SUBINTEGRALITY, INVERTIBLE MODULES AND LAURENT POLYNOMIAL EXTENSIONS

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Abstract. Let $A \subseteq B$ be a commutative ring extension. Let $I(A,B)$ be the multiplicative group of invertible $A$-submodules of $B$. In this article, we extend Sadhu and Singh result by finding a necessary and sufficient condition on $A \subseteq B$, so that the natural map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ is an isomorphism. We also discuss some properties of the cokernel of the natural map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ in general case.

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Introduction

In [3], Roberts and Singh have introduced the group $I(A,B)$ to generalize a result of Dayton. The relation between the group $I(A,B)$ and subintegral extensions has been investigated by Reid, Roberts and Singh in a series of papers. Recently in [4], Sadhu and Singh have proved that $A$ is subintegrally closed in $B$ if and only if the canonical map $I(A,B) \to I(A[X],B[X])$ is an isomorphism. It is easy to see that $I(A[X],B[X]) = I(A,B) \oplus NI(A,B)$. So the result of [4], just mentioned amounts to saying that $NI(A,B) = 0$ if and only if $A$ is subintegrally closed in $B$.

The primary goal of this paper is to extend Sadhu and Singh result of [4] just mentioned above by finding a necessary and sufficient condition on $A \subseteq B$, so that the natural map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ is an isomorphism. It is easy to see that the map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ is always injective. The secondary goal is to investigate the cokernel of the natural map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ in general case. This cokernel will be denoted by $MI(A,B)$.

In Section 1, we mainly give basic definitions and notations.

In Section 2, we discuss conditions on $A \subseteq B$ under which the map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ is an isomorphism. We show that for an integral, birational one dimensional domain extension $A \subseteq B$, the map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ is an isomorphism if and only if $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal. We give an example to show that the map $I(A,B) \to I(A[X,X^{-1}],B[X,X^{-1}])$ need not be
an isomorphism for a 2 dimensional extension even if $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal.

In Section 3, we discuss the surjectivity of the natural map $\varphi(A, C, B) : \mathcal{I}(A, B) \to \mathcal{I}(C, B)$ is given by $\varphi(A, C, B)(I) = IC$ for any ring extensions $A \subseteq C \subseteq B$. We show that the map $\varphi(A, C, B)$ is surjective if $C$ is subintegral over $A$. We show further that if $C$ subintegral over $A$, then the sequence

$$1 \to M\mathcal{I}(A, C) \to M\mathcal{I}(A, B) \to M\mathcal{I}(C, B) \to 1$$

is exact. We conclude this section by discussing some properties of the group $M\mathcal{I}(A, B)$.

1. Basic definitions and Notations

All of the rings we consider are commutative with 1, and all ring homomorphisms are unitary. Let $X$, $T$ be indeterminates.

An **elementary subintegral** extension is an extension of the form $A \subseteq B$ with $B = A[b]$ for some $b \in B$ such that $b^2, b^3 \in A$. An extension $A \subseteq B$ is **subintegral** if it is a filtered union of elementary subintegral extensions; that is, for each $b \in B$ there is a finite sequence $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$ of ring extensions such that $b \in C_r$ and $C_{i-1} \subseteq C_i$ is elementary subintegral for each $i$, $1 \leq i \leq r$. We say that $A$ is **subintegrally closed** in $B$ if whenever $b \in B$ and $b^2, b^3 \in B$ then $b \in A$. The ring $A$ is **seminormal** if the following condition holds: $b, c \in A$ and $b^3 = c^2$ imply that there exists $a \in A$ with $b = a^2$ and $c = a^3$. A seminormal ring is necessarily reduced and is subintegrally closed in every reduced overring. It is easily seen that if $A$ is subintegrally closed in $B$ with $B$ seminormal then $A$ is seminormal. For details see [6, 7].

For a ring $A$ we denote by:

- $U(A)$: The groups of units of $A$.
- $H^0(A) = H^0(\text{Spec}A, \mathbb{Z})$: The group of continuous maps from $\text{Spec}(A)$ to $\mathbb{Z}$.
- $\text{Pic}A$: The Picard group of $A$.
- $KU(A)$: Cokernel of the natural map $U(A) \to U(A[X])$.
- $MU(A)$: Cokernel of the natural map $U(A) \to U(A[X, X^{-1}])$.
- $NU(A)$: Kernel of the map $U(A[X]) \to U(A)$.
- $K\text{Pic}A$: Cokernel of the natural map $\text{Pic}A \to \text{Pic}A[X]$.
- $M\text{Pic}A$: Cokernel of the natural map $\text{Pic}A \to \text{Pic}A[X, X^{-1}]$.
- $N\text{Pic}A$: Kernel of the map $\text{Pic}A[X] \to \text{Pic}A$.
- $L\text{Pic}A$: Cokernel of the map $\text{Pic}A[X] \times \text{Pic}A[X^{-1}] \to \text{Pic}A[X, X^{-1}]$.

Let $A \subseteq B$ be a ring extension. Then we denote by

- $\mathcal{I}(A, B)$: The group of invertible $A$-submodules of $B$. 
It is easily seen that \( \mathcal{I} \) is a functor from extensions of rings to abelian groups. Some properties of \( \mathcal{I}(A, B) \) can be found in [3, Section 2].

\( K\mathcal{I}(A, B) \): Cokernel of the natural map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X]) \).

\( M\mathcal{I}(A, B) \): Cokernel of the natural map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \).

\( N\mathcal{I}(A, B) \): Kernel of the map \( \mathcal{I}(A[X], B[X]) \to \mathcal{I}(A, B) \) (Here the map is induced by the B-algebra homomorphism \( B[X] \to B \) given by \( X \mapsto 0 \)).

Recall from [3, Section 2] that for any commutative ring extension \( A \subseteq B \), we have the exact sequence

\[
1 \to U(A) \to U(B) \to \mathcal{I}(A, B) \to \text{Pic } A \to \text{Pic } B.
\]

Applying \( M, K \) we obtain the chain complexes:

\[
(1.0) \quad 1 \to MU(A) \to MU(B) \to M\mathcal{I}(A, B) \to MPic A \to MPic B.
\]

and

\[
(1.1) \quad 1 \to KU(A) \to KU(B) \to K\mathcal{I}(A, B) \to K\text{Pic } A \to K\text{Pic } B.
\]

2. **The map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \)**

In this section we examine some conditions on \( A \subseteq B \) under which the natural map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \) is an isomorphism (i.e \( M\mathcal{I}(A, B) = 0 \)). For this we consider the notions of quasinormal and anodal extensions.

Let \( A \subseteq B \) be a ring extension. We say that \( A \) is **quasinormal** in \( B \) if the natural map \( MPic A \to MPic B \) is injective. For properties see [2].

The following result is due to Sadhu and Singh [4] which we use frequently throughout this paper:

**Lemma 2.1.** Let \( A \subseteq B \) be a ring extension. Then \( A \) is subintegrally closed in \( B \) if and only if the canonical map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X]) \) is an isomorphism.

**Proof.** See Theorem 1.5 of [4].

One can restate the above result in the following way: \( A \) is subintegrally closed in \( B \) if and only if \( K\mathcal{I}(A, B) = 0 \) if and only if \( N\mathcal{I}(A, B) = 0 \).

The following result is due to Weibel [8]

**Lemma 2.2.** There is a natural decomposition

\[
\text{Pic } A[X, X^{-1}] \cong \text{Pic } A \oplus \text{NPic } A \oplus \text{NPic } A \oplus \text{LPic } A
\]

for any commutative ring \( A \).

**Proof.** See Theorem 5.2 of [8].
Remark 2.3. By Swan Theorem \[6\], $\text{NPic} A = 0$ if and only if $A_{\text{red}}$ is seminormal. So for a seminormal ring $A$, $L\text{Pic} A \cong \text{MPic} A$.

Lemma 2.4. There is a natural decomposition

$$U(A[X, X^{-1}]) \cong U(A) \oplus N(U(A)) \oplus N(U(A)) \oplus H^0(A)$$

for any commutative ring $A$.

Proof. See Exercise 3.17 of [9] in page 30. \qed

Remark 2.5. So for a reduced ring $A$, $H^0(A) \cong \text{MU}(A)$.

Lemma 2.6. The natural map $\phi : I(A, B) \to I(A[X, X^{-1}], B[X, X^{-1}])$, given by $I \to I[A[X, X^{-1}]]$ is injective.

Proof. Let $I = (b_1, b_2, ..., b_r) A \in \text{Ker} \phi$, where $b_i \in B$. Then $IA[X, X^{-1}] = A[X, X^{-1}]$. This implies that $b_i \in A[X, X^{-1}] \cap B = A$, for all $i$. So $I \subseteq A$. Similarly $I^{-1} \subseteq A$. Hence $I = A$. \qed

Lemma 2.7. The sequence (1.0) [resp. (1.1)] is exact, except possibly at the place $M\text{Pic} A$ [resp. $K\text{Pic} A$]. It is exact there too if the map $\text{Pic} A \to \text{Pic} B$ is surjective.

Proof. We have the following commutative diagram

where first two rows are exact and each column is exact. Now result follows by chasing this diagram. \qed

Lemma 2.8. Let $A \subseteq B$ be a ring extension. The map $\text{Pic} A \to \text{Pic} B$ is surjective if any one of the following conditions holds

(1) $A \subseteq B$ is subintegral.
(2) $A \subseteq B$ is a birational integral extension of domains, with $\text{dim} A = 1$. 

Proof. (1) See Proposition 7 of [1].
(2) Let $K$ be the quotient field of $A$ and $B$. We have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{I}(A, K) & \longrightarrow & \text{Pic } A \\
\theta(A, B, K) & \downarrow \varphi & \downarrow \\
\mathcal{I}(B, K) & \longrightarrow & \text{Pic } B
\end{array}
$$

where $\theta(A, B, K)$ is surjective by Proposition 2.3 of [4]. Hence $\varphi$ is surjective.

\[\square\]

Lemma 2.9. Let $A \subseteq B$ be a ring extension with $B$ a domain. Then
(1) If $A$ is quasinormal in $B$ then $M\mathcal{I}(A, B) = 0$ .
(2) Suppose the extension $A \subseteq B$ is integral and birational with $\dim A \leq 1$ and $M\mathcal{I}(A, B) = 0$. Then $A$ is quasinormal in $B$.

Proof. (1) Since $A$ and $B$ are domains, $MU(A) = MU(B) \cong \mathbb{Z}$. By (1.0), $\text{Im} \eta \subseteq \ker \varphi$. As $A$ is quasinormal in $B$, $\ker \varphi = 0$. This implies that $\text{Im} \eta = 0$. We get $M\mathcal{I}(A, B) = 0$.

(2) By Lemma 2.8(2) and Lemma 2.7, the sequence (1.0) is exact at $M\text{Pic } A$ also. Since $M\mathcal{I}(A, B) = 0$, we get the result.

\[\square\]

Lemma 2.10. (cf. [2], Lemma 1.4.) Let $A \subseteq B$ be a ring extension with $B$ reduced and $A$ quasinormal in $B$. Then $A$ is subintegrally closed in $B$.

Proof. We have not assumed $B$ to be a domain. By Lemma 2.1, it is enough to show that $K\mathcal{I}(A, B) = 0$. We have the sequence

$$1 \to KU(A) \to KU(B) \to K\mathcal{I}(A, B) \xrightarrow{\alpha} KPic A \xrightarrow{\beta} KPic B.$$

which is exact except possibly at the place $KPic A$. Since $A$ and $B$ are reduced, $KU(A) = 0$ and $KU(B) = 0$. In the proof of Lemma 1.4 [2], it is shown that the map $KPic A \to KPic B$ is injective i.e $\ker \beta = 0$. We have $\text{im} \alpha \subseteq \ker \beta$. Hence $K\mathcal{I}(A, B) = 0$.

\[\square\]

An inclusion $A \subseteq B$ of rings is called anodal or an anodal extension, if every $b \in B$ such that $(b^2 - b) \in A$ and $(b^3 - b^2) \in A$ belongs to $A$.

Lemma 2.11. Let $A \subseteq C \subseteq B$ be extensions of rings. Then
(1) If $A$ is anodal in $B$, then so is $A$ in $C$. 

(2) If $A$ is anodal in $C$ and $C$ is anodal in $B$, then so is $A$ in $B$.

Proof. Clear from the definition. \hfill \Box

**Proposition 2.12.** Let $A \subseteq B$ be a ring extension. If $A \subseteq B$ is subintegral, then it is anodal.

Proof. Assume first that $A \subseteq B$ is an elementary subintegral extension i.e $A \subseteq B = A[b]$ such that $b^2, b^3 \in A$. Let $f \in B$ such that $f^2 - f, f^3 - f^2 \in A$. We have to show that $f \in A$. Clearly $f$ is of the form $a + \lambda b$ where $a, \lambda \in A$. So it is enough to show that $\lambda b \in A$. Since $\lambda b(2a - 1), \lambda b(3a^2 - 1) \in A$, $\lambda b = \lambda b.1 = \lambda b[(6a + 3)(2a - 1) - 4(3a^2 - 1)] \in A$. Hence $f \in A$.

In the general case, for $f \in B$ there exists a finite sequence $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$ of extensions such that $C_i \subseteq C_{i+1}$ is an elementary subintegral extension for each $i, 0 \leq i \leq r - 1$ and $f \in C_r$. So by the above argument $C_i \subseteq C_{i+1}$ is anodal for each $i$. Now the result follows from Lemma 2.11(2). \hfill \Box

**Lemma 2.13.** (1) The diagram

\[
\begin{array}{ccc}
\mathcal{I}(A, B) & \xrightarrow{\theta_1} & \mathcal{I}(A[X], B[X]) \\
\downarrow \theta & & \downarrow \theta_2 \\
\mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & & 
\end{array}
\]

is commutative.

(2) the maps $\theta, \theta_1$ and $\theta_2$ are injective.

(3) $\theta$ is an isomorphism if and only if $\theta_1$ and $\theta_2$ are isomorphisms.

(4) If $\theta$ is an isomorphism i.e $M\mathcal{I}(A, B) = 0$ then $A$ is subintegally closed in $B$.

Proof. (1) Since the maps are natural, the diagram is commutative.

(2) $\theta$ is injective by Lemma 2.6. The injectivity of $\theta_1$ and $\theta_2$ follows by similar argument as Lemma 2.6.

(3) If $\theta_1$ and $\theta_2$ are isomorphisms then clearly $\theta$ is an isomorphism. Conversely, suppose $\theta$ is an isomorphism. Then by simple diagram chasing we get that $\theta_1$ and $\theta_2$ are isomorphisms.

(4) If $\theta$ is an isomorphism then $\theta_1$ is an isomorphism. Hence by Lemma 2.1, $A$ is subintegally closed in $B$. \hfill \Box

**Lemma 2.14.** Let $\mathfrak{a}$ be a $B$-ideal contained in $A$. Then the homomorphism $M\mathcal{I}(A, B) \rightarrow M\mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$ is an isomorphism.
Let Corollary 2.16.

2.9(1).

domains. Then the following are equivalent:
by applying $M$ to the unit-Pic sequence (see Theorem 3.10 , [9]) ,

(1) By Lemma 2.13(4),

Proof. [2], it is enough to show that for every intermediate ring

$A \subseteq C$ is a finite

$A$ module, the map $\text{MPic } A \to \text{MPic } A/\mathfrak{c} \times \text{MPic } C$ is injective, where $\mathfrak{c}$ is the conductor of $C$ in $A$. We first claim that the map $\phi : M\mathcal{I}(A, C) \to M\mathcal{I}(A, B)$ is injective, where $C$ is any intermediate ring between $A$ and $B$.

We have the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, C) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \phi & & \\
1 & \longrightarrow & \mathcal{I}(A, B) & \xrightarrow{\beta} & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, B) & \longrightarrow & 1 \\
\end{array}
$$

where the first two vertical arrows are natural inclusions (because any invertible $A$-submodule of $C$ is also an invertible $A$-submodule of $B$).

Let $\bar{J} \in \ker \phi$, where $J \in \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}])$. Then $J \in \text{im} \beta$ and there exist $J_1 \in \mathcal{I}(A, B)$ such that $J_1 A[X, X^{-1}] = J$. Let $J_1 = (b_1, b_2, ..., b_r)A$ and $J = (f_1, f_2, ..., f_s)A[X, X^{-1}]$ where $b_i \in B$ and $f_i \in C[X, X^{-1}]$. Then clearly $b_i \in B \cap C[X, X^{-1}] = C$ for all $i$. So $J_1 \subseteq C$. Also $J_1^{-1} \subseteq C$. This implies that $J_1 \in \mathcal{I}(A, C)$. So $\bar{J} = 0$. This proves the claim.

Since $M\mathcal{I}(A, B) = 0$, $M\mathcal{I}(A, C) = 0$. By Lemma 2.14, $M\mathcal{I}(A/\mathfrak{c}, C/\mathfrak{c}) = 0$, where $\mathfrak{c}$ is the conductor of $C$ in $A$. By (1.0), we have $MU(A) \cong MU(C)$ and $MU(A/\mathfrak{c}) \cong MU(C/\mathfrak{c})$. Now the result follows from the following exact sequence which we obtain by applying $M$ to the unit-Pic sequence (see Theorem 3.10 , [9]) ,

$$MU(A) \to MU(A/\mathfrak{c}) \times MU(C) \to MU(C/\mathfrak{c}) \to \text{MPic } A \to \text{MPic } A/\mathfrak{c} \times \text{MPic } C$$

(2) By Theorem 1.13 of [2], $A$ is quasinormal in $B$. Now the result follows from Lemma 2.15(1).

Corollary 2.16. Let $A \subseteq B$ be an integral, birational extension of one dimensional domains. Then the following are equivalent:

(1) $A$ is quasinormal in $B$. 

□
(2) $A \subseteq B$ anodal and $A$ is subintegrally closed in $B$.

(3) $MI(A, B) = 0$.

**Proof.** (1) $\iff$ (3) This is Lemma 2.9.
(2) $\Rightarrow$ (3) This is Theorem 2.15(2).
(3) $\Rightarrow$ (2) This is follows from Lemma 2.13(4) and Theorem 2.15(1). □

The statement of Theorem 2.15(2) need not be true for dimension greater than 1. This is seen by considering Example 3.5 of [8]. In that example $A$ is a 2-dimensional noetherian domain whose integral closure is $B = K[X, Y]$, where $K$ is a field. So $A \subseteq B$ is an integral, birational extension. By Proposition 3.5.2 of [8], $A \subseteq B$ is anodal and $A$ is subintegrally closed in $B$. Since $B$ is a UFD, $\text{Pic } B = \text{Pic } B[T, T^{-1}] = 0$ and we have the exact sequence

$$1 \to MU(A) \to MU(B) \to MI(A, B) \to MPic A \to 0.$$ 

As $A, B$ are domains, $MU(A) = MU(B) \cong \mathbb{Z}$. So $MI(A, B) \cong MPic A$. By Remark 2.3 $LPic A \cong MPic A$. Hence by Proposition 3.5.2 of [8], $MI(A, B) \neq 0$.

3. **Some observations on** $MI(A, B)$

Recall from [5, Section 3] that for any extensions $A \subseteq C \subseteq B$ of rings, we have the exact sequence

$$1 \to \mathcal{I}(A, C) \to \mathcal{I}(A, B) \xrightarrow{\varphi(A,C,B)} \mathcal{I}(C, B)$$

where the map $\varphi(A, C, B)$ is given by $\varphi(A, C, B)(I) = IC$.

Now it is natural to ask under what conditions on $A \subseteq B$ the map $\varphi(A, C, B)$ is surjective. In [3], Singh has proved that if $B$ is subintegral over $A$ then the map $\varphi(A, C, B)$ is surjective. In the next Proposition we generalize Singh’s result as follows:

**Proposition 3.1.** For all rings $C$ between $A$ and $B$ such that $C$ is subintegral over $A$, the map $\varphi(A, C, B)$ is surjective.

**Proof.** We have the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \to & U(A) & \to & U(B) & \to & \mathcal{I}(A, B) & \to & \text{Pic } A & \to & \text{Pic } B \\
& & \downarrow & & \downarrow \cong & & \downarrow \varphi(A,C,B) & & \downarrow \theta & & \downarrow \cong \\
1 & \to & U(C) & \to & U(B) & \to & \mathcal{I}(C, B) & \to & \text{Pic } C & \to & \text{Pic } B
\end{array}
$$

Since $\theta$ is surjective by Lemma 2.8(1), the result follows by chasing the diagram. □
Recall that a local ring $A$ is **hensel** if every finite $A$- algebra $B$ is a direct product of local rings.

The following result gives another case where the map $\varphi(A, C, B)$ is surjective.

**Proposition 3.2.** Let $A \subseteq B$ be an integral extension with $A$ hensel local. Then for all rings $C$ with $A \subseteq C \subseteq B$ the map $\varphi(A, C, B)$ is surjective.

**Proof.** By Lemma 2.2 of [4], it is enough to show that $\varphi(A, D, B)$ is surjective for every subring $D$ of $C$ containing $A$ such that $D$ is finitely generated as an $A$-algebra. Let such a ring $D$ be given. Since $D$ is integral over $A$, $D$ is a finite $A$-algebra. As $A$ is hensel, $D$ is a finite direct product of local rings. Then Pic $A$ and Pic $D$ are both trivial. This implies that $I(A, B) = U(B)/U(A)$, $I(D, B) = U(B)/U(D)$ and clearly $\varphi(A, D, B)$ is surjective. □

**Proposition 3.3.** Let $A \subseteq C \subseteq B$ be extensions of rings with $A \subseteq C$ subintegral. Then the sequence

$$1 \rightarrow M(I(A, C)) \rightarrow M(I(A, B)) \rightarrow M(I(C, B)) \rightarrow 1$$

is exact.

**Proof.** Consider the commutative diagram

where the rows are clearly exact. Since $A \subseteq C$ is subintegral, so is $A[X, X^{-1}] \subseteq C[X, X^{-1}]$. Therefore by Proposition 3.1, the first two columns are exact. Hence exactness of the last column follows by chasing the diagram. □

**Corollary 3.4.** Let $A \subseteq B$ be a ring extension and let $^+A$ denote the subintegral closure of $A$ in $B$. Then the sequence

$$1 \rightarrow M(I(A, ^+A)) \rightarrow M(I(A, B)) \rightarrow M(I(^+A, B)) \rightarrow 1$$

is exact.
Proof. We have \(A \subseteq^+ A \subseteq B\) where \(A \subseteq^+ A\) is subintegral and \(^+ A\) is subintegrally closed in \(B\). By Proposition 3.1, we have the exact sequence

\[
1 \rightarrow \mathcal{I}(A, ^+ A) \rightarrow \mathcal{I}(A, B) \xrightarrow{\phi(A, ^+ A, B)} \mathcal{I}(A, B) \rightarrow 1
\]

Applying \(M\) we also get the following exact sequence by Proposition 3.3,

\[
1 \rightarrow M\mathcal{I}(A, ^+ A) \rightarrow M\mathcal{I}(A, B) \rightarrow M\mathcal{I}(A, B) \rightarrow 1
\]

Hence the proof. \(\Box\)

Proposition 3.5. Let \(A \subseteq B\) be a ring extension. Assume that \(A\) is subintegrally closed in \(B\). Then

1. \(M\mathcal{I}(A, B) \cong M\mathcal{I}(A[T], B[T])\).
2. \(M\mathcal{I}(A, B)\) is a torsion free abelian group if \(B\) is a seminormal ring.
3. \(M\mathcal{I}(A, B)\) is a free abelian group if \(B\) is a seminormal ring and \(A\) is hensel local.
4. \(M\mathcal{I}(A, B) = 0\) if \(B\) is a seminormal domain and \(A\) is hensel local.

Proof. (1) Since \(A\) is subintegrally closed in \(B\), \(A[X]\) is subintegrally closed in \(B[X]\) by Corollary 1.6 of [4]. Therefore \(A[X, X^{-1}]\) is subintegrally closed in \(B[X, X^{-1}]\). We have the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mathcal{I}(A, B) & \rightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \rightarrow & M\mathcal{I}(A, B) & \rightarrow & 1 \\
& & \downarrow \beta & & \downarrow \theta & & & & \\
1 & \rightarrow & \mathcal{I}(A[T], B[T]) & \rightarrow & \mathcal{I}(A[T][X, X^{-1}], B[T][X, X^{-1}]) & \rightarrow & M\mathcal{I}(A[T], B[T]) & \rightarrow & 1
\end{array}
\]

where \(\beta\) and \(\theta\) are isomorphisms by Lemma 2.1. Hence we get the result.

(2) As \(A\) is subintegrally closed in \(B\) and \(B\) is a seminormal ring, \(A\) is seminormal. Then by Remark 2.3, \(LPic A \cong MPic A\). Since seminormal ring is reduced, \(MU(A) = H^0(A)\) and \(MU(B) = H^0(B)\) by Remark 2.5. Now, we have the exact sequence

\[
1 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A
\]

where \(H^0(A)\) and \(H^0(B)\) are always free abelian groups by Construction 1.2.1 of [8] and by Corollary 2.3.1 of [8], \(MPic A\) is a torsion free abelian group. Let \(T = Coker[H^0(A) \rightarrow H^0(B)]\). Then

\[
1 \rightarrow T \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A
\]

is exact and \(T\) is a free abelian group by Proposition 1.3 of [8]. Therefore \(M\mathcal{I}(A, B)\) is a torsion free abelian group.
By Theorem 2.5 of \[8\], \(\text{LPic} \, A = 0\). Since \(A\) is seminormal, \(\text{MPic} \, A = 0\). Then we have the exact sequence
\[
1 \to H^0(A) \to H^0(B) \to \mathcal{M}(A, B) \to 1
\]
and \(\mathcal{M}(A, B) = \text{Coker}[H^0(A) \to H^0(B)]\) is a free abelian group by Proposition 1.3 of \[8\].

Since \(B\) is a domain, \(H^0(A) = H^0(B) \cong \mathbb{Z}\). So \(\mathcal{M}(A, B) = 0\).

\[\square\]

**Lemma 3.6.** Let \(A \subseteq B\) be a subintegral extension. Then the map \(\text{LPic} \, A \to \text{LPic} \, B\) is surjective.

**Proof.** Since \(A \subseteq B\) is subintegral, so are \(A[X] \subseteq B[X]\) and \(A[X, X^{-1}] \subseteq B[X, X^{-1}]\). Then the maps \(\text{Pic} \, A[X] \times \text{Pic} \, A[X^{-1}] \to \text{Pic} \, B[X] \times \text{Pic} \, B[X^{-1}]\) and \(\text{Pic} \, A[X, X^{-1}] \to \text{Pic} \, B[X, X^{-1}]\) are surjective by Lemma 2.8(1). Hence we get the result by chasing the following commutative diagram

\[
\begin{array}{ccc}
\text{Pic} \, A[X] \times \text{Pic} \, A[X^{-1}] & \longrightarrow & \text{Pic} \, A[X, X^{-1}] \\
\uparrow & & \uparrow \\
\text{Pic} \, B[X] \times \text{Pic} \, B[X^{-1}] & \longrightarrow & \text{Pic} \, B[X, X^{-1}] \\
\uparrow & & \uparrow \\
1 & & 1 \\
\end{array}
\]


\[\square\]

**Theorem 3.7.** Let \(A \subseteq B\) be a ring extension with \(A\) hensel local and \(B\) seminormal. Then \(\mathcal{M}(A, B) \cong \mathcal{M}(A, +A) \oplus \mathcal{M}(\bar{A}, B)\) where \(\bar{A}\) is the subintegral closure of \(A\) in \(B\).

**Proof.** By Lemma 3.6 \(\text{LPic} \, A \to \text{LPic} \, \bar{A}\) is surjective. Since \(A\) is hensel local, \(\text{LPic} \, A = 0\) by Theorem 2.5 of \[8\]. Therefore \(\text{LPic} \, \bar{A} = 0\) and \(\text{MPic} \, \bar{A} = 0\) because \(\bar{A}\) is seminormal. Then by same argument as Proposition 3.5(3), \(\mathcal{M}(\bar{A}, B)\) is a free abelian group. Now the result follows from the exact sequence
\[
1 \to \mathcal{M}(A, +A) \to \mathcal{M}(A, B) \to \mathcal{M}(\bar{A}, B) \to 1
\]

\[\square\]

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SUBINTEGRALITY, INVERTIBLE MODULES AND LAURENT POLYNOMIAL EXTENSIONS

VIVEK SADHU

Abstract. Let $A \subseteq B$ be a commutative ring extension. Let $I(A, B)$ be the multiplicative group of invertible $A$-submodules of $B$. In this article, we extend a result of Sadhu and Singh by finding a necessary and sufficient condition on an integral birational extension $A \subseteq B$ of integral domains with $\dim A \leq 1$, so that the natural map $I(A, B) \to I(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. In the same situation, we show that if $\dim A \geq 2$ then the condition is necessary but not sufficient. We also discuss some properties of the cokernel of the natural map $I(A, B) \to I(A[X, X^{-1}], B[X, X^{-1}])$ in the general case.

Keywords: Subintegral extensions, Seminormal rings, Invertible modules

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Introduction

In [4], Roberts and Singh have introduced the group $I(A, B)$ to generalize a result of Dayton. The relation between the group $I(A, B)$ and subintegral extensions has been investigated by Reid, Roberts and Singh in a series of papers. Recently in [5], Sadhu and Singh have proved that $A$ is subintegrally closed in $B$ if and only if the canonical map $I(A, B) \to I(A[X], B[X])$ is an isomorphism. It is easy to see that the map is injective and that $I(A[X], B[X]) = I(A, B) \oplus NI(A, B)$, where $NI(A, B)$ denotes the kernel of the map $I(A[X], B[X]) \xrightarrow{X\rightarrow0} I(A, B)$. So the result of [5], just mentioned, amounts to saying that $NI(A, B) = 0$ if and only if $A$ is subintegrally closed in $B$.

The primary goal of this paper is to extend the result of Sadhu and Singh in [5] just mentioned above by finding a necessary and sufficient condition on $A \subseteq B$, so that the natural map $I(A, B) \to I(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. It is easy to see that the map $I(A, B) \to I(A[X, X^{-1}], B[X, X^{-1}])$ is always injective (see Lemma 2.3). Thus the problem reduces to the investigation of conditions for the cokernel of the above map to be zero. This cokernel will be denoted by $MI(A, B)$. The secondary goal will be to investigate properties of the cokernel $MI(A, B)$ in the general case.

In Section 1, we mainly give basic definitions and notations.

In Section 2, we discuss conditions on $A \subseteq B$ under which the map $I(A, B) \to I(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. We are able to prove some results in the
situation when $A \subseteq B$ is an integral birational extension of domains. First, if $\dim A \leq 1$ then by using a result of Onoda-Yoshida ([3], Theorem 1.13), we prove the following

**Theorem 2.14.** Let $A \subseteq B$ be an integral, birational extension of domains with $\dim A \leq 1$. Then $MI(A, B) = 0$ if and only if $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal.

For higher dimension, we show that the above conditions are necessary but not sufficient. More precisely, we prove the following

**Theorem 2.17.** Let $A \subseteq B$ be an integral, birational extension of domains. Suppose $MI(A, B) = 0$. Then $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal.

That the conditions are not sufficient is shown by an example of C. Weibel (see Remark 2.18). We note that for any ring extension $A \subseteq B$, the condition $MI(A, B) = 0$ implies easily that $A$ is subintegrally closed in $B$ (see Lemma 2.15(4)).

In Section 3, we examine the cokernel $MI(A, B)$ in the general case. In order to do this, we first discuss the surjectivity of the natural map $\varphi(A, C, B) : I(A, B) \rightarrow I(C, B)$ given by $\varphi(A, C, B)(I) = IC$ for any ring extensions $A \subseteq C \subseteq B$. We show that the map $\varphi(A, C, B)$ is surjective in two cases: (1) $C$ is subintegral over $A$, (2) $A \subseteq B$ is an integral extension with $A$ Hensel local (see Propositions 3.1 and 3.2). We show further that if $C$ is subintegral over $A$, then the sequence

$$1 \rightarrow MI(A, C) \rightarrow MI(A, B) \rightarrow MI(C, B) \rightarrow 1$$

is exact (see Proposition 3.3). Finally we prove the following

**Theorem 3.7.** Let $A \subseteq B$ be a ring extension with $A$ Hensel local and $B$ seminormal. Then $MI(A, B) \cong MI(A, 'A) \oplus MI('A, B)$, where $'A$ is the subintegral closure of $A$ in $B$.

In this section we also observe that if $A$ is subintegrally closed in $B$ with $B$ a seminormal domain and $A$ Hensel local then $MI(A, B) = 0$ (see Proposition 3.5(4)).

1. **Basic definitions and Notations**

All the rings we consider are commutative with 1, and all ring homomorphisms are unitary. Let $X, T$ be indeterminates.

An elementary subintegral extension is an extension of the form $A \subseteq B$ with $B = A[b]$ for some $b \in B$ such that $b^2, b^3 \in A$. An extension $A \subseteq B$ is subintegral if it is a filtered union of elementary subintegral extensions; that is, for each $b \in B$ there is a finite sequence $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$ of ring extensions such that $b \in C_r$ and $C_{i-1} \subseteq C_i$ is elementary subintegral for each $i, 1 \leq i \leq r$. We say that $A$ is subintegarly closed in $B$ if whenever $b \in B$ and $b^2, b^3 \in B$ then $b \in A$. The ring $A$
is **seminormal** if the following condition holds: \( b, c \in A \) and \( b^3 = c^2 \) imply that there exists \( a \in A \) with \( b = a^2 \) and \( c = a^3 \). A seminormal ring is necessarily reduced and is subintegribly closed in every reduced overring. It is easily seen that if \( A \) is subintegribly closed in \( B \) with \( B \) seminormal then \( A \) is seminormal. For details see [7, 8].

For a ring \( A \) we denote by:

- \( U(A) \): The groups of units of \( A \).
- \( H^0(A) = H^0(\text{Spec}(A), \mathbb{Z}) \): The group of continuous maps from \( \text{Spec}(A) \) to \( \mathbb{Z} \).
- \( \text{Pic} A \): The Picard group of \( A \).
- \( KU(A) \): Cokernel of the natural map \( U(A) \to U(A[X]) \).
- \( MU(A) \): Cokernel of the natural map \( U(A) \to U(A[X, X^{-1}]) \).
- \( NU(A) \): Kernel of the map \( U(A[X]) \to U(A) \).
- \( K\text{Pic} A \): Cokernel of the natural map \( \text{Pic} A \to \text{Pic} A[X] \).
- \( M\text{Pic} A \): Cokernel of the natural map \( \text{Pic} A \to \text{Pic} A[X, X^{-1}] \).
- \( N\text{Pic} A \): Kernel of the map \( \text{Pic} A[X] \to \text{Pic} A \) (Here the map is induced by the \( B \)-algebra homomorphism \( B[X] \to B \) given by \( X \mapsto 0 \)).
- \( L\text{Pic} A \): Cokernel of the map \( \text{Pic} A[X] \times \text{Pic} A[X^{-1}] \to \text{Pic} A[X, X^{-1}] \).

Let \( A \subseteq B \) be a ring extension. Then we denote by

- \( \mathcal{I}(A, B) \): The group of invertible \( A \)-submodules of \( B \).

It is easily seen that \( \mathcal{I} \) is a functor from extensions of rings to abelian groups. Some properties of \( \mathcal{I}(A, B) \) can be found in [4, Section 2].

- \( K\mathcal{I}(A, B) \): Cokernel of the natural map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X]) \).
- \( M\mathcal{I}(A, B) \): Cokernel of the natural map \( \mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \).
- \( N\mathcal{I}(A, B) \): Kernel of the map \( \mathcal{I}(A[X], B[X]) \to \mathcal{I}(A, B) \) (Here the map is induced by the \( B \)-algebra homomorphism \( B[X] \to B \) given by \( X \mapsto 0 \)).

Recall from [4, Section 2] that for any commutative ring extension \( A \subseteq B \), we have the exact sequence

\[
1 \to U(A) \to U(B) \to \mathcal{I}(A, B) \to \text{Pic} A \to \text{Pic} B.
\]

Applying \( M, K \) we obtain the chain complexes:

\[(1.0)\quad 1 \to MU(A) \to MU(B) \to M\mathcal{I}(A, B) \xrightarrow{\eta} M\text{Pic} A \xrightarrow{\bar{\beta}} M\text{Pic} B\]

and

\[(1.1)\quad 1 \to KU(A) \to KU(B) \to K\mathcal{I}(A, B) \xrightarrow{\alpha} K\text{Pic} A \xrightarrow{\beta} K\text{Pic} B.\]
2. The map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$

In this section we examine some conditions on $A \subseteq B$ under which the natural map $\mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ is an isomorphism. For this we consider the notions of quasinormal and anodal extensions (or $u$-closed).

Let $A \subseteq B$ be a ring extension. We say that $A$ is quasinormal in $B$ if the natural map $M_{\text{Pic}}(A) \to M_{\text{Pic}}(B)$ is injective. For properties of quasinormal extensions see [3].

An inclusion $A \subseteq B$ of rings is called anodal or an anodal extension, if every $b \in B$ such that $(b^2 - b) \in A$ and $(b^3 - b^2) \in A$ belongs to $A$. This notion was first introduced by Asanuma and Onoda-Yoshida in [3], and they called this notion ‘$u$-closed’. Some related details can be found in [1, 3, 9].

We first show in Proposition 2.2 below that a subintegral extension is always an anodal extension, which is perhaps a result of independent interest.

Lemma 2.1. Let $A \subseteq C \subseteq B$ be extensions of rings. Then the following statements hold:

1. If $A$ is anodal in $B$, then so is $A$ in $C$.
2. If $A$ is anodal in $C$ and $C$ is anodal in $B$, then so is $A$ in $B$.

Proof. Clear from the definition. □

Proposition 2.2. Let $A \subseteq B$ be a ring extension. If $A \subseteq B$ is subintegral, then it is anodal.

Proof. Assume first that $A \subseteq B$ is an elementary subintegral extension, i.e., $A \subseteq B = A[b]$ such that $b^2, b^3 \in A$. Let $f \in B$ such that $f^2 - f, f^3 - f^2 \in A$. We have to show that $f \in A$. Clearly $f$ is of the form $a + \lambda b$ where $a, \lambda \in A$. So it is enough to show that $\lambda b \in A$. Since $\lambda b(2a - 1), \lambda b(3a^2 - 1) \in A$, $\lambda b = \lambda b.1 = \lambda b[(6a + 3)(2a - 1) - 4(3a^2 - 1)] \in A$. Hence $f \in A$.

In the general case, for $f \in B$ there exists a finite sequence $A = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_r \subseteq B$ of extensions such that $C_i \subseteq C_{i+1}$ is an elementary subintegral extension for each $i, 0 \leq i \leq r - 1$ and $f \in C_r$. So by the above argument $C_i \subseteq C_{i+1}$ is anodal for each $i$. Now the result follows from Lemma 2.1(2). □

The following result is due to Sadhu and Singh ([5], Theorem 1.5) which we use frequently throughout this paper:

Lemma 2.3. Let $A \subseteq B$ be a ring extension. Then $A$ is subintegrally closed in $B$ if and only if the canonical map $\mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X])$ is an isomorphism. □

One can restate the above result in the following way: $A$ is subintegrally closed in $B$ $\iff K\mathcal{I}(A, B) = 0 \iff N\mathcal{I}(A, B) = 0$. 
The following result is due to Weibel ([9], Theorem 5.2).

**Lemma 2.4.** There is a natural decomposition
\[
\text{Pic } A[X, X^{-1}] \cong \text{Pic } A \oplus \text{NPic } A \oplus \text{NPic } A \oplus \text{LPic } A
\]
for any commutative ring \(A\).

**Remark 2.5.** By Swan Theorem [7], \(\text{NPic } A = 0\) if and only if \(A_{red}\) is seminormal. So for a seminormal ring \(A\), \(\text{LPic } A \cong \text{MPic } A\).

The next result is given in ([10], Exercise 3.17, Page 30).

**Lemma 2.6.** There is a natural decomposition
\[
U(A[X, X^{-1}]) \cong U(A) \oplus Nu(A) \oplus Nu(A) \oplus H^0(A)
\]
for any commutative ring \(A\).

**Remark 2.7.** It follows that for a reduced ring \(A\), \(H^0(A) \cong MU(A)\).

**Lemma 2.8.** The natural map \(\phi : \mathcal{I}(A, B) \to \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])\), given by \(I \to IA[X, X^{-1}]\), is injective. Thus, \(\phi\) is an isomorphism if and only if \(MI(A, B) = 0\).

**Proof.** Let \(I = (b_1, b_2, \ldots, b_r)A \in \ker \phi\), where \(b_i \in B\). Then \(IA[X, X^{-1}] = A[X, X^{-1}]\). This implies that \(b_i \in A[X, X^{-1}] \cap B = A\), for all \(i\). So \(I \subseteq A\). Similarly \(I^{-1} \subseteq A\). Hence \(I = A\).

**Lemma 2.9.** The sequence \((1.0)\) [respectively \((1.1)\)] is exact, except possibly at the place \(\text{MPic } A\) [respectively \(K\text{Pic } A]\). It is exact there too if the map \(\text{Pic } A \to \text{Pic } B\) is surjective.

**Proof.** We have the following commutative diagram
Lemma 2.10. Let $A \subseteq B$ be a ring extension. The map $\text{Pic } A \rightarrow \text{Pic } B$ is surjective if any one of the following conditions holds:

1. $A \subseteq B$ is subintegral.
2. $A \subseteq B$ is an integral, birational extension of domains with $\dim A \leq 1$.

Proof. (1) See Proposition 7 of [2].

(2) Let $K$ be the quotient field of $A$ and $B$. We have the commutative diagram

\[
\begin{array}{cccccc}
\mathcal{I}(A,K) & \longrightarrow & \text{Pic } A & \longrightarrow & 0 \\
\varphi(A,B,K) \downarrow & & \rho \downarrow & & \\
\mathcal{I}(B,K) & \longrightarrow & \text{Pic } B & \longrightarrow & 0
\end{array}
\]

where $\varphi(A,B,K)$ is surjective by Proposition 2.3 of [3]. Hence $\rho$ is surjective. \qed

Lemma 2.11. (cf. [3], Lemma 1.4.) Let $A \subseteq B$ be a ring extension with $B$ reduced and $A$ quasinormal in $B$. Then $A$ is subintegrally closed in $B$.

Proof. We have not assumed $B$ to be a domain. By Lemma 2.13 it is enough to show that $K\mathcal{I}(A,B) = 0$. We have the sequence

\[1 \rightarrow KU(A) \rightarrow KU(B) \rightarrow K\mathcal{I}(A,B) \xrightarrow{\alpha} K\text{Pic } A \xrightarrow{\beta} K\text{Pic } B\]

which is exact except possibly at the place $K\text{Pic } A$. Since $A$ and $B$ are reduced, $KU(A) = 0$ and $KU(B) = 0$. In the proof of Lemma 1.4 of [3], it is shown that the map $K\text{Pic } A \rightarrow K\text{Pic } B$ is injective, i.e., $\ker \beta = 0$. We have $\text{im } \alpha \subseteq \ker \beta$. Hence $K\mathcal{I}(A,B) = 0$. \qed

Remark 2.12. In the above lemma we cannot drop the condition that $B$ is reduced. For example, consider the extension $A = K \not\subseteq B = K[b]$ with $b^2 = 0$, where $K$ is any field. Since $M\text{Pic } K = 0$, clearly $A$ is quasinormal in $B$. But $A$ is not subintegrally closed in $B$, because $b^2 = b^3 = 0 \in K$, $b \notin K$.

Lemma 2.13. Let $A \subseteq B$ be a ring extension with $B$ a domain. Then the following statements hold:

1. If $A$ is quasinormal in $B$ then $M\mathcal{I}(A,B) = 0$.
2. Suppose the extension $A \subseteq B$ is integral and birational with $\dim A \leq 1$, and $M\mathcal{I}(A,B) = 0$. Then $A$ is quasinormal in $B$. 

where the first two rows are exact and each column is exact. Now the result follows by chasing this diagram. \qed
Proof. (1) Since $A$ and $B$ are domains, $MU(A) = MU(B) \cong \mathbb{Z}$. By (1.0), $\text{im} \eta \subseteq \ker \varphi$. As $A$ is quasinormal in $B$, $\ker \varphi = 0$. This implies that $\text{im} \eta = 0$. We get $\mathcal{MI}(A, B) = 0$.

(2) By Lemma 2.10(2) and Lemma 2.9, the sequence (1.0) is exact at $MPic A$ also. Since $\mathcal{MI}(A, B) = 0$, we get the result.

Theorem 2.14. Let $A \subseteq B$ be an integral, birational extension of domains with $\dim A \leq 1$. Then $\mathcal{MI}(A, B) = 0$ if and only if $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal.

Proof. If $\dim A = 0$ then $A = B$ and the assertion holds trivially in this case. If $\dim A = 1$ then by Theorem 1.13 of [3], $A$ is quasinormal in $B$ if and only if $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal. We also have $A$ is quasinormal in $B$ if and only if $\mathcal{MI}(A, B) = 0$ by Lemma 2.13. Combining these two results we get the assertion.

Next, in Theorem 2.17 and Remark 2.18, we show that in general, the conditions $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal are necessary but not sufficient.

Lemma 2.15. (1) The diagram

$$
\begin{array}{c}
\mathcal{I}(A, B) \\
\downarrow \phi \\
\mathcal{I}(A[X], B[X]) \\
\downarrow \theta \\
\mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])
\end{array}
\xrightarrow{\psi} \mathcal{I}(A[X], B[X])
$$

is commutative.

(2) The maps $\phi$, $\psi$ and $\theta$ are injective.

(3) $\phi$ is an isomorphism if and only if $\psi$ and $\theta$ are isomorphisms.

(4) If $\phi$ is an isomorphism, i.e., $\mathcal{MI}(A, B) = 0$, then $A$ is subintegrally closed in $B$.

Proof. (1) Since the maps are natural, the diagram is commutative.

(2) $\phi$ is injective by Lemma 2.8. The injectivity of $\psi$ and $\theta$ follows by a similar argument as in Lemma 2.8.

(3) If $\psi$ and $\theta$ are isomorphisms then clearly $\phi$ is an isomorphism. Conversely, suppose $\phi$ is an isomorphism. Then by simple diagram chasing we get that $\psi$ and $\theta$ are isomorphisms.

(4) If $\phi$ is an isomorphism then $\psi$ is an isomorphism. Hence by Lemma 2.3, $A$ is subintegrally closed in $B$.

Lemma 2.16. Let $a$ be a $B$-ideal contained in $A$. Then the homomorphism $\mathcal{MI}(A, B) \to \mathcal{MI}(A/a, B/a)$ is an isomorphism.
Proof. Clearly, $a[X, X^{-1}]$ is a $B[X, X^{-1}]$-ideal contained in $A[X, X^{-1}]$. We have $\mathcal{I}(A, B) \cong \mathcal{I}(A/a, B/a)$ and $\mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \cong \mathcal{I}(A/a[X, X^{-1}], B/a[X, X^{-1}])$ by Proposition 2.6 of [4]. Now by chasing a suitable diagram we get the result. \[\square\]

**Theorem 2.17.** Let $A \subseteq B$ be an integral, birational extension of domains. Suppose $\mathcal{M}(A, B) = 0$. Then $A$ is subintegrally closed in $B$ and $A \subseteq B$ is anodal.

**Proof.** By Lemma 2.15(4), $A$ is subintegrally closed in $B$. To prove $A \subseteq B$ is anodal, by Lemma 1.10 of [3], it is enough to show that for every intermediate ring $C$ between $A$ and $B$ such that $C$ is a finite $A$-module, the map $\text{MPic} A \to \text{MPic} (A/c) \times \text{MPic} C$ is injective, where $c$ is the conductor of $C$ in $A$. We first claim that the map $\tau : \mathcal{M}(A, C) \to \mathcal{M}(A, B)$ is injective, where $C$ is any intermediate ring between $A$ and $B$.

We have the commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & \mathcal{M}(A, C) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & \mathcal{M}(A, B) & \longrightarrow & 1
\end{array}
\]

where the first two vertical arrows are natural inclusions (because any invertible $A$-submodule of $C$ is also an invertible $A$-submodule of $B$).

Let $J \in \ker \tau$, where $J \in \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}])$. Then $J \in \text{im} \phi$ and there exists $J_1 \in \mathcal{I}(A, B)$ such that $J_1 A[X, X^{-1}] = J$. Let $J_1 = (b_1, b_2, ..., b_r)A$ and $J = (f_1, f_2, ..., f_s)A[X, X^{-1}]$ where $b_i \in B$ and $f_i \in C[X, X^{-1}]$. Then clearly $b_i \in B \cap C[X, X^{-1}] = C$ for all $i$. So $J_1 \subseteq C$. Also $J_1^{-1} \subseteq C$. This implies that $J_1 \in \mathcal{I}(A, C)$. So $J = 0$. This proves the claim.

Since $\mathcal{M}(A, B) = 0$, $\mathcal{M}(A, C) = 0$. By Lemma 2.16, $\mathcal{M}(A/c, C/c) = 0$, where $c$ is the conductor of $C$ in $A$. By (1.0), we have $\text{MU}(A) \cong \text{MU}(C)$ and $\text{MU}(A/c) \cong \text{MU}(C/c)$. Now the result follows from the following exact sequence which we obtain by applying M to the unit-Pic sequence ([10], Theorem 3.10),

\[\text{MU}(A) \to \text{MU}(A/c) \times \text{MU}(C) \to \text{MU}(C/c) \to \text{MPic} A \to \text{MPic} (A/c) \times \text{MPic} C\]

\[\square\]

**Remark 2.18.** The converse of the above theorem holds for $\dim A \leq 1$ as seen in Theorem 2.14. In general, the converse does not hold. This is seen by considering Example 3.5 of C. Weibel [9]. In that example $A$ is a 2-dimensional noetherian domain whose integral closure is $B = K[X, Y]$, where $K$ is a field. So $A \subseteq B$ is an integral, birational extension. By Proposition 3.5.2 of [9], $A \subseteq B$ is anodal and $A$ is subintegrally
closed in $B$. Since $B$ is a UFD, $\text{Pic } B = \text{Pic } B[T, T^{-1}] = 0$. Then we get the exact sequence

$$1 \to MU(A) \to MU(B) \to MI(A, B) \to M\text{Pic } A \to 0.$$ As $A, B$ are domains, $MU(A) = MU(B) \cong \mathbb{Z}$. So $MI(A, B) \cong M\text{Pic } A$. By Remark 2.5, $L\text{Pic } A \cong M\text{Pic } A$. Hence by Proposition 3.5.2 of [9], $MI(A, B) \neq 0$.

3. Some observations on $MI(A, B)$

In this section we discuss some properties of the cokernel $MI(A, B)$ in the general case.

Recall from [6, Section 3] that for any extensions $A \subseteq C \subseteq B$ of rings, we have the exact sequence

$$1 \to I(A, C) \to I(A, B) \xrightarrow{\varphi(A, C, B)} I(C, B)$$

where the map $\varphi(A, C, B)$ is given by $\varphi(A, C, B)(I) = IC$.

Now it is natural to ask under what conditions on $A \subseteq B$ the map $\varphi(A, C, B)$ is surjective. In [4], Singh has proved that if $B$ is subintegral over $A$ then the map $\varphi(A, C, B)$ is surjective. In the next Proposition we generalize Singh’s result as follows:

**Proposition 3.1.** For all rings $C$ between $A$ and $B$ such that $C$ is subintegral over $A$, the map $\varphi(A, C, B)$ is surjective.

**Proof.** We have the commutative diagram

$$
\begin{array}{cccccc}
1 & \to & U(A) & \to & U(B) & \to & I(A, B) & \to & \text{Pic } A & \to & \text{Pic } B \\
\downarrow & & \downarrow= & & \downarrow{\varphi(A, C, B)} & & \downarrow{\rho} & & \downarrow= \\
1 & \to & U(C) & \to & U(B) & \to & I(C, B) & \to & \text{Pic } C & \to & \text{Pic } B \\
\end{array}
$$

Since $\rho$ is surjective by Lemma 2.10(1), the result follows by chasing the diagram. \(\square\)

The following result gives another case where the map $\varphi(A, C, B)$ is surjective.

Recall that a local ring $A$ is **Hensel** if every finite $A$-algebra $B$ is a direct product of local rings.

**Proposition 3.2.** Let $A \subseteq B$ be an integral extension with $A$ Hensel local. Then for all rings $C$ with $A \subseteq C \subseteq B$ the map $\varphi(A, C, B)$ is surjective.

**Proof.** By Lemma 2.2 of [3], it is enough to show that $\varphi(A, D, B)$ is surjective for every subring $D$ of $C$ containing $A$ such that $D$ is finitely generated as an $A$-algebra. Let such a ring $D$ be given. Since $D$ is integral over $A$, $D$ is a finite $A$-algebra. As $A$ is Hensel, $D$ is a finite direct product of local rings. Then $\text{Pic } A$ and $\text{Pic } D$ are both trivial. This
implies that \( \mathcal{I}(A, B) = U(B)/U(A) \), \( \mathcal{I}(D, B) = U(B)/U(D) \) and clearly \( \varphi(A, D, B) \) is surjective. \( \square \)

**Proposition 3.3.** Let \( A \subseteq C \subseteq B \) be extensions of rings with \( A \subseteq C \) subintegral. Then the sequence

\[
1 \to M\mathcal{I}(A, C) \to M\mathcal{I}(A, B) \to M\mathcal{I}(C, B) \to 1
\]

is exact.

**Proof.** Consider the commutative diagram

\[
\begin{array}{cccccc}
1 & & 1 & & 1 & \\
\downarrow & & \downarrow & & \downarrow & \\
1 & \to & \mathcal{I}(A, C) & \to & \mathcal{I}[A, X^{-1}, C[X, X^{-1}]] & \to & M\mathcal{I}(A, C) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \to & \mathcal{I}(A, B) & \to & \mathcal{I}[A, X^{-1}, B[X, X^{-1}]] & \to & M\mathcal{I}(A, B) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \to & \mathcal{I}(C, B) & \to & \mathcal{I}[C, X^{-1}, B[X, X^{-1}]] & \to & M\mathcal{I}(C, B) & \to & 1 \\
\end{array}
\]

where the rows are clearly exact. Since \( A \subseteq C \) is subintegral, so is \( A[X, X^{-1}] \subseteq C[X, X^{-1}] \). Therefore by Proposition 3.1, the first two columns are exact. Hence exactness of the last column follows by chasing the diagram. \( \square \)

**Corollary 3.4.** Let \( A \subseteq B \) be a ring extension and let \( \text{^+}A \) denote the subintegral closure of \( A \) in \( B \). Then the sequence

\[
1 \to M\mathcal{I}(A, \text{^+}A) \to M\mathcal{I}(A, B) \to M\mathcal{I}(\text{^+}A, B) \to 1
\]

is exact.

**Proof.** Immediate from Proposition 3.3. \( \square \)

**Proposition 3.5.** Let \( A \subseteq B \) be a ring extension. Assume that \( A \) is subintegrally closed in \( B \). Then

1. \( M\mathcal{I}(A, B) \cong M\mathcal{I}(A[T], B[T]) \).
2. \( M\mathcal{I}(A, B) \) is a torsion-free abelian group if \( B \) is a seminormal ring.
3. \( M\mathcal{I}(A, B) \) is a free abelian group if \( B \) is a seminormal ring and \( A \) is Hensel local.
4. \( M\mathcal{I}(A, B) = 0 \) if \( B \) is a seminormal domain and \( A \) is Hensel local.
Proof. (1) Since $A$ is subintegrally closed in $B$, $A[X]$ is subintegrally closed in $B[X]$ by Corollary 1.6 of [5]. Therefore $A[X, X^{-1}]$ is subintegrally closed in $B[X, X^{-1}]$. We have the commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & I(A, B) & \longrightarrow & I(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & MI(A, B) & \longrightarrow & 1 \\
& & \downarrow \psi & & \downarrow \xi & & & & \\
1 & \longrightarrow & I(A[T], B[T]) & \longrightarrow & I(A[T][X, X^{-1}], B[T][X, X^{-1}]) & \longrightarrow & MI(A[T], B[T]) & \longrightarrow & 1
\end{array}
$$

where $\psi$ and $\xi$ are isomorphisms by Lemma 2.3. Hence we get the result.

(2) As $A$ is subintegrally closed in $B$ and $B$ is a seminormal ring, $A$ is seminormal. Then by Remark 2.5, $LPic A \cong MPic A$. Since any seminormal ring is reduced, $MU(A) = H^0(A)$ and $MU(B) = H^0(B)$ by Remark 2.7. Now, from (1.0), we have the exact sequence

$$
1 \to H^0(A) \to H^0(B) \to MI(A, B) \to MPic A
$$

where $MPic A$ is a torsion-free abelian group by Corollary 2.3.1 of [9]. Let $T$ be the cokernel of the map $H^0(A) \to H^0(B)$. Then

$$
1 \to T \to MI(A, B) \to MPic A
$$

is exact and $T$ is a free abelian group by Proposition 1.3 of [9]. Therefore $MI(A, B)$ is a torsion-free abelian group.

(3) By Theorem 2.5 of [9], $LPic A = 0$. Since $A$ is seminormal, $MPic A = 0$. Then we have the exact sequence

$$
1 \to H^0(A) \to H^0(B) \to MI(A, B) \to 1
$$

and $MI(A, B) = \text{Coker}[H^0(A) \to H^0(B)]$ is a free abelian group by Proposition 1.3 of [9].

(4) Since $B$ is a domain, $H^0(A) = H^0(B) \cong \mathbb{Z}$. So $MI(A, B) = 0$.  

Lemma 3.6. Let $A \subseteq B$ be a subintegral extension. Then the map $LPic A \to LPic B$ is surjective.

Proof. Since $A \subseteq B$ is subintegral, so are $A[X] \subseteq B[X]$ and $A[X, X^{-1}] \subseteq B[X, X^{-1}]$. Then the maps $Pic A[X] \times Pic A[X^{-1}] \to Pic B[X] \times Pic B[X^{-1}]$ and $Pic A[X, X^{-1}] \to Pic B[X, X^{-1}]$ are surjective by Lemma 2.10(1). Hence we get the result by chasing the
following commutative diagram

\[
\begin{array}{ccccccccc}
\text{Pic } A[X] \times \text{Pic } A[X^{-1}] & \rightarrow & \text{Pic } A[X, X^{-1}] & \rightarrow & \text{LPic } A & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Pic } B[X] \times \text{Pic } B[X^{-1}] & \rightarrow & \text{Pic } B[X, X^{-1}] & \rightarrow & \text{LPic } B & \rightarrow & 1 \\
1 & & 1 & & \\
\end{array}
\]

**Theorem 3.7.** Let \( A \subseteq B \) be a ring extension with \( A \) Hensel local and \( B \) seminormal. Then \( M_{\mathcal{I}}(A, B) \cong M_{\mathcal{I}}(\mathring{A}, A) \oplus M_{\mathcal{I}}(\mathring{A}, B) \), where \( \mathring{A} \) is the subintegral closure of \( A \) in \( B \).

**Proof.** By Lemma 3.6, \( \text{LPic } A \rightarrow \text{LPic } \mathring{A} \) is surjective. Since \( A \) is Hensel local, \( \text{LPic } A = 0 \) by Theorem 2.5 of [9]. Therefore \( \text{LPic } \mathring{A} = 0 \) and \( \text{MPic } \mathring{A} = 0 \) because \( \mathring{A} \) is seminormal. Then by the same argument as Proposition 3.5(3), \( M_{\mathcal{I}}(\mathring{A}, B) \) is a free abelian group. Now the result follows from the following exact sequence (Corollary 3.4)

\[
1 \rightarrow M_{\mathcal{I}}(\mathring{A}, A) \rightarrow M_{\mathcal{I}}(A, B) \rightarrow M_{\mathcal{I}}(\mathring{A}, B) \rightarrow 1
\]

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