NONTRIVIALITY OF EQUATIONS AND EXPLICIT TENSORS IN $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ OF BORDER RANK AT LEAST $2m - 1$

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Abstract. For odd $m$, I write down tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank at least $2m - 1$, showing the non-triviality of the Young-flattening equations of [6] that vanish on the matrix multiplication tensor. I also study the border rank of the tensors of [1] and [3]. I show the tensors $T_{2k} \in \mathbb{C}^k \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of [1], despite having rank equal to $2k + 1 - 1$, have border rank equal to $2k$. I show the equations for border rank of [3] on $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ are trivial in the case of border rank $2m - 1$ and determine their precise non-vanishing on the matrix multiplication tensor.

1. Results and context

Let $A, B, C$ be complex vector spaces of dimensions $a, b, c$. A tensor $T \in A \otimes B \otimes C$ is said to have rank one if $T = a \otimes b \otimes c$ for some $a \in A, b \in B, c \in C$. More generally the rank of a tensor $T \in A \otimes B \otimes C$ is the smallest $r$ such that $T$ may be written as the sum of $r$ rank one tensors. Let $\hat{\sigma}_r \subset A \otimes B \otimes C$ denote the set of tensors of rank at most $r$. This set is not closed (under taking limits or in the Zariski topology) so let $\tilde{\sigma}_r$ denote its closure (the closure is the same in the Euclidean or Zariski topology). The variety $\tilde{\sigma}_r$ is familiar in algebraic geometry, it is cone over the $r$-th secant variety of the Segre variety, but we won’t need that in what follows. The rank and border rank of a tensor are measures of its complexity. While rank is natural to complexity theory, border rank is more natural from the perspective of geometry, as one can obtain lower bounds on border rank via polynomials. Let $R(T), \underline{R}(T)$ respectively denote the rank and border rank of $T$.

The maximum rank of a tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is at most $m^2$, although it is not known in general if this actually occurs. The maximum border rank of a tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is $\lceil \frac{m^3}{3m - 2} \rceil$ for all $m \neq 3$ and five when $m = 3$, see [9, 8]. It is an important problem to find explicit tensors of high rank and border rank, and to develop tests that bound the rank and border rank from below. For the border rank, such tests are in the form of polynomials that vanish on $\tilde{\sigma}_r$. For rank the study is more complicated. All lower bounds for rank that I am aware of arise from first proving a lower bound on border rank, and then taking advantage of special structure of a particular tensor to show its rank is higher than its border rank.

Perhaps the most important tensor for this study is the matrix multiplication tensor, where one considers matrix multiplication

$$M_n : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$$

as a tensor $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} = A \otimes B \otimes C$. In [6], G. Ottaviani and I proved the bound $R(M_n) \geq 2n^2 - n$ by finding polynomials that vanished on $\hat{\sigma}_{2n^2 - n - 1}$ and showing these polynomials did not vanish on $M_n$. At the same time we found additional polynomials that vanished on $\hat{\sigma}_{2n^2 - k}$ for $k = n, \ldots, 2$, but these polynomials also vanished on $M_n$. However we did not know whether or not these additional polynomials were identically zero. The motivation for this paper was to show these additional polynomials are in fact not identically zero. To do this I

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write down an explicit sequence tensors on which the polynomials do not vanish, see Theorem 1.2. These are the first proven nontrivial polynomials for border rank in $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^m$ beyond $2m - \sqrt{m}$. Since matrix multiplication satisfies these polynomials, the result raises the intriguing possibility that the border rank of matrix multiplication could be far less than I had previously expected.

My first hope had been to use the tensors of [1], as they had been shown to have high rank, but it turns out, see Proposition 1.5 below, that they have low border rank.

A referee for an earlier version of this paper wrote that [3] contains equations for $\hat{\sigma}_r$ in the range $m + 1 \leq r \leq 2m - 1$. This turned out to be erroneous - the author of [3] had only claimed the equations were potentially nontrivial in this range. Since the equations are presented indirectly, it was difficult to determine their non-triviality in general (see §5 for a discussion), but I do show:

**Proposition 1.1.** Let $\dim A = a$, $\dim B = \dim C = m$. Then Griesser’s equations of [3] for $\hat{\sigma}_r$ have the following properties:

1. They are trivial for $r = 2m - 1$ and all $a$.
2. They are trivial for $r = 2m - 2$, $a = m$ and $m \leq 4$.
3. Setting $m = n^2$, matrix multiplication $M_n$ fails to satisfy the equations for $r \leq \frac{3}{2}n^2 - 1$

   when $n$ is even and $r \leq \frac{3}{2}n^2 + \frac{1}{2} - 2$ when $n$ is odd, and satisfies the equations for all larger $r$.

I was unable to determine whether or not the equations are trivial for $r = 2m - 2$, $a = m$ and $m > 4$. If they are nontrivial for even $m$, they would give equations beyond the equations of [6].

In [3] the equations are only shown to be nontrivial on matrix multiplication for $r \leq n\left\lceil \frac{3n}{2} \right\rceil - 2$ and their non-triviality in general was not examined. Note that the bound for $n$ odd that (3) gives is $R(M_n) \geq \frac{3}{2}n^2 + \frac{1}{2}n - 1$, which equals Lickteig’s bound of [7] which held the “world record” for over twenty years.

The equations of [6] are special cases of equations obtained via Young flattenings defined in [5], which I now review. The classical flattenings (which date back at least to Macaulay and Sylvester) arise by viewing $T \in A \otimes B \otimes C$ as a linear map $T : B^* \to A \otimes C$, and taking the size $(r + 1)$ minors (i.e., the determinants of the $(r + 1) \times (r + 1)$-submatrices), which give equations for $r \leq \min(b,ac)$. These do not give all the equations and the idea behind Young flattenings is to pass from multi-linear algebra to linear algebra in more sophisticated ways. The particular Young flattening used in [6] may be described as follows:

Let $A^p A \subset A^\otimes p$ denote the skew-symmetric tensors. Let $Id_{A^p A} : A^p A \to A^p A$ denote the identity map, and consider, assuming $p \leq \left\lceil \frac{a}{2} \right\rceil$, the map $T \otimes Id_{A^p A} : B^* \otimes A^p A \to A^p A \otimes A \otimes C$.

Compose this map with the skew-symmetrization map to get a map

$$T_A^{\wedge p} : A^p A \otimes B^* \to A^{p+1} A \otimes C.$$  

(1.1)

If $R(T) = 1$, then the linear map $T_A^{\wedge p}$ has rank $\left(\frac{a-1}{p}\right)$. More precisely, if $T = a \otimes b \otimes c$, then the image of $T_A^{\wedge p}$ is the image of $A^{p} A \otimes a \otimes c$ under the skew-symmetrization map $A^p A \otimes A \otimes C \to A^{p+1} A \otimes C$. Thus if $R(T) \leq r$, then the size $\left(\frac{a-1}{p}\right)r + 1$ minors of $T_A^{\wedge p}$ will be zero. These minors are the equations used in [6] to bound the border rank of matrix multiplication.

Now let $a = b = c = m$, so when dealing with matrix multiplication, $m = n^2$. The Young flattenings (potentially) give the best lower bounds when $p = \left\lceil \frac{a}{2} \right\rceil$ so I examine them in that range. If $m = 2p + 1$, $\frac{m^2}{m-2} = 2m - 2 + \frac{1}{p+1}$ and if $m = 2p + 2$, then $\frac{m^2}{m-2} = 2m - 4 + \frac{4}{p+2}$, so the minors of (1.1) potentially give equations for $\hat{\sigma}_r$ up to $r = 2m - 2$ when $m$ is odd and $r = 2m - 4$ when $m$ is even. In [6] it was shown these equations are nontrivial (i.e., do not
vanish identically) when \( m \) is a square up to \( r = 2m - \sqrt{m} \) by showing they did not vanish on
the matrix multiplication tensor \( M_n \in \mathbb{C}^{n^3} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \).

**Theorem 1.2.** When \( m \) is odd and equal to \( 2p + 1 \), the maximal minors of (1.1) give nontrivial
equations for \( \delta, \gamma, \zeta \subseteq \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \), the tensors of border rank at most \( r \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \), up to
\( r = 2m - 2 \). They give equations up to \( r = 2m - 4 \) when \( m \) is even. The maximal minors do not
vanish on the explicit tensors \( T_m(\lambda) \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) of (2.1).

Thus the equations may be used to show that \( R(T) \geq 2m - 1 \) when \( m \) is odd and \( R(T) \geq 2m - 3 \) when \( m \) is even. These are the largest values of border rank we know how to test for.

**Corollary 1.3.** Let \( M_n \in \mathbb{C}^{n^3} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \) denote the matrix multiplication operator. Then \( M_n \)
satisfies nontrivial equations for the variety of tensors of border rank \( 2n^2 - n + 1 \).

In [1] (also see [10]), setting \( m = 2^k \), they give an explicit sequence of tensors \( T_m \in \mathbb{C}^{k+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) of rank \( 2m - 1 \), see (3.1), and explicit tensors \( T_{m+1} \in \mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^m \) of rank \( 3(m + 1) - k - 4 \), see §4. Their tensors may be defined over an arbitrary field.

**Proposition 1.4.** Let \( m = 2^k \). The tensors \( T_m \in \mathbb{C}^{k+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) of (3.1) have border rank \( m \),
i.e., \( R(T_m) = m \). \( R(T_{m+1}) = 2m - 1 \).

**Proposition 1.5.** Let \( m = 2^k \). The tensors \( T'_{m+1} \in \mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1} \otimes \mathbb{C}^{m+1} \) of §4 satisfy \( m + 2 \leq R(T'_{m+1}) \leq 2(m + 1) - 2 - k < R(T'_{m+1}) = 3(m + 1) - 4 - k \).

I expect the actual border rank to be close to the lower bound as many of the Young flattening
equations vanish, even in the \( p = 1 \) case.

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suggestions, in particular a referee pointed out a much simpler proof of the border rank of the
tensors in [1] than the one I had given.

2. Proof of Theorem 1.2

Let \( a_{-p}, \ldots, a_p \) be a basis of \( A = \mathbb{C}^{2p+1}, b_1, \ldots, b_m \) a basis of \( B = \mathbb{C}^m \), and \( c_1, \ldots, c_m \) a basis
of \( C = \mathbb{C}^m \).

Let \( \lambda_{i,u} \) be numbers satisfying open conditions to be specified below. (They may be chosen
to be e.g., \( \lambda_{i,u} = 2^{2i+u} + 2u \).) Consider

\[
T_{m,p}(\lambda) := \sum_{j=-p}^{1} a_j \otimes \left( \sum_{\alpha=1}^{m-p-j-1} \lambda_{-j,\alpha} b_{j+p+1+\alpha} \otimes c_\alpha \right) + \sum_{j=0}^{p} a_j \otimes \left( \sum_{\beta=1}^{m-j} b_\beta \otimes c_{j+\beta} \right)
\]

When \( m = 2p + 1 \), write \( T_m(\lambda) = T_{2p+1,p}(\lambda) \).

For example, in matrices, when \( p = 1 \) and \( m = 3 \)

\[
T_3(\lambda) = a_{-1} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_{1,1} & 0 \\ 0 & \lambda_{1,2} & 0 \end{pmatrix} + a_0 \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + a_1 \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

Write \( T = \sum a_j \otimes X_j \), where \( X_j \in B \otimes C \), and use \( X_0 = \sum_{\alpha=1}^{m} b_\alpha \otimes c_\alpha : B^* \to C \) to identify \( C 
with \( B^* \), so \( X_0 \) becomes the identity matrix in \( \mathfrak{gl}(B) \), the space of endomorphisms of \( B \). Let \( \{ [X_i, X_j], i, j \in \{-p, \ldots, -1, 1, \ldots, p \} \} \) denote the \( 2mp \times 2mp \) block matrix, whose \((i, j)\)-th block is
the commutator \( [X_i, X_j] = X_i X_j - X_j X_i \). By [4, p. 4], (1.1) is injective if and only if

\[
\det_{2mp}([X_i, X_j]) \neq 0.
\]
Remark 2.1. Despite the simplicity of the equation (2.2), I do not know how to prove it is related to border rank other than by expressing (1.1) in coordinates, making a choice of \( a_0 \), and applying elementary identities regarding determinants. It would be desirable to have a direct explanation.

The choice of bases and \( X_0 \) gives a grading to \( \mathfrak{gl}(B) \). That is, one has a vector space decomposition \( \mathfrak{gl}(B) = \bigoplus_{j=m}^{m} \mathfrak{gl}(B)_j \), such that the commutators satisfy \( \mathfrak{gl}(B)_i \mathfrak{gl}(B)_j \subset \mathfrak{gl}(B)_{i+j} \). Here \( \mathfrak{gl}(B)_j \) consists of the matrices that are zero except on the \( j \)-th diagonal (with \( j = 0 \) being the main diagonal). With this grading, taking \( T = T_{m,p}(\lambda), X_j \in \mathfrak{gl}(B)_j \). In particular, omitting the zero index, \( [X_i, X_j] \) is an \( m \times m \) matrix that is zero if \( i, j \) are greater than zero, and otherwise zero except for the \((i + j)\)-th diagonal, all of whose entries are nonzero as long as for each fixed \( i \), the \( \lambda_{i,u} \) are distinct as \( u \) varies. Writing the \( 2mp \times 2mp \) matrix, ordered \( p, p-1, \ldots, 1, -1, \ldots, -p \), as four equal size square blocks, the first block, consisting of the upper left \( mp \times mp \) submatrix, is zero, so the determinant of \(([X_i, X_j])\) is, up to sign, the square of the determinant of the lower left \( mp \times mp \) submatrix.

I will show that this determinant may be written as a product of smaller determinants of sizes \( 1, 2, \ldots, p-1, p, \ldots, p-1, p-2, \ldots, 2, 1 \).

For example, when \( p = 2 \) and \( m = 5 \) we get (blocking 2 1, 1 1, 2 1 for both rows and columns) the lower left block consists of four smaller blocks which are:

\[
[X_2, X_{-1}] = \begin{pmatrix}
0 & \lambda_{1,2} & 0 & 0 & 0 \\
0 & 0 & \lambda_{1,3} - \lambda_{1,1} & 0 & 0 \\
0 & 0 & 0 & \lambda_{1,4} - \lambda_{1,2} & 0 \\
0 & 0 & 0 & 0 & -\lambda_{1,3}
\end{pmatrix}
\]

\[
[X_2, X_{-2}] = \begin{pmatrix}
\lambda_{2,1} & 0 & 0 & 0 & 0 \\
0 & \lambda_{2,2} & 0 & 0 & 0 \\
0 & 0 & \lambda_{2,3} - \lambda_{2,1} & 0 & 0 \\
0 & 0 & 0 & -\lambda_{2,2} & 0 \\
0 & 0 & 0 & 0 & -\lambda_{2,3}
\end{pmatrix}
\]

\[
[X_1, X_{-1}] = \begin{pmatrix}
\lambda_{1,1} & 0 & 0 & 0 & 0 \\
0 & \lambda_{1,2} - \lambda_{1,1} & 0 & 0 & 0 \\
0 & 0 & \lambda_{1,3} - \lambda_{1,2} & 0 & 0 \\
0 & 0 & 0 & \lambda_{1,4} - \lambda_{1,3} & 0 \\
0 & 0 & 0 & 0 & -\lambda_{1,4}
\end{pmatrix}
\]

\[
[X_1, X_{-2}] = \begin{pmatrix}
\lambda_{2,1} & 0 & 0 & 0 & 0 \\
0 & \lambda_{2,2} - \lambda_{2,1} & 0 & 0 & 0 \\
0 & 0 & \lambda_{2,3} - \lambda_{2,2} & 0 & 0 \\
0 & 0 & 0 & \lambda_{2,4} - \lambda_{2,3} & 0 \\
0 & 0 & 0 & 0 & -\lambda_{2,4}
\end{pmatrix}
\]

In this case the determinant of the lower left block is

\[
(\lambda_{2,1}) \det \begin{pmatrix}
\lambda_{1,2} & \lambda_{1,1} \\
\lambda_{2,2} & \lambda_{2,1}
\end{pmatrix} \det \begin{pmatrix}
\lambda_{1,3} - \lambda_{1,1} & \lambda_{1,2} - \lambda_{1,1} \\
\lambda_{2,3} - \lambda_{2,1} & \lambda_{2,2} - \lambda_{2,1}
\end{pmatrix} \\
\cdot \det \begin{pmatrix}
\lambda_{1,4} - \lambda_{1,2} & -\lambda_{2,2} \\
\lambda_{1,3} - \lambda_{1,2} & \lambda_{2,3} - \lambda_{2,2}
\end{pmatrix} \det \begin{pmatrix}
-\lambda_{1,4} & -\lambda_{2,3} \\
\lambda_{1,4} - \lambda_{1,3} & -\lambda_{2,1}
\end{pmatrix} (-\lambda_{1,4}).
\]

Returning to the general case, consider the first column of the \( mp \times mp \) matrix \(([X_i, X_{-k}])\), \( 1 \leq i, k \leq p \). All the entries are zero except the first entry of the lowest block, i.e., the entry in the slot \(((p-1)m + 1, 1)\), which is \( \lambda_{p,1} \).
Now consider the second column. There are two nonzero entries - the first entry of the second lowest block, which is \( \lambda_{p-1,2} \), and the second entry of the last block, which is \( \lambda_{p,2} \). The \((m+1)\)-st column (first column of the second block) also has two nonzero entries, and they occur at the same heights, the entries are respectively \( \lambda_{p-1,1} \) and \( \lambda_{p,1} \). Thus these two columns contribute
\[
\det\left( \begin{array}{ccc} \lambda_{p-1,2} & \lambda_{p-1,1} \\ \lambda_{p,2} & \lambda_{p,1} \end{array} \right)
\]
to the determinant.

Consider the third column. If \( p > 2 \), there are three nonzero entries, the first entry of the third to last block, the second entry of the second to last block, and the third entry of the last block. The second column of the second block and the third column of the third block all have the same nonzero entries. The result is a contribution of
\[
\det\left( \begin{array}{ccc} \lambda_{p-2,3} & \lambda_{p-2,2} & \lambda_{p-2,1} \\ \lambda_{p-1,3} & \lambda_{p-1,2} & \lambda_{p-1,1} \\ \lambda_{p,3} & \lambda_{p,2} & \lambda_{p,1} \end{array} \right)
\]
to the determinant.

In general, considering the \( i \)-th column, for \( i \leq p \), there is a contribution of the determinant of an \( i \times i \) matrix whose \((s,t)\)-th entry is \( \lambda_{p-i+1+s,i+t} \), or \( \lambda_{p-i+1+s,i+t} - \lambda_{p-i+1+s,i+t+1} \), or \( -\lambda_{p-i+1+s,i+t+1} \). For \( i \geq p \) one gets \( p \times p \) matrices until they start shrinking in size.

In all cases, for any given minor of size \( f \) that appears, it will have a unique term with coefficient plus or minus one on \( \Pi_{s=1}^{f} \lambda_{p-i+1-s,i+s} \), so for a generic choice of \( \lambda \) it will not vanish. Thus for a generic choice no minors will vanish, which means that their product, the determinant, will not vanish either, proving the theorem. To have an explicit matrix, one could take e.g., \( \lambda_{i,j} = 2^{i+j} + 2^{j} \) to assure a single monomial in each minor will dominate the expression.

3. THE TENSORS \( T_m \) OF [1]

I restrict to the case \( m = 2^k \) because the other cases are similar only padded with zeros. In [1] they define tensors \( T_m \in \mathbb{C}^{k+1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m = A \otimes B \otimes C \) by
\[
(3.1) \quad T_m := a_0 \otimes \left( \sum_{\beta=1}^{m} b_{\beta} \otimes c_{\beta} \right) + \sum_{j=1}^{k} a_j \otimes \left( \sum_{\alpha=1}^{2^{j-1}} b_{\alpha} \otimes c_{m-2^\alpha-1+1} \right)
\]
Here I have changed the indices slightly from [1].

For example, when \( k = 3 \), in matrices, this is:
\[
T_8(A^*) = \begin{pmatrix} a_0 & a_0 & a_0 \\ a_3 & a_0 & a_0 \\ a_2 & a_3 & a_0 \\ a_1 & a_2 & a_3 \end{pmatrix}.
\]

If we reorder the basis of \( B \) and write the tensor as
\[
(3.2) \quad T_m = a_0 \otimes \left( \sum_{\beta=1}^{m} \tilde{b}_{m-\beta} \otimes c_{\beta} \right) + \sum_{j=1}^{k} a_j \otimes \left( \sum_{\alpha=1}^{2^{j-1}} \tilde{b}_{m-\alpha} \otimes c_{m-2^\alpha-1+1} \right)
\]
We see this is a specialization of the multiplication tensor in \( \mathbb{C}[X]/(X^m) \) whose border rank is \( m \) (see, e.g., [2, Ex. 15.20]). This proves Proposition 1.4.
4. The tensors $T_{m+1}'$ of [1]

In [1], they also define tensors in $\mathbb{C}^{m+1} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m+1}$ by enlarging the matrices to have size $m \times (m+1)$ and adding vectors in the last column. For example, when $k = 3$ (so $m = 8$), one gets the $8 \times 9$ matrix

$$T_9'(A^*) := \begin{pmatrix} a_0 & a_0 & \cdots & a_0 & a_4 \\ a_3 & a_3 & \cdots & a_3 & a_7 \\ a_2 & a_2 & \cdots & a_2 & a_8 \\ a_1 & a_1 & \cdots & a_1 & a_0 \end{pmatrix}$$

which they express as a tensor in $\mathbb{C}^{m+1} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m+1}$ by adding zeros. These tensors have rank close to $3m$, to be precise $R(T'_m) = 3m - 2H(m - 1) - \lfloor \log_2(m - 1) \rfloor - 2$, where $H(m)$ is the number of 1’s in the binary expansion of $m$, so the rank is best if $m - 1 = 2^k$, in which case $R(T'_{2^k+1}) = 3(2^k + 1) - 4 - k$. The border rank is smaller. Write $T'_{m+1} = T_m + T''_m$ where $T''_m = (a_{k+1} \otimes b_1 + a_{k+2} \otimes b_2 + \cdots + a_m \otimes b_{m-k}) \otimes e_{m+1}$. Thus $R(T'_{m+1}) \leq R(T_m) + R(T''_m) = m + m - k$. One obtains the lower bound of $m + 2$ for the border rank (as opposed to the trivial $m + 1$) because the map $T''_m$ has a kernel of size $2^{k-1} = \frac{m-1}{2}$. Since this kernel is still quite large, I expect the actual border rank to be close to the lower bound.

5. The equations of [3]

Given $T = \sum_{j=0}^{a-1} a_j \otimes X_j$ with $a_j$ a basis of $A$, $X_j \in B \otimes C$, $\dim A = a$ and $\dim B = \dim C = m$, assume $X_0$ is of full rank and use it to identify $C$ with $B^*$. The equations in [3] are stated as: if the border rank of $T$ is at most $r$, with $m + 1 \leq r \leq 2m - 1$, then the space of endomorphisms $\langle \langle X_1, X_2, \ldots, [X_1, X_{a-1}] \rangle \rangle \subset \mathfrak{sl}(B)$ is such that there exists $E \in G(2m - r, B)$, with $\dim(\langle \langle X_1, X_2, \ldots, [X_1, X_{a-1}] \rangle \rangle(E)) \leq r - m$. (Here $\langle \langle \ldots \rangle \rangle$ denotes the linear span and $G(k, B)$ the Grassmannian of $k$ planes in $B$.) Compared with the equations (2.2), here one is just examining the last block column of the matrix appearing in (2.2), but one is extracting apparently more refined information from it.

Assuming $T$ is sufficiently generic, we may choose $X_1$ to be diagonal with distinct entries on the diagonal (a general element of $\mathfrak{sl}(B)$, the space of traceless endomorphisms, is diagonalizable with distinct eigenvalues), and this is a generic choice of $X_1$. Let $\mathfrak{s}(B)_R$ denote the matrices with zero on the diagonal (the sum of the root spaces). Then $ad(X_1) : \mathfrak{s}(B)_R \to \mathfrak{s}(B)_R$, given by $Y \mapsto [X, Y]$, is a linear isomorphism, and $ad(X_1)$ kills the diagonal matrices. Write $U_j = [X_1, X_j]$, so the $U_j$ will be matrices with zero on the diagonal, and by picking $T$ generically we can have any such matrices, and this is the most general choice of $T$ possible, so if the equations vanish for a generic choice of $U_j$, they vanish identically.

Proof of Proposition 1.1. Proof of (1): In the case $r = 2m - 1$, so $r - m = m - 1$ and $a \leq m + 1$ the equations are trivial as we only have $a - 2 \leq m - 1$ linear maps. When $a \geq m + 2$ a naïve dimension count makes it possible for the equations to be non-trivial, the equations are that $\dim(U_{2v}, \ldots, U_{a-1}v) \leq m - 1$. However, with our normalizations of $X_0 = I_d$ and $X_1$ diagonal with distinct entries on the diagonal, taking $v = (1, 0, \ldots, 0)^T$ (the superscript $T$ denotes transpose), the $U_jv$ will be contained in the hyperplane of vectors with their first entry zero. Since we only made genericity assumptions, we conclude.
Proof of (2): In the case \( r = 2m - 2 \), the equations will be nontrivial if and only if there exist \( U_2, \ldots, U_{m-1} \) such that for all linearly independent \( v, w \) \( \dim\langle U_2v, \ldots, U_{m-1}v, U_2w, \ldots, U_{m-1}w \rangle \geq m-1 \). For \( a = m \), we saw we could have \( U_2v, \ldots, U_{m-1}v \) linearly independent, so the non triviality condition is that for some \( j, U_jw \notin \langle U_2v, \ldots, U_{m-1}v \rangle 
\).

First observe that \( U_j, w \in \langle U_2v, \ldots, U_{m-1}v \rangle \) mod \( U_j \hat{v} \) (where \( \hat{v} \) is the line determined by \( v \)) means \( w = \sum_{k \neq j} a_{j,k} U_j^{-1} U_k v \) for some constants \( a_{j,k} \). (We are working with generic \( U_j \) so we may assume they are invertible.) Thus we must have \( v \) and constants \( s_{i,j}, t_{i,j} \), such that
\[
s_{i,j} \sum_{k \neq j} a_{j,k} U_j^{-1} U_k v = t_{i,j} \sum_{l \neq i} a_{l,i} U_l^{-1} U_i v, \]
i.e., \( U_2, \ldots, U_{m-1} \) must be such that there exist constants \( s_{i,j}, t_{i,j} \) for \( i < j \), and \( a_{j,k} \) for \( j \neq k \) such that
\[
\det(s_{i,j} \sum_{k \neq j} a_{j,k} U_j^{-1} U_k - t_{i,j} \sum_{l \neq i} a_{l,i} U_l^{-1} U_i) = 0.
\]

When \( m = 4 \), the \( s, t \) are irrelevant and we need \( a_{2,3} U_2^{-1} U_3 v = a_{3,2} U_3^{-1} U_2 v \), i.e., that for some choice of \( [a_{2,3}, a_{3,2}] \in \mathbb{P}^1 \), the linear map \( a_{2,3} U_2^{-1} U_3 - a_{3,2} U_3^{-1} U_2 \) has a kernel. But every \( \mathbb{P}^1 \) of matrices intersects the hypersurface \( \det m = 0 \) so we conclude.

**Remark 5.1.** I expect the equations are non-trivial for \( m \geq 5 \) but I was unable to show this, even for \( m = 5 \). The \( r = 2m - 1 \) case shows that one should be cautious. Consider the \( m = 5 \) case. The equations would be trivial if for all \( U_2, U_3, U_4 \in \mathfrak{s}l(B)_R \), one could choose \( ([a_{2,3}, a_{2,4}], [a_{2,3}, a_{2,4}]], [s_{2,3}, t_{2,3}], [s_{2,4}, t_{2,4}]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) such that the linear maps \( s_{2,3} (a_{2,3} U_2^{-1} U_3 + a_{2,4} U_2^{-1} U_4) - t_{2,3} (a_{2,3} U_2^{-1} U_3 + a_{2,4} U_2^{-1} U_4) \) and \( s_{2,4} (a_{2,3} U_2^{-1} U_3 + a_{2,4} U_2^{-1} U_4) - t_{2,4} (a_{2,3} U_2^{-1} U_3 + a_{2,4} U_2^{-1} U_4) \) have a common kernel. If we consider the variety \( \Sigma_m \subset G(m - 3, \mathbb{C}^{m^2}) \) defined by
\[
\Sigma_m := \{ E \in G(m - 3, \mathbb{C}^{m^2}) \mid \exists v \in V \setminus 0 \text{ such that } e.v = 0 \forall e \in E \},
\]
then \( \dim \Sigma_m = (m - 1) + (m - 3)(m^2 - m - (m - 3)) \) (as for each point in \( \mathbb{P} V \) there is an \( m^2 - m \) dimensional space of endomorphisms with the line in the kernel, and we have the Grassmannian of \( m - 3 \) planes in that space of endomorphisms). So in the \( m = 5 \) case a general four dimensional subvariety of the Grassmanian will fail to intersect \( \Sigma_5 \), but our four dimensional subvariety is not general.

Proof of (3): Consider matrix multiplication \( M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} = A \otimes B \otimes C \). With a judicious choice of bases, \( M_n(A) \) is block diagonal
\[
\begin{pmatrix}
x & \\ & \ddots & \\ & & x
\end{pmatrix}
\]
where \( x = (x^i_j) \) is \( n \times n \). In particular, the image is closed under brackets. Choose \( X_0 \) so it is the identity. We may not have \( X_1 \) diagonal with distinct entries on the diagonal, the best we can do is for \( X_1 \) to be block diagonal with each block having the same \( n \) distinct entries. For a subspace \( E \) of dimension \( 2m - r = dn + e \) (recall \( m = n^2 \)) with \( 0 \leq e \leq n - 1 \), the image of a generic choice of \([X_1, X_2], \ldots, [X_1, X_{n^2-1}]\) applied to \( E \) is of dimension at least \( (d + 1)n \) if \( e \geq 2 \), at least \( (d + 1)n - 1 \) if \( e = 1 \) and \( dn \) if \( e = 0 \), and equality will hold if we choose \( E \) to be, e.g., the span of the first \( 2m - r \) basis vectors of \( B \). (This is because the \( [X_1, X_j] \) will span the entries of type (5.1) with zeros on the diagonal.) If \( n \) is even, taking \( 2m - r = \frac{n^2}{2} + 1 \), so \( r = \frac{3n^2}{2} - 1 \), the image occupies a space of dimension \( \frac{n^2}{2} + n - 1 > \frac{n^2}{2} - 1 = r - m \). If one takes \( 2m - r = \frac{n^2}{2} \), so \( r = \frac{3n^2}{2} \), the image occupies a space of dimension \( \frac{n^2}{2} > r - m = \frac{n^2}{2} + \frac{n}{2} - 1 \), and taking \( 2m - r = \frac{4n^2}{2} - \frac{n}{2} + 1 \) the
image can have dimension \( \frac{n^2}{2} - \frac{n}{2} + (n - 1) = r - m \), so the equations vanish for this and all larger \( r \). Thus Griesser’s equations for \( n \) odd give Lickteig’s bound \( R(M_n) \geq \frac{3n^2}{2} + \frac{3n}{2} - 1 \). 

\[ \square \]

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