Gauge theory of Gravity based on the correspondence between the 1\textsuperscript{st} and the 2\textsuperscript{nd} order formalisms

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A covariant canonical gauge theory of gravity free from torsion is studied. Using a metric conjugate momentum and a connection conjugate momentum, which takes the form of the Riemann tensor, a gauge theory of gravity is formulated, with form-invariant Hamiltonian. Through the introduction of the metric conjugate momenta, a correspondence between the Affine-Palatini formalism and the metric formalism is established. For, when the dynamical gravitational Hamiltonian \( \tilde{H}_{D\alpha} \) does not depend on the metric conjugate momenta, a metric compatibility is obtained from the equation of motions and the energy momentum is covariant conserved. When the gravitational Hamiltonian \( H_{D\alpha} \) depends on the metric conjugate momentum, an extension to the metric compatibility comes from the equation of motion and the energy momentum covariant conservation is violated. For a sample of the \( H_{D\alpha} \) which consists of a quadratic term of the connection conjugate momentum, the effective Lagrangian has the Einstein Hilbert term with a quadratic Riemann term in the second order formalism. A bouncing inflation is briefly discussed in the context of cosmological solutions of this action.

Keywords: gauge theory of gravity - Affine formalism - Hamiltonian

I. INTRODUCTION

General Relativity is one of the well tested theories in physics, with many excellent predictions. A search of a rigorous derivation of General Relativity on the basis of the action principle and the requirement that the description of any system should be form-invariant under general space time transformations has been constructed in the framework of the Covariant Canonical Gauge theory of Gravity.

The Covariant Canonical Gauge theory of Gravity [1,2] is formulated within the framework of the covariant Hamiltonian formalism of classical field theories. The latter ensures by construction that the action principle is maintained in its form requiring all transformations of a given system to be canonical. The imposed requirement of invariance of the original action integral with respect to local transformations in curved space time is achieved by introducing additional degrees of freedom, the gauge fields. In the basis of the formulation there are two independent fields: the metric \( g^{\alpha \beta} \), which contains the information about lengths and angles of the space time, and the connection \( \gamma^a_{\alpha \beta} \), which contains the information how a vector transforms under parallel displacement. In this formulation, these two fields are assumed to be independent dynamical quantities in the action and referred to as the Affine-Palatini formalism (or the 1\textsuperscript{st} order formalism).

The basic equation of motion that was calculated from an action with non-zero torsion — which was set to zero subsequently. As was discussed in [3], this formulation differs from the standard 1\textsuperscript{st} order formalism, which does not include torsion right from the outset, and hence assumes the connection to be symmetric. The standard 1\textsuperscript{st} order formalism has a special feature: the covariant derivative of the gravitational energy momentum tensor is not necessarily covariantly conserved [4], while for the 2\textsuperscript{nd} order formalism the metric energy momentum tensor is always covariantly conserved. In this paper we investigate the complete zero-torsion equation of motion for the covariant canonical gauge theory of gravity and show the impact on the energy momentum conservation: For a theory with a metric compatibility, the energy momentum tensor is covariantly conserved even the starting point is the affine 1\textsuperscript{st} order formalism, which does not enforce a covariant conservation of the stress energy momentum tensor. If the theory breaks the metric compatibility, a violation of the energy momentum covariant conservation could ensue.

II. A BASIC FORMULATION

The Covariant Canonical Gauge theory of Gravity is a well defined formulation derived from the canonical transformation theory in the covariant Hamiltonian picture of classical field theories [1]. It identifies two independent fundamental fields, which form the basis for a description of gravity: the metric \( g^{\alpha \beta} \) and the connection \( \gamma^a_{\alpha \beta} \). In the Hamiltonian description, any fundamental field has a conjugate momentum: the metric conjugate momentum is \( \tilde{k}^{\alpha \beta} \) and the connection conjugate momentum is \( \tilde{q}^{a}_{\alpha \beta} \).

\[
S = \int_{R} \left( \tilde{k}^{\alpha \beta} g_{\alpha \gamma} \gamma_\gamma - \frac{1}{2} \tilde{q}^{a}_{\alpha \beta} \gamma_\gamma^{a}_{\alpha \beta} - \tilde{H}_0 \right) d^4x \tag{1}
\]

where the “tilde” sign denotes a tensor density, which multiplies the tensor with \( \sqrt{-g} \). As the conjugate momentum components of the fields are the duals of the complete set of the derivatives of the field, the formulation is referred to as “covariant canonical”. A closed description of the coupled dynamics of fields and space-time geometry has been derived in
where the gauge formalism yields:

\[ S = \int_R \left( \kappa^\alpha\beta\gamma g_{\alpha,\beta} - \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} - \tilde{H}_{\text{Dyn}}(\tilde{q}, \tilde{k}, g) \right) d^4x \quad (2) \]

As a result of the gauge procedure, all partial derivatives of tensors in Eq. (1) reappear as covariant derivatives. The partial derivative of the (non-tensorial) connection changes into the tensor \( R^\eta_{\alpha\beta\gamma} \), which was shown to be the Riemann-Christoffel curvature tensor:

\[ R^\eta_{\alpha\beta\gamma} = \frac{\partial \gamma^\eta_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial \gamma^\eta_{\alpha\gamma}}{\partial x^\beta} + \gamma^\eta_{\alpha\beta} \gamma^\rho_{\gamma\rho} - \gamma^\eta_{\alpha\rho} \gamma^\rho_{\beta\gamma}. \quad (3) \]

The “dynamics” Hamiltonian \( \tilde{H}_{\text{Dyn}} \) — which is supposed to describe the dynamics of the free (uncoupled) gravitational field — is to be built from a combination of the metric conjugate momentum, the connection conjugate momentum, and the metric itself.

### III. A CORRESPONDENCE BETWEEN THE 1st AND THE 2nd ORDER FORMALISM

In addition to the foundations of the gauge theory of gravity, it turned out that the part of the action: \( \kappa^\alpha\beta\gamma g_{\alpha,\beta,\gamma} \), which contains the metric conjugate momentum, has a strong impact as a connector between the affine-Palatini formalism (or the 1st order formalism) and the metric formalism (or the 2nd order formalism):

\[ \mathcal{L}(g, \gamma)_{1\text{st}} + \kappa^\alpha\beta\gamma g_{\alpha,\beta,\gamma} \Rightarrow \mathcal{L}(g)_{2\text{nd}} \quad (4) \]

In the 1st order formalism, one assumes that there are two independent fields: the metric \( g^{\mu\nu} \) and the connection \( \gamma^\mu_{\nu} \). In contrast to that, in the 2nd order formalism the connection is assumed to be the Levi Civita or Christoffel symbol:

\[ \gamma^\mu_{\nu} \bigg\{ \frac{\rho}{\mu \nu} \bigg\} = \frac{1}{2} g^{\alpha\beta} (g_{\alpha,\nu} + g_{\alpha,\mu} - g_{\mu,\alpha}) \quad (5) \]

and appears in the action directly in this way. In general, only for Lovelock theories [5], which includes Einstein Hilbert action, both formulations will yield the same equations of motion and the connection will be in both cases the Christoffel symbol [6].

In Ref. [7], it was proved that for any general action which starts in the 1st order formalism in addition to the term \( \kappa^\alpha\beta\gamma g_{\alpha,\beta,\gamma} \) the energy momentum tensor will be the same as it would be calculated in the 2nd order formalism. The main reason for that correspondence is the metric compatibility constraint. The variation with respect to \( \kappa^\alpha\beta\gamma \) gives the metricity condition:

\[ g_{\alpha,\beta,\gamma} = 0 \Rightarrow \gamma^\mu_{\nu} \bigg\{ \frac{\rho}{\mu \nu} \bigg\}, \quad (6) \]

which cause the connection to be the Christoffel symbol. The variation with respect to the connection gives the tensors:

\[ \frac{\delta}{\delta \gamma^\mu_{\nu}} \kappa^\alpha\beta\gamma g_{\alpha,\beta,\gamma} = -k^{\mu\nu} g_{\rho\alpha} - k^\alpha\gamma g_{\rho\alpha} \quad (7) \]

with a symmetrization between the components \( \mu \) and \( \nu \). The variation with respect to the metric is:

\[ \frac{\delta}{\delta g_{\mu\nu}} \kappa^\alpha\beta\gamma g_{\alpha,\beta,\gamma} = -k^{\mu\nu} \quad (8) \]

Because of the new contribution to the field equation \( k^{\mu\nu} \), the complete field equation will contains additional terms which make the first order field equations to be equivalent to the field equation under the second order formalism. Indeed, isolating the tensor \( k^{\mu\nu} \) and inserting it back into Eq. (8) gives the relation:

\[ \frac{\partial \mathcal{L}(\kappa)}{\partial g_{\mu\nu}} = \frac{1}{2} \nabla_{\mu} \left( g^{\rho\sigma} \frac{\partial \mathcal{L}(\kappa)}{\partial g^\rho_{\mu\nu}} + g^{\rho\sigma} \frac{\partial \mathcal{L}(\kappa)}{\partial g^\rho_{\mu\nu}} - g^{\rho\sigma} \frac{\partial \mathcal{L}(\kappa)}{\partial g^\rho_{\nu\nu}} \right) \quad (9) \]

where \( \mathcal{L}(\kappa) = \kappa^\alpha\beta\gamma g_{\alpha,\beta,\gamma} \). The terms in the right hand side represents the additional terms that appear in the second order formalism. One option for obtain the contributions into the field equation is to solve \( \kappa^\alpha\beta\gamma \). The direct way is by using this equation, that gives the new contributions for the second order formalism into the field equation, from the variation with respect to the connection \( \gamma^\mu_{\nu} \). An application for this correspondence is from the Covariant Canonical Gauge theory of gravity action [2].

### IV. THE STRESS ENERGY MOMENTUM TENSOR

As was discussed in [4], in the 1st order formalism the stress energy momentum tensor does not have a covariant conservation, in contrast to the 2nd where the gravitational stress energy momentum tensor is covariantly conserved for all actions. From the correspondence between the 1st and the 2nd order formalisms theorem, we obtain a basic link between the dependence of the \( \tilde{H}_{\text{Dyn}} \) with the metric conjugate momentum \( \kappa^\alpha\beta\gamma \) and the conservation of the metric energy momentum tensor. In the first case, \( \tilde{H}_{\text{Dyn}} \) does not depend on the metric conjugate momentum \( \kappa^\alpha\beta\gamma \):

\[ S_A = \int_R \left( \kappa^\alpha\beta\gamma g_{\alpha,\beta} - \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} - \tilde{H}_{\text{Dyn}}(\tilde{q}, \tilde{k}, g) \right) d^4x \quad (10) \]

A variation with respect to the metric conjugate momentum \( \kappa^\alpha\beta\gamma \) gives the metric compatibility condition. According to the theorem [4] the gravitational energy momentum tensor is the same as the gravitational energy momentum tensor in the second order formalism, which promise the covariant conservation of this gravitational energy momentum tensor:

\[ \nabla_{\mu} G^\mu_\nu = 0 \quad , \quad G^\mu_\nu = -\frac{2}{\sqrt{-g}} \delta S_A \quad (11) \]

In the second case \( \tilde{H}_{\text{Dyn}} \) does depend on the metric conjugate momentum \( \kappa^\alpha\beta\gamma \):

\[ S_B = \int_R \left( \kappa^\alpha\beta\gamma g_{\alpha,\beta} - \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} - \tilde{H}_{\text{Dyn}}(\tilde{q}, \tilde{k}, g) \right) d^4x \quad (12) \]

A variation with respect to the metric conjugate momentum
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FIG. 1: A flowchart that summarizes the link between the formulation of the theory to covariant conservation of the stress energy momentum tensor, where \( k \) is the conjugate momentum of the metric.

\[ \tilde{k}^{\alpha\beta\gamma} \] breaks the metric compatibility condition, and according to the [4], the gravitational stress energy tensor does not have to be covariantly conserved. This basic framework is not a special feature only for the Covariant Canonical Gauge Theory of Gravity, but leads to a fundamental correlation for many options for \( H_{\text{Dyn}} \). In Fig. 1, we summarize the link between the formulation of the theory and the covariant conservation of the stress energy momentum tensor.

V. COUPLED ACTION WITH MATTER

In analogy to the definition of the metric energy-momentum tensor density of the given system Hamiltonian, the metric energy-momentum tensor density is being define as the variation of the \( \tilde{L}_m \) with respect to the metric:

\[ T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial \tilde{L}_m}{\partial g_{\mu\nu}} \]  

(13)

Therefore the complete action takes the form:

\[ S = \int \left[ \tilde{k}^{\alpha\beta\gamma} g_{\alpha\beta\gamma} - \frac{1}{2} \tilde{q}^{\alpha\beta\gamma} R_{\alpha\beta\gamma} - \tilde{H}_{\text{Dyn}}(\tilde{q}, \tilde{k}, g) + \tilde{L}_m \right] d^4x \]  

(14)

The variation with respect to the metric conjugate momenta:

\[ g_{\alpha\beta\gamma} = \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{k}^{\alpha\beta\gamma}} \]  

(15)

which presents the existence of non-metricity if \( H_{\text{Dyn}} \) depends on \( \tilde{k}^{\alpha\beta\gamma} \). The second variation is the variation with respect to the connection:

\[ -\left( \tilde{k}^{\alpha\mu\nu} + \tilde{k}^{\alpha\nu\mu} \right) g_{\alpha\beta} = \frac{1}{2} \nabla_\beta \left( \tilde{q}^{\mu\nu} + \tilde{q}^{\nu\mu} \right), \]  

(16)

which contracts the relation between the momenta of the metric and the connection. The third variation is with respect to the connection conjugate momentum \( \tilde{q}^{\mu\nu}_{\rho\sigma} \), which gives:

\[ \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{q}^{\mu\nu}_{\rho\sigma}} = -\frac{1}{2} R^{\mu\nu}_{\rho\sigma\rho\sigma} \]  

(17)

If \( \tilde{H}_{\text{Dyn}} \) is not depend on \( \tilde{q} \), the Riemann tensor will be zero. Therefore the contribution for the stress energy tensor comes from the Dynamical Hamiltonian and from the metric conjugate momenta:

\[ T^{\mu\nu} = g^{\alpha\beta\gamma} \left( \tilde{k}^{\alpha\beta\gamma} g_{\alpha\beta\gamma} - \frac{1}{2} \tilde{q}^{\alpha\beta\gamma} R_{\alpha\beta\gamma} - \tilde{H}_{\text{Dyn}}(\tilde{q}, \tilde{k}, g) + \tilde{L}_m \right) \]  

(18)

From the variation with respect the connection (16), the value of the momentum \( k^{\alpha\beta\gamma} \). As an example, we consider a Dynamical Hamiltonian which has no dependence with the metric conjugate momentum.

VI. SAMPLE \( \tilde{H}_{\text{Dyn}} \) WITHOUT BREAKING METRICITY

In order to see implication for those abstract theorems about the covariant conservation of the metric energy momentum tensor, our starting point is a dynamical Hamiltonian with the connection conjugate momentum up to the second order, without a dependence on the metric conjugate momentum:

\[ \tilde{H}_{\text{Dyn}} = \frac{1}{4 g_1} q_a^{\alpha\beta\gamma\delta} q_r^{\alpha\beta\gamma\delta} - g_2 q_{\alpha\beta\gamma\delta} q_{\alpha\beta\gamma\delta} + g_3 \sqrt{-g} \]  

(19)

This Hamiltonian was investigated in [8] under the original formalism for non-zero torsion (which is finally set to zero), and led to resolving the cosmological constant problem. In our case, assuming that there is no torsion, the formalism demands that the energy momentum tensor is covariantly conserved, as is supposed to be in the second order formalism.
The variation with respect to the metric conjugate momenta $\tilde g_{\mu
u}$ gives the metricity condition:

$$g_{\alpha\beta} = 0 \quad \Rightarrow \quad \gamma_{\alpha\beta} = \left\{ \lambda \right\}_{\alpha\beta}$$  \hspace{1cm} (20)

The variation with respect to the connection conjugate momenta $\tilde q_{\sigma\mu\nu}$ gives

$$q_{\mu\nu\beta} = g_1(R_{\mu\nu\beta} - \tilde R_{\mu\nu\beta})$$  \hspace{1cm} (21)

where:

$$\tilde R_{\mu\nu\beta} = g_2(g_{\mu\nu}g_{\beta\alpha} - g_{\mu\beta}g_{\nu\alpha})$$  \hspace{1cm} (22)

refers to the ground state geometry of space-time which is the de Sitter (dS) or the anti-de Sitter (AdS) space-time for the positive or the negative sign of $g_2$, respectively. The last variation is with respect to the metric. In order to isolate the tensor $k_{\mu\nu}$ one can use the following process: First, we multiply by the metric $g^{\sigma\nu}$ and sum over the index $\sigma$:

$$- \tilde k^{\mu\nu} - k^{\mu\nu} = \frac{1}{2} \nabla_a (\tilde q^{\nu\mu\alpha\sigma} + q^{\nu\alpha\mu\sigma})$$  \hspace{1cm} (23)

Switching the indices $\sigma \leftrightarrow \nu$:

$$- \tilde k^{\nu\mu} - k^{\nu\mu} = \frac{1}{2} \nabla_a (\tilde q^{\mu\nu\alpha\sigma} + q^{\mu\alpha\nu\sigma})$$  \hspace{1cm} (24)

and the indices $\mu \leftrightarrow \nu$:

$$- \tilde k^{\mu\nu} - k^{\mu\nu} = \frac{1}{2} \nabla_a (\tilde q^{\nu\mu\alpha\sigma} + q^{\nu\alpha\mu\sigma})$$  \hspace{1cm} (25)

gives a new combination of the $k^{\mu\nu}$ tensor. By summing the Eqs. (23) + (24) - (25), the isolated value gives:

$$- k^{\nu\mu} = \frac{1}{2} \nabla_a (\tilde q^{\nu\mu\alpha\sigma} + q^{\nu\alpha\mu\sigma})$$  \hspace{1cm} (26)

Therefore, the contribution for the stress energy momentum comes from the covariant derivative of (20):

$$T^{\mu\nu} = - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} \tilde k^{\rho\sigma} + \nabla_{\nu} (q^{\rho\sigma\mu\alpha} + q^{\sigma\mu\rho\alpha}) + \frac{2}{\sqrt{-g}} \frac{\partial H_{\text{Dyn}}}{g_{\mu\nu}}$$  \hspace{1cm} (27)

By plugging in the explicit value of the Dynamical Hamiltonian, the stress energy momentum tensor takes on the form:

$$T^{\mu\nu} = \frac{1}{g_1} \left\{ g^\beta \epsilon_{\mu\nu\rho\sigma} q^\rho q^\sigma \epsilon_{\sigma\alpha\beta\lambda} - \frac{1}{4} g^{\rho\sigma} (q_{\rho\mu\nu} q_{\sigma\lambda} + q_{\rho\nu\mu} q_{\sigma\lambda}) \right\}$$

$$- g_2 (q_\mu g^{\nu\beta} + q_\nu g^{\mu\beta}) + g_3 + \nabla_\mu \nabla_\nu (q^{\rho\sigma\mu\alpha} + q^{\sigma\mu\rho\alpha}).$$  \hspace{1cm} (28)

Plugging in the value of the tensor $q_{\alpha\beta\gamma\delta}$ from Eq. (21) gives the result:

$$T^{\mu\nu} = \frac{1}{8\pi G} G^{\mu\nu} + g_1 Q^{\mu\nu} + g_1 S^{\mu\nu} + g^{\mu\nu} \Lambda$$  \hspace{1cm} (29)

where the tensor:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$  \hspace{1cm} (30)

is the Einstein tensor, and

$$Q^{\mu\nu} = R^{\mu\nu\rho\sigma} R_{\rho\sigma} - \frac{1}{4} g^{\mu\nu} R^{\rho\sigma\delta\epsilon} R_{\rho\sigma\delta\epsilon},$$  \hspace{1cm} (31)

is an additional quadratic Riemann term,

$$S^{\mu\nu} = (\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha) R^{\mu\nu\rho\sigma},$$  \hspace{1cm} (32)

while the last part is the derivative of Riemann tensor. The coupling constants relate to the physical quantities with the relations:

$$g_1 g_2 = \frac{1}{16\pi G}, \quad 6g_1 g_2^2 + g_3 = \frac{\Lambda}{8\pi G}$$  \hspace{1cm} (33)

This stress energy momentum tensor is exactly the same metric energy momentum tensor if our starting point was the effective Lagrangian:

$$\mathcal{L} = g_1 R^{\rho\sigma\delta\epsilon} R_{\rho\sigma\delta\epsilon} - \frac{1}{16\pi G} (R - 2\Lambda)$$  \hspace{1cm} (34)

and the stress energy momentum tensor is the same stress energy momentum tensor for this Lagrangian in the second order formalism. Because of this fact, we conclude that the stress energy tensor is covariantly conserved:

$$\nabla_\mu T^{\mu\nu} = 0$$  \hspace{1cm} (35)

as promised for any action in the second order formalism. This result coincides with the condition we formulated before [11]. One from the big benefits of this formulation is common in many gauge theories of gravity [9], where the starting point is with additional variables with no higher derivatives in the action, and the equations of motion are equivalent to actions with higher derivatives of metric. In this specific case, the starting point is with the quartic momentum $q$ and at the end is equivalent to an action with quadratic Riemann term. The hyperbolicity of the quadratic Riemann term discussed in [10].

\section*{VII. COSMOLOGICAL VACUUM SOLUTION}

In order to study the physical consequences of the contributions of a quadratic Riemann term, we examine the vacuum cosmological solutions. Several combinations of curvature invariants, like $R^2$, $R_{\mu\nu}R^{\mu\nu}$ could be considered. However the combination of quadratic Riemann term emerges from the Covariant Canonical Gauge theory of Gravity. With the Strobinski modification [11], which produces an inflation model, we expect a different scenario of inflation for the additional quadratic Riemann term. Clearly, in the low energy regime, General Relativity has to be recovered and then the quadratic Riemann term. Essentially, starting from very early epochs,
one has to recover first the Starobinsky model and then the Einstein regime.

The (FLRW) Friedman-Lemaitre-Robertson-Walker ansatz is the standard model of cosmology dynamics based on the assumption of a homogeneous and isotropic universe at any point, commonly referred to as the cosmological principle. The symmetry considerations lead to the FLRW metric

$$\text{d}s^2 = -n^2 \text{d}t^2 + a^2 (\text{d}x^2 + \text{d}y^2 + \text{d}z^2)$$

(36)

where \(n(t)\) is the lapse function, \(a(t)\) defines the dimensionless cosmological scale factor. For simplicity we take the spatial curvature \(k\) to be zero. The definition of the Hubble parameter \(H = \frac{\dot{a}}{a}\) is used later.

In the following, we determine the expansion factor dynamics \(a(t)\) by means of our generalized effective Lagrangian (34).

With this ansatz we obtain the effective Lagrangian (where \(8\pi G = 1\)):

$$\mathcal{L} = \frac{12g_1a\dot{a}^2}{n^2} + \frac{3a^2\dot{a}}{n} + \frac{12g_1a^2\dot{a}^2}{n^2} + \frac{12g_1a^4}{an^3}$$

$$- \frac{3a^2\dot{a}}{n} + \frac{3a^2}{n} - \frac{24g_1a\dot{a}\ddot{a}}{n^3} + \Lambda a^3n$$

(37)

An interesting feature that is used also in an earlier publication [12] is using the \(g_1\) constant to rescale the time with the time \(\tau = \sqrt{g_1}\). The physical time is rescaled according to \(t := \tau/\tau\). We can rewrite the action:

$$\mathcal{L} = \frac{12a\dot{a}^2\ddot{a}}{n^2} - \hat{n}\left(\frac{3a^2\dot{a}}{n^2} + \frac{24a\dot{a}\ddot{a}}{n^3}\right)$$

$$+ \frac{3a^2\ddot{a}}{n} + \frac{3a^2}{n} + \frac{12}{n^3}(a\ddot{a}^2 + \frac{\dot{a}^4}{a}) - n\Lambda a^3$$

(38)

The physical parameters which appear in the matter energy momentum tensor is modeled as a perfect fluid, with the form of:

$$T^{\mu}_{\nu} = \text{diag}(\rho, -p, -p, -p).$$

(39)

The density, \(\rho\), and the pressure, \(p\), refer to all types of matter present in the universe. From variations with respect to the lapse function and the scale parameters we obtain the Friedmann equations:

$$\rho = 3H^2 - \Lambda + 24H\dot{H} + 72H^2\dot{H} - 12H^2$$

(40)

FIG. 2: A bouncing inflation for a vacuum solution under slow roll assumption.

In order to see the inflationary vacuum solution we set the density to zero. Under the assumption of slow roll inflation the dimensionless parameters have to be very small:

$$\left| \frac{\dot{H}}{H^2} \right| \ll 1, \quad \left| \frac{\ddot{H}}{H\dot{H}} \right| \ll 1$$

(43)

Plugging in this assumption into the density equation gives the differential equation:

$$\frac{\Lambda}{3} = H^2(1 + 24\dot{H})$$

(44)

A simple integration leads to the closed relation:

$$\tanh^{-1}\left(\frac{H}{\sqrt{\Lambda/3}}\right) - \frac{H}{\sqrt{\Lambda/3}} = \frac{t}{\sqrt{\Lambda/3}}$$

(45)

For tracking the physical behavior, we plot a parametrized of

| Name | Critical point | Eigenvalues | stability |
|------|---------------|-------------|-----------|
| A    | \((4\Lambda, -\sqrt{3})\) | \(\lambda_s = \frac{1}{2}(\sqrt{3\Lambda + 3\Lambda - 1})\) | saddle point |
| B    | \((4\Lambda, +\sqrt{3})\) | \(\lambda_s = \frac{1}{2}(\sqrt{3\Lambda + 3\Lambda - 1})\) | stable |

TABLE I: The properties of the critical points for the vacuum solution, where the axis are \((R, H)\).
the solution in Fig. (2). The solution starts from a bounce and finishes at $H = \sqrt{\Lambda}/3$ as the traditional inflation. In order to see the attractor of the vacuum solution, we investigate the autonomous system by means of a variation on the effective Lagrangian. The vacuum density equation can be written as:

$$H = \frac{R}{6} - 2H^2, \quad \dot{R} = \frac{R^2}{12H} - HR - \frac{3}{4}H + \frac{\Lambda}{4H} \quad (46)$$

where $R$ is the Ricci scalar and $H$ is the Hubble parameter. The analysis for this dynamical system gives two points as described in table (1). The stable point A has the value of $H = \sqrt{\Lambda}/3$ as we have shown in the plot of analytic solution (2). The physical interpretation for those vacuum solutions is the start of the bounce and the stable end of the traditional inflation, where the Hubble parameter is constant. Besides the stability of the vacuum solution, the theory has to be worked out in order to select reliable constraint to be compared with data. In a forthcoming paper, the comparison with data will be addressed in detail.

VIII. DISCUSSION

In this paper we investigated the formulation of the covariant canonical gauge theory of gravity free from torsion. Diffemorphisms appear as canonical transformations. A tensor field which plays the role of the canonical conjugate of the metric is introduced. It enforces the metricity condition provided that the “Dynamics” Hamiltonian does not depend on this field. The resulting theory has a direct correspondence with our recent work concerning the correspondence between the first order formalism and the second order formalism through the introduction of a Lagrange multiplier field which in this case corresponds with the field that is used to provide the metric with a canonically conjugate momentum. The procedure is exemplified by using a “Dynamics” Hamiltonian which consists of a quadratic term of the connection conjugate momentum. The effective stress energy momentum tensor that emerged from the canonical equations of motion were equivalent to Einstein Hilbert tensor in addition to quadratic Riemann term.

The resulting dynamics is described by an action with the Riemann tensor squared and formulated in the context of the standard second order formalism. The cosmological solutions avoid the initial Big Bang singularity, which is replaced by a bounce. The universe then goes into an inflationary phase asymptotically.

In the future, other combinations for the “Dynamics” Hamiltonian should be investigated. A test particle with a non-metricity or a formulation which uses a torsion (without set it to zero) has to be studied. A vacuum solution with a spherically symmetric feature could be also solved and predicts a different scenarios for black holes. In addition, the theory has to be worked out in order to select reliable models eligible to be compared with data. So in the future, this inflationary scenario should be tested from the CMB anisotropies.

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