Active Hypothesis Testing on A Tree: Anomaly Detection under Hierarchical Observations

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Abstract

The problem of detecting a few anomalous processes among a large number of $M$ processes is considered. At each time, aggregated observations can be taken from a chosen subset of processes, where the chosen subset conforms to a given binary tree structure. The random observations are i.i.d. over time with a general distribution that may depend on the size of the chosen subset and the number of anomalous processes in the subset. The objective is a sequential search strategy that minimizes the sample complexity (i.e., the expected number of observations which represents detection delay) subject to a reliability constraint. A sequential test that results in a biased random walk on the tree is developed and is shown to be asymptotically optimal in terms of detection accuracy. Furthermore, it achieves the optimal logarithmic-order sample complexity in $M$ provided that the Kullback-Liebler divergence between aggregated observations in the presence and the absence of anomalous processes are bounded away from zero at all levels of the tree structure as $M$ approaches infinity. Sufficient conditions on the decaying rate of the aggregated observations to pure noise under which a sublinear scaling in $M$ is preserved are also identified for the Bernoulli case.

Index Terms—Sequential design of experiments, active hypothesis testing, random walk, anomaly detection, noisy group testing.

I. INTRODUCTION

A. Searching for the Rare

The problem of searching for a few rare events of interest among a massive number of possibilities is ubiquitous. The rare events may represent opportunities with exceptional returns or anomalies associated with high costs or potential catastrophic consequences. Examples include financial trading opportunities and transmission opportunities in dynamic spectrum access, endogenous extreme events or exogenous attacks in communication and computer networks, structural anomalies on bridges or buildings, and high-risk contingencies in power systems that may lead to cascading failures.

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Regardless of the application domain, the problem of searching for the rare has the following defining features:
(i) the massive search space; (ii) the need for high detection accuracy, especially in terms of missing a rare event; (iii) the time sensitivity of the problem, either due to the transient nature of opportunities or the urgency for taking recourse measures in the face of anomalies. The goal is thus to detect the rare events as quickly and as reliably as possible when the total number of hypotheses is large and the observations are probabilistic thus inherently ambiguous. The performance measure of interest is sample complexity (the total number of observations which represents the detection delay) with respect to the size of the search space and the required detection accuracy.

A question of particular interest is while achieving the optimal scaling with respect to detection accuracy, whether a sublinear scaling of the sample complexity with respect to the search space is feasible. In other words, whether accurate detection can be achieved by examining only a diminishing fraction of the search space as the search space grows.

The key to a sublinear scaling in the problem size is to exploit the hierarchical structure of the search space inherent to many applications. For example, financial transactions can be aggregated at different temporal and geographic scales [1]. In computer vision applications such as bridge inspection by UAVs with limited battery capacity [2], sequentially determining areas to zoom in or zoom out can quickly locate anomalies by avoiding giving each pixel equal attention. In heavy hitter detection for Internet traffic monitoring, traffic flows follow a natural hierarchy based on prefix aggregation of the source or destination IP addresses. Indeed, recent advances in software defined networking (SDN) allow programmable routers to count aggregated flows that match a given IP prefix (by installing a wildcard rule in a TCAM entry) [4]. The search space of all traffic flows thus follows a binary tree structure.

B. Main Results

We consider the following prototypical model for searching for the rare in a binary-tree structured search space. Consider a large number $M$ of processes, among which $L$ are anomalous. The decision maker is allowed to search for the anomalous processes by taking (aggregated) observations from a subset of processes, where the chosen subset conforms to a given binary tree structure. The random observations are i.i.d. over time with a general distribution that may depend on the size of the chosen subset and the number of anomalies in the subset. The objective is a sequential search strategy that adaptively determines which node on the tree to probe at each time and when to terminate the search in order to minimize the sample complexity under a constraint on the error probability.

We develop an active search strategy and show that its sample complexity is asymptotically optimal in detection accuracy. In other words, it offers the optimal error exponent. Furthermore, we show that the proposed search strategy achieves a logarithmic-order sample complexity in $M$ provided that the Kullback-Liebler (KL) divergence between aggregated observations in the presence and the absence of anomalous processes are bounded away from

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1It is a common observation that Internet traffic flows are either “elephants” (heavy hitters) or “mice” (normal flows). A small percentage of high-volume flows account for most of the total traffic [3]. Heavy hitters can be defined as the top flows in terms of weight in total network traffic or flows with a weight exceeding a given threshold.
zero at all levels of the tree structure as \( M \) approaches infinity. It is thus order optimal in \( M \) as determined by the information theoretic lower bound. Using Bernoulli distribution as a case study, we also examine scenarios where higher level observations decay to pure noise as \( M \) grows. We establish sufficient conditions on the decaying rate of the quality of the hierarchical observations under which the proposed strategy achieves a sublinear sample complexity in \( M \).

The proposed search strategy is deterministic with search actions explicitly specified at each given time. It involves little online computation beyond calculating the sum log-likelihood ratio and performing simple comparisons. The analysis of its sample complexity in terms of both \( M \) and the detection accuracy is based on analyzing a biased random walk on the tree resulted from the search strategy. The desired scaling with \( M \) and the detection accuracy is achieved by ensuring that the random walk, initiated at the root of the tree, has a higher probability of moving toward than moving away from the anomalous processes at the leaf level of the tree.

**C. Related Work**

The anomaly detection problem considered here falls into the general class of sequential design of experiments pioneered by Chernoff in 1959 [5] in which he posed a binary (i.e., \( M = 2 \) for the problem at hand) active hypothesis testing problem. Compared with the classic sequential hypothesis testing pioneered by Wald [6] where the observation model under each hypothesis is fixed, active hypothesis testing has a control aspect that allows the decision maker to choose different experiments (associated with different observation models) at each time. Chernoff proposed a randomized strategy and showed that it is asymptotically optimal as the error probability approaches zero. Known as the Chernoff test, this randomized strategy chooses, at each time, a probability distribution governing the selection of experiments based on all past actions and observations. The probability distribution is given as a solution to a maxmin problem that can be difficult to solve, especially when the number \( M \) of hypotheses and/or the number of experiments (which is also \( M \) for the problem at hand) is large. Furthermore, the Chernoff test does not address the scaling in \( M \) and results in a linear sample complexity in \( M \) when applied to the problem considered here. Variations and extensions of Chernoff’s randomized test were given in [7]–[11]. In particular, in [10], Naghshvar and Javidi developed a randomized test that achieves the optimal logarithmic order of the detection delay in the number of hypotheses under certain implicit conditions. These conditions, however, do not hold for the problem considered here. Furthermore, similar to the Chernoff test, this randomized test is specified only implicitly as solutions to a sequence of maxmin problems that can be intractable for general observation distributions and large problem size.

The problem considered here shares similarity with the classic group testing problem [12]–[14]. In group testing, the objective is to identify defective items in a large population by performing tests on subsets of items that reveal whether the tested group contains any defective items. Most work on group testing assumes error-free test outcomes. The issue of sample complexity in terms of detection accuracy is absent in the basic formulation. There are several recent studies on noisy group testing that assume the presence of one-sided noise (either false alarms or miss detections but not both) [15]–[18] or the symmetric case with equal false alarm and miss detection probabilities that are independent of the size of the testing group [19]–[21]. We address in this paper general observation
models that go beyond these special cases of Bernoulli distributions. Another key difference is that these existing
results on noisy group testing focus on non-adaptive open-loop strategies that determine all actions in one shot \textit{a priori}. In other words, all group tests are predetermined and can be carried out in parallel. The problem can be
mapped to a channel coding problem, and achievable lower bounds on the sample complexity were established
using information-theoretic techniques such as the random coding argument [15], [16].

Order-optimal test algorithms with acceptable decoding complexity were also constructed in [18]–[20]. These
one-shot non-adaptive group testing problems are fundamentally different from the \textit{sequential} and \textit{active} hypothesis
testing problem studied in this work.

The problem of detecting anomalies or outlying sequences has been studied under different formulations, as-
sumptions, and objectives (see an excellent survey in [22] and references therein). These studies, in general, do not
address the optimal scaling in both the detection accuracy and the size of the search space.

II. Problem Formulation

Consider the problem of detecting \( L \) anomalous processes among \( M \) processes. We adopt a binary tree observation
model as illustrated in Fig. 1 where the leafs represent the individual processes. We start by considering the case
of \( L = 1 \). The case of multiple anomalous processes is discussed in Section [V]

Let \( g_0 \) and \( f_0 \) denote, respectively, the distributions of the anomalous process and the normal processes. Let
\( g_l \) (\( l = 1, \ldots, \log_2 M \)) denote the distribution of the measurements that aggregate the anomalous process and
\( 2^l - 1 \) normal processes, and \( f_l \) (\( l = 1, \ldots, \log_2 M \)) the distribution of the measurements that aggregate \( 2^l \) normal
processes (see Fig. 1). The relation between \( \{ g_l, f_l \} \) and \( \{ g_0, f_0 \} \) depends on the specific application. For example,
in the case of heavy hitter detection where the measurements are packet counts of an aggregated flow, \( g_l \) and \( f_l \)
are given by multi-fold convolutions of \( f_0 \) and \( g_0 \). For Poisson flows, \( g_l \) and \( f_l \) are also Poisson with mean values
given by the sum of the mean values of their children at the leaf level. As is the case in practically all applications,
we expect that observations from each individual process are more informative than aggregated observations. More
precisely, we expect \( D(g_0\|f_0) \geq D(g_l\|f_l) \) and \( D(f_0\|g_0) \geq D(f_l\|g_l) \) for all \( l > 0 \), where \( D(\cdot\|\cdot) \) denotes the KL
divergence between two distributions.

![Fig. 1. A binary tree observation model.](image)
We aim to develop an active search strategy that sequentially determines whether to terminate the search and if not, which node on the tree to probe next. Specifically, an active search strategy \( \Gamma = (\{ \phi(t) \}_{t \geq 1}, \tau, \delta) \) consists of a sequence of selection rules \( \{ \phi(t) \}_{t \geq 1} \) governing which node to probe at each time, a stopping rule \( \tau \) deciding when to terminate the search, and a declaration rule \( \delta \) deciding which leaf node is the target at the time of stopping.

We adopt a Bayesian approach as in Chernoff’s original work [5] and assign a cost of \( c \) for each observation and a loss of 1 for a wrong declaration. Let \( \pi_m \) denote the a priori probability that process \( m \) is anomalous, which is referred to as hypothesis \( H_m \). Let \( P_e(\Gamma) = \sum_{m=1}^{M} \pi_m \alpha_m(\Gamma) \) be the probability of error under strategy \( \Gamma \), where \( \alpha_m(\Gamma) = \Pr_m(\delta \neq m|\Gamma) \) is the probability of declaring \( \delta \neq m \) when \( H_m \) is true. Let \( E[\tau|\Gamma] = \sum_{m=1}^{M} \pi_m E_m[\tau|\Gamma] \) be the average sample complexity of \( \Gamma \). The average Bayes risk under strategy \( \Gamma \) is then given by

\[
R(\Gamma) = P_e(\Gamma) + cE[\tau|\Gamma].
\] (1)

The objective is to find a strategy \( \Gamma \) that achieves the lower bound of the Bayes risk:

\[
R^* = \inf_{\Gamma} R(\Gamma).
\] (2)

We are interested in test strategies that offer the optimal scaling in both \( c \) (characterizing the detection accuracy) and \( M \). A test \( \Gamma \) is said to be asymptotically optimal in \( c \) if, for fixed \( M \),

\[
\lim_{c \to 0} \frac{R(\Gamma)}{R^*} = 1.
\] (3)

A shorthand notation \( f \sim g \) will be used for \( \lim_{c \to 0} f/g = 1 \). A test \( \Gamma \) is said to be order optimal in \( c \) if, for fixed \( M \),

\[
\lim_{c \to 0} \frac{R(\Gamma)}{R^*} = O(1).
\] (4)

The asymptotic and order optimalities in \( M \) are similarly defined as the limit of \( M \) approaching infinity for all fixed \( c \).

A dual formulation of the problem is to minimize the detection delay subject to an error constraint \( \varepsilon \), i.e.,

\[
\Gamma^* = \arg \inf_{\Gamma} E[\tau|\Gamma], \quad \text{s.t.} \quad P_e(\Gamma) \leq \varepsilon.
\] (5)

In the Bayes risk given in (1), \( c \) can be viewed as the inverse of the Lagrange multiplier, thus controls the detection accuracy of the test that achieves the minimum Bayes risk. Following the same lines of argument in [23], [24], one can obtain the solution of (5) once the solution of the Bayesian formulation is found.

### III. A Deterministic Active Search Strategy

In this section, we propose a deterministic active search strategy and analyze its sample complexity and Bayes risk in terms of both \( M \) and \( c \).
A. A Biased Random Walk on the Tree

Referred to as Random Walk on the Tree (RWT), the proposed policy starts at the root node (i.e., level \( l = \log_2 M \)) as illustrated in Fig. 1 and walks on the tree based on random observations. Specifically, assume that the policy is currently at node \( i \) on level \( l > 0 \) (i.e., an upper level above the leaves). The following steps are taken:

- \( K_l \) samples are taken from each of the two children of node \( i \).
- The sum log-likelihood ratio (SLLR) of each child is computed from these \( K_l \) samples \( \{y(n)\}_{n=1}^{K_l} \):
  \[
  \sum_{n=1}^{K_l} \log \frac{g_{l-1}(y(n))}{f_{l-1}(y(n))}. \tag{6}
  \]
- If the SLLRs of both children are negative, go back to the parent of node \( i \); otherwise, zoom into the child whose SLLR is larger. Note that we define the parent of the root node as itself.
- Repeat until arriving at a leaf node.

Once arriving at a leaf node, say node \( m \) (\( m = 1, \ldots, M \)), samples are drawn one by one and the SLLR \( S_m(t) \) of node \( m \) is updated with each new sample based on \((g_0, f_0)\). The policy continues sampling node \( m \) as long as \( S_m(t) \geq 0 \). The moment \( S_m(t) \) becomes negative, the policy goes back to the parent of node \( m \) and carries out the steps specified above for upper level nodes. The policy terminates when the SLLR \( S_m(t) \) of the currently probed leaf node exceeds the threshold of \( \log_2 \frac{M}{c} \), and declare this node as anomalous.

Note that due to the probabilistic observations, it is possible that the policy zooms into a node that contains no target. There is thus a mechanism built into the policy to correct such mistakes. Specifically, when the SLLRs of both children of the current node are negative (indicating none of the decedents of the current node is likely to be the target), the policy moves back to the previous level. A similar correcting mechanism is in place for leaf nodes by checking the sign of the SLLR \( S_m(t) \).

The number \( K_l \) of samples taken from children of the node on level \( l \) for deciding whether to zoom in or zoom out is determined by \( g_{l-1} \) and \( f_{l-1} \) that reflect the quality of the aggregated measurements. It is chosen to ensure
Algorithm 1 RWT Policy

**Input:** SLLR threshold $\log \frac{1}{e} M$; $K_l \in \mathbb{Z}$ for $l = 1, 2, \ldots, \log_2 M$.

**Output:** Index of the target cell.

1: **Initialize:** Level index $l \leftarrow \log_2 M$; SLLR of cell $i$: $S_i \leftarrow 0$ for $i = 1, 2, \ldots, M$.
2: while $\max_i \{S_i \}_{i=1}^M \leq \log \frac{1}{e} M$ do
3:   if $l = \log_2 M$ (root level) then
4:     Take $K_l$ samples from each child; calculate the SLLRs, $S_L$ and $S_R$, as given in (6).
5:     Zoom into the child with a larger SLLR, $l \leftarrow l - 1$.
6:   else if $1 \leq l < \log_2 M$ (internal level) then
7:     Take $K_l$ samples from each child; calculate the SLLRs, $S_L$ and $S_R$, as given in (6).
8:     if $S_L < 0$ and $S_R < 0$ then
9:       Go back to the previous level, $l \leftarrow l + 1$.
10:    else
11:      Zoom into the child with a larger SLLR, $l \leftarrow l - 1$.
12:    end if
13:   else if $l = 0$ (cell $m$ in the leaf level) then
14:     Take one sample from cell $m$ and update $S_m$.
15:     if $S_m < 0$ then
16:       Go back to the previous level, $l \leftarrow l + 1$.
17:    end if
18: end if
19: end while
20: **return** $\arg \max_i \{S_i \}_{i=1}^M$.

that the random walk has a higher probability of moving toward than moving away from the target. Specifically, as illustrated in Fig. 2, at each non-leaf node, the random walk may go up to its parent node, go to its left child node, or go to its right child node. The probabilities for each of the three events are determined by the relative location of this node to the target and $g_{l-1}$ and $f_{l-1}$ (observation distributions of its children). In particular, at level $l$, the probability of moving closer to the target is either $p_l^{(g)}$ or $p_l^{(f)}$ depending on whether this node contains the target or not (see Fig. 2). Let $Y_n$ and $Z_n$ denote i.i.d. random variables with distribution $g_{l-1}$ and $f_{l-1}$, respectively. It is not difficult to show that $p_l^{(g)}$ and $p_l^{(f)}$ are given by

$$p_l^{(g)} = \Pr \left( \sum_{n=1}^{K_l} \log \frac{g_{l-1}(Y_n)}{f_{l-1}(Y_n)} > \max \left( \sum_{n=1}^{K_l} \log \frac{g_{l-1}(Z_n)}{f_{l-1}(Z_n)}, 0 \right) \right),$$

$$p_l^{(f)} = \left[ \Pr \left( \sum_{n=1}^{K_l} \log \frac{g_{l-1}(Z_n)}{f_{l-1}(Z_n)} < 0 \right) \right]^2.$$

(7)
The parameter $K_l (l = 1, 2, \ldots, \log_2 M)$ is chosen as the minimum value that ensures $p_l^{(g)} > \frac{1}{2}$ and $p_l^{(f)} > \frac{1}{2}$. Note that the value of $K_l$ can be computed offline and simple upper bounds suffice.

B. Performance Analysis

We now analyze the scaling behavior of the sample complexity and the Bayes risk of the RWT policy in terms of both $M$ and $c$.

The key to the analysis of RWT lies in the realization that the test leads to a biased random walk on the tree with random sojourn times at leaf nodes. The random walk starts at the root node and terminates at a leaf node. At each leaf node, the random walk stays for a random sojourn time and then either moves to its parent node (when the SLLR becomes negative) or terminates (when the SLLR exceeds $\log \log_2 M$). Let $D_g$ and $D_f$ denote, respectively, the sojourn times at the target and a normal leaf node; they have different distributions determined by $g_0$ and $f_0$, respectively.

The sample complexity of the RWT policy is analyzed by examining the trajectory of the resulting random walk. As expected, with high probability, the random walk will concentrate on a smaller and smaller portion of the tree containing the target and eventually probes the target only. Our approach is to partition the tree into $\log_2 M + 1$ half trees $T_{\log_2 M}, T_{\log_2 M - 1}, \ldots, T_0$ with decreasing size, and bound the time the random walk spent in each half tree. As illustrated in Fig. 2 for $M = 8$, $T_l$ is the half tree (including the root) rooted at level $l$ ($l = \log_2 M, \log_2 M - 1, \ldots, 1$) that does not contain the target and $T_0$ consists of only the target node. The entire search process, or equivalently, each sample path of the resulting random walk, is then partitioned into $\log_2 M + 1$ stages by the successively defined last passage time to each of the half trees in the shrinking sequence. In particular, the first stage with length $\tau_{\log_2 M}$ starts at the beginning of the search process and ends at the last passage time to the first half tree $T_{\log_2 M}$ in the sequence, the second stage with length $\tau_{\log_2 M - 1}$ starts at $\tau_{\log_2 M} + 1$ and ends at the last passage time to $T_{\log_2 M - 1}$, and so on. Note that if the random walk terminates at a half tree $T_l$ with $l > 0$ (i.e., a detection error occurs), then $\tau_j = 0$ for $j = l - 1, \ldots, 0$ by definition. It is easy to see that, for each sample path, we have the total time of the random walk equal to $\sum_{l=0}^{\log_2 M} \tau_l$.

Next, we consider two different scenarios regarding the quality of aggregated observations and provide the sample complexity analysis based on the approach outlined above.

C. Informative Observations at All Levels

We first consider the scenario where the KL divergence between aggregated observations in the presence and the absence of anomalous processes is bounded away from zero at all levels of the tree structure, i.e., there exists a constant $\delta > 0$ independent of $M$ such that $D(g_l||f_l) > \delta$ and $D(f_l||g_l) > \delta$ for all $l = 1, 2, \ldots, \log_2 M$ and for all $M$. The theorem below characterizes the Bayes risk of the RWT policy.

**Theorem 1:** Suppose that $D(g_l||f_l)$ and $D(f_l||g_l)$ are bounded away from zero for all $l$. For all $M$ and $c$, we have

$$R(\Gamma_{\text{RWT}}) \leq cB \log_2 M + \frac{c \log \log_2 M}{D(g_0||f_0)} + O(c),$$  \hspace{1cm} (8)
where $B$ is a constant independent of $c$ and $M$. Furthermore, the Bayes risk of RWT is order optimal in $M$ for all $c$ and asymptotically optimal in $c$ for all $M$ greater than a finite constant $M_0$.

Proof: The basic idea to prove (3) is to show that $E[\tau_l]$ on the half tree $\mathcal{J}_l$ for all $l = 1, 2, \ldots, \log_2 M$ introduced in Section III-B is upper bounded by a constant. Also, because of the condition on the KL divergence, the higher level observations are of sufficient quality such that $K_{\text{max}} = \max_{l=1,\ldots,\log_2 M} K_l$ is upper bounded by a constant. Then sample complexity in the first log$_2 M$ stages are in logarithmic-order. For the last stage on $\mathcal{J}_0$, it can be shown that

$$E[\tau_0] = \frac{\log \log_2 M}{D(g_0||f_0)} + O(1).$$

The error probability satisfies $P_e = O(c)$. Details of the proof can be found in Appendix [A].

The order optimality in $M$ results from the information theoretic lower bound [10]. The asymptotic optimality in $c$ results from that $-\frac{c \log c}{D(g_0||f_0)}$ is an asymptotic lower bound on the achievable Bayes risk, which can be established by following similar lines of proof as given in [25].

The sample complexity of the RWT policy is upper bounded as

$$E(\tau|\Gamma_{\text{RWT}}) \leq B \log_2 M + \frac{\log \log_2 M}{D(g_0||f_0)} + O(1).$$

(9)

Following the similar lines of proofs as given in [23], [24], with the error constraint $P_e = O(c)$, for $M > M_0$, the RWT policy has a sample complexity that is asymptotically optimal in $c$:

$$E(\tau|\Gamma_{\text{RWT}}) \sim -\frac{\log c}{D(g_0||f_0)}.$$

For all $c$, the RWT policy has a sample complexity that is order optimal in $M$,

$$E(\tau|\Gamma_{\text{RWT}}) = O(\log_2 M).$$

D. Aggregated Observations Decaying to Pure Noise

Using Bernoulli distribution as a case study, we also examine the scenario where higher level observations decay to pure noise as $M$ grows. We focus here on the sample complexity, and $P_e = O(c)$ still holds (see Appendix). We establish sufficient conditions on the decaying rate of the quality of the hierarchical observations under which the proposed strategy achieves a sublinear sample complexity in $M$.

We assume $g_l$ follows a Bernoulli distribution which takes the value 1 with probability of $1 - \mu_l$ and the value 0 with probability of $\mu_l$ (false negative rate); $f_l$ follows a Bernoulli distribution which takes the value 0 with probability of $1 - \mu_l$ and the value 1 with probability of $\mu_l$ (false positive rate). We further assume that $\mu_l$ increases with $l$ and is less than 0.5 for all $l = 0, 1, 2, \ldots, \log_2 M$. The KL divergence between $g_l$ and $f_l$ is $D(g_l||f_l) = (1 - 2\mu_l) \log \frac{1-\mu_l}{\mu_l}$. When $\mu_l$ converges to 0.5 as $M$ approaches infinity, the KL divergence between $g_l$ and $f_l$ will converge to zero, which leads to unbounded $K_l$. The following two theorems characterize the detection delay of RWT policy when $\mu_l$ converges to 0.5 in polynomial-order and exponential-order, respectively.

Theorem 2: Assume $\mu_l = 0.5 - (0.5 - \mu_0) \cdot (l + 1)^{-\tau}$ for $l = 0, 1, 2, \ldots, \log_2 M$, where $\tau \in \mathbb{Z}^+$ is a constant and $\mu_0 < 0.5$. The sample complexity of the RWT policy is upper bounded by:

$$E(\tau|\Gamma_{\text{RWT}}) \leq O((\log_2 M)^{2\tau+1}) + \frac{\log \log_2 M}{D(g_0||f_0)} + O(1).$$

(10)
The sample complexity of the RWT policy is upper bounded by:

\[ \mathbb{E}(\tau|\Gamma_{\text{RWT}}) = O((\log_2 M)^{2t+1}) = o(M), \quad \text{for } t \in \mathbb{Z}^+. \]

**Theorem 3:** Assume \( \mu_t = 0.5 - (0.5 - \mu_0) \cdot t^{-l} \) for \( l = 0, 1, 2, \ldots, \log_2 M \), where \( t > 1 \) is a constant and \( \mu_0 < 0.5 \). The sample complexity of the RWT policy is upper bounded by:

\[ \mathbb{E}(\tau|\Gamma_{\text{RWT}}) \leq \hat{B}M^\log_2 t^2 + \frac{\log \frac{\log_2 M}{c}}{D(g_0||f_0)} + O(1), \]

where \( \hat{B} \) is a constant independent of \( c \) and \( M \).

**Proof:** See Appendix B.

From Theorem 3, we can find that, for any fixed \( c \), the RWT policy has a sample complexity that is sublinear in \( M \), as long as \( 1 < t < \sqrt{2} \):

\[ \mathbb{E}(\tau|\Gamma_{\text{RWT}}) = O(M^{\log_2 t^2}) = o(M), \quad \text{for } 1 < t < \sqrt{2}. \]

When \( t = \sqrt{2} \), \( \mathbb{E}(\tau|\Gamma_{\text{RWT}}) \) is linear in \( M \). When \( t > \sqrt{2} \), \( \mathbb{E}(\tau|\Gamma_{\text{RWT}}) \) is superlinear order in \( M \).

**IV. Detecting Multiple Anomalies**

In this section, we extend the RWT policy to cases with \( L > 1 \). The idea is to locate the \( L \) targets sequentially.

Let \( g_l^{(d)} \) (\( l = 0, 1, 2, \ldots, \log_2 M \), \( d \leq \min\{L, 2^l\} \)) denote the distribution of the measurements that aggregate \( d \) anomalous processes \( g_0 \) and \( 2^l - d \) normal processes \( f_0 \). For a node at level \( l \), let \( d^* \) denote the number of targets in the subtree rooted at the current node. It is easy to find that the distribution of measurements taken at this node will be \( g_l^{(d^*)} \). As is the case in practical applications, we expect \( D(g_l^{(d+k)}\|g_l^{(d+1)}) \leq D(g_l^{(d+k)}\|g_l^{(d)}) \) for all \( 1 \leq k \leq \min\{L, 2^l\} - d \) and \( D(g_l^{(d-k)}\|g_l^{(d+1)}) \geq D(g_l^{(d-k)}\|g_l^{(d)}) \) for all \( 0 \leq k \leq d \).

The basic structure of the policy remains the same. One difference is in the calculation of the SLLR at an upper level node. Suppose that the policy is currently at node \( i \) on level \( l > 0 \). Let \( \hat{d} \) denote the number of already-declared targets of one of the child nodes. After taking \( K_l^{(\hat{d})} \) samples \( \{y(n)\}_{n=1}^{K_l^{(\hat{d})}} \) from this child, the SLLR is computed as

\[ \sum_{n=1}^{K_l^{(\hat{d})}} \log \frac{g_l^{(d+1)}(y(n))}{g_l^{(d)}(y(n))}, \]

where \( K_l^{(\hat{d})} \) is chosen to ensure the probability of moving closer to the target is greater than \( \frac{1}{\hat{d}} \). Notice that since now \( K_l^{(\hat{d})} \) is a function of \( \hat{d} \), it may be different for the two children. It is easy to see that, when \( \hat{d} = 2^{l-1} \), there is no need to take samples from that child node, since all its leaf nodes are already declared as targets.

Similar to the case with \( L = 1 \), if the SLLRs of the both children are negative, the policy goes back to the parent of node \( i \). Otherwise, the policy zooms into the child node whose SLLR is larger. The policy repeats this procedure until a leaf node is declared as a target. Then the search process restarts from the root node. The test terminates until \( L \) nodes have been declared as targets.
In this section, we compare the RWT policy with the Chernoff test and the DGF test developed in [25]. In this example, we consider heavy hitter detection among Poisson flows and the measurements are exponentially distributed inter-arrival times. For the leaf-node, \( g_0 \) and \( f_0 \) are exponential distributions with parameter \( \lambda_g = 10 \) and \( \lambda_f = 0.1 \), respectively. We assume there is only one heavy hitter (\( L = 1 \)). For all the upper level nodes, \( g_i \) and \( f_i \) follow the exponential distributions with parameters \( \lambda_g + (2^i - 1)\lambda_f \) and \( 2^i\lambda_f \), respectively.

The Chernoff test works as follows. When \( D(g_0\parallel f_0) \geq D(f_0\parallel g_0)/(M - 1) \), the Chernoff test always probes the leaf node who currently has the largest SLLR. When \( D(g_0\parallel f_0) < D(f_0\parallel g_0)/(M - 1) \), the Chernoff test randomly selects one node from all the leaf nodes except the one with the largest SLLR with equal probability to probe. The DGF test conducts the same action as the Chernoff test when \( D(g_0\parallel f_0) \geq D(f_0\parallel g_0)/(M - 1) \). However, when \( D(g_0\parallel f_0) < D(f_0\parallel g_0)/(M - 1) \), the DGF test selects the leaf node with the second largest SLLR to probe. Both the Chernoff test and the DGF test update the SLLR of leaf nodes with each corresponding sample and terminate when the SLLR difference between the largest and the second largest ones exceeds the threshold \( -\log c \); then declare the node with the largest SLLR as target.

Fig. 3 shows a simulation example comparing the expected detection delays of the Chernoff test, the DGF policy, and the RWT policy as a function of \( M \). The detection delays of the Chernoff test and the DGF test increase linearly with \( M \), while the delay of the RWT policy increases in a logarithmic-order with \( M \). The advantage of the RWT policy in terms of the detection delay is significant when \( M \) is large.

VI. CONCLUSION

In the paper, we considered the problem of detection a few anomalous processes among a large number \( M \) of processes under a binary tree observation model. The average detection delay of the proposed RWT policy is shown
to be order-optimal with the search space size $M$ when the aggregated observations are of sufficient quality and asymptotically optimal in terms of detection reliability.

Appendix

A. Proof for Theorem 1

We now prove Theorem 1. Without loss of generality (due to the symmetry of the binary tree structure), we assume that the left-most leaf is the target.

The basic idea of the proof is introduced in Section III-B. The entire search process is partitioned into $\log_2 M + 1$ stages. We first prove a uniform upper bound of $E[\tau_l]$ for all $l = 1, 2, \ldots, \log_2 M$. After that we compute $E[\tau_0]$ for the last stage on $T_0$ and show the upper bound of the error probability. Therefore, the upper bound of the Bayesian risk can be easily proved.

The upper bound for $E[\tau_l]$ for all $l = 1, 2, \ldots, \log_2 M$ is proved by the structure of the random walk on the tree. The random walk on the tree has two cases. The first case is the random walk on upper level nodes of the binary tree. In this case, at each time, after taking $K_l$ samples, we either zoom-in to one child node or zoom-out to the parent node. i.e., the distance between the current node to the target is defined as the sum of the discrete distance to the target node on the tree and the threshold $\log_2 \frac{M}{c}$, which will either minus one (zoom-in) or plus one (zoom-out) after every $2K_l$ samples from the children. Once arriving at a leaf node, the test arrives at the second case, where samples are taken one by one from the current node until the cumulative SLLR exceeds the threshold or becomes negative. The cumulative SLLR can be viewed as a discrete time random walk with random continuous step size which is the log-likelihood ratio of each sample. During the search process, these two cases happen consecutively before the final stage of the RWT policy.

For all the non-target leaf-nodes, we define the distance between the node to the target as the sum of the discrete distance on the tree, the cumulative SLLR of current node, and the threshold. For the target node, we define the distance to the target as the difference between the threshold and the current cumulative SLLR of the target node.

Let $W_n$ denote the random variable of the step size of the random walk at time $n$. When the RWT is in the first case (random walk on the high-level nodes), depending on the current level $l > 0$, $W_n$ will have the distribution

$$
\Pr(W_n) = \begin{cases} 
1 - p_l^{(g)} & \text{for } W_n = 1 \\
1 - p_l^{(f)} & \text{for } W_n = 1
\end{cases}
$$

if the node is located at a sub-tree contains the target, or

$$
\Pr(W_n) = \begin{cases} 
p_l^{(g)} & \text{for } W_n = 1 \\
1 - p_l^{(f)} & \text{for } W_n = 1
\end{cases}
$$

if the node is located at a sub-tree does not contain the target. Since $p_l^{(g)} > 0.5$ and $p_l^{(f)} > 0.5$ for all $l = 1, 2, \ldots, \log_2 M$, we have

$$
E[W_n] = 1 - 2p_l^{(g)} \text{ or } 1 - 2p_l^{(f)},
$$
which are both negative.

For the second case, let \( Y_0 \) and \( Z_0 \) denote the random variables with the distributions \( g_0 \) and \( f_0 \), respectively. The log-likelihood ration (LLR) will be either \( -\log \frac{g_0(Y_0)}{f_0(Y_0)} \) or \( -\log \frac{g_0(Z_0)}{f_0(Z_0)} \). It is not difficult to see that for the target node, we have
\[
E[W_n] = E\left[-\log \frac{g_0(Y_0)}{f_0(Y_0)}\right] = -D(g_0\|f_0) < 0,
\]
and for all the non-target node, we have
\[
E[W_n] = E\left[\log \frac{g_0(Z_0)}{f_0(Z_0)}\right] = -D(f_0\|g_0) < 0.
\]

We further assume that the distribution of \(-\log \frac{g_0(Y_0)}{f_0(Y_0)}\) and \( \log \frac{g_0(Z_0)}{f_0(Z_0)} \) are light-tailed.

Now we are ready to present the following lemma that characterizes the distributions of \( \tau_\ell \).

**Lemma 1:** For all \( \tau_\ell \) with \( \ell = 1, \ldots, \log_2 M \), there exist an \( \alpha > 0 \) and a \( \gamma > 0 \) which are independent of \( M \) and \( c \), such that
\[
\Pr(\tau_\ell \geq n) \leq \alpha e^{-\gamma n}, \quad \forall n \geq 0.
\]

**Proof:** We first prove this lemma for \( \tau_{\log_2 M} \) which is the last passage time of the sub-tree at the root that does not contain the target, i.e., \( \tau_{\log_2 M} \). Because of the recursive definitions of \( \tau_1, \tau_2, \ldots, \tau_{\log_2 M} \), the proofs of all the other \( \tau_\ell \) will follow the same procedure, and end up with the same result.

Let \( S_t \) denote the distance to the target at time \( t \). The RWT policy starts at the root node, therefore the initial distance to the target is \( S_0 = \log_2 M + \log \frac{\log_2 M}{c} \). Define
\[
\tau^* = \sup \{ t \geq 0 : S_t \geq S_0 \}
\]
as the last time when the search approach has the distance to the target greater than \( S_0 \). It is not difficult to see that
\[
\tau_{\log_2 M} \leq \tau^*.
\]

Therefore, we have
\[
\Pr(\tau_{\log_2 M} \geq n) \leq \Pr(\tau^* \geq n).
\]

Based on the definition of \( \tau^* \), we have
\[
\Pr(\tau^* > n) = \Pr(\sup \{ t \geq 0 : S_t \geq S_0 \} > n) \leq \sum_{t=n}^{\infty} \Pr(S_t \geq S_0) = \sum_{t=n}^{\infty} \Pr \left( \sum_{j=1}^{t} W_j \geq 0 \right).
\]

Let \( \mu_j \) denote the mean value for each \( W_j \), where \( \mu_j < 0 \) for all \( j = 1, 2, \ldots, t \). Applying the Chernoff bound to the sum of independent random variables \( \sum_{j=1}^{t} W_j \), we have
\[
\Pr \left( \sum_{j=1}^{t} W_j \geq 0 \right) \leq E \left[ e^{s \sum_{j=1}^{t} W_j} \right] = \prod_{j=1}^{t} E \left[ e^{s W_j} \right], \text{ for all } s > 0.
\]

Note that a moment generating function (MGF) of each \( W_j \) is equal to one at \( s = 0 \). Furthermore, since \( E[W_j] < 0 \) is strictly negative for all \( j \geq 1 \), differentiating the MGFs of all \( W_j \) with respect to \( s \) yields strictly negative derivatives.
at \(s = 0\). Because all \(W_j\)'s are light-tailed distributions, as a result, for all possible distributions of \(W_j\), there exist \(s > 0\) and \(\gamma > 0\) such that \(E[e^{sW_j}]\) is strictly less than \(e^{-\gamma} < 1\). Hence, from (19), we have

\[
\Pr \left( \sum_{j=1}^{t} W_j \geq 0 \right) \leq e^{-\gamma t}.
\]  

(20)

Due to (18), we have

\[
\Pr(\tau^* > n) \leq \sum_{t=n}^{\infty} \Pr(\tau \geq n) \leq \sum_{t=n}^{\infty} e^{-\gamma t} = \frac{e^{-\gamma n}}{1-e^{-\gamma}}.
\]  

(21)

Let \(\alpha = \frac{1}{1-e^{-\gamma}}\), with (17), we eventually get Lemma 1 proved for \(\tau_{\log_2 M}\).

Based on Lemma 1 we can easily get the following lemma that characterizes the expected value of \(\tau_l\) for all \(l = 1, \ldots, \log_2 M\), with a constant \(\beta > 0\), such that

\[
E[\tau_l] \leq \beta.
\]  

(22)

**Proof:** Based on the the tail-sum formula of expectation of the non-negative random variables, we have

\[
E[\tau_l] = \sum_{n=0}^{\infty} \Pr[\tau_l > n] \leq \sum_{n=0}^{\infty} \alpha e^{-\gamma n} = \frac{\alpha}{1-e^{-\gamma}} = \frac{1}{(1-e^{-\gamma})^2} = \beta.
\]  

(23)

Now we are ready to prove Theorem 1.

Base on Lemma 2 it is not difficult to show that

\[
E[\tau] \leq 2K_{\max} \sum_{l=1}^{\log_2 M} E[\tau_l] + E[\tau_0] \leq 2\beta K_{\max} \log_2 M + E[\tau_0].
\]  

(24)

Since in Theorem 1, \(K_{\max}\) is assumed to be bounded by a constant, the first term in (24) is upper bounded by \(B \log_2 M\), where \(B\) is a constant greater than \(2\beta K_{\max}\).

For the last stage, \(\tau_0\) is a stopping time with respect to the i.i.d. sequence of the log-likelihood ratio

\[
\left\{ \log \frac{g_0(X_n)}{f_0(X_n)} : n \geq 1 \right\},
\]

where \(X_n\) denote i.i.d. random variable with distribution \(g_0\).

Due to the Wald’s Equation [26], we have

\[
E\left[ \sum_{n=1}^{\tau_0} \log \frac{g_0(X_n)}{f_0(X_n)} \right] = E[\tau_0] E\left[ \log \frac{g_0(X_n)}{f_0(X_n)} \right].
\]  

(25)

i.e.,

\[
\log \frac{\log_2 M}{c} + R_b = E[\tau_0] D(g_0||f_0),
\]  

(26)

where \(R_b\) is the overshooting at the threshold. Due to Lorden’s inequality [27], we have

\[
E[R_b] \leq \frac{E\left[ \log \frac{\log_2 M}{c} \right]^2}{E\left[ \log \frac{g_0(X_n)}{f_0(X_n)} \right]^2}.
\]  

(27)

Assuming that the first two moments of log-likelihood ration are finite, then we have

\[
E[\tau_0] = \frac{\log \frac{\log_2 M}{c} + O(1)}{D(g_0||f_0)}.
\]  

(28)
The following lemma characterizes the error probability of the RWT policy.

**Lemma 3:** The error probability of the RWT policy is upper bounded by:

\[ P_e \leq \beta c = O(c). \] (29)

**Proof:** When the RWT policy arrives a non-target node, say node \( j \), the probability of error (accepting \( H_j \)) equals to \( \Pr(S_j \geq \log \frac{\log_2 M}{c}) \). The Wald’s approximation [6] gives

\[ \Pr(S_j \geq \log \frac{\log_2 M}{c}) \leq \exp \left[ -\log \frac{\log_2 M}{c} \right] = \frac{c}{\log_2 M}. \] (30)

Let \( N \) denote the random number of times of visiting these non-target leaf nodes in the RWT policy. The conditional error probability is upper bounded by \( \frac{N c}{\log_2 M} \). Based on the proof of Theorem [1] the expected value of \( N \) is upper bounded by \( \beta \log_2 M \). Therefore, by taking expectation, the error probability is bounded by

\[ P_e \leq \frac{c}{\log_2 M} \cdot \mathbb{E}[N] \leq \frac{c}{\log_2 M} \cdot \beta \log_2 M = \beta c = O(c). \] (31)

We thus arrive at Theorem [1]

**B. Proofs for Theorem 2 and 3**

We now prove Theorem [2] and Theorem [3]. The basic idea of the proof is based on what was proven for Theorem [1]. We already have the upper bound of \( \mathbb{E}[\tau] \) shown in Lemma [2]. Next we need to show the upper bound for the sample complexity of the first \( \log_2 M \) stages of the search process. For the \( l \)th stage (\( l = 1, 2, \ldots, \log_2 M \)), the upper bound of the sample complexity is determined by \( \mathbb{E}[\tau] \) and \( K_l \). For the last stage, (28) still holds.

The proof uses the following lemma established in [28].

**Lemma 4:** Let \( X_1, \ldots, X_n \) be independent Poisson trials such that \( \Pr(X_i) = p_i \). Let \( X = \sum_{i=1}^n X_i \) and \( \nu = \mathbb{E}[X] \). Then the following Chernoff bounds hold for \( 0 < \delta \leq 1, \)

\[ \Pr(X \geq (1 + \delta)\nu) \leq e^{-\nu \delta^2 / 3}; \] (32)

\[ \Pr(X \leq (1 - \delta)\nu) \leq e^{-\nu \delta^2 / 3}. \] (33)

Based on Lemma [4] we can show that in order to have \( p_l^{(e)} \) and \( p_l^{(f)} \) define in (7) greater than 0.5 for all \( l = 1, 2, \ldots, \log_2 M \), we can choose \( K_l \) to satisfy

\[ K_l \geq \max \left\{ \frac{12(1 - \mu_{l-1}) \log(1 - \eta)^{-1}}{(1 - 2\mu_{l-1})^2}, \frac{12\mu_{l-1} \log(1 - \lambda)^{-1}}{(1 - 2\mu_{l-1})^2} \right\}, \] (34)

where \( \eta \) and \( \lambda \) can be any value in \( (\frac{1}{\sqrt{2}}, 1) \) such that \( \eta \cdot \lambda > 0.5 \) and \( \lambda^2 > 0.5 \). Since \( \mu_l < 0.5 \), w.l.o.g., we choose

\[ K_l = \frac{12(1 - \mu_{l-1}) \log(1 - \eta)^{-1}}{(1 - 2\mu_{l-1})^2}. \] (35)

It is not difficult to find that \( K_l \) increases with \( \mu_{l-1} \). For any stage \( l \), when \( l = 1, 2, \ldots, \log_2 M \), the sample complexity in this stage is upper bounded by \( 2K_l \cdot \mathbb{E}[\tau] \). Based on Lemma [2] the total sample complexity from Stage 1 to Stage \( \log_2 M \) is thus upper bounded by

\[ \mathbb{E}[\tau] \leq \sum_{l=1}^{\log_2 M} 2K_l \cdot \mathbb{E}[\tau] \leq \sum_{l=1}^{\log_2 M} 2\beta K_l. \] (36)
For Theorem 2 if \( \mu_l = 0.5 - (0.5 - \mu_0) - (l + 1)^{-l} \), from (35) and (36), we have
\[
\mathbb{E}[\tau] \leq B' \log_2 M \sum_{l=1}^{M} t^{2l},
\] (37)
where \( B' = \frac{6\sqrt{\log(1-n)}}{(0.5 - \mu_0)^2} \) is a constant. By using the Faulhaber’s formula [29], we have
\[
\sum_{l=1}^{M} t^{2l} = O((\log_2 M)^{2r+1}),
\]
which leads to Theorem 2.

Similarly, for Theorem 3 if \( \mu_l = 0.5 - (0.5 - \mu_0) - l^{-1} \), we have
\[
\mathbb{E}[\tau] \leq B' \sum_{l=1}^{M} t^{2(l-1)},
\] (38)
By summing up the geometric terms in (38), we can show that
\[
\mathbb{E}[\tau] \leq \tilde{B}(t^2 \log_2 M) = \tilde{B}M^{\frac{1}{2(l-1)}} = \tilde{B}M^{\log_2 t^2},
\] (39)
where \( \tilde{B} = \frac{1}{t-1}B' \). Thus Theorem 3 is proved.

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