SUPERSTRINGS ON CURVED SPACETIMES * †

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ABSTRACT

In this lecture I summarize recent developments on strings propagating in curved spacetime. Exact conformal field theories that describe gravitational backgrounds such as black holes and more intricate gravitational singularities have been discovered and investigated at the classical and quantum level. These models are described by gauged Wess-Zumino-Witten models, or equivalently current algebra G/H coset models based on non-compact groups, with a single time coordinate. The classification of such models for all dimensions is complete. Furthermore the heterotic superstrings in curved spacetime based on non-compact groups have also been constructed. For many of the \(d \leq 4\) models the gravitational geometry described by a sigma model has been determined. Some general results outlined here include a global analysis of the geometry and the exact classical geodesics for any G/H model. Moreover, in the quantized theory, the conformally exact metric and dilaton are obtained for all orders in an expansion of \(k\) (the central extension). All such models have large-small (or mirror) duality properties which we reformulate as an inversion in group space. To illustrate model building techniques a specific 4-dimensional heterotic string in curved spacetime is presented. Finally the methods for investigating the quantum theory are outlined. The construction and analysis of these models at the classical and quantum level involve some aspects of noncompact groups which are not yet sufficiently well understood. Some of the open problems in the physics and mathematics areas are outlined.

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1. Introduction

I would like to begin this talk by explaining why non-compact current algebra coset models, or equivalently gauged Wess-Zumino-Witten (WZW) models based on certain non-compact groups, are of great interest in string theory on curved spacetime.

The first quantized version of string theory envisages a general sigma model-like action that describes a string $X^\mu(\tau, \sigma)$ propagating in a general background that includes a metric $G_{\mu\nu}(X)$, dilaton $\Phi(X)$, antisymmetric field $B_{\mu\nu}(X)$, tachyon $T(X)$, as well as all fields representing all other massive modes of the string. These background fields are imagined to be determined dynamically in the fully interacting string theory, including non-perturbative effects. Conformal invariance is a “kinematical” property of string theory. However, it is such a strong constraint that it generates dynamical equations for these background fields. One may hope that a solution to these dynamical equations provides a “classical vacuum” to string theory that captures some of the important dynamical properties of the theory. If one could really determine the vacuum state one would hope that it explains all the essential properties of low energy physics, including the facts that we live in four dimensions ($D = 4$), that the color-electroweak gauge group is $SU(3) \times SU(2) \times U(1)$, that there are three families of quarks and leptons, etc.. In addition, one may derive a quantum cosmological history of the universe and learn about the mechanism for mass generation. If all this is accomplished new predictions should follow.

In the past the main tool for imposing conformal invariance in string theory in curved spacetime has been the beta function conditions of Fradkin and Tseytlin and Callan et. al. These conditions have the form of Einstein’s equations modified by the presence of a dilaton and some additional fields such as an antisymmetric tensor, etc. In fact they are identical to the equations of motion of the low energy (perturbative) effective action of string theory. These equations represent only a perturbative version of string theory in two senses: (i) they apply only in the absence of string loops (i.e. they apply only on the topology of the sphere, not the torus, pretzel, etc.) and (ii) the non-linear sigma model interactions are considered only in lowest orders. There is not much we can do about (i) at this stage (except for the non-perturbative matrix model type of treatment of two dimensional gravity); one maintains the hope that the string loop expansion is useful to extract at least part of the physics, in the same sense that perturbative expansions have been useful in field theories such as QED and QCD. On the other hand there is much we can and should do to improve (ii) since the basic symmetry of conformal invariance is satisfied only perturbatively by the solutions of the beta function equations.

Fortunately, conformal invariance is exactly satisfied in a class of models based on current algebra cosets or equivalently gauged WZW models (still excluding string loops). The beta function equations are then automatically solved in the perturbative limit of the model. But more significantly, conformal invariance is satisfied non-perturbatively to all orders of the central extension $k$ (equivalently to all orders of the non-linear sigma model interactions). When a non-compact group is used there are automatically both spacelike as well as timelike coordinates. Only one timelike coordinate can be tolerated.
since, roughly, the conformal constraints (Virasoro constraints) are sufficient to remove the negative norm states generated by only one timelike coordinate - this is analogous to the naive counting that leads to the no-ghost theorem in flat spacetime. To achieve the single time requirement [1] one must restrict the possible non-compact groups and their cosets to the following list [2, 3]:

\[
\begin{align*}
SO(d - 1, 2)/SO(d - 1, 1) & \quad SO(d, 1)/SO(d - 1, 1) \\
SU(n, m)/SU(n) \times SU(m) & \quad SO(n, 2)/SO(n) \\
SO(2n^*)/SU(n) & \quad Sp(2n^*)/SU(n) \\
E_6^*/SO(10) & \quad E_7^*/E_6
\end{align*}
\]

(1.1)

This list, which contains only simple groups, may be extended with direct products of simple groups \(G_1 \times G_2 \times \cdots\) including \(U(1)\) or \(\mathbb{R}\) factors, or their cosets, so long as the additional factors do not introduce additional time coordinates [1, 2, 3]. While these models represent only a small subset of all possible curved spacetime models described by the general sigma model, they have the advantage of being solvable in principle thanks to the current algebra formulation. Thus a lot more can be said about the spectrum, correlation functions, etc. of the quantum string theory based on these models. Furthermore, it has been realized that the special geometries described by these non-compact groups are relevant to gravitational singularities such as black holes and cosmological Big Bang. For these reasons this class of models has received considerable attention during the past year and a half [4, 5, 6]. It is hoped that through such solvable models new light will be shed on unresolved gravitational issues, in string theory as well as general relativity, such as singularities, quantization and finiteness or renormalizability in curved spacetime, the question of Euclidean-Minkowski continuation, spectrum of low energy particles and excited string states in the presence of curvature, etc..

The other important question for string theory is the nature and content of the low energy matter it is supposed to predict in the form of quarks, leptons, gauge bosons, etc.. The new models have opened up the possibility of heterotic superstring theories in four spacetime dimensions (with or without additional compactified dimensions) [7]. This is possible because the \(c = 26\) (or \(c = 15\) with supersymmetry) condition can be satisfied in fewer dimensions provided the space is curved. For example it has been possible to construct consistent purely four dimensional heterotic string theories based on non-compact current algebra cosets [8] as will be illustrated in the next section. The gauge groups that emerge fall within a remarkably narrow range and include the desirable low energy color-electroweak symmetry of \(SU(3) \times SU(2) \times U(1)\). The quark and lepton states, which come in color triplets and \(SU(2)\) doublets, are expected to emerge in several families. Compared

\footnote{The star \(*\) in the last two lines of (1.1) means that one must take the non-compact version of \(G\) such that the maximal compact subgroup is \(H \times U(1)\), and \(H\) is the subgroup that appears in the denominator of \(G^*/H\).}
to the popular approach of four flat dimensions plus compactified dimensions, the gauge
groups are either the same or closely related. This gives the hope that the spectrum of a
curved purely four dimensional heterotic superstring that describes the very early universe
may be closely related to the quarks and leptons that survive to the present times.

Thus, it is of interest to investigate this spectrum for a purely four dimensional het-
erotic superstring theory in curved spacetime with or without additional compactified
dimensions. One advantage of pure four dimensions is the expected tighter predictions
since the multitude of string “vacua” associated with the higher compactified dimensions
would be avoided. This approach does not explain the deeper issue of why there are four
dimensions in the physical world, but allows the exploration of the kinds of results that
would emerge if the Universe has in fact only four dimensions at all times

The study of specific four dimensional manifolds that emerge in these models has
proceeded through some rather special group theoretical methods. Some of it should be
of interest in classical gravitational physics and would seem rather remarkable to general
relativists. For example, it has been possible to determine the global properties of the
singular geometries and solve completely for all particle geodesics. These global
manifolds contain several copies of the same “world” and geodesics continue smoothly from
one world to the next by going through curvature singularities. The general solution to
\textit{string geodesics} has also been outlined.

The quantum investigations have also gone well beyond the one loop approximations.
For example, it has been possible to determine the conformally exact metric and dilaton

\footnote{From the point of view of perturbative conformal conditions there seems to be a problem
with the incompatibility of Poincaré invariance and \( c = 26 \) in purely four dimensions (e.g. \( c =
26 \) is possible for an asymptotically flat spacetime in four dimensions provided the dilaton is
asymptotically linear). This problem may or may not be resolved with a better understanding
of conformal invariance and non-perturbative effects such as phase transitions and the relation of
the observed physical flat universe to the early curved string universe. Presumably the curved
spacetime string theory \textit{must} undergo some physical phase transitions, including giving a mass
to the dilaton and the usual inflationary scenario, before it can be connected to the observed
flat, homogeneous and isotropic Universe. In the final stage one expects an effective field theory
of the massless particles that describes the low energy physics below the Planck scale. This is
no longer the full string theory, and it is at this stage that one requires a flat, homogeneous and
isotropic universe after inflating a small part of the early universe into our present Universe. Thus,
there may be a resolution for conformal invariance on the one hand and four flat dimensions on
the other, after taking into consideration the physical situation. If this proves to be impossible
then one must accept more than four dimensions and consider the non-compact group models in
higher dimensions and/or direct products with an internal conformal field theory. The methods
for computing the low energy spectrum are basically the same in a model with either pure four
dimensions or with higher compactified dimensions included.}
to all orders in the sigma model interactions. The new algebraic methods, which will be described below apply to all gauged WZW models.

2. Curved Spacetime and Non-compact Groups

2.1. Time Coordinate

Let us consider a WZW model based on a non-compact group. Let us parametrize the group element by $X^A(\tau, \sigma)$, where $A$ is an index in the adjoint representation. The left or right moving currents take the form $J^A = \partial X^A + \cdots$, where the dots stand for non-linear terms in an expansion in powers of $X$. The Fourier components of these currents $J^A_n$ satisfy a Kac-Moody algebra

$$[J^A_n, J^B_m] = i f^{AB}_C J^C_{n+m} - k \frac{n}{2} \eta^{AB} \delta_{n+m,0}$$

(2.1)

where $k$ is the central extension and $\eta^{AB}$ is proportional to the Killing metric. In an appropriate basis one can choose a diagonal $\eta^{AB} = \text{diag}(1, \cdots, 1, -1, \cdots -1)$ with $+1$ entries corresponding to compact generators and $-1$ entries to non-compact ones. For example, for $SL(2, \mathbb{R})$ with currents $(J^0, J^1, J^2)$, one has the Minkowski metric in $2 + 1$ dimensions: $\eta^{AB} = \text{diag}(1, -1, -1)$.

Let us consider the large positive $k$ limit of the WZW model and examine the commutation rules of its canonical currents. It is convenient to define the rescaled currents $\alpha^A_n = \sqrt{2} J^A_n$. When $k \to \infty$ these behave like the free field oscillators of the flat string theory (either left or right movers)

$$[\alpha^A_n, \alpha^B_m] = -n \eta^{AB} \delta_{n+m,0}.$$

(2.2)

We see that, in the large $k$ limit, we have free field degrees of freedom $X^A \sim \sum_n \frac{1}{n!} \alpha^A_n z^n + \cdots$, that behave like time coordinates when $A$ corresponds to compact generators and like space coordinates when $A$ corresponds to non-compact generators. The signature of the coordinates are the same for finite positive $k$. This is seen by specializing the commutation rules (2.1) to $A = B$ for which the structure constant of the Lie algebra drops out.

In a string theory one can tolerate only one time coordinate. This is because, by naive counting, the Virasoro constraints $L_n \sim 0$ can eliminate only the ghosts generated by the negative norm of one time-like oscillator $\alpha^0_n$, just like string theory in flat spacetime. Therefore, one must put constraints that set to zero the unwanted compact generators, except for one of them. However, first class constraints must close to form an algebra. Therefore, the currents that are set equal to zero ($J^a \sim 0$ weakly on states) must form a subalgebra corresponding to a subgroup of the non-compact group $H \subset G$. The subalgebra may include compact and non-compact generators. The remaining currents $J^\mu$, $\mu = 0, 1, 2, \cdots (d - 1)$ stand in one-to-one correspondence to the coset coordinates $X^\mu$ that include just one time coordinate. Thus, one must choose a subgroup $H$ such that the coset
$G/H$ has the signature of Minkowski space in $d$ dimensions. It is well known that this set of constraints defines an exact conformal field theory that fits the algebraic framework of GKO. The new ingredient is that one must take an appropriate non-compact coset $G/H$. The only simple groups that give a single time coordinate were classified in [2] and are listed in (1.1).

There is another way to see the same result by using a Lagrangian method at the classical level rather than the algebraic Hamiltonian argument given above at the quantum level. A GKO theory corresponds to a gauged WZW model with the subgroup $H$ local. Using the gauge invariance one can eat-up $\dim(H)$ degrees of freedom, leaving behind $\dim(G/H)$ group parameters that contain just one timelike coordinate. Since the gauge fields are non-dynamical they can be integrated out. This leaves behind a sigma model type theory with the desired signature. The large $k$ limit of this theory has free field quantum oscillators with a single time coordinate.

Both the Hamiltonian and Lagrangian arguments were first given by Bars and Nemeschansky [1]. The Hamiltonian approach was given more weight in [1] where several examples, including $SL(2, \mathbb{R})/\mathbb{R}$ at $k = 9/4$, were investigated. The Lagrangian method was explicitly carried out for $SL(2, \mathbb{R})/\mathbb{R}$ by Witten [4] who interpreted the sigma model metric as a black hole. With the realization that non-compact group coset methods generate singular geometries there has been a flurry of activity to determine the geometries of higher dimensional cosets [5][6][7][3] as discussed in the following sections.

2.2. Action for Heterotic Superstring in Curved Spacetime

There are additional ingredients that must be included in a physical model. A good model must not have a tachyon. The large $k$ limit that reduces to a flat string theory provides a guide for how to eliminate the tachyon state. Namely, one must start with a string theory that has at least $\tilde{N} = 1$ supersymmetry on the world sheet, and then impose the GSO projection [12] on the spectrum $(-1)^F = 1$. This is achieved for any $k$ in curved spacetime by starting from a Kazama-Suzuki type model based on non-compact cosets [3]. Thus, the super coset may be given in the form

$$\frac{G_{-k} \times SO^*(\dim(G/H))_1}{H_{-k+g-h}}$$

(2.3)

where $SO^*$ is a non-compact version of $SO$. Moreover, a physical model must be a heterotic theory that includes gauge groups. This is done by taking the coset (2.3) with central charge $c = 15$ for left movers and the coset $G_{-k}/H_{-k} \times (gauge\ group)$ with $c = 26$ for right movers. We can then search for all possibilities that satisfy these requirements. In what follows, for definiteness, we restrict ourselves to cosets $G/H$ that have only four bosonic dimensions. It is evident that this restriction is not a priori justified in our formalism and evidently more general models with higher dimensions are possible. However, as emphasized above it is of interest to find out the behaviour of purely four dimensional models of this type (see footnote 2).
At this point let us construct the gauged WZW action for a heterotic superstring in curved spacetime. Here we first repeat the $SO(3,2)/SO(3,1)$ example worked out in [1] and emphasize a few important points. In the conformal gauge the action has four parts $S = S_0 + S_1 + S_2 + S_3$ with

\[
S_0(g) = \frac{k}{8\pi} \int_M d^2\sigma \Tr (g^{-1} \partial_+ g^{-1} \partial_- g) - \frac{k}{24\pi^2} \int_B Tr (g^{-1} dg g^{-1} dg g^{-1} dg)
\]

\[
S_1(g, A) = -\frac{k}{4\pi} \int_M d^2\sigma \Tr (A_- \partial_+ g^{-1} - \tilde{A}_+ g^{-1} \partial_- g + A_- g \tilde{A}_+ g^{-1} - A_- A_+ )
\]

\[
S_2(\psi_+, A_-) = -\frac{k}{4\pi} \int_M d^2\sigma \ \psi_\mu^\dagger (iD_- \psi_+) \nu \eta_{\mu\nu}, \quad S_3(\chi_-) = \frac{k}{4\pi} \int_M d^2\sigma \sum_{a=1}^{22} \chi_a^a i \partial_+ \chi_a^a
\]  

(2.4)

In addition, there are ghost actions $S_4(b_L, c_L, \beta_L, \gamma_L)$ for left movers and $S_5(b_R, c_R)$ for right movers that are added due to the superconformal or conformal gauge fixing respectively. This action has $(1,0)$ superconformal symmetry (see below) and is appropriate for the heterotic string. The type-II string requires $(1,1)$ superconformal symmetry. Its action follows if $\chi^a$ is removed and replaced by $\psi^\mu_+$ that appears with a gauge covariant kinetic term just like $\psi^\mu_-$. Then $S_3, S_5$ are replaced by $S_3(\psi_-, A_+) \quad \text{and} \quad S_5(b_R, c_R, \beta_R, \gamma_R)$.

In the above, $S_0$ is the global WZW model [13] with $g(\sigma^+, \sigma^-) \in SO(3,2)$. By itself this piece has $SO(3,2)_L \times SO(3,2)_R$ symmetry. Since $SO(3,2)$ has a non-Abelian compact subgroup $SO(3)$ the quantum path integral could be defined uniquely only for $k = integer$ (this was not a restriction for $d = 2, 3$). Indeed, we take $k = 5$ which is the value required by the total Virasoro central charge for the supersymmetric left movers [1]

\[
c_L = \frac{3kd}{2(k-d+1)} = 15 \quad \text{for} \quad d = 4, \quad k = 5.
\]  

(2.5)

c_L is cancelled by the super ghost system of $S_4$. For type-II the central charge of the supersymmetric right-movers is also $c_R = 15$. However, for the heterotic string the bosonic part $SO(3,2)_{-k}/SO(3,1)_{-k}$ gives

\[
c_R(\text{bose}) = \frac{10k}{k-3} - \frac{6k}{k-2} = 15
\]  

(2.6)

for the special value $k = 5$ (already fixed in the action). Since the ghosts in $S_5(b_R, c_R)$ contribute $-26$ we require a $c_R(\chi) = 11$ contribution from the free fermions $\chi^a_-$. Therefore the action $S_3$ contains 22 free fermions. This action could be viewed as giving rise to

\[\footnote{The easiest way to see this point is to write $g$ in parametric form $g = abc$ with $a \in SO(3), b \in SO(2)$ and $c \in SO(3,2)/SO(3) \times SO(2)$ and apply the Polyakov-Wiegman formula [14]. Then $S_0(g)$ decomposes into several pieces one of which is $S_0(a)$ that can be defined only for integer $k$ since $SO(3)$ is compact [13]. The remaining pieces do not present a problem.}
SO(22)_1 current algebra theory for right movers. There are many other ways of obtaining \( c_R = 11 \) as exact conformal theories based on current algebras. 

The second piece in the action \( S_1 \) gauges \[15\] the Lorentz subgroup \( H = SO(3, 1) \) which is embedded in \( SO(3, 2)_L \times SO(3, 2)_R \) with a deformation. As explained in \[3\] the action of the gauge group could be deformed on the left or the right of the group element \( g \). If the matrix representation of the gauged Lorentz algebra on the left is \( t_a \) and the one on the right is \( \tilde{t}_a \) then gauge invariance is satisfied by \( \tilde{t}_a = g_0^{-1} t_a g_0 \) or \( \tilde{t}_a = g_0^{-1} (-t_a)^T g_0 \), where \( g_0 \) is any constant group element in complexified \( SO(3, 2) \) (including \( g_0 \)'s not continuously connected to the identity) and \( t^T \) is the transpose of the matrix. In this notation the action \( S_1 \) is expressed in terms of \( A_\pm = A^a_\pm t_a \) and \( \tilde{A}_\pm = A^a_\pm \tilde{t}_a \) with the same \( SO(3, 1) \) gauge potential \( A^a_\pm (\sigma^+, \sigma^-) \). The simplest case of \( \tilde{t}_a = t_a \) corresponds to the standard vector subgroup. The remaining cases generalize the vector/axial gauging options that were first noticed for the 2d black hole \[16\] \[17\] \[18\] \[19\] and thus provide a generalization of the concept of duality. Examples are given in \[1\].

The action \( S_2 \) contains the fermions \( \psi_+^\mu \) with \( \mu = 0, 1, 2, 3 \) that belongs to the coset \( SO(3, 2)/SO(3, 1) \). The flat Minkowski metric \( \eta_{\mu\nu} = \text{diag} (1, -1, -1, -1) \) is used to contract the Lorentz indices. As shown in \[21\] coset fermions lead to \( N = 1 \) superconformal symmetry. Indeed the super coset scheme \( SO(3, 2)_{-5} \times SO(3, 1)_1/\text{SO}(3, 1)_{-4} \) for left movers requires that they appear with gauge covariant derivatives \( D_- \psi_+^\mu = \partial_- \psi_+^\mu - (A_-)^\mu_\nu \psi_+^\nu \). The explicit supersymmetry transformations are written more conveniently in terms of the 5 \( \times \) 5 matrix \( \psi_+ = \begin{pmatrix} 0 & -\psi_+^\mu \\ \psi_+^\mu & 0 \end{pmatrix} \) that belongs to the \( G/H \) part of the Lie algebra \[21\].

\[
\delta g = i\epsilon_- \psi_+ g, \quad \delta \psi_+ = \epsilon_- (gD_+ g^{-1})_{G/H}, \quad \delta \chi^-_a = 0, \quad \delta A_\pm = 0, \tag{2.7}
\]
with \( \partial_- \epsilon_- (\sigma^+) = 0 \). In a type-II theory \( \psi_+^\mu \) also mixes under supersymmetry with the group element \( g^{-1} \) with a transformation similar to the one above. The independent right-moving supersymmetry parameter in this case is \( \epsilon_+ (\sigma^-) \).

This theory is supplemented with the original GSO projection \[12\] adapted to four dimensions. Namely, we construct the operator \( (-1)^F \) with the same prescriptions as \[12\] and project onto the states \( (-1)^F = 1 \). Let us describe the effect on the ground states in the Neveu-Schwarz and Ramond sectors for left movers. In our coset scheme these are conveniently labelled by the scalar, vector and the two spinor representations of the fermionic \( SO(3, 1)_1 \). The GSO projection eliminates the scalar and one of the spinor representations so that the tachyon is eliminated from the theory. The remaining vector and Weyl spinor form the representations \( (\frac{1}{2}, \frac{1}{2}) \) and \( (\frac{1}{2}, 0) \) of the Lorentz group in four dimensions. As is well known this is a covariant space-time supersymmetric vector multiplet and therefore

\[\text{Some examples are } [(E_8)_1 \times SU(4)_1], [(E_7)_1 \times SU(5)_1], [(E_7)_1 \times SU(3)_1 \times SU(2)_1 \times U(1)], [(E_6)_1 \times SO(10)_1], [(E_6)_1 \times SU(4)_2], [SO(10)_2 \times SU(3)_1], \text{etc.}\]
signals the possibility of space-time supersymmetry in our heterotic model. The GSO projections for the type-II theory can be chosen such that the remaining ground state Weyl spinors for the left movers and right movers have either the opposite or the same chirality. Accordingly the theory will be called type-IIA or type-IIB respectively. The important aspect of the GSO projection is to eliminate the tachyon. After the projection one is automatically left over with an equal number of fermionic and bosonic states. It is not clear whether these fall into supermultiplets of some spacetime supersymmetry. To see whether these theories are supersymmetric in curved space-time the target space supercharges have to be constructed explicitly by a curved space-time modification of the analysis of [22].

2.3. Classification of 4D Heterotic Models

We can now ask, what other heterotic models can be constructed with the non-compact group method? Among the cosets listed in (1.1), the only ones that lead to models in four curved spacetime dimensions \((D = 4)\) always include \(SO(d - 1, 2)/SO(d - 1, 1)\) for \(d \leq 4\). The remaining cosets always give models in more than four dimensions. In this lecture we will concentrate on four dimensions and therefore use only \(SO(d - 1, 2)\) for \(d = 2, 3, 4\). For \(D = d = 4\) there are no other bosonic coordinates. When \(d \leq 3\), then \(D - d = 4 - d\) additional bosonic coordinates are supplied by taking direct products with other groups (including space-like \(U(1)\) or \(\mathbb{R}\) factors) and then gauging an appropriate subgroup. Furthermore, we include in our list the possibility of a time-like bosonic coordinate and denote it by a factor of \(T\) instead of \(\mathbb{R}\). All possibilities are listed in Table-1 in the column labelled “right movers”.

| #  | left movers with N=1 SUSY | right movers |
|----|---------------------------|--------------|
| 1  | \(SO(3, 2)_{-k} \times SO(3, 1)_{1}/SO(3, 1)_{-k+1}\) | \(SO(3, 2)_{-k}/SO(3, 1)_{-k}\) |
| 2  | \(\frac{SL(2, \mathbb{R})_{-k_1} \times SL(2, \mathbb{R})_{-k_2} \times SO(3, 1)_1}{SL(2, \mathbb{R})_{-k_1} \times SL(2, \mathbb{R})_{-k_2}} \times \mathbb{R}\) | \(\frac{SL(2, \mathbb{R})_{-k_1} \times SL(2, \mathbb{R})_{-k_2} \times SO(3, 1)_1}{SL(2, \mathbb{R})_{-k_1} \times SL(2, \mathbb{R})_{-k_2}} \times \mathbb{R}\) |
| 3  | \((SO(2, 2)_{-k} \times SO(3, 1)_{1}/SO(2, 1)_{-k+2}) \times \mathbb{R}\) | \((SO(2, 2)_{-k}/SO(2, 1)_{-k}) \times \mathbb{R}\) |
| 4  | \(SL(2, \mathbb{R})_{-k} \times SO(3, 1)_{1} \times \mathbb{R}\) | \(SL(2, \mathbb{R})_{-k} \times \mathbb{R}\) |
| 5  | \(\frac{SL(2, \mathbb{R})_{-k_1} \times SL(2, \mathbb{R})_{-k_2} \times SO(3, 1)_1}{\mathbb{R}^2}\) | \(\frac{SL(2, \mathbb{R})_{-k_1} \times SL(2, \mathbb{R})_{-k_2}}{\mathbb{R}^2}\) |
| 6  | \(SL(2, \mathbb{R})_{-k_1} \times SU(2)_{k_2} \times SO(3, 1)_{1}/\mathbb{R}^2\) | \(SL(2, \mathbb{R})_{-k_1} \times SU(2)_{k_2}/\mathbb{R}^2\) |
| 7  | \((SL(2, \mathbb{R})_{-k} \times \mathbb{R}^2 \times SO(3, 1)_1)/\mathbb{R}\) | \((SL(2, \mathbb{R})_{-k} \times \mathbb{R}^2)/\mathbb{R}\) |
| 8  | \(\mathbb{R}^3 \times \mathbb{R}_Q \times SO(3, 1)_{1}\) | \(\mathbb{R}^3 \times \mathbb{R}_Q\) |

Table-1. Current algebraic description of left movers and right movers.

For brevity we used \(\mathbb{R}\) where we could have used either \(\mathbb{R}\) or \(U(1)\). Case 3 is obtained from case 2 in the limit \(k_1 = k_2 = k\), while case 4 is the \(k_1 = k, k_2 = \infty\) limit of either case 2 or 5. In case 8, the notation \(\mathbb{R}_Q\) is used to denote a free boson with background charge \(Q\) that contributes to the central charge \(c_Q = 1 + 12Q^2\) just like a Liouville field.
The factors of $I_R$ in cases 2, 3 could also be allowed to have a background charge, but we will assume that it is zero in order to keep the discussion as simple as possible.

The cases 5, 6, 7 which contain an $I_R$ factor in the denominator may further be generalized by multiplying both numerator and denominator by a factor $I_R^n$. What this implies is that there are many possible ways of gauging the $I_R$ factors by taking linear combinations. This may lead to models that have different spacetime dynamics, however since the central charges remain unchanged, this generalization does not alter the gauge symmetry results given in Table-2.

The heterotic string will have a supersymmetric left-moving sector and a non-supersymmetric right-moving sector. The cosets above describe the four dimensional space-time part of the right-moving sector. This contributes $c_R(4D)$ toward the Virasoro central charge. After we analyse the central charge of the supersymmetric left movers and fix it to be $c_L = 15$ in only four dimensions, we will see that $c_R(4D)$ will be fixed to some value less than 26. Therefore, for the mathematical consistency of the theory, we must require that the right moving sector contains an additional “internal” part which makes up for the difference, i.e. $c_R(int) + c_R(4D) = 26$. One of the aims is to compute $c_R(int)$ in each model and then find gauge symmetry groups that precisely give this value. This procedure will allow us to discover the gauge symmetries that are possible in these curved spacetime string models.

To construct a heterotic string we introduce four left moving coset fermions $\psi^\mu$ that are classified under $H$ as $G/H$ and form a $N = 1$ supermultiplet together with the four bosons. The action that possesses the superconformal symmetry has the form of $S_2$ in (2.4) as given in [7]. The left moving fermions $\psi^\mu$ are coupled to the gauge bosons in $H$. In the Hamiltonian language, the left moving stress tensor is expressed in the form of current algebra cosets $[20|8]$ as listed in Table-1, where $SO(3,1)_1$ represents the fermions.

This algebraic formulation allows an easy computation of the Virasoro central charges for left movers $c_L$ as well as the right movers $c_R(4D)$. For a consistent theory we must set $c_L = 15$. This condition puts restrictions on the various central extensions $k$ and/or background charge $Q$, as listed in Table-2 (assuming $Q = 0$ for cases 2, 3, 4). After inserting these in $c_R(4D)$ we find the deficit from the critical value of 26, i.e. $c_R(int) = 26 - c_R(4D)$. As seen in the table, the resulting values for $c_R(int)$ fall within a narrow range. For case 2 or 3 it is possible to change the central charge within the range $11\frac{1}{2} < c_R(int) < 13$ by varying $k_1 + k_2$. For the remaining cases it is not possible to change $c_R(int)$ by using the remaining freedom with the $k's$. 

10
conditions for \( c_L = 15 \)

| # | \( k \) | \( c_R(int) \) | gauge group, right movers |
|---|---|---|---|
| 1 | 5 | 11 | \((E_7)_1 \times SU(5)_1\) |
| 2 | \( k_1 - 2 = \frac{k_2-2}{2}(-1 + \sqrt{\frac{3k_2}{k_2-8}}) \) | 13 - \( \delta \) | \( \delta = \frac{12}{(k_1+k_2-4)(k_1+k_2-2)} \) |
| 3 | 3 | 11 \( \frac{1}{2} \) | \((E_7)_1 \times SU(3)_1 \times SU(2)_2 \times U(1)_1\) |
| 4 | 8/3 | 13 | \((E_8)_1 \times SO(10)_1\) |
| 5 | \( k_1 = \frac{8k_2-20}{3k_2-8}, \ k_1, k_2 > \frac{2}{3} \) | 13 | \((E_8)_1 \times SO(10)_1\) |
| 6 | \( k_1 = \frac{8k_2+20}{3k_2+8}, \ k_1 = 1, 2, 3, \ldots \) | 13 | \((E_8)_1 \times SO(10)_1\) |
| 7 | 8/3 | 13 | \((E_8)_1 \times SO(10)_1\) |
| 8 | \( Q_0^2 = \frac{1}{4} \) | 13 | \((E_8)_1 \times SO(10)_1\) |

Table-2. Conditions for \( c_L = 15 \) and examples of symmetries that give \( c_R = 26 \).

The value of \( c_R(int) = 13 \) that occurs for most of the cases is the same as the deficit for the popular heterotic string models that have four flat dimensions plus compactified dimensions described by a \( c = 9 \), \( N = 2 \) superconformal theory (i.e. \( 4 + 9 + 13 = 26 \)). Hence, for these cases, the appearance of \((E_8)_1 \times SO(10)_1\) as the gauge group has precisely the same explanation as the usual approach. For the remaining cases we give an example of a gauge symmetry that will make up the deficit, as listed in Table-2. Other gauge groups are clearly possible just on the basis of \( c_R(int) \); for example for case 1 see footnote 4.

The gauge symmetry is associated with a conformal theory of right movers. This additional part of the action may be constructed (as \( S_3 \) in (2.4)) from right moving free fermions with appropriate boundary conditions, or by using other devices that are quite familiar. We can think of this part as another current algebra associated with the gauge group, and with the central extensions that are given in Table-2. This final step completes the action for the model. Further discussion of the model is required to determine the symmetries consistent with modular invariance. At this stage it is encouraging to note that the desirable low energy symmetries, including \( SU(3) \times SU(2) \times U(1) \), are contained in these curved space string models that have only four dimensions.

### 3. Geometry of the Manifold

A gauged WZW model can be rewritten in the form of a non-linear sigma model by choosing a unitary gauge that eliminates some of the degrees of freedom from the group element, and then integrating out the non-propagating gauge fields \[ \Box \] \[ \Box \]. The remaining degrees of freedom are identified with the string coordinates \( X^\mu(\tau, \sigma) \). The resulting action exhibits a gravitational metric \( G_{\mu\nu}(X) \) and an antisymmetric tensor \( B_{\mu\nu}(X) \) at the classical level. At the one loop level there is also a dilaton \( \Phi(X) \). These fields govern the spacetime geometry of the manifold on which the string propagates. Conformal invariance
at one loop level demands that they satisfy coupled Einstein’s equations. Thanks to the exact conformal properties of the model these equations are automatically satisfied. Therefore, any of our non-compact gauged WZW models can be viewed as generating automatically a solution of these rather unyielding equations. One only needs to do some straightforward algebra to extract the explicit forms of $G_{\mu\nu}, B_{\mu\nu}, \Phi$.

This algebra can be carried out by starting from the Lagrangian, such as in (2.4), and has been done for all the models in four dimensions listed in Table-1. The first case was $SL(2, \mathbb{R})/\mathbb{R}$ which was interpreted by Witten [4] as the geometry of a 2D black hole. The higher dimensional cases yield more intricate but singular geometries [5] [6] [7] [3]. Although the Lagrangian method is straightforward, it has a number of drawbacks. First, it yields the geometry only in a patch that is closely connected to a particular choice of a unitary gauge. The remaining patches of the global geometry can be recovered only in other unitary gauges and may have no resemblance to the analytic form of the metric, dilaton, etc. in another unitary gauge. To overcome this problem we have introduced global coordinates [9] on the complete geometry. The global coordinates are gauge invariant. The second problem with the Lagrangian method is that it yields the semi-classical geometry up to one loop in an expansion in powers of $1/k$. However, since the gauged WZW model is conformally exact one would rather obtain the conformally exact geometry by using alternative methods. It turns out that the Hamiltonian method that utilizes the GKO construction solves both of these problems simultaneously and yields an exact metric and dilaton to all orders in $1/k$ [11]. Therefore in this lecture we concentrate on the Hamiltonian approach.

With the Hamiltonian approach one can compute the gravitational metric and dilaton backgrounds to all orders in the quantum theory (all orders in the central extension $k$) at the “classical” level (i.e. no string loops). We have managed to obtain these quantities for bosonic, type-II supersymmetric, and heterotic string theories in $d \leq 4$. It turns out that the geometry of the heterotic and type-II superstrings are obtained by deforming the geometry of the purely bosonic string by definite shifts in the exact $k$-dependence. Therefore, it is sufficient to first concentrate on the purely bosonic string. The following relations have been proven for $G/H = SO(d - 1, 2)/SO(d - 1, 1)$ which is relevant to string theory [11]: (i) For type-II superstrings the conformally exact metric and dilaton are identical to those of the non-supersymmetric semi-classical bosonic model except for an overall renormalization of the metric obtained by $k \to k - g$. (ii) The exact expressions for the heterotic superstring are derived from their exact bosonic string counterparts by shifting the central extension $k \to 2k - h$ (but an overall factor $(k - g)$ remains unshifted). (iii) The combination $e^{\Phi} \sqrt{-G}$ is independent of $k$ and therefore can be computed in lowest order perturbation theory. Cases 2,5,6 in Table-1 are a bit more complicated because of the two central extensions, but the results that relate the bosonic string to superstrings are analogous. Case 6 is explicitly discussed in [10], and the others are just analytic continuations of this one.

The main idea is the following. For the bosonic string the conformally exact Hamiltonian is the sum of left and right Virasoro generators $L_0^L + L_0^R$. They may be written purely
in terms of Casimir operators of $G$ and $H$ when acting on a state $T(X)$ at the tachyon level. The exact dependence on the central extension $k$ is included in this form by using the GKO formalism in terms of currents. For example for the left-movers

$$L^L_0 T = \left( \frac{\Delta^L_G}{k-g} - \frac{\Delta^L_H}{k-h} \right) T$$

$$\Delta^L_G \equiv \text{Tr}(J^L_G)^2, \quad \Delta^L_H \equiv \text{Tr}(J^L_H)^2 ,$$

(3.1)

The exact quantum eigenstate $T(X) = \langle X | T >$ can be analyzed in $X$-space. Then the Casimir operators become Laplacians constructed as differential operators in group parameter space ($\text{dim}G$). Consider a state $T(X)$ which is a singlet under the gauge group $H$ (acting simultaneously on left and right movers)

$$(J^R_H + J^R_H) T = 0 .$$

(3.2)

Because of the $\text{dim}H$ conditions $T(X)$ can depend only on $d = \text{dim}(G/H)$ parameters, $X^a$ (string coordinates), which are $H$-invariants constructed from group parameters (see below). The fact that there are exactly $\text{dim}(G/H)$ such independent invariants is not immediately obvious but it should become apparent to the reader by considering a few specific examples. As discussed in [9] these are in fact the coordinates that globally describe the sigma model geometry. Consequently, using the chain rule, we reduce the derivatives in (3.1) to only derivatives with respect to the $d$ string coordinates $X^a$. In this way we can write the conformally exact Hamiltonian $L^L_0 + L^R_0$ as a Laplacian differential operator in the global curved space-time manifold involving only the string coordinates $X^a$. By comparing to the expected general form

$$(L^L_0 + L^R_0) T = \frac{-1}{e^\Phi \sqrt{-G}} \partial_a (e^\Phi \sqrt{-GG^{ab} \partial_b T})$$

(3.3)

for the singlet $T$, we read off the exact global metric and dilaton.

We have applied this program to all the models in Table-1 and obtained the exact geometry to all orders in $1/k$. The large $k$ limit of our results agree with the semi-classical computations of the Lagrangian method. In the special case of two dimensions we also agree with another previous derivation of the exact metric and dilaton for the $SL(2, \mathbb{R})/\mathbb{R}$ bosonic string [23]. We summarize here the global and conformally exact results for the metric and dilaton in the case of $SO(d-1, 2)_k/ SO(d-1, 1)_k$ for $d=2,3,4$ [11]. Due to the more complex expressions we refer the reader to the original literature for the remaining cases [24] [10]. The group element $g$ for $SO(d-1, 2)/SO(d-1, 1)$ can be parametrized as a $(d+1) \times (d+1)$ matrix in the form

5 Here $J^L_G$ and $J^L_H$ are antihermitian group and subgroup generators obeying the appropriate Lie algebras, and $g$, $h$ are the Coxeter numbers for the group and the subgroup. For the cases of interest in this paper $g = d - 1$, $h = d - 2$ for $d \geq 3$, and $g = 2$, $h = 0$ for $d = 2$.
\[ g = \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{1+a})_{\mu}^{\nu} \end{pmatrix} \left( \begin{array}{c} b \\ -(b+1)x_{\mu} \end{array} \right) \left( \begin{array}{c} (b+1)x^{\nu} \\ (\eta_{\mu}^{\nu} - (b+1)x_{\mu}x^{\nu}) \end{array} \right), \] (3.4)

where \( b = \frac{1-x^2}{1+x^2} \). The \( d \) parameters \( x_{\mu} \) and \( d(d-1)/2 \) parameters \( a_{\mu\nu} \) transform as vector and antisymmetric tensor respectively under the Lorentz subgroup \( H = SO(d-1,1) \) which acts on both sides of the matrix as \( g \to gh^{-1} \). By considering the infinitesimal left transformations \( \delta_L g = \epsilon_L g \) we can read off the generators that form an \( SO(d-1,2) \) algebra for left transformations.

\[ J^L_{\mu\nu} = \frac{1}{2}(1 + a)_{\mu\alpha}(1 + a)_{\nu\beta} \frac{\partial}{\partial a_{\alpha\beta}} \]
\[ J^L_{\mu} = -\frac{1}{2}(1 + x^2)(\frac{1}{1-a})_{\mu}^{\nu} \frac{\partial}{\partial x^{\nu}} + \frac{1}{2}(1 + a)_{\mu\alpha}(1 + a)_{\beta\gamma}x^{\gamma} \frac{\partial}{\partial a_{\alpha\beta}}. \] (3.5)

If we consider instead, the infinitesimal right transformations \( \delta_R g = g\epsilon_R \) we find the following expressions for the generators of right transformations

\[ J^R_{\mu\nu} = -\frac{1}{2}(1 - a)_{\mu\alpha}(1 - a)_{\nu\beta} \frac{\partial}{\partial a_{\alpha\beta}} - x^{[\mu} \frac{\partial}{\partial x^{\nu}]} \]
\[ J^R_{\mu} = \frac{1}{2}(x^2 - 1) \frac{\partial}{\partial x^{\mu}} - x_{\mu}x^{\nu} \frac{\partial}{\partial x^{\nu}} - \frac{1}{2}(1 - a)_{\mu\alpha}(1 - a)_{\beta\gamma}x^{\gamma} \frac{\partial}{\partial a_{\alpha\beta}}. \] (3.6)

The \( J^R \) currents obey the same commutation rules as \( J^L \) and moreover commute with each other \( [J^L, J^R] = 0 \). The quadratic Casimirs for the group and subgroup on either the left or the right are obtained by squaring these currents. For the explicit expressions see [11].

As argued above the global parametrization of the manifold is given in terms of \( H \)-invariants, i.e. Lorentz invariants in the present case. In order to obtain a diagonal metric on the manifold one must find \( d \) convenient combinations of these Lorentz invariants in \( d \) dimensions. We give here the basis that diagonalizes the semi-classical metric at large \( k \). One of the natural invariants already occurs in the construction of the group element for every \( d \), namely \( b = \frac{1-x^2}{1+x^2} \).

### 3.1. Two dimensions

For \( d = 2 \) the antisymmetric tensor is Lorentz invariant \( a_{\mu\nu} = ae_{\mu\nu} \), and it is convenient to parametrize \( a = tanh(t) \) or \( coth(t) \). Then the global string coordinates can be taken as \( X^a = (t,b) \). Given all possible values for \( (a,x^\mu) \) the ranges of the two invariants cover the entire plane \(-\infty < t, b < +\infty \). The metric is given by the line element

\[ ds^2 = 2(k-2)(\frac{db^2}{4(b^2-1)} - \beta(b) \frac{b-1}{b+1} dt^2), \quad \beta^{-1}(b) = 1 - \frac{2}{k} \frac{b-1}{b+1}. \] (3.7)

For the dilaton the corresponding expression is
\[ \Phi = \ln \left( \frac{b + 1}{\sqrt{\beta(b)}} \right) + \text{const} \tag{3.8} \]

The scalar curvature for this metric is
\[ R = \frac{2k}{k - 2} \frac{(k - 2)b + k - 4}{((k - 2)b + k + 2)^2} \tag{3.9} \]

The curvature is singular at \( b = -(k + 2)/(k - 2) \), which is also where \( \beta(b) = \infty \). These are the properties of the exact 2d metric. The semi-classical metric is obtained by taking the large \( k \) limit, for which \( \beta = 1 \). The signature of the space is \((+ -)\) or \((- +)\) depending on the region in the \((t, b)\) plane as indicated in Fig-2 of [7]. The signature is understood by examining the semi-classical metric. To see the connection to the Kruskal coordinates used by Witten let \( b = 1 - 2uv \) and \( u^2 = e^{2t}|b-1|/2 \), \( v^2 = e^{-2t}|b-1|/2 \).

There are asymptotically flat regions which are displayed by the change of coordinates \( b = \pm \cosh \sqrt{2/(k-2)} z_1 \), \( t = z_0 \sqrt{2k} \). For large \( z_1 \to \pm \infty \) and any \( z_0 \) the exact metric and dilaton have the asymptotic forms
\[ ds^2 = dz_1^2 - dz_0^2, \quad \Phi = \sqrt{\frac{2}{k - 2}} |z_1|, \tag{3.10} \]

displaying a dilaton which is asymptotically linear in the space direction, just like a Liouville field in 2d quantum gravity with a background charge. Despite the flat metric there is no Poincaré invariance due to the linear dilaton. Note that both the region outside the horizon \((b \to +\infty)\) and the naked singularity region \((b \to -\infty)\) are asymptotically flat.

### 3.2. Three dimensions

For \( d = 3 \) the antisymmetric tensor is equivalent to a pseudo-vector \( a_{\mu\nu} = \epsilon_{\mu\nu\lambda} y^\lambda \), from which we construct two convenient invariants \( v = 2/(1 + y^2) \) and \( u = -v(x \cdot y)^2/x^2 \), which together with \( b \) provide a basis for the string coordinates \( X^a = (v, u, b) \). Given all possible values taken by \((x^\mu, y^\mu)\) the allowed ranges for the invariants are
\[ (+ - +) \text{ or } (- + +) \quad \{b^2 > 1 \& uv > 0\}, \tag{3.11} \]
\[ (+ + -) \quad \{b^2 < 1 \& uv < 0\}, \quad \text{except} \quad 0 < v < u + 2 < 2. \]

The 3d conformally exact metric is given by the line element [11]
\[ ds^2 = 2(k - 2)(G_{bb}db^2 + G_{vv}dv^2 + G_{uu}du^2 + 2G_{vu}dvdu) \tag{3.12} \]
where
\[
G_{bb} = \frac{1}{4(b^2 - 1)},
\]
\[
G_{vv} = -\frac{\beta(v, u, b)}{4v(v - u - 2)}\left(\frac{b + 1}{b - 1} - \frac{1}{k - 1} \frac{u + 2}{v - u - 2}\right),
\]
\[
G_{uu} = \frac{\beta(v, u, b)}{4u(v - u - 2)}\left(\frac{b - 1}{b + 1} + \frac{1}{k - 1} \frac{v - 2}{v - u - 2}\right),
\]
\[
G_{vu} = \frac{1}{4(k - 1)} \frac{\beta(v, u, b)}{(v - u - 2)^2},
\]
and
\[
\beta^{-1}(v, u, b) = 1 + \frac{1}{k - 1} \frac{1}{v - u - 2} \left(\frac{b - 1}{b + 1}(u + 2) - \frac{b + 1}{b - 1}(v - 2) - \frac{2}{k - 1}\right). \quad (3.14)
\]
The exact dilaton is
\[
\Phi = \ln \frac{(b^2 - 1)(v - u - 2)}{\sqrt{\beta(v, u, b)}} + \Phi_0, \quad (3.15)
\]
In the large \(k\) limit one obtains the global version of a semi-classical metric derived in [9] with Lagrangian methods
\[
\frac{ds^2}{2(k - 2)} \bigg|_{k \to \infty} = \frac{db^2}{4(b^2 - 1)} - \frac{1}{v - u - 2} \left(\frac{b + 1}{b - 1} \frac{dv^2}{4v} - \frac{b - 1}{b + 1} \frac{du^2}{4u}\right). \quad (3.16)
\]
The signature \((+++)\), or \((-++)\), or \((++-)\) depends on the region and is indicated in Fig-1 of [9]. A three dimensional view of this metric is given in Figs-4 of [9]. The surface is where the scalar curvature blows up. This coincides with the location where the dilaton blows up in the large \(k\) limit as seen from the above expression. The space has two topological sectors denoted by the sign of a conserved "charge" \(\pm = \text{sign}(v(b + 1)) = \text{sign}(u(b - 1))\). The sign never changes along geodesics. A more intuitive view of the space is obtained in another set of coordinates for the plus sector \((b, \lambda_+, \sigma_+)\) and the minus sector \((b, \lambda_-, \sigma_-)\), which are given by \(\lambda_+^2 = \pm v(b + 1)\) and \(\sigma_+^2 = \pm u(b - 1)\). Then the singularity surface is shown in Figs-3 of [9]. In the plus region the singularity surface has the topology of the double trousers with pinches in the legs. In the minus region we have the topology of two sheets that divide the space into three regions.

There are asymptotically flat regions that may be displayed by a change of variables to \(b = \pm \cosh \frac{1}{\sqrt{3(k - 2)}}(2z_1 - z_0), u = (\pm)\cosh \frac{1}{\sqrt{3(k - 2)}}(-z_1 + 2z_0)\cosh^2 z_2, v = (\pm)\cosh \frac{1}{\sqrt{3(k - 2)}}(-z_1 + 2z_0)\sinh^2 z_2\). For large values of \(z_1 \to \pm \infty\), and finite values of \((z_0, z_2)\), the semiclassical metric and dilaton take the form
\[
ds^2 = -dz_0^2 + dz_1^2 + dz_2^2, \quad \Phi = \sqrt{\frac{6}{k - 2} |z_1'|}, \quad (3.17)
\]
showing that the dilaton is linear in a space-like direction \( z'_1 = \frac{5}{3}z_1 - \frac{4}{3}z_0 \) in the asymptotically flat region. Then \( z'_1 \) behaves just like a Liouville field. The asymptotic metric may be written as \( ds^2 = -(dz'_0)^2 + (dz'_1)^2 + dz'_2^2 \) in terms of the Lorentz transformed \( z'_1 \) and \( z'_0 = \frac{5}{3}z_0 - \frac{4}{3}z_1 \). The exact metric is not flat when only \( |z_1| \) is large. To display its asymptotically flat region one requires somewhat different coordinates.

### 3.3. Four dimensions

For \( d = 4 \) one can construct the Lorentz invariants

\[
x^2, \quad z_1 = \frac{1}{4} Tr(a^2), \quad z_2 = \frac{1}{4} Tr(a^*a), \quad z_3 = xa^2x/x^2, \tag{3.18}
\]

where \( a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} a^{\alpha\beta} \) is the dual of \( a_{\mu\nu} \). However, the semi-classical metric is diagonal for a different set of four invariants \( X^a = (v, u, w, b) \) given by

\[
b = \frac{1 - x^2}{1 + x^2}, \quad u = \frac{1 + z_2^2 + 2(z_1 - z_3)}{1 - 2z_1 - z_2^2}, \quad \frac{1 + z_1 + \sqrt{z_1^2 + z_2^2}}{1 - z_1 - \sqrt{z_1^2 + z_2^2}}, \quad w = \frac{1 + z_1 - \sqrt{z_1^2 + z_2^2}}{1 - z_1 + \sqrt{z_1^2 + z_2^2}}. \tag{3.19}
\]

To find the ranges in which the above global coordinates take their values we consider a Lorentz frame that can cover all possibilities without loss of generality. First we notice that by Lorentz transformations the antisymmetric matrix \( a_{\mu\nu} \) can always be transformed to a block diagonal matrix with the non-zero elements

\[
a_{01} = \tanh t \text{ or } \coth t, \quad a_{23} = \tan \phi. \tag{3.20}
\]

Then using (3.19) one can deduce the form of the global variables: \( v = \pm \cosh 2t, \quad w = \cos 2\phi, \quad \text{and} \quad u = \frac{1}{2x} (w(x_0^2 - x_1^2) - v(x_2^2 + x_3^2)) \). Therefore the string variables can take values in the following regions with the signature in the \((v, u, w, b)\) basis

\[
\begin{align*}
(-+++): & \quad b^2 > 1, \quad \{-1 < w < u < 1 < v \text{ or } v < -1 < u < w < 1 \} \\
& \text{or} \quad -1 < w < 1 < u < v, \\
\end{align*}
\]

\[
(+-+-): \quad b^2 > 1, \quad \{-1 < w < 1 < v < u \text{ or } u < v < -1 < w < 1 \} \\
\]

\[
(++++): \quad b^2 < 1, \quad \{u < w < 11 < v \text{ or } v < -1 < w < u \text{ or } v < u < -1 < w < 1 \}. \tag{3.21}
\]

With this set of coordinates we compute the conformally exact dilaton and metric as before. The dilaton field is

\[
\Phi = \ln \left( \frac{(b^2 - 1)(b - 1)(v - u)(w - u)}{\sqrt{\beta(b, u, v, w)}} \right) + \Phi_0. \tag{3.22}
\]
and the metric is given by

\[
\begin{aligned}
ds^2 &= 2(k - 3) \left( G_{bb} db^2 + G_{uu} du^2 + G_{vv} dv^2 + G_{ww} dw^2 \\
&\quad + 2G_{uv} dudv + 2G_{uw} du dw + 2G_{vw} dv dw \right),
\end{aligned}
\]

where

\[
G_{bb} = \frac{1}{4(b^2 - 1)},
\]

\[
G_{uu} = \frac{\beta(b, u, v, w)}{4(u - w)(v - u)} \left( \frac{b - 1}{b + 1} - \frac{1}{k - 2} \frac{(v - w)^2}{(v - u)(u - w)} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \right) \right),
\]

\[
G_{vv} = \frac{(v - w)\beta(b, u, v, w)}{4(u^2 - 1)(v - u)} \left( \frac{b + 1}{b - 1} - \frac{1}{k - 2} \frac{1}{(v - u)(u - w)} \left[ 1 - u^2 + \right. \right. \]
\[\left. \left. \left( \frac{b + 1}{b - 1} \right)^2 (v - u)(v - w) + \frac{1}{k - 2} \frac{b + 1}{b - 1} (1 + v^2)(u + v) - 2v(1 + uv) \right] \right),
\]

\[
G_{ww} = \frac{(v - w)\beta(b, u, v, w)}{4(1 - u^2)(u - w)} \left( \frac{b + 1}{b - 1} - \frac{1}{k - 2} \frac{1}{(v - u)(u - w)} \left[ 1 - u^2 + \right. \right. \]
\[\left. \left. \left( \frac{b + 1}{b - 1} \right)^2 (u - w)(v - u) - \frac{1}{k - 2} \frac{b + 1}{b - 1} (1 + u^2)(u + v) - 2u(1 + uv) \right] \right),
\]

\[
G_{uw} = \frac{\beta(b, u, v, w)}{4(k - 2)(v - u)^2} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{v - w}{u - w} \right),
\]

\[
G_{uw} = \frac{\beta(b, u, v, w)}{4(k - 2)(u - w)^2} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{v - w}{v - u} \right),
\]

\[
G_{vw} = \frac{1}{(k - 2)^2 b - 1} \frac{b + 1}{4(v - u)(u - w)} \beta(b, u, v, w),
\]

and the function \( \beta(b, u, v, w) \) is defined by

\[
\beta^{-1}(b, u, v, w) = 1 + \frac{1}{k - 2} \frac{(v - w)^2}{(v - u)(w - u)} \left( \frac{b + 1}{b - 1} + \frac{b - 1}{b + 1} \frac{1 - u^2}{(v - w)^2} \right) + \frac{1}{k - 2} \frac{vw + u(v + w) - 3}{(v - w)^2} - \left( \frac{b + 1}{b - 1} \right)^2 + \frac{2}{(k - 2)^3} \frac{b + 1}{b - 1} \frac{vw - 1}{(v - u)(u - w)}. \]

The large \( k \) limit of these expressions reduce to the semiclassical dilaton and metric that follow from the Lagrangian approach

\[
\frac{ds^2}{2(k - 2)} \bigg|_{k \to \infty} = \frac{db^2}{4(b^2 - 1)} + \frac{b - 1}{b + 1} \frac{du^2}{4(v - u)(u - w)} + \frac{b + 1}{b - 1} \frac{dv^2}{(4(1 - w^2)(u - w) - \frac{dv^2}{4(v^2 - 1)(v - u)}}.
\]
We can see that the signature of the semiclassical metric for different ranges of the parameters (3.21) is precisely as required by the group parameter space which led to (3.21). However, for the exact metric $\beta(u, v, w, b)$ must remain positive to keep $-\det(G)$ positive. This implies that part of the regions in (3.21) are screened out by quantum effects for the exact geometry. This screening phenomenon is true for every dimension $d = 2, 3, 4$ and the screened regions must be interpreted in the quantum theory as tunneling or decay regions for probability amplitudes (such as the tachyon wavefunction). Under any circumstances the manifold cannot go outside of the range (3.21) dictated by the group theory.

As in the previous $d = 2, 3$ cases, we can check that our explicit expressions for the dilaton and metric give the $k$-independent combination $\sqrt{-G} e^\Phi$. Therefore this quantity takes the same value for either the exact metric and dilaton or the semiclassical metric and dilaton. Since it is unrenormalized by quantum effects (other than one loop), it may be computed in lowest order perturbation theory. This combination appears in the Dalmambertian and is also closely related to the integration measure in the path integral. Through group theoretical arguments given in [5][7] it was possible to guess that this combination should remain unrenormalized by quantum effects. So far there has not been a more satisfactory explanation of the non-renormalization of this quantity.

Similar to the $d = 2, 3$ cases the 4d manifold has an asymptotically flat region, but it will not be discussed here.

3.4. Particle and String Geodesics

Having global coordinates and a global geometry is not sufficient to get a feeling of the geometry, one also needs to know the behavior of the geodesics. However, for the complicated metrics that are displayed above the geodesic equation seems to be completely unmanageable. Fortunately, we have developed a procedure that relies on group theory and managed to solve for all particle geodesics. The trick is to take advantage of the fact that the global coordinates are gauge invariant under $H$-transformations. Then we may solve the equations of motion for the group element $g$ in any gauge, and use the solution to construct the $H$-invariant combinations that form the global coordinates of the geometry. In fact, there is an axial gauge in which $g$ is solved easily [9]. For a point particle (string shrunk to a point) it is given as a function of proper time

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6 In previous papers [3][4] we erroneously stated that the path integral for the gauged WZW model needed an additional factor $F = e^\Phi/\sqrt{-G}$. This point was a mistake because our argument did not take an additional anomaly factor into account. I thank A. Tseytlin for discussions on this issue. The correct measure for the gauged WZW model is just the Haar measure for the group element times the naive measure for the gauge fields. In a unitary gauge that reduces the action to a sigma model, the integration over the gauge fields produces a determinant and an anomaly factor such that, when combined with the Haar measure and a Faddeev-Popov determinant, the effective measure takes the form of the volume element in curved space, i.e. $D^dX \sqrt{-G}$, which is the expected result [25].

19
\begin{equation}
g(\tau) = e^{\alpha \tau} g_0 e^{(p - \alpha) \tau},
\end{equation}

where $g_0$ is a constant group element at initial proper time $\tau$, and $\alpha, p$ are constant matrices in the Lie algebras of $H$ and $G/H$ respectively. The equations of motion require that these constants satisfy a constraint

\begin{equation}
(g_0(p - \alpha)g_0^{-1})_H + \alpha = 0,
\end{equation}

where the subscript $H$ implies a projection to the Lie algebra of $H$. This solution applies to any group and subgroup. As shown in [3] the standard geodesics equations for the geometries displayed above are automatically solved when the $H$-invariants are constructed from the solution (3.27)(3.28). In this way all light-like, space-like and time-like geodesic solutions are obtained.

With the point geodesics at hand we have learned a number of additional interesting properties about the $d = 2, 3, 4$ manifolds [3] which generalize to other non-compact gauged WZW models as well. The most striking feature is that the manifolds that are pictured in the figures have many copies and the complete manifold must include all the copies. The gauge invariant coordinates (e.g. $(b, t)$ for $d = 2$) are not sufficient to fully describe the structure. There are additional discrete gauge invariants constructed from the group element $g$ that label the copies of the manifold. This can be seen easily in our examples since the gauge subgroup is just the Lorentz group and its properties are well known. In this case the invariants are Lorentz dot products constructed from a vector $x^\mu$ and a tensor $a^{\mu\nu}$. Let us consider the invariant $b = (1 - x^2)/(1 + x^2)$, say in the region $x^2 > 0$. It is known that the time component $x^0$ could be either positive or negative and that a Lorentz transformation cannot change this sign. Therefore, the sign of $x^0$ is a discrete gauge invariant which does not show up in the metric or dilaton that characterized the manifolds discussed above. However, the model as whole knows about this sign through the group element $g$. Such discrete invariants are present in every non-compact gauged WZW model and they label copies of the manifolds described above. We may then ask whether these copies communicate with each other? The answer is yes, they do, and this can be seen by following the behaviour of a particle geodesic. The full information about the particle geodesic is contained in the solution for $g$ in (3.27)(3.28). From this it can be verified that at the proper time that a particle touches a curvature singularity the discrete invariant switches sign and then the particle continues its journey smoothly from one copy of the manifold to the next. For example, in the 2d black hole case this happens for a time-like geodesic (i.e. massive particle) in a finite amount of proper time (on the other hand, a light-like geodesic takes an infinite amount of proper time to reach the singularity and therefore ends its journey without changing copies of the manifold). This behavior is present in all non-compact models in this paper as well as other models (e.g. we have verified it in the $SL(2, \mathbb{R}) \times SU(2)/\mathbb{R}^2$ model). It is reminiscent of the Reissner-Nordtsrom black hole in which geodesics move on to other worlds. The difference is that in our case
this happens at the singularity itself. When quantum corrections are included and the exact metric considered, then the singularity and the transition to other worlds no longer seem to be at the same place, at least this is the case for the 2d black hole. The spectrum of the discrete invariant depends on the group representation and therefore one expects different numbers of copies in different quantum states. The number of copies is infinite for quantum states with non-fractional quantum numbers, which is typical in unitary non-holomorphic representations of non-compact groups. When the number of copies is infinite the particle can never come back to the same world, but for a finite number of copies the particle returns to the original world by emerging from a white singularity.

So far we have discussed particle geodesics that correspond to a string collapsed to a single point. We may also investigate string geodesics in the same manifolds. That is we are also interested in solutions for the strings moving in curved spacetime, just like one has a complete solution in flat spacetime in terms of harmonic oscillator normal modes. This problem has been solved in principle for the non-compact gauged WZW models in [5]. There the solution for the group element $g(\tau, \sigma)$ has been obtained explicitly in terms of normal modes. This is the analog of (3.27) above. There remains to construct the appropriate dot products to form the invariants, which in turn are the solutions to the string geodesics. This last part has not yet been performed explicitly, but it is only a matter of straightforward algebra of the kind performed for the particle geodesics in [9]. This procedure gives all the solutions in curved spacetime and can answer questions of the type “what happens when a string falls into a black hole?”

3.5. Duality

Due to the lack of space we have not covered other interesting topics such as duality properties of these manifolds. It was shown in [5,9] that there is a dynamical duality that generalizes the $R \to 1/R$ duality properties of conformal field theories based on tori. This is related to the axial/vector duality that is present in the 2d black hole. It was shown in [5] that the duality transformation is equivalent to an inversion in group parameter space $(x_\mu, a_{\mu\nu})$ given in (3.4). This inversion generates discrete leaps for the group parameter that corresponds to interchanging different regions of the geometrical manifold. For details the reader is referred to [9]. This duality property is closely related to mirror symmetry of the kind discussed for Calabi-Yau manifolds, as will be explained elsewhere. The duality symmetry mentioned here is different than the one discussed in recent months by Verlinde, Giveon, Rocek and others.

4. Some Open problems

The special property of the models constructed in this lecture is that they can be further investigated by using current algebra techniques. The simplest model is case 8, since it is essentially flat, its quantum theory reduces to the manipulation of harmonic oscillators. For the remaining models the spectrum of low energy particles is obtained
by computing the quadratic Casimir operators of the non-compact groups that define the model. The computation of the spectrum will be reported in a future publication. Since the flavor groups such as $SU(3) \times SU(2) \times U(1)$ or $SU(5)$, $SO(10)$, etc. appear at level 1, it is already evident that the quark and lepton type of matter will appear in color triplets and singlets and $SU(2)$ doublets and singlets.

To compute the spectrum one needs to know all the representations of the non-compact groups that appear in Table-1. An inspection of the table shows that one needs the following representations

(i) $SL(2, \mathbb{R})$ in the basis in which one of the non-compact generators is diagonal. This basis is labelled by $|j \mu>$, the range of the allowed values of $j$ are known in a unitary representation, but the literature is not too clear on the allowed values of $\mu$.

(ii) $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ in the basis in which the diagonal subgroup $SL(2, \mathbb{R})$ labels the states. This basis has the form $|j_1, j_2; j, m>$, just like in the problem of addition of angular momentum. The allowed values of $j_1, j_2$ in unitary representations of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ are the standard ones. For given $j_1, j_2$ the allowed values of $j$ are not fully classified.

(iii) $SO(3, 2)$ in the basis in which $SO(3, 1)$ is diagonal. This basis has the form $|a, b; c, k; j, m>$ where $(a, b)$ label the quadratic and quartic Casimirs of $SO(3, 2)$, $(c, k)$ label the two Casimirs of the Lorentz group $SO(3, 1)$ and $(j, m)$ labels the $SO(3)$ rotation subgroup. In unitary representations the allowed ranges for each one of these pairs are known, however, for given $(a, b)$ it is not generally known what would be the allowed values of $(c, k)$.

Evidently there is some mathematics to be developed to obtain the full spectrum. After determining the required ranges of quantum numbers, the strategy is to determine those states that satisfy the conformal conditions for the Virasoro generators in the Neveu-Schwarz and Ramond sectors of the theory (i.e. $L_0 = 1$ in the bosonic sector, etc.). These states will include particles of various spins, chirality and gauge quantum numbers. The ones that are relevant to low energy physics are those protected from getting masses by chiral invariance and gauge invariance. In particular the spin one bosons will fall into the adjoint representation of the gauge groups in Table-2, while the chiral fermions will be the candidates for quarks and leptons. These matter multiplets are expected to appear in the fundamental representation of the gauge group when the Kac-Moody level is exactly one. One of the interesting questions is the number of families that will emerge in this computation. I speculate that the number of distinct non-compact group representations that satisfy the same eigenvalue condition for $L_0$ may be interpreted as the number of families. This may have some relation to the geometrical or topological or duality properties of the manifold. This program for computing the spectrum is underway.

We have only scratched the surface of the subject of non-compact gauged WZW models. We have shown that this approach is very useful for learning about strings in curved spacetime that may be relevant for the early part of the Universe. It is during this era that string theory should be relevant and it is during this era that the matter we know
was formed. Therefore, in trying to solve the puzzles of the Standard Model with respect to the spectrum of matter and gauge bosons we may hope that a string theory in curved spacetime may guide us. For this reason I believe that it is valuable to study in great detail the models presented in Table-1. These are solvable models that should direct us toward a realistic unified theory.
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