The peak of the solution of elliptic equations

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Abstract

A counter example of inheritance of convexity of domain of positive solution of Dirichlet boundary value problem and the hot spot problem that proposed by J. Rauch is given. The difficulty of these two problems is that the critical points of the solutions is not singleton but a level curve. However, using Pohozaev identity locally, partial answer to both problems can be derived.

Keywords: Critical points, inheritance of convexity, Neumann boundary value problems, hot spot problem.

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1 Introduction

This article concerns the local behavior of semi-linear elliptic equation

\[- \Delta u = f(u), \quad x \in D\]  \hspace{1cm} (1)
with either Neumann boundary condition
\[ \frac{\partial u}{\partial n} \bigg|_{\partial D} = 0, \quad (2) \]
or Dirichlet boundary value problem
\[ u \bigg|_{\partial D} = 0. \quad (3) \]
The solution of equations (1) are the steady states of both heat transferring problem and the standing wave of sound propagation. For heat transferring problem, the maximum of the solution is expected to occur at the boundary. This problem was proposed by Rauch [1]. The conclusion of this problem is not necessary true for standing wave problem. In fact, the counter example below looks more like a standing wave solution than the solution of heat transferring problem. Perhaps this is the reason why that there are many counter examples as [2, 3] and example 1 of this article. After [2, 3], this problem then adjusted to consider over convex domain. For nonlinear problem, similar behavior of the solution is called spike layer [4]. Spike layer problem attracts many research attentions, for example [5, 6, 7, 8, 9]. In this article, the location of the maximum of the solution of Neumann boundary value and the uniqueness of local extrema of Dirichlet boundary value problem are studied. Example 1 is the counter example of both the hot spot problem and the inheritance of convexity of positive solution. It was believed that the positive solution of Dirichlet boundary value problem of equation (1) will inherit the convexity of the domain. However, counter examples were found by Koreeva [11], Cabré and Chanillo [12] and Hamel et al. [13] over convex domain. Thus, instead of studying the inheritance of convexity of the solution of elliptic equation, the uniqueness of local maximum of the solution will be considered in this article. Although the condition of the solution that preserve convexity of domain remains unknown, the condition of positive solution to have a unique local maximum is obtained.
At first glance, these two problems seem to be irrelevant because their boundary conditions are different although they share the same equation. However, the behavior of these two solutions of the contrary proposition of the corresponding problems is similar. For Dirichlet boundary condition problem, the level curves should be a closed curves unless there are more then one isolated critical points that imply the existence of saddle point. On the other hand, the level curves of the solution of Neumann boundary condition should be orthogonal to the boundary or the boundary itself is a collection of critical points. Excluding the boundary being a collection of critical points, the level curves of Neumann boundary condition should not form a closed curves inside the domain unless there exists some other isolated critical point in the interior of the domain that implies the existence of saddle point. Thus, to derive the right condition of these problems, it is necessary to classify the critical point. The common method of classification critical points refers to Hessian matrix. Unfortunately, the solution of a differential equation is not a definite function therefore it is not possible to adept the Hessian matrix to it. To this end, a new testing method is introduced which involves the following definitions of critical points.

**Definition 1.** An isolated critical point \( p \) of function \( u \) is a local maximum (minimum) if for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \nabla u(x) \cdot (x - p) < 0 \) (>) and 
\[
0 < \|\nabla u(x)\| \leq \epsilon \quad \text{for} \quad x \in B_\delta(p) \cap D \backslash \{p\}; \quad \text{otherwise,} \quad p \text{ is a saddle point.}
\]

**Definition 2.** If \( s^i_t \subset S_t \) is a simply connected component of level curve \( S_t \) and \( \nabla u(x) = 0 \) for all \( x \in s^i_t \) then \( x \) is called a non-isolated critical point of function \( u \).

If \( p \) is a local maximum of solution \( u \) of equation (1) then \( f(u(p)) \geq 0 \). To see this, we integrate equation (1) over \( B_\delta(p) \) and it yields,
\[
- \int_{\partial B_\delta(p) \cap D} \frac{\partial u}{\partial n} ds = \int_{B_\delta(p) \cap D} f(u(x)) dx.
\]
Since $p$ is a local maximum and the outward normal $n(x) = -\frac{x-p}{\delta}$ along $\partial B_\delta(p)$, it gives $\frac{\partial u}{\partial n} \leq 0$. If $p \in \partial D$ then $\frac{\partial u}{\partial n} = 0$. Notice that, $\delta$ is arbitrary, therefore $f(u(p)) \geq 0$ and hence the following conclusion is derived.

**Proposition 3.** If $p$ is a local maximum (minimum) of the solution $u$ of (1) then $f(u(p)) \geq 0$ ($f(u(p)) \leq 0$).

Notice that definition 1 and 2 can be applied to classify the degenerate critical point such as $(0, 0)$ of $x^3 - y^3$ or the critical points at the boundary of the domain.

Before discussing the main issues, a counter example will be given first. The counter example is constructed over disks because disk posses both convex and symmetry properties. Thus if the counter example of these problems were found over disk then it seems that other kinds of necessary conditions are needed so that these conjectures hold. In fact, Lemma 5 and the counter example below seem to suggest that the nonlinearity of reaction term is the key to solve these two problems. For instance, the power $m$ of the nonlinear reaction term of counter example satisfies $m < 1$; however, the condition of Lemma 5 (Pohozeav’s identity) says that if $f(u) = u^m$ then $m > 1$. On the other hand, if $m = 1$ then (1) turns to eigen-value problem and the location of hot spot of eigen-function depends on the domain.

**Example 4.** $u(x, y) = \frac{(x^2+y^2)^2}{4} - (x^2 + y^2)^\frac{3}{2} + (x^2 + y^2)$, which is the symmetry solution over domain $D = B_{r_1}(0)$ of equation (4) below, where $B_r(p)$ is the ball of radius $r$ centered at point $p$ where

$$-\Delta u = 3 - \sqrt{1 + 2\sqrt{u} + 8\sqrt{u}}, \quad x \in D,$$

and its graph is as follows:
If domain $D = B_1(0)$ then $u$ satisfies Neumann boundary condition and all points of the boundary are maximum of $u$ which does agree with hot spot conjecture. However, if $D = B_2(0)$ then $u$ satisfies both Neumann and Dirichlet boundary condition; moreover, $u$ is positive but not convex and still $\nabla u = 0$ along $B_1(0)$. Thus $u$ conflicts with both hot spot of heat transferring on convex domain and the inheritance of convexity of positive solution.

In this article, we use $S_t$ to denote the level curves of a function. However, it is possible that the level curves contains more than one components. Therefore, we use $s^j_t \subset S_t = \bigcup_{j=1}^{k} s^j_t$ to denote the simply connected component of $S_t$. Here $s^j_t$ may be a singleton $\{x_0\}$, if $x_0$ is a local extrema.

2 Main results

Throughout this article we assume that $D$ is a smooth open bounded simply connected convex domain of $R^2$ satisfying interior spherical condition and $f$ is monotone with respect to $u$ or $\frac{df}{du} > 0$. As usual, we say that $D$ is smooth if $\partial D$ is smooth. $\bar{D}$ is the closure of domain $D$ and $\overset{0}{D}$ is the interior of $D$. We shall note that all proofs of this article are dimensionless; therefore, all results are expected to be true for higher dimension $N > 2$.

In the proof below, it involves with the critical point at boundary; therefore,
we introduce the following notations

\[ B_\delta(p) = B_\delta(p) \cap D, \]

\[ B_\delta^+(p) = \{ x \in B_\delta(p) : \nabla u(x) \cdot (x - p) > 0 \}, \]

and

\[ B_\delta^-(p) = \{ x \in B_\delta(p) : \nabla u(x) \cdot (x - p) < 0 \}. \]

Without ambiguity, \( p \in \partial D \) and \( \nabla u(p) = 0 \) if and only if \( \lim_{x \to p} \nabla u = 0 \) where \( x \in \bar{D} \). In particular, if \( p \in \partial D \) but \( \nabla u(x) \cdot (x - p) \) remains constant sign, for all \( x \in \bar{D} \), then we still say that \( p \) is a local extrema.

\subsection{Neumann boundary value problem}

From Neumann boundary condition, we have a natural constraint:

\[ \int_D f(u) dx = 0, \quad (5) \]

therefore \( f(u) \) must change its sign, say at \( u_0 \). As usual, we let \( F(u) = \int f(u) du \). The assumption \( \frac{df}{du} > 0 \) implies that \( F(t) \) concaves upward with respect to \( t \) and hence \( u_0 \) is the absolute minimum of \( F \). To explore the behavior of solution \( u \) of Neumann boundary value problem, we denote \( D^+ = \{ x \in D : f(u(x)) > 0 \} \), \( D^- = \{ x \in D : f(u(x)) < 0 \} \) and \( m = \min_{x \in D} u(x) \) and \( M = \max_{x \in D} u(x) \).

Most of the results of this article are based on the following hypothesis:

\[ u \cdot f(u) - 2F(u) > 0, \quad \frac{df}{du} > 0. \quad (A) \]

First, we consider equation (1) with Neumann boundary value problem.

\textbf{Lemma 5.} If \( u \) is the smooth solution of (1) satisfying hypothesis (A), \( F(t) > 0 \) and if \( p \in \bar{D} \) is an isolated critical point then \( p \) is a local extrema.
Proof. The lemma will be proved by deriving a contradiction. Without loss of

generality, we assume that $p \in \bar{D}^+$ such that $\nabla u \cdot (x - p)$ changes its sign.

To derive the results, we apply Phozeav identity locally. Multiplying $\nabla u(x) \cdot (x - p)$ to equation (1) and integrating over $B^+_δ(p)$, it yields

$$\int_{B^+_δ(p)} -\Delta u(\nabla u(x) \cdot (x - p))dx = -\int_{\partial B^+_δ(p)} \frac{\partial u}{\partial n}(\nabla u(x) \cdot (x - p))ds + \int_{B^+_δ(p)} \nabla u \cdot \nabla(\nabla u(x) \cdot (x - p))dx.$$  \hspace{1cm} (6)

Replacing $-\Delta u$ by $f(u)$, the left hand side of (6) yields

$$\int_{B^+_δ(p)} -\Delta u(\nabla u(x) \cdot (x - p))dx = \int_{B^+_δ(p)} f(u)(\nabla u \cdot (x - p))dx,$$

where

$$\int_{B^+_δ(p)} f(u)(\nabla u \cdot (x - p))dx = \int_{B^+_δ(p)} \nabla F(u) \cdot (x - p)dx,$$  \hspace{1cm} (8)

and

$$\int_{B^+_δ(p)} \nabla F(u) \cdot (x - p)dx = \int_{\partial B^+_δ(p)} F(u) \cdot ((x - p) \cdot n)ds - 2\int_{B^+_δ(p)} F(u)dx.$$  \hspace{1cm} (9)

Calculating the right hand side of (9) and by $D \subset R^2$ it yields

$$\int_{B^+_δ(p)} \nabla u \cdot \nabla(\nabla u \cdot (x - p))dx = \int_{\partial B^+_δ(p)} \frac{|\nabla u|^2}{2}((x - p) \cdot n)ds - \int_{B^+_δ(p)} \|\nabla u\|^2 dx.$$  \hspace{1cm} (10)

Replacing $\int_{B^+_δ(p)} \|\nabla u\|^2 dx$ by $\int_{\partial B^+_δ(p)} \frac{\partial u}{\partial n}uds + \int_{B^+_δ(p)} f(u)udx$ and then adding all together, we get

$$0 < \int_{B^+_δ(p)} -2F(u) + f(u)udx = \int_{\partial B^+_δ(p)} (\frac{|\nabla u|^2}{2} - F(u))((x - p) \cdot n) - \frac{\partial u}{\partial n}(\nabla u(x) \cdot (x - p) + u)ds.$$  \hspace{1cm} (11)

Let $\partial B^+_δ(p) = \mathcal{N} \cup \mathcal{D} \cup B$ where $\mathcal{N} = \{x \in B^+_δ(p)|\nabla u(x) \cdot (x - p) = 0\}$, $\mathcal{D} = B^+_δ(p) \cap \partial D$ and $B = \partial B_δ(p) \cap \partial B^+_δ(p)$. If $p \in \partial D$ then $\mathcal{D} = \emptyset$ otherwise it is an empty set.

Along $\mathcal{N}$, $x - p$ is parallel to the tangent of the curve therefore $(x - p) \cdot n = 0$. On the other hand, $\nabla u \cdot (x - p) \geq 0$ over $B^+_δ(p)$ therefore $\frac{\partial u}{\partial n} \geq 0$. Along $B$,
$x - p \cdot n = \delta$ and $\nabla u \cdot (x - p) \geq 0$ therefore $\frac{\partial u}{\partial n} \geq 0$. Along $D$, $\frac{\partial u}{\partial n} = 0$ ($u = 0$ for Dirichlet boundary value problem) and $(x - p) \cdot n = \delta$. Since $F(u(p)) > 0$, (11) yields

$$0 < \int_{B_\delta^+(p)} -2F(u) + f(u)udx$$
$$= \int_{\partial B_\delta^+(p)} F(u)((x - p) \cdot n) - \frac{\partial u}{\partial n} u - \frac{\delta ||\nabla u||^2}{2} ds$$
$$= \int_B -\frac{\delta ||\nabla u||^2}{2} - \delta F(u) - \frac{\partial u}{\partial n} u ds - \int_N \frac{\partial u}{\partial n} u dx, \quad (12)$$

Every terms on the right hand side of equation (12) are negative, a contradiction. Therefore $p$ must be a local extrema. The proof is completed. □

From the conclusion of Lemma 5 and implicit function theorem, the level curves of $u$ are either the union of disjoint simply connected smooth curves or singletons. Thus we have the following conclusion.

**Corollary 6.** If hypothesis (A) holds, $F > 0$ and if $S_t = \cup_{j=1}^k s^j_t$ then for all $t$ $s^j_t \cap s^l_t = \emptyset$.

**Theorem 7.** If hypothesis (A) holds, $F > 0$, and if $S_{u_0} = s_{u_0}$ contains only one component then $S_{u_0} \cap \partial D \neq \emptyset$, $\nabla u \neq 0$ along $S_{u_0}$ and there exists a unique $p_\pm \in \partial D^\pm$ such that $u(p_+) = \max_{x \in \partial D^+} u(x)$ and $u(p_-) = \min_{x \in \partial D^-} u(x)$, respectively.

**Proof.** If on the contrary $S_{u_0} \cap \partial D = \emptyset$ then the sign of $f(u)$ along $\partial D$ remains constant. If $u|_{\partial D} = C$ then $\nabla u(x) \cdot T(x) = 0$ where $T(x)$ is the unit tangent vector at $x$ along $\partial D$. Thus $\nabla u(x) = 0$ which contradicts Lemma 5. Hence $u$ cannot be a constant along $\partial D$. Let $u(p) = \max_{x \in \partial D} u(x)$ and $u(q) = \min_{x \in \partial D} u(x)$ then by Proposition 4, $f(u(q)) \leq 0$, a contradiction. Thus $S_{u_0} \cap \partial D = \emptyset$. By Lemma 5, $\nabla u \neq 0$ along $S_{u_0}$.

To prove the uniqueness of local maximum along $\partial D^+$, we let $p_i \in \partial D^+$ such that $u(p_i) = \max_{x \in \partial D^+} u(x)$. Let $\xi \subset \partial D^+$ be the arc containing all the points lie in between $p_i$. By mean value theorem, there exists at least a critical point $p_0$ lies
in between \( p_i \). By Lemma 5, \( p_0 \) cannot be a saddle point. Therefore \( p_0 \) is a local minimum which contradicts proposition 4.

If the condition \( F(u) > 0 \) may relax to \( F(u) \geq 0 \) but with assumption \( F(t) = 0 \) only when \( t = u_0 \), then Lemma 5 still holds. Thus \( u \) contains interior maximum provided that \( S_{u_0} \) contains more than one component.

### 2.2 Dirichlet boundary value problem

Lemma 5 is a local property of the solution of equation (1) therefore it remains true for Dirichlet boundary value problem. With the conclusion of Lemma 5 and mean value theorem, we may derive that the positive solution of (1) has a unique local maximum provided that the domain is convex. From the proof of Lemma 5, we see that the necessary condition of it is that \( F > 0 \). The positiveness of the solution and the conditions \( \frac{df}{du} > 0 \) and \( f(t) > 0 \) for \( t > 0 \) imply that \( F(u) > 0 \) over \( \bar{D} \). Thus the following conclusion holds.

**Theorem 8.** If \( u \) is the smooth positive solution of (1) satisfying Dirichlet boundary condition with hypothesis (A), \( f(0) \geq 0 \) then \( u \) has a unique local maximum.

**Proof.** If \( p, q \) are both local maximum of \( u \) then by mean value theorem there must another critical point \( p_0 \) which is either a saddle or a local minimum. If \( p_0 \in \bar{D} \) then \( p_0 \) cannot be a local minimum because \( f(u) > 0 \) which contradicts proposition 4. \( p_0 \) cannot be a saddle because that will contradict Lemma 5. Next, if \( p_0 \in \partial D \) and if it is a saddle or local minimum then there is a subset \( B^+ \subset B_\delta(p) \cap D \) such that \( \nabla u \cdot (x - p_0) > 0 \), if \( x \neq p_0 \), which contradicts Lemma 5. Thus the interior local extrema is unique. The proof is completed.

**Remark 9.** The assumptions \( f(u)u > 2F(u) \) and \( F(u) > 0 \) of hypothesis (A) indicate that if \( f(u) = u^p \) then \( p > 1 \) which coincides with counter example 1. The
condition $\frac{df}{du} > 0$ does fit the first non-constant eigenfunction of Laplacian with Neumann boundary condition. However, excluding the constant eigenfunction, the second non-constant eigenfunction $\cos(x)\cos(y)$ on $[0, 2\pi] \times [0, 2\pi]$ has an interior critical point and level curve $S_0$ contains two components. Therefore, the location of the local extrema seems not only depends on the convexity of the domain but the structure of the level curve of $u_0$ as well.

References

[1] J. Rauch, Five problems: an introduction to the qualitative theory of partial differential equations, *Partial differential equations and related topics*, Lecture Notes in Math., Vol. 446, Springer, 1975, pp. 355–369.

[2] K. Burdzy and W. Werner, *A counter example to the “hot spots” conjecture*, Ann. of Math. (2) 149 (1999), no. 1, 309–317.

[3] K. Burdzy, *The hot spots problem in planar domains with on hole*, Duke Math. J. 129 (2005), no. 3, 481–502.

[4] C.-S. Lin, W. M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations 72 (1988), 1-27.

[5] C. S. Lin and W. M. Ni, On the diffusion coefficient of a semilinear Neumann problem, *Calculus of variations and partial differential equations*, Lecture Notes in Math., vol. 1340, Springer, 1986, pp. 160-174.

[6] W. M. Ni and I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. 44 (1991), No. 7, 819-851.

[7] W. M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semi-linear Neumann problem*, Duke Math. J. 70 (1993), 247-281.
[8] Y. Miyamoto, *An instability criterion for activator-inhibitor systems in a two dimensional ball*, J. Differential Equations 229 (2006), 494-508.

[9] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations 134 (1997), 104-133.

[10] Y. Lou and W. M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Diff. Eqns 131, (1996), 79-131.

[11] N. Korevaar, *Capillary surface convexity about convex domain*, Indiana Univ. Math. J., 32 (1983), 73-82.

[12] X. Cabré and S. Chanillo, *Stable solutions of semilinear elliptic problems in convex domains*, Sel. Math. 4 (1998), 1-10.

[13] F. Hamel and N. Nadirashvili and Y. Sire, *Convexity of level set for elliptic problems in convex domains or convex rings:two counterexamples*, arXiv:1304.3355.

[14] J. Spruck and Y. Yang, *Charged Cosmological Dust Solutions of the Coupled Einstein and Maxwell Equations*, Discrete and Continuous Dynamical Systems (in honor of Louis Nirenberg) 28 (2010) 567-589.

[15] Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer-Verlag, New York 2001.