Gravitational turbulent instability of anti-de Sitter space

Óscar J C Dias\textsuperscript{1,2}, Gary T Horowitz\textsuperscript{3} and Jorge E Santos\textsuperscript{3}

\textsuperscript{1} DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
\textsuperscript{2} Institut de Physique Théorique, CEA Saclay, CNRS URA 2306, F-91191 Gif-sur-Yvette, France
\textsuperscript{3} Department of Physics, University of California, Santa Barbara, CA 93106, USA

E-mail: oscar.dias@cea.fr, gary@physics.ucsb.edu and jss55@physics.ucsb.edu

Received 12 January 2012, in final form 24 February 2012
Published 29 August 2012
Online at stacks.iop.org/CQG/29/194002

Abstract
Bizon and Rostworowski have recently suggested that anti-de Sitter spacetime might be nonlinearly unstable to transferring energy to smaller and smaller scales and eventually forming a small black hole. We consider pure gravity with a negative cosmological constant and find strong support for this idea. While one can start with certain linearized modes and add higher order corrections to construct nonlinear geons, this is not possible starting with a linear combination of two or more modes. One is forced to add higher frequency modes with growing amplitude. The implications of this turbulent instability for the dual-field theory are discussed.

PACS numbers: 04.20.−q, 04.60.Cf, 04.70.Bw

Introduction
At the linearized level, anti-de Sitter (AdS) spacetime looks just as stable as Minkowski spacetime or de Sitter spacetime. There is a complete set of modes that oscillate and do not grow in time. However, at the nonlinear level, it has been suggested that AdS might behave very differently, whereas Minkowski space and de Sitter space have been shown to be stable under small but finite perturbations [1, 2]; this has never been shown for AdS. In fact the following intuitive argument suggests that it will not be the case. AdS boundary conditions act like a confining box. Any finite excitation that is added to this box might be expected to eventually explore all configurations consistent with the conserved quantities. This includes small black holes. In other words, one might conjecture that after a sufficiently long time, any finite excitation of AdS eventually finds itself inside its Schwarzschild radius and collapses to a black hole\textsuperscript{4}.

\textsuperscript{4} We first became aware of this possibility in an early version of [5].
In a very interesting paper [3], Bizon and Rostworowski have recently found evidence in favour of this conjecture. They considered the spherically symmetric collapse of a massless scalar field in AdS. No matter how small they made the initial amplitude for the scalar field, their numerical evolution always produced a black hole. They also did a perturbative analysis of the problem and found a possible loophole in the above argument. If they started with a single linearized mode of the scalar field in AdS, they could add nonlinear corrections systematically in a way that suggested that no singularity will form. However, if they started with a general superposition of linearized modes and tried to add higher order corrections, they had to add higher frequency modes. They also showed that under evolution the energy in the initial modes decrease, while the energy in the higher frequency modes grow. They argued that this is analogous to a turbulent instability in which energy is transferred from larger to smaller scales. This suggests that a generic finite perturbation of AdS will indeed eventually collapse to a black hole.

One drawback of their analysis was that it was restricted to spherical symmetry, so gravitational degrees of freedom were not excited. In addition, one might wonder if an angular momentum barrier will prevent a similar collapse at late times. We study the purely gravitational problem of the nonlinear stability of AdS and find results very similar to [3]. Starting with certain linearized modes, we compute higher order perturbative corrections and find no obstruction. However, as soon as one has a superposition of linearized gravitational modes, higher order corrections force one to add higher frequency modes with growing amplitudes. Although we do not follow the evolution to see whether a black hole eventually forms, the following argument suggests that it will. One of the singularity theorems shows that closed universes are generically singular [4]. This theorem does not apply to AdS directly (e.g. because spacelike surfaces are not compact) but, morally speaking, the negative cosmological constant acts like a confining box for fields inside. So one expects that generic solutions will be singular.

There is no contradiction between this nonlinear instability of AdS and the positive energy theorem. The latter says that if you start with AdS, it will not decay or evolve to anything else since it is the unique solution with zero energy. The question of whether small amounts of energy can collapse to small black holes is very different and usually ruled out by arguing that waves disperse at late time. This does not happen in AdS.

The nonlinear generalizations of individual perturbative modes are geons, i.e. lumps of gravitational energy that are held together by their own self-gravity. They are nonsingular, asymptotically globally AdS, and can be viewed as gravitational analogues of boson stars. However, these solutions are all special in that they each have one (helical) Killing field. They are exactly periodic in time (answering a question raised in [5]). The geons are stable to linearized perturbations, but will be unstable at higher order to the turbulent instability. We will argue that one can put small rotating black holes inside the geons and obtain black holes with only a single Killing field. (These are purely gravitational analogues of the black holes constructed in [6].) Thus, Kerr AdS is not the only stationary, asymptotically AdS black hole. This had previously been argued for based on the superradiant instability of Kerr AdS. Our small black holes sitting inside large geons provide another class of examples.

Physically, one can view the turbulent instability as a result of one geon focussing the energy of the other. It is perhaps analogous to the collision of plane gravitational waves that results in singularities in finite time due to similar focussing. Of course, one difference is that plane waves produce singularities after one collision. In our case, it takes many collisions of the geons before one expects singularities to form.
Perturbation theory

We shall focus on the four-dimensional case, described by the following action:

$$S = \int d^4x \sqrt{-\bar{g}} \left( R + \frac{6}{L^2} \right),$$

(1)

where $L$ is the AdS length scale. We perturb the equations of motion derived from (1) by expanding the metric about the AdS background, i.e., $\bar{g} = g + \sum h^{(i)} \epsilon^i$, where $\epsilon$ is a perturbation parameter whose physical meaning will be discussed later, and $\bar{g}$ is the metric of AdS written in global coordinates

$$\bar{g} = -\left(1 + \frac{r^2}{L^2} \right) dr^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).$$

(2)

At each order in perturbation theory, the Einstein equations yield

$$\Delta_i h^{(i)}_{ab} = T^{(i)}_{ab},$$

(3)

where $T^{(i)}$ is a function of $[h^{(j)} \leq i-1]$ and their derivatives and $\Delta_i$ is a second-order operator constructed from $\bar{g}$:

$$2\Delta_i h^{(i)}_{ab} = -\nabla^2 h^{(i)}_{ab} - 2\bar{R}_{a}{}^{c}g_{bc}h^{(i)}_{cd} - \bar{\nabla}_{a}\bar{\nabla}_{b}h^{(i)} + 2\bar{\nabla}_{a}\bar{\nabla}_{c}h^{(i)}_{bc}.$$  

(4)

Here, $h^{(i)} \equiv \bar{g}^{ab}h_{ab}^{(i)}$, and $\bar{R}_{abcd}$ is the AdS Riemann tensor. As a consequence of the Bianchi identities, $\nabla^{a}T^{(i)}_{ab} = 0$ for each $i$.

We will construct regular finite energy and angular momentum solutions of (3) for $i \geq 1$. In [7], the $SO(3)$ symmetry of AdS was used to show that any regular two-tensor, say $A$, can be expressed as an infinite sum of two simple building blocks:

$$A = \sum_{\ell, m} A_{\ell, m}^{(v)} + \sum_{\ell, m} A_{\ell, m}^{(s)} + \cos \phi \leftrightarrow \sin \phi,$$

(5)

where $A^{(v)}$ ($A^{(s)}$) represent scalar- (vector-) type modes, which are symmetric two-tensors built from scalar (vector) harmonics on the two-sphere. Here, $(\ell, m)$ are the usual labels for spherical harmonics, and vector harmonics on $S^2$ are of the form $\bar{\nabla} \phi_{\ell, m}$. The last term in (5) accounts for the fact that we are using a real representation for the spherical harmonics, so both the $\sin \phi$ and $\cos \phi$ terms need to be included. Each $A_{\ell, m}^{(v)}$ ($A_{\ell, m}^{(s)}$) is parametrized by seven (three) arbitrary functions of $t$ and $r$.

It is possible to show, by applying the expansion (5) to both $h^{(i)}$ and $T^{(i)}$ in (3), that any solution of (3) is described by two decoupled PDEs of the form

$$\Box_{\ell, m} \Phi^{(i)}_{\ell, m}(t, r) + V_{\ell}(r) \Phi^{(i)}_{\ell, m}(t, r) = T^{(i)}_{\ell, m}(t, r),$$

(6)

where $\Phi^{(i)}_{\ell, m}(t, r)$ is a Kodama–Ishibashi-like (gauge-invariant) variable from which $h^{(i)}_{\ell, m}$ can be recovered (in a particular gauge) through a linear differential map [7, 8]. One of the equations governs vector-type modes, while the other is related to scalar-type modes. Regular solutions of (6) are in one-to-one correspondence with smooth solutions of (3). Here, $T^{(i)}_{\ell, m}(t, r)$ is a scalar source term that can be expressed as a function of the components of $T^{(i)}$ and its derivatives; $V_{\ell}(r)$ is a potential that can be found in [7] and $\Box_{\ell}$ is the d’Alambertian associated with the auxiliary orbit space $d\nu^2 = -(1 + r^2/L^2) \, dt^2 + L^2 \, dr^2/(L^2 + r^2)$.

At each order in perturbation theory, we will impose regularity of $h^{(i)}_{\ell, m}$, seen as a tensor on a fixed AdS background. This in turn induces very strong regularity conditions on any solution of (6). In order for $h^{(i)}_{\ell, m}$ to have a regular centre, $\Phi^{(i)}_{\ell, m} \sim O(r^\ell)$, as $r \to 0$. At asymptotic

5 We have performed analogous calculations in five dimensions, and the results are qualitatively similar.
infinity, we require the metric to be asymptotically globally AdS. The asymptotic form of \( \tilde{T}_{L,m}^{(i)} \) is such that any solution of (6) behaves as

\[
\Phi_{L,m}^{(i)} \sim R_{L,m}(t) + \frac{S_{L,m}(t)}{r} + O(r^{-2}),
\]

where \( R_{L,m} \) and \( S_{L,m} \) are the arbitrary functions of \( t \). Note that the condition of asymptotically globally AdS is imposed on \( h^{(i)}_{L,m} \) not on \( \Phi^{(i)}_{L,m} \) directly. This means that we must reconstruct \( h^{(i)}_{L,m} \) before imposing the boundary conditions on \( \Phi^{(i)}_{L,m} \). It turns out that if we want our geometry to approach the Einstein static universe at the boundary, we must keep the leading term and choose \( S_{L,m}(t) = 0 \).

A couple of comments about equation (6) are in order: (i) we need to compute all the previous \( h^{(i)}_{L,m} \) in order to calculate the source term in (6) at the next order and (ii) if the time dependence of the source term is an overall multiplicative factor of the form \( \cos(\omega_t t) \), (6) can be solved by assuming a separable form for \( \Phi_{L,m}^{(i)}(t, r) = \cos(\omega_t t)R(r) \). The only exception to (ii) arises if and only if \( \omega_t \) coincides with any of the AdS gravitational normal frequencies described below. In this case, the modes are said to be resonant, and the general solution of (6) now grows with time:

\[
\Phi_{L,m}^{(i)}(t, r) = \cos(\omega_t t)R_{\omega_t, L,m}(r) + t \sin(\omega_t t)L_{\omega_t, L,m}(r).
\]

At first order, \( \tilde{T}_{L,m}^{(1)} \equiv 0 \), and one can Fourier transform equation (6) in time, with the Fourier parameter \( \omega_t \), reducing the problem to the study of a single ODE of the Sturm–Liouville type. For simplicity, we focus on scalar-type modes. Requiring a regular centre and the solution to (ii) arises if and only if \( \omega_t \) coincides with any of the AdS gravitational normal frequencies calculated the energy as a function of the angular momentum to fifth order in \( \epsilon \):

\[
E_\epsilon \equiv \frac{3J_x}{2L} \left( 1 - \frac{4901 J_x}{17920} \right), \quad \omega_2 \equiv \frac{3}{L} \left( 1 - \frac{4901 J_x}{3780 \pi L^2} \right).
\]

We then calculate \( T^{(2)} \) and rewrite it as in (5), with a total of six independent terms, corresponding to six values for the pair \((\ell, m)\). This means we have to solve for a total of six PDEs of the form (6). The solutions can be found and rendered regular both at asymptotic infinity and at the centre of AdS. At third order, we find that \( T^{(3)} \) can also be expressed as a sum of six terms, five of which behave as the second-order terms, but one, with \( \ell = m = 2 \) and \( L \omega_2 = 3 \), is resonant. This could lead to the secular behavior exhibited in (8). However, it turns out that we can set \( L_3 \omega_2 (r) = 0 \) by promoting the frequency of our initial data to be a function of \( \epsilon^2 \): \( L \omega_2 = 3 - 14703 \epsilon^2/17920 \). Thus, to third order, the solution is regular everywhere, with no growing modes in time. It is invariant under a Killing vector that is a slightly shifted version of the symmetry of the linearized mode (10): \( K \equiv \partial_t + \frac{\omega_2}{\pi} \partial_\phi \). This property is best understood if we recall that the time dependence of \( T^{(3)}_\ell \), for instance, occurs through factors of the form \( \cos(\omega_2 t - 2\phi) \) and \( \cos^3(\omega_2 t - 2\phi) \). We have computed \( T^{(4)} \) and calculated the energy as a function of the angular momentum to fifth order in \( \epsilon \):
where we defined $\epsilon$ by $J_\epsilon = \frac{27}{128}\pi L^2 \epsilon^2$. It can also be checked that this solution obeys the first law of thermodynamics $dE_\epsilon = \frac{\epsilon}{\pi} dJ_\epsilon$ to $O(J_\epsilon^3)$. We expect the situation at higher orders to be similar. It is easy to show that there are no resonant terms at any even order. Furthermore, we have carried out the perturbative construction to fifth order, and we find that two modes are resonant. However, one of the modes is regular even if the growing mode $L_{\omega, \ell, m}(r)$ is zero, while the second resonant mode can be eliminated by correcting the frequency with a quartic term in $\epsilon$. Thus, it appears that one can remove the resonances at each odd order by adjusting the frequency and construct a (nonlinear) geon.

The two-mode initial data behaves quite differently. At the linear level, we start with a composite mode $\Phi_{2,2}^{(1)}(t, r) + \Phi_{4,4}^{(1)}(t, r)$. At second order, we find that $T^{(2)}$ is now expanded as a sum of 17 terms, none of which are resonant. For each of these terms, one can calculate the right-hand side of (6) and a regular solution can always be found. The situation changes at third order. Here, we find that $T^{(3)}$ can be expressed as a sum of 36 harmonics, 32 of which behave just as the second-order perturbations. The remaining four correspond to resonant modes. Two of the resonant modes, with $L_{\omega_{0,2}} = 3$, $m_\ell = \ell_s = 2$ and $L_{\omega_{4,4}} = 5$, $m_\ell = \ell_s = 4$, can be made regular at infinity and the origin by a suitable change in the frequencies of the initial data, just like we did for the single-mode case. For the resonant mode with $L_{\omega_{0,0}} = 1$, $m_\ell = \ell_s = 0$, the solution is already regular with $L_{1,0,0}(r) = 0$ in (8). The interesting behavior occurs for the resonant mode with the highest possible frequency, $L_{\omega_{0,0}} = 7$, $m_\ell = \ell_s = 6$. In this case, the solution can only be made regular if $L_{2,6,6}(r)$ in (8) is nonzero, leading to a power-law growth in time of the initial perturbation and, thus, to an instability. It is significant that although there are four resonant modes at third order, only the highest frequency one leads to a growing mode. This is the expected behavior for a turbulent flow, for which the amplitude of higher frequency modes should become larger as time passes by. We expect that when this term is large enough, it will create other resonant modes with higher frequency and that this process will end with the formation of a rotating black hole. The fact that the breakdown of perturbation theory occurs at third order has important consequences for the timescale, say $T_{\text{inst}}$, for which we expect nonlinear interactions to be important. This will occur whenever the cubic term that grows linearly in time is of the same order of our initial perturbation, i.e. when $T_{\text{inst}} \sim O(\epsilon^{-1})$.

We see this as providing a good estimate for the time it takes to form the final rotating black hole. Indeed, this is the time to form a black hole in the spherically symmetric case [3].

**Black holes with only one Killing field**

We now argue that one can place a small Kerr AdS black hole in the core of the geon without disturbing it. Since the geon has only one Killing field, the resulting black hole will have only a single (helical) Killing field. The argument is based on earlier studies where explicit examples of scalar hairy black holes were constructed. When the black holes are small, their leading order thermodynamics is accurately reproduced by a non-interacting mixture of two components. In one study, the components were charged black holes and charged solitons [9]. In another, they were rotating black holes and rotating boson stars [6]. Similarly, we assume that a small black hole does not interact with a large geon. The absence of interaction means that the charges $E$ and $J$ of the final black hole are simply the sum of the charges of its individual constituents: $E = E_K + E_\epsilon$ and $J = J_K + J_\epsilon$. The Kerr component controls the entropy and the temperature of the final black hole: the geon has no entropy and has undefined temperature. Since the geon has only one Killing field and we place a Kerr black hole with a Killing horizon at its centre, the geon’s Killing field must coincide with the horizon generator of the black hole. That is, the angular velocity of the horizon must be $\Omega_H = \frac{\mathcal{H}}{\mathcal{R}}$. This thermodynamic equilibrium condition also follows from maximizing the entropy for a given total energy and angular momentum.

\[ \mathcal{H} = 2\mathcal{R}^3 \eta |\omega_H| = 2\mathcal{R}^3 \eta |\omega| = 2\mathcal{R}^3 \eta |\omega| \]

\[ \mathcal{T} = \frac{\mathcal{H}}{2\mathcal{R}^3} \eta \]
Combining this condition with the thermodynamics of the two components, it follows that at leading order, the geon carries all the angular momentum of the system.

Another way to construct black holes with only a single Killing field is through the superradiant instability of Kerr AdS black holes with $L \Omega H > 1$ [10]. This instability causes certain modes to grow outside the horizon, extracting energy and angular momentum from the black hole. If you perturb a black hole by a single unstable mode, then it will grow and eventually settle down to a black hole with ‘gravitational hair’ outside the horizon. For small black holes, the onset of this instability (for a mode with azimuthal quantum number $m$) can be determined by simply setting $\Omega H = \frac{\omega}{m}$, where $\omega$ is the frequency of the linearized geon.

(It suffices to consider the linearized geon since the ‘hair’ is very weak near the onset of the instability.) This condition is motivated both by the Killing field argument above and the fact that it saturates the condition for a superradiant instability: $\omega \leq \frac{m}{L \Omega H}$. One finds

$$E|_{\text{onset}} \simeq \frac{r_+}{2} + \frac{r_+^3}{2L^2} \left( 1 + \frac{\omega^2 L^2}{m^2} \right), \quad J|_{\text{onset}} \simeq \frac{1}{2} \frac{r_+}{m} \omega.$$  \hspace{1cm} (12)

This simple argument reproduces the known results for black holes with scalar hair. In [6], it was shown that in a phase diagram of $E$ versus $J$, black holes with scalar hair exist in the region below the onset of super-radiance and above the rotating boson star curve. Similarly, we expect that vacuum black holes with a single Killing field exist between (12) and the geon curve (11).

**Discussion**

There is a connection between the turbulent instability and the superradiant instability. Superradiance causes modes to grow. If you perturb a black hole by a single unstable mode, then it will eventually settle down to a hairy black hole. However, if you perturb the black hole with two unstable modes, both modes will start to grow. Their interaction will trigger the turbulent instability, which will transfer energy to higher and higher frequency.

Our discussion so far has been entirely classical. Quantum mechanically, we expect that the transfer of energy from large to small scales will still occur, but now will be cut off at a frequency equal to the initial energy. So, if one starts with an initial state consisting of a large number of low-energy quanta with the total energy less than the Planck energy, one will never form a black hole.

We now comment on the dual-field theory interpretation of our results using gauge/gravity duality. Since we have only considered gravity in the bulk, any field theory with a gravity dual must exhibit the same turbulent instability and transfer energy from large to small scales.

The fact that this is so universal seems surprising, although the final outcome of a small black hole can be viewed as thermalization in a microcanonical ensemble. It is even more surprising considering the fact that our dual system is $(2 + 1)$-dimensional, and in this case, classical turbulence causes energy to flow from small to large scales [11]. Our results indicate that (at least at large $N$) strongly coupled $(2 + 1)$-dimensional quantum theories in finite volume behave very differently. We have found similar results in AdS$_5$, which might appear to contradict the idea that the large $N$ limit of super Yang–Mills is integrable. However, the discussion of integrability does not usually include states with energy of order $N^2$ needed for finite backreaction.

Our nonlinear geons also have a dual interpretation. Since the field theory is on a sphere, at zero temperature it is in a confined phase in the sense that the free energy is of order 1 and not a power of $N$. A linearized graviton can be viewed as a spin-2 excitation (e.g. a glueball).
The nonlinear geon is dual to a bose condensate of these excitations. It is interesting that
(at large $N$ and strong coupling) these high-energy states do not thermalize.

We have argued that asymptotically AdS solutions are generically singular. It would be
of great interest to prove a singularity theorem establishing this rigorously.

Acknowledgments

We thank E Martinec, D Marolf and J Polchinski for discussions. This work was supported in
part by the National Science Foundation under grant no PHY08-55415. OJCD acknowledges
financial support from a European Marie Curie contract.

References

[1] Christodoulou D and Klainerman S 1993 The Global Nonlinear Stability of the Minkowski Space (Princeton,
NJ: Princeton University Press)
[2] Friedrich H 1986 On the existence of $n$-geodesically complete or future complete solutions of Einstein’s field
equations with smooth asymptotic structure Comm. Math. Phys. 107 587
[3] Bizon P and Rostworowski A 2011 On weakly turbulent instability of anti-de Sitter space Phys. Rev.
Lett. 107 031102 (arXiv:1104.3702 [gr-qc])
[4] Hawking S-W and Penrose R 1970 The singularities of gravitational collapse and cosmology Proc. R. Soc.
A 314 529–48
[5] Anderson M T 2006 On the uniqueness and global dynamics of AdS spacetimes Class. Quantum
Grav. 23 6935–54 (arXiv:hep-th/0605293)
[6] Dias O J C, Horowitz G T and Santos J E 2011 Black holes with only one Killing field J. High Energy
Phys. JHEP07(2011)115 (arXiv:1105.4167 [hep-th])
[7] Kodama H and Ishibashi A 2003 A master equation for gravitational perturbations of maximally symmetric
black holes in higher dimensions Prog. Theor. Phys. 110 701 (arXiv:hep-th/0305147)
[8] Kodama H and Ishibashi A 2004 Master equations for perturbations of generalized static black holes with charge
in higher dimensions Prog. Theor. Phys. 111 29 (arXiv:hep-th/0308128)
[9] Basu P, Bhattacharyya J, Bhattacharyya S, Loganayagan R, Minwalla S and Umesh V 2010 Small hairy black
holes in global AdS spacetime J. High Energy Phys. JHEP10(2010)045 (arXiv:1003.3232 [hep-th])
[10] Kunduri H K, Lucietti J and Reall H S 2006 Gravitational perturbations of higher dimensional rotating black
holes: tensor perturbations Phys. Rev. D 74 084021 (arXiv:hep-th/0606076)
[11] Gkioulekas E and Tung K 2006 Recent developments in understanding two-dimensional turbulence and the
Nastrom–Gage spectrum J. Low Temp. Phys. 145 25 (arXiv:nlin/0608064)