Quantum degrees of polarization

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Abstract

We discuss different proposals for the degree of polarization of quantum fields. The simplest approach, namely making a direct analogy with the classical description via the Stokes operators, is known to produce unsatisfactory results. Still, we argue that these operators and their properties should be basic for any measure of polarization. We compare alternative quantum degrees and put forth that they order various states differently. This is to be expected, since, despite being rooted in the Stokes operators, each of these measures only captures certain characteristics. Therefore, it is likely that several quantum degrees of polarization will coexist, each one having its specific domain of usefulness.

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1. Introduction

Far from a source, all propagating electromagnetic fields can be treated, to a very good approximation, like plane waves. As a consequence, for a monochromatic component at a fixed space point, the tip of the electric field vector describes an ellipse in the plane transverse to the propagating direction. This geometric observation has led to the concept of light polarization, which was laid down already in 1852 in the seminal work of Stokes \cite{stokes1852}

Polarization is important in a variety of classical optical phenomena, which are used in many applications including remote sensing \cite{remote_sensing}, light scattering \cite{light_scattering}, thin-film ellipsometry \cite{thin_film_ellipsometry}, and near-field microscopy \cite{near_field_microscopy}, to cite only some relevant examples.

The description of polarization for quantum fields has also attracted a great deal of attention in recent years \cite{quantum_polarization_1, quantum_polarization_2, quantum_polarization_3, quantum_polarization_4}, mainly due to the rapid growth of quantum information science. Light is an excellent information carrier as the coded information remains relatively intact upon propagation, since photons are very resilient against (unwanted) interactions with the environment. For example, for visible light at room temperature, the ratio $\hbar \omega / k_B T$ is approximately equal to 80, so thermal noise is negligible. In addition, in an optical fiber, the absorption is only about 50% per 10 km of propagation distance at wavelengths around 1.55 µm.

Since photon polarization is a property that can be accurately, rapidly, and almost losslessly manipulated, it is the variable of choice in many experiments and demonstrations in quantum optics. Examples include quantum key distribution \cite{quantum_key_distribution}, quantum dense coding \cite{quantum_dense_coding}, polarization entanglement \cite{polarization_entanglement}, quantum teleportation \cite{quantum_teleportation}, quantum tomography \cite{quantum_tomography}, rotationally invariant states \cite{rotationally_invariant_states}, and phase super-resolution \cite{phase_super_resolution}.

All this seems to call for a proper description and quantification of polarization for quantum fields: our aim here is to make, at least, a rudimentary overview of such recent developments.

The paper is organized as follows. In Section 2, we describe the simplest “translation” from a classical to a quantum description in terms of Stokes operators. We point out the problems arising in this approach, mainly due to the fact that a large set of states are classified as unpolarized, although they carry some polarization information. In consequence, we delineate the conditions for the appearance of this “hidden” polarization. In Section 3, we discuss criteria and desiderata for any quantum measure of polarization. We apply them in Section 4 to some distance-based measures, examining how they may be modified to avoid potential shortcomings. In terms of these new measures, we investigate the degree of polarization for maximally polarized pure states. For completeness, we also treat other non-distance-based degrees of polarization. In Section 5, we speculate about how nonlinear transformations would affect some aspects of this picture. Finally, in Section 6, we round off the exposé with some general remarks and conclusions.

2. Stokes description of polarization

2.1. Stokes parameters and operators

Let us start by briefly discussing some basic concepts about classical and quantum polarization. We assume a monochromatic plane wave, whose electric field lies in the plane perpendicular to its direction of propagation. Under these conditions, the field can be represented by two complex amplitudes denoted by $E_H$ and $E_V$ when using the basis of linear horizontal and vertical polarizations. The Stokes parameters are then defined as

\begin{align*}
S_0 &= E_H^\ast E_H + E_V^\ast E_V = |E_H|^2 + |E_V|^2 = 1,
S_1 &= E_H^\ast E_V + E_V^\ast E_H = 2\text{Re}(E_H^\ast E_V),
S_2 &= (E_H^\ast E_H - E_V^\ast E_V) / 2,
S_3 &= (E_H^\ast E_V - E_V^\ast E_H) / 2,
S_4 &= E_H^\ast E_H - E_V^\ast E_V.
\end{align*}
\[
S_0 = E_H^* E_H + E_V^* E_V, \quad S_x = E_H E_V + E_V^* E_H, \quad S_z = E_H E_H - E_V^* E_V.
\]

(1)

In case of stochastic fields, one usually uses the average values given by the corresponding statistical mixture of deterministic waves. For quantum fields, the amplitudes \( E_H \) and \( E_V \) are represented by complex amplitude operators, denoted by \( \hat{a}_H \) and \( \hat{a}_V \). They obey the bosonic commutation relations

\[
[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}, \quad j, k \in \{H, V\}.
\]

(2)

The Stokes operators are subsequently introduced as the quantum counterparts of the classical variables, namely

\[
\hat{S}_0 = \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V, \quad \hat{S}_x = \hat{a}_H \hat{a}_V^\dagger + \hat{a}_V^\dagger \hat{a}_H, \quad \hat{S}_z = \hat{a}_H \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V^\dagger,
\]

(3)

and their mean values correspond to the Stokes parameters \( \langle \hat{S}_0 \rangle, \langle \hat{S}_x \rangle, \langle \hat{S}_z \rangle \). The Stokes operators satisfy the SU(2)-like commutation relations:

\[
[\hat{S}_x, \hat{S}_y] = 2i\hat{S}_z,
\]

(4)

and cyclic permutations. The noncommutability of these operators precludes the simultaneous exact measurement of the corresponding physical quantities. Among other consequences, this implies that no field state (leaving aside the two-mode vacuum) can have definite nonfluctuating values of all the Stokes operators simultaneously. This is expressed by the uncertainty relation

\[
(\Delta S)^2 = (\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \geq 2\langle \hat{S}_0 \rangle,
\]

(5)

where the variances are \( \langle \Delta X \rangle^2 = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \). This reflects the fact that, contrary to the classical optics description, the electric field of a monochromatic field never describes a definite ellipse in its quantized description.

Using Stokes operators, the standard degree of polarization employed in classical optics can be generalized to quantum fields through the definition

\[
P_S = \frac{\langle |\hat{S}_0\rangle |^2 + \langle |\hat{S}_x\rangle |^2 + \langle |\hat{S}_z\rangle |^2}{\langle \hat{S}_0 \rangle}.
\]

(6)

We will refer to this definition as the Stokes degree of polarization, since it is the length of the normalized Stokes vector (so \( 0 \leq P_S \leq 1 \)). Expression (6) is undefined for \( \langle \hat{S}_0 \rangle = 0 \), i.e., when both modes are in the vacuum state. However, in order to simplify our discussion below, we complement definition (6) with \( P_S \equiv 0 \) for the two-mode vacuum. We also note in passing that \( P_S \) depends exclusively on the first moments of the Stokes operators. However, it follows from the relation

\[
\hat{S}^2 = \hat{S}_0(\hat{S}_0 + 2)
\]

that \( P_S \) can be recast as

\[
P_S = \frac{\sqrt{\langle \hat{S}_0(\hat{S}_0 + 2) \rangle} - (\Delta S)^2}{\langle \hat{S}_0 \rangle},
\]

(8)

which shows that it can be expressed either in terms of the average Stokes vector or its fluctuations.

We observe that \( \hat{S}_0 = \hat{N}_H + \hat{N}_V \), where \( \hat{N}_H \) (\( \hat{N}_V \)) is the photon number operator in mode \( H \) (\( V \)) and that

\[
[\hat{S}_0, \hat{S}_i] = 0,
\]

(9)

so each energy manifold can be treated separately. To bring out this point more clearly, it is advantageous to relabel the standard two-mode Fock basis as

\[
|k, N - k \rangle = |k \rangle_H \otimes |N - k \rangle_V, \quad k = 0, 1, \ldots, N,
\]

(10)

so that, for each fixed total number of photons \( N \), these states span an SU(2) invariant subspace of dimension \( N + 1 \).

2.2. Hidden polarization

As noticed early on, there are problems with the definition (6). For example, this approach assigns zero degree of polarization to pure fields that carry polarization information. This is referred to as “hidden polarization” [6, 7], but perhaps it would be better to say that such states have higher-order polarization. Classical fields can also have significant higher-order polarization correlations. There are, for example, stochastic classical fields that can be seen as statistical mixtures of fully polarized states, and simultaneously be unpolarized according to its average Stokes vector. In order to fully characterize such classical mixtures, one would need higher-order moments of the Stokes operators.

In the literature it becomes quite clear that the vast majority of physicists view the classical counterpart of Eq. (6) as the degree of polarization of a plane-wave classical field. In the quantum physics community it has been common to measure higher-order moments, and hence, the inadequacies of the definition (6) has been more visible in this community. However, \( P_S \) assigns a relevant degree of polarization to every pure state in classical optics, whereas this is not the case in quantum optics.

For a state to have \( P_S = 0 \), the expectation values of the Stokes vector \( \hat{S} \) must vanish. To derive the set of pure \( N \)-photon states that are unpolarized according to the Stokes definition, let

\[
|\Psi_N \rangle = \sum_{k=0}^{N} c_k |k, N - k \rangle,
\]

(11)

denote a general, normalized, pure \( N \)-photon state. Since \( \langle \hat{a}_H \hat{a}_V^\dagger \rangle = \langle \hat{a}_H \hat{a}_V^\dagger \rangle^\dagger \), we find that

\[
\langle \Psi_N | \hat{S}_i | \Psi_N \rangle = \sum_{k=0}^{N-1} c_k c_{k+1} \sqrt{(k + 1)(N - k)} = 0
\]

(12)

is a necessary and sufficient condition for \( \langle \hat{S}_x \rangle \) and \( \langle \hat{S}_z \rangle \) to vanish simultaneously. To achieve \( P_S \equiv 0 \), we must also have

\[
\langle \Psi_N | \hat{S}_0 | \Psi_N \rangle = \sum_{k=0}^{N} |c_k|^2 (2k - N) = 0.
\]

(13)
Equations (12–13) are thus necessary and sufficient conditions for the Stokes degree of polarization of a pure N-photon state to vanish. Clearly, N-photon states that have photon-distribution probabilities with the horizontal-vertical symmetry \( |c_{\pm 1/2}|^2 = |c_{-1/2}|^2 \) satisfy \( \langle \hat{S}_z \rangle = 0 \). Examples of Stokes unpolarized states with this symmetry in any odd manifold \( N \geq 5 \) and any even manifold are given by states satisfying

\[
c_{N-k} = \pm (-1)^k i c_k^*,
\]

where the upper or lower sign is used for all \( k \), and \( c_{(N \pm 1)/2} = 0 \) for odd \( N \). For even \( N \), the solutions corresponding to the upper and lower sign imply \( \arg c_{N/2} = (N - 1 \pm 2)\pi/4 \) and \( \arg c_{N/2} = (N + 1 \pm 2)\pi/4 \), respectively.

In excitation manifold \( N = 0 \) there exists only one state, the two-mode vacuum state \( |0, 0 \rangle \), and it fulfills Eqs. (12) and (13) and thus has \( P_3 = 0 \) in accordance with our complement to definition 6.

All pure single-photon states lie on the surface of the Poincaré sphere. That is, the corresponding vectors \( \langle \hat{S} \rangle \) have unit length and are thus fully polarized (\( P_3 = 1 \)). Hence, no Stokes unpolarized pure state exists in manifold \( N = 1 \).

Since an overall phase factor has no physical significance, Eqs. (12) and (13) imply that any Stokes unpolarized pure state in manifold \( N = 2 \) can be written as

\[
\alpha e^{i\theta} |0, 2 \rangle + \alpha^* e^{-i\theta} |1, 1 \rangle + \alpha e^{-i\theta} |2, 0 \rangle,
\]

where \( \alpha \) and \( \theta \) are real numbers and \( 0 \leq \alpha \leq 1/\sqrt{2} \). That is, they are of the form (13). Although all unpolarized states have vanishing Stokes parameters according to the definition of \( P_3 \), the corresponding fluctuations are, in general, anisotropic. Explicitly, the state (15) has the variances

\[
\begin{align*}
(\Delta S)_{y}^2 &= 4 - 4\alpha^2(1 - \cos 2\theta), \quad (16a) \\
(\Delta S)_{z}^2 &= 4 - 4\alpha^2(1 + \cos 2\theta), \quad (16b) \\
(\Delta S)_{x}^2 &= 8\alpha^2. \quad (16c)
\end{align*}
\]

This shows that Stokes unpolarized states can have “hidden” polarization properties that are not quantified by the corresponding degree of polarization. As exemplified above, also pure states can carry hidden polarization in quantum optics, which is in contrast to classical optics.

When the Stokes parameters \( \langle \hat{S}_x \rangle \), \( \langle \hat{S}_y \rangle \), and \( \langle \hat{S}_z \rangle \) are all zero, it also follows from relation (7) that Stokes unpolarized N-photon states satisfy

\[
(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 = N(N + 2).
\]

The only states of the form (15) that have isotropic fluctuations, i.e., satisfy \( (\Delta S_x)^2 = (\Delta S_y)^2 = (\Delta S_z)^2 = 8/3 \), are seen to be those characterized by \( (a, \theta) = (1/\sqrt{3}, (2m + 1)\pi/4) \), where \( m \in \{0, 1, 2, 3\} \). These are equipartition states in the considered basis, and consequently can be seen as relative-phase eigenstates \( 22, 23 \). We also note that the \( x \), \( y \), and \( z \)-variances vanish for \( (a, \theta) = (1/\sqrt{2}, \pi/2) \), \( (a, \theta) = (1/\sqrt{3}, 0) \), and \( a = 0 \), respectively. Hence, these states only have one or two nonvanishing components in the horizontal-vertical Fock basis.

Figure 1: Permissible state probability amplitudes for pure three-photon states that are Stokes unpolarized.

From the examples given so far, one may be led to believe that Stokes unpolarized states have a symmetry with respect to permutation of the horizontally and vertically polarized modes. This is a chimera, however, as is seen in excitation manifold \( N = 3 \). In this manifold, there exists no unpolarized pure state with only one nonzero probability amplitude or exactly one vanishing probability amplitude. In order to find the unpolarized states, let us express the probability amplitudes as \( c_k = a_k \exp(i\theta_k) \), where \( a_k \) and \( \theta_k \) are real, \( a_k \) is non-negative, and we set \( \theta_0 = 0 \) to remove an unimportant overall phase. From Eqs. (12) and (13), one can then derive the relations

\[
a_1 = \frac{\sqrt{3 - 6a_0^2 - 2a_0^2}}{2}, \quad a_3 = \frac{\sqrt{1 + 2a_0^2 - 2a_0^2}}{2}.
\]

These equations can be used to delineate limits for \( a_0 \) and \( a_2 \) by looking at the values for which \( a_1 = 0 \) and \( a_3 = 0 \), respectively. These limits are shown in Fig. 1 where the axes represent the additional limits \( a_0 = 0 \) and \( a_2 = 0 \). To satisfy Eq. (12), one can view the terms \( \sqrt{3}a_0a_1 \exp(-i\theta_1), 2a_1a_2 \exp[i(\theta_1 - \theta_2)], \) and \( \sqrt{3}a_2a_3 \exp[i(\theta_2 - \theta_3)] \) as three vectors in the complex plane forming a triangle when Eq. (12) is fulfilled. This is only possible if the triangle inequality is satisfied, i.e., the length of any one vector cannot be larger than the sum of the lengths of the remaining two vectors. The borders of these inequalities can be written as

\[
a_2 = \sqrt{3}a_0, \quad (19a)
\]
\[
a_2 = -\sqrt{3}a_0 - \sqrt{6} \cos \left( \frac{2\pi + \arccos \sqrt{\frac{3}{5}} a_0}{3} \right), \quad (19b)
\]
\[
a_2 = \sqrt{3}a_0 - \sqrt{6} \cos \left( \frac{\pi + \arccos \sqrt{\frac{3}{5}} a_0}{3} \right). \quad (19c)
\]

In Fig. 1 these borders form the innermost, sail-shaped “triangle” with vertices \((0, 0), (1/\sqrt{2}, 0), \) and \((1/2, \sqrt{3}/2) \). This area comprises the allowed values for \( a_0 \) and \( a_2 \), ultimately given
by Eq. (13). For any permissible pair \((a_0, a_2)\), one can obtain the values of \(a_1\) and \(a_3\) through the relations (13). Then one can arbitrarily chose \(\theta_1\) and subsequently find the four pairs of values of \(\theta_1 - \theta_2\) and \(\theta_2 - \theta_3\) that make the vectors corresponding to the three terms in Eq. (12) form a triangle. When one of the triangle inequalities is exactly satisfied, i.e., when we are on one of the borders of the sail-shaped area in Fig. 1 there are only two solutions for \(\theta_1\) and \(\theta_3\) once \(\theta_1\) is chosen, namely \((\theta_2, \theta_3) = (2\theta_1 + \pi, 3\theta_1 + \pi)\) and \((2\theta_1 - \pi, 3\theta_1 - \pi)\). For example, on the left border described by Eq. (19a), the states take the form

\[
a_0|0, 3\rangle + e^{i\theta_0} \sqrt{3} \left(\frac{1}{4} - a_0^2\right)|1, 2\rangle
\]

\[
\pm \left( e^{i\theta_0} \sqrt{3} a_0|2, 1\rangle + e^{i\theta_0} \frac{1}{4} - a_0^2 |3, 0\rangle \right),
\]

(20)

where \(0 \leq a_0 \leq 1/2\).

For an arbitrary three-photon unpolarized pure state, the variances of the Stokes operators are

\[
(\Delta S_{x,y})^2 = 3 + 4(a_1^2 + a_2^2) \pm 4 \sqrt{3} [a_0a_2 \cos \theta_2 + a_1a_3 \cos (\theta_1 - \theta_3)],
\]

(21a)

\[
(\Delta S_z)^2 = 9 - 8(a_1^2 + a_2^2),
\]

(21b)

where \(x\) and \(y\) correspond to the plus and minus sign, respectively. Equations (21b) and (13) give the curves

\[
a_2 = \sqrt{3a_0^2 - (\Delta S_z)^2 - 3} \frac{4}{4}
\]

(22)

on which \(\Delta S_z\) is constant. In particular, the states (20) correspond to \((\Delta S_z)^2 = 3\).

Using relations (13), it can also be verified that the symmetry condition \(a_k = a_{N-k}\) is equivalent to \(a_0^2 + a_2^2 = 1/2\). Hence, states whose probability amplitudes have this symmetry are located on the dashed circle arc in Fig. 1 which has the vertex \((a_0, a_2) = (1/\sqrt{2}, 0)\) as one of its end points.

We note that if \(c_k = a_k\), \(\forall k\), is a solution of Eqs. (12)- (13), then \(c_k = a^*_k\), \(\forall k\), is a solution too. In manifold \(N = 3\), any such pair of solutions that do not have the symmetry \(a_k = a_{N-k}\) will correspond to one point on each side of the dashed circle arc. For example, the states corresponding to the two vertices \((0, 0)\) and \((1/2, \sqrt{3}/2)\) can be seen as such “mirror images” of each other. It is clear from Fig. 1 that the vertices of the sail-shaped area correspond to states with only two nonvanishing components in the used basis. The states corresponding to the mirror-image vertices are given by Eq. (20), and the states corresponding to the remaining vertex are \([0, 3] + \exp(i\theta_0)|3, 0\rangle]/\sqrt{2}\), which indeed have the discussed symmetry.

Above, we have seen that states that are unpolarized according to the Stokes definition can have anisotropic polarization fluctuations. Perhaps a more dramatic example is demonstrated by the unpolarized state \(|1, 1\rangle\) corresponding to \(a = 0\) in Eq. (15). A rotation by 45 degrees around its axis of propagation transforms this state into \(((\sqrt{2} + i)|0, 2\rangle + ((\sqrt{2} - i)|2, 0\rangle)/\sqrt{6},

which is orthogonal to the original \(|1, 1\rangle\) \(24\). Hence, despite being unpolarized, the state \(|1, 1\rangle\) can be transformed into a perfectly distinguishable state by a simple geometrical rotation. This is due to the fact that this change cannot be detected by any linear combination of the Stokes operators, as it requires higher-order field correlation measurements. The classification of states according to Stokes degree of polarization is hence insufficient already in excitation manifold \(N = 2\).

3. Desiderata for a quantum degree of polarization

3.1. SU(2)-invariant quantum states

As we have shown in the previous section, one is ill advised to describe polarization properties of quantum fields by a direct analogy with the classical description. A different starting point is needed.

In this respect, we recall that (linear) polarization transformations are generated by the Stokes operators \(\hat{S}\). However, \(\hat{S}_0\) induces only a common phase shift to all the states in any given subspace, and below we will argue that such phases do not change the polarization and can thus be omitted. Therefore, we restrict ourselves to the SU(2) transformations, generated by \(\hat{S}_x, \hat{S}_y\), and \(\hat{S}_z\). In fact, since each of these operators is proportional to the commutator of the others, two generators suffice. It is well known that \(\hat{S}_y\) generates rotations around the direction of propagation, whereas \(\hat{S}_z\) represents differential phase shifts between the modes. Any polarization transformation is thus photon-number preserving and can be expressed as

\[
\hat{U}_{pol}(\alpha, \beta, \gamma) = e^{-i\alpha \hat{S}_0^y} e^{-i\beta \hat{S}_0^z} e^{-i\gamma \hat{S}_0^z}.
\]

(23)

This also means that they can be realized with linear optics. Experimentally, birefringent plates in rotation mounts are the only components needed, and consequently these transformations can be simply and inexpensively achieved in a laboratory.

There is a consensus that the SU(2)-invariant states, which satisfy \(\hat{S}_0^x \hat{\sigma} \hat{S}_0^x = \hat{\sigma}\), are unpolarized. These states are known to be of the form \(25, 26, 27\):

\[
\hat{\sigma} = \bigoplus_{N=0}^{\infty} \pi_N \hat{\sigma}_N,
\]

(24)

where \(\pi_N\) is the probability of finding the state \(\hat{\sigma}\) in excitation manifold \(N\), and \(\hat{\sigma}_N\) is the only unpolarized \(N\)-photon state

\[
\hat{\sigma}_N = \sum_{N=0}^{1} \hat{U}_N.
\]

(25)

Here, \(\hat{U}_N\) is the projector onto the \(N\)-photon subspace, namely

\[
\hat{U}_N = \sum_{k=0}^{N} |k, N-k\rangle \langle k, N-k|.
\]

(26)

One notices that if a pure state is written as \(\sum_{k=0}^{N} c_{N,k} |k, N-k\rangle\), then coherence terms of the form

\[
c_{N,k} c_{N,k}' |k, N-k\rangle \langle k', N'-k'|
\]

(27)

for \(N \neq N'\), can neither be induced nor measured by the Stokes operators. In consequence, \(\hat{\sigma}\) appears as a direct sum over the excitation manifolds in Eq. (24) and any common phase to all the states in any given excitation manifold is inconsequential for any polarization characteristics.
3.2. Requirements for polarization measures

Before discussing specific quantum measures of polarization, it is worthwhile to look at requirements and desiderata for such measures.

**Requirement 1.** A first requirement for any reasonable degree of polarization $\mathbb{P}$ is

$$\mathbb{P} (\hat{\rho}) = 0 \Leftrightarrow \hat{\rho} \text{ is unpolarized}. \quad (28)$$

This immediately rules out the possibility of defining the degree of polarization as a function of the purity $\text{Tr}(\hat{\rho}^2)$. The state $|0,0\rangle$ is pure and unpolarized, while a two-mode thermal state (with the same mean photon number in each mode) is maximally mixed (under the constraint of a fixed average number of photon number) and likewise unpolarized. Also, any state $p_0|0,0\rangle + p_N |N\rangle/(N + 1)$, where $p_0$ and $p_N$ are both non-vanishing, is unpolarized and mixed, but not maximally mixed (under the same constraint). Hence, unpolarized quantum states span the whole purity scale.

**Requirement 2.** A second requirement is SU(2) invariance

$$\mathbb{P}(\hat{\rho}) = \mathbb{P}(\hat{U}_{pol} \hat{\rho} \hat{U}_{pol}^\dagger). \quad (29)$$

Hence, the measure is invariant under polarization transformations. For instance, the Stokes degree of polarization $(6)$ and the fluctuations along the polarization coordinate that gives the induced ordering of states is more important than the numerical value, especially when the measure does not have a clear operational meaning.

**Requirement 3.** A third requirement that has been put forward is that the measure should not depend on the coherences between different manifolds $\{11\}$. The basis for this requirement is that since $\hat{S}_0$ commutes with all Stokes operators, a polarization measurement (a measurement of any linear combination of the Stokes operators) on an arbitrary state

$$\hat{\rho} = \sum_{k,N=0}^{\infty} \sum_{k'=0}^{N} q_{N,k} |k,N-k\rangle\langle k',N'-k'|, \quad (30)$$

does not on average alter the photon-number distribution

$$p_N = \sum_{k=0}^{N} q_{N,k} \quad (31)$$

and the measurement outcome will not depend on any coherences between the manifolds.

On the other hand, a von Neumann measurement of the number of photons gives an outcome $N$ with probability $p_N$ and, at the same time, the state $\hat{\rho}$ collapses into the $N$-photon state

$$\hat{\rho}_N = \frac{1}{p_N} \sum_{k,k'=0}^{N} q_{N,k} |k,N-k\rangle\langle k',N'-k'|. \quad (32)$$

Considering all possible outcomes, we obtain the block-diagonal state

$$\mathbb{B}[\hat{\rho}] = \bigoplus_{N=0}^{\infty} p_N \hat{\rho}_N, \quad (33)$$

where the ideal non-selective measurement of the total photon number is described by the map

$$\mathbb{B} : \hat{\rho} \mapsto \sum_{N=0}^{\infty} \hat{I}_N \hat{\rho} \hat{I}_N. \quad (34)$$

This is a quantum channel $\{28\}$ preserving both the polarization properties and the photon-number distribution of the state $\hat{\rho}$, and provides an operational meaning for the channel. Alternatively, the map $\mathbb{B}$ can be viewed as randomization of the phases between superpositions of states in different excitation manifolds. Using this map, requirement 3 can be expressed as

$$\mathbb{P}(\hat{\rho}) = \mathbb{P}[\mathbb{B}[\hat{\rho}]]. \quad (35)$$

Polarization measures that depend on coherences between different manifolds but fulfill requirement 2 can be made to fulfill Eq. (35) by applying the measure to the channel output as will be done below in Eq. (39).

Notice that some polarization-measure candidates (such as the entropy $\mathcal{S}$) are only positive semidefinite, so that $0 \leq \mathcal{S}(\hat{\rho}) < \infty$. In this case, a common “remedy” is to normalize the measure through the transformation $\mathbb{P} = \mathcal{S}/(1+\mathcal{S})$, which guarantees the condition

$$0 \leq \mathbb{P}(\hat{\rho}) \leq 1. \quad (36)$$

Such a rescaling keeps the “ordering” of states intact. Indeed, the induced ordering of states is more important than the numerical value, especially when the measure does not have a clear operational meaning.

The requirements 1-3 can be supplemented by a number of desiderata. The most common, in particular among experimentalists, is that $\mathbb{P}$ should be operational and easily measurable. Theoreticians, on the other hand, desire that the measure is easy to compute. Unfortunately, in general, these wishes are conflicting.

From an experimental point of view, the measure may favor a number of different operational characteristics. One could, e.g., quantify the maximum visibility achievable in a polarization interference measurement $\{29\}$. Such a measure would fulfill all three requirements and would also have a direct operational meaning, but it would not be easily measurable, in general, as one would not know what are the polarization transformations that yield the maximum and minimum interference intensity.

One could alternatively determine how close a given state is to a polarization minimum uncertainty state $\{30\}$. Such a measure would also fulfill the requirements and have a relatively clear operational meaning, but it would require polarization tomography, a complicated measurement procedure, to be determined.

Another possibility is to evaluate the polarization fluctuations. In this case, it would probably make sense to assess the fluctuations along the polarization coordinate that gives the smallest fluctuations. This would give an idea about the smallest detectable polarization transformations and hence have an operational meaning. However, for a general state, the measure
would be difficult to determine and, in general, also difficult to compute.

In summary, we see that there are many possibilities of defining a measure of polarization. We have argued that our three requirements are reasonable conditions for any such a measure. Note that all are closely related to the properties of the Stokes operators, which we have taken as our starting point. Any ensuing degree of polarization will have their particular merits and drawbacks.

4. Quantum degrees of polarization

4.1. Distance-based measures

After our discussion in Section 3.3, it seems sensible to define the degree of polarization as the shortest distance between the considered state and the set \( \mathcal{U} \) of unpolarized states \( \hat{\sigma} \) given in Eq. (24). Similar notions have been successfully applied to other key concepts such as nonclassicality [31, 32, 33], entanglement [34] and quantum information [35, 36, 37].

Several distance measures have been proposed, such as the Hilbert-Schmidt and Bures distances [10]. We also include the Chernoff distance, recently used to quantify the nonclassicality of Gaussian states [39] and polarization [40]. For an arbitrary state \( \hat{\rho} \), these measures are given by

\[
\begin{align*}
\mathcal{P}_{\text{HS}}(\hat{\rho}) &= \inf_{\hat{\sigma} \in \mathcal{U}} \text{Tr}(\hat{\rho} - \hat{\sigma})^2, \\
\mathcal{P}_{\text{B}}(\hat{\rho}) &= 1 - \sup_{\hat{\sigma} \in \mathcal{U}} \sqrt{F(\hat{\rho}, \hat{\sigma})}, \\
\mathcal{P}_{\text{C}}(\hat{\rho}) &= 1 - \sup_{s(0,1)} \text{inf} \left[ \text{Tr}(\hat{\rho}^{s^{-1}}) \right],
\end{align*}
\]

where the infimum in Eq. (37) is taken over a function that is continuous with respect to \( s \) [41], and the fidelity is

\[
F(\hat{\rho}, \hat{\sigma}) = (\text{Tr}(\hat{\rho}^{1/2} \hat{\sigma}^{1/2} \hat{\rho}^{1/2} \hat{\sigma}^{1/2}))^2.
\]

While all these definitions seem sensible, they do not satisfy requirement 3; that is, they are sensitive to coherences between different excitation manifolds [11]. To bypass this drawback, we apply requirement 3, i.e., we replace the states by the corresponding block-diagonal density matrices:

\[
\mathcal{P}_{\text{B}}(\hat{\rho}) = \mathcal{P}_{Z}(\mathcal{B}[\hat{\rho}]), \quad Z \in \{\text{HS}, \text{B}, \text{C}\}.
\]

These measures can thus be seen as applying the original measures on the block-diagonal output state of the photon-number measurement channel [34], whose input state is \( \hat{\rho} \). Using the fact that \( \mathcal{B}[\hat{\rho}] \) and \( \hat{\sigma} \) commute, we find the following general formulas:

\[
\begin{align*}
\mathcal{P}_{\text{HS}}(\hat{\rho}) &= \sum_{N=0}^{\infty} \frac{p_N}{N + 1} \left( \frac{\epsilon_N^{(1)}}{N + 1} - 1 \right), \\
\mathcal{P}_{\text{B}}(\hat{\rho}) &= 1 - \sum_{N=0}^{\infty} \frac{p_N}{N + 1} \left( \frac{\epsilon_N^{(1)}}{N + 1} \right)^{1/2}, \\
\mathcal{P}_{\text{C}}(\hat{\rho}) &= 1 - \inf_{x \in [0,1]} \left[ \sum_{N=0}^{\infty} p_N (N + 1)^{1 - x} \left( \frac{\epsilon_N^{(1)}}{N + 1} \right)^{x/2} \right],
\end{align*}
\]

where \( \lambda_{N,\nu} \) are the eigenvalues of \( \hat{\rho}_N \) and \( \epsilon_N^{(1)} \equiv \sum_{\nu=0}^{\infty} \lambda_{N,\nu}^2 \). These measures fulfill our three requirements for a degree of polarization. Obviously, \( \mathcal{P}_{\text{B}}(\hat{\rho}) \leq \mathcal{P}_{\text{C}}(\hat{\rho}) \).

Any pure state \( \hat{\rho} = |\Psi\rangle\langle\Psi| \) satisfies \( \mathcal{P}_{\text{B}}(\hat{\rho}) = 1 \) in each manifold with nonzero excitation probability. Hence, for any such state, one of the eigenvalues \( \lambda_{N,\nu} \) equals unity and the rest of them vanish. The degrees of polarization are thus given by

\[
\begin{align*}
\mathcal{P}_{\text{HS}}(|\Psi\rangle\langle\Psi|) &= \sum_{N=0}^{\infty} \frac{p_N^2}{N + 1}, \\
\mathcal{P}_{\text{B}}(|\Psi\rangle\langle\Psi|) &= 1 - \sum_{N=0}^{\infty} \frac{p_N}{N + 1}^{1/2}, \\
\mathcal{P}_{\text{C}}(|\Psi\rangle\langle\Psi|) &= 1 - \inf_{x \in [0,1]} \left[ \sum_{N=0}^{\infty} p_N (N + 1)^{1 - 1/2} \right]^{x/2}.
\end{align*}
\]

These expressions involve only the excitation probabilities \( p_N \). Thus, for pure states, the block-diagonal distance degrees of polarization are insensitive to the form(s) of \( \hat{\rho}_N \). In the special case of a pure \( N \)-photon state \( \hat{\rho} = |\Psi_N\rangle\langle\Psi_N| \), the above expressions simplify to \( \mathcal{P}_{\text{HS}}(|\Psi_N\rangle\langle\Psi_N|) = \mathcal{P}_{\text{C}}(|\Psi_N\rangle\langle\Psi_N|) = N/(N + 1) \) and \( \mathcal{P}_{\text{B}}(|\Psi_N\rangle\langle\Psi_N|) = 1 - (N + 1)^{-1/2} \). All of them tend to unity for large \( N \).

It is clear that for a fixed excitation manifold \( N \), pure states have a higher degree of polarization than any mixed state, as one would intuitively expect. Using Lagrange multipliers and numerical optimization, we have derived the block-diagonal states that for a given average photon number \( \bar{N} \) have the highest degrees of polarization [42]. For the Hilbert-Schmidt measure, these maximally polarized states are of the form

\[
\frac{[\bar{N} - 1]}{[\bar{N}]} [0,0,0,0] + \frac{\bar{N}}{[\bar{N}]} [\Psi(\hat{N})] [\Psi(\hat{N})]
\]

if \( \bar{N} \geq \sqrt{[N][N] + 2} \), and of the form

\[
\lim_{M \to \infty} \frac{M - \bar{N}}{M - [\bar{N}]} [\Psi(\hat{N})] [\Psi(\hat{N})] + \frac{\bar{N} - [\bar{N}]}{M - [\bar{N}]} [\Psi_M] [\Psi_M], 
\]

if \( \bar{N} \leq \sqrt{[N][N] + 2} \). Here, \( [\bar{N}] \) denotes the smallest integer larger than or equal to \( \bar{N} \), whereas \( [\bar{N}] \) is the largest integer smaller than or equal to \( \bar{N} \). Hence, the maximal polarization degree is given by the somewhat “rounded” staircase function

\[
\mathcal{P}_{\text{HS}}^{\text{max}} = \begin{cases} 
\frac{[\bar{N}]}{[\bar{N}]} + 1, & \bar{N} \leq \sqrt{[N][N] + 2}, \\
0, & \bar{N} \geq \sqrt{[N][N] + 2}.
\end{cases}
\]

If instead the Bures or Chernoff measure is used, the maximally polarized states are given by

\[
\left(\frac{[\bar{N}]}{[\bar{N}] - 1}\right) [\Psi(\bar{N} - 1)] [\Psi(\bar{N} - 1)] + \left(1 - \frac{[\bar{N}]}{[\bar{N}] - 1}\right) [\Psi(\bar{N})] [\Psi(\bar{N})]
\]

and the corresponding degrees of polarization are

\[
\begin{align*}
\mathcal{P}_{\text{B}}^{\text{max}} &= 1 - \frac{2[\bar{N}] - \bar{N}}{[\bar{N}][\bar{N}]}, \\
\mathcal{P}_{\text{C}}^{\text{max}} &= 1 - \inf_{x \in [0,1]} \left[ \frac{[\bar{N}]}{[\bar{N}]} [\bar{N}] - \bar{N} \right]^{x/2} + (\bar{N} + 1)^{1/2} (1 + \bar{N} - [\bar{N}]).
\end{align*}
\]
Plots of $P_{\text{HSh}}^\text{max}$, $P_{\text{B}}^\text{max}$, and $P_{\text{Cb}}^\text{max}$ are shown in Fig. 2. We note that any pure $N$-photon state $|\Psi_N\rangle$ is maximally polarized according to any of these three measures. We also note that $P_{\text{HSh}}^\text{max}$ has a strictly positive derivative with respect to $N$, whereas $P_{\text{B}}^\text{max}$ and $P_{\text{Cb}}^\text{max}$ have non-negative but discontinuous derivatives.

4.2. Other quantum polarization measures

Several other quantum degrees of polarization have been proposed. One of them is based on the SU(2) $Q$ function [30], which is defined as

$$Q_d(\Omega) = \sum_{k=0}^{N} \int \frac{N+1}{4\pi} \langle N; \Omega | \bar{N} | N; \Omega \rangle = Q_{\text{B}}(\Omega). \quad (48)$$

Here, $\Omega = (\theta, \varphi)$ and $\theta$ and $\varphi$ are the polar and azimuthal angles over the unit 2-sphere $S^2$, and $|N; \Omega\rangle$ are the $N$-photon SU(2) coherent states

$$|N; \Omega\rangle = \sum_{k=0}^{N} \binom{N}{k} \cos^k \left( \frac{\theta}{2} \right) \sin^k \left( \frac{\theta}{2} \right) e^{i\varphi k} |N-k\rangle. \quad (49)$$

For any unpolarized state [24], the $Q$ function takes the constant value $(4\pi)^{-1}$. Apart from the unpolarized vacuum state, any SU(2) coherent state has a $Q$ function that is highly peaked around some angle $\Omega_0$. For example, for a SU(2) coherent state centered around $\bar{\theta} = 0$, that is, the state $|N; 0\rangle$, we have

$$Q_{|N; 0\rangle}(\Omega) = \frac{N+1}{4\pi} \left( \cos \frac{\theta}{2} \right)^{2N}. \quad (50)$$

The idea behind a $Q$ function-based measure is to assess the spread of $Q$ over the sphere by comparing with a uniform distribution:

$$D_Q(\hat{\Omega}) = 4\pi \left( \int Q_\Omega(\Omega) - \frac{1}{4\pi} \right)^2 d\Omega = 4\pi \int Q_d^2(\Omega) d\Omega - 1. \quad (51)$$

However, since $0 \leq D_Q(\hat{\Omega}) < \infty$, the associated degree of polarization is defined as

$$P_Q(\hat{\Omega}) = P_Q(\mathcal{B}[\hat{\Omega}]) = \frac{D_Q(\hat{\Omega})}{D_Q(\hat{\Omega}) + 1}. \quad (52)$$

This degree favors polarization minimum uncertainty states as it measures the “area” of $Q$, but is insensitive to its shape or orientation. The measure can be obtained experimentally, but only through rather involved polarization tomography.

For an SU(2) coherent state, $P_Q(|N; \Omega\rangle) = [N/(N+1)]^2$. As the SU(2) coherent states are polarization minimum uncertainty states they are maximally polarized according to the definition of $P_Q$. For an average photon number $\bar{N}$, the superposition, or mixture, of the SU(2) coherent states $|n-1; \Omega\rangle$ and $|n; \Omega\rangle$, with probabilities $p_{n-1} = n - \bar{N}$ and $p_n = 1 + \bar{N} - n$, respectively, are the states with the maximal $P_Q$-degree of polarization.

Another proposed polarization measure is given in [29]. Using $\text{Tr}(\hat{Q}_d \hat{\Omega})$ as the overlap for mixed states, the definition for pure states in Ref. [23] is generalized to

$$P_d(\hat{\Omega}) = \frac{\text{Tr}(\mathcal{B}[\hat{\Omega}])}{\text{Tr}(\mathcal{B}[\hat{\Omega}],\hat{\Omega})} = \left[ 1 - \inf_{\hat{\Omega}, N} \sum_{n=0}^{\infty} p_n \text{Tr}(\hat{\Omega}_N \hat{U}_\text{pol} \hat{\Omega}_N \hat{U}_\text{pol}) \right]^{1/2}. \quad (53)$$

This definition is based on the probability averaged minimal overlap between a state and all of its SU(2) transformed states. Hence, it gives the (square root of the) maximum visibility one can achieve by using a polarization interferometer. The problem in this case is to find and implement the polarization projection and the subsequent polarization transformation that achieves the maximum polarimetric visibility. In contrast to the previous measures in this section, $P_d$ may assign the degree of polarization unity for states with a finite average excitation. It has been shown that any pure state having an odd photon number is maximally polarized in the sense that $P_d = 1$ [43]. One may conjecture that pure states with an even number of photons (excluding the vacuum state) also are maximally polarized according to this definition, but to the best of our knowledge no proof thereof exists, except for $N = 2$.

The last speculation makes it tempting to define a degree of polarization in terms of the state purities in every excitation.
manifold, as follows:
\[ P_p(\hat{\mathcal{B}}) = P_p(\hat{\mathcal{B}}[\hat{\mathcal{B}}]) = \sum_{N=1}^{\infty} p_N \frac{(N + 1) \text{Tr}(\hat{\mathcal{B}}^2) - 1}{N}, \]
where we need the additional definition \( P_p((0, 0)) \equiv 0 \). Again, the maximally polarized states are the pure states in any excitation manifold or any mixture, or superposition, thereof. The measure makes no direct use of the Stokes operators (except for being a direct sum over manifolds). This indicates that the measure quantifies a distinguishability under a general energy-preserving unitary transformation rather than the distinguishability under the more restrictive unitary polarization transformations \( \hat{U}_{\text{pol}} \). Should the conjecture in the previous paragraph prove false, this measure seems questionable. A measurement will unfortunately be difficult since the purity essentially must be assessed through polarization tomography. In Fig. 3 the maximum degree of polarization for the measures \( P_Q, P_d \), and \( P_p \) are plotted.

5. Nonlinear polarization transformations

Above we have defined the set of “proper” polarization transformations in Eq. 23 as all linear transformations generated by the Stokes operators. Such a viewpoint has a basis both in classical and quantum optics. Of course, one could think in more general terms and allow nonlinear (energy-preserving) transformations, which can be represented as
\[ \hat{U}_{\text{nl}} = e^{-i g \hat{\mathcal{B}} \hat{\mathcal{S}}}, \]
where \( g \) is an arbitrary nonlinear function. Such a set of transformations includes a variety of effects such as polarization squeezing and excitation manifold-dependent transformations. However, one could argue that both polarization squeezing and manifold-dependent transformations can change a state’s degree of polarization, as we shall give an example of below. Another reason for excluding such transformations is that they are very difficult to implement experimentally.

If one allows nonlinear polarization transformations, \( P_Q \) will no longer fulfill Eq. 29. For example, the state \( |\Psi_1\rangle = (|N, 0\rangle + |N', 0\rangle) / \sqrt{2} \), where \( N, N' \neq 0 \) has a \( Q \) function that is concentrated on the north pole and whose dispersion is
\[ D_Q(|\Psi_1\rangle) = \frac{1}{4} \left[ \frac{(N + 1)^2}{2N + 1} + \frac{(N' + 1)^2}{2N' + 1} \right] + 2 \left( \frac{(N + 1)(N' + 1)}{N + N' + 1} \right) - 1. \]

With a nonlinear polarization transformation it is possible to transform the state \( |N', 0\rangle \) to \( |0, N'\rangle \), that is, to rotate this state to the south pole of the representation sphere without affecting the state \( |N, 0\rangle \). However, the state \( |\Psi_2\rangle = (|N, 0\rangle + |0, N'\rangle) / \sqrt{2} \) has the dispersion
\[ D_Q(|\Psi_2\rangle) = \frac{1}{4} \left[ \frac{(N + 1)^2}{2N + 1} + \frac{(N' + 1)^2}{2N' + 1} + 2 \frac{(N + 1)(N' + 1)!}{(N + N' + 1)!} \right] - 1. \]

which is close to half the value of \( D_Q(|\Psi_1\rangle) \) when \( N \approx N' \gg 1 \). Hence, \( P_Q(|\Psi_1\rangle) > P_Q(|\Psi_2\rangle) \).

Perhaps, superior future technology will make it natural to view also nonlinear Stokes operator induced transformations as “proper” polarization transformations, in contrast to our definition. Such a view would distance the quantum description of polarization even further from the classical one.

6. Discussion and conclusions

As we have seen, defining a quantitative measure of polarization for quantum fields is a task without any obvious or unique solution. As a consequence, no universally accepted view on how to quantify the polarization of such fields exists, and the prospects of this happening seem bleak.

In this paper, we have advocated the view that the Stokes operators should be central to any quantum polarization theory. Three of the central requirements for a quantitative degree of polarization are based on their properties. Adhering to this view would ascertain at least partial correspondence between classical and quantum concepts and descriptions of polarization. A consequence of this view is the definition of a polarization transformation. This is a unitary transformation generated by any linear combination of the Stokes operators. Another rather unavoidable consequence is the definition of an unpolarized quantum state as a state where each excitation manifold is invariant under any polarization transformation.

It is clear from our discussion of maximally polarized states that the proposed measures order the degree of polarization for states differently. This is to be expected since each measure focuses on one specific polarization property. For example, a state that can become self-orthogonal under a polarization transformation, thus having \( P_d = 1 \), may not be even close to a polarization minimum uncertainty state, which are states for which \( P_Q \) is large. We therefore conjecture that different degrees of polarization will coexist, and that they will find applications in different polarization contexts.

We have also shown in Section 4 that allowing nonlinear transformations as “proper” polarization transformations will lead to profound differences in the way we view quantum polarization. As long as such transformations are essentially outside the realm of what is experimentally realizable for few photon states, it seems reasonable to stick with the set of linear transformations.

We have only discussed polarization properties for two-mode fields. In principle, the formalism above will apply to any two harmonic oscillators, but if we want to retain some connection to the classical concepts of polarization, the two modes should be monochromatic, co-propagating, approximately plane waves in approximately the same temporal modes. Of course one could, in analogy with the development in classical optics, start to define polarization concepts and degrees of polarization for three-dimensional fields \( \text{e.g., in strongly focused beams of light) or for polarization-entangled, four-mode states. Attempts in this direction have been made (48, 49, 50, 51). However, such generalizations of the basic concepts are often} \)
difficult to interpret and to give an operational meaning. Hence, it is probably more fruitful to refer to such general multimode characteristics as field- or mode-correlations, without using the word polarization.

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References

[1] G. G. Stokes, Trans. Cambridge Philos. Soc. 9 (1852) 399.
[2] J. R. Schott, Fundamentals of Polarimetric Remote Sensing, SPIE, Bellingham, 2009.
[3] L. D. Barron, Molecular Light Scattering and Optical Activity, Cambridge University Press, Cambridge, UK, 2004.
[4] R. M. A. Azzam, N. M. Bashara, Ellipsometry and Polarized Light, Elsevier, Amsterdam, 1987.
[5] S. Werner, O. Rudow, C. Mihalcea, E. Oesterschulze, Appl. Phys. A 66 (1998) 5367.
[6] D. N. Klyshko, Phys. Lett. A 163 (1992) 349.
[7] D. N. Klyshko, Sov. Phys. JETP 84 (1997) 1065.
[8] A. P. Alodjants, S. M. Arakelian, A. S. Chirkin, Quant. Semiclass. Opt. 9 (1997) 311.
[9] A. Luis, L. L. Sánchez-Soto, in: E. Wolf (Ed.), Progress in Optics, vol. 41, Elsevier, Amsterdam, 2000, p. 421.
[10] A. B. Klimov, L. L. Sánchez-Soto, E. C. Yustas, J. Söderholm, G. Björk, Phys. Rev. A 72 (2005) 033813.
[11] A. Luis, Opt. Commun. 273 (2007) 173.
[12] C. H. Bennett, F. Bessette, G. Brassard, L. Salvail, J. Smolin, J. Cryptology 5 (1992) 3.
[13] A. Muller, J. Breguet, N. Gisin, Europhys. Lett. 23 (1993) 383.
[14] K. Mattle, H. Weinfurter, P. G. Kwiat, A. Zeilinger, Phys. Rev. Lett. 76 (1996) 4656.
[15] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, Y. Shih, Phys. Rev. Lett. 75 (1995) 4337.
[16] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, A. Zeilinger, Nature 390 (1997) 375.
[17] M. Barbieri, F. De Martini, G. Di Napoli, P. Mataloni, G. M. D’Ariano, C. Macchiavello, Phys. Rev. Lett. 91 (2003) 227901.
[18] M. Rádmark, M. Zukowski, M. Bourennane, New J. Phys. 11 (2009) 103016.
[19] K. J. Resch, K. L. Pregnell, R. Prevedel, A. Gilchrist, G. J. Pryde, J. L. O’Brien, A. G. White, Phys. Rev. Lett. 98 (2007) 223601.
[20] E. Collett, Am. J. Phys. 38 (1970) 563.
[21] C. Brosseau, Fundamentals of Polarized Light: A Statistical Optics Approach, Wiley, New York, 1999.
[22] A. Luis, L. L. Sánchez-Soto, Phys. Rev. A 48 (1993) 4702.
[23] G. Björk, J. Söderholm, A. Trifonov, Opt. Commun. 193 (2001) 161.
[24] H. Prakash, N. Chandra, Phys. Rev. A 4 (1971) 796
[25] G. S. Agarwal, Lett. Nuovo Cimento 1 (1971) 53.
[26] J. Söderholm, G. Björk, A. Trifonov, Opt. Spectrosc. (USSR) 91 (2001) 532.
[27] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, UK, 2000.
[28] G. Björk, J. Söderholm, A. Trifonov, P. Usachev, L. L. Sánchez-Soto, A. B. Klimov, Proc. SPIE 4750 (2002) 1.