INCIDENCE GEOMETRY IN THE PROJECTIVE PLANE VIA ALMOST-PRINCIPAL MINORS OF SYMMETRIC MATRICES

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Abstract. We present an encoding of a polynomial system into vanishing and non-vanishing constraints on almost-principal minors of a symmetric, principally regular matrix, such that the solvability of the system over some field is equivalent to the satisfiability of the constraints over that field. This implies two complexity results about Gaussian conditional independence structures. First, all real algebraic numbers are necessary to construct inhabitants of non-empty Gaussian statistical models defined by conditional independence and dependence constraints. This gives a negative answer to a question of Petr Šimeček. Second, we prove that the implication problem for Gaussian CI is polynomial-time equivalent to the existential theory of the reals.

1. Introduction

Matroids were conceived by Whitney in the 1930s as combinatorial abstractions of the common properties of independence relations in vector spaces and graphs. Today, matroid theory is a broad and active field of research with an extensive corpus of theorems and constructions. Among the jewels of this theory are the universality theorems of Sturmfels and Mnëv: to every affine variety \( V \) over an algebraically closed field, there exists a rank-3 matroid whose projective realization space over this field is birationally isomorphic to \( V \) [BS89, Theorem 4.30]; and to every basic primary semialgebraic set \( K \) over a real-closed field, a rank-3 oriented matroid may be found whose realization space is stably equivalent to \( K \) [BLS+99, Theorem 8.6.6], [RG97].

Interpreting the term of “universality” more liberally, the idea is to show that certain objects or properties associated to the combinatorial structure of a matroid can be arbitrarily complicated, within the confines of an obvious upper bound and modulo a notion of equivalence which blurs the concrete object but not its complexity. In this looser sense, the NP-completeness of orientability of rank-3 matroids by Richter-Gebert [RG99] may count as a universality result. Notably, also one of the first contributions to matroid theory, in a paper by MacLane [Mac36], gives a universality result: it shows how to construct for every finite algebraic extension \( K/Q \) a rank-3 matroid which is realizable over a field \( L/Q \) if and only if there is an embedding of fields \( K \hookrightarrow L \).

In one way or another, all of these theorems rest on the ability of simple rank-3 matroids to capture, prescribe and forbid incidence relations between points and lines in the projective plane. To be precise, consider a \( 3 \times n \) matrix \( A \) realizing a simple rank-3 matroid. Its columns contain the homogeneous coordinates of \( n \) points in the projective plane. The non-bases of the matroid are those triplets \( pqr \) where the \( 3 \times 3 \) subdeterminant of \( A \) with columns \( p, q \) and \( r \), denoted by the bracket \([pq r]\), vanishes. This is equivalent for the indexed points \( pqr \) to be collinear in the projective plane. The collinearity predicate on point triplets is a geometric primitive which allows the design of arbitrary planar incidence relations. For example, to construct the intersection point \( r \) of two distinct lines, given by two pairs of distinct points \((p, q)\) and \((p', q')\), one requires that \([ pqr ] = 0 \) and \([ p'q'r ] = 0 \), i.e. \( r \) is on the line \( p \lor q \) and also on \( p' \lor q' \). Oriented matroids refine this to a left-right-or-on relation between points and lines, such as studied by Knuth [Knu92].
Building on these constructions with a “projective ruler” one can execute the von Staudt constructions and gain access, through the synthetic geometry language of matroids, to the algebraic structure underlying the projective plane. Figure 1 shows two configurations of lines and (intersection) points whose projective incidence relation (also taking into account parallelity via intersection points at infinity) can only be realized over fields K/Q containing $\sqrt{5}$ or $\sqrt{2}$, respectively. The von Staudt technique is a systematic way to design incidence relations for which constructibility requires solutions to arbitrary polynomial systems with integer coefficients.

The theory of conditional independence (CI) structures follows the spirit of matroid theory, in studying common properties of conditional independence relations among random variables in probability theory, abstracting away the probability distributions, like matroids abstract from vector spaces. One branch of CI structure theory is concerned with discrete random variables. In this context, the works of Fero Matúš are in particular to highlight, in which a connection to matroid theory is drawn explicitly. In [Mat94], (simple) matroids are interpreted as a subset of special CI structures called semimatroids [Mat94], which are in turn semigraphoids. This hierarchy of abstractions is the combinatorial shadow of the hierarchy of representations: rank functions of hyperplane arrangements are rank functions of subspace arrangements, which are in turn entropy functions of discrete random variables, as far as the independence relations are concerned; cf. [Mat97, Section 6]. Through this connection, essentially probabilistic questions can be raised about matroid representations [Mat99b] and the related rank inequalities are of interest in information theory and adjacent fields; see the survey [Cha11]. In the opposite direction, combinatorial or geometric ideas from matroid theory have inspired developments in the theory of discrete CI structures [Mat04, Stu20].

This article transfers the technique of von Staudt constructions to encode polynomial systems to the realm of Gaussian instead of discrete CI structures. Gaussian CI studies the sets of vanishing almost-principal minors of (positive-definite) symmetric matrices. The similarity to matroid theory, which studies sets of vanishing maximal minors of general matrices, is apparent, but it is a different similarity from that which embeds linear matroids into discrete CI structures. In Section 3 we show how to use almost-principal minor constraints, similar to bases and non-bases in matroid theory, to encode plane projective incidence relations which ultimately drive the von Staudt encoding of a polynomial system. Based on this technique, we obtain an analogue of the universality result of MacLane in Section 4 and answer a question posed by Petr Šimeček in [Šim06b] negatively: there exist non-empty Gaussian CI models which contain no rational point. In fact, the minimum algebraic degree of any point in such a model may be arbitrarily high. Finally, in Section 5, we show that the implication problem for Gaussian CI statements is polynomial-time equivalent to the existential theory of the reals. Possible extensions and future directions are discussed in Section 6.

**Notation.** An introduction to the (real) projective plane including a thorough treatment of the von Staudt constructions, which are at the heart of this paper, can be found in [RG11]. The symbol $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of vectors and $\langle \langle \cdot, \cdot \rangle \rangle$ another symmetric bilinear form whose definition will depend on context. We work with homogeneous coordinates of points $p$ and lines $\ell$ in the projective plane $\mathbb{P}\mathbb{K}^2$ over a field $\mathbb{K}$. The set of points on the line $\ell$ is $\ell^\perp := \{p \in \mathbb{P}\mathbb{K}^2 : \langle p, \ell \rangle = 0 \}$. 

![Figure 1](image-url)}
We adopt the convention that sans-serif symbols $i, j, k, l, p$ denote elements and $E, K, L, P$ subsets of an implicit ground set $N$ which indexes the rows and columns of a symmetric matrix. Elements and singleton subsets are not distinguished and juxtaposition abbreviates set union. For example $iK$ is short for $\{i\} \cup K \subseteq N$ and $ij$ denotes the two-element set $\{i, j\}$ and equals $ji$.

If $\Sigma$ is a symmetric matrix indexed by $N$, then $\Sigma_{ij}$ addresses an entry of $\Sigma$ with a well-defined value.

2. Preliminaries on Gaussian conditional independence

Let $\xi$ denote a vector of random variables indexed by $N$ with a joint distribution. A conditional independence statement $\xi_i \indep \xi_j \mid \xi_K$ asserts that the entries $\xi_i$ and $\xi_j$ are stochastically independent under the distribution conditioned on the subvector $\xi_K$. Informally, this means that any dependency between $\xi_i$ and $\xi_j$ is explained by the entries of $\xi_K$. In the theory of conditional independence one studies the sets of all symbols $(ij|K)$, denoting CI statements, which can simultaneously occur for a given class of distributions or other objects for which “$(ij|K)$” can be given an interpretation. For a proper introduction to the theory of (probabilistic) conditional independence structures, the reader is referred to [Stu05].

An $N$-variate regular Gaussian distribution is given by its mean vector $\mu$ and its positive-definite covariance matrix $\Sigma$. Checking whether a CI statement $(ij|K)$ holds is equivalent to evaluating a special kind of subdeterminant of the covariance matrix, called an almost-principal minor:

\[(\indep) \quad (ij|K) \text{ holds } \iff \Sigma[|j|K] := \det \Sigma_{iK,jK} = 0.
\]

This definition readily generalizes to symmetric matrices over arbitrary fields, where positive-definiteness is replaced by principal regularity, the condition that the principal minors $\Sigma[|K] := \det \Sigma_K$ do not vanish. Principal regularity and its semidefinite extension, positive-definiteness, may be seen as technical conditions which make the interpretation of the CI symbols $(ij|K)$ more well-behaved, providing the theory with a notion of minors, duality, symmetries and combinatorial constructions; cf. [Boe20, Section 3]. The generalization of the structure of positive-definite matrices to $\mathbb{C}$ was first undertaken by Matúš [Mat05] when he derived the axioms of gaussoids [LM07]. While there exists a (combinatorially more involved) definition of conditional independence for positive-semidefinite matrices, we work with principally regular matrices in this paper and defer the treatment of semidefinite matrices over $\mathbb{R}$ to Section 6.1.

**Definition 2.1.** A set of CI constraints is a collection $\mathcal{I}$ of conditional independence $(ij|K)$ or dependence statements $\neg(ji|K)$ over a ground set $N$. For any field $K$, the model $\mathcal{V}_K(\mathcal{I})$ of $\mathcal{I}$ consists of all principally regular, symmetric $N \times N$ matrices $\Sigma$ which satisfy $\mathcal{I}$ as per $(\indep)$. For ordered fields $K$ the positive model $K_+(\mathcal{I})$ is defined analogously as a subset of the positive-definite matrices.

Principal regularity imposes $\Sigma[|K] \neq 0$ whereas positive-definiteness imposes $\Sigma[|K] > 0$ on the model, in addition to the CI equations and inequations. Thus our models are first-order definable sets in the theory of $K$. That is, if $K$ is any field $\mathcal{V}_K$ is a constructible subset of the space of symmetric matrices, and if $K$ is ordered, $K_+$ is a semialgebraic subset, defined by polynomials with integer coefficients. The language of CI constraints allows to impose vanishing and non-vanishing conditions on almost-principal minors of a matrix. The $(ij|K)$ almost-principal submatrix of a matrix $\Sigma$ contains the invertible $K \times K$ principal submatrix. A Schur complement expansion of the determinant with respect to this block yields

\[(\leq) \quad \Sigma[|j|K] = \Sigma[|K] \left( \Sigma_{ji} - \Sigma_{iK} \Sigma_{Kj}^{-1} \Sigma_{Kj} \right)^{-1} = 0.
\]

Thus prescribing a conditional independence $(ij|K)$ effectively stores the scalar product of the row vector $\Sigma_{iK}$ and the column vector $\Sigma_{Kj}$ with respect to the Gram matrix $\Sigma_{Kj}^{-1}$ in the entry $\Sigma_{ij}$ of every matrix $\Sigma$ in the model of $(ij|K)$.
3. Polynomial systems as CI constraints

3.1. Polynomial systems as ruler constructions. The universality theorems for matroids rest on an encoding of arbitrary polynomial systems in the bases and non-bases of a matroid. The relation of the solution set of the polynomial system to the realization space of the matroid depends on the technical finesse of this encoding and its proof. For the results of this paper, we restrict ourselves to solvability questions: the task is to convert a system of polynomial equations to a set of CI constraints such that the constraints have a model if and only if the system has a solution. Polynomial systems, in turn, are modeled according to von Staudt by certain incidence relations in the projective plane. His classical constructions rely only on a basis of the projective plane and the ruler as a construction tool. So the encoding of certain plane projective ruler constructions is our gateway to universality.

Definition 3.1. The standard projective basis consists of the infinite point on the x-axis $\infty_x = [1 : 0 : 0]$, the infinite point on the y-axis $\infty_y = [0 : 1 : 0]$, the origin $0 = [0 : 0 : 1]$ and the point of units $1 = [1 : 1 : 1]$.

From these points, the x- and y-axes $\ell_x$ and $\ell_y$, unit points on the axes $1_x$ and $1_y$ and the line at infinity $\ell_\infty$ can be constructed, which complete the framework in which ruler constructions are carried out. The standard basis has favorable properties for the constructions in the next section, notably its shape can be prescribed easily using CI constraints, which is why we insist on it.

Definition 3.2. A ruler construction over a field $\mathbb{K}$ is a finite list of instructions which constructs a set of points and lines in $\mathbb{P}\mathbb{K}^2$ from a given set of points including the standard projective basis using the computational primitives of (a) joining two already constructed, distinct points to form the line through them, and (b) meeting two already constructed, distinct lines to form their intersection point. The construction algorithm may receive parameters in the form of indeterminate points which are placed on the x-axis $\ell_x$.

Ruler constructions are required to be deterministic: by stipulating the distinctness of joined points and met lines in $\mathbb{P}\mathbb{K}^2$, the resulting line or point is uniquely defined as the one-dimensional space of solutions to two independent linear equations in $\mathbb{K}^3$. For instance, the line $\ell$ through two distinct points $p, p'$ is given by $\langle p, \ell \rangle = 0$ and $\langle p', \ell \rangle = 0$. Usage of the indeterminate points and all objects constructed from them is permitted as long as all joins and meets are provably between distinct objects in every instantiation of the indeterminates. In this case, the join $\ell$ of the distinct points $p, p'$ can be immediately computed by the cross product

$$[p^x : p'^x : p^z] \times [p^y : p'^y : p'^z] := \left[ \det \begin{pmatrix} p^y & p'^y & p^z \\ p^x & p'^x & p'^z \\ p^z & p'^z & p^y \end{pmatrix} : -\det \begin{pmatrix} p^x & p'^x & p^z \\ p^z & p'^z & p^y \\ p^y & p'^y & p^z \end{pmatrix} : \det \begin{pmatrix} p^x & p'^x & p^z \\ p^y & p'^y & p^z \\ p^z & p'^z & p^y \end{pmatrix} \right].$$

The same operation computes, dually, the coordinates of the meet of two distinct lines.

Problem (F). Given a system $F = \{f_1, \ldots, f_r\}$ of integer polynomials $f_i \in \mathbb{Z}[t_1, \ldots, t_k]$, construct with a ruler, starting from the standard projective basis and an indeterminate point $t_i = [t_i : 0 : 1]$ for each unknown $t_i$, the points $f_i = [f_i(t_1, \ldots, t_k) : 0 : 1]$.

By introducing more equalities and variables, the polynomial system can be assumed to contain only atomic equations using the elementary arithmetic operations of addition and multiplication: $t_i = t_j + t_k$ and $t_i = t_j \cdot t_k$, as well as a distinguished variable $t_1 = 1$ for the unit [KPY20, Section 3.2]. The von Staudt constructions implement precisely addition and multiplication of points on the x-axis. Before we describe these algorithms, we construct a larger projective framework out of the standard basis, containing points and lines which are used in both:

\[
\begin{align*}
(1) \quad \ell_x &= 0 \times \infty_x = [0 : 1 : 0] \\
(2) \quad \ell_y &= 0 \times \infty_y = [1 : 0 : 0] \\
(3) \quad \ell_\infty &= \infty_x \times \infty_y = [0 : 0 : 1] \\
(4) \quad \ell_{1x} &= 1 \times \infty_x = [0 : -1 : 1] \\
(5) \quad \ell_{1y} &= 1 \times \infty_y = [-1 : 0 : 1] \\
(6) \quad 1_x &= \ell_{1y} \times \ell_x = [1 : 0 : 1] \\
(7) \quad 1_y &= \ell_{1x} \times \ell_y = [0 : 1 : 1]
\end{align*}
\]
Figure 2. Von Staudt constructions in two affine pictures. The solid points are given, the hollow ones are helper points in the construction of the square target points. The axes are displayed as solid lines, helper lines are dotted and the dashed lines, which are parallel to the dotted ones, yield the target points.

Figure 2 contains pictures of the von Staudt constructions for addition and multiplication of indeterminate points in the affine \(xy\)-plane by projective ruler constructions from the standard basis. The pictures join points, meet lines and construct the parallel to a line through another point. This last affine operation can be performed by the projective ruler using the line at infinity not pictured here. Full descriptions of these classical constructions are given in [RG11, Section 5.6] and with emphasis on matroids (over skew fields) in [KPY20]. Here we give the algorithms with indeterminates \(x = [x: 0: 1]\) and \(y = [y: 0: 1]\) using cross products and using the same notation as in Figure 2:

**Addition:**

1. \(g := 1_y \times \infty_x = [0: -1: 1]\)
2. \(h := x \times \infty_y = [-1: 0: x]\)
3. \(q := g \times h = [x: 1: 1]\)
4. \(j := y \times 1_y = [-1: -y: y]\)
5. \(\infty_j := j \times \ell_\infty = [-y: 1: 0]\)
6. \(j' := q \times \infty_j = [1: -y: x + y]\)
7. \(x + y := j' \times \ell_x = [x + y: 0: 1]\)

**Multiplication:**

1. \(g := 1_x \times 1_y = [-1: -1: 1]\)
2. \(h := x \times 1_y = [-1: -x: x]\)
3. \(\infty_y := g \times \ell_\infty = [-1: 1: 0]\)
4. \(\infty_h := h \times \ell_\infty = [-x: 1: 0]\)
5. \(g' := \infty_j \times y = [1: 1: -y]\)
6. \(q := g' \times \ell_y = [0: y: 1]\)
7. \(h' := q \times \infty_h = [1: x: -x \cdot y]\)
8. \(x \cdot y := h' \times \ell_x = [x \cdot y: 0: 1]\)

**Lemma 3.3.** Given the standard basis, the von Staudt constructions solve Problem (F).

This very analytic treatment of the construction is required to observe the following subtle point which will be important in Section 3.2:

**Lemma 3.4.** All meet and join operations in the von Staudt construction are between distinct points and lines, independently of the positions of the indeterminates \(t_i\) on the \(x\)-axis. Moreover, for every point and line needed in the construction, one homogeneous coordinate can be given which is non-zero, also independently of the indeterminates.

### 3.2. Ruler constructions as CI constraints

To model a ruler construction as CI constraints, we work over a ground set \(N = \text{PLE}\) which decomposes into sets \(P = \{p_1, p_2, \ldots\}\) and \(L = \{l_1, l_2, \ldots\}\) for labeling the points and lines which are used during the algorithm, respectively, and \(E = \{x, y, z\}\) which indexes the homogeneous coordinates of the points and lines. Instead of implementing the join and meet primitives via collinearity of points, as matroids do (and where all lines are implicit), or by the cross product, we use the scalar product interpretation of CI constraints given in \((\mathcal{A})\). Namely, given homogeneous coordinates of a point \(p\) and of a line \(\ell\), the incidence \(p \in \ell^\perp\) is equivalent to the vanishing of the standard scalar product \(\langle p, \ell \rangle\). To construct the line \(\ell_{pq}\) joining two already constructed, distinct points \(p, q\) labeled by \(p, q \in P\), introduce a new variable \(l_{pq}\) into the set \(L\) and require the incidence of the point \(p\) and the point \(q\) to the new line \(l_{pq}\).
The complete encoding of the von Staudt constructions of a polynomial system into a set of CI constraints is given in Definition 3.5 below. Up to some implementation details, the basic ideas behind this definition are:

— The homogeneous coordinates of the point indexed by \( p \in P \) are stored in the entries \( \Sigma_{pe} \) of a model \( \Sigma \), for \( e \in E = \{x, y, z\} \). Likewise for lines \( l \in L \).

— For every pair of distinct points and/or lines \( a, b \in PL \), we impose the CI statement \((ab|xyz)\) in order to store the scalar product of their homogeneous coordinates in the entry \( \Sigma_{ab} \). This scalar product is with respect to the inverse block matrix \( \Sigma_{E}^{-1} \), according to \( (\mathcal{L}) \).

— The desired orthogonalities between \( p \in P \) and \( l \in L \) which assert incidence relationships can then be prescribed with CI constraints \((pl)\).

**Definition 3.5.** Let \( F = \{f_1, \ldots, f_r\} \subseteq \mathbb{Z}[t_1, \ldots, t_k] \). Consider the von Staudt construction of these polynomials making reference to points labeled \( P = \{t_1, \ldots, t_k, f_1, \ldots, f_r, \} \) and lines labeled \( L = \{l_1, \ldots, l_m\} \), where the \( t_i \) represent the indeterminate points and \( f_i \) represent the values of the \( f_i \) in the construction. Define a set of CI constraints \( \tilde{I}(F) \) over the ground set \( PLE \) consisting of:

(\( I.i \)) \((pe)\) or \(\neg(pe)\) for all points \( p \) corresponding to the standard projective basis and \( e \in E \), depending on whether the \( e \)-coordinate of the point is zero or not.

(\( I.ii \)) \((ty)\) and \(\neg(tz)\) for indeterminate points \( t = t_1, \ldots, t_k \).

(\( I.iii \)) \(\neg(ae)\) for each \( a \in PL \) and one of the coordinates \( e \in E \) on which the point or line labeled \( a \) is non-zero, which can be deduced by Lemma 3.4.

(\( I.iv \)) \((ab|xyz)\) for all distinct \( a, b \in PL \).

(\( I.v \)) \((pl)\) for any incidence relationship between \( p \in P \) and \( l \in L \) which is required to express a join or meet operation of the construction.

Figure 3 shows the generic matrix satisfying constraint type \( (I.iv) \). The constraints \( \tilde{I}(F) \) emulate the incidence relations behind the von Staudt construction. Every matrix which satisfies \( \tilde{I}(F) \) gives values to the parameters \( t_1, \ldots, t_k \) and all other points and lines such that the same incidence relations hold, which forces the \( f_i \) to assume the evaluation of \( f_i(t_1, \ldots, t_k) \) up to various scalings. The caveat, however, is that each model starts the construction with coordinates of points which are not necessarily the standard basis and executes the ruler construction with a possibly non-standard notion of “incidence” which comes from \( (\mathcal{L}) \) and constraint type \( (I.iv) \): the linear system defining incidence \( p \in \ell \) switches from \( \langle p, \ell \rangle = 0 \) to \( \langle p, \ell \rangle \neq 0 \), where \( \langle \cdot, \cdot \rangle \) is a non-degenerate symmetric bilinear form defined by the inverse of \( \Sigma_{E} \) in the matrix \( \Sigma \) thought of as executing the ruler construction.

Figure 3. The generic matrix satisfying the encoding of incidence relations among points \( p_1, \ldots, p_n \) and lines \( l_1, \ldots, l_m \) in the projective plane, according to Definition 3.5. The scalar product \( \langle \cdot, \cdot \rangle \) is given by the inverse of the \( \Sigma_{E} \) block.
Lemma 3.6. In the notation of Definition 3.5, let \( \Sigma \in \mathcal{V}_E(F) := \mathcal{V}_E(\tilde{f}(F)) \).

(1) \( \Sigma \) contains the homogeneous coordinates of points \( t_i, f_i, p_i \) and lines \( \ell_i \) in \( \mathbb{P}_K \mathbb{P}^2 \) in the entries \( \mathbb{P} \times \mathbb{E} \) and \( \mathbb{L} \times \mathbb{E} \). The image of the projective basis coincides with the standard projective basis, except for \( \tilde{I} = [s_x : s_y : 1] \), which may be different from \( 1 \). The \( x \)-axis, the \( y \)-axis and the line at infinity are the same as with the standard projective basis. The points \( t_i \) and \( f_i \) lie on the \( x \)-axis.

(2) With the non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) defined by \( \Sigma_{E^{-1}} \), an incidence \( p_i \in \mathcal{I}^1 \) imposed in the construction implies \( \langle p_i, \ell_j \rangle = 0 \), i.e. \( p_i \in \Sigma_E(\ell_j^\perp) \).

(3) The \( f_i, p_i \) and \( \ell_i \) are uniquely determined as points in \( \mathbb{P}_K \mathbb{P}^2 \) by the points \( t_i \), the scalings \( s_x, s_y \) and the \( \Sigma_E \) block. All other off-diagonal entries of \( \Sigma \) are functions of these homogeneous coordinates.

Proof. (1) By the relations \((\mathcal{L}.i)\), we have \( \infty_x = [1 : 0 : 0] \), \( \infty_y = [0 : 1 : 0] \), \( \tilde{0} = [0 : 0 : 1] \) and \( \tilde{I} = [s_x : s_y : 1] \) as points in the projective plane, with \( s_x, s_y \neq 0 \). This is still a projective basis and the \( x \)-axis, the \( y \)-axis and the line at infinity remain the same. This is consistent with constraints \((\mathcal{L}.ii)\), proving that indeterminate points are on the \( x \)-axis. The \( f_i \) are constructed by von Staudt as intersection points with \( \ell_x \), so they remain on the \( x \)-axis. Because of constraints \((\mathcal{L}.iii)\) all homogeneous coordinate vectors are non-zero and hence valid points/lines in \( \mathbb{P}_K \mathbb{P}^2 \).

(2) Denote by \( \langle v, w \rangle := v^T \Sigma_{E^{-1}} w \) the non-degenerate symmetric bilinear form defined by \( \Sigma_{E^{-1}} = \Sigma_{xyz}^{-1} \). The relations \( (p_i)_{xyz} \) of type \((\mathcal{L}.iv)\) are equivalent to

\[
\Sigma_{p_i} = \Sigma_{p_i}^{-1} \Sigma_{xyz}^{-1} \Sigma_{xyz} = \langle p_i, \ell_j \rangle
\]

and then type \((\mathcal{L}.v)\) makes this scalar product vanish, for every relation \( p_i \in \mathcal{I}^1 \) requested.

(3) Since all points and lines are valid objects in \( \mathbb{P}_K \mathbb{P}^2 \) — in particular due to type \((\mathcal{L}.iii)\), the zero vector is never permissible as a vector of homogeneous coordinates (even though it satisfies all incidence relations it may be involved in), thus the construction never degenerates — , the uniqueness of the result of the von Staudt construction in Lemma 3.4 proves that all points and lines are uniquely determined by the starting points, which are the projective basis and the indeterminates, as well as the definition of incidence. Relations \((\mathcal{L}.iv)\) then fix all off-diagonal entries on \( \mathbb{P}_L \times \mathbb{P}_L \) as functions of the homogeneous coordinates, as per \((\mathcal{L})\). \( \square \)

Lemma 3.6 shows that constraints \((\mathcal{L}.i)\) for the standard projective basis fix the standard projective basis in every model up to a scaling of the \( x \)- and \( y \)-axis by non-zero quantities \( s_x \) and \( s_y \). The points \( f_i \) which correspond to the evaluations of polynomials \( f_i \) end up on the \( x \)-axis and their location is uniquely determined by the scalings, the bilinear form and the locations of \( t_i \). The next lemma makes this more precise:

Lemma 3.7. Let \( \Sigma \in \mathcal{V}_E(F) \). Denote by \( t_i = [t_i : 0 : 1] \) and \( f_i = [f_i : 0 : 1] \) the points in \( \mathbb{P}_K \mathbb{P}^2 \) determined by \( \Sigma \) according to Lemma 3.6. Then \( f_i^\perp = s_x f_i(t_1/s_x, \ldots, t_k/s_x) \).

Proof. Let \( S = \left( \begin{array}{ccc} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{array} \right) \) be the scaling which maps the standard projective basis to the one contained in \( \Sigma \). Use \( p' \) and \( \ell' \) to refer to points constructed with the von Staudt algorithm and the standard basis and use \( p \) and \( \ell \) for the same objects constructed by \( \Sigma \) with the scaled basis and non-standard scalar product. Consider also the points \( t_i' = [t_i/s_x : 0 : 1] \). Then we obtain the projective basis and the indeterminates of \( \Sigma \) as images under \( S \) of the standard basis and the indeterminates \( t_i' \). It is straightforward to show by induction on the steps of the ruler construction using Lemma 3.6 (2) that:

- If \( \ell \) is constructed from \( p_1 \) and \( p_2 \) with \( p_i = Sp'_i \), then \( \ell = \Sigma_E S^{-1} \ell' \).
- If \( q \) is constructed in turn from \( \ell_1 \) and \( \ell_2 \) with \( \ell_i = \Sigma_E S^{-1} \ell'_i \), then \( q = Sq' \).

Thus, \( f_i = SF'_i \) and \( f_i^\perp = s_x f_i^\perp = s_x f_i(t_1/s_x, \ldots, t_k/s_x) \) by Lemma 3.3. \( \square \)
Lemma 3.8. Let $a_1, \ldots, a_k \in \mathbb{K}$ be arbitrary. If $\mathbb{K}$ is infinite, then $\tilde{V}_\mathbb{K}(F)$ has a model which gives indeterminates the value $t_i = [a_i : 0 : 1]$ and evaluations $f_i = [f_i(a_1, \ldots, a_k) : 0 : 1]$. Likewise for $\tilde{K}_\mathbb{K}(F)$ if $\mathbb{K}$ is ordered.

Proof. A model $\Sigma$ can be constructed based on Figure 3. Fill the coordinates of points of $P$ corresponding to the standard projective basis with the actual standard projective basis, set the indeterminate points $t_i$ as required and finally set $\Sigma_\mathbb{E}$ to be the identity matrix. Then for every incidence $p \in \ell^I$ demanded by the von Staudt construction we have, by Lemma 3.6 (2) with $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$ that indeed $\langle p, \ell \rangle = 0$, i.e. $p \in \ell^I$. Execute the von Staudt construction from these settings and all off-diagonal entries will be filled to satisfy the constraints.

This gives values to all entries of $\Sigma$ except for the diagonals $p^*_i$ and $\ell^*_i$ as shown in Figure 3. These diagonals are used to make $\Sigma$ principally regular. Consider any principal submatrix of $\Sigma$:

$$\Sigma_{UV} = \begin{vmatrix} u & \lor \\ A & B \\ \vert \\ B^T & C \end{vmatrix},$$

with $U \subseteq PL$ and $V \subseteq E$.

The $C$ block is a principal submatrix of $\Sigma_E$, which is the identity matrix, so $C$ is an identity as well. Then it suffices to show that the Schur complement $A - BB^T$ of $C$ is regular. The diagonals $p^*_i$ and $\ell^*_i$ appear only in $A$. Similarly to [Boe20, Lemma 4.1] one can show that a generic choice of diagonal elements over an infinite field makes the determinant of this Schur complement non-zero. If $\mathbb{K}$ is ordered, the same argumentation yields positivity — $C$ is positive-definite and one may even choose the diagonals of $A$ so large that the Schur complement becomes diagonally dominant and therefore positive-definite.

Definition 3.9. Let $F = \{f_1, \ldots, f_r\} \subseteq \mathbb{Z}[t_1, \ldots, t_k]$. Denote by $I(F)$ the extension of $\tilde{I}(F)$ by $(I.vi)$ $(fx)$ for all polynomial value symbols $f = f_1, \ldots, f_r$ and let $V_\mathbb{K}(F) := V_\mathbb{K}(I(F))$ and $K_\mathbb{K}(F) := K_\mathbb{K}(I(F))$ over ordered fields.

Proposition 3.10. Let $F = \{f_1, \ldots, f_r\} \subseteq \mathbb{Z}[t_1, \ldots, t_k]$. The set $I(F)$ of CI constraints has a (principal regular) model over an infinite field $\mathbb{K}$ if and only the variety of $F$ (over the algebraic closure $\overline{\mathbb{K}}$) has a $\mathbb{K}$-rational point. The same is true for ordered fields and positive models.

Proof. Suppose that there exists a model $\Sigma \in V_\mathbb{K}(F)$ which contains points $\tilde{1} = [s_x : s_y : 1]$ as well as $t_i = [t_i : 0 : 1]$ and $f_i = [f_i^x : 0 : 1]$. The homogeneous coordinates are unique up to a scalar from $\mathbb{K}$. Then by Lemma 3.7 and constraint $(I.vi)$, we have $0 = f_i^x = s_x f_i(t_1/s_x, \ldots, t_k/s_x)$. Since $s_x \neq 0$ and $t_i/s_x \in \mathbb{K}$, these define a solution to $F$ in $\mathbb{K}$.

Conversely, let $a_1, \ldots, a_k \in \mathbb{K}$ be a solution to $F$. Then they define a model of $\tilde{V}(F)$ by Lemma 3.8. Moreover, since the $a_i$ are roots of the polynomials, the constraints $(I.vi)$ are satisfied by Lemma 3.8. The proof works likewise for $\mathbb{K}$.

4. Field extensions and a question of Šimeček

In this section we apply Proposition 3.10 to prove a universality result for Gaussian CI constraints and field extensions which can be used to answer a question of Petr Šimeček. We work in the Gaussian setting, which means ordered fields below $\mathbb{R}$. The central notion is that of *semialgebraic degree*, which measures the algebraic complexity of semidefinite Gaussian models.

Satisfying a set of CI constraints is equivalent to solving a system of polynomial equations, inequations and inequalities with integer coefficients. Tarski’s transfer principle [MT03, Chapter 2.4] implies that if a model exists over some ordered field $\mathbb{K}$, then there exists a model in the real algebraic numbers $\overline{\mathbb{Q}} \cap \mathbb{R}$. Since every entry of such a model has finite algebraic degree over $\mathbb{Q}$, a model can already be found in a finite real extension of $\mathbb{Q}$. Therefore it is sensible to measure the algebraic complexity of a CI model in terms of field extension degrees. This quantity is finite whenever the constraints are satisfiable at all.
Definition 4.1. The semialgebraic degree of $I$ is the minimal extension degree over $\mathbb{Q}$ which is required to satisfy $I$:

$$\text{sdeg} I := \min_{\Sigma \in K_R(I)} \max_{ij} \deg_Q \Sigma_{ij}.$$ 

The reference to $\mathbb{R}$ in this definition is natural in the context of statistics, but the notion remains the same if $\mathbb{R}$ is replaced by any real-closed field such as $\mathbb{Q} \cap \mathbb{R}$. Notice that both, conditional independence and dependence statements, are necessary to make this notion interesting: without dependence statements, the identity matrix satisfies the constraints, whereas without independence statements, any generic rational positive-definite matrix does.

Petr Šimeček in [Šim06b] asked if every non-empty Gaussian CI model has a rational point. For the special case of regular Gaussians, we can rephrase this question as follows:

Šimeček’s Question. If the semialgebraic degree of a model is finite, then is it always one?

Using the von Staudt constructions of Proposition 3.10, we are able to recreate the proof of MacLane [Mac36], which strongly implies a negative answer to this question.

Theorem 4.2. Let $K/\mathbb{Q}$ be a real, finite field extension. There exists a set $I$ of CI constraints such that $K_L(I)$ is non-empty for some extension $L/\mathbb{Q}$ if and only if there exists an embedding of fields $K \hookrightarrow L$.

Proof. The prime field $\mathbb{Q}$ is perfect and hence by the primitive element theorem the finite extension $K$ has a primitive element $\alpha$ over $\mathbb{Q}$ with minimal polynomial $f \in \mathbb{Z}[t]$. Application of Proposition 3.10 to $F = \{f\}$ produces a constraint set $I(f)$ which has a positive model over $L$ if and only if $L$ contains a root of $f$. By standard facts about field extensions, this implies that $K$ is contained in any field $L$ which has a model of $I(f)$. □

Corollary 4.3. All real algebraic numbers are necessary to witness the non-emptiness of regular Gaussian CI models. □

5. Hardness of the implication problem

A set of CI constraints $I$ decomposes into two sets of CI statements $L$ and $M$ such that $I = L \cup \neg M$, where the negation is applied to every element of $M$. The model $K_R(I)$ is the set of counterexamples to the validity of the conditional independence inference formula

$$\varphi(I) : \bigwedge L \Rightarrow \bigvee M,$$

which expresses that every regular Gaussian distribution which satisfies all independencies in $L$ must satisfy at least one of the independencies in $M$. Denote the decision problem about the validity (with respect to regular Gaussians) of an input inference formula by $\text{GCI}$. Since the set of counterexamples $K_R(I)$ is semialgebraic, $\text{GCI}$ reduces to $\text{ETR}$, the decision problem for the existential theory of the reals; see [BPR06, Chapter 13] for a definition and algorithms.

In the previous sections we have cast incidence geometry in the plane into Gaussian CI constraints. Reasoning about incidence statements in the plane is hard because the valid incidence theorems are the axioms of linear rank-3 matroids, and so we expect that reasoning about Gaussian conditional independence inference is hard as well. This is the main result of this section:

Theorem 5.1. There exists a polynomial-time algorithm to compute, for a basic semialgebraic set $K$ defined by polynomial constraints with integer coefficients, two sets $L$ and $M$ of CI statements such that $\bigwedge L \Rightarrow \bigvee M$ is valid for regular Gaussians if and only if $K$ is empty.

We have to explain the input encoding of a system of polynomials to make sense of the “polynomial-time algorithm” assertion. The size of a system of polynomials is the number of symbols used to write it down in the rudimentary language of ordered fields, i.e. using the constants 0 and 1, the operations $+, -, \cdot$, variables and the relations $=, \neq, <, \leq, \geq, >$. One should note that by increasing the number of variables and equations, it is possible to assign powers of two to certain variables and then encode coefficients and exponents efficiently with
binary coding length, if desired. This encoding of a polynomial system is the same that was
applied in Section 3.1 to model a system of equations via the von Staudt constructions for
addition and multiplication.

Proof of Theorem 5.1. Let any system of polynomial constraints $f_i \bowtie 0$ be given, where $f_i \in \mathbb{Z}[t_1, \ldots, t_k]$ and $\bowtie \in \{=, \neq, <, \leq, \geq, >\}$. A standard construction transforms this system into
an equisatisfiable system of equations by introducing for each non-equality constraint a new
variable $y$ and:

$\neq$: replace $f \neq 0$ by $yf - 1 = 0$.

$>$: replace $f > 0$ by $y^2f - 1 = 0$.

$\geq$: replace $f \geq 0$ by $f - y^2 = 0$.

This procedure takes polynomial time and produces an equivalent system over Euclidean fields,
since we replace a positivity constraint by an equality to a square. In particular real-closed fields
like $\mathbb{R}$ are Euclidean. The resulting equations are encoded into CI constraints via Proposition
noting that the number of points constructed and constraints $\mathcal{I}$ emitted by the algorithm is
polynomial in the number of basic arithmetic operations. Hence $\mathcal{I}$ has polynomial size in the
coding length of the original system. As explained at the beginning of this section, the points
satisfying the polynomial system are precisely the counterexamples to the validity of $\varphi(\mathcal{I})$. □

Remark 5.2. In the opposite direction, GCI can be reduced to ETR in polynomial time as well
by writing out the CI constraints as a polynomial system for a generic positive-definite matrix.
Care has to be taken to not allow the polynomials in the system to grow too large. For example,
writing out an almost-principal minor as the determinant of a generic $iK \times jK$ matrix using the
Leibniz formula takes at least $(|K| + 1)!$ steps which is not polynomial in the coding length of a
CI statement $(ij|K)$, which we may suppose to be $|ijK|$.

A polynomial-time procedure to write $\det \Sigma$ as a system of equations employs Gaussian elimi-
nation simultaneously on rows and columns, producing an
$LDU$ decomposition of $\Sigma$, so that
its determinant is the product of the final diagonal elements. After the elimination of a row and
column, new variables for the remaining, modified entries of the matrix must be introduced to
avoid accumulating long polynomials.

6. Remarks

The proofs of universality results for regular Gaussian CI in Theorem 4.2 and Theorem 5.1
can be further generalized, as the tools developed in Section 3 are largely independent of the
field. The existence proof of principally regular or positive-definite models in Lemma 3.8 requires
infinite fields. In the following we address some directions for generalization and partial results.

6.1. Semidefinite models. Positive-semidefinite matrices are the covariance matrices of singu-
lar Gaussian distributions, whose probability mass is concentrated on a proper subspace of $\mathbb{R}^N$.
This class of distributions has a more involved definition of conditional independence [Sim07]:

$(\perp \preceq)$

$(ij|K)$ holds $\iff$ $\Sigma[ij|L] = 0$,

for any subset $L \subseteq K$ with $\Sigma[L] \neq 0$ and $\text{rk} \Sigma_L = \text{rk} \Sigma_K$. The vanishing of this almost-principal
minor is independent of the choice of $L$; see [Stu05, Section A.8.3] and its references. This
definition generalizes $(\perp)$ for regular Gaussians. The model $\mathcal{K}_F(N)$ of a set of CI constraints
are then all positive-semidefinite matrices which satisfy them. This model is still semialgebraic,
because for every assignment of ranks to the principal minors of a generic symmetric matrix, there
is a clear interpretation of every CI statement as an algebraic equation. Since rank conditions are
algebraic as well, the model is described by a union of basic semialgebraic sets, one for each rank
assignment. However, the combinatorics of vanishing principal and almost-principal minors and,
depending on it, the interpretation of CI are much more involved than in the regular Gaussian
or principally regular case.
The CI constraints of Definition 3.5 and Definition 3.9 are special in this regard as the only ones which do not directly address entries of the matrix are of type (I.iv). Therefore the only principal submatrix whose singularity would change the interpretation of the constraints is $\Sigma_E$. Indeed, if this matrix was singular, the proofs in Section 3.2 would break down because joins and meets would no longer be unique. As an undeserved extra of the encoding of the standard projective basis and the following harmless additional relations, this never happens:

(I.vii) $(pq)$ or $\neg(pq)$ for all points $p, q$ of the standard projective basis depending on whether $(p, q) = 0$ or not, respectively.

These constraints for pairs of points from $\{\infty_x, \infty_y, 0\}$ imply that $\Sigma_E$ is diagonal, which does not impair the existence proofs in Lemma 3.8. The remaining combinations with $1$ ensure that $\Sigma_E$ is invertible:

**Lemma 6.1.** Let $F$ be a polynomial system and $K$ an ordered field. If $\Sigma \in K_{\Sigma}(F)$ satisfies the additional constraints (I.vii), then $\Sigma_E$ is invertible.

**Proof.** Consider the generic matrix on $\{\infty_x, \infty_y, 0, 1, x, y, z\}$ corresponding to the projective basis and the homogeneous coordinates which satisfy the constraints (I.i), (I.iv) and (I.vii):

$$
\begin{pmatrix}
\infty_x & \infty_y & 0 & 1 & x & y & z \\
\infty_x & * & 0 & 0 & * & 0 & 0 \\
\infty_y & 0 & * & 0 & * & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & * \\
x & * & 0 & 0 & * & * & * \\
y & 0 & * & 0 & * & * & * \\
z & 0 & 0 & * & 0 & 0 & 0 \\
\end{pmatrix}
$$

where all occurring * symbols denote non-zero numbers. The submatrix $\Sigma_E$ may have rank 0, 1, 2 or 3. If the rank is zero, then $(ab|xyz)$ is equivalent to $(ab)$ by ($\perp$), but this contradicts $\neg((01)\land (01)|xyz)$. If the rank is one, we may assume by the symmetry of the projective basis that $\Sigma_x$ is an invertible principal submatrix of full rank in $\Sigma_E$. Then $(ab|xyz)$ is equivalent to $(ab|x)$. But $(01|xyz) \Leftrightarrow (01|x)$ means

$$
\Sigma[01|x] = \det_0 x \begin{pmatrix} 1 & x \\ * & * & \Sigma_x \end{pmatrix} \not= 0,
$$

which is a contradiction to $\neg((01))$ and $\Sigma_x$ having rank one. Lastly, if the rank is two and $\Sigma_{xy}$ is, without loss of generality, an invertible principal submatrix of full rank, then the same contradiction arises from $(01|xyz) \Leftrightarrow (01|xy)$ and $\neg((01))$, as the almost-principal minor

$$
\Sigma[01|xy] = \det_y x \begin{pmatrix} 1 & x & y \\ * & 0 & 0 \\
* & * & \Sigma_{xy} \Sigma_{xy} \end{pmatrix} \not= 0
$$

again factors into the non-zero $\Sigma[xy]$ and the non-zero 01-entry. Hence $\Sigma_E$ has full rank. $
$\end{proof}

It follows that every model of the extended constraints interprets the CI statements exactly like the principally regular models and performs the incidence geometry with a non-degenerate symmetric bilinear form. We recover Lemma 3.7 and together with Lemma 3.8, which holds a fortiori, this proves Proposition 3.10 for the semidefinite model. This answers Šimeček’s Question also in the semidefinite setting:

**Corollary 6.2.** All real algebraic numbers are necessary to witness the non-emptiness of (semidefinite) Gaussian CI models.

**Example 6.3** (Šimeček’s model № 85). The paper [Šim06b] in which Petr Šimeček posed his rationality question is concerned with the CI structures realizable by (not necessarily regular) Gaussian distributions on four variables. For all but one of the models, № 85, he identified a rational covariance matrix realizing it.
The CI structure \( \mathcal{N} 85 \) is \{ (12|4), (14|3), (14|23), (24|3), (24|13), (34|12) \}. The above corollary shows that there are semidefinite CI models without rational points, but \( \mathcal{N} 85 \) is not one of them. The matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & -1/17 & -49/51 & -7/17 \\
-1/17 & 1 & 1/3 & 1/7 \\
-49/51 & 1/3 & 1 & 3/7 \\
-7/17 & 1/7 & 3/7 & 1
\end{pmatrix}
\]

is positive-semidefinite and its vanishing principal minors are \( \Sigma[123] \) and \( \Sigma[1234] \), which do not affect the interpretation of any CI statement via \( (\perp^\perp) \), and the CI structure realized by this matrix is \( \mathcal{N} 85 \). The matrix was found (quickly) by Wolfram Mathematica [Wol] after optimistically imposing the mentioned rank constraints and simplifying the resulting polynomial system by hand.

6.2. Finite fields. Proposition 3.10 does not apply to finite fields because principal regularity is a restrictive condition for very small fields. For instance, the identity matrix is the only principally regular matrix over GF(2). Principal regularity is part of the definition of CI model because it ensures a well-defined interpretation of the CI statement \( (ij|K) \), but afterwards this property is almost trivially enforced in Lemma 3.8 and plays no role for encoding of point and line configurations. For positive-semidefinite matrices, there exists the more intricate definition \( (\perp^\perp) \), but this does not generalize to arbitrary symmetric matrices:

**Example 6.4.** Consider the symmetric (not positive-semidefinite) matrix

\[
\Sigma = \begin{pmatrix}
i & j & x & y & z \\
i & 1 & -1 & 0 & 0 \\
j & 1 & -1 & -1 & 0 \\
x & -1 & -1 & 1 & 0 \\
y & 0 & 0 & 0 & 2 \\
z & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

The interpretation of the CI symbol \( (ij|xyz) \) according to \( (\perp^\perp) \) is inconsistent. It depends on the choice of full-rank subset of \( xyz \):

\[
\Sigma[xyz] = 0, \quad \Sigma[xy] = 2 \neq 0, \quad \Sigma[yz] = 2 \neq 0, \quad \Sigma[xz] = 0, \\
\Sigma[ij|xy] = 2 - 2 = 0, \quad \Sigma[ij|yz] = 2 \neq 0.
\]

The proof of the well-definedness of \( (\perp^\perp) \) for semidefinite matrices needs the well-definedness of the generalized Schur complement [Stu05, Section A.8.1] which in turn rests on the fact that for every block decomposition of a semidefinite matrix \( \Sigma = (A B) \) the colspan(\( B \)) is contained in colspan(\( A \)), i.e. there exists \( B' \) such that \( B = AB' \). This is shown in [Zha05, Theorem 1.19]. This colspan property on every block decomposition, for which \( \Sigma \) which may be called diagonally spanning, is entirely linear-algebraic and it can be defined over every field, where it ensures the well-definedness of \( (ij|K) \) via \( (\perp^\perp) \). The matrix in Example 6.4 is symmetric but not diagonally spanning as witnessed by \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin \text{colspan} \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \). It is not obvious that the analogue of Lemma 3.8 holds for models of diagonally spanning matrices over finite fields.

**Question 6.5.** Is it possible to define an interpretation of \( (ij|K) \) for general symmetric matrices which is consistent with semidefinite matrices? Can the proof of Proposition 3.10 be generalized?

Another attempt at generalization would explicitly involve vanishing and non-vanishing statements about principal minors in the constraint set. For results and open questions about the polynomial relations among principal and almost-principal minors of a generic symmetric matrix, see [BDKS19].

6.3. Algebraic degrees. The *algebraic degree* of a constraint set \( \mathcal{I} \) with respect to some characteristic \( k \) (zero or a prime number) is \( \deg_k \mathcal{I} := \min_{\Sigma \in \mathcal{I}(\mathcal{H})} \max_{ij} \deg_k \Sigma_{ij} \), where \( k \) is the prime
field of characteristic $k$ and $\mathbb{F}$ its algebraic closure. The Lefschetz principle states that the emptiness of a constructible set defined by integer polynomials over algebraically closed fields does not depend on the particular field but only on the characteristic; for a proof see [MT03, Chapter 2.4]. Analogous to Tarski’s transfer principle for real-closed fields, it ensures the finiteness of this degree whenever a model over some field of the given characteristic exists.

Our techniques apply only to infinite prime fields, i.e. the rational numbers, but they suffice to settle the algebraic analogue of Šimeček’s Question posed in [Boc20, Question 6.7]:

**Corollary 6.6.** All algebraic numbers are necessary to witness the non-emptiness of principally regular CI models over $\mathbb{Q}$. In particular, for every step in the chain $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, there exist CI structures which are realizable over the next but not over the previous field. □

6.4. **Existential theories.** Denote by $\text{ET}_K$ the problem to decide whether a constructible set defined by integer polynomials over $K$ is empty or not and by $\text{ET}_K \leq$ the corresponding problem for semialgebraic sets over an ordered field. We have analogous versions of the implication problem $\text{GCI}(K)$ and $\text{GCI}(K \leq)$.

**Corollary 6.7.** (1) $\text{ET}_K$ reduces to $\text{GCI}(K)$ when $K$ is infinite. (2) $\text{ET}_K \leq$ reduces to $\text{GCI}(K \leq)$ whenever $K$ is a Euclidean ordered field. (3) $\text{ET}_K$ and $\text{ET}_K \leq$ and hence $\text{GCI}(K)$ and $\text{GCI}(K \leq)$ are decidable over algebraically and real-closed fields, respectively. □

This implies that the classification problem attached to Šimeček’s question, i.e. to decide $\text{GCI}(\mathbb{Q})$, is famously open, as it is equivalent to Hilbert’s 10th problem over $\mathbb{Q}$. For a survey of this problem, see [Shl11].

It is remarkable that the upper bound of 3 can be placed on the size of conditioning sets $K$ in every CI symbol $(ij|K)$ required in the reduction of $\text{ETR}$ to $\text{GCI}$ in Theorem 5.1. On the other hand, the construction requires an unbounded number of each, antecedents $\mathcal{L}$ and consequents $\mathcal{M}$: antecedents to enforce incidence relations ($I.\text{iv}$) and ($I.\text{v}$) and consequents to ensure that every point defined by the von Staudt construction is a valid point in projective space with constraint type ($I.\text{iii}$). Prior research into infinite families of Gaussian inference rules has usually targeted single-consequent formulas [Sul09, Šim06a].

**Question 6.8.** Is there a polynomial-time reduction of $\text{ETR}$ to $\text{GCI}$ for which there is a universal upper bound on the number of consequents in the constructed inference formulas?

6.5. **Future work.** Given the encoding of projective plane incidence relations over fields presented here, a natural direction for future work concerns skew fields with the same ambitions as in [KPY20]. The more fine-grained universality theorems mentioned in the introduction are an attractive target as well. These concern topological properties of the realization spaces of matroids. The analogue of matroids in this context would be gaussoids [BDKS19], which are “complete” sets of CI constraints in the sense that each CI statement appears in them either negated or non-negated. The extension to oriented gaussoids allows to impose sign constraints on conditional dependencies. Statistically, this prescribes the signs of correlations among random variables, and in our model of the plane it would make the orientation of points with respect to lines expressible.

Recent work on discrete random variables [CMM20] considers related applications of point and line configurations to the decomposition of conditional independence ideals, in particular with hidden variables. At the Algebraic Statistics Online Seminar where this work was presented, Thomas Kahle suggested that a universality theorem like Theorem 4.2 for discrete CI would settle a question of Matúš, which is entirely parallel to and older than that of Šimeček:

**Question 6.9 ([Mat99a]).** Does every CI structure which is realizable by discrete random variables have a realization where every atomic probability is rational?

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