String–Localized Covariant Quantum Fields

Jens Mund

Instituto de Física, Universidade de São Paulo, CP 66 318, 05315 - 970 São Paulo, SP, Brazil mund@fma.if.usp.br

Summary. We present a construction of string–localized covariant free quantum fields for a large class of irreducible (ray) representations of the Poincaré group. Among these are the representations of mass zero and infinite spin, which are known to be incompatible with point-like localized fields. (Based on joint work with B. Schroer and J. Yngvason [13].)

1 Introduction

The principles of relativistic quantum physics admit certain “exotic” particle types which do not allow for point–localized quantum fields, namely the massless “infinite spin” representations found by Wigner [17] and anyons [19]. However, it is known [3, 12] that all Wigner particle types\(^1\) do allow for localization, in a certain sense, in spacetime regions which extend to infinity in some space–like direction.

In this contribution, we present the construction of free Wightman type fields for the massless “infinite spin” particles, which are localized in semi–infinite strings extending to space–like infinity. This result solves the old problem [6,18,20] of reconciling these representations with the principle of causality. It has been obtained in collaboration with B. Schroer and J. Yngvason and partly published in [13]. The details will be presented in [14]. Here, we emphasize the relation with the work of Bros et al. [2], as appropriate for the occasion.

The construction also works for the usual, “non–exotic”, particle types. Our motivation to consider such fields, dispite the fact that they generate the same algebras as the corresponding point–like localized free fields, is the hope that they may serve as a starting point for the construction of interacting string–localized quantum fields.

\(^1\)By Wigner particle type, we mean here an irreducible unitary ray representation of the Poincaré group with positive energy.
Let us make precise what we mean by a string–localized covariant free quantum field for a given particle type. The “string” is a ray which extends from a point \( x \in \mathbb{R}^d \) to infinity in a space–like direction. That is to say, it is of the form \( x + \mathbb{R}^+ e \), where \( e \) is in the manifold of space–like directions \( \mathcal{H}^{d-1} := \{ e \in \mathbb{R}^d : e \cdot e = -1 \} \).

Let now \( U \) be a unitary ray representation of the Poincaré group acting on a Hilbert space \( \mathcal{H} \) with positive energy and a unique invariant vector \( \Omega \), which contains an irreducible ray representation \( U^{(1)} \) acting on \( \mathcal{H}^{(1)} \subset \mathcal{H} \).

**Definition 1.** A string–localized covariant quantum field for \( U^{(1)} \) is an operator valued distribution \( \varphi(x,e) \) over \( \mathbb{R}^d \times \mathcal{H}^{d-1} \) acting on \( \mathcal{H}^{(1)} \) such that the following requirements are satisfied.

0) Reeh–Schlieder property: \( \Omega \) is cyclic for the polynomial algebra of fields \( \varphi(f,h) \) with \( \text{supp} f \times \text{supp} h \) in a fixed region in \( \mathbb{R}^d \times \mathcal{H}^{d-1} \).

i) Covariance: For all \((a, \Lambda) \in \mathcal{P}_+^1 \) and \((x, e) \in \mathbb{R}^d \times \mathcal{H}^{d-1} \) holds

\[
U(a, \Lambda)\varphi(x,e)U(a, \Lambda)^{-1} = \varphi(\Lambda x + a, \Lambda e).
\]

ii) String–locality: If the strings \( x_1 + \mathbb{R}^+ e_1' \) and \( x_2 + \mathbb{R}^+ e_2 \) are space–like separated for all \( e_1' \) in some open neighborhood of \( e_1 \), then

\[
[\varphi(x_1, e_1), \varphi(x_2, e_2)] = 0.
\]

The field is called free, if it creates only single particle states from the vacuum vector, \( \varphi(f,h)\Omega \in \mathcal{H}^{(1)} \).

Our construction of such fields in [13, 14] is reduced to a single particle problem. Namely, consider the single particle vector \( \psi(x,e) := \varphi(x,e)\Omega \) if a free field \( \varphi(x,e) \) as above is given. It enjoys certain specific properties reflecting the covariance and locality of the field. The crucial point is that these properties are intrinsic to the representation \( U^{(1)} \) and can be formulated without reference to the field, using the concept of a modular localization structure [3, 5, 12] based on Tomita–Takesaki modular theory. We will call a \( \mathcal{H}^{(1)} \)-valued distribution satisfying the ensuing properties a string–localized covariant wave function for \( U^{(1)} \), cf. Definition. Our strategy is to reverse the route, namely to construct such a \( \mathcal{H}^{(1)} \)-valued distribution \( \psi(x,e) \) for given \( U^{(1)} \) and then to obtain the field via second quantization.

The idea of the construction of \( \psi(x,e) \) is as follows. Recall that an irreducible representation \( U^{(1)} \) of the Poincaré group is induced by a representation \( D \) of a subgroup \( G \) of the Lorentz group. If \( V \) is an extension of \( D \) to the Lorentz group, then \( U^{(1)} \) is contained in \( U_0 \otimes V \), where \( U_0 \) is the scalar representation. Thus the problem can be separated. The \( U_0 \) part is solved by

\[\text{[Our notion of a string–localized covariant quantum field is a generalization of the generalized Wightman fields of Steinmann [16].]}\]
Fourier transformation. Now Bros et al. exhibit in [2] a suitable representation for which they (implicitly) construct a localized covariant wave function living on $H^{d-1}$. Consider then the tensor product of a wave function localized at $x$ for $U_0$ and a wave function localized at $e \in H^{d-1}$ for $V$. Our basic result is that the projection onto $U^{(1)}$ of this vector turns out to be a vector which is localized for $U^{(1)}$ in the string with initial point $x$ and direction $e$.

We recall the relevant representations $U^{(1)}$ of the Poincaré group and the concept of a modular localization structure in Sections 2 and 3, respectively. We will concentrate on the bosonic representations with positive mass and for those with zero mass and infinite spin, in dimension $d = 3$ and 4. In Section 4 we present the (definition and) construction of a string–localized covariant wave function, as sketched above. In Section 5 we summarize our results and give a brief outlook.

2 Wigner Particles

Following Wigner [17], the state space of an elementary relativistic particle corresponds to an irreducible ray representation of the Poincaré group with positive energy. We recall the relevant representations here for spacetime dimension $d = 3$ and 4, restricting to proper representations since we are at the moment only interested in bosons. We denote the proper orthochronous Poincaré and Lorentz groups by $P^+_\uparrow$ and $L^+_\uparrow$, respectively. Reflecting the semidirect product structure $P^+_\uparrow = \mathbb{R}^d \rtimes L^+_\uparrow$, elements of the Poincaré group will be denoted $g = (a, \Lambda)$.

An irreducible positive energy representation $U^{(1)}$ of $P^+_\uparrow$ is characterized by two data. The first one is the mass value $m \geq 0$, determining the energy–momentum spectrum of the corresponding particle as the mass hyperboloid

$$H^+_m := \{ p \in \mathbb{R}^d : p \cdot p = m^2, p_0 > 0 \}. \quad (4)$$

Given $m$, one fixes a base point $\bar{p} \in H^+_m$, and considers the stabilizer subgroup, within $L^+_\uparrow$, of this point. This so–called “little group” will be denoted $G_{\bar{p}}$ in the sequel. Then the second characteristic of $U^{(1)}$ is a unitary irreducible representation $V$ of $G_{\bar{p}}$, acting in a Hilbert space $\mathfrak{h}$. The representation $U^{(1)}$ fixed by these data is said to be induced from $D$. It acts on

$$\mathcal{H}^{(1)} := L^2(H^+_m, d\mu) \otimes \mathfrak{h}, \quad (5)$$

which we identify with $L^2(H^+_m, d\mu; \mathfrak{h})$, according to

$$(U^{(1)}(a, \Lambda)(\phi \otimes \varphi))(p) = e^{i a \cdot p} \phi(\Lambda^{-1} p) D(R(A, p)) \varphi. \quad (6)$$

Here, $R(A, p)$ is the so–called Wigner rotation, defined by

$$R(A, p) := A_{\bar{p}}^{-1} A A_{A^{-1} p}, \quad (7)$$
where \( (A_p, p \in H^+_m) \) is a section of the bundle \( \mathcal{L}^+_m \to H^+_m \), i.e. \( A_p \) maps \( \bar{p} \) to \( p \).

The little groups \( G_{\bar{p}} \) can be conveniently determined as follows. Let

\[
\Gamma_{\bar{p}} := \{ q \in H^+_0 : q \cdot \bar{p} = 1 \}. \tag{8}
\]

Then \( G_{\bar{p}} \) is precisely the (unit component of the) isometry group of \( \Gamma_{\bar{p}} \). But \( \Gamma_{\bar{p}} \), with the induced metric from ambient Minkowski space, is isometric to the sphere \( S^{d-2} \) for \( m > 0 \), and to \( \mathbb{R}^{d-2} \) for \( m = 0 \). (Eg. for \( m = 0 \) and \( d = 4 \), the map \( \xi : \mathbb{R}^2 \to \Gamma_{\bar{p}} \), with \( \bar{p} = \frac{1}{2}(1,1,0,0) \), defined by

\[
\xi(z) := (|z|^2 + 1, |z|^2 - 1, z_1, z_2) \tag{9}
\]

can be checked to be an isometric diffeomorphism.) It follows that the little group \( G_{\bar{p}} \) is for \( m > 0 \) isomorphic to \( SO(d-1) \), and for \( m = 0 \) isomorphic to the euclidean group in \( d - 2 \) dimensions, i.e. \( G_{\bar{p}} \cong E(2) \) in \( d = 4 \) and \( G_{\bar{p}} \cong \mathbb{R} \) in \( d = 3 \). Now faithful representations of \( E(2) \) are infinite dimensional. Owing to this fact, a representation \( U^{(1)} \) resulting from \( m = 0 \) and a faithful representation of \( G_{\bar{p}} \) is called a massless infinite spin representation. The faithful representations \( D \) of \( E(2) \) are labelled by a strictly positive number \( \kappa > 0 \), and \( D = D_{(\kappa)} \) acts on \( L^2(\mathbb{R}^2, \delta(|k|^2 - \kappa^2)) \) according to

\[
(D_{(\kappa)}(c,R)\phi)(k) := \exp(ic \cdot k) \phi(R^{-1}k), \quad (c,R) \in E(2). \tag{10}
\]

3 Modular Localization Structure for \( U^{(1)} \)

As the first step in our construction, the single particle space is endowed with a family of so–called Tomita operators, labelled by a specific class of spacetime regions. This family will be called a modular localization structure for the single particle space.

The basic geometrical ingredient is the family of wedge regions. A wedge is a region in Minkowski space which arises by a Poincaré transformation from the “standard wedge”

\[
W_0 := \{ x \in \mathbb{R}^d : |x^0| < x^1 \}.
\]

Associated with each wedge \( W \) is the one–parameter group of Lorentz boosts \( A_W(t) \) leaving \( W \) invariant, and the reflection \( j_W \) about the edge of \( W \). More precisely, for the standard wedge \( W_0 \) the boosts \( A_{W_0}(t) \) act on the coordinates \( x^0, x^1 \) as

\[
\begin{pmatrix}
\cosh(t) & \sinh(t) \\
\sinh(t) & \cosh(t)
\end{pmatrix}, \tag{11}
\]

and the reflection \( j_{W_0} \) inverts the sign of the coordinates \( x^0, x^1 \) and leaves the other coordinate(s) invariant. For a general wedge \( W = gW_0, g \in P^+_1 \), the
boosts and reflection are defined as\textsuperscript{3}
\begin{align}
A_{gw_0}(t) &:= g A_{w_0}(t) g^{-1}, \tag{12} \\
 j_{gw} &:= g j_{w_0} g^{-1}. \tag{13}
\end{align}

Let now \(U\) be an (anti-) unitary representation of the proper Poincaré group acting in some Hilbert space \(\mathcal{H}\). Then there is, in particular, for each wedge \(W\) an anti-unitary representer \(U(j_W)\) of the reflection \(j_W\). Let further \(K_W\) denote the self-adjoint generator of the unitary group representing the corresponding boosts, i.e. \(K_W\) is defined by \(\exp(itK_W) = U(A_W(t))\) for all \(t \in \mathbb{R}\). Then we define an anti-linear operator associated with \(W\) by
\begin{equation}
S_U(W) := U(j_W) \exp(-\pi K_W). \tag{14}
\end{equation}

Owing to the group relations, it is an antilinear involution, \(S_U(W)^2 \subset 1\), i.e. a so-called Tomita operator.

We now consider the class of causally complete, convex spacetime regions, which we denote by \(\mathcal{C}\). It is known \cite{3} that each \(C \in \mathcal{C}\) coincides with the intersection of all wedges which contain \(C\). Typical regions belonging to this class are double cones, space–like cones, and wedges. For each \(C \in \mathcal{C}\) we now define the subspace of vectors which are “localized in \(C\)” by
\begin{equation}
\mathcal{D}_U(C) := \{ \psi \in \bigcap_{W \supset C} \text{dom} S_U(W), S_U(W) \psi \text{ independent of } W \}. \tag{15}
\end{equation}

Brunetti et al. have shown \cite{3} that if \(U\) has positive energy, then \(\mathcal{D}_U(C)\) is dense in \(\mathcal{H}\) if \(C\) contains a space–like cone. On this domain we define a closed anti-linear involution \(S_U(C)\) by\textsuperscript{4}
\begin{equation}
S_U(C) \psi := S_U(W) \psi, \quad \forall C. \tag{16}
\end{equation}

The family of these anti–linear involutions satisfies isotony \cite{3}, \(S_U(C_1) \subset S_U(C_2)\) for \(C_1 \subset C_2\), and covariance, \(U(g)S_U(C)U(g)^{-1} = S_U(gC)\). It has further a property \cite{3} which will soon turn out to correspond to locality:

**Lemma 1.** If \(C_1\) and \(C_2\) are causally disjoint, then
\begin{equation}
S_U(C_1) \subset S_U(C_2)^*. \tag{17}
\end{equation}

**Proof.** Choose a wedge \(W\) which contains \(C_1\) and whose causal complement \(W'\) contains \(C_2\). The group relations \(A_{W'}(t) = A_{W}(-t)\) and \(j_{W'} = j_{W}\), cf. \cite{3}, imply that \(S(W') = U(j_W) \exp(\pi K_W)\). On the other hand, \(A_{W}(t)\) commutes with \(j_W\), hence \(U(j_W) \exp(\pi K_W) = \exp(-\pi K_W)U(j_W)\equiv S(W)^*\). Hence \(S(W') = S(W)^*\). Therefore
\begin{equation*}
S(C_1) \subset S(W) = S(W')^* \subset S(C_2)^*,
\end{equation*}
which proves the claim. \hfill \(\Box\)

\textsuperscript{3}This definition is consistent because every Poincaré transformation which leaves \(W_0\) invariant commutes with \(A_{W_0}(t)\) and \(j_{W_0}\), cf. \cite{3}.

\textsuperscript{4}We shall skip the index \(U\) when no confusion can arise.
All these properties motivate us to call the family \( S_U(C), C \in \mathcal{C} \), a modular localization structure\(^5\) for the representation \( U \). They allow the construction of a local and covariant theory for a given particle type from the single particle space via second quantization as follows. Given the corresponding irreducible representation \( U^{(1)} \) of \( \mathcal{P}_+^\uparrow \), extend it to \( \mathcal{P}_+ \) as eg. in Appendix\(^5\), and define \( \mathcal{D}(C) = \mathcal{D}_{U^{(1)}}(C) \) as above. Let \( a^*(\psi) \) and \( a(\psi) \), for \( \psi \in \mathcal{H}^{(1)} \), denote the creation and annihilation operators acting on the symmetrized Fock space over \( \mathcal{H}^{(1)} \). Then define, for \( \psi \in \mathcal{D}(C) \),

\[
\Phi(\psi) := a^*(\psi) + a(S(C)\psi).
\]

These operators generate a covariant and local theory \([3]\), the locality property coming about as follows. For \( \psi_1 \in \mathcal{D}(C_1), \psi_2 \in \mathcal{D}(C_2) \), the commutator \([\Phi(\psi_1), \Phi(\psi_2)]\) equals \((S(C_1)\psi_1, \psi_2) - (S(C_2)\psi_2, \psi_1)\). But if \( C_1 \) is causally disjoint from \( C_2 \), this expression vanishes by Lemma\(^5\) hence

\[
[\Phi(\psi_1), \Phi(\psi_2)] = 0 \quad (19)
\]

in this case. Thus, the property (17) of our modular localization structure implies the locality property of the second quantization.

The motivation to construct the modular localization structure from the representation \( U \) is the Bisognano-Wichmann theorem \([1, 11]\). This theorem states that for a large class of local relativistic quantum fields \( \varphi(f) \) the so-called modular covariance property holds:

\[
S(C)\varphi(f)\Omega = \varphi(f)^*\Omega \quad \text{if } C \supset \text{supp } f,
\]

where \( S(C) \) is constructed as above, cf. \([14, 15]\), from the representation \( U \) under which the field is covariant. Thus, given a local quantum field \( \varphi(f) \), the vectors \( \varphi(f)\Omega, \text{supp } f \subset C \), are the prototypes for elements of the subspace \( \mathcal{D}_{U}(C) \).

### 4 String–Localized Covariant Wave Functions

In view of the above discussion, our task of constructing a string–localized covariant free quantum field for a given particle type reduces to the first-quantized version of the problem: Namely, the construction of the spaces \( \mathcal{D}(C) \)

\(^5\)Note that our notion of a modular localization structure is equivalent to the usual one, as formulated e.g. in \([3, 5, 12, 13]\). There, one considers for each \( C \in \mathcal{C} \) the real subspace

\[
K_U(C) := \bigcap_{W \supset C} \{ \phi \in \text{dom} S_U(W) : S_U(W)\phi = \phi \}.
\]

These are precisely the +1 eigenspaces of our Tomita operators \([15, 16]\). But these eigenspaces are well–known \([8, 15]\) to be in one-to-one correspondence with the latter.
in terms of “covariant string–localized wave functions” as mentioned in the introduction. These are defined as follows. Let $U^{(1)}$ be the corresponding representation, acting on $\mathcal{H}^{(1)}$.

Definition 2. A string–localized covariant wave function for $U^{(1)}$ is a weak $\mathcal{H}^{(1)}$-valued distribution $\psi(x,e)$ on $\mathbb{R}^d \times H^{d-1}$ satisfying the following requirements.

0) The set of $\psi(f,h)$, with $\text{supp} f \times \text{supp} h$ in a fixed compact region in $\mathbb{R}^d \times H^{d-1}$, is dense in $\mathcal{H}^{(1)}$.

i) Covariance: For all $(a,\Lambda) \in P^+_\uparrow$ and $(x,e) \in \mathbb{R}^d \times H^{d-1}$ holds

$$U^{(1)}(a,\Lambda)\psi(x,e) = \psi(\Lambda x + a, \Lambda e).$$ (20)

ii) String–locality: If $\text{supp} f + \mathbb{R}^+ \text{supp} h \subset C \in \mathcal{C}$, then $\psi(f,h)$ is in $\mathcal{D}_{U^{(1)}}(C)$.

Given such $\psi(x,e)$, one verifies that

$$\varphi(x,e) := \Phi(\psi(x,e)),$$ (21)

with $\Phi(\psi)$ as in (18), is a string–localized covariant free quantum field in the sense of Definition 1. (Locality (3) follows from (19).)

Example 1. To illustrate the concept, we consider the scalar irreducible unitary representation $U_0$ with mass $m \geq 0$. (Scalar means that the little group is represented trivially.) For $f \in \mathcal{S}(\mathbb{R}^d)$, let $Ff$ denote the restriction of the Fourier transform of $f$ to the mass shell $H^+_m$. This map enjoys the covariance properties

$$U_0(g) Ff = Fg_\ast f, \quad g \in P^+_\uparrow,$$ (22)

$$U_0(j) Ff = Fj_\ast \tilde{f}, \quad j \in P^+_\downarrow,$$ (23)

where $(g_\ast f)(x) := f(g^{-1}x)$. Further, if $f$ has compact support contained in some wedge $W$, then

$$\exp(-\pi (K_0)_W) Ff = F(j_W)_* f,$$ (24)

where $(K_0)_W$ is the generator of $U_0(\Lambda_W(t))$. The basic fact underlying this identity is that for $x \in W$, the analytic function $t \mapsto \Lambda_W(t)x$ has imaginary part in the forward light cone for $t$ in the strip $\mathbb{R} + i(0, \pi)$, and goes to $j_W x$ if $t$ goes to $i\pi$. Lemma 5 in the appendix then implies (24). It follows that $S_{U_0}(W) Ff = F\tilde{f}$, hence $Ff \in \mathcal{D}_{U_0}(O)$ if supp $f \subset O$. Consequently the map $f \mapsto Ff$ is, in analogy to the above definition, a (point-) localized covariant wave function for $U_0$. Note that the definition (21), namely $\varphi(f) := \Phi(Ff)$ with $\Phi(\cdot)$ as in (18), then coincides with the usual scalar free field.

We now turn to the construction of a string–localized covariant wave function for arbitrary $U^{(1)}$ with mass $m \geq 0$ and faithful (or scalar) inducing representation $D$ of the little group $G_\tilde{p}$. Bros et al. exhibit in [2] a family of unitary
irreducible representations $V^\alpha$ of the Lorentz group $\mathcal{L}_+$, labelled by a complex number $\alpha$ with real part $-(d-2)/2$. As we show in Lemma 3, the inducing representation $D$ is contained in the restriction of $V^\alpha$ to $G_{\tilde{\mathcal{P}}}$, namely as a subrepresentation if $m > 0$ and in a direct integral decomposition if $m = 0$. This implies that $U^{(1)}$ is contained in the representation induced by $V^\alpha|_{G_{\tilde{\mathcal{P}}}}$. But the latter is equivalent to the representation $U_0 \otimes V^\alpha$, hence $U^{(1)}$ is contained in $U_0 \otimes V^\alpha$. More precisely, there is a map $R^\alpha$ from (a dense domain in) the tensor product of the representation spaces of $U_0$ and $V^\alpha$ into the representation space of $U^{(1)}$ satisfying the intertwiner relation

$$U^{(1)}(a, \Lambda) \circ R^\alpha = R^\alpha \circ U_0(a, \Lambda) \otimes V^\alpha(\Lambda), \quad (a, \Lambda) \in \mathcal{P}_+^1,$$

(25)

on its domain. We write down a suitable intertwiner $R^\alpha$ in Lemma 3 which turns out to satisfy also

$$U^{(1)}(j) \circ R^\alpha = R^\alpha \circ U_0(j) \otimes V^\alpha(j), \quad j \in \mathcal{L}_+^1.$$

(26)

Thus, the problem of finding a string–localized wave function can now be separated. For $U_0$ we already have a localized wave function, cf. Example 1. Now for $V^\alpha$, Bros et al. [2] construct implicitly a “localized covariant wave function” on $H^{d-1}$, in the following sense:

**Example 2.** There is a continuous linear map $F^\alpha$ from $\mathcal{D}(H^{d-1})$ into the representation space of $V^\alpha$ with the following properties:

(i) The set of $F^\alpha h$, with $\text{supp} h$ in a fixed region in $H^{d-1}$, is dense.

$$V^\alpha(\Lambda) F^\alpha h = F^\alpha \Lambda_+ h, \quad \Lambda \in \mathcal{L}_+^1,$$

(27)

$$V^\alpha(j) F^\alpha h = F^\alpha j_+ h, \quad j \in \mathcal{L}_+^1.$$

(28)

(ii) For a wedge $W$ whose edge contains the origin, let $K^\alpha_W$ be the generator of $V^\alpha(\Lambda_W(t))$. Then for all $h \in \mathcal{D}(H^{d-1})$ with $\text{supp} h \subset W \cap H^{d-1}$, the vector $F^\alpha h$ is in the domain of $\exp(-\pi K^\alpha_W)$, and

$$\exp(-\pi K^\alpha_W) F^\alpha h = F^\alpha(j_W)_+ h.$$

(29)

(We recall the definition of $F^\alpha$ in the appendix, cf. 49, and show in Lemma 4 that the mentioned properties are implicitly contained in [2].)

All this implies that a good candidate for a covariant string–localized wave function in the sense of Definition 2 is given by

$$\psi^\alpha(f, h) := R^\alpha(F f \otimes F^\alpha h),$$

(30)

with $f \in \mathcal{D}(\mathbb{R}^d)$ and $h \in \mathcal{D}(H^{d-1})$. We have in fact the following result.

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6 $V^\alpha$ is in fact equivalent to the irreducible principal series representation corresponding to the value $-|\alpha|^2$ of the Casimir operator.
Proposition 1. Equation (30) defines a string–localized covariant wave function for $U^{(1)}$ in the sense of Definition 2. Moreover, for $C \in \mathcal{C}$ the Tomita operator $S(C)$ acts as follows. Let $\mathcal{O} \subset \mathbb{R}^d$ and $\mathcal{O}^\circ \subset H^{d-1}$ be such that $\mathcal{O} + \mathbb{R}^+ \mathcal{O} \subset C$. Then for all $f$ with supp$f \subset \mathcal{O}$ and $h$ with supp$h \subset \mathcal{O}$,

$$S(C) \psi^\alpha(f, h) = \psi^\alpha(f, \bar{h}).$$

(31)

For $m = 0$, $\psi^\alpha(f, h)$ has the explicit form

$$\psi^\alpha(f, h)(p) = Ff(p) u^\alpha(h, p),$$

(32)

where $h \mapsto u^\alpha(h, p)$ is the $h$-valued distribution on $H^{d-1}$ with kernel

$$u^\alpha(e, p)(k) = e^{-ip\alpha/2} \int_{\mathbb{R}^{d-2}} d^2z e^{i\mathbf{k} \cdot \mathbf{z}} (\xi(z) \cdot A_p^{-1} e)^\alpha.$$  

(33)

Here $z \mapsto \xi(z)$ is the isometry from $\mathbb{R}^{d-2}$ onto $\Gamma_p$ exhibited in (9).

Proof. We first consider $m > 0$, in which case the intertwiner $R^\alpha$ is a partial isometry defined on the whole Hilbert space, cf. Lemma 3. Then the covariance condition $i)$ of Definition 2 is satisfied by construction, cf. (22), (25) and (27). The “Reeh-Schlieder” property $0)$ of Definition 2 follows from the well-known Reeh–Schlieder property of $ff$ and that of $F^\alpha h$, cf. $0)$ of Example 2. It remains to prove (31), which then implies the locality property $ii)$. As a first step, let $f$ and $h$ be such that supp$f + \mathbb{R}^+ \text{supp}h$ is contained in the standard wedge $W_0$. Then it follows that supp$f$ is contained in $W_0$ and supp$h$ in its closure. Suppose first that supp$h \subset W_0$. Then from (24) and (29) we know that the vectors $ff$ and $F^\alpha h$ are in the domains of the corresponding “modular operators” $\exp(-\pi K_{W_0})$ and that the latter maps them to $F(j_0)_0 f$ and $F^\alpha(j_0)_0 h$, respectively. From the intertwining property (26) of $R^\alpha$ and its continuity it follows (eg. using Lemma 5) that $\psi^\alpha(f, h)$ is in the domain of $\exp(-\pi K_{W_0})$ and that

$$\exp(-\pi K_{W_0}) \psi^\alpha(f, h) = \psi^\alpha((j_0)_0 f, (j_0)_0 h).$$

(34)

Further, (25), (26) and (28) imply that $U^{(1)}(j) \psi^\alpha(f, h) = \psi^\alpha(j_0 f, j_0 h)$ for $j \in \mathcal{L}^*_+$. Now the last two equations imply that

$$S(W_0) \psi^\alpha(f, h) = \psi^\alpha(\bar{f}, \bar{h}).$$

(35)

If, on the other hand, supp$h$ meets the boundary of $W_0$ (but is contained in its closure), then one finds a sequence $h_n \rightarrow h$ so that supp$h_n \subset W_0$ for all $n$. Then $S(W_0)\psi^\alpha(f, h_n)$ goes to $\psi^\alpha(f, \bar{h})$, and (34) also holds in this case because $S(W_0)$ is closed. By covariance, it follows that for any wedge $W$, the operator $S(W)$ acts as in (35) if supp$f + \mathbb{R}^+ \text{supp}h \subset W$. This proves (31). The continuity property follows from that of $f$, $F^\alpha$ and $R^\alpha$. The proof is complete for $m > 0$. 


For $m = 0$, we show in [14] the following facts. $R^\alpha$ is well-defined on vectors of the form $F f \otimes F^\alpha h$, leading to the formula (32), (33), and the intertwining properties (25) and (26) hold on these vectors. Further, if $h$ has support in a wedge $W$, then for almost all $p$ the $h$-valued function $t \mapsto u^\alpha(A_W(t), h, p)$ is analytic on the strip $\mathbb{R} + i(0, \pi)$ and weakly continuous on its closure. It is uniformly bounded in $p$ and $t$, for $p$ in a dense set of $H_d^+$ and for $t$ in any compact subset of the closure of the strip. As $t$ goes to $i\pi$, it goes to $u((jW)_*, h, p)$. This implies (34), eg. using Lemma 5 (details are spelled out in [14]). The proof of (31) is then completed as in the case $m > 0$. Finally, we show in [14] an analyticity and growth property in $e$ of $u^\alpha(e, p)$ which implies continuity of $\psi^\alpha(f, h)$.

\section{5 Summary and Outlook}

We have constructed, for each $\alpha \in \mathbb{C}$ with $\Re \alpha = -(d - 2)/2$ and each massless “infinite spin” representation $U^{(1)}$, a $\mathcal{H}^{(1)}$-valued distribution on $\mathbb{R}^d \times H^{d-1}$ with certain specific properties, which motivate our name “string–localized covariant wave function”. They guarantee that second quantization of these objects leads to a string–localized covariant free quantum field, cf. Definition 2 and discussion thereafter. Summarizing, and using the explicit formula (32), we have as our main result:

**Theorem 1.** Let $\varphi(x, e)$ be the operator–valued distribution given by

$$\varphi^\alpha(x, e) = \int_{H^+_d} d\mu(p) \left\{ e^{ip \cdot x} u^\alpha(e, p) \circ a^*(p) + e^{-ip \cdot x} \overline{u^\alpha(e, p)} \circ a(p) \right\}, \quad (36)$$

with $u^\alpha$ as in (33). Then $\varphi(x, e)$ is a string–localized covariant free quantum field for $U^{(1)}$ in the sense of Definition 1.

It turns out [14] that the formula works for all $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$, and that for a certain range of values the fields need not be smeared in the directional variable $e$. It also works with $u^\alpha$ and $\overline{u^\alpha}$ replaced by $F(p \cdot e)u^\alpha(e, p)$ and $\overline{F(-p \cdot e)}u^\alpha(e, p)$, respectively, where $F$ is the distributional boundary value of a suitable function which is analytic on the upper half plane. The resulting fields are all relatively “string–local” to each other. It is shown in [14] that every string–localized covariant free field, in the sense of Definition 1, is of the above form.

An important open problem for our infinite spin fields is the existence of local observables. These are operators which are localized in bounded regions, in the sense that they commute with field operators localized causally disjoint from the respective region. In this sense, a local expression for the energy

\footnote{We use the symbolic notation $\overline{a^*(\psi)} =: \int d\mu(p) \overline{\psi(p)} \circ a^*(p)$ and $a(\psi) =: \int d\mu(p) \psi(p) \circ a(p)$.}
density is of particular interest, since it would be valuable for a discussion of the thermodynamic properties of the KMS states [13] of our fields.

We have performed the above construction also for massive bosons with arbitrary spin, and similar constructions work for fermions with half–integer spin and for photons [14]. Our photon field \(A_\mu(x,e)\) is a string–localized covariant version of the “axial gauge”, acting on the physical photon Hilbert space. The resulting fields in all these cases are strictly string–localized, but relatively local to the corresponding standard point–localized free fields. (In fact, they can be written as certain line integrals over the latter [14].)

The reason why these fields nevertheless have the potential for applications is that they might serve as ingredients for the construction of interacting models with string–like localization. Recall that the results of [3, 4] support the viewpoint that localization of charged quantum fields in space–like cones (the idealizations of which are our strings) is a natural concept, yet there is so far a lack of rigorous model realizations\(^8\). There are two reasons to believe that our free fields are good starting points for a construction of interacting fields with strict string–localization. Firstly, since the obstruction to point–like localization is due to the charge, which is already carried by the single particle states, one should expect that already the latter are strictly string–localized. That is to say, the single particle states \(E^{(1)}(\varphi\Omega)\), where \(E^{(1)}\) denotes the projection onto the single particle space and \(\varphi\) is an interacting field, are string–like (but not point–like) localized in the sense of (15). But then the LSZ relations imply that the corresponding incoming and outgoing free fields are also strictly string–localized. Therefore, our fields might represent the in– and out–fields of such a model, in contrast to the usual point–localized free fields. Secondly, the distributional character of our free fields is less singular than that of the point–localized free fields, as is made precise in [14], even more so in the direction of the localization string. This fact should lead to a larger class of admissible interactions in a perturbative construction, as compared to taking the standard point–localized free fields as starting point.

### A Extension of the Representations to \(\mathcal{P}_+\)

The proper Poincaré group \(\mathcal{P}_+\) is generated by \(\mathcal{P}_+^\uparrow\) and any single element \(j_0\) in \(\mathcal{P}_+^\downarrow\). We choose

\[
j_0 := j_{e0}.
\]

(Note that \(-j_0\) is in \(\mathcal{P}_+^\downarrow\) and hence leaves each mass shell \(H^+_m, m \geq 0\), invariant.) As to the irreducible representation \(U^{(1)}\), we choose the base point \(\bar{p} \in H^+_m\) so that

\[
-j_0\bar{p} = \bar{p}.
\]

\(^8\)apart from non–Lorentz covariant infra–vacua models as in [7] and lattice models as in [9, 10].
Then the section \( p \to A_p \) of the bundle \( L^+_+ \to L^+_+/G_\bar{\rho} = H^+_m \) can be chosen [14] so that it transforms under the adjoint action of \( j_0 \) as
\[
j_0 A_p j_0 = A_{-j_0 p}.
\] (39)

Let \( D(j_0) \) be the anti-unitary involution from Lemma 2 below. Then, by virtue of (39) and (41), the anti-unitary involution defined by
\[
(U^{(1)}(j_0)\psi)(p) = D(j_0) \psi(-j_0 p)
\] (40)
extends \( U^{(1)} \) to an (anti-)unitary representation of \( P^+ \) within the same Hilbert space \( \mathcal{H}^{(1)} = L^2(H_0^+, d\mu) \otimes \mathfrak{h} \). Due to irreducibility, \( U^{(1)}(j_0) \) is fixed up to a phase factor.

**Lemma 2.** There is an anti-unitary involution \( D(j_0) \) acting on \( \mathfrak{h} \) satisfying the representation properties
\[
D(j_0)^2 = 1 \quad \text{and} \quad D(j_0)D(A)D(j_0) = D(j_0 A j_0), \quad A \in G_\bar{\rho}.
\] (41)

The existence of such a representer is established in Lemma 3. Note that the adjoint action of \( j_0 \) leaves \( G_\bar{\rho} \) invariant due to (38), hence the Lemma states that \( D \) extends to a representation of the subgroup of \( P^+ \) generated by \( G_\bar{\rho} \) and \( j_0 \).

### B Intertwiners and Localization Structure for the Principal Series Representations

We recall the representation of the Lorentz group presented by Bros et al. in [2]. Fix a complex number \( \alpha \) with real part \(-\frac{d-2}{2}\). Let \( H_0^+ \) denote the mantle of the forward light cone in \( \mathbb{R}^d \) as before, and let \( C^\alpha(H_0^+) \) denote the space of continuous \( \mathbb{C} \)-valued functions on \( H_0^+ \) which are homogenous of degree \( \alpha \), i.e.

\[
C^\alpha(H_0^+) := \{ \psi \in C(H_0^+) : \psi(tp) = t^\alpha \psi(p), \quad t > 0 \}.
\]

Consider the maps \( V^\alpha(A), A \in \mathcal{L}_+ \), defined on \( C(H_0^+) \) by
\[
(V^\alpha(A)\psi)(p) := \psi(A^{-1} p), \quad A \in \mathcal{L}_+, \quad \alpha \in \mathbb{C}.
\] (42)
\[
(V^\alpha(j)\psi)(p) := \psi(-jp), \quad j \in \mathcal{L}_+^\perp.
\] (43)

Clearly, \( V^\alpha|\mathcal{L}_+^\perp \) establishes a representation of \( \mathcal{L}_+^\perp \) in \( C^\alpha(H_0^+) \), while \( V^\alpha|\mathcal{L}_+ \) maps \( C^\alpha(H_0^+) \) onto \( C^\alpha(H_0^+) \), and the pair \( V^\alpha, V^{\bar{\alpha}} \) satisfies the following representation property:
\[
V^{\bar{\alpha}}(j_1)V^\alpha(A)V^\alpha(j_2) = V^\alpha(j_1 A j_2), \quad A \in \mathcal{L}_+, j_1, j_2 \in \mathcal{L}_+^\perp.
\] (44)
Let now \( \Gamma \) be any \((d-2)\)-dimensional cycle which encloses the origin. Then \( C^\alpha(H_0^m) \) can (and will) be identified with \( C(\Gamma) \). Let \( d\nu_\Gamma \) be the restriction of the Lorentz invariant measure \( d\nu \) on \( H_0^+ \) to \( \Gamma \), and define a scalar product on \( C(\Gamma) \) by

\[
(\psi, \psi') := \int_\Gamma \overline{\psi(p)} \psi'(p) \, d\nu_\Gamma(p). \tag{45}
\]

As Bros and Moschella point out [2], the representation \( V^\alpha \) of the Lorentz group is unitary w.r.t. this scalar product. The corresponding Hilbert space completion of \( C(\Gamma) \) will be denoted by \( \mathcal{H}' \), and the extension of \( V^\alpha \) to this space will be denoted by the same symbol. It is equivalent to the irreducible principal series representation corresponding to the value \(-|\alpha|^2\) of the Casimir operator [2].

**Lemma 3.** i) Let \( D \) be a faithful irreducible representation of \( G_{\bar{p}} \), and let \( \Re \alpha = -(d-2)/2 \). Then \( V^\alpha|G_{\bar{p}} \) contains \( D \), i.e. there is a map \( T \) from a dense domain in \( \mathcal{H}' \) onto a dense subspace of \( \mathcal{H} \) which intertwines the representations \( V^\alpha|G_{\bar{p}} \) and \( D \) in the sense that

\[
D(\Lambda) \circ T = T \circ V^\alpha(\Lambda), \quad \Lambda \in G_{\bar{p}}, \tag{46}
\]

holds on its domain. In the case \( m > 0 \), \( D \) is a subrepresentation of \( V^\alpha \), while for \( m = 0 \), \( D \) occurs in a direct integral decomposition of \( V^\alpha \). \( T \) also intertwines \( V^\alpha(j_0) \), in the sense of \([40]\), with an anti–unitary operator \( D(j_0) \) satisfying the representation properties \([11]\).

ii) Let \( R^\alpha \) be the map from (a dense domain in) \( L^2(H_m^+) \otimes \mathcal{H}' \) into \( \mathcal{H}^{(1)} = L^2(H_m^+) \otimes \mathcal{H} \) defined by

\[
(R^\alpha(\phi \otimes \varphi))(p) := \phi(p) TV^\alpha(A_{\bar{p}}^{-1})\varphi. \tag{47}
\]

Then \( R^\alpha \) satisfies on its domain the intertwiner relations \([26]\) and \([28]\).

**Proof.** Ad i). We choose the cycle \( \Gamma \) conveniently as \( \Gamma := \Gamma_{\bar{p}} \) defined in \([3]\). As mentioned, the cycle \( \Gamma = \Gamma_{\bar{p}} \) is isometric to the sphere \( S^{d-2} \) for \( m > 0 \), and to \( \mathbb{R}^{d-2} \) for \( m = 0 \), and its isometry group coincides with \( G_{\bar{p}} \). Hence the action of \( G_{\bar{p}} \) on \( \Gamma \) corresponds to the natural action of \( SO(d-1) \) on \( S^{d-2} \) for \( m > 0 \), and to the natural action of \( E(d-2) \) on \( \mathbb{R}^{d-2} \) for \( m = 0 \). It also follows that the invariant measure \( d\nu_\Gamma \) goes over into the \( SO(d-1) \) invariant measure \( d\Omega \) on \( S^{d-2} \) or the Lebesgue measure \( dz \) on \( \mathbb{R}^{d-2} \), respectively. In the case \( m > 0 \), it follows that \( \mathcal{H}' \) is naturally isomorphic to \( L^2(S^{d-2}, d\Omega) \), and \( V^\alpha|G_{\bar{p}} \) acts as the push–forward representation. As is well-known, this representation decomposes into the direct sum of all irreducible representations \( D(\alpha) \) of \( SO(d-1) \). (In \( d = 4 \), \( s \) runs through \( \mathbb{N}_0 \) and the irreducible subspaces

\^Note that \( \Gamma_{\bar{p}} \) is invariant under \( G_{\bar{p}} \) and \(-j_0\), which implies that the restriction of \( V^\alpha \) to the subgroup of \( \mathcal{L}_+ \) generated by \( G_{\bar{p}} \) and \( j_0 \) does not depend on \( \alpha \).
are spanned by the spherical harmonics $Y_{s,m}$, and in $d=3$, $s$ runs through $\mathbb{Z}$ and the irreducible subspaces are spanned by $\theta \mapsto \exp(i\theta)$. Hence, for each $s$ there is a partial isometry $T = T_{(s)}$ with the claimed property \(11\). Further, under the mentioned equivalence $\Gamma \cong S^{d-2}$ the representer of $j_0$ acts as $(V^\alpha(j_0)\varphi)(n) = \varphi(I_0n)$, where $I_0$ corresponds to $-j_0$ and is hence in $O(d-1)$. Since the spherical harmonics $\{Y_{s,m}, m = -s, \ldots, s\}$ for given $s \in \mathbb{N}$ are invariant under $O(3)$ as well as under complex conjugation, it follows that $V^\alpha(j_0)$ leaves each irreducible subrepresentation invariant in $d=4$. This implies that $V^\alpha(j_0)$ is intertwined by $T$ with an (anti-unitary) operator $D(j_0)$ satisfying \(11\), as claimed. In $d=3$, the same conclusion follows from the facts that $I_0$ is an orientation reversing isometry of the circle, hence $SO(2)$-conjugate to $\theta \mapsto -\theta$, and that $\exp(i\theta) = \exp(i\theta)$.

Similarly, in the case $m = 0$, $\mathfrak{h}'$ is naturally isomorphic to $L^2(\mathbb{R}^{d-2}, dz)$, and $V^\alpha|G_{\tilde{p}}$ acts as the push–forward representation. Via Fourier transformation, this representation decomposes into a direct integral of irreducible representations $D_{(\kappa)}$, where $\kappa$ runs through $\mathbb{R}$ for $d = 3$, and through $\mathbb{R}^+$ for $d = 4$. Thus there is a densely defined intertwiner $T$ satisfying \(40\) on its domain: $T\varphi$ is the restriction of the Fourier transform of $\varphi$ to the circle with radius $\kappa$ for $d = 4$, respectively its value at $\kappa$ for $d = 3$. Further, under the mentioned equivalence $\Gamma \cong \mathbb{R}^{d-2}$ the representer of $j_0$ acts as $(V^\alpha(j_0)\varphi)(z) = \varphi(I_0z)$, where $I_0$ corresponds to $-j_0$. With our explicit formula \(41\), $I_0$ coincides with the reflection $z \mapsto -z$. The identity $TV^\alpha(j_0)\varphi = T\varphi$ then implies that $V^\alpha(j_0)$ leaves the kernel of $T$ invariant. Hence

$$D(j_0)T\varphi := T V^\alpha(j_0)\varphi,$$

defines an anti–unitary operator $D(j_0)$ on the image of $T$, which also has the representation property \(11\), as claimed.

Ad ii). The intertwiner relations \(25\) and \(26\) follow from part i), \(39\) and \(41\). 

\[\square\]

We now discuss the map $F^\alpha$ defined by Bros et al. \cite{2}, which we used in Example \(2\). It is the Fourier-Helgason type transformation given by

$$F^\alpha : \mathcal{D}(H^{d-1}) \rightarrow C^\alpha(H^+_0) \subset \mathfrak{h},$$

$$F^\alpha h)(p) := e^{-i\pi\alpha/2} \int_{H_{d-1}} d\sigma(e) h(e) (e \cdot p)^\alpha. \quad (49)$$

Here, $e \cdot p$ denotes the scalar product in $d$-dimensional Minkowski space, of which $H^{d-1}$ and $H^+_0$ are considered submanifolds. The power $t^\alpha$ is defined via the branch of the logarithm on $\mathbb{R} \setminus \mathbb{R}_0^+$ with $\ln 1 = 0$, and as $\lim_{t \to 0^+}(t + i\varepsilon)^\alpha$ for $t < 0$. Further, $d\sigma$ denotes the Lorentz invariant measure on $H^{d-1}$. In our context, the upshot of this transformation is the following.

\[10\] Using another isometric diffeomorphism yields the same $I_0$ up to conjugation with a euclidean transformation, leading to the same conclusion.
Lemma 4 (Bros et al.). The map $h \mapsto F^\alpha h$ establishes a “localized covariant wave function” on $H^{d-1}$ in the sense of the properties 0)...ii) listed in Example 2.

Proof. The transformation $F^\alpha$ has been taken over from [2] in such a way that $F^\alpha h = \phi(h)\Omega$, where $\phi(\cdot)$ is the free field of [2], cf. [2, eq. (4.30)]. In this context, property 0) of our Example 2 is the Reeh-Schlieder property, Proposition 5.4 of [2]. The covariance property i) corresponds to the covariance of the field $\phi(\cdot)$ (and also follows directly from the definitions). Finally, the geometrical KMS condition [2, Prop. 2.3] enjoyed by the two-point function of $\phi(\cdot)$ implies that $F^\alpha h$ is in the domain of the Tomita operator $\exp(-\pi K^\alpha_w)$. The antipodal condition [2, Prop. 2.4] then shows that this operator acts on $F^\alpha h$ as in (29). This proves ii). □

We finally mention a standard result, which we have used occasionally in the context of our modular operators.

Lemma 5. Let $U_t$ be a continuous unitary one-paramter group, with generator $K$. Then $\psi$ is in the domain of $\exp(-\pi K)$ if, and only if, the vector-valued map

$$t \mapsto U_t \psi$$

is analytic in the strip $\mathbb{R} + i(0, \pi)$ and weakly continuous on the closure of that strip. In this case, $\exp(-\pi K)\psi$ coincides with the analytic continuation of $U_t \psi$ into $t = i\pi$.

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