Quantally fed steady-state domain distributions in Stochastic Inflation

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Within the framework of stochastic inflationary cosmology we derive steady-state distributions $P_c(V)$ of domains in comoving coordinates, under the assumption of slow-rolling and for two specific choices of the coarse-grained inflaton potential $V(\Phi)$. We model the process as a Starobinsky-like equation in $V$-space plus a time-independent source term $P_w(V)$ which carries (phenomenologically) quantum-mechanical information drawn from either of two known solutions of the Wheeler-De Witt equation: Hartle-Hawking’s and Vilenkin’s wave functions. The presence of the source term leads to the existence of nontrivial steady-state distributions $P_w^c(V)$. The relative efficiencies of both mechanisms at different scales are compared for the proposed potentials.

Since the differential microwave radiometer (DMR) mounted on the COBE satellite first detected temperature anisotropies in the cosmic background radiation, we have the possibility to directly probe the initial density perturbation. The fact that the resulting energy density fluctuations ($\delta\rho/\rho \approx 5 \times 10^{-5}$) fit the scaling spectrum predicted by the inflation model, suggests that they had indeed their origin in the quantum fluctuations of the “inflaton” scalar field during the inflationary era. Although this problem is in principle of a quantum-mechanical nature, the fact that under certain conditions (which are made precise in Ref. [1]) the inflaton field can be considered as classical largely simplifies the approach, by allowing a Langevin-like stochastic treatment. Stochastic Inflation branched off and grew up from within the context of chaotic inflation [2]. It was Starobinsky [3] the first one to derive, from a Langevin-like equation for the dynamics of the coarse-grained inflaton field $\Phi$ and in the absence of back-reaction from the metric, a Fokker-Planck equation for the transition probabilities $P_c(\Phi, t|\Phi', t')$ in comoving coordinates. The latter provide us with statistical information about the relative number of spatial domains that evolve in a time interval $t-t'$ from having a typical value $\Phi'$ (assumed constant throughout the domain) towards a new configuration with a typical value $\Phi$. This approach is purely phenomenological and keeps very little information about its quantum origin, since the coarse-graining procedure erases quantum correlations. Hence, the issue of integrating quantum-mechanical information to the inflationary evolution is still a pending assignment.

In a recent paper [4] we have offered a phenomenologically drawn example that takes into account—if not properly back-reaction—at least the possibility that Hubble’s constant $H$ depend on $\Phi$. Our approach disregarded instead the boundary conditions at Planck’s and exit-of-inflation $\Phi$ regimes and, in particular, domain injection from the purely quantum-mechanical phase. As we know, the Wheeler-De Witt (WDW) equation is a possible way to describe the fully quantum-mechanical behaviour of the Universe. However, this equation has infinitely many solutions: hence, there have been several attempts to solve it by starting from different initial conditions, each providing a different mechanism whereby domains with a typical size $a = H^{-1}$ can enter the inflationary stage. Two very popular ones among them are quantum creation from “nothing” [5] and the no boundary proposal [6].

It is our aim in this work to investigate the inflationary stage in the light of the information provided by solutions to the WDW equation. Our (still phenomenological) approach will be to enlarge the scope of Starobinsky’s equation by adding a time-independent source term. We shall take this (extrinsic) domain-injection probability per unit time to be proportional to the squared modulus of the aforementioned solutions. Moreover, we shall assume that during the inflationary phase the slow-rolling condition is satisfied. Under these assumptions, we obtain phenomenologically meaningful stationary domain distributions in $V$-space for two physically appealing forms of the inflaton potential $V(\Phi)$.

For the role that quantum cosmology is intended to play in this work (i.e. to act as an effective source of homogeneous horizon-sized domains) it suffices to consider the minisuperspace version of the WDW equation [7]:

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\[-(3\pi M_p^2)^{-1} \frac{\partial^2}{\partial a^2} + \frac{3\pi M_p^2}{4} a^2 + (2\pi a)^{-3} \frac{\partial^2}{\partial \Phi^2} - 2\pi^2 a^4 V(\Phi) \right] \Psi(a, \Phi) = 0 \tag{1}\]

This is the cornerstone for a quantum-mechanical description of the pre-inflationary era, and is derived under the assumption that the metric is of the Friedmann-Robertson-Walker (FRW) type

\[ ds^2 = \sigma^2 [N^2(t)dt^2 - a^2(t)d\Omega_3^2] \tag{2}\]

and that the only quantum variables in this stage are the FRW scale factor $a$ and the inflaton field $\Phi$. In eq. \(2\) $N(t)$ is the lapse function, $d\Omega_3$ the metric on a unit three-sphere and $\sigma^2 = 2G/3\pi$ a normalizing factor chosen for convenience. Equation \(2\) arises as a constraint when varying the Hamiltonian with respect to $N$.

The asymptotic solution of eq. \(2\) with the boundary condition $\Psi(a \to \infty) \to 0$ was obtained by Vilenkin [5], in analogy with the decay of a metastable state. Taking $a = H^{-1} \langle \Phi \rangle$, its squared modulus

\[ P_v(V) = \exp(-3M_p^4/8V) \tag{3}\]

describes the nucleation of universes (domains in our phenomenological model) with a given value of the inflaton potential $V$ in the inflationary phase through quantum tunneling, and the corresponding process is known as “quantum creation from nothing”. On the other hand, the celebrated Hartle-Hawking (HH) solution [3] leads to

\[ P_h(V) = \exp(3M_p^4/8V). \tag{4}\]

We find that eqs. \(3\) and eq. \(4\) differ in the sign of the exponential. In the following—and unless otherwise specified—we shall denote the domain-injection probability arising from any solution to the WDW equation as $P_w(V)$. Obviously, given a specific functional form $V(\Phi)$ one immediately obtains $P_w(\Phi)$.

Now we want to incorporate this information into the equation for $P_c(V, t)$ (the probability distribution of domains in comoving coordinates during the inflationary stage). After choosing units such that $M_p = 1$, changing variables in the Fokker-Planck equation for $P_c(\Phi, t)$ and recalling that $\frac{\partial}{\partial \Phi} = V' \frac{\partial}{\partial V}$ (the prime denotes $\frac{d}{dV}$), one obtains

\[ \frac{\partial}{\partial t} P^w_c(V, t) = \sqrt{\frac{2}{3\pi} V'} \frac{\partial}{\partial V} \left[ V^{\frac{3}{2}(1-\beta)} V' \frac{\partial}{\partial V} \left( V^{\frac{1}{2}+\beta} P^w_c \right) + \frac{3V'}{8V^{1/2}} P^w_c \right] + C^w P^w_c, \]

where the time-independent domain distribution $P^w_c(V)$ acts as a source term that accounts for domain injection into the inflationary phase, and $C^w$ is a (unknown) constant. $\beta = 1/2$ for the Stratonovich prescription and $\beta = 0$ for Itô’s one.

In this work we shall restrict ourselves to the stationary equation which, by defining

\[ G^w(V) = V^{\frac{3}{2}(1-\beta)} V' \frac{d}{dV} \left( V^{\frac{1}{2}+\beta} P^w_c \right) + \frac{3V'}{8V^{1/2}} P^w_c, \tag{5}\]

can be rewritten as

\[ \frac{d}{dV} G^{sol}_1(V) = -C^w \frac{P^w_c(V)}{V'}, \tag{6}\]

with $C^w = 6\sqrt{6\pi}C^w$, and $G^{sol}(V)$ is the function of $V$ solution of the last equation. Once we solve eq. \(6\) we can obtain $P^w_c(V)$ from eq. \(5\), setting $\beta = 1$ as in Ref. [3]:

\[ P^w_c(V) = K_o V^{-3/2} \exp \left( \frac{3}{8V} \right) \left[ B + \int dV' \exp \left( -\frac{3}{8V'} \frac{G^{sol}_1(V)}{V'} \right) \right]. \tag{7}\]

Here $B$ is an integration constant and $K_o = V^{-1}_c - 1$ (with $V_c = V(\Phi_c)$) comes from normalizing at the exit-of-inflation value $\Phi_c$ of the inflaton field.

In the remaining of this letter—and with the aim to illustrate the procedure—we derive the steady-state domain distributions $P^w_c(V)$ and $P^w_h(V)$ arising from both domain-injection mechanisms, for two specific choices of $V(\Phi)$.

1. Let us first consider the case of a quadratic potential $V(\Phi) = \frac{m^2}{2} \Phi^2$, so that $V' = (2m^2V)^{1/2}$:
(a) If the domains are produced by quantum tunneling, \( P_w(V) \) is Vilenkin's distribution eq.(3). For simplicity we choose \( C_w^1 = -1 \). This gives

\[
G_{\text{sol}}^w(V) = \frac{1}{m} \left[ \frac{1}{\sqrt{2V}} \exp \left( \frac{3}{8V} \right) + \frac{1}{2} \sqrt{3\pi} \text{Erf} \left( \sqrt{\frac{3}{8V}} \right) \right],
\]

where \( \text{Erf} \) is the error function. The general solution of eq.(7) is

\[
P_v^w(V) = K_o \exp \left( \frac{3}{8V} \right) \left[ B_v^{(1)} - f_1(V) \right],
\]

where \( B_v^{(1)} \) is a constant of integration and

\[
f_1(V) = -\frac{1}{m} \int dV \exp \left( -\frac{3}{8V} \right) \left[ \frac{1}{\sqrt{2V}} \exp \left( -\frac{3}{8V} \right) + \frac{1}{2} \sqrt{3\pi} \text{Erf} \left( \sqrt{\frac{3}{8V}} \right) \right].
\]

From the requirement that \( P_v^w(V) \) be \textit{positive in the inflationary regime} it follows that \( B_v^{(1)} \geq |\min[f_1(V)]| \). We need another condition in order to fully specify the integration constants. Considering our ignorance, we will require both solutions to \textit{agree at the Planck scale} i.e. that \( P_v^w(V_p) = P_v(V_p) \), where \( V_p = V(\Phi_p) \). Then

\[
B_v^{(1)} = \frac{e^{-81}}{V_e - 1} + f_1(V_p).
\]

(b) In the case of a HH source term \[6\] and choosing \( C_h^1 = 1 \), it results

\[
G_{\text{sol}}^h(V) = -\frac{1}{m} \left[ \frac{1}{\sqrt{2V}} \exp \left( -\frac{3}{8V} \right) + \frac{1}{2} \sqrt{3\pi} \text{Erf} \left( \sqrt{\frac{3}{8V}} \right) \right].
\]

The steady-state domain distribution in comoving coordinates is then

\[
P_v^h(V) = K_o \exp \left( \frac{3}{8V} \right) \left[ B_h^{(1)} - g_1(V) \right],
\]

where

\[
g_1(V) = \frac{1}{m} \int dV \exp \left( -\frac{3}{8V} \right) \left[ \frac{1}{\sqrt{2V}} \exp \left( -\frac{3}{8V} \right) - \frac{1}{2} \sqrt{3\pi} \text{Erf} \left( \sqrt{\frac{3}{8V}} \right) \right]
\]

and \( B_h^{(1)} \) is an integration constant required now to be \( B_h^{(1)} \geq |\min[g_1(V)]| \). Imposing as before \( P_v^w(V_p) = P_v(V_p) \),

\[
B_h^{(1)} = \frac{1}{V_e - 1} + g_1(V_p).
\]

The functions \( f_1(V) \) and \( g_1(V) \)—describing the domain flow from the quantum sector towards the inflationary regime—must be evaluated numerically in this example.

2. The following is an exactly solvable example:

(a) Let us consider in equation \[8\] the choice \( P_v(V)/V' = V^{-2} \), and choose also \( C_t^w = -1 \). The resulting scalar potential is

\[
V(\Phi) = \frac{3}{8 \ln \left( \frac{3}{8} (\Phi_p - \Phi) \right)}.
\]
The condition $V(\Phi) \geq 0$ imposes the restriction $\Phi_0 - \Phi \geq 8/3$. Since, on the other hand $G^\text{sol}_c(V) = -V^{-1}$, it is possible to find $P^c_c(V)$ from equation (10):

$$P^c_c(V) = K_c V^{-3/2} \exp \left( \frac{3}{8V} \right)$$

$$\times \left[ B^{(2)}_c - \exp \left( -\frac{3}{8V} \right) V^{1/2} \left( \frac{2}{3} V - \frac{1}{2} \right) + \frac{\sqrt{6\pi}}{8} \text{Erf} \left( \sqrt{\frac{3}{8V}} \right) \right].$$

(10)

Here $B^{(2)}_c$ is again a constant ensuring that the distribution $P^c_c(V)$ be positive in the whole inflationary domain. By choosing $P^c_c(V_p) = P_c(V_p)$ and taking $V_p = 1$, it results to be

$$B^{(2)}_c = \frac{V_c}{(1-V_c)} + \frac{1}{40} \left[ 23 e^{3/8} - \sqrt{6\pi} \text{Erf} \left( \sqrt{\frac{3}{8}} \right) \right].$$

(b) When the source term is HH-like, choosing $P^h(V)/V' = V^{-2} \exp \left( \frac{3}{4V} \right)$—so that $G^\text{sol}_h(V) = \frac{4}{3} \exp \left( \frac{1}{3V} \right)$—the solution for $P^g_c$ (with $C^h_p = 1$) results:

$$P^h_c(V) = K_c V^{-3/2} \exp \left( \frac{3}{8V} \right)$$

$$\times \left[ B^{(2)}_h + \frac{1}{5} \exp \left( \frac{3}{8V} \right) V^{1/2} \left( \frac{8}{3} V^2 + \frac{2}{3} V + \frac{1}{2} \right) - \frac{\sqrt{6\pi}}{40} \text{Erf} \left( \sqrt{\frac{3}{8V}} \right) \right].$$

(11)

By choosing $P^c_c(V_p) = P_c(V_p)$,

$$B^{(2)}_h = \frac{V_c}{(1-V_c)} e^{-3/4} - \frac{1}{6} e^{-3/8} + \frac{\sqrt{6\pi}}{8} \text{Erf} \left( \sqrt{\frac{3}{8}} \right),$$

Figure 1 compares the ratio $Z(V) = P^c_c(V)/P^h_c(V)$ as a function of $V$ for the two cases, between the values $V_c \approx 0$ and $V_p = 1$. Although both curves are rather similar in form when looked in their own scale, $Z(V)$ for the logarithmic potential looks very flat when compared with the quadratic one. Moreover—except for $V < 0.1$—the quadratic potential predicts larger values of $Z(V)$. In fact, whereas for the quadratic potential and $V > 0.1$ it is $Z(V) > 1$ (with a maximum larger than 3 at $V \approx 0.3$)—thus implying a higher efficiency of the nucleation mechanism as compared to the HH one—for the logarithmic one it is always $Z(V) < 1$ (thus implying a higher efficiency of the HH mechanism) and this relative efficiency passes through a (less dramatic) minimum at $V \approx 0.2$.

Summing up, in this work we have studied the effect of the domain-injection mechanism on the (stationary) domain distribution at the end of the inflationary era, for two phenomenologically appealing proposals of inflaton potential forms: quadratic and logarithmic, respectively. The domain-injection mechanisms are provided in our approach by known solutions to the WDW equation. Although the ratio $Z(V)$ remains of order one and looks similar for both potential forms, it is concluded that the relative efficiency of each mechanism depends on the form of the inflaton potential.

This letter is a first step towards studying the possible appearance of stability in Stochastic Inflation due to quantum sources of inflationary domains, deserving the matter further research and debate. We are aware of the lack of naturalness of our approach, that combines such different descriptions as quantum cosmology and (gravitationally classical) stochastic inflation. The transition from quantum to classical behaviour in quantum cosmology, and particularly the conditions under which the wave function of the Universe becomes semiclassical, are far from being understood. Thus, considering the possibility that the exit of the quantum phase is not a well defined event, a stochastic interplay between both classical and quantum descriptions seems plausible.

[1] M. Bellini, H. Casini, R. Montemayor and P. Sisterna, Phys. Rev. D 54, 7172 (1996).
FIG. 1. The ratio \( Z(V) = \frac{P_v^c(V)}{P_h^c(V)} \) between the steady-state distributions arising from Vilenkin's and Hartle-Hawking's source terms, for the case of a quadratic potential (upper curve) and for a logarithmic one (lower curve).