In this study we use the conformable fractional reduced differential transform (CFRDTM) method to compute solutions for systems of nonlinear conformable fractional PDEs. The proposed method yields a numerical approximate solution in the form of an infinite series that converges to a closed form solution, which is in many cases the exact solution. We inspect its efficiency in solving systems of CFPDEs by working on four different nonlinear systems. The results show that CFRDTM gave similar solutions to exact solutions, confirming its proficiency as a competent technique for solving CFPDEs systems. It required very little computational work and hence consumed much less time compared to other numerical methods.

1. Introduction

Fractional partial differential equations (FPDEs) are mathematical tools employed to depict a broad variety of phenomena that arise in distinct areas of science and engineering like biology, physics, fluid mechanics, chemistry, and so on [1–4]. Recently investigators have centered their attention on studying the solution of partial differential equations of fractional order. However, till now not a single definition of fractional order derivative has been found [5]. Therefore, researchers raced to set definitions for fractional order derivatives and consequently many definitions, mostly complicated, were proposed such as the Riemann-Liouville, Caputo, Hadamard, and others [5]. However, in 2014 an intriguing perspective to the subject was presented by Khalil [6] who came up with the conformable fractional derivative, an extension of the classical simple limit definition of derivative we all know. This new definition unlocked a whole new chapter in the world of research of FPDEs, in which many recent works have been done. To fully grasp these problems and to be able to analyze those phenomena, it is a necessity to find solutions for those systems of FPDEs or CFPDEs. But then again, finding a numerical or analytical solution for those systems is a very tough task to achieve, and exact solutions for them are not easily obtained.

To tackle this problem, researchers worked on finding adequate methods for those systems of FPDEs. One of the greatest methods found by researchers is the RDTM. What distinguishes this method is that it provides us with analytical approximations, which in many cases are exact solutions, through convergent series with terms that are relatively easy to compute. Not to mention its ability in solving a huge variety of problems, which makes it a very significant mathematical tool. For example, in 2006, Bildik et al. [7] solved different partial differential equations using the RDM and ADM and performed a comparison with approximate solution and analytic solutions. Guptav [8] obtained the approximate solution of Benney-Lin equation through the RDTM and the HPM. Acan and Baleanu [9] handled three linear and nonlinear equations by CRDTM and CADM. For more applications of the present method, see [9–12] and some references cited therein. In this paper we propose the conformable fractional reduced differential transform method (CFRDTM) as a reliable method to solve systems of nonlinear CFPDEs.

This paper is arranged as follows. In the second section we demonstrate the properties and theorems of the CFRDTM, in addition to reviewing some of the properties of the conformable derivative which are needed in the following work. In Section 3, we solve 4 systems by using CFRDTM
to test its efficiency. In Section 4, we conclude this article by analyzing the results we have obtained briefly.

2. CFRDTM

In this section, we introduce the CFRDTM which we will be using to solve the system of conformable FPDEs in this paper. Throughout this section, let \( x \in \mathbb{R}^n \) and \( q \in (0, 1] \).

For more details about the definitions and properties given in this section, the reader can refer to [13–16] and the references therein.

**Definition 1.** Given a function \( u: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R} \), the conformable fractional partial derivative of \( u(x, t) \) of order \( q \) with respect to \( t \) is given as

\[
\frac{\partial^q_t u(x, t)}{\partial t^q} = \lim_{\varepsilon \to 0} \frac{u(x, t + \varepsilon t^q) - u(x, t)}{\varepsilon}
\]

provided that this limit exists as a finite number.

**Definition 2.** Let \( u(x, t) \) be an infinitely partially q-differentiable function near 0 with respect to \( t \) of order \( q \); then the CFRDT of \( u(x, t) \) is given as

\[
U^q_k(x) = \frac{1}{q^k k!} \left[ (\partial_t^q)^k u(x, t) \right]_{t=0}
\]

where \((\partial_t^q)^k u(x, t)\) resembles applying the conformable partial fractional derivative \( k \)-times, and \( U^q_k(x) \) is the CFRDT function.

**Definition 3.** Let \( U^q_k(x) \) be the CFRDT of \( u(x, t) \). Then the inverse CFRDT of \( U^q_k(x) \) is given as

\[
u(x, t) = \sum_{k=0}^{\infty} U^q_k(x) t^k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{q^k k!} \left[ (\partial_t^q)^k u(x, t) \right]_{t=0} t^k.
\]

**Definition 4.** The CFRDT of \( u(x, t) \) of the initial conditions is defined as

\[
U^q_k(x) = \begin{cases} \frac{1}{(q^k k!)} \left[ (\partial_t^q)^k u \right]_{t=0} & \text{if } q^k k \in \mathbb{Z}^+ \\ 0 & \text{if } q^k k \notin \mathbb{Z}^+ \end{cases}
\]

for \( k = 0, 1, 2, \ldots, (n/q - 1) \), where \( n \) is the order of CFDE.

**Theorem 5.** Let \( u(x, t), v(x, t) \), and \( w(x, t): \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R} \) be partially q-differentiable at a point \( t > 0 \) and \( a, b \in \mathbb{R} \). Then, the following is obtained.

1. If \( u(x, t) = v(x, t)w(x, t) \) then \( U^q_k(x) = \sum_{s=0}^{k} V^q_k(x, s)W^q_{k-s} \).
2. If \( v(x, t) = u^q_k(x, t) \) then \( V^q_k(x) = q(k + 1)U^q_k(x) \).
3. If \( u(x, t) = av(x, t) \pm bw(x, t) \) then \( U^q_k(x) = aV^q_k(x) \pm bW^q_k(x) \).

In general for \( u(x, t) = v_1(x, t)v_2(x, t) \cdots v_n(x, t) \) then we have

\[
U^q_k(x) = \sum_{k_1=0}^{k} \cdots \sum_{k_n=0}^{k} V^q_{1,k_1}V^q_{2,k_2} \cdots V^q_{n,k_n}V^q_{m,k_{n-1}}V^q_{m,k_{n-2}} \cdots V^q_{m,k_1}
\]

(4) If \( u(x, t) = t^m h(x) \) then \( U^q_k(x) = \delta(k - m/q)h(x) \), where

\[
\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}
\]

3. Applications

To demonstrate the excellent performance of this method, we apply it on four different systems in this section.

**Example 6.** Consider the nonlinear system of CFDEs

\[
\partial_t^q u = u^2 v - 2u + \frac{1}{4}(\partial_x^2 u + \partial_y^2 u)
\]

\[
\partial_t^q v = u - u^2 v + \frac{1}{4}(\partial_x^2 v + \partial_y^2 v)
\]

with initial conditions

\[
u(x, y, 0) = e^{-x-y};
\]

\[
v(x, y, 0) = e^{x+y}.
\]

By applying CFRDTM on (7), we obtain the two recurrence relations

\[
q(k+1)U^q_{k+1} = \sum_{i=0}^{k} \sum_{j=0}^{k} U^q_i \cdot U^q_j \cdot V^q_{k-i} - 2U^q_k + \frac{1}{4}(\partial_x^2 U^q_k + \partial_y^2 U^q_k)
\]

(9) \[
q(k+1)V^q_{k+1} = U^q_k - \sum_{i=0}^{k} \sum_{j=0}^{k} U^q_i \cdot V^q_j \cdot V^q_{k-i} + \frac{1}{4}(\partial_x^2 V^q_k + \partial_y^2 V^q_k),
\]

where the I.C (8) can be transformed at \( t = 0 \) to

\[
U^q_0 = e^{-x-y}
\]

(10) \[
V^q_0 = e^{x+y}.
\]

Now, substitute \( k = 0 \) into (9), one obtains

\[
U^q_1 = \frac{-1}{2q}e^{-x-y}
\]

(11) \[
V^q_1 = \frac{1}{2q}e^{x+y}.
\]
For \( k = 1 \) and \( k = 2 \), we get

\[
U^q_2 = \frac{1}{8q^2} e^{x-y}, \quad V^q_2 = \frac{1}{8q^2} e^{x+y}, \quad W^q_2 = \frac{1}{2^2 \cdot 2!q^2} e^{x+y},
\]

and

\[
U^q_3 = \frac{-1}{48q^3} e^{x-y}, \quad V^q_3 = \frac{1}{48q^3} e^{x+y}, \quad W^q_3 = \frac{1}{2 \cdot 3 \cdot 3!q^3} e^{x+y}.
\]

Carrying on the iterative calculations in a similar way for other values of \( k \), we obtain the general terms:

\[
U^q_n = \frac{(-1)^n}{2^n n!q^n} e^{-x-y},
\]

\[
V^q_n = \frac{1}{2^n n!q^n} e^{x+y}.
\]

Applying the inverse transformation of CFRDTM, the solution of (7) is given as

\[
\begin{align*}
\text{if } k = 1, \quad & u(x, y, t) = \sum_{k=0}^{\infty} U^q_k (x, y, t) t^k = e^{-x-y-t^q/2q}, \\
\text{if } k = 2, \quad & v(x, y, t) = \sum_{k=0}^{\infty} V^q_k (x, y, t) t^k = e^{x+y-t^q/2q}.
\end{align*}
\]

**Example 7.** Consider the following system of CFDEs

\[
\begin{align*}
\partial_t^q u &= -\partial_x u \partial_x v - \partial_y u \partial_y v - u \\
\partial_t^q v &= -\partial_x v \partial_x w + \partial_y v \partial_y w + v \\
\partial_t^q w &= -\partial_x w \partial_x u - \partial_y w \partial_y u - w,
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
u(x, y, 0) &= e^{x+y}; \\
w(x, y, 0) &= e^{-x+y}.
\end{align*}
\]

Applying theorems of CFRDTM on (16) we obtain the following system of recurrence relations:

\[
\begin{align*}
(k + 1) U^q_{k+1} &= -\sum_{r=0}^{k} \partial_x U^q_k \partial_x V^q_k - \sum_{r=0}^{k} \partial_y U^q_k \partial_y V^q_k - U^q_k \\
(k + 1) V^q_{k+1} &= -\sum_{r=0}^{k} \partial_x V^q_k \partial_x W^q_k + \sum_{r=0}^{k} \partial_y V^q_k \partial_y W^q_k + V^q_k \\
(k + 1) W^q_{k+1} &= -\sum_{r=0}^{k} \partial_x W^q_k \partial_x U^q_k - \sum_{r=0}^{k} \partial_y W^q_k \partial_y U^q_k - W^q_k,
\end{align*}
\]

where the I.C. can be transformed at \( t = 0 \) to

\[
\begin{align*}
U^q_0 &= e^{x+y} \\
V^q_0 &= e^{x-y} \\
W^q_0 &= e^{-x+y}.
\end{align*}
\]

Substituting \( k = 0 \) into (16) we obtain

\[
\begin{align*}
U^q_0 &= \frac{-1}{q} e^{x+y} \\
V^q_0 &= \frac{1}{q} e^{x-y} \\
W^q_0 &= \frac{-1}{q} e^{-x+y}.
\end{align*}
\]

Similarly for \( k = 1 \) and \( k = 2 \), we get

\[
\begin{align*}
U^q_2 &= \frac{1}{2q^2} e^{x+y} \\
V^q_2 &= \frac{1}{2q^2} e^{x-y} \\
W^q_2 &= \frac{1}{2q^2} e^{-x+y}.
\end{align*}
\]

Continuing the iterative calculations in a similar way for other values of \( k \), we obtain the general terms:

\[
\begin{align*}
U^q_n &= \frac{(-1)^n}{n!q^n} e^{x+y} \\
V^q_n &= \frac{1}{n!q^n} e^{x-y} \\
W^q_n &= \frac{(-1)^n}{n!q^n} e^{-x+y}.
\end{align*}
\]

Applying the inverse transformation of CFRDTM, the solution of (16) is given as

\[
\begin{align*}
u(x, y, t) &= \sum_{k=0}^{\infty} V^q_k (x, y, t) t^k = e^{-x-y-t^q/2q} \\
v(x, y, t) &= \sum_{k=0}^{\infty} V^q_k (x, y, t) t^k = e^{x+y-t^q/2q} \\
w(x, y, t) &= \sum_{k=0}^{\infty} W^q_k (x, y, t) t^k = e^{-x+y-t^q/2q}.
\end{align*}
\]
Example 8. Consider the following system of CFDEs
\[
\begin{align*}
\partial^q_t u + u \partial_x u + v \partial_y u &= \partial^2_{xx} u + \partial^2_{yy} u, \\
\partial^q_t v + u \partial_x v + v \partial_y v &= \partial^2_{xx} v + \partial^2_{yy} v,
\end{align*}
\]
subject to the initial conditions
\[
\begin{align*}
u(x, y, 0) &= x + y; \\
u(x, y, 0) &= x - y.
\end{align*}
\]
Applying theorems of CFRDTM on (25) we obtain the following system of recurrence relations:
\[
\begin{align*}
q(k + 1) U_{k+1}^q + \sum_{r=0}^{k} U_{k-r}^q \partial_x U_r^q + \sum_{r=0}^{k} V_{k-r}^q \partial_y U_r^q \\
= \partial^2_{xx} U_k^q + \partial^2_{yy} U_k^q,
\end{align*}
\]
where the I.C. can be transformed at \( t = 0 \) to
\[
\begin{align*}
U_0^q &= x + y \\
V_0^q &= x - y.
\end{align*}
\]
Substituting \( k = 0 \) into (25) we obtain
\[
\begin{align*}
U_0^q &= -\frac{2}{q} x \\
V_0^q &= -\frac{2}{q} y.
\end{align*}
\]
Similarly for \( k = 1, k = 2, k = 3, \) and \( k = 4 \) we get, respectively,
\[
\begin{align*}
U_2^q &= \frac{2}{q^4} (x + y) \\
V_2^q &= \frac{2}{q^4} (x - y), \\
U_3^q &= -\frac{4}{q^3} x \\
V_3^q &= -\frac{4}{q^3} y, \\
U_4^q &= \frac{4}{q^4} (x + y) \\
V_4^q &= -\frac{4}{q^4} (x - y),
\end{align*}
\]
and
\[
\begin{align*}
U_5^q &= -\frac{8}{q^5} x \\
V_5^q &= -\frac{8}{q^5} y.
\end{align*}
\]
Applying the inverse transformation of CFRDTM, the solution of (25) is given as
\[
\begin{align*}
u(x, y, t) &= \sum_{k=0}^{\infty} V_k^q (x, y, t) t^k \\
&= x + y - \frac{2}{q} x t + \frac{2}{q^2} (x + y) t^2 - \frac{4}{q^3} x t^3 + \frac{4}{q^4} (x + y) t^4 - \frac{8}{q^5} x t^5 + \cdots
\end{align*}
\]
Example 9. Consider the following system of CFDEs
\[
\begin{align*}
\partial^q_t u + u \partial_x u + \partial_t \partial_x u + 1 - x + y + \frac{\rho_1}{q} \\
\partial^q_t v &= u \partial_x v + \partial_t \partial_y v + 1 - x - y - \frac{\rho_1}{q},
\end{align*}
\]
subject to the initial conditions
\[
\begin{align*}
u(x, y, 0) &= x + y - 1; \\
u(x, y, 0) &= x - y + 1.
\end{align*}
\]
Applying theorems of CFRDTM on (25) we obtain the following system of recurrence relations:
\[
\begin{align*}
q(k + 1) U_{k+1}^q + \sum_{r=0}^{k} V_{k-r}^q \partial_x U_r^q &+ \sum_{r=0}^{k} V_{k-r}^q \partial_y U_r^q \\
&\quad + \sum_{r=0}^{k} q(k + 1) V_{k-r}^q \partial_x V_r^q + \delta(k) \\
= \partial^2_{xx} U_k^q + \partial^2_{yy} U_k^q - x + y - 1 + \delta(k - 1),
\end{align*}
\]
and
\[
\begin{align*}
q(k + 1) V_{k+1}^q &+ \sum_{r=0}^{k} V_{k-r}^q \partial_x V_r^q \\
&\quad + \sum_{r=0}^{k} q(k + 1) V_{k-r}^q \partial_y V_r^q + \delta(k) \\
= \partial^2_{xx} V_k^q + \partial^2_{yy} V_k^q - x - y - 1 + \delta(k - 1),
\end{align*}
\]
where the I.C. can be transformed at $t = 0$ to

$$U_0^q = x + y - 1$$

$$V_0^q = x - y + 1.$$ 

Substituting $k = 0$ into (36) we obtain

$$U_1^q = \frac{1}{q}$$

$$V_1^q = \frac{-1}{q}.$$ 

Similarly for $k = 1$ and $k = 2$, we get

$$U_2^q = -\frac{x}{2q}$$

$$V_2^q = \frac{1}{2q} - \frac{y - 1}{2q}.$$ 

and

$$U_3^q = \frac{-2q^2 - x}{6q^2} + \frac{1 - 5q}{12q^3}$$

$$V_3^q = \frac{-y}{3q} - \frac{q + 1}{12q^3}.$$ 

Therefore, the approximate analytical solution of (36) is given as

$$u = x + y - 1 + \frac{t^q}{q} - \frac{x^2q^2}{2q} + \frac{-2q^2 - x}{6q^2} + \frac{1 - 5q}{12q^3} + \cdots$$

$$v = x - y + 1 - \frac{t^q}{q} + \frac{1}{q} - \frac{y - 1}{2q} + \frac{y - 3q}{3q} + \cdots.$$ 

Since this solution does not equal the exact one, we use mathematica program to show its accuracy. In Figures 1 and 3, we plot the approximate solution for Example 9 for $q = 0.99$ at $t = 0.1$ and at $t = 0.2$, respectively. The graphs in Figures 2 and 4 illustrate the exact solution for Example 9 for $q = 0.99$ at $t = 0.1$ and at $t = 0.2$, respectively. We can clearly observe at the time $t = 0.1$ and $t = 0.2$ that the values of the approximate solution by CFRDFTM and the values of the exact solution, obtained by other methods, are extremely close. In other words, solution provided by CFRDFTM is highly precise and accurate. Particularly CFRDFTM is a successful technique for solving systems of conformable FPDEs.
4. Conclusion

In this paper, our main purpose was to inspect the competence of the CFRDTM as a valid technique to solve systems of nonlinear CFPDEs. We successfully applied it to four distinct systems of nonlinear conformable time and space FPDEs. CFRDTM produced a numerical approximate solution having the form of an infinite series that converged to a closed form solution, which coincided with the exact solution in the first 3 applications. In the 4th application, however, we obtained an approximate solution that turned out to be extremely close to the exact solution as we have interpreted from the attached Figures 1–4. On this basis, we conclude that the CFRDTM is a powerful tool and a facile approach for obtaining solutions for systems of CFPDEs. It reduces the amount of required computational work when compared to other techniques used for this purpose and gives solutions that are in outstanding agreement with the exact ones. This provides a good starting point for further research via CFRDTM.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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