Theta vocabulary II. Multidimensional case

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Abstract

It is shown that the Jacobi and Riemann identities of degree four for the multidimensional theta functions as well as the Weierstrass identities emerge as algebraic consequences of the fundamental multidimensional binary identities connecting the theta functions with Riemann matrices \( \tau \) and \( 2\tau \).

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1 Introduction

In the previous paper [KZ], we have shown that all known identities for Jacobi theta functions of one variable with half-integer characteristics (of the Jacobi, Weierstrass and Riemann type) are algebraic corollaries of the six fundamental 3-term binary identities that connect theta functions with modular parameters \( \tau \) and \( 2\tau \). In this paper we give a multidimensional extension of these results.

Let \( g \in \mathbb{N}, \langle ., . \rangle : \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{C} \) be the standard scalar product. Let \( u := (u_1, \ldots, u_g) \in \mathbb{C}^g \), and let \( \tau \) be a symmetric \( g \times g \) matrix with positively definite imaginary part. For any \( a, b \in \mathbb{R}^g \) the \( g \)-dimensional theta functions with characteristics are defined as follows:

\[
\theta_{[a,b]}(u|\tau) = \sum_{k \in \mathbb{Z}^g} \exp\{\pi i (\langle k + a, k + a \rangle + 2\pi i \langle k + a, u + b \rangle)\}. \tag{1.1}
\]

The claim is that the binary relations (due to Schröter [Sch])

\[
\theta_{[u_1,b_1]}(u_1|\tau)\theta_{[u_2,b_2]}(u_2|\tau) = \sum_{p \in \mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[u_1+u_2,p+b_1+b_2]}(u_1 + u_2|2\tau)\theta_{[u_1+u_2,p+b_1-b_2]}(u_1 - u_2|2\tau) \tag{1.2}
\]

are fundamental ones in the sense that all 4th order identities (Jacobi and Riemann) and Weierstrass identities of order \( 2^g + 2 \) can be derived from the Schröter relations. We also show that the identities of the Jacobi and Riemann types are equivalent.

To formulate the main statement, it is necessary to introduce some notations. Let \( x_k \in \mathbb{C}^g, \ k = 1, 2, 3, 4 \) be \( g \)-dimensional vectors. Define the Whittaker-Watson dual vectors \( x_k' \) as follows [WW]:

\[
\begin{align*}
x_1' &= \frac{1}{2}(-x_1 + x_2 + x_3 + x_4), \\
x_2' &= \frac{1}{2}(x_1 - x_2 + x_3 + x_4), \\
x_3' &= \frac{1}{2}(x_1 + x_2 - x_3 + x_4), \\
x_4' &= \frac{1}{2}(x_1 + x_2 + x_3 - x_4). \tag{1.3}
\end{align*}
\]

### Proposition 1.1

(i) As a consequence of (1.2), the following multidimensional Jacobi identities hold:

\[
\sum_{q \in \mathbb{Z}^g/2\mathbb{Z}^g} \prod_{k=1}^{4} e^{-\pi i (a_k \cdot q)} \theta_{[a_k,b_k+q]}(u_k|\tau) = \sum_{q \in \mathbb{Z}^g/2\mathbb{Z}^g} \prod_{k=1}^{4} e^{-\pi i (a_k' \cdot q)} \theta_{[a_k',b_k+q]}(u_k'|\tau), \tag{1.4}
\]

where the dual variables are defined by (1.3).

(ii) Similarly, there are Jacobi identities:

\[
\sum_{p \in \mathbb{Z}^g/2\mathbb{Z}^g} \prod_{k=1}^{4} \theta_{[a_k+p,b_k]}(u_k|\tau) = \sum_{p \in \mathbb{Z}^g/2\mathbb{Z}^g} \prod_{k=1}^{4} \theta_{[a_k'+p,b_k']} (u_k'|\tau). \tag{1.5}
\]
(iii) As a consequence of (1.4) or (1.5), there are multidimensional Riemann identities:

$$\prod_{k=1}^{4} \theta_{a_k}^{u_k}(u_k'|\tau) = 2^{-g} \sum_{p,q \in \mathbb{Z}^g/2\mathbb{Z}^g} e^{-\pi i(p,q)} \prod_{k=1}^{4} e^{-\pi i(a_k,q)} \theta_{a_k + \frac{p}{2} b_k + \frac{q}{2}}(u_k'|\tau).$$  \hspace{1cm} (1.6)$$

(iv) The identities (1.4), (1.5), and (1.6) are equivalent.

(v) Let $\theta(u|\tau)$ be an arbitrary $g$-dimensional odd theta function. Then, by virtue of (1.2), the following Weierstrass identity holds:

$$\text{Pf}||\theta(w_i + w_j)\theta(w_i - w_j)||_{i,j=1}^{2g+2} = 0,$$

where $\text{Pf}||a_{i,j}||$ is the Pfaffian of the skew-symmetric matrix $||a_{i,j}||$.

Let us make some historical remarks. Among the huge diversity of non-trivial identities satisfied by the theta functions, the binary identities are the oldest ones. The history can be traced to Jacobi’s paper [J1] published in 1828. Namely, Jacobi wrote the identity $H(x,q)\Theta(X,q) - H(X,q)\Theta(x,q) = H(\frac{x+X}{2}, \sqrt{q})H(\frac{x-x}{2}, \sqrt{q})$ and noted that l’équation remarquable et aisément à démontrer au moyen des premiers éléments de la trigonométrie [J1] p. 305]. The general one-dimensional binary identities are due to Schröter [Sch] (1854) who wrote the formulas for the product of theta functions with modular parameters $n_1 \tau$ and $n_2 \tau$, ($n_1, n_2 \in \mathbb{N}$) (see below). One-dimensional four- and five-term identities of degree 4 were derived by Jacobi [J2] in 1835-1836. The three-term identity of degree 4 (the addition formula) have been obtained by Weierstrass [We2] in 1862. It turns out that all one-dimensional identities of degree 4 are equivalent (see [K] and [KZ]).

Multidimensional binary identities have been written by Königsberger in 1864 (without proof) [Kö]. A generalization of 4th order five-term Jacobi identities was first given by Prym [Pr]. Prym named the multidimensional identities the Riemann theta formulas. In the preface of his book, Prym wrote that he learned of the formulas during his meeting with Riemann in Pisa in early 1865, and stressed that he wrote down a proof following Riemann’s suggestions. In [Wel], Weierstrass presented the multidimensional generalizations of both four-term and five-term identities of the 4th order in his dissertation which has not been

1 The Jacobi functions $H(x,q)$ and $\Theta(x,q)$ (where $q := e^{i\pi\tau}$) correspond to the standard functions $\theta_1(x|\tau)$ and $\theta_4(x|\tau)$, respectively.

2 Im Frühjahr 1865 war mir das Glück zu Theil geworden, bei meinem hoch verehrten Lehrer Riemann in Pisa, wo derselbe sich seiner Gesundheit wegen aufhielt, einige Wochen zu bringen zu können. Ich war damals mit gewissen Untersuchungen aus der Theorie der hyperelliptischen Functionen beschäftigt, deren Anfänge ich schon in meiner in Jahre 1864 den Denkschriften der Weiner Akademie erschienenen Arbeit "Neue Theorie der ultraelliptischen Functionen" anhangsweise veröfentlicht hatte, und es bildete während meines Aufenthaltes in Pisa unter anderem auch das Additionstheorem der hyperelliptischen Functionen einen Gegenstand meiner Studien. Bei dieser Gelegenheit wurde mir von Riemann eine Formel (Formel (12) der ersten Arbeit) mitgetheilt, die für die Theorie der Thetafunctionen als eine fundamentale anzusehen ist, und ich verfasste auf seine Anregung hin einen Beweis für diese Formel, dessen Gange auch die Zustimmung meines Lehrers fand. Zu einer Verwerthung der erwähnten Formel gelangte ich aber damals nicht, einmal, weil eine Verschlimmerung in dem Befinden Riemann’s weitere Besprechungen unmöglich machte, dann aber auch, weil eingehendere auf die Charakteristiken der Thetaeihen bezügliche Untersuchungen, deren vorherige Durchführung mir nothwendig erschien, mich ganz in Anspruch nahmen. [Pr] p. V]
published at that time (‘due to typographical difficulties’). He also mentioned that the Prym’s results are the particular case of his formulas. Quite general multidimensional identities can be found in [M1], [M2].

The generalization of the addition formula to multidimensional case in terms of Pfaffian is due to Weierstrass [We2].

The main goal of the paper is to represent the detailed proof of Proposition 1.1 thus establishing the fact that the binary relations are fundamental ones to compare with all other higher order identities.

In section 2, we give the definition of the multidimensional theta functions and discuss their properties which are essential for the proof.

In section 3, we represent the detailed derivation of the general multidimensional Schröter identities which describe the product \( \theta_{[a_1;b_1]}(u_1|n_1\tau)\theta_{[a_2;b_2]}(u_2|n_2\tau) \) as a linear combination of appropriate products of theta functions with Riemann matrices \((n_1+n_2)\tau\) and \(n_1 n_2(n_1+n_2)\tau\), where \(n_1, n_2 \in \mathbb{N}\).

In section 4 and 5, we prove that the Jacobi identities of both types (1.4), (1.5) as well as the Riemann identities (1.6), are the simple corollaries of the standard Schröter relations \((n_1 = n_2 = 1)\). For completeness, we represent all identities both in terms of Whittaker-Watson variables \(u_k'\) defined by (1.3) and Jacobi ones, \(\tilde{u}_k\) [J2 p.503] (see definition (1.7) below) which are also widely used (see [M1] p. 212, [M2] p.102).

The equivalence of identities (1.4), (1.5), and (1.6) is proved in section 6. It is shown that there are equivalent relations (see (6.3) below) which can be considered as a certain generalization of Weierstrass addition formulas (see complete list for \(g = 1\) in [KZ]). These naïve Weierstrass identities of degree 4 relate the products of theta function written in terms three sets of variables \(u_k, u_k', \tilde{u}_k, (k = 1, 2, 3, 4)\).

Finally, in section 7, we present the simple proof of original Weierstrass identities (1.7) of degree \(2g + 2\) [We2] with the help of multidimensional Schröter’s relations (1.2).

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\(^3\) Specielle Falle der im Vorstehenden entwickelten Gleichungen (I), (II), (III), (IV.) sind bereits von Andern, namentlich von Herrn Prym, behandelt worden. Ich habe meine Abhandlung bereits vor einer Reihe von Jahren der Akademie vorgelegt, dieselbe konnte jedoch damals wegen typographischer Schwierigkeiten nicht veröffentlicht werden. Ich mochte aber noch bemerken, dass die Gleichungen (II), (III), (IV.) aus der Gleichung (I.) auch dadurch erhalten werden können, dass man die darin vorkommenden Argumente um gewisse Constanten (halbe Perioden der zu den vorkommenden Ï–Functionen gehorenden Abelschen Integrale erster Art) vernehmt, wie dies in dem Falle \(n = 1\) von Jacobi geschehen ist, während bei meiner Ableitung die Kenntniss jener Constanten nicht erforderlich ist. [We1] p. 137.
2 Multidimensional theta functions

Let $g \in \mathbb{N}$, $(\ldots) : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ be the standard scalar product. Let $u := (u_1, \ldots, u_g) \in \mathbb{C}^g$, and symmetric $g \times g$ matrix $\tau$ has positive definite imaginary part. Consider the infinite series [M1]:

$$\theta_{a,b}(u|\tau) = \sum_{k \in \mathbb{Z}^g} \exp\{\pi i (\tau + a, k + a) + 2\pi i (k + a, u + b)\}, \quad (2.1)$$

where $i = \sqrt{-1}$ and $a, b \in \mathbb{R}^g$. The series is absolutely convergent for any $u \in \mathbb{C}$ and defines the entire function $\theta_{a,b}(u|\tau)$. It is called $g$-dimensional theta function with characteristics $[a; b]$.

It directly follows from definition that the following relations hold:

$$\theta_{a+m,b+n}(u|\tau) = e^{2\pi i (a,n)} \theta_{a,b}(u|\tau), \quad m, n \in \mathbb{Z}^g, \quad (2.2a)$$

$$\theta_{-a-b}(-u|\tau) = \theta_{a,b}(u|\tau). \quad (2.2b)$$

By virtue of (2.2), it is sufficient to consider the theta functions with characteristics $[a; b]$ such that $0 \leq a_j, b_j < 1, j = 1, \ldots, g$.

The characteristics $[a, b]$ for which all components $a_j, b_j$ are $0$ or $\frac{1}{2}$ are called half-periods. A half-period $[a, b]$ is said to be even if $4(a, b) = 0$ (mod 2) and odd otherwise. From (2.2) it follows that $\theta_{a,b}(-u|\tau) = e^{4\pi i (a,b)} \theta_{a,b}(u|\tau)$, $(a, b \in \frac{1}{2}\mathbb{Z}^g / \mathbb{Z}^g)$, i.e. the function $\theta_{a,b}(u|\tau)$ is even or odd according to whether $[a; b]$ is even or odd half-periods. It is well-known (see [M1] for example) that there are $2^{g-1}(2^g + 1)$ even half-periods and $2^{g-1}(2^g - 1)$ odd ones.

Note that the functions (2.1) with different characteristics are connected by the relations

$$\theta_{a,b}(u + \tau a' + b'|\tau) = e^{-\pi i (\tau a', a') - 2\pi i (a', u + b' + b)} \theta_{a+a', b+b}(u|\tau) \quad (2.3)$$

for all $a', b' \in \mathbb{R}^g$. In particular, by virtue of (2.2a),

$$\theta_{a,b}(u + n|\tau) = e^{2\pi i (a,n)} \theta_{a,b}(u|\tau), \quad (2.4)$$

Hence, the functions $\theta_{a,b}(u|\tau)$ are quasiperiodic with (quasi)periods $n$ and $\tau n$, $n \in \mathbb{Z}^g$.

There is a plethora of relations satisfied by the theta functions. The simplest ones are well-known linear identities which relate the theta functions with the Riemann matrices $\tau$ and $n^2\tau, n \in \mathbb{N}$.

**Proposition 2.1** Let $n \in \mathbb{N}$. The following identities hold:

$$\theta_{a,b}(u|\tau) = \sum_{p \in \mathbb{Z}^g / n\mathbb{Z}^g} \theta_{a+pa, b+pb}(nu|n^2\tau), \quad (2.5a)$$

$$\theta_{a,b}(nu|n^2\tau) = n^{-g} \sum_{q \in \mathbb{Z}^g / n\mathbb{Z}^g} e^{-2\pi i (a,q)} \theta_{a+qa, b+qb}(u|\tau). \quad (2.5b)$$
Proof. Starting with definition (2.1), one can represent the summation over \( k \in \mathbb{Z}^g \) as follows: \( k = nl + p, \ l \in \mathbb{Z}^p, \ p \in \mathbb{Z}^p / m \mathbb{Z}^p \). Then \( \sum_{k \in \mathbb{Z}}(\ldots) = \sum_{p \in \mathbb{Z}^p / m \mathbb{Z}^p} \sum_{l \in \mathbb{Z}^p}(\ldots) \) and (2.5a) is proved. To prove (2.5b), perform the shift \( b \to b + \frac{a}{m}, \ q \in \mathbb{Z}^q / m \mathbb{Z}^q \) in (2.5a). By virtue of (2.2a), one obtains the relation
\[
e^{-\frac{2\pi i}{m} (p,q)} \theta_{(a,b+\frac{a}{m})} (u|\tau) = \sum_{p \in \mathbb{Z}^p / m \mathbb{Z}^p} e^{\frac{2\pi i}{m}(p,q)} \theta_{(a,b,m)} (nu|n^2\tau). \tag{2.6}
\]
Performing the summation over \( q \in \mathbb{Z}^q / m \mathbb{Z}^q \) and using the identity
\[
\sum_{q \in \mathbb{Z}^q / m \mathbb{Z}^q} e^{\pm \frac{2\pi i}{m} (p,q)} = n^g \delta_{p,0},
\]
which holds for any \( p \in \mathbb{Z}^p / m \mathbb{Z}^p \), one arrives to (2.5b). \qed

Below, we derive more complicated identities of higher order.

3 Schröder binary identities

In this section, we represent the simple derivation of multidimensional binary relations which are the building blocks for higher order identities. The material of this section is not new, but we present all the details for completeness.

Proposition 3.1 [Sch]. Let \( n_1, n_2 \in \mathbb{N} \). The following binary identities hold:

\[
\theta_{[a_1;b_1]} (u_1|n_1\tau) \theta_{[a_2;b_2]} (u_2|n_2\tau) = \sum_{p \in \mathbb{Z}^p / (n_1 + n_2) \mathbb{Z}^p} \left( \theta \frac{n_1 a_1 + n_2 a_2 + n_1 v_{b_1 + b_2}}{n_1 + n_2} \right) (u_1 + u_2) (n_1 + n_2) \tau \times \theta \frac{n_1 a_1 - n_2 a_2}{n_1 + n_2} (n_2 u_1 - n_1 u_2) n_1 n_2 (n_1 + n_2) \tau. \tag{3.1}
\]

Proof. The proof is essentially the one given in [Sch]. Consider the simplest case of zero characteristics \( a_k = b_k = 0, \ k = 1, 2 \). One has the product \( \theta_{[0;0]} (u_1|n_1\tau) \theta_{[0;0]} (u_2|n_2\tau) = \sum_{k_1, k_2 \in \mathbb{Z}^p} e^{\pi i F_2 + 2\pi i F_1} \), where \( F_2 := n_1 \tau k_1, k_1 \) \( n_2 \tau k_2, k_2 \), \( F_1 := \langle k_1, u_1 \rangle + \langle k_2, u_2 \rangle \). First we transform the quadratic part \( F_2 \). The whole procedure consists of three simple steps.

Changing \( k_1 := l_1 + k_2, l_2 \in \mathbb{Z}^p \), we have: \( F_2 = n_1 \tau (l_1 + k_2), l_1 + k_2 \) \( n_2 \tau k_2, k_2 \). This can be identically written as \( F_2 = n_2 n_1 (n_1 + n_2) \tau (l_1 + k_1) + (n_1 + n_2) \tau (k_2 + \frac{n_1}{n_1 + n_2} l_1) + (n_1 + n_2) \tau (k_2 + n_1 m_1 + \frac{n_1 p}{n_1 + n_2}). \)

Next, one can divide the sum over \( l_1 \) into two sums setting \( l_1 := (n_1 + n_2) m_1 + p, \ m_1 \in \mathbb{Z}^p, \ p \in \mathbb{Z}^p / (n_1 + n_2) \mathbb{Z}^p \). Thus \( F_2 = n_2 n_1 (n_1 + n_2) \tau (m_1 + \frac{p}{n_1 + n_2}), m_1 + \frac{p}{n_1 + n_2} + (n_1 + n_2) \tau (k_2 + n_1 m_1 + \frac{n_1 p}{n_1 + n_2}) \), \( k_2 + n_1 m_1 + \frac{n_1 p}{n_1 + n_2} \).

Changing the summation index \( k_2 := m_2 - n_1 m_1, m_2 \in \mathbb{Z}^p \), the term \( F_2 \) acquires the final form: \( F_2 = n_2 n_1 (n_1 + n_2) \tau (m_1 + \frac{p}{n_1 + n_2}), m_1 + \frac{p}{n_1 + n_2} + (n_1 + n_2) \tau (m_2 + \frac{n_1 p}{n_1 + n_2}), m_2 + \frac{n_1 p}{n_1 + n_2} \), where \( m_1, m_2 \in \mathbb{Z}^p, p \in \mathbb{Z}^p / (n_1 + n_2) \mathbb{Z}^p \).
Using the identity (2.7), one arrives to the standard binary identities
\[ F_1 = \langle m_1 + \frac{p}{n_1+n_2}, n_2u_1 - n_1u_2 \rangle + \langle m_2 + \frac{np}{n_1+n_2}, u_1 + u_2 \rangle. \]
Thus one proves the identity
\[ \theta_{[0;0]}(u_1|n_1\tau)\theta_{[0;0]}(u_2|n_2\tau) \]
\[ = \sum_{p \in \mathbb{Z}^3/(n_1+n_2)\mathbb{Z}^3} \theta_{\{1-n_1p\}_{1+n_2|n_1+n_2|0}}(u_1 + u_2|(n_1 + n_2)\tau)\theta_{\{1-n_2p\}_{1+n_2|n_1+n_2|0}}(n_2u_1 - n_1u_2|n_1n_2(n_1 + n_2)\tau). \]  
(3.2)

Using the relation (2.3), one arrives to the general formula (3.1).

Consider the particular case \( n_1 = n_2 := n \). Changing \( n\tau \rightarrow \tau \), one arrives to the relation:
\[ \theta_{[a_1;b_1]}(u_1|\tau)\theta_{[a_2;b_2]}(u_2|\tau) \]
\[ = \sum_{p \in \mathbb{Z}^3/2n\mathbb{Z}^3} \theta_{[a_1+a_2+p, b_1+b_2]}(u_1 + u_2|2\tau)\theta_{[a_1-a_2+p, b_1-b_2]}(n(u_1 - u_2)|2n^2\tau). \]  
(3.3)

Let us show in detail that the relation (3.3) can be reduced to the standard form with \( n = 1 \). Indeed, using the linear identity (2.5b), one can write
\[ \theta_{[a_1-a_2+p, b_1-b_2]}(u_1 - u_2|\tau). \]
Substituting this identity to (3.3) and letting \( p := 2r + p' \), \( (r \in \mathbb{Z}^3/n\mathbb{Z}^3, p' \in \mathbb{Z}^3/2\mathbb{Z}^3) \), one has by virtue of (2.2a):
\[ \theta_{[a_1;b_1]}(u_1|\tau)\theta_{[a_2;b_2]}(u_2|\tau) = \sum_{p' \in \mathbb{Z}^3/2\mathbb{Z}^3} \theta_{[a_1+a_2+p', b_1+b_2]}(u_1 + u_2|2\tau) \sum_{r \in \mathbb{Z}^3/n\mathbb{Z}^3} e^{-2\pi i (r,q)}. \]  
(3.4)

Using the identity (2.7), one arrives to the standard binary identities [M1], [D]:
\[ \theta_{[a_1;b_1]}(u_1|\tau)\theta_{[a_2;b_2]}(u_2|\tau) = \sum_{p \in \mathbb{Z}^3/2\mathbb{Z}^3} \theta_{[a_1+a_2+p, b_1+b_2]}(u_1 + u_2|2\tau)\theta_{[a_1-a_2+p, b_1-b_2]}(u_1 - u_2|2\tau). \]  
(3.5)

In the rest of the paper, we shall deal only with the standard binary identities (3.5).

**Corollary 3.1** The inverse binary relations are:
\[ \theta_{[a_1,b_1]}(u_1 + u_2|2\tau)\theta_{[a_2,b_2]}(u_1 - u_2|2\tau) = 2^{-g} \sum_{p \in \mathbb{Z}^3/2\mathbb{Z}^3} e^{-2\pi i (a_1,p)}\theta_{[a_1+a_2+b_1+b_2]}(u_1|\tau)\theta_{[a_1-a_2+b_1+b_2]}(u_2|\tau). \]  
(3.6)

**Proof.** In (3.5), perform the shifts \( b_k \rightarrow b_k + \frac{q}{2} \), where by definition \( q \in \mathbb{Z}^3 \). Due to (2.2a), the product of theta functions in the right hand side of (3.5) acquires the form \( e^{\pi i (a_1+a_2+p,q)} \theta_{[a_1+a_2+p, b_1+b_2]}(u_1 + u_2|2\tau)\theta_{[a_1-a_2+p, b_1-b_2]}(u_1 - u_2|2\tau) \). Thus we have:
\[ e^{-\pi i (a_1+a_2,q)}\theta_{[a_1,b_1]}(u_1|\tau)\theta_{[a_2,b_2]}(u_2|\tau) \]
\[ = \sum_{p \in \mathbb{Z}^3/2\mathbb{Z}^3} e^{\pi i (p,q)}\theta_{[a_1+a_2+p, b_1+b_2]}(u_1 + u_2|2\tau)\theta_{[a_1-a_2+p, b_1-b_2]}(u_1 - u_2|2\tau). \]  
(3.7)
Note that the right hand side of (3.7) depends on the vector \( q \in \mathbb{Z}^9 \) in a very simple manner. In particular, performing the summation over \( q \in \mathbb{Z}^9/2\mathbb{Z}^9 \) in (3.6) and using the identity

\[
\sum_{q \in \mathbb{Z}^9/2\mathbb{Z}^9} e^{\pi i(p,q)} = 2^9 \delta_{p,0}, \quad p \in \mathbb{Z}^9/2\mathbb{Z}^9,
\]

one arrives to (3.6).

**Remark 3.1** The derivation of Jacobi and Riemann identities below is essentially the same as derivation of (3.6): taking the appropriate product of theta functions, the only thing is to perform the relevant shifts of characteristics \([a;b]\) and take into account the simple properties (2.2). Then all identities arise by virtue of identity (3.8).

### 4 Jacobi identities

In one-dimensional case \((g = 1)\), the four-term identities of order 4 were essentially obtained by Jacobi [J2, p. 507] (see also [WW, pp. 468, 488]). For example, there is the identity

\[
\prod_{k=1}^{4} \theta_{\frac{1}{4}\tau}(u_k | \tau) + \prod_{k=1}^{4} \theta_{\frac{3}{4}\tau}(u_k | \tau) = \prod_{k=1}^{4} \theta_{\frac{1}{4}\tau}(u'_k | \tau) + \prod_{k=1}^{4} \theta_{\frac{3}{4}\tau}(u'_k | \tau),
\]

where the primed variables \( u'_k, k = 1, 2, 3, 4 \) are defined by (1.3). The problem is to generalize the relations of type (4.1) to multidimensional case. In this section we derive the identities (1.4) and (1.5) from (3.5) and (3.6), respectively.

The derivation of (1.4) is as follows. Shifting in (3.5) \( b_k \to b_k + \frac{q}{2}, \ q \in \mathbb{Z}^9/2\mathbb{Z}^9 \) and taking into account (2.2a), one has:

\[
\prod_{k=1}^{4} e^{-\pi i(a_k \cdot q)} \theta_{\frac{1}{4}\tau}(u_k | \tau)
\]

\[
= \sum_{p,p'} e^{\pi i(p+p' \cdot q)} \theta_{\frac{a_1+a_2+p}{2},b_1+b_2}(u_1 + u_2 | 2\tau) \theta_{\frac{a_1-a_2+p}{2},b_1-b_2}(u_1 - u_2 | 2\tau)
\]

\[
\times \theta_{a_3+a_4+p',b_3+b_4}(u_3 + u_4 | 2\tau) \theta_{a_3-a_4+p',b_3-b_4}(u_3 - u_4 | 2\tau).
\]

Note that the shift \( b_k \to b_k + \frac{q}{2} \) implies the transformation \( b'_k \to (b'_k + \frac{q}{2}, \frac{q}{2}), \) where the Whittaker-Watson dual vectors \( b'_k \) are defined in accordance with (1.3). Therefore, by virtue of (2.2) one has:

\[
\prod_{k=1}^{4} e^{-\pi i(a'_k \cdot q)} \theta_{\frac{1}{4}\tau}(u'_k | \tau)
\]

\[
= \sum_{p,p'} e^{\pi i(p+p' \cdot q)} \theta_{\frac{a_1+a_2+p}{2},b_1+b_2}(u_1 + u_2 | 2\tau) \theta_{\frac{a_1-a_2+p}{2},b_1-b_2}(u_1 - u_2 | 2\tau)
\]

\[
\times \theta_{\frac{a_3+a_4+p}{2},b_3+b_4}(u_3 + u_4 | 2\tau) \theta_{\frac{a_3-a_4+p}{2},b_3-b_4}(u_3 - u_4 | 2\tau).
\]
Performing the summation over \( q \in \mathbb{Z}/2\mathbb{Z} \) in (4.2), (4.3) and using the identity (3.8), one arrives to the following expressions:

\[
\sum_{q \in \mathbb{Z}/2\mathbb{Z}} \prod_{k=1}^{4} e^{-\pi i (a_k q + \frac{1}{2})} (u_k | \tau) = 2^q \sum_{p \in \mathbb{Z}/2\mathbb{Z}} \theta_{\left(\frac{a_1 + a_2 + p}{2}; b_1 + b_2\right)} (u_1 + u_2 | 2\tau) \theta_{\left(\frac{a_1 - a_2 + p}{2}; b_1 - b_2\right)} (u_1 - u_2 | 2\tau) \\
\times \theta_{\left(\frac{a_3 + a_4 + p}{2}; b_3 + b_4\right)} (u_3 + u_4 | 2\tau) \theta_{\left(\frac{a_3 - a_4 + p}{2}; b_3 - b_4\right)} (u_3 - u_4 | 2\tau),
\]

(4.4)

\[
\sum_{q \in \mathbb{Z}/2\mathbb{Z}} \prod_{k=1}^{4} e^{-\pi i (a'_k q + \frac{1}{2})} (u'_k | \tau) = 2^q \sum_{p \in \mathbb{Z}/2\mathbb{Z}} \theta_{\left(\frac{a'_1 + a'_2 + p}{2}; b_1 + b_2\right)} (u_1 + u_2 | 2\tau) \theta_{\left(\frac{a'_1 - a'_2 + p}{2}; b_1 - b_2\right)} (u_1 - u_2 | 2\tau) \\
\times \theta_{\left(\frac{a'_3 + a'_4 + p}{2}; b_3 + b_4\right)} (u_3 + u_4 | 2\tau) \theta_{\left(\frac{a'_3 - a'_4 + p}{2}; b_3 - b_4\right)} (u_3 - u_4 | 2\tau).
\]

(4.5)

Finally, the right hand sides of (4.4) and (4.5) coincide by virtue of (2.2a) and identities (1.3) do hold.

Quite similarly, one can derive identities (1.5) staring with the Schröter relations (3.6) written in the form

\[
\theta_{[a_1; b_1]} (u_1 | \tau) \theta_{[a_2; b_2]} (u_2 | \tau) = 2^{-q} \sum_{p \in \mathbb{Z}/2\mathbb{Z}} e^{-2\pi i (a_1 p)} \theta_{[a_1 + a_2; b_1 + b_2]} \left(\frac{u_1 + u_2}{2} | \tau\right) \theta_{[a_1 - a_2; b_1 - b_2]} \left(\frac{u_1 - u_2}{2} | \tau\right).
\]

(4.6)

Hence, the relations (1.4), (1.5) are simple algebraic consequences of the binary Schröter identities. Below we prove the equivalence of (1.4) and (1.5). \(\square\)

For completeness, we represent the Jacobi identities in a slightly different form. In accordance with original work [J2, p. 503], introduce the Jacobi dual variables as follows:

\[
\tilde{x}_1 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4), \\
\tilde{x}_2 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4), \\
\tilde{x}_3 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4), \\
\tilde{x}_4 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4).
\]

(4.7)

These variables have been used, for example, by Prym [P1, p. 6] in his monograph on multidimensional Riemann identities. The same notations are used also in [M1] p. 212, [M2] p. 102.
Evidently, the change $x_1 \to -x_1$ in (1.3) leads to the transformation of the Whittaker-Watson dual variables to the Jacobi ones: $x'_1 \to \bar{x}_1, x'_2 \to -\bar{x}_2, x'_3 \to -\bar{x}_3, x'_4 \to -\bar{x}_4$. As a consequence, it is easy to see that the Jacobi identities can be written in the form:

$$\sum_{p,q} \prod_{k=1}^4 \theta_{[a_k + \frac{p}{2}, b_k]}(u_k | \tau) = \sum_{p,q} \prod_{k=1}^4 \theta_{[\bar{a}_k + \frac{q}{2}, \bar{b}_k]}(\bar{u}_k | \tau),$$

(4.8a)

$$\sum_{p,q} \prod_{k=1}^4 e^{-\pi i (a_k, q)} \theta_{[a_k + \frac{p}{2}, b_k + \frac{q}{2}]}(u_k | \tau) = \sum_{p,q} \prod_{k=1}^4 e^{-\pi i (\bar{a}_k, q)} \theta_{[\bar{a}_k + \frac{p}{2}, \bar{b}_k + \frac{q}{2}]}(\bar{u}_k | \tau).$$

(4.8b)

Indeed, one can change $(a_1, b_1, u_1) \to (-a_1, -b_1, -u_1)$ in (1.4). Then, by virtue of (2.2b) and (2.2a), the identity (1.4) gives (4.8a). Similarly, under the same change, the identity (4.8b) follows from (1.5).

## 5 Riemann identities

One can derive the Riemann identities (1.6) from the Jacobi’s ones (1.4) or (1.5). For example, perform the following shifts in (1.5): $b_1 \to b_1 - \frac{q}{2}, b_2 \to b_2 + \frac{p}{2}, b_3 \to b_3 + \frac{q}{2}, b_4 \to b_4 + \frac{p}{2},$ where $q \in \mathbb{Z}^g / 2\mathbb{Z}^g$. According to the definition of the Whittaker-Watson dual variables (1.3), one has the corresponding transformations $b'_1 \to b'_1 + q, b'_2 \to b'_2, b'_3 \to b'_3, b'_4 \to b'_4$. Taking into account (2.2a), the Jacobi identity (1.5) acquires the form:

$$e^{-2\pi i (a_1 + a'_1, q)} \sum_{p \in \mathbb{Z}^g / 2\mathbb{Z}^g} e^{-\pi i (p, q)} \prod_{k=1}^4 \theta_{[a_k + \frac{p}{2}, b_k + \frac{q}{2}]}(u_k | \tau)$$

(5.1)

$$= \sum_{p \in \mathbb{Z}^g / 2\mathbb{Z}^g} e^{-\pi i (p, q)} \prod_{k=1}^4 \theta_{[a'_k + \frac{p}{2}, b'_k + \frac{q}{2}]}(u'_k | \tau).$$

Performing the summation over $q \in \mathbb{Z}^g / 2\mathbb{Z}^g$ and using identity (3.8), one arrives at the relation

$$2^g \prod_{k=1}^4 \theta_{[a_k, b_k]}(u'_k | \tau) = \sum_{p, q \in \mathbb{Z}^g / 2\mathbb{Z}^g} e^{-\pi i (p + 2a_1 + 2a'_1, q)} \prod_{k=1}^4 \theta_{[a_k + \frac{p}{2}, b_k + \frac{q}{2}]}(u_k | \tau).$$

(5.2)

Noting that in accordance with definition (1.3) $2a_1 + 2a'_1 = a_1 + a_2 + a_3 + a_4$, the relation (5.2) gives the Riemann identity (1.6). Quite similarly, the same identity can be derived from (1.4) performing the shifts $a_1 \to a_1 - \frac{p}{2}, a_2 \to a_2 + \frac{p}{2}, a_3 \to a_3 + \frac{p}{2}, a_4 \to a_4 + \frac{p}{2}$, where $p \in \mathbb{Z}^g / 2\mathbb{Z}^g$.

It is easy to see that the connection between $u_k$ and $u'_k$ (1.3) is a reciprocal one. Hence, the inverse Riemann identities are:

$$\prod_{k=1}^4 \theta_{[a_k, b_k]}(u_k | \tau) = 2^{-g} \sum_{p, q \in \mathbb{Z}^g / 2\mathbb{Z}^g} e^{-\pi i (p, q)} \prod_{k=1}^4 e^{-\pi i (a_k', q)} \theta_{[a_k' + \frac{p}{2}, b_k' + \frac{q}{2}]}(u'_k | \tau).$$

(5.3)
As a trivial corollary of (5.5) and (6.2), the following identities hold:

\[
\prod_{k=1}^{4} \theta_{[\tilde{a}_k, \tilde{b}_k]}(\tilde{u}_k | \tau) = 2^{-g} \sum_{p,q \in \mathbb{Z}^g/2\mathbb{Z}^g} \prod_{k=1}^{4} e^{-\pi i (a_k, q)} \theta_{[a_k + \frac{g}{2}, b_k + \frac{g}{2}]}(u_k | \tau).
\] (5.4)

Indeed, (5.4) can be easily obtained from (1.6) by the change \((a_1, b_1, u_1) \rightarrow (-a_1, -b_1, -u_1)\). Finally, it is easy to see that the connection between \(u_k\) and \(\tilde{u}_k\) (4.7) is a reciprocal one. Hence, the inverse formula to (5.4) is:

\[
\prod_{k=1}^{4} \theta_{[a_k, b_k]}(u_k | \tau) = 2^{-g} \sum_{p,q \in \mathbb{Z}^g/2\mathbb{Z}^g} \prod_{k=1}^{4} e^{-\pi i (\tilde{a}_k, q)} \theta_{[\tilde{a}_k + \frac{g}{2}, \tilde{b}_k + \frac{g}{2}]}(\tilde{u}_k | \tau).
\] (5.5)

### 6 Equivalence of the Jacobi and Riemann identities

In the previous section, we have obtained the Riemann identities (1.6) from the Jacobi identities of the 'second kind' (1.5). It is easy to show that (1.6) implies both (1.4) and (1.3). For example, perform the shifts \(a_k \rightarrow a_k + r\delta_{k,1}, \ k = 1, 2, 3, 4, \) where \(r \in \mathbb{Z}^g/2\mathbb{Z}^g\). Then \(a_k' \rightarrow a_k' + (-1)^{dk,1} \frac{g}{2}\), and by virtue of (2.2a) the identity (1.6) acquires the form:

\[
\prod_{k=1}^{4} \theta_{[a_k', \frac{g}{2}, b_k']} (u_k' | \tau) = 2^{-g} \sum_{p,q \in \mathbb{Z}^g/2\mathbb{Z}^g} e^{-\pi i (p,q)} \prod_{k=1}^{4} e^{-\pi i (a_k, q)} \theta_{[a_k + \frac{g}{2}, b_k + \frac{g}{2}]}(u_k | \tau).
\] (6.1)

Performing in (6.1) the summation over \(r \in \mathbb{Z}^g/2\mathbb{Z}^g\) and using (3.8), one arrives to the Jacobi identity (1.5). Similarly, performing the shifts \(b_k \rightarrow b_k + r\delta_{k,1}, \ k = 1, 2, 3, 4, \) in (1.6), one arrives to (1.4). Thus the Jacobi and Riemann identities are equivalent.

Finally, one can represent the Riemann identities in slightly different form which relates the products of theta functions in Whittaker-Watson and Jacobi variables. Changing in the left hand side of (5.5) \((a_k, b_k, u_k) \rightarrow (a_k', b_k', u_k')\), one obtains the following transformations of all variables in the right hand side of (5.5): \(\tilde{x}_1 \rightarrow \tilde{x}_1, \tilde{x}_2 \rightarrow -\tilde{x}_2, \tilde{x}_3 \rightarrow -\tilde{x}_3, \tilde{x}_4 \rightarrow -\tilde{x}_4\). Therefore, by virtue of (2.2), one can rewrite (5.5) as follows:

\[
\prod_{k=1}^{4} \theta_{[a_k', \frac{g}{2}, b_k']} (u_k' | \tau) = 2^{-g} \sum_{p,q \in \mathbb{Z}^g/2\mathbb{Z}^g} e^{-\pi i (p,q)} \prod_{k=1}^{4} e^{-\pi i (\tilde{a}_k, q)} \theta_{[\tilde{a}_k + \frac{g}{2}, \tilde{b}_k + \frac{g}{2}]}(\tilde{u}_k | \tau).
\] (6.2)

As a trivial corollary of (5.5) and (6.2), the following identities hold:

\[
\prod_{k=1}^{4} \theta_{[a_k, b_k]}(u_k | \tau) - \prod_{k=1}^{4} \theta_{[a_k', b_k']} (u_k' | \tau) = 2^{-g} \sum_{p,q \in \mathbb{Z}^g/2\mathbb{Z}^g} (1 - e^{-\pi i (p,q)}) \prod_{k=1}^{4} e^{-\pi i (\tilde{a}_k, q)} \theta_{[\tilde{a}_k + \frac{g}{2}, \tilde{b}_k + \frac{g}{2}]}(\tilde{u}_k | \tau).
\] (6.3)
Certainly, the relations (6.3) are equivalent to the Jacobi identities (1.4), (1.5). Indeed, perform the shifts $a_k \to a_k + \frac{r}{2}, r \in \mathbb{Z}/2\mathbb{Z}$. Then in accordance with (1.3), (4.7), $a'_k \to a'_k + \frac{r}{2}$, $\tilde{a}_k \to \tilde{a}_k + r\delta_{k,1}$, and by virtue of (2.2a) the relation (4.7) acquires the form

$$\prod_{k=1}^{4} \theta_{[a_k + \frac{r}{2}, b_k]}(u_k | \tau) - \prod_{k=1}^{4} \theta_{[a'_k + \frac{r}{2}, b'_k]}(u'_k | \tau) = 2^{-g} \sum_{p, q \in \mathbb{Z}^g/2\mathbb{Z}^g} (1 - e^{-\pi i(p, q)}) e^{-\pi i(r, q)} \prod_{k=1}^{4} e^{-\pi i(\tilde{a}_k, q)} \theta_{[\tilde{a}_k + \frac{r}{2}, \tilde{b}_k + \frac{r}{2}]}(\tilde{u}_k | \tau).$$  \hspace{1cm} (6.4)

Performing summation over $r \in \mathbb{Z}/2\mathbb{Z}$ it is easy to see that the right hand side of (6.4) vanishes by virtue of (3.8). Hence, one arrives to the Jacobi identities (1.5).

**Example 6.1** In one-dimensional case ($g = 1$) the general formulas (6.3) read:

$$\prod_{k=1}^{4} \theta_{[a_k, b_k]}(u_k | \tau) - \prod_{k=1}^{4} \theta_{[a'_k, b'_k]}(u'_k | \tau) = e^{-2\pi i a_1} \prod_{k=1}^{4} \theta_{[\tilde{a}_k + \frac{1}{2}, \tilde{b}_k]}(\tilde{u}_k | \tau),$$  \hspace{1cm} (6.5)

which describe all Weierstrass addition formulas listed in [KZ]. In particular, letting $a_k = b_k = \frac{1}{2}$ and using the standard notation $\theta_1(u | \tau) := -\theta_{\frac{1}{2}, \frac{1}{2}}(u | \tau)$, one has:

$$\prod_{k=1}^{4} \theta_1(u_k) - \prod_{k=1}^{4} \theta_1(u'_k) = \prod_{k=1}^{4} \theta_1(\tilde{u}_k).$$  \hspace{1cm} (6.6)

To rewrite (6.6) in a more explicit form, set $u_1 = w_1 + w_2$, $u_2 = w_1 - w_2$, $u_3 = w_3 + w_4$, $u_4 = w_3 - w_4$. Then the relation (6.6) has the form

$$\theta_1(w_1 + w_2)\theta_1(w_1 - w_2)\theta_1(w_3 + w_4)\theta_1(w_3 - w_4)$$
$$-\theta_1(w_3 - w_2)\theta_1(w_3 + w_2)\theta_1(w_1 - w_4)\theta_1(w_1 + w_4)$$
$$= \theta_1(w_1 + w_3)\theta_1(w_1 - w_3)\theta_1(w_2 + w_4)\theta_1(w_2 - w_4).$$  \hspace{1cm} (6.7)

This is the famous Weierstrass identity for the function $\sigma(u)$ [We1]. In preface of his paper, Weierstrass pointed out that identity (6.7) is essentially different from the Jacobi ones since the latter contain two or more theta functions. \footnote{"Diese Gleichung ist wesentlich anderer Art als die von Jacobi entdeckten, auf S. 507 des ersten Bandes der "Gesammelten Werke" vollstandig augestellten Relationen unter Producten von je vier $\theta$-Functionen; sie enthalten nur eine Function, wahrend in jeder der Jacobi'schen Gleichungen, die sich ubrigens aus ihr ableiten lassen, zwei oder mehrere $\theta$-Functionen vorkommen". [We2] p. 155}

Nevertheless it turns out [K], [KZ] that in the one-dimensional case the Jacobi and Weierstrass identities are equivalent. In particular, the addition formula (6.7) is a simple corollary of Jacobi identity (4.1) (see [KZ] for detail).
7 Weierstrass identities

The problem raised by Weierstrass \[\text{We2}\] is to find the identity for the single multidimensional theta function thus obtaining the direct generalization of identity (6.7). From this point of view, the naive multidimensional identities \[\text{6.3}\] do not fit the scheme.

Let \([a; b]\) be half-periods, i.e the components of \(g\)-dimensional vectors \(a\) and \(b\) are 0 or \(\frac{1}{2}\).

Proposition 7.1 \[\text{We2}\] Let \(\theta(u)\) be any \(g\)-dimensional odd theta function and \(w_i, i = 1, \ldots, 2^g + 2\) be arbitrary variables. Then the Pfaffian of \((2^g + 2) \times (2^g + 2)\) skew-symmetric matrix \(A = ||\theta(w_i + w_j)\theta(w_i - w_j)||_{i,j=1}^{2^g+2}\) identically vanishes:

\[
Pf ||\theta(w_i + w_j)\theta(w_i - w_j)||_{i,j=1}^{2^g+2} = 0.
\]

The proof of Proposition \[7.1\] is a simple corollary of the following Lemma:

Lemma 7.1 Let \(\theta(u)\) be an arbitrary odd \(g\)-dimensional theta function. Then

\[
\theta(w_1 + w_2)\theta(w_1 - w_2) = \sum_{k=1}^{2^{g-1}} \left\{ A_k(w_1)B_k(w_2) - B_k(w_1)A_k(w_2) \right\},
\]

where \(A_k(w), B_k(w), k = 1, \ldots, 2^{g-1}\) are appropriate theta functions of type \(\theta^{[c]}(2w|2\tau), c \in \frac{1}{2}\mathbb{Z}^g/2\mathbb{Z}^g\).

Proof of Lemma \[7.1\] Consider the multidimensional Schröter identities \[3.5\]. For any half-periods \([a; b]\) one has the particular relations

\[
\theta_{[a; b]}(w_1 + w_2|\tau)\theta_{[a; b]}(w_1 - w_2|\tau) = e^{4\pi i(a, b)} \sum_{p \in \frac{1}{2} \mathbb{Z}^g} e^{2\pi i(p, b)} \theta_{[a + \frac{p}{2};0]}(2w_1|2\tau)\theta_{[\frac{p}{2};0]}(2w_2|2\tau)
\]

which hold by virtue of \[2.2a\]. The sum in \[7.3\] contains \(2^g\) terms, and for any given expression \(e^{2\pi i(p, b)} \theta_{[a + \frac{p}{2};0]}(2w_1|2\tau)\theta_{[\frac{p}{2};0]}(2w_2|2\tau)\) there is the corresponding term

\[
e^{2\pi i(p+2a, b)} \theta_{[a + \frac{p + 2a}{2};0]}(2w_1|2\tau)\theta_{[\frac{p + 2a}{2};0]}(2w_2|2\tau)
\]

in the sum \[7.3\]. Hence for the odd periods \(4(a, b) = 1 (\text{mod} 2)\) one has the structure \[7.2\] and Lemma \[7.1\] is proved.

Now the proof of Proposition \[7.1\] is very simple. It is easy to see that the determinant of \((2n) \times (2n)\) skew-symmetric matrix \(||a_{i,j}||\) identically vanishes provided the structure

\[
a_{i,j} = \sum_{k=1}^{n-1} \{ A_k(w_1)B_k(w_2) - B_k(w_1)A_k(w_2) \}.
\]

Example 7.1 The Pfaffian of \(4 \times 4\) skew-symmetric matrix \(||a_{i,j}||\) is:

\[
Pf ||a_{i,j}|| = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}.
\]

Letting \(a_{i,j} = \theta_1(w_i + w_j)\theta_1(w_i - w_j)\), one arrives to the Weierstrass identity \[6.7\] in the case \(g = 1\).
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