Separability conditions based on local fine-grained uncertainty relations

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Many protocols of quantum information processing use entangled states. Hence, separability criteria are of great importance. We propose new separability conditions for a bipartite finite-dimensional system. They are derived by using fine-grained uncertainty relations. Fine-grained uncertainty relations can be obtained by consideration of the spectral norms of certain positive matrices. One of possible approaches to separability conditions is connected with upper bounds on the sum of maximal probabilities. Separability conditions are often formulated for measurements that have a special structure. For instance, mutually unbiased bases and mutually unbiased measurements can be utilized for such purposes. Using resolution of the identity for each subsystem of a bipartite system, we construct some resolution of the identity in the product of Hilbert spaces. Separability conditions are then formulated in terms of maximal probabilities for a collection of specific outcomes. The presented conditions are compared with some previous formulations. Our results are exemplified with entangled states of a two-qutrit system.

Keywords: uncertainty principle, mutually unbiased bases, spectral norm, separable states

I. INTRODUCTION

The concept of entanglement plays a principal role in foundations and applications of quantum mechanics. Entangled states provide a tool for basic protocols of quantum information processing [1]. The quantum parallelism of Deutsch [2] cannot be implemented without the use of entangled states. Studies of quantum entanglement has a long history. An existence of purely quantum correlations was first emphasized in the Schrödinger “cat paradox” paper [3] and in the Einstein–Podolsky–Rosen paper [4]. A role of such correlations is brightly shown in specified experiments similar to Bohm’s version of the EPR experiment [5]. Historically, studies of these conceptual questions not only provide better understanding foundations of quantum physics [6]. Today, we see active development of technological applications of rather sophisticated quantum-mechanical effects.

Due to progress in quantum information processing, both the detection and quantification of entanglement are very important. Despite of many efforts, these problems are still the subject of active research [7, 8]. For entanglement detection, we may use conditions that are satisfied by all separable states. The violation of such conditions will be sufficient for detection. There exist several criteria of entanglement. Among them, the positive partial transpose (PPT) criterion [9] and the reduction criterion [10] are well known. Detection of entanglement beyond the PPT criterion can sometimes be realized with the realignment criterion [11] or the computable cross-norm (CCN) criterion [12]. The criteria mentioned above are formulated in terms of transformations of the given density matrix.

Some separability conditions are immediately connected with special measurement schemes. As was shown in a series of works [13–17], separability conditions can be based on uncertainty relations of various forms. The author of [18] pointed out connections of such criteria with the correlation matrix criterion. Recently, entanglement properties were studied in the context of classical correlations between outcomes of complementary measurements [19]. Since the Heisenberg famous paper [20] appeared, much many studies of uncertainty relations were accomplished [21]. Traditionally, uncertainty relations are formulated in terms of the standard deviations within the Robertson approach [22]. Entropic uncertainty relations are currently the subject of active research [23, 24]. Entropic formulations of the uncertainty principle allowed to overcome some doubts connected with the traditional approach [25, 26].

Uncertainty relations of the Landau–Pollak type differ from entropic ones. Although the original formulation of Landau and Pollak [27] is related to signal analysis, it can be treated quantum mechanically [28]. Entropic-like uncertainty relations based on this approach were considered in [29]. The Landau–Pollak uncertainty relation can be treated as an example of fine-grained uncertainty relations. Such relations have been proposed and motivated in [29]. Indeed, entropic bounds cannot distinguish the uncertainty inherent in obtaining a particular combination of the outcomes [29]. Fine-grained uncertainty relations for some special quantum measurements were derived in [30, 31]. Using the Landau–Pollak uncertainty relation, the authors of [32] examined separability conditions for a bipartite system of qubits. So, it is natural to ask for separability conditions based on fine-grained uncertainty relations.

Separability conditions can be derived for a scheme with local measurements of the special type. Using measurements with a complete set of mutually unbiased bases, the authors of [32] proposed the so-called correlation measure for entanglement detection. Similarly, the entanglement detection can be realized with a symmetric informationally complete measurement [33]. The correlation measure can also be introduced with mutually unbiased measurements [34, 35] and with a general symmetric informationally complete measurement [36]. The latter is based on the exact purity-based expression for the sum of squared probabilities [37]. Mutually unbiased measurements have been proposed as a
quantum system is described by the density matrix $\rho$. To each quantum system, the observer can adopt several complementary experiments. For instance, the spin-1/2 system was examined in [41]. It is possible to formulate unbiased measurements under weaker requirements. The authors of [38] introduced a construction of maximal numbers of MUBs [44]. It is based on properties of prime powers and weakly extension of usual symmetric informationally complete measurements [39].

In quantum theory, measurements play a key role in the sense that without them the mathematical formalism would have no physical meaning. A measurement stage is one of central questions in quantum information processing. Quantum measurements are generally described within the POVM formalism [42]. Let $\mathcal{M} = \{M_i\}$ be a set of elements $M_i \in \mathcal{L}(\mathcal{H})$ that satisfy the completeness relation

$$\sum_{i \in \Omega(\mathcal{M})} M_i = \mathbb{1}.$$  \hspace{1cm} (2.2)

Here, $\mathbb{1}$ is the identity operator on $\mathcal{H}$, and $\Omega(\mathcal{M})$ is the set of possible measurement outcomes. For the pre-measurement state $\rho$, the probability of $i$-th outcome is represented as

$$p_i(\mathcal{M} | \rho) = \text{Tr}(M_i \rho).$$  \hspace{1cm} (2.3)

For the given $\mathcal{M}$ and $\rho$, the maximal probability will be denoted by $p_{\text{max}}(\mathcal{M} | \rho)$. To estimate a state of the given quantum system, the observer can adopt several complementary experiments. For instance, the spin-1/2 system can be dealt with measurements of the three orthogonal components of spin [43]. Eigenbases of the three Pauli matrices give an example of a complete set of mutually unbiased bases.

Let $\mathcal{E} = \{|e_i\rangle\}$ and $\mathcal{F} = \{|f_j\rangle\}$ be two orthonormal bases in a $d$-dimensional Hilbert space $\mathcal{H}$. They are said to be mutually unbiased if and only if for all $i$ and $j$,

$$|\langle e_i | f_j \rangle|^2 = \frac{1}{d}. $$  \hspace{1cm} (2.4)

Several orthonormal bases form a set of mutually unbiased bases (MUBs), when each two of them are mutually unbiased. Mutually unbiased bases have found use in many questions of quantum information theory (see [44] and references therein). The states within MUBs are indistinguishable in the following sense. If the two observables have unbiased eigenbases, then the measurement of one observable reveals no information about possible outcomes of the measurement of other. This property is utilized in some schemes of quantum key distribution. When $d$ is a prime power, we certainly have a construction of $d + 1$ MUBs [44]. It is based on properties of prime powers and corresponding finite fields [45, 46]. In general, however, the maximal number of MUBs in $d$ dimensions is still an open question [44]. Approaches to MUBs beyond prime power dimensionalities are considered in [47, 48].

It is possible to formulate unbiased measurements under weaker requirements. The authors of [38] introduced mutually unbiased measurements (MUMs). They show that a complete set of $d + 1$ measurements exists for all $d$. Let
us consider two POVM measurements \( \mathcal{P} = \{P_i\} \) and \( \mathcal{Q} = \{Q_j\} \). Each of them contains \( d \) elements such that

\[
\text{Tr}(P_i) = \text{Tr}(Q_j) = 1 ,
\]

\[
\text{Tr}(P_iQ_j) = \frac{1}{d} .
\]

The POVM elements are all of trace one, but now not of rank one. The formula (2.6) replaces the square of (2.4). Further, two different elements from the same POVM obey

\[
\text{Tr}(P_iP_j) = \delta_{ij} \kappa + (1 - \delta_{ij}) \frac{1 - \kappa}{d - 1} ,
\]

where \( \kappa \) is the efficiency parameter [38]. The same condition is imposed on the elements of \( \mathcal{Q} \). The efficiency parameter characterizes a closeness of the POVM elements to rank-one projectors [38]. This parameter satisfies [38]

\[
\frac{1}{d} < \kappa \leq 1 .
\]

The case \( \kappa = 1/d \) should be excluded, as it gives \( P_i = 1/d \) for all \( i \). The value \( \kappa = 1 \), if possible, leads to the standard case of mutually unbiased bases. We can only say that the maximal efficiency is reached for prime power \( d \). More precise bounds on \( \kappa \) depend on an explicit construction of POVM elements [38]. The Brukner–Zeilinger approach to quantum information can be realized with MUMs instead of MUBs [49].

We will study separability conditions that can be derived from fine-grained uncertainty relations. In this regard, one result of the paper [50] should be recalled. In our notation, it is written as follows. Let \( \{\mathcal{M}^{(1)}, \ldots, \mathcal{M}^{(N)}\} \) be a set of \( N \) POVMs, and let some index \( i(t) \in \Omega(\mathcal{M}^{(t)}) \) be assigned to each \( t = 1, \ldots, N \). For arbitrary \( \rho \), we have [50]

\[
\sum_{t=1}^{N} p_{i(t)}(\mathcal{M}^{(t)} | \rho) \leq 1 + \left( \sum_{s \neq t} \left\| \sqrt{\mathcal{M}^{(s)}} - \sqrt{\mathcal{M}^{(t)}} \right\|_\infty^2 \right)^{1/2} .
\]

This upper bound generalizes a weak version of the Landau–Pollak uncertainty relation to the case of more than two measurements. Recall that Landau and Pollak prove their uncertainty relation in the context of signal analysis. Reformulation in quantum-mechanical terms was mentioned in [26]. The authors of [15] discussed separability conditions based on a weak version of the Landau–Pollak relation. In general, the use of (2.9) is not very immediate since the sum in its right-hand side demands additional calculation or estimation. For a set of MUBs, however, this sum is exactly calculated.

We now consider a bipartite system of two \( d \)-dimensional subsystems. Its Hilbert space is the product \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) of two spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Let us choose the orthonormal basis \( \{|i_S\} \) where \( S = A, B \), for each of the two spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \). A maximally entangled pure state is then expressed as

\[
|\Phi_{AB}^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i_A\rangle \otimes |i_B\rangle .
\]

Entangled states are a basic resource in quantum information. Hence, the problem of efficient detection of entanglement is of great importance [8]. Let us recall basic notions related to separability. A product state is any state of the form \( \rho_A \otimes \rho_B \) [31]. When both the matrices \( \rho_A \) and \( \rho_B \) are rank-one, we have a pure product state. A bipartite mixed state is called separable, when its density matrix \( \rho_{AB} \) can be represented as a convex combination of product states [52]. That is, there exist a probability distribution \( \{q(n)\} \) and two sets \( \{\rho_A(n)\} \) and \( \{\rho_B(n)\} \) such that

\[
\rho_{AB} = \sum_n q(n) \rho_A(n) \otimes \rho_B(n) .
\]

Note that each separable state can also be expressed as a convex combination of only pure product states. This fact easily follows from (2.11) by substitution of the corresponding spectral decompositions. When representations of the form (2.11) are not possible, the state is called entangled [52].

### III. SEPARABILITY CONDITIONS

In this section, we derive separability conditions from local fine-grained uncertainty relations. Criteria based on local uncertainty relations can be motivated as follows. Such criteria are sometimes able to detect entanglement of
states that escape detection by both the CCN and PPT criteria. To realize entanglement detection, we will use MUBs and MUMs. Fine-grained uncertainty relations for them were derived in \[31\]. Separability conditions will be formulated in terms of maximal probabilities for POVMs on a total system. For a bipartite system of qubits, the authors of \[13\] gave separability conditions in terms of maximal probabilities for observables. Their result is based on lemma \(1\) of the paper \[13\]. Before presenting separability conditions, we shall formulate a similar statement about maximal probabilities in measurements. This statement holds under an additional condition, which we impose on a total POVM built of local ones. Let us begin with the corresponding definition. We give it for a bipartite system, since an extension to the multipartite case is obvious.

**Definition 1** Let \( \mathcal{P} = \{P_i\} \) with \( i \in \Omega(\mathcal{P}) \) be a POVM in \( \mathcal{H}_A \), and let \( \mathcal{Q} = \{Q_j\} \) with \( j \in \Omega(\mathcal{Q}) \) be a POVM in \( \mathcal{H}_B \). Let subsets \( \varpi(k) \) form a partition of the Cartesian product \( \Omega(\mathcal{P}) \times \Omega(\mathcal{Q}) \) with the following property. For each subset, two different pairs \((i_1,j_1) \in \varpi(k) \) and \((i_2,j_2) \in \varpi(k) \) do not intersect, namely \( i_1 \neq i_2 \) and \( j_1 \neq j_2 \). We say that a POVM \( \mathcal{M} = \{M_k\} \) in \( \mathcal{H}_A \otimes \mathcal{H}_B \) with \( k \in \Omega(\mathcal{M}) \) is built of \( \mathcal{P} \) and \( \mathcal{Q} \) according to the family \( \{\varpi(k)\} \), in signs \( \mathcal{M}(\mathcal{P}, \mathcal{Q}) \), when

\[
M_k = \sum_{(i,j) \in \varpi(k)} P_i \otimes Q_j. \tag{3.1}
\]

Note that a particular local value \( i \in \Omega(\mathcal{P}) \) \( (j \in \Omega(\mathcal{Q})) \) cannot appear twice or more in the same subset of ordered pairs. We will also use this definition with two orthonormal bases in \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. Then the right-hand side of (3.1) involves the corresponding rank-one projectors. Using probabilities, we will focus on POVMs rather than on observables. The following statement takes place.

**Proposition 2** Let a POVM \( \mathcal{M}(\mathcal{P}, \mathcal{Q}) \) in \( \mathcal{H}_A \otimes \mathcal{H}_B \) be built of local POVMs \( \mathcal{P} \) and \( \mathcal{Q} \) according to Definition 1. For all product states, we then have

\[
p_{\text{max}}(\mathcal{M}(\mathcal{P}, \mathcal{Q})|\rho_A \otimes \rho_B) \leq p_{\text{max}}(\mathcal{P}|\rho_A), \tag{3.2}
\]
\[
p_{\text{max}}(\mathcal{M}(\mathcal{P}, \mathcal{Q})|\rho_A \otimes \rho_B) \leq p_{\text{max}}(\mathcal{Q}|\rho_B). \tag{3.3}
\]

For each separable state \( (2.7) \), the quantity \( p_{\text{max}}(\mathcal{M}(\mathcal{P}, \mathcal{Q})|\rho_{AB}) \) is no greater than

\[
\max_{\rho_A(n)} p_{\text{max}}(\mathcal{P}|\rho_A(n)), \quad \max_{\rho_B(n)} p_{\text{max}}(\mathcal{Q}|\rho_B(n)). \tag{3.4}
\]

**Proof.** Since \( p_i(\mathcal{P}|\rho_A) \leq p_{\text{max}}(\mathcal{P}|\rho_A) \) for all \( i \in \Omega(\mathcal{P}) \), we write

\[
p_k(\mathcal{M}(\mathcal{P}, \mathcal{Q})|\rho_A \otimes \rho_B) = \sum_{(i,j) \in \varpi(k)} p_i(\mathcal{P}|\rho_A) p_j(\mathcal{Q}|\rho_B) \leq p_{\text{max}}(\mathcal{P}|\rho_A) \sum_{j \in \Omega(\mathcal{Q})} p_j(\mathcal{Q}|\rho_B). \tag{3.5}
\]

By \( \theta(k) \), we mean here the set of all those \( j \in \Omega(\mathcal{Q}) \) that appear in ordered pairs of \( \varpi(k) \). As any \( j \) never appears twice or more, the normalization condition completes the proof of (3.2). By a parallel argument, we get (3.3). Finally, the claim (3.4) follows from (3.2) and the linearity of the trace. \( \square \)

The statement of Proposition 2 could be compared with lemma \(1\) of the paper \[13\]. For a pair of local observables with non-zero eigenvalues, there exists some majorization relation between probability distributions \[13\]. We refrain from presenting the details here, since the results \(3.2\) and \(3.3\) are quite sufficient for our purposes. The formulas \(3.2\) and \(3.3\) can be violated, when subsets \( \varpi(k) \) contain intersecting pairs. We exemplify the claim with a system of two qubits, both in the completely mixed state. Measuring each of qubits in some basis, say \( \{|0\rangle, |1\rangle\} \), we then obtain the uniform distribution with two outcomes. For any local measurement, the maximum probability is equal to \(1/2\). Let us build a total POVM with respect to the two subsets \( \{(0,0), (1,0), (1,1)\} \) and \( \{(0,1)\} \). As the former contains intersecting pairs, this measurement on the two-qubit system does not share Definition 1. For the first subset, we have the probability \(3/4 > 1/2\). Thus, the condition of Definition 1 should be kept in constructing POVMs of local measurements. In principle, each of the formulas \(3.2\) and \(3.3\) taken separately holds under weaker conditions. Say, the former can be derived, when different pairs of the same subset are allowed to intersect in their first entry but not in the second. In the following, such measurements are not used.

For entangled states, the statement of Proposition 2 does not hold in general. Let us consider a system of two qubits in the pure state

\[
|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{3.6}
\]
We further take the observable \( \sigma_z \otimes \sigma_z \), where \( \sigma_z \) is diagonal in the basis \( \{ |0\rangle, |1\rangle \} \). In the notation of Definition 1, the measurement is based on the subsets \( \{ (0,0), (1,1) \} \) and \( \{ (0,1), (1,0) \} \). As the state (3.9) is an eigenstate of \( \sigma_z \otimes \sigma_z \), the maximal probability is equal to 1. Further, the state of each of two qubits is completely mixed. Measuring \( \sigma_z \) on a single qubit will then result in the uniform distribution with the maximal probability \( 1/2 \). The latter is less than for the combined system.

Together with Proposition 2, we will also use another property of the maximal probability. For any POVM \( \mathcal{M} \) and a convex combination of density matrices \( \rho \) and \( \varrho \), we have

\[
 p_{\text{max}}(\mathcal{M} | \lambda \rho + (1 - \lambda) \varrho) \leq \lambda p_{\text{max}}(\mathcal{M} | \rho) + (1 - \lambda) p_{\text{max}}(\mathcal{M} | \varrho),
\]

where \( \lambda \in [0;1] \). This property immediately follows from (2.3) and the linearity of the trace. Our first collection of separability conditions is posed as follows.

**Proposition 3** Let \( \{ \mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(N)} \} \) be a set of \( N \) MUBs in the space \( \mathcal{H}_A \), and let \( \{ \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(N)} \} \) be a set of \( N \) MUBs in the space \( \mathcal{H}_B \). Let \( N \) POVMs \( \mathcal{M}^{(t)}(\mathcal{E}^{(t)}, \mathcal{F}^{(t)}) \) in \( \mathcal{H}_A \otimes \mathcal{H}_B \) be constructed from these MUBs according to Definition 1. If the density matrix \( \rho_{AB} \) is separable, then

\[
 \sum_{t=1}^{N} p_{\text{max}}(\mathcal{M}^{(t)}(\mathcal{E}^{(t)}, \mathcal{F}^{(t)}) | \rho_{AB}) \leq \frac{N}{d_S} \left( 1 + \frac{d_S - 1}{\sqrt{N}} \right),
\]

where \( S = A, B \).

**Proof.** The following statement was proved in [31]. Let \( \{ \mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(N)} \} \) be a set of MUBs in \( d_A \)-dimensional Hilbert space \( \mathcal{H}_A \). Let some index \( i(t) \in \Omega(\mathcal{E}^{(t)}) \) be assigned to each \( t = 1, \ldots, N \). We then have

\[
 \sum_{t=1}^{N} p(i(t)) | \mathcal{E}^{(t)} | \rho_A \leq \frac{N}{d_A} \left( 1 + \frac{d_A - 1}{\sqrt{N}} \right).
\]

It is important here that the inequality (3.9) holds for arbitrary choice of indices \( i(t) \). Hence, we can replace \( p(i(t)) | \mathcal{E}^{(t)} | \rho_A \) with \( p_{\text{max}}(\mathcal{E}^{(t)} | \rho_A) \) for all \( t = 1, \ldots, N \). For all density matrices of the form (2.11), we write

\[
 \sum_{t=1}^{N} p_{\text{max}}(\mathcal{M}^{(t)}(\mathcal{E}^{(t)}, \mathcal{F}^{(t)}) | \rho_{AB}) \leq \sum_{n} q(n) \sum_{t=1}^{N} p_{\text{max}}(\mathcal{M}^{(t)}(\mathcal{E}^{(t)}, \mathcal{F}^{(t)}) | \rho_A(n) \otimes \rho_B(n)) \leq \sum_{n} q(n) \sum_{t=1}^{N} p_{\text{max}}(\mathcal{E}^{(t)} | \rho_A(n)) .
\]

Here, the step (3.10) follows from (3.7), the step (3.11) follows from (3.2). Combining (3.11) with (3.9) and the normalization condition \( \sum_n q(n) = 1 \), we obtain (3.8) for \( S = A \). Similarly, we complete the proof for \( S = B \).

The statement of Proposition 3 provides necessary conditions of separability. In principle, this result can be used for bipartite systems combined of two quantum systems of different dimensionality. It leads to a family of schemes of entanglement detection with the use of mutually unbiased bases. Another approach follows from the uncertainty relation (2.3). For a set of MUBs, the sum in the right-hand side of (2.3) is easily calculated. Indeed, the square root of any projector is projector again, so that each summand becomes equal to 1/d. Using (2.3) instead of (3.8) at the step (3.11) finally gives

\[
 \sum_{t=1}^{N} p_{\text{max}}(\mathcal{M}^{(t)}(\mathcal{E}^{(t)}, \mathcal{F}^{(t)}) | \rho_{AB}) \leq \frac{N}{d_S} \left( 1 + \sqrt{N^2 - N} \frac{1}{d_S} \right),
\]

where \( S = A, B \) and \( \rho_{AB} \) is separable. In general, the condition (3.12) seems to be weaker than (3.8). On the other hand, the condition (3.12) may sometimes lead to better restriction. It will be exemplified below. Thus, both the results (3.8) and (3.12) are of interest. In each concrete case, we should choose more restrictive condition.

The violation of any of the relations (3.8) and (3.12) implies that the measured state is entangled. To increase an efficiency of detection, we try to use as many different MUBs as possible. Various constructions of POVMs
\( \mathcal{M}(t)(\mathcal{E}^{(i)}, \mathcal{F}^{(i)}) \) in \( \mathcal{H}_A \otimes \mathcal{H}_B \) may be considered. A general scheme for a bipartite system of two \( d \)-dimensional subsystems will be described in the next section. In the case of two subsystems of the same dimensionality, we will omit subscript \( S = A, B \) in the conditions (3.8) and (3.12).

The authors of [15] obtained separability conditions for a bipartite system of qubits on the base of the Landau–Pollak uncertainty relation. Their conditions are formulated in terms of a tensor product of the Pauli observables. Substituting \( N = 2 \) and \( d = 2 \) into (3.8), for each separable \( \rho_{AB} \) we have

\[
\sum_{t=1}^{2} p_{\text{max}}(\mathcal{M}(t)|\rho_{AB}) \leq 1 + \frac{1}{\sqrt{2}} \approx 1.707 .
\]

(3.13)

The right-hand side of (3.13) also follows from the Landau–Pollak uncertainty relation. Thus, the result (3.8) includes one of the separability conditions of [15] as a particular case. Our approach differs in the following respects. First, it holds for all those POVMs that can be built of local MUBs with respect to Definition 1. Second, it is not formulated in terms of observables. Note also that the bound (3.8) is not tightest [15, 53].

When \( N = 3 \) and \( d = 2 \), for separable \( \rho_{AB} \) the formula (3.8) gives

\[
\sum_{t=1}^{3} p_{\text{max}}(\mathcal{M}(t)|\rho_{AB}) \leq \frac{3}{2} \left( 1 + \frac{1}{\sqrt{3}} \right) \approx 2.366 .
\]

(3.14)

Of course, for particular choices of concrete POVM elements the general bound (3.14) may be improved. Indeed, even if we obtain some tight bound on the sum of maximal probabilities for local measurements, we still do not have a tight bound for the sum of maximal probabilities for the total system. Particularly, a direct maximization could be used in simple cases. The authors of [12] provided an example of three observables on a qubit pair, when the sum of three probabilities is bounded from above by 2. The latter was obtained by performing a direct maximization of the sum of probabilities in product states [15]. For complementary qubit observables, a similar approach was used in studying entropic uncertainty and certainty relations [54, 55]. This direction was originally initiated in the papers [50, 57]. On the other hand, a direct optimization becomes hardly appropriate with growth of the dimensionality. The bound (3.14) is a particular case of the result (3.8), which has been proved for arbitrary finite \( d \). It is rather natural that bounds of the form (3.8) are not tight.

For prime power \( d \), we can build a set of \( d + 1 \) MUBs for each subsystem. This case is apparently of most practical interest, since in practice we will rather deal with systems of qubits or qutrits. The condition (3.8) then gives

\[
\sum_{t=1}^{d+1} p_{\text{max}}(\mathcal{M}(t)(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})|\rho_{AB}) \leq \frac{d+1}{d} \left( 1 + \frac{d-1}{\sqrt{d+1}} \right) ,
\]

(3.15)

where \( \rho_{AB} \) is separable. In the same case, the condition (3.12) reads

\[
\sum_{t=1}^{d+1} p_{\text{max}}(\mathcal{M}(t)(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})|\rho_{AB}) \leq 1 + \sqrt{d+1} .
\]

(3.16)

In this especially important case, the condition (3.8) is better than (3.12). Indeed, for all \( d \geq 2 \) the right-hand side of (3.15) is strictly less than the right-hand side of (3.16). For \( d = 2 \), the former is approximately 2.366, whereas the latter is approximately 2.732. A relative difference is more than 15 %. This difference decreases with growth of \( d \). For sufficiently large \( d \), the conditions (3.15) and (3.16) are almost the same. For two MUBs in the qubit case, when \( N = 2 \) and \( d = 2 \), we respectively obtain \( 1 + 1/\sqrt{2} \approx 1.707 \) from (3.8) and \( 2 \) from (3.12). A relative difference is more than 17 %. For \( N = 2 \) and sufficiently small \( d \), the right-hand side of (3.8) is also less than the right-hand side of (3.12). With growth of \( d \) at \( N = 2 \), however, we observe some domain in which the condition (3.12) is stronger. Here, a relative difference is firstly small, say, about 2 % for \( d = 10 \). This fact exemplifies that the conditions (3.8) and (3.12) are both of interest. At the same time, the result (3.8) is more relevant with respect to the case of most practical interest.

Except for prime power \( d \), the maximal number of MUBs in \( d \) dimensions is still an open question [44]. Today, the answer is unknown even for \( d = 6 \). In this sense, alternate approaches are of interest. The authors of [38] introduced the concept of mutually unbiased measurements. Elements of such measurements are not of rank one. This approach with lesser measurement efficiency is easy to construct. A complete set of \( d + 1 \) MUBs can be given explicitly for arbitrary finite \( d \) [38]. In the paper [31], we derived fine-grained uncertainty relations for a set of MUMs. These relations lead to a collection of separability conditions.
Proposition 4 Let \( \{P^{(1)}, \ldots, P^{(N)}\} \) be a set of \( N \) MUMs of the efficiency \( \kappa_A \) in \( H_A \), and let \( \{Q^{(1)}, \ldots, Q^{(N)}\} \) be a set of \( N \) MUMs of the efficiency \( \kappa_B \) in \( H_B \). Let \( N \) POVMs \( M^{(t)}(P^{(t)}, Q^{(t)}) \) in \( H_A \otimes H_B \) be constructed from these MUMs according to Definition 1. If the density matrix \( \rho_{AB} \) is separable, then

\[
\sum_{t=1}^{N} p_{\text{max}}(M^{(t)}(P^{(t)}, Q^{(t)})|\rho_{AB}) \leq \frac{N}{d_S} \left( 1 + \sqrt{\frac{(d_S-1)(\kappa_S d_S - 1)}{N}} \right),
\]

where \( S = A, B \).

**Proof.** The following statement was proved in [31]. Let \( \{P^{(1)}, \ldots, P^{(N)}\} \) be a set of MUMs of the efficiency \( \kappa_A \) in \( d_A \)-dimensional Hilbert space \( H_A \). Let some index \( i(t) \in \Omega(P^{(t)}) \) be assigned to each \( t = 1, \ldots, N \). We then have

\[
\sum_{t=1}^{N} p_{i(t)}(P^{(t)}|\rho_A) \leq \frac{N}{d_A} \left( 1 + \sqrt{\frac{(d_A-1)(\kappa_A d_A - 1)}{N}} \right).
\]

This inequality also holds for arbitrary choice of indices \( i(t) \). Remaining steps are written similarly to the proof of Proposition 3. \( \blacksquare \)

The statement of Proposition 4 generalizes (3.8) to the case of mutually unbiased measurements. In the paper [31], fine-grained uncertainty relations for MUMs were based only on the formulas (2.5) and (2.6). Hence, the separability condition (3.17) holds irrespectively to the explicit construction for MUMs given in [38]. Thus, entanglement detection can be realized with mutually unbiased measurements, at least in principle. We may use a complete set of \( d + 1 \) of MUMs for those \( d \), for which \( d + 1 \) MUBs are not available. For the efficiency \( \kappa_S = 1 \), the condition (3.17) is reduced to (3.8). We may also ask for a similar extension of (3.12). In the case of MUMs, the sum in the right-hand side of (2.9) is difficult to evaluate. At present, we can give only the condition (3.17) based on the results of [31].

Finally, we shortly mention applications of the separability conditions (3.8), (3.12), and (3.17) to multipartite systems. Various approaches to study multipartite entanglement are considered in [58–62]. In general, the problem becomes more complicated [58]. It can be illustrated with the case of tripartite systems. Here, we should distinguish between fully separable states and biseparable states [63]. Fully separable states are mixtures of product states of the form \( \rho_A \otimes \rho_B \otimes \rho_C \). Biseparable states are mixtures of the form \( \rho_A \otimes \rho_{BC} \), in which \( \rho_{BC} \) is not separable. For multipartite system, Definition 1 should be reformulated accordingly. Then the conditions of this section can be treated as biseparability conditions. For a multipartite system, we will group the subsystems into two larger groups. Further, the described schemes can be used. In principle, this issue may be a theme of separate investigation.

IV. APPLICATIONS TO SOME STATES OF A TWO-QUTRIT SYSTEM

In this section, we apply some of the above separability conditions to states whose separability limits are already known. As separability of qubit systems are well studied in the literature, we will focus on systems of qutrits. Such system are also of great interest since the qutrit can be implemented by a biphoton [63].

Bipartite separability conditions are typically illustrated with density matrices of the form

\[
(1-s)\rho_{\text{sep}} + s |\Psi\rangle \langle \Psi|.
\]

Here, the density matrix \( \rho_{\text{sep}} \) is separable, \( |\Psi\rangle \) is a maximally entangled state, and \( s \in [0,1] \). Taking \( \rho_{\text{sep}} \) to be the completely mixed state, the form (4.1) leads to the bipartite case of Werner states [64]. That is, we consider mixtures of the completely mixed state and a maximally entangled pure state. A bipartite Werner state is separable if and only if [65]

\[
s \leq \frac{1}{d + 1}.
\]

The authors of [65] also gave a necessary and sufficient condition for multipartite Werner states. For a bipartite system of two qutrits, the inequality (4.2) reads \( s \leq 1/4 \). We will exemplify separability conditions of the previous section by applying them to states (4.1) of a system of two qutrits.

We will use the result (3.8) formulated for the scheme with MUBs. Let us describe briefly our construction for arbitrary \( d \). In any base, the number index of its elements runs integers from 0 up to \( d - 1 \). To each basis \( \mathcal{E}^{(t)} \), we assign the operator \( A^{(t)} \). It is taken as diagonal in that basis and represented as

\[
A^{(t)} = \text{diag}(1, \omega, \ldots, \omega^{d-1}),
\]
where $\omega = \exp(\pm 2\pi i/d)$ is a primitive root of the unit. That is, each $A^{(t)}$ is taken in own eigenbasis as the corresponding
generalized Pauli operator. Similarly, we assign the operator $B^{(t)}$ to each basis $F^{(t)}$ so that it is described by
the right-hand side of (4.3) in this basis. The spectrum of $A^{(t)} \otimes B^{(t)}$ also contains $d$ powers of $\omega$. For all $t = 1, \ldots, N$, we write

$$A^{(t)} \otimes B^{(t)} = \sum_{k=0}^{d-1} \omega^k A_k^{(t)}.$$  \hspace{1cm} (4.4)

In terms of vectors of the bases $E^{(t)}$ and $F^{(t)}$, one gives

$$A_k^{(t)} = \sum_{i=0}^{d-1} |e_i^{(t)} \rangle \langle e_i^{(t)}| \otimes |f_{k+i}^{(t)} \rangle \langle f_{k+i}^{(t)}|,$$  \hspace{1cm} (4.5)

where the sign “$\ominus$” denotes the subtraction in $\mathbb{Z}/d$. In the described scheme, we finally have $M^{(t)} = \{A_k^{(t)}\}$. For $d = 2$, the operators (4.3) and (4.4) are Hermitian. It is not the case for $d > 2$. In effect, the operators (4.3) and (4.4) are only auxiliary in order to get the projection operators (4.5). In a certain sense, the above scheme is an extension
of the case of two qubits considered in [15].

In the case of qubit system, we have $\omega = \exp(\pm 2\pi i/3)$. Four MUBs in the Hilbert space of qubit can be written as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega \\ 1 \end{pmatrix}, \begin{pmatrix} \omega^2 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (4.6)

Here, $\omega^* = \omega^2$ is the complex conjugation of $\omega$. When one of MUBs is taken as the standard base, other MUBs can be described in terms of complex Hadamard matrices. This observation was used for classification of MUBs in low dimensions [60]. The bases (4.6) are respectively the eigenbases of the generalized Pauli operators

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.8)

The bases (4.7) are respectively the eigenbases of the operators $ZX$ and $ZX^2$. In each base, the kets are arranged according to the order of eigenvalues $1, \omega, \omega^*$. It is easy to see that $ZX = \omega^*XZ$. The bases (4.6)–(4.7) can be used in studying higher-dimensional quantum key distribution protocols [67,68].

Detecting entanglement of states of the form (4.1) generally depends on the choice of $|\Psi\rangle$ with respect to the taken
measurement projectors. For a system of two qutrits, we consider four projective measurements corresponding to the normal operators

$$Z \otimes X, \quad X \otimes Z, \quad ZX \otimes ZX, \quad ZX^2 \otimes ZX^2.$$  \hspace{1cm} (4.9)

The first three operators are mutually commuting, as we see from $ZX = \omega XZ$. Formulating separability conditions in terms of maximal probabilities, these probabilities should be sufficiently larger. For example, such conditions are useful, when $|\Psi\rangle$ is a common eigenstates of the three commuting operators. We further take a maximally entangled state as

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \left( |z_0x_0\rangle + |z_1x_2\rangle + |z_2x_1\rangle \right).$$  \hspace{1cm} (4.10)

By $|z_i\rangle$ and $|x_j\rangle$, we respectively mean the eigenstates of $Z$ and $X$ so that the subscript shows the power of $\omega$. Constructing the projection operators according to the formulas (4.3)–(4.5) gives the following. To the operator $Z \otimes X$, we assign the three projectors

$$A_0^{(1)} = |z_0x_0\rangle \langle z_0x_0| + |z_1x_2\rangle \langle z_1x_2| + |z_2x_1\rangle \langle z_2x_1|,$$  \hspace{1cm} (4.11)

$$A_1^{(1)} = |z_0x_1\rangle \langle z_0x_1| + |z_1x_0\rangle \langle z_1x_0| + |z_2x_2\rangle \langle z_2x_2|,$$  \hspace{1cm} (4.12)

$$A_2^{(1)} = |z_0x_2\rangle \langle z_0x_2| + |z_1x_1\rangle \langle z_1x_1| + |z_2x_0\rangle \langle z_2x_0|.$$  \hspace{1cm} (4.13)
They project a bipartite state on the eigenspaces, which correspond to the eigenvalues $1, \omega, \omega^*$. For other three bases, projectors on the product space are constructed in a similar way. They can directly be obtained from the spectral decomposition of the operators $X \otimes Z$, $ZX \otimes ZX$, and $ZX^2 \otimes ZX^2$. We will denote the outcomes as the eigenvalues, though these operators are not Hermitian. Recall that generalized Pauli operators are used for convenience of describing the scheme (4.3)–(4.5).

Let us consider probability distributions for the pre-measurement state $|\Psi\rangle$. In three of the all four cases, the state (4.10) is an eigenstate, namely

\[
(Z \otimes X)|\Psi\rangle = |\Psi\rangle, \quad (X \otimes Z)|\Psi\rangle = |\Psi\rangle, \quad (ZX \otimes ZX)|\Psi\rangle = \omega|\Psi\rangle. \tag{4.14}
\]

In the three cases, we therefore obtain a deterministic probability distribution. For the operator $ZX^2 \otimes ZX^2$, we have the uniform distribution with three outcomes. This fact is easily derived immediately.

Let us take $\rho_{sep}$ to be the completely mixed state. In each of the four cases, the three projectors has the same trace equal to 3. So, the completely mixed state always leads to the uniform distribution for the three outcomes denoted by $1, \omega, \omega^*$. Using these facts, we find the desired maximal probabilities for the state (4.1). The result (3.8) gives a collection of separability conditions. Since the right-hand side of (3.8) is equal to 1 for $N = 1$, several measurements should be involved. To study entanglement of the given state, we will rather use as more complementary measurements.

Hence, we have arrived at a conclusion. The separability condition (4.16) detects entanglement when

\[
s > \frac{1}{\sqrt{3}} \cos \frac{\pi}{18} \approx 0.569. \tag{4.17}
\]

Similarly, we can consider other forms of $\rho_{sep}$, say, the matrix

\[
\frac{1}{3} \left( |z_0z_0\rangle\langle z_0z_0| + |z_1z_1\rangle\langle z_1z_1| + |z_2z_2\rangle\langle z_2z_2| \right). \tag{4.18}
\]

For all the four measurements, this density matrix also leads to the uniform distribution with three outcomes. Thus, the separability condition (4.16) again detects entanglement when the parameter $s$ obeys (4.17).

To reach more efficient criteria, we should perform a direct maximization of the sum of probabilities in product states of a pair of qutrit. This procedure is not obvious and rather difficult. As we already mentioned, detection of entanglement of the states (4.11) within the considered scheme depends on the choice of $|\Psi\rangle$. Let us consider the state (4.10), in which (4.10) is replaced with

\[
|\Phi\rangle = \frac{1}{\sqrt{3}} \left( |z_0z_0\rangle + |z_1z_1\rangle + |z_2z_2\rangle \right). \tag{4.19}
\]

This ket is not an eigenstate of any of the operators (4.3). In all the four measurements, the state (4.19) leads to the uniform distribution with three outcomes. To reach entanglement detection with the use of (3.8), we should locally rotate the bases (4.6)–(4.7) with respect to the standard one. This is a limitation of the scheme with considering maximal probabilities. On the other hand, no universal entanglement criteria are now known. Say, the PPT criterion is necessary and sufficient for $2 \times 2$ and $2 \times 3$ systems, but in higher dimensional systems some entangled states escape detection. Thus, separability conditions of the considered type can be useful, at least as additional to other criteria.
V. CONCLUSION

We have derived a collection of separability conditions for a bipartite quantum system. The presented separability conditions are obtained from local fine-grained uncertainty relations. They are based on considering maximal probabilities for a set of measurements. Separability criteria are often formulated for measurements that have a special structure. For example, such measurements can be built of mutually unbiased bases or mutually unbiased measurements on subsystems. The considered schemes allow a freedom in constructing a total measurement. Separable states inevitably fulfill upper bounds that follow from local uncertainty relations for a particular subsystem. Entangled states sometimes violate such conditions. Separability conditions are obtained for the measurement schemes based on MUBs as well as on MUMs. Of course, we usually try to use as many different measurements as possible. Actually, an efficiency of detection within the described schemes depends on orientation of local measurement bases and number of involved measurements. Main findings are exemplified with some entangled states of a bipartite system of two qutrits. Separability conditions of the considered type can be used in entanglement detection together with other criteria.

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