STRICLY FLAT CYCLIC FRÉCHET MODULES
AND APPROXIMATE IDENTITIES

A. YU. PIRKOVSKIY

Abstract. Let $A$ be a locally $m$-convex Fréchet algebra. We give a necessary and sufficient condition for a cyclic Fréchet $A$-module $X = A_+ / I$ to be strictly flat, generalizing thereby a criterion of Helemskii and Sheinberg [7]. To this end, we introduce a notion of locally bounded approximate identity (a.i.), and we show that $X$ is strictly flat if and only if the ideal $I$ has a right locally bounded a.i. An example is given of a commutative locally $m$-convex Fréchet algebra that has a locally bounded a.i., but does not have a bounded a.i. On the other hand, we show that a quasinormable locally $m$-convex Fréchet algebra has a locally bounded a.i. if and only if it has a bounded a.i. Some applications to amenable Fréchet algebras are also given.

1. Introduction

Flat Banach modules over Banach algebras were introduced by Helemskii by analogy with pure algebra. An important fact which explains the rôle of flat modules in Functional Analysis is that flatness is closely related to amenability. More exactly, amenable Banach algebras can be characterized in terms of flat Banach modules (see [7 VII.2]; cf. also Section A below).

The definition of flat Banach module readily extends to Fréchet modules over Fréchet algebras. By a Fréchet algebra we mean a complete, Hausdorff, metrizable locally convex $\mathbb{C}$-algebra. If $A$ is a Fréchet algebra, then a left Fréchet $A$-module is a Fréchet space $X$ together with the structure of a left $A$-module such that the product map $A \times X \to X$ is continuous. Right Fréchet $A$-modules are defined similarly. The category of all left (respectively, right) Fréchet $A$-modules will be denoted by $A$-mod (respectively, mod-$A$). Recall from [7] that a chain complex $Y_\bullet$ of Fréchet $A$-modules is admissible if it splits in the category of Fréchet spaces, i.e., if it has a contracting homotopy consisting of continuous linear maps.

Definition (Helemskii). A left Fréchet $A$-module $X$ is said to be flat (respectively, strictly flat) if for each admissible complex (respectively, for each exact complex) $Y_\bullet$ in mod-$A$ the sequence $Y_\bullet \otimes_A X$ is exact.

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Suppose now that $X$ is a cyclic left Fréchet $A$-module, i.e., $X \cong A_+/I$ for a closed left ideal $I \subset A_+$. Is is natural to ask when $X$ is flat or strictly flat. This question is important, for example, because of its connection with amenability (see [7, VII.2] and Section 6 below).

In the case of Banach algebras, the answer is as follows (see [7, VII.1.4]).

**Theorem (Helemskii, Sheinberg).** Let $A$ be a Banach algebra, and let $I \subset A_+$ be a closed left ideal. Then the following conditions are equivalent:

(i) $A_+/I$ is strictly flat;

(ii) $I$ has a right b.a.i.

If, in addition, $I$ is weakly complemented in $A_+$ (i.e., if the annihilator of $I$ is complemented in $A_+^*\otimes A$), then (i) and (ii) are equivalent to

(iii) $A_+/I$ is flat.

**Remark 1.** In the original form of the Helemskii-Sheinberg theorem, the phrase “$X$ is flat” (respectively, strictly flat) actually means “$X$ is flat (respectively, strictly flat) as a Banach $A$-module”, i.e., for each admissible (respectively, exact) complex $Y_\bullet$ of right Banach $A$-modules the sequence $Y_\bullet \otimes_A X$ is exact. However, this automatically implies the exactness of $Y_\bullet \otimes_A X$ for each admissible (respectively, exact) complex $Y_\bullet$ of right Fréchet $A$-modules, as the following proposition suggests.

**Proposition 1.** Let $A$ be a Banach algebra, and let $X$ be a left Banach $A$-module. Suppose that $X$ is flat (respectively, strictly flat) when considered as a Banach $A$-module. Then $X$ is flat (respectively, strictly flat) as a Fréchet $A$-module.

The aim of this paper is to generalize the Helemskii-Sheinberg theorem to Fréchet-Arens-Michael algebras (i.e., to locally $m$-convex Fréchet algebras).

**Remark 2.** Related results were independently obtained by C. Podara.

## 2. The main result

Let $A$ be an Arens-Michael algebra, i.e., a complete locally $m$-convex algebra. Recall (see [4] or [5]) that $A$ is isomorphic to a projective limit of Banach algebras. More exactly, let $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed family of submultiplicative seminorms generating the topology of $A$. Given $\lambda, \mu \in \Lambda$, we write $\lambda \prec \mu$ if $\|a\|_\lambda \leq \|a\|_\mu$ for all $a \in A$. For each $\lambda \in \Lambda$ we set $N_\lambda = \{a \in A : \|a\|_\lambda = 0\}$. Since $\|\cdot\|_\lambda$ is submultiplicative, we see that $N_\lambda$ is a two-sided ideal of $A$, so that $A/N_\lambda$ is an algebra in a natural way. Moreover, the seminorm $\|\cdot\|_\lambda$ determines a submultiplicative norm on $A/N_\lambda$. Hence the completion of $A/N_\lambda$ with respect to this norm is a Banach algebra. This algebra is denoted by $A_\lambda$ and is called the accompanying Banach algebra of $A$ corresponding to the seminorm $\|\cdot\|_\lambda$. The canonical homomorphism $A \to A_\lambda, a \mapsto a + N_\lambda$, will be denoted by $\tau_\lambda$. If $\lambda, \mu \in \Lambda$ and $\lambda \prec \mu$, then there is a unique continuous homomorphism $\tau_\lambda^\mu : A_\mu \to A_\lambda$ such that $\tau_\lambda = \tau_\lambda^\mu \tau_\mu$. The family $\{\tau_\lambda : \lambda \in \Lambda\}$ determines a continuous homomorphism from $A$ to the projective limit $\lim_{\leftarrow} \{A_\lambda, \tau_\lambda^\mu\}$, and the Arens-Michael decomposition theorem states that this homomorphism is a
The situation described above is usually expressed by the phrase "Let $A$ be an Arens-Michael algebra, and let $A = \varprojlim A_\lambda$ be the Arens-Michael decomposition of $A$.

For each $\lambda \in \Lambda$ the homomorphism $\tau_\lambda : A \to A_\lambda$ uniquely extends to a unital homomorphism $\tau_\lambda^+ : A_+ \to (A_\lambda)_+$, and a similar statement is true for all the connecting homomorphisms $\tau^\mu_\lambda (\lambda \prec \mu)$. Obviously, $A_+ \cong \varprojlim (A_\lambda)_+$. Given a closed left ideal $I \subset A_+$, we set $I_\lambda = \tau_\lambda^+(I) \subset (A_\lambda)_+$. Evidently, $I_\lambda$ is a closed left ideal of $(A_\lambda)_+$. Note that $\tau_\mu^\lambda (I_\mu) \subset I_\lambda$ for each $\lambda \prec \mu$, and that $I \cong \varprojlim I_\lambda$.

From now on, we suppose that $A$ is a Fréchet-Arens-Michael algebra. Without loss of generality, we may assume that $(\Lambda, \prec) = (\mathbb{N}, \leq)$.

**Theorem 1.** The following conditions are equivalent:

(i) $A_+/I$ is strictly flat;

(ii) for each $n \in \mathbb{N}$, $I_n$ has a right b.a.i.

**Remark 3.** The implication (i)$\implies$(ii) of the above theorem holds for any Arens-Michael algebra. However, we do not know whether (ii) implies (i) without the metrizability condition.

It is easy to see that if $I$ has a right b.a.i., then so does $I_n$ for each $n \in \mathbb{N}$. A natural question is then whether the converse is also true. As we shall see later, this need not be the case in general, but this is the case under some additional linear topological assumptions on $I$. Another natural question is whether it is possible to formulate condition (ii) intrinsically, i.e., without referring to the accompanying Banach algebras. Let us start by answering the latter question.

### 3. Locally bounded approximate identities

The following theorem is well known in the case of Banach algebras (see, e.g., [2, 4, 12]), but it readily extends to arbitrary topological algebras.

**Theorem.** Let $A$ be a topological algebra. Then

(i) $A$ has a right a.i. if and only if for each finite subset $F \subset A$ and each $0$-neighbourhood $U \subset A$ there exists $b \in A$ such that $a - ab \in U$ for all $a \in F$;

(ii) $A$ has a right b.a.i. if and only if there exists a bounded subset $S \subset A$ such that for each finite subset $F \subset A$ and each $0$-neighbourhood $U \subset A$ there exists $b \in S$ such that $a - ab \in U$ for all $a \in F$.

For our purposes, it is convenient to reformulate the above theorem in the language of seminorms.

**Theorem.** Let $A$ be a locally convex topological algebra, and let $\{\| \cdot \|_\lambda : \lambda \in \Lambda\}$ be a directed family of seminorms generating the topology of $A$. Then

(i) $A$ has a right a.i. if and only if for each finite subset $F \subset A$, each $\lambda \in \Lambda$, and each $\varepsilon > 0$ there exists $b \in A$ such that $\|a - ab\|_\lambda < \varepsilon$ for all $a \in F$;

(ii) $A$ has a right b.a.i. if and only if there exists a family $\{C_\lambda : \lambda \in \Lambda\}$ of positive reals such that for each finite subset $F \subset A$, each $\lambda \in \Lambda$, and each $\varepsilon > 0$ there exists $b \in A$ such that $\|a - ab\|_\lambda < \varepsilon$ for all $a \in F$, and

(ii) $A$ has a right a.i. if and only if for each finite subset $F \subset A$, each $\lambda \in \Lambda$, and each $\varepsilon > 0$ there exists $b \in A$ such that $\|a - ab\|_\lambda < \varepsilon$ for all $a \in F$.
Now let us relax condition (ii$_2$) as follows.

**Definition 1.** Let $A$ be a locally convex topological algebra, and let $\{\| \cdot \|_\lambda : \lambda \in \Lambda\}$ be a directed family of seminorms generating the topology of $A$. We say that $A$ has a right locally bounded a.i. if there exists a family $\{C_\lambda : \lambda \in \Lambda\}$ of positive reals such that for each finite subset $F \subset A$, each $\lambda \in \Lambda$, and each $\varepsilon > 0$ there exists $b \in A$ such that

1. $\|a - ab\|_\lambda < \varepsilon$ for all $a \in F$, and
2. $\|b\|_\lambda \leq C_\lambda$.

**Remark 4.** It is easy to see that the above definition does not depend on the choice of the defining family of seminorms.

**Proposition 2.** Let $A$ be an Arens-Michael algebra, and let $A = \lim\limits\leftarrow A_\lambda$ be the Arens-Michael decomposition of $A$. Then the following conditions are equivalent:

1. $A$ has a right locally bounded a.i.;
2. for each $\lambda \in \Lambda$, $A_\lambda$ has a right b.a.i.

Now we can reformulate Theorem 1 in a more elegant way.

**Theorem 2.** Let $A$ be a Fréchet-Arens-Michael algebra, and let $I \subset A_+$ be a closed left ideal. Then the following conditions are equivalent:

1. $A_+/I$ is strictly flat;
2. $I$ has a right locally bounded a.i.

### 4. Quasinormable Fréchet Algebras

It is clear that if a locally convex algebra $A$ has a locally bounded a.i., then it has a bounded a.i. It is natural to ask whether the converse is true. Before answering this question, let us recall the following definition.

**Definition (Grothendieck [5]).** A locally convex space $E$ is quasinormable if for each 0-neighbourhood $U \subset E$ there exists a 0-neighbourhood $V \subset E$ such that for each $\varepsilon > 0$ there exists a bounded set $B \subset E$ such that $V \subset B + \varepsilon U$.

Many naturally arising Fréchet spaces are quasinormable. Clearly, all Banach spaces and all Schwartz spaces [5] are quasinormable. It is also true that each quojection (i.e., the projective limit of a sequence of Banach spaces and surjective mappings) is quasinormable [3]. This implies that the space $C(X)$ (where $X$ is a locally compact Hausdorff topological space, countable at infinity) and, more generally, each Fréchet locally $C^*$-algebra is quasinormable. Standard examples of non-quasinormable Fréchet spaces belong to the class of Köthe sequence spaces [5] (see also [10, Section 27]). Within the class of function spaces, a number of concrete examples were found in [1].

**Theorem 3.** Let $A$ be a quasinormable Fréchet-Arens-Michael algebra with a right locally bounded a.i. Then $A$ has a right b.a.i.
5. A counterexample

Now we present an example of a commutative Fréchet-Arens-Michael algebra with a locally bounded a.i., but without a b.a.i. Together with Theorem \cite{2} this will show that Helemskii’s theorem does not extend \textit{verbatim} to Fréchet-Arens-Michael algebras.

A “building block” for our example is the following Banach algebra. In the sequel, for each \( i \in \mathbb{N} \) we set \( e_i = (0, \ldots, 0, 1, 0, \ldots) \), where the single nonzero entry is in the \( i \)th slot. It is easy to see that there is a unique continuous multiplication on \( \ell^1 \) such that \( e_ie_j = e_{\min\{i,j\}} \) for all \( i, j \in \mathbb{N} \). The resulting Banach algebra will be denoted by \( A_i \). Let us remark that \( A_i \) is topologically isomorphic to the sequence algebra \( bv_0 \) consisting of all sequences converging to 0 and having bounded variation (see \cite{2}). An explicit isomorphism \( A_i \to bv_0 \) is given by \( e_n \mapsto e_1 + \ldots + e_n \) \((n \in \mathbb{N})\). This implies that \( \{e_n : n \in \mathbb{N}\} \) is a b.a.i. for \( A_i \).

We shall need the following generalization of \( A_i \). Let \( P \) be a family of real-valued sequences such that \( p_i \geq 1 \) for all \( p \in P \) and all \( i \in \mathbb{N} \). Suppose also that \( P \) is directed, i.e., for each \( p, q \in P \) there exists \( r \in P \) such that \( r_i \geq \max\{p_i, q_i\} \) for all \( i \in \mathbb{N} \). Then it is easy to see that there exists a unique multiplication on the Köthe space

\[
\lambda(P) = \left\{ a = (a_i)_{i \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \|a\|_P = \sum_i |a_i|p_i < \infty \text{ \forall } p \in P \right\}
\]

such that \( e_ie_j = e_{\min\{i,j\}} \) for all \( i, j \in \mathbb{N} \). Moreover, we have \( \|ab\|_P \leq \|a\|_P\|b\|_P \) for all \( a, b \in \lambda(P) \) and all \( p \in P \), so that \( \lambda(P) \) becomes as Arens-Michael algebra with respect to the above multiplication. Let us denote this algebra by \( A(P) \).

Given \( p \in P \), the accompanying Banach algebra \( A(P)_p \) can be described in much the same way as \( A_i \) (see above), by replacing \( \ell^1 \) with the weighted space \( \ell^1(p) = \{x = (x_i) : \|x\| = \sum_i |x_i|p_i < \infty\} \). Together with Proposition \cite{2} this easily implies the following.

\textbf{Lemma 1.} \textit{(i)} Suppose that each sequence \( p \in P \) has a bounded subsequence. Then \( A(P) \) has a locally bounded a.i.

\textit{(ii)} Suppose that there exists an infinite increasing sequence \( \{n_k : k \in \mathbb{N}\} \) of positive integers such that the sequence \( \{p_{nk} : k \in \mathbb{N}\} \) is bounded for each \( p \in P \). Then \( A(P) \) has a b.a.i.

Unfortunately, we do not know whether condition (ii) is necessary for \( A \) to have a b.a.i. In order to formulate a necessary condition, let us introduce some notation. Given \( a \in A(P) \), we set \( w(a) = \sum_i |a_i| \) (note that this number is finite because \( p_i \geq 1 \) for all \( p \in P \) and all \( i \in \mathbb{N} \)). If \( a \neq 0 \), then we set \( \ell(a) = \min\{k \in \mathbb{N} : a_k \neq 0\} \).

\textbf{Lemma 2.} Suppose that \( A(P) \) has a b.a.i. Then there exists a bounded sequence \( \{x_n\} \subset A(P) \) such that

\[
\ell(x_n) \to \infty \quad \text{as} \quad n \to \infty, \quad \text{and} \quad \inf_n w(x_n) > 0. \quad (1)
\]

Thus our aim is to find a countable family \( P \) that satisfies condition (i) of Lemma \cite{1} but not Lemma \cite{2}. 

For each $k \in \mathbb{N}$ we define an infinite matrix $\alpha^{(k)} = (\alpha^{(k)}_{ij})_{i,j \in \mathbb{N}}$ by setting
\[
\alpha^{(k)}_{ij} = \begin{cases} 
ij, & i \leq k \\
i, & i > k.
\end{cases}
\]
Fix a bijection $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that $\varphi(i,j) < \varphi(k,l)$ whenever $i + j < k + l$. For each $k \in \mathbb{N}$ define a sequence $p^{(k)} = (p^{(k)}_n)_{n \in \mathbb{N}}$ by $p^{(k)}_n = \alpha^{(k)}_{\varphi^{-1}(n)}$. Finally, set $P = \{p^{(k)} : k \in \mathbb{N}\}$.

**Theorem 4.** The set $P$ has the following properties:

(i) each sequence $p \in P$ has a bounded subsequence;
(ii) each sequence $\{x_n\} \subset A(P)$ satisfying (1) is unbounded.

Therefore the algebra $A(P)$ has a locally bounded a.i., but does not have a b.a.i.

Together with Theorem 2 this gives the following.

**Corollary 1.** There exists a commutative Fréchet-Arens-Michael algebra $A$ such that the trivial Fréchet $A$-module $C = A_+ / A$ is strictly flat, but $A$ does not have a b.a.i.

### 6. Remarks on flat cyclic Fréchet modules and amenable Fréchet-Arens-Michael algebras

The second part of the Helemskii-Sheinberg theorem can be generalized as follows.

**Theorem 5.** Let $A$ be a Fréchet-Arens-Michael algebra, let $A = \lim_{\leftarrow} A_n$ be the Arens-Michael decomposition of $A$, and let $I \subset A_+$ be a closed left ideal. For each $n \in \mathbb{N}$, denote by $I_n \subset (A_n)_+$ the closure of the image of $I$ under the canonical map $A_+ \to (A_n)_+$. Suppose that $I_n$ is weakly complemented in $(A_n)_+$ for each $n \in \mathbb{N}$. Then the following conditions are equivalent:

(i) $A_+ / I$ is flat;
(ii) $A_+ / I$ is strictly flat;
(iii) $I$ has a right locally bounded a.i.

The condition “$I_n$ is weakly complemented in $(A_n)_+$ for each $n \in \mathbb{N}$” looks rather unnatural, but we do not know how to put it into a more reasonable form. In particular, we do not know the answers to the following questions:

**Open problems.** (1) Does the above condition depend on the choice of a sequence of submultiplicative seminorms that gives the Arens-Michael decomposition of $A$?

(2) Suppose that $I$ is weakly complemented in $A_+$. Does it follow that $I_n$ is weakly complemented in $(A_n)_+$ for each $n \in \mathbb{N}$?

(3) Conversely, suppose that $I_n$ is weakly complemented in $(A_n)_+$ for each $n \in \mathbb{N}$. Does it follow that $I$ is weakly complemented in $A_+$?

(4), (5), (6). The same as (1), (2), (3), with “weakly complemented” replaced by “complemented”.

Fortunately, there is an important situation where the above-mentioned difficulties disappear. First let us recall some standard notation [7]. Given a Fréchet algebra $A$, the algebra opposite to $A$ is denoted by $A^{\text{op}}$. Set $A^c = \ldots$
A_+ \otimes A^e_+ \text{ and denote by } \pi: A^e \rightarrow A_+ \text{ the linear continuous map uniquely determined by } a \otimes b \mapsto ab. \text{ The kernel of this map is a complemented closed left ideal of } A^e. \text{ It is denoted by } I^\Delta \text{ and is called the \textit{diagonal ideal} of } A^e.

Recall that a Fréchet algebra } A \text{ is said to be \textit{amenable} if } A_+ \text{ is a flat Fréchet } A\text{-bimodule (see [7]). Recall also that the category of Fréchet } A\text{-bimodules is isomorphic to the category of left unital Fréchet } A^e\text{-modules. The canonical morphism } \pi: A^e \rightarrow A_+ \text{ determines an isomorphism between } A^e/I^\Delta \text{ and } A_+ \text{ in } A^e\text{-mod}. \text{ Therefore the question of whether or not } A \text{ is amenable is equivalent to the question of whether or not the left cyclic Fréchet } A^e\text{-module } A^e/I^\Delta \text{ is flat.}

Now suppose that } A \text{ is a Fréchet-Arens-Michael algebra and } A = \varprojlim A_n \text{ is its Arens-Michael decomposition. Then we have } A^e = \varprojlim A^e_n \text{ (see, e.g., [9])}, \text{ and it is easy to see that the closure of the canonical image of } I^\Delta \text{ in } A^e_n \text{ is precisely the diagonal ideal of } A^e_n. \text{ Since the latter is complemented in } A^e_n, \text{ Theorem 5 implies the following.}

\textbf{Theorem 6.} \textit{Let } A \text{ be a Fréchet-Arens-Michael algebra. The following conditions are equivalent:}

(i) } A \text{ is amenable;}

(ii) } I^\Delta \text{ has a right locally bounded a.i.}

\text{If, in addition, } A \text{ is quasinormable, then (i) and (ii) are equivalent to}

(iii) } I^\Delta \text{ has a right b.a.i.}

\textbf{Open problems.} (7) Does there exist an amenable Fréchet-Arens-Michael algebra such that } I^\Delta \text{ does not have a right b.a.i.}?

(8) Does there exist a quasinormable amenable Fréchet-Arens-Michael algebra with this property?

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Department of Nonlinear Analysis and Optimization, Faculty of Science, Peoples’ Friendship University of Russia, Mikluho-Maklaya 6, 117198 Moscow, Russia

E-mail address: pirkosha@sci.pfu.edu.ru, pirkosha@online.ru