Generalized solution of a mixed problem for linear hyperbolic system

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Abstract

In the first part of this article, we will prove an existence-uniqueness result for generalized solutions of a mixed problem for linear hyperbolic system in the Colombeau algebra. In the second part, we apply this result to a wave propagation problem in a discontinuous environment.

1 Introduction

In 1982, Colombeau introduced an algebra $G$ of generalized functions to deal with the multiplication problem for distributions, see Colombeau [1, 2]. This algebra $G$ is a differential algebra which contains the space $D'$ of distributions. Furthermore, nonlinear operations more general than the multiplication make sense in the algebra $G$. Therefore the algebra $G$ is a very convenient one to find and study solutions of nonlinear differential equations with singular data and coefficients.

Consider the mixed problem for the linear hyperbolic system in two variables

\[
\begin{aligned}
&\left( \partial_t + \Lambda (x,t) \partial_x \right) U = F(x,t)U + A(x,t) \quad (x,t) \in (\mathbb{R}_+^*)^2 \\
&U (x,0) = U_0 (x) \quad x \in \mathbb{R}_+ \\
&U_i (0,t) = \sum_{k=r+1}^n v_{ik} (t) U_k (0,t) + H_i (t) \quad i = 1, \ldots, r \quad t \geq 0
\end{aligned}
\]

(1)

where $\Lambda$, $F$ and $V$ are $(n \times n)$ matrices whose terms are discontinuous functions. The matrix $\Lambda$ is real and diagonal such that

\[\Lambda_1 > \Lambda_2 > \cdots > \Lambda_r > 0 > \Lambda_{r+1} > \cdots > \Lambda_n\]

In the case where $\Lambda \in L^\infty (\mathbb{R}_+^*)$ and $F \in W^{-1,\infty}_{\text{loc}} (\mathbb{R}_+^*)$, multiplicative products of distributions appear in system (1), and so there is no general way of giving a meaning to system (1) in the sense of distribution. This hyperbolic system even when it is in the form of a system of conservation laws does not admit any solutions distributions in general see [3]. Our approach is to study (1) in Colombeau’s algebra $G (\mathbb{R}_+^*)$, and under some hypotheses on $\Lambda$, $F$, $\nu$ and $H$, the system (1) admits an unique solution in $G (\mathbb{R}_+^*)$. This result completes work already made in the global case by M. Oberguggenberger [4].

The second part of this article, we will apply this result to the wave propagation problem in a discontinuous...
environment, the following system

\[
\begin{align*}
\left( \partial_t + c(x) \partial_x \right) u(x,t) &= 0 \quad (x,t) \in (\mathbb{R}_+)^2 \\
\left( \partial_t - c(x) \partial_x \right) v(x,t) &= 0 \quad (x,t) \in (\mathbb{R}_+)^2 \\
u(x,0) &= u_0(x) \quad x \geq 0 \\
v(x,0) &= v_0(x) \quad x \geq 0 \\
u(0,t) &= h(t) \quad v(0,t) + b(t) \quad t \geq 0
\end{align*}
\] (2)

with

\[c(x) = \begin{cases} c_R & \text{if } x > x_0 \\ c_L & \text{if } 0 < x < x_0 \end{cases}\]

c_R and c_L are real constants, u_0 and v_0 are continuous almost everywhere.

For this problem one can find a classical solution on \(\{0 \leq x < x_0 : t \geq 0\}\) and \(\{x > x_0 : t \geq 0\}\), so imposing a transmission condition in \(x = x_0\): the continuity of \(u\) and \(v\), one will have a classical solution on \(\{x \geq 0, t \geq 0\}\).

Further if \((u_0, v_0)\) are generalized functions, one can show that the problem (2) has a unique solution \((U,V) \in G(R^2_+) \times G(R^2_+)\), without having us need of the passage conditions, in the same way one shows that this solution admits an associated distribution that is equal to the classical solution by adjusting.

## 2 Existence and uniqueness

We recall some definitions from the theory of generalized functions which we need in the sequel.

We define the algebra \(G(R^m)\) as follows

\[A_q(R) = \\{ \chi \in D(R) : \int_R \chi(x) \, dx = 1 \text{ and } \int_R x^k \chi(x) \, dx = 0 \text{ for } 1 \leq k \leq q \}\]

and

\[A_q(R^m) = \\{ \phi(x_1, \ldots, x_m) = \prod_{j=1}^m (\chi(x_j)) \}\]

Let \(E[R^m]\) be the set of functions on \(A_1(R^m) \times C^\infty(R^m)\) with values in \(C^\infty\) which are \(C^\infty\) to second variable. Obviously \(E[R^m]\) with point wise multiplication is an algebra but \(C^\infty(R^m)\) is not a subalgebra.

Then given \(\phi \in A_1(R^m)\) and \(\varepsilon \in [0, 1]\), we define a function \(\phi_\varepsilon\) by

\[\phi_\varepsilon(x) = \varepsilon^{-m} \phi\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in R^m\]

An element of \(E[R^m]\) is called ”moderate” if for every compact subset \(K\) of \(R^m\) and every differential operator \(D = \partial_{x_1}^{k_1} \ldots, \partial_{x_m}^{k_m}\) there is \(N \in \mathbb{N}\) such that the following holds

\[
\begin{cases}
\forall \phi \in A_N(R^m), \exists C, \exists \eta > 0 \text{ such that} \\
\sup_{x \in K} |D u(\phi_\varepsilon, x)| \leq C \varepsilon^{-N} \\
\text{if } 0 < \varepsilon < \eta
\end{cases}
\]

\(E_M[R^m]\) denotes the subset of moderate elements where the index \(M\) stands for ”Moderate”. We define an ideal \(N'[R^m]\) of \(E_M[R^m]\) as follows:
\( u \in \mathcal{N} [\mathbb{R}^m] \) if for every compact subset \( K \) of \( \mathbb{R}^m \) and every differential operator \( D \), there is \( N \in \mathbb{N} \) such that:

\[
\forall q \geq N, \forall \varphi \in \mathcal{A}_q (\mathbb{R}^m), \exists C, \exists \eta > 0 \quad \text{such that} \quad \sup_{x \in K} |D u (\varphi_{\varepsilon}, x)| \leq C \varepsilon^{q-N} \quad \text{if } 0 < \varepsilon < \eta
\]

Finally the algebra \( \mathcal{G} (\mathbb{R}^m) \) is defined as the quotient of \( \mathcal{E}_M [\mathbb{R}^m] \) with respect to \( \mathcal{N} [\mathbb{R}^m] \).

In what follows, the elements of \( \mathcal{G} (\mathbb{R}^2) \) will be written with capital letters and their representatives in \( \mathcal{E}_M [\mathbb{R}^2] \) with small letters. Furthermore we use the following simplified notations:

\[
u (\varphi_{\varepsilon}, x) = u^\varepsilon (x)
\]

In our work we need a subset of \( \mathcal{E}_M [\mathbb{R}^2_+] \) that contains elements \( u \) satisfying the following properties:

(a) \( \exists N \in \mathbb{N} \) such that for all \( \varphi \in \mathcal{A}_N (\mathbb{R}^2_+) \)

\[
\exists c > 0 \quad \eta > 0 : \sup_{y \in \mathbb{R}^2_+} |u (\varphi_{\varepsilon}, y)| \leq c \quad \text{if } 0 < \varepsilon < \eta
\]

(b) For every compact subset \( K \) of \( \mathbb{R}^2_+ \), \( \exists N \in \mathbb{N} \) such that \( \forall \varphi \in \mathcal{A}_N (\mathbb{R}^2_+) \)

\[
\exists c > 0 \quad \exists \eta > 0 : \sup_{y \in K} |u (\varphi_{\varepsilon}, y)| \leq N \log \left( \frac{c}{\varepsilon} \right) \quad \text{if } 0 < \varepsilon < \eta
\]

**Definition 1** A generalized function \( U \in \mathcal{G} (\mathbb{R}^2_+) \) admitting a representative \( u \) with the property (a) (respectively (b)) is called globally bounded (respectively locally logarithmic growth).

**Definition 2** the system [1] satisfies the compatibility conditions in \( \mathcal{G} (\mathbb{R}^2_+) \) if there exist \( u^0_0, \lambda^\varepsilon, f^\varepsilon, h^\varepsilon \), \( v^\varepsilon \) et \( a^\varepsilon \) the representatives of \( U_0, \Lambda, F, H, V \) and \( A \) that satisfy to the classic conditions compatibility in order to have a \( C^\infty \) solution for the classic problem.

**Theorem 1** Let \( F, \Lambda \) and \( A \) be \( n \times n \) matrices with coefficients in \( \mathcal{G} (\mathbb{R}^2_+) \), suppose that: there exists \( r \) as:

\[
\Lambda_1 > \Lambda_2 > \cdots > \Lambda_r > 0 > \Lambda_{r+1} > \cdots > \Lambda_n
\]

\( \Lambda_i \) (\( i = 1, \ldots, n \)) are globally bounded, \( \partial_x \Lambda_i \) and \( F_i \) are locally logarithmic growth, so for an initial data \( U_0 \) in \( \mathcal{G} (\mathbb{R}^+ \right), \) \( V \) an element in \( \mathcal{G} (\mathbb{R}^+ \right) \) globally bounded and \( H_1 \) in \( \mathcal{G} (\mathbb{R}^+) \), then the problem [2] has an unique solution in \( \mathcal{G} (\mathbb{R}^2_+) \).

**Proof**: The proof of the theorem is an adaptation to the demonstration of the theorem 1.2 in [3], therefore one is going to give the big lines right.

Let \( \lambda \) a representative of \( \Lambda \) in \( \mathcal{G} (\mathbb{R}^2_+) \) such that

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0 > \lambda_{r+1} > \cdots > \lambda_n
\]

with \( \lambda_i \) satisfies the property (a) and \( \partial_x \lambda_i \) satisfies the property (b).

Let \( f \) and \( a \) are any representatives of \( F \) and \( A \) in \( \mathcal{G} (\mathbb{R}^2_+) \) with \( f \) satisfies (b).

\( v, h \) and \( u_0 \) are any representatives of \( V, H \) and \( U_0 \) in \( \mathcal{G} (\mathbb{R}^+) \) with \( v \) satisfies (a).
so Let’s consider the following problem

\[
\begin{aligned}
&
\left\{ \\
&\quad (\partial_t + \lambda_i^\varepsilon(x,t)\partial_x) u_i^\varepsilon = \sum_{k=1}^{n} f_{ik}^\varepsilon(x,t) u_k^\varepsilon(x,t) + a_i^\varepsilon(x,t) \quad (x,t) \in \mathbb{R}^2_+ \\
&\quad u_i^\varepsilon(x,0) = u_{i0}^\varepsilon(x) \\
&\quad u_i^\varepsilon(0,t) = \sum_{k=r+1}^{n} v_{ik}^\varepsilon(t) u_k^\varepsilon(0,t) + h_i^\varepsilon(t) \\
&\end{aligned}
\]

\[ (L) \]

if we denote \( \gamma_i^\varepsilon \) the corresponding characteristic curve to \( \lambda_i^\varepsilon \) then the problem \((L)\) admits an unique solution \( u_i^\varepsilon \), \( u_i^\varepsilon \in C^\infty(\mathbb{R}^2_+) \) given by

for \( i = r + 1, \ldots, n \)

\[
u_i^\varepsilon(x,t) = u_{i0}^\varepsilon(\gamma_i^\varepsilon(x,t,0)) + \int_0^t \left[ \sum_{k=1}^{n} f_{ik}^\varepsilon(\gamma_i^\varepsilon(x,t,\tau),\tau) u_k^\varepsilon(\gamma_i^\varepsilon(x,t,\tau),\tau) \right. \\
\left. + a_i^\varepsilon(\gamma_i^\varepsilon(x,t,\tau),\tau) \right] d\tau 
\]

for \( i = 1, \ldots, r \)

\[
u_i^\varepsilon(x,t) = \sum_{k=r+1}^{n} v_{ik}^\varepsilon(t) \int_0^t \sum_{s=1}^{n} \left[ f_{ks}^\varepsilon(\gamma_k^\varepsilon(0,t,\tau),\tau) u_k^\varepsilon(\gamma_k^\varepsilon(0,t,\tau),\tau) \right. \\
\left. + a_k^\varepsilon(\gamma_k^\varepsilon(0,t,\tau),\tau) \right] d\tau \\
+ \int_0^t \sum_{k=1}^{n} f_{ik}^\varepsilon(\gamma_i^\varepsilon(x,t,\tau),\tau) u_k^\varepsilon(\gamma_i^\varepsilon(x,t,\tau),\tau) d\tau \\
+ \int_0^t a_i^\varepsilon(\gamma_i^\varepsilon(x,t,\tau),\tau) d\tau \\
+ \sum_{k=r+1}^{n} v_{ik}^\varepsilon(t_0) \int_{t_0}^{t} a_k^\varepsilon(\gamma_k^\varepsilon(0,t,\tau),\tau) d\tau \\
+ \sum_{k=r+1}^{n} v_{ik}^\varepsilon(t_0) u_{i0}^\varepsilon(\gamma_k^\varepsilon(0,t,0)) + h_i^\varepsilon(t_0) 
\]

where \( t_0 \) is such that the curve \( \gamma_i \) cuts the axis \((0t)\) at a point \( P_i(0,t_0) \). \( u_i^\varepsilon \) is \( C^\infty \) function, so it remains to show therefore that \( u_i^\varepsilon \) is moderate growth.

from assumptions, we have

\[
\exists M > 0 \text{ such that } \left| \frac{d\gamma_i^\varepsilon(x,t,\tau)}{d\tau} \right| < M \quad \forall (x,t) \in \mathbb{R}_+^2 \quad \forall i = 1, \ldots, n \\
\exists M_1 > 0 \text{ such that } \max_{i,j} \left| v_{ij}^\varepsilon(y) \right| < M_1 \quad \forall y \in \mathbb{R}_+ 
\]

Let \( K_0 \) be a compact in \( \mathbb{R}_+ \), we draw the straight line with a slope \(-M\), the determination domain \( K_T \) of the solution \( u_i^\varepsilon \) does not depend on \( \varepsilon \). 

Lemma 1 Let $u^\varepsilon$ a solution of problem $(I_\varepsilon)$ then $u^\varepsilon_i$ verified

$$\sup_{(x,t) \in K_T} |u^\varepsilon_i(x,t)| \leq M_2 \left[ \sup_{k} \sup_{(x,t) \in K_T} |a^\varepsilon_k(x,t)| \cdot T + \right.$$

$$\left. \sup_{k} \sup_{x \in K_0} |u^\varepsilon_0(x)| + \sup_{k} \sup_{t \in [0,T]} |h^\varepsilon_k(t)| \right] \times$$

$$\exp \left( nM_2 \sup_{i,k} \sup_{(x,t) \in K_T} |f^\varepsilon_{ik}(x,t)| \cdot T \right)$$

with

$$M_2 = \max (nM_1, 1)$$

Proof:

for $i = 1, \ldots, r$, and from the integral equation that verified by $u^\varepsilon_i$ we have

$$\sup_{(x,t) \in K_T} |u^\varepsilon_i(x,t)| \leq M_2 \left[ T \sup_{(x,t) \in K_T} |a^\varepsilon_k(x,t)| + \sup_{k} \sup_{x \in K_0} |u^\varepsilon_0(x)| + $$

$$\sup_{k} \sup_{t \in [0,T]} |h^\varepsilon_k(t)| \right] +$$

$$nM_2 \int_0^T \sup_{(x,t) \in K_T} |f^\varepsilon(x,t)| \sup_{k} \sup_{(x,t) \in K_T} |u^\varepsilon_k(x,t)| \, d\tau$$

and the proof is completed by applying the Gronwall’s lemma to the function

$$s \rightarrow \max_{k} \sup_{(x,t) \in K_s} |u^\varepsilon_k(x,t)|$$

For $i = r + 1, \ldots, n$ it is the same way with $t_0 = 0, v = 0, h = 0$. 

□

the next of the proof of theorem 1, we have

$$\exists N_1 \in \mathbb{N} \text{ such that : } \forall \phi \in \mathcal{A}_{N_1}(\mathbb{R}_+)$$

$$\exists C_1 > 0 \ \exists \eta > 0 : \sup_{(x,t) \in K_T} |a^\varepsilon(x,t)| \leq C_1 \varepsilon^{-N_1} \text{ if } 0 < \varepsilon < \eta$$
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\[ \exists N_2 \in \mathbb{N} \text{ such that: } \forall \phi \in A_{N_2}(\mathbb{R}_+) \]

\[ \exists C_2 > 0, \exists \eta > 0 : \sup_{x \in K_0} |u_0(x)| \leq C_2 \varepsilon^{-N_2} \text{ if } 0 < \varepsilon < \eta \]

\[ \exists N_3 \in \mathbb{N} \text{ such that: } \forall \phi \in A_{N_3}(\mathbb{R}_+) \]

\[ \exists C_3 > 0, \exists \eta > 0 : \sup_{t \in [0,T]} |h^\varepsilon (t)| \leq C_3 \varepsilon^{-N_3} \text{ if } 0 < \varepsilon < \eta \]

\[ \exists N_4 \in \mathbb{N} \text{ such that: } \forall \phi \in A_{N_4}(\mathbb{R}_+) \]

\[ \exists C_4 > 0, \exists \eta > 0 : \sup_{(x,t) \in K_T} |f^\varepsilon (x,t)| \leq N_4 \log \left( \frac{C_4}{\varepsilon} \right) \text{ if } 0 < \varepsilon < \eta \]

therefore according to the lemma, we have

\[ \forall \phi \in A_{N_5}, \exists C > 0, \exists \eta > 0 : \sup_{(x,t) \in K_T} |u_\varepsilon^\phi (x,t)| \leq C_5 \varepsilon^{-N_5} \text{ if } 0 < \varepsilon < \eta \]

with

\[ N_5 = E (N_1 + N_2 + N_3 + NTC_4N_4) + 1 \]

for the other derivatives, differentiating the system (I_ε) for example with regard to x, one gets a system similar to the first. And because \( \partial_x \Lambda \) is locally logarithmic growth one gets the same estimation as before, \( \ldots \), then one has

\[ u_\varepsilon^\phi \in \mathcal{E}_M(\mathbb{R}_+) \quad i = 1, \ldots, n \]

either the existence of the solution for the problem (1) is in \( \mathcal{G}(\mathbb{R}_+) \).

Uniqueness

Let U, V two solutions in \( \mathcal{G}(\mathbb{R}_+) \) of the problem (I_ε), with the same initial data and the same boundary values. One must show that so \( u_\varepsilon \) is a representative of U and \( \mathcal{G}(\mathbb{R}_+) \) and if \( v_\varepsilon \) is a representative of V in \( \mathcal{G}(\mathbb{R}_+) \) then \( u_\varepsilon - v_\varepsilon \in \mathcal{A}(\mathbb{R}_+) \) see [2].

indeed : \( u_\varepsilon - v_\varepsilon \) verifies the same problem that previously and therefore the demonstration is the same. Then one has

\[ u_\varepsilon - v_\varepsilon = O (\varepsilon^q) \quad \forall q \]

Remark 1 To get the solution in the case where \( \Lambda \in L^\infty(\mathbb{R}_+) \), \( F \in W^{-1,\infty}(\mathbb{R}_+) \), one uses the following result. see [4, proposition 2]

Proposition 1

a) Let \( \omega \in W^{-1,\infty}_{loc}(\mathbb{R}_+) \) then there exist \( U \in \mathcal{G}(\mathbb{R}^2) \) such that: \( U \) is associated to \( \omega \) and \( U \) is locally logarithmic growth.

b) Let \( \omega \in L^\infty(\mathbb{R}^2) \) then there exist \( U \in \mathcal{G}(\mathbb{R}^2) \) such that: \( U \) is associated to \( \omega \) and \( U \) is globally bounded, and \( \partial^\alpha U \) is locally logarithmic growth. \( \alpha = (\alpha_1, \alpha_2) \quad \text{such that} \quad |\alpha| = \alpha_1 + \alpha_2 = 1 \)

Remark 2 For \( g \in L^\infty(\mathbb{R}_+) \) one can find \( G \in \mathcal{G}(\mathbb{R}_+) \) such that:

\[ G \approx g \]

and there exist a representative \( g^\varepsilon \) of \( G \) such that \( g^\varepsilon \) is nil at the neighborhood of 0 for all \( \varepsilon \).
Application

Consider the problem (2)

\[
\begin{cases}
\left(\partial_t + c(x)\partial_x\right)u(x,t) = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\
\left(\partial_t - c(x)\partial_x\right)v(x,t) = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\
u(x,0) = u_0(x) & x \geq 0 \\
v(x,0) = v_0(x) & x \geq 0 \\
u(0,t) = v(0,t) & t \geq 0
\end{cases}
\]

+ Compatibility conditions

with

\[c(x) = \begin{cases} 
  c_R & \text{if } x > x_0 \\
  c_L & \text{if } 0 < x < x_0
\end{cases}\]

For the initials data \(u_0, v_0\) continuous almost everywhere, and nil at neighborhood of 0. the problem (2) admits a classic solution for

\[\{0 < x < x_0 : t \geq 0\} \quad \text{and} \quad \{x > x_0 : t \geq 0\}\]

and while imposing a passage condition on the \(x_0\) (continuity of \(u\) and \(v\) at the point \(x_0\)) then one will have a solution on

\[\{x \geq 0 : t \geq 0\}\]

defined by

\[
v(x,t) = v_0(\gamma_2(x,t,0)) \\
u(x,t) = \begin{cases} 
  u_0(\gamma_1(x,t,0)) & \text{on (I)} \\
n(0,t) & \text{on (II)}
\end{cases}
\]

so one designates by \(\Gamma\) the characteristic curve comes from of \((0,0)\) the part (I) designates the set of \((x,t) \in \mathbb{R}_+^2\) below \(\Gamma\). and the part (II) the set the points \((x,t)\) over \(\Gamma\) (see the figure (2)).

\(\gamma_1\) the connected curve characteristic corresponding to \(c\).
\(\gamma_2\) the connected curve characteristic corresponding to \(-c\).

![Figure 2](image)

**Proposition 2** given \(u_0, v_0\) two continuous functions nearly everywhere, bounded and nil at the neighborhood of 0 then the problem (2) admit an unique solution \(U, V\) in \(G(\mathbb{R}_+^2)\) besides one has:

\[U \approx u \quad \text{et} \quad V \approx v\]

with \(u\) et \(v\) are the distributions solutions of the same problem obtained by imposing a passage condition.
Proof $c \in L^\infty(\mathbb{R}_+)$, from the proposition (1) there exists $C \in \mathcal{G}(\mathbb{R}_+)$ such that

$$C \approx c$$

$c$ is globally bounded and $\partial_x C$ is locally logarithmic growth. And so, from the theorem 1, there exists an unique solution $U, V$ in $\mathcal{G}(\mathbb{R}^2_+)$ of the problem (2).

To show that

$$U \approx u$$

we suppose that $(x, t)$ belongs to the region limited by the broken characteristic curve $\Gamma$ comes from the origin and the axis $(ox)$ which we note (region I).

If $(x, t)$ is over of this curve, the demonstration is identical but with reflection (region II) and for $(x, t) \in \Gamma$ (the characteristic curve comes from the origin) this set is negligible.

let $c^\varepsilon$ a representative of $C$ in $\mathcal{G}(\mathbb{R}_+)$

$u_0^\varepsilon$ a representative of $U_0$ in $\mathcal{G}(\mathbb{R}_+)$

$v_0^\varepsilon$ a representative of $V_0$ in $\mathcal{G}(\mathbb{R}_+)$

considering then the following problem

$$\left\{ \begin{array}{ll}
(\partial_t + c^\varepsilon \partial_x) u^\varepsilon = 0 & (x, t) \in (\mathbb{R}^*_+)^2 \\
(\partial_t - c^\varepsilon \partial_x) v^\varepsilon = 0 & (x, t) \in (\mathbb{R}^*_+)^2 \\
u^\varepsilon(x, 0) = u_0^\varepsilon(x) & x \in \mathbb{R}_+ \\
v^\varepsilon(x, 0) = v_0^\varepsilon(x) & x \in \mathbb{R}_+ \\
u^\varepsilon(0, t) = v^\varepsilon(0, t) & t \in \mathbb{R}_+
\end{array} \right.$$ 

This problem admits an unique solution $u^\varepsilon, v^\varepsilon$ in $C^\infty(\mathbb{R}^2_+)$. 

taking

$$\gamma_1^\varepsilon = \gamma_1 \ast \phi_{\eta^\varepsilon}$$

with $\phi \in \mathcal{D}(\mathbb{R}^+)$ such that

$$\int_{\mathbb{R}^+} \phi(\lambda)d\lambda = 1 \quad \text{supp } \phi_{\eta^\varepsilon} \subset [x_0 - \eta^\varepsilon, x_0 + \eta^\varepsilon] \quad \eta^\varepsilon = |\log \varepsilon|^{-1}$$

it is evident that for all $(x, t)$ in (region I)

$$u^\varepsilon(x, t) = u_0^\varepsilon(\gamma_1^\varepsilon(x, t, 0))$$

then to show that $U \approx u$ it is necessary and sufficient to show that : $\forall \psi \in \mathcal{D}(\mathbb{R}^2_+)$

$$\lim_{\varepsilon \to 0} \int_{\text{region I}} \left( u_0^\varepsilon(\gamma_1^\varepsilon(x, t, 0)) - u_0(\gamma_1(x, t, 0)) \right) \psi(x, t)dxdt = 0$$

we have

$$\int \left( u_0^\varepsilon(\gamma_1^\varepsilon(x, t, 0)) - u_0(\gamma_1(x, t, 0)) \right) \psi(x, t)dxdt =$$

$$\int \left( u_0^\varepsilon(\gamma_1^\varepsilon(x, t, 0)) - u_0(\gamma_1^\varepsilon(x, t, 0)) \right) \psi(x, t)dxdt$$

$$+ \int \left( u_0(\gamma_1^\varepsilon(x, t, 0)) - u_0(\gamma_1(x, t, 0)) \right) \psi(x, t)dxdt$$
but

\[
\int \left( u_0^\varepsilon (\gamma_1^\varepsilon (x, t, 0)) - u_0 (\gamma_1^\varepsilon (x, t, 0)) \right) \psi(x, t) dx dt
\]

\[
\leq \sup_{x \in \mathbb{R}^+} |u_0 * \phi_x - u_0| \int_{\mathbb{R}^+_e} \psi(x, t) dx dt
\]

so

\[
\lim_{\varepsilon \to 0} \int \left( u_0^\varepsilon (\gamma_1^\varepsilon (x, t, 0)) - u_0 (\gamma_1^\varepsilon (x, t, 0)) \right) \psi(x, t) dx dt = 0
\]

to show that

\[
\lim_{\varepsilon \to 0} \int \left( u_0 (\gamma_1^\varepsilon (x, t, 0)) - u_0 (\gamma_1 (x, t, 0)) \right) \psi(x, t) dx dt = 0
\]

it is sufficient to show that

\[
\lim_{\varepsilon \to 0} \left( \gamma_1^\varepsilon (x, t, 0) - \gamma_1 (x, t, 0) \right) = 0
\]

or \( c \) is globally bounded, then

\[
\exists M > 0 \sup_{x \in \mathbb{R}^+} |c^\varepsilon (x)| < M
\]

so we can to surround the curve \( \gamma_1^\varepsilon \) between two broken curves, (see the figure 3).

and taking the intersection of these two curves with the axis \((0x)\), it gives us two points

\[
x_1 = c_L \left( \frac{2 \eta \varepsilon}{M} - \frac{x_0 + \eta - x}{c_R} - t \right) - \eta + x_0
\]

\[
x_2 = -c_L \left( \frac{2 \eta \varepsilon}{M} + \frac{x_0 + \eta - x}{c_R} + t \right) - \eta + x_0
\]

such that

\[
x_1 \leq \gamma_1^\varepsilon (x, t, 0) \leq x_2
\]

Figure 3
hence
\[
\lim_{\varepsilon \to 0} \gamma_1^{\varepsilon}(x, t, 0) = -c_L t + \frac{c_L}{c_R}(x - x_0) + x_0
\]
\[
= \gamma_1(x, t, 0)
\]
then
\[
U \approx u
\]
for \( v \), the demonstration is the same.

References

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