THE MANIN-MUMFORD CONJECTURE AND THE TATE-VOLOCH
CONJECTURE FOR A PRODUCT OF SIEGEL MODULI SPACES

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Abstract. We use perfectoid spaces associated to abelian varieties and Siegel moduli spaces to study torsion points and ordinary CM points. We reprove the Manin-Mumford conjecture, i.e. Raynaud’s theorem. We also prove the Tate-Voloch conjecture for a product of Siegel moduli spaces, namely ordinary CM points outside a closed subvariety can not be $p$-adically too close to it.

1. Introduction

We use the theory of perfectoid spaces to study torsion points in abelian varieties and ordinary CM points in Siegel moduli spaces. The use of perfectoid spaces is inspired by Xie’s recent work [34].

1.1. Tate-Voloch conjecture. Our main new result is about ordinary CM points. Let $p$ be a prime number, $L$ the complete maximal unramified extension of $\mathbb{Q}_p$. Let $X$ be a product of Siegel moduli spaces over $L$ with arbitrary level structures.

Theorem 1.1.1. Let $Z$ be a closed subvariety of $X_L$. There exists a constant $c > 0$ such that for every ordinary CM point $x \in X(\bar{L})$, if the distance $d(x, Z)$ from $x$ to $Z$ satisfies $d(x, Z) \leq c$, then $x \in Z$.

The distance $d(x, Z)$ is defined as follows. Let $\| \cdot \|$ be a $p$-adic norm on $\bar{L}$. Let $X$ be an integral model of $X$ over $\mathcal{O}_L$. Let $\{U_1, \ldots, U_n\}$ be a finite open cover of $X$ by affine schemes flat over $\mathcal{O}_L$. Define $d(x, Z)$ to be the supremum of $\|f(x)\|$’s where $U_i$ contains $x$ and $f \in \mathcal{O}_X(U_i)$ vanishing on $Z \cap U_i$. The definition of $d(x, Z)$ depends on the choices of the integral model and the cover. However, the truth of Theorem 1.1.1 does not depend on these choices, see §2.2.2. Moreover, we show that Theorem 1.1.1 holds for formal subschemes of $X$ (with maximal level at $p$), see Theorem 6.2.2. And for CM points which are canonical liftings, we prove an “almost effective” version, see Theorem 6.2.7.

It is clear that the same statement in Theorem 1.1.1 is true replacing $X_L$ by a closed subvariety. In particular, Theorem 1.1.1 is in fact equivalent the same statement for $X$ being a single Siegel moduli space, by embedding a product of Siegel moduli spaces into a larger Siegel moduli space.

Remark 1.1.2. (1) For a power of the modular curve without level structure, Theorem 1.1.1 was proved by Habegger [5] by a different method. However, Habegger’s proof relies on a result of Pila [16] (see also [5, Theorem 8]) concerning Zariski closure of a Hecke orbit. As far as we know, it is not available for Siegel moduli spaces yet. Moreover, Habegger’s method seems not applicable to formal schemes.
(2) Habegger [5] also showed that the ordinary condition is necessary.
(3) The original Tate-Voloch conjecture [32] states that in a semi-abelian variety, torsion points outside a closed subvariety cannot be $p$-adically too close to it. This conjecture was proved by Scanlon [22] [23] when the semi-abelian variety is defined over $\mathbb{Q}_p$. Xie [34] proved dynamic analogs of Tate-Voloch conjecture for projective spaces.

1.1.1. Idea of the proof of Theorem 1.1.1. It is not hard to reduce Theorem 1.1.1 to the case that $X$ has maximal level at $p$, see Lemma 2.2.12. We sketch the proof of Theorem 1.1.1 in this case. Relative to the canonical lifting of an ordinary point $x$ in the reduction of $X$, ordinary CM points in $X$ with reduction $x$ are like $p$-primary roots of unity relative to 1 in the open unit disc around 1 (see Proposition 5.2.1). This is the Serre-Tate theory. If we only consider one such disc, Theorem 1.1.1 follows from a result of Serban [27]. In general, we need to study all infinitely many Serre-Tate deformation spaces together. In characteristic $p$, this can be achieved by Chai’s global Serre-Tate theory [4] (see 5.4). To prove Theorem 1.1.1, we at first prove a Tate-Voloch type result in a family characteristic $p$ (see 6.1). Then we use the ordinary perfectoid Siegel space associated to $X$ and the perfectoid universal covers of Serre-Tate deformation spaces to translate this result to the desired Theorem 1.1.1.

1.1.2. Possible generalizations. For Shimura varieties of Hodge type, the ordinary locus in the usual sense could be empty. In this case, we consider the notion of $\mu$-ordinariness (see [33]). Then following our strategy, we need three ingredients. At first, a theory of Serre-Tate coordinates for $\mu$-ordinary CM points. For Shimura varieties of Hodge type, see [7] and [30]. Secondly, a global theory of Serre-Tate coordinates in characteristic $p$. For Shimura varieties of PEL type, such results should be known to experts. Thirdly, $\mu$-ordinary perfectoid Shimura varieties. Following [25], certain perfectoid Shimura varieties of abelian type are constructed in [31]. For universal abelian varieties over Shimura varieties of PEL type, we expect a Tate-Voloch type result for torsion points in fibers over $\mu$-ordinary CM points. Still, we need analogs of the above three ingredients.

1.2. Manin-Mumford conjecture. For torsion points in abelian varieties, we reprove Raynaud’s theorem [21], which is also known as the Manin-Mumford conjecture.

Theorem 1.2.1 (Raynaud [21]). Let $F$ be a number field. Let $A$ be an abelian variety over $F$ and $V$ a closed subvariety of $A$. If $V$ contains a dense subset of torsion points of $A$, then $V$ is the translate of an abelian subvariety of $A$ by a torsion point.

1.2.1. Idea of the proof of Theorem 1.2.1. We simply consider the case when $V$ does not contain any translate of a nontrivial abelian subvariety. Suppose that $A$ has good reduction at a place of $F$ unramified over a prime number $p$. Let $[p] : A \to A$ be the morphism multiplication by $p$. Let $\Lambda_n$ be a suitable set of reductions of torsions in $[p^n]^{-1}(V)$, and $\Lambda_n^{\text{Zar}}$ its Zariski closure in the base change to $\overline{\mathbb{F}}_p$ of the reduction of $A$. Use the $p$-adic perfectoid universal cover of $A$ to lift $\Lambda_n^{\text{Zar}}$ to $A$. A variant of Scholze’s approximation lemma [24] shows that as $n$ get larger, the liftings are closer to $V$ (see Proposition 2.4.5). A result of Scanlon [22] on the Tate-Voloch conjecture for prime-to-$p$ torsions implies that the prime-to-$p$ torsions of these points are in $V$ for $n$ large enough (see Proposition 4.2.1). Assume that $\Lambda_n$ is infinite and we deduce a contradiction as follows. A result of Poonen [19] (see Theorem 4.1.1) shows that the size of the set of prime-to-$p$ torsions in $\Lambda_n^{\text{Zar}}$ is not small. Then the liftings give a lower bound on the size of the set of prime-to-$p$ torsions in $V$ (see Proposition 4.2.2). Now consider the $l$-adic perfectoid space associated to $A$. By the same approach, we can repeatedly improve such lower bounds. Finally we get a contradiction as $A$ is of finite dimensional.

Remark 1.2.2. The proofs of Poonen’s result and Scanlon’s result are independent of Theorem 1.2.1.
1.3. Organization of the Paper. The preliminaries on adic spaces and perfectoid spaces are given in Section 2. We introduce the perfectoid universal cover of an abelian scheme in 3.1. The reader may skip these materials and only come back for references. We set up notations for the proof of Theorem 1.2.1 in 3.2, then prove Theorem 1.2.1 in Section 4. We introduce the ordinary perfectoid Siegel space and set up notations for the proof of Theorem 1.1.1 in Section 5. Then we prove Theorem 1.1.1 in Section 6.

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2. Adic spaces and perfectoid spaces

We briefly recall the theory of adic spaces due to Huber [8][9][10][11], and the generalization by Scholze-Weinstein [26]. Then we define tube neighborhoods in adic spaces and distance functions. Finally we recall the theory of perfectoid spaces of Scholze [24] and an approximation lemma due to Scholze.

Let $K$ be a non-archimedean field, i.e. a complete nondiscrete topological field whose topology is induced by a non-archimedean norm $\|\cdot\|_K$ (short for $\|\cdot\|$). Define $K^\circ = \{ x \in K : \|x\| \leq 1 \}$, $K^{\circ\circ} = \{ x \in K : \|x\| < 1 \}$. Let $\varpi \in K^{\circ\circ} - \{0\}$.

2.1. Adic generic fibers of certain formal schemes.

2.1.1. Adic spaces. Let $R$ be a complete Tate $K$-algebra, i.e. a complete topological $K$-algebra with a subring $R_0 \subset R$ such that $\{aR_0 : a \in K^\times \}$ forms a basis of open neighborhoods of 0. A subset of $R$ is called bounded if it is contained in a certain $aR_0$. Let $R^\circ$ be the subring of power bounded elements, i.e. $x \in R^\circ$ if and only if the set of all powers of $x$ form a bounded subset of $R$. Let $R^+ \subset R^\circ$ be an open integrally closed subring. Such a pair $(R, R^+)$ is called an affinoid $K$-algebra. Let $\text{Spa}(R, R^+)$ be the topological space whose underlying set is the set of equivalent classes of continuous valuations $|\cdot(x)|$ on $R$ such that $|f(x)| \leq 1$ for every $f \in R^+$ and topology is generated by the subsets of the form

$$U(\frac{f_1, \ldots, f_n}{g}) := \{ x \in \text{Spa}(R, R^+) : \forall i, |f_i(x)| \leq |g(x)| \}$$

such that $(f_1, \ldots, f_n) = R$. There is a natural presheaf on $\text{Spa}(R, R^+)$ (see [10, p 519]). If this presheaf is a sheaf, then the affinoid $K$-algebra $(R, R^+)$ is called sheafy, and $\text{Spa}(R, R^+)$ is called an affinoid adic space over $K$.

Assumption 2.1.1. If $K^\circ \subset R^+$, for every $x \in \text{Spa}(R, R^+)$, we always choose a representative $|\cdot(x)|$ in the equivalence class of $x$ such that $|f(x)| = \|f\|_K$ for every $f \in K$.

Define a category $(V)$ as in [24, Definition 2.7]. Objects in $(V)$ are triples $(X, \mathcal{O}_X, \{ |\cdot(x)| : x \in X \})$ where $(X, \mathcal{O}_X)$ is a locally ringed topological space whose structure sheaf is a sheaf of complete topological $K$-algebras, and $|\cdot(x)|$ is an equivalence class of continuous valuations on the stalk of $\mathcal{O}_X$ at $x$. Morphisms in $(V)$ are morphisms of locally ringed topological spaces which are continuous $K$-algebra morphisms on the structure sheaves, and compatible with the valuations on the stalks in the obvious sense.

Definition 2.1.2. An adic space $X$ over $K$ is an object in $(V)$ which is locally on $X$ an affinoid adic space over $K$. An adic space over $\text{Spa}(K, K^\circ)$ is an adic space over $K$ with a morphism to $\text{Spa}(K, K^\circ)$. 
A morphism between two adic spaces over Spa$(K, K^o)$ is a morphism in $(V)$ compatible the morphisms to Spa$(K, K^o)$. The set of morphisms Spa$(K, K^o) \to \mathcal{X}$ is denoted by $\mathcal{X}(K, K^o)$.

There is a natural inclusion $\mathcal{X}(K, K^o) \hookrightarrow \mathcal{X}$ by mapping a morphism Spa$(K, K^o) \to \mathcal{X}$ to its image. We always identify $\mathcal{X}(K, K^o)$ as a subset of $\mathcal{X}$ by this inclusion.

2.1.2. Adic generic fibers of certain formal schemes. A Tate $K$-algebra $R$ is called of topologically finite type (tft for short) if $R$ is a quotient of $K(T_1, T_2, \ldots, T_n)$. In particular, it is equipped with the $\mathcal{O}$-adic topology. Similarly define $K^o$-algebras of tft. By [2, 5.2.6. Theorem 1] and [10, Theorem 2.5], if $R$ is of tft, then an affinoid $K$-algebra $(R, R^+)$ is sheafy. Similar to the rigid analytic generic fibers of formal schemes over $K^o$ [1, 7.4], we naturally have a functor from the category of formal schemes over $K^o$ locally of tft to adic spaces over Spa$(K, K^o)$ such that the image of Spf$A$ is Spa$(A[\frac{1}{m}], A^c)$ where $A^c$ is the integral closure of $A$ in $A[\frac{1}{m}]$. The image of a formal scheme under this functor is called its adic generic fiber.

We are interested in certain infinite covers of abelian schemes and Siegel moduli spaces. They are not of tft. We need to generalize the adic generic fiber functor. In [26], the category of adic spaces over Spa$(K, K^o)$ is enlarged in a sheaf-theoretical way. Moreover, the adic generic fiber functor extends to the category of formal schemes over $K^o$ locally admitting a finitely generated ideal of definition.

For our purpose, we only need the following special case. Let $\mathfrak{X}$ be a formal $K^o$-scheme which is covered by affine open formal subschemes $\{\text{Spf}A_i : i \in I\}$, where $I$ is an index set, such that each affinoid $K$-algebra $(A_i[\frac{1}{m}], A_i^c)$ is sheafy. Then the adic generic fiber $\mathcal{X}$ of $\mathfrak{X}$ is an adic space over Spa$(K, K^o)$ in the sense of Definition 2.1.2. Indeed, $\mathcal{X}$ is the obtained by glueing the affinoid adic spaces Spa$(A_i[\frac{1}{m}], A_i^c)$’s in the obvious way. We have an easy consequence.

Lemma 2.1.3. Let $\mathcal{X}$ be the adic generic fiber $\mathfrak{X}$. Then there is a natural bijection $\mathfrak{X}(K^o) \simeq \mathcal{X}(K, K^o)$.

2.2. Tube neighborhoods and distance functions.

2.2.1. Tube neighborhoods. Let $\mathfrak{X} = \text{Spf}B$, where $B$ is a flat $K^o$-algebra of tft. Let $\mathfrak{Z}$ be a closed formal subscheme defined by a closed ideal $I$. Let $\mathcal{X}$ be the adic generic fiber of $\mathfrak{X}$. Then $\mathcal{X} = \text{Spa}(R, R^+)$ where $R = B[\frac{1}{m}]$ and $R^+$ is the integral closure of $B$ in $R$.

Definition 2.2.1. For $\epsilon \in K^\times$, the $\epsilon$-neighborhood of $\mathfrak{Z}$ in $\mathcal{X}$ is defined to be the subset $Z_\epsilon := \{x \in \mathcal{X} : |f(x)| \leq |\epsilon(x)| \text{ for every } f \in I\}$.

Remark 2.2.2. Note that $Z_\epsilon$ may not be open in $\mathcal{X}$. If $I$ is generated by $\{f_1, \ldots, f_n\}$, then $Z_\epsilon = U(I, f, \epsilon) = U(\frac{f_1, \ldots, f_n}{\epsilon})$ is naturally an open adic subspace of $\mathcal{X}$. In fact, for our applications, we only use this case.

Definition 2.2.1 immediately implies the following lemmas.

Lemma 2.2.3. Let $\mathfrak{Z} = \bigcap_{i=1}^m \mathfrak{Z}_i$, where each $\mathfrak{Z}_i$ is a closed formal subschemes of $\mathfrak{X}$. For $\epsilon \in K^\times$, let $Z_{\epsilon, i}$ be the $\epsilon$-neighborhood of $\mathfrak{Z}_i$. Then $Z_\epsilon = \bigcap_{i=1}^m Z_{\epsilon, i}$.

Lemma 2.2.4. Let $\mathfrak{Z} = \mathfrak{Z}_1 \cup \mathfrak{Z}_2$, where $\mathfrak{Z}_1, \mathfrak{Z}_2$ are closed formal subschemes of $\mathfrak{X}$.

1) Then $Z_{\epsilon, 1} \subset Z_\epsilon$.

2) Suppose that there exists $\delta \in K^\times - \{0\}$ which vanishes on $\mathfrak{Z}_2$. Then $Z_\epsilon \subset Z_{\epsilon, 1/\delta}$.

Let $X$ be a $K^o$-scheme locally of finite type, and $\mathfrak{X}$ the $\mathcal{O}$-adic formal completion of $X$. Let $\mathcal{X}$ be the adic generic fiber of $\mathfrak{X}$. We also call $\mathcal{X}$ the adic generic fiber of $X$. Let $Z$ be a closed subscheme of $X_K$. We define tube neighborhoods of $Z$ in $\mathcal{X}$ as follows (see also [24, Proposition 8.7]).

Suppose that $X$ is affine. Let $\mathfrak{Z} \subset \mathfrak{X}$ be the closed formal subscheme associated to the schematic closure of $Z$. 


Definition 2.2.5. For $\epsilon \in K^\times$, the $\epsilon$-neighborhood of $Z$ in $X$ is defined to be the $\epsilon$-neighborhood of $\bar{Z}$ in $\bar{X}$.

Remark 2.2.6. If the schematic closure of $Z$ has empty special fiber, then $Z_\epsilon$ is empty.

To define tube neighborhoods in general, we need to glue affinoid pieces. We consider the following relative situation. Let $Y$ be another affine $K^\circ$-scheme of finite type, and $\Phi : Y \to X$ a $K^\circ$-morphism. Let $W$ be the preimage of $Z$ which is a closed subscheme of $Y_K$, and $W_i$ its $\epsilon$-neighborhood. By the functoriality of formal completion and taking adic generic fibers, we have an induced morphism $\Psi : Y \to X$. From the fact that schematic image is compatible with flat base change (see [3, 2.5, Proposition 2]), we easily deduce the following lemma.

Lemma 2.2.7. If $\Phi : Y \to X$ is flat, then $\Psi^{-1}(Z_\epsilon) = W_\epsilon$. In particular, if $Y \subset X$ is an open $K^\circ$-scheme, $W_\epsilon = Z_\epsilon \cap Y$ under the natural inclusion $Y \hookrightarrow X$.

Now we turn to the general case. Let $X$ be an $K^\circ$-scheme locally of finite type. For an open subscheme $U \subset X$, let $Z_U$ be the restriction of $Z$ to $U$. Let $S = \{ U_i : i \in I \}$ be an affine open cover of $X$, where $I$ is an index set and each $U_i$ is of finite type over $K^\circ$. Let $Z_{U_i,\epsilon}$ be the $\epsilon$-neighborhood of $Z_{U_i}$ in the adic generic fiber $U_i$ of $U_i$. Note that each $U_i$ is naturally an open adic subspace of $X$.

Definition 2.2.8. Define the $\epsilon$-neighborhood of $Z$ in $X$ by $Z_\epsilon := \bigcup_{U \in S} Z_{U,\epsilon}$.

As a corollary of Lemma 2.2.7, this definition is independent of the choice of the cover $U$.

2.2.2. Distance functions. Let $U$ be an affine open subset of $X$ which is flat over $K^\circ$. Let $I$ be an ideal of the coordinate ring of $U$. For $x \in \bar{U}(K)$, define $d_U(x, I) := \sup \{ \| f(x) \| : f \in I \}$. Let $I$ be the ideal sheaf of the schematic closure of $Z$ in $X$.

Assume that $X$ is of finite type over $K^\circ$. Let $U := \{ U_1, ..., U_n \}$ be a finite affine open cover of $X$ such that each $U_i$ is flat over $K^\circ$. For $x \in X(K)$, define $d^U(x, Z)$ to be the maximum of $d_{U_i}(x, I)$ over all $i$'s such that $x \in U_i$.

Let $x^0 \in X(K^\circ)$ and $x$ the generic point of $x^0$. Regard $x$ as a point in $\mathcal{X}(K, K^\circ)$ via Lemma 2.1.3. Let $U$ be an affine open subset of $X$ flat over $K^\circ$ such that $x^0 \in U(K^\circ)$. We have a tautological relation between the distance function and tube neighborhoods.

Lemma 2.2.9. Let $\epsilon \in K^\times$. Then $x \in Z_{U,\epsilon}$ if and only if $d_U(x, I(U)) \leq \| \epsilon \|$.

By Lemma 2.2.7, the number $d_U(x, I(U))$ does not depend on the choice of $U$. Define $d(x, Z) := d_U(x, I(U))$.

Then $d(x, Z) = d^U(x, Z)$ for every finite affine open cover $U$ of $X$. Our distance function coincides with the one in the end of [22, Section 1], which is defined globally.

A finite extension of $K$ has a natural structure of a non-archimedean field (see [2]). Let $\bar{K}$ be an algebraic closure of $K$. The above discussion is naturally generalized to $x \in X(\bar{K})$ and $Z \subset X_{\bar{K}}$.

2.2.3. Tate-Voloch type sets. Let $X$ be of finite type over $K^\circ$.

Definition 2.2.10. Fix an arbitrary finite affine open cover $U$ of $X$ by subschemes flat over $K^\circ$. A set $T \subset X(\bar{K})$ is of Tate-Voloch type if for every closed subscheme $Z$ of $X_{\bar{K}}$, there exists a constant $c > 0$ such that for every $x \in T$, if $d^U(x, Z) \leq c$, then $x \in Z(\bar{K})$.

Remark 2.2.11. Is there always a set of Tate-Voloch type? Let $C \subset X$ be irreducible and flat over $K^\circ$ of relative dimension 1. Choose one point in each residue disk in $C$. Easy to check that this set of points of $X$ is of Tate-Voloch type. Moreover, we can choose points in residue disks in $C$ chose degrees are unbounded. The following questions are more meaningful. Is there always a Tate-Voloch type set
which is Zariski dense in $X$? Can the points in this set have unbounded the degrees over $K$? Indeed, the Tate-Voloch type sets in Theorem 1.1.1 and in the results of Habegger, Scanlon and Xie give positive answers to these two questions.

Let $Y$ be a $K^o$-scheme of finite type, and $\pi : Y \to X$ a finite schematically dominant morphism.

**Lemma 2.2.12.** Let $T \subset X(\bar{K})$ be of Tate-Voloch type and $T' = \pi^{-1}(T) \subset Y(\bar{K})$. Then $T'$ is of Tate-Voloch type.

**Proof.** We may assume that $Y = \text{Spec}B$ and $X = \text{Spec}A$ where $A$ is a subring of $B$. Let $L$ be a finite extension of $K$. Let $Z'$ be a closed subscheme of $Y_L$. We need to show that $d(x', Z')$ has a positive lower bound for $x' \in T' - Z'(\bar{K})$. Define the dimension of $Z'$ to be the maximal dimension of the irreducible components of $Z'$. We allow $Z'$ to be empty, in which case we define its dimension to be $-1$. We do induction on the dimension of $Z'$. Then the dimension $-1$ case is trivial. Now we consider the general case with the hypothesis that the lemma holds for all lower dimensions.

Suppose such a lower bound does not exists, then there exists a sequence of $x'_n \in T' - Z'(\bar{K})$ such that $d(x'_n, Z') \to 0$ as $n \to \infty$. We will find a contradiction. Let $Z$ be the schematic image of $Z'$ by $\pi$, $x_n = \pi(x'_n)$. Let the schematic closure of $Z$ in $X_{L'}$ be defined by an ideal $J \subset A \otimes L^o$ (resp. $I \subset B \otimes L^o$). Then $I \otimes L \supset JB \otimes L$. Since $JB \otimes L^o$ is finitely generated, there exists a positive integer $s$ such that $I \supset w^s JB \otimes L^o$. Thus $d(x_n, Z) \to 0$ as $n \to \infty$. Since $T$ is of Tate-Voloch type, $x_n \in Z(\bar{K})$ for $n$ large enough. We may assume that every $x_n \in Z(\bar{K})$. Since $x'_n \notin Z'$, $\pi^{-1}(Z) = Z' \cap Z_l$ where $Z_l$ is a closed subscheme of $Y_L$ not containing $Z'$ but containing all $x'_n$. Claim: $d(x'_n, Z' \cap Z_l) \to 0$ as $n \to \infty$. This contradicts the induction hypothesis. Thus $d(x', Z')$ has a positive lower bound for $x' \in T' - Z'(\bar{K})$.

Now we prove the claim. Let the schematic closure of $Z_l$ in $Y_{L'}$ be defined by an ideal $I_1 \subset B \otimes L^o$. Then the schematic closure of $Z' \cap Z_l$ is defined by the following ideal of $B \otimes L^o$:

$I_2 := (I_1 \otimes L + I' \otimes L) \bigcap B \otimes L^o = (I_1 + I') \otimes L \bigcap B \otimes L^o$,

which is finitely generated. Thus there exists a positive integer $s$ such that $(I_1 + I') \supset w^s I_2$. Now the claim follows from that $d(x'_n, Z') \to 0$ and $x'_n \in Z_l$.

2.3. **Perfectoid spaces.**

2.3.1. **Two perfectoid fields.** Instead of recalling the definition of perfectoid fields (see [24, Definition 3.1]), we consider two examples and use them through out this paper.

Let $k = \mathbb{F}_p$, $W = W(k)$ the ring of Witt vectors, and $L = W[1/p]$. For each integer $n \geq 0$, let $\mu_{pn}$ be a primitive $p^n$-th root of unity in $\bar{L}$ such that $\mu_{pn}^p = \mu_{pn}$. Let

$L^\text{cycl} := \bigcup_{n=1}^\infty L(\mu_{pn})$.

Let $\varpi = \mu_p - 1$, and $K$ the $\varpi$-adic completion of $L^\text{cycl}$. Then $K$ is a perfectoid field in the sense that

$K^o/\varpi \to K^o/\varpi, \ x \mapsto x^p$

is surjective (see [24, Definition 3.1]). Let

$K^b = k((t^{1/p^n}))$

be the $t$-adic completion of $\bigcup_{n=1}^\infty k((t))(t^{1/p^n})$. Then $K^b$ is a perfectoid field. Let $\varpi^b = t^{1/p}$. Equip $K^b$ with the nonarchimedean norm $\| \cdot \|_K^b$ such that $\|\varpi^b\|_{K^b} = \|\varpi\|_K$. Consider the morphism

(2.1) $K^o/\varpi \to K^o/\varpi^b, \ \mu_{pn} - 1 \mapsto t^{1/p^n}$.
This morphism is well-defined since
\[(\mu_p^n - 1)^{p^m} \simeq \mu_{p^n - m} - 1 \pmod{\varpi}\]
for \(m < n\). Easy to check this morphism is an isomorphism. We call \(K^\flat\) the tilt of \(K\).

2.3.2. **Perfectoid spaces.** The most important property of a perfectoid \(K\)-algebra \(R\) is that
\[R^\varpi/\varpi \to R^\varpi/\varpi, \; x \mapsto x^p\]
is surjective (see \([24, \text{Definition 5.1}]\)). An affinoid \(K\)-algebra \((R, R^+)\) is called perfectoid if \(R\) is perfectoid. By \([24, \text{Theorem 6.3}]\), an affinoid \(K\)-algebra \((R, R^+)\) is sheafy. Define a perfectoid space over \(K\) to be an adic space over \(K\) locally isomorphic to \(\text{Spa}(R, R^+)\), where \((R, R^+)\) is a perfectoid affinoid \(K\)-algebra.

By \([24, \text{Theorem 5.2}]\), there is an equivalence between the categories of perfectoid \(K\)-algebras and perfectoid \(K^\flat\)-algebras. By \([24, \text{Lemma 6.2}]\), and \([24, \text{Proposition 6.17}]\), this category equivalence induces an equivalence between the categories of perfectoid affinoid \(K\)-algebras and perfectoid affinoid \(K^\flat\)-algebras, as well as an equivalence between the categories of perfectoid spaces over \(K\) and perfectoid spaces over \(K^\flat\).

The image of an object or a morphism in the category of perfectoid \(K\)-algebras, perfectoid affinoid \(K\)-algebras, or perfectoid spaces over \(K\) is called its tilt.

2.3.3. **Two important maps \(\sharp\) and \(\rho\).** Let \(R\) be perfectoid \(K\)-algebra and \(R^\flat\) its tilt. By \([24, \text{Proposition 5.17}]\), there is a multiplicative homeomorphism \(R^\flat \simeq \varprojlim_{x \to x^p} R\). Denote the projection to the first component by
\[R^\flat \to R, \; f \mapsto f^\sharp.\]
Let \((R, R^+)\) be perfectoid affinoid \(K\)-algebra and \((R^\flat, R^\flat^+)\) its tilt. For \(x \in \text{Spa}(R, R^+)\), let \(\rho(x) \in \text{Spa}(R^\flat, R^\flat^+)\) be the valuation \(|f(\rho(x))| = |f^\sharp(x)|\) for \(f \in R^\flat\). This defines a map between sets
\[\rho : \text{Spa}(R, R^+) \to \text{Spa}(R^\flat, R^\flat^+).\]
Note that \(\text{Spa}(R^\flat, R^\flat^+)\) is the tilt of \(\text{Spa}(R, R^+)\). The definition of \(\rho\) glued and we have a map
\[\rho_X : |X| \simeq |X^\flat|\]
between the underlying sets of a perfectoid space \(X\) over \(K\) and its tilt \(X^\flat\).

**Lemma 2.3.1.** (1) Let \(\phi : R \to S\) be a morphism between perfectoid \(K\)-algebras, and \(\phi^\flat : R^\flat \to S^\flat\) its tilt. Then for every \(f \in R^\flat\), we have \(\phi^\flat(f^\sharp) = \phi(f^\sharp)\).

(2) Let \(\Phi : X \to Y\) be a morphism between perfectoid spaces over \(K\) and \(\Phi^\flat\) its tilt. Then as maps between topological spaces, we have
\[\rho_Y \circ \Phi = \Phi^\flat \circ \rho_X.\]

**Proof.** (1) follows from the definition of the \(\sharp\)-map and \([24, \text{Theorem 5.2}]\). (2) follows from (1). \(\square\)

By (2), the restriction of \(\rho_X\) to \(\chi(K, K^\circ)\) gives the functorial bijection \(\chi(K, K^\circ) \simeq \chi^\flat(K^\flat, K^\flat^\circ)\), which we also denote by \(\rho_X\). In the next two paragraphs, we compute \(\rho_X\) in two cases.

2.3.4. **Tilting and reduction.** Let \((R, R^+)\) be a perfectoid affinoid \(K\)-algebra and \((R^\flat, R^\flat^+)\) its tilt. Suppose there exists a flat \(W\)-algebra \(S\) such that
\[1 \quad R^+ = \varpi\text{-adic completion of } S \otimes_W K^\circ,\]
\[2 \quad R^\flat^+ = \varpi^\flat\text{-adic completion of } S_k \otimes_k K^\flat^\circ.\]
Let \(\phi : S \to W\) be a \(W\)-algebra morphism, \(\phi_k : S_k \to k\) be its base change. Then \(\phi\) induces a map \(\psi : R^+ \to K^\circ\) which further induces a point \(x\) of \(\text{Spa}(R, R^+)\). Similarly, \(\phi_k\) induces a map \(\psi' : R^\flat^+ \to K^\flat^\circ\) which further induces a point \(x'\) of \(\text{Spa}(R^\flat, R^\flat^+)\). Then \(\psi/\varpi = \psi'/\varpi^\flat\) under the isomorphism \(R^+ / \varpi \simeq R^\flat^+ / \varpi^\flat\). By \([24, \text{Theorem 5.2}]\), \(\phi'\) is the tilt of \(\phi\) and thus we have the following lemma.
Lemma 2.3.2. We have \( \rho_{\text{Spec}(R,R^+)}(x) = x' \).

2.3.5. An example: the perfectoid closed unit disc. Let \( R = K\langle T_1^{1/p}, T_2^{-1/p} \rangle \), the \( \varpi \)-adic completion of \( \bigcup_{c \in \mathbb{Z}_{\geq 0}} K(T_1^{1/p^c}, T_2^{-1/p^c}) \). Then \( R \) is perfectoid. The tilt \( R^t \) of \( R \) is \( K\langle T_1^{1/p^c}, T_2^{-1/p^c} \rangle \). Let \( G_{\text{perf}} := \text{Spa}(R, R^t) \). Then \( G_{\text{perf}} \) is a perfectoid space over \( \text{Spa}(K, K^\circ) \), and \( G_{\text{perf,}\circ} := \text{Spa}(R^t, R^\circ) \) is its tilt.

Let \( c \in \mathbb{Z}_p \) and \( m \in \mathbb{Z}_{>0} \). The \( K^\circ \)-morphism \( R^c \to K^\circ \) defined by
\[
T_1^{1/p^m} \to \mu_{p^{m+n}}^c
\]
gives a point \( x \in G_{\text{perf}}(K, K^\circ) \). The \( K^\circ \)-morphism \( R_{\text{perf}} \to K^\circ \) defined by
\[
T_1^{1/p^m} \to (1 + t_1^{1/p^m})^c
\]
gives a point \( x' \in G_{\text{perf},\circ}(K^b, K^\circ) \). The following lemma follows from (2.1) and [24, Theorem 5.2].

Lemma 2.3.3. We have \( \rho_{G_{\text{perf}}}(x) = x' \).

Similar result holds for \( G_{\text{perf}}^t := \text{Spa}(R, R^t) \) where
\[
R = K\langle T_1^{1/p^m}, T_2^{-1/p^m}, ..., T_l^{1/p^m}, T_l^{-1/p^m} \rangle,
\]
and its tilt \( G_{\text{perf},\circ}^t := \text{Spa}(R^t, R^\circ) \) where
\[
R^t = K^b\langle T_1^{1/p^m}, T_2^{-1/p^m}, ..., T_l^{1/p^m}, T_l^{-1/p^m} \rangle.
\]

2.4. A variant of Scholze’s approximation lemma. The perfectoid fields \( K, K^b \) and related notations are as in 2.3.1. Let \( (R, R^+) \) be a perfectoid affinoid \( (K, K^\circ) \)-algebra with tilt \( (R^t, R^{t+}) \). Let \( X = \text{Spa}(R, R^+) \) with tilt \( X^\circ = \text{Spa}(R^t, R^{t+}) \). For \( f, g \in R \), define \(|f(x) - g(x)|\) to be \(|(f - g)(x)|\). The following approximation lemma plays an important role in Scholze’s work [24].

Lemma 2.4.1 ([24, Corollary 6.7 (1)]). Let \( f \in R^+ \). Then for every \( c \geq 0 \), there exists \( g \in R^{t+} \) such that for every \( x \in X^\circ \), we have
\[
|f(x) - g^2(x)| \leq \|\varpi\|^\frac{c}{p} \max\{|f(x)|, \|\varpi\|^c\} = \|\varpi\|^\frac{c}{p} \max\{|g^2(x)|, \|\varpi\|^c\}.
\]

Here the map \( \sharp \) is as in 2.3.3 (i.e. \(|g(\rho(x))| = |g^\sharp(x)|\)), and we use \( \| \cdot \| \) to denote \( \| \cdot \|_K \).

Recall that \( k = \bar{F}_p \). Assume that there exists a \( k \)-algebra \( S \), such that \( R^{t+} \) is the \( \varpi^\circ \)-adic completion of \( S \otimes K^{\varpi} \). Then we have natural maps
\[
\text{Hom}_k(S, k) \to \text{Hom}_{K^{\varpi}}(S \otimes K^{\varpi}, K^{\varpi}) \simeq X^\circ(K^b, K^{\varpi}).
\]
Thus we regard \( \text{Spec}(S)(k) \) as a subset of \( X^\circ \).

Lemma 2.4.2. Continue to use the notations in Lemma 2.4.1. Assume that \( c \in \mathbb{Z}[\frac{1}{p}] \). There exists a finite sum
\[
g_c = \sum_{i \in \mathbb{Z}[\frac{1}{p}], i < \frac{1}{p} + c} g_{c,i} \cdot (\varpi^\circ)^i
\]
with \( g_{c,i} \in S \) and only finitely many \( g_{c,i} \neq 0 \), such that
\[
g - g_c \in (\varpi^\circ)^{\frac{1}{p} + c} R^{t+}.
\]

Proof. There exists a finite sum \( g' = \sum s_j a_j \in S \otimes K^{\varpi} \), where \( s_j \in S \) and \( a_j \in K^{\varpi} \), such that \( g - g' \in (\varpi^\circ)^{\frac{1}{p} + c} R^{t+} \). Claim: let \( a \in K^{\varpi} \), then there exists a positive integer \( N \) such that
\[
a - \sum_{\substack{h \in (1/p)_{\geq 0}, \ h < \frac{1}{p} + c}} \alpha_h \cdot (\varpi^\circ)^h \in (\varpi^\circ)^{\frac{1}{p} + c} K^{\varpi}
\]
for certain $\alpha_h \in k$. Indeed, the claim follows from that $K^{p\circ}$ is the $\varpi^p$-adic completion of $\bigcup_{n=1}^{\infty} k[[t]][(\varpi^p)^{1/p^n}]$.

Note that $\{h \in (\frac{1}{p^n})_{\geq 0}, h < \frac{1}{p} + c\}$ is finite set. So there exists a finite sum

$$g_c = \sum_{i \in \mathbb{Z}[\frac{1}{l}]_{\geq 0}, i < \frac{1}{p} + c} g_{c,i} \cdot (\varpi^p)^i$$

with $g_{c,i} \in S$ such that $g' - g_c \in (\varpi^p)^{\frac{1}{p} + c} R^\circ$. Then $g - g_c \in (\varpi^p)^{\frac{1}{p} + c} R^\circ$. \hfill $\square$

**Lemma 2.4.3.** Let $g_c$ be as in Lemma 2.4.2 and $x \in (\text{Spec}S)(k)$. Regarding $x \in \mathcal{X}^n(K^\circ, K^{p\circ})$ via the inclusion above. If $|g_c(x)| \leq \|\varpi\|^{\frac{1}{p} + c}$, then $g_{c,i}(x) = 0$ for all $i$.

**Proof.** Since $x \in (\text{Spec}S)(k)$, if $g_{c,i}(x) \neq 0$, then $|g_{c,i}(x)| = 1$. Let $i_0 < \frac{1}{p} + c$ be the minimal $i$ such that $|g_{c,i}(x)| = 1$. Then $|g_c(x)| = \|\varpi^p\|^{i_0} > \|\varpi\|^{\frac{1}{p} + c}$, a contradiction. \hfill $\square$

### 2.4.1 Profinite setting

**Assumption 2.4.4.** There are $k$-algebras $S_0 \subset S_1 \subset \ldots$ such that $S = \bigcup S_n$.

Let $\mathcal{X}_n$ is the adic generic fiber of $\text{Spec}S_n \otimes K^{p\circ}$. Then we have a natural morphism

$$\pi_n : \mathcal{X}^n \to \mathcal{X}_n.$$ 

We also use $\pi_n$ to denote the morphism $(\text{Spec}S)(k) \to (\text{Spec}S_n)(k)$. We have natural maps

$$(\text{Spec}S_n)(k) \hookrightarrow \text{Hom}_{K^{p\circ}}(S_n \otimes K^{p\circ}, K^{p\circ}) \simeq \mathcal{X}_n(K^\circ, K^{p\circ})$$

by which we regard $(\text{Spec}S_n)(k)$ as a subset of $\mathcal{X}_n$. For each $n$, let $\Lambda_n \subset (\text{Spec}S_n)(k)$ be a set of $k$-points, and $\Lambda_n^{\text{zar}}$ the Zariski closure of $\Lambda_n$ in $\text{Spec}S_n$. We have the following maps and inclusions between sets:

$$|\mathcal{X}| \xrightarrow{\Delta} |\mathcal{X}| \xrightarrow{\pi_n} |\mathcal{X}| \supset \Lambda_n^{\text{zar}}(k) \supset \Lambda_n,$$

where $\rho$ is as in 2.3.3.

Let $f \in R^\circ$, and $\Xi := \{x \in \mathcal{X} : |f(x)| = 0\}$. We have the following variant of Lemma 2.4.1.

**Proposition 2.4.5.** Assume that $\Lambda_n \subset \pi_n(\rho(\Xi))$ for each $n$. Then for each $\epsilon \in K^\times$, there exists a positive integer $n$ such that $|f(x)| \leq \|\epsilon\|_K$ for every $x \in (\pi_n \circ \rho)^{-1}(\Lambda_n^{\text{zar}}(k))$.

**Proof.** Choose $c \in \mathbb{Z}_{\geq 0}$ large enough such that $\|\varpi\|^{\frac{1}{p} + c} < \|\epsilon\|_K$, choose $g$ as in Lemma 2.4.1 and choose a finite sum

$$g_c = \sum_{i \in \mathbb{Z}[\frac{1}{l}]_{\geq 0}, i < \frac{1}{p} + c} g_{c,i} \cdot (\varpi^p)^i$$

as in Lemma 2.4.2 where $g_{c,i} \in S$ for all $i$. There exists a positive integer $n(c)$ such that $g_{c,i} \in S_{n(c)}$ for all $i$ by the finiteness of the sum. By the assumption, every element $x \in \Lambda_{n(c)}$ can be written as $\pi_{n(c)} \circ \rho(y)$ where $y \in \Xi$. By (2.2) and (2.3), $|g_c(\rho(y))| \leq \|\varpi\|^{\frac{1}{p} + c}$. Then by Lemma 2.4.3 and that $\rho(y) \in (\text{Spec}S)(k)$, $g_{c,i}(\rho(y)) = 0$. Since $g_{c,i} \in S_{n(c)}$, $g_{c,i}(x) = 0$. Thus $g_{c,i}$ lies in the ideal defining $\Lambda_{n(c)}^{\text{zar}}$. So $g_{c,i}(x) = 0$, and thus $g_c(x) = 0$, for every $x \in \Lambda_{n(c)}^{\text{zar}}(k)$. By (2.2) and (2.3), for every $x \in \Lambda_{n(c)}^{\text{zar}}(k)$, we have

$$|f \left(\rho^{-1}(\pi_{n(c)}^{-1}(x))\right)| \leq \|\varpi\|^{\frac{1}{p} + c} \leq \|\epsilon\|.$$ 

\hfill $\square$
Moreover, \( (3.2) \) consider the following commutative diagram for \( \mathfrak{A} \) and its tilt constructed in [18, Lemme A.16]. Then we study the relation between tilting and reduction.

### 3. Perfectoid universal cover of an abelian scheme

Let \( K \) be the perfectoid field in 2.3.1 and \( K^\flat \) its tilt. Let \( \mathfrak{A} \) be a formal abelian scheme over \( K^\circ \). We first recall the perfectoid universal cover of \( \mathfrak{A} \) and its tilt constructed in [18, Lemme A.16]. Then we study the relation between tilting and reduction.

#### 3.1. Perfectoid universal cover of an abelian scheme

Let \( \mathfrak{A}' \) be a formal abelian scheme over \( \text{Spf} K^\flat \). Assume that there is an isomorphism

\[
(3.1) \quad \mathfrak{A} \otimes K^\circ / \varpi \simeq \mathfrak{A}' \otimes K^{\flat\circ} / \varpi^b
\]

of abelian schemes over \( K^\circ / \varpi \simeq K^{\flat\circ} / \varpi^b \). Let

\[
\tilde{\mathfrak{A}} := \lim_{\leftarrow [p]} \mathfrak{A}, \quad \tilde{\mathfrak{A}}' := \lim_{\leftarrow [p]} \mathfrak{A}'.
\]

Here the transition maps \([p]\) are the morphism multiplication by \( p \) and inverse limits exist in the categories of \( \varpi \)-adic and \( \varpi^b \)-adic formal schemes (see [18, Lemme A.15]). Index the inverse systems by \( \mathbb{Z}_{\geq 0} \). Let \( \text{Spf} \mathfrak{A}'_0 \subset \mathfrak{A} \) be an affine open formal subscheme. Let \( \mathfrak{A}'_i \) be the coordinate ring of \( \mathfrak{A}'_0 \), in other words, \( \text{Spf} \mathfrak{A}'_i = ([p])^{-1} \text{Spf} \mathfrak{A}'_0 \). Let \( \mathfrak{A}' = \mathfrak{A}'_0 \cup \bigcup_{i=0}^\infty \mathfrak{A}'_i \) be an affine open formal subscheme such that the restriction of \((3.1)\) to \( \text{Spf} \mathfrak{A}'_0 \otimes K^\circ / \varpi \) is an isomorphism to \( \text{Spf} \mathfrak{A}'_0 \otimes K^{\flat\circ} / \varpi^b \). We similarly define \( \mathfrak{A}'_0, \mathfrak{A}'_+ \) and \( \mathfrak{A}'_+ \).

**Lemma 3.1.1** ([18, Lemme A.16]). The affinoid \( K^\flat \)-algebra \( (\mathfrak{A}', \mathfrak{A}_+) \) is perfectoid. So is \( (\mathfrak{A}, \mathfrak{A}_+) \). Moreover, \( (\mathfrak{A}', \mathfrak{A}_+) \) is the tilt of \( (\mathfrak{A}, \mathfrak{A}_+) \).

Thus the adic generic fiber \( \mathfrak{A}_{\text{perf}} \) (resp. \( \mathfrak{A}'_{\text{perf}} \)) of \( \mathfrak{A} \) (resp. \( \tilde{\mathfrak{A}} \)) is a perfectoid space. Moreover, \( \mathfrak{A}'_{\text{perf}} \) is the tilt of \( \mathfrak{A}_{\text{perf}} \). Thus we use \( \mathfrak{A}_{\text{perf}} \) to denote \( \mathfrak{A}'_{\text{perf}} \). We call \( \mathfrak{A}_{\text{perf}} \) (resp. \( \mathfrak{A}'_{\text{perf}} \)) the perfectoid universal cover of \( \mathfrak{A} \) (resp. \( \mathfrak{A}' \)). By Lemma 2.1.3, there are natural bijections

\[
\tilde{\mathfrak{A}}(K^\circ) \simeq \mathfrak{A}_{\text{perf}}(K, K^\circ), \quad \tilde{\mathfrak{A}}'(K^{\flat\circ}) \simeq \mathfrak{A}'_{\text{perf}}(K^\flat, K^{\flat\circ}).
\]

Let \( \mathfrak{A} \) (resp. \( \mathfrak{A}' \)) be the adic generic fiber of \( \mathfrak{A} \) (resp. \( \mathfrak{A}' \)). By Lemma 2.1.3, we have natural bijections

\[
\mathfrak{A}(K^\circ) \simeq \mathfrak{A}(K, K^\circ), \quad \mathfrak{A}'(K^{\flat\circ}) \simeq \mathfrak{A}'(K^\flat, K^{\flat\circ}).
\]

**Definition 3.1.2.** The group structures on \( \mathfrak{A}(K, K^\circ), \mathfrak{A}_{\text{perf}}(K, K^\circ), \mathfrak{A}'(K^\flat, K^{\flat\circ}), \text{and} \mathfrak{A}'_{\text{perf}}(K^\flat, K^{\flat\circ}) \) are defined to be the ones induced from the natural bijections above.

By the functoriality of taking adic generic fibers, we have morphisms

\[
\pi_n : \mathfrak{A}_{\text{perf}} \to \mathfrak{A}, \quad \pi'_n : \mathfrak{A}'_{\text{perf}} \to \mathfrak{A}'
\]

for \( n \in \mathbb{Z}_{\geq 0} \), and morphisms

\[
[p] : \mathfrak{A} \to \mathfrak{A}, \quad [p] : \mathfrak{A}' \to \mathfrak{A}'.
\]

Consider the following commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathfrak{A}}(K^\circ) & \xrightarrow{\simeq} & \lim_{\leftarrow [p]} \mathfrak{A}(K^\circ) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathfrak{A}_{\text{perf}}(K, K^\circ) & \xrightarrow{\lim_{\leftarrow [p]}} & \lim_{\leftarrow [p]} \mathfrak{A}(K, K^\circ)
\end{array}
\]

where the bottom map is given by \( \pi_n \)'s. We immediately have the following lemma.
Lemma 3.1.3. The bottom map in (3.2) is a group isomorphism.

Remark 3.1.4. Indeed, $A^\text{perf}$ serves as certain “limit” of the inverse system $\varprojlim A$ in the sense of [26, Definition 2.4.1] by [26, Proposition 2.4.2]. Then Lemma 3.1.3 also follows from [26, Proposition 2.4.5].

Now we study torsion points in the inverse limit. We set up some group theoretical convention once for all. Let $G$ be an abelian group. We denote by $G[n]$ the subgroup of elements of orders dividing $n$ and by $G_{\text{tor}}$ the subgroup of torsion elements. For a prime $p$, we use $G[p^\infty]$ to denote the subgroup of $p$-primary torsion points, and $G[p^\prime - \text{tor}]$ to denote the subgroup of prime-to-$p$ torsion points. If $H$ is a subset of $G$, $H_{\text{tor}}$ and $H[p^\prime - \text{tor}]$ to denote the subset $H \cap G_{\text{tor}}$ and $H \cap G[p^\prime - \text{tor}]$ when both the definitions of $H$ and $G$ are clear from the context. The following lemma is elementary.

Lemma 3.1.5. Let $G$ be an abelian group, then

$$\left(\varprojlim G\right)[p^\prime - \text{tor}] \simeq \varprojlim G[p^\prime - \text{tor}].$$

Lemma 3.1.6. There are group isomorphisms

$$A^\text{perf}(K,K^\circ)_{p^\prime - \text{tor}} \simeq \varprojlim A(K,K^\circ)_{p^\prime - \text{tor}} \simeq A(K,K^\circ)_{p^\prime - \text{tor}}$$

where the second isomorphism is the restriction of $\pi_n$. Similar result holds for $A'$ and $A^\text{perf}$.

Proof. The first isomorphism is from Lemma 3.1.3 and 3.1.5. Since $A(K,K^\circ)[n] \simeq A(K^\circ)[n]$ is a finite group, $[p]$ is an isomorphism on $A(K,K^\circ)[n]$ for every natural number $n$ coprime to $p$. The second isomorphism follows.

Proposition 3.1.7. The functorial bijection

$$\rho = \rho_{A^\text{perf}} : A^\text{perf}(K,K^\circ) \simeq A^\text{perf}(K^b,K^{b\circ})$$

(see 2.3.3) is a group isomorphism.

Proof. We only show the compatibility of $\rho$ with the multiplication maps, i.e. we show that the following diagram is commutative:

$$\begin{array}{ccc}
A^\text{perf}(K,K^\circ) \times A^\text{perf}(K,K^\circ) & \xrightarrow{\rho \times \rho} & A^\text{perf}(K^b,K^{b\circ}) \times A^\text{perf}(K^b,K^{b\circ}) \\
\downarrow & & \downarrow \\
A^\text{perf}(K,K^\circ) & \xrightarrow{\rho} & A^\text{perf}(K^b,K^{b\circ}).
\end{array}$$

Here the vertical maps are the multiplication maps on corresponding groups.

Consider the formal abelian schemes $\mathcal{B} = \mathfrak{A} \times \mathfrak{A}$ and $\mathcal{B}' = \mathfrak{A}' \times \mathfrak{A}'$. We do the same construction to get their perfectoid universal covers $B^\text{perf}$ and $B^\text{perf}$. The multiplication morphism $\mathcal{B} \to \mathfrak{A}$ induces $m : B^\text{perf} \to A^\text{perf}$. The multiplication morphism $\mathcal{B}' \to \mathfrak{A}'$ induces $m' : B^\text{perf} \to A^\text{perf}$. By (3.1) and [24, Theorem 5.2], $m' = m^b$. By functoriality, we have a commutative diagram

$$\begin{array}{ccc}
B^\text{perf}(K,K^\circ) & \xrightarrow{\rho_{B^\text{perf}}} & B^\text{perf}(K^b,K^{b\circ}) \\
m & & m^b \\
A^\text{perf}(K,K^\circ) & \xrightarrow{\rho_{A^\text{perf}}} & A^\text{perf}(K^b,K^{b\circ}).
\end{array}$$
We only need to show that this diagram can be identified with the diagram we want. For example we show that the top horizontal maps in the two diagrams coincide, i.e. a commutative diagram

\[
\begin{array}{ccc}
B_{\text{perf}}(K, K^\circ) & \xrightarrow{\rho_{\text{perf}}} & B_{\text{perf}}(K^\flat, K^{\circ\circ}) \\
\cong & & \cong \\
A_{\text{perf}}(K, K^\circ) \times A_{\text{perf}}(K, K^\circ) & \xrightarrow{\rho \times \rho} & A_{\text{perf}}(K, K^\circ) \times A_{\text{perf}}(K, K^\circ).
\end{array}
\]

The projection $\mathcal{B} = \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ to the $i$-th component, $i = 1, 2$, induces $p_i : B_{\text{perf}} \to A_{\text{perf}}$. Easy to check that

\[p_1 \times p_2 : B_{\text{perf}}(K, K^\circ) \to A_{\text{perf}}(K, K^\circ) \times A_{\text{perf}}(K, K^\circ)\]

is a group isomorphism by passing to formal schemes. Similarly we have an isomorphism

\[p'_1 \times p'_2 : B_{\text{perf}}(K^\flat, K^{\circ\circ}) \to A_{\text{perf}}(K, K^\circ) \times A_{\text{perf}}(K, K^\circ).\]

The commutativity is implied by that $p'_i = p_i^{\flat}$, which is from (3.1) and [24, Theorem 5.2].

3.2. Tilting and reduction. Let $k = \overline{\mathbb{F}_p}$ and let $W = W(k)$ be the ring of Witt vectors. Let $A$ be an abelian scheme over $W$, $A_{K^s}$ be its base change to $K^s$, $A$ be the adic generic fiber of $A_{K^s}$. Let $A_k$ be the special fiber of $A$, and $A'$ be the base change $A_k \otimes K^{\circ\circ}$ with adic generic fiber $A'$. Since

\[A_{K^s} \otimes (K^\circ / \varpi) \simeq A \otimes_W k \otimes_k (K^\circ / \varpi) \simeq A' \otimes_{K^{\circ\circ}} (K^{\circ\circ} / \varpi),\]

we can apply the construction in Lemma 3.1.1 to the formal completions of $A_k \otimes_k K^\circ$ and $A'$. Then we have the perfectoid universal cover $A_{\text{perf}}$ of the $\varpi$-adic formal completion of $A_{K^s}$, the perfectoid universal cover $A_{\text{perf}}^{\flat}$ of the $\varpi^0$-adic formal completion of $A_{K^{\circ\circ}}$, and the morphisms $\pi_n : A_{\text{perf}} \to A$, $\pi_n' : A_{\text{perf}}^{\flat} \to A'$ for each $n \in \mathbb{Z}_{\geq 0}$. The following well-known results can be deduced from [28].

**Lemma 3.2.1.** (1) The inclusion $A(W) \hookrightarrow A(K^\circ)$ gives an isomorphism $A(W)_{p' - \text{tor}} \simeq A(k)_{p' - \text{tor}}$.

(2) The reduction map gives an isomorphism

\[\text{red} : A(W)_{p' - \text{tor}} \simeq A(k)_{p' - \text{tor}}\]

(3) The natural inclusion $A(k) \hookrightarrow A_{K^{\circ\circ}}(K^{\circ\circ})$ gives an isomorphism $A(k)_{p' - \text{tor}} \simeq A_{K^{\circ\circ}}(K^{\circ\circ})_{p' - \text{tor}}$.

Now we relate reduction and tilting.

**Lemma 3.2.2.** Let the unindexed maps in the following diagram be the naturals ones:

\[
\begin{array}{ccccccccc}
A(K, K^\circ)_{p' - \text{tor}} & \xrightarrow{\pi_n} & A_{\text{perf}}(K, K^\circ)_{p' - \text{tor}} & \xrightarrow{\rho} & A_{\text{perf}}^{\flat}(K^\flat, K^{\circ\circ})_{p' - \text{tor}} & \xrightarrow{\pi_n'} & A'(K^\flat, K^{\circ\circ})_{p' - \text{tor}} \\
A_{K^s}(K^\circ)_{p' - \text{tor}} & \xrightarrow{\text{red}} & A(W)_{p' - \text{tor}} & \xrightarrow{\text{red}} & A(k)_{p' - \text{tor}} & \xrightarrow{\text{red}} & A_{K^{\circ\circ}}(K^{\circ\circ})_{p' - \text{tor}}
\end{array}
\]

Then each map is a group isomorphism, and the diagram is commutative (up to inverting the arrows).

**Proof.** We may assume $n = 0$. Definition 3.1.2, Lemma 3.1.6, Proposition 3.1.7 and Lemma 3.2.1 give the isomorphisms. We only need to check the commutativity. And we only need to check the two maps from $A(W)_{p' - \text{tor}}$ to $A_{\text{perf}}^{\flat}(K^\flat, K^{\circ\circ})_{p' - \text{tor}}$ are the same. This follows from Lemma 2.3.2. \(\square\)
Similarly, we have the following commutative diagram:

\[
\begin{array}{c}
\mathcal{A}^{\text{perf}}(K, K^0) \xrightarrow{\rho} \mathcal{A}^{\text{perf}}(K^p, K^{30}) \\
\xrightarrow{\iota} \xrightarrow{\sim} \\
\lim_{\frac{\kappa}{\kappa}} A(W) \xrightarrow{\text{red}} \lim_{\frac{\kappa}{\kappa}} A(k) \xrightarrow{\pi'_n} \lim_{\frac{\kappa}{\kappa}} A'(K^p, K^{30}) \\
\bigcap_{i=0} A(W) \xrightarrow{\text{red}} A(k) \xrightarrow{\pi'_n} A'(K^p, K^{30}) \\
A(W) \xrightarrow{\text{red}} A(k) \xrightarrow{\pi'_n} A'(K^0, K^{30}).
\end{array}
\]

Here \( \iota \) is induced from the inclusion \( A(W) \hookrightarrow \mathcal{A}^{\text{perf}}(K, K^0) \) and the isomorphism \( \mathcal{A}^{\text{perf}}(K, K^0) \cong \lim_{\frac{\kappa}{\kappa}} A(K, K^0) \) (see Lemma 3.1.3). Here and from now on we regard \( \lim_{\frac{\kappa}{\kappa}} A(W) \) as a subset of \( \mathcal{A}^{\text{perf}}(K, K^0) \) via \( \iota \), \( A(k) \) as a subset \( A'(K^p, K^{30}) \), and \( \lim_{\frac{\kappa}{\kappa}} A(k) \) as a subset of \( \mathcal{A}^{\text{perf}}(K^0, K^{30}) \).

4. Proof of Theorem 1.2.1

In this section, we at first prove a lower bound on prime-to-\( p \) torsion points in a subvariety. Then we prove Theorem 1.2.1. Let \( k = \mathbb{F}_p \), \( W = W(k) \) the ring of Witt vectors, and \( L = W[\frac{1}{p}] \).

4.1. Results of Poonen, Raynaud and Scanlon.

**Theorem 4.1.1** (Poonen [19]). Let \( B \) be an abelian variety defined over \( k \), and \( V \) an irreducible closed subvariety of \( B \). Let \( S \) be a finite set of primes. Suppose that \( V \) generates \( B \), then the composition of

\[
V(k) \hookrightarrow B(k) \xrightarrow{\bigoplus_{i \in S} \text{pr}_i} \bigoplus_{i \in S} B[l^\infty]
\]

is surjective, where \( \text{pr}_i \) is the projection to the \( l \)-primary component.

Let \( A \) be an abelian scheme over \( W \). Let \( T = \bigcap_{n=0}^\infty p^n(A(L)[p^\infty]) \), the maximal divisible subgroup of \( A(L)[p^\infty] \). Though not needed, as an illustration, we note that by [20, Exemples 5.2.3], \( T = 0 \) if the \( p \)-rank of \( A_k \) is 0 or if \( A \) is a “general ordinary abelian variety”, and \( T = A(L)[p^\infty] \cong L/[p^\dim A_k] \) if \( A \) is the canonical lifting in Serre-Tate theory, see 5.2.

**Lemma 4.1.2** (Raynaud [20, Lemma 5.2.1]).

1. Let \( T_o \) be the subgroup of \( A(\bar{L})[p^\infty] \) coming from the connected component of the \( p \)-divisible group of \( A \), then \( T_o \cap T = 0 \).
2. As a subgroup of \( A(\bar{L})[p^\infty] \), \( T \) is a \( \text{Gal}(\bar{L}/L) \)-direct summand.

Note that

\[
\bigcap_{n=0}^\infty p^n(A(W)_{\text{tor}}) = A(W)_{p^\prime - \text{tor}} \bigoplus \bigcap_{n=0}^\infty p^n(A(W)[p^\infty]).
\]
Corollary 4.1.3. The following reduction map is injective
\[
\text{red} : \bigcap_{n=0}^{\infty} p^n(A(W)_{\text{tor}}) \to A(k).
\]

Let \( Z \subset A_L \) be a closed subvariety.

Lemma 4.1.4 (Raynaud [21, 8.2]). Let \( T' \) be a \( \text{Gal}(L/L) \)-direct summand such that \( \text{Gal}(L/L) \)-modules
\[
A(L)_{\text{tor}} = A(L)_{p' \text{-tor}} \bigoplus T \bigoplus T'.
\]
If \( Z \) does not contain any translate of a nontrivial abelian subvariety of \( A_L \), there exists a positive integer \( N \) such that the order of the \( T' \)-component of every element in \( Z(L)_{\text{tor}} \) divides \( p^N \).

Remark 4.1.5. Lemma 4.1.4 is used by Raynaud [21] to reduce the Manin-Mumford conjecture to a theorem (see [21, Theorem 3.5.1]) obtained by studying \( p \)-adic rigid analytic properties of universal vector extension of an abelian variety.

Let \( K \) and \( K' \) be the perfectoid fields in 2.3.1. Let \( A \) be the adic generic fiber of \( A_{K'} \). Let \( Z_{\text{Zar}} \) be the Zariski closure of \( Z \) in \( A \), and \( Z \) the adic generic fiber of \( Z_{\text{Zar}} \). For \( \epsilon \in K^\times \), let \( Z_\epsilon \) be the \( \epsilon \)-neighborhood of \( Z_K \) in \( A \) as in Definition 2.2.8. By Lemma 2.2.9, a result of Scanlon [22] on the Tate-Voloch conjecture implies the following lemma.

Lemma 4.1.6 (Scanlon [22]). There exists \( \epsilon \in K^\times \), such that \( A(K, K')_{p' \text{-tor}} \cap Z_\epsilon \subset Z \).

Remark 4.1.7. The proofs of Poonen’s result and Scanlon’s result are independent of Theorem 1.2.1.

4.2. A lower bound. Define

\[
\Lambda := Z^{\text{Zar}}(W) \cap \bigcap_{n=0}^{\infty} p^n(A(W)_{\text{tor}}), \quad \Lambda_{\infty} := \iota \left( \pi_{0}^{-1}(\Lambda) \right),
\]
where \( \pi_0 \) and \( \iota \) are as in the left column of diagram (3.3). Then \( \rho(\Lambda_{\infty}) \) is contained in (the image of) \( \lim \) \( A(k) \) by diagram (3.3). Now let \( \Lambda_n := \pi_n'(\rho(\Lambda_{\infty})) \). Then \( \Lambda_n \) is contained in (the image of) \( A(k) \). Let \( \Lambda_n^{\text{Zar}} \) be the Zariski closure of \( \Lambda_n \) in \( A_k \).

Proposition 4.2.1. There exists a positive integer \( n \) such that
\[
\pi_0 \left( \rho^{-1} \left( \pi_{n}^{-1} \left( \Lambda_n^{\text{Zar}}(k)_{p' \text{-tor}} \right) \right) \right) \bigcap A(K, K')_{p' \text{-tor}} \subset Z.
\]

Proof. Let \( U \) be a finite affine open cover of \( A \) by affine open subschemes flat over \( W \). Let \( U \in \mathcal{U} \). The restriction of \( A_{\text{perf}} \) over the adic generic fiber of \( U_{K'} \) is a perfectoid space \( \mathcal{X} = \text{Spa}(R, R^+) \) whose tilt satisfies Assumption 2.4.4 (see Lemma 3.1.1 and the discussion above it). Let \( \mathcal{I} \) be the ideal sheaf of \( Z_{\text{Zar}} \). Let \( f \in \mathcal{I}(U) \). Regard \( f \) as in \( R \). By definition of \( \Lambda_n \), we can apply Proposition 2.4.5 to \( f \) and \( \Lambda_n \). Varying \( U \) in \( \mathcal{U} \) and varying \( f \) in a finite set of generators of \( \mathcal{I}(U) \), Proposition 2.4.5 implies that for every \( \epsilon \in K^\times \), there exists a positive integer \( n \) such that
\[
\pi_0 \left( \rho^{-1} \left( \pi_{n}^{-1} (\Lambda_n^{\text{Zar}}(k)) \right) \right) \subset Z_\epsilon.
\]
Then Proposition 4.2.1 follows from Lemma 4.1.6. \( \square \)

Our lower bound on the size of the set of prime-to-\( p \) torsions in \( Z \) is as follows.

Proposition 4.2.2. Let \( p > 2 \). Assume that \( Z \) contains the unit \( 0 \in A_L \).

1. Assume \( \Lambda \) is infinite. For every prime number \( l \neq p \), the image of the composition of

\[
Z^{\text{Zar}}(W)_{p' \text{-tor}} \hookrightarrow A(W)_{p' \text{-tor}} \xrightarrow{\text{pr}_l} A(W)[l^\infty]
\]

(4.3)
contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least 2. Here the map $\text{pr}_l$ is the projection to the $l$-primary component.

(2) Assume that the image of the composition of
\[ \Lambda \hookrightarrow A(W)_{\text{tor}} \xrightarrow{\text{pr}_l} A(W)[p^\infty] \]
contains a translate of a free $L/\mathbb{Z}_p$-submodule of rank $r$. For every prime number $l \neq p$, the image of the composition of (4.3) contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank $2r$.

Proof. Fix a large $n$ such that
\[ \pi_0 \left( \rho^{-1} \left( \pi_n^{-1} (\Lambda_{n}^{\text{Zar}}(k)_{p' - \text{tor}}) \right) \right) \cap A(K, K^\circ)_{p' - \text{tor}} \subset Z(K, K^\circ)_{p' - \text{tor}} \]
as in Proposition 4.2.1. Let $X$ be the image of the left hand side of (4.4) via the composition of
\[ A(K, K^\circ)_{p' - \text{tor}} \cong A(W)_{p' - \text{tor}} \xrightarrow{\text{pr}_l} A(W)[l^\infty] \]
Then $X$ is contained in the image of the composition of (4.3).

To prove (1), we only need to prove the following claim: $X$ contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least 2 for every $l$.

By diagram (3.3), we have $\Lambda_0 = \text{red}(\Lambda)$. Since $p > 2$, by Corollary 4.1.3, $\Lambda_0$ is infinite. Since $\Lambda_0 = [p]^n(\Lambda_n)$, $\Lambda_n$ is infinite. There exists $a \in A(k)$ such that an irreducible component of $\Lambda_n^{\text{Zar}} + a$ (is contained and) generates a nontrivial abelian subvariety $A'$ of $A_k$. Since $Z$ contains the unit $0 \in A_L$, $\Lambda_n^{\text{Zar}}$ contains the unit $0 \in A_k$ and $A'$ contains $a$. Let $a_p$ be the $p$-primary part of $a$ and $a_{p'} = a - a_p$. By Theorem 4.1.1 (for $\Lambda_n^{\text{Zar}} + a \subset A'$ and $S = \{p, l\}$), the image of
\[ \Lambda_n^{\text{Zar}}(k) + a \xrightarrow{\text{pr}_l \bigoplus \text{pr}_p} A(k)[l^\infty] \bigoplus A(k)[p^\infty] \]
contains $M \bigoplus \{a_p\}$, where $M$ is a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(k)[l^\infty]$ of rank at least 2. Thus
\[ \left( \text{pr}_l \bigoplus \text{pr}_p \right) \left( \Lambda_n^{\text{Zar}}(k) + a_{p'} \right) \supset M \bigoplus \{0\}. \]
We claim:
\[ \text{pr}_l \left( \left( \Lambda_n^{\text{Zar}}(k) + a_{p'} \right)_{p' - \text{tor}} \right) \supset M. \]
Indeed, write $b \in \Lambda_n^{\text{Zar}}(k) + a_{p'}$ as the sum $b_p + b_{p'}$ of $p$-primary part and prime-to-$p$ part. Then
\[ \left( \text{pr}_l \bigoplus \text{pr}_p \right) b = \text{pr}_l(b_p) + b_p. \]
If this is $x + 0 \in M \bigoplus \{0\}$, then $b_p = 0$, and $b = b_{p'}$. Thus $\text{pr}_l(b) = x \in M$.
The claim is proved. By the claim,
\[ M - \text{pr}_l(a_{p'}) \supset Y := \text{pr}_l \left( \Lambda_n^{\text{Zar}}(k)_{p' - \text{tor}} \right). \]
By Lemma 3.2.2, $X$ contains the preimage of $[p]^n(Y)$ under the isomorphism $\text{red} : A(W)_{p' - \text{tor}} \cong A(K)_{p' - \text{tor}}$. Thus we proved the claim above.

To prove (2), we only need to prove the following claim: $X$ contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least $2r$ for every $l$.

By diagram (3.3), we have $\Lambda_0 = \text{red}(\Lambda)$. Since $p > 2$, by Corollary 4.1.3 and the assumption on $A$, $\text{pr}_p(\Lambda_0)$ contains a translate of a free $L/\mathbb{Z}_p$-submodule of rank $r$. Let $V_1, \ldots, V_m$ be the irreducible components of $\Lambda_0^{\text{Zar}}$. Let $A_i$ be the minimal abelian subvariety of $A_k$ such that a certain translate of $A_i$ contains $V_i$. Since the $p$-rank of $A_i$ is at most its dimension, at least one $A_i$ is of dimension at least $r$. Since $\Lambda_0 = [p]^n(\Lambda_n)$, there exists $a \in A(k)$ such that an irreducible component of $\Lambda_n^{\text{Zar}} + a$ generates an abelian subvariety of $A_k$ of dimension at least $r$. Then we prove (2) by copying the proof of (1) above, starting from the sentence containing (4.5). The only modification needed is that the rank of $M$ should be at least $2r$. \qed
4.3. The proof of Theorem 1.2.1. Now we prove Theorem 1.2.1. By the argument in [17], we only need to prove the following weaker theorem. We save the symbol $A$ for the proof.

**Theorem 4.3.1.** Let $F$ be number field. Let $B$ be an abelian variety over $F$ and $V$ a closed subvariety of $B$. If $V$ does not contain any translate of an abelian subvariety of $B$ of positive dimension, then $V$ contains only finitely many torsion points of $B$.

*Proof.* We only need to prove the theorem up to replacing $V$ by a multiple.

Let $v$ be a place of $F$ unramified over a prime number $p > 2$ such that $B$ has good reduction. Let $A$ be the base change to $W$ of the integral smooth model of $B$ over $\mathcal{O}_F$. Let $Z = V_L \subset A$. By (4.1) and Lemma 4.1.4, up to replacing $V$ by $[p^N]V$ for $N$ large enough, we may assume that $Z^{\text{Zar}}(W)_{\text{tor}} \subset \bigcap_{n=0}^{\infty} p^n(A(W)_{\text{tor}})$. Thus $\Lambda = Z^{\text{Zar}}(W)_{\text{tor}}$, where $\Lambda$ is defined as in (4.2). Suppose that $V$ contains infinitely many torsion points. Then $\Lambda$ is infinite. Up to replacing $V$ by $[p^N]V$, we may assume that $Z$ contains the unit $0 \in A_L$. Now we want to find a contradiction. By Proposition 4.2.2 (1), for every prime number $l \neq p$, the composition

$$Z(L)_{l\text{,tor}} \hookrightarrow A(L)_{l\text{,tor}} \xrightarrow{\text{pr}_1} A(L)[l^\infty]$$

contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(L)[l^\infty]$ of rank 2.

Let $u$ be another place of $F$, unramified over an odd prime number $l \neq p$, such that $B$ has good reduction at $u$. Let $B_u$ be the reduction. Let $M$ be the completion of the maximal unramified extension of $F_u$ and $\bar{M}$ its algebraic closure. Then the composition

$$V(\bar{M})_{\text{tor}} \hookrightarrow B(\bar{M})_{\text{tor}} \xrightarrow{\text{pr}_1} B(\bar{M})[l^\infty]$$

contains a translate $G$ of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $B(\bar{M})[l^\infty]$ of rank 2. Let $T = \bigcap_{n=0}^{\infty} l^n(B(M)[l^\infty])$. By Lemma 4.1.4 (applied to $l$, $M$ instead of $p$, $L$), up to replacing $V$ by $[l^N]V$ for $N$ large enough, $G$ is contained in $T$. By (4.1) (applied to $l$, $M^\circ$ instead of $p$, $W$), the image of the composition of

$$V(M) \bigcap_{n=0}^{\infty} l^n(B(M)_{\text{tor}}) \hookrightarrow B(M) \xrightarrow{\text{pr}_1} B(M)[l^\infty]$$

contains $G$. By Proposition 4.2.2 (2) (applied to $l$, $M^\circ$ instead of $p$, $W$), for every prime number $q \neq l$, the composition

$$V(M)_{l\text{,tor}} \hookrightarrow B(M)_{l\text{,tor}} \xrightarrow{\text{pr}_1} B(M)[q^\infty]$$

contains a translate of a free $\mathbb{Q}_q/\mathbb{Z}_q$-submodule of rank 4. Repeating this process (use more places or only work at $v$ and $u$), we get a contradiction as $A$ is of finite dimension. \qed

5. Ordinary perfectoid Siegel space and Serre-Tate theory

Let $A_f$ be the ring of finite adeles of $\mathbb{Q}$, $U^p \subset \text{GSp}_{2g}(A_f^p)$ an open compact subgroup contained in the congruence subgroup of level-$N$ for some $N \geq 3$ prime to $p$. Let $X = X_{g,U^p}$ over $\mathbb{Z}_p$ be the Siegel moduli space of principally polarized $g$-dimensional abelian varieties over $\mathbb{Z}_p$-schemes with level-$U^p$-structure. Let $\mathcal{X}_o$ be special fiber of $X$. We will use the perfectoid fields defined in 2.3.1. We briefly recall some notations. Let $k = \mathbb{F}_p$, $W = W(k)$ the ring of Witt vectors, $L$ the fraction field of $W$, and $L^{\text{cycl}}$ the field extension of $L$ by adjoining all $p$-power-th roots of unity. Let $K$ be the $p$-adic completion of $L^{\text{cycl}}$ which is a perfectoid field. Then $K^p = k((1/p^n))$ is the tilt of $K$. Fix a primitive $p^n$-th root of unity $\mu_{p^n}$ for every positive integer $n$ such that $\mu_{p^n}^{p^n+1} = \mu_{p^n}$. 
5.1. **Ordinary perfectoid Siegel space.** Let $X_o(0) \subset X_o$ be the ordinary locus. Let $\bar{X}(0)$ over $\mathbb{Z}_p$ be the open formal subscheme of the formal completion of $X$ along $X_o$ defined by the condition that every local lifting of the Hasse invariant is invertible (see [25, Definition 3.2.12, Lemma 3.2.13]). Then $\bar{X}(0)/p = X_o(0)$ (see [25, Lemma 3.2.5]). Let $X_o(0)_{K^{1+}}$ be the $\varpi^b$-adic formal completion of $X_o(0)_{K^{1+}}$. Let $X(0)$ and $X'(0)$ be the adic generic fibers of $X(0)_{K^{1+}}$ and $X_o(0)_{K^{1+}}$, respectively.

Let $\text{Fr}: X_o(0) \to X_o(0)$ be the (relative) Frobenius morphism (note that $X_o(0)$ is defined over $\mathbb{F}_p$). Let $\text{Fr}_{\text{can}} : \bar{X}(0) \to \bar{X}(0)$ be given by the functor sending an abelian scheme $A$ to its quotient by the connected subgroup scheme of $A[p]$. Then $\text{Fr}_{\text{can}}/p = \text{Fr}$. We also use $\text{Fr}_{\text{can}}$ and $\text{Fr}$ to denote their base changes to $\mathbb{K}^0$ and $K^{1+}$ respectively. Let

$$\hat{\bar{X}}(0) := \lim_{\leftarrow} \bar{X}(0)_{K^{1+}}, \quad \hat{X}'(0) := \lim_{\leftarrow} \bar{X}(0)_{K^{1+}},$$

where the inverse limits are taken in the categories of $\varpi$-adic and $\varpi^b$-adic formal schemes respectively. Here $\varpi = \mu_p - 1$ and $\varpi^b = e^{1/p}$. By [25, Corollary 3.2.19], the corresponding adic generic fibers $X(0)_{K^{1+}}$ and $X'(0)_{K^{1+}}$ of $\hat{\bar{X}}(0)$ and $\hat{X}'(0)$ are perfectoid spaces. Moreover, $X'(0)_{K^{1+}} = X(0)_{K^{1+}}$, the tilt of $X(0)_{K^{1+}}$. Then we have the natural projections

$$\pi : X(0)_{K^{1+}} \to X(0), \quad \pi' : X'(0)_{K^{1+}} \to X'(0).$$

We also have a natural map between the underlying sets defined in 2.3.3

$$\rho_{X(0)_{K^{1+}}} : |X(0)_{K^{1+}}| \to |X'(0)_{K^{1+}}|.\] (The map $\rho_{X(0)_{K^{1+}}}$ is in fact a homeomorphism and we do not need this fact.)

5.2. **Classical Serre-Tate theory.** We use the adjective “classical” to indicate the Serre-Tate theory [14] discussed in this subsection, compared with Chai’s global Serre-Tate theory to be discussed in 5.4.

Let $R$ be an Artinian local ring with maximal ideal $m$ and residue field $k$. Let $A/\text{Spec} R$ be an abelian scheme with ordinary special fiber $A_k$. Let $A_k^\vee$ be the dual abelian variety of $A_k$. There is a $\mathbb{Z}_p$-module morphism from the product of Tate-modules $T_p A_k \otimes T_p A_k^\vee$ to $1 + m$ constructed by Katz [14]. We call this morphism the classical Serre-Tate coordinate system for $A/\text{Spec} R$. If $A/\text{Spec} R$ is moreover a principally polarized abelian scheme, the Serre-Tate coordinate system for $A/\text{Spec} R$ is a $\mathbb{Z}_p$-module morphism

$$q_{A/\text{Spec} R} : \text{Sym}^2(T_p A_k) \to 1 + m.$$

Let $x \in X_o(0)(k)$, and let $A_x$ be the corresponding principally polarized abelian variety. Let $M_x$ be the formal completion of $X$ at $x$, and $M/M_x$ the formal universal deformation of $A_x$. Then as part of the construction of $q_{A/\text{Spec} R}$, there is an isomorphism of formal schemes over $W$:

$$M_x \simeq \text{Hom}_{\mathbb{Z}_p}(\text{Sym}^2(T_p A_x), \hat{G}_m),$$

where $\hat{G}_m$ is the formal completion of the multiplicative group scheme over $W$ along the unit section. In particular, $M_x$ has a formal torus structure. Moreover, if $A_k \simeq A_x$ in (5.3), then (5.3) is the value of (5.4) at the morphism $\text{Spec} R \to M_x$ induced by $A$. Let $\mathcal{O}(M_x)$ be the coordinate ring of $M_x$, and let $m_x$ be the maximal ideal of $\mathcal{O}(M_x)$. From (5.4), we have a morphism of $\mathbb{Z}_p$-modules:

$$q = q_{M/M_x} : \text{Sym}^2(T_p A_x) \to 1 + m_x.$$

Fix a basis $\xi_1, ..., \xi_{g(g+1)/2}$ of $\text{Sym}^2(T_p A_x)$.

**Proposition 5.2.1** ([12, 3.2]). Let $F$ be a finite extension of $L$ with ring of integers $F^\circ$. Let $y^\circ \in X(F^\circ)$ with generic fiber $y$. Suppose that $y^\circ \in M_x(F^\circ)$. Then $y$ is a CM point if and only if $q(\xi_i)(y)$ is a $p$-primary root of unity for $i = 1, ..., g(g + 1)/2$. 


Thus every ordinary CM point is contained in $X(L^\text{cycl})$. For an ordinary CM point $y \in X(L^\text{cycl})$, there is a unique $y^o \in X(K^o)$ whose generic fiber is $y_K \in X(K)$. We regard $y^o$ as a point in $X(0)(K^o)$ and $y_K$ as a point in $X(K,K^o)$ via Lemma 2.1.3.

**Definition 5.2.2.** Let $a = (a^{(1)},...,a^{(g(g+1)/2)}) \in \mathbb{Z}^{g(g+1)/2}$.

1. An ordinary CM point $y \in X(L^\text{cycl})$ with reduction $x$ is called of order $p^a$ w.r.t. the basis $\xi_1,\ldots,\xi_{g(g+1)/2}$ if $q(\xi_i)(y^o)$ is a primitive $p^{a^{(i)}}$-th root of unity for each $i = 1,\ldots,g(g+1)/2$. If moreover $q(\xi_i)(y^o) = \mu_p^{a^{(i)}}$, $y$ is called a $p$-generator w.r.t. the basis $\xi_1,\ldots,\xi_{g(g+1)/2}$.

2. Assume that $a$ is non-increasing so that $q(\xi_i+1)(y^o)$ is an $r^{(i)}$-th power of $q(\xi_i)(y^o)$ for some (non-unique) $r^{(i)} \in \mathbb{Z}_p$, $i = 1,\ldots,g(g+1)/2 - 1$. We call $(r^{(1)},\ldots,r^{(g(g+1)/2-1)}) \in \mathbb{Z}_p^{g(g+1)/2-1}$ a ratio of $y$ w.r.t. the basis $\xi_1,\ldots,\xi_{g(g+1)/2}$.

It is clear that if $a$ is non-increasing, then the usual $p$-adic absolute value $|r^{(i)}|_p = p^{a^{(i)} - r^{(i)}}$.

Let $T_i = q(\xi_i) - 1 \in m_x$ Then we have an isomorphism

$$O(\mathfrak{M}_x) \simeq W[[T_1,\ldots,T_{g(g+1)/2}]].$$

Let $\tilde{X}_o(0)_{/x}$ be the formal completion of $X_o(0)$ at $x$. Restricted to $\tilde{X}_o(0)_{/x}$, (5.5) gives an isomorphism

$$O(\tilde{X}_o(0)_{/x}) \simeq k[[T_1,\ldots,T_{g(g+1)/2}]].$$

Let $U \to X_o(0)$ be an étale morphism, $z \in U(k)$ with image $x$. Then (5.6) gives an isomorphism

$$O(\tilde{U}_{/z}) \simeq k[[T_1,\ldots,T_{g(g+1)/2}]].$$

Let $A_z$ be the pullback of $A_x$ at $z$. Then we naturally have $T_p A_x \simeq T_p A_z$. Thus we also regard $\xi_1,\ldots,\xi_{g(g+1)/2}$ as a basis of $\text{Sym}^2(T_p A_z)$.

**Definition 5.2.3.** We call (5.7) the realization of the classical Serre-Tate coordinate system of $\tilde{U}_{/z}$ at the basis $\xi_1,\ldots,\xi_{g(g+1)/2}$ of $\text{Sym}^2(T_p A_z)$.

We have another description of (5.7). Let $I_n$ be a descending sequence of open ideals of $O(\tilde{U}_{/z})$ defining the topology of $O(\tilde{U}_{/z})$. Let $R_n := O(\tilde{U}_{/z})/I_n$, let $A_n$ be the pullback of the formal universal principally polarized abelian scheme over $\mathfrak{M}_z$ to $\text{Spec} R_n$ with special fiber $A_z$. Let

$q_{A_n/\text{Spec} R_n} : \text{Sym}^2(T_p A_n^{\text{univ}}) \to R_n^\times$

be the classical Serre-Tate coordinate system of $A_n/R_n$. Then $q_{A_n/\text{Spec} R_n}(\xi_i - 1) = T_i(\text{mod } I_n)$. Thus the sequence $\{q_{A_n/\text{Spec} R_n}(\xi_i - 1)\}n$ gives an element in $O(\tilde{U}_{/z}) \simeq \varprojlim R_n$, which equals $T_i$.

### 5.3. Tilts of ordinary CM points

Let $\tilde{X}_o(0)_{/x}$ be the formal completion of $X_o(0)_{K^o}$ at $x$. By (5.6), we have

$$O\left(\tilde{X}_o(0)_{K^o/\times}\right) \simeq K^{\text{univ}}[[T_1,\ldots,T_{g(g+1)/2}]].$$

Let $D_z$ be the adic generic fiber of $\tilde{X}_o(0)_{K^o/\times}$. Then $D_z$ is an adic subspace of $\mathcal{X}'(0)$ in the sense of Definition 2.1.2. Moreover, (5.8) and Lemma 2.1.3 imply an isomorphism

$$D_z(K^o, K^{\text{univ}}) \simeq K^{\text{univ}, g(g+1)/2}. $$

**Lemma 5.3.1.** Let $y \in X(L^\text{cycl})$ be an ordinary CM point with reduction $x$.

1. For every $\tilde{y} \in \pi^{-1}(y_K) \subset X(0)^{\text{perf}}$, we have

$$\pi' \circ \rho_{X(0)^{\text{perf}}} (\tilde{y}) \in D_z.$$
(2) Let \( a = (a^{(1)}, \ldots, a^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2} \) and \( I \subset \{1, 2, \ldots, g(g + 1)/2\} \) the subset of \( i \)'s such that \( a^{(i)} = 0 \). Let \( y \) be a \( \mu \)-generator of order \( p^n \) w.r.t. the basis \( \xi_1, \ldots, \xi_{g(g+1)/2} \) (see Definition 5.2.2). There exists \( \tilde{y} \in \pi^{-1}(y_K) \) such that via the isomorphism (5.9), the \( i \)-th coordinate of \( \pi' \circ \rho_{\lambda(0)\text{perf}}(\tilde{y}) \) is 0 for \( i \in I \) and is \( t^{1/p^{(i)}} \) for \( i \notin I \).

Proof. We recall the effect of \( \text{Fr}^\text{can} \) on \( \mathcal{M}_x \) (see [14, 4.1]). Denote \( \mathfrak{M}_x \) by \( \mathfrak{M}_{A_x} \). Let \( \sigma \in \text{Aut}(k) \) be the Frobenius. Let \( A_x^{(\sigma)} := A_x \otimes_{k, \sigma} k \) be the base change by \( \sigma \). Then \( \text{Fr}^\text{can} \), restricted to \( \mathfrak{M}_{A_x} \), gives a morphism \( \text{Fr}^\text{can} \) : \( \mathfrak{M}_{A_x} \rightarrow \mathfrak{M}_{A_x^{(\sigma)}} \) over \( W \) [14, p 171]. Let \( \sigma(\xi_1), \ldots, \sigma(\xi_{g(g+1)/2}) \) be the induced basis of \( \text{Sym}^2(T_p A_x^{(\sigma)}[p^\infty]) \). Then [14, Lemma 4.1.2] implies that

\[
\text{Fr}^\text{can,*}(q(\sigma(\xi_i))) = q(\xi_i)^p.
\]

We associate a perfectoid space to \( \mathfrak{M}_x \). Let

\[
\mathfrak{M}_x := \varinjlim \mathfrak{M}_{A_x^{(\sigma-n)}}.
\]

By a similar (and easier) proof as the one for [25, Corollary 3.2.19], the adic generic fiber \( \mathcal{M}_x^{\text{perf}} \) of \( \mathfrak{M}_x, K^\text{e} \) is a perfectoid space. Moreover, let \( \mathcal{M}_x^{\text{perf}} \) be the adic generic fiber of \( \varprojlim X_0(0)/\text{Fr} \) \( \mathfrak{M}_{A_x^{(\sigma-n)}}/\mathfrak{M}_{A_x} \). Then \( \mathcal{M}_x^{\text{perf}} \) is the tilt of \( \mathcal{M}_x^{\text{perf}} \). By Lemma 2.3.1, the tilting process commutes with restriction to an open subspace. Thus to prove Lemma 5.3.1, we only need to consider the tilting between \( \mathcal{M}_x^{\text{perf}} \) and \( \mathcal{M}_{x'}^{\text{perf}} \). Then Lemma 5.3.1 follows from the cases \( c = 0 \) and \( c = 1 \) of Lemma 2.3.3 (which deals with closed units discs while here we are dealing with open unit discs so that we apply 2.3.1 again). \( \square \)

5.4. Global Serre-Tate theory.

5.4.1. The algebraic and geometric formulations. Now we review Chai’s globalization of Serre-Tate coordinate system in characteristic \( p \) [4]. Let \( U \) be a \( \mathbb{F}_p \)-scheme. Let \( A/U \) be an abelian scheme whose relative dimensions on connected components of \( U \) are the same. Define

\[
\nu_U = \varprojlim_n \text{Coker}(p^n : \mathbb{G}_m \rightarrow \mathbb{G}_m),
\]

which is a \( \mathbb{Z}_p \)-sheaf on \( U^\text{et} \).

Example 5.4.1. (1) Let \( m \geq n \) be positive integers, and \( U_0 = \text{Spec}k[T]/T^m \). Then the \( p^n \)-th power of an element in \((k[T]/T^m)^\times\) with constant term \( b \) is \( b^{p^n} \). Thus

\[
\nu_{U_0}(U_0) = (k[T]/T^m)^\times/k^\times \simeq 1 + T(k[T]/T^m).
\]

(2) Let \( B \) be an \( \mathbb{F}_p \)-algebra, \( U = \text{Spec}B \) and \( U' = \text{Spec}B[T]/T^m \). For \( m \geq n \), consider the map

\[
B^\times/(B^\times)^p \bigoplus (1 + TB[T]/T^m) \rightarrow (B[T]/T^m)^\times/((B[T]/T^m)^\times)^{p^n}
\]

defined by \((a, f) \mapsto af\). Easy to check that this is a group isomorphism. In particular,

\[
\nu_{U'}(U') \simeq \nu_U(U) \bigoplus (1 + TB[T]/T^m).
\]

(3) For every \( z \in U(k) \), \( \{z\} \times_U U' \simeq U_0 \). Then the restriction of the isomorphism (5.12) at \( z \) is the isomorphism (5.11).

Suppose \( A/U \) is ordinary. Let \( T_p A[p^\infty]^{\text{et}} \) be the Tate module attached to the maximal étale quotient of the \( p \)-divisible group \( A[p^\infty] \). The global Serre-Tate coordinate system for \( A/U \) is a homomorphism of \( \mathbb{Z}_p \)-sheaves

\[
q_{A/U} : T_p A[p^\infty]^{\text{et}} \otimes T_p A[v]^{\text{et}} \rightarrow \nu_U
\]
constructed by Chai [4, 2.5]. Let \( U_0 = \text{Spec} k[T]/T^n \). Let \( A/U_0 \) be an ordinary abelian scheme, and \( A_k \) the special fiber of \( A \). Then
\[
T_pA[p^\infty]^\text{et} \otimes T_pA^\vee[p^\infty]^\text{et} \cong T_pA_k[p^\infty] \otimes T_pA_k^\vee[p^\infty],
\]
where the right hand side is regarded as a constant sheaf.

**Lemma 5.4.2 ([4, (2.5.1)])**. The morphism of \( \mathbb{Z}_p \)-modules
\[
T_pA_k[p^\infty] \otimes T_pA_k^\vee[p^\infty] \to \nu_U(U) \cong 1 + T(k[T]/T^n)
\]
induced from \( q_{A/U_0} \) via (5.13) coincides with the classical Serre-Tate coordinate system (see (5.3)).

The geometric formulation of global Serre-Tate coordinate system is as follows. Let \( A^{\text{univ}} \) be the universal principally polarized abelian scheme over \( X_o(0) \), and \( A^{\text{univ}} \) the formal completion of \( A^{\text{univ}} \) along the zero section which is a formal torus over \( X_o(0) \). Then the sheaf of polarization-preserving \( \mathbb{Z}_p \)-homomorphisms between \( T_pA^{\text{univ}}[p^\infty]^\text{et} \) and \( A^{\text{univ}} \) is a formal torus over \( X_o(0) \) of dimension \( \frac{g(g+1)}{2} \).

Let us call it \( \mathcal{T}_1 \). Let \( \Delta \) be the diagonal embedding of \( X_o(0) \) into \( X_o(0) \times X_o(0) \), and let \( \mathcal{T}_2 \) be the formal completion of \( X_o(0) \times X_o(0) \) along this embedding.

**Proposition 5.4.3 ([4, Proposition 5.4]).** There is a canonical isomorphism \( \mathcal{T}_1 \cong \mathcal{T}_2 \). In particular, \( \mathcal{T}_2 \) has a formal torus structure over the first \( X_o(0) \).

**5.4.2. Igusa tower.** In order to have sections of the étale \( \mathbb{Z}_p \)-sheaf \( T_pA[p^\infty]^\text{et} \) over \( U \), or equivalently to trivialize the formal torus, we need to pass to the Igusa tower, defined as follows. For \( n = 0, 1, \ldots, \infty \), let \( \mathcal{J}_n \) be the functor assigning to every \( k \)-algebra \( R \) the set of isomorphism classes of pairs
\[
\{(A, \varepsilon) : A \in X_o(0)(R), \varepsilon : A[p^n] \cong \hat{\mathcal{O}}_{m,R}[p^n]\}.
\]

By [6, 8.1.1], for \( n < \infty \) (resp. \( n = \infty \)) the functor \( \mathcal{J}_n \) is represented by a \( k \)-scheme (which we still denote by \( \mathcal{J}_n \)) finite (resp. profinite) Galois over \( X_o(0) \) with Galois group \( \text{GL}_g(\mathbb{Z}/p^n\mathbb{Z}) \) (resp. \( \text{GL}_g(\mathbb{Z}_p) \)). And \( \mathcal{J}_n \) is known as the Igusa scheme of level \( n \).

**5.4.3. Realization of the global Serre-Tate coordinate system at a basis.** Let \( U_0 \) be the affine open sub-scheme of \( X_o(0) \). Let \( U = \text{Spec} B := \mathcal{J}_\infty|_{U_0} \). Let \( \Delta \) be the diagonal of \( U \times U \). We have two projection maps \( \text{pr}_1, \text{pr}_2 \) from \( U \times U / \Delta \) to the first and second \( U \). For \( z \in U(k) \), the restriction of \( \text{pr}_2 \) induces
\[
(5.14)
\]

Let \( \mathcal{O} \left( \hat{U} / \hat{U} / \Delta \right) \) be the coordinate ring of \( 
\hat{U} / \hat{U} / \Delta \). Endow \( \mathcal{O} \left( \hat{U} / \hat{U} / \Delta \right) \) a \( B \)-algebra structure via \( \text{pr}_1 \). By Proposition 5.4.3, we have a (non-unique) \( B \)-algebra isomorphism
\[
(5.15)
\]

Let \( A/\hat{U} \times \hat{U} / \Delta \) be the pullback of \( A^{\text{univ}}|_{U_0} \). Assume that \( \text{Sym}^2(T_pA^{\text{univ}}[p^\infty]^\text{et})(U) \) is a free \( \mathbb{Z}_p \)-modules of rank \( g(g+1)/2 \). Let \( \xi_1, \ldots, \xi_{g(g+1)/2} \) be a basis of \( \text{Sym}^2(T_pA^{\text{univ}}[p^\infty]^\text{et})(U) \) (whose existence follows from the definition of \( \mathcal{J}_\infty \) and the polarization). The realization of the global Serre-Tate coordinate system of \( A/\hat{U} \times \hat{U} / \Delta \) at the basis \( \xi_1, \ldots, \xi_{g(g+1)/2} \) is a construction of an isomorphism (5.15) as follows.

For the simplicity of notations, let us assume \( g = 1 \). The general case can be dealt in the same way. Let \( \xi = \xi_1 \) and \( T = T_1 \). Let
\[
U' = \hat{U} / \hat{U} / \Delta / T^p \cong \text{Spec} B[[T]]/T^p
\]
and let \( A_n \) be the restriction of \( A \) to \( U' \). The global Serre-Tate coordinate system of \( A_n / U' \) is a homomorphism of \( \mathbb{Z}_p \)-sheaves over \( U_{2,et} \)
\[
q_{A_n/U'} : \text{Sym}^2(T_pA_n[p^\infty]^\text{et}) \to \nu_{U'}.
\]
Note that \( \xi \) gives a basis \( \xi_n \) of \( \text{Sym}^2(T_p A_n[p^{\infty}]^t(U')) \). Then we have
\[
q_{A_n/U'}(\xi_n) \in \nu(U') \simeq \nu(U) \bigoplus (1 + TB[T]/T^n)
\]
where the second isomorphism is (5.12). Consider the morphism
\[
\phi_n : \nu(U') \to (1 + TB[T]/T^n) \to B[T]/T^n
\]
where the first map is the projection and second map is the natural inclusion. Let \( T_n^{ST} \in B[T]/T^n \) be \( \phi_n(q_{A_n/U'}(\xi_n)) - 1 \). As \( n \) varies, \( T_n^{ST} \)’s give an element
\[
T^{ST} \in \mathcal{O}\left(\hat{U} \times \hat{U}/\Delta\right) \simeq \varprojlim_n B[T]/T^n.
\]
We compare the above construction with the realization of the classical Serre-Tate coordinate system. Let \( z \in U(k) \). The restriction of \( A \) to \( \text{pr}_1^{-1}\{\{z\}\} \) is pullback \( A^\text{univ}|_{\hat{U}/z} \) of \( A^\text{univ}|_{U_0} \) to \( \hat{U}/z \) via (5.14).
(Thus we may regard \( A \) as the family \( \{A^\text{univ}|_{\hat{U}/z} : z \in U(k)\}\).) The realization of the classical Serre-Tate coordinate system of \( \hat{U}/z \) at \( \xi_z \) (the restriction of \( \xi \) at \( z \)) gives an element \( T_z^{ST} \in \hat{U}/z \) and an isomorphism \( \hat{U}/z \simeq \text{Spf}k[[T_z^{ST}]] \) (see Definition 5.2.3). Here and below, the subscript \( c \) indicates “classical”.

**Lemma 5.4.4.** The restriction of \( T^{ST} \) to \( \text{pr}_1^{-1}\{\{z\}\} \) \( \simeq \hat{U}/z \) is \( T_z^{ST} \). In particular,
\[
\mathcal{O}\left(\hat{U} \times \hat{U}/\Delta\right) = B[[T^{ST}]].
\]

**Proof.** The restriction of (5.15) to \( \text{pr}_1^{-1}\{\{z\}\} \) \( \simeq \hat{U}/z \) via (5.14) gives an isomorphism \( \mathcal{O}(\hat{U}/z) \simeq k[[T]] \). Let
\[
q^c_z = q^c_{A^\text{univ}|_{\hat{U}/z}/T^n} : \text{Sym}^2(T_p A^\text{univ}_z[p^{\infty}]) \to 1 + T k[[T]]/T^n
\]
be the classical Serre-Tate coordinate system of \( A^\text{univ}|_{\hat{U}/z}/T^n \) (see (5.3)). Then the image of \( T_z^{ST} \) in \( k[[T]]/T^n \) is \( q^c_z(\xi_z) - 1 \). By Example 5.4.1 (3) and Lemma 5.4.2, \( q^c_z(\xi_z) \) equals the restriction of \( \phi_n(q_{A_n/U'}(\xi_n)) \) at \( z \). Thus the first statement follows. The second statement follows from the first one.

\[\square\]

6. Proof of Theorem 1.1.1

In this section, we at first prove a Tate-Voloch type result in a family in characteristic \( p \). Combined with the results in Section 5, we prove Theorem 1.1.1. We continue to use the notations in Section 5.

6.1. Tate-Voloch type result in a family in characteristic \( p \). Recall that \( k = \bar{\mathbb{F}}_p \) and \( K^\infty = k((t^{1/p^n})) \). In the proof of Lemma 2.4.3, we used the following simple fact: let \( S \) a \( k \)-algebra, \( g \in S \) and \( x \in (\text{Spec}S)(k) \), then \( g(x) = 0 \) or \( |g(x)|_k = 1 \) where the valuation \( |\cdot|_k \) on \( k \) takes value \( 0 \) on \( k \) and \( 1 \) on \( k^\times \). This fact can be naively regarded as an analog of the Tate-Voloch conjecture over \( k \). We want to consider this analog in a family. We need some notations.

Let \( l \) be a positive integer. For \( d = (d^{(1)}, \ldots, d^{(l)}) \in (p^{\infty})^l \), define \( t^d := (t^{d^{(1)}}, \ldots, t^{d^{(l)}}) \in (K^{\infty})^l \). For \( c = (c^{(1)}, \ldots, c^{(l)}) \in (\mathbb{Z}_p)^l \), define
\[
(1 + t^d)^c - 1 := \left( (1 + t^{d^{(1)}})^{c^{(1)}} - 1, \ldots, (1 + t^{d^{(l)}})^{c^{(l)}} - 1 \right) \in (K^{\infty})^l.
\]
Fix a sequence \( \{d_n\}_{n=1}^\infty \) of elements in \( (p^{\infty})^l \) and a sequence \( \{c_n\}_{n=1}^\infty \) of elements in \( (\mathbb{Z}_p)^l \). Let
\[
y_n = (1 + t^{d_n})^{c_n} - 1 \in (K^{\infty})^l \subset \text{Spec}K^{\infty}[[T_1, \ldots, T_l]].
\]
Let \( \mathbb{N} = \{1, 2, \ldots\} \) the sequence of positive integers. For \( \delta \in (0, 1) \) and the given sequence \( \{d_n\}_{n=1}^\infty \), let
\[
\mathbb{N}(\delta) = \{n \in \mathbb{N} : d_n^{(i)} / d_n^{(i+1)} < \delta\}.\]
If \( l = 1 \), we understand \( \mathbb{N}(\delta) \) as \( \mathbb{N} \).

**Proposition 6.1.1.** Let \( A \) be a reduced \( k \)-algebra and \( V = \text{Spec} A \). Let \( \{z_n\}_{n=1}^{\infty} \) be a sequence of (not necessarily distinct) points in \( V(k) \). Let \( f \in A[[T_1, ..., T_l]] \) and let \( f_{z_n} \in k[[T_1, ..., T_l]] \) be the restriction of \( f \) at \( z_n \). Assume that

\[
(*) \text{ for every infinite subset } \mathbb{N'} \subset \mathbb{N}, \text{ the set } \{z_n : n \in \mathbb{N'}\} \text{ is Zariski dense in } V.
\]

If \( f \neq 0 \), then there exists \( D_0 \in \mathbb{R}_{>0} \) and \( \delta_0 \in (0, 1) \) such that for every \( D \geq D_0 \) and \( \delta \leq \delta_0 \), the following set is finite

\[
(6.2) \quad \{n \in \mathbb{N}(\delta) : \|f_{z_n}(y_n)\| < \|T_l(y_n)\|^D\}.
\]

Here \( T_l(y_n) \) is, by definition, the \( l \)-th coordinate of \( y_n \).

**Proof.** We do induction on \( l \).

The case \( l = 1 \) is proved as follows. Let \( f = \sum_{m=0} a_m T^m \) where \( a_m \in A \). Regard \( a_m \) as a function on \( V \) so that \( a_m(z_n) \in k \). Claim: there exists some \( m \) such that \( a_m(z_n) \neq 0 \) for \( n \) large enough. Let \( m_0 \) be the smallest such \( m \). Then

\[
\|f_{z_n}(y_n)\| = \|a_{m_0}(y_n)\|
\]

for \( n \) large enough. Let \( D_0 = m_0 \) and we are done. Now we prove the claim by contradiction. Assume that for every \( m \), \( a_m(z_n) = 0 \) for infinitely many \( n \). By assumption \((*)\) and the reducedness of \( A \), \( a_m = 0 \). Thus \( f = 0 \). This is a contradiction.

Now we do the induction. Let \( l > 1 \). We prepare some notations. Let \( d'_n, y'_n \) be the first \( l-1 \) components of \( d_n, y_n \) respectively. For \( \delta \in (0, 1) \), we have a subsequence \( \mathbb{N}(\delta)' \subset \mathbb{N} \) defined using the sequence \( \{d'_n\}_{i=1}^{\infty} \). Then \( \mathbb{N}(\delta)' \supset \mathbb{N}(\delta) \).

Assume that \( f \neq 0 \). Write \( f = T_l^{m_1}(g_1 + f_1) \) where \( g_1 \in A[[T_1, ..., T_{l-1}]] \setminus \{0\} \) and \( f_1 \in T_l A[[T_1, ..., T_l]] \).

Below, to lighten notation, we abbreviate the subscript \( z_n \). Then for \( n \) in the set \((6.2)\), with \( D \) and \( \delta \) to be determined, we have

\[
\|g_1(y_n) + f_1(y_n)\| = \|T_l(y_n)\|^{-m_1}\|f(y_n)\| < \|T_l(y_n)\|^D - m_1.
\]

If \( D \geq m_1 + 1 \), then

\[
\|g_1(y_n)\| \leq \|g_1(y_n) + f_1(y_n)\| + \|f_1(y_n)\| \leq \|T_l(y_n)\|.
\]

Since \( \|T_l(y_n)\| < \|T_{l-1}(y'_n)\|^{1/\delta} \) and \( \|g_1(y'_n)\| = \|g_1(y_n)\| \), we have

\[
(6.3) \quad \|g_1(y_n')\| < \|T_{l-1}(y'_n)\|^{1/\delta}.
\]

By the induction hypothesis, there exists \( D' > 0 \) and \( \delta'_0 \in (0, 1) \) such that if \( \delta \leq 1/D' \) and \( \delta \leq \delta'_0 \), \( \{n \in \mathbb{N}(\delta)' : (6.3) \text{ holds}\} \) is finite. Then \((6.2)\) is finite by choosing \( \delta_0 = \min\{1/D', \delta'_0\} \).

**Remark 6.1.2.** (1) \( D_0 \) and \( \delta_0 \) are uniform for all choices of \( \{c_n\}_{n=1}^{\infty} \). We do not need this fact later.

(2) The proposition is inspired by [27, Lemma 2.10]. In the proof of [27, Lemma 2.10], there is a minor imprecision. The following modification is suggested by Serban. Define \( T_3 \) in [27, Lemma 2.10] to be the first set in the intersection but not the entire intersection, so that the statement \((2)\) in loc. cit. is about \( T_3 \cap S_{q^{-1+c}} \). The 3rd displayed formula in the proof of [27, Lemma 2.10] should be removed. Then, on can still get the 5th displayed formula in that proof with slightly more effort.

**Lemma 6.1.3.** Let \( \{B_i\}_{i=0}^{\infty} \) be a system of rings and \( B = \varinjlim B_i \). Let \( f_i : \text{Spec} B \to \text{Spec} B_i \) be the natural morphism. Let \( \Lambda \subset \text{Spec} B \) be a subset and \( \Lambda_i = f_i(\Lambda) \subset \text{Spec} B_i \). We have the following relation between Zariski closures:

\[
(6.4) \quad \Lambda^{\text{Zar}} = \bigcap_{i=0}^{\infty} f_{i}^{-1}\left(\Lambda_i^{\text{Zar}}\right).
\]
Proof. The ideal \( I \subset B \) defining \( \Lambda^\text{Zar} \), with reduced induced structure as a closed subscheme, is generated by the union of the images \( I_i \) in \( B \), where \( I_i \subset B_i \) is the ideal of elements whose image in \( B \) vanishes on \( \Lambda^\text{Zar} \). By the definition of \( \Lambda_i \), \( I_i \) is the ideal defining \( \Lambda_i^\text{Zar} \). Then (6.4) follows. \( \square \)

Let \( f : U \to U_0 \) be a surjective morphism of schemes. Let \( \Lambda_0 \subset U_0 \) be a subset with Zariski closure \( \Lambda_0^\text{Zar} \) in \( U_0 \). For \( s \in \Lambda_0 \), choose \( z_s \in f^{-1}(s) \). Let \( \Lambda = \{ z_s : s \in \Lambda_0 \} \) with Zariski closure \( \Lambda^\text{Zar} \) in \( U \).

**Lemma 6.1.4.** Assume that \( f \) is closed.

1. The image of \( \Lambda^\text{Zar} \) in \( U_0 \) is \( \Lambda_0^\text{Zar} \).
2. Assume that \( \Lambda_0^\text{Zar} \) is irreducible and \( U \) is noetherian. There exists a choice of \( \Lambda \), such that \( \Lambda^\text{Zar} \) is irreducible.
3. In (2), further assume that \( f \) is finite and the Zariski closure of every infinite subset of \( \Lambda_0 \) is \( \Lambda_0^\text{Zar} \). Then the Zariski closure of every infinite subset of \( \Lambda \) is \( \Lambda^\text{Zar} \).

Proof. (1) is easy and the proof is omitted.

(2) For every member of the finitely many irreducible (so closed) components of \( f^{-1}(\Lambda_0^\text{Zar}) \), its image in \( \Lambda_0^\text{Zar} \) is a closed subscheme. By the irreducibility of \( \Lambda_0^\text{Zar} \), some irreducible component of \( f^{-1}(\Lambda_0^\text{Zar}) \) is surjective to \( \Lambda_0^\text{Zar} \). We choose all \( z_s \)'s in this component.

(3) Note that a finite surjective morphism preserves dimension, and a proper closed subscheme of a noetherian irreducible scheme has a strictly smaller dimension. Then (3) follows from (1) and counting dimensions. \( \square \)

The last two lemmas imply the following corollary.

**Corollary 6.1.5.** Let \( B, B_i \)'s be as in Lemma 6.1.3. Let \( U = \text{Spec} B \) (not necessary noetherian), \( U_0 = \text{Spec} B_0 \) and \( f = f_0 \). Assume that each \( B_i \) is noetherian and the transition morphisms \( \text{Spec} B_j \to \text{Spec} B_i \) are finite surjective. Assume that the Zariski closure of every infinite subset of \( \Lambda_0 \) is \( \Lambda_0^\text{Zar} \). There exists a choice of \( \Lambda \), such that the Zariski closure of every infinite subset of \( \Lambda \) is \( \Lambda^\text{Zar} \).

To fulfill the second assumption of the corollary, we use the following lemma.

**Lemma 6.1.6.** Let \( U_0 \) be a noetherian scheme. For every infinite subset \( Y \subset U_0 \), there is an infinite subset \( \Lambda_0 \subset Y \) such that the Zariski closure of every infinite subset of \( \Lambda_0 \) is \( \Lambda_0^\text{Zar} \).

Proof. By the noetherianess of \( U_0 \), there exists a closed subscheme \( V \) of \( U_0 \) containing an infinite subset \( \Lambda_0 \) of \( Y \) such that every proper closed subscheme of \( V \) only contains finitely many elements in \( Y \). \( \square \)

6.2. **Proof of Theorem 1.1.1.** Let \( X \) be a product of Siegel moduli spaces over \( \mathbb{Z}_p \) with certain level structures away from \( p \). By Lemma 2.2.12, Theorem 1.1.1 follows from the following theorem.

**Theorem 6.2.1.** Let \( Z \) be a closed subvariety of \( X_L \). There exists a constant \( c > 0 \) such that for every ordinary CM point \( x \in X(L^{\text{cyc}}) \), if \( d(x, Z) \leq c \), then \( x \in Z \).

Here the distance function \( d(x, Z) \) is defined as in 2.2.2 using the integral model \( X \).

**Proof.** We prove Theorem 6.2.1 when \( X \) is a single Siegel moduli space. The general case is proved in the same way or by embedding a product of Siegel moduli spaces into a bigger one. We continue to use the notations in Section 5. In particular, the fields \( L, L^{\text{cyc}}, K \) and \( K^0 \) below are as in the beginning of Section 5; the formal scheme \( \mathfrak{X}(0) \), the adic locus \( \mathfrak{X}(0) \), the perfectoid spaces \( \mathfrak{X}(0)^{\text{per}^L}, \mathfrak{X}(0)^{\text{per}^L} \) and Frobenius morphism \( \text{Fr}^\text{can} \) below are as in 5.1. For an ordinary CM point \( x \in X(L^{\text{cyc}}) \), we take the same notation \( x \) to denote its base change in \( X(K) \). Let \( x^0 \) be the unique \( \mathcal{E}^0 \)-point in \( X \) whose generic fiber is \( x \).

Suppose that \( Z \) is defined over a finite Galois extension \( F \) of \( L \). Let \( \mathcal{I} \) be the ideal sheaf of the schematic closure of \( Z \) in \( X_F \). Let \( \mathcal{U} \) be an affine open subscheme of \( X_W \), of finite type over \( W \). (This is the only use of a calligraphic font not representing an adic space in this paper.) We only need to find
a constant $c$ such that, if an ordinary CM point $x \in X(L^\cyc)$ satisfies $x^0 \in U(K^\cyc)$ and $d_{U,\kappa}(x_K, I) < c$, then $x \in Z$. Here the distance function is as in 2.2.2.

We at first have the following simplification on $F$. Let $K' = FK$. Suppose $\mathcal{I}(U_{F^\infty})$ is generated by $f_i$, $i = 1, \ldots, n$. For $\sigma \in G := \text{Gal}(K'/K)$, $f_i^\sigma$ is in the coordinate ring of $U_{K^\infty}$ and $\|f_i(x_{K'})\| = \|f_i^\sigma(x_{K'})\|$. Let $I$ be the ideal of the coordinate ring of $U_{K^\infty}$ generated by $\prod_{\sigma \in G} f_i^\sigma$, $i = 1, \ldots, n$. Then

$$d_{U,\kappa}(x_K, I) = d_{U,\kappa}(x_{K'}, I_{K^\infty}(U_{K^\infty}))^{|G|}.$$  

Thus we may assume that $F \subseteq K$. Equivalently, $F \subseteq L^{\cyc}$.

Now we reduce Theorem 6.2.1 to Theorem 6.2.2 below, which is formulated with affine formal schemes. For an ordinary CM point $x \in X(K)$, we also use $x$ to denote the corresponding point $x(0)(K, K^\cyc)$. Let $\mathcal{I}$ be the restriction of the $\mathfrak{m}$-adic formal completion of $U$ to $X(0)$. By Lemma 2.2.9, Theorem 6.2.1 is deduced from Theorem 6.2.2.

**Theorem 6.2.2.** Let $\mathfrak{F}$ be an irreducible closed formal subscheme of $\mathcal{I}U_{F^\infty}$. For a sequence $\{x_n\}_{n=1}^\infty$ of ordinary CM points such that $x_n$ is in the $\epsilon_n$-neighborhood of $\mathfrak{F}$ and with $\|\epsilon_n\| \to 0$, we have $x_n \in Z$ for infinitely many $n$'s.

The proof of Theorem 6.2.2 consists of two bulks: one involvs perfectoid spaces and one does not. The perfectoid one is more technical and proves results to be used in the second one. The non-perfectoid one concludes Theorem 6.2.2. We will present the on-perfectoid one first, in 6.2.1 and 6.2.2.

A canonical lifting is an ordinary CM points of order $1$ w.r.t. a (equivalently every) basis, see Definition 5.2.2 (1). The following lemma will be proved in Theorem 6.2.7 using perfectoid spaces.

**Lemma 6.2.3.** Theorem 6.2.2 holds if we replace “ordinary CM points” by “canonical liftings”.

6.2.1. Global Serre-Tate coordinate. Before we proceed to the proof of Theorem 6.2.2, let us recall the realization of the global Serre-Tate coordinate system in 5.4.3.

Let $U_0$ be the special fiber of $\mathcal{I}$. Let $U = \text{Spec} B$ be the profinite Galois cover of $U_0$ defined in 5.4.3 (and coming from the infinite level Igusa scheme) such that $\text{Sym}^2(T_p A^{\univ}[p^\infty]^e)(U)$ is a free $\mathbb{Z}_p$-modules of rank $g(g+1)/2$. Let $\Delta$ be the diagonal of $U \times U$. Then by Lemma 5.4.4, for a basis

$$\xi_1, \ldots, \xi_{g(g+1)/2}$$

of $\text{Sym}^2(T_p A^{\univ}[p^\infty]^e)(U)$, we have the realization of the global Serre-Tate coordinate system

$$O \left( \widehat{U \times U/\Delta} \right) = B[[T_{1ST}^{ST}, \ldots, T_{g(g+1)/2}^{ST}]],$$

which has the following property. For every $z \in U(k)$, we have an isomorphism

$$pr_1^{-1}(\{z\}) \overset{pr_2}{\cong} \widehat{U}/z$$

as in (5.14), and the corresponding isomorphism

$$\widehat{U}/z \cong \text{Spf} k[[T_{1ST}^{ST}, \ldots, T_{g(g+1)/2}^{ST}]],$$

Let $T_{i,z}^{ST}$ be the restriction of $T_{i,z}^{ST}$ to $pr_1^{-1}(\{z\}) \cong \widehat{U}/z$. Let

$$\xi_{1,1}, \ldots, \xi_{g(g+1)/2}$$

be the restriction of $\xi_1, \ldots, \xi_{g(g+1)/2}$. Then (6.7) coincides with the realization of the classical Serre-Tate coordinate system of $\widehat{U}/z$ at $\xi_{1,1}, \ldots, \xi_{g(g+1)/2}$, see Definition 5.2.3.
6.2.2. Proof of Theorem 6.2.2. After passing to an infinite subsequence, we may assume that \( \{ \text{red}(x_n) \}_{n=1}^{\infty} \) is a sequence of the same point or pairwisely different points. Let \( z_n \in U(k) \) be over \( \text{red}(x_n) \in U_0(k) \). By Corollary 6.1.5 and Lemma 6.1.6, after passing to an infinite subsequence, we may assume the following.

**Assumption 6.2.4.** For every infinite subset \( \mathbb{N}' \subset \mathbb{N} \), the Zariski closure of the set \( \{ z_n : n \in \mathbb{N}' \} \) in \( U \) is the Zariski closure of the set \( \{ z_n : n \in \mathbb{N} \} \).

We regarded the basis (6.8) for \( z = z_n \) as a basis of \( \text{Sym}^2(T_p A x_n) \) naturally. Let \( x_n \) be of order \( p^{a_n} \) w.r.t. (6.8) (see Definition 5.2.2 (1)) where \( a_n = (a_n^{(1)}, \ldots, a_n^{(g(g+1)/2)}) \in \mathbb{Z}^{g(g+1)/2} \). After passing to an infinite subsequence and permuting the basis (6.5) of \( \text{Sym}^2(T_p A^{\text{univ}}[p^{\infty}]^{(l)})(U) \), we may assume that every \( a_n \) is non-increasing (see Definition 5.2.2 (2)). Let \( l \leq g(g+1)/2 \) be a non-negative integer such that for every \( n \), if \( i > l \), then \( a_n^{(i)} = 0 \). For example, if \( l = g(g+1)/2 \), the assumption automatically holds; if \( l = 0 \), we are in the situation of Lemma 6.2.5.

We will reduce Theorem 6.2.2 to the case \( l = 0 \) by using Lemma 6.2.5 below. We need the “upper triangular change of variables” argument following [27]. By “upper triangular change of variables”, we indeed mean changing the first \( l \)-element of the basis (6.5) of \( \text{Sym}^2(T_p A^{\text{univ}}[p^{\infty}]^{(l)})(U) \) via an upper triangular matrix as follows. For \( C \in \text{GL}_l(\mathbb{Z}_p) \), \( (\xi_1, \ldots, \xi_l)C \) combined with \( (\xi_{l+1}, \ldots, \xi_{g(g+1)/2}) \) gives a new basis of \( \text{Sym}^2(T_p A^{\text{univ}}[p^{\infty}]^{(l)})(U) \). Thus by restriction as in (6.8), we have a new basis of \( \text{Sym}^2(T_p A x_n) \) for every \( n \). Let \( x_n \) be of order \( p^{a_n}(C) \) w.r.t. this new basis, where \( a_n(C) \in \mathbb{Z}^{g(g+1)/2} \). Then for \( C \) upper triangular, \( a_n(C) \) is still non-increasing.

**Lemma 6.2.5.** Assume Assumption 6.2.4. Assume that for every upper triangular matrix \( C \in \text{GL}_l(\mathbb{Z}_p) \), the \( l \)-th component (so the \( i \)-th component for \( i = 1, \ldots, l \) as well) of \( a_n(C) \) goes to \( \infty \) as \( n \to \infty \). Then \( x_n \in \mathcal{Z} \) for all \( n \in \mathbb{N} \).

We postpone the proof of Lemma 6.2.5.

We finish the proof of Theorem 6.2.2 by induction on the dimension of \( \mathfrak{z} \). If \( \mathfrak{z} \) is empty, define its dimension to be \(-1\). When \( \mathfrak{z} \) is of dimension \(-1\), the theorem is trivial. The induction hypothesis is that the theorem holds for lower dimensions, and it will only be used in the proof of Lemma 6.2.6 (2) below.

By Lemma 6.2.5 and passing to an infinite subsequence, we may assume that for an upper triangular matrix \( C \in \text{GL}_l(\mathbb{Z}_p) \), the \( l \)-th component of \( a_n(C) \) is bounded. Replacing the basis (6.5) by the new basis that is \( (\xi_1, \ldots, \xi_l)C \) combined with \( (\xi_{l+1}, \ldots, \xi_{g(g+1)/2}) \), we may assume that there is a non-negative integer \( m \) such that for every \( n \), \( a_n^{(i)} \leq p^m \). The fact that \( a_n^{(i)} = 0 \) for \( i > l \) does not change.

**Lemma 6.2.6.** Let \( m \) be a non-negative integer. Then the following hold.

1. The adic generic fiber of \( (\text{Fr}^{\text{can}})^m(x_n^o) \) is in the \( \epsilon_n \)-neighborhood of the scheme theoretic image \( (\text{Fr}^{\text{can}})^m(\mathfrak{z}) \) (see [13, 2.3]).

2. Assume that \( (\text{Fr}^{\text{can}})^m(x_n^o) \in (\text{Fr}^{\text{can}})^m(\mathfrak{z}(K^0)) \) for infinitely many \( n \)'s, then \( x_n^o \in \mathfrak{z}(K^0) \) for infinitely many \( n \).

**Proof.** To lighten the notations, assume that \( m = 1 \).

Consider the closed formal subscheme \( (\text{Fr}^{\text{can}})^{-1}(\text{Fr}^{\text{can}}(3)) \) of \( \mathfrak{z} \) which contains \( \mathfrak{z} \). Then \( x_n^o \) is contained in the \( \epsilon_n \)-neighborhood of \( (\text{Fr}^{\text{can}})^{-1}(\text{Fr}^{\text{can}}(3)) \) by Lemma 2.2.4 (1). Then (1) follows from the analog of Lemma 2.2.7 for formal schemes (which directly follows from Definition 2.2.1).

For (2), we prove it by contradiction. Let \( \{ n_i \} \subset \mathbb{N} \) be an infinite subsequence such that \( \text{Fr}^{\text{can}}(x_n^o) \in \text{Fr}^{\text{can}}(\mathfrak{z}(K^0)) \) and \( x_{n_i}^o \notin \mathfrak{z}(K^0) \) for \( n_i \) large enough. In particular, \( (\text{Fr}^{\text{can}})^{-1}(\text{Fr}^{\text{can}}(3)) \neq \emptyset \). Thus by [13, Proposition 2.10], it is not hard to show that

\[
(\text{Fr}^{\text{can}})^{-1}(\text{Fr}^{\text{can}}(3)) = \mathfrak{z} \cup \mathfrak{z}'
\]

such that \( \mathfrak{z}' \) does not contain \( \mathfrak{z} \) and \( x_{n_i}^o \in \mathfrak{z}'(K^0) \). By Lemma 2.2.3, every \( x_n \) is contained in the \( \epsilon_n \)-neighborhood of \( \mathfrak{z}(K^0) \). Let \( \mathfrak{z}_1 \) be the union of irreducible components of \( \mathfrak{z}(K^0) \) which dominate
Spf$F^\circ$. By Lemma 2.2.4 (2), there exists $\delta \in K^\circ \setminus \{0\}$ such that every $x_{n_i}$ is contained in the $\epsilon_{n_i}/\delta$-neighborhood of $Z_1$. Since every irreducible component of $Z_1$ has dimension less than the dimension of $S$, by the induction hypothesis, we have $x_{n_i} \in Z_1(K^\circ) \subset Z(K^\circ)$. This is a contradiction. \hfill $\Box$

By (5.10) and Lemma 6.2.6, after passing to an infinite subsequence, we may assume that for every $n$, if $i \geq l$, then $a_{n_i}^{\ell} = 0$, i.e. we may replace $l$ by $l - 1$. Continue this process, we may assume that $l = 0$, i.e. $a_{n_i}^{\ell} = 0$ for every $n$ and $i$. Now Theorem 6.2.2 follows from Lemma 6.2.5.

6.2.3. Canonical liftings and perfectoid strategy. Now our remaining tasks are: proof of Lemma 6.2.3 and proof of Lemma 6.2.5. For Lemma 6.2.3, we prove an “almost effective” version of Theorem 6.2.2 for canonical liftings. In the proof, we use the ordinary perfectoid Siegel space and Scholze’s approximation lemma, following a strategy of Xie [34]. Our later proof of Lemma 6.2.5 involves a more complicated version of this proof (which in particular uses the global Serre-Tate coordinate).

Let $X$ be the restriction of $\mathcal{X}(0)^{\per}$ to the adic generic fiber of $\mathfrak{X}_{K^\circ}$. Then $X = \text{Spa}(R, R^\per)$ where $(R, R^\per)$ is a perfectoid affine $((K, K^\circ))$-algebra (there is no need to specify $R$ though it is easy to do so). The restriction of $\mathcal{X}(0)^{\per}$ to the adic generic fiber of $U_0 \otimes K^\circ$ is $X^\circ = \text{Spa}(R^\circ, R^\per^\circ)$, the tilt of $X$. More concretely, it is given as follows: let $S_0$ be the coordinate ring of $(R^\per^m)^{-1}(U_0)$ with the natural inclusion $S_{m-1} \to S_0$, and $S = \bigcup_m S_m$, then $R^\per$ is the $\mathfrak{w}^b$-adic completion of $S \otimes K^\circ$. Let $X_m$ be the adic generic fiber of $\text{Spec} S_m \otimes K^\circ$, and $\pi_m : X^\circ \to X_m$ the natural projection. Recall $\pi$ and $\pi'$ as defined in (5.1). Then $\pi_0 = \pi'|_{X^\circ}$ (which has image in $X_0$).

We abbreviate $\pi|_{X^\circ}$ as $\pi$ (which has image in the adic generic fiber of $\mathfrak{X}_{K^\circ}$). Let $\rho$ be the restriction of $\rho_{\mathcal{X}(0)^{\per}}$ (see (5.2)) to $X$.

For $f \in \mathcal{O}(\mathfrak{X})$ in the defining ideal of $3$, regard $f$ as an element of $R^\per$ by the inclusion $\mathcal{O}(\mathfrak{X}) \subset R^\per$. For $c \in \mathbb{Z}_{>0}$, choose $g$ as in Lemma 2.4.1 (w.r.t. $f$) and choose a finite sum

$$g_c = \sum_{i \in \mathbb{Z}_{\frac{1}{b} + c}, \frac{1}{b} + c < i} g_{c,i} \cdot (\mathfrak{w}^b)^i$$

as in Lemma 2.4.2 where $g_{c,i} \in S$ for all $i$. There exists a positive integer $m(c)$ such that $g_{c,i} \in S_{m(c)}$ for all $i$ by the finiteness of the sum. Let $G_c := g_c^{m(c)}$. Then we have the finite sum

$$(6.9) \quad G_c = \sum_{i \in \mathbb{Z}_{\frac{1}{b} + c}, \frac{1}{b} + c < i} G_{c,i} \cdot (\mathfrak{w}^b)^{m(c)i},$$

where $G_{c,i} = g_{c,i}^{m(c)}$. By the construction of $S_n$’s, we have $G_{c,i} \in S_0$. Let $I_c$ be the ideal of $S_0$ generated by $\{G_{c,i} : i \in \mathbb{Z}_{\frac{1}{b} + c}, i < \frac{1}{b} + c\}$. By the noetherianness of $S_0$, there exists a positive integer $M$ such that

$$(6.10) \quad \sum_{c=1}^{\infty} I_c = \sum_{c=1}^{M} I_c.$$

For $y \in \mathcal{X}(0)$ and $\tilde{y} \in \pi^{-1}(y) \subset X$, $|f(\tilde{y})| = ||f(y)||$. If $||f(y)|| \leq ||\mathfrak{w}||^{\frac{1}{b} + M}$, by (2.2) and (2.3), we have $|g_{c}(\pi_{m(c)}(\rho(\tilde{y})))| \leq ||\mathfrak{w}||^{\frac{1}{b} + c}$ for $c = 1, \ldots, M$. So for $c = 1, \ldots, M$, we have

$$(6.11) \quad |G_{c}(\pi_0(\rho(\tilde{y})))| = |G_{c}(\pi_{m(c)}(\rho(\tilde{y})))| \leq ||\mathfrak{w}||^{\frac{1}{b} + c})^{m(c)}.$$

Theorem 6.2.7. Assume that $\{f_1, \ldots, f_t\} \subset \mathcal{O}(\mathfrak{X})$ generates the ideal defining $3$. For each $f_j$, let $M_j$ be the $M$ as in (6.10) with $f$ replaced by $f_j$. Let $\mathbb{M} = \max\{M_j : j = 1, \ldots, t\}$. Let $y$ be a canonical lifting in the $\mathfrak{w}^{\frac{1}{b} + \mathbb{M}}$-neighborhood of $3$. Then $y \in \mathcal{Z}$. 


Proof. Apply Lemma 5.3.1 (2) to \( y \) with \( a = 1 \). Choose \( \tilde{y} \in \pi^{-1}(y) \) to be as in Lemma 5.3.1 (2). Then \( \pi_0(\rho(\tilde{y})) = \text{red}(\tilde{y}) \in U_0(k) \), where we understand \( U_0(k) \) as a subset of \( \mathfrak{X}(K^0,K^{\text{ad}}) \) naturally. Let \( f = f_j \) and \( M = M_j \) for some \( j \). Then \( |f(\tilde{y})| \leq \|\pi\|^\frac{1}{2}+M \) and thus we have (6.11). Similar to Lemma 2.4.3, by (6.11) and (6.9), we have \( G_{c,i}(\pi_0(\rho(\tilde{y}))) = 0 \) for every \( c = 1,...,M \) and corresponding \( i \)’s. By (6.10), \( L_c(\pi_0(\rho(\tilde{y}))) = \{0\} \) for every \( c \in \mathbb{Z}_{>0} \). So

\[
G_{c}(\pi_{m(c)}(\rho(\tilde{y}))) = G_{c}(\pi_0(\rho(\tilde{y}))) = 0
\]

for every \( c \in \mathbb{Z}_{>0} \). Thus \( g_c(\pi_{m(c)}(\rho(\tilde{y}))) = 0 \). By (2.2) and (2.3), \(|f(\tilde{y})| \leq \|\pi\|^\frac{1}{2}+c \) for every \( c \in \mathbb{Z}_{>0} \). Thus \(|f(\tilde{y})| = 0\). \( \square \)

Remark 6.2.8. The effectivity of \( M \) is essentially determined by the effectivity of the determination of the approximating function \( g \) in Lemma 2.4.1. However, Scholze’s proof of lemma 2.4.1 uses “almost ring theory” and is not effective. It is meaningful to ask if Lemma 2.4.1 can be made effective.

6.2.4. Toward the proof of Lemma 6.2.5. This paragraph closely mimics the proof of Theorem 6.2.7. Let notations be as above Theorem 6.2.7 and let \( y = x_n \). For every \( c = 1,...,M \) and a corresponding \( i \), we want to show that \( G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = 0 \). Then by (6.10), \( L_c(\pi_0(\rho(\tilde{x}_n))) = \{0\} \) for every \( c \in \mathbb{Z}_{>0} \). So \( G_c(\pi_{m(c)}(\rho(\tilde{x}_n))) = G_c(\pi_0(\rho(\tilde{x}_n))) = 0 \) for every \( c \in \mathbb{Z}_{>0} \). Thus \( g_c(\pi_{m(c)}(\rho(\tilde{x}_n))) = 0 \). By (2.2) and (2.3), \(|f(\tilde{x}_n)| \leq \|\pi\|^\frac{1}{2}+c \) for every \( c \in \mathbb{Z}_{>0} \). Thus \(|f(\tilde{x}_n)| = 0 \). Let \( f \) run over a finite set of generators of the defining ideal of \( \mathfrak{I} \) and choose infinite subsequences successively, we have \( x_n \in \mathfrak{I} \) for infinitely many \( n \)’s.

6.2.5. Spaces. For \( x \in U_0(k) \) (resp. \( U(k) \)), let \( D_x \) be the adic generic fiber of the formal completion of \( U_0 \otimes K^{\text{ad}} \) (resp. \( U \otimes K^{\text{ad}} \)) at \( x \). (This coincides with the definition in 5.3.) Equivalently, \( D_x \) is the adic generic fiber of the formal completion of \( \tilde{U}/x \otimes K^{\text{ad}} \) (resp. \( \tilde{U}/x ) \otimes K^{\text{ad}} \). The following two diagrams summarize the adic spaces/k-schemes and morphisms between them that we use:

\[
\begin{align*}
X & \xrightarrow{\rho} X^e \\
\pi & \downarrow \quad \pi_0 \\
X(0) & \xrightarrow{(1)} \prod_{x \in U_0(k)} D_x \quad \xrightarrow{(2)} \prod_{z \in U(k)} D_z \\
U_0 \xrightarrow{(1')} \prod_{x \in U_0(k)} \tilde{U}_0/x & \xrightarrow{(2')} \prod_{z \in U(k)} \tilde{U}/z \quad \xrightarrow{(6.6)} \prod_{z \in U(k)} \text{pr}^{-1}_1(\{z\}) \xrightarrow{(3)} \tilde{U} \times \tilde{U}/\Delta
\end{align*}
\]

Here the morphisms (1) (1’) (3) are the natural inclusions. And the morphism (2), when restricted to \( D_z, z \in U(k) \), is the natural isomorphism \( D_z \simeq D_x \) where \( x \in U_0(k) \) is the image of \( z \). We have the parallel statement for (2’).

6.2.6. Functions. Let \( H_{c,i} \) be the image of \( G_{c,i} \) in \( B \) under the morphism \( S_0 = \mathcal{O}(U_0) \to B = \mathcal{O}(U) \), and \( H_{c,i,z_n} = \mathcal{O}(\tilde{U}/z_n) \) the image of \( H_{c,i} \) under the morphism \( B = \mathcal{O}(U) \to \mathcal{O}(\tilde{U}/z_n) \).

For \( \tilde{x}_n \in \pi^{-1}(x_n) \subset X \), by Lemma 5.3.1 (1), \( \pi_0(\rho(\tilde{x}_n)) \in D_{\text{red}(x_n)} \). Let \( y_n \) be the preimage of \( \pi_0(\rho(\tilde{x}_n)) \) in \( D_{z_n} \) via the natural isomorphism \( D_{z_n} \simeq D_{\text{red}(x_n)} \). Then as elements in \( K^{\text{ad}} \), we have

\[
H_{c,i,z_n}(y_n) = H_{c,i}(y_n) = G_{c,i}(\pi_0(\rho(\tilde{x}_n))).
\]

Lemma 6.2.9. There is a constant \( h_{c,i} < 1 \) such that \( \|H_{c,i,z_n}(y_{n_m})\| < h_{c,i} \).

Proof: If the lemma is not true, let \( i_0 \) be the smallest \( i \) appearing in the finite sum (6.9) such that \( \|H_{c,i,z_n}(y_{n_m})\| \to 1 \) for a subsequence \( \{n_m\}_{m=1}^{\infty} \subset \mathbb{N} \). Then (6.9) implies that \( \|G_c(\pi_0(\rho(\tilde{x}_{n_m})))\| \to \|\pi\|^m_{0}m(c) \), which contradicts (6.11). \( \square \)
Let $\phi$ be the composition of

$$
\phi : B = \mathcal{O}(U) \to B \otimes B \to \mathcal{O} \left( \overline{U \times U} / \Delta \right) = B[[T_1^{ST}, \ldots, T_{g(g+1)/2}^{ST}]]
$$

where the first morphism is $b \mapsto 1 \otimes b$. I.e. $\phi$ gives the projection $pr_2 : \overline{U \times U} / \Delta$ to the second $U$. Tracking the second diagram of (6.12), we have the following lemma.

**Lemma 6.2.10.** The restriction of $\phi(H_{c,i})$ to $pr_1^{-1}(\{z_n\}) \overset{pr_2}{\to} \overline{U_{1/z_n}}$ in (6.6) is $H_{c,i,z_n} \in \mathcal{O}(\overline{U_{1/z_n}})$.

6.2.7. **Proof of Lemma 6.2.5.** We need some notations. For an open subset $O \subset \mathbb{Z}_p^{-1}$, let $\mathbb{N}(O) \subset \mathbb{N}$ be the subsequence such that the first $l-1$ components of a ratio of $x_n$ w.r.t. this basis (see Definition 5.2.2) is in $O$. If $l = 1$, we understand $\mathbb{N}(O)$ as the whole $\mathbb{N}$ (and we will not need the case $l = 0$). For $r \in \mathbb{Z}_p^{-1}$ and $\delta \in (0, 1)$, let $\mathbb{N}(r, \delta) = \mathbb{N}(O(r, \delta))$ where $O(r, \delta)$ is the $p$-adic closed disc centered at $r$ of radius $\delta$.

Now we start to prove Lemma 6.2.5. By the discussion in 6.2.4, we only need to prove that for every $n \in \mathbb{N}$, $G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = 0$. Let $SpecA \subset SpecB$ be the Zariski closure of the set $\{z_n : n \in \mathbb{N}\}$. Let $f$ be the image of $\phi(H_{c,i})$ under $B[[T_1^{ST}, \ldots, T_{g(g+1)/2}^{ST}]] \to A[[T_1^{ST}, \ldots, T_{g(g+1)/2}^{ST}]]$. By Lemma 6.2.10, we have

$$
(6.13) 
G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = H_{c,i,z_n}(y_n) = f(y_n).
$$

We prove the stronger result $f = 0$ by contradiction.

Assume that $f \neq 0$. We want to apply Proposition 6.1.1 to $f$ and $y_n$’s. We check the conditions in Proposition 6.1.1. First, by the compatibility between the Global and classical Serre-Tate coordinates as in the end of 6.2.1, we use Lemma 5.3.1 (2) to conclude that $y_n$’s are as in (6.1) above Proposition 6.1.1. Second, the assumption $(* )$ in Proposition 6.1.1 holds by Assumption 6.2.4. By the assumption that $a_n$ goes to $\infty$ as $n \to \infty$ in Lemma 6.2.5, Lemma 6.2.9 and the second “$=$” of (6.13), for $n$ large enough, $n$ satisfies the inequality in (6.2) of Proposition 6.1.1 (for every $D$). Then by Proposition 6.1.1, there exists $\delta_0 \in (0, 1)$ such that $\mathbb{N}(0, \delta_0)$ is finite. For a general $r \in \mathbb{Z}_p^{-1}$, by [27, Lemma 2.7], after an “upper triangular change of variables” (as defined above Lemma 6.2.5), we may use the same proof for $r = 0$ to conclude that there exists $\delta_r \in (0, 1)$ such that $\mathbb{N}(r, \delta_r)$ is finite. By its compactness, $\mathbb{Z}_p^{-1}$ is the union of $p$-adic closed discs centered at $r$ of radius $\delta_r$ for finitely many $r$’s. Then the infinite set $\mathbb{N}$ is the union of the finite sets $\mathbb{N}(r, \delta_r)$’s for these finitely many $r$’s. This is a contradiction.

**References**

[1] Bosch, Siegfried. Lectures on formal and rigid geometry. Vol. 2105. Berlin/Heidelberg/New York: Springer, 2014.

[2] Bosch, Siegfried, Ulrich Günther, and Reinhold Remmert. Non-Archimedean analysis, volume 261 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. (1984).

[3] Bosch, Siegfried, Werner Lütkebohmert, and Michel Raynaud. Néron models. Vol. 21. Springer Science & Business Media, 2012.

[4] Chai, Ching-Li. Families of ordinary abelian varieties: canonical coordinates, $p$-adic monodromy. Tate-linear subvarieties and Hecke orbits (2003).

[5] Habegger, Philipp. The Tate-Voloch Conjecture in a Power of a Modular Curve. International Mathematics Research Notices 2014.12 (2013): 3303-3339.

[6] Hida, Haruzo. p-adic automorphic forms on Shimura varieties. Springer Science & Business Media, 2012.

[7] Hong, Serin. On the Hodge-Newton filtration for $p$-divisible groups of Hodge type. Mathematische Zeitschrift (2016): 1-25.

[8] Huber, Roland. Continuous valuations. Math. Z 212.3 (1993): 455-477.

[9] Huber, Roland. Bewertungsspektrum und rigide Geometrie. Fakultät für Mathematik, 1993.

[10] Huber, Roland. A generalization of formal schemes and rigid analytic varieties. Mathematische Zeitschrift 217.1 (1994): 513-551.

[11] Huber, Roland. Étale Cohomology of Rigid Analytic Varieties and Adic Spaces. Aspects of Mathematics 30(1996).

[12] De Jong, Johan, and Rutger Noot. Jacobians with complex multiplication. Arithmetic algebraic geometry. Birkhäuser, Boston, MA, 1991. 177-192.

[13] Kappen, Christian. Néron models of formally finite type. International Mathematics Research Notices 2013.22 (2013): 5059-5147.
[14] Katz, Nicholas. Serre-Tate local moduli. Surfaces algébriques. Springer, Berlin, Heidelberg, 1981. 138-202.
[15] Oort, Frans. Canonical liftings and dense sets of CM-points. Arithmetic geometry (Cortona, 1994) 37 (1997): 228-234.
[16] Pila, Jonathan. Special point problems with elliptic modular surfaces. Mathematika 60.1 (2014): 1-31.
[17] Pila, Jonathan, and Umberto Zannier. Rational points in periodic analytic sets and the Manin-Mumford conjecture. Rendiconti Lincei-Matematica e Applicazioni 19.2 (2008): 149-162.
[18] Pilloni, Vincent, and Benoît Stroh. Cohomologie cohérente et représentations Galoisiennes. Annales mathématiques du Québec (2015): 1-36.
[19] Poonen, Bjorn. Multiples of subvarieties in algebraic groups over finite fields. International Mathematics Research Notices 2005.24 (2005): 1487-1498.
[20] Raynaud, Michel. Courbes sur une variété abélienne et points de torsion. Inventiones mathematicae 71.1 (1983): 207-233.
[21] Scanlon, Thomas. p-adic distance from torsion points of semi-Abelian varieties. Journal für die Reine und Angewandte Mathematik (1998): 225-225.
[22] Scanlon, Thomas. The conjecture of Tate and Voloch on p-adic proximity to torsion. International Mathematics Research Notices 1999.17 (1999): 909-914.
[23] Scholze, Peter. Perfectoid spaces. Publications mathématiques de l’IHÉS 116.1 (2012): 245-313.
[24] Scholze, Peter. On torsion in the cohomology of locally symmetric varieties. Annals of Mathematics (2) 182 (2015), no. 3, 945-1066.
[25] Scholze, Peter, and Jared Weinstein. Moduli of p-divisible groups, Cambridge Journal of Mathematics 1 (2013), 145-237.
[26] Serban, Vlad. An infinitesimal p-adic multiplicative Manin-Mumford Conjecture. Journal de Théorie des Nombres de Bordeaux 30.2 (2018): 393-408.
[27] Serre, Jean-Pierre, and John Tate. Good reduction of abelian varieties. Annals of Mathematics (1968): 492-517.
[28] Shankar, Ananth N., and Jacob Tsimerman. Unlikely intersections in finite characteristic. Forum of Mathematics, Sigma. Vol. 6. Cambridge University Press, 2018.
[29] Shankar, Ananth N., and Rong Zhou. Serre-Tate theory for Shimura varieties of PEL type. arXiv preprint arXiv:1612.06456 (2016).
[30] Shen, Xu. Perfectoid Shimura varieties of abelian type. International Mathematics Research Notices 2017.21 (2016): 6599-6653.
[31] Tate, John. Perfectoid Shimura varieties of abelian type. International Mathematics Research Notices 2017.21 (2016): 6599-6653.
[32] Tate, John, and José Felipe Voloch. Linear forms in p-adic roots of unity. International Mathematics Research Notices 1996.12 (1996): 589-601.
[33] Wedhorn, Torsten. Ordinariness in good reductions of Shimura varieties of PEL-type. Annales scientifiques de l’Ecole normale supérieure. Vol. 32. No. 5. Paris: Gauthiers-Villars [1864]-, 1999.
[34] Xie, Junyi. Algebraic dynamics of the lifts of Frobenius. Algebra & Number Theory 12.7 (2018): 1715-1748.