POINCARE ALGEBRA AND SPACE-TIME CRITICAL DIMENSIONS FOR PARASPINNING STRINGS*

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Abstract

In this paper, we paraquantize the spinning string theory in the Neuveu-Shwarz model. Both the center of mass variables and the excitation modes of the string verify para-commutation relations. Except the \([p^\mu, p^\nu]\) commutator, the two other commutators of Poincaré algebra are satisfied. With the sole use of trilinear relations we find existence possibilities of spinning strings at space-time dimensions other than \(D = 10\).
1 Introduction

In the parabosonic string case [1], a critical study of the Poincaré algebra was done and space-time critical dimensions \( D \) as functions of the order of the paraquantization \( Q \) were obtained. This work consist in doing an extention of all these questions to the paraspinning string case in the Neuveu-Shwarz model. To set the notations, we begin with a brief summary of some familiar results in spinning string theory [2] [3] [4].

The action is postulated as:

\[
S = -\frac{1}{2\pi} \int d\sigma d\tau \left( \partial_a X^\mu \partial^a X_\mu - \overline{\psi}^a \rho^a \partial_a \psi \right)
\]  

with

\[
\psi^\mu = \begin{pmatrix} \psi_0^\mu \\ \psi_1^\mu \end{pmatrix} ; \quad \overline{\psi} = \psi^\mu \rho^0
\]

and

\[
\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} ; \quad \{\rho^a, \rho^b\} = -2\eta^{ab}
\]

The solutions are:

\[
X^\mu (\sigma, \tau) = x^\mu + p^\mu \tau + \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu (0) \exp (-im\tau) \cos n\sigma
\]  

where \( x^\mu \) and \( p^\mu \) are respectively the "center of mass" coordinates and the total energy momentum of the string. In the Neuveu-Shwarz model:

\[
\begin{align*}
\psi_0^\mu (\sigma, \tau) &= \frac{1}{\sqrt{2}} \sum_{r \in (Z+\frac{1}{2})} b_r^\mu \exp [-ir(\tau - \sigma)] \\
\psi_1^\mu (\sigma, \tau) &= \frac{1}{\sqrt{2}} \sum_{r \in (Z+\frac{1}{2})} b_r^\mu \exp [-ir(\tau + \sigma)]
\end{align*}
\]  

The total angular momentum is given by:

\[
M^{\mu\nu} = M_0^{\mu\nu}(x) + K^{\mu\nu}
\]  

where \( M_0^{\mu\nu}(x) \) is the bosonic part given by:

\[
M_0^{\mu\nu}(x) = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu)
\]  

and

\[
K^{\mu\nu} = -\frac{i}{2} \sum_{r=-\infty}^{\infty} (b_{-r}^\mu, b_r^\nu - b_r^{\nu -} - b_{-r}^\nu, b_r^\mu)
\]
In the same way as in the parabosonic case, a first study of the paraquantum Poincaré algebra was done by F. Ardalan and F. Mansouri [5]. These authors paraquantize the excitation modes of the string and impose to the center of mass variables to satisfy the ordinary quantum commutation relations. This is done by the choice of a specific direction in the paraspace of the Green components [6], [7], [8], which requires relative paracommutation relations between the center of mass variables and the excitation modes of the string. In order that the theory is Poincaré invariant, the space-time dimension $D$ and the order of the paraquantization $Q$ are related by the relation $D = 2 + \frac{8}{Q}$. Here, we investigate the paraquantization of the spinning string theory in the Neveu-Shwarz model without the Ardalan and Mansouri hypothesis on the center of mass variables [5]. We construct the paraspinning string formalism and we discuss the three commutators of the Poincaré algebra. We find that, except the $[p^\mu, p^\nu]$ commutator, the other results are the same as in the Ardalan and Mansouri work [5]. In particular, the relation between the space-time dimensions $D$ and the order of the paraquantization $Q$ is again $D = 2 + \frac{8}{Q}$.

2 Paraquantum formalism of spinning string

2.1 Covariant gauge

The paraquantization of the theory is carried out by reinterpreting the classical dynamical variables $\alpha^\mu_n$, $p^\mu$, $x^\mu$ and $b^\mu_r$ as operators satisfying the so-called trilinear commutation relations:

\[
\begin{align*}
[x^\mu, [p^\nu, A]] &= 2ig^{\mu\nu} A \\
[x^\mu, [p^\nu, p^\rho]] &= 2i (g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu) \\
[\alpha^\mu_n, [\alpha^\nu_m, \alpha^\rho_l]] &= 2 (g^{\mu\nu} n\delta_{n+m,0} \alpha^\rho_l + g^{\mu\rho} n\delta_{n+l,0} \alpha^\nu_m) \\
[\alpha^\mu_n, [\alpha^\nu_m, B]] &= 2ng^{\mu\nu} \delta_{n+m,0} B \\
[b^\mu_r, [b^\nu_s, b^\rho_q]] &= 2 (g^{\mu\nu} \delta_{r+s,0} b^\rho_q - g^{\mu\rho} \delta_{r+q,0} b^\nu_s) \\
[b^\mu_r, [b^\nu_s, C]] &= 2g^{\mu\nu} \delta_{r+s,0} C
\end{align*}
\]  

(7-a) (7-b) (7-c) (7-d) (7-e) (7-f)

and all the other commutators are null. Here $l, n \in \mathbb{Z}$ and $r, s, q \in \left(\mathbb{Z} + \frac{1}{2}\right)$ and A, B, and C represent the following operators:

- $A=\alpha^\mu_n$, $x^\mu$ or $b^\mu_r$
- $B= p^\rho$, $x^\rho$ or $b^\rho_r$
- $C= p^\rho$, $x^\rho$ or $\alpha^\rho_n$
2.2 Transverse gauge

In this gauge, the paraquantum operators \( x^-, p^+, x^i, p^i, \alpha^i_n \) and \( b^i_r \) verify the trilinear relations:

\[
\begin{align*}
\left[b^i_r, [b^j_s, b^k_q]_+ \right] &= 2 \left( \delta^{ij} \delta_{r+s} b^k_q - \delta^{ik} \delta_{r+q} b^j_s \right) \quad (8-a) \\
\left[\alpha^i_n, [\alpha^j_m, \alpha^k_l]_+ \right] &= 2 \left( \delta^{ij} n \delta_{n+m,0} \alpha^k_l + n \delta^{ik} \delta_{n+l,0} \alpha^j_m \right) \quad (8-b) \\
\left[x^i, [p^j, p^k]_+ \right] &= 2i \left( \delta^{ij} pk + \delta^{ik} p^j \right) \quad (8-c) \\
\left[\alpha^i_n, [\alpha^j_m, D]_+ \right] &= 2 \delta^{ij} n \delta_{n+m} D \quad (8-d) \\
\left[b^i_s, [b^j_s, E]_+ \right] &= 2 \delta^{ij} \delta_{r+s} E \quad (8-e) \\
\left[x^i, [p^j, F]_+ \right] &= 2i \delta^{ij} F \quad (8-f) \\
\left[x^-, [p^+, G]_+ \right] &= 2i G \quad (8-g)
\end{align*}
\]

and all the others commutators are null. Here \( D, E, F, \) and \( G \) represent the following operators :

- \( D = x^-, p^+, x^k, p^k \) or \( b^k_r \)
- \( E = x^-, p^+, x^k, p^k \) or \( \alpha^k_n \)
- \( F = x^-, p^+, x^k, \alpha^k_n \) or \( b^k_r \)
- \( G = x^-, x^k, p^k \alpha^k_n \) or \( b^k_r \)

3 Paraquantum Poincaré Algebra

In view of the form of the relations (7), the quantum form of the Poincaré algebra generators \( M^{\mu\nu}(4,5,6) \) presents an order ambiguity problem so that there must be rewritten on the basis of an adequate symmetrisation which takes the form (4) where now :

\[
M^0_{\mu\nu}(x) = l^{\mu\nu} + E^{\mu\nu} \quad (9)
\]

with

\[
l^{\mu\nu} = \frac{1}{2} [x^\mu, p^\nu]_+ - \frac{1}{2} [x^\nu, p^\mu]_+ \quad (10)
\]

and

\[
E^{\mu\nu} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha^\mu_{-n}, \alpha^\nu_n]_+ - [\alpha^\nu_{-n}, \alpha^\mu_n]_+ \right) \quad (11)
\]

and

\[
K^{\mu\nu} = -\frac{1}{4} \sum_{r=-\infty}^{\infty} \left( [b^\mu_{-r}, b^\nu_r]_- - [b^\nu_{-r}, b^\mu_r]_- \right) \quad (12)
\]
With a direct application of the trilinear relations (7), let us perform the second and the third commutators of the algebra:

\[ [p^\mu, M^{\nu\rho}] = [p^\mu, M^{\nu\rho}_0] + [p^\mu, K^{\nu\rho}] \quad (13) \]

By the use of (7-a,b), it is easy to see that:

\[ [p^\mu, K^{\nu\rho}] = 0 \quad (14) \]

and

\[ [p^\mu, M^{\nu\rho}_0] = i(g^{\mu\rho}p^\nu - g^{\mu\nu}p^\rho) \quad (15) \]

so that

\[ [p^\mu, M^{\nu\rho}] = i(g^{\mu\rho}p^\nu - g^{\mu\nu}p^\rho) \quad (16) \]

Now, for the third commutator

\[ [M^{\mu\nu}, M^{\alpha\beta}] = [M^{\mu\nu}_0(x), M^{\alpha\beta}_0(x)] + [M^{\mu\nu}_0(x), K^{\alpha\beta}] + [K^{\mu\nu}, M^{\alpha\beta}_0(x)] + [K^{\mu\nu}, K^{\alpha\beta}] \quad (17) \]

the first term is given by [1]:

\[ [M^{\mu\nu}_0, M^{\alpha\beta}_0] = ig^{\nu\rho}M^{\alpha\mu}_0 - ig^{\mu\alpha}M^{\rho\beta}_0 - ig^{\rho\beta}M^{\mu\alpha}_0 + ig^{\mu\rho}M^{\nu\beta}_0 \quad (18) \]

It is again clear from (7) that:

\[ [M^{\mu\nu}_0(x), K^{\alpha\beta}] = [K^{\mu\nu}, M^{\alpha\beta}_0(x)] = 0 \quad (19) \]

Let us now consider the commutator:

\[ [K^{\mu\nu}, K^{\alpha\beta}] = -\frac{1}{16} \sum_{r,s} \{ [[b^\mu_{-r}, b^\nu_r], [b^\alpha_{-s}, b^\beta_s]] - (\mu, \nu, \alpha \leftrightarrow \beta) \]

\[ (\mu \leftrightarrow \nu, \alpha, \beta) + (\mu \leftrightarrow \nu, \alpha \leftrightarrow \beta) \} \quad (20) \]

The first term gives:

\[ A = \sum_{r,s} \{ [[b^\mu_{-r}, b^\nu_r], [b^\alpha_{-s}, b^\beta_s]] \]

\[ = \sum_{r,s} \{ b^\mu_{-r} [b^\nu_r, b^\alpha_{-s}, b^\beta_s] + [b^\mu_{-r}, b^\alpha_{-s}, b^\beta_s] b^\nu_r - b^\mu_r [b^\nu_r, b^\alpha_{-s}, b^\beta_s] - b^\nu_r [b^\mu_r, b^\alpha_{-s}, b^\beta_s] \} \]

With the use of (7-e), (21) becomes:

\[ A = 2 \sum_{r=-\infty}^{+\infty} \left( g^{\nu\alpha} [b^\mu_{-r}, b^\beta_r] - g^{\mu\beta} [b^\nu_{-r}, b^\alpha_r] - g^{\nu\beta} [b^\mu_{-r}, b^\alpha_r] + g^{\mu\alpha} [b^\beta_{-r}, b^\nu_r] \right) \quad (22) \]
In the same way, one can perform the other terms of (20) and obtain:

\[
[K_{\mu\nu}, K^{\alpha\beta}] = i \sum_{r=\pm\infty} \left\{ -\frac{i}{4} \left( g^{\nu\alpha} \left( [b^\beta_{-r}, b^\mu_{r}] - [b^\mu_{-r}, b^\beta_{r}] \right) - g^{\mu\beta} \left( [b^\alpha_{-r}, b^\nu_{r}] - [b^\nu_{-r}, b^\alpha_{r}] \right) \right) - g^{\nu\beta} \left( [b^\alpha_{-r}, b^\mu_{r}] - [b^\mu_{-r}, b^\alpha_{r}] \right) + g^{\mu\alpha} \left( [b^\nu_{-r}, b^\beta_{r}] - [b^\beta_{-r}, b^\nu_{r}] \right) \right\} \quad (23)
\]

then

\[
[K_{\mu\nu}, K^{\alpha\beta}] = i \left( g^{\nu\alpha} K^{\beta\mu} - g^{\mu\alpha} K^{\nu\beta} + g^{\mu\beta} K^{\alpha\nu} + g^{\nu\beta} K^{\alpha\mu} \right) \quad (24)
\]

When combined with (18) and (19), (17) gives:

\[
[M_{\mu\nu}, M^{\alpha\beta}] = i \left( g^{\nu\alpha} M^{\beta\mu} - g^{\mu\alpha} M^{\nu\beta} + g^{\mu\beta} M^{\alpha\nu} + g^{\nu\beta} M^{\alpha\mu} \right) \quad (25)
\]

Now, for the first commutator \([p^\mu, p^\nu]\) of the algebra, one can only write \([p^\mu, [p^\nu, p^\sigma]] = 0\)
and not \([p^\mu, p^\nu] = 0\)!

### 4 Space-time critical dimensions

As in the ordinary case, one can obtain the space-time critical dimension by performing, in
the transverse gauge, the commutator \([M^i, M^j]\).

Let us introduce, in the transverse gauge, the generators \(M^i\) in the form:

\[
M^i = M_0^i (x) + K^i \quad (26)
\]

where

\[
M_0^i (x) = l^i + E^i \quad (27)
\]

\[
l^i = \frac{1}{2} \left[ x^i, \frac{1}{p^+} \right]_+ \alpha_0 - \frac{1}{2} \left[ x^-, p^i \right]_+ \quad (28)
\]

\[
E^i = -\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( \left[ \alpha^i_{-n}, \frac{1}{p^+} \right]_+ + \alpha^-_n \left[ \alpha^i_n, \frac{1}{p^+} \right]_+ \right) \quad (29)
\]

with

\[
\alpha^-_n = -\frac{1}{4} \sum_{l=\pm\infty}^{\infty} \left[ \alpha^i_{n-l}, \alpha^i_l \right]_+ - \frac{1}{4} \sum_{r=\pm\infty}^{\infty} \left( r - \frac{n}{2} \right) \left[ b^i_{n-r}, b^i_r \right]_+ - \frac{a}{2} \delta_{n0} \quad (30)
\]

and

\[
K^i = -\frac{i}{4} \sum_{r=\pm\infty}^{\infty} \left( \left[ b^i_{-r}, \frac{1}{p^+} \right]_+ G_r - G_{-r} \left[ b^i_r, \frac{1}{p^+} \right]_+ \right) \quad (31)
\]
with
\[ G_r = \frac{1}{2} \sum_{n=-\infty}^{+\infty} [\alpha_r^n, b_r^j]_+ \] (32)

Let us now perform this commutator:
\[ [M_i^-, M_j^-] = [M_i^0, M_j^0] + [M_i^-, K_j^+] + [K_i^-, M_j^0] + [K_i^-, K_j^+] \] (33)

Projecting the equation \([M_i^-, M_j^-] = 0\) on the physical states \(\alpha^k_m |0\rangle\) and \(b^k_s |0\rangle\), one can write:
\[ \langle 0 | \alpha^l_m [M_i^-, M_j^-] \alpha^k_m |0\rangle + \langle 0 | b^l_s [M_i^-, M_j^-] b^k_s |0\rangle = 0 \] (34)
We begin by computing the first commutator of (33). First, notice that one can write:
\[ \langle 0 | b^l_s [M_i^0, M_j^0] b^k_s |0\rangle \equiv 0 \] (35)

For the other mean value on the physical states \(\alpha^k_m |0\rangle\), except the term which is done by [9] :
\[ C_4 = \frac{1}{4} \sum_{n,n'} \langle 0 | \alpha^l_m \left[ \frac{\alpha_j^{n} - \alpha_i^{n'}}{n - n' + p^+}, \frac{1}{n - n' + p^+} \right] + \alpha_i^{-n} \alpha^{-n'} \left[ \frac{\alpha_j^{-n'} - \alpha_i^{-n}}{n' - n + p^+}, \frac{1}{n' - n + p^+} \right] + \alpha_k^m |0\rangle \]
\[ = \frac{1}{(p^+)^2} \delta^{ij} \delta^{jk} \left( Q \frac{D - 2}{8} m (m^2 - 1) + 2ma \right) \] (36)

all the other terms are analog to parabosonic case [1]. Then, we obtain:
\[ [M_i^0, M_j^0] = - \frac{1}{2 (p^+)^2} \sum_{n=1}^{\infty} \left( [\alpha_i^{-n}, \alpha_j^{n}] - [\alpha_i^{n}, \alpha_j^{-n}] \right) \times \left[ -2n + Q \frac{D - 2}{8} \left( n - \frac{1}{n} \right) + 2a \right] \] (37)

For the second and the third terms of (33), one can write:
\[ [M_i^0, K_j^+] + [K_i^-, M_j^0] = [l_i^-, E_i^-, K_j^+] - (i \leftrightarrow j) \] (38)

Before computing the mean value of the commutator \([l_i^-, K_j^+]\), we first transform it as follows: we notice that by the use of the relations (8-a,b,d,e), one can verify that:
\[ [\alpha_i^{-n}, b_r^j] = \left( r + \frac{n}{2} \right) b_{r+n}^j \] (39)
\[ [\alpha_i^{-n}, G_r] = \left( r - \frac{n}{2} \right) G_{r+n} \] (40)

so that:
\[ \left[ \alpha_i^{-n}, \left[ b_r^j, \frac{1}{p^+} \right] + G_r \right] = -r \left[ b_r^{-n}, \frac{1}{p^+} \right] + G_r + \left[ \frac{1}{p^+}, b_r^j \right] + rG_r = 0 \] (41)
On the other hand, and again by the use of (8-f,g), one can verify that:

\[
[x^i, G_r] = i b^i_r
\]

\[
[x^-, \left(\frac{1}{p^+}, H\right)_+] = -\frac{2i}{(p^+)^2} H
\]

where the operator \( H = x^-, x^k, p^k \alpha^k_n \) or \( b^k_r \). Then we obtain:

\[
\left[ l^{i-}, K^{j+} \right] = \frac{1}{4} \sum_{r=\frac{1}{2}}^{\infty} \left( \left[ \frac{1}{p^+}, b^j_{-r} \right] + \left[ \frac{1}{p^+}, b^i_{-r} \right] - \left[ \frac{1}{p^+}, b^j_{-r} \right] + \left[ \frac{1}{p^+}, b^i_{-r} \right] \right) \alpha^r_0
\]

\[
- \frac{1}{4} \left( \frac{1}{p^+} \right)^2 \sum_{r=\frac{1}{2}}^{\infty} \left( [b^i_{-r}, p^j]_+ G_r - G_{-r} [b^i_{-r}, p^j]_+ \right)
\]

It is easy to see that:

\[
\langle 0 | \alpha^l_m \left[ l^{i-}, K^{j+} \right] \alpha^{k}_{-m} | 0 \rangle = 0
\]

Now, for the other mean value, one can prove that:

\[
\langle 0 | b^l_s \left[ l^{i-}, K^{j+} \right] b^k_{-s} | 0 \rangle = \frac{1}{2 (p^+)^2} \sum_{r=\frac{1}{2}}^{\infty} \langle 0 | b^l_s r \left( [b^j_{-r}, b^i_r] - [b^i_{-r}, b^j_r] \right) b^k_{-s} | 0 \rangle
\]

\[
- \frac{1}{2 (p^+)^2} \delta^{lj} p^i p^k + \frac{1}{2 (p^+)^2} \delta^{jk} p^i p^l
\]

Notice that one can again write:

\[
\langle 0 | b^l_s \left( \left[ l^{i-}, K^{j+} \right] \right. + \left. \left[ K^{i-}, l^{j+} \right] \right) b^k_{-s} | 0 \rangle = 2 \langle 0 | b^l_s \left[ l^{i-}, K^{j+} \right] b^k_{-s} | 0 \rangle
\]

In the same way, for the commutator \( [E^{i-}, K^{j-}] \), we compute the mean value:

\[
\langle 0 | \alpha^l_m \left[ E^{i-}, K^{j-} \right] \alpha^{k}_{-m} | 0 \rangle + \langle 0 | b^l_s \left[ E^{i-}, K^{j-} \right] b^k_{-s} | 0 \rangle
\]

To do this, we need to perform the following non null expressions, which give the results

\[
H_1 = -\frac{1}{4} \sum_{r=\frac{1}{2}}^{\infty} \sum_{n=1}^{\infty} \langle 0 | \alpha^l_m \left[ \frac{1}{p^+}, \alpha^i_n \right] + \alpha^r_0 \left[ \frac{1}{p^+}, b^j_{-r} \right] + G_r \alpha^k_{-m} | 0 \rangle
\]

\[
= \frac{3m^2}{4 (p^+)^2} \delta^{li} \delta^{jk}
\]

\[
H_2 = \frac{1}{4} \sum_{r,n} \langle 0 | b^l_s \left[ \frac{\alpha^l_n}{n}, \frac{1}{p^+} \right] + \alpha^r_0 \left[ b^i_r, \frac{1}{p^+} \right] b^k_{-s} | 0 \rangle
\]

\[
= \delta^{jk} \delta^{li} \frac{1}{2 (p^+)^2} \left( s^2 - \frac{1}{4} + s \right) + \frac{1}{(p^+)^2} \delta^{jk} p^i p^l
\]
\[ H_3 = -\frac{1}{4} \sum_{r,n} \langle 0 | b'_r \alpha_{-n} \left[ \frac{\alpha^2}{n}, \frac{1}{p^+} \right] + \left[ b'_{-n}, \frac{1}{p^+} \right] G_r b^k_{-s} | 0 \rangle = \frac{1}{2 (p^+)^2} \delta^{ij} \delta^{ik} \left( s^2 - \frac{1}{4} \right) \] (51)

It follows that:
\[
\langle 0 | \alpha'_m \left[ E^{i-}, K^{j-} \right] \alpha^k_{-m} | 0 \rangle + \langle 0 | b'_s \left[ E^{i-}, K^{j-} \right] b^k_{-s} | 0 \rangle = -\frac{1}{2 (p^+)^2} \left\{ \left( \frac{3}{4} n^3 - s^2 + \frac{1}{4} \right) \left( \delta^{ij} \delta^{ik} - \delta^{ij} \delta^{ik} \right) + \frac{1}{2} \delta^{jk} \delta^{li} \left( s^2 - \frac{1}{4} + s \right) + \delta^{jk} \delta^{li} \right\} \] (52)

Then we can write the result:
\[
\langle 0 | \alpha'_m \left( \left[ E^{i-}, K^{j-} \right] + \left[ K^{i-}, E^{j-} \right] \right) \alpha^k_{-m} | 0 \rangle + \langle 0 | b'_s \left( \left[ E^{i-}, K^{j-} \right] + \left[ K^{i-}, E^{j-} \right] \right) b^k_{-s} | 0 \rangle = \frac{1}{(p^+)^2} \left\{ \left( \frac{3}{4} m^3 - s^2 + \frac{1}{8} + \frac{s}{2} \right) \left( \delta^{ij} \delta^{ik} - \delta^{ij} \delta^{ik} \right) + \delta^{jk} \delta^{li} \left( s^2 - \frac{1}{4} + s \right) + \delta^{jk} \delta^{li} \right\} \] (53)

Lastly, one can perform the mean value of the last commutator of (33) and obtain (Appendix A):
\[
\langle 0 | b'_s \left[ K^{i-}, K^{j-} \right] b^k_{-s} | 0 \rangle + \langle 0 | \alpha'_m \left[ K^{i-}, K^{j-} \right] \alpha^k_{-m} | 0 \rangle = \frac{1}{4 (p^+)^2} \left( \delta^{ij} \delta^{ik} - \delta^{ij} \delta^{ik} \right) \left[ \left( s^2 - \frac{1}{4} \right) + s - \frac{Q}{2} \left( D - 2 \right) \left( s^2 - \frac{1}{4} \right) - 8a - m^3 \right] + \frac{1}{4 (p^+)^2} \left( \delta^{ij} p^j p^k - \delta^{ij} p^j p^k + \delta^{jk} p^j p^i - \delta^{ik} p^j p^i \right) \] (54)

Now by regrouping all these results, in terms of operators, one obtain:
\[
\left[ M^{i-}, M^{j-} \right] = \frac{1}{2 (p^+)^2} \sum_{n=1}^{\infty} \left( \left[ \alpha^i_{-n}, \alpha^j_{-n} \right] - \left[ \alpha^j_{-n}, \alpha^i_{-n} \right] \right) \left( Q \frac{D - 2}{8} \left( n - \frac{1}{n} \right) + 2a - n \right)
- \frac{1}{2 (p^+)^2} \sum_{r=1}^{\infty} \left( \left[ b'^i_{-r}, b'^j_{-r} \right] - \left[ b'^j_{-r}, b'^i_{-r} \right] \right) \left( \left( Q \frac{D - 2}{8} - 1 \right) \left( r^2 - \frac{1}{4} \right) + 2a - 1 \right) \] (55)

We are thus led to conclude that, in order to have \( \left[ M^{i-}, M^{j-} \right] = 0 \), one must have:
\[
\begin{align*}
D &= 2 + \frac{8}{Q} \\
a &= \frac{1}{2}
\end{align*}
\] (56)
5 Conclusion

This work consist in paraquantizing the spinning string by imposing paracommutation relations to the classical variables $X^\mu(\sigma, \tau)$, $P^\mu(\sigma, \tau)$ and $\psi^\mu_A(\sigma, \tau)$. Unlike in Ardalan and Mansouri work [5], this requires that both the center of mass variables and the excitation modes of the string verify paracommutation relations. To satisfy this, one must have $[X^{\mu(\alpha)}(\sigma, \tau), P^{\mu(\alpha)}(\sigma', \tau)] = i g^\mu^\nu \delta(\sigma - \sigma')$ [1]. But the Ardalan and Mansouri hypothesis, characterized by the anzatz $x^{\mu(\beta)} = x^{\mu} \delta_{\beta 1}$ and $p^{\mu(\beta)} = p^{\mu} \delta_{\beta 1}$, leads to the result $[X^{\mu(\alpha)}(\sigma, \tau), P^{\nu(\alpha)}(\sigma', \tau)] = i g^{\mu\nu} [\delta(\sigma - \sigma') - (1 - \delta_{\alpha 1})]$, which is different from the latter. With the only use of the trilinear relations (7), one can prove that:

$$[p^\mu, M^{\mu\rho}] = -i g^{\mu\nu} p^\rho + i g^{\rho\sigma} p^\mu$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i g^{\rho\sigma} M^{\sigma\mu} - i g^{\mu\sigma} M^{\rho\nu} - i g^{\mu\rho} M^{\nu\sigma} + i g^{\rho\sigma} M^{\mu\nu}$$

In the transverse gauge, in order to have $[M^{i-}, M^{j-}] = 0$, the space-time critical dimension $D$ is calculated with the only use of the trilinear relations (8). Like in Ardalan and Mansouri work [5], $D$ is again given as function of the paraquantization order through the relation $D = 2 + \frac{8}{Q}$. Thus, one can have paraspinning strings with critical dimensions $D = 10, 6, 4, 3$ (respectively in orders $Q = 1, 2, 4, 8$). This coincide with the dimensions in which fractional superstrings can be formulated [10], [11].

Some questions arises; in order to satisfy $([p^\mu, p^\nu] = 0)$, one adopt the Ardalan and Mansouri anzatz, then the question is can one speak about the paraquantum formalism on the classical variables $X^\mu(\sigma, \tau)$, $P^\mu(\sigma, \tau)$ and $\psi^\mu_A(\sigma, \tau)$? On the other hand, if we paraquantize these classical variables by imposing them to satisfy paracommutation relations, what we may only write is $[p^\mu, [p^\nu, p^\rho]_+] = 0$. Then, what about the space-time properties?
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Appendix
Let us set

\[ \langle 0 | \alpha_n^l K^i - K^j - \alpha_m^k | 0 \rangle + \langle 0 | b_s^l K^i - K^j - b_s^k | 0 \rangle = \sum_{i=1}^{4} D_i \]  

(A-1)

computing

\[ D_1 = \frac{1}{4} \sum_{r,r'} \langle 0 | b_r^l \left[ b_{-r}^l + \frac{1}{p^+} \right] G_r \left[ b_{-r}^l + \frac{1}{p^+} \right] G_r b_{-s}^k | 0 \rangle \]

\[ = \frac{1}{(p^+)^2} \delta^{il} \delta^{jk} \frac{1}{2} \left( s^2 - \frac{1}{4} \right) + \frac{1}{(p^+)^2} \delta^{il} p' p^k \]  

(A-2)

\[ D_2 = -\frac{1}{4} \sum_{r,r'} \langle 0 | b_r^l \left[ b_{-r}^l + \frac{1}{p^+} \right] G_r G_{-r'} \left[ b_{-r'}^l + \frac{1}{p^+} \right] b_{-s}^k | 0 \rangle \]

\[ = -\frac{1}{(p^+)^2} \delta^{il} \delta^{jk} \left[ \frac{Q}{2} (D - 2) \left( s^2 - \frac{1}{4} \right) + 2a \right] \]  

(A-3)

\[ D_3 = -\frac{1}{4} \sum_{r,r'} \langle 0 | \alpha_m^l G_{-r} \left[ b_r^i + \frac{1}{p^+} \right] G_{-r'} \left[ b_{-r'}^i + \frac{1}{p^+} \right] G_r \alpha_m^k | 0 \rangle \]

\[ = \frac{m^3}{(p^+)^2} \delta^{ij} \delta^{ik} \]  

(A-4)

\[ D_4 = \frac{1}{4} \sum_{r,r'} \langle 0 | b_s^l G_{-r} \left[ \frac{1}{p^+}, b_r^i \right] G_{-r'} \left[ b_{-r'}^i + \frac{1}{p^+} \right] b_{-s}^k | 0 \rangle \]

\[ = \frac{1}{2 (p^+)^2} \delta^{ik} \delta^{ij} \frac{1}{2} \left( s^2 - \frac{1}{4} \right) + \frac{1}{(p^+)^2} \delta^{ik} p' p^i \]  

(A-5)

it is then straightforward to obtain the relation (54).
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