$U_{\hbar}(g, r)$ Invariant Quantization on Some Homogeneous Manifolds

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Abstract

We consider a class of homogeneous manifolds over a simple Lie group which appears in the problem of classification of homogeneous manifolds with reductive subgroups of maximal rank as stabilizer of a point. We prove that any manifold of this class possesses a Poisson bracket admitting a quantization invariant (equivariant) with respect to the corresponding quantum group.
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1 Introduction

A quantum homogeneous manifold is obtained from the usual homogeneous manifold by replacing the original commutative function algebra with a deformed non-commutative algebra. Since a homogeneous manifold is equipped with a Lie group action, it is natural to look for deformation quantizations of the function algebra which are invariant with respect to the action of the corresponding quantum group.

Let \( G \) be a simple connected Lie group over \( \mathbb{C} \), \( \mathfrak{g} \) its Lie algebra, \( K \) a closed subgroup of \( G \), \( \mathfrak{k} \) the Lie algebra of \( K \). Denote by \( M = G/K \), \( \mu \) the commutative multiplication in \( C^\infty(M) \). A quantization of \( M \) is a formal deformation \( \mu_\hbar = \mu + \hbar \mu_1 + \hbar^2 \mu_2 + \cdots \) of \( \mu \) which defines an associative multiplication in the space \( C^\infty[\hbar] \) of formal power series in \( \hbar \) with smooth functions on \( M \) as their coefficients. Let \( U_\hbar(\mathfrak{g}) = (U(\mathfrak{g}), \Delta_\hbar) \) be a quantization of \( \mathfrak{g} \). We say that \( \mu_\hbar \) is invariant with respect to \( U_\hbar(\mathfrak{g}) \) if \( x.\mu_\hbar(a, b) = \mu_\hbar(\Delta_\hbar(x) \cdot (a \otimes b)) \) for any \( a, b \in C^\infty(M) \) and \( x \in U_\hbar(\mathfrak{g}) \). One can assume from the beginning that the bilinear mapping \( \mu_1 \) is skew symmetric. The associativity of the deformed multiplication \( \mu_\hbar \) implies the Jacobi identity for \( \mu_1 \). Thus \( \mu_1 \) is essentially a Poisson bracket on \( M \). The first question we investigate is whether there exist Poisson brackets on the homogeneous manifold \( M = G/K \) admissible for \( U_\hbar(\mathfrak{g}) \) invariant quantization. The second question we consider is whether there exists a deformed multiplication \( \mu_\hbar \) which is invariant under the quantum group action.

It turns out that brackets which admit invariant quantization are of special form which we describe now. The action on \( M \) determines a homomorphism from \( \mathfrak{g} \) to the Lie algebra \( \text{Vect}(M) \) of vector fields on \( M \) and so it induces a linear map \( \rho : \wedge^2 \mathfrak{g} \rightarrow \wedge^2 \text{Vect}(M) \). Consider a bivector \( r \in \wedge^2 \mathfrak{g} \) such that the Schouten bracket \( [[\rho(r), \rho(r)]] \) is \( \mathfrak{g} \) invariant. Such an element is called a Belavin–Drinfeld classical r-matrix. Since the algebra \( \mathfrak{g} \) is simple, there exists a \( \mathfrak{g} \) invariant 3-vector \( \varphi \in \wedge^3 \mathfrak{g} \) unique up to constant multiple. We normalize \( r \) so that \( [r, r] = \varphi \) and thus \( [\rho(r), \rho(r)] = \rho(\varphi) \). Each Belavin–Drinfeld classical r-matrix \( r \) defines a Lie bialgebra structure on \( \mathfrak{g} \). P. Etingof and D. Kazhdan have proven \( \cite{etingof} \) that any Lie bialgebra can be quantized. In particular, if such a structure is determined by \( r \), there exists a quantum group \( U_\hbar(\mathfrak{g}, r) \) whose multiplication is preserved from \( U(\mathfrak{g}) \) and the comultiplication is of the form \( \Delta_\hbar = \Delta + \hbar r + \cdots \) where \( \Delta \) denotes the original comultiplication in \( U(\mathfrak{g}) \). If \( \rho(\varphi) = 0 \) then \( \rho(r) \) is a Poisson bracket on \( M \) which is called an r-matrix Poisson bracket. It can happen,
however, that $\rho(\varphi) \neq 0$ on $M$, but there exists a $G$ invariant bivector field $s$ on $M$ such that

$$\langle s, s \rangle = -\rho(\varphi)$$  \hspace{1cm} (1)

We call such $s$ a $\varphi$-Poisson bracket on $M$. Thus $\varphi$-Poisson bracket is a skew symmetric bracket obeying the Leibniz rule and the weak version of the Jacobi identity expressed by equation (1).

For a $\varphi$-Poisson bracket $s$, the sum $s + \rho(r)$ is a Poisson bracket on $M$, but it is not $g$ invariant. We call Poisson brackets of this form admissible. J. Donin, D. Gurevich and S. Shnider proved that if Poisson bracket on a homogeneous manifold admits a $U_\hbar(g, r)$ invariant quantization, it is of necessity admissible, $s$ and $r$ satisfying (1). If such a Poisson bracket exists on $M$, the question arises whether it can be quantized in such a way that the deformed multiplication $\mu_\hbar = \mu + \hbar s + \hbar^2 \mu_2 + \cdots$ is $U_\hbar(g, r)$ invariant. We call an admissible Poisson bracket quantizable if there exists some its $U_\hbar(g, r)$ invariant quantization.

The problem of invariant quantization of Poisson brackets was considered by many authors, among them are J. Donin, D. Gurevich, S. Khoroshkin, Sh. Majid, A. Radul, V. Rubtsov, S. Shnider and others.

J. Donin, D. Gurevich and Sh. Majid have considered so called Drinfeld–Jimbo classical r-matrix which is a particular case of Belavin–Drinfeld r-matrix. They proved that if the Lie subalgebra $\mathfrak{f}$ contains a maximal nilpotent subalgebra, then the Drinfeld–Jimbo classical r-matrix generates a Poisson bracket on $M = G/K$, and that there exists a quantization of this bracket invariant with respect to the Drinfeld–Jimbo quantum group $U_\hbar(g)$. J. Donin and D. Gurevich proved that on a semi-simple orbit of coadjoint representation the Poisson bracket coming from the Sklyanin–Drinfeld bracket on $G$ is quantizable. J. Donin and S. Shnider proved that the Drinfeld–Jimbo classical r-matrix generates a Poisson bracket on any symmetric space, and solved the problem of quantization for this bracket.

As we mentioned above, J. Donin, D. Gurevich and S. Shnider proved that if Poisson bracket on a homogeneous manifold admits a $U_\hbar(g, r)$ invariant quantization, it is of necessity of the form $s + \rho(r)$ with $s$ and $r$ satisfying (1). They have shown that almost any such bracket can be quantized invariantly on any semi-simple orbit of the coadjoint representation. In the present work, we introduce another class of homogeneous manifolds containing manifolds which are not necessarily orbits of the coadjoint repre-
sentation of $\mathfrak{g}$. We prove that any manifold in of this class can be equipped with an essentially unique Poisson bracket which admits a $U_h(\mathfrak{g}, r)$ invariant quantization.

To describe this class of homogeneous manifolds, fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, a simple root base for the corresponding root system and an integer $l \geq 2$. Take a simple root $\alpha$ and denote by $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ generated by the Cartan subalgebra and by all roots for which the coefficient of $\alpha$ is divisible by $l$. Denote by $K$ the subgroup of $G$ corresponding to $\mathfrak{k}$ and set $M_{l\alpha} = G/K$. The significance of the manifolds $M_{l\alpha}$ is in the fact that any quotient of $G$ by a reductive subgroup of maximal rank can be obtained by taking consequent quotients of homogeneous manifolds of this type, see [13].

We prove that for any $M_{l\alpha}$ there exists a $\mathfrak{g}$ invariant bivector field $s$ satisfying (1). We prove then that for any Belavin–Drinfeld $r$-matrix $r$, the Poisson bracket $\rho(r) + s$ can be quantized in $U_h(\mathfrak{g}, r)$ invariant way. We produce this quantization in two steps. First, using methods developed from the techniques of [8], we prove that the $\varphi$-Poisson bracket $s$ can be quantized in such a way that the deformed multiplication is invariant under the action of the group $G$ and obeys some deformed associativity constraint. Then, using $\rho(r)$, we correct the above deformed multiplication, turning it into a new multiplication satisfying the usual associativity law. This new multiplication is invariant under the action of the quantum group $U_h(\mathfrak{g}, r)$. The reason why we first pass to the category with non-trivial associativity is that while losing the associativity, we gain $G$ invariance. The next step, passing to the quantum group symmetry, is achieved by an equivalence of categories. To construct this equivalence, we prove that there exists an invertible element $F_h \in U(\mathfrak{g})^{\otimes 2}[\hbar]$ such that

$$\Delta_h(x) = F_h^{-1} \cdot \Delta(x) \cdot F_h$$

for any $x \in U(\mathfrak{g})$. Here $\Delta$ is the standard comultiplication in $U(\mathfrak{g})$ and $\Delta_h$ is the comultiplication in the quantum group $U_h(\mathfrak{g}, r)$. This proof is essentially based on the work [14] by P. Etingof and D. Kazhdan. This two step method allows us to reduce $U_h(\mathfrak{g}, r)$ invariant quantizations of brackets $s + \rho(r)$ for different Belavin–Drinfeld $r$-matrix $r$ to a single quantization of the $\varphi$-Poisson bracket $s$ in the category with non-trivial associativity.

The work is organized as follows. In Section 2 we recall some facts about Hochschild complexes, with special attention to those aspects which are important for our purposes. In particular, we consider some symmetry properties of Hochschild complexes.
In Section 3 we explain how to reduce the problem of associative $U_h(\mathfrak{g}, r)$ invariant quantization to the problem of non-associative $G$ invariant quantization. Namely, for any Belavin—Drinfeld $t$-matrix, we construct the element $F_h$ which follows (2) and gives the equivalence of the categories with trivial and non-trivial associativity.

In Section 4 we study properties of infinitesimals for $G$ invariant quantization on homogeneous $G$ manifolds in the category with the non-trivial associativity. We introduce the notion of $\varphi$-Poisson bracket and show that, similarly to the case of associative deformation, any $G$ invariant deformation with non-trivial associativity has a $\varphi$-Poisson bracket as its infinitesimal.

In Section 5 we prove a general theorem which gives a sufficient condition for quantizability of $\varphi$-Poisson brackets.

In Section 6 we introduce the homogeneous manifolds $M_{l\alpha}$. We prove that all manifolds $M_{l\alpha}$ posses $G$ invariant $\varphi$-Poisson brackets and give explicit form for all these brackets. We consider cochain complexes generated by these brackets and prove that dimensions of some cohomologies is equal to zero. This allows us to apply the general theorem of Section 5 and prove that these $\varphi$-Poisson brackets can be quantized $G$ invariantly in the category with non-trivial associativity. Applying the results of Section 3, we conclude that any Poisson bracket $s + \rho(r)$ can be quantized invariantly with respect to the quantum group $U_h(\mathfrak{g}, r)$ action.
2 Poisson brackets as infinitesimals for algebra deformations

In this section we give basic definitions of deformation theory of commutative algebras, with special attention to the function algebras on manifolds. We show that the linear term of a formal deformation of commutative algebra is a Poisson bracket on this algebra. We recall some facts about Hochschild complexes, stressing their symmetry properties.

Fix an associative commutative C algebra $A$ with unity and denote by $\mu$ the multiplication in $A$. We denote by $\mathbb{C}[\hbar]$ the C algebra of formal power series in a variable $\hbar$. All tensor products over $\mathbb{C}[\hbar]$ are assumed to be completed in the $\hbar$-adic topology.

2.1 Deformations of commutative algebras

Definition 2.1 A $\mathbb{C}[\hbar]$ algebra $(A_\hbar, \mu_\hbar)$ is called a (formal) deformation of $(A, \mu)$ if

(i) $A_\hbar$ is equal to $A \otimes \mathbb{C}[\hbar]$ as a topological $\mathbb{C}[\hbar]$ module;

(ii) $A_\hbar/\hbar A_\hbar \cong A$ as $\mathbb{C}$ algebras.

Thus a deformation of $A$ is a topologically free $\mathbb{C}[\hbar]$ module $A[\hbar]$ equipped with a $\mathbb{C}[\hbar]$ linear mapping $\mu_\hbar : A[\hbar] \otimes_{\mathbb{C}[\hbar]} A[\hbar] \to A[\hbar]$ obeying the associativity law. One can think about $\mu_\hbar$ as of a formal series $\mu_0 + \mu_1 \hbar + \mu_2 \hbar^2 + \mu_3 \hbar^3 + \cdots$ where $\mu_i$ is the $\mathbb{C}[\hbar]$ linear extension of a $\mathbb{C}$ linear mapping $A \otimes A \to A$. Usually, $\mu_0$ is called the initial term of $\mu_\hbar$ and $\mu_1$ is called the infinitesimal of $\mu_\hbar$. The second condition in the definition means that $\mu_0 = \mu$, the original multiplication in $A$. We are interested in a special kind of deformations which are called quantizations. Before giving the definition, we examine some important properties of infinitesimals.

Let $(A_\hbar, \mu_\hbar)$ be a deformation of $(A, \mu)$. The associativity of $\mu_\hbar = \mu + \hbar \mu_1 + \hbar^2 \mu_2 + \cdots$ is equivalent to the following infinite system of equations:

$$\sum_{i+j=n \atop i,j \geq 0} (\mu_i \circ (\mu_j \otimes \text{id}) - \mu_i \circ (\text{id} \otimes \mu_j)) = 0, \quad n = 0, 1, 2, \ldots.$$ 

Separating the terms with $i = 0$ and $j = 0$, one obtains the above equation
in the following form:

\[
\mu \circ (\text{id} \otimes \mu_n) - \mu_n \circ (\mu \otimes \text{id}) + \mu_n \circ (\text{id} \otimes \mu) - \mu \circ (\mu_n \otimes \text{id}) = \\
\sum_{i+j=n, \ i,j \geq 1} \left( \mu_i \circ (\mu_j \otimes \text{id}) - \mu_i \circ (\text{id} \otimes \mu_j) \right).
\]

The expression in the left hand side is known as the Hochschild coboundary of \( \mu_n \), and is denoted by \( d\mu_n \). This gives an equivalent form for the associativity of \( (A_\hbar, \mu_\hbar) \):

\[
d\mu_n = \sum_{i+j=n, \ i,j \geq 1} \left( \mu_i \circ (\mu_j \otimes \text{id}) - \mu_i \circ (\text{id} \otimes \mu_j) \right).
\]

For \( n = 1 \) one has \( d\mu_1 = 0 \), i.e. the infinitesimal of any deformation is always a Hochschild cocycle.

**Definition 2.2** Two deformations, \( (A[\hbar], \mu_\hbar) \) and \( (A[\hbar], \mu'_\hbar) \), are equivalent if there exists a \( \mathbb{C}[\hbar] \) module automorphism \( u_\hbar \) of \( A[\hbar] \) such that

(i) the restriction \( u_0 \) of \( u_\hbar \) to \( A \) is the identity map;

(ii) the following diagram is commutative:

\[
\begin{array}{ccc}
A[\hbar] \otimes A[\hbar] & \xrightarrow{\mu_\hbar} & A[\hbar] \\
\downarrow u_\hbar \otimes u_\hbar & & \downarrow u_\hbar \\
A[\hbar] \otimes A[\hbar] & \xrightarrow{\mu'_\hbar} & A[\hbar]
\end{array}
\]

2.2 The Hochschild complex

The obstruction theory, which links formal deformations of algebras to Hochschild complexes, was developed by Murray Gerstenhaber in [15], [16], [17] and [18]. We give here some properties of Hochschild cocycles on commutative algebras. For a given commutative \( \mathbb{C} \) algebra \( (A, \mu) \), the Hochschild
complex on $A$ with coefficients in $A$ is the graded $\mathbb{C}$ vector space
\[ C^\bullet(A; A) = \bigoplus_{p \geq 0} \text{Hom}(A^\otimes p, A) \]
together with the coboundary operator $d : C^p(A; A) \to C^{p+1}(A; A)$ defined by
\[ d\xi = \mu \circ (\text{id} \otimes \xi) + \sum_{k=1}^{p} (-1)^k \xi \circ (\text{id}^\otimes k-1 \otimes \mu \otimes \text{id}^\otimes p-k) + (-1)^{p+1} \mu \circ (\xi \otimes \text{id}). \quad (4) \]

There are two useful operators on $C^\bullet(A; A)$. The first one we denote by $\tau$ and define as
\[ (\tau.\xi)(a_1, a_2, \ldots, a_p) = (-1)^{\frac{p(p+1)}{2}} \xi(a_p, a_{p-1}, \ldots, a_1) \]
for $\xi \in C^p(A; A)$.

**Proposition 2.1** The map $\tau$ commutes with the coboundary operator on $C^\bullet(A; A)$ and thus induces a cochain complex morphism, which we denote also by $\tau$. This morphism splits the complex into the direct sum of two sub-complexes, $C^\bullet(A; A) = C^\bullet_+(A; A) \oplus C^\bullet_-(A; A)$, where $C^\bullet_+(A; A)$ consists of all elements $\xi$ with $\tau.\xi = \xi$ and $C^\bullet_-(A; A)$ consists of all elements $\xi$ such that $\tau.\xi = -\xi$.

**Proof:** Straightforward. \[ \square \]

These sub-complexes are called the *even* and the *odd* part of $C^\bullet(A; A)$ respectively. Sometimes we refer to elements of $C^\bullet_+(A; A)$ as of *even parity* and to to elements of $C^\bullet_-(A; A)$ as of *odd parity*.

The second operator on $C^\bullet(A; A)$ the *alternation* $\text{Alt}$ is defined as
\[ \text{Alt} \xi = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma \xi \circ \mathcal{P}_\sigma \quad (5) \]
where $\xi \in C^p(A; A)$, $\mathfrak{S}_p$ is the permutation group of order $p$, $(-1)^\sigma$ is equal to $+1$ if the permutation $\sigma$ is even and $-1$ if $\sigma$ is odd, $\mathcal{P}$ is the representation of $\mathfrak{S}_p$ in the vector space $A^\otimes p$ defined by $\mathcal{P}_\sigma(a_1 \otimes \ldots \otimes a_p) = a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(p)}$.

An element $w \in C^1(A; A)$ is called *derivation* if $w(ab) = aw(b) + bw(a)$ for all $a, b \in A$ (Leibniz rule). Any cocycle in $C^1(A; A)$ is a derivation of $A$ and vice versa.

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Definition 2.3 An element \( w \in C^p(A; A) \) is called \( p \)-derivation if it obeys the Leibniz rule for each variable, i.e.
\[
w(a_1, \ldots, a'_i a''_i, \ldots, a_p) = a'_i w(a_1, \ldots, a''_i, \ldots, a_p) + a''_i w(a_1, \ldots, a'_i, \ldots, a_p)
\]
for all \( i = 1, \ldots, p \).

Proposition 2.2 Any \( p \)-derivation is a Hochschild cocycle.

PROOF: Straightforward computation. ■

Proposition 2.3 Let \( A \) be a commutative algebra, \( \xi \in C^p(A; A) \). Then \( Alt(d\xi) = 0 \).

PROOF: Using definition (4) and the additivity of operator \( Alt \), one has:
\[
Alt(d\xi) = Alt\left(\mu \circ (id \otimes \xi) + (-1)^{n+1} \mu \circ (\xi \otimes id)\right) + \sum_{k=1}^{n} (-1)^k Alt\left(\xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k})\right).
\]
The symmetry of \( \mu \) implies:
\[
\xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k}) = \xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k}) \circ \mathcal{P}_{(k,k+1)}
\]
where \( (k, k+1) \) is the permutation switching the \( k \)-th and \( (k+1) \)-th elements and leaving all the others unchanged. Then
\[
Alt\left(\xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k})\right) = Alt\left(\xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k}) \circ \mathcal{P}_{(k,k+1)}\right) = -Alt\left(\xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k})\right)
\]
so \( Alt\left(\xi \circ (id^{\otimes k-1} \otimes \mu \otimes id^{\otimes p-k})\right) = 0 \) for every \( k = 1, \ldots, n \).

It is left to consider the terms \( Alt\left(\mu \circ (id \otimes \xi) + (-1)^{p+1} \mu \circ (\xi \otimes id)\right) \).
Using the symmetry of \( \mu \), one has
\[
\mu \circ (\xi \otimes id)(a_1, \ldots, a_{p+1}) = \mu \circ (id \otimes \xi)(a_{p+1}, a_1, \ldots, a_p) = \mu \circ (id \otimes \xi) \circ \mathcal{P}_\pi(a_1, \ldots, a_{p+1})
\]
where \( \pi = (1 2 \ldots p + 1) \), the cyclic permutation. Since \((-1)^\pi = (-1)^p\), one has \( Alt(\mu \circ (id \otimes \xi) + (-1)^{p+1} \mu \circ (\xi \otimes id)) = 0 \). ■
Proposition 2.4 Any skew symmetric Hochschild cocycle is a polyderivation.

Proof: Let $\xi \in C^p(A; A)$ be skew symmetric and suppose $d\xi = 0$. Since $\xi$ is skew symmetric, it suffices to prove that $\xi(a_1a_2, a_3, \ldots, a_{p+1}) = a_1\xi(a_2, a_3, \ldots, a_{p+1}) + a_2\xi(a_1, a_3, \ldots, a_{p+1})$ for any $a_1, \ldots, a_{p+1} \in A$. Direct computations show that

\[
d\xi(a_1, a_2, a_3, \ldots, a_{p+1}) + d\xi(a_2, a_1, a_3, \ldots, a_{p+1}) =
\]

\[
= a_1\xi(a_2, a_3, \ldots, a_{p+1}) + a_2\xi(a_1, a_3, \ldots, a_{p+1}) - 2\xi(a_1a_2, a_3, \ldots, a_{p+1}) +
\]

\[
+ \xi(a_1, a_2a_3, \ldots, a_{p+1}) + \xi(a_2, a_1a_3, \ldots, a_{p+1})
\]

and

\[
\left( d\xi(a_1, a_2, a_{p+1}, a_3, \ldots, a_p) + d\xi(a_2, a_1, a_{p+1}, a_3, \ldots, a_p) \right) +
\]

\[
+ \left( d\xi(a_{p+1}, a_1, a_2, a_3, \ldots, a_p) + d\xi(a_{p+1}, a_2, a_1, a_3, \ldots, a_p) \right) =
\]

\[
= (-1)^p \left( a_1\xi(a_2, a_3, \ldots, a_{p+1}) + a_2\xi(a_1, a_3, \ldots, a_{p+1}) -
\]

\[
- \xi(a_1, a_2a_3, \ldots, a_{p+1}) - \xi(a_2, a_1a_3, \ldots, a_{p+1}) \right).
\]

Therefore

\[
0 = d\xi(a_1, a_2, a_3, \ldots, a_{p+1}) + d\xi(a_2, a_1, a_3, \ldots, a_{p+1}) +
\]

\[
+ (-1)^p \left( d\xi(a_1, a_2, a_{p+1}, a_3, \ldots, a_p) + d\xi(a_2, a_1, a_{p+1}, a_3, \ldots, a_p) +
\]

\[
+ d\xi(a_{p+1}, a_1, a_2, a_3, \ldots, a_p) + d\xi(a_{p+1}, a_2, a_1, a_3, \ldots, a_p) \right) =
\]

\[
= 2 \left( a_1\xi(a_2, a_3, \ldots, a_{p+1}) + a_2\xi(a_1, a_3, \ldots, a_{p+1}) - \xi(a_1a_2, a_3, \ldots, a_{p+1}) \right).
\]

\]

Later (see Section 3.4) we shall also use the Hochschild complex for a non-commutative algebra with coefficients in two-sided modules.
2.3 The local Hochschild complex

Recall that the ring of differential operators on $A = \mathbb{C}^\infty(M)$ is the associative algebra generated over $A$ by all derivations $A \to A$ and all multiplication operators $L_a : A \to A, b \mapsto ab$. A map $A^\otimes_p \to A$ is called a $p$-differential operator on $A$ if it is a differential operator $A \to A$ with respect to every its variable. Denote by $\tilde{C}^p(A; A)$ the $\mathbb{C}$ space of $p$-differential operators on $A$. The direct sum of these spaces forms the sub-complex $(\tilde{C}^\bullet(A; A), d)$ of the complex $(C^\bullet(A; A), d)$, it is called the local Hochschild complex of $A$.

We need the local Hochschild complex because we deal actually with the whole sheaf of function algebras on $M$ rather than with the algebra $A$ of global sections. Clearly, all propositions of Section 2.2 are valid for the local Hochschild complex. In particular, this complex can be decomposed into the sum of even and odd parts, $\tilde{C}^\bullet(A; A) = \tilde{C}^\bullet(A; A) \oplus \tilde{C}^\bullet(A; A)$ where $\tilde{C}^\bullet(A; A) = C^\bullet(A; A) \cap \tilde{C}^\bullet(A; A)$.

Denote by $\Lambda_p(M)$ the space of all $p$-vector fields on $M$, i.e. all $p$-derivations $A^\otimes_p \to A$.

**Theorem 2.1** Let $A = \mathbb{C}^\infty(M)$, $\theta \in Z^p(\tilde{C}^\bullet(A; A), d)$. Then

(I) Alt $\theta$ is a skew symmetric $p$-derivation on $A$;

(II) The difference $\theta - \text{Alt} \theta$ is a Hochschild coboundary, i.e. there exists $\xi \in \tilde{C}^{p-1}(A; A)$ such that $\theta - \text{Alt} \theta = d\xi$;

(III) The map $\text{Alt} : Z^p(\tilde{C}^\bullet(A; A), d) \to \Lambda_p(M)$ induces a vector space isomorphism $H^p(\tilde{C}^\bullet(A; A), d) \to \Lambda_p(M)$.

**Proof:** The algebraic version of (I) and (II) has been proven by G. Hochschild, B. Kostant and A. Rosenberg [21]. A proof for $A = \mathbb{C}^\infty(M)$ see in [23], 4.6.1.1. Claim (III) has been proven by J. Vey [8], see also [26]. □

**Corollary 2.1** Let $\theta \in Z^p(\tilde{C}^\bullet(A; A), d)$, then $\text{Alt} \theta = 0$ implies that $\theta$ is a Hochschild coboundary.

2.4 Infinitesimal of a deformation

It was shown above that the associativity of a deformation $\mu_\hbar = \mu + \hbar \mu_1 + \cdots$ forces the infinitesimal $\mu_1$ to be a Hochschild cocycle. It can be decomposed as $\mu_1 = \mu_1^+ + \mu_1^- \in C^2_+(A; A) \oplus C^2_-(A; A)$ (see Proposition 2.1) with $\mu_1^- = \text{Alt} \mu_1$ and $\mu_1^+ = \mu_1 - \mu_1^-$. 

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Proposition 2.5 If $d\mu = 0$ then both $d\mu^+ = 0$ and $d\mu^- = 0$.

**Proof:** Clearly, $d\mu_1 = 0$ implies $d(\mu_1 \circ P_{(12)}) = 0$. On the other hand, $\mu^+_1 = \frac{1}{2}(\mu_1 + \mu_1 \circ P_{(12)})$ and $\mu^-_1 = \frac{1}{2}(\mu_1 - \mu_1 \circ P_{(12)})$. ■

The symmetric part, $\mu^+_1$, is responsible for commutative deformations of $(A,\mu)$. For $A = C^\infty(M)$ any deformation with $\mu^+_1 + \mu^-_1$ as the infinitesimal is equivalent to a deformation with infinitesimal $\mu^-_1$ as the infinitesimal. Therefore one can assume from the beginning that $\mu^+_1 = 0$ i.e. that the infinitesimal $\mu_1$ is skew symmetric. By Proposition 2.4, $d\mu_1 = 0$ implies that $\mu_1$ is a bivector field.

To simplify our considerations, throughout the text we put the following restriction on any deformation $\mu_h$.

**Definition 2.4** We say that a deformation $\mu_0 + \mu_1 h + \mu_2 h^2 + \mu_3 h^3 + \cdots$ obeys the Parity Convention if $\mu_{2k} \in \widetilde{C}^2_-(A;A)$ and $\mu_{2k+1} \in \widetilde{C}^2_+(A;A)$.

**Lemma 2.1** Let $(A_h,\mu_h)$ be a deformation of commutative algebra $A$ obeying the Parity Convention. Then its infinitesimal $\mu_1$ satisfies the Jacobi identity.

**Proof:** For $n = 2$ equation (3) takes the form

$$\mu_1 \circ (\mu_1 \otimes \text{id}) - \mu_1 \circ (\text{id} \otimes \mu_1) = d\mu_2. \quad (6)$$

The right hand side of (3) is a 3-coboundary. Straightforward computation with the use of commutativity of $A$ and symmetry of $\mu_2$ proves that $d\mu_2(a,b,c) + \text{Cycl} = 0$ for any $a,b,c \in A$, where $\text{Cycl}$ denotes the terms obtained by taking of all the cyclic permutations of $a,b,c$. For the left hand side of (3), direct computations with the use of the skew symmetry of $\mu_1$ show that the expression $\mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) + \text{Cycl}$ is equal to the left hand side of the Jacobi identity. ■

**Definition 2.5** A quantization of a commutative algebra $(A,\mu)$ with a given element $b \in C^2_+(A;A)$ is a non-commutative deformation of $(A,\mu)$ with $b$ as the infinitesimal.

The arguments of this section show us that the infinitesimal of any quantization is of necessity a Poisson bracket.
3 Equivalence of monoidal categories related to Drinfeld algebras

In this section we reduce the problem of associative quantization invariant under the Etingof—Kazhdan quantum group $U_\hbar(g, r)$ to the problem of non-associative $G$ invariant quantization. To do this, we consider two monoidal categories associated to the problem of invariant quantization of function algebras, and, for any Belavin—Drinfeld $r$-matrix, we construct the equivalence for these two categories.

Throughout this section, $R$ denotes a commutative associative ring with unity of characteristic 0 (we shall be interested in cases $R = \mathbb{C}$ and $R = \mathbb{C}[\hbar]$). All algebras and modules are defined over $R$.

3.1 Monoidal categories and Drinfeld algebras

The concept of monoidal category was introduced by J. Bénabou [4] and S. Mac Lane [24], see [25], Chapter VII.

**Definition 3.1** A monoidal category is a 6-tuple $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ where $\mathcal{C}$ is a category, $\otimes$ is a functor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, $a$ is a natural isomorphism of functors $(U, V, W) \mapsto (U \otimes V) \otimes W$ and $(U, V, W) \mapsto U \otimes (V \otimes W)$, $\mathbf{1}$ is a natural isomorphism of the functor $V \mapsto \mathbf{1} \otimes V$ and the identity functor $\text{Id}$, $r$ is a natural isomorphism of functors $V \mapsto V \otimes \mathbf{1}$ and $\text{Id}$, and the following two diagrams are commutative for any $U, V, W, S \in \text{Obj}(\mathcal{C})$:

\[
\begin{array}{ccc}
(U \otimes V) \otimes W \otimes S & \xrightarrow{a} & (U \otimes V) \otimes (W \otimes S) \\
\downarrow{a \otimes \text{Id}} & & \downarrow{\text{Id} \otimes a} \\
(U \otimes (V \otimes W)) \otimes S & \xrightarrow{a} & U \otimes ((V \otimes W) \otimes S)
\end{array}
\]
where \(a\) over the left upper arrow means \(a_{U \otimes V,W,S}\) and so on. The isomorphism \(a\) is called the associativity constraint of \(\mathcal{C}\).

**Definition 3.2** A monoidal functor from a monoidal category \((\mathcal{C}, \otimes, a)\) to a monoidal category \((\mathcal{D}, \boxtimes, b)\) is a pair \((L, u)\) where \(L\) is a functor \(\mathcal{C} \to \mathcal{D}\), \(u\) is a natural transformation from the functor \((U,V) \mapsto L(U \otimes V)\) to \((U,V) \mapsto L(U) \boxtimes L(V)\) such that \(u \circ L(l_C) = l_D\), \(u \circ L(r_C) = r_D\) and the following diagram is commutative:

\[
\begin{array}{ccc}
L((U \otimes V) \otimes W) & \xrightarrow{u} & L(U \otimes V) \boxtimes L(W) & \xrightarrow{u \boxtimes \text{id}} & (L(U) \boxtimes L(V)) \boxtimes L(W) \\
L(a) \downarrow & & & & b \\
L(U \otimes (V \otimes W)) & \xrightarrow{u} & L(U) \boxtimes L(V \otimes W) & \xrightarrow{\text{id} \boxtimes u} & L(U) \boxtimes (L(V) \boxtimes L(W))
\end{array}
\]

**Example 3.1** Let \((B, \Delta, \varepsilon)\) be a bialgebra over \(\mathbb{C}\). Consider the category \(\mathcal{C}_B\) of all modules over \(B\) which are of finite dimension as \(\mathbb{C}\) vector spaces. The category \(\mathcal{C}_B\) possesses a monoidal structure: for given \(B\) modules \(M\) and \(N\) one can introduce a \(B\) module structure on their tensor product \(M \otimes N\) over \(\mathbb{C}\) by putting \(b(m \otimes n) = \sum b'm \otimes b'n\) where \(b \in B\), \(m \otimes n \in M \otimes N\) and \(\Delta(b) = \sum b' \otimes b''\).

V. G. Drinfeld has generalized the above example in the following way.

**Definition 3.3** A Drinfeld algebra is a 6-tuple \((B, m, \iota, \Delta, \varepsilon, \Phi)\) where \((B, m, \iota)\) is an associative \(R\) algebra with a unit map \(\iota : R \to B\), \((B, \Delta, \varepsilon)\) is a coalgebra with comultiplication \(\Delta : B \to B \otimes_R B\) and counit \(\varepsilon : B \to R\) are
algebra morphisms, $\Phi \in B \otimes_R B \otimes_R B$ is an invertible element, and the following conditions are obeyed, where we use the notation $a \cdot b = m(a,b)$:

(i) $(\text{id} \otimes \Delta) \circ \Delta(b) \cdot \Phi = \Phi \cdot (\Delta \otimes \text{id}) \circ \Delta(b)$ for any $b \in B$;
(ii) $(\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\Phi) = (1 \otimes \Phi) \cdot (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \cdot (\Phi \otimes 1);
(iii) $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta$;
(iv) $(\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) = 1$.

The element $\Phi$ is called the Drinfeld associator of $B$. The condition (ii) is called the pentagon or Mac Lane identity for $\Phi$.

**Example 3.2** The trivial associator, $\Phi = 1 \otimes 1 \otimes 1$, obeys all the conditions of Definition 3.3. In this case $B$ is a usual coassociative bialgebra.

**Remark 3.1** We did not put the antipode ([1], Chapter 2, Subsection 1.2) in the above definition, because we will not referring to it in any of our results. Yet in the main example, the QUE Drinfeld algebra (Definition 3.9) the antipode exists as a deformation of the standard antipode on $U(\mathfrak{g})$.

Drinfeld algebras were introduced in [12] under the name quasi Hopf algebras. For a given Drinfeld algebra $(B, \Phi)$ over $R$, denote by $\mathcal{D}_B$ the category of $B$ modules which are free $R$ modules of finite rank. Note that $\mathcal{D}_B$ is a sub-category of the category $\mathcal{D}_R$. In particular, one can consider in $\mathcal{D}_B$ tensor products over $R$.

**Proposition 3.1** The category $\mathcal{D}_B$ possesses a monoidal structure with associativity constraint defined by $\Phi$.

**Proof:** Consider the usual tensor product $(U, V) \mapsto U \otimes_R V$ over $R$ in $\mathcal{D}_B$. One can equip $U \otimes_R V$ with a $B$-action by setting $b.(u \otimes v) = \Delta(b) \cdot (u \otimes v)$ for any $b \in B$, $u \in U$, $v \in V$. Denote this $B$ module by $U \otimes_B V$.

Write $\Phi$ as $\sum \Phi' \otimes \Phi'' \otimes \Phi'''$ and for any $u \in U$, $v \in V$, $w \in W$ define

$$a_{UVW}((u \otimes v) \otimes w) = \sum (\Phi' \cdot u) \otimes ((\Phi'' \cdot v) \otimes (\Phi''' \cdot w)).$$

One can check that $(\mathcal{D}_B, \otimes_B, a)$ is a monoidal category. ■

The difference between Drinfeld algebras and the usual bialgebras is that the associativity constraint in the category of modules over a Drinfeld algebra is in general non-trivial. We shall quantize in a category over a Drinfeld algebra, i.e. we shall look for a $\Phi$-associative deformed multiplication rather than for a multiplication with the ordinary associativity.
Definition 3.4 Monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are called equivalent if there exist two monoidal functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that the both compositions of $F$ and $G$ are naturally equivalent to the corresponding identity functors.

Theorem 3.1 Let $(B, \Delta, \Phi)$ be a Drinfeld algebra, $F \in B \otimes_R B$ an invertible element. Define $\tilde{\Delta}$ and $\tilde{\Phi}$ by $\tilde{\Delta}(b) = F \cdot \Delta(b) \cdot F^{-1}$ for all $b \in D$ and $\tilde{\Phi} = (1 \otimes F) \cdot (\text{id} \otimes \Delta)(F) \cdot \Phi \cdot (\Delta \otimes \text{id})(F^{-1}) \cdot (F \otimes 1)^{-1}$. Then $(B, m, \iota, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is also a Drinfeld algebra and the categories $\mathcal{D}_B$ and $\mathcal{D}_{\tilde{B}}$ are monoidally equivalent.

Proof: See [12]. □

Note that gauge transform is called in [12] the twisting by $F$.

Definition 3.5 The Drinfeld algebra $(B, m, \iota, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ of Theorem 3.1 is called a gauge transformation of $(B, m, \iota, \Delta, \varepsilon, \Phi)$ determined by $F$. Two Drinfeld algebras are called gauge equivalent if there exists a gauge transformation from one to the other.

Theorem 3.1 states that if two Drinfeld algebras are gauge equivalent then the corresponding monoidal categories are equivalent. The equivalence is given by the pair $(\text{Id}, F)$.

3.2 Monoids in a category and invariant multiplications

The monoidal structure in a category makes it possible to construct algebra-like objects in it. In the category $\mathcal{D}_B$, the corresponding multiplication is called $B$ invariant.

Definition 3.6 A monoid in a monoidal category $(\mathcal{C}, \otimes, 1, a)$ is a triple $(A, \mu, \eta)$ where $A \in \text{Obj}(\mathcal{C})$, $\mu : A \otimes A \to A$, and $\eta : 1 \to A$ are arrows of $\mathcal{C}$, and the following three diagrams are commutative:
Example 3.3 Let $B$ be a bialgebra. Then a monoid $(A, \mu)$ in the category $\mathcal{D}_B$ is a usual associative $B$ algebra with invariance property $x.\mu(a, b) = \mu(x' a, x'' b)$, where $\Delta(x) = \sum x' \otimes x''$.

Example 3.4 Let $(B, \Delta, \Phi)$ be a Drinfeld algebra where $\Phi = \sum \Phi' \otimes \Phi'' \otimes \Phi'''$. Then a monoid $(A, \mu)$ in $\mathcal{D}_B$ is a $B$ algebra with a multiplication $a \cdot b = \mu(a \otimes b)$ obeying

$$(a \cdot b) \cdot c = \sum \Phi' a \cdot (\Phi'' b \cdot \Phi''' c)$$

rather than the usual associativity. Also this can be written as $\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu) \circ \Phi$ where $\Phi$ is considered as a left multiplication operator. We call the multiplication law of such $A$ a $B$ invariant multiplication or $\Phi$-associative multiplication.

Definition 3.7 Let $\mathfrak{g}$ be a Lie algebra, $\text{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, $\Phi \in \text{U}(\mathfrak{g})^\otimes$ an element satisfying the conditions of Definition 3.3. Any multiplication in monoid in the category $\mathcal{D}_{(\text{U}(\mathfrak{g}), \Phi)}$ we call $\mathfrak{g}$ invariant.

Example 3.5 Consider $\text{U}(\mathfrak{g})$ as a Drinfeld algebra with the trivial associator. The algebra $(A, \mu)$ of $\mathcal{C}^\infty$ functions on $M$ is $\text{U}(\mathfrak{g})$ invariant if and
only if \( x \circ \mu = \mu \circ \Delta(x) \) for any \( x \in U(g) \) considering as a differential operator on \( A \). Since the comultiplication \( \Delta \) is an algebra \((U(g), m)\) morphism, it suffices to check the above identity on elements of any set of generators of \( U(g) \). For instance, one can take elements of the Lie algebra \( g \) itself: 
\[
x(ab) = \mu(\Delta(x)(a \otimes b)) = \mu((x \otimes 1 + 1 \otimes x)(a \otimes b)) = x(a)b + ax(b).
\]
Thus the multiplication \( \mu \) is \( g \) invariant if and only if elements of \( g \) act on \( A \) by derivations.

### 3.3 Belavin—Drinfeld r-matrices and QUE Drinfeld algebras

Let \( g \) be a complex simple Lie algebra. It is well known that the \( g \) invariant elements of \( \bigwedge^3 g \) form a one dimensional subspace. A base is given by the element \( \varphi \) representing \( x \otimes y \otimes z \mapsto ([x, y], z) \) where \( (\cdot, \cdot) \) is the Killing form in \( g \).

**Definition 3.8** The equation \([ [r, r] ] = \varphi\) is called the modified classical Yang—Baxter equation. Any solution of it is called a Belavin—Drinfeld r-matrix on \( g \).

**Example 3.6** Let \( \Omega^+ \) be a system of positive roots for \( g \), \( X_\alpha \) corresponding root vectors satisfying \((X_\alpha, X_{-\alpha}) = 1\). Then the element of \( \bigwedge^2 g \) defined by
\[
r = \sum_{\alpha \in \Omega^+} X_\alpha \wedge X_{-\alpha}
\]
is a Belavin—Drinfeld r-matrix. This element is called the Drinfeld—Jimbo r-matrix and is the most important example of a Belavin—Drinfeld r-matrix.

All solutions for the modified classical Yang—Baxter equation were found by A. A. Belavin and V. G. Drinfeld [3], [4]. The significance of Belavin—Drinfeld r-matrices in the fact [11] that any such element \( r \in \bigwedge^2 g \) determines a Lie algebra structure on the dual vector space \( g^* \). This structure is consistent with the Lie bracket on \( g \), i.e. \( r \) is a Chevalley–Eilenberg 2-coboundary. The pair \((g, r)\) is called a Belavin—Drinfeld Lie bialgebra. A Belavin—Drinfeld r-matrix is a natural initial term for a coalgebra deformation, as a Poisson structure is a natural initial term for an algebra deformation.
Remark 3.2 It follows from the M. Gerstenhaber’s obstruction theory \[13\] that since \(H^2(U(\mathfrak{g}); U(\mathfrak{g})) = 0\) for \(\mathfrak{g}\) semisimple, there are no non-trivial deformations of the multiplication in the universal enveloping algebra \(U(\mathfrak{g})\).

Definition 3.9 A deformation \((U(\mathfrak{g})[\hbar], m, \iota, \Delta_\hbar, \varepsilon, \Phi_\hbar)\) of the Drinfeld algebra \((U(\mathfrak{g}), m, \iota, \Delta, \varepsilon, 1 \otimes 1 \otimes 1)\) is called a quantized universal enveloping (QUE) Drinfeld algebra. A quantization of a Belavin—Drinfeld Lie bialgebra \((\mathfrak{g}, r)\) is a QUE Drinfeld algebra with the deformed comultiplication of the form \(\Delta_\hbar(x) = \Delta(x) + \hbar[\Delta(x), r] + \cdots\).

Note that we have a normalized form in which the multiplication is undeformed, which is always possible by Remark 3.2.

3.4 Uniqueness of Drinfeld quantization for \(\mathfrak{g}\) simple and existence of \(F_\hbar\) for any Belavin–Drinfeld infinitesimal

For a given Lie algebra \(\mathfrak{g}\), denote by \(H^*_{CE}(\mathfrak{g}; U(\mathfrak{g}) \otimes 2)\) its Chevalley–Eilenberg cohomology module and by \(H^*_H(U(\mathfrak{g}); U(\mathfrak{g}) \otimes 2)\) its Hochschild cohomology module. To construct the Chevalley–Eilenberg complex, one uses the action \((X, v) \mapsto [\Delta(X), v] = \Delta(X)v - v\Delta(X)\) of \(\mathfrak{g}\) on \(U(\mathfrak{g}) \otimes U(\mathfrak{g})\), while in the Hochschild complex the right and left actions determined by the standard comultiplication \(\Delta\) on \(U(\mathfrak{g})\) are utilized.

Lemma 3.1 Let \(\mathfrak{g}\) be a finite dimensional Lie algebra. Then \(H^1_{CE}(\mathfrak{g}; U(\mathfrak{g}) \otimes 2) = 0\) implies \(H^1_H(U(\mathfrak{g}); U(\mathfrak{g}) \otimes 2) = 0\).

Proof: The restriction \(\tilde{\xi}\) to \(\mathfrak{g}\) of a Hochschild 1-cocycle \(\xi : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})\) is a Chevalley–Eilenberg cocycle. Indeed, \(d_H\xi = 0\) implies
\[
\xi(XY) = \Delta(X)\xi(Y) + \xi(X)\Delta(Y) \quad \text{for any } X, Y \in \mathfrak{g}. \tag{7}
\]
The relation \(XY - YX - [X,Y] = 0\) in \(U(\mathfrak{g})\) (where \([X,Y]\) is the Lie bracket in \(\mathfrak{g}\)) implies \(\xi([X,Y]) = \xi(XY) - \xi(YX)\). Substituting this expression with (7), one has
\[
\xi([X,Y]) = \Delta(X)\xi(Y) - \xi(Y)\Delta(X) + \xi(X)\Delta(Y) - \Delta(Y)\xi(X)
\]
for any \(X, Y \in \mathfrak{g}\), that is, \(\xi\) is a Chevalley–Eilenberg cocycle.

The above considerations together with the condition \(H_{CE}^1\left(U(\mathfrak{g}) \otimes U(\mathfrak{g})\right) = 0\) imply that there exists \(\eta \in U(\mathfrak{g}) \otimes U(\mathfrak{g})\) such that \(d_{CE}\eta = \tilde{\xi}\) i.e. such that \(\Delta(X)\eta - \eta\Delta(X) = \tilde{\xi}(X)\) for any \(X \in \mathfrak{g}\). The element \(\eta\) can be considered also as a Hochschild 0-cocycle with \(d_H\eta(x) = \Delta(x)\eta - \eta\Delta(x)\). Therefore \(d_H\eta(X) = d_{CE}\eta(X)\), and also \(d_H\eta(X) = \xi(X)\) for any \(X \in \mathfrak{g}\).

Since both \(\xi\) and \(d_H\eta\) are Hochschild cocycles, they obey the condition \(\mathfrak{g}\). Combining \(\mathfrak{g}\) with the fact that \(\mathfrak{g}\) generates the associative algebra \(U(\mathfrak{g})\), one concludes that \(\tilde{\xi}\) can be extended uniquely to a Hochschild cocycle, and that \(d_H\eta(x) = \xi(x)\) for all \(x \in U(\mathfrak{g})\). Thus any cocycle in \(C^1_{Hoch}\left(U(\mathfrak{g}); U(\mathfrak{g})^{\otimes 2}\right)\) can be resolved. ■

**Lemma 3.2** Let \(\delta : A \rightarrow B\) is a homomorphism of two algebras, \(\delta_h : A[h] \rightarrow B[h]\) a deformation of \(\delta = \delta_0\). Consider the structure of two-sided \(A\) module on \(B\) determined by \(\delta\) and suppose \(H_{Hoch}^1(A; B) = 0\). Then there exists an invertible element \(F_h \in B[h]\) such that \(\delta(x) = F_h \cdot \delta_h(x) \cdot F_h^{-1}\) for any \(x \in A[h]\).

**Proof:** Obviously, the element \(1 + \hbar k f_k\) is invertible in \(B[h]\) for any \(f_k \in B\). Put \(\delta^{(1)} = \delta_h\) and prove that for any \(k = 1, 2, \ldots\) one can find an element \(f_k \in B\) such that each algebra homomorphism \(A[h] \rightarrow B[h]\) defined by \(\delta^{(k+1)} = (1 + \hbar k f_k)\delta^{(k)}(a)(1 + \hbar f_k)^{-1}\) is of the form \(\delta + \hbar k + \hbar\delta_k + \hbar^2\delta_k + \cdots\). Indeed, gathering all the terms with \(\hbar k\) the equation \((1 + \hbar k f_k)(\delta(a) + \hbar k \delta_k + \hbar^2 \delta_k + \cdots) = (\delta(a) + \hbar k \delta_k + \hbar^2 \delta_k + \cdots)(1 + \hbar f_k)\), one obtains the equation \(f_k \delta(a) - \delta(a) f_k = -\delta_k(a)\) for \(f_k\). One checks readily that the left hand side of this equation is equal to \(-df_k(a)\) where \(d\) is the Hochschild coboundary operator. Thus the equation for \(f_k\) is of the form \(df_k = \delta_k\). Since \(H_{Hoch}^1(A; B) = 0\), the equation has a solution if \(\delta_k\) is a Hochschild cocycle, which is the case since \(\delta^{(k)} = \delta + \hbar k \delta_k + \cdots\) is multiplicative, i.e. \(\delta^{(k)}(ab) = \delta^{(k)}(a)\delta^{(k)}(b)\) for any \(a, b \in A\). For \(\delta'_k\), this gives the property \(\delta_k(ab) = \delta(a)\delta'_k(b) + \delta'_k(a)\delta(b)\) which exactly means that \(\delta'_k\) is a Hochschild cocycle. Finally, denote by \(F_h = \Pi(1 \otimes 1 + \hbar^i f_i)\), the infinite product on the left. ■

**Proposition 3.2** Let \(\mathfrak{g}\) be a semisimple complex Lie algebra, \(\Delta_h : U(\mathfrak{g})[h] \rightarrow U(\mathfrak{g})^{\otimes 2}[h]\) a deformation of the standard comultiplication \(\Delta\) on \(U(\mathfrak{g})\). Then
there exists an invertible element $F_h \in \mathcal{U}(\mathfrak{g}) \otimes [\hbar] \mathcal{U}(\mathfrak{g}) \otimes [\hbar]$ such that $\Delta(x) = F_h^{-1} \cdot \Delta(x) \cdot F_h$ for any $x \in \mathcal{U}(\mathfrak{g}) \otimes [\hbar]$.

**Proof:** It is well known that for $\mathfrak{g}$ semi-simple and for any $\mathfrak{g}$ module $V$ one has $H^1_{CE}(\mathfrak{g}; V) = 0$ (see for instance [30], Lecture 19, Corollary 1). Therefore $H^1_{CE}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}) \otimes [\hbar]) = 0$, and by Lemma 3.1, this implies $H^1_{Hoch}(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g}) \otimes [\hbar]) = 0$. Now one can apply Lemma 3.2 to the case when $B = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, $\delta = \Delta$ and $\delta \hbar = \Delta \hbar$.

P. Etingof and D. Kazhdan have proven [14] that any Lie bialgebra can be quantized. For a Belavin–Drinfeld Lie bialgebra $(\mathfrak{g}, r)$ this means that there exists a bialgebra deformation $\mathcal{U}(\mathfrak{g}, r) = (\mathcal{U}(\mathfrak{g}) \otimes [\hbar], \Delta \hbar)$ with undeformed multiplication (see Remark 3.2) and comultiplication $\Delta \hbar$ of the form $\Delta \hbar(x) = \Delta(x) + \hbar[\Delta(x), r] + \cdots$. We shall call $\mathcal{U}(\mathfrak{g}, r)$ the Etingof–Kazhdan quantum group corresponding to $r$. In this case, the element $F \hbar$ of Proposition 3.2 is of the form $F \hbar = 1 \otimes 1 + \frac{1}{2} \hbar r + \cdots$.

**Proposition 3.3** Let $\Phi \hbar \in \mathcal{U}(\mathfrak{g}) \otimes [\hbar]$ be an invertible element $\Phi \hbar \equiv 1 \pmod{\hbar^2}$ satisfying the Pentagon Identity. Then there exists a gauge transformation making $\Phi \hbar$ into $\Phi \hbar \equiv 1 + \hbar^2 \varphi \pmod{\hbar^3}$ for some $\varphi \in \wedge^3 \mathfrak{g}$. If $\Phi \hbar$ satisfies $\Phi \hbar \cdot \Delta_3(x) \cdot \Phi \hbar^{-1} = \Delta_3(x)$ for the standard comultiplication $\Delta$ and for all $x \in \mathcal{U}(\mathfrak{g})$ then the element $\varphi$ is $\mathfrak{g}$ invariant.

**Proof:** We use the Cartier cochain complex

$$\mathbb{C} \rightarrow \mathcal{U}(\mathfrak{g}) \xrightarrow{d_C} \mathcal{U}(\mathfrak{g}) \otimes [\hbar] \mathcal{U}(\mathfrak{g}) \xrightarrow{d_C} \mathcal{U}(\mathfrak{g}) \otimes [\hbar] \mathcal{U}(\mathfrak{g}) \otimes [\hbar] \rightarrow \cdots,$$

$$d_C = 1 \otimes \text{id} \otimes \varphi + \sum_{k=1}^p (-1)^k \text{id} \otimes (k-1) \otimes \Delta \otimes \text{id} \otimes k + (-1)^{k+1} \text{id} \otimes \varphi \otimes 1.$$  

Let $\Phi \hbar = 1 + \hbar^2 \psi + \cdots$. Putting together all the terms of the pentagon identity for $(\Delta \hbar, \Phi \hbar)$ which are quadratic in $\hbar$, and using $\Delta \hbar(1) = 1 \otimes 1$, one obtains $d_C \psi = 0$. It is well known (see for example [12], Proposition 2.2) that the cohomology module for the Cartier complex is equal to $\Lambda(\mathfrak{g})$. Thus there exist $\varphi \in \wedge^3 \mathfrak{g}$ and $f \in \mathcal{U}(\mathfrak{g}) \otimes [\hbar]$ such that $\psi = \varphi + d_C f$. Take $\widetilde{\Phi} \hbar = 1 + \hbar^2 f$ and $\widetilde{\Phi} \hbar = (\widetilde{\Phi} \hbar \otimes 1) \cdot (\Delta_3 \otimes \text{id})(\widetilde{\Phi} \hbar) \cdot (\text{id} \otimes \Delta \hbar)(\widetilde{\Phi} \hbar^{-1}) \cdot (1 \otimes \widetilde{\Phi} \hbar)^{-1}$. Obviously, $\widetilde{\Phi} \hbar \equiv 1 + \hbar^2 \varphi \pmod{\hbar^3}$.
Suppose now that \( \Phi_h \cdot \Delta_3(x) \cdot \Phi_h^{-1} = \Delta_3(x) \) for all \( x \in U(\mathfrak{g}) \). Taking the terms with \( h^2 \) in this equation, one obtains \( \varphi \cdot \Delta_3(x) = \Delta_3(x) \cdot \varphi \) for any \( x \in U(\mathfrak{g}) \) which is an expression for the \( \mathfrak{g} \) invariance of \( \varphi \). 

The following theorem follows from results in \([12]\).

**Theorem 3.2** Let \( r \) be a Belavin–Drinfeld \( r \)-matrix, \( U(\mathfrak{g}, r) \) the corresponding Etingof–Kazhdan quantum group, \( m \) and \( \Delta \) the standard multiplication and comultiplication on \( U(\mathfrak{g}) \) respectively. Then there exists an invertible \( \mathfrak{g} \) invariant element \( \Phi_h \in U(\mathfrak{g})^\otimes 3[\hbar] \) such that

1. \((U(\mathfrak{g})[\hbar], m, \Delta, \Phi_h)\) is a Drinfeld algebra;
2. \( \Phi_h \equiv 1 + h^2 \varphi \pmod{h^3} \) where \( 0 \neq \varphi \in (\wedge^3 \mathfrak{g})^\mathfrak{g} \);
3. There exists an element \( F_h = 1 + hr + \cdots \) which defines a gauge equivalence between \( U(\mathfrak{g}, r) \) and \((U(\mathfrak{g})[\hbar], m, \Delta, \Phi_h)\). Therefore the module categories corresponding to \( U(\mathfrak{g}, r) \) and \((U(\mathfrak{g})[\hbar], m, \Delta, \Phi_h)\) are equivalent as monoidal categories.

**Proof:** First we prove that there exists an element \( \Phi_h \in U(\mathfrak{g})^\otimes 3[\hbar] \) satisfying \( \Phi_h \equiv 1 \pmod{h^2} \) and such that \((U(\mathfrak{g})[\hbar], \Delta, \Phi_h)\) is a Drinfeld algebra, gauge equivalent to \( U(\mathfrak{g}, r) \).

By Proposition \([32]\), there exists \( F_h \) such that \( F_h^{-1} \cdot \Delta(x) \cdot F_h = \Delta_h(x) \). In our case, \( \Delta_h \) is coassociative, that is,

\[
(\Delta_h \otimes \text{id}) \circ \Delta_h = (\text{id} \otimes \Delta_h) \circ \Delta_h. \tag{8}
\]

Using multiplicativity of \( \Delta \), one has for the left hand side of (8):

\[
(\Delta_h \otimes \text{id})(\Delta_h(x)) = (F_h \otimes 1)^{-1} \cdot (\Delta \otimes \text{id})(\Delta_h(x)) \cdot (F_h \otimes 1) =
\]

\[
= (F_h \otimes 1)^{-1} \cdot (\Delta \otimes \text{id})(F_h^{-1} \cdot \Delta(x) \cdot F_h) \cdot (F_h \otimes 1) =
\]

\[
= (F_h \otimes 1)^{-1} \cdot (\Delta \otimes \text{id})(F_h^{-1}) \cdot \Delta_3(x) \cdot (\Delta \otimes \text{id})(F_h) \cdot (F_h \otimes 1)
\]

where the notation \( \Delta_3 = (\Delta \otimes \text{id}) \circ \Delta \) is used. Similarly, the right hand side of (8) is of the form \((1 \otimes F_h)^{-1} \cdot (\text{id} \otimes \Delta)(F_h^{-1} \cdot \Delta_3(x) \cdot (\text{id} \otimes \Delta)(F_h) \cdot (1 \otimes F_h))\) since \( (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \). Put \( \Phi_h = (\text{id} \otimes \Delta)(F_h) \cdot (1 \otimes F_h) \cdot (F_h \otimes 1)^{-1} \cdot (\Delta \otimes \text{id})(F_h^{-1}) \). Clearly, \( \Phi_h \) satisfies \( \Phi_h \cdot \Delta_3(x) \cdot \Phi_h^{-1} = \Delta_3(x) \) for all \( x \in U(\mathfrak{g})[\hbar] \). Considering \((U(\mathfrak{g})[\hbar], \Delta_h)\) as a Drinfeld algebra with the trivial associator and using Theorem \([31]\), one concludes that \((U(\mathfrak{g})[\hbar], \Delta, \Phi_h)\) is also a Drinfeld algebra.
We prove that $\Phi_h$ has no linear term in $h$. Recall that $F_h \equiv 1 \otimes 1 + hr \pmod{h^2}$, thus $F_h^{-1} \equiv 1 \otimes 1 - hr \pmod{h^2}$. Using the standard notation $1 \otimes r = r^{23}$, $r \otimes 1 = r^{12}$ and $r^{13} = \sum a_i \otimes 1 \otimes b_i$ for $r = \sum a_i \otimes b_i$ and the fact that $r \in g \otimes g$, one has $(\text{id} \otimes \Delta)(r) = r^{12} + r^{13}$ and $(\Delta \otimes \text{id})(r) = r^{13} + r^{23}$. Hence the linear in $h$ term of $\Phi_h$ is equal to $r^{12} + r^{13} + r^{23} - r^{12} - r^{13} - r^{23} = 0$.

Combining this with Proposition 3.2, Proposition 3.3 and Theorem 3.1, one completes the proof. ■

**Remark 3.3** As we mentioned above, the gauge transformation $F_h = 1 \otimes 1 + hr + \cdots$ from $U_h(g, r)$ to $(U(g)[h], m, \Delta, \Phi_h)$ establishes a monoidal equivalence between $C = \mathcal{D}(U(g)[h], m, \Delta, \Phi_h)$ and $C' = \mathcal{D}(U(g)[h], m, \Delta_h, \text{id} \otimes \Delta)$. If $(A[h], \mu_h)$ is an algebra in $C$ then the corresponding algebra in $C'$ is $(A[h], \mu_h \circ F_h)$. In explicit terms, if $\mu_h = \mu + h\mu_1 + \cdots$ then $\mu_h \circ F_h = \mu + h(\mu_1 + r) + \cdots$. 29
4 Infinitesimals for $U_{\hbar}(\mathfrak{g}, r)$ invariant quantizations

Let $G$ be a connected simple Lie group over $\mathbb{C}$, $\mathfrak{g}$ the Lie algebra of $G$. In this section we study properties of infinitesimals for $G$ invariant quantization of function algebras on homogeneous manifolds in the category with Drinfeld associator $\Phi_\hbar$. We show that, similarly to the case of associative deformation, any $\Phi_\hbar$-associative deformation has as infinitesimal a bracket obeying the Leibniz rule and a weak version of the Jacobi identity.

Throughout the section, $M$ denotes a homogeneous $G$ manifold, thus it is isomorphic to $G/K$ for some closed Lie subgroup $K$ of $G$. We fix a reductive closed subgroup $K$ and denote by $\mathfrak{k}$ its Lie algebra. By $\Lambda(M) = \bigoplus_{p \geq 0} \Lambda^p(M)$ we denote the exterior $C^\infty(M)$ algebra of vector fields on $M$. For a $\mathfrak{g}$ module $V$, we denote by $V^\mathfrak{g}$ the set of all $\mathfrak{g}$ invariant elements of $V$, i.e. $V^\mathfrak{g} = \{ v \in V | X.v = 0 \text{ for all } X \in \mathfrak{g} \}$.

4.1 Invariant polyvector fields on a homogeneous manifold

We show here how to construct polyvector fields on $M = G/K$ starting from elements of $\Lambda^p \mathfrak{g}$ and $\Lambda^p \mathfrak{g}/\mathfrak{k}$.

Taking $\overline{\theta} \in (\Lambda^p \mathfrak{g}/\mathfrak{k})^\mathfrak{t}$, one can construct a $p$-vector field $\lambda_G(\overline{\theta})$ on $G$ in the following way. Using reductivity of $\mathfrak{t}$, we choose a $\mathfrak{t}$ invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$, complement to $\mathfrak{t}$. Then lift $\overline{\theta}$ to $\theta \in (\Lambda^p \mathfrak{g})^\mathfrak{t}$ and put $(\lambda_G(\theta))_g = (L_g)_* \theta$, where $L_g : G \rightarrow G$ is the left translation $x \mapsto gx$. The field $\lambda_G(\theta)$ is left $G$ invariant. It is also right $K$ invariant, thus it is projectable on $M = G/K$. Set $\lambda_M(\theta) = \pi_*(\lambda_G(\theta))$ where $\pi$ is the natural projection $G \rightarrow G/K$. The field $\lambda_M(\theta)$ on $M$ is $G$ invariant. We shall denote the $C^\infty(M)$ module of all $\mathfrak{g}$ invariant $p$-vector fields on $M$ by $\Lambda^p_\mathfrak{g}(M)$.

**Proposition 4.1** Any $\mathfrak{g}$ invariant polyvector field on $M$ is of the form $\lambda_M(\theta)$ for some $\theta \in (\Lambda^p \mathfrak{g}/\mathfrak{k})^\mathfrak{t}$.

**Proof:** A $\mathfrak{g}$ invariant polyvector field is fully determined by its value at an arbitrary point $m \in M$. There is a natural vector space isomorphism between the quotient $\mathfrak{g}/\mathfrak{k}$ and the tangent space $T_m M$. This isomorphism induces an isomorphism between $\Lambda^p \mathfrak{g}/\mathfrak{k}$ and $\Lambda^p T_m M$. Thus any $\mathfrak{g}$ invariant
A $p$-vector field is generated by some element of $\bigwedge^p \mathfrak{g}/\mathfrak{k}$. On the other hand, to be projectable, this element of $\bigwedge^p \mathfrak{g}/\mathfrak{k}$ must be $\mathfrak{k}$ invariant. (See also [29, 1.4.6].)

In a similar way, consider an element $\psi \in \bigwedge^p \mathfrak{g}$ and put $(\rho_{G}(\psi))_g = (R_g)_* \psi$ for any $g \in G$, where $R_g : G \rightarrow G$ is the right translation $x \mapsto xg$. By its definition, $\rho_{G}(\psi)$ is $G$ invariant from the right. In particular, it is $K$ invariant from the right. This makes it projectable on $M$, denote the corresponding polyvector field by $\rho_M(\psi)$. Note that the field $\rho_M(\psi)$ is not necessarily $\mathfrak{g}$ invariant.

### 4.2 The Schouten bracket

**Definition 4.1** Let $\Lambda(\mathfrak{a})$ be the exterior algebra of a Lie algebra $\mathfrak{a}$. The Schouten bracket on $\Lambda(\mathfrak{a})$ is a $\mathbb{C}$ bilinear mapping $\Lambda_p(\mathfrak{a}) \times \Lambda_q(\mathfrak{a}) \rightarrow \Lambda_{p+q-1}(\mathfrak{a})$ defined by

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \hat{X}_i \ldots \hat{Y}_j \ldots \wedge Y_q$$

where $[X,Y]$ is the Lie bracket on $\mathfrak{a}$ and the notation $\hat{X}_i$ means that $X_i$ is omitted in the summand.

In particular, putting $\mathfrak{a} = \text{Vect}(M)$, the Lie algebra of vector fields on $M$, one obtains the bracket $\Lambda_p(M) \times \Lambda_q(M) \rightarrow \Lambda_{p+q-1}(M)$ of polyvector fields, introduced by J. A. Schouten in [31]. Interpreting polyvectors on $M$ as polyderivations of the function algebra $A = C^\infty(M)$, one can give an equivalent definition of their Schouten bracket [27], [34]. We give here the corresponding formulas for two particular cases which are important for us.

Let $\xi, \eta \in \Lambda_2(M)$, then the Schouten bracket $[[\xi, \eta]]$ is a 3-vector field acting on a triple $a \otimes b \otimes c \in A^\otimes 3$ as follows:

$$[[\xi, \eta]](a,b,c) = \xi(\eta(a,b)c) + \eta(\xi(a,b),c) + \text{Cycl}$$

where Cycl denotes taking of all the cyclic permutations of $a, b, c$. Using this presentation of Schouten bracket, we prove that

$$[[\xi, \eta]] = 3 \text{Alt} \left( \xi \circ (\eta \otimes \text{id}) + \eta \circ (\xi \otimes \text{id}) \right). \tag{9}$$
Indeed, by using the definition of the operator Alt and the skew symmetry of $\xi$ and $\eta$, one obtains:

$$\text{Alt} \left( \xi \circ (\eta \otimes \text{id}) \right)(a, b, c) = \frac{1}{6} \left( \xi (\eta(a, b), c) - \xi (\eta(b, a), c) + \xi (\eta(b, c), a) - \xi (\eta(c, b), a) + \xi (\eta(c, a), b) - \xi (\eta(a, c), b) \right) =$$

$$= \frac{1}{3} \left( \xi (\eta(a, b), c) + \xi (\eta(b, c), a) + \xi (\eta(c, a), b) \right) = \frac{1}{3} \left( \xi (\eta(a, b), c) + \text{Cycl} \right).$$

In the same way, $\text{Alt} \left( \eta \circ (\xi \otimes \text{id}) \right)(a, b, c) = \frac{1}{3} \left( \eta (\xi(a, b), c) + \text{Cycl} \right)$, which proves (I).

For $\xi \in \Lambda_2(M)$, $\nu \in \Lambda_3(M)$, one has

$$[[\xi, \nu]] = 2 \text{Alt} \left( \xi \circ (\nu \otimes \text{id}) - \nu \circ (\xi \otimes \text{id} \otimes \text{id}) + \nu \circ (\text{id} \otimes \xi \otimes \text{id}) - \nu \circ (\text{id} \otimes \text{id} \otimes \xi) + \xi \circ (\text{id} \otimes \nu) \right).$$

(10)

The following proposition will be useful for cohomological calculations.

**Proposition 4.2**

(I) Let $\overline{\theta}, \overline{\nu} \in \Lambda^\ell(\mathfrak{g}/\mathfrak{k})$, then $[[\lambda_M(\overline{\theta}), \lambda_M(\overline{\nu})]] = \lambda_M([\overline{\theta}, \overline{\nu}])$.

(II) Let $\theta, \nu \in \Lambda(\mathfrak{g})$, then $[[\rho_M(\theta), \rho_M(\nu)]] = -\rho_M([\theta, \nu])$.

(III) Let $\theta \in \Lambda(\mathfrak{g})$, $\nu \in \Lambda^\ell(\mathfrak{g})$, then $[[\rho_M(\theta), \lambda_M(\nu)]] = 0$.

The left-hand side of each formula is the **Schouten** bracket of polyvector fields on $M$ while the right-hand side contains the **Schouten** bracket on $\Lambda(\mathfrak{g})$.

**Proof:** First we consider polyvector fields generated on the group $G$ and then pass to the manifold $M = G/K$. Using Definition $\ref{defn}$, one sees that the **Schouten** brackets in $\Lambda(\mathfrak{g})$ and in $\Lambda(G)$ defined by the **Lie** bracket in $\mathfrak{g}$ and the commutator of left invariant vector fields on $G$ correspondingly. Since the **Lie** bracket in $\mathfrak{g}$ generated by the commutator of left invariant vector fields on $G$, one has $\lambda_G([\overline{X}, \overline{Y}]) = [\lambda_G(\overline{X}), \lambda_G(\overline{Y})]$ for $\overline{X}, \overline{Y} \in \mathfrak{g}/\mathfrak{k}$ and $\rho_G([X, Y]) = -[\rho_G(X), \rho_G(Y)]$ for $X, Y \in \mathfrak{g}$. Thus $\lambda_G([\overline{\theta}, \overline{\nu}]) = [\lambda_G(\overline{\theta}), \lambda_G(\overline{\nu})]$ for $\overline{\theta}, \overline{\nu} \in \Lambda^\ell(\mathfrak{g}/\mathfrak{k})$ and $\rho_G([\theta, \nu]) = -[\rho_G(\theta), \rho_G(\nu)]$ for $\theta, \nu \in \Lambda(\mathfrak{g})$. This implies the
second claim of the proposition immediately. To complete the proof of the first claim, note that if $\theta$ and $\upsilon$ are $k$-invariant then the same true for $[\theta, \upsilon]$. Indeed, it is easy to check that for any $X \in k$ one has $X.[\theta, \upsilon] = [X, \theta, \upsilon] + [\theta, X, \upsilon]$. Thus for the polyvector fields $\lambda_G(\theta)$ and $\lambda_G(\upsilon)$ projectable their Schouten bracket $[\rho_G(\theta), \rho_G(\upsilon)]$ is projectable too. This completes the proof for (I).

The third claim is obvious since left and right invariant vector fields on $G$ are always commuting. ■

### 4.3 $\varphi$-Poisson brackets

We showed in Section 2 that the infinitesimal element of an algebra deformation of $C^\infty(M)$ is necessarily a bivector field on $M$. A bivector field obeying the Jacobi identity is a Poisson bracket. We introduce here a bracket on a homogeneous manifold which is an analog of Poisson bracket for the category with non-trivial associator $\Phi$. Fix a non-zero element $\tilde{\varphi} \in (\bigwedge^3 g)^g$ and put $\varphi = \lambda_M(\tilde{\varphi})$.

**Definition 4.2** Let $M$ be a $G$-manifold. A skew symmetric biderivation $\{\cdot, \cdot\} : C^\infty(M)^\otimes 2 \rightarrow C^\infty(M)$ is called a $\varphi$-Poisson bracket on $M$ if for any $f, g, h \in C^\infty(M)$ one has

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = \varphi(f, g, h).$$

**Proposition 4.3** Let $\upsilon \in \Lambda_2(M)$, then the expression $[\upsilon, \upsilon]$ is the left-hand side of the Jacobi identity for the bracket corresponding to $\upsilon$.

**Proof:** Straightforward computation. ■

Thus $\upsilon \in \Lambda_2(M)$ corresponds a $\varphi$-Poisson bracket on $M$ if and only if $[\upsilon, \upsilon] = \varphi$. It determines a Poisson structure if and only if $\varphi = \lambda_M(\tilde{\varphi}) = 0$. It will be shown in Section 3 that using an appropriate r-matrix bivector field, one can turn any manifold with $\varphi$-Poisson bracket into a Poisson manifold. (It is crucial that the 3-vector fields $\lambda_M(\tilde{\varphi})$ and $\rho_M(\tilde{\varphi})$ coincide and $G$ invariant.) Moreover, J. Donin, D. Gurevich and S. Shnider recently proved ([10], Proposition 2.2.) that the infinitesimal of any $U_\hbar(g, r)$ invariant quantization is always of the form $s + r$ where $s$ is a $\varphi$-Poisson bracket, $r = \rho_M(\tilde{r})$ and $\tilde{r}$ is a Belavin–Drinfeld r-matrix.
5 Quantization of $\varphi$-Poisson manifolds

In this section we give a sufficient condition for existence for a given $\varphi$-Poisson bracket $s$ on $M$ a $G$ invariant $\Phi_\hbar$-associative deformation of the multiplication in $C^\infty(M)$ with $s$ as the infinitesimal.

As above, $G$ denotes a simple Lie algebra over $\mathbb{C}$ with the Lie algebra, $K$ a connected closed Lie subgroup of $G$, $\mathfrak{k}$ the Lie algebra of $K$. By $s$ we always denote a $\varphi$-Poisson bracket on $M = G/K$ generated by an element $\tilde{s} \in (\wedge^2 \mathfrak{g})^\mathfrak{k}$ as it was explained in Section 4.1.

5.1 The complex $(\tilde{\Lambda}(M), d_s)$

For a $\varphi$-Poisson bracket $s \in \Lambda^\mathfrak{k}_\mathfrak{g}(M)$ set $d_s(v) = [s, v], v \in \Lambda^\mathfrak{g}(M)$.

**Proposition 5.1** $(\Lambda^\mathfrak{g}(M), d_s)$ is a cochain complex.

**Proof:** We need to prove that $d_s \circ d_s = 0$. Take $v \in \Lambda^\mathfrak{g}_\mathfrak{g}(M)$, then $v = \lambda_M(\tilde{v})$ for some $\tilde{v} \in (\wedge^p \mathfrak{g}/\mathfrak{k})^\mathfrak{k}$ (see Section 4.1). We denote by $[[\cdot,\cdot]]$ the Schouten brackets both on $\Lambda^\mathfrak{g}(M)$ and on $\Lambda^\mathfrak{g}(\mathfrak{g}/\mathfrak{k})$. Thus $d_s(v) = \lambda_M([\tilde{s}, \tilde{v}])$, so it suffices to prove that $[[\tilde{s}, [\tilde{s}, \tilde{v}]] = 0$ for any $\tilde{v} \in (\wedge^p \mathfrak{g})^\mathfrak{k}$. One checks easily that $[[\tilde{s}, [\tilde{s}, \tilde{v}]] = \frac{1}{2}[[[\tilde{s}, \tilde{s}]], \tilde{v}] = \frac{1}{2}[\tilde{\varphi}, \tilde{v}]$ where $\tilde{\varphi}$ a $\mathfrak{g}$ invariant element of $\wedge^3 \mathfrak{g}$. All this implies that $d_s(d_s(v)) = \frac{1}{2}[[\varphi, v]]$ where $\varphi = \lambda_M(\tilde{\varphi})$. Note that also $\varphi = \rho_M(\tilde{\varphi})$ thus, by Proposition 4.2(III), $[[\varphi, v]] = 0$. \hfill \blacksquare

Now, take a $G$ invariant field $v = \lambda_M(\tilde{v})$ on $M$, then one has $d_s(v) = \lambda_M([\tilde{s}, \tilde{v}])$. Thus the cochain complexes $(\Lambda^\mathfrak{g}(M), d_s)$ and $(\Lambda^\mathfrak{g}(\mathfrak{g}/\mathfrak{k}), [\tilde{s}, \cdot])$ are isomorphic.

Suppose $\Lambda(\mathfrak{g})$ possesses an involution $\theta : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ such that $\theta(\tilde{\omega}) = (-1)^{p+1}\tilde{\omega}$ if $\tilde{\omega} \in \wedge^p \mathfrak{g}$. Since $\theta(\tilde{s}) = -s$, the set of all elements with property $\theta(\tilde{\omega}) = \tilde{\omega}$ forms a sub-complex of $(\Lambda^\mathfrak{g}(\mathfrak{g}/\mathfrak{k}), d_s)$ which we denote by $(\Lambda^{\mathfrak{g}\theta}(\mathfrak{g}/\mathfrak{k}), d_s)$. The corresponding sub-complex of $(\Lambda^\mathfrak{g}(M), d_s)$ we denote by $(\tilde{\Lambda}(M), d_s)$.

5.2 Effect of Drinfeld associator

We prove here that the presence of the Drinfeld associator $\Phi_\hbar$ of Theorem 3.2 does not affect substantially the cohomological construction of the deformed multiplication $\mu_\hbar$. The reason is basically that $\Phi_\hbar$ begins with $1 \otimes 1 \otimes 1$ and has no term with $\hbar$, so it does not change the infinitesimal.
Let \( \mu_h^{-1} \) be a \( g \) invariant \( \mathbb{C}[h] \) linear mapping \( A[h] \otimes_R A[h] \to A[h] \) of the form \( \mu_h^{-1} = \mu + \sum_{k=1} h^k \mu_k \). (The pair \( (A[h], \mu_h^{-1}) \) is a not necessary associative algebra over \( \mathbb{C}[h] \).

**Lemma 5.1** Set \( B_\Phi(\mu) = \mu \circ (\mu \otimes \text{id}) - \mu \circ (\text{id} \otimes \mu) \circ \Phi \) and suppose that \( B_{\Phi_h}(\mu_h^{-1}) \equiv 0 \ (\text{mod } h^n) \). Then the following congruences are valid modulo \( h^{n+2} \):

\[
B_{\Phi_h}(\mu_h^{-1}) \circ \Phi_h^{-1} \equiv B_{\Phi_h}(\mu_h^{-1}); \\
B_{\Phi_h}(\mu_h^{-1}) \circ (\text{id} \otimes \Delta \otimes \text{id})(\Phi_h) \circ (\Phi \otimes \text{id}) \equiv B_{\Phi_h}(\mu_h^{-1}); \\
B_{\Phi_h}(\mu_h^{-1}) \circ (\text{id} \otimes \mu_h^{-1} \otimes \text{id}) \circ (\Phi_h \otimes \text{id}) \equiv B_{\Phi_h}(\mu_h^{-1}) \circ (\text{id} \otimes \mu_h^{-1} \otimes \text{id}); \\
B_{\Phi_h}(\mu_h^{-1}) \circ (\text{id} \otimes \text{id} \otimes \mu_h^{-1}) \circ (\Delta \otimes \text{id} \otimes \text{id})(\Phi_h) \equiv B_{\Phi_h}(\mu_h^{-1}) \circ (\text{id} \otimes \text{id} \otimes \mu_h^{-1}).
\]

**Proof:** The congruence \( B_{\Phi_h}(\mu_h^{-1}) \equiv 0 \ (\text{mod } h^n) \) implies

\[
B_{\Phi_h}(\mu_h^{-1}) \equiv h^n \eta + h^{n+1} \xi \quad (\text{mod } h^{n+2})
\]

for some \( \eta, \xi \in \tilde{\mathcal{C}}^3(A; A) \). The formal power series \( \Phi_h \) has \( 1 \otimes 1 \otimes 1 \) as the initial term, and it has no linear term. Its formal inverse, \( \Phi_h^{-1} \), is of the same form. Thus

\[
(h^n \eta + h^{n+1} \xi) \circ \Phi_h \equiv h^n \eta + h^{n+1} \xi \quad (\text{mod } h^{n+2})
\]

\[
(h^n \eta + h^{n+1} \xi) \circ \Phi_h^{-1} \equiv h^n \eta + h^{n+1} \xi \quad (\text{mod } h^{n+2}).
\]

Hence the left hand side of (11) will also not be changed after taking the composition with any algebraic expression containing \( \Phi_h \) or \( \Phi_h^{-1} \). \( \blacksquare \)

Denote by \( \overline{d} \) the operator \( A[h] \otimes_R A[h] \to A[h] \otimes_A (p+1) \) defined by formula (4) with \( \mu_h^{-1} \) instead of \( \mu \). It is a polynomial in \( h \) with the HOCHSCHILD coboundary operator \( d \) (defined by \( \mu \)) as the initial term: \( \overline{d} = d + h d_1 + h^2 d_2 + \cdots + h^{n-1} d_{n-1} \). In what follows we need the explicit formula for \( d_1 \):

\[
d_1 \xi = s \circ (\text{id} \otimes \xi) + \sum_{k=1}^p (-1)^k \xi \circ (\text{id} \otimes (k-1) \otimes \text{id} \otimes (p-k)) + (-1)^{p+1} s \circ (\xi \otimes \text{id}).
\]

The operator \( \overline{d} \) is not a differential, since \( \overline{d} \circ \overline{d} \neq 0 \). However, using the fact that \( \mu_h^{-1} \) is associative modulo \( h^2 \), we prove the following lemma.

**Lemma 5.2** Let \( \mu_h^{-1} \) be \( g \) invariant and let \( B_{\Phi_h}(\mu_h^{-1}) \equiv 0 \ (\text{mod } h^n) \). Then \( \overline{d} B_{\Phi_h}(\mu_h^{-1}) \equiv 0 \ (\text{mod } h^{n+2}) \).
Proof: For this proof we set the following notation: $B = B_{\Phi_h}(\mu_n^{n-1})$, $\overline{\pi} = \mu_n^{n-1}$ and $\Phi_h = \Phi$. The sign $\equiv$ will denote the congruence modulo $\hbar_n^{n+2}$.

By the definition of $d$,

$$d B = \overline{\pi} \circ (\text{id} \otimes B) - B \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) + B \circ (\text{id} \otimes \overline{\mu} \otimes \text{id}) - B \circ (\text{id} \otimes \text{id} \otimes \overline{\mu}) + \overline{\pi} \circ (B \otimes \text{id}).$$

(13)

Compute each term of the right hand side of (13). The following two identities can be checked directly: $\text{id} \otimes (\overline{\pi} \circ (\overline{\pi} \otimes \text{id})) = (\text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \overline{\pi} \otimes \text{id})$ and $\text{id} \otimes (\overline{\pi} \circ (\text{id} \otimes \overline{\pi}) \circ \Phi) = (\text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \Phi)$. Thus

$$\overline{\pi} \circ (\text{id} \otimes B) = \overline{\pi} \circ (\text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \overline{\pi} \otimes \text{id}) - \overline{\pi} \circ (\text{id} \otimes \text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \Phi).$$

(14)

In analogous way, using the identities $(\overline{\pi} \circ (\overline{\pi} \otimes \text{id})) \otimes \text{id} = (\overline{\pi} \otimes \text{id}) \circ (\overline{\pi} \otimes \text{id} \otimes \text{id})$ and $(\overline{\pi} \circ (\text{id} \otimes \overline{\pi}) \circ \Phi) \otimes \text{id} = (\overline{\pi} \otimes \text{id}) \circ (\text{id} \otimes \overline{\pi} \otimes \text{id}) \circ (\Phi \otimes \text{id})$, one obtains

$$\overline{\pi} \circ (B \otimes \text{id}) = \overline{\pi} \circ (\overline{\pi} \otimes \text{id}) \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) - \overline{\pi} \circ (\overline{\pi} \otimes \text{id}) \otimes (\text{id} \otimes \overline{\pi} \otimes \text{id}) \circ (\text{id} \otimes \Phi).$$

(15)

To proceed with the remaining three terms of (13), recall that $\overline{\pi}$ is $g$ invariant, i.e. $\overline{\pi} \circ \overline{\pi} = \overline{\pi} \circ \Delta(\overline{\pi})$ for all $\overline{\pi} \in \overline{U}$ (see Definition 3.7). Since $\Phi \in \overline{U}^{\otimes 3}$, this implies the following three equations:

$$\Phi \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) = (\overline{\pi} \otimes \text{id} \otimes \text{id}) \circ (\Delta \otimes \text{id} \otimes \text{id})(\Phi);$$

$$\Phi \circ (\text{id} \otimes \overline{\pi} \otimes \text{id}) = (\text{id} \otimes \overline{\pi} \otimes \text{id}) \circ (\text{id} \otimes \Delta \otimes \text{id})(\Phi);$$

$$\Phi \circ (\text{id} \otimes \text{id} \otimes \overline{\pi}) = (\text{id} \otimes \text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \text{id} \otimes \Delta)(\Phi).$$

Using these equations, one obtains

$$B \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) = \overline{\pi} \circ (\overline{\pi} \otimes \text{id}) \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) - \overline{\pi} \circ (\text{id} \otimes \overline{\pi}) \circ \Phi \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) =$$

$$= \overline{\pi} \circ (\overline{\pi} \otimes \text{id}) \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) - \overline{\pi} \circ (\text{id} \otimes \overline{\pi}) \circ (\overline{\pi} \otimes \text{id} \otimes \text{id}) \circ (\Delta \otimes \text{id} \otimes \text{id})(\Phi);$$

(16)

$$B \circ (\text{id} \otimes \overline{\pi} \otimes \text{id}) = \overline{\pi} \circ (\overline{\pi} \otimes \text{id}) \circ (\text{id} \otimes \overline{\pi} \otimes \text{id}) - \overline{\pi} \circ (\text{id} \otimes \overline{\pi}) \circ (\text{id} \otimes \overline{\pi} \otimes \text{id}) \circ (\text{id} \otimes \Delta \otimes \text{id})(\Phi);$$

(17)
\[ B \circ (\text{id} \otimes \text{id} \otimes \mu) = \overline{\mu} \circ (\overline{\mu} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \overline{\mu}) \]
\[ - \mu \circ (\text{id} \otimes \overline{\mu}) \circ (\text{id} \otimes \text{id} \otimes \mu) \circ (\text{id} \otimes \text{id} \otimes \Delta)(\Phi). \]

Take the composition from the right of both sides of (14) with \((\text{id} \otimes \Delta \otimes \text{id})(\Phi) \circ (\Phi \otimes \text{id})\). By Lemma 5.1, \[ B \circ (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \circ (\Phi \otimes \text{id}) \equiv B. \]
Thus one obtains the left hand side of (14) unchanged, and so
\[
\mu \circ (\text{id} \otimes B) \equiv \mu \circ (\text{id} \otimes \mu) \circ (\text{id} \otimes \mu \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \circ (\Phi \otimes \text{id})
\]
Similarly, taking the composition from the right of both sides of (17) with \(\Phi \otimes \text{id}\) and both sides of (18) with \((\Delta \otimes \text{id} \otimes \text{id})(\Phi)\), one obtains:
\[
B \circ (\text{id} \otimes \overline{\mu} \otimes \text{id}) \equiv \overline{\mu} \circ (\overline{\mu} \otimes \text{id}) \circ (\text{id} \otimes \overline{\mu} \otimes \text{id}) \circ (\Phi \otimes \text{id})
\]
\[
- \overline{\mu} \circ (\text{id} \otimes \overline{\mu}) \circ (\text{id} \otimes \text{id} \otimes \overline{\mu}) \circ (\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \circ (\Phi \otimes \text{id})
\]
\[
B \circ (\text{id} \otimes \text{id} \otimes \overline{\mu}) \equiv \overline{\mu} \circ (\overline{\mu} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \overline{\mu}) \circ (\Delta \otimes \text{id} \otimes \text{id})(\Phi)
\]
\[
- \overline{\mu} \circ (\text{id} \otimes \overline{\mu}) \circ (\text{id} \otimes \text{id} \otimes \overline{\mu}) \circ (\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \circ (\Delta \otimes \text{id} \otimes \text{id})(\Phi)
\]
Finally, put (19), (16), (20), (21) and (15) into (13). Using the identity \((\overline{\mu} \otimes \text{id}) \circ (\text{id} \otimes \overline{\mu}) = \overline{\mu} \otimes \overline{\mu} = (\overline{\mu} \otimes \text{id}) \circ (\overline{\mu} \otimes \text{id} \otimes \text{id})\) and the Pentagon Identity, one concludes that \(\overline{d}B \equiv 0\). \[\blacksquare\]

5.3 Quantization in the category with \(\Phi_h\)-associativity

Let \(\varphi\) and \(s\) be as above, \(\tilde{C}^*(M, d)\) the local Hochschild complex for \(A = C^\infty(M), \Phi_h\) as in Theorem 3.2. Our goal is to prove that there exists a noncommutative \(\Phi_h\-associative formal deformation \((A[h], \mu_h)\) of the associative commutative algebra \((A, \mu)\) with the original associative multiplication \(\mu\) as the initial term and \(s\) as the infinitesimal. Recall (Example 3.4) that \(\Phi_h\) associativity means \(\mu_h \circ (\mu_h \otimes \text{id}) = \mu_h \circ (\text{id} \otimes \mu_h) \circ \Phi_h\). This can be written as
\[
B_{\Phi_h}(\mu_h) = 0. \quad (22)
\]
Recall also that \(g\) invariance means \(x \circ \mu_h = \mu_h \circ \Delta(x)\) for any \(x \in U(g)\) considering as a differential operator on \(A[h]\).
Suppose that $\Lambda(\mathfrak{g})$ is equipped with an involution $\theta$. Recall (see Section 5.1) that the subset $\tilde{\Lambda}(M)$ of $\Lambda(g)$ consisting of elements whose homogeneous components $\tilde{\omega} \in \bigwedge^p g$ satisfy $\theta(\tilde{\omega}) = (-1)^{p+1}\tilde{\omega}$ forms a sub-complex.

**Definition 5.1** We call a $\varphi$-Poisson bracket $s \Phi_\hbar$-quantizable if there exists a $\Phi_\hbar$-associative commutative $\mathfrak{g}$ invariant multiplication $\mu_\hbar$ on $C^\infty(M)[[\hbar]]$ of the form $\mu_\hbar = \mu + \hbar s + \cdots$.

**Theorem 5.1** Let $s$ be a $\mathfrak{g}$ invariant bivector field on $M$ such that $[s, s] = \varphi$. Suppose that $H^3\left(\tilde{\Lambda}(M), d_s\right) = 0$. Then $s$ is $\Phi_\hbar$-quantizable.

This theorem is a generalization of Proposition 4 of [8]. The difference of our considerations from those of [8] is that we fix the first order term for $\mu_\hbar$ and we do not assume $\lambda_M(\varphi) = 0$. Since $H^3(C^\ast(M), d) \neq 0$ in general, one can not apply the arguments of [8] directly to the case. Instead, we combine these arguments with the techniques of O. M. NEROSLAVSKY and A. T. VLASSOV [28]. A clear exposition of their method in the context of obstruction theory is given in [27].

**Proof:** Starting with $\mu_1^\hbar = \mu + \hbar s$, we find subsequent approximations of $\mu_\hbar$ by $\mathfrak{g}$ invariant multiplications

$$\mu^n_\hbar = \mu + \hbar s + \sum_{i=2}^n \mu_i \hbar^i$$

with $\mu_i$'s obeying the Parity Convention: $\mu_{2k} \in C^2_-(A; A)$ and $\mu_{2k+1} \in C^2_+(A; A)$.

To proceed by induction on $n$, we prove first that the linear approximation, $\mu_1^\hbar = \mu + \hbar s$, obeys (22) modulo $\hbar^2$, i.e.

$$B_{\Phi_\hbar}(\mu_1^\hbar) = \mu_1^\hbar \circ (\mu_1^\hbar \otimes \text{id}) - \mu_1^\hbar \circ ($$

Opening the brackets and leaving the constant and linear on $\hbar$ terms only and using the definition (4) of the HOCHSCHILD coboundary operator, one obtains $B_{\Phi_\hbar}(\mu_1^\hbar) = d\mu + \hbar d s$. Simple computation shows that $d\mu = 0$. We prove that $d s = 0$. Using the definition (4) of operator $d$ and the LEIBNIZ rule for each argument of $s$, one has for any $a, b, c \in A$:

$$d s(a, b, c) = as(b, c) - s(ab, c) + s(a, bc) - s(a, b)c =$$

$$= as(b, c) - as(b, c) - bs(a, c) + s(a, c)b + s(a, b)c - s(a, b)c = 0,$$
where \(ab = \mu(a, b)\). Note that \(\Phi_h\) still played no rôle in our arguments.

Assume we found \(\mu_i\)'s for all \(i < n\), such that \(\mu_{n-1}^n\) is \(\Phi_h\)-associative modulo \(\hbar^n\):

\[
B_{\Phi_h}(\mu_{n-1}^n) = \mu_{n-1}^n \circ (\mu_{n-1}^n \otimes \text{id}) - \mu_{n-1}^n \circ (\text{id} \otimes \mu_{n-1}^n) \circ \Phi_h \equiv 0 \pmod{\hbar^n} \quad (23)
\]

This means that

\[
B_{\Phi_h}(\mu_{n-1}^n) \equiv \hbar^n \eta_n \pmod{\hbar^{n+1}} \quad (24)
\]

for some \(\eta_n \in \tilde{C}^2(A; A)\). We are looking for a \(\mu_n \in \tilde{C}^2(A; A)\) which will cancel \(\eta_n\), i.e. such that \(\eta_n = 0\) modulo \(\hbar^n+1\):

\[
\eta_n = \sum_{\substack{i+j=n \\ 0<i,j<n \ \ i+j+2k=n \ \ 0<i,j<n \ \ k \geq 0}} \mu_i \circ (\mu_j \otimes \text{id}) - \mu_i \circ (\text{id} \otimes \mu_j) \circ \varphi_{2k}, \quad (25)
\]

where \(\varphi_{2k}\) as in Theorem 3.2. Note that the obstruction \(\eta_2\) is \(g\) invariant, for this is true for \(\mu\) and \(\mu_1^1 = s\). Thus we assume that \(\eta_n\) is \(g\) invariant and prove that \(\mu_n\) can be chosen to be \(g\) invariant as well.

**Lemma 5.3**

(I) \(d \eta_n = 0\);

(II) \(d_s(\text{Alt} \eta_n) = 0\).

**Proof:** Since \(B_{\Phi_h}(\mu_{n-1}^n) \equiv 0 \pmod{\hbar^n}\), one has

\[
B_{\Phi_h}(\mu_{n-1}^n) \equiv \hbar^n \eta_n + \hbar^{n+1} \xi \pmod{\hbar^{n+2}}
\]

for some \(\xi\). By Lemma 5.2, \(\partial B_{\Phi_h}(\mu_{n-1}^n) \equiv 0 \pmod{\hbar^{n+2}}\), thus

\[
\partial B_{\Phi_h}(\mu_{n-1}^n) = (d + \hbar d_1 + \cdots) (\hbar^n \eta_n + \hbar^{n+1} \xi + \cdots) = \hbar^n d \eta_n + \hbar^{n+1} (d_1 \eta_n + d \xi) + \cdots \equiv 0 \pmod{\hbar^{n+2}}
\]

where \(d\) is the Hochschild coboundary for the algebra \((A, \mu)\), and \(d_1\) is as in (12). Equating to zero the coefficients before \(\hbar^n\) and \(\hbar^{n+1}\) respectively, one obtains two equations: \(d \eta_n = 0\) and \(d_1 \eta_n + d \xi = 0\). The former equation
proves (I). Taking the alternation of the both sides of the latter equation, one obtains \( \text{Alt}(d_1 \eta_n) = 0 \) since \( \text{Alt}(d \xi) = 0 \) by Proposition 2.3. Using \( d \eta_n = 0 \) and Theorem 2.1, one concludes that \( \eta_n \) is a 3-vector field on \( M \). Using direct computation, one can prove that \( 2 \text{Alt}(d_1 \eta_n) = [s, \text{Alt} \eta_n] \) (see formula (10) and the proof for formula (9)). This proves (II).

In classical obstruction theory, when the third cohomology space is equal to zero, all obstructions vanish, but we have \( H^3(\mathcal{C}(M), d) \neq 0 \). To eliminate the obstruction \( \eta_n \), one can try to correct the previous term, \( \mu_{n-1} h^{n-1} \). First prove that the parity of \( \eta_n \) is the opposite to the parity of the integer \( n \), i.e. \( \eta_{2k} \in C^3_-(A; A) \) and \( \eta_{2k+1} \in C^3_+(A; A) \).

**Lemma 5.4** \( \eta_n(c, b, a) = (-1)^{n+1} \eta_n(a, b, c) \).

**Proof:** By Lemma 5.1, \( B_{\Phi_h}(\mu_h^{n-1}) \circ \Phi_h^{-1} \equiv B_{\Phi_h}(\mu_h^{n-1}) \pmod{h^{n+1}} \). This implies

\[
\mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id}) - \mu_h^{n-1} \circ (\text{id} \otimes \mu_h^{n-1}) \circ \Phi_h \equiv \mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id}) \circ \Phi_h^{-1} - \mu_h^{n-1} \circ (\text{id} \otimes \mu_h^{n-1}) \pmod{h^{n+1}}
\]

(26)

For a formal power series \( \xi = \sum_{k \geq 0} \xi_k h^k \), set \( c_n(\xi) = \xi_n \). Clearly, \( c_n(\xi) = 0 \) if \( \xi \equiv 0 \pmod{h^{n+1}} \). Therefore (26) implies that

\[
\eta_n = c_n(\mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id}) - \mu_h^{n-1} \circ (\text{id} \otimes \mu_h^{n-1}) \circ \Phi_h) = \mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id}) \circ \Phi_h^{-1} - \mu_h^{n-1} \circ (\text{id} \otimes \mu_h^{n-1})
\]

(27)

Assume \( n \) is even, then the parity of \( i \) and \( j \) in each term of (27) is the same. Consequently, \( \mu_i(\mu_j(a, b, c)) = \mu_i(c, \mu_j(b, a)) \), and thus

\[
\mu_h^n(\mu_h^n(a, b, c)) = \mu_h^n(c, \mu_h^n(b, a))
\]

for any \( a, b, c \in A \). Using (27) and the property \( \Phi_{h}^{321} = \Phi_{h}^{-1} \), one obtains:

\[
\eta_n(c, b, a) = c_n\left(\mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id}) \circ \Phi_h^{321}(c, b, a) - \mu_h^{n-1} \circ (\text{id} \otimes \mu_h^{n-1})(c, b, a)\right).
\]

(28)

The second summand in (28) is equal to \( \mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id})(a, b, c) \). To consider the first summand, write \( \Phi_h(a, b, c) \) as \( \Phi'(a) \otimes \Phi''(b) \otimes \Phi'''(c) \), then \( \Phi^{321}(c, b, a) = \Phi''(c) \otimes \Phi'''(b) \otimes \Phi'(a) \), and thus

\[
\mu_h^{n-1} \circ (\mu_h^{n-1} \otimes \text{id}) \circ \Phi_h^{321}(c, b, a) = \mu_h^{n-1}(\Phi'(a), \mu_h^{n-1}(\Phi''(b), \Phi'''(c))) = \mu_h^{n-1} \circ (\text{id} \otimes \mu_h^{n-1}) \circ \Phi_h(a, b, c).
\]
One concludes that $\eta_n(c, b, a) = -\eta_n(a, b, c)$.

Similarly, when $n$ is odd, then the indices $i$ and $j$ are of the opposite parity for each term in (25). Thus $\mu_n^c (\mu_n^a(b), c) = - \mu_n^a(c, \mu_n^b(b), a)$ which is followed by $\eta_n(c, b, a) = \eta_n(a, b, c)$.

By Lemma 5.3, $\eta_n$ is a Hochschild cocycle. Then by Theorem 2.1, $\text{Alt} \eta_n$ is a 3-vector field on $M$, and

$$\eta_n = \text{Alt} \eta_n + d\nu$$

(29)

for some bidifferential operator $\nu$. Note that $\text{Alt} \eta_n$ is $g$ invariant.

Assume $n$ is odd. By Lemma 5.4, $\eta_n \in \tilde{C}_3(A; A)$, i.e. it is even, and thus $\text{Alt} \eta_n$ is equal to zero, for this is an odd 3-cochain. Thus in this case $\eta_n = d\nu$ for some bidifferential operator $\nu \in \tilde{C}^2(A; A)$. Proposition 2.1 shows that $\nu$ in (29) can be chosen skew symmetric.

**Lemma 5.5** Let $\xi = d_H\nu$ be a $g$ invariant Hochschild $p$-coboundary, $\nu \in \tilde{C}^{p-1}(A; A)$. Then there exists a $g$ invariant $\nu^o \in \tilde{C}^{p-1}(A; A)$ such that $\xi = d_H\nu^o$.

**Proof:** Consider the spaces $\tilde{C}^p(A; A)$ as $g$ modules. The modules $\tilde{C}^p(A; A)$ can be decomposed into direct sum of finite dimensional highest weight spaces. The sum of the spaces of highest weight zero is equal exactly to the subspace $\tilde{C}^p(A; A)^g$ of $g$ invariant cochains. For $\tilde{C}^p = \tilde{C}^p(A; A)$ and a weight $\lambda$ of $g$, denote by $V^\lambda(\tilde{C}^{p-1})$ the direct sum of all components of the highest weight $\lambda$. In particular, $V^0(\tilde{C}^p) = (\tilde{C}^p)^g$. It is easy to check that the Hochschild coboundary operator $d_H$ commutes with the $g$ action. Thus

$$d_H \left( V^\lambda(\tilde{C}^{p-1}) \right) \subset V^\lambda(\tilde{C}^p).$$

(30)

Decompose $\nu = \nu^o + \nu'$ where $\nu^o \in V^0(\tilde{C}^{p-1})$ and $\nu' \in \bigoplus_{\lambda \neq 0} V^\lambda(\tilde{C}^{p-1})$. Then, by (30), $\xi \in V^0(\tilde{C}^p)$ implies $d_H\nu' = 0$, and thus $\xi = d_H\nu^o$. ■

This lemma implies that $\nu$ can be chosen $g$ invariant. The polynomial $\mu_n^c = \mu_n^{n-1} + h^n \nu$ is $\Phi_h$-associative modulo $h^{n+1}$ and it is $g$ invariant. Hence for an odd $n$, a $g$ invariant term $\mu_n h^n$ obeying the parity convention can always be constructed.

Consider the case when $n$ is even. By Lemma 5.4, $\eta_n$ is odd, thus $\text{Alt} \eta_n$ is not necessary zero. Proposition 2.1 and Lemma 5.5 show that $\nu$ in (29)
can be chosen symmetric and $\mathfrak{g}$ invariant. By Lemma 5.3, $d_s(\text{Alt } \eta_n) = 0$. Together with $H^3\left(\tilde{\Lambda}(M), d_s\right) = 0$, this implies that $\text{Alt } \eta_n = d_s \zeta$ for some $\zeta \in \tilde{\Lambda}_2(M)$.

Put

$$\tilde{\mu}_n = \mu_n - 3 h^{n-1} \zeta.$$

Since $\zeta$ is skew symmetric, $\tilde{\mu}_n^{-1}$ obeys the parity convention. Thus for the obstruction corresponding to $\tilde{\mu}_n^{-1}$, i.e. a bidifferential operator $\tilde{\eta}_n$ such that $B_{\Phi_n}(\tilde{\mu}_n) \equiv h^n \tilde{\eta}_n \pmod{h^{n+1}}$, the statements of Lemma 5.3 (II) and Lemma 5.4 are still valid, that is, $\text{Alt } \eta_n(c,b,a) = -\text{Alt } \eta_n(a,b,c)$ and $d_s \tilde{\eta}_n = 0$.

Direct computation shows that

$$\tilde{\eta}_n = \eta_n + \frac{3}{2} d \zeta - \frac{3}{2} \left( \zeta \circ (s \otimes \text{id}) + s \circ (\zeta \otimes \text{id}) - \zeta \circ (\text{id} \otimes s) - s \circ (\text{id} \otimes \zeta) \right).$$

(31)

The Hochschild coboundary of any biderivation is equal to zero, hence $d \zeta = 0$. Using the skew symmetry of $s$ and $\zeta$, one checks directly that

$$\text{Alt } \left( \zeta \circ (\text{id} \otimes s) + s \circ (\text{id} \otimes \zeta) \right) = - \text{Alt } \left( \zeta \circ (s \otimes \text{id}) + s \circ (\zeta \otimes \text{id}) \right).$$

Thus taking the alternation of the both sides of (31) and using formula (9), Section 4.2, one obtains $\text{Alt } \tilde{\eta}_n = \text{Alt } \eta_n - \{ s, \zeta \} = \theta - d_s \zeta = \theta - \theta = 0$. By Theorem 2.1, this implies that $\tilde{\eta}_n = d v$ for some bidifferential operator $v$. As it was mentioned above, $v$ can be chosen to be symmetric and $\mathfrak{g}$ invariant. Finally, put $\mu_n = v$ and $\mu_n^{-1} = \tilde{\mu}_n^{-1} + h^n v$. It is left to prove that $B_{\Phi_n}(\mu_n) \equiv 0 \pmod{h^{n+1}}$. Using the definition of $\tilde{\eta}_n$ and equality $\tilde{\eta}_n = dv$, one has $B_{\Phi_n}(\mu_n^{-1}) \equiv h^n dv \pmod{h^{n+1}}$. Therefore

$$B_{\Phi_n}(\mu_n) = B_{\Phi_n}(\tilde{\mu}_n^{-1}) + h^n \left( \mu \circ (v \otimes \text{id}) + v \circ (\mu \otimes \text{id}) \right) - h^n \left( \mu \circ (\text{id} \otimes v) + v \circ (\text{id} \otimes \mu) \right) \circ \Phi_n \equiv$$

$$\equiv B_{\Phi_n}(\tilde{\mu}_n^{-1}) + h^n \left( \mu \circ (v \otimes \text{id}) + v \circ (\mu \otimes \text{id}) - \mu \circ (\text{id} \otimes v) - v \circ (\text{id} \otimes \mu) \right) \equiv$$

$$\equiv dv - dv \equiv 0 \pmod{h^{n+1}},$$

where the congruence $h^n \mu \circ (\text{id} \otimes v) \circ \Phi_n \equiv h^n \mu \circ (\text{id} \otimes v) \pmod{h^{n+1}}$ was used (see Lemma 5.1). Theorem 5.1 is proven. ■
6  A class of homogeneous manifolds with quantizable Poisson brackets

In this section we introduce a class of homogeneous manifolds, \( M_{l_\alpha} \), which closely related to manifolds appearing in the problem of classification of quotients of \( G \) by a reductive subgroup of maximal rank. We prove that all the manifolds \( M_{l_\alpha} \) possess \( G \) invariant \( \varphi \)-Poisson brackets and present their explicit forms. It turns out that these brackets are essentially unique. Any \( \varphi \)-Poisson bracket \( s \) determines a cochain complex. By computing the corresponding cohomologies of this complex, we prove that \( s \) can be quantized in such a way that after quantization we obtain a \( G \) invariant \( \Phi_\hbar \)-associative multiplication. Due to results of Section 3, this implies that any Poisson bracket on \( M_{l_\alpha} \) of the form \( s + r \) can be quantized invariantly with respect to the quantum group \( U_\hbar(\mathfrak{g}, r) \) action.

6.1 Poisson brackets generated by Belavin–Drinfeld \( r \)-matrices

Let \( G \) be a connected simple Lie group over \( \mathbb{C} \), \( \mathfrak{g} \) the Lie algebra of \( G \), \( K \) a connected Lie subgroup of \( G \), \( \mathfrak{k} \) the Lie algebra of \( K \). Denote by \( M \) the homogeneous \( G \) manifold \( G/K \). Recall that the vector space \( \mathfrak{m} = \mathfrak{g}/\mathfrak{k} \) is isomorphic to the tangent space to \( M \) at the point fixed by \( K \). In Section 4.3 we introduced the notion of \( \varphi \)-Poisson bracket. Here we explain how to pass from a \( \varphi \)-Poisson bracket to a usual Poisson bracket.

Let \( s \) be a \( \varphi \)-Poisson bracket on \( M \). By Proposition 4.1, \( s = \lambda_M(\tilde{s}) \) for some \( \tilde{s} \in (\wedge^2 \mathfrak{m})^\mathfrak{k} \) satisfying \([\tilde{s}, \tilde{s}] = \tilde{\varphi} \), where \( \tilde{\varphi} \) is a non-zero \( \mathfrak{g} \) invariant element of \( \wedge^3 \mathfrak{g} \). Take a Belavin—Drinfeld \( r \)-matrix \( \tilde{r} \) with \([\tilde{r}, \tilde{r}] = \tilde{\varphi} \) for the same \( \tilde{\varphi} \), and put \( r = \rho_M(\tilde{r}) \). Then, by Proposition 4.2, one has \([s, s] = \varphi, [r, r] = -\varphi \) and \([s, r] = 0 \), thus \([s + r, s + r] = 0 \).

As in Section 3.4, we denote by \( U_\hbar(\mathfrak{g}, r) \) the Etingof–Kazhdan quantum group determined by \( r \). Recall that \( \Phi_\hbar \)-quantizability of \( s \) means that there exists a \( \Phi_\hbar \)-associative \( \mathfrak{g} \) invariant multiplication \( \mu_\hbar \) on \( C^\infty(M)[\hbar] \) of the form \( \widehat{\mu}_\hbar = \mu + \hbar s + \cdots \) where \( \Phi_\hbar \) is given by Theorem 3.2.

Theorem 6.1 Let \( M \) be a smooth manifold with the above \( \varphi \)-bracket \( s \) and Belavin—Drinfeld bivector field. Then

(I) \( s + r \) is a Poisson bracket on \( M \);
If \( s \) is \( \Phi_\hbar \)-quantizable, then there exists an associative multiplication \( \mu_\hbar \) on \( \mathcal{C}^\infty(M)[\hbar] \) of the form \( \mu_\hbar = \mu + \hbar(s + r) + \cdots \) which is invariant under the action of the quantum group \( \mathcal{U}_\hbar(\mathfrak{g},r) \).

**Proof:** (I). It was mentioned above that \( [s + r, s + r] = 0 \). By Proposition [4.3], this means that the sum \( s + r \) obeys the Jacobi identity.

(II). Suppose that there exists a \( \Phi_\hbar \)-associative multiplication of the form \( \tilde{\mu}_\hbar = \mu + \hbar(s) + \cdots \). By Theorem [3.2], there exists an invertible power series \( F_\hbar = \text{id}^{\otimes 2} + h r + \cdots \in \mathcal{U}_\hbar(\mathfrak{g})[\hbar] \) such that the composition \( \mu_\hbar = \tilde{\mu}_\hbar \circ F_\hbar \) is a (strictly) associative multiplication. Expanding the composition in powers of \( h \), one obtains \( \mu_\hbar = \mu + \hbar(s + r) + \cdots \). (See also Remark [3.3].)

---

6.2 Homogeneous manifolds related to regular subalgebras of \( \mathfrak{g} \)

Fix a Cartan subalgebra \( \mathfrak{h} \) of the simple Lie algebra \( \mathfrak{g} \), denote by \( \Omega \) the corresponding root system and fix a set of simple roots for \( \Omega \).

In what follows, all tensor products and dimensions are taken over \( \mathbb{C} \).

**Definition 6.1** We call a Lie subalgebra \( \mathfrak{k} \) regular if it is reductive and contains a Cartan subalgebra.

Since all Cartan subalgebras of \( \mathfrak{g} \) are conjugate, we can consider only those regular subalgebras which contain the fixed Cartan subalgebra \( \mathfrak{h} \). Choose a subset \( P \subset \Omega \) and denote by \( \Gamma(\Omega) \) and \( \Gamma(P) \) the \( \mathbb{Z} \) lattices generated by \( \Omega \) and \( P \) respectively. Set \( \Omega_P = \Gamma(P) \cap \Omega \) and denote by \( \mathfrak{k} \) the Lie subalgebra in \( \mathfrak{g} \) of the form

\[
\mathfrak{k} = \mathfrak{h} \oplus \left( \bigoplus_{\beta \in \Omega_P} \mathfrak{g}^\beta \right)
\]

where \( \mathfrak{g}^\beta \) is the one dimensional root space in \( \mathfrak{g} \) corresponding to \( \beta \in \Omega \), and \( \oplus \) denotes the direct sum of vector spaces. Then \( \mathfrak{k} \) is regular, and any regular Lie subalgebra of \( \mathfrak{g} \) appears in this way, see [10], Chapter 6, § 1. Denote by \( K \) the Lie subgroup of \( G \) corresponding to the Lie algebra \( \mathfrak{k} \). Since \( K \) corresponds to a regular Lie algebra, it is connected and closed. Set \( M = G/K \) and

\[
\mathfrak{m} = \bigoplus_{\beta \in \Omega \setminus \Omega_P} \mathfrak{g}^\beta.
\]
It is a \( k \) module, and it is easy to see that \( m \) is the orthogonal complement to \( k \) with respect to the Killing form \((\cdot, \cdot)\). Thus one can identify \( m \) with the quotient \( g/k \) and treat it as the tangent space to the homogeneous \( M \) at the point fixed by \( K \).

Denote by \( \overline{\Omega} \) the image without zero of \( \Omega \) under the canonical epimorphism \( \Gamma(\Omega) \to \Gamma(\Omega)/\Gamma(P) \). We denote the image of \( \beta \in \Omega \) in \( \overline{\Omega} \) by \( \overline{\beta} \), and call the elements of \( \overline{\Omega} \) quasi-roots. Quasi-roots are convenient labels for some important irreducible representations of Lie algebra \( k \). Namely, put

\[
m_{\overline{\beta}} = \bigoplus_{\gamma \in \overline{\beta}} g^{\gamma}.
\]

To prove that all \( k \) modules \( m_{\overline{\beta}} \) are simple, we will use the following lemma which is proven in [10].

**Lemma 6.1** Let \( \overline{\beta} = \overline{\beta}' \), then there exist \( \alpha_1, \ldots, \alpha_k \in \Omega_P \) such that \( \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_k = \beta' \) and \( \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_i \) is a root for every \( i \leq k \).

**Proof:** We use the following fact about root systems of simple Lie algebras (see [32], Chap. IV, Prop. 3). If \((\alpha, \beta) > 0\) for some roots \( \alpha, \beta \) then \( \alpha - \beta \) is a root as well. The equality \( \overline{\beta}' = \overline{\beta} \) means that \( \beta' = \beta + \gamma_1 + \cdots + \gamma_m \) for some (not necessary different) roots \( \gamma_i \in P \). If \( (\beta', \beta) > 0 \) then \( \beta' - \beta \) is a root in \( \Omega_P \), and everything is done. In the opposite case, \( (\beta', \beta) \leq 0 \), one uses induction by \( m \). Namely, since \( (\beta', \beta') > 0 \), there exists a root \( \gamma_i \) such that \( (\beta', \gamma_i) > 0 \). Changing labels, one can assume \( (\beta', \gamma_m) > 0 \). Thus \( \beta' - \gamma_m = \beta + \gamma_1 + \cdots + \gamma_{m-1} \) is a root, and \( \overline{\beta}' - \gamma_m = \overline{\beta} \). By the induction hypothesis, there exist \( \alpha_1, \ldots, \alpha_{m-1} \in \Omega_P \) such that \( \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_{m-1} = \beta' - \gamma_m \) and \( \beta + \alpha_1 + \alpha_2 + \cdots + \alpha_i \) is a root for every \( i \leq m - 1 \). Taking \( \alpha_m = \gamma_m \), one completes the proof. \( \blacksquare \)

**Corollary 6.1** For any \( \overline{\beta} \in \overline{\Omega} \), \( m_{\overline{\beta}} \) is an irreducible \( \mathfrak{k} \) module.

**Proof:** Chose a base \( \{X_\alpha\} \) of weight vectors for \( m_{\overline{\beta}} \) such that \( X_\alpha \in g^\alpha \), and take \( \beta, \beta' \) and \( \alpha_i \) as in Lemma 6.1. Then \( X_{\alpha_i} \in \mathfrak{k} \), and \( X_{\beta'} = \text{ad} X_{\alpha_k} \cdots \text{ad} X_{\alpha_2} \text{ad} X_{\alpha_1} (X_{\beta}) \). This implies that any element of a basis of weight vectors of \( m_{\overline{\beta}} \) can be mapped into another arbitrary element with the help of a composition of operators from \( \text{ad} \mathfrak{k} \). Thus all the weight spaces are in same irreducible component. \( \blacksquare \)

The following lemma is also proven in [10].

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Lemma 6.2 Let \( \overline{\beta}, \overline{\beta}_1, \ldots, \overline{\beta}_m \in \Omega \) such that \( \overline{\beta} = \sum_{i=1}^m \overline{\beta}_i \). Then there exist roots \( \gamma, \gamma_i \in \Omega \) such that \( \gamma \in \overline{\beta}, \gamma_i \in \overline{\beta}_i \) and \( \sum_{i=1}^m \gamma_i = \gamma \).

Proof: The equality \( \overline{\beta} = \sum_{i=1}^m \overline{\beta}_i \) means that \( \beta = \sum_{i=1}^m \beta_i + \sum_{j=1}^n \alpha_j \) for some \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega_{\beta} \). Let \( (\beta, \alpha_j) > 0 \) for some \( \alpha_j \), then \( \beta' = \beta - \alpha_j \) is a root representing \( \overline{\beta} \). This procedure can be repeated if necessary several times. Thus one can assume that \( (\beta, \alpha_j) \leq 0 \) for all \( j \)’s. We use the induction on \( m \) to find the representatives \( \gamma_i \). For \( m = 1 \), there is nothing to prove. Assume that the lemma is proven for all sums of the form \( \overline{\beta} = \sum_{i=1}^{m-1} \overline{\beta}_i \).

Let \( \beta = \sum_{i=1}^m \beta_i + \sum_{j=1}^n \alpha_j \) and \( (\beta, \alpha_j) \leq 0 \) for all \( j \)’s. Since \( (\beta, \beta) = \sum_{i=1}^m (\beta, \beta_i) + \sum_{j=1}^n (\beta, \alpha_j) > 0 \), there exists \( i \), say \( i = m \), such that \( (\beta, \beta_m) > 0 \). Therefore \( \beta - \beta_m \) is a root. Put \( \beta'' = \beta - \beta_m \), then \( \overline{\beta''} = \sum_{i=1}^{m-1} \overline{\beta}_i \), i.e. \( \overline{\beta''} \) is a sum of \( m - 1 \) quasi-roots, and the induction hypothesis applies. \( \blacksquare \)

Corollary 6.2 \( [m_{\overline{\beta}_1}, m_{\overline{\beta}_2}] = m_{\overline{\beta}_1 + \overline{\beta}_2} \)

Proof: The inclusion \( [m_{\overline{\beta}_1}, m_{\overline{\beta}_2}] \subset m_{\overline{\beta}_1 + \overline{\beta}_2} \) is obvious. We prove that \( [m_{\overline{\beta}_1}, m_{\overline{\beta}_2}] \supset m_{\overline{\beta}_1 + \overline{\beta}_2} \). By Lemma 6.2, \( \overline{\beta}_1 + \overline{\beta}_2 \in \Omega \) implies that there exist representatives \( \gamma_1 \in \overline{\beta}_1 \) and \( \gamma_2 \in \overline{\beta}_2 \) such that \( \gamma_1 + \gamma_2 \in \overline{\beta}_1 + \overline{\beta}_2 \) is a root. Then the space \( [g^{\gamma_1}, g^{\gamma_2}] \) is non-zero, and it is contained in \( m_{\overline{\beta}_1 + \overline{\beta}_2} \). By Corollary 6.1, the latter is irreducible. This proves the claim. \( \blacksquare \)

Since \( \mathfrak{g} \) is reductive, the \( \mathfrak{g} \) module \( m \) can be decomposed into direct sum of irreducible modules.

Lemma 6.3 Let \( \overline{\beta}_1 + \overline{\beta}_2 \in \Omega \), then \( m_{\overline{\beta}_1 + \overline{\beta}_2} \) is a multiplicity free irreducible component of \( \mathfrak{g} \) module \( m_{\overline{\beta}_1} \otimes m_{\overline{\beta}_2} \).

Proof: First we prove that the \( \mathfrak{g} \) module \( m_{\overline{\beta}_1 + \overline{\beta}_2} \) appears as a component of \( m_{\overline{\beta}_1} \otimes m_{\overline{\beta}_2} \). Consider the mapping \( \mathcal{L} : m_{\overline{\beta}_1} \otimes m_{\overline{\beta}_2} \rightarrow [m_{\overline{\beta}_1}, m_{\overline{\beta}_2}], X \otimes Y \mapsto [X, Y] \). It is a \( \mathfrak{g} \) module homomorphism. By Corollary 6.2, \( \mathcal{L} \) has image \( m_{\overline{\beta}_1 + \overline{\beta}_2} \). The Lie algebra \( \mathfrak{g} \) is reductive, thus the \( \mathfrak{g} \) module \( m_{\overline{\beta}_1} \otimes m_{\overline{\beta}_2} \) is decomposed into direct sum of irreducible components. Among them, there exists an irreducible sub-module, say \( \mathfrak{p} \), such that the restriction \( \mathcal{L}|_{\mathfrak{p}} \) is non-zero. By Schur’s Lemma, it is an isomorphism of \( \mathfrak{g} \) modules. The fact that \( m_{\overline{\beta}_1 + \overline{\beta}_2} \) is multiplicity free follows from [4], Chap. VIII, §9, Ex. 14). \( \blacksquare \)
Recall that if $V$ is a module over an arbitrary Lie algebra $\mathfrak{a}$, the dual space $V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C})$ is endowed with an $\mathfrak{a}$ module structure: $(X\xi)(v) = -\xi(Xv)$ for $v \in V$, $\xi \in V^*$.

**Lemma 6.4** For any $\beta \in \Omega$, $\mathfrak{t}$ modules $m_{-\beta}$ and $m_{\beta}^*$ are isomorphic.

**Proof:** One has $(g^\beta, g^\gamma) = 0$ if and only if $\beta + \gamma \neq 0$. Hence the restriction of the Killing form $(\cdot, \cdot)$ to $m_\beta \otimes m_{-\beta}$ is non-degenerate, since $m_\beta \otimes m_{-\beta} = \sum g^{\gamma_1} \otimes g^{\gamma_2}$. The restriction is $\mathfrak{t}$ invariant, thus it defines an isomorphism $m_{-\beta} \cong m_{\beta}^*$.

**Corollary 6.3** For any $\beta \in \Omega$, one has $\dim (m_\beta \otimes m_{-\beta})^\mathfrak{t} = 1$.

**Proof:** First note that if $U$ and $V$ are modules over $\mathfrak{t}$, then the $\mathfrak{t}$ modules $(V^* \otimes U)^\mathfrak{t}$ and $\text{Hom}_\mathfrak{t}(V, U)$ are isomorphic. Indeed take $\xi \otimes u \in (V^* \otimes U)^\mathfrak{t}$ and set $L(v) = \xi(v)u$ for any $v \in V$. The operator $L$ is $\mathfrak{t}$ linear since $X(\xi \otimes u) = 0$ for any $X \in \mathfrak{t}$.

By Lemma 6.4, $m_{-\beta}$ and $m_{\beta}^*$ are isomorphic, thus the $\mathfrak{t}$ module $(m_\beta \otimes m_{-\beta})^\mathfrak{t}$ is isomorphic to $\text{End}_\mathfrak{t}(m_\beta)$. By Corollary 6.1, $m_\beta$ is irreducible, thus one can apply Schur's Lemma and conclude that all elements of $\text{End}_\mathfrak{t}(m_\beta)$ are multiples of the unity operator.

**Corollary 6.4** Let $\beta$ be a quasi-root such that $\beta = -\beta$, then $(\bigwedge^2 m_\beta)^\mathfrak{t} = 0$.

**Proof:** By Lemma 6.4, $(m_\beta \otimes m_{-\beta})^\mathfrak{t}$ is isomorphic as $\mathfrak{t}$ module to $\text{End}_\mathfrak{t}(m_\beta)$. Thus $\dim (m_\beta \otimes m_{-\beta})^\mathfrak{t} = 1$. However, $(m_\beta \otimes m_{-\beta})^\mathfrak{t}$ contains a non-trivial element, that is, the Killing form which is symmetric. Hence there is no skew symmetric elements in $(m_\beta \otimes m_{-\beta})^\mathfrak{t}$.

In what follows we need the existence of one well-known base for the Lie algebra $\mathfrak{g}$.

**Proposition 6.1** For a semi-simple Lie algebra $\mathfrak{g}$ there exists a base $\{X_\alpha\}_{\alpha \in \Omega}$, $\{H_\beta\}_{\beta \in \Pi}$ with $(X_\alpha, X_{-\alpha}) = 1$ such that corresponding structural constants $N_{\alpha,\beta}$ have the following properties:
(I) $N_{\alpha\beta} \neq 0$ if and only if $\alpha + \beta \in \Omega$;
(II) $N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}$ for $\alpha + \beta + \gamma = 0$;
(III) $N_{\beta\alpha} = -N_{\alpha\beta}$;
(IV) $N_{-\alpha, -\beta} = -N_{\alpha\beta}$.

**Proof:** See [20], Lemma III.5.1 and Theorem III.5.5. ■

We are interested in $G$ invariant polyvector fields on $M$. They are of the form $\lambda_M(\tilde{\upsilon})$, $\tilde{\upsilon} \in (\bigwedge^p m)^\mathfrak{g}$ (see Section 4.1). We describe the spaces $(\bigwedge^p m)^\mathfrak{g}$ for $p = 1, 2, 3$. Obviously, there are only trivial $\mathfrak{g}$ invariant vector fields on $M$ since $\mathfrak{g}$ contains a Cartan subalgebra. The following lemma describes invariant 2-vector fields on $M$.

**Lemma 6.5** Any element of $(\bigwedge^2 m)^\mathfrak{g}$ is of the form

$$\tilde{\upsilon} = \sum_{\pi \in \Pi} c_\pi \left( \sum_{\beta \in \pi} X_\beta \wedge X_{-\beta} \right)$$

where $c_{-\pi} = -c_\pi$.

**Proof:** We prove first that any element of this form is $\mathfrak{g}$ invariant. It suffices to check that for any $\gamma \in \Omega_p$ and any quasi-root $\pi \in \Pi$ one has $X_\gamma.\tilde{\upsilon} = 0$ where

$$\tilde{\eta} = \sum_{\beta \in \pi} X_\beta \wedge X_{-\beta}.$$  

Choose $\beta \in \pi$. If $\beta + \gamma$ and $\gamma - \beta$ are not roots, then $X_\gamma(X_\beta \wedge X_{-\beta})$ does not contribute to $X_\gamma.\tilde{\eta}$. If $\beta + \gamma$ is a root, then $X_\gamma.\tilde{\eta}$ contains the terms $N_{\gamma\beta}X_{\beta+\gamma} \wedge X_{-\beta}$ and $N_{\gamma,-(\beta+\gamma)}X_{\beta+\gamma} \wedge X_{-\beta}$ which cancel one another, see Proposition 6.1. In the same way one considers the case when $\gamma - \beta$ is a root.

Note that any expression of the form $\sum_{\beta \in \pi} c_\beta X_\beta \wedge X_{-\beta}$ is invariant with respect to the Cartan subalgebra $\mathfrak{h}$. We prove that invariance by $\mathfrak{g}$ implies $c_\beta = c_{\beta'}$ if $\beta = \beta'$. Set

$$\tilde{\psi} = \sum_{\beta \in \pi} c_\beta X_\beta \wedge X_{-\beta}.$$  

By Lemma 6.1, if $\beta \in \pi$ then $\beta = \alpha + \zeta_1 + \cdots \zeta_k$ for some $\zeta_i \in \Omega_p$, and $\alpha + \zeta_1 + \cdots + \zeta_j$ is a root for any $j \leq k$. Let $\beta = \alpha + \zeta_1 + \cdots + \zeta_{j-1}$, $\zeta = \zeta_j$, $\beta' = \beta + \zeta$ and consider the element $X_\zeta.\tilde{\psi}$. Since $\beta + \zeta$ is a root, $X_\zeta.\tilde{\psi}$
contains the term \(X_{\beta+\zeta} \wedge X_{-\beta}\). Direct computations show that its coefficient is equal to \(c_{\beta}N_{\zeta\beta} + c_{\beta+\zeta}N_{\zeta,-(\beta+\zeta)}\). Using the properties of \(N_{\zeta\beta}\), one reduces this coefficient to the form \((c_{\beta} - c_{\beta+\zeta})N_{\zeta\beta}\). Thus \(X_{\zeta}\tilde{\psi} = 0\) implies \(c_{\beta+\zeta} = c_{\beta}\). □

Consider the space \((\wedge^3 m)^{\xi}\) of \(\xi\) invariant 3-vector fields on \(M\). First, note that both \((\wedge^3 m)^{\xi}\) and any its subspace of the form \((m_\alpha \otimes m_\beta \otimes m_\gamma)^{\xi}\) are invariant under the action of Cartan subalgebra \(\mathfrak{h}\). This implies that any of their elements must be of weight zero, i.e. it is a linear combination of monomials \(X_\alpha \otimes X_\beta \otimes X_\gamma\) with \(\alpha + \beta + \gamma = 0\). In particular, if \(\bar{\gamma} \neq -\bar{\alpha} - \bar{\beta}\) then \((m_\alpha \otimes m_\beta \otimes m_\gamma)^{\xi} = 0\). Indeed, it is easy to see that if \(\bar{\gamma} \neq -\bar{\alpha} - \bar{\beta}\) then \(\alpha + \beta + \gamma \neq 0\) for any \(\alpha \in \bar{\alpha}, \beta \in \bar{\beta}\) and \(\gamma \in \bar{\gamma}\).

**Lemma 6.6** The dimension of \((m_\alpha \otimes m_\beta \otimes m_{-(\alpha+\beta)})^{\xi}\) is one.

**Proof:** Set \(V = m_\alpha \otimes m_\beta\) and consider this space with the usual \(\xi\) module structure. By Lemma 6.4, \(m_{-(\alpha+\beta)}\) is isomorphic as \(\xi\) module to \(m^*_{\alpha+\beta}\). Thus the \(\xi\) modules \((V \otimes m_{-(\alpha+\beta)})^{\xi}\) and \(\text{Hom}_\xi(m_{\alpha+\beta}, V)\) are isomorphic (see proof for Corollary 6.3). Since \(\xi\) is reductive, \(V\) is a direct sum of simple \(\xi\) modules. By Lemma 5.3, the module \(m_{\alpha+\beta}\) is simple and multiplicity free in \(m_\alpha \otimes m_\beta\). Thus \(\text{Hom}_\xi(m_{\alpha+\beta}, V)\) and \(\text{End}_\xi(m_{\alpha+\beta})\) are isomorphic, and \(\dim \text{End}_\xi(m_{\alpha+\beta}) = 1\). □

**Lemma 6.7** The dimension of \((\wedge^3 m)^{\xi}\) is equal to the number of unordered pairs \(\{\bar{\alpha}, \bar{\beta}\}\) such that \(\bar{\alpha} + \bar{\beta} \in \bar{\Omega}\).

**Proof:** The dimension of \((\wedge^3 m)^{\xi}\) is equal to the number of distinct subspaces of the form \((m_\alpha \otimes m_\beta \otimes m_{-(\alpha+\beta)})^{\xi}\). □

Now we describe the elements of \((\wedge^2 m)^{\xi}\) which are \(\varphi\)-Poisson brackets.

**Lemma 6.8** \([\tilde{\nu}, \tilde{\nu}] = \kappa^2 \tilde{\varphi}\) if and only if the coefficients \(c_{\beta}\) from Lemma 6.3 satisfy the following condition: if \(\bar{\alpha} + \bar{\beta} \in \bar{\Omega}\) then

\[
c_{\alpha+\beta} = \frac{c_{\alpha}c_{\beta} + \kappa^2}{c_{\alpha} + c_{\beta}}.
\]
Proof: Let \( \tilde{\omega} = \sum \alpha b_\alpha X_\alpha \wedge X_{-\alpha} \) and \( \tilde{\nu} = \sum \alpha c_\alpha X_\alpha \wedge X_{-\alpha} \) be elements of \( (\wedge^2 m)^G \). We compute the coefficient before monomial \( X_{\alpha+\beta} \wedge X_{-\alpha} \wedge X_{-\beta} \) in \([\tilde{\omega}, \tilde{\nu}]\). This monomial appears six times, with the coefficients \( N_{\alpha\beta} c_\alpha b_\beta \), \( -N_{\beta,(\alpha+\beta)} c_\beta b_\alpha \), \( N_{\alpha,(\alpha+\beta)} c_\alpha b_\beta \), \( -N_{\beta\alpha} c_\beta b_\alpha \) and \( -N_{(\alpha+\beta)\beta} c_\beta b_\alpha \). Using the properties of \( N_{\alpha\beta} \) (Proposition 6.1), one obtains the sum of the above coefficients:

\[
N_{\alpha\beta} \cdot (c_\alpha b_\beta - c_{\alpha+\beta} b_\beta - c_\beta b_{\alpha+\beta} - c_\alpha b_{\alpha+\beta} + c_\beta b_\alpha - c_{\alpha+\beta} b_\alpha).
\]

(32)

The element \( \tilde{\varphi} \) is of the form

\[
\tilde{\varphi} = \sum_{\alpha \in \pi} N_{\alpha\beta} X_\alpha \wedge X_\beta \wedge X_{-(\alpha+\beta)}.
\]

(33)

Thus the coefficient before \( X_{\alpha+\beta} \wedge X_{-\alpha} \wedge X_{-\beta} \) in \( \kappa^2 \tilde{\varphi} \) is \( -\kappa^2 N_{\alpha\beta} \). and, by replacing in (32) \( b_\alpha \)'s with \( c_\alpha \)'s, one completes the proof. (See also [22]).

6.3 The manifold \( M_{l\alpha} \) and its quantization

We preserve the notation of Section 6.2. Fix a simple root \( \alpha \) and a positive integer \( l \) not greater than the multiplicity of \( \alpha \) in the highest root in \( \Omega \). Consider the set \( \Omega_{l\alpha} \) of all roots in \( \Omega \) whose coefficients before \( \alpha \) is divisible by \( l \) (in Section 6.2 we used the notation \( \Omega_p \)). We denote by \( \Omega \) the set of quasi-roots for \( m_\pi \), i.e. the image of \( \Omega_{l\alpha} \) in \( \Gamma(\Omega)/\Gamma(\Omega_{l\alpha}) \), by \( M_{l\alpha} \) the homogeneous manifold whose stabilizer \( K \) is generated by \( \Gamma(\Omega_{l\alpha}) \cap \Omega \). Clearly, \( \Omega = \{\alpha, 2\alpha, \ldots, (l-1)\alpha\} \). It follows from the classification of semi-simple Lie algebras over \( \mathbb{C} \), that \( l \leq 6 \).

The significance of the manifolds \( M_{l\alpha} \) is in fact that their quantization can be considered as the first step to resolving the following more general problem. Let \( G \) be a simple connected Lie group over \( \mathbb{C} \), \( \mathfrak{g} \) its Lie algebra, \( \mathfrak{k} \subset \mathfrak{g} \) a reductive Lie subalgebra, \( K \) the corresponding Lie subgroup of \( G \). E. B. Dynkin [13] has proven that the homogeneous manifold \( M = G/K \) can be obtained by taking consequent quotients of direct products of manifolds \( M_{l\alpha} \).

Recall that the quotient \( \mathfrak{m} = \mathfrak{g}/\mathfrak{k} \) is isomorphic to the tangent space to \( M_{l\alpha} \) at the point fixed by \( K \). We calculate the dimensions of the cohomology spaces which figured in Theorem 5.1.

For a given \( \varepsilon \in \mathbb{R} \), denote by \( [\varepsilon] \) the largest integer \( n \) such that \( n \leq \varepsilon \).
Lemma 6.9 Let \( m = g/\mathfrak{k} \) be the tangent space of \( M_\alpha \), then \( \dim (\bigwedge^2 m)^\mathfrak{k} = \left[ \frac{l-1}{2} \right] \).

**Proof:** According to Corollary 6.4 and Lemma 6.5, this dimension is equal to the number of quasi-roots \( \alpha \) such that \( \alpha \neq -\alpha \). The \( \mathfrak{k} \) invariance reduces that to the number of unordered pairs \( \{\alpha, -\alpha\} \) which is equal to \( \left[ \frac{l-1}{2} \right] \). \( \blacksquare \)

Note that all elements of \( (\bigwedge^2 m)^\mathfrak{k} \) are skew invariant with respect to the Cartan involution defined by

\[ \theta : X_\alpha \mapsto -X_{-\alpha} \]  

(34)

The image of the coboundary map \( [\mathfrak{s}, \cdot] : (\bigwedge^2 m)^\mathfrak{k} \to (\bigwedge^3 m)^\mathfrak{k} \) consists of \( \theta \) invariant elements, because, if \( \tilde{v} \in (\bigwedge^p m)^\mathfrak{k} \) obeys \( \theta(\tilde{v}) = (-1)^{p+1}\tilde{v} \) then the element \([\mathfrak{s}, \tilde{v}]\) is of the opposite parity: \( \theta([\mathfrak{s}, \tilde{v}]) = (-1)^p[\mathfrak{s}, \tilde{v}] \). Therefore one can consider the sub-complex of \( \theta \) invariant \( p \)-vector fields for \( p \) odd and skew \( \theta \) invariant \( p \)-vector field for \( p \) even. Recall that we denote this sub-complex by \((\bigwedge^p m)^{\mathfrak{k},\theta}\). Note that the element \( \tilde{\varphi} \) given by formula (33) belongs to \((\bigwedge^3 m)^{\mathfrak{k},\theta}\).

Now, we compute the dimension of \((\bigwedge^3 m)^{\mathfrak{k},\theta}\). For quasi-roots \( i\alpha, j\alpha \) such that \( j\alpha \neq -i\alpha \), denote by \( v(i, j) \) the subspace of \((\bigwedge^3 m)^{\mathfrak{k},\theta}\) generated by the image of \( \mathbb{C} \) linear mapping

\[ (m_{i\alpha} \otimes m_{j\alpha} \otimes m_{-(i+j)\alpha}) \to (3 \bigwedge m)^{\mathfrak{k},\theta} \]  

(35)

\[ X_\beta \otimes X_\gamma \otimes X_{-\beta-\gamma} \mapsto X_\beta \wedge X_\gamma \wedge X_{-\beta-\gamma} = X_{-\beta} \wedge X_{-\gamma} \wedge X_{\beta+\gamma}. \]

It is easy to see that

\[ v(i, j) = v(j, i) = v(i, l - i - j) = v(j, l - i - j), \]

thus only spaces \( v(i, j) \) with \( i \leq j < \frac{l}{2} \) should be taken into consideration. Hence one has

\[ (3 \bigwedge m)^{\mathfrak{k},\theta} = \bigoplus_{1 \leq i \leq j < \frac{l}{2}} v(i, j). \]  

(36)

Lemma 6.10 The subspace \( v(i, j) \) has dimension one.
Proof: By Lemma 6.6, \( \dim(\mathfrak{m}_i \otimes \mathfrak{m}_j \otimes \mathfrak{m}_k) = 1 \). On the other hand, the mapping (35) is non-zero, since there exist \( \beta \in \mathfrak{m}_i \) and \( \gamma \in \mathfrak{m}_j \) such that \( \beta + \gamma \) is a root. Thus the image \( v(i, j) \) of that mapping is of dimension one.

\[ \Box \]

Lemma 6.11 The dimension of \( (\wedge^3 \mathfrak{m})^{t, \theta} \) is equal to the number of its subspaces \( v(i, j) \) with \( i \leq j < \frac{l}{2} \).

Proof: Direct consequence of the formula (36) and Lemma 6.10. See, also, Lemma 6.7. \[ \Box \]

We are interested with the third cohomology space of the sub-complex \( \tilde{\Lambda}(M_{t \alpha}) \) of \( \mathfrak{g} \) and \( \theta \) invariant polyvector fields on \( M \), see Theorem 5.1. The dimension of \( \tilde{\Lambda}_3(M_{t \alpha}) \) is equal to \( \dim(\wedge^3 \mathfrak{m})^{t, \theta} \). By Lemma 6.11, the latter dimension is equal to the number of pairs \( (i, j) \) with \( 1 \leq i \leq j < \frac{l}{2} \). For \( l = 2 \) there are no pairs \( (i, j) \) satisfying \( 1 \leq i \leq j < \frac{l}{2} \), thus \( \dim \tilde{\Lambda}_3(M_{2 \alpha}) = 0 \).

For \( l = 3 \) and \( l = 4 \) one has one subspace in \( (\wedge^3 \mathfrak{m})^{t, \theta} \), it is \( v(1, 1) \), thus \( \dim \tilde{\Lambda}_3(M_{3 \alpha}) = \dim \tilde{\Lambda}_3(M_{4 \alpha}) = 1 \). For \( l = 5 \) there are two subspaces in \( (\wedge^3 \mathfrak{m})^{t, \theta} \), \( v(1, 1) \) and \( v(1, 2) \), thus \( \dim \tilde{\Lambda}_3(M_{5 \alpha}) = 2 \). For \( l = 6 \) there are three subspaces in \( (\wedge^3 \mathfrak{m})^{t, \theta} \), \( v(1, 1) \), \( v(1, 2) \) and \( v(2, 2) \), thus \( \dim \tilde{\Lambda}_3(M_{6 \alpha}) = 3 \).

All calculated dimensions are presented in the table on page 59. Note that the dimension of space \( (\wedge^3 \mathfrak{m})^{t, \theta} \) is equal to the number of 3-partitions of the integer \( l \), and that the dimension of space \( (\wedge^2 \mathfrak{m})^{t, \theta} \) is equal to the number of 2-partitions of \( l \) with non-equal components.

Theorem 6.2

(I) Any manifold \( M = M_{t \alpha} \), \( 2 \leq l \leq 6 \), possesses a \( \varphi \)-Poisson bracket.

(II) For any \( \varphi \)-Poisson bracket \( s \) on \( M = M_{t \alpha} \), \( l \geq 2 \), the cohomology spaces \( H^2 \left( \tilde{\Lambda}(M), d_s \right) \) and \( H^3 \left( \tilde{\Lambda}(M), d_s \right) \) are trivial.

Proof: We consider each case \( l = 2, \ldots, 6 \) separately. First prove that for \( M = M_{2 \alpha} \) one has \( (\wedge^3 \mathfrak{m})^{t, \theta} = 0 \). Indeed, take an element \( X_{\beta_1} \wedge X_{\beta_2} \wedge X_{\beta_3} \in \wedge^3 \mathfrak{m} \), \( \beta_1, \beta_2, \beta_3 \notin \Omega_{2 \alpha} \). If \( \beta_1 + \beta_2 + \beta_3 \in \Gamma(\Omega_{2 \alpha}) \) then at least one of the \( \beta_i \)'s contains the root \( \alpha \) with an even coefficient, and therefore the element \( X_{\beta_1} \wedge X_{\beta_2} \wedge X_{\beta_3} \) is actually equal to zero.
In particular, the 3-vector field $\varphi$ is equal to zero on $M$. The second consequence from $(\Lambda^3 m)^b = 0$ is that $H^3 (\tilde{\Lambda}(M_{2\alpha}), \delta_s) = 0$.

Let $M = M_{3\alpha}$ or $M_{4\alpha}$, then, by Lemma 3.9 (see also the table on page 59), dim $\tilde{\Lambda}_2 (M) = 1$. Thus $(\Lambda^2 m)^t$ contains a non-trivial element $\tilde{s} = c_\alpha \sum_{\beta \in \alpha} X_\beta \wedge X_\beta$. Direct calculation shows that $[\tilde{s}, \tilde{s}]$ is proportional to the invariant element $\varphi$, (33), thus $s = \lambda (\tilde{s})$ is a $\varphi$-Poisson bracket on $M$. The vector space $(\Lambda^3 m)^{t, \theta}$ is also of dimension one (see table on p. 59). It is generated by the element $\varphi$. Since $d_s (s) = \varphi$, one has $H^3 (\tilde{\Lambda}(M), \delta_s) = 0$.

On the hand, $d_s : \tilde{\Lambda}_2 (M) \rightarrow \tilde{\Lambda}_3 (M)$ is a non-trivial linear operator from one dimensional vector space to another one dimensional vector space. Thus it is non-degenerate, which proves that $H^2 (\tilde{\Lambda}(M), \delta_s) = 0$.

Let $M = M_{5\alpha}$, then, by Lemma 6.9, dim $(\Lambda^2 m)^{t, \theta} = 2$. Take a non-trivial element

$$\tilde{s} = c_\alpha \sum_{\beta \in \alpha} X_\beta \wedge X_\beta + c_{2\alpha} \sum_{\beta \in 2\alpha} X_\beta \wedge X_\beta \in \left( \Lambda^2 m \right)^t$$

and prove that the multiples $c_\alpha$ and $c_{2\alpha}$ can be chosen in such a way that $[\tilde{s}, \tilde{s}] = \varphi$. It follows from Lemma 6.8 and the identities $\overline{\alpha} + \overline{\alpha} = 2\overline{\alpha}$ and $2\overline{\alpha} + 2\overline{\alpha} = -\overline{\alpha}$ that such the coefficients $c_\alpha$ and $c_{2\alpha}$ should satisfy the system of equations

$$\begin{cases} 5(c_\alpha)^4 + 10(c_\alpha)^2 \kappa^2 + \kappa^4 = 0 \\ c_{2\alpha} = \frac{(c_\alpha)^2 + \kappa^2}{2c_\alpha}. \end{cases}$$

(37)

This system has two solutions (see table on p. 59).

**Lemma 6.12** Let $M = M_{5\alpha}$. Then the coboundary operator $\delta : (\Lambda^2 m)^t \rightarrow (\Lambda^3 m)^{t, \theta}$ is injective.

**Proof:** Any element $\tilde{u}$ of $(\Lambda^2 m)^t$ is of the form

$$\tilde{u} = b_\alpha \sum_{\beta \in \alpha} X_\beta \wedge X_\beta + b_{2\alpha} \sum_{\beta \in 2\alpha} X_\beta \wedge X_\beta.$$
Let $\tilde{d}(\tilde{v}) = 0$, then the identities $\overline{\alpha} + \overline{\alpha} = 2\overline{\alpha}$ and $2\overline{\alpha} + 2\overline{\alpha} = -\overline{\alpha}$ and formula (32) imply the following system of equations for $b_\alpha$ and $b_{2\alpha}$:

\[
\begin{cases}
  b_\alpha(c_\alpha + c_{2\alpha}) = c_\alpha b_{2\alpha} \\
  b_{2\alpha}(c_\alpha - c_{2\alpha}) = c_{2\alpha} b_{2\alpha},
\end{cases}
\]

where $c_\alpha$ and $c_{2\alpha}$ are solutions for (37). From the condition for the existence of a non-trivial to these homogeneous equations and formula (37) one comes to the following inconsistent system of equations:

\[
\begin{cases}
  5c_\alpha^4 + 10c_{2\alpha}^2 \kappa^2 + \kappa^4 = 0 \\
  -c_\alpha^4 + 4c_{2\alpha}^2 \kappa^2 + \kappa^4 = 0
\end{cases}
\]

Therefore the only possibility is $b_\alpha = b_{2\alpha} = 0$. ■

The immediate consequence of this lemma is that $H^2\left(\tilde{\Lambda}(M_{5\alpha}), d_s\right) = 0$. On the other hand, together with the equality $\dim (\bigwedge^2 m)^{e} = \dim (\bigwedge^3 m)^{e,\theta}$, it implies that $H^3\left(\tilde{\Lambda}(M_{5\alpha}), d_s\right) = H^3\left(\bigoplus_{p \geq 0} (\bigwedge^p m)^{e,\theta}, \tilde{d}\right) = 0$.

Let $M = M_{6\alpha}$ then the table on page 59 shows that $\dim (\bigwedge^2 m)^{e,\theta} = 2$. There exists a $g$ invariant $\varphi$-Poisson bracket on $M$ generated by an element $\tilde{s}$ of the form

$$
\tilde{s} = c_\alpha \sum_{\beta \in \alpha} X_\beta \wedge X_{-\beta} + c_{2\alpha} \sum_{\beta \in 2\alpha} X_\beta \wedge X_{-\beta}
$$

with coefficients satisfying a certain equation which we obtain now. It follows from Lemma 6.8 and the identities $2\overline{\alpha} + 2\overline{\alpha} = -2\overline{\alpha}$, $\overline{\alpha} + \overline{\alpha} = 2\overline{\alpha}$, $\overline{\alpha} + 2\overline{\alpha} = 3\overline{\alpha}$ and $c_{3\alpha} = 0$ that the coefficients $c_\alpha$ and $c_{2\alpha}$ should satisfy the equation

$$
c_\alpha^2 - 9c_{2\alpha}^2 = 0. \tag{38}
$$

This equation has one non-trivial solution (see table on p. 59).

**Lemma 6.13** Let $M = M_{6\alpha}$. Then the coboundary operator $\tilde{d}: (\bigwedge^2 m)^e \rightarrow (\bigwedge^3 m)^{e,\theta}$ is injective.
**Proof:** Any element $\bar{\nu}$ of $(\bigwedge^2 m)^k$ is of the form

$$\bar{\nu} = b_\pi \sum_{\beta \in \pi} X_\beta \wedge X_{-\beta} + b_{2\pi} \sum_{\beta \in 2\pi} X_\beta \wedge X_{-\beta}.$$ 

Let $\bar{d}(\bar{\nu}) = 0$, then the identities $2\pi + 2\pi = -2\pi$ and $\pi + \pi = 2\pi$ and equations \(32\) and \(38\) imply $b_\pi = b_{2\pi} = 0$. ■

Thus $H^2\left(\bar{\Lambda}(M_{6\alpha}), d_s\right) = 0$.

**Lemma 6.14** Let $M = M_{6\alpha}$, then there exists $\bar{\nu} \in (\bigwedge^3 m)^{t,0}$ such that $\bar{d}(\bar{\nu}) \neq 0$.

**Proof:** Note that there is only one root system with the highest root containing coefficient 6 in the simple root decomposition. This is the root system $E_8$, and the corresponding simple root is the following:

$$\alpha = \left(\begin{array}{cccccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}\right)$$

Thus we can set $g$ as a Lie algebra of type $E_8$ and the simple root $\alpha$ as determined by the above weighted Dynkin graph.

Take the element

$$\bar{\nu} = \sum_{\beta, \gamma \in \pi} N_{\beta \gamma} X_\beta \wedge X_\gamma \wedge X_{-(\beta + \gamma)}$$

and prove that $\bar{d}(\bar{\nu}) \neq 0$. Note that $\bar{\nu}$ is the projection of $\bar{\varphi}$ onto subspace $v(1, 1) \subset (\bigwedge^3 m)^{t,0}$.

Choose roots $\beta, \gamma, \epsilon, \zeta \in E_8$ satisfying the properties: $\beta, \gamma \in 2\pi$, $\epsilon \in 3\pi$, $\zeta \in 5\pi$, $\beta + \gamma + \epsilon + \zeta = 0$; $\beta + \gamma$, $\beta + \zeta$ and $\gamma + \epsilon$ are roots; $\gamma + \zeta$ and $\beta + \epsilon$ are not roots. One can take for example the following set of roots:

$$\beta = \left(\begin{array}{cccccccccc} 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\
1
\end{array}\right) \quad \gamma = \left(\begin{array}{cccccccccc} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
1
\end{array}\right)$$
The monomial \( X_\beta \wedge X_\gamma \wedge X_\varepsilon \wedge X_\zeta \) is contained in the image \( \mathfrak{d}(\tilde{v}) \). We prove that it is present in \( \mathfrak{d}(\tilde{v}) \) with a non-zero coefficient.

We introduce the following temporary definition. We say that a root vector \( X_\beta \) is of type \( i \) if the root \( \beta \) contains the simple root \( \alpha \) with multiplicity \( i \). Obviously, \( -6 < i < 6 \). It is clear how to extend this definition to any exterior monomial. For instance, \( X_\beta \wedge X_\gamma \wedge X_\varepsilon \wedge X_\zeta \) for the above roots is of type \((2, 2, -3, -1)\).

The vector space \( \mathfrak{v}(1,1) \) consists of certain sums of exterior 3-monomials. The type of each monomial is a triple of integers ranging from \(-5\) to \(5\). Since \( \mathfrak{v}(1,1) \) is invariant by the Cartan subalgebra \( \mathfrak{h} \), the sum of integers in each triple is equal to zero. The triples satisfying these two conditions are \((1, 1, -2), (1, -1, 2), (1, -5, 4) \) and \((-1, 5, -4)\).

We describe all possible 3-monomials of elements in \( \mathfrak{v}(1,1) \) and all possible 2-monomials \( X_\sigma \wedge X_{-\sigma} \) such that their Schouten bracket contains the given 4-monomial \( X_\beta \wedge X_\gamma \wedge X_\varepsilon \wedge X_\zeta \). First note that all monomials in the Schouten bracket of monomials of types \((i,j,k)\) and \((d,-d)\) are of types \((*,*,*,d)\) or \((*,*,*, -d)\). Since we want to obtain a monomial of type \((2,2,-3,-1)\), this means that \( d = 1, 2 \) or \( 3 \). The case \( d = 3 \) is excluded by Corollary 6.4.

Consider the case \( d = 1 \). We want to obtain a monomial of type \((2,2,-3,-1)\) by taking the Schouten bracket of monomials of types \((a,b,c)\), \( a+b+c = 0 \), and \((1, -1)\). Since \([X_\alpha, X_\beta] = N_{\alpha\beta}X_{\alpha+\beta}\), one concludes that each of \( a, b \) or \( c \) is equal to either \( 2 \) or \(-3\). Thus either \((a, b, c) = (1, 2, -3)\) or \((a, b, c) = (2, 2, -4)\). However, as it was pointed out above, there is no element of \( \mathfrak{v}(1,1) \) containing monomials of these two types. Let \( d = 2 \), then the same arguments show that \( a, b \) and \( c \) are taken from the set of integers \( 2, -3, -1 \). From the above list of possible types for \( \mathfrak{v}(1,1) \), it can be seen that only the triple \((a, b, c) = (2, -1, -1)\) fits.

Thus the monomial \( X_\beta \wedge X_\gamma \wedge X_\varepsilon \wedge X_\zeta \) can be obtained as the Schouten bracket of monomials of types \((2, -1, -1)\) and \((2, -2)\). This implies that the latter monomial is equal either to \( c_{\gamma\beta}X_\beta \wedge X_{-\beta} \) or to \( c_{\gamma\lambda}X_\gamma \wedge X_{-\gamma} \). Since \( \tilde{s} \neq 0 \), one has \( c_{\gamma\beta} \neq 0 \) (see table on p. 59). The former monomial is either of the form

\[
\varepsilon = -\left( \begin{array}{ccccc}
1 & 2 & 3 & 3 & 2 & 1 \\
2 & & & & & \\
\end{array} \right) \\
\zeta = -\left( \begin{array}{ccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & & & & & & \\
\end{array} \right)
\]
\[-6N_{\gamma\zeta}X_\gamma \wedge X_{\beta+\varepsilon} \wedge X_\zeta \text{ or } -6N_{\beta\zeta}X_\beta \wedge X_{\gamma+\varepsilon} \wedge X_\zeta \text{ correspondingly. However, } \gamma+\zeta \text{ is not a root, thus the only way to obtain the monomial } X_\beta \wedge X_\gamma \wedge X_{\varepsilon} \wedge X_\zeta \text{ is to take the Schouten bracket } [[-6N_{\beta\zeta}X_\beta \wedge X_{\gamma+\varepsilon} \wedge X_\zeta, c_{2\alpha}X_\gamma \wedge X_{-\gamma}]]. \text{ The corresponding coefficient is } -6N_{\beta\zeta}c_{2\alpha}, \text{ and it differs from zero since } \beta+\zeta \text{ is a root.} \]

Now, we prove that \( H^3(\widetilde{\Lambda}(M_{6\alpha}), d_s) = 0. \) Indeed, the complex \( (\widetilde{\Lambda}(M_{6\alpha}), d_s) \) is isomorphic to the complex \( \left( \bigoplus_{p \geq 0}(\Lambda^p \mathfrak{m})^{t,\theta}, d_{\tilde{d}} \right) \). Lemma 6.13 implies that

\[
\dim B^3(\widetilde{\Lambda}(M_{6\alpha}), d_s) = \dim \widetilde{\Lambda}_2(M_{6\alpha}) = 2.
\]

On the other hand, it follows from Lemma 6.14 that

\[
2 \leq \dim Z^3(\widetilde{\Lambda}(M_{6\alpha}), d_s) = \dim \left( \bigwedge^3 \mathfrak{m} \right)^{t,\theta} - \dim \left( \text{Im } d_{\tilde{d}} \right) \leq 2.
\]

Therefore

\[
\dim B^3(\widetilde{\Lambda}(M_{6\alpha}), d_s) = \dim Z^3(\widetilde{\Lambda}(M_{6\alpha}), d_s).
\]

Theorem 6.2 is proven.

Note that \( H^2(\widetilde{\Lambda}(M_{t\alpha}), d_s) \) and \( H^3(\widetilde{\Lambda}(M_{t\alpha}), d_s) \) coincide with the corresponding topological cohomologies (see [3]).

In the following table we put together the computed dimensions and give the explicit formulas for coefficients of \( \varphi \)-Poisson brackets:
| $l$ | $\varphi$-Poisson brackets | $\dim (\bigwedge^2 \mathfrak{m})^\ell$ | $\dim (\bigwedge^3 \mathfrak{m})^{\ell, \theta}$ |
|-----|-----------------------------|-----------------|-----------------|
| 2   | 0                           | 0               | 0               |
| 3   | $c_\pi = \pm \frac{i}{\sqrt{3}} \kappa$ | 1               | 1               |
| 4   | $c_\pi = \pm i \kappa$      | 1               | 1               |
| 5   | $c_\pi = \frac{i}{\sqrt{3}} \left(\sqrt{5} \pm 2\right)^{\frac{1}{2}} \kappa$ | 2               | 2               |
|     | $c_{2\pi} = \pm \frac{i}{\sqrt{3}} \left(\sqrt{5} \pm 2\right)^{-\frac{1}{2}} \kappa$ (the signs are consistent) |               |                  |
| 6   | $c_\pi = \pm i \sqrt{3} \kappa$ | 2               | 3               |
|     | $c_{2\pi} = \pm \frac{i}{\sqrt{3}} \kappa$ (the signs are consistent) |               |                  |

Note that for $l = 2, 3, 4, 6$ there exists a $\varphi$-Poisson bracket on $M_{l\alpha}$ unique up to a scalar multiple, and for $l = 5$ there are two such brackets.

Combining Theorems 6.2, 6.1 and 5.1, one obtains the main result of the present work:

**Theorem 6.3** Let $r$ be a bivector field on $M_{l\alpha}$ generated by a Belavin–Drinfeld classical $r$-matrix. Then there exists a $\mathfrak{g}$ invariant $\varphi$-Poisson bracket $s$ on $M_{l\alpha}$ such that $s + r$ is a Poisson bracket, and this bracket has a $U_h(\mathfrak{g}, r)$ invariant quantization.

Note that despite the fact that the bracket $s + r$ is not $\mathfrak{g}$ invariant, its quantization is invariant under the quantum group $U_h(\mathfrak{g}, r)$ action.
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