Twisted complexes on a ringed space as a dg-enhancement of perfect complexes

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Abstract
In this paper we study twisted complexes on a ringed space and prove that it gives a new dg-enhancement of the derived category of perfect complexes on that space. A twisted complex is a collection of locally defined sheaves together with the homotopic gluing data. In this paper we construct a functor from twisted complexes to perfect complexes, which turns out to be a dg-enhancement. This new enhancement has the advantage of being completely geometric and it comes directly from the definition of perfect complex.

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1 INTRODUCTION

The derived categories of perfect complexes and pseudo-coherent complexes on ringed topoi were introduced in SGA 6 [ber71]. They have played an important role in mathematics ever since. Nevertheless we would like to consider the differential graded (dg)-enhancement of these derived category. More precisely we have the following definition.

**Definition 1.1.** Let $C$ be a triangulated category. A dg-enhancement of $C$ is a pair $(B, \varepsilon)$ where $B$ is a pre-triangulated dg-category and $\varepsilon : \text{Ho} B \xrightarrow{\sim} C$ is an equivalence of triangulated categories. Here $\text{Ho} B$ is the homotopy category of $B$.

For the derived category $D_{\text{perf}}(X)$ on a ringed space $X$ we have the classical injective enhancement, which consists of perfect complexes with bounded below injective components, see [LS14] Section 3.1.

Although very useful, the injective resolution has its drawback that the modules are too "large" and the construction is not geometric. Therefore we are seeking for a new, more geometric dg-enhancement.

In the later 1970's Toledo and Tong [TT78] introduced twisted complexes as a way to get their hands on perfect complexes of sheaves on a complex manifold and implicitly they recognized this was a dg-model for the derived category of perfect complexes. In this paper we prove in all details that twisted complexes form a dg-model for categories of perfect complexes (and more generally pseudo-coherent complexes) of sheaves on a ringed space.

Let us first give an informal description to illustrate the idea of twisted complex.

Recall that a complex of sheaves $S^\bullet$ on $X$ is perfect if for any point $x \in X$, there exists an open neighborhood $x \in U \subset X$, a two-side bounded complex of finitely generated locally free sheaves $E^\bullet_U$ on $U$ together with a quasi-isomorphism

$\theta_U : E^\bullet_U \xrightarrow{\sim} S^\bullet|_U$.

For two different open subsets $U_i$ and $U_j$ we have two quasi-isomorphisms $\theta_i : E^\bullet_{U_i} \xrightarrow{\sim} S^\bullet|_{U_i}$. 


and
\[ \theta_j : E^*_U \sim \rightarrow S^*_U. \]

For simplicity we denote \( E^*_U \) by \( E^*_i \) and \( U_i \cap U_j \) by \( U_{ij} \). Hence on \( U_{ij} \) we have
\[
\begin{array}{ccc}
E^*_i |_{U_{ij}} & \sim & E^*_j |_{U_{ij}} \\
\theta_i & \sim & \theta_j \\
S^*_i |_{U_{ij}} & \sim & S^*_j |_{U_{ij}}
\end{array}
\]

Since \( E^*_i \) and \( E^*_j \) are bounded and locally free, we can refine the open cover if necessary and lift the identity map on \( S^*_i |_{U_{ij}} \) (under some assumptions on \( S^* \), see Lemma 2.7 below) to a map \( a_{ij} : E^*_j \rightarrow E^*_i \), i.e. the following diagram
\[
\begin{array}{ccc}
E^*_i |_{U_{ij}} & \sim & E^*_j |_{U_{ij}} \\
\theta_i & \sim & \theta_j \\
S^*_i |_{U_{ij}} & \sim & S^*_j |_{U_{ij}}
\end{array}
\]
commutes up to homotopy.

Then we consider a third open subset \( U_k \) together with \( E^*_k \) on it. According to the discussion above, the following diagram
\[
\begin{array}{ccc}
E^*_i |_{U_{ijk}} & \sim & E^*_j |_{U_{ijk}} \\
\theta_i & \sim & \theta_j \\
E^*_k |_{U_{ijk}} & \sim & E^*_k |_{U_{ijk}}
\end{array}
\]
commutes up to homotopy. More precisely, we have a degree \(-1\) map \( a_{kji} : E^*_i \rightarrow E^*_k^{-1} \) on \( U_{ijk} \) such that
\[ a_{ki} - a_{kj}a_{ji} = da_{kji} + a_{kji}d. \]

Hence we cannot simply use the \( a_{ji} \)'s to glue the \( E^*_i \)'s into a complex of locally free sheaves on \( X \). On the other hand, since the \( E^*_i \)'s come from the same complex of sheaves \( S^* \), we expect that the homotopy operator \( a_{kji} \)'s satisfy higher compatible relations up to higher homotopies, and so on.

Toledo and Tong show that all these compatibility data together satisfy the Maurer-Cartan equation
\[ \delta a + a \cdot a = 0 \quad (1) \]
where \( \delta \) is a Čech-like differential. They call a collection \( E^*_i \) together with such maps \( a \)'s a twisted complex or twisted cochain. In this paper we call them twisted perfect complexes and keep the term twisted complex for a more general concept. (For precise definition, see Section 2 of this paper)

Moreover, O’Brian, Toledo and Tong have proved that every perfect complex has a twisted resolution, see Proposition 1.2.3 in [OTT85] or Proposition 3.8 in this paper. This result is closely related to the essential-surjectivity part of a dg-enhancement. Nevertheless, they have not attempted to build any equivalence between the categories.

In this paper we construct a sheafification functor
\[ S : \text{Tw}_{\text{perf}}(X) \rightarrow \text{Qcoh}_{\text{perf}}(X) \]
where \( \text{Tw}_{\text{perf}}(X) \) denotes the dg-category of twisted perfect complexes on \( X \) and \( \text{Qcoh}_{\text{perf}}(X) \) denotes the dg-category of perfect complexes of quasi-coherent sheaves on \( X \).

We will prove that the functor \( S \) gives the expected dg-enhancement.
Theorem 1.1. [See Theorem 3.15 below] Under reasonable conditions, the sheafification functor induces an equivalence of categories
\[ S : \text{HoTw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X)). \] (2)

We would also like to consider perfect complexes of general \( O_X \)-modules rather than quasi-coherent modules. Actually we have

Theorem 1.2. [See Theorem 3.17 below] Under some additional conditions, the sheafification functor induces an equivalence of categories
\[ S : \text{HoTw}_{\text{perf}}(X) \to D_{\text{perf}}(X). \] (3)

Here we briefly mention the strategy of the proof. We extend the dg-category of twisted perfect complexes to a more general twisted complexes on \( X \) and define a twisting functor
\[ T : \text{Sh}(X) \to \text{Tw}(X) \]
where \( \text{Sh}(X) \) denotes the dg-category of sheaves on \( X \) and \( \text{Tw}(X) \) denotes the dg-category of twisted complexes on \( X \). The essential-surjectivity and fully faithfulness of \( S \) can be achieved by a careful study of the relations between \( S \) and \( T \).

The construction and proof are inspired by [Blo10] Section 4. In fact Block gives a Dolbeault-theoretic dg-enhancement of perfect complexes in [Blo10], while our construction can be considered as a Čech-theoretic enhancement.

This paper is organized as follows: In Section 2 we give the definition of twisted (perfect) complexes. We show that these dg-categories have a pre-triangulated structure. Moreover we introduce weak equivalence between twisted complexes.

In Section 3 we construct the dg-enhancement. In more details, we construct the sheafification functor \( S \) in Section 3.1. In Section 3.2 we prove that the image of a twisted perfect complex under \( S \) is really a perfect complex. In Section 3.3 we prove that \( S \) is essentially surjective and in Section 3.4 we prove that \( S \) is fully faithful. Hence \( S \) gives the dg-enhancement.

In Section 4 we talk about some applications of twisted complexes. In particular we talk about the application in descent theory.

In Section 5 we talk about a more general construction: the twisted coherent complex and prove that they form a dg-enhancement of the derived category of bounded above complexes of coherent sheaves. Actually the proofs are the same as those for twisted perfect complexes.

In Section 6 we make a digression and discuss the degenerate twisted complexes and how they give splitting of idempotents.

In Section 7 we discuss an approach which we do not yet taken in this paper: We wish to put a suitable model structure on twisted complexes and view \( S \) and \( T \) in terms of Quillen adjunctions.

In Appendix A we compare coherent complex and pseudo-coherent complex. Moreover we study the relation between quasi-coherent modules and general \( O_X \)-modules.

To ensure Theorem 1.1 we need that the open cover \( \{ U_i \} \) of \( X \) is fine enough, in Appendix B we discuss good covers of a ringed space \( X \).

2 A REVIEW OF TWISTED COMPLEXES

2.1 A quick review of perfect complexes

Before talking about twisted complexes, we give a quick review of the derived category of perfect complexes in this subsection. For more details see [TT90] and [Sta15].
Remark 3. Let \((X, \mathcal{O}_X)\) be a locally ringed space. A complex \(S^\bullet\) is strictly perfect if \(S^i\) is zero for all but finitely many \(i\) and \(S^i\) is a direct summand of a finite free \(\mathcal{O}_X\)-module for all \(i\). The second condition is equivalent to that \(S^i\) is a finite locally free \(\mathcal{O}_X\)-module for all \(i\).

Moreover, a complex \(S^\bullet\) of \(\mathcal{O}_X\)-modules is perfect if for any point \(x \in X\), there exists an open neighborhood \(U\) of \(x\) and a strictly perfect complex \(E_U^\bullet\) on \(U\) such that the restriction \(S^\bullet|_U\) is isomorphic to \(E_U^\bullet\) in \(D(\mathcal{O}_U - \text{mod})\), the derived category of sheaves of \(\mathcal{O}_X\)-modules on \(U\).

**Caution 1.** If we did not assume that \(X\) is a locally ringed space, then it may not be true that a direct summand of a finite free \(\mathcal{O}_X\)-module is finite locally free. See [Sta15, Tag 08C3].

**Remark 1.** In fact, the definition of perfect complex is equivalent to the stronger requirement that for any point \(x \in X\), there exists an open neighborhood \(U\) of \(x\) and a bounded complex of finite rank locally free sheaves \(E_U^\bullet\) on \(U\) together with a quasi-isomorphism

\[
E_U^\bullet \simto S^\bullet|_U.
\]

See [TT90] Lemma 2.2.9 for details.

**Remark 2.** It is obvious that a strictly perfect complex must be perfect. On the other hand, if \(X\) is a projective scheme and \(\mathcal{O}_X\) is the structure sheaf of \(X\), then a perfect complex must also be strictly perfect, see [ber71] Exposé II or [TT90] Section 2. Nevertheless for general \((X, \mathcal{O}_X)\) a perfect complexes is not necessarily strictly perfect.

We consider the following categories.

**Definition 2.2.** Let \(\text{Sh}(X)\) be the dg-category of complexes of \(\mathcal{O}_X\)-modules on \(X\). Let \(\text{Sh}_{\text{perf}}(X)\) be the full dg-subcategory of perfect complexes on \(X\).

Let \(K(X)\) be the homotopy category of complexes of \(\mathcal{O}_X\)-modules on \(X\). Then \(K_{\text{perf}}(X)\) is the triangulated subcategories of \(K(X)\) which consists of perfect complexes of \(\mathcal{O}_X\)-module.

Moreover let \(D(X)\) be the derived category of complexes of \(\mathcal{O}_X\)-modules on \(X\). Then \(D_{\text{perf}}(X)\) is the triangulated subcategory of \(D(X)\) which consists of perfect complexes \(\mathcal{O}_X\)-modules.

We need to also consider the complexes of quasi-coherent sheaves on \(X\) and we have the following definition.

**Definition 2.3.** Let \(\text{Qcoh}(X)\) be the dg-category of complexes of quasi-coherent sheaves on \(X\). It is clear that \(\text{Qcoh}(X)\) is a full dg-subcategory of \(\text{Sh}(X)\). Let \(\text{Qcoh}_{\text{perf}}(X)\) be the full dg-subcategory of \(\text{Qcoh}(X)\) which consists of perfect complexes of quasi-coherent sheaves. \(\text{Qcoh}_{\text{perf}}(X)\) is also a full dg-subcategory of \(\text{Sh}_{\text{perf}}(X)\).

Let \(K(\text{Qcoh}(X))\) be the homotopy category of complexes of quasi-coherent \(\mathcal{O}_X\)-modules on \(X\) and \(D(\text{Qcoh}(X))\) be its derived category. Similarly we have \(K_{\text{perf}}(\text{Qcoh}(X))\) and \(D_{\text{perf}}(\text{Qcoh}(X))\).

**Remark 3.** We have the natural inclusion \(i : \text{Qcoh}(X) \to \text{Sh}(X)\) which induces a functor

\[
i : D(\text{Qcoh}(X)) \to D_{\text{Qcoh}}(X),
\]

where \(D_{\text{Qcoh}}(X)\) is the derived category of complexes of \(\mathcal{O}_X\)-modules with quasi-coherent cohomologies. However for general \((X, \mathcal{O}_X)\) the functor \(i\) is not always essentially surjective nor fully faithful. As a result we need to distinguish complexes of quasi-coherent modules and complexes of general \(\mathcal{O}_X\)-modules. This issue will be discussed further in Appendix [A].
2.2 Notations of bicomplexes and sign conventions

In this subsection we introduce some notations which are necessary in the definition of twisted complexes, for reference see [OTT81] Section 1.

Let \((X, \mathcal{O}_X)\) be a locally ringed paracompact space and \(\mathcal{U} = \{U_i\}\) be an locally finite open cover of \(X\). Let \(U_{i_0 \ldots i_n}\) denote the intersection \(U_{i_0} \cap \ldots \cap U_{i_n}\).

Remark 4. [TT78], [OTT81] and [OTT85] consider the special case that \(X\) is a complex manifold and \(\mathcal{O}_X\) is the sheaf of holomorphic functions on \(X\). In this paper we consider general \((X, \mathcal{O}_X)\).

For each \(U_{i_k}\), let \(E_{i_k}^\bullet\) be a graded sheaf of \(\mathcal{O}_X\)-modules on \(U_{i_k}\). Let

\[
C^\bullet(\mathcal{U}, E^\bullet) = \prod_{p,q} C^p(\mathcal{U}, E^q)
\]

be the bigraded complexes of \(E^\bullet\). More precisely, an element \(c_{i_0 \ldots i_p}^{p,q}\) of \(C^p(\mathcal{U}, E^q)\) consists of a section \(c_{i_0 \ldots i_p}^{p,q}\) of \(E^q\) over each non-empty intersection \(U_{i_0 \ldots i_p}\). If \(U_{i_0 \ldots i_p} = \emptyset\), simply let the component on it be zero.

Now if another graded sheaf \(F_{i_k}^\bullet\) of \(\mathcal{O}_X\)-modules is given on each \(U_{i_k}\), then we can consider the map

\[
C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, F)) = \prod_{p,q} C^p(\mathcal{U}, \text{Hom}^q(E, F)).
\]

An element \(u_{i_0 \ldots i_p}^{p,q}\) of \(C^p(\mathcal{U}, \text{Hom}^q(E, F))\) gives a section \(u_{i_0 \ldots i_p}^{p,q}\) of \(\text{Hom}^q_{\mathcal{O}_X}(E_{i_0}^\bullet, F_{i_p}^\bullet)\), i.e. a degree \(q\) map from \(E_{i_0}^\bullet\) to \(F_{i_p}^\bullet\) over the non-empty intersection \(U_{i_0 \ldots i_p}\). Notice that we require \(u_{i_0 \ldots i_p}^{p,q}\) to be a map from the \(F\) on the last subscript of \(U_{i_0 \ldots i_p}\) to the \(E\) on the first subscript of \(U_{i_0 \ldots i_p}\). Again, if \(U_{i_0 \ldots i_p} = \emptyset\), let the component on it be zero.

Remark 5. In this paper when we talk about degree \((p, q)\), the first index always indicates the Čech degree while the second index always indicates the sheaf degree.

We need to study the compositions of \(C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, F))\). Let \(\{G_{i_k}^\bullet\}\) be a third graded sheaf of \(\mathcal{O}_X\)-modules. There is a composition map

\[
C^\bullet(\mathcal{U}, \text{Hom}^\bullet(F, G)) \times C^\bullet(\mathcal{U}, E^\bullet) \rightarrow C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, G)).
\]

In fact, for \(u_{i_0 \ldots i_p}^{p,q} \in C^p(\mathcal{U}, \text{Hom}^q(E, F))\) and \(v_{i_p \ldots i_{p+r}}^{r,s} \in C^r(\mathcal{U}, \text{Hom}^s(F, G))\), their composition \((u \cdot v)^{p+r,q+s}\) is given by (see [OTT81] Equation (1.1))

\[
(u \cdot v)^{p+r,q+s}_{i_0 \ldots i_{p+r}} = (-1)^{qr} u_{i_0 \ldots i_p}^{p,q} v_{i_p \ldots i_{p+r}}^{r,s}
\]

where the right hand side is the composition of sheaf maps.

In particular \(C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, E))\) becomes an associative algebra under this composition (It is easy but tedious to check the associativity). We also notice that \(C^\bullet(\mathcal{U}, E^\bullet)\) becomes a left module over this algebra. In fact the action

\[
C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, E)) \times C^\bullet(\mathcal{U}, E^\bullet) \rightarrow C^\bullet(\mathcal{U}, E^\bullet)
\]

is given by \((u^{p,q}, c^{r,s}) \mapsto (u \cdot c)^{p+r,q+s}\) where the action is given by (see [OTT81] Equation (1.2))

\[
(u \cdot c)^{p+r,q+s}_{i_0 \ldots i_{p+r}} = (-1)^{qr} u_{i_0 \ldots i_p}^{p,q} c_{i_p \ldots i_{p+r}}^{r,s}
\]

where the right hand side is given by evaluation.
There are also Čech-style differential operator $\delta$ on $C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, F))$ and $C^\bullet(\mathcal{U}, E^\bullet)$ of bidegree $(1, 0)$ given by the formula

$$\left(\delta u_{i_0 \cdots i_{p+1}}\right)^{p+1, q}_{i_0 \cdots i_{p+1}} = \sum_{k=1}^{p} (-1)^{k} u_{i_0 \cdots i_{k} \cdots i_{p+1}}$$

for $u^{p,q} \in C^p(\mathcal{U}, \text{Hom}^q(E, F))$ (8)

and

$$\left(\delta c_{i_0 \cdots i_{p+1}}\right)^{p+1, q}_{i_0 \cdots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^{k} c_{i_0 \cdots i_{k} \cdots i_{p+1}}$$

for $c^{p,q} \in C^p(\mathcal{U}, E)$.

(9)

Caution 2. Notice that the map $\delta$ defined above is different from the usual Čech differential since we do not include the zeroth index.

Proposition 2.1. The differential satisfies the Leibniz rule. More precisely we have

$$\delta(u \cdot v) = (\delta u) \cdot v + (-1)^{|u|} u \cdot (\delta v)$$

and

$$\delta(u \cdot c) = (\delta u) \cdot c + (-1)^{|u|} u \cdot (\delta c)$$

where $|u|$ is the total degree of $u$.

Proof. This is a routine check. \hfill \Box

2.3 The definition of twisted complex

Now we can define twisted complexes.

Definition 2.4. [Twisted complexes] Let $(X, \mathcal{O}_X)$ be a locally ringed paracompact space and $\mathcal{U} = \{U_i\}$ be a locally finite open cover of $X$. A twisted complex consists of a graded sheaves $E_i$ of $\mathcal{O}_X$-modules on each $U_i$ together with an

$$a = \sum_{k \geq 0} a^{k,1-k}$$

where $a^{k,1-k} \in C^k(\mathcal{U}, \text{Hom}^{1-k}(E, E))$ such that they satisfy the Maurer-Cartan equation

$$\delta a + a \cdot a = 0.$$ (10)

More explicitly, for $k \geq 0$

$$\delta a^{k-1,2-k} + \sum_{i=0}^{k} a^{1,1-i} \cdot a^{k-i,1-k+i} = 0.$$ (11)

Moreover we impose the following non-degenerate condition: for each $i$, the chain map

$$a_i^{1,0} : (E_i^\bullet, a_i^{0,1}) \to (E_i^\bullet, a_i^{0,1})$$

is chain homotopic to id.

(12)

The twisted complexes on $(X, \mathcal{O}_X, \{U_i\})$ form a dg-category: the objects are the twisted complexes $E = (E_i^\bullet, a)$ and the morphism from $E = (E_i^\bullet, a)$ to $F = (F_i^\bullet, b)$ are $C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, F))$ with the total degree. Moreover, the differential on a morphism $\phi$ is given by

$$d\phi = \delta \phi + b \cdot \phi - (-1)^{|\phi|} \phi \cdot a.$$ (13)

We denote the dg-category of twisted complexes on $(X, \mathcal{O}_X, \{U_i\})$ by $\text{Tw}(X, \mathcal{O}_X, \{U_i\})$. If there is no danger of confusion we can simply denote it by $\text{Tw}(X)$.
Actually the first few terms of the Maurer-Cartan equation (11) can be written as

\[
\begin{align*}
  a^{0,1}_i \cdot a^{0,1}_i &= 0 \\
  a^{0,1}_i \cdot a^{1,0}_i + a^{1,0}_i \cdot a^{0,1}_i &= 0 \\
  -a^{1,0}_{ik} + a^{1,0}_{ij} \cdot a^{1,0}_{jk} &= 0 \\
  a^{1,0}_{ij} + a^{1,0}_{ik} \cdot a^{2,-1}_i + a^{2,-1}_j \cdot a^{1,0}_k &= 0 \\
  \cdots 
\end{align*}
\]  

(14)

Let us explain the meaning of these equations. The first equation tells us that for each \( i \), \((E^\bullet_i, a^{0,1}_i)\) is a chain complex. The second equation, together with the sign convention in Equation (6), tells us that \( a^{1,0}_{ij} \) gives a chain map \((E^\bullet_j, a^{0,1}_j) \to (E^\bullet_i, a^{0,1}_i)\). The third equation says that we have the cocycle condition up to homotopy with the homotopy operator \( a^{2,-1}_{ijk} \).

**Remark 6.** The twisted perfect complex in this paper is almost the same as the twisted cochain in [OTT81]. The only difference between our definition and theirs is that we do not require that for any \( i \)

\[
a^{1,0}_{ii} = id_{E^\bullet_i}
\]

on the nose.

Our definition guarantees that the mapping cone exists in the category \( \text{Tw}(X) \), see Definition 2.8 below.

**Caution 3.** Notice that a twisted complex itself is not a complex of sheaves on \( X \).

For our purpose we need the following smaller dg-categories

**Definition 2.5.** A twisted perfect complex \( E = (E^\bullet_i, a) \) is the same as twisted complex except that each \( E^\bullet_i \) is a strictly perfect complex on \( U_i \).

The twisted perfect complexes form a dg-category and we denote it by \( \text{Tw}_{\text{perf}}(X, O_X, \{U_i\}) \) or simply \( \text{Tw}_{\text{perf}}(X) \). Obviously \( \text{Tw}_{\text{perf}}(X) \) is a full dg-subcategory of \( \text{Tw}(X) \).

**Remark 7.** We would like to mention some related topics here.

- Our construction is very similar to the twisted complex in [BK90]. For example both constructions involve the Maurer-Cartan equation. The main difference is that the differential of the Maurer-Cartan equation in Bondal and Kapranov’s twisted complex is the differential in the dg-category, while our differential is the Čech differential \( \delta \).

- The construction of twisted complexes is very similar to the \( dg\)-nerve as in [Lur] 1.3.1.6 or Definition 2.3 in [BS14]. It is worthwhile to find the deeper relations.

- We expect the dg-category \( \text{Tw}_{\text{perf}}(X) \) gives an explicit realization of the homotopy limit of \( L(U_i) \), the dg-categories of locally free finitely generated sheaves on \( U_i \). We notice that the simplicial resolution of dg-categories is defined in [Hol14] A.2 and we hope that we can use it to construct the homotopy limit in the future.

**Definition 2.6.** For a twisted complex \((E^\bullet, a)\), we can define an operator \( \delta_a \) on \( C^\bullet(U, E^\bullet) \) of total degree 1 by

\[
\delta_a c = \delta c + a \cdot c.
\]

(15)

The Maurer-Cartan equation \( \delta a + a \cdot a = 0 \) implies that \( \delta^2_a = 0 \), i.e. \( \delta_a \) is a differential on \( C^\bullet(U, E^\bullet) \). We have the same construction when we restrict to \( \text{Tw}_{\text{perf}}(X) \).
2.4 Further study of the non-degeneracy condition of twisted complexes

Recall that for each $i$, the $(0, 1)$ component $a_{i}^{0,1} : E_{i}^{n} \to E_{i}^{n+1}$ is a differential of $\mathcal{O}_{X}$-modules on $U_{i}$, hence we get a complex $(E_{i}^{n}, a_{i}^{0,1})$ on $U_{i}$. Remember that the map is the dot multiplication of $a_{i}^{0,1}$ as in Equation (7).

Now we consider the map $a_{ii}^{1,0} : E_{j}^{n} \to E_{i}^{n}$, the Maurer-Cartan equation \( (\ref{eq:MC}) \) in the $k = 1$ case tells us
\[
a_{ii}^{1,0} \cdot a_{i}^{0,1} + a_{i}^{0,1} \cdot a_{ii}^{1,0} = 0.
\]
Actually the sign convention in Equation \( (\ref{eq:sign}) \) makes the above equation to
\[
a_{ii}^{1,0} a_{i}^{0,1} - a_{i}^{0,1} a_{ii}^{1,0} = 0.
\]
In other words, $a_{ii}^{1,0}$ gives a chain map $(E_{i}^{n}, a_{i}^{0,1}) \to (E_{i}^{n}, a_{i}^{0,1})$.

Similar to $K(X)$, let us denote the homotopy category of complexes of $\mathcal{O}_{X}$-modules on $U_{i}$ by $K(U_{i})$. Then we have the following lemma.

**Lemma 2.2.** If the $a_{k,1-k}$'s satisfy the Maurer-Cartan equation, then $a_{ii}^{1,0} : (E_{i}^{n}, a_{i}^{0,1}) \to (E_{i}^{n}, a_{i}^{0,1})$ is an idempotent map in the homotopy category $K(U_{i})$, i.e. $(a_{ii}^{1,0})^{2} = a_{ii}^{1,0}$ up to chain homotopy.

**Proof.** The $k = 2$ case of the Maurer-Cartan equation \( (\ref{eq:MC}) \) gives us
\[
-a_{ii}^{1,0} + a_{i}^{0,1} \cdot a_{ii}^{1,0} + a_{i}^{0,1} \cdot a_{ii}^{2,0} - a_{ii}^{2,0} \cdot a_{i}^{0,1} = 0.
\]
We take $a_{ii}^{2,0}$ to be the homotopy operator and this immediately gives what we want. \( \square \)

**Lemma 2.3.** If the $a_{k,1-k}$'s satisfy the Maurer-Cartan equation, then $a_{ii}^{1,0} : (E_{i}^{n}, a_{i}^{0,1}) \to (E_{i}^{n}, a_{i}^{0,1})$ is homotopic to $id$ if and only if it is homotopic invertible.

**Proof.** By Lemma 2.2 we know that $(a_{ii}^{1,0})^{2} = a_{ii}^{1,0}$ up to chain homotopy. Then the result is obvious. \( \square \)

We will discuss the non-degeneracy condition further in Section 6.

2.5 Pre-triangulated structure on Tw($X$)

The dg-category Tw($X$) has a natural shift-by-1 functor and a mapping cone construction as follows.

**Definition 2.7.** [Shift] Let $\mathcal{E} = (E^{\bullet}_{i}, a)$ be a twisted complex. We define the shift by 1 of $\mathcal{E}[1]$ to be $\mathcal{E}[1] = (E^{\bullet}[1], a[1])$ where
\[
E^{1}_{i}[1] = E^{*+1}_{i} \quad \text{and} \quad a[1]^{k,1-k} = (-1)^{k-1}a^{k,1-k}.
\]
Moreover, let $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism. We define the shift $\phi[1]$ by
\[
\phi[1]^{p,q} = (-1)^{q}\phi^{p,q}.
\]

**Definition 2.8.** [Mapping cone] Let $\phi^{*,*}$ be a closed degree zero map between twisted complexes $\mathcal{E} = (E^{\bullet}, a^{*,1-*})$ and $\mathcal{F} = (F^{\bullet}, b^{*,1-*})$, we can define the mapping cone $\mathcal{G} = (G^{*}, c)$ of $\phi$ as follows (see [OTT83] Section 1.1):
\[
G^{n}_{i} := E_{i}^{n+1} \oplus F_{i}^{n}.
\]
and
\[
c_{i_{0}...i_{k}}^{k,1-k} = \begin{pmatrix}
(-1)^{k-1}a_{i_{0}...i_{k}}^{k,1-k} & 0 \\
(-1)^{k}b_{i_{0}...i_{k}}^{k,1-k} & b_{i_{0}...i_{k}}^{k,1-k}
\end{pmatrix}.
\]
Remark 8. As a special case of Equation (18) we get
\[ c_{1,0}^{i,i} = \begin{pmatrix} a_{1,0}^{i,i} & 0 \\ -\phi_{1,1}^{i,i} & b_{1,0}^{i,i} \end{pmatrix}. \]

It is clear that \( c_{1,0}^{i,i} \neq \text{id} \) even if both \( a_{1,0}^{i,i} \) and \( b_{1,0}^{i,i} \) equal to \( \text{id} \) since we cannot assume that \( \phi_{1,1}^{i,i} = 0 \) for any \( i \). This is the main technical reason that we drop the requirement \( a_{1,0}^{i,i} = \text{id} \) in the definition of twisted complex, see Remark 8 after Definition 2.4.

Nevertheless, we can prove that the mapping cone satisfies the non-degeneracy condition in Definition 2.1.

Lemma 2.4. Let \( \phi^{ullet,-ullet} \) be a closed degree zero map between twisted complexes \( E = (E^{ullet}, a^{ullet,1-ullet}) \) and \( F = (F^{ullet}, b^{ullet,1-ullet}) \). Let \( G = (G, c) \) be the mapping cone of \( \phi \). Then
\[ c_{1,0}^{i,i} : (G^{ullet}_i, c_i^{0,1}) \to (G^{ullet}_i, c_i^{0,1}) \]
is chain homotopic to \( \text{id} \).

Proof. By Lemma 2.3 we know that \( a_{1,0}^{i,i} \) and \( b_{1,0}^{i,i} \) are homotopic invertible, hence
\[ c_{1,0}^{i,i} = \begin{pmatrix} a_{1,0}^{i,i} & 0 \\ -\phi_{1,1}^{i,i} & b_{1,0}^{i,i} \end{pmatrix} \]
is also homotopic invertible since it is a lower block triangular matrix.

On the other hand the \( c_{k,1-k}^{i,i} \)'s satisfy the Maurer-Cartan equation. Again by Lemma 2.3 we know that \( c_{1,0}^{i,i} \) is chain homotopic to \( \text{id} \).

Proposition 2.5. \( \text{Tw}(X) \) is a pre-triangulated dg-category and \( \text{Tw}(X) \) is a pre-triangulated dg-subcategory of \( \text{Tw}(X) \). Therefore the category \( \text{HoTw}(X) \) is triangulated.

The same result holds for \( \text{Tw}_{\text{perf}}(X) \).

Proof. It is easy to check this result.

Caution 4. The degree and sign convention in the definition of mapping cones in this paper are slightly different to those in [OTT85] Section 1.1.

2.6 Weak equivalence in \( \text{Tw}(X) \)

In this subsection we specify the class of weak equivalences in \( \text{Tw}(X) \), which is very important in our latter constructions.

Definition 2.9. [Weak equivalence] Let \( E = (E^{ullet}, a^{ullet,1-ullet}) \) and \( F = (F^{ullet}, b^{ullet,1-ullet}) \) be two objects in \( \text{Tw}(X) \). A closed degree zero morphism \( \phi^{ullet,-ullet} : E \to F \) is called a weak equivalence if its \((0,0)\) component
\[ \phi_0^{i,0} : (E_i^{ullet}, a_i^{0,1}) \to (F_i^{ullet}, b_i^{0,1}) \]
is a quasi-isomorphism of complexes of \( \mathcal{O}_X \)-modules on \( U_i \) for each \( i \).

Remark 9. The definition of weak equivalence between twisted complexes is first introduced in [Gil86].
Lemma 2.6. Let $U$ be a subset of $X$ which satisfies $H^k(U, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$ on $U$ and any $k \geq 1$. Let $E^\bullet$ be a complex of finitely generated locally free sheaves on $U$ and $G^\bullet$ be an acyclic complex of quasi-coherent modules on $U$, then the Hom complex $\text{Hom}^\bullet(E, G)$ is acyclic.

Proof. We have a filtration on $\text{Hom}^\bullet(E, G)$ given by the $E^\bullet$ degree. More explicitly let

$$F^k\text{Hom}^\bullet(E, G) = \{ \phi \in \text{Hom}^\bullet(E, G) | \deg(e) = 0 \text{ if } \deg(e) < k \}.$$ 

By a simple spectral sequence argument, it is sufficient to prove that $(F^k\text{Hom}^\bullet(E, G)/F^{k+1}\text{Hom}^\bullet(E, G), d_{\text{Hom}})$ is acyclic. We notice that

$$(F^k\text{Hom}^\bullet(E, G)/F^{k+1}\text{Hom}^\bullet(E, G), d_{\text{Hom}}) \cong (\text{Hom}(E^k, G^\bullet), d_G).$$

We know that $(G^\bullet, d_G)$ is acyclic. On the other hand $E^k$ is locally free finitely generated hence Condition 2 in Definition [B.1] guarantees that $\text{Hom}(E^k, \cdot)$ is exact, hence we get the acyclicity.

Lemma 2.7. Let $U$ be a subset of $X$ which satisfies $H^k(U, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$ on $U$ and any $k \geq 1$. Suppose we have chain maps $r : E^\bullet \to F^\bullet$ and $s : G^\bullet \to F^\bullet$ between complexes of sheaves on $U$, where $E^\bullet$ is a complex of finitely generated locally free sheaves, and $F^\bullet$ and $G^\bullet$ are quasi-coherent. Moreover $s$ is a quasi-isomorphism. Then $r$ factors through $s$, i.e. there exists a chain map $r' : E^\bullet \to G^\bullet$ such that $s \circ r'$ is homotopic to $r$.

Proof. We can take the mapping cone of $s$, which is acyclic, then the result is a simple corollary of Lemma 2.6.

With this definition we have the following result for twisted perfect complexes.

Proposition 2.8. Let the cover $\{U_i\}$ satisfies $H^k(U_i, \mathcal{F}) = 0$ for any $i$, any quasi-coherent sheaf $\mathcal{F}$ on $U_i$ and any $k \geq 1$. If $\mathcal{E}$ and $\mathcal{F}$ are both in the subcategory $\text{Tw}_{\text{perf}}(X)$, then a closed degree zero homomorphism $\phi$ between twisted complexes $\mathcal{E}$ and $\mathcal{F}$ is a weak equivalence if and only if $\phi$ is invertible in the homotopy category $\text{Tw}_{\text{perf}}(X)$.

Proof. It is obvious that homotopy invertible implies weak equivalent.

For the other direction, we know $\phi$ is a weak equivalence, hence $\phi_i^{0,0} : E_i^\bullet \to F_i^\bullet$ is a quasi-isomorphism for each $i$. Since $F_i^\bullet$ is a bounded complex of finitely generated locally free sheaves, we apply Lemma 2.7 and get

$$\psi_i^{0,0} : F_i^\bullet \to E_i^\bullet$$

such that $\phi_i^{0,0} \circ \psi_i^{0,0}$ is homotopy to $id_{F_i^\bullet}$. It is clear that $\psi_i^{0,0}$ is also a quasi-isomorphism and gives the two-side homotopy inverse of $\phi_i^{0,0}$.

The remaining task is to extend the $\psi_i^{0,0}$’s to a degree zero cocycle $\psi^*_{\cdot, \cdot}$ in $\text{Tw}(X)$ and to show that it gives the homotopy inverse of $\phi^*_{\cdot, \cdot}$. This is a simple spectral sequence argument which is the same as the proof of Proposition 2.9 in [Blo10].

Remark 10. The result Proposition 2.8 is no longer true if one of the $\mathcal{E}$ and $\mathcal{F}$ is not a twisted perfect complex.

We also have the following result.

Proposition 2.9. Let $\mathcal{E}$ be a twisted perfect complex and $\mathcal{F}$ be twisted complexes consists of quasi-coherent sheaves on each $U_i$. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a weak equivalence. Then any closed morphism $\phi : \mathcal{E} \to \mathcal{F}$ factors through $\varphi$, i.e. there exists a chain map $\phi' : \mathcal{E} \to \mathcal{G}$ such that $\varphi \circ \phi'$ is homotopic to $\phi$.

Proof. Apply Lemma 2.7 repeatedly and we can get this result.

Remark 11. Proposition 2.8 and 2.9 are not explicitly given in [TT78], [OTT81], [OTT85].
3  TWISTED COMPLEXES AND THE DG-ENHANCEMENT OF $D_{pert}(X)$

3.1 The sheafification functor $S$

We fix a locally finite open cover $\mathcal{U} = \{U_i\}$ of $X$. As we noticed in Caution[3], a twisted complex $\mathcal{E} = (E_i^*, a)$ is not a complex of sheaves. Nevertheless in this subsection we associate a complex of sheaves to each twisted complex on $X$.

First we introduce a variation of the notations in Equation (4) and (5). Let $E_i^{\bullet} = \{E_i^r\}_{r \in \mathbb{Z}}$ be a graded sheaf of $\mathcal{O}_X$-modules on $U_{i_k}$ as before. For $V$ an open subset of $X$, let

$$C^\bullet(\mathcal{U}, E^{\bullet}; V) = \prod_{p,q} C^p(\mathcal{U}, E^q; V)$$

be the bigraded complex on $V$. More precisely, an element $c^{p,q}$ of $C_p(\mathcal{U}, E^q; V)$ consists of a section $c_{i_0\ldots i_p}^{p,q}$ of $E_{i_0}^q$ over each non-empty intersection $U_{i_0\ldots i_n} \cap V$. If $U_{i_0\ldots i_n} \cap V = \emptyset$, let the component on $U_{i_0\ldots i_n} \cap V$ simply be zero.

Similarly if another graded sheaf $F_{i_k}$ of $\mathcal{O}_X$-modules is given on each $U_{i_k}$, and $V$ is an open subset of $X$, we can consider the map

$$C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E,F); V) = \prod_{p,q} C^p(\mathcal{U}, \text{Hom}^q(E,F); V).$$

An element $u^{p,q}$ of $C_p(\mathcal{U}, \text{Hom}^q(E,F); V)$ gives a section $u_{i_0\ldots i_p}^{p,q}$ of $\text{Hom}^q_{\mathcal{O}_X-\text{Mod}}(E_{i_p}^r, F_{i_0}^r)$ over each non-empty intersection $U_{i_0\ldots i_n} \cap V$. If $U_{i_0\ldots i_n} \cap V = \emptyset$, let the component on $U_{i_0\ldots i_n} \cap V$ simply be zero.

Moreover, let $\mathcal{E} = (E_i^*, a)$ be a twisted complex, recall that in Definition 2.6 we defined a differential

$$\delta_a = \delta + a$$

on $C^\bullet(\mathcal{U}, E^{\bullet})$. Now let $V$ be an open subset of $X$, we can restrict $\delta_a$ to $V$ to get a differential on $C^\bullet(\mathcal{U}, E^{\bullet}; V)$.

With all these notations, we can introduce the following definition.

**Definition 3.1.** For a twisted complex $\mathcal{E} = (E_i^*, a)$, we define the associated complex of sheaves $S(\mathcal{E})$ as follows: for each $n$, the degree $n$ part $S^n(\mathcal{E})$ is a sheaf on $X$ such that for any open subset $V$ of $X$

$$S^n(\mathcal{E})(V) = \prod_{p+q=n} C^p(\mathcal{U}, E^q; V).$$

The differential on $S^\bullet(\mathcal{E})$ is defined to be the sheafification of $\delta_a = \delta + a$. More precisely, for each open subset $V$ of $X$, the differential

$$S^n(\mathcal{E})(V) \to S^{n+1}(\mathcal{E})(V)$$

is given by $\delta + a$ restricted to $V$. We still denote it by $\delta_a$ since there is no danger of confusion.

It is obvious that $S^n(\mathcal{E})$ is a sheaf of $\mathcal{O}_X$-module for each $n$ and $\delta_a : S^n(\mathcal{E}) \to S^{n+1}(\mathcal{E})$ is a map of $\mathcal{O}_X$-modules.

Now we turn to the morphisms. Let $\phi : \mathcal{E} \to \mathcal{F}$ be a degree $n$ morphism in Tw$(X)$. We can define the associated sheaf morphism

$$S(\phi) : S^\bullet(\mathcal{E}) \to S^{n+\bullet}(\mathcal{F}).$$

in the same spirit as Definition 3.1, i.e. by restricting to each of the $C^p(\mathcal{U}, E^q; V)$’s.

In fact we can view $S^\bullet(\mathcal{E})$ in another way. For this we recall some definitions in sheaf theory. Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_X$-modules on $X$ and $U$ be an open subset of $X$ with $j : U \to X$ be the inclusion map. We denote the restriction sheaf of $\mathcal{F}$ on $U$ by $\mathcal{F}|_U$. The pushforward of $\mathcal{F}|_U$, $j_*(\mathcal{F}|_U)$, will be a sheaf of $\mathcal{O}_X$-modules on $X$ again and we also denote it by $\mathcal{F}|_U$ if there is no confusion.
Remark 12. We do not use the fancy pushforward \( j! \) in our construction.

Then we have
\[
S^n(\mathcal{E}) = \prod_{p+q=n} E^q_{i_0}|_{U_{i_0} \circ \ldots \circ U_{i_p}}
\]
as a sheaf and the differential \( \delta_q = \delta + a \) and the morphism \( \phi \) are defined likewise by restriction.

In conclusion we have the following definition.

Definition 3.2. [The sheafification functor] The above construction defines a dg-functor
\[
S : \text{Tw}(X) \to \text{Sh}(X)
\]
and we call it the sheafification functor.

Remark 13. Since the complexes \( E^\bullet \) are bounded and the cover \( \{U_j\} \) is locally finite, it is easy to see that the product in \( S^n(\mathcal{E}) = \prod_{p+q=n} E^q_{i_0}|_{U_{i_0} \circ \ldots \circ U_{i_p}} \) is locally finite, hence the image of a twisted perfect complex under \( S \) actually consists of quasi-coherent sheaves. In other words, the sheafification functor restricts to \( \text{Tw}_{\text{perf}}(X) \) and gives
\[
S : \text{Tw}_{\text{perf}}(X) \to \text{Qcoh}(X).
\]

Further study of the sheafification of twisted perfect complexes will be given in the next subsection.

3.2 The sheafification of twisted perfect complexes

Let \( \mathcal{E} \) be a twisted perfect complex, we want to show that the associated complex of sheaves \( (S^\bullet(\mathcal{E}), \delta_a) \) is perfect. In fact in this subsection we will get a more general result. The next proposition, which is important in our work, says that locally \( (S^\bullet(\mathcal{E}), \delta_a) \) contains the same information as \( (E^\bullet_j, a^0_{0,1}) \) for each \( j \).

Proposition 3.1. [The local property of \( S \)] Let \( \mathcal{E} = (E^\bullet, a) \) be a twisted complex and \( (S^\bullet(\mathcal{E}), \delta_a) \) be the associative complex of sheaves. Then \( (S^\bullet(\mathcal{E}), \delta_a)|_{U_j} \) is chain homotopy equivalent to \( (E^\bullet_j, a^0_{0,1}) \), i.e. we have two morphisms
\[
f : (S^\bullet(\mathcal{E}), \delta_a)|_{U_j} \to (E^\bullet_j, a^0_{0,1})
\]
and
\[
g : (E^\bullet_j, a^0_{0,1}) \to (S^\bullet(\mathcal{E}), \delta_a)|_{U_j}
\]
such that
\[
f \circ g = \text{id}_{E^\bullet_j} \text{ and } g \circ f = \text{id}_{S^\bullet(\mathcal{E})|_{U_j}} \text{ up to chain homotopy.}
\]

Proof. The proof is long and involves several technical lemmas.

First we can construct the chain map
\[
f : (S^\bullet(\mathcal{E})(V), \delta_a) \to (E^\bullet_j(V), a^0_{0,1})
\]
for \( V \subset U_j \) by projecting to the \((0, n)\) component. In more details, we know that \( S^n(\mathcal{E})(V) = \prod_{p+q=n} C^p(U, E^q; V) \). The \((0, n)\) component \( C^0(U, E^0; V) \) has a further decomposition
\[
C^0(U, E^0; V) = \prod_{i_0} E^0_{i_0}(V \cap U_{i_0}).
\]

We also notice that \( j \) appears in one of the \( i_0 \)'s. Then \( f : (S^\bullet(\mathcal{E})(V), \delta_a) \to (E^\bullet_j(V), a^0_{0,1}) \) is given by first projecting to the \((0, n)\) component and then projecting to the \( j \) component. It is easy to see that \( f \) is a chain map.
The construction of the other map

\[ g : (E^*_j(V), a^{0,1}_j) \to (S^*(E)(V), \delta_a) \]

is more complicated. We first introduce the following auxiliary morphism

\[ \epsilon^p_{i_0\ldots i_p} : E^*_{i_0}(U_{i_0\ldots i_p}) \to E^*_{i_0}(U_{i_0\ldots i_p} \cap V) \quad (26) \]

as

\[ \epsilon^p_{i_0\ldots i_p} = (-1)^p id. \]

Sometimes we simply denote it by \( \epsilon^p \). Since \( V \subseteq U_j \), we have \( U_{i_0\ldots i_p} \cap V \subseteq U_{i_0\ldots i_p} \cap V \) hence the above formula makes sense.

Notice that the identity map \( E^{n-p-1}_{i_0}(U_{i_0\ldots i_p} \cap V) \to E^{n-p}_{i_0}(U_{i_0\ldots i_p} \cap V) \) shifts the Čech degree by \(-1\) and hence we introduce the factor \((-1)^p\) to compensate it.

We have the following property of the maps \( \epsilon^p \)'s.

**Lemma 3.2.** The \( \epsilon^p \)'s anti-commute with \( \alpha \) and \( \delta \). More precisely, for a multi-index \( i_0, \ldots, i_{p+q} \), we have

\[ a^{p,1-p}_{i_0\ldots i_p} c = (-1)^{q+1} \epsilon^p_{i_0\ldots i_p} a^{p,1-p}_{i_0\ldots i_p} c \quad (27) \]

where both sides are considered as maps

\[ E^*_{i_p}(U_{i_p\ldots i_{p+q}} \cap V) \to E^*_{i_0}(U_{i_0\ldots i_{p+q}} \cap V). \]

As for \( \delta \), we introduce a map \( \delta \) on \( U_{i_0\ldots i_p} \cap V \) as

\[ (\delta c)_{i_0\ldots i_p} = \sum_{k=1}^{p} (-1)^k \epsilon_{i_0\ldots i_k} \cap \epsilon_{i_k\ldots i_p} c. \]

Then we have

\[ \delta \epsilon^p = -\epsilon^{p+1} \tilde{\delta}. \quad (28) \]

**Proof of Lemma 3.2.** First we prove Equation (27). Let \( c \in E^*_{i_p}(U_{i_p\ldots i_{p+q}} \cap V) \) with Čech degree \( q+1 \). By definition

\[ \epsilon^q_{i_p\ldots i_{p+q}} c = (-1)^q c \in E^*_{i_p}(U_{i_p\ldots i_{p+q}} \cap V) \]

with Čech degree \( q \). Then according to the sign convention in Equation (7) we have

\[ a^{p,1-p}_{i_0\ldots i_p} c = (-1)^q a^{p,1-p}_{i_0\ldots i_p} c = (-1)^q (-1)^{1-p} q a^{p,1-p}_{i_0\ldots i_p} c = (-1)^q a^{p,1-p}_{i_0\ldots i_p} c. \]

On the other hand we have

\[ a^{p,1-p}_{i_0\ldots i_p} c = (-1)^{1-p}(1+q) a^{p,1-p}_{i_0\ldots i_p} c \]

hence

\[ \epsilon^{p+q}_{i_0\ldots i_{p+q}} a^{p,1-p}_{i_0\ldots i_p} c = (-1)^{p+q} (-1)^{(1-p)(1+q)} a^{p,1-p}_{i_0\ldots i_p} c = (-1)^{1+pq} a^{p,1-p}_{i_0\ldots i_p} c. \]

Compare the two sides we get

\[ a^{p,1-p}_{i_0\ldots i_p} = -\epsilon^{p+q}_{i_0\ldots i_{p+q}} a^{p,1-p}_{i_0\ldots i_p}. \]

Equation (28) follows similarly and we leave it to the reader. This finishes the proof of Lemma 3.2.
We move on to the definition of $g$. Recall that
\[ S^n(E)(V) = \prod_{p+q=n} C^p(U, E^q; V) = \prod_{p \geq 0} \prod_{i_0 \cdots i_p} E^n_{i_0 \cdots i_p}(U_{i_0 \cdots i_p} \cap V) \]
and it is sufficient to define the projection of $g$ to each component. With the help of the map $\epsilon^p$ we define that component to be
\[ \epsilon^p \circ a^{p+1,-p}_{i_0 \cdots i_p} : E^n_{i_0 \cdots i_p}(V) \to E^n_{i_0 \cdots i_p}(U_{i_0 \cdots i_p} \cap V), \ p \geq 0. \tag{29} \]
Remember this map is the dot multiplication of $a^{p+1,-p}_{i_0 \cdots i_p}$ as in Equation (7).

**Lemma 3.3.** The map $g : (E^*_j(V), a^0,j) \to (S^*(E)(V), \delta_a)$ defined above is a chain map.

**Proof of Lemma 3.3.** It is a consequence of the Maurer-Cartan equation
\[ \delta a^{k-1,2-k} + \sum_{i=0}^k a^{i,1-i} \cdot a^{k-i,1+k+i} = 0 \]
together with the anti-commute properties in Lemma 3.2.

Now we need to prove that $f$ and $g$ satisfy the relations in Equation (49). First it is obvious that
\[ f \circ g = a^{1,0}_{i,j} : (E^*_j(V), a^0,j) \to (E^*_j(V), a^0,j). \]
By definition $a^{1,0}_{i,j} = id_{E^*_j}$ up to homotopy hence we get $f \circ g = id_{E^*_j}$ up to homotopy.

The other half is more complicated. We need to define maps
\[ h : S^*(E)(V) \to S^{*-1}(E)(V) \]
such that
\[ g \circ f - id = \delta_a h + h \delta_a. \tag{30} \]
In fact we define $h$ as
\[ (hc)_{i_0 \cdots i_k} := (-1)^k c_{i_0 \cdots i_k,j}. \tag{31} \]
Clearly $h$ is a sheaf map with degree $-1$. Moreover we have
\[ (\delta_a hc)_{i_0 \cdots i_k} = (\delta (hc))_{i_0 \cdots i_k} + (a \cdot (hc))_{i_0 \cdots i_k} \]
\[ = \sum_{l=1}^k (-1)^l (hc)_{i_0 \cdots i_l \cdots i_k} + \sum_{l=0}^k a^{l,1-l}_{i_0 \cdots i_l} \cdot (hc)_{i_l \cdots i_k} \]
\[ = \sum_{l=1}^k (-1)^l (-1)^{k-l} c_{i_0 \cdots i_l \cdots i_k,j} + \sum_{l=0}^k a^{l,1-l}_{i_0 \cdots i_l} \cdot (hc)_{i_l \cdots i_k,j}. \]
For the second term $a^{l,1-l}_{i_0 \cdots i_l} \cdot (hc)_{i_l \cdots i_k}$ we need to be more careful. We know that $(hc)_{i_l \cdots i_k}$ has Čech degree $n - l$ hence
\[ a^{l,1-l}_{i_0 \cdots i_l} \cdot (hc)_{i_l \cdots i_k} \]
\[ = (-1)^{(l-1)(k-l)} a^{l,1-l}_{i_0 \cdots i_l} \circ (hc)_{i_l \cdots i_k} \]
\[ = (-1)^{(l-1)(k-l)} (-1)^{k-l} a^{l,1-l}_{i_0 \cdots i_l} \circ c_{i_l \cdots i_k,j} \]
\[ = (-1)^{l-k} a^{l,1-l}_{i_0 \cdots i_l} \circ c_{i_l \cdots i_k,j}. \]
In conclusion we have

\[(\delta_a hc)_{i_0 \ldots i_k} = \sum_{l=1}^{k} (-1)^{k+l-1} c_{i_0 \ldots \hat{i} \ldots i_k} + \sum_{l=0}^{k} (-1)^{l} a_{i_0 \ldots i_l} \circ c_{i_l \ldots i_k}. \tag{32}\]

On the other hand we have

\[(h\delta_a c)_{i_0 \ldots i_k} = (-1)^{k}(\delta_a c)_{i_0 \ldots i_k, j} \]
\[= (-1)^{k}[(\delta c) + (a \cdot c)]_{i_0 \ldots i_k, j} \]
\[= (-1)^{k} \sum_{l=1}^{k} (-1)^{l} c_{i_0 \ldots \hat{i} \ldots i_k} + (-1)^{k+1} c_{i_0 \ldots i_k} + \sum_{l=0}^{k} a_{i_0 \ldots i_l} c_{i_l \ldots i_k} + a_{i_0 \ldots i_{k-l}} c_j \]
\[= (-1)^{k} \sum_{l=1}^{k} (-1)^{l} c_{i_0 \ldots \hat{i} \ldots i_k} + (-1)^{k+1} c_{i_0 \ldots i_k} + \sum_{l=0}^{k} (-1)^{l+1} a_{i_0 \ldots i_l} c_{i_l \ldots i_k} + (-1)^{k} a_{i_0 \ldots i_{k-l}} c_j \]
\[= \sum_{l=1}^{k} (-1)^{k+l} c_{i_0 \ldots \hat{i} \ldots i_k} - c_{i_0 \ldots i_k} + \sum_{l=0}^{k} (-1)^{l+1} a_{i_0 \ldots i_l} \circ c_{i_l \ldots i_k} + (-1)^{k} a_{i_0 \ldots i_{k-l}} \circ c_j. \tag{33}\]

In short we have

\[(h\delta_a c)_{i_0 \ldots i_k} = \sum_{l=1}^{k} (-1)^{k+l} c_{i_0 \ldots \hat{i} \ldots i_k} - c_{i_0 \ldots i_k} + \sum_{l=0}^{k} (-1)^{l+1} a_{i_0 \ldots i_l} \circ c_{i_l \ldots i_k} + (-1)^{k} a_{i_0 \ldots i_{k-l}} \circ c_j. \tag{33}\]

Compare Equation (32) and (33) we get

\[[\delta_a hc + h\delta_a c]_{i_0 \ldots i_k} = -c_{i_0 \ldots i_k} + (-1)^{k} a_{i_0 \ldots i_{k-l}} \circ c_j. \tag{34}\]

Recall that \(fc = c_j\) and

\[g(fc)_{i_0 \ldots i_k} = c_j a_{i_0 \ldots i_{k-l}} \circ c_j = (-1)^{k} a_{i_0 \ldots i_{k-l}} \circ c_j \]

hence we get the desired result

\[[\delta_a hc + h\delta_a c]_{i_0 \ldots i_k} = -c_{i_0 \ldots i_k} + g(fc)_{i_0 \ldots i_k}. \]

This finishes the proof of Proposition 3.1.

The perfectness now is a simple corollary of Proposition 3.1.

**Corollary 3.4.** If \(\mathcal{E} = (E^*, a)\) is a twisted perfect complex, then the sheafification \(\mathcal{S}^*(\mathcal{E})\) is a perfect complex on \((X, \mathcal{O}_X)\). In other words the sheafification functor \(\mathcal{S}\) restricts to \(\text{Tw}_{\text{perf}}(X)\) and gives the following functor

\[S : \text{Tw}_{\text{perf}}(X) \rightarrow \text{Sh}_{\text{perf}}(X). \tag{35}\]

**Proof.** Proposition 3.1 tells us that \(\mathcal{S}^*(\mathcal{E})|_{U_j}\) is isomorphic to \((E^*_j, a^0_{ij})\) in \(K(U_j)\) hence by definition it is perfect on \(U_j\). Moreover this is true for any member \(U_j\) of the open cover, therefore \(\mathcal{S}^*(\mathcal{E})\) is a perfect complex of sheaves on \((X, \mathcal{O}_X)\). \(\square\)
Remark 14. Corollary 3.4 together with Remark 13 tells us that actually we have a functor \( S : \text{Tw}_{\text{perf}}(X) \to \text{Qcoh}_{\text{perf}}(X) \).

Another consequence of Proposition 3.1 is the following criterion of weak equivalence. Recall that by Definition 2.9 a closed degree zero morphism \( \phi^* \circ \phi : E \to F \) is called a weak equivalence if its \((0,0)\) component \( \phi_{ij}^{0,0} : (E_i^*, a_{ij}^{0,1}) \to (F_i^*, b_{ij}^{0,1}) \) is a quasi-isomorphism of complexes of \( \mathcal{O}_X \)-modules on \( U_i \) for each \( i \).

**Corollary 3.5.** [Criterion of weak equivalence] A degree 0 cocycle \( \phi^* \circ \phi : E \to F \) in \( \text{Tw}(X) \) is a weak equivalence if and only if its sheafification \( S(\phi) : S(E) \to S(F) \) is a quasi-isomorphism.

**Proof.** First we fix a \( U_j \). It is obvious that the quasi-isomorphism \( f : S^*(E)|_{U_j} \sim \to E_j^* \) is functorial hence we have the following commutative diagram

\[
S^*(E)|_{U_j} \xrightarrow{S(\phi)|_{U_j}} S^*(F)|_{U_j} \\
\sim \downarrow \quad \sim \\
E_j^* \xrightarrow{\phi_{ij}^{0,0}} F_j^*.
\]

Now the claim is obviously true. \( \square \)

### 3.3 Essential surjectivity of \( S \)

#### 3.3.1 The twisting functor \( T \) and some generalities

Remark 14 after Corollary 3.4 ensures that we have the functor \( S : \text{Tw}_{\text{perf}}(X) \to \text{Qcoh}_{\text{perf}}(X) \) which induces a functor \( S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X)) \).

In this subsection we will show that this functor is essentially surjective under some mild condition. Moreover we will show that the functor \( S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(X) \) is essentially surjective under some additional conditions.

First we define a natural dg-functor from \( \text{Sh}(X) \) to \( \text{Tw}(X) \) as follows

**Definition 3.3.** Let \( (S^*, d) \) be a complex of \( \mathcal{O}_X \)-modules. We define its associated twisted complex, \( \mathcal{T}(S) \), by restricting to the \( U_i \)'s. In more details let \( (E^*, a) = \mathcal{T}(S) \) then

\[
E^n_i = S^n|_{U_i}
\]

and

\[
a_i^{0,1} = d|_{U_i}, \quad a_{ij}^{1,0} = id \quad \text{and} \quad a^{k,1-k} = 0 \text{ for } k \geq 2.
\]

The \( \mathcal{T} \) of morphisms is defined in a similar way.

We call the functor \( \mathcal{T} : \text{Sh}(X) \to \text{Tw}(X) \) the **twisting functor**.
We would like to find the relation between the functors $S$ and $T$. First we have the following result.

**Proposition 3.6.** Let $P = (S^\bullet, d)$ be a complex of $O_X$-modules, the natural map
\[ \tau_P : P \rightarrow ST(P) \] (36)
is a quasi-isomorphism. Hence $\tau : id \rightarrow ST$ gives a natural isomorphism between functors.

**Proof.** By definition $T(P)$ is a double complex and $ST(P)$ is the total complex of that double complex. Hence it is sufficient to prove that the Čech direction of the double complex is acyclic. But we know that the Čech complex (without taking global sections) is always acyclic.

On the other hand let $E = (E, a)$ be a twisted complex, we would like to define a closed degree 0 morphism
\[ \gamma_E : TS(E) \rightarrow E. \]
Actually for each $U_{i_0 \ldots i_p}$ we need to construct a map
\[ (\gamma_E)^{p-n}_{i_0 \ldots i_p} : S^\bullet(E)|_{U_{i_0 \ldots i_p}} \rightarrow E^{\bullet-n}_{i_0 \ldots i_p}. \]
Recall that $S^\bullet(E) = \prod_{j_0 \ldots j_k} E^{\bullet-k}_{j_0 \ldots j_k}|_{U_{j_0 \ldots j_k}}$, then $(\gamma_E)^{p-n}_{i_0 \ldots i_p}$ is defined to be projecting to the component $i_0 \ldots i_p$. In particular $(\gamma_E)^{0-0}_j$ is the map $f$ in Proposition 3.1. It is easy to verify that $\gamma_E$ commutes with the differentials.

**Proposition 3.7.** The map
\[ \gamma_E : TS(E) \rightarrow E \]
is a weakly equivalence, hence $\gamma : TS \rightarrow id$ is a natural weak equivalence of functors.

**Proof.** This is a direct corollary of Proposition 3.1.

### 3.3.2 The twisted resolution and the essential surjectivity on quasi-coherent sheaves

Let $P = (S^\bullet, d)$ be a perfect complex. There is no guarantee that its associated twisted complex $T(P)$ is a twisted perfect complex on the nose, even if we assume $P$ consists of quasi-coherent sheaves. Nevertheless we have a quasi-isomorphic result. First we need to introduce the following definitions.

**Definition 3.4.** A locally ringed space $(U, O_U)$ is called $p$-good if it satisfies the following two conditions

1. For every perfect complex $P^\bullet$ on $U$ which consists of quasi-coherent sheaves, there exists a strictly perfect complex $E^\bullet$ on $U$ together with a quasi-isomorphism $u : E^\bullet \sim P^\bullet$.
2. The higher cohomologies of quasi-coherent sheaves vanish, i.e. $H^k(U, F) = 0$ for any quasi-coherent sheaf $F$ on $U$ and any $k \geq 1$.

Then we can define $p$-good cover of a ringed space.

**Definition 3.5** ($p$-good cover). Let $(X, O_X)$ be a locally ringed space, an open cover $\{U_i\}$ of $X$ is called a $p$-good cover if $(U_i, O_X|_{U_i})$ is a $p$-good space for any finite intersection $U_I$ of the open cover.

**Remark 15.** We introduce $p$-good covers mainly because we need to fix a cover which works for any complex of quasi-coherent sheaves on $X$. Actually we may refine the open cover and consider the refinement of twisted complexes and get a direct limit
\[ \lim_{\text{refinement of } \{U_i\}} \text{Tw}(X, O_X, \{U_i\}). \]
Nevertheless in this paper we do not take the above approach and just fix a $p$-good cover.
A lot of "reasonable" ringed space have p-good covers. For example we have

- \((X, \mathcal{O}_X)\) is a separated scheme, then any affine cover is p-good.
- \((X, \mathcal{O}_X)\) is a complex manifold with \(\mathcal{O}_X\) the sheaf of holomorphic functions. In these case a Stein cover is p-good.
- \((X, \mathcal{O}_X)\) is a paracompact topological space with soft structure sheaf \(\mathcal{O}_X\). Then any contractible open cover is p-good.

Further discussions of p-good covers will be given in Appendix B.

With the notion of p-good covers we can state and prove the following important proposition.

**Proposition 3.8.** [Twisted resolution, see [OTT85] Proposition 1.2.3] Assume the cover \(\{U_i\}\) is p-good. Let \(P = (S^\bullet, d_S)\) be a perfect complex which consists of quasi-coherent modules, then there exists a twisted perfect complex \(E^\bullet\) together with a weak equivalence (Definition 2.9)

\[ \phi : E^\bullet \sim \to T(P). \]

**Proof.** This proposition and its proof are essentially the same as Proposition 1.2.3 in [OTT85]. For completeness we give the proof here in our terminology.

First we can assume that for each perfect complex \(P = (S^\bullet, d_S)\), there exists a strictly perfect complex \(E^\bullet_i\) on each \(U_i\) together with a quasi-isomorphism \(\phi^0_{0,i} : E^\bullet_i \sim \to S^\bullet|_{U_i}\).

Let us denote the differential of the chain complex \(E^\bullet_i\) by \(a^0,1_i\). Now we need to do the following two constructions:

1. Find maps \(a^{0,k-1}\)'s for \(k \geq 1\) such that they and the \(a^{0,1}_i\)'s together make \(E^\bullet_i\) a twisted complex.
2. Extend the map \(\phi^0_{0,i}\)'s to get a morphism \((E^\bullet, a) \to S^\bullet\) in \(\text{Tw}(X)\).

Actually we can construct the two kinds of maps simultaneously. Let \(L^\bullet_i\) be the mapping cone of \(\phi^0_{0,i}\) (So far \(L^\bullet_i\) is not the mapping cone of any twisted complexes), which is a complex of (not necessarily locally free) sheaves on each open cover \(U_i\) and we denote its differential by \(A^0,1_i\). In fact we have

\[ L^n_i = \bigoplus_{k} E^{n+1}_i. \]

and

\[ A^0,1_i = \left( \begin{array}{cc} -a^{0,1}_i & 0 \\ \phi^0_{0,i} & d_S|_{U_i} \end{array} \right) \]

We want to construct \(A^{k,1-k}\) in \(C^k(U, \text{Hom}^{1-k}(L, L))\) which make \(L\) into a twisted complex. Moreover, we want \((L, A)\) to be the mapping cone of a closed degree zero morphism \(\phi : E \to T(P)\) which extends the \(\phi^0_{0,i}\). More precisely, we have the following two conditions on \(A^{k,1-k}\):

1. \(A\) satisfies the Maurer-Cartan equation

\[ \delta A + A \cdot A = 0. \]
2. We have
\[ A^{0,1}_{ij} = \left( -a_{ij}^{0,1} 0 \right), \quad A^{1,0}_{ij} = \left( * 0 \right), \quad dS|_{U_i}, \quad \right) \]
and for \( k \geq 2 \), \( A^{k,1-k} \) is of the form
\[ \left( * 0 \right) \].

The construction involves the previous Lemmas 2.6 and 2.7. For convenience we rephrase them here.

**Lemma 3.9** (Lemma 2.6). Let \( U \) be a subset of \( X \) which satisfies \( H^k(U, F) = 0 \) for any quasi-coherent sheaf \( F \) and any \( k \geq 1 \). Let \( E^* \) be a complex of finitely generated locally free sheaves on \( U \) and \( F^* \) be an acyclic complex of quasi-coherent modules on \( U \), then the Hom complex \( \text{Hom}^*(E, F) \) is acyclic.

**Lemma 3.9** (Lemma 2.7). Let \( U \) be a subset of \( X \) which satisfies \( H^k(U, F) = 0 \) for any quasi-coherent sheaf \( F \) and any \( k \geq 1 \). Suppose we have chain maps \( r : E^* \rightarrow F^* \) and \( s : G^* \rightarrow F^* \) between complexes of sheaves on \( U \), where \( E^* \) is a complex of finitely generated locally free sheaves, and \( F^* \) and \( G^* \) are quasi-coherent. Moreover s is a quasi-isomorphism. Then \( r \) factors through \( s \), i.e. there exists a chain map \( r' : E^* \rightarrow G^* \) such that \( s \circ r' \) is homotopic to \( r \).

Notice that \( S^n \) is quasi-coherent for each \( n \), with small Lemma 2.7 to the case \( U = U_{ij} \), \( r = \phi_{0,0} \) : \( E^*_i|_{U_{ij}} \rightarrow S^*|_{U_{ij}} \) and \( s = \phi_{0,0} \) : \( E^*_i|_{U_{ij}} \rightarrow S^*|_{U_{ij}} \) and we obtain a chain map \( r' : E^*_j|_{U_{ij}} \rightarrow E^*_i|_{U_{ij}} \) together with a homotopy \( h : E^*_j|_{U_{ij}} \rightarrow S^{*-1}|_{U_{ij}} \) such that
\[ a_{ij}^{1,0} r' - \phi_{0,0} = dS h + h a_{ij}^{1,1}. \]

Hence we get
\[ a_{ij}^{1,0} r' = r' \quad \text{and} \quad \phi_{ij}^{1,-1} = h. \] (37)

Moreover let
\[ A^{1,0}_{ij} = \left( a_{ij}^{1,0} 0 \right) - \phi_{ij}^{1,-1} \quad \text{id}|_{U_{ij}}. \] (38)

It is clear that \( A^{1,0} \) satisfies
\[ A^{1,0} \cdot A^{0,1} + A^{0,1} \cdot A^{1,0} = 0. \]

The \( A^{k,1-k} \) for \( k \geq 2 \) are constructed by induction. Let \( D \) denote the differential on \( \text{Hom}^*(L^*_i, L^*_i) \).

We need to find \( A^{k,1-k}_{i_0...i_k} \) on \( U_{i_0...i_k} \) satisfying
1. \[ (−1)^{k+1} D(A^{k,1-k}_{i_0...i_k}) = [δ A^{k-1,2-k} + \sum_{l=1}^{k-1} A^{l,1-l} \cdot A^{k-l,1+l-k}_{i_0...i_k}, \] (39)
2. \( A^{k,1-k}_{i_0...i_k} \) vanishes on the component \( S^*|_{U_i} \) of \( L^*_i \).

Keep in mind that \( F^n_i = E^{n+1}_i \oplus F^n_i \), Condition 2. is equivalent to the fact that \( A^{k,1-k}_{i_0...i_k} \) lies in the subcomplex \( \text{Hom}^*(E^*_i+L^*_i, \text{Hom}^*(L^*_i, L^*_i)) \).

It is easy to verify that \( [δ A^{1,0} + A^{1,0}]_{i_0...i_k} \) lies in \( \text{Hom}^*(E^*_i+L^*_i, \text{Hom}^*(L^*_i, L^*_i)) \). Hence by induction we know that the right hand side of Equation (39), \( [δ A^{k-1,2-k} + \sum_{l=1}^{k-1} A^{l,1-l} \cdot A^{k-l,1+l-k}_{i_0...i_k}, \] lies in \( \text{Hom}^*(E^*_i+L^*_i, \text{Hom}^*(L^*_i, L^*_i)) \) for \( k \geq 2 \). Also by induction we can show that it is a cocycle under the differential \( D \). By Lemma 2.6 we know that \( \text{Hom}^*(E^*_i+L^*_i) \) is acyclic, hence the \( A^{k,1-k}_{i_0...i_k} \) in \( \text{Hom}^*(E^*_i+L^*_i) \) which satisfies Equation (39) exists. By induction we construct the desired \( (L, A) \).
With the help of Proposition 3.8 we can prove the essential surjectivity of the sheafification functor $S$.

**Corollary 3.11.** *Essential surjectivity*] If the cover $\{U_i\}$ is $p$-good, then the sheafification functor

$$S : Tw_{perf}(X) \to Qcoh_{perf}(X)$$

induces an essentially surjective functor

$$S : HoTw_{perf}(X) \to D_{perf}(Qcoh(X)).$$

**Proof.** Let $P = (S^*, d)$ be an object in $Qcoh_{perf}(X)$. Consider the associated twisted complex $T(P)$, by Proposition 3.8 there exists a twisted complex $E$ together with a weak equivalence

$$\phi : E \xrightarrow{\sim} T(P).$$

Then by Corollary 3.5 we get a quasi-isomorphism

$$S(\phi) : S(E) \xrightarrow{\sim} ST(P).$$

On the other hand Proposition 3.6 provides us another quasi-isomorphism

$$\tau_P : P \xrightarrow{\sim} ST(P).$$

Therefore $S(E)$ is quasi-isomorphic to $P$, which finishes the proof of Corollary 3.11.

**3.3.3 Essential surjectivity on complexes of $O_X$-modules**

Now we want to show that the following functor

$$S : HoTw_{perf}(X) \to D_{perf}(X)$$

is essentially surjective. For this we need the following additional condition on the ringed space $(X, O_X)$.

**Definition 3.6.** We say a locally ringed space $(X, O_X)$ satisfies the perfect-onto condition if the natural map

$$D_{perf}(Qcoh(X)) \to D_{perf}(X)$$

is essentially surjective.

Further discussions of perfect-onto condition will be given in Appendix A. In particular we can show that any quasi-compact and semi-separated scheme or any Noetherian scheme satisfies the perfect-onto condition.

With Definition 3.6 we have the following result.

**Corollary 3.12.** *Essential surjectivity*] If $X$ satisfies the perfect-onto condition and the cover $\{U_i\}$ is $p$-good, then the functor

$$S : HoTw_{perf}(X) \to D_{perf}(X)$$

is essentially surjective.

**Proof.** It is a direct corollary of Corollary 3.11 and Definition 3.6.

\[ \square \]
3.4 Fully faithfulness of the sheafification functor

3.4.1 Full faithful on complexes of quasi-coherent sheaves

We want to show that the sheafification functor $S$ induces a fully faithful functor

$$S : \text{HoTw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X)).$$

First we have the following proposition.

**Proposition 3.13.** Let the cover $\{U_i\}$ satisfy $H^k(U_i, F) = 0$ for any $i$, any quasi-coherent sheaf $F$ on $U_i$ and any $k \geq 1$. If $E$ and $F$ are both in the subcategory $\text{Tw}_{\text{perf}}(X)$, then $S(\phi) : S(E) \to S(F)$ is a quasi-isomorphism if and only if $\phi : E \to F$ is invertible in $\text{HoTw}_{\text{perf}}(X)$.

**Proof.** We first use Proposition 2.8 which claims that $\phi : E \to F$ is invertible in $\text{HoTw}(X)$ if and only if $\phi$ is a weak equivalence. Moreover Corollary 3.5 tells us $\phi$ is a weak equivalence if and only if $S(\phi) : S(E) \to S(F)$ is a quasi-isomorphism, hence we get the result.

**Corollary 3.14.** [Fully faithful] If the cover $\{U_i\}$ is $p$-good, then the functor $S : \text{HoTw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X))$ is fully faithful.

**Proof.** Let $A$ and $B$ be two objects in $D_{\text{perf}}(\text{Qcoh}(X))$ and $\varphi : A \to B$ be a chain map between them. Proposition 3.8 gives us two twisted resolutions $E \xrightarrow{\sim} T(A)$ and $F \xrightarrow{\sim} T(B)$. Apply Proposition 2.9 we get a morphism $\hat{\phi} : E \to F$ such that the following diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\phi} & F \\
\sim \downarrow & & \sim \downarrow \\
T(A) & \xrightarrow{T(\varphi)} & T(B)
\end{array}
$$

commutes up to homotopy. Hence we get

$$
\begin{array}{ccc}
S(E) & \xrightarrow{S(\phi)} & S(F) \\
\sim \downarrow & & \sim \downarrow \\
S T(A) & \xrightarrow{S T(\varphi)} & S T(B). 
\end{array}
$$

On the other hand by Proposition 3.6 we know that $id \to ST$ is a natural isomorphism of functors hence we have

$$
\begin{array}{ccc}
S(E) & \xrightarrow{S(\phi)} & S(F) \\
\sim \downarrow & & \sim \downarrow \\
S T(A) & \xrightarrow{S T(\varphi)} & S T(B) \\
\sim \uparrow & & \sim \uparrow \\
A & \xrightarrow{\varphi} & B.
\end{array}
$$

Moreover by Proposition 3.13 we invert the same morphism on $\text{HoTw}_{\text{perf}}(X)$ and $D_{\text{perf}}(\text{Qcoh}(X))$. Hence we know that $S$ is fully faithful.
Remark 16. The great advantage of twisted complexes is that we have more flexibility on morphisms. For example when \((X, \mathcal{O}_X)\) is a projective scheme, then it is well-known that any perfect complex on \(X\) is strictly perfect. In other words let \(L(X)\) be the dg-category of two-side bounded complexes of finitely generated locally free sheaves on \(X\). Then the natural functor \(\text{Ho}L(X) \to D_{\text{coh}}(\text{Qcoh}(X))\) is essentially surjective but not necessarily fully faithful.

In fact let \(E\) and \(F\) be two objects in \(L(X)\) and \(\phi : E \xrightarrow{\sim} F\) be a quasi-isomorphism. Then in general \(\phi\) does not have an inverse in \(\text{Ho}L(X)\). Nevertheless the inverse of \(\phi\) exists in \(\text{Ho}\text{Tw}_{\text{perf}}(X)\) if we consider \(E\) and \(F\) as twisted perfect complexes through the twisting functor \(\mathcal{T}\) and the cover is p-good.

Finally we reach our main theorem of this paper

**Theorem 3.15.** [dg-enhancement, see Theorem 1.2 in the Introduction] If the cover \(\{U_i\}\) is p-good, then the sheafification functor \(S : \text{Tw}_{\text{perf}}(X) \to \text{Qcoh}_{\text{perf}}(X)\) gives an equivalence of categories

\[
S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X))
\]  

**Proof.** This is a immediate consequence of Corollary 3.11 and Corollary 3.14. 

**Example 1.** We have the following cases which we can apply Theorem 3.15. In fact we only need to verify the following spaces have p-good covers. For more discussion see Appendix B.

- Let \((X, \mathcal{O}_X)\) be a separated scheme, then we have an equivalence of categories \(S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X))\).

- Let \(X\) be a complex manifold with the structure sheaf of holomorphic functions, then we have an equivalence of categories \(S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X))\).

- Let \(X\) be a smooth manifold with the structure sheaf of smooth functions, then we have an equivalence of categories \(S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(\text{Qcoh}(X))\).

### 3.4.2 Fully faithful on complexes of \(\mathcal{O}_X\)-modules

Similar to the discussion in Section 3.3.3 we can add certain conditions on \(X\) and get the fully faithfulness on perfect complexes of arbitrary \(\mathcal{O}_X\)-modules.

**Corollary 3.16.** [Fully faithful] If \(X\) satisfies the perfect-onto condition and the cover \(\{U_i\}\) is p-good, then the functor

\[
S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(X)
\]

is fully faithful.

**Proof.** The proof is similar to that of Corollary 3.14. First by Corollary 3.12 the functor \(S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(X)\) is essentially surjective. Then by Proposition 3.13 we invert the same morphisms in \(\text{Ho}\text{Tw}_{\text{perf}}(X)\) and \(D_{\text{perf}}(X)\). On the other hand we know that \(S\) is fully faithful on the level of dg-categories, therefore \(S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(X)\) is fully faithful. 

**Theorem 3.17.** [dg-enhancement, see Theorem 1.2 in the Introduction] If \(X\) satisfies the perfect-onto condition and the cover \(\{U_i\}\) is p-good, then the sheafification functor \(S : \text{Tw}_{\text{perf}}(X) \to \text{Sh}_{\text{perf}}(X)\) gives an equivalence of categories

\[
S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(X)
\]

**Proof.** This is a immediate consequence of Corollary 3.12 and Corollary 3.16. 

**Example 2.** The application of Theorem 3.17 is more restrictive than Theorem 3.15 since we need to verify the perfect-onto condition. Nevertheless it contains the following important cases: Let \((X, \mathcal{O}_X)\) be a quasi-compact and semi-separated or Noetherian scheme, then we have an equivalence of categories \(S : \text{Ho}\text{Tw}_{\text{perf}}(X) \to D_{\text{perf}}(X)\). See Appendix A Corollary A.3.
4 APPLICATIONS OF TWISTED COMPLEXES

Twisted complex has various applications. For example in [OT185] twisted complex is used to formulate and prove a Grothendieck-Riemann-Roch theorem for perfect complexes and in [Gil86] it is used to compute the higher algebraic K-theory of schemes.

Remark 17. Neither of the above works uses the fact that twisted perfect complexes is a dg-enhancement of perfect complexes.

In this paper we talk about the application of twisted complexes in descent theory. One of the drawback of derived categories is that they do not satisfies descent. In more details, let $X$ be a scheme and $U$, $V$ be an open cove of $X$, then the natural functor

$$D_{\text{perf}}(X) \to D_{\text{perf}}(U) \times_{D_{\text{perf}}(U \cap V)} D_{\text{perf}}(V)$$

is not an equivalence even in the case that $X = \mathbb{P}^1$ and $U$, $V$ are the upper and lower hemispheres.

This problem can be solved in the framework of dg-categories. In face Tabuada in [Lab10] gives an explicit construction of path object in dg-categories, which leads to the following definition of homotopy fiber product of dg-categories.

Definition 4.1. ([BBB13] Section 4) Let $A$, $B$, $C$ be dg-categories and $\phi : A \to C$, $\theta : B \to C$ be dg-functors. Then the homotopy fiber product $A \times_B^h C$ is a dg-category with objects

$$\text{ob}(A \times_B^h C) = \{M, N, f| M \in \text{ob}(A), N \in \text{ob}(B), f : \phi(M) \to \theta(N) \text{ closed of degree 0 and invertible in } H^0(C)\}.$$

The degree $k$ morphisms between $(M_1, N_1, f_1)$ and $(M_2, N_2, f_2)$ are given by

$$(\mu, \nu, \tau) \in A^k(M_1, M_2) \oplus B^k(N_1, N_2) \oplus C^{k-1}(\phi(M_1), \theta(N_2))$$

with composition given by

$$(\mu', \nu', \tau')(\mu, \nu, \tau) = (\mu' \mu, \nu' \nu, \tau' \phi(\mu) + \theta(\nu') \tau).$$

The differential on the morphisms is given by

$$d(\mu, \nu, \tau) = (d\mu, d\nu, d\tau + f_2 \phi(\mu) - (-1)^k \theta(\nu) f_1).$$

Now we move on to the descent problem. Let $X$ be a separated scheme and $X = U \cup V$ be two open subsets. For simplicity let us consider the case that $U$ and $V$ are affine. Then $U \cap V$ is affine too. Moreover, $\{U, V\}$ gives an affine (hence p-good) open cover of $X$ and we have $\text{Tw}_{\text{perf}}(X, \mathcal{O}_X, \{U, V\})$.

It is clear that $\text{Tw}_{\text{perf}}(U, \mathcal{O}_U, \{U\})$ is exactly the dg-category of strictly perfect complexes on $U$. The same assertion holds for $\text{Tw}_{\text{perf}}(V, \mathcal{O}_V, \{V\})$ and $\text{Tw}_{\text{perf}}(U \cap V, \mathcal{O}_{U \cap V}, \{U \cap V\})$. There are natural dg-functors

$$\phi : \text{Tw}_{\text{perf}}(U, \mathcal{O}_U, \{U\}) \to \text{Tw}_{\text{perf}}(U \cap V, \mathcal{O}_{U \cap V}, \{U \cap V\})$$

and

$$\theta : \text{Tw}_{\text{perf}}(V, \mathcal{O}_V, \{V\}) \to \text{Tw}_{\text{perf}}(U \cap V, \mathcal{O}_{U \cap V}, \{U \cap V\})$$

given by restriction.

We omit open covers and structure rings in the notation of the twisted perfect complexes and we have the following descent property.


Proposition 4.1. Let $X$, $U$, $V$ be as above, then we have an quasi-equivalence of dg-categories

$$\text{Tw}_{\text{perf}}(X) \xrightarrow{\sim} \text{Tw}_{\text{perf}}(U) \times_{\text{Tw}_{\text{perf}}(U \cap V)} \text{Tw}_{\text{perf}}(V).$$  (42)

Proof. The proof is to untangle the definition of homotopy fiber product. Let $E$ be an object in $\text{Tw}_{\text{perf}}(X)$. Then it gives $E^\bullet_U$ on $U$ and $E^\bullet_V$ on $V$.

It is clear $E^\bullet_U$ together with $a^{0,1}_U$ give an object in $\text{Tw}_{\text{perf}}(U)$ and we denote it by $\mathcal{M}$. Similarly $E^\bullet_V$ together with $a^{0,1}_V$ give an object $\mathcal{N}$ in $\text{Tw}_{\text{perf}}(V)$. Moreover the map $a_{V,U}^{1,0}$ gives the morphism

$$f: \mathcal{M} \to \mathcal{N}$$

and it is homotopically invertible since the $a$’s satisfies the Maurer-Cartan equation.

Hence we get a dg-functor

$$R: \text{Tw}_{\text{perf}}(X) \to \text{Tw}_{\text{perf}}(U) \times_{\text{Tw}_{\text{perf}}(U \cap V)} \text{Tw}_{\text{perf}}(V).$$

It is clear that $R$ is essentially surjective. By the same method as in the proof of Proposition 3.8 we can prove it is also quasi-fully faithful.

Remark 18. The same idea works for the general case where $U$ and $V$ are not affine. Nevertheless we need an explicitly construction of homotopy limit of dg-categories and this topic will be treated in another paper.

5 TWISTED COHERENT COMPLEXES

In this section we consider a generalization of twisted perfect complex, where the two-side bounded complexes are replaced by bounded above complexes. We omit most of the proofs since they are the same of the corresponding proofs for the twisted perfect complexes.

5.1 The derived category of bounded above coherent complexes

First we review the relevant derived categories. We have a definition of coherent complex.

Definition 5.1. Let $(X, \mathcal{O}_X)$ be a Noetherian scheme. A complex $S^\bullet$ of $\mathcal{O}_X$-modules is bounded above and coherent if for any point $x \in X$, there exists an open neighborhood $U$ of $x$ and a bounded above complex of finite rank locally free sheaves $\mathcal{E}^\bullet_U$ on $U$ such that the restriction $S^\bullet|_U$ is isomorphic to $\mathcal{E}^\bullet_U$ in $D(\mathcal{O}_X|_U - \text{mod})$, the derived category of sheaves of $\mathcal{O}_X$-modules on $U$.

Remark 19. If $X$ is not a separated Noetherian scheme then the category of bounded above coherent complexes does not behave well. In fact a more standard notion is the pseudo-coherent complex on a ringed space, see [ber71] Exposé I or [IT90] Section 2. Nevertheless, pseudo-coherent coincides with our definition of coherent if $X$ is a Noetherian scheme as in Appendix A. In this paper we will stick to our definition of coherent complex.

In this section we assume $X$ is a separated Noetherian scheme.

We consider the following categories.

Definition 5.2. Let $\text{Sh}_{\text{coh}}(X)$ be the full dg-subcategory of $\text{Sh}(X)$ which consists of bounded above coherent complexes on $X$.

Similarly we have $K^+_{\text{coh}}(X)$, $D^+_{\text{coh}}(X)$, $K^+_{\text{coh}}(\text{Qcoh}(X))$, and $D^+_{\text{coh}}(\text{Qcoh}(X))$.
5.2 Twisted coherent complexes

We have the following definition which is similar to Definition 2.5.

**Definition 5.3.** A twisted coherent complex $\mathcal{E} = (E^\bullet, a)$ is the same as twisted complex except that $E^\bullet$ are bounded above graded finitely generated locally free $\mathcal{O}_X$-modules.

The twisted perfect complexes form a dg-category and we denote it by $\text{Tw}_{\text{coh}}^-(X, \mathcal{O}_X, \{U_i\})$ or simply $\text{Tw}_{\text{coh}}^-(X)$. Obviously $\text{Tw}_{\text{coh}}^-(X)$ is a full dg-subcategory of $\text{Tw}(X)$ while $\text{Tw}_{\text{perf}}(X)$ is a full dg-subcategory of $\text{Tw}_{\text{coh}}^-(X)$.

The differential $\delta_a$, shift functor, mapping cone and weak equivalence as in Section 2.5 and 2.6 can be defined on $\text{Tw}_{\text{coh}}^-(X)$ without any change. Moreover we have the same result as in Proposition 2.8.

**Proposition 5.1.** Let the cover $\{U_i\}$ satisfy $H^k(U_i, \mathcal{F}) = 0$ for any $i$, any quasi-coherent sheaf $\mathcal{F}$ on $U_i$ and any $k \geq 0$. If $\mathcal{E}$ and $\mathcal{F}$ are both in the subcategory $\text{Tw}_{\text{coh}}^-(X)$, then a closed degree zero morphism $\phi$ between twisted complexes $\mathcal{E}$ and $\mathcal{F}$ is a weak equivalence if and only if $\phi$ is invertible in the homotopy category $\text{HoTw}_{\text{coh}}^-(X)$.

**Proof.** Notice that in the proof of Proposition 2.8 we do not use the boundedness of the complexes hence the same proof works for $\text{HoTw}_{\text{coh}}^-(X)$. □

5.3 The sheafification functor on twisted coherent complexes

We can restrict the sheafification functor in Definition 3.2 to twisted coherent complexes and get a functor

$$S : \text{Tw}_{\text{coh}}^-(X) \to \text{Sh}(X).$$

(43)

Actually we have a result which is similar to Remark 13.

**Proposition 5.2.** The functor $S$ maps $\text{Tw}_{\text{coh}}^-(X)$ to complexes of quasi-coherent sheaves, i.e. we have

$$S : \text{Tw}_{\text{coh}}^-(X) \to \text{Qcoh}(X).$$

(44)

**Proof.** Recall that

$$S^n(\mathcal{E}) = \prod_{p+q=n} E^p_{iq}|_{U_{i_0 \ldots i_p}}$$

and we know that each $E^q_{ip}|_{U_{i_0 \ldots i_p}}$ is quasi-coherent on $X$. Now the product is not necessarily finite. Nevertheless we have the following lemma.

**Lemma 5.3.** Let $X$ be a quasi-compact and quasi-separated scheme, then the category $\text{Qcoh}(X)$ has all limits.

**Proof.** See [TT90] Lemma B.12. □

Since our scheme $X$ is quasi-compact and separated, $\text{Qcoh}(X)$ has infinite product, hence the result follows. □

Keep in mind that Proposition 3.1 works for any twisted complexes, hence it works for twisted coherent complexes. Moreover we also have the same result as in Corollary 3.4.

**Proposition 5.4.** If $\mathcal{E} = (E^\bullet, a)$ is a twisted coherent complex, then the sheafification $S^*(\mathcal{E})$ is a coherent complex of sheaves on $(X, \mathcal{O}_X)$. In other words the sheafification functor $S$ restricts to $\text{Tw}_{\text{coh}}^-(X)$ and gives the following functor

$$S : \text{Tw}_{\text{coh}}^-(X) \to \text{Sh}_{\text{coh}}^-(X).$$

(45)

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Proof. The proof is the same as that of Corollary 3.4.

Similar to Remark 3.4, we know that we actually have
\[ S : Tw_{coh}(X) \to Qcoh_{coh}(X). \]  
(46)

5.4 Essential surjectivity on coherent complexes

Similar to the discussion in Section 3.3, the functor
\[ S : Tw_{coh}(X) \to Qcoh_{coh}(X) \]
induces a functor
\[ S : HoTw_{coh}(X) \to D_{coh}(Qcoh(X)). \]

In this subsection we will show that this functor is essentially surjective under some mild condition. Moreover we will show that the functor
\[ S : HoTw_{coh}(X) \to D_{coh}(X) \]
is essentially surjective under some additional conditions.

First we have the following definitions which are similar to Definition 3.4 and 3.5.

**Definition 5.4.** A locally ringed space \((U, O_U)\) is called c-good if it satisfies
a. For every coherent complex \(C^\bullet\) on \(U\) which consists of quasi-coherent sheaves, there exists a bounded above complex of finitely generated locally free sheaves \(E^\bullet\) together with a quasi-isomorphism \(v : E^\bullet \simeq C^\bullet\).

b. The higher cohomologies of quasi-coherent sheaves vanish, i.e. \(H^k(U, F) = 0\) for any quasi-coherent sheaf \(F\) on \(U\) and any \(k \geq 1\).

**Definition 5.5.** Let \((X, O_X)\) be a locally ringed space, an open cover \(\{U_i\}\) of \(X\) is called a c-good cover if \((U_I, O_X|_{U_I})\) is a c-good space for any finite intersection \(U_I\) of the open cover.

Then we have the coherent version of twisted resolution (Proposition 3.8).

**Proposition 5.5.** Assume the cover \(\{U_i\}\) is c-good. Let \(P = (S^\bullet, d_S)\) be a bounded above coherent complex which consists of quasi-coherent modules, then there exists a twisted coherent complex \(E\) together with a weak equivalence
\[ \phi : E \simeq T(P). \]

Proof. The proof is the same as that of Proposition 3.8.

Hence we have the following essential surjectivity.

**Corollary 5.6.** If the cover \(\{U_i\}\) is c-good, then the sheafification functor
\[ S : Tw_{coh}(X) \to Qcoh_{coh}(X) \]
induces an essentially surjective functor
\[ S : HoTw_{coh}(X) \to D_{coh}(Qcoh(X)). \]

Proof. The proof is the same as that of Corollary 3.11.
The essential surjectivity on arbitrary $\mathcal{O}_X$-modules involves the following definition.

**Definition 5.6.** We say a locally ringed space $(X, \mathcal{O}_X)$ satisfies the coherent-onto condition if the natural map

$$D^-(\text{Qcoh}(X)) \to D^-(X)$$

is essentially surjective.

Actually we can show that any Noetherian scheme with finite Krull dimension satisfies the coherent-onto condition, see Appendix [A] Corollary [A.5].

**Corollary 5.7.** If $X$ satisfies the coherent-onto condition and the cover $\{U_i\}$ is c-good, then the functor

$$S : H\text{O}Tw^-(X) \to D^-(X)$$

is essentially surjective.

**Proof.** It is obvious from Corollary 5.6 and Definition 5.6.

5.5 Fully Faithfulness on coherent complexes

**Proposition 5.8.** Let the cover $\{U_i\}$ satisfy $H^k(U_i, \mathcal{O}_X|_{U_i}) = 0$ for any $i$ and any $k \geq 0$. If $\mathcal{E}$ and $\mathcal{F}$ are both in the subcategory $\text{Tw}^-(X)$, then $S(\phi) : S(E) \to S(F)$ is a quasi-isomorphism if and only if $\phi : \mathcal{E} \to \mathcal{F}$ is invertible in $H\text{O}Tw^-(X)$.

**Proof.** Since we have Proposition 5.1, the proof is the same as that of Proposition 3.13.

**Corollary 5.9.** If the cover $\{U_i\}$ is c-good, then the functor $S : \text{Ho}Tw^-(X) \to D^-(\text{Qcoh}(X))$ is fully faithful.

**Proof.** The proof is the same as that of Corollary 3.14.

**Theorem 5.10.** If the cover $\{U_i\}$ is c-good, then the sheafification functor $S : \text{Tw}^-(X) \to \text{Qcoh}^-(X)$ gives an equivalence of categories

$$S : \text{Ho}Tw^-(X) \to D^-(\text{Qcoh}(X))$$

**Proof.** It is a immediate consequence of Corollary 5.6 and 5.9.

**Example 3.** If $X$ is a separated Noetherian scheme, then we have an equivalence of categories $S : \text{Ho}Tw^-(X) \to D^-(\text{Qcoh}(X))$.

Then we consider the coherent complexes of arbitrary $\mathcal{O}_X$-modules.

**Corollary 5.11.** If $X$ satisfies the coherent-onto condition and the cover $\{U_i\}$ is c-good, then the functor

$$S : \text{Ho}Tw^-(X) \to D^-(X)$$

is fully faithful.

**Proof.** The proof is the same as that of Corollary 3.16.

**Theorem 5.12.** If $X$ satisfies the coherent-onto condition and the cover $\{U_i\}$ is c-good, then the sheafification functor $S : \text{Tw}^-(X) \to \text{Sh}^-(X)$ gives an equivalence of categories

$$S : \text{Ho}Tw^-(X) \to D^-(\text{Qcoh}(X))$$

**Proof.** This is a immediate consequence of Corollary 5.7 and Corollary 5.11.

**Example 4.** If $X$ is a separated Noetherian scheme with finite Krull dimension, then we have an equivalence of categories $S : \text{Ho}Tw^-(X) \to D^-(\text{Qcoh}(X))$. See Appendix [A] Corollary [A.5].
6 DEGENERATE TWISTED COMPLEXES AND THE SPLITTING OF IDEMPOTENT

Recall that in the definition of twisted complex we have the non-degenerate condition which requires that for each \( i \) we have
\[
a_{ii}^{1,0} = id
\]
up to homotopy.

It is interesting to drop the non-degenerate condition and have the following definition.

**Definition 6.1.** A generalized twisted complex is the same as a twisted complex except that we do not require \( a_{ii}^{1,0} = id \) up to homotopy.

Similarly we have generalized twisted perfect complexes and generalized twisted coherent complexes.

We denote the dg-category of generalized twisted complexes by \( gTw(X) \).

Similarly we have \( gTw_{\text{perf}}(X) \) and \( gTw_{\text{coh}}(X) \).

**Example 5.** For given \( E_i^\bullet \)'s, we get set all \( a_{i-1}^k \)'s to be 0. It definitely satisfies the Maurer-Cartan equation
\[
\delta a + a \cdot a = 0
\] hence it gives a generalized twisted complex but not a twisted complex unless the \( E_i^\bullet \)'s are all zero.

For generalized twisted complexes we have the following obvious observations

1. \( Tw(X) \) is a full dg-subcategory of \( gTw(X) \), \( Tw_{\text{perf}}(X) \) is a full dg-subcategory of \( gTw_{\text{perf}}(X) \) and \( Tw_{\text{coh}}(X) \) is a full dg-subcategory of \( gTw_{\text{coh}}(X) \).

2. Nevertheless there is no inclusion relation between \( gTw_{\text{perf}}(X) \) and \( Tw(X) \) nor between \( gTw_{\text{coh}}(X) \) and \( Tw(X) \).

3. The pre-triangulated structure as in Section 2.5 can be defined on \( gTw(X) \), \( gTw_{\text{perf}}(X) \) and \( gTw_{\text{coh}}(X) \) without any change.

4. The weak equivalence in \( gTw(X) \) is exactly the same as in Section 2.6 and Proposition 2.8 still holds for generalized twisted complexes.

5. We can define the sheafification functor
\[
\mathcal{S} : gTw(X) \to Sh(X)
\]
in the same way as Section 3.1 Definition 3.1 and 3.2.

It is not obvious that \( \mathcal{S} \) maps a generalized twisted perfect/coherent complex to a perfect/coherent complex. Actually we need some more work. Recall Lemma 2.2 claims that if the \( a^{k,1-k} \)'s satisfy the Maurer-Cartan equation, then \( a_{i-1}^{1,0} : (E^n_i, a^0_i) \to (E^n_i, a^0_i) \) is an idempotent map in the homotopy category \( K(U_i) \), i.e. \( (a_{i-1}^{1,0})^2 = a_{i-1}^{1,0} \) up to chain homotopy.

It is a classical result that the category \( K(U_i) \) is *idempotent complete* ([BN93] Proposition 3.2), i.e. for any object \( S \) of \( K(U_i) \) and any idempotent \( \alpha : S \to S \), there exists a splitting of \( \alpha \). More precisely there exists a \( T \) in \( K(U_i) \) together with \( i : T \to S \) and \( p : S \to T \) such that
\[
i \circ p = id_T \text{ and } ip = \alpha.
\]

Intuitively such a splitting \( T \) can be considered as the image of the map \( \alpha \). However in general \( T \) is not the naive image of \( \alpha \) in the chain complex.

The following proposition gives an explicit construction of the splitting.
Proposition 6.1. Let $\mathcal{E} = (E^*_E, a)$ be a twisted complex and $(S^*(\mathcal{E}), \delta_a)$ be the associative complex of sheaves. Then $(S^*(\mathcal{E}), \delta_a)|_{U_j}$ is a splitting of the idempotent $\alpha_{j,0}^{1,0} : (E^*_E, a^{0,1}_j) \rightarrow (E^*_E, a^{0,1}_j)$, i.e. we have two morphisms
\[ f : (S^*(\mathcal{E}), \delta_a)|_{U_j} \rightarrow (E^*_E, a^{0,1}_j) \]
and
\[ g : (E^*_E, a^{0,1}_j) \rightarrow (S^*(\mathcal{E}), \delta_a)|_{U_j} \]
such that
\[ f \circ g = \alpha_{j,0}^{1,0} \text{ and } g \circ f = \text{id}_{S^*(\mathcal{E})|_{U_j}} \text{ up to chain homotopy.} \quad (49) \]

Proof. The proof is exactly the same as that of Proposition 3.1 except that here $f \circ g = \alpha_{j,0}^{1,0}$ does not necessarily equal to $\text{id}$, not even up to homotopy.

With the help of Proposition 6.1 we can get the following result.

Corollary 6.2. If $\mathcal{E} = (E^*_E, a)$ is a generalized twisted perfect (or twisted coherent) complex, then the sheafification $S^*(\mathcal{E})$ is a perfect (or coherent, respectively) complex of sheaves on $(X, \mathcal{O}_X)$. In other words the sheafification functor $S$ restricts to $g\mathcal{T}_\text{perf}(X)$ (or $g\mathcal{T}_\text{coh}(X)$, respectively) and gives the following functor
\[ S : g\mathcal{T}_\text{perf}(X) \rightarrow \mathcal{Q}_\text{coh}(X). \quad (50) \]
and
\[ S : g\mathcal{T}_\text{coh}(X) \rightarrow \mathcal{Q}_\text{coh}(X). \quad (51) \]

Proof. Since $\mathcal{E} = (E^*_E, a)$ is a generalized twisted perfect complex, for each $U_j$ the complex $(E^*_E, a^{0,1}_j)$ is a two-side bounded complex which consists of locally free finitely generated $\mathcal{O}_X$-modules, i.e. $(E^*_E, a^{0,1}_j)$ is an object in $K_{\text{perf}}(U_j)$. We know that $K_{\text{perf}}(U_j)$ is also idempotent complete since it consists of compact objects in $K(U_j)$. Proposition 6.1 tells us that $S^*(\mathcal{E})|_{U_j}$ is a splitting of idempotent $\alpha_{j,0}^{1,0}$ hence $S^*(\mathcal{E})|_{U_j}$ is perfect on $U_j$. Moreover this is true for any member $U_j$ of the open cover, therefore $S^*(\mathcal{E})$ is a perfect complex of sheaves on $(X, \mathcal{O}_X)$.

The same proof works for twisted coherent complexes.

Corollary 6.3. a. If the cover $\{U_i\}$ is p-good, then the functor
\[ S : \text{Ho}(g\mathcal{T}_\text{perf}(X)) \rightarrow D_{\text{perf}}(\mathcal{Q}_\text{coh}(X)) \]
is essentially surjective.

b. If the cover $\{U_i\}$ is c-good, then the functor
\[ S : \text{Ho}(g\mathcal{T}_\text{coh}(X)) \rightarrow D_{\text{coh}}(\mathcal{Q}_\text{coh}(X)) \]
is essentially surjective.

Proof. By Corollary 3.11 we already know that $S : \text{Ho}\mathcal{T}_\text{perf}(X) \rightarrow D_{\text{perf}}(\mathcal{Q}_\text{coh}(X))$ is essentially surjective. Since $\mathcal{T}_\text{perf}(X)$ is a subcategory of $g\mathcal{T}_\text{perf}(X)$ and the functor $S$ coincide on $\mathcal{T}_\text{perf}(X)$, the claim is obviously true.

The same proof works for twisted coherent complexes.
However, $S$ does not induce a fully faithful functor

$$S: Ho(gTw_{perf}(X)) \rightarrow D_{perf}(Qcoh(X))$$

nor

$$S: Ho(gTw_{coh}(X)) \rightarrow D_{coh}(Qcoh(X)).$$

The main reason of the failure is that we no longer have the same result as in Corollary 3.5 for generalized twisted complex hence Proposition 2.8 does not hold for generalized twisted complex either.

In fact, if $E$ and $F$ are generalized twisted coherent complexes, then the fact that $S(\phi) : S(E) \rightarrow S(F)$ is a quasi-isomorphism does not imply $\phi : E \rightarrow F$ is invertible in the homotopy category.

Example 6. For a counter-example, let $E = (E_i^*, 0)$ be non-zero, two-side bounded graded locally free finitely generated $O_X$-modules on each $U_i$ with all $a$’s equal to 0. Let $F$ simply be $0$ and $\phi$ be the zero map. It is clear that $\phi_i^{0,0} : E_i^* \rightarrow 0$ is not a quasi-isomorphism hence $\phi$ cannot be invertible in $Ho(gTw_{perf}(X)).$ However by Proposition 6.1 it is not difficult to show that $S(E)$ is an acyclic complex hence $S(\phi) = 0 : S(E) \rightarrow 0$ is a quasi-isomorphism.

The above discussion tells us that $(gTw_{perf}(X), S)$ (or $(gTw_{coh}(X), S)$) is not a dg-enhancement of $D_{perf}(Qcoh(X))$ (or $D_{coh}(Qcoh(X))$ respectively). Nevertheless, $gTw(X)$ has its own interests and may be further studied in the future.

7 FURTHER TOPIC: QUILLEN ADJUNCTION

The proof of dg-enhancement in this paper is more or less a by-hand proof. Nevertheless in this section we would like to briefly mention a more categorical approach to the result in this paper.

We have defined two functors

$$S: Tw(X) \rightarrow Sh(X)$$

and

$$T: Sh(X) \rightarrow Tw(X).$$

So far we know that $S$ and $T$ quasi-inverse to each other by Proposition 3.6 and Proposition 3.7.

On the other hand we have the injective and projective model structure on $Sh(X)$, see [Hov01]. Moreover in Definition 2.9 we already have a notion of weak equivalence in $Tw(X)$ and we wish to further construct a suitable model structure on $Tw(X)$ with the weak equivalence as above, which, together with the suitable model structure on $Sh(X)$, makes $S$ and $T$ a Quillen adjunction

$$S: Tw(X) \rightleftarrows Sh(X) : T.$$ (52)

The Quillen adjunction, if exists, will reveal deeper information on twisted complexes. It is also hoped that the dg-enhancement result can be also proved in this approach.

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Appendices

A SOME DISCUSSIONS ON COMPLEXES OF SHEAVES

A.1 Pseodo-coherent complexes and coherent complexes

Recall that we have a definition of coherent complexes.

Definition A.1 (Definition 5.1). Let \((X, \mathcal{O}_X)\) be a locally ringed space. A complex \(S^\bullet\) of \(\mathcal{O}_X\)-modules is coherent if for any point \(x \in X\), there exists an open neighborhood \(U\) of \(x\) and a bounded above complex of finite rank locally free sheaves \(E^\bullet_U\) on \(U\) such that the restriction \(S^\bullet|_U\) is isomorphic to \(E^\bullet_U\) in \(D(\mathcal{O}_X|_U - \text{mod})\), the derived category of sheaves of \(\mathcal{O}_X\)-modules on \(U\).

For general locally ringed spaces \((X, \mathcal{O}_X)\), this version of coherent complex does not behave well and we have the following definition.

Definition A.2. [[TT90] Definition 2.1.1, 2.2.6 or [ber71] Exposé I, 2.1, 2.3]

a. For any integer \(m\), a complex \(E^\bullet\) of \(\mathcal{O}_X\)-modules on \(X\) is called strictly \(m\)-pseudo-coherent if \(E^i\) is a locally free finitely generated \(\mathcal{O}_X\)-module for \(i \geq m\) and \(E^i = 0\) for \(i\) sufficiently large.

b. A complex \(E^\bullet\) of \(\mathcal{O}_X\)-modules on \(X\) is called strictly pseudo-coherent if it is \(m\)-strictly-pseudo-coherent for all \(m\), i.e. it is a bounded above complex of locally free finitely generated \(\mathcal{O}_X\)-modules.

c. For any integer \(m\), a complex \(E^\bullet\) of \(\mathcal{O}_X\)-modules on \(X\) is called \(m\)-pseudo-coherent if for any point \(x \in X\) there exists an open neighborhood \(x \in U \subset X\) and a morphism of complexes \(\alpha : P^\bullet_U \to E^\bullet|_U\) where \(P_U\) is strictly \(m\)-pseudo-coherent on \(U\) and \(\alpha\) is a quasi-isomorphism on \(U\).

d. We say \(E^\bullet\) is pseudo-coherent if it is \(m\)-pseudo-coherent for all \(m\).

We may hope that a pseudo-coherent complex is locally quasi-isomorphic to a strictly pseudo-coherent complex. However according to [[TT90] 2.2.7:

For a pseudo-coherent complex of general \(\mathcal{O}_X\)-modules, there will locally be \(n\)-quasi-isomorphisms with a strictly pseudo-coherent complex, but the local neighborhoods where the \(n\)-quasi-isomorphisms are defined may shrink as \(n\) goes to \(-\infty\), and so may fail to exist in the limit. So there may not be a local quasi-isomorphism with a strict pseudo-coherent complex.

As a result, the definition of pseudo-coherent complex and our definition of coherent complex are not equivalent in general. Nevertheless if we assume that \(X\) is a Noetherian scheme, then we have the following proposition.

Proposition A.1 ([[TT90] 2.2.8, [ber71] Exposé I Section 3). A complex \(E^\bullet\) of \(\mathcal{O}_X\)-modules on a Noetherian scheme \(X\) is pseudo-coherent if and only if \(E^\bullet\) is cohomologically bounded above and all the \(H^k(E^\bullet)\) are coherent \(\mathcal{O}_X\)-modules, i.e. \(E^\bullet\) is pseudo-coherent if and only if \(E^\bullet \in D_{\text{coh}}^-(X)\).

Proof. See [[TT90] 2.2.8 or [ber71] Exposé I Section 3. □
A.2 Quasi-coherent modules v.s. arbitrary $\mathcal{O}_X$-modules

It is a subtle but important question that whether we could replace a complex of $\mathcal{O}_X$-modules by a complex of quasi-coherent modules in the derived categories. In this subsection we collect some result in [TT90] Appendix B and [ber71] Exposé II.

**Definition A.3.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called quasi-coherent if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus_{j \in J} \mathcal{O}_X|_U \to \bigoplus_{i \in I} \mathcal{O}_X|_U.$$ 

**Remark 20.** If $(X, \mathcal{O}_X)$ is a complex manifold, then we need the category of Fréchet quasi-coherent sheaves, which is a variation of the category of quasi-coherent sheaves, see [EP96] Section 4.3 for more details.

The natural inclusion $i : \text{Qcoh}(X) \to \text{Sh}(X)$ induces a natural functor

$$\tilde{i} : D(\text{Qcoh}(X)) \to D_{\text{Qcoh}}(X)$$

where $D_{\text{Qcoh}}(X)$ is the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomologies. However, the functor $\tilde{i}$ is not necessarily essentially surjective nor fully faithful for general $(X, \mathcal{O}_X)$. The same is true when we restrict to certain subcategories such as perfect complexes or coherent complexes.

Since $\tilde{i} : D(\text{Qcoh}(X)) \to D_{\text{Qcoh}}(X)$ is not an equivalence in general, we need to impose some condition on the locally ringed space $(X, \mathcal{O}_X)$ for our purpose. Here are some definitions we used.

**Definition A.4.** [See Definition 3.6 and Definition 5.6]

a. We say a locally ringed space $(X, \mathcal{O}_X)$ satisfies the **perfect-onto condition** if the natural map

$$D_{\text{perf}}(\text{Qcoh}(X)) \to D_{\text{perf}}(X)$$

is essentially surjective.

b. We say a locally ringed space $(X, \mathcal{O}_X)$ satisfies the **coherent-onto condition** if the natural map

$$D_{\text{coh}}^-(\text{Qcoh}(X)) \to D_{\text{coh}}^-(X)$$

is essentially surjective.

It is important to verify for which $X$ the above condition holds. In fact, we have the following result.

**Proposition A.2.** ([TT90] Proposition B.16, [ber71] Exposé II 3.5) Let $X$ be either a quasi-compact and semi-separated scheme, or else a Noetherian scheme. Then the functor

$$\tilde{i} : D^+(\text{Qcoh}(X)) \to D^+_{\text{Qcoh}}(X)$$

is an equivalence, where $D^+(\text{Qcoh}(X))$ is the derived category of complexes of quasi-coherent modules with bounded below cohomologies, and $D^+_{\text{Qcoh}}(X)$ is the derived category of complexes of $\mathcal{O}_X$-modules with bounded below and quasi-coherent cohomologies.

**Proof.** See the proof of [TT90] Proposition B.16. \(\square\)
Corollary A.3. Any quasi-compact and semi-separated or Noetherian scheme satisfies the perfect-onto condition.

Proof. On a quasi-compact scheme, any perfect complex has bounded below cohomology, hence the perfect-onto condition is satisfied by Proposition A.2.

However a bounded above coherent complex is not necessarily bounded below hence we can no longer use Proposition A.2. Nevertheless we have the same result under additional conditions.

Proposition A.4. [(TT90) B.17] Let $X$ be either a Noetherian scheme of finite Krull dimension or a semi-separated scheme with underlying space a Noetherian space of finite Krull dimension. Then the functor 
\[
\tilde{i} : D(Qcoh(X)) \to D_{Qcoh}(X)
\]
is an equivalence.

Proof. See [TT90] B.17.

Corollary A.5. Any Noetherian scheme of finite Krull dimension or a semi-separated scheme with underlying space a Noetherian space of finite Krull dimension satisfies the coherent-onto condition.

Proof. It is a direct corollary of Proposition A.4.

Remark 21. The result in Proposition A.2 and Proposition A.4 is stronger than we need. Hence it is desirable to find more precise criteria for the perfect-onto and coherent-onto conditions.

B GOOD COVERS OF LOCALLY RINGED SPACES

We discussion good covers of locally ringed spaces in this appendix. Recall that we have the following definitions.

Definition B.1. [Definition 3.4]

a. A locally ringed space $(U, O_U)$ is called $p$-good if it satisfies the following two conditions

1. For every perfect complex $P^\bullet$ on $U$ which consists of quasi-coherent sheaves, there exists a strictly perfect complex $E^\bullet$ together with a quasi-isomorphism $u : E^\bullet \to P^\bullet$.
2. The higher cohomologies of quasi-coherent sheaves vanish, i.e. $H^k(U, F) = 0$ for any quasi-coherent sheaf $F$ on $U$ and any $k \geq 1$.

b. A locally ringed space $(U, O_U)$ is called $c$-good if the first condition above is replaced by For every coherent complex $C^\bullet$ on $U$ which consists of quasi-coherent sheaves, there exists a bounded above complex of finitely generated locally free sheaves $E^\bullet$ together with a quasi-isomorphism $v : E^\bullet \to C^\bullet$.

The second condition remains the same.

Definition B.2. [Definition 3.5] Let $(X, O_X)$ be a locally ringed space, an open cover $\{U_i\}$ of $X$ is called a $p$-good cover (or $c$-good cover) if $(U_i, O_X|_{U_i})$ is a $p$-good space (or $c$-good space, respectively) for any finite intersection $U_I$ of the open cover.

The definition of good cover is not too restrictive since we have the following examples of ringed spaces with good covers.
• \((X, \mathcal{O}_X)\) is a separated scheme, then any affine cover is both p-good and c-good. In fact on a separated scheme the intersection of two affine open subsets is still affine hence Condition 2. in Definition B.1 is obviously satisfied and Condition 1. is proved in [TT90] Proposition 2.3.1.

• \((X, \mathcal{O}_X)\) is a complex manifold with \(\mathcal{O}_X\) the sheaf of holomorphic functions. In these case a Stein cover is both p-good and c-good. Actually on complex manifolds we should use the definition of Fréchet quasi-coherent sheaves, which is a variation of ordinary quasi-coherent sheaves, see [EP96] Section 4. A Stein manifold satisfies Condition 2. by Proposition 4.3.3 in [EP96], and Condition 1. can be proved in the same way as the argument in [TT90] Section 2.

• \((X, \mathcal{O}_X)\) is a paracompact topological space with soft structure sheaf \(\mathcal{O}_X\). Then any contractible open cover is both p-good and c-good.

References

[BBB13] Oren Ben-Bassat and Jonathan Block. Milnor descent for cohesive dg-categories. Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology, 12(03):433–459, 2013.

[ber71] Théorie des intersections et théorème de Riemann-Roch. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin-New York, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie.

[BK90] A. I. Bondal and M. M. Kapranov. Framed triangulated categories. Mat. Sb., 181(5):669–683, 1990.

[Blo10] Jonathan Block. Duality and equivalence of module categories in noncommutative geometry. In A celebration of the mathematical legacy of Raoul Bott, volume 50 of CRM Proc. Lecture Notes, pages 311–339. Amer. Math. Soc., Providence, RI, 2010.

[BN93] Marcel Bökstedt and Amnon Neeman. Homotopy limits in triangulated categories. Compositio Math., 86(2):209–234, 1993.

[BS14] Jonathan Block and Aaron M. Smith. The higher Riemann-Hilbert correspondence. Adv. Math., 252:382–405, 2014.

[EP96] Jörg Eschmeier and Mihai Putinar. Spectral decompositions and analytic sheaves, volume 10 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[Gil86] Henri Gillet. The \(K\)-theory of twisted complexes. In Applications of algebraic \(K\)-theory to algebraic geometry and number theory. Part I, II (Boulder, Colo., 1983), volume 55 of Contemp. Math., pages 159–191. Amer. Math. Soc., Providence, RI, 1986.

[Hol14] Julian Victor Sebastian Holstein. Morita cohomology. PhD thesis, University of Cambridge, 2014.

[Hov01] Mark Hovey. Model category structures on chain complexes of sheaves. Trans. Amer. Math. Soc., 353(6):2441–2457 (electronic), 2001.

[LS14] Valery A Lunts and Olaf M Schnürer. New enhancements of derived categories of coherent sheaves and applications. arXiv preprint arXiv:1406.7559, 2014.
[Lur] Jacob Lurie. Higher algebra. 2012. Preprint, available at http://www.math.harvard.edu/~lurie.

[OTT81] Nigel R. O’Brian, Domingo Toledo, and Yue Lin L. Tong. The trace map and characteristic classes for coherent sheaves. Amer. J. Math., 103(2):225–252, 1981.

[OTT85] Nigel R. O’Brian, Domingo Toledo, and Yue Lin L. Tong. A Grothendieck-Riemann-Roch formula for maps of complex manifolds. Math. Ann., 271(4):493–526, 1985.

[Sta15] The Stacks Project Authors. stacks project. http://stacks.math.columbia.edu, 2015.

[Tab10] Gonçalo Tabuada. Homotopy theory of dg categories via localizing pairs and Drinfeld’s dg quotient. Homology, Homotopy Appl., 12(1):187–219, 2010.

[TT78] Domingo Toledo and Yue Lin L. Tong. Duality and intersection theory in complex manifolds. I. Math. Ann., 237(1):41–77, 1978.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In The Grothendieck Festschrift, Vol. III, volume 88 of Progr. Math., pages 247–435. Birkhäuser Boston, Boston, MA, 1990.