Fat realization and Segal’s classifying space

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Abstract

In this paper, we give a new proof of a well-known theorem due to tom Dieck that the fat realization and Segal’s classifying space of an internal category in the category of topological spaces are homotopy equivalent.

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1 Introduction

Motivated by bundle theory, foliation theory, and delooping theory, classifying spaces of topological groups and groupoids were intensively studied during the 60s, 70s and 80s. Since then, many different constructions of classifying spaces of topological groups and groupoids have been introduced, for example, the Milnor construction, the Segal construction, fat realization, geometric realization and so on [Mil56a, Mil56b, DL59, Sta63a, Sta63b, Sta63c, Seg68, Hae71, Bot72, Seg74, tD74]. Some of them have even been generalized to any internal categories in $\mathbf{Top}$, the category of topological spaces. For topological groups, most of the constructions give rise to homotopy equivalent spaces. However, for general internal categories in $\mathbf{Top}$, the relation between them is not always clear. In this paper, we shall focus on the comparison between the Segal construction and fat realization of internal categories in $\mathbf{Top}$.

Main results

tom Dieck’s theorem [tD74, Proposition 2] asserts that, given any simplicial space $X$, the projection

$$\pi : \|X \times S\| \to \|X\|$$  (1)

is a homotopy equivalence, where $S$ is the semi-simplicial set given by strictly increasing sequences of natural numbers. The idea of his proof is to construct a map $\|X\| \to \|X \times S\|$ and show it is a homotopy inverse to the projection.
\[ \pi \]. However, to find a well-defined map from \( \|X\| \) to \( \|X \times S\| \) and show it is a homotopy inverse to \( \pi \) turn out to be quite difficult and complicated. To get around this inconvenience, we employ Quillen’s theorem A. Our approach is more conceptual, but it works only when \( X_k \) has the homotopy type of a CW-complex and \( X = \text{Ner} \mathcal{C} \), where \( \text{Ner} \mathcal{C} \) is the nerve of an internal category in \( \text{Top} \).

**Theorem 1.1.** Given \( \mathcal{C} \) an internal category in \( \text{Top} \) such that its nerve \( \text{Ner} \mathcal{C} \) has the homotopy type of a CW-complex at each degree, then the canonical projection

\[
\pi : \|\text{Ner} \mathcal{C} \times S\| \to \|\text{Ner} \mathcal{C}\|
\]

is a homotopy equivalence.

Combining Theorem 1.1 with the fact that Segal’s classifying space \( |\text{Ner} \mathcal{C}^\mathbb{N}| \) is homeomorphic to the space \( \|\text{Ner} \mathcal{C} \times S\| \), where \( \mathcal{C}^\mathbb{N} \) is Segal’s unraveled category of \( \mathcal{C} \) over the natural numbers \( \mathbb{N} \), one can easily deduce the following useful corollary.

**Corollary 1.2.** If, in addition to the assumptions in Theorem 1.1, the simplicial space \( \text{Ner} \mathcal{C} \) is proper, then the forgetful functor \( \mathcal{C}^\mathbb{N} \to \mathcal{C} \) induces a homotopy equivalence

\[
|\text{Ner} \mathcal{C}^\mathbb{N}| \to |\text{Ner} \mathcal{C}|.
\]

The space \( |\text{Ner} \mathcal{C}^\mathbb{N}| \) is more natural from the point of view of bundle theory, whereas, category-theoretically, the geometric realization \( |\text{Ner} \mathcal{C}| \) is easier to handle.

In the third section, following Stasheff’s approach [Bot72, Appendices B and C], we work out a detailed proof of a classification theorem for bundles with structures in a topological groupoid.

**Theorem 1.3.** Given a topological groupoid \( \mathcal{G} \) and a topological space \( X \), there exists a 1-1 correspondence between the set of homotopy classes of maps from \( X \) to \( |\text{Ner} \mathcal{G}^\mathbb{N}| \) and the set of homotopy classes of numerical \( \mathcal{G} \)-structures on \( X \).

It is not a new theorem, and we also claim no originality for the approach presented here as it is essentially the proof of Theorem D, a special case of Theorem 1.3 in [Bot72, Appendix C]. In fact, Stasheff has indicated that his method can be applied to more general cases (see [Mos76, p.126] and [Bot72 Theorem E in Appendix C]). A similar classification theorem in terms of Milnor’s construction can be found in [Hae71]. It is because we need a classification theorem in terms of Segal’s classifying space in [Wan17], Theorem 1.3 is discussed in details here.

1The homotopy inverse \( \rho \) given in [tD74] appears not to be well-defined. See the appendix for more details.
Outline of the paper

In the second section, we review some background notions on topological groupoids, denoted by $G$, and $G$-structures on topological spaces. The third section discusses how Stasheff’s approach [Bot72, Appendix B and C] can be generalized to arbitrary topological groupoids. The forth section, where the novelty of the paper is, is independent of the previous two sections, and we shall apply Quillen’s theorem A to prove Theorem 1.1 there.

As the homotopy inverse $\rho : ||X|| \to ||X \times S||$ in [tD74] appears not to be well-defined, in the appendix, we use a different construction, which is due to S. Goette, to find a well-defined map $\tau : ||X|| \to ||X \times S||$ and explain why it is a promising candidate for a homotopy inverse to $\pi$.

Notation and convention

Throughout the paper, we shall use the Quillen equivalences between the model category of simplicial sets $sSets$ and the Quillen model category of topological spaces $Top$ given by the singular functor and geometric realization functor:

$$|-| : sSets \rightleftarrows Top : Sing.$$

A simplicial space is proper if and only of it is a cofibrant object in the Reedy model category associated to the Strøm model structure on $Top$.

Given $X, Y \in Top$, $[X, Y]$ denotes the set of homotopy classes of maps from $X$ to $Y$.

Recall that the natural numbers $\mathbb{N}$ is an ordered set and hence can be viewed as a category.

We have chosen to work with the category of topological spaces, but the results in this paper hold for other convenient categories of topological spaces such as the category of $k$-spaces (Kelly spaces) or the category of weakly Hausdorff $k$-spaces.

Acknowledgment

I would like to thank S. Goette for suggesting the approach using Quillen’s theorem A. The construction of the map $\tau$ in Appendix is also due to him.

2 Topological groupoids and $G$-cocycles

Definition 2.1. A topological groupoid $G$ is an internal category in $Top$ equipped with the inverse map $i : M(G) \to M(G)$ and the identity-assigning map $e : O(G) \to M(G)$ such that $s \circ i = t : M(G) \to O(G)$ and $t \circ i = s : M(G) \to O(G)$; and the diagrams below commute

\[
\begin{array}{ccc}
M(G) & \xrightarrow{D} & M(G) \times M(G) \\
\downarrow{s} & & \downarrow{} \circ \ (id, i) \\
O(G) & \xrightarrow{c} & M(G)
\end{array}
\]
where $s$ and $t$ are the source and target maps, respectively, $D$ is the diagonal map $D(x) = (x, x)$ and $\circ : M(\mathcal{G}) \times O(\mathcal{G}) M(\mathcal{G}) \to M(\mathcal{G})$ is the composition map.

Given a topological groupoid $\mathcal{G}$ and a topological space $X$, we can define a $\mathcal{G}$-structure on $X$.

**Definition 2.2.** 1. A $\mathcal{G}$-cocycle on $X$ is a collection $\{ U_\alpha; f_{\beta\alpha} \}_{\alpha, \beta \in I}$, where $\{ U_\alpha \}_{\alpha \in I}$ is an open cover of $X$ and $f_{\beta\alpha}$ is a map $f_{\beta\alpha} : U_\alpha \cap U_\beta \to M(\mathcal{G})$ that satisfies

$$f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}.$$ 

In particular, $f_{\alpha\alpha}$ factors through $O(\mathcal{G})$, meaning

$$f_{\alpha\alpha} : U_\alpha \to O(\mathcal{G}) \hookrightarrow M(\mathcal{G}).$$

Hence we often omit the repetition of $\alpha$ and just write $f_\alpha$, thinking of it as a map from $U_\alpha$ to $O(\mathcal{G})$.

2. Two $\mathcal{G}$-cocycles $\{ U_\alpha; f_{\beta\alpha} \}_{\alpha, \beta \in I}$ and $\{ V_\gamma; g_{\delta\gamma} \}_{\gamma, \delta \in J}$ are isomorphic if and only if there exists a map $\phi_{\gamma\alpha} : U_\alpha \cap V_\gamma \to M(\mathcal{G})$, for each $\alpha \in I$ and $\gamma \in J$, such that

$$g_{\delta\gamma} \circ \phi_{\gamma\alpha} = \phi_{\delta\beta} \circ f_{\beta\alpha},$$

or equivalently, the union

$$\{ U_\alpha; V_\gamma; f_{\beta\alpha}, g_{\delta\gamma}, \phi_{\gamma\alpha} \}_{\alpha, \beta \in I, \gamma, \delta \in J}$$

constitutes a $\mathcal{G}$-cocycle on $X$. Note that the index sets $I$ and $J$ are often omitted when there is no risk of confusion.

An isomorphism class of $\mathcal{G}$-cocycles on $X$ is called a $\mathcal{G}$-structure, and the set of $\mathcal{G}$-structures on $X$ is denoted by $H^1(X, \mathcal{G})$.

3. Two $\mathcal{G}$-structures $u, v \in H^1(X, \mathcal{G})$ are said to be homotopic if and only if there exists a $\mathcal{G}$-structure $w \in H^1(X \times I, \mathcal{G})$ such that $i_0^* w = u$ and $i_1^* w = v$, where $i_0 : X = X \times \{ 0 \} \hookrightarrow X \times I$ and $i_1 : X = X \times \{ 1 \} \hookrightarrow X \times I$.

$\mathcal{G}(X)$ denotes the set of homotopy classes of $\mathcal{G}$-structures, and it is a contravariant functor from Top to Sets, the category of sets.

**Remark 2.3.** In this remark, we shall expand on the definition above.
1. Isomorphisms of \( G \)-cocycles constitute an equivalence relation on the set of \( G \)-cocycles. Suppose the \( G \)-cocycles \( \{ U_\alpha; f_{\beta\alpha} \} \) and \( \{ U_\gamma; g_{\delta\gamma} \} \) are iso-
morphic through \( \phi_{\gamma\alpha} \) and the \( G \)-cocycles \( \{ V_\gamma; g_{\delta\gamma} \} \) and \( \{ W_\epsilon; h_{\eta\epsilon} \} \) are iso-
morphic through \( \psi_{\gamma\epsilon} \)—namely, \[
\{ U_\alpha; f_{\beta\alpha} \} \overset{\phi_{\gamma\alpha}}{\cong} \{ U_\gamma; g_{\delta\gamma} \} \overset{\psi_{\gamma\epsilon}}{\cong} \{ W_\epsilon; h_{\eta\epsilon} \}.
\]

Then we can define \( \rho_{\epsilon\alpha,\gamma} := \psi_{\epsilon\gamma} \circ \phi_{\gamma\alpha} : U_\alpha \cap W_\epsilon \cap V_\gamma \to M(G) \)
for each \( \alpha, \gamma \) and \( \epsilon \). Since they are compatible when \( \gamma \) varies, there is a
well-defined map \( \rho_{\epsilon\alpha} : U_\alpha \cap W_\epsilon \to M(G) \).

On the other hand, from the definition of \( \rho_{\epsilon\alpha,\gamma} \), we have the identity \( h_{\eta\epsilon} \circ \rho_{\epsilon\alpha,\delta} = \rho_{\eta\beta,\gamma} \circ f_{\beta\alpha} \)
on \( W_\eta \cap U_\alpha \cap W_\epsilon \cap V_\gamma \cap V_\delta \), for all \( \gamma \) and \( \delta \), and hence \( \{ \rho_{\epsilon\alpha} \} \) constitute an isomorphism between the
\( G \)-cocycles \( \{ U_\alpha; f_{\beta\alpha} \} \) and \( \{ W_\epsilon; h_{\eta\epsilon} \} \).

2. The notion of homotopy between \( G \)-structures on \( X \) gives an equivalence
relation on \( H^1(X, G) \). To see this, it suffices to observe that, for any two
isomorphic \( G \)-cocycles
\[
\{ U_\alpha; f_{\beta\alpha} \}_{\alpha, \beta \in I} \overset{\phi_{\gamma\alpha}}{\cong} \{ U_\gamma'; f'_{\gamma\delta} \}_{\gamma, \delta \in J},
\]
there is a \( G \)-cocycle \( \{ \hat{U}_\mu; \hat{f}_{\nu\mu} \}_{\nu, \mu \in I \cup J} \) on \( X \times I \) which is given by
\[
\hat{U}_\alpha := U_\alpha \times (1/3, 1] \quad \text{and} \quad \hat{U}_\gamma := U_\gamma' \times [0, 2/3).
\]

and
\[
\hat{f}_{\beta\alpha}(x, t) := f_{\beta\alpha}(x) \quad t > 1/3 \text{ on } \hat{U}_\alpha \cap \hat{U}_\beta,
\hat{f}_{\gamma\delta}(x, t) := f_{\gamma\delta}(x) \quad t < 2/3 \text{ on } \hat{U}_\delta \cap \hat{U}_\gamma,
\hat{\phi}_{\gamma\alpha}(x, t) := \phi_{\gamma\alpha}(x) \quad 1/3 < t < 2/3 \text{ on } \hat{U}_\alpha \cap \hat{U}_\gamma'.
\]

To define numerable \( G \)-structures on a topological space. We first recall the
definition of a partition of unity.

**Definition 2.4.** Given a topological space \( X \) and an open cover \( \{ U_\alpha \}_{\alpha \in I} \), then
\( \{ \lambda_\alpha \}_{\alpha \in I} \) is a partition of unity subordinate to the open cover \( \{ U_\alpha \}_{\alpha \in I} \) if and
only if

1. \( \lambda_\alpha \) is a map \( \lambda_\alpha : X \to [0, 1] \) with \( \text{supp}(\lambda_\alpha) \subset U_\alpha \), for each \( \alpha \in I \).

2. For every point \( x \in X \), there exists a neighborhood \( U_x \) of \( x \) such that,
when restricted to this neighborhood, \( \lambda_\alpha = 0 \) for all but finite \( \alpha \in I \).
3. For every \( x \in X \),
\[
\sum_{\alpha \in I} \lambda_\alpha(x) = 1.
\]

A numerable open cover is an open cover that admits a partition of unity subordinate to it.

Not every open cover admits a partition of unity, for example, the line with two origins.

**Definition 2.5.**

1. A numerable \( G \)-cocycle on \( X \) is a \( G \)-cocycle \( \{U_\alpha; f_{\beta\alpha}\} \) with \( \{U_\alpha\} \) a numerable open cover.

2. Two numerable \( G \)-cocycles are isomorphic if and only if they are isomorphic as \( G \)-cocycles. An isomorphism class of numerable \( G \)-cocycles is called a numerable \( G \)-structure on \( X \). The set of \( G \)-structures on \( X \) is denoted by \( H^1_{nu}(X,G) \).

3. Two numerable \( G \)-structures are homotopic if and only if they are homotopic as \( G \)-structures via a numerable \( G \)-structure on \( X \times I \). The set of homotopy classes of numerable \( G \)-structures is denoted by \( G_{nu}(X) \), and it is a contravariant functor from \( \text{Top} \) to \( \text{Sets} \).

The following lemmas imply that, when \( G \) is a topological group, \( G_{nu}(X) = H^1_{nu}(X,G) \).

**Lemma 2.6.** Let \( G \) be a topological group and assume \( w \) is a numerable principal \( G \)-bundle on \( X \times I \). Then there is a bundle morphism \( w \to \pi^*w \), where \( \pi \) is the composition \( X \times I \xrightarrow{(x,t)\mapsto(x,0)} X \times \{0\} \hookrightarrow X \times I \).

*Proof.* See [Hus94, Theorem 9.6 in Chapter 4].

**Lemma 2.7.** Let \( G \) be a topological group. Then there is a 1-1 correspondence between \( H^1(X,G) \) and \( \{ \text{principle} \ G \text{-bundles} \}/\text{iso} \).

*Proof.* See [Swi02, Theorem 11.16].

The next lemma explains why it is called “numerable open cover”. This technical lemma is very useful in simplifying proofs.

**Lemma 2.8.** Given a numerable open cover \( \{V_j\} \) of a topological space \( X \), there exists a countable numerable open cover \( \{W_n\}_{n\in\mathbb{N}} \) such that, for each \( n \), \( W_n \) is an union of some open sets, each of which is contained in some members of the original open cover \( \{V_j\} \).

*Proof.* This lemma is due to Milnor (see [Jam84, Thm.7.27] or [Hus94, Proposition 12.1] for a detailed proof).

A \( G \)-cocycle \( \{U_\alpha; f_{\beta\alpha}\} \) on \( X \) is countable and numerable if and only if \( \{U_\alpha\} \) has countably many members and is numerable. Two countable numerable \( G \)-cocycles are isomorphic if and only if they are isomorphic as \( G \)-cocycles. Two countable numerable \( G \)-structures—isomorphism classes of countable numerable \( G \)-cocycles—are homotopic if and only if they are homotopic through a countable numerable \( G \)-structure on \( X \times I \).
Corollary 2.9. 1. Given a numerable $G$-structure on $X$, there exists a countable numerable $G$-cocycle representing this $G$-structure.

2. Given two homotopic numerable $G$-structures, there exists a countable numerable $G$-cocycle on $X \times I$ such that its restrictions to $X \times \{0\}$ and $X \times \{1\}$ represent the two given $G$-structures.

In particular, the set of homotopy classes of numerable $G$-structures is the same as the set of homotopy classes of countable numerable $G$-structures.

On the other hand, in most cases, there is no loss of generality by assuming the open cover in a $G$-cocycle is numerable. The ensuing corollary results from the fact that every open cover of a paracompact Hausdorff space admits a subordinate partition of unity.

Corollary 2.10. Suppose $X$ is paracompact Hausdorff, then every $G$-cocycle on $X$ is a numerable $G$-cocycle, and any two numerable $G$-structures on $X$ are homotopic if and only if they are homotopic through a numerable $G$-structure on $X \times I$. In other words, classifying $G$-cocycles on a paracompact Hausdorff space $X$ is equivalent to classifying numerable $G$-cocycles on $X$.

Proof. It is because the product of a paracompact space and a compact space is paracompact.

3 A classification theorem

This section discusses a classification theorem for numerable $G$-structures, and the method presented here is taken from [Bot72, Theorem D in Appendix C], where Stasheff classifies Haefliger’s structures and indicates that his approach can be applied to any topological groupoids.

Recall first the construction of classifying spaces in [Bot72, Appendix B].

Definition 3.1. Given $\mathcal{C}$ an internal category in $\text{Top}$, the associated classifying space is defined to be the quotient space

$$BC := \prod_{\alpha:|k| \to \mathbb{N}} \text{Ner}_k \mathcal{C} \times \Delta^k_\alpha / \sim,$$

where $\alpha$ can be viewed a strictly increasing sequence of natural numbers $\{i_0 < i_1 < \ldots < i_k\}$, $\Delta^k_\alpha$ is the triangle in the infinite triangle $\Delta^\infty = [\text{Ner}, \mathbb{N}]$ with vertices $\{i_0, i_1, \ldots, i_k\}$, and the relation $\sim$ is given by

$$(f_{i_0}, \ldots, f_{k-1}; t_0, \ldots, t_k) \sim (f_{i_0}, \ldots, f_{i_{j+1}} \circ f_{i_{j+1}}; t_0, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k),$$
$$(f_{i_0}, \ldots, f_{k-1}; t_0, \ldots, t_k) \sim (f_{i_0}, \ldots, f_{i_{k-1}}; t_1, \ldots, t_k),$$
$$(f_{i_0}, \ldots, f_{k-1}; t_0, \ldots, t_{k-1}, 0) \sim (f_{i_0}, \ldots, f_{i_{k-1}}; t_0, \ldots, t_{k-1}).$$

Theorem 3.2. Given a topological groupoid $\mathcal{G}$, there is a 1-1 correspondence:

$$[X, BG] \leftrightarrow \mathcal{G}_{nu}(X),$$

for every $X \in \text{Top}$. 

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Proof. Firstly, we note there is a canonical open cover of $BG$ given by the preimage $U_i := t_j^{-1}((0, 1])$, where $t_j$ is induced by the projection
\[ \prod_{i_0 < \ldots < i_k} \text{Ner}_k \mathcal{G} \times \Delta^k_{i_0 < \ldots < i_k} \xrightarrow{t_{i_k}} [0, 1]. \]
The collection $\{t_j\}_{j \in \mathbb{N}}$ is not locally finite and hence not a partition of unity. To construct a partition of unity with respect to $\{U_i\}$, we consider maps $w_i; v_i: U_i \times [0, 1] \rightarrow [0, 1]$ defined by
\[ w_i(x, s) := \max\{0, t_i(x) - s \sum_{j < i} t_j(x)\} \]
and
\[ v_i(x, s) := \frac{w_i(x, s)}{\sum_{j=0}^{\infty} w_j(x, s)}, \tag{2} \]
respectively. Observe that $v_i(0, x) = t_i$ and $\{v_i(1, x)\}$ is locally finite and constitutes a partition of unity subordinate to the open cover $\{U_i\}$. The universal $G$-cocycle on $BG$ is then given as follows:
\[ \gamma_{i_0, i_k} : \prod_{i_0 < \ldots < i_k} \text{Ner}_k \mathcal{G} \times \Delta^k_{i_0 < \ldots < i_k} \rightarrow M(\mathcal{G}) \]
\[ (g_{i_0, i_1}, \ldots, g_{i_k, i_k+1} : t_{i_0}, \ldots, t_{i_k}) \mapsto \begin{cases} g_{i_0, i_1} \circ \ldots \circ g_{i_j, i_{j+1}} & \text{for } i_j < i_k, \\ s(g_{i_j, i_{j+1}}) & \text{for } i_j = i_k, \\ (g_{i_j, i_{j+1}} \circ \ldots \circ g_{i_k, i_k+1})^{-1} & \text{for } i_j > i_k. \end{cases} \]
On the other hand, given a countable numerable open cover $\{U_n\}$ of a topological space $X$ and a subordinate partition of unity $\{\lambda_n\}$, one can define the space
\[ X_\mu := \prod_{\alpha: \{k\} \rightarrow \mathbb{N}} U_\alpha \times \Delta^k_\alpha / \sim, \]
where $\alpha = \{i_0 < \ldots < i_k\} \subset \mathbb{N}$ and the equivalence relation is
\[ (x; i_0, i_1, \ldots, i_k) \sim (x; i_0, i_{i_1}, i_{i_2}, \ldots, i_k), \]
for any $x \in U_{i_0,\ldots,i_k}$. There is a homotopy equivalence $\lambda$ from $X$ to $X_\mu$ given by
\[ \lambda : X \rightarrow X_\mu \]
\[ x \in U_{i_0,\ldots,i_k} \mapsto (x; \lambda_{i_0}(x), \ldots, \lambda_{i_k}(x)), \]
whose homotopy inverse is the canonical projection
\[ p : X_\mu \rightarrow X. \]
It is clear that $p \circ \lambda = \text{id}_X$ and there is an obvious linear homotopy connecting $\text{id}_{X_\mu}$ and $\lambda \circ p$. In this way, we see that the homotopy type of $X_\mu$ is independent of partition of unities.

Now, in view of Corollary 2.10, we may assume all numerable $G$-cocycles on $X$ are countable, and they are homotopic if they are homotopic through a
countable numerable \( G \)-cocycles on \( X \times I \). Given a countable numerable \( G \)-cocycle \( u = \{ U_\alpha, g_\beta \alpha, \lambda_\alpha \} \), we have the composition

\[
X \xrightarrow{\lambda} X_\mu \xrightarrow{Bu} B\mathcal{G},
\]

where the map \( Bu \) is induced from the assignment

\[
(x; t_{i_0}, ..., t_{i_k}) \mapsto (g_{i_1 i_0}(x), g_{i_2 i_1}(x), ..., g_{i_k i_{k-1}}(x); t_{i_0}, t_{i_1}, ..., t_{i_k}).
\]

Suppose two \( G \)-cocycles \( u = \{ U_\alpha, g_\beta \alpha, \lambda_\alpha \} \) and \( v = \{ V_\gamma, f_\delta \gamma, \mu_\gamma \} \) on \( X \) are homotopic, then their induced maps \( Bu \circ \lambda \) and \( Bv \circ \mu \) are also homotopic. This can be seen from the diagram below:

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda} & X_\mu \\
\downarrow t_0 & & \downarrow Bu \\
X \times I & \xrightarrow{\nu} & X_\nu \\
\downarrow t_1 & & \downarrow Bw \\
X & \xrightarrow{\mu} & X_\nu \\
\end{array}
\]

where \( w = \{ W_\epsilon, h_\eta \epsilon, \nu_\epsilon \} \) is a countably numerable \( G \)-cocycle on \( X \times I \) connecting \( u \) and \( v \), meaning \( \iota_0^*w = u \) and \( \iota_1^*w = v \) (see Remark 2.2). We may also assume that \( \{ \iota_0^*\nu_\epsilon \} = \{ \lambda_\alpha \} \) and \( \{ \iota_1^*\nu_\epsilon \} = \{ \mu_\alpha \} \).

Thus, there is a well-defined map of sets

\[
\Psi : \mathcal{G}_{nu}(X) \to [X, B\mathcal{G}].
\]

Since \( \mathcal{G}_{nu}(X) \) is a contravariant functor, given any map \( X \to B\mathcal{G} \), by pulling back the universal \( G \)-cocycle on \( B\mathcal{G} \), one obtains a \( G \)-cocycle on \( X \). It is also clear that pullback \( G \)-cocycles along homotopic maps are homotopic, and hence there is a well-defined map of sets

\[
\Phi : [X, B\mathcal{G}] \to \mathcal{G}_{nu}(X).
\]

To see \( \Phi \) is the inverse of \( \Psi \), we first note that \( \Psi \circ \Phi = \text{id} \) is obvious as the collection \( \{ v_i(x, s) \} \) defined in equation (2) connects \( \{ t_i \} \) and \( \{ v_i(1, -) \} \) and hence gives the homotopy between

\[
\Psi \circ \Phi(f)(x) = (g_{i_1 i_0}(x), ..., g_{i_k i_{k-1}}(x); v_{i_0}(1, x), ..., v_{i_k}(1, x))
\]

and

\[
f(x) = (g_{i_1 i_0}(x), ..., g_{i_k i_{k-1}}(x); t_{i_0}(x), ..., t_{i_k}(x)),
\]

for any map \( f : X \to B\mathcal{G} \). Secondly, we observe that, given \( u = \{ U_\alpha, g_\beta \alpha, \lambda_\alpha \} \) a \( G \)-cocycle on \( X \), the pullback \( \mathcal{G} \)-cocycle along \( Bu \circ \lambda \) is simply a restriction of \( u \), namely that the pullback open cover is a subcover of \( \{ U_\alpha \} \), and thus \( u \) and \((Bu \circ \lambda)^*u \) are isomorphic. \( \square \)
4 Fat realization and Segal’s classifying space

In this section, we shall employ a variant of Quillen’s theorem [Wal83, Section 1.4] to show Segal’s classifying space and the fat realization of an internal category in $\text{Top}$ are homotopy equivalent under some mild conditions.

We first recall the construction of Segal’s classifying space [Seg68, Section 3].

**Definition 4.1.** Let $C$ be an internal category in $\text{Top}$. Then the associated unraveled category $C^N$ is defined as the subcategory of $C \times N$ obtained by deleting those morphisms $(f, i \leq i)$ when $f \neq \text{id}$; and Segal’s classifying space of $C$ is the geometric realization of the associated unraveled category $C^N$, namely $|\text{Ner } C^N|$. 

**Lemma 4.2.** Let $S$ denote the semi-simplicial space given by

$$S_k := \{\text{strictly increasing maps from } [k] \text{ to } N\}.$$ 

Then there are homeomorphisms

$$\left|\text{Ner } C \times S\right| \simeq B C \simeq |\text{Ner } C^N|.$$ 

**Proof.** The first homeomorphism (left) is clear as $S$ is just another way of interpreting triangles $\{\Delta^n_{\alpha}\}$, where $\alpha$ is a strictly increasing sequence of natural numbers of length $k + 1$.

For the second homeomorphism, we observe that the inclusion

$$\coprod_{\alpha: [k] \to N} \text{Ner}_k C \times \Delta^k_{\alpha} \hookrightarrow \coprod_{k} \text{Ner}_k C^N \times \Delta^k$$

$$(c_0 \to ... \to c_k, t, i_0 < ... < i_k) \mapsto ((c_0, i_0) \to ... \to (c_k, i_k), t)$$

descends to a homeomorphism

$$B C \xrightarrow{\simeq} |\text{Ner } C^N|.$$ 

Now we can state our main theorem (compare with [D74, Proposition 2]).

**Theorem 4.3.** The canonical projection

$$B C \simeq |\text{Ner } C^N| \simeq \left|\text{Ner } C \times S\right| \xrightarrow{\simeq} \left|\text{Ner } C\right|$$

is a homotopy equivalence, provided $\text{Ner}_k C$ has the homotopy type of a CW-complex, for every $k$.

**Proof.** Firstly, we observe that there is a commutative diagram for any simplicial space $X$ with $X_k$ having the homotopy type of a CW-complex, for each $k$: .
where arrows with the symbol $\sim$ stand for homotopy equivalences and the (semi-) simplicial space $Y^p$ is the properization of a (semi-) simplicial space $Y$, namely level-wisely applying the singular functor and geometric realization to $Y$:

$$Y^p_k := |\operatorname{Sing} Y_k|.$$  

The simplicial space $Y^N$ in the diagram above is given by

$$Y^N_n := \prod_{k_0 \leq \cdots \leq k_n} Y_{l},$$

where $l$ is the number of the distinct members in $\{k_0, \ldots, k_n\}$. The degenerate map $s^N_i : Y^N_{n-1} \to Y^N_n$ is given by identities, sending the copy of $Y_l$ indexed by $k_0 \leq \cdots \leq k_i \leq \cdots \leq k_n$ to another copy indexed by $k_0 \leq \cdots \leq k_i = k_i \leq \cdots \leq k_n$. To define its face maps, we first group together the members in $\{k_0, \ldots, k_n\}$ that are the same. Meaning, given a sequence of increasing sequence $k_0 \leq \cdots \leq k_n$ that contains $l$ distinct numbers, we partition it into $l$ groups:

$$\begin{array}{ccccccccc}
1 & \ldots & < & 2 & \ldots & < & \cdots & < & l & \ldots.
\end{array}$$

Then we define

$$d_i^N : Y^N_n \to Y^N_{n-1}$$

to be

$$d_i^N |_{Y_l} := \text{id}$$

when $k_i$ belongs to the group of more than one member, or otherwise

$$d_i^N |_{Y_l} := d_j : Y_l \to Y_{l-1},$$

where $Y_l$ is indexed by the given sequence and $k_i$ belongs to the $j$-the group in figure 4. It is not difficult to see from the construction that $Y \times S$ can be obtained by throwing away the degenerate part of $Y^N$—namely, those components indexed by non-strictly increasing numbers. Furthermore, if $Y = \text{Ner} \, C$, we have $Y^N = \text{Ner} \, C^N$.

Now, we should expand on the homotopy equivalences in diagram 4. Firstly, since $X^N$ is proper (with respect to the Strøm model structure) and both $X^N_k$...
and $X^{N,p}_k$ have the homotopy type of CW-complexes, the upper right arrow is a homotopy equivalence \cite[VII, Proposition 3.6]{GJ99}, and therefore, in view of the homeomorphism $|Y^N| \simeq ||Y \times S||$ for any simplicial space $Y$, we immediately get the upper left arrow is also a homotopy equivalence. Secondly, following from the fact that $X^p \to X$ is a level-wise homotopy equivalence and Proposition A.1 in \cite{Seg74}, we have the map $||X^p|| \to ||X||$ is also a homotopy equivalence. Hence, in view of diagram (11), it suffices to show the map

$$\gamma : |X^{N,p}| \to |X^p|$$

(6)

is a homotopy equivalence. Because both simplicial spaces involved in the map (6) are proper, it is enough to prove the map

$$\text{Sing}_k X^N \to \text{Sing}_k X$$

induces a homotopy equivalence, for every $k$. Now, in the case where $X = \text{Ner} \, C$, we have $X^N$ is the nerve of $C^N$ and the map (6) is given by the natural forgetful functor

$$C^N \to C$$

$$(c, k) \mapsto c$$

$$(c \to d, k \leq l) \mapsto c \to d$$

Because the nerve $(\text{Ner.})$ and unraveling $(C \mapsto C^N)$ constructions commute with the singular functor, the map

$$\text{Sing}_k \, \text{Ner} \, C^N \to \text{Sing}_k \, \text{Ner} \, C$$

is identical to

$$\text{Ner} \, \text{Sing}_k C^N \to \text{Ner} \, \text{Sing}_k C.$$

Therefore, if one can show the functor

$$C^N \to C$$

induces a homotopy equivalence, for any discrete category $C$, then we are done.

Let’s pause for a moment and recall the variant of Quillen’s theorem A in \cite[Sec.1.4]{Wal83}: Given a map of simplicial space $f : X \to Y$, if, for any $y : \Delta^n \to Y$, the space $|f/(\Delta^n, y)|$ is contractible, then $f$ induces a homotopy equivalence, where $f/(\Delta^n, y)$ is the pullback of

$$\Delta^n \to Y \leftarrow X.$$

Using this version of Quillen’s theorem A, we know if one can prove the space

$$|\gamma/(\Delta^n, y)|$$

is contractible, for every $y : \Delta^n \to \text{Ner} \, C$, then the theorem follows.

To show this, we note first that every simplex $y : \Delta^n \to \text{Ner} \, C$ factors through a non-degenerate one as illustrated below:
In view of commutative diagram (7) and the fact that \( p \) is a trivial fibration and the category of simplicial sets \( s\text{Sets} \) is a proper model category, we may assume \( y \) is non-degenerate. In this case, \( y : \Delta^n \to \text{Ner} C \) is induced from a functor \( [n] \to C \), and hence, the pullback simplicial set \( \gamma/(\Delta^n, y) \) can be identified with \( \text{Ner} ([n]) \) and the map \( \gamma/(\Delta^n, y) \to \Delta^n \) can be realized by the forgetful functor \( [n]^N \to [n] \).

Now, we claim the forgetful functor \( [n]^N \to [n] \) induces a homotopy equivalence. Consider the full subcategory \( [n]^{N,\prime} \) of \( [n]^N \) which consists of objects \((k, l)\) with \( k \leq l \). There is a natural projection \( \pi_1 : [n]^N \to [n]^{N,\prime} \),

\[
(k, l) \mapsto (k, k) \\
(k, l) \mapsto (k, l)
\]

Suppose \( \iota : [n]^{N,\prime} \to [n]^N \) is the canonical inclusion, then \( \pi_1 \circ \iota = \text{id} \) is obvious, and on the other hand, there is a natural transformation \( \phi_1 : \text{id} \mapsto \iota \circ \pi_1 \) given by

\[
\phi_1(k, l) : (k, k) \to (k, l) \\
(k, l) \mapsto (k, l)
\]

Thus, the functors \( \iota \) and \( \pi_1 \) are inverse equivalences of categories.

Similarly, there is a natural projection \( \pi_2 : [n]^{N,\prime} \to [n] \)

\[
(k, l) \mapsto k,
\]

which, along with the canonical inclusion

\[
\iota : [n] \to [n]^{N,\prime} \\
k \mapsto (k, k),
\]

gives an equivalence of categories. More precisely, we have \( \pi_2 \circ \iota = \text{id} \) and the natural transformation \( \phi_2 : \iota \circ \pi_2 \mapsto \text{id} \) given by

\[
\phi_2(k, l) : (k, k) \to (k, l) \\
(k, k) \mapsto (k, l)
\]

for every \((k, l) \in [n]^{N,\prime}\).

Thus, we have shown the commutative diagram of equivalences of categories:
and in particular, the space \(| \mathrm{Ner} [\pi]^n |\) is contractible.

\[ \]
where $S_d \triangle^n$ stands for the set of $k$-simplices in $S_d \triangle^n$, $t$ is a point in a $k$-simplex, and $|A_i|$ is the size of $A_i$. The map $u^*$ is given by the assignment

$$[k] \rightarrow [n]$$

$$(1, 2, ..., k) \mapsto (\max(A_0), \max(A_1), ..., \max(A_k)) \subset [n],$$

and $\max(A_i)$ stands for the maximal element in the set $A_i$. It is clear that $\{\tau_{n,k}\}_{k \in [n]}$ induces a map

$$\tau_n : X_n \times S_d \triangle^n \rightarrow \|X \times S\|.$$

To see it respects the face maps in $X$, we assume $y = d^*_i x \in X_{n-1}$ and express the image of the simplex

$$(x, A_0 \subset A_1 \subset ... \subset A_k)$$

with $A_k \subset [n] \setminus \{i\}$ under $\tau_n$ in $\|X \times S\|$ by

$$(u^* x, |A_0| < ... < |A_k|)$$

and the image of the simplex

$$(y, B_0, ..., B_k)$$

with $d_i|B_j : B_j \xrightarrow{\simeq} A_j$ for each $j$ under $\tau_{n-1}$ in $\|X \times S\|$ by

$$(v^* y, |B_0| < ... < |B_k|).$$

Now, the second components in simplices (9) and (10) are clearly the same, in view of the assumption $d_i|B_j : B_j \xrightarrow{\simeq} A_j$, and the same assumption also implies the compositions

$$(1, 2, ..., k) \mapsto (\max(B_0), ..., \max(B_k)) \subset [n-1] \xrightarrow{d_i} [n];$$

$$(1, 2, ..., k) \mapsto (\max(A_0), ..., \max(A_k)) \subset [n]$$

are identical. Hence, we have $u^* x = v^* d_i^* x = v^* y$, meaning that the first components in simplices (9) and (10) are also identical.

The homotopy between $\pi \circ \tau \simeq \text{id}$ is very easy to describe. It is given by the linear homotopy from the identity to the last vertex map. Pictorially, it looks like the following:
Conjecture. Is $\tau$ a homotopy inverse to the map $\pi$?

Remark. Though the idea is not so complicated, we find it very hard to write down the homotopy between $\tau \circ \pi$ and $\text{id}$ in details. Observe first that, given any simplicial space $Y$, there is a natural filtration $\emptyset \subset ||Y||_{(0)} \subset ... \subset ||Y||_{(k)} \subset ... \subset ||Y||$ given by truncating the simplicial space $Y$. Our strategy is to define a homotopy $h_n : ||X \times S||_{(n)} \times I \to ||X \times S||_{(n+1)}$ between $\tau \circ \pi : ||X \times S||_{(n)}$ and $\text{id}$, for each $n$ such that the diagram below commutes:

\[
\begin{array}{ccc}
||X \times S||_{(n)} \times I & \longrightarrow & ||X \times S||_{(n+1)} \\
\downarrow & & \downarrow \\
||X \times S||_{(n+1)} \times I & \longrightarrow & ||X \times S||_{(n+2)}
\end{array}
\]  

(12)

Then, passing to the colimit, we get the required homotopy.

The homotopy $h_0$ is simply the homotopy given by the 1-simplex $(x, 0 < k)$, where $x \in X_0$. For general $n$, we decompose the homotopy $h_n$ into two parts. Given a non-degenerate $x \in X_n$, there is a canonical embedding $\Delta^n \times [n, +\infty) \leftrightarrow ||X \times S||$, and the first part of $h_n$ is the linear homotopy to the projection $\Delta^n \times [n+\epsilon, +\infty) \to \Delta^n \times \{n+\epsilon\}$ with any thing below $\{n+\epsilon\}$ intact, where $\epsilon > 0$. The following illustrates the case $n = 1$.

The second part of $h_n$ is nothing but a thickened version of the homotopy illustrated in figure (11) except that instead of moving the vertices simultaneously, we start with the vertex of largest depth and down to the one of the smallest depth. The figure below illustrates the case $n = 1$. 

The second part of $h_n$ is nothing but a thickened version of the homotopy illustrated in figure (11) except that instead of moving the vertices simultaneously, we start with the vertex of largest depth and down to the one of the smallest depth. The figure below illustrates the case $n = 1$. 

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One still needs to take care of degenerate simplices in $X$, and it seems to be a cumbersome task to write down the homotopy of those degenerate simplices, even though it is possible to describe it in low-dimension. We are hoping for a better way to approach this problem.

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