On statistically stationary homogeneous shear turbulence

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A statistically stationary turbulence with a mean shear gradient is realized in a flow driven by suitable body forces. The flow domain is periodic in downstream and spanwise directions and bounded by stress free surfaces in the normal direction. Except for small layers near the surfaces the flow is homogeneous. The fluctuations in turbulent energy are less violent than in the simulations using remeshing, but the anisotropy on small scales as measured by the skewness of derivatives is similar and decays weakly with increasing Reynolds number.

Introduction.— Although most flows in nature and laboratory are anisotropic on large scales the statistical behaviour on small scales is expected to become isotropic. This seems to be supported by experiment and numerical analysis on the level of second order moments. However, motivated by analogous behaviour in the passive scalar problem, Pumir and Shraiman suggested that higher order moments might remain anisotropic even for large Reynolds number. It would seem that a natural situation in which to investigate this problem is that of a homogeneous shear flow, in which the time averages of the flow \[11,7\]. Our aim here is to present an alternative numerical approach to simulations of homogeneous shear turbulence.

Numerical Implementation.— With lengths measured in units of the gap width \(d\), and times in units of \(S^{-1}\), the dimensionless form of the equations for an incompressible Navier–Stokes fluid become

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + \frac{1}{Re_s} \nabla^2 \mathbf{u} + \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

where \(p(x,t)\) is the pressure, \(\mathbf{u}(x,t)\) the velocity field. The shear Reynolds number is \(Re_s = S \delta^2 / \nu\) with \(\nu\) as kinematic viscosity. In the \(x\) (streamwise) and \(z\) (spanwise) directions periodic boundary conditions apply. In the other direction the flow domain is bounded by two parallel flat surfaces that are assumed to be impenetrable and stress-free, i.e. \(u_y = \partial_x u_x = \partial_y u_z = 0\). The effects of the free slip surfaces at \(y = 0\) and \(y = d\) on the bulk behaviour are much weaker than those of rigid walls, since only the wall-normal component is forced to vanish. The resulting boundary layer in the wall normal component is of the order \(\nu / u'_{y, \text{rms}}\) where \(u'_{y, \text{rms}} = (\langle u'^2 \rangle)^{1/2}\) denotes the root mean square fluctuations of the turbulent wall normal velocity. The statistical properties of the tangential components are not affected by this boundary layer.

The mean shear and turbulence are maintained by a suitable body force \(\mathbf{f}(x,t)\). A linear mean profile \(\langle u_x \rangle(y) = (1/2 - y)\) for \(y \in [0,1]\) can be approximated by a finite Fourier sum of cosines

\[
\langle u_x \rangle(y) \approx \frac{4}{\pi^2} \sum_{n=0}^{5} \cos[(2n + 1)\pi y/(2n + 1)]
\]

The external forcing \(\mathbf{f}\) was chosen such that the six modes used in eq. remained constant in time, i.e. \(\partial_t \text{Re} \langle u_x(q,t) \rangle = 0\) for Fourier modes with \(q = (2n + 1)\pi d\) for \(n = 0\) to 5. At sufficiently high shear rates the flow is unstable and turbulence sets in. Then the force \(\mathbf{f}\) fluctuates in time as well.

The free slip boundary conditions allow for efficient numerical simulations with Fourier modes for the velocity components. Nevertheless, the data presented here amount to approximately 360 CPU hours of computing time on a Cray T-90. The equations are integrated by means of a pseudospectral technique using a 2/3-rule dealizing. The integration domain has an aspect ratio \(l_x : l_y : l_z = 2\pi d : d : 2\pi d\) and is resolved by \(256 \times 65 \times 256\) Fourier modes.

Stationarity.— When averaged in downstream and spanwise direction as well as in time, the mean velocity components show the expected shear flow behaviour, \(\langle u_y \rangle = \langle u_z \rangle = 0\) and \(\langle u_x \rangle \sim -y\) (see fig. ). The downstream profile differs from the linear shear flow only in a small region near the surfaces; this region decreases as the Reynolds number increases (inset of fig. ). Before starting the statistical analysis a forward integration over a period of \(ST \geq 20\) (time is measured in units of the shear
rate $S$) with the full spectral resolution was always performed to guarantee relaxation to the turbulent state.

The fluctuations in the velocity field are defined as $u'_i = u_i - \langle u_i \rangle$ for $i = x, y, z$. Figure 2 shows the time evolution of the total kinetic energy in the fluctuations, $q^2(t) = \langle (u'_i)^2 \rangle_V$, where $\langle \cdot \rangle_V$ denotes an average over the volume. The amplitude of the variations in kinetic energy decreases with increasing Reynolds number. This seems to be connected to the fragmentation of coherent streaks and vortices with increasing Re and the reduced downstream correlation. This is shown by volume surface plots of the turbulent streamwise velocity component at the lowest (see fig. 3) and the highest (see fig. 4) of our Reynolds numbers. The periods of the oscillations in total kinetic energy are surprisingly large, thus requiring very long time integrations for converged time averages. Compared to the long time simulations of Pumir [7] the very long time integrations for converged time averages, $\langle \cdot \rangle$ with the full spectral resolution was always performed to guarantee relaxation to the turbulent state.

Finally, statistical stationarity implies balancing of the turbulent kinetic energy in the mean,

$$0 = -\nu \langle \partial_x u'_j (\partial_i u'_j + \partial_j u'_i) \rangle - \langle u'_x u'_y \rangle \partial_y \langle u_x \rangle + \langle u'_x f'_z \rangle. \quad (5)$$

The first term on the right side is the energy dissipation rate $\epsilon$ and the second term is the turbulent energy production rate $P$. The last term, is the energy injection due to the applied volume forcing. In the bulk, outside the small boundary layers near the top and bottom surfaces, we find that the production and dissipation differ by less than 4% and that the energy injection from the volume forcing is negligibly small. Here we took volume and time averages for the Reynolds stress component in the production term, for the energy dissipation rate and the energy injection term. We used an averaging time in shear rate units, i.e. $ST$, of 75 for the lower $R_s$ and 50 as well as 100 for the highest $R_s$.

**Higher order statistical moments.—** Data on statistical moments from the experiment of Garg and Warhaft [5], two sets of direct numerical simulations [6] and [7] and our simulations are collected in table I. The data sets cover almost the same range of parameters with similar behaviour, despite the different realizations of the mean shear. The dimensionless $S^* = Sq^2/\epsilon$ does not seem to vary with $Re_s$, but $S(\nu/\epsilon)^{1/2}$ decreases for our range of $Re_s$. This agrees with the observations of Pumir and Shraiman [8].

| Exp. [5] | DNS [6] | DNS [7] | $Re_s = 500$ | $Re_s = 1000$ | $Re_s = 2000$ |
|----------|---------|---------|---------------|---------------|---------------|
| $(\langle u_i'^2 \rangle)/q^2$ | 0.53 | 0.53 | 0.57 | 0.55 | 0.52 (0.52) |
| $S_{\omega}/S$ | 3.31 | 4.58 | 3.98 | 4.17 | 4.72 (4.26) |
| $S(\nu/\epsilon)^{1/2}$ | 0.04 | 0.11 | 0.38 | 0.25 | 0.18 (0.17) |
| $P/\epsilon$ | 0.96 | – | 1.04 | 0.99 | 1.02 (0.98) |
| $R_s$ | 310 | 73 | 90 | 59 | 87 | 94 (106) |
| $S_{\omega}^2$ | – | – | – | – | – |
| $K_{xx}$ | – | – | – | – | – |
| $S_{\omega_x}/\nu$ | 0.5 | 0.87 | 0.96 | 0.90 | 0.82 (0.84) |
| $K_{\omega_x}/\nu$ | 8.6 | – | – | – | – |

**TABLE I.** Comparison between experiment and different simulations on homogeneous shear turbulence. The Taylor-Reynolds number is calculated from $R_s = \langle (u_i')^2 \rangle/\nu \langle (\partial_x u_x')^2 \rangle^{1/2}$. The skewness and kurtosis of a field $\phi$ are defined as $S_\phi = \langle \phi^3 \rangle/\langle \phi^2 \rangle^{3/2}$ and $K_\phi = \langle \phi^4 \rangle/\langle \phi^2 \rangle^2$, respectively. Note that here the total kinetic energy $q^2 = \langle (u_i')^2 \rangle$ is also averaged in time. The values given in parentheses in the last column are obtained by averaging over a time interval twice as long.

Most quantities in table I change monotonically with $Re_s$. We attribute the few exceptions to peculiarities of the simulations at $Re_s = 500$ which has the strongest coherent structures and is perhaps most strongly affected by the downstream periodicity. Coherent structures (predominantly streaks and downstream vortices) show up in all our simulations, even at the lowest Reynolds number. In the simulations of [7] that used remeshing they were observed for higher values of $S^*$ only where the flow became strongly non-stationary. The effects of vortices and downstream periodicity may also explain the rather large skewness values $S_{\omega_x}$ and $S_{\omega_x}/\nu$ for $Re_s = 500$. Both quantities decrease steadily with $R_s$, while they remained constant ($\sim R_s^0$) in the simulations of Ref. [5]. The shear flow experiments [5] have been fitted to $\sim R_s^{-0.6}$ for $150 \lesssim R_s \lesssim 400$. The statistical error of the data may be gauged by comparing the results for two different averaging times. The last two columns in table I, obtained by averaging over $ST = 50$ and $ST = 100$, respectively, show small variation. In particular, the trend observed for skewness and kurtosis is weak but lies outside these statistical variations. A similar slow decay of anisotropy
effects was also found recently in a systematic analysis of the anisotropic scaling contributions to high-order structure functions in a turbulent atmospheric boundary layer with $R_\lambda \sim 2000$ [18].

The effects of the surfaces on the higher order moments are limited to the boundary layers very close to the surfaces. The spatially resolved plots in fig. 5 show the shear direction dependence of the skewness and kurtosis of the spanwise vorticity, $\omega_z = \partial_x u'_y - \partial_y u'_x$, and the shear gradient $\partial_y u'_x$. The variations across the shear layer become smaller with increasing Reynolds number.

**Summary** — The simulations for the statistically stationary shear flows bounded by free slip surfaces show that in the central region an approximately homogeneous shear flow with statistically stationary properties develops. The moments of the velocity field are compatible with previous experimental and numerical findings. The most noticeable difference to the long time simulations by Pumir [7] is the absence of violent bursts in turbulent energy. Further investigations of the statistical properties of this model for almost homogeneous shear flows are in progress.

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FIG. 1. The mean velocity profiles $\langle u_i(y) \rangle$ for the three components $i = x, y$ and $z$, averaged in time and over planes normal to the mean shear, for $Re_s = 500, 1000$ and $2000$. The solid line is the idealized linear function in dimensionless form. The inset resolves the small layer near the surface where the mean shear is not constant.

FIG. 2. Time traces of the turbulent kinetic energy $q^2(t)/(Sd)^2$ and of the separate contributions from the turbulent velocity components $\langle (u'_i)^2 \rangle_V/(Sd)^2$ for the three components $i = x, y$ and $z$.

FIG. 3. Isosurfaces of the streamwise turbulent velocity component $u'_x$ for $Re_s = 500$ at the levels $u'_x/(Sd) = \pm 0.48$. The positive value is brighter.

FIG. 4. Same as fig. 3, but for $Re_s = 2000$ with levels $u'_x/(Sd) = \pm 0.46$. At this higher Reynolds number the flow structures are more fragmented and smaller.

FIG. 5. Profiles of skewness and kurtosis in normal direction for two values of $Re_s$. The datapoints closest to the boundary surfaces (indicated by dotted lines) are excluded from the statistical analysis that leads to the entry in table. I.
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