Spontaneous Jamming in One-Dimensional Systems

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Abstract

We study the phenomenon of jamming in driven diffusive systems. We introduce a simple microscopic model in which jamming of a conserved driven species is mediated by the presence of a non-conserved quantity, causing an effective long range interaction of the driven species. We study the model analytically and numerically, providing strong evidence that jamming occurs; however, this proceeds via a strict phase transition (with spontaneous symmetry breaking) only in a prescribed limit. Outside this limit, the nearby transition (characterised by an essential singularity) induces sharp crossovers and transient coarsening phenomena. We discuss the relevance of the model to two physical situations: the clustering of buses, and the clogging of a suspension forced along a pipe.

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Many non-equilibrium physical situations can be modelled as driven diffusive systems\[1\]. An intriguing feature of certain driven systems is their propensity to jam – in traffic flow\[2\] jamming behaviour is a fact of modern life and in colloid rheology the phenomenon of shear thickening (dilatancy) is widely studied\[3\].

One-dimensional (1\(d\)) driven systems exhibit a wide variety of interesting phenomena, including phase transitions and spontaneous symmetry breaking\[4\], which are precluded from 1\(d\) equilibrium systems (in the absence of long range interactions). This suggests that the physics of jamming might be captured in simple 1\(d\) models. In previous studies of simple 1\(d\) non-equilibrium models, jamming arises because of the presence of disorder or inhomogeneities such as defect sites \[5\]. In contrast, the model we introduce below is homogeneous; the jamming emerges via spontaneous symmetry breaking. Jamming arises through a mechanism in which a non-conserved quantity in the dynamics mediates an effective long range interaction of a conserved quantity (driven species), even though the microscopic dynamics is local and stochastic.

We now define the microscopic model we study, which we refer to as the Bus Route Model (BRM) for reasons to become clear. The BRM is defined on a 1\(d\) periodic lattice with \(L\) sites. Site \(i\) has two variables \(\tau_i\) and \(\phi_i\) associated with it, each of which can be either 1 or 0. When a site is occupied by a “bus”, \(\tau_i\) is 1 and if \(\phi_i\) is 1 the site is said to have “passengers” on it\[6\]. \(\tau_i\) and \(\phi_i\) cannot both be 1 simultaneously. There are \(M\) buses in total and the bus density \(\rho = M/L\) is a conserved quantity. However, the total number of sites with passengers is not conserved.

In updating the system, a site \(i\) is chosen at random. If both \(\tau_i\) and \(\phi_i\) are 0, then \(\phi_i \rightarrow 1\) with probability \(\lambda\). If \(\tau_i = 1\) and \(\tau_{i+1} = 0\), then the bus at site \(i\) hops forward...
with probability $1 - (1 - \beta)\phi_{i+1}$. If the bus hops, $\phi_{i+1}$ becomes 0. Thus, a bus hops with probability 1 onto a site without passengers, and probability $\beta$ onto a site with passengers thereby removing them. The probability that passengers arrive at an empty site is $\lambda$. We generally take $\beta < 1$, reflecting the fact that buses are slowed down by having to pick up passengers. Buses are forbidden from overtaking each other but relaxing this condition will have no significant effect. We remark that the dynamics is local and does not satisfy detailed balance.

At this point it is useful to discuss two scenarios which illustrate possible applications of the model and highlight the roles of the conserved and non-conserved quantities. The first and most obvious example is that of buses moving along a bus-route. Clearly, the ideal situation is that the buses are evenly spaced so that they pick up roughly equal numbers of passengers. However, what commonly occurs is that a bus falls behind the one in front and consequently has more passengers awaiting it. Thus the bus becomes further delayed and at the same time, following buses catch up with it, leading to a cluster of buses. The number of passengers awaiting a bus gives an indication of the elapsed time since the last bus went past and in this way communicates information between the two buses, resulting in an effective long range interaction.

We now turn to an alternative interpretation of the model describing a system of driven particles, each of which can exist in two states of mobility. Each time a bus hops to the right in the BRM, a vacancy moves to the left. In the new interpretation of the model, which can be thought of as the dual of the BRM, the vacancies become “particles” and the non-conserved variable is the mobility (hopping probability) of a particle, which is either 1 or $\beta$. A possible application of this dual model is to the phenomenon of clogging. A simple scenario is the flow of particles suspended in a fluid being forced through a pipe. The pipe is narrow enough to prevent the particles passing each other and stationary particles may become weakly attached to the pipe (with rate $\lambda$), reducing their mobility from 1 to $\beta$. At high density, individual particles move more slowly and therefore are more likely to become attached to the pipe, thus impeding the motion of the following particles and encouraging them to attach. Hence clogging ensues. Although set up as a strictly 1d model (requiring the diameters of the particles and the pipe to be comparable), a similar scenario could affect the flow of any heterogeneous material with a tendency to solidify when at rest.

From our study of the BRM we provide strong evidence, both numerical and analytical, that a true jamming phase transition does occur, but only in the limit $\lambda \to 0^+$ with $\lambda L \to \infty$. The transition is from a low density “ jammed” phase to a high density homogeneous phase. When $\lambda$ is small but finite, we find two strong signatures of the transition. Firstly, the transition is rounded to a crossover; but this is exponentially sharp in $1/\lambda$. Secondly, apparent coarsening occurs where over long time scales, the system separates into jammed regions of finite size with long but finite lifetimes.

We first present some simulation results for the BRM. Figure 1 shows a space-time plot of the system at low density and small $\lambda$. As passengers enter the system, one sees the large inter-bus gaps increasing in size until the system comprises several distinct clusters (or “jams”) of buses. The system then coarsens via coalescence of the bus clusters until finally, only a single large cluster remains. For high densities, we find that the system is homogeneous – a snapshot of the system as whole resembles the high density final cluster in fig. 1. Figure 2 shows a space-time plot for the same system as in fig. 1, with the exception
that now $\lambda = 0.1$. While small, transient clusters of buses do appear, the “phase-separation” seen for $\lambda = 0.02$ does not occur. Figure 2 shows plots of bus velocity $v$ (average rate of hopping forward) against bus density $\rho$. For the two larger values of $\lambda$, velocity decreases smoothly with increasing density. However, for $\lambda = 0.02$, $v(\rho)$ has an apparent cusp at an intermediate value of the density, suggesting the presence of a phase transition.

We now show that the BRM exhibits a phase transition in the limit $\lambda \to 0$ with $\lambda L \to \infty$. To see this, consider a system comprising a single large cluster (as in fig. 1). If $\lambda L \to \infty$, then the site in front of the leading bus has passengers with probability one (because the time since that site was last visited by a bus is $\propto L$). Hence, the leading bus hops forward with probability $\beta$. Since all of the gaps within the cluster are finite, there are no passengers within the cluster as $\lambda \to 0$; the buses within the cluster hop with probability one into unoccupied sites. The velocity (average rate of hopping forward) of these buses is $1 - \rho_c$, where $\rho_c$ is the density of buses in the cluster. For the cluster to be stable, this velocity must equal that of the leading bus and so we have $\rho_c = 1 - \beta$. For overall bus densities greater than $\rho_c$, the system becomes homogeneous with all gaps finite. Therefore, we identify $\rho_c$ as the critical density.

This shows that the BRM exhibits a phase transition in the limit of $\lambda \to 0$. We now present a two-particle approximation to the problem which suggests that there is no strict transition for non-zero $\lambda$. First, let us approximate the probability that a bus hops into a gap of size $x$ by $u(x) = f(x) + \beta(1 - f(x))$, where $f(x)$ is an estimate of the probability that there are no passengers on the first site of the gap. The average time since a bus last left this site is $x/v$ (where $v$ is the average velocity in the system), so we estimate

Figure 1: Space-time plot of bus positions for $\lambda = 0.02$, $\rho = 0.2$, $\beta = 0.5$ and $L = 500$. There are 10 time-steps between each snapshot on the time axis. Initially, the buses are positioned randomly and there are no passengers.
Figure 2: Space-time plot of bus positions for the same parameters as in fig. \[\text{fig.1}\] with the exception that here, $\lambda = 0.1$.

Figure 3: The velocity as a function of bus density for $\beta = 0.5$ and various values of $\lambda$. The symbols are simulation results for the BRM with $L = 10000$ and the lines are mean-field model (see below) results in the thermodynamic limit.
\[ f(x) = \exp(-\lambda x/v) \]
to give
\[ u(x) = \beta + (1 - \beta) \exp(-\lambda x/v) \quad \text{for} \quad x > 0 \tag{1} \]
with exclusion requiring \( u(0) = 0 \). This is in the spirit of a mean-field approximation for the BRM, the nature of which is to replace the “induced” interaction between buses (which is subject to stochastic variation) with a deterministic one.

Now consider a “jammed” system as in fig. 1, with the large gap in front of the leading bus in the cluster having size \( kL \) (where \( k \) is independent of \( L \)). We denote the size of the gap between the leading two buses by \( x \) so that, using the mean-field hopping rate in (1), we may write a Langevin equation for the dynamics of this gap size as
\[
\dot{x} = u(kL) - u(x) + \eta(t) \equiv -\frac{d\Phi}{dx} + \eta(t). \tag{2}
\]
where \( \eta(t) \) is a noise term (say white noise of unit variance[12]). The gap size \( x \) has the dynamics of a particle diffusing in a potential well \( \Phi(x) \) given by (1, 2). The potential has a maximum at \( x^* = kL \) so that when \( x > x^* \), the particle has escaped from the well, or equivalently, the leading bus has left the cluster. We denote the average time for this break-up to occur by \( \tau \), which is given by \( \exp[\Phi(x^*) - \Phi(0)] \) to a good approximation[13]. In the limit \( L \to \infty \), this becomes
\[ \tau \sim \exp\left[\frac{\beta(1 - \beta)}{\lambda}\right] \tag{3} \]
which is finite for \( \lambda > 0 \), implying that a jam is not a stable object and will eventually break up. However, when \( \lambda \to 0 \), the jam becomes stable in agreement with our previous argument. When \( \lambda \) is small but non-zero, \( \tau \) is exponentially large in \( 1/\lambda \) and it can appear that a jam is stable when in fact it has a finite lifetime. Thus, we do not expect true phase-separation to occur for non-zero \( \lambda \).

Let us now move beyond the two-particle picture described above. Consider a model of hopping particles where the hopping rate of a particle is a function \( u(x) \) of the size of the gap \( x \) in front of that particle. By using the mean-field expression for \( u(x) \) given in (1), one defines a new model which we call the mean-field model (MFM). The (rigorous) solution and analysis of the steady state[14] of the MFM can be found in [11]; here we present some selected results.

The MFM exhibits no phase transition for non-zero \( \lambda \) in agreement with our two-particle argument but there is indeed a transition in the limit \( \lambda \to 0 \) with \( \lambda L \to \infty \). Figure 3 compares velocity as a function of density in the MFM (solved analytically) and the BRM (simulated); the agreement is quite good. For \( \lambda = 0.02 \) in both models, \( v(\rho) \) has an apparent cusp at an intermediate value of the density. We know that for the MFM, \( v(\rho) \) is in fact non-singular since there is rigorously no transition for non-zero \( \lambda \). Since we believe that the MFM captures the essential physics of the BRM, we expect that likewise there is no transition for non-zero \( \lambda \) in the BRM. When \( \lambda \) is small there is, however, a very sharp crossover between a low density “jammed” regime with \( v \simeq \beta \), and a high density “congested” regime where \( v \) decreases roughly linearly with increasing density.

To quantify the sharpness of the crossover for \( \lambda \) close to zero in the MFM, we calculated \( \kappa_{\text{max}} \), the maximum curvature of \( v(\rho) \). For \( \lambda \) small (less than about 0.02), we found [11]...
Figure 4: Current as a function of density for the dual model for several values of $\lambda$. $L = 10000$ for all simulation data and MFM results are in the thermodynamic limit. The uppermost solid curve is the MFM result in the limit $\lambda \to 0$ with $\lambda L \to \infty$. The dashed curve is the exact result when $\lambda$ is set equal to zero before the thermodynamic limit is taken. The latter two curves are identical for $\rho < 0.5$.

that $\kappa_{\text{max}}$ varies as $\exp(a/\lambda)$, where $a$ depends on $\beta$. Therefore, although a strict phase transition occurs only in the limit $\lambda \to 0$, the crossover is exponentially sharp in $1/\lambda$ for small $\lambda$.

We now comment on the occurrence of apparent coarsening (see fig. 1) in a system which, according to the above discussion, does not strictly phase-separate. (On the one hand, we have argued that large clusters are ultimately unstable while on the other hand, fig. 1 appears to show a fully phase-separated system.) We believe[11] that sufficiently large systems coarsen up to some finite length scale which is exponentially large in $1/\lambda$. For the system in fig. 1, this length scale is much larger than the system size.

Let us now return to the dual model defined earlier and interpret our findings in that context. Since an inter-bus gap in the BRM corresponds to a cluster of particles in the dual model, jamming is now a high density phenomenon, characterised by the presence of large clusters of particles. This restores to the word “jamming” a meaning closer to that used in everyday life. In the limit $\lambda \to 0$, a phase transition arises from a low density homogeneous phase in which the particles move quickly, to a high density jammed phase which is characterised by macroscopic clusters of particles and a slow flow. An infinitesimal rate $\lambda$ can result in macroscopic inhomogeneity and decrease in flow. This is illustrated in fig. 4 which shows the current (velocity times density) as a function of particle density for the dual model.

A different interpretation of the dual model is as a model of stop-start traffic flow with the particles representing cars. The longer a car is at rest, the more likely it is that the driver will be slow to react when it is possible to move again. This is related to several
“slow-to-start” cellular automaton traffic models studied recently [13].

In conclusion, we have found that the BRM exhibits a jamming transition from a high density homogeneous phase to a low density jammed phase. There is a spontaneously broken symmetry in the jammed phase: one bus is selected over all others to head the jam, even though all buses are identical. We have argued, however, that a strict phase transition occurs only in the limit $\lambda \to 0$ with $\lambda L \to \infty$ and that for non-zero $\lambda$, one sees crossover behaviour which is exponentially sharp in $1/\lambda$. Thus the model exhibits an essential singularity at $\lambda = 0$ which causes, alongside the dramatic crossover, the transient coarsening behaviour observed (see fig. [1] and [11]) for small, positive $\lambda$. If similar phenomena were to arise in other models, this could easily be interpreted as signifying a true phase transition where in fact none exists. Such phenomena may indeed arise in certain cellular automata models of traffic [10].

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[6] While we have taken the passenger variable $\phi_i$ to be binary, this does not to forbid the presence of more than one passenger at a site; we merely require that the extra passengers have no further effect on the dynamics.

[7] This is because the interchange of a fast-moving bus with a slower-moving bus in front also interchanges their velocities. This contrasts sharply with $1d$ models where jamming is induced by quenched disorder [3].

[8] The relevance of the BRM to real buses is discussed further in [11].

[9] In an exact mapping from the dual model to the the BRM, a particle attempts to hop to the left when the site to its left is updated. However, a more natural dynamics with no significant difference in behaviour is to update only particles [11].
Note that this violates Galilean invariance, requiring a pipe or some other fixed reference frame to be present.

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The variance of the noise should strictly depend on \( \beta \) but since we are primarily interested in the effect of \( \lambda \) on the dynamics of the gap, we ignore this dependence.

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