A quasi-particle description of the $\mathcal{M}(3,p)$ models

P. Jacob and P. Mathieu
Department of Mathematical Sciences, University of Durham, Durham, DH1 3LE, UK
and
Département de physique, Université Laval, Québec, Canada G1K 7P4

Abstract

The $\mathcal{M}(3,p)$ minimal models are reconsidered from the point of view of the extended algebra whose generators are the energy-momentum tensor and the primary field $\phi_{2,1}$ of dimension $(p - 2)/4$. Within this framework, we provide a quasi-particle description of these models, in which all states are expressed solely in terms of the $\phi_{2,1}$-modes. More precisely, we show that all the states can be written in terms of $\phi_{2,1}$-type highest-weight states and their $\phi_{2,1}$-descendants. We further demonstrate that the conformal dimension of these highest-weight states can be calculated from the $\phi_{2,1}$ commutation relations, the highest-weight conditions and associativity. For the simplest models ($p = 5, 7$), the full spectrum is explicitly reconstructed along these lines. For $p$ odd, the commutation relations between the $\phi_{2,1}$ modes take the form of infinite sums, i.e., of generalized commutation relations akin to parafermionic models. In that case, an unexpected operator, generalizing the Witten index, is unravelled in the OPE of $\phi_{2,1}$ with itself. A quasi-particle basis formulated in terms of the sole $\phi_{1,2}$ modes is studied for all allowed values of $p$. We argue that it is governed by jagged-type partitions further subject a difference 2 condition at distance 2. We demonstrate the correctness of this basis by constructing its generating function, from which the proper fermionic expression of the combination of the Virasoro irreducible characters $\chi_{1,s}$ and $\chi_{1,p-s}$ (for $1 \leq s \leq [p/3] + 1$) are recovered. As an aside, a practical technique for implementing associativity at the level of mode computations is presented, together with a general discussion of the relation between associativity and the Jacobi identities.
1. Introduction

1.1. Quasi-particle description of the minimal models: extended algebra vs spinon-type formulation

Fermionic-type character expressions are known for all Virasoro minimal models. However, most of these Virasoro characters, as well as the underlying basis of states, have not been explained within a conformal field theoretical set up. To look for such an intrinsic understanding of the fermionic characters is a fundamental quest. So far, only the minimal models \( \mathcal{M}(2,p) \) have been successfully addressed from that perspective [4, 12]. But is there any natural lines of attack for handling this question? Two potential avenues can readily been identified: a reinterpretation in terms of an extended algebra or a spinon-type reformulation.

All minimal models have a hidden extended conformal symmetry. The extension is obtained by adding one generator to the usual energy-momentum tensor. This extra generator is \( \phi_{p-1,1} \), with fusion rule

\[
\phi_{p',-1,1} \times \phi_{p',-1,1} = \phi_{1,1} = I .
\]

 Reformulating the minimal models from the perspective of this two-generator extended algebra has the obvious disadvantage that the defining extended algebra differs from one model to the other. But on the other hand, that the algebraic formalism is tailor-made for each model strongly suggests the existence of an underlying basis, generated by the \( \phi_{p',-1,1} \) modes, that would be free of singular vectors, i.e., a quasi-particle basis.

An alternative formulation might be considered, in which the fundamental spectrum generating field is either \( \phi_{1,2} \) or \( \phi_{2,1} \). In such a formulation, the extended algebra (incorporating one of \( \phi_{1,2} \) and \( \phi_{2,1} \)) would generically be ‘non-abelian’ (that is, multi-channels), e.g., since

\[
\phi_{2,1} \times \phi_{2,1} = \phi_{1,1} + \phi_{3,1} .
\]

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1 A brief review of the origin of fermionic characters in conformal field theory, together with an extended list of references (focusing mainly on the early works), is presented in the introduction of [1]. Note that the first fermionic character formulas appeared in mathematics [2] and its interpretation in terms of an exclusion principle seems to go back to [3]. Another early reference on fermionic formulas is [4]. This subject has somewhat exploded with [5] where various fermionic-sum formulas were proposed and interpreted in terms of a generalized exclusion principle. More formulas pertaining to the Virasoro minimal models were conjectured and proved in [6] and [7]. The works of the last reference rely heavily on a special truncation of the spin chain XXZ spectrum. A different method for proving the Virasoro fermionic formulas has been developed in [8], using the connection with the RSOS models [9]. The recent work [10] contains exhaustive results and many references to other works. A different approach to the construction of fermionic formulas has been initiated in [11] and vigorously extended recently by Feigin and collaborators.

2 The field \( \phi_{p',-1,1} \) has dimension \( h_{p',-1,1} = (p - 2)(p' - 2)/4 \), which is integer when either \( p \) or \( p' \) is of the form \( 2 + 4m \). In that case, this extended algebra is at the roots of the corresponding A-D type block-diagonal modular invariants [13] (for this interpretation, see for instance [14]). Notice that even for this case, the minimal models have not been reformulated in terms of the representation theory of this extended algebra. But we stress that such a reformulation does not requires \( h_{p',-1,1} \) to be integer (a point that is plainly illustrated in this work).
with the two fields on the right-hand side having dimensions that do not differ by an integer. In that framework, the added algebra generators would be the set of fields \( \{ \phi_{r,1} \} \) in which \( \phi_{2,1} \) plays the role of a basic generator. This is much like the parafermionic algebra \([16]\) which is spanned by the parafermionic fields \( \psi_n \), with \( 0 \leq n \leq k - 1 \); the \( \psi_n \)'s can all be obtained from the multiple product of \( \psi_1 \) with itself. However, the parafermionic algebra is single-channel (that is, \( \psi_n \times \psi_m = \psi_{n+m} \)), in contradistinction with the above \( \{ \phi_{r,1} \} \) algebra. This yet-to-be-defined \( \phi_{2,1} \)-type reformulation of the minimal models is actually closer to the spinon description of the \( \hat{su}(2)_k \) WZW models (see \([17,18]\) for \( k = 1 \) and \([19]\) for \( k > 1 \)). In the WZW context, the spinon field refers to the primary doublet of spin \( j = 1/2 \), denoted \( \phi_{1/2} \) and the analog of the above fusion rule is \( \phi_{1/2} \times \phi_{1/2} = \phi_0 + \phi_1 \) (\( \phi_1 \) being absent for \( k = 1 \)). Because of this analogy, this approach will be referred to as a spinon-type reformulation.

Extended algebras associated to a set of OPE including multi-channel ones of the sort (1.2) appear rather difficult to analyze, however. But that the field \( \phi_{1,2} \) could be regarded as a primary field in a formulation based on an extended algebra containing \( \phi_{2,1} \) (or the inverse) would fit nicely the following suggestive formula that would necessarily results from such a construction

\[
\frac{2h_{1,2}h_{2,1}}{c_{p',p}} = \frac{1}{8}.
\]

This expression is valid for all minimal models \( \mathcal{M}(p', p) \), with

\[
h_{r,s} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'} \quad \text{and} \quad c_{p',p} = 1 - \frac{6(p - p')^2}{pp'} ,
\]

\((1 \leq r \leq p' - 1 \text{ and } 1 \leq s \leq p - 1, \text{ with } p' < p)\).

Whether this spinon-type approach can be worked out in general remains to be seen. In this work, we consider the special case for which this second method reduces to the first one, namely the \( \mathcal{M}(3, p) \) minimal models. When \( p' = 3 \), the fusion of \( \phi_{2,1} \) with itself yields the identity:

\[
\phi_{2,1} \times \phi_{2,1} = I .
\]

### 1.2. Generalities on the structure of the \( \phi \)-algebra

Our aim here is thus to describe the \( \mathcal{M}(3, p) \) models in terms of the extended algebra associated to the OPEs

\[
\phi(z)\phi(w) = \frac{1}{(z-w)^2k} \left[ I + (z-w)^2 \frac{2h}{c} T(w) + \cdots \right] S ,
\]

\[
T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{(z-w)} + \cdots
\]

\[
T(z)T(w) = \frac{c_{3,p}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots
\]

with

\[
\phi \equiv \phi_{2,1} \quad \text{and} \quad h \equiv h_{1,2} = \frac{p - 2}{4} ,
\]

\(^3\) They differ by an integer when \( p' = 2 \) but in that case \( \phi_{3,1} \) is a descendant of \( \phi_{1,1} \) \([15]\).
instead of through the irreducible representations of the Virasoro algebra. In the first OPE, we have included an operator $\mathcal{S}$ which is enforced (see below) by the mutual locality of $\phi$ for the cases where $h \notin \mathbb{Z}^+ / 2$, that is, for $p$ odd; $\mathcal{S}$ anticommutes with $\phi$ and commutes with $T$. That this algebra is associative [20], at least for the value central charge $c_{3,p}$, is guaranteed by the associativity of the operator algebra of minimal models [21, 22].

Note that, as for the WZW models [23] or the parafermionic models [16], the Virasoro algebra lives in the ‘enveloping algebra’ of $\phi$. This is well-known for the free-fermionic formulation of the Ising model, which corresponds to the $p = 4$ case of the above construction (i.e., $\psi = \phi_{1,3} = \phi_{2,1}$). That $T$ might not occur in a (pole-type) singular term in the OPE $\phi(z)\phi(w)$ (which is the case for $p < 6$) is no obstruction to the fact that $T$ can always be written as a bilinear in the field $\phi$. In that sense, the ‘$\phi$-algebra’ is essentially defined by the first OPE in (1.6) and it will be understood as such.

We stress that, in writing (1.6), we do not consider the generic form of the OPE $\phi(z)\phi(w)$ [20]. In fact, we have imposed an implicit restriction on those terms that can appear on the right-hand side of (1.6): these are solely the descendants of the Virasoro identity. In particular, no bilinear composite of $\phi$ does occur. This condition is restrictive; it actually fixes $c$ to finitely many values (and in particular, $c$ cannot be free). For instance, when $\phi$ has dimension $h = 3/2$, there are only two solutions to the associativity conditions: $c = 7/10$ and $c = -21/4$.

The structure of the $\phi$-algebra (1.6) depends crucially upon the parity of $p$. For $p$ even, $2h$ is integer so that $\mathcal{S} = I$. There are then only integer powers of $z - w$ on the right-hand side of (1.6). This results into ordinary (anti)commutation relations. The basic data for the first few models with $p$ even are:

\[
\begin{align*}
\mathcal{M}(3,8) : & \quad h = 3/2, \quad c = -21/4 : \simeq \mathcal{S}M(2,8), \\
\mathcal{M}(3,10) : & \quad h = 2, \quad c = -44/5 : \simeq [M(2,5)]^2, \\
\mathcal{M}(3,14) : & \quad h = 3, \quad c = -114/7 : \simeq W_3(3,7), \\
\mathcal{M}(3,16) : & \quad h = 7/2, \quad c = -161/8, \\
\mathcal{M}(3,20) : & \quad h = 9/2, \quad c = -279/10, \\
\mathcal{M}(3,22) : & \quad h = 5, \quad c = -350/11.
\end{align*}
\]

$\mathcal{S}M$ stands for a superconformal minimal model. That associativity could fix the central charge in spite of the fact that these relations might satisfy the Jacobi identities for generic values of the central charge, as in the first three cases [20], is a point that is clarified in the appendix. The solution displayed in the last three cases is among those few (3 for $h = 7/2$ and 5 for the other two cases) found in an investigation of low-dimensional two-generators extended algebras [24].

On the other hand, for $p$ odd, $h = \pm 1/4 \mod 1$. That case presents at once a severe complication in that the commutation relations resulting from (1.6) take the form of infinite sums, as for parafermionic models [16] (cf. section 2). Our analysis of the first two models in this class, $h = 3/4$ and $h = 5/4$, shows that in these cases $c$ is uniquely determined and it agrees with $c_{3,5}$ and $c_{3,7}$ respectively. But that two solutions are found for the next case ($h = 9/4$) suggests that this uniqueness is not generic.

As previously mentioned, the analysis of the $p$ odd case also reveals that the first OPE in (1.6) requires the introduction of the operator $\mathcal{S}$, anticommuting with $\phi$. This operator can be viewed as a generalization of the Witten operator $(-1)^F$ which anticommutates with fermions [25]. Here we have $\mathcal{S} = (-1)^{pF}$ with

\[
\phi \times \phi = (-1)^{pF} I \quad \text{with} \quad (-1)^{pF} \phi = (-1)^{pF} \phi = (-1)^{pF} \phi.
\]
This is quite similar to the phase factor introduced in the $\widehat{su}(2)_1$ commutation relations for the spinon fields (for which $h = 1/4$) in [19] (cf. eq. (3) there). The existence of this operator is definitely forced by associativity. Note that it does not appear in the OPE of the physical field $\Phi(z, \bar{z}) = \phi(z)\phi(\bar{z})$ since it squares to 1. Since $T$ is bilinear in $\phi$, it commutes with $S$.

1.3. The spinon-type reformulation of the $\mathcal{M}(3, p)$ models: setting the problem

What does a $\phi$-algebra reformulation of the $\mathcal{M}(3, p)$ models should amount to? It should certainly allow us to fix completely the spectrum of the model for a given $p$, that is, to determine the highest-weight states and their conformal dimension. The highest-weight states turn out to be completely characterized by an integer $\ell$ such that $0 \leq \ell \leq (p - 2)/2$. The highest-weight state conditions are formulated directly in terms of the $\phi$-modes and they read

$$\phi_{-h-n+\frac{1}{2}}|\sigma_\ell\rangle = 0 \quad n > 0.$$  \hspace{1cm} (1.10)

We then observe that the conformal dimension of some highest-weight states $|\sigma_\ell\rangle$ are directly obtained from the commutation relations deduced from (1.6) and the highest-weight conditions (1.10). However, it is generally necessary to invoke associativity to fix the full spectrum. This analysis is performed in section 2 for the two simplest $\mathcal{M}(3, p)$ models apart from the Ising case, namely $\mathcal{M}(3, 5)$ and $\mathcal{M}(3, 7)$. The obtained conformal dimensions confirm the following identification with the Virasoro highest-weight states: $|\phi_{1,s}\rangle = |\sigma_{s-1}\rangle$ for $1 \leq s < p/2$.

A second required ingredient is a complete characterization of the descendant states in terms of the action of the $\phi$-modes. Note in that regard that a single $\phi$-module generically decompose into the direct sum of two Virasoro modules. Each Virasoro component will be singled out from the subset of descendant states containing an even or odd number of $\phi$-modes acting on the highest-weight state. This feature is well known for those $\mathcal{M}(3, p)$ models that have previously been formulated from that perspective, namely the $\mathcal{M}(3, 4)$, $\mathcal{M}(3, 5)$ and $\mathcal{M}(3, 8)$ models, for which $\phi$ is respectively the free fermion, the fundamental graded parafermion [26] and the superpartner of the energy-momentum tensor. In the present case, the Virasoro highest-weight states $|\phi_{1,s}\rangle = |\sigma_{s-1}\rangle$ and $|\phi_{1,p-s}\rangle$ (1 \leq s < p/2) are combined into a single $\phi$-module since $|\phi_{1,p-s}\rangle$ is a descendant of $|\phi_{1,s}\rangle$:

$$|\phi_{1,p-s}\rangle = \phi_{-h+(s-1)/2}|\phi_{1,s}\rangle.$$  \hspace{1cm} (1.11)

This matches the relation

$$h_{1,p-s} - h_{1,s} = h - \frac{(s - 1)}{2}.$$  \hspace{1cm} (1.12)

We stress that we invoke here only very general aspects of a yet-to-be-defined representation theory of the $\phi$-algebra. Actually, we make use the notion of a highest-weight state, that of $\phi$-lowering operators and associativity.

Ideally, we would also like to look for a complete description of the irreducible modules, i.e., a basis of states. There are two natural possibilities for such a basis, one formulated in terms of the combination of the Virasoro modes and the $\phi$-modes or one formulated solely in terms of the $\phi$-modes. In this work, we focus on the second possibility. In that case, the basis is expected to be of a quasi-particle type. A quasi-particle basis entails the construction of the Hilbert space by the action of the quasi-particle creation operators subject to a restriction rule, i.e., a filling process controlled by an exclusion principle. The obtention of this basis is our main result.
1.4. Quasi-particle basis of states

In the quest for a quasi-particle basis, we were guided by our previous construction [26] of the quasi-particle basis for the graded parafermions [27] in terms of jagged partitions. Such partitions differ from standard partitions, for which parts are non-increasing from left to right, in that a possible increase between parts at short distance is allowed.

Manipulations with the generalized commutation relations derived from (1.6) together with simple considerations on the structure of the highest-weight modules lead us to infer the general form of a candidate basis (section 3). It takes the following form. The highest-weight modules $|\sigma\rangle$ are described by the successive action of the lowering $\phi$-modes subject to specific constraints. In the $N$-particle sector, with strings of lowering modes written in the form

$$\phi_{-h+\frac{1}{2}+\frac{(N-1)}{2}}\phi_{-h+\frac{1}{2}+\frac{(N-2)}{2}}\cdots\phi_{-h+\frac{1}{2}+\frac{1}{2}}\phi_{-h+\frac{1}{2}}|\sigma\rangle,$$

(1.13)

these constraints are:

$$n_i \geq n_{i+1} - r + 1 \quad n_i \geq n_{i+2} + 2$$

$$n_{N-1} \geq \ell - r \quad n_N \geq 0,$$

(1.14)

where

$$2r = p - 5.$$  

(1.15)

The $n_i$’s are integers for $p$ odd and alternate between integer and half-integer values when $p$ is even (from right to left). The condition $0 \leq \ell \leq k$ is linked to the boundary condition on $n_{N-1}$ that appears to be ‘complete’ (i.e., to represent the full set of conditions that singles out the different modules) only in these cases.

This quasi-particle basis of states was known in at least three cases: $p = 4, 5$ and $8$. In each case, it reduces to (1.14). For $p = 4$, the quasi-particle basis is that of a free fermion:

$$b_{-s_1} \cdots b_{-s_N}|0\rangle \quad \text{with} \quad s_i \geq s_{i+1} + 1.$$  

(1.16)

To rephrase this in terms of our previous notation, we set

$$s_i = n_i + \frac{1}{2} - \frac{(N-i)}{2} \quad \Rightarrow \quad n_i \geq n_{i+1} + \frac{3}{2},$$

(1.17)

which is indeed the first condition in (1.14) when $r = -1/2$. In this special case, the second condition is a consequence of the first one.

For $p = 5$, as mentioned previously, the model is the simplest example of a graded $\mathbb{Z}_k$ parafermion [26], corresponding to the value $k = 1$. The quasi-particle basis in that special case, when reformulated in terms of the $n_i$’s reads (see [26], end of section 5):

$$n_i \geq n_{i+1} + 1.$$  

(1.18)

This is again the first condition in (1.14) for $r = 0$; here again it implies the second condition of (1.14).
For \( p = 8 \), it has been pointed out that \( \mathcal{M}(3,8) \simeq \mathcal{SM}(2,8) \). Now, we have recently obtained the quasi-particle basis of superconformal models \( \mathcal{SM}(2,4\kappa) \) in [28]. It is expressed solely in terms of \( G \) modes (\( G \) being the superpartner of \( T \)) and for \( \kappa = 4 \), it takes the form

\[
G_{-s_1} \cdots G_{-s_n} |0\rangle \quad \text{with} \quad s_i \geq s_{i+1} - 1 \quad \text{and} \quad s_i \geq s_{i+2} + 1 ,
\]

with all \( s_i \) half-integers (cf. [28] eqs (13) and (17)). The relation between \( s_i \) and \( n_i \) being \( s_i = n_i + 3/2 - (N - i)/2 \), the above conditions translate into

\[
n_i \geq n_{i+1} - \frac{1}{2} \quad \text{and} \quad n_i \geq n_{i+2} + 2 .
\]

We again recover (1.14) for \( r = 3/2 \).

1.5. Fermionic characters

The complete module of \( |\sigma_\ell\rangle \) is obtained by summing over all these states (1.13) satisfying (1.14) and all values of \( N \). Granting the correctness of this basis, one can then enumerate states in highest-weight modules (section 4). This leads to a fermionic expression of the (normalized) character that takes the simple form

\[
\hat{\chi}_\ell(q) = \sum_{m_1, m_2, \ldots, m_k \geq 0} \frac{q^m B^m C^m}{(q)_{m_1} \cdots (q)_{m_k}} ,
\]

where the matrices \( B \) and \( C \) are defined in section 4. In terms of the Virasoro characters, \( \hat{\chi}_\ell \) decomposes as follows:

\[
\hat{\chi}_\ell(q) = q^{-h_1, s + c/24} \left[ \chi_{1,s}^{\text{Vir}}(q) + q^{h_{1, p-s} - h_{1, s}} \chi_{1,p-s}^{\text{Vir}}(q) \right] \quad (\ell = s - 10 \leq \ell \leq \frac{p}{3}) .
\]

We recover in this way the fermionic sums given in [29], where these expressions where first conjectured. The expression of some of these characters can also be found in [7, 30, 31] (see also the second reference of [5]). Their derivation from the general expressions in [10] is presented in [32]. This equivalence confirms the correctness of the basis (1.13)-(1.14).

Obtaining the fermionic character amounts to finding the generating function for all the states (1.13)-(1.14). But finding such generating functions is in general a difficult problem. In the present case, we modify the characterization of our states in order to make use of a related generating function derived in [33]. Given that we read off our generating function (up to boundary terms) from [33] and that this latter article explicitly deals with an algebra related to the \( \mathcal{M}(3, p) \) models, it is appropriate to clarify the relation between this work and the present one. The authors of [33] construct a vertex operator algebra out of the product of the \( \mathcal{M}(3, p) \) model and a free boson. The algebra is generated by the two local fields: \( a(z) = V_1 \phi_{2,1} \) and \( a^*(z) = V_{-1} \phi_{2,1} \), where \( V_1 \) and \( V_{-1} \) are vertex operators with dimension such that \( a, a^* \) have respective dimension 1 and \( p - 3 \). The monomial basis underlying this vertex operator algebra is spanned by the modes \( a_\lambda = (a_{\lambda_1}, \ldots, a_{\lambda_N}) \), with the \( \lambda_i \)'s subject to \( \lambda_i \geq \lambda_{i+2} + 2r \). The origin of the exclusion here is rooted in polynomial relations of the type \( a^p \phi = 0 \) for \( 0 \leq n \leq p - 3 \) (plus an additional one). In view of our results, by stripping off the contribution of the free boson, we end up with the jagged-type basis (1.14). In preparing the revised version of this work, we became aware of [34] where the translation of these results to the \( \mathcal{M}(3, p) \) models is performed and the resulting basis (cf. Lemma 5.5 there) agrees perfectly with ours for \( \ell = 0 \).
2. The $\mathcal{M}(3,p)$ algebra for $p$ odd

2.1. Generalized commutation relations

Let us first justify the necessity of the operator $S$ for $p$ odd by invoking associativity. Consider a correlator of the form $\langle \phi(z_1)\phi(z_2)\phi(z_3)\cdots \rangle$. The first OPE in (1.6) shows that moving $\phi(z_2)$ and then $\phi(z_3)$ in front of $\phi(z_1)$ induces a negative phase (since $2h$ is half-integer for $p$ odd):

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\cdots \rangle = -\langle \phi(z_2)\phi(z_3)\phi(z_1)\cdots \rangle \sim -\frac{1}{z^{2\ell}} \langle (S + \cdots) \phi(z_1)\cdots \rangle,$$

which is to be compared with

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\cdots \rangle \sim \frac{1}{z^{2\ell}} \langle \phi(z_1)(S + \cdots)\cdots \rangle.$$

The compatibility of these expressions forces

$$S \phi(z) = -\phi(z) S,$$

which captures the whole effect of $S$ and allows for the identification with $(-1)^{pF}$ previously displayed in (1.9). For the $\mathcal{M}(3,5)$ case, this operator also appears in the description of the model as a graded parafermionic theory. This is briefly reviewed in the following subsection.

Consider now the commutation relations associated to (1.6). The mode decomposition of $\phi$ acting on a state of ‘charge’ $\ell$ (or, in the sector specified by the integer $\ell$) is:

$$\phi(z) = \sum_{n \in \mathbb{Z}} z^{-n-\frac{2}{\ell}} \phi_{-h+\frac{2}{\ell}+n}.$$

The field $\phi$ itself has charge 1. A first useful commutation relation is obtained by evaluating the following integral:

$$\frac{1}{(2\pi i)^2} \oint dw \oint dz \phi(z) \phi(w) z^{\frac{2}{\ell}+n} w^{\frac{2}{\ell}+m-1} (z-w)^{2h-1},$$

in two different ways [16]. That yields

$$\sum_{t=0}^{\infty} C_{2h-1}^{(t)} \phi_{n-t+h}^{\ell} \phi_{\frac{2}{\ell}+m+t-h} + \phi_{\frac{2}{\ell}+m-1-t+h} \phi_{\frac{2}{\ell}+n+1+t-h} = S \delta_{n+m+\ell,0},$$

where

$$C_{u}^{(t)} = \frac{\Gamma(t-u)}{t!\Gamma(-u)}.$$

If we replace

$$w^{\frac{2}{\ell}+m-1}(z-w)^{2h-1} \rightarrow w^{\frac{2}{\ell}+m+1}(z-w)^{2h-3}$$

in (2.5), in order to pick up the contribution of $T$ (the change in the power of $w$ being purely conventional), we get instead

$$\sum_{t=0}^{\infty} C_{2h-3}^{(t)} \phi_{n-2-t+h}^{\ell} \phi_{\frac{2}{\ell}+m+t+2} + \phi_{\frac{2}{\ell}+m-1-t+h} \phi_{\frac{2}{\ell}+n+1+t-h}$$

$$= \frac{1}{2} \left( \frac{\ell}{2} + n \right) \left( \frac{\ell}{2} + n - 1 \right) S \delta_{n+m+\ell,0} + \frac{2h}{c} \delta_{n+m+\ell} S.$$
In the following, we will refer to the above two generalized commutation relations as follows:

\begin{align}
(2.6) & \leftrightarrow I_{n,m,\ell} \\
(2.9) & \leftrightarrow \Pi_{n,m,\ell}.
\end{align}

Let us first test the relative sign for the two terms in the infinite sum (2.9)\(^4\) by acting with both sides of (2.9) on the vacuum state \(|0\rangle\) (so that \(\ell = 0\)) which is such that

\[ \phi_{-h+n}|0\rangle = 0 \quad n > 0. \tag{2.11} \]

With \(n = 2\) and \(m = -2\), only one term contributes from the first sum and none from the second sum, so that

\[ \phi_h \phi_{-h}|0\rangle = \mathcal{S}|0\rangle. \tag{2.12} \]

Taking instead \(n = m = 0\), only one term of the second sum contributes and we get the same result, confirming thus the positive relative sign.

We have just stated in (2.11) the highest-weight condition pertaining to the vacuum state. Let us denote by \(|\sigma_\ell\rangle\) the highest-weight state in the sector labeled by \(\ell\). Its highest-weight state characterization is

\[ \phi_{-h-n+\ell}|\sigma_\ell\rangle = 0 \quad n > 0. \tag{2.13} \]

Note that in order for the dimension of the first descendant \(\phi_{-h+\ell}|\sigma_\ell\rangle\) to be non-negative, we require

\[ 0 \leq \ell \leq \frac{p-2}{2}. \tag{2.14} \]

This bound will be assumed to hold from now on. The action of \(\mathcal{S}\) on a highest-weight state \(|\sigma_\ell\rangle\) is normalized as

\[ \mathcal{S}|\sigma_\ell\rangle = |\sigma_\ell\rangle. \tag{2.15} \]

A somewhat remarkable feature of the \(\phi\)-algebra for \(p\) odd is that the dimension of the highest-weight state \(|\sigma_1\rangle\) follows directly from (2.9) and (2.13), exactly as for all parafermionic highest-weight states \([16]\). (This, of course, is also true for \(\sigma_0 = I\).) Applying both sides of (2.9) on \(|\sigma_1\rangle\) with \(n = m = 0\), we see that no term contributes on the left-hand side, so that using (2.15) and \(L_0|\sigma_1\rangle = h_1|\sigma_1\rangle\) we obtain

\[ \frac{2h_1}{c} = \frac{1}{8}, \tag{2.16} \]

which is a special case of (1.3) when \(c = c_{3,p}\) (and recall that \(h = h_{2,1}\)). In other words, with \(c = c_{3,p}\), this relation fixes the value of \(h_1\) to \(h_{1,2}\).

As previously pointed out, within the framework of this reformulation of the \(\mathcal{M}(3,p)\) models in terms of the \(\phi\)-algebra, a highest-weight \(\phi\)-module is generically a combination of two Virasoro highest-weight modules (and this holds true irrespectively of the parity of \(p\)). Indeed, take for instance the vacuum state \(|0\rangle = |\sigma_0\rangle = |\phi_{1,1}\rangle\); its first descendant will be \(\phi_{-h}|0\rangle\), which is itself the Virasoro highest-weight state \(|\phi_{1,p-1}\rangle = |\phi_{2,1}\rangle\). More generally, the states \(|\phi_{1,s}\rangle\) and \(|\phi_{1,p-s}\rangle\) will be combined into a single module since \(|\phi_{1,p-s}\rangle\) is a descendant of \(|\phi_{1,s}\rangle\) - cf. (1.11). Note also that

\[ \mathcal{S}|\phi_{1,p-s}\rangle = (-1)^s|\phi_{1,p-s}\rangle, \tag{2.17} \]

still with the understanding that \(s = \ell + 1 < p/2\). The only case for which the \(\phi\)-module reduces to a single Virasoro module is when \(p\) is even and \(s = p/2\).

\(^4\) Note that this sign is correlated to that in (2.6). This computation verifies the ‘bosonic’ nature of the field \(\phi\) within the present framework, i.e., that the interchange of the two fields in (2.5) does not generate a minus sign.
2.2. The $\mathcal{M}(3,5)$ model

Let us first demonstrate, in a very explicit way, the necessity of the operator $S$ anticommuting with $\phi$. For this, we evaluate $\phi_+^1 \phi_-^{1/4} \phi_-^{3/4}|0\rangle$ in two different ways, symbolically written as:\footnote{This is an example of a mode-formulated associativity computation – cf. the appendix for more detail.}

\[
\phi_+^1 \phi_-^{1/4} \phi_-^{3/4}|0\rangle = \phi_+^{1/4} \phi_-^1 \phi_-^{3/4}|0\rangle ,
\]

using the notation introduced in (2.10). This relation means that we commute the first two terms using (2.6) with $n = -1$, $m = 0$, $\ell = 1$ ($\ell = 1$ because we act on a state with $\ell = 1$, namely $\phi_-^{1/4}|0\rangle$) and we compare this with the result of commuting the second and third terms using again (2.6) but now with $n = -1$, $m = 0$, $\ell = 0$. That leads to

\[
\phi_-^{1/4} \phi_-^{3/4}|0\rangle + S\phi_-^{1/4}|0\rangle = 0 ,
\]

(i.e., $\phi_-^{1/4} \phi_-^{3/4}|0\rangle = 0$ follows directly from $L_{-1,0,0}$ and this is in agreement with the absence of a level-one descendant in the vacuum module: $L_{-1}|0\rangle = 0$). Next we use $I_{0,0,0}$ to obtain

\[
\phi_+^{1/4} \phi_-^{1/4}|0\rangle = S|0\rangle .
\]

The relation (2.19) becomes then

\[
[\phi_-^{1/4} S + S\phi_-^{1/4}] |0\rangle = 0 ,
\]

which is precisely what we wanted to establish: $S$ anticommutates with the modes of $\phi$. On the other hand, there is no way of fixing the eigenvalue of $S$ on $|0\rangle$ and it is thus chosen to be 1.

The central charge is obtained by evaluating

\[
\phi_+^{1/4} \phi_-^{1/4} \phi_-^{3/4}|0\rangle = \phi_+^{1/4} \phi_-^1 \phi_-^{3/4}|0\rangle .
\]

This leads to

\[
-\frac{3}{8c} (5c + 3) \phi_-^{3/4}|0\rangle = 0 ,
\]

fixing $c = -3/5$ as expected. Associativity is further tested by computing the central charge as

\[
\phi_-^{1/4} \phi_+^{1/4} \phi_-^{3/4}|0\rangle = \phi_-^{1/4} \phi_+^{1/4} \phi_-^{3/4}|0\rangle .
\]

This gives the equivalent result:

\[
-\frac{1}{8c} (7c + 9) \phi_-^{3/4}|0\rangle = \phi_-^{3/4}|0\rangle .
\]

Consider now the spectrum of the model. Since $p = 5$, the bound (2.14) yields $0 \leq \ell \leq 1$. But we have already obtained the general dimension of the primary field $\sigma_1$ in (2.16). With $p = 5$ and $c = -3/5$, this yields $h_1 = -1/20$, which identifies $\sigma_1$ to $\phi_{1,2}$. Its first descendant is

\[
\phi_-^{3/4} |\sigma_1\rangle = \phi_-^{3/4} |\sigma_1\rangle \simeq |\phi_{1,3}\rangle ,
\]
of dimension 1/5. With $\sigma_0 \simeq \phi_{1,1}$ and
\[ \phi_{-\frac{3}{4}} |\sigma_0\rangle \simeq |\phi_{1,4}\rangle , \] (2.27)
with dimension 3/4, the spectrum of the $\mathcal{M}(3,5)$ model is completely recovered.

Let us briefly comment on the origin of the operator $S$ in the context of the graded parafermionic models, with coset representation $osp(1,2)_{\Lambda}/u(1)$ [27]. The $\mathcal{M}(3,5)$ model corresponds to $k = 1$ [26]. Let $\psi_{\frac{1}{4}}$ be the fundamental parafermionic field of dimension $1 - 1/4k$, which satisfies $(\psi_{\frac{1}{4}})^{2k} \sim I$. Denote by $\psi_1$ the parafermion of dimension $1 - 1/k$. We have $(\psi_1)^2 \sim \psi_1$. For $k = 1$ however, $\psi_1 \sim I$. But this is true up to a zero mode. Indeed, if we denote by $B$ and $A$ the respective modes of $\psi_{\frac{1}{4}}$ and $\psi_1$, then we have
\[ B_{\frac{1}{4}} A_{-1} |0\rangle = 0 = [A_0 B_{-\frac{1}{4}} + B_{-\frac{1}{4}} A_0] |0\rangle . \] (2.28)
We thus recover (2.21) with $B_{-\frac{1}{4}} \simeq \phi_{-\frac{1}{4}}$ and $A_0 \simeq S$.

### 2.3. The $\mathcal{M}(3,7)$ model

In the present case, the central charge is readily computed from
\[ \phi_{-\frac{1}{2}} \phi_{\frac{1}{2}} \phi_{-\frac{1}{2}} |0\rangle = \phi_{-\frac{1}{2}} \phi_{\frac{1}{2}} \phi_{-\frac{1}{2}} |0\rangle , \] (2.29)
leading to
\[ -\frac{1}{16c} (7c + 25) \phi_{-\frac{1}{4}} |0\rangle = 0 , \] (2.30)
with solution $c = -25/7$.

The most interesting aspect to consider here, compared to the previous case, is the determination of the spectrum. In the $\mathcal{M}(3,5)$ case, we had two primary field: $\sigma_0 = I$ and $\sigma_1$, whose dimension are directly determined by the commutation relations. With $p = 7$, we have three primary fields: $\sigma_0, \sigma_1$ and $\sigma_2$. Again the dimension of $\sigma_1$ results from (2.9): $h_1 = -5/28 (= h_{1,2})$. Here the difficulty lies in the determination of $h_2$, the dimension of $\sigma_2$, which does not follow from a direct application of the generalized commutation relations on $|\sigma_2\rangle$. This dimension can be fixed by associativity, however. For instance, by comparing :
\[ \phi_{-\frac{1}{2}} \phi_{\frac{1}{2}} \phi_{-\frac{1}{2}} |\sigma_2\rangle = \phi_{-\frac{1}{2}} \phi_{\frac{1}{2}} \phi_{-\frac{1}{2}} |\sigma_2\rangle , \] (2.31)
we get
\[ \left\{ \frac{1}{16} - \frac{5}{4c} \left( h_2 + \frac{1}{4} \right) \right\} \phi_{-\frac{1}{4}} |\sigma_2\rangle = \frac{5h_2}{2c} \phi_{-\frac{1}{4}} |\sigma_2\rangle , \] (2.32)
with solution $h_2 = -1/7$. We have thus
\[ |\sigma_0\rangle \simeq |\phi_{1,1}\rangle \quad |\sigma_1\rangle \simeq |\phi_{1,2}\rangle \quad |\sigma_2\rangle \simeq |\phi_{1,3}\rangle \]
\[ \phi_{-\frac{1}{4}} |\sigma_0\rangle \simeq |\phi_{1,6}\rangle \quad \phi_{-\frac{1}{4}} |\sigma_1\rangle \simeq |\phi_{1,5}\rangle \quad \phi_{-\frac{1}{4}} |\sigma_2\rangle \simeq |\phi_{1,4}\rangle , \] (2.33)
and this complete the analysis of the $\mathcal{M}(3,7)$ model.
3. The $\mathcal{M}(3,p)$ quasi-particle basis

3.1. The general form of the basis

Let us now turn to a description of the basis of states for the $\mathcal{M}(3,p)$ models, with $p$ of both parities.

In the highest-weight module of $|\sigma_\ell\rangle$, the different states in the $N$-particle sector are of the form:

$$
\phi_{-h+\frac{p}{2}+\frac{p-1}{2}^N-n_1} \cdots \phi_{-h+\frac{p}{2}+\frac{p-1}{2}^i-n_i} \cdots \phi_{-h+\frac{p}{2}+\frac{p-1}{2}^N-n_N} |\sigma_\ell\rangle,
$$

(3.1)

with some constraints on the $n_i$'s to be specified below. Note the cumulative contribution of the $\phi$ charge. The highest-weight module is obtained by summing over all possible particle-sector $N$. The indices $n_i$ are integers for $p$ odd and alternate between integer and half-integer values when $p$ is even:

$$
n_{N-2i} \in \mathbb{Z}, \quad n_{N-2i-1} \in \mathbb{Z} + \frac{p-1}{2}.
$$

(3.2)

It is convenient to represent the string (3.1) by a sequence whose $N$ entries (in the $N$-particle sector) are minus the modes $n_i$'s, i.e.,

$$
\phi_{-h+\frac{p}{2}+\frac{p-1}{2}^N-n_1} \cdots \phi_{-h+\frac{p}{2}+\frac{p-1}{2}^i-n_i} \cdots \phi_{-h+\frac{p}{2}+\frac{p-1}{2}^N-n_N} \leftrightarrow (n_1, \cdots, n_{N-1}, n_N),
$$

(3.3)

We will define our quasi-particle basis in terms of a filling process on a ground state. As argued below, this ground state is described by the sequence

$$
(\cdots, 6, -r+5, 4, -r+3, 2, -r+1, 0).
$$

(3.4)

where we have introduced the notation

$$
r = \frac{p-5}{2}.
$$

(3.5)

Notice the increase of 2 units at distance 2 (from left to right). The filling process amounts to add states corresponding to ordinary partitions on this ground state and sum over all particle sectors. The resulting sequences are not genuine partitions but merely 'jagged partitions' [26, 35, 36] which satisfy

$$
n_i \geq n_{i+1} - r + 1, \quad n_i \geq n_{i+2} + 2, \quad n_N \geq 0.
$$

(3.6)

These conditions are directly read off (3.4). Note that for $r = 0$ ($p = 5$), these are standard partitions with distinct parts. For $r = 1$ ($p = 7$), (3.6) describes standard partitions with a difference 2 condition between parts separated by the distance 2. In the generic case $r > 1$, there is a possible increase of $r - 1$ between the part $n_{N-2i-1}$ and its right nearest neighbour $n_{N-2i}$. Note that the difference 2 at distance 2 implies a further possible increase of $r + 1$ between $n_{N-2i}$ and $n_{N-2i+1}$ (again this is a direct consequence of (3.4): for $i = 1$, $r + 1$ is the difference between 2 and $-r + 1$). For instance, with $p = 14$, so that $r = 9/2$, our candidate-basis in the vacuum module is built on the ground state $(\cdots, 6, 1/2, 4, -3/2, 2, -7/2, 0)$, that is:

$$
\cdots \phi_{-3+3-6} \phi_{-3+3/2} \phi_{-3+2-4} \phi_{-3+3/2} \phi_{-3+1-2} \phi_{-3+3/2} \phi_{-3+0}|0\rangle = \cdots \phi_{-6} \phi_{-1} \phi_{-5} |0\rangle.
$$

(3.7)
(Note in particular that $\phi_1 \phi_{-3} |0 \rangle \propto L_{-2} |0 \rangle$).\footnote{Recall that $\mathcal{M}(3,14) \simeq W_3(3,7)$ model, so that $\phi = W$ in this context. Whether this basis can be lifted to a basis for the whole class of $W_3(3,p)$ models remains to be seen, however.}

Arguments supporting (3.6) are presented in the following subsection.

The characterization of the basis of states is not quite complete since yet there is no way of distinguishing the different highest-weight modules. Indeed, for $\ell \geq 2$, a further restriction has to be imposed on the parts. The origin of this sort of boundary condition is simply that the lowest state in the 2-particle sector of the $|\sigma_\ell \rangle$ module must be of the form $\phi_{-h+\frac{1}{2}+\frac{1}{2}-m_0} \phi_{-h+\frac{1}{2}} |\sigma_\ell \rangle$ of dimension $2h - \ell - 1/2 + m_0$. But this state has to be proportional to $L_{-1} |\sigma_\ell \rangle$, which forces $m_0 = \ell - r$. Since $m_0$ is the lowest value that $n_{N-1}$ can take, we have

$$n_{N-1} \geq \ell - r .$$

Our hypothesis would be that this is the whole set of boundary conditions. This is conformed by state counting at low levels.

Let us illustrate the conditions (3.6) as well as the boundary condition (3.8) by listing the states of the $\ell = 4$ module of the $\mathcal{M}(3,11)$ model at the first levels. We do this by writing the corresponding sequences $(n_1, \cdots, n_N)$. We also restrict ourself to the Virasoro module $|\phi_{1,5} \rangle$ which means that we only consider states that contain an even number of $\phi$ modes. The boundary condition (3.8) requires $n_{N-1} \geq 1$. The descendant states up to level 6 are:

| Level | States |
|-------|--------|
| 1     | (1,0)  |
| 2     | (2,0), (1,1) |
| 3     | (3,0), (2,1) |
| 4     | (4,0), (3,1), (2,2), (1,3), (3,2,1,0) |
| 5     | (5,0), (4,1), (3,2), (2,3), (4,2,1,0), (3,3,1,0) |
| 6     | (6,0), (5,1), (4,2), (3,3), (2,4), (5,2,1,0), (4,3,1,0), (3,4,1,0), (4,2,2,0), (3,3,1,0) |

For instance, the two sequences (2,4) and (5,2,1,0) correspond to the states

$$\begin{align*}
(2,4) : & \phi_{-\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-4} |\sigma_4 \rangle , \\
(5,2,1,0) : & \phi_{-\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+1-2} \phi_{-\frac{1}{4}+\frac{1}{4}+\frac{1}{2}-1} \phi_{-\frac{1}{4}-0} |\sigma_4 \rangle ,
\end{align*}$$

which indeed both have level 6. The first state containing 6 modes $\phi$ occurs at level 9 and it is associated to the sequence (5,4,3,2,1,0).

### 3.2. The rationale for the condition (3.6)

The aim of this section is to justify the basis (3.6) from conformal field theory. Our argument is, to a large extend, maintained at a sketchy level but we expect that the main points can be recovered by a more rigorous analysis.

To investigate the structure of the $\phi$-type quasi-particle basis, we consider the counting of independent states by treating successively the different particle sectors (recall that the ‘particle sector’ is the number of
φ-modes acting on the highest-weight state). In the first stage of the analysis, we only invoke the generic features of the φ-algebra. Only in the later steps do we require this to be also a Virasoro minimal model.

We will stick to the vacuum module (ℓ = 0) for simplicity. It will prove convenient to rewrite the states under the form

\[ \cdots \phi_{h-bN-3} \phi_{h-aN-2} \phi_{h-bN-1} \phi_{h-aN} |0\rangle. \]

Observe the specific choice made for the sign in front of \( h \) within the modes which is designed to facilitate the use of the commutation relations. In this notation, the cumulative charge of the φ-mode is absorbed into the \( a_i \) and \( b_i \) labels.

In the one-particle sector, the states are of the form \( \phi_{h-n} |0\rangle \). The only constraint is the highest-weight condition (2.11) which forces \( n \geq 0 \).

Consider next the two-particle sector. The basic constraint here comes from the commutation relation (2.6). This relation certainly continues to hold true also for the generic version of the algebra under consideration, which does not affect the leading term of the OPE.\(^7\) It is rather immediate to see that all the states \( \phi_{h-m} \phi_{h-n} |0\rangle \) which do not satisfy \( m \geq n + 2 \) can be reexpressed in terms of those that do satisfy this constraint.

Note that \( \phi_{h-m} \phi_{h-n} |0\rangle \) with \( m \geq n + 2 \) is equivalent to \( \phi_{h+1/2-n_1} \phi_{h-n_2} |0\rangle \) with \( n_1 \geq n_2 - r + 1 \). This analysis of the two-particle sector can be directly transposed to a sequence of two adjacent modes within the bulk of a string of φ-modes, where the condition takes the form \( n_i \geq n_{i+1} - r + 1 \).

So far, we have succeeded in explaining the jagged nature of the sequences of \( n_i \)'s associated to the string of modes in descendant states. Next, we have to consider the independent states in the three-particle sector. We consider states of the form

\[ \phi_{h-n'} \phi_{h-m} \phi_{h-n} |0\rangle, \]

and look for a constraint relating \( n' \) to \( n \). At this point, we move away from the study of the most general φ-type free basis and take into account the constraints coming from the fact that the models we consider are also Virasoro minimal models. As a result, every regular field appearing in the OPE \( \phi(z)\phi(w) \) has to be reexpressible in terms of the energy-momentum tensor only.\(^8\) This means that when we compute the commutation relations between the φ modes, we can pick up regular terms of arbitrary order and still only obtain Virasoro modes on the right-hand side of the commutation relations. This in turns has the obvious consequence that any state made of Virasoro modes can be expressed in terms of φ modes (in even number) and vice-versa. This is the advantage we take into account in our next step.

---

\(^7\) We stress that this commutation relation holds for both parities of \( p \) if we assume that \( S = I \) for \( p \) even (in which case the infinite sum truncates to a finite one). More generally, it holds for all value of \( p' \) with the understanding that \( \phi = \phi_{p'-1,1} \) and \( h = h_{p'-1,1} \).

\(^8\) To be plain, this immersion amounts to a reduction of the allowed fields appearing in the OPE \( \phi(z)\phi(w) \). For instance, \( (\partial GG)(w) \) is the first field so removed for those superconformal models that are also Virasoro minimal models, namely for the \( \mathcal{M}(3,8) \) and \( \mathcal{M}(4,5) \) models.
We stress that our present objective is to determine some ordering condition on trilinear states by a recursive process. Given a target ordering condition, we need to show that states which do not satisfy this ordering condition can be expressed in terms of those which do satisfy it. With this in mind, we now introduce a simplifying trick.

Let us return for a moment to the left-hand side of the commutation relations (2.6) and (2.9). Given the way these relations are derived, we see that the more terms we pick up in the OPE \( \phi(z) \phi(w) \), the higher is the gap between the left-most modes of the first sum and the right-most modes of the second sum. Knowing that we can pick up any regular term (since they are all expressible as combinations of \( T \)), let us suppose that we select a regular term of sufficiently high order such that the second sum contains only ordered states already considered so far in our analysis (that is, states that are more ordered than those under consideration at a given recursive step). For the purpose of the present discussion, we can thus ignore the contribution of this second sum. That leads to a simplified version of the commutation relations (where the equality is to be understood as modulo terms previously considered):

\[
\sum_{i=0}^{\infty} \phi^{(i)} \phi_{-n'-i} \phi_{h-m+i} = [L]_{-n'-m} + [LL]_{-n'-m} + \cdots
\]

(3.13)

where in this notation, the \( \phi^{(i)} \) are unspecified constants and \( [L \cdots L]_{-N} \) (with \( n \) factors of \( L \)) represents a given linear combination of \( n \) \( L \) modes at level \( N \). All the coefficients are fixed by the proper choice of commutation relations and the values of \( n' \) and \( m \).

Let us now assume at this point that the terms \( n' < n-1 \) in (3.12) can be reorganized in terms of those with \( n' < n \). Using (3.13) in (3.12) for \( n' = n-1 \), we obtain:

\[
\phi_{-h-n+1} \phi_{h-m} \phi_{-h-n} |0\rangle
\]

\[
= \left( [L]_{-n-m+1} + [LL]_{-n-m+1} + \cdots + \sum_{i=1}^{\infty} \phi^{(i)} \phi_{-h-n-i+1} \phi_{h-m+i} \right) \phi_{-h-n} |0\rangle
\]

\[
= a \phi_{-h-2n-m+1} |0\rangle + \phi_{-h-n} ( [L]_{-n-m+1} + [LL]_{-n-m+1} + \cdots ) |0\rangle + \sum_{i=1}^{\infty} \phi^{(i)} \phi_{-h-n-i+1} \phi_{h-m+i} \phi_{-h-n} |0\rangle
\]

\[
= a \phi_{-h-2n-m+1} |0\rangle + \phi_{-h-n} \sum_{i=0}^{\infty} \phi^{(i)} \phi_{h-n-m+i} \phi_{-h-i} |0\rangle + \sum_{i=1}^{\infty} \phi^{(i)} \phi_{-h-n-i+1} \phi_{h-m+i} \phi_{-h-n} |0\rangle
\]

(3.14)

where \( a \) is some constant. In the second equality, \( \phi_{-h-n} \) has been commuted with the Virasoro modes, while in last one, the Virasoro modes have been reexpressed in terms of the \( \phi \) modes.

We see that most of the \( \phi \)-trilinear states can be now written in the more ordered form \( n' \geq n \). If we get rid of the states already considered so far, we see that our initial state can be expressed in terms of the remaining non-ordered terms as:

\[
\phi_{-h-n+1} \phi_{h-m} \phi_{-h-n} |0\rangle = [\phi_{-h-n} \phi_{h-m+1} + \phi_{-h-n+1} \phi_{h-m+2} + \cdots ] |0\rangle
\]

(3.15)

The sum has to stop at some point as we can use the reordering of the bilinear terms in \( \phi \). Now we can repeat this for these new non-ordered states until we ultimately run out of such non-ordered terms. In other
words, by starting with a given \( n' = n - i \), a recursive process allows us to obtain states of the form \( n' > n - i \) for \( i > 0 \).

If we try to apply the same trick to eliminate states of the form \( n' > n \), we end up with relations linking the \( n' > n \) states to \( n' < n \) states. By consistency, we expect these relations to be the same ones we have first obtained for \( n' < n \).

Finally, if we consider the states with \( n' = n \), we now notice that these states with \( n' = n \) reappear along the derivation. Without explicitly calculating every coefficients in front of the states, we do not know whether some of the \( n' = n \) states can be eliminated or not.

We can thus naturally expect that the conditions on the trilinear terms in \( \phi \) will lie somewhere in between the conditions \( n' \geq n \) and \( n' > n \). Note that the above analysis can be applied to all \( \mathcal{M}(p, p') \) models if we replace \( \phi \) by \( \phi_{p'-1,1} \). From low-level state-counting checks, we can verify that the previous property is indeed verified for any \( \mathcal{M}(p', p) \) model with \( p' \geq 3 \) (which ensures that \( \phi_{p'-1,1} \neq \phi_{1,1} \)). The most restricted case corresponds to \( p' = 3 \) because the \( \phi \) singular vector arises at the lowest possible level, which is 2. This case lies exactly on the upper-bound constraint, that is, \( n' > n \). In order to get the least restricted case, we have to examine the models for which the singular vector appears as deeply as possible in the module. As \( \phi_{p'-1,1} \) has its first singular vector at level \( p' - 1 \), it corresponds to cases where \( p' \) is large. It appears that the trilinear terms in those cases are actually more restricted than \( n' \geq n \), indicating that this lower-bound condition is not saturated.

We have just seen how the condition \( n' \geq n \) had to be respected. Now we will try to be slightly more explicit about the possible restrictions for \( n' = n \). Let us consider (3.13) once again along with (3.12). Depending on the value of \( m \), we do not always need to pick up the same regular term in order to eliminate the second sum in the commutation relations. The higher the value of \( m \), the greater is the number of possibilities we have to write the commutation relations. Using all these different choices of commutation relations, we get a number of linearly independent relations taking all the form of the first equality in (3.14), with the coefficients before each terms differing from one choice to the other. If one could keep track of all the conditions coming form these different equalities, this could lead us to two possible outcomes. On the one hand, the result might be some intermediate ‘basis’ between the spanning set of states and the complete sought-for basis, upon which we would have to apply the restrictions coming from the removal of the first null-field in \( \phi \). On the other hand, the result could actually take care of all possible constraints. If this second possibility is the actual one, it would mean that the immersion of the \( \phi \)-extended conformal field theories into the \( \mathcal{M}(p', p) \) models fixes completely the Virasoro singular-vector structure.

We end up this section by displaying a sample computation supporting the later alternative. The example to be considered is the one for which \( \phi \) has dimension 3/2. The \( \phi \)-algebra is thus a superconformal algebra. Considered also as a Virasoro minimal model (which embodies a truncation of the space of states), this is associative for two values of \( c \), corresponding to the \( \mathcal{M}(3, 8) \) and \( \mathcal{M}(4, 5) \) models. We will show that in the former case, the state \( \phi_{-\frac{3}{2}, \phi_{-\frac{3}{4}}} ^ {\phi_{-\frac{3}{4}}} |0\rangle \) can be eliminated.

The anticommutation relation for \( \phi = G \) takes the form

\[
\{ G_{n+\frac{1}{2}}, G_{m-\frac{1}{2}} \} = \frac{n(n+1)}{2} \delta_{n+m,0} + \frac{c}{3} L_{m+n} .
\] (3.16)
This is obtained by considering only the singular terms in the OPE $G(z)G(w)$. But if, instead, we pick up the contribution of more terms, in particular, up to and including the level-two descendants of $T$, we then get

$$
\sum_{t \geq 0} C_{-2}^{(t)} [G_{n+\frac{3}{2}}G_{m-\frac{3}{2}} + G_{m-\frac{3}{2}}G_{n+\frac{3}{2}}] + \frac{(n+3)!}{24(n-1)!} \delta_{n+m,0} + \left[ \frac{3(n+3)(-m)}{2c} + \beta_1 (n+m+3)(n+m+2) \right] L_{n+m} + \beta_2 \sum_{t \geq 0} [L_{-2-t}L_{n+m+2+t} + L_{n+m+1-t}L_{-1+t}],
$$

with $\beta_1$ and $\beta_2$ being the coefficients of $T''$ and $(TT)$ respectively. These constants are fixed by associativity to the values

$$
\beta_1 = \frac{9(c+1)}{4c(22+5c)} \quad \text{and} \quad \beta_2 = \frac{51}{2c(22+5c)}.
$$

Further calculations show that the central charge must be restricted to the two values $7/10$ and $-21/4$ as previously said. Using (3.16), we can write

$$
G_{-\frac{3}{2}}G_{-\frac{3}{2}}G_{-\frac{3}{2}} = \frac{3}{c} G_{-\frac{3}{2}}L_{-2} = \frac{3}{c} \left[ G_{-\frac{3}{2}} + L_{-2}G_{-\frac{3}{2}} \right].
$$

Using now the commutation relations (3.17) with $n = m = -1$ in order to get an expression for $L_{-2}$ (as given by the first term on the right-hand side of (3.17)) acting on $G_{-3/2}$, we find that

$$
L_{-2}G_{-\frac{3}{2}} |0\rangle = \frac{3}{c} \left[ \left( -3\beta_2 - \frac{6}{c} \right) L_{-2}G_{-\frac{3}{2}} + \left( -2\beta_2 + 5 - \frac{3}{2c} \right) G_{-\frac{3}{2}} \right] |0\rangle.
$$

For $c = 7/10$, the second coefficient on the right-hand side vanishes while the first one reduces to 1; in other words, we end up with the identity $L_{-2}G_{-\frac{3}{2}} |0\rangle = L_{-2}G_{-\frac{3}{2}} |0\rangle$. Therefore, in that case, there is no relation between $L_{-2}G_{-\frac{3}{2}} |0\rangle$ and $G_{-\frac{3}{2}} |0\rangle$. But for the other allowed value of $c$, which corresponds to that of the $\mathcal{M}(3,8)$ model, there is one such relation. (Actually, we have recovered here the expression for the $\phi_{2,1}$ singular vector.) It implies that, in this precise case, we can eliminate the state $G_{-\frac{3}{2}}G_{-\frac{3}{2}}G_{-\frac{3}{2}} |0\rangle$.

Higher order terms can be treated along these lines. But clearly, going deeper in the modules requires the computation of more and more terms in $\phi(z)\phi(w)$. These computations are thus rather complicated, in addition to being model-dependent. But they provide independent verifications of the stated conditions (3.6).

4. The $\mathcal{M}(3,p)$ fermionic-type characters

Our main assumption is that (3.6) provides a basis. This has been supported by heuristic considerations, some explicit computations and the comparison with known bases for small values of $p$. However, establishing (3.6) rigorously is a hard mathematical problem. We circumvent this by showing that these conditions do indeed lead us to the expected characters. More precisely, we demonstrate here that the enumeration of states subject to the conditions (3.6) together with the boundary condition (3.8), reproduces the known expressions for the $\mathcal{M}(3,p)$ characters in their fermionic form.
In view of enumerating all the states in a given module (i.e., constructing its character), it is convenient to transform the ground state into one for which the parts do not increase from left to right. Let us then add to the ground state (3.4) the staircase of \((r - 1)\)-height step:

\[
(\cdots, 5r - 4, 4r - 3, 3r - 2, 2r - 1, r, 1). \tag{4.1}
\]

The shifted ground state is thus

\[
(\cdots, 4r + 1, 4r + 1, 2r + 1, 2r + 1, 1, 1). \tag{4.2}
\]

Partitions defined on this shifted ground state can be characterized as follows. These are partitions \((\lambda_1, \lambda_2, \cdots, \lambda_N)\) of length \(N\),

\[
\lambda_i \geq \lambda_{i+1} , \quad \lambda_N \geq 1 , \tag{4.3}
\]

satisfying the supplementary condition

\[
\lambda_i \geq \lambda_{i+2} + 2r . \tag{4.4}
\]

These conditions follow from (3.6) and

\[
\lambda_i = n_i + (N - i)(r - 1) + 1 . \tag{4.5}
\]

In other words, parts in \((\lambda_1, \cdots, \lambda_N)\) that are separated by the distance 2 must then differ by at least a \(2r\), with \(2r = p - 5\). To these conditions, we need to add the boundary condition:

\[
\lambda_{N-1} \geq \ell . \tag{4.6}
\]

Let \(p_{r,\ell}(w, N)\) be the number of partitions of length \(N\) and weight \(w\) (that is, \(w = \sum \lambda_i\)) satisfying (4.4) and (4.6). Denote the corresponding generating function by

\[
G_{r,\ell}(z, q) = \sum_{w,N \geq 0} p_{r,\ell}(w, N) q^w z^N . \tag{4.7}
\]

For \(0 \leq \ell \leq k\), this function can be obtained in closed form as a \(k\)-multiple sum:

\[
G_{r,\ell}(z, q) = \sum_{m_1, m_2, \cdots, m_k \geq 0} q^{m \tilde{B} m + \tilde{C} \cdot m} z^{2(m_1 + \cdots + m_{k-1}) + m_k} (q)_{m_1} \cdots (q)_{m_k} , \tag{4.8}
\]

where \(k\) is related to \(p\) by

\[
k = \left[ \frac{p}{3} \right] , \tag{4.9}
\]

(where \([x]\) stands for the integer part of \(x\)) and it is understood that

\[
m \tilde{B} m = \sum_{i,j=1}^{k} m_i \tilde{B}_{ij} m_j, \quad \tilde{C} \cdot m = \sum_{i=1}^{k} \tilde{C}_i m_i . \tag{4.10}
\]

\[\text{We stress that this shifting is simply a relabeling of the ground state. Note also if the ground state (3.4) involves both integers and half-integers for \(p\) even, the shifting process generates only integer parts since \(2r\) is always integer.}\]

\[\text{If, instead, we subtract from the ground state (3.4) the staircase \((\cdots, -r + 2, -r + 1, -r)\), it becomes \((\cdots, 0, r, 0, r, 0, r)\). This is the ground state of jagged partitions of type \(0r\) in the terminology of [36].}\]
The $k \times k$ symmetric matrix $\tilde{B}$ reads

$$
\tilde{B} = \begin{pmatrix}
2r & 2r & \cdots & 2r & r \\
2r & 2r+1 & \cdots & 2r+1 & r + \frac{1}{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2r & 2r+1 & \cdots & 2r+k-2 & r - 1 + \frac{k}{2} \\
r & r + \frac{1}{2} & \cdots & r - 1 + \frac{k}{2} & k - 1
\end{pmatrix},
$$

(4.11)

while the vector $\tilde{C}$ takes the form

$$
\tilde{C}_j = -2r + j + 1 + \max(\ell - j, 0) \quad \text{for} \quad j < k \quad \text{and} \quad \tilde{C}_k = -k + 2.
$$

(4.12)

Finally, in the denominator of (4.8), we made use of the notation

$$
(q)_n = \prod_{i=1}^{n}(1 - q^i).
$$

(4.13)

The generating function (4.8) for $\ell = 0$ (which turns out to hold also for $\ell = 1$) has been found in [33] (cf. Theorem 5.14 with $M_\ell, N \rightarrow \infty$). For $\ell \neq 0$ but $\ell \leq k$, the boundary condition (4.6) induces a modification by terms linear in the $m_i$’s and these are easily fixed by looking at the lowest partitions in low ($N = 1, 2$) particle sectors. This is taken care by the second term in $C_j$, as demonstrated in appendix B. The latter is a verification proof and it breaks for $\ell > k$. A further analysis of these boundary terms is presented in appendix C. It is shown there that $G_{r,\ell}$ can be recovered from $G_{r,0}$ recursively, for all $1 \leq \ell \leq p/2 - 1$. However, it appears that it is only for $1 \leq \ell \leq k$ that the $G_{r,\ell}$ can be reconstructed in closed form, as a single fermionic multisum.

Our goal is to count partitions built on the ground state (3.4) subject to (3.8) and weight them by their proper conformal dimension, constructing thereby the character of the $|\sigma_\ell\rangle$ module. We thus start with $G_{r,\ell}(z, q)$ and enforce $z^N$ to be equal to a certain power of $q$ adjusted in order to: 1- undo the ground state shifting by taking out the staircase contribution (4.1), whose weight is denoted $w_{\text{stair}}$; and 2- add the fractional part, with weight $w_{\text{frac}}$, to correct for the fact that (3.4) does not take into account the fractional part of the $\phi$-modes. These numbers are easily computed. On the one hand, the staircase $((N-1)r - (N-2), \cdots, 2r-1, r, 1)$ has weight

$$
w_{\text{stair}} = \frac{N}{2} \left[(N-1)(r-1) + 2\right].
$$

(4.14)

On the other hand, the fractional part has the following dimension

$$
w_{\text{frac}} = N \left[h - \frac{\ell}{2} - \frac{(N-1)}{4}\right] = \frac{N}{4} \left(2r + 4 - 2\ell - N\right),
$$

(4.15)

where the third term in the square bracket comes from the cumulative contribution of the $\phi$ charges. We thus replace $z^N$ in (4.8) by $q^{w_{\text{frac}} - w_{\text{stair}}}$, where

$$
N = 2(m_1 + \cdots + m_{k-1}) + m_k.
$$

(4.16)
This leads to the following expression for the character \( \hat{\chi}_\ell \) (normalized such that its leading \( q \) power is \( q^0 \), hence the hat),

\[
\hat{\chi}_\ell(q) = \sum_{m_1, m_2, \ldots, m_k \geq 0} \frac{q^{m_B m + C \cdot m}}{(q)_{m_1} \cdots (q)_{m_k}},
\]

where, with \( 1 \leq i, j \leq k - 1 \)

\[
B_{ij} = \min(i, j), \quad B_{jk} = B_{kj} = j/2, \quad B_{kk} = \frac{k + 1 - \epsilon}{4},
\]

and \( C \) reads

\[
C_j = \max(j - \ell, 0), \quad C_k = \frac{k - 1 + \epsilon - \ell}{2},
\]

with \( \epsilon = 0, 1 \) defined by

\[
p = 3k + 1 + \epsilon.
\]

In terms of the Virasoro characters, \( \hat{\chi}_\ell \) decomposes as follows

\[
\hat{\chi}_{s-1}(q) = q^{-\tilde{h}_{1,s} + c/24} \left[ \chi_{1,s}^{\text{Vir}}(q) + q^{\tilde{h}_{1,p-s}-\tilde{h}_{1,s}} \chi_{1,p-s}^{\text{Vir}}(q) \right].
\]

Recall that in our construction, the Virasoro primary field \( \phi_{1,p-s} \) (for \( p > 2s \)) is a \( \phi \)-descendant of \( \phi_{1,s} \). The two Virasoro characters can be separated by the parity of \( N \), which is the same as that of \( m_k \): with \( m_k \) even (odd), we obtain \( \chi_{1,s} \) (\( \chi_{1,p-s} \) respectively).

We recover thus the fermionic sums given in [29] in a form similar to that displayed here. Their complete proof is presented in [10] and their reexpression in the above form is worked out in [32].

5. Conclusion

In this work, we have considered the reformulation of the minimal models \( \mathcal{M}(3,p) \) in terms of the algebra spanned by \( \phi \equiv \phi_{2,1} \) and defined by the OPE (1.6). The structure of this algebra differs somewhat according to the parity of \( p \): for \( p \) even, \( 2\hbar \in \mathbb{Z}_+ \) while for \( p \) odd, \( \hbar \in \mathbb{Z}_+ \pm 1/4 \). In the latter case, the associativity analysis forces the introduction of a Witten-type operator anticommuting with \( \phi \). A similar operator (but presented differently) has been found in the spinon formulation of the \( \hat{su}(2)_1 \) model [19]. It seems to characterize non-local algebras with generators of dimension \( \hbar \in \mathbb{Z}_+ \pm 1/4 \).

The Hilbert spaces (highest-weight states and their descendants) have been completely described in terms of the \( \phi \)-algebra. In particular, the modules are described by the successive action of the lowering \( \phi \)-modes subject to specific constraints. In the \( N \)-particle sector, with strings of lowering modes written in the form (1.13), these constraints are given in (1.14). The highest-weight states themselves are distinguished by the integer \( \ell \) whose range is \( 0 \leq \ell \leq p/2 - 1 \). Moreover, the \( \ell \)-dependence of the descendant states is fully captured by the condition \( n_{\ell+1} \geq \ell - r \). The obtained basis agrees with those previously found for \( p = 4, 5 \) and 8 and the one derived in [33, 34] for \( \ell = 0 \).

In absence of a complete argumentation underlying the derivation of this basis, our considerations have been supplemented by general arguments and explicit computations relying on the observation that the fine
structure of the defining $M(3, p)$ OPE $\phi(z)\phi(w)$ encodes the complete information on the models, including its quasi-particle basis. Note that this analysis does not mimic that of the $M(2, p)$ and $SM(2, 4\kappa)$ cases. In these cases, the spanning basis of states is first obtained and a set of restrictions, arising from the identity null field, is then imposed. The spanning set of states for the $\phi$-algebra, on which one could impose constraints such as the level-two $\phi$ null field, has not been found yet.

The simplest way of verifying the correctness of the basis of states is to derive the character of the irreducible module of $|\sigma_\ell\rangle$. This is obtained by enumerating all these states (1.13)-(1.14) and summing over all values of $N$. The character is explicitly given by (4.17) when $0 \leq \ell \leq \lfloor p/3 \rfloor$. This agrees with the known fermionic form of these characters [29, 32]. Although for $\lfloor p/3 \rfloor < \ell \leq p/2 - 1$, the character has not been found in closed form, it is shown in appendix C how it can be obtained recursively from $G_{r,0}$. But we stress that the validity of the boundary term in this range has been tested by writing explicitly the states at the first few levels of various modules and comparing their enumeration with that given by the usual bosonic formula.

The $M(3, p)$ models, due to the presence of an extra symmetry generator, are somewhat similar to the superconformal models. For the special class of $SM(2, 4\kappa)$ models, we can either choose to write the quasi-particle basis either in terms of the modes of $G$ together with the Virasoro modes or solely in terms of the $G$ modes [28]. This suggests that one could look for an alternative quasi-particle basis for the $M(3, p)$ models, this one formulated in term of an ordered set of Virasoro modes acting on an ordered set of $\phi$ modes as

$$L^{-n_1} \cdots L^{-n_k} \phi^{-m_1} \cdots \phi^{-m_k'} |0\rangle \quad (5.1)$$

with some constraints on the numbers $n_i$ and $m_i$. A basis of that type has indeed been found; this result will be presented elsewhere [37].

Another natural axis for extension follows from the observation that an analysis similar to the present one should be applicable to all extended algebras having the essential simplifying property of being single-channel.

Let us conclude by emphasizing the fact that there is a relatively small number of conformal field theories for which the fermionic characters are described in terms of a basis derived by intrinsic conformal field theoretical methods. We have already mentioned that this is so for the $M(2, p)$ minimal models [4] together with their superconformal analogues, the $SM(2, 4\kappa)$ models [28]. But there are few other examples, like the $\hat{su}(2)_k$ models [18,19], particular higher-rank WZW models [38, 39], the parafermionic models [1], their graded version [26, 35] and some higher-rank formulations [40]. The present analysis is a step toward the addition of a further example to this list, the $M(3, p)$ minimal models.

Appendix A. Associativity and Jacobi identities

The associativity conditions for the symmetry generators of an extended conformal algebra are sometimes loosely viewed as being equivalent to the Jacobi identities for the mode-generators. For fields with (half-)integer dimension that associativity implies the Jacobi identity can indeed be derived in a rather direct
way.\textsuperscript{11} But the reverse is not true. As the following considerations will illustrate, associativity contains more information than the mere Jacobi identities for the modes.

Consider a free fermion, whose OPE and mode decomposition (in the NS sector) read:

\[
\psi(z)\psi(w) \sim \frac{1}{z-w}, \quad \psi(z) = \sum_{n \in \mathbb{Z}+1/2} b_n z^{-n-1/2}. \tag{A.1}
\]

By evaluating the integral

\[
\frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{n-1/2} w^{m-1/2} \psi(z) \psi(w), \tag{A.2}
\]

we find the usual anti-commutation relations:

\[
\{b_n, b_m\} = \delta_{m+n,0}. \tag{A.3}
\]

Considering this anticommutator together with the Virasoro commutation relations and

\[
[L_n, b_m] = -\left(\frac{n}{2} + m\right) b_{n+m}, \tag{A.4}
\]

it is simple to convince oneself that the central charge is not fixed by the Jacobi identity. However, if we consider the following version of the OPE

\[
\psi(z)\psi(w) = \frac{1}{z-w} + \frac{z-w}{c} T(w) + \cdots, \tag{A.5}
\]

(given that the $\beta^{(2)}$ coefficient is $2\hbar \psi/c = 1/c$) and the integral

\[
\frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz \frac{z^{n+1/2} w^{m+1/2}}{(z-w)^2} \psi(z) \psi(w), \tag{A.6}
\]

\textsuperscript{11} Given three integer-dimension operators $A$, $B$, $C$ with ordinary commutation relations, let us consider the action of $A_n B_m C_p$ on an arbitrary state $|h\rangle$. The associativity requirement retranscribed at the level of modes implies that this state can be evaluated by commuting the first two terms or the last two ones without affecting the result. This in turn implies the usual form of the Jacobi identity as we now show. Commuting $A$ and $B$ and reexpressing the result in terms of the state $C_p B_m A_n |h\rangle$ yields

\[
([A_n, B_m], C_p) + C_p[A_n, B_m] + B_m[A_n, C_p] + [B_m, C_p] A_n + C_p B_m A_n \mid h\rangle.
\]

Commuting $B$ and $C$ and singling out again the state $C_p B_m A_n |h\rangle$ gives

\[
([A_n, [B_m, C_p]] + [B_m, C_p] A_n + [A_n, C_p] B_m + C_p[A_n, B_m] + C_p B_m A_n) \mid h\rangle.
\]

The comparison of these two expressions implies the identity

\[
([A_n, B_m], C_p) + [C_p, A_n], B_m] + [B_n, C_p], A_n\rangle) |h\rangle = 0.
\]

This is the Jacobi identity; more precisely, this is the Jacobi identity modulo a singular vector of $|h\rangle$. For fields with half-integer dimension, this is also true but with an appropriate graded version of the Jacobi identity.
we get the following generalized relations:

\[ \sum_{l \geq 0} l \left[ b_{n-1-l}b_{m+1+l} + b_{m-1-l}b_{n+1+l} \right] = \frac{(n-1/2)(n+1/2)}{2} \delta_{n+m,0} + \frac{1}{c} L_{n+m}, \quad (A.7) \]

or equivalently (cf. [41], App. C)

\[ \sum_{l \geq 0} (l+1) \left[ b_{n-3/2-l}b_{m+3/2+l} + b_{m-1/2-l}b_{n+1/2+l} \right] = \frac{n(n-1)}{2} \delta_{n+m,0} + \frac{1}{c} L_{n+m}. \quad (A.8) \]

These relations capture the expression of the energy-momentum of the free fermion in terms of modes (but as a function of the yet-to-be-fixed central charge). In other words, the relation between \( T \) and \( \psi \) is already coded in the OPE and the conformal invariance.\(^{12}\) Note that the four-point function

\[ \langle \psi_1(z_1)\psi_2(z_2)\psi_3(z_3)\psi_4(z_4) \rangle = \frac{1}{z_{12}z_{34}} - \frac{1}{z_{13}z_{24}} + \frac{1}{z_{14}z_{23}}, \quad (A.9) \]

which is evaluated solely from the knowledge of the singular terms (through meromorphicity, cf. [42]), contains the information on the central charge when viewed in the light of the OPE (A.5). It is extracted by evaluating the correlator in the limit \( z_1 \to z_2 \). This does not contradict the previous conclusion because the correlation function encompasses the whole content of the involved OPE.

The free-boson theory offers another simple illustration of the gap between the information obtained from associativity and the Jacobi identities. The OPE

\[ i\partial z \varphi(z) i\partial z \varphi(w) = \frac{1}{(z-w)^2} + \frac{2}{c} T(w) + \cdots, \quad (A.10) \]

and mode decomposition

\[ i\partial z \varphi = \sum_n a_n z^{-n-1}, \quad (A.11) \]

lead to the standard mode commutation relation as

\[ [a_n, a_m] = \frac{1}{(2\pi i)^2} \oint dw \oint w \frac{dz}{z-w} z^n w^m i\partial z \varphi(z) i\partial z \varphi(w) = n\delta_{n+m,0}, \quad (A.12) \]

for which the Jacobi identity is trivial. If we consider instead the integral

\[ \frac{1}{(2\pi i)^2} \oint dw \oint w \frac{dz}{z-w} z^n w^m i\partial z \varphi(z) i\partial z \varphi(w), \quad (A.13) \]

we end up with a very different form of the commutation relations, namely,

\[ \sum_{l \geq 0} [a_{n-l-1}a_{m+1+l} + a_{m-l}a_{n+l}] = \frac{n(n-1)}{2} \delta_{n+m,0} + \frac{2}{c} L_{n+m}. \quad (A.14) \]

\(^{12}\) This, of course, is a totally standard statement since the OPE (A.5) can also be written as

\[ \psi(z)\psi(w) = \frac{1}{z-w} + (\psi(z)\psi(w)) = \frac{1}{z-w} + (z-w)(\partial z \psi)(w) + \cdots, \]

from which we read that

\[ T(w) = c(\partial z \psi)(w) = -c(\psi \partial z \psi)(w), \]

and this corresponds to the usual expression when \( c = 1/2 \).
It is easily checked that $c$ is fixed to 1 by associativity. This last expression is thus seen to be the mode expression of the energy-momentum of the free boson, i.e.,

$$L_n = \sum_{l \in \mathbb{Z}} a_l a_{n-l} \quad (n \neq 0) \quad L_0 = \sum_{l \geq 0} a_{-l} a_l + \frac{1}{2} a_0^2 . \quad (A.15)$$

This observation is of course true in general: picking up the energy-momentum as the single pole, yields directly the expression of the Virasoro modes in terms of the conserved-current ones.

These simple considerations show that the OPE contains more information than a particular form of commutation relation derived from it. To recover the complete information which is contained in the OPE, we need to consider the infinite family of commutation relations that follows from evaluating (A.6) with $(z - w)^2 \to (z - w)^p$, for all values of $p$. In practice, however, only few values of $p$ should be sufficient.

The associativity of the four-point functions $\langle ABCD \rangle$ is equivalent to the statement that the state $A_n B_m C_n \mid h \rangle$ where $h$ is the dimension of $D$, is independent of the way it is evaluated. Generically, the resulting constraints are independent of the state $\mid h \rangle$ (unless an equality is true modulo a singular vector) and in practice it can be replaced by the vacuum. To formulate the mode version of the associativity constraints, we introduce the notation

$$\text{AB} \equiv [A,B]_p + BAC , \quad (A.16)$$

where $[A,B]_p$ stands for the commutator evaluated in terms of the generalized commutation relations that follow from evaluating the integral:

$$\frac{1}{(2\pi i)^2} \int_0 dw \int_w d\frac{z}{z-w} (z-w)^p A(z) B(w) , \quad (A.17)$$

(the values of $n'$ and $m'$ being adapted to the choice of $p$ in order to recover a final commutator in standard form – cf. (A.2) vs (A.6) and (A.12) vs (A.13)). Now, mode-associativity boils down to the following conditions:

$$\text{AB} \equiv [A,B]_p \equiv [A,B]_q \equiv [A,B]_r . \quad (A.18)$$

In principle, these conditions should be tested for all values of $p$ and $q$ and all combinations of modes. However, in practice, a small number of computations of this type are needed to fix the whole structure of the models under consideration.

As a simple illustration, let us show how we can fix the central charge of the free-fermion theory by enforcing the mode-associativity of $b_{-\frac{1}{2}} b_{\frac{1}{2}} b_{-\frac{1}{2}} (0).$ For this we compare

$$b_{-\frac{1}{2}} b_{\frac{1}{2}} b_{-\frac{1}{2}} (0) = b_{-\frac{1}{2}} (0) , \quad (A.19)$$

(the $p = 0$ commutator being (A.2)) to

$$b_{-\frac{1}{2}} b_{\frac{1}{2}} b_{-\frac{1}{2}} (0) = \frac{1}{c} L_0 b_{-\frac{1}{2}} (0) = \frac{1}{2c} b_{-\frac{1}{2}} (0) , \quad (A.20)$$
(the \( p = 2 \) commutator being (A.6)) to find that \( c = 1/2 \). Note that in this case, we could also compute the central charge by comparing the last result with

\[
b_{-1/2} \underbrace{b_{1/2} b_{-1/2}}_{p=2} |0 \rangle = b_{-1/2} |0 \rangle ,
\]

(A.21)

(in the first case, we generate a term proportional to a Virasoro mode, hence containing \( c \), and in the second case the remaining contribution is the delta term, independent of \( c \)).

Appendix B. Boundary terms in the generating function

Our goal is to derive the modification of the generating function \( G_{r,0}(z, q) \) (4.8), that counts the partitions \( (\lambda_1, \cdots, \lambda_N) \) with \( \lambda_1 \geq \lambda_{i+2} + 2r \) with \( \lambda_N \geq 1 \) [33], which results from the further condition \( \lambda_{N-1} \geq \ell \). We start with the assumption that the boundary terms are represented by linear factors in the exponent, i.e., are accounted by a correction of the form \( q \sum_{j=1}^{k} m_{j}a_{j} \). Denote the modified form as \( G_{r,\ell}(z, q) \). Since the boundary condition is well localized i.e., it concerns only the second term at the right, it suffices to consider the \( N = 1, 2 \) sectors to fix the \( a_{j} \). Recall that \( N = 2 \sum_{j=1}^{k} m_{j} + m_{k} \). Therefore, \( N = 1 \) corresponds to \( m_{k} = 1 \) and all the others \( m_{j} = 0 \). The generating function \( G_{r,0} \) multiplied by \( q \) reads

\[
G_{r,\ell}(q, 1) = \frac{q^{1+a_{k}}}{(q)_{1}} \quad (N = 1).
\]

But the partitions that are to be counted are simply those with a single part and their generating function is \( q/(q)_{1} \). This fixes \( a_{k} = 0 \).

Consider next \( N = 2 \) which requires either that a single mode \( m_{j} = 1 \) for \( 1 \leq j \leq k - 1 \) with all other modes zero or that \( m_{k} = 2 \), again with all other modes vanishing. This results into

\[
G_{r,\ell}(q, 1) = \frac{1}{(q)_{1}} \sum_{j=1}^{k-1} q^{2j+a_{j}} + \frac{q^{2k}}{(q)_{2}} \quad (N = 2)
\]

In order to fix the \( a_{j} \), we must determine the generating function enumerating partitions with two parts \( (\lambda_1, \lambda_2) \) and satisfying

\[
\lambda_1 \geq \lambda_2 \geq 1 \quad \text{and} \quad \lambda_1 \geq \ell.
\]

This can be done by means of the MacMahon method [43] (cf. vol 2 Sect. VIII, chap. 1 and see also [44], Sect.. 11.2), by projecting the following expression

\[
\frac{a_{3}^{\ell}a_{2}^{-1}}{(1 - a_{1}a_{3}q)(1 - a_{2}q/a_{1})} = \sum_{\lambda_{1}, \lambda_{2} \geq 0} a_{1}^{\lambda_{1} - \lambda_{2}} a_{2}^{\lambda_{2} - 1} a_{3}^{\lambda_{1} - \ell} q^{\lambda_{1} + \lambda_{2}},
\]

onto positive powers of the \( a_{i} \)’s, ensuring thereby the three inequalities: \( \lambda_{1} \geq \lambda_{2}, \lambda_{2} \geq 1 \) and \( \lambda_{1} \geq \ell \). It is convenient to introduce the MacMahon projection symbol \( \Omega \), defined by

\[
\Omega \geq \sum_{n=-\infty}^{\infty} c_{n}a^{n} = \sum_{n \geq 0} c_{n}a^{n} \bigg|_{a=1} = \sum_{n \geq 0} c_{n}
\]

(B.5)
and make use of identities of the following type:

\[
\frac{a}{\Omega} \frac{1}{(1 - aq)(1 - a^{-1}q^2)} = \frac{a}{(1 - q^2)} \left( \frac{1}{1 - aq} + \frac{a^{-1}q}{1 - a^{-1}q^2} \right) = \frac{1}{(1 - q)(1 - q^2)}.
\] (B.6)

The first two projections are rather direct

\[
\frac{a_3 a_2 a_1}{\Omega} \frac{a^{-\ell} q^2}{(1 - a_1 a_3 q)(1 - a_2 q/a_1)} = \frac{a_3 a_2}{\Omega} \frac{a^{-\ell} q^2}{(1 - a_3 q)(1 - a_2 a_3 q^2)} = \frac{a_3}{\Omega} \frac{a^{-\ell} q^2}{(1 - a_3 q)(1 - a_3 q^2)},
\] (B.7)

and for the third one, we have

\[
\frac{a}{\Omega} \frac{1}{(1 - aq)(1 - aq^2)} = \frac{a\ell q^2}{(1 - aq)(1 - aq^2)} \left( 1 - (1 - aq)(1 - aq^2) \sum_{p=0}^{\ell-2} \sum_{j=0}^{P} a^p q^{p+j} \right) \bigg|_{a=1}.
\] (B.8)

The \( a_j \) are then fixed by comparing (B.2) and (B.8), the solution of which being

\[
 a_j = \max(\ell - j, 0),
\] (B.9)
as announced (cf. (4.12)). This is valid for \( 0 \leq \ell \leq k \).

**Appendix C. The analysis of boundary terms via recurrence relations for generating functions**

We reconsider the construction of the generating functions for partitions \( \lambda = (\lambda_1, \cdots, \lambda_N) \) into \( N \) parts satisfying \( \lambda_i \geq \lambda_{i+1} \) and \( \lambda_i \geq \lambda_{i+2} + 2r \), together with the boundary condition \( \lambda_{N-1} \geq \ell \). The set of such partitions can be described schematically as (see e.g., [36])

\[
\cdots (2r + \ell)(2r + 1)(\ell)1^+,
\] (C.1)

indicating that we build up these restricted partitions on the above ground state and the + sign indicates the position from which we start the building up process (from right to left) by addition of ordinary partitions. We then use this pictorial representation to write down the recurrence relation between sets with different boundary conditions:

\[
\cdots (2r + \ell)(2r + 1)(\ell)1^+ - \cdots (2r + \ell + 1)(2r + 1)(\ell + 1)1^+
= \cdots (2r + \ell)(2r + 1)^+(\ell)1 + \cdots (2r + \ell)(2r + 2)^+(\ell)2
\] (C.2)

\[
+ \cdots + \cdots (2r + \ell)(2r + \ell + 1)^+(\ell)(\ell)
\]

The difference on the left-hand side generates the set of partitions for which the penultimate part is \( \ell \) and this set is then broken, on the right-hand side, into sets with prescribed values of the last two entries. Denote by \( p_{r}(w, N) \) the number of partitions of weight \( w \) (where \( w = \sum \lambda_i \)) with \( N \) parts in the set (C.1). The above recurrence relation can be translated into the following condition

\[
p_{r,\ell}(w, N) - p_{r,\ell+1}(w, N) = \sum_{s=0}^{\ell-1} p_{r,\ell-s}(w - (2r + s)(N - 2) - \ell - s - 1, N - 2)
\] (C.3)
On the left-hand side, we have used the observation that the set with the tail \((\ell)(s+1)\) (for \(0 \leq s \leq \ell - 1\)) is in one-to-one correspondence with the set obtained by deleting the last two parts \((\ell)(s+1)\) and subtracting \(2r+s\) from each of the \(N-2\) remaining parts, whose cardinality is thereby given by \(p_{r,\ell-s}(w-(2r+s)(N-2)-\ell-s-1,N-2)\). In terms of the generating function (4.7), the above recurrence relation implies:

\[
G_{\ell+1}(z,q) = G_\ell(z,q) - z^2 \sum_{s=0}^{\ell-1} q^{s+1} G_{\ell-s}(zq^{2r+s},q).
\]  

(C.4)

To this recurrence relation, we add the boundary condition \(G_1 = G_0\), with \(G_0\) given in [33]. In this way, we can construct \(G_\ell\) recursively out of the known expression for \(G_0\). In the following, we will only need the specialized version at \(z = 1\):

\[
G_{r,\ell+1}(1,q) = G_{r,\ell}(1,q) - \sum_{s=0}^{\ell-1} q^{s+1} G_{r,\ell-s}(q^{2r+s},q).
\]  

(C.5)

Let us now show that for \(0 \leq \ell \leq k\), this relation leads to the expression already presented in (4.8), (4.11) and (4.12). Take some \(\ell + 1 \leq k\) and start by considering the difference\[
G_{r,\ell}^{(0)}(1,q) = G_{r,\ell}(1,q) - q^{\ell+1} G_{r,\ell}(q^{2r},q).
\]  

(C.6)

We first break the multisum expression for \(G_\ell(1,q)\) into two parts: one with \(m_1 = 0\) and the other with \(m_1 > 0\). In the second sum, we redefine \(m_1 = m_1' + 1\). The resulting expression has the same numerator as \(q^{\ell+1} G_{r,\ell}(q^{2r},q)\) so that their difference is easily computed. Recombining the result with the first multisum corresponding to \(m_1 = 0\) yields an expression, denoted \(G_{r,\ell}^{(0)}(1,q)\) above, that is identical to \(G_{r,\ell}(1,q)\) except that the linear coefficient of \(m_1\) has been increased by 1. Next let us consider the difference between this expression \(G_{r,\ell}^{(0)}(1,q)\) and the second term \((s = 1)\) of the sum of the right-hand side of (C.5):

\[
G_{r,\ell}^{(1)}(1,q) = G_{r,\ell}^{(0)}(1,q) - q^{\ell+2} G_{r,\ell}(q^{2r+1},q).
\]  

(C.7)

The same manipulations but with \(m_1\) replaced by \(m_2\) gives back \(G_{r,\ell}^{(0)}(1,q)\) except that the linear coefficient of \(m_2\) has been increased by 1. Proceeding in this way by successively taking into account the different terms of the sum, we end up with the prescription that going from \(G_{r,\ell}\) to \(G_{r,\ell+1}\), amount to increase by 1 all the linear coefficients of \(m_1, \ldots, m_{\ell-1}\) and \(m_\ell\) by 1 without modifying the other mode coefficients. This demonstrates that the difference between the linear term without boundary condition, pertaining to \(\ell = 0, 1\), and those with \(\lambda_{N-1} \geq \ell\), with \(1 \leq \ell \leq k\), is precisely given by the term \(\max(\ell - j)\) in (4.12).

By construction, this recombination of the right-hand side of (C.5) works for \(\ell + 1 \leq k\). This bound can be justified by the argument presented in appendix B concerning the independence of the coefficient of \(m_k\) upon \(\ell\): there are thus no more coefficient available to resuffle. To see explicitly how this recombination process breaks down for \(\ell > k\), it suffices to consider \(\ell = 3\) and \(p = 8\) (so that \(k = 2\)). This suggests that for \(\ell > k\), the generating function \(G_{r,\ell}(z,q)\) might not be expressible in terms of a single multiple sum.

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