Orthogonal basis spreading sequence for optimal cdma

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Received

Abstract
Recently, new spreading sequences have been proposed to multiplex the capacity of users. In particular, Weyl spreading sequences have the larger capacity of users than the Gold code. This paper shows that Weyl spreading sequences appear in bit recovering model and they are orthogonal basis. This result shows the reason why they have the large capacity and that any spreading sequence is expressed as the sum of Weyl spreading sequences.

Keywords CDMA, spreading sequence, orthogonal basis, Weyl sequence, SNR

Research Activity Group Applied Chaos

1. Introduction
The code division multiple access called CDMA is used for 3G mobile communication system. There are some way to communicate, for example, frequency-division multiple access and time-division multiple access. In CDMA system, we use spreading sequence as codes to communicate. CDMA provides a new axis, code-division. Recently, OFDM is used as 4G. The number of users has been increased and users have not only one device. In 5G, it is necessary to multiplex to communicate among a number of devices. The frequency spectrum band we can use is limited. It is difficult to increase the capacity. To increase the capacity, we focus on CDMA again.

The current spreading sequence is the Gold code [1]. The new types of spreading sequences have been proposed, for example, the chaos spreading sequence [2], whose SNR equals to the Gold code. The Weyl spreading sequence [3] is based on the Weyl sequence [4], used in Quasi-Monte Carlo Method. Its SNR is higher than the Gold code.

In this paper, we show that Weyl spreading sequences appear as orthogonal basis in the CDMA model. This is the reason why the Weyl spreading sequences have low interference noise. In section 2, we show the Weyl spreading sequences and their features. In section 3, we show the bit recovering model. To recover bit, the value of the cross-correlation is used. This model is generally model because we consider the value of bits. We divide the case whether consecutive bits are same or different. In both of cases, Weyl spreading sequences appear as orthogonal basis. In section 4, we show the new way to decompose spreading sequences. The interference noise is simply expressed with their coefficients.

2. Weyl Spreading Sequence
In [3], the new spreading sequence $w_k[n]$, the Weyl spreading sequence is proposed by the following formula.

$$w_k[n] = \exp(2\pi j(n-1)(\frac{k}{N}+\sigma)) \quad (n = 1, 2, \ldots, N), \quad (1)$$

where the parameter $\sigma$ satisfies $0 \leq \sigma < 1$, $j$ is a unit imaginary number, $k$ ($1 \leq k \leq K$) is an integer parameter assigned to a user, and $N$ is the length of the spreading sequence. This sequence is based on the Weyl sequence [4]. We define the cross-correlation function $\hat{C}_{k_1,k_2}(N)$ between the different sequences as follows:

$$\hat{C}_{k_1,k_2}(N) = \frac{1}{N} \sum_{n=1}^{N} w_{k_1}[n] w_{k_2}[n]. \quad (2)$$

This sequence has a low cross-correlation function with different $k$ [3]. The absolute value satisfies the orthogonal relation as

$$|\hat{C}_{k_1,k_2}(N)| = 0 \quad \text{(3)}$$

for $k_1 \neq k_2$. This result shows that $w_k[n]$ are orthogonal to each other.

It is known [5] that the spreading sequence based on the Weyl sequence has the order of cross-correlation $O(\frac{1}{N^2})$. The absolute value of cross-correlation with the current spreading sequence follows $O(\frac{1}{\sqrt{N}})$. It is desirable that cross-correlation is low because cross-correlation is treated as an interference noise. When the length $N$ gets large, the cross-correlation of the Weyl spreading sequences converges to 0 faster than the cross-correlation of the Gold codes. In [3], the SNR of the Gold code and the SNR of the Weyl spreading sequence are shown respectively.

$$\text{SNR}_{\text{Gold}} = \left\{ \frac{K-1}{3N} \right\}^{-1/2}, \quad (4)$$

$$\text{SNR}_{\text{Weyl}} \geq \left\{ \frac{K-1}{6N} \right\}^{-1/2}. \quad (4)$$
Figure 1 shows numerically obtained BER (Bit Error Rate) of the Weyl spreading sequence and the Gold code with the length 63. The BER value \(10^{-3}\) is considered the standard threshold BER, below the number of users can communicate. Thus, the capacity of users using the Weyl spreading sequence can be seen about twice larger than the system using the Gold code. Weyl spreading sequences realize the larger multiplexing.

In the next section, we shows that such Weyl spreading sequence can be mathematically treated in the specific CDMA model as the orthogonal basis for the specific CDMA model.

3. Derivation Orthogonal Basis

Define the set of all spreading sequences \(S\) with constant power condition:

\[
S := \{ s \mid s \in \mathbb{C}^N, |s[n]| = 1 \quad (n = 1, 2, \ldots, N) \}
\]

\[
s = [s_1, s_2, \ldots, s[N]]^T
\]

The Weyl sequence \(w_k(\sigma) = [w_k[1], w_k[2], \ldots, w_k[N]]^T\) belongs to \(S\).

Assume that the user \(\kappa\) has the spreading sequence \(s_\kappa\). We consider the chip-synchronous CDMA model. It is easy to extend this model to the asynchronous model because the interference noise in the asynchronous model is expressed as the sum of the two terms of the interference noise in the chip-synchronous model [6]. In CDMA system, despreading process is necessary to recover the bit. Figure 2 shows the model of despreading. The symbol \(b_{k,0}, b_{k,-1} \in \{-1, 1\}\) are the bits sent by user \(k\). They are spread with the spreading sequence \(s_k\) as in Figure 2. \(l\) is the gap length between the beginning of \(s_k\) and the one of \(s_i\).

\[
\begin{array}{c|c}
  b_{k,-1}s_k & b_{k,0}s_k \\
\hline
  l & N-l \\
  \text{S}_i & \\
\end{array}
\]

Define \(W_{i,k}(l)\) as the cross-correlation between \(s_i\) and \(s_k\). \(W_{i,k}(l)\) is written as

\[
W_{i,k}(l) = T \exp(j\phi_k) \left\{ b_{k,-1} \sum_{n=1}^{l} s_{i,n}s_k[N-n+l] + b_{k,0} \sum_{n=1}^{N-l} s_{i,n+l}s_k[n] \right\},
\]

where \(\phi_k \in [0, 2\pi)\) is the phase of user \(k\)’s carrier and \(T\) is the width of each chip. For simplicity, we set \(\phi_k = 0\) and \(T = 1\). Eq. (6) can be expressed as the following quadratic form:

\[
W_{i,k}(l) = s_i^* B^{(l)}_{b_{k,-1}, b_{k,0}} s_k,
\]

where

\[
B^{(l)}_{b_{k,-1}, b_{k,0}} = \begin{pmatrix} O & b_{k,-1}E_l \\ b_{k,0}E_{N-l} & O \end{pmatrix},
\]

and \(E_l\) is the \(l\) dimensional identity matrix. Note that \(B^{(l)}_{b_{k,-1}, b_{k,0}}\) is a normal matrix. This matrix is diagonalizable by a unitary matrix. It is clear that

\[
B^{(l)}_{1,1} = -B^{(l)}_{-1,-1},
\]

\[
B^{(l)}_{-1,1} = -B^{(l)}_{1,-1}.
\]

\(B^{(l)}_{1,1}\) and \(B^{(l)}_{-1,-1}\) show that user \(k\) sends the different bits \(B^{(l)}_{1,1}\) and \(B^{(l)}_{-1,-1}\) show that he sends the same bits. It is sufficient to consider only \(B^{(l)}_{1,1}\) and \(B^{(l)}_{-1,-1}\).

(A) \(B^{(l)}_{1,1}\)

Set \(\lambda\) and \(\mathbf{x}\) the eigenvalue and the eigenvector of \(B^{(l)}_{1,1}\). They satisfy

\[
B^{(l)}_{1,1} \mathbf{x} = \lambda \mathbf{x}.
\]

From Eq. (8),

\[
\begin{pmatrix}
  O & E_l \\
  E_{N-l} & O
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{N-l+1}
\end{pmatrix}
= 
\begin{pmatrix}
  x_{N-l+1} \\
  \vdots \\
  x_1 \\
  \vdots \\
  x_N \\
  x_{N-l}
\end{pmatrix}.
\]

Note that \(B^{(l)}_{1,1}\) is a permutation matrix. About \(B^{(l)}_{1,1}\), the following equation is satisfied for an integer \(1 \leq l \leq N\):

\[
B^{(l)}_{1,1} = B^{(l-1)}_{1,1} B^{(l-1)}_{1,1}.
\]

Define \(\mu^{(1)}\) and \(\mathbf{v}^{(1)}\) as the eigenvalue and the eigenvector of \(B^{(1)}_{1,1}\). From the above equation, we obtain

\[
B^{(l)}_{1,1} \mathbf{v}^{(1)} = B^{(l-1)}_{1,1} B^{(l-1)}_{1,1} \mathbf{v}^{(1)} = (B^{(1)}_{1,1}) \mathbf{v}^{(1)} = (\mu^{(1)}) \mathbf{v}^{(1)}.
\]

The matrix \(B^{(l)}_{1,1}\) (\(1 \leq l \leq N\)) have the same eigenvectors whose eigenvalues are expressed as products of \(\mu^{(1)}\).
\( \mu^{(1)} \) can be expressed as
\[
\mu^{(1)} = \exp(-2\pi j_{m}^{N}) \quad (1 \leq m \leq N).
\] (14)

Define the eigenvalue \( \lambda^{(l)}_{m} \) and the eigenvector \( v_{m} \) as
\[
\lambda^{(l)}_{m} = \exp(-2\pi j_{m}^{l}N) \quad (m = 1, 2, \ldots, N),
\] (15)

\[
B_{l,1}^{(l)}v_{m} = \lambda^{(l)}_{m}v_{m}.
\]

Then, the eigenvector \( v_{m} \) is expressed as
\[
v_{m} = \frac{1}{\sqrt{N}} \begin{pmatrix}
\exp(2\pi j_{m}^{N}) \\
\exp(2\pi j_{m+1}^{N}) \\
\vdots \\
\exp(2\pi j_{(N-1)m}^{N})
\end{pmatrix} = \frac{1}{\sqrt{N}}w_{m}(0).
\] (16)

In the above equation, Weyl spreading sequence \( w_{m}(0) \) appears as the eigenvector. From the above result, we can decompose the matrix \( B_{1,1}^{(l)} \) and transform Eq. (7) as
\[
W_{i,k}(l) = s_{k}^{i}vA^{(l)}V^{*}s_{k},
\] (17)

where
\[
V = (v_{1} \quad v_{2} \quad \cdots \quad v_{N}),
\]

\[
A^{(l)} = \begin{pmatrix}
\lambda^{(l)}_{1} & 0 & \cdots & 0 \\
0 & \lambda^{(l)}_{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda^{(l)}_{N}
\end{pmatrix}.
\] (18)

From Eq. (16) and Eq. (17), \( W_{i,k}(l) \) gets low if we choose \( w_{k}(\sigma) \) as \( s_{k} \) and \( i \neq k \). In particular, \( W_{i,k}(l) = 0 \) if we choose \( w_{k}(0) \) as \( s_{k} \).

**B) \( B_{l-1,1}^{(l)} \)**

Similarly to (A), set \( \hat{\lambda} \) and \( \hat{x} \) the eigenvalue and the eigenvector of \( B_{l-1,1}^{(l)} \) satisfying
\[
B_{l-1,1}^{(l)}\hat{x} = \hat{\lambda}\hat{x}.
\] (19)

From Eq. (8),
\[
\begin{pmatrix}
O & -E_{l} \\
E_{N-l} & O
\end{pmatrix}
\begin{pmatrix}
\hat{x}_{1} \\
\vdots \\
\hat{x}_{N-l} \\
\hat{x}_{N-l+1} \\
\vdots \\
\hat{x}_{N}
\end{pmatrix}
= \begin{pmatrix}
-\hat{x}_{N-l+1} \\
\vdots \\
-\hat{x}_{N} \\
\hat{x}_{1} \\
\vdots \\
\hat{x}_{N-l}
\end{pmatrix}.
\] (20)

About \( B_{l-1,1}^{(l)} \), the following equation is satisfied for an integer \( 1 \leq l \leq N \):
\[
B_{l-1,1}^{(l)} = B_{l-1,1}^{(l)}B_{l-1,1}^{(l-1)}.
\] (21)

Define \( \tilde{\mu}^{(1)} \) and \( \tilde{V}^{(1)} \) as the eigenvalue and the eigenvector of \( B_{1,1}^{1} \). From the above equation, we obtain
\[
B_{1,1}^{(l)}\tilde{V}^{(1)} = B_{1,1}^{(l)}B_{l,1}^{(l-1)}\tilde{V}^{(1)} = (B_{l,1}^{(l)}\tilde{V}^{(1)}) = (\tilde{\mu}^{(1)})\tilde{V}^{(1)}.
\] (22)

The matrix \( B_{l,1}^{(l)} \) (\( 1 \leq l \leq N \)) have the same eigenvectors whose eigenvalues are expressed as products of \( \tilde{\mu}^{(1)} \).

Define the eigenvalue \( \hat{\lambda}^{(l)}_{m} \) and the eigenvector \( \hat{v}_{m} \) of \( B_{l-1,1}^{(l)} \). They are expressed as
\[
\hat{\lambda}^{(l)}_{m} = \exp(-2\pi j_{m}^{l}(\frac{m}{N} + \frac{1}{2N})) \quad (1 \leq m \leq N),
\] (23)

\[
\hat{v}_{m} = \frac{1}{\sqrt{N}}w_{m}(1, 2, \ldots, N),
\] (24)

In the above equation, the Weyl spreading sequence \( w_{m}(\frac{1}{2N}) \) appears as the eigenvector. From the above result, we can decompose the matrix \( B_{l-1,1}^{(l)} \) and transform Eq. (7) as
\[
W_{i,k}(l) = s_{k}^{i}\hat{V}\hat{\Lambda}^{(l)}\hat{V}^{*}s_{k},
\] (25)

where
\[
\hat{V} = (\hat{v}_{1} \quad \hat{v}_{2} \quad \cdots \quad \hat{v}_{N}),
\]

\[
\hat{\Lambda}^{(l)} = \begin{pmatrix}
\hat{\lambda}^{(l)}_{1} & 0 & \cdots & 0 \\
0 & \hat{\lambda}^{(l)}_{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{\lambda}^{(l)}_{N}
\end{pmatrix}.
\] (26)

From Eq. (24) and Eq. (25), \( W_{i,k}(l) \) gets low if we choose \( w_{k}(\sigma) \) as \( s_{k} \) and \( i \neq k \). In particular, \( W_{i,k}(l) = 0 \) if we choose \( w_{k}(\frac{1}{2N}) \) as \( s_{k} \).

**4. Expression of Spreading Sequence**

We obtain the orthogonal basis \( \frac{1}{\sqrt{N}}w_{m}(0) \) and \( \frac{1}{\sqrt{N}}w_{m}(\frac{1}{2N}) \). We see that any spreading sequence can be expressed as the sum of them. Define a spreading sequence \( s \in S \). The sequence \( s \) is expressed as,
\[
s = \frac{1}{\sqrt{N}}\sum_{m=1}^{N} \alpha_{m}w_{m}(0),
\]
\[
= \frac{1}{\sqrt{N}}\sum_{m=1}^{N} \beta_{m}w_{m}(\frac{1}{2N}),
\] (27)
where $\alpha_m$ and $\beta_m \in \mathbb{C}$ are the coefficients. They satisfy
\[
\alpha_m = \frac{1}{\sqrt{N}} \langle w_m(0), s \rangle, \\
\beta_m = \frac{1}{\sqrt{N}} \langle w_m(1), s \rangle,
\]
where
\[
\langle s_i, s_k \rangle = \sum_{n=1}^{N} s_i(n) s_k(n). \tag{29}
\]
Define the matrices $\Phi$ and $\hat{\Phi}$
\[
\Phi = \frac{1}{N} \begin{pmatrix}
\phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,N} \\
\phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,N} \\
\vdots & \vdots & & \vdots \\
\phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,N}
\end{pmatrix}, \\
\hat{\Phi} = \frac{1}{N} \begin{pmatrix}
\hat{\phi}_{1,1} & \hat{\phi}_{1,2} & \cdots & \hat{\phi}_{1,N} \\
\hat{\phi}_{2,1} & \hat{\phi}_{2,2} & \cdots & \hat{\phi}_{2,N} \\
\vdots & \vdots & & \vdots \\
\hat{\phi}_{N,1} & \hat{\phi}_{N,2} & \cdots & \hat{\phi}_{N,N}
\end{pmatrix}, \tag{30}
\]
where
\[
\phi_{p,q} = \langle w_p(0), w_q \left( \frac{1}{2N} \right) \rangle = \frac{2}{1 - \exp(2\pi j (\frac{2p + 1}{2N} + \frac{2q}{N}))}, \\
\hat{\phi}_{p,q} = \langle w_p \left( \frac{1}{2N} \right), w_q(0) \rangle = \frac{2}{1 - \exp(2\pi j (\frac{2p + 1}{2N} - \frac{2q}{N}))},
\]
Between $\alpha_m$ and $\beta_m$, there is the relationship
\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_N
\end{pmatrix} = \Phi \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_N
\end{pmatrix}, \\
\hat{\Phi} \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_N
\end{pmatrix} = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_N
\end{pmatrix}.
\]
With the above expression, we can easily calculate the interference noise $W_{i,k}(l)$ and the auto-correlation $W_{i,i}(l)$. Assume that user $i$ and $k$ have the spreading sequence $s_i$ and $s_k$. Their coefficients are $\alpha_m^{(i)}$, $\beta_m^{(i)}$ and $\alpha_m^{(k)}$, $\beta_m^{(k)}$. When user $k$ sends the same bits, for example 1 and 1,
\[
W_{i,k}(l) = s_i^* B_{i,1}^l s_k \\
= s_i^* V A^l V^* s_k \\
= \sum_{m=1}^{N} \lambda_m^{(i)} \alpha_m^{(i)} \beta_m^{(k)}. \tag{32}
\]
When user $k$ sends the different bits, for example $-1$ and 1,
\[
W_{i,k}(l) = s_i^* B_{-1,1}^l s_k \\
= s_i^* V A^l V^* s_k \\
= \sum_{m=1}^{N} \lambda_m^{(i)} \alpha_m^{(i)} \beta_m^{(k)}. \tag{33}
\]
$\lambda_n^{(i)}$ and $\lambda_m^{(i)}$ are rotation terms. They are different for each $m$. In the both of the cases, the interference noise and auto-correlation function can be expressed the weighted sum of the coefficients.

5. Conclusion

In the chip-synchronous CDMA model, we show that the Weyl spreading sequences appear as the orthogonal basis. From this result, the interference noise always gets low when we use the Weyl spreading sequence as a spreading sequence $s$. However, it shows that auto-correlation functions get high in all $l$ when we use $w_m(x)$. The absolute value of the auto-correlation function is expressed as $|W_{i,i}(l)|$. For example, consider $B_{1,1}$ and $w_i(0)$,
\[
W_{i,i}(l) = w_i(0)^* B_{1,1}^l w_i(0) \\
= w_i(0)^* V A^l V^* w_i(0) \\
= e_i^T A^l e_i \\
= \lambda_i^l,
\]
where $e_i$ is the unit vector whose $i$-th element is 1 and the other element are 0. The eigenvalues satisfy that $|\lambda_i^l| = 1$ and $\lambda_i^l$ rotate when $l$ changes. Therefore, $|W_{i,i}(l)|$ get high in all $l$.

Any spreading sequence can be expressed as the weighted sum of $w_m(0)$ or $w_m(1)$. The interference noise $W_{i,k}(l)$ can be the expressed as the weighted sum of $\alpha_m^{(i)} \alpha_m^{(k)}$ or $\beta_m^{(i)} \beta_m^{(k)}$. This expression is useful for designing new spreading sequences because their coefficients naturally appear in the interference noise and auto-correlation which are the key clue for designing.

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