Zero modes, entropy bounds
and partition functions.

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Abstract

Some recent finite temperature calculations arising in the investigation
of the Verlinde-Cardy relation are re-analysed. Some remarks are also
made about temperature inversion symmetry.
1. Introduction

In a recent article Brevik, Milton and Odintsov, [1], have evaluated the thermodynamical quantities for quantised scalars, spinors and vectors on the Einstein universe. The results are compared with the earlier evaluations of partitions sums by Kutasov and Larsen, [2] and by Klemm, Petkou and Siopsis, [3], one objective being the investigation of the validity of the Verlinde–Cardy relation. Some discrepancies were found.

The evaluation of finite temperature field theoretic quantities on the Einstein universe was actually undertaken earlier, [4,5], in the area of quantum field theory on curved spaces and the various high and low temperature limits determined. The present article draws upon these earlier investigations to compare with the more recent analyses. The specific point we wish to look at is the apparent appearance of a term in the internal energy proportional to the temperature that, [1], spoils the Verlinde–Cardy relation.

2. Thermal scalar on the circle.

We begin with the low dimensional case of the two-dimensional space–time $T \times S^1$, which is very well investigated. The case of scalars is quite adequate for our purposes and was considered in [1]. In some ways the mechanism will turn out to be generic, which is why it is given here.

In two dimensions, conformal scalars are minimal scalars. The eigenvalues of $-\nabla$ are $\omega_n^2 = n^2/a^2$, ($n = 0, \pm 1, \pm 2, \ldots$) and we note the existence of a zero mode $n = 0$. The point is that, in such a case, it is not correct simply to ignore the zero mode, when evaluating the internal energy for example. This would be correct for ordinary multi–particle statistical mechanics, but in field theory, the existence of a zero mode implies an infinite tower of states ‘attached’ to each non-zero level. The effect, [6], is to add a term $b_0 kT$ to the internal energy where $b_0$ is the number of zero modes (here $b_0 = 1$).

There is no need to go through the general theory, e.g. [7,8], and the expression for the internal energy is just written down,

$$E = \frac{-1}{12a} + kT + \frac{2}{a} \sum_{n=1}^{\infty} \frac{n}{e^{n\beta/a} - 1}, \quad \beta = 1/kT. \quad (1)$$

The first term is the zero temperature (Casimir) value, the third is the standard statistical sum over the non-zero modes and the second is the quantised effect of the zero mode which is the term omitted in [1], but not, of course, in Cardy [9].
The high temperature form of this expression can be found in various standard ways and one obtains,

\[ E \sim \frac{1}{12a} (2\pi a T)^2 , \]

the term linear in \( T \) disappears and the objections of [1] to the analysis of Klemm et al [3], who mention this term and then argue it away, are no longer valid.

In slightly more detail, in the application of Poisson summation,

\[ \sum_{n=1}^{\infty} f(n) = -\frac{1}{2} f(0) + \int_{0}^{\infty} f(t) \, dt + 2 \sum_{m=1}^{\infty} \text{Re} \int_{0}^{\infty} f(t) e^{2im\pi t} \, dt , \tag{2} \]

which was also used in [5], it is the term \(-f(0)/2\) that cancels the \( kT \) in (1). This is the basic mechanism.

As a simple observation, which will be useful later, we relate the number of zero modes to the value of the \( \zeta \)–function on a \( d \)–dimensional spatial section by the standard result,

\[ \zeta(0) = C_{d/2} - b_0 , \tag{3} \]

where \( C_{d/2} \) is the constant term in the short–time expansion of the heat-kernel expansion. If \( d \) is odd, and space is closed (e.g. \( S^d \)), this term is zero and the value of \( \zeta(0) \) is due purely to any zero modes. On the circle the \( \zeta \)–function is the Riemann \( \zeta \)–function, \( \zeta(s) = 2^{2s} \zeta_R(s) \) which yields \( b_0 = 1 \).

3. p–forms in higher dimensions.

On the Einstein universe, \( T \times S^3 \), the only problem is posed by the gauge invariance of the electromagnetic field but, before giving explicit expressions, it is better to analyse the extension of photon theory to higher dimensions in order to get the correct field content.

For a \( p \)–form on the space-time \( T \times M \), it is easy to show that that the Obhukov, [10], combination of ghosts reduces to a simple alternating sum of \( p \)–forms on \( M \). We may express this schematically as,

\[
\sum_{k=0}^{p} (-1)^k (k + 1) \phi_{T \times M}(p - k) = \sum_{k=0}^{p} (-1)^k (k + 1) \left( \phi_M(p - k) \oplus \phi_M(p - k - 1) \right) = \sum_{k=0}^{p} (-1)^k \phi_M(p - k),
\]
where $\phi(p)$ stands for a $p$–form.

It is now possible to use ‘télescoping’ to reduce this sum further for any spectral quantity such as the $\zeta$–function. If one makes the Hodge decomposition of the forms on $\mathcal{M}$ into coexact, exact and harmonic types, and then uses the isomorphism between exact $p$–forms and coexact $(p - 1)$–forms, only the coexact $p$–form quantity survives the summation, together with the alternating combination of all the harmonic contributions. The coexact $p$–form on $\mathcal{M}$ describes the physical degrees of freedom of the system.

The field content is then, schematically,

$$\phi^{\text{tot}}(p) = \phi^{\text{CE}}(p) \oplus \sum_{k=0}^{p} (-1)^k \phi^{\text{H}}(p - k),$$

where CE refers to a coexact form and H to a harmonic one, of which, we assume, there is only a finite number.

For the spatial section, $S^d$ with $d$ odd, the $p = (d - 1)/2$ theory is conformally invariant when propagated by the de Rham Laplacian and this constitutes the appropriate generalisation of the photon field.

General and specific information on $p$–forms on spheres exists in Copeland and Toms [11] and Dowker and Kirsten [7] and in the more mathematical references therein. Interesting comments regarding heat–kernel expansions are contained in Elizalde et al [12]. See also Cappelli and D’Appollonio, [13].

The only non-zero Betti numbers on the sphere are $\beta_0 = \beta_d = 1$ and it is then straightforward to check, from the $\zeta$–function for the coexact form for example, e.g. [7], that there is no constant term in the short-time expansion of the heat-kernel for the complete field system, the two terms in (4) cancelling.

It is worthwhile giving a few analytical details. By specialising to the middle dimension, $p = (d - 1)/2$, case ($d$ odd), the coexact $\zeta$–function, is

$$\zeta^\text{CE}_p(s) = a^{2s} \frac{2}{p!^2} \sum_{n=p+1}^{\infty} \frac{(n^2 - p^2)(n^2 - (p - 1)^2)\ldots(n^2 - 1)}{n^{2s}},$$

$a$ being the sphere radius.

Although we shall not make great use of it, the alternative expression for this coexact sphere $\zeta$–function derived in [7] is worth recording here,

$$\zeta^\text{CE}_p(s) = (-1)^p \sum_{k=0}^{p} (-1)^k \binom{2k}{k} \zeta_{2k+1}^{2k+1}(s),$$

(6)
where \( \zeta^k_0 \) is the \( \zeta \)–function on the sphere \( S^k \) for a scalar field conformal in \( 1 + k \) dimensions which itself can be represented as the sum of two Barnes \( \zeta \)–functions,

\[
\zeta^k_0(s) = \zeta_B(2s; (k + 1)/2 \mid 1_k) + \zeta_B(2s; (k - 1)/2 \mid 1_k),
\]

(7)
corresponding to the sum of Dirichlet and Neumann hemisphere quantities, \([14]\).

Another way of writing the coexact \( \zeta \)–function is, curiously,

\[
\zeta^{\text{CE}}_p(s) = \left( \frac{p + 1}{a^2} \right) \left( \frac{2p + 2}{p + 1} \right) \zeta^{2p+3}_0(s + 1),
\]

(8)
which can be derived from simple manipulation with the degeneracies. Together with (7) it provides a route to closed, \( i.e. \) non–summed, forms for required quantities. However this will not be pursued here.

Returning to (5), the summation can be started at \( n = 1 \) and then the coexact \( \zeta \)–function expanded in Riemann \( \zeta \)–functions using, say, Stirling numbers. It immediately follows that \( \zeta^{\text{CE}}_p(0) = (-1)^{(d+1)/2} \), as needed to cancel the term from the harmonic piece. This can be seen from equation (3) with no true zero modes for the coexact \( \zeta \)–function, (5). The harmonic mode contributes an opposite \( (-1)^{(d-1)/2} \) to the constant term in the heat-kernel expansion.

The situation is really exactly like that on the circle, \( S^1 \), discussed previously. One can incorporate the harmonic piece into the coexact one by including \( n = 0 \) in the spectrum and splitting the positive and negative values, \( n = 0, \pm 1, \pm 2, \ldots \). The language one would then use is that there is a zero mode (at \( n = 0 \)) which comes in either positively (if \( p \) is even) or negatively (if \( p \) is odd).

Since this ‘zero mode’ is a Fock space state it is, presumably, subject to the same thermalisation as a normal zero mode. This process is actually built into the construction of the finite temperature \( \zeta \)–function which is just a \( \zeta \)–function on the manifold \( S^1 \times M \), \([8,15,16]\),

\[
\zeta(s, \beta) = \frac{i}{\beta} \sum_{m = -\infty}^{\infty} \frac{d_n}{(\omega_n^2 + 4\pi^2 m^2 / \beta^2)^s},
\]

(9)
the \( \omega_n^2 \) and \( d_n \) being the relevant eigenvalues and degeneracies on \( M \) and the dash signifying that the denominator should not be zero. In the present case the specific \( d_n \) and \( \omega_n \) can be read off from the zero temperature \( \zeta \)–function, (5) which we shall now write as

\[
\zeta_M(s) = \frac{1}{p!^2} \sum_{n = -\infty}^{\infty} \frac{(n^2 - p^2)(n^2 - (p - 1)^2)\ldots(n^2 - 1)}{(n^2/a^2)^s},
\]

(10)
extended to $p = 0$, when the numerator is 1.

The finite temperature $\zeta$–function, (9), because both summations are now double sided, can be related to the Epstein $\zeta$–function by differentiation with respect to $1/a^2$, which is sometimes useful.

One can again appeal to the general theory as given in [8] to write the total internal energy as,

$$ E = E_0 + (-1)^p kT + \sum_{\omega_n \neq 0} \frac{d_n \omega_n}{e^{\beta \omega_n} - 1}. $$

The second term is the effect of the zero mode and $E_0$ is the zero temperature value,

$$ E_0 = \frac{1}{2} \zeta_M(-1/2), $$

which is easily evaluated in terms of Bernoulli polynomials, for example. The very general development given in [7] could be used, or equation (8), but I prefer here to work directly and, instead of finding an expression valid for all $p$, treat the problem as a purely arithmetic one, $p$ by $p$. Numerical evaluation is easily implemented.

$E_0$ alternates in sign with $p$ and the values, $-1/12a$ for $p = 0$ and $11/120a$ for $p = 1$, agree with known ones. Incidentally, in the limit $p \to \infty$ one finds the following result,

$$ E_0(p) \to (-1)^{p+1} \frac{1}{\pi^2 a}, \quad p \to \infty. $$

Setting $p$ to 1 reproduces the results given explicitly in [4,5] on the Einstein universe and so will not be reproduced here.

4. High temperature expansions.

The important high temperature expansions of the various thermodynamic quantities have been given in [15,8] in terms of standard heat-kernel coefficients for which we can treat (10) as a bone fide $\zeta$–function for evaluation purposes. The crucial point, at the moment, is that the $C_{3/2}$ coefficient is zero, according to our previous discussion, and so there cannot be a term proportional to $kT$ in the expansion of the energy. This is confirmed by looking at the expression, (11), for the internal energy directly and applying Poisson summation, for example.

It is also clear from the alternative representation, (6), that the zero mode resides only in the $S^1$ part and so the discussion of the 0–form on the two-torus, regarding the effect of this mode, is actually generic.
There is, however, a price to pay for this ‘simplification’. Zero modes, on the spatial section, give rise to a temperature dependent pole in the free energy, [8], and one has to decide what to do about it. Equivalently, there is a dependence on the scaling length which implies a contribution to the conformal anomaly. If the quantum field theory is formally conformally invariant, this will be the only contribution and the trace condition is the equation of state,

$$P_V = \frac{1}{d}(E - b_0T),$$

which is an equation purely in terms of the non-zero modes, since the zero mode contribution to $F$, being volume independent, does not affect the pressure. However the entropy suffers from the same drawbacks as the classical Sackur–Tetrode expression, which is a problem this author does not know how to solve within the present formalism.

Clearly the coefficients of the powers of $kT$ in the expansions can be readily determined quite generally but this will not be done here. The expressions (5), (6), (7) also show that there are only a finite number of terms in the high temperature expansions, which is the same as the old statement that the conformal heat–kernel expansion terminates on odd spheres. A neat way of seeing this is to relate the theories on spheres of different dimensions by differentiation, starting with the flat $S^1$, e.g. [17–19]. For even spheres, the heat-kernel expansion does not terminate. This behaviour is related to the non-existence of a Huyghens principle in odd dimensional space-times signaled by the appearance of Bessel functions in massless propagators. To increase the dimension of the sphere by one, a fractional derivative is needed.

5. High and low temperature relations.

Closely related is the connection between the ‘Planck’ high temperature expansion and the ‘Casimir’ low temperature form of, say, the total internal energy. Such a relation was derived and used by Brown and Maclay [20] in their elegant treatment of the finite temperature Casimir effect between plates. The analysis in their paper formed the basis for our work on the Einstein universe, $T \times S^3$, presented in [4,5] where this relation was also discussed. Some further remarks and facts were also discussed in Candelas and Dowker [21] which was concerned with the relation between vacuum averaged stress-energy tensors in conformal space-times.
It is clear from the structure of the Epstein $\zeta$–functions that such a temperature inversion symmetry will hold, in some form, for cavities of rectangular shape and their periodic counterparts, the tori. A number of calculations have explicitly verified these rather elementary facts, [22–25]. The symmetry applies for the other flat space forms as well, [15,26–28].

In two dimensions, in statistical contexts, this symmetry is referred to as modular invariance and corresponds, more or less, to the interchange of the two cycles on the torus, $S^1 \times S^1$. It has been very widely analysed, especially in connection with conformal field theories and an attempt was made by Cardy [9] to extend the notion to higher dimensions by considering, in particular, free field theories on $T \times S^d$.

Because conformal eigenvalues are perfect squares, the relevant quantities, such as the $\zeta$–function, have a general torus appearance, apart from the degeneracies, and therefore there is the possibility of a high–$aT$ low–$aT$ duality which was realised explicitly in [4,5,21] for the three–sphere.

The existence of zero modes is a minor annoyance in this topic and so one may restrict most attention to scalar fields. Although we have discussed the case of the Einstein universe before, it is worth looking at again as a very specific case.

6. Thermal on Einstein universe.

There are a number of starting points for a discussion of temperature inversion symmetry on the Einstein universe. One way is to take the final double sum forms for the various thermodynamic quantities, as in [20,4,5], and look at them. Or one can begin with the thermal $\zeta$–function, as mentioned in [21],

$$\zeta(s, \beta) = \frac{i}{2\beta} \sum_{n=0}^{\infty} \frac{n^2}{(n^2/a^2 + 4\pi^2m^2/\beta^2)^s}$$  \hspace{1cm} (12)

and force it to be symmetrical. Before proceeding with this, a necessary technical point has to be brought forward. The free energy is given as the limit

$$F = -\frac{i}{2} \lim_{s \to 0} \frac{\zeta(s, \beta)}{s}$$  \hspace{1cm} (13)

and in the present case, this is finite because (12) is just a derivative, with respect to $1/a^2$, of an Epstein $\zeta$–function which equals $-1$ at $s = 0$. Hence we don’t need to take the limit of (13) any further in order to analyse the general properties of $F$. 

7
A useful dimensionless variable is \( \xi = 2\pi a/\beta \), and

\[
\frac{1}{\xi} a F = \frac{1}{4} \lim_{s \to 0} \frac{1}{s} \sum_{n = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} \frac{n^2}{(n^2 + \xi^2 m^2)^s}.
\]

The symmetrisation of this quantity amounts to an evaluation of the internal energy \( E = \partial(\beta F)/\partial \beta \) or,

\[
\frac{1}{\xi^2} e(\xi) = -\frac{\partial}{\partial \xi} \left( \frac{f(\xi)}{\xi} \right),
\]

where \( e \) and \( f \) are defined by the rescaling,

\[
f(\xi) = a F, \quad e(\xi) = a E,
\]

and we easily see that

\[
\frac{1}{\xi^2} e(\xi) = \xi^2 e\left( \frac{1}{\xi} \right),
\]

which is the desired temperature inversion of the total internal energy and allows high \( T \) and low \( T \) to be related. To obtain the precise values, the calculation has to be taken a bit further, [4,5]. It is perhaps fortuitous that the quantity, the energy, that satisfies a simple temperature inversion relation is a standard thermodynamical one.

One can view the differentiation that occurs in (14) as equivalent to the one that raises the dimension of a sphere by two, as was mentioned earlier. That is, the thermal circle has been turned into a ‘thermal three–sphere’ so that the total manifold becomes symmetrical, effectively \( S^3 \times S^3 \), allowing an inversion symmetry to manifest itself.

The reason why the relation (15) is so simple is that the conformal heat–kernel expansion on the three-sphere terminates with the first, volume or Weyl, term and the high temperature expansion has only one term – the Planck \( T^4 \) one. Things are slightly different for the higher spheres. However, before briefly looking at these, it is worth examining what appears to be another route to temperature inversion symmetry. This time we take the limit in (13) seriously. Evaluation using heat kernels, [15], or the functional equation for the Epstein \( \zeta \)–function, [29], can produce the standard statistical mechanical expression,

\[
e(\xi) = e_0 + \sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/\xi} - 1}.
\]
where $e_0 = aE_0$ is the zero temperature Casimir constant first evaluated by Ford, $e_0 = 1/240 = -B_4/8$ ($B_4$ is a Bernoulli number).

One now examines the properties of the sum in (16) directly. Early papers in which this was done are by Marlukar [30] and Glaisher [31]. Let us define the sum under consideration as

$$S(\alpha, q) = \sum_{n=1}^{\infty} \frac{n^q}{e^{2\alpha n} - 1}.$$  

Then Marlukar shows analytically that,

$$S(\alpha, 3) = \left(\frac{\alpha}{\beta}\right)^2 S(\beta, 3) + \frac{B_4}{8} \left(1 - \left(\frac{\alpha}{\beta}\right)^2\right),$$  

where

$$\alpha \beta = \pi^2,$$

which is precisely the inversion (15) for the total energy.

There are various ways of organising the information, usually distinguished by the ways the factors in $S^1 \times \mathcal{M}$ are combined analytically. ($S^1$ here stands for the thermalising imaginary time circle.) One can use heat-kernels to encode the mode information, or, possibly $\zeta$–functions. In the latter case there is a standard and informative way of obtaining the high temperature expansion of the thermodynamic quantities which is usually stated for $\log Z$, defined generally by

$$\ln Z(\beta) = -\sum_m \ln \left(1 - e^{-\beta \omega_m}\right),$$

for zero chemical potential. This is to write the Mellin transform,

$$\ln Z(\beta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s) \zeta_M(s/2) \zeta_R(1 + s) \beta^{-s}, \quad \text{Re } c > d,$$

which arises from the relation between $\zeta$–functions on the product $\mathcal{M}_1 \times \mathcal{M}_2$ and on the factors.

Rather than $\beta F_1 = -\log Z$, we concentrate on the finite temperature energy correction, $E_1$,

$$E_1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s+1) \zeta_M(s/2) \zeta_R(1 + s) \beta^{-s-1}, \quad \text{Re } c > d,$$

which can be looked upon as a continuous expansion in the temperature. The asymptotic behaviour is determined in an elegant fashion by pushing the inverse
Mellin contour around, usually to the left, and using the analytical properties of the \( \zeta \)–functions. For example, there is always the volume, or Weyl, pole in \( \zeta_M(s) \) at \( s = d/2 \) giving the Planck term. There is also a pole at \( s = -1 \) from the \( \Gamma \)–function which yields the contribution \(-\zeta_M(-1/2)/2\), assuming this quantity is finite, and this is minus the vacuum Casimir energy, \( E_0 \), \( [15,16,26] \).

For conformal scalars on the three–sphere, \( \zeta_M(s) = a^{2s} \zeta_R(2s-2) \) and the high temperature series is rapidly found (as it is in the other methods). Being exact, (18) can be used to investigate the the temperature inversion question somewhat explicitly. The quantity of interest is

\[
e_1(\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s+1) \zeta_R(s-2) \zeta_R(s+1) \left( \frac{\xi}{2\pi} \right)^{s+1}, \quad \text{Re} \; c > 3, \quad (20)
\]

the poles of the integrand at \( s = 3 \) and \( s = -1 \) corresponding, as mentioned, to the Planck and (minus) the Casimir term, \(-e_0\). The total energy is \( e(\xi) = e_0 + e_1(\xi) \), (16).

Simple manipulation with the functional equation for the Riemann \( \zeta \)–function quickly yields the equivalent form,

\[
e_1(\xi) = \xi^4 \frac{1}{2\pi i} \int_{2-c-i\infty}^{2-c+i\infty} ds \Gamma(s+1) \zeta_R(s-2) \zeta_R(s+1) \left( \frac{1}{2\pi \xi} \right)^{s+1}, \quad \text{Re} \; c > 3, \quad (21)
\]

and, translating the contours in (20) and (21) into coincidence, the inversion symmetry (15) follows immediately.

This method, which is also employed by Cardy [9] and by Kutasov and Larsen [2], is entirely equivalent to the one involving heat–kernels and theta functions. It has a certain general disadvantage exemplified by its application to the case when \( \mathcal{M} \) is a torus. The relevant \( \zeta \)–function, \( \zeta_M(s) \), is of Epstein form, and the standard functional equation is not adequate since in the step to (21), the two Riemann \( \zeta \)–functions are switched. The interchange of the thermal circle with one of the factors of the torus is brought out most clearly in the thermal \( \zeta \)–function method, as in (12). There is a relevant functional relation for the Epstein function based on Rosenhain’s generalisation of the Jacobi transformation for theta functions that allows one to invert part of the modulus, rather than all of it (which yields the standard functional relation) but this route seems unnecessarily complicated.
7. Higher spheres.

The $\zeta$–function on spheres is an old topic and well investigated. It is clear from the form of the standard degeneracies that, in order to obtain a symmetrical quantity, further differentiations with respect to $\xi$ must be performed, effectively turning the thermal circle into a thermal $d$–sphere. Alternatively, the $\zeta$–function, $\zeta_M$, can be expressed as a sum of Riemann $\zeta$–functions and the above analysis performed for each piece, producing a sum of terms with different inversion properties. This piecemeal approach, adopted by Kutasov and Larsen, [2] has practical uses.

8. Conclusion.

The implications of these results for the validity of the Verlinde–Cardy formula are unclear as the dependence on the scaling length when a zero mode is present appears to render the value of the entropy subject to some uncertainty. These issues will be discussed in a later communication.

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