Cooling-heating phase transition for dS/AdS Bardeen Black Holes with consistent 4D Gauss-Bonnet gravity theory

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Abstract

Instead of the work \(^{[1]}\) which in accordance to the Lovelock theorem could not applicable for all types of 4D curved spacetimes of Einstein Gauss Bonnet (EGB) gravity, a well defined model is formulated recently in \(^{[2]}\) by applying break of diffeomorphism property. We use the latter model by adding a nonlinear electromagnetic field lagrangian density to obtain a 4D \(dS/AdS\) GB Bardeen black hole metric. We solved metric field equations numerically by Runge Kutta method and after to determine the black hole metric and position of the exterior horizons we investigate to study its thermodynamic. In this study we obtained that AdS GB Bardeen black hole participates in the small to large black hole phase transition and also Joule-Thomson expansion phase transition at constant enthalpy. While dS GB Bardeen black hole participates just in the Hawking-Page phase transition where an evaporating black hole reaches to a vacuum dS space finally and so does not participate in the JT expansion phase transition.

1 Introduction

From point of theoretical view we know the black holes are made from metric solutions of the Einstein’s equation with no temperature and so they are not supposed to show any thermodynamic behavior. For the first time, Hawking presented an important theorem where the event horizon of the black holes should never be decreased because all objects are absorbed by them.
[3]. This is called now as the Hawking’s area theorem. After this presentation Bekenstein suggested that for the black hole should be assigned an entropy appropriate to the area of its horizon[4]. In analogy with the thermodynamics rules of the ordinary systems, four laws proposed for the black holes thermodynamics. But by considering this analogy, there was obtained a problem for thermodynamics of the black holes as follows. Actually the first law of the thermodynamics of the black holes lacks the pressure and volume components. Because there is no a clear concept for thermodynamic volume and pressure of the black hole. The first idea to solve the pressure problem led to the consideration of a negative cosmological constant [5, 6, 7, 8, 9] which it is called conjugate variable for the thermodynamic volume. There are done a lot of research where the pressure-volume (PV) criticality of thermodynamics of AdS black holes mimic thermodynamics behavior of the well known Van der Waals ordinary gases[10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. Recently Glavan and Lin released a paper [29] in which an alternative generally covariant gravity theory is defined which in 4D curved space-times can propagates just massless gravitons by bypassing the Lovelock’s theorem. This alternative higher order derivatives gravity theory has two correction terms called as Gauss-Bonnet topological invariant and cosmological constant respectively. In 4D curved spacetimes the Gauss-Bonnet coupling constant parameter diverges to an infinite value. In this singular limit the Gauss-Bonnet topological invariant term gives rise non-trivial contributions to the gravitational dynamics, while preserving the number of degrees of freedom of graviton and being free from Ostrogradsky instability. They reported some appealing corrections to the dispersion relation of cosmological tensor-scalar modes in the cosmological space times and also singularity resolution in the spherically symmetric space times. As an spherically symmetric static black hole metric solutions of this model, authors of the work [30] obtained a Bardeen type of the black hole solutions and generated its thermodynamic variables via the horizon calculation. They obtained a critical location for the black hole horizon where the corresponding Hawking temperature raises to a maximum value for which a second-order phase transition is happened because the heat capacity diverges to infinity. Existence of these appealing thermodynamic behavior encourages us to study Joule-Thomson adiabatic free expansion phenomena for this black hole given in the AdS background. If we want to describe this expansion briefly, this is down as follows. In fact this expansion is happened when a gas is allowed to move from a high pressure region to a low pressure one.
without to change its enthalpy. As it established in the above, the black hole mass would be taken as enthalpy in an extended thermodynamic phase space, so during the Joule Thomson expansion phenomena the mass remains constant (isentropic process). In presence of this expansion, the black hole usually could reach to one of two the heating or cooling phases finally. After to present the pioneer works about the black hole Joule Thomson expansion given by [31, 32, 33], many other scientists investigated this phenomena for several black holes interacting with many types of the material fields [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. To study this phenomena one can usually investigate the inversion curves which mimics behavior of the van der Waals fluid. In general there are several AdS black holes which behave as different with respect to the van der Waals fluid and they do not mimic completely with the inversion curves (see for instance [30, 31]). In this work, we want to examine the possibility of the emergence of the expansion phenomenon of Joule Thomson for $dS/AdS$ GB Bardeen black hole. In fact we will see importance of parameters of this black hole namely the magnetic charge and the GB coupling constant into its heating-cooling phase transition. The paper is organized as follows.

In section 2 we define briefly the consistent 4D GB gravity model given by [2] for Bardeen black hole nonlinear electromagnetic field lagrangian density. In section 3 we generate metric field equations of $dS/AdS$ GB Bardeen black hole and solve them numerically by Runge Kutta method with best fitting functions. In section 4 we calculate the horizon equation, the Hawking temperature and the equation of state. In section 5 we present the JT expansion phenomena for the black hole under consideration and we seek possibility of its cooling-heating phase transition by calculating the JT coefficient versus the temperature. Section 6 denotes to conclusion of the work and outlook.

2 Consistent 4D Einstein Gauss Bonnet gravity

By according to the work [2] we define consistent Einstein Gauss Bonnet gravity in $D \rightarrow 4$ limit with the first term of the following lagrangian density and $L_{\text{matter}}$ in the second term of the following action is additional matter part.

$$I = \frac{1}{16\pi G} \int dt d^3x \sqrt{g} \left( L_{\text{EGB}}^{4D} + L_{\text{matter}} \right), \quad (2.1)$$
where
\[ \mathcal{L}_{\text{EGB}}^{4D} = 2R - 2\Lambda - \mathcal{M} \] (2.2)
\[ + \frac{\tilde{\alpha}}{2} [8R^2 - 4RM - \mathcal{M}^2 - \frac{8}{3} (8R_{ij}R^{ij} - 4R_{ij}M^{ij} - M_{ij}M^{ij})], \]
and \( G \) is the Newton’s gravitational coupling constant. \( R \) and \( R_{ij} \) are the Ricci scalar and the Ricci tensor of the spatial 3-metric \( \gamma_{ij} \) respectively. In the definition (2.2) we have
\[ M_{ij} = R_{ij} + K^k_k K_{ij} - K_{ik} K^k_j, \quad \mathcal{M} = \mathcal{M}_i \] (2.3)
with
\[ K_{ij} = \frac{1}{2N} (\dot{\gamma}_{ij} - 2D_i N_j - 2D_j N_i - \gamma_{ij} D_k D^k \lambda_{GF}). \] (2.4)
Here, dot \( \dot{} \) denotes derivative with respect to the time \( t \) and all the effects of the constraint stemming from the gauge fixing (GF) are now encoded in lagrange multiplier \( \lambda_{GF} \). \( D_i \) is spatial covariant derivative and re-scaled regular EGB coupling constant \( \tilde{\alpha} \) is defined versus the irregular GB coupling constant \( \alpha_{GB} \) such that \( \tilde{\alpha} = (D - 4)\alpha_{GB} \) which in limits of \( D \to 4 \) dimensions become finite. The above EGB gravity action functional satisfies the following gauge condition for all spherically symmetric and cosmological backgrounds (see [2] and [50]).
\[ \sqrt{\gamma} D_k D^k (\pi^{ij} \gamma_{ij} / \sqrt{\gamma}) \approx 0. \] (2.5)
In fact the above EGB action functional is generated from ADM decomposition of the 4D background metric as \( 1 + 3 \) dimensions such that
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \] (2.6)
where \( N, N_i, \gamma_{ij} \) are the lapse function, the shift vector, and the spatial metric respectively. \( \gamma \) factor in the action functional (2.1) is absolute value of determinant of the spatial 3-metric \( \gamma_{ij} \). This ADM decomposition is done on the background metric to remove divergent boundary term of the higher order metric derivatives in the GB term of the action functional (2.1) in general 4 dimensional form [2]. First term in the theory defined by (2.1) has the time re-parametrization symmetry \( t \to t = t(t') \). We now set the matter source \( I_{\text{matter}} \) to be action of a nonlinear electromagnetic antisymmetric Maxwell field \( F_{\mu\nu} \) with Ayon Beato Garcia form of the lagrangian density as follows.
\[ \mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{ABG}}(F) = \frac{12 \mathcal{M}_{\text{ADM}}}{Q^3} \left( \frac{\sqrt{2Q^2 F}}{1 + \sqrt{2Q^2 F}} \right)^{\frac{3}{2}} \] (2.7)
where \( F = F_{\mu\nu}F^{\mu\nu} \), \( M_{ADM} \) is ADM mass of the black hole and \( Q \) is Bardeen magnetic charge (see eq. (8) in ref. [51]).

### 3 4D dS/AdS GB Bardeen Black Holes

By comparing the metric line element (2.6) with general form of a spherically symmetric state 4D metric line element

\[
\text{d}s^2 = -e^{2A(r)} \left( 1 - \frac{2M(r)}{r} \right) \text{d}t^2 + \frac{\text{d}r^2}{1 - \frac{2M(r)}{r}} + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{d}\phi^2
\]  

(3.1)

we infer that the lapse function, and the shift vector and the spatial metric components and lagrange multiplier should be \( r \) dependent so that we can write

\[
N = e^{A(r)} \sqrt{1 - \frac{2M(r)}{r}}, \quad N_{r,\theta,\phi} = 0,
\]

(3.2)

\[
\gamma_{rr} = \frac{1}{1 - \frac{2M(r)}{r}}, \quad \gamma_{\theta\theta} = r^2, \quad \gamma_{\phi\phi} = r^2 \sin^2 \theta, \quad \lambda_{GF} = \lambda_{GF}(r).
\]

By substituting (3.1) into (2.2) we obtain

\[
\mathcal{L}_{EGB}^{4D} = R(\gamma_{ij}) - 2\Lambda + 12q^2 + \frac{\bar{g}}{2} \left[ 3R^2(\gamma) + \frac{88}{3} q^2 R(\gamma) - 272q^4 - 8R_{ij}(\gamma)R^{ij}(\gamma) \right]
\]

(3.3)

in which

\[
q(r) = \frac{e^{-A(r)}}{r^2} \left[ r^2 \lambda_{GF}'(r) \right]', \quad K_{ij} = -q\gamma_{ij}
\]

(3.4)

\[
R(\gamma) = -\frac{4M'}{r^2}, \quad R_{ij}(\gamma)R^{ij}(\gamma) = \frac{6}{r^6}(M - rM')^2
\]

(3.5)

and \( t \) denotes derivative with respect to \( r \). From the Maxwell equations, we can prove that the magnetic field has the form

\[
F_{\theta\phi}(\theta) = Q \sin \theta
\]

(3.6)

which for the metric equation (3.1) the EM lagrangian density become

\[
F = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{Q^2}{2r^4}
\]

(3.7)
for which (2.7) leads to the following form \([51]\).

\[
\mathcal{L}_{\text{Bardeen}}(r) = \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^\frac{2}{3}.
\]  

(3.8)

By adding (3.8) and by substituting (3.3) and (3.5) and by integrating the action functional (2.1) on the 2-sphere \(0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\) we obtain

\[
I = \frac{1}{4G} \int dt \int dr r^2 e^{A(r)} \left\{ -\frac{4M'}{r^2} - 2\Lambda + 12q^2 + \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^\frac{2}{3} \right\} - \tilde{\alpha} \left[ \frac{176M'q^2}{3r^2} + 136q^4 + \frac{24M^2}{r^6} - \frac{48MM'}{r^5} \right].
\]  

(3.9)

Euler Lagrange equation for \(q\) reads

\[
q \left[ 12 - \tilde{\alpha} \left( \frac{176M'}{3r^2} + 136q^2 \right) \right] = 0
\]  

(3.10)

which has two different solutions as

\[
q_1 = 0, \quad q_2 = \pm \frac{1}{\sqrt{136}} \sqrt{12 \tilde{\alpha} - \frac{176M'}{3r^2}}.
\]  

(3.11)

By substituting these two different gauge fixing conditions into the equation (3.4) and the action functional (3.9) we obtain

\[
\lambda_{GF}^{(1)}(r) \sim \frac{1}{r}
\]  

(3.12)

\[
\lambda_{GF}^{(2)}(r) = \int r \frac{dr'}{r'^2} \int r'^2 q_2(r'', e^{A(r''')}dr''\right)
\]  

(3.13)

and

\[
I_1 = I_2 = \frac{1}{4G} \int dt \int dr r^2 e^{A(r)} \left\{ -\frac{4M'}{r^2} - 2\Lambda + \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2 + Q^2} \right)^\frac{2}{3} \right\} + \tilde{\alpha} \left[ -\frac{24M^2}{r^6} + \frac{48MM'}{r^5} \right].
\]  

(3.14)
This shows that two different gauge fixing conditions \( q_{1,2} \) reach to similar action functional and so similar metric solutions. The Euler Lagrange equations for the function \( A(r) \) and the mass distribution function \( M(r) \) reduce to the following relations respectively.

\[
\frac{4M'}{r^2} = \frac{12M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2+Q^2} \right)^{\frac{2}{3}} - 2\Lambda - \frac{2\tilde{\alpha}M^2(r)}{r^6} \quad (3.15)
\]

and

\[
A'(r) = \frac{-24\tilde{\alpha}M(r)}{1 - \frac{12\tilde{\alpha}M(r)}{r^4}}. \quad (3.16)
\]

Now we are in position to solve the above nonlinear differential equations which via numerical Runge Kutta method. To do so we need physical boundary conditions as follows. We know that the mass distribution function \( M(r) \) is related to the matter density function \( \rho_M \) as

\[
M(r) = \int_0^r 4\pi r^2 \rho_M(r) dr \quad (3.17)
\]

for which we can write

\[
8\pi \rho_M(r) = \frac{2M'}{r^2} = \frac{6M_{ADM}}{Q^3} \left( \frac{Q^2}{r^2+Q^2} \right)^{\frac{2}{3}} - \Lambda - \frac{12\tilde{\alpha}M^2(r)}{r^6}. \quad (3.18)
\]

For arbitrary form of the density function \( \rho_M(r) \) the mass integral equation \((3.17)\) shows the following general boundary condition.

\[
M(0) = 0 \quad (3.19)
\]

while to obtain position of radius of the compact object \( R \) we should substitute \( M(R) = M_{ADM} = M_{total} \) with \( \rho_M(R) = M'(R) = 0 \) in the equation \((3.18)\) such that

\[
m_{\pm}(x) = \frac{x^6}{4(1+x^2)^{\frac{3}{2}}} \left[ 1 \pm \sqrt{1 - \frac{4\epsilon(1+x^2)^4}{3x^6}} \right] \quad (3.20)
\]

where we defined dimensionless quantities such that

\[
x = \frac{R}{Q}, \quad m(x) = \frac{\tilde{\alpha}M(R)}{Q^3}, \quad \epsilon = \tilde{\alpha}\Lambda. \quad (3.21)
\]
If this compact object to be a black hole we should set its exterior horizon with \( R = r_H = 2M(R) \) which by substituting (3.21) we obtain

\[
\zeta(x) = \frac{\alpha}{Q^2} = \frac{2m(x)}{x} = \frac{x^5}{2(1 + x^2)^{\frac{5}{2}}} \left[ 1 \pm \sqrt{1 - \frac{4\epsilon(1 + x^2)^4}{3x^6}} \right].
\]  

(3.22)

One can check easily that \( \zeta(x) \) and \( m(x) \) take real numeric values just for

\[
\epsilon \leq \frac{3x^6}{4(1 + x^2)^4}.
\]  

(3.23)

By substituting the dimensionless variables

\[
y = \frac{r}{Q}, \quad m(y) = \frac{\tilde{\alpha}M(r)}{Q^3}
\]  

(3.24)

into the equations (3.15) and (3.16) we obtain their dimensionless forms as follows.

\[
\dot{m}(y) = \frac{3m(x)y^2}{(1 + y^2)^{\frac{5}{2}}} - \frac{\epsilon(x)y^2}{2} - \frac{6m^2(y)}{y^4}, \quad |y| \leq x
\]  

(3.25)

and

\[
\dot{A}(y) = -\frac{24m(y)}{y^3} - \frac{12m(y)^4}{y^6}, \quad |y| \leq x
\]  

(3.26)

where \( \cdot \) denotes to derivative with respect to \( y \) and the parameters \( m(x) \) and \( \epsilon(x) \) should be fixed via (3.20) and (3.23). We choose particular choice of boundary values in what follows such that

\[
\epsilon(x) = \frac{3x^6}{4(1 + x^2)^4}, \quad m(x) = \frac{x^6}{4(1 + x^2)^{\frac{5}{2}}}, \quad \zeta(x) = \frac{x^5}{2(1 + x^2)^{\frac{5}{2}}}
\]  

(3.27)

and solve the dynamical equation (3.25) numerically via Runge Kutta. By looking at the figure 1 we see that for the horizon radius \( x = 0.25 \) the mass function solution \( m(y) \) has take on raising form vs \( y \) with maximum value at \( x = 0.25 \) and for horizons larger than \( x > 0.25 \) the mass function diagram is bending to some smaller masses after passing from a maximum value. These figures show that 1-d is un-physical because of positivity property on the mass function. Hence we use numerical solutions given in the figure 1-a and solve numerically the equation (3.26) just for \( 0 < y \leq 0.25 \). These numeric solutions and corresponding metric components are collected in the table 1 and the figure 2.
Table 1: Numerical metric solutions for inside of the Bardeen GB black hole

| y   | m(y)     | A(y)     | g''(y) | gυυ(y) |
|-----|----------|----------|--------|--------|
| 0.00| 0        | 0        | 1.     | -1.    |
| 0.05| 3.4 x 10^{-9} | -6.6 x 10^{-7} | 1.     | -0.999998 |
| 0.10| 2.4 x 10^{-8}  | -0.00029 | 0.999988 | -0.999408 |
| 0.15| 6.6 x 10^{-8}  | -0.00016 | 0.999978 | -0.999658 |
| 0.20| 1.1 x 10^{-7}  | -0.000083 | 0.999973 | -0.999807 |
| 0.25| 1.3 x 10^{-7}  | -0.0000399 | 0.999974 | -0.999894 |

In the next section we investigate thermodynamic behavior of the dS/AdS GB Bardeen black holes metric.

4 dS/AdS GB Bardeen black holes thermodynamics

By substituting $M'(r_H) = 0$ into the equation (3.15) we obtain enthalpy of the black hole $H$ for dS and AdS sector of space time as follows.

$$dS/AdS: \quad M_{ADM} = H = PV + U$$ (4.1)

where thermodynamic volume $V$ and dS/AdS pressure $P$ and internal energy $U$ are defined respectively as follows.

$$P = -\frac{\Lambda}{8\pi}, \quad V = -\frac{4\pi}{3} Q^3 \left(1 + \frac{r_H^2}{Q^2}\right)^{\frac{5}{2}} > 0, \quad Q < 0$$ (4.2)

for AdS sector and

$$P = \frac{\Lambda}{8\pi}, \quad V = \frac{4\pi}{3} Q^3 \left(1 + \frac{r_H^2}{Q^2}\right)^{\frac{5}{2}} > 0, \quad Q > 0$$ (4.3)

for dS sector respectively with internal energies

$$U = \frac{\tilde{\alpha} Q^3}{2 r_H^4} \left(1 + \frac{r_H^2}{Q^2}\right)^{\frac{5}{2}}.$$ (4.4)

We know that the Hawking temperature of the black hole is given vs the exterior horizon surface gravity which for metric field (3.1) reads.

$$T = -\frac{1}{4\pi} \frac{d}{dr} [g_{\mu\nu}(r)]_{r=r_H=2M(r_H)} = \frac{e^{2A(r_H)}}{4\pi r_H^2}. \quad (4.5)$$
By substituting $r_H = Qx, \epsilon(x) = \tilde{\alpha} \Lambda$ into the quantities $P = \frac{\tilde{\alpha}}{8\pi}, V(r_H)$ and $T(r_H)$ given in the above formulae we obtain dimensionless parametric equation of state for AdS and dS sectors of space time as follows.

In AdS:

$$p(x) = \tilde{\alpha}P = \frac{-3x^6}{32\pi(1 + x^2)^4},$$  \hspace{1cm} (4.6)

$$v(x) = -\frac{V}{Q^3} = \frac{4\pi}{3}(1 + x^2)^{\frac{5}{2}}, \quad t(x) = QT = \frac{1}{4\pi x}, \quad Q < 0$$

and

In dS:

$$p(x) = \tilde{\alpha}P = \frac{3x^6}{32\pi(1 + x^2)^4},$$  \hspace{1cm} (4.7)

$$v(x) = \frac{V}{Q^3} = \frac{4\pi}{3}(1 + x^2)^{\frac{5}{2}}, \quad t(x) = QT = \frac{1}{4\pi x}, \quad Q > 0$$

where by looking at the third column of the table 1 we substituted $A(r_H) \approx 0$ in the above dimensionless temperature. We show phase diagrams in the figure 2 for dS/AdS GB Bardeen black holes. In figure 2-a one can infer that the p-v diagram is similar to an ordinary Wan der Waals gas-fluid containing a minimum point in which AdS GB Bardeen black holes participate in the small to large phase transition. While the figure 2-b shows a Hawking-Page phase transition for dS GB Bardeen black hole at the maximum point of the p-v curve where the unstable black hole reaches to dS vacuum because of Hawking thermal radiation. Other diagrams in this figure have similar results for thermal black holes. We now investigate to possibility of Joule-Thomson expansion of the black hole as follows.

5 Joule Thomson expansion phenomena

In adiabatic free expansion of a gas called as Joule Thomson or Joule Kelvin expansion, a gas expand from an initial smaller volume $V_1$ at high pressure $P_1$ and temperature $T_1$ to a final larger volume $V_2$ with lower pressure $P_2$ at the temperature $T_2$ where its internal energy $U$ dose not changed because the heat absorbed by the gas and the work done by the gas during the adiabatic expansion are both zero \[52, 53\]. One of the most important issues in this gas expansion is whether the temperature changes? Answer of this question is ‘No’ just for an ideal gas because for which the internal energy is dependent just to the temperature but for real hydrodynamic systems
for instance the van der Waals fluid its answer is not known. In fact in real hydrodynamic systems the internal energy has a relationship with the temperature and at least one of physical characteristics. In short careful measurement show \( T_2 \neq T_1 \). If \( T_2 < T_1 \)\( (T_2 > T_1) \) the gas reaches to a cooling (heating) phase at final state. Since the process is adiabatic thus from the first law of thermodynamic we can write for internal energy \( U \):

\[
\Delta U = U_2 - U_1 = P_1 V_1 - P_2 V_2 = 0,
\]

which can be rewritten as

\[
U_2 + P_2 V_2 = U_1 + P_1 V_1.
\]

By definition the enthalpy as \( U + PV = H \), one can infer that the above relation indicates \( H_1 = H_2 \) which means that the JT expansion is isenthalpic. One of applicable thermodynamic parameters to study cooling-heating phase transition of the gas under this adiabatic expansion is called the JT coefficient as follows.

\[
\mu = \left( \frac{\partial T}{\partial P} \right)_H,
\]

in which the black hole mass is considered as enthalpy \( M = H \) and it will be constant throughout the adiabatic expansion. There is more experiments where sign of this coefficient is changed from positive to negative values and vice versa at the particular temperature \( T_i \) called as inversion temperature. This inversion temperature is particular one where the JT coefficient vanishes \( \mu(T_i) = 0 \). In fact if the initial temperature \( T_1 \) be higher than the inversion temperature \( T_i \), then the final temperature \( T_2 \) rises with respect to \( T_1 \) (heating phase) and if the initial temperature be lower than \( T_i \), the final temperature would be lower than the initial one (cooling phase). Experiments show that \( T_i \) is dependent to the pressure. To obtain its relationship with the thermodynamic variables of the gas we can use equation of state of any gas system by keeping the enthalpy \( H = H(T, P) \) as a constant for which we can obtain other form for the JT coefficient \[54\] (see Appendix) such that

\[
\mu = T \left( \frac{\partial V}{\partial T} \right)_P - V \frac{C_P}{C_P},
\]

where \( C_P = \left( \frac{\partial H}{\partial T} \right)_P \) is heat capacity at constant pressure. It is easy to check that the equation \( \mu(T_i) = 0 \) has a solution at the following inversion temper-
nature.

\[ T_i = V \left( \frac{\partial T}{\partial V} \right)_P. \] (5.5)

By substituting (5.5) the equation (5.3) reads

\[ \mu = \frac{V}{C_P} \left( \frac{T}{T_i} - 1 \right) \] (5.6)

which will be have negative (positive) values when \( T < T_i(T > T_i) \). If \( \mu \) reaches to some negative (positive) values the gas exhibits with a cooling (heating) phase. In fact inversion temperature is a borderline between cooling and heating of thermodynamic system. This can be studied schematically, by plotting the isenthalpic \( P - T \) curves where the inversion temperature line crosses some points which their slope are changed. For instance we can see some pioneer work given by [32] (see also [33]) where the authors are studied thermodynamics of charged black hole in AdS space. Authors of the work [32] specified its cooling and heating regimes by plotting the \( P - T \) diagrams for various values of charge and mass of the black hole. In AdS space, mass of black holes is considered as enthalpy [6], so isenthalpic curve will be diagrams with constant mass of the black holes. In fact what should be studied is the evolution of black hole temperature versus the pressure changes at constant mass of the black hole. To do so one can try to plot \( T - P \) diagrams for inversion curve and isenthalpic curve so that they are crossed with each other in an particular inversion point (see for instance [32] [33] [55] [56]). By according to this general description about the JT expansion of a real gas we now study this for metric solution of the black hole under consideration as follows. To calculate (5.3) we must be first obtain equation of the state \( P = P(T, M) \). This is done by eliminating \( r_H \) between the temperature \( T = 1/4\pi r_H \) given by the equation (4.5) with assumption \( A(r_H) \approx 0 \) (see the table 1) and the enthalpy equation (4.1) such that

\[ AdS : \quad P = \frac{96\pi^3 \alpha T^4 - 768\pi^4 M_{ADM}}{Q^3} \left( \frac{16\pi^2 Q^2 T^2 + 1}{Q^2 T^2} \right)^{-\frac{3}{2}}, \quad Q < 0 \] (5.7)

and

\[ dS : \quad P = -\frac{96\pi^3 \alpha T^4 + 768\pi^4 M_{ADM}}{Q^3} \left( \frac{16\pi^2 Q^2 T^2 + 1}{Q^2 T^2} \right)^{-\frac{3}{2}}, \quad Q > 0. \] (5.8)
By substituting the above equations into the JT coefficient (5.3) we obtain

\[
\mu_{AdS} = \frac{(16\pi^2 Q^2 T^2 + 1)^3}{384\tilde{\alpha}\pi^3 T^3} \left(4096\pi^6 Q^6 T^6 + \frac{768\pi^4 Q^4 T^4 + 48\pi^2 Q^2 T^2 + 1 - \frac{10\pi Q^2 M_{ADM} T}{\tilde{\alpha}\sqrt{16\pi^2 Q^2 T^2 + 1}}}{\tilde{\alpha}\sqrt{16\pi^2 Q^2 T^2 + 1}}\right)^{-1}, Q < 0
\]

and

\[
\mu_{dS} = -\frac{(16\pi^2 Q^2 T^2 + 1)^3}{384\tilde{\alpha}\pi^3 T^3} \left(4096\pi^6 Q^6 T^6 + \frac{768\pi^4 Q^4 T^4 + 48\pi^2 Q^2 T^2 + 1 - \frac{10\pi Q^2 M_{ADM} T}{\tilde{\alpha}\sqrt{16\pi^2 Q^2 T^2 + 1}}}{\tilde{\alpha}\sqrt{16\pi^2 Q^2 T^2 + 1}}\right)^{-1}, Q > 0.
\]

To plot diagrams of the above JT coefficients vs the temperature for simplification, we set \(Q = -1 (+1)\) for AdS(dS) sector of space time and substitute the following numeric initial conditions for \(\zeta\) and \(m\) which are obtained by substituting \(x = 0.25\) into the equation (3.27).

\[
\zeta(0.25) = 0.00042 = \frac{\tilde{\alpha}}{Q^2}, \quad m(0.25) = 5.2 \times 5.2 \times 10^{-5} = \frac{\tilde{\alpha} M_{ADM}}{Q^3}, \quad (5.11)
\]

These numeric values give us the following initial conditions.

\[
AdS: \quad \tilde{\alpha} = 0.00042, \quad M_{ADM} = -0.1238095238 \quad (5.12)
\]

and

\[
dS: \quad \tilde{\alpha} = 0.00042, \quad M_{ADM} = 0.1238095238. \quad (5.13)
\]

Negative (positive) sign for ADM mass \(M_{ADM}\) or the black hole enthalpy in AdS(dS) sector of space time means that the black hole is exothermic (endothermic). Because by according to ordinary thermodynamic systems we know that for positive enthalpy, the reaction is endothermic, that is heat is absorbed by the system due to the products of the reaction having a greater enthalpy than the reactants. On the other hand, if the enthalpy is negative, the reaction is exothermic, that is the overall decrease in enthalpy is achieved by the generation of heat. By regarding the numeric setting (5.12) and (5.13), the diagrams of the JT coefficients (5.9) and (5.10) are plotted for dS and AdS GB Bardeen black holes in figure 2-f. This diagram shows that for dS GB Bardeen black hole the JT coefficient has negative sign absolutely.
for all temperatures but for AdS GB Bardeen black hole the JT coefficient exhibit change of sign which means that the latter black hole participates in cooling-heating black hole phase transition but the former one does not. This diagram shows inversion temperature is $T_i = 0.2230159633$ in which sign of the JT coefficient is changed and so an AdS GB Bardeen black hole is exothermic for $T < T_i$ and is endothermic for $T > T_i$. In case $T = T_i$ the black hole has unknown degenerate state.

6 Conclusion

In this work we used Einstein Gauss Bonnet gravity to study thermodynamics of 4D $dS/AdS$ GB Bardeen black hole. Metric source of this type of black hole is nonlinear electromagnetic fields with a non-vanishing magnetic charge. Physical importance of this type of black holes is nonsingular property which have and they are applicable to study black hole structure of center of galaxies. We solved numerically the metric fields via Runge Kutta method and obtained the black hole mass distribution. We calculated exterior horizons of the black holes and the Hawking temperature. At last by calculating the equation of state in presence of both dS and AdS space pressures we investigated the black hole thermodynamics by focusing on the phase transitions. In this way we saw that the magnetic charge plays important role in these phase transitions. Positivity condition on the thermodynamic volume causes to be valid negative (positive) magnetic charge for AdS (dS) space pressure. Our mathematical calculations predict that 4D AdS GB Bardeen black hole takes on small to large phase transition where its pressure-volume phase diagrams is similar to an ordinary Van der Waals fluid but 4D dS GB Bardeen black hole participates in the Hawking-Page phase transition where evaporating black hole reaches to a vacuum dS space finally. Also by calculating the Joule-Thomson coefficient and by determining its sign we investigated cooling-heating phase transition for the obtained $dS/AdS$ black holes where we saw that AdS GB Bardeen black hole participates in the JT adiabatic expansion (cooling-heating phase transition) but dS type of the black hole does not participated. Method of our study for JT expansion of the black hole in this work is different with respect to our previous work [33] because in this work the metric field equations have not analytic solutions and So we had to solve the equations numerically. This case that we analyzed cooling-heating phase transition by plotting diagram of the JT coefficient vs the temperature.
at constant enthalpy, while in the previous work we plotted P-V and inversion temperature diagrams at constant enthalpy and seek their crossing points as position of the cooling and heating phase transition. As an future work we like investigate JT expansion of other types of black holes for instance given by [57, 58].

Appendix

As we mentioned previously the JT expansion occurs at constant enthalpy $H = U + PV$ for which we can write

$$dH = TdS + VdP$$  \hspace{1cm} (6.1)

where we used

$$TdS = dQ = dU + PdV.$$  \hspace{1cm} (6.2)

Applying $dH = 0$ the equation (6.1) reduces to the form $0 = TdS + VdP$ which can be rewritten as

$$\frac{dH}{dP} = 0 = T\left(\frac{\partial S}{\partial P}\right)_H + V.$$  \hspace{1cm} (6.3)

If we assume that the entropy depends to the temperature $T$ and the pressure $P$ as $S = S(T, P)$ then we can write $dS = \left(\frac{\partial S}{\partial P}\right)_T dP + \left(\frac{\partial S}{\partial T}\right)_P dT$ for which we will have

$$\left(\frac{\partial S}{\partial P}\right)_H = \left(\frac{\partial S}{\partial P}\right)_T + \left(\frac{\partial S}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_H.$$  \hspace{1cm} (6.4)

Substituting (6.4) into the equation (6.3) we obtain

$$0 = -T\left(\frac{\partial V}{\partial T}\right)_P + C_P\left(\frac{\partial T}{\partial P}\right)_H + V.$$  \hspace{1cm} (6.5)

where we used $C_P = T\left(\frac{\partial S}{\partial T}\right)_P$ and the Maxwell relationship $\left(\frac{\partial S}{\partial T}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$ which by solving $\left(\frac{\partial T}{\partial P}\right)_H$ one can obtain the JT coefficient [5.3].

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Figure 1: Numeric solutions for mass function and the internal metric of the $dS/AdS$ GB Bardeen black hole.
Figure 2: Thermodynamic phase diagrams of $dS/AdS$ GB Bardeen black holes and diagram of JT coefficient $\mu$ vs $T$