KHAVINSON CONJECTURE FOR HYPERBOLIC HARMONIC FUNCTIONS ON THE UNIT BALL

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Abstract. In this paper, we prove the Khavinson conjecture for hyperbolic harmonic functions on the unit ball. This conjecture was partially solved in [5].

1. INTRODUCTION

For $n \geq 2$, let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. We use $B^n$ and $S^{n-1}$ to denote the unit ball and the unit sphere in $\mathbb{R}^n$, respectively.

A mapping $u \in C^2(B^n, \mathbb{R})$ is said to be hyperbolic harmonic if $\Delta_h u = 0$, where $\Delta_h$ is the hyperbolic Laplacian operator defined by

$$\Delta_h u(x) = (1 - |x|^2)^2 \Delta u + 2(n - 2)(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x),$$

here $\Delta$ denotes the Laplacian on $\mathbb{R}^n$.

Clearly for $n = 2$, hyperbolic harmonic and harmonic functions coincide.

If $\phi \in L^1(S^{n-1}, \mathbb{R})$, we define the invariant Poisson integral of $\phi$ in $B^n$

$$P_h[\phi](x) = \int_{S^{n-1}} P_h(x, \zeta) \phi(\zeta) d\sigma(\zeta),$$

where

$$P_h(x, \zeta) = \left( \frac{1 - |x|^2}{|x - \zeta|^2} \right)^{n-1}$$

is the Poisson kernel with respective to $\Delta_h$ satisfying

$$\int_{S^{n-1}} P_h(x, \zeta) d\sigma(\zeta) = 1.$$

For more information about hyperbolic harmonic functions we refer to Stoll [21] and Burgeth [3, 4].

2. KHAVINSON PROBLEM

Let $p \in (1, \infty]$ and let $q$ be its conjugate. Assume that $u = P_h[\phi]$, where $\phi \in L^p(S^{n-1}, \mathbb{R})$. For $x \in B^n \setminus \{0\}$ and $\ell \in S^{n-1}$, let $C_p(x)$ and $C_p(x; \ell)$ denote the optimal numbers such that

$$|\nabla u(x)| \leq C_p(x) \|\phi\|_p,$$

and

$$|\langle \nabla u(x), \ell \rangle| \leq C_p(x; \ell) \|\phi\|_p.$$
Since $|\nabla u(x)| = \sup_{\ell \in S^{n-1}} |\langle \nabla u(x), \ell \rangle|$, clearly we obtain
\[ C_p(x) = \sup_{\ell \in S^{n-1}} C_p(x; \ell). \]

We prove the Khavinson conjecture for hyperbolic harmonic functions, partially solved in [5].

**Conjecture 1.** Let $p \in (1, \infty]$, $n \geq 3$ and $x \in \mathbb{B}^n \setminus \{0\}$. Then
\[ C_p(x) = \begin{cases} C_p(x; n_x) & \text{if } 1 < p < n, \\ C_p(x; t_x) & \text{if } p > n, \end{cases} \]
where $n_x = \frac{x}{|x|}$, and $t_x$ is any unit vector such that $\langle t_x, x \rangle = 0$.

Moreover, if $p = n$ or $p = \infty$, then $C_p(x, \ell)$ does not depend on $\ell$.

In the planar case, i.e., $n = 2$, this conjecture was solved by Kalaj and Marković, see [9, Theorem 1.1].

Khavinson [12] obtained a sharp pointwise estimate for the radial derivative of bounded harmonic functions on the unit ball of $\mathbb{R}^3$ and conjectured that the same estimate holds for the norm of the gradient of bounded harmonic functions. For harmonic functions this conjecture was formulated by Kresin and Maz’ya in [14] and in [15] considered the half-space analogue of the above conjecture. See [16, Chapter 6] for various Khavinson-type extremal problems for harmonic functions. Kalaj [8] showed that the conjecture for $n = 4$ and Melentijević [19] confirmed the conjecture in $\mathbb{R}^3$. Marković [18] solved the Khavinson problem for points near the boundary of the unit ball. The general conjecture was recently proved by Liu [17].

By computing the gradient of the Poisson-Szegö kernel and using the Möbius transformation as a substitution, we obtain the following integral representation

**Lemma 1.** [5] For any $p \in (1, \infty]$, $x \in \mathbb{B}^n$ and $\ell \in S^{n-1}$, we have
\[ C_p(x; \ell) = \frac{2(n-1)}{(1 - |x|^2)^{\frac{n(q-1)+1}{q}}} \left( \int_{S^{n-1}} |\eta - x|^{2(n-1)(q-1)} |\langle \eta, \ell \rangle|^q \, d\sigma(\eta) \right)^{\frac{1}{q}}. \quad (2.1) \]

Moreover, one can easily deduce the following

**Lemma 2.** [5] For any $p \in (1, \infty]$, $x \in \mathbb{B}^n$, $\ell \in S^{n-1}$ and unitary transformation $A$ in $\mathbb{R}^n$, we have
\[ C_p(x; \ell) = C_p(Ax; A\ell). \quad (2.2) \]

For $p \in (1, \infty]$ and $\ell \in S^{n-1}$, let
\[ K_p(x; \ell) = \int_{S^{n-1}} |\eta - x|^{2(n-1)(q-1)} |\langle \eta, \ell \rangle|^q \, d\sigma(\eta). \quad (2.3) \]
So in view of (2.1), we have
\[ C_p(x; \ell) = \frac{2(n-1)}{(1 - |x|^2)^{\frac{n(q-1)+1}{q}}} (K_p(x; \ell))^{\frac{1}{q}}. \quad (2.4) \]

Our main result is the following theorem solving the Khavinson conjecture for hyperbolic harmonic functions.
Theorem 1. Let \( p \in (1, \infty] \), \( n \geq 3 \) and \( x \in \mathbb{B}^n \setminus \{0\} \). Then

\[
\max_{\ell \in S^{n-1}} C_p(x; \ell) = \begin{cases} C_p(x; n_x) & \text{if } 1 < p < n, \\ C_p(x; t_x) & \text{if } p > n. \end{cases}
\]

\[
\min_{\ell \in S^{n-1}} C_p(x; \ell) = \begin{cases} C_p(x; t_x) & \text{if } 1 < p < n, \\ C_p(x; n_x) & \text{if } p > n. \end{cases}
\]

If \( p = n \) or \( p = \infty \), then \( \ell \mapsto C_p(x; \ell) \) is constant.

One of our main tools is the method of slice integration on spheres.

Theorem A. [2, Theorem A.5] Let \( f \) be a Borel measurable, integrable function on \( S^{n-1} \). If \( 1 \leq k < n \), then

\[
\int_{S^{n-1}} f d\sigma_n = \frac{k V(B^k)}{n V(B^n)} \int_{B^{n-k}} (1 - |z|^2)^{\frac{k-2}{2}} \int_{S^{k-1}} f(x, \sqrt{1 - |z|^2} \zeta) d\sigma_k(\zeta) dV_{n-k}(x),
\]

where \( V(B^n) \) denotes the volume of the ball, which is given by

\[
V(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)},
\]

and \( \sigma_n \) denotes the normalized measure on the sphere \( S^{n-1} \).

We will consider two special cases for \( k = n - 1 \) and \( k = n - 2 \). The corresponding formulas are useful when the integrand function \( f \) depends only on one or two variables.

Corollary B. Let \( \eta = (\eta_1, \ldots, \eta_n) \in S^{n-1} \) and \( f(\eta) \) be a Borel measurable, integrable function on \( S^{n-1} \).

1. If \( n \geq 2 \) and \( f(\eta) \) depends only on the first variable \( \eta_1 \), then

\[
\int_{S^{n-1}} f(\eta_1) d\sigma_n(\eta) = \frac{n - 1}{n} V(B^{n-1}) V(B^n) \int_{-1}^1 (1 - t^2)^{\frac{n-4}{2}} f(t) dt.
\]

2. If \( n \geq 3 \) and \( f(\eta) \) depends only on the first two variables \( \eta_1, \eta_2 \), then

\[
\int_{S^{n-1}} f(\eta_1, \eta_2) d\sigma_n(\eta) = \frac{n - 2}{2 \pi} \int_{B^2} (1 - |z|^2)^{\frac{n-4}{2}} f(z) dA(z).
\]

where \( dA(z) \) denotes the Lebesgue measure on the unit disc \( B^2 \).

Using the invariance of \( K_p \) by unitary transformations, see Lemma 2, we may assume that:

\[ x = |x| e_1 \text{ and } \ell = \ell_\gamma = \cos \gamma e_1 + \sin \gamma e_2, \text{ with } \gamma \in [0, \pi]. \]

Let \( \eta = (\eta_1, \ldots, \eta_n) \in S^{n-1} \). Then

\[ |\eta - x|^2 = 1 + |x|^2 - 2|x|\eta_1, \]

and

\[ \langle \eta, \ell_\gamma \rangle = \eta_1 \cos \gamma + \eta_2 \sin \gamma. \]
For \( r, \rho \in (0, 1) \), introduce the notation
\[
J_q(r, \rho; \gamma) = \int_{-\pi}^{\pi} (1 + \rho^2 - 2\rho r \cos \theta)^{(n-1)(q-1)} |\cos(\theta - \gamma)|^q d\theta. \tag{2.9}
\]

**Lemma 3.** Let \( x \in \mathbb{B}^n \), \( 1 < p < \infty \) and \( q \) its conjugate. Then \[
K_p(x; \ell) = K_p(|x|e_1; \ell_\gamma) = \frac{n - 2}{2\pi} \int_0^1 (1 - r^2)^{\frac{n-4}{2} - \frac{q}{r}^q + 1} J_q(r, |x|; \gamma) dr.
\]

**Proof.** Using the invariance of \( K_p \) by unitary transformations, we may assume that \( x = |x|e_1 \) and \( \ell = \ell_\gamma = \cos \gamma e_1 + \sin \gamma e_2 \), with \( \gamma \in [0, \pi] \).
\[
K_p(|x|e_1; \ell_\gamma) = \int_{S^{n-1}} (1 + |x|^2 - 2|\eta_1|^{(n-1)(q-1)} |\eta_1 \cos \gamma + \eta_2 \sin \gamma|^q d\sigma(\eta). \tag{2.10}
\]

As the integrand function depends only on \( \eta_1 \) and \( \eta_2 \), the method of slice integration on spheres reduces an integral on the sphere to some integral on the unit disc \( \mathbb{B}^2 \). Using polar coordinates on the unit disc, let us denote \( \eta_1 = r \cos \theta \) and \( \eta_2 = r \sin \theta \). Thus \[
\langle \eta, \ell_\gamma \rangle = \eta_1 \cos \gamma + \eta_2 \sin \gamma = r \cos(\theta - \gamma). \quad \Box
\]

To find the extreme values of \( J_q(r, \rho; \gamma) \), we will consider the following more general integral
\[
I_{a,b}(\gamma) = \int_{-\pi}^{\pi} (A - B \cos \theta)^a |\cos(\theta - \gamma)|^b d\theta. \tag{2.11}
\]

The function \( I_{a,b} \) has the following properties
(1) \( I_{a,b} \) is \( \pi \)-periodic.
(2) \( I_{a,b} \) is an even function.

Thus, we will consider the behaviour of \( I_{a,b} \) only on \([0, \frac{\pi}{2}]\) and we show that \( I_{a,b} \) is a monotonic on \([0, \frac{\pi}{2}]\), thus the extreme values are reached at \( \gamma = 0 \) and \( \gamma = \frac{\pi}{2} \). A special case was considered in [3 Lemma 2.1].

**Lemma 4.** Let \( A, B \) be positive numbers such that \( 0 < B < A \), and \( a, b \) are real numbers such that \( b > 0 \).

(1) If \( a = 0 \) or \( a = 1 \), then \( \gamma \mapsto I_{a,b}(a, b; \gamma) \) is constant.
(2) If \( a \in (0, 1) \), then \( \gamma \mapsto I_{a,b}(a, b; \gamma) \) is increasing on \([0, \pi/2]\). Thus \[
\max_{\gamma \in [0, \pi/2]} I_{a,b}(a, b; \gamma) = I_{a,b}(a, b; \pi/2).
\]
(3) If \( a > 1 \), then \( \gamma \mapsto I_{a,b}(a, b; \gamma) \) is decreasing on \([0, \pi/2]\). Thus \[
\max_{\gamma \in [0, \pi/2]} I_{a,b}(a, b; \gamma) = I_{a,b}(a, b; 0).
\]

**Proof.** As the integrand function is \( 2\pi \) periodic with respect to \( \theta \), we deduce that
\[
I_{a,b}(\gamma) = \int_{-\pi}^{\pi} (A - B \cos(\theta + \gamma))^a |\cos \theta|^b d\theta.
\]

Therefore the mapping is differentiable and \[
I'_{a,b}(\gamma) = aB \int_{-\pi}^{\pi} \sin(\theta + \gamma)(A - B \cos(\theta + \gamma))^{a-1} |\cos \theta|^b d\theta.
\]
Again, using the $2\pi$-periodicity of the integrand, we obtain

$$I'_a,b(\gamma) = aB \int_{-\pi}^{\pi} \sin(\theta) (A - B \cos \theta)^{a-1} |\cos(\theta - \gamma)|^b \, d\theta.$$ 

Next, we split the integral into two parts from 0 to $\pi$ and from $\pi$ to $2\pi$. Using a substitution, we obtain

$$I'_a,b(\gamma) = aB \int_0^{\pi} \sin \theta \left[ (A - B \cos \theta)^{a-1} - (A + B \cos \theta)^{a-1} \right] \cos(\theta - \gamma)^b \, d\theta.$$ 

By considering the substitution $u = \theta - \pi/2$, we get

$$I'_a,b(\gamma) = aB \int_{-\pi/2}^{\pi/2} \cos \theta \left[ (A + B \sin \theta)^{a-1} - (A - B \sin \theta)^{a-1} \right] \sin(\theta - \gamma)^b \, d\theta.$$ 

Next, we split the integral into two parts from $0$ to $\pi/2$ and from $\pi/2$ to $\pi$. Using the substitution $u = \pi - \theta$, we obtain

$$I'_a,b(\gamma) = aB \int_0^{\pi/2} \cos \theta \left[ (A + B \sin \theta)^{a-1} - (A - B \sin \theta)^{a-1} \right] \left[ \sin(\theta - \gamma)^b - \sin(\theta + \gamma)^b \right] \, d\theta.$$ 

Clearly, if $a = 0$ or $a = 1$, then $I'_a,b = 0$ and the function $I_{a,b}$ is constant and

$I'_a,b(0) = I'_a,b(\pi/2) = 0.$

For $\gamma, \theta \in (0, \pi/2)$ and $b > 0$, then

$$|\sin(\theta - \gamma)^b - \sin(\theta + \gamma)^b| < 0.$$ 

Indeed, $\sin(\theta - \gamma) - \sin(\theta + \gamma) = -2 \cos(\theta) \sin(\gamma) < 0$. Therefore,

1. If $a \in (0, 1)$ and $\gamma \in (0, \pi/2)$, then $I'_a,b(\gamma) > 0$ and the mapping $I_{a,b}$ is strictly increasing on $[0, \pi/2]$;
2. If $a \in (1, \infty)$ and $\gamma \in (0, \pi/2)$, then $I'_a,b(\gamma) < 0$ and the mapping $I_{a,b}$ is strictly decreasing on $[0, \pi/2]$.

As a consequence, we get

**Corollary 2.1.** Let $r, \rho \in (0, 1)$, and $q \geq 1$. Then the mapping $\gamma \mapsto J_q(r, \rho; \gamma)$ on $[0, \pi/2]$ is

1. constant for $q = 1$ or $q = \frac{n}{n-1}$;
2. strictly increasing for $1 < q < \frac{n}{n-1}$;
3. strictly decreasing for $q > \frac{n}{n-1}$.

**Corollary 2.2.** Let $p \in (1, \infty]$ and $x \in \mathbb{R}^n \setminus \{0\}$. Then

1. If $p = n$ or $p = \infty$, then $\gamma \mapsto K_p(x; \ell, \gamma)$ is constant.
If \( p \in (1, n) \), then
\[
\max_{\gamma \in [0, \pi/2]} K_p(x; \ell_\gamma) = K_p(x; \ell_0) = K_p(x; n_x).
\]

If \( p \in (n, \infty) \), then
\[
\max_{\gamma \in [0, \pi/2]} K_p(x; \ell_\gamma) = K_p(x; \ell_{\pi/2}) = K_p(x; t_x).
\]

Thus we obtain our main theorem.

3. Computation of \( C_p(x) \)

We will start with two particular cases \( p = n \) or \( p = \infty \).

3.1. Case \( p = \infty \).

In this case, the mapping \( \gamma \mapsto K_\infty(x, \ell_\gamma) \) is constant, hence
\[
K_\infty(x, \ell_0) = K_\infty(x, \ell_{\pi/2}) = K_\infty(x; t_x).
\]

Using the substitution \( u = 1 - t^2 \), we deduce
\[
K_\infty(x, \ell_0) = \frac{(n-1) V(B^{n-1})}{V(B^n)} \int_0^1 u^{n-3} du.
\]

By (2.4), we have
\[
C_\infty(x; \ell) = \frac{2(n-1) \Gamma(n/2)}{1 - |x|^2} K_\infty(x; \ell)
\]
\[= \frac{2(n-1) V(B^{n-1})}{V(B^n)} \int_0^1 u^{n-3} du.
\]

Hence, if \( u = \mathcal{P}_h[\phi] \), where \( \phi \in L^\infty(S^{n-1}, \mathbb{R}) \), then
\[
|\nabla u(x)| \leq \frac{2(n-1) \Gamma(n/2)}{\sqrt{n} \Gamma(n/2 + 1)} \frac{1}{1 - |x|^2} \|\phi\|_\infty.
\]

We should mention that the sharp inequality (3.4) can also be obtained as follows. According to [11, Corollary 1.2], see also [3], if \( u \) is a bounded hyperbolic harmonic function, with \( |u| < 1 \), then
\[
|\nabla u(0)| \leq \frac{4(n-1) V(B^{n-1})}{n V(B^n)}.
\]

Let \( x \in B^n \) and \( \varphi_x \) be the M"obius transformation such that \( \varphi_x(0) = x \), see [21], p. 7 (2.1.4)]. By the M"obius invariance of \( \Delta_h \), the function \( u \circ \varphi_x \) is also bounded hyperbolic harmonic function with \( \nabla(u \circ \varphi_x)(0) = -(1 - |x|^2) \nabla u(x) \). Hence (3.4) follows by considering \( u \circ \varphi_x \) in (3.5).
3.2. Case $p = n$.

The conjugate of $n$ is $q = \frac{n}{n - 1}$.

\[ J_q(r, \rho; \gamma) = \int_{-\pi}^{\pi} (1 + \rho^2 - 2\rho r \cos \theta) |\cos(\theta - \gamma)|^q d\theta. \]

\[ = (1 + \rho^2) \int_{-\pi}^{\pi} |\cos \theta|^q d\theta \]

\[ = 4(1 + \rho^2) \int_{0}^{\pi/2} \cos^q \theta d\theta. \quad (3.6) \]

Using

\[ K_n(x; \ell; \gamma) = \frac{n - 2}{2\pi} \int_0^1 (1 - r^2)^{\frac{n-4}{2} + 1} J_q(r, |x|; \gamma) dr \]

\[ = \frac{n - 2}{2\pi} \left( \int_0^1 (1 - r^2)^{\frac{n-4}{2} + 1} dr \right) \left( 4(1 + |x|^2) \int_0^{\pi/2} \cos^q \theta d\theta \right) \]

\[ = \frac{n - 2}{\pi} \left( \int_0^1 (1 - r)^{\frac{n-4}{2}} dr \right) \left( \int_0^{\pi/2} \cos^q \theta d\theta \right) (1 + |x|^2). \quad (3.7) \]

Recall some properties of the beta function. Let $a, b > 0$

\[ B(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} dt = 2 \int_0^{\pi/2} (\sin t)^{2a-1} (\cos t)^{2b-1} \cos^q \theta d\theta = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}. \quad (3.8) \]

Therefore,

\[ K_n(x; \ell; \gamma) = \frac{n - 2}{2\pi} B\left( \frac{n}{2} - 1, \frac{q}{2} + 1 \right) \Gamma\left( \frac{1 + q}{2} \right) \Gamma\left( \frac{1}{2} \right) (1 + |x|^2) \]

\[ = \frac{n - 2}{2\sqrt{\pi}} \frac{\Gamma\left( \frac{n}{2} - 1 \right) \Gamma\left( \frac{q+1}{2} \right) \Gamma\left( \frac{1}{2} \right)}{\Gamma\left( \frac{n+q}{2} \right)} (1 + |x|^2). \quad (3.9) \]

3.3. Case $1 < p < n$.

The following lemma is useful to compute $K_p(x; \ell_0)$.

Lemma 5. For $a > -1$ and $b > -1$

\[ (1) \int_{-1}^{1} |t|^a (1 - t^2)^b dt = B\left( \frac{a + 1}{2}, b + 1 \right). \]

\[ (2) \text{If } |u| < 1 \text{ and } \alpha \in \mathbb{R}, \text{ then} \]

\[ \int_{-1}^{1} (1 - ut)^{-\alpha} |t|^a (1 - t^2)^b dt = B\left( \frac{a + 1}{2}, b + 1 \right) \frac{\alpha + 1}{2} \frac{a + 1}{2} \frac{1}{2} \frac{a + 3}{2} + b; u^2. \]

Proof. (1) Using the substitution $t = \sin \theta$ and (3.8), we get

\[ \int_{-1}^{1} |t|^a (1 - t^2)^b dt = 2 \int_{0}^{\pi/2} t^a (1 - t^2)^b dt \]

\[ = 2 \int_{0}^{\pi/2} (\sin \theta)^a (\cos \theta)^{2b+1} \cos \theta d\theta \]

\[ = B\left( \frac{a + 1}{2}, b + 1 \right). \]
(2) Since $|ut| \leq |u| < 1$, for $|t| \leq 1$, we have

$$(1 - ut)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} u^k t^k,$$

and this series converges uniformly in $[-1, 1]$, thus

$$\int_{-1}^{1} (1 - ut)^{-\alpha} |t|^\alpha (1 - t^2)^b \, dt = \sum_{k=0}^{\infty} \int_{-1}^{1} \frac{(\alpha)_k}{k!} u^k t^k |t|^\alpha (1 - t^2)^b \, dt$$

$$= \sum_{k=0}^{\infty} \int_{-1}^{1} \frac{(\alpha)_2k}{(2k)!} t^{2k} |t|^\alpha (1 - t^2)^b \, dt$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha)_2k}{(2k)!} \left( \int_{-1}^{1} t^{2k} |t|^\alpha (1 - t^2)^b \, dt \right) u^{2k}. \quad (3.10)$$

Using Lemma 5 (1), we have

$$\int_{-1}^{1} t^{2k} |t|^\alpha (1 - t^2)^b \, dt = \frac{\Gamma(k + \frac{a+1}{2}) \Gamma(b + 1)}{\Gamma(k + \frac{a+3}{2} + b)}.$$

Thus

$$\sum_{k=0}^{\infty} \frac{(\alpha)_2k}{(2k)!} \left( \int_{-1}^{1} t^{2k} |t|^\alpha (1 - t^2)^b \, dt \right) u^{2k} = \sum_{k=0}^{\infty} \frac{(\alpha)_2k}{(2k)!} \frac{\Gamma(k + \frac{a+1}{2}) \Gamma(b + 1)}{\Gamma(k + \frac{a+3}{2} + b)} u^{2k}.$$

Using

$$(\alpha)_2k = 2^{2k} \frac{\alpha}{2} k (\frac{\alpha + 1}{2})_k,$$

$$(2k)! = \Gamma(2(k + 1/2)) = \sqrt{\pi} \frac{2^{2k} \Gamma(k + 1/2)k!}{\sqrt{\pi}} = 2^{2k} (1/2)_k k!,$$

and

$$(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)}, \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}_-,$$

we get

$$\frac{(\alpha)_2k}{(2k)!} = \frac{(\alpha/2)_k (\alpha + 1/2)_k}{(1/2)_k k!}. \quad (3.11)$$

Therefore

$$\sum_{k=0}^{\infty} \frac{(\alpha)_2k}{(2k)!} \left( \int_{-1}^{1} t^{2k} |t|^\alpha (1 - t^2)^b \, dt \right) u^{2k} = \sum_{k=0}^{\infty} \frac{(\alpha/2)_k (\alpha + 1/2)_k \Gamma(k + \frac{a+1}{2}) \Gamma(b + 1)}{\Gamma(k + \frac{a+3}{2} + b)} \frac{\Gamma(n/2)}{(1/2)_k k!} u^{2k}$$

$$= \frac{\Gamma(n/2) \Gamma(b + 1)}{\Gamma(n/2 + b)} \sum_{k=0}^{\infty} \frac{(\alpha/2)_k (\alpha + 1/2)_k}{(a+3/2)_k (a+1/2)_k} u^{2k}$$

$$= \text{B}(\frac{a+1}{2}; b + 1) \text{ } \frac{\Gamma(n/2; a+1; a+1/2; a+3/2; a+3/2; a+1/2; a+3/2; a+3/2; b; u^2).}$$

Recall that

$$\mathcal{K}_p(x; \ell_0) = \int_{S^{n-1}} |\eta - x|^{2(n-1)(q-1)} |\langle \eta, \ell \rangle|^q d\sigma(\eta),$$
with \( \ell_0 = e_1 = \frac{x}{|x|} \); \( x = |x|e_1 \).

We have \( |\eta - x|^2 = 1 + |x|^2 - 2|x|\eta \) and \( \langle \eta, \ell_0 \rangle = |\eta| \).

Let \( \alpha = (n - 1)(1 - q) \) and \( u = \frac{2|x|}{1 + |x|^2} \).

\[
\mathcal{K}_p(x; \ell_0) = \int_{S^{n-1}} (1 + |x|^2 - 2|x|\eta_1) - |\eta_1|^2 d\sigma(\eta)
= (1 + |x|^2) - \alpha \int_{S^{n-1}} \left( 1 - 2 \frac{|x|}{1 + |x|^2} \eta_1 \right) - |\eta_1|^2 d\sigma(\eta)
= (1 + |x|^2) - \alpha \int_{S^{n-1}} (1 - u\eta_1) - |\eta_1|^2 d\sigma(\eta).
\]

As we integrate a function which depends on one variable on \( S^{n-1} \), then by the slice integration on spheres, we have

\[
\int_{S^{n-1}} (1 - u\eta_1) - |\eta_1|^2 d\sigma(\eta) = \frac{n - 1}{n} V(B_n^{n-1}) V(B_n) \int_{-1}^1 (1 - ut) - |t|^q (1 - t^2)^{\frac{n-3}{2}} dt.
\]

By Lemma 6 (2), it yields

\[
\int_{-1}^1 (1 - ut)^{(n-1)(q-1)}|t|^q (1 - t^2)^{\frac{n-3}{2}} dt
= \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{q+n}{2})} 3F_2 \left( \frac{(n-1)(1-q)}{2}, \frac{(n-1)(1-q)}{2}, \frac{q+1}{2}, \frac{q+n}{2}, \frac{4|x|^2}{(1 + |x|^2)} \right).
\]

Therefore

\[
\int_{S^{n-1}} (1 - u\eta_1)^{(n-1)(q-1)}|\eta_1|^q d\sigma(\eta)
= \frac{n - 1}{n} V(B_n^{n-1}) V(B_n) \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{q+n}{2})} 3F_2 \left( \frac{(n-1)(1-q)}{2}, \frac{(n-1)(1-q)}{2}, \frac{q+1}{2}, \frac{q+n}{2}, \frac{4|x|^2}{(1 + |x|^2)} \right)
= \frac{2}{\pi^{\frac{3}{2}}} n^{\frac{n+1}{2}} 3F_2 \left( \frac{(n-1)(1-q)}{2}, \frac{(n-1)(1-q)}{2}, \frac{q+1}{2}, \frac{q+n}{2}, \frac{4|x|^2}{(1 + |x|^2)} \right)
= \frac{1}{\pi^{\frac{3}{2}}} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{q+n}{2})} 3F_2 \left( \frac{(n-1)(1-q)}{2}, \frac{(n-1)(1-q)}{2}, \frac{q+1}{2}, \frac{q+n}{2}, \frac{4|x|^2}{(1 + |x|^2)} \right).
\]

Finally,

\[
\mathcal{K}_p(x; \ell_0)
= (1 + |x|^2)^{(n-1)(q-1)} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{q+n}{2})\sqrt{n}} 3F_2 \left( \frac{(n-1)(1-q)}{2}, \frac{(n-1)(1-q)}{2}, \frac{q+1}{2}, \frac{q+n}{2}, \frac{4|x|^2}{(1 + |x|^2)} \right).
\]

3.4. Case \( p > n \). First, we will need the following lemma
Lemma 6. Let $p \in \mathbb{N}$ and $q$ be a positive real number and $n \geq 3$. Then

\begin{align}
(1) \quad & \int_{\mathbb{S}^{n-1}} \eta_1^{2p+1} |\eta_2|^q d\sigma(\eta) = 0. \\
(2) \quad & \int_{\mathbb{S}^{n-1}} \eta_1^{2p} |\eta_2|^q d\sigma(\eta) = \frac{n-2}{\pi} B\left(\frac{n}{2} - 1, p + \frac{q}{2} + 1\right) B\left(p + \frac{1}{2}, \frac{q+1}{2}\right). 
\end{align}

Proof. Using (2.8) we have

\[ \int_{\mathbb{S}^{n-1}} \eta_1^k |\eta_2|^q d\sigma(\eta) = \frac{n-2}{2\pi} \left( \int_0^{1} (1 - r^2) \frac{n-2}{2} r^{k+q+1} dr \right) \left( \int_0^{2\pi} \cos^k \theta |\sin \theta|^q d\theta \right). \]

One can check that \( \int_0^{2\pi} \cos^k \theta |\sin \theta|^q d\theta = 0 \) if $k$ is odd and if $k = 2p$, then by (3.13), we get

\begin{align}
\int_{\mathbb{S}^{n-1}} \eta_1^{2p} |\eta_2|^q d\sigma(\eta) &= \frac{n-2}{\pi} B\left(\frac{n}{2} - 1, p + \frac{q}{2} + 1\right) B\left(p + \frac{1}{2}, \frac{q+1}{2}\right) \\
&= \frac{n-2}{\pi} \Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{q}{2} + 1\right) \Gamma\left(p + \frac{1}{2}\right) \Gamma\left(p + \frac{q+1}{2}\right).
\end{align}

\[ \square \]

Remark 3.1. For $n = 2$ we obtain

\[ \int_{\mathbb{S}^1} \eta_1^{2p} |\eta_2|^q d\sigma(\eta) = B\left(p + \frac{1}{2}, \frac{q+1}{2}\right). \]

In the case $p > n$, the maximum of $K_p(x, \ell_\gamma)$ is reached in the tangential direction, i.e., $\gamma = \frac{\pi}{2}$ and

\[ K_p(x; \ell_{\pi/2}) = \int_{\mathbb{S}^{n-1}} (1 + |x|^2 - 2|x|\eta_1)^{(n-1)(q-1)} |\eta_2|^q d\sigma(\eta). \]

To simplify the notation, we consider

\[ u = \frac{2|x|}{1 + |x|^2} \quad \text{and} \quad \alpha = (n-1)(1-q). \]

Thus we obtain

\[ K_p(x; \ell_{\pi/2}) = (1 + |x|^2)^{-\alpha} \int_{\mathbb{S}^{n-1}} (1 - u\eta_1)^{-\alpha} |\eta_2|^q d\sigma(\eta) \\
= (1 + |x|^2)^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \int_{\mathbb{S}^{n-1}} \eta_1^k |\eta_2|^q d\sigma(\eta) \\
= \frac{n-2}{\pi} \Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{q+1}{2}\right) (1 + |x|^2)^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(2k)!} \Gamma\left(\frac{k+\frac{q+1}{2}}{2}\right) u^{2k}. \]

Since

\[ \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(2k)!} \Gamma\left(\frac{k+\frac{q+1}{2}}{2}\right) u^{2k} = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{q+n}{2}\right)} \sum_{k=0}^{\infty} \frac{\binom{\alpha+1}{2k}}{(q+n)_k!} u^{2k} \]

\[ = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{q+n}{2}\right)} 2F_1\left(\frac{\alpha+1}{2}, q+n+1, \frac{q+n}{2}, u^2\right). \]
Therefore,

\[ K_p(x; \ell_{\pi/2}) = \frac{n - 2}{{\sqrt{\pi}}} \frac{\Gamma \left(\frac{n-2}{2}\right) \Gamma \left(\frac{q+1}{2}\right)}{\Gamma \left(\frac{n+q}{2}\right)} (1 + |x|^2)^{-\alpha} \, _2F_1 \left(\frac{\alpha}{2}, \frac{\alpha + 1}{2}; \frac{q + n}{2}; u^2 \right). \]

Using the following well-known transformation formula due to Kummer

\[ _2F_1 \left(a, a + \frac{1}{2}, \frac{4v}{(1+v)^2} \right) = (1+v)^{2a} \, _2F_1 \left(2a, 2a - c + 1; c; v \right), \]

which is respectively a slight variation of the one given in [1, Section 15.3 (20)].

It yields

\[ K_p(x; \ell_{\pi/2}) = \frac{n - 2}{{\sqrt{\pi}}} \frac{\Gamma \left(\frac{n-2}{2}\right) \Gamma \left(\frac{q+1}{2}\right)}{\Gamma \left(\frac{n+q}{2}\right)} \, _2F_1 \left(\alpha - \frac{q + n}{2} + 1; \frac{q + n}{2}; |x|^2 \right) \] (3.17)

\[ = \frac{n - 2}{{\sqrt{\pi}}} \frac{\Gamma \left(\frac{n-2}{2}\right) \Gamma \left(\frac{q+1}{2}\right)}{\Gamma \left(\frac{n+q}{2}\right)} \, _2F_1 \left(\frac{(n-1)(1-q)}{2}, \frac{n}{2} + q \left(\frac{1}{2} - n\right); \frac{q + n}{2}; |x|^2 \right). \]

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