QUANTUM TORSORS AND HOPF-GALOIS OBJECTS

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Abstract. We prove that every faithfully flat Hopf-Galois object is a quantum torsor in the sense of Grunspan.

1. Introduction

The main result of this short note is to complete the comparison between the notion of a quantum torsor recently introduced by Grunspan [3], and the older notion of a Hopf-Galois object.

An $H$-Galois object for a $k$-Hopf algebra $H$ is a right $H$-comodule algebra $A$ whose coinvariant subalgebra is the base ring $k$ and for which the canonical map

$$\beta := \left( A \otimes A \xrightarrow{A \otimes \rho} A \otimes A \otimes H \xrightarrow{\nabla \otimes H} A \otimes H \right)$$

is a bijection (where $\nabla$ is the multiplication map of $A$, and $\rho: A \to A \otimes H$ is the coaction of $H$ on $A$). The notion appears in this generality in [4]; we refer to Montgomery’s book [5] for background. If one specializes $A$ and $H$ to be affine commutative algebras, then they correspond to an affine scheme and an affine group scheme, respectively, and the definition recovers the definition of a $G$-torsor with structure group $G = \text{Spec}(H)$, in other words the affine algebraic version of a principal fiber bundle.

In Grunspan’s definition a quantum torsor is an algebra $T$ equipped with certain structure maps $\mu: T \to T \otimes T^{\text{op}} \otimes T$ and $\theta: T \to T$ which are required to fulfill a set of axioms that we shall recall below. The definition is also inspired by results in classical algebraic geometry, going back to work of Baer [1]; we refer to [3] for more literature. Notably, if we again specify $T$ to be an affine commutative algebra, then the definition (which now does not need the map $\theta$) is known to characterize torsors, without requiring any prior specification of a structure group; in fact two structure groups can be constructed from the torsor rather than having to be given in advance. In addition to being group-free, this characterization has advantages when additional structures, notably Poisson structures, come into play: In the latter situation one cannot expect the canonical map $\beta$ in the definition of a Hopf-Galois extension to be maps of Poisson algebras, while the structure maps of a torsor are; thus the definition of a Poisson torsor becomes more natural when given in the group-free form.

Generalizing the results on commutative torsors, Grunspan shows that any torsor $T$ in the sense of his definition has the structure of an $L$-$H$-bi-Galois extension for two naturally constructed Hopf algebras $L = H_L(T)$ and $H = H_R(T)$. Thus, as in the commutative case, a torsor is a quantum group-free way to define a quantum...
principal homogeneous space (with trivial base), with quantum structure group(s) that can be constructed afterwards.

The following natural question is left open (or rather, asked explicitly) in [3]: Are there Hopf-Galois objects that do not arise from quantum torsors? Or, on the contrary, does every Hopf-Galois object have a quantum torsor structure?

We shall prove the latter (under the mild assumptions that Hopf algebras should have bijective antipodes, and Hopf-Galois objects should be faithfully flat). Thus Grunspan’s quantum torsors are seen to be an equivalent characterization of Hopf-(bi)-Galois objects, without reference to the Hopf algebras involved, parallel to the commutative case. On the other hand, the group Tor(H) of quantum torsors associated to a Hopf algebra H in [3] coincides with the group BiGal(H, H) of H-H-bi-Galois objects introduced in [6].

2. Notations

Throughout the paper, we work over a commutative base ring k.

We denote multiplication in an algebra A by \( \nabla = \nabla_A \), and comultiplication in a coalgebra C by \( \Delta = \Delta_C \); we will write \( \Delta(x) =: \delta(x_1) \otimes \delta(x_2) \). We will write \( \rho: V \rightarrow V \otimes C \) for the structure map of a right C-comodule V, and \( \rho(v) =: v^{(0)} \otimes v^{(1)} \).

Let H be a k-(faithfully) flat k-Hopf algebra, with antipode S. A right H-comodule algebra T is an algebra T which is a right H-comodule whose structure map \( \rho: T \rightarrow T \otimes H \) is an algebra map. We say that T is an H-Galois extension of its coinvariant subalgebra \( T^{\text{co}H} := \{ t \in T | \rho(t) = t \otimes 1 \} \) if the canonical map \( \beta: T \otimes T^{\text{co}H} \rightarrow T \otimes H \) given by \( \beta(x \otimes y) = xy^{(0)} \otimes y^{(1)} \) is a bijection. We will call an H-Galois extension T whose coinvariant subalgebra is the base ring an H-Galois object for short. In most of this paper we will be interested in faithfully flat (i.e. faithfully flat as k-module) H-Galois objects. For an H-Galois object T, we define \( \gamma: H \rightarrow T \otimes T \) by \( \gamma(h) := \beta^{-1}(1 \otimes h) \), and write \( \gamma(h) =: h^{[1]} \otimes h^{[2]} \). The following facts on \( \gamma \) can be found in [8]: For all \( x \in T \), \( g, h \in H \) we have

\[
\begin{align*}
(2.1) & \quad x^{(0)} x^{[1]} \otimes x^{[2]} = 1 \otimes x \\
(2.2) & \quad h^{[1]} h^{[2]} = \varepsilon(h) \cdot 1 \\
(2.3) & \quad h^{[1]} \otimes h^{[2]}_{(0)} \otimes h^{[2]}_{(1)} = h^{(1)} \otimes h^{(1)}_{(2)} \otimes h^{(2)} \\
(2.4) & \quad h^{[1]}_{(0)} \otimes h^{[2]} \otimes h^{[1]}_{(1)} = h^{(2)}_{(1)} \otimes h^{(2)}_{(2)} \otimes \varepsilon(h^{(1)}) \\
(2.5) & \quad (gh)^{[1]} \otimes (gh)^{[2]} = h^{[1]} g^{[1]} \otimes g^{[2]} h^{[2]} \\
(2.6) & \quad 1^{[1]} \otimes 1^{[2]} = 1 \otimes 1
\end{align*}
\]

In particular, the last two equations say that \( \gamma: H \rightarrow T^{\text{op}} \otimes T \) is an algebra map.

We now recall Grunspan’s definition of a quantum torsor [3]: A quantum torsor \( (T, \nabla, 1, \mu, \theta) \) consists of a faithfully flat k-algebra \( (T, \nabla, 1) \), an algebra map \( \mu: T \rightarrow T \otimes T^{\text{op}} \otimes T \), and an algebra automorphism \( \theta: T \rightarrow T \) satisfying, for all \( x \in T \):

\[
\begin{align*}
(2.7) & \quad (T \otimes \nabla)\mu(x) = x \otimes 1 \\
(2.8) & \quad (\nabla \otimes T)\mu(x) = 1 \otimes x \\
(2.9) & \quad (T \otimes T^{\text{op}} \otimes \mu)\mu = (\mu \otimes T^{\text{op}} \otimes T)\mu \\
(2.10) & \quad (T \otimes T^{\text{op}} \otimes \theta \otimes T^{\text{op}} \otimes T)(\mu \otimes T^{\text{op}} \otimes T)\mu = (T \otimes \mu^{\text{op}} \otimes T)\mu \\
(2.11) & \quad (\theta \otimes \theta \otimes \theta)\mu = \mu \theta,
\end{align*}
\]
Lemma 3.1. Let \( H \) occur in our calculations can be written with the rightmost tensor factors taken to for all \( x \in T \), and
\[
S(x_{(1)})^{[1]} \otimes x_{(0)}S(x_{(1)})^{[2]} \in T \otimes k \subset T \otimes T
\]
for all \( x \in T \), and
\[
h_{(1)}^{[1]} \otimes S(h_{(2)})^{[1]} \otimes h_{(1)}^{[2]}S(h_{(2)})^{[2]} \in T \otimes T \otimes k \subset T \otimes T \otimes T
\]
for all \( h \in H \).

Proof. For \( x \in T \) we have
\[
S(x_{(1)})^{[1]} \otimes \rho(x_{(0)}S(x_{(1)})^{[2]})
= S(x_{(2)})^{[1]} \otimes x_{(0)}S(x_{(2)})^{[2]}(0) \otimes x_{(1)}S(x_{(2)})^{[2]}(1)
\]
\[
= S(x_{(3)})^{[1]} \otimes x_{(0)}S(x_{(3)})^{[2]} \otimes x_{(1)}S(x_{(2)})
= S(x_{(1)})^{[1]} \otimes x_{(0)}S(x_{(1)})^{[2]} \otimes 1
\]
in \( T \otimes T \otimes H \). Since \( T^{\text{co} H} = k \) and \( T \) is flat over \( k \), this proves the first claim. Similarly, for \( h \in H \) we have
\[
h_{(1)}^{[1]} \otimes S(h_{(2)})^{[1]} \otimes \rho(h_{(1)}^{[2]}S(h_{(2)})^{[2]})
\]
\[
= h_{(1)}^{[1]} \otimes S(h_{(3)})^{[1]} \otimes h_{(1)}^{[2]}S(h_{(3)})^{[2]} \otimes h_{(2)}S(h_{(3)})^{[2]} \otimes h_{(2)}S(h_{(3)})^{[1]}
\]
Proving the second claim, again by flatness of \( T \).

Abusing Sweedler notation, the Lemma says that the “elements” \( x_{(0)}S(x_{(1)})^{[2]} \) and \( h_{(1)}^{[2]}S(h_{(2)})^{[2]} \) are scalars. We will use this by moving these elements around freely in any \( k \)-multilinear expression in calculations below, sometimes indicating our plans by putting parentheses around the “scalar” before moving it.

Theorem 3.2. Let \( T \) be a faithfully flat \( H \)-Galois object, where \( H \) is a Hopf algebra with bijective antipode. Then \( (T, \mu, \theta) \) is a quantum torsor, with
\[
\mu(x) = (T \otimes \gamma)\rho(x) = x_{(0)} \otimes x_{(1)}^{[1]} \otimes x_{(1)}^{[2]}
\]
\[
\theta(x) = (x_{(0)}S(x_{(1)})^{[2]}S(x_{(1)})^{[1]} = S(x_{(1)})^{[1]}(x_{(0)}S(x_{(1)})^{[2]})
\]
Proof. For all calculations, we let \( x, y \in T \) and \( h \in H \).

Since \( \rho \) and \( \gamma \) are algebra maps, so is \( \mu \). We have

\[
(T \otimes \nabla) \mu(x) = x(0) \otimes \nabla \gamma(x(1)) = x(0) \otimes \varepsilon(x(1))1 = x \otimes 1
\]

by (2.2), and \((\nabla \otimes T) \mu(x) = x(0) x(1)^{[1]} \otimes x(1)^{[2]} = 1 \otimes x\) by (2.1). Next

\[
\begin{align*}
(T \otimes \text{T} \otimes \mu) \mu(x) &= x(0) \otimes x(1)^{[1]} \otimes \mu(x(1)^{[2]}) \\
&= x(0) \otimes x(1)^{[1]} \otimes x(1)^{[2]} \otimes \gamma(x(1)^{[2]}(1)) \\
&\quad \stackrel{(2.3)}{=} x(0) \otimes x(1)^{[1]} \otimes x(1)^{[2]} \otimes \gamma(x(2)) \\
&= \mu(x(0)) \otimes \gamma(x(1)) \\
&= (\mu \otimes \text{T} \otimes T) \mu(x)
\end{align*}
\]

proves (2.9). It is clear that \( \theta(1) = 1 \). For \( x, y \in T \) we have

\[
\begin{align*}
\theta(xy) &= x(0)y(0)S(x(1)y(1))^{[2]}S(x(1)y(1))^{[1]} \\
&= x(0)y(0)(S(y(1))S(x(1)))^{[2]}(S(y(1))S(x(1)))^{[1]} \\
&\quad \stackrel{(2.5)}{=} x(0)(y(0)S(y(1))^ {[2]}S(x(1))^ {[1]}S(y(1))^ {[1]} \\
&\quad \gamma \; x(0)S(x(1))^ {[2]}S(x(1))^ {[1]}(y(0)S(y(1))^ {[2]}S(y(1))^ {[1]} \\
&= \theta(x)\theta(y),
\end{align*}
\]

so \( \theta \) is an algebra map.

For \( h \in H \) we have

\[
(3.3) \quad h^{[1]} \otimes \theta(h^{[2]}) = S(h)^{[2]} \otimes S(h)^{[1]}
\]

by the calculation

\[
\begin{align*}
h^{[1]} \otimes \theta(h^{[2]}) &= h^{[1]} \otimes h^{[2]}(0)S(h^{[2]}(1))^{[2]}S(h^{[2]}(1))^{[1]} \\
&\quad \stackrel{(2.3)}{=} h^{[1]} \otimes (h^{[1]}^{[2]}S(h(2))^{[2]}S(h(2))^{[1]} \\
&\quad \gamma \; h^{[1]} \otimes (h^{[1]}^{[2]}S(h(2))^{[2]} \otimes S(h(2))^{[2]} \\
&\quad \gamma \; (2.2) \; S(h)^{[2]} \otimes S(h)^{[1]}.
\end{align*}
\]

We conclude that

\[
(T \otimes \text{T} \otimes \theta) \mu(x) = x(0) \otimes x(1)^{[1]} \otimes \theta(x(1)^{[2]}) \quad \stackrel{(3.3)}{=} \quad x(0) \otimes S(x(1))^{[2]} \otimes S(x(1))^{[1]},
\]

hence

\[
\begin{align*}
(T \otimes \text{T} \otimes \theta \otimes \text{T} \otimes \text{T}) \mu(x) &= (T \otimes \text{T} \otimes \theta) \mu(x(0)) \otimes \gamma(x(1)) \\
&= x(0) \otimes S(x(1))^{[2]} \otimes S(x(1))^{[1]} \otimes \gamma(x(2)),
\end{align*}
\]

and on the other hand

\[
\begin{align*}
(T \otimes \text{m} \otimes \text{T}) \mu(x) &= x(0) \otimes \text{m}(x(1)^{[1]}) \otimes x(1)^{[2]} \\
&= x(0) \otimes x(1)^{[1]} \otimes x(1)^{[2]} \otimes x(1)^{[4]}(1) \otimes x(1)^{[1]}(0) \otimes x(1)^{[2]} \\
&\quad \stackrel{(2.4)}{=} x(0) \otimes S(x(1))^{[2]} \otimes S(x(1))^{[1]} \otimes x(2)^{[1]} \otimes x(2)^{[2]},
\end{align*}
\]
proving (2.10). To prove (2.11) we first check

\[(3.4) \quad \rho \theta(x) = \theta(x(0)) \otimes S^2(x(1)), \]

by the calculation

\[
\rho \theta(x) \overset{(3.1)}{=} (x(0)S(x(1))^{[2]})\rho(S(x(1))^{[1]}) \\
\overset{(2.4)}{=} x(0)S(x(1))^{[2]}S(x(1))^{[2]} \otimes S(S(x(1))^{[1]}) \\
= x(0)S(x(1))^{[2]}S(x(1))^{[1]} \otimes S^2(x(2)) \\
= \theta(x(0)) \otimes S^2(x(2)).
\]

Using this, we find

\[(\theta \otimes \theta \otimes \theta) \mu(x) = \theta(x(0)) \otimes \theta(x(1)) \otimes \theta(x(1))^{[2]} \]

\[
\overset{(3.3)}{=} \theta(x(0)) \otimes \theta(S(x(1))^{[2]} \otimes S(x(1))^{[1]} \\
\overset{(3.3)}{=} \theta(x(0)) \otimes S^2(x(1))^{[1]} \otimes S^2(x(1))^{[2]} \\
= \theta(x(0)) \otimes \gamma(S^2(x(1))) \\
\overset{(3.4)}{=} \theta(x(0)) \otimes \gamma(\theta(x(1))) = \mu \theta(x).
\]

It remains to check that \( \theta \) is a bijection. Now we have seen that \( \theta \) is an algebra map, and colinear, provided that the codomain copy of \( T \) is endowed with the comodule structure restricted along the Hopf algebra automorphism \( S^2 \) of \( H \). Of course \( T \) with this new comodule algebra structure is also \( H \)-Galois. It is known [7, Rem.3.11.(1)] that every comodule algebra homomorphism between nonzero \( H \)-Galois objects is a bijection. \( \square \)

**Remark 3.3.** Obviously, if we drop the requirement that \( \theta \) be bijective from the definition of a quantum torsor, we can do without bijectivity of the antipode of \( H \) in the proof. More precisely, the proof shows that \( \theta \) is bijective if and only if \( S \) is.

By the results of Grunspan, any quantum torsor \( T \) has associated to it two Hopf algebras \( H_1(T) \) and \( H_r(T) \), which make it into an \( H_1(T)-H_r(T) \)-bi-Galois object in the sense of [6]. That is, \( T \) is a right \( H_r(T) \)-Galois object in the sense recalled above, and at the same time a left \( H_1(T) \)-Galois object (i.e. the same as a right Galois object, with sides switched in the definition), in such a way that the two comodule structures involved make it into an \( H_1(T)-H_r(T) \)-bicomodule. Together with these constructions, Theorem 3.2 shows that the notions of a quantum torsor and of a Hopf-bi-Galois extension are equivalent, provided that we complete the picture by proving the following:

**Proposition 3.4.**

1. Let \( T \) be a faithfully flat \( H \)-Galois object, and consider the torsor associated to it as in Theorem 3.2. Then \( H_r(T) \cong H \), and \( H_1(T) \cong L(T,H) \), where the latter is the Hopf algebra making \( T \) an \( L(T,H)-H \)-bi-Galois object, see [6].

2. Let \( T \) be a quantum torsor. Then the quantum torsor associated as in Theorem 3.2 to the \( H_r(T) \)-Galois object \( T \) coincides with \( T \).

**Proof.** By the results in [6], each of the two one-sided Hopf-Galois structures in an \( L-H \)-bi-Galois object determines the other (along with the other Hopf algebra). Thus to prove (1), it suffices to check that \( L(T,H) \cong H_1(T) \), and the isomorphism
is compatible with the left coactions. Now let $\xi \in T \otimes T^{\text{op}}$. We write formally $\xi = x \otimes y$ even though we do not assume $\xi$ to be a decomposable tensor. According to the definition of $H_\ell(T) \subseteq T \otimes T^{\text{op}}$ in [3], we have

$$\xi \in H_\ell(T) \iff (T \otimes T^{\text{op}} \otimes T \otimes \theta) \mu(x) \otimes y = x \otimes \mu^{\text{op}}(y)$$

$$\iff x(0) \otimes (x(1)[1] \otimes \theta(x(1)[2])) \otimes y = x \otimes \mu^{\text{op}}(y)$$

$$\iff x(0) \otimes \mathcal{S}(x(1))(2) \otimes \mathcal{S}(x(1))(1) \otimes y = x \otimes y(1)[2] \otimes y(1)[1] \otimes y(0)$$

$$\iff x(0) \otimes \mathcal{S}(x(1)) \otimes y = x \otimes y(1) \otimes y(0)$$

$$\iff \xi \in (T \otimes T)^{\text{co}H}$$

where in the last step $T \otimes T$ is endowed with the codiagonal comodule structure, and we have used a version of [7, Lem.3.1]. By the definition of $L(T, H)$ in [6], this shows $L(T, H) = H_\ell(T)$ as algebras. A look at the respective definitions of comultiplication in $L(T, H)$ and $H_\ell(T)$ and of their coactions on $T$ shows that these also agree.

To show (2), we use the following results on $H_r(T)$ from [3]: $H_r(T)$ is some subalgebra of $T^{\text{op}} \otimes T$, the right $H_r(T)$-comodule algebra structure of $T$ maps $x \in T$ to $x(0) \otimes x(1) := \mu(x) \in T \otimes H_r(T) \subseteq T \otimes T^{\text{op}} \otimes T$, and $T$ is in fact $H_r(T)$-Galois, that is, the canonical map $\beta: T \otimes T \to T \otimes H$ is bijective. Now the torsor structure $(T, \mu', \theta')$ induced on $T$ by its Hopf-Galois structure as in Theorem 3.2 satisfies $\mu'(x) = x(0) \otimes x(1)[1] \otimes x(1)[2]$. To check $\mu = \mu'$, we apply $\beta$ to the two right tensor factors. Writing $\mu(x) := x(1) \otimes x(2) \otimes x(3)$, we have

$$(T \otimes \beta) \mu(x) = x(1) \otimes \beta(x(2) \otimes x(3))$$

$$= x(1) \otimes x(2) x(3)(0) \otimes x(3)(1)$$

$$= x(1) \otimes x(2) x(3)(1) \otimes x(3)(2) \otimes x(3)(3)$$

$$\stackrel{(2.7)}{=} x(1)(1) \otimes x(1)(2) x(1)(3) \otimes x(2) \otimes x(3)$$

$$\stackrel{(2.9)}{=} x(1) \otimes 1 \otimes x(2) \otimes x(3)$$

$$= x(0) \otimes 1 \otimes x(1)$$

$$= x(0) \otimes \beta(x(1)[1] \otimes x(1)[2])$$

$$= (T \otimes \beta) \mu'(x)$$

Since $\theta$ is determined by $\mu$, we are done. \hfill \Box

As a result of the Proposition, the construction $L(T, H)$ for a Hopf-Galois object $T$ coincides with the construction of $H_\ell(T)$ as in [3] for the quantum torsor associated to the Hopf-Galois object $T$ as in Theorem 3.2. Finally

**Corollary 3.5.** The group Tor$(H)$ of isomorphism classes of quantum torsors $T$ equipped with specified isomorphisms $H \cong H_\ell(T) \cong H_r(T)$ was observed by Grun span to be a subgroup of the group BiGal$(H)$ of $H$-$H$-bi-Galois objects defined in [6]. We see that the two groups in fact coincide.

4. **Ribbon transformations and the Miyashita-Ulbrich action**

The proof we gave for Theorem 3.2 is rather direct. One can shorten it slightly, and perhaps provide some partial explanation for the behavior of the $\theta$ map by using
the Miyashita-Ulbrich action [10, 2] and the notion of a ribbon transformation of monoidal functors introduced by Sommerhäuser [9]. To discuss this, we assume again that $H$ has bijective antipode.

Recall that a right-right Yetter-Drinfeld module $V \in \mathcal{YD}_H^H$ is a right $H$-module (with action denoted $\leftarrow$) and $H$-comodule such that

$$v(0) \leftarrow h(1) \otimes v(1) h(2) = (v \leftarrow h(2))(0) \otimes h(1)(v \leftarrow h(2))(1),$$

or equivalently $\rho(v \leftarrow h) = v(0) \leftarrow h(2) \otimes S(h(1))v(1)h(2)$ holds for all $v \in V$. The category $\mathcal{YD}_H^H$ is a braided monoidal category. The tensor product of Yetter-Drinfeld modules is their tensor product over $k$ with the (co)diagonal action and coaction, the braiding $\sigma$ is given by

$$\sigma_{VW} : V \otimes W \ni v \otimes w \mapsto w(0) \otimes v \leftarrow w(1) \in V \otimes W$$

for $V, W \in \mathcal{YD}_H^H$, its inverse by $\sigma_{VW}^{-1}(w \otimes v) = v \leftarrow S^{-1}(w(1)) \otimes w(0)$.

Let $T$ be a faithfully flat $H$-Galois object. The Miyashita-Ulbrich action of $H$ on $T$ is defined by $x \leftarrow h := h[1]xh[2]$ for $x \in T$ and $h \in H$. It is proved in [10, 2] (without the terminology) that $T$ with its $H$-comodule structure and the Miyashita-Ulbrich action is a Yetter-Drinfeld module algebra, that is, an algebra in $\mathcal{YD}_H^H$. This means that it is a module algebra (it is a comodule algebra to begin with), and a Yetter-Drinfeld module. Moreover, $T$ is commutative in the braided monoidal category $\mathcal{YD}_H^H$, which means that we have $\nabla \sigma_{TT} = \nabla$, that is $xy = y(x \leftarrow y(1))$ for all $x, y \in T$.

An endofunctor $F$ of $\mathcal{YD}_H^H$ is defined by letting $F(V)$ be the $k$-module $V$, equipped with the new right coaction $v \mapsto v(0) \otimes S^{-2}(v(1))$ and right action $v \otimes h \mapsto v \leftarrow S^2(h)$. The functor $F$ preserves the tensor product as well as the braiding of $\mathcal{YD}_H^H$.

According to Sommerhäuser, a ribbon transformation $\theta : Id \to F$ is a natural transformation such that $\theta_V \otimes \theta_W = \theta_{V \otimes W} \sigma_{VW} \sigma_{VW}$ holds for all $V, W \in \mathcal{YD}_H^H$ (moreover, we should have $\theta_k = id_k$). The example of a ribbon transformation we will use is essentially in [9], up to a switch of sides. It generalizes the map $\theta$ in the proof of Theorem 3.2, and is defined by $\theta_V(v) = v(0) \leftarrow S(v(1))$ for $V \in \mathcal{YD}_H^H$ and $v \in V$. This is surely natural, and also a morphism in $\mathcal{YD}_H^H$, that is, $H$-linear and $H$-colinear according to the formulas

$$\rho \theta_V(v) = \theta_V(v(0)) \otimes S^2(v(1)) \quad \theta_V(v) \leftarrow h = \theta_V(v \leftarrow S^{-2}(h)),$$

the first of which was used in our proof of Theorem 3.2; we’ll omit the proofs. Since for all $v \in V \in \mathcal{YD}_H^H$ and $w \in W \in \mathcal{YD}_H^H$ we find

$$\theta_{W \otimes V} \sigma(v \otimes w) = \sigma(v \otimes w)(0) \leftarrow S(\sigma(v \otimes w))$$

$$= \sigma((v \otimes w)(0)) \leftarrow S((v \otimes w)(1))$$

$$= (w(0) \otimes v(0) \leftarrow w(1)) \leftarrow S(v(1)w(2))$$

$$= w(0) \leftarrow S(v(2)w(3)) \otimes v(0) \leftarrow w(1)S(v(1)w(2))$$

$$= w(0) \leftarrow S(v(2)w(1)) \otimes v(0) \leftarrow S(v(1))$$

$$= \theta(w) \leftarrow S(v(1)) \otimes \theta(v(0))$$

$$= \theta(w \leftarrow S^{-1}(v(1))) \otimes \theta(v(0))$$

$$= (\theta_W \otimes \theta_V) \sigma^{-1}(v \otimes w),$$

$\theta$ is a ribbon transformation.
Given the results on the ribbon transformation \( \theta \) (which we could have taken by side-switching from [9]), it is almost obvious that \( \theta_T \) is an algebra map: 

\[
\nabla \theta_T \otimes T = \nabla (\theta_T \otimes \theta_T) = \nabla \sigma^{-2} = \nabla \theta_T \otimes \theta_T = \nabla T
\]

Using naturality of \( \theta \), the ribbon property, naturality of \( \sigma \), and braided commutativity of \( T \).

There is also a formula for the inverse of \( \theta \) in [9], namely \( \theta^{-1}(v) = v(0) \leftarrow S^{-2}(v(1)) \). We compute for completeness:

\[
\theta \theta^{-1}(v) = \theta(v(0) \leftarrow S^{-2}(v(1))) = v(1) = v(0) \leftarrow S(v(1))v(2) = v
\]

and

\[
\theta^{-1} \theta(v) = \theta^{-1}(\theta(v)(0) \otimes S^{-2}(\theta(v)(1))) = v(1) = v(0) \leftarrow S(v(1))v(2)
\]

Our final shortcut is not dependent on any results on ribbon transformations or Miyashita-Ulbrich actions, but rather on bijectivity of the antipode, and its consequence that \( \theta \) is bijective. The morphism \( \mu: T \to T \otimes T^{op} \otimes T \) constructed for Theorem 3.2 depends only on the \( H \)-comodule algebra structure of \( H \), but does not contain \( H \), so that it surely does not change if we replace the \( H \)-comodule structure by the \( H \)-comodule structure induced along \( S^2 \). But since \( \theta: T \to T \) is colinear between these two comodule structures, and an algebra isomorphism, it follows that \( \theta \) also preserves \( \mu \), that is, axiom (2.11) holds.

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