Robustness of cluster states and surface code states against random local fields

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In ideal quantum circuits, qubits are tacitly assumed to be uniformly fabricated and operated by prescribed signals. In reality, however, we must cope with different control signals to adjust individual qubits, which requires a large overhead of control circuits. Here, we theoretically investigate how random local fields affect cluster states and surface code states which constitute the key highly entangled states in quantum computation. We find similar behavior of temporal degradation of the fidelity for both cluster states and surface code states for the number of qubits up to ten. We find that the effect of local field fluctuations is greatly mitigated if the magnitude of fluctuations can be suppressed below 10% of the energy gap \(\Delta\) for both cluster states and surface code states. If the magnitude of fluctuations exceeds \(\Delta/2\), the fidelity for both states deteriorates dramatically. A simple estimation based on the average fidelity up to \(t \sim 2\hbar/\Delta\) shows that the maximum number of qubits that can be corrected with the 1% error threshold is less than 31 for surface code states and 27 for cluster states. This means that the error correction should be carried out during a time shorter than \(2\hbar/\Delta\).

I. INTRODUCTION

Quantum computation has seen a surge of interest since quantum annealing machines based on superconducting qubits were commercialized by D-wave Systems Inc. Moreover, IBM has made small-size quantum computers available to wide users through internet connection.\(^3,4\) The number of qubits in quantum computing devices is currently below one hundred, but it is expected to increase significantly in the years to follow. Thus, it will become increasingly more important to investigate quantum computing systems from a viewpoint of engineering reliability as in semiconductor devices. It seems not long before quantum computing systems can be prepared with near-unity fidelity; then we will face the problem of how robustly they can operate against local field fluctuations caused by defects and ambient electromagnetic fields due to crosstalk, wiring, etc. It is therefore important to examine the effect of local field fluctuations on the state of a system whose number of qubits exceeds 50 where the quantum advantage becomes experimentally feasible.\(^5\)

Here, we numerically investigate the effect of unavoidable local field fluctuations on otherwise perfect qubit states. In integrated circuits, qubits are supposed to operate uniformly where qubits change their quantum state with some prescribed signals. If there are additional local fields, qubit states change from their ideal points, resulting in errors. Thus, the additional local fields require an additional overhead to the circuits. Even after the technologies of qubits advances, it will remain a major challenge to perfectly control the fabrication process of the qubit system and realize the perfect qubit system.\(^5\) Besides, there are ambient electromagnetic fields ranging from low to high frequencies. Unless the electric circuits are not shielded from external electromagnetic environments, they suffer electromagnetic noises caused by unavoidable defects or trap sites.

At present, solid-state qubits are fabricated based on materials such as Si, SiGe/Si, GaAs/AlGaAs and superconducting materials. Table 1 lists the defect density and the defect-free area for solid-state qubits. Let \(n_{\text{imp}}\) \(\text{cm}^{-2}\) be the number density of defects. If the area \(S\) of a qubit is larger than the \(S_c \equiv 1/n_{\text{imp}}\), it has on average more than one defect. The bottom row of Table 1 shows the maximum defect-free area in units of 100 \(\text{nm}^2\). For example, a Si-qubit with an area larger than 100 \(\text{nm} \times 100 \text{ nm}\), has a significant probability of including one defect in its conducting area. The integrated circuits which include thousands of qubits will inevitably involve numerous defects and suffer fluctuations caused by them. This presents a major challenge for solid-state quantum computers which require highly entangled states for information processes. Thus, it is vital to investigate the effect of the local fluctuating fields on qubit systems for near-future quantum computation.

| Material | SiGe/Si | Si/SiO\(_2\) | GaAs/AlGaAs | Al/AlO\(_x\) |
|----------|---------|-------------|-------------|-------------|
| defect density \(\text{cm}^{-2}\) | \(\sim 10^8\) | \(\sim 10^{10}\) | \(\sim 10^{11}\) | \(\sim 10^4\) |
| defect-free area \(/(100 \text{ nm}^2)\) | \(\sim 10^4\) | \(\sim 10^2\) | \(\sim 10\) | \(\sim 10^8\) |

Table 1: Defect density and the defect-free device area for solid-state qubits. Data are taken from Ref.\(^6\) (SiGe/Si), Ref.\(^6\) (Si/SiO\(_2\)), Ref.\(^7\) (GaAs/AlGaAs), and Ref.\(^8\) (Al/AlO\(_x\)).
state systems using a series of emitted photons\textsuperscript{15–17} and continuous variable cluster states\textsuperscript{18,19}. One of the authors (T.T.) has proposed how to produce cluster states from solid-state qubits\textsuperscript{20,21}. In any qubit system, it will not be practical to ignore unexpected local fluctuations because we cannot prepare a perfect qubit lattice. The surface code has been studied by a number of researches both theoretically\textsuperscript{22–25} and experimentally\textsuperscript{26,27}. The qubit system consists of data qubits and measurement qubits where the data qubits form a logical qubit state and errors are detected and corrected by the measurement qubits. In the standard setup, the measurement qubits are placed close to the data qubits. In general, it is assumed that qubit-qubit interactions are switched on when necessary. Because a number of operations have to be applied to the measurement qubits next to the data qubits, unexpected dynamical noises affect the data qubits. Thus, it is important to consider the effect of local random fields on the qubit state of the surface code.

The standard approach to the analysis of the decoherence and degradation of qubits is to use master equations\textsuperscript{28,29} which are very effective to describe ensemble-averaged characteristics of the qubit system. However, it is not easy to include local field fluctuations. There are other sophisticated approaches to understand the noise properties\textsuperscript{30–35}. Here, we take a direct and simple approach to local fluctuations. We numerically investigate the effect of local field fluctuations caused by internal and external electromagnetic fields by adding the corresponding terms to an ideal Hamiltonian. We describe how the fidelity deteriorates as the number of fluctuating qubits and the magnitude of fluctuations increase. Specifically, we assume an ideal cluster state or a surface code state initially, and numerically solve the Schrödinger equation of the Hamiltonian $H_0 + H_1$, where $H_0$ describes an ideal cluster or surface code state, and $H_1$ describes the effect of local field fluctuations. Fowler et al\textsuperscript{28} show that a 1 % error can be corrected by the quantum error correction in the surface code. Thus, we regard the 1 % degradation of the fidelity as the threshold above which fidelity can be improved by the standard quantum error-corrections.

Because of the constraint of our computer resources, the number of qubits is limited up to $N = 8$ for the cluster state and $N = 9$ for the surface-code Hamiltonian. In this paper, we consider the effect of local fluctuations by examining the time-dependent solutions of the Schrödinger equations. In the first part, we add the fluctuating local fields to all qubits. We show that when the fluctuations are small, the fidelity of wave functions remains close to unity. We find that for both cluster states and surface code states the effects of high-frequency local fields are similar to those of static local fields. We also investigate a situation in which fluctuating fields are added to some part of qubits to understand the relationship between the number of qubits with the local fluctuations and the fidelity. Because both cluster states and surface code states are highly entangled, one might expect that a single local fluctuating field drastically changes the fidelity and the magnitude of the local fluctuating fields are of secondary importance to the fidelity. However, our numerical result shows that the degradation of the fidelity depends critically on it and, furthermore, it is proportional to the number of the qubits in the presence of fluctuating fields. This is similar to the classical devices: the reduction in the number of noise sources directly improves the device performance.

Because there is no dissipation in the present framework, the wave functions is a periodic function of time. When the coherence time is short compared with the period of local field fluctuations, we will have to consider the effect of dissipation in addition to the random qubit system. In this paper, we take into account the effect of dissipation by introducing an optical potential\textsuperscript{37} which is a non-Hermitian term added to the original Hamiltonian to describe various decay phenomena\textsuperscript{38}.

The rest of this paper is organized as follows. In Sec. II we briefly review the cluster state and the surface code state. We also discuss the cases of two qubits that can be solved analytically. In Sec. III we show our numerical results about the effects of locally random fluctuations. In Sec. IV we show numerical results for the effect of the optical potential. In Sec. V we briefly discuss the origin of the local fluctuation term, and estimate the order of an operation time discussed in this paper. In Sec. VI we provide a summary and conclusions of this paper.

II. FORMULATION

The Hamiltonian of the system is given by $H = H_0^\alpha + H_1$ ($\alpha = c,s$) where the superscript $\alpha = c$ and $s$ refer to the cluster state and the surface code state, respectively, and $H_1$ expresses the effects of local field fluctuations given by

\begin{equation}
H_1 = \sum_{i=1}^{N} [g_i(t)X_i + h_i(t)Z_i],
\end{equation}

where $N$ is the number of qubits, $g_i(t) = g_i \cos(\omega_i^c t)$ and $h_i(t) = h_i \cos(\omega_i^c t)$ are randomly oscillating fields, and $X_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the $x$ and $z$ components of the Pauli matrices.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Minimal cluster (a) and surface code (b) constituted from qubits. The white circles show the data qubits and the colored ones show the measurement qubits, where $X$ and $Z$ denote the $x$ and $z$ components of the Pauli matrices.}
\end{figure}
z components of the Pauli matrices, respectively. The amplitudes (gi and hi) and the frequencies (ωz and ωz) are chosen from random numbers such that |hi(t)| ≤ h_ave, |gi(t)| ≤ g_ave, |ωz| ≤ ωz_ave, and |ωz| ≤ ωz_ave for given h_ave, g_ave, ωz_ave, and ωz_ave. To extract the effect of the local fluctuation, dissipation is not included at this level. Thus, the effect of the additional term alters the phase of the ideal wave function, causing the fidelity of the system to decrease from unity.

A. Cluster state

As described in Ref.11, it is straightforward to analyze the eigenstates of cluster states. The cluster state |ΦC⟩ is an eigenstate of the stabilizer operator

\[ K_i = X_i \prod_{j \in \text{nbhd}(i)} Z_j, \]

such that \( K_i |\Phi_C⟩ = (-)^{κ_i} |\Phi_C⟩ \) (κ_i = {0, 1}), where nbhd(i) denotes the nearest neighborhood of qubit i. The corresponding cluster-state Hamiltonian is given by

\[ H_0^C = -\Delta \sum_i K_i, \]

where Δ characterizes the energy gap of cluster states. We consider the lowest-lying cluster state, |ΦC⟩, such as \( K_i |Φ_C⟩ = |Φ_C⟩ \).

In the cluster states, the excited states are determined from the action of \( Z_i \) on the ground state. Because \( H_0^C |Φ_C⟩ / Δ = -\sum_j K_j |Φ_C⟩ = -N |Φ_C⟩ \), we have

\[ H Z_i |Φ_C⟩ / Δ = -(K_i + \sum_{j \neq i} K_j) |Φ_C⟩ = -Z_i(-K_i + N - 1) |Φ_C⟩ = -Z_i(N - 2) |Φ_C⟩. \]

The second excited states are given by \( Z_i Z_j |Φ_C⟩ \) and similarly the n-th excited states are given by \( \prod_{i=1}^n Z_i |Φ_C⟩ \) (n ≤ N). For N = 4, there is a unique ground state of energy \(-4Δ\), and two first excited states of energy \(-2Δ\), the next excited state of energy \(2Δ\), and the highest excited state of energy \(4Δ\). Thus, we can explicitly write down the matrix elements for the fluctuating term \( \sum_i h_i Z_i \) by expressing the Hilbert space in terms of \( \prod_{i=1}^n Z_i |Φ_C⟩ \). We can also explicitly write down the matrix elements for the fluctuating term \( \prod_{i=1}^n X_i |Φ_C⟩ \) by using the characteristics of the stabilizer code. For the case of N = 4, we have

\[ X_1 |Ψ_2^C⟩ = X_1(X_1 Z_2 Z_3) |Ψ_2^C⟩ = Z_2 Z_1 |Ψ_2^C⟩. \]

Thus, the effect of the x-perturbation brings each state into two higher eigenstates.

B. Surface code

The surface code, which is defined as a qubit system on a planar square lattice, is one of the stabilizer codes which are simultaneous eigenspaces of check operators. The check operators of the surface code consist of products of Pauli matrices \( X_i \) and \( Z_i \), and the corresponding Hamiltonian is given by

\[ H_0^S = -\Delta \sum_i \left( \sum_{l \in \text{star}(i)} X_l + \sum_{l' \in \text{boundary}(i)} Z_{l'} \right). \]

A realistic qubit system for the surface code consists of data qubits and measurement qubits. The data qubits constitute the logical system of Eq. (3), and the measurements qubits play the role of error-correcting operations on the data qubits. Here we identify the number of the data qubits as the number of qubits N in our calculations. The minimum surface code includes four qubits, two of which are data qubits (N=2) and the other two are measurement qubits that control the logical qubits (Fig. 1(b)). The next larger system includes five data qubits (N = 5), and four measurement qubits as shown in Fig. 2(c). The N = 5 surface-code Hamiltonian is given by

\[ H_0^S = -\Delta [X_1 X_3 X_4 + X_2 X_3 X_5 + Z_1 Z_2 Z_3 + Z_2 Z_1 Z_5]. \]

The ground state of the surface code is nondegenerate, and the eigenenergy of the N = 5 system is \(-4Δ\). The eigenenergies of the excited states are given by \(-2Δ, 0, 2Δ\), and \(4Δ\). For the surface-code Hamiltonian, there is no useful formula unlike for the cluster states, and we directly construct the system Hamiltonian and solve eigenvalue problems numerically. Note that, to realistically correct quantum errors, the minimum size of the surface code is N = 2524. However, because of the limitation of computational resources, we consider a qubit system for N ≤ 8 surface codes.

C. Time-dependent fidelity

We numerically solve the Schrödinger equation

\[ i\hbar \frac{\partial |Φ⟩}{\partial t} = H |Φ⟩ \]

by using the Runge-Kutta method in the Lapack subroutines. We consider three types of qubit configurations for the cluster state and the surface code state as shown in Fig. 1. For cluster states, we consider N = 2, 4, 9 qubits, and, for surface code states, we consider N = 2, 5, 8 qubits.

The fidelity is defined as

\[ F(t) = \langle \Psi_0(t) | Φ(t) \rangle^2, \]

where \( |Ψ_0(t)⟩ \) is a time-dependent wave function of \( H_0 \).
The minimum surface-code Hamiltonian is given by

$$H_{\text{ave}} = \Phi_{\text{ave}} = \frac{1}{2} \omega + i \frac{1}{2} \eta$$

The ground state of the minimum surface code of $N=2$ is given by a singlet state of $(0|1⟩ - |1⟩0)/\sqrt{2}$. The Hamiltonian of the minimum cluster state is given by

$$H_0 = -\Delta (X_1 Z_2 + Z_1 X_2). \quad (10)$$

The three excited states are given by $|\Phi_{1}⟩ = Z_1 |\Phi_0⟩$, $|\Phi_{1}⟩ = Z_2 |\Phi_0⟩$, and $|\Phi_{2}⟩ = Z_1 Z_2 |\Phi_0⟩$, whose eigenenergies are, 0, 0, and $2\Delta$, respectively. The total Hamiltonian is given by

$$H = \begin{pmatrix} -2\Delta & h_1 + g_2 & h_2 + g_1 & 0 \\ h_1 + g_2 & 0 & 0 & h_2 - g_1 \\ h_2 + g_1 & 0 & 0 & h_1 - g_2 \\ 0 & h_2 - g_1 & h_1 - g_2 & 2\Delta \end{pmatrix}. \quad (11)$$

The eigenvalues of the simplest case of $g_1 = g_2 = 0$ is given by

$$E = 2\Delta^2 + h_1^2 + h_2^2 + 2\sqrt{(\Delta^2 + h_1^2)(\Delta^2 + h_2^2)}. \quad (12)$$

Thus, the effect of the fluctuations is to shift the eigenenergy when the magnitude of the fluctuation is small compared with $\Delta$. The ground state is given by

$$|\Phi_{0}⟩ = b^c|\Phi_{0}⟩ + c^c|\Phi_{0}⟩ + d^c|\Phi_{0}⟩ + |\Phi_{0}⟩$$

where $a^c = (q_1 + \Delta)(q_2 + \Delta)/\sqrt{Dc}$, $b^c = h_1 h_2/\sqrt{Dc}$, and $c^c = -h_2(q_1 + \Delta)/\sqrt{Dc}$, using $q_i = \sqrt{\Delta^2 + h_i^2}$ (i = 1, 2) and

$$D^c = (q_1 + \Delta)^2(q_2 + \Delta)^2 + h_1^2 h_2^2.$$
function of the ground state is given by \( b^s |01⟩ - c^s |10⟩ \), where \( b^s = \Delta / \sqrt{2\sqrt{D^2(\sqrt{D^2} + h_m)}} \) and \( c^s = (\Delta + \sqrt{\Delta^2}) / \sqrt{2\sqrt{D^2(\sqrt{D^2} + h_m)}} \) with \( D^s = \Delta^2 + h_m^s \). Thus, the wave function of the surface code slightly changes from the original singlet state.

III. NUMERICAL RESULTS

The random numbers are taken from \([-1, 1]\) and multiplied by \( g_{ave}, h_{ave}, \omega^x_{ave}, \) and \( \omega^z_{ave} \). All calculations are repeated ten times to examine the randomized effects. Thus, each figure includes ten curves. Figure 3 shows the numerical results of the time-dependent fidelities for \( N = 4 \) cluster states (a-c) and \( N = 5 \) surface code states (d-f). For small static fluctuations of \( g_{ave} = h_{ave} = 0.1\Delta \), the fidelity is stable with no large variations. When the magnitude of the fluctuations is increased to \( 2\Delta \), we can see similar behavior for the cluster state. That is, for small static fluctuations of \( g_{ave} = h_{ave} = 0.1\Delta \), the fidelity remains stable, but as the magnitude of the fluctuations is increased to \( \Delta / 2 \), the modulations of wave functions become larger and larger. For the cluster state, the fidelity of \( N = 9 \) deteriorates after \( t \sim 3\Delta \). During \( t < 3\Delta \), we can see similar behavior for the cluster state. It is striking that the behavior of the surface code shown in Fig. 3 is very similar to that in Fig. 4 despite the fact that the number of qubits is almost doubled. However, as we show later in Fig. 8, the fidelity becomes smaller with increasing \( N \). The similarity between Fig. 3 and Fig. 4 indicates that the degradation rate is not large even as the size of the system increases.

Figure 5 shows the numerical results of the time-dependent fidelity for \( N = 9 \) cluster states (a-c) and \( N = 8 \) surface code states (d-f). For the surface code state, we can see a tendency similar to Fig. 3. That is, for small static fluctuations of \( g_{ave} = h_{ave} = 0.1\Delta \), the fidelity remains stable, but as the magnitude of the fluctuations is increased to \( \Delta / 2 \), the modulations of wave functions become larger and larger. For the cluster state, the fidelity of \( N = 9 \) deteriorates after \( t \sim 3\Delta \). During \( t < 3\Delta \), we can see similar behavior for the cluster state. It is striking that the behavior of the surface code shown in Fig. 3 is very similar to that in Fig. 4 despite the fact that the number of qubits is almost doubled. However, as we show later in Fig. 8, the fidelity becomes smaller with increasing \( N \). The similarity between Fig. 3 and Fig. 4 indicates that the degradation rate is not large even as the size of the system increases.

Figure 6 shows the numerical results of the time-dependent fidelity for the \( N = 5 \) surface code states (d-f) and the \( N = 5 \) surface code states (d-f) with oscillating fluctuations by finite \( h_{ave} \). Figure 5 shows the numerical results of the time-dependent fidelity for the \( N = 5 \) surface code states (d-f) with oscillating fluctuations by finite \( h_{ave} \). Figure 6 shows the numerical results of the time-dependent fidelity for the \( N = 5 \) surface code states (d-f) with oscillating fluctuations by finite \( h_{ave} \). When these figures are compared with those of the static local fields (Figs. 3 and 4), there is no notable difference. Thus, we are led to conclude that the effect of the local variations is mainly governed by the magnitudes of the local fluctuations. Figure 7 shows the time evolution of the fidelity of the oscillating local fields with high frequencies \( \omega^x_{ave} = \omega^z_{ave} = 5\Delta \) and \( g_{ave} = h_{ave} = 0.1\Delta \). From these results, we can again say that the degradation of the fidelity is mainly determined by the amplitude of the local fields.

Next, we estimate the magnitudes of small local fields below which quantum error-correction can be effective. Wang et al. show that a 1% error can be corrected by the quantum error correction. As mentioned above, our numerical results show that if the magnitude of local fluctuations is less than \( 0.1\Delta \), the fidelity remains close to unity. To examine the possibility of the quantum error correction, we replot the average fidelity over a long time period compared with \( h / \Delta \), as a function of the average magnitude of the local fields, \( \delta \):

\[
\delta = g_{ave} = h_{ave} = \omega^x_{ave} = \omega^z_{ave},
\]

Figure 8 shows the average fidelity during (a) \( 0 < t < 2h / \Delta \) and (b) \( 0 < t < 6h / \Delta \) for \( N = 9 \) and \( N = 4 \) cluster states and for \( N = 8 \) and \( N = 5 \) surface code states. For all cases, we can see some critical magnitudes around \( \delta \sim 0.1\Delta \) beyond which the average fidelity rapidly decreases. In other word, both the cluster states and the surface code states are robust against small local fluctuations to the extent that quantum error correction can be carried out. In Fig. 8 we can also see that as the number of qubits \( N \) increases, the fidelity decreases faster for cluster states than surface-code states. To see this scaling effect more clearly, we choose the variation \( \delta / \Delta \) at which the fidelity is 0.99 in Fig. 8 and plot them as a function of the number of qubits in Fig. 9. Although the number of the qubits in the cluster states is not the same as that of the surface code states, we think that we can see a general trend by this scaling. The results
From Fig. 9(a), we have that correction is meaningful when all qubits are affected by local fluctuations over a time interval of the order of $2\hbar/\Delta$. However, as can be seen from Figs. 8(b) and 9(d) for $N=5$ surface code state the infidelity is approximately proportional to $N_{\text{defect}}$. That is, the reliability of the system linearly depends on the density of local fluctuations. One might expect that a single local fluctuating field drastically changes the fidelity, because both the cluster state and the surface code state are highly entangled. However, our numerical finding that the degradation of the fidelity is proportional to the number of fluctuating fields shows that the fidelity can be improved if the number of fluctuating sites is reduced.

Combining with the results of Figs. 8 in order to construct a large qubit system whose number is larger than 50 to gain quantum advantage, we have to fine-tune local fluctuations one by one or reduce the number of defects. In the former case, an extra overhead of control circuits is inevitable. In the latter case, we should reduce the area shown in Table 1. Next, we investigate the relationship between the fidelity and the number of the fluctuating sites by assuming that we can make qubits smaller and smaller. Figure 10 shows the infidelity as a function of the defect sites $N_{\text{defect}}$. The numerical results shown above correspond to those of $N_{\text{defect}}=N$. From Fig. 10, the infidelity is approximately proportional to $N_{\text{defect}}$. Thus far, the number of the fluctuating sites are the same as that of qubits, that is, all qubits are assumed to be affected by local fluctuating fields. This is the case where the active qubit area is larger than the defect-free area shown in Table 1. Next, we investigate the relationship between the fidelity and the number of the fluctuating sites by assuming that we can make qubits smaller and smaller. Figure 10 shows the infidelity as a function of the defect sites $N_{\text{defect}}$. The numerical results shown above correspond to those of $N_{\text{defect}}=N$. From Fig. 10, the infidelity is approximately proportional to $N_{\text{defect}}$. That is, the reliability of the system linearly depends on the density of local fluctuations. One might expect that a single local fluctuating field drastically changes the fidelity, because both the cluster state and the surface code state are highly entangled. However, our numerical finding that the degradation of the fidelity is proportional to the number of fluctuating fields shows that the fidelity can be improved if the number of fluctuating sites is reduced.

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FIG. 7. Time evolution of the fidelity of the $N=9$ cluster state (a-c), and that of the $N=8$ surface code state (d-f) under oscillating local fields. (a) and (d) for $g_{\text{ave}}=h_{\text{ave}}=0.1\Delta$, and $\omega_{\text{ave}}=\omega_{\text{ave}}=0.1\Delta$. (b) and (e) for $g_{\text{ave}}=h_{\text{ave}}=0.5\Delta$, and $\omega_{\text{ave}}=\omega_{\text{ave}}=0.5\Delta$. (c) and (f) for $g_{\text{ave}}=h_{\text{ave}}=\Delta$, and $\omega_{\text{ave}}=\omega_{\text{ave}}=\Delta$.

FIG. 6. Time evolution of the fidelity of the $N=9$ cluster state (a-c), and that of the $N=8$ surface code state (d-f) under oscillating local fields. (a) and (d) for $g_{\text{ave}}=h_{\text{ave}}=0.1\Delta$, and $\omega_{\text{ave}}=\omega_{\text{ave}}=0.1\Delta$. (b) and (e) for $g_{\text{ave}}=h_{\text{ave}}=0.5\Delta$, and $\omega_{\text{ave}}=\omega_{\text{ave}}=0.5\Delta$. (c) and (f) for $g_{\text{ave}}=h_{\text{ave}}=\Delta$, and $\omega_{\text{ave}}=\omega_{\text{ave}}=\Delta$.
IV. EFFECT OF DISSIPATION

We can use an optical potential to describe the effect of dissipation phenomenologically. The Hamiltonian of the optical potential is given by $H_D = i\alpha$, where $\alpha$ is related to the lifetime $\tau$ of the system given by $\alpha = 1/\tau$. By the optical potential, the fidelity decrease exponentially. Figure 11 shows time evolution of the fidelity in the presence of an optical potential. The fidelity decreases similarly for the cluster state and the surface code state.

V. DISCUSSION

In this paper, we assume that the cluster-state Hamiltonian $H_S^\alpha$ and the surface code Hamiltonian $H_0^\alpha$ are given. However, the construction of the target Hamiltonian is also an important problem. For solid-state systems, the most natural interaction between qubits is two-body. On the other hand, both the cluster-state Hamiltonian and the surface-code Hamiltonian includes more than three-body interactions. Thus, we have to artificially construct the corresponding Hamiltonians to keep those states as eigenstates. In Ref. 39, we have shown how to derive the cluster-state Hamiltonian and the surface-code Hamiltonian starting from the general Hamiltonian of two-body interactions. In those methods, there remain residual terms other than target interactions. Thus, when we construct the target Hamiltonian, we will have to include many interaction terms other than $H_1$ in this paper. The investigation of the effects of these terms will be a future problem.

Let us estimate the time scale in this paper. When we construct the surface-code Hamiltonian on the basis of the method of Ref. 39, $\Delta$ can be estimated by using the original coupling strength between two qubits. From Ref. 39, if we use 10 MHz $< \Delta < 100$GHz, the time scale expressed by $\sim \Delta^{-1}$ is in the range of 10ps-100us. Thus, depending on the measurement time, the effect of the fluctuations described here should be able to be detected.

VI. CONCLUSION

We have numerically investigated the effect of random local fields on cluster states and surface code states. We have estimated how the fidelity is affected depending on the magnitude of local fluctuations for both cluster states and surface code states. We have shown that time evolution of the fidelity looks similar between the cluster states and the surface code states. For small number of qubits ($N < 10$), we find that the effect of the local fluctuations rapidly decreases to reach the 1% error when the fluctuation magnitude is 10% of the energy gap $\Delta$ for both the
cluster states and the surface code states when all qubits are subject to fluctuating fields. We also find that, if the magnitude of fluctuations exceeds $\Delta/2$, the fidelity for both entangled states deteriorates dramatically. Although the number of data is small, it is found that the maximum number of qubits that can be corrected (1% error threshold) is estimated to be less than 31 for surface code states and 27 for cluster states, when the fidelity is averaged during $t < 2\hbar/\Delta$. This means that when there are local fluctuations the error-correction should be carried out during a time shorter than $\hbar/\Delta$. We can also improve the fidelity by reducing the number of qubits that are subject to local field fluctuations. More elaborate calculations will be needed to estimate the maximum number of qubits during a long time of the order of $\hbar/\Delta$. The quantum advantage requires at least 50 qubits, and our results suggest that local fluctuations place severe restrictions to go way above this threshold.

1. A.D. King, J. Carrauquilla, J. Raymond, I. Ozfidan, E. Andryash, A. Berkley, M. Reis, T. Lanting, R. Harris, F. Altomare, K. Boothby, P.I. Bunyik, C. Enderud, A. Fréchette, E. Hoskinson, N. Ladizinsky, T. Oh, G. Poulin-Lamarre, C. Rich, Y. Sato, A.Yu. Smirnov, L.J. Swenson, M.H. Volkman, J. Whittaker, J. Yao, E. Ladizinsky, M.W. Johnson, J. Hilton, and M.H. Amini, Nature 560, 456 (2018).

2. A. Chiesa, F. Tacchino, M. Grossi, P. Santini, I. Tavernelli, D. Gerace and S. Carretta, Nat. Physics 15, 455 (2019).

3. S. Boixo, V.N. Isakov, V.N. Smelyanskiy, R. Babbush, N. Ding, Z. Jiang, M.J. Bremner, J.M. Martinis, and H. Neven, Nat. Phys. 14, 1 (2018).

4. B. Villalonga, D. Lyakh, S. Boixo, H. Neven, T.S. Humble, R. Biewas, E.G. Rieffel, A. Ho, and S. Mandrà, arXiv:1905.00444.

5. H.J. Queisser and E. E. Haller, Science 281, 945 (1998).

6. S.W. Bedell, K. Fogel, D.K. Sadana, H. Chen, and A. Domenicucci, Appl. Phys. Letts. 85 2493 (2004).

7. M.L. Green, E.P. Gusev, R. Degraeve, and E.L. Garfunkel, J. Appl. Phys. 90, 2057 (2001).

8. R. Magno, and M.G. Spencer, J. Appl. Phys. 75, 368 (1994).

9. M.J.A. Stoutimore, M.S. Khalil, C.J. Lomb, and K.D. Osborn, Appl. Phys. Lett. 101, 062602 (2012).

10. R. Raussendorf and H.J. Briegel, Phys. Rev. Lett. 86, 5188 (2001); R. Raussendorf, D.E. Browne, and H.J. Briegel, Phys. Rev. A 68, 022312 (2003).

11. M. Heim, W. Dr. J. Eisert, R. Raussendorf, M. Van den Nest, H.-J. Briegel, arXiv:quant-ph/0602096 in the Proceedings of the International School of Physics “Enrico Fermi” on “Quantum Computers, Algorithms and Chaos”, Varenna, Italy, July, 2005.

12. M.S. Tame, R. Prevedel, M. Paternostro, P. Bíglí, M.S. Kim, and A. Zeilinger, Phys. Rev. Lett. 98, 140501 (2007).

13. H. Wunderlich, G. Vallone, P. Mataloni3, and M.B. Plenio, New J. Phys. 13 033033 (2010).

14. W.B. Gao, X.C. Yao, J.M. Cai, H. Lu, P. Xu, T. Yang, C.Y. Lu, Y.A. Chen, Z.B. Chen, and J.W. Pan, Nat. Photonics 5, 117 (2011).

15. N. H. Lindner and T. Rudolph, Phys. Rev. Lett. 103 113602 (2009).

16. M. Gimeno-Segovia, T. Rudolph, and S.E. Economou, arXiv:1801.02599.

17. T. Nutz, A. Milne, P. Shadbolt, and T. Rudolph, APL Photonics 2, 066103 (2017).

18. N.C. Menicucci, P. Loock, M. Gu, C. Weedbrook, T.C. Ralph, and M.A. Nielsen, Phys. Rev. Lett. 97, 110501 (2006).

19. D.W. Moore, O. Houhou, and A. Ferraro, Phys. Rev. A 96, 022305 (2017).

20. T. Tanamoto, Y. X. Liu, S. Fujita, X. Hu, and F. Nori, Phys. Rev. Lett. 97 230501 (2006); J.Q. You, X. Wang, T. Tanamoto, and F. Nori, Phys. Rev. A 75, 052319 (2007); T. Tanamoto, Y. Liu, X. Hu, and F. Nori, Phys. Rev. Lett. 102, 100501 (2009).

21. T. Tanamoto, Jpn. J. Appl. Phys. 49 04DJ08 (2012).

22. S. Bravyi, and A.Y. Kitaev, ArXiv:quant-ph/9811052.

23. D. Dennis, A.Y. Kitaev, Y.A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002).

24. A.G. Fowler, M. Mariantoni, J.M. Martinis, and A.N. Cleland, Phys. Rev. A 86, 032324 (2012).

25. D.S. Wang, A.G. Fowler, and L.C.L. Hollenberg, Phys. Rev. A 83, 020302 (2011).

26. R. Barends, J. Kelly, A. Megrant, A. Veitia, D. Sank, E. Jeffrey, T.C. White, J. Mutus, A.G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, C. Neill, P. Ö. Malley, P. Roushan, A. Vainsencher, J. Wenner, A.N. Korotkov, A.N. Cleland, and J.M. Martinis, Nature 508, 500 (2014).

27. J. Kelly, R. Barends, A.G. Fowler, A. Megrant, E. Jeffrey, T.C. White, D. Sank, J.Y. Mutus, B. Campbell, Yu Chen, Z. Chen, B. Chiaro, A. Dunsworth, L.-C. Hoi, C. Neill, P.J. J. O’Malley, C. Quintana, P. Roushan, A. Vainsencher, J. Wenner, A.N. Korotkov, A.N. Cleland, and J.M. Martinis, Nature 519, p66 (2015).

28. A. Hutter and D. Loss, Phys. Rev. A 89, 042334 (2014).

29. C. Daniel Freeman, C.M. Herdman, Dylan J Gorman, K. B. Whaley, Phys. Rev. B 90, 134302 (2014).

30. O. Viyuela, A. Rivas and M.A. Martin-Delgado, New J. Phys. 14 033044 (2012).

31. D. Jennings, A. Dragan, S.D. Barrett, S.D. Bartlett, and T. Rudolph, Phys. Rev. A 80, 032328 (2009).

32. F. Pastawski, A. Kay, N. Schuch, and I. Cirac, Quantum Inf. Comput. 10, 580 (2010).

33. A. Hamma and D.A. Lidar, Phys. Rev. Lett. 100, 030502 (2008).

34. D. Dusuel, M. Kamfor, R. Orús, K.P. Schmidt, and J. Vidal, Phys. Rev. Lett. 106, 107203 (2011).

35. D. Amaro, M. Miller and A.K. Pat, [arXiv:1907.13161].

36. T. Tanamoto, V.M. Stojanović, C. Bruder, and D. Becker, Phys. Rev. A 87 052305 (2013).

37. Y. Zolta, and T. Tanamoto, J. Appl. Phys. 74, 6996 (1993).

38. Y. Ashida, S. Furukawa, and M. Ueda, Nat. Comms 8, 15791 (2017).

39. P. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito and K. Sema, Nat. Phys. 13, 44 (2017).