Two-body problem on spaces of constant curvature

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Abstract

The two-body problem with a central interaction on simply connected constant curvature spaces of an arbitrary dimension is considered. The explicit expression for the quantum two-body Hamiltonian via a radial differential operator and generators of the isometry group is found. We construct a self-adjoint extension of this Hamiltonian. Some its exact spectral series are calculated for several potential in the space $S^3$.

We describe also the reduced classical mechanical system on a homogeneous space of a Lie group in terms of the coadjoint action of this group. Using this approach the description of the reduced classical two-body problem on constant curvature spaces is given.

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1 Introduction

The simply connected constant curvature spaces $S^n$ and $H^n$ posses isometry groups as wide as the isometry group for the Euclidean space $E^n$ and have no selected points or directions [1]. The one-body classical and quantum problems in the central field on these spaces were studied in many papers, among which we indicate basic ones [2]–[13].

In contrast to the Euclidean case, the configuration spaces $S^n \times S^n$, $H^n \times H^n$ of the two-body problem on spaces $S^n$ and $H^n$ are not spaces of a constant curvature. Only space isometries that preserve an interaction potential enter in the symmetry group of such problem a priori. This group does not suffice to ensure the integrability of the two-body problem. At the same time, no "hidden" symmetries or other integrability tools are known for nontrivial potential. Numerical experiments for the reduced classical two-body problem in spaces $S^n$ and $H^n$, $n = 2, 3$ [14] show the soft chaos in this system for some natural interactive potentials. Numerical experiments ([15]) and analytical results ([16]–[18]) for the restricted classical two-body problem on $S^2$ and $H^2$ also prove its nonintegrability.

The classical mechanical two-body problem in constant curvature spaces was first considered in [19], where the method of the Hamiltonian reduction of systems with symmetries [20] was used to exclude the motion of a system as a whole. The description of reduced mechanical systems, their classification, and conditions for existence of a global dynamic were obtained using explicit analytic coordinate calculations on a computer. In [22], an analogous quantum mechanical system was considered in the two-dimensional case, i.e. on the spaces $S^2$ $H^2$. There, the quantum mechanical two-body Hamiltonian was expressed through isometry group generators and a radial differential operator. The structure of this expression is similar to the structure of reduced Hamilton function. The idea arises

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to seek a general procedure for simultaneous simplification both classical and quantum problems without performing cumbersome calculations. We present such a procedure in the present paper. The obtained expression for the quantum two-body Hamiltonian is useful for solving at least three problems.

First it enables us to prove that the two-body Hamiltonian with the proper domain is self-adjoint. Secondly, using this expression, one can reduce the spectral problem for the two-body Hamiltonian to a sequence of systems of ordinary differential equations enumerated by irreducible representations of the isometry group. In the case of the sphere $S^3$ we found all separate differential equations for spectral values, which can be solved in an explicit form for some interaction potentials. Such a situation appears in so called quasi exactly solvable models \[23\]-\[25\]. There are two essential differences however. First, usually quasi exactly solvable models are artificially constructed. Second exact energy levels are obtained, as a rule, from one differential equation with the specially selected potential, but this equation has also other unknown spectral values. Conversely, in the problem under consideration we select a separate differential equation from systems of ordinary differential equations and find all its spectral values.

Finally, from the obtained expression for the two-body Hamiltonian we derive the Hamilton function of the reduced two-body classical mechanical system, using the description of a reduced classical mechanical system on a homogeneous space in terms of coadjoint orbits of the corresponding Lie group, founded in section 6.

This paper is an essential revision of papers \[26\] and \[27\], made by the first author. Some results in those papers were not properly grounded. This was later done in \[28\]. Proofs in the present paper are extended and in some technically difficult cases they contain references to \[28\]. More serious revision concerns the theorem 5 of the present paper. Because of a confusion between left and right shifts on a group, the statement in \[27\] corresponding to this theorem contains some additional erroneous eigenvectors.

## 2 Notations

Consider the sphere $S^n$ as the space $\mathbb{R}^n \cup \{\infty\}$ with the metric

$$g_s = \left( 4R^2 \sum_{i=1}^{n} dx_i^2 \right) / \left( 1 + \sum_{i=1}^{n} x_i^2 \right)^2,$$ \hspace{1cm} (1)

where $x_i$, $i = 1, \ldots, n$ are Cartesian coordinates in $\mathbb{R}^n$ and $R$ is the curvature radius. Let $\rho_s(\cdot, \cdot)$ denotes the distance between two points in $S^n$. The identity component of a whole isometry group for $S^n$, acting from the left, is $\text{SO}(n+1)$. The set

$$X_{ij}^s = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i < j \leq n,$$

$$Y_s^i = \frac{1}{2} \left( 1 - \sum_{j=1}^{n} x_j^2 \right) \frac{\partial}{\partial x_i} + x_i \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n,$$ \hspace{1cm} (2)

is a base of Killing vector fields on $S^n$. It corresponds to some base in the Lie algebra $\text{so}(n+1)$.

Consider the hyperbolic space $H^n$ as the unit ball $D^n \subset \mathbb{R}^n$ with the metric

$$g_h = \left( 4R^2 \sum_{i=1}^{n} dx_i^2 \right) / \left( 1 - \sum_{i=1}^{n} x_i^2 \right)^2, \quad \sum_{i=1}^{n} x_i^2 < 1.$$ \hspace{1cm} (3)
Denote the distance between two points in $\mathbf{H}^n$ as $\rho^h(\cdot, \cdot)$. Let $O_0(1, n)$ be an identity component of a whole isometry group for $\mathbf{H}^n$, acting from the left. Its Lie algebra is $\mathfrak{so}(1, n)$. The set
\[
X^h_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leqslant i < j \leqslant n,
\]
and
\[
Y^h_i = \frac{1}{2} \left( 1 + \sum_{j=1}^n x_j^2 \right) \frac{\partial}{\partial x_i} - x_i \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n
\]
is a base of Killing vector fields on $\mathbf{H}^n$.

### 3 Special forms of free Hamiltonians

Let $Q_s = S^n \times S^n$ and $Q_h = H^n \times H^n$ be the configuration spaces of the two-body problems in $S^n$ and $H^n$. The corresponding Hamiltonians are defined as
\[
\hat{H}_{s,h} = -\frac{1}{2m_1} \triangle_1 - \frac{1}{2m_2} \triangle_2 + U(\rho^{s,h}) \equiv \hat{H}_0^{s,h} + U(\rho^{s,h}),
\]
where $\triangle_1$ and $\triangle_2$ are Laplace-Beltrami operators in the direct factors of $S^n \times S^n$ and $H^n \times H^n$, corresponding to the first and the second particles, and $U$ is a central potential. Here and below the subscript ”$s$” corresponds to the spherical case and the subscript ”$h$” corresponds to the hyperbolic case.

According to the general concept of quantum mechanics (29), a domain of the operator $\hat{H}_{s,h}$ must be a proper everywhere dense subspace in the space $L^2(Q_{s,h}, d\mu_{s,h})$ of all square integrable functions on $Q_{s,h}$. This subspace is chosen in such a way that the operator $\hat{H}_{s,h}$ becomes self-adjoint; the corresponding measure $d\mu_{s,h}$ is the product of measures on the direct factors of $S^n \times S^n$ ($H^n \times H^n$), invariant w.r.t. the group $\text{SO}(n+1)$ ($O_0(1, n)$).

To express the total Hamiltonian $\hat{H}_{s,h}$ through the radial differential operator and generators of the isometry group, it is suffices to find such an expression for the free Hamiltonian. Recall (see, for example, (30)) that the Laplace-Beltrami operator $\triangle$ on spaces $S^n$ and $H^n$ is respectively self-adjoint with domains
\[
W^{2,2}_s := \{ \phi \in L^2(S^n, d\mu_s) \mid \triangle \phi \in L^2(S^n, d\mu_s) \},
\]
\[
W^{2,2}_h := \{ \phi \in L^2(H^n, d\mu_h) \mid \triangle \phi \in L^2(H^n, d\mu_h) \}.
\]
The action of the operator $\triangle$ is considered in the sense of distributions. The operator $\triangle$ on $S^n$ is essentially self-adjoint on the space $C^\infty(S^n)$ of smooth functions, and the operator $\triangle$ on $H^n$ is essentially self-adjoint on the space of finite smooth functions $C^\infty_0(H^n)$. Hence the free Hamiltonian $\hat{H}_0^{s,h}$ is self-adjoint on the product
\[
W_{s,h} := W^{2,2}_{s,h} \otimes W^{2,2}_{s,h}
\]
of two copies of the space $W^{2,2}_{s,h}$ corresponding respectively to the first and the second particles.

Let $F^{s,h}_r$ be submanifolds of the space $Q_{s,h}$, corresponding to the constant value $r$ of the function $\text{tan}(\rho^h/(2R))$ for the space $Q_s$ and the function $\text{tanh}(\rho^h/(2R))$ for the space $Q_h$. The submanifolds $F^{s}_0, F_{\infty}^s$ are diffeomorphic to $S^n$ (the value $r = \infty$ corresponds to two diametrically opposite points of the sphere $S^n$) and $F^h_0$ is diffeomorphic to $H^n$. For $0 < r < \infty$ the submanifold $F^s_r$ is a homogeneous Riemannian space of the group $\text{SO}(n+1)$ with the stationary subgroup $K \cong \text{SO}(n - 1)$. For $0 < r < 1$ the submanifold $F^h_r$ is a homogeneous Riemannian space of the group $O_0(1, n)$ with the stationary subgroup $K$. 


Up to a zero measure set it holds $Q_s = \mathbb{R}_+ \times (SO(n+1)/K)$, where $\mathbb{R}_+ = (0, \infty)$ and also $Q_h = I \times (O(0,1)/K)$, where $I = (0, 1)$. The operators $-\tilde{\mathcal{H}}^{s,h}_q$ are Laplace-Beltrami ones for the metric $\tilde{g}_{s,h} = 2m_1 g^{(1)}_{s,h} + 2m_2 g^{(2)}_{s,h}$ on $Q_{s,h}$, where the metrics $g^{(1)}_{s,h}$ and $g^{(2)}_{s,h}$ have either form $\mathbf{1}$ or $\mathbf{3}$ on different copies of the spaces $S^n$ and $H^n$, corresponding to particles 1 and 2.

### 3.1 Two-particle Hamiltonian on $S^n \times S^n$

Given the point $x_0 \in F_r$ one can identify the submanifold $F_r$ with the factor space $SO(n+1)/SO(n-1)$ by the formula $x = g K x_0$, where $gK$ is the left coset of the element $g$ in the group $SO(n+1)$ w.r.t. its subgroup $K$. Let $(r, y_1, \ldots, y_{2n-1})$ be local coordinates in some neighborhood $W$ of the point $x_0 \in Q_s$ and then $(y_1, \ldots, y_{2n-1})$ are the coordinates in the open subset $W \cap F_r$ of the submanifold $F_r$. Then the metric $\tilde{g}_s$ in $W$ becomes

$$
\tilde{g}_s = g_{rr}(r) dr^2 + \sum_{i,j=1}^{2n-1} g_{ij}(r, y_1, \ldots, y_{2n-1}) dy_i dy_j.
$$

The second term in this formula is the restriction of a metric $g_f$ from the submanifold $F_r$ onto the domain $W \cap F_r$. Using the standard expression for the Laplace-Beltrami operator through local coordinates, one gets

$$
\Delta_{\tilde{g}_s} = (g_{rr} \det g_{ij})^{-1/2} \frac{\partial}{\partial r} \left( \sqrt{g_{rr} \det g_{ij}} \frac{\partial}{\partial r} \right) + \Delta_{g_f}.
$$

To express the operator $\Delta_{g_f}$ on $F_r$ through the generators of the Lie group SO($n+1$) one can use the following construction from $\mathbf{23}$.

Let $\Gamma$ be a Lie group and $\Gamma_0$ be its subgroup. The group $\Gamma$ acts from the left on the homogeneous space $\Gamma/\Gamma_0$. Left-invariant differential operators on the space $\Gamma/\Gamma_0$ can be represented by left-invariant differential operators on the group $\Gamma$ that are simultaneously invariant w.r.t. the right action of the group $\Gamma_0$. This representation is one to one up to operators summands, vanishing while acting onto functions that are invariant w.r.t. right $\Gamma_0$-shifts.

Indeed, functions on the factor space $\Gamma/\Gamma_0$ are in one to one correspondence with functions on the group $\Gamma$ that are invariant w.r.t. right $\Gamma_0$-shifts. This correspondence is defined by the formula $\lambda : f \to \hat{f} := f \circ \pi$, where $\pi : \Gamma \to \Gamma/\Gamma_0$ is the canonical projection and $f$ is a function on the factor space $\Gamma/\Gamma_0$. Let $D$ be a differential operator on $\Gamma$ that is invariant w.r.t. left $\Gamma$-shifts and right $\Gamma_0$-shifts. Let $f$ be a smooth function on the factor space $\Gamma/\Gamma_0$. Then the formula $\hat{D} u f = D \hat{f}$ defines the correspondence $D \to D_u$, where $D_u$ is a differential operator on the space $\Gamma/\Gamma_0$, invariant w.r.t. left $\Gamma$-shifts.

Let $e_1, \ldots, e_N$ be a base of the Lie algebra of the group $\Gamma$, $N := \dim \Gamma$ and let $L_\gamma$ and $R_\gamma$ denote respectively the left and right shifts by the element $\gamma \in \Gamma$. The algebra of left invariant differential operators on the group $\Gamma$ is generated over $\mathbb{R}$ by left invariant vector fields $e_1^\gamma, \ldots, e_N^\gamma$, where $e_i^\gamma(\gamma) = dL_\gamma(e_i), \gamma \in \Gamma, i = 1, \ldots, N$. Let $\Gamma = SO(n+1), \Gamma_0 = K \cong SO(n-1), e_i^\gamma(\gamma) = dR_\gamma(e_i), i = 1, \ldots, N, N = (n+1)(n+2)/2, x_0 = (r_1, 0, \ldots, 0, r_2, 0, \ldots, 0) \in S^n \times S^n$, where

$$
r_1 = \tan \left( \frac{m_2}{m_1 + m_2} \arctan r \right), \quad r_2 = -\tan \left( \frac{m_1}{m_1 + m_2} \arctan r \right).
$$

$^1$We suppose that this metric does not contain summands proportional to $dr dy_1$. It is valid for the parametrization $\mathbf{3}$ below. More detailed consideration for a more general parametrization and an arbitrary two-point homogeneous spaces $M$ can be found in $\mathbf{24}$. The connection of the two-body Hamiltonian with the algebra of invariant differential operators on the unit sphere bundle over $M$ was described in $\mathbf{22}$. 


The set of Killing vector fields $X_{ij}^a, Y_{ij}^a$, $i, j = 1, \ldots, n$ on the space $S^n \times S^n$, corresponding to (2), coincides (up to a permutation) with the set

$$\left\{ \tilde{e}_i(\gamma x_0) = \frac{d}{dt}\bigg|_{t=0} \exp(\tau e_i)\gamma x_0 \right\}_{i=1}^N, \quad x_0 = x_0(r), \quad 0 < r < \infty, \quad (9)$$

under the proper choice of the basis $e_1, \ldots, e_N$. Let $\Delta_f$ be a second order differential operator on the group $\Gamma$ such that $(\Delta_f)^n = \Delta_{g_f}$. Then it is left invariant and can be expressed in the form\(^2\)

$$\Delta_f|_e = \sum_{i,j=1}^N c^{ij} e^j_i \circ e^j_j|_e + \sum_{i=1}^N e^j_i|_e,$$

where $c^{ij}, e^j_i$ are constant on the submanifold $F$. Let $e$ be the unit element of the group $\Gamma$. Obviously, $e^j_i|_e = e^j_i|_e$, $i = 1, \ldots, N$ and

$$\Delta_f|_e = \sum_{i,j=1}^N c^{ij} e^j_i \circ e^j_j|_e + \sum_{i=1}^N e^j_i|_e. \quad (10)$$

It yields

$$\Delta_{g_f}|_{x_0} = \sum_{i,j=1}^N c^{ij} \tilde{e}_i^j \circ \tilde{e}_i^j|_{x_0} + \sum_{i=1}^N \tilde{e}_i|_{x_0} =: \Delta^{(2)}_{g_f}|_{x_0} + \Delta^{(1)}_{g_f}|_{x_0}. \quad (11)$$

One can find coefficients $c^{ij}$ in the following way. Consider the ordered set of vectors $\left\{ Y_1^a|_{x_0}, \ldots, Y_n^a|_{x_0}, X_{12}^a|_{x_0}, \ldots, X_{nn}^a|_{x_0} \right\}$ as a base in the linear space $T_{x_0}F$. Let $\{Y^1, \ldots, Y^n, X^2, \ldots, X^n\}$ be the dual basis. Then

$$g_{f|_{x_0}} = \sum_{i=2}^n \left[ Y^i \otimes (\alpha_i Y^i + \beta_i X^i) + \sum_{j=2}^n (\alpha_{ij} Y^i \otimes Y^j + \beta_{ij} X^i \otimes X^j + \gamma_{ij} Y^i \otimes X^j) \right]$$

$$+ a Y^1 \otimes Y^1,$$

where

$$a = \tilde{g}(Y^a_1, Y^a_1)|_{x_0} = 2R^2(m_1 + m_2),$$

$$\alpha_i = \tilde{g}(Y^a_i, Y^a_i)|_{x_0} = 0,$$

$$\beta_i = \tilde{g}(Y^a_i, X^a_i)|_{x_0} = 0, \quad i = 2, \ldots, n,$$

$$\alpha_{ij} = \tilde{g}(Y^a_i, Y^a_j)|_{x_0} = 2R^2 \sum_{k=1}^2 m_k \frac{(1 - r^2_k)^2}{(1 + r^2_k)^2} \delta_{ij},$$

$$\beta_{ij} = \tilde{g}(X^a_i, X^a_j)|_{x_0} = 8R^2 \sum_{k=1}^2 m_k r^2_k \delta_{ij},$$

$$\gamma_{ij} = \tilde{g}(Y^a_i, X^a_j)|_{x_0} = 4R^2 \sum_{k=1}^2 m_k r^2_k (1 - r^2_k) \delta_{ij}, \quad i, j = 2, \ldots, n. \quad (11)$$

\(^2\)Here we consider vector fields as differential operators of the first order.
Therefore one gets
\[
\triangle_{g|}^{(2)} \mid_{x_0} = \frac{1}{a} (Y^s)^2 \mid_{x_0} + \frac{1}{2} \sum_{k=2}^{n} \left[A_s (X^s_{1k})^2 + C_s (Y^s)^2 - B_s (X^s_{1k}, Y^s)\right] \mid_{x_0},
\]
(12)
where \{\cdot, \cdot\} denotes the anticommutator and functions \(A_s, B_s, C_s\) have the form
\[
A_s = \frac{m_1(1-r_1^2)(1+r_2^2) + m_2(1+r_1^2)(1-r_2^2)}{4R^2 m_1 m_2 (r_1 - r_2)^2 (1 + r_1 r_2)^2},
\]
\[
B_s = \frac{m_1 r_1 (1-r_1^2)(1+r_2^2) + m_2 r_2 (1-r_2^2)(1+r_1^2)}{2R^2 m_1 m_2 (r_1 - r_2)^2 (1 + r_1 r_2)^2},
\]
\[
C_s = \frac{m_1 r_1^2 (1-r_2^2) + m_2 r_2^2 (1+r_1^2)}{R^2 m_1 m_2 (r_1 - r_2)^2 (1 + r_1 r_2)^2}.
\]
These functions can be expressed also through the coordinate \(r\):
\[
A_s(r) = \frac{1}{R^2} \left(\frac{(1+r^2)^2}{8mr^2} - \frac{1-r^4}{8mr^2} \cos \zeta + \frac{1+r^2}{4m_1 m_2 r} (m_1 - m_2) \sin \zeta\right),
\]
\[
B_s(r) = -\frac{1}{4R^2} \left(\frac{m_2 - m_1}{m_1 m_2 r} (1+r^2) \cos \zeta + \frac{1-r^4}{2mr^2} \sin \zeta\right),
\]
\[
C_s(r) = \frac{1}{R^2} \left(\frac{(1+r^2)^2}{8mr^2} - \frac{1-r^4}{8mr^2} \cos \zeta - \frac{1+r^2}{4m_1 m_2 r} (m_1 - m_2) \sin \zeta\right),
\]
\[
\zeta := 2m_1 m_2 \arctan r, \quad m := \frac{m_1 m_2}{m_1 + m_2}.
\]

The operators \(\triangle_{g|}^{(1)} \mid_{x_0}\) and \(\triangle_{g|}^{(2)} \mid_{x_0}\) (see [12]) are invariant w.r.t. reflections \(T_k : x_k \rightarrow -x_k, x_j \rightarrow x_j, j \neq k, j = 1, \ldots, n, k = 2, \ldots, n\) of the sphere \(S^n\). Therefore the operator \(\triangle_{g|}^{(1)} \mid_{x_0}\) is also invariant w.r.t. these reflections. Reflections \(T_k\) changes signs of the vector fields \(X^s_{1k} \mid_{x_0}, Y^s \mid_{x_0}\), \(k = 2, \ldots, n\), therefore the operator \(\triangle_{g|}^{(1)} \mid_{x_0}\) is proportional to \(Y^s \mid_{x_0}\).

The more accurate analysis (see [23]) shows that the operator \(\triangle_{g|}^{(1)}\) vanishes.

If we denote by \(Y_{1}^{s,l}, X_{1}^{s,l}, Y_{k}^{s,l}, X_{k}^{s,l}\) left invariant vector fields on the group \(SO(n+1)\), corresponding to vectors \(Y_{1}^{s} \mid_{x_0}, X_{1}^{s} \mid_{x_0}, Y_{k}^{s} \mid_{x_0}, X_{k}^{s} \mid_{x_0}\), \(k = 2, \ldots, n\), we get
\[
\triangle_f = \frac{1}{a} D_0^2 + \frac{1}{2} A_1 D_2 + \frac{1}{2} C_1 D_1 + B_1 D_3,
\]
(13)
where operators \(D_0, D_1, D_2\) and \(D_3\) have the form
\[
D_0 = Y_{1}^{s,l}, \quad D_1 = \sum_{k=2}^{n} (Y_{k}^{s,l})^2, \quad D_2 = \sum_{k=2}^{n} (X_{k}^{s})^2, \quad D_3 = -\frac{1}{2} \sum_{k=2}^{n} \{X_{k}^{s,l}, Y_{k}^{s,l}\}.
\]

By direct calculations in the universal enveloping algebra \(U(so(n+1))\), one can get (see [32]) the following commutator relations for the operators \(D_0, \ldots, D_3\)
\[
[D_0, D_1] = -2D_3, \quad [D_0, D_2] = 2D_3, \quad [D_0, D_3] = D_1 - D_2, \quad [D_1, D_2] = -2[D_0, D_3], \quad [D_1, D_3] = -1 [D_0, D_1] + \frac{(n-1)(n-3)}{2} D_0, \quad [D_2, D_3] = D_0 D_2 - \frac{(n-1)(n-3)}{2} D_0.
\]
(14)

Thus we found the operator \(\triangle_f\) up to summands, annihilated by functions that are invariant w.r.t. right \(\Gamma_0\)-shifts.
We are to find now the first term in expression (7). At the point \( x_0 \) one has
\[
\frac{\partial}{\partial r} = \frac{m_2}{m_1 + m_2} \frac{1 + r^2}{1 + r^2} \frac{\partial}{\partial r_1} - \frac{m_1}{m_1 + m_2} \frac{1 + r^2}{1 + r^2} \frac{\partial}{\partial r_2}
\]
and therefore
\[
g_{rr} = \tilde{g} \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = \frac{8R^2m_1m_2}{(m_1 + m_2)(1 + r^2)^2}.
\]
Due to formulas (11) one gets
\[
\Delta \tilde{g}_s = \left(1 + r^2\right)^n \frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(1 + r^2)^{n-2}} \frac{\partial}{\partial r} \right) + \Lambda g_f,
\]
where the first term is the radial part of the one particle Hamiltonian with the mass \( m \).

Direct calculation at the point \( x_0 \) gives for the measure \( d\mu_s \), corresponding to the metric \( \tilde{g}_s \) in the space \( Q_s \), the following expression
\[
d\mu_s|_{x_0} = \frac{r^{n-1}}{(1 + r^2)^n} dr \wedge Y^1 \wedge \cdots \wedge Y^n \wedge X^2 \wedge \cdots \wedge X^n
\]
up to a constant factor.

The measure \( d\mu_s \) is left invariant w.r.t. the group \( SO(n + 1) \) and therefore it can be represented in the form \( d\mu_s = dv_s \otimes d\mu_f \), where \( dv_s = r^{n-1} dr/(1 + r^2)^n \) is a measure on \( \mathbb{R}_+ = (0, \infty) \), the same as for the one particle case, and \( d\mu_f \) is a measure on \( SO(n + 1)/K \) left invariant w.r.t. the group \( SO(n + 1) \).

Each Lie group admits unique (up to a constant factor) left-invariant and right-invariant measures (Haar measures [34]). For the groups under consideration \( SO(n + 1) \) and \( O_0(1, n) \) such measures are two-side invariant. Hence there exist a unique two-side invariant measure \( d\eta_s \) on the group \( SO(n + 1) \) such that the integral of any integrable function \( f \) on the space \( SO(n + 1)/K \) w.r.t. the measure \( d\mu_f \) equals the integral of the function \( \tilde{f} \) on the group \( SO(n + 1) \) w.r.t. the measure \( d\eta_s \).

**Definition 1.** For a subgroup \( \Gamma_0 \) of a Lie group \( \Gamma \) denote by \( \mathcal{L}^2(\Gamma, \Gamma_0, d\eta) \) the space of square-integrable functions on the group \( \Gamma \) (w.r.t. the measure \( d\eta \) on \( \Gamma \)) that are right invariant w.r.t. \( \Gamma_0 \)-shifts.

**Theorem 1.** The free two-particle Hamiltonian on the sphere \( S^n \) can be considered as the self-adjoint differential operator (on the manifold \( \tilde{Q}_s = \mathbb{R}_+ \times SO(n + 1) \))
\[
\tilde{H}_0 = -\frac{1}{8mR^2n^{n-1}} \frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(1 + r^2)^{n-2}} \frac{\partial}{\partial r} \right) - \Lambda f,
\]
with the domain
\[
D_s := D_s^{(1)} \otimes D_s^{(2)} \subset \mathcal{H}_s := \mathcal{L}^2(\mathbb{R}_+, dv_s) \otimes \mathcal{L}^2(SO(n + 1), K, d\eta_s),
\]
where
\[
D_s^{(1)} := \left\{ \phi \in \mathcal{L}^2(\mathbb{R}_+, dv_s) \mid \triangle_s^{(1)} \phi \in \mathcal{L}^2(\mathbb{R}_+, dv_s) \right\},
\]
\[
D_s^{(2)} := \left\{ \phi \in \mathcal{L}^2(SO(n + 1), K, d\eta_s) \mid \Lambda f \phi \in \mathcal{L}^2(SO(n + 1), K, d\eta_s) \right\},
\]
\[
\triangle_s^{(1)} := -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(1 + r^2)^{n-2}} \frac{\partial}{\partial r} \right),
\]
the subgroup \( K \) is isomorphic to the group \( SO(n - 1) \), and \( d\eta_s \) is a unique (up to a constant factor) two-side invariant measure on the group \( SO(n + 1) \). It means that there exists an isometry of the initial space of functions \( \mathcal{L}^2(Q_s, d\mu_s) \) onto the space \( \mathcal{H}_s \) that generates the isomorphism of Hamiltonians. The space \( D_s \) is everywhere dense in \( \mathcal{H}_s \).
The formal substitution \( x \rightarrow \dot{x} \) Hamiltonian on the space \( H \) also \([19]\). Therefore from results of the previous section one can obtain the two-particle \( \text{SO}(n) \) generates the isometry \( \lambda \) transforms operator \([16]\) into operator \([10]\) and the space \( W \) into the space \( D \).

\[
\mathcal{L}^2(Q, d\mu) = \mathcal{L}^2(\mathbb{R}^n, dv_n) \otimes \mathcal{L}^2(\text{SO}(n+1)/\text{SO}(n-1), d\mu_f).
\]

The isometry \( \lambda : f \rightarrow \dot{f} \) of spaces

\[
\mathcal{L}^2(\text{SO}(n+1)/\text{SO}(n-1), d\mu_f) \text{ and } \mathcal{L}^2(\text{SO}(n+1), d\mu_f)
\]

generates the isometry \( \text{id} \otimes \lambda \) of spaces \( \mathcal{L}^2(\mathbb{R}^n, dv_n) \otimes \mathcal{L}^2(\text{SO}(n+1)/\text{SO}(n-1), d\mu_f) \) and \( H \). Calculations above imply that the isometry \( \text{id} \otimes \lambda \) transforms operator \([17]\) into operator \([16]\) and the space \( W \) into the space \( D \).

**Remark 1.** In the case \( n = 2 \) this result can be obtained by treating a basis in the Lie algebra \( \mathfrak{so}(3) \) as a moving frame on the submanifold \( F \). For \( n > 2 \) this is impossible since the \( \text{SO}(n+1) \)-action on \( F \) is not free and the projection of left-invariant vector fields from \( \text{SO}(n+1) \) onto \( \text{SO}(n+1)/\text{SO}(n-1) \) is not well defined. By lifting the Hamiltonian onto the symmetry group one can express it through group generators.

### 3.2 Two-particle Hamiltonian on the space \( H^n \times H^n \)

The formal substitution \( x_j \rightarrow -ix_j, r \rightarrow -ir, R \rightarrow iR, j = 1, \ldots, n \), (here \( i \) is the imaginary unit) transforms objects on the sphere \( S^n \) into objects on the hyperbolic space \( H^n \) (see also \([13]\)). Therefore from results of the previous section one can obtain the two-particle Hamiltonian on the space \( H^n \times H^n \):

\[
\tilde{H}_0^h = -\frac{(1 - r^2)^n}{8mR^2r^{n-1}} \frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(1 - r^2)^{n-2}} \frac{\partial}{\partial r} \right) - \frac{1}{a} \bar{D}_0^2 - \frac{1}{2} A_h \bar{D}_2 - \frac{1}{2} C_h \bar{D}_1 - B_h \bar{D}_3,
\]

where \( 0 < r < 1 \),

\[
\bar{D}_0 = Y_{i,k}, \bar{D}_1 = \sum_{k=2}^{n} \left( \bar{Y}_{i,k} \right)^2, \bar{D}_2 = \sum_{k=2}^{n} \left( \bar{X}_{i,k} \right)^2, \bar{D}_3 = -\frac{1}{2} \sum_{k=2}^{n} \{ \bar{X}_{i,k}, \bar{Y}_{i,k} \},
\]

vector fields \( \bar{X}_{i,k}, \bar{Y}_{i,k} \) relate to vector fields \([4]\) in the same way as vector fields \( X_{i,k}, Y_{i,k} \) relate to vector fields \([4]\), and

\[
A_h(r) = \frac{1}{R^2} \left( \frac{(1 - r^2)^2}{8mr^2} + \frac{1 - r^4}{8mr^2} \cosh \zeta - \frac{1 - r^2}{4m_1m_2r} (m_1 - m_2) \sinh \zeta \right),
\]

\[
B_h(r) = -\frac{1}{4R^2} \left( \frac{(m_2 - m_1)}{m_1m_2r} (1 - r^2) \cosh \zeta + \frac{1 - r^4}{2mr^2} \sinh \zeta \right),
\]

\[
C_h(r) = -\frac{1}{R^2} \left( \frac{(1 - r^2)^2}{8mr^2} + \frac{1 - r^4}{8mr^2} \cosh \zeta - \frac{1 - r^2}{4m_1m_2r} (m_1 - m_2) \sinh \zeta \right),
\]

\[
\zeta := \frac{2m_1 - m_2}{m_1 + m_2} \arctanh r.
\]

Commutator relations for operators \( \bar{D}_0, \bar{D}_1, \bar{D}_2, \bar{D}_3 \) have the form (see \([32]\))

\[
[\bar{D}_0, \bar{D}_1] = 2\bar{D}_3, [\bar{D}_0, \bar{D}_2] = 2\bar{D}_3, [\bar{D}_0, \bar{D}_3] = \bar{D}_1 + \bar{D}_2, [\bar{D}_1, \bar{D}_2] = -2(\bar{D}_0, \bar{D}_3),
\]

\[
[\bar{D}_1, \bar{D}_3] = -\{ \bar{D}_0, \bar{D}_1 \} - \frac{(n-1)(n-3)}{2} \bar{D}_0, [\bar{D}_2, \bar{D}_3] = \{ \bar{D}_0, \bar{D}_2 \} - \frac{(n-1)(n-3)}{2} \bar{D}_0.
\]
Theorem 2. The free two-particle Hamiltonian on the space $H^n$ can be considered as the self-adjoint differential operator \((17)\) on the manifold $Q_h = I \times O_0(1, n)$ with the domain

$$D_h := D_h^{(1)} \otimes D_h^{(2)} \subset \mathcal{H}_h := \mathcal{L}^2 (\mathbb{R}_+, d\nu_h) \otimes \mathcal{L}^2 (O_0(1, n), K, d\eta_h),$$

where $K = SO(n-1), \quad D_h^{(1)} := \{ \phi \in \mathcal{L}^2 (\mathbb{R}_+, d\nu_h) \mid \bigtriangleup_h^{(1)} \phi \in \mathcal{L}^2 (\mathbb{R}_+, d\nu_h) \}, \quad D_h^{(2)} := \{ \phi \in \mathcal{L}^2 (O_0(1, n), K, d\eta_h) \mid \bigtriangleup_h \phi \in \mathcal{L}^2 (O_0(1, n), K, d\eta_h) \},

$$\bigtriangleup_h^{(1)} := -\frac{(1-r^2)^n}{r^{n-1}} \frac{\partial}{\partial r} \left( \frac{r^{n-1}}{(1-r^2)^{n-2}} \frac{\partial}{\partial r} \right), \quad d\nu_h = r^{n-1} dr, \quad \bigtriangleup_h := -\frac{1}{a} \hat{D}_0^2 - \frac{1}{2} A_h \hat{D}_2 - \frac{1}{2} C_h \hat{D}_1 - B_h \hat{D}_3,$$

and $d\eta_h$ is a unique (up to a constant factor) two-side invariant measure on the group $O_0(1, n)$.

The proof is analogous to the proof of theorem \([1]\).

4 Self-adjointness of two-particle Hamiltonians

In Euclidean space the self-adjointness of many-particle Hamiltonians with pairwise interacting particles is usually proved using the Galilei invariance, which has no analog in the spaces $S^n$ and $H^n$ \([35]\). The self-adjointness of one-particle Hamiltonians with singular potential unbounded from below can be proved using the perturbation theory for the corresponding quadratic forms. The key point of the proof is the following estimate, called the "uncertainty principle" in \([35]\):

$$\langle U \psi, \psi \rangle \leq \| \nabla \psi \|^2,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the space $\mathcal{L}^2 (\mathbb{R}^3)$, $\nabla$ is the gradient operator, and $U$ is a potential. To derive this estimate for spaces $S^n$ and $H^n$ one needs some modification of the proof w.r.t. the Euclidean case. Instead of tending to full generality, we are mostly interested in physically significant potentials.

From the self-adjointness of the free two-particle Hamiltonian with domain \((17)\) we shall prove the self-adjointness of the two-particle Hamiltonian with an interaction using the perturbation theory for the quadratic forms.

Let $\langle \cdot, \cdot \rangle$ be the scalar product on fibers of the cotangent bundle $T^*Q_s$, generated by the metric $\tilde{g}_s$, $\| \cdot \|$ and $\nabla$ be respectively the corresponding norm and the gradient operator. Let also $f, \psi \in C^\infty (Q_s)$ be real functions\(^3\) such that $f$ is constant on submanifolds $F^s_r$, i.e. $f = f(r)$. Then it holds

$$\| \nabla \psi \|^2 = \frac{\| \nabla (f \psi) \|}{f} - \frac{\psi \nabla f}{f} \| \nabla \psi \|^2 \geq \frac{\psi^2 \| \nabla f \|^2}{f^2} - \frac{2 \psi \langle \nabla f, \nabla (f \psi) \rangle}{f^2} = \frac{\psi^2 \tilde{g}_s r^2}{f^2} (f')^2 - \frac{2 \psi}{f^2} \tilde{g}_s r f' \frac{\partial}{\partial r} (f \psi)$$

Equation \((15)\) implies that $\tilde{g}_s r^2 = (1+r^2)^2/(8R^2m)$. Integrating over $Q_s$ with the measure(Book p.9)\(^3\)

---

\(^3\)All functional spaces are assumed to comprise complex-valued functions.
\[ \int_{Q_s} \| \nabla \psi \|^2 d\mu_s \geq \frac{1}{8R^2m} \int_{Q_s} \left[ \frac{(f')^2}{f^2} - \frac{2 \psi f f' \partial}{f^2} \right] (1 + r^2)^2 d\mu_s \]

\[ = \frac{1}{8R^2m} \int_{Q_s} \frac{(f')^2}{f^2} (1 + r^2)^2 d\mu_s - \frac{1}{4R^2m} \int_{F_s'} \int_0^\infty \psi f'^n - \partial (f') d\mu f. \]  

We now want to find a function \( f \) such that for every smooth function \( \psi \), the function

\[ \frac{\psi f'^n - \partial (f')}{{f^2}(1 + r^2)^{n-2}} (f') \]

has the form \( \frac{\partial}{\partial r} (\phi(r) \psi^2) \). Then the last integral in (19) can vanish identically. Solving the system of equations

\[ \frac{(f')^2 r^{n-1}}{f^2(1 + r^2)^{n-2}} = \phi', \quad \frac{f'^n - \partial}{f(1 + r^2)^n} = 2 \phi, \]

one gets

\[ \phi(r) = - \left[ \frac{4}{\{1 + r^2\}^{n-2}} dr \right]^{-1}, \quad \phi(r) \sim \begin{cases} (n - 2) r^{n-2}/4, n \geq 3 \\ (4|\ln r|)^{-1}, n = 2 \end{cases}, r \to 0, \]

\[ \phi(r) \sim \begin{cases} - (n - 2) r^2 - n/4, n \geq 3 \\ - (4|\ln r|)^{-1}, n = 2 \end{cases}, r \to \infty, \quad (20) \]

It is easy to verify that for any choice of the integration constant the function \( \phi(r) \) is discontinuous at some point, which does not occur in the Euclidean case for \( n \geq 3 \). Let \( \omega_s := \{ x \in Q_s | r(x) < \delta \} \) and \( \omega'_s := \{ x \in Q_s | r(x) > \delta^{-1} \} \). Choose \( \delta > 0 \) in such a way that the function \( \phi(r) \) is continuous in the domain \( (\omega_{2\delta} \cup \omega'_{2\delta}) \setminus (F_0 \cup F_{\infty}) \). On the space \( Q_s \setminus (F_0 \cup F_{\infty}) \) the continuity domains of the functions \( f' f \) and \( \phi(r) \) coincide; moreover both functions are nonzero for \( r \neq 0, \infty \). We set

\[ u_n(r) = \begin{cases} r^{-2} + r^2, n \geq 3 \\ (r^{-2} + r^2)/\ln^2 r, n = 2 \end{cases}, \]

and choose the constant \( \kappa > 0 \) such that the inequality

\[ \kappa u_n(r) \leq \frac{(1 + r^2)^2 (f')^2}{8R^2m f^2} \]

holds. Then due to (19) and (20) one gets the following inequality

\[ \kappa \int_{Q_s} u_n(r) |\psi|^2 d\mu_s \leq \int_{Q_s} \| \nabla \psi \|^2 d\mu_s \]  

for the function \( \psi \in C^\infty (Q_s) \) with \( \text{supp } \psi \subset \omega_s \cup \omega'_s \). Writing inequality (21) separately for the real and imaginary parts of a complex-valued function, we obtain (21) for an arbitrary function \( \psi \in C^\infty (Q_s) \) with \( \text{supp } \psi \subset \omega_s \cup \omega'_s \).
Theorem 3. Let $U$ be a real function that is smooth in the domain $Q_s \setminus (F_0^s \cup F_\infty^s)$ and satisfies the estimate $U = o(u_n(r))$ as $r \to 0, \infty$, uniformly w.r.t. coordinates on the second factor of the direct product $\mathbb{R}_+ \times SO(n + 1)/SO(n - 1)$, representing $Q_s$ up to the zero measure set $F_0^s \cup F_\infty^s$. Then the two-particle Hamiltonian $\hat{H}_s$ is essentially self-adjoint in any domain of essential self-adjointness of the free Hamiltonian $\hat{H}_0^s$. In particular, $\hat{H}_s$ is essentially self-adjoint in the domain $C^\infty(Q_s)$.

Proof. Theorem X.17 from [35] states that it is suffice to prove that the inequality

$$\int_{Q_s} |U||\psi|^2 d\mu_s \leq a \int_{Q_s} \psi^*_0 \psi d\mu_s + b \int_{Q_s} |\psi|^2 d\mu_s,$$

is valid for all $\psi \in C^\infty(Q_s)$, where $0 < a < 1, b \in \mathbb{R}$.Fixing a constant $a \in (0,1)$, we choose the function $\chi \in C^\infty(Q_s)$ such that $\supp \chi \subset \omega_\delta \cup \omega_\delta'$, $\chi|_{\omega_{\delta/2} \cup \omega_{\delta/2}' \equiv 1}, 0 \leq \chi \leq 1$. Now let $0 < \varepsilon \leq \delta/2$ such that $|U(x)| \leq a \kappa u_n(r(x))/2, x \in \omega_\varepsilon \cup \omega_\varepsilon'$. Let also

$$c := \sup_{x \in Q_s \setminus (\omega_\varepsilon \cup \omega_\varepsilon')} |U(x)| \text{ and } \psi \in C^\infty(Q_s).$$

Then

$$\int_{Q_s} |U||\psi|^2 d\mu_s \leq \frac{\kappa}{2} \int_{\omega_\varepsilon \cup \omega_\varepsilon'} u_n(x)|\chi\psi|^2 d\mu_s + c \int_{Q_s} |\psi|^2 d\mu_s.$$

Applying estimate (21) to the first integral, one gets

$$\kappa \int_{\omega_\varepsilon \cup \omega_\varepsilon'} u_n(x)|\chi\psi|^2 d\mu_s \leq \int \|\nabla(\chi\psi)\|^2 d\mu_s \leq 2 \left( \|\nabla\psi\|^2 + \|\nabla\psi|^2 |\psi|^2 \right) d\mu_s.$$

Hence

$$\int_{Q_s} |U||\psi|^2 d\mu_s \leq a \int_{Q_s} \|\nabla\psi\|^2 d\mu_s + b \int_{Q_s} |\psi|^2 d\mu_s = a \int_{Q_s} \psi^*_0 \psi d\mu_s + b \int_{Q_s} |\psi|^2 d\mu_s,$$

where $b = c + 2 \sup_{Q_s} (|\nabla\psi|^2) \square$. 

An analogous result is valid for the space $H^n$.

Theorem 4. Let $U$ be a real function that is smooth in the domain $Q_h \setminus F_0^h$ and satisfies the estimate $U = o(u_n(r))$ as $r \to 0, \infty$, uniformly w.r.t. coordinates on the second factor of the direct product $I \times O_0(1, n)/SO(n - 1)$, representing $Q_h$ up to the zero measure set $F_0^h$. Then the two-particle Hamiltonian $\hat{H}_h$ is essentially self-adjoint in any domain of essential self-adjointness of the free Hamiltonian $\hat{H}_0^h$. In particular, $\hat{H}_h$ is essentially self-adjoint in the domain $C^\infty(Q_h)$.

5 The spectrum of the operator $\hat{H}_s$

It is known (see, for instance, [36]) that all (enumerable set) irreducible representations of a compact Lie group $\Gamma$ are finite dimensional and are contained in its regular representation by left or right shifts in the space $L^2(\Gamma, d\eta)$, where $\eta$ is a two-side invariant measure on the group $\Gamma$. Let

$$L^2(SO(n + 1), d\eta_k) = \bigoplus_k T_k$$

(22)
be the decomposition of the right regular representation of the group $\text{SO}(n + 1)$ into irreducible ones. Restricting decomposition \([22]\) onto the subspace $L^2(\text{SO}(n + 1), K, d\eta_s)$ of the space $L^2(\text{SO}(n + 1), dt_h)$ one gets

$$L^2(\text{SO}(n + 1), K, d\eta_s) = \bigoplus_k T'_k,$$

where subspaces $T'_k \subset T_k$ consist of functions annulled by left invariant vector fields on $\text{SO}(n + 1)$ generated by elements from the Lie algebra $\mathfrak{t}$ of the group $K \cong \text{SO}(n - 1)$. Differential operators in the space $L^2(\text{SO}(n + 1), K, d\eta_s)$, invariant w.r.t. left $\text{SO}(n + 1)$-shifts, conserve subspaces $T'_k$.

Explicit (and rather cumbersome) expressions for the action of infinitesimal generators of the group $\text{SO}(n)$ on basis elements (described by the Gelfand-Tsetlin schemes) of its irreducible representations were found in [37, 38].

Thus the matrix ordinary differential operator that corresponds to the restriction of $\hat{H}_s$ onto the subspace $L^2(\mathbb{R}^+, d\nu_s) \otimes T'_k$ can be found explicitly. It is interesting to find one-dimensional subspaces $\hat{T}'_k \subset T'_k$ for which the spaces $L^2(\mathbb{R}^+, d\nu_s) \otimes \hat{T}'_k$ are invariant w.r.t. the operator $\hat{H}_s$. In this case one can found separate spectral ordinary differential equations for the two-particle Hamiltonian on the space $S^3$ that can be solved explicitly for some potentials $U(r)$.

In the case $m_1 \neq m_2$ formulas [38] and [16] imply that for the realization of this program it is sufficient to find common eigenfunctions for operators $D_0^2, D_1, D_2, D_3$ in the space $T_k$. In the case $m_1 = m_2$ one has $B_4(r) \equiv 0$ and it is sufficient to find common eigenfunctions for operators $D_0^2, D_1, D_2$.

Due to the lack of a general methods for finding common eigenvectors of noncommutative operators, we restrict ourselves with the case of the three dimensional sphere $S^3$. Recall that the two dimensional sphere $S^2$ was considered from this point of view in [22].

The base $L_1, L_2, L_3, G_1, G_2, G_3$ of the Lie algebra $\mathfrak{so}(4)$ defined as

$$L_1 = \frac{1}{2}(X^1_2 + Y^2_1), \quad L_2 = \frac{1}{2}(X^1_3 + Y^3_1), \quad L_3 = \frac{1}{2}(X^1_2 + Y^2_3),$$

$$G_1 = \frac{1}{2}(X^3_2 - Y^2_3), \quad G_2 = \frac{1}{2}(X^3_3 - Y^3_2), \quad G_3 = \frac{1}{2}(X^3_2 - Y^2_3)$$

(4)

corresponds to the decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The corresponding commutator relations are

$$[L_l, L_j] = \sum_{k=1}^{3} \epsilon_{ijk} L_k, \quad [G_l, G_j] = \sum_{k=1}^{3} \epsilon_{ijk} G_k, \quad [L_l, G_j] = 0, \quad l, j = 1, 2, 3,$$

where $\epsilon_{ijk}$ is a totally antisymmetric tensor such that $\epsilon_{123} = 1$. Let

$$T_\pm = iL_2 \pm L_3, \quad T_0 = -iL_1, \quad W_\pm = iG_2 \pm G_3, \quad W_0 = -iG_1$$

be a base in the complexification of the Lie algebra $\mathfrak{so}(4)$, where $i$ is the imaginary unit. Then one has the following commutative relations

$$[T_0, T_+] = T_+, \quad [T_0, T_-] = -T_-, \quad [T_+, T_-] = 2T_0,$$

$$[W_0, W_+] = W_+, \quad [W_0, W_-] = -W_-, \quad [W_+, W_-] = 2W_0.$$

Evidently, operators from different triples commute with each other.

---

4 Commutative relations for elements of a Lie algebra and commutative relations of corresponding Killing vector fields (considered as differential operators of the first order) differ by a sign for a left action of an isometry group.
Since the group SU(2) \times SU(2) is the double covering of the group SO(4) the representation theory for the latter one can be derived from the representation theory for the group SU(2).

Let \( U_{\ell_1}, \ell_1 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \) be the unitary space of irreducible representation \( \mathcal{T}_{\ell_1} \) for the group SU(2), generated by elements \( L_1, L_2, L_3 \). This space has a base \( \psi_{n_1}^{\ell_1}, n_1 = -\ell_1, -\ell_1 + 1, \ldots, \ell_1 + 1, \ell_1, \) satisfying relations (39)

\[
T_0 \psi_{n_1}^{\ell_1} = n_1 \psi_{n_1}^{\ell_1}, \quad T_+ \psi_{n_1}^{\ell_1} = -\sqrt{(\ell_1 - n_1)(\ell_1 + n_1 + 1)} \psi_{n_1+1}^{\ell_1},
\]

\[
T_- \psi_{n_1}^{\ell_1} = -\sqrt{(\ell_1 + n_1)(\ell_1 - n_1 + 1)} \psi_{n_1-1}^{\ell_1}.
\]

Let also \( \phi_{n_2}^{\ell_2}, n_2 = -\ell_2, -\ell_2 + 1, \ldots, \ell_2 + 1, \ell_2 \) be an analogous base in the unitary space \( \mathcal{V}_{\ell_2}, \ell_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \) of irreducible representation \( \mathcal{T}_{\ell_2} \) of another copy of the group SU(2), generated by \( G_1, G_2, G_3 \), with similar relations

\[
W_0 \phi_{n_2}^{\ell_2} = n_2 \phi_{n_2}^{\ell_2}, \quad W_+ \phi_{n_2}^{\ell_2} = -\sqrt{(\ell_2 - n_2)(\ell_2 + n_2 + 1)} \phi_{n_2+1}^{\ell_2},
\]

\[
W_- \phi_{n_2}^{\ell_2} = -\sqrt{(\ell_2 + n_2)(\ell_2 - n_2 + 1)} \phi_{n_2-1}^{\ell_2}.
\]

Here we identify operators \( T_{\pm}, T_0 \) and \( W_{\pm}, W_0 \) with their restrictions onto \( U_{\ell_1} \) and \( \mathcal{V}_{\ell_2} \). An every irreducible representation of SO(4) is isomorphic to the product \( \mathcal{T}_{\ell_1} \otimes \mathcal{T}_{\ell_2} \), where \( \ell_1, \ell_2 \) are simultaneously integer or half-integer numbers.

Since the group \( K \cong SO(2) \) is one-dimensional and its algebra is generated by the element \( X_2^2 = L_1 + G_1 = i(T_0 + W_0) \), subspaces

\[
\mathcal{T}_{(\ell_1, \ell_2)} \subset \mathcal{T}_{(\ell_1, \ell_2)} := \mathcal{T}_{\ell_1} \otimes \mathcal{T}_{\ell_2}
\]

of decomposition (28) for the group SO(4) and \( k = (\ell_1, \ell_2) \) are generated by vectors

\[
\chi^\ell_j := \psi_j^{\ell_1} \otimes \phi_j^{\ell_2}, \quad \ell = (\ell_1, \ell_2), -\min(\ell_1, \ell_2) \leq j \leq \min(\ell_1, \ell_2). \tag{25}
\]

The dimension of \( \mathcal{T}_{(\ell_1, \ell_2)} \) equals

\[
2 \min(\ell_1, \ell_2) + 1 = \ell_1 + \ell_2 - |\ell_1 - \ell_2| + 1.
\]

One should find all common eigenvectors of operators \( D_0^2, D_1, D_2 \) and optionally \( D_3 \) in the space \( \mathcal{T}_{(\ell_1, \ell_2)} \).

Evidently, eigenvectors of the operator \( D_0^2 = -(T_0 - W_0)^2 \) are

\[
\chi_0^\ell, c_+ \chi_j^\ell + c_- \chi_{-j}^\ell, \quad j = 1, 2, \ldots, \min(\ell_1, \ell_2), \tag{26}
\]

if \( \ell_1, \ell_2 \) are integer and

\[
c_+ \chi_j^\ell + c_- \chi_{-j}^\ell, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, \min(\ell_1, \ell_2), \tag{27}
\]

if \( \ell_1, \ell_2 \) are half-integer. The corresponding eigenvalues are \(-4j^2\).

Since

\[
D_1 = -\frac{1}{2} (\{T_+, T_-\} + \{W_+, W_-\}) - T_+ W_- - T_+ W_+,
\]

\[
D_2 = -\frac{1}{2} (\{T_+, T_-\} + \{W_+, W_-\}) + T_+ W_- + T_+ W_+,
\]

one should choose eigenvectors of the operators \( T_+ W_- + T_- W_+ \) and \( \{T_+, T_-\} + \{W_+, W_-\} \) from (26) and (27).
The base \((25)\) consists of eigenvectors of the operator
\[
\{T_+, T_-\} + \{W_+, W_-\} = -(T_0 - W_0)^2 - D_0^2 - D_1 - D_2.
\]
In fact
\[
(\{T_+, T_-\} + \{W_+, W_-\}) \chi^j_\ell = 2(\ell_1(\ell_1 + 1) + \ell_2(\ell_2 + 1) - 2j^2)\chi^j_\ell,
\]
therefore it is enough to choose eigenvectors of the operators \(T_+W_- + T_-W_+\) from \((26)\) and \((27)\). Since
\[
(T_+W_- + T_-W_+) \chi^j_0 = \sqrt{\ell_1 \ell_2(\ell_1 + 1)(\ell_2 + 1)}(\chi^j_\ell + \chi^j_{-\ell}),
\]
one gets common eigenvectors \(\chi^{(\ell_1,0)}_0, \chi^{(0,\ell_2)}_0\), where \(\ell_1\) and \(\ell_2\) are integer.

Let \(\varepsilon_\ell := 1\) if \(\ell_1, \ell_2\) are integer and \(\varepsilon_\ell := \frac{1}{2}\) if \(\ell_1, \ell_2\) are half-integer. Since for \(j = \varepsilon_\ell \varepsilon_\ell + 1, \ldots, \min(\ell_1, \ell_2)\) it holds
\[
(T_+W_- + T_-W_+) (c_+\chi^j_\ell + c_-\chi^j_{-\ell}) = \sqrt{(\ell_1 - j)(\ell_1 + j + 1)(\ell_2 - j)(\ell_2 + j + 1)}(c_+\chi^j_{\ell+1}
+c_-\chi^j_{-\ell-1}) + \sqrt{(\ell_1 + j)(\ell_1 + j + 1)(\ell_2 + j)(\ell_2 - j + 1)}(c_+\chi^j_{\ell-1} + c_-\chi^j_{-\ell+1}),
\]
the requirement
\[
(T_+W_- + T_-W_+) (c_+\chi^j_\ell + c_-\chi^j_{-\ell}) \sim (c_+\chi^j_\ell + c_-\chi^j_{-\ell})
\]
implies \((\ell_1 - j)(\ell_1 + j + 1)(\ell_2 - j)(\ell_2 + j + 1) = 0\), that gives two cases: \(\varepsilon_\ell \leq j = \ell_1 \leq \ell_2\) and \(\varepsilon_\ell \leq j = \ell_2 \leq \ell_1\).

In the first case one obtains
\[
\sqrt{2\ell_1(\ell_2 + 1)(\ell_2 + \ell_1 + 1)}(c_+\chi^j_{\ell_1,1} + c_-\chi^j_{-\ell_1,-1}) = c(c_+\chi^j_{\ell_1,1} + c_-\chi^j_{-\ell_1,-1}), \ c \in \mathbb{C},
\]
that means either \(\ell_1 - 1 = -\ell_1\) or \(\ell_1 - 1 = -\ell_1 + 1 = 0, c_+ + c_- = 0\). This gives the following eigenvectors \(\chi^{(\ell_1,\ell_2)}_{1/2} \pm \chi^{(\ell_1,\ell_2)}_{-1/2}, c = (\ell_2 + \frac{1}{2})\) and \(\chi^{(1,\ell_2)}_1 - \chi^{(1,\ell_2)}_{-1}, c = 0\).

In the second case one similarly gets eigenvectors of the operator \(T_+W_- + T_-W_+\) in the form:
\[
(T_+W_- + T_-W_+) (\chi^{(\ell_1,\ell_2)}_{1/2} \pm \chi^{(\ell_1,\ell_2)}_{-1/2}) = \pm(\ell_1 + \frac{1}{2})(\chi^{(\ell_1,\ell_2)}_{1/2} \pm \chi^{(\ell_1,\ell_2)}_{-1/2}),
\]
\[
(T_+W_- + T_-W_+) (\chi^{(1,\ell_1)}_{\ell_2} - \chi^{(1,\ell_1)}_{-\ell_2}) = 0.
\]

Only two first vectors \(\chi^{(\ell_1,0)}_0, \chi^{(0,\ell_2)}_0\) found above are eigenvectors of the operator \(D_3 = i(T_+W_- - T_-W_+)\).

This consideration is summarized in the following theorem.

**Theorem 5.** In the space \(L^2(SO(4), SO(2), d\eta_\alpha)\) there are eight partially overlapping series of common eigenvectors for operators \(D_0^2, D_1\) and \(D_2\):

1. \(D_0^{(\ell_1,0)} = D_3\chi^{(\ell_1,0)}_0 = 0, D_1\chi^{(\ell_1,0)}_0 = D_2\chi^{(\ell_1,0)}_0 = -\ell_1(\ell_1 + 1)\chi^{(\ell_1,0)}_0, \ell_1 = 0, 1, 2, \ldots;\)

2. \(D_0^{(0,\ell_2)} = D_3\chi^{(0,\ell_2)}_0 = 0, D_1\chi^{(0,\ell_2)}_0 = D_2\chi^{(0,\ell_2)}_0 = -\ell_2(\ell_2 + 1)\chi^{(0,\ell_2)}_0, \ell_2 = 0, 1, 2, \ldots;\)

3. \(D_0^{(1,1)} \chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}) = -(\chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}),\)
\(D_1(\chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}) = -(\ell_2 + 2\ell_2 + \frac{3}{2})(\chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}),\)
\(D_2(\chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}) = -(\ell_2 - \frac{1}{2})(\chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}),\)
\(D_3(\chi^{(\ell_1,\ell_2)}_{1/2} + \chi^{(\ell_1,\ell_2)}_{-1/2}) = -i(\ell_2 + \frac{1}{2})(\chi^{(\ell_1,\ell_2)}_{1/2} - \chi^{(\ell_1,\ell_2)}_{-1/2}), \ell_2 = \frac{1}{2}, \frac{3}{2}, \ldots;\)
from theorem 5, one gets spectral equation for the two-body problem on the sphere

\[ D_3(\ell_1, \ell_2) + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)} = -(\ell_1 \frac{\ell_1}{2} + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)}), \]

\[ D_1(\ell_1, \ell_2) + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)} = -(\ell_1^2 + 2\ell_1 + \frac{3}{4})(\ell_1 \frac{\ell_1}{2} + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)}), \]

\[ D_2(\ell_1, \ell_2) + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)} = -(\ell_1^2 + \frac{1}{2})(\ell_1 \frac{\ell_1}{2} + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)}), \]

\[ D_3(\ell_1, \ell_2) + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)} = -i(\ell_1 + \frac{1}{2})(\ell_1 \frac{\ell_1}{2} + \chi_{-\frac{1}{2}}^{(\ell_1, \ell_2)}), \]

Only the first and the second vectors are also eigenvectors for the operator \( D_3 \).

Seeking an eigenfunction of the operator \( \tilde{H} \), in the form \( f(r)\psi \), where \( \psi \) is some vector from theorem 5, one gets spectral equation for the two-body problem on the sphere \( S^3 \) in the form

\[ -\frac{(1 + r^2)^3}{8mR^2 \partial^2 \partial r} \left( \frac{r^2}{1 + r^2} f' \right) + \left( \frac{1}{mR^2} \left( \frac{a}{r^2} + b + cr^2 \right) + U(r) - E \right) f = 0. \]  \( (28) \)

The first and the second case of theorem 5 correspond to arbitrary particle masses \( m_1, m_2 \) and equalities

\[ a = c = \frac{\ell(\ell + 1)}{8}, \quad b = \frac{\ell(\ell + 1)}{4}, \quad \ell = 0, 1, 2, \ldots \]

In other cases both particle masses equals \( 2m \) and it holds

\[ a = \frac{1}{8}(\ell^2 - \frac{1}{4}), \quad b = \frac{1}{4}(\ell^2 + \ell + \frac{3}{4}), \quad c = \frac{1}{8}(\ell^2 + 2\ell + \frac{3}{4}), \quad \ell = \frac{1}{2}, \frac{3}{2}, \ldots \]  in cases 3 and 4;

\[ a = \frac{1}{8}(\ell^2 + 2\ell + \frac{3}{4}), \quad b = \frac{1}{4}(\ell^2 + \ell + \frac{3}{4}), \quad c = \frac{1}{8}(\ell^2 - \frac{1}{4}), \quad \ell = \frac{1}{2}, \frac{3}{2}, \ldots \]  in cases 5 and 6;

\[ a = c = \frac{\ell(\ell + 1)}{8}, \quad b = \frac{\ell^2 + \ell + 2}{4}, \quad \ell = 1, 2, 3, \ldots \]  in cases 7 and 8.
Note that the spectral one-particle equation for the radial component \( \psi(r) \) of an eigenfunction has the form
\[
- \frac{(1 + r^2)^3}{8mR^2r^2} \frac{\partial}{\partial r} \left( \frac{r^2}{1 + r^2} f' \right) + \left( (l(l + 1) - \frac{1}{8mR^2} (r^2 + 2 + r^2) + U - E \right) f(r) = 0, \quad l = 0, 1, 2, \ldots
\]
(29)
Therefore energy levels can be exactly found from equation (28) for \( a = c \) iff they can be found for the one-particle problem with the same potential.

Usually, the spectrum of an ordinary differential operator can be exactly found if the corresponding equation can be solved in elementary functions or it can be reduced to the hypergeometric equation (or its limiting cases). The hypergeometric equation is a particular case of the Riemann equation, while the latter can be reduced to the former by well-known linear transformations of a dependent variable [39].

The equation
\[
- \frac{(1 + r^2)^3}{8mR^2r^2} \frac{\partial}{\partial r} \left( \frac{r^2}{1 + r^2} f' \right) + \left( \eta r^{-2} + \nu r^2 - E \right) f(r) = 0, \quad \eta, \nu = \text{const}
\]
(30)
is the Riemannian one w.r.t. the independent variable \( \xi = r^2 \). For \( \eta, \nu > 0 \) the corresponding differential operator admits the Friedrichs self-adjoint extension [22], and the energy levels are
\[
E_k = \mu \left[ k(k+1) - \frac{5}{8} + (2k+1) \left( \sqrt{\frac{1}{16} + \frac{\eta}{\mu}} + \sqrt{\frac{1}{16} + \frac{\nu}{\mu}} \right) \right]
+ 2 \sqrt{\frac{1}{16} + \frac{\eta}{\mu}} \sqrt{\frac{1}{16} + \frac{\nu}{\mu}}, \quad \mu = \frac{1}{2mR^2}.
\]
(31)

By the obvious change of variables in (31) one can easily find energy levels for the equation (28) with the potential \( U = \alpha r^{-2} + \beta r^2 \), \( \alpha, \beta \geq 0 \).

For the sphere \( S^n \) the analogs of the Coulomb and oscillator potentials are [8]
\[
U_q = \frac{\gamma}{2R} \left( r - \frac{1}{r} \right), \quad U_o = \frac{2\omega^2 R^2 r^2}{(1-r^2)^2}.
\]
(32)
All trajectories of a classical one-particle motion are closed for these potentials. Equation (29) for potentials [52] can be reduced to the Riemann one by changing the independent variable \( r \to u = (1 - r^2)/r \) for the Coulomb potential and \( r \to v = u^2 \) for the oscillator one.

However the coefficients of equation (30) for these potentials are rational in the independent variables \( u \) and \( v \) only for \( \eta = \nu \). This way therefore allows us to reduce equation (28) with potentials \( U = U_q \) and \( U = U_o \) to the Riemann equation only for \( a = c \), i.e. in cases 1, 2, 7, 8 of theorem 4.

Theorem 4 implies the self-adjointness of the operator \( \tilde{H}_s \) with \( U = U_q \) for any \( n \geq 2 \).

For the operator \( \tilde{H}_s \) with \( U = U_q \) we use the Friedrichs self-adjoint extension. Energy levels for equation (29) with \( U = U_q \) are (see, for example, [7, 8]):
\[
E_k = - \frac{1}{2mR^2} + \frac{(k+1)^2}{2mR^2} - \frac{m\gamma^2}{2(k+l)^2}, \quad k = 1, 2, 3, \ldots
\]
Changing coefficients one can find energy levels for equation (28) in cases 1, 2, 7, 8 of theorem 3 with \( U = U_q \)
\[
E_k = \frac{1}{mR^2} \left( \frac{1}{2} (k^2 - k + 1) - \frac{3}{4} + 2c + b + \frac{2k-1}{4} \sqrt{1 + 32c} \right)
- \frac{2m\gamma^2}{(\sqrt{1 + 32c} + 2k - 1)^2}, \quad k \in \mathbb{N}.
\]
Similarly, the formula

$$E_k = -\frac{1}{2mR^2} \left( \frac{3}{4} - \left(2k + l + \frac{3}{2}\right)^2 \right) + \frac{\omega(2k + l + \frac{3}{2})}{\sqrt{m}} \sqrt{1 + \frac{1}{4\omega^2 R^4 m^2}}, \quad k = 0, 1, 2, \ldots$$

for energy levels of equation \(28\) with \(U = U_0\) implies the following energy levels for equation \(28\) in cases 1, 2, 7, 8 of theorem 5 with \(U = U_0\)

$$E_k = \frac{1}{8mR^2} \left( (4k + 2 + \sqrt{1 + 32c})^2 - 16c + 8b - 3 \right) + \frac{\omega}{2\sqrt{m}} (4k + 2 + \sqrt{1 + 32c}) \sqrt{1 + \frac{1}{4R^4 m^2}}, \quad k = 0, 1, 2, \ldots$$

6 Reduction of cotangent bundles of homogeneous manifolds

Results of this section will be used below for the two-body problem on constant curvature spaces.

Recall that an action of a Lie group \(\Gamma\) on a smooth manifold \(M\) is proper, if for the map \(\Gamma \times M \rightarrow M \times M, (g, x) \rightarrow (gx, x)\) preimages of all compact sets are compact. If additionally this action is free, then the orbit space \(\tilde{O}\) for any element \(O\) is therefore the set of all \(\tilde{O}\)-actions, additionally this action is free, then the orbit space \(\tilde{O}\) for any element \(O\) is therefore the set of all \(\tilde{O}\)-actions, and the formula

$$\tilde{O} = \{\text{preimages of all compact sets are compact} \land \text{preimages of all compact sets are compact} \land \text{preimages of all compact sets are compact}\}$$

for any element \(Y_0' \in g_0\). It means that the formula \(\tilde{O}(\tilde{X}, \tilde{Y}) = \omega(d\pi^{-1}\tilde{X}, d\pi^{-1}\tilde{Y})\) defines the 2-form \(\tilde{\omega}\) on \(T\tilde{O}_{\beta}\), for \(\tilde{X} \in T\pi_\beta\tilde{O}_{\beta}, \tilde{Y} \in T\pi_\beta\tilde{O}_{\beta}\).

**Theorem 6.** Suppose that the orbit \(O_{\beta_0}\) is transversal to the subspace \(\text{ann} g_0 \subset g^*\) and therefore the set \(O'_{\beta_0}\) is a submanifold of the orbit \(O_{\beta_0}\). Let also the \(Ad_{\Gamma_0}^*\)-action on the
space \( O'_{\beta_0} \) be free and proper. Then the reduced phase space \( \tilde{M}_{\beta_0} \), corresponding to the value \( \beta_0 \) of the momentum map, is symplectomorphic to the symplectic space \( (O_{\beta_0}, \tilde{\omega}) \).

**Proof.** Consider a point \( x \in M_{\beta_0} \) of the level set
\[
M_{\beta_0} := \Phi^{-1}(\beta_0) \subset M
\]
for the momentum map as the orbit \( O_{\gamma'} \) of some point \( x' = (\gamma, p) \in T^*\Gamma, \gamma \in \Gamma, p \in T^*_\gamma \Gamma \) under right \( \Gamma_0 \)-shifts on \( T^*\Gamma \). To avoid cumbersome notations we preserve symbols \( L_{\gamma_1} \) and \( R_{\gamma_1} \) respectively for the left \( (\gamma, p) \rightarrow (\gamma_{\gamma_1}, L^*_{\gamma_{\gamma_1}}p) \) and the right \( (\gamma, p) \rightarrow (\gamma_{\gamma_1}, R^*_{\gamma_{\gamma_1}}p) \) actions of an element \( \gamma_1 \in \Gamma \) on \( T^*\Gamma \). Due to the definition of the momentum map [21] for a vector
\[
X = \left. \frac{d}{dt} \right|_{t=0} L_{\exp(tX')} \gamma, \ X' \in g, X \in T^*_\gamma \Gamma
\]
it holds \( p(X) = \beta_0(X'), \) i.e. \( p = R^*_{\gamma_{\gamma_1}} \beta_0 \). If additionally \( X' \in \text{Ad}^*_g \gamma_0 \), then \( X \in d\gamma_1(T_x O_{\gamma_0}') \), where \( \gamma_1 : T^*\Gamma \rightarrow \Gamma \) is the canonical projection, and \( p(X) = 0 \). Thus one gets \( \text{Ad}^*_g \beta_0 \big|_{g_0} = 0 \).

Denote \( O = \left\{ x' = (\gamma, p) \in T^*\Gamma \mid \text{Ad}^*_g \beta_0 \big|_{g_0} = 0, p = R^*_{\gamma_{\gamma_1}} \beta_0 \right\} \). Due to the theorem assumptions the set \( O \) is a submanifold of \( T^*\Gamma \). An action of a Lie subgroup on the whole Lie group (or its submanifolds) by shifts is always proper. Therefore the quotient manifold \( O/\Gamma_0 \) coincides with the set \( M_{\beta_0} \), which is therefore a submanifold of the space \( M \). Let \( \tau : O \rightarrow g^* \equiv T^*\Gamma \) be a map defined by the formula \( \tau(\gamma, p) = L^*_\gamma p = \text{Ad}^*_\gamma \beta_0 \). The following diagram is commutative [21]
\[
\begin{array}{ccc}
T^*\Gamma & \xrightarrow{L_{\gamma_{\gamma_1}}^{-1}} & T^*\Gamma \\
\Phi & & \Phi \\
g^* & \xrightarrow{\text{Ad}^*_g} & g^*
\end{array}
\]
 Consequently two points of the manifold \( O \) are mapped by \( \tau \) into one point iff they lie on one \( \Gamma_{\beta_0} \)-orbit w.r.t. left \( \Gamma_{\beta_0} \)-shifts on the manifold \( O \). By definition of \( O \) it holds \( \tau(O) = O'_{\beta_0} \), and therefore \( \tau \) is the quotient map \( O \rightarrow \Gamma_{\beta_0} \backslash O = O'_{\beta_0} \).

The point \( (\gamma, p) \) is mapped by \( \tau \) into \( \text{Ad}^*_g \gamma_0 \beta_0 \), so the point \( R_{\gamma_{\gamma_1}}(\gamma, p) \) is mapped into \( \text{Ad}^*_g \gamma_0 \beta_0 = \text{Ad}^*_\gamma \text{Ad}^*_g \gamma_0 \beta_0 \). Thus \( \Gamma_0 \)-orbits in \( O \) w.r.t. right shifts are mapped into \( \text{Ad}^*_\gamma \)-orbits in \( O'_{\beta_0} \).

By definition the \( \text{Ad}^*_\gamma \)-action in \( O'_{\beta_0} \) is free and proper, therefore the intersection of \( L_{\Gamma_{\beta_0}} \)- and \( R_{\Gamma_{\beta_0}} \)-orbits in \( O \) consists of no more than of one point. This implies that \( L_{\Gamma_{\beta_0}} \)-action on the manifold \( M_{\beta_0} = O/\Gamma_0 \) is free and the reduced space \( \tilde{M}_{\beta_0} := \Gamma_{\beta_0} \backslash M_{\beta_0} \) is a quotient manifold.

Hence the map \( \tau \) induces the diffeomorphism
\[
\phi : \tilde{M}_{\beta_0} = \Gamma_{\beta_0} \backslash M_{\beta_0} = \Gamma_{\beta_0} \backslash (O/\Gamma_0) = (\Gamma_{\beta_0} \backslash O) / \Gamma_0 \rightarrow O'_{\beta_0} / \text{Ad}^*_\gamma \Gamma_0 = \tilde{\phi}_{\beta_0}.
\]

Finally we need to prove that the symplectic form \( \tilde{\omega} \) on \( \tilde{M}_{\beta_0} \) is mapped by \( \phi \) into the form \( -\tilde{\omega} \). However this fact is an easy consequence of its particular case for \( \Gamma_0 = \{e\} \) [21], the possibility to represent vectors tangent to the space \( \tilde{M}_{\beta_0} \) via vectors tangent to \( O \), and the commutativity of the following diagram
\[
\begin{array}{ccc}
O & \xrightarrow{R_{\beta_0}} & O \\
\tau & & \tau \\
O'_{\beta_0} & \xrightarrow{R_{\beta_0}} & O'_{\beta_0}
\end{array}
\]
for any \( \gamma_0 \in \Gamma_0 \).

The form \( \tilde{\omega} \) is symplectic, therefore one gets

**Corollary 1.** The form \( \tilde{\omega} \) on \( \tilde{\mathcal{O}}_{\beta_0} \) is symplectic, i.e. it is nondegenerate and closed.

## 7 Hamiltonian reduction of the two-body problem on constant curvature spaces

We adjust Poisson brackets with a symplectic structure in the following way. Let \( X_h \) be a Hamiltonian vector field on a symplectic space \( M \), corresponding to a Hamilton function \( h \), then

\[
dh = \omega(\cdot, X_h) \equiv -i_{X_h} \omega,
\]

where \( i_{X} \omega \) is the contraction of the vector field \( X \) and the symplectic form \( \omega \). The Poisson brackets of functions \( f \) and \( h \) on \( M \) are

\[
[f, h]_P := -\omega(X_f, X_h) = -dh(X_f) = df(X_h).
\]

(34)

It was noted in [19] that the classical two-body problem on spaces \( H^n \) and \( S^n, n \geq 3 \) reaches its full generality at \( n = 3 \), since for \( n > 3 \) any two elements from the space \( T^*H^n (T^*S^n) \) are in some subspace \( T^*H^3 \subset T^*H^n (T^*S^3 \subset T^*S^n) \). Therefore two particles with a central interaction will always stay in some subspace \( H^3 \) (in \( S^3 \)). Below we consider the case \( n = 3 \).

### 7.1 Two body problem on \( S^3 \)

Let the space \( M = T^*Q \), is endowed with the standard symplectic structure of a cotangent bundle. Then due to section 3.1 one can represent the manifold \( M \) in the form

\[
T^*\mathbb{R}_+ \times T^* (\text{SO}(4)/\text{SO}(2)).
\]

(35)

up to a zero measure set. The symmetry group SO(4) acts only onto the second factor of the product (35), therefore the construction from section 6 can be easily generalize for the case under consideration. The reduced phase space for (35) is

\[
\tilde{M}_{\beta_0} = T^*\mathbb{R}_+ \times \tilde{\mathcal{O}}_{\beta_0},
\]

where the space \( \tilde{\mathcal{O}}_{\beta_0} \) is constructed for the groups \( \Gamma = \text{SO}(4), \Gamma_0 = \text{SO}(2) \) as in section 6.

Below we shall introduce coordinates in the space \( \tilde{M}_{\beta_0} \) and express the reduced two-body Hamilton function through these coordinates using formula (16).

For \( n = 3 \) the Killing vector fields (2) are \( X^{12}, X^{31}, X^{23}, Y^1, Y^2, Y^3 \). In the present section for simplicity we use the same notations for the corresponding basis in \( \mathfrak{so}(4) \) (omitting the superscript "s") in accordance with [9]. Let

\[
L^1 = X^{23} + Y^1, L^2 = X^{31} + Y^2, L^3 = X^{12} + Y^3,
G^1 = X^{23} - Y^1, G^2 = X^{31} - Y^2, G^3 = X^{12} - Y^3
\]

be the base in \( \mathfrak{so}^*(4) \), dual to (24). Let also

\[
p = p_1X^{23} + p_2X^{31} + p_3X^{12} + p_4Y^1 + p_5Y^2 + p_6Y^3 = \sum_{i=1}^{3} (u_iL^i + v_iG^i)
\]

(36)

be an arbitrary element from the space \( \mathfrak{so}^*(4) \).
In order to avoid cumbersome calculations, similar to calculations in section 3.1, we pass from the quantum case to the classical one changing a filtered operator algebra by the corresponding graded one. In particular, commutator relations turn into Poisson brackets. Formulas (13) and (16) lead to the following expression

\[ H_s = \frac{(1 + r^2)^2}{8mR^2} p_r^2 + \frac{1}{a} p_r^2 + \frac{1}{2} A_s \left( p_r^2 + p_3^2 \right) + \frac{1}{2} C_s \left( p_r^2 + p_6^2 \right) - B_s \left( p_3 p_5 - p_2 p_6 \right) + U(r) \]

for the classical Hamilton function, where \( p_r \) is the momentum, corresponding to the coordinate \( r \).

Expressions

\[ P_0 := p_4, \quad P_1 := p_5^2 + p_6^2, \quad P_2 := p_2^2 + p_3^2, \quad P_3 := -p_3 p_5 + p_2 p_6 \]

correspond to SO(4)-invariant functions on the space \( T^* (SO(4)/SO(2)) \). The substitution \( D_k \to P_k \) and the subsequent rejection of summands with a degree less than \( \deg D_k + \deg D_j - 1 \) transform commutator relations \( [D_k, D_j] \) (see (14)) into Poisson brackets \([P_k, P_j]_P\). Thus one gets

\[ [P_0, P_1]_P = -2P_3, \quad [P_0, P_2]_P = 2P_3, \quad [P_0, P_3]_P = P_1 - P_2, \]
\[ [P_1, P_2]_P = -4P_0 P_3, \quad [P_1, P_3]_P = -2P_0 P_1, \quad [P_2, P_3]_P = 2P_0 P_2. \] (37)

Changing variables as \( p_i = u_i + v_i, \quad p_{3+i} = u_i - v_i, \quad i = 1, 2, 3 \) we obtain the following form of the two-body Hamilton function

\[ H_s = \frac{(1 + r^2)^2}{8mR^2} p_r^2 + \frac{1}{a} \left( u_1 - v_1 \right)^2 + \frac{1}{2} A_s \left( (u_2 + v_2)^2 + (u_3 + v_3)^2 \right) + \frac{1}{2} C_s \left( (u_2 - v_2)^2 + (u_3 - v_3)^2 \right) - 2B_s (u_2 v_3 - v_2 u_3) + U(r). \]

We now construct canonical conjugate coordinates on the space \( \tilde{O}_{\beta_0} \). Due to the special choice of the point \( x_0 \) in the submanifold \( F_r \) (see section 3.1) its stabilizer \( K \cong SO(2) \) is generated by \( X_{23} \). It is well known that coadjoint orbits of the group \( SO(3) \) are two dimensional spheres. The Kirillov form on these spheres coincides with their area forms. Therefore the orbit \( O_{\beta_0} \) is a set of points \( \tilde{O}_{\beta_0} \) such that their coordinates \( u_i, \quad v_i, \quad i = 1, 2, 3 \) satisfy the following equations

\[ u_1^2 + u_2^2 + u_3^2 = \mu^2, \quad v_1^2 + v_2^2 + v_3^2 = \nu^2, \] (38)

where \( \mu, \nu \) are nonnegative real numbers.

The subset \( O'_{\beta_0} \subset O_{\beta_0} \) consists of elements from \( O_{\beta_0} \) that are annulled by the vector \( X_{23} \) and for description of the subset \( O'_{\beta_0} \) one must add the condition \( p_1 = u_1 + v_1 = 0 \) to equations (38).

Let us verify the first assumption of theorem 6, i.e. whether the orbit \( O_{\beta_0} \) is transversal to the subspace \( \text{ann} X_{23} \subset \mathfrak{so}^*(4) \). Consider a point \( z \in O'_{\beta_0} \) with coordinates

\[ (u_1, u_2, u_3, v_1, v_2, v_3 = -u_1). \]

First let \( \mu, \nu > 0 \). A vector

\[ Z = \sum_{i=1}^{3} (y_i L^i + z_i G^i) \]

is tangent to the orbit \( O_{\beta_0} \) iff

\[ u_1 y_1 + u_2 y_2 + u_3 y_3 = 0, \quad -u_1 z_1 + v_2 z_2 + v_3 z_3 = 0. \] (39)
Since \( \dim \text{ann } X_{23} = 5 \), the orbit \( O_{\beta_0} \) is not transversal to the subspace \( \text{ann } X_{23} \) at the point \( z \) iff \( T_zO_{\beta_0} \subset \text{ann } X_{23} \). On the coordinate level the latter condition means that equations (39) imply the equality
\[
y_1 = 0.
\]
If \( \mu > 0, \nu > 0 \), then \( u_1 = v_1 = v_2 = v_3 = 0 \) and a vector \( Z = y_1L^1 + y_2L^2 + y_3L^3 \) is tangent to the orbit \( O_{\beta_0} \) iff
\[
u_2y_2 + u_3y_3 = 0.
\]
(40)

Since equation (40) does not restrict values of \( y_1 \), the orbit \( O_{\beta_0} \) is again transversal to the subspace \( X_{23} \). The case \( \mu = 0, \nu > 0 \) is completely similar.

Thus the orbit \( O_{\beta_0} \) is transversal to the subspace \( \text{ann } X_{23} \subset \mathfrak{so}^*(4) \) iff \( \mu \neq \nu \).

Consider the cases \( \mu \neq \nu \) separately.

1. Let \( \mu \neq \nu \).

(a) First consider the subcase \( \mu, \nu > 0 \). Let \( u, \psi, \chi \) be coordinates on the space \( O'_{\beta_0} \), defined by the following equations
\[
u_1 = -v_1 = u, \quad u_2 = \sqrt{\mu^2 - u^2}\sin \psi, \quad u_3 = \sqrt{\mu^2 - u^2}\cos \psi,
\]
\[
u_2 = \sqrt{\nu^2 - u^2}\sin \chi, \quad v_3 = \sqrt{\nu^2 - u^2}\cos \chi, \quad -\min\{\mu, \nu\} \leq u \leq \min\{\mu, \nu\}.
\]

The restriction of the Kirillov form from \( O_{\beta_0} \) onto \( O'_{\beta_0} \) is
\[
\omega = \frac{1}{\mu^2} (u_1du_2 \land du_3 + u_2du_3 \land du_1 + u_3du_1 \land du_2)
+ \frac{1}{\nu^2} (v_1dv_2 \land dv_3 + v_2dv_3 \land dv_1 + v_3dv_1 \land dv_2) = du \land d(\psi - \chi).
\]

Formulas \( u \to u, \psi \to \psi + \xi, \chi \to \chi + \xi, 0 \leq \xi < 2\pi \) describe the \( \text{Ad}_K^* \)-action in \( O'_{\beta_0} \). This action is free and proper. Therefore \( \tilde{O}_{\beta_0} = O'_{\beta_0} / \text{Ad}_K^* \) is a quotient manifold with canonical conjugate coordinates \( \phi = \psi - \chi, p_\phi = u \).

For \( \mu > \nu > 0 \) an arbitrary \( \text{Ad}_K^* \)-orbit in \( \tilde{O}_{\beta_0} \) contains a unique point with coordinates
\[
u_1 = u, \quad u_2 = 0, \quad u_3 = \sqrt{\mu^2 - u^2}, \quad v_1, v_2, v_3 = -u
\]
such that
\[
u_i^2 + \nu_j^2 + \nu_k^2 = \nu^2.
\]
This implies that the space \( \tilde{O}_{\beta_0} \) is diffeomorphic to the sphere \( S^2 \). Similarly, for \( \nu > \mu > 0 \) the space \( \tilde{O}_{\beta_0} \) is also diffeomorphic to the sphere \( S^2 \).

The coordinate system \( p_\phi, \phi \) has a singularity at the points \( p_\phi = \pm \min\{\mu, \nu\} \).

It differs from the coordinate system on the reduced space in [14]. The reduced Hamilton function is
\[
\tilde{H}_s = \frac{(1 + r^2)^2}{8mR^2} p_r^2 + \frac{4p_\phi^2}{a} + \frac{1}{2} A_s \left( \mu^2 + \nu^2 - 2p_\phi^2 + 2\sqrt{\mu^2 - p_\phi^2} \sqrt{\nu^2 - p_\phi^2} \cos \phi \right)
+ \frac{1}{2} C_s \left( \mu^2 + \nu^2 - 2p_\phi^2 - 2\sqrt{\mu^2 - p_\phi^2} \sqrt{\nu^2 - p_\phi^2} \cos \phi \right)
- 2B_s \sqrt{\mu^2 - p_\phi^2} \sqrt{\nu^2 - p_\phi^2} \sin \phi + U(r).
\]
(b) In the subcase $\mu = 0, \nu > 0$ (or $\nu = 0, \mu > 0$) the orbit $O^\prime_{\beta_0}$ is defined by equations $u_1 = u_2 = u_3 = v_1 = 0$. Therefore it holds $O^\prime_{\beta_0} = S^1$ and $\tilde{O}_{\beta_0} = \text{pt}$.

The reduced phase space is $T^*\mathbb{R}_+^* \times O_{\beta_0}$, with the reduced Hamilton function

$$\tilde{H}_s = \frac{(1 + r^2)^2}{8mR^2} \left( p_r^2 + \frac{\nu^2}{r^2} \right) + U(r),$$

(42)

corresponding to an integrable system.

2. Let $\nu = \mu$. This case corresponds to particle motion along a two dimensional sphere $S^2 \subset S^3$ (see [19], proposition 1). Therefore one can assume $\Gamma = SO(3)$ and $\Gamma_0 = \{e\}$. Obviously, the requirements of theorem 6 are satisfied.

(a) First consider the subcase $\beta_0 \neq 0$. In accordance with theorem 6 the reduced phase space $\tilde{M}_{\beta_0}$ of the two-body problem is diffeomorphic to the space $T^*\mathbb{R}_+^* \times O_{\beta_0}$, where $O_{\beta_0} \cong S^2 \subset \mathfrak{so}^*(3)$. The reduced Hamilton function has the form

$$\tilde{H}_s = \frac{(1 + r^2)^2}{8mR^2} \left( p_r^2 + \frac{1}{a} p_3^2 + \frac{1}{2} A_s p_3^2 + \frac{1}{2} C_s p_5^2 - B_s p_3 p_5 + U(r).$$

The orbit $O_{\beta_0}$ is defined by the equation

$$p_3^2 + p_4^2 + p_5^2 = \beta_0^2$$

and it holds

$$[p_3, p_4]_P = p_5, \ [p_4, p_5]_P = p_3, \ [p_5, p_3]_P = p_4.$$

(b) The last subcase $\nu = \mu = 0$ corresponds to particle motion along a common geodesic $S^1$ (see [19], proposition 2). Here $O_{\beta_0} = \text{pt}$ and one gets

$$\tilde{M}_0 = T^*\mathbb{R}_+^* \times \tilde{O}_{\beta_0},$$

$$\tilde{H}_s = \frac{(1 + r^2)^2}{8mR^2} \left( p_r^2 + U(r).$$

7.2 Two body problem on $H^3$

After excluding the diagonal from the space $Q_h = H^3 \times H^3$ one gets the phase space of the two-body problem in the form

$$T^*I \times T^* (O_0(1, 3)/SO(2)).$$

(43)

The symmetry group $O_0(1, 3)$ acts only onto the second factor of the product, therefore the Hamiltonian reduction leads to the reduced space

$$\tilde{M}_{\beta_0} = T^*I \times \tilde{O}_{\beta_0},$$

where $\tilde{O}_{\beta_0}$ is constructed for the groups $\Gamma = O_0(1, 3), \Gamma_0 = SO(2)$ as in section 6.

Since the Lie algebra $\mathfrak{so}(1, 3)$ is simple, one can not represent $\text{Ad}^*_\mathfrak{so}(1, 3)$-orbits as direct products contrary to section 4. Nevertheless dynamic systems on the sphere $S^3$ and the hyperbolic space $H^3$ are connected by the formal substitution (see section 3.2 and [19]). This motivates the following construction.

Let $L_1 = X_{23}, \ L_2 = X_{31}, \ L_3 = X_{12}, \ Y_1, Y_2, Y_3$ be the basis in the Lie algebra $\mathfrak{so}(1, 3)$, corresponding to Killing vector fields (13), and $L^1, L^2, L^3, Y^1, Y^2, Y^3$ be the dual basis in
\( \mathfrak{so}^* (1, 3) \). Let \( p = p_1 L^1 + p_2 L^2 + p_3 L^3 + p_4 Y^1 + p_5 Y^2 + p_6 Y^3 \) be an arbitrary element from \( \mathfrak{so}^* (1, 3) \). Direct calculation shows that the expressions

\[
 I_1 = p_1^2 + p_2^2 + p_3^2 - p_4^2 - p_5^2 - p_6^2, \quad I_2 = p_1 p_4 + p_2 p_5 + p_3 p_6
\]

are invariants of \( \text{Ad}^*_O(1, 3) \)-action.

Similarly to section 7.1 one gets the following expression of the two-body Hamilton function

\[
 H_h = \frac{(1 - r^2)^2}{8mR^2} P_r^2 + 1 - \frac{1}{a} P^2 + \frac{1}{2} A_h P_2 + \frac{1}{2} C_h P_1 - B_h P_3 + U(r), \quad 0 < r < 1,
\]

where expressions

\[
 P_0 := p_4, \quad P_1 := p_5^2 + p_6^2, \quad P_2 := p_2^2 + p_3^2, \quad P_3 := -3p_5 + p_2 p_6
\]

correspond to \( O(1, 3) \)-invariant functions on the space \( T^* (O(1, 3)/SO(2)) \). One can derive Poisson brackets [\( [P_k, P_j]_\mu \)] from commutator relations \( [18] \) in full analogy with \( [13] \)

\[
 [P_0, P_1]_\mu = 2P_3, \quad [P_0, P_2]_\mu = 2P_3, \quad [P_0, P_3]_\mu = P_1 + P_2,
\]

\[
 [P_1, P_2]_\mu = -4P_0 P_3, \quad [P_1, P_3]_\mu = -2P_0 P_1, \quad [P_2, P_3]_\mu = 2P_0 P_2.
\]

Let \( O_{\beta_0} \) be an \( \text{Ad}^*_O(1, 3) \)-orbit defined by equations \( I_1 = \mu, I_2 = \nu \neq 0, \mu, \nu \in \mathbb{R} \). Therefore the subset \( O_{\beta_0} \) is defined by equations \( I_1 = \mu, I_2 = \nu, p_1 = 0 \). The stationary subgroup \( K \simeq SO(2) \) of the point \( x_0 \in F_r \) is generated by the element \( L_1 \) and the \( \text{Ad}^*_K \)-action coincides with the simultaneous rotation in coordinate planes \( (p_2, p_3) \) and \( (p_5, p_6) \). Likewise in section 7.1 one can verify that for \( \nu \neq 0 \) the orbit \( O_{\beta_0} \) is transversal to \( \text{ann} L_1 \subset \mathfrak{so}^* (1, 3) \) and the first assumption of theorem \( [16] \) is valid.

1. Let \( \nu \neq 0 \). The following formulas define coordinates \( p_4, \psi, \chi \) on the manifold \( O_{\beta_0} \)

\[
 p_2 = u \cosh \psi \cos \chi + v \sinh \psi \sin \chi, \quad p_3 = v \sinh \psi \cos \chi - u \cosh \psi \sin \chi, \\
 p_5 = v \cosh \psi \cos \chi - u \sinh \psi \sin \chi, \quad p_6 = -u \sinh \psi \cos \chi - v \cosh \psi \sin \chi,
\]

where \( p_4, \psi, \chi \in \mathbb{R}, \psi \in \mathbb{R} \mod 2\pi \) and values \( u, v \) are defined by equations

\[
 u^2 - v^2 = \mu + p_4^2, \quad uv = \nu.
\]

Two solutions of \( [16] \) differ in sign and it suffice to choose either of them. The \( \text{Ad}^*_K \)-action is free, proper and corresponds to the rotation \( \chi \to \chi + \xi \). Thus theorem \( [16] \) is applicable.

An every \( \text{Ad}^*_K \)-orbit in \( O_{\beta_0} \) contains a unique point with coordinates

\[
 p_1 = p_2 = 0, \quad p_3 > 0, \quad p_4, p_5, p_6 = \frac{\nu}{p_3}
\]

such that

\[
 p_4^2 + p_5^2 = p_3^2 - \frac{\nu^2}{p_3^2} - \mu.
\]

This equation defines a unique positive \( p_3 \), therefore the space \( \tilde{O}_{\beta_0} \) is diffeomorphic to the plane \( \mathbb{R}^2 \) with global coordinates \( p_4, p_5 \).

Thus for \( \nu \neq 0 \) the reduced phase space \( \tilde{M}_{\beta_0} \) of the two-body problem in \( \mathbb{H}^3 \) is diffeomorphic to the space

\[
 T^* I \times \mathbb{R}^2.
\]
It is well known that Ad$_G^\ast$-orbits of an arbitrary Lie group $G$ coincides with symplectic leaves of the canonical Poisson structure in the space $g^\ast$. Let $\{e_i\}_{i=1}^n$ be a basis in the Lie algebra $g$, $[e_i, e_j] = c_{ij}^k e_k$ and $\{x_i\}_{i=1}^n$ be coordinates on $g^\ast$, corresponding to the dual basis $\{e^i\}_{i=1}^n$. Let also $f_1, f_2$ be arbitrary smooth functions on $g^\ast$. Then their Poisson brackets has the form

$$[f_1, f_2]_p = \sum_{i,j,k=1}^n c_{ij}^k x_k \frac{\partial f_1}{\partial x_i} \frac{\partial f_2}{\partial x_j}.$$ 

The choice of a sing in this equation is defined by the correspondence between commutators of differential operators and Poisson brackets of corresponding functions.

We shall use Poisson brackets on $so^\ast(1, 3)$ for construction of canonical conjugate coordinates on the space $\hat{O}_{\hat{\beta}_0}$. Formulas

$$\psi = \frac{1}{4} \ln \left( \frac{(p_2 - p_6)^2 + (p_5 + p_3)^2}{(p_2 + p_6)^2 + (p_5 - p_3)^2} \right),$$

$$\chi = \frac{1}{2} \left( \arctan \left( \frac{p_5 - p_3}{p_2 + p_6} \right) - \arctan \left( \frac{p_5 + p_3}{p_2 - p_6} \right) \right),$$

$$[L_i, L_j] = \sum_{k=1}^3 \varepsilon_{ijk} L_k, \quad [Y_i, Y_j] = -\sum_{k=1}^3 \varepsilon_{ijk} Y_k,$$

yield the following relations

$$[p_4, \psi]_p = -1, \quad [p_4, \chi]_p = 0, \quad [\psi, \chi]_p = 0.$$

Therefore equations (33) and (34) imply that the symplectic structure on the space $\hat{O}_{\hat{\beta}_0}$ is defined by $dp_4 \wedge d\psi$. From (35) one gets

$$p_2^2 + p_3^2 = \frac{1}{2} \left( \mu + p_4^2 + \sqrt{(\mu + p_4^2)^2 + 4\nu^2 \cosh 2\psi} \right),$$

$$p_5^2 + p_6^2 = \frac{1}{2} \left( -\mu - p_4^2 + \sqrt{(\mu + p_4^2)^2 + 4\nu^2 \cosh 2\psi} \right),$$

$$p_3 p_5 - p_2 p_6 = \frac{1}{2} \sqrt{(\mu + p_4^2)^2 + 4\nu^2 \sinh 2\psi}.$$

Introducing the new canonical conjugate coordinates $p_\phi = p_4/2$, $\phi = 2\psi$, one gets from (36) the final form of the reduced Hamilton function

$$\bar{H}_h = \frac{(1 - \nu^2)^2}{8mR^2} p_\phi^2 + \frac{4p_\phi^2}{a} + \frac{1}{2} A_h \left( \frac{\mu}{2} + 2p_\phi^2 + 2\sqrt{\left( \frac{\mu}{4} + p_\phi^2 \right)^2 + \frac{\nu^2}{4} \cosh \phi} \right)$$

$$- \frac{1}{2} C_h \left( \frac{\mu}{2} + 2p_\phi^2 - 2\sqrt{\left( \frac{\mu}{4} + p_\phi^2 \right)^2 + \frac{\nu^2}{4} \cosh \phi} \right)$$

$$- 2B_h \sqrt{\left( \frac{\mu}{4} + p_\phi^2 \right)^2 + \frac{\nu^2}{4} \sinh \phi} + U(r).$$

2. The case $\nu = 0$ corresponds to the particle motion along a hyperbolic plane $H^2 \subset H^3$ (see [19], proposition 1). Thus one can assume $\Gamma = O_0(1, 2)$ and $\Gamma_0 = \{e\}$. Obviously, requirements of theorem [2] are satisfied.
(a) First consider the subcase $\beta_0 \neq 0$. In this case according to theorem 6 the reduced phase space $\tilde{\mathcal{M}}_{\beta_0}$ of the two-body problem is diffeomorphic to the space 

$$T^*I \times \mathcal{O}_{\beta_0},$$

and the reduced Hamilton function has the form

$$\tilde{H}_r = \frac{(1 - r^2)^2}{8mR^2} p_r^2 + \frac{1}{a}p_4^2 + \frac{1}{2}A_{h} p_3^2 + \frac{1}{2}C_{h} p_5^2 + B_{h} p_3 p_5 + U(r).$$

Here $p_3, p_4, p_5$ are coordinates on the space $so^*(1, 2)$ and

$$[p_3, p_4]_P = p_5, \quad [p_4, p_5]_P = -p_3, \quad [p_5, p_3]_P = p_4.$$

The orbit $\mathcal{O}_{\beta_0}$ is defined by the equation

$$p_3^2 - p_4^2 - p_5^2 = \mu.$$

For $\mu > 0$ the orbit $\mathcal{O}_{\beta_0}$ is a one sheet of a two-sheet hyperboloid (diffeomorphic to the plane $\mathbb{R}^2$), for $\mu = 0$ it is the cone without vertex (diffeomorphic to $\mathbb{R}^2 \setminus \text{pt}$), and for $\mu < 0$ it is a one-sheet hyperboloid (diffeomorphic to the cylinder $\mathbb{R} \times S^1$).

(b) The last subcase $\beta_0 = 0$ corresponds to particle motion along a common geodesic (see [19], proposition 2). Here $\mathcal{O}_{\beta_0} = \text{pt}$ and one gets

$$\tilde{M}_0 = T^*I, \quad \tilde{H}_s = \frac{(1 - r^2)^2}{8mR^2} p_r^2 + U(r).$$

8 Conclusion

In the present paper we have found the expression of the two-body Hamiltonian on spaces $S^n$ and $H^n$ through a radial differential operator and invariant differential operators on a homogeneous spaces of isometry groups. This expression enables to find some explicit series of energy levels for two particles on the sphere $S^3$. The most part of these series corresponds to equal particle masses. Probably, the quasi-exactly solvability of this quantum problem is connected with the existence of some closed trajectories of the corresponding classical system. Clearly, it is not difficult to find circular trajectories, when the distance between particles with equal masses is constant.

A connection of closed trajectories of some non-integrable classical mechanical problem with the spectrum of the corresponding quantum mechanical system was studied in many papers (see the overview and references in [41]). It would be interesting to find such a connection in the problem under consideration and also to calculate some exact spectral series for the two-body problem on $S^n$, $n \geq 4$.

It was conjectured in [22] that the two-body Hamiltonian on the hyperbolic plane $H^2$ has no discrete energy levels. The same seems to be valid for spaces $H^n$, $n \geq 3$.

The explicit form of the Hamilton function for the reduced two-body problem in constant curvature spaces, founded in [19] with a help of computer algebraic calculations, was used there to prove the absence of particles collision. In the present paper we have derived the explicit form of the reduced Hamilton function without computer calculations and clarify its connection with the two-body quantum Hamiltonian. This form of the Hamiltonian reduction seems to be the most natural from the geometric point of view, since the "radial" degree of freedom, invariant w.r.t. the isometry group, is isolated as a direct factor and another direct factor corresponds to the cotangent bundle over a homogeneous manifold of the isometry group. The only a priori integrable case of the reduced classical two-body problem with a central interaction on constant curvature spaces, different from particles movement along a common geodesic, corresponds to the reduced Hamilton function [42].
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