Shape effects in doubly clamped bridge structures at large deflections

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Abstract. The shape of a doubly clamped bridge structure depends on its deflection. At large deflections, where the system exhibits nonlinear behaviour, the shape effect becomes significant. We present a general method, based on variational analysis, for computing corrections to the nominal linear regime shape function. The method is used to compute the first non-trivial correction and quantify the corresponding improvement in the large deflection regime. The model obtained is also validated using FEM simulations.

1. Introduction
The interest in nonlinear mechanical structures for vibrational energy harvesting applications has increased over the past decade. Advantages compared to linear resonator systems include increased bandwidth, limited deflection and improved structural durability [1–5]. In addition, bi-stable modes of operation can be used to target low frequency and amplitude excitations [6].

A model system featuring mechanical nonlinearities is the doubly clamped bridge with a central proof mass shown in Fig. 1. The boundary conditions induce stretching of the two beams which results in a contribution to the restoring force nonlinear in the proof mass displacement \( \delta \).

In this paper we present a model for the bridge system which accounts for the dependence on \( \delta \) of the shape of the beams, defined as the functional coordinate dependence of the displacement field. The model uses variational analysis to compute corrections to the nominal shape function, which is valid at infinitesimal deflections where the bridge is described by pure bending, by minimizing the strain energy. We explicitly compute the first non-trivial correction and the corresponding strain energy reduction. Furthermore, we consider an example of a bridge structure and validate the predictions of our model using FEM simulations.

Although the present paper is concerned with a purely mechanical model of the doubly clamped structure, an electromechanical model of a piezoelectric harvester could be constructed based on the derived shape function, extending the work by Gafforelli et al. [7].

2. Theoretical models
2.1. Large deflection model
At large displacements \( \delta \), the total strain in the beams is the sum of bending and stretching strains according to Green-Lagrange

\[
S_{33} = -x_1 \frac{\partial^2 w}{\partial x_3^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x_3} \right)^2,
\]  
\( (1) \)
Figure 1. Illustration of the doubly clamped bridge structure with definitions of proof mass displacement $\delta$, beam length $L$ and coordinates $x_1$ and $x_3$.

where $w$ is the $x_1$ component of the displacement field and $S_{33}$ is the total strain in the beam.

Introducing the dimensionless coordinate $\xi = x_3/L$, the total strain energy $V$ of the structure can be obtained by integrating the energy density $c_{33}S_{33}^2$, where $c_{33}$ is the elastic modulus, over the total system volume and enforcing the doubly clamped boundary conditions

$$V = A_b \int_0^1 \left( \frac{\partial^2 w}{\partial \xi^2} \right)^2 d\xi + A_s \int_0^1 \left( \frac{\partial w}{\partial \xi} \right)^4 d\xi. \quad (2)$$

The constants $A_b$ and $A_s$, corresponding respectively to bending and stretching, are given in terms of moments of area for the beam cross-section $A$ as

$$A_b = \frac{1}{L^3} \int_A c_{33} x_3^2 dA, \quad A_s = \frac{1}{4L^3} \int_A c_{33} dA. \quad (3)$$

The shape function $w = w(\xi, \delta)$ is the function that minimizes the strain energy $V$. To describe the shape function we use a power series Ansatz for $w$

$$w(\xi, \delta) = \sum_{i=0}^{\infty} a_n(\delta) \xi^n, \quad (4)$$

where the coefficients $a_n(\delta)$ are determined by the doubly clamped boundary conditions, the symmetry with respect to $\xi = 1/2$ and the condition that $w$ extremizes the functional $V[w]$

$$\frac{\delta V}{\delta w} = 0. \quad (5)$$

In particular, the boundary conditions imply the vanishing of the coefficients $a_0$ and $a_1$.

Given the shape function, and corresponding strain energy, the dynamics of the bridge system is governed by the Euler-Lagrange equation of motion for $\delta$.

2.2. Shape function approximations

Approximate expressions for the shape function are obtained by truncating the series at some power $N$, where the symmetry of the problem implies that it is sufficient to consider $N$ odd.

The lowest order for which all boundary conditions can be satisfied is $N = 3$, for which the unique solution is

$$a_2 = 3\delta, \quad a_3 = -2\delta. \quad (6)$$

The minimization requirement is trivial for $N = 3$, in the sense that all coefficients $a_n$ are uniquely determined by the structural boundary conditions\(^1\).

\(^1\) We also note that for $N = 3$ the shape function is the exact solution for the doubly clamped structure in Euler-Bernoulli theory describing infinitesimal displacements $\delta$. 

The first non-trivial correction to the shape function is obtained for $N = 5$, where the boundary conditions and symmetry constraints can be used to solve for the first four coefficients

$$a_2 = 3\delta - \frac{1}{2}a_5, \quad a_3 = -2\delta + 2a_5, \quad a_4 = -\frac{5}{2}a_5.$$  

The remaining coefficient $a_5$ can then be determined by the variational condition (5), which amounts to solving the equation $\partial V / \partial a_5 = 0$.

The method described above can be applied to compute corrections to arbitrary order $N$. Although the power series is not a perturbation series expansion in the ordinary sense, the strain energy $V$ is a decreasing function of $N$ since a solution of the same energy as for $N - 2$ is obtained by setting the added coefficients to zero.

2.3. Validation model

In order to validate the model we consider a bridge structure with beams of length $L = 5$ mm, width $W = 3$ mm and thickness $H = 5\mu m$, chosen to comply with MEMS manufacturing constraints while being sufficiently weak. The proof mass is taken to be a $2 \times 3 \times 0.6$ mm$^3$ block of tungsten to obtain a low frequency system which can be expected to operate in the large deflection regime when subjected to moderate excitations. The theoretical approximations are evaluated and compared to the results obtained using a large deflection FEM simulation model accounting for geometric nonlinearities.

3. Results and simulations

3.1. Shape function approximations

The solution to the $N = 5$ system can be presented in a normalized way, which makes it independent of the mechanical and geometrical properties of the bridge structure, by introducing the length scale $\lambda$ according to $\lambda^2 = A_b/A_s$. From the definition of $\lambda$ it follows that the characteristic length scale of large deflection effects is the thickness of the beam $[5]$. The normalized coefficient $a_5$ and the reduction of the strain energy in the $N = 5$ model relative to the $N = 3$ model, as functions of the normalized proof mass displacement $\delta$, are shown in Fig. 2. As expected, at $\delta \approx \lambda$ the coefficient $a_5$ and the strain energy reduction become significant.

![Figure 2](image)

**Figure 2.** (a) Normalized model coefficient $a_5$ in the $N = 5$ model as a function of normalized proof mass displacement. (b) Relative strain energy reduction in the $N = 5$ model.

3.2. Validation model results

The validation structure is evaluated using the $N = 3$ and $N = 5$ approximations and the resulting shape functions are compared to the FEM simulation in Fig. 3. Similarly, the total strain energy of the structure at static proof mass displacement is shown in Fig. 4a. The results show that at large deflection the $N = 5$ model gives significant improvement over the
nominal $N = 3$ case. For example, at $\delta = 2H$ the relative error in strain energy compared to simulation is reduced from 27% in the $N = 3$ model to 12% in the $N = 5$ model. Finally, in order to exemplify the impact on the dynamics of the structure, the frequency of unforced and undamped oscillations is computed from the potential strain energy and shown in Fig. 4b as a function of amplitude. Again, the results demonstrate improved agreement with the FEM simulation results in the $N = 5$ model.

![Figure 3](image1.png)

**Figure 3.** Shape function $w(\xi)$ for (a) $\delta = 5\ \mu m$, (b) $\delta = 15\ \mu m$ and (c) $\delta = 25\ \mu m$.

![Figure 4](image2.png)

**Figure 4.** (a) Total strain energy of the structure as a function of proof mass displacement. (b) Frequency of unforced oscillations as a function of amplitude.

4. Conclusions

We have presented a method for computing corrections to the shape of a doubly clamped bridge structure at large deflections. The lowest order corrections have been computed and shown to provide significant improvements of both static and dynamic properties of the structure.

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