General affine differential geometry of surfaces in affine space $A^3$, I: the elliptical case∗

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Abstract
In this paper we study the general affine differential geometry of surfaces in affine space $A^3$. For a regular elliptical surface we define a moving frame of minimal order and get the complete system of differential invariants. As an application we classify regular elliptical surfaces of constant curvatures up to affine congruence.

Keywords: General affine differential geometry, Regular elliptical surfaces, Moving frame, Invariant differential operator, Curvature.
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1 Introduction

The affine geometry was founded by Blaschke, Pick, Radon, Berwald and Thomsen among others in the period from 1916 to 1923. For accounts and expository books appeared on the subject, see Blaschke[1], Guggenheimer[6] and Spivak’s[10]. In these work affine geometry means the equi-affine geometry. That is the Kleinian geometry of the affine transformation group which preserves volume. But for general affine geometry, i.e. the Kleinian geometry of general affine transformation group, there is little work. All the work we can find is Weise[13][14], Kllingenberg[7][8], Svec[11], Wilkinson[15] and Weiner[12]. The property of volume preserving makes things easier to be dealt with. Compared with the equi-affine geometry, the study of general affine geometry is more difficult.

In Zhao and Gao[16] we constructed the theory of general affine differential geometry of plane curves. In this paper we will consider the case for space surfaces. In the study of classical differential geometry, it seems that the geometry and analysis are vital. We will show in this paper that neither geometry nor analysis but algebra is essential in general Kleinian differential geometry. The algebraic methods exhibit their strength in our work. The viewpoint in this paper makes the structure of a general Kleinian differential geometry theory more transparent.

Our strategies consist of three steps:

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1. We determine the action of the affine transformation group on jet spaces of surfaces. Then
the differential invariants of surfaces are converted to algebraic invariants of jets.

2. We choose the standard forms of jets of surfaces. By Fels and Olver[3][4][5], we have an
equivariant moving frame.

3. Based on steps 1 and 2, by using Cartan’s moving frame method we construct the total
affine differential geometry theory.

In fact this strategy is valid for a general Kleinian differential geometry.

This paper is a part of the project to study the algebraic structure of general Kleinian differen-
tial geometry. For simplicity, we only consider the regular elliptical surfaces here. The hyperbolic
and parabolic cases will be considered elsewhere.

The contents of this paper are as follows. In section 2, we give an introduction to jet spaces and
equivariant moving frame method. In section 3, we determine the standard forms of jets of surfaces
under the space affine transformation group. In section 4 we construct an equivariant moving frame
of minimal order and compute the moving equations. As results we obtain the complete system of
differential invariants. In section 5 we construct the differential geometry theory of regular elliptical
surfaces. As an application we classify the space elliptical surfaces with constant curvatures up to
affine congruence. Section 6 is an appendix which gives the details of the procedure to solve the
compatible equations.

2 Jet spaces and the equivariant moving frame method

This section is an introduction to jet spaces and Fels and Olver’s equivariant moving frame method.
See Fels and Olver[3][4][5] for details.

2.1 Jet spaces

Let $M$ be a smooth manifold, the differential geometry of a submanifold $N$ at a point $P$ is deter-
mined by the shape of $N$ at an arbitrary small neighborhood. So all the local differential geometric
properties and invariants of $N$ are determined by the local data of $N$. It is useful to isolate the
information of $N$ at $P$. This idea hints the concept of jet of submanifolds. The gene-
ral definition
of jet of submanifolds was given by Ehresmann in 1950s for an ambient manifold $M$.

For the smooth manifold $M$ of dimension $n$, let $SM^d_P(M)$ be the set of all smooth $d$-dimensional
submanifolds of $M$ that contain the point $P$ and $SM^d(M) = \bigcup_{P \in M} SM^d_P(M)$. For integer $r \geq 0$,
define an equivalence relation $\sim$ on $SM^d_P(M)$ such that $N_1 \sim N_2$ iff $N_1$ and $N_2$ have contact of
order $r$. The jet space of $d$-dimensional submanifolds of $M$ at a point $P$ of order $r$ is the quotient
set $J^{d,r}_P(M) = SM^d_P(M)/\sim$. And $J^{d,r}(M) = \bigcup_{P \in M} J^{d,r}_P(M)$ is the jet space of $d$-dimensional
submanifolds of $M$ of order $r$. An element of jet space $J^{d,r}(M)$ is called a $d$-jet of order $r$. $J^{d,r}(M)$
is a smooth manifold of dimension $d + (n - d)\binom{d + r}{r}$. In this paper we use the jet space $J^{2,r}(A^3)$
for $r \geq 0$ in the study of surfaces in affine space $A^3$.

Jet space can be regarded as the finite dimension cut-off of infinite dimension space of all
submanifolds with fixed dimension. The use of jet spaces separates the study of local differential
geometry into algebra part and analysis part. So it makes the structure of local differential geometry theory more transparent.

2.2 Fels and Olver’s moving frame method

**Definition 2.1.** Let \( \phi : G \times M \to M \) be a smooth action of Lie group \( G \) on smooth manifold \( M \). A moving frame on \( M \) is a smooth, \( G \)-equivariant map \( \rho : M \to G \).

There are two types of moving frames.

\[
\begin{align*}
\rho(gz) &= g\rho(z), \quad \text{left moving frame}; \\
\rho(gz) &= \rho(z)g^{-1}, \quad \text{right moving frame}.
\end{align*}
\]

**Theorem 2.1.** (Fels and Olver) A moving frame exists on \( M \) if and only if \( G \) acts freely and regularly on \( M \).

The explicit construction of a moving frame is based on Cartan’s normalization procedure. Let \( G \) act freely and regularly on \( M \) and \( K \) be a cross-section to the group orbits, that is a submanifold \( K \) which transversally intersects each orbit once. Let \( g \) be the unique group element which maps \( P \) into the cross-section \( K \), then \( \rho : M \to G, P \mapsto g \) is a right moving frame. And \( \rho : M \to G, P \mapsto g^{-1} \) is a left moving frame. The unique intersection point of the orbit of \( P \) and \( K \) can be regarded as the standard form of \( P \), as prescribed by the cross-section \( K \).

If a moving frame is in hand, the determination of the invariants is routine. The specification of a moving frame by choosing a cross-section induces a canonical procedure to map functions to invariants.

**Definition 2.2.** The invariantization of a function \( F : M \to \mathbb{R} \) is the unique invariant function \( \iota(F) \) that coincides with \( F \) on the cross-section, that is \( \iota(F)|_{K} = F|_{K} \).

Invariantization defines a projection from the space of smooth functions to the space of invariants. The fundamental differential invariants are obtained by invariantization of coordinate functions on jet space.

3 The action of \( Aff(3) \) on \( J^{2,r}(A^{3}) \)

Let \( Aff(3) \) be the general affine transformation group of affine space \( A^{3} \). The general affine geometry is the Kleinian geometry given by the natural group action \( Aff(3) \times A^{3} \to A^{3} \). Under the affine coordinates \( x, y, z \) on \( A^{3} \), a general affine transformation \( T \) has the form

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}
+ \begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}.
\]

or

\[
\begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix}
= \begin{pmatrix}
b_{11} & b_{12} & b_{13} & x_0 \\
b_{21} & b_{22} & b_{23} & y_0 \\
b_{31} & b_{32} & b_{33} & z_0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}.
\]
where \( \det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \neq 0 \). Hence the coordinates on \( Aff(3) \) are \( b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}, x_0, y_0, z_0 \).

In the following we assume that a smooth surface has the form \( z = u(x, y) \) locally. The local coordinates of the jet space \( J^{2,3}(A^3) \) are \( x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}, z_{xxx}, z_{xxy}, z_{xyy}, z_{yyy}, z_{yyyy} \). The action of \( Aff(3) \) on \( A^3 \) induces an action on \( SM^2(A^3) \) and hence on \( J^{2,3}(A^3) \).

We expand \( z = u(x, y) \) at \((x_0, y_0)\) as

\[
z(x, y) = z_0 + a_{10}(x - x_0) + a_{01}(y - y_0) + \frac{1}{2}(a_{20}(x - x_0)^2 + 2a_{11}(x - x_0)(y - y_0) + a_{02}(y - y_0)^2)
+ \frac{1}{6}(a_{30}(x - x_0)^3 + 3a_{21}(x - x_0)^2(y - y_0) + 3a_{12}(x - x_0)(y - y_0)^2 + a_{03}(y - y_0)^3) + \cdots. \tag{1}
\]

We identify the symbols \( z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}, z_{xxx}, z_{xxy}, z_{xyy}, z_{yyy}, \cdots \) with \( a_0, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}, \cdots \).

In the following lemmas, we will use affine transformations to transform a jet of surface \( S : z = u(x, y) \) at \( P(x_0, y_0, z_0) \) to a jet of surface \( S' : z' = u'(x', y') \) at \( P'(x'_0, y'_0, z'_0) \) step by step, such that the final jet of surface has certain standard form. To simplify our notations, we often use the same symbols \( x, y, z \) instead of \( x', y', z' \) after the affine transformations.

First we observe that a jet as in Equation 1 can be transformed to a jet at \((0,0,0)\) which has the form

\[
z = a_{10}x + a_{01}y + \frac{1}{2}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) + \cdots. \tag{2}
\]

Let \( T_1 \) be an affine transformation \( x = b_{11}x' + b_{12}y' + b_{13}z', y = b_{21}x' + b_{22}y' + b_{23}z', z = b_{31}x' + b_{32}y' + b_{33}z' \) preserving the point \( O(0,0,0) \).

**Lemma 3.1.** Under the affine transformation \( T_1 \), if a jet of the form in Equation 2 is transformed to the form

\[
z' = a'_{10}x' + a'_{01}y' + \frac{1}{2}(a'_{20}x'^2 + 2a'_{11}x'y' + a'_{02}y'^2) + \cdots,
\]

then we have

\[
(b_{33} - a_{10}b_{13} - a_{01}b_{23})(a'_{10}, a'_{01}) = (b_{31}, b_{32}) - (a_{10}, a_{01}) \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{22} \\ b_{32} \\ b_{33} \end{pmatrix}.
\]

**Proof:** This is a direct computation from the affine transformation \( T_1 \) and Equation 2.

**Lemma 3.2.** (1) There is an affine transformation \( T_1 \) such that under which a jet of the form in Equation 2 is transformed to a form with \( a_{00} = a_{10} = a_{01} = 0 \)

(2) The subgroup \( G_1 \) of \( Aff(3) \) keeping the condition \( a_{00} = a_{10} = a_{01} = 0 \) is

\[
G_1 = \left\{ \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \mid (b_{11}b_{22} - b_{12}b_{21})b_{33} \neq 0 \right\}.
\]
Lemma 3.3. Under the affine transformation \( T_2 \), if a jet of the form
\[
z = u(x, y) = \frac{1}{2}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) + \cdots
\]
is transformed to the form
\[
z' = u'(x', y') = \frac{1}{2}(a'_{20}x'^2 + 2a'_{11}x'y' + a'_{02}y'^2) + \cdots,
\]
then we have
\[
b_{33} \begin{pmatrix} a'_{20} & a'_{11} \\ a'_{11} & a'_{02} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\]
(5)

Proof: This follows directly from \( T_2 \) and Equation 4.

By rotating in the \( xy \)-plane and re-scaling on axes \( x \) and \( y \), the quadratic items of a jet can be transformed to one of the four standard forms \( \frac{1}{2}(x^2 + y^2) \), \( \frac{1}{2}(x^2 - y^2) \), \( \frac{1}{2}x^2 \) or 0.

Definition 3.1. We call the corresponding four jet types the elliptical, hyperbolic, parabolic and degenerate types.

In jet space \( J^2,3(A^3) \), all jets of elliptical type form an open submanifold. We call a point \( P \) in a surface \( S \) is elliptical if the jet of surface \( S \) at \( P \) is elliptical. In this paper, we only consider elliptical surfaces, i.e. surfaces whose points are all elliptical.

Corollary 3.1. The jet of elliptical surface \( S \) can be transformed to the standard form of
\[
z = u(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3) + \cdots
\]

Lemma 3.4. Let \( G_2 \) be the subgroup of \( G_1 \) keeping the condition \( a_{00} = a_{10} = a_{01} = a_{11} = 0, a_{20} = a_{02} = 1 \), then \( G_2 = \left\{ \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \mid b_{11}^2 + b_{21}^2 = b_{12}^2 + b_{22}^2 = b_{33}, b_{11}b_{12} + b_{21}b_{22} = 0 \right\} \).

Proof: This follows from Lemma 3.3. In the present case \( a_{11} = a'_{11} = 0, a_{20} = a_{02} = a'_{20} = a'_{02} = 1 \).

Let \( T_3 \) be an affine transformation \( x = b_{11}x' + b_{12}y' + b_{13}z', y = b_{21}x' + b_{22}y' + b_{33}z', z = b_{33}z' \) which is in \( G_2 \).

Lemma 3.5. Under the affine transformation \( T_3 \), if a jet of the form
\[
z = u(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3) + \cdots
\]
(6)
is transformed to the form
\[ z' = u'(x', y') = \frac{1}{2}(x'^2 + y'^2) + \frac{1}{6}(a'_{30}x'^3 + 3a'_{21}x'^2y' + 3a'_{12}x'y'^2 + a'_{03}y'^3) + \cdots, \]
then we have
\begin{align*}
    b_{33}a'_{30} &= a_{30}b_{11}^2 + 3a_{21}b_{11}b_{21} + 3a_{12}b_{11}b_{21} + a_{03}b_{21}^2 + 3b_{11}b_{13} + 3b_{12}b_{23} \\
    b_{33}a'_{12} &= a_{30}b_{11}b_{12} + a_{21}b_{11}(b_{11}b_{22} + 2b_{12}b_{21}) + a_{12}b_{21}(2b_{11}b_{22} + b_{12}b_{21}) + a_{03}b_{21}b_{22} + b_{11}b_{13} + b_{22}b_{23} \\
    b_{33}a'_{03} &= a_{30}b_{12}^2 + 3a_{21}b_{12}^2 + 3a_{12}b_{12}b_{22} + a_{03}b_{22}^2 + 3b_{12}b_{13} + 3b_{22}b_{23}
\end{align*}

**Proof:** This follows directly from \( T_3 \) and Equation 5.

We now come to the main result of this section, i.e. the standard forms of elliptical jets.

**Theorem 3.1.** An elliptical jet can be transformed by affine transformations to the following standard form
\[ z = u(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3) + \cdots \]
Or
\[ z = u(x, y) = \frac{1}{2}(x^2 + y^2) + 0 + \cdots \]
We call the two cases the regular case and the degenerate case respectively.

**Proof:** By Corollary 3.1, we can assume the elliptical jet is of the form
\[ z = u(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3) + \cdots \]
Let \( T_4 \in G_2 \) be the transformation given by
\[
    x = \sqrt{b_{33}}x' \cos t + \sqrt{b_{33}}y' \sin t + b_{13}z', \quad x = -\sqrt{b_{33}}x' \sin t + \sqrt{b_{33}}y' \cos t + b_{23}z', \quad z = b_{33}z'.
\]
By setting \( a'_{12} = -a'_{30}, a'_{21} = a'_{03} \) and using Equations 7 and 9, Equations 8 and 10, we obtain two linear equations with unknowns \( b_{13} \) and \( b_{23} \). Thus \( b_{13}, b_{23} \) can be determined. This means that we can transform the jet to a form
\[ z' = u'(x', y') = \frac{1}{2}(x'^2 + y'^2) + \frac{1}{6}(a'_{30}(x'^3 - 3x'y'^2) + a'_{03}(3x'^2y' + y'^3)) + \cdots. \]
By substituting \( b_{13}, b_{23} \) into the Equations 7 and 10, we obtain
\begin{align*}
    a'_{30} &= \sqrt{b_{33}}(\frac{1}{4}a_{30} \cos 3t - \frac{3}{4}a_{21} \sin 3t - \frac{3}{4}a_{12} \cos 3t + \frac{1}{4}a_{03} \sin 3t) \\
    a'_{03} &= \sqrt{b_{33}}(\frac{1}{2}a_{30} \sin 3t + \frac{3}{2}a_{21} \cos 3t - \frac{3}{2}a_{12} \sin 3t - \frac{1}{2}a_{03} \cos 3t)
\end{align*}
If we already have \( a_{12} = -a_{30} \) and \( a_{21} = a_{03} \), then Equations 13 and 14 become to
\[ a'_{30} = \sqrt{b_{33}}(a_{30} \cos 3t - \frac{a_{03}}{2} \sin 3t), \quad a'_{03} = \sqrt{b_{33}}(a_{30} \sin 3t + \frac{a_{03}}{2} \cos 3t). \]
It is obvious that besides the case both \( a_{30} \) and \( a_{03} \) are zero we can choose \( t \) and \( b_{33} > 0 \) such that \( a'_{30} = 1 \) and \( a'_{03} = 0 \). This proves the theorem.
Corollary 3.2. Let $G_3$ be the subgroup of $G_2$ which fixes the jets with $a_{00} = a_{10} = a_{01} = a_{11} = a_{21} = a_{03} = 0, a_{20} = a_{02} = a_{30} = -a_{12} = 1$, then $G_3$ is the symmetry group $D_3$ of equilateral triangle with vertices $(1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Proof: For the affine transformation $T \in G_3$, by Equations 7, 8, 9 and 10, we have

\[
\begin{align*}
0 &= b_{11}^2 - 2b_{11}b_{21} - b_{21}^2 + b_{12}b_{21} + b_{22}\b_{23} \\
-b_{33} &= b_{11}b_{12} - b_{11}b_{22} - 2b_{12}b_{21} + b_{11}b_{13} + b_{12}b_{23} \\
0 &= b_{12}^3 - 3b_{12}b_{22}^2 + 3b_{12}b_{23} + b_{22}^3
\end{align*}
\]

By canceling $b_{13}$ and $b_{23}$, we get

\[
\begin{align*}
4b_{33} &= b_{11}^3 - 3b_{11}b_{12}^2 - 3b_{11}b_{21}^2 + 3b_{11}b_{22}^2 + 6b_{12}b_{21}b_{22} \\
0 &= -3b_{11}^2 + 6b_{12}b_{21} + b_{12}^2 + 3b_{11}b_{22} - 3b_{22}^2
\end{align*}
\]

Since the transformation $T$ is in $G_2$, we must have

\[
\begin{pmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22}
\end{pmatrix}
= \begin{pmatrix}
 1 & 0 \\
 0 & \pm 1
\end{pmatrix}
\begin{pmatrix}
 \cos \frac{3\pi}{4} & -\sin \frac{3\pi}{4} \\
 -\sin \frac{3\pi}{4} & \cos \frac{3\pi}{4}
\end{pmatrix}
\begin{pmatrix}
 \pm \sin \frac{\pi}{2} & \pm \cos \frac{\pi}{4} \\
 \pm \cos \frac{\pi}{4} & \pm \sin \frac{\pi}{2}
\end{pmatrix}
\begin{pmatrix}
 \cos t & -\sin t \\
 \pm \sin t & \pm \cos t
\end{pmatrix}
\]

The generators of $D_3$ are the $\frac{2\pi}{3}$ rotation $\sigma$ and the involution $\tau$ changing the sign of coordinate $y$.

Their actions on coordinates $x, y, z$ are given by

\[
\sigma : x = -\frac{1}{2}x' - \frac{\sqrt{3}}{2}y', y = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y', z = z'; \quad \tau : x = x, y = -y', z = z'.
\]

4 Equivariant moving frame and its moving equations

In this section, we only discuss the regular elliptical surfaces.

4.1 Differential invariants and invariant forms

Definition 4.1. A smooth function $f : J^{2,r}(A^3) \to \mathbb{R}$ which is invariant under the action of $Af(f(3))$ is called a differential invariant of surfaces in $A^3$ of order not greater than $r$.

Up to affine congruence the jet of a regular elliptical surface $S$ at a point $P$ can be written as a form

\[
z = \frac{1}{2}(x^2 + y^2) + \frac{1}{6}(x^3 - 3xy^2) + \sum_{n=4}^{\infty} \frac{1}{n!} \sum_{i+j=n} a_{ij} \binom{n}{i} x^i y^j.
\]

It is obvious that the coefficients $a_{ij}, i + j \geq 4$ are all affine differential invariants on surface $S$. Recall that in Definition 2.2, we defined the invariantization $\iota(F)$ of a function $F$. Now we denote
the invariantization of $a_{ij}$ by $I_{ij}$, i.e. $\iota(a_{ij}) = I_{ij}$. The 1-forms $dx, dy$ can also be invariantized to affine invariant differential forms $w_1, w_2$. The corresponding dual invariant differential operators $D^1, D^2$ are defined by $df = D^1 f w_1 + D^2 f w_2$. For details see [5].

By Theorem 3.1, the jet of $S$ at a point $(x_0, y_0, z_0)$ can be transformed to the standard form by an affine transformation $T$. Let the transformation $T$ be of the form

$$
\begin{pmatrix}
  x \\
  y \\
  z \\
  1
\end{pmatrix} =
\begin{pmatrix}
  b_{11} & b_{12} & b_{13} & x_0 \\
  b_{21} & b_{22} & b_{23} & y_0 \\
  b_{31} & b_{32} & b_{33} & z_0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z \\
  1
\end{pmatrix}
$$

We set $e_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$, $e_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$, $e_3 = \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix}$, $r = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$. $(e_1, e_2, e_3, r)$ gives a smooth affine moving frame on surface $S$. The moving equation of the moving frame has the form

$$
d(e_1, e_2, e_3, r) = (e_1, e_2, e_3, r)\Omega \tag{18}
$$

where $\Omega$ is a matrix of 1-forms.

Since we have chosen $a_{00} = a_{10} = a_{01} = 0$, which means that the fundamental vector fields $e_1, e_2$ dual to $w_1, w_2$ are tangent to $S$ and $dr = w_1 e_1 + w_2 e_2$. So $\Omega$ must be of form

$$
\Omega =
\begin{pmatrix}
  w_{11} & w_{12} & w_{13} & w_1 \\
  w_{21} & w_{22} & w_{23} & w_2 \\
  w_{31} & w_{32} & w_{33} & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
$$

The Maurer-Cartan invariants $R_{ij}^k, 1 \leq i, j \leq 3, 1 \leq k \leq 2$ are defined by

$$
w_{ij} = \sum_{k=1}^{2} R_{ij}^k w_k. \tag{19}
$$

For differential manifold $M$, let $Diff(M)$ be its diffeomorphism group and $\mathcal{X}(M)$ be the Lie algebra of smooth vector fields. The action of $Aff(3)$ on $A^3$ can be written as a homomorphism $Aff(3) \xrightarrow{\Phi} Diff(A^3)$. $\Phi$ induces the Lie algebra homomorphism $aff(3) \xrightarrow{\phi} \mathcal{X}(A^3)$. The Lie algebra $aff(3)$ has a natural basis $X_i, X_{ij}, 1 \leq i, j \leq 3$. Under the homomorphism $\phi$ the basis corresponds to vector fields

\begin{align*}
X_1 &= \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \\
X_{11} &= x \frac{\partial}{\partial x}, \quad X_{12} = y \frac{\partial}{\partial y}, \quad X_{13} = z \frac{\partial}{\partial x}, \\
X_{21} &= x \frac{\partial}{\partial y}, \quad X_{22} = y \frac{\partial}{\partial y}, \quad X_{23} = z \frac{\partial}{\partial y}, \\
X_{31} &= x \frac{\partial}{\partial z}, \quad X_{32} = y \frac{\partial}{\partial z}, \quad X_{33} = z \frac{\partial}{\partial z}.
\end{align*}
Similarly the induced action of \(\text{Aff}(3)\) on \(J^{2,3}(A^3)\) can be written as a homomorphism \(\text{Aff}(3) \to Diff(J^{2,3}(A^3))\). It induces the Lie algebra homomorphism \(\text{aff}(3) \to \mathfrak{X}(J^{2,3}(A^3))\). Under the homomorphism \(\psi\), the vector fields corresponding to the basis \(X_i, X_{ij}, 1 \leq i, j \leq 3\) are the prolongation of the above vector fields. The prolongation vector fields can be calculated by the Theorem 4.16 in Olver\[9\].

Computation shows that on \(J^{2,3}(A^3)\) these vector fields are given by

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial a_{00}},
\]

\[
X_{11} = x \frac{\partial}{\partial x} - a_{10} \frac{\partial}{\partial a_{10}} - 2a_{20} \frac{\partial}{\partial a_{20}} - a_{11} \frac{\partial}{\partial a_{11}} - 3a_{30} \frac{\partial}{\partial a_{30}} - 2a_{21} \frac{\partial}{\partial a_{21}} - a_{12} \frac{\partial}{\partial a_{12}},
\]

\[
X_{12} = y \frac{\partial}{\partial x} - a_{10} \frac{\partial}{\partial a_{01}} - a_{20} \frac{\partial}{\partial a_{11}} - 2a_{11} \frac{\partial}{\partial a_{02}} - a_{30} \frac{\partial}{\partial a_{21}} - 2a_{21} \frac{\partial}{\partial a_{12}} - 3a_{12} \frac{\partial}{\partial a_{03}},
\]

\[
X_{13} = a_{00} \frac{\partial}{\partial x} - a_{10} \frac{\partial}{\partial a_{10}} - a_{10} a_{01} \frac{\partial}{\partial a_{01}} - 3a_{10} a_{20} \frac{\partial}{\partial a_{20}} - (a_{01} a_{20} + 2a_{10} a_{11}) \frac{\partial}{\partial a_{11}} - (2a_{01} a_{11} + a_{10} a_{02}) \frac{\partial}{\partial a_{02}} - (4a_{10} a_{30} + 3a_{20}^2) \frac{\partial}{\partial a_{30}} - (a_{01} a_{30} + 3a_{10} a_{21} + 3a_{20} a_{11}) \frac{\partial}{\partial a_{21}} - (2a_{01} a_{21} + 2a_{20} a_{02} + 2a_{10} a_{12} + 2a_{11}^2) \frac{\partial}{\partial a_{12}} \nonumber
\]

\[
+(3a_{01} a_{12} + 3a_{11} a_{02} + a_{10} a_{03}) \frac{\partial}{\partial a_{03}},
\]

\[
X_{21} = x \frac{\partial}{\partial y} - a_{01} \frac{\partial}{\partial a_{10}} - 2a_{11} \frac{\partial}{\partial a_{02}} - a_{21} \frac{\partial}{\partial a_{11}} - 3a_{21} \frac{\partial}{\partial a_{30}} - 2a_{12} \frac{\partial}{\partial a_{21}} - a_{03} \frac{\partial}{\partial a_{12}},
\]

\[
X_{22} = y \frac{\partial}{\partial y} - a_{01} \frac{\partial}{\partial a_{01}} - a_{11} \frac{\partial}{\partial a_{11}} - 2a_{20} \frac{\partial}{\partial a_{02}} - a_{21} \frac{\partial}{\partial a_{21}} - 2a_{12} \frac{\partial}{\partial a_{12}} - 3a_{03} \frac{\partial}{\partial a_{03}},
\]

\[
X_{23} = a_{00} \frac{\partial}{\partial y} - a_{01} a_{10} \frac{\partial}{\partial a_{10}} - a_{01} \frac{\partial}{\partial a_{01}} - (a_{01} a_{20} + 2a_{10} a_{11}) \frac{\partial}{\partial a_{20}} - (2a_{01} a_{11} + a_{10} a_{02}) \frac{\partial}{\partial a_{11}} - 3a_{01} a_{02} \frac{\partial}{\partial a_{02}} \nonumber
\]

\[
-(a_{01} a_{30} + 3a_{10} a_{21} + 3a_{20} a_{11}) \frac{\partial}{\partial a_{30}} - (2a_{01} a_{21} + 2a_{20} a_{02} + 2a_{10} a_{12} + 2a_{11}^2) \frac{\partial}{\partial a_{21}} \nonumber
\]

\[
-(3a_{01} a_{12} + 3a_{11} a_{02} + a_{10} a_{03}) \frac{\partial}{\partial a_{12}} - (4a_{01} a_{03} + 3a_{20}^2) \frac{\partial}{\partial a_{03}},
\]

\[
X_{31} = x \frac{\partial}{\partial a_{00}} + \frac{\partial}{\partial a_{10}},
\]

\[
X_{32} = y \frac{\partial}{\partial a_{00}} + \frac{\partial}{\partial a_{01}},
\]

\[
X_{33} = a_{00} \frac{\partial}{\partial a_{00}} + a_{10} \frac{\partial}{\partial a_{10}} + a_{01} \frac{\partial}{\partial a_{01}} + a_{20} \frac{\partial}{\partial a_{20}} + a_{11} \frac{\partial}{\partial a_{11}} + a_{02} \frac{\partial}{\partial a_{02}} \nonumber
\]

\[
+a_{30} \frac{\partial}{\partial a_{30}} + a_{21} \frac{\partial}{\partial a_{21}} + a_{12} \frac{\partial}{\partial a_{12}} + a_{03} \frac{\partial}{\partial a_{03}}.
\]

The Maurer-Cartan invariants can be computed by the following theorem of Fels and Olver\[3\],\[4\].
Theorem 4.1. Let $F$ be a differential function on jet spaces $J^{2,\tau}(A^3)$ and $\iota(F)$ be its invariantization. Then
\[ D_k(\iota(F)) = \iota(D_k(F)) - \sum_{i,j=1}^{3} \iota(X_{ij}F)R^k_{ij}, \]

where $R^k_{ij}, 1 \leq i, j \leq 3, 1 \leq k \leq 2$ are the Maurer-Cartan differential invariants.

In this theorem, $D_k$ represents the partial differential operator on jet space. For example $D^1(a_{21}) = a_{31}$ and $D^2(a_{21}) = a_{22}$.

Lemma 4.1. The Maurer-Cartan invariants are
\[
\begin{pmatrix}
R_{11}^1 & R_{12}^1 & R_{13}^1 \\
R_{21}^1 & R_{22}^1 & R_{23}^1 \\
R_{31}^1 & R_{32}^1 & R_{33}^1
\end{pmatrix} = \begin{pmatrix}
-\frac{I_{40}}{4} + \frac{3I_{44}}{4} - \frac{1}{2} & -\frac{I_{43}}{4} + \frac{3I_{43}}{2} & -\frac{I_{40}}{4} + \frac{I_{42}}{4} + \frac{1}{2} \\
\frac{I_{43}}{4} - \frac{I_{44}}{2} & 0 & -\frac{I_{42}}{4} + \frac{I_{43}}{2} \\
0 & 1 & -\frac{I_{41}}{2} + \frac{3I_{44}}{2}
\end{pmatrix}
\]
\[
\begin{pmatrix}
R_{11}^2 & R_{12}^2 & R_{13}^2 \\
R_{21}^2 & R_{22}^2 & R_{23}^2 \\
R_{31}^2 & R_{32}^2 & R_{33}^2
\end{pmatrix} = \begin{pmatrix}
-\frac{I_{41}}{4} & -\frac{I_{42}}{4} & -\frac{I_{43}}{4} - \frac{I_{44}}{4} + \frac{1}{2} \\
\frac{I_{42}}{4} - \frac{I_{44}}{2} + \frac{1}{2} & 0 & -\frac{I_{43}}{4} - \frac{I_{44}}{4} + \frac{1}{2} \\
0 & 1 & -\frac{I_{41}}{2} + \frac{3I_{44}}{2}
\end{pmatrix}
\]

Proof: By the expressions of vector fields $X_{ij}$ and Theorem 4.1, under the invariantization condition $a_{10} = a_{01} = a_{11} = 0, a_{20} = a_{02} = 1$, we have
\[
R_{31}^1 - \iota(a_{20}) = 0, \quad R_{32}^1 - \iota(a_{11}) = 0
\]
\[
R_{32}^1 - \iota(a_{02}) = 0, \quad -2R_{11}^1 + R_{33}^1 - \iota(a_{30}) = 0
\]
\[
-R_{12}^1 - \iota(a_{21}) = 0, \quad -2R_{11}^2 + R_{33}^2 - \iota(a_{21}) = 0
\]
\[
-R_{12}^1 - R_{21}^1 - \iota(a_{12}) = 0, \quad -2R_{12}^2 + R_{23}^2 - \iota(a_{03}) = 0
\]
By solving these equations we have
\[
R_{31}^1 = 1, R_{32}^1 = 0, R_{11}^1 = \frac{R_{33}^1 - \frac{1}{2}}{2}, R_{22}^1 = \frac{R_{33}^1 + \frac{1}{2}}{2}, R_{21}^1 = -R_{12}^1;
\]
\[
R_{31}^2 = 0, R_{32}^2 = 1, R_{11}^2 = \frac{R_{33}^2}{2}, R_{22}^2 = \frac{R_{33}^2 - \frac{1}{2}}{2}, R_{21}^2 = -R_{12}^2 + 1.
\]
The conditions $a_{30} = -a_{12} = 1$, and $a_{21} = a_{03} = 0$ give the equations
\[
-3R_{11}^1 - 3R_{13}^1 + R_{33}^1 - \iota(a_{40}) = 0
\]
\[
-R_{12}^1 + 2R_{21}^1 - R_{13}^1 - \iota(a_{31}) = 0
\]
\[
R_{11}^1 - R_{13}^1 + 2R_{22}^1 - R_{33}^1 - \iota(a_{22}) = 0
\]
\[
3R_{12}^1 - 3R_{23}^1 - \iota(a_{13}) = 0
\]
and
\[
-3R_{11}^2 - 3R_{13}^2 + R_{33}^2 - \iota(a_{31}) = 0
\]
\[
-R_{12}^2 + 2R_{21}^2 - R_{23}^2 - \iota(a_{22}) = 0
\]
\[
R_{11}^2 - R_{13}^2 + 2R_{22}^2 - R_{33}^2 - \iota(a_{13}) = 0
\]
\[
3R_{12}^2 - 3R_{23}^2 - \iota(a_{04}) = 0
\]
By solving these equations, we prove the lemma.
Theorem 4.2. Let S be a regular elliptical surface, then the 1-forms \( w_1, w_2 \) and \( I_{40}, I_{31}, I_{22} \) are complete affine differential invariants.

Proof: By Cartan [2], the moving equations of the moving frame determine the surface up to congruence. Hence by Equations 18, 19 and Lemma 4.1, \( w_1, w_2 \) and \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) are complete differential invariants. Define invariants \( Y_1, Y_2 \) by \( dw_1 = Y_1 w_1 \wedge w_2, dw_2 = Y_2 w_1 \wedge w_2 \). Computation shows \( Y_1 = R_{11}^2 - R_{12}^1 = \frac{2I_{13}}{3} \) and \( Y_2 = R_{21}^2 - R_{22}^1 = \frac{I_{40} - I_{22}}{4} - \frac{I_{04}}{12} \). Since \( Y_1, Y_2 \) are invariants determined by \( w_1, w_2 \), and \( I_{13}, I_{04} \) can be computed from \( Y_1, Y_2 \) and \( I_{40}, I_{31}, I_{22} \), this proves the theorem.

Definition 4.2. We call the invariants \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) the fundamental curvatures of elliptical surfaces.

Definition 4.3. The first fundamental forms of a regular elliptical surface are defined to be the 1-forms \( \Phi^1_1 = w_1 \) and \( \Phi^1_2 = w_2 \). The second, third and fourth fundamental forms of a regular elliptical surface are \( \Phi_2 = w_1^2 + w_2^2, \Phi_3 = w_1^3 - 3w_1w_2^2, \Phi_4 = I_{40}w_1^4 + 4I_{31}w_1^3w_2 + 6I_{22}w_1^2w_2^2 + 4I_{13}w_1w_2^3 + I_{04}w_2^4 \).

Theorem 4.3. Let S be a regular elliptical surface, then the second, third and fourth fundamental forms \( \Phi_2, \Phi_3, \Phi_4 \) are complete differential invariants.

Proof: We note that if \( w_1, w_2 \) is a solution of equations \( w_1^2 + w_2^2 = \Phi_2, w_1^3 - 3w_1w_2^2 = \Phi_3, \) then other solutions are given by the action of symmetry group \( D_3 \) in Corollary 3.2 on this solution. Hence we can solve \( w_1, w_2 \) from \( \Phi_2, \Phi_3 \). By expanding \( \Phi_4 \) with respect to \( w_1, w_2 \), we can determine \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \). So \( \Phi_2, \Phi_3, \Phi_4 \) determine \( w_1, w_2 \) and \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) up to the action of \( D_3 \). This proves the theorem.

4.2 The compatible conditions

By Theorem 4.2, the 1-forms \( w_1, w_2 \) and \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) are complete differential invariants. But not all sets of \( w_1, w_2 \) and \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) can be realized as the invariants of some surfaces. They must satisfy some compatible conditions which are given by the Cartan’s structure equation \( d\Omega + \Omega \wedge \Omega = 0 \). The components of this equation are

1. \( dw_{ij} = -w_{ik} \wedge w_{kj}, 1 \leq i, j \leq 3 \).
2. \( dw_1 = -w_{11} \wedge w_1 - w_{12} \wedge w_2, dw_2 = -w_{21} \wedge w_1 - w_{22} \wedge w_2 \).

For \( dw_{ij} = -w_{ik} \wedge w_{kj}, \) the left side of this equation is
\[
dw_{ij} = d(R^k_{ij}w_k) = d(R^{ik}_{ij}) \wedge w_k + R^k_{ij}dw_k, \\
dw_{ij} = (D^1R^k_{ij}w_1 + D^2R^k_{ij}w_2)w_k + R^k_{ij}Y_kw_1 \wedge w_2, \\
dw_{ij} = (D^1R^2_{ij} - D^2R^1_{ij} + R^1_{ij}Y_1 + R^2_{ij}Y_2)w_1 \wedge w_2, \\
\]
The right side of the equation is
\[
-w_{ik} \wedge w_{kj} = -R^i_{ik}w_1 \wedge R^m_{kj}w_m = (R^2_{ik}R^1_{kj} - R^1_{ik}R^2_{kj})w_1 \wedge w_2. \\
\]

Hence the structure equations give
\[
D^1R^2_{ij} - D^2R^1_{ij} + R^1_{ij}Y_1 + R^2_{ij}Y_2 = R^2_{ik}R^1_{kj} - R^1_{ik}R^2_{kj}. \\
\]

Theorem 4.4. The compatible conditions are given by
\[ 36D^2 I_{31} - 36D^1 I_{22} - 12D^2 I_{13} + 12D^1 I_{04} - 9I_{40} + 3I_{40} I_{04} - 24I_{31} I_{13} + 18I_{22}^2 - 3I_{22} I_{04} + 8I_{13}^2 - I_{04}^2 - 18I_{40} - 36I_{22} - 18I_{04} = 0. \]
\[ 12D^2 I_{40} - 12D^1 I_{31} + 12D^2 I_{22} - 12D^1 I_{13} - 3I_{40} I_{31} - 23I_{40} I_{13} + 5I_{31} I_{04} - 6I_{22} I_{13} + 3I_{13} I_{04} + 48I_{13} = 0. \]
\[ 12D^2 I_{31} - 12D^1 I_{22} + 12D^2 I_{13} - 12D^1 I_{04} - 3I_{40} I_{22} - 7I_{40} I_{04} - 16I_{31} I_{13} + 18I_{22}^2 + 5I_{22} I_{04} - 16I_{13}^2 + I_{04}^2 + 18I_{40} - 36I_{22} - 6I_{04} = 0. \]
\[ 12D^2 I_{40} + 12D^1 I_{31} - 36D^2 I_{22} + 36D^1 I_{13} - 3I_{40} I_{31} + I_{40} I_{13} + 6I_{31} I_{22} + 3I_{31} I_{04} + 6I_{22} I_{13} - 3I_{13} I_{04} = 0. \]
\[ dw_1 = Y_1 w_1 \wedge w_2. \]
\[ dw_2 = Y_2 w_1 \wedge w_2. \]

**Proof:** This is computed directly from Equation 20 and Lemma 4.1. The first compatible condition corresponds to \( i = 1, j = 2 \) in Equation 20, the second corresponds to \( i = 1, j = 3 \), the third corresponds to \( i = 2, j = 3 \), the fourth corresponds to \( i = 3, j = 3 \), the fifth corresponds to \( i = 3, j = 1 \) and the sixth corresponds to \( i = 3, j = 2 \). And the other compatible conditions give no essentially new relation.

**Corollary 4.1.** Let \( S \) be a regular elliptical surface with constant curvatures \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \), then these invariants satisfy

\[
(-23I_{40} - 6I_{22} + I_{04} + 48)I_{13} + (-3I_{40} + 18I_{22} + 5I_{04})I_{31} = 0 \tag{21}
\]

\[(I_{40} + 6I_{22} - 3I_{04})I_{13} + (-3I_{40} + 6I_{22} + I_{04})I_{31} = 0 \tag{22}
\]

\[-9I_{40} I_{22} + 3I_{40} I_{04} - 24I_{31} I_{13} + 18I_{22}^2 - 3I_{22} I_{04} + 8I_{13}^2 - I_{04}^2 - 18I_{40} - 36I_{22} - 18I_{04} = 0 \tag{23}
\]

\[-3I_{40} I_{22} - 7I_{40} I_{04} - 16I_{31} I_{13} + 18I_{22}^2 + 15I_{22} I_{04} - 16I_{13}^2 + I_{04}^2 + 18I_{40} - 36I_{22} - 6I_{04} = 0 \tag{24}
\]

**Proof:** The Equations 21, 22, 23 and 24 are derived directly from the second, the fourth, the first and the third equations of Theorem 4.4 respectively.

## 5 Classification of elliptical surfaces with constant curvatures

In this section we classify the elliptical surfaces with constant curvatures. At first we solve the solutions of Equations 21, 22, 23 and 24.

**Theorem 5.1.** Let \( S \) be a regular elliptical surface with constant curvatures, then \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) must be of the forms in the following table.

| \( I_{40} \) | \( I_{31} \) | \( I_{22} \) | \( I_{13} \) | \( I_{04} \) |
|--------|--------|--------|--------|--------|
| 1      | \( k \) | 0      | -1     | 0      | 3      |
| 2      | -3k^2 + 6k | 0    | -k^2 - 2k + 2 | 0    | -3k^2 - 6k |
| 3      | \( k \) | \( l(2k - 3) \) | \( k \) | 0      | -3k    |
| 4      | \( k \) | \( \pm \sqrt{k(3 - k)} \) | -k + 2 | \( \pm \sqrt{k(3 - k)} \) | \( k \) |
| 5      | -3k^2 + \frac{9}{4}k + \frac{9}{4} | -3\sqrt{3}(3k + 1) | -k^2 + \frac{1}{4}k - \frac{1}{4} | -3\sqrt{3}(k + 1) | -3k^2 + \frac{9}{4}k + \frac{9}{4} |
| 6      | \( k \) | \( \pm 3\sqrt{2}(2k - 3) \) | 3k - 1 | \( \pm 3\sqrt{3}(2k - 1) \) | 9k - 6 |

In case 4, we have 0 < \( k < 3 \).
As the proof of this theorem involves quite tedious computation, we write it as an appendix.

To classify the surfaces with constant curvatures, we need to consider the action of $D_3$ on $(I_{40}, I_{31}, I_{22}, I_{13}, I_{04})$.

**Lemma 5.1.** The induced action of $D_3$ on $(I_{40}, I_{31}, I_{22}, I_{13}, I_{04})$ given by $\sigma$ is

$$(I'_{40}, I'_{31}, I'_{22}, I'_{13}, I'_{04}) = \frac{1}{16} (I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) \begin{pmatrix} 1 & \sqrt{3} & 3 & 3\sqrt{3} & 9 \\ -4\sqrt{3} & -8 & -4\sqrt{3} & 0 & 12\sqrt{3} \\ 18 & 6\sqrt{3} & -2 & -6\sqrt{3} & 18 \\ -12\sqrt{3} & 0 & 4\sqrt{3} & -8 & 4\sqrt{3} \\ 9 & -3\sqrt{3} & 3 & -\sqrt{3} & 1 \end{pmatrix}.$$  

The induced action given by $\tau$ is

$$(I'_{40}, I'_{31}, I'_{22}, I'_{13}, I'_{04}) = (I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

**Proof:** This is computed from Equation 17 and the terms of the fourth power of Equation 11.

**Theorem 5.2.** Let $S$ be a regular elliptical surface with constant curvatures, then up to $D_3$ action, the curvature $(I_{40}, I_{31}, I_{22}, I_{13}, I_{04})$ of $S$ has one of the following forms.

1. $A_k^1 = (k, 0, -1, 0, 3)$, $A_k^2 = (-3k^2 + 6k, 0, -k^2 - 2k + 2, 0, -3k^2 - 6k)$ or $A_k^3 = (k, 0, k, 0, -3k)$.
2. $B_{k,l} = (k, l(2k - 3), k, 0, -3k)$, where $l(2k - 3) > 0$.
3. $C_k = (k, \sqrt{k(3 - k)}, -k + 2, -\sqrt{k(3-k)}, k)$.

**Proof:** This is computed from the Theorem 5.1 by considering the $D_3$ action. The $A_k^1$ form comes from the first case of Theorem 5.1, the $A_k^2$ form comes from the second case and the $A_k^3$ form comes from the third case with $l = 0$. The $B_{k,l}$ form comes from the third case of Theorem 5.1 with $l \neq 0$. The $C_k$ form comes from the fourth case. Note that under the action of $D_3$, cases 1 and 6 as well as cases 2 and 5 can be transformed to each other by the action of $\sigma$ in Lemma 5.1.

We need the following lemma to finish the classification of regular elliptical surfaces with constant curvatures.

**Lemma 5.2.** If $w_1, w_2$ are 1-forms satisfying the equations

$$dw_1 = \lambda w_1 \wedge w_2, dw_2 = \mu w_1 \wedge w_2, w_1 \wedge w_2 \neq 0.$$  

where $(\lambda, \mu) \in \mathbb{R}^2$ are constants. Then $w_1$ and $w_2$ are determined.

**Proof:** Case 1. If $(\lambda, \mu) = (0, 0)$, the equations have the local solutions $w_1 = df, w_2 = dg$ with $df \wedge dg \neq 0$, and $w_1, w_2$ are the pull back of $du, dv$ by the local diffeomorphism $u = f(x, y), v = g(x, y)$.

Case 2. If $(\lambda, \mu) = (\lambda, 0), \lambda \neq 0$, the solutions of these equations must be of the forms $w_1 = a df, w_2 = dg$ for certain $f, g$ and $a$ with $adf \wedge dg \neq 0$. Let $u = f(x, y), v = g(x, y)$, we get $-\frac{\partial a}{\partial v} = \lambda a$. Hence we have $w_1 = e^{-\lambda v} du, w_2 = dv$ up to local diffeomorphisms.
Case 3. If \( \mu \neq 0 \), define \( w'_1 = \frac{1}{\mu} w_2 \), \( w'_2 = -\mu w_1 + \lambda w_2 \), then it is easy to check that \( dw'_1 = w'_1 \wedge w'_2, dw'_2 = 0 \), \( w'_1 \wedge w'_2 = w_1 \wedge w_2 \neq 0 \). So \( w'_1, w'_2 \) can be determined by Case 2. It follows that \( w_1 \) and \( w_2 \) are determined up to local diffeomorphisms.

**Theorem 5.3.** A regular elliptical surface is determined by the invariants \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) up to affine congruence.

**Proof:** Let \( S \) be a regular elliptical surface with constant curvatures \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \), then \( w_1, w_2 \) satisfy Equation 25 with \( (\lambda, \mu) = (Y_1, Y_2) \), where \( Y_1 = \frac{2I_{13}}{3}, Y_2 = \frac{I_{40}}{4} - \frac{I_{22}}{2} - \frac{I_{04}}{12} \) are defined in Section 4. By Lemma 5.2, \( w_1, w_2 \) are determined up to local diffeomorphisms by \( I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \). Since \( w_1, w_2, I_{40}, I_{31}, I_{22}, I_{13}, I_{04} \) are complete differential invariants, the theorem is proved.

### 6 Appendix: the solutions of the compatible equations

To prove Theorem 5.1, we need to solve the compatible equations in Corollary 4.1. Denote

\[
\begin{align*}
J_1 &:= (-23I_{40} - 6I_{22} + I_{04} + 48)I_{13} + (-3I_{40} + 18I_{22} + 5I_{04})I_{31} = 0 \\
J_2 &:= (I_{40} + 6I_{22} - 3I_{04})I_{13} + (-3I_{40} + 6I_{22} + I_{04})I_{31} = 0 \\
K_1 &:= -9I_{40}I_{22} + 3I_{40}I_{04} - 24I_{31}I_{13} + 18I_{22}I_{04} + 8I_{13}I_{31} - I_{04}^2 - 18I_{40} - 36I_{22} - 18I_{04} = 0 \\
K_2 &:= -3I_{40}I_{22} - 7I_{40}I_{04} - 16I_{31}I_{13} + 18I_{22}I_{04} - 16I_{13}I_{31} + I_{04}^2 + 18I_{40} - 36I_{22} - 6I_{04} = 0 \\
H_{11} &:= -23I_{40} - 6I_{22} + I_{04} + 48, \quad H_{12} := -3I_{40} + 18I_{22} + 5I_{04} \\
H_{21} &:= I_{40} + 6I_{22} - 3I_{04}, \quad H_{22} := -3I_{40} + 6I_{22} + I_{04} \\
K_3 &:= H_{11}H_{22} - H_{12}H_{21} = 9I_{40}^2 - 15I_{40}I_{22} - 5I_{40}I_{04} - 18I_{22}I_{04} + 2I_{04}^2 - 18I_{40} + 36I_{22} + 6I_{04} \\
K_2 &:= K_2 - K_1 = (6I_{40} + 18I_{04})I_{22} - 10I_{40}I_{04} + 8I_{31}I_{13} - 24I_{22}I_{04} + 2I_{04}^2 + 12I_{40} + 12I_{04} = 0.
\end{align*}
\]

Case 1. If \( K_3 \neq 0 \), then by Equations 21 and 22, we must have \( I_{13} = I_{31} = 0 \). By substituting \( I_{13} = I_{31} = 0 \) in \( K_2 \), we obtain

\[
(3I_{40} + 9I_{04})I_{22} - 5I_{40}I_{04} + I_{04}^2 + 18I_{40} + 6I_{04} = 0 \tag{26}
\]

i) If \( I_{40} + 3I_{04} = 0 \), then \(-5I_{40}I_{04} + I_{04}^2 + 18I_{40} + 6I_{04} = 16I_{04}^2 = 48I_{04} = 0 \). We have \((I_{40}, I_{04}) = (0, 0) \) or \((-9, 3)\).

If \((I_{40}, I_{04}) = (0, 0)\), then by \( K_1 = 0 \) we get \( I_{22} = 0 \) or 2. In this case
\((I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (0, 0, 0, 0, 0) \) or \((0, 0, 2, 0, 0)\).

These two solutions are included in the third and the second cases of the table in Theorem 5.1 respectively.

If \((I_{40}, I_{04}) = (-9, 3)\), then by \( K_1 = 0 \) we get \( I_{22} = -1 \). In this case
\((I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (-9, 0, -1, 0, 3)\).

This solution is included in the first case of the table in Theorem 5.1.

ii) If \( I_{40} + 3I_{04} \neq 0 \), then by Equation 26, we have
\[
I_{22} = \frac{-5I_{40}I_{04} + I_{04}^2 + 18I_{40} + 6I_{04}}{3(I_{40} + 3I_{04})}. \tag{27}
\]
By substituting the expression of $I_{22}$ into $K_1$, we get

$$K_1 = -\frac{4(I_{04} - 3)(3I_{40} + I_{04})(I_{40}^2 - 2I_{40}I_{04} + I_{04}^2 + 24I_{40} + 24I_{04})}{(I_{40} + 3I_{04})^2}.$$ 

Hence $I_{04} = 3, 3I_{40} + I_{04} = 0$ or $I_{40}^2 - 2I_{40}I_{04} + I_{04}^2 + 24I_{40} + 24I_{04} = 0$.

If $I_{04} = 3$, then by Equation 27 we get $I_{22} = -1$. Set $I_{40} = k$, we have $(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (k, 0, -1, 0, 3)$.

This solution is included in the first case of the table in Theorem 5.1.

If $3I_{40} + I_{04} = 0$, then $I_{04} = -3I_{40}$. By Equation 27 we get $I_{22} = I_{40}$. Set $I_{40} = k$, we have $(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (k, 0, k, 0, -3k)$.

This solution is included in the third case of the table in Theorem 5.1.

If $I_{40}^2 - 2I_{40}I_{04} + I_{04}^2 + 24I_{40} + 24I_{04} = 0$, then by regarding this equation as a parabola, we can parameterize it as $I_{40} = -3k^2 + 6k, I_{04} = -3k^2 - 6k$. By Equation 27 we get $I_{22} = -k^2 - 2k + 2$.

Hence

$$(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (-3k^2 + 6k, 0, -k^2 - 2k + 2, 0, -3k^2 - 6k).$$

This solution is included in the second case of the table in Theorem 5.1.

Case 2. If $K_3 = 0$, then $I_{13}$ or $I_{31}$ can be nonzero.

i) If $H_{11} = -23I_{40} - 6I_{22} + I_{04} + 48 = 0$ and $H_{12} = -3I_{40} + 18I_{22} + 5I_{04} = 0$, then $I_{04} = 9I_{40} - 18, I_{22} = \frac{7}{3}I_{40} + 5$. Equation 22 can be written as $H_{21}I_{13} + H_{22}I_{31} = 0$. As in this case it is easy to check that $H_{21}$ and $H_{22}$ cannot be both zero. Then $I_{13} = kH_{22} = k(-3I_{40} + 6I_{22} + I_{04}), I_{31} = -kH_{21} = -k(I_{40} + 6I_{22} - 3I_{04})$, by substituting the expression into $K_1$ and $K_2$ we get

$$K_1 = 4(2I_{40} - 3)(1024I_{40}k^2 - 2112k^2 + 16I_{40} - 45) = 0,$$

$$K_2 = 8(64k^2 - 3)(2I_{40} - 3)(4I_{40} - 9) = 0.$$ 

By solving these two equations we get solutions $(I_{40}, k) = (\frac{3}{2}, k), (\frac{9}{4}, \pm \frac{\sqrt{3}}{2}).$ Hence

$$(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (\frac{3}{2}, -24k, \frac{3}{2}, 0, -\frac{9}{2}) \ 	ext{or} \ (\frac{9}{4}, \frac{\pm 3\sqrt{3}}{4}, \frac{-1}{4}, \frac{\pm 3\sqrt{3}}{4}, \frac{9}{4}).$$

These two solutions are included in the third and the fourth cases of the table in Theorem 5.1 respectively.

ii) If at least one of $H_{11}, H_{12}$ is not zero. Equation 21 can be written as $H_{11}I_{13} + H_{12}I_{31} = 0$. Then $I_{13} = kH_{12} = k(-3I_{40} + 18I_{22} + 5I_{04}), I_{31} = -kH_{11} = -k(-23I_{40} - 6I_{22} + I_{04} + 48)$. By substituting the expression into $K_1, K_2$ and $K_3$ we get

$$K_1 = k^2(1728I_{40}^2 - 10368I_{40}I_{22} - 3072I_{40}I_{04} + 1152I_{22}I_{04} + 320I_{04}^2 - 3456I_{40} - 20736I_{22} + 5760I_{04})$$

$$-9I_{40}I_{22} + 3I_{40}I_{04} + 18I_{22}^2 - 3I_{22}I_{04} - I_{04}^2 - 18I_{40} - 36I_{22} - 18I_{04} = 0,$$

$$K_2 = k^2(960I_{40}^2 - 4608I_{40}I_{22} + 6912I_{40}I_{04} - 6912I_{22}I_{04} + 320I_{04}^2 - 2304I_{40} + 13824I_{22}$$

$$+ 3840I_{04}) - 3I_{40}I_{22} + 7I_{40}I_{04} + 18I_{22}^2 + 15I_{22}I_{04} + I_{04}^2 + 18I_{40} - 36I_{22} - 6I_{04} = 0,$$

$$K_3 = 9I_{40}^2 - 15I_{40}I_{22} - 5I_{40}I_{04} - 18I_{22}^2 + 3I_{22}I_{04} + 12I_{04}^2 - 18I_{40} + 36I_{22} + 6I_{04}.$$ 

Rewrite $K_1$ and $K_2$ by $K_1 = k^2G_{11} + G_{12}$ and $K_2 = k^2G_{21} + G_{22}$, where $G_{11}, G_{12}, G_{21}, G_{22}$ are polynomials in $I_{40}, I_{22}, I_{04}$. We set $L = G_{11}G_{22} - G_{12}G_{21}$. Then $K_1 = 0$ and $K_2 = 0$ make $L = 0$. Computation shows
\[ L = 384(-3I_{40} + 18I_{22} + 5I_{04})((-3I_{40}^2I_{22} + 13I_{40}^2I_{04} - 21I_{40}I_{22}^2 - 24I_{40}I_{22}I_{04} - 3I_{40}I_{04}^2 + 18I_{22}^3 + 3I_{22}^2I_{04} + I_{22}I_{04}^2 - 42I_{40}^2 + 15I_{40}I_{22} - 33I_{40}I_{04} - 18I_{22}^2 + 21I_{22}I_{04} + I_{04}^2 + 90I_{40} - 36I_{22} + 18I_{04}) = 0 \]

Set \( M = -3I_{40}^2I_{22} + 13I_{40}^2I_{04} - 21I_{40}I_{22}^2 - 24I_{40}I_{22}I_{04} - 3I_{40}I_{04}^2 + 18I_{22}^3 + 3I_{22}^2I_{04} + I_{22}I_{04}^2 - 42I_{40}^2 + 15I_{40}I_{22} - 33I_{40}I_{04} - 18I_{22}^2 + 21I_{22}I_{04} + I_{04}^2 + 90I_{40} - 36I_{22} + 18I_{04} \).

a) If \( H_{12} = -3I_{40} + 18I_{22} + 5I_{04} = 0 \), then \( H_{11} = -23I_{40} - 6I_{22} + 9I_{04} = 48 \neq 0 \). Since \( J_1 = 0 \) we have \( I_{13} = 0 \). And \( I_{22} = \frac{3I_{40} - 5I_{04}}{18} \); By substituting \( I_{22} \) into \( K_1, K_2 \) and \( K_3 \), we have

\[ K_1 = \frac{1}{9}(3I_{40} + I_{04})(-3I_{40} + 11I_{04} - 72) = 0 \]
\[ K_2 = -\frac{1}{9}(3I_{40} + I_{04})(4I_{04} - 9) = 0 \]
\[ K_3 = -\frac{1}{9}(3I_{40} + I_{04})(-9I_{40} + I_{04} + 18) = 0 \]

Since \( K_1 = K_2 = K_3 = 0 \), we must have \( 3I_{40} + I_{04} = 0 \). And we have a solution of the form \( (I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (I_{40}, (2I_{40} - 3), I_{40}, 0, -3I_{40}) \). This solution is included in the third case of the table in Theorem 5.1.

b) If \( M = 0 \), then by \( K_3 = 0 \) we have

\[ I_{04} = \frac{9}{2}I_{40} - \frac{15}{2}I_{40}I_{22} + \frac{5}{2}I_{40}I_{04} + 9I_{22}^2 - \frac{3}{2}I_{22}I_{04} + 9I_{40} - 18I_{22} - 3I_{04}. \]

By substituting the expression of \( I_{04} \) into \( M = 0 \), we have

\[ M = \frac{1}{2}(-I_{40} + 3I_{22} + 3)(-27I_{40}^2 - 21I_{40}I_{22} - 11I_{40}I_{04} + 18I_{22}^2 + I_{22}I_{04} + 66I_{40} - 36I_{22} + 10I_{04}) = 0 \]

If \( -I_{40} + 3I_{22} + 3 = 0 \), then \( I_{40} = 3I_{22} + 3 \). By substituting \( I_{40} \) into \( K_3 \), we get \( K_3 = 18I_{22}^2 - 12I_{22}I_{04} + 21I_{04}^2 + 99I_{22} - 9I_{04} + 27 = 0 \). Regarding this equation as a parabola equation, it can be parameterized as

\[ I_{04} = -\frac{3}{8}(2l^2 - 11l + 3), I_{22} = -\frac{1}{8}(-2l^2 - 3l + 3). \]

Inserting the expression of \( I_{40}, I_{22}, I_{04} \) into \( K_1, K_2 \), we have \( K_1 = -9(l - 3)(256(l - 1)^2k^2 - 3) \) and \( K_2 = -9(l - 1)(l - 3)(256(l - 1)^2k^2 - 3) \). Hence \( l = 3 \) or \( 256(l - 1)^2k^2 = 3 \). And we have

\[ \left(\frac{3(768k^2 \pm 8\sqrt{3}k - 3)}{1024k^2}, \frac{3\sqrt{3}(\pm 32k - 3\sqrt{3})}{128k}\right), \frac{(256k^2 \pm 8\sqrt{3}k - 3)}{1024k^2}, \frac{3\sqrt{3}(\pm 32k - 3\sqrt{3})}{128k}, \frac{3(768k^2 \pm 56\sqrt{3}k - 3)}{1024k^2}\right). \]

We replace \( k \) by \( \frac{\sqrt{3}}{32k} \) then the later case can be written as

\[ (I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (-3k^2 \pm \frac{3}{4}k + \frac{9}{4}, -\frac{3\sqrt{3}}{4}(3k \mp 1), -k^2 \pm \frac{k}{4} - \frac{1}{4}, -\frac{3\sqrt{3}}{4}(k \pm 1), -3k^2 \pm \frac{21}{4}k + \frac{9}{4}). \]

These solutions are included in the third and the fifth cases of the table in Theorem 5.1 respectively.

If \( -27I_{40}^2 - 21I_{40}I_{22} - 11I_{40}I_{04} + 18I_{22}^2 + I_{22}I_{04} + 66I_{40} - 36I_{22} + 10I_{04} = 0 \), then by adding \( K_3 = 9I_{40}^2 - 15I_{40}I_{22} - 5I_{40}I_{04} - 18I_{22}^2 + 3I_{22}I_{04} + 2I_{04}^2 - 18I_{40} + 36I_{22} + 6I_{04} = 0 \) to both sides of the above equation, we have \((-36I_{40} + 4I_{04})I_{22} - 18I_{20}^2 - 164I_{40}I_{04} + 21I_{04}^2 + 48I_{40} + 16I_{04} = 0 \).

If \( -36I_{40} + 4I_{04} = 0 \), then we get \( (I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (0, -48k, 0, 0, 0) \).

This solution is included in the third case of the table in Theorem 5.1.
If $-36I_{40} + 4I_{04} \neq 0$, then we have
\[ I_{22} = -\frac{-9I_{40}^2 - 8I_{40}I_{04} + I_{04}^2 + 24I_{40} + 8I_{04}}{-18I_{40} + 2I_{04}}. \]

By substituting $I_{22}$ into $K_3 = 0$, we have
\[ K_3 = \frac{-4(-I_{40} + I_{04})(3I_{40} + I_{04})(-9I_{40} + I_{04} + 6)(-9I_{40} + I_{04} + 18)}{(-9I_{40} + I_{04})^2} = 0. \]

If $I_{40} - I_{04} = 0$, then we get
\[(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (I_{40}, 4k(4I_{40} - 9), -I_{40} + 2, -4k(4I_{40} - 9), I_{40}).\]
By $K_1 = 0$, we have $k^2 = -\frac{1}{16}I_{40}(I_{40} - 3)/(4I_{40} - 9)^2$. This solution is the fourth case of the table in Theorem 5.1.

If $3I_{40} + I_{04} = 0$, then we get
\[(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (I_{40}, 16k(2I_{40} - 3), I_{40}, 0, -3I_{40}).\]
This solution is included in the third case of the table in Theorem 5.1.

If $-9I_{40} + 6 + I_{04} = 0$, then we get
\[(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (\frac{1}{2}, -32k, \frac{1}{2}, 0, -\frac{3}{2}) \text{ or } (I_{40}, \pm\sqrt{3}/2(2I_{40} - 3), 3I_{40} - 1, \pm3\sqrt{3}/2(2I_{40} - 1), 9I_{40} - 6).\]
These solutions are included in the third and the sixth cases of the table in Theorem 5.1.

If $-9I_{40} + 18 + I_{04} = 0$, then we get
\[(I_{40}, I_{31}, I_{22}, I_{13}, I_{04}) = (\frac{3}{2}, 0, \frac{3}{2}, 0, -\frac{9}{2}).\]
This solution is included in the third case of the table in Theorem 5.1.

These are all the possible cases for regular elliptical surfaces with constant curvatures.

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