The graph and range singularity spectra of random wavelet series built from Gibbs measures

Xiong Jin

INRIA Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France
E-mail: xiongjin82@gmail.com

Received 12 November 2009, in final form 10 March 2010
Published 13 May 2010
Online at stacks.iop.org/Non/23/1449

Recommended by J A Glazier

Abstract
We consider the random wavelet series built from Gibbs measures, and study the Hausdorff dimension of the graph and range of these functions restricted to their iso-Hölder sets. To obtain the Hausdorff dimension of these sets, we apply the potential theoretic method to families of Gibbs measures defined on a sequence of topologically transitive subshift of finite type whose Hausdorff distance to the set of zeros of the mother wavelet tends to 0.

Mathematics Subject Classification: 26A30, 28A78, 28A80, 37D35, 37B10

1. Introduction

1.1. The graph and range singularity spectra

In the study of geometric properties of a non-smooth function, there are two main points of view. The first one consists in computing the Hausdorff dimension of the graph and range of the function restricted to certain subset of its domain, where given a function \( f : [0, 1] \to \mathbb{R} \) and a subset \( E \subset [0, 1] \), the graph and range of \( f \) restricted to \( E \) are defined as

\[
G_f(E) = \{(x, f(x)) : x \in E\} \quad \text{and} \quad R_f(E) = \{f(x) : x \in E\}.
\]

It is classical in probability and geometric measure theories to study this problem for non-smooth functions. The first work on this subject could be traced back to Lévy [36] and Taylor [50], regarding the Hausdorff dimension and Hausdorff measure of the range of Brownian motion. Since then, much progress has been made in the cases of fractional Brownian motions, stable Lévy processes and many other processes and functions [6–9, 13, 15, 22–24, 34, 35, 38, 40, 42, 43, 46, 51, 52]. As a typical example, Kahane [34] studied this problem...
for fractional Brownian motion \((X(t))_{t \in \mathbb{R}_+}\) of Hurst parameter \(\beta \in (0, 1)\). He shows that for any compact set \(E \subset \mathbb{R}_+\), almost surely (here \(\dim_H\) stands for Hausdorff dimension)

\[
\dim_H G_X(E) = \frac{\dim_H E}{\beta} \land (\dim_H E + 1 - \beta) \quad \text{and} \quad \dim_H R_X(E) = \frac{\dim_H E}{\beta} \land 1.
\]

The second point of view is that of the multifractal analysis, which consists in computing the Hausdorff dimension of the iso-Hölder sets

\[E_f(h) := \{x \in [0, 1] : h_f(x) = h\}, \quad h \geq 0,\]

where at \(x_0 \in [0, 1]\), the pointwise Hölder exponent \(h_f(x_0)\) is given by

\[h_f(x_0) = \sup\{h > 0 : \exists P \in \mathbb{R}[x], |f(x) - P(x)| = O(|x - x_0|^h), x \to x_0\}.
\]

The function

\[d_f : h \geq 0 \mapsto \dim_H E_f(h)\]

is called the singularity spectrum of \(f\). The singularity spectrum was first introduced by physicists [17, 19, 20] to study the intermittent phenomenon in fully developed turbulence, and it has been computed rigorously by mathematicians for Riemann’s nowhere differentiable function, Lévy processes, Lévy processes in multifractal time, wavelet series, self-similar functions and generic functions in certain Besov or Sobolev spaces, as well as indefinite integrals of positive measures [1, 3–5, 11, 12, 16, 21, 25–31, 41, 44, 49]. As a typical example, when \((X(t))_{t \in \mathbb{R}_+}\) is a Lévy process having no Brownian component and of upper index \(\beta \in (0, 2)\), Jaffard [29] shows that almost surely

\[d_X(h) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta], \\ -\infty & \text{otherwise.} \end{cases}\]

In this paper, we will consider these two general problems simultaneously, by studying the Hausdorff dimension of the graph and range of the function restricted to its iso-Hölder sets:

\[G_f(h) := G_f(E_f(h)) \quad \text{and} \quad R_f(h) := R_f(E_f(h)), \quad h \geq 0.
\]

We call the functions

\[d_f^G : h \geq 0 \mapsto \dim_H G_f(h) \quad \text{and} \quad d_f^R : h \geq 0 \mapsto \dim_H R_f(h)\]

the graph and range singularity spectra of \(f\). It is of interest to study these singularity spectra for non-smooth functions, and such a study can be considered as a new type of multifractal analysis. In fact, when \(f\) is a stochastic process, with respect to the pointwise Hölder exponent, the set \(E_f(h)\) is a collection of times, and the sets \(G_f(h)\) and \(R_f(h)\) are collections of points on the sample path and range.

Normally it is difficult to calculate the Hausdorff dimension of non-trivial subsets on the graph and range of non-smooth functions. The problem would be easier if we consider random functions. In [33] we studied the graph and range singularity spectra for a class of random multifractal functions, namely the \(b\)-adic independent cascade functions, which is a generalization to functions of the Mandelbrot measures introduced in [37]. In this paper we will study these singularity spectra for another large class of random multifractal functions constructed in [4]: the random wavelet series built from Gibbs measures. Before stating our main result, let us first give some background and notation on the wavelet series and multifractal analysis.
1.2. Orthogonal wavelet basis and multifractal analysis

Let $\psi$ be an $r_0$-smooth mother wavelet on $\mathbb{R}$, with $r_0 \in \mathbb{N}^*$, so that the functions $\{\psi_{j,k} = \psi(2^j \cdot -k)\}_{(j,k) \in \mathbb{Z}^2}$ form an orthogonal wavelet basis of $L^2(\mathbb{R})$ (see [39] for instance for definition and construction). Each function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{(j,k) \in \mathbb{Z}^2} d_{j,k} \cdot \psi_{j,k}(x),$$

where the wavelet coefficient $d_{j,k}$ is given by

$$d_{j,k} = 2^j \int_{\mathbb{R}} f(t) \cdot \psi_{j,k}(t) \, dt.$$

It is known that the asymptotic behaviour of the wavelet coefficients provides fine information on the Hölder regularity of the function. For example, due to proposition 4 in [31], if there exist constants $C_0 > 0$, $\epsilon_0 \in (0, 1)$ such that $|d_{j,k}| \leq C_0 2^{-\epsilon_0 j}$ holds for each $j \geq 0$ and $k \in \mathbb{Z}$, then $f$ is $\epsilon_0$-Hölder continuous. Moreover, once $f$ is $\epsilon_0$-Hölder continuous for some $\epsilon_0 > 0$, one can also obtain the pointwise Hölder exponent of $f$ from its wavelet leaders. For each $(j, k) \in \mathbb{Z}^2$ we denote by $I_{j,k}$ the dyadic interval $[k2^{-j}, (k+1)2^{-j})$; the wavelet leader is then given by

$$L_{j,k} = \sup\{|d_{j',k'}| : (j', k') \in \mathbb{Z}^2, I_{j',k'} \subset I_{j,k}\}.$$

For any $x_0 \in \mathbb{R}$ and $j \geq 0$ let

$$L_j(x_0) = \sup\{L_{j,k} : k \in \mathbb{Z}, 3I_{j,k} = [(k-1)2^{-j}, (k+2)2^{-j}] \ni x_0\}.$$

Due to corollary 1 in [31] one has if $|h_f(x_0)| \leq r_0$, then

$$h_f(x_0) = \liminf_{j \to +\infty} -j^{-1} \log_2 L_j(x_0). \quad (1)$$

Equality (1) shows that wavelet is an effective tool to study the local regularity of a Hölder continuous function. It is also connected to the singularity spectrum as follows: define the scaling function of $f$ as

$$\xi_f(q) = \liminf_{j \to +\infty} -j^{-1} \log_2 \sum_{k \in \mathbb{Z} : I_{j,k} \subset [0, 1]} 1_{\{L_{j,k} \neq 0\}} |L_{j,k}|^q, \quad q \in \mathbb{R}. \quad (2)$$

If $r_0$ is large enough so that $h_f(x) \leq r_0$ holds for all $x \in [0, 1]$, then one has (see [31, 32])

$$d_f(h) \leq \xi_f^*(h) := \inf_{q \in \mathbb{R}} q \cdot h - \xi_f(q), \quad \forall \ h > 0, \quad (3)$$

where a negative dimension means that the set is empty. One can say that the restriction of $f$ to $[0, 1]$ fulfils the multifractal formalism at $h > 0$ if (3) is an equality.

1.3. Random wavelet series built from multifractal measure

In general, it is difficult to precisely control the wavelet coefficients of a non-smooth function. Thus, to construct good examples of wavelet series whose singularity spectrum can be calculated, it is convenient to directly prescribe the wavelet coefficients. In [4], Barral and Seuret construct a class of multifractal wavelet series whose wavelet coefficients are directly built from a given positive Borel measure, thus the singularity spectrum of the wavelet series is transferred to the singularity spectrum of the measure. To be more precise, let $\mu$ be a positive Borel measure carried by $[0, 1]$, fix two positive numbers $s_0$, $p_0 > 0$ such that $s_0 - 1/p_0 > 0$, then the sequence of wavelet coefficients built from $\mu$ is defined as

$$|d_{j,k}| : j \geq 0, \ 0 \leq k \leq 2^j - 1, \ |d_{j,k}| \equiv 2^{-j(s_0-1/p_0)} \mu(I_{j,k})^{1/p_0},$$

where $I_{j,k}$ is the dyadic interval $[k2^{-j}, (k+1)2^{-j})$. If $h_f(x_0)$ is $\epsilon_0$-Hölder continuous for some $\epsilon_0 > 0$, one can also obtain the pointwise Hölder exponent of $f$ from its wavelet leaders. For each $(j, k) \in \mathbb{Z}^2$ we denote by $I_{j,k}$ the dyadic interval $[k2^{-j}, (k+1)2^{-j})$; the wavelet leader is then given by

$$L_{j,k} = \sup\{|d_{j',k'}| : (j', k') \in \mathbb{Z}^2, I_{j',k'} \subset I_{j,k}\}.$$

For any $x_0 \in \mathbb{R}$ and $j \geq 0$ let

$$L_j(x_0) = \sup\{L_{j,k} : k \in \mathbb{Z}, 3I_{j,k} = [(k-1)2^{-j}, (k+2)2^{-j}] \ni x_0\}.$$
and the corresponding wavelet series is given by
\[ F_\mu(x) = \sum_{j \geq 0} \sum_{k=0}^{2^j-1} d_{j,k} \cdot \psi_{j,k}(x). \]

Due to (2), one has
\[ \xi_{F_\mu}(q) = (s_0 - 1/p_0) \cdot q + \tau_\mu(q/p_0), \quad q \in \mathbb{R}, \] (4)
where \( \tau_\mu \) is the \( L^q \) spectrum of \( \mu \) and it is defined as
\[ \tau_\mu(q) = \liminf_{n \to \infty} -j^{-1} \log \sum_{k=0}^{2^j-1} \mathbf{1}_{[\mu(I_{j,k}) \neq 0]} \cdot \mu(I_{j,k})^q, \quad q \in \mathbb{R}. \] (5)

By construction, \( F_\mu \) is \((s_0 - 1/p_0)\)-Hölder continuous and belongs to the Besov space \( B_{s_0,\infty}^{p_0}(\mathbb{R}) \) if \( s_0 < r_0 \). Moreover, it is shown in [4] that if \( \mu \) fulfills the multifractal formalism for measures at \( \alpha \geq 0 \) (in the sense of [11]), then the restriction of \( F_\mu \) to \([0, 1]\) fulfills the multifractal formalism at \( h = s_0 - 1/p_0 + \alpha/p_0 \) when \( |h| \leq r_0 \).

Typical examples of measures which fulfill the multifractal formalism are the random measures obtained as the limit of multiplicative chaos (such as Mandelbrot measure) and quasi-Bernoulli measure (such as Gibbs measure). There is also a general condition on \( \mu \) which ensures the validity of the multifractal formalism of \( F_\mu \), see [4] for details.

In [4], Barral and Seuret also considered the random perturbation of \( F_\mu \): let \( (\Omega_1, \mathcal{A}, \mathbb{P}) \) be the probability space and take a sequence of independent random variables \( \{\pi_{j,k} : j \geq 0, k = 0, 1, \ldots, 2^j - 1\} \).

The random perturbation of the wavelet coefficients with respect to \( \{\pi_{j,k}\} \) is defined as
\[ d_{j,k}^{\text{pert}} = \pi_{j,k} \cdot d_{j,k}, \quad j \geq 0, k = 0, 1, \ldots, 2^j - 1, \]
and the corresponding random perturbation of the wavelet series is given by
\[ F_{\mu}^{\text{pert}}(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k}^{\text{pert}} \cdot \psi_{j,k}(x). \]

Under certain conditions on the moments of \( \pi_{j,k} \), for example,
A1 For any \( q \in \mathbb{R} \) we have \( \sup_{j \geq 0} \sup_{k=0,1,\ldots,2^j-1} \mathbb{E}(|\pi_{j,k}|^q) < \infty \),
they show that almost surely
\[ \xi_{F_{\mu}^{\text{pert}}}(q) = \xi_{F_\mu}(q) \quad \text{for all } q \in \mathbb{R} \quad \text{and} \quad h_{F_{\mu}^{\text{pert}}}(x) = h_{F_\mu}(x) \quad \text{for all } x \in [0, 1]. \]

As a direct consequence, if \( F_\mu \) (restricted to \([0, 1]\)) fulfills the multifractal formalism, then so does \( \mathbb{P}\)-almost every \( F_{\mu}^{\text{pert}} \), i.e. almost surely for all \( h > 0 \) such that \( \xi_{F_{\mu}^{\text{pert}}}(h) > 0 \),
\[ \dim_H E_{F_{\mu}^{\text{pert}}}(h) = \xi_{F_{\mu}^{\text{pert}}}(h) = \xi_{F_\mu}(h). \]

1.4. Main result

In this paper we calculate the graph and range singularity spectra of random wavelet series \( F_{\mu}^{\text{pert}} \) built from \( \mu \), where \( \mu \) is the canonical image on \([0, 1]\) of the Gibbs measure \( \mu_\varphi \) associated with a Hölder potential \( \varphi \) on the dyadic symbolic space \( \Sigma \) (see sections 2 and 3.1 for precise definitions). We expect to obtain a similar result when \( \mu \) is a random cascade measure.
in a further work. Such a result requires a technique different from the 'subshift of finite type' used in this paper to avoid the set of zeros of $\psi$.

The random perturbation of $F_\mu$ is essential to our approach based on the potential theoretic method (see chapter 4 in [14]). The efficiency of the combination between randomness and the potential theoretic method has been used to compute the Hausdorff dimension of the whole graph of classical processes [14], random Weierstrass function [24] and random wavelet series [46] (see also [28, 45] for questions related to the dimensions of the whole graph of wavelet series).

In addition to A1, we shall assume the following:

A2 The mother wavelet $\psi$ has only finitely many zeros on $[0, 1]$.

A3 Each random variable $\pi_{j,k}$ has a bounded density function $f_{j,k}$ and for any $\epsilon > 0$ we have

$$\sum_{j \geq 0} (\sup_{k=0, \ldots, 2^{j-1}} \| f_{j,k} \|_\infty) \cdot 2^{-j\epsilon} < \infty.$$ 

Under these assumptions, we prove the following result:

**Theorem 1.1.** Almost surely for all $h \in (0, 1)$ such that $\xi_{F_\mu}^+(h) > 0$,

$$d_{G_{F_\mu}}^+(h) \equiv \frac{\xi_{F_\mu}^+(h)}{h} \land (\xi_{F_\mu}^+(h) + 1 - h),$$

$$d_{R_{F_\mu}}^+(h) = \frac{\xi_{F_\mu}^+(h)}{h} \land 1.$$

**Remark 1.1.** Notice that our result is uniform: it is valid almost surely for all $h \in (0, 1)$ such that $\xi_{F_\mu}^+(h) > 0$, not for each $h \in (0, 1)$ almost surely. Since it is a general fact that $\dim_H G_{F_\mu}^-(h) = \dim_H E_{F_\mu}^-(h)$ when $h \geq 1$, when $r_0$ is large enough so that $h_{F_\mu}(x) \leq r_0$ holds for all $x \in [0, 1]$, theorem 1.1 actually provides us with a complete graph singularity spectrum almost surely:

$$d_{G_{F_\mu}}^-(h) = \left( \frac{\xi_{F_\mu}^-(h)}{h} \land (\xi_{F_\mu}^-(h) + 1 - h) \right) \lor \xi_{F_\mu}^-(h), \quad \forall h > 0.$$ 

But we have no results for the range singularity spectrum when $h \geq 1$.

Let us roughly explain our strategy to prove theorem 1.1. The upper bound is easy to deduce, we just need to find good coverings of $G_{F_\mu}^+(h)$ and $R_{F_\mu}^+(h)$ with respect to the coverings of $E_{F_\mu}^+(h)$ and the H"older exponent $h$. For the lower bound, we apply the potential theoretic method to families of images of Gibbs measures on the graph and range of $F_{F_\mu}$. We consider the restrictions of the potentials $(q\varphi)_{q \in \mathbb{R}}$ on a sequence $\{X_k\}_{k \geq 1}$ of subshifts of finite type whose canonical projection $\tilde{X}_k$ in $[0, 1]$ has a positive Hausdorff distance to the set of zeros of $\psi$, and the distance tends to 0 as $k \to \infty$. We also need to consider the canonical projections on $[0, 1]$ of the equilibrium states of these restricted potentials, which we denote by $\{ (\mu^{(k)}_q)_{q \in \mathbb{R}} \}_{k \geq 1}$. Then for $k$ large enough, there exists an interval $J_k$ such that for each $q \in J_k$, there exist an exponent $h^{(k)}_q \in (0, 1)$, a subset $E_{F_\mu}^{(k)}(h^{(k)}_q) \subset E_{F_\mu}^-(h^{(k)}_q) \cap \tilde{X}_k$ such that $\mu^{(k)}_q(E_{F_\mu}^{(k)}(h^{(k)}_q)) > 0$ and two numbers $r_{q,G}^{(k)}$, $r_{q,R}^{(k)} > 0$ such that for any $\delta > 0$ small enough, almost surely for all $q \in J_k$,

$$\int \int_{E_{F_\mu}^{(k)}} \frac{d\mu^{(k)}_q(s) d\mu^{(k)}_q(t)}{|F_{F_\mu}^+(s) - F_{F_\mu}^+(t)|^2 + |s - t|^{2 \gamma_{\mu,s} - \delta/2}} < \infty.$$
for proving the finiteness of the above integrals, we have to control from below the increment
be a sequence of real-valued random variables whose laws are absolutely continuous with
this, together with A3 and the definition of
T

for details). The reason why we consider subshift of finite type that avoids zeros of

and by letting \( k \) tend to \( \infty \) we can get the sharp lower bound in theorem 1.1 (see section 3.4
for details). The reason why we consider subshift of finite type that avoids zeros of \( \psi \) is that
for proving the finiteness of the above integrals, we have to control from below the increment
of potential,

here \( \psi (s) - \psi (t) \) when \( s \in E^{(k)}_q \) and \( t \) is far away from \( s \). This is possible only if \( s \) is never too

close to the zeros of \( \psi \).

Here we must mention the result of Roueff in [46] which deals with the Hausdorff
dimension of whole graph of random wavelet series. Briefly speaking, let \( \{ c_{j,k} \}_{j \geq 0, k = 0, \ldots, 2^{-1}} \)
be a sequence of real-valued random variables whose laws are absolutely continuous with
respect to Lebesgue measure. Let \( T(c_{j,k}) \) stand for the \( L^\infty \) norm of the density of \( c_{j,k} \). Roueff
proves that (theorem 1 in [46]) if \( \psi \) has finitely many zeros on \( [0, 1] \), then the Hausdorff
dimension of the graph of the random wavelet series

\[
F(x) = \sum_{j \geq 0} \sum_{k = 0}^{2^{j-1}} c_{j,k} \cdot \psi_{j,k}(x)
\]

restricted to \([0, 1]\) is almost surely larger than or equal to

\[
\lim_{j \to \infty} \lim_{j \to \infty} \frac{\log \min_{i=1}^{2j-1} \left[ \sum_{k=0}^{2^{j}-1} \min\{1, T(c_{i,k}) \cdot 2^{-j} \cdot v(I_{j,k})^2\} \right]}{-j \log 2},
\]

where \( v \) can be chosen as any probability measure on \([0, 1]\) being independent of
\( \{c_{j,k}\}_{j \geq 0, k = 0, \ldots, 2^{j-1}} \) such that there exist constants \( C \) and \( s \) such that for any Borel sets
\( A \subset [0, 1]\) and \( B \subset A \) such that \( v(A) > 0 \), one has \( v(B)/v(A) \leq C |B|/|A|^s \), where \( |B|, |A| \)
stand for the diameters of \( A \) and \( B \). Due to the scaling properties of the equilibrium state
\( \mu_q \) of potential \( q \psi \), it is natural to try using Roueff’s approach to our problem: for \( j \geq 0 \) and
\( k = 0, \ldots, 2^j - 1 \) we take

\[
c_{j,k} = \pi_{j,k} \cdot d_{j,k},
\]

where \( |d_{j,k}| = 2^{-j(h_{j,k} - 1/p_0)} \mu(I_{j,k})^{1/p_0} \),

this, together with A3 and the definition of \( T(c_{j,k}) \), gives us

\[
T(c_{j,k}) = \| f_{j,k} \| \cdot |d_{j,k}|^{-1}.
\]

For \( q \in \mathbb{R} \) and \( \epsilon > 0 \) we define the subset

\[
E_q(q, \epsilon) = \left\{ x \in [0, 1] : \forall j \geq n, \left[ |d_{j,k_{j,n}}| \in [2^{-j(h_{j,n} + \epsilon)}, 2^{-j(h_{j,n} - \epsilon)}], \right. \left. (\mu_q(I_{j,k_{j,n}}) \in [2^{-j(\xi^{q}_p(h_q) + \epsilon)), 2^{-j(\xi^{q}_p(h_q) - \epsilon)})] \right\},
\]

where \( \mu_q \) is the canonical projection of \( \mu_{qq} \) on \([0, 1]\), \( k_{j,n} \) is the unique integer \( k \) such that
\( x \in I_{j,k} \) and \( h_q = s_0 - 1/p_0 + \xi^{q}_p(q)/p_0 \). Due to sections 2.3 and 3.1 we have for
each \( \epsilon > 0 \), \( \mu_q(\lim_{n \to \infty} E_q(q, \epsilon)) = 1 \). Thus by continuity we can find an integer \( N_q, \epsilon \) \( \mu_q(\lim_{n \to \infty} E_{N_q}(q, \epsilon)) = 1 \). Now we define \( v \) by \( v(B) = \mu_q(B \cap E_{N_q}) \) for each Borel subset of
\([0, 1]\) (here \( v(B)/v(A) \leq C |B|/|A|^s \) holds with some constant \( C, s = \xi^{q}_p(h_q) - \epsilon \), and \( A \),
B dyadic intervals. Then for \( j > N_{q, \epsilon} \), we have
\[
\sum_{k=0}^{2^j - 1} \min\{1, T(e_j, k) \cdot 2^{-j}\} \cdot v(I_{e_j, k})^2
\]
\[
= \sum_{k=0}^{2^j - 1} \min\{1, f_{e_j, k} \sup\{1, 2^{-j(1-h_q-\epsilon)}\} \cdot 2^{-2js} \cdot 1_{\{\mu_q(I_{e_j, k}) \geq 2^{-j(\epsilon+2\epsilon)}\}} \}
\]
\[
\leq \sum_{k=0}^{2^j - 1} \min\{1, f_{e_j, k} \sup\{1, 2^{-j(1-h_q-\epsilon)}\} \cdot 2^{-2js} \cdot 2^{j(s+2\epsilon)} \cdot \mu_q(I_{e_j, k}) \}
\]
\[
\leq \min\{2^{j(1-h_q-\epsilon)}, \sup_{k=0, \ldots, 2^j - 1} f_{e_j, k} \sup\{1, 2^{-j(s+1-h_q-4\epsilon)}\} \sum_{k=0}^{2^j - 1} \mu_q(I_{e_j, k}) \}
\]
\[
\leq (\sup_{k=0, \ldots, 2^j - 1} f_{e_j, k} \sup\{1, 2^{-j(s+1-h_q-4\epsilon)}\})^2.
\]
Then under assumptions A1-3, due to the fact that \( \mu_q \) is carried by the set \( E_{p, \rho}(h_q) \) almost surely, Roueff’s result implies that if \( \xi_{p, \rho}(h_q) + 1 - h_q - 4\epsilon > 1 \) (this is essential in his proof), then almost surely
\[
d_{G_{p, \rho}}(h_q) \geq \xi_{p, \rho}(h_q) + 1 - h_q - 4\epsilon.
\]
By taking a sequence of \( \epsilon \) tending to 0, we get the sharp lower bound for \( d_{G_{p, \rho}}(h_q) \) given by theorem 1.1 when \( d_{G_{p, \rho}}(h_q) > 1 \). But this result holds only ‘for each \( q \in \mathbb{R} \) almost surely’, so it is not uniform, and it seems that Roueff’s method cannot yield a uniform result, nor the value of \( d_{G_{p, \rho}}(h_q) \) when \( d_{G_{p, \rho}}(h_q) \leq 1 \) and \( h_q < 1 \).

The rest of the paper is organized as follows: in section 2 we give some definitions and notation on the subshift of finite type, Gibbs measure and its multifractal analysis. In section 3 we prove theorem 1.1 thanks to an intermediate result, theorem 3.1, whose proof is postponed to section 4 and the proof is divided into three parts: in section 4.1 we give the main proof of theorem 3.1 with the help of the essential tool of this paper, proposition 4.1; in section 4.2 we prove proposition 4.1 by using two technical lemmas; and finally in section 4.3 we give the proofs of these two technical lemmas.

2. Subshift of finite type, Gibbs measure and multifractal analysis

2.1. Subshift of finite type

Let \( \Sigma = \{0, 1\}^\infty \) and \( \Sigma_n = \bigcup_{n \geq 0} \Sigma_n \), where \( \Sigma_0 = \{\varnothing\} \) and \( \Sigma_n = \{0, 1\}^n \) for \( n \geq 1 \).

Denote the length of \( w \) by \(|w| = n \) if \( w \in \Sigma_n \), \( n \geq 0 \) and \(|w| = \infty \) if \( w \in \Sigma \).

For \( w \in \Sigma_n \) and \( t \in \Sigma_n \bigcup \Sigma \), the concatenation of \( w \) and \( t \) is denoted by \( w \cdot t \) or \( w t \).

For \( w \in \Sigma_\infty \), the cylinder with root \( w \), i.e. \( \{w \cdot u : u \in \Sigma\} \) is denoted by \([w]\).

The set \( \Sigma \) is endowed with the standard metric distance
\[
\rho(s, t) = \inf\{2^{-n} : n \geq 0, \exists w \in \Sigma_n \text{ such that } s, t \in [w]\}.
\]
Then \((\Sigma, \rho)\) is a compact metric space. Denote by \( \mathcal{B} \) the Borel \( \sigma \)-algebra with respect to \( \rho \). Clearly \( \mathcal{B} \) can be generated by the cylinders \( \{[w] : w \in \Sigma_\infty\} \).
If $n \geq 1$ and $w = w_1 \cdots w_n \in \Sigma_n$, then for any $0 \leq i \leq n$ we write $w_i = w_1 \ldots w_i$, with the convention $w_0 = \emptyset$. Also, for any infinite word $t = t_1 t_2 \cdots \in \Sigma$ and $i \geq 0$, we write $t_i = t_i \ldots t_i$, with the convention $t_0 = \emptyset$.

The left-side shift mapping $\sigma$ on $\Sigma$ is defined as

$$\sigma : t = t_1 t_2 \cdots \in \Sigma \mapsto \sigma(t) = t_2 t_3 \cdots \in \Sigma.$$ 

A subshift is a $\sigma$-invariant compact set $X \subset \Sigma$, that is $\sigma(X) \subset X$.

A subshift $X$ is said to be of finite type if there is a subset $A \subset \Sigma_1^n$ for some $n \geq 2$ such that

$$X = \{ t \in \Sigma : \sigma^m(t)|_n \in A, \forall m \geq 0 \}.$$ 

The set $A$ is usually called an admissible set and it induces a transition matrix $B : \Sigma_1^{n-1} \times \Sigma_1^{n-1} \mapsto \{0, 1\}$ with $B(a_1 \cdots a_{n-1}, a_2 \cdots a_n) = 1$ if $a_1 a_2 \cdots a_n \in A$, and $B(i, j) = 0$ otherwise. With respect to the transition matrix $B$, $X$ can be redefined as

$$X = \{ t \in \Sigma : B(\sigma^m(t)|_{n-1}, \sigma^{m+1}(t)|_{n-1}) = 1, \forall m \geq 0 \}.$$ 

The dynamical system $(X, \sigma)$ is called topologically transitive (respectively mixing) if $B$ is irreducible, that is for any $i, j \in \Sigma_1^{n-1}$ there is a $k \geq 1$ such that $B^k(i, j) > 0$ (respectively if $B$ is primitive, that is there is a $k \geq 1$ such that $B^k(i, j) > 0$ for all $i, j \in \Sigma_1^{n-1}$).

2.2. Gibbs measure on topologically transitive subshift of finite type

Let $\psi$ be a Hölder continuous function defined on $\Sigma$, which will be mentioned as a Hölder potential in the following.

Let $(X, \sigma)$ be a topologically transitive subshift of finite type.

For $n \geq 1$ the $n$th-order Birkhoff sum of $\psi$ over $\sigma$ is defined as

$$S_n\psi(t) = \sum_{i=0}^{n-1} \psi \circ \sigma^i(t), \quad t \in \Sigma.$$ 

The topological pressure of $\psi$ on $X$ is given by

$$P_X(\psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \Sigma_1^n : w \cap X \neq \emptyset} \exp \left( \max_{t \in [w]} S_n \psi(t) \right),$$

where the existence of the limit is ensured by the sub-additive property of the logarithm on the right-hand side.

It follows from the thermodynamic formalism developed by Sinai, Ruelle, Bowen and Walters [10, 47] that there exists a constant $C(\psi)$ (independent of $X$), as well as a unique ergodic measure $\mu_\psi$ on $(X, \sigma)$, namely the equilibrium state or Gibbs measure of $\psi$ restricted to $X$, such that for any $t \in X$, $n \geq 0$ and $t' \in [t|_n]$,

$$C(\psi)^{-1} \leq \frac{\mu_\psi([t|_n])}{\exp(S_n \psi(t') - n P_X(\psi))} \leq C(\psi),$$

and $\mu_\psi$ possesses the quasi-Bernoulli property:

$$C(\psi)^{-1} \mu_\psi([w]) \mu_\psi([u]) \leq \mu_\psi([wu]) \leq C(\psi) \mu_\psi([w]) \mu_\psi([u])$$

holds for all $w, u \in \Sigma_*$ such that $[wu] \cap X \neq \emptyset$. 

2.3. Multifractal analysis of Gibbs measure

Here we follow [2, 44]. Consider a topologically transitive subshift $X$ of finite type and a Hölder potential $\varphi$ on $X$. Denote by $\mu_\varphi$ the equilibrium state on $(X, \sigma)$ with potential $\varphi$. Recall that the $L^q$ spectrum of $\mu_\varphi$ is defined as

$$\tau_{\mu_\varphi}(q) = \liminf_{n \to \infty} -\frac{1}{n} \log_2 \sum_{w \in \Sigma_n \mu_\varphi([w]) \neq 0} \mu_\varphi([w])^q, \quad q \in \mathbb{R}. \quad (9)$$

It is easy to deduce from (6) and (7) that the above limit inferior is in fact a limit, and it is equal to

$$\tau_{\mu_\varphi}(q) = (\log 2)^{-1} \cdot (P_X(\varphi) \cdot q - P_X(q\varphi)). \quad (10)$$

Due to corollary 5.27 in [47], if $(X, \sigma)$ is topologically transitive, then $q \mapsto P_X(q\varphi)$ is a convex analytic function on $\mathbb{R}$, thus $\tau_{\mu_\varphi}$ is a concave analytic function on $\mathbb{R}$ and

$$\tau'_{\mu_\varphi}(q) = (\log 2)^{-1} \cdot \left( P_X(\varphi) - \frac{d}{dq} P_X(q\varphi) \right). \quad (11)$$

Denote by $\tau^*_{\mu_\varphi} : \alpha \in \mathbb{R} \mapsto \inf_{\varphi \in \mathcal{X}} \alpha - \tau_{\mu_\varphi}(q)$ the Legendre transform of $\tau_{\mu_\varphi}$. Since $\tau_{\mu_\varphi}$ is concave and analytic over $\mathbb{R}$, we have for any $q \in \mathbb{R}$,

$$\tau^*_{\mu_\varphi}(\tau_{\mu_\varphi}(q)) = q \tau_{\mu_\varphi}(q) - \tau_{\mu_\varphi}(q) = -q \frac{d}{dq} P_X(q\varphi) + P_X(q\varphi). \quad (12)$$

For $\alpha \in \{\tau^*_{\mu_\varphi}(q) : q \in \mathbb{R}\}$ define the set

$$E_{\mu_\varphi}(\alpha) = \left\{ t \in X : \lim_{n \to \infty} \frac{\log \mu_\varphi([t|n])}{\log 2^{-n}} = \alpha \right\}$$

and

$$\tilde{E}_{\mu_\varphi}(\alpha) = \left\{ t \in X : \lim_{n \to \infty} \frac{\log \mu_\varphi([t|n])}{\log 2^{-n}} = \lim_{n \to \infty} \frac{\log \max_{w \in X([t|n])} \mu_\varphi([w])}{\log 2^{-n}} = \alpha \right\},$$

where for any $w \in \Sigma_n$ define the set of neighbour words of $w$ by

$$\mathcal{N}(w) = \left\{ u \in \Sigma_{|w|} : \sum_{i=1}^{|w|} (u_i - w_i) \cdot 2^{-i} \leq 2^{-|w|} \right\}. \quad (13)$$

By using (9), it is standard to check that for $\alpha \in \{\tau^*_{\mu_\varphi}(q) : q \in \mathbb{R}\}$ one has

$$\dim_H \tilde{E}_{\mu_\varphi}(\alpha) \leq \dim_H E_{\mu_\varphi}(\alpha) \leq \tau^*_{\mu_\varphi}(\alpha).$$

Moreover, one can prove that this is actually an equality: for $q \in \mathbb{R}$ denote by $\mu_{q\varphi}$ the equilibrium state of $q\varphi$ restricted to $X$. Then applying (7) to $\mu_\varphi$ and $\mu_{q\varphi}$, together with (10), we can easily get for any $t \in X$ and $n \geq 1$,

$$(C(\varphi)\varphi(C(q\varphi))^{-1} \mu_\varphi([t|n])^q e^{-n\tau_{\mu_\varphi}(q)} \leq \mu_{q\varphi}([t|n]) \leq C(\varphi)^q C(q\varphi) \mu_\varphi([t|n])^q e^{-n\tau_{\mu_\varphi}(q)}.$$

Then due to [2] and the fact that $\mu_\varphi$ is quasi-Bernoulli, we get for $q\varphi$-almost every $t \in X$,

$$\lim_{n \to \infty} \frac{\log \mu_\varphi([t|n])}{\log 2^{-n}} = \lim_{n \to \infty} \frac{\log \max_{w \in X([t|n])} \mu_\varphi([w])}{\log 2^{-n}} = \tau_{\mu_\varphi}(q)$$

and

$$\lim_{n \to \infty} \frac{\log \mu_{q\varphi}([t|n])}{\log 2^{-n}} = \tau^*_{\mu_\varphi}(\tau_{\mu_\varphi}(q)).$$

Due to the mass distribution principle, this implies that for any $q \in \mathbb{R}$,

$$\dim_H \tilde{E}_{\mu_\varphi}(\tau_{\mu_\varphi}(q)) \geq \dim_H (\mu_{q\varphi}) \geq \tau^*_{\mu_\varphi}(\tau_{\mu_\varphi}(q)). \quad (14)$$

where for any positive Borel measure $\mu$ defined on a compact metric space, the lower Hausdorff dimension of $\mu$ is given by $\dim_H(\mu) = \inf \{ \dim_H E : \mu(E) > 0 \}$. 

The graph and range singularity spectra of random wavelet series 1457
3. Proof of theorem 1.1

From now on we fix a Hölder potential \( \varphi \) on the \( \Sigma \) and denote by \( \mu \) the Gibbs measure on \( (\Sigma, \sigma) \) with potential \( \varphi \). We avoid the trivial case that \( \varphi \) is a constant function.

3.1. The multifractal analysis of \( F_\mu \) and \( F_\mu^{pert} \)

Denote the canonical mapping by

\[
\lambda : w \in \Sigma^* \cup \Sigma ^* \mapsto \lambda (w) = \sum_{i=1}^{[w]} w_i \cdot 2^{-i} \in [0, 1].
\]

For \( w \in \Sigma_w \) let \( T_w : x \in \mathbb{R} \mapsto 2^{-[w]} \cdot x + \lambda (w) \) and \( \psi_w = \psi \circ T_w^{-1} \). Consider the wavelet series

\[
F_\mu(x) = \sum_{w \in \Sigma^*} d_w \cdot \psi_w(x), \quad \text{where } |d_w| = 2^{-[w](s_0-1/p_0)} \mu([w])^{1/p_0}. \quad (15)
\]

Up to the formal replacement of dyadic intervals by the cylinders of \( \Sigma \), this is the wavelet series built from the image measure \( \mu \circ \lambda^{-1} \) in section 1.3. Recall (see (4)) that

\[
\xi_{F_\mu}(q) = q (s_0 - 1/p_0) + \tau_\mu(q/p_0), \quad q \in \mathbb{R}.
\]

So for \( \alpha \geq 0 \) and \( h = s_0 - 1/p_0 + \alpha/p_0 \) such that \( h \leq r_0 \), we have

\[
d_{F_\mu}(h) \leq \xi_{F_\mu}^+(h) = \tau_\mu^+(\alpha). \quad (16)
\]

For \( q \in \mathbb{R} \) denote by \( \mu_q \) the equilibrium state of the potential \( q \varphi \) on \( (\Sigma, \sigma) \). Applying the results in section 2.3 we have for any \( q \in \mathbb{R} \), for \( \mu_q \)-almost every \( t \in \Sigma \),

\[
\lim_{n \to \infty} \frac{\log \mu([t_{[a]}])}{\log 2^{-n}} = \lim_{n \to \infty} \frac{\log \max_{w \in N([t_{[a]}])} \mu([w])}{\log 2^{-n}} = \tau_\mu^+(q).
\]

Together with theorem 1 of [4], this implies that \( \mu_q \) is carried by the iso-Hölder set \( E_{F_\mu}(s_0 - 1/p_0 + \tau_\mu^+(q)/p_0) \). Then due to (16) and (14), we get that \( F_\mu \) obeys the multifractal formalism at each \( h = s_0 - 1/p_0 + \alpha/p_0 \) such that \( \alpha \in \{\tau_\mu^+(q) : q \in \mathbb{R}\} \) and \( h \leq r_0 \):

\[
d_{F_\mu}(h) = \xi_{F_\mu}^+(h) = \tau_\mu^+(\alpha). \quad (17)
\]

The random perturbation \( F_\mu^{pert} \) is obtained from \( F_\mu \) and a sequence of independent random variables \( \{\pi_w : w \in \Sigma_w\} \) as

\[
F_\mu^{pert}(x) = \sum_{w \in \Sigma_w} \pi_w \cdot d_w \cdot \psi_w(x),
\]

and our assumption A1 is for any \( q \in \mathbb{R} \) we have \( \sup_{w \in \Sigma_w} E(|\pi_w|^q) < \infty \). We have seen in section 1.3 that this implies that almost surely

\[
\xi_{F_\mu^{pert}}(q) = \xi_{F_\mu}(q) \quad \text{for all } q \in \mathbb{R} \quad \text{and} \quad h_{F_\mu^{pert}}(x) = h_{F_\mu}(x) \quad \text{for all } x \in [0, 1]. \quad (18)
\]

Thus, almost surely \( F_\mu^{pert} \) fulfils the multifractal formalism at each \( h = s_0 - 1/p_0 + \alpha/p_0 \) such that \( \alpha \in \{\tau_\mu^+(q) : q \in \mathbb{R}\} \) and \( h \leq r_0 \).
3.2. Upper bound estimation

For \( x \in [0, 1] \) we define the pointwise oscillation exponent of \( F_{\mu}^\ast \) at \( x \) by

\[
O_{F_{\mu}^\ast}(x) = \liminf_{r \to 0^+} \frac{1}{\log r} \log \left( \sup_{s,t \in [x-r,x+r)} \left| F_{\mu}(s) - F_{\mu}(t) \right| \right).
\]

By definition one has \( O_{F_{\mu}^\ast}(x) \leq h_{F_{\mu}^\ast}(x) \) for any \( x \in [0, 1] \), and they are equal when neither of them is an integer. Due to (18), this implies that almost surely for all \( h \in (0, 1) \)

\[
\{ x \in [0, 1] : O_{F_{\mu}^\ast}(x) = h \} = E_{F_{\mu}^\ast}(h) = E_{F_{\mu}}(h).
\]  

(19)

Recall in [33] that we have proved the following theorem.

**Theorem A.** Let \( f : [0, 1] \to \mathbb{R} \) be a real-valued function and let \( E \) be any subset of \([0, 1]\]. Suppose that \( \inf_{x \in E} O_f(x) = h > 0 \). Then

\[
\dim_h G_f(E) \leq \left( \frac{\dim_h E}{h} \wedge (\dim_h E + 1 - h) \right) \vee \dim_h E,
\]

\[
\dim_h R_f(E) \leq \frac{\dim_h E}{h} \wedge 1.
\]

Together with (17) and (19), theorem A implies that almost surely for all \( h \in (0, 1) \)

\[
d^G_{F_{\mu}^\ast}(h) \leq \frac{\xi^\ast_{F_{\mu}^\ast}(h)}{h} \wedge \left( \frac{\xi^\ast_{F_{\mu}^\ast}(h)}{h} + 1 - h \right)
\]

and \( d^R_{F_{\mu}^\ast}(h) \leq \frac{\xi^\ast_{F_{\mu}^\ast}(h)}{h} \wedge 1 \).

3.3. Topologically transitive subshifts of finite type avoiding the set of zeros of \( \psi \)

Let \( \mathcal{Z} = \psi^{-1}(\{0\}) \cap [0, 1] \). We have assumed in A.2 that \( \mathcal{Z} \) is finite.

For \( k \geq 0 \) and \( x \in [0, 1] \), let \( x_k \) be the unique word \( w \in \Sigma_k \) such that

\[
\lambda(w) \leq x < \lambda(w) + 2^{-k},
\]

as well as \( 1_k = 1 \cdots 1 \) for \( k \geq 1 \). For \( k \geq 2 \), the set of forbidden words with respect to \( \mathcal{Z} \) is given by \( \mathcal{F}_k = \bigcup_{z \in \mathcal{Z}} \mathcal{F}_k(z) \), where

\[
\mathcal{F}_k(z) = \begin{cases} \{x_k\}, & \text{if } x \notin \lambda(\Sigma_k), \\ \{w \in \Sigma_k : 0 \leq \lambda(x_k) - \lambda(w) \leq 2^{-k}\}, & \text{otherwise}. 
\end{cases}
\]

The subshift of finite type \( X_k \) with the set of forbidden words \( \mathcal{F}_k \) is defined as

\[
X_k = \{ t \in \Sigma : a^m(t)|_k \notin \mathcal{F}_k, \ \forall \ m \geq 0 \}.
\]

Clearly for small \( k \), \( X_k \) might be an empty set. But, since \( \mathcal{Z} \) is a finite set, it is easy to see that \( X_k \) is not empty for \( k \) large enough. In fact, denote by \( \delta = \min\{|x-y| : x, y \in \mathcal{Z}, x \neq y\} > 0 \) and \( k_0 = \lceil -\log_2 \delta \rceil + 3 \). Then for any \( x, y \in \mathcal{Z} \) with \( x < y \), there exists at least one word \( w \in \Sigma_{k_0-1} \) such that \( x < \lambda(w) < y \), thus \( \lambda(x|_{k_0-1}) < \lambda(w) < \lambda(y|_{k_0-1}) \), since \( y - x \geq \delta \geq 2^{-(k_0-2)} \).

This ensures that for \( k \geq k_0 \), for all \( w \in \mathcal{F}_k \), his brother \( w' \) (the unique \( w' \in \Sigma_k \) such that \( w'|_{k-1} = w|_{k-1} \), \( w' \neq w \)) is allowed. Thus for any \( u \in \Sigma_{k-1} \), at least one of \( u0, u1 \) is allowed in \( X_k \), which also implies that for each \( u \in \Sigma_{k-1} \), there exists an infinite word \( t \in X_k \) such that \( t|_{k-1} = u0 \) or \( t|_{k-1} = u1 \). So for \( k \geq k_0 \), the Hausdorff distance between \( X_k \) and \( \Sigma \) is not greater than \( 2^{-k} \), that is

\[
\text{dist}_H(X_k, \Sigma) := \max \left\{ \sup_{s \in X_k} \inf_{t \in \Sigma} \rho(s, t), \sup_{s \in \Sigma} \inf_{t \in X_k} \rho(s, t) \right\} \leq 2^{-k},
\]

(20)

thus it converges to 0 when \( k \to \infty \).
Since $X_k$ is an increasing sequence (it is easy to see that $\Sigma \setminus X_k \supset \Sigma \setminus X_{k+1}$), $\dim B X_k$ increases and converges to 1 as $k \to \infty$. Otherwise we will get $\dim B \bigcup_k X_k < 1$, thus $\bigcup_k X_k$ is not dense in $\Sigma$, which is in contradiction with (20). Here $\dim B$ is the upper box-counting dimension (see [14] for definition and properties).

It is known that any subshift of finite type can be decomposed into several disjoint closed sets $X_{k,1}, \ldots, X_{k,m}, m \geq 1$, and each of them is a topologically transitive subshift of finite type. This can be deduced from the non-negative matrix analysis that one can always decompose a reducible matrix into several irreducible pieces.

The finite stability of $\dim B$ (see section 3.2 in [14]) implies
\[
\dim B X_k = \max_{i=1, \ldots, m} \dim B X_{k,i},
\]
so we can choose one of the $X_{k,i}$ such that $\dim B X_{k,i} = \dim B X_k$ and also denote it as $X_k$. Then we obtain a sequence of topologically transitive subshift of finite type $(X_k)_{k \geq 1}$ such that the upper box-counting dimension $\dim B X_k$ converges to 1. We prove that this sequence converge to $\Sigma$ in the Hausdorff distance:

Suppose that it is not the case, then there exist an $\epsilon > 0$ and a subsequence $(X_{k_j})_{j \geq 1}$ such that $\text{dist}_H(X_{k_j}, \Sigma) \geq \epsilon$ for $j \geq 1$. Fix an integer $N > -\log_2 \epsilon + 1$. Then $\text{dist}_H(X_{k_j}, \Sigma) \geq \epsilon$ implies that there exists a $w_j \in \Sigma_N$ such that $X_{k_j} \cap \{w_j\} = \emptyset$. Since $\#\Sigma_N = 2^N$ is finite, then there exist $w_s \in \Sigma_N$ and a subsequence $(X_{k_j})_{j \geq 1}$ of $(X_{k_j})_{j \geq 1}$ such that $X_{k_j} \cap \{w_s\} = \emptyset$ for $j \geq 1$.

Since $X_{k_j}$ is a subshift of finite type, $X_{k_j} \cap \{w_s\} = \emptyset$ implies that
\[
\lambda_{k_j} < X_s := \{ t \in \Sigma : \sigma^m(t) \neq w_s, \forall m \geq 0 \}.
\]
Denote by $B_s$ the transition matrix of $X_s$ and $\lambda_s$ the maximal eigenvalue of $B_s$. Due to the standard Perron–Frobenius theory ( [48], theorem 1.1), $\lambda_s$ is strictly less than the maximal eigenvalue of the transition matrix of the full shift, which is equal to 2. This yields that $\dim B X_s = \log \lambda_s / \log 2 < 1$, which is in contradiction with the fact that $\dim B X_k \geq \lim_{j \to \infty} \dim B X_{k_j} = 1$.

To end this section, since $\psi$ is $r_0$-smooth, for each $k \geq k_0$ we can easily find a constant $c_{\psi,k} > 0$ such that for each $t \in X_k$,
\[
|\psi(\lambda(\sigma^m(t)))| \leq c_{\psi,k}, \quad \forall m \geq 0, \quad n \geq k.
\]
This is the main property required in our proof, which clearly would not hold if we considered any $t \in \Sigma$.

### 3.4. Lower bound estimation

For $k \geq k_0$ and $q \in \mathbb{R}$, denote by $\mu^{(k)} q$ the Gibbs measure on $(X_k, \sigma)$ with potential $q \varphi$. Apply (7) to $\mu^{(k)} q$ and $\mu$, together with (10) we have for any $t \in X_k$ and $n \geq 1$,
\[
(C(\varphi)^{q} C(q\varphi))^{-1} \mu([I_{\varphi}])^q e^{-n \tau(q) \varphi} e^{-n \log 2^{-1}(P_{X_k}(q\varphi) - P_{X_k}(\varphi))} \leq \mu^{(k)} q(I_{\varphi})^q \mu([I_{\varphi}])^q e^{-n \tau(q) \varphi} e^{-n \log 2^{-1}(P_{X_k}(q\varphi) - P_{X_k}(\varphi))}.
\]
By using the large deviation method as in [2], it is standard to prove that for $\mu^{(k)} q$-almost every $t \in X_k$,
\[
\lim_{n \to \infty} \log \mu([I_{\varphi}]) / \log 2^{-n} = \lim_{n \to \infty} \log \max_{u \in N(I_{\varphi})} \mu([u]) / \log 2^{-n} = \tau'_q(q) + (\log 2)^{-1} \frac{d}{dq} (P_{X_k}(q\varphi) - P_{X_k}(\varphi)) := a^{(k)} q,
\]
\[\tag{23}\]
The graph and range singularity spectra of random wavelet series 1461

Theorem 3.1. With probability 1 for all \( q \in \mathbb{R} \) with \( 0 < h_q^{(k)} < 1 \), we have

\[
\dim_H(\mu_{q,G}^{(k)}) \geq \gamma_{q,G}^{(k)} := \frac{D_q^{(k)}}{h_q^{(k)}} \wedge \left( 1 - h_q^{(k)} + D_q^{(k)} \right),
\]

\[
\dim_H(\mu_{q,R}^{(k)}) \geq \gamma_{q,R}^{(k)} := \frac{D_q^{(k)}}{h_q^{(k)}} \wedge 1.
\]

Let us show how it makes it possible to conclude. For \( k \geq k_0 \) let \( I^{(k)} = \{ h_q^{(k)} : q \in \mathbb{R} \} \cap (0, 1) \) and \( J^{(k)} = \bigcap_{\beta > k} J^{(\beta)} \).

For \( S \in [G, R] \) define the function \( f_S^{(k)} : h_q^{(k)} \in I^{(k)} \mapsto \gamma_{q,S}^{(k)}. \) Due to (19) and (25), \( \mu_{q,S}^{(k)} \) is carried by \( S_{F_{\mu}^{\text{pen}}}(h_q^{(k)}) \). Thus theorem 3.1 implies that, for each \( k \geq k_0 \), with probability 1 for all \( h \in J^{(k)}, \)

\[
d_{F_{\mu}^{\text{pen}}}(h) \geq f_S^{(p)}(h), \quad \forall p \geq k.
\]

Then to end the proof of theorem 1.1, it only remains to show that for each \( q \in \mathbb{R}, \)

\[
h_q^{(k)} \to h_q = s_0 - 1/p_0 + \tau_p^{(q)}(q)/p_0 \quad \text{and} \quad D_q^{(k)} \to D_q = \tau_p^{(q)}(q), \quad \text{as} \ k \to \infty,
\]

which implies that \( \bigcup_{k \geq k_0} J^{(k)} = \{ s_0 - 1/p_0 + \tau_p^{(q)}(q)/p_0 : q \in \mathbb{R} \} \cap (0, 1) \) and for any compact subset \( I \subset \bigcup_{k \geq k_0} J^{(k)} \), the functions \( f_S^{(k)}, S \in [G, R] \) restricted to \( I \) converge uniformly to \( f_S : h_q \in I \mapsto \gamma_{q,S} \), where

\[
\gamma_{q,G} := \frac{D_q}{h_q} \wedge (1 - h_q + D_q) \quad \text{and} \quad \gamma_{q,R} := \frac{D_q}{h_q} \wedge 1.
\]

This implies that almost surely for all \( h \in I \) and \( \alpha = hp_0 + 1 - so p_0, \)

\[
d_{F_{\mu}^{\text{pen}}}(h) \geq \frac{\tau_p^{(\alpha)}}{h} \wedge (1 - h + \tau_p^{(\alpha)}) \quad \text{and} \quad d_{F_{\mu}^{\text{pen}}}(h) \geq \frac{\tau_p^{(\alpha)}}{h} \wedge 1.
\]
Together with section 3.2, we get the conclusion by taking a sequence of $I$ converging to $\{s_0 - 1/p_0 + t_0^{(q)}/p_0 : q \in \mathbb{R} \} \cap (0, 1)$.

Now we prove (26). This can be done due to (23), (24) and the following lemma.

**Lemma 3.1.** Given $q \in \mathbb{R}$, we have $\lim_{k \to \infty} P_{X_k}(q \varphi) = P_{\mathcal{C}}(q \varphi)$. Consequently, since these functions are convex and analytic, $P_{X_k}(q \varphi)$ and $\lim_{k \to \infty} P_{X_k}(q \varphi)$ converge uniformly on any compact interval to $P_{\mathcal{C}}(q \varphi)$ and $\lim_{k \to \infty} P_{X_k}(q \varphi)$, respectively.

**Proof.** The idea is borrowed from the proof of proposition 2 in [18].

Assume that this is not the case for some $q \in \mathbb{R}$. Since $P_{X_k}(q \varphi) \leq P_{\mathcal{C}}(q \varphi)$, let $P_{\varphi}(q \varphi) = \liminf_{k \to \infty} P_{X_k}(q \varphi)$ and $\delta = P_{\mathcal{C}}(q \varphi) - P_{\varphi}(q \varphi) > 0$.

Take a subsequence $\{\mu_q^{(k)}\}_{k \geq 1}$ converging to some probability measure $\mu^*_q$ in the weak*-topology. From section 2.2, for any $t \in \Sigma$ and $n \geq 1$ we have

$$(C(q \varphi)|C(q \varphi))^{-1} \leq \frac{\mu_q(|I|_n)}{\text{exp}(S_n q \varphi(t) - n P_{\mathcal{C}}(q \varphi))} \leq (C(q \varphi)|C(q \varphi)).$$

Since $X_k$ converges to $\Sigma$ in sense of Hausdorff distance, then for all $k$ large enough, we have $X_k \cap |I|_n \neq \emptyset$, thus

$$\mu_q(|I|_n) \leq (C(q \varphi)|C(q \varphi))^{-(C(q \varphi)|C(q \varphi))} \cdot e^{-n\delta/2}.$$

Taking $n$ large enough so that $(C(q \varphi)|C(q \varphi))^{-(C(q \varphi)|C(q \varphi))} \cdot e^{-n\delta/2} < 1$, this is in contradiction with the fact that both $\mu_q$ and $\mu^*_q$ are probability measures.

4. Proof of theorem 3.1

4.1. Main proof

**Proof.** From now on we fix a $k \geq k_0$, so that $X_k \neq \emptyset$. Recall that the set of neighbour words of $w \in \Sigma$, is given by

$$\mathcal{N}(w) = \left\{ u \in \Sigma_{|w|} : |\lambda(u) - \lambda(w)| \leq 2^{-|w|} \right\}.$$ 

For $p \geq 1$ let $\mathcal{P}_p$ be the subset of pairs of elements of $\Sigma_{p+1}$ defined as

$$\mathcal{P}_p = \{(u, v) \in \Sigma_{p+1} \times \Sigma_{p+1} : v \in \mathcal{N}(u | p), v \notin \mathcal{N}(u)\}.$$ (27)

Then for any $s, t \in \Sigma$ with $|s - t| > 0$, there exists a unique $p \geq 1$ such that $I_p(s, t) = 1$, where the indicator function is defined by

$$I_p(s, t) = \begin{cases} 1, & \text{if } (s | p+1, t | p+1) \in \mathcal{P}_p, \\ 0, & \text{otherwise}. \end{cases}$$

By construction we know that if $I_p(s, t) = 1$, then

$$\inf_{x' \in |r|_{p+1}, r \in |t|_{p+1}} |x' - t'| \geq 2^{-p-1} \quad \text{and} \quad \sup_{x' \in |s|_{p+1}, r \in |t|_{p+1}} |x' - t'| \leq 2^{-p+1}.$$ 

Recall for any $w \in \Sigma$, the mapping

$$T_w : x \in \mathbb{R} \mapsto 2^{-|w|} \cdot x + \lambda(w).$$
Then for any \( s,t \in \Sigma \) with \( I_p(s,t) = 1 \), and for any \( m \geq 1 \), we have
\[
T_{|p|+m}^{-1}(\lambda(s)) \in [0,1] \text{ and } \left| T_{|p|+m}^{-1}(\lambda(s)) - T_{|p|+m}^{-1}(\lambda(t)) \right| \geq 2^m.
\]
Since \( \psi \) decays at infinity, due to (21), there exists a large enough \( N_{\psi,k} \) such that for any \( s,t \in X_k \) with \( I_p(s,t) = 1 \), for any \( m \geq 1 \) and \( n \geq k \)
\[
\left| \gamma(s,t) \right|^n \geq \frac{c_{\psi,k}}{2}.
\]
Recall (23) and \( h_q = s_0 - 1/p_0 + a_q^k / p_0 \). Let \( I_k = \{ q \in \mathbb{R} : 0 < h_q < 1 \} \).
For any \( q \in I_k \), \( \epsilon > 0 \) and \( w \in \Sigma_k \), define the following three indicator functions:
\[
I_k^{(a)}(q,\epsilon) = 1 \left\{ \left| \left| \lambda(s) \right| - \left| \lambda(t) \right| \right| < 2^{-1/2 - \epsilon} \right\},
\]
\[
I_k^{(b)}(q,\epsilon) = 1 \left\{ \left| \left| \lambda(s) \right| - \left| \lambda(t) \right| \right| < 2^{-1/2} \right\},
\]
\[
I_k^{(c)}(q,\epsilon) = 1 \left\{ \left| \left| \lambda(s) \right| - \left| \lambda(t) \right| \right| < 2^{-1/2 - \epsilon} \right\}.
\]
For \( n \geq 1 \) define
\[
\Sigma_k^n(q,\epsilon) = \{ w \in \Sigma_k : [w] \cap X_k \neq \emptyset \text{ and } I_k^{(a)}(q,\epsilon) \cdot I_k^{(b)}(q,\epsilon) \cdot I_k^{(c)}(q,\epsilon) = 1 \}.
\]
(In fact, \( I_k^{(b)}(q,\epsilon) = 1 \) implies \( [w] \cap X_k \neq \emptyset \).) Then let
\[
E_k^n(q,\epsilon) = \bigcap_{p \geq 1} \bigcup_{n \in \Sigma_k^n(q,\epsilon)} [w] \quad \text{and} \quad E_k(q) = \lim_{\epsilon \to 0^+} \lim_{n \to \infty} E_k^n(q,\epsilon).
\]
Due to (18), (23) and (24), we have almost surely for all \( q \in I_k \),
\[
\lambda(E_k(q)) \subset \lambda(X_k) \cap E_{1} = (h_q^k) \quad \text{and} \quad \mu_q^k(E_k(q)) = 1.
\]
For \( \gamma > 0 \) define the Riesz-like kernel: for \( s,t \in \Sigma_k \cup \Sigma \),
\[
K_k^\gamma(s,t) = \begin{cases} \left| \lambda(s) - \lambda(t) \right|^\gamma \wedge 1, & \text{if } \gamma \geq 1, \\ \left| \lambda(s) - \lambda(t) \right|^\gamma \vee 1, & \text{if } \gamma < 1. \end{cases}
\]
For \( q \in I_k \) recall that
\[
\gamma_{q,k}^{(k)}(D_q^k) = \frac{D_q^k}{h_q^k} \wedge \left( D_q^k + 1 - h_q^k \right) \quad \text{and} \quad \gamma_{q,k}^{(k)} = \frac{D_q^k}{h_q^k} \wedge 1.
\]
For \( q \in I_k \), \( \delta > 0 \) and \( \epsilon > 0 \) we define the \( n \)th energy for \( n \geq 1 \) and \( S \in \{ G, R \} \):
\[
I_{n,\delta}(q,\epsilon) = \int_{t \in E_{n}^{G}(q,\epsilon), s \notin q} K_{\gamma_{q,k}^{(k)}}^{(k)}(s,t) \, d\mu_{q,k}^{(k)}(s) \, d\mu_{q,k}^{(k)}(t).
\]
Let \( K \) be any compact subinterval of \( J_k \). We assume for a while that we have proved that for any \( \delta \) small enough, there exists \( \epsilon_0 > 0 \) such that for any \( n \geq 1 \), \( \epsilon \in (0,\epsilon_0) \) and \( S \in \{ G, R \} \),
\[
\mathbb{E} \left( \sup_{q \in K} I_{n,\delta}(q,\epsilon) \right) < \infty.
\]
The following lemma is a slight modification of theorem 4.13 in [14] regarding the Hausdorff dimension estimate through the potential theoretic method.
Lemma 4.1. Let \( \nu \) be a Borel measure on \( \mathbb{R}^d \) and let \( E \subset \mathbb{R}^d \) be a Borel set such that \( \nu(E) > 0 \). For any \( \gamma > 0 \), if

\[
\int \int_{x,y \in E, x \neq y} |x - y|^{-\gamma} \vee 1 \, d\nu(x)d\nu(y) < \infty,
\]

then

\[
\nu \left( \left\{ x \in E : \dim_{\text{loc}} \nu(x) = \liminf_{r \to 0^+} \frac{\log \nu(B(x,r))}{\log r} > \gamma \right\} \right) = 0.
\]

Then, it easily follows from (35) and lemma 4.1 that, almost surely for all \( q \in K \):

- For \( \mu^{(k)}_{q,G} \)-almost every \( x \in \left\{ \lambda(t), F^{\text{pert}}_\mu(\lambda(t)) : t \in E^{(k)}(q) \right\} \),

\[
\dim_{\text{loc}} \mu^{(k)}_{q,G}(x) \geq \gamma_{q,G}^{(k)} - \delta.
\]

- For \( \mu^{(k)}_{q,R} \)-almost every \( y \in \left\{ F^{\text{pert}}_\mu(\lambda(t)) : t \in E^{(k)}(q) \right\} \),

\[
\dim_{\text{loc}} \mu^{(k)}_{q,R}(y) \geq \gamma_{q,R}^{(k)} - \delta.
\]

Since \( \delta \) can be taken arbitrarily small, we get the conclusion by taking a countable sequence of compact subintervals \( K_j \subset J_k \) such that \( \bigcup K_j = J_k \).

Now we prove (35).

For any \( \bar{q} \in K \) and \( \epsilon > 0 \) we define the neighbourhood of \( \bar{q} \) in \( K \):

\[
U_\epsilon(\bar{q}) = \left\{ q \in K : \max \left\{ |q - \bar{q}|, |a^{(k)}_{q} - a^{(k)}_{\bar{q}}|, |b^{(k)}_{q} - b^{(k)}_{\bar{q}}|, |D^{(k)}_{q} - D^{(k)}_{\bar{q}}|, |\gamma^{(k)}_{q,G} - \gamma^{(k)}_{\bar{q},G}|, |\gamma^{(k)}_{q,R} - \gamma^{(k)}_{\bar{q},R}| \right\} < \epsilon \right\}.
\]  

(36)

By continuity of these functions, the set \( U_\epsilon(\bar{q}) \) is open in \( K \).

Note that for \( q \in K, \delta > 0 \) and \( S \in \{ G, R \} \) the Riesz-like kernels \( K_{\gamma^{(k)}_{q,S},\delta - \delta} \) are positive functions and, moreover, by the continuity of \( F^{\text{pert}}_\mu \) we have for any \( s, t \in \Sigma \),

\[
\lim_{m \to \infty} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(s_{m}, t_{m}) = K_{\gamma^{(k)}_{q,S},\delta}(s, t).
\]

Then by applying Fatou’s lemma we get for any \( q \in U_\epsilon(\bar{q}) \),

\[
I_{n,\delta}^{(k)}(q, \epsilon) = \int \int_{s,t \in E^{(k)}(q, \epsilon)} \lim_{m \to \infty} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(s_{m}, t_{m}) \, d\mu^{(k)}_{q}(s) \, d\mu^{(k)}_{q}(t)
\]

\[
= \sum_{p \geq 1} \sum_{s,t \in E^{(k)}(q, \epsilon)} \lim_{m \to \infty} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(s_{m}, t_{m}) \, d\mu^{(k)}_{q}(s) \, d\mu^{(k)}_{q}(t)
\]

\[
\leq \sum_{p \geq 1} \liminf_{m \to \infty} \int \int_{s,t \in E^{(k)}(q, \epsilon)} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(s_{m}, t_{m}) \, d\mu^{(k)}_{q}(s) \, d\mu^{(k)}_{q}(t)
\]

\[
= \sum_{p \geq 1} \liminf_{m \to \infty} \sum_{u,v \in E^{(k)}(q, \epsilon)} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(u, v) \cdot \mu^{(k)}_{q}(u \cap E^{(k)}_{n}(q, \epsilon)) \mu^{(k)}_{q}(v \cap E^{(k)}_{n}(q, \epsilon))
\]

\[
\leq \sum_{p \geq 1} \liminf_{m \to \infty} \sum_{u,v \in E^{(k)}(q, \epsilon)} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(u, v) \cdot \mu^{(k)}_{q}(u \cap E^{(k)}_{n}(q, \epsilon)) \mu^{(k)}_{q}(v \cap E^{(k)}_{n}(q, \epsilon))
\]

\[
\leq \sum_{p \geq 1} \liminf_{m \to \infty} \sum_{u,v \in E^{(k)}(q, \epsilon)} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(u, v) \cdot \mu^{(k)}_{q}(u \cap E^{(k)}_{n}(q, \epsilon)) \mu^{(k)}_{q}(v \cap E^{(k)}_{n}(q, \epsilon))
\]

\[
\leq \sum_{p \geq 1} \liminf_{m \to \infty} \sum_{u,v \in E^{(k)}(q, \epsilon)} K_{\gamma^{(k)}_{q,S},\delta}^{(k)}(u, v) \cdot \mu^{(k)}_{q}(u \cap E^{(k)}_{n}(q, \epsilon)) \mu^{(k)}_{q}(v \cap E^{(k)}_{n}(q, \epsilon)),
\]
where the last inequality comes from the fact that due to (33) and (36), for any \( \bar{q} \in K, \epsilon > 0 \) and \( u, v \in \Sigma_s \), we have \( \forall \bar{q} \in U_\epsilon(q) \), \( \mathcal{K}_{\bar{q},s,q}^{\alpha,\beta}(u, v) \leq \mathcal{K}_{\bar{q},s,q}^{\alpha,\beta}(u, v) \). Let

\[
A_{p,m} = \sum_{\bar{q} \in U_\epsilon(q)} \mathcal{K}_{\bar{q},s,q}^{\alpha,\beta}(u, v) \cdot \mu_q(k) ([u] \cap E_n^k(q, \epsilon)) \mu_q(k) ([v] \cap E_n^k(q, \epsilon)).
\]

Then,

\[
\sup_{q \in U_\epsilon(q)} J^S_{n,q}(q, \epsilon) \leq \sup_{q \in U_\epsilon(q)} \sum_{p \geq 1} \liminf_{m \to \infty} A_{p,m} \\
\leq \sup_{q \in U_\epsilon(q)} \left( \sum_{p \geq 1} \left( A_{p,m_p} + \sum_{m \geq m_p} |A_{p,m+1} - A_{p,m}| \right) \right) \\
\leq \sum_{p \geq 1} \left( \sup_{q \in U_\epsilon(q)} A_{p,m_p} + \sum_{m \geq m_p} \sup_{q \in U_\epsilon(q)} |A_{p,m+1} - A_{p,m}| \right),
\]

(37)

where for \( p \geq 1 \), we can choose \( m_p \geq 2 \) to be any integer. We have

\[
\sup_{q \in U_\epsilon(q)} A_{p,m} \leq B_{p,m} \quad \text{and} \quad \sup_{q \in U_\epsilon(q)} |A_{p,m+1} - A_{p,m}| \leq \Delta B_{p,m},
\]

(38)

where

\[
B_{p,m} = \sum_{\bar{q} \in U_\epsilon(q)} \mathcal{K}_{\bar{q},s,q}^{\alpha,\beta}(u, v) \sup_{q \in U_\epsilon(q)} \mu_q(k) ([u] \cap E_n^k(q, \epsilon)) \mu_q(k) ([v] \cap E_n^k(q, \epsilon)),
\]

\[
\Delta B_{p,m} = \sum_{\bar{q} \in U_\epsilon(q)} \left| \mathcal{K}_{\bar{q},s,q}^{\alpha,\beta}(uu', vv') - \mathcal{K}_{\bar{q},s,q}^{\alpha,\beta}(u, v) \right| \\
\cdot \sup_{q \in U_\epsilon(q)} \mu_q(k) ([uu'] \cap E_n^k(q, \epsilon)) \mu_q(k) ([vv'] \cap E_n^k(q, \epsilon)),
\]

and we have used the equality \( \mu_q(k) ([u] \cap E_n^k(q, \epsilon)) = \sum_{u' \in [0,1]} \mu_q(k) ([uu'] \cap E_n^k(q, \epsilon)) \) to get the second inequality.

**Remark 4.1.** For technical reasons, we need to divide \( J_k \) into two parts, in which \( K \) will be chosen:

\[
J'_k = \{ q \in J_k : \gamma_{q,G}^{(k)} \geq 1 \} \quad \text{and} \quad J''_k = \{ q \in J_k : \gamma_{q,G}^{(k)} \leq 1 \}.
\]

Then, due to (34), we have

\[
\gamma_{q,G}^{(k)} = \begin{cases} D_q^{(k)} + 1 & \text{if } q \in J'_k, \\
D_q^{(k)}/h_q^{(k)} & \text{if } q \in J''_k, \end{cases} \quad \text{and} \quad \gamma_{q,R}^{(k)} = \begin{cases} 1 & \text{if } q \in J'_k, \\
D_q^{(k)}/h_q^{(k)} & \text{if } q \in J''_k. \end{cases}
\]

For any compact subinterval \( K \) of \( J_k \) there exists \( c_K > 0 \) such that for any \( \epsilon < c_K \) and \( q \in K, \gamma_{q,G}^{(k)} - \epsilon > 0 \) if \( K \subset J'_k \) and \( \gamma_{q,R}^{(k)} - \epsilon > 0 \) if \( K \subset J''_k \).

Let \( \delta_k = \epsilon_k = c_k/2 \). We have the following key proposition.

**Proposition 4.1.** Let \( S \in [G, R] \). Suppose that \( K \) is a compact subinterval of \( J'_k \) or \( J''_k \). For any \( 0 < \delta < \delta_k \) we can find constants \( c_1, c_2 > 0, k_1, k_2, \eta_1, \eta_2 > 0 \) and \( \epsilon_0 > 0 \) such that for any \( \bar{q} \in K, 0 < \epsilon < \epsilon_0, n \geq 1, p \geq 1 \) and \( m \geq (p \lor n) + 1 + N_{q,k} + k \),

\[
\mathbb{E} \left( B_{p,m} \right) \leq c_1 \cdot C_{p,n} \cdot 2^n \cdot 2^{\gamma_{q,G}^{(k)}} - p \cdot 2^{-n_1} - \eta_1 - k, \\
\mathbb{E} \left( \Delta B_{p,m} \right) \leq c_1 \cdot C_{p,n} \cdot 2^n \cdot 2^{\gamma_{q,R}^{(k)}} - p \cdot 2^{-n_1} - \eta_1 - m,
\]

where \( C_{p,n} = \sup_{q \in \Sigma_{p+n+1,k}} ||f_q||_\infty \); here \( f_q \) is just the formal replacement of the bounded density function \( f_{j,k} \) given in A3.
Now fix any \( n \geq 1 \), and choose \( m_p = \frac{\epsilon_2 + \frac{1}{2} \delta_1}{\eta_2} \cdot (p \lor n) \) (by slightly modifying \( \eta_2 \) we can always assume that \( \frac{\epsilon_2 + \frac{1}{2} \delta_1}{\eta_2} \cdot (p \lor n) > (p \lor n) + 1 + N_{\varphi,k} + k \) and \( \epsilon_2 = \epsilon_2 \wedge \frac{1}{2} \delta_1 \)). Then by using Proposition 4.1 and (37), (38), for any \( \delta < \delta_1 \), \( \bar{q} \in K \), \( \epsilon < \epsilon_2 \) and \( S \in \{G, R\} \) we have

\[
E \left( \sup_{q \in U_i(q)} T_{n,q}^S(q, \epsilon) \right) \\
\leq \sum_{p \geq 1} \left( E \left( B_{p,m} \right) + \sum_{m \geq m_p} E \left( \Delta B_{p,m} \right) \right) \\
\leq \sum_{p \geq 1} c_1 \cdot C_{p,n} \cdot 2^{c_2 \cdot \left( (p \lor n) - p \right)} \cdot \left( 2^n \cdot 2^{-\eta_1 \delta_p \epsilon \cdot \epsilon_1 m_p} \sum_{m \geq m_p} 2^{c_2 \cdot p - \eta_2 m_p} \right) \\
= c_1 \cdot \sum_{p \geq 1} C_{p,n} \cdot 2^{c_2 \cdot \left( (p \lor n) - p \right)} \cdot 2^n \cdot 2^{-\eta_1 \delta_p \epsilon \cdot \epsilon_1 m_p} + 2^{c_2 \cdot p - \eta_2 m_p} \cdot \frac{1}{1 - 2^{-\eta_2}} \\
\leq \frac{c_1}{1 - 2^{-\eta_2}} \sum_{p \geq 1} C_{p,n} \cdot 2^{c_2 \cdot \left( (p \lor n) - p \right)} \cdot 2^n \cdot 2^{-\frac{1}{2} \eta_1 \delta_p \epsilon_1 m_p} \\
= \frac{2^{n+1} c_1}{1 - 2^{-\eta_2}} \left( \left( \sup_{u \in \Sigma_{\varphi,w} \eta_2} \| f_u \|_{\infty} \right) \cdot \sum_{p=1}^{n} 2^{c_2 \cdot \frac{1}{\eta_2} \delta_p \epsilon_1 (n-p)} \cdot 2^{-\frac{1}{2} \eta_1 \delta_p \epsilon_1 m_p} \right) \\
+ 2^{\frac{1}{2} \eta_1 \delta_1 (4+n_{\varphi,k})} \cdot \sum_{p=n+1}^{\infty} \left( \sup_{u \in \Sigma_{\varphi,w} \eta_2} \| f_u \|_{\infty} \right) \cdot 2^{-\frac{1}{2} \eta_1 \delta_p \epsilon_1 (p+4+n_{\varphi,k})} \right) < \infty,
\]

where the finiteness is ensured by assumption A3. Since for any \( 0 < \epsilon < \epsilon_2 \), the family \( \{U_i(q)\}_{q \in K} \) forms an open covering of \( K \), there exist \( q_1, \ldots, q_N \) such that \( \{U_i(q_i)\}_{1 \leq i \leq N} \) also covers \( K \). This gives us the conclusion.

4.2. Proof of Proposition 4.1

**Proof.** Due to (36) we always have \( \bigcup_{q \in U_i(q)} E_n^{(k)}(q, \epsilon) \subset E_n^{(k)}(\bar{q}, 2\epsilon) \). Then due to (30) we get

\[
\sup_{q \in U_i(q)} \mu_q^{(k)}(\{u \mid E_n^{(k)}(q, \epsilon)\}) \cdot \mu_q^{(k)}(\{v \mid E_n^{(k)}(q, \epsilon)\}) \\
\leq \sup_{q \in U_i(q)} \left[ \mu_q^{(k)}(\{u \mid E_n^{(k)}(q, \epsilon)\}) \cdot \mu_q^{(k)}(\{v \mid E_n^{(k)}(q, \epsilon)\}) \right] \\
\leq 1 \left[ \mu_q^{(k)}(\{u \mid E_n^{(k)}(\bar{q}, 2\epsilon)\}) \cdot \mu_q^{(k)}(\{v \mid E_n^{(k)}(\bar{q}, 2\epsilon)\}) \right] \cdot 2^{-(\frac{m(D_u^{(k)}))}{\lambda} - 2\epsilon}. \\
\]

This gives us

\[
B_{p,m} \leq 2^{-2m(D_u^{(k)}))} \cdot \sum_{\mu \in \Sigma_{\varphi,w} \eta_2} K_{\eta_2}^{(k)}(\mu) \cdot 1_{\{u \mid E_n^{(k)}(\bar{q}, 2\epsilon)\}} \cdot 1_{\{v \mid E_n^{(k)}(\bar{q}, 2\epsilon)\}}. 
\]
\[
\Delta B_{p,m} \leq 2^{-2(m+1)D_{p}^{*} - 2\epsilon},
\]
\[
\sum_{u,v \in \Sigma_{n}, u'v' \in [0,1], 1_p(u,v) = 1} \left| K_{\gamma,\delta_k^{-\delta - \epsilon}}(uu', vv') - K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \right|
\cdot \mathbf{1}_{\left\{ uu' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} \mathbf{1}_{\left\{ vv' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}}.
\] (40)

Now we deal with each term of the above sums individually.

Fix \( p \) and \( n \) in \( \mathbb{N}^* \), let \( r = p \lor n \), and fix \( m \geq r + 1 + N_{\gamma, \delta_k} + k \).

Fix a pair \( u, v \in \Sigma_{m} \) with \( 1_p(u,v) = 1 \), so \( (u|_{p+1}, v|_{p+1}) \in P_{p} \).

Let
\[
\begin{align*}
V :&= K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \cdot \mathbf{1}_{\left\{ uu' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} \mathbf{1}_{\left\{ vv' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} , \\
\Delta V :&= K_{\gamma,\delta_k^{-\delta - \epsilon}}(uu', vv') - K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \cdot \mathbf{1}_{\left\{ uu' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} \mathbf{1}_{\left\{ vv' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}}.
\end{align*}
\]

Due to (29), (30) and (31), if \([ u'] \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \), then for \( l = r, \ldots , m \) we have
\[
\mathbf{1}^{(a)}_{u|_{l}}(\tilde{q},2\epsilon) \cdot \mathbf{1}^{(b)}_{u|_{l}}(\tilde{q},2\epsilon) \cdot \mathbf{1}^{(c)}_{u|_{l}}(\tilde{q},2\epsilon) = 1.
\]

Define
\[
\begin{align*}
\mathbf{1}^{\text{ran}}_{u, v}(\tilde{q}, \epsilon) &= \mathbf{1}^{(a)}_{u|_{l}}(\tilde{q}, 2\epsilon) \cdot \mathbf{1}^{(c)}_{u|_{l}}(\tilde{q}, 2\epsilon) \cdot \mathbf{1}^{(c)}_{u|_{l}}(\tilde{q}, 2\epsilon) \cdot \mathbf{1}^{(c)}_{u|_{l}}(\tilde{q}, 2\epsilon), \\
\mathbf{1}^{\text{det}}_{u, v}(\tilde{q}, \epsilon) &= \mathbf{1}^{(a)}_{u|_{l}}(\tilde{q}, 2\epsilon) \cdot \mathbf{1}^{(b)}_{u|_{l}}(\tilde{q}, 2\epsilon) \cdot \mathbf{1}^{(b)}_{u|_{l}}(\tilde{q}, 2\epsilon) \cdot \mathbf{1}^{(b)}_{u|_{l}}(\tilde{q}, 2\epsilon),
\end{align*}
\] (41)

where ‘ran’ stands for random and ‘det’ stands for deterministic.

Since \([ u'] \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \) implies \([ u'] \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \), we have
\[
\begin{align*}
\mathbf{1}_{\left\{ uu' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} \mathbf{1}_{\left\{ vv' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} &\leq \mathbf{1}_{\left\{ uu' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} \mathbf{1}_{\left\{ vv' \cap E_{\mu}^{(\delta_k)}(\tilde{q},2\epsilon) \neq \emptyset \right\}} \leq \mathbf{1}^{\text{ran}}_{u, v}(\tilde{q}, \epsilon) \cdot \mathbf{1}^{\text{det}}_{u, v}(\tilde{q}, \epsilon).
\end{align*}
\]

This implies
\[
\begin{align*}
V \leq K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \cdot \mathbf{1}^{\text{det}}_{u|_{l}}(\tilde{q}, \epsilon) \quad \text{and} \quad \Delta V \leq \Delta K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \cdot \mathbf{1}^{\text{det}}_{u|_{l}}(\tilde{q}, \epsilon),
\end{align*}
\]

where
\[
\begin{align*}
K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) &= K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \cdot \mathbf{1}^{\text{ran}}_{u|_{l}}(\tilde{q}, \epsilon), \\
\Delta K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) &= K_{\gamma,\delta_k^{-\delta - \epsilon}}(uu', vv') - K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \cdot \mathbf{1}^{\text{ran}}_{u|_{l}}(\tilde{q}, \epsilon).
\end{align*}
\] (43)

Since \( \mathbf{1}^{\text{det}}_{u|_{l}}(\tilde{q}, \epsilon) \) is deterministic, we have
\[
\mathbb{E}(V) \leq \mathbb{E}\left( K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \right) \cdot \mathbf{1}^{\text{det}}_{u|_{l}}(\tilde{q}, \epsilon) \quad \text{and} \quad \mathbb{E}(\Delta V) \leq \mathbb{E}\left( \Delta K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) \right) \cdot \mathbf{1}^{\text{det}}_{u|_{l}}(\tilde{q}, \epsilon).
\]

Recall that in remark 4.1 we distinguished the cases \( K \subset J_{\mu}^{r} \) and \( K \subset J_{\mu}^{d} \) according to whether or not the corresponding power on the kernel is greater than 1. Then, due to (33), once we have taken \( \delta < \delta_k \) and \( \epsilon < \epsilon_k \), only two situations are left:
\[
K_{\gamma,\delta_k^{-\delta - \epsilon}}(u,v) = \begin{cases} 
\left( \lvert F^{\text{pert}}_{\mu}(\lambda(u)) - F^{\text{pert}}_{\mu}(\lambda(v)) \rvert \right)^2 + \lvert \lambda(u) - \lambda(v) \rvert^2 \lor 1, & \text{if } \gamma > 1, \\
\lvert F^{\text{pert}}_{\mu}(\lambda(u)) - F^{\text{pert}}_{\mu}(\lambda(v)) \rvert^2 \lor 1, & \text{if } \gamma < 1,
\end{cases}
\]

where \( \gamma = \gamma_{\delta_k}^{-\delta - \delta - \epsilon} \). Note that when we take \( K \) a compact subinterval of \( J_{\mu}^{r} \) or \( J_{\mu}^{d} \), \( \gamma \) could never be equal to 1.
Recall that \( C_r = \sup_{w \in \Sigma_{u,v,N}} \| f_w \|_\infty \), where \( f_w \) is the bounded density function of \( \pi_w \) given in A3. We have the following two lemmas.

**Lemma 4.2.** There exists a constant \( c_\gamma > 0 \) such that

\[
\mathbb{E}(\mathcal{L}_\gamma(u, v)) \leq c_\gamma \cdot C_r \cdot \begin{cases} 
2^{r(h_q^{(1)}+2\varepsilon)-p(1-\gamma)}, & \text{if } \gamma > 1, \\
2^n, & \text{if } \gamma < 1,
\end{cases}
\]

\[
\mathbb{E}(\Delta \mathcal{L}_\gamma(u, v)) \leq c_\gamma \cdot C_r \cdot \begin{cases} 
2^{p(3-m)(h_q^{(1)}-2\varepsilon)}, & \text{if } \gamma > 1, \\
2^{2-m}(h_q^{(1)}-2\varepsilon), & \text{if } \gamma < 1.
\end{cases}
\]

**Lemma 4.3.** One has

\[
\sum_{u,v \in \Sigma_{u,v}} 1_{p(u,v)} \cdot 1_{\det(N\psi,k)}(\tilde{q}, \epsilon) \leq 3^{t-p+1} \cdot 2^{2m(D_q^{(1)}+2\varepsilon)-r(D_q^{(1)}-2\varepsilon)}.
\]

Now, due to remark 4.1, we have the following three expressions of \( \gamma \):

\[
\begin{align*}
\gamma &= D_q^{(1)} + 1 - h_q^{(k)} - \delta - \epsilon > 1, & h_q^{(k)} < D_q^{(1)}, & \text{case(i)}, \\
\gamma &= D_q^{(1)} / h_q^{(k)} - \delta - \epsilon < 1, & h_q^{(k)} > D_q^{(1)}, & \text{case(ii)}, \\
\gamma &= 1 - \delta - \epsilon < 1, & h_q^{(k)} < D_q^{(1)}, & \text{case(iii)}.
\end{align*}
\]

Then, due to (39), (40), lemma 4.2 and Lemma 4.3, we have

\[
\mathbb{E}(B_{p,m}) \leq c_\gamma \cdot C_r \cdot 3^{t-p+1} \cdot \begin{cases} 
2^{2(r(h_q^{(1)}+2\varepsilon))-p(h_q^{(1)}-D_q^{(1)}+6\epsilon+6)}, & \text{if} \\
2^n, & \text{if} \\
2^{2(r(h_q^{(1)}-h_q^{(k)}(\delta+\epsilon)+4\epsilon)+4\epsilon)}, & \text{if} \\
2^n, & \text{if} \\
2^{2(r(h_q^{(1)}-h_q^{(k)}(\delta+\epsilon)+4\epsilon)+4\epsilon)}, & \text{if} \\
2^n, & \text{if} \\
2^{2(r(h_q^{(1)}-h_q^{(k)}(\delta+\epsilon)+4\epsilon)+4\epsilon)}, & \text{if} \\
2^n, & \text{if} \\
2^{2(r(h_q^{(1)}-h_q^{(k)}(\delta+\epsilon)+4\epsilon)+4\epsilon)}, & \text{if}
\end{cases}
\]

\[
\leq c_\gamma \cdot C_r \cdot 3 \cdot 2^{r-p(\log_2 3+2)} \cdot 2^n \cdot \begin{cases} 
2^{2(r-h_q^{(1)}-3\epsilon)+8\epsilon m}, & \text{if} \\
2^{2(r-h_q^{(1)}-3\epsilon)+8\epsilon m}, & \text{if} \\
2^{2(r-h_q^{(1)}-3\epsilon)+8\epsilon m}, & \text{if}
\end{cases}
\]

The upper bound of \( \mathbb{E}(\Delta B_{p,m}) \) is simpler, in all cases we have

\[
\mathbb{E}(\Delta B_{p,m}) \leq c_\gamma \cdot C_r \cdot 3^{t-p+1} \cdot 4 \cdot 2^{2m(D_q^{(1)}+2\varepsilon)-r(D_q^{(1)}-2\varepsilon)} 
\begin{cases} 
2^{2(r-p)+3-m(h_q^{(1)}-2\varepsilon)}, & \text{if} \\
2^{2(r-p)+3-m(h_q^{(1)}-2\varepsilon)}, & \text{if}
\end{cases}
\]

\[
\leq c_\gamma \cdot C_r \cdot 3 \cdot 2^{r-p(\log_2 3+3)} \cdot 2^{2p-3-m(h_q^{(1)}-10\epsilon)}.
\]

Note that by construction we always have \( h_q^{(k)} \geq s_0 - 1/p_0 > 0 \), then the existences of the parameters \( c_1, c_2 > 0, \kappa_1, \kappa_2, \eta_1, \eta_2 > 0 \) and \( \epsilon_* > 0 \) are direct consequences of what we have obtained.
4.3. Proofs of lemmas 4.2 and 4.3

4.3.1. Proof of lemma 4.2.

Proof. Let \( l = r + 1 + N_{\psi,k} \). Due to (15), we have
\[
F^\text{pert}_\mu(\lambda(u)) - F^\text{pert}_\mu(\lambda(v)) = \sum_{w \in \Sigma_u} \pi_w \cdot d_w \cdot (\psi_w(\lambda(u)) - \psi_w(\lambda(v)))
\]
where
\[
A = d_{a_{\psi}} \cdot (\psi_{a_{\psi}}(\lambda(u)) - \psi_{a_{\psi}}(\lambda(v))).
\]
\[
B = \sum_{w \in \Sigma_u - \{a_{\psi}\}} \pi_w \cdot d_w \cdot (\psi_w(\lambda(u)) - \psi_w(\lambda(v))).
\]

By construction \( A \) is deterministic, and \( \pi_{a_{\psi}} \) and \( B \) are independent.

Since when \( \mathbf{1}^{(b)}(q, 2\epsilon) = 1 \) we have \( \mu(k)(\{u\}) \neq 0 \), then (28) and the fact that \( |u| = m \geq l + k \) yield
\[
\left| \psi_{a_{\psi}}(\lambda(u)) - \psi_{a_{\psi}}(\lambda(v)) \right| \geq \frac{c_{\psi,k}}{2}. \tag{46}
\]

For \( u', v' \in [0, 1) \) we can write
\[
F^\text{pert}_\mu(\lambda(uu')) - F^\text{pert}_\mu(\lambda(vv')) \]
\[
= \eta \cdot \left( F^\text{pert}_\mu(\lambda(u)) - F^\text{pert}_\mu(\lambda(v)) \right) + D = \eta \cdot (\pi_{a_{\psi}} \cdot A + B) + D, \tag{47}
\]
where
\[
\eta = \psi_{a_{\psi}}(\lambda(uu')) - \psi_{a_{\psi}}(\lambda(vv')) \]
\[
D = \sum_{w \in \Sigma_u - \{a_{\psi}\}} \pi_w \cdot d_w \cdot \left( \psi_w(\lambda(uu')) - \psi_w(\lambda(vv')) - \eta \cdot (\psi_w(\lambda(u)) - \psi_w(\lambda(v))) \right).
\]

We have that \( \eta \) is deterministic, and \( \pi_{a_{\psi}} \) and \( D \) are independent. Moreover, since \( \psi \) is \( r_0 \)-smooth, there exists a constant \( C_{\psi} \) such that for any \( x, y \in \mathbb{R} \) we have \( |\psi(x) - \psi(y)| \leq C_{\psi}|x - y| \). Due to (46), this implies
\[
|\eta - 1| = \left| \frac{\psi_{a_{\psi}}(\lambda(uu')) - \psi_{a_{\psi}}(\lambda(u)) + \psi_{a_{\psi}}(\lambda(v)) - \psi_{a_{\psi}}(\lambda(vv'))}{\psi_{a_{\psi}}(\lambda(u)) - \psi_{a_{\psi}}(\lambda(v))} \right| \]
\[
\leq \frac{2C_{\psi}}{c_{\psi,k}} \left( \left| T_{a_{\psi}}^{-1}(\lambda(uu')) - T_{a_{\psi}}^{-1}(\lambda(u)) \right| + \left| T_{a_{\psi}}^{-1}(\lambda(v)) - T_{a_{\psi}}^{-1}(\lambda(vv')) \right| \right) \]
\[
\leq \frac{2C_{\psi}}{c_{\psi,k}} \cdot 2^l \cdot 2^{-m}, \tag{48}
\]
where we have used \( |\lambda(u) - \lambda(uu')| \lor |\lambda(v) - \lambda(vv')| \leq 2^{-m} \).

For \( w \in \Sigma_u \) define the \( \sigma \)-algebra \( \mathcal{A}_w = \sigma(\pi_u : u \in \Sigma_u \setminus \{w\}) \).

By construction, \( B \) and \( D \) are \( \mathcal{A}_{a_{\psi}} \)-measurable, thus are constant given \( \mathcal{A}_{a_{\psi}} \).

From assumption A3 we know \( \pi_{a_{\psi}} \) has a bounded density function \( f_{a_{\psi}} \).

From (29), (45) and (46), we have
\[
\mathbf{1}^{(b)}(q, 2\epsilon) \cdot |A|^{-1} \leq \frac{2}{c_{\psi,k}} \cdot 2^{l(\log_2 + 2\epsilon)} \tag{49}
\]

When \( u, v \in \Sigma_u \) and \( \mathbf{1}_p(u, v) = 1 \), we have
\[
|\lambda(u) - \lambda(v)| \lor |\lambda(uu') - \lambda(vv')| \leq 2 \cdot 2^{-m}
\]
\[
|\lambda(u) - \lambda(v)| \lor |\lambda(uu') - \lambda(vv')| \geq 2^{-p-1}. \tag{50}
\]
Since $1_p(u, v) = 1$ implies $v|_p \in \mathcal{N}(u|_p)$, by (31) we have

$$1_u^{(c)}(\widehat{q}, 2\epsilon) \cdot \sup_{s, t \in U_{\mathcal{X} \times \mathcal{Y} | \mathcal{W}}} |F_{\mu}^{\text{pert}}(\lambda(s)) - F_{\mu}^{\text{pert}}(\lambda(t))| \leq 2^{-p(h^{(i)}_{\widehat{q}} - 2\epsilon)}.$$

This implies, when $1_u^{(c)}(\widehat{q}, 2\epsilon) = 1$,

$$\left( |F_{\mu}^{\text{pert}}(\lambda(u)) - F_{\mu}^{\text{pert}}(\lambda(v))| \vee |F_{\mu}^{\text{pert}}(\lambda(uu')) - F_{\mu}^{\text{pert}}(\lambda(vv'))| \right) \wedge 1 \leq 2^{-1_{\|\lambda\|_{\infty} > r(h^{(i)}_{\widehat{q}} - 2\epsilon)}} := \alpha. \quad (51)$$

Also, for the same reason, when $1_u^{(c)}(\widehat{q}, 2\epsilon) \cdot 1_v^{(c)}(\widehat{q}, 2\epsilon) = 1$,

$$\left( |F_{\mu}^{\text{pert}}(\lambda(u)) - F_{\mu}^{\text{pert}}(\lambda(uu'))| \vee |F_{\mu}^{\text{pert}}(t_u) - F_{\mu}^{\text{pert}}(\lambda(vv'))| \right) \leq 2^{-m(h^{(i)}_{\widehat{q}} - 2\epsilon)} := \beta. \quad (52)$$

These two inequalities with (42), (47) and (48) imply that when $1_u^{(c)}(\widehat{q}, \epsilon) = 1$,

$$|D| \leq |F_{\mu}^{\text{pert}}(\lambda(uu')) - F_{\mu}^{\text{pert}}(t_u) + F_{\mu}^{\text{pert}}(t_v) - F_{\mu}^{\text{pert}}(\lambda(vv'))| + |\eta| - 1 \cdot |F_{\mu}^{\text{pert}}(t_u) - F_{\mu}^{\text{pert}}(t_v)|$$

$$\leq 2\beta + 2\frac{C\psi_r^{(c)}}{c_{\psi, k}} \cdot 2^l \cdot 2^{-m} \cdot \alpha$$

$$= 2 \cdot 2^{-m(h^{(i)}_{\widehat{q}} - 2\epsilon)} + 2\frac{C\psi_r^{(c)}}{c_{\psi, k}} \cdot 2^l \cdot 2^{-m} \cdot 2^{-1_{\|\lambda\|_{\infty} > r(h^{(i)}_{\widehat{q}} - 2\epsilon)}}$$

$$\leq 2\left(2\frac{C\psi_r^{(c)}}{c_{\psi, k}} \cdot 2^{l+N_{\psi, \lambda}}\right) \cdot 2^{-m(h^{(i)}_{\widehat{q}} - 2\epsilon)} + 2^{-1_{\|\lambda\|_{\infty} > (h^{(i)}_{\widehat{q}} - 2\epsilon)}}$$

$$\leq C_D \cdot 2^{-m(h^{(i)}_{\widehat{q}} - 2\epsilon)}r, \quad (53)$$

where $C_D = 2\left(2\frac{C\psi_r^{(c)}}{c_{\psi, k}} \cdot 2^{l+N_{\psi, \lambda}}\right)$ and we have used $h^{(i)}_{\widehat{q}} - 2\epsilon \in (0, 1)$.

Recall that $l = r + 1 + N_{\psi, \lambda}$. Now we have the following:

(I) When $\gamma > 1$ (also $\gamma \leq 2$), due to (43), (33), (49) and (50),

$$\mathbb{E}\left(\mathcal{T}_\nu(u, v) | A_{u|v}\right)$$

$$\leq \int_{\mathbb{R}} 1_{\mathcal{T}_\nu(u, v) = 1} \cdot f_{u|v}(x) \ dx + \int_{\mathbb{R}} \frac{1^{\text{nom}}_{u, \nu}^{(c)}(\widehat{q}, \epsilon) \cdot f_{u|v}(x)}{|A : x + B|^2 + |\lambda(u) - \lambda(v)|^2} \ dx$$

$$\leq 1 + \int_{\mathbb{R}} 1^{\text{nom}}_{u, \nu}^{(c)}(\widehat{q}, \epsilon) \cdot |A|^{-1} |\lambda(u) - \lambda(v)|^{1 - \gamma} \cdot \frac{f_{u|v}(x)}{|(\|z\|^2 + 1)^{\gamma/2}} \ dz$$

$$\leq 1 + \int_{\mathbb{R}} \frac{2^{l+N_{\psi, \lambda}} \cdot 2^{(l+N_{\psi, \lambda})h^{(i)}_{\widehat{q}} - 2\epsilon} \cdot \|f_{u|v}\|_{\infty} \cdot \int_{\mathbb{R}} \frac{1}{(\|z\|^2 + 1)^{\gamma/2}} \ dz}{2 \cdot c_{\psi, k}} \cdot \|f_{u|v}\|_{\infty} \cdot 2^{(l+N_{\psi, \lambda})h^{(i)}_{\widehat{q}} - 2\epsilon} \cdot \|f_{u|v}\|_{\infty} \cdot 2^{-p(h^{(i)}_{\widehat{q}} - 2\epsilon) - (1 - \gamma)}$$

$$\leq 2\left(\int_{\mathbb{R}} \frac{d\nu}{(\|z\|^2 + 1)^{\gamma/2}} \cdot \frac{2^{(l+N_{\psi, \lambda})h^{(i)}_{\widehat{q}} - 2\epsilon} \cdot \|f_{u|v}\|_{\infty} \cdot 2^{-p(h^{(i)}_{\widehat{q}} - 2\epsilon) - (1 - \gamma)}}{c_{\psi, k}} \right) \cdot C_r \cdot 2^{(l+N_{\psi, \lambda})h^{(i)}_{\widehat{q}} - 2\epsilon} - (1 - \gamma),$$

where we recall that $C_r = \sup_{u \in \mathcal{X}} \|f_{u|v}\|_{\infty}$.
The graph and range singularity spectra of random wavelet series

(II) When $\gamma > 1$, let

$$\phi_\gamma(x, y) = \left( \frac{F_{\text{per}}^\gamma(\lambda(u)) - F_{\text{per}}^\gamma(\lambda(v)) + x}{\partial_x \phi_\gamma(x)} + \frac{\lambda(u) - \lambda(v) + y}{\partial_y \phi_\gamma(x)} \right)^{-\gamma/2}.$$  

Then due to (43), (33), (52) and (50), we have

$$\Delta K_\gamma(u, v) \leq \int_{|y| \leq 2^{-n}} \left| \frac{\partial}{\partial y} \phi_\gamma(0, y) \right| \, dy + \int_{|x| \leq \beta} \sup_{|y| \leq 2^{-n}} \left| \frac{\partial}{\partial x} \phi_\gamma(x, y) \right| \, dx,$$

where we have used that $|a + 1 - b + 1| \leq |a - b|$ for any $a, b \geq 0$. It is not difficult to check that

$$\left| \frac{\partial}{\partial y} \phi_\gamma(0, y) \right| \vee \left| \frac{\partial}{\partial x} \phi_\gamma(x, y) \right| \leq \gamma \cdot |\lambda(u) - \lambda(v)| + y^{-\gamma - 1}.$$

In fact, we have

$$\left| \frac{\partial}{\partial y} \phi_\gamma(0, y) \right| \leq \gamma \cdot \left( \frac{\lambda(u) - \lambda(v) + y}{\partial_x \phi_\gamma(x)} + \frac{\lambda(u) - \lambda(v) + y}{\partial_y \phi_\gamma(x)} \right)^{1/\gamma}$$

and

$$\left| \frac{\partial}{\partial x} \phi_\gamma(x, y) \right| \leq \gamma \cdot \left( \frac{\lambda(u) - \lambda(v) + x}{\partial_x \phi_\gamma(x)} + \frac{\lambda(u) - \lambda(v) + y}{\partial_y \phi_\gamma(x)} \right) \cdot \frac{1}{2|\lambda(u) - \lambda(v)| + y} \leq \gamma \cdot \left( \frac{\lambda(u) - \lambda(v) + y}{\partial_x \phi_\gamma(x)} + \frac{\lambda(u) - \lambda(v) + y}{\partial_y \phi_\gamma(x)} \right) \cdot \frac{1}{2|\lambda(u) - \lambda(v)| + y} \leq \gamma \cdot |\lambda(u) - \lambda(v)| + y^{-1 - \gamma},$$

where we have used that $\frac{a}{\partial_x \phi_\gamma(x)} \leq \frac{b}{\partial_y \phi_\gamma(x)}$ for any $a, b \geq 0$. This together with $|\lambda(u) - \lambda(v)| \geq 1 - p - 1$, $m > p + 1$ and $\gamma \leq 2$ yields

$$\Delta K_\gamma(u, v) \leq \gamma \cdot \left( 2^{-p - 1} - 2 \cdot 2^{-m} \right)^{-\gamma - 1} \cdot \left( 2 \cdot 2^{-m} + 2\gamma \right)$$

$$\leq \gamma \cdot 2^{(p + 2)(\gamma + 1)} \cdot \left( 2 \cdot 2^{-m} + 2 \cdot 2^{-m(h_\mu^{(0)} - 2a)}) \right)$$

$$\leq \gamma \cdot 2^{(p + 2)(\gamma + 1)} \cdot \left( 4 \cdot 2^{-m(h_\mu^{(0)} - 2a)} \right) \quad \text{(since } h_\mu^{(k)} - 2a < 1)$$

$$\leq (4\gamma 2^{2(\gamma + 1)} \cdot C_\rho \cdot 2^{p+3-m(h_\mu^{(0)} - 2a)}.$$
(III) When \( \gamma < 1 \), due to (43), (33), (49) and (51),
\[
\mathbb{E}(\mathcal{K}_\gamma(u, v) | A_{u,v}) \\
\leq \int_{|A + B| \leq \alpha} 1_{\mathcal{K}_\gamma(u, v) = 1} \cdot f_{u,v}(x) \, dx + \int_{|A + B| \leq \alpha} 1_{\mathcal{K}_\gamma(u, v) > 1} \cdot f_{u,v}(x) \, dx \\
\leq 1 + \int_{|z| \leq \alpha} 1_{\mathcal{K}_\gamma(u, v) > 1} \cdot |A|^{-1} \cdot \frac{f_{u,v}(\frac{z - B}{A})}{|z|^\gamma} \, dz \\
\leq 1 + \frac{2}{c_{\psi,k}} \cdot 2^{(h_{\psi,k}^0 + 2\epsilon)} \cdot \|f_{u,v}\|_{\infty} \cdot \int_{|z| \leq \alpha} \frac{1}{|z|^\gamma} \, dz \\
= 1 + \frac{2}{c_{\psi,k}} \cdot 2^{(h_{\psi,k}^0 + 2\epsilon)} \cdot \|f_{u,v}\|_{\infty} \cdot 2^{\alpha(1 - \gamma)} \\
\leq 2 \left( 2^{1+ N_{\psi,k} h_{\psi,k}^0 + 2\epsilon} \right) \cdot \|f_{u,v}\|_{\infty} \cdot 2^{t(h_{\psi,k}^0 + \gamma + 2\epsilon)} \\
\leq 2 \left( 2^{1+ N_{\psi,k} h_{\psi,k}^0 + 2\epsilon} \right) \cdot C_v \cdot 2^n \cdot 2^{r(h_{\psi,k}^0 + \gamma + 4\epsilon)}.
\]

(IV) When \( \gamma < 1 \), due to (43), (33), (49) and (51), by using again \( |a \lor 1 - b \lor 1| \leq |a - b| \) for any \( a, b \geq 0 \), we have
\[
\mathbb{E}(\Delta \mathcal{K}_\gamma(u, v) | A_{u,v}) \\
\leq \int_{\mathbb{R}} 1_{\mathcal{K}_\gamma(u, v) > 1} \cdot \left| \frac{1}{|\eta(A \cdot x + B) + D|^\gamma} - \frac{1}{|A \cdot x + B|^\gamma} \right| f_{u,v}(x) \, dx \\
= \int_{\mathbb{R}} 1_{\mathcal{K}_\gamma(u, v) > 1} \cdot |A|^{-1} \cdot |D|^{1 - \gamma} \cdot \left| \frac{1}{|\eta \cdot z + 1|^\gamma} - \frac{1}{|z|^\gamma} \right| f_{u,v}(D \cdot z - B) \, dz \\
\leq \frac{2}{c_{\psi,k}} \cdot 2^{(h_{\psi,k}^0 + 2\epsilon)} \cdot (C_v \cdot 2^{-m(h_{\psi,k}^0 - 2\epsilon)} + 1 - \gamma) \cdot \|f_{u,v}\|_{\infty} \\
\cdot \int_{\mathbb{R}} \left| \frac{1}{|\eta \cdot z + 1|^\gamma} - \frac{1}{|z|^\gamma} \right| \, dz \\
= \left( \frac{2^{1+ (1 + N_{\psi,k} h_{\psi,k}^0 + 2\epsilon)} C_v}{c_{\psi,k}} \right) \int_{\mathbb{R}} \left| \frac{1}{|\eta \cdot z + 1|^\gamma} - \frac{1}{|z|^\gamma} \right| \, dz \\
\cdot \|f_{u,v}\|_{\infty} \cdot 2^{t(h_{\psi,k}^0 + \gamma + 2\epsilon - m(h_{\psi,k}^0 - 2\epsilon)(1 - \gamma))} \\
\leq \left( \frac{2^{1+ (1 + N_{\psi,k} h_{\psi,k}^0 + 2\epsilon)} C_v}{c_{\psi,k}} \right) \int_{\mathbb{R}} \left| \frac{1}{|\eta \cdot z + 1|^\gamma} - \frac{1}{|z|^\gamma} \right| \, dz \cdot C_v \cdot 2^{r(3 - m(h_{\psi,k}^0 - 2\epsilon)).}
\]

Now, since
\[
\int_{\mathbb{R}} \frac{dz}{(|z|^2 + 1)^{\gamma/2}} (\gamma > 1) \quad \text{and} \quad \int_{\mathbb{R}} \left| \frac{1}{|\eta \cdot z + 1|^\gamma} - \frac{1}{|z|^\gamma} \right| \, dz (\gamma < 1)
\]
are both finite (note that \( \eta \) is bounded away from 0 and infinity uniformly), and \( h^{(k)}_\psi \) are chosen between \( s_0 - 1/p_0 \) and 1, we can easily find a constant \( c_\gamma \) such that

\[
\max \left\{ \frac{2}{(4\gamma 2^{2(p+1)})}, \left( \frac{2}{c_{\psi,k}} \right)^{1-\gamma}, \left( \frac{2}{c_{\psi,k}} \right)^{1-\gamma} \right\} \leq c_\gamma.
\]

This gives us the conclusion.

**4.3.2. Proof of lemma 4.3.**

**Proof.** Recall (see (42)) that

\[
I_{u,v}^{\psi}(\bar{\gamma}, \epsilon) = I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon).
\]

Let

\[
S_{p,m} = \sum_{u, v \in \Sigma_m} I_p(u, v) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon).
\]

Recall that \( r = p \lor n \). For any \( u \in \Sigma_m \) we write \( u = u_{|r} \cdot u' \) with \( u' \in \Sigma_{m-r} \). Since \( I_p(u, v) \) only depends on \( u_{|r}, v_{|r} \), we can write

\[
S_{p,m} = \sum_{u, v \in \Sigma_r} I_p(u, v) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot \sum_{u', v' \in \Sigma_{m-r}} I_{u', v'}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u', v'}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon).
\]

Recall (see (30)) that

\[
I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) = \frac{1}{|\bar{\gamma}|} \left( \frac{1}{|\bar{\gamma}|^{2|\gamma|}} \right).
\]

Thus

\[
I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \leq 2^{m(D_\psi^{(\Sigma_1)}+2\epsilon)} \cdot \mu^{(k)}_{\bar{\gamma}}(|u_{|r} \cdot u'|).
\]

This implies that

\[
\sum_{u', v' \in \Sigma_{m-r}} I_{u', v'}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u', v'}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \leq 2^{m(D_\psi^{(\Sigma_1)}+2\epsilon)} \cdot \mu^{(k)}_{\bar{\gamma}}(|u_{|r} \cdot u'|) \cdot \mu^{(k)}_{\bar{\gamma}}(|v_{|r} \cdot v'|)
\]

\[
= 2^{m(D_\psi^{(\Sigma_1)}+2\epsilon)} \cdot \mu^{(k)}_{\bar{\gamma}}(|u_{|r}|) \cdot \mu^{(k)}_{\bar{\gamma}}(|v_{|r}|).
\]

Thus by the fact that given \( u_{|r} \) in \( \Sigma_r \), there are at most \( 3^{r-p+1} \) many \( v_{|r} \) in \( \Sigma_r \) such that \( I_p(u_{|r}, v_{|r}) = 1 \), we have

\[
S_{p,m} \leq 2^{m(D_\psi^{(\Sigma_1)}+2\epsilon)} \cdot \sum_{u_{|r}, v_{|r} \in \Sigma_r} I_p(u_{|r}, v_{|r}) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon) \cdot I_{u,v}^{(\Sigma_1)}(\bar{\gamma}, 2\epsilon)
\]

\[
\leq 2^{m(D_\psi^{(\Sigma_1)}+2\epsilon)} \cdot \sum_{u_{|r}, v_{|r} \in \Sigma_r} I_p(u_{|r}, v_{|r}) \cdot \mu^{(k)}_{\bar{\gamma}}(|u_{|r}|)
\]

\[
\leq 2^{m(D_\psi^{(\Sigma_1)}+2\epsilon)} \cdot \sum_{u_{|r} \in \Sigma_r} \mu^{(k)}_{\bar{\gamma}}(|u_{|r}|) \cdot 3^{r-p+1}.
\]
Acknowledgments

The author would like to gratefully thank his supervisor Professor Julien Barral for having suggested him to study the graph and range singularity spectra of random wavelet series, and for his help in achieving this paper. He would also like to thank Doctor Yanhui Qu for some valuable discussions.

References

[1] Aubry J M and Jaffard S 2002 Random wavelet series Commun. Math. Phys. 227 483–514
[2] Barral J, Nasr F B and Peyrière J 2003 Comparing multifractal formalisms: the neighboring condition Asian J. Math. 7 149–66
[3] Barral J and Jin X 2009 Multifractal analysis of complex random cascades Commun. Math. Phys. (arXiv:0906.1501) at press
[4] Barral J and Seuret S 2005 From multifractal measures to multifractal wavelet series J. Fourier Anal. Appl. 11 589–614
[5] Barral J and Seuret S 2007 The singularity spectrum of Lévy processes in multifractal time Adv. Math. 214 437–68
[6] Bedford T and Urbanski M 1990 The box and Hausdorff dimension of self-affine sets Ergod. Theory Dyn. Syst. 10 627–44
[7] Berman S M 1972 Gaussian sample functions: uniform dimension and Hölder conditions nowhere Nagoya Math. J. 46 63–86
[8] Blumenthal R M and Getoor R K 1961 Sample functions of stochastic processes with stationary independent increments J. Math. Mech. 10 493–516
[9] Blumenthal R M and Getoor R K 1962 The dimension of the set of zeroes and the graph of a symmetric stable process Illinois J. Math. 6 308–16
[10] Bowen R 1975 Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics vol 470) (Berlin: Springer)
[11] Brown G, Michon G and Peyrière J 1992 On the multifractal analysis of measures J. Stat. Phys. 66 775–90
[12] Buczolich Z and Nagy J 2000 Hölder spectrum of typical monotone continuous functions Real Anal. Exch. 26 133–56
[13] Demichel Y and Falconer K J 2007 The Hausdorff dimension of pulse-sum graphs Math. Proc. Camb. Phil. Soc. 143 145–55
[14] Falconer K J 2003 Fractal Geometry: Mathematical Foundations and Applications 2nd edn (New York: Wiley)
[15] Feng D J 2005 The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers Adv. Math. 195 24–101
[16] Fraysse A and Jaffard S 2006 How smooth is almost every function in a Sobolev space? Rev. Mat. Iberoamericana 22 663–82
[17] Frisch U and Parisi G 1985 Fully developed turbulence and intermittency Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics ed M Ghil, R Benzi and G Parisi (Amsterdam: North Holland) pp 84–8
[18] Gelfert K and Rams M 2009 The Lyapunov spectrum of some parabolic systems Ergod. Theory Dyn. Syst. 29 919–40
[19] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 Fractal measures and their singularities: the characterization of strange sets Phys. Rev. A 33 1141–51
[20] Hentschel H G and Procaccia I 1983 The infinite number of generalized dimensions of fractals and strange attractors Physica D 8 435–44
[21] Holley R and Waymire E C 1992 Multifractal dimensions and scaling exponents for strongly bounded random fractals Ann. Appl. Probab. 2 819–45
[22] Horowitz J 1968 The Hausdorff dimension of the sample path of a subordinator Israel J. Math. 6 176–82
[23] Hu T Y and Lau K S 1993 Fractal dimensions and singularities of the Weierstrass type functions Trans. Am. Math. Soc. 335 649–65
[24] Hunt B 1998 The Hausdorff dimension of graphs of Weierstrass functions Proc. Am. Math. Soc. 126 791–800
[25] Jaffard S 1996 The spectrum of singularities of Riemann’s function Rev. Mat. Iberoamericana 12 441–60
[26] Jaffard S 1997 Old friends revisited: the multifractal nature of some classical functions J. Fourier Anal. Appl. 3 1–22
[27] Jaffard S 1997 Multifractal formalism for functions: I. Results valid for all functions SIAM J. Math. Anal. 28 944–70
The graph and range singularity spectra of random wavelet series

Jaffard S 1997 Multifractal formalism for functions: II. Self-similar functions SIAM J. Math. Anal. 28 971–98
[28] Jaffard S 1998 Oscillations spaces: properties and applications to fractal and multifractal functions J. Math. Phys. 39 4129–41
[29] Jaffard S 1999 The multifractal nature of Lévy processes Probab. Theory Relat. Fields 114 207–27
[30] Jaffard S 2000 On the Frisch–Parisi conjecture J. Math. Pures Appl. 79 525–52
[31] Jaffard S 2004 Wavelets techniques in multifractal analysis Fractal Geometry and Applications: a Jubilee of Benoît Mandelbrot (Proc. Symp. Pure Math. vol 72) ed M L Lapidus and M van Frankenhuijsen (Providence, RI: American Mathematical Society) pp 91–151
[32] Jaffard S 2004 Beyond Besov spaces: I. Distributions of wavelet coefficients J. Fourier Anal. Appl. 10 221–46
[33] Jin X 2009 The graph, range and level sets singularity spectra of b-adic independent cascade functions (arXiv:0911.1289v2)
[34] Kahan E J P 1985 Some Random Series of Functions 2nd edn (New York: Cambridge University Press)
[35] Khoshnevisan G and Xiao Y M 2005 Lévy processes: capacity and Hausdorff dimension Ann. Probab. 33 841–78
[36] Lévy P 1953 La mesure de Hausdorff de la courbe du mouvement brownien G. Ist. Ital. Attuari 16 1–7
[37] Mandelbrot B 1974 Intermittent turbulence in self-similar cascades: divergence of height moments and dimension of the carrier J. Fluid. Mech. 62 331–58
[38] Mauldin R D and Williams S C 1986 On the Hausdorff dimension of some graphs Trans. Am. Math. Soc. 298 793–803
[39] Meyer Y 1990 Ondelettes et Opérateurs (Paris: Hermann)
[40] Millar P W 1971 Path behavior of processes with stationary independent increments Z. Wahrscheinlichkeitstheoret. Verwandte Geb. 17 53–73
[41] Pesin Y 1997 Dimension theory in dynamical systems: Contemporary Views and Applications (Chicago Lectures in Mathematics) (Chicago: The University of Chicago Press)
[42] Pruitt W E 1969 The Hausdorff dimension of the range of a process with stationary independent increments J. Math. Mech. 19 371–8
[43] Przytycki F and Urbanski M 1989 On the Hausdorff dimension of some fractal sets Stud. Math. 93 155–86
[44] Rand D A 1989 The singularity spectrum f(α) for cookie-cutters Ergod. Theory Dyn. Syst. 9 527–41
[45] Roueff F 2003 New upper bounds of the Hausdorff dimensions of graphs of continuous functions Math. Proc. Camb. Phil. Soc. 135 219–37
[46] Roueff F 2003 Almost sure Hausdorff dimension of graphs of random wavelet series J. Fourier Anal. Appl. 9 237–60
[47] Ruelle D 2004 Thermodynamic Formalism 2nd edn (New York: Cambridge University Press)
[48] Seneta E 1981 Non-Negative Matrices and Markov Chains (New York: Springer)
[49] Seuret S 2009 On multifractality and time subordination for continuous functions Adv. Math. 220 936–63
[50] Taylor S J 1953 The Hausdorff ξ-dimensional measure of Brownian paths in n-space Proc. Camb. Phil. Soc. 49 31–9
[51] Urbanski M 1990 The Hausdorff dimension of the graphs of continuous self-affine functions Proc. Am. Math. Soc. 108 921–30
[52] Xiao Y M 2004 Random fractals and Markov processes Fractal Geometry and Applications: a Jubilee of Benoît Mandelbrot (Proc. Symp. Pure Math. vol 72) ed M L Lapidus and M van Frankenhuijsen (Providence, RI: American Mathematical Society) pp 261–338