Social Network Reciprocity as a Phase Transition in Evolutionary Cooperation.

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In Evolutionary Dynamics the understanding of cooperative phenomena in natural and social systems has been the subject of intense research during decades. We focus attention here on the so-called "Lattice Reciprocity" mechanisms that enhance evolutionary survival of the cooperative phenotype in the Prisoner’s Dilemma game when the population of darwinian replicators interact through a fixed network of social contacts. Exact results on a "Dipole Model" are presented, along with a mean-field analysis as well as results from extensive numerical Monte Carlo simulations. The theoretical framework used is that of standard Statistical Mechanics of macroscopic systems, but with no energy considerations. We illustrate the power of this perspective on social modeling, by consistently interpreting the onset of lattice reciprocity as a thermodynamical phase transition that, moreover, cannot be captured by a purely mean-field approach.

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I. INTRODUCTION

Is the term "social temperature" just a rhetoric figure (suggestive metaphor), or on the contrary, could it be given a precise meaning? By working out in detail the Evolutionary Dynamics of the most studied social dilemma (the Prisoner’s Dilemma) on a simple kind of artificial social networks we will show here that the formal framework of Equilibrium Statistical Mechanics is, to a large extent, applicable to the rigorous description of the asymptotic behavior of strategic evolution, thus providing the key for a formal quantitative meaning of the term social "temperature" in these contexts.

Evolutionary game theory, in contrast with classical game theory that focusses on the decision making process of (rational) agents, is concerned with entire populations of agents programmed to use some strategy in their interactions with other agents. The agents are replicators, i.e. entities which have the means of making copies of themselves (by inheritance, learning, infection, imitation, etc...), whose reproductive success depends on the payoff obtained during interaction. As the payoff depends on the current composition of strategies among the interacting agents, this yields a feedback loop that drives the evolution of the strategic state of the population [1,2,8,4].

This Darwinian feedback (frequency-dependent fitness) dynamics depends strongly not only on the particular game, and on the specifics of the way strategies spread, but also on the (social) structure of connections describing the interactions. Under the assumption of a well-mixed population (social panmixia assumption), the temporal evolution of the proportion of strategies among the population is governed by a differential equation named replicator equation (see below). Well-known celebrated folk’s theorems (see, e.g. [3]) establish a connection between the asymptotic behavior of this equation and the powerful concepts of classical game theory based on the notion of best reply (Nash). However, if the social panmixia assumption is abandoned, and individuals only interact with their neighbors in a social network, the asymptotic of evolutionary dynamics generically differ in a substantial way from this "well-mixed population" description. The social structure of strategic interactions turns out to be of importance regarding the evolutionary outcome of the strategic competition.

We will consider here the Prisoner’s Dilemma (PD), a two-players-two-strategies game, where each player chooses one of the two available strategies, cooperation or defection: A cooperator receives $R$ when playing with a cooperator, and $S$ when playing with a defector, while a defector earns $P$ when playing with a defector, and $T$ (temptation) against a cooperator. When $T > R > P > S$, the game is a PD (while if $T > R > S > P$ it is called Snowdrift game, also "Chicken" or "Hawks and Doves"). Given the payoff’s ordering, whatever the value of the prior assign of probability to the co-player’s strategy is, the expected payoff is higher for defection, and that is what a rational agent should choose. In the PD game only the defective strategy is a strict best response to itself and to cooperation, thus it is an easy example of game with an unbeatable [4] strategy. Still, though there is no difficulty in the making of the strategic decision from Nash analysis, two cooperators are better off than two defectors, hence the Social Dilemma.

In graph-structured populations, a large body of research [5,6,7,8,10,11,12] (and references therein) on evolutionary dynamics of the PD game has convincingly show the so-called lattice reciprocity effects: The cooperative phenotype can take advantage of the topology of the social net, so that clusters of cooperators are often resilient to invasion by the (continuum-unbeatble) defective phenotype. This enhancement of asymptotic macroscopic levels of cooperation due to the structure and topology of strategic interactions includes, but it is far more general than, the so-called space reciprocity mechanisms, where social nets are discretizations (solid state
and simple thermodynamical predictions are obtained within
ancies are evidenced: while mean-field prediction does not
fects of lattice reciprocity. These effects are seen to operate
 mutating population, and signals the onset of macroscopic ef-
tion IV A and IV B. A simple thermodynamical interpreta-
tions of the macroscopic behavior, is provided by the theorem
jects by picking up at random a neighbor, \( j \), and comparing their respective payoffs \( P_i \) and \( P_j \). If
\( P_i > P_j \), nothing happens and \( i \) keeps the same strategy for the next
generation. On the contrary, if \( P_j > P_i \), with probability \( \Pi_{i \rightarrow j} = \beta(P_j - P_i) \), \( i \) adopts the strategy of its neighbor \( j \) for the
round robin with its neighbors, before which all payoffs are reset to zero. Here \( \beta \) is a number small enough to make \( \Pi_{i \rightarrow j} \) an acceptable probability; its physical meaning is related to the characteristic inverse time scale: the larger it is, the faster evolution takes place.

From a theoretical point of view, this specific choice of the dynamics has the virtue of leading directly (see, e.g., (2)), under the hypothesis of a well-mixed population and very large population size, to the celebrated replicator equation for the frequencies \( p_{\alpha} \) of strategies \( \alpha (= \text{C or D}) \) in the population:

\[
p_{\alpha} = p_{\alpha}(f_{\alpha} - \bar{f})
\]

(1)

where \( f_{\alpha} \) is the payoff of an \( \alpha \)-strategist and \( \bar{f} \) is the average payoff for the whole population. Note that time unit in equation (1) is scaled to \( \beta^{-1} \).

For the payoffs of the Prisoner’s Dilemma the asymptotic frequency of cooperators, from the replicator equation, is driven to extinction, \( p_{C} = 0 \), while for the Hawks and Doves game, its asymptotic value is \( \epsilon/(b - 1 + \epsilon) \). As stated in the introductory section, we will be concerned here mainly with

II. NATURAL STRATEGIC SELECTION ON GRAPHS.

We specify here the evolutionary game dynamics scenario, meaning the game parametrization, the microscopic strategic dynamics (replication mechanism or strategic updating rule), and the social structure of contacts that we will consider along the paper.

We normalize the PD payoffs to the reward for cooperating, \( R = 1 \), and fix the null payoff at punishment \( P = 0 \). Note that provided the (differential or relative) selective advantage among two individuals depends on their payoff’s difference (see below), one can arbitrarily fix the zero payoff level. Then only two parameters \( T = b > 1 \) and \( R = \epsilon < 0 \) are tuned. Note that the range \( \epsilon > 0 \) defines a game named Hawks and Doves (also Chicken and Snowdrift) when punishment and sucker’s payoff have the reverse order. We will occasionally comment on this range of parameters.

Moreover, we do not restrict our computations to \( 2R > T + S \). This restriction means that the total payoff for the two players is higher if both cooperate \((2R)\) than if one cooperates and the other defects \((T + S)\), and is usually incorporated in iterated games studies of the PD to prevent agents taking turns at defection and then sharing the payoffs. For the specifics of the replicator dynamics (memory-less, markovian) in the next paragraph, one should not expect that this restriction qualitatively matters.

Regarding the replication mechanism, we implement the finite population \((N \gg 1)\) analogue of replicator dynamics \([17, 21]\). At each time step \( t \), which represents one generation of the discrete evolutionary time, each agent \( i \) plays once with each one of the agents in its neighborhood and accumulates the obtained payoffs, \( PI \). Then, the individuals, \( i \), update synchronously their strategies by picking up at random a neighbor, \( j \), and comparing their respective payoffs \( P_i \) and \( P_j \). If \( P_i > P_j \), nothing happens and \( i \) keeps the same strategy for the next generation. On the contrary, if \( P_j > P_i \), with probability \( \Pi_{i \rightarrow j} = \beta(P_j - P_i) \), \( i \) adopts the strategy of its neighbor \( j \) for the next round robin with its neighbors, before which all payoffs are reset to zero. Here \( \beta \) is a number small enough to make \( \Pi_{i \rightarrow j} \) an acceptable probability; its physical meaning is related to the characteristic inverse time scale: the larger it is, the faster evolution takes place.

The concluding section \([V]\) tries to call interdisciplinary attention on the wide and utmost interesting prospectives for Statistical Physics “concepts and methods” in current studies on Evolutionary Dynamics and Social Systems modelling in general.
populations that are not well-mixed, where predictions based on this nonlinear differential equation are often of little use.

Regarding the structure of connections between interacting agents, we will consider here that it is given by a fixed graph (i.e., connections between players do not change by rewiring) where agents are represented by nodes, and a link between nodes indicates that they interact (play). If \( k_i \) is the number of neighbors of agent \( i \) (connectivity or degree), and \( \Delta \) is the maximal possible one-shot-payoff difference (\( \Delta = \max\{b, b−\epsilon\} \)), we will assume \( \beta = (\max\{k_i, k_j\} \Delta)^{-1} \) for the specification of the probability \( \Pi_{i→j} \) of invasion of node \( i \) by the strategy of neighbor \( j \). This simple choice, introduced in [17], assures that \( \Pi_{i→j} < 1 \); in heterogeneous networks it has also the effect of slowing down the invasion processes from or to highly connected nodes, with respect to the rate of invasion processes between poorly connected nodes, a feature not without consequences. [22].

We now introduce some notation, which is familiar to statistical physicists: The configuration (strategic microstate) \( l \) of a population of \( N \) agents at time \( t \) is specified by the sequence \( l = \{s_i(t)\} (i = 1, ..., N) \), where \( s_i(t) = 1 \) (or 0) denotes that node \( i \) is at this time a cooperator (resp. defector). The set of all possible \( 2^N \) configurations is called the phase space. Stationary probability densities of microstates \( \mathcal{P}(l) \) (\( l = 1, ..., 2^N \)) are then representatives of strategic macro-states. The average cooperation \( c_t \) of microstate \( l \) is defined as

\[
c_t = \frac{1}{N} \sum_{i} s_i
\]

We denote by \( \Pi_{ll'} \) the probability that the strategic microstate of the population at time \( t + 1 \) is \( l' \), provided that it is \( l \) at time \( t \). Note that \( \sum_{l'} \Pi_{ll'} = 1 \). A microstate \( l \) is a frozen equilibrium configuration if the probability that it changes in one time step is null, and then \( \Pi_{ll} = 1 \) and \( \Pi_{ll'} = 0 \) if \( l' \neq l \). We will assume generic real values (irrational) of the payoff parameters, so that if a configuration contains a C-D link it cannot be a frozen configuration. The only possible frozen equilibrium configurations are all-C and all-D. However, for a very wide class of graphs, and a wide range of model parameters they are not the only possible stationary probability measures.

We now illustrate by means of easy examples the evolution of PD on graphs. Our first and simplest example is a star-shaped graph consisting of a central node connected to \( N − 1 \) peripheral nodes. It is straightforward to check that any initial condition with cooperators at the central node and (at least) at \( \text{Int}((b−\epsilon(N−1))/((1−\epsilon)) + 1 \) peripheral nodes has a positive probability of evolving in one time step to a configuration with a higher number of cooperators, and a null probability of evolving towards less cooperators. Thus, all those configurations evolve asymptotically to the all-C equilibrium. The rest of configurations evolve towards the all-D equilibrium. Therefore, if \( N > (b−\epsilon +2) \) both equilibria are attractors (absorbing states), in the sense that some configurations different from themselves evolve to them; the phase space is partitioned into two basins of attraction. If \( N < b−\epsilon +2 \), only the all-D frozen equilibrium is attractor. The stationary probability densities \( \mathcal{P}^*(l) \) of the star are pure point measures (two- or one-Dirac delta peaks) in the thermodynamical limit \( N → \infty \).

Now take a star and add some arbitrary number of links between its peripheral nodes. We call this network a crown, whose head is the central node. If the head is occupied at \( t_0 \) by a defector, it will remain so forever, because the payoff of a peripheral cooperator is strictly lower than head’s payoff. Sooner or later the head (center) of the crown will be imitated by the whole crown, and the evolution will stop when everybody is defecting. But, what happens to a cooperator on the head? The answer is dependent on both, the net topology of the crown periphery and the cooperators disposition there: To ensure fixation of cooperation at the head node, it suffices that a subset \( C \) of peripheral nodes occupied by cooperators, and with no direct links to the rest of the periphery, have a size \( n_C > b k_{max} − \epsilon(N − n_C − 1) \), where \( k_{max} \) is the maximal degree in the rest of the periphery. Under this proviso all-C is the unique absorbing microstate of all corresponding initial conditions.

Finally consider the graph schematized in Fig. I composed of the following:

(a) A component \( \mathcal{F} \) of \( n_F \) nodes with arbitrary connections among them.

(b) A node, say node 1, that is connected to all the nodes in \( \mathcal{F} \) and has no other links.

(c) A component \( \mathcal{C} \) of \( n_C \) nodes with arbitrary connections among them.

(d) A node, say node 2, that is connected to all the nodes in \( \mathcal{F} \) and \( \mathcal{C} \), but not to node 1.

This is what we will call a Dipole Model Network. It is a two-headed (nodes 1 and 2) crown (with periphery \( \mathcal{F} \)) plus a tail \( \mathcal{C} \) hanging on head 2. To strength the special status of the head nodes, let us nickname them as “Big Brothers”. They certainly enjoy a sort of omnipresence that fits well with the character of Orwell’s famous social sci-fiction novel 1984. In the following section we prove that for this simple network there exists a non-trivial stationary probability density of microstates \( \mathcal{P}^*(l) \) for the strategic evolution of the PD game.

### III. THE DIPOLE MODEL

The analysis of Evolutionary Dynamics of the PD on the Dipole network shows that there is a non-trivial invariant measure in phase space. Let us consider the set \( \mathcal{I} \) of initial conditions defined by: (i) Big Brother 1 is a defector, (ii) Big Brother 2 is a cooperator, and (iii) all nodes in component \( \mathcal{C} \) are cooperators. Note that this set contains \( 2^{n_F} \) different configurations. We now prove that, provided some sufficient conditions, this is a minimally invariant set of the evolutionary dynamics.

First, one realizes that Big Brother 1 cannot be invaded by the cooperative strategy: The payoff of a cooperator node \( i \) in \( \mathcal{F} \) is \( P^c_i = k_i^c + 1 + \epsilon(k_i - k_i^c + 1) \), where \( k_i \) is the number of its neighbors in \( \mathcal{F} \) and \( k_i^c \leq k_i \) is the number of those
that are cooperators. The payoff of Big Brother 1 (BB1) is then \( P_1 \geq (k^c_i + 1)b \). For the PD game, where \( \epsilon \leq 0 \), the inequality \( P_1 > P^c_F \) always holds, so that BB 1 will always be a defector. (Note also that for the Hawks and Doves game, a sufficient condition for \( P_1 > P^c_F \) is \( b > 1 + \epsilon (k_F + 1) \), where \( k_F < n_F \) is the maximal degree in component \( F \), i.e. the maximal number of links that a node in \( F \) shares within \( F \).) We thus conclude that defection is fixed at BB 1.

Second, thanks to its interaction with set \( C \), Big Brother 2 resists invasion, provided its size \( n_C \) is above a threshold: The payoff of a defector node \( i \) in \( F \) is \( P^d_F = (k^c_i + 1)b \), where \( k^c_i \) is the number of its cooperator neighbors in \( F \), while the payoff of Big Brother 2 (BB2) is \( P_2 = n_C + n_F \epsilon + n_F (1 - \epsilon) \), where \( n_F \leq n_F \) is the number of cooperators in \( F \). Thus, a sufficient condition for \( P_2 > P^d_F \) is \( n_C > \text{Int}(b(k_F + 1) - n_F \epsilon) \). With this proviso, BB2 will always be a cooperator, which in turn implies that all the nodes in the component \( C \) will remain always cooperators.

The previous argument proves that provided the sufficient conditions \( n_C > \text{Int}(b(k_F + 1) - \epsilon n_F) \) and \( b > 1 + \epsilon (k_F + 1) \) hold, the subset \( I \) of phase space defined by (i), (ii), and (iii) is an invariant set. As this set does not contain equilibria, no stochastic trajectory evolves from it to a frozen equilibrium configuration.

Finally, one realizes that \( I \) is indeed minimal, because at any time, a defector in \( F \) has a positive probability to be invaded by the cooperation strategy (from BB2), and a cooperator in \( F \) has a positive probability of being invaded by the defection strategy (from BB1). Therefore, any strategic configuration of the set \( I \) is reachable in one time step from any other, i.e. for all pairs \( (l, l') \) of microstates in \( I \), the transition probability \( \Pi_{l \rightarrow l'} \) > 0. Consequently, \( I \) does not contain proper invariant subsets: it is minimally invariant. Moreover, following Perron-Frobenius theorem, there exists a unique equilibrium macro-state \( \mathcal{P}^*(l) \). This provides a rigorous framework for the interpretation of results from numerical Monte Carlo simulation studies in Evolutionary Dynamics on Dipole models, provided the sufficient conditions above.

While nodes in \( C \) and Big Brother 2 are permanent cooperators, and Big Brother 1 is a permanent defector, nodes in \( F \) are forced to fluctuate. This partition of the network into sets of nodes where each particular strategy is fixed forever, and a set of fluctuating nodes, turns out to be a generic feature of the discrete replicator dynamics (neighbor imitation proportional to payoffs difference) on many network settings [18, 19]. The simplicity of the Dipole Network model allows on it an easy formal proof of existence of this partition, so providing an illustration of both, its origins and generic character. It also shows the formal applicability of Equilibrium Statistical Physics formalism to characterize the asymptotic behavior of Evolutionary Dynamics on these graphs. This will be made in the next section for specific choices of structural traits for the subgraph \( F \).

The name dipole for this structure of connections is suggested by the strategic polar \( (C - F - D) \) aspect of the whole graph. Note also that the number of \( C \) and \( F \) and \( F - D \) connections scales linearly with the size \( n_F \) of the fluctuating interior, that is to say that the poles (C and D) act as an externally imposed (AC) field on \( F \), whose strength is proportional to the internal levels of cooperation. As the cooperation (and then the fitness) levels are self-sustained (as proved by the previous theorem), this is a closed macroscopic system with a non-trivial self-sustained social activity of cooperation at evolutionary equilibrium.

The interest of the Dipole Model is by no means restricted to a mere academic illustration: First of all, we can make a technical use of it in macroscopic stability analysis studies of PD-evolution on highly heterogeneous complex networks. Indeed, the fluctuations inside the subset \( F \) are the effect of the competition for invasion among two non-neighboring hubs (hugely connected nodes), where opposite pure strategies have reached fixation, in their common neighborhood. This is a local strategic configuration that mimics those that are often observed in stochastic simulations of evolutionary dynamics in highly heterogeneous (scale-free) networks [18, 19]. Simple multipolar network models can easily be constructed (e.g. by establishing direct links from \( C \) to \( F \) in a way that simple sufficient conditions guarantee that the theorem still holds), that are indeed indistinguishable from typical strategic patterns found in the numerical simulations on scale-free networks. This makes the Dipole net a very useful technical device to analyze the stability mechanisms of the cooperator clusters [18, 19] in scale-free structured populations, as well as the kind of temporal fluctuations of cooperation that one should expect in the fluctuating set of nodes.

Regarding potentialities for Econo-Socio-Physics applications of the Dipole model, it could be viewed as a sort of schematic (then simplistic, cartoon-like) model for the competition for influence of two powerful superstructural institutions (e.g. like “mass media”, political parties, or lobbies) on a target population, in strongly polarized strategic contexts. The analysis rigorously provides sufficient conditions for the parameter values where fixation of strategic traits is proved impossible, so that temporal fluctuations dominate forever the target population of social imitators \( F \). The influence on each individual of the two competing institutions is simulated here through the omnipresent ("Big Brother" nodes 1 and 2) neigh-
bors, whose own high appeal for imitation (the strength of Big Brother’s influence) is in turn conditioned by the strategic composition of the target population. Here the interest could well be the study of the influence that metric and topological network characteristics of the social structure have on the strategic macro-state, and thus on the quantitative values of social indicators. We address some aspects of this issue in the next section.

At a more general level, the design of experiments in Social Sciences as well as theoretical studies of Artificial Societies could greatly benefit from having at hand simple but non-trivial “exactly soluble statistical-mechanical models” that may provide safe guides to develop further intuitions on social phenomena that demands more comprehension.

IV. THE ROLE OF SOCIAL STRUCTURE IN BIG BROTHERS COMPETITION.

In this section we present some analytical and numerical results on the evolutionary dynamics of games in the Dipole Model for different choices of topologies of the fluctuating set \( \mathcal{F} \). The sufficient conditions stated in the previous section are assumed hereafter. We are interested in the situation where \( n_F \gg 1 \), i.e. large size of the fluctuating population. First we will analyze in \( \text{[IVA]} \) the straightforward limiting case when the macroscopic set \( \mathcal{F} \) is a fully connected set. This is the well-mixed population limit, where replicator equation is an exact description. Next in subsection \( \text{[IVB]} \) we will explicitly solve the opposite trivial case of disconnected \( \mathcal{F} \) set (\( k_F = 0 \)), which turns out to reduce to the standard textbook ideal two-states model of Statistical Physics. After that, in subsection \( \text{[IVC]} \) the “random regular” network structure for \( \mathcal{F} \) is seen to be amenable to a plausible mean-field approach, but insufficient to explain the phenomenology shown by Monte Carlo numerical results. These show beyond any doubt a critical behavior, a transition point separating two qualitatively different types of social macro-states. This transition is sensibly interpreted as the onset of lattice reciprocity. In other words, lattice reciprocity is a true critical social phenomenon.

A. Well-mixed population limit.

We now show that the well-mixed population assumption in the macroscopic set \( \mathcal{F} \) leads in the thermodynamic limit to the standard replicator equation: Assume that each node in \( \mathcal{F} \) is connected to all nodes in \( \mathcal{F} \), and to BB1 and BB2. The degree of each of them is thus \( n_F + 1 \), and the parameter \( k_F = n_F - 1 \).

The sufficient conditions for fixation of defection at node 1 and of cooperation at node 2 are respectively, \( b > 1 + cn_F \) (that always hold for the PD), and \( n_C > \text{Int}(n_F(b - \epsilon)) \). The payoffs of polar nodes 1 and 2 are given by

\[
P_1 = bcn_F \quad , \quad P_2 = n_C + cn_F + \epsilon (1 - c) n_F \quad , \quad (3)
\]

while the payoffs of a cooperator node and a defector node in \( \mathcal{F} \) are

\[
P_c = cn_F + \epsilon (n_F - cn_F + 1) \quad , \quad P_d = (cn_F + 1) b \quad . \quad (4)
\]

To compute the invasion probabilities, \( Q_{DC} \) and \( Q_{CD} \), one first easily realizes that \( P_c < P_d \), provided the sufficient condition \( (b > 1 + cn_F) \) for fixation of defection at node 1. Thus the (one time step) probability \( Q_{DC} \) of invasion of a cooperator node in \( \mathcal{F} \) is

\[
Q_{DC} = \frac{1}{(n_F + 1) \Delta (n_F + 1)} P_1 - P_c + \frac{(1 - c)n_F}{(n_F + 1) \Delta (n_F + 1)} P_d - P_c , \quad (5)
\]

and the probability \( Q_{CD} \) of invasion of a defector node in \( \mathcal{F} \) is

\[
Q_{CD} = \frac{1}{(n_F + 1) \Delta (n_F + n_C)} . \quad (6)
\]

At time \( t + 1 \), the expected fraction of cooperators is:

\[
c(t + 1) = c(t)(1 - Q_{DC} + (1 - c(t))Q_{CD} . \quad (7)
\]

Assuming that the size of \( \mathcal{F} \) is macroscopic, \( n_F \gg 1 \), the fraction of cooperators \( c \) in \( \mathcal{F} \) evolves according to the differential equation

\[
\frac{dc}{dt} = (1 - c)Q_{CD} - cQ_{DC} . \quad (8)
\]

Now, if \( n_F \gg 1 \), and \( n_C/(n_F)^2 \rightarrow 0 \), then both \( Q_{CD} \) and the first term in the right-hand side of \( (8) \) vanish, and we arrive to the differential equation

\[
\frac{dc}{dt} = \frac{c(1 - c)}{\Delta}(c(1 - c) - (b - 1)c) . \quad (9)
\]

As expected, with a simple re-scaling of time, equation \( (9) \) is no other than the replicator equation \( (1) \): note that in the limit \( n_F \gg 1 \) that we have considered, the probability that a node in \( \mathcal{F} \) picks up a Big Brother when updating its strategy is negligible, and then the evolution inside the complete graph \( \mathcal{F} \) is overwhelmingly determined by the internal connections, and thus by the replicator equation. In other words, in this limit of maximal possible connectivity, BB1 and BB2 are no longer bigger than the nodes in \( \mathcal{F} \) and their influence on the fluctuating set is negligibly small in the thermodynamical limit.

Note however that as far as \( n_F \) is finite, the Theorem in section \( \text{[III]} \) still hold, and a chance for the fluctuating activity inside \( \mathcal{F} \) remains. We now turn attention to situations where \( k_F \ll n_F \), far from the social panmixia.
B. $\mathcal{F}$ is a disconnected graph (ideal-gas)

Let us now obtain some explicit results for one of the simplest choices for the topology of connections inside the fluctuating set, namely $k_F=0$. In this case each node in $\mathcal{F}$ is only connected to Big Brothers. This is in fact an effective single node problem, where homogeneity assumption in $\mathcal{F}$ is exact; in other words, the absence of internal interactions in the set $\mathcal{F}$ is a sort of ideal-gas condition easy to deal with in the large size limit.

1. A differential equation for $c$

Note that the sufficient conditions for fixation of defection at BB1 and of cooperation at BB2 are respectively, $b > 1 + \epsilon$, and $n_C > b - c n_F$. Denoting by $c(t)$ the instantaneous fraction of cooperators in $\mathcal{F}$, the payoffs of Big Brothers are given by (3), and the payoffs of a cooperator node and a defector node in $\mathcal{F}$ are, respectively,

$$P_c = 1 + \epsilon \quad , \quad P_d = b \ . \quad (10)$$

Then one finds for the (one time step) probability $Q_{DC}$ of invasion of a cooperator node in $\mathcal{F}$

$$Q_{DC} = \frac{c b - (1 + \epsilon)/n_F}{2 \Delta} \ , \quad (11)$$

and using the notation $A = \epsilon + (n_C - b)/n_F$ and $B = 1 + n_C/n_F$

$$Q_{CD} = \frac{A + c(1 - \epsilon)}{2 \Delta B} \ , \quad (12)$$

for the probability of invasion of a defector node in $\mathcal{F}$. Note that $A > 0$ due to the non-invasion of BB2 (sufficient) condition.

Provided $n_F \gg 1$, the fraction of cooperators $c$ in $\mathcal{F}$ evolves according to the differential equation (8), which after insertion of expressions (11) and (12), and re-scaling of time, becomes

$$\dot{c} = f(c) \equiv A_0 + A_1 c + A_2 c^2 \ , \quad (13)$$

where the coefficients are

$$A_0 = A \ , \quad (14)$$

$$A_1 = 1 - \epsilon - A + B(1 + \epsilon)/n_F \ , \quad (15)$$

$$A_2 = -(1 - \epsilon + bB) \ , \quad (16)$$

One can easily check ($A_0 > 0$ and $A_2 < 0$) that there is always one positive root $c^*$ of $f(c)$, which is the asymptotic value for any initial condition $0 \leq c(0) \leq 1$ of equation (13). In this asymptotic regime, the one step invasion probabilities, $Q_{DC}$ and $Q_{CD}$, become time independent and one can then compute the probability that the cooperation strategy remains for a time $\tau_c \geq 1$ (permanence time of cooperation) at a fluctuating node, simply as $P(\tau_c) = Q_{DC}(1 - Q_{DC})^{\tau_c - 1}$. In a similar way, the distribution density $P(\tau_d)$ of defection permanence times is obtained as $P(\tau_d) = Q_{CD}(1 - Q_{CD})^{\tau_d - 1}$. Thus the distribution densities of both strategies permanence time are exponentially decreasing.

For $\epsilon = 0$, in the so called weak PD game (*i.e.* at the border between the PD and the Hawks and Doves game), if one further assumes that the relative size $\mu(F)$ of the component F is large enough, *i.e.* $\mu(F) \to 1$, and $\mu(C) \to 0$, one easily obtains that the stationary solution of equation (13) behaves as $c^* \simeq (b + 1)^{-1}$ near the limit $\mu(F) \to 1$. The distribution density $P(\tau_c)$ of the cooperation permanence times of a fluctuating node, as a function of the parameter $b$ is thus

$$P(\tau_c) = (2b + 1)^{-1} \left( \frac{2b + 1}{2b + 2} \right)^{\tau_c} \ , \quad (17)$$

and the distribution density $P(\tau_d)$ of defection permanence times

$$P(\tau_d) = (2b(2b + 1) - 1)^{-1} \left( \frac{2b(b + 1)}{2b(b + 1)} \right)^{\tau_d} \ . \quad (18)$$

These distribution densities characterize the pace of invasion cycles at a fluctuating node in the (asymptotic) equilibrium strategic macroscopic state.

2. A formal thermodynamical approach

From the point of view of the set $\mathcal{F}$, when $n_F \gg 1$, the model corresponds to a non-interacting (ideal) set of independent phenotypic strategists that fluctuate due to a polar field (Big Brothers influence) whose strength is self-consistently determined by the average cooperation $c$. This problem is equivalent to that of an ideal paramagnetic salt in a noisy (telegraphic) magnetic AC field of intensity proportional to the average magnetization.

A typical and correct statistical-physicists approach “from scratch” to this two-states model is the familiar micro-canonical setting: At (dynamical) macroscopic equilibrium, the probability of each strategic micro-state $l \in \{s_i\}$ of fixed value of $c_l = c$ is uniform

$$P_l = \Omega^{-1} \ , \quad (19)$$

where $\Omega = n_F!/(\{(cn_F)!(n_F - cn_F)\})$ is their number. The lack of information $S = \ln \Omega$ of the macro-state as a function of global cooperation $n_F c$, *i.e.* the relation $S(n_F c)$, can be regarded as the micro-canonical fundamental “thermodynamical” relation, and its first derivative is the intensive parameter $\beta$, that after using Stirling’s approximation is easily obtained as

$$\beta = \ln \left( \frac{1 - c}{c} \right) \ . \quad (20)$$
This is an equation of state, which simply expresses the connection of the equilibrium value of the macroscopic cooperation level \( c \) to the "entropic" intensive parameter \( \beta \). Note that \( c \) is determined by the balance condition \( \dot{c} = 0 \):

\[
\frac{1 - c}{c} = \frac{Q_{DC}}{Q_{CD}},
\]

from where equation of state \((20)\) determines \( \beta \) as a function of model parameters \( i.e. b, c, \) and \( n_C/n_F \). For example, when \( \epsilon = 0, \beta = \ln b > 0 \), indicating that the disorder of the activity increases with increasing cooperation. The maximal value of \( \beta \to \infty \) corresponds to zero disorder \( (b \to \infty) \), while its minimal zero value corresponds to highest possible value \((at b = 1)\) of cooperation \((c = (1/2))\). Note that lower values of \( b \) \((b < 1)\) correspond to negative \( \beta \) values, where entropy decreases with increasing values of cooperation, outside the PD domain (Stag Hunt game domain, see \([23]\)).

An alternative (and equivalent in the thermodynamic limit) setting is to consider the whole space of \( 2^n_F \) configurations \( l = \{s_i\}_{i=1}^{n_F}\), of unrestricted \( c_l \), but under the condition that the average value \( c = \sum_d P(d) c_l \) is fixed. That is the familiar canonical setting. The normalization factor \( Z = \sum_l \exp(-\beta c_l) \) is the canonical partition function (Boltzmann’s Zustandsumme), that due to independence \((k = 0)\) is easily factorized as \((1 + \exp(-\beta))^{n_F}\).

In the canonical setting a most informative macroscopic quantity is the "heat capacity" analog: The fluctuations of \( c_l \) along representative (typical) stochastic trajectories at equilibrium under the evolutionary dynamics of the game are, following the standard thermodynamical formalism given by \( \partial c / \partial (\beta^{-1}) \), so that this quantitative social indicator detects very precisely sudden variations of macroscopic cooperation with payoff’s parameters. In this ideal-gas kind of case there are no critical points and fluctuations do not diverge. For example, for \( \epsilon = 0 \) they are given by the (Bernouillian) binomial variance \( n_Fc(1 - c) = n_Fb/(b + 1)^2 \).

**C. \( \mathcal{F} \) is a random regular graph**

Random regular networks are random networks of fixed degree \( k \). All nodes being thus equivalent, a sensible approach is to assume (mean-field like) that the fraction of instantaneous cooperators in the neighborhood of a node is the fraction \( c \) of the whole set \( \mathcal{F} \). In other words, one neglects local fluctuations of \( c \). The contribution of the internal interactions to the variation of \( c \) is then of the "replicator equation" type, as discussed above for the complete graph case. The difference here is that if \( k_F \ll n_F \) the contribution of the interactions with Big Brothers cannot be longer neglected.

**1. Mean-field approximation**

The payoffs of Big Brothers BB1 and BB2 are given by equation \((3)\), and the payoffs of a cooperator node and a defection node at \( \mathcal{F} \) under the mean-field assumption are:

\[
P_e = ck + 1 + \epsilon (k(1 - c) + 1) , \quad P_d = (ck + 1)b .
\]

The differential equation for \( c \) is then

\[
\dot{c} = \frac{(1 - c)(P_e - P_d)}{(k + 2)n_F}\frac{c(1 - P_e)}{(k + 2)n_F} + \frac{(1 - c)ck(P_e - P_d)}{(k + 2)^2}\frac{c(1 - P_e)}{(k + 2)n_F},
\]

which under the assumption \( kb \ll n_F \), takes the form

\[
\dot{c} = f(c) = \frac{1}{(k + 2)^2B}\left(A_0' + A_1' c + A_2' c^2 + A_3' c^3\right),
\]

where the coefficients are

\[
\begin{align*}
A_0' &= (k + 2)(B - 1 + \epsilon) , \\
A_1' &= 2(2(1 - \epsilon) - B) + k(2(1 - \epsilon) - B(b - \epsilon)) + k^2Bc , \\
A_2' &= 2(\epsilon - 1 - Bb) + k(\epsilon - 1 - B(1 + \epsilon)) + k^2B(1 - b - 2\epsilon) , \\
A_3' &= k^2B(b - 1 + \epsilon) ,
\end{align*}
\]

Note that the assumption \( n_C > b - n_Fc \) (i.e. the condition for Big Brother 2 to be a permanent cooperator) implies that \( A_0' > 0 \), so that \( \dot{c}(0) > 0 \) and one positive root, say \( c^* \), of \( f(c) \) is then ensured, in agreement with the theorem of section \( \[\text{III}\] \).

**2. A social phase transition, and the mean-field failure.**

We now adopt the statistical mechanics formal perspective, and proceed to explicitly compute the equilibrium macrostate, i.e. the stationary probability distribution density \( P_e^* \), inside the mean-field approximation.

![FIG. 2: Macroscopic cooperation in a random regular graph structure for the set \( \mathcal{F} \), with \( k = 4 \), and \( n_F = 4000 \), and \( \epsilon = 0 \). A decreasing sequence of \( n_C/n_F \), as indicated in figure, has been used. Symbols represent numerical Monte Carlo results, and the different lines represent the mean-field predictions as given by the solution \((\dot{c} = 0)\) of Eq. \((24)\).]
Let us consider two different (arbitrary) strategic microstates \( l = \{s_i\} (i = 1,...,n_F) \), and \( l' = \{s'_i\} \), of the fluctuating set. For any pair of microstates \((l,l')\) we define the following numbers:

\[
\begin{align*}
n_{11} &= \sum_i \delta_{s_i,s'_i}\delta_{s'_i,1} , \\
n_{10} &= \sum_i (1 - \delta_{s_i,s'_i})\delta_{s'_i,0} , \\
n_{00} &= \sum_i \delta_{s_i,s'_i}\delta_{s'_i,0} , \\
n_{01} &= \sum_i (1 - \delta_{s_i,s'_i})\delta_{s'_i,1} ,
\end{align*}
\]

i.e., \( n_{11} \) is the number of nodes that are cooperators in both microstates, \( n_{10} \) that of the nodes that are cooperators in \( l \) but defectors in \( l' \), etc... Using equation (2) it is straightforward to obtain

\[
e_l - e_{l'} = \frac{1}{n_F} (n_{10} - n_{01}) .
\]

Now, let us assume that the probabilities that a node \( i \) changes strategy are independent of node \( i \) (homogeneity assumption, mean-field), and denote them by \( Q_{CD} \) (transition from defector to cooperator) and \( Q_{DC} \) (for the transition from cooperator to defector). Then we can easily see that the transition probabilities between the microstates \( l \) and \( l' \) are given by

\[
\begin{align*}
\Pi_{l,l'} &= (1 - Q_{DC})^{n_{11}}(1 - Q_{CD})^{n_{00}}Q_{DC}^{n_{10}}Q_{CD}^{n_{01}}, \\
\Pi_{l,l'} &= (1 - Q_{DC})^{n_{11}}(1 - Q_{CD})^{n_{00}}Q_{DC}^{n_{10}}Q_{CD}^{n_{01}}.
\end{align*}
\]

Henceforth, denoting \( \exp(-\beta) = Q_{CD}/Q_{DC} \), one easily obtains the expression:

\[
\Pi_{l,l'} \exp(-\beta c_l n_F) = \Pi_{l,l'} \exp(-\beta c_l n_F) ,
\]

from where the unique solution to the fixed point equation

\[
\Pi_{l,l'} P_{l'}^* = P_l^* ,
\]

is easily found to be:

\[
P_l^* = Z^{-1} \exp(-\beta c_l n_F) ,
\]

where \( Z \) is the canonical partition function

\[
Z = \left[ \frac{Q_{CD} + Q_{DC}}{Q_{DC}} \right]^{n_F} .
\]

As it is well-known \[24], the canonical probability distribution density \[38\] is the unique density that maximizes the lack of information (entropy), \( S = -\sum_i P_l \ln P_l \), among those (compatible) densities that share a common value for the macroscopic average of cooperation \( c = \sum_i P_l c_l \). This provides a standard thermodynamic meaning to the parameter \( \beta \): it is no other than the intensive entropic parameter associated to cooperation, that is, the Lagrange multiplier \[25, 26\] associated to the restriction \( c = \sum_l P_l c_l \), namely \( \beta = \frac{\partial S}{n_F \partial c} \).

The parameter \( \beta \) simply measures how fast the entropy of the equilibrium macro-state increases versus global cooperation variations. Its formal role is that of inverse thermodynamical temperature. The fluctuations of the micro-states cooperation \( c_l \), namely \( \langle n_F \rangle^2 [\sum_l (P_l c_l^2) - (\sum_l P_l c_l)^2] \) are given by \( n_F c_l(1 - c) \). The dependence on the game and network parameters \( b, c, n_C/n_F, k \) of the fluctuations of cooperation is obtained by solving for the cooperation equilibrium value \( c = 0 \) in \[24\], and plotted in figure for \( k = 4, \epsilon = 0 \), and decreasing values of the ratio \( n_C/n_F \).
To which extent the mean-field prediction fails for low values of the parameter $n_C/n_F$, can be appreciated by confronting the prediction above with the results from Monte Carlo simulations. There in figure we see how a peak in cooperation fluctuations is revealed, when $n_C/n_F \to 0$, signaling the occurrence of a phase transition between two qualitatively different macroscopic behaviors, that correspond to low and high temptation regimes. The mean-field assumption is thus only valid if the payoff received from $C$ by Big Brother 2 is not negligible versus the size $n_F$.

The reasons for the failure of the mean-field approximation rely on the lattice reciprocity of internal interactions, which is totally absent in the mean field approximation. Let us remind here our remark above on the replicator-equation-type of effect of internal interactions in equation (13) because of the mean-field assumption. The transition signaled by the divergence of fluctuations at $b^*$ reveals the onset of internal lattice reciprocity, a conclusion that we now substantiate (see also appendix 1 below).

For $b > b^*$, say in the low-temperature (high temptation) phase, the macro-state is dominated by fast defection invasions on the relatively few nodes that are instantaneous cooperators due to sporadic interactions with Big Brother 2. In the appendix 1 we show that, in the low $c$ and low $n_C/n_F$ regime, the BB-imitation events in a given node are typically separated by intervals of time of about $c^{-1}$ time units large. In those large intervals when Big Brother 2’s influence is null, the very few and mostly isolated instantaneous cooperators are quickly invaded by defector internal neighbors. In this regime lattice reciprocity has no chance to develop, and cooperation is only weakly sustained by the sporadic influence of BB2.

On the contrary, for $b < b^*$ (high temperature, or low temptation phase) the local fluctuations of the neighbors strategic field favor the building up of clusters of cooperators that resist invasions during time intervals that are comparable to the characteristic time intervals between BB-imitation events. Under these circumstances the "extra payoff" that BB2 receives from $C$ does not anymore need to be high in order to sustain high levels of cooperation. Internal lattice reciprocity enhances the probability of highly cooperative micro-states, so that the macro-states below transition differs substantially from those of the high-temperature phase. This was not captured by the mean-field approximation, for these effects require a sizable likelihood of occurrence for the local fluctuations of the strategic field, and the neglect of them is all a mean-field approach is based upon.

To summarize the discussion of the results shown in figure, a random regular structure of interactions inside $\mathcal{F}$ is enough to support lattice reciprocity mechanisms that cannot be captured by a simple mean-field approach. The onset of lattice reciprocity is furthermore interpreted as a "thermodynamical" phase transition, in a rigorous formal sense (divergence of the fluctuations of an extensive parameter, the cooperation $c$). One is then lead to a sensible and precise formal framework where such a term as "social temperature" is not a vague metaphor, but it denotes a truly quantitative parameter, a legitimate (measurable, observable) social indicator.

V. PROSPECTIVE REMARKS

The plausibility of a thermodynamical perspective on Evolutionary game dynamics studies is not a new issue, for it is somehow implicit (or at least connatural) to a body of research literature on statistical mechanics of strategic interactions [11,27]. What our simple analysis here shows is that it can sometimes be strengthened up to a formal interpretation of quantitative macroscopic social indicators as thermodynamic quantities. In the extent that it helps to understand and to quantitatively characterize the phenomenology of social and economical models, it should be recognized as a powerful theoretical perspective. What is even more important, this perspective emphasizes the central role of quantitative (experimental, observational) studies in social sciences, and could provide, in those contexts, alternate valuable meanings to quantitative social indicators and even suggestions for new and better ones.

Any "general-physics" trained scientist recognizes that entropy reasoning is an extraordinary powerful tool for the analysis of macroscopic behavior in (material) traditional-physics systems. It turns out that some of the models (at least a bunch of interesting ones) of social phenomena are to a large extent amenable to a macroscopic description where thermodynamical concepts have proved to be essential. Of course, some notions like e.g. "First Law of Thermodynamics" could be often absent in these new contexts. However we emphasize that the absence of energy as a variable in social models is not a shortcoming for the applicability mutatis mutandis of many aspects of the thermodynamical formalism to these models. A word of caution is nevertheless worth here regarding typical system sizes in controlled social experiments, where finite size effects could be hugely determinant.

Nowadays, it is somewhat generally accepted that Physics in general, and Statistical Physics in particular, offers a powerful tool-box for problem solving in Social Sciences and many other areas. Recent trends in cognitive science [28] have correctly emphasized the power of the "diversity of perspectives" in problem solving, so it does not come as a surprise that adding physical perspectives to social models may sometimes pave the way to the needed breakthrough. Perhaps one should also wonder about the possibility of reverse flow in these interdisciplinary approaches to Social Sciences. After all, the proper use of a tool helps to its reshaping, and one could perhaps expect some kind of feedback. In other words, is there any new physics that we can learn from the study of Social and Economic Complex Systems? Only the recourse to empirical and quantitative methods in the study of social phenomena may likely give clues for sensible answers to this question.

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1. Low $c$ approximation

In order to simplify expressions we assume hereafter $\epsilon = 0$ and $k = 4$, and denote $\delta = n_{C}/n_{F}$. For the case of a random regular graph structure of the fluctuating set $\mathcal{F}$, the probability $\Pi_{C-D}^{BB}$ that an instantaneous defector node chooses to imitate Big Brother 2 (invasion event from BB2) is, to first order in $n_{F}^{-1}$,

$$\Pi_{C-D}^{BB} = \frac{1}{(k+2)} \frac{c + \delta}{(1 + \delta)b}$$

(41)

while the probability $\Pi_{D-C}^{BB}$ of an invasion event from BB1 to an instantaneous cooperator node in $\mathcal{F}$ is, to first order in $n_{F}^{-1}$,

$$\Pi_{D-C}^{BB} = \frac{c}{(k+2)}$$

(42)

Thus, for $\delta \leq c$, typical intervals between invasion events from Big Brothers in a node are (of the order of) $c^{-1}$ time units large. For large values of the temptation, where the value of $c$ is expected to be very small, the dynamics is consequently dominated, for typically very large intervals of time, by internal strategic interactions. Let us analyze them.

The internal neighbors of a cooperator $i$ are overwhelmingly likely instantaneous defectors in this “low $c$” regime, so that $i$ will be quickly invaded by them. The only chance for it to resist invasion would be that its instantaneous neighborhood microstate had at least two cooperator neighbors and that $b < (3/2)$ (note that in this strategic configuration, the pay-off of $i$ is $P_{i} = 3$ and that of its typical defector neighbors is $2b$). These neighborhood microstates (cooperative clusters) are so rare fluctuations that low values of the temptation $b$ are necessary for their non-negligible occurrence. Provided $b$ is below the transition value, the resilience to invasion (lattice reciprocity) of cooperative clusters enhances the likelihood of these fluctuations, which in turn reinforces the clusters resilience, and so on. This positive feedback mechanism of cooperative fluctuations enhancement is thus what triggers the transition to highly cooperative macro-states, and qualifies lattice reciprocity as a critical social phenomenon.

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