Chirality through classical physics

Chris L Lin

Department of Physics, University of Houston, Houston, TX 77204-5005, United States of America

Received 21 February 2020, revised 2 April 2020
Accepted for publication 15 April 2020
Published 5 June 2020

Abstract

Chirality, or handedness, is a topic that is common in biology and chemistry, yet is rarely discussed in physics courses. We provide a way of introducing the topic in classical physics, and demonstrate the merits of its inclusion—such as a simple way to visually introduce the concept of symmetries in physical law—along with giving some simple proofs using only basic matrix operations, thereby avoiding the full formalism of the three-dimensional point group.

Keywords: chirality, knots, point group, reflections

1. Introduction

Chirality is a topic that spans several sciences, from the helicity of DNA in biology to the existence of organic enantiomers in chemistry [1], to the optical rotation of liquid crystal displays [2, 3] and the handedness of radioactive decay in physics [4], to name a few. It is a common phenomena of everyday life, from right-handed threaded screws and twisted ropes to the design choices made in the manufacture of objects catered to a majority right-handed population. As an illustration of the importance of the topic, it has appeared multiple times in the popular BBC-televised annual Christmas lectures given by the Royal Institution of Great Britain [5, 6].

A natural starting point to discuss chirality is in optics when discussing the plane mirror, as chirality can broadly be defined as the study of the effects of replacing an object by its mirror image. The perennial question of a plane mirror’s left–right inversion can be used to define chirality and introduce reflection and rotation matrices: this is done in section 2. Having introduced these matrices, in section 3 we prove some simple facts about reflections and rotations using matrix multiplication, and introduce improper rotations. In section 4, we consider the effect of chirality on interactions, using knots as an example, and make a distinction between the behavior of chiral objects in physics versus whether the laws of physics are themselves chiral by discussing the violation of parity in the weak interaction. In section 5, we summarize the merits of introducing chirality prior to a course in quantum mechanics, along with conclusions.

2. Chirality definition

In geometric optics, it is commonly derived that the image is the same height as the object, upright, and at the same distance from the mirror as the object. Nevertheless it would be a
mistake to say that the image is unchanged by the mirror. The question of the left–right inver-
sion of a plane mirror, which has been discussed extensively in the literature [7–13], can be
modeled with matrices, where \( R_i \) represents a reflection in the plane perpendicular to axis \( i \),
and \( R_i(\theta) \) represents a rotation by angle \( \theta \) about axis \( i \):

\[
R_xR_y(\pi) = R_z \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \pi & 0 & \sin \pi \\
0 & 1 & 0 \\
-\sin \pi & 0 & \cos \pi
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]

(1)

which mathematically models the sequence of operations shown in figure 1. The frontal inver-
sion \( z \rightarrow -z \) on the RHS is the simplest mathematical way to get an object to face ourselves.
However, the most common physical way to get an object to face us is to rotate the object
180 degrees about the \( y \)-axis. We have a keen sense of symmetry and notice that the result
after such a rotation differs from the mirror image by a reflection in the \( x \)-direction, hence the
nomenclature that the mirror inverts left–right\(^1\).

We note that in figure 1 the object can be distinguished from its mirror image, i.e. cannot be
superposed on its mirror image after any rotation. Such an object is called chiral, a term first
coined by Lord Kelvin [14]. In chemistry the pair comprising the object and its mirror image
are called enantiomers.

If an object cannot be distinguished from its mirror image, then it is called achiral. For
example, if the person in figure 1 rests his right-hand at his side, then the object is achiral. In
the next section we will derive the most general condition for an object to be achiral, but one
can see that for this particular example, i.e. for the person with his hand at his side which we

\(^1\) For example, in chemistry, one configuration of a molecule may be designated as right-handed, and its mirror image
is then left-handed.
Figure 2. Left- and right-handed helices, demonstrated with twisted rope. The left-handed helix on the left cannot be rotated into its mirror image, the right-handed helix on the right, hence is chiral.

represent as a collection of points \( X_{\text{person}} = \{(x, y, z)\}^T \), then due to the bilateral symmetry of the person,

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix},
\]

i.e. the reflection does not change the set of points that comprise the person, and so \( R_y R_z X_{\text{person}} = R_z X_{\text{person}} \) becomes just \( R_y(\pi) X_{\text{person}} = R_z X_{\text{person}} \), or that the mirror image (\( R_z X_{\text{person}} \)) is just a rotation of the object (\( R_y(\pi) X_{\text{person}} \)), hence the object is achiral. Any object that has a plane of symmetry is achiral, but not all achiral objects have a plane of symmetry.

In the next section we proceed to derive the most general condition for an object to be achiral: if an object is invariant under a rotation about an axis followed by a reflection in the plane perpendicular to the same axis, a combined operation specified by a single axis called an improper rotation, then it is achiral.

The quintessential example of a chiral object that often appears in science is the helix. In undergraduate physics, examples include screws, twisted rope, solenoids, kinematics of a charge particles in magnetic fields, circular polarization, and the very definition of right-handed coordinate systems. A helix is right-handed if curling the fingers of your right-hand around the turns advances you in the direction of your thumb. Alternatively, borrowing from the rope and textile sector, a helix is right-handed if when laid vertically and facing you, the turns are moving from the bottom left to the upper right, which is called a ‘Z’ twist due to the shape of the middle of that letter (see figure 2). A left-handed helix is called an ‘S’ twist for similar reasons.

3. Rotations and reflections

Group theory is generally not part of the standard undergraduate curriculum. However, most students are familiar with rotation matrices. Such matrices allows one to be precise about the
relation between the object and its mirror image instead of relying on descriptors such as left, right, turn, and indeed were already used for that purpose in the previous section, where the slipperiness of words are replaced with the clarity of maths.

The familiar rotation matrices are given by

\[ R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \quad R_y(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}, \quad R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (3)

each of which has determinant equal to one. We construct a general rotation of angle \( \alpha \) about an axis in the \((\theta, \phi)\) direction, where \( \theta \) is the polar angle and \( \phi \) the azimuthal angle, using a similarity transformation:

\[ R_{\theta, \phi}(\alpha) = R_z(\phi)R_y(\theta)R_z(\alpha)R_x(-\theta)R_y(-\phi), \] (4)

which can be understood as rotating the \((\theta, \phi)\) axis (along with the point that is rotating about this axis) so that \((\theta, \phi)\) aligns with the z-axis, then performing the rotation of \( \alpha \) about the z-axis, and then undoing the initial rotation. The determinant of such a matrix is the product of determinants each of which has a value equal to 1, so we have proven that any rotation has its determinant equal to 1.

The reflection matrices are as follows:

\[ R_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \] (5)

each of which has determinant equal to negative one compared to their counterparts in (3). The fact that reflections have a different determinant than rotations shows that these are in general different operations, although as we have seen, acting on achiral objects they give the same effect.

A general reflection about a plane whose normal is in the \((\theta, \phi)\) direction is given by an argument similar to (4):

\[ R_{\theta, \phi} = R_z(\phi)R_y(\theta)R_z(\alpha)R_y(-\theta)R_z(-\phi). \] (6)

The determinant of such a matrix is the product of determinants which is \(-1\), so we have proven that any reflection has its determinant equal to \(-1\).

A useful identity relates the parity operator \( P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) to reflections:

\[ PR_{\theta, \phi}(180) = R_{\theta, \phi}. \] (7)

We can now state the most general condition for an object \( X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) to be achiral. If a reflection of the object is equivalent to a rotation

\[ R_{\theta, \phi}X = R_{\theta, \phi}(\alpha')X, \] (8)
then clearly $R_{\theta,\phi}X$ can be superposed on $X$ via the subsequent rotation $R_{\theta,\phi}^{-1}(\alpha') = R_{\theta,\phi}(-\alpha')$, so that the object is achiral. However, the common way of defining achirality is if there exists an axis such that rotation about this axis, followed by a reflection in the plane perpendicular to this same axis, leaves the object invariant. This is known as an improper- or roto-rotation. This can again be proven with just matrices. Starting with condition (8) and using $R_{\theta,\phi}^2 = 1$:

$$X = R_{\theta,\phi}R_{\theta,\phi}(\alpha')X$$

$$= PR_{\theta,\phi}(180)R_{\theta,\phi}(\alpha')X$$

$$= PR_{\theta,\phi}(\alpha'' + 180)X$$

$$= R_{\theta,\phi}R_{\theta,\phi}(\beta)X.$$  (9)

Some particular cases of equation (9): if $\beta = 0$, then the object has a plane of symmetry\(^2\). If $\beta = 180$, then the object is symmetric under parity (see figure 3(a)). The reason achirality is defined as an invariance relation on $X$ is because the subset of transformations that leave an object $X$ invariant forms a subgroup, the isometry group, and one can categorize the shape of objects (such as molecules) based on the maximal isometry group to which they belong. Such considerations show that for equation (9), $\beta = 360/n$, where $n$ is a positive integer.

A useful way to think about all this is to view a reflection acting on a chiral object as creating a new object $R_{\theta,\phi}X$ that lives in a mirror world, where further rotation $R_{\theta,\phi}R_{\theta,\phi}X$ keeps the object in this mirror world as the determinant is negative one, i.e. $R_{\theta,\phi}R_{\theta,\phi} \neq R_{\theta,\phi}$, and therefore cannot be attained physically with rotation. Upon another reflection one can exit the mirror world, as a product containing two reflections has determinant positive one, i.e. $R_{\theta,\phi}R_{\theta,\phi}R_{\theta,\phi}X = R_{\theta,\phi}X$. A common demonstration is to show that a left-handed glove, when turned inside-out, becomes a right-handed glove: both a reflection in the $z$-direction (inside-out) and a reflection in the $x$-direction (left–right) move you into the mirror world.

\(^2\) It should be noted that any 2D planar object is achiral, where the symmetry plane is the plane of the object.
Figure 4. Although the square and granny knot look similar, they are not mirror reflections of each other, and therefore can (and do) behave differently. (a) A square knot, with a right-handed helix on the bottom and a left-handed helix on top. From left to right, the tension jumps from 0 to $T_L$ as vertex A is crossed, and from $T_R$ to 0 as vertex B is crossed. (b) The granny knot, with a right-handed helix on the bottom and a right-handed helix on top. Although the top is the mirror image of the top of figure 4(a), the bottom is not, so the entire knot is not the mirror image of figure 4(a). (c) The mirror image of figure 4(a). Due to the invariance of classical physics under reflection, this knot performs the same as figure 4(a).

where they are related by rotation. Indeed, left–right, top–down, and front–back reflections are all related by rotations.

We note that equation (9), which takes the form $X = \hat{O}X$, is abstracted to define symmetries in physical law. The set of all transformations $\hat{O}$ that leaves the law of physics $X$ invariant, forms a group. For example, in classical physics, $\hat{O}$ represents the Galilean group which includes translations, rotations, and boosts, and $X$ can represent Newton’s laws so that $\hat{O}X$ are Newton’s laws seen in the translated, rotated, and boosted frames, respectively. Indeed, modern physics is often done by selecting a set of symmetries $\hat{O}$ that we believe nature respects, and finding laws of physics $X$ that do not change under the symmetry transformations.

4. Symmetries and interactions

Now that we have defined chiral objects, we move on to interactions. But before we do, we note that chirality is important in chemistry, where for example the left-handed molecule limonene gives an orange its characteristic smell while its right-handed version gives a lemon its smell \[6\]. The fact that we can olfactorily distinguish an orange from a lemon means that the detector in our nose, which interacts with the molecule, is itself handed\(^3\). We have symmetry of physical

\(^3\)Reflecting a handed object along with an achiral detector produces the opposite-handed object with the same detector which, as we will see, if the laws are symmetric under reflection, will produce the same measurement so that one cannot distinguish the chirality of the object.
law under reflection only if we swap the handedness of both the molecule and detector. Stated another way, only relative chirality can be detected in an interaction. An analogy in physics is the difference in binding between a square knot and a granny knot, which we now proceed to discuss.

The square or reef knot (figure 4(a)), which is most commonly used to tie shoe-laces, binds two strings (or two ends of the same string) together by having a twist of one chirality followed by one of opposite chirality\(^4\). By contrast, the granny knot (figure 4(b)) has two twists of the same chirality\(^5\). Both of these knots look mechanically similar, but it is well-known that the granny knot unravels more easily, although only recently was this explained theoretically in detail\(^15\). Given that the difference between the two knots are the relative chirality of the top and bottom twists (like–unlike vs like–like), we choose to frame discussion as an interaction between two chiral objects.

We begin by considering the unravelling of the top helix in figure 4(a) due to \(T_{\text{load}}\). Starting from vertex A, friction from the left rope (i.e. the rope whose free end is on the left) of tension \(T_L\) wound around the right rope at vertex B will allow the tension at the loose end of the right rope (i.e. to the right of B) to be zero without this rope sliding to the left due to the tension \(T_R\), thereby unbinding. To calculate the friction needed to prevent slipping we approximate the right rope at vertex B as a pulley around which the left rope is wound (see figure A1), and use the famous capstan formula for the friction of a rope wound around a pulley with a wrap angle of 180\(^\circ\).\(^6\) The friction is \(T_L e^{\pi \mu} - T_L\), which is derived in the appendix. We get

\[
T_R - 0 \leq T_L e^{\pi \mu} - T_L
\]

\[\mu \geq 0.22,\]

where we used the symmetry of the square knot to set \(T_L = T_R\). This is in good numerical agreement with\(^16\). The same analysis would give the same value of friction for the granny knot. However, this only considers friction from sliding through the tightened loops and not the twisting of each strand due to friction at the vertices. We will find that such twisting can be seen as an interaction between the top and bottom helices. To illustrate this, consider figure 5, which shows the torque exerted at a vertex, with the local twisting of the rope given by the dotted arrow using the right-hand rule. In figure 6, the bottom of the left rope is pulled to the left and the resulting velocity of the left rope at various points is given by the solid arrow. The resulting torque (dotted arrow) on the right rope will twist this rope locally in the direction

---

\(^4\)To tie this knot a common mnemonic is ‘right over left, left over right, makes a reef knot both tidy and tight’.

\(^5\)To tie this knot ‘right over left’ is followed by another ‘right over left’.

\(^6\)For different methods of approximation and analysis, see\(^{21, 22}\).

Figure 6. The solid vector denotes the velocity of parts of the left rope when its bottom end is pulled to the left, while the dotted vector is the local twist on the right rope caused by friction due to the rubbing of the left rope on the right rope due to the initial pulling motion. The image to the right of each knot is the right rope unwrapped to show the induced torsion (or lack thereof). (a) Twisting of a square knot. (b) Twisting of a granny knot.

Figure 7. Decay of spin-up colbat-60, where the electron is represented by the small circle, and the antineutrino is not shown. The image on the right is related to the one on the left by parity. However, the right image, if realized in the physical world, would have a different probability. Therefore parity is violated.

given by the right-hand rule. One can see from figure 6 that for the square knot, the net effect is a clockwise twist on the top part of the rope and a counterclockwise twist on the bottom part of the rope. This type of strain, where two ends are twisted in opposite directions, is known as torsion\textsuperscript{7}, and this resists the initial pull velocity. The granny knot is twisted in the same direction at both ends, so it does not cost anything energetically for the rope to start slipping.

That the knots in figures 4(a) and (c) have the same strength is due to symmetry of classical physics under reflections. We would say that the same laws of physics are obeyed in the mirror world, by which is meant if we were to actually construct figure 4(c) in the real world and pull it, then the result would be the same as figure 4(a) viewed in a mirror. More generally, if the laws of physics are symmetric under a transformation, then one can perform the transformation first and then evolve, or time evolve and then perform the transformation: time evolution commutes with the symmetry transformation. This interpretation of symmetry is well-known in both classical and quantum mechanics\textsuperscript{17}.

\textsuperscript{7}Torsion is similar to tension where a bar is subjected to opposite pulling forces at its ends, except the opposite forces are replaced by opposite torques. A rotational version of Hooke’s law is obeyed that tries to restore the system to its original angle, which provides additional stability to the square knot.
An example of a symmetry not being obeyed is the parity violation experiment of Wu et al [18]. A spin up Co$^{60}$ decays via the weak interaction, ejecting an electron; see figure 7. Spin is invariant under parity, but momentum reverses sign, so that the mirror image of Co$^{60}$’s ejection of an electron with momentum $\vec{p}$ is its ejection of an electron with momentum $-\vec{p}$. Therefore a difference in the decay rate between $\vec{p}$ and $-\vec{p}$ in the real world would indicate a violation of parity$^8$, and such a difference is indeed found. This stunning result, predicted by Lee and Yang [19], shows that as far as weak interactions are concerned, symmetry arguments such as those used to equate the knots of figures 4(a) and (c) are invalid and instead each knot must be checked individually, as was done above.

It was mentioned that chirality can only be detected by a chiral detector$^9$, and if one were to swap the chirality of both the object and detector, the same result would occur: in other words, only relative chirality can be detected. However, the violation of parity means one can detect chirality absolutely: left and right can be distinguished in an absolute sense.

5. Conclusions

The importance of chirality is reflected in the fact that it receives its own section in introductory organic chemistry [20]. By contrast, in physics, examples of chiral objects are scattered. It is only in quantum mechanics where some formal organization is given to the topic with the introduction of the parity operator. In this paper we argued the merits of having a dedicated section on this topic much earlier than quantum mechanics, to provide a connection between otherwise disparate topics such as the inversion of a plane mirror, solenoid handedness, various right-hand rules, circular polarizations, and threaded ropes and fasteners. We have also shown that reflections are a particularly simple transformation, which makes them ideal for serving as preparation for more abstract concepts such as similarity transformations, group actions, and symmetry in physical law with its connection to commutativity. Both mirrors and knots have the additional merit that they visually display chirality. Finally, the fact that not just certain objects, but that nature itself is chiral (due to the weak interaction), has intellectual interest beyond particle physics.

Appendix. The Capstan equation

To find the tension in a cord wrapped around a rough pulley, equilibrium requires that the sum of the forces in the horizontal and vertical directions is zero (see figure A1). Taylor expanding the tension and keeping terms only to order $d\theta$ one gets

$$\frac{dT}{d\theta} d\theta - \mu_s dN = 0$$
$$dN - T d\theta = 0.$$  \hspace{1cm} (11)

Solving the second line for $dN$ and substituting into the first line gives

$$\frac{dT}{d\theta} = \mu_s T$$
$$T(\theta) = T(0)e^{\mu_s \theta},$$  \hspace{1cm} (12)

$^8$See [4] for a detailed calculation of the Co$^{60}$ experiment.

$^9$An example of a chiral detector is an enzyme whose shape precludes it from binding to a molecule of the opposite-handedness.
Figure A1. A rope wrapped around a pulley. The rope in the figure represents the left rope in figure 4(a), and the pulley in the figure represents the circular cross-section of the right rope in figure 4(a) at vertex B. The size $d\theta$ has been exaggerated for clarity.

which is the famous capstan equation [23, 24], which states that a tension of $T(0)e^{\mu \theta}$ at one end can be supported with a smaller tension $T(0)$ at the other end, where $\theta$ is the wrap angle around the pulley.

From equations (11) and (12), we now have the distribution of normal force $dN(\theta) = T(0)e^{\mu \theta}d\theta$ of the left rope wound around the right rope, so that a maximum bound can be placed on the friction exerted on the right rope:

$$f = \int \mu_s dN(\theta) = T(0)e^{\pi \mu} - T(0). \quad (13)$$

We emphasize that this frictional force is what prevents the pulley in figure A1 from sliding into or out of the page: the tighter the rope is wound around the pulley, the more difficult it is for the pulley (which represents the cross-section of the other rope) to slide beneath the rope.

ORCID iDs

Chris L Lin © https://orcid.org/0000-0001-5198-3420

References

[1] Hegstrom R A and Kondepudi D K 1990 *Sci. Am.* 262 108
[2] Ondris-Crawford R J, Crawford G P and Doane J W 1995 *Am. J. Phys.* 63 781
[3] Vekstein G E 1996 *Am. J. Phys.* 64 607
[4] Hong R, Sternberg M G and Garcia A 1995 *Am. J. Phys.* 63 45
[5] Laithwaite E 1974 *The Engineer through the Looking Glass* (Christmas Lectures) (London: Royal Institution)
[6] Stirling C J M 1992 *Our World through the Looking Glass* (Christmas Lectures) (London: Royal Institution)
[7] Horsfield E C 1991 *Eur. J. Phys.* 12 207
[8] Gee J K 1988 *Phys. Educ.* 23 300
[9] Kalmus P I P 1989 *Phys. Educ.* 24 122
[10] Galili I, Goldberg F and Bendall S 1991 *Phys. Teach.* 29 471
[11] Ford K W 1975 Phys. Teach. 13 228
[12] Nams V O 1993 Eur. J. Phys. 14 44
[13] Yates D J 1992 Eur. J. Phys. 13 152
[14] Kelvin W 1894 The Molecular Tactics of a Crystal (Robert Boyle lecture) (Oxford: Clarendon)
[15] Patil V P, Sandt J D, Kolle M and Dunkel J 2020 Science 367 71
[16] Maddocks J H and Keller J B 1987 SIAM J. Appl. Math. 47 1185
[17] Shankar R 2012 Principles of Quantum Mechanics (Berlin: Springer)
[18] Wu C S, Ambler E, Hayward R W, Hoppes D D and Hudson R P 1957 Phys. Rev. 105 1413
[19] Lee T D and Yang C N 1956 Phys. Rev. 104 254
[20] Wade L 2012 Organic Chemistry (Boston, MA: Pearson)
[21] Bayman B F 1977 Am. J. Phys. 45 185
[22] Jawed M K, Dieleman P, Audoly B and Reis P M 2015 Phys. Rev. Lett. 115 118302
[23] Hazelton G L 1976 Phys. Teach. 14 432
[24] Levin E 1991 Am. J. Phys. 59 80