FLUCTUATIONS OF THE WINDING NUMBER OF A DIRECTED POLYMER ON A CYLINDER

YU GU AND TOMASZ KOMOROWSKI

ABSTRACT. We prove a central limit theorem for the winding number of a directed polymer on a cylinder, which is equivalent with proving the Gaussian fluctuations of the endpoint of the directed polymer in a spatial periodic environment.

KEYWORDS: directed polymer, central limit theorem, homogenization.

1. INTRODUCTION

1.1. Main result. We consider the problem of a directed polymer on a cylinder and study the fluctuations of the winding number, that is, the algebraic number of turns the polymer path does around the cylinder. The problem is equivalent to studying the fluctuations of the endpoint of a directed polymer in a random periodic environment. To state the main result, we first give an informal description of the model. The reference path measure is chosen to be the Wiener measure, and the random environment is modeled by a Gaussian space-time white noise \( \xi(t,x) \) on \( \mathbb{R}_+ \times [0,1] \), with periodic boundary condition, and we periodically extend it to \( \mathbb{R}_+ \times \mathbb{R} \).

For each realization of the random environment, the partition function of the directed polymer is given by

\[
Z_T = \mathbb{E} \exp(\beta \int_0^T \xi(t,w_t)dt),
\]

where \( \{w_t\}_{t \geq 0} \) is a one-dimensional standard Brownian motion starting from the origin, independent of \( \xi \), and \( \mathbb{E} \) is the expectation over the realizations of the Brownian motion \( w \). Here \( \beta > 0 \) is a fixed parameter playing the role of the inverse temperature. Since \( \xi \) is a space-time white noise, the above expression should be interpreted carefully, see Section 2 below for more details.

The quenched density of the polymer endpoint \( w_T \) is then given by

\[
\rho(T,x) = \frac{\mathbb{E} \exp(\beta \int_0^T \xi(t,w_t)dt)\delta(w_T-x)}{\mathbb{E} \exp(\beta \int_0^T \xi(t,w_t)dt)}.
\]

Since the random environment \( \xi \) is periodic in space, another perspective is to view the polymer path as lying on a cylinder, in which case it is the
trajecory of \( \{w_t - [w_t]\}_{t \geq 0} \) we are tracking. The winding number of the polymer path around the cylinder, denoted by \( W_T \), then equals to
\[
(1.3) \quad W_T = [w_T] \mathbb{1}_{w_T \geq 0} + [w_T] \mathbb{1}_{w_T < 0}.
\]

Here \([\cdot]\) and \([\cdot]\) denote the floor and ceiling functions, respectively. Thus, to study the large time behavior of \( w_T \) is equivalent to that of \( W_T \). Denote the quenched probability measure by \( \hat{P}_T \) and the expectation with respect to it by \( \hat{E}_T \), so for any bounded function \( f : \mathbb{R} \to \mathbb{R} \), we have \( \hat{E}_T f(w_T) = \int_{\mathbb{R}} f(x) \rho(T, x) dx \). Let \( P, E \) be the probability and expectation with respect to the noise \( \xi \). Now we can state the main result of the paper, namely, under the annealed polymer measure \( P \otimes \hat{P}_T, \{\frac{w_T}{\sqrt{T}}\}_{T > 0} \), or equivalently, \( \{\frac{W_T}{\sqrt{T}}\}_{T > 0} \), satisfies a central limit theorem.

**Theorem 1.1.** There exists \( \sigma_{\text{eff}}^2 \in (0, \infty) \), given in (3.27) below, such that for any \( \theta \in \mathbb{R} \), we have
\[
E\hat{E}_T \exp(i\theta \frac{w_T}{\sqrt{T}}) \to \exp(-\frac{1}{2} \sigma_{\text{eff}}^2 \theta^2), \quad \text{as } T \to \infty.
\]

### 1.2. Context and motivation.

Our study of the winding number is motivated by the work of Brunet [2], where the same problem was investigated by the replica method. What is particularly interesting is the exact formula he derived for \( \sigma_{\text{eff}}^2 \) and how it depends on the size of the period, see [2, Eq. (19)-(20)]. It is not hard to convince oneself that Theorem 1.1 holds for any spatial period, with the effective diffusion constant depending on the size of the cell – we chose the length \( L = 1 \) only to simplify the notations. Denote the corresponding variance by \( \sigma_{\text{eff}}^2(L) \). It is known that the polymer path is super-diffusive with the exponent 2/3 when \( L = \infty \), i.e., if there is no periodic structure, \( T^{-2/3} w_T \) is of order \( O(1) \) for \( T \gg 1 \), see [5, Theorem 1.11] for relevant results on this particular model. To go from the diffusive to super-diffusive scaling as \( L \to \infty \), it is natural to expect \( \sigma_{\text{eff}}^2(L) \) to blow up. This was indeed predicted in [2]: as \( L \to \infty \), \( \sigma_{\text{eff}}^2(L) \sim \sqrt{L} \). The blow up rate is related to the 2/3 super-diffusion exponent, and here is a heuristic explanation: for cells of size \( L \), the displacement of the endpoint \( w_T \) is of the order \( \sigma_{\text{eff}}(L) \sqrt{T} \), provided that \( L \sim O(1) \) and \( T \gg 1 \). As we keep \( T \gg 1 \) fixed and slowly increase \( L \), the polymer path would still visit many cells provided that \( \sigma_{\text{eff}}(L) \sqrt{T} \gg L \). In this case we still expect to see a homogenization phenomenon and the central limit theorem as in Theorem 1.1 holds. So, it is natural to guess that the critical scale comes from balancing the two terms, \( \sigma_{\text{eff}}(L) \sqrt{T} \) and \( L \). This leads to \( L \sim T^{2/3} \), under the assumption of \( \sigma_{\text{eff}}^2(L) \sim \sqrt{L} \).

It was our hope to prove the above heuristics rigorously, and to confirm (or disprove) the replica calculations in [2]. Theorem 1.1 can be viewed as a small step towards this goal, through which we confirmed the diffusive scaling and the Gaussian fluctuations. The formula derived for \( \sigma_{\text{eff}}^2 \), see (3.27) below, is of Green-Kubo type which involves the integral of some covariance
function, and is too implicit to perform any asymptotic analysis. This is not surprising though, since the homogenization constant is usually given by the solution to some cell problem and precise estimates on it are not easy to obtain (see a very recent contribution along this line for a different model of diffusion in random environment [4]). Guided by the same philosophy, a similar study was carried out for the fluctuations of the free energy $\log Z_T$, which leads to the optimal size of fluctuations in certain regimes where $L, T$ go to infinity together, see [7].

For the connections between the winding number of the directed polymer in random environment and other models in statistical physics, such as vortices in superconductors and strongly correlated fermions, we refer to [2] and the references cited there.

One can also formulate the problem as a diffusion in a distribution-valued random environment and study the corresponding SDE with a singular drift, see e.g. [6, 3, 10] and the references therein. In this framework, making sense of the singular diffusion is already highly nontrivial, and is intimately related to the study of singular SPDE [12, 13, 11]. For our specific problem of directed polymer, one can view it as a passive scalar with the velocity field given by the solution of a stochastic Burgers equation, see [6, Theorem 31] which gives a rigorous meaning of it. Although the velocity field is spatial periodic, which is sometimes viewed as the simple case in the study of homogenization or invariance principle of diffusion in a random environment, the problem does not fall into any classical framework. It might be possible to employ the tools developed for singular diffusions and combine with homogenization type of arguments to study similar problems and to prove central limit type results. For this particular problem, we make use of the structure of the Gibbs measure and give a proof using a classical argument for the central limit theorem for weakly dependent random variables.

1.3. Sketch of proof. Our approach relies heavily on the previous work of studying the periodic KPZ equation [8], where we showed the endpoint distribution of the directed polymer on a cylinder mixes exponentially fast. The proof in [8] was inspired by the classical work of Sinai [16], which was on the stochastic Burgers equation on the torus. Similar results were also obtained in [15], using a random version of the Krein-Rutman theorem. As mentioned previously, one could view the polymer path as lying on the cylinder by considering the path $\{w_t - [w_t]\}_{t \geq 0}$. Assuming that at each integer time $k$, the position of the path is $x_k$, i.e., $w_k - [w_k] = x_k$, our previous result implies a strong mixing property of $\{x_k\}_{k \geq 1}$ under the polymer measure. If we denote $\eta_k$ the winding number of the polymer path accumulated during the interval $[k - 1, k]$, then the total winding number is simply $\sum_k \eta_k$. It is not hard to deduce that, given the positions of $\{x_k\}_{k \geq 1}$, the sequence of random variables $\{\eta_k\}_{k \geq 1}$ are independent, so, the correlation only comes from the correlation in those $x_k$. Our strategy will be to first consider the
case when the starting and ending points $x_0$ and $x_T$ are both sampled from the stationary measure so that $\{\eta_k\}_{k \geq 1}$ is a sequence of stationary random variables, and we will prove a $\rho$-mixing (correlation mixing) condition to apply the general central limit theorem for the sum of stationary random variables. Then, to finish the proof, we will show that the error induced by resampling $x_0$ and $x_T$ is asymptotically small, again using the strong mixing property of $\{x_k\}_{k \geq 1}$.

The same proof applies verbatim to the high dimensional setting when the random environment is assumed to be white in time and smooth in space.

Organization of the paper. In Section 2, we formulate the problem rigorously, define the endpoint distribution through a stochastic heat equation, and construct a Markov chain which keeps tracking the winding number of the polymer path as time increases. Sections 3 and 4 are devoted to proving the main result, by first reducing it to the stationary setting, then proving the $\rho$-mixing condition for the stationary sequence $\{\eta_k\}$. In Section 5, we prove the nondegeneracy of the variance $\sigma^2_{\text{eff}}$. Some further discussion is left in Section 6.

Notations. We will sometimes use the shorthand integral notation $\int$ when the domain of integration is clear from the context. If we do not specify the range of the summation in $\sum_j$, it stands for $\sum_{j \in \mathbb{Z}}$.

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2. Preparations

2.1. Stochastic heat equation and endpoint density on $\mathbb{R}$. As mentioned previously, the expression $Z_T = \mathbb{E} \exp(\int_0^T \xi(t, w_t) dt)$ is only formal since $\xi$ is a space-time white noise, and we actually need to consider the so-called Wick exponential. In this section, we define the endpoint density $\rho$ rigorously, through the stochastic heat equation (SHE). For an excellent introduction to the theory of the stochastic heat equation, we refer to the monograph [14].

Consider the equation of the form

$$
\begin{align*}
\partial_t u(t, x; \nu) &= \frac{1}{2} \Delta u(t, x; \nu) + \beta \xi(t, x) u(t, x; \nu), \\
u(0, dx) &= \nu(dx),
\end{align*}
$$

where $\beta > 0$, the product between $u$ and $\xi$ is interpreted in the Ito-Walsh sense, and $\nu \in \mathcal{M}_1(\mathbb{R})$ - the set of Borel probability measures on $\mathbb{R}$. Denote
the propagator of the above equation by \( Z_{t,s}(x,y) \), i.e. for each \((s,y) \in \mathbb{R}_+ \times \mathbb{R}\) fixed, we have
\[
\begin{align*}
\partial_t Z_{t,s}(x,y) &= \frac{1}{2} \Delta_x Z_{t,s}(x,y) + \beta \xi(t,x) Z_{t,s}(x,y), \quad t > s, x \in \mathbb{R}, \\
Z_{s,s}(x,y) &= \delta_y(x).
\end{align*}
\]

Due to the 1-periodicity of the noise, we obviously have
\[
Z_{t,s}(x+j,y+j) = Z_{t,s}(x,y), \quad j \in \mathbb{Z}, x, y \in \mathbb{R}, t > s.
\]

Let \( T = [0,1] \) be the unit torus with the end points identified in the usual way. Since \( \xi \) is periodic, we can consider the same equation on \( T \) with the periodic boundary condition. Then its propagator is given by
\[
G_{t,s}(x,y) = \sum_j Z_{t,s}(x+j,y) = \sum_j Z_{t,s}(x,y-j), \quad x, y \in T.
\]

In other words, \( G_{t,s}(x,y) \) is the periodic solution to (2.2) with the initial data \( G_{s,s}(x,y) = \sum_j \delta_{y-j}(x) \).

With the above notations, the random density \( \rho \), which is the density of \( w_T \) under the quenched polymer measure \( \hat{P}_T \) and was formally defined in (1.2), takes the form
\[
\rho(T,x;\nu) = \frac{u(T,x;\nu)}{\int_T u(T,x';\nu) dx'}.
\]

From now on, we choose \( \nu(dx) \) to be the Dirac measure at the origin. To simplify the notation, we will omit the dependence on \( \nu \) when there is no confusion.

2.2. **Endpoint density on the cylinder.** Besides studying the polymer endpoint on the whole line, we also consider its periodic counterpart:
\[
\rho_{\text{per}}(t,x;\nu) = \frac{v(t,x;\nu)}{\int_T v(t,x';\nu) dx'},
\]
where \( \nu \in \mathcal{M}_1(T) \) and \( v \) solves the equation
\[
\begin{align*}
\partial_t v(t,x;\nu) &= \frac{1}{2} \Delta v(t,x;\nu) + \beta \xi(t,x) v(t,x;\nu), \quad t > 0, x \in T, \\
v(0, dx) &= \nu(dx).
\end{align*}
\]

Using the propagator, the solution can be written as
\[
v(t,x;\nu) = \int_T G_{t,0}(x,y) \nu(dy).
\]

It turns out, see [8, Lemma 2.2], that \( \{ \rho_{\text{per}}(t) \}_{t \geq 0} = \{ \rho_{\text{per}}(t,.;\nu) \}_{t \geq 0, \nu \in \mathcal{M}_1} \) is a Markov family. For any \( t > 0 \), the random element \( \rho_{\text{per}}(t,.;\nu) \) takes values in \( \mathbb{D}_c(T) \), which we use to denote the space of continuous probability densities on \( T \).
To simplify the notation, for any $t > s$, we define the forward and backward polymer densities starting from $\nu$ by

\begin{align}
\rho_{\text{per}}(t, x; s, \nu) &= \frac{\int_{T} G_{t,s}(x, y)\nu(dy)}{\int_{T^2} G_{t,s}(x', y')\nu(dy')dx'}, \\
\tilde{\rho}_{\text{per}}(t, \nu; s, x) &= \frac{\int_{T} G_{t,s}(y, x)\nu(dy)}{\int_{T^2} G_{t,s}(y', x')\nu(dy')dx'}.
\end{align}

(2.7)

We have $\rho_{\text{per}}(t, x; \nu) = \rho_{\text{per}}(t, x; 0, \nu)$. By the time reversal of the space-time white noise, for any $t > s$ and $\nu$ fixed, we have

$$\{\rho_{\text{per}}(t, x; \nu)\}_{x \in T} \overset{\text{law}}{=} \{\tilde{\rho}_{\text{per}}(t, \nu; s, x)\}_{x \in T}.$$

We emphasize that, throughout the paper, the notation $\rho(\cdot; \nu)$ is the endpoint density of the polymer on the whole line, while the notations $\rho_{\text{per}}(t, \cdot; s, \nu)$ and $\tilde{\rho}_{\text{per}}(t, \nu; s, \cdot)$ are reserved for the endpoint density on the torus.

Now we summarize a few results concerning the properties of the Markov family $\{\rho_{\text{per}}(t)\}_{t \geq 0}$, see [8, Theorem 2.3, Eq. (4.17), Lemma 4.1, Proposition 4.6].

**Proposition 2.1.** There exists a unique invariant measure $\pi_{\infty}$ for $\{\rho_{\text{per}}(t)\}_{t \geq 0}$, supported on $\mathbb{D}_c(T)$. For any $p \geq 1$, there exists $C, \lambda > 0$ such that for all $t > 1$,

$$\mathbb{E} \sup_{\nu, \nu' \in M_1(T)} \sup_{x \in T} |\rho_{\text{per}}(t, x; \nu) - \rho_{\text{per}}(t, x; \nu')|^p \leq Ce^{-\lambda t},$$

(2.8) and

$$\mathbb{E} \sup_{\nu \in M_1(T)} \sup_{x \in \mathbb{R}} (\rho_{\text{per}}(t, x; \nu)^p + \rho_{\text{per}}(t, x; \nu)^{-p}) \leq C.$$

(2.9)

**2.3. A Markov chain for the winding number.** As the random environment is periodic in space, to study the displacement of the polymer endpoint $u_T$, it is equivalent to studying the winding number of the polymer path when we view it as lying on a cylinder by considering the trajectory $\{w_t - [w_t]\}_{t \in [0, T]}$. This is the perspective we will take from now on. The idea is to first sample the trajectory of the polymer path on the cylinder at integer times, then consider the winding of the path between successive integer times.

For any $N \in \mathbb{Z}_+$, consider $u(N, j_N + x_N)$ where $x_N \in [0, 1)$ and $j_N \in \mathbb{Z}$. By the definition of the propagator, we have

$$u(N, j_N + x_N) = \int_{\mathbb{R}} Z_{N,N-1}(j_N + x_N, y)u(N-1, y)dy$$

$$= \sum_{j_{N-1}} \int_{T} Z_{N,N-1}(j_n + x_N, j_N - 1 + x_{N-1})u(N-1, j_{N-1} + x_{N-1})dx_{N-1}.$$
We construct an integer-valued, time inhomogeneous Markov chain

\[ u(N, j_N + x_N) = \sum_{j_1, \ldots, j_{N-1}} \int_{\mathbb{T}^N} \prod_{k=1}^N Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1}) dx_{1,N-1}, \]

where we used the simplified notation \( dx_{1,N-1} = dx_1 \ldots dx_{N-1} \) and the convention \( j_0 = x_0 = 0 \). In other words, in the above integration, we have decomposed the domain \( \mathbb{R} \) as \( \mathbb{R} = \cup_{j} [j, j + 1) \), then integrate in each interval and sum them up. One should think of the variable \( j_k + x_k \) as representing the location of the polymer path at time \( k \), with \( j_k \) the integer part and \( x_k \) the fractional part, i.e., \( j_k = \lfloor w_k \rfloor \) and \( x_k = w_k - \lfloor w_k \rfloor \).

Now we make use of the periodicity and observe that

\[ \sum_{j_k} Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1}) = \sum_{j_k} Z_{k,k-1}(j_k - j_{k-1} + x_k, x_{k-1}) \]

Iterate the above relation, we reach at (recall that \( u(0, x) = \delta(x) \))

\[ u(N, j_N + x_N) = \sum_{j_1, \ldots, j_{N-1}} \int_{\mathbb{T}^N} \prod_{k=1}^N Z_{k,k-1}(j_k + x_k + j_{k-1} + x_{k-1}) dx_{1,N-1}, \]

where \( G \) is the periodic propagator defined in (2.4). Then we can write

\[ u(N, j_N + x_N) = \int_{\mathbb{T}^N} \left( \prod_{j_1, \ldots, j_{N-1}} \frac{Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1})}{G_{k,k-1}(x_k, x_{k-1})} \right) \prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1}) dx_{1,N-1}. \]

We construct an integer-valued, time inhomogeneous Markov chain \( \{Y_j\}_{j=1}^N \) with

\[ \mathbb{P}_x[Y_1 = j_1] = \frac{Z_{1,0}(j_1 + x_1, x_0)}{G_{1,0}(x_1, x_0)}, \]

\[ \mathbb{P}_x[Y_2 = j_2|Y_1 = j_1] = \frac{Z_{2,1}(j_2 + x_2, j_1 + x_1)}{G_{2,1}(x_2, x_1)}, \]

\[ \vdots \]

\[ \mathbb{P}_x[Y_N = j_N|Y_{N-1} = j_{N-1}] = \frac{Z_{N,N-1}(j_N + x_N, j_{N-1} + x_{N-1})}{G_{N,N-1}(x_N, x_{N-1})}. \]

With the Markov chain, one can write the summation in (2.11) as

\[ \sum_{j_1, \ldots, j_{N-1}} \frac{\prod_{k=1}^N Z_{k,k-1}(j_k + x_k, j_{k-1} + x_{k-1})}{\prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1})} = \mathbb{P}_x[Y_N = j_N], \]

where, to emphasize the dependence of the Markov chain on \( x_0, x_1, \ldots, x_N \), we have denoted the probability by \( \mathbb{P}_x \).
In this way, (2.11) is rewritten as

\[
(2.14) \quad u(N, j_N + x_N) = \int_{[0,1]^{N-1}} \mathbb{P}_x[Y_N = j_N] \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) dx_{1,N-1}.
\]

Recall that \( \hat{\mathbb{P}}_N \) is the quenched probability of the polymer measure on paths of length \( N \), and \( [w_N] \) is the integer part of the endpoint \( w_N \). Then

\[
(2.15) \quad \hat{\mathbb{P}}_N([w_N] = j_N) = \int_{x} \rho(N, j_N + x_N) dx_N = \frac{\int_{T} u(N, j_N + x_N) dx_N}{\int_{R} u(N, x') dx'}
\]

\[
= \frac{\int_{T_N} \mathbb{P}_x [Y_N = j_N] \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) dx_{1,N}}{\int_{T_N} \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) dx_{1,N}}.
\]

In other words, the quenched distribution of \( [w_N] \) is a weighted average of the distribution of \( Y_N \) (the average is over the \( x \) variable).

We introduce another notation: suppose that \( f, g \in \mathbb{D}_{c}(T) \), define

\[
(2.16) \quad \mu_N(x; f, g) := \frac{f(x_N) \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) g(x_0)}{G_{N,0}(f, g)},
\]

with \( x = (x_0, \ldots, x_N) \) and the normalization factor

\[
(2.17) \quad G_{N,0}(f, g) := \int_{T^{N+1}} f(x_N) \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) g(x_0) dx_{0,N},
\]

where \( dx_{0,N} := dx_0 \ldots dx_N \). For each realization of the random environment, one should view \( \mu_N(x; f, g) \) as the joint density of the polymer on the cylinder, evaluated at \( (0, x_0), (1, x_1), \ldots, (N, x_N) \), with the starting and ending points sampled from the densities \( g, f \) respectively. For any \( \nu, \nu' \in \mathcal{M}_1(T) \), we abuse the notation and write \( \mu_N(x; \nu, \nu') \) as well, meaning that the starting and ending points are sampled from \( \nu', \nu \). In this case, \( G_{N,0}(\nu, \nu') \) equals to

\[
G_{N,0}(\nu, \nu') := \int_{T^{N+1}} \prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) \nu(dx_0) dx_{1,N-1,\nu}(dx_N).
\]

With the above new notation, we can rewrite

\[
(2.18) \quad \hat{\mathbb{P}}_N([w_N] = j_N) = \int_{T^{N+1}} \mathbb{P}_x [Y_N = j_N] \mu_N(x; m, \delta_0) dx_{0,N},
\]

where \( m \) is the Lebesgue measure on \( T \) (note that in (2.15), the convention is \( x_0 = 0 \)).

By (2.10) and (2.13), it is clear that \( Y_N \) is a sum of independent random variables, for each fixed realization of the noise and \( x \). We rewrite it as

\[
(2.19) \quad Y_N = \sum_{k=1}^{N} \eta_k, \quad \eta_k = Y_k - Y_{k-1}, \quad Y_0 = 0.
\]
One should interpret $\eta_k$ as the winding number accumulated during the time interval $[k-1, k]$, and we have

$$\mathbb{P}_x[\eta_k = j] = \frac{Z_{k,k-1}(x_k + j, x_{k-1})}{G_{k,k-1}(x_k, x_{k-1})}, \quad j \in \mathbb{Z}. \quad (2.20)$$

To prove Theorem 1.1 for the winding number $W_N$, see (1.3), or the endpoint $w_N$, it is equivalent to proving it for $\lfloor w_N \rfloor$. From now on, we will focus on the law of $\lfloor w_N \rfloor$ and the rest of the analysis starts from the representation (2.18).

### 3. Proof of the central limit theorem

The goal is to prove the central limit theorem for $\frac{w_T}{\sqrt{T}}$ under the annealed polymer measure $\mathbf{P} \otimes \hat{\mathbb{P}}_T$. For $\theta \in \mathbb{R}$, define

$$\varphi_T(\theta) := \mathbb{E} \hat{\mathbb{P}}_T e^{i\theta w_T/\sqrt{T}} = \mathbb{E} \int_{\mathbb{R}} \exp \left\{ \frac{i\theta x}{\sqrt{T}} \right\} \rho(T, x) dx,$$

where $\rho$ was defined in (1.2) with $\nu$ chosen to be the Dirac measure at the origin. In this section, we will consider those $T$ taking integer values, and the main goal is to show

**Theorem 3.1.** We have

$$\lim_{N \to \infty} \varphi_N(\theta) = \exp \left\{ -\frac{(\sigma_{\text{eff}} \theta)^2}{2} \right\}, \quad \theta \in \mathbb{R},$$

with $\sigma_{\text{eff}}$ given by (3.27) below.

To show the above theorem it suffices to consider the integer part of $w_N$. Define

$$\psi_N(\theta) = \mathbb{E} \int_{\mathbb{Z}} \exp \left\{ \frac{i\theta x}{\sqrt{N}} \right\} \rho(N, x) dx. \quad (3.2)$$

We focus on finding the limit of $\psi_N(\theta)$. From the construction of the Markov chain in Section 2.3 and (2.18), we have

$$\psi_N(\theta) = \mathbb{E} \int_{\mathbb{T}^N} \mathbb{E}_x \exp \left\{ \frac{i\theta Y_N}{\sqrt{N}} \right\} \mu_N(x; m, \delta_0) dx_{0,N}$$

$$= \mathbb{E} \int_{\mathbb{T}^N} \mathbb{E}_x \exp \left\{ \sum_{k=1}^{N} \frac{i\theta \eta_k}{\sqrt{N}} \right\} \mu_N(x; m, \delta_0) dx_{0,N}$$

$$= \mathbb{E} \int_{\mathbb{T}^N} \prod_{k=1}^{N} \mathbb{E}_x \exp \left\{ \frac{i\theta \eta_k}{\sqrt{N}} \right\} \mu_N(x; m, \delta_0) dx_{0,N}.$$

Here we used $\mathbb{E}_x$ to denote the expectation with respect to $\mathbb{P}_x$. 
3.1. Estimates of the moments of increments. For any \( \nu, \nu' \in \mathcal{M}_1(\mathbb{T}) \) and \( p \geq 1 \), define
\[
E^{(p)}_{N,k}(\nu, \nu') := E \int_{\mathbb{T}^{N+1}} (|E_x|\eta_k|^p) \mu_N(x; \nu, \nu') \, dx_{0,N}.
\]
Recall that \( \eta_k \) was defined in (2.19) and its law is given by (2.20). The following result holds.

**Lemma 3.2.** For any \( p > 1 \), we have
\[
\mathcal{E}_p := \sup_{\nu, \nu' \in \mathcal{M}_1(\mathbb{T})} \sup_{N \geq 1} \sup_{k=1}^{\cdots N} E^{(p)}_{N,k}(\nu, \nu') < \infty.
\]

**Proof.** By the definition we have
\[
E^{(p)}_{N,k}(\nu, \nu') = E \int_{\mathbb{T}^2} dx_k dx_{k-1} \sum_j |j|^p Z_{k,k-1}(x_k + j, x_{k-1})
\times \frac{G_{N,k}(\nu'; x_k)G_{k-1,0}(x_{k-1}; \nu')}{\int_{\mathbb{T}^2} G_{N,k}(\nu'; y)G_{k-1,0}(y; \nu') \, dy \, dy'.}
\]
Here we used the simplified notation \( G_{t,s}(x; \nu) = \int_{\mathbb{T}} G_{t,s}(x, y) \nu(dy) \) and \( G_{t,s}(\nu; x) = \int_{\mathbb{T}} G_{t,s}(y, x) \nu(dy) \). The above expression is bounded from above by
\[
E\left\{ \left( \inf_{z, z'} G_{k,k-1}(z', z) \right)^{-1} \int_{\mathbb{T}^2} dx_k dx_{k-1} \sum_j |j|^p Z_{k,k-1}(x_k + j, x_{k-1})
\times \frac{G_{N,k}(\nu'; x_k)G_{k-1,0}(x_{k-1}; \nu')}{\int_{\mathbb{T}^2} G_{N,k}(\nu'; y)G_{k-1,0}(y; \nu') \, dy \, dy'} \right\}
= \int_{\mathbb{T}^2} E\{ F_k(x_k, x_{k-1}) \rho_{\text{per}}(N, \nu; k, x_k) \rho_{\text{per}}(k-1, x_{k-1}; \nu') \} \, dx_k dx_{k-1}.
\]
Here
\[
F_k(x_k, x_{k-1}) := \left( \inf_{z, z'} G_{k,k-1}(z', z) \right)^{-1} \left( \sum_j |j|^p Z_{k,k-1}(x_k + j, x_{k-1}) \right).
\]
Note that \( F_k, \rho_{\text{per}}(N, \nu; k, x_k) \) and \( \rho_{\text{per}}(k-1, x_{k-1}; \nu') \) are independent. Therefore, we have
\[
E^{(p)}_{N,k}(\nu, \nu') \leq \int_{\mathbb{T}^2} E F_k(x_k, x_{k-1}) E \rho_{\text{per}}(N, \nu; k, x_k) E \rho_{\text{per}}(k-1, x_{k-1}; \nu') \, dx_k dx_{k-1}.
\]
For any \( q > 1 \), we conclude by the Hölder inequality
\[
EF_k(x_k, x_{k-1}) = \sum_j |j|^p E\left\{ \left( \inf_{z, z'} G_{k,k-1}(z', z) \right)^{-1} Z_{k,k-1}(x_k + j, x_{k-1}) \right\}
\leq \sum_j |j|^p \left\{ E\left( \inf_{z, z'} G_{k,k-1}(z', z) \right)^{-q} \right\}^{1/q} \left\{ EZ_{k,k-1}^q(x_k + j, x_{k-1}) \right\}^{1/q'},
\]
with $1/q + 1/q' = 1$. There exists a constant $C_q > 0$ depending only on $q$ and such that
\[
E \left( \inf_{z,z'} G_{k,k-1}(z', z) \right)^{-q} \leq C_q, \\
\sup_{x',x} \left\{ E Z_{k,k-1}^q(x' + j, x) \right\}^{1/q'} \leq C_q \exp \left\{ -\frac{j^2}{C_q} \right\},
\]
see [8, Lemma 4.1] and Lemma 3.3 below. Hence
\[
\bar{F} := \sup_{k \geq 1} \sup_{x',x} E F_k(x', x) < \infty
\]
and $E^{(p)}_{N,k}(\nu, \nu') \leq \bar{F}$. This completes the proof of the lemma. □

The following lemma, concerning the moments estimate of the propagator of SHE, is quite standard. For completeness sake we present its proof in Section A.

**Lemma 3.3.** For any $p \geq 1$, there exists $C_p > 0$ such that
\[
EZ_{t,0}(x, 0)^p \leq \frac{C_p}{t^{p/2}} \exp \left\{ -\frac{x^2}{C_p t} \right\}, \quad \text{for all } t \in (0, 2], x \in \mathbb{R}.
\]

### 3.2. Characteristic function at equilibrium
Recall from Section 2.2 that, there exists a unique probability measure $\pi_\infty$ on the space $D_c(T)$ - continuous probability densities on $T$ - that is invariant under the dynamics of the polymer endpoint process.

Suppose that $\varrho$ and $\tilde{\varrho}$ are two independent copies of $D_c(T)$-valued random fields, distributed according to $\pi_\infty$. They are also assumed to be independent of the noise $\xi$. We will use $E_\varrho, E_{\tilde{\varrho}}$ to denote the expectation with respect to them respectively. From [8, Theorem 2.3] and (2.9), we know that (2.9) also holds for $\varrho$, i.e. for any $p \geq 1$ we have
\[
\mathfrak{N}_p := E_\varrho \sup_x \{ \varrho(x)^p + \varrho(x)^{-p} \} < +\infty.
\]

Define
\[
\tilde{\psi}_N(\theta) = E_\varrho E_{\tilde{\varrho}} E \int_{T^{N+1}} E_x \exp \left\{ \sum_{k=1}^N i \eta_k \theta \right\}^N \mu_N(x; \varrho, \tilde{\varrho}, \theta) d\mu_0N,
\]
and recall that
\[
\psi_N(\theta) = E \int_{T^{N+1}} E_x \exp \left\{ \sum_{k=1}^N i \eta_k \theta \right\}^N \mu_N(x; m, \delta_0) d\mu_0N.
\]

The only difference between $\psi_N$ and $\tilde{\psi}_N$ comes from the distributions of the starting and ending points of the directed polymer on the cylinder: for $\psi_N$, the starting point is the origin and the ending point is “free” and distributed according to the Lebesgue measure on $T$, while for $\tilde{\psi}_N$, the starting and ending points are sampled independently from the stationary distribution.
The purpose of the present section is to show the following proposition, which reduces the proof of central limit theorem to the stationary setting.

**Proposition 3.4.** For any \( \theta \in \mathbb{R} \) we have

\[
\lim_{N \to \infty} \left[ \tilde{\psi}_N(\theta) - \psi_N(\theta) \right] = 0. \tag{3.7}
\]

**Proof.** Consider a sequence \( \{k_N\}_N \) such that \( k_N \to \infty \) and \( \frac{k_N}{\sqrt{N}} \to 0 \). Define

\[
\psi_{N,o}(\theta) := \mathbb{E} \int_{T^{N+1}} \mathbb{E}_x \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} \mu_N(x; m, \delta_0) dx_{0,N},
\]

\[
\tilde{\psi}_{N,o}(\theta) := \mathbb{E}_e \mathbb{E}_e \mathbb{E} \int_{T^{N+1}} \mathbb{E}_x \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} \mu_N(x; \bar{\theta}, \bar{\varrho}) dx_{0,N}.
\]

We have

\[
\left| \psi_{N,o}(\theta) - \psi_N(\theta) \right| \\
\leq \mathbb{E} \int_{T^{N+1}} \mathbb{E}_x \left| \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} - \exp \left\{ \sum_{k=1}^{N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} \right| \mu_N(x; m, \delta_0) dx_{0,N} \\
\leq \mathbb{E} \int_{T^{N+1}} \left( \sum_{k=1}^{k_N-1} + \sum_{k=k_N-k_N+1}^{N} \right) \mathbb{E}_x \left| \frac{\theta \eta_k}{\sqrt{N}} \right| \mu_N(x; m, \delta_0) dx_{0,N} \\
\leq \frac{2k_N|\theta|\sqrt{\mathcal{E}_2}}{\sqrt{N}} \to 0,
\]

as \( N \to \infty \). In the last step, we have applied Lemma 3.2 with \( \mathcal{E}_2 \) defined in (3.4). By the same proof, we have \( |\tilde{\psi}_{N,o}(\theta) - \tilde{\psi}_N(\theta)| \to 0 \). Thus, to prove the proposition, it suffices to show that

\[
\left| \tilde{\psi}_{N,o}(\theta) - \psi_{N,o}(\theta) \right| \to 0. \tag{3.9}
\]

Recall that for any \( t > s \) and \( \nu \in \mathcal{M}_1(\mathbb{T}) \), \( \rho_{\text{per}}(t, s; \nu) \) and \( \tilde{\rho}_{\text{per}}(t; \nu; \cdot, \cdot) \) were defined in (2.7). We also have \( \rho_{\text{per}}(t; \cdot; \nu) = \rho_{\text{per}}(t; \cdot; 0, \nu) \). In \( \mu_N(x; m, \delta_0) \), we integrate out the variables

\[
x_0, \ldots, x_{k_N-2}, x_{N-k_N+1}, \ldots, x_N
\]

to obtain

\[
\int \mu_N(x; m, \delta_0) dx_{0,k_N-2} dx_{N-k_N+1,N} \\
\tilde{\rho}_{\text{per}}(N, m; N - k_N, x_{N-k_N}) \left( \prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1}) \right) \rho_{\text{per}}(k_N - 1, x_{N-k_N}; \delta_0) \\
\frac{\mathcal{G}_{N-k_N,k_N}(m, \delta_0)},
\]

with

\[
\mathcal{G}_{m_2,m_1}(\nu, \nu') := \int \tilde{\rho}_{\text{per}}(N, \nu; m_2, x) G_{m_2,m_1-1}(x, y) \rho_{\text{per}}(m_1 - 1, y; \nu') dxdy. \tag{3.10}
\]
This leads to the following expression
\begin{equation}
\psi_{N,o}(\theta) = E \int \tilde{\rho}_{\text{per}}(N, m; N - k_N, x_{N-k_N}) \frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(m, \delta_0)} \times \rho_{\text{per}}(k_N - 1, x_{k_N-1}; \delta_0) E_x \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} d x_{k_N-1, N-k_N}.
\end{equation}

For brevity sake we write $d x_{m,M} = dx_m \ldots dx_M$ for any $m \leq M$. With the above notations, we can also write $\tilde{\psi}_{N,o}$ as
\begin{equation}
\tilde{\psi}_{N,o}(\theta) = E_{\theta} E_{\tilde{\theta}} E \int \tilde{\rho}_{\text{per}}(N, \tilde{\theta}; N - k_N, x_{N-k_N}) \frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(\tilde{\theta}, \theta)} \times \rho_{\text{per}}(k_N - 1, x_{k_N-1}; \theta) E_x \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} d x_{k_N-1, N-k_N}.
\end{equation}

The idea is that, since $k_N \gg 1$, we expect $\rho_{\text{per}}(k_N - 1; \cdot; \delta_0)$ and $\tilde{\rho}_{\text{per}}(N, m; N - k_N; \cdot)$ to be close to the stationary distribution, so that in the expression of $\psi_{N,o}$, we can first “replace” $\delta_0$ by $\tilde{\theta}$, then “replace” $m$ by $\tilde{\theta}$. In this way, we reach at the expression of $\tilde{\psi}_{N,o}$.

We define an intermediate version:
\begin{equation}
\psi^{(1)}_{N,o}(\theta) = E_{\tilde{\theta}} E \int \tilde{\rho}_{\text{per}}(N, \tilde{\theta}; N - k_N, x_{N-k_N}) \frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(\tilde{\theta}, \theta)} \times \rho_{\text{per}}(k_N - 1, x_{k_N-1}; \theta) E_x \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} d x_{k_N-1, N-k_N}.
\end{equation}

Compare (3.13) with (3.11), the only difference comes from replacing $\delta_0$ with the stationary measure $\tilde{\theta}$. Next we write
\[
\psi_{N,o}(\theta) - \psi^{(1)}_{N,o}(\theta) = I_N + \Pi_N,
\]
where
\begin{equation}
I_N = E_{\tilde{\theta}} E \int \tilde{\rho}_{\text{per}}(N, m; N - k_N, x_{N-k_N}) \frac{\prod_{k=k_N}^{N-k_N} G_{k,k-1}(x_k, x_{k-1})}{\mathcal{G}_{N-k_N,k_N}(m, \tilde{\theta}_0)} \times \mathcal{E}_1(k_N - 1, x_{k_N-1}; \delta_0, \tilde{\theta}) E_x \exp \left\{ \sum_{k=k_N}^{N-k_N} \frac{i \theta \eta_k}{\sqrt{N}} \right\} d x_{k_N-1, N-k_N},
\end{equation}
with
\begin{equation}
\mathcal{E}_1(k_N - 1, \cdot; \delta_0, \tilde{\theta}) := \rho_{\text{per}}(k_N - 1, \cdot; \delta_0) - \rho_{\text{per}}(k_N - 1, \cdot; \tilde{\theta}),
\end{equation}
According to Proposition 2.1, there exist constants with definition of above expression is bounded by here we used the simplified notation. In the following, we will estimate \( I_N \) and \( \Pi_N \) separately.

Estimates on \( I_N \). First we note that the term \( \| \mathbb{E}_x \exp \left\{ \sum_{k=1}^{N-k_N} \frac{i\theta k}{\sqrt{N}} \right\} \| \leq 1 \). Secondly, by the definition of \( \mathcal{G} \) in (3.10), it is straightforward to check that

\[
\frac{\tilde{\rho}_{\text{per}}(N, m; N - k_N, x_{N-k_N})}{\mathcal{G}_{N-k_N,k_N}(m, \delta_0)} \leq \sup_x \tilde{\rho}_{\text{per}}(N, m; N - k_N, x) \inf_x \rho_{\text{per}}(k_N - 1, x; \delta_0) \times \frac{1}{\mathcal{G}_{N-k_N,k_N}(m, m)}.
\]

Here we used the simplified notation \( G_{t,s}(\nu, \nu') = \int G_{t,s}(x, y)\nu(dx)\nu'(dy) \).

Thus, we have

\[
|I_N| \leq \mathbb{E}_x \mathbb{E} \left[ \frac{\sup_x \tilde{\rho}_{\text{per}}(N, m; N - k_N, x)}{\inf_x \rho_{\text{per}}(N, m; N - k_N, x) \sup_x \rho_{\text{per}}(k_N - 1, x; \delta_0)} \sup_x |\mathcal{E}_1(k_N - 1, x; \delta_0, \theta)| \right]
\]

\[
\leq \sqrt{\mathbb{E}_x \mathbb{E} \left[ \frac{\sup_x \tilde{\rho}_{\text{per}}(N, m; N - k_N, x)}{\inf_x \rho_{\text{per}}(N, m; N - k_N, x) \sup_x \rho_{\text{per}}(k_N - 1, x; \delta_0)} \right]^2}
\]

\[
\times \sqrt{\mathbb{E}_x \mathbb{E} \sup_x |\mathcal{E}_1(k_N - 1, x; \delta_0, \theta)|^2}.
\]

According to Proposition 2.1, there exist constants \( C, \lambda > 0 \) such that the above expression is bounded by

\[
|I_N| \leq Ce^{-\lambda k_N}.
\]

Estimates on \( \Pi_N \). The proof is similar to that of \( I_N \). By (3.10) and the definition of \( \mathcal{E}_1 \) in (3.15), we have

\[
\mathcal{E}_2(k_N - 1; \delta_0, \theta) = - \int \tilde{\rho}_{\text{per}}(N, m; N - k_N, x_{N-k_N}) \frac{\mathcal{G}_{N-k_N,k_N-1}(x_{N-k_N}, x_{k_N-1})}{\mathcal{G}_{N-k_N,k_N}(m, \delta_0) \mathcal{G}_{N-k_N,k_N}(m, \theta)}
\]

\[
\times \mathcal{E}_1(k_N - 1, x_{k_N-1}; \delta_0, \theta) dx_{N-k_N} dx_{k_N-1},
\]

with

\[
\mathcal{E}_2(k_N - 1; \delta_0, \theta) := \mathcal{G}_{N-k_N,k_N}(m, \delta_0)^{-1} - \mathcal{G}_{N-k_N,k_N}(m, \theta)^{-1}.
\]
which implies that
\[ |\mathcal{E}_2(k_N - 1; \delta_0, \varrho)| \leq \sup_x \tilde{\rho}_\text{per}(N, m; N - k_N, x) \sup_x |\mathcal{E}_1(k_N - 1, x; \delta_0, \varrho)| \]
\[ \frac{(\inf_x \tilde{\rho}_\text{per}(N, m; N - k_N, x))^2 \inf_x \rho_\text{per}(k_N - 1, x; \delta_0) \inf_x \rho_\text{per}(k_N - 1, x; \varrho)}{1 \times G_{N-k_N,k_N-1}(m, m)}. \]

This leads to
\[ |\Pi_N| \leq \mathbb{P}_\sigma \mathbb{E} \left( (\sup_x \tilde{\rho}_\text{per}(N, m; N - k_N, x))^2 \sup_x \rho_\text{per}(k_N - 1, x; \varrho) \sup_x |\mathcal{E}_1(k_N - 1, x; \delta_0, \varrho)| \right). \]

Applying Hölder inequality as before, we also obtain that
\[ |\Pi_N| \leq C e^{-\lambda k_N}, \quad N = 1, 2, \ldots \]

Thus, we have
\[ |\psi_{N,0}(\theta) - \psi_{N,0}^{(1)}(\theta)| \leq C e^{-\lambda k_N} \to 0 \]
as \( N \to \infty. \)

It remains to show that \( \psi_{N,0}(\theta) - \psi_{N,0}^{(1)}(\theta) \to 0 \) as \( N \to \infty. \) Comparing (3.12) and (3.13), the only difference is \( m \) being replaced by \( \bar{\varrho}. \) By following the same proof for \( \psi_{N,0}(\theta) - \psi_{N,0}^{(1)}(\theta) \) verbatim, we conclude the proof of the proposition. \( \square \)

3.3. Construction of the path measure. Recall that by the construction of the Markov chain in Section 2.3, the study of the winding number \( W_N, \) see (1.3), or equivalently \( |w_N|, \) reduces to that of \( Y_N = \sum_{k=1}^N \eta_k: \)
\[ \tilde{\mathbb{P}}_N[|w_N| = j] = \int_{\mathbb{Z}} \mathbb{P}_x[Y_N = j] \mu_N(x; m, \delta_0) dx_{0,N}, \quad \text{for all } j \in \mathbb{Z}. \]

By the result in Section 3.2, to study the law of \( |w_N|, \) we can further replace \( \mu_N(x; m, \delta_0) \) in the above expression by the stationary density \( \mu_N(x; \bar{\varrho}, \varrho). \) In this section, we construct a path measure to realize \( \{\eta_k\}_{k \in \mathbb{Z}} \) as a sequence of stationary random variables, and the proof of the central limit theorem for \( |w_N| \) reduces to that of \( \sum_{k=1}^N \eta_k. \)

The space \( \mathbb{Z}^\mathbb{Z} \) consists of all functions \( \sigma: \mathbb{Z} \to \mathbb{Z}. \) For any \( k \in \mathbb{Z}, \) we denote by \( \eta_k: \mathbb{Z}^\mathbb{Z} \to \mathbb{Z} \) the \( k \)-th coordinate map, i.e. \( \eta_k(\sigma) := \sigma(k). \) Recall that for any \( f, g \in \mathcal{D}(\mathbb{T}), \) we have defined
\[ \mu_N(x; f, g) = \frac{f(x_N) \prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1}) g(x_0)}{\int_{\mathbb{Z}} f(x_N') \prod_{k=1}^N G_{k,k-1}(x_k', x_{k-1}') g(x_0') dx_{0,N}'}. \]

In the following, we construct a probability measure \( \mathcal{P} \) on \( \mathbb{Z}^\mathbb{Z} \) such that
The right hand side of (3.19) equals
\[ \ell, \]

Here \( \bar{\rho}_{\text{per}}(N + \ell, \bar{q}, N, \cdot) \) is the reverse time, polymer endpoint process starting at stationarity, see the definition of \( \bar{\rho}_{\text{per}} \) in (2.7). So we know that \( \bar{\rho}_{\text{per}}(N + \ell, \bar{q}, N, \cdot) \) has the same law as \( \bar{q} \) and is independent of \( q \) and the random...
environment in the interval \([0, N]\). We can therefore write that the right hand side of (3.19) equals

\[
E \varrho \frac{E \tilde{\varrho}}{\text{integral.disp}} \int dx_0, N \prod_{k=1}^{N} \frac{Z_{k,k-1}(j_k + x_k, x_{k-1})}{G_{k,k-1}(x_k, x_{k-1})} \times \frac{\prod_{k=1}^{N} G_{k,k-1}(x_k, x_{k-1}) \varrho(x_N) \varrho(x_0)}{\text{product.disp} k=1 \sum_{k=1}^{N} \frac{G_{k,k-1}(x_0, x_{k-1}) \varrho(x_N) \varrho(x_{k-1})}{\text{product.disp} k=1 \sum_{k=1}^{N} \frac{G_{k,k-1}(x_0, x_{k-1}) \varrho(x_N) \varrho(x_{k-1})}{\text{product.disp} k=1 \sum_{k=1}^{N} \frac{G_{k,k-1}(x_0, x_{k-1}) \varrho(x_N) \varrho(x_{k-1})}}},
\]

which proves (3.19). The argument for (3.18) is similar (for \(\ell \geq 1\)). This way we construct a stationary measure \(P\) on \(\mathbb{Z}^N\). Its extension to \(\mathbb{Z}^Z\) is standard.

From now on, we use \(E\) to denote the expectation with respect to \(P\).

**Proposition 3.5.** We have

\[
(3.20) \quad E\eta_j = 0,
\]

\[
(3.21) \quad E\eta_j^2 < \infty, \quad \text{for all } j \in \mathbb{Z}.
\]

**Proof.** First, (3.21) is a direct consequence of Lemma 3.2. Now we prove (3.20). Define

\[
M_N := E \int \sum_{k=1}^{N} \frac{\eta_k}{N} \mu_N(x; m, \delta_0) dx_{0, N} = E \int \mu_N(x; m, \delta_0) dx_{0, N},
\]

\[
\tilde{M}_N := E \int \sum_{k=1}^{N} \frac{\eta_k}{N} \mu_N(x; \tilde{\varrho}, \varrho) dx_{0, N} = E\eta_1.
\]

By following the proof of Proposition 3.4 and applying Lemma 3.2, we have \(M_N - \tilde{M}_N \to 0\) as \(N \to \infty\). On the other hand, note that \(M_N = N^{-1} E \bar{w}_N w_N\). By symmetry we have \(E \bar{w}_N w_N = 0\), which implies that \(M_N \to 0\) as \(N \to \infty\), since \(w_N - 1 < |w_N| \leq w_N\). This further implies that \(\tilde{M}_N = E\eta_1 = 0\). The proof is complete. \(\square\)

### 3.4. The correlation mixing.

In this section, we will show that the stationary sequence \(\{\eta_k\}_{k \in \mathbb{Z}}\) constructed in Section 3.3 satisfies a central limit theorem:

\[
(3.23) \quad \frac{\sum_{k=1}^{N} \eta_k}{\sqrt{N}} \Rightarrow N(0, \sigma_{eff}^2), \quad \text{as } N \to \infty.
\]

With (3.23), Proposition 3.4, we conclude the proof of Theorem 3.1.

Consider the probability space space \((\mathbb{Z}^Z, \sigma(C), P)\). Let \(F_j\) and \(F^j\) be the \(\sigma\)-algebras generated by \(\{\eta_k\}_{k \leq j}\) and \(\{\eta_k\}_{k \geq j}\), respectively. Define the correlation coefficient between two square integrable random variables \(X, Y\) as

\[
\text{corr}[X, Y] = \frac{\text{Cov}[X, Y]}{(E X^2)^{1/2}(E Y^2)^{1/2}}.
\]

We have the following definition of the \(\rho\)-mixing coefficient:
Definition 3.6. ($\rho$-mixing coefficients, see [1, Section 19]) The $\rho$-mixing coefficients for the stationary sequence $\{\eta_k\}_{k \in \mathbb{Z}}$ are defined as
\begin{equation}
 r(n) := \sup \{|\text{corr}[F,G]| : F \in \mathcal{F}_j, G \in \mathcal{F}_{j+n}\}, \quad n = 1, 2, \ldots.
\end{equation}

Note that, due to the stationarity, the definition of $r(n)$ in (3.24) does not depend on $j$. The main result of this section is the following:

Proposition 3.7. There exist $C, \lambda > 0$ such that
\begin{equation}
 |\text{Cov}[F,G]| \leq Ce^{-\lambda n}\|F\|_{L^2}\|G\|_{L^2}
\end{equation}
for any $n \geq 1$ and $F, G$ that are $\mathcal{F}_j$ and $\mathcal{F}_{j+n}$ measurable respectively. As a consequence, we have
\begin{equation}
 r(n) \leq Ce^{-\lambda n}.
\end{equation}
From (3.26), Proposition 3.5 and [1, Theorem 19.2], we immediately conclude the proof of (3.23), with
\begin{equation}
 \sigma^2_{\text{eff}} = \sum_{j \in \mathbb{Z}} E[\eta_0 \eta_j].
\end{equation}

Note that $\sigma^2_{\text{eff}} < \infty$ is a direct consequence of (3.26) and (3.21). We will show $\sigma^2_{\text{eff}} > 0$ in Section 5 below.

The rest of the section is devoted to the proof of Proposition 3.7.

Proof. Recall that $E$ denote the expectation with respect to $\mathcal{P}$. Suppose that
\begin{align*}
 F &= f(\eta_1, \ldots, \eta_{m_1}), \quad G = g(\eta_{m_1+n}, \ldots, \eta_{m_2+n})
\end{align*}
for some $m_1, m_2 \in \mathbb{N}$ and Borel measurable functions
\begin{align*}
 f : \mathbb{R}^{m_1} \to \mathbb{R}, \quad g : \mathbb{R}^{m_2-m_1+1} \to \mathbb{R}.
\end{align*}
Throughout the proof, to simplify the notation, define
\begin{align*}
 x = (x_0, \ldots, x_N), \quad N = m_2 + n.
\end{align*}
We have
\begin{equation}
 E[FG] = E_\varrho E \int \mathbb{E}_x[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1+n}, \ldots, \eta_{m_2+n})] \mu_N(x; \varrho_1, \varrho_2) dx_{0,N},
\end{equation}
where $\varrho_1, \varrho_2$ are sampled independently from $\pi_\infty$, also independent from $\xi$, and $\mathbb{E}_\varrho$ is the expectation on them. We recall that
\begin{align*}
 \mu_N(x; \varrho_1, \varrho_2) := \varrho_1(x_N) \prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1}) G_{N,0}(\varrho_1, \varrho_2) \varrho_2(x_0)
\end{align*}
is a density function, and
\begin{align*}
 G_{N,0}(\varrho_1, \varrho_2) = \int \varrho_1(x_N) \prod_{k=1}^N G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0) dx_{0,N}.
\end{align*}
The goal is to show that, when \( N \) is large, the density \( \mu_N(x; q_1, q_2) \) factorizes into two independent ones, with an error that is exponentially small in \( n \). The proof consists of several steps.

**Step 1. Rewriting \( E[FG] \).** Since \( E_{x}[f(\eta_1, \ldots, \eta_{m_1})g(\eta_{m_1+n}, \ldots, \eta_{m_2+n})] \) only depends on the variables \( x_{0,m_1}, x_{m_1+n-1,N} \), we will first integrate out other variables in \( \mu_N(\cdot; q_1, q_2) \). We keep a “middle” one for a future purpose: define

\[
\ell = m_1 + \lceil n/2 \rceil.
\]

After integrating out the variables \( x_{m_1+1, \ell-1}, x_{\ell+1,m_1+n-2} \), we obtain (3.29)

\[
\int \mu_N(x; q_1, q_2) dx_{m_1+1, \ell-1} dx_{\ell+1,m_1+n-2}
\]

\[
= G_{N,0}(q_1, q_2)^{-1} q_1(x_N) \prod_{k=m_1+n} G_{k,k-1}(x_k, x_{k-1}) G_{m_1+n-1, \ell}(x_{m_1+n-1, x_{\ell}})
\]

\[
\times G_{\ell,m_1}(x_{\ell}, x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) p_2(x_0).
\]

Recall the definition of forward and backward density in (2.7), we rewrite the two factors in (3.29) that contain \( x_\ell \) as

\[
G_{m_1+n-1, \ell}(\cdot, x_{\ell}) = \rho_{\text{per}}(m_1 + n - 1, \cdot; \ell, \delta_{x_{\ell}}) \int G_{m_1+n-1, \ell}(y, x_{\ell}) dy,
\]

\[
G_{\ell,m_1}(x_{\ell}, \cdot) = \tilde{\rho}_{\text{per}}(\ell, \delta_{x_{\ell}}; m_1, \cdot) \int G_{\ell,m_1}(x_{\ell}, y) dy.
\]

Further define the normalization constant (3.31)

\[
h_1(x_{\ell}) = \int q_1(x_N) \prod_{k=m_1+n} G_{k,k-1}(x_k, x_{k-1}) \rho_{\text{per}}(m_1 + n - 1, x_{m_1+n-1}; \ell, \delta_{x_{\ell}}) dx_{m_1+n-1,N},
\]

\[
h_2(x_{\ell}) = \int \tilde{\rho}_{\text{per}}(\ell, \delta_{x_{\ell}}; m_1, x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) p_2(x_0) dx_{0,m_1},
\]

and the densities (3.32)

\[
p_1(x_{m_1+n-1,N}, x_{\ell}) = h_1(x_{\ell})^{-1} q_1(x_N) \prod_{k=m_1+n} G_{k,k-1}(x_k, x_{k-1})
\]

\[
\times \rho_{\text{per}}(m_1 + n - 1, x_{m_1+n-1}; \ell, \delta_{x_{\ell}}),
\]

\[
p_2(x_{\ell}, x_{0,m_1}) = h_2(x_{\ell})^{-1} \tilde{\rho}_{\text{per}}(\ell, \delta_{x_{\ell}}; m_1, x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) p_2(x_0).
\]

Using the above notations, one can rewrite (3.29) as

\[
\int \mu_N(x; q_1, q_2) dx_{m_1+1, \ell-1} dx_{\ell+1,m_1+n-2}
\]

\[
= p_1(x_{m_1+n-1,N}, x_{\ell}) p_2(x_{\ell}, x_{0,m_1}) p(x_{\ell}),
\]
where \( p(\cdot) \) is a density that takes the form

\[
p(x_\ell) = G_{N,0}(\varrho_1, \varrho_2)^{-1} h_1(x_\ell) h_2(x_\ell) \times \left( \int G_{m_1+n-1, \ell}(y, x_\ell) dy \right) \left( \int G_{\ell, m_1}(x_\ell, y) dy \right).
\]

It is clear that \( p \) is the marginal density of \( x_\ell \), since

\[
\int \mu_N(x; \varrho_1, \varrho_2) dx_{0, \ell-1} dx_{\ell+1, N} = p(x_\ell).
\]

In this way, the expectation of the product is rewritten as (3.33)

\[
E[FG] = \mathbb{E}_g \mathbb{E}_f \mathbb{E}_x \mathbb{E}_y \left[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1+n}, \ldots, \eta_{m_2+n}) \right] \times p_1(x_{m_1+n-1, N}) p_2(x_\ell, x_{0, m_1}) p(x_\ell) dx_{m_1+n-1, N} dx_{0, m_1} dx_\ell.
\]

Step 2. Rewriting \( E[F] E[G] \). Let \( \varrho_3, \varrho_4 \) be sampled independently from \( \pi_\infty \), which are also independent from \( \varrho_1, \varrho_2 \) and the random environment. Define (3.34)

\[
p_3(x_{m_1+n-1, N}) = \frac{\varrho_1(x_N) \prod_{k=1}^{N} G_{k, k-1}(x_k, x_{k-1}) \rho_{\text{per}}(m_1 + n - 1, x_{m_1+n-1}; \ell, \varrho_3)}{\int \varrho_1(x_N') \prod_{k=1}^{N} G_{k, k-1}(x_k', x_{k-1}) \rho_{\text{per}}(m_1 + n - 1, x_{m_1+n-1}'; \ell, \varrho_3) dx_{m_1+n-1, N}},
\]

\[
p_4(x_{0, m_1}) = \frac{\hat{\rho}_{\text{per}}(\ell, \varrho_4; m_1, x_{m_1}) \prod_{k=1}^{m_1} G_{k, k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \hat{\rho}_{\text{per}}(\ell, \varrho_4; m_1, x_{m_1}') \prod_{k=1}^{m_1} G_{k, k-1}(x_k', x_{k-1}') \varrho_2(x_0') dx_{0, m_1}}.
\]

In other words, in the expressions of \( p_1, p_2 \), we have replaced \( \delta_x \) with \( \varrho_3, \varrho_4 \) to obtain \( p_3, p_4 \) respectively. Now it is straightforward to check that (3.35)

\[
E[F] E[G] = \mathbb{E}_g \mathbb{E}_f \mathbb{E}_x \left[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1+n}, \ldots, \eta_{m_2+n}) \right] \times p_3(x_{m_1+n-1, N}) p_4(x_{0, m_1}) p(x_\ell) dx_{m_1+n-1, N} dx_{0, m_1} dx_\ell.
\]

In the above expression, the term \( p(x_\ell) \) actually plays no role since one can integrate it out and \( \int p(x_\ell) dx_\ell = 1 \) – we kept it there to compare with the expression of \( E[FG] \).

Combining (3.33) and (3.35), we have (3.36)

\[
\text{Cov}[F, G] = \mathbb{E}_g \mathbb{E}_f \mathbb{E}_x \left[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1+n}, \ldots, \eta_{m_2+n}) \right] \times \left[ p_1(x_{m_1+n-1, N}) p_2(x_\ell, x_{0, m_1}) - p_3(x_{m_1+n-1, N}) p_4(x_{0, m_1}) \right] \times p(x_\ell) dx_{m_1+n-1, N} dx_{0, m_1} dx_\ell.
\]
Step 3. Approximation. Now we decompose \( \text{Cov}[F, G] = \text{Err}_1(n) + \text{Err}_2(n) \), with

\[
\text{Err}_1(n) = \mathbb{E}_g \mathbb{E} \int \mathbb{E}_\mathbf{x} \left[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1 + n}, \ldots, \eta_{m_2 + n}) \right] \times \left[ p_1(\mathbf{x}_{m_1 + n - 1, N}, x_\ell) - p_3(\mathbf{x}_{m_1 + n - 1, N}) \right] \mathbf{p}(x_\ell) d\mathbf{x}_{m_1 + n - 1, N} d\mathbf{x}_{0, m_1} d\mathbf{x}_\ell,
\]
and

\[
\text{Err}_2(n) = \mathbb{E}_g \mathbb{E} \int \mathbb{E}_\mathbf{x} \left[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1 + n}, \ldots, \eta_{m_2 + n}) \right] \times p_3(\mathbf{x}_{m_1 + n - 1, N}) \left[ p_2(x_\ell, \mathbf{x}_{0, m_1}) - p_4(\mathbf{x}_{0, m_1}) \right] \mathbf{p}(x_\ell) d\mathbf{x}_{m_1 + n - 1, N} d\mathbf{x}_{0, m_1} d\mathbf{x}_\ell.
\]

It suffices to show that \( |\text{Err}_i(n)| \leq C e^{-\lambda n} \| F \|_{L^2} \| G \|_{L^2} \) for \( i = 1, 2 \). The two cases are handled in the same way, so we will only focus on \( \text{Err}_1(n) \).

The rest of the proof is very similar to that of Proposition 3.4. First, from (3.32) and (3.34) we have

\[
|p_1(\mathbf{x}_{m_1 + n - 1, N}, x_\ell) - p_3(\mathbf{x}_{m_1 + n - 1, N})| \leq I_1 + I_2,
\]
with

\[
I_1 = \frac{\varrho_1(x_N) \prod_{k=m_1+n}^N G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x_N') \prod_{k=m_1+n}^N G_{k,k-1}(x_{k}', x_{k-1}') d\mathbf{x}_{m_1 + n - 1, N} \inf_y \rho_{\text{per}}(m_1 + n - 1, y; \ell, \delta_x)} \times \sup_y \rho_{\text{per}}(m_1 + n - 1, y; \ell, \delta_x),
\]

\[
I_2 = \frac{\varrho_1(x_N) \prod_{k=m_1+n}^N G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x_N') \prod_{k=m_1+n}^N G_{k,k-1}(x_{k}', x_{k-1}') d\mathbf{x}_{m_1 + n - 1, N} \inf_y \rho_{\text{per}}(m_1 + n - 1, y; \ell, \delta_x)} \times \sup_y \rho_{\text{per}}(m_1 + n - 1, y; \ell, \delta_x),
\]

Thus, \( \text{Err}_1(n) \) can be bounded from above by

\[
\text{Err}_1(n) \leq \mathbb{E}_g \mathbb{E} \int \mathbb{E}_\mathbf{x} \left[ f(\eta_1, \ldots, \eta_{m_1}) g(\eta_{m_1 + n}, \ldots, \eta_{m_2 + n}) \right] \times \left[ I_1 + I_2 \right] p_2(x_\ell, \mathbf{x}_{0, m_1}) \mathbf{p}(x_\ell) d\mathbf{x}_{m_1 + n - 1, N} d\mathbf{x}_{0, m_1} d\mathbf{x}_\ell =: J_1 + J_2.
\]

Consider the term \( J_1 \). We first bound \( I_1 \) by

\[
I_1 \leq \frac{\varrho_1(x_N) \prod_{k=m_1+n}^N G_{k,k-1}(x_k, x_{k-1})}{\int \varrho_1(x_N') \prod_{k=m_1+n}^N G_{k,k-1}(x_{k}', x_{k-1}') d\mathbf{x}_{m_1 + n - 1, N} \inf_y \rho_{\text{per}}(m_1 + n - 1, y; \ell, \delta_x)} \times \sup_{y, x} \rho_{\text{per}}(m_1 + n - 1, y; \ell, \delta_x),
\]

Using (3.32), we bound \( p_2(x_\ell, \mathbf{x}_{0, m_1}) \) by

\[
p_2(x_\ell, \mathbf{x}_{0, m_1}) \leq \frac{\prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) \varrho_2(x_0)}{\int \prod_{k=1}^{m_1} G_{k,k-1}(x_k', x_{k-1}') \varrho_2(x_0') d\mathbf{x}_{0, m_1}} \times \sup_{x,y} \varrho_{\text{per}}(\ell, \delta_x; m_1, y).
\]
Applying Proposition 2.1, we have

Applying the Cauchy-Schwarz inequality, we have

In this way we got rid of the dependence on $x_{\ell}$ in all other terms except for $p(x_{\ell})$ which can be integrated out. By the independence of the random environment in separate time intervals, we obtain $J_1 \leq \prod_{i=1}^{4} K_i$, with

$$K_1 := E_0 E \int \mathbb{E}_x |f(\eta_1, \ldots, \eta_{m_1})| \frac{\prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) g_2(x_0)}{\prod_{k=1}^{m_1} G_{k,k-1}(x_k', x_{k-1}') g_2(x_0') dx'_{0,m_1}} dx_{0,m_1},$$

$$K_2 := E_0 E \int \mathbb{E}_x |g(\eta_{m1+n}, \ldots, \eta_{m2+n})| \times \frac{g_1(x_N) \prod_{k=m1+n}^{N} G_{k,k-1}(x_k, x_{k-1})}{\int g_1(x'_N) \prod_{k=m1+n}^{N} G_{k,k-1}(x'_k, x'_{k-1}) dx'_{m1+n-1,N}} dx_{m1+n-1,N},$$

$$K_3 := E_0 E \sup_{y,x} |\rho_{per}(m_1 + n - 1, y; \ell, \delta_2) - \rho_{per}(m_1 + n - 1, y; \ell, \delta_2)|,$$

$$K_4 := E_0 E \sup_{x,y} \bar{\rho}_{per}(\ell, \delta_2; m_1, y) \bar{\inf}_{x,y} \bar{\rho}_{per}(\ell, \delta_2; m_1, y).$$

Applying Proposition 2.1, we have $K_3 \leq C e^{-\lambda n}$ and $K_4 \leq C$. For $K_1$, we can bound it from above by

$$K_1 \leq E_0 E \int \mathbb{E}_x |f(\eta_1, \ldots, \eta_{m_1})| \times \frac{g_4(x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) g_2(x_0)}{\int g_4(x'_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) g_2(x_0') dx'_{0,m_1}} dx_{0,m_1} \times \sup_y g_4(y).$$

Applying the Cauchy-Schwarz inequality, we have

$$K_1^2 \leq C E_0 E \left( \int \mathbb{E}_x |f(\eta_1, \ldots, \eta_{m_1})| \times \frac{g_4(x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) g_2(x_0)}{\int g_4(x'_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) g_2(x_0') dx'_{0,m_1}} dx_{0,m_1} \right)^2.$$

The r.h.s. can be further bounded from above by

$$C E_0 E \int \mathbb{E}_x |f(\eta_1, \ldots, \eta_{m_1})|^2 \times \frac{g_4(x_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x_k, x_{k-1}) g_2(x_0)}{\int g_4(x'_{m_1}) \prod_{k=1}^{m_1} G_{k,k-1}(x'_k, x'_{k-1}) g_2(x_0') dx'_{0,m_1}} dx_{0,m_1} = C \| F \|^2_{L^2}.$$

The same proof shows that $K_2 \leq C \| G \|_{L^2}$. So we have $J_1 \leq C e^{-\lambda n} \| F \|_{L^2} \| G \|_{L^2}$. The term $J_2$ is dealt with in the same way, and this combines to show that $Err_1(n) \leq C e^{-\lambda n} \| F \|_{L^2} \| G \|_{L^2}$. Since $Err_2(n)$ is proved in exactly the same way, we complete the proof of the proposition. \(\square\)

4. Proof of Theorem 1.1

Recall the goal was to show that, as $T \to \infty$, $\frac{V}{\sqrt{F}}$ converges to a nondegenerate Gaussian distribution under the measure $P \otimes \hat{P}_T$. As we have already
observed in Section 3.4 the laws of \( \frac{w_N}{\sqrt{N}} \) under the measure \( P \otimes \hat{\mathbb{P}}_N \) weakly converge to \( N(0, \sigma^2_{\text{eff}}) \), as \( N \to \infty \). In this section we show how to extend the conclusion to the laws of \( \frac{w_T}{\sqrt{T}} \), under the measure \( P \otimes \hat{\mathbb{P}}_T \), when \( T \to \infty \) (not necessarily taking integer values).

For a general \( T > 0 \), define \( N := \lfloor T \rfloor \). Let \( X_1(T) \) and \( X_2(T) \) be random variables with the same laws as \( w_N \) under \( P \otimes \hat{\mathbb{P}}_N \) and \( w_T \) under \( P \otimes \hat{\mathbb{P}}_T \), respectively. We have the following lemma, which is the last piece that is needed to complete the proof of Theorem 1.1.

**Lemma 4.1.** There exists \( C > 0 \) such that for any \( T \geq 1 \) one can find a coupling \( \left( X_1(T), X_2(T) \right) \) such that

\[
E \left| X_2(T) - X_1(T) \right|^2 \leq C.
\]

**Proof.** To lighten the notation we write \( X_1 \) and \( X_2 \), instead of \( X_1(T) \) and \( X_2(T) \). Recall that \( \rho(T, \cdot) \) is the density of \( w_T \) under \( \hat{\mathbb{P}}_T \), and we have the relation

\[
\rho(T, x) = \int Z_{T,N}(x, y) \rho(N, y) dy \int Z_{T,N}(x', y') \rho(N, y') dx' dy'.
\]

From (4.2), we know that, given the value of \( X_1 \), if we sample \( X_2 \) from the density \( \int Z_{T,N}(x, \cdot) dx \), then \( X_2 \) has the same law as \( w_T \) under \( P \otimes \hat{\mathbb{P}}_T \). By the construction of \( X_1, X_2 \), we have for any \( y \in \mathbb{R} \) that

\[
E[|X_2 - X_1|^2 | X_1 = y] = E \frac{\int (x-y)^2 Z_{T,N}(x, y) dx}{\int Z_{T,N}(x', y) dx'}.
\]

By the Cauchy-Schwarz inequality, we have

\[
E[|X_2 - X_1|^2 | X_1 = y] \leq \int (x-y)^2 \sqrt{EZ_{T,N}(x, y)^2} dx \sqrt{E(\int Z_{T,N}(x', y) dx')^{-2}}.
\]

For the first term on the r.h.s., we apply Lemma 3.3 to derive (note that \( T - N < 1 \))

\[
\int (x-y)^2 \sqrt{EZ_{T,N}(x, y)^2} dx \leq C.
\]

For the second term, we have for each fixed \( y \) that

\[
\int Z_{T,N}(x', y) dx' \text{law} \equiv v_1(T - N, 0),
\]

with \( v_1 \) solving the SHE (2.6), with the initial data \( v_1(0, x) \equiv 1 \). Thus, applying [8, Lemma B.7], we have \( E(\int Z_{T,N}(x', y) dx')^{-2} \leq C \). The proof is complete. \( \square \)
5. NONDEGENERACY OF THE DIFFUSION CONSTANT

The goal of this section is to show that \( \sigma^2_{\kappa_\ell} \neq 0 \), which is a nontrivial fact. In general, it needs not be true that \( \sum_{j \in \mathbb{Z}} r(j) > 0 \) when \( r(\cdot) \) is the covariance function of a stationary sequence. We will need to make use of some specific structure of our model.

We first show a stability result on the approximation of the diffusion constant. Define

\[
\sigma^2_N := \frac{1}{N} \mathbb{E} \left( \sum_{k=1}^{N} \eta_k \right)^2 \mu_N(x; m, \delta_0) dx_{0,N} = \frac{1}{N} \mathbb{E} \hat{\mathbb{E}}_N |w_N|^2,
\]

and

\[
\tilde{\sigma}^2_N := \frac{1}{N} \mathbb{E}_\rho \hat{\mathbb{E}}_\theta \mathbb{E} \left( \sum_{k=1}^{N} \eta_k \right)^2 \mu_N(x; \tilde{\rho}, \theta) dx_{0,N} = \frac{1}{N} \mathbb{E} \left( \sum_{k=1}^{N} \eta_k \right)^2.
\]

We have

**Proposition 5.1.** \( \sigma^2_N - \tilde{\sigma}^2_N \to 0 \) as \( N \to \infty \).

**Proof.** The proof is similar to that of Propositions 3.4 and 3.7, so we only sketch the argument here.

First, for some \( k_N \to \infty \), yet to be determined, we decompose \( \sigma^2_N = \sum_{\ell=1}^{4} I_\ell \) and \( \tilde{\sigma}^2_N = \sum_{\ell=1}^{4} J_\ell \), with

\[
I_\ell = \frac{1}{N} \sum_{(i,j) \in A_\ell} \mathbb{E} \int_{T^{N+1}} \left( \mathbb{E}_\kappa \eta_i \eta_j \right) \mu_N(x; m, \delta_0) dx_{0,N},
\]

\[
J_\ell = \frac{1}{N} \sum_{(i,j) \in A_\ell} \mathbb{E}_\rho \hat{\mathbb{E}}_\theta \mathbb{E} \int_{T^{N+1}} \left( \mathbb{E}_\kappa \eta_i \eta_j \right) \mu_N(x; \tilde{\rho}, \theta) dx_{0,N},
\]

with

\[
A_1 = B_1 \times B_1, \quad A_2 = B_2 \times B_2,
\]

\[
A_3 = B_1 \times B_2, \quad A_4 = B_2 \times B_1,
\]

and \( B_1 = [k_N, N-k_N] \) and \( B_2 = [1, k_N-1] \cup [N-k_N+1, N] \). In the following, we will show that \( I_\ell - J_\ell \to 0 \), for each \( \ell = 1, \ldots, 4 \).

**The case of** \( \ell = 1 \). By following closely the proof of Proposition 3.4 and with the help of Lemma 3.2, we derive that, for each \( (i, j) \in A_1 \),

\[
\left| \mathbb{E} \int \left( \mathbb{E}_\kappa \eta_i \eta_j \right) \mu_N(x; m, \delta_0) dx_{0,N} - \mathbb{E}_\rho \hat{\mathbb{E}}_\theta \mathbb{E} \int \left( \mathbb{E}_\kappa \eta_i \eta_j \right) \mu_N(x; \tilde{\rho}, \theta) dx_{0,N} \right| \leq C e^{-\lambda k_N},
\]

which implies that

\[
|I_1 - J_1| \leq CN^{-1}N^2 e^{-\lambda k_N} = CN e^{-\lambda k_N}.
\]

**The case of** \( \ell = 2 \). In this case, we directly apply Lemma 3.2 to derive that

\[
|I_2 - J_2| \leq CN^{-1}k_N^2.
\]
The case of $\ell = 3$ and 4. By symmetry, we only need to consider $\ell = 3$. First, for $J_3$, we have

$$J_3 = \frac{1}{N} \sum_{(i,j) \in A_3} E(\eta_i \eta_j)$$

By Proposition 3.7, we know that $|E(\eta_i \eta_j)| \leq Ce^{-\lambda|i-j|}$, and this implies that

$$|J_3| \leq CN^{-1} \sum_{i \in B_1, j \in B_2} e^{-\lambda|i-j|} \leq CN^{-1} k_N.$$ 

It remains to study $I_3$, which we rewrite as

$$I_3 = \frac{1}{N} \sum_{(i,j) \in A_3} \left( E \int (E_x \eta_i \eta_j) \mu_N(x; m, \delta_0) dx_{0,N} - \prod_{k=i,j} E \int (E_x \eta_k) \mu_N(x; m, \delta_0) dx_{0,N} \right)$$

$$+ \frac{1}{N} \sum_{(i,j) \in A_3} \prod_{k=i,j} (E_x \eta_k) \mu_N(x; m, \delta_0) dx_{0,N} =: I_{31} + I_{32}.$$ 

For the term $I_{31}$, by following the proof for Proposition 3.7, one can show that

$$|I_{31}| \leq CN^{-1} \sum_{(i,j) \in A_3} e^{-\lambda|i-j|} \leq CN^{-1} k_N.$$ 

For the other term, since $(i, j) \in A_3$, we have $i \in [k_N, N - k_N]$. By following the proof for Proposition 3.4, one can show that

$$|E \int (E_x \eta_i) \mu_N(x; m, \delta_0) dx_{0,N} - E \int (E_x \eta_i) \mu_N(x; \tilde{q}, q) dx_{0,N}| \leq Ce^{-\lambda k_N}.$$ 

We have

$$E \int (E_x \eta_i) \mu_N(x; \tilde{q}, q) dx_{0,N} = E \eta_i = 0.$$ 

We can apply Lemma 3.2 again and conclude

$$|I_{32}| \leq CN^{-1} \sum_{(i,j) \in A_3} e^{-\lambda k_N} \leq C k_N e^{-\lambda k_N}.$$ 

To summarize, we have

$$|\sigma_N^2 - \sigma_N^{2, \text{eff}}| \leq \sum_{\ell=1}^4 |I_\ell - J_\ell| \leq C \left( Ne^{-\lambda k_N} + N^{-1} k_N^2 + N^{-1} k_N + k_N e^{-\lambda k_N} \right).$$

Choosing $k_N = N^\alpha$ for any $\alpha \in (0, 1/2)$, the proof is complete. \(\square\)

Since

$$\sigma_N^2 = \frac{1}{N} E(\sum_{k=1}^N \eta_k)^2 \to \sigma_{\text{eff}}^2, \quad \text{as} \ N \to \infty,$$

applying Proposition 5.1, we derive that, as $N \to \infty$,

$$\sigma_N^2 \to \sigma_{\text{eff}}^2.$$ 

The following proposition completes the proof of the nondegeneracy of $\sigma_{\text{eff}}^2$.
Proposition 5.2. We have
\[
\lim\inf_{N \to \infty} \sigma_N^2 \geq 1.
\]
As a result, \(\sigma_{\text{eff}}^2 \geq 1\).

Proof. By definition, we have
\[
\sigma_N^2 = \frac{1}{N} \mathbb{E} \hat{\mathbb{E}}_N [w_N]^2.
\]
Since \(|w_N| \leq \|w_N\| + 1\), we have via a triangle inequality that
\[
\sqrt{\frac{1}{N} \mathbb{E} \hat{\mathbb{E}}_N [w_N]^2} \leq \sqrt{\frac{1}{N} \mathbb{E} \hat{\mathbb{E}}_N [w_N]^2} + \sqrt{\frac{1}{N}}.
\]
Sending \(N \to \infty\) and applying Lemma 5.3 below, the proof is complete. \(\square\)

Lemma 5.3. For any \(N \in \mathbb{Z}_+\), we have
\[
(5.3) \quad \mathbb{E} \hat{\mathbb{E}}_N w_N^2 \geq N.
\]

Proof. Recall that \(\rho(N, \cdot)\) is the density of \(w_N\) under the quenched polymer measure \(\hat{\mathbb{P}}_N\), we have
\[
\mathbb{E} \hat{\mathbb{E}}_N w_N^2 = \mathbb{E} \int x^2 \rho(N, x) dx \geq \mathbb{E} \left[ \int x^2 \rho(N, x) dx - (\int x \rho(N, x) dx)^2 \right].
\]
Note that the last expression is just the average of the quenched variance. It remains to show
\[
(5.4) \quad \mathbb{E} \left[ \int x^2 \rho(N, x) dx - (\int x \rho(N, x) dx)^2 \right] = N,
\]
which is a standard folklore for the directed polymer when the random environment is statistically invariant under shear transformations. We sketch the argument below.

For any \(\theta \in \mathbb{R}\), consider the solution to SHE
\[
\begin{align*}
\partial_t Z_\theta(t, x) &= \frac{1}{2} \Delta Z_\theta(t, x) + \beta \xi(t, x) Z_\theta(t, x), \quad t > 0, \\
Z_\theta(0, x) &= e^{\theta x},
\end{align*}
\]
then we know that, for fixed \(N\),
\[
\int x^2 \rho(N, x) dx - (\int x \rho(N, x) dx)^2 \xrightarrow{\text{law}} \partial_\theta^2 \log \mathbb{E} \hat{\mathbb{E}}_N [w_N] \bigg|_{\theta = 0}.
\]
For a proof of this fact, we refer to e.g. [9, Eq. (2.15)]. Since \(\xi\) is a Gaussian process that is white in time and stationary in space, we have
\[
(5.6) \quad \{\xi(t, x)\}_{t,x} \xrightarrow{\text{law}} \{\xi(t, x + \theta t)\}_{t,x},
\]
and this implies
\[
(5.7) \quad \{Z_\theta(t, x)\}_{t,x} \xrightarrow{\text{law}} \{Z_0(t, x + \theta t) e^{\theta x + \frac{1}{2} \theta^2 t}\}_{t,x},
\]
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which comes from the fact that \( Z_0(t, x + \theta t) e^{\theta x + \frac{1}{2} \theta^2 t} \) solves (5.5) with \( \{\xi(t, x)\} \) replaced by \( \{\xi(t, x + \theta t)\} \). Therefore, we have

\[
E \log Z_\theta(N, 0) = \frac{1}{2} \theta^2 N + E \log Z_0(N, \theta N)
\]

\[
= \frac{1}{2} \theta^2 N + E \log Z_0(N, 0),
\]

where the last step comes from the fact that \( Z_0(N, \cdot) \) is a stationary random field. Therefore, we have

\[
E \partial^2_\theta \log Z_\theta(N, 0)|_{\theta=0} = N.
\]

The proof is complete. \( \square \)

6. FURTHER DISCUSSION

We list two problems here.

Quenched behavior. Theorem 1.1 concerns the behavior of \( w_T \) under the annealed measure \( P \otimes \hat{P}_T \), and our approach does not give information on the quenched behavior. For the problem on the whole line with no periodic structure, the annealed and quenched behaviors are quite different, due to the localization phenomenon. It would be interesting to study the quenched asymptotics of \( w_T \) in our setting.

Relation between two diffusion constants. Recall that \( Z_T \) is the partition function formally defined in (1.1). It was shown in [8], under the same assumption as here, that the free energy \( \log Z_T \) satisfies a central limit theorem: there exists \( \gamma, \Sigma > 0 \) such that under \( P \),

\[
\frac{\log Z_T + \gamma T}{\sqrt{T}} \Rightarrow N(0, \Sigma), \quad \text{as } T \to \infty.
\]

Different expressions of \( \Sigma \) were derived, see [8, Eq. (5.58)] which involves the solution to an abstract cell problem and [7, Eq. (2.10)] which takes the form of an average of a Brownian bridge functional. A surprising relation between \( \Sigma \) and \( \sigma^2_{\text{eff}} \) was derived by Brunet through the replica method, see [2, Eq. (20)]. It is unclear at all why they should be related, and we believe it is an important problem to unravel the connection here.

APPENDIX A. PROOF OF LEMMA 3.3

We start with the following.

Lemma A.1. For each \( t > s \) and \( y \in \mathbb{R} \), the process \( \left\{ Z_{t,s}(x, y)/q_{t-s}(x-y) \right\}_{x \in \mathbb{R}} \) is stationary.
Proof. The argument is rather standard, so we only sketch the proof. We consider first the case when $\xi_R(t, x)$ is a 1-periodic Gaussian noise that is white in time and colored in space, with the covariance function $R(\cdot) \in C^\infty(\mathbb{T})$. Recall that $q_t(x) = (2\pi t)^{-1/2} \exp(-x^2/2t)$ denotes the standard heat kernel. The propagator of equation (2.1), corresponding to this noise, shall be denoted by $Z_{t,s}^{(R)}(x, y)$ and is given by the formula

$$Z_{t,s}^{(R)}(x, y) = \frac{q_t(x) - q_t(y)}{2\beta} \mathbb{E} \left[ \exp \left\{ \beta \int_s^t \xi_R(\sigma, B_{s,t}^{y,x}(\sigma)) \, d\sigma - \frac{1}{2} \beta^2 R(0)(t - s) \right\} \right],$$

where $(B_{s,t}^{y,x}(\sigma))_{s \leq \sigma \leq t}$ is the Brownian bridge between $(s, y)$ and $(t, x)$. It is clear from the above formula that $\{Z_{t,s}^{(R)}(x, y)/q_t(x - y)\}_{x \in \mathbb{R}}$ is stationary, using the fact that $\xi_R$ also satisfies the relation (5.6). The conclusion can be extended to the case of the Gaussian space-time white noise by approximation of $\delta(x - y)$ by a sequence of $R_n(\cdot) \in C^\infty(\mathbb{T})$ as $n \to \infty$. □

The end of the proof of Lemma 3.3. Using Lemma A.1 and the definition of the propagator of the SHE on a torus, see (2.4), we can write

$$\mathbb{E} Z_{t,0}(x, 0)^p = \left( \frac{q_t(x)}{q_t(0)} \right)^p \mathbb{E} Z_{t,0}(0, 0)^p \leq \left( \frac{q_t(x)}{q_t(0)} \right)^p \mathbb{E} G_{t,0}(0, 0)^p.$$

Applying [8, Lemma B.1], we complete the proof.

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(Yu Gu) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA

(Tomasz Komorowski) INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, ul. Śniadeckich 8, 00-656, WARSAW, POLAND