Abstract. Recently Alexandrov, Banerjee, Manschot and Pioline constructed generalizations of Zwegers theta functions for lattices of signature \((n - 2, 2)\). Their functions, which depend on two pairs of time like vectors, are obtained by ‘completing’ a non-modular holomorphic generating series by means of a non-holomorphic theta type series involving generalized error functions. We show that their completed modular series arises as integrals of the 2-form valued theta functions, defined in old joint work of the author and John Millson, over a surface \(S\) determined by the pairs of time like vectors. This gives an alternative construction of such series and a conceptual basis for their modularity. The holomorphic generating series is interpreted as the series of intersection numbers of the surface \(S\) with complex divisors associated to positive lattice vectors.

1. Introduction

Recently Alexandrov, Banerjee, Manschot and Pioline [1] introduced and investigated certain generalized error functions and used them to construct theta series for indefinite lattices of signature \((n - 2, 2)\). Their result provides an analogue of the work of Zwegers [12] for the case of signature \((n - 1, 1)\). Roughly speaking, in each case, the choice of a suitable collection of negative (timelike) vectors allows one to cut out a certain cone of positive vectors and to define a convergent theta type series as the sum over lattice vectors in this cone. Unfortunately, these nice holomorphic \(q\)-series are not yet modular forms in the variable \(q = e^{2\pi i \tau}\). A main result of [12], for signature \((n - 1, 1)\), and of [1], for signature \((n - 2, 2)\), is that these \(q\)-series can be completed by the addition of a certain non-holomorphic series, obtained as a sum over all lattice vectors, so that the resulting function is a (non-holomorphic) modular form of weight \(n/2\) in \(\tau\). The completion in Zwegers involves the classical error function \(E_1(u)\), while the completion in [1] depends on the new generalized error function \(E_2(\alpha; u_1, u_2)\) introduced there.

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Following our habit, we have taken signature \((n - 2, 2)\) in place of signature \((2, n - 2)\) used in [1].
On the other hand, as a very special case of the results of old joint work with John Millson [5,6] and [7], there are theta forms \( \theta(\tau, \varphi_{KM}^{(n-r,r)}) \) associated to a lattice \( L \) of signature \( (n-r, r) \). These non-holomorphic series take values in closed \( r \)-forms on the associated symmetric space, are invariant under a finite index subgroup of the isometry group of the lattice, and have a modular transformation law of weight \( n/2 \) in \( \tau \). This transformation law follows from the classical Poisson summation argument due to Siegel [10], and hence arises in a natural way. The image \([\theta(\tau, \varphi_{KM}^{(n-r,r)})]\) of \( \theta(\tau, \varphi_{KM}^{(n-r,r)}) \) in cohomology inherits the modular transformation law of the theta form and is often a holomorphic series [7].

In the case of a lattice \( L \) in an indefinite inner product space \( V \) of signature \((n-1, 1)\), Zwegers’ theta series depends on the choice of a pair of negative vectors \( \{C, C'\} \) lying in the same component of the negative cone in \( V \). The pair \( \{C, C'\} \) determines a geodesic \( \gamma_{C,C'} \) joining the points \( C||C||^{-1} \) and \( C'||C'||^{-1} \) on the hyperboloid of vectors \( v \) with \((v, v) = -1\) in that component. Then, up to sign, Zwegers’ completed theta series coincides with the integral of the theta form over this geodesic [4,8],

\[
\theta_{\text{Zwegers}}(\tau; \{C, C'\}) = -\int_{\gamma(C,C')} \theta(\tau, \varphi_{KM}^{(n-1,1)}).
\]

In the present note, we show that the non-holomorphic theta series constructed in [1] for lattices \( L \) of signature \((n-2, 2)\) can also be constructed as an integral of the theta form \( \theta(\tau, \varphi_{KM}^{(n-2,2)}) \).

Let \( D \) be the space of oriented negative 2-planes in \( V = L \otimes_{\mathbb{Z}} \mathbb{R} \), so that \( \theta(\tau, \varphi_{KM}^{(n-2,2)}) \) is a closed 2-form on \( D \). The main step is to define a certain oriented 2-cycle \( S \) in \( D \) over which to integrate this form. The basic data consists of two pairs \( \{C_1, C_2\} \) and \( \{C'_1, C'_2\} \) of negative vectors satisfying certain incidence conditions, \((3.1), (3.2), (3.3), \) and \((3.12)\) in Sect. 3. The relation of these conditions to those introduced in [1] is discussed in Remark 1.1 (ii). For convenience we write \( C \) for the collection \( \{\{C_1, C'_1\}, \{C_2, C'_2\}\} \). Due to our incidence conditions, there are 4 oriented negative 2-planes, i.e., 4 points in \( D \), given by

\[
\begin{align*}
z_{12} &= \text{span}\{C_1, C_2\}_{\text{p.o.}}, \\
z_{12'} &= \text{span}\{C_1, C'_2\}_{\text{p.o.}}, \\
z_{1'2} &= \text{span}\{C'_1, C_2\}_{\text{p.o.}}, \\
z_{1'2'} &= \text{span}\{C'_1, C'_2\}_{\text{p.o.}},
\end{align*}
\]

lying on one component, say \( D^+ \) of \( D \). Here the subscript ‘p.o.’ indicates that the given ordered pair of vectors defines the orientation. Any negative vector \( C \) defines a copy of hyperbolic space of dimension \( n-2 \) in \( D \) consisting of the oriented negative 2-planes containing \( C \). If we write \( H_1, H_2, H'_1 \) and \( H'_2 \) for the hyperbolic subspaces defined by \( C_1, C_2, C'_1 \) and \( C'_2 \), then these subspace intersect in precisely 8 points, 4 on each component of \( D \). The intersection points on \( D^+ \) are precisely the points \((1.1)\), for example, \( z_{12} = H_1 \cap H_2 \cap D^+ \), etc. The points \( z_{12} \) and \( z_{12'} \) lie on \( H_1 \) and are joined by a unique geodesic \( \gamma_1 \) in \( H_1 \). This and the analogously defined geodesics \( \gamma_2', \gamma_2 \) and \( \gamma_2'' \) form a quadrilateral loop in \( D \). Let \( S(C) \) be the geodesic surface in \( D \) filling in this quadrilateral.
Recall that the theta form is defined as follows. For \( x \in V \), let
\[
\varphi_{KM}^0(x) = 2(\omega_1(x) \wedge \omega_2(x) - \frac{1}{4\pi} \Omega) e^{-2\pi R(x, z)},
\]
and, for \( \tau = u + iv \) in the upper half-plane, let
\[
\varphi_{KM}(\tau, x) = q^{\frac{1}{2}(x, x)} \varphi_{KM}^0(x) e^{i\tau},
\]
where we write \( \varphi_{KM} \) in place of \( \varphi_{KM}^{(n-2,2)} \), since the signature is fixed from now on. Here \( R(x, z) = -(pr_z(x), pr_z(x)) \), where \( pr_z(x) \) is the orthogonal projection of \( x \) to the negative 2-plane \( z, \omega_1(x) \) and \( \omega_2(x) \) are 1-forms on \( D \) determined by \( x \), and \( \Omega \) is an invariant 2 form on \( D \), independent of \( x \). Let \( L^\vee \supset L \) be the dual lattice of \( L \). For \( \mu \in L^\vee / L \), the theta form is the closed 2-form given by
\[
\theta_{\mu}(\tau; \varphi_{KM}) = \sum_{x \in L+\mu} \varphi_{KM}(\tau, x).
\]

Since \( S(C) \) is compact, we can compute the integral of \( \theta_{\mu}(\tau; \varphi_{KM}) \) over \( S(C) \) termwise and so the essential result is the evaluation of the integral of \( \varphi_{KM}^0(x) \) over \( S(C) \).

In fact, it is natural to introduce the following slight generalization. Suppose that \( C = \{\{C_1, C_1'\}, \{C_2, C_2'\}\} \) is a collection of negative vectors. For \( s \) and \( t \in \mathbb{R} \), let
\[
B_1(s) = (1 - s)C_1 + sC_1', \quad B_2(t) = (1 - t)C_2 + tC_2'.
\]
We say that \( C \) is in good position if these vectors span a properly oriented negative 2-plane span\{\(B_1(s), B_2(t)\)\}_{p.o.} for all \( s \) and \( t \in [0, 1] \). The incidence conditions (3.1), (3.2), (3.3) imply that \( C \) is in good position, but the notion of good position is more general and allows even the most degenerate case where \( C_1 = C_1', C_2 = C_2' \) and \( \{C_1, C_2\}_{p.o.} \) is a negative 2-plane. If \( C \) is in good position, we can define a singular 2-cycle
\[
\phi_C : [0, 1]^2 \to D, \quad [s, t] \mapsto \text{span}\{B_1(s), B_2(t)\}_{p.o.}
\]
and the integral
\[
I(x, C) = \int_{S(C)} \varphi_{KM}^0(x) := \int_{[0, 1]^2} \phi_C^*(\varphi_{KM}^0(x))
\]
is well defined. Of course, this integral vanishes in the most degenerate case.

**Theorem A.** Suppose that \( C = \{\{C_1, C_2\}, \{C_1', C_2'\}\} \) is in good position. (a) For any \( x \in V \),
\[
I(x/\sqrt{2}; C) = -\frac{1}{4} \left( E_2(C_1, C_2; x) - E_2(C_1, C_2'; x) - E_2(C_1', C_2; x) + E_2(C_1', C_2'; x) \right),
\]
where \( E_2(C, C'; x) \) is the ‘boosted’ generalized error function defined in (3.38) of [1]. (b) In particular,
\[ I(0; C) = -\frac{1}{2\pi} \arctan \left( \frac{(C_1, C_2)}{\sqrt{\Delta_{12}}} \right) + \frac{1}{2\pi} \arctan \left( \frac{(C_1, C_2')}{\sqrt{\Delta_{12}'}} \right) \]
\[ + \frac{1}{2\pi} \arctan \left( \frac{(C_1', C_2)}{\sqrt{\Delta_{12}}} \right) - \frac{1}{2\pi} \arctan \left( \frac{(C_1', C_2')}{\sqrt{\Delta_{12}'}} \right). \]

Here
\[ \Delta(C, C') = (C, C)(C', C') - (C, C')^2, \]
and \( \Delta_{12} = \Delta(C_1, C_2), \) etc.

This result is first proved in the ‘generic’ case, where \( C \) satisfies the incidence conditions (3.1), (3.2), (3.3) and (3.12) and where \((x, C) \neq 0\) for all \( C \in \{C_1, C_2, C_1', C_2'\},\) by an application of Stokes’ theorem. Recall that for any nonzero \( x \in V, \) there is a submanifold
\[ D_x = \{ z \in D \mid x \in z^\perp \} \]
of \( D, \) which is the zero locus of the function \( R(x, \cdot) \) and is empty unless \((x, x) > 0.\) If \((x, x) > 0,\) then \( D_x \) has codimension 2. Indeed, in the hermitian model, it is either empty or a complex divisor on \( D. \) On the set \( D - D_x, \) we define a 1-form \( \psi(x) \) with
\[ d\psi(x) = \varphi_{KM}(x). \]

As in [1], let
\[ \Phi_2(x, C) = \frac{1}{4} [ \text{sgn}(x, C_1) - \text{sgn}(x, C_1')] [\text{sgn}(x, C_2) - \text{sgn}(x, C_2')], \quad (1.6) \]
where we take \( \text{sgn}(0) = 0.\) Then, for \( x \) generic, \( D_x \cap S(C) \) is non-empty if and only if \( \Phi_2(x, C) \neq 0! \) To apply Stokes’ theorem, we must cut out an \( \epsilon \)-disk around \( D_x \cap S(C). \) It turns out that
\[ \int_{\partial S(C)} \psi(x) \]
is easy to evaluate and leads immediately to the function
\[ f(a, b) := -\frac{1}{2\pi} b e^{-2\pi b^2} \int_0^a \frac{e^{-2\pi t^2}}{b^2 + t^2} dt. \]
But this function is given, in turn, as
\[ f(a, b) = -\frac{1}{4} \hat{e}_2(a \sqrt{2}, b \sqrt{2}), \]
where \( \hat{e}_2 \) is one of the building blocks in [1] in the generalized error function. Now using the identity (4.7) relating \( \hat{e}_2 \) and \( E_2, \) we obtain the expression in (i) of Theorem A together with an extra term \( \Phi_2(x, C), \) cf. Corollary 4.9. This extra term is cancelled by the contribution of the excised \( \epsilon \)-disk! The case of non-generic \( x \) is obtained by continuity, where we note that the singularities arising from \( \psi(x) \) were not present in the original integral, which depends smoothly on \( x \) and \( C. \) Then a second application of continuity yields the case of any \( C \) in good position.

Our main global result is then the following.
**Theorem B.** Suppose that \( C \) is in good position.

(a) The series
\[
I_\mu(\tau; C) := \int_{S(C)} \theta_\mu(\tau; \varphi_{KM}) = \sum_{x \in L + \mu} I(\sqrt{v} x; C) q^{\frac{1}{2}(x, x)}
\]
(1.7)
is a non-holomorphic modular form of weight \( n/2 \) with the same transformation law as \( \theta_\mu(\tau, \varphi_{KM}) \).

(b) For \( \mu \in L^\vee/L \), the q-series
\[
\sum_{x \in L + \mu} \Phi_2(x, C) q^{\frac{1}{2}(x, x)}
\]
(1.8)
is termwise absolutely convergent.

(c) The modular form \( I_\mu(\tau; C) \) is its modular completion\(^2\), as defined in [1]. In particular,
\[
I_\mu(\tau; C) = -\theta_{ABMP}(\tau; C_1, C_2, C_1', C_2'; \mu),
\]
(1.9)
where \( \theta_{ABMP}(\tau; C_1, C_2, C_1', C_2'; \mu) \) is the series defined in [1].

**Remark 1.1.** (i) Since the theta form is closed, the integral on the right side of (1.7) does not depend on the choice of the oriented surface \( S \) with boundary \( \gamma_1 + \gamma_2' - \gamma_1' - \gamma_2 \).

(ii) Our incidence conditions (3.1), (3.2), and (3.12) are a subset of those imposed in [1], (4.2) and (4.6), and, together with (3.3), they are the natural conditions giving rise to the 4 points \( \{z_{12}, z_{1'2}, z_{12'}, z_{1'2'}\} \) on \( D^+ \) and the boundary frame of geodesics connecting them. It is a pleasant bonus that our conditions are sufficient to ensure the convergence of the q-series (1.8). Their relation to the set of conditions [1] (4.2) alone is not clear. Indeed even the weaker condition of being in good position suffices to prove convergence.

(iii) Of course, part (c) is just a restatement of the result of [1] relating the expression \( \Phi_2(x, C) \) and the quantity \( I(x; S(C)) \).

(iv) Note that, in passing from Theorem A to the theta series in Theorem B, the at first sight anomalous \( \sqrt{2} \) is precisely what is introduced by the \( \sqrt{2} \tau_2 \) in the argument of \( \Phi \) in the definition of the theta series in (2.1) of [1].

Our construction also provides the following geometric interpretation of the quantity \( \Phi_2(x, C) \).

**Proposition 1.2.** Suppose that \( C \) is in good position and that \( x \in V \) with \( \Phi_2(x, C) \neq 0 \). Then \( D_x \cap S(C) \) consists of a single point \( \phi_C(s_0, t_0) \), \( \phi_C \) is an immersion at \([s_0, t_0] \in [0, 1]^2 \), and
\[
\Phi_2(x, C) = I(S(C), D_x)
\]
where \( I(S(C), D_x) \) is the (local) intersection number at this point of the (oriented) cycles \( S(C) \) and \( D_x \) in \( D \).

\(^2\) We assume that \( L \) is even integral, so that the characteristic vector \( p \) used in (2.1) of [1] can be taken to be 0 and hence does not appear in our theta series. This is only done to simplify the notation; our results hold in the general case.
Corollary 1.3. The holomorphic part (1.8) of the indefinite theta function $I_{\mu}(\tau; S(\mathcal{C}))$ can be written as

$$
\sum_{x \in L+\mu} I(S(C), D_x) q^{\frac{1}{2}(x,x)}.
$$

As is evident from this description, our result technically depends only on the special functions results of Sect. 3 of [1], i.e., in our approach, the completed theta series is constructed directly as an integral of the theta form and the results about generalized error functions, in particular (4.7), etc., then identify it as a completion of the $q$-series (1.8). But of course, the whole business depends entirely on the original ideas of [1] about constructing a signature $(n-2, 2)$ analogue of Zwegers’ theta series, the ingenious choice of the correct input data $\mathcal{C} = \{\{C_1, C_2\}, \{C_1', C_2'\}\}$, the function $\Phi_2(x, \mathcal{C})$, etc. Aside from the connection with the theta form, our main addition is the introduction of the surface $S(\mathcal{C})$ over which we take our integral and the interpretation of $\Phi_2(x, \mathcal{C})$ as an intersection number. Hopefully, these ideas will prove useful in the construction of examples for more general signatures proposed in section 6 of [1]. The theta forms $\theta(\tau, \Phi_{KM}^{(n-r,r)})$ are available in this situation and can undoubtedly be used in the same way, cf. the discussion in Sect. 5.2. Moreover, it is clear that the theta forms can be integrated over various compact and, with suitable care about convergence, non-compact regions in $D$. Results of [1] suggest that in certain cases such integrals can be connected to Appell–Lerch sums, so much work remains to be done.

Here is a brief description of the contents of the paper. In Sect. 2, we review the construction of the 2-form $\varphi_{KM}(x)$ and show that, for $x \neq 0$ and away from the set $D_x$, there is an explicitly constructed 1-form $\psi(x)$ with $d\psi(x) = \varphi_{KM}(x)$. In Sect. 3, we describe how the data $\{C_1, C_2\}, \{C_1', C_2'\}$ satisfying the incidence conditions of [1] gives rise to a quadrilateral of geodesics and a spanning surface $S$. In Sect. 4, we compute the basic integral of $\varphi_{KM}(x)$ over the surface $S$. This is done using Stokes’ theorem and the 1-form $\psi(x)$, where the singularity along $D_x$ necessitates a limit procedure when $D_x \cap S$ is non-empty. Also in this section, we prove Proposition 1.2. In Sect. 4.4, we take care of the cases where the regularity condition fails. In Sect. 5.1, we discuss how the computation of the shadows i.e., the image of $I_{\mu}(\tau; \mathcal{C})$ under the lowering operator, can be done in our language, and in Sect. 5.2 we sketch how things should go for other signatures. Here the surface $S(\mathcal{C})$ is generalized to a (singular) hypercube, and the convergence of the associated $q$-series is proved. Convenient coordinates for our computation are explained in an Appendix, Sect. 6.

Since the first version of this paper was written, related work by several authors has appeared. Westerholt-Raum [11] described a construction of indefinite theta series using Gaussian convolutions of piecewise linear functions. Following the method suggested in section 6 of [1], Nazaroglu [9] has extended their construction to the case of general signature. The second version of [1] mentions unpublished work of S. Zwegers on indefinite theta series for signature $(2, n-2)$ and forthcoming work of Zwegers and Zagier on the case of general signature.

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1.1. Notation

Following the conventions of [1], we write

$$\Delta(C, C') = (C, C')(C', C') - (C, C')^2 = \det \begin{pmatrix} (C, C) & (C, C') \\ (C', C) & (C', C') \end{pmatrix},$$  \hspace{1cm} (1.10)

$$\Delta_{12} = \Delta(C_1, C_2),$$ etc. More generally, for any subset $I$ of $\{1, 2, 1', 2'\}$, $\Delta_I$ is the determinant of the matrix of inner products $((C_i, C_j))_{i,j \in I}$. In addition, write

$$C_{1\perp 2} = C_1 - \frac{(C_1, C_2)}{(C_2, C_2)} C_2,$$ \hspace{1cm} (1.11)

Note that

$$(C_2, C_2)(C_{i\perp 2}, C_{j\perp 2}) = (C_2, C_2)(C_i, C_j) - (C_i, C_2)(C_j, C_2) =: \Delta_{ij\perp 2}. $$ \hspace{1cm} (1.12)

For a vector $C$ with $(C, C) \neq 0$, we write

$$||C|| = |(C, C)|^{\frac{1}{2}}, \quad C = \frac{C}{||C||}.$$ 

We use the convention $\text{sgn}(0) = 0$ to extend the sign function $\text{sgn}$ from $\mathbb{R}^\times$ to $\mathbb{R}$.

For collections $x = [x_1, \ldots, x_r] \in V^r$ and $y = [y_1, \ldots, y_s] \in V^s$, we let

$$(x, y) = ((x_i, y_j)) \in M_{r,s}(\mathbb{R}).$$

Note that for $a \in \text{GL}_r(\mathbb{R})$ and $b \in \text{GL}_s(\mathbb{R})$,

$$(xa, yb) = 'a(x, y)b.$$ 

2. The basic setup

Let $V$, $(\cdot, \cdot)$ be an indefinite inner product space of signature $(n-2, 2)$ and let $D$ be the space of oriented negative 2-planes in $V$. Let

$$\text{OFD} = \{ \zeta = [\zeta_1, \zeta_2] \in V^2 \mid (\zeta, \zeta) = -1_2 \}$$

be the space of oriented orthonormal frames and let $\pi : \text{OFD} \rightarrow D$ be the projection sending an oriented frame to the oriented 2-plane it spans.

For $z \in D$ with properly oriented orthonormal basis $\zeta = [\zeta_1, \zeta_2]$, the orthogonal projection of $x \in V$ to $z$ is given by

$$\text{pr}_z(x) = \zeta (\zeta, \zeta)^{-1} (\zeta, x).$$
Then
\[ R(x, z) = -(\text{pr}_z(x), \text{pr}_z(x)) = (x, \xi)(\xi, x), \]
and the majorant is
\[ (x, x)_z = (x, x) + 2R(x, z). \]
Note that \( z \in D_x \), i.e., \( z \) is in \( x^\perp \) if and only if \( R(x, z) = 0 \).

2.1. The Schwartz form

We recall the definition and basic properties of the Schwartz form constructed in [5, 6].

Define a Schwartz function on \( V \) valued in 2-forms on \( D \) as follows. For \( \eta \) and \( \mu \in T_z(D) = \text{Hom}(z, z^\perp) \), we choose \( \zeta \in \text{OFD} \) with \( \pi(\zeta) = z \) and write
\[ [\eta_1, \eta_2] = [\eta(\xi_1), \eta(\xi_2)], \quad [\mu_1, \mu_2] = [\mu(\xi_1), \mu(\xi_2)] \in U(z)^2. \]
Here, for convenience, we write \( z^\perp = U(z) \). Then let
\[ \omega_1(x) \wedge \omega_2(x)(\eta, \mu) = (x, \eta_1)(x, \mu_2) - (x, \eta_2)(x, \mu_1) = \det \begin{pmatrix} (x, \eta_1) & (x, \eta_2) \\ (x, \mu_1) & (x, \mu_2) \end{pmatrix}, \]
and
\[ \Omega(\eta, \mu) = (\eta_1, \mu_2) - (\eta_2, \mu_1). \]

These expressions are independent of the choice of \( \zeta \in \text{OFD} \) and define 2-forms on \( D \). The basic Schwartz form\(^3\)
\[ \varphi_{KM}(x, z) = 2 \left( \omega_1(x) \wedge \omega_2(x) - \frac{1}{4\pi} \Omega \right) \exp(-\pi(x, x)_z), \]
where \((x, x)_z\) is the majorant, is a closed 2-form on \( D \). This is a very special case of the forms defined in [5] and [6]. Note that
\[ \varphi_{KM}(0) = -\frac{1}{2\pi} \Omega. \]
For convenience, since \( x \) will often be fixed in our discussion, we write
\[ \varphi_{KM}(x) = e^{-\pi(x, x)} \varphi_{KM}^0(x). \]

\(^3\) Note that we have included an extra factor of 2 here compared with the expression given in Proposition 5.1 of [5]. This is done to compensate for the factor \( 2^{-q/2} \) that occurs in the fiber integral in Proposition 6.2 of [6] and hence to correctly normalize the Poincaré dual form.
Note that the pullback of \( \varphi_{KM}^o(x) \) under \( \pi : \text{OFD} \to D \) is

\[
\pi^*(\varphi_{KM}^o(x)) = 2 \left[ (x, d\xi_1) \wedge (x, d\xi_2) - \frac{1}{4\pi} (d\xi_1, d\xi_2) - \frac{1}{2} dR \wedge (\zeta_2, d\xi_1) \right] e^{-2\pi R}.
\]

On the set \( D - Dx \) define a 1-form \( \psi(x) \) whose pullback to \( \text{OFD} \) is given by

\[
\pi^*(\psi(x)) = -\frac{1}{2\pi} e^{-2\pi R} \left( R^{-1}((x, \xi_1)(x, d\xi_2) - (x, \xi_2)(x, d\xi_1)) - (\zeta_2, d\xi_1) \right).
\]

The proof of the following proposition is given in Appendix 6.2.

**Proposition 2.1.** On the set \( D - Dx \),

\[
\varphi_{KM}(x) = d\psi(x) e^{-\pi(x,x)}.
\]

### 3. The spanning surface of a frame

In this section, we explain how to obtain an oriented 2-cycle in \( D \) from the basic data introduced in [1] consisting of two pairs of negative\(^4\) vectors \( \{C_1, C_2\} \) and \( \{C'_1, C'_2\} \) subject to a certain collection of incidence conditions. We explain these conditions step by step, observing how the geometry we want emerges.

With the notation of [1], as explained in Sect. 1.1, we impose the following incidence conditions\(^5\)

\[
\Delta_{12}, \Delta_{1'2'}, \Delta_{1'2'} > 0, \quad (3.1)
\]
\[
\Delta_{22'\perp 1}, \Delta_{22'\perp 1'}, \Delta_{11'\perp 2}, \Delta_{11'\perp 2'} > 0, \quad (3.2)
\]
\[
2(C_1, C_1')(C_2, C_2') - (C_1, C_2)(C_1', C_2') - (C_1, C_2')(C_1', C_2) \geq 0. \quad (3.3)
\]

Here recall that, for example,

\[
\Delta_{12} = (C_1, C_1)(C_2, C_2) - (C_1, C_2)^2,
\]

and

\[
\Delta_{22'\perp 1} = (C_1, C_1)(C_2, C_2') - (C_1, C_2)(C_1', C_2') - (C_1, C_1')(C_2, C_2'), \quad (C_1', C_1)(C_2, C_2') = (C_1, C_1')(C_2, C_2'),
\]

\(^4\) Note that our quadratic form is the negative of theirs, so for us, the term ‘timelike’ refers to vectors \( C \) with \( Q(C) = (C, C) < 0 \).

\(^5\) Notice that replacing signature \((2, n - 2)\) with signature \((n - 2, 2)\) amounts to changing the sign of the quadratic form. The quantities \( \Delta(C, C') \) are invariant under this change, but the quantities \( (C, C') \) change sign. This gives the equivalence of our conditions (3.1) and (3.2) with the conditions (4.6) in [1]. Condition (3.3) was pointed out to me by Jens Funke.
etc. Note that condition (3.2) is equivalent to the condition:

\[(C_{2\perp 1}, C_{2'\perp 1}), (C_{2\perp 1}', C_{2'\perp 1}'), (C_{1\perp 2}, C_{1'\perp 2}), (C_{1\perp 2}', C_{1'\perp 2}') < 0. \quad (3.4)\]

We next explain the meaning of these conditions. Condition (3.1) is equivalent to the requirement at the pairs \(\{ C_1, C_2 \}, \{ C_1', C_2 \}, \{ C_1, C_2' \} \) and \( \{ C_1', C_2' \} \) span negative 2-planes

\[P_{12}, P_{1'2}, P_{12'}, P_{1'2'}, \quad (3.5)\]

where, for the moment we ignore the orientations.

For any negative vector \( C \in V \), we let

\[H_C = \{ z \in D \mid C \in z \}, \]

and \( V_C = C_\perp \). Then \( V_C \) has signature \((n - 2, 1)\) and we have an isomorphism

\[
\{ v \in V_C \mid (v, v) = -1 \} \xrightarrow{\sim} H_C, \quad v \mapsto \text{span}\{C, v\}_{\text{p.o.}}.
\]

Here the subscript ‘p.o.’ indicates that the given ordered frame is properly oriented. In particular, \( H_C \) is a copy of hyperbolic \( n \)-space in \( D \). Note that \( H_C \) has two components and that vectors \( v \) and \( v' \) in \( V_C \) with \((v, v) = -1 = (v', v')\) lie in the same component precisely when \((v, v') < 0\).

From the given set of negative vectors \( \{ C_1, C_2, C_{1'}, C_{2'} \} \) we obtain hyperbolic subspaces \( H_1, H_{1'}, H_2 \) and \( H_{2'} \) in \( D \), where we write \( H_1 = H_{C_1}, V_1 = V_{C_1} \), etc. Assuming condition (3.1), the intersections \( H_1 \cap H_2, H_1 \cap H_{2'}, H_{1'} \cap H_2 \), and \( H_{1'} \cap H_{2'} \) of these hyperbolic subspaces are easily determined. For example,

\[H_1 \cap H_2 = \{ \text{span}\{C_1, C_2\}_{\text{p.o.}}, \text{span}\{C_1, -C_2\}_{\text{p.o.}} \}. \]

Since \( \text{span}\{C_1, C_2\} = \text{span}\{C_1, C_{2\perp 1}\} \) is a negative 2-plane, it follows that \( C_{2\perp 1} \) is a negative vector in \( V_1 \) as is \( C_{2'\perp 1} \), by the same argument. Since \( V_1 \) has signature \((n - 2, 1)\) the inner product\(^6\) \( (C_{2\perp 1}, C_{2'\perp 1}) \neq 0 \), and condition (3.2), or, equivalently (3.4), implies that the vectors \( C_{2\perp 1} \) and \( C_{2'\perp 1} \) lie in the same component of the negative cone in \( V_1 \). The same discussion applies to the other pairs of projections, \( C_{1\perp 2}, C_{1'\perp 2} \), etc.

Since \( C_{2\perp 1} \) and \( C_{2'\perp 1} \) lie in the same component of the negative cone in \( V_1 \), there is a unique geodesic in \( H_1 \) joining the points \( z_{12} \) and \( z_{12'} \) given by\(^7\)

\[\gamma_1(t) = \text{span}\{C_1, (1 - t)C_2 + tC_2'\}_{\text{p.o.}} = \text{span}\{C_1, B_{2\perp 1}(t)\}_{\text{p.o.}}, \quad t \in [0, 1], \quad (3.7)\]

where

\[B_{2\perp 1}(t) = (1 - t)C_{2\perp 1} + tC_{2'\perp 1}, \quad \frac{B_{2\perp 1}(t)}{||B_{2\perp 1}(t)||}, \quad C_1 = \frac{C_1}{||C_1||}. \]

\(^6\) This is why we have replaced the \( \geq 0 \) in (4.6b)–(4.6e) with \( < 0 \) in (3.2)

\(^7\) We make no claim that \( t \) is the arc-length parameter.
Note that the second expression gives an orthonormal basis of \( \gamma_1(t) \) whose second component lies in \( V_1 \). In particular, \( \gamma_1 \) is the image in \( D \) via (3.6) of the geodesic joining the points \( C_{2\perp 1} \) and \( C_{2'\perp 1} \) in \( \{ v \in V_1 \mid (v, v) = -1 \} \).

There are analogous geodesics \( \gamma_1' \) from \( z_{1'2} \) to \( z_{1'2'} \), \( \gamma_2 \) from \( z_{12} \) to \( z_{1'2} \) and \( \gamma_{2'} \) from \( z_{1'2'} \) to \( z_{1'2'} \). Explicitly, these are given by

\[
\gamma_1'(t) = \text{span}\{C_{1'}, B_{2\perp 1'}(t)\}_\text{p.o.}, \quad t \in [0, 1],
\]
\[
\gamma_2(s) = \text{span}\{B_{1\perp 2}(s), C_2\}_\text{p.o.}, \quad s \in [0, 1],
\]
\[
\gamma_{2'}(s) = \text{span}\{B_{1\perp 2'}(s), C_{2'}\}_\text{p.o.}, \quad s \in [0, 1],
\]

where

\[
B_{2\perp 1'}(t) = (1 - t)C_{2\perp 1'} + tC_{2'\perp 1'},
\]
\[
B_{1\perp 2}(s) = (1 - s)C_{1\perp 2} + sC_{1'\perp 2}
\]
\[
B_{1\perp 2'}(s) = (1 - s)C_{1\perp 2'} + sC_{1'\perp 2'}.
\]

Altogether, they form a closed loop in \( D \). Note that the first component is fixed for \( \gamma_1 \) and \( \gamma_1' \) and the second component is fixed for \( \gamma_2 \) and \( \gamma_{2'} \). This will result in a sign change later in the calculation.

We want to fill this loop with an oriented 2-cycle, and this is where the condition (3.3) comes into play. As in (1.3), let

\[
B_1(s) = (1 - s)C_1 + sC_{1'}, \quad B_2(t) = (1 - t)C_2 + tC_{2'}.
\] (3.8)

**Lemma 3.1.** If a collection \( \mathcal{C} = \{ \{ C_1, C_{1'} \}, \{ C_2, C_{2'} \}\} \) of negative vectors satisfies conditions (3.1), (3.2) and (3.3), then

\[
\text{span}\{B_1(s), B_2(t)\}_\text{p.o.} \in D
\]

for all \((s, t) \in [0, 1]^2\).

**Proof.** Note that, by a short computation, \( (C_2, C_2)(B_1(s), B_1(s)) = (1 - s)^2 \Delta_{12} + 2s(1 - s)\Delta_{1'\perp 2} + s^2\Delta_{1'2} \)

\[
+ \left( (1 - s)(C_1, C_2) + s(C_{1'}, C_{2'}) \right)^2 > 0,
\] (3.9)

so that \( B_1(s) \) is a negative vector for all \( s \in [0, 1] \). Thus, we need to show that, under the given conditions,

\[
\Delta(s, t) := \det \begin{pmatrix}
(B_1(s), B_1(s)) & (B_1(s), B_2(t)) \\
(B_2(t), B_1(s)) & (B_2(t), B_2(t))
\end{pmatrix} > 0,
\]

for all \( s, t \in [0, 1] \).

(3.10)
A short calculation\textsuperscript{8} yields
\[
\Delta(s, t) = (1 - s)^2 (1 - t)^2 \Delta_{12} + (1 - s)^2 t^2 \Delta_{12'} + s^2 (1 - t)^2 \Delta_{1'2} + s^2 t^2 \Delta_{1'2'} \\
+ 2(1 - s)^2 t(1 - t) \Delta_{22' - 1} + 2s^2 t(1 - t) \Delta_{22' - 1'} + 2s(1 - s)(1 - t)^2 \\
\Delta_{11' - 12} + 2s(1 - s)t^2 \Delta_{11' - 12'} \\
+ 2s(1 - s)t(1 - t) \left[ 2(C_1, C_1')(C_2, C_2') \\
- (C_1, C_2)(C_1', C_2') - (C_1, C_2')(C_1', C_2) \right],
\]
and so, (3.3) implies (3.10), as claimed. \hfill \Box

**Definition 3.2.** We say that a collection \( C = \{ \{ C_1, C_1' \}, \{ C_2, C_2' \} \} \) of negative vectors is in **good position** if for all \( s, t \in [0, 1] \), the vectors \( B_1(s) \) and \( B_2(t) \) span a negative 2-plane in \( V \).

**Lemma 3.3.** A collection \( C \) is in good position if and only if it satisfies conditions (3.1), (3.2), and (3.10).

**Proof.** If \( C \) satisfies (3.1) and (3.2), then by (3.9) the vector \( B_1(s) \) is negative for all \( s \in [0, 1] \). The additional condition (3.10) implies that span\{\( B_1(s), B_2(t) \)\}_p.o. is a negative 2-plane for all \( s \) and \( t \in [0, 1] \). Conversely, if \( C \) is in good position, then (3.10) is immediate and (3.1) is its specialization to the corners while (3.2) follows from its specialization to the edges. For example, the fact that
\[
\text{span}\{B_1(0), B_2(t)\}_p.o. = \text{span}\{C_1, (1 - t) C_{2\perp 1} + t C_{2'\perp 1}\}_p.o.
\]
is a negative 2-plane for all \( t \in [0, 1] \) implies that \( (1 - t)C_{2\perp 1} + t C_{2'\perp 1} \) is a negative vector in \( V_1 \) and hence
\[
(C_{2\perp 1}, (1 - t)C_{2\perp 1} + t C_{2'\perp 1}) \neq 0.
\]
Thus \( (C_{2\perp 1}, C_{2\perp 1}) \) and \( (C_{2\perp 1}, C_{2'\perp 1}) \) are both negative, so (3.2) holds. \hfill \Box

**Remark 3.4.** By Lemma 3.1, if \( C \) satisfies the incidence conditions (3.1), (3.2), and (3.3), then \( C \) is in good position. We do not know if the converse is true.

For a collection \( C \) in good position, we parametrize a singular 2-cycle \( \phi_C \) in \( D \) as in (1.4). This oriented 2-cycle fills in the frame and has oriented boundary
\[
\partial S(C) = \gamma_1 + \gamma_2' - \gamma_1' - \gamma_2.
\]
Here it will sometimes be useful to write
\[
\gamma = \gamma_C = \gamma_1 + \gamma_2' - \gamma_1' - \gamma_2, \tag{3.11}
\]
to emphasize the dependence on the given data.

\textsuperscript{8} Pointed out to me by Jens Funke.
Finally, if we impose the additional condition
\[
\Delta_{11/22'} > 0, \tag{3.12}
\]
i.e., condition (4.2b) in [1], then the space \( U = \text{span}\{C_1, C_2, C_1', C_2'\} \) is a 4 dimensional subspace of \( V \) of signature \((2, 2)\). It follows that the negative 2-planes in (3.5) are distinct. Thus, under conditions (3.1), (3.2), (3.3), and (3.12), we obtain 8 points in \( D \), the 4 listed in (1.1) and the 4 obtained from them by reversing the orientations.

**Remark 3.5.** Note that our conditions (3.1), (3.2), and (3.12) are a subset of the incidence conditions (4.2) and (4.6) imposed in [1]. We did not check whether their additional conditions are consequences of the ones we impose.

### 4. The cycle integral

Given a collection \( C = \{\{C_1, C_1'\}, \{C_2, C_2'\}\} \) in good position and a vector \( x \in V \), our main goal is to compute the integral (1.5).

#### 4.1. Some geometry

The first step is to determine the intersection of \( S(C) \) with the singular set \( D_x \) of \( \psi(x) \). For the moment, we consider only the generic case.

**Definition 4.1.** A vector \( x \) is **regular** with respect to \( C \) if \((C, x) \neq 0\) for all \( C \in \{C_1, C_1', C_2, C_2'\} \).

**Lemma 4.2.** Let \( C \) be a collection in good position. (i) Suppose that \( x \) is regular with respect to \( C \). Then the set \( D_x \cap S(C) \) is non-empty if and only if \( \Phi_2(x, C) \neq 0 \), and, in this case,
\[
D_x \cap S(C) = \text{span}\{B_1(s_0), B_2(t_0)\}_{\text{p.o.}},
\]
where
\[
s_0 = \frac{(x, C_1)}{(x, C_1) - (x, C_1')}, \quad t_0 = \frac{(x, C_2)}{(x, C_2) - (x, C_2')}.
\tag{4.1}
\]

(ii) Suppose that \( x \in V \) is any vector with \( \Phi_2(x, C) \neq 0 \). Then, \( D_x \cap S(C) \) consists of a single point given by (4.1).

**Proof.** A point \( \phi(s, t) = \text{span}\{B_1(s), B_2(t)\}_{\text{p.o.}} \) lies in \( D_x \cap S(C) \) precisely when
\[
(1 - s)(x, C_1) + s(x, C_1') = 0 = (1 - t)(x, C_2) + t(x, C_2'), \quad s, t \in [0, 1].
\tag{4.2}
\]

Under the regularity assumption, if \((x, C_1) \neq (x, C_1')\) and \((x, C_2) \neq (x, C_2')\), the given pair \( s_0, t_0 \) is the unique solution. These lie in \([0, 1]\) precisely when \( \Phi_2(x, C) \neq 0 \). Indeed, under the regularity assumption, they both lie in \((0, 1)\). If either \((x, C_1) = (x, C_1')\) or \((x, C_2) = (x, C_2')\), then there is no solution due to regularity. This proves (i). For statement (ii), note that \( \Phi_2(x, C) \neq 0 \) implies that \((x, C_1) \neq (x, C_1')\) and \((x, C_2) \neq (x, C_2')\), and hence \( s = s_0, t = t_0 \) is the unique solution of (4.2). \(\square\)
Note that if, for example $(x, C_1) = (x, C_1') = 0$ and \( \text{sgn}(x, C_2) \neq \text{sgn}(x, C_2') \), then \( \Phi_2(x, C) = 0 \) and the image of the set $[0, 1] \times t_0$ lies in $D_x \cap S$. This shows that the conditions in the lemma are sharp.

If \( \Phi_1(x, C) \neq 0 \) and for \( \epsilon > 0 \) sufficiently small so that \( s_0 \pm \epsilon \) and \( t_0 \pm \epsilon \) lie in \( (0, 1) \), let

\[
C_1^e(x) = B_1(s_0 - \epsilon) \quad C_1'(x) = B_1(s_0 + \epsilon) \\
C_2^e(x) = B_2(t_0 - \epsilon) \quad C_2'(x) = B_2(t_0 + \epsilon).
\]

Consider the collection of negative vectors \( C^e(x) := \{ [C_1^e(x), C_1'(x)], [C_2^e(x), C_2'(x)] \} \). In what follows, if \( x \) is fixed, we will omit it from the notation and write \( C^e_C = C^e(x) \), etc. For \( \sigma \) and \( \tau \in [0, 1] \), let

\[
B_1^e(\sigma) = (1 - \sigma)C_1^e + \sigma C_1' \\
B_2^e(\tau) = (1 - \tau)C_2^e + \tau C_2'.
\]

Since

\[
B_1^e(\sigma) = B_1(s_0 - \epsilon + 2\epsilon\sigma), \quad B_2^e(\tau) = B_2(t_0 - \epsilon + 2\epsilon\tau),
\]

the collection \( C^e \) is again in good position and, indeed,

\[
\phi^e_C(\sigma, \tau) = \phi_C(s_0 - \epsilon + 2\epsilon\sigma, t_0 - \epsilon + 2\epsilon\tau). \tag{4.3}
\]

Thus, the collection \( C^e(x) \) determines a singular 2-cell \( S^e(x) \) with

\[
S^e(x) := S(C^e(x)) \subset S(C).
\]

where the set \( S^e(x) \) contains the point \( D_x \cap S(C) = \phi_C(s_0, t_0) \).

In the discussion so far, we have only assumed that \( C \) is in good position so that the singular 2-cell \( \phi_C \) is well defined, but possibly degenerate. We now assume that \( C \) also satisfies (3.12) so that, in particular, our collection is linearly independent. The following fact is proved in Sect. 6.3 of the Appendix.

**Lemma 4.3.** If a collection \( C \) is in good position and satisfies (3.12), then the map \( \phi_C \) is an embedding.

**Definition 4.4.** A collection \( C \) is in **very good position** if it is in good position and the map \( \phi_C \) is an embedding.

In particular, if \( C \) is in good position and satisfies (3.12), then by Lemma 4.3 it is in very good position.

We now suppose that \( C \) is in very good position, so that \( \phi_C \) parametrizes a surface \( S(C) \) in \( D \). Suppose that \( x \in V \) with \( \Phi_2(x, C) \neq 0 \). Then \( C^e(x) \) is also in
very good position\(^9\), by (4.3), and \(\phi_C\) parametrizes a closed neighborhood \(S^\epsilon(x)\) of the point \(D_x \cap S\). We write the boundary of \(S^\epsilon(x)\) as
\[
\partial S^\epsilon(x) = \gamma_1^\epsilon(x) + \gamma_2^\epsilon(x) - \gamma_1^\epsilon(x) - \gamma_2^\epsilon(x),
\]
for curves defined by the analogues of (3.7), etc. Thus
\[
\partial S^\epsilon(x) = \gamma_{\mathcal{C}^\epsilon} =: \gamma^\epsilon(x),
\]
in the notation introduced in (3.11).

For \(\mathcal{C}\) in very good position and for \(x \in V\) regular with respect to \(\mathcal{C}\), let
\[
S'_\epsilon = \begin{cases} S(\mathcal{C}) \setminus \text{int} S^\epsilon(x) & \text{if } \Phi_2(x, \mathcal{C}) \neq 0 \\ S(\mathcal{C}) & \text{if } \Phi_2(x, \mathcal{C}) = 0, \end{cases}
\]
so that
\[
\partial S'_\epsilon = \begin{cases} \partial S(\mathcal{C}) - \gamma^\epsilon(x) & \text{if } \Phi_2(x, \mathcal{C}) \neq 0 \\ \partial S(\mathcal{C}) & \text{if } \Phi_2(x, \mathcal{C}) = 0. \end{cases}
\]

Then by Stokes’ theorem we have
\[
\int_{S'_\epsilon} \varphi^\mu_{KM}(x) = \int_{\partial S'_\epsilon} \psi(x).
\]

4.2. The contribution of \(\partial S(\mathcal{C})\)

We continue to assume that \(\mathcal{C}\) is in very good position and that \(x \in V\) is regular with respect to \(\mathcal{C}\). We compute the integral of the 1-form \(\psi(x)\) around the boundary \(\gamma = \partial S(\mathcal{C})\).

First consider the integral over \(\gamma_1\). The second expression in (3.7) gives a lift of \(\gamma_1\) to a curve in OFD with tangent vector
\[
\eta(t) = [\eta_1(t), \eta_2(t)] = [\dot{\mathcal{C}}_1, \dot{\mathcal{B}}_{2\perp 1}(t)] = [0, \dot{\mathcal{B}}_{2\perp 1}(t)].
\]

Note that
\[
(C_1, \dot{\mathcal{B}}_{2\perp 1}(t)) = 0, \quad \text{and } (B_{2\perp 1}(t), \dot{\mathcal{B}}_{2\perp 1}(t)) = 0,
\]
since \((B_{2\perp 1}(t), B_{2\perp 1}(t)) = -1\). Thus the term in \(\psi(x)\) involving \((\xi_2, d\xi_1)\) vanishes along \(\gamma_1\), and we have simply
\[
\int_{\gamma_1} \psi(x) = -\frac{1}{2\pi} \int_0^1 e^{-2\pi R} (x, C_1)(x, \dot{\mathcal{B}}_{2\perp 1}(t))^2 dt, \quad (4.4)
\]
where
\[
R = (x, C_1)^2 + (x, B_{2\perp 1}(t))^2.
\]

\(^9\) Notice that the geometric conditions ‘good position’ and ‘very good position’ are easier to handle than the incidence conditions.
Let \( u = (x, B_{2\perp 1}(t)) \) so that \( du = (x, \dot{B}_{2\perp 1}(t)) \, dt \), and we have

\[
\int_{\gamma_1} \psi(x) = -\frac{1}{2\pi} (x, C_1) e^{-2\pi (x, C_1)^2} \int_{\alpha'} e^{-2\pi u^2} (x, C_1)^2 + u^2 \, du.
\]

where \( \alpha = (x, C_{2\perp 1}) \) and \( \alpha' = (x, C_{2'\perp 1}) \).

Now, recalling the function

\[
\tilde{e}_2(a, b) = \frac{2}{\pi} b e^{-\pi b^2} \int_0^a e^{-\pi v^2} (b^2 + v^2)^{-1} \, dv,
\]

defined in [1], (3.25), we can write

\[
4 \int_{\gamma_1} \psi(x) = \tilde{e}_2((x, C_{2\perp 1})\sqrt{2}, (x, C_1)\sqrt{2}) - \tilde{e}_2((x, C_{2'\perp 1})\sqrt{2}, (x, C_1)\sqrt{2}).
\]

(4.6)

The values of the other integrals are obtained by permutation of indices. Note that there will be an additional change in sign in the \( \gamma_2 \) and \( \gamma_{2'} \) integrals due to the fact noted above that the ‘fixed component’ is then the second component.

**Proposition 4.5.** For \( \mathcal{C} \) in very good position and \( x \in V \) regular with respect to \( \mathcal{C} \),

\[
4 \int_{\partial S(\mathcal{C})} \psi(x/\sqrt{2}) = \tilde{e}_2((x, C_{2\perp 1}), (x, C_1)) - \tilde{e}_2((x, C_{2'\perp 1}), (x, C_1))
\]

\[
- \tilde{e}_2((x, C_{1\perp 2}), (x, C_{2'})) + \tilde{e}_2((x, C_{1'\perp 2}), (x, C_{2'}))
\]

\[
- \tilde{e}_2((x, C_{2\perp 1'}), (x, C_{1'})) + \tilde{e}_2((x, C_{2'\perp 1'}), (x, C_{1'}))
\]

\[
+ \tilde{e}_2((x, C_{1\perp 2'}), (x, C_2)) - \tilde{e}_2((x, C_{1'\perp 2'}), (x, C_2)).
\]

In order to compare this quantity with what occurs in [1], we note the identity which follows from equation (3.24) there once the ‘boosting’ is taken into account:

\[
E_2(C_1, C_2; x) = -\tilde{e}_2((x, C_{1\perp 2}), (x, C_2)) - \tilde{e}_2((x, C_{2\perp 1}), (x, C_1))
\]

\[
+ \text{sgn}((x, C_2)) \text{ sgn}((x, C_1)).
\]

(4.7)

Using this, we obtain a nice expression for our integral.

**Corollary 4.6.** For \( \mathcal{C} \) and \( x \) as in Proposition 4.5,

\[
\int_{\partial S(\mathcal{C})} \psi(x/\sqrt{2}) = \Phi_2(x, \mathcal{C}) - \frac{1}{4} \left( E_2(C_1, C_2; x) - E_2(C_1, C_{2'}; x)
\right.
\]

\[
- E_2(C_{1'}, C_2; x) + E_2(C_{1'}, C_{2'}; x) \bigg).
\]
4.3. The contribution of the singular point

Now suppose that $C$ is in very good position, that $x$ is regular with respect to $C$ and that $\Phi_2(x, C) \neq 0$.

The integral of $\psi(x)$ around $\partial S^e(x)$ is given by the expression in Corollary 4.6 with the collection $C$ replaced by $C^e(x)$. Thus we have

$$
\int_{\partial S^e(x)} \psi(x/\sqrt{2}) = \Phi_2(x, C^e) - \frac{1}{4} \left( E_2(C_1^e, C_2^e; x) - E_2(C_1^e, C_2^e; x) \right)
$$

where we have written $C_i^e$ in place of $C_i^e(x)$ etc., as before. Note that at $\epsilon = 0$ we have $C_1^0 = C_1^0 = B_1(s_0)$ and $C_2^0 = C_2^0 = B_2(t_0)$. Thus the sum of the $E_2$ terms vanishes in the limit as $\epsilon$ goes to 0. Also, as $\epsilon$ runs over $[s_0, 0)$, the vector $C_i^e(x)$ runs from $C_1$ to $C_1^0(x)$ and the quantity $(x, C_1^e(x))$ does not vanish and, in particular, does not change sign along this path. Similarly for the other corners. Thus $\lim_{\epsilon \downarrow 0} \Phi_2(x, C^e) = \Phi_2(x, C)$, and we have the following.

**Corollary 4.7.**

$$
\lim_{\epsilon \downarrow 0} \int_{\partial S^e(x)} \psi(x/\sqrt{2}) = \Phi_2(x, C).
$$

Combining Corollaries 4.6 and 4.7, we obtain the following result.

**Corollary 4.8.** For $C$ in very good position and for $x$ regular with respect to $C$,

$$
\int_{S(C)} \varphi_{KM}^e(x/\sqrt{2}) = -\frac{1}{4} \left( E_2(C_1, C_2; x) - E_2(C_1, C_2; x) \right)
$$

$$
- E_2(C_1', C_2; x) + E_2(C_1', C_2; x)
$$

4.4. Irregular and good position cases

In this section, we show that the identity in Corollary 4.8 holds for any $C$ in good position and for any $x \in V$. This follows by continuity as we now explain in more detail.

First suppose that $C$ is in very good position but that $x$ is not regular with respect to the collection $C$. Fix a vector $y \in V$ so that for all $\epsilon > 0$ sufficiently small, $x^e = x + \epsilon y$ is regular with respect to $C$. Then, by Corollary 4.8, we have

$$
\int_S \varphi_{KM}^e(x^e/\sqrt{2}) = -\frac{1}{4} \left( E_2(C_1, C_2; x^e) - E_2(C_1, C_2; x^e) \right)
$$

$$
- E_2(C_1', C_2; x^e) + E_2(C_1', C_2; x^e)
$$

(4.8)

But $\varphi_{KM}^e(x)$ is a smooth 2-form on $D$ which depends smoothly on the vector $x \in V$. Also, $E(C, C'; x)$ is a smooth function on the space of parameters $\{x, C, C'\}$ where
\(x \in V\) and the negative vectors \(C, C'\) span a negative 2-plane. This follows from (3.38), (3.23) and Proposition 3.7 (ii) in [1]. Thus, passing to the limit as \(\epsilon \downarrow 0\) in (4.8), we obtain the identity of Corollary 4.8 for all \(x \in V\).

Next we show how to relax the ‘very good position’ condition on \(C\). Let \(V^-\) be the set of vectors \(y \in V\) with \((y, y) < 0\) and let \(\text{GP}\) (resp. \(\text{GP}\)) be the set of collections \(C = \{C_1, C_1', C_2, C_2'\}\) in very good position (resp. good position). Then

\[
\text{VGP} \subset \text{GP} \subset V^- \subset V^4
\]

and the set \(\text{GP}\) is an open subset of \(V^4\), since it is the set of elements of \(V^-\) satisfying conditions (3.1), (3.2), and (3.10). Now suppose that \(C\) is in good position. Since \(\Delta_{11'/22'}\) is the determinant of the matrix of inner products of the vectors in \(C\), if \(\Delta_{11'/22'} \neq 0\), then the subspace \(U\) spanned by these vectors is of dimension 4 and is non-degenerate with respect to the restriction of \((,\cdot)\). Since \(U\) contains a negative 2-plane, it must have signature (2, 2) and hence \(\Delta_{11'/22'} > 0\). By Lemma 4.3 it follows that

\[
(\text{GP}\setminus \text{VGP}) \subset (\text{GP} \cap \{\Delta_{11'/22'} = 0\}).
\]

Now suppose that \(C \in (\text{GP}\setminus \text{VGP})\) and take \(y = (y_1, y_1', y_2, y_2') \in V^4\) such that, with the obvious notation\(^{10}\),

\[
\Delta_{11'/22'}(C_1 + y_1, C_1' + y_1', C_2 + y_2, C_2' + y_2') \neq 0.
\]

The function

\[
f(t) = \Delta_{11'/22'}(C_1 + ty_1, C_1' + ty_1', C_2 + ty_2, C_2' + ty_2')
\]

is a nonzero polynomial in \(t\) of degree at most 8 with \(f(0) = 0\). Thus there is an \(\eta > 0\) such that for \(t \in (-\eta, \eta),\)

\[
C' := \{C_1 + ty_1, C_1' + ty_1', C_2 + ty_2, C_2' + ty_2'\} \in \text{GP},
\]

and, for \(t \in (-\eta, \eta), t \neq 0\), we have \(f(t) \neq 0\). It follows that \(C' \in \text{VGP}\) for \(t \in (-\eta, \eta), t \neq 0\), and we have, for any \(x \in V,\)

\[
\int_{S(C')} \varphi^0_{KM}(x) = -\frac{1}{4} \left( E_2(C_1 + ty_1, C_2 + ty_2; x) - E_2(C_1 + ty_1, C_2' + ty_2'; x) - E_2(C_1' + ty_1', C_2 + ty_2'; x) + E_2(C_1' + ty_1', C_2' + ty_2'; x) \right).
\]

(4.9)

The left side here is

\[
\int_{S(C')} \varphi^0_{KM}(x) = \int_{[0,1]^2} \phi^*_{C'}(\varphi^0_{KM}(x)),
\]

and the 2-form \(\phi^*_{C'}(\varphi^0_{KM}(x))\) is a continuous function of \(t\). Since both sides of (4.9) are continuous functions of \(t\), this identity also holds for \(t = 0\). Thus the identity

\(^{10}\) This argument requires \(n \geq 4\). For \(n = 3\), enlarge \(V\) by adding a positive line.
of Corollary 4.8 holds for all $\mathcal{C}$ in good position and all $x \in V$. This completes the proof of Theorem A(a) under the weaker hypothesis that $\mathcal{C}$ is in good position.

Next note that

$$E_2(C, C'; 0) = \frac{2}{\pi} \arctan \left( \frac{(C, C')}{\sqrt{\Delta(C, C')}} \right).$$

This follows from [1], (3.23) together with the fact that $e_2(0, 0) = 0$ and the boosting procedure. As a consequence, we obtain the following nice formula.

**Corollary 4.9.**

$$-\int_S \varphi_{KM}^0 (0) = \frac{1}{2\pi} \int_S \Omega = \frac{1}{2\pi} \arctan \left( \frac{(C_1, C_2)}{\sqrt{\Delta_{12}}} \right) - \frac{1}{2\pi} \arctan \left( \frac{(C_1, C_2')}{\sqrt{\Delta_{12}'}'} \right) - \frac{1}{2\pi} \arctan \left( \frac{(C_1', C_2)}{\sqrt{\Delta_{12}'}} \right) + \frac{1}{2\pi} \arctan \left( \frac{(C_1', C_2')}{\sqrt{\Delta_{12}''}} \right).$$

**Remark 4.10.** This formula can be seen as an incarnation of the Gauss-Bonnet formula in the following way. On the Grassmannian $\text{Gr}_2(V)$ of oriented 2-planes in $V$ there is a tautological rank 2 vector bundle $\text{Taut}_2 \to \text{Gr}_2(V)$. Its restriction to $D$, which we denote by the same symbol, has a natural metric given on a 2-plane $z$ by the negative of the restriction to $z$ of the form $(\ , \ )$ on $V$. Our bundle $\text{OFD} \to D$ is the associated principal $\text{SO}(2)$-bundle with connection described in Sect. 6.1. It is not difficult to check that $\Omega$ is the Euler form for this connection. The pullback $\phi^*\text{Taut}_2$ is a rank 2 oriented vector bundle with connection on $[0, 1]^2$ whose Euler form is the pullback $\phi^*\Omega$. We identify this bundle with the standard basis $e_1$ and $e_2$ in $\mathbb{R}^2$, viewed as global sections of the tangent bundle, with the global sections $B_1(s)$ and $B_2(t)$ of $\phi^*\text{Taut}_2$. The metric on $\text{Taut}_2$ then gives a Riemannian structure on $[0, 1]^2$ with metric

$$g = g(s, t) = \begin{pmatrix} (B_1(s), B_1(s)) & (B_1(s), B_2(t)) \\ (B_2(t), B_1(s)) & (B_2(t), B_2(t)) \end{pmatrix}.$$ 

Note that this construction extends to an open neighborhood $\mathcal{U}$ of $[0, 1]^2$ in $\mathbb{R}^2$. It is not difficult to check that the identity given in Corollary 4.9 is then the Gauss-Bonnet formula for the integral of the Gaussian curvature over $[0, 1]^2$, provided that the contribution of the integral of the geodesic curvature of the edges vanishes. Since, in general, the edges are not geodesics for the given metric, the vanishing of this contribution is a non-trivial identity for which we did not see an independent proof.

### 4.5. An intersection number

In this section, we give the proof of Proposition 1.2. Suppose that $\mathcal{C}$ is in good position. If $x \in V$ with $\Phi_2(x, \mathcal{C}) \neq 0$, by (ii) of Lemma 4.2, $D_x \cap S(\mathcal{C})$ consists of a single point $z_0$. If $\zeta \in \text{OFD}$ with $\pi(\zeta) = z_0$ and we write $U(z_0) = z_0^\perp$, then
\[ T_{z_0}(D) \simeq U(z_0)^2, \quad T_{z_0}(D_x) \simeq (U(z_0) \cap x^\perp)^2, \]

and the normal to \( D_x \) at \( z_0 \) is the subspace \((\mathbb{R}^x)^2 \subset U(z_0)^2\). If \( e_1, \ldots, e_{n-3} \) is a basis for \( U(z_0) \cap x^\perp \), then the orientation of \( D \) is given by the basis vector \([e_1, 0] \wedge [0, e_1] \wedge \cdots \wedge [e_{n-3}, 0] \wedge [0, e_{n-3}] \wedge [x, 0] \wedge [0, x] \in \wedge^{2(n-2)} T_{z_0}(D)\).

Note that this is the orientation determined by the complex structure in the hermitian model of \( D \). Write

\[ \phi_*(\frac{\partial}{\partial s})_{z_0} = [\eta_1, \eta_2], \quad \phi_*(\frac{\partial}{\partial t})_{z_0} = [\mu_1, \mu_2]. \]

Explicitly, write \( B = [B_1(s), B_2(t)] \) and write \(- (B, B) = P^2 \) with \( P \in \text{Sym}_2(\mathbb{R}) \) positive definite. Let

\[ B = [B_1(s, t), B_2(s, t)] = [B_1(s), B_2(t)] P^{-1}. \]

Then

\[ \phi_*(\frac{\partial}{\partial s})_{z_0} = [-C_1 + C_1', 0] P^{-1} + [B_1(s_0), B_2(t_0)] \frac{\partial}{\partial s}(P^{-1}) \]

\[ \phi_*(\frac{\partial}{\partial t})_{z_0} = [0, -C_2 + C_2'] P^{-1} + [B_1(s_0), B_2(t_0)] \frac{\partial}{\partial t}(P^{-1}) \]

hence

\[ (x, \phi_*(\frac{\partial}{\partial s})_{z_0}) = [(x, -C_1 + C_1'), 0] P^{-1}, \]

\[ (x, \phi_*(\frac{\partial}{\partial t})_{z_0}) = [0, (x, -C_2 + C_2')] P^{-1}, \]

where we have used the fact that \((x, B_1(s_0)) = (x, B_2(t_0)) = 0\). This gives

\[ \text{sgn det} \begin{pmatrix} (x, -C_1 + C_1') & 0 \\ 0 & (x, -C_2 + C_2') \end{pmatrix} \]

\[ = \text{sgn}((x, C_1') - (x, C_1)) \text{sgn}((x, C_2') - (x, C_2)) \]

\[ = \frac{1}{4} \left[ \text{sgn}(x, C_1) - \text{sgn}(x, C_1') \right] \left[ \text{sgn}(x, C_2) - \text{sgn}(x, C_2') \right] = \Phi_2(x, C). \]

Since we are assuming that \( \Phi_2(x, C) \neq 0 \), it follows that \( \phi_C \) an immersion at the point \([s_0, t_0]\), and the quantity

\[ I(D_x, S(C)) = \text{sgn}([x, 0] \wedge [0, x], \phi_*(\frac{\partial}{\partial s})_{z_0} \wedge \phi_*(\frac{\partial}{\partial t})_{z_0}) \]

\[ = \text{sgn det} \begin{pmatrix} (x, \eta_1) & (x, \eta_2) \\ (x, \mu_1) & (x, \mu_2) \end{pmatrix} = \Phi_2(x, C) \]

is the intersection number of \( D_x \) and \( S(C) \) at this point, as claimed in Proposition 1.2.
4.6. Convergence of the holomorphic generating series

Suppose that the collection of negative vectors \( C = \{ \{ C_1, C_{1'} \}, \{ C_2, C_{2'} \} \} \) is in good position and let \( S = S(C) = \phi_C([0, 1]^2) \) be resulting subset of \( D \).

**Lemma 4.11.** There exists a positive definite inner product \(( , )_S\) on \( V \) such that

\[(x, x)_z \geq (x, x)_S, \quad \forall z \in S, \quad \forall x \in V.\]

**Proof.** Let \(( , )_+\) be any positive definite inner product on \( V \) with unit sphere \( B_+ \). Define a continuous function

\[f : D \rightarrow \mathbb{R}_{>0}, \quad f(z) = \min_{x \in B_+} (x, x)_z.\]

The image \( f(S) \) of the compact set \( S \) is compact and has a lower bound \( v \in \mathbb{R} > 0 \). Take \(( , )_S = v( , )_+\). \(\square\)

**Lemma 4.12.** Suppose that \( x \in V \) with \( \Phi_2(x, C) \neq 0 \). Then

\[(x, x) \geq (x, x)_S.\]

**Proof.** By Lemma 4.2, the condition, \( \Phi_2(x, C) \neq 0 \) implies that \( D_x \cap S \) is a single point \( z_0 \). Thus \( R(x, z_0) = 0 \) and we have

\[(x, x) = (x, x)_{z_0} - 2R(x, z_0) = (x, x)_{z_0} \geq (x, x)_S,\]

as claimed. \(\square\)

**Proposition 4.13.** Assume that the collection \( C \) is in good position. Then the series

\[\sum_{x \in L+\mu} \Phi_2(x, C) q^{\frac{1}{2}(x, x)}\]

is termwise absolutely convergent.

**Proof.** By the previous lemma, this series is dominated by the convergent series

\[\sum_{x \in L+\mu} \exp(-\pi v(x, x)_S).\]

\(\square\)

**Remark 4.14.** This result shows that our incidence conditions (3.1), (3.2), and (3.3), which imply good position, are sufficient to establish the convergence obtained from conditions (4.2), Theorem 4.1 in [1].
5. Additional remarks

5.1. Shadows

We suppose that \( C \) is in good position. It is natural to consider the image of the non-holomorphic modular form \( I_\mu(\tau; C) \) under the lowering operator \( L_k = -2iv^2 \frac{\partial}{\partial \bar{\tau}} \) or, equivalently, under the \( \xi \)-operator. This is done in [1]. In this section, we explain how this image can be calculated using our theta integrals.

We begin with the identity

\[
-2i \ v^2 \frac{\partial}{\partial \bar{\tau}} \{ \varphi_{KM}(\tau, x) \} = v^2 \ q^{\frac{1}{2}(x, x)} d\left( \frac{\partial}{\partial v} \{ \psi(x, \sqrt{v}) \} \right).
\]

Recalling (2.1), on OFD we have

\[
v^2 \frac{\partial}{\partial v} \{ \psi(x, \sqrt{v}) \} = v^2 \ e^{-2\pi vR} \left( (x, \zeta_1)(x, d\zeta_2) - (x, \zeta_2)(x, d\zeta_1) \right) - R(\zeta_2, d\zeta_1)
\]

and its restriction to the connection subspace can be written as

\[
v^2 \frac{\partial}{\partial v} \{ \psi(x, \sqrt{v}) \} = v \ \Psi_M^a(x, \sqrt{v}),
\]

where

\[
\Psi_M^a(x) = e^{-2\pi R} \left( (x, \zeta_1)(x, d\zeta_2) - (x, \zeta_2)(x, d\zeta_1) \right).
\]

Let

\[
\Psi_M(\tau, x) = q^{\frac{1}{2}(x, x)} \Psi_M^a(x, \sqrt{v}),
\]

a Schwartz function on \( V \) valued in 1-forms on \( D \). Define the associated theta form

\[
\theta_\mu(\tau, \Psi_M) = \sum_{x \in L+\mu} \Psi_M(\tau, x).
\]

Then

\[
-2i \ v^2 \frac{\partial}{\partial \bar{\tau}} \{ \theta_\mu(\tau, \varphi_{KM}) \} = v d \theta_\mu(\tau, \Psi_M),
\]

and hence the image of \( I_\mu(\tau; C) \) under lowering can be written as an integral over the boundary of \( S(C) \),

\[
L_{\frac{\partial}{\partial \bar{\tau}}} I_\mu(\tau; C) = -2i \ v^2 \frac{\partial}{\partial \bar{\tau}} \{ I_\mu(\tau; C) \} = v \int_{\partial S(C)} \theta_\mu(\tau, \Psi_M).
\]

(5.1)

We could, of course, now integrate termwise using the parametrization of \( \partial S(C) \) as before and arrive at the formulas for the shadow given in [1]. However, it is more enlightening to continue using the geometry. Recall that \( \partial S(C) = \gamma_1 + \gamma_2' - \gamma_1' - \gamma_2 \).
We note that, by construction, the path $\gamma_1$ is the image in $D$ of a geodesic in the hyperbolic space

$$D(V_1) = \{ v \in V_1 \mid (v, v) = -1 \}$$

in $V_1$ under the inclusion

$$j_1 : D(V_1) \rightarrow D, \quad v \mapsto \text{span}(C_1, v)_{\text{p.o.}}.$$ 

For the decomposition

$$V = \mathbb{R}C_1 + V_1,$$

we write

$$x = x_0 + x_1, \quad x_0 = - (x, C_1) C_1,$$

and we note that

$$R(x, j_1(v)) = (x, C_1)^2 + R_1(x_1, v).$$

where

$$R_1(x_1, v) = (x_1, v)^2.$$

Lemma 5.1. Let $\varphi^{(n-2,1)}_{KM}(\tau, x_1)$ be the Schwartz form associated to the space $V_1$ of signature $(n-2, 1)$. It is a closed 1-form on $D(V_1)$ whose value on a tangent vector $\eta \in T_v(D(V_1))$ is

$$\varphi^{(n-2,1)}_{KM}(\tau, x_1)(\eta) = v_1^2 (x_1, \eta) q_1^2 (x_1, x_1) e^{-2\pi v R_1(x_1, v)}.$$

Then

$$v j_1^* \Psi_M(\tau, x) = v_1^2 (x_0, C_1) (\bar{q})^{-\frac{1}{2}(x_0, x_0)} [\varphi^{(n-2,1)}_{KM}(\tau, x_1)]. \quad (5.2)$$

Proof. If $v \in D(V_1)$, there is an identification

$$T_v(D(V_1)) = \{ \eta \in V_1 \mid (v, \eta) = 0 \}.$$

The image of the tangent vector $\eta \in T_v(H_1)$ in $T_{j_1(v)}(D)$ is the vector $[0, \eta]$ where we have used the point $[C_1, v] \in \text{OFD}$ in the identification of $T_{j_1(v)}(D) \cong U(j_1(v))^2$. Then

$$j_1^* \Psi_M(\tau, x)(\eta) = q_1^2 (x_0, x_0) + \frac{1}{2}(x_1, x_1) e^{2\pi v [R_1(x_1, v)]} v_1^2 (x, C_1) v_1^2 (x, \eta).$$
Now assume that $C_1$ is a rational vector and let
\[ L_0 = L \cap \mathbb{Q}C_1, \quad L_1 = L \cap V_1, \]
so that
\[ L_0 + L_1 \subset L \subset L^\vee \subset L_0^\vee + L_1^\vee. \]
Then, we can write any coset as
\[ \mu + L = \bigcup_{\lambda = \lambda_0 + \lambda_1} (\lambda_0 + L_0) + (\lambda_1 + L_1), \]
and we have
\[ \nu \int_{\gamma_1} \theta_\mu(\tau, \Psi_M) = \sum_\lambda \nu \frac{1}{2} \theta_{0,\lambda_0}(\tau) \theta_{\lambda_1}(\tau, \varphi_{KM}^{(n-2,1)}). \]
Here
\[ \theta_{0,\lambda_0}(\tau) = \sum_{x_0 \in \lambda_0 + L_0} (x_0, C_1) q^{-\frac{1}{2}(x_0,x_0)} \]
is the theta series of weight $\frac{3}{2}$ attached to the coset $\lambda_0 + L_0$ and the positive
definite unary form $-(x_0, x_0)$. The form $\nu \frac{1}{2} \theta_{0}(\tau)$ then has weight $-3/2$, while
$\theta_{\lambda_1}(\tau, \varphi_{KM}^{(n-2,1)})$ has weight $(n-1)/2$ so the product has weight $n/2 - 2$, as required.
But now
\[ \nu \int_{\gamma_1} \theta_\mu(\tau, \Psi_M) = \sum_\lambda \nu \frac{1}{2} \theta_{0,\lambda_0}(\tau) \int_{\gamma(C_{2,1}, C_{2'}_{1,1})} \theta_{\lambda_1}(\tau, \varphi_{KM}^{(n-2,1)}), \quad (5.3) \]
where the right side is a linear combination of products of weight $-\frac{3}{2}$ antiholo-
morphic unary theta series times the Zwegers' indefinite theta series which arise
as integrals of the $\theta_{\lambda_1}(\tau, \varphi_{KM}^{(n-2,1)})'$s along the geodesic $\gamma(C_{2,1}, C_{2'}_{1,1})$ in $D(V_1)$.
The remaining terms in the expression (5.1) for $L_n/2 I_\mu(\tau; C)$ as an integral over
$\partial S(C)$ have the same form. This formula reveals the inductive nature of the shadow.

5.2. Other signatures

The case in which $V$ has signature $(n - r, r)$ is considered in section 6 of [1]. As
remarked in the introduction, the relevant Schwartz forms $\varphi_{KM}^{(n-r,r)}(x)$ are consid-
ered in [5,6] and [7]. In the present section, we indicated the first steps in general-
izing the constructions of earlier sections. Complete details and proofs will be given
in a forthcoming paper with Jens Funke [3].

Let $D$ be the space of oriented negative $r$-planes in $V$.

We suppose that a collection of $r$ pairs of negative vectors $\{C_1, C'_1\}, \ldots, \{C_r, C'_r\}$
is given with the following incidence properties. For a subset $I \subset \{1, \ldots, r\}$, let $C_I$
be the ordered set of $r$ vectors where we take $C'_j \in C_I$ if $j \in I$ and $C_j \in C_I$ if $j \notin I$
and the vectors are ordered according to the index $j$. Thus, $C_\emptyset = \{C_1, \ldots, C_r\}$,
etc. We require
(i) Each collection $C_I$ spans a negative $r$-plane

$$P_I = \text{span}\{C_I\}.$$ 

(ii) These planes are distinct, so that there are $2^r$ of them.

(iii) The oriented negative $r$-planes

$$z_I = \text{span}\{C_I\}_{p.o.}$$

all lie on the same component $D^+$ of $D$.

We will not attempt express these in terms of determinants of minors of Gram matrices, although this is clearly possible. Note that the conditions can be expressed inductively. We use the notation $C_j' = C'_j$, as before. For example, for $r = 3$, the collection of vectors $\{C_2 \perp, C_2' \perp, C_3 \perp, C_3' \perp\}$ should satisfy the incidence conditions analogous to (3.1), (3.2) and (3.3), and similarly for the other 5 ‘projections’. Thus, the set of $2^r$ points $\{z_I\}$ form the vertices of a geodesic hypercube $S$ in $D$. Assuming that $C$ is in good position, this cube can be parametrized by

$$\phi : [0, 1]^r \sim S \subset D, \quad s = [s_1, \ldots, s_r] \mapsto \text{span}\{B_1(s_1), \ldots, B_r(s_r)\},$$

where

$$B_j(s_j) = (1 - s_j)C_j + s_j C_j'.$$

For a vector $x \in V$, let

$$\Phi_r(x, C) = \frac{1}{2^r} [\text{sgn}(x, C_1) - \text{sgn}(x, C_1')] \ldots [\text{sgn}(x, C_r) - \text{sgn}(x, C_r')]$$

as in (6.5) of [1]. Lemma 4.2 and the arguments of Sect. 4.6 carry over immediately and yield the following. Here

$$D_x = \{ z \in D \mid x \perp z \}$$

is an oriented$^{11}$ totally geodesic submanifold of $D$ of codimension $r$ and $L \subset \text{L}^\vee$ is an integral lattice in $V$.

**Proposition 5.2.** (i) Suppose that $\Phi_r(x, C) \neq 0$. Then $D_x \cap S$ consists of a single point. (ii) The $q$-series

$$\sum_{x \in L + \mu} \Phi_r(x, C) e^{\frac{1}{2}(x,x)}, \quad \mu \in \text{L}^\vee / \text{L},$$

is termwise absolutely convergent.

$^{11}$ The orientation is defined as follows: Suppose that an orientation of $D$ has been fixed. Then, at a point $z \in D_x$, we have $T_z(D) = T_z(D_x) + N_z(D_x)$ where, under the identification $T_z(D) = \text{Hom}(z, z^\perp)$, the normal space $N_z(D_x) = \text{Hom}(z, \mathbb{R}x)$. Since $z \in D$ is oriented, the basis vector $x$ in $\mathbb{R}x$ determines an orientation of $N_z(D_x)$ and hence of $T_z(D_x)$. 

The theta form

$$\theta_{\mu}(\tau, \varphi_{KM}) = \sum_{x \in L + \mu} \varphi_{KM}(\tau, x)$$

is a closed $r$-form on $D$ and we define

$$I_{\mu}(\tau; S) = \int_{S} \theta_{\mu}(\tau, \varphi_{KM}).$$

As before $I_{\mu}(\tau; S)$ is a non-holomorphic modular form of weight $n/2$ and transformation law inherited from that of $\theta_{\mu}(\tau; \varphi_{KM})$. We expect that $I_{\mu}(\tau; S)$ is again the modular completion of the $q$-series (5.4). A solution of the equation

$$\frac{\partial}{\partial \tau} \varphi_{KM}(\tau, x) = d\Psi_{M}(\tau, x)$$

is defined in section 8 of [7] and the analogue of the form $\psi(x)$ used above can be obtained from it. It remains to do the boundary calculations explicitly and to determine the relation with the generalized error functions defined in Sect. 6 of [1]. In particular, there will be an analogue of the inductive relation (5.3) for the shadow in general [3].

6. Appendix: some computations and details

6.1. Differential forms on $D$

In this section, we review the conventions used above concerning tangent spaces and differential forms. The basic idea is to simplify computations by pulling back to the frame bundle and then viewing this bundle as an open subset of $V^2$.

We use the following construction. Let

$$FD = \{ \zeta = [\zeta_1, \zeta_2] \in V^2 \mid (\zeta, \zeta) < 0 \}$$

and

$$OFD = \{ \zeta = [\zeta_1, \zeta_2] \in V^2 \mid (\zeta, \zeta) = -1_2 \}$$

be the spaces of oriented negative frames and oriented orthogonal negative frames respectively. We have natural projections to $D$ sending a frame to the 2-plane it spans. Note that $FD \subset V^2$ is an open subset so that there is a natural identification $T_{\zeta}(FD) = V^2$. Also, FD (resp. OFD) is a $GL_2(\mathbb{R})^+$ (resp. $SO(2)$) torsor over $D$. For convenience, for $z \in D$ we write $U(z) = z^\perp$. If $\zeta \in FD$ with span $\pi(\zeta) = z$, we obtain an isomorphism

$$d\pi_{\zeta} : U(z)^2 \cong T_{\zeta}(D). \quad (6.1)$$
satisfying the equivariance condition

\[
\begin{array}{c}
U(z)^2 \xrightarrow{d\pi_{\xi}} T_\xi(D) \\
\downarrow R_g \\
U(z)^2 \xrightarrow{d\pi_{\xi'}} T_\xi(D),
\end{array}
\]

where \(g \in \text{GL}_2(\mathbb{R})^+\) and \(\xi' = \xi g\). Thus, the spaces \(U(z)^2\) define a connection on the \(\text{GL}_2(\mathbb{R})^+\) principal bundle \(FD\) and provide a convenient way to describe differential forms on \(D\). Note that, for a choice of \(\xi\) with \(\pi(\xi) = z\), the natural isomorphism

\[\text{Hom}(z, z^\perp) \sim T_z(D), \quad (6.2)\]

can be written as the composition of the isomorphism

\[\text{Hom}(z, z^\perp) \xrightarrow{\sim} U(z)^2, \quad \lambda \mapsto [\lambda(\xi_1), \lambda(\xi_2)],\]

given by the basis \(\xi\), with the map (6.1).

For a point \(\xi \in OFD \subset FD\), we have

\[V^2 = T_\xi(FD) \supset T_\xi(OFD) = \{ \eta = [\eta_1, \eta_2] | (\eta, \xi) + (\xi, \eta) = 0 \},\]

i.e., \((\eta_1, \xi_1) = 0, (\eta_2, \xi_2) = 0, (\eta_1, \xi_2) = -(\eta_2, \xi_1)\). This contains the connection subspace \(U(z)^2\) and the additional vertical subspace \(\mathbb{R}[\xi_2, -\xi_1] \subset z^2\).

Suppose that \(\gamma : [0, 1] \longrightarrow D\) is a smooth curve given explicitly by a lift

\[\gamma(s) = \text{span}[\xi(s)], \quad \xi(s) = [\xi_1(s), \xi_2(s)],\]

to a smooth curve in \(OFD\), i.e., with \((\xi(s), \xi(s)) = -1\). The vector field along the lift \(\xi(s)\) obtained by pushing forward \(\frac{d}{ds}\) is given by

\[d\xi_*\left(\frac{d}{ds}\right) = [\dot{\xi}_1(s), \dot{\xi}_2(s)],\]

and the image along \(\gamma\) is

\[d\gamma_*\left(\frac{d}{ds}\right) = d\pi_{\xi}([\dot{\xi}_1(s), \dot{\xi}_2(s)]).\]

Under the isomorphism (6.2), this corresponds to the vector

\[\xi_1 \otimes \text{pr}_{U(z)}\dot{\xi}_1(s) + \xi_2 \otimes \text{pr}_{U(z)}\dot{\xi}_2(s) \in z \otimes z^\perp,\]

where \(\text{pr}_{U(z)}\) is the orthogonal projection of \(V\) to \(U(z)\).
Remark 6.1. There is one subtle point here. The condition \((\zeta(s), \zeta(s)) = -1_2\) implies that

\[0 = (\dot{\zeta}(s), \zeta(s)) + (\zeta(s), \dot{\zeta}(s)).\]

But the component \((\dot{\zeta}_1(s), \zeta_2(s))\) need not be zero and so the \(\dot{\zeta}_j(s)\)’s may not lie in \(U(z)\), hence the need for the projection. On the other hand, if we integrate \(\pi^*(\psi(x))\) along the lift of a curve to OFD, then the pullback will vanish on the vertical component of the tangent vector, and so we do not need to take any projection.

Remark 6.2. It will be convenient to compute differential forms on \(D\) by working with their pullbacks to OFD. In the end, we obtain forms on \(D\) by restricting to the connection subspace. This restriction cannot, of course, be done at the various intermediate steps!

6.2. Proof of Proposition 2.1

On OFD, we have

\[
\pi^*(\psi(x)) = -\frac{1}{2\pi} e^{-2\pi R} \left( R^{-1} ((x, \xi_1)(x, d\xi_2) - (x, \xi_2)(x, d\xi_1)) - (\xi_2, d\xi_1) \right).
\]

Note that since \((\xi, \xi) = -1_2\) on OFD, we have

\[R = (x, \xi)(\xi, x), \quad dR = 2(x, \xi)(d\xi, x) = 2(x, \xi_1)(x, d\xi_1) + 2(x, \xi_2)(x, d\xi_2),\]

and relations \((\xi_1, d\xi_1) = 0, (\xi_2, d\xi_2) = 0,\) and \((\xi_2, d\xi_1) = -(\xi_1, d\xi_2)\). Now we calculate

\[
d\pi^*(\psi(x)) = e^{-2\pi R} \left[ R^{-1} + \frac{1}{2\pi} R^{-2} \right] dR \wedge ((x, \xi_1)(x, d\xi_2) - (x, \xi_2)(x, d\xi_1)) \]

\[\quad - e^{-2\pi R} dR \wedge (\xi_2, d\xi_1) - \frac{1}{2\pi} e^{-2\pi R} R^{-1} 2 (x, d\xi_1) \wedge (x, d\xi_2)\]

\[\quad - \frac{1}{2\pi} e^{-2\pi R} (d\xi_1, d\xi_2).
\]

But we have

\[dR \wedge ((x, \xi_1)(x, d\xi_2) - (x, \xi_2)(x, d\xi_1)) = 2R (x, d\xi_1) \wedge (x, d\xi_2),\]

so that the second term in the first line cancels the second term in the second line and we are left with

\[e^{-2\pi R} \left[ 2(x, d\xi_1) \wedge (x, d\xi_2) - dR \wedge (\xi_2, d\xi_1) - \frac{1}{2\pi} e^{-2\pi R} (d\xi_1, d\xi_2) \right],\]

as required.
6.3. Proof of Lemma 4.3

In this section, we assume that $C$ is in good position and satisfies condition (3.12). In particular the vectors in $C$ are linearly independent.

We first show that $\phi = \phi_C$ is an immersion. Let $B(s, t) = [B_1(s), B_2(t)]$ and let $P^2 = -(B, B)$ with $P$ positive definite and symmetric, as in Sect. 4.5. Let $\zeta(s, t) = B(s, t) P^{-1}$ so that the map $\zeta : [0, 1]^2 \to OFD$ lifts $\phi$. Then

$$\zeta_\ast \left( \frac{\partial}{\partial s} \right) = [-C_1 + C_1', 0] P^{-1} - \zeta \dot{P}^s P^{-1}$$

and

$$\zeta_\ast \left( \frac{\partial}{\partial t} \right) = [0, -C_2 + C_2'] P^{-1} - \zeta \dot{P}^t P^{-1}.$$

As explained in Sect. 6.1, the corresponding horizontal vectors in $T_\zeta(OFD)$ are obtained by taking the projection to $U(z)$ and so are given by

$$[\text{pr}_{U(z)}(-C_1 + C_1'), 0] P^{-1}$$

and

$$[0, \text{pr}_{U(z)}(-C_2 + C_2')] P^{-1}.$$

These will be linearly dependent only if one of $\text{pr}_{U(z)}(-C_1 + C_1')$ or $\text{pr}_{U(z)}(-C_2 + C_2')$ is zero. But $-C_1 + C_1' \in z$ if and only if there is a relation

$$-C_1 + C_1' = \alpha((1 - s)C_1 + sC_1') + \beta((1 - t)C_2 + tC_2'),$$

which clearly has no solutions, and similarly for $-C_2 + C_2'$.

Next we show that $\phi$ is injective. Suppose that $\phi(s, t) = \phi(s', t')$, i.e., that there is a relation

$$[B_1(s'), B_2(t')] = [B_1(s), B_2(t)] g, \quad \det g > 0.$$

By linear independence, this immediately forces $s = s', t = t'$ and $g = 1_2$.

This completes the proof of Lemma 4.3.

6.4. The generalized error function

For completeness, in this section, we record the definitions of the generalized error function and its ‘boosted’ version given in [1].

The classical error function and its complementary version, in the normalization of [1], are given by

$$E_1(u) = \text{sgn}(u) \text{Erf}(|u|\sqrt{\pi}) = 2u \int_0^1 e^{-\pi u^2 v^2} dv$$

$$M_1(u) = -\text{sgn}(u) \text{Erfc}(|u|\sqrt{\pi}) = -\frac{2}{\pi} \text{sgn}(u) \int_0^\infty e^{-\pi u^2(v^2+1)} (v^2 + 1)^{-1} dv.$$
The auxiliary function used to define the generalized error function is then

\[ e_2(u_1, u_2) = 2u_1 \int_0^1 e^{-\pi t^2 u_1^2} E_1(tu_1) \, dt \]

\[ = 4u_1 u_2 \int_0^1 \int_0^1 e^{-\pi t^2 (u_1^2 + u_2^2)} \, t \, dt \, dv \]

\[ = \frac{2u_1 u_2}{\pi} \int_0^1 \left( 1 - e^{-\pi (u_1^2 + u_2^2)} \right) (u_1^2 + u_2^2)^{-1} \, dv \]

\[ = \frac{2}{\pi} \text{Arctan} \left( \frac{u_1}{u_2} \right) - \frac{2}{u_2} e^{-\pi u_2^2} \int_0^u \left( u_1 \right) e^{-\pi v^2} \left( u_1^2 + v^2 \right)^{-1} \, dv \]

\[ = \frac{2}{\pi} \text{Arctan} \left( \frac{u_1}{u_2} \right) - \tilde{e}_2(u_1, u_2), \]

where the last line defines \( \tilde{e}_2(u_1, u_2) \), cf. (4.5). The generalized error function \( E_2 \) can be expressed as, cf. (3.24) of [1],

\[ E_2(\alpha; u_1, u_2) = -\tilde{e}_2(u_1, u_2) - \tilde{e}_2(u_1', u_2') + \text{sgn}(u_2) \text{sgn}(u_2'), \quad (6.4) \]

where

\[ u_1' = \frac{u_2 - \alpha u_1}{\sqrt{1 + \alpha^2}}, \quad u_2' = \frac{u_1 + \alpha u_2}{\sqrt{1 + \alpha^2}}. \]

It has, of course, a more elegant definition given in section 3.1 of [1], but (6.4) is all we will need. Now the ‘boosted’ version, Definition 3.14 of [1], is defined for a pair of negative vectors \( C_1, C_2 \) with \( \Delta(C_1, C_2) > 0 \) and a vector \( x \in V \) by

\[ E_2(C_1, C_2; x) = E_2 \left( \frac{(C_1, C_2)}{\sqrt{\Delta(C_1, C_2)}}, \frac{(C_1 \perp x)}{\sqrt{|(C_1 \perp x)\Delta(C_1, C_2)|}}, \frac{(C_2, x)}{\sqrt{|(C_2, C_2)|}} \right). \]

After a short calculation using (6.4), this yields (4.7).

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