SUPERSYMMETRIC $W$-ALGEBRAS

ALEXANDER MOLEV, ERIC RAGOUCY, AND UHI RINN SUH

ABSTRACT. We develop a general theory of $W$-algebras in the context of supersymmetric vertex algebras. We describe the structure of $W$-algebras associated with odd nilpotent elements of Lie superalgebras in terms of their free generating sets. As an application, we produce explicit free generators of the $W$-algebra associated with the odd principal nilpotent element of the Lie superalgebra $\mathfrak{gl}(n+1|n)$.

1. Introduction

The $W$-algebras first appeared in relation with the conformal field theory in the work of Zamolodchikov [23] and Fateev and Lukyanov [10]. These algebras were studied intensively by physicists, both at the classical level through Hamiltonian reduction of Wess–Zumino–Novikov–Witten models and their connection with affine Lie algebras, see e.g. [4, 11, 13], but also using BRST formalism [6, 7]. For an extensive review on physicists works, see [5] and references therein. A definition of the $W$-algebras in the context of the vertex algebra theory and quantized Drinfeld–Sokolov reduction was given by Feigin and Frenkel [12]; see also the book by Frenkel and D. Ben-Zvi [14, Ch. 15]. A more general family of $W$-algebras $W^k(g, f)$ was introduced by Kac, Roan and Wakimoto [20], which depends on a simple Lie (super)algebra $g$, an (even) nilpotent element $f \in g$ and the level $k \in \mathbb{C}$. In the particular case of the principal nilpotent element $f = f_{\text{prin}}$ this reduces to the definition of [12]; see also a recent expository article by Arakawa [1] where basic structure theorems and representation theory of $W$-algebras are reviewed.

In the present paper we will be concerned with supersymmetric counterparts of the $W$-algebras which can be defined by analogy with [14, Ch. 15]. Such $W$-algebras have already been studied, mostly in the physics literature; see [9, 16, 17]. Moreover, a supersymmetric quantum hamiltonian reduction approach was developed in the work of Madsen and the second author [22]. We will rely on this work and the supersymmetric vertex algebra theory developed by Heluani and Kac [15, 18] to describe the structure of the $W$-algebras associated with odd nilpotent elements of Lie superalgebras. Our
main structural result is Theorem 4.11 which describes free generating sets of the $W$-algebras.

We will then apply the main result to the case of the general linear Lie superalgebras. It is well-known that the Lie superalgebra $\mathfrak{gl}(m|n)$ contains an odd principal nilpotent element if and only if $m = n \pm 1$. We take $m = n + 1$ (this can be done without a real loss of generality) and produce explicit free generators of the $W$-algebra as coefficients of a certain noncommutative characteristic polynomial (Theorems 5.1 and 5.3). These formulas can be regarded as supersymmetric analogues of the generators of the principal $W$-algebra associated with the Lie algebra $\mathfrak{gl}(n)$ produced by Arakawa and the first author [2]. Furthermore, we show that the Miura transformation used in [2] can also be applied in the supersymmetric context to recover the generators of the $W$-algebra appeared in [9, 16, 17].

The second author wishes to thank the School of Mathematics and Statistics at the University of Sydney for the hospitality and warm atmosphere during his visit, as the work on this project was under way. The work of the third author was supported by NRF Grant # 2016R1C1B1010721.

2. Supersymmetric Vertex Algebras

In this section, we introduce supersymmetric vertex algebras following [15] and [18]. Proofs and additional details can be found in these references. Note that in the terminology of the paper [15] these objects are called $N_K = 1$ supersymmetric vertex algebras.

2.1. Notation and basic definitions. We will be considering two couples of coordinates

$$Z = (z, \theta), \quad W = (w, \zeta),$$

where $z$ and $w$ are even and $\theta$ and $\zeta$ are odd. Introduce the notation

$$\mathbb{C}[Z] := \mathbb{C}[z] \otimes \mathbb{C}[\theta], \quad \mathbb{C}((Z)) := \mathbb{C}((z)) \otimes \mathbb{C}[\theta].$$

Since $\theta^2 = 0$ we have $\mathbb{C}[\theta] = \mathbb{C} \oplus \mathbb{C}\theta$. Similarly,

$$\mathbb{C}[Z, Z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\theta], \quad \mathbb{C}[Z, Z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\theta].$$

Furthermore, set

$$Z - W := (z - w - \theta \zeta, \theta - \zeta),$$

$$Z^{j_0 j_1} := z^{j_0} \theta^{j_1} \quad \text{for} \quad j_0 \in \mathbb{Z}, \quad j_1 = 0, 1,$$

$$(Z - W)^{j_0 j_1} := (z - w - \theta \zeta)^{j_0} (\theta - \zeta)^{j_1}.$$
Let $\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space which we will also call a vector superspace. Accordingly, elements $a \in \mathcal{U}_0$ (resp. $a \in \mathcal{U}_1$) are called even (resp. odd) with the parity $p(a) = 0$ (resp. $p(a) = 1$). The corresponding endomorphism algebra $\text{End} \mathcal{U} = (\text{End} \mathcal{U})_0 \oplus (\text{End} \mathcal{U})_1$ is a superalgebra, where

$$f \in (\text{End} \mathcal{U})_i \iff f((\text{End} \mathcal{U})_j) \subset (\text{End} \mathcal{U})_{i+j}$$

for any $i, j \in \mathbb{Z}/2\mathbb{Z}$.

Any element of the vector superspace $\mathcal{U}[Z, Z^{-1}] := \mathcal{U} \otimes \mathbb{C}[Z, Z^{-1}]$ is called a $\mathcal{U}$-valued formal distribution. It has the form

$$a(Z) = \sum_{j_0 \in \mathbb{Z}, j_1 = 0, 1} Z^{j_0|j_1} a_{j_0|j_1} \in \mathcal{U}[Z, Z^{-1}], \quad a_{j_0|j_1} \in \mathcal{U}. \quad (2.1)$$

The super residue of a formal distribution $a(Z)$ is defined by

$$\text{res}_Z a(Z) := a_{-1|1} \in \mathcal{U}.$$ 

Since $\text{res}_Z Z^{j_0|j_1} a(Z) = a_{-1-j_0|1-j_1}$, it is convenient to use the notation

$$a_{(j_0|j_1)} := \text{res}_Z Z^{j_0|j_1} a(Z)$$

so that $a_{j_0|j_1} = a_{(-1-j_0|1-j_1)}$ and the distribution $a(Z)$ in (2.1) takes the form

$$a(Z) = \sum_{j_0 \in \mathbb{Z}, j_1 = 0, 1} Z^{-1-j_0|1-j_1} a_{(j_0|j_1)}.$$ 

An $\text{End} \mathcal{U}$-valued formal distribution $a(Z)$ is called a super field if for any given $v \in \mathcal{U}$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$a_{(j_0|j_1)} v = 0 \quad \text{for all} \quad j_0 \geq N, \quad j_1 = 0, 1.$$ 

Similarly, a $\mathcal{U}$-valued formal distribution in two variables is an element of the vector superspace $\mathcal{U}[Z, Z^{-1}, W, W^{-1}]$:

$$a(Z, W) = \sum_{j_0, k_0 \in \mathbb{Z}, j_1, k_1 = 0, 1} Z^{j_0|j_1} W^{k_0|k_1} a_{j_0|j_1, k_0|k_1} \in \mathcal{U}[Z, Z^{-1}, W, W^{-1}]$$ 

with $a_{j_0|j_1, k_0|k_1} \in \mathcal{U}$. A formal distribution $a(Z, W)$ is called local if

$$(z - w)^n a(Z, W) = 0$$

for some $n \in \mathbb{Z}_{\geq 0}$. We let the formal $\delta$-distribution be defined by

$$\delta(Z, W) = (\theta - \zeta) \sum_{n \in \mathbb{Z}} z^n w^{-n-1}.$$
Note that for any \( f \in \mathcal{U}[Z, Z^{-1}] \) we have
\[
\text{res}_{Z} \delta(Z, W) f(Z) = f(W).
\]
Since \((z - w) \delta(Z, W) = 0\), the formal \(\delta\)-distribution is local.

The differential operators \(\partial_z\), \(\partial_\theta\), \(\partial_w\) and \(\partial_\zeta\) act naturally on \(\mathbb{C}[Z, Z^{-1}, W, W^{-1}]\).

Consider two more odd differential operators
\[
D_Z = \partial_\theta + \theta \partial_z, \quad D_W = \partial_\zeta + \zeta \partial_w.
\]
Then \([D_Z, D_Z] = 2 \partial_z\). Set
\[
D_{j_0|j_1}^Z = \partial_{j_0} D_{j_1}^Z, \quad D_{j_0|j_1}^W = (-1)^{j_1} \frac{1}{j_0!} D_{j_0|j_1}^Z.
\]

**Lemma 2.1.** Let \(a(Z, W)\) be a local formal distribution. Then
\[
a(Z, W) = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1 = 0, 1} D_{j_0|j_1}^W \delta(Z, W) c_{j_0|j_1}(W),
\]
where the sum is finite, and
\[
c_{j_0|j_1}(W) = \text{res}_{Z} (Z - W)^{j_0|j_1} a(Z, W).
\]

**Definition 2.2.** A supersymmetric vertex algebra is a tuple \((V, |0\rangle, S, Y)\) where \(V\) is a vector superspace, \(|0\rangle \in V\) is a vacuum vector, \(S\) is an odd endomorphism of \(V\), and the state-field correspondence \(Y\) is a parity preserving linear map from \(V\) to the space of \(\text{End} V\)-valued super fields
\[
Y : V \to \text{End} V[Z, Z^{-1}], \quad a \mapsto a(Z)
\]
satisfying the following axioms:

- (vaccum) \(a(Z) |0\rangle |_{z=0, \theta=0} = a, \ S |0\rangle = 0\),
- (translation covariance) \([S, a(Z)] = (\partial_\theta - \theta \partial_z) a(Z),\)
- (locality) for any \(a, b \in V\) there exists \(N \in \mathbb{Z}_+\) such that
\[
(z - w)^N [a(Z), b(W)] = 0.
\]

By Lemma 2.1, the locality axiom implies a finite sum decomposition
\[
[a(Z), b(W)] = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1 = 0, 1} (D_{j_0|j_1}^W \delta(Z, W)) a(W)_{(j_0|j_1)} b(W)
\]
for \(a(W)_{(j_0|j_1)} b(W) := \text{res}_{Z} (Z - W)^{j_0|j_1} [a(Z), b(W)]\). The expression \(a(W)_{(j_0|j_1)} b(W)\) is called the \((j_0|j_1)\)-th product of the super fields \(a(W)\) and \(b(W)\).
**Definition 2.3.** (1) The *normally ordered product* of two End $V$-valued formal distributions $a(Z)$ and $b(Z)$ is defined by

$$a(Z)b(Z) := a_+(Z)b(Z) + (-1)^{p(a)p(b)}b(Z)a_-(Z),$$

where

$$a_+(Z) = \sum_{j_0 \in \mathbb{Z}, j_1 = 0, 1} Z^{j_0|j_1} a_{j_0|j_1} \quad \text{and} \quad a_-(Z) = \sum_{j_0 \in \mathbb{Z}<0, j_1 = 0, 1} Z^{j_0|j_1} a_{j_0|j_1}.$$ 

(2) If $j_0 \leq -2$ and $j_1 = 0, 1$, or $j_0 = -1$ and $j_1 = 0$, then $a(Z)_{(j_0|j_1)}b(Z)$ is given by

$$a(Z)_{(j_0|j_1)}b(Z) = (-1)^{1-j_1} :D_Z^{(-1-j_0|1-j_1)}a(Z)b(Z):.$$ 

**Remark 2.4.** One can check that

$$a(Z)b(Z) :|0\rangle |z=0, \theta=0 = a_{(-1|1)}b$$

and

$$a(Z)_{(j_0|j_1)}b(Z) |0\rangle |z=0, \theta=0 = a_{(j_0|j_1)}b$$

for $(j_0, j_1)$ as in part (2) of Definition 2.3.

**Lemma 2.5** (Dong’s lemma). Let $a(Z), b(Z), c(Z)$ be pairwise local formal distributions. Then $(a(Z), (b_{(j_0|j_1)}c)(Z))$ is local for any $j_0 \in \mathbb{Z}$ and $j_1 = 0, 1$.

**Lemma 2.6** (Uniqueness lemma). Let $V$ be a supersymmetric vertex algebra. If $a(Z)$ is a super field such that $(a(Z), b(Z))$ is local for every $b \in V$ and $a(Z) |0\rangle = 0$ then $a(Z) = 0$.

By the uniqueness lemma and Remark 2.4,

$$a(Z)_{(j_0|j_1)}b(Z) = (a_{(j_0|j_1)}b)(Z),$$

and we set

$$ab := a_{(-1|1)}b = :a(Z)b(Z) :|0\rangle |z=0, \theta=0.$$ 

Note that for a given supersymmetric vertex algebra $V$, the state-field correspondence map

$$Y : V \to (\text{End } V)[Z, Z^{-1}], \quad a \mapsto a(Z),$$

is injective. Hence a supersymmetric vertex algebra $V$ can be considered as a set of super fields $Y(V)$. In the following theorem, we construct a vertex algebra as a set of super fields.
Theorem 2.7 (Existence theorem). Let $V$ be a vector superspace and $\hat{V}$ be a set of pairwise local $\text{End} V$-valued super fields. Suppose $\text{Id} \in \hat{V}$ is the constant field and $\hat{V}$ is invariant under the operator $D = \partial_\theta + \theta \partial_z$ and all $(j_0|j_1)$-products. Then the superspace $V$ with the vacuum vector $\text{Id}$, the operator $S$ given by $Sa(Z) = D(a(Z))$ and the $(j_0|j_1)$-products is a supersymmetric vertex algebra.

2.2. Supersymmetric Lie conformal algebras. Recall that a Lie conformal algebra (LCA) $R$ gives rise to a vertex algebra called a universal enveloping vertex algebra $V(R)$ [3, 18]. Now we introduce its supersymmetric analogue: that is, a supersymmetric LCA and the corresponding universal enveloping supersymmetric vertex algebra. Consider two superalgebras:

- Let $L$ be the associative superalgebra generated by a pair of elements $\Lambda = (\lambda, \chi)$, where $\lambda$ is even and $\chi$ is odd, such that $[\lambda, \chi] = 0$, $[\chi, \chi] = 2\chi^2 = -2\lambda$.
- Let $K$ be another associative superalgebra generated by a pair of elements $\nabla = (T, S)$, where $T$ is even and $S$ is odd, such that $[T, S] = 0$, $[S, S] = 2S^2 = 2T$.

Set $(Z - W)\Lambda = (z - w - \theta\zeta)\lambda + (\theta - \zeta)\chi$.

Given a formal distribution $a(Z, W)$ of two variables $Z$ and $W$, consider the formal Fourier transformation

$$\mathcal{F}^\Lambda_{Z,W} a(Z, W) = \text{res}_Z \exp((Z - W)\Lambda) a(Z, W)$$

which can be expanded as

$$\mathcal{F}^\Lambda_{Z,W} a(Z, W) = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1 = 0,1} (-1)^{j_1} \Lambda^{(j_0|j_1)} c_{j_0|j_1}(W),$$

where

$$\Lambda^{(j_0|j_1)} = (-1)^{j_1} \frac{\lambda^{j_0} \chi^{j_1}}{j_0!}$$

and $c_{j_0|j_1}(W)$ is defined in Lemma 2.1.

Define the $\Lambda$-bracket $(a, b) \rightarrow [a\Lambda b]$ of a local pair $(a(Z), b(Z))$ by

$$[a\Lambda b](W) := \mathcal{F}^\Lambda_{Z,W} [a(Z), b(W)].$$

Proposition 2.8. The $\Lambda$-bracket satisfies the following properties for all pairwise local distributions $(a(Z), b(Z), c(Z))$:
(1) (sesquilinearity)
\[ [Sa_Ab] = \chi [a_Ab], \quad [a_A Sb] = -(-1)^{p(a)} (S + \chi) [a_Ab]; \]

(2) (skew-symmetry)
\[ [b_Aa] = (-1)^{p(a)p(b)} [a_{-A}-\nabla b], \]

where
\[ [a_{-A}-\nabla b] = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1=0,1} (-1)^{j_1} (-\Lambda - \nabla)^{(j_0j_1)} a_{(j_0j_1)} b \]
for \(-\Lambda - \nabla = (-\lambda - T, -\chi - S)\) with
\[ [\chi, S] = 2\lambda \quad \text{and} \quad [\chi, T] = [\lambda, T] = [\lambda, S] = 0; \]

(3) (Jacobi identity)
\[ [a_{\Lambda}[b_{\Gamma}c]] = -(-1)^{p(a)} [[a_{\Lambda}b]_{\Lambda+\Gamma c}] + (-1)^{(p(a)+1)(p(b)+1)} [b_{\Gamma}[a_{\Lambda}c]], \]

where
\[(i) \Gamma = (\gamma, \eta) \text{ with } [\gamma, \eta] = [\gamma, \gamma] = 0 \text{ and } [\eta, \eta] = -2\gamma, \]
\[(ii) \Lambda + \Gamma = (\lambda + \gamma, \zeta + \eta) \text{ with } [\lambda, \gamma] = [\zeta, \gamma] = [\zeta, \eta] = 0. \]

This motivates the following definition.

**Definition 2.9.** A supersymmetric Lie conformal algebra (LCA) \( \mathcal{R} \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \mathcal{K} \)-module endowed with odd bilinear map \( \mathcal{R} \otimes \mathcal{R} \to \mathcal{L} \otimes \mathcal{R} \), called \( \Lambda \)-bracket, given by a finite sum expansion
\[ a \otimes b \mapsto [a_{\Lambda}b] = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1=0,1} (-1)^{j_1} A^{(j_0j_1)} a_{(j_0j_1)} b \]
with \( a_{(j_0j_1)} b \in \mathcal{R} \), satisfying the following properties:

1. (sesquilinearity) In \( \mathcal{L} \otimes \mathcal{R} \) we have
\[ [Sa_Ab] = \chi [a_Ab], \quad [a_A Sb] = -(-1)^{p(a)} (S + \chi) [a_Ab], \]

where \( S \) and \( \chi \) obey the relation \([S, \chi] = 2\lambda; \)

2. (skew-symmetry) In \( \mathcal{L} \otimes \mathcal{R} \) we have
\[ [b_Aa] = (-1)^{p(a)p(b)} [a_{-A}-\nabla b], \]

where
\[ [a_{-A}-\nabla b] = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1=0,1} (-1)^{j_1} (-\Lambda - \nabla)^{(j_0j_1)} a_{(j_0j_1)} b \]
for \(-\Lambda - \nabla = (-\lambda - T, -\chi - S)\) satisfying
\[
[\chi, S] = 2\lambda \quad \text{and} \quad [\chi, T] = [\lambda, T] = [\lambda, S] = 0;
\]

(3) (Jacobi-identity) In \(L \otimes L' \otimes R\) we have
\[
[a_\Lambda [b_T c]] = \sum_{j \geq 1} (-1)^{j+1} \lambda^j \frac{z}{j!} [a_\Lambda b]_{(j-1)1} c + (-1)^{(j+1)p(b)+1} [b_T [a_\Lambda c]],
\]
where
(i) \(\Gamma = (\gamma, \eta)\) such that \([\gamma, \eta] = [\gamma, \gamma] = 0\) and \([\eta, \eta] = -2\gamma,
(ii) \(\Lambda + \Gamma = (\lambda + \gamma, \zeta + \eta)\) such that \([\lambda, \eta] = [\lambda, \gamma] = [\zeta, \gamma] = [\zeta, \eta] = 0.

The next theorem provides an equivalent definition of supersymmetric vertex algebras in terms of \(\Lambda\)-brackets; cf. [19, Thm. 4.1].

\begin{theorem}
A supersymmetric vertex algebra is a tuple \((V, S, [\Lambda], |0\rangle, :: :)\) such that

(i) \((V, S, [\Lambda])\) is a supersymmetric Lie conformal algebra.

(ii) \((V, S, |0\rangle, :: :)\) is a unital differential superalgebra, where \(S\) is an odd derivation of the product ::, and the following properties hold:

\[
: ab : = (-1)^{p(a)p(b)} : ba := (-1)^{p(a)p(b)} \sum_{j \geq 1} \frac{(-T)^j}{j!} (b_{(-1+j|1)} a),
\]

\[
:: ab : c = - : a : bc := \sum_{j \geq 0} a_{(-2-j|1)} (b_{(j|1)} c) + (-1)^{p(a)p(b)} \sum_{j \geq 0} b_{(-2-j|1)} (a_{(j|1)} c).
\]

(iii) The \(\Lambda\)-bracket and the product :: are related by the non-commutative Wick formula:

\[
[a_\Lambda : bc :] = \sum_{k \geq 0} \frac{\lambda^k}{k!} [a_\Lambda b]_{(k-1)1} c + (-1)^{(p(a)+1)p(b)} : b [a_\Lambda c] :.
\]

The properties (2.2) of the product :: are referred to as the \textit{quasi-commutativity} and \textit{quasi-associativity}, respectively.

\begin{definition}
(1) A set \(\mathcal{B} = \{a_i \mid i \in I\}\) of elements in a supersymmetric vertex algebra \(V\) strongly generates \(V\) if the set of monomials \(\{ : a_{j_1} a_{j_2} \ldots a_{j_s} : \mid j_1, \ldots, j_s \in I, s \in \mathbb{Z}_{\geq 0}\}\) spans \(V\). If \(s = 0\), the monomial is understood as \(|0\rangle\). For \(s > 2\) the product in the monomial is applied consecutively from right to left.
\end{definition}
(2) An ordered set \( \mathcal{B} = \{a_i \mid i \in I\} \subset V \) freely generates a supersymmetric vertex algebra \( V \) if the set of monomials
\[
\{ : a_{j_1} a_{j_2} \ldots a_{j_s} : \mid j_r \leq j_{r+1} \text{ and } j_r < j_{r+1} \text{ if } p(a_{j_r}) = \bar{1} \}
\]
forms a basis of \( V \) over \( \mathbb{C} \).

**Theorem 2.12.** Let \( \mathcal{R} \) be a supersymmetric Lie conformal algebra with an ordered \( \mathbb{C} \)-basis \( \mathcal{B} = \{a_i \mid i \in I\} \). Then there exists a unique supersymmetric vertex algebra \( V(\mathcal{R}) \) such that
(i) \( V(\mathcal{R}) \) is freely generated by \( \mathcal{B} \),
(ii) the operator \( S \) on \( V(\mathcal{R}) \) is defined by \( S(: ab :) =: (Sa)b : + (-1)^{p(a)} : a(Sb) : \),
(iii) the \( \Lambda \)-bracket on \( \mathcal{R} \) extends to the \( \Lambda \)-bracket on \( V(\mathcal{R}) \) via the Wick formula (2.3).

**Definition 2.13.** For a given supersymmetric Lie conformal algebra \( \mathcal{R} \), the supersymmetric vertex algebra \( V(\mathcal{R}) \) in Theorem 2.12 is called the universal enveloping supersymmetric vertex algebra associated to \( \mathcal{R} \).

**2.3. Supersymmetric nonlinear LCAs.** In this section we follow Section 3 of \([8]\) to introduce nonlinear supersymmetric LCAs. We omit the arguments which are straightforward supersymmetric analogues of those in \([8]\).

For a positive integer \( n \), consider a \( \mathcal{K} \)-module \( \mathcal{R} = \bigoplus_{\zeta \in \mathbb{N}/n} \mathcal{R}_\zeta \) with \((\mathbb{N}/n)\)-grading so that \( \text{gr}(a) = \zeta \) for \( a \in \mathcal{R}_\zeta \). The grading \( \text{gr} \) is naturally extended to the grading of the tensor algebra \( T(\mathcal{R}) \) by
\[
\text{gr}(a \otimes b) = \text{gr}(a) + \text{gr}(b).
\]
Set
\[
T(\mathcal{R})_{(\zeta) -} = \bigoplus_{\zeta' < \zeta} T(\mathcal{R})_{\zeta'}.
\]

**Definition 2.14.** Suppose that \( \mathcal{R} \) is endowed with a nonlinear \( \Lambda \)-bracket
\[
[\mathcal{R}_\zeta \Lambda \mathcal{R}_{\zeta'}] \subset \mathcal{L} \otimes T(\mathcal{R})_{(\zeta + \zeta') -},
\]
satisfying skew-symmetry, sesquilinearity and Jacobi identity in Definition 2.9. Then \( \mathcal{R} \) is called supersymmetric nonlinear Lie conformal algebra.

**Proposition 2.15.** Let \( \mathcal{R} \) be a supersymmetric nonlinear LCA. Then the normally ordered product and \( \Lambda \)-bracket admit unique extensions to the linear maps
\[
T(\mathcal{R}) \otimes T(\mathcal{R}) \to T(\mathcal{R}), \quad A \otimes B \mapsto AB ;,
\]
\[
T(\mathcal{R}) \otimes T(\mathcal{R}) \to \mathcal{L} \otimes T(\mathcal{R}), \quad A \otimes B \mapsto [A_\Lambda B],
\]
in such a way that for any \( a, b \in \mathcal{R} \) and \( A, B, C \in T(\mathcal{R}) \) we have
(i) \([a \Lambda b]\) is defined by the \(\Lambda\)-bracket on \(\mathcal{R}\),
(ii) \(aB := a \otimes B\),
(iii) \(1A := A_1 := A\),
(iv) \((a \otimes B)C : - : a : BC ::\) is defined by the quasi-associativity,
(v) \([A\Lambda (b \otimes C)]\) and \([(a \otimes B)\Lambda C]\) are defined by the Wick formula.

For a given supersymmetric nonlinear LCA \(\mathcal{R}\), consider the two-sided ideal \(\mathcal{J}(\mathcal{R})\) of \(\mathcal{T}(\mathcal{R})\) generated by elements of the form

\[
(ab) : -(-1)^{p(a)p(b)} : ba : ) - (-1)^{p(a)p(b)} \sum_{j \geq 1} \frac{(-1)^j}{j!} b_{(-1+j|1)a},
\]

where

\[
[b_\Lambda a] = \sum_{j_0 \in \mathbb{Z}_{\geq 0}, j_1 = 0, 1} (-1)^{j_1} \Lambda^{(j_0|j_1)} b_{(j_0|j_1)} a.
\]

Then the \(\Lambda\)-bracket and the product \(:\) on \(\mathcal{T}(\mathcal{R})\) induce a well-defined \(\Lambda\)-bracket and product on the quotient

\[
V(\mathcal{R}) = \mathcal{T}(\mathcal{R})/\mathcal{J}(\mathcal{R}).
\]

Since \(V(\mathcal{R})\) satisfies quasi-commutativity, quasi-associativity and Wick formula, it is a supersymmetric vertex algebra which is called the \textit{universal enveloping supersymmetric vertex algebra} of \(\mathcal{R}\); cf. Definition 2.13.

**Proposition 2.16.** For a given ordered basis \(\mathcal{B}\) of \(\mathcal{R}\), the supersymmetric vertex algebra \(V(\mathcal{R})\) is freely generated by \(\mathcal{B}\).

### 3. Good filtered complexes of supersymmetric nonlinear LCAs

Here we reproduce some useful facts about bigraded complexes. Proofs can be obtained by suitable supersymmetric versions of the arguments in [8, Sec. 4]. Introduce the notation

\[
\Gamma = \frac{\mathbb{Z}}{2}, \quad \Gamma_+ = \frac{\mathbb{Z}_{\geq 0}}{2}, \quad \Gamma'_+ = \frac{\mathbb{Z}_{> 0}}{2}.
\]

Let \(\mathfrak{g}\) be a graded vector superspace and \(\mathcal{R} = \mathcal{K} \otimes \mathfrak{g}\) be a nonlinear Lie conformal algebra such that

\[
\mathfrak{g} = \bigoplus_{p,q \in \Gamma, p+q = \mathbb{Z}_+} \mathfrak{g}^{p,q}[\Delta], \quad \mathcal{R} = \bigoplus_{p,q \in \Gamma, p+q = \mathbb{Z}_+, \Delta \in \Gamma'_+} \mathcal{R}^{p,q}[\Delta],
\]

where

\[
\mathcal{R}^{p,q}[\Delta] = \bigoplus_{n \geq 0} S^n \otimes \mathfrak{g}^{p,q}[\Delta - \frac{n}{2}).
\]
The universal enveloping supersymmetric vertex algebra $V(\mathcal{R})$, which is strongly generated by a basis $\{a_i \mid i \in I\}$ of $\mathcal{R}$, has the $\Gamma'_+'$-grading

$$V(\mathcal{R}) = \bigoplus_{\Delta \in \Gamma'_+'} V(\mathcal{R})[\Delta]$$

where

$$V(\mathcal{R})[\Delta] = \text{span}_{\mathbb{C}} \{ a_{i_1} a_{i_2} \ldots a_{i_s} : i_k \in I, a_{i_k} \in \mathcal{R}[\Delta_k], \sum_{k=1}^s \Delta_k = \Delta \}.$$ 

We assume that

$$V(\mathcal{R})[\Delta_1(n_0,n_1)] V(\mathcal{R})[\Delta_2] \subset V(\mathcal{R})[\Delta_1 + \Delta_2 - n_0 - \frac{n_1}{2} - \frac{1}{2}].$$

Consider a $\Gamma$-filtration and a $\mathbb{Z}$-grading of $\mathcal{R}$ induced from (3.1)

$$F^p \mathcal{R} = \bigoplus_{p' \geq p, q \Delta} \mathcal{R}^{p',q}[\Delta], \quad \mathcal{R}^n = \bigoplus_{p+q=n} \mathcal{R}^{p,q},$$

and the corresponding filtration and $\mathbb{Z}$-grading of $V(\mathcal{R})$ defined by

$$V(\mathcal{R})^n = \text{span}_{\mathbb{C}} \{ a_{i_1} a_{i_2} \ldots a_{i_s} : |i_k| \leq I, a_{i_k} \in \mathcal{R}^{p_k,q_k}, \sum_{k=1}^s (p_k + q_k) = n \},$$

$$F^p V(\mathcal{R}) = \text{span}_{\mathbb{C}} \{ a_{i_1} a_{i_2} \ldots a_{i_s} : |i_k| \leq I, a_{i_k} \in \mathcal{R}^{p_k,q_k}, \sum_{k=1}^s p_k \geq p \}.$$ 

Set

$$F^p V(\mathcal{R})^n = F^p V(\mathcal{R}) \cap V(\mathcal{R})^n, \quad F^p V(\mathcal{R})^n[\Delta] = F^p V(\mathcal{R})^n \cap V(\mathcal{R})[\Delta]$$

and consider the associated graded algebra

$$\text{gr} V(\mathcal{R}) = \bigoplus_{p',q \in \Gamma} \text{gr}^{p',q} V(\mathcal{R}),$$

where

$$\text{gr}^{p',q} V(\mathcal{R})[\Delta] = F^{p'} V(\mathcal{R})^{p',q+1}[\Delta]/F^{p'+1} V(\mathcal{R})^{p'+1}[\Delta],$$

$$\text{gr}^{p',q} V(\mathcal{R}) = F^{p'} V(\mathcal{R})^{p'+q}/F^{p'+1} V(\mathcal{R})^{p'+q} = \bigoplus_{\Delta \in \Gamma'_+'} \text{gr}^{p',q} V(\mathcal{R})[\Delta].$$

Suppose a differential map $d : V(\mathcal{R}) \to V(\mathcal{R})$ satisfies

$$d(F^p V(\mathcal{R})^n) \subset F^p V(\mathcal{R})^{n+1}, \quad d(V(\mathcal{R})[\Delta]) \subset V(\mathcal{R})[\Delta].$$

Then we set for the cohomology spaces

$$F^p H^n V(\mathcal{R}), d) = \text{Ker}(d|_{F^p V(\mathcal{R})^n})/\text{Im} d \cap F^p V(\mathcal{R})^n,$$

$$\text{gr}^{p',q} H V(\mathcal{R}), d) = F^{p'} H^{p'+q} V(\mathcal{R}), d)/F^{p'+1} H^{p'+q} (V(\mathcal{R}), d).$$
In addition, for the graded differential map \( d^{gr} : \text{gr} V(\mathcal{R}) \to \text{gr} V(\mathcal{R}) \) induced from \( d \), we define cohomology spaces by

\[
H^{p,q}(\text{gr} V(\mathcal{R}), d^{gr}) = \text{Ker} \ d^{gr}|_{\text{gr}^{p,q} V(\mathcal{R})} / \text{Im} \ d^{gr} \cap \text{gr}^{p,q} V(\mathcal{R}).
\]

**Definition 3.1.** Let \( d \) be a differential on \( V(\mathcal{R}) \) satisfying (3.2).

1. We say \( d \) is an almost linear differential of \( \mathcal{R} \) if
   \[
d^{gr}(g^{p,q} [\Delta]) \subset g^{p,q+1}[\Delta];
   \]
or, equivalently, \( d(g^{p,q} [\Delta]) \subset g^{p,q+1}[\Delta] \oplus F^{p+\frac{q}{2}} V(\mathcal{R})^{p+q+1} \).
2. A differential \( d \) is called a good almost linear differential of \( \mathcal{R} \) if
   \[
   H^{p,q}(g, d^{gr}) = 0 \quad \text{if} \quad p + q \neq 0.
   \]

In the rest of this section we assume that \( V(\mathcal{R})[\Delta] \) has finite dimension for any \( \Delta \in \Gamma'_{+} \) and \( d \) is a good almost linear differential of \( \mathcal{R} \). Take bases

\[
B_{g}^{p}[\Delta] = \{ e_{i} | i \in I_{g}^{p}[\Delta] \} \quad \text{for some index sets} \quad I_{g}^{p}[\Delta],
\]

\[
B_{\mathcal{R}}^{p}[\Delta] = \{ e_{(i,n)} | e_{(i,n)} = S^{n}e_{i}, \ e_{i} \in B_{g}^{p}[\Delta'], \ \Delta' + \frac{n}{2} = \Delta \},
\]

of \( g^{p,-p}[\Delta] \cap \text{Ker} \ d^{gr} \) and \( \mathcal{R}^{p,-p}[\Delta] \cap \text{Ker} \ d^{gr} = H^{p,-p}(\text{gr} \mathcal{R}, d^{gr})[\Delta], \) respectively. Then

\[
B_{\mathcal{R}} := \bigsqcup_{\Delta \in \Gamma'_{+}, p \in \Gamma} B_{\mathcal{R}}^{p}[\Delta] = \{ e_{(i,n)} | e_{(i,n)} = S^{n}e_{i}, \ i \in I_{g} \}
\]
is a basis of \( H(\text{gr} \mathcal{R}, d^{gr}) \), where

\[
I_{g} := \bigsqcup_{\Delta \in \Gamma'_{+}, p \in \Gamma} I_{g}^{p}[\Delta].
\]

**Proposition 3.2.**

1. \( H(\text{gr} V(\mathcal{R}), d^{gr}) \) is freely generated by \( B_{\mathcal{R}} \).
2. \( H^{p,-p}(\text{gr} V(\mathcal{R}), d^{gr})[\Delta] \) has the basis
   \[
   B_{V(\mathcal{R})}^{p}[\Delta] = \{ e_{(i_{1},n_{1})}e_{(i_{2},n_{2})} \cdots e_{(i_{k},n_{k})} : \}
   \]
   where the sets of indices \( (i_{t}, n_{t}) \in I_{g}^{n}[\Delta_{t}] \times \mathbb{Z}_{\geq 0} \) satisfy the conditions:
   (i) \( (i_{t}, n_{t}) \leq (i_{t+1}, n_{t+1}) \),
   (ii) if \( e_{(i,n)} \) and \( e_{(i_{t+1},n_{t+1})} \) are odd then \( (i_{t}, n_{t}) < (i_{t+1}, n_{t+1}) \),
   (iii) \( \sum_{t=1}^{k} i_{t} = p \),
   (iv) \( \sum_{t=1}^{k} (\Delta_{t} + \frac{n_{t}}{2}) = \Delta \).
For $e_i \in \mathfrak{g}^{p-\frac{1}{2}}[\Delta] \cap \text{Ker } d^{gr}$ there exists an element $f_i \in F^{p+\frac{1}{2}} V(\mathcal{R})^0[\Delta]$ such that $E_i = e_i + f_i \in F^p V(\mathcal{R})^0[\Delta] \cap \text{Ker } d$. Set

$$H^{p-\frac{1}{2}}(\mathfrak{g}, d)[\Delta] = \text{span} \{ E_i \mid i \in T_0^p[\Delta] \}, \quad H(\mathfrak{g}, d)[\Delta] = \bigoplus_{p \in \Gamma} H^{p-\frac{1}{2}}(\mathfrak{g}, d)[\Delta].$$

**Theorem 3.3.**

(1) $H(V(\mathcal{R}), d) = H^0(V(\mathcal{R}), d)$.

(2) If the $\mathcal{K}$-module $H(\mathcal{R}, d) = \mathcal{K} \otimes H(\mathfrak{g}, d)$ admits a nonlinear supersymmetric LCA structure, then

$$H(V(\mathcal{R}), d) \simeq V(H(\mathcal{R}, d)).$$

4. BRST COHOMOLOGY

We are now in a position to define supersymmetric $W$-algebras via BRST cohomology following [22]. We will rely on the supersymmetric vertex algebra theory developed by Heluani and Kac [15, 18] to describe the structure of the $W$-algebras associated with odd nilpotent elements of Lie superalgebras.

4.1. BRST COMPLEX. Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra with a $(\frac{1}{2}\mathbb{Z})$-grading $\mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}(i)$ satisfying the following conditions:

(i) There exists $h \in \mathfrak{g}_0$ such that $\mathfrak{g}(i) = \{ a \in \mathfrak{g} \mid \frac{1}{2}[h, a] = ia \}$.

(ii) There are odd elements $f_{\text{odd}} \in \mathfrak{g}(-\frac{1}{2})$ and $e_{\text{odd}} \in \mathfrak{g}(\frac{1}{2})$ such that

$$\text{span}\{ e, e_{\text{odd}}, h, f_{\text{odd}}, f \} \simeq \mathfrak{osp}(1|2),$$

where $(e, h, f)$ is an $\mathfrak{sl}_2$-triple.

We will suppose that $\mathfrak{g}$ is equipped with a nondegenerate invariant bilinear form $(\mid)$ normalized by the conditions $(e\mid f) = \frac{1}{2}(h\mid h) = 1$.

Introduce two supersymmetric vertex algebras.

(1) Let $\overline{\mathfrak{g}} = \{ \overline{a} \mid a \in \mathfrak{g} \}$ be the vector superspace defined by $\overline{\mathfrak{g}}_1 = \mathfrak{g}_0$ and $\overline{\mathfrak{g}}_0 = \mathfrak{g}_1$. The supersymmetric current nonlinear LCA is

$$\mathcal{R}_{\text{cur}} := \mathcal{K} \otimes \overline{\mathfrak{g}}$$

endowed with the $\Lambda$-bracket

$$[\overline{a} \Lambda \overline{b}] = (-1)^{p(a)p(b)}[a, b] + k \chi(a|b).$$

(2) Set $n = \bigoplus_{i>0} \mathfrak{g}(i)$ and $n_- = \bigoplus_{i<0} \mathfrak{g}(i)$. Then there are bases

$$\{ u_\alpha \mid \alpha \in I_+ \} \quad \text{and} \quad \{ u^\alpha \mid \alpha \in I_+ \}$$
of \( n \) and \( n_- \), respectively, parameterized by a certain index set \( I_+ \), such that \((u^\alpha|u_\beta) = \delta_{\alpha,\beta}\). Introduce two vector superspaces
\[
\phi_n \simeq n \subset g, \quad \phi_{n_-} \simeq n_- \subset \overline{g},
\]
spanned by the respective families of elements \( \phi_b \) and \( \phi_{\overline{b}} \) with \( b \in n \) and \( \overline{b} \in n_- \). Consider the supersymmetric nonlinear LCA \( R_{\text{ch}} = K \otimes (\phi_n \oplus \phi_{n_-}) \) endowed with the \( \Lambda \)-bracket
\[
[\phi^{\overline{a}} \Lambda \phi_b] = [\phi_b \Lambda \phi^{\overline{a}}] = (a|b).
\]
Due to the results of Section 2.3, the two above supersymmetric nonlinear LCAs give rise to respective universal enveloping supersymmetric vertex algebras \( V(R_{\text{cur}}) \) and \( V(R_{\text{ch}}) \). Their tensor product
\[
C(\overline{g}, f_{\text{odd}}, k) = V(R_{\text{cur}}) \otimes V(R_{\text{ch}})
\]
also carries a supersymmetric vertex algebra structure. Introduce the element \( d \) by
\[
d = \sum_{\alpha \in I_+} : (\overline{u}_\alpha - (f_{\text{odd}}|u_\alpha)) \phi^\alpha : + \frac{1}{2} \sum_{\alpha, \beta \in I_+} (-1)^{p(\alpha)p(\overline{\beta})} : \phi_{[u_\alpha, u_\beta]} \phi^{\overline{\beta}} \phi^\alpha : ,
\]
where \( \phi^\alpha = \phi^{\overline{u}_\alpha} \), \( \phi_\alpha = \phi_{u_\alpha} \), \( p(\alpha) = p(u_\alpha) \) and \( p(\overline{\alpha}) = p(\overline{u}_\alpha) \).

**Proposition 4.1.** The \( \Lambda \)-brackets between \( d \) and elements in \( C(\overline{g}, f_{\text{odd}}, k) \) have the form:
\[
[d \Lambda a] = \sum_{\alpha \in I_+} (-1)^{p(\overline{\beta})p(\alpha)} : \phi^\alpha [u_\alpha, a] : + \sum_{\alpha \in I_+} (-1)^{p(\overline{\alpha})p(\beta)} k(\chi + S) \phi^\alpha (u_\alpha|a),
\]
\[
[d \Lambda \phi^\alpha] = \frac{1}{2} \sum_{\alpha, \beta \in I_+} (-1)^{p(\overline{\alpha})p(\beta)} : \phi^\beta \phi_{[u_\beta, u_\alpha]} : ,
\]
\[
[d \Lambda \phi_\alpha] = (-1)^{p(\overline{\alpha})} u_\alpha - (f_{\text{odd}}|u_\alpha) + \sum_{\beta \in I_+} (-1)^{p(\overline{\beta})p(\beta)} : \phi^\beta \phi_{[u_\beta, u_\alpha]} : .
\]

**Proof.** The formulas are verified by a direct calculation in the same way as for the supersymmetric classical \( W \)-algebras; see [21]. \( \Box \)

Set \( Q := d_{(0|0)} \). Then, by the Wick formula (2.3), we have
\[
Q(: AB :) = : Q(A) B : + (-1)^{p(A)} : A Q(B) : .
\]

**Proposition 4.2.** The linear map \( Q \) on \( C(\overline{g}, f_{\text{odd}}, k) \) satisfies \( Q^2 = 0 \).

**Proof.** This follows by a direct computation with the use of Proposition 4.1 and property (4.2). \( \Box \)
By taking the cohomology of the BRST complex $C(\mathfrak{g}, f_{\text{odd}}, k)$ with the differential $Q$, we can now define the corresponding supersymmetric $W$-algebra as in [22]; cf. [1] and [14, Ch. 15].

**Definition 4.3.** The **supersymmetric $W$-algebra associated to** $\mathfrak{g}, f_{\text{odd}}$ and $k \in \mathbb{C}$ is

$$W(\mathfrak{g}, f_{\text{odd}}, k) = H(C(\mathfrak{g}, f_{\text{odd}}, k), Q).$$

**Proposition 4.4.** Let $A, B \in C(\mathfrak{g}, f_{\text{odd}}, k)$ satisfy $Q(A) = Q(B) = 0$ and $C$ be any element in $C(\mathfrak{g}, f_{\text{odd}}, k)$. Then the following holds:

1. $Q(SA) = Q(AB) = 0$ and $Q([A_B]) = 0$;
2. $S(QC), :Q(C)B:$ and $[Q(C)_A B]$ belong to the image of $Q$.

**Proof.** By sesquilinearity of supersymmetric LCAs, for any $X \in C(\mathfrak{g}, f_{\text{odd}}, k)$ we have $S(QX) = Q(SX)$. Hence the first properties in (1) and (2) hold. The second properties follow from (4.2). By the Jacobi identity of supersymmetric LCAs, for $X, Y \in C(\mathfrak{g}, f_{\text{odd}}, k)$ we have

$$Q([X_Y]) = -Q(X)_A Y + (-1)^{p(X)+1}[X_A Q(Y)]$$

which gives the third properties in (1) and (2).

**Corollary 4.5.** The **supersymmetric $W$-algebra** $W(\mathfrak{g}, f_{\text{odd}}, k)$ is a supersymmetric vertex algebra.

**4.2. Building blocks of supersymmetric $W$-algebras.** For any $\bar{a} \in \bar{\mathfrak{g}}$ set

$$J_{\bar{a}} = \bar{a} + \sum_{\beta \in I_+} (-1)^{p(\bar{a})p(\beta)} :\phi^\beta \phi_{[\bar{u}_\beta, \bar{a}]} : \in C(\mathfrak{g}, f_{\text{odd}}, k).$$

**Proposition 4.6.** For the element $d$ defined in (4.1) we have

$$[d_A J_{\bar{a}}] = \sum_{\beta \in I_+} (-1)^{p(\bar{\pi})p(\beta)} :\phi^\beta (J_{\pi_{\bar{a}}[u_\beta, a]} + (f_{\text{odd}}[u_\beta, a])) : + \sum_{\beta \in I_+} (-1)^{\beta k (S + \chi)} \phi^\beta (u_\beta | a),$$

where $\pi_{\bar{a}} : \mathfrak{g} \rightarrow \bigoplus_{i \in \bar{\mathfrak{g}}} \bar{\mathfrak{g}}(i)$ is the projection map with the kernel $\bigoplus_{i > 0} \mathfrak{g}(i)$. 
Proof. By the Wick formula,
\[
[d_{\Lambda} J_\alpha] = [d_{\Lambda} \bar{a}] + \sum_{\beta \in I_+} (-1)^p(\bar{a})p(\bar{\beta}) [d_{\Lambda} : \phi^\beta \phi_{[u,\beta,a]} : ]
\]
(4.3)

\[
= [d_{\Lambda} \bar{a}] + \sum_{\beta \in I_+} (-1)^p(\bar{a})p(\bar{\beta}) : [d_{\Lambda} \phi^\beta] \phi_{[u,\beta,a]} : 
\]

(4.4)

\[
+ \sum_{\beta,\gamma \in I_+, k \geq 1} \frac{\Lambda^k}{2k!} (-1)^p(\bar{\beta})p(\gamma)+p(a)+1 \left( : \phi^\gamma \phi^{[u,\gamma,u\beta]} : \right)_{(k-1|1)} \phi_{[u,\beta,a]} 
\]

\[
+ \sum_{\beta \in I_+} (-1)^p(\bar{a})p(\bar{\beta}) : \phi^\beta [d_{\Lambda} \phi_{[u,\beta,a]}] : 
\]

Since the coefficients of $\Lambda^j \chi$ in $[\phi_{[u,\beta,a]} : \phi^\gamma \phi^{[u,\gamma,u\beta]} : ]$ are all zero, the coefficients of $\Lambda^j \chi$ in $[ : \phi^\gamma \phi^{[u,\gamma,u\beta]} : \Lambda \phi_{[u,\beta,a]} = (-1)^p(\bar{\beta})p(\gamma)+p(a)+1 \left( \phi^\gamma \phi^{[u,\gamma,u\beta]} \right)_{(k-1|1)} \phi_{[u,\beta,a]} ]$ are also 0 so that the expression in (4.4) vanishes. The second term in (4.3) equals

\[
\sum_{\beta,\gamma \in I_+, k \geq 1} \frac{1}{2} (-1)^p(\bar{\beta})p(\gamma)+p(a)+1 \left( : \phi^\gamma \phi^{[u,\gamma,u\beta]} : \right)_{(k-1|1)} \phi_{[u,\beta,a]} : 
\]

By the quasi-associativity in (2.2) and the fact that $\phi^n_{(j|1)} \phi_m = 0$ for any $n \in n$ and $m \in n_-$ with $j \geq 0$, we have

\[
[ : \phi^\gamma \phi^{[u,\gamma,u\beta]} : \phi_{[u,\beta,a]} ] = [ : \phi^\gamma \phi^{[u,\gamma,u\beta]} : ] 
\]

The remaining computations are straightforward, they are analogous to the classical case in [21].

Proposition 4.7. If $a, b \in \bigoplus_{i \leq 0} g(i)$ or $a, b \in \bigoplus_{i > 0} g$ then

\[
[J_a \Lambda J_b] = (-1)^p(a)p(b) \int_{[a,b]} + k(S + \chi)(a|b). 
\]

Proof. This is verified by a direct computation. □

Introduce the vector superspaces

\[
r_+ = \phi_n \oplus J_\bar{n} \quad \text{and} \quad r_- = J_{\bar{n} \geq 0} \oplus \phi^{-\bar{n}},
\]

where

\[
J_n = \text{span} \{ J_b \mid b \in \bar{n} \} \quad \text{and} \quad J_{\bar{n} \geq 0} = \text{span} \{ J_\bar{a} \mid a \in \bigoplus_{i \in Z_{\leq 0}} g(i) \}.
\]
It is not difficult to see that both \( R_+ = K \otimes r_+ \) and \( R_- = K \otimes r_- \) are supersymmetric nonlinear LCAs and that \( C(\mathfrak{g}, f_{\text{odd}}, k) \) decomposes into the tensor product of supersymmetric vertex subalgebras:

\[
C(\mathfrak{g}, f_{\text{odd}}, k) = V(R_+) \otimes V(R_-).
\]

**Lemma 4.8** (Künneth lemma). Let \( V_1 \) and \( V_2 \) be vector superspaces and \( d_i : V_i \to V_i \), \( i = 1, 2 \), be differentials. If \( d : V_1 \otimes V_2 \to V_1 \otimes V_2 \) is defined by

\[
d(a \otimes b) = d_1(a) \otimes b + (-1)^{p(a)}a \otimes d_2(b)
\]

then

\[
H(V, d) \simeq H(V_1, d_1) \otimes H(V_2, d_2).
\]

**Proposition 4.9.** The differential \( Q \) has the properties

\[
(4.5) \quad Q(V(R_+)) \subset V(R_+) \quad \text{and} \quad Q(V(R_-)) \subset V(R_-),
\]

so that

\*

(4.6) \quad W(\mathfrak{g}, f_{\text{odd}}, k) = H(V(R_+), Q) \otimes H(V(R_-), Q).
*

**Proof.** The inclusions (4.5) follow from Propositions 4.1 and 4.6. The decomposition (4.6) is then implied by the Künneth lemma. \( \square \)

**4.3. Generators of supersymmetric \( W \)-algebras.** We now aim to describe the cohomologies \( H(V(R_+), Q) \) and \( H(V(R_-), Q) \).

**Proposition 4.10.** We have \( H(V(R_+), Q) = \mathbb{C} \) so that \( W(\mathfrak{g}, f_{\text{odd}}, k) = H(V(R_-), Q) \).

**Proof.** Set \( K_{\bar{n}} = (-1)^{p(\bar{n})}j_{\bar{n}} - (f_{\text{odd}}|n) \) for \( n \in \mathfrak{n} \) and introduce the superspace

\[
r'_+ = \phi_{\mathfrak{n}} \oplus K_{\bar{n}}, \quad K_{\bar{n}} = \text{span} \{ K_{\bar{n}} | \bar{n} \in \bar{n} \}.
\]

Then \( R_+ = K \otimes r'_+ \). Define the conformal weight \( \Delta \) and the bigrading on \( r'_+ \) by

\[
\Delta(\phi_n) = \Delta(K_{\bar{n}}) = j_n, \quad \text{gr}(\phi_n) = (j_n - 1, -j_n), \quad \text{gr}(K_{\bar{n}}) = (j_n - 1, -j_n + 1),
\]

assuming that \( n \in \mathfrak{g}(j_n) \). The graded differential \( Q^{gr} \) associated with \( Q \) is good almost linear (see Section 3) and

\[
H(r'_+, Q^{gr}) = 0.
\]

By Theorem 3.3, we have \( H(V(R_+), Q) = \mathbb{C} \). \( \square \)
To describe $H(V(\mathcal{R}_-), Q)$, recall that

$$Q(J_a) = \sum_{\beta \in I_+} (-1)^{p(\alpha)p(\beta)} : \phi^\beta (J_{\pi_{\leq 0}[u_\beta, a]} + (f_{\text{odd}}[u_\beta, a])) :$$

(4.7)

and

$$Q(\phi^m) = \frac{1}{2} \sum_{\beta \in I_+} (-1)^{p(m)p(\beta)} : \phi^\beta [u_\beta, m] :.$$  

(4.8)

Consider the conformal weight $\Delta$ and the bigrading on $r_-$ satisfying

$$\Delta(J_a) = \frac{1}{2} - j_a, \quad \Delta(\phi^m) = -j_m,$$

$$\text{gr}(J_a) = (j_a, -j_a), \quad \text{gr}(\phi^m) = (j_m + \frac{1}{2}, -j_m + \frac{1}{2}),$$

where $a \in g(j_a)$ and $m \in g(j_m)$ for $j_a \leq 0$ and $j_m < 0$. Note that

$$\Delta(\phi^\beta) = j_\beta, \quad \text{gr}(\phi^\beta) = (-j_\beta + \frac{1}{2}, j_\beta + \frac{1}{2}),$$

where $u^\beta \in g(-j_\beta)$. Since $\Delta(S) = \frac{1}{2}$ and $\text{gr}(S) = (0, 0)$. Every term in (4.7) has conformal weight $\frac{1}{2} - j_a$ and every term in (4.8) has conformal weight $-j_m$. The bigradings of terms in (4.7) are given by

$$\text{gr}(\phi^\beta J_{\pi_{\leq 0}[u_\beta, a]}) = (j_a + \frac{1}{2} - j_a + \frac{1}{2}),$$

(4.9)

$$\text{gr}(\phi^\beta (f_{\text{odd}}[u_\beta, a])) = (j_a, -j_a + 1),$$

$$\text{gr}(S \phi^\beta (u_\beta | a)) = (j_a + \frac{1}{2}, -j_a + \frac{1}{2}).$$

The bigradings of terms in (4.8) are

$$\text{gr}(\phi^m) = (j_m + \frac{1}{2}, -j_m + \frac{1}{2}), \quad \text{gr}(: \phi^\beta [u_\beta, m] :) = (j_m + 1, -j_m + 1).$$

(4.10)

**Theorem 4.11.** Let $\text{Ker}(\text{ad} f_{\text{odd}}) = \{ u_\alpha \mid \alpha \in \mathcal{J} \}$ with an index set $\mathcal{J}$. Then

1. $W(\mathfrak{g}, f_{\text{odd}}, k)$ is freely generated by $|\mathcal{J}|$ elements as a differential algebra,
2. there exists a free generating set of the form

$$\{ u_\alpha + A_\alpha \mid \alpha \in \mathcal{J} \},$$

where $A_\alpha \in F^{j_\alpha + \frac{1}{2}} V(\mathcal{R}_-) [\frac{1}{2} - j_\alpha]$ for $u_\alpha \in g(j_\alpha)$. 


Proof. Since we know that \( W(\mathfrak{g}, f_{\text{odd}}, k) = H(V(\mathcal{R}_-), Q) \), it is enough to show (1) and (2) for \( H(V(\mathcal{R}_-), Q) \). The conformal weight and bigrading on \( r_- \) induce those on \( V(\mathcal{R}_-) \). With respect to the conformal weight and bigrading, \( Q \) induces the graded differential \( Q^{gr} \). The bigradings listed in (4.9) and (4.10) show that

\[
Q^{gr}(J_a) = \sum_{\beta \in I_+} (-1)^{p(\beta)p(a)} \phi^\beta (f_{\text{odd}})[u_\beta, a], \quad Q^{gr}(\phi^m) = 0.
\]

Note that \( V(\mathcal{R}_-)^0 \cap r_- = J_{\mathfrak{g} \in C} \) and \( V(\mathcal{R}_-) \cap r_- = \phi^0 \). Since \( Q^{gr}(r_-) = \phi^0 \), we have \( H^{p,q}(r_-, Q^{gr}) = 0 \) when \( p + q \neq 0 \) and so \( Q \) is a good almost linear differential map. Furthermore, \( \text{Ker}(Q^{gr}|_{r_-}) = \{ J_a | a \in \text{Ker}(\text{ad} f_{\text{odd}}) \} \oplus \phi^0 \), hence

\[
H(r_-, Q^{gr}) = \{ J_a | a \in \text{Ker}(\text{ad} f_{\text{odd}}) \}.
\]

Thus, using Theorem 3.3, we arrive at (1) and (2). \( \square \)

5. Generators of \( W(\mathfrak{g}, f_{\text{prin}}, k) \) for \( \mathfrak{g} = \mathfrak{gl}(n+1|n) \)

Consider the Lie superalgebra \( \mathfrak{g} = \mathfrak{gl}(n+1|n) \) with the basis \( \{ E_{i,j} | i, j = 1, \ldots, 2n+1 \} \) and the \( \mathbb{Z}/2\mathbb{Z} \)-grading defined by \( p(E_{i,j}) = i+j \mod 2 \) with the commutation relations

\[
[E_{i,j}, E_{i',j'}] = \delta_{j,j'} E_{i,j'} - (-1)^{(i+j)(i'+j')} \delta_{i,i'} E_{i',j}.
\]

Take the odd principal nilpotent element in the form

\[
f_{\text{prin}} = \sum_{p=1}^{2n} E_{p+1,p}.
\]

By Proposition 4.6, for \( C(\mathfrak{g}, f_{\text{prin}}, k) \) and any \( m \geq l \), we have

\[
Q(J_{m,l}) = (-1)^m k S \phi^{l,m} + \sum_{j=l+1}^m (-1)^{l+j+1} : \phi^{l,j} J_{m,j} : + \sum_{i=l}^{m-1} (-1)^{(i+m)(m-l+1)} : \phi^{i,m} J_{l,i} : + (-1)^l \phi^{l,m+1} + (-1)^m \phi^{l-1,m},
\]

where we set \( \phi^{i,j} = (-1)^{i+1} \phi^{i,j}_{\text{Eij}} \) for \( i > j \) and \( J_{i,j} = J_{\text{Eij}} \) for \( i \geq j \).

We will be working with operators on \( C(\mathfrak{g}, f_{\text{prin}}, k) \) of the form \( \sum_{t=0}^N A_t S^t \) with \( A_t \in C(\mathfrak{g}, f_{\text{prin}}, k) \), which act on an arbitrary element \( X \in C(\mathfrak{g}, f_{\text{prin}}, k) \) by the rule

\[
\sum_{t=0}^N A_t S^t(X) = \sum_{t=0}^N : A_t(S^t(X)) :.
\]
Theorem 5.1.

All elements $A_{i,j} = \delta_{ij} k S + (-1)^{i+1} J_{i,j}$ on $C(\mathfrak{g}, f_{\text{prin}}, k)$ we have

$$A_{i,j}(X) = \delta_{ij} k S(X) + (-1)^{i+1} : J_{i,j} X : .$$

Consider the $(2n+1) \times (2n+1)$ matrix

$$A := \begin{bmatrix}
A_{1,1} & -1 & 0 & \cdots & \cdots & 0 \\
A_{2,1} & A_{2,2} & -1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
A_{2n,1} & A_{2n,2} & A_{2n,3} & \cdots & A_{2n,2n} & -1 \\
A_{2n+1,1} & A_{2n+1,2} & A_{2n+1,3} & \cdots & A_{2n+1,2n} & A_{2n+1,2n+1}
\end{bmatrix}$$

whose entries are operators on $C(\mathfrak{g}, f_{\text{prin}}, k)$. Then the column (or row) determinant of $A$ is given by the formula

$$\text{cdet } A = \sum_{N=0}^{2n} \sum_{0=i_0 < i_1 < \cdots < i_{N+1}=2n+1} A_{i_1,i_0+1} A_{i_2,i_1+1} \cdots A_{i_{N+1},i_N+1}.$$

Write

$$\text{cdet } A = W_0 + W_1 S + \cdots + W_{2n+1} S^{2n+1}$$

for certain elements $W_p \in C(\mathfrak{g}, f_{\text{prin}}, k)$. Clearly, $W_{2n+1} = k^{2n+1}$.

**Theorem 5.1.** All elements $W_1, \ldots, W_{2n}$ belong to the $W$-algebra $W(\mathfrak{g}, f_{\text{prin}}, k)$.

**Proof.** One readily verifies that

$$Q \sum_{p=0}^{2n+1} W_p S^p = \sum_{p=0}^{2n+1} Q(W_p) S^p - W_p S^p Q$$

so that $QA_{m,i} = (-1)^{m+i+1} A_{m,i} Q + (-1)^{m+1} Q(J_{m,i})$. Therefore,

$$QA_{i_1,i_0+1} \cdots A_{i_{p+1},i_p+1} \cdots A_{i_{N+1},i_N+1}$$

$$= \sum_{p=0}^{N} (-1)^i (A_{i_{1,i_{0}}+1} \cdots ((-1)^{i_{p+1}} Q(J_{i_{p+1},i_{p+1}})) \cdots A_{i_{N+1},i_{N+1}})$$

$$- A_{i_{1,i_{0}}+1} \cdots A_{i_{p+1},i_{p+1}} \cdots A_{i_{N+1},i_{N+1}} Q.$$

Hence the property $W_p \in W(\mathfrak{g}, f_{\text{prin}}, k)$ will follow if we show that $\sum_{N=0}^{2n} B_N = 0$, where we set

$$B_N = \sum_{p=0}^{N} (-1)^i (A_{i_{1,i_{0}}+1} \cdots ((-1)^{i_{p+1}} Q(J_{i_{p+1},i_{p+1}})) \cdots A_{i_{N+1},i_{N+1}}).$$
Using the relations

\[ J_{i,j} = (-1)^{i+1}(A_{i,j} - \delta_{i,j} kS) \quad \text{and} \quad \phi^{i,j} J_{i',j'} := (-1)^{(i+j+1)(i'+j'+1)} : J_{i',j'} \phi^{i,j} : \]

we find that

\[
(−1)^{i+p+1} Q(J_{i+1,i+p+1}) \\
= -kS(\phi^{i+p+1,i+p+1}) + \sum_{j=i_p+2}^{i_{p+1}} (−1)^{i_p+j} \phi^{i+p+1,j}(A_{i_p+1,j} - \delta_{i_p+1,j} kS) \\
+ \sum_{i=i_p+1}^{i_{p+1}-1} (−1)^{i_p+i}(A_{i,i_p+1} - \delta_{i,i_p+1} kS) \phi^{i,i_p+1} + (−1)^{i_p+i_{p+1}} \phi^{i+1,i_{p+1}+1} - \phi^{i_{p+1}}
\]

and

\[
−kS(\phi^{i_p+1,i_{p+1}}) + (−1)^{i_p+i_{p+1}+1} \phi^{i_p+1,i_{p+1}} S + S \phi^{i+1,i_{p+1}} = 0.
\]

Therefore,

\[
(−1)^{i+p+1} Q(J_{i_p+1,i_p+1}) = \sum_{j=i_p+2}^{i_{p+1}} (−1)^{i_p+j} \phi^{i+p+1,j} A_{i_p+1,j} \\
+ \sum_{i=i_p+1}^{i_{p+1}-1} (−1)^{i_p+i} A_{i,i_p+1} \phi^{i,i_p+1} + (−1)^{i_p+i_{p+1}} \phi^{i+1,i_{p+1}+1} - \phi^{i_{p+1}}
\]

so that \( B_N \) can be expressed as

\[
\sum_{p=0}^{N} A_{i_1,i_0+1} \ldots A_{i_p,i_p-1+1} \left[ \left( \sum_{j=i_p+2}^{i_{p+1}} (−1)^{j} \phi^{i+p+1,j} A_{i_p+1,j} + (−1)^{i_p+1} \phi^{i+1,i_{p+1}+1} \right) \\
+ \left( \sum_{i=i_p+1}^{i_{p+1}-1} (−1)^{i} A_{i,i_p+1} \phi^{i,i_p+1} - (−1)^{i} \phi^{i,i_{p+1}} \right) \right] A_{i_{p+2},i_{p+1}+1} \ldots A_{i_{N+1},i_N+1}.
\]

By the quasi-associativity property, we have

\[
(\phi^{i+p+1,j} A_{i_p+1,j})(A_{i_{p+2},i_{p+1}+1} \ldots A_{i_{N+1},i_N+1}) = \phi^{i+p+1,j}(A_{i_p+1,j}(A_{i_{p+2},i_{p+1}+1} \ldots A_{i_{N+1},i_N+1})),
\]

\[
(A_{i,i_p+1} \phi^{i,i_{p+1}})(A_{i_{p+2},i_{p+1}+1} \ldots A_{i_{N+1},i_N+1}) = A_{i,i_{p+1}}(\phi^{i,i_{p+1}}(A_{i_{p+2},i_{p+1}+1} \ldots A_{i_{N+1},i_N+1})),
\]

for \( j = i_p+2, \ldots, i_{p+1} \) and \( i = i_p+1, \ldots, i_{p+1} \), so that vanishing of the telescoping sum implies that \( \sum_{N=0}^{2n} B_N = 0 \).

\[\square\]

**Lemma 5.2.** Suppose that \( \{v_p \mid p = 0, \ldots, 2n\} \) is a basis of \( \text{Ker}(\text{ad} f_{\text{odd}}) \) such that \( \Delta_{J_{v_p}} = \frac{1}{2}(2n+1-p) \). Take \( V_p \in W(\overline{\mathfrak{g}}, f_{\text{prim}}, k) \) of the form \( V_p = J_{v_p} + w_p \) satisfying the conditions...
(i) \( V_p \) and \( w_p \) have the conformal weight \( \frac{1}{2}(2n + 1 - p) \),
(ii) \( w_p \) lies in the differential algebra generated by \( J_{\bar{a}} \) for \( \Delta J_{\bar{a}} < \Delta V_p \).

Then the set \( \{ V_p \mid p = 0, \ldots, 2n \} \) freely generates the \( W \)-algebra \( W(\bar{g}, f_{\text{prin}}, k) \).

**Proof.** A generating set of the form \( \{ V'_p = J_{\bar{v}} + w'_p \mid p = 0, \ldots, 2n \} \) satisfying the required conditions (i) and (ii) exists by Theorem 4.11. Set

\[
W_m := \text{subalgebra freely generated by} \{ V_m, V_{m+1}, \ldots, V_{2n} \},
\]

\[
W'_m := \text{subalgebra freely generated by} \{ V'_m, V'_{m+1}, \ldots, V'_{2n} \}.
\]

We will show by a (reverse) induction that \( W_m = W'_m \) for all \( m = 0, \ldots, 2n \). Note that \( W_{2n} = W'_{2n} \), since \( w_{2n} \) and \( w'_{2n} \) are constants. Now suppose that \( W_p = W'_p \) for some \( p \leq 2n \). Then \( V_{p-1} - V'_{p-1} \in W_p = W'_p \) by condition (ii). Hence we can conclude that \( V'_{p-1} = V_{p-1} + (w'_p - w_p) \in W_{p-1} \) and, similarly, \( V_{p-1} \in W'_{p-1} \). This shows that \( W_{p-1} = W'_{p-1} \). Thus, \( W'_0 = W_0 \) and since \( W(\bar{g}, f_{\text{prin}}, k) = W'_0 \), the lemma follows. \( \square \)

**Theorem 5.3.** The set of coefficients \( \{ W_p \mid p = 0, \ldots, 2n \} \) of \( \text{cdet} \ A \) freely generates \( W(\bar{g}, f_{\text{prin}}, k) \) as a differential algebra.

**Proof.** Note that for \( i \geq j \) we have

\[
\Delta A_{i,j}(X) = \frac{1}{2}(i - j + 1) + \Delta X,
\]

and each term in (5.1) satisfies

\[
\Delta A_{i_1,i_2,i_3,\ldots,i_{N+1};i_{N+1}}(X) = \frac{2n + 1}{2} + \Delta X.
\]

A direct calculation gives

\[
W_{2n-k} = \sum_{l=1}^{2n+1-k} (-1)^{kl} J_{k+l,l} + w_{2n-k} \quad \text{for} \quad k = 0, 1, \ldots, 2n,
\]

where \( \Delta_{2n-k} = \frac{2n+1}{2} - \frac{2n-k}{2} \) and \( w_{2n-k} \) can be expressed as a normally ordered product of the elements \( J_{i,j} \) with \( 0 \leq i - j \leq k \) and their derivatives. It remains to apply Lemma 5.2. \( \square \)

**Example 5.4.** Let \( g = \text{gl}(2|1) \). Then \( f_{\text{prin}} = E_{21} + E_{32} \) and

\[
A = \begin{bmatrix}
A_{1,1} & -1 & 0 \\
A_{2,1} & A_{2,2} & -1 \\
A_{3,1} & A_{3,2} & A_{3,3}
\end{bmatrix}.
\]
The column determinant of $\mathcal{A}$ is
\[
\text{cdet } \mathcal{A} = A_{1,1}A_{2,2}A_{3,3} + A_{3,1} + A_{2,1}A_{3,3} + A_{1,1}A_{3,2} \\
= (kS)^3 + W_2S^2 + W_1S + W_0.
\]
where
\[
W_2 = k^2 (J_{1,1} + J_{2,2} + J_{3,3}), \\
W_1 = k(-J_{1,1}J_{2,2} - J_{1,1}J_{3,3} - J_{2,2}J_{3,3} - J_{2,1} + J_{3,2} - kJ_{2,2}'), \\
W_0 = -J_{1,1}J_{2,2}J_{3,3} - J_{2,1}J_{3,3} + J_{1,1}J_{3,2} + J_{3,1} \\
+ kJ_{3,2}' + kJ_{3,1}J_{3,3}' - kJ_{2,2}'J_{3,3} + kJ_{2,2}J_{3,3}' + k^2 J_{3,3}'',
\]
and $X' := [S, X]$. Hence $W(\mathfrak{g}, f_{\text{prin}}, k)$ is freely generated by $W_0, W_1$ and $W_2$. \qed

As in [2], by taking the quotient of the $W$-algebra $W(\mathfrak{g}, f_{\text{prin}}, k)$ over the supersymmetric vertex algebra ideal generated by the elements $J_{i,j}$ with $i > j$ we recover the presentation of the $W$-algebra via the Miura transformation; cf. [9, 16, 17]:
\[
\text{cdet } \mathcal{A} \mapsto (kS + J_{1,1})(kS - J_{2,2})(kS + J_{3,3}) \ldots (kS - J_{2n,2n})(kS + J_{2n+1,2n+1}).
\]

**References**

[1] T. Arakawa, *Introduction to $W$-algebras and their representation theory*, in “Perspectives in Lie theory”, pp. 179–250, Springer INdAM Ser., 19, Springer, Cham, 2017.

[2] T. Arakawa and A. Molev, *Explicit generators in rectangular affine $W$-algebras of type A*, Lett. Math. Phys. 107 (2017), 47–59.

[3] B. Bakalov and V. G. Kac, *Field Algebras*, Int. Math. Res. Not. 3 (2003), 123–159.

[4] P. Bouwknegt, *Extended conformal algebras from Kac–Moody algebras*, in: “Infinite-dimensional Lie Algebras and Lie Groups”, ed. V. Kac, Proc. CIRM-Luminy Conf., 1988 (World Scientific, Singapore, 1989); Adv. Ser. Math. Phys. 7 (1988), 527.

[5] P. Bouwknegt and K. Schoutens, *$W$-symmetry in conformal field theory*, Phys. Rep. 223 (1993), 183–276.

[6] J. de Boer, F. Harmsze and T. Tjin, *Non-linear finite $W$-symmetries and applications in elementary systems*, Phys. Rep. 272 (1996), 139–214.

[7] J. de Boer and T. Tjin *The relation between quantum $W$ algebras and Lie algebras*, Comm. Math. Phys. 160 (1994), 317–332.

[8] A. De Sole and V. Kac, *Finite vs affine $W$-algebras*, Jpn. J. Math. 1 (2006), 137–261.

[9] J. Evans and T. Hollowood, *Supersymmetric Toda field theories*, Nucl. Phys. B352 (1991), 723–768.

[10] V. A. Fateev and S. L. Lukyanov, *The models of two-dimensional conformal quantum field theory with $Z_n$ symmetry*, Internat. J. Modern Phys. A 3 (1988), 507–520.

[11] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *On hamiltonian reductions of the Wess–Zumino–Novikov–Witten theories*, Phys. Rep. 222 (1992), 1–64.

[12] B. Feigin and E. Frenkel, *Quantization of the Drinfeld–Sokolov reduction*, Phys. Lett. B 246 (1990), 75–81.
[13] L. Frappat, E. Ragoucy and P. Sorba, \textit{W-algebras and superalgebras from constrained WZW models: a group theoretical classification}, Comm. Math. Phys. \textbf{157} (1993) 499–548.

[14] E. Frenkel and D. Ben-Zvi, \textit{Vertex algebras and algebraic curves}, Second edition. Mathematical Surveys and Monographs, 88. AMS, Providence, RI, 2004.

[15] R. Heluani and V. G. Kac, \textit{Supersymmetric vertex algebras}, Comm. Math. Phys. \textbf{271} (2007), 103–178.

[16] K. Ito, \textit{Quantum hamiltonian reduction and $N = 2$ coset models}, Phys. Lett. B \textbf{259} (1991), 73–78.

[17] K. Ito, \textit{$N = 2$ superconformal CP$_n$ model}, Nucl. Phys. B\textbf{370} (1992), 123–142.

[18] V. Kac, \textit{Vertex algebras for beginners}, Second edition. University Lecture Series, 10. AMS, Providence, RI, 1998.

[19] V. Kac, \textit{Introduction to vertex algebras, Poisson vertex algebras, and integrable Hamiltonian PDE}, in “Perspectives in Lie theory”, pp. 3–72, Springer INdAM Ser., 19, Springer, Cham, 2017.

[20] V. Kac, S.-S. Roan and M. Wakimoto, \textit{Quantum reduction for affine superalgebras}, Comm. Math. Phys. \textbf{241} (2003), 307–342.

[21] V. Kac and U.R. Suh, \textit{Supersymmetric classical W-algebras}, in preparation.

[22] J. O. Madsen and E. Ragoucy, \textit{Quantum hamiltonian reduction in superspace formalism} Nucl. Phys. B\textbf{429} (1994), 277–290.

[23] A. B. Zamolodchikov, \textit{Infinite extra symmetries in two-dimensional conformal quantum field theory}, Teoret. Mat. Fiz. \textbf{65} (1985), 347–359.

(A. Molev) School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
\textit{E-mail address: alexander.molev@sydney.edu.au}

(E. Ragoucy) Laboratoire de Physique Théorique LAPTh, CNRS, Université Savoie Mont Blanc and U.G.A., BP 110, 74941 Annecy-le-Vieux Cedex, France
\textit{E-mail address: eric.ragoucy@lapth.cnrs.fr}

(U.R. Suh) Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, GwanAkRo 1, Gwanak-Gu, Seoul 08826, Korea
\textit{E-mail address: uhrisu1@snu.ac.kr}