No-hair theorems for non-canonical self-gravitating static multiple scalar fields

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We prove under certain assumptions no-hair theorems for non-canonical self-gravitating static multiple scalar fields in spherically symmetric spacetimes. It is shown that the only static, spherically symmetric and asymptotically flat black hole solutions consist of the Schwarzschild metric and a constant multi-scalar map. We also prove that there are no static, horizonless, asymptotically flat, spherically symmetric solutions with static scalar fields and a regular center. The last theorem shows that the static, asymptotically flat, spherically symmetric reflecting compact objects with Neumann boundary conditions can not support a non-trivial self-gravitating non-canonical (and canonical) multi-scalar map in their exterior spacetime regions. In order to prove the no-hair theorems we derive a new divergence identity.

I. INTRODUCTION

Many theories beyond the standard model physics predict the existence of additional scalar fields. The natural question that arises is whether there exist black holes supporting scalar hair beyond those existing in General relativity. This is an old problem in gravitational physics going back to the classical paper [1]. In the case of a single scalar field, minimally or non-minimally coupled to gravity, with a non-negative potential, there are no-hair theorems which rule out the existence of stationary asymptotically flat black holes with a scalar hair under some certain conditions. We refer the reader to the recent review [2] for exhaustive discussion of the black holes with single scalar hair. Another interesting and important problem is the extension of the no-hair theorems to the case of a single non-canonical scalar field. No-hair theorems for a non-canonical single scalar field were presented in [3]–[5].

It is of obvious theoretical interest to see how far the above results can be extended in the case of both canonical and non-canonical multiple scalar fields. Some results in this direction have also been established for canonical multiple scalar fields. It was proven in [6] that there exist no static and spherically symmetric asymptotically flat black holes with multiple scalar hair when the scalar fields are static and their potential is non-negative. Similar result holds for static, horizonless, asymptotically flat, spherically symmetric solutions with static scalar fields [6]. These results can be extended to the case of stationary (rotating) solutions with time independent harmonic mappings [7]. In general the situation with the multiple scalar fields is much more complicated in comparison with the case of a single scalar field. For example, if we allow the scalar fields to be

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time dependent in a certain way so that the effective scalar fields energy-momentum tensor to be
time independent, the picture can change drastically. It was shown in [8] that the Kerr black hole
can support hair of two time dependent scalar fields. Generalizations of [8] were presented in [9].
Moreover there exist also soliton-like (horizoless) solutions with multiple scalar fields [10], [11].

The purpose of the present paper is to prove several no-hair theorems for non-canonical self-
gravitating static multiple scalar fields in spherically symmetric spacetimes. Under certain as-
sumptions we prove that there are no non-trivial static, asymptotically flat and spherically sym-
metric black holes and particle-like solutions with static non-canonical multiple scalar hair. We
also show that the static, spherically symmetric reflecting objects with Neumann boundary con-
ditions in an asymptotically flat spacetime can not support static hair consisting of non-canonical
scalar fields. We prove our no-hair theorems by first deriving a new divergence identity. In the
particular case of canonical scalar fields our approach gives a new independent proof of the the-
orems of [6].

II. BASIC EQUATIONS

Let \((M, g)\) is the 4-dimensional spacetime and \((\mathcal{E}_N, \gamma)\) is a N-dimensional Riemannian mani-
fold with metric \(\gamma\). We consider a map \(\varphi : (M, g) \to (\mathcal{E}_N, \gamma)\) and its deferential \(d\varphi\) induces a map
between the tangent spaces of \(M\) and \(\mathcal{E}_N\), \(d\varphi : TM \to T\mathcal{E}_N\). The norm of the differential will be
denoted by \(<d\varphi, d\varphi>\). In local coordinate patches on \(M\) and \(\mathcal{E}_N\) we have

\[
<d\varphi, d\varphi> = g^{\mu\nu}(x) \gamma_{ab}(\varphi(x)) \partial_\mu \varphi^a(x) \partial_\nu \varphi^b(x).
\]

The multi-scalar theories we focus on are self-gravitating theories with non-canonical scalar
fields given by the action

\[
S = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} [R - f_*(\varphi, K)],
\]

where \(R\) is the Ricci scalar curvature, \(K = 2 <d\varphi, d\varphi>\) and \(f_*(\varphi, K)\) is at least \(C^2\) - function
in \(\varphi\) and \(K\). Some mild restrictions on \(f_*(\varphi, K)\) will be discussed below. The canonical scalar
fields correspond to the particular choice \(f_*(\varphi, K) = K + 4V(\varphi)\), where \(V(\varphi)\) is the scalar fields
potential. The field equations corresponding to the action (2) and written in local coordinate
patches of \(M\) and \(\mathcal{E}_N\) we have the following

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 \frac{\partial f_*}{\partial K} \gamma_{ab}(\varphi) \nabla_\mu \varphi^a \nabla_\nu \varphi^b - \frac{1}{2} f_* g_{\mu\nu},
\]

\[
\nabla_\mu \left( \frac{\partial f_*}{\partial K} \nabla^\mu \varphi^a \right) = - \frac{\partial f_*}{\partial K} \gamma_{cd}^{a}(\varphi) \nabla_\mu \varphi^c \nabla_\nu \varphi^d + \frac{1}{4} \gamma^{ab}(\varphi) \frac{\partial f_*}{\partial \varphi^b},
\]

where \(\nabla_\mu\) is the covariant derivative with respect to the spacetime metric \(g_{\mu\nu}\) and \(\gamma_{cd}^{a}(\varphi)\) are the
Christoffel symbols with respect to the target space metric \(\gamma_{ab}(\varphi)\).

Let us discuss the restrictions on the function \(f_*(\varphi, K)\). As in the case of a single non-canonical
scalar field, the classical and quantum stability of the theory is guaranteed when \(\frac{\partial f_*}{\partial K} \geq 0\) and
\[ \frac{\partial f_s}{\partial K} + 2K \frac{\partial^2 f_s}{\partial K^2} \geq 0 \] These conditions ensure also that the initial value formulation of the theory is well-posed \cite{12}. Another condition which is natural from a physical point of view is \( \lim_{K \to 0} \frac{\partial f_s}{\partial K} = 1 \) which means that that in the weak field regime we recover the canonical case. One more condition on \( f_s \) playing crucial role in proving the no-hair theorems, will be imposed in Sec. 3.

From now on we shall focus on the static and spherically symmetric case with a metric
\[ ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 s_{ij} dx^i dx^j, \] where \( s_{ij} \) is the metric on the unit 2D sphere, namely \( s_{ij} dx^i dx^j = d\theta^2 + \sin^2 \theta d\phi^2 \).

We require the scalar fields to be also static, \( L_\xi \phi^a = 0 \), i.e. the Lie derivative of the scalar fields along the timelike Killing vector \( \xi = \frac{\partial}{\partial t} \) to be zero. This condition automatically insures that the effective energy-momentum tensor of the scalar fields is static, \( L_\xi T_{\mu \nu}^\phi = 0 \). Contrary to the static symmetry, the scalar fields are not required to respect the spherical symmetry, i.e. to depend only on the radial coordinate \( r \). They can depend on the angular variables \( \theta \) and \( \phi \) in such way that the energy-momentum tensor respects the spherical symmetry (see for example \cite{13, 14}).

Under our assumptions the dimensionally reduced field equations are the following
\begin{align*}
\frac{2}{r} e^{-2\Lambda} \Lambda' + \frac{1}{r^2} \left( 1 - e^{-2\Lambda} \right) &= \frac{1}{2} f_s, \\
\frac{2}{r} e^{-2\Lambda} \Phi' - \frac{1}{r^2} \left( 1 - e^{-2\Lambda} \right) &= 2 \frac{\partial f_s}{\partial K} e^{-2\Lambda} \gamma_{ab}(\phi) \partial_r \phi^a \partial_r \phi^b - \frac{1}{2} f_s, \\
e^{-2\Lambda} \left[ \Phi'' + \left( \Phi' + \frac{1}{r} \right) (\Phi' - \Lambda') \right] r^2 s_{ij} &= 2 \frac{\partial f_s}{\partial K} \gamma_{ab}(\phi) \partial_r \phi^a \partial_j \phi^b - \frac{1}{2} f_s r^2 s_{ij}, \\
\partial_r \left( e^{\Phi - \Lambda} r^2 \frac{\partial f_s}{\partial K} \gamma_{ab}(\phi) \partial_r \phi^b \right) + \frac{\Phi + \Lambda}{\sqrt{s}} \partial_i \left( \sqrt{s} s_{ij} \frac{\partial f_s}{\partial K} \gamma_{ab}(\phi) \partial_j \phi^b \right) &= \frac{\Phi + \Lambda}{4} r^2 \left( \frac{\partial f_s}{\partial \phi^a} + \frac{\partial f_s}{\partial K} \frac{\partial \phi^a}{\partial \phi^b} \right).
\end{align*}

From the spherical symmetry via the dimensionally reduced field equations it follows that \( f_s, \frac{\partial f_s}{\partial K} \gamma_{ab}(\phi) \partial_r \phi^a \partial_r \phi^b \) and \( \frac{\partial f_s}{\partial K} s_{ij} \gamma_{ab}(\phi) \partial_i \phi^a \partial_j \phi^b \) are functions of \( r \) only. We define
\[ P^2(r) = \frac{\partial f_s}{\partial K} \gamma_{ab}(\phi) \partial_r \phi^a \partial_r \phi^b, \quad H^2(r) = \frac{\partial f_s}{\partial K} s_{ij} \gamma_{ab}(\phi) \partial_i \phi^a \partial_j \phi^b. \]

In the present paper we consider only asymptotically flat spacetimes. In this case, by using the dimensionally reduced field equations, it is not difficult one to show that \( \lim_{r \to \infty} f_s(r) = 0, \lim_{r \to \infty} P^2(r) = 0 \) and \( \lim_{r \to \infty} H^2(r) = 0 \). More precisely \( f_s(r), P^2(r) \) and \( H^2(r) \) drop off at least as
\[ f_s(r) \sim \frac{1}{r^4}, \quad P^2(r) \sim \frac{1}{r^4}, \quad H^2(r) \sim \frac{1}{r^2}. \]
\] for \( r \to \infty \).

**III. DIVERGENCE IDENTITY**

In this section we derive a divergence identity which plays a central role in proving the no-hair theorems. In fact the potential application of our divergence identity is beyond the no-hair theorems. It can be used to study the quantitative and qualitative properties of static and spherically symmetric vacuum solutions in theories with multiple scalar fields as those presented in \cite{13-15}. 

In order to derive the desired divergence identity we multiply the equation (9) for $q^a$ by $\partial_r q^a$ and after some algebra we obtain

$$
\partial_r \left( e^{\Phi - \Lambda} r^2 p^2(r) \right) - e^{\Phi - \Lambda} r^2 \frac{\partial f_s}{\partial K} \gamma_{ab}(q) \partial_r q^b \partial_r q^a + e^{\Phi + \Lambda} \partial_r q^a D_i j_i^a = \frac{e^{\Phi + \Lambda}}{4} r^2 \left( \frac{\partial f_s}{\partial q^a} + \frac{\partial f_s}{\partial K} \frac{\partial K}{\partial q^a} \right) \partial_r q^a
$$

(12)

where we have defined $j_i^a = \frac{\partial f_s}{\partial K} \gamma_{ab}(q) D_i q^b$ and $D_i$ is the covariant derivative with respect to $s_i$. The next step is to express $e^{\Phi - \Lambda} r^2 \frac{\partial f_s}{\partial K} \gamma_{ab}(q)$ again from (9) and to substitute into the above equation (12). We find

$$
\partial_r \left( e^{\Phi - \Lambda} r^2 p^2(r) \right) + \partial_r \left( e^{\Phi - \Lambda} r^2 \frac{\partial f_s}{\partial K} \gamma_{ab}(q) \right) \partial_r q^a \partial_r q^b + 2e^{\Phi + \Lambda} \partial_r q^a D_i j_i^a = \frac{e^{\Phi + \Lambda}}{2} r^2 \left( \frac{\partial f_s}{\partial q^a} + \frac{\partial f_s}{\partial K} \frac{\partial K}{\partial q^a} \right) \partial_r q^a.
$$

(13)

Proceeding further we can work out the term $2e^{\Phi + \Lambda} \partial_r q^a D_i j_i^a$. Using the dimensional reduced field equations and after long manipulations one can show that

$$
2e^{\Phi + \Lambda} \partial_r q^a D_i j_i^a = -e^{\Phi + \Lambda} \partial_r H^2(r) + e^{\Phi + \Lambda} D_i D^i q^a \partial_r \left( \frac{\partial f_s}{\partial K} \gamma_{ab}(q) \right).
$$

(14)

Substituting this expression back into (13) and after long algebra we get

$$
\frac{d}{dr} \left[ e^{\Phi + \Lambda} H^2(r) + \frac{1}{2} e^{\Phi + \Lambda} r^2 (f_s - K \frac{\partial f_s}{\partial K}) - e^{\Phi - \Lambda} r^2 p^2(r) \right] = e^{\Phi - \Lambda} r^2 \left( \Phi' - \Lambda' + \frac{2}{r} \right) p^2(r) + e^{\Phi + \Lambda} \left( \Phi' + \Lambda' \right) \left[ H^2(r) + \frac{1}{2} \left( f_s - K \frac{\partial f_s}{\partial K} \right) r^2 \right] + re^{\Phi + \Lambda} \left( f_s - K \frac{\partial f_s}{\partial K} \right).
$$

(15)

The last step is to express $(\Phi' + \Lambda')$ and $(\Phi' - \Lambda' + \frac{2}{r})$ form the dimensionally reduced equations, namely

$$
\Phi' + \Lambda' = r p^2(r),
$$

(16)

$$
(\Phi' - \Lambda' + \frac{2}{r}) e^{-2\Lambda} = \frac{1 + e^{-2\Lambda}}{r} + r e^{-2\Lambda} p^2(r) - \frac{1}{2} r f_s
$$

(17)

and to substitute them back into (15). Doing so we finally obtain the desired divergence identity

$$
\frac{d}{dr} \left[ e^{\Phi + \Lambda} H^2(r) + \frac{1}{2} e^{\Phi + \Lambda} r^2 (f_s - K \frac{\partial f_s}{\partial K}) - e^{\Phi - \Lambda} r^2 p^2(r) \right] = re^{\Phi + \Lambda} \left[ (1 + e^{-2\Lambda}) p^2(r) + (f_s - K \frac{\partial f_s}{\partial K}) \right].
$$

(18)
In what follows we shall consider theories with

\[ f_s - K \frac{\partial f_s}{\partial K} \geq 0. \tag{19} \]

When this condition is satisfied, the right hand side of the divergence identity is non-negative. In the case of canonical scalar fields, i.e. for \( f_s = K + 4V(\varphi) \), our condition (19) reduces just to \( V(\varphi) \geq 0 \). It is also instructive to present the divergence identity in the particular case of canonical scalar fields. Then the divergent identity takes the form

\[
\frac{d}{dr} \left[ e^{\Phi + \Lambda} H^2(r) + 2e^{\Phi + \Lambda} r^2 V(\varphi) - e^{\Phi - \Lambda} r^2 P^2(r) \right] = re^{\Phi + \Lambda} \left[ (1 + e^{-2\Lambda}) P^2(r) + 4V(\varphi) \right]. \tag{20}
\]

**IV. NO-HAIR THEOREMS**

In this section we present and prove our no-hair theorems. The first no-hair theorem is a theorem which states that there do not exist static, asymptotically flat and spherically symmetric black holes with non-trivial non-canonical static multiple scalar fields. More precisely we have:

**Theorem 1** Let us consider self-gravitating non-canonical multi-scalar map \( \varphi : (M, g) \to (\mathcal{E}_N, \gamma) \) with the action (2) and let the function \( f_s(\varphi, K) \) satisfy the inequality \( f_s - K \frac{\partial f_s}{\partial K} \geq 0 \). Then every static and spherically symmetric black hole solution to the field equations (3)-(4) with static multi-scalar map \( \varphi (\mathcal{L}_E \varphi = 0) \) and regular horizon consists of the Schwarzschild solution and a constant map \( \varphi_0 \) with \( f_s(\varphi_0, K) = 0 \).

The key role in proving the theorem plays the divergence identity (18) derived in the previous section. Integrating this identity from the horizon \( r = r_h \) to infinity we get

\[
-e^{(\Phi + \Lambda)h} \left[ H^2(r_h) + \frac{1}{2} \frac{1}{h}(f_s - K \frac{\partial f_s}{\partial K})h \right] = \int_{r_h}^{\infty} dr e^{(\Phi + \Lambda)} \left[ (1 + e^{-2\Lambda}) P^2(r) + (f_s - K \frac{\partial f_s}{\partial K}) \right], \tag{21}
\]

where we have taken into account that \( \lim_{r \to \infty} H^2(r) = 0, \lim_{r \to \infty} r^2 P^2(r) = 0, \lim_{r \to \infty} r^2(f_s - K \frac{\partial f_s}{\partial K}) = 0 \) as well as \( (e^{\Phi - \Lambda} r^2 P^2(r))h = 0 \). Note that \( (\Phi + \Lambda)h \) is finite for regular horizons. The right hand side of (21) is non-negative while the left hand side is non-positive. Therefore we can conclude that both sides vanish. Consequently we have \( P^2(r) = 0 \) and \( f_s - K \frac{\partial f_s}{\partial K} = 0 \) for every \( r \in [r_h, \infty) \) as well as \( H^2(r_h) = 0 \). Turning again to the divergence identity (18) and using the fact that \( P^2(r) = 0 \) and \( f_s - K \frac{\partial f_s}{\partial K} = 0 \) we find that \( H^2(r) = 0 \) for every \( r \in [r_h, \infty) \). The fact that \( P^2(r) = H^2(r) = 0 \) implies \( K(r) = 0 \) which in turn means \( f_s(\varphi, K = 0) = 0 \). As a consequence of all this we conclude that the map \( \varphi \) is a constant map \( \varphi = \varphi_0 \) with \( f_s(\varphi_0, K = 0) = 0 \).

All the above results show that the right hand side of the dimensionally reduced field equations (5)-(9) vanish and the equations are reduced to the vacuum, static and spherically symmetric Einstein equations whose unique black hole solution with regular horizon is the Schwarzschild one. This concludes the proof of the theorem.

The next theorem deals with static, horizonless, asymptotically flat, spherically symmetric solutions with static scalar fields and a regular center. Before formulating the theorem let us discuss
the regularity conditions at the center. The geometry of the \( t = \text{const} \) hypersurfaces has to be locally Euclidean in a small vicinity of the center \( r = 0 \) which means \( \lim_{r \to 0} e^{2 \lambda} = 1. \)

**Theorem 2** Let us consider self-gravitating non-canonical multi-scalar map \( \varphi : (M, g) \to (E_N, \gamma) \) with the action (2) and let the function \( f_\ast(\varphi, K) \) satisfy the inequality \( f_\ast - K \frac{\partial f_\ast}{\partial K} \geq 0. \) Then every static, asymptotically flat, spherically symmetric solution to the field equations (3.4) with static scalar fields and a regular center consists of the flat metric and a constant map \( \varphi_0 \) with \( f_\ast(\varphi_0, K = 0) = 0. \)

The proof is as follows. The regularity at the center and the dimensionally reduced field equations imply \( \lim_{r \to 0} e^{\Phi + \Lambda} H^2 (r) = \lim_{r \to 0} r^2 e^{\Phi - \Lambda} p^2 (r) = \lim_{r \to 0} r^2 e^{\Phi + \Lambda} \left( f_\ast - K \frac{\partial f_\ast}{\partial K} \right) = 0. \) Using this fact and integrating the divergence identity from \( r = 0 \) to infinity we obtain

\[
\int_0^\infty dr r e^{\Phi + \Lambda} \left[ (1 + e^{-2 \Lambda}) p^2 (r) + (f_\ast - K \frac{\partial f_\ast}{\partial K}) \right] = 0. \tag{22}
\]

Therefore, we have \( p^2 (r) = 0 \) and \( f_\ast - K \frac{\partial f_\ast}{\partial K} = 0 \) for every \( r \in [0, \infty) \) which substituted back in (13) give \( H^2 (r) = 0 \) for \( r \in [0, \infty). \) As in the previous case we can conclude that the map \( \varphi \) is a constant map \( \varphi = \varphi_0 \) with \( f_\ast(\varphi_0, K = 0) = 0. \) Then equations (6-9) reduce again to the vacuum, static, and spherically symmetric Einstein equations whose unique asymptotically flat solution with a regular center is the flat metric.

Let us mention that our theorems proven above in the particular case of canonical scalar fields recover the results of [6], however the method of proving is completely different from the method of [6].

The last no-hair theorem is related to the so-called reflecting compact objects (stars) [16],[17]. More precisely we would like to answer the following question: Can regular compact, spherically symmetric reflecting objects (which possess no event horizons) support nontrivial, self-gravitating non-canonical (and canonical) static multi scalar field configurations in their exterior spacetime regions? A rigorous answer to this question can be given only for Neumann boundary conditions. In other words we consider a reflecting compact object with a coordinate radius \( \mathcal{R} \) at which we have \( \frac{\partial \varphi}{\partial t} \big|_\mathcal{R} = 0. \)

**Theorem 3** Let us consider self-gravitating non-canonical multi-scalar map \( \varphi : (M, g) \to (E_N, \gamma) \) with the action (2) and let the function \( f_\ast(\varphi, K) \) satisfy the inequality \( f_\ast - K \frac{\partial f_\ast}{\partial K} \geq 0. \) Then the static, asymptotically flat, spherically symmetric reflecting compact objects with Neumann boundary conditions can not support a non-trivial self-gravitating non-canonical (and canonical) multi-scalar map \( \varphi : (M, g) \to (E_N, \gamma) \) in their exterior spacetime regions.

The proof of the theorem is the same as in the case of the theorem for black holes with the only difference that the integration is from the surface \( r = \mathcal{R} \) of the reflecting object to infinity.

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