Kinematic Description of Ricci Solitons in Fluid Spacetimes

Umber Sheikh*
Department of Applied Sciences, National Textile University,
Faisalabad-37610, Pakistan.

Abstract

We consider the kinematics of specific fluid spacetimes admitting timelike congruences of Ricci Solitons. These fluids includes string cloud, string fluid, perfect fluid, radially symmetric fluid, anisotropic fluid and relativistic magneto-fluid. Results are obtained and important physical aspects are discussed.

Keywords: Ricci Solitons; String Cloud; String Fluid; Perfect Fluid; Radially Symmetric Fluid; Anisotropic Fluid; Relativistic Magneto-fluid; timelike congruences.

1 Introduction

The concepts of symmetry and beauty are interrelated. In-fact, invariants are the deep truths, i.e., things that do not change. Invariants are defined by symmetries which in turn define properties of nature are conserved. Selfsame symmetries like conformal collineations, Ricci and matter inheritance collineations etc., are those which appeal to the senses in art and natural forms like snowflakes and galaxies. Thus, the fundamental truths are raised on basis of symmetries which contains deep kind of beauty in them.

The symmetries are a part of geometry and thus reveals the physics. According to Wheeler, Spacetime tells matter how to move; matter

*umbersheikh@gmail.com
tells spacetime how to curve. There are many symmetries regarding the spacetime geometry and matter. The metric symmetries are important as they simplify solutions to many problems. Their main application in general relativity is that they classify solutions of Einstein Field Equations. One of these symmetries is Ricci Solitons associated with the Ricci flow of spacetime geometry.

Ricci flow is a process of deformation of metric in directions of its Ricci curvature. The Ricci flow can be defined by following geometric evolution equation [1]

\[
\partial_t g_{ij} = -2R_{ij} \tag{1.1}
\]

The stretch or contraction of metric depends on positive or negative Ricci curvature. The faster deformation of the metric depends upon the strength of curvature.

The most interesting problem in Ricci flow is to determine its self similar solutions of Ricci flow namely Ricci Solitons. These are natural generalization of Einstein metrics [2]. Ricci solitons are fixed points of Ricci flow in a dynamical system. These are the self similar solutions to the Ricci flow. Ricci solitons model the formulation of singularities in the Ricci flow. It is worth mentioning here that the mathematical notion of Ricci soliton should not be confused with the notion of soliton solutions, which arise in several areas of mathematical and theoretical physics and its applications.

Ricci flow is important because it can help in understanding the concepts of energy and entropy in general relativity. The most interesting property of the Ricci flow is its tendency to lose memory of initial conditions. It leads to the property that the irregularities of metric can be evolved into some regular form. This property is the same as that of heat equation due to which an isolated system loses the heat for a thermal equilibrium. Ricci solitons are the points at which the curvature obeys a self similarity.

Most of recent literature is found regarding the geometry of Ricci flow and Ricci Solitons. Hamilton [3] was the frontier to introduce Ricci flow. Ivey [4] discussed the geometry of Ricci Solitons on compact manifolds. Catino et al. [5] found that complete gradient expanding Ricci soliton with nonnegative Ricci curvature is isometric to a quotient of the three dimensional Gaussian soliton. Munteanu [6] discussed the curvature behavior of four dimensional shrinking gradient Ricci solitons. Kroencke [7] showed that though complex projective spaces of even complex dimension have infinitesimal solitonic deformations, they are rigid as Ricci solitons. Catino et al. [8] provided some
necessary integrability conditions for the existence of gradient Ricci solitons in conformal Einstein manifolds. Catino et al. [9] proved some of the classification results for generic shrinking Ricci solitons. Kroencke [10] discussed the stability of Ricci solitons with respect to Perelman’s shrinker entropy. Jablonski [11] proved that homogeneous Ricci soliton spaces are algebraic spaces. Wei and Wu [12] discussed the conditions for Euclidean volume growth of Ricci solitons. Woolgar [13] considered some applications of Ricci flows in static metrics of general relativity. He also took Ricci solitons in context. Akbar and Woolgar [14] showed an explicit family of complete expanding solitons as a Ricci flow for a complete Lorentzian metric.

This article is devoted to find the kinematics of spacetimes admitting Ricci solitons. Section 2 comprises of matter tensors of different fluid spacetimes and respective Ricci tensors. Section 3 consists of basic kinematics regarding to the congruences in fluids. The same section contains basic information of Ricci solitons as spacetime congruences. Section 4 contains the mathematical expressions modeling the kinematical properties of specific fluid spacetimes admitting Ricci solitons. Section 5 contains the summary of results.

2 Matter Tensors of Different Fluid Spacetimes

The famous Einstein field equations relate geometry of spacetimes with their physics via following expression:

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab} \]  

where \( G_{ab} \) is the Einstein tensor. Substituting \( \kappa = 1 \)

\[ R_{ab} - \frac{1}{2} R g_{ab} = T_{ab} \]  

The Einstein tensor is entirely made up of Ricci tensor components or containing the geometry of spacetime. The matter tensor describes the physics of spacetime.

Consider a fluid spacetime admitting the unit spacelike vector \( x^a \) perpendicular to unit timelike vector \( u^a \). Then, \( u_a u^a = -1 \), \( x_a x^a = 1 \), \( u^a x_a = 0 \).
The signature of the spacetime be (+,−,−,−). Let $T_{ab}$ be the matter tensor of this fluid spacetime. Using Eq.(2.2) we can evaluate the Ricci tensor as 

$$R_{ab} = T_{ab} - \frac{1}{2}T g_{ab}$$  \hspace{1cm} (2.3)

where $T = T_a^a$ is the trace of matter tensor.

The matter tensor for the string cloud can be written as

$$T_{ab} = \rho u_a u_b - \lambda x_a x_b$$  \hspace{1cm} (2.4)

where $\rho$ is the density of the fluid and $\lambda$ is string tension. Eq.(2.3) provides us

$$R_{ab} = \rho u_a u_b - \lambda x_a x_b + \frac{g_{ab}}{2}(\rho + \lambda).$$  \hspace{1cm} (2.5)

The matter tensor for the string fluid can be expressed as

$$T_{ab} = (\rho + q)(u_a u_b - x_a x_b) + q g_{ab}$$  \hspace{1cm} (2.6)

where $\rho$ is density of the fluid and $q$ is the pressure on the strings. Eq.(2.3) gives

$$R_{ab} = (\rho + q)(u_a u_b - x_a x_b) + \rho g_{ab}.$$  \hspace{1cm} (2.7)

The matter tensor for perfect fluid can be written as

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}$$  \hspace{1cm} (2.8)

where $\rho$ is density of the fluid and $p$ is the pressure. The use of Eq.(2.3) leads to

$$R_{ab} = (\rho + p)(u_a u_b - x_a x_b) + \frac{1}{2}(\rho - p) g_{ab}.$$  \hspace{1cm} (2.9)

The matter tensor for anisotropic fluid can be expressed as

$$T_{ab} = \rho u_a u_b + p x_a x_b$$  \hspace{1cm} (2.10)

where $\rho$ is density of the fluid, $p$ is principal pressure and transverse pressures measured in orthogonal direction to $x^a$ are all vanishing. Its Ricci tensor can be written as

$$R_{ab} = \rho u_a u_b + p x_a x_b + \frac{1}{2}(\rho - p) g_{ab}.$$  \hspace{1cm} (2.11)

The matter tensor for imperfect fluid is

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q x_a x_b$$  \hspace{1cm} (2.12)
where $\rho$ and $p$ are respectively density and pressure of the fluid, and $q$ is the shear of the fluid in $x$-direction. The Ricci tensor for this fluid is

$$R_{ab} = \rho u_a u_b + qx_a x_b + \frac{1}{2}(\rho - p - q)g_{ab}. \quad (2.13)$$

The matter tensor for relativistic magneto-fluid can be written as

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + \mu \left\{ |b|^2 \left( u_a u_b + \frac{1}{2} g_{ab} \right) - b_a b_b \right\} \quad (2.14)$$

where $\rho$ is the density of fluid, $p$ is the pressure, $\mu$ is the magnetic permeability and $b_a$ is magnetic flux vector such that $u^a b_a = 0$. Since, the magnetic field is spacelike, therefore $b_a = |b| x_a$. The strength of magnetic field can be determined by $b^a b_a = |b|^2 = B$. Eq. (2.3) implies the following Ricci tensor

$$R_{ab} = (\mu B + \rho + p)u_a u_b + qx_a x_b + \frac{1}{2}(\mu B + \rho - p)g_{ab} - \mu b_a b_b. \quad (2.15)$$

### 3 Basic Kinematics of Spacetime Congruences

Consider the decomposition of a timelike tidal tensor

$$u_{a:b} = \frac{\theta}{3} h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b. \quad (3.16)$$

Here $\dot{u}_a$ is the acceleration vector of the flow. The projective tensor $h_{ab} = g_{ab} + u_a u_b$ projects a tangent vector perpendicular to $u^a$. The rate of separation of flowlines, from a timelike curve tangent to $u_a$ can be expressed by expansion tensor $\theta_{ab} = h^c_a h^d_b u_{(c;d)}$. The volume expansion of the flow lines is $\theta = h^{ab} \theta_{ab}$. The shear between the flow lines can be measured by

$$\sigma_{ab} = \theta_{ab} - \frac{\theta}{3} h_{ab} \Rightarrow \theta_{ab} = \sigma_{ab} + \frac{\theta}{3} h_{ab}. \quad (3.17)$$

The vorticity tensor $\omega_{ab}$ and expansion tensor $\theta_{ab}$ added to result in

$$u_{a;b} + \dot{u}_a u_b = \theta_{ab} + \omega_{ab}$$

where $\omega_{ab} = h^c_a h^d_b u_{[c;d]}$. 

5
For any Riemann manifold, the Ricci Soliton Eq. (1.1) can be written as

\[ R_{ab} - \frac{1}{2} \xi g_{ab} = kg_{ab} \]  \hspace{1cm} (3.1)

It can also be written as

\[ L_{\xi}g_{ab} = 2(R_{ab} - kg_{ab}). \] \hspace{1cm} (3.2)

The solitons may be

- shrinking for \( k > 0 \),
- steady for \( k = 0 \),
- expanding for \( k > 0 \).

The component form of the equation becomes

\[ \xi_{a;b} + \xi_{b;a} = 2(R_{ab} - kg_{ab}). \] \hspace{1cm} (3.3)

If \( \xi \) is taken to be a timelike vector, then \( \xi^a = \xi u^a \) where \( u^a \) is a timelike unit vector. Thus, the above equation becomes

\[ u_{a;b} + u_{b;a} + (\ln \xi)_a u_b + (\ln \xi)_b u_a = 2\xi^{-1}(R_{ab} - kg_{ab}). \] \hspace{1cm} (3.4)

4 Properties of Fluid Spacetimes Admitting Ricci Solitons

The aim of this research is to identify the kinematical properties of different spacetime fluids containing Ricci Solitons. In this respect, we have calculated some restrictions and specifications on such fluid spacetimes for the existence of Ricci solitons.

**Theorem 1:**

A 4-dimensional string cloud spacetime possesses a timelike Ricci soliton \( \xi \), parallel to fluid flow velocity vector field \( u \), i.e., \( \xi_a = \xi u_a, \xi > 0 \) iff

C1

\[ (\ln \dot{\xi}) = \xi^{-1} \left( \frac{\lambda - \rho}{2} - k \right) \] \hspace{1cm} (4.1)
\[ \dot{u}_a = (\ln \xi)_a + (\ln \xi) u_a = (\ln \xi)_a + \xi^{-1} \left( \frac{\lambda - \rho}{2} - k \right) \] (4.2)

\[ \theta = \frac{3\xi^{-1}}{2} \left( \rho + \frac{\lambda}{3} - 2k \right) \] (4.3)

\[ \sigma_{ab} = \xi^{-1} \lambda \left( \frac{1}{3} h_{ab} - x_a x_b \right) \] (4.4)

\[ \theta_{ab} = \xi^{-1} \left\{ \left( \frac{\lambda}{6} - \frac{\rho}{2} + k \right) h_{ab} - \lambda x_a x_b \right\} \] (4.5)

\[ \omega_{ab} = \xi^{-1} \left\{ \left( \rho + \frac{\lambda}{3} \right) h_{ab} - 2k g_{ab} \right\} \] (4.6)

**Proof:**

The Ricci soliton equation Eq.(3.4) for a string cloud becomes

\[ u_{a;b} + u_{b;a} + (\ln \xi)_a u_b + (\ln \xi)_b u_a = 2\xi^{-1} \left\{ \rho u_a u_b - \lambda x_a x_b + g_{ab} \left( \frac{\rho + \lambda}{2} - k \right) \right\} \] (4.7)

**C1** Contracting Eq.(4.7) with \( u^a u^b \) gives Eq.(4.5).

**C2** Contracting Eq.(4.7) with \( u^a h^b_c \) we have

\[ \dot{u}_c = (\ln \xi)_c + (\ln \xi) u_c. \] (4.8)

Substituting the value of \((\ln \xi)\) from Eq.(4.5) gives Eq.(4.2).

**C3** Contracting Eq.(4.7) with \( h_{ab} \) gives Eq.(4.3).

**C4** Contracting Eq.(4.7) with \( h^a_c h^b_d - \frac{1}{3} h^{ab} h_{cd} \) gives Eq.(4.4).

**C5** Substituting values from Eq.(4.3) and Eq.(4.4) in Eq.(3.17) gives Eq.(4.5).
C6 The left hand side of Eq.(3.3) can be expressed as

\[ \xi_{a;b} + \xi_{b;a} = \xi(u_{a;b} + u_{b;a} + (\ln \xi)_a u_b + (\ln \xi)_b u_a) \]

Putting value of \( \dot{u}_c \) from Eq.(4.8) gives

\[ \xi_{a;b} + \xi_{b;a} = \xi(u_{a;b} + u_{b;a} + \dot{u}_a u_b + \dot{u}_b u_a - 2(\ln \xi) u_a u_b) \]

\[ = 2\xi(\theta_{ab} + \omega_{ab} - (\ln \xi) u_a u_b) \]

\[ \Rightarrow 2(\theta_{ab} + \omega_{ab}) = \xi^{-1}(\xi_{a;b} + \xi_{b;a}) + 2(\ln \xi) u_a u_b \]

\[ \Rightarrow 2\omega_{ab} = \xi^{-1}(\xi_{a;b} + \xi_{b;a}) + 2(\ln \xi) u_a u_b - 2\theta_{ab} \]

\[ \Rightarrow \omega_{ab} = \xi^{-1}(R_{ab} - k g_{ab}) + (\ln \xi) u_a u_b - \theta_{ab} \] (4.9)

Substituting values from Eqs.(2.5),(4.5) and (4.5), the vorticity tensor for string cloud becomes

\[ \omega_{ab} = \xi^{-1} \left[ \left( \rho + \frac{\lambda}{3} \right) h_{ab} - 2k g_{ab} \right] \]

Substituting values from Eqs.(4.1)-(4.6) satisfies Eq.(4.7). In the similar manner, the kinematic quantities related to string fluid, perfect fluid, anisotropic fluid, imperfect fluid and relativistic magnetofluid can be obtained.

**Theorem 2:**

A 4-dimensional string fluid spacetime possesses a timelike Ricci soliton \( \xi \), parallel to fluid flow velocity vector field \( u \), i.e., \( \xi_a = \xi u_a, \xi > 0 \) iff

C1

\[ (\ln \xi) = -\xi^{-1}(k + q) \] (4.10)

C2

\[ \dot{u}_a = (\ln \xi)_a + (\ln \xi) u_a = (\ln \xi)_a - \xi^{-1}(k + q) u_a \] (4.11)

C3

\[ \theta = \xi^{-1}(2\rho - q - 3k) \] (4.12)

C4

\[ \sigma_{ab} = \xi^{-1}(\rho + q) \left( \frac{1}{3} h_{ab} - x_a x_b \right) \] (4.13)
Theorem 3:

A 4-dimensional perfect fluid spacetime possesses a timelike Ricci soliton \( \xi \), parallel to the fluid velocity flow vector field \( u \), i.e., \( \xi_a = \xi u_a, \xi > 0 \) iff

\[\begin{align*}
\text{C1} & \quad (\ln \xi)_a = -\xi^{-1} \left( \frac{\rho + 3p}{2} + k \right), \\
\text{C2} & \quad \dot{u}_a = (\ln \xi)_a - \xi^{-1} \left( \frac{\rho + 3p}{2} + k \right) u_a, \\
\text{C3} & \quad \theta = 3\xi^{-1} \left( \frac{\rho - p}{2} - k \right), \\
\text{C4} & \quad \sigma_{ab} = 0, \\
\text{C5} & \quad \theta_{ab} = \xi^{-1} \left( \frac{\rho - p}{2} - k \right) h_{ab}, \\
\text{C6} & \quad \omega_{ab} = 0.
\end{align*}\]

Theorem 4:

A 4-dimensional anisotropic fluid spacetime possesses a timelike Ricci soliton \( \xi \), parallel to fluid flow velocity vector field \( u \), i.e., \( \xi_a = \xi u_a, \xi > 0 \) iff

\[\begin{align*}
\text{C1} & \quad (\ln \xi)_a = -\xi^{-1} \left( k + \frac{\rho + p}{2} \right),
\end{align*}\]
\[ \dot{u}_a = (\ln \xi)_a - \xi^{-1} \left( k + \frac{\rho + p + q}{2} \right) u_a, \quad (4.23) \]

\[ \theta = \xi^{-1} \left( \frac{3\rho - p}{2} - \frac{3p - q}{2} \right), \quad (4.24) \]

\[ \sigma_{ab} = \xi^{-1} p \left( x_a x_b - \frac{1}{3} h_{ab} \right), \quad (4.25) \]

\[ \theta_{ab} = \xi^{-1} \left\{ \left( \frac{\rho - p}{2} - k \right) h_{ab} + p x_a x_b \right\}, \quad (4.26) \]

\[ \omega_{ab} = 0. \quad (4.27) \]

**Theorem 5:**

A 4-dimensional imperfect fluid spacetime possesses a timelike Ricci soliton \( \xi \), parallel to fluid flow velocity vector field \( u \), i.e., \( \xi_a = \xi u_a, \xi > 0 \) iff

\[ (\ln \xi) = -\xi^{-1} \left( \frac{\rho + p + q}{2} + k \right), \quad (4.28) \]

\[ \dot{u}_a = (\ln \xi)_a - \xi^{-1} \left( \frac{\rho + p + q}{2} + k \right) u_a, \quad (4.29) \]

\[ \theta = \xi^{-1} \left( \frac{3\rho - p}{2} - \frac{3p - q}{2} + 2k \right), \quad (4.30) \]

\[ \sigma_{ab} = -\xi^{-1} q \left( \frac{1}{3} h_{ab} - x_a x_b \right), \quad (4.31) \]

\[ \theta_{ab} = \xi^{-1} \left\{ \left( \frac{\rho - p}{2} - \frac{2k}{3} \right) h_{ab} + q x_a x_b \right\}, \quad (4.32) \]
A 4-dimensional relativistic magneto-fluid spacetime possesses a timelike Ricci soliton \( \xi \), parallel to fluid flow velocity vector field \( u \), i.e., \( \xi_a = \xi u_a, \xi > 0 \) iff

\[ (\ln \xi) = -\xi^{-1} \left( \frac{\mu B + \rho + 3p}{2} + k \right), \]

\[ \xi_a = (\ln \xi)_a - \xi^{-1} \left( \frac{\mu B + \rho + 3p}{2} + k \right) u_a, \]

\[ \theta = \xi^{-1} \left( \frac{3\rho - 3p + \mu B}{2} - 3k \right), \]

\[ \sigma_{ab} = \xi^{-1} \mu B \left( \frac{1}{3} h_{ab} - x_a x_b \right) = \xi^{-1} \mu \left( B \frac{1}{3} h_{ab} - b_a b_b \right), \]

\[ \theta_{ab} = \xi^{-1} \left\{ \left( \frac{\mu B + \rho - p}{2} - k \right) h_{ab} - \mu b_a b_b \right\}, \]

\[ \omega_{ab} = 0. \]

5 Summary

A fluid spacetime admitting a timelike Ricci soliton \( \xi^a = \xi u^a \) can possess some kinematic properties. The specifications obtained may be summerized as under:
• In all the mentioned fluid spacetimes, the timelike congruences have an acceleration vector dependent on time derivative of natural logarithm of $\xi$ as

$$\dot{u}_a = (\ln \xi)_a + (\ln \xi) u_a.$$  

Thus, the fluid exert a net force proportional to the acceleration.

• In string cloud spacetime, the shear tensor of Ricci soliton congruences depend on the tension of strings.

• The string fluid admits a Ricci soliton congruence with a shear tensor dependent on density of the strings as well as the pressure between the strings.

• The Ricci soliton flow lines in a perfect fluid do not possess any shear.

• For anisotropic fluid, the shear tensor components of Ricci soliton congruences are parallel pressure dependent whereas in the case of imperfect fluid, these are viscosity dependent.

• In relativistic magnetofluid, the shear of Ricci soliton congruences is dependent on magnetic field strength and magnetic permeability.

• A perfect fluid admitting a Ricci soliton congruence has volume expansion proportional to its expansion tensor.

• In cases of string fluid, perfect fluid and relativistic magnetofluid, the congruences of Ricci solitons have no vorticity, i.e, the congruences have no common point of rotation.

• The vorticity tensor of an imperfect fluid is totally dependent on $k$, i.e, in the case of shrinking solitons the vorticity is negative, for steady solitons there is no vorticity in the congruences whereas expanding solitons admit positive vorticity.

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