Gyrokinetic simulations of plasma turbulence in a Z-pinch configuration using a moment approach and advanced collision operators

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(Received xx; revised xx; accepted xx)

The first nonlinear gyrokinetic simulations obtained with a moment approach based on the Hermite-Laguerre decomposition of the distribution function and the use of advanced collision operator models are presented. Turbulence in a Z-pinch configuration is considered within a flux tube configuration. In the collisionless regime, our moment approach shows very good agreement with nonlinear simulations carried out with the continuum gyrokinetic code GENE, even with a reduced number of moments than the one necessary for the convergence of the linear growth rate. By using advanced linear collision operators, the role of collisions in setting the level of turbulent transport is then analyzed. The choice of the collision operator model is shown to have a crucial impact when turbulence is quenched by the presence of zonal flows. The convergence properties of the moment approach improve when collisions are included.

1. Introduction

In order to efficiently study the dynamics of a plasma in the parameter regime of the tokamak boundary, while bridging the gap between fluid models (see, e.g., Dudson et al. (2009); Stegmeir et al. (2018); Bufferand et al. (2019); Giacomin et al. (2022)) and more expensive gyrokinetic (GK) simulations (see, e.g., Grandgirard et al. (2016); Chang et al. (2017); Dorf & Dorr (2020); Mandell et al. (2020); Michels et al. (2021)), Frei et al. (2020) extend the drift-kinetic model in Jorge et al. (2017a) and develop a GK model based on the projection of the velocity space dependence of the distribution function on a Hermite-Laguerre polynomial basis. This approach, also pursued by Mandell et al. (2018a) with the GX code (Mandell et al. 2018b), can be interpreted as an extension of the gyrofluid model (Beer & Hammett 1996; Ribeiro & Scott 2008; Held et al. 2016) to an arbitrary number of moments, since it yields an infinite set of fluid equations for the basis coefficients, the gyromoments, which describe the deviations of the distribution function from a Maxwellian distribution. The set of fluid equations converges, as the number of moments increases, to the description of the evolution of the distribution function provided by the full GK Boltzmann equation. Advanced collision operators, such as the full nonlinear Coulomb collision operator (Jorge et al. 2019), as well as linear operators (Frei et al. 2021) are expressed in this framework. The efficiency of the gyromoment approach is demonstrated by Frei et al. (2022b) focusing on the linear properties of the ion temperature gradient (ITG) instability in the local limit, as well as in a flux-tube geometry (Frei et al. 2022a), including the use of the linearized GK Coulomb collision operator. These works demonstrate the improvement of convergence properties of the gyromoments method with collisions, when deviations

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from a Maxwellian distribution function are reduced. In addition, even in the collisionless case, it is shown that the number of gyromoments needed for convergence is not larger than the number of grid points necessary for convergence in continuum GK codes.

In order to extend previous investigations based on the gyromoment approach to the nonlinear regime, we consider a simplified setting, such as the Z-pinch geometry, which is characterized by a cylindrical symmetric plasma confined by a purely azimuthal, radially dependent, magnetic field with equilibrium temperature and density radial gradients. Despite the simplicity of this configuration, which ultimately allows for substantial analytical progress, turbulence in the Z-pinch geometry displays a complex dynamics reminiscent of more complex magnetic topologies.

In the presence of a background density gradient, the entropy mode (Ricci et al. 2006b) develops in a Z-pinch, which can be modelled by using a local $\delta f$ GK approach with a kinetic treatment of the electrons. The first nonlinear simulations in a Z-pinch, presented by Ricci et al. (2006a), study the level of transport induced by the entropy mode depending on the density gradient, showing that zonal flows (ZF) regulate the level of turbulent transport. However, ZF can be destroyed either by strong collisional damping (modelled in Ricci et al. (2006a) through a drift–kinetic (DK) Lorentz operator), or by a tertiary Kelvin–Helmholtz instability (KHI), destabilized in scenarios characterized by a sufficiently large density gradient drive. These results are confirmed by Kobayashi & Rogers (2012) using a GK Lorentz collision operator (Barnes et al. 2009). At low density gradient drive, i.e. under the tertiary KHI instability threshold, transport regimes characterised by bursts rising from the competition between ZF collisional damping and quenching of the primary instability are identified and modeled with a predator–prey cycle by Kobayashi et al. (2015). In order to explore the mechanisms behind the ZF formation and damping, Ivanov et al. (2020) use a fluid diffusive collision operator obtained by the integration of the full linearized Coulomb collision operator and derive a three-field two-dimensional fluid model directly from the GK equation in a Z-pinch geometry, later extended to three dimensions (Ivanov et al. 2022). This model includes first-order finite Larmor radius (FLR) effects in the long wavelength, cold ion limits and allows exploring the ZF dynamics within an analytical framework. The simulations show good qualitative agreement with modified Hasegawa-Wakatani simulations (Qi et al. 2020). Similarly, Hallenbert & Plunk (2021) derive a fluid model in a Z-pinch geometry in the collisionless limit, including FLR effects up to the second order. Then, they predict numerically the Dimits transition, being the transport strongly reduced by the presence of ZF when the density is below a density gradient threshold ()). These predictions are confirmed by using the state-of-the-art gyrokinetic continuum code GENE (Jenko et al. 2000).

The present work reports on the first nonlinear GK simulations carried out with the gyromoment approach and advanced collision operators. These simulations include nonlinear $E \times B$ advection, FLR effects at arbitrary order, kinetic electrons and, leveraging the work in Frei et al. (2021), a set of advanced linear GK collision operators. These operators include the single-species Dougherty model (Dougherty 1964), the multi-species Sugama model (Sugama et al. 2009), the single-species pitch-angle scattering operator with a restoring momentum term, denoted as Lorentz operator (Helander & Sigmar 2002), and the Landau form of the multi-species Fokker–Planck model that we denote as Coulomb operator (Rosenbluth & Longmire 1957; Hazeltine & Meiss 2003). We consider a $\delta f$ flux-tube approach that separates equilibrium and fluctuating quantities, assuming constant equilibrium gradients across the domain. By imposing $k_{\parallel} = 0$, we evolve the turbulent dynamics on a perpendicular plane.

Our results demonstrate, first, the ability of the gyromoment approach to retrieve
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linear and nonlinear collisionless results obtained with the GK continuum code GENE. In particular, we observe that the number of gyromoments needed for convergence increases while approaching marginal stability conditions, and that under-resolved collisionless simulations present predator-prey cycles, typically observed in collisional GK simulations (Kobayashi et al. 2015) and fluid reduced model Qi et al. (2020). The same dynamics is observed when increasing drastically the numerical dissipation acting on the velocity space in GENE. Second, we present a set of simulations where we vary the instability drive in the collisionless limit and in the presence of collisions, which are modelled using the Dougherty, Sugama, Lorentz and Coulomb collision operators. The particle flux reveals a typical Dimits threshold in the collisionless limit. At finite collisionality we observe negligible differences between collisionless and collisional results for gradient above the Dimits threshold, independently of the collision operator we use. In this parameter regime, shear flow stabilization effects are negligible and turbulence is fully developed. The transport is well approximated by a mixing length argument \( \Gamma_x \sim \gamma_p^2 \) (Ricci et al. 2006a), where \( \Gamma_x \) is the saturated particle transport level along the radial direction, while \( \gamma_p \) is the peak linear growth rate of the entropy mode. Below the Dimits threshold, the mixing length argument is not sufficient to estimate and explain the nonlinear saturated transport level. In this case, turbulence is quenched by ZF, which may be damped by collisions and a dependence of the level of transport on the collision models emerges. A ZF collisional damping study explains the differences observed between the collision operators.

The reminder of the present paper is organized as follow. In Sec. 2, we briefly describe the nonlinear GK model in Z-pinch geometry and develop the gyromoment approach in this configuration. Section 3 presents linear and nonlinear benchmarks of the gyromoment approach with GENE in the collisionless limit. The dependence of the transport level with the instability drive and the role of collisions is investigated in Section 4. The conclusions follow in Sec. 5.

2. Gyrokinetic model of a Z-pinch configuration based on the gyromoment model

In this section we present, first, the gyrokinetic (GK) model in the Z-pinch geometry, considering the \( \delta f \) flux-tube limit. Second, we project the Z-pinch GK equation on a Hermite-Laguerre polynomial basis in velocity space, thus obtaining an infinite set of two-dimensional equations for the gyromoments. Finally, we present the numerical implementation of the hierarchy of equations for the gyromoments.

2.1. Gyrokinetic model in a Z-pinch configuration

We consider the GK approach (Hazeltine & Meiss 2003) to study turbulence in a Z-pinch geometry. Using the standard \( \delta f \) approach, we decompose the gyrocenter distribution function of species \( a, a = \{e, i\} \), as the sum of an equilibrium Maxwellian component and a perturbation, \( f_a = F_{aM} + \delta f_a \), where \( \delta f_a \ll F_{aM} \) and the Maxwellian distribution for a species \( a \) is defined as \( F_{aM} = N_a/(\pi^{1/2}v_{tha})^3 \exp(-m_a v_{\parallel}^2/2T_a - \mu B/T_a) \), with \( B = B_b \) the equilibrium magnetic field, \( N_a \) the background density, \( T_a \) the background temperature, \( m_a \) the particle mass, \( v_{\parallel} \) the velocity parallel to the magnetic field line, \( \mu = m_a v_{\perp}^2/B \) the magnetic moment and \( v_{tha}^2 = 2T_a/m_a \) the thermal velocity. The electrostatic GK Boltzmann equation determining the evolution of the Fourier modes
of the perturbed gyrocenter distribution function,
\[
g_a(k, v_{\parallel}, v_{\perp}, t) := \int \delta f_a(R, v_{\parallel}, v_{\perp}, t) e^{-ikR} dR,
\]  
(2.1)
with \( k = k_\perp + k_\parallel b \), writes (Brizard & Hahm 2007)
\[
\partial_t g_a + i\omega_B a g_a + \frac{1}{B} \{ g_a + F_{aM}, J_0 \phi \} = \sum_{a'} C_{aa'}.
\]  
(2.2)
where the Poisson bracket operator, \( \{ f, g \} = b \cdot (\nabla f \times \nabla g) \), is introduced to describe the effect of the background density and temperature gradients and of the quadratic nonlinearities rising from the \( E \times B \) drift. In Eq. (2.2) the magnetic drift frequency \( i\omega_B a \) contains the magnetic curvature and gradient drifts, i.e.
\[
\omega_B a = b \times \frac{1}{\Omega_a} \left[ v_{\parallel}^2 (b \cdot \nabla) b + v_{\perp}^2 \nabla \ln B \right] \cdot k.
\]  
(2.3)
where we introduce the cyclotron frequency \( \Omega_a = q_a B/m_a, q_a \) being the particle charge. In addition, following previous work, in Eq. (2.2) we assume \( k_\parallel = 0 \) (Ricci et al. 2006a; Ivanov et al. 2020), and therefore we consider a two-dimensional domain that extends perpendicularly to the magnetic field line. The electrostatic potential \( \phi \) is evaluated at the gyrocenter position through the gyroaveraging operator expressed in Fourier space with the Bessel function of the first kind, \( J_0(b) \) with \( b = k_\perp v_{\perp}/\Omega_a \), representing FLR effects at all orders. Finally \( C_{a,a'} \) is the collision operator between species \( a \) and \( a' \).

The electrostatic Poisson’s equation, in the quasi neutrality limit, allows us to close the system by expressing the fluctuation of the electrostatic potential according to
\[
\sum_a q_a^2 \frac{g_a}{\tau_a} \left( 1 - \frac{q_a^2}{2} e^{-b_a^{2}/2} \right) \phi = \sum_a q_a \int d\mathbf{v} J_0 a g_a
\]  
(2.4)
where the right-hand side of Eq. (2.4) represents the gyroaveraged charge density and \( I_n \) is the modified Bessel function.

Equation (2.2) is now simplified considering the Z-pinch magnetic field and geometry (Ricci et al. 2006b). Using local field-aligned coordinates \( (x, y, z) \), the Z-pinch magnetic field can be expressed as \( B = B_e z \). The magnetic field presents a radial gradient, \( \nabla B/B = -1/L_B e_x \), and curvature, \( (b \cdot \nabla) b = -1/L_B e_x \), which are assumed constant within the flux-tube approach. Similarly, we also consider constant background density and temperature gradients, \( \nabla N_a/N_a = -1/L_N e_x \) and \( \nabla T_a/T_a = -1/L_T e_x \), for both electrons and ions. The GK equations for \( g_a \), Eq. (2.2), writes now in the Z-pinch geometry as
\[
\partial_t g_a + (ik_x J_0 \phi) * (ik_y g_a) - (ik_y J_0 \phi) * (ik_x g_a) + \frac{\tau_a}{z_a} \left[ s_{||a}^2 + \frac{1}{2} x_{a}^2 \right] i k_y g_a
\]  
+ \( \kappa_N \left[ 1 + \eta \left( s_{||a}^2 + x_{a} - \frac{3}{2} \right) + \frac{1}{\kappa_N} \left( 2s_{||a} + x_{a} \right) \right] J_0 i k_y \phi = \sum_{a'} C_{aa'}
\]  
(2.5)
where the convolution between two fields is introduced, \( f \ast g := \int d\mathbf{k}' f(\mathbf{k} - \mathbf{k}') g(\mathbf{k}') \). We close our system by writing the Poisson equation, i.e.
\[
\sum_a \frac{z_a^2}{\tau_a} \left( 1 - \frac{q_a^2}{2} e^{-b_a^{2}/2} \right) \phi = \sum_a z_a \int ds_{||a} dx_a J_0 a g_a.
\]  
(2.6)
In Eqs. (2.5) and (2.6), as well as in the following of the paper, we use dimensionless
units. The dimensionless parallel and perpendicular velocity coordinates are defined by \( s_{\parallel a} = v_{\parallel}/v_{th,a} \) and \( x_a = \mu B/T_a \), respectively. The perpendicular spatial scales are normalized to the electron sound Larmor radius \( \rho_{se} = c_s/\Omega_e \) with \( c_s = \sqrt{T_{e}/m_e} \) the sound speed. Time is normalized to \( L_B/c_s \). The electrostatic potential is normalized to \( T_e/e \) with \( e \) the elementary charge which allows us to define the normalized particle charge, \( z_a = q_a/e \), as well. We define the temperature and mass ratio \( \tau_a = T_a/T_e \) and \( \sigma_a = \sqrt{m_0/m_i} \), respectively, and we introduce the dimensionless density gradient drive, \( \kappa_N = L_B/L_N \), and density temperature gradients ratio, \( \eta = |\nabla \ln T|/|\nabla \ln N| \).

### 2.2. Nonlinear gyromoments hierarchy

In order to solve Eq. (2.5) in the gyromoments framework, we expand the distribution function on a Hermite-Laguerre polynomial basis (Jorge et al. 2017b; Frei et al. 2020), i.e.

\[
g_a(k, s_{\parallel a}, s_{\perp a}, t) = \sum_{p,j} N^{p,j}_a(k, t) H_p(s_{\parallel a}) L_j(x_a) F_{aM}(s_{\parallel a}, s_{\perp a})
\]

(2.7)

where

\[
H_p(s_{\parallel a}) = \frac{(-1)^p}{\sqrt{2^p p!}} e^{s_{\parallel a}^2} \frac{d^p}{ds_{\parallel a}^p} e^{-s_{\parallel a}^2}
\]

(2.8)

and

\[
L_j(x_a) = e^{-x_a} \frac{d^j}{dx_a^j} e^{x_a}
\]

(2.9)

denote the probabilistic Hermite polynomial of order \( p \) and Laguerre polynomial of order \( j \), respectively. The Hermite polynomials of Eq. (2.8) are normalized such that \( \int ds_{\parallel a} H_p H_{p'} e^{-s_{\parallel a}^2} = \delta_{pp'} \). Similarly, the Laguerre polynomials follow the orthogonality relation \( \int dx_a L_j L_{j'} e^{-x_a} = \delta_{jj'} \).

We now project the Boltzmann GK equation, Eq. (2.5), onto the Hermite-Laguerre basis by introducing the projection operator \( ||f||^2_a = \int f H_p(s_{\parallel a}) L_j(x_a) ds_{\parallel a} dx_a \) for any generic function of the phase space, \( f \). We define the gyromoment of order \( (p,j) \) of the distribution \( g_a \) as \( N^{p,j}_a(k, t) = ||g_a||^2_a \) and we expand the Bessel function of the first kind in terms of Laguerre polynomials as

\[
J_0 = J_0(\sqrt{l_a x_a}) = \sum_{n=0}^{\infty} K_n(l_a) L_n(x_a)
\]

(2.10)

with the kernel functions \( K_n(l_a) = l_a^n e^{-l_a}/n! \), being \( l_a = \sigma_a^2 \tau_a k_{\perp}^2/2 \) (Frei et al. 2020). The projection of Eq. (2.5) yields the gyromoment nonlinear hierarchy in a Z-pinch configuration, which can be expressed as

\[
\partial_t N^{pj}_a + M^{pj}_a + D^{pj}_a = S^{pj}_a + C^{pj}_a
\]

(2.11)

where

\[
M^{pj}_a = \frac{\tau_a}{z_a} i k_y \left[ \sqrt{(p+1)(p+2)n_a^{p+2,j}} + (2p + 1)n_a^{p,j} + \sqrt{p(p-1)}n_a^{p-2,j} \right]
\]

\[
+ \frac{\tau_a}{z_a} i k_y \left( (2j + 1)n_a^{p,j} - (j + 1)n_a^{p,j+1} - j n_a^{p,j-1} \right)
\]

(2.12)
and
\[ D^p_j = -\kappa_N i k_y \phi [\mathcal{K}_j \delta^0_p + \eta J_j \left( \frac{1}{\sqrt{2}} \delta^2_j - \frac{1}{2} \delta^0_j \right) + \eta ((2j+1)K_j - [j+1]K_{j+1} - jK_{j-1})\delta^0_j], \]
\[ \text{(2.13)} \]

having introduced the non-adiabatic gyromoments,
\[ n_{a}^{pj}(k, t) = ||g_a + \tau_a / q_a \lambda_0 \phi||_{a}^{pj} = N_{a}^{pj} + \tau_a / q_a \mathcal{K}_j \delta^0_j. \]
\[ \text{(2.14)} \]

The Hermite polynomial product rule, \( s_{a}H_p = (p+1)H_{p+1} + \sqrt{2}H_{p-1} \), and the Laguerre polynomial product rule, \( x_a L_j = (2j+1)L_j - jL_j - (j+1)L_j \), as well as the derivative rule, \( H'_p = \sqrt{2p}H_{p-1} \), are used to deduce Eqs. (2.12) and (2.13).

The nonlinear term related to the \( E \times B \) drift is expressed in terms of gyromoments by using the Bessel-Laguerre decomposition in Eq. (2.10), which yields
\[ S_{a}^{pj} = \sum_{n=0}^{\infty} (ik_x \mathcal{K}_n \phi) * \left( ik_y \sum_{s=0}^{n+j} d_{njs} N_{a}^{ps} \right) - \sum_{n=0}^{n+j} (ik_y \mathcal{K}_n \phi) * \left( ik_x \sum_{s=0}^{n+j} d_{njs} N_{a}^{ps} \right). \]
\[ \text{(2.15)} \]

To obtain Eq. (2.15), we express the product of two Laguerre polynomials as sum of single polynomials using the identity
\[ L_j L_n = \sum_{s=0}^{n+j} d_{njs} L_s \]
\[ \text{(2.16)} \]
with
\[ d_{njs} = \sum_{n_1=0}^{n} \sum_{j_1=0}^{j} \sum_{s_1=0}^{s} \frac{(-1)^{n_1+j_1+s_1}}{n_1! j_1! s_1!} \binom{n}{n_1} \binom{j}{j_1} \binom{s}{s_1}. \]
\[ \text{(2.17)} \]

Finally, the \( C_{a}^{pj} \) term in Eq. (2.11) represents the effect of collisions. The details of the GK Dougherty, GK Sugama, GK Lorentz and the GK Coulomb operators can be found in Frei et al. (2021). Here, we note that we set the intensity of the collisions through the normalized ion-ion collision frequency \( \nu \). The collision frequencies among the different species are thus given by \( \nu_{ii} = \nu \), \( \nu_{ee} = \sigma_e \tau_e^{3/2} \nu \), \( \nu_{ei} = \nu \) and \( \nu_{ee} = \sigma_e \tau_e^{3/2} \nu \).

The Poisson equation, Eq. (2.6), is also projected onto the Hermite-Laguerre basis. This yields (Frei et al. 2020)
\[ \left[ \sum_{a} z_a^2 \mathcal{K}_n^2 \right] \phi = \sum_{a} z_a \sum_{n=0}^{\infty} \mathcal{K}_n N_{a}^{0n}, \]
\[ \text{(2.18)} \]
where the quasi-neutrality approximation is used, i.e. \( k_y \lambda_D \ll 1 \) with \( \lambda_D \) the Debye length.

To conclude, we note that we characterize the turbulent transport in a Z-pinch by considering the dimensionless ion particle flux, \( \Gamma = \delta n_i \times v_{E \times B} \), with \( v_{E \times B} = -\nabla \phi \times b \) the \( E \times B \) velocity and \( \delta n_i = \sum_{n=0}^{\infty} \mathcal{K}_n N_{i}^{0n} \) the ion particle density. In the following, we analyse the time series of the spatially averaged radial ion particle flux, \( \Gamma_x(t) = \langle \Gamma \cdot e_x \rangle y \), which can be expressed, using the Fourier modes of the gyromoments, as
\[ \Gamma_x(t) = \frac{1}{4\pi^2} \int (ik_y \phi)^* \sum_{n=0}^{\infty} \mathcal{K}_n N_{i}^{0n} d \mathbf{k}. \]
\[ \text{(2.19)} \]
2.3. Numerical approach

In order to solve Eq. (2.11), we evolve a finite set of gyromoments $N^p_j(k, t)$ with $0 \leq p \leq P$ and $0 \leq j \leq J$ and consider the Fourier modes with $k_x = m \Delta k_x$, with $0 \leq m \leq M$, and $k_y = n \Delta k_y$, with $-N/2+1 \leq n \leq N/2$, using a standard explicit fourth-order Runge-Kutta time-stepping scheme. In the Z-pinch geometry, the gyromoments hierarchy decouples odd and even Hermite moments, which is a consequence of the $k_{\parallel} = 0$ assumption. This allows us to evolve only the even moments $N^p_{a,j}$, with $p = 2n$, $n \in \mathbb{N}$. The hierarchy is closed using a simple truncation, i.e. $N^p_{a,j} = 0$ for all $p > P$ or $j > J$. The use and analysis of more advanced closure scheme, e.g. semi-collisional closure proposed by Zocco & Schekochihin (2011) and Loureiro et al. (2016), is left for future work.

Focusing on the nonlinear term in Eq. (2.15), we first observe that truncation of the sum over $s$ in Eq. (2.15) must be avoided in order to prevent polynomial aliasing. Hence, we truncate the sum over $n$ in Eq. (2.15) to $n \leq J - j$, to guarantee the exact Laguerre product identity in Eq. (2.16). Second, we note that the computation of the $d_{njs}$ coefficients is challenging since they involve the sum and differences of large numbers. To avoid the overflow of the floating point representation, we use an arbitrary precision library for our calculations (Smith 1991). Finally, we remark that the convolutions in Fourier space are treated with a conventional pseudo-spectral method, i.e. the backward fast Fourier transform (Frigo & Johnson 2005) of the fields to convolve, the multiplication in real space and the forward fast Fourier transform of the result including the usual 2/3 Orszag rule for anti-aliasing (Orszag 1971).

Regarding the collision operators, the GK Dougherty operator, which is computationally light, is directly implemented in the moment hierarchy. On the other hand, the evaluation of the GK Sugama, the GK Lorentz pitch-angle and GK Coulomb collision terms is reduced to a fourth-dimensional matrix vector operation, i.e. the $p,j$-th collision term is written as $C^{pj}_a = \sum_{p'=0}^{P} \sum_{j'=0}^{J} C^{pj,p'j'}_{aa'} N^{p'j'}_{a'}$ with the collision matrix $C^{pj,p'j'}_{aa'}$ of size $(P_a \times J_a) \times (P_{a'} \times J_{a'})$. The projection of the collision operators on the Hermite-Laguerre basis and the details of the computation of the matrix coefficients for each collision operator mentioned can be found in Frei et al. (2021). It is worth noting that the GK corrections create a $k_{\perp}$ dependency of the matrix coefficients that calls for the precomputation of the coefficients for each $k_{\perp}$ present in the simulations.

3. Collisionless limit and benchmark with the GENE code

In the present section we analyse the results of the gyromoment simulations in the collisionless limit and demonstrate the ability of this approach to retrieve the results of the continuum gyrokinetic code GENE in a collisionless Z-pinch configuration considering an isothermal equilibrium plasma, $\tau = 1$, and a realistic mass ratio, $\sigma = 0.023$.

At density gradients below the magneto-hydrodynamic (MHD) interchange instability threshold, a small-scale non-MHD instability, the entropy mode, can be destabilized in the Z-pinch configuration. The theory and region of stability of the entropy mode is presented in Ricci et al. (2006b). Our analysis focuses on density gradient values $1.6 \leq \kappa_N \leq 2.5$, while the temperature density gradient ratio is constant, $\eta = 0.25$. This parameter encompasses the an unstable region of the entropy mode, which extends, indeed, from $\kappa_N \simeq 2.5$, where the ideal MHD interchange mode is destabilized, to $\kappa_N \simeq 1.6$, which is close to marginal stability.

We start the analysis of the collisionless case by focusing on the linear growth rate of the entropy mode. Obtained by zeroing out the nonlinear terms in Eq. (2.15), the growth rate of the entropy mode is shown in Fig. 1. First, we note the gyromoment approach
retrieves, given a sufficiently large polynomial basis, the converged results obtained with GENE. The convergence properties of the gyromoment model depend on the strength of the gradients and improve at steep gradients, confirming previous results obtained for the slab (Frei et al. 2022) and the toroidal ITG instability (Frei et al. 2022a). Second, the results obtained with a number of polynomials below convergence show a stabilization of the high $k_y$ tail of the entropy mode and a larger peak growth rate. Third, it is worth noting that, independently of the polynomial resolution, the growth rates obtained with a small number of polynomials agree with the converged results in the long wavelength limit, $k_y \ll 1$, highlighting the fact that the gyromoment method retrieves the fluid limit, even when a small set of gyromoments is used.

Let us now consider the nonlinear case by including the $E \times B$ term in Eq. (2.15). GENE results are used to benchmark our implementation (the GENE simulations performed here closely recall those presented by Hallenbert & Plunk (2022)). We focus on three values of the density background gradient, $\kappa_N = 1.6, 2.0$ and 2.5, with $\eta = 0.25$ and $\nu = 0$. The system is evolved in a periodic box of dimension $L_x \times L_y = 120 \times 80$ for the lowest gradient value and $L_x \times L_y = 400 \times 240$ for the highest gradient value. In terms of spatial resolution, we consider a Fourier grid with $N = 128$ and $M = 32$ Fourier modes along the $x$ and $y$ direction, respectively, except for the steepest gradient case where we increase the resolution to $N = 256$ and $M = 128$ in order to reduce the need of artificial numerical dissipation. The velocity space is represented by a finite Hermite-Laguerre set with $(P, J) = (4, 2)$ extended up to $(P, J) = (20, 10)$ at the lowest gradient. GENE results are obtained using the same Fourier grid as the gyromoment simulation and a velocity grid resolution of $32 \times 12$ points in the $(v_\parallel, \mu)$ velocity space in a box of dimension $L_{v_\parallel} = 6$ and $L_\mu = 4$, which ensure convergence of GENE results. This resolution is presented as the minimal necessary not to compromise key results by Hallenbert & Plunk (2022) which is confirmed by observing spurious predator-prey cycles when running lower resolution simulations at $\kappa_N = 1.6$.

When running the collisionless cases, GENE uses a kinetic artificial diffusion term, $\nu_v (\Delta v_\parallel/2)^4 \partial_{v_\parallel}^4 g_a$, with the diffusion parameter fixed to $\nu_v = 0.2$ (Pueschel et al. 2010). Both codes use a spatial fourth order hyperdiffusion term $\mu_{HD}(k/k_{max})^4$ with $0.5 \leq \mu_{HD} \leq 5.0$, adjusted on the drive level in order to avoid energy pile-up without compromising the accuracy of results. For the intermediate gradient level, we run GENE with both constant and adaptive numerical diffusion (see Hallenbert & Plunk (2022)). This test allows us to confirm that the level of transport is resilient to spatial hyperdiffusion. While a comparison of the computational cost of the two approaches is not straightforward, we note that the number of gyromoments evolved is given by $N_{P,J} = (P/2 + \ldots$
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Figure 2. Comparison of the time-averaged transport level, $\Gamma_x^\infty$, obtained with the gyromoment (GM) approach for $(P, J) = (4, 2)$ (blue), $(10, 5)$ (red) and $(20, 10)$ (yellow) and GENE, in the collisionless case.

Figure 3. Spectrum of the radial particle flux, $\langle |\Gamma_x(k_x = 0, k_y)| \rangle_t$ for the highest resolution simulations presented in Fig. 5 for $\kappa_N = 2.5$ (solid squares), $\kappa_N = 2.0$ (dashed diamonds) and $\kappa_N = 1.6$ (dotted circles).

$1) \times (J + 1)$ (we take into account that only the even $p$ gyromoments are evolved in the Z pinch geometry). Therefore, 9 and 25 gyromoments are evolved in the $(P, J) = (4, 2)$ and $(20, 10)$ simulations, respectively. This compares with the $10^2$ velocity grid points used by GENE.

Focusing on the turbulent state that is established after an initial transient following the initialization of the simulation, the gyromoment approach retrieves the time-averaged turbulent transport level, $\Gamma_x^\infty = \langle \Gamma_x \rangle_t$, obtained by GENE for all drive values, given a sufficient number of moments, over four order of magnitudes (see Fig. 2). As for the linear case, faster convergence with the number of gyromoments is observed in the case of the strongest gradient, with a set of $(P, J) = (4, 2)$ gyromoments being sufficient for convergence. This result might be surprising, considering the linear growth rate obtained with the same gyromoment resolution, significantly broader and showing a higher peak value than the converged value (see Fig. 1). Even the results obtained with $(P, J) = (4, 2)$ at the lowest gradient are surprisingly accurate when considering the accuracy of the linear growth rate.

One can explain the faster convergence of the nonlinear simulations with respect to the evolution of the linear growth rate by considering the Fourier spectrum of the radial particle transport, $\langle |\Gamma \cdot e_x| \rangle_t$ at $k_x = 0$ (see Fig. 3). It is found that the transport is driven by fluctuations that occur on scale lengths that are larger than the ones at the peak growth rate of the entropy mode (see Fig. 1). The peak of the transport spectrum shifts towards smaller wavelengths when the density gradient is reduced. In addition, the turbulent quasi steady state at low driving gradients is dominated by ZF which result from the growth of a KHI rising from $E \times B$ shear flow produced by the primary instability. The growth rate of the KHI typically peaks at wavelength that are twice as long as the primary instability (Rogers & Dorland 2005), thus pushing the dynamics towards larger spatial scales, where convergence of the gyromoment approach is easier.

Comparing the convergence of the entropy mode growth rate (Fig. 1), the convergence of saturated transport level (Fig. 2) and the spectrum of the radial particle transport (Fig. 3), one can infer that the gyromoment simulations yield an accurate nonlinear transport level when the linear growth rate of the entropy mode is converged at the wavenumber of the transport spectrum peak. For example, considering the $\kappa_N = 2.0$ case, we note that the transport spectrum peaks at $k_y \simeq 0.3$. The $(P, J) = (4, 2)$ gyromoment result
provides accurate growth rate for $k_y \lesssim 0.2$, thus yielding an inaccurate transport level. On the other hand, the $(P, J) = (10, 5)$ gyromoment set is linearly accurate for $k_y \lesssim 0.5$, which explains the correct saturated transport result.

For a finer analysis of our simulation results, we study the time dependence of the turbulent transport for the three equilibrium gradient values (see Fig. 4). For instance, as a confirmation, we note that a negligible variation of the transport level with respect to the hyperdiffusion parameter, which mostly affect small scale fluctuations (Ricci et al. 2006a; Hallenbert & Plunk 2022). At large gradient, $\kappa_N = 2.5$, the gyromoment approach qualitatively and quantitatively agrees with GENE, showing an approximately constant transport. The analysis of the turbulent eddies show fully developed turbulence, with a negligible role of ZF (see Fig. 5). At the intermediate gradient value, $\kappa_N = 2.0$, time intervals of high turbulent transport level ($\Gamma_x \sim 1$) alternates with quiescent periods ($\Gamma_x \ll 1$), as shown in Fig. 5. The $(P, J) = (10, 5)$ simulation is in a good agreement with GENE results, while the $(P, J) = (4, 2)$ results underestimate the average $\Gamma_x$ value because of longer low-transport intervals and lower burst level. However, our numerical tests show that the level of agreement between the gyromoment approach and GENE is within an uncertainty similar to the one related to the use of a constant hyperdiffusion or an adaptive numerical diffusion algorithm in GENE. Finally, at the lowest gradient value considered, $\kappa_N = 1.6$, the system is dominated by strong ZF that quench the turbulence reducing drastically the transport (see Fig. 5). As expected, in this case, the gyromoment method shows the largest discrepancies with respect to GENE. GENE simulation results in a transport with small amplitude fluctuations occurring on long time scales around a plateau value, $\Gamma_x^\infty \sim 10^{-2}$. On the other hand, the gyromoment approach shows bursts related to the damping of the ZF, for both the $(P, J) = (4, 2)$ and
Electrostatic potential (left) and ion density (right) in the collisionless case at the three drive values considered, i.e. $\kappa_N = 2.5$ (top) with $(P, J) = (4, 2)$, $\kappa_N = 2.0$ with $(P, J) = (10, 5)$ (middle) and $\kappa_N = 1.6$ with $(P, J) = (20, 10)$ (bottom).

(10, 5) resolutions. Hence, despite the fact that the $(P, J) = (10, 5)$ simulation results in an averaged transport level similar to GENE, an accurate description of the turbulent dynamics requires a larger number of moments. This is demonstrated by a $(P, J) = (20, 10)$ simulation (see yellow line in Fig. 7), which agrees better with GENE results and does not produce the spurious bursts observed when a lower number of gyromoments is used.

We note that bursts can also be obtained with GENE by reducing the $(v_{\parallel}, \mu)$ velocity grid resolution to $16 \times 8$, keeping $\nu_v = 0.2$. Bursts are also obtained with a $32 \times 16$ resolution when the velocity diffusion parameter is increased, i.e. $\nu_v = 2^4 \times 0.2$. Thus, predator-prey cycles appear as soon as a sufficient level of diffusion in the velocity space is present. This diffusion can also be introduced through simple collision models, such as the Lenard-Bernstein operator (Lenard & Bernstein 1958). Moreover, it demonstrates that the effect of using a reduced number of gyromoments is comparable to the presence of diffusion in velocity space, with the level of diffusion that depends on the highest gyromoment considered.

Since the representation of the velocity dependence of the distribution functions
differs fundamentally between moments and continuum approaches, we compare the
time-averaged velocity distribution functions obtained by the gyromoments and GENE
codes. Within the gyromoments method, one can reconstruct the
distribution function by using the gyromoments as coefficients of the Hermite Laguerre basis. This yields for
the averaged velocity distribution
\[
\langle g_a(s\|a, x_a, t) \rangle_{k_x, k_y} = \sum_{p=0}^{P} \sum_{j=0}^{J} \langle N^p_j(k, t) \rangle_{k_x, k_y} H_p(s\|a) L_j(x_a) F_{aM}.
\]
The results are presented in Figs. 6 and 7 for \(\kappa_N = 2.5\) and 1.6, respectively. As for the
transport properties, the agreement between both codes depends on the gradient value.
At all gradient values considered, the \((P, J) = (4, 2)\) gyromoment simulations leads to a
smoothing of the distribution functions, reducing the sharp feature that appears around
the thermal velocity (see Fig. 6, around \(s\|a = 1\)). This feature can be seen also in the
lowest drive simulation in Fig. 7 where the \((P, J) = (10, 5)\) gyromoment results present
finer structure than the \((4, 2)\) resolution. This smoothing effect confirms the hypothesis
that the use of a reduced number of gyromoments yields an effective diffusion in the
velocity space.

As a conclusion, we remark that the gyromoment method shows its ability to simulate
the Z-pinch nonlinear turbulent dynamics in the collisionless limit, which represents the
most challenging regime for this approach. Valid results are obtained at large gradient
drive with a velocity space represented by only 9 gyromoments per species, compared
to the \(10^2\) velocity grid points used in GENE simulations. On the other hand, the
weakest gradient drive studied, convergence is obtained with a number of gyromoments
approximately equal to the number of points used by GENE. However, results obtained
with a lower number of moments still provide a reasonable prediction of the time-averaged
level of transport.

4. Collisional turbulent transport

Building on the benchmark of the gyromoment method with the GENE code in
the collisionless limit, we now investigate the properties of turbulence in a Z-pinch by
including collisional processes modelled through different linear collision operators at a
fixed collision rate, \(\nu = 0.1\).

Figure 8 shows the impact of collisions on the entropy mode linear growth rate. Collisions stabilize the tail of the entropy mode present at high \(k_y\) in the collisionless regime because of diffusion in phase space, as observed in Ricci et al. (2006b). This effect is recovered by all the operators considered here, which also include gyrokinetic effects
that induce a strong damping for \(k_y \gtrsim 1\). Consequently, collisions help significantly the
convergence of the gyromoment method in the linear case. In particular, low \(k_y\) modes
that provide the largest contribution to the nonlinear transport are well resolved with the
\((P, J) = (4, 2)\) polynomial basis on the whole range of equilibrium gradients considered
in the present study.

At low \(k_y\) and in the proximity of its peak value, one can observe that the growth rate
is affected by collisions differently depending on the collision model. On the one hand,
large scale fluctuations are destabilized by collisional effects in the case of the Dougherty
and Sugama collision operators for \(\kappa_N = 2.0\) and \(\kappa_N = 2.5\). In this case, an increase
of the growth rate at \(k_y \sim 0.5\) is observed. This effect is similar to the one observed in
instabilities that have a fluid nature, such as the drift waves, which are destabilized by resistivity (Goldston & Rutherford 1995). On the other hand, the collisional growth rate
Figure 6. Time-averaged velocity distribution function for ions (top) and electrons (bottom). The results from GENE (left) and from the gyromoment approach \((P, J) = (4, 2)\) (middle), \((P, J) = (10, 5)\) (right), are presented for \(\kappa N = 2.5\), \(\eta = 0.25\) and \(\nu = 0\).

Figure 7. Time-averaged velocity distribution function for ions (top) and electrons (bottom). The results from GENE (left) and from the gyromoment approach \((P, J) = (4, 2)\) (middle), \((P, J) = (10, 5)\) (right), are presented for \(\kappa N = 1.6\), \(\eta = 0.25\) and \(\nu = 0\).

Figure 8. Linear growth rate of the entropy mode for different collision models and comparison with the collisionless results (black) for three different drive values, \(\kappa N = 1.6\) (left), \(\kappa N = 2.0\) (middle) and \(\kappa N = 2.5\) (right), with \(\eta = 0.25\). The different lines denote the Dougherty (red), Sugama (blue), Lorentz (yellow) and Coulomb (green) operators used in the gyromoment approach with a \((4, 2)\) Hermite-Laguerre basis.

is always smaller or equal to the collisionless case with the Coulomb and Lorentz collision operators for all values of \(\kappa N\) and \(k_y\) considered.

We now turn to the nonlinear results that include finite collisionality, and we discuss a scan of simulations where the drive value is varied from \(\kappa N = 1.6\) to \(\kappa N = 2.5\), being \(\eta = 0.25\). The scan is also performed considering the collisionless limit, observing a Dimits threshold value \(\kappa \simeq 2\), similarly to Hallenbert & Plunk (2022), below which ZF suppress turbulence. Convergence tests show that collisions reduce greatly the number of gyromoments necessary for a correct estimate of the saturated transport level, as observed for the linear growth rate. This allows us to obtain accurate results using a
Figure 9. Collisional saturated transport level for different collision operators: Dougherty (red triangles), Sugama (blue squares), modified Sugama (light blue squares), Coulomb (green diamonds) and Lorentz (yellow triangles). The collisionless results are also reported (black stars) with the mixing length estimate $\Gamma_{\infty} \sim \gamma^2$ (dashed black line).

reduced polynomial basis, $(P, J) = (4, 2)$, for most of the results presented in Fig. 9. For example, a $(P, J) = (10, 5)$ gyromoment simulation provides approximately same saturated transport level as a $(4, 2)$ simulation for $\kappa_N = 1.7$ with the Sugama operator.

The results shown in Fig. 9 reveal that the effect of collisions vanishes at large drive values, where the ZF do not play a crucial role, in agreement with the observations in Ricci et al. (2006a). This suggests that the effect of collisions is mostly related to the ZF dynamics and, as we show later, through their damping and related weakening of the associated transport barrier.

When turbulence is fully developed, the amplitude of the fluctuations can be estimated considering a balance between the nonlinear saturating terms and the linear drive, $\partial_t \sim v_E \times B \cdot \nabla$. This yields $\gamma \sim k^2 \phi$ and, thus, $\phi \sim \gamma$ for a given $k$. Using Poisson equation, one observes that the particle density scales with the potential fluctuations, $n \sim \phi$, which leads to the estimate of the particle transport $\Gamma \sim \gamma^2$. This scaling, based on the collisionless peak value of the entropy mode instability, is shown in Fig. 9, revealing that it captures well the dependence of $\Gamma$ at strong gradients, where the effect of ZF is weak. On the other hand, the reduction of the transport by the ZF can not be captured by this mixing-length estimate at low gradient value.

At medium and low drive levels where zonal flows are present, the different collision models lead to significantly different results. In particular the Sugama and Dougherty operators tend to differ from the Lorentz and Coulomb ones. The difference cannot be explained solely in terms of linear growth rate since the Coulomb operator linear results differ from the Lorentz results at lower gradient values (see Fig. 8). Thus, the ZF quenching of the turbulence (Kobayashi & Rogers 2012) has a strong dependence on the collision model, regardless of the features of the unstable modes. The results obtained with the Dougherty operator appear to most closely approach the collisionless case, with Dougherty being the only operator that shows a Dimits shift at $\kappa_N \simeq 2.1$. This similarity can be explained by the simplicity of Dougherty model, which is mainly composed of kinetic and spatial diffusion terms that are present, albeit at smaller amplitude and for numerical reasons, also in the collisionless case. With respect to the collisionless case, we expect that the slight reduction of transport at the lowest drive level is due to the reduced linear drive.

Similarly to the Dougherty operator, the Sugama operator yields a regime of suppressed transport at low gradient and a regime of fully developed turbulence at large gradient.
However, at intermediate gradient level, transport is remarkably larger in comparison to the collisionless results and Dougherty operator, with the oscillations between quiescent and turbulent periods (see Fig. 2) being replaced by fluctuations around a plateau value with persistent ZF structures. This feature, also observed with the Lorentz and Coulomb operators, can be explained by a ZF damping sufficiently strong to continuously allow fluctuations to grow in the ZF zero shear region (Ivanov et al. 2020). In the context of the predator-prey cycles, this case corresponds to an overlap of bursts. In fact, we note that, by reducing the collision frequency to $\nu = 0.01$ at $\kappa N = 1.6$, a cyclic transport dynamics, previously identified by Kobayashi & Gürcan (2015) and shown in Fig. 10, is obtained. The frequency of these bursts is directly related to the ZF damping rate due to the collision operator that dissipate the ZF structures (highlighted by the decreasing phase of the zonal energy, blue line of Fig. 10), and the primary instability growth rate (underlined by the slope of the increasing part of the non-zonal energy, red line in Fig. 10).

The reduction of the transport level with respect to the mixing length estimate at the lowest gradients is less pronounced with the Lorentz operator than with the Dougherty and Sugama operators. The differences between Sugama and the Lorentz model is mainly due to the energy diffusion term contained in the field part of the former collision operator. In fact, the pitch-angle scattering Lorentz operator does not contain any energy diffusion term, while Sugama uses an ad-hoc energy diffusion term (on the other hand, the spatial diffusion term of the Lorentz and Coulomb operators coincide). Confirming the importance of the energy diffusion, we note that tests at low drive values, where we modify the Sugama operator by zeroing out the ad-hoc energy diffusion term, show a significant increase of the transport level (light blue squares in Fig. 9).

The Coulomb collision operator simulations do not show remarkable differences with respect to Lorentz collision operator results. Both operators maintain a high level of transport, even at low gradient values. It is worth noting that the Coulomb collision operator induces the largest level of transport than all other collision operators for almost every $\kappa N$, which can be surprising considering that the related linear growth rate is smaller than the one yielded by the other collision operators (see Fig. 8).

Confirming that collisions regulate transport through the ZF damping, we now describe a detailed study of this mechanism, as induced by the different collision models. In order isolate the damping of the ZF, we consider the nonlinear collisionless saturated states for $\kappa N = 1.6$, 2.0 and 2.5 (see Fig. 4) as initial conditions for a set of simulations that use our different collision models. We remove the entropy mode drive by setting $\kappa N = 0$, and we use a $(P, J) = (4, 2)$ gyromoment set with $\nu = 0.1$. We let the system evolve and
follow the damping of the ZF profile. The results of this numerical experiment can be first observed qualitatively in Fig. 11 where the averaged radial ZF profile, $\langle \partial_z \phi \rangle_y$, is plotted as a function of time. Fig. 11 reveals that the effect of collisions on the ZF profile is highly dependent on the operator model. The Dougherty model does not affect significantly the ZF structure, while the Sugama operator leads to their damping. The Lorentz operator filters the initial ZF structure, decreasing the amplitude of short wavelength ZF, while a long wavelength ZF mode survives. Finally, the Coulomb operator strongly damps the ZF at all wavelength. Thus, confirming our hypothesis that the different ZF damping is responsible of the different level of transport, we observe that the operators that let the smallest scale of the ZF structure survive yield the smallest transport values observed on Fig. 9.

As a further confirmation and a more quantitative analysis of the results shown in Fig. 11, we define normalized zonal wave amplitude, i.e.

$$ A^2_{ZF}(t) = \frac{\int dx \langle \phi \rangle^2_y(t)}{\int dx \langle \phi \rangle^2_y(0)} \quad (4.1) $$

and study its time evolution for all collision operators in Fig. 12. We consider the ZF obtained at the low, medium and high level of gradient drive of the collisionless simulations as initial conditions. Focusing on the damping at early times, $\partial_t A^2_{ZF}|_{t=0}$, (the growth of the linear instability alters the ZF damping at timescales $1/\gamma_p \sim 10$) this analysis unveils a clear difference between Dougherty and Sugama operators, while these operators provide very similar linear growth rates. We observe also that the Lorentz and Coulomb operators yield a similar damping, corresponding to a similar transport level in the nonlinear simulations. Thus, we can deduce that the saturated transport level is directly related to the ZF damping rate.

5. Conclusions

In the present paper, the first nonlinear gyrokinetic simulations carried out using a gyromoment approach and including advanced collision models are presented. By implementing the moment hierarchy in Eq. (2.11), turbulence in a two-dimensional Z-
pinch geometry is studied. We present a benchmark with the continuum GK code GENE that demonstrates the ability of the gyromoment approach to simulate accurately the nonlinear evolution of the entropy mode, even in the collisionless limit. We show that the convergence behavior of the nonlinear results follow the same trend as the linear ones, i.e., the convergence properties improve with the increase of the gradient strength. However, only the linear growth rate of the modes developing at large scales need to be accurately resolved to obtain accurate nonlinear results.

We then extend the nonlinear results, adding collisions with the use of four different collision operator models. We observe that the gyromoment simulations converge with a lower number of moments than in the collisionless case. With a Dimits threshold identified around $\kappa_N \sim 2$ in the collisionless case, the influence of collision on transport level appears only for gradients below this change of regime. This shows that collisional effects are mainly related to the dynamics of the ZF. Our results highlight the disagreement between Dougherty, Sugama, Lorentz and Landau collision models, in the linear growth rate and, even more, in nonlinear simulations. We show that the analysis of the linear results is not sufficient to predict the difference observed in the saturated transport level. However, we observe a direct link between ZF damping and transport level, which could be used to make predictions of the transport level in a future work.

In conclusion, this work is a step forward in the simulation of the tokamak boundary based on the use of the gyromoment approach, which presents the potential of bridging the gap between kinetic and fluid models. We remark that the truncation of the moment hierarchy is still an open question and advanced closure models may improve the efficiency of our simulations with respect to the number of evolved moments.

Acknowledgements

The authors acknowledge helpful discussions with A. Cerfon, S. Brunner, J. Ball, L. Villard, A. Hallenbert, A. Volčokas, G. Van Parys, L. Driever and L. Lechot. This research has been carried out within the framework of the EUROfusion Consortium and has received funding from the European Union via the Euratom Research and Training Programme (Grant Agreement No101052200 — EUROfusion). The views and opinions expressed herein do not necessarily reflect those of the European Commission. The simulations presented herein were carried out in part on the CINECA Marconi supercomputer under the TSVVT421 project and in part at CSCS (Swiss National
Supercomputing Center). This work was supported in part by the Swiss National Science Foundation.

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