THE SIZE OF THE JULIA SET OF MEROMORPHIC FUNCTIONS

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Abstract. We give a lower bound of the hyperbolic and the Hausdorff dimension of the Julia set of meromorphic functions of finite order under very general conditions.

1. Introduction

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set $\mathcal{F}_f$ is the set of points $z \in \mathbb{C}$ such that all the iterates $f^n$ of $f$ are well defined and normal in some neighborhood of $z$. The complement $\mathcal{J}_f = \hat{\mathbb{C}} \setminus \mathcal{F}_f$ is the Julia set which is the part of the sphere where the dynamics of the function are complicated. However, some of the points $z \in \mathcal{J}_f$ have the nice property that a family of neighborhoods $D(z, r_j), r_j \to 0$, can be zoomed conformally and with bounded distortion to a definite size without going to infinity. These points built the so called set of conical points $\Lambda_c$ (sometimes denoted by $\mathcal{J}_r$) and its Hausdorff dimension is the hyperbolic dimension $\text{HypDim}(f)$ of the function $f$.

In this paper we show the following general theorem for a lower estimate of the Hausdorff and hyperbolic dimension of the Julia set of meromorphic functions of finite order. Concrete applications are given in the last section.

Theorem 1.1. Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function of finite order $\rho$ and suppose that:

(i) $f$ has (at least) one pole $b \in f(\mathbb{C})$ which is not in the closure of the singular values $\text{sing}(f^{-1})$. Let $q$ be the multiplicity of the pole $b$.

(ii) There is a neighborhood $D$ of $b$ and constants $K > 0, \alpha_1 > -1 - 1/q$ such that

$$|f'(z)| \leq K|z|^{\alpha_1} \quad \text{for } z \in f^{-1}(D), \ |z| \to \infty.$$ 

Then

$$\text{Hdim}(\mathcal{J}_f) \geq \text{HypDim}(f) \geq \frac{\rho}{\alpha_1 + 1 + 1/q}.$$ 

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It has been shown by G.M. Stallard [St] that the Hausdorff and the hyperbolic dimension of any meromorphic function is strictly positive and also that there are transcendental meromorphic functions with Julia set of Hausdorff dimension arbitrarily close to zero.

Our result is in the spirit of, and generalizes, work of J. Kotus and M. Urbański [K, KU]. The main difference with [K] is that we need only a condition on one single pole of the function and that we need not suppose that the post critical set stands away from poles. Also, the condition (ii) is trivially satisfied for all periodic functions (with $\alpha_1 = 0$). Henceforth, our result applies especially to all elliptic functions and in this case our theorem reduces to the content of [KU]. This and various other applications are discussed in the last section of the paper.

Our result is also important for the new theory of thermodynamical formalism for meromorphic functions of finite order in [MyUr2, MyUr3] since it yields an estimation for the topological pressure which ensures the existence of geometric conformal measures (see in particular Section 7 in [MyUr2]).

We conclude this introduction in presenting now a first corollary of our result.

1.1. **Exponential elliptic functions and continuity of Hausdorff dimension.**

Compositions of exponential with elliptic functions are considered in [MyUr1] and it is shown there that they all have hyperbolic dimension two. These exponential elliptic functions do not fit directly into our context. Indeed, our methods do not apply to essential singularities. However they apply to

\[ f_d(z) = \lambda \left(1 + \frac{f(z)^d}{d}\right), \quad z \in \mathbb{C}, \]

with $f$ any elliptic function, $d \geq 1$ entire and $\lambda \in \mathbb{C} \setminus \{0\}$. Such a function $f_d$ is of order $\rho = 2$, the number $\alpha_1$ is again zero and the maximal multiplicity of poles is $dq$ with $q$ the maximal multiplicity of the poles of $f$.

**Corollary 1.2.** The estimation

\[ Hdim(J_{f_d}) \geq HypDim(f_d) \geq \frac{2dq}{dq+1} \]

holds,

\[ \lim_{d \to \infty} Hdim(J_{f_d}) = Hdim(J_{\lambda \exp \circ f}) = 2 \]

and the same relation is true for the hyperbolic dimension.

2. **Meromorphic Functions**

The reader may consult, for example, [Nev] or [HI] for a detailed exposition on meromorphic functions and [B] for their dynamical aspects. We collect here the properties of interest for our concerns.

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. We always suppose that $f$ is of finite order $\rho = \rho(f)$. If $a \in \hat{\mathbb{C}}$ is not an omitted value, then we name the $a$–points \( \{z_n(a), n \in \mathbb{N}\} = f^{-1}(a) \). The *exponent of convergence* $\rho_c(f, a)$ of the series

\[ \Sigma(t, a) = \sum_n |z_n(a)|^{-t} \]
is defined by $\rho(f,a) = \inf\{t > 0 : \Sigma(t,a) < \infty\}$. A Theorem of Borel ([Bo], see also Corollary 2, p. 231 of [H1]) states that

$$\rho(f,a) = \rho$$

for all but at most two values $a \in \hat{\mathbb{C}}$.

As usual we denote $J_f$ the Julia set and by $sing(f^{-1})$ the set of singular values, i.e. critical and asymptotic values of $f$. The following classes of functions are used (see [B, EL]):

- $S$ the functions $f$ with $sing(f^{-1})$ a finite set.
- $B$ the functions with $sing(f^{-1})$ a bounded subset of the plane.

We also remark that a meromorphic function which is not entire has either exactly one pole $b \in f^{-1}(\infty)$ which is an omitted value (in which case the function is conjugate to a function of the punctured plane) or some iterate of $f$ (actually $f^3$) has infinitely many poles. The reason is Picard’s Theorem.

### 3. Iterated function systems and the Proof of Theorem 1.1

Similar to [KU, MyUr1], the main argument of the proof of Theorem 1.1 is that the function $f$ induces an iterated function system for which we have a well developed theory thanks to Mauldin - Urbański [MdU].

Let $\Omega \subset \mathbb{C}$ be a simply connected domain with smooth boundary and suppose that $\varphi_j : \Omega \to \Omega_j = \varphi_j(\Omega) \subset \Omega$, $j \in \mathbb{N}$, are conformal mappings that are uniformly contracting and which have uniformly bounded distortion property. Suppose further that $\Omega_j \cap \Omega_i = \emptyset$ for all $i \neq j$. Then the limit set $L(S)$ of the iterated function system $S = \{\varphi_j : \Omega \to \Omega_j ; j \in \mathbb{N}\}$ is defined by

$$L(S) = \bigcap_{n=1}^{\infty} \bigcup_{j_1,\ldots,j_n} \varphi_{j_1} \circ \cdots \circ \varphi_{j_n}(\Omega) .$$

Our proof of Theorem 1.1 relies on the following:

**Theorem 3.15.** of [MdU]: For the Hausdorff dimension of the limit set $L(S)$ we have the relation

$$Hdim(L(S)) \geq \theta$$

where $\theta = \inf\{t > 0 : \sum_{j \in \mathbb{N}} |\varphi_j'(z_0)|^t < \infty\}$, $z_0$ any point of $\Omega$.

**Proof of theorem 1.1** Let $f$ be meromorphic with pole $b$ of multiplicity $q$ and let $D$ be a neighborhood of $b$ as in the conditions (i) and (ii) of Theorem 1.1. Note that, near the pole $b$, the function $f$ has the form $f(z) = \frac{g(z)}{(z-b)^q}$ with $g$ a holomorphic function defined near $b$ and $g(b) \neq 0$. We may suppose that $D = \mathbb{D}(b,r)$ is a disk centered at $b$ and with radius $r > 0$ sufficiently small such that firstly no singular value of $f$ belongs to $D^* = \mathbb{D}(b,2r)$ and secondly

$$|f'(z)| \simeq \frac{1}{|z-b|^{q+1}} \simeq |f(z)|^{1+1/q} \quad \text{for } z \in D^* \setminus \{b\} .$$
Here and in the rest of the paper $A \simeq B$ means that $\frac{A}{B}$ stands away from 0 and $\infty$ independently of the variables involved. We remark that the last equation implies in particular that no critical point of $f$ belongs to $D^*$. Denote $V = f(D \setminus \{ b \})$ and let $R > 0$ such that $\{ |z| > R \} \subset V$.

Consider now the $b$-points $z_n = z_n(b) \in f^{-1}(b) \cap V$ and denote $\varphi_n$ the inverse branch of $f$ defined on $D$ by $\varphi_n(b) = z_n$.

**Claim 3.1.** There is $n_0 \geq 1$ such that $\varphi_n(D) \subset U$ for all $n \geq n_0$.

**Proof.** Suppose $n \in \mathbb{N}$ such that there exists $y = \varphi_n(x) \in \varphi_n(D) \cap \{ |z| = R \}$ and such that $|z_n| = |\varphi_n(b)| \geq 3R$. Denote $B_n = \varphi_n(\mathbb{D}(x, r))$. By Koebe’s distortion theorem this set contains a disk $\mathbb{D}(y, r|\varphi_n'(x)|/4)$. The same argument shows that

$$|\varphi_n'(x)| \simeq |\varphi_n'(b)| \simeq \frac{\text{diam}(\varphi_n(D))}{\text{diam}(D)} \geq \frac{|z_n| - R}{2r} \geq \frac{R}{r}.$$ 

Therefore, there is $t = t(r, R) > 0$ such that $B_n \subset \mathbb{D}(y, t)$ which shows that

$$\varphi_n(D^*) \cap \{ |z| = R \}$$

contains an arc of length at least $t$. But this can happen only for finitely many $n \in \mathbb{N}$, the sets $\varphi_n(D^*)$, $n \in \mathbb{N}$, being disjoint. The claim follows. \[\square\]

Since $f : D \setminus \{ b \} \to V$ is an unbranched covering there is, for any $n \geq n_0$, an inverse branch $\psi_n$ of $f$ defined on $\varphi_n(D)$ with

$$D_n = \Phi_n(D) = \psi_n \circ \varphi_n(D)$$

compactly contained in $D$. Call $w_n = \psi_n(z_n)$. We now have

$$S = \{ \Phi_n : D \to D_n ; n \geq n_0 \}$$

a conformal iterated function system according to [MdU] and it remains to estimate the exponent of convergence of the associated Poincaré series.

Suppose $w \in D$ such that $f^2(w) = b$ and set $z = f(w) \in V$. Then it follows from condition (ii) that

$$|(f^2)'(w)| = |f'(w)||f'(z)| \lesssim |f(w)|^{1+1/q}|z|^{\alpha_1} = |z|^{\alpha_1+1+1/q}.$$ 

Therefore

$$\Sigma_t = \sum_{n \geq n_0} |\Phi_n'(w)| = \sum_{n \geq n_0} |(f^2)'(w_n)|^{-t} \gtrsim \sum_{n \geq n_0} |z_n|^{-t(\alpha_1+1+1/q)}.$$ 

Because of Koebe’s distortion theorem we may suppose that the pole $b$ is a Borel point, meaning that the exponent of convergence of $\sum |z_n|^{-t}$ is $\rho$ the order of $f$. We showed the following estimation for the critical exponent $\theta$ of $\Sigma_t$:

$$\theta \geq \frac{\rho}{\alpha_1 + 1 + 1/q}.$$
and conclude that the limit set of the system $S$ has Hausdorff dimension at least $\theta$. The limit set of $S$ being a subset of the set of conical points of $f$ the Theorem is proven. □.

**Remark 3.2.** In many cases and especially in the examples given in the next section the series $\Sigma_t$ is divergent at the critical exponent $t = \theta$. The system $S$ is then called hereditarily regular and it is shown in [MdU] that the estimation in Theorem 1.1 can be sharpened to

$$Hdim(J_f) \geq HypDim(f) > \frac{\rho}{\alpha_1 + 1 + 1/q}.$$ 

4. Applications of Theorem 1.1

We now discuss the conditions of Theorem 1.1 and give some applications and explicit examples. We will focus our attention on the essential condition which is (ii). The first one (i) is often automatically fulfilled. For example this is the case if $f \in B$, meaning that $sing(f^{-1})$ a bounded subset of the plane, and if $f$ has infinity many poles. All the following examples are of this kind.

4.1. Periodic functions. The polynomial growth condition (ii) on the derivative is always satisfied for all periodic functions. Indeed, suppose $f : \mathbb{C} \to \hat{\mathbb{C}}$ is periodic, $b \in \mathbb{C} \setminus sing(f^{-1})$ is a pole and $D = \mathbb{D}(b, r)$ a disk free of singular values. Then it follows immediately from the periodicity that $|f'|$ is almost constant on $f^{-1}(D)$. Therefore, condition (ii) is satisfied with $\alpha_1 = 0$.

Consider, as a first example, the 1–periodic family

$$f_\lambda(z) = \lambda(tan \ z)^m, \ m \in \mathbb{N} \ and \ \lambda \in \mathbb{C}^*.$$ 

All these functions are of order $\rho = 1$ and the (infinitely many) poles are all of order $q = m$ which implies

$$Hdim(J_{f_\lambda}) \geq HypDim(f_\lambda) \geq \frac{1}{1 + 1/m} = \frac{m}{m + 1} \ for \ all \ \lambda \in \mathbb{C}^*.$$ 

This is precisely the lower bound found in [K] and it is shown there that, for this family, this bound is sharp.

Suppose now that $f : \mathbb{C} \to \hat{\mathbb{C}}$ is an elliptic function, i.e. a meromorphic function for which there exists $\omega_1, \omega_2 \in \mathbb{C}$ with $\mathfrak{z}(\frac{\omega_1}{\omega_2}) > 0$ and such that

$$f(a) = f(b) \ if \ and \ only \ if \ a = b + n\omega_1 + m\omega_2$$

for some $n, m \in \mathbb{Z}$. These functions are of order $\rho = 2$ and Theorem 1.1 gives as the next Corollary the main result of [KU]:

**Corollary 4.1.** Suppose $f : \mathbb{C} \to \hat{\mathbb{C}}$ is an elliptic function and $q$ the maximal multiplicity of its poles. Then

$$Hdim(J_{f}) \geq HypDim(f) \geq \frac{2}{1 + 1/q} = \frac{2q}{1 + q}.$$
In the same way one can consider functions like \( g(z) = f(z) + f(\sqrt{2}z) \), \( f \) elliptic, which are no longer periodic but the same conclusion do hold. An other family of examples are functions of the form \( g = f \circ P \) with again \( f : \mathbb{C} \to \hat{\mathbb{C}} \) an elliptic function and \( P \) a polynomial of degree \( d \). Then \( g \) is of order \( \rho = 2d \) and condition (ii) is satisfied with \( \alpha_1 = d - 1 \). We therefore get the estimation
\[
H\dim(J_g) \geq H\text{ypDim}(g) \geq \frac{2d}{d+1/p}.
\]

4.2. Solutions of differential equations. Our result does also apply to solutions of suitable differential equations. Let us illustrate this by inspecting the following Ricatti equation
\[
(4.1) \quad w' = P_0(z) + P_1(z)w + w^2
\]
were \( P_0, P_1 \) are polynomials of degree \( d_0, d_1 \) respectively.

Suppose \( f : \mathbb{C} \to \hat{\mathbb{C}} \) is a meromorphic solution of (4.1) having infinitely many poles and at least one of them outside \( \text{sing}(f^{-1}) \). Then, by a result of Wittich (see [H2, Thm 4.6.3]), \( f \) is of finite order \( \rho = 1 + \max\{\frac{1}{2}d_0, d_1\} \). Moreover, the growth condition (ii) of Theorem 1.1 is satisfied with \( \alpha_1 = \max\{d_0, d_1\} \). It easily follows then that
\[
H\dim(J_f) \geq H\text{ypDim}(f) \geq \rho - d \geq 1/2
\]
for any such function.

4.3. Meromorphic functions with polynomial Schwarzian derivative. The exponential and tangent functions are examples for which the Schwarzian derivative
\[
S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2,
\]
is constant. By Möbius invariance of \( S(f) \), functions like
\[
\frac{e^z}{\lambda e^z + e^{-z}} \quad \text{and} \quad \frac{\lambda e^z}{e^z - e^{-z}}
\]
also have constant Schwarzian derivative. Examples for which \( S(f) \) is a polynomial are
\[
f(z) = \int_0^z \exp(Q(\xi)) \, d\xi, \quad Q \text{ a polynomial},
\]
and also
\[
f(z) = \frac{a Ai(z) + b Bi(z)}{c Ai(z) + d Bi(z)} \quad \text{with } ad - bc \neq 0
\]
and with \( Ai \) and \( Bi \) the Airy functions of the first and second kind.

The asymptotic behavior of a general meromorphic solution \( f \) of
\[
S(f) = P, \quad P \text{ a polynomial},
\]
are well known and it turns out that such a function satisfies the conditions of Theorem 1.1 with \( \rho = d/2+1 \) and \( \alpha_1 = d/2 \) where \( d = \deg(P) \). Moreover, \( f \) is of divergence
type and has only simple poles (details can be found in Section 3 of [MyUr3]). Consequently,

\[ H\text{dim}(\mathcal{J}_f) \geq H\text{ypDim}(f) > \frac{d + 2}{d + 4} \geq \frac{1}{2}. \]

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