Bell-state Analysis for Logic Qubits Entanglement

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Decoherence is one of the main obstacles in long-distance quantum communication. Recently, the theoretical work of Fröwis and W. Dür (Phys. Rev. Lett. 106, 110402 (2011)) and the experiment of Lu et al. (Nat. Photon. 8, 364 (2014)) both showed that the logic qubits entanglement say the concatenated Greenberger-Horne-Zeilinger (C-GHZ) state is more robust under decoherence. In this paper, we describe a protocol for Bell-state analysis for this logic qubits entanglement. This protocol can also be extended to the multipartite C-GHZ state analysis. Also, we discuss its application in the quantum teleportation of a unknown logic qubit and in the entanglement swapping of logic Bell states. As the logic qubits entanglement is more robust under decoherence, our protocol shows that it is possible to realize the long-distance quantum communication based on logic qubits entanglement.

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I. INTRODUCTION

Quantum entanglement provides a valuable resource for many important applications in quantum communication, quantum computation and quantum metrology. For example, quantum teleportation [1], quantum key distribution [2], and other quantum communication protocols all require the entanglement to set up the quantum channel. In one-way quantum computation, they require to create the cluster state to perform the task [3]. However, entanglement is generally fragile. In a practical noisy environment, the entanglement will decohere and lose the quantum features. In current quantum information processing, one of the main goals is to protect the entanglement against influences from the uncontrollable environment. In long-distance quantum communication, the main approach is to protect the entanglement encoded in the physical qubits directly. For example, the approaches of quantum repeaters [4] and photon noiseless linear amplification [5] are used to extend the distance of particle number in a noisy environment [6]. It is called concatenated GHZ (C-GHZ) state of the form

\[ |\Phi^\pm_N,m\rangle = \frac{1}{\sqrt{2}}(|GHZ^+_m\rangle^\otimes N \pm |GHZ^-_m\rangle^\otimes N), \]

with \(|GHZ^+_m\rangle = \frac{1}{\sqrt{2}}(|0\rangle^\otimes m \pm |1\rangle^\otimes m)\). The degree of such logics qubits entanglement decreases polynomially with particle number in \(N\) and \(m\). Ding et al. described a way of creating the C-GHZ state with cross-Kerr nonlinearity [11]. Lu et al. reported the experimental realization of a C-GHZ state in optical system with \(m = 2\) and \(N = 3\) [12]. They also demonstrated that the C-GHZ state is more robust than the conventional GHZ state.

On one hand, up to now, though several groups discussed the C-GHZ state in both theory and experiment, such logic entangled state has not been studied in current entanglement based quantum communication. On the other hand, as the C-GHZ state is more robust, setting up the entanglement channel with logic qubits entanglement rather than the physical qubit entanglement may be an alternative way to resist the noisy environment. In this paper, we will discuss one of the most important two-qubit measurement, say the Bell-state measurement [13]. The Bell-state measurement enables many important applications in quantum information processing, such as teleportation [1], quantum key distribution [2], and so on [4]. Different from the previous Bell-state analysis, we describe the logic Bell-state analysis (LBBSA). Here, the logic Bell state is the state in Eq. \(1\) with \(N = m = 2\).
It is shown that such state can be deterministic distinguished with the help of controlled-not (CNOT) gate. Moreover, the approach for LBSA can also be extended to distinguish the C-GHZ state with arbitrary N and m.

This paper is organized as follows: In Sec. II, we will describe the approach of the LBSA. In Sec. III, we will extend this approach for distinguishing arbitrary C-GHZ state. Our protocol reveals that the logic Bell-State and the C-GHZ state analysis can be simplified to the conventional Bell-state and GHZ analysis, respectively. In Sec. IV, we will present a discussion. It is shown that, with the help of LBSA, we can teleportate an unknown logic qubit. We also show that we can perform the complete logic entanglement swapping. In Sec. V, we will make a conclusion.

II. LOGIC BELL-STATE ANALYSIS

The four logic Bell states can be described as

\[
|\Phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|\phi^+\rangle_A|\phi^+\rangle_B \pm |\phi^-\rangle_A|\phi^-\rangle_B),
\]

\[
|\Psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|\phi^+\rangle_A|\phi^+\rangle_B \pm |\phi^-\rangle_A|\phi^-\rangle_B).
\] (2)

Here

\[
|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle),
\]

\[
|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle),
\] (3)

with |0\rangle and |1\rangle are the physical qubit, respectively.

|\Phi^+\rangle_{AB} essentially is the state with m = N = 2 in Eq. (1). In Fig. 1, four physical qubits comprise two logic qubits A and B. The four physical qubits are in the spatial mode a1, a2, b1 and b2 respectively.

From Eqs. (5) and (6), one can find that the qubits a1 and b1 disentangle with the qubits a2 and b2. Interestingly, by removing the qubits a1 and b1, the remained qubits a2 and b2 essentially are the standard Bell states shown in Eq. (4). That is the |\Phi^\pm\rangle_{AB} become |\phi^\pm\rangle_{AB} and |\Psi^\pm\rangle_{AB} become |\psi^\pm\rangle_{AB}, respectively. The standard Bell states can be well distinguished with the CNOT gate and the Hadamard gate. From Fig. 1, the qubits a2 and b2 pass through the CNOT gate in a second time, and |\phi^\pm\rangle_{AB} and |\psi^\pm\rangle_{AB} will become

\[
|\phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle|b_2\rangle \pm |1\rangle|1\rangle|b_2\rangle),
\]

\[
\rightarrow \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle|b_2\rangle),
\]

\[
|\psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle|b_2\rangle \pm |1\rangle|0\rangle|b_2\rangle),
\]

\[
\rightarrow \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle|b_2\rangle).
\] (7)
Finally, after the qubit $a_2$ passing through the Hadamard gate, both qubits are measured in the basis $\{0, 1\}$. If the measurement result is $|0\rangle$, the original state is $|\Phi^+\rangle_{AB}$. If the measurement result is $|1\rangle$, the original state is $|\Phi^-\rangle_{AB}$. On the other hand, if the measurement result is $|0\rangle$ or $|1\rangle$, the original state is $|\Psi^+\rangle_{AB}$ or $|\Psi^-\rangle_{AB}$, respectively. In this way, the LBSA is completed.

### III. Entanglement Analysis for Arbitrary Concatenated Entangled State

In this section, we show that the way of distinguishing the LBSA can be used to analyze the arbitrary C-GHZ state of the form

$$|\Phi^\pm\rangle_{N,m} = \frac{1}{\sqrt{2}}(|\text{GHZ}_1^+\rangle \pm |\text{GHZ}_1^-\rangle)^\otimes N.$$  

From Eq. (8), the logic qubits denote as $|\text{GHZ}_m^\pm\rangle$. The C-GHZ state analysis for the states in Eq. (9) can be simplified to distinguish the states

$$|\Phi^+_1\rangle_{N,2} = \frac{1}{\sqrt{2}}(|\phi^+\rangle \pm |\phi^-\rangle)^\otimes N,$$

$$|\Phi^+_2\rangle_{N,2} = \frac{1}{\sqrt{2}}(|\phi^-\rangle |\phi^+\rangle)^\otimes N - 1 \pm |\phi^-\rangle |\phi^+\rangle)^\otimes N - 1,$$

$$|\Phi^\pm_{N-1}\rangle_{N,2} = \frac{1}{\sqrt{2}}(|\phi^+\rangle \pm |\phi^-\rangle)^\otimes N - 1 |\phi^-\rangle)^\otimes N - 1 |\phi^+\rangle.$$  

(9)

In C-GHZ state analysis, we only need to distinguish the states in the logic qubit level, and do not need to care about the exact information in each logic qubit. Before we start this protocol, we first transform the states $|\Phi^\pm_{1}\rangle_{N,m}$ to $|\phi^\pm_{1}\rangle_{N,m}$, $|\Phi^\pm_{2}\rangle_{N,m}$ to $|\phi^\pm_{2}\rangle_{N,m}$, $|\Phi^\pm_{N-1}\rangle_{N,m}$ to $|\phi^\pm_{N-1}\rangle_{N,m}$, respectively. Such transformation can be completed by performing $N - 2$ Hadamard operations on each qubit and measuring these $N - 2$ qubits in $\{0, 1\}$ basis. From the measurement results, if the number of $|1\rangle$ is even, the states $|\Phi^\pm_{N,m}\rangle$, $|\Phi^\pm_{N,m}\rangle$, $|\Phi^\pm_{N-1,m}\rangle$, $|\Phi^\pm_{N-1,m}\rangle$ have fully transformed to $|\phi^\pm_{1,2}\rangle_{N,m}$, $|\phi^\pm_{1,2}\rangle_{N,m}$, $|\phi^\pm_{1,2}\rangle_{N,m}$, $|\phi^\pm_{1,2}\rangle_{N,m}$, respectively. In this way, we should perform a phase-flip operation to transform the states $|\Phi^\pm_{1,2}\rangle_{N,2}$, $|\Phi^\pm_{1,2}\rangle_{N,2}$, $|\Phi^\pm_{1,2}\rangle_{N,2}$, $|\Phi^\pm_{1,2}\rangle_{N,2}$ to $|\phi^\pm_{1,2}\rangle_{N,2}$, $|\phi^\pm_{1,2}\rangle_{N,2}$, $|\phi^\pm_{1,2}\rangle_{N,2}$, $|\phi^\pm_{1,2}\rangle_{N,2}$, respectively. In Fig. 2, we denote each logic qubit $|\phi^\pm\rangle$ as $A, B, \cdots$, etc. We first perform the Hadamard operation on each physical qubit, and make the states in Eq. (10) becomes

$$|\Phi^+_1\rangle_{N,2} = \frac{1}{\sqrt{2}}(|\phi^+\rangle |\phi^-\rangle)^\otimes N - 1 |\phi^-\rangle)^\otimes N - 1 |\phi^+\rangle,$$

$$|\Phi^+_2\rangle_{N,2} = \frac{1}{\sqrt{2}}(|\phi^-\rangle |\phi^+\rangle)^\otimes N - 1 |\phi^-\rangle)^\otimes N - 1 |\phi^+\rangle.$$  

(10)

After passing through the CNOT gates, the states in Eq. (10) can be written as

$$|\Phi^+_1\rangle_{N,2} \rightarrow \frac{1}{\sqrt{2}}(|\phi^+\rangle |\phi^-\rangle)^\otimes N - 1 |\phi^-\rangle)^\otimes N - 1 |\phi^+\rangle,$$

$$|\Phi^+_2\rangle_{N,2} \rightarrow \frac{1}{\sqrt{2}}(|\phi^-\rangle |\phi^+\rangle)^\otimes N - 1 |\phi^-\rangle)^\otimes N - 1 |\phi^+\rangle.$$  

(11)

From Eq. (11), all the first physical qubits $a_1, b_1, \cdots, n_1$ in each logic qubits disentangle with the second one. By removing these qubits, the states in Eq. (11) can be
written as the standard $N$-particle GHZ state of the form
\[ |\Phi^\pm_{N,2}\rangle \rightarrow |\Phi^\pm_{N}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^\otimes N \pm |1\rangle^\otimes N), \]
\[ |\Phi^\pm_{N,2}\rangle \rightarrow |\Phi^\pm_{2}\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle^\otimes N-1 \pm |0\rangle|1\rangle^\otimes N-1), \]
\[ \cdots \]
\[ |\Phi^\pm_{2N-1}\rangle \rightarrow |\Phi^\pm_{2N}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^\otimes N-1|1\rangle \pm |1\rangle^\otimes N-1|0\rangle). \]

(12)

The $N$ particles are denoted as $a_2, b_2, \cdots, n_2$, as shown in Fig. 2. Therefore, the discrimination of the C-GHZ state equals to the discrimination of the standard $N$-particle GHZ state. The next step can be described as follows. We first perform the CNOT gate between the neighboring qubits $(n-1)_2$ and $n_2$. In Fig. 2, we denote $k \equiv n - 1$. The qubit $k_2$ is the source qubit and the qubit $n_2$ is the target qubit. After performing the CNOT operation, we let the qubit $(n-2)_2$ be the source qubit and the qubit $k_2$ $(n-1)_2$ be the target qubit and perform the CNOT operation again. In each round, we let the $l_2 (l_2=1,2,\cdots, N-1)$ qubit be the source qubit and the $(l+1)_2$ qubit be the target qubit and perform the CNOT operation. After performing $N-1$ CNOT operations, the states in Eq. (12) become
\[ |\Phi^\pm_{1}\rangle \rightarrow \frac{1}{\sqrt{2}}|\pm\rangle|0\rangle^\otimes N-1, \]
\[ |\Phi^\pm_{2}\rangle \rightarrow \frac{1}{\sqrt{2}}|\pm\rangle|1\rangle|0\rangle^\otimes N-2, \]
\[ \cdots \]
\[ |\Phi^\pm_{2N}\rangle \rightarrow \frac{1}{\sqrt{2}}|\pm\rangle|0\rangle^\otimes N-2|1\rangle. \]

(13)

Finally, after performing the Hadmard operation on the first qubit $a_2$ to transform $|+\rangle$ to $|0\rangle$ and $|-\rangle$ to $|1\rangle$, all the states in Eq. (8) can be distinguished deterministically by measuring all the qubits in the basis \{|0\rangle, |1\rangle\}. For example, if the measurement results are $|0\rangle|0\rangle|0\rangle\cdots|0\rangle$, the original state is $|\Phi^+_2\rangle_{N,m}$. If the measurement results are $|1\rangle|0\rangle|0\rangle\cdots|0\rangle$, the original state is $|\Phi^-_1\rangle_{N,m}$. In this way, we can distinguish arbitrary C-GHZ state.

IV. DISCUSSION

So far, we have fully explained the LBSA and the C-GHZ state analysis. It is interesting to discuss some applications of such state analysis. Quantum teleportation [1] and entanglement swapping [3] are two unique techniques in quantum communication. The former can transmit an unknown state of the information encoded in a particle to a remote point without distributing the particle itself. The later can be used to extend the distance of quantum communication. We will show that the unknown logic qubit can also be teleported and entanglement swapping based on the logic entanglement can also be performed in principle.

A. Logic qubit teleportation

Suppose that an arbitrary logic qubit in Alice’s laboratory can be defined as
\[ |\varphi\rangle_A = \alpha|GHZ^+_m\rangle_A + \beta|GHZ^-_m\rangle_A, \]
with $|\alpha|^2 + |\beta|^2 = 1$. Alice and Bob share the logic qubits entanglement in the channel BC of the form
\[ |\Psi\rangle_{BC} = \frac{1}{\sqrt{2}}(|GHZ^+_m\rangle_B|GHZ^+_m\rangle_C + |GHZ^-_m\rangle_B|GHZ^-_m\rangle_C) \]

(14)

Alice performs the LBSA on her two logic qubits A and B. The whole state can be described as
\[ |\varphi\rangle_A \otimes |\Psi\rangle_{BC} = (\alpha|GHZ^+_m\rangle_A + \beta|GHZ^-_m\rangle_A) \]

\[ \otimes \frac{1}{\sqrt{2}}(|GHZ^+_m\rangle_B|GHZ^+_m\rangle_C + |GHZ^-_m\rangle_B|GHZ^-_m\rangle_C) \]

\[ = \frac{1}{2} \left( |GHZ^+_m\rangle_A|GHZ^+_m\rangle_B + |GHZ^-_m\rangle_A|GHZ^-_m\rangle_B \right) \]

\[ \otimes (\alpha|GHZ^+_m\rangle_C + \beta|GHZ^-_m\rangle_C) \]

\[ + \frac{1}{2} \left( |GHZ^+_m\rangle_A|GHZ^+_m\rangle_B - |GHZ^-_m\rangle_A|GHZ^-_m\rangle_B \right) \]

\[ \otimes (\alpha|GHZ^-_m\rangle_C - \beta|GHZ^+_m\rangle_C) \]

(15)

Obviously, from Eq. (15), according to the LBSA, Alice can teleportate the arbitrary logic qubit $|\varphi\rangle_A$ to Bob.

B. Logic qubit entanglement swapping

Let pairs AB and CD be in the following logic entangled states
\[ |\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|GHZ^+_m\rangle_A|GHZ^+_m\rangle_B \]

\[ + |GHZ^-_m\rangle_A|GHZ^-_m\rangle_B), \]

(17)

and
\[ |\Psi\rangle_{CD} = \frac{1}{\sqrt{2}}(|GHZ^+_m\rangle_C|GHZ^+_m\rangle_D \]

\[ + |GHZ^-_m\rangle_C|GHZ^-_m\rangle_D). \]

(18)

If we perform a logic Bell-state measurement between the logic qubits B and C, the whole system can be described as
\[ |\Psi\rangle_{AB} \otimes |\Psi\rangle_{CD} = \frac{1}{\sqrt{2}}(|GHZ^+_m\rangle_A|GHZ^+_m\rangle_B \]

\[ + |GHZ^-_m\rangle_A|GHZ^-_m\rangle_B \]

\[ \otimes (|GHZ^+_m\rangle_C|GHZ^+_m\rangle_D + |GHZ^-_m\rangle_C|GHZ^-_m\rangle_D). \]
the probability of $|1\rangle$, logic qubits A and D will collapse to the same state with

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|GHZ_m^+\rangle_A |GHZ_m^+\rangle_B + |GHZ_m^-\rangle_A |GHZ_m^-\rangle_B)$$

\[ \text{with the probability of } \frac{1}{2}. \]

On the other hand, if the measurement results of B and C are the other states, we can also obtain the corresponding entangled states, which can be transformed to $|\Psi\rangle_{AD}$ deterministically.

\section{Conclusion}

In conclusion, we have presented a protocol for LBSA. We exploit the CNOT gate, Hadamard gate and single qubit measurement to complete the task. We also showed that arbitrary C-GHZ state can also be well distinguished. It is shown that the number of physical qubit in each logic qubit does not affect the analysis of C-GHZ state. Therefore, the analysis of C-GHZ state with arbitrary $N$ and $m$ equals to the analysis of the C-GHZ state with $m = 2$. Both the LBSA and C-GHZ state can be simplified to the standard Bell-state analysis and GHZ state analysis, respectively, which can be well distinguished with the help of CNOT gate. As the C-GHZ state is more robust than the normal GHZ states in a noisy environment \[10, 12\], our LBSA shows that it is possible to perform the long-distance quantum communication based on the logic qubits rather than the physical qubits directly.

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