Double Aztec Rectangles.

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Abstract

We introduce a new region obtained by attaching two different Aztec rectangles, which we call a double Aztec rectangle. We prove that the tilings of a double Aztec rectangle are enumerated by a simple product formula that is related to MacMahon’s formula on the number of plane partitions fit in a given box. The result implies the numbers of perfect matchings of several new families Aztec rectangle graphs with vertices removed. In addition, we investigate the generating function of the tilings of a double Aztec rectangle with two statistics introduced in the Aztec diamond theorem by Elkies, Kuperberg, Larson and Propp.

Keywords: Aztec diamonds, dominoes, tilings, perfect matchings

1 Introduction

A lattice divides the plane into disjoint fundamental regions. We call the union of any two fundamental regions sharing an edge a tile. We are interested in how many different ways to cover a region by tiles so that there are no gaps or overlaps; and we call those coverings tilings. Denote by $M(R)$ the number of tilings of a region $R$. The simplest (and probably most popular) tilings are the domino and the lozenge tilings on the square and the triangular lattices, respectively.

A classical result in enumeration of lozenge tilings is MacMahon’s formula [17], which states that the number of lozenge tilings of a (centrally symmetric) hexagon $H_{a,b,c}$ of sides $a, b, c, a, b, c$ (in cyclic order, start from the north side) equals

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{t=1}^{c} \frac{t + j + t - 1}{t + j + t - 2}.$$  \hfill (1)

Figure 1 shows the hexagon $H_{4,7,3}$ and one of its lozenge tilings. The lozenge tilings of $H_{a,b,c}$ are in bijection with the plane partitions fit in an $a \times b \times c$ box (see [20]).

In enumeration of domino tilings, the Aztec diamonds and its variants (e.g., Aztec rectangles, Aztec dungeons [6], Aztec dragons [6], Aztec castle [16]) have the central role. Here, the Aztec diamond of order $n$ is the union of all unit squares lying inside the contour $|x| + |y| = n + 1$ (see Figure 2(a)). It has been proven that the number of tilings of the Aztec diamond of order
Figure 1: The hexagon $H_{4,7,3}$ (a) and one of its lozenge tilings (b).

Figure 2: The Aztec diamond $\mathcal{AD}_4$ (a), the Aztec rectangle $\mathcal{AR}_{4,6}$ (b), and the Aztec rectangle $\mathcal{AR}_{5,4}$ (c).
Figure 3: The double rectangle $\mathcal{DR}_{m_1,n_1}^{4,7}$, $\mathcal{DR}_{m_2,n_2}^{3,6}$ is obtained by matching two Aztec rectangles $\mathcal{AR}_{4,7}$ (white) and $\mathcal{AR}_{3,6}$ (shaded).

$n$ equals $2^{n(n+1)/2}$ by Elkies, Kuperberg, Larsen and Propp in [8]. Many further proofs followed (see e.g., [1], [2], [9], [11], [13], [19]).

By extending an Aztec diamond to the southwest or the southeast of it, we have an Aztec rectangle (see Figures 2(b) and (c) for examples). We use the notations $\mathcal{AD}_n$ and $\mathcal{AR}_{m,n}$ for the Aztec diamond of order $n$ and the Aztec rectangle of order $m \times n$, respectively. One can view the Aztec rectangle as a generalization of the Aztec diamond in the sense that $\mathcal{AD}_n$ is exactly $\mathcal{AR}_{n,n}$.

The Aztec rectangle $\mathcal{AR}_{m,n}$ does not have any domino tilings when $m \neq n$. However, when removing several squares along a long side of the region, the number of tilings is given by a simple product formula (see [18], [4], or [10]). This number of domino tilings is closely related to the number of lozenge tilings a semihexagon with dents, i.e. the upper half of a lozenge hexagon with some triangles removed from the bottom (see Lemmas 1 and 2 in [10]). We notice that the lozenge tilings of the latter region correspond to Gelfand patterns (see [7]), column-strict plane partitions, and semi-standard Young tableau (see [3] and [20]).

In this paper, we give a new example for the interesting relation between the domino and the lozenge tilings as follows. Consider two Aztec rectangles $\mathcal{AR}_{m_1,n_1}$ and $\mathcal{AR}_{m_2,n_2}$ with $m_1 \leq n_2$ and $m_2 \leq n_2$. We match the lower long side of $\mathcal{AR}_{m_1,n_1}$ to the upper long side of $\mathcal{AR}_{m_2,n_2}$ so that the first square on the lower side of $\mathcal{AR}_{m_1,n_1}$ stays immediately on the right of the $(k+1)$-th square on the upper side of $\mathcal{AR}_{m_2,n_2}$. We denote by $\mathcal{DR}_{m_1,n_1,k}$ the resulting region (see Figure 3), and call it a double Aztec rectangle. In addition, we call the parts of the double Aztec rectangle corresponding to the Aztec rectangles $\mathcal{AR}_{m_1,n_1}$ and $\mathcal{AR}_{m_2,n_2}$ the upper and lower parts, respectively.

Interestingly, the number of domino tilings of a double Aztec rectangle equals the product
Theorem 1. Assume that \( m_1, n_1, m_2, n_2 \) are four positive integers, and that \( k \) is a non-negative integer so that \( m_1 \leq n_1 \), \( m_2 \leq n_2 \), and \( k \leq \min(m_2, n_2 - 1) \).

(a) \( M(\mathcal{D}R_{m_1, n_1, k}) = 0 \) unless \( n_1 - m_1 = n_2 - m_2 \).

(b) If \( n_1 - m_1 = n_2 - m_2 \), then

\[
M(\mathcal{D}R_{m_1, n_1, k}) = 2^{m_1(m_1+1)/2 + m_2(m_2+1)/2} \prod_{i=1}^{n_1-m_1} \prod_{j=1}^{m_2-k+1} \prod_{t=1}^{m_1+k} \frac{i + j + t - 1}{i + j + k - 2}.
\]

Next, we consider a similar attaching-process with three different Aztec rectangles as follows. Given three Aztec rectangles \( \mathcal{A}R_{m_1, n_1} \), \( \mathcal{A}R_{m_2, n_2} \) and \( \mathcal{A}R_{m_3, n_3} \) with \( n_1 = n_3 \) and \( m_i \leq n_i \), for \( i = 1, 2, 3 \). We match the lower long side of \( \mathcal{A}R_{m_i, n_i} \) to the upper long side of \( \mathcal{A}R_{m_{i+1}, n_{i+1}} \), for \( i = 1, 2 \), so that the first square on the lower long side of \( \mathcal{A}R_{m_1, n_1} \) lies immediately on the right of the \( (k+1) \)-th square on the upper long side of \( \mathcal{A}R_{m_2, n_2} \), and the first square on the upper long side of \( \mathcal{A}R_{m_3, n_3} \) stays right below the \( (k+1) \)-th square on the lower long side of \( \mathcal{A}R_{m_2, n_2} \). Similar to the case of double Aztec rectangles, we assume in addition that \( k \leq \min(m_2, n_2 - 1) \). Denote by \( TR_{m_1, n_1, k} \) the resulting region; and we call it a *triple Aztec rectangle* (see Figure 4). The portions of the triple Aztec rectangle corresponding to the Aztec rectangles \( \mathcal{A}R_{m_1, n_1} \), \( \mathcal{A}R_{m_2, n_2} \), and \( \mathcal{A}R_{m_3, n_3} \) are called the *upper*, *lower*, and *lower parts*, respectively.
Let \( x, y, t \) be positive integers, and \( z \) a non-negative integer so that \( 2t + 1 \geq 2y + z - 1 \geq t \) and \( x + t \leq 2y + z \). Define

\[
Q(x, y, z, t) := 2^{(2y+z)+(t+1)(t-2y-z+1)} \times \prod_{i=1}^{t+1}(i-1)! \prod_{i=y+z+1}^{t+1}(i-1)! \prod_{i=1}^{2t-2y-z+2}(x-i-2)!/(t-x+i+2)!,
\]

(3)

where the product \( \prod_{i=y+z+1}^{t+1}(i-1)! \) has to interpreted as 1 if \( t = y+z-1 \) and as 0 if \( t < y+z-1 \), and similarly for the other products. It has been shown that \( Q(x, y, z, t) \) gives the number of tilings of a certain Aztec rectangle with some squares removed (see Theorem 16 in [12]).

**Theorem 2.** Assume that \( m_i \) and \( n_i \) are positive integers, for \( i = 1, 2, 3 \), and that \( k \) is a non-negative integer so that \( m_i \leq n_i, n_1 = n_3, \) and \( k \leq \min(m_2, n_2 - 1) \).

(a) If \( (n_1 - m_1) + (n_3 - m_3) \neq n_2 - m_2 \), then \( M(TR_{m_1, n_1, k}) = 0 \).

(b) If \( (n_1 - m_1) + (n_3 - m_3) = n_2 - m_2 \), then

\[
M(TR_{m_2, n_2, m_3}) = 2^{\binom{m_1}{2} + \binom{m_2}{2} + \binom{m_3}{2} - \binom{m_1 + m_3 + 1}{2} - \binom{m_2 + m_3 + 1}{2}} \times Q(m_2 - k + 2, n_2 + m_3 + 1, m_1 - m_3, m_2 + n_1).
\]

(4)

This paper is organize as follows. We present the proofs of Theorems 1 and 2 in Section 2. Our proofs use several preliminary results in a series of papers written by the author ([14] and [15]) about hybrid domino-lozenge tilings. However, the proofs will be presented in certain way so that ones do not need to know the (long and complicated) definitions of regions in those papers, such as generalized Douglas regions, quasi-hexagonal regions, and quasi-octagonal regions. Section 3 investigates several new families of graphs inspired by the double and triple Aztec rectangles. In Section 4, we consider the generating function of tilings of a double Aztec rectangle with two statistics introduced in the Aztec diamond theorem (see [8](part II), pp. 224; or [11], Theorem 1).

## 2 Proofs of Theorems 1 and 2

A **perfect matching** of a graph \( G \) is a collection of edges so that each vertex in \( G \) is incident to precisely one edge in the collection. The tilings of a region \( R \) are in bijection with the perfect matchings of its dual graph (i.e., the graph whose vertices are fundamental regions of \( R \) and whose edges connect precisely two fundamental regions sharing an edge). In the view of this, we denote by \( M(G) \) the number of perfect matchings of the graph \( G \). The dual graph of the Aztec rectangle \( AR_{m,n} \) rotated by 45° is denoted by \( AR_{m,n} \) and called an Aztec rectangle graph. The baseless Aztec rectangle graph \( AR_{m - \frac{1}{2}, n} \) is the graph obtained from \( AR_{m,n} \) by removing its bottommost vertices.

The following result was proved in [14] (see Lemma 3.6).

**Lemma 3** (Graph Splitting Lemma). Let \( G \) be a bipartite graph, and \( V_1 \) and \( V_2 \) the two vertex classes. Let \( H \) be an induced subgraph of \( G \).

(a) Assume that \( H \) satisfies the following two conditions:

(i) The separating condition: there are no edges of \( G \) connecting a vertex in \( V(H) \cap V_1 \) and a vertex in \( V(G - H) \),
Figure 5: The transformations in Lemma 4. The white circles indicate the vertices $v_1, v_2, \ldots, v_n$.

(ii) The balancing condition: $|V(H) \cap V_1| = |V(H) \cap V_2|$.

Then

$$M(G) = M(H) M(G - H).$$

(5)

(b) If $H$ satisfies the separating condition and but $|V(H) \cap V_1| > |V(H) \cap V_2|$, then $M(G) = 0$.

The semihexagon $SH_{a,b}$ is the upper half of the lozenge hexagon $H_{b,a,a}$. We denote $SH_{a,b}$ the dual graph of the semihexagon.

The connected sum $G \# G'$ of two disjoint graphs $G$ and $G'$ along the ordered sets of vertices $\{v_1, \ldots, v_n\} \subset V(G)$ and $\{v'_1, \ldots, v'_n\} \subset V(G')$ is the graph obtained from $G$ and $G'$ by identifying vertices $v_i$ and $v'_i$, for $i = 1, \ldots, n$.

Lemma 4. Assume that $m$ and $n$ are two positive integers so that $m < n$. Let $G$ be a graph, and $\{v_1, v_2, \ldots, v_n\}$ an ordered set of its vertex. Then

(a) $$M(G \# AR_{m,n}) = 2^m M(G \# |AR_{m-1/2,n-1/2}),$$

where $|AR_{m-1,2,n-1}$ is the graph obtained by appending $n$ vertical edges to the bottommost vertices of the baseless Aztec rectangle $AR_{m-1/2,n-1}$ (see Figures 5(a) and (b)).

(b) $$M(G \# AR_{m,n}) = 2^{m(n+1)/2} M(G \# SH_{m,n-m}),$$

where $|SH_{m,n-m}$ is the graph obtained from the graph $SH_{m,n-m}$ by appending $n$ vertical edges to its bottommost vertices (see Figures 5(a) and (d)).

In both parts (a) and (b), the connected sum acts on $G$ along $\{v_1, v_2, \ldots, v_n\}$ and on other summands along their bottommost vertices ordered from left to right.

Proof. Part (a) was proved in [14] (see Lemma 3.5 (in reverse)). For part (b), we apply part (a) repeatedly as shown in Figures 5(a)–(d).

Proof of Theorem 1. Color the double Aztec rectangle black and white so that two squares sharing an edge have opposite colors. If the region admits a tiling, then the numbers of black and white squares must be the same. A region satisfying such condition is said balanced; and part (a) follows from the balance of the double Aztec rectangle.

Assume that $n_1 - m_1 = n_2 - m_2$. If $m_1 = n_1$ and $m_2 = n_2$, then we divide the dual graph of the region into two Aztec diamond graphs graph $AD_{m_1}$ and $AD_{m_2}$ (i.e. the dual graph of $AD_{m_1}$ and $AD_{m_2}$ rotated by $45^0$) as they appear from top to bottom by a dotted cut as in Figure 6. The graph $AD_{m_1}$ satisfies the conditions in Graph-splitting Lemma 3(a), so we have

$$M(DR_{m_1,n_1,k}^{m_2,n_2}) = M(AD_{m_1}) M(AD_{m_2}) = 2^{m_1(m_1+1)/2 + m_2(m_2+1)/2},$$

(8)
Figure 6: Illustrating the proof of Theorem 1 for the case $m_1 = n_1$ and $m_2 = n_2$.

Figure 7: Illustrating the proof of Theorem 1 for the case $m_1 < n_1$ and $m_2 < n_2$. 
and the theorem follows in this case.

Next, we prove the theorem for the case when \( n_1 - m_1 = n_2 - m_2 > 0 \). Divide the dual graph \( G \) of the double Aztec rectangle into three part by two dotted lines as in Figure 7. Apply the transformation in Lemma 4(b) separately to the upper and lower parts (see the portions above and below the dotted lines Figure 7, respectively) of \( G \). In particular, we replace upper part by a the graph \( |SH_{m_1,n_1-m_1} \); and the lower part is replaced by the graph \( |SH_{m_2,n_2-m_2} \) reflected about its base. Denote the resulting graph by \( \overline{G} \), we have

\[
M(\mathcal{DR}_{m_2,n_2}^{m_1,n_1,k}) = 2^{m_1(m_1+1)/2+m_2(m_2+1)/2} M(\overline{G}).
\]  

(9)

However, the dual graph of the lozenge hexagon \( H_{n_1-m_1,m_2-k+1,m_1+k} \) is obtained from \( \overline{G} \) by removing vertical forced edges (see bold edges in Figure 7(b)). By (9), we obtain

\[
M(\mathcal{DR}_{m_2,n_2}^{m_1,n_1,k}) = 2^{m_2(m_1+1)/2+m_2(m_2+1)/2} \prod_{i=1}^{n_1-m_1} \prod_{j=1}^{m_2-m_1-k+1} \prod_{t=1}^{m_1+k} \frac{i+j+t-1}{i+j+t-2},
\]  

(10)

which implies the theorem.

\[ \square \]

**Lemma 5.** Assume that \( m < n \) are two positive integers. Let \( G \) be a graph, and \( \{v_1, v_2, \ldots, v_{2n}\} \) an ordered set of its vertices. Then

(a) \[ M(G\#AR_{m,n}) = 2^m M(G\#K), \]

(11)

where \( K \) is the graph obtained from the Aztec rectangle graph \( AR_{m+1,n-1} \) by removing all topmost and bottommost vertices, and appending a vertical edge to each topmost and each bottommost vertex of the resulting graph (see Figures 8(a) and (b)).

(b) \[ M(G\#AR_{m,n}) = 2^{m(m+1)} M(G\#K'), \]

(12)

where \( K' \) is the “butterfly-shaped” graph obtained by taking connected sum \( |SH_{m,n-m} \) by itself along the topmost vertices ordered from left to right (see Figures 8(a) and (d)).

In both parts, the connected sum acts on \( G \) along \( \{v_1, v_2, \ldots, v_{2n}\} \) and on other summands along its topmost from left to right, and along its bottommost vertices from left to right.
Proof. Part (a) was proved in [15] (see Lemma 3.1(b)). Part (b) is obtained by applying part (a) \( m \) times as in Figure 8.

Proof of Theorem 2. Similar to Theorem 1, part (a) follows from the balance of the triple Aztec rectangle

Assume that \((n_1 - m_1) + (n_3 - m_3) = n_2 - m_2\). If \(n_2 = m_2\) (so \(n_1 = n_3 = m_1 = m_3\)), then we apply to dotted cuts as in Figure 9(a) to divide the dual graph of \(\mathcal{TR}_{m_1,n_1,k}^{m_2,n_2,m_3}\) into three Aztec diamond graphs \(AD_{m_1}, AD_{m_2}\) and \(AD_{m_3}\) as they appear from top to bottom. Then this case follows from the Graph-splitting Lemma 3(a).

If \(n_2 > m_2\) and \(n_1 = n_3 = \max(m_1,m_3)\), say \(n_1 = m_1\), we cut the dual graph as in Figure 9(b). The top part of the dual graph is still \(AD_{m_1}\). The lower part is now the dual graph of a double Aztec diamond \(\mathcal{DR}_{m_2,n_2}^{m_3,n_3,k}\) flipped about a horizontal line. Then by the Graph-splitting Lemma again

\[
M(\mathcal{TR}_{m_1,n_1,k}^{m_2,n_2,m_3}) = 2^{m_1(m_1+1)/2} M(\mathcal{DR}_{m_3,n_3,k}^{m_2,n_2}),
\]

then part(b) follows from Theorem 1.

Next, we assume that \(n_i > m_i\), for \(i = 1,2,3\). Divide the dual graph of the region into five parts by four dotted lines as in Figure 10(a). We apply separately the transformation in Lemma 4(b) to the top and bottom parts of the graph, and apply the transformation in Lemma 5(b) to the middle part. In particular, we replace the top and bottom parts by the graph \(\mathcal{SH}_{m_1,n_1-m_1}\) and the graph \(\mathcal{SH}_{m_3,n_3}\) reflected about its base; and the middle part by the butterfly-shaped graph which is isomorphic to \(\mathcal{SH}_{m_2,n_2-m_2}\# \mathcal{SH}_{m_2,n_2-m_2}\). Next, we remove vertical forced edges (indicated by bold edges in Figure 10(b)). This way, the dual graph of the region is transformed into a “double honeycomb graph” as in Lemma 3.2 in [15]. By this lemma, the latter graph

Figure 9: Illustrating the proof of Theorem 2.
in turn can be transformed into the Aztec rectangle graph with several vertices removed\textsuperscript{1} in Theorem 14 of [12]. Therefore, the theorem follows from Lemma 3.2 in [15] and Theorem 14 of [12].

3 Aztec rectangle graphs with holes on boundary.

We are interested in the number perfect matchings of an Aztec rectangle graph, where some vertices along its sides have been removed. We usually call the removed vertices holes. As mentioned in Section 1, the number of perfect matchings of an Aztec rectangle graph with holes on one side is given by a simple product formula.

**Theorem 6.** Assume $m$ and $n$ are two positive integers so that $m < n$. The number of perfect matchings of the Aztec rectangle graph $AR_{m,n}$, where all bottommost vertices, except for the $a_1$-st, the $a_2$-nd, \ldots, and the $a_m$-th, have been removed (see Figure 11(b) for example when $m = 3$, $n = 5$, $a_1 = 1$, $a_2 = 3$, and $a_3 = 5$), equals

$$2^{m(m+1)/2} \prod_{1 \leq i < j \leq n} \frac{a_j - a_i}{j - i}. \quad (14)$$

\textsuperscript{1}This graph was called \textit{holey Aztec rectangle} in [12]
Theorem 6 was essentially proved by Mills, Robbins and Rumsey (see Theorem 2 in [18]). We have a variant of the above result.

**Theorem 7.** (Lemma 2 in [10]) Assume $m$ and $n$ are two positive integers so that $m < n$. The number of perfect matchings of the baseless Aztec rectangle graph $AR_{m-\frac{1}{2},n}$, where the $a_1$-st, the $a_2$-nd, ..., and the $a_m$-th bottommost vertices have been removed (see Figure 11(c) for example when $m = 3$, $n = 5$, $a_1 = 3$, $a_2 = 4$, and $a_3 = 5$), equals

\[
2^{m(m-1)/2} \prod_{1 \leq i < j \leq n} \frac{a_j - a_i}{j - i}.
\]

We notice that the product $\prod_{1 \leq i < j \leq n} \frac{a_j - a_i}{j - i}$ in Theorems 6 and 7 is exactly the number of lozenge tilings of the semihexagon $SH_{m,n-m}$, where the unit triangles at positions $a_1, a_2, \ldots, a_m$ have been removed from the base (see Proposition 2.1 in [7] or Lemma 2 in [10]).

Next, we consider two new families of Aztec rectangle graphs with holes.

Consider the Aztec rectangle $AR_{m,a+b}$, where the first, the second, ..., and the $a$-th topmost and bottommost vertices, and all the leftmost vertices have been removed. Denote by $S_{a,b}$ the resulting graph (see Figure 12(b) for $S_{3,3}$; the dotted line should be ignored at this point).

Now we remove the $(a+1)$-st, the $(a+2)$-nd, ..., and the $(a+b)$-th topmost and bottommost vertices from the the Aztec rectangle $AR_{a-b+c,a+b+c}$. Denote by $Q_{a,b,c}$ the resulting graph (see Figure 12(b) for $Q_{2,2,2}$; the dotted lines should be ignored at this stage).

Denote by $\mathbb{H}(n) := 0!1!2! \ldots (n-1)!$ ($\mathbb{H}(0) := 1$) the hyperfactorial function. The numbers of perfect matchings of the above new graphs can be written in terms of $\mathbb{H}(n)$ as follows.

**Theorem 8.** Assume that $m, a, b$ are three positive integers. Then

(a) $M(S_{a,b}^m) = 0$ unless $a = b \leq m$.
(b) If $a = b \leq m$, then

\[
M(S_{a,a}^m) = 2^a \mathbb{H}(a)^2 \mathbb{H}(m-a) \mathbb{H}(m+a) \mathbb{H}(m)^2 \mathbb{H}(2a).
\]

Proof. If $a$ or $b$ greater than $m$, say $a > m$, then $M(S_{a,b}^m) = 0$ by applying Graph-splitting Lemma 3(b) to the induced subgraph on the left the dotted line in Figure 12(b). For $a, b \leq m$ the graph $S_{a,b}^m$ is isomorphic to the dual graph of the double Aztec rectangle $DR_{b-1,m-1,1}$, and the theorem follows from Theorem 1.

\[\square\]
Theorem 9. Assume that $a, b, c$ are four positive integers, so that $m = a - b + c > 0$. Then

(a) If $m < \max(a, b, c)$, then $\text{M}(Q_{a,b,c}) = 0$.

(b) If $m \geq \max(a, b, c)$, then

$$\text{M}(Q_{a,b,c}) = 2^{\binom{a-b+c+1}{2}+b^2} \frac{H(a+c)^2 H(c-b) H(a-b) H(b)^2}{H(a+b) H(b+c) H(a-b+c)^2}. \quad (17)$$

Proof. Assume that one of $a, b, c$ greater then $m$, say $b < 0$. We apply the Graph-splitting Lemma 3(b) to the induced subgraph between two dotted lines in Figure 12(c) and get implies that $\text{M}(Q_{a,b,c}) = 0$.

For $m \geq \max(a, b, c)$ the graph $Q_{a,b,c}$ is isomorphic to the dual graph of the triple Aztec rectangle $\mathcal{T} R_{a,m-1,n-1}^b$; and the theorem follows from Theorem 2.

We are also interested in the following families of Aztec rectangle graphs with holes, which are inspired by $S_{a,b}^m$ and $Q_{a,b,c}$.

Consider the Aztec rectangle graph $AR_{m,2n}$. We first remove the first, the second, \ldots, and the $n$-th topmost vertices, and all the leftmost vertices. Label the bottommost vertices of the resulting graph by $1, 2, \ldots, 2n$. Remove all the even vertices, we obtain the graph denoted by $K^e_{m,n}$ (see Figure 13(a) for an example). Repeat the process, with the difference that we now remove the odd vertices, and obtain a new graph denoted by $K^o_{m,n}$ (see Figure 13(b)).

Next, we consider a variation of the above $K$-graphs. We apply similarly the above vertex-removing process to the baseless Aztec rectangle graph $AR_{m-1,2n}$ (instead of the Aztec rectangle graph $AR_{m,2n}$). Denote by $K^e_{m,n}$ and $K^o_{m,n}$ the new Aztec rectangle graphs with holes corresponding to $K^e_{m,n}$ and $K^o_{m,n}$, respectively (illustrated in Figure 14).

The perfect matchings the above families of Aztec rectangle graphs with holes are enumerated in the following theorem.
Theorem 10. Assume that $m, n$ are positive integers. (a) For $2m \geq n$ and $k = \left\lfloor \frac{2m-n}{2} \right\rfloor$

$$M(K_{m,n+1}^e) = M(K_{m,n}^o) = 2^{m(2m-1)-2k(n+k)}$$

$$\times \prod_{i=1}^{k} \frac{(2m - n - 2i)!(2n + 4i - 2)!(2n + 4i - 1)!}{(n + 2i - 1)!^2(2m + n + 2i - 1)!}. \quad (18)$$

(b) For $2m - 1 \geq n$ and $k = \left\lfloor \frac{2m-1-n}{2} \right\rfloor$

$$M(K_{m,n+1}^o) = M(K_{m,n}^e) = 2^{(m-1)(2m-1)-2k(n+k)}$$

$$\times \prod_{i=1}^{k} \frac{(2m - 1 - n - 2i)!(2n + 4i - 2)!(2n + 4i - 1)!}{(n + 2i - 1)!^2(2m + n + 2i - 2)!}. \quad (19)$$

Next, we consider the Aztec rectangle $AR_{m,a+b+c}$, where $a - b + c = 2m \geq \max(a, b, c)$ (this implies that $b \leq \min(a, c)$). We remove the $(a+1)$-st, the $(a+2)$-nd, \ldots, and the $(a+b)$-th topmost vertices from the graph. Removing also all the odd or all the even bottommost vertices of the resulting graph, we get two new Aztec rectangle graphs with holes, which we denote by $L_{a,b,c}^o$ and $L_{a,b,c}^e$, respectively (illustrated in Figure 15).

Consider a variant of the above L-type graphs by doing similarly with the baseless Aztec rectangle graph $AR_{m-\frac{1}{2},a+b+c}$, where $a - b + c = 2m - 1 \geq \max(a, b, c)$ (instead of $AR_{m,a+b+c}$). We obtain two new graphs, denoted by $\overline{L}_{a,b,c}^o$ and $\overline{L}_{a,b,c}^e$ (shown in Figure 16).

Theorem 11. Assume that $a, b, c, m$ are positive integers so that the $L$ and $\overline{L}$-graphs are well defined. (a) If $a - b = 2k$, for some $k \geq 0$, then

$$M(L_{a-1,b,c+1}^e) = M(L_{a,b,c}^o)$$

$$= 2^{n^2} M(K_{m,b}^e) \prod_{i=1}^{k} \frac{(a + 2i - b)!(c + 2i - 1 + b)!}{(c + 2i - b - 1)!(a - 2i + b)!}. \quad (20)$$
and

\[ M(\mathcal{T}_{a-1,b,c+1}^o) = M(\mathcal{T}_{a,b,c}^e) \]

\[ = 2^{m(m-1)} M(\mathcal{K}_{m,b}^o) \prod_{i=1}^{k} \frac{(a + 2i - b)!(c + 2i - 1 + b)!}{(c + 2i - b - 1)!(a - 2i + b)!} \]  

\[ (21) \]

(b) If \( a - b = 2k + 1 \), for some \( k \geq 1 \), then

\[ M(L_{a-1,b,c+1}^e) = M(L_{a,b,c}^o) \]

\[ = 2^{m^2} M(\mathcal{K}_{m,b}^o) \prod_{i=1}^{k} \frac{(a + 2i - b)!(c + 2i - 1 + b)!}{(c + 2i - b - 1)!(a - 2i + b)!} \]  

\[ (22) \]

and

\[ M(\mathcal{T}_{a-1,b,c+1}^o) = M(\mathcal{T}_{a,b,c}^e) \]

\[ = 2^{m(m-1)} M(\mathcal{K}_{m,b}^e) \prod_{i=1}^{k} \frac{(a + 2i - b)!(c + 2i - 1 + b)!}{(c + 2i - b - 1)!(a - 2i + b)!} \]  

\[ (23) \]

Then we obtain the explicit formulas for the numbers of perfect matchings of the \( L^- \) and \( \overline{L} \)-type graphs from Theorem 10.

We also enumerated in [8] the tilings of a “half” of an Aztec diamond defined as follows. Consider a zigzag cut passing the center of an Aztec diamond as in Figure 17. The cut divides the region into two isomorphic parts called halved Aztec diamonds. It has been shown that the number of tilings of halved Aztec diamonds is always given by a power of 2. In particular, denoting by \( SD_n \) the halved Aztec diamond of order \( n \), we have

\[ M(SD_n) = 2^{n^2/4}, \text{ for even } n \]  

\[ (24) \]

and

\[ M(SD_n) = 2^{(n^2-1)/4}, \text{ for odd } n. \]  

\[ (25) \]
Next, we state a factorization theorem due to Ciucu [4].

Given a reflectively symmetric weighted graph $G$ that is also bipartite. Without loss of generality, we always assume that the symmetry axis $\ell$ of $G$ is horizontal. Assume that the vertices of $G$ lying on $\ell$ form a cut set. Color the vertices in two vertex classes of the bipartite graph $G$ black and white, so that the leftmost vertex of $G$ lying on $\ell$ is black. We define two subgraphs $G^+$ and $G^-$ of $G$ as follows.

It is easy to see that if $G$ has perfect matchings, then $G$ must have an even number of vertices on $\ell$. Therefore, we assume that the number of vertices of $G$ lying on $\ell$ is $2w(G)$, for some integer $w(G)$ ($w(G)$ was called the width of $G$ in [4]). Assume $a_1, b_1, a_2, b_2, \ldots, a_{w(G)}, b_{w(G)}$ are the vertices of $G$ lying on $\ell$ as they occur from left to right. Going along the line $\ell$ from left to right, we delete the edges of $G$ that touch white $a_i$ ’s and black $b_i$’s from above, and delete the edges of $G$ that touch black $a_i$’s and white $b_i$’s from below (see Figure 18 for an example).

Next, we reduce the weights of the edges of $G$ lying on $\ell$ by half, and leave the weights of other edges unchanged. Since the vertices of $G$ on $\ell$ form a cut set, the resulting graph is disconnected into two parts. Denote by $G^+$ and $G^-$ the part above and below $\ell$, respectively.

**Theorem 12 (Factorization Theorem, Ciucu [4]).** Let $G$ be a bipartite weighted symmetric graph separated by its symmetry axis. Then

$$M(G) = 2^{w(G)} M(G^+) M(G^-).$$  \hfill (26)

We are now ready to prove Theorems 10 and 11.

**Proof of Theorem 10.** If $2m = n$, then we cut the graph $K_{m,2m}^o$ into two parts, which are isomorphic to the dual graph of some halved Aztec diamonds (see the graph restricted by bold contours in Figure 19; the cut is illustrated by the dotted line; and the white circles indicate the hole on the bottom of the graph). By Graph-splitting Lemma 3(a), we have

$$M(K_{m,2m}^o) = M(SD_{2m}) M(SD_{2m-1}) = 2^{m^2} 2^{m(m-1)} = 2^{m(2m-1)}.$$  \hfill (27)
Removing forced edges in $K^e_{m,n}$, we get the graph $K^o_{m,n-1}$ (illustrated by the bold edges in Figure 20). Then thus

$$M(K^e_{m,n+1}) = M(K^o_{m,n}).$$  \hfill (28)

Next, we apply the edge-trimming process in Factorization Theorem to the graph $G := \mathcal{S}^{2m}_{n+1,n+1}$. We get $G^+$ is isomorphic to $K^e_{m,n+1}$, and $G^-$ is isomorphic to $K^o_{m,n+1}$ (see Figure 20). Thus

$$M(\mathcal{S}^{2m}_{n+1,n+1}) = 2^{n+1} M(K^o_{m,n+1}) M(K^e_{m,n+1})$$

$$= 2^{n+1} M(K^o_{m,n+1}) M(K^o_{m,n}).$$  \hfill (29)

In particular, by letting $n + 1 = 2m$ in (29), we deduce the number of perfect matchings of $K^o_{m,2m-1}$ as

$$M(K^o_{m,2m-1}) = \frac{M(\mathcal{S}^{2m}_{2m,2m})}{2^{2m} M(K^o_{m,2m})} = 2^{m(2m-1)}$$  \hfill (30)

(so $M(K^o_{m,2m-1}) = M(K^o_{m,2m})$).

The equality (29) also implies the following recurrence

$$\frac{M(K^o_{m,n})}{M(K^o_{m,n+2})} = 2 \frac{M(\mathcal{S}^{2m}_{n+1,n+1})}{M(\mathcal{S}^{2m}_{n+2,n+2})}. \quad \hfill (31)$$
for any \( n \leq 2m - 2 \). Therefore, repeated application of the recurrence (31) implies that

\[
\frac{M(K^o_{m,n})}{M(K^o_{m,2m})} = 2^k \prod_{i=1}^{k} \frac{M(S^2_{n+2i-1,n+2i-1})}{M(S^2_{n+2i,n+2i})} = 2^{-2k(n+k)} \frac{(2m - n - 2i!)(2n + 4i - 2)!(2m + 4i - 1)!}{(n + 2i - 1)!^2(2n + n + 2i - 1)!},
\]

(32)

where \( n + 2k = 2m \).

By (27), we obtain the number of perfect matchings of \( K^o_{m,n} \), for \( n + 2k = 2m \), as

\[
M(K^o_{m,n}) = 2^{m(2m-1)-2k(n+k)} \prod_{i=1}^{k} \frac{(2m - n - 2i)!}{(n + 2i - 1)!^2(2m + n + 2i - 1)!}.
\]

(33)

Similarly, for \( n = 2m - 2k - 1 \), the equalities (29) and (30) imply that

\[
M(K^o_{m,n}) = 2^{m(2m-1)-2k(n+k)} \prod_{i=1}^{k} \frac{(2m - n - 2i)!}{(n + 2i - 1)!^2(2m + n + 2i - 1)!}.
\]

(34)

Then part (a) follows from the equalities (28), (27) and (30).

Part (b) can be proved by a perfectly analogous manner.

\[\square\]

**Proof of Theorem 11.** We only prove (20) and (22); as (21) and (23) can be obtained similarly.

Assume that \( 2m = a - b + c \). Without loss of generality, we can assume that \( c \geq a \). By our assumption, we get \( c = \max(a, b, c) \).
Similar to Theorem 10, the Graph-splitting Lemma deduces

\[ M(L_{b,b,2m}^0) = M(SD_{2m}) M(K_{m,b}^e) = 2^{m^2} M(K_{m,b}^e) \]  

(see Figure 21), which in turn implies our statement for the case \( c = 2m \). Therefore, we can assume that \( 2m > c = \max(a, b, c) \).

By considering forced edges, one readily sees that

\[ M(L_{a,b,c}^0) = M(L_{a-1,b,c+1}^e). \]  

(36)

Applying Factorization Theorem to \( Q_{a-1,b,c+1} \), we obtain

\[ M(Q_{a-1,b,c+1}) = 2^{(a+b+c)/2} M(L_{a-1,b,c+1}^e) M(L_{a-1,b,c+1}^o) \]  

(37)

(see Figure 22).

In particular, by letting \( a = b + 1 \) and \( c = 2m - 1 \) in (37), we can find the number of perfect matching of \( L_{b+1,b,2m-1}^o \) as

\[ M(L_{b+1,b,2m-1}^o) = \frac{M(Q_{b,b,2m})}{2^{m+b} M(L_{b,b,2m}^o)} \]  

\[ = \frac{2^{m(2m+1)} M(S_{b,b}^{2m})}{2^{m+b+m^2} M(K_{m,b}^e)} \]  

\[ = \frac{2^{m(2m+1)}2^b M(K_{m,b}^e) M(K_{m,b}^o)}{2^{m+b+m^2} M(K_{m,b}^e)} \]  

\[ = 2^{m^2} M(K_{m,b}^o), \]  

(38)

where the second and third equality signs follow from (13) and (29), respectively.

The equality (37) also deduces the following recurrence

\[ \frac{M(L_{a,b,c}^o)}{M(L_{a-2,b,c+2}^o)} = \frac{M(TQ_{a-1,b,c+1})}{M(Q_{a-2,b,c+2})}, \]  

(42)

where \( c + 2 \leq 2m \); and its repeated application deduces

\[ \frac{M(L_{a,b,c}^o)}{M(L_{b,b,2m}^o)} = \prod_{i=1}^{k} \frac{M(Q_{a-2i+1,b,c+2i-1})}{M(Q_{a-2i,b,c+2i})} \]  

\[ = \prod_{i=1}^{k} \frac{(a + 2i - b)!(c + 2i - 1 + b)!}{(c + 2i - b - 1)!((a - 2i + b))!}, \]  

(43)

where \( a - b = 2k \). We now use the equality (35), to imply the number of perfect matchings of \( L_{a,b,c}^o \) as

\[ M(L_{a,b,c}^o) = 2^{m^2} M(K_{b,m}^e) \prod_{i=1}^{k} \frac{(a + 2i - b)!(c + 2i - 1 + b)!}{(c + 2i - b - 1)!((a - 2i + b))!} \]  

(45)
Figure 23: The elementary moves: rotation of a 2×2 block of two vertical or horizontal dominoes.

Then (36) implies (20).

Similarly, if \( a - b = 2k + 1 \), we also get

\[
\frac{M(L\alpha_{a,b,c})}{M(L\alpha_{b+1,b,2m-1})} = \prod_{i=1}^{k} \frac{M(Qa-2i+1,b,c+2i-1)}{M(Qa-2i,b,c+2i)}
\]

(46)

\[
= \prod_{i=1}^{k} \frac{(a+2i-b)!(c+2i-1+b)!}{(c+2i-b-1)!(a-2i+b)!},
\]

(47)

and \((22)\) follows from \((36)\) and \((38)\).

4 Weighted double Aztec rectangle

As mentioned before, the lozenge tilings of the hexagon \( H_{a,b,c} \) are in bijection with the plane partitions fit in an \( a \times b \times c \) box (see Figure 33(a)). MacMahon [17] obtained also the following weighted version of the formula introduced in Section 1:

\[
\sum_{\mu} q^{\mu} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{t=1}^{c} \frac{1 - q^{i+j+t-1}}{1 - q^{i+j+t-2}},
\]

(48)

where the sum on left hand side is taken over all the plane partitions \( \mu \) fit in an \( a \times b \times c \) box, and where \( |\mu| \) is the volume of \( \mu \)–the number of unit cubes in \( \mu \). One can see that it we take the sum of all labels of white rhombi in Figure 33(a), we get the volume of the corresponding plane partitions.

Next, we introduce also a weighted version of the Aztec diamond theorem. Given a tiling \( T \) of the Aztec diamond \( \mathcal{AD}_n \). We denote by \( v(T) \) haft number of vertical dominoes in \( T \), and \( r(T) \) the rank of \( T \), which is defined as follows. The tiling \( T_0 \) consisting of all horizontal dominoes has rank 0; and the rank \( r(T) \) of \( T \) is the minimal number of elementary moves required to reach \( T \) from \( T_0 \) (see Figure 23 for two types of the elementary moves). We call \( T_0 \) the minimal tiling of the region.

The authors of [8] actually proved several different versions of the Aztec diamond theorem including the following Theorem.

**Theorem 13** (Aztec Diamond Theorem with two statistics).

\[
\sum_{T} t^{v(T)} q^{r(T)} = \prod_{k=0}^{n-1} (1 + t q^{2k+1})^{n-k},
\]

(49)

where the sum is taken over all tilings \( T \) of the Aztec diamond of order \( n \).

The goal of this section is to investigate a similar generating function of the domino tilings of a double Aztec rectangle.
In general, a double Aztec rectangle does not admit the tiling consisting of all horizontal dominoes. Thus, we need to find a suitable “nominee” for the role of minimal tiling of the double Aztec rectangle. We consider the tiling of the double Aztec rectangle \( DR_{m_1,n_1,k}^{m_2,n_2} \) as shown in Figure 24(b). In particular, we divide the region into 3 disjoint parts as follows. The first part is an Aztec diamond of order \( m_1 \) on the top of the region; the second part is an Aztec rectangle \( AR_{m_1+k,n_1-m_1} \), where the top and bottom are shrunk to size 1; and the third part is the remaining (see Figure 24(a)). We cover the first and the second parts by vertical dominoes, and the third part by horizontal dominoes (see Figure 24(b)). We define the resulting tiling to be the minimal tiling of the region, and still use the notation \( T_0 \) for it. Similar to the case of Aztec diamonds, we now define the rank \( r(T) \) of a tiling \( T \) to be the minimum number of elementary moves to obtain \( T \) from \( T_0 \), and set \( r(T_0) := 0 \). We also use the notation \( v(T) \) for half of the number of vertical dominoes in \( T \).

**Theorem 14.** Assume that \( m_1, m_2, n_1, n_2 \) are positive integers, and \( k \) is a non-negative integer so that \( m_1 \leq n_1, m_2 \leq n_2, k \leq \min(m_2, n_2 - 1) \), and \( n_1 - m_1 = m_2 - n_2 \). Then

\[
\sum_T r(T)q^v(T) = \ell\left(\frac{m_1+1}{2}\right)+\left(\frac{m_2+1}{2}\right)+(n_1-m_1)(m_1+k)/2qN+(n_1-m_1)(m_1+k)+A
\]

\[\times \prod_{i=0}^{m_1-1}(\star_i)^{m_1-i} \prod_{i=0}^{m_2-1}(\star_i)^{m_2-i} \prod_{i=1}^{n_1-m_1} \prod_{j=1}^{m_2-k+1} \prod_{t=1}^{m_1+k} \left(1 - \frac{q^{2(i+j+t-1)}}{1 - q^{2(i+j+t-2)}}\right), \tag{50}\]

where the sum is taken over all tilings \( T \) of the double Aztec rectangle \( DR_{m_1,n_1,k}^{m_2,n_2} \), where

\[N = (m_1(m_1+1)(n_1-1) - m_2(m_2+1)(n_2-1))\]

\[+ (n_1-m_1)(2m_2^2 + m_2m_1 + m_2n_1 + k^2 + 2km_1 + m_1n_1 + k - m_2),\]

and

\[A = \frac{2m_2(m_2-1)(m_2+1)}{3} + (m_2-k+1)(m_2+n_2-1)(n_1-m_1) + \frac{m_1(m_1+1)(2k + 2m_1 + 2n_2 - 1)}{2},\]

and where \( \star_i = q^{2m_2+2n_2-3}(1 + t^{-1}q^{-2i-1}) \) and \( \star_i' = q^{2m_2+2k+1}(1 + t^{-1}q^{2i+1}) \).

In the next three paragraphs, we show that the minimal tiling \( T_0 \) of a double Aztec rectangle is indeed the smallest one in some sense.

Label the centers of vertical steps on the lower south-western boundary of the region by \( u_1, u_2, \ldots, u_{m_2} \), and the centers of vertical steps on the upper north-western boundary by \( u_{m_2+1}, \ldots, u_{m_2+n_1} \). Next, we label the centers of vertical steps on the lower south-eastern boundary by \( v_1, v_2, \ldots, v_{n_2} \), and the centers of vertical steps on the upper north-eastern boundary by \( v_{n_2+1}, v_{n_2+2}, \ldots, v_{n_2+m_1} \) (see Figure 24).

In this section, we only consider the double Aztec rectangles \( DR_{m_1,n_1,k}^{m_2,n_2} \), which satisfy the balancing condition \( n_1 - m_1 = n_2 - m_2 \). We color the region black and white so that two squares sharing an edge have opposite colors, and the squares on the lower south-western boundaries are black. Given a tiling \( T \) of the double Aztec rectangle, we decorate the dominoes as in Figure 25. Then the tiling \( T \) corresponds to a family of non-intersecting Schröder paths \( P = (P_1, P_2, \ldots, P_{n_2}) \), where \( P_i \) connects \( u_i \) and \( v_i \) (see Figure 26).

As shown in Figure 27, each elementary move rising the rank of the tiling \( T \) by one gives a deformation of some path in \( P \) increasing the underneath area by one (here, the “ground” is
Figure 24: The minimal tiling of the double Aztec rectangle $\mathcal{DR}_{3,6,2}$.  

Figure 25: Decorating the dominoes by steps of Schröder paths.

Figure 26: The correspondence between tilings and families of non-intersecting Schröder paths.

Figure 27: An elementary move raises the rank of the tiling $T$ by one (left-to-right, respectively) if and only if the corresponding family $P$ of non-intersecting Schröder paths increases the underneath area by one.
Next, we assign weights to the dominoes of a double Aztec rectangle as follows. Assume that $a, b, c, d, q$ are five positive numbers. We assign each odd horizontal domino (see Figure 25 for four types of dominoes) a weight $b$, each even vertical domino a weight $a$, each even horizontal domino on the $k$ level a weight $cq^{k-1}$ (the bottom of the region is on the level 0), and each odd vertical domino a weight $dq^k$. Denote by $\text{wt} := \text{wt}_{a,b}(q)$ the weight assignment.

In the weighted case, we use the notation $M(R)$ for generating function of the tilings of $R$ (i.e., the sum of weights of all tilings, where the weight a tiling is the product of weights of its tiles). The notation $M(G)$ is used for the generating function of the perfect matchings of the graph $G$ in the same fashion.

We notice that, the each edge of the dual graph is assigned the weight of its corresponding tile. Similar to the un-weighted case, the generating function of the tilings of the region is the same as that of the perfect matchings of its dual graph.

If an Aztec rectangle $\mathcal{AR}_{m,n}$ is assigned the weights as above, then we denote by $\mathcal{AR}_{m,n}(a, b, c, d, q)$ the corresponding weighted version of its dual graph (see Figure 29(a) for an example). In this section, we always present the graph $\mathcal{AR}_{m,n}$ so that the edge on its lower-left corner corresponds to the bottom horizontal domino of the region $\mathcal{AR}_{m,n}$.

Before investigating the main theorem we introduce several simple transformations as follows.

**Lemma 15** (Vertex-Splitting Lemma). Let $G$ be a graph, $v$ be a vertex of it, and denote the set of neighbors of $v$ by $N(v)$. For any disjoint union $N(v) = H \cup K$, let $G'$ be the graph obtained from $G \setminus v$ by including three new vertices $v', v''$ and $x$ so that $N(v') = H \cup \{x\}$, $N(v'') = K \cup \{x\}$, and $N(x) = \{v', v''\}$. Then $M(G) = M(G')$.

**Lemma 16** (Star Lemma). Let $G$ be a weighted graph, and let $v$ be a vertex of $G$. Let $G'$ be the graph obtained from $G$ by multiplying the weights of all edges incident to $v$ by $t > 0$. Then $M(G') = t M(G)$.

The following result is a generalization (due to Propp) of the “urban renewal” trick first observed by Kuperberg.

**Lemma 17** (Spider Lemma). Let $G$ be a weighted graph containing the subgraph $K$ shown on the left in Figure 28 (the labels indicate weights, unlabeled edges have weight 1). Suppose in
Figure 29: The transformation in Lemma 18.

addition that the four inner black vertices in the subgraph $K$, different from $A, B, C, D$, have no neighbors outside $K$. Let $G'$ be the graph obtained from $G$ by replacing $K$ by the graph $K$ shown on right in Figure 28, where the dashed lines indicate new edges, weighted as shown. Then $M(G) = (xz + yt) M(G')$.

We have a weighted version of Lemma 4(a):

**Lemma 18.** Let $G$ be a weighted graph, and $\{v_1, v_2, \ldots, v_n\}$ an ordered set of its vertices. Then

$$M(G \# AR_{m,n}(a, b, c, d, q)) = (ad + bc)^m q^{m(n-1)+\binom{m}{2}} M(G \# |AR_{m-\frac{1}{2},n-\frac{1}{2}}(a/q, b, c, d)), \quad (51)$$

where $|AR_{m-\frac{1}{2},n-\frac{1}{2}}(a/q, b, c, d, q)$ is obtained from the graph $AR_{m,n-1}(a/q, b, c, d, q)$ by removing the bottommost vertices, and appending vertical edges to the bottom of the resulting graph; and where the connected sum acts on $G$ along the ordered set $\{v_1, v_2, \ldots, v_n\}$, and on $AR_{m,n}(a, b, c, d, q)$ and $|AR_{m-\frac{1}{2},n-\frac{1}{2}}(a/q, b, c, d)$ along their bottommost vertices ordered from left to right (see Figure 29).

**Proof.** The proof is illustrated in the Figure 30, for $m = 3$ and $n = 4$. First, we apply Vertex-splitting Lemma to all vertices of $AR_{m,n}(a, b, c, d, q)$ as in Figures 30(a) and (b). Apply the Spider Lemma around $mn$ shaded cells, and remove all edges incident to a vertex of degree 1, which is forced edges (see Figure 30(b)). This way, $G \# AR_{m,n}(a, b, c, d, q)$ is transformed into $G \# |AR_{m-\frac{1}{2},n-\frac{1}{2}}$, where $|AR_{m-\frac{1}{2},n-\frac{1}{2}}$ is the weighted version of the graph $AR_{m-\frac{1}{2},n-\frac{1}{2}}$ illustrated in Figure 30(c). Next, we divide the graph $|AR_{m-\frac{1}{2},n-\frac{1}{2}}$, except for vertical edges, into $m(n-1)$ subgraphs restricted in dotted squares as in Figure 30(c). Apply Star Lemma with factor $t = \Delta q^{i+j-2}$ to the center vertex of the dotted square in row $i$ (from bottom to top) and column $j$ (from left to right). Finally, we get the graph $G \# |AR_{m-\frac{1}{2},n-\frac{1}{2}}(a/q, b, c, d)$. By Spider, Graph
Figure 30: Illustrating the proof of Lemma 18, where $\Delta := ad + bc$.

Splitting, and Star Lemmas, we get

\[
M(G\#AR_{m,n}(a, b, c, d, q)) = \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n} \Delta q^{i+j-2} \cdot M(G\#AR_{m-\frac{1}{2},n-\frac{1}{2}}) \quad (52)
\]

\[
= \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n} \Delta q^{i+j-2} \quad (53)
\]

\[
\times \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n-1} (\Delta q^{i+j-2})^{-1} \cdot M(G\#AR_{m-\frac{1}{2},n-\frac{1}{2}}(a/q, b, c, d)), \quad (54)
\]

which implies the lemma.

Consider the double Aztec rectangle $\mathcal{D}R_{m_1,n_1,k}^{m_2,n_2}$, where the weight assignment $wt := wt_{e,d}^{a,b}(q)$ is applied to its dominoes. To specify, we denote by $\mathcal{D}R_{m_1,n_1,k}^{m_2,n_2}(a, b, c, d, q)$ the corresponding weighted double Aztec rectangle. The generating function of its tilings is given by the theorem stated below.

**Theorem 19.** Assume that $a, b, c, d, q$ are positive numbers. Assume in addition that $m_1, m_2, n_1, n_2, k$ are positive integers so that $m_1 \leq n_1$, $m_2 \leq n_2$, $k \leq \min(m_2, n_2 - 1)$, and $n_1 - m_1 = n_2 - m_2$. 

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Figure 31: Illustrating the proof of Theorem 14. The shaded diamond with label $x$ have edge-weight $a, b, dq^x, cq^x$ (in cyclic order, start from the northwest side); and the baseless triangle with label $y$ have the left and right edge-weights $cq^y$ and $dq^y$, respectively.

Figure 32: Three types of rhombi.

Then

$$M \left( DR_{m_2, n_2}^{m_1, n_1, k} (a, b, c, d, q) \right) = c^{(m_2-k+1)(n_1-m_1)}d^{(m_1-k)(n_2-m_1)} \prod_{i=0}^{m_1-1} (\Delta_i')^{m_1-i} \prod_{i=0}^{m_2-1} (\Delta_i)^{m_2-i} q^{N/2} \times \prod_{i=1}^{n_1-m_1} \prod_{j=1}^{m_2-k+1} \prod_{t=1}^{m_1+k} \frac{1 - q^{i+j+t-1}}{1 - q^{i+j+t-2}},$$

where $N$ is defined as in Theorem 14, and where $\Delta_i' = q^{k+m_2+i}(adq^{-i} + bcq)$ and $\Delta_i = q^{m_2+n_2-2-i}(adq^i + bc)$.

Proof. Consider the dual graph $G$ of $DR_{m_1, n_1, k}^{m_2, n_2} (a, b, c, d, q)$. Divide the graph $G$ into three parts by two dotted horizontal lines as in Figure 31 (a).

Apply the transformation in Lemma 18 to the top part of $G$, which is isomorphic to $AR_{m_1, n_1} (a, b, cq^{k+m_2+1}, dq^{k+m_2}, q)$. In particular, we replace this portion of $G$ by the graph $|AR_{m_1-\frac{1}{2}, n_1-\frac{1}{2}} (a/q, b, cq^{k+m_2+1}, dq^{k+m_2}, q)$. Keep applying this process as in the proof of Lemma 4(b) (see Figure 5). This way we transform the upper part into a dual graph of a weighted semi-hexagon (see Figure 31).

Similarly, we transform the bottom part of $G$, which is isomorphic to the weighted Aztec rectangle graph $AR_{m_2, n_2} (a, b, cq^{m_2+n_2-2}, dq^{m_2+n_2-2}, q^{-1})$ rotated by $180^0$, into the dual graph of some weighted semi-hexagon rotated by $180^0$. Finally, we remove forced vertical edges in the resulting graph (as in the proof of Theorem 1). This way, the dual graph $G$ of the region is
transformed in to the dual graph of a weighted lozenge hexagon $H := H_{n_1-m_1,m_2-k+1,m_1+k}$ (see Figure 31(b)). In particular, the rhombi in $H$ are weighted as follows. The vertical rhombi are weighted by 1. The left rhombi, which have its left side $i\sqrt{3}/2$ units on the left of the south-west side of the hexagon, are weighted by $cd^{m_2+i}$. The right rhombi, which have its lower-left corner $i\sqrt{3}/2$ units on the left of the south-west side of the hexagon, are weighted by $dq^{m_2+i}$ (see illustration in Figure 33(c); each left rhombus with label $x$ is weighted by $cq^x$; and each right rhombus with label $y$ is weighted by $dq^y$). Moreover, by Lemma 18, we have

$$M(G) = \prod_{i=0}^{m_1-1} (\Delta'_i)^{m_1-i} q^{(m_1-i)(n_1-i-1)+\binom{m_1-i}{2}} \prod_{i=0}^{m_2-1} (\Delta_i)^{m_2-i} q^{-(m_2-i)(n_2-i-1)+\binom{m_2-i}{2}} \cdot M(H),$$

where $\Delta'_i = q^{k+m_2+i}(adq^{-i} + bcq)$ and $\Delta_i = q^{m_2+n_2-2-i}(adq^{-i} + bc)$.

Now each lozenge tiling $T$ of the lozenge hexagon $H$ corresponds to family of $n_1-m_1$ disjoint paths of left and right rhombi $P = (P_1, P_2, \ldots, P_{n-m})$ (see Figure 32 for three orientations of a unit rhombus), where $P_i$ starts from the rhombi containing the $i$-th triangle on the north side and ends at the rhombi containing the $i$-th triangle on the south side (see Figure 33(b)).

We notice that all rhombi, which are not on the paths $P_i$'s, are vertical and have weight 1. Thus, the weight of the tiling $T$ equals $wt(P) = \prod_{i=1}^{n_1-m_1} wt(P_i)$, where $wt(P_i)$ is the product of weights of its rhombi.

Each path $P_i$ has $m_2 - k$ left rhombi and $m_1 + k$ right rhombi. Moreover, it does not matter where the left rhombi appear in $P_i$, the $j$-th right rhombi (from the top) has weight $dq^{m_1+m_2+k+i-j}$ (see Figure 33(c)). Next, we re-assign all right rhombi a weight 1, and divide weights of left rhombi in $P_i$ by $cq^{m_2+i-1}$. We get a new weight assignment $wt'$ and

$$wt(P_i) = wt'(P_i)c^{m_2-k}q^{(m_2+i-1)(m_2-k)}d^{m_1+k} \prod_{j=1}^{m_1+k} q^{m_1+m_2+k+i-j},$$

where $wt'(P_i)$ is the new weight of $P_i$. Thus, by taking product of all weights of $P_i$, for $i = 1, 2, \ldots, n_1-m_1$, we obtain

$$wt(P) = wt'(P)c^{(n_1-m_1)(m_2-k)}q^{(n_1-m_1)m_2(m_2-k)+(m_2-k)(n_1-m_1)}d^{(m_1+k)(n_1-m_1)} \times \prod_{i=1}^{n_1-m_1} \prod_{j=1}^{m_1+k} q^{m_1+m_2+k+i-j}. \quad (58)$$

(see Figure 33(d); each left rhombus with label $x$ is weighted by $q^x$).

We can see that in the weight assignment $wt'$, the weight of $P$ is $wt'(P) = q^x$, where $x$ is the sum of all labels of left rhombi as in Figure 33(d). One readily see that the sum $x$ is exactly the number of unit cubes in the complement $c(\mu)$ of the plane partition $\mu$. Taking sum over all tilings $T$, by (58), we get

$$\sum_T wt(T) = \sum_P wt(P) = K \sum_P wt'(P) = K \prod_\mu q^{c(\mu)} = K \prod_\mu q^{l(\mu)} \quad (59)$$

$$= K \prod_\mu q^{c(\mu)} = K \prod_\mu q^{l(\mu)} \quad (60)$$

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Figure 33: Illustrating the proof of Theorem 14 (cont.). (a) Each lozenge tiling of the lozenge hexagon $H$ corresponds to a plane partition fit in an $(n_1 - m_1) \times (m_2 - k + 1) \times (m_1 + k)$ box. A white rhombus indicates the upper face of a unit cubes, and its label shows the number of unit cubes in the column. (b) Each lozenge tiling corresponds to a family of $n_1 - m_1$ distinct rhombi paths. (c) The weight assignments of rhombi of the hexagon obtained from transforming the weighted double Aztec rectangle. Each right rhombus with label $x$ has weight $dq^x$, and each left rhombus with label $x$ has weight $cq^x$. (d) The weight assignment after changing the weights of left and right rhombi. Each rhombus with label $x$ has weight $q^x$. 
where the products are taken over all plane partitions fit in the \((n_1-m_1) \times (m_2-k+1) \times (m_1+k)\) box, and where

\[
K = c^{(m_2-k+1)(n_1-m_1)}q^{(m_1+k)(n_1-m_1)}q^{(n_1-m_1)/2}(2m_2^2+m_2m_1+m_2n_1+k^2+2km_1+m_1n_1+k-m_2).
\]

By MacMahon’s \(q\)-formula, we have

\[
\sum_{T} wt(T) = K \prod_{i=1}^{n_1-m_1} \prod_{j=1}^{m_2-k+1} \prod_{t=1}^{m_1+k} \frac{1-q^{i+j+t-1}}{1-q^{i+j+t-2}}.
\]

Then the theorem follows from (56).

\(\Box\)

**Proof of Theorem 14.** We assign the weight assignment \(wt_{1,1}^t(q^2)\) to the dominoes of the double Aztec rectangle. Consider a family of non-interesting Schröder paths \(P\) corresponding to a tiling \(T\) of the region. Denote by \(\alpha(T)\) the exponent of \(q\) in the expression of \(wt(P)\). We have from Figure 27 that an elementary move rises the underneath area by one if and only if it rises the value \(\alpha(T)\) by one. It means we have \(r(T) - \alpha(T)\) is a constant that does not depend on \(T\). Thus,

\[
r(T) - \alpha(T) = r(T_0) - \alpha(T_0) = -\alpha(T_0).
\]

By explicit evaluation, one gets

\[
\alpha(T_0) = \frac{2m_2(m_2-1)(m_2+1)}{3} + (m_2-k+1)(m_2+n_2-1)(n_2-m_2)
+ \sum_{i=0}^{m_1-1} (2(m_1-i)(k+n_2+2i) + (m_1-i)^2)
= \frac{2m_2(m_2-1)(m_2+1)}{3} + (m_2-k+1)(m_2+n_2-1)(n_2-m_2)
+ \frac{m_1(m_1+1)(2k+2m_1+2n_2-1)}{2}.
\]

Consider the numbers of up, down and level steps in \(P\) (denoted by \(up(P)\), \(down(P)\), and \(level(P)\), respectively), we have

\[
up(P) + down(P) + 2level(P) = m_2(m_2+1) + 2(n_1-m_1)(m_2-k+1) + (n_1-m_1)(m_1+k) + m_1(m_1+1),
\]

so

\[
v(T) = m_2(m_2+1)/2 + (n_1-m_1)(m_2-k+1) + (n_1-m_1)(m_1+k)/2 + m_1(m_1+1)/2 - level(P).
\]

Thus,

\[
\sum_{T} t^{v(T)}q^{r(T)} = \sum_{P} t^{m_2(m_2+1)/2 + (n_1-m_1)(m_2-k+1) + (n_1-m_1)(m_1+k)/2 + m_1(m_1+1)/2} q^{\alpha(T)} \sum_{P} t^{-level(P)}q^{\alpha(T)}
= \sum_{P} t^{m_2(m_2+1)/2 + (n_1-m_1)(m_2-k+1) + (n_1-m_1)(m_1+k)/2 + m_1(m_1+1)/2} q^{\alpha(T_0)}
\times M(D\mathcal{R}_{m_1,n_1,k}(1,1,t^{-1},q,q^2)).
\]

Then the theorem follows from Theorem 19.  

\(\Box\)
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