Fidelity at Berezinskii-Kosterlitz-Thouless quantum phase transitions

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We clarify the long-standing controversy concerning the behavior of the ground state fidelity in the vicinity of a quantum phase transition of the Berezinskii-Kosterlitz-Thouless type in one-dimensional systems. Contrary to the prediction based on the Luttinger liquid approach it is shown that the fidelity susceptibility does not diverge at the transition, and numerical claims of its logarithmic divergence with the system size (or temperature) are explained by logarithmic corrections due to marginal operators.

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The ground state fidelity[1,2], a concept stemming from quantum information theory, is the overlap $F(\lambda, \lambda + \delta \lambda) = \langle \psi_0(\lambda) | \psi_0(\lambda + \delta \lambda) \rangle$ between two ground state wave functions of the Hamiltonian $\hat{H}(\lambda) = H_0 + \lambda \hat{V}$ at different values of the coupling parameter $\lambda$. It is widely used as an unbiased indicator of quantum phase transitions[2,3], especially in one-dimensional (1D) systems where a very accurate numerical calculation of the ground state wave function is possible thanks to the well-developed density matrix renormalization group (DMRG) technique[4,7]. Hereafter, we restrict the discussion to the 1D case.

Fidelity vanishes exponentially with the system size $L$. The fidelity susceptibility per site (FS)

$$\chi_{\lambda} = \left(1/L\right)\lim_{\delta \lambda \to 0} \left[-2 \ln |F(\lambda, \lambda + \delta \lambda)|/|\delta \lambda|^2\right]$$

is an intensive quantity expected to diverge in the thermodynamic limit $L \to \infty$ at the phase transition point $\lambda = \lambda_c$, due to nonanalyticity in the ground state. For finite $L$, this divergence translates into a presence of a peak in $\chi_{\lambda}(\lambda)$ at $\lambda = \lambda_c$, with $\chi_{\lambda}(\lambda_c) \to \infty$ at $L \to \infty$. Assuming a translational invariant system with the unique ground state, perturbed by a local operator $\hat{V} = \delta_\lambda \hat{H} = \sum_x \hat{V}(x)$, one obtains[2] the following connection between the FS and the reduced correlation function $G(x, \tau) = \langle \hat{V}(x, \tau) \hat{V}(0, 0) \rangle = \langle \hat{V}(x, \tau) \hat{V}(0, 0) \rangle - \langle \hat{V}(0, 0) \rangle^2$:

$$\chi_{\lambda} = \int_a^L dx \int_0^\infty d\tau \tau G(x, \tau),$$

where the imaginary time evolution is defined by $\hat{V}(x, \tau) = e^{\tau \hat{H}} \hat{V}(x) e^{-\tau \hat{H}}$, averages are taken in the ground state $|\psi_0(\lambda)\rangle$, and $a$ is the short-range (lattice) cutoff. Expression [1] diverges at $L \to \infty$ as $\chi \propto L^{d + 2\Delta_V - 2 \Delta \nu}$, where $\Delta_V$ is the scaling dimension of $\hat{V}(x)$ at the critical point and $z$ is the dynamic exponent, as long as $\Delta_V > z + 1/2$. At $\Delta_V = z + 1/2$ there is only a logarithmic divergence[3], and with the further increase of $\Delta_V$ the FS remains finite at the critical point.

The above arguments[2] show that the FS must be insensitive not only against marginal and irrelevant perturbations ($\Delta_V \geq z + 1$), but even against relevant perturbations with $z + 1/2 < \Delta_V < z + 1$. In the case of the Berezinskii-Kosterlitz-Thouless (BKT) phase transition, $z = 1$ and $\Delta_V = 2$, so it has been initially concluded[2,3,9] that transitions of this type cannot be detected by means of the finite-size scaling analysis of the FS. A prominent example of the BKT transition is the transition at the isotropic point $\lambda = 1$ in the so-called XXZ spin-$1/2$ chain defined by the Hamiltonian

$$\hat{H}_{XXZ} = \sum_n \left\{ S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \lambda S_n^z S_{n+1}^z \right\},$$

(2)

where $S^a_n$ are spin-$1/2$ operators at site $n$.

This conclusion has been apparently defeated by Yang[10] and Fjærestad[11]. Their approach uses the fact that the low-energy effective theory of the model[2] (as well as of many other gapless 1D systems), obtained by the Abelian bosonization[12], is the so-called Luttinger liquid (LL) described by the Hamiltonian

$$\hat{H}_{\text{eff}} = \frac{v}{2} \int dx \left\{ \Pi^2 + \frac{1}{K} (\partial_x \Phi)^2 \right\},$$

(3)

where $\Phi$ is the compact bosonic field ($\Phi = \Phi + \sqrt{\pi}$), and $\Pi$ is its conjugate momentum. The velocity $v = v(\lambda)$ and the LL parameter $K = K(\lambda)$ are generally functions of the original coupling $\lambda$ that have to be obtained as fixed points of the renormalization group (RG) flow equations. Alternatively, they can be extracted from the knowledge of exact long-distance behavior of correlation functions. For the XXZ model[2], exact correlator asymptotics is known from the Bethe ansatz, which yields $K = \pi/[2(\pi - \text{arccos} \lambda)]$ and $v = \pi \sqrt{1 - \lambda^2}/(2 \text{arccos} \lambda)$[13]. Since the effective model[14] is quadratic, one can explicitly calculate the fidelity $F(K, K + \delta K)$ and obtain for the FS in the thermodynamic limit

$$\chi_{\lambda=\infty} = (\partial_\lambda K)^2/(8 a K^2).$$

(4)
The dependence $K(\lambda)$ is singular at $\lambda = 1$, which leads to the divergence of the FS $\chi \propto (1 - \lambda)^{-1}$.

This direct calculation of overlaps might seem questionable since the connection between the wave functions of the initial model and its fixed-point low-energy theory is not so clear. Instead, one can use an alternative derivation due to Sirker [14] based on the relation (1). Indeed, using the effective Hamiltonian $\hat{H}$, the perturbation $\hat{V} = \partial_\lambda \hat{H}$ can be represented as

$$
\hat{V} = \frac{\partial V}{\partial \lambda} \hat{H}(\lambda) + \frac{V \partial^2 \lambda}{2} \int dx \, \hat{W}(x),$$

$$\hat{W}(x) = K \Pi^2 - (1/K)(\partial_x \Phi)^2. \tag{5}
$$

The first term in $\hat{H}$ commutes with $\hat{H}(\lambda)$ and thus gives no contribution into $\chi_L$ [15], but the second term contributes the prefactor leading exactly to the form $\hat{H}$. The scaling dimension of the second term is $\Delta = 1$, so the integral in (1) is finite at $L \to \infty$ and the divergence originates solely from the prefactor.

One can show that this singular behavior of $K(\lambda)$ is a generic property of any BKT transition. The general theory of a BKT transition is given by the Hamiltonian $\hat{H}$ perturbed by the cosine term $- (u/a^2) \cos(\sqrt{16\pi} \Phi)$, where $a$ is the lattice cutoff and $u$ is the dimensionless coupling. The proximity to the transition is controlled by the parameter $(\lambda_c - \lambda) \approx K - 1/2 - \pi u/2$. It follows from the RG equations [16, 17] that in the thermodynamic limit the fixed point behavior is $K - 1/2 \approx \sqrt{\pi u/(\lambda_c - \lambda)}$, so the FS diverges. In a finite system, however, the RG flow should be stopped at the RG scale $l \propto \ln(L/a)$, and the derivative of $K$ behaves as $(\partial K/\partial \lambda) \approx (4\pi u/3)l$ at $l \to \infty$, so according to $\hat{H}$ the FS at the transition should scale as

$$\chi_L \propto u^2 \ln^2(L/a). \tag{6}
$$

Indeed, numerical results show that in the vicinity of the isotropic point the FS exhibits a peak [18] located at $\lambda = 1$, which moves very slowly towards $\lambda = 1$ with increasing $L$, and whose height grows with $L$ much faster than corrections in powers of $1/L$ could explain [19]. A similar behavior has been reported [14] for the finite-temperature FS, and the results were claimed to be consistent with the $\ln^2(T_0/T)$ behavior, essentially of the same origin as Eq. (6) [11] (in the case of an infinite system at finite temperature, the RG flow is stopped at the scale $l \sim \ln(T_0/T)$ where $T_0$ is some nonuniversal energy scale of the order of the spin exchange energy). Recently, for the XXZ chain a divergent FS similar to [11] has been claimed [20] on the basis of the real-space quantum renormalization group. On the other hand, other authors did not see any divergence in the FS at BKT-type transitions in the spin $\frac{1}{2}$ XXZ chain [9] and in bosonic Hubbard model [21]. Further, comparison of the FS calculated according to [14] with the numerical results shows [14] that in order to fit the data one has to assume the ultraviolet cutoff $a$ to be strongly $\lambda$-dependent even in the vicinity of the free fermion point $\lambda = 0$. This controversy is aggravated by the fact that a logarithmic growth is difficult to distinguish numerically from logarithmic finite-size corrections, known to be a markant feature of BKT transitions.

There is a subtle problem with the above derivation of a diverging FS, immediately revealed by a closer look at the representation $\hat{H}$: recalling the original model (2), we see that $\hat{V} = \sum_n S_n^z S_{n+1}^z$, so $\hat{V}(x)$ is a bounded operator, while in Eq. (5) the bounded operator $\hat{W}(x)$ carries a prefactor that diverges at $\lambda \to 1$ independently of any ultraviolet cutoff. This indicates that the divergence might be an artefact of the effective representation $\hat{H}$, built on the Abelian bosonization and becoming inapplicable in the vicinity of the transition point. A similar problem is known for the amplitudes of correlation functions calculated by Lukyanov [22]: the amplitudes explode at $\lambda \to 1$, meaning that the applicability of the correspondent asymptotics is pushed to larger and larger distances. The integral (6) is convergent at $L \to \infty$, so divergences coming from such prefactors may be compensated by the subleading terms.

FIG. 1. (Color online). (a) The fidelity susceptibility for spin-$\frac{1}{2}$ XXZ chain [2] of up to $L = 500$ spins, as a function of the anisotropy $\lambda$; symbols denote DMRG results, and lines are guide to the eye. (b) The finite-size scaling of the peak position $\lambda_c$ and amplitude $\chi_L(\lambda_c)$; lines are results of fits to (11) and (12), see text for details.
Assuming, from the boundedness of \( \hat{V}(x) \), that the amplitude of the correlator \( G(x, \tau) \sim A(\lambda) v^2 (x^2 + \nu^2 \tau^2)^{-2} \) in (1) remains finite at \( \lambda \to 1 \), one returns to the initial conclusion that the FS at the transition stays finite as well. Finite-size corrections to \( \chi \) might be naively estimated in a standard way by means of the conformal substitution \( (x \pm i \nu \tau) \to (L/\pi) \sinh \left[ \frac{\pi}{2} (\nu \tau \pm ix) \right] \) in the correlator, yielding

\[
\chi_L = \chi_0(1 - \pi a L + O(L^{-2})), \quad \chi_0(\lambda) \sim -\pi A(\lambda)/2a. \tag{7}
\]

Although numerical results \([2, 3, 20]\) for the model (2) in the gapless phase \( | \lambda | < 1 \) are indeed consistent with (7), this type of finite-size scaling certainly breaks down close to the observed peak in the FS \([2, 3, 18]\). The logarithmic enhancement of the correlator \( G(x, 0) \to \ln(x) G(x, 0) \), which can take place at the isotropic point \( \lambda = 1 \), would affect the finite-size corrections in (7) only by changing them from \( L^{-1} \) to \( L^{-1} \ln(L) \), not solving the problem.

To obtain the correct finite-size scaling of \( \chi \) close to the SU(2)-symmetric BKT transition point \( \lambda = 1 \), it is convenient to use non-Abelian bosonization \([23]\). The low-energy theory of the model (2) is described by level-1 SU(2) Wess-Zumino-Witten (WZW) theory \([24]\) perturbed by marginal current-current term

\[
\mathcal{H} = \mathcal{H}_{\text{WZW}} - \int \frac{dx}{2\pi} \left\{ g_{\parallel} J_0 \tilde{J}_0 + \frac{g_{\perp}}{2} (J_+ \tilde{J}_- + J_- \tilde{J}_+) \right\}, \tag{8}
\]

where \( g_{\parallel} \) and \( g_{\perp} \) are running couplings governed by BKT-type RG equations \([25]\), and currents \( J \) and \( \tilde{J} \) are holomorphic functions of complex coordinates \( z = x \pm i \tau \), respectively. It is well known that finite-size corrections from marginal operators are only suppressed logarithmically \([26, 27]\). At the SU(2) point \( \lambda = 1 \) one has \( g_{\perp} = g_{\parallel} > 0 \) and the fidelity-changing perturbation \( \tilde{V} = \sum_n S_n^x S_n^x \) can be represented as \( \tilde{V} = \mathcal{H}_{\text{XXZ}}^{(\lambda=1)} - \sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) \to \mathcal{H} - \hat{O} \), where

\[
\hat{O} \sim \int dx (J_+ \tilde{J}_+ + J_- \tilde{J}_-) + \text{h.c.} \tag{9}
\]

Evaluating the correlator \( G(\tau) = \langle \hat{O}(\tau) \hat{O}(0) \rangle \) in the perturbed WZW model \([8]\), and calculating \( \chi_L = \frac{1}{\mathcal{Z} J_0} \int_0^\infty d\tau G(\tau) \), to the lowest order in \( g_{\parallel}, g_{\perp} \) one obtains

\[
\chi_L = \chi_0 - g_{\parallel} \chi_1 + O(g^2), \tag{10}
\]

where \( \chi_0 \sim \int \frac{d^2(x) d^2(x')}{|x-x'|} \) is given in (7), and \( \chi_1 \sim \int \frac{d^2(x) d^2(x') d^2(\tau) d^2(\tau')}{|x-x'|} \) is determined by three-point functions \([26, 27]\) of marginal operators of the WZW model, such as \( \int d^2x d^2\tau \langle J_+(z) \tilde{J}_-(z') J_0(z') \tilde{J}_0(0) J_0(0) \rangle \). At the SU(2) point, in the spirit of Ref. \([26]\), we replace the running coupling \( g_{\parallel}(l) \) by its “RG-improved” value \( g_{\parallel}(l) \sim 1/\ln(L/a) \) taken at the length scale \( l_L \sim \ln(L/a) \), thus obtaining

\[
\chi_L \sim \chi_0 - \chi_1/\ln(L/a) + O\left[1/\ln^2(L/a)\right]. \tag{11}
\]

We expect such scaling of the FS to be valid at BKT transitions in other 1D models as well, since any BKT transition point exhibits an enhanced SU(2) symmetry.

Off the SU(2) point into the gapless region, one can expect log corrections in (11) to be replaced by power laws. Indeed, close to the SU(2) point the amplitude of the leading correction to FS in Eq. (10) can be RG-improved as \( g_{\parallel}(l = \infty) = 2 - K^{-1} \) is the fixed point value of \( g_{\parallel} \) coupling. Thus, for \( K < \frac{\alpha}{\pi} (\lambda > \cos \frac{\pi}{8} \approx 0.81 \) in the XXZ chain) the leading finite size correction to \( \chi_L \) is \(-\lambda(L)^{8K-4} \), transforming into log corrections in the SU(2) limit \( K \to \infty \). For \( K > \frac{\alpha}{\pi} \) the leading corrections fall back to the usual behavior \( \chi_L \sim \chi_0 \sim -1/\ln(T_0/T) \) following from (7). This analysis carries over to the low-temperature behavior of FS \([21]\) by the replacement \( (a/L) \to (T/T_0) \); particularly, at the isotropic point \( \chi(T) \sim \chi_0 \sim 1/\ln(T_0/T) \) at \( T \to 0 \).

To support our conclusions, we have performed DMRG calculations \([3, 7]\) (in matrix product formulation \([32]\)) of the FS for the model (2), for large open chains of up to \( L = 500 \) sites \([33]\). The resulting FS as a function of \( \lambda \) is shown in Fig. 1(a): as observed in previous studies for small systems, there is a peak with the height slowly growing and the position slowly converging with the increase of \( L \). We start by fitting the finite-size dependence of the peak position \( \lambda_* \) with the help of the ansatz

\[
\lambda_* \simeq \lambda_0 + \lambda_1/\ln^2(L/a) + \cdots, \tag{12}
\]

which can be extracted by using standard scaling arguments \([34]\) on the gapped side of the BKT transition: since the infinite-system correlation length in the vicinity of the transition behaves as \( \xi \sim a e^{-B/\sqrt{\lambda_0 - \lambda_*}} \), Eq. (12) is obtained by postulating that \( \xi \sim L \) at \( \lambda \sim \lambda_* \). Further, when fitting the peak position \( \lambda_* \) according to (12), we fix \( \lambda_0 = 1 \), which allows us to extract the cutoff \( a \). Subsequently, we use the extracted value of the cutoff when fitting the peak value \( \chi_L(\lambda_*) \) of the FS according to our result (11). The results of those fits, shown in Fig. 1, demonstrate good agreement with the theory.

The above picture of non-diverging FS with strong logarithmic corrections due to marginal operators should be a generic feature of any BKT transition. To demonstrate that, we present here results of numerical studies for two more models containing such transitions. The first model is the anisotropic spin-1 chain defined by the Hamiltonian

\[
\hat{\mathcal{H}} = \sum_n \left( J^z_n(S_n \cdot S_{n+1}) + J^z_n(S_n \cdot S_{n+1})^2 + D(S_n^z)^2 \right), \tag{13}
\]

where \( S_n \) are spin-1 operators at site \( n \), and \( J_n^z = \cos \theta \) are exchange constants, and \( D > 0 \) is the single-ion anisotropy. For \( \theta \in [\pi, \frac{3\pi}{2}] \) with the increase of \( D \) this model exhibits a BKT transition from the gapless ferromagnetic XY phase to the gapped large-D phase, recently studied \([35]\) in the context of spinor
the numerical data is again consistent with the scaling
finite-size scaling results presented in Fig. 2(b) show that
of a slowly growing and poorly converging peak, and the
for the FS as function of $h$
For $h < -0.85\pi$, the same for the peak position $h_\star$ and value $\chi(L,h_\star)$ of the FS with respect to
the staggered field $h$ in the spin-$1$ anisotropic chain described
by the model (14) at $\lambda = -0.9$. Symbols show numerical
(DMRG) results, and lines represent fits to the scaling laws (11), (12).

One more model corresponds to the spin-$\frac{1}{2}$ ferromag-
netic anisotropic chain defined by the Hamiltonian (2)
with $-1 < \lambda < 0$, perturbed by staggered field $h$:

$$\hat{H} = \hat{H}_{XXZ} + h \sum_n (-1)^n S_n^z.$$  \hspace{1cm} (14)

For $-1 < \lambda < -1/\sqrt{2}$, this model exhibits a BKT-type
phase transition [24] to a gapped antiferromagnetic phase
at a finite value of the field $h$. Our DMRG calculations
for the FS as function of $h$ show the same typical behavior
of a slowly growing and poorly converging peak, and the
finite-size scaling results presented in Fig. 2(b) show that
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laws (11), (12).

In summary, we have shown that FS does not di-
verge at Berezinskii-Kosterlitz-Thouless-type quantum
phase transitions in one spatial dimension. Instead, it
merely exhibits a finite-amplitude peak in the vicinity of
the transition, with logarithmic finite-size scaling correc-
tions of the form (11), (12) which are too easy to con-
fuse numerically with a logarithmic growth of the peak.
The same is true for the finite-temperature FS, which
instead of the claimed [14] divergence $\chi \propto \ln^2(T_0/T)$
at $T \to 0$ should contain log corrections of the form
$\chi \sim \chi_0 - \chi_1/\ln(T_0/T)$.

The would-be divergence of the FS, originally proposed
[10] on the basis of mapping to Luttinger liquid (LL), and
in the meantime enjoying the status of an established re-
sult [14] 24 53 39, is an artefact due to the catastrophic
shrinking of the applicability range for using the LL cor-
relators when calculating the FS of the original micro-
scopic model as one approaches the BKT transition. To
identify the correct scaling of the FS in a specific model
at the BKT transition, it is crucial to take into account
properly marginal corrections to the LL Hamiltonian be-
ond the mere renormalization of LL parameters.

On the practical side, our results indicate that using
the FS as a tool to detect BKT transitions is extremely
inconvenient, since the uncertainties of logarithmic fits
remain too strong, even if one goes to the largest nu-
merically tractable system sizes. Using other detection
methods, e.g., looking at discontinuities of fidelity, as pro-
posed in Ref. 40, might be a better alternative.

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\[\begin{align*}
\chi &= \chi_0 - \frac{\chi_1}{\ln(T_0/T)}, \\
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\end{align*}\]

For $\lambda < -1/\sqrt{2}$, this model exhibits a BKT-type
phase transition (27) to a gapped antiferromagnetic phase
at a finite value of the field $h$. Our DMRG calculations
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\[\begin{align*}
\chi &= \chi_0 - \frac{\chi_1}{\ln(T_0/T)}, \\
\chi &= \frac{\chi_0}{\ln(T_0/T)}.
\end{align*}\]
This term does contribute to the finite-temperature FS \[14\], but its contribution vanishes exactly for the ground state FS even in finite-size systems.

J. M. Kosterlitz, J. Phys. C 7, 1046 (1974).

A. Pelissetto and E. Vicari, Phys. Rev. E 87, 032105 (2013).

B. Wang, M. Feng, and Z.-Q. Chen, Phys. Rev. A 81, 064301 (2010).

The authors of Ref. \[18\] attempted a power-law fitting of the FS which resulted in an anomalously low power \[\chi \propto L^{0.025}\] usually indicating the presence of log corrections.

A. Langari and A. T. Rezakhani, New J. Phys. 14, 053014 (2012).

J. Carrasquilla, S. R. Manmana, and M. Rigol, Phys. Rev. A 87, 043606 (2013).

S. Lukyanov, Phys. Rev. B 59, 11163 (1999).

I. Affleck, Phys. Rev. Lett. 55, 1355 (1985).

P. Di Francesco, P. Mathieu, and D. Sénéchal, \textit{Conformal Field Theory} (Springer, New York 1997).

S. Lukyanov, Nucl. Phys. B 522, 522 (1998).

J. L. Cardy, J. Phys. A: Math. Gen. 19, 511, L1093 (1986).

H. W. Blote, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. 56, 742 (1986).

I. Affleck, Phys. Rev. Lett. 56, 746 (1986).

S. Eggert, I. Affleck, and M. Takahashi, Phys. Rev. Lett. 73, 332 (1994).

I. Affleck, D. Gepner, H. J. Schulz and T. Ziman, J. Phys. A: Math. Gen. 22, 511 (1989).

We adopt the definition of the finite-temperature FS \[1\] based on \[1\] with ground state averages replaced by thermal ones, and the upper integration limit in \[\tau\] set to \[1/(2T)\]. Note that in this case the conformal substitution in the correlator leads only to corrections of the type \[\chi - \chi_0 \propto -T^2\], but there is another contribution \[14\] to \[\chi\] which is positive and \[\propto T\], due to the term in the perturbation that is proportional to the Hamiltonian, see \[15\].

F. Verstraete, J. J. Garcia-Ripoll, and J. I. Cirac, Phys. Rev. Lett. 93, 207204 (2004).

In our calculations of the FS, we have chosen the step \[\delta \lambda = 10^{-3}\] and checked that difference between results with \[\delta \lambda = 10^{-4}\] is negligible. We have taken care that for any system size \(L\), a convergence in the matrix dimension \(m\) had been reached; about \(m \lesssim 1000\) were typically sufficient to achieve good accuracy.