A LINEAR INDEPENDENCE RESULT FOR $p$-ADIC $L$-VALUES

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Abstract. The celebrated theorem of Ball–Rivoal gives a lower bound for the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} + \mathbb{Q}\zeta(3) + ... + \mathbb{Q}\zeta(s)$ spanned by the odd positive zeta values between 3 and $s$. In particular, they have proven for the first time the existence of infinitely many irrational odd zeta values. Although there are still many open questions about the structure of odd zeta values, even less is known about the values of the $p$-adic zeta function at positive integers or more generally about $p$-adic $L$-values: Aside from asymptotic results on $p$-adic Hurwitz zeta values due to Bel, we only know the irrationality of $\zeta_p(3)$ for $p = 2, 3$ and of $L_2(2, \chi)$ for $\chi$ of conductor 4 thanks to Beukers and Calegari. The aim of this paper is to provide an analogue of the Ball–Rivoal theorem for $p$-adic $L$-values of Dirichlet characters: More precisely, we prove

$$\dim_{\mathbb{Q}} (\mathbb{Q}(\chi) + \sum_{i=2}^{s+1} L_p(i, \chi^{1-i})\mathbb{Q}(\chi)) \geq C \log(s)$$

for an explicit constant $C$.

1. Introduction

The values of the Riemann zeta function at even positive integers are non-zero rational multiples of powers of $\pi$ and thereby transcendental. The question about the structure of the odd positive zeta values is much harder and widely open. The first result in this direction has been given by Apéry, [Apé79]. He has proven the irrationality of $\zeta(3)$. While this is still the only particular zeta value known to be irrational, the celebrated theorem of Ball and Rivoal gives an asymptotic lower bound for the dimension of the $\mathbb{Q}$-vector space spanned by the first odd zeta values in the interval $[3, s]$:

**Theorem (Ball–Rivoal (2001), [Riv00, BR01]).** For any $\epsilon > 0$, there exists an integer $s_0$ such that for all odd integers $s \geq s_0$:

$$\dim_{\mathbb{Q}} (\mathbb{Q} + \zeta(3)\mathbb{Q} + \zeta(5)\mathbb{Q} + ... + \zeta(s)\mathbb{Q}) \geq \frac{\log(s)(1 - \epsilon)}{1 + \log(2)}.$$

The lower bound on the dimension has not been significantly improved since then. Nevertheless, if one is only interested on irrationality of odd zeta values and not their linear independence, the lower bound can be further improved:

**Theorem (Fischler–Sprang–Zudilin (2018), [FSZ18]).** For any $\epsilon > 0$, there exists an integer $s_0$ such that for all odd integers $s \geq s_0$ at least

$$2^{(1-\epsilon)} \left( \frac{\log(s)}{\log(\log(s))} \right)$$

among the numbers $\zeta(3), \zeta(5), ..., \zeta(s)$ are irrational.

Quite recently, S. Fischler has given a unified approach to both of the above theorems using a variant of Shidlovsky’s Lemma and even more importantly, he has extended both results to $L$-values of Dirichlet characters and more generally periodic functions on $\mathbb{Z}/N\mathbb{Z}$, [Fis18]. His result

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can be seen as the state of the art result on asymptotic irrationality and linear independence results on $L$-values.

Although we are still far away from having a good understanding of the nature of odd zeta values, even less is known about the values of the $p$-adic zeta function at positive integers: Building on a beautiful re-interpretation of Apery’s proof by Beukers in terms of modular forms, Calegari succeeded to prove the irrationality of $\zeta_p(3)$ for $p = 2, 3$ and of $L_3(2, \chi)$ with $\chi$ the Dirichlet character of conductor 4, [Cal05]. Shortly afterwards, Beukers gave a new proof of Calegari’s results and furthermore provided irrationality proofs for certain values of the $p$-adic Hurwitz zeta function at small integers and for small primes $p$, [Beu08]. Partial results towards a $p$-adic analogue of the Ball–Rivoal Theorem have been given by Bel, [Bel10]: He has given linear independence results for the $p$-adic Hurwitz zeta values

$$1, \zeta_p(2, x), \ldots, \zeta_p(s, x)$$

but unfortunately his proof is limited to the case that the modulus $x \in \mathbb{Q}_p$ has sufficiently large $p$-adic norm.

Let us recall, that we have the following $p$-adic interpolation property for the $p$-adic $L$-function of a Dirichlet character $\chi$ (here, $\omega$ is the Teichmüller character):

$$L_p(i, \chi \omega^{1-i}) = L(i, \chi), \quad i \in \mathbb{Z}_{<0}.$$

This formula suggests to ask about the structure of the $p$-adic integers $L_p(i, \chi \omega^{1-i})$ for positive integers $i \in \mathbb{Z}$. In this paper, we prove the following theorem:

**Theorem.** Let $\chi$ be a Dirichlet character of conductor $f$ and $K$ be a finite extension of $\mathbb{Q}$ containing all values of the Dirichlet character $\chi$. There is an integer $s_0$ depending on $\chi$ and $p$ such that for all $s \geq s_0$ a power of $p$:

$$\dim K \left( K + \sum_{i=2}^{s+1} L_p(i, \chi \omega^{1-i})K \right) \geq \frac{\log s}{[K : \mathbb{Q}] (\log(2) + 2 + 2f)}.$$

Our strategy is roughly, to construct linear forms of $p$-adic $L$-values using certain linear combinations of linear forms of $p$-adic Hurwitz $L$-values with related coefficients. The technique of using linear forms of (classical) Hurwitz $L$-values has been initiated in [Zud18], developed in [Spr18] and fruitfully applied in [FSZ18].

Although the ideas of our paper are motivated by Bel’s paper, our proof is somehow different to the one in [Bel10]: Bel starts with the construction of linear forms in Hurwitz zeta values for the classical Hurwitz zeta function via Padé approximation. Then he observes, that a variant of the Euler–Maclaurin formula

$$\zeta(s, x) = \frac{x^{1-s}}{s-1} - \sum_{j=1}^{k} \left( \frac{-s}{j-1} \right) B_j \frac{x^{1-s-j}}{j} + O(x^{-s-j}), \quad \text{for } x \to \infty$$

is also valid $p$-adically. This allows him to transfer the linear forms in values of the classical Hurwitz zeta function to the $p$-adic world. In this way he obtains sufficiently many linearly independent families of linear forms and he can finally apply a $p$-adic variant of Siegel’s criterion. In this paper, we try to avoid the passage from the Archimedean to the non-Archimedean world and work intrinsically with $p$-adic methods.

To motivate our approach, let us briefly recall that the most important source of linear forms in Hurwitz zeta values is provided by sums of the form

$$\sum_{k \geq 0} R_n(k + x) = L_n(1, \zeta(3, x), \ldots, \zeta(s, x)), \quad x \in \mathbb{Q},$$
with $R_n(t) \in \mathbb{Q}(t)$ a rational function of degree less then 1 having only poles of maximal order $s$ at negative integers. Indeed, this construction appears in all of the above mentioned results on zeta values. Using partial fraction decomposition together with the formula

$$\omega(x)^{-s} \zeta_p(s, x) = \frac{1}{s-1} \int_{\mathbb{Z}_p} \frac{1}{(x+t)^{s-1}} dt := \lim_{n \to \infty} \frac{1}{p^n} \sum_{1 \leq k < p^n} \frac{1}{(x+k)^{s-1}}$$

suggests that Volkenborn integration over rational functions is the right $p$-adic analogue to the above summation process.

Finally, let us give an overview over the content of the paper: After recalling basic facts about $p$-adic Hurwitz zeta functions and Volkenborn integration, we will use Volkenborn integration over certain rational functions $R_n(t, x) \in \mathbb{Q}(t, x)$ to construct families of linear forms in $p$-adic Hurwitz zeta values (see section 4):

$$(\ast) \quad \int_{\mathbb{Z}_p} R_n(t + y, x) dt = \rho_0^{(y)} + \sum_{i=1}^{s} \rho_i \omega(x)^{-i} \zeta_p (i + 1, x + y).$$

Here, $x, y \in \mathbb{Q}$ are rational numbers with $|x + y|_p$ sufficiently large. This yields families of linear forms with related coefficients in the sense that the higher coefficients $\rho_i, i \geq 1$ are independent of the parameter $y \in \mathbb{Q}_p$. In section 4.1 we will study the arithmetic properties of the coefficients $\rho_i$. In section 4.2 we will discuss the Archimedean growth of the coefficients $\rho_i$ and section 4.3 contains the $p$-adic convergence properties of the linear forms. Since the higher coefficients are independent of $y$, certain linear combinations of $\ast$ yield linear forms in $p$-adic $L$-values. Finally we will apply in section 5 a variant (due to Chantanasiri) of Nesterenko’s $p$-adic linear independence criterion to conclude the proof. The appendix provides the details of Chantanasiri’s proof of Nesterenko’s criterion for number fields, Chantanasiri’s work only discusses the case $K = \mathbb{Q}$ in detail.

We hope that the methods developed in this paper, especially the technique of Volkenborn integration over suitable rational functions, proves to be useful in further investigations of $p$-adic $L$-values.

2. Statement of the main result

In the following, $p$ will denote a prime and we will write $\mathbb{C}_p$ for the completion of the algebraic closure of the field of $p$-adic numbers $\mathbb{Q}_p$. Let us once and for all fix an embedding $\bar{\mathbb{Q}} \subseteq \mathbb{C}_p$. In particular, this allows us to view Dirichlet characters $\chi : \mathbb{Z} \to \bar{\mathbb{Q}}$ as having values in $\mathbb{C}_p$. For a Dirichlet character $\chi$ let us write $\mathbb{Q}(\chi) \subseteq \bar{\mathbb{Q}}$ for the smallest subfield of $\bar{\mathbb{Q}}$ containing all values of $\chi$.

**Theorem 2.1.** Let $\chi$ be a Dirichlet character of conductor $f$ and $K$ be a finite extension of $\mathbb{Q}$ satisfying $\mathbb{Q}(\chi) \subseteq K$. Then there is an integers $s_0$ depending on $\chi$ and $p$ such that for all $s \geq s_0$ which are powers of $p$,

$$\dim_K \left( K + \sum_{i=2}^{s+1} L_p(i, \chi \omega^{1-i})K \right) \geq \frac{\log s}{[K : \mathbb{Q}]} (\log(2) + 2 + 2f).$$

**Remark 2.2.** Usually, the lower bound in similar statements like the theorem of Rivoal and Ball involves a factor $(1 - \epsilon)$. Let us observe, that the lower bound does not depend on the choice of an $\epsilon > 0$. Let us further remark that the proof of the theorem uses the construction of linear forms with related coefficients as in [FSZ18] but it does not make use of the elimination method from [FSZ18].
3. Preliminaries and notation

3.1. Volkenborn integrals. Let \( f : \mathbb{Z}_p \to \mathbb{C}_p \) be a uniformly continuous function on \( \mathbb{Z}_p \). Then, the limit
\[
\int_{\mathbb{Z}_p} f(t) dt := \lim_{n \to \infty} \frac{1}{p^n} \sum_{0 \leq k < p^n} f(k)
\]
events and is called the Volkenborn integral of \( f \), see \( \text{[Vol72, Satz 5]} \). In particular, this applies to convergent power series, i.e. \( f = \sum_{n \geq 0} a_n t^n \in \mathbb{C}_p[[t]] \) with \( |a_n|_p \to 0 \) as \( n \to \infty \). For a convergent power series \( f = \sum_{n \geq 0} a_n t^n \in \mathbb{C}_p[[t]] \) let us write \( \|f\| := \sup_n |a_n|_p \). The following result is due to Volkenborn:

**Proposition 3.1** (\( \text{[Vol72, Satz 6]} \)). Let \( f \in \mathbb{C}_p[[t]] \) be a convergent power series, then
\[
| \int_{\mathbb{Z}_p} f(t) dt |_p \leq p \|f\|.
\]

As an immediate Corollary, we obtain:

**Corollary 3.2.** Let \( f, g \in \mathbb{Z}_p[t] \) be two convergent power series with \( f - g \in p^{l_0+1}\mathbb{Z}_p[t] \) for some \( l_0 \in \mathbb{Z}_{\geq 1} \), then
\[
\int_{\mathbb{Z}_p} f(t) dt - \int_{\mathbb{Z}_p} g(t) dt \in p^{l_0}\mathbb{Z}_p.
\]

The Volkenborn integral is not translation invariant, indeed it satisfies:

**Proposition 3.3** (\( \text{[Vol72, Satz 3]} \)). Let \( m \in \mathbb{Z}_{\geq 0} \) and \( f \in \mathbb{C}_p[[t]] \) be a convergent power series, then
\[
\int_{\mathbb{Z}_p} f(u + m) du = \int_{\mathbb{Z}_p} f(u) du + \sum_{i=0}^{m-1} f'(i).
\]

3.2. The Hurwitz zeta function. Let us recall the definition of the Hurwitz zeta function. For a positive real number \( x > 0 \) the Hurwitz zeta function is defined by the convergent series
\[
\zeta(s, x) := \sum_{n \geq 0} \frac{1}{(x+n)^s}, \quad \text{for } s > 1
\]
and can be continued to a holomorphic function on \( \mathbb{C} \setminus \{1\} \) with a simple pole at \( s = 1 \). Its values at the negative integers can be described in terms of the Bernoulli polynomials \( B_n(x) \). More precisely, it is given by the formula
\[
\zeta(1-n, x) = -\frac{B_n(x)}{n}, \quad n \in \mathbb{Z}_{\geq 1}.
\]
In particular, note that for \( n \in \mathbb{Z}_{\geq 1} \) and \( x \in \mathbb{Q}_{>0} \) the value \( \zeta(1-n, x) \) is rational. For \( x \in \mathbb{Q} \) with large \( p \)-adic norm, it is possible to interpolate the values of the Hurwitz zeta function at negative integers \( p \)-adically:

3.3. The \( p \)-adic Hurwitz zeta function. Let \( p \) be a prime number and define
\[
q_p := \begin{cases} p & p \neq 2 \\ 4 & p = 2 \end{cases}.
\]

Let us write \( \nu_p : \mathbb{Q}_p \to \mathbb{Z} \) for the \( p \)-adic valuation with the normalization \( \nu_p(p) = 1 \) and \( |x|_p := p^{-\nu_p(x)} \) for the \( p \)-adic norm on \( \mathbb{Q}_p \). The units \( \mathbb{Z}_p^\times \) of the \( p \)-adic integers decompose canonically:
\[
\mathbb{Z}_p^\times \cong \mu_{p \cdot q_p}(\mathbb{Z}_p) \times (1 + q_p \mathbb{Z}_p).
\]
Here, $\mu_n(R)$ denotes the group of $n$-th roots of unity in a ring $R$ and $\phi(n)$ denotes Euler’s totient function. The canonical projection
\[ \omega: \mathbb{Z}_p^\times \to \mu_{\phi(q_p)}(\mathbb{Z}_p) \]
is called the Teichmüller character. Let us extend the Teichmüller character to a map
\[ \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times \]
by setting
\[ \omega(x) := p^{r_p(x)}\omega(x/p^{r_p(x)}) \]
and define $\langle x \rangle := \frac{x}{\omega(x)}$ for $x \in \mathbb{Q}_p^\times$. Let $x \in \mathbb{Q}_p$ be given with $|x|_p \geq q_p$. It can be shown that there is a unique $p$-adic meromorphic function $\zeta_p(s, x)$ on
\[ \{ s \in \mathbb{C}_p \mid |s|_p < q_p^{p^{-1/(p-1)}} \} \]
such that
\[ \zeta_p(1 - n, x) = -\omega(x)^{-n} \frac{B_n(x)}{n} , \]
see [Coh07, §11.2]. Explicitly, the function $\zeta_p(s, x)$ can be defined by a Volkenborn integral [Coh07, Def. 11.2.5]:
\[ \zeta_p(s, x) = \frac{1}{s - 1} \int_{\mathbb{Z}_p} \langle x + t \rangle^{1-s} dt. \]

This can be reformulated as follows:

**Lemma 3.4.** Let $s > 1$ be an integer and $x \in \mathbb{Q}_p$ with $|x|_p \geq q_p$, then
\[ \omega(x)^{1-s}\zeta_p(s, x) = \frac{1}{s - 1} \int_{\mathbb{Z}_p} (x + t)^{1-s} dt. \]

**Proof.** For $m \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$ with $|x|_p \geq q_p$ we get $\omega(x) = \omega(x + m)$ using the formula
\[ \zeta_p(s, x) = \frac{1}{s - 1} \lim_{r \to \infty} p^{-r} \sum_{0 \leq m < p^r} \langle x + m \rangle^{1-s} \]
we conclude
\[ \omega(x)^{1-s}\zeta_p(s, x) = \omega(x)^{1-s} \frac{1}{s - 1} \lim_{r \to \infty} p^{-r} \sum_{0 \leq m < p^r} \langle x + m \rangle^{1-s} \]
\[ = \frac{1}{s - 1} \lim_{r \to \infty} p^{-r} \sum_{0 \leq m < p^r} \omega(x + m)^{1-s}(x + m)^{1-s} \]
\[ = \frac{1}{s - 1} \lim_{r \to \infty} p^{-r} \sum_{0 \leq m < p^r} (x + m)^{1-s} \]
\[ \square \]

Finally, let us recall that $p$-adic $L$-values of a Dirichlet character $\chi$ of conductor $f$ can be expressed through $p$-adic Hurwitz zeta values. Indeed, let $s \in \mathbb{Z} \setminus \{1\}$ and $F$ be a multiple of $f$ satisfying $q_p \mid F$, then:
\[ L_p(s, \chi) = \frac{(F)^{1-s}}{F} \sum_{1 \leq j \leq F \atop \gcd(j, p) = 1} \chi(j) \zeta_p \left(s, \frac{j}{F} \right) . \]
This formula will be important to write $p$-adic $L$-values of Dirichlet characters as linear combinations of $p$-adic Hurwitz zeta values.
3.4. Nesterenko’s $p$-adic linear independence criterion. In [Nes12] Nesterenko proves a $p$-adic analogue of his well-known linear independence criterion. In the following, we will use a variant of Nesterenko’s criterion due to Chantanasiri, [Cha12]: Let us recall Chantanasiri’s version of Nesterenko’s linear independence criterion. Let $K$ be a number field with ring of integers $\mathcal{O}_K$ and let us write $N_K: K \to \mathbb{Q}$ for the norm map. For a linear form

$$L(X_0, ..., X_m) = l_0 X_0 + ... + l_m X_m, \quad l_i \in \mathcal{O}_K$$

with coefficients in $\mathcal{O}_K$, let us define

$$H_K(L) := \max_{0 \leq i \leq m} |N_K(l_i)|.$$

Note, that $H_K(L)$ depends on the field $K$. For $K \subseteq K'$ we have $H_{K'}(L) = H_K(L)^{|K':K|}$.

**Theorem 3.5** (Nesterenko, Chantanasiri). Let $\tau_1, \tau_2$ be positive real numbers and $\sigma(n)$ a non-decreasing positive function satisfying $\lim_{n \to \infty} \sigma(n) = \infty$ and $\lim_{n \to \infty} \frac{\sigma(n+1)}{\sigma(n)}$. Let $\overrightarrow{\theta} = (\theta_1, ..., \theta_m) \in \mathbb{C}^m$. Assume that, for all sufficiently large integers $n$, there exists a linear form with coefficients in $\mathcal{O}_K$ in $m+1$ variables

$$L_n(X) = l_{0,n} X_0 + l_{1,n} X_1 + ... + l_{m,n} X_m, \quad l_{i,n} \in \mathcal{O}_K$$

satisfying

$$H_K(L_n) \leq e^{\sigma(n)} \quad \text{and} \quad e^{-(\tau_1+\alpha(1))\sigma(n)} \leq |L_n(1, \overrightarrow{\theta})|_p \leq e^{-(\tau_2+\alpha(1))\sigma(n)},$$

then $\dim_K(K + K\theta_1 + ... + K\theta_m) \geq \frac{\tau_2}{1+\tau_1-\tau_2}$.

The above version differs slightly from the original formulation of Nesterenko by the normalization of the $p$-adic norm of the linear forms. The above formulation appears in Chantanasiri’s work [Cha12] for the base field $K = \mathbb{Q}$. But the proof generalizes directly to the case of arbitrary base fields. For the convenience of the reader and for completeness, we decided to include the proof for arbitrary base fields in the appendix.

4. Linear forms in $p$-adic Hurwitz zeta values with related coefficients

An important source for linear forms in values of the classical Riemann zeta function is provided by summation of rational functions rational function of degree less than one having only poles of maximal order $s$ over the positive integers. The formula

$$\omega(x)^{1-s}\zeta_p(s, x) = \frac{1}{s-1} \int_{\mathbb{Z}_p} (x+t)^{1-s} dt$$

together with the partial fraction decomposition of rational functions suggests that Volkenborn integration over rational functions might be a good substitute in the $p$-adic case.

Let $x \in \mathbb{Q}_p$ and let $n, q \geq 1$ and $s \geq 2$ be positive integers with $q \leq s$. Let us define the following rational function

$$R_n(t, x) := n^{s-1} \frac{(t)_{n+1}}{(t+x)^s (t+x+n)^q}.$$

Here, we write $(t)_n = t(t+1)...(t+n-1)$ for the rising factorial. Observe, that the assumptions on $s, n$ and $q$ imply that $R_n$ has negative degree. Of course, the definition of $R_n(t, x)$ does also depend on $s$ and $q$ but later $s$ and $q$ will be fixed and it will be convenient to drop them from the notation.
**Proposition 4.1.** Let \( n, q \geq 1 \) and \( s \geq 2 \) be as above. For every \( x \in \mathbb{Q}_p \setminus \mathbb{Z}_p \) there exist \( \rho_i \in \mathbb{Q}_p \), \( 1 \leq i \leq s \) such that for every \( y \in \mathbb{Q}_p \) with \( |x + y|_p \geq q_p \) there exists \( \rho_0^{(y)} \in \mathbb{Q}_p \) with:

\[
\int_{\mathbb{Z}_p} R_n(t + y, x) dt = \rho_0^{(y)} + \sum_{i=1}^{s} \rho_i \omega(x + y)^{-i} \zeta_p(i + 1, x + y).
\]

The \( p \)-adic numbers \( \rho_i \) for \( 1 \leq i \leq s \) depend on \( x \) but they are independent of the parameter \( y \in \mathbb{Q}_p \). If \( x, y \in \mathbb{Q} \) then also \( \rho_0^{(y)} \) and \( \rho_i \) \( (1 \leq i \leq s) \) are rational numbers.

**Proof.** Let us first observe, that the condition \( x \in \mathbb{Q}_p \setminus \mathbb{Z}_p \) implies that \( R_n \) is a rational function without poles in \( \mathbb{Z}_p \) and thus is uniformly continuous on \( \mathbb{Z}_p \). The partial fraction decomposition of \( R_n(t, x) \) with respect to \( t + x \) gives

\[
R_n(t, x) = \sum_{i=1}^{s} \sum_{k=0}^{n} \frac{r_{i,k}}{(t + x + k)^i},
\]

with

\[
 r_{i,k} = \frac{(-1)^{s-i}}{(s-i)!} \left( \frac{d}{dt} \right)^{s-i} \left[ R_n(-t - x, (k - t)^s) \right]_{t=k}.
\]

The Volkenborn integral can be computed as follows:

\[
\int_{\mathbb{Z}_p} R_n(t + y, x) dt = \int_{\mathbb{Z}_p} \sum_{i=1}^{s} \sum_{k=0}^{n} \frac{r_{i,k}}{(t + y + x + k)^i} dt
\]

\[= \sum_{i=1}^{s} \sum_{k=0}^{n} r_{i,k} \int_{\mathbb{Z}_p} (t + y + x + k)^{-i} dt\]

\[= \sum_{i=1}^{s} \sum_{k=0}^{n} r_{i,k} \left[ \int_{\mathbb{Z}_p} (t + y + x)^{-i} dt - i \sum_{\nu=0}^{k-1} (\nu + y + x)^{-i-1} \right] - \sum_{i=1}^{s} \sum_{k=0}^{n} \sum_{\nu=0}^{k-1} i \cdot r_{i,k} (\nu + y + x)^{-i-1} - \sum_{i=1}^{s} \sum_{k=0}^{n} i \cdot r_{i,k} \omega(x + y)^{-i} \zeta_p(i + 1, y + x)\]

By setting

\[
\rho_0^{(y)} := - \sum_{i=1}^{s} \sum_{k=0}^{n} \sum_{\nu=0}^{k-1} i \cdot r_{i,k} (\nu + y + x)^{-i-1}
\]

\[
\rho_i := \sum_{k=0}^{n} i \cdot r_{i,k}, \quad 1 \leq i \leq s
\]

we get the desired formula

\[
\int_{\mathbb{Z}_p} R_n(t + y, x) dt = \rho_0^{(y)} + \sum_{i=1}^{s} \rho_i \omega(x + y)^{-i} \zeta_p(i + 1, x + y).
\]

\(\square\)

**Remark 4.2.** Although we use the same rational function as in Bel’s paper [Bel10], the resulting linear forms are somewhat different from Bel’s approach using Padé approximation.
Later we will consider for a given Dirichlet character \( \chi \) of conductor \( f \) and for suitable choices of \( j_0 \) and \( l \) linear combinations of Volkenborn integrals of the form:

\[
S_{\chi,l}^n := \sum_{1 \leq j \leq f \cdot p^l, \gcd(j,p) = 1} \chi(j) \int_{Z_p} R_n(t + \frac{j - j_0}{f \cdot p^l}, \frac{j_0}{f \cdot p^l}) dt.
\]

The above proposition combined with (1) allows us to write these sums as linear combinations of \( p \)-adic \( L \)-values of the Dirichlet character \( \chi \):

\[
S_{\chi,l}^n = \left( \sum_{1 \leq j \leq f \cdot p^l, \gcd(j,p) = 1} \chi(j) \rho_0^{(1 - j_0/p^l)} \right) + \sum_{i = 2}^{s + 1} \rho_{i - 1} f \cdot p^l L_p(i, \chi \omega^{1 - i}).
\]

4.1. Arithmetic properties of the linear forms. In the following, we will study the arithmetic properties of the coefficients \( \rho_0^{(y)} \) and \( \rho_i, 1 \leq i \leq s \) for \( y \in \frac{D}{D} \mathbb{Z} \) with \( D \) a positive integer. For \( D \in \mathbb{Z}_{>0} \) let us define

\[
\mu_n(D) := D^n \prod_{\tilde{p} | D} q^{\frac{n}{\tilde{p} - 1}}
\]

where \( \tilde{p} \) runs over all primes dividing \( D \).

**Lemma 4.3** ([Bel10, Lemme 5.4]). Let \( x \in \frac{1}{D} \mathbb{Z} \) and \( n \) be a positive integer, then

\[
\mu_n(D) \frac{(x)_n}{n!} \in \mathbb{Z}
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} \log |\mu_n(D)| = \log D + \sum_{\tilde{p} | D} \log \frac{\tilde{p}}{\tilde{p} - 1}.
\]

Let \( D, n, s \geq 1 \) as above and \( x, y \in \frac{1}{D} \mathbb{Z} \) satisfying \( |x|_p > 1 \) and \( |x + y|_p \geq q_p \). Recall, that \( \rho_i, 1 \leq i \leq s \) and \( \rho_0^{(y)} \) have been defined by the formula

\[
\int_{Z_p} R_n(t + y, x) dt = \rho_0^{(y)} + \sum_{i = 1}^{s} \rho_i \omega(x)^{-i} \zeta_p(i + 1, x + y)
\]

with the rational function

\[
R_n(t, x) := n^{s+1} \frac{(t)_{n+1}}{(t + x)^n(t + x + n)^q}.
\]

**Proposition 4.4.** Let \( x, y \in \frac{1}{D} \mathbb{Z} \) with \( |x|_p > 1, |x + y|_p \geq q_p \) and \( 0 < |x + y| < 1 \).

(a) For \( 1 \leq i \leq s \) we have

\[
d_n^{s-i} D \mu_n(D) \rho_i \in \mathbb{Z}
\]

(b) For \( i = 0 \) we get the formula

\[
d_n^s D^n \mu_n(D) \rho_0^{(y)} \in \mathbb{Z}.
\]

**Proof.** Although our linear forms differ from Bel’s, we can proceed similar as in [Bel10 Propo-
(a): Let us recall the formula

\[ \rho_i := \sum_{k=0}^{n} i \cdot r_{i,k}, \quad 1 \leq i \leq s \]

with

\[ r_{i,k} = \frac{(-1)^{s-i}}{(s-i)!} \left( \frac{d}{dt} \right)^{s-i} \left[ R_n(-t - x)(k-t)^s \right]_{t=k} \]

Let us fix \( k \) with \( 0 \leq k \leq n \) and write

\[ R_n(-t - x)(k-t)^s = n^{s-1} \frac{(-t - x)^{n+1}}{(-t)^n} (-t + n)^q (k - t)^s \]

\[ = H(t)G(t)^{s-1}F(t) \]

with

\[ F(t) := \frac{\prod_{j=0}^{n-1}(-t - j + 1)}{\prod_{i=0}^{n}(-t + i)} (k - t), \quad G(t) := \frac{n!}{\prod_{i=0}^{n}(-t + i)} (k - t) \]

\[ H(t) := (n - t - x) (-t + n)^{s-q} \]

First, observe that \( D \cdot H(t) \in \mathbb{Z}[t] \). The functions \( F \) and \( G \) have the following partial fraction decompositions:

\[ G(t) = \sum_{m=0}^{n} \frac{(-1)^m (k - m) \binom{n}{m}}{t + m}, \quad F(t) = 1 + \sum_{m=0}^{n} \frac{(k - m) f_m}{t + m} \]

with

\[ f_m = (-1)^m \binom{n}{m} \frac{(-m - x)_n}{n!}. \]

By lemma \ref{lem:mu} we have \( \mu_n(D) f_m \in \mathbb{Z} \). Let us write \( \partial^{(\lambda)} := \frac{1}{\lambda!} \left( \frac{d}{dt} \right)^{\lambda} \). For \( \lambda \geq 1 \) we obtain:

\[ \left( \mu_n(D) d_n^{\lambda} \right) \cdot \partial^{(\lambda)} F(t)|_{t=k} = -\left( \mu_n(D) d_n^{\lambda} \right) \sum_{0 \leq m \leq n, m \neq k} \frac{f_m}{(m - k)^{\lambda}} \in \mathbb{Z} \]

\[ d_n^{\lambda} \cdot \partial^{(\lambda)} G(t)|_{t=k} = -d_n^{\lambda} \sum_{0 \leq m \leq n, m \neq k} \frac{(-1)^m \binom{n}{m}}{(m - k)^{\lambda}} \in \mathbb{Z}. \]

Together with the Leibniz rule, we get:

\[ (D\mu_n(D) d_n^{s-i}) r_{i,k} = (D\mu_n(D) d_n^{s-i}) (-1)^{s-i} \partial^{(s-i)} [R_n(-t - x)(k-t)^s]|_{t=k} \]

\[ = d_n^{s-i} \sum_{|\lambda|=s-i} (D\partial^{(\lambda_0)} H)|_{t=k} \cdot (\partial^{(\lambda_1)} G)|_{t=k} \cdot \ldots \cdot (\partial^{(\lambda_{s-1})} G)|_{t=k} \cdot (\mu_n(D) \cdot \partial^{(\lambda_s)} F)|_{t=k} \]

For the proof of (b) we proceed similarly. We have

\[ \rho^{(y)}_0 := -\sum_{i=1}^{s} \sum_{k=0}^{n} \sum_{\nu=0}^{k-1} i \cdot r_{i,k} (\nu + y + x)^{-i-1}. \]
and compute for $k \in \{0, \ldots, n\}$:

\[ \sum_{r_1, \ldots, r_k} \sum_{i=0}^{k-1} i \cdot (\nu + y + x)^{-i-1} = \sum_{i=1}^{s} \frac{(-1)^{s-i}}{(s-i)!} \left( \frac{d}{dt} \right)^{s-i} \left[ R_n(-t-x)(k-t)^s \right] \bigg|_{t=k} \]

\[ \times \frac{(-1)^{i-1}}{(i-1)!} \left( \frac{d}{dt} \right)^{i-1} \left[ \sum_{\nu=1}^{k} (t-\nu+y+x)^{-2} \right] \bigg|_{t=k} = \frac{(-1)^{s-1}}{(s-1)!} \left( \frac{d}{dt} \right)^{s-1} \left[ R_n(-t-x)(k-t)^s \sum_{\nu=1}^{k} (t-\nu+y+x)^{-2} \right] \bigg|_{t=k}. \]

This gives

\[ \rho_0^{(y)} = \sum_{k=0}^{n} \frac{(-1)^{s-1}}{(s-1)!} \left( \frac{d}{dt} \right)^{s-1} \left[ R_n(-t-x)(k-t)^s \sum_{\nu=1}^{k} \frac{1}{(t-\nu+y+x)^2} \right] \bigg|_{t=k}. \]

For $0 \leq \nu \leq k \leq n$ let us decompose

\[ R_n(-t-x,x)(k-t)^s \frac{1}{(t-\nu+y+x)^2} = H(t)G(t)^{s-1} \frac{1}{(t-\nu+y+x)^2} F(t). \]

with:

\[ \tilde{F}(t) := \frac{1}{(t-\nu+y+x)^2} \prod_{i=0}^{n-1} \frac{(-t-x+i)}{(t-\nu+x+i)}(k-t), \quad G(t) := \frac{n!}{\prod_{i=0}^{n}(-t+i)}(k-t) \]

\[ H(t) := (n-t+x)(x+t+n)^{s-q} \]

Then

\[ \tilde{F}(t) = \sum_{\substack{m=0 \atop m \neq k}}^{n} \frac{(k-m)\tilde{f}_m}{m-t} \]

with

\[ \tilde{f}_m = (-1)^m \frac{(-m-x)m+1}{n!(m-\nu+y+x)^2} \binom{n}{m}. \]

Since $D(m-\nu+y+x) \in \mathbb{Z}$ and $0 < |D(m-\nu+y+x)| \leq Dn$, we deduce from lemma 4.3 the integrality of $d_{Dn\mu_n(D)}^{\lambda}(\tilde{F}(t)) \big|_{t=k}$. For $\lambda \geq 1$ we deduce

\[ d_{Dn\mu_n(D)}^{\lambda} \partial^{(\lambda)} \left( \tilde{F}(t) \right) \bigg|_{t=k} = d_{Dn\mu_n(D)}^{\lambda} \sum_{\substack{m=0 \atop m \neq k}}^{n} \frac{\tilde{f}_m}{(m-k)^{\lambda}} \in \mathbb{Z} \]

and we conclude again by using the Leibniz formula. \hfill \square

Let $\chi$ be a Dirichlet character of conductor $f$ and write $K = \mathbb{Q}(\chi)$ for its field of definition. For positive $j_0, l \in \mathbb{Z}$ we have defined

\[ S_n^{\chi,l} := \sum_{1 \leq j \leq f \cdot p' \atop \gcd(j,p) = 1} \chi(j) \int_{\mathbb{Z}_p} R_n(t + \frac{j-j_0}{fp'}, \frac{j_0}{fp'}) dt. \]

and we have already seen

\[ S_n^{\chi,l} = \left( \sum_{1 \leq j \leq f \cdot p' \atop \gcd(j,p) = 1} \chi(j)\rho_0^{\left(\frac{j-j_0}{fp'}\right)} \right) + \sum_{i=2}^{s+1} \rho_{i-1}(fp')^i L_p(i, \chi \overline{\omega}^{1-i}) \]

\[ \text{for positive } j \in \mathbb{Z} \]
Let us define $C(\chi, l, n, s) := f p^l d^*_n d^2_{f p^l} \mu_n(f p^l)$

$$\lambda_{0,n} := C(\chi, l, n, s) \sum_{j=1}^{p^l} \chi(j) \rho_0^{(2-jp)}$$

$$\lambda_{i,n} := C(\chi, l, n, s) (f p^l)^{i+1} \rho_i.$$ and the linear form $\Lambda_n(\underline{z}) := \lambda_{0,n} + \sum_{i=1}^{s} \lambda_{i,n} x_i$. According to proposition 4.4 we have $\lambda_{i,n} \in \mathcal{O}_K$ and

$$C(\chi, l, n, s) \cdot S^{\chi, l, n, s} = \Lambda_n \left((L_p(i + 1, \chi \omega^{-i}))_{i=1}^{s}\right).$$

4.2. Asymptotic growth of the coefficients. Let $x, y \in \mathbb{Q}$ with $|x|_p > 1$ and $|x + y|_p \geq q_p$. Recall the formula

$$\int_{\mathbb{Z}_p} R_n(t + y, x) dt = \rho_0^{(y)} + \sum_{i=1}^{s} \rho_i \omega(x + y)^{-i} \zeta_p(i + 1, x + y).$$

In proposition 4.4 we have seen that $\rho_0^{(y)}, \rho_i \in \mathbb{Q}$ for $1 \leq i \leq s$. Our next aim is to study the Archimedean norm of the coefficients $\rho_i$:

**Proposition 4.5.** We have

$$\limsup_{n \to \infty} |\rho_0^{(y)}|^{1/n} \leq 2^{s-1} \quad \text{and} \quad \limsup_{n \to \infty} |\rho_i|^{1/n} \leq 2^{s-1}.$$ 

**Proof.** By

$$|\rho_i| \leq \sum_{k=0}^{n} i |r_{i,k}|, \quad |\rho_0^{(y)}| \leq \sum_{k=0}^{n} \sum_{i=1}^{s} \sum_{\nu=0}^{k-1} i \cdot |\nu + y + x|^{-i-1} |r_{i,k}|$$

it suffices to prove

$$\limsup_{n \to \infty} |r_{i,k}|^{1/n} \leq 2^{s-1}.$$ 

We have

$$r_{i,k} = \frac{1}{2\pi i} \int_{|t+k+x| = \frac{1}{2}} R_n(t)(t + k + x)^{i-1} dt$$

$$= \frac{1}{2\pi i} \int_{|t+k+x| = \frac{1}{2}} n^{i-1} \frac{(t)_{n+1}}{(t+x)^{n}(t+x+n)^{q}}(t + k + x)^{i-1} dt.$$ 

From here on, we can follow the proof of [Bel10, Proposition 5.7] quite closely: We deduce

$$|r_{i,k}| \leq 2^{-i} \sup_{|t+k+x| = \frac{1}{2}} \left(n^{i-1} \frac{|(t)_{n+1}|}{|(t+x)^{n}(t+x+n)^{q}|}\right).$$

Let $m$ be an integer with $|x| + \frac{1}{2} \leq m$, for $t \in \mathbb{C}$ with $|t + k + x| = \frac{1}{2}$ we get:

$$|(t)_{n+1}| = \prod_{t=0}^{n} |t + k + x - k - x + t| \leq$$

$$\leq \prod_{t=0}^{n} \left(\frac{1}{2} + |x| + |k - t|\right)$$

$$\leq \prod_{t=0}^{n} (m + |k - t|) \leq (m + k)! \cdot (m + n - k)!.$$
For \( t \in \mathbb{C} \) with \( |t + k + x| = \frac{1}{2} \) we get

\[
|\( t + x \)| = \prod_{\nu=0}^{n-1} |t + \nu + x|
\geq \prod_{\nu=0}^{n-1} \left| -\frac{1}{2} + |\nu - k| \right|
\]

If \( |\nu - k| > 1 \) we can estimate \( -\frac{1}{2} + |\nu - k| \geq |\nu - k| - 1 \) and otherwise \( -\frac{1}{2} + |\nu - k| \geq \frac{1}{2} \). We obtain

\[
|\( t + x \)| \geq \frac{1}{8n^3} k!(n-k)!
\]

Combining (6) and (7) gives

\[
\frac{|\( (t)_{n+1} \)|}{|\( (t + x)_n \)|} \leq 8n^3 \prod_{\nu=1}^{m} (\nu + k)(\nu + n - k) \leq 8n^3(n + m)^{2m}.
\]

Furthermore, we can estimate

\[
|(t + x + n)^q| = |t + x + k + n|^q \geq \frac{1}{2q}.
\]

By (5), (7) and (8) we get

\[
|r_{i,k}| \leq 2^{-i+q+3s}(n + m)^{2m-3s} \binom{n}{k}^{s-1}.
\]

Since \( \binom{n}{k} \leq 2^n \) we get as desired:

\[
\lim_{n \to \infty} |r_{i,k}|^{1/n} \leq 2^{s-1}.
\]

Above, we have shown that Volkenborn integration over a certain rational function gives linear forms in \( p \)-adic \( L \)-values:

\[
C(\chi, l, n, s) \sum_{\substack{1 \leq j \leq f p^l \times \nu, \text{gcd}(j, p^l) = 1}} \chi(j) \int_{\mathbb{Z}_p} R_n(t + \frac{j - j_0}{f p^l}, \frac{j_0}{f p^l}) dt = \Lambda_n \left( (L_p(i + 1, \chi\omega^{-i}) \right)_{i=1}^s
\]

where \( \Lambda_n(x) := \lambda_{0,n} + \sum_{i=1}^s \lambda_{i,n} x_i \) is a linear form with coefficients in \( K = \mathbb{Q}(\chi) \). As a corollary of the above result we get:

**Corollary 4.6.** For \( \epsilon > 0 \) there exists an integer \( n_0 \) such that for all \( n \geq n_0 \):

\[
H_K(\Lambda_n) \leq \exp \left( n[K : \mathbb{Q}] \left[ \log(2)(s - 1) + s + 2f p^l + \log(f p^l) + \sum_{\nu|p} \frac{\log \nu}{p - 1} + \epsilon \right] \right).
\]

**Proof.** Let us first recall the definitions

\[
H_K(\Lambda_n) := \max_{0 \leq i \leq s} |N_K(\lambda_{i,n})|
\]
and
\[ C(\chi, l, n, s) := f p^l d_1^s d_2^s \mu_n(f p^l). \]

We have
\[ |N_K(C(\chi, l, n, s)\rho_i)| = |C(\chi, l, n, s)\rho_i|[K:Q]\]
and
\[
\left| N_K \left( C(\chi, l, n, s) \sum_{1 \leq j \leq f p^l} \chi(j) \rho_0 \left( \frac{\ell - j}{f p^l} \right) \right) \right| = \prod_{\sigma : K \rightarrow C} \left| C(\chi, l, n, s) \sum_{1 \leq j \leq f p^l} \sigma(\chi(j)) \rho_0 \left( \frac{\ell - j}{f p^l} \right) \right| \\
\leq \left( \sum_{1 \leq j \leq f p^l} \left| C(\chi, l, n, s) \rho_0 \left( \frac{\ell - j}{f p^l} \right) \right| \right)^{[K:Q]}.
\]

Now proposition 4.5 gives
\[
\limsup_{n \to \infty} \frac{1}{n} \log H_K(\Lambda_n) \leq (s - 1)[K : Q] \log 2.
\]

Further, according to lemma 4.3 we have
\[
\lim_{n \to \infty} \frac{1}{n} \log |\mu_n(f p^l)| = \log(f p^l) + \sum_{\tilde{p}^M} \frac{\log(\tilde{p})}{\tilde{p} - 1}
\]
and by the prime number theorem:
\[
\lim_{n \to \infty} \frac{1}{n} \log |d_n| = 1.
\]

Combining everything gives
\[
\limsup_{n \to \infty} \frac{1}{n} \log H_K(\Lambda_n) \leq \left[ K : Q \right] \left( \log(2)(s - 1) + s + 2f p^l + \log(f p^l) + \sum_{\tilde{p}^M} \frac{\log(\tilde{p})}{\tilde{p} - 1} \right).
\]
and the corollary follows. \(\square\)

4.3. \textit{p-adic convergence of the linear forms}. The aim of this section is to control the \(p\)-adic convergence of the involved linear forms. Let us first recall the following well-known fact about generalized Bernoulli numbers: Let \(\chi\) be a Dirichlet character of conductor \(f\). Let us define
\[ \delta := \begin{cases} 0, & \chi(-1) = 1 \\ 1, & \chi(-1) = -1. \end{cases} \]

\textbf{Lemma 4.7.} Let \(\chi\) be a Dirichlet character of conductor \(f\).

(a) For positive integers \(k, l\) we have the following congruence
\[
(1 - \chi(p)p^{k-1})B_{k,\chi} \equiv \frac{1}{f} \cdot p^{r+1} \sum_{1 \leq j \leq p^l+1, \gcd(j,p) = 1} \chi(j) j^k \mod p^l.
\]

(b) For a positive integer \(k\) we have \(B_{k,\chi} \neq 0\) if \(k \equiv \delta \mod 2\).

\textit{Proof.} For the proof of (a) see e.g. [Was82 7.11]. For (b), recall
\[ L(1 - k, \chi) = \frac{B_{k,\chi}}{k}. \]
Thus, it is enough to prove $L(1-k,\chi(0)) \neq 0$ for $k \equiv \delta \mod 2$. Let $X := \langle \chi \rangle$ be group of Dirichlet characters generated by $\chi$ and $K$ be the abelian field extension corresponding to $X$. We have

$$\zeta_K(s) = \prod_{\chi \in X} L(s, \chi).$$

Now, the claim follows from the functional equation of $\zeta_K$ and the non-vanishing of $\zeta_K$ at positive integers.

**Lemma 4.8.** Let $\chi$ be a Dirichlet character of conductor $f$.

(a) Let $p \neq 2$ be a prime. There exists $j_0 \in \{1, \ldots, 2p\}$ co-prime to $p$ and $l_0 \in \mathbb{Z}_{>0}$ such that for all $l > l_0$:

$$\sum_{1 \leq j \leq pl, j \not\equiv j_0, 0 \mod p} \chi(j)(j - j_0) j^{1+\delta} \not\equiv 0 \mod pl.$$

and

$$\sum_{1 \leq j \leq pl, j \not\equiv j_0, 0 \mod p} \chi(j)(j - j_0) j^{1+\delta} \equiv 0 \mod pl - 1.$$

(b) There exists $j_0 \in \{1, \ldots, 8\}$ co-prime to 2 and $l_0 \in \mathbb{Z}_{>0}$ such that for all $l > l_0$:

$$\sum_{1 \leq j \leq 2l, j \not\equiv j_0, 0 \mod 4} \chi(j)(j - j_0) j^{1+\delta} \not\equiv 0 \mod 2l.$$

and

$$\sum_{1 \leq j \leq 2l, j \not\equiv j_0, 0 \mod 4} \chi(j)(j - j_0) j^{1+\delta} \equiv 0 \mod 2l - 1.$$

**Proof.** We give the proof in the case $p \neq 2$, the case $p = 2$ is a straightforward adaptation of the case $p \neq 2$.

**Claim 1:** Let $k \in \mathbb{Z}_{>0}$ and chose an integer $\tilde{j} \in \mathbb{Z}$ which is co-prime to $p$. For all $l \geq 1$ we have the congruence

$$\sum_{1 \leq j \leq pl, j \equiv \tilde{j} \mod p} \chi(j)j^k \equiv \sum_{1 \leq j \leq pl+1, j \equiv \tilde{j} \mod p} \chi(j)j^k \mod pl - 1.$$

**Proof of Claim 1:** We prove the claim by induction. For $k = 0$ the sum is independent of $l \geq 1$, indeed

$$\sum_{1 \leq j \leq pl, j \equiv \tilde{j} \mod p} \chi(j) = \sum_{1 \leq j \leq pl+1, j \equiv \tilde{j} \mod p} \chi(j).$$

Assume that we already know the Claim for $k - 1$, in particular we have

$$\sum_{1 \leq j \leq pl, j \equiv \tilde{j} \mod p} \chi(j)j^{k-1} \equiv 0 \mod pl - 1.$$
Then we compute

\[
\frac{1}{f^{l+1}} \sum_{1 \leq j \leq p^l, \ j \equiv j \mod p} \chi(j)j^k = \frac{1}{f^{l+1}} \sum_{0 \leq \alpha \leq p-1} \sum_{1 \leq j \leq p^l, \ j \equiv j \mod p} \chi(j)(j + pf \alpha)^k = \frac{1}{f^l} \left( \sum_{1 \leq j \leq p^l, \ j \equiv j \mod p} \chi(j)j^k \right)
\]

\[+ \left( \frac{k}{1} \right) \frac{(p - 1)}{2} \left( \sum_{1 \leq j \leq p^l, \ j \equiv j \mod p} \chi(j)j^{k-1} \right)
\]

\[+ \frac{1}{p} \sum_{i=2}^{k} \left( \frac{k}{i} \right) (pf)^{i-1} \left( \sum_{0 \leq \alpha \leq p-1} \alpha^i \right) \left( \sum_{1 \leq j \leq p^l, \ j \equiv j \mod p} \chi(j)j^{k-i} \right).
\]

The term in the second line is congruent \( \equiv 0 \mod p^{l-1} \) by the induction hypothesis and the terms in the last line are all divisible by \( p^{l-1} \).

**Claim 2:** There is an integer \( \tilde{j} \in \{1, \ldots, p - 1\} \) such that

\[
\lim_{l \to \infty} \frac{1}{p^l} \sum_{1 \leq j \leq p^l, \ j \equiv \tilde{j} \mod p} \chi(j)^{2+\delta} \neq 0.
\]

**Proof of Claim 2:** The series converges according to Claim 1 and by lemma 4.7 we have

\[
\lim_{l \to \infty} \frac{1}{p^l} \sum_{1 \leq j \leq p^l, \ j \equiv \tilde{j} \mod p} \chi(j)^{2+\delta} = (1 - p^{1+\delta} \chi(p))B_{2+\delta, \chi}.
\]

The parity of \( 2 + \delta \) is chosen in a way that \( B_{2+\delta, \chi} \neq 0 \). Assume that

\[
\lim_{l \to \infty} \frac{1}{p^l} \sum_{1 \leq j \leq p^l, \ j \equiv \tilde{j} \mod p} \chi(j)^{2+\delta} = 0
\]

for all \( \tilde{j} \in \{1, \ldots, p - 1\} \). Then we would have

\[
0 = \sum_{j=1}^{p-1} \lim_{l \to \infty} \frac{1}{p^l} \sum_{1 \leq j \leq p^l, \ j \equiv \tilde{j} \mod p} \chi(j)^{2+\delta} = (p - 1)(1 - p^{1+\delta} \chi(p))B_{2+\delta, \chi} \neq 0,
\]

which is an obvious contradiction. This proves the second Claim.

**Conclusion:** Define

\[
\lambda := \lim_{l \to \infty} \frac{1}{p^l} \sum_{1 \leq j \leq p^l, \ j \equiv \tilde{j} \mod p} \chi(j)^{2+\delta} \in \mathbb{Q}_p \setminus \{0\},
\]

\[
\mu := \lim_{l \to \infty} \frac{1}{p^l} \sum_{1 \leq j \leq p^l, \ j \equiv \tilde{j} \mod p} \chi(j)^{1+\delta} \in \mathbb{Q}_p.
\]

Since \( \lambda \neq 0 \) by Claim 2, the linear equation

\[
\lambda - x\mu, \ x \in \mathbb{Q}_p
\]
has at most one solution in $\mathbb{Q}_p$. This allows us to pick $j_0 \in \{1, \ldots, 2p\}$ such that $j_0 \equiv \bar{j} \mod p$ and $\lambda - j_0 \mu \neq 0$. Set $l_0 := \nu_p(\lambda + j_0 \mu) - 1$. Then we have for all $l > l_0$

$$\frac{1}{fp^l} \sum_{\substack{1 \leq j \leq fp^l \mod p \\ j \neq j_0, 0}} \chi(j)(j - j_0)^{j+\delta} \not\equiv 0 \mod p^{l_0}.$$ 

\[ \square \]

For a given Dirichlet character $\chi$ of conductor $f$ and a prime $p \neq 2$ let us chose $j_0$ and $l_0$ as in lemma \ref{lem:congruences}. Let us define $q := p^{l_0+2} - p^{l_0+1} - \delta - 1$. For positive integers $s, n, l$ with $s > p^{l_0+2} - p^{l_0+1}$ and $l \geq l_0 + 2$ let us consider

$$S_{n,l} := \sum_{\substack{1 \leq j \leq fp^l \\ \gcd(j, p) = 1}} \chi(j) \int_{\mathbb{Z}_p} R_n(t + \frac{j - j_0}{fp^l}, \frac{j_0}{fp^l}) \, dt.$$ 

Where the rational function $R_n(t, x)$ is defined as above by

$$R_n(t, x) = n^{s-1} \frac{(f \cdot p^l)^{s-1}(x + t)^{n+1}}{(x + t)^{n}(x + n)^q}.$$ 

Let us introduce the function $\sigma' : \mathbb{Z} \to \mathbb{Z}$ defined by

$$\sigma'(n) := (p^{l_0+2} - p^{l_0+1}) \cdot n.$$ 

**Proposition 4.9.** Let $l \geq l_0 + 2$ be an integer, then

$$\nu_p \left(S_{\sigma'(n)}^{\chi,l} \right) = \begin{cases} (s - 1) \nu_p(\sigma'(n))! + (l + \nu_p(f)) \lfloor \sigma'(n)(s - 1) + q \rfloor + l_0 - 1, & p \neq 2 \\ (s - 1) \nu_p(\sigma'(n))! + (l + \nu_p(f)) \lfloor \sigma'(n)(s - 1) + q \rfloor + \sigma'(n) + l_0 - 1, & p = 2 \end{cases}$$

**Proof.** Let us first assume $p \neq 2$. We have

$$S_{\sigma'(n)}^{\chi,l} = \sum_{\substack{1 \leq j \leq fp^l \\ \gcd(j, p) = 1}} \chi(j) \int_{\mathbb{Z}_p} \frac{\prod_{i=0}^{\sigma'(n)} (f \cdot p^l t + f p^l \bar{j} + j - j_0)}{(f \cdot p^l)^{\sigma'(n)}(f \cdot p^l t + f p^l \bar{j} + j - j_0)} \, dt.$$ 

Thus, the claim follows if we prove

$$\frac{1}{fp^l} \sum_{\substack{1 \leq j \leq fp^l \\ \gcd(j, p) = 1}} \chi(j) \int_{\mathbb{Z}_p} \frac{\prod_{i=0}^{\sigma'(n)} (f \cdot p^l t + f p^l \bar{j} + j - j_0)}{(fp^l (t + \sigma'(n)) + j)^q \prod_{i=0}^{\sigma'(n)-1} (fp^l (t + \bar{j}) + j)^s} \, dt \not\equiv 0 \mod p^{l_0}$$

and

$$\frac{1}{fp^l} \sum_{\substack{1 \leq j \leq fp^l \\ \gcd(j, p) = 1}} \chi(j) \int_{\mathbb{Z}_p} \frac{\prod_{i=0}^{\sigma'(n)} (f \cdot p^l t + f p^l \bar{j} + j - j_0)}{(fp^l (t + \sigma'(n)) + j)^q \prod_{i=0}^{\sigma'(n)-1} (fp^l (t + \bar{j}) + j)^s} \, dt \equiv 0 \mod p^{l_0-1}.$$ 

We reduce these two congruences to the congruences proven in lemma \ref{lem:congruences}. Let us first observe the congruences

$$\sigma'(n) \equiv 0 \mod p^{l_0+1},$$

$$\sigma'(n) \equiv 0 \mod p^{l_0+2} - p^{l_0+1} = \phi(p^{l_0+2}).$$
For fixed $j \in \mathbb{Z}$ co-prime to $p$ let us compute modulo $p^{l_0+1}\mathbb{Z}_p[t]$:

\[
\prod_{i=0}^{\sigma'(n)} (fp^l t + fp^l t + j - j_0) \equiv (j - 1)^{\sigma'(n)+j_0} + (j - j_0)^{\sigma'(n)} fp^l \sum_{i=0}^{\sigma'(n)} (t + i)
\]

\[
\equiv \begin{cases} (j - j_0) + fp^l t, & p \mid (j - j_0) \\ 0, & p \nmid (j - j_0). \end{cases}
\]

and

\[
(fp^l(t + \sigma'(n)) + j)^{-q} \prod_{i=0}^{\sigma'(n)-1} (fp^l(t + i) + j)^{q} \equiv j^{-\sigma'(n)q} \equiv j^{1+\delta}
\]

Combining the above equations yields modulo $p^{l_0+1}\mathbb{Z}_p[t]$

\[
\frac{1}{fp^l} \sum_{1 \leq j \leq fp^l \atop \gcd(j, p) = 1} \chi(j) \prod_{i=0}^{\sigma'(n)-1} (f \cdot p^l t + fp^l t + j - j_0)
\]

\[
\equiv \frac{1}{fp^l} \sum_{1 \leq j \leq fp^l \atop j \equiv j_0, \mod p} \chi(j)(j - j_0) j^{1+\delta}
\]

Thus, by corollary 3.2 it is enough to prove

\[
\frac{1}{fp^l} \sum_{1 \leq j \leq fp^l \atop j \equiv j_0, \mod p} \chi(j)(j - j_0) j^{1+\delta} \not\equiv 0 \mod p^{l_0}
\]

and

\[
\frac{1}{fp^l} \sum_{1 \leq j \leq fp^l \atop j \equiv j_0, \mod p} \chi(j)(j - j_0) j^{1+\delta} \equiv 0 \mod p^{l_0-1}
\]

but this is exactly the content of lemma 4.8.

In the case $p = 2$ let us consider

\[
S_{\sigma'(n)}^{l_0} = \sum_{1 \leq j \leq 2^l \atop \gcd(j, 2) = 1} \chi(j) \int_{\mathbb{Z}_2} R_{\sigma'(n)}(t + \frac{j - j_0}{f \cdot p^l}, \frac{j_0}{f \cdot p^l}) dt
\]

\[
= \sigma'(n)^{l_0} \frac{(f \cdot 2^{l-1})^{\sigma'(n)+q}}{f^{2l} \sigma'(n)} \frac{1}{2^{2l}} \sum_{1 \leq j \leq 2^l \atop \gcd(j, 2) = 1} \chi(j) \int_{\mathbb{Z}_2} \frac{\prod_{i=0}^{\sigma'(n)-1} (f \cdot 2^{l-1} t + 2^{l-1} + \frac{j - j_0}{2}) dt.}
\]

Similarly, as above we reduce the claim to

\[
\frac{1}{2^{2l}} \sum_{1 \leq j \leq 2^l \atop j \equiv j_0, \mod 2} \chi(j)(\frac{j - j_0}{2}) j^{1+\delta} \not\equiv 0 \mod 2^{l_0}
\]
and
\[ \frac{1}{f^2} \sum_{1 \leq j \leq f^2 \mod 4} \chi(j) \left( \frac{j - j_0}{2} \right) j^{1+\delta} \equiv 0 \mod 2^{j_0-1} \]
which has been proven in part (b) of lemma \[4.8\].

**Corollary 4.10.** For every \( \epsilon > 0 \) there is an \( n_0 \) such that for all \( n \geq n_0 \) we have
\[ \exp \left( -\sigma'(n) \left( \tau'(p, s) + \epsilon \right) \right) \leq \left| C(\chi, l, \sigma'(n), s) \cdot S_{\sigma'(n)}^{\chi,l} \right|_p \leq \exp \left( -\sigma'(n) \left( \tau'(p, s) - \epsilon \right) \right) \]
with
\[ \tau'(p, s) = \begin{cases} s \log p \left( l + \nu_p(f) + \frac{1}{p-1} \right), & p \neq 2 \\ s \log p \left( l + \nu_p(f) + 1 + \frac{1}{s} \right), & p = 2. \end{cases} \]

**Proof.** For a positive integer \( m \) we have by Legendre’s formula
\[ \frac{m}{p-1} - \log_p(m) \leq \nu_p(ml) \leq \frac{m}{p-1}. \]
In particular, for every \( \epsilon > 0 \) there exists some \( m_0 \) such that for \( m \geq m_0 \)
\[ \exp \left( -m \left( \frac{\log p}{p-1} \right) \right) \leq |ml|_p \leq \exp \left( -m \left( \frac{\log p}{p-1} - \epsilon \right) \right). \]
Further, we have \( |d_n|_p = p^{-\frac{\log n}{\log p}} \) and \( |\mu_n(f^l)|_p = p^{-n(l+\nu_p(f))} \cdot p^{-\frac{\log n}{\log p}} \), thus putting everything together, we find for every \( \epsilon > 0 \) a positive integer \( n_0 \) such that for all \( n \geq n_0 \):
\[ \exp \left( -\sigma'(n) \left( \tau'(p, s) + \epsilon \right) \right) \leq \left| C(\chi, l, \sigma'(n), s) \cdot S_{\sigma'(n)}^{\chi,l} \right|_p \leq \exp \left( -\sigma'(n) \left( \tau'(p, s) - \epsilon \right) \right) \]
with
\[ \tau'(p, s) = \begin{cases} s \log p \left( l + \nu_p(f) + \frac{1}{p-1} \right), & p \neq 2 \\ s \log p \left( l + \nu_p(f) + 1 + \frac{1}{s} \right), & p = 2. \end{cases} \]

**5. Proof of the main result**

For the given Dirichlet character \( \chi \) and the given prime \( p \), let us chose \( j_0 \) and \( l_0 \) as in lemma \[4.8\].
Let us define \( q := q := p^{l_0+2} - p^{l_0+1} - 1 \). For positive integers \( s, n, l \) with \( s > p^{l_0+2} - p^{l_0+1} \)
and \( l \geq l_0 + 2 \) let us consider
\[ S_{n,l}^{\chi,i} := \sum_{1 \leq j \leq f^l \atop \gcd(j,f^l) = 1} \chi(j) \int_{\mathbb{Z}_p} R_n(t + j \frac{j_0}{f^l}, \frac{j_0}{f^l}) dt, \]
where the rational function \( R_n(t, x) \) has been defined by
\[ R_n(t, x) = n^{l-1} \frac{(t)_{n+1}}{(t+x)_{n}(t+x+n)^q}. \]
Assume that \( s \geq p^{l_0+2} \) and define \( l := \lfloor \frac{\log s}{\log p} \rfloor \). We have already seen that this yields a linear form with integral coefficients in \( p \)-adic \( L \)-values:
\[ C(\chi, l, n, s) S_{n,l}^{\chi,i} = \lambda_{0,n} + \sum_{i=1}^{s} \lambda_{i,n} L_p(i + 1, \chi \omega^{-i}) \]
Let us recall that the function $\sigma': \mathbb{Z} \to \mathbb{Z}$ is given by

$$
\sigma'(n) := (p^{l_0+2} - p^{l_0+1}) \cdot n.
$$

Let $\epsilon > 0$. By corollary 4.6 we can find an $n_0$ such that for all $n \geq n_0$:

$$
H_K(A_{\sigma'(n)}) \leq \exp (\sigma(n)).
$$

with

$$
\sigma(n) := \sigma'(n)[K : \mathbb{Q}] \left[ \log(2)(s - 1) + s + 2fp^l + \log(fp^l) + \sum_{q|fp^l} \frac{\log q}{q - 1} + \epsilon \right].
$$

According to corollary 4.10 we get for all $n \geq n_0$ (after maybe increasing $n_0$):

$$
\exp (-\sigma(n)(\tau(p, s) + \epsilon)) \leq |C(\chi, l, \sigma'(n), s)S_{\sigma'(n)}^{\chi, l}|_p \leq \exp (-\sigma(n)(\tau(p, s) - \epsilon))
$$

with

$$
\tau(p, s) = \begin{cases} 
\frac{s \log p(l + \nu_p(f) + \frac{1}{p-1})}{[K : \mathbb{Q}] \left[ \log(2)(s - 1) + s + 2fp^l + \log(fp^l) + \sum_{q|fp^l} \frac{\log q}{q - 1} + \epsilon \right]}, & p \neq 2 \\
\frac{s \log p(l + \nu_p(f) + 1 + \frac{1}{p-1})}{[K : \mathbb{Q}] \left[ \log(2)(s - 1) + s + 2fp^l + \log(fp^l) + \sum_{q|fp^l} \frac{\log q}{q - 1} + \epsilon \right]}, & p = 2.
\end{cases}
$$

To simplify notation, let us only consider the case $p \neq 2$, the case $p = 2$ is analogous: After applying Nesterenko’s criterion (theorem 3.6) we obtain

$$
\dim_K \left( K + \sum_{i=2}^{s+1} L_p(i, \chi \omega^{1-i})K \right) \geq \tau(p, s).
$$

Since $\epsilon$ was arbitrary we deduce

$$
\dim_K \left( K + \sum_{i=2}^{s+1} L_p(i, \chi \omega^{1-i})K \right) \geq \frac{s \log p \left( l + \nu_p(f) + \frac{1}{p-1} \right)}{[K : \mathbb{Q}] \left[ \log(2)(s - 1) + s + 2fp^l + \log(fp^l) + \sum_{q|fp^l} \frac{\log q}{q - 1} \right]}.
$$

Recall that $l$ was defined by

$$
l := \left\lfloor \frac{\log s}{\log p} \right\rfloor
$$

and hence

$$
\frac{1}{p} \leq \frac{l}{s} < 1.
$$

Further, we have

$$
l \geq \frac{\log s}{\log p} - 1 \quad \text{and if } s = p^l, \text{ we get } l = \frac{\log s}{\log p}.
$$

For $s$ sufficiently large, we may assume

$$
\frac{1}{s} \left( \log(fp^l) + \sum_{\hat{p}|fp^l} \frac{\log \hat{p}}{\hat{p} - 1} \right) < 1.
$$

This allows us to conclude

$$
\frac{s \log p \left( l + \nu_p(f) + \frac{1}{p-1} \right)}{[K : \mathbb{Q}] \left[ \log(2)(s - 1) + s + 2fp^l + \log(fp^l) + \sum_{\hat{p}|fp^l} \frac{\log \hat{p}}{\hat{p} - 1} \right]} \geq \frac{\log s - \log p}{[K : \mathbb{Q}] (\log(2) + 2 + 2f)}.
$$
for $s$ sufficiently large. If $s = p^l$ we can slightly improve this to

$$s \log p \left( t + \nu_p(f) + \frac{1}{p-1} \right)$$

$$\frac{[K : \mathbb{Q}]}{[\mathbb{Q}]} \left( \log(2)(s - 1) + s + 2fp^l + \log(fp^l) + \sum_{\bar{p}/f} \frac{\log p^l}{p-1} \right) \geq \frac{\log s}{[K : \mathbb{Q}]} (\log(2) + 2 + 2f)$$

6. Appendix A: Chantanasiri’s approach to Nesterenko’s criterion

The aim of this appendix is to provide a proof for the following variant of Nesterenko’s $p$-adic linear independence criterion. The main difference to the statement in [Nes12] is the missing renormalization of the $p$-adic norm of the linear form. The present criterion is proven in the case $K = \mathbb{Q}$ in [Cha12] and it is indicated that the proof works for arbitrary number fields. We do not claim any originality in this appendix. For the convenience of the reader, we just work out the details of Chantanasiri’s proof for arbitrary number fields:

**Theorem 6.1** (Nesterenko, Chantanasiri). Let $\tau_1, \tau_2$ be positive real numbers and $\sigma(n)$ a non-decreasing positive function with $\lim_{n \to \infty} \sigma(n) = \infty$ and $\lim_{n \to \infty} \frac{\sigma(n)}{\sigma(n+1)} = 1$. Let $\theta = (\theta_1, ..., \theta_m) \in \mathbb{C}_p^m$. Assume that, for all sufficiently large integers $n$, there exists a linear form with coefficients in $\mathcal{O}_K$ in $m + 1$ variables

$$L_n(X) = l_{0,n}X_0 + l_{1,n}X_1 + ... + l_{m,n}X_m, \quad l_{i,n} \in \mathcal{O}_K$$

satisfying

$$H_K(L_n) \leq e^{\sigma(n)} \quad \text{and} \quad e^{-(\tau_1+o(1))\sigma(n)} \leq |L_n(1, \theta)|_p \leq e^{-(\tau_2+o(1))\sigma(n)},$$

then $\dim_K(K + K\theta_1 + ... + K\theta_m) \geq \frac{\sigma}{\tau_1 - \tau_2}$.

Let us recall that $H_K(L_n)$ was defined as follows

$$H_K(L) := \max_{0 \leq i \leq m} |N_K(l_{i,n})|.$$ 

More generally, for $0 \leq s \leq r$ let us set for any $M = (m_{i,j})_{0 \leq i \leq s, 0 \leq j \leq r} \in M_{(s+1) \times (r+1)}(K)$

$$H_K(M) := \max_J |N_K(\det M_J)|$$

where $J$ runs through all subsets $J \subseteq \{0, ..., r\}$ of cardinality $s + 1$ and $M_J$ denotes the matrix $M_J = (m_{i,j})_{0 \leq i \leq s, j \in J}$. The following Lemma follows immediately from the definitions of $H_K$ and $\Delta$, cf. [Cha12] Lemme 1.4:

**Lemma 6.2** (Chantanasiri, cf. [Cha12] Lemme 1.4). The above defined height satisfies the following properties:

(a) If $s = 0$ and $M = (m_0, ..., m_r)$ then

$$H_K(M) = \max_{0 \leq i \leq r} |N_K(a_i)|.$$ 

(b) If $s = r$, i.e. $M \in M_{(r+1) \times (r+1)}(K)$ then

$$H_K(M) = |N_K(\det(M))|.$$ 

(c) If $s \leq r - 1$ and $M \in M_{(s+1) \times (r+1)}(K)$, $L \in M_{1 \times (r+1)}(K)$ let us denote by $M \oplus L$ the matrix obtained by appending $L$ to $M$ as the last row, then

$$H_K(M \oplus L) \leq (s + 2)H_K(M)H_K(L).$$
Let us define
\[ H_p(M) := \max_J |\det M_J|_p \]
where \( J \) runs through all subsets \( J \subseteq \{0, \ldots, r\} \) of cardinality \( s + 1 \). For \( \xi = (\xi_0, \ldots, \xi_r) \in \mathbb{C}_p^{r+1} \) and \( J' \subseteq \{0, \ldots, r\} \) with \(|J'| = s\) let us write \( M_{J'\xi} \in M_{(s+1)\times(s+1)}(K) \) for the matrix obtained by appending \( M_{J'} \xi \) to the matrix \( M_{J'} \) and define
\[ \Delta_p(M) := \max_J |\det M_{J'}|_p \]
where \( J' \subset \{0, \ldots, r\} \) runs through the subsets of \( \{0, \ldots, r\} \) with \( s \) elements. Let us observe the following properties of \( H_p \) and \( \Delta_p \):

**Lemma 6.3** (Chantanasiri, cf. \[Cha12\], Lemme 1.5}). The above defined \( p \)-adic height satisfies the following properties:

(a) If \( s = r \), i.e. \( M \in M_{(r+1)\times(r+1)}(K) \) then
\[ H_p(M) = |\det(M)|_p \text{ and } \Delta_p(M) = |\det M|_p \cdot \max_{0 \leq i \leq r} |\xi_i|_p. \]

(b) If \( s \leq r \) and \( \Delta_p(M) \neq 0 \), then \( H_p(M) \neq 0 \) and \( H_p(M) \geq \frac{1}{H_K(M)} \).

(c) If \( s \leq r - 1 \) and \( M \in M_{(s+1)\times(r+1)}(K) \), \( L \in M_{1\times(r+1)}(K) \), then
\[ H_p(M \oplus L) \leq (s + 2)H_p(M)H_p(L). \]

If furthermore \( H_p(M)\Delta_p(L) > H_p(L)\Delta_p(M) \), then
\[ \Delta_p(M \oplus L) = H_p(M)\Delta_p(L). \]

**Theorem 6.4** (Chantanasiri,\[Cha12\], Thm. 1.7). Let \( \xi = (\xi_0, \ldots, \xi_r) \in \mathbb{C}_p^{r+1} \) and let \( (A_n)_{n \geq n_0}, \ldots, (Q_n)_{n \geq n_0} \) be sequences of real positive numbers. Assume that the sequence \( (B_nQ_{n-1})_{n > n_0} \) is increasing and that \( (A_n)_{n \geq n_0} \) and \( (B_nQ_{n-1})_{n > n_0} \) tend to infinity. For a sequence of linear forms
\[ \Lambda_n(X) = \lambda_{0,n}X_0 + \ldots + \lambda_{r,n}X_r, \quad \lambda_{i,n} \in \mathcal{O}_K \]
with
\[ H_K(\Lambda_n) \leq Q_n, \quad 0 < |\Lambda_n(\xi)|_p \leq \frac{1}{A_n} \quad \text{and} \quad \frac{|\Lambda_{n-1}(\xi)|_p}{|\Lambda_n(\xi)|_p} \leq B_n, \]
there exists a positive constant \( c > 0 \) such that for all sufficiently large integers \( n \):
\[ c \cdot A_n \leq (r + 1)!B_nQ_{n-1} \cdot Q_n. \]

**Proof.** Let us sketch the proof following \[Cha12\], Thm. 1.7]: By abuse of notation, let us denote the \( 1 \times (r + 1) \)-matrix \((\lambda_0,\ldots,\lambda_r)\) associated to the linear form \( \Lambda_n \) again by \( \Lambda_n \). For a sufficiently large positive number \( n \) and \( s \leq r \) let us inductively construct a sequence
\[ n = k_0 > k_1 > \ldots > k_s \geq n_0, \]
such that the matrix
\[ M_n^{(s)} := \Lambda_{k_0} \oplus \ldots \oplus \Lambda_{k_s} \]
satisfies
\[ 0 < \Delta_p(M_n^{(s)}) \leq B_{k_1+1} \cdot \ldots \cdot B_{k_s+1} \cdot \frac{1}{A_n}. \]

For \( s = 0 \) let us set \( k_0 = n \) and we have by the hypothesis of Theorem 6.3 the inequality \( \Delta_p(M_n^{(0)}) = |\Lambda_n(\xi)|_p \leq \frac{1}{A_n}. \) For \( 0 \leq s \leq r - 1 \) suppose that \( k_0, \ldots, k_s \) have already been chosen. According to Lemma 6.2(6) and the hypothesis we have
\[ H_K(M_n^{(s)}) \leq (s + 1)!Q_{k_0} \cdot \ldots \cdot Q_{k_s}. \]
Let us first assume that
\[ \Delta_p(M_n^{(s)}) H_p(\Lambda_{n0}) \geq |\Lambda_{n0}(\xi)|_p H_p(M_n^{(s)}). \tag{11} \]
In this case we can skip the inductive construction and argue directly as follows: From (9) and (11) we get:
\[ A_n \leq B_{k1+1} \cdots B_{k_s+1} \frac{H_p(\Lambda_{n0})}{|\Lambda_{n0}(\xi)|_p H_p(M_n^{(s)})}. \]
Now we can make us of (10) and lemma 6.3 (b)
\[ \frac{1}{H_p(M_n^{(s)})} \leq H_K(M_n^{(s)}) \leq (s + 1)!Q_{k0} \cdots Q_{k_s} \]
and deduce
\[ A_n \leq (s + 1)!B_{k1+1} \cdots B_{k_s+1}Q_{k0} \cdots Q_{k_s} \frac{H_p(\Lambda_{n0})}{|\Lambda_{n0}(\xi)|_p}. \]
By the hypothesis of the Theorem the sequence \( B_nQ_{n-1} \) is increasing and in this case we conclude
\[ \frac{|\Lambda_{n0}(\xi)|_p}{H_p(\Lambda_{n0})} A_n \leq (s + 1)!Q_n(B_nQ_{n-1}). \]
Now let us resume our inductive construction of \( k_{s+1} \) under the assumption that (11) does not hold. In this case, the set
\[ S = \left\{ k \in \mathbb{Z} \cap [n_0, k_s] \mid \Delta_p(M_n^{(s)}) H_p(\Lambda_k) < |\Lambda_k(\xi)|_p H_p(M_n^{(s)}) \right\} \]
is not empty since it contains \( n_0 \). Let us set \( k_{s+1} := \max(S) \) and \( M_n^{(s+1)} = M_n^{(s)} \oplus \Lambda_{k_{s+1}} \).
Lemma 6.3(c) gives
\[ \Delta_p(M_n^{(s+1)}) = H_p(M_n^{(s)}) |\Lambda_{k_{s+1}}(\xi)|_p. \]
By the choice of \( k_{s+1} \) we get the inequality
\[ |\Lambda_{k_{s+1}}(\xi)|_p H_p(M_n^{(s)}) \leq \Delta_p(M_n^{(s)}) H_p(\Lambda_{k_{s+1}+1}). \]
Combining the last two inequality and observing \( H_p(\Lambda_{k_{s+1}+1}) \leq 1 \) gives
\[ \Delta_p(M_n^{(s+1)}) \leq \Delta_p(M_n^{(s)}) H_p(\Lambda_{k_{s+1}+1}) \frac{|\Lambda_{k_{s+1}}(\xi)|_p}{|\Lambda_{k_{s+1}+1}(\xi)|_p} \leq \Delta_p(M_n^{(s)}) B_{k_{s+1}+1}. \]
This implies that the condition (9) also for \( s + 1 \) and finishes the inductive construction. Thus, we may assume that we have constructed
\[ n = k_0 > k_1 > \ldots > k_r \geq n_0, \]
such that
\[ 0 < \Delta_p(M_n^{(r)}) \leq B_{k1+1} \cdots B_{k_r+1} \frac{1}{A_n}. \]
According to Lemma 6.3(a) we have \( \Delta_p(M_n^{(r)}) = |\det M_n^{(r)}|_p (\max_i |\xi_i|_p) \). Since \( \det M_n^{(r)} \in K^\times \), we get by the product-formula for normed places of number fields
\[ |\det M_n^{(r)}|_p \geq \frac{1}{|N_K(\det M_n^{(r)})|}. \]
Now we make use of Lemma 6.3(b) and (10) and get
\[ |N_K(\det M_n^{(r)})| = H_K(M_n^{(r)}) \leq (r + 1)!Q_{k0} \cdots Q_{k_r}. \]
and then
\[ \max_i |\xi_i|p A_n \leq (r + 1)!Q_n (B_{k_1+1}Q_{k_1}) \cdots (B_{k_r+1}Q_{k_r}). \]
Since \((B_{k+1}Q_k)_{k \geq n_0}\) is increasing, we get
\[ \max_i |\xi_i|p A_n \leq (r + 1)!Q_n (B_nQ_{n-1})^r \]
as desired. \qed

Let us finally deduce Theorem 3.3 from Theorem 6.4:

**Proof of theorem 3.3.** Let \(\{\xi_0, ... , \xi_r\}\) with \(\xi_0 = 1\) be a \(K\)-basis of the \(K\)-vector space spanned by \(\theta_0 := 1, \theta_1, ..., \theta_m\). Then there is an integer \(d \in \mathbb{Z}\) and for \(1 \leq i \leq m\) coefficients \(c_{i,0}, ..., c_{i,r} \in \mathcal{O}_K\) such that
\[ d\theta_i = \sum_{j=0}^r c_{i,j} \xi_j. \]

Let us define the linear form
\[ \Lambda_n(Y_0, ..., Y_r) := L_n(\sum_{j=0}^r c_{0,j} Y_j, ..., \sum_{j=0}^r c_{m,j} Y_j). \]

Let us choose \(\tau'_1 > \tau_1\) and \(\tau'_2 < \tau_2\) and apply theorem 6.4 to \(\xi_i, \Lambda_n\) and
\[ Q_n = Ce^{\sigma(n)}, A_n = e^{\tau'_2 \sigma(n)} \quad \text{and} \quad B_n = e^{\tau'_1 \sigma(n) - \tau'_2 \sigma(n)} \]
where \(C > 0\) is a suitable constant. Theorem 6.4 gives us a constant \(c > 0\) such that for all sufficiently large \(n\):
\[ ce^{\tau'_2 \sigma(n)} \leq (r + 1)!C^{\sigma + 1} \cdot (e^{\tau'_1 \sigma(n) - \tau'_2 \sigma(n) + \sigma(n-1)})^r e^{\sigma(n)}. \]
By the hypothesis we have \(\lim_{n \to \infty} \sigma(n) = \infty\) and \(\lim_{n \to \infty} \frac{\sigma(n)}{\sigma(n+1)} = 1\) and deduce \(\tau'_1 \leq (r + 1)(\tau'_1 - \tau'_2 + 1)\). Since \(\tau'_1 > \tau_1\) and \(\tau'_2 < \tau_2\) where arbitrary, we obtain the desired estimate for \(\dim_K(K + \sum_{i=1}^m \theta_i K) = r + 1\). \qed

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