T+1} \right) = P \left( \frac{1}{T+1} \sum_{t=1}^{T+1} 1 \left\{ \hat{V}_{t} (Y_{T+1}) \geq \hat{V}_{t} (Y_{T+1}) \right\} > \alpha \mid X_{T+1} \right) = P \left( 1 - \hat{G}(\hat{V}_{T+1}) > \alpha \mid X_{T+1} \right).
\]
C.4 Proof of Lemma 2

We proceed in three steps.

**Step 1:** show \( Q_\psi (1-\alpha) = (1-\alpha)/2 \).

By the same argument as in Lemma S1 (proved later),
\[
P \left( |F(Y_t, X_t) - b(X_t, \alpha) - (1-\alpha)/2| \leq \frac{1-\alpha}{2} \mid X_t \right) = 1 - \alpha.
\]

Therefore,
\[
P \left( |F(Y_t, X_t) - b(X_t, \alpha) - (1-\alpha)/2| \leq \frac{1-\alpha}{2} \right) = 1 - \alpha.
\]

In other words, \( Q_\psi (1-\alpha) = (1-\alpha)/2 \).

**Step 2:** show \( \mu \left( C^\text{opt}_{(1-\alpha)}(X_{T+1}) \right) = \mu \left( C^\text{conf}_{(1-\alpha)}(X_{T+1}) \right) \).

By the definition of \( C^\text{opt}_{(1-\alpha)}(X_{T+1}) \),
\[
\mu \left( C^\text{opt}_{(1-\alpha)}(X_{T+1}) \right) = \min_{F(z_2, X_{T+1}) - F(z_1, X_{T+1}) \geq 1-\alpha} z_2 - z_1.
\]

Since \( F(\cdot, X_{T+1}) \) is a continuous function, we have that
\[
\min_{F(z_2, X_{T+1}) - F(z_1, X_{T+1}) \geq 1-\alpha} z_2 - z_1 = \min_{F(z_2, X_{T+1}) - F(z_1, X_{T+1}) = 1-\alpha} z_2 - z_1.
\]

We can see this by contradiction. Let \((z_1^*, z_2^*)\) be the solution to the optimization in (24). Suppose that \( F(z_2^*, X_{T+1}) - F(z_1^*, X_{T+1}) > 1 - \alpha \). Notice that the mapping \( g(z) = F(z, X_{T+1}) - F(z_1^*, X_{T+1}) \) is continuous in \(z\). Since \( g(z_2^*) > 1 - \alpha \) and \( g(z_1^*) = 0 < 1 - \alpha \). By the intermediate value theorem, there exists \( z_2^{**} \in (z_1^*, z_2^*) \) such that \( g(z_2^{**}) = 1 - \alpha \). Thus, \( F(z_2^{**}, X_{T+1}) - F(z_1^*, X_{T+1}) \geq 1 - \alpha \) and \( z_2^{**} - z_1^* < z_2^* - z_1^* \), contradicting the assumption that \((z_1^*, z_2^*)\) is the solution to the optimization in (24). Therefore, \( F(z_2^*, X_{T+1}) - F(z_1^*, X_{T+1}) = 1 - \alpha \). Therefore, we have that
\[
\mu \left( C^\text{opt}_{(1-\alpha)}(X_{T+1}) \right) = \min_{F(z_2, X_{T+1}) - F(z_1, X_{T+1}) = 1-\alpha} z_2 - z_1.
\]

Since \( F(z_2, X_{T+1}) - F(z_1, X_{T+1}) = 1 - \alpha \), we can write \( F(z_2, X_{T+1}) = F(z_1, X_{T+1}) + 1 - \alpha \), which means \( z_2 = Q(F(z_1, X_{T+1}) + 1 - \alpha, X_{T+1}) \). Since \( F(z_1, X_{T+1}) + 1 - \alpha \leq 1 \), we have \( F(z_1, X_{T+1}) \leq \alpha \). Therefore,
\[
\mu \left( C^\text{opt}_{(1-\alpha)}(X_{T+1}) \right) = \min_{F(z_1, X_{T+1}) \in [0, \alpha]} Q(F(z_1, X_{T+1}) + 1 - \alpha, X_{T+1}) - z_1.
\]
where (i) follows by a change of variables $w = F(z_1, X_{T+1})$ (and thus $z_1 = Q(w, X_{T+1})$).

We notice that
\[
C_{(1-\alpha)}(X_{T+1}) = \{ y : |F(y, X_{T+1}) - b(X_{T+1}, \alpha) - (1-\alpha)/2| \leq Q_\psi(1-\alpha) \}
\]
\[
= \{ y : |F(y, X_{T+1}) - b(X_{T+1}, \alpha) - (1-\alpha)/2| \leq (1-\alpha)/2 \}
\]
\[
= \{ y : b(X_{T+1}, \alpha) \leq F(y, X_{T+1}) \leq b(X_{T+1}, \alpha) + 1 - \alpha \}
\]
\[
= [Q(b(X_{T+1}, \alpha), X_{T+1}), Q(b(X_{T+1}, \alpha) + 1 - \alpha, X_{T+1})],
\]
where (i) follows by $Q_\psi = (1-\alpha)/2$. Thus,
\[
\mu \left( C_{(1-\alpha)}(X_{T+1}) \right) = \min_{w \in [0, \alpha]} Q(w + 1 - \alpha, X_{T+1}) - Q(w, X_{T+1}),
\]
where (i) follows by the assumption that $b(x, \alpha) \in \text{arg min}_{w \in [0, \alpha]} Q(w + 1 - \alpha, x) - Q(w, x)$ for any $x \in X$. By (26), $\mu \left( C_{(1-\alpha)}^\text{opt}(X_{T+1}) \right) = \mu \left( C_{(1-\alpha)}^\text{conf}(X_{T+1}) \right)$.

**Step 3:** show that if $C_{(1-\alpha)}^\text{opt}(x)$ is uniquely defined for any $x \in X$, then $C_{(1-\alpha)}^\text{opt}(X_{T+1}) = C_{(1-\alpha)}^\text{conf}(X_{T+1})$.

Notice that $C_{(1-\alpha)}^\text{opt}(x) = [r_1(x, \alpha), r_2(x, \alpha)]$, where the pair $(r_1(x, \alpha), r_2(x, \alpha))$ uniquely solves
\[
\min_{z_1, z_2} z_2 - z_1 \quad \text{s.t.} \quad F(z_2, x) - F(z_1, x) \geq 1 - \alpha.
\]

By the argument in (25), the pair $(r_1(x, \alpha), r_2(x, \alpha))$ uniquely solves
\[
\min_{z_1, z_2} z_2 - z_1 \quad \text{s.t.} \quad F(z_2, x) - F(z_1, x) = 1 - \alpha.
\]

By the same change of variables in (26), $r_1(x, \alpha)$ uniquely solves
\[
\min_{z_1} Q(F(z_1, x) + 1 - \alpha, x) - z_1 \quad \text{s.t.} \quad F(z_1, X_{T+1}) \leq \alpha
\]
and $r_2(x, \alpha) = Q(F(r_1(x, \alpha), x) + 1 - \alpha, x)$. Similar to (26), this can be rewritten as an optimization problem on $[0, \alpha]$. Since $b(x, \alpha)$ solves $\min_{w \in [0, \alpha]} Q(w + 1 - \alpha, x) - Q(w, x)$, we have
\[
r_1(x, \alpha) = Q(b(x, \alpha), x)
\]
and $r_2(x, \alpha) = Q(b(x, \alpha)+1-\alpha, x)$. Thus, $C_{(1-\alpha)}^\text{opt}(x) = [r_1(x, \alpha), r_2(x, \alpha)] = [Q(b(x, \alpha), x), Q(b(x, \alpha)+1-\alpha, x)]$. By the same argument as in (27), $C_{(1-\alpha)}^\text{conf}(x) = [Q(b(x, \alpha), x), Q(b(x, \alpha)+1-\alpha, x)]$. Therefore, $C_{(1-\alpha)}^\text{opt}(x) = C_{(1-\alpha)}^\text{conf}(x)$. Since this holds for any $x \in X$, we have completed the proof.
C.5 Proof of Theorem 4

We first prove three auxiliary lemmas.

**Lemma S1.** Let Assumption S1 hold. Let $\tilde{V}_t^* = F(Y_t, X_t) - \hat{b}(X_t) - (1 - \alpha)/2$ for $t \in T_2$. Then

$$P \left( |\tilde{V}_t^*| \leq \frac{1 - \alpha}{2} | X_t \right) = 1 - \alpha.$$ 

Moreover, for any non-random $\delta \in [-\alpha, \alpha]$,

$$P \left( |\tilde{V}_t^*| \leq \frac{1 - \alpha}{2} + \delta \right) - (1 - \alpha) \leq \delta \quad \text{if } \delta \in [-\alpha, 0]$$

and

$$P \left( |\tilde{V}_t^*| \leq \frac{1 - \alpha}{2} + \delta \right) - (1 - \alpha) \geq \delta \quad \text{if } \delta \in [0, \alpha].$$

**Proof.** We show the two claims in two steps.

**Step 1:** show the first claim.

We observe that

$$P \left( |\tilde{V}_t^*| \leq \frac{1 - \alpha}{2} | X_t \right) = P \left( |U_t - \hat{b}(X_t) - (1 - \alpha)/2| \leq \frac{1 - \alpha}{2} | X_t \right)$$

$$= P \left( \hat{b}(X_t) \leq U_t \leq \hat{b}(X_t) + 1 - \alpha | X_t \right).$$

Recall that $U_t = F(Y_t, X_t)$ is independent of $X_t$ and has the uniform distribution on $[0,1]$. Since $t \in T_2$, $(U_t, X_t)$ is independent of $\hat{b}(\cdot)$. Since $\hat{b}(X_t) \in [0, \alpha]$, we have that $[\hat{b}(X_t), \hat{b}(X_t) + 1 - \alpha] \subseteq [0, 1]$. Therefore,

$$P \left( |\tilde{V}_t^*| \leq \frac{1 - \alpha}{2} | X_t \right) = P \left( \hat{b}(X_t) \leq U_t \leq \hat{b}(X_t) + 1 - \alpha | X_t \right)$$

$$= \left( \hat{b}(X_t) + 1 - \alpha \right) - \hat{b}(X_t) = 1 - \alpha.$$

**Step 2:** show the second claim.

By the same argument as in Step 1, we have

$$P \left( |\tilde{V}_t^*| \leq \frac{1 - \alpha}{2} + \delta | X_t \right) = P \left( |U_t - \hat{b}(X_t) - (1 - \alpha)/2| \leq \frac{1 - \alpha}{2} + \delta | X_t \right)$$

$$= P \left( \hat{b}(X_t) - \delta \leq U_t \leq \hat{b}(X_t) + 1 - \alpha + \delta | X_t \right)$$

$$\overset{(i)}{=} P \left( \max\{\hat{b}(X_t) - \delta, 0\} \leq U_t \leq \min\{\hat{b}(X_t) + 1 - \alpha + \delta, 1\} | X_t \right)$$

$$\overset{(ii)}{=} \min\{\hat{b}(X_t) + 1 - \alpha + \delta, 1\} - \max\{\hat{b}(X_t) - \delta, 0\}.$$
where (i) and (ii) follow by the fact that $U_t$ has the uniform distribution on $[0,1]$ and is independent of $X_t$ and $\hat{b}(\cdot)$. Thus,

\[
\begin{align*}
P \left( |\hat{V}_t^\ast| \leq \frac{1-\alpha}{2} + \delta \mid X_t \right) &= (1-\alpha) \\
&= \min\{\hat{b}(X_t) + 1 - \alpha + \delta, 1\} + \min\{\delta - \hat{b}(X_t), 0\} - (1-\alpha) \\
&= \min\{\hat{b}(X_t) + \delta, \alpha\} + \min\{\delta - \hat{b}(X_t), 0\} \\
&= \min\{\delta, \alpha - \hat{b}(X_t)\} + \hat{b}(X_t) + \min\{\delta, \hat{b}(X_t)\} - \hat{b}(X_t) \\
&= \min\{\delta, \alpha - \hat{b}(X_t)\} + \min\{\delta, \hat{b}(X_t)\}. \quad (28)
\end{align*}
\]

We now consider the random mapping $\delta \mapsto g(\delta) = \min\{\delta, \alpha - \hat{b}(X_t)\} + \min\{\delta, \hat{b}(X_t)\}$, where the randomness is from the randomness of $X_t$. Clearly,

\[ Eg(\delta) = \delta + \delta = 2\delta \leq \delta \quad \text{for } \delta \in [-\alpha, 0]. \]

For $\delta \in [0, \max\{\alpha - \hat{b}(X_t), \hat{b}(X_t)\}]$, we have $g(\delta) \geq \delta$. For $\delta \in [\max\{\alpha - \hat{b}(X_t), \hat{b}(X_t)\}, \alpha]$, we have that

\[ g(\delta) = \alpha \geq \delta. \]

Hence, for any $\delta \in [0, \alpha]$, we have $g(\delta) \geq \delta$, which implies $Eg(\delta) \geq \delta$. The proof is complete. \qed

**Lemma S2.** Let Assumption S1 hold. Then $\sup_{(a,x)\in[0,1]\times\mathcal{X}} |\hat{Q}(a, x) - Q(a, x)| = o_P(1)$ and $\sup_{x \in \mathcal{X}} |\hat{L}(x) - L(x)| = o_P(1)$.

**Proof.** Let $\varepsilon = \sup_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}(x)} |\hat{F}(y, x) - F(y, x)|$. Fix an arbitrary $\delta > 0$ and $x \in \mathcal{X}$. Since $\hat{F}(\cdot, x)$ is right-continuous and $\hat{Q}(a, x) = \inf\{y : \hat{F}(y, x) \geq a\}$ for any $a \in [0,1]$, we have that $\hat{F}(\hat{Q}(a, x), x) = a$. For simplicity, we write $\hat{F}(y, x), \hat{Q}(a, x), F(y, x)$ and $Q(a, x)$ as $\hat{F}(y), \hat{Q}(a)$, respectively whenever no confusion arises.

We consider the event $\{Q(a) > \hat{Q}(a) + \delta\}$:

\[
\{Q(a) > \hat{Q}(a) + \delta\} \subseteq \left\{ F(Q(a)) \geq F(\hat{Q}(a) + \delta) \right\} = \left\{ a \geq F(\hat{Q}(a) + \delta) \right\} \subseteq \left\{ a \geq F(\hat{Q}(a)) + C_1\delta \right\} \subseteq \left\{ a \geq \hat{F}(\hat{Q}(a)) - \varepsilon + C_1\delta \right\} \quad \text{(ii)},
\]

where (i) follows by the fact that $F(b + \delta, x) = F(b, x) + \int_b^{b+\delta} f(z, x)dz \geq F(b, x) + \int_b^{b+\delta} C_1dz = F(b, x) + C_1\delta$ and (ii) follows by $\hat{F}(\hat{Q}(a)) = a$. Similarly, we observe that

35
\[
\{ \hat{Q}(a) > Q(a) + \delta \} \subseteq \{ \hat{F}(\hat{Q}(a)) \geq \hat{F}(Q(a) + \delta) \} = \{ a \geq \hat{F}(Q(a) + \delta) \} \\
\subseteq \{ a \geq F(Q(a) + \delta) - \varepsilon \} \subseteq \{ a \geq F(Q(a)) + C_1 \delta - \varepsilon \} = \{ \varepsilon \geq C_1 \delta \}.
\]

By the above two displays, we have that
\[
\{ \left| \hat{Q}(a) - Q(a) \right| > \delta \} = \{ Q(a) > \hat{Q}(a) + \delta \} \cup \{ \hat{Q}(a) > Q(a) + \delta \} \subseteq \{ \varepsilon \geq C_1 \delta \}.
\]

Notice that the right-hand side \( \{ \varepsilon \geq C_1 \delta \} \) does not depend on \( x \) or \( a \). Therefore,
\[
\sup_{(a,x) \in [0,1] \times X} \left| \hat{Q}(a,x) - Q(a,x) \right| > \delta \subseteq \{ \varepsilon \geq C_1 \delta \}.
\]

Hence,
\[
P \left( \sup_{(a,x) \in [0,1] \times X} \left| \hat{Q}(a,x) - Q(a,x) \right| > \delta \right) \leq P (\varepsilon \geq C_1 \delta) \overset{(i)}{=} o(1),
\]

where (i) follows by \( \varepsilon = o_P(1) \). Since \( \delta > 0 \) is arbitrary, we have proved \( \sup_{(a,x) \in [0,1] \times X} |\hat{Q}(a,x) - Q(a,x)| = o_P(1) \).

To show \( \sup_{x \in X} |\hat{L}(x) - L(x)| = o_P(1) \), we define \( \eta = \sup_{(a,x) \in [0,1] \times X} |\hat{Q}(a,x) - Q(a,x)| \).

We observe that
\[
\hat{L}(x) = \min_{z \in [0,\alpha]} \hat{Q}(z + 1 - \alpha, x) - \hat{Q}(z, x) \\
\leq \min_{z \in [0,\alpha]} (Q(z + 1 - \alpha, x) - Q(z, x) + 2\eta) = L(x) + 2\eta
\]

and
\[
\hat{L}(x) = \min_{z \in [0,\alpha]} \hat{Q}(z + 1 - \alpha, x) - \hat{Q}(z, x) \\
\geq \min_{z \in [0,\alpha]} (Q(z + 1 - \alpha, x) - Q(z, x) - 2\eta) = L(x) - 2\eta.
\]

Thus, \( |\hat{L}(x) - L(x)| \leq 2\eta \). Since this holds for any \( x \in X \), we have \( \sup_{x \in X} |\hat{L}(x) - L(x)| \leq 2\eta \). Because we have proved \( \eta = o_P(1) \), it follows that \( \sup_{x \in X} |\hat{L}(x) - L(x)| = o_P(1) \). The proof is complete.

\textbf{Lemma S3.} Let Assumption S1 hold. Then \( \hat{Q}_T^* = (1 - \alpha)/2 + o_P(1) \).

\textit{Proof.} Fix an arbitrary \( \delta \in (0, \alpha) \). Define the event

36
$\mathcal{A} = \left\{ \max_{t \in T_2} |\hat{V}_t^* - \tilde{V}_t^*| \leq \delta/2 \right\}$

$$\bigcap \left\{ \left| T_2 \right|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \} - P \left( |\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \right) \leq \delta/4 \right\}$$

$$\bigcap \left\{ \left| T_2 \right|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2 \} - P \left( |\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2 \right) \leq \delta/4 \right\}.$$  

Since $\hat{Q}_{T_2}^*$ is the $(1 - \alpha)(1 + |T_2|^{-1})$ sample quantile of $\{|\hat{V}_t^*|\}_{t \in T_2}$, we have that

$$(1 - \alpha)(1 + |T_2|^{-1}) - |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq \hat{Q}_{T_2}^* \} \leq (1 - \alpha)(1 + |T_2|^{-1}) + |T_2|^{-1}. \tag{29}$$

We consider the two events $\mathcal{M}_1 = \{ \hat{Q}_{T_2}^* > (1 - \alpha)/2 + \delta \}$ and $\mathcal{M}_2 = \{ \hat{Q}_{T_2}^* < (1 - \alpha)/2 - \delta \}$. We will show three claims: $P(\mathcal{A}^c) = o(1)$, $P(\mathcal{M}_1 \cap \mathcal{A}) = o(1)$ and $P(\mathcal{M}_2 \cap \mathcal{A}) = o(1)$. Then the desired result follows by $P(|\hat{Q}_{T_2}^* - (1 - \alpha)/2| > \delta) = P(\mathcal{M}_1 \cup \mathcal{M}_2)$ together with

$$P \left( \mathcal{M}_1 \bigcup \mathcal{M}_2 \right) \leq P(\mathcal{A}^c) + P \left( (\mathcal{M}_1 \bigcup \mathcal{M}_2) \cap \mathcal{A} \right) \leq P(\mathcal{A}^c) + P(\mathcal{M}_1 \cap \mathcal{A}) + P(\mathcal{M}_2 \cap \mathcal{A}).$$

We show these three claims in three steps.

**Step 1:** show $P(\mathcal{A}^c) = o(1)$.

Since $\{|\hat{V}_t^*|\}_{t \in T_2}$ is independent, we have

$$E \left[ \left| T_2 \right|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \} - P \left( |\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \right) \right]^2$$

$$= \left| T_2 \right|^{-2} \sum_{t \in T_2} E \left[ 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \} - P \left( |\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \right) \right]^2 \leq \frac{1}{4|T_2|},$$

where (i) follows by the fact that $E(Z - P(Z = 1))^2 = P(Z = 1) \cdot (1 - P(Z = 1)) \leq \max_{x \in [0,1]} x(1 - x) \leq 1/4$ for any Bernoulli variable $Z$. Thus,

$$P \left( \left| T_2 \right|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \} - P \left( |\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2 \right) \right) > \delta/4 \leq \frac{1}{\delta|T_2|} = o(1).$$

Similarly, we can show that

$$P \left( \left| T_2 \right|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2 \} - P \left( |\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2 \right) \right) > \delta/4 \leq \frac{1}{\delta|T_2|} = o(1).$$

We notice that $|\hat{V}_t^* - \tilde{V}_t^*| \leq |\hat{F}(Y_t, X_t) - F(Y_t, X_t)|$ and thus

$$P \left( \max_{t \in T_2} |\hat{V}_t^* - \tilde{V}_t^*| > \delta/2 \right) \leq P \left( \sup_{x \in X} \sup_{y \in Y(x)} |\hat{F}(y, x) - F(y, x)| > \delta/2 \right) = o(1).$$
Therefore, $P(A^c) = o(1)$.

**Step 2:** show $P(M_1 \cap A) \to 0$.

On the event $M_1 \cap A$, we have that

\[
|T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2\} \\
= |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \hat{V}_t^* - |\hat{V}_t^*|\} \\
\leq |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2\} \\
\leq |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq \hat{Q}_T^*\} \leq (1 - \alpha)(1 + |T_2|^{-1}) + |T_2|^{-1},
\]

where (i) follows by the definition of $A$, (ii) follows by the definition of $M_1$ and (iii) follows by (29). On the event $A$, we have

\[
|T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2\} \geq P\left(|\hat{V}_t^*| \leq (1 - \alpha)/2 + \delta/2\right) - \delta/4 \geq (1 - \alpha) + \delta/2 - \delta/4 = (1 - \alpha) + \delta/4,
\]

where (i) follows by Lemma S1.

The above two displays imply that on the event $M_1 \cap A$, $(1 - \alpha) + \delta/4 \leq (1 - \alpha)(1 + |T_2|^{-1}) + |T_2|^{-1}$, which is $\delta \leq 4(2 - \alpha)/|T_2|$. Since $|T_2| \to \infty$ and $\delta > 0$ is fixed, we have

\[
P(M_1 \cap A) \leq 1\{\delta \leq 4(2 - \alpha)/|T_2|\} = o(1).
\]

**Step 3:** show $P(M_2 \cap A) \to 0$.

The argument is similar to Step 2. On the event $M_2 \cap A$, we have that

\[
|T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2\} \\
= |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2 + |\hat{V}_t^*| - |\hat{V}_t^*|\} \\
\geq |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2\} \\
\geq |T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq \hat{Q}_T^*\} \geq (1 - \alpha)(1 + |T_2|^{-1}) - |T_2|^{-1}.
\]

On the event $A$, we have

\[
|T_2|^{-1} \sum_{t \in T_2} 1\{|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2\} \leq P\left(|\hat{V}_t^*| \leq (1 - \alpha)/2 - \delta/2\right) + \delta/4
\]

38
where (i) follows by Lemma S1.

The above two displays imply that on the event $\mathcal{M}_2 \cap \mathcal{A}$, $(1 - \alpha) - \delta/4 \geq (1 - \alpha)(1 + |\mathcal{T}_2|^{-1}) - |\mathcal{T}_2|^{-1}$, which is $\delta/4 \leq \alpha/|\mathcal{T}_2|$. Since $|\mathcal{T}_2| \to \infty$ and $\delta > 0$ is fixed, we have

$$P(\mathcal{M}_2 \cap \mathcal{A}) \leq 1\{\delta \leq 4\alpha/|\mathcal{T}_2|\} = o(1).$$

The proof is complete. $\square$

We are now ready to prove Theorem 4.

**Proof of Theorem 4.** Let $\varepsilon_1 = \hat{Q}_\mathcal{T}_2 - (1 - \alpha)/2$, $\varepsilon_2 = \sup_{y,x} |\hat{F}(y, x) - F(y, x)|$, $\varepsilon_3 = \sup_{(a,x) \in [0,1] \times \mathcal{X}} |\hat{Q}(a, x) - Q(a, x)|$ and $\varepsilon_4 = \sup_{x \in \mathcal{X}} |\hat{L}(x) - L(x)|$. For simplicity, we write $\hat{C}_{\text{conf}}^{(1 - \alpha)}$ instead of $\hat{C}_{\text{conf}}^{(1 - \alpha)}(X_{T+1})$. We proceed in two steps.

**Step 1:** show asymptotic conditional validity.

To show $P\left(Y_{T+1} \in \hat{C}_{\text{conf}}^{(1 - \alpha)} | X_{T+1}\right) = 1 - \alpha + o_P(1)$, it suffices to verify that $P(|\hat{V}_{T+1}| \leq \hat{Q}_\mathcal{T}_2 | X_{T+1}) = 1 - \alpha + o_P(1)$. We notice that

$$P\left(|\hat{V}_{T+1}| \leq \hat{Q}_\mathcal{T}_2 | X_{T+1}\right) \leq (1 - \alpha) - \delta/2 + \delta/4 = (1 - \alpha) - \delta/4,$$

(30)

Observe that

$$P\left(F(Y_{T+1}, X_{T+1}) < \hat{b}(X_{T+1}) - \varepsilon_1 - \varepsilon_2 | X_{T+1}\right) \leq P\left(F(Y_{T+1}, X_{T+1}) < \hat{b}(X_{T+1}) - \varepsilon_1 | X_{T+1}\right)$$

$$\leq P\left(F(Y_{T+1}, X_{T+1}) < \hat{b}(X_{T+1}) - \varepsilon_1 + \varepsilon_2 | X_{T+1}\right).$$

Since $F(Y_{T+1}, X_{T+1})$ is independent of $(\varepsilon_1, \varepsilon_2, \hat{b}(X_{T+1}), X_{T+1})$ and has the uniform distribution on $[0,1]$, it follows that

$$\beta\left(\hat{b}(X_{T+1}) - \varepsilon_1 - \varepsilon_2\right) \leq P\left(F(Y_{T+1}, X_{T+1}) < \hat{b}(X_{T+1}) - \varepsilon_1 | X_{T+1}\right) \leq \beta\left(\hat{b}(X_{T+1}) - \varepsilon_1 + \varepsilon_2\right),$$

39
where
\[
\beta(z) = \begin{cases} 
1 & \text{if } z > 1 \\
0 & \text{if } z < 0 \\
z & \text{otherwise.}
\end{cases}
\]

Clearly, \(|\beta(z_1) - \beta(z_2)| \leq |z_1 - z_2|\) for any \(z_1, z_2 \in \mathbb{R}\). Thus, \(|\beta(\hat{b}(X_{T+1}) - \varepsilon_1 + \varepsilon_2) - \beta(\hat{b}(X_{T+1}))| \leq |\varepsilon_1 + \varepsilon_2|\) and \(|\beta(\hat{b}(X_{T+1}) - \varepsilon_1 - \varepsilon_2) - \beta(\hat{b}(X_{T+1}))| \leq |\varepsilon_1 - \varepsilon_2|\). This means that
\[
|P\left(\hat{F}(Y_{T+1}, X_{T+1}) < \hat{b}(X_{T+1}) - \varepsilon_1 \mid X_{T+1}\right) - \beta(\hat{b}(X_{T+1}))| \leq |\varepsilon_1| + |\varepsilon_2|.
\]

Similarly,
\[
|P\left(\hat{F}(Y_{T+1}, X_{T+1}) \leq \hat{b}(X_{T+1}) + 1 - \alpha + \varepsilon_1 \mid X_{T+1}\right) - \beta(\hat{b}(X_{T+1}) + 1 - \alpha)| \leq |\varepsilon_1| + |\varepsilon_2|.
\]

By \(\hat{b}(X_{T+1}) \in [0, \alpha]\), we have \(\beta(\hat{b}(X_{T+1})) = \hat{b}(X_{T+1})\) and \(\beta(\hat{b}(X_{T+1}) + 1 - \alpha) = \hat{b}(X_{T+1}) + 1 - \alpha\). Hence, the above two displays imply that
\[
|P\left(\hat{F}(Y_{T+1}, X_{T+1}) \leq \hat{b}(X_{T+1}) + (1 - \alpha) + \varepsilon_1 \mid X_{T+1}\right) - P\left(\hat{F}(Y_{T+1}, X_{T+1}) < \hat{b}(X_{T+1}) - \varepsilon_1 \mid X_{T+1}\right) - (1 - \alpha)| \leq 2|\varepsilon_1| + 2\varepsilon_2.
\]

By (30), we have
\[
|P\left(|\hat{V}_{T+1}^*| \leq \hat{Q}_{T_2}^* \mid X_{T+1}\right) - (1 - \alpha)| \leq 2|\varepsilon_1| + 2\varepsilon_2.
\]

Since \(\varepsilon_1\) and \(\varepsilon_2\) are \(o_P(1)\), we have \(|P\left(|\hat{V}_{T+1}^*| \leq \hat{Q}_{T_2}^* \mid X_{T+1}\right) - (1 - \alpha)| = o_P(1)\).

**Step 2:** show asymptotic efficiency.

We can rewrite the interval \(\bar{C}^{\text{conf}}_{(1-\alpha)} = \{y : |\hat{F}(y, X_{T+1}) - \hat{b}(X_{T+1}) - (1 - \alpha)/2| \leq \hat{Q}_{T_2}^*\}\) as
\[
\bar{C}^{\text{conf}}_{(1-\alpha)} = \{y : \hat{b}(X_{T+1}) + (1 - \alpha)/2 - \hat{Q}_{T_2}^* \leq \hat{F}(y, X_{T+1}) \leq \hat{b}(X_{T+1}) + (1 - \alpha)/2 + \hat{Q}_{T_2}^*\}.
\]

In other words, we can write it as
\[
\bar{C}^{\text{conf}}_{(1-\alpha)} = \left[\hat{Q}\left(\beta(\hat{b}(X_{T+1}) + (1 - \alpha)/2 - \hat{Q}_{T_2}^*), X_{T+1}\right), \hat{Q}\left(\beta(\hat{b}(X_{T+1}) + (1 - \alpha)/2 + \hat{Q}_{T_2}^*), X_{T+1}\right)\right].
\]

We can now compute the length of \(\bar{C}^{\text{conf}}_{(1-\alpha)}\). We observe
\[
\mu\left(\bar{C}^{\text{conf}}_{(1-\alpha)}\right) = \hat{Q}\left(\beta(\hat{b}(X_{T+1}) + (1 - \alpha)/2 + \hat{Q}_{T_2}^*), X_{T+1}\right) - \hat{Q}\left(\beta(\hat{b}(X_{T+1}) + (1 - \alpha)/2 - \hat{Q}_{T_2}^*), X_{T+1}\right)
\]
\[
\overset{(i)}{=} \hat{Q}\left(\beta(\hat{b}(X_{T+1}) + 1 - \alpha + \varepsilon_1), X_{T+1}\right) - \hat{Q}\left(\beta(\hat{b}(X_{T+1}) - \varepsilon_1), X_{T+1}\right)
\]

40
\[ Q\left(\beta(\hat{b}(X_{T+1}) + 1 - \alpha + \varepsilon_1), X_{T+1}\right) - Q\left(\beta(\hat{b}(X_{T+1}) - \varepsilon_1), X_{T+1}\right) + 2\varepsilon_3, \]

where (i) follows by \( \hat{Q}\overline{^*}_{T_2} = (1 - \alpha)/2 + \varepsilon_1 \).

We notice that for any \( a_1, a_2 \in [0, 1] \) with \( a_1 > a_2 \) and for any \( x \in \mathcal{X} \),
\[ Q(a_1, x) - Q(a_2, x) = \int_{a_2}^{a_1} \left( \frac{\partial Q(z, x)}{\partial z} \right) dz = \int_{a_2}^{a_1} \left( \frac{1}{f(Q(z, x))} \right) dz \leq \int_{a_2}^{a_1} \left( \frac{1}{C_1} \right) dz = (a_1 - a_2)/C_1. \]

Therefore,
\[ \sup_{a_1, a_2 \in [0, 1], a_1 \neq a_2} \sup_{x \in \mathcal{X}} \left| \frac{Q(a_1, x) - Q(a_2, x)}{a_1 - a_2} \right| \leq 1/C_1. \]

Since \( |\beta(z_1) - \beta(z_2)| \leq |z_1 - z_2| \) for any \( z_1, z_2 \in \mathbb{R} \), it follows that
\[ |Q\left(\beta(\hat{b}(X_{T+1}) + 1 - \alpha + \varepsilon_1), X_{T+1}\right) - Q\left(\beta(\hat{b}(X_{T+1}) + 1 - \alpha), X_{T+1}\right)| \leq \beta(\hat{b}(X_{T+1}) + 1 - \alpha + \varepsilon_1) - \beta(\hat{b}(X_{T+1}) + 1 - \alpha) \leq |\varepsilon_1|/C_1 \]
and
\[ |Q\left(\beta(\hat{b}(X_{T+1}) - \varepsilon_1), X_{T+1}\right) - Q\left(\beta(\hat{b}(X_{T+1})), X_{T+1}\right)| \leq |\varepsilon_1|/C_1. \]

The above two displays and (31) imply
\[ \mu\left(\mathcal{C}_{(1-\alpha)}^{\text{conf}}\right) \leq 2|\varepsilon_1|/C_1 + 2\varepsilon_3 + Q\left(\beta(\hat{b}(X_{T+1}) + 1 - \alpha), X_{T+1}\right) - Q\left(\beta(\hat{b}(X_{T+1})), X_{T+1}\right) \]
\[ \overset{(i)}{=} 2|\varepsilon_1|/C_1 + 2\varepsilon_3 + Q\left(\hat{b}(X_{T+1}) + 1 - \alpha, X_{T+1}\right) - Q\left(\hat{b}(X_{T+1}), X_{T+1}\right) \]
\[ \leq 2|\varepsilon_1|/C_1 + 4\varepsilon_3 + \hat{Q}\left(\hat{b}(X_{T+1}) + 1 - \alpha, X_{T+1}\right) - \hat{Q}\left(\hat{b}(X_{T+1}), X_{T+1}\right) \]
\[ = 2|\varepsilon_1|/C_1 + 4\varepsilon_3 + \hat{L}(X_{T+1}) \]
\[ \leq 2|\varepsilon_1|/C_1 + 4\varepsilon_3 + \varepsilon_4 + L(X_{T+1}) \overset{(ii)}{=} L(X_{T+1}) + o_P(1), \]

where (i) follows by \( \hat{b}(X_{T+1}) \in [0, \alpha] \) and (ii) follows by \( \varepsilon_3 = o_P(1) \) and \( \varepsilon_4 = o_P(1) \) (Lemma S2) as well as \( \varepsilon_1 = o_P(1) \) (Lemma S3). The desired result follows by
\[ \mu\left(\mathcal{C}_{(1-\alpha)}^{\text{opt}}(X_{T+1})\right) = \min_{F(z_1, X_{T+1}) - F(z_2, X_{T+1}) \geq 1-\alpha} z_1 - z_2 \]
\[ = \min_{F(z_1, X_{T+1}) - F(z_2, X_{T+1}) = 1-\alpha} z_1 - z_2 \]
\[ = \min_{z \in [0, \alpha]} Q(z + 1 - \alpha, X_{T+1}) - Q(z, X_{T+1}) \]
\[ = L(X_{T+1}). \]
C.6 Proof of Theorem 5

For simplicity, we may omit $X_{T+1}$ and $\alpha$ when no confusion can arise. For example, we write $F(y)$, $Q(y)$, $f(y)$ and $b$ rather than $F(y, X_{T+1})$, $Q(y, X_{T+1})$, $f(y, X_{T+1})$ and $b(X_{T+1}, \alpha)$, respectively.

By Lemma 2, we have that

$$C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) = C_{(1-\alpha)}^{\text{conf}}(X_{T+1}) = \left\{ y : \left| F(y) - b - \frac{1-\alpha}{2} \right| \leq Q_\psi(1-\alpha) \right\},$$

where $Q_\psi(1-\alpha)$ is the $(1-\alpha)$ quantile of $V_t^* = F(Y_t, X_t) - b(X_t, \alpha) - \frac{1-\alpha}{2}$. Again by Lemma 2, $Q_\psi(1-\alpha) = \frac{1-\alpha}{2}$. Therefore,

$$C_{(1-\alpha)}^{\text{opt}}(X_{T+1}) = \{ y : b \leq F(y) \leq b + 1 - \alpha \}.$$  \hspace{1cm} (32)

On the other hand,

$$C_{(1-\alpha)}^{\text{conf}}(X_{T+1}) = \left\{ y : \hat{b}(X_{T+1}, \alpha) + \frac{1-\alpha}{2} - \hat{Q}_{\tau_2}^* \leq \hat{F}(y, X_{T+1}) \leq \hat{b}(X_{T+1}, \alpha) + \frac{1-\alpha}{2} + \hat{Q}_{\tau_2}^* \right\} = \left\{ y : b + \varepsilon_1(y) \leq F(y, X_{T+1}) \leq b + 1 - \alpha + \varepsilon_2(y) \right\},$$

where $\varepsilon_1(y) = \hat{b}(X_{T+1}, \alpha) - b + \frac{1-\alpha}{2} - \hat{Q}_{\tau_2}^* + F(y, X_{T+1}) - \hat{F}(y, X_{T+1})$ and $\varepsilon_2(y) = \varepsilon_1(y) + 2\hat{Q}_{\tau_2}^* - (1-\alpha)$.

The rest of the proof proceeds in two steps.

**Step 1:** show that $\hat{Q}_{\tau_2}^* = (1-\alpha)/2 + o_P(1)$.

Notice that

$$\left| \hat{V}_t^* - \tilde{V}_t^* \right| \leq \left| \hat{F}(Y_t, X_t) - F(Y_t, X_t) \right| + \left| \hat{b}(X_t, \alpha) - b(X_t, \alpha) \right|.$$

By the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have that

$$\left| \tau_2 \right|^{-1} \sum_{t \in \tau_2} \left( \left| \hat{V}_t^* - \tilde{V}_t^* \right| \right)^2 \leq 2\left| \tau_2 \right|^{-1} \sum_{t \in \tau_2} \left( \hat{F}(Y_t, X_t) - F(Y_t, X_t) \right)^2 + 2\left| \tau_2 \right|^{-1} \sum_{t \in \tau_2} \left( \hat{b}(X_t, \alpha) - b(X_t, \alpha) \right)^2 = o_P(1).$$

We now show that $G_\ast(\cdot)$ is Lipschitz. Fix any $y_1, y_2$ in the support of $V_t^*$ such that $y_1 < y_2$. Notice that

$$P(y_1 \leq \left| V_t^* \right| \leq y_2 \mid X_t)$$

$$= P \left( y_1 \leq \left| U_t - b(X_t, \alpha) - \frac{1}{2}(1-\alpha) \right| \leq y_2 \mid X_t \right)$$

42
\[ P \left( y_1 \leq U_t - b(X_t, \alpha) - \frac{1}{2}(1 - \alpha) \leq y_2 \mid X_t \right) + P \left( y_1 \leq - \left[ U_t - b(X_t, \alpha) - \frac{1}{2}(1 - \alpha) \right] \leq y_2 \mid X_t \right) \]

\[(y_2 - y_1) + (y_2 - y_1) \leq 2(y_2 - y_1), \]

where (i) follows by the fact that conditional on \( X_t, U_t \) follows the uniform distribution on \((0, 1)\). Thus,

\[
G_\ast(y_2) - G_\ast(y_1) = P(y_1 \leq |V_t^\ast| \leq y_2) \leq 2(y_2 - y_1).
\]

Therefore, \( \sup_{y_1 \neq y_2} |G_\ast(y_2) - G_\ast(y_1)|/|y_2 - y_1| \leq 2 \). By the same argument as the in the proof of Lemma 1,

\[
\sup_{v \in \mathbb{R}} \left| \hat{G}_\ast(v) - G_\ast(v) \right| = o_P(1).
\]

By the continuity of \( G_\ast(\cdot) \), we have that \( \hat{Q}_{t_2}^\ast = G_{-1}^\ast(1 - \alpha) + o_P(1) \) (since \( \hat{Q}_{t_2}^\ast \) is the \((1 - \alpha)(1 + 1/|T_2|)\) quantile of \( \hat{G}_\ast(\cdot) \)). Notice that \( G_{-1}^\ast(1 - \alpha) = Q_\psi(1 - \alpha) \). By Lemma 2, \( Q_\psi(1 - \alpha) = (1 - \alpha)/2 \). This means that \( \hat{Q}_{t_2}^\ast = (1 - \alpha)/2 + o_P(1) \).

**Step 2:** derive the final result.

By Step 1 and the assumptions that \( \hat{b}(X_{T+1}, \alpha) - b = o_P(1) \) and \( \sup_{y \in \mathbb{R}} \left| \hat{F}(y, X_{T+1}) - F(y, X_{T+1}) \right| = o_P(1) \), we have that \( \tilde{\varepsilon}_1 := \sup_{y \in \mathbb{R}} |\varepsilon_1(y)| = o_P(1) \) and \( \tilde{\varepsilon}_2 := \sup_{y \in \mathbb{R}} |\varepsilon_2(y)| = o_P(1) \). Define \( H_1 = \{ y : b - \tilde{\varepsilon}_1 \leq F(y) \leq b + 1 - \alpha + \tilde{\varepsilon}_2 \} \) and \( H_2 = \{ y : b + \tilde{\varepsilon}_1 \leq F(y) \leq b + 1 - \alpha - \tilde{\varepsilon}_2 \} \). Clearly,

\[
H_2 \subseteq \overline{C_{\ast(1-\alpha)}^\text{conf}(X_{T+1})} \subseteq H_1 \quad \text{almost surely.} \tag{33}
\]

On the other hand, we observe that \( H_1 \) is an interval that can be written as

\[
H_1 = \left[ Q \left( \max \{ b - \tilde{\varepsilon}_1, 0 \} \right), \ Q \left( \min \{ b + 1 - \alpha + \tilde{\varepsilon}_2, 1 \} \right) \right].
\]

Since \( P(|Y_{T+1}| \leq C_2 \mid X_{T+1}) = 1 \) and \( F(\cdot) \) is strictly increasing, \( Q(0) \) and \( Q(1) \) are well defined and satisfy \( \max \{|Q(0)|, |Q(1)|\} \leq C_2 \) almost surely.

By (32), we can write \( C_{\ast(1-\alpha)}^\text{opt}(X_{T+1}) \) as an interval

\[
C_{\ast(1-\alpha)}^\text{opt}(X_{T+1}) = \left[ Q(b), Q(b + 1 - \alpha) \right].
\]

Therefore,

\[
\mu \left( H_1 \triangle C_{\ast(1-\alpha)}^\text{opt}(X_{T+1}) \right) \leq |Q \left( \max \{ b - \tilde{\varepsilon}_1, 0 \} \right) - Q(b)| + |Q \left( \min \{ b + 1 - \alpha + \tilde{\varepsilon}_2, 1 \} \right) - Q(b + 1 - \alpha)|.
\]

Notice that \( dQ(u)/du = 1/f(Q(u)) \). By assumption, the density is bounded below by \( C_1 \) on the support of \( Y_{T+1} \mid X_{T+1} \). It follows that \( |dQ(u)/du| \) is uniformly bounded by \( 1/C_1 \). Thus,

\[
|Q \left( \max \{ b - \tilde{\varepsilon}_1, 0 \} \right) - Q(b)| \leq \frac{1}{C_1} \cdot \max \{ b - \tilde{\varepsilon}_1, 0 \} - b \leq \frac{1}{C_1} \cdot \tilde{\varepsilon}_1
\]
and similarly
\[ |Q(\min\{b+1-\alpha+\bar{\varepsilon}_2, 1\}) - Q(b+1-\alpha)| \leq \frac{1}{C_1} \cdot \bar{\varepsilon}_2. \]

The above three displays imply
\[ \mu\left(H_1 \triangle C_{(1-\alpha)}^{opt}(X_{T+1})\right) \leq (\bar{\varepsilon}_1 + \bar{\varepsilon}_2)C_1^{-1}. \]

Similarly, we can show that
\[ \mu\left(H_2 \triangle C_{(1-\alpha)}^{opt}(X_{T+1})\right) \leq (\bar{\varepsilon}_1 + \bar{\varepsilon}_2)C_1^{-1}. \]

By (33), we have that, almost surely
\[ \hat{C}_{conf}(1-\alpha)(X_{T+1}) \triangle C_{opt}(1-\alpha)(X_{T+1}) \subseteq \left(H_1 \triangle C_{(1-\alpha)}^{opt}(X_{T+1})\right) \cup \left(H_2 \triangle C_{(1-\alpha)}^{opt}(X_{T+1})\right). \]

The above three displays imply
\[ \mu\left(\hat{C}_{conf}^{(1-\alpha)}(X_{T+1}) \triangle C_{(1-\alpha)}^{opt}(X_{T+1})\right) \leq 2(\bar{\varepsilon}_1 + \bar{\varepsilon}_2)C_1^{-1}. \]

The desired result follows by \( \bar{\varepsilon}_1 = o_P(1) \) and \( \bar{\varepsilon}_2 = o_P(1) \).

D Time series discussion

For time series data \( \{Z_t\}_{t=1}^{T+1} \) with \( Z_t = (X_t, Y_t) \), it is often plausible that these \( T + 1 \) observations are not independent. Here, we assume that data is strictly stationary, i.e., for any \( m > 1 \), the distribution \( (Z_{t-m}, Z_{t-m+1}, \ldots, Z_{t-1}) \) does not depend on \( t \). This is a common assumption in the time series literature. Although the data is not independent, it is often not strongly dependent either. Usually, we work with various notions of weak dependence. A popular way of defining weak dependence is in terms of mixing conditions. There are numerous mixing conditions, see, for example, Bradley (2005, 2007); Dedecker et al. (2007). We focus on the \( \beta \)-mixing condition (also known as the absolute regularity condition): for any \( m > 1 \),

\[ \beta(m) = \frac{1}{2}\|P\{Z_t\}_{t \leq s}, \{Z_t\}_{t \geq s+m} - P\{Z_t\}_{t \leq s} \otimes P\{Z_t\}_{t \geq s+m}\|_{TV}, \]

where \( P\{Z_t\}_{t \leq s} \) denotes the probability measure of \( \{Z_t\}_{t \leq s} \), \( P\{Z_t\}_{t \geq s+m} \) denotes the probability measure of \( \{Z_t\}_{t \geq s+m} \), and \( P\{Z_t\}_{t \leq s} \otimes P\{Z_t\}_{t \geq s+m} \) denotes the probability measure of the joint random components \( \{(Z_t)_{t \leq s}, (Z_t)_{t \geq s+m}\} \). Here, \( \otimes \) denotes the product measure and \( \|\cdot\|_{TV} \) is the total-variation norm. Since the data is strictly stationary, the above definition does
not depend on \( s \). We borrow the above definition of Section 1.6 of Rio (2017), but equivalent definitions can be found in Bradley (2005, 2007) among others.\(^{13}\)

We say that the sequence \( \{Z_t\} \) is \( \beta \)-mixing if \( \beta(m) \to 0 \) as \( m \to \infty \). The \( \beta \)-mixing condition captures the idea that observations that are far apart in time become nearly independent. As \( m \) increases, \( \{Z_t\}_{t \leq s} \) and \( \{Z_t\}_{t \geq s+m} \) become more independent, in the sense that the joint distribution \( P_{\{Z_t\}_{t \leq s}} \otimes P_{\{Z_t\}_{t \geq s+m}} \) is close to the product measure of the marginal distributions \( P_{\{Z_t\}_{t \leq s}} \otimes P_{\{Z_t\}_{t \geq s+m}} \).

The \( \beta \)-mixing condition is satisfied for a large class of stochastic processes. The simplest examples are perhaps \( m \)-dependent processes, which satisfy that \( \{Z_j\}_{j \leq t} \) and \( \{Z_j\}_{j \geq s} \) are independent as long as \( s - t \geq m \) for some fixed \( m \). Moving average processes are \( m \)-dependent. Autoregressive moving average (ARMA) processes with independent errors are \( \beta \)-mixing. In general, strictly stationary Markov chains that are Harris recurrent and aperiodic are \( \beta \)-mixing (e.g., Bradley, 2005; Meyn and Tweedie, 2012). Several stochastic volatility models for asset returns, including the popular generalized autoregressive conditionally heteroskedastic (GARCH) models are also \( \beta \)-mixing with \( \beta(m) \) decaying exponentially with \( m \) (e.g., Boussama, 1998; Carrasco and Chen, 2002; Francq and Zakoian, 2006).

Now we consider the problem of empirical risk minimization mentioned in Section 3. Let \( \mathcal{F} \) be a model, i.e., a class of functions of \( Z_t = (X_t, Y_t) \). Define \( F^* = \arg \min_{f \in \mathcal{F}} R_{T+1}(f) \), where \( R_{T+1}(f) = (T+1)^{-1} \sum_{t=1}^{T+1} E[L(Z_t, f)] \), where \( L \) is a loss function. Let \( \hat{F} = \arg \min_{f \in \mathcal{F}} \hat{R}_{T+1}(f) \), where \( \hat{R}_{T+1}(f) = (T + 1)^{-1} \sum_{t=1}^{T+1} L(Z_t, f) \). Suppose that the following entropy condition with brackets holds:

\[
\int_0^1 \sqrt{\varepsilon^{-1} \log N([\varepsilon, L(\mathcal{F})], \| \cdot \|_{1,P})} d\varepsilon < \infty,
\]

where \( N([\varepsilon, L(\mathcal{F})]) \) is the class \( \{L(Z_t, f) : f \in \mathcal{F}\} \) and \( \| \cdot \|_{1,P} \) is the \( L_1 \)-norm \( \|f\|_{1,P} = E|f(Z_t)| \). By Theorem 8.3 of Rio (2017), \( \sup_{f \in \mathcal{F}} |\hat{R}_{T+1}(f) - R_{T+1}(f)| = O_P(T^{-1/2}) \) as long as \( \sum_{m=1}^{\infty} \beta(m) < \infty \). (Similar results for empirical processes of dependent data can be found in Dedecker and Louhichi (2002).) By the usual arguments, it follows that \( 0 \leq R_{T+1}(\hat{F}) - R_{T+1}(F^*) \leq o_P(1) \).

Suppose that the risk function is convex in a neighborhood of \( F^* \): there exist \( C_1, C_2 > 0 \) such that \( R_{T+1}(f) - R_{T+1}(F^*) \geq C_2 \|f - F^*\|_{\sup}^2 \) whenever \( \|f - F^*\|_{\sup} \leq C_1 \) and \( f \in \mathcal{F} \), where \( \|f\|_{\sup} = \sup_x |f(z)| \). Then \( \sup_{y,x} |\hat{F}(y, x) - F^*(y, x)| = \sup_{y} |\hat{F}(z) - F^*(z)| = o_p(1) \). This implies the consistency requirement in Assumption 1. Importantly, \( F^* \) does not need to be the true CDF \( F \) because \( F \) may or may not be in \( \mathcal{F} \).

For the popular linear QR model, we establish a more concrete result; similar results can

\(^{13}\)To see that these definitions are equivalent, one can find details in Theorem 3.29 of Bradley (2007).
be established for DR. Suppose that $X_t \in \mathbb{R}^d$ for a fixed $d$. Let $\hat{\gamma}(u) = \arg \min_{\gamma \in \Gamma} \sum_{t=1}^{T+1} \rho_u(Y_t - X_t^\top \gamma)$ for $u \in [c_T, 1-c_T]$, where $\Gamma \subset \mathbb{R}^d$ is a compact set and $c_T > 0$ is either a small constant or a sequence tending to zero. Define

$$\hat{F}(y, x) = c_T + \int_{c_T}^{1-c_T} 1\{x^\top \hat{\gamma}(u) \leq y\} du.$$ 

Let $\| \cdot \|_2$ denote the Euclidean norm in $\mathbb{R}^d$. We have the following result.

**Theorem S1.** Assume that the data $(X_t, Y_t)$ is strictly stationary. Let $\gamma^*(u) = \arg \min_{\gamma \in \Gamma} E\rho_u(Y_t - X_t^\top \gamma)$ with $\rho_u(a) = a(u - 1\{a \leq 0\})$. Define

$$F^*(y, x) = c_T + \int_{c_T}^{1-c_T} 1\{x^\top \gamma^*(u) \leq y\} du.$$ 

Suppose that the following conditions hold:

1. There exists a constant $C_1 > 0$ such that $\|X_t\|_2 \leq C_1$ and $|Y_t| \leq C_1$ almost surely.
2. The $\beta$-mixing coefficient of $(X_t, Y_t)$ satisfies $\sum_{m=1}^{\infty} \beta(m) < \infty$.
3. There exists a function $h(\cdot)$ such that $\lim_{\delta \to 0} h(\delta) = 0$ and $|F^*(y_1, x) - F^*(y_2, x)| \leq h(|y_1 - y_2|)$ for any $(y_1, y_2)$ and any $x$ with $\|x\|_2 \leq C_1$.
4. $f(y, x) = \partial F(y, x)/\partial y$ exists and there exists a constant $C_2 > 0$ such that $f(y, x) \geq C_2$ for any $x$ and any $y \in [s_1(x), s_2(x)]$, where $[s_1(x), s_2(x)]$ is the support of the distribution $Y_t \mid X_t = x$.
5. the smallest eigenvalue of $E(X_tX_t^\top)$ is bounded below by a constant $C_3 > 0$.

Then $\sup_{y \in \mathbb{R}, \|x\|_2 \leq C_1} |\hat{F}(y, x) - F^*(y, x)| = o_P(1)$.

Theorem S1 establishes the uniform consistency of $\hat{F}$, which guarantees the consistency requirement in Assumption 1. Notice that Theorem S1 does not assume that $F^*$ is the true conditional distribution function $F$. It thus generalizes the analysis of QR under misspecification in Angrist et al. (2006) to time series settings.

The assumptions of Theorem S1 are relatively mild. The boundedness of $X_t$ and $Y_t$ can be relaxed with extra technical arguments. The summability of $\beta$-mixing coefficients holds if $\beta(m)$ decays exponentially. The third assumption says that $F^*(y, x)$ is uniformly continuous in $y$. The last assumption states that the true conditional density of $Y_t \mid X_t$ is bounded away from zero on the support.
Proof of Theorem S1. We proceed in two steps.

Step 1: show that \( \sup_{u \in [c_T, 1-c_T]} \| \hat{\gamma}(u) - \gamma^*(u) \|_2 = o_P(1) \).

Let \( R(\gamma, u) = E\rho_u(Y_t - X_t^\top \gamma) \) and \( \hat{R}(\gamma, u) = (T + 1)^{-1} \sum_{t=1}^{T+1} \rho_u(Y_t - X_t^\top \gamma) \). For any \( u_1, u_2 \in [c_T, 1-c_T] \) and \( \gamma_1, \gamma_2 \in \Gamma \), we observe

\[
\left| \rho_{u_1}(y - x^\top \gamma_1) - \rho_{u_2}(y - x^\top \gamma_2) \right| \\
\leq \left| \rho_{u_1}(y - x^\top \gamma_1) - \rho_{u_1}(y - x^\top \gamma_2) \right| + \left| \rho_{u_1}(y - x^\top \gamma_2) - \rho_{u_2}(y - x^\top \gamma_2) \right| \\
\leq \max\{u_1, 1 - u_1\} \cdot |x^\top \gamma_1 - x^\top \gamma_2| + |u_1 - u_2| \cdot |y - x^\top \gamma_2| \\
\leq C_1 \| \gamma_1 - \gamma_2 \|_2 + \left( 1 + \sup_{\gamma \in \Gamma} \| \gamma \|_2 \right) C_1 |u_1 - u_2|,
\]

where (i) follows by \( |y - x^\top \gamma_2| \leq C_1 + C_1 \sup_{\gamma \in \Gamma} \| \gamma \|_2 \). Since \( \Gamma \) is compact and \( d \) is fixed, \( \sup_{\gamma \in \Gamma} \| \gamma \|_2 \) is bounded by a positive constant. Hence, \( (\gamma, u) \mapsto \rho_u(y - x^\top \gamma) \) is Lipschitz. By Theorem 2.7.11 of van der Vaart and Wellner (1996) and the usual covering number bounds for Euclidean balls (e.g., Corollary 4.2.13 of Vershynin (2018)), we have that for any norm \( \| \cdot \| \), the bracketing number satisfies

\[
N_{[\varepsilon]}(\varepsilon, G, \| \cdot \|) \leq (K_1/\varepsilon)^d,
\]

where \( K_1 \geq 1 \) is a constant depending only on \( C_1, \sup_{\gamma \in \Gamma} \| \gamma \|_2 \) and \( d \) and \( G \) is the class of functions \( \rho_u(y - x^\top \gamma) \) with \( (\gamma, u) \in \Gamma \times [c_1, 1-c_2] \). Notice that

\[
\int_0^1 \sqrt{\varepsilon^{-1} \log N_{[\varepsilon]}(\varepsilon, G, \| \cdot \|_1, \rho)} d\varepsilon \leq \int_0^1 \sqrt{\varepsilon^{-1} \log ((K_1/\varepsilon)^d)} d\varepsilon \\
= \int_0^1 \sqrt{d \varepsilon^{-1} (\log K_1 - \log \varepsilon)} d\varepsilon \\
\leq \int_0^1 \sqrt{(d \log K_1) \varepsilon^{-1} d\varepsilon} + \sqrt{-d \varepsilon^{-1} \log \varepsilon d\varepsilon} < \infty.
\]

Therefore, it follows, by Theorem 8.3 of Rio (2017), that \( \delta_T := \sup_{(\gamma, u) \in \Gamma \times [c_1, 1-c_2]} |\hat{R}(\gamma, u) - R(\gamma, u)| = O_P(T^{-1/2}). \)

By the definition of \( \gamma^*(u) \) and \( \hat{\gamma}(u) \), we observe that \( R(\gamma^*(u), u) \leq R(\hat{\gamma}(u), u) \leq \hat{R}(\hat{\gamma}(u), u) + \delta_T \leq \hat{R}(\gamma^*(u), u) + \delta_T \leq R(\gamma^*(u), u) + 2\delta_T \). Hence,

\[
0 \leq R(\hat{\gamma}(u), u) - R(\gamma^*(u), u) \leq 2\delta_T. \tag{34}
\]

For any \( \gamma \in \Gamma \), we observe that

\[
E \left[ \rho_u(Y_t - X_t^\top \gamma) - \rho_u(Y_t - X_t^\top \gamma^*(u)) \mid X_t = x \right]
\]

47
\[
\int f(y, x) \left( \rho_u(y - x^\top \gamma) - \rho_u(y - x^\top \gamma^*(u)) \right) dy \\
\overset{(i)}{=} \int f(y, x) \left[ -x^\top (\gamma - \gamma^*(u)) \left( u - \mathbf{1} \{ y - x^\top \gamma^*(u) \leq 0 \} \right) \right] dy \\
+ \int f(y, x) \left[ \int_0^{x^\top (\gamma - \gamma^*(u))} \left( \mathbf{1} \{ y - x^\top \gamma^*(u) \leq s \} - \mathbf{1} \{ y - x^\top \gamma^*(u) \leq 0 \} \right) ds \right] dy \\
= (F(x^\top \gamma^*(u), x) - u)x^\top (\gamma - \gamma^*(u)) + \int_0^{x^\top (\gamma - \gamma^*(u))} (F(s + x^\top \gamma^*(u), x) - F(x^\top \gamma^*(u), x)) ds,
\]
where (i) follows by Equation (4.3) of Koenker (2005b). By the optimality condition of \( \gamma^*(u) = \arg \min_{\gamma \in \Gamma} \mathcal{E}_u(Y_t - X_t^\top \gamma) \), we have \( \mathcal{E}(X_t^\top \gamma^*(u), X_t) - u)X_t^\top = 0 \). Thus, the above display implies that for any \( \gamma \in \Gamma \),
\[
R(\gamma, u) - R(\gamma^*(u), u) \\
= \mathcal{E} \left[ \int_0^{X_t^\top (\gamma - \gamma^*(u))} (F(s + X_t^\top \gamma^*(u), x) - F(X_t^\top \gamma^*(u), x)) ds \right] \\
\overset{(i)}{\geq} \frac{1}{2} C_2 \mathcal{E} \left( X_t^\top (\gamma - \gamma^*(u)) \right)^2 \geq \frac{1}{2} C_2 C_3 \| \gamma - \gamma^*(u) \|_2^2,
\]
where (i) follows by \( f(y, x) \geq C_2 \) for \( y \in [s_1(x), s_2(x)] \). By (34) and the above display,
\[
\frac{1}{2} C_2 C_3 \| \hat{\gamma}(u) - \gamma^*(u) \|_2^2 \leq R(\hat{\gamma}(u), u) - R(\gamma^*(u), u) \leq 2\delta_T.
\]
Since this bound holds for any \( u \), we have that
\[
\sup_{u \in [c_1, 1-c_1]} \| \hat{\gamma}(u) - \gamma^*(u) \|_2^2 \leq 4\delta_T/(C_2 C_3).
\]
Since we have proved \( \delta_T = o_P(1) \), we have \( \sup_{u \in [c_T, 1-c_T]} \| \hat{\gamma}(u) - \gamma^*(u) \|_2 = o_P(1) \).

**Step 2**: show the desired result.

Let \( \varepsilon_T = \sup_{u \in [c_T, 1-c_T]} \| \hat{\gamma}(u) - \gamma^*(u) \|_2 \). Then
\[
\sup_{\| x \|_2 \leq C_1, u \in [c_1, 1-c_1]} | x^\top \hat{\gamma}(u) - x^\top \gamma^*(u) | \leq C_1 \varepsilon_T. \tag{35}
\]

We observe that for any \( x \in \mathbb{R}^d \) with \( \| x \|_2 \leq C_1 \),
\[
\hat{F}(y, x) = c_T + \int_{c_T}^{1-c_T} \mathbf{1} \{ x^\top \hat{\gamma}(u) \leq y \} du \overset{(i)}{\leq} c_T + \int_{c_T}^{1-c_T} \mathbf{1} \{ x^\top \gamma^*(u) - C_1 \varepsilon_T \leq y \} du = F^*(y+C_1 \varepsilon_T, x),
\]
where (i) follows by (35). Similarly, we can show that \( \hat{F}(y, x) \geq F^*(y - C_1 \varepsilon_T, x) \). Therefore,
\[
| \hat{F}(y, x) - F^*(y, x) | \leq \max \{ F^*(y+C_1 \varepsilon_T, x) - F^*(y, x), F^*(y, x) - F^*(y - C_1 \varepsilon_T, x) \} \leq h(C_1 \varepsilon_T).
\]

Since this bounds holds for any \( y \) and \( x \), we have that \( \sup_{\| x \|_2 \leq C_1} \sup_{y \in \mathbb{R}} | \hat{F}(y, x) - F^*(y, x) | \leq h(C_1 \varepsilon_T) \).
By \( \varepsilon_T = o_P(1) \) and \( \lim_{\delta \to 0} h(\delta) = 0 \), the desired result follows. \( \square \)
E Additional figures

Figure 5: Histograms of estimated conditional coverage probability. Vertical line at nominal coverage of \(1 - \alpha = 0.9\).