Pion–Pion and Pion–Nucleon Interactions in Chiral Perturbation Theory*

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Abstract. Elastic pion–pion and pion–nucleon scattering are reviewed in the context of chiral perturbation theory. Theoretical results from systematic low–energy expansions to $O(p^6)$ for $\pi\pi$ and to $O(p^3)$ for $\pi N$ are compared with experimental data. Possible future developments are outlined.

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1 Overview

Elastic $\pi\pi$ and $\pi N$ scattering are the classical scattering processes of hadron physics, with a long history both in experiment and in theory. The purpose of this talk is to review the present status of these processes within the effective field theory of the standard model at low energies, chiral perturbation theory (CHPT) (Weinberg 1979; Gasser and Leutwyler 1984, 1985; Leutwyler 1994).

With pions and nucleons only, chiral $SU(2)$ is the appropriate setting. For this review, isospin conservation is assumed and the electromagnetic interaction is turned off. The framework will be standard CHPT (Gasser and Leutwyler 1984, 1985), but I will contrast the standard CHPT calculations for $\pi\pi$ scattering (Bijnens et al. 1996, 1997) with the extensive work performed in the framework of generalized CHPT (Knecht et al. 1995, 1996). For elastic $\pi N$ scattering, the only complete calculation to $O(p^3)$ (Mojžiš 1997) is based on the standard scheme.

Comparing the two reactions from the chiral point of view, the differences are more apparent than the similarities. Like any non–Goldstone degrees of freedom, the nucleons are less restricted by chiral symmetry than the pions. In addition, the chiral expansion proceeds in steps of $p$ in the pion–nucleon case rather than $p^2$ in the purely mesonic case, with $p$ as usual a generic small momentum. Both facts explain why the presently available amplitudes involve about the same number of low–energy constants for the two processes, even though we have reached $O(p^6)$ for $\pi\pi$, but only $O(p^3)$ for $\pi N$. The $\pi\pi$ amplitude is now known up to and including the two–loop level. The $\pi N$ amplitude, on the other hand, is still not completely known even to one–loop accuracy as long as the $O(p^4)$ part is missing.

For chiral $SU(2)$, one may expect good convergence (in the usual sense of CHPT) of the low–energy expansion near threshold. Anticipating the summary, the convergence is by now very satisfactory for $\pi\pi$, but not sufficient yet for $\pi N$. This is in contrast to the experimental situation. The available data for $\pi\pi$ scattering near threshold are scarce and not precise enough for definitive tests of the CHPT predictions. However, much of the recent theoretical activity
in this field has been motivated by the experimental prospects for significant improvements in the near future, at DAΦNE (Lee-Franzini 1997), BNL (Lowe 1997) and CERN (Schacher 1997). For elastic πN scattering, on the other hand, data are abundant. Here the precision seems almost too good for some of the quantities extracted from experiment, given the inconsistencies in the existing data sample (Höhler 1997; Pavan 1997).

2 Pion–Pion Scattering

In the isospin limit, the scattering amplitude for

\[ \pi^a(p_1) + \pi^b(p_2) \rightarrow \pi^c(p_3) + \pi^d(p_4) \]  

(1)
is determined by a single scalar function \( A(s, t, u) \) of the usual Mandelstam variables \( s, t, u \):

\[
T_{ab,cd}(s, t, u) = \delta_{ab}\delta_{cd}A(s, t, u) + \delta_{ac}\delta_{bd}A(t, s, u) + \delta_{ad}\delta_{bc}A(u, t, s)
\]

(2)

The amplitudes of definite isospin \((I = 0, 1, 2)\) in the \( s \)-channel are decomposed into partial–wave amplitudes \( t^I_l(s) \). In the elastic region \( 4M^2_{\pi} \leq s \leq 16M^2_{\pi} \), the partial–wave amplitudes can be described by real phase shifts \( \delta^I_l(s) \):

\[
t^I_l(s) = (1 - \frac{4M^2_{\pi}}{s})^{-1/2} \exp \left( i\delta^I_l(s) \sin \delta^I_l(s) \right).
\]

(3)
The behaviour of the partial waves near threshold is of the form

\[
\Re t^I_l(s) = q^{2l}\left(a^I_l + q^2b^I_l + O(q^4)\right),
\]

(4)

with \( q \) the center–of–mass momentum. The quantities \( a^I_l \) and \( b^I_l \) are referred to as scattering lengths and slope parameters, respectively.

The amplitude \( A(s, t, u) \) was calculated to \( O(p^2) \) by Weinberg (1966) and to \( O(p^4) \) by Gasser and Leutwyler (1983). To next–to–next–to–leading order, \( O(p^6) \), the amplitude is now available in two different forms as described in the following two subsections.

2.1 Dispersive Calculation

Unitarity for the partial–wave amplitudes,

\[
\Im t^I_l(s) = (1 - \frac{4M^2_{\pi}}{s})^{1/2}|t^I_l(s)|^2 + \text{inelastic contributions (for } s > 16M^2_{\pi}),
\]

(5)

leads to the following consequences:

1. If \( t^I_l(s) \) is known to \( O(p^{2n}) \), \( \Im t^I_l(s) \) can be calculated in the elastic region to \( O(p^{2n+2}) \) from the unitarity relation (5).
ii. Since the lowest–order amplitude of $O(p^2)$ corresponds to partial waves with $l = 0, 1$ only, Eq. (5) implies

$$\Im \Sigma_l(s) = O(p^8) \quad \text{for } l \geq 2.$$  

Inelastic contributions enter at $O(p^8)$ only.

Given the amplitude to $O(p^4)$, unitarity and analyticity therefore allow for a dispersive calculation (Knecht et al. 1995) of $A(s, t, u)$ to $O(p^6)$ up to a crossing symmetric subtraction polynomial. The analytically nontrivial part has a relatively simple form (Stern et al. 1993) that can be expressed in terms of up to third powers of the standard one–loop function. The subtraction polynomial depends on six parameters four of which can be obtained from sum rules involving high–energy $\pi\pi$ data (Knecht et al. 1996).

The general form of the scattering amplitude given by Knecht et al. (1995) is valid for both the standard and the generalized (Stern 1997) picture. The differences are all contained in the six subtraction constants.

2.2 Field Theoretic Calculation

The diagrammatic calculation of $A(s, t, u)$ to $O(p^6)$ (Bijnens et al. 1996, 1997) in the standard framework is quite involved. The main features for a comparison with the dispersive approach are the following:

– The analytically nontrivial part of the dispersive result is reproduced as well as the general structure of the polynomial piece.
– To arrive at the final renormalized amplitude, one needs in addition the following quantities to $O(p^6)$: the pion wave function renormalization constant (Bürgi 1996), the pion mass (Bürgi 1996) and the pion decay constant (Bijnens et al. 1996, 1997).
– In the notation of Bijnens et al. (1996, 1997), the subtraction polynomial is expressed in terms of six dimensionless parameters $b_i$ ($i = 1, \ldots, 6$). The field theoretic approach produces these parameters as functions

$$b_i(M_\pi/F_\pi, M_\pi/\mu, l^\mu_i(\mu), k^\mu_i(\mu)),$$  

where $\mu$ is an arbitrary renormalization scale, the $l^\mu_i$ ($i = 1, \ldots, 4$) are the relevant low–energy constants of $O(p^4)$ (Gasser and Leutwyler 1984) and the $k^\mu_i$ are six combinations of the corresponding constants of $O(p^6)$ in the $SU(2)$ version of the chiral Lagrangian of Fearing and Scherer (1996).

Compared to the dispersive approach, the diagrammatic method offers the following advantages:

i. The full infrared structure is exhibited to $O(p^6)$. In particular, the $b_i$ contain chiral logs of the form $(\ln M_\pi/\mu)^n$ ($n \leq 2$) that are known to be numerically important, especially for the infrared–dominated parameters $b_1$ and $b_2$. On the other hand, $b_3, \ldots, b_6$ are more sensitive to the “high–energy” structure. The latter are precisely the four parameters for which sum rule estimates exist (Knecht et al. 1996).
ii. The explicit dependence on low–energy constants makes phenomenological determinations of these constants and comparison with other processes possible. This is especially relevant for determining $l_1^r$, $l_2^r$ to $O(p^6)$ accuracy (Colangelo et al. 1997b).

iii. The fully known dependence on the pion mass allows one to evaluate the amplitude even at unphysical values of the quark mass (remember that we assume $m_u = m_d$). One possible application is to confront the CHPT amplitude with (unquenched) lattice calculations (Colangelo 1997a).

2.3 Results

In the standard picture, the $\pi\pi$ amplitude depends on four low–energy constants of $O(p^4)$ and on six combinations of $O(p^6)$ couplings. The latter have been estimated with meson resonance exchange that is known to account for the dominant features of the $O(p^4)$ constants (Ecker et al. 1989). Referring to Bijnens et al. (1997) for details, the inherent uncertainties of this approximation induce small (bigger) uncertainties for the low (higher) partial waves. The main reason is that the higher partial waves are more sensitive to the high–energy structure.

Concerning the low–energy constants of $O(p^4)$, the $\pi\pi$ amplitude of $O(p^6)$ will lead eventually to a more precise determination of some of those constants (Colangelo et al. 1997b) than presently available. For the time being, one can investigate the sensitivity of the amplitude to the $\bar{l}_i$, the scale independent couplings introduced by Gasser and Leutwyler (1984). One obvious choice is based on the original analysis to $O(p^4)$ (Gasser and Leutwyler 1984) supplemented by a more recent investigation of $K_{e4}$ decays including dispersive estimates of higher–order effects (Bijnens et al. 1994). The following values are referred to as set I:

$$
\bar{l}_1 = -1.7 , \bar{l}_2 = 6.1 ,
$$
$$
\bar{l}_3 = 2.9 , \bar{l}_4 = 4.3 .
$$

As first emphasized by Girlanda et al. (1997), the amplitude for set I leads to $D$–wave scattering lengths that do not agree well with the values extracted from experiment (see Table 1 below). For set II we have therefore updated the procedure of Gasser and Leutwyler (1984) to fix $\bar{l}_1, \bar{l}_2$ from the $D$–wave scattering lengths $a_2^0, a_2^+$, now to $O(p^6)$ accuracy:

$$
\bar{l}_1 = -1.5 , \bar{l}_2 = 4.5 ,
$$

leaving $\bar{l}_3, \bar{l}_4$ unchanged. Although $\bar{l}_1$ is practically unchanged, the associated error is large because the $D$–wave scattering lengths depend only weakly on $\bar{l}_1$. On the other hand, the decrease of $\bar{l}_2$ from set I to set II is more pronounced. In fact, there are some independent indications in favour of such a smaller value of $\bar{l}_2$ (Pennington and Portolés 1995; Ananthanarayan and Büttiker 1996; Wanders 1997).
The dependence on the low-energy constants is contained in the parameters $b_i$. It turns out (Bijnens et al. 1997) that $b_1$ and $b_2$ are rather insensitive to the precise values of the $\bar{l}_i$ whereas $b_3, \ldots, b_6$ change substantially between sets I and II. This is of course in line with the previous observation that $b_1, b_2$ are infrared dominated while the other $b_i$ are more sensitive to the high-energy structure.

In Figs. 1, 2 the phase shift difference $\delta_0^0 - \delta_1^1$ and the $I = 2$ $S$-wave phase shift $\delta_2^0$ are plotted as functions of the center-of-mass energy and compared with the available low-energy data. The two-loop phase shifts describe the $K^e4$ data (Rosselet et al. 1977) very well for both sets of $\bar{l}_i$, with a small preference for set I. The $I = 2$ $S$ wave, on the other hand, seems to prefer set II.

The lower value of $\bar{l}_2$ in set II raises the question whether such a value is still compatible with the idea of resonance saturation of the low-energy constants of $O(p^4)$. To illustrate this point, I take the resonance dominance values for the $l'_r(\mu)$ as given in Ecker et al. (1989) and choose the renormalization scale $\mu = M_\eta$. This gives rise to set III:

$$
\bar{l}_1 = -0.7, \quad \bar{l}_2 = 5.0, \\
\bar{l}_3 = 1.9, \quad \bar{l}_4 = 3.7.
$$

In Table 1, the threshold parameters for the low partial waves are displayed at
Fig. 2. Phase shift $\delta_2^0$ at $O(p^2)$, $O(p^4)$ and $O(p^6)$ (set I and II) from Bijnens et al. (1997).

$O(p^2)$, $O(p^4)$ (set I) and to $O(p^6)$ for the three sets I, II and III. The experimental values are taken from the compilation of Dumbrajs et al. (1983). Referring again to Bijnens et al. (1997) for additional discussion, let me emphasize two relevant points here:

- The $S$–wave threshold parameters are very stable, especially the $I = 0$ scattering length, while the higher partial waves are more sensitive to the choice of low–energy constants of $O(p^4)$ (and of $O(p^6)$, for that matter).
- The resonance dominance prediction (set III) is in perfect agreement with the data although the agreement becomes less impressive for $\mu > M_\eta$.

Table 1 also documents that more work is needed to extract the low–energy constants from the data and to assign credible theoretical errors to phase shifts and threshold parameters. Such an analysis based on the Roy equation machinery (Roy 1971) is well under way (Colangelo et al. 1997b). A similar approach has been used by Knecht et al. (1996) to determine the parameters $b_3, \ldots, b_6$ via sum rules.

Without anticipating the results of such an analysis, it is clear from Table 1 that the $I = 0$ $S$–wave scattering length is very stable with respect to reasonable
Table 1. Threshold parameters in units of $M_{\pi^+}$ for three sets of low–energy constants $\bar{l}_i$ defined in Eqs. (8,9,10). The values of $O(p^2)$ correspond to set I. The experimental values are from Dumbrajs et al. (1983).

|       | $O(p^2)$ | $O(p^4)$ | $O(p^6)$ set I | $O(p^6)$ set II | $O(p^6)$ set III | experiment |
|-------|----------|----------|----------------|----------------|-----------------|------------|
| $a_0^0$ | 0.16     | 0.20     | 0.217          | 0.206          | 0.209           | 0.26 ± 0.05 |
| $b_0^0$ | 0.18     | 0.25     | 0.275          | 0.249          | 0.261           | 0.25 ± 0.03 |
| $-10 a_0^2$ | 0.45     | 0.42     | 0.413          | 0.443          | 0.415           | 0.28 ± 0.12 |
| $-10 b_0^2$ | 0.91     | 0.73     | 0.72           | 0.80           | 0.75            | 0.82 ± 0.08 |
| $10 a_1^1$ | 0.30     | 0.37     | 0.40           | 0.38           | 0.37            | 0.38 ± 0.02 |
| $10^2 b_1^1$ | 0        | 0.48     | 0.79           | 0.54           | 0.60            |            |
| $10^3 a_2^2$ | 0        | 0.21     | 0.23           | input          | 0.19            | 0.17 ± 0.03 |
| $10^3 b_2^2$ | 0        |          |                | input          | 0.16            | 0.13 ± 0.30 |

changes of the $\bar{l}_i$ and that a value

$$a_0^0 = 0.21 \div 0.22$$

(11)
is strongly favoured. Moreover, as shown in Fig. 1, the corresponding phase shift gives an excellent description of the $K_e4$ data, especially for set I (the curve for set III not shown explicitly lies between those for sets I and II). This makes one wonder why the mean value of the usually quoted $a_0^0 = 0.26 \pm 0.05$ (Froggatt and Petersen 1977) is substantially bigger than (11), although of course consistent within the error.

To make the point that with the present experimental precision $a_0^0$ depends crucially on the fitting procedure, I analyze the data of Rosselet et al. (1977) in Fig. 1 in two different ways. The first method is the one employed by the experimentalists themselves who used the approximate formula

$$\sin 2(\delta_0^0 - \delta_1^1) = 2 \sqrt{1 - \frac{4M_{\pi^+}^2}{s}} (a_0^0 + q^2 b).$$

(12)

Fitting the data with free parameters $a_0^0$ and $b$ produces a mean value $a_0^0 = 0.31$. Employing a relation between $a_0^0$ and $b$ attributed to Basdevant et al. (1974), the final value of Rosselet et al. (1977) is $a_0^0 = 0.28 \pm 0.05$. The second method for extracting $a_0^0$ uses the full CHPT amplitude to $O(p^6)$. To perform a one–parameter fit, I choose for definiteness set I but leave $\bar{l}_2$ as a free parameter. The fit has practically the same $\chi^2$ as the previous one and gives rise to

$$a_0^0 = 0.220 \pm 0.012$$

$$\bar{l}_2 = 6.4 \pm 1.6.$$
The main reason for the different values of $a_0^0$ extracted from the same data is that the scattering length only dominates in a small neighbourhood of the threshold (Leutwyler 1997). Even for the limited range in $E_{\pi\pi}$ covered by the experiment of Rosselet et al. (1977), higher–order terms in the threshold expansion (4) are important. It goes without saying that the full chiral amplitude to $O(p^n)$ has a superior theoretical status than the simple approximation (12) on all accounts. Therefore, I conclude that the CHPT prediction (11) is in perfect agreement with the available data and that there is no indication for an unusually small quark condensate (Stern 1997) on the basis of existing $\pi\pi$ data. By no means, this is meant to imply that we do not need more precise experimental information to really pin down the quark condensate from $\pi\pi$ scattering (Lee-Franzini 1997; Lowe 1997; Schacher 1997).

3 Pion–Nucleon Scattering

The amplitude for pion–nucleon scattering

$$\pi^a(q_1) + N(p_1) \to \pi^b(q_2) + N(p_2)$$

(14)
can be expressed in terms of four invariant amplitudes $D^\pm$, $B^\pm$:

$$T_{ab} = T^+ \delta_{ab} - T^- i \varepsilon_{abc} \tau_c$$

(15)

$$T^\pm = \bar{u}(p_2) \left[ D^\pm(\nu, t) + \frac{i}{2m_N} \sigma^{\mu\nu} q_2 \sigma_{\mu\nu} B^\pm(\nu, t) \right] u(p_1)$$

with

$$s = (p_1 + q_1)^2, t = (q_1 - q_2)^2, u = (p_1 - q_2)^2, \nu = \frac{s - u}{4m_N}.$$  

(16)

With the choice of invariant amplitudes $D^\pm$, $B^\pm$, the low–energy expansion is straightforward: to determine the scattering amplitude to $O(p^n)$, one has to calculate $D^\pm$ to $O(p^n)$ and $B^\pm$ to $O(p^{n-2})$.

In the framework of chiral perturbation theory, the first systematic calculation of pion–nucleon scattering was performed by Gasser et al. (1988). To make the relation between chiral and loop expansions more transparent than in the original relativistic formulation, the so–called “nonrelativistic” approach of heavy baryon chiral perturbation theory (HBCHPT) (Jenkins and Manohar 1991; Bernard et al. 1992) uses a simultaneous expansion in $p/4\pi F_\pi$ and $p/m_N$.

In HBCHPT, the pion–nucleon scattering amplitude is not directly obtained in the relativistic form (15) but rather as (Mojžiš 1997)

$$\bar{u}(p_2) P_v^+ \left[ \alpha^\pm + i \varepsilon^{\mu\nu\rho\sigma} q_1 q_2 v_\nu S_\sigma \beta^\pm \right] P_v^+ u(p_1)$$

(17)

$$\nu^2 = 1, \quad P_v^+ = (1 + \gamma^\nu) / 2, \quad S^{\nu} = i/2\gamma_5 \sigma^{\mu\nu} v_\nu.$$
The HBCHPT amplitudes $\alpha^\pm, \beta^\pm$ depend on the choice of the arbitrary time–like four–vector $v$. A natural and convenient choice is the initial nucleon rest frame with $v = p_1/ m_N$. In this frame, the relativistic amplitudes are given in terms of the HBCHPT amplitudes as (Ecker and Mojžiš 1997)

\begin{align}
D^\pm &= \alpha^\pm + \frac{vt}{4m_N} \beta^\pm \\
B^\pm &= -m_N \left( 1 - \frac{t}{4m_N^2} \right) \beta^\pm .
\end{align}

(18)

Also the amplitudes $D^\pm, B^\pm$ will in general depend on the specific frame associated with $v$. However, since the chiral pion–nucleon Lagrangian in HBCHPT can be constructed from a fully relativistic Lagrangian, the amplitudes $D^\pm, B^\pm$ obtained from (18) are guaranteed (Ecker and Mojžiš 1996) to be Lorentz invariant except for terms of at least $O(p^{n+1})$ if the HBCHPT amplitude (17) has been calculated to $O(p^n)$. The same conclusion can be drawn on the basis of reparametrization invariance (Luke and Manohar 1992).

The first complete calculation of $\pi N$ scattering to $O(p^3)$ in the framework of HBCHPT has recently been performed by Mojžiš (1997). The one–loop amplitude of $O(p^3)$ has also been calculated by Bernard et al. (1997), in agreement with Mojžiš (1997). Up to $O(p^2)$, only tree diagrams with vertices from the Lagrangians of $O(p)$ and $O(p^2)$ appear. In the isospin limit, four low–energy constants of $O(p^2)$ enter the scattering amplitude in addition to the single coupling constant $g_A$ of the lowest–order Lagrangian. In the notation of Ecker and Mojžiš (1996), those four dimensionless constants are called $a_1, a_2, a_3$ and $a_5$. The one–loop amplitude with only lowest–order vertices comes in at $O(p^3)$. There are at this order also irreducible and reducible tree diagrams involving vertices up to $O(p^3)$. In addition to the couplings already present at $O(p^2)$, five more combinations of low–energy constants of $O(p^3)$ contribute to the amplitudes denoted $b_1 + b_2, b_3, b_6, b_{15} - b_{16}$ and $b_{19}$. Thus, the final renormalized amplitudes depend on $\nu, t, M_\pi, F_\pi, m_N, g_A$ and on 9 combinations of low–energy constants of $O(p^2)$ and $O(p^3)$. As a technical side–remark, let me mention that nucleon wave function renormalization in HBCHPT turns out to be momentum dependent in general (Ecker and Mojžiš 1997).

The invariant amplitudes $D^\pm, B^\pm$ can be projected onto partial–wave amplitudes $f^\pm_{l\pm}(s)$ (Höhler 1983). Threshold parameters are defined as in Eq. (4):

\begin{equation}
\Re f^\pm_{l\pm}(s) = q^{2l} \{ a^\pm_{l\pm} + q^2 b^\pm_{l\pm} + O(q^4) \} .
\end{equation}

(19)

To confront the chiral amplitude with experiment, Mojžiš (1997) has compared 16 of these threshold parameters with the corresponding values extrapolated from experimental data on the basis of the Karlsruhe–Helsinki phase–shift analysis (Koch and Pietarinen 1980).

Six of the threshold parameters ($D$ and $F$ waves) are independent of all low–energy constants of $O(p^2)$ and $O(p^3)$. The results are shown$^1$ in Table 2 and compared with Koch and Pietarinen (1980).

$^1$I am grateful to Martin Mojžiš for providing me with partly unpublished results
Table 2. Comparison of two D–wave and four F–wave threshold parameters up to the first, second and third order (the two columns differ by higher–order terms) with (extrapolated) experimental values (Koch and Pietarinen 1980). The theoretical values are based on the calculation of Mojžiš (1997). Units are appropriate powers of GeV$^{-1}$.

|       | $O(p)$ | $O(p^2)$ | $O(p^3)$ | HBCHPT $O(p^3)$ | exp.  |
|-------|--------|----------|----------|-----------------|-------|
| $a_{2+}$ | 0      | −48      | −35      | −36             | −36 ± 7 |
| $a_{4+}$ | 0      | 48       | 56       | 56              | 64 ± 3  |
| $a_{3+}$ | 0      | 0        | 226      | 280             | 440 ± 140 |
| $a_{4-}$ | 0      | 14       | 26       | 31              | 160 ± 120 |
| $a_{2+}$ | 0      | 0        | −158     | −210            | −260 ± 20 |
| $a_{3-}$ | 0      | −14      | 65       | 57              | 100 ± 20  |

Before discussing the results, let me explain the chiral counting in Tables 2, 3. As emphasized before, each chiral order contains all contributions up to the given order, but it may also contain some higher–order contributions depending on the chosen frame (through the choice of the four–vector $v$). The numerical values in Tables 2, 3 correspond to the initial nucleon rest frame employed by Mojžiš (1997). The column denoted HBCHPT $O(p^3)$ is based on the amplitudes $\alpha^\pm, \beta^\pm$ calculated to $O(p^3)$ in the initial nucleon rest frame. These amplitudes are then inserted in Eq. (18) to determine the relativistic amplitudes $D^\pm, B^\pm$. The relations (18), which are exact to all orders, introduce also contributions of higher orders than $p^3$. For the column denoted $O(p^3)$, those higher–order terms in $D^\pm, B^\pm$ have been eliminated. The difference between the two columns is therefore an indication of the sensitivity of the respective threshold parameter to higher–order contributions. At least for the parameters listed in Table 2, these differences are within reasonable limits.

The entries of $O(p)$ and $O(p^2)$ in Tables 2 and 3 differ from those listed in Mojžiš (1997). The reason is that Mojžiš originally used a different set of invariant amplitudes ($A^\pm, B^\pm$) that are less suited for a proper chiral counting.

The main conclusion from Table 2 is a definite improvement seen at $O(p^3)$. Since there are no low–energy constants involved (except, of course, $M_\pi, F_\pi, m_N$ and $g_A$), this is clear evidence for the relevance of loop effects. The numbers shown in Table 2 are based on the calculation of Mojžiš (1997), but essentially the same results have been obtained by Bernard et al. (1997).

The remaining 10 threshold parameters do depend on the 9 low–energy constants $O(p^2)$ and $O(p^3)$. To fit these 9 constants, Mojžiš (1997) has included appearing in Tables 2, 3.
**Table 3.** Fitted values of ten \(\pi N\) threshold parameters up to the first, second and third order (the two columns differ by higher–order terms), compared with (extrapolated) experimental values (Koch and Pietarinen 1980). The theoretical values are based on the calculation of Mojziš (1997). Units are appropriate powers of GeV\(^{-1}\).

|       | \(O(p)\) | \(O(p^2)\) | \(O(p^3)\) | HBCHPT \(O(p^3)\) | exp. |
|-------|----------|------------|------------|-------------------|------|
| \(a_0^+\) | 0 | -0.13 | -0.07 | -0.07 ± 0.09 | -0.07 ± 0.01 |
| \(b_0^+\) | 6.8 | -15.5 | -11.5 | -13.9 ± 3.0 | -16.9 ± 2.5 |
| \(a_0^-\) | 0.55 | 0.55 | 0.67 | 0.67 ± 0.10 | 0.66 ± 0.01 |
| \(b_0^-\) | 2.6 | 2.6 | 6.7 | 5.5 ± 6.7 | 5.1 ± 2.3 |
| \(a_{1+}^+\) | 15.4 | 48.4 | 49.6 | 50.4 ± 1.1 | 50.5 ± 0.5 |
| \(a_{1-}^+\) | -37.5 | -4.5 | -22.4 | -21.6 ± 1.8 | -21.6 ± 0.5 |
| \(a_{1+}^-\) | -15.4 | -27.5 | -31.4 | -31.0 ± 0.8 | -31.0 ± 0.6 |
| \(a_{1-}^-\) | -15.5 | 8.6 | -4.9 | -4.5 ± 1.0 | -4.4 ± 0.4 |
| \(a_{2+}^-\) | -4.4 | 6.9 | 28.8 | 31.2 ± 0.3 | 44 ± 7 |
| \(a_{2-}^-\) | 4.4 | -12.8 | -0.1 | -5.0 ± 0.2 | 2 ± 3 |

The nucleon \(\sigma\)–term and the \(\pi N\) coupling constant \(g_{\pi N}\) that depend on the same constants. The results of the fit for the threshold parameters are shown in Table 3. Two immediate conclusions are:

– Not too surprisingly with 9 parameters for 12 observables, the “experimental” values can be reproduced with the chiral amplitude to \(O(p^3)\). Incidentally, the fitted value of the \(\sigma\)–term tends to be larger (Mojziš 1997) than the canonical value (Gasser et al. 1991). The discrepancy becomes smaller when increasing the errors for the threshold parameters.

– In many cases, the corrections of \(O(p^3)\) are sizable and definitely bigger than what naive chiral order–of–magnitude estimates would suggest. For instance, at threshold the third–order corrections \(\alpha_3\), \(\beta_3\) due to the low–energy constants \(b_i\) of \(O(p^3)\) are some 30% of the leading–order amplitudes.

At least as interesting as the fitted values of the threshold parameters are the corresponding values of the low–energy constants shown in Table 4. The first observation is that most of these constants are of \(O(1)\) as expected from naive chiral dimensional analysis. Looking a little closer into the calculation of Mojziš (1997), one finds that in some amplitudes the low–energy constants of \(O(p^3)\) interfere constructively near threshold so that their overall effect is larger than the naive estimate.
The values for the low–energy constants $a_i$ in Table 4 agree very well with an independent analysis (although some of the input data are the same) of Bernard et al. (1997). Moreover, these authors have shown that the specific values of $a_1$, $a_2$ and $a_5$ can be understood as being mainly due to $\Delta$ and $\rho$ exchange, whereas $a_3$ (appearing in the $\sigma$ term) is compatible with scalar resonance dominance. I conclude that the low–energy constants of $O(p^2)$ in the pion–nucleon Lagrangian are under good control, both numerically and conceptually. A similar analysis is not yet available for the constants $b_i$.

Table 4. Values of low–energy constants of $O(p^2)$ and $O(p^3)$ from fitting (Mojziš 1997) ten $\pi N$ threshold parameters, the nucleon $\sigma$–term and the Goldberger–Treiman discrepancy. The $\tilde{b}_i$ are scale independent versions of the $b_i$.

| $a_i$  | Value          |
|-------|----------------|
| $a_1$ | $-2.60 \pm 0.03$ |
| $a_2$ | $1.40 \pm 0.05$  |
| $a_3$ | $-1.00 \pm 0.06$ |
| $a_5$ | $3.30 \pm 0.05$  |
| $b_1 + b_2$ | $2.4 \pm 0.3$  |
| $b_3$  | $-2.8 \pm 0.6$  |
| $b_6$  | $1.4 \pm 0.3$   |
| $b_{16} - b_{15}$ | $6.1 \pm 0.6$  |
| $b_{19}$ | $-2.4 \pm 0.4$ |

4 Summary and Outlook

4.1 $\pi\pi \rightarrow \pi\pi$

Unlike for most other processes discussed during this Workshop, the available CHPT calculations to $O(p^6)$ are amply sufficient, even in view of the forthcoming precision experiments. However, while waiting for the results from KLOE at DAΦNE (Lee-Franzini 1997), DIRAC at CERN (Schacher 1997) and E865 at BNL (Lowe 1997), several things remain to be done. Among the most interesting topics are the following:

- There is a complementarity between the field theoretic calculation (Bijnens et al. 1996, 1997) and the dispersive one (Knecht et al. 1995, 1996). While the first method fully accounts for the infrared structure, the latter encompasses the high–energy information via sum rules for the relevant parameters. It
remains to bring the two ingredients together in an optimal way through a Roy–type analysis (Colangelo et al. 1997b) to extract especially the $S$–wave scattering lengths $a_0^0$, $a_2^0$ and the low–energy constants $l_1$, $l_2$ with reliable errors.

– Isospin violation and electromagnetic corrections have to be included. First results are already available (Meißner et al. 1997; Knecht and Urech 1997).

Concerning the $I = 0$ $S$–wave scattering length, I conclude that a value

$$a_0^0 = 0.21 \div 0.22$$

is well established on the basis of existing CHPT calculations in the standard scheme. Such a value is in perfect agreement with the available experimental information. Thus, there is at present no indication from pion–pion scattering against the standard scenario of chiral symmetry breaking with a dominant quark condensate.

### 4.2 $\pi N \rightarrow \pi N$

The first complete analysis of $\pi N$ scattering to $O(p^3)$ by Mojžiš (1997) has produced very encouraging results. However, we are still far from the theoretical precision attained in $\pi\pi$ scattering. Among the most urgent tasks are the following:

– The chiral amplitude should be confronted with extrapolated and/or real phase shifts to check the range of validity of the chiral expansion and to control the stability of the low–energy constants involved. In a recent paper that has appeared after the Workshop, Ellis and Tang (1997) have actually calculated the phase shifts to $O(p^3)$ and compared them with experiment, using a somewhat different approach that is claimed to be equivalent to HBCHPT.

– The threshold parameters of Koch and Pietarinen (1980) are derived from a data sample parts of which are claimed to be inconsistent (Pavan 1997). An update on the basis of generally accepted experimental input would be highly welcome for meaningful tests of chiral perturbation theory.

– As the low–energy constants of $O(p^2)$ are now well understood, both phenomenologically and theoretically, a similar analysis for the constants of $O(p^3)$ is called for. The results should eventually be contrasted with an alternative analysis based on the $\varepsilon$–expansion with the $\Delta$–isobar as explicit degree of freedom (Kambor 1997).

– Finally, in spite of the encouraging results to $O(p^3)$, the $O(p^4)$ calculation is absolutely necessary to complete the pion–nucleon amplitude to one–loop accuracy.
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References

Ananthanarayan, B., Büttiker, P. (1996): Phys. Rev. D54, 5501
Basdevant, J.L., Froggatt, C.D., Petersen, J.L. (1974): Nucl. Phys. B72, 413
Bernard, V., Kaiser, N., Kambor, J., Meißner, U.-G. (1992): Nucl. Phys. B388, 315
Bernard, V., Kaiser, N., Meißner, U.-G. (1997): Nucl. Phys. A615, 483
Bijnens, J., Colangelo, G., Gasser, J. (1994): Nucl. Phys. B427, 427
Bijnens, J., Colangelo, G., Ecker, G., Gasser, J., Sainio, M.E. (1996): Phys. Lett. B374, 210
Bijnens, J., Colangelo, G., Ecker, G., Gasser, J., Sainio, M.E. (1997): Pion–pion scattering at low energy, hep-ph/9707291, Nucl. Phys. B (in print)
Bürgi, U. (1996): Phys. Lett. B377, 147; Nucl. Phys. B479, 392
Colangelo, G. (1997a): Phys. Lett. B395, 289
Colangelo, G., Gasser, J., Leutwyler, H., Wanders, G. (1997b): in preparation
Dumbrajs, O. et al. (1983): Nucl. Phys. B216, 277
Ecker, G., Gasser, J., Pich, A., de Rafael, E. (1989): Nucl. Phys. B321, 311
Ecker, G., Možiš, M. (1996): Phys. Lett. B365, 312
Ecker, G., Možiš, M. (1997): Wave function renormalization in heavy baryon chiral perturbation theory, hep-ph/9705216, Phys. Lett. B (in print)
Ellis, P.J., Tang, H.-B. (1997): Pion–nucleon scattering in a new approach to chiral perturbation theory, hep-ph/9709354
Fearing, H.W., Scherer, S. (1996): Phys. Rev. D53, 315
Froggatt, C.D., Petersen, J.L. (1977): Nucl. Phys. B129, 89
Gasser, J., Leutwyler, H. (1983): Phys. Lett. B125, 325
Gasser, J., Leutwyler, H. (1984): Ann. Phys. (N.Y.) 158, 142
Gasser, J., Leutwyler, H. (1985): Nucl. Phys. B250, 465
Gasser, J., Sainio, M.E., Švarc, A. (1988): Nucl. Phys. B307, 779
Gasser, J., Leutwyler, H., Sainio, M.E. (1991): Phys. Lett. B253, 252, 260
Girlanda, L., Knecht, M., Moussallam, B., Stern, J. (1997): Comment on the prediction of two–loop standard chiral perturbation theory for low–energy ππ scattering, hep-ph/9703448, Phys. Lett. B (in print)
Höhler, G. (1983): in Landolt–Börnstein, vol. 9 b2, Ed. H. Schopper (Springer, Berlin)
Höhler, G. (1997): Contribution to this Workshop (Working Group 2)
Jenkins, E., Manohar, A.V. (1991): Phys. Lett. B255, 558
Kambor, J. (1997): These Proceedings and references therein
Knecht, M., Moussallam, B., Stern, J., Fuchs, N.H. (1995): Nucl. Phys. B457, 513
Knecht, M., Moussallam, B., Stern, J., Fuchs, N.H. (1996): Nucl. Phys. B471, 445
Knecht, M., Urech, R. (1997): Virtual photons in low–energy ππ scattering, hep-ph/9709348
Koch, R., Pietarinen, E. (1980): Nucl. Phys. A336, 331
Lee–Franzini, J. (KLOE) (1997): Contribution to this Workshop (Working Group 4) presented by G. Colangelo
Leutwyler, H. (1994): Ann. Phys. (N.Y.) 235, 165
Leutwyler, H. (1997): Probing the quark condensate by means of $\pi\pi$ scattering, Talk given at the DAΦNE Workshop, Frascati, Nov. 1996, hep-ph/9709406
Lowe, J. (BNL-E865) (1997): Contribution to this Workshop (Working Group 2)
Luke, M., Manohar, A.V. (1992): Phys. Lett. B286, 348
Meißner, U.-G., Müller, G., Steininger, S. (1997): Phys. Lett. B406, 154 and preprint hep-ph/9704377
Mojžiš, M. (1997): Elastic $\pi N$ scattering to $O(p^3)$ in heavy baryon chiral perturbation theory, hep-ph/9704415, Z. Phys. C (in print)
Pavan, M. (1997): Contribution to this Workshop (Working Group 2)
Pennington, M.R., Portolés, J. (1995): Phys. Lett. B344, 399
Rosselet, L. et al. (1977): Phys. Rev. D15, 574
Roy, S.M. (1971): Phys. Lett. 36B, 353
Schacher, J. (DIRAC) (1997): Contribution to this Workshop (Working Group 2)
Stern, J., Sazdjian, H., Fuchs, N.H. (1993): Phys. Rev. D47, 3814
Stern, J. (1997): These Proceedings and references therein
Wanders, G. (1997): Chiral two–loop pion–pion scattering parameters from crossing–symmetric constraints, hep-ph/9705323
Weinberg, S. (1966): Phys. Rev. Lett. 17, 616
Weinberg, S. (1979): Physica 96A, 327