Hamiltonian Renormalisation VII: Free fermions and doubler free kernels

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Abstract

The Hamiltonian renormalisation programme motivated by constructive QFT and Osterwalder-Schrader reconstruction which was recently launched for bosonic field theories is extended to fermions. As fermion quantisation is not in terms of measures, the scheme has to be mildly modified accordingly.

We exemplify the scheme for free fermions both for compact and non-compact spatial topologies respectively (i.e. with and without IR cut-off) and demonstrate that the convenient Dirichlet or Shannon coarse graining kernels recently advertised in a companion paper lead to a manifestly doubler free flow.

1 Introduction

The Hamiltonian or canonical approach to quantum gravity [1] aims at implementing the constraints as operators on a Hilbert space. In the classical theory, the constraints generate the Einstein equations via the Hamiltonian equations of motion [2]. They underlie the numerical implementation of the initial value formulation of Einstein’s equations e.g. in black hole merger and gravitational wave template codes [3].

The mathematically sound construction of canonical quantum gravity is a hard problem because the constraints are non-polynomial expressions in the elementary fields and in that sense much more non-linear than even the most complicated interacting QFT on Minkowski space such as QCD whose Hamiltonian is still polynomial in gluon and quark fields. As the theory is non-renormalisable and thus believed to exist only non-perturbatively, the Loop Quantum Gravity (LQG) approach has systematically developed such a non-perturbative programme [4]. LQG derives its name from the fact that it uses a connection rather than metric based formulation, hence it is phrased in the language of Yang-Mills type gauge fields and thus benefits from the non-perturbative technology introduced for such theories, specifically gauge invariant Wilson loop variables [5].

The current status of LQG can be described as follows: While the quantum constraints can indeed be implemented in a Hilbert space representation [6] of the canonical (anti-) commutation and adjointness relations as densely defined operators [7] and while its commutator algebra is mathematically consistent in the sense that it closes, it closes with the wrong structure “functions”. The inverted commas refer to the fact that the classical constraints do not form a Lie Poisson algebra because for a Lie algebra it is required that one has structure constants. By contrast, here we have non-trivial structure functions in the classical theory which are dictated by the fundamental hypersurface deformation algebra [8] and in the quantum theory they become operators themselves and are not simply constant multiples of the identity operator. We therefore call them structure operators.

The most important missing step in LQG is therefore to correct those structure operators. It is for this reason that more recently Hamiltonian renormalisation techniques were considered [9]. There one actually works with a 1-parameter family of gauge fixed versions of the theory [10] so that the constraints no longer appear and are traded for a Hamiltonian which drives that one parameter evolution. The reason for doing is are twofold: On the one hand, working with the gauge fixed version means solving the constraints classically and saves the work to determine quantum kernel and Hilbert space structure on it. On the other hand, the techniques of [9] were...
derived from Osterwalder-Schrader reconstruction [11] which deals with theories whose dynamics is driven by an actual Hamiltonian rather than constraints (see however [12]). Still, that Hamiltonian uniquely descends from the constraints and therefore its quantisation implicitly depends on the quantisation of the constraints. Therefore, the quantum constrains and their structure operators are implicitly also present in the gauge fixed version. In addition, in [14] we have shown that the techniques of [9] can be “abused” also for constrained quantum theories in the sense that the renormalisation steps to be carried out can be performed independently for all constraints “as if they were actual Hamiltonians”, even if the corresponding operators are not bounded from below. In that sense the methods of [9] complement those of [16] where the correction of the structure operators is approached by exploiting the spatial diffeomorphism invariance of the classical theory in an even more non-linear fashion than it was already done in [7].

The programme of [9] rests on the following observation: In quantising an interacting classical field theory one cannot proceed directly but rather has to introduce at least an UV cut-off $M$ where we may think of $M^{-1}$ as a spatial resolution. Introducing $M$ produces quantisation ambiguities which are encoded in a set of parameters depending on $M$. Almost all points in that set do not define consistent theories where a consistent theory is defined to be one in which the theory at resolution $M$ is the same as the theory at higher resolution $M' > M$ after “integrating out” the extra degrees of freedom. Renormalisation introduces a flow on these parameters whose fixed or critical points define consistent theories. In this way, the correct structure operators or algebra of constraints referred to above are also believed to be found, either explicitly or implicitly. In [14] we have shown that this is what actually happens for the much simpler case of 2d parametrised field theory [15] whose quantum hypersurface deformation algebra coincides with the Virasoro algebra. The lesson learnt from this is that the quantum constraint algebra must not even close at any finite resolution even if the continuum algebra closes with the correct structure operators. In other words, it is physically correct that the finite resolution constraints are anomalous while the actual continuum theory is anomaly free.

In [17] we have further tested [9] for free bosons (scalars and vector fields). Theories with fermions were not considered so far. In this paper we close this gap, see also [18] for a closely related formulation.

The architecture of this paper is as follows:

In section 2 we briefly recall the bosonic theory from [9].
In section 3 we adapt the bosonic theory to the fermionic setting.
In section 4 we test the fermionic Hamiltonian renormalisation theory for free Dirac-Weyl fermions both with and without IR cut-off using the Dirichlet-Weyl kernel and confirm a manifestly doubler free spectrum at each resolution $M$ at the fixed point. The Nielsen-Ninomiya theorem [19] is evaded because the finite resolution Hamiltonians are spatially non-local as it is usually the case when one “blocks from the continuum” i.e. computes the “perfect Hamiltonian”. A similar observation was made in the context of QCD in the Euclidian action approach [20].

In section 5 we summarise and conclude.

2 Review of Hamiltonian renormalisation for bosons

To be specific will consider the theory either with IR cut-off so that space is a d-torus $T^d$ or without IR cut-off so that space is d-dimensional Euclidian space $\mathbb{R}^d$ and it will be sufficient to consider one coordinate direction as both spaces are a Cartesian products. Thus $X = [0, 1)$ or $X = \mathbb{R}$ in what follows.

Thus for simplicity we consider a bosonic, scalar quantum field $\Phi$ (operator valued distribution) with conjugate momentum $\Pi$ on $X$ with canonical commutation and adjointness relations (in natural units $\hbar = 1$)

$$[\Pi(x), \Phi(y)] = i \delta(x, y), \quad \Phi(x)^* = \Phi(x), \quad \Pi^*(x) = \Pi(x)$$  \hspace{1cm} (2.1)

where

$$\delta(x, y) = \sum_{n \in \mathbb{Z}} e_n(x) e_n(y)^*, \quad e_n(x) = e^{2\pi i n x}$$  \hspace{1cm} (2.2)
is the periodic $\delta$ distribution on the torus or

$$\delta(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi} e_k(x) e_k(y)^*, \ e_k(x) = e^{ikx} \tag{2.3}$$
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on the real line respectively. It is customary to work with the bounded Weyl operators

$$w[f, g] = \exp(i[\Phi(f) + \Pi(g)]), \ \Phi(f) = \int_X dx \ f(x) \Phi(x), \ \Pi(g) = \int_X dx \ g(x) \Pi(x) \tag{2.4}$$

with $f, g \in L = L_2(X, dx)$ test functions or smearing functions usually with some additional properties such as differentiability or even smoothness. For tensor fields of higher degree a similar procedure can be followed (see [9]).

Since the space $L$ enters the stage naturally we use multi resolution analysis (MRA) language [21] familiar from wavelet theory [23] to define a renormalisation group flow. MRA’s serve as a powerful organising principle to define renormalisation flows in terms of coarse graining kernels and while the choice of the kernel should intuitively not have much influence on the fixed point or continuum theory (at least in presence of universality) the examples of [14, 22] show that generic features such as smoothness can have an impact.

In the most general sense an MRA is a nested sequence of Hilbert subspaces $V_M \subset L$ indexed by $M \in \mathcal{M}$ where $\mathcal{M}$ is partially ordered and directed by $\leq$. That is, one has $V_M \subset V_{M'}$ for $M \leq M'$ and $\bigcup_{M \in \mathcal{M}} V_M$ is dense in $L$. Pick an ONB $d(M)^{1/2} \chi^M_m$ for $V_M$ where $m$ is from a countably finite (infinite) index set $Z_M$ for $X = [0, 1)$ $(X = \mathbb{R})$ respectively and $d(M)$ is a finite number. In case that $X = [0, 1)$ typically $Z_M$ is the lattice $x^M_m$, $m/d(M)$ and $d(M) = \dim(V_M)$ the number of points in it. Let $L_M = l_2(Z_M)$ be the Hilbert space of square summable sequences indexed by $Z_M$ with inner product

$$<f_M, g_M> := d(M)^{-1} \sum_{m \in Z_M} f^*_M(m) g_M(m) \tag{2.5}$$

This scalar product offers the interpretation of $f_M(m) := f(x^M_m), \ x^M_m := \frac{m}{d(M)}$ and similar for $g_M$ as the discretised values of some functions $f, g \in L$ in which case (2.5) is the Riemann sum approximant of $<f, g>_L$. It is for this reason that we did not normalise the $\chi^M_m$.

What follows works for any such choice of ONB indexed by $M$. However, to reduce the amount of arbitrariness and to give additional structure to MRA’s one requires, both in wavelet theory and renormalisation, in addition that the ONB’s descend from a few mother scaling functions $\phi$ by dilatations depending on $M$ and translations depending on $m$. In wavelet theory on the real line one is rather specific about the concrete descendance. In particular, there is only one mother scaling function, the $\chi^M_m$ and $\phi$ are linearly related, $\mathcal{M}$ just consists of the powers $M = 2^N, \ N \in \mathbb{Z}$ and $\chi^M_m = \phi(M x - m)$. As advertised in [21] we allow a more general descendence and thus accept a finite, fixed number of mother scaling functions and that the $\chi^M_m$ are dilatations and translations of a rational function of those mother scaling functions. This keeps the central idea of providing minimal structure to an MRA while increasing flexibility.

The spaces $V_M, L_M$ are in bijection via

$$I_M : L_M \to L, \ f_M \mapsto \sum_m f_M(m) \chi^M_m \tag{2.6}$$

Note that (2.6) has range in $V_M \subset L$ only. Its adjoint $I^*_M : L \to L_M$ is defined by

$$<I^*_M f, f_M>_{L_M} := <f, I_M f_M>_L \tag{2.7}$$

so that

$$(I^*_M f)(m) = d(M) <\chi^M_m, f>_L \tag{2.8}$$

Clearly

$$(I^*_M I_M f_M)(m) = d(M) <\chi^M_m, I_M f_M>_L = f_M(m) \tag{2.9}$$
i.e. $I_M^\dagger I_M = 1_{L_M}$ while

$$
(I_M I_M^\dagger f)(x) = d(M) \sum_m \chi_m^M(x) < \chi_m^M, f_M >_{L_M} = (p_M f)(x) \tag{2.10}
$$

is the projection $P_M : L \mapsto V_M$.

Given $M \leq M'$ we define the coarse graining map

$$
I_{MM'} := I_{M'}^\dagger I_M : L_M \mapsto L_{M'} \tag{2.11}
$$

It obeys

$$
I_M : I_{MM'} = p_{M'} I_M = I_M \tag{2.12}
$$

because $I_M$ has range in $V_M \subset V_{M'}$ for $M \leq M'$. This is the place where the MRA property of the nested set of subspaces $V_M$ was important. Next for $M_1 \leq M_2 \leq M_3$ we have

$$
I_{M_2 M_3} I_{M_1 M_2} = I_{M_3}^\dagger p_{M_2} I_{M_1} = I_{M_3}^\dagger I_{M_1} = I_{M_1 M_3} \tag{2.13}
$$

for the same reason. This is called the condition of cylindrical consistency which is crucial for the renormalisation group flow.

To see the importance of (2.13) we consider a probability measure $\nu$ on the space $F$ of field configurations $\Phi$ which define a Hilbert space $H = L_2(F, d\nu)$ and a representations space for the Weyl algebra $\mathcal{A}$ generated from the Weyl elements (2.4). We set $w[f] := w[f, g = 0]$ and define the generating functional of moments of $\nu$ by

$$
\nu(f) := \nu(w[f]) \tag{2.14}
$$

If we restrict $f$ to $V_M$ we obtain an effective measure on the space of discretised quantum fields $\Phi_M = I_M^\dagger \Phi$ via

$$
w[I_M f_M] = w_M[f_M] = e^{i\Phi_M(f_M)}, \Phi_M(f_M) = f_M, \Phi_M >_{L_M} \tag{2.15}
$$

and

$$
\nu_M(f_M) := \nu(w[I_M f_M]) = \nu_M(w_M[f_M]) \tag{2.16}
$$

The measures $\nu_M$ on the spaces $F_M$ of fields $\Phi_M$ are consistently defined by construction

$$
\nu_M'(I_{MM'} f_M) = \nu_M(f_M) \tag{2.17}
$$

for any $M < M'$ since the $\nu_M$ descend from a continuum measure. Conversely, given a family of measures $\nu_M$ satisfying (2.17) a continuum measure $\nu$ can be constructed known as the projective limit of the $\nu_M$ under mild technical assumptions [24]. To see the importance of (2.13) for this to be the case, suppose we write $f \in L$ in two equivalent ways $f = I_M f_M = I_{M_2} g_{M_2}$ then we should have $\nu_{M_1}(f_{M_1}) = \nu_{M_2}(g_{M_2})$. Now while $M_1, M_2$ may not be in relation, as $M$ is directed we find $M_1, M_2 \leq M_3$. Applying $I_{M_3}^\dagger$ we conclude $I_{M_1 M_3} f_{M_1} = I_{M_2 M_3} g_{M_2}$ thus due to (2.17) indeed

$$
\nu_{M_1}(f_1) = \nu_{M_3}(I_{M_1 M_3} f_{M_1}) = \nu_{M_3}(I_{M_2 M_3} g_{M_2}) = \nu_{M_2}(g_{M_2}) \tag{2.18}
$$

In CQFT the task is to construct a representation of the Weyl algebra $\mathcal{A}$ with additional properties such as allowing for the implementation of a Hamiltonian operator $H = H[\Phi, \Pi]$ which imposes severe restrictions on the Hilbert space representation. One may start with discretised Hamiltonians

$$
H_M^{(0)} : [\Phi_M, \Pi_M] := H(p_M \Phi, p_M \Pi) \tag{2.19}
$$

on $H_M^{(0)} := L_2(F_M, \nu_M^{(0)})$ where $\nu_M^{(0)}$ is any probability measure to begin with, for instance a Gaussian measure or a measure constructed from the ground state $\Omega_M^{(0)}$ of the Hamiltonian $H_M^{(0)}$. The point of using the IR cut-off is that there are only finitely many, namely $d(M)$ degrees of freedom $\Phi_M, \Pi_M$ which are conjugate

$$
[\Pi_M(m), \Phi(m')] = i d(M) \delta(m, m'), \Phi_M(m)^* = \Phi_M(m), \Pi_M^*(m) = \Pi_M(m) \tag{2.20}
$$
so that construction of $\nu_M^{(0)}$ does not pose any problems. In case that there is no IR cut-off it is significantly harder to show that the theories even at finite UV cut-off exist. Assuming this to be the case, one fixes for each $M \in \mathcal{M}$ an element $M \leq M'(M) \in \mathcal{M}$ and defines isometric injections
\[ J_{MM'}^{(n+1)} : \mathcal{H}_M^{(n+1)} \rightarrow \mathcal{H}_{M'}^{(n)} ; \quad \mathcal{H}_M^{(n)} := L_2(F_M, d\nu_M^{(n)}) \]
via
\[ \nu_M^{(n+1)}(f_M) := \nu_M^{(n)}(I_{MM'}(M)f_M) \]
and with these the flow of Hamiltonians
\[ H_M^{(n+1)} := J_{MM'}^{(1)} H_{M'}^{(n)} J_{MM'}^{(1)}(M) \]
The isometry of the injections relies on the assumption that the span of the $w_M[f_M]$ is dense in $\mathcal{H}_M^{(0)}$ which is typically the case.

This defines a sequence or flow (indexed by $n$) of families (indexed by $M$) of theories $\mathcal{H}_M^{(n)}, H_M^{(n)}$. At a critical or fixed point of this flow the consistency condition (2.17) is satisfied (at first in the linearly ordered sets of $\mathcal{M}(M) := \{(M')^N(M), N \in \mathbb{N}_0\}$ and then usually for all of $\mathcal{M}$ by universality) and one obtains a consistent family $\{\mathcal{H}_M, H_M\}$. This family defines a continuum theory $\{\mathcal{H}, H\}$ as one obtains inductive limit isometric injections $J_M : \mathcal{H}_M \rightarrow \mathcal{H}$ such that $J_{M'}J_{MM'} = J_M, M \leq M'$ thanks to the fixed point identity $J_{M_2M_3}J_{M_1M_2} = J_{M_1M_3}, M_1 \leq M_2 \leq M_3$ and such that
\[ H_M = J_M^\dagger H J_M \]
is a consistent family of quadratic forms $H_M = J_{MM'}^\dagger H_{M'} J_{MM'}, M \leq M'$.

We conclude this section by noting that wavelet theory actually also seeks to decompose the spaces as $V_{MM'} = V_M \oplus W_M$ where $W_M$ is the orthogonal complement of $V_M$ in $V_{MM'}$, $M \leq M'$, to provide an ONB for the $W_M$ and to require that this basis descends from a mother wavelet $\psi$ concretely related to the scaling function in the same specific way as outlined above for the scaling function. For the purpose of renormalisation this additional structure is not essential, thus we will not go into further details. We remark however that in [21] we also generalised the notion of wavelets in the same way as for the scaling function which again keeps the central idea of structurising the MRA and showed that the Dirichlet and Shannon kernels are non-trivial realisations of that more general definition.

3 Hamiltonian renormalisation for Fermions

To distinguish the bosonic field $\Phi$ from the previous section form the present fermionic field we use the notation $\xi_B$ for a chiral (or Weyl) fermion where $B = 1, 2$ transforming in one of the two fundamental representations of $SL(2,\mathbb{C})$. Its Majorana conjugate $\epsilon \xi^\ast$ with $\epsilon = i\sigma_2$ (Pauli matrix) of opposite chirality then transforms in the dual fundamental representation. We have the fundamental simultaneous canonical anti-commutation relations (CAR)
\[ [\xi_B(x), \xi_C(y)^\ast]_+ := \xi_B(x) \xi_C(y)^\ast + \xi_C(y)\ast \xi_B(x) = \delta_{BC} \delta(x, y) \] (3.1)
all other anti-commutators vanishing. Dirac fermions and Majorana fermions can be considered as usual by using direct sum $SL(2,\mathbb{C})$ representations of of independent Weyl fermions of opposite chirality or of the direct sum of a fermion with its Majorana conjugate. It will be sufficient to consider a single Weyl fermion species $\xi_B$ for what follows.

The measure theoretic language given for bosons of the previous section cannot apply because of several reasons: i. the “Weyl elements” $w[f] := \exp(i \cdot (\xi < f, \xi)^\dagger)$ are not mutually commuting, ii. the $w[f] \Omega$ are not dense in the Fock space $\mathcal{H}$ defined by $\xi < f, \xi > \Omega = 0$ because in fact $w[f] = 1_H + i \cdot (\xi < f, \xi)^\dagger$ due to nil-potency and iii. $w[f]$ is not unitary. To avoid this one can formally work with Berezin “integrals” [25] and anti-commuting smearing fields $f$ but then we cannot immediately transfer the functional analytic properties
of the commuting test functions from the bosonic theory and apart from serving as compact organising tool, anticommuting smearing functions do not have any advantage over what we say below.

One of the motivations to work with Weyl elements rather than say \( \Phi(f), \Pi(f) \) in the bosonic case is that the Weyl elements are bounded operators. However, the operators \( \xi(f) := \langle \tilde{f}, \xi \rangle_L, \xi(f)^\dagger \) are already bounded by \( ||f||_L \) as follows from the CAR

\[
[\xi(f), \xi(f)^\dagger]_+ = ||f||_L^2 1_H \Rightarrow ||\xi(f) \psi||_H^2, ||\xi(f) \psi||_H^2 \leq ||f||_L^2 ||\psi||^2
\]

(3.2)

The derivation of the renormalisation scheme given in [9] in fact covers both the bosonic and fermionic case but the practical implementation for bosons used measures \[17\]. We thus adapt the bosonic renormalisation scheme by reformulating it in an equivalent way which then extends to the fermionic case:

Given cyclic vectors \( \Omega^{(n)}_M \) for the algebra generated by the

\[
\xi_M(f_M) := \langle I_M f_M, \xi \rangle_L = \langle f_M, I_M^\dagger \xi \rangle_{L_M} = \frac{1}{d(M)} \sum_{B,m \in Z_M} [f^B_M(m)]^* \xi_{M,B}(m)
\]

(3.3)

and their adjoints (perhaps the vacua of the Hamiltonians \( H^{(n)}_M \)) we define the flow of isometric injections (e.g. for \( M' = M'(M) \))

\[
j^{(n+1)}_{MM'} \Omega^{(n)}_M := \Omega^{(n)}_{M'}, \ j^{(n+1)}_{MM'} \Xi_M(F_{M,1})..\Xi_M(F_{M,N}) \Omega^{(n+1)}_M := \Xi_{M'}(I_{MM'}F_{M,1})^*..\Xi_{M'}(I_{MM'}F_{M,N})^* \Omega^{(n)}_{M'}
\]

(3.4)

Note that \( \xi_M = d(M) I_M^\dagger \xi \) preserve the CAR in the sense that

\[
[\xi_M(m), [\xi_M(m')]^*]_+ = d(M) M \delta_{mm'}
\]

(3.5)

and \( \Xi(F) = \sum_B \langle f_B, \xi_B \rangle_L + \langle \tilde{f}_B, \xi_B \rangle^* \) where we have collected four independent smearing functions \( f_B, \tilde{f}_B, B = 1, 2 \) into one symbol \( F \). The same notation was used in (3.4) for the \( M \) dependent quantities.

With these we define the flow of Hamiltonian quadratic forms as

\[
H^{(n+1)}_M := [j^{(n+1)}_{MM'}] H^{(n)}_M j^{(n+1)}_{MM'}
\]

(3.6)

These formulas are even simpler than in the bosonic case because there is no fermionic Gaussian measure and corresponding covariance to consider. However, as in the bosonic case, one has to give initial data for this flow. This can be done, e.g. by defining

\[
H^{(0)}_M[\xi_M, \xi_M] := H[p_M \xi, (p_M \xi)^*] :
\]

(3.7)

where \( (p_M \xi)_B := I_M I_M^\dagger \xi_B, H \) is the classical Hamiltonian and \( :, \dots : \) denotes normal ordering with respect to a Fock space \( H^{(0)}_M \) with cyclic Fock vacuum \( \Omega^{(0)}_M \) annihilated by \( A^{(0)}_{M,M} \) assembled from \( \xi_{M,B}, \xi^*_M \) as suggested by the form of \( H[p_M \xi, (p_M \xi)^*] \). As in the bosonic case, the fields \( \xi_{M,B} \) do not depend on the sequence label \( n \) while the annihilators \( A^{(n)}_{M,M} \) do as one obtains them from the \( \xi_{M,B} \) using extra discretised structure that depends on \( M \), typically lattice derivatives and more complicated aggregates made from those (Dirac-Weyl operators, Laplacians,...).

## 4 Hamiltonian renormalisation of free fermions and fermion doubling

In this section we will concretely choose the renormalisation structure as follows (see [21] for more details): \( Z_M \) will be the lattice of points \( x^M_m \) with \( m \in \mathbb{Z} \) if \( X = \mathbb{R} \) and \( m \in \mathbb{Z}_M := \{0, 1, 2, \ldots, M - 1\} \) if \( X = [0, 1) \) respectively and \( d(M) = M \). The set \( \mathcal{M} \) consists of the odd naturals with partial order \( M \leq M' \) iff \( M'/M \in \mathbb{N} \). The renormalisation sequence will be constructed using \( M'(M) = 3M \) for simplicity. The MRA’s are based on the Shannon [25] and [26] Dirichlet kernels respectively, that is,

\[
\chi^M_M(x) = \begin{cases} \sin(M \pi (x-x^M_0)) / M \pi (x-x^M_0) & X = \mathbb{R} \\ \sin(M \pi (x-x^M_0)) / M \sin(M \pi (x-x^M_0)) & X = [0, 1) \end{cases}
\]

(4.1)
Their span is dense in $V_M$ and they are mutually orthogonal with norm $M^{-1}$. The Dirichlet kernel is 1-periodic as it should be. Both have maximal value 1 at $x = x_m^M$, are symmetric about this point and (slowly) decay away from it, thus display some position space locality. They are real valued and smooth and have compact momentum support $k \in [-\pi M, \pi M]$ and $k = 2\pi n$, $n \in \mathbb{Z}_M = \{-M^{-1}_2, -M^{-1}_2 + 1, \ldots, M^{-1}_2\}$ respectively.

Recall the following facts about the topologies of position space and momentum space via the Fourier transform where we denote by $M$ the spatial resolution of the lattice $x_m^M$ with either $m \in \mathbb{Z}$ or $m \in \mathbb{Z}_M = \{0, 1, 2, \ldots, M - 1\}$ where for $M$ odd we set $\mathbb{Z}_M = \{-M^{-1}_2, \ldots, M^{-1}_2\}$ (c: compact, nc: non-compact, d: discrete, nd: non-discrete (continuous)):

| space – topology | momentum – topology | Fourier – function |
|------------------|---------------------|-------------------|
| nc, nd : $\mathbb{R}$ | nc, nd : $\mathbb{R}$ | $e_k(x) = e^{i k x}$ |
| nc, d : $\frac{1}{M} \cdot \mathbb{Z}$ | c, nd : $[-M \pi, M \pi)$ | $e_k^M(m) = e^{i k x_m^M}$ |
| c, nd : $[0, 1)$ | nc, d : $\mathbb{Z}$ | $e_n(x) = e^{2 \pi i n x}$ |
| c, d : $\frac{1}{M} \cdot \mathbb{Z}_M$ | c, d : $\hat{\mathbb{Z}}_M$ | $e_n^M(m) = e^{2 \pi i n x_m^M}$ |

Accordingly, in the non-compact and compact case respectively, the space of Schwartz test functions is a suitable subspace of $L = L_2(\mathbb{R}, dx)$ and $L = L_2([0, 1), dx)$ respectively which have momentum support in $2\pi \mathbb{R}$ and $2\pi \cdot \mathbb{Z}$ respectively. Upon discretising space into cells of width $1/M$ the momentum support $\mathbb{R}$ and $\mathbb{Z}$ respectively gets confined to the Brillouin zones $[-\pi M, \pi M)$ and $\hat{\mathbb{Z}}_M$ respectively.

The corresponding completeness relations or resolutions of the identity read

$$\delta_{\mathbb{R}}(x, x') = \int_{\mathbb{R}} \frac{dk}{2\pi} e_k(x - x')$$

$$M \delta_{\mathbb{Z}}(m, m') = \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k^M(m - m')$$

$$\delta_{[0, 1)}(x, x') = \sum_{n \in \mathbb{Z}} e_n(x - x')$$

$$M \delta_{\mathbb{Z}_M}(m, m') = \sum_{n \in \mathbb{Z}_M} e_n^M(m - m')$$

While the first and third relation in (4.3) define the $\delta$ distribution on $\mathbb{R}$ and $[0, 1)$ respectively, the second and fourth relation in (4.3) are the restrictions to the lattice of the regular functions

$$\delta_{\mathbb{R}, M}(x) = \int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x) = \frac{\sin(\pi M x)}{\pi x}$$

$$\delta_{[0, 1), M}(x) = \sum_{n \in \mathbb{Z}_M} e_n(x) = \frac{\sin(\pi M x)}{\sin(\pi x)}$$

which we recognise as the Shannon (sinc) and Dirichlet kernel respectively. After dividing and dilating them by $M$ and translating them by $m$ we obtain precisely the functions (4.1). These kernels can be considered as regularisations of the afore mentioned $\delta$ distributions in the sense that the momentum integral $k \in \mathbb{R}$ or momentum sum $n \in \mathbb{Z}$ has been confined to $|k| < \pi M$ and $|n| < M^{-1}$ respectively. Both are real valued, smooth, strongly peaked at $x = 0$ and have compact momentum support. The Shannon kernel like the Dirichlet kernel is an $L_2$ function but it is not of rapid decay with respect to position.

The simplest possible action for fermions is the massless, chiral theory in 2d Minkowski space

$$S = i \int_{\mathbb{R}} dt \int_X dx \, \bar{\xi} \frac{\partial}{\partial x} \xi$$

Here $X = \mathbb{R}$ or $X = [0, 1)$. The 2d Clifford algebra with signature $(-1, +1)$ is generated by $\gamma^0 = \epsilon = i \sigma_2$, $\gamma^1 = \sigma_1$ where $\sigma_1, \sigma_2, \sigma_3 = \epsilon \gamma_1$ are the Pauli matrices. Then $\phi = \gamma^\mu \partial_\mu$, $x^0 = t, x^1 = x$ and $\xi = (\xi^*)^T \gamma^0$. Due to


\((\gamma^{0,\alpha}\mu)^*) = \gamma^{0,\mu}\) the action is real valued. Generalisations to higher dimensions, massive theories, with more species or higher spin are immediate and just require the corresponding Clifford algebras.

Then \(i[\xi^A]\) \(A = 1, 2\) is canonically conjugate to \(\xi^A\) which results in the non vanishing canonical anti commutation relations (CAR)

\[ [\xi^A(x), (\xi^B)^*(y)]_+ = \delta^{AB} \delta(x,y) \]  

and the Hamiltonian is

\[ H = -i \int_X dx \{ [\xi^T] \sigma^s \xi^s'(x) \} \]  

with \(\xi^s' = \partial \xi / \partial x\) which is linear in spatial derivatives. Indeed the Dirac-Weyl equation \(\bar{\phi} = 0\) is reproduced by the Heisenberg equation of (4.7)

\[ i\dot{\xi} = [H, \xi] = i\sigma_3,\xi^T \Leftrightarrow \epsilon\dot{\xi} - \epsilon\sigma_3,\xi^T = \bar{\phi} = 0 \]  

As (4.7) is indefinite as it stands we introduce the self-adjoint projections on \(L = L_2(X, dx)\) with \(s = \pm 1\)

\[ Q_s = \frac{1}{2} [1_L + i s \frac{\partial}{\partial \omega}] Q, Q = 1_L - 1 < 1_L / ||1||_L^2, \omega = \sqrt{-\partial^2}, i\partial Q_s = s \omega Q_s \]  

Note \(Q = 1_L \) for \(X = \mathbb{R}\). We then rewrite the Hamiltonian as

\[ -H = \langle \xi_1, [Q_+ - Q_-] \omega \xi_1 \rangle_{L} - \langle \xi_2, [Q_+ - Q_-] \omega \xi_2 \rangle_{L} \]  

\[ = \langle Q_+ \xi_1, \omega Q_+ \xi_1 \rangle_{L} - \langle Q_- \xi_1, \omega Q_- \xi_1 \rangle_{L} - \langle Q_+ \xi_2, \omega Q_+ \xi_2 \rangle_{L} + \langle Q_- \xi_2, \omega Q_- \xi_2 \rangle_{L} \]  

Thus we declare

\[ A_{1,+} := (Q_+ \xi_1)^*, A_{1,-} := Q_- \xi_1, A_{2,+} := (Q_- \xi_2)^*, A_{2,-} := Q_+ \xi_2 \]  

as annihilators and obtain the normal ordered, positive semi-definite Hamiltonian

\[ : H : = \sum_{B = 1, \sigma = \pm} \int_X dx \frac{A^*_B \omega A_B, \sigma}{\partial x} \]  

where the \(A_B, \sigma\) obey the CAR

\[ [A_B, \sigma(x) , A_{B', \sigma'}(x')]_+ = \delta_{B,B'} \delta_{\sigma, \sigma'} Q_s(x,x') \]  

where \(Q_s(x,x')\) is the integral kernel \((Q_s f)(x) = \int_X dx' Q_s(x,x') f(x')\). Note that the zero modes of \(\xi_B\) do not not contribute to \(H\) so we have to quantise them without guidance from the form of the Hamiltonian. With \(Q^\perp = 1_L - Q\) we define \(A_{B,0} := Q^- \xi_B\) as the annihilation operator which is non vanishing only for \(X = [0, 1]\).

From this perspective, the problem of the fermion doublers on the lattice \(M \mathbb{Z}\) or \(\frac{1}{M} \mathbb{Z}\) for \(X = \mathbb{R}\) and \(X = [0, 1]\) respectively is encoded in the way one discretises the partial derivative \(\partial\) that appears in the projections \(Q_s\) (in Hamiltonian renormalisation the time variable and time derivatives are kept continuous). For scalar theories, \(\partial\) appears only quadratically in the Laplacian \(\Delta = -\partial^2\) while for fermions it appears linearly. This problem is therefore not only present for fermions but for all theories in which besides the Laplacian also the partial derivatives themselves are involved in the quantisation process. One such example is parametrised field theory which shares many features with string theory.

Alternatively, this problem shows up in the discretisation of the 2-point functions of the theory (as the theory is free, the two point function determines all higher N-point functions). To compute them from the current Hamiltonian setting we use the CAR to compute the Heisenberg time evolution of the annihilators (from now on normal ordering is being understood)

\[ A_{B, \sigma}(t, x) = e^{-itH} A_{B, \sigma}(x) e^{itH} = [e^{it\omega} A_{B, \sigma}](x) \]  

(4.14)
where $Q_{a}A_{B,a} = A_{B,a}$ was used. Then the non-vanishing two point functions are

$$
\langle \Omega, \xi_B(s, x) \xi_C(t, y)^* \Omega \rangle = \langle \Omega, (\{Q_+ + Q_- + Q^\perp|\xi_B\}(s, x) (\{Q_+ + Q_- + Q^\perp|\xi_C\}(t, y)^* \Omega

= \langle \Omega, \{\delta_{B,1}[A^1_{1,+} + A_{1,-} + A_{1,0}] + \delta_{B,2}[A^2_{2,+} + A_{2,-} + A_{2,0}](s, x) \times
\{\delta_{C,1}[A_{1,+} + A^1_{1,-} + A_{1,0}] + \delta_{C,2}[A^2_{2,+} + A_{2,-} + A_{2,0}](t, y) \Omega

= \langle \Omega, \{\delta_{B,1}[A_{1,+} - A_{1,0}] + \delta_{B,2}[A_{2,+} - A_{2,0}](s, x) \{\delta_{C,1}[A^1_{1,+} - A_{1,-} + A_{1,0}] + \delta_{C,2}[A^2_{2,+} - A_{2,-} + A_{2,0}](t, y) \Omega

= \langle \Omega, e^{i\omega_x - i\omega_y} \{\delta_{B,C}(Q_- - Q^\perp)Q_+(x, y) + Q^\perp(x, y) + \delta_{B,2}Q_+(x, y) + Q^\perp(x, y)\}

= \frac{1}{2} e^{i\omega_x - i\omega_y} \{\delta_{B,C}(1 + Q^\perp) - i [\sigma_3]_{BC} \frac{\partial}{\partial \omega_x} \delta(x, y)

= \frac{\delta_{BC}}{2 ||1||^2} + \int \frac{dk}{2 \pi 2 \omega(k)} e^{i\omega_x(k) - t(k - y)}[\omega(k) 1_2 - k \sigma_3]_{BC}

= \frac{\delta_{BC}}{2 ||1||^2} + \int \frac{dk}{2 \pi 2 \omega(k)} e^{-i K \cdot (X - Y)}[K^0 (1 + Q^\perp)1_2 - K^0 \sigma_3]_{BC}

= \frac{\delta_{BC}}{2 ||1||^2} - i [\sigma_3]_{BC} \frac{\partial}{\partial X^0} \int \frac{dk}{2 \pi 2 \omega(k)} e^{-i K \cdot (X - Y)}

= \frac{\delta_{BC}}{2 ||1||^2} + i ([\sigma_3]_{BC} \frac{\partial}{\partial X^1} \epsilon)_{BC} \Delta_+(x - y)

= \frac{\delta_{BC}}{2 ||1||^2} + i [\partial_X \epsilon]_{BC} \Delta_+(X - Y) (4.15)

with $K^0 := \omega(k) = |k|$, $K^1 = k$ and $X^0 = s$, $X^1 = t$, $Y^0 = y$ and $K \cdot X = -K^0 X^0 + K^1 X^1$. Here $\Delta_+$ is the Wightman two point function of a free massless Klein-Gordon field in 2d Minkowski space

$$
\Delta_+(X - Y) = \int \frac{dk}{2 \pi 2 \omega(k)} e^{-i K \cdot (X - Y)} (4.16)

A similar computation yields $(X, Y)$ and $B, C$ and $Q_+, Q_-$ switch and the contribution from $A_{B,0}$ is missing leading to $-\delta_{BC}$ in the final result)

$$
\langle \Omega, \xi_C(t, y)^* \xi_B(s, x) \Omega \rangle = \langle \Omega, [\epsilon \partial Y^0 - \sigma_3 \partial Y^1]_{CB} \Delta_+(Y - X) = -\frac{\delta_{BC}}{2 ||1||^2} + i [\epsilon \phi Y]_{CB} \Delta_+(Y - X) (4.17)

Using the conjugate spinor $\bar{\xi} = [\xi^*]^T \epsilon$ we may rewrite (4.16), (4.17) as

$$
\langle \Omega, \xi(X) \otimes \bar{\xi}(Y) \Omega \rangle > x = \frac{\epsilon}{2 ||1||^2} + i \phi_X \Delta_+(X - Y),

\langle \Omega, \bar{\xi}(Y) \otimes \xi(X) \Omega \rangle > x = -\frac{\epsilon}{2 ||1||^2} + i \phi_Y \Delta_+(Y - X) (4.18)

which gives the time ordered 2 point function or Feynman propagator

$$
D_F(X - Y) := \langle \Omega, T[\xi(X) \otimes \bar{\xi}(Y)] \Omega \rangle >

\theta(Y^0 - X^0) < \Omega, \xi(X) \otimes \bar{\xi}(Y) \Omega > -\theta(Y^0 - X^0) < \Omega, \bar{\xi}(Y) \otimes \xi(X) \Omega >

= \phi_X \Delta_F(X - Y) (4.19)

where

$$
\Delta_F(X - Y) = -i \lim_{\epsilon \to 0^+} \int \frac{d^2 K}{(2\pi)^2} \frac{e^{-iK \cdot (X - Y)}}{-K \cdot K - i\epsilon} (4.20)

is the Feynman propagator of the 2d massless Klein Gordon field. We see that $\phi_X D_F(X - Y) = i\delta^{(2)}(X - Y)$ due to $\phi^2 = 0$, i.e. $D_F = i\phi^{-1}$. In Hamiltonian renormalisation one discretises only $x, \partial_x$ and confines only $|K^1| < \pi M$ while in the Euclidian approach one discretises also $t, \partial_t$ and confines $|K^0| < \pi M$. In any case we see that it is the projections $Q_{a}$ that directly translate into $\phi$ which is linear in the derivatives. If the propagator
is to keep the property to invert the Dirac-Weyl operator $\partial$ then we are forced to write the momentum expression of (4.19), say in the Hamiltonian approach, as

$$
\frac{\epsilon K_0 + \sigma_1 \lambda_M(K_1)}{K_0^2 - \lambda_M(K_1)^2 - i\epsilon}
$$

where $[\partial_M e_{K_1}](X^1) = i\lambda_M(K_1) e_{K_1}(X^1)$, $X^1 \in \mathbb{Z}/M$ defines the eigenvalues of the discrete derivative and indices are moved with the Minkowski metric.

The case $X = [0, 1]$ is literally the same, just that we must sum over $k = K^1 = 2\pi n$, $n \in \mathbb{Z}$ rather than integrating over $K^1 \in \mathbb{R}$ with measure $dK^1/(2\pi)$. Also the $Q^\pm$ contribution is now non-trivial but cancels in the Feynman propagator. That is, all expressions remain the same except that we must replace $\Delta_+ , \Delta_F$ by

$$
\Delta_+(X - Y) = \sum_{n \in \mathbb{Z}} \frac{1}{2\omega(n)} e^{-i K \cdot (X - Y)} , \omega(n) = 2\pi |n| , \ K_1 = 2\pi n
$$

$$
\Delta_F(X - Y) = -i \int \frac{dK^0}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{-i K \cdot (X - Y)}}{-K \cdot K - i\epsilon} , \ K_1 = 2\pi n
$$

In the so-called “naive” discretisation one writes

$$
(\partial_M f_M)(m) := \frac{M}{2} [f_M(m + 1) - f_M(m - 1)]
$$

for $f_M \in L_M$ the Hilbert space of square symmable sequences on the lattice. Using the Fourier functions $f_M(m) = e_k^M(m) = e_k(x_m^M)$ with $|k| < \pi M$ for $X = \mathbb{R}$ and $f_M(m) = e_k^M(m) = e_{2\pi n}(x_m^M)$ with $|n| \leq \frac{M-1}{2}$ and $x_m^M = \frac{m}{M}$ with $m \in \mathbb{Z}$ or $m \in \mathbb{Z}_M$ respectively we find the eigenvalues $\lambda_M(k)$ given by $i M \sin(\frac{\pi}{M})$ and $i M \sin(\frac{2\pi n}{M})$ respectively. These vanish in the allowed domain of $k$ and $n$ respectively at $k = 0$, $k = \pm \pi M$ and $n = 0, n = \frac{M}{2}$ if $M$ is even, otherwise only at $n = 0$ with corresponding doubler pole in the propagator when $K^0 = 0$. We see that there are no doublers in the compact case for lattices with odd numbers of points even with respect to the naive discretisation of the discrete derivative. Still, even in the compact case, and for odd $M$ the eigenvalue $i M \sin(\pi M^{-1} - \frac{1}{2}) = -i M \sin(\pi/M)$ for $n = \frac{M-1}{2}$ approaches $-i\pi$ for large $M$ while most other eigenvalues are large of order $M$ and thus $n = \pm (M-1)/2$ can be considered as an “almost” doubler mode.

We now show that the spectrum of $\partial_M$ is doubler free if we do not pick the naive discretisation but rather the natural discretisation provided by the maps $I_M, I_M^1$ in terms of which the renormalisation flow is defined. This discretisation is defined by

$$
\partial_M := I_M^1 \partial I_M
$$

for both $X = \mathbb{R}$ and $X = [0, 1)$ and is well defined whenever the MRA functions $\chi_m^M$ are at least $C^1$. Note that with this definition $\partial_M$ is automatically anti-symmetric since $\partial$ is. In fact, for the Haar flow which is not $C^1$ we formally find

$$
\partial_M f_M(m) = M \sum_m < \chi_m^M , [\chi_m^M]'_L > f_M(m) = -M \sum_m < [\chi_m^M]' , \chi_m^M >_L f_M(m) = \frac{M}{2} (f_M(m+1) - f_M(m-1))
$$

i.e. precisely the naive derivative where we have formally integrated by parts in between and used that $\chi_m^M$ is of compact support for $X = \mathbb{R}$ and periodic for $X = [0, 1)$ respectively. Thus the Haar flow results in the naive discretisation which yields the doubler troubled spectrum.

Note that the map $I_M : L_M \to L$ has range in $V_M$ and in fact $I_M^1 : L \to L_M$ restricts to the inverse as $I_M^1 I_I = 1_{LM}$ i.e. $L_M, V_M$ are in bijection. Thus, if in fact $\partial$ preserves $V_M$ then the spectrum of $\partial_M$ will simply coincide with that of $\partial$ except that $k$ will be restricted from $\mathbb{R}$ to $[-\pi M, \pi M]$ and $n$ from $\mathbb{Z}$ to $\mathbb{Z}_M$. This is precisely what happens for both the Shannon and Dirichlet kernel as we will now confirm.
For the Shannon kernel in the case $X = \mathbb{R}$ we compute

\[(\partial_M f_M)(m) = M \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) < x_m^M, \partial \chi_m^M >_L (\partial_M f_M)(m)\]

\[= M \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) \int_{\pi M}^{\pi M} \frac{dk}{2\pi} (ik) < x_m^M, e_k >_L < e_k, x_m^M >_L\]

\[= M \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) \int_{\pi M}^{\pi M} \frac{dk}{2\pi} (ik) e_k(x_m^M - x_m^M)\]

\[= \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) [\partial_x \chi_m^M(x)|_{x=x_m^M}\]

\[= \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) \left[ y \cos(M\pi y) - (M\pi)^{-1} \sin(\pi My) y \right] \frac{2\pi i \tilde{m}}{2\pi} \sum_{\tilde{m} \in \mathbb{Z}_M} e_{k-q}(x_m^M \tilde{m}) = ik e^M_k(m) (4.26)\]

which displays the non-local nature of the discrete derivative as all points $\tilde{m} \in \mathbb{Z}$ contribute. However, \[(4.26)\] vanishes at $m = \tilde{m}$ and takes the maximal value $\mp M$ at $m - \tilde{m} = \pm 1$ which shows that it approximates the naive derivative in the vicinity of $m$. On the other hand, for $f_M = e^M_k$ we find the exact eigenfunctions

\[(\partial_M e^M_M)(m) = M \int_{\pi M}^{\pi M} \frac{dq}{2\pi} (iq) e_q(x_m^M) \sum_{\tilde{m} \in \mathbb{Z}_M} e_{k-q}(x_m^M \tilde{m}) = ik e^M_k(m) (4.27)\]

with manifestly doubler free spectrum.

For the Dirichlet kernel in the case $X = [0, 1)$ the computations are completely analogous

\[(\partial_M f_M)(m) = M \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) < x_m^M, \partial \chi_m^M >_L (\partial_M f_M)(m)\]

\[= M \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) \sum_{|n| \leq M^{-1}} (2\pi i n) < x_m^M, e_{2\pi n} >_L < e_{2\pi n}, x_m^M >_L\]

\[= M \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) \sum_{|n| \leq M^{-1}} (2\pi i n) e_{2\pi n}(x_m^M - x_m^M)\]

\[= \sum_{\tilde{m} \in \mathbb{Z}_M} f_M(\tilde{m}) \sin(\pi y) \cos(M\pi y) - M^{-1} \sin(\pi My) \cos(\pi y) \frac{2\pi i \tilde{m}}{2\pi} \sum_{\tilde{m} \in \mathbb{Z}_M} e_{k-q}(x_m^M \tilde{m}) = 2\pi i n e^M_M(m) (4.28)\]

which displays the non-local nature of the discrete derivative as all points $\tilde{m} \in \mathbb{Z}_M$ contribute. However, \[(4.28)\] vanishes at $m = \tilde{m}$ and takes the maximal value $\mp M$ at $m - \tilde{m} = \pm 1$ which shows that approximates the naive derivative in the vicinity of $m$. On the other hand, for $f_M = e^M_n$ we find the exact eigenfunctions

\[(\partial_M e^M_n)(m) = M \sum_{\tilde{m} \in \mathbb{Z}_M} e^M_n(m) \sum_{|n| \leq M^{-1}} (2\pi i n) e^M_n(m - \tilde{m}) = 2\pi i n e^M_M(m) (4.29)\]

with manifestly doubler free spectrum.

We now study the Shannon or Dirichlet flow of the (non-)compact theory. We start with some initial discretisation $\hat{\rho}_{M,0}, \omega_{M,0} = \sqrt{-[\hat{\rho}_{M,0}]^2}, Q_{M,0} = \frac{1}{2}[1_{LM} + i s \frac{\hat{\rho}_{M,0}}{\omega_{M,0}}]$ which determines the annihilators in analogy to (4.11)

\[A_{M,1,+}^{(0)} := (Q_{M,+,\xi M,1})^*, A_{M,1,-}^{(0)} := Q_{M,-\xi M,1}, A_{M,2,-}^{(0)} := (Q_{M,-\xi M,2})^* A_{M,2,+}^{(0)} := Q_{M,+\xi M,2} (4.30)\]
the vacuum $\Omega^{(0)}_M$, the Fock space $\mathcal{H}^{(0)}_M$ and the initial Hamiltonian family

$$H^{(0)}_M = \sum_{m \in \mathbb{Z}} \sum_{B,\sigma} [A^{(0)}_{M,B,\sigma}]^* \omega^{(0)}_M A^{(0)}_{M,B,\sigma} \tag{4.31}$$

and similar for the compact case with the restriction $m \in \mathbb{Z}_M$.

We can encode the flow (3.4), (3.6) into a single quantity $\partial^{(n)}_M$ in terms of which we define analogously

$$\omega^{(n)}_M = \sqrt{-(\partial^{(n)}_M)^2}, \quad Q^{(n)}_{M,s} = \frac{1}{2}[1 + \text{i} s \frac{\partial^{(n)}_M}{\omega^{(n)}_M}]$$

as well as

$$A^{(n)}_{M,1,+} := (Q^{(n)}_{M,+} + \xi_{M,1})^*, ~ A^{(n)}_{M,1,-} := Q^{(n)}_{M,-} - \xi_{M,1}, ~ A^{(n)}_{M,2,-} := (Q^{(n)}_{M,-} - \xi_{M,2})^*, ~ A^{(n)}_{M,2,+} := Q^{(n)}_{M,+} + \xi_{M,2} \tag{4.32}$$

and the initial Hamiltonian family

$$H^{(n)}_M = \sum_{m \in \mathbb{Z}} \sum_{B,\sigma} [A^{(n)}_{M,B,\sigma}]^* \omega^{(n)}_M A^{(n)}_{M,B,\sigma} \tag{4.33}$$

and again for the compact case we just restrict to $m \in \mathbb{Z}_M$.

To see that this indeed possible we note that in the corresponding Fock spaces it is sufficient to check isometry on vectors of the form

$$\Psi^{(n)}_{M'}(I_{MM'} F_{M,1}, \ldots, I_{MM'} F_{M,N}) := A^{(n)}_{M'}(I_{MM'} F_{M,1})^* \ldots A^{(n)}_{M'}(I_{MM'} F_{M,N})^* \Omega^{(n)}_{M'}, \quad A^{(n)}_{M'}(F_M)$$

$$:= \sum_{B,\sigma} <F_{M,B,\sigma}, A^{(n)}_{M,B,\sigma}>_{L_M} \tag{4.34}$$

These give the inner products

$$<\Psi^{(n)}_{M'}(I_{MM'} F_{M,1}, \ldots, I_{MM'} F_{M,N}), \Psi^{(n)}_{M'}(I_{MM'} G_{M,1}, \ldots, I_{MM'} G_{M,N})>_{\mathcal{H}^{(n)}_{M'}} = \delta_{N,N'} \det([<Q^{(n)}_{M'} I_{MM'} F_{M,k}, Q^{(n)}_{M'} I_{MM'} G_{M,l}>_{L_{M'}}, 1]) \tag{4.35}$$

where

$$<Q^{(n)}_{M'} I_{MM'} F_{M}, Q^{(n)}_{M'} I_{MM'} G_{M}>_{L_{M'}} = \sum_{B,\sigma} <I_{MM'} F_{M,B,\sigma}, Q^{(n)}_{M',\sigma} I_{MM'} G_{M,B,\sigma}>_{L_{M'}}$$

$$= \sum_{B,\sigma} <F_{M,B,\sigma}, [I_{MM'}^\dagger, Q^{(n)}_{M',\sigma} I_{MM'}] G_{M,B,\sigma}>_{L_{M'}} \tag{4.36}$$

We used that, whatever $\partial^{(n)}_M$ is, the corresponding operators $Q^{(n)}_{M,s}$ are orthogonal projections and that the $B = 1, 2$ species anti-commute. Comparing with

$$<\Psi^{(n+1)}_{M}(F_{M,1}, \ldots, F_{M,N}), \Psi^{(n+1)}_{M}(G_{M,1}, \ldots, G_{M,N})>_{\mathcal{H}^{(n+1)}_{M}} \tag{4.37}$$

we obtain isometry iff

$$Q^{(n+1)}_{M,\sigma} = I_{MM'}^\dagger Q^{(n)}_{M',\sigma} I_{MM'} \tag{4.38}$$

Similarly, since

$$[H^{(n)}_{M'} [A^{(n)}_{M'}(I_{MM'} F_{M})]^*] = -[A^{(n)}_{M'}(\omega^{(n)}_M Q^{(n)}_{M,M'} I_{MM'}) F_{M}]^* \tag{4.39}$$

we get match between the matrix elements of Hamiltonians iff

$$\omega^{(n+1)}_M = I_{MM'}^\dagger \omega^{(n)}_M I_{MM'} \tag{4.40}$$

where we used that by construction $[\omega^{(n)}_M, Q^{(n)}_{M,s}] = 0$.

We now ask under what conditions on the coarse graining kernel $I_M$ both (4.38) and (4.40) are implied by

$$\partial^{(n+1)}_M := I_{MM'}^\dagger \partial^{(n)}_M I_{MM'} \tag{4.41}$$
Theorem 4.1.
Suppose that $\partial_M^{(0)} := I_M^D \partial I_M$ is the natural discrete derivative w.r.t. a coarse graining kernel $I_M : L_M \to L$ and such that $[\partial, I_M I_M^D] = 0$. Then (4.41) implies both (4.38) and (4.40).

Proof:
By (4.41) we have
\begin{equation}
\partial_M^{(1)} = [I_{MM'}]^{\frac{1}{2}} \partial_M^{(0)} I_{MM'} = I_M^D \partial I_M = \partial_M^{(0)}
\end{equation}

since by construction $I_M = I_M^D I_{MM'}$. Thus by iteration $\partial_M^{(n)} = \partial_M^{(0)} = \partial_M$ is already fixed pointed, no matter what the coarse graining maps $I_M$ are as long as they descend from an MRA.

It follows
\begin{equation}
\partial_M^{N} = I_M^D (\partial [I_M I_M^D]^{N-1} \partial I_M)
\end{equation}

While $I_M^D I_M = 1_{LM}$ by isometry, $p_M := I_M I_M^D$ is a projection in $L$ (onto the subspace $V_M$ of the MRA). Thus, if $[\partial, p_M] = 0$ we find $\partial_M^{N} = I_M^D \partial^N I_M$. The claim then follows from the spectral theorem (functional calculus).

To see that both the Shannon and Dirichlet kernel satisfy the assumption of the theorem it suffices to remark that they only depend on the difference $x - y$, i.e. they are translation invariant. Explicitly, since the $\chi^M_m$ with $m \in \mathbb{Z}$ and $m \in \mathbb{Z}_M$ respectively are an ONB of $V_M$ just as are the $e_k$, $|k| \leq \pi M$ and $e_{2\pi n}$, $|n| \leq \frac{M-1}{2}$ respectively
\begin{equation}
(p_M f)(x) = \sum_m \chi^M_m(x) < \chi^M_m, f > = \int_X dy [\int_{-\pi M}^{\pi M} \frac{dk}{2\pi} e_k(x - y)] f(y) X = \mathbb{R}
\end{equation}

and integration by parts does not lead to boundary terms due to the support properties of $f$ or by periodicity respectively.

It follows that using the natural discretisation the theory is already at its fixed point and the fixed point family member at resolution $M$ coincides with the the continuum theory blocked from the continuum to resolution $M$, that is, by simply dropping the superscript $(n)$ we have
\begin{equation}
J_M \Omega_M = \Omega, \quad J_M A_M (F_{M,1})^*..A_M (F_{M,N})^* \Omega_M = A(I_M F_{M,1})^*..A(I_M F_{M,N})^* \Omega, \quad H_M = J_M^J H J_M
\end{equation}

Remark:
Thus translation invariance of the Shannon and Dirichlet kernel respectively is, besides smoothness, another important difference with the Haar kernel [27]
\begin{equation}
\sum_m \chi^M_m(x) \chi^M_m(y) = = \sum_m \chi_{\frac{m}{M}, \frac{m+1}{M}}(x) \chi_{\frac{m}{M}, \frac{m+1}{M}}(y)
\end{equation}

which is not translation invariant. Therefore in this case the flows of $\omega_M$ or $\omega_M^{-1}$ are not simply related by $\omega_M = I_M^D \omega I_M$, $\omega_M^{-1} = I_M^D \omega^{-1} I_M$ and thus one must define $\omega_M$ as the inverse of the covariance $\omega_M^{-1}$. As $M \to \infty$ this difference disappears but at finite $M$ it is present and makes the study of the flow with respect to a non-translation invariant kernel much more and unnecessarily involved.

5 Conclusion and Outlook

In this paper we have extended the definition of Hamiltonian renormalisation in the sense of [9] from the bosonic to the fermionic case. The definition given in [9] in fact covers both cases but the practical implementation for bosons was in terms of measures [17] which cannot be used for fermions. We have tested the scheme for massless 2d chiral fermion theories, the extension to the massive and higher dimensional case being immediate, just requiring the higher dimensional Clifford algebra. In particular we showed that using the smooth local Shannon-Dirichlet kernel for renormalisation and discretisation results in simple flow, an easy computable fixed point theory which coincides with the known continuum theory and has manifestly doubler free spectrum even at
finite resolution due to the inherent non-locality with respect to the chosen finite resolution microscopes based on those kernels.

An immediate extension of the current paper that suggests itself is to apply the current framework to the known solvable 2d interacting fermion theories [29].

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