Asymptotically isometric metrics on relatively hyperbolic groups and marked length spectrum

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Abstract

We prove asymptotically isometric, coarsely geodesic metrics on a toral relatively hyperbolic group are coarsely equal. The theorem applies to all lattices in $SO(n,1)$. This partly verifies a conjecture by Margulis. In the case of hyperbolic groups/spaces, our result generalizes a theorem by Furman and a theorem by Krat.

We discuss an application to the isospectral problem for the length spectrum of Riemannian manifolds. The positive answer to this problem has been known for several cases. All of them have hyperbolic fundamental groups. We do not solve the isospectral problem in the original sense, but prove the universal covers are $(1, C)$-quasi-isometric if the fundamental group is a toral relatively hyperbolic group.

1 Introduction

1.1 Asymptotically isometric metrics

Suppose a group $G$ acts on a space $X$ with two metrics $d_1$ and $d_2$ that are $G$-invariant. We say $(X, d_1)$ and $(X, d_2)$ are

1. coarsely equal if there exists $C$ such that for all $x, y \in X$,

\[ |d_1(x, y) − d_2(x, y)| \leq C. \]

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2. asymptotically isometric (or just asymptotic) if \( d_1(x, y) \to \infty \) if and only if \( d_2(x, y) \to \infty \), and

\[
\frac{d_1(x, y)}{d_2(x, y)} \to 1 \quad \text{as} \quad d_2(x, y) \to \infty.
\]

3. weakly asymptotically isometric (or weakly asymptotic) if \((X, d_1)\) and \((X, d_2)\) are quasi-isometric, \(g \in G\) is hyperbolic for \(d_1\) if and only if it is hyperbolic for \(d_2\), and that for every \(g \in G\) that is hyperbolic, we have

\[
\lim_{n \to \infty} \frac{d_1(x, g^n(x))}{d_2(x, g^n(x))} = 1.
\]

This property does not depend on the choice of \(x\).

4. \(d_1\) and \(d_2\) have the same marked length spectrum with respect to the \(G\)-actions if \(|g|_1 = |g|_2\) for all \(g \in G\), where \(|g|_i\) is the translation length of \(g\) for \(d_i\) defined by

\[
|g|_i = \lim_{n \to \infty} \frac{d_i(x, g^n(x))}{n}.
\]

\(g\) is hyperbolic on \((X, d_i)\) if and only if \(|g|_i > 0\).

Clearly, \((1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4)\). We discuss the other implications in this paper. Some remarks are in order. \((1)\) is same as the identity map is a \((1, C)\)-quasi-isometry. This is stronger than \(d_1\) and \(d_2\) are \((1, C)\)-quasi-isometric. Even if \(d_1\) and \(d_2\) are isometric, \((1)\) may not hold. \((2)\) implies that the identity map is a quasi-isometry. In view of that, if we concern the implication from \((3)\) to \((1)\) or \((2)\), then we should look at \(G\)-equivariant maps from \((X, d_1)\) to \((X, d_2)\), not the identity map.

We write \(A \sim_C B\) if \(|A - B| \leq C\). A metric space \((X, d)\) is \(C\)-coarsely geodesic if for any \(x, y \in X\), there is a path \(\gamma\) from \(x\) to \(y\) such that for all \(t, s\), we have \(|t - s| \sim_C |\gamma(t) - \gamma(s)|\). \(\gamma\) does not have to be continuous. If \(C = 0\), \(X\) is geodesic. We may suppress the constant \(C\) and just say coarsely geodesic.

We write \(d_1 \sim_C d_2\) if for all \(x, y\) we have \(d_1(x, y) \sim_C d_2(x, y)\). We may simply write \(d_1 \sim d_2\). This is nothing but they are coarsely equal.

Recall that a map \(f : X \to Y\) between two metric spaces \((X, d_X)\) and \((Y, d_Y)\) is called a \((L, C)\)-quasi-isometry if \(L d_X(a, b) - C \leq d_Y(f(a), f(b)) \leq L d_X(a, b) + C\).
\[ d_X(a, b)/L + C, \text{ for all } a, b \in X, \text{ and every } y \in Y \text{ is at distance at most } C \]
from some element of \( f(X) \). If such \( f \) exists for some \( L \geq 1, C \geq 0 \), then \( X \) and \( Y \) are quasi-isometric.

Burago [7] proved for \( G = \mathbb{Z}^n \) and \( X = \mathbb{R}^n \), \( (2) \Rightarrow (1) \) for \( G \)-invariant Riemannian metrics. His argument applies to a pair of coarsely geodesic metrics on \( \mathbb{Z}^n \) (Corollary 3.3).

Krat [18] proved an analogous result when \( X \) is \( \delta \)-hyperbolic, in particular, which implies \( (2) \Rightarrow (1) \) for two left invariant metrics on a hyperbolic group \( G \) that are quasi-isometric to a word metric. Furman [14] proved \( (3) \Rightarrow (1) \) in the same setting. His argument is different from hers. We will modify her argument and prove \( (3) \Rightarrow (1) \) for toral relatively hyperbolic groups, which are more general than hyperbolic groups.

Abels and Margulis [1] proved \( (2) \Rightarrow (1) \) for “word metrics” on reductive Lie groups. It is asked by Margulis in [19] if \( (2) \Rightarrow (1) \) holds in general on a finitely generated group \( G \). Breuillard [4] answered this question in the negative. He found a counter example, which are two word-metrics on \( H_3(\mathbb{Z}) \times \mathbb{Z} \), where \( H_3(\mathbb{Z}) \) is the three dimensional discrete Heisenberg group. In fact, the two metrics are not coarsely equal on some cyclic (undistorted) subgroup in his example. Also, those two metrics are not even \((1, C)\)-quasi-isometric to each other for any \( C \), [5]. It was known that \( (2) \Rightarrow (1) \) on \( H_3(\mathbb{Z}) \), [18].

### 1.2 Main results

An isometric action on a metric space \((X, d)\) by a group \( G \) is **cobounded** if there exists a bounded set \( B \) in \( X \) such that \( G.B = X \), and **proper** if for any \( x \in X \) and \( R > 0 \) there exist at most finitely many \( g \in G \) with \( d(x, g.x) \leq R \).

The following is the main result.

**Theorem 3.1** Assume \((G, \mathcal{H})\) is a relatively hyperbolic group such that for each \( H_i \in \mathcal{H} \), \( H_i \) contains \( \mathbb{Z}^{n_i} \) as a finite index subgroup. Assume \( G \) acts on \( X \) properly and co-boundedly by isometries for geodesic metrics \( d_1, d_2 \) (or more generally, \( d_2 \) is a coarsely geodesic metric). If they are weakly asymptotically isometric, then they are coarsely equal.

The following are examples of toral relatively hyperbolic groups (see [17] and Theorem 1.2.1 therein):

- all lattices in \( SO(n, 1) \) (uniform ones are hyperbolic).
• CAT(0) groups with isolated flats ([17]). In particular, the fundamental group of a closed, irreducible 3-manifold such that each piece of its JSJ-decomposition is atoroidal (namely, hyperbolic).

• Limit groups in the sense of Sela.

It seems it is an open question if the conclusion of the theorem holds for non-uniform lattices in the Lie group $SU(n, 1)$. See the discussion in Section 3.3.

A merit to show (3) $\Rightarrow$ (1) is it has an application to the marked length spectrum problem since (3) $\Leftrightarrow$ (4).

**Corollary 4.2.** Let $(M_1, d_1), (M_2, d_2)$ be closed Riemannian manifolds with the isomorphic fundamental group $G$ that is toral relatively hyperbolic. Assume they have the same marked length spectrum. Then there is a $G$-equivariant $(1, C)$- quasi-isometry map $f : \tilde{M}_1 \to \tilde{M}_2$.

It is easy to see that $M_1$ and $M_2$ have the same marked length spectrum if the conclusion in the corollary holds. Therefore we rephrase the marked length spectrum problem as follows. If there is a $G$-equivariant $(1, C)$- quasi-isometry map from $\tilde{M}_1$ to $\tilde{M}_2$, then is $M_1$ isometric to $M_2$? Notice that if $C = 0$, then $M_1$ and $M_2$ are isometric.

The iso-spectral problems for the marked length spectrum has been solved for several families of Riemannian manifolds, but in all of those cases, the fundamental group is hyperbolic (see Section 4). The novel part of our result is that we put the marked length spectrum problem into context for a broader class of groups.

We close the introduction with a discussion on a question by Gromov. In [16, 2C2(c)] he asks if the Hausdorff distance of two manifolds $X_1$ and $X_2$ is finite if they are acted by $G$ properly and co-compactly and that “$AL \operatorname{Dist}(X_1, X_2) = 0$”. Here, $AL$ is for asymptotically Lipschitz and $AL \operatorname{Dist}(X_1, X_2) = 0$ if for any $a > 0$ there is a $(1 + a, C_a)$-quasi-isometry $f_a$ between $X_1$ and $X_2$ for some $C_a$. Our property (2) implies $AL \operatorname{Dist}(X_1, X_2) = 0$ since we can take the identity map as $f_a$ for any $a > 0$ with sufficiently large $C_a$. (1) implies that the Hausdorff distance between $X_1$ and $X_2$ is at most $C$ via the identity map. Our result affirmatively answers a version of the question by Gromov with both the assumption and the conclusion $G$-equivariant when $G$ is toral relatively hyperbolic.
2 The case of hyperbolic spaces

We first prove the following. Krat [18] proved the result under the assumption that $d_1, d_2$ are asymptotic, but our assumption is weaker.

**Theorem 2.1.** Let $d_1, d_2$ be coarsely geodesic metrics on $X$ on which $G$ acts by isometries, co-boundedly with respect to both $d_1$ and $d_2$. Suppose $(X, d_1)$ is $\delta$-hyperbolic. Assume that $d_1$ and $d_2$ are weakly asymptotically isometric. Then, $d_1 \sim d_2$.

**Proof.** The outline of our argument is same as the one by Krat. Define $\Delta(x, y) = d_1(x, y) - d_2(x, y)$. To argue by contradiction, assume $\Delta$ is not bounded. We will prove that then $\lim \inf_n \Delta(x, g^n(x))/d_2(x, g^n(x)) \not= 0$ as $n \to \infty$ for some $g \in G$. This contradicts to the assumption.

Here are two elementary lemmas. The first one is straightforward from the triangle inequality for $d_1$ and $d_2$.

**Lemma 2.2.** Let $x, y, z$ be points in $X$. Then,

$$|\Delta(x, y) - \Delta(x, z)| \leq d_1(y, z) + d_2(y, z).$$

**Lemma 2.3.** Let $\gamma_1$ be a $d_1$-geodesic and $\gamma_2$ a $d_2$-geodesic from $x$ to $y$. Let $z \in \gamma_1$ be a point such that there exists $z' \in \gamma_2$ with $d_2(z, z') \leq C$. Then, $\Delta(x, z) + \Delta(z, y) \sim_{2C} \Delta(x, y)$. Moreover, the conclusion holds if $\gamma_1$ and $\gamma_2$ are $L$-coarse geodesics (with a constant that is larger than $2C$ depending on $L$).

We say that $\Delta$ is almost additive at $z$.

**Proof.** Suppose $\gamma_1, \gamma_2$ are geodesics. $\Delta(x, z) = d_1(x, z) - d_2(x, z) \sim_C d_1(x, z) - d_2(x, z')$, and similarly $\Delta(z, y) \sim_C d_1(z, y) - d_2(z', y)$. Now, $\Delta(x, z) + \Delta(z, y) \sim_{2C} d_1(x, z) - d_2(x, z') + (d_1(z, y) - d_2(z', y)) = d_1(x, y) - d_2(x, y) = \Delta(x, y)$.

A similar argument applies when $\gamma_i$ are coarse geodesics and we omit details. \qed

We go back to the proof of the theorem. In the following we assume that $d_1$ and $d_2$ are geodesic metrics. We can easily modify each argument when they are only coarsely geodesic with extra constants, but we leave it to the readers.

Since $d_1$ and $d_2$ are asymptotic, therefore quasi-isometric to each other, any $d_1$-geodesic is a $d_2$-quasi-geodesic with a controlled quasi-isometric constants, vice-versa. There exists a constant $C$ (by the Morse lemma, [6, Theorem 1.7]) such that a $d_1$-geodesic and a $d_2$-geodesic with the same endpoints
are in the $C$-neighborhood (for both $d_1$ and $d_2$) of each other. Therefore by Lemma 2.3, $\Delta$ is almost additive, for a uniform constant, on any $d_1$-geodesic at any point.

By assumption, a metric ball of radius, say, $D$ covers $X$ by the $G$-action. Fix a base point $x$ and we write $g(x)$ as $g$. Write $\Delta(1,g)$ as $\Delta(g)$. Notice $\Delta(g) = \Delta(g^{-1})$. By assumption there is $g$ with $\Delta(g) >> \delta, C, D$. Take a $d_1$-geodesic $\gamma$ from 1 to $g$, then there is $h \in G$ with $\Delta(h)$ is approximately $\Delta(g)/2$. This is possible since $\Delta$ is almost continuous on a geodesic. Set $k = h^{-1}g$. Then $\Delta(k)$ is approximately $\Delta(g)/2$ by Lemma 2.3 and Lemma 2.2.

Let $[p, q]$ denote a $d_1$-geodesic from $p$ to $q$. For $g \in G$, let $g^*.x$ denote the piecewise geodesic $\cup_{n \geq 0} [g^n.x, g^{n+1}.x]$ starting at $x$.

**Claim.** At least one of the following piecewise geodesic is a $(2, 10\delta + D)$-quasi-geodesic on $X$: $k^*.x$ or $h^*.x$ or $(hk)^*.x$.

This is a standard fact. See Lemma 8.1.A in [15]. Since the points $1, h, g = hk$ is almost on a geodesic, $d_1(x, g.x) \sim d_1(x, h.x) + d_1(h.x, g.x)$ and both $d_1(x, h.x)$ and $d_1(h.x, g.x)$ are large, since $\Delta(g)$ is large. This is enough to apply Lemma 8.1.

This claim is the new ingredient than the argument by Krat, who stated that a piecewise geodesic of a similar property, not necessarily periodic, exists (Lemma 1.12). The periodicity is crucial under our assumption.

Let $f^*.x$ be one of the paths we obtain from the claim. On the path, $\Delta(x, f^n.x)$ grows roughly linearly on $n$ since on each geodesic piece it increases at least by $\Delta(h) - 2C$ or $\Delta(g) - 2C$, depending on $f^*.x$, by Lemma 2.3. We used that $\Delta(h) \sim \Delta(k) >> \delta, C, D$. It follows $\lim_{n} \inf \Delta(x, f^n(x))/d_1(x, f^n(x)) > 0$. \[ \square \]

### 3 Toral relatively hyperbolic groups

In this section we will generalize Theorem 2.1 to relatively hyperbolic groups.

#### 3.1 Asymptotically tree graded spaces and relatively hyperbolic groups

We review a few key notions and results from [11]. Let $X$ be a complete geodesic metric space and let $P$ be a collection of closed geodesic subsets (called pieces) with the following properties ([11]):
(T1) every two different pieces have at most one common point.

(T2) Every simple geodesic triangle (a geodesic triangle that is a simple loop) in $X$ is contained in one piece.

Let $X$ be a metric space and $A$ a collection of subsets in $X$. $X$ is asymptotically tree-graded with respect to $A$ if every asymptotic cone, $\text{Con}_\omega(X)$, of $X$ is tree-graded with respect to a certain collection of subsets $A_\omega$, which are defined from the collection $A$ (see [11, Definition 3.19] for the precise definition). Here, $\omega$ is an ultra filter. See the definition of the asymptotic cone in [11, Definition 3.8]. In this paper, we do not use the definitions of (asymptotically) tree-graded spaces, but only quote geometric properties of those spaces from [11] and [22]. We will state them later.

There is more than one way to define relatively hyperbolic groups. The following definition is one of the main theorems in [11]. We say a finitely generated group $G$ is relatively hyperbolic with respect to a collection of subgroups $H = \{H_1, \ldots, H_m\}$ if $G$ is asymptotically tree-graded with respect to the subgroups $H$, namely, the Cayley graph of $G$, $\Gamma(G, S)$, with respect to some (and any) finite set, $S$, of generators is asymptotically tree-graded with respect to the collection of left cosets $\{gH_i|g \in G, i = 1, \ldots, m\}$. The subgroups $H_i$ are called peripheral subgroups. If they are all finitely generated virtually abelian groups, $G$ is called a toral relatively hyperbolic group. We do not assume that a toral relatively hyperbolic group is torsion-free.

Farb [12] defined that $G$ is weakly relatively hyperbolic with respect to $H$ if the Cayley graph $\Gamma(G, S \cup H)$ is hyperbolic, where $S \cup H$ means the union of the elements in $S$ and the elements in $H_i$. Drutu-Sapir [11, Theorem 8.6] proved the relative hyperbolicity in the above sense implies the weak relative hyperbolicity.

### 3.2 Toral relatively hyperbolic groups

We generalize Theorem 2.1 to toral relatively hyperbolic groups.

**Theorem 3.1.** Assume $(G, H)$ is a relatively hyperbolic group such that for each $H_i \in H$, $H_i$ contains $\mathbb{Z}^{n_i}$ as a finite index subgroup. Assume $G$ acts on $X$ properly and co-boundedly by isometries for geodesic metrics $d_1, d_2$ (or more generally, $d_2$ is a coarsely geodesic metric). If they are weakly asymptotically isometric, then they are coarsely equal.

Since $d_2$ may be coarsely geodesic, the theorem for example applies to two word metrics on $G$ (take the Cayley graph for one of the two metrics as...
We assume \( d_1 \) is geodesic since we apply 11 and 22 to \( d_1 \). They only discuss geodesic metrics. It looks likely that the results we use from those two papers hold for coarsely geodesic metrics.

Notice that \( \Gamma(G,S) \) and \( X \) are quasi-isometric with respect to both \( d_1, d_2 \) (6, Proposition 8.19). The statement therein is for co-compact group actions on a geodesic space, but the argument uses only co-boundedness, and also it applies to a coarse geodesic space).

In the proof of the theorem we will first apply the theorem by Burago [7] to \( (H,d_1|H), (H,d_2|H) \), and want to conclude \( d_1 \sim d_2 \) on \( H \). A little issue is that \( d_i|H \) is not a geodesic metric. So we first modify his result then argue that the modification is enough for us.

Theorem 3.2. Let \( d \) be a coarsely geodesic metric on \( \mathbb{Z}^n \) that is invariant by the left action of \( \mathbb{Z}^n \). Then

1. for each \( g \in \mathbb{Z}^n \), \( \lim_{n \to \infty} \frac{d((1,g^n))}{n} \) exists. We write the limit by \( |g|_d \).
2. There is a constant \( C \) such that for all \( g \), \( |d(1,g) - |g|_d| \leq C \).
3. \( |d(1,g^n) - n|g|_d| \leq C \) for all \( n > 0 \) and \( g \).

Proof. We first prove that the limit exists. The argument is a modification of the proof of [7, Theorem 1]. We explain the change we need. We only need to modify Lemma 4 in [7].

Embed \( \mathbb{Z}^n < \mathbb{R}^n \) as a subgroup. Fix \( x \in \mathbb{Z}^n \). The claim of Lemma 4 is for \( h \in \mathbb{Z}^n \), \( 2d(x, h.x) \sim d(x, h^2(x)) \), where a bound does not depend on \( h \). To prove it, join \( x, h^2.x \) by a coarse geodesic \( \gamma : [0,L] \to \mathbb{Z}^n \). Approximate it by a continuous path \( \gamma' : [0,L] \to \mathbb{R}^n \) such that at each time, the points on the two paths are boundedly apart such that for each point \( g = \gamma(t) \) on \( \gamma \), there exists \( t' \) such that \( \gamma(t') = \gamma'(t') \) (also \( |t - t'| \) is bounded). In other words, \( \gamma' \) and \( \gamma \) visit the same points in \( \mathbb{Z}^n \) at the same time (for each point).

Apply Lemma 2 in [7] to \( \gamma' \) and divide it into at most \( n \) segments at points in \( \mathbb{Z}^n \) and rearrange, then get a path \( \gamma'' : [0,L] \to \mathbb{R}^n \) from \( x \) to \( h^2.x \) such that the distance from \( h.x \) to \( \gamma'' \) is bounded. In the original case it exactly passes \( h.x \) since we can divide the path anywhere. In our setting, we approximate the original dividing points by nearby points in \( \mathbb{Z}^n \). Since \( \mathbb{Z}^n \) is commutative and we cut only at most \( n \) times, we get a uniform bound. From \( \gamma'' \), by approximating it by nearby points in \( \mathbb{Z}^n \), we get a coarse geodesic \( \gamma''' : [0,L] \to \mathbb{Z}^n \) from \( x \) to \( h^2(x) \). It follows that \( 2d(x,h.x) = d(x,h.x) + \).
\(d(h.x, h^2.x) \sim h(x, h^2(x))\). Lemma 3 in [7] is proved, therefore (1) and (2) are proved as in [7].

For (3), notice that by the definition of \(|g|_d\), we have \(|g^n|_d = |n||g|_d\), therefore, \(|d(1, g^n) - n|g|_d| \leq C\) for all \(n > 0\) and \(g\).

**Corollary 3.3.** Let \(d_1, d_2\) be a coarsely geodesic metric on \(\mathbb{Z}^n\) that is invariant by the left action of \(\mathbb{Z}^n\). Assume that if \(d_2(1, g^n)\) or \(d_1(1, g^n) \to \infty\) as \(n \to \infty\), then \(\lim_{n \to \infty} \frac{d_1(1, g^n)}{d_2(1, g^n)} = 1\). Then \(d_1 \sim d_2\).

Moreover, the result is true if we replace \(\mathbb{Z}^n\) by a group which contains \(\mathbb{Z}^n\) as a finite index subgroup.

In other words, if \(d_1, d_2\) are weakly asymptotically isometric, then they are coarsely equal.

**Proof.** By Theorem 3.2, there is \(C\) such that for all \(n > 0, g\), we have \(|d_1(1, g^n) - n|g|_{d_1}| \leq C\) and \(|d_2(1, g^n) - n|g|_{d_2}| \leq C\). Now by our assumption, \(|g|_{d_1} = |g|_{d_2}\).

Again, by the theorem, for all \(g\), \(|d_1(1, g) - d_2(1, g)| \leq 2C\).

Since both \(d_1, d_2\) are \(\mathbb{Z}^2\) invariant, we have \(d_1 \sim d_2\).

For the moreover part, we already know \(d_1 \sim d_2\) on \(\mathbb{Z}^n\). But any element in \(H\) is at bounded distance, say \(D\), from a subgroup isomorphic to \(\mathbb{Z}^n\) (for both \(d_1, d_2\)), therefore \(d_1 \sim_{C+2D} d_2\) on \(H\).

Now, here is a lemma that will assure that \(d|_H\) is coarsely geodesic.

**Lemma 3.4.** Assume \(H\) acts on a coarsely geodesic space \(X\) with a point \(x \in X\). Define a metric on \(H\) by \(d(a, b) = d_X(a.x, b.x)\). Assume that there is \(C\) such that for any \(a, b \in H\), there is a \(X\)-geodesic between \(a.x, b.x\) which is in the \(C\)-neighborhood of \(H.x\). Then \(d\) is a coarse geodesic metric on \(H\).

**Proof.** This is straightforward from the definition. First assume that \(X\) is geodesic. Let \(\gamma(t), 0 \leq t \leq L\) be a geodesic in \(X\) from \(a.x\) to \(b.x\). For each \(t\), choose \(h_t \in H\) with \(d(\gamma(t), h_t.x) \leq C\). Define a path \(\alpha\) in \(H\) from \(a\) to \(b\) by \(\alpha(t) = h_t, 0 \leq t \leq L\). Now for any \(0 \leq t \leq s \leq L\), we have \(d(\alpha(t), \alpha(s)) = d(h_t, h_s) = d_X(h_t.x, h_s.x) \sim_{2C} d_X(\gamma(t), \gamma(s)) = |t - s|\). Therefore \(\alpha\) is a \(2C\)-coarsely geodesic. If \(X\) is coarsely geodesic, then start with a coarse geodesic \(\gamma\) and argue.

We quote several results on asymptotically tree graded spaces and relatively hyperbolic groups. First, a peripheral subgroup is almost convex, therefore Lemma 3.4 applies to \(d|_H\).
Lemma 3.5. [11, Lemma 4.3] Let $(G, \mathcal{H})$ be relatively hyperbolic. Then each $H \in \mathcal{H}$ is almost convex in $G$, in the sense that any $(K, L)$-quasi-geodesic joining two points of $H$ in $\Gamma(G, S)$ is in the $C$-neighborhood of $H$, where $C$ depends on $K, L$ but not on the quasi-geodesic.

We remark that in [11] the lemma is stated only for geodesics, which is sufficient to apply Lemma 3.4, but the lemma holds for quasi-geodesics. The reason is that two distinct $aH$ and $bH'$ stay close only in a bounded set (Lemma 3.7), so that the claim follows using the next lemma.

Two quasi-geodesics in $X$ with common end points stay close to each other in the following sense. We say that the two quasi-geodesics fellow travels. This is a version of Morse lemma for asymptotically tree graded spaces.

Lemma 3.6. [11, Theorem 1.12] Let $x, y \in X$, and $\alpha$ a $(K, L)$-quasi-geodesic and $\gamma$ a geodesic both between $x, y$ in $(X, d_1)$. Then there exists a constant $C(K, L)$ such that they $C$-fellow travel up to $aH.x$’s, $(H \in \mathcal{H})$. More precisely, $\alpha$ is in the $C$-neighborhood of $\gamma$ except for the union of some (long) sub-quasi-geodesics of $\alpha$ each of which is contained in the $C$-neighborhood of some $aH.x$ such that $aH.x$ is distance at most $C$ from $\gamma$. In this case, the end points of each of the sub-quasi-geodesics are in the $C$-neighborhood of $\gamma$.

In [11] the above lemma is stated for $\Gamma(G, S)$, but $\Gamma(G, S)$ and $(X, d_1)$, as well as $(X, d_2)$, are quasi-isometric therefore the results hold for $(X, d_1), (X, d_2)$ as well.

In the above lemma, we may assume the long subpaths of $\alpha$ are disjoint by the following lemma.

Lemma 3.7. [11, Lemma 4.7] Fix $x \in X$. For each $D$, the diameter of the intersection of the $D$-neighborhood of $aH_1.x$ and the $D$-neighborhood of $bH_2.x$ in $X$ is uniformly bounded unless $aH_1 = bH_2$. The bound depends only on $D$.

Following [11], the (almost) projection to $aH$, $\pi_{aH}$, in $\Gamma(G, S)$ is defined as follows for $H \in \mathcal{H}$: for $g \in G$, $\pi_{aH}(g)$ is the subset of points in $aH$ whose distance from $g$ is less than $d(g, aH) + 1$.

The following result also holds for the projection to $aH.x$ in $X$ as well (the proof is same).
Lemma 3.8. [22, Lemma 1.13 (1) and Theorem 2.14] Let \((G, \mathcal{H})\) be relatively hyperbolic. Let \(H \in \mathcal{H}\) and \(\pi\) be the projection to \(aH\) in the Cayley graph with \(a \in G\). Then

1. Any \((K, L)\)-quasi geodesic from a point \(g\) in \(\Gamma(G, S)\) to a point in \(aH\) passes the \(C\)-neighborhood of \(\pi(g)\). The constant \(C\) depends on \(K, L\), but not on \(a, H\) and \(g\).

2. The diameter of \(\pi(g)\) is uniformly bounded. The bound does not depend on \(a, H\) and \(g\).

We may write \(\Gamma(G, S \cup \mathcal{H})\) as \(G'\). We denote the distance on \(\Gamma(G, S \cup \mathcal{H})\) by \(d_{G'}\). \(\Gamma(G, S \cup \mathcal{H})\) is hyperbolic (see [11, Section 8]) and \(G\) acts on it. Each edge in \(\Gamma(G, S \cup \mathcal{H})\) that is not in \(\Gamma(G, S)\) joins two points in \(aH\) for some \(H \in \mathcal{H}\). Given a geodesic \(\gamma\) in \(\Gamma(G, S \cup \mathcal{H})\), a lift is a path in \(\Gamma(G, S)\) obtained by replacing each edge of \(\gamma\) that is not in \(\Gamma(G, S)\) by a geodesic in \(\Gamma(G, S)\) connecting the two end points of the edge.

Lemma 3.9. [22, Prop 1.14] If \(\gamma\) is a geodesic in \(\Gamma(G, S \cup \mathcal{H})\) then its lift is a quasi-geodesic in \(\Gamma(G, S)\) with uniform quasi-geodesic constants.

The above lemma does not hold for quasi-geodesics \(\gamma\) in general.

Lemma 3.10. [11, Prop 8.25] Let \(\gamma\) be a quasi-geodesic in \(\Gamma(G, S)\) between \(x, y\) and \(\gamma'\) a quasi-geodesic in \(\Gamma(G, S \cup \mathcal{H})\) between \(x, y\). Then they are in a bounded Hausdorff-distance in \(\Gamma(G, S \cup \mathcal{H})\). The bound depends only on the quasi-geodesic constants.

We start the proof of Theorem 3.1.

Proof. Fix \(x \in X\). Define a left invariant (pseudo-)metric \(d_1(g, h) = d_1(g.x, h.x)\) on \(G\), and also \(d_2\) in the same way. Both \((G, d_1), (G, d_2)\) are quasi-isometric to \(\Gamma(G, S)\). As before set \(\Delta = d_1 - d_2\) on \(X\) and \(G\). Note that \(\Delta(g, h) = \Delta(g.x, h.x)\).

We summarize what we know on each peripheral subgroup \(H\) by now. On \(H\), \(d_1\) and \(d_2\) are coarsely geodesic metrics by Lemma 3.4 and Lemma 3.5. By assumption they are weakly asymptotic on \(H\). Therefore \(d_1 \sim d_2\) on \(H\) by Corollary 3.3. The conclusion holds for \(aH\) as well.

Lemma 3.11. There exists \(L\) such that on each \(aH.x\) and \(z \in X\), \(\Delta(z, y)\) varies at most \(L\) for \(y \in aH.x\). \(L\) does not depend on \(a, H\) and \(z\).
Moreover, the statement holds when \( y \) is in the \( K \)-neighborhood of \( aH.x \) (\( L \) depends on \( K \)).

Also, the statement holds on \( G \), namely, on each \( aH \) and \( g \in G \), \( \Delta(g, h) \) varies at most \( L \) for \( h \) that is in the \( K \)-neighborhood of \( aH \).

**Proof.** As usual we assume \( d_2 \) is geodesic. We omit details for the coarse geodesic case. Fix a point \( A \in \pi_{aH.x}(x) \), where \( \pi_{aH.x} \) is defined in \( (X, d_1) \).

We claim that \( \Delta(x, y) \sim \Delta(x, A) \) for all \( y \in aH.x \) such that the constant for \( \sim \) does not depend on \( y \) or \( aH \). Let \( \gamma_1, \gamma_2 \) be geodesics from \( x \) to \( y \) for \( d_1, d_2 \). Notice that \( \gamma_2 \) is a \( d_1 \)-quasi-geodesic with controlled constants. By Lemma 3.8, they pass \( C \)-neighborhood (in both \( d_1 \) and \( d_2 \), which are quasi-isometric to each other) of \( A \). \( C \) does not depend on \( a, H, y \). Now take \( q_1, q_2 \) on each geodesic with \( d_i(q_1, A) \leq C \) and \( d_i(q_2, A) \leq C \) with \( i = 1, 2 \). Then, by Lemma 2.3 and Lemma 2.2 we have \( \Delta(x, y) \sim 4C \Delta(x, q_1) + \Delta(q_1, y) \sim 8C \Delta(x, A) + \Delta(A, y) \).

But we already know \( d_1 \sim d_2 \) on \( aH \), therefore, we have \( \Delta(A, y) \sim 0 \). It follows \( \Delta(x, y) \sim \Delta(x, A) \). The argument is complete when \( y \in aH.x \). The moreover part now follows from Lemma 2.2. The argument for the group \( G \) is same.

Here is a consequence.

**Lemma 3.12.** There exists \( P \) such that for any \( x, y, z \in G \), we have

\[
|\Delta(x, y) - \Delta(x, z)| \leq P d_G(y, z).
\]

**Proof.** Let \( N = d_G(y, z) \) and \( \gamma \) a geodesic from \( y \) to \( z \) in \( \Gamma(G, S \cup H) \) with vertices \( y = y_0, y_1, \ldots, y_N = z \). \( y_n \) and \( y_{n+1} \) are joined by an edge. There exists \( K \) such that \( |\Delta(x, y_n) - \Delta(x, y_{n+1})| \leq K \) if the edge is in \( \Gamma(G, S) \). On the other hand, if the edge is not in \( \Gamma(G, S) \), then \( y_n, y_{n+1} \) are in some \( aH \), therefore \( |\Delta(x, y_n) - \Delta(x, y_{n+1})| \leq L \) by Lemma 3.11. Now it follows that \( |\Delta(x, y) - \Delta(x, z)| \leq (L + K)N \). Set \( P = L + K \).

**Lemma 3.13.** Let \( \gamma \) be a \( d_2 \)-quasi-geodesic in \( X \). Then, \( \Delta \) is almost additive on \( \gamma \). Namely, let \( x, z, y \) be points on \( \gamma \) in this order, then \( \Delta(x, z) + \Delta(z, y) \sim_B \Delta(x, y) \), where \( B \) depends on the quasi-geodesic constants of \( \gamma \).

The statement holds for quasi-geodesics on \( \Gamma(G, S) \) as well.

**Proof.** By assumption \( \gamma \) is a \( d_2 \)-quasi-geodesic with controlled constants. Then it is a \( d_1 \)-quasi-geodesic with controlled constants as well. Let \( \gamma' \) be a \( d_1 \)-geodesic in \( X \) from \( x \) to \( y \). By Lemma 3.6 there exists \( C \) that depends
on the quasi-geodesic constants such that \( \gamma \) and \( \gamma' \) stay close to each other except for subsegments in \( \gamma \) each of which stays in the \( C \)-neighborhood of one \( aH.x \) for a long time, but the end points of those segments are \( C \)-close to \( \gamma' \). If \( z \) is outside of those segments, then \( z \) is close to \( \gamma' \), which implies the almost additivity at \( z \) by Lemma 2.3.

Now assume \( z \) is contained in one of the subsegments, say, \([z_1, z_2]\). Each \( z_i \) is \( C \)-close to \( \gamma' \), therefore by Lemma 2.3, we have \( \Delta(x, y) \sim \Delta(x, z_1) + \Delta(z_1, y) \). On the other hand, since \( z, z_1, z_2 \) is in the \( C \)-neighborhood of \( aH.x \), by Lemma 3.11 and Lemma 2.2, we have \( \Delta(x, z_1) + \Delta(z_1, y) \sim \Delta(x, z) + \Delta(z, y) \) (Lemma 3.11 applies to \( \Delta(\ast, y) \)). Combining them, \( \Delta \) is almost additive at \( z \). The argument is complete.

Since \( \Gamma(G, S) \) and \( X \) are quasi-isometric, we also have the almost additivity in \( \Gamma(G, S) \).

We go back to the proof of the theorem. We want to show that \( \Delta(x, y) \) is bounded on \( X \), which is equivalent to that \( \Delta(g, h) \) is bounded on \( G \). To argue by contradiction, assume not. We will find \( f \in G \) such that \( \Delta(x, f^n.x) = \Delta(1, f^n) \) grows roughly linearly on \( n \), which will be a contradiction since \( d_1 \) and \( d_2 \) are weakly asymptotic.

Take \( g \) such that \( \Delta(1, g) \) is very large. Let \( \gamma \) be a geodesic from 1 to \( g \) in \( \Gamma(G, S) \), and let \( h \in G \) be such that \( h \) is on \( \gamma \) and \( \Delta(1, h) \) is approximately \( \Delta(1, g)/2 \). This is possible since \( \Delta(1, y) \) is almost continuous when we vary \( y \) on \( \gamma \).

Let \( \alpha \) be a geodesic from 1 to \( g \) in \( \Gamma(G, S \cup H) \). By Lemma 3.10, the Hausdorff distance between \( \gamma \) and \( \alpha \) in \( \Gamma(G, S \cup H) \) is bounded. In particular, \( h \) is at bounded distance from \( \alpha \) in \( \Gamma(G, S \cup H) \). Notice that \( d_{G'}(1, h), d_{G'}(h, g), d_{G'}(1, g) \) are all large since \( \Delta(1, h), \Delta(h, g), \Delta(1, g) \) are all large (use Lemma 3.12). Since \( \Gamma(G, S \cup H) \) is hyperbolic, as before, one of the paths \( h^*, k^*, (hk)^* \) (we use the same notation as in the proof of Theorem 2.1 with \( x = 1 \)) is a quasi-geodesic in \( \Gamma(G, S \cup H) \) with uniform quasi-geodesic constants. Denote it by \( \gamma = f^* \).

First, we assume that \( \gamma \) is a geodesic, and argue. Take a lift, \( \beta \), of \( \gamma \) in \( \Gamma(G, S) \). Then \( \beta \) is a quasi-geodesic in \( \Gamma(G, S) \) with controlled constants by Lemma 3.9. Since \( \Delta(1, f) \) is very large (approximately \( \Delta(1, g) \) or \( \Delta(1, g)/2 \)), by Lemma 3.13, \( \Delta(1, f^n) \) grows roughly linearly on \( n \) (apply the lemma at each point \( f^n \)).

In general, \( \gamma \) is only a quasi-geodesic in \( \Gamma(G, S \cup H) \) with the quasi-geodesic constants controlled. In this case, for each \( N > 0 \), take a geodesic \( \gamma' \) from 1 to \( f^N \) in \( \Gamma(G, S \cup H) \), so that \( 1, f, \ldots, f^N \) are in a bounded neigh-
borhood of $\gamma'$ in $\Gamma(G, S \cup \mathcal{H})$. This is because $\Gamma(G, S \cup \mathcal{H})$ is hyperbolic. For each $f^n$, let $y_n \in \gamma'$ be a closest point on $\gamma'$ for $d_{G'}$. Take $y_0 = 1, y_N = f^N$. By Lemma 3.12, for each $n$, $\Delta(y_n, y_{n+1}) \sim \Delta(f^n, f^{n+1}) = \Delta(1, f)$. Take a lift of $\gamma'$, denoted by $\beta$, which is a quasi-geodesic in $\Gamma(G, S)$ with controlled constants. The points $y_n$ are on $\beta$, and by Lemma 3.13, $\Delta(y_0, y_n)$ grows roughly linearly on $n$ (roughly the slope is $\Delta(1, f)$, which is much larger than any constants for $\sim$ in the above argument). Again by Lemma 3.12, $\Delta(1, f^n)$ grows roughly linearly for $0 \leq n \leq N$, with the slope roughly $\Delta(1, f)$. Since $N$ was arbitrary, $\Delta(1, f^n)$ grows roughly linearly on $0 \leq n$. This finish the argument.

3.3 More general case

In the proof of Theorem 3.1, what we need from the peripheral subgroups $H$ is the property that any two weakly asymptotic, coarsely geodesic metrics on $H$ are coarsely equal (see the discussion in the beginning of the proof). We verified this property in Corollary 3.3 for virtually abelian groups. We restate Theorem 3.1 in this more general form. The proof is identical and we omit it.

**Theorem 3.14.** Let $(G, \mathcal{H})$ be a relatively hyperbolic group. Assume that each $H_i \in \mathcal{H}$ satisfies the property such that any two weakly asymptotic, coarsely geodesic metrics on $H_i$ are coarsely equal. Suppose $G$ acts on $X$ properly and co-boundedly by isometries for geodesic metrics $d_1, d_2$ (or more generally, $d_2$ is a coarsely geodesic metric). If $d_1$ and $d_2$ are weakly asymptotically isometric on $X$, then they are coarsely equal.

One potential application would be to non-uniform lattices in the Lie group $SU(n, 1)$. It is known that such lattice is relatively hyperbolic with peripheral subgroups virtually nilpotent. As we mentioned in the introduction, the assumption in the theorem does not hold in general for all nilpotent groups.

4 Marked length spectrum

In this section we discuss an application to the marked length spectrum problem. We state a variant of Theorem 3.1.
Theorem 4.1. Let $(X_1, d_1), (X_2, d_2)$ be geodesic metric spaces on which $G$ acts by isometries, co-boundedly and properly with respect to both $d_1$ and $d_2$. Suppose the action on $X_1$ is free. We allow $X_1$ to be coarsely geodesic. Assume $|g|_1 = |g|_2$ for any hyperbolic element $g \in G$. If $G$ is toral relatively hyperbolic group, then there exists a $G$-equivariant, $(1, C)$-quasi-isometry map $f : X_1 \to X_2$.

Proof. We follow the argument for Theorem 3.1. We fix points $x_1 \in X_1$ and $x_2 \in X_2$, and define $\Delta(g, h) = d_1(g.x_1, h.x_1) - d_2(g.x_2, h.x_2)$ for $g, h \in G$. We claim that $\Delta$ is bounded. We will prove this later, but once this is known, then there exists a desired map $f$. In other words, $\Delta(p, q) = d_1(p, q) - d_2(f(p), f(q))$ is bounded. Indeed, set $f(g.x_1) = g.x_2$ for each $g \in G$ (use the $G$-action on $X_1$ is free). Then, $f$ is $G$-equivariant and $\Delta(p, q)$ is bounded for $p, q \in G.x_1$. Moreover, we can extend $f$ to $X_1$, $G$-equivariantly, such that $\Delta$ is bounded. Indeed, choose a point $p$ on each $G$-orbit in $X_1$. There is $g \in G$ such that $d_1(p, g.x_1)$ is bounded. Define $f(p) = g.x_2$, and extend $f$, $G$-equivariantly, to the $G$-orbit of $p$. It is clear that $\Delta$ is bounded on $X_1$.

We are left to show $\Delta$ is bounded on $G.x_1$. Again, we repeat the argument for Theorem 3.1. When needed, we replace $X$ with $X_2$ and map any objects in $X_1$ to $X_2$ by $f$ and argue on $X_2$. Notice that the map $f$ defined above is a quasi-isometry from $X_1$ to $X_2$. For example, in the proof of Lemma 3.11, $\gamma_1$ is a geodesic from $x_1$ to $y_1 = ah.x_1 \in aH.x_1$, and $\gamma_2$ is a geodesic from $x_2$ to $y_2 = ah.x_2 \in aH.x_2$. Then $f(\gamma_1)$ is a quasi-geodesic from $x_2$ to $y_2$. Apply the argument to $\gamma_2$ and $f(\gamma_1)$. Lemma 3.13 is similar. We omit details. \hfill $\square$

We apply the result to the marked length spectrum problem for manifolds.

Corollary 4.2. Let $(M_1, d_1), (M_2, d_2)$ be closed Riemannian manifolds with the isomorphic fundamental group $G$ that is toral relatively hyperbolic. Assume they have the same marked length spectrum. Then there is a $G$-equivariant $(1, C')$- quasi-isometry $f : \tilde{M}_1 \to \tilde{M}_2$. Moreover, if there is a homeomorphism $H : M_1 \to M_2$ that induces the isomorphism on $G$ and lifts to a $G$-equivariant homeomorphism $h : \tilde{M}_1 \to \tilde{M}_2$, then $h$ is a $(1, C')$- quasi-isometry.

Proof. Let $X_i$ be the universal cover of $M_i$, respectively. Each of them have the lift of $d_i$, which we also write by $d_i$. Each action by $G$ on $X_i$ is free. By assumption, we can apply Theorem 4.1 and we get a desired map $f$. For the moreover part, it suffices to show that there is a constant $L$ such that for any point $p \in \tilde{M}_1, d_2(f(p), h(p)) \leq L$. To see this, fix $x_1 \in X_1$ and set...
Let \( \pi : X_1 \to M_1 \) be the covering map. Given \( p \in X_1 \), join \( \pi(x_1) \) to \( \pi(p) \) by a shortest geodesic \( \gamma \) in \( M_1 \). Lift it to a geodesic \( \tilde{\gamma} \) from \( p \) to \( g.x_1 \) with \( g \in G \). Then \( h(\tilde{\gamma}) \) is a path from \( h(p) \) to \( h(g.x_1) = g.(h(x_1)) = g.x_2 \). By the argument for Theorem 4.1, \( f(p) = g.x_2 \) (to be precise, we can extend \( f \) in this way). That means that \( h(p) \) and \( f(p) \) are joined by \( h(\tilde{\gamma}) \), but the length of this path is bounded since the length of \( \tilde{\gamma} \) is bounded and \( h \) is continuous and \( G \)-equivariant.

For example, if \( M_1 \) has non-positive curvature, dimension is not 3 nor 4, and \( M_2 \) is aspherical (for example \( M_2 \) has non-positive curvature), then each isomorphism of \( G \) is induced by a homeomorphism \( H \) by Farrell-Jones (see [13]), therefore its lift \( h \) is a \((1, C)\)-quasi-isometry map.

The case where \( G \) is word-hyperbolic in Corollary 4.2 is proved by Furman [14, Theorem 2]. His argument is different from ours, and uses Patterson-Sullivan measures for hyperbolic groups with respect to word metrics constructed by Coornaert [9], and does not seem to apply to prove Theorem 4.1.

If \( C = 0 \) for some \( f \) in Theorem 4.2, \( M_1 \) and \( M_2 \) are isometric. This is the conclusion that the marked length spectrum problem/conjecture concerns (see the conjecture in [8, 3.1] for negative curvature case). Several cases are known to have the positive answer, for example for surfaces of negative curvature [21]. Interestingly, Bonahon [3] gave examples of geodesic metrics \( d_1, d_2 \) on a hyperbolic surface such that \( d_1 \) is Riemannian with constant negative curvature and that they have the same marked length spectrum, but are not isometric to each other. Theorem 4.1 applies to his example. For higher dimension there is a result by Hamenstädt (\( M_1 \) is a rank-1 locally symmetric space and \( M_2 \) is negatively curved) using a theorem of Besson-Courtois-Gallot [2] on the volume entropy. In all of those cases, the fundamental group is hyperbolic. Our result put the marked length spectrum problem (for manifolds) into context for a broader class of groups. Although this is not for manifolds, another case where the isospectral length problem is solved is \( \mathbb{R} \)-trees under the assumption that the action is minimal and semi-simple, [10].

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