AN SPECTRAL CONDITION FOR GLOBAL EQUIVALENCE OF PLANAR MAPS

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Abstract. The spectral condition on the differentiable maps, of the Euclidean plane \( \mathbb{R}^2 \) into itself, is the assumption that their Jacobian eigenvalues are all equal to one (unipotent maps). It is demonstrated that a \( C^1 \)-unipotent map is globally equivalent to the linear translation \( \tau(x,y) = (x+1,y) \), as long as the map is fixed point free (i.e. \( G(q) \neq q, \forall q \) implies \( \varphi \circ G \circ \varphi^{-1} = \tau \), for some homeomorphism \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)). Similarly, it is proved not only that the fixed point set induced by a \( C^1 \)-unipotent has no isolated elements, but that a \( C^1 \)-unipotent map has no periodic points. The relation with the existence of global attractors in \( \mathbb{R}^2 \), by using a global bifurcation on unipotent maps, is also studied.

1. Introduction

A differentiable map \( G : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), not necessarily continuously differentiable, is called unipotent, provided that its spectrum \( \text{Spc}(G) = \{ \text{Eigenvalues of } DG_z : z \in \mathbb{R}^2 \} \subset \mathbb{C} \) is the set \( \{1\} \); other words, their Jacobian eigenvalues are all equal to one. Evidently, the unipotent maps are always orientation preserving maps, as the simplest examples given by the rules \( \tau(x,y) = (x+1,y) \) (linear translation) and \( \text{Id}(x,y) = (x,y) \) (identity map). In these circumstances, it is acceptable and also reasonable to ask if the identity map and the linear translation describe the global behavior of the unipotent maps, in the sense that a nonlinear unipotent map is equivalent to either the linear translation or the identity map.

To describe the precise equivalence under consideration, the global map \( G : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is said to be globally equivalent to \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), if there exists a homeomorphism \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that the composition \( \varphi \circ G \circ \varphi^{-1} = F \). This homeomorphism \( \varphi \) is denominated a topological conjugacy between \( G \) and \( F \). In the particular case that the map \( F \) is linear, the conjugacy \( \varphi \) is called a linearisation of the map \( G \). For instance, in [CK94] the authors present a modern description of the Theorem of Kerékjártó which justifies the existence of a global linearisation of a \( m \)-periodic map (i.e. \( F^m = \text{Id} \), for some positive integer \( m > 0 \)). This is also studied in the recent paper [CGMo19], where the authors study the existence of a smooth linearisation of planar periodic maps.

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It is proved that a $C^1$-unipotent map $G$ is globally equivalent to the linear translation, as long as $G$ is fixed point free. This means that, its fixed point set $\text{Fix}(G) = \{(x, y) \in \mathbb{R}^2 : G(x, y) = (x, y)\}$ is empty, as in [Fra92]. This result is obtained after a global characterization of each fixed point free unipotent map, by using a linear conjugacy induced by a rotation (Proposition 5.1). In the complementary case, $\text{Fix}(G) \neq \emptyset$, it is demonstrated not only that the fixed point set induced by a $C^1$-unipotent has no isolated elements, but that a $C^1$-unipotent map has no periodic points. Specifically, these maps are free, in the sense of the paper [Bro85]. In the proofs, the existent normal form of $C^1$-unipotent maps is important. For instance, it is utilised in the last section in order to describe a global bifurcation on unipotent maps, related with the existence of global attractor fixed points.

This paper is organized as follows. Section 2 presents the notably normal form for each $C^1$-unipotent two-dimensional map, as proved in the paper [Cam00]; this important characterization on these bijective maps is used in the other sections of the paper, in order to present a global description of the continuously differentiable unipotent maps. Section 3 describes the unipotent maps with a fixed point. Theorem 3.2 implies that for each nonlinear unipotent map with a fixed point at the origin, there exists a linear rotation $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ such that $(R_\theta \circ G \circ R_{-\theta})(x, y) = (x + \psi(y), y)$, where $\psi$ is a nonlinear function with a fixed point located at zero. Section 4 includes the definition of a topological disk in $\mathbb{R}^2$ as the subsets $D$ of $\mathbb{R}^2$ that are homeomorphic to the compact set $\overline{D}_1 = \{z \in ||z|| \leq 1\}$. In this context, Theorem 4.6 proves that a nonlinear unipotent map induces a simple dynamics, in the sense that

$$G(D) \cap D = \emptyset \Rightarrow G^p(D) \cap G^q(D) = \emptyset,$$

for each disk topological $D$ in $\mathbb{R}^2$ and integers $p \neq q$. As usual, the symbol $G^m$ denotes the composition $G \circ \cdots \circ G$ (respectively $G^{-1} \circ \cdots \circ G^{-1}$) $m > 0$ (respectively $m < 0$) times, and $G^0$ is the identity map. Consequently, a unipotent map has no periodic points as long as it is continuously differentiable. Section 5 characterizes the fixed point free unipotent maps, and includes the proof that a $C^1$-unipotent map has no isolated fixed points. Section 6 presents the proof that a fixed point free unipotent map is globally equivalent to the translation $\tau(x, y) = (x + 1, y)$ as long as it is continuously differentiable. Section 7 presents one parameter families whose fixed points change its stability, between to be a global repellor or to be a global attractor. This is related with discrete Markus-Yamabe question [CGMn99, vdE00]; where the authors study the existence of spectral conditions on Euclidean maps $\mathbb{R}^n \mapsto \mathbb{R}^n$, in order to obtain a global attractor fixed point.
2. Normal form of two-dimensional unipotent maps

It is accepted that, under different circumstances, the unipotent maps in dimension two have been classified and also globally described. For instance, in the recent paper [Cha00], Chamberland proves that a real–analytic map of the Euclidean plane into itself has an inverse as long as it is unipotent (see also [Che99, Rab10]). An interesting proof for $C^1$–maps appears in the contemporary paper [Cam00], where Campbell presents an important normal form of unipotent maps, and also observes that it has an explicit global inverse. This impressively result may be enunciated as follows.

**Theorem 2.1** (Campbell). Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be $C^1$. Then $G$ is unipotent if $G$ is of the form

\[
G(x, y) = (x + b\phi(ax + by) + c, y - a\phi(ax + by) + d),
\]

for some constants $a, b, c, d \in \mathbb{R}$ and some function $\phi$ of a single variable. If that is the case, then $G$ has an explicit global inverse. Conversely, if $G$ is $C^1$ and unipotent, then $G$ is of the form above for a $\phi$ that is $C^1$.

This normal form has been analysed in [Cha03], in the case of maps whose Jacobian matrices have equal eigenvalues, not necessarily one. Unfortunately, this normal form does not exist outside the unipotent maps as shown the interesting real–analytic map, presented in [Cha03 Theorem 1.3].

3. Unipotent maps with a fixed point

In this context, a $C^1$–map $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is unipotent, if and only if, it has the form (2.1), where $a, b, c$ and $d$ are real constants, and $\phi$ is a $C^1$–function on a single variable. Consequently,

- $\phi(0) = 0$ implies that $G(0, 0) = (c, d)$.
- $(0, 0) = (a, b)$ means that $G(x, y) = (x + c, y + d)$, it is a linear map.

**Proposition 3.1.** Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a $C^1$–map. Then the following statements are equivalent

1. $G$ is a non-linear unipotent map such that $G(0, 0) = (0, 0)$.
2. There are $\phi$, a non-linear $C^1$–function of a single variable joint to $(b, -a)$, a non-zero constant vector such that

\[
G(x, y) = \left( x + b\phi(ax + by), y - a\phi(ax + by) + a\phi(0) \right).
\]

3. There are $\psi$, a non-linear $C^1$–function of a single variable with $\psi(0) = 0$ joint to $(\beta, -\alpha)$, a unitary constant vector such that

\[
G(x, y) = \left( x + \beta\psi(\alpha x + \beta y), y - \alpha\psi(\alpha x + \beta y) \right).
\]
Proof. To demonstrate that (1) implies (2), let $G: \mathbb{R}^2 \to \mathbb{R}^2$ refer the $C^1$-map in the first statement. The existence of the non-linear map $\phi$ and the non-zero vector $(b, -a)$, described in (2), is a consequence of the normal form (2.1). In addition, $G(0, 0) = (0, 0)$ implies that $0 = c + b\phi(0)$ and $0 = d - a\phi(0)$. Therefore, announcement (2) holds.

In order to obtain that (2) implies (3), the initial observation is that (2) gives the existence of the unitary vector $(\beta, -\alpha) \in \mathbb{R}^2$, defined by

$$\alpha = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \beta = \frac{b}{\sqrt{a^2 + b^2}}.$$

Similarly, $\phi$ induces the map given by

$$\psi(t) = \sqrt{a^2 + b^2}\left(\phi(t\sqrt{a^2 + b^2}) - \phi(0)\right).$$

This map $\psi$ satisfies $\psi(0) = 0$ and $\psi(ax + \beta y) = \sqrt{a^2 + b^2}\left(\phi(ax + by) - \phi(0)\right)$. Consequently,

$$\beta \psi(ax + \beta y) = b\phi(ax - by) - b\phi(0),$$

$$-\alpha \psi(ax + \beta y) = -a\phi(ax - by) + a\phi(0).$$

Therefore, (2) implies (3).

Finally, as the non-linear map $\psi$ satisfies $\psi(0) = 0$, the map $G$ is non-linear and satisfies $G(0, 0) = (0, 0)$. In addition, a direct computation shows that $G(x, y) = (x + \beta \psi(\alpha x + \beta y), y - \alpha \psi(\alpha x + \beta y))$ is unipotent, and consequently (3) implies (1). Therefore, this proposition holds. \hfill \Box

For each $\theta \in \mathbb{R}$, let $R_\theta$ denote the linear rotation.

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

**Theorem 3.2.** Let $G: \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^1$-map. Then the following are equivalent

1. $G$ is a non-linear unipotent map such that $G(0, 0) = (0, 0)$.
2. There is a rotation $R_\theta$ such that

$$(R_\theta \circ G \circ R_{-\theta})(x, y) = (x + \psi(y), y),$$

where $\psi$ is a non-linear $C^1$-function such that $\psi(0) = 0$.

**Proof.** In statement (1) conditions, Proposition 3.1 establishes the existence of $\psi$, a non-linear $C^1$-function with $\psi(0) = 0$ joint to $(\beta, -\alpha)$, a unitary constant vector such that

$$G(u, v) = \left(u + \beta \psi(\alpha u + \beta v), v - \alpha \psi(\alpha u + \beta v)\right).$$

In this situation, the well defined rotation

$$R_\theta = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}.$$
not only sends the unitary vector \((\beta, -\alpha)\) into the vector \((1, 0)\), but \((\alpha, \beta)\) into \((0, 1)\). In addition, the inverse \(R_{-\theta}\) satisfies \(R_{-\theta}(x, y) = (\beta x + \alpha y, -\alpha x + \beta y)\) and then \((G \circ R_{-\theta})(x, y) = (\beta x + \alpha y + \beta \psi(y), -\alpha x + \beta y - \alpha \psi(y))\). To be precise,
\[
(G \circ R_{-\theta})(x, y) = (\beta x + \alpha y, -\alpha x + \beta y) + \psi(\beta, -\alpha).
\]
Therefore, \((R_{\theta} \circ G \circ R_{-\theta})\) satisfies (2). It concludes that (I) implies (2).

Finally, (2) shows that the Jacobian matrix \(DG\) has the form
\[
R_{-\theta} \circ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \circ R_{\theta}.
\]
Thus, \(G\) is unipotent and (I) is true. Therefore this proposition holds. \(\square\)

Remark 3.3. In the case that \(G\) is a linear unipotent map, the standard Jordan Form Theory directly gives the existence of a linear isomorphism \(T : \mathbb{R}^2 \to \mathbb{R}^2\) such that \((T \circ G \circ T^{-1})(x, y) = (x + By, y)\), where \(B \in \mathbb{R}^2\) is a constant.

4. Dynamics of unipotent maps with a fixed point located at the origin

The characterization presented Theorem 3.2 is used in this section in order to describe the dynamical properties of nonlinear unipotent maps with a fixed point. It is motivated by the important description presented in [Bro85], where the author introduce a new class of homeomorphisms, called free homeomorphisms (Theorem 4.6).

Theorem 4.1. Let \(G : \mathbb{R}^2 \to \mathbb{R}^2\) be a non-linear unipotent \(C^1\)-map also with \(G(0, 0) = (0, 0)\). That is, there is a rotation \(R_{\theta}\) such that \((R_{\theta} \circ G)(y, y) = (x + \psi(y), y)\), where \(\psi\) is a non-linear \(C^1\)-function satisfying \(\psi(0) = 0\). Then, for each open interval \(I \subset \{y \in \mathbb{R} : \psi(y) \neq 0\}\), the vertical segment \(\Delta_x = \{x\} \times I\) satisfies
\[
G_{\theta}^{-1}(\Delta_x) \cap \Delta_x = \emptyset \quad \text{and} \quad \Delta_x \cap G_{\theta}(\Delta_x) = \emptyset,
\]
where \(G_{\theta} = R_{\theta} \circ G \circ R_{-\theta}\) with \(G_{\theta}^{-1}\) its inverse, and \(x \in \mathbb{R}\). In addition, when such an interval \(I\) is maximal (a connected component), then for every endpoint, say \(a \in \mathbb{R}\), the image satisfies
\[
G_{\theta}(x, a) = (x, a), \quad \forall x \in \mathbb{R}.
\]

Proof. In the open interval, the map does not change its sign. So without loss of generality, the interval has the following form \(I \subset \{y \in \mathbb{R} : \psi(y) > 0\}\). Thus, if \((x, y) \in \Delta_x\) the image \(G_{\theta}(x, y)\) and its inverse \(G_{\theta}^{-1}(x, y)\) satisfy
\[
G_{\theta}(x, y) - (x, y) = (x, y) - G_{\theta}^{-1}(x, y).
\]
These differences are different from zero. Therefore, the first part of the theorem holds.

To conclude, let \(a \in \mathbb{R}\) be an endpoint of a maximal interval \(I\), given by \(\{y \in \mathbb{R} : \psi(y) > 0\}\). As the continuous function \(\psi\) is defined in the whole \(\mathbb{R}\), and \(I\)
maximal, the value $\psi(a) = 0$, consequently $G_\theta(x, a) = (x, a)$. This complete the proof. □

**Example 4.2.** The illustrative unipotent maps

$$(x, y) \mapsto (x + y^3, y) \quad \text{and} \quad (x, y) \mapsto (x + y^2, y)$$

have two different behaviors around the horizontal axis, where both maps have all their fixed points.

The next corollary considers the notations of Theorem 4.1. Furthermore, for each positive integer $m > 0$, the symbol $G_m$ denotes the composition $G \circ \cdots \circ G$, $m$ times ($G^0$ is the identity map).

**Corollary 4.3.** Let $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map, as in Theorem 4.1. If $I \subset \mathbb{R}$ is the respective maximal interval. Then for any $z \in R_\theta^{-1}(\mathbb{R} \times I)$ the sequence $\{G^n(z) : n \geq 0\}$, induced by compositions is well defined and it is divergent, in the sense that

$$\lim_{n \to +\infty} ||G^n(z)|| = +\infty.$$  

Otherwise, $G(z) = z$.

*Proof.* If $z \in R_\theta^{-1}(\mathbb{R} \times I)$ the point $(x, y) = R_\theta(z)$ satisfies that

$$G_\theta^n(x, y) = (x + n\psi(y), y) \quad \forall n \geq 0.$$  

This sequence is unbounded, because $y \in I$, where the value $\psi(y) \neq 0$. Finally, it is enough to observe that

$$G^n = R_{-\theta} \circ G_\theta^n \circ R_\theta$$  

Therefore, the first part of the corollary holds.

The second part follows, since the complement set of

$$\bigcup \{ R_\theta^{-1}(\mathbb{R} \times I) : I \text{ is maximal} \},$$

is contained in $Fix(G) = \{ p \in \mathbb{R}^2 : G(p) = p \}$, as shown in the last part of Theorem 4.1. This concludes the proof. □

**Corollary 4.4.** Let $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map, as in Theorem 4.1. If $I \subset \mathbb{R}$ is the respective maximal interval. Then for any $w, z \in R_\theta^{-1}(\mathbb{R} \times I)$ there exists a compact segment $\Lambda \subset R_\theta^{-1}(\mathbb{R} \times I)$ whose endpoints are exactly $z$ and $w$ such that

$$\lim_{n \to +\infty} G^n(\Lambda) = \infty \quad \text{and} \quad \lim_{m \to +\infty} G^{-m}(\Lambda) = \infty.$$  

It means that for every compact set $K \subset \mathbb{R}^2$ there is a natural number $\tilde{n} \in \mathbb{N}$ such that $G^n(\Lambda) \cap K = \emptyset$ when $|n| > \tilde{n}$. 

Proof. The points \( w, z \in R_\theta^{-1}(\mathbb{R} \times I) \) means that \( R_\theta(z) \) and \( R_\theta(w) \) belong to the band \( \mathbb{R} \times I \). Thus, the compact segment \( \{tR_\theta(z) + (1 - t)R_\theta(w) : 0 \leq t \leq 1\} \) is not only contained in \( \mathbb{R} \times I \), but its image

\[
\Lambda = R_\theta^{-1}(\{tR_\theta(z) + (1 - t)R_\theta(w) : 0 \leq t \leq 1\})
\]

is the compact segment whose endpoints are \( z \) and \( w \). This \( \Lambda \) induces, by a projection, the compact interval \( \{y \in \mathbb{R} : R_{-\theta}(x, y) \in \Lambda \} \) where the restriction of \( \psi \) has its maximum and minimum, both different from zero, and with the same sign. Therefore, this \( \Lambda \) satisfies the conditions of the corollary. □

Remark 4.5. Under the notations of Theorem 4.1, the connected components of the open set

\[
\bigcup \{R_\theta^{-1}(\mathbb{R} \times I) : I \text{ is maximal}\} \neq \emptyset
\]

are fundamental regions, in the sense of [And65]. These fundamental regions are invariant sets, where the inclusion of a simple closed curve (Jordan Curve) implies the inclusion of the open set enclosed by it.

In the next theorem, a subset \( D \) of \( \mathbb{R}^2 \) is called a topological disc in \( \mathbb{R}^2 \) when it is homeomorphic to \( D_1 = \{z \in \|z\| \leq 1\} \).

**Theorem 4.6.** Let \( G : \mathbb{R}^2 \to \mathbb{R}^2 \) be a unipotent \( C^1 \)-map, with \( G(0, 0) = (0, 0) \). Then

\[
G(D) \cap D = \emptyset \implies G^p(D) \cap G^q(D) = \emptyset,
\]

for each disk topological \( D \) in \( \mathbb{R}^2 \) and integers \( p \neq q \).

Proof. In the linear case, this map becomes the identity map. In the non-linear case, Theorem 3.2 shows that there is no loss of generality by writing:

\[
G(x, y) = (x + \psi(y), y),
\]

for some \( C^1 \)-function \( \psi \), with \( \psi(0) = 0 \). In this context, the fixed point set \( \{z \in \mathbb{R}^2 : G(z) = z\} \) are the horizontal lines \( y = \gamma \), where \( \psi(\gamma) = 0 \). In a different situation, that is \( G(x, y) \neq (x, y) \), the sequence

\[
G^m(x, y) = (x + m\psi(y), y), \quad \forall m \in \mathbb{Z}.
\]

Under these conditions, the assumption \( G(D) \cap D = \emptyset \) implies that the closed set \( D \) is contained in an open band of the form

\[
\mathbb{R} \times I \quad \text{where} \quad I \subset \{y \in \mathbb{R} : \psi(y) \neq 0\}.
\]

Consequently

\[
G(D) \cap D = \emptyset \implies G^m(D) \cap G(D) = \emptyset, \quad \forall m > 0.
\]

Thus, the result is obtained by using either \( m = p - q \) or \( m = q - p \), the positive one. Therefore, this theorem holds. □
Remark 4.7. In the terminology of [Bro85], the maps in Theorem 4.6 are free. Consequently, the conclusions of Theorem 4.6 remain correct with $D$ replaced by a continuum, this is a compact and connected subset of $\mathbb{R}^2$. For instance, each one point set $\{z\} \subset \mathbb{R}^2$ is a continuum.

**Theorem 4.8.** Let $G: \mathbb{R}^2 \to \mathbb{R}^2$ be a unipotent $C^1$-map, with $G(0,0) = (0,0)$. When $z \in \mathbb{R}^2$, and $G(z) \neq z$, then the exist a line $\ell_z \subset \mathbb{R}^2$ such that $G^{-1}(\ell_z) \cap \ell_z = \emptyset$ and $\ell_z \cap G(\ell_z) = \emptyset$.

**Proof.** As in the proof of Theorem 4.6 there is no loss of generality by assuming that: $G(x,y) = (x + \psi(y), y)$, for some $C^1$-function $\psi$, with $\psi(0) = 0$. Thus, $\ell_z$ corresponds to the vertical line passing through $z \in \mathbb{R}^2$. Therefore, this theorem holds.

A an element $(x, y) \in \mathbb{R}^2$ is called periodic point of $G$ provided the existence of some integer $p > 1$ such that

$$G^p(x, y) = (x, y) \quad \text{but} \quad G^m(x, y) = (x, y), \quad \forall 1 \leq m \leq p - 1.$$ 

Notice that, this periodic point is also a fixed point of $G^p$.

**Theorem 4.9.** A $C^1$-unipotent map $G: \mathbb{R}^2 \to \mathbb{R}^2$ has no periodic points.

**Proof.** There is no loss of generality by assuming that: $G(x, y) = (x + \psi(y), y)$, for some $C^1$-function $\psi$, with $\psi(0) = 0$. Consequently, the condition $G(x, y) \neq (x, y)$, that meas $\psi(y) \neq 0$, directly implies that

$$G^m(x, y) - (x, y) = (m\psi(y), 0) \neq 0, \quad \forall m \in \mathbb{Z} \setminus \{0\}.$$ 

Therefore, $G$ has no fixed points.

It should be mentioned that the results not only give a description of the full dynamics, but it presented a smooth conjugacy of the system with systems of the form $(x, y) \mapsto (x + \psi(y), y)$, where, clearly, the degree of the map $(x, y) \mapsto (\psi(y), 0)$ is different from one.

5. **Characterization of fixed point free unipotent maps**

In order to present a complete characterization of the unipotent maps, they are described in two different cases. The existence of the normal form in Theorem 2.1 remains correct with $\phi$ exchanged by $t \mapsto \phi(t) - \phi(0)$. Therefore, it is not difficult to see that a $C^1$-map $G: \mathbb{R}^2 \to \mathbb{R}^2$ is unipotent, if and only if, it has the form

$$G(x, y) = (x + b\phi(ax + by) + c, y - a\phi(ax + by) + d),$$

where $a, b, c$ and $d$ are real constants, and $\phi$ is a $C^1$-function such that $\phi$ sends zero into zero.
In this context, there is no ambiguity in the presentation of the following sets. Specifically,

\[ \mathcal{UP}_1 = \left\{ G : \mathbb{R}^2 \to \mathbb{R}^2 : \text{In (5.1), } (c, d) = (0, 0) \text{ and } \phi(0) = 0 \right\}, \]

and

\[ \mathcal{UP}_2 = \left\{ G : \mathbb{R}^2 \to \mathbb{R}^2 : \text{In (5.1), } (c, d) \neq (0, 0) \text{ and } \phi(0) = 0 \right\}. \]

Notice that, the union \( \mathcal{UP}_1 \cup \mathcal{UP}_2 \) coincides with the set of all the \( C^1 \)-unipotent maps of \( \mathbb{R}^2 \) into itself. In addition,

\[ (5.2) \quad G \in \mathcal{UP}_1 \implies G(0, 0) = (0, 0). \]

Therefore, \( \mathcal{UP}_1 \) has no fixed point free maps.

**Proposition 5.1.** Let \( G : \mathbb{R}^2 \to \mathbb{R}^2 \) be \( C^1 \)-unipotent map of the form

\[ G(x, y) = (x + b\phi(ax + by) + c, y - a\phi(ax + by) + d) \quad \forall x, y \in \mathbb{R}, \]

where \( a, b, c, d \in \mathbb{R} \) and \( \phi \) is a \( C^1 \)-function such that \( \phi(0) = 0 \). Then the following are equivalent.

1. The non-linear map \( G \in \mathcal{UP}_2 \) is fixed point free.
2. There is a rotation \( R_\theta \) such that

\[ (R_\theta \circ G \circ R_{-\theta})(x, y) = (x + \psi(y), y) + R_\theta(c, d), \]

where \( R_\theta(c, d) \) is a non-zero constant vector, and \( \psi \) is a \( C^1 \)-function with \( \psi(0) = 0 \) such that

\[ \begin{bmatrix} \psi(t)\sqrt{a^2 + b^2} + (bc - ad) \\ ac + bd \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall t \in \mathbb{R}. \]

**Proof.** The map \( G \in \mathcal{UP}_2 \) is non-linear. Consequently, the constants \( \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \beta = \frac{b}{\sqrt{a^2 + b^2}} \) and the map \( \psi_2(t) = \sqrt{a^2 + b^2}\phi(t\sqrt{a^2 + b^2}) \) are well defined. Under these conditions,

\[ G(u, v) = (u + \beta\psi_2(\alpha u + \beta v) + c, v - \alpha\psi_2(\alpha u + \beta v) + d) \]

with \( \alpha^2 + \beta^2 = 1 \), \( \psi_2(0) = 0 \) and \( (c, d) \neq (0, 0) \).

The proposition is obtained the rotation \( R_\theta = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix} \) whose inverse can be write as \( R_{-\theta}(x, y) = (\beta x + \alpha y, -\alpha x + \beta y) \). Consequently, \( (G \circ R_{-\theta})(x, y) \) is equal to \( (\beta x + \alpha y, \beta y - \alpha x) + (\beta\psi_2(y), -\alpha\psi_2(y)) + (c, d) \) and then

\[ (R_\theta \circ G \circ R_{-\theta})(x, y) = (x + \psi(y), y) + (c\beta - ad, \alpha c + \beta d). \]

Therefore, (2) is true.

The reverse conclusion is obtained by a direct computation and, therefore, this proposition holds. \( \square \)
Remark 5.2. The linear unipotent maps in \( \mathcal{UP}_2 \) have cases with a similar form in Theorem 5.1. In this situation, when \( G \in \mathcal{UP}_2 \), there exist a linear isomorphism \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that

\[
(T \circ G \circ T^{-1})(u, v) = (u + Bv, v) + (C, D),
\]

where \( B \in \mathbb{R} \) and \( (C, D) \neq (0, 0) \). In addition, the rotation \( R_{\Theta} = \frac{1}{C^2 + D^2} \begin{bmatrix} C & D \\ -D & C \end{bmatrix} \) satisfies

\[
(R_{\Theta} \circ G \circ R_{-\Theta})(x, y) = (x + 1, y) + \frac{B}{C^2 + D^2} \begin{bmatrix} CD & C^2 \\ -D^2 & -DC \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

The particular condition, \( D = 0 \) implies

\[
(R_{\Theta} \circ G \circ R_{-\Theta})(x, y) = (x + 1, y),
\]

because \( B \neq 0 \) induces a fixed point as long as \( D = 0 \).

The general case \( D \neq 0 \) might be studied in (5.3) with \( B \neq 0 \). In this case, the projection into the vertical axes is different from zero. Thus \( \mathbb{R}^2 \) is a fundamental region and concludes that (5.3) is conjugated to a linear translation \( (x, y) \mapsto (x + 1, y) \) [And65].

**Theorem 5.3.** A \( C^1 \)-unipotent map \( G: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) has no isolated fixed points.

**Proof.** In the linear case, the map becomes the identity map, as long as it has fixed points. In the non-linear case, \( G \) satisfies Proposition 3.1. So the fixed points appear in a line of the form \( \alpha x + \beta y = 0 \). Thus, this theorem holds.

\[\square\]

6. Dynamics of fixed point free \( C^1 \)-unipotent maps

In the global description of \( C^1 \)-unipotent maps, the initial observation follows from the proof of Theorem 3.2. To be precise, any \( C^1 \)-unipotent map admits a decomposition of \( \mathbb{R}^2 \) in a family of parallel lines such that the map preserves each such a line, and sends it homeomorphically into itself.

**Proposition 6.1.** Let \( G \in \mathcal{UP}_2 \) be a fixed point free map. Then, for any pair of points \( z, w \in \mathbb{R}^2 \) the compact connected segment \( \Lambda = \{ tz + (1 - t)w : 0 \leq t \leq 1 \} \) satisfies

\[
\lim_{n \rightarrow +\infty} G^n(\Lambda) = \infty \quad \text{and} \quad \lim_{m \rightarrow +\infty} G^{-m}(\Lambda) = \infty.
\]

**Proof.** The linear case has been analyzed in Remark 5.2. In the non-linear case, Proposition 5.1 shows that there is no loss of generality by writing:

\[
G(x, y) = (x + \psi(y) + C, y + D),
\]

for some \( C^1 \)-function \( \psi \) such that \( \psi(0) = 0 \), and \( (\psi(y) + C, D) \neq (0, 0) \), for all \( y \in \mathbb{R} \). If \( D = 0 \), the condition \( \psi(y) + C \neq 0 \) implies that the compact connected
segment $\Lambda = \{tz + (1 - t)w: 0 \leq t \leq 1\}$ satisfies the requested conditions. Therefore, in this first case the proposition holds.

If $D \neq 0$, the projection of $G(x, y) - (x, y)$ into the vertical axis is different from zero. Consequently, its sign in the connected set $\{y: (x, y) \in \Lambda\}$ is constant. Thus, $\Lambda$ satisfies the limits. Therefore, this proposition holds. □

As usual, the homeomorphisms $F$ and $G$ of $\mathbb{R}^2$ into itself, are **conjugated** if there exists an homeomorphisms $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\varphi \circ F = G \circ \varphi$.

**Theorem 6.2.** If the $C^1$–unipotent map $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is fixed point free, then $G$ is topologically conjugated to the global translation $\tau(x, y) = (x + 1, y)$.

**Proof.** This theorem is proved by using [And65]. In this paper, the author studies the orientation preserving homeomorphisms $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose $\text{Fix}(G) = \emptyset$, and demonstrates that $G$ is topologically conjugated to the global translation $\tau(x, y) = (x + 1, y)$ if the whole plane is a fundamental region. Thus, this theorem follows by Proposition 6.1, where is proved that $\mathbb{R}^2$ is a fundamental region, in the sense of [And65]. Therefore, the theorem holds. □

7. Families where the fixed point changes its stability

This section, motivated by [SP87], is concerned with the description of the simplest patterns according to which unipotent maps of the form $G_{\mu}(x, y) = G(x, y) - (\mu x, \mu y)$, where $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a non–linear unipotent map with $G(0, 0) = (0, 0)$, change its stability – bifurcate – under perturbations of the parameter $\mu$, in a small open interval centred at zero.

The next definitions, presented in [AGnG08], will be needed.

**Definition 7.1.** Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a topological embedding; that is, a globally injective local homeomorphism.

- Let $p \in \mathbb{R}^2$. The $\omega$–**limit set** of $p$ is $\omega(p) = \left\{ z \in \mathbb{R}^2: \exists 0 < n_k \in \mathbb{N}, \text{ such that } \lim_{n_k \rightarrow \infty} F^{n_k}(p) = z \right\}$.

- The origin $(0, 0)$ is a **local attractor** (resp. **local repellor**) for $F$ if there exist a topological disc $D$, which is contained in the domain of definition of $F$ (resp. $F^{-1}$), that is a neighbourhood of $(0, 0)$ such that $F(D) \subset \text{Int}(D)$ (resp. $F^{-1}(D) \subset \text{Int}(D)$) and $\cap_{n=1}^{\infty} F^{-n}(D) = \{(0, 0)\}$ (resp. $\cap_{n=1}^{\infty} F^{-n}(D) = \{(0, 0)\}$).

- The origin $(0, 0)$ is a **global attractor** for $F$ if $(0, 0)$ is a local attractor for $F$ and $\omega(p) = \{(0, 0)\}$ for all $p \in \mathbb{R}^2$.

**Lemma 7.2.** Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving $C^1$–embedding. Assume that $F(x, y) = (\lambda x + \psi(y), \lambda y)$, where the constant $0 < \lambda < 1$ and the function $\psi$ has a fixed point at zero. Then $(0, 0)$ is a global attractor for $F$. 

Proof. The origin \((0, 0)\) is an hyperbolic attractor. In addition,
\[
F^n(x, y) = \left( \lambda^n x + \sum_{k=0}^{n-1} \lambda^{n-1-k} \psi(y\lambda^k), \lambda^n y \right), \quad \forall n \geq 1.
\]
In this context,
\[
\left| \sum_{k=0}^{n-1} \lambda^{n-1-k} \psi(y\lambda^k) \right| \leq \max_{0 \leq k \leq n-1} |\psi(y\lambda^k)| \frac{1}{1 - \lambda}.
\]
Thus, \(\lim_{n \to \infty} F^n(x, y) = (0, 0)\), because \(\psi(0) = 0\), and then \(\omega(x, y) = \{(0, 0)\}\) for all \((x, y) \in \mathbb{R}^2\). Therefore, \((0, 0)\) is a global attractor for \(F\). \(\square\)

Definition 7.3. Let \(F: \mathbb{R}^2 \to \mathbb{R}^2\) be a homeomorphism. The fixed point \((0, 0)\) is a global repellor for \(F\) if \(\omega(x, y) = \{(0, 0)\}\) for all \((x, y) \in \mathbb{R}^2\). Therefore, \((0, 0)\) is a global attractor for \(F\).

The next theorem presents a family where the fixed point changes its stability.

Theorem 7.4. Let \(G: \mathbb{R}^2 \to \mathbb{R}^2\) be a \(C^1\)-map with \(G(0, 0) = (0, 0)\). If the map \(G\) is non-linear and unipotent, then there exists \(\varepsilon > 0\) such that the family of maps
\[
\{G_\mu(x, y) = G(x, y) - (\mu x, \mu y): -\varepsilon < \mu < \varepsilon\}
\]
satisfies the following two conditions:

(a) For \(\mu > 0\) the map \(G_\mu\) has a global attractor at \((0, 0)\), and for \(\mu < 0\) the map \(G_\mu\) has a global repellor at \((0, 0)\).

(b) The map \(G_\mu\) has no periodic points in \(\mathbb{R}^2\) for \(-\varepsilon < \mu < \varepsilon\).

Proof. Set \(0 < \varepsilon \leq 1\). Theorem 3.2 implies that the local diffeomorphisms \(G_\mu\) are proper, consequently the family only includes diffeomorphisms.

If \(\mu > 0\), the spectrum \(\text{Spc}(G_\mu) = \{1 - \mu\}\) satisfies \(0 < 1 - \mu < 1\), \(G_\mu(0, 0) = (0, 0)\). Thus, Theorem 3.2 and Lemma 7.2 imply that \(G_\mu\) has a global attractor at \((0, 0)\). If \(\mu < 0\), Theorem 3.2 implies that
\[
G_\mu^{-1}(u, v) = \left( \frac{u}{1 - \mu} + \psi\left( \frac{v}{1 - \mu} \right), \frac{v}{1 - \mu} \right),
\]
where \(0 < \frac{1}{1 - \mu} < 1\) and \(\psi(0) = 0\). Lemma 7.2 and Definition 7.3 show that \(G_\mu\) has a global repellor at \((0, 0)\). Therefore, statement (a) holds.

The item (b) follow by using (a) and Theorem 4.9. \(\square\)

Remark 7.5. In [AGnG08] appears a smooth diffeomorphims \(F: \mathbb{R}^2 \to \mathbb{R}^2\) which has an order four periodic point, and is such that \(F(0, 0) = (0, 0)\), its spectrum \(\text{Spc}(F) \subset \{z \in \mathbb{C}: ||z|| < 1\}\), and \(\infty\) is a repellor in the sense that \(\infty\) is a local repellor of the natural extension to a homeomorphism \(F: \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}\) of the Riemann sphere, with \(F(\infty) = \infty\).
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**References**

[AGnG08] Begoña Alarcón, Víctor Guínez, and Carlos Gutiérrez, *Planar embeddings with a globally attracting fixed point*, Nonlinear Anal. **69** (2008), no. 1, 140–150. MR 2417859

[And65] S. A. Andrea, *On homeomorphisms of the plane, and their embedding in flows*, Bull. Amer. Math. Soc. **71** (1965), 381–383. MR 172258

[Bro85] M. Brown, *Homeomorphisms of two-dimensional manifolds*, Houston J. Math. **11** (1985), no. 4, 455–469. MR 837985

[Cam00] L. Andrew Campbell, *Unipotent Jacobian matrices and univalent maps*, Combinatorial and computational algebra (Hong Kong, 1999), Contemp. Math., vol. 264, Amer. Math. Soc., Providence, RI, 2000, pp. 157–177. MR 1800694

[CGMn99] Anna Cima, Armengol Gasull, and Francesc Mañosas, *The discrete Markus-Yamabe problem*, Nonlinear Anal. **35** (1999), no. 3, Ser. A: Theory Methods, 343–354. MR 1643454

[CGMnO19] A. Cima, A. Gasull, F. Mañosas, and R. Ortega, *Smooth linearisation of planar periodic maps*, Math. Proc. Cambridge Philos. Soc. **167** (2019), no. 2, 295–320. MR 3991373

[Cha00] Marc Chamberland, *Diffeomorphic real-analytic maps and the Jacobian conjecture*, Math. Comput. Modelling **32** (2000), no. 5-6, 727–732, Boundary value problems and related topics. MR 1791178

[Cha03] ______, *Characterizing two-dimensional maps whose Jacobians have constant eigenvalues*, Canad. Math. Bull. **46** (2003), no. 3, 323–331. MR 1994859

[Che99] Yu Qing Chen, *A note on holomorphic maps with unipotent Jacobian matrices*, Proc. Amer. Math. Soc. **127** (1999), no. 7, 2041–2044. MR 1485463

[CK94] Adrian Constantin and Boris Kolev, *The theorem of Kerékjártó on periodic homeomorphisms of the disc and the sphere*, Enseign. Math. (2) **40** (1994), no. 3-4, 193–204. MR 1309126

[Fra92] John Franks, *A new proof of the Brouwer plane translation theorem*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 217–226. MR 1176619

[Rab10] Roland Rabanal, *On differentiable area-preserving maps of the plane*, Bull. Braz. Math. Soc. (N.S.) **41** (2010), no. 1, 73–82. MR 2609212

[SP87] J. Sotomayor and R. Paterlini, *Bifurcations of polynomial vector fields in the plane, Oscillations, bifurcation and chaos* (Toronto, Ont., 1986), CMS Conf. Proc., vol. 8, Amer. Math. Soc., Providence, RI, 1987, pp. 665–685. MR 909943

[vdE00] Arno van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, vol. 190, Birkhäuser Verlag, Basel, 2000. MR 1790619

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