Abstract. We discuss stationary discs for generic CR manifolds and apply them to the problem of finite jet determination for CR mappings. We prove that a CR diffeomorphism of two finitely smooth strictly pseudoconvex Levi generating CR manifolds is uniquely determined by its 2-jet at a given point. A new key element of the proof is the existence of non-defective stationary discs.

Key words: Finite jet determination, CR mapping, Stationary disc.

1. Introduction

We discuss stationary discs for generic CR manifolds and apply them to the problem of finite jet determination for CR mappings.

Let $M \subset \mathbb{C}^n$ be a generic real submanifold of real codimension $k$. Then $M$ has CR dimension $m = n - k$. We introduce coordinates $(z, w) \in \mathbb{C}^n$, $z = x + iy \in \mathbb{C}^k$, $w \in \mathbb{C}^m$, so that $M$ has a local equation of the form

$$x = h(y, w),$$

where $h = (h_1, \ldots, h_k)$ is a smooth real vector function with $h(0) = 0$, $dh(0) = 0$. Furthermore, one can choose the coordinates in such a way that each term in the Taylor expansion of $h$ contains both $w$ and $\overline{w}$ variables. Then the equations of $M$ take the form

$$x_j = h_j(y, w) = \langle A_j w, \overline{w} \rangle + O(|y|^3 + |w|^3), \quad 1 \leq j \leq k.$$ 

Here $A_j$ are hermitian matrices, and $\langle a, b \rangle = \sum a_i b_i$. The matrices $A_j$ can be regarded as the components of the vector valued Levi form of $M$ at 0.

- We say $M$ is **Levi generating** at 0 if the matrices $A_j$ are linearly independent.
- We say $M$ is **Levi nondegenerate** at 0 if $\langle A_j u, \overline{v} \rangle = 0$ for all $j$ and $u$ implies $v = 0$.
- We say $M$ is **strongly Levi nondegenerate** at 0 if there is $c \in \mathbb{R}^k$ such that $\det (\sum c_j A_j) \neq 0$. This condition means that $M$ lies on a Levi nondegenerate hypersurface.
- We say $M$ is **strongly pseudoconvex** at 0 if there is $c \in \mathbb{R}^k$ such that $\sum c_j A_j > 0$. This condition means that $M$ lies on a strongly pseudoconvex hypersurface.
We are concerned with the problem whether a CR diffeomorphism between two manifolds is uniquely determined by its finite jet at a point. This problem is called the problem of finite jet determination, and it has been a subject of work by many authors (Baouendi, Beloshapka, Bertrand, Blanc-Centi, Ebenfelt, Kim, Lamel, Meylan, Mir, Rothschild, Zaitsev, etc., see [2]). We are interested in the problem for generic manifolds of higher codimension. We restrict to the Levi generating case here.

Beloshapka [1] proved that a real analytic CR automorphism of a real analytic generic Levi generating and Levi nondegenerate manifold is determined by its 2-jet at a point. This result leads to the question whether it still holds for finitely smooth manifolds.

Bertrand, Blanc-Centi and Meylan [2] prove 2-jet determination for $C^3$-smooth CR automorphisms of $C^4$-smooth generic strongly Levi nondegenerate manifold $M$ under the additional assumption that there is $v \in \mathbb{C}^m$ such that the vectors $\{A_jv : 1 \leq j \leq k\}$ are $\mathbb{C}$-linearly independent. This condition is quite restrictive, in particular, it implies that $k \leq m$ whereas the natural restriction imposed by the Levi generating condition is $k \leq m^2$.

Our main result is the following.

**Theorem 1.1.** Let $M_1$ and $M_2$ be $C^4$-smooth generic strongly pseudoconvex Levi generating manifolds in $\mathbb{C}^n$. Let $p \in M_1$. Then every germ at $p$ of a $C^2$-smooth CR diffeomorphism $f : M_1 \to M_2$ is uniquely determined by its 2-jet at $p$.

Our proof as well as the proof in [2] is based on stationary discs. Lempert [4] introduced extremal and stationary discs for strongly convex domains and applied them to various problems. The author [7] developed a local version of the theory of extremal and stationary discs in higher codimension and applied it to the regularity of CR mappings.

Stationary discs form a family invariant under CR mappings. They depend on finitely many, namely $4n$, real parameters. To apply stationary discs to the problem of finite jet determination, one would like to know that they cover a sufficiently large set. This will be the case if there exists a non-defective stationary disc through the given point.

Defective discs arise in the problem of wedge extendibility of CR functions [6] and can be characterized as critical points of the evaluation map $\phi \mapsto \phi(0)$ defined on the set of all complex discs $\phi$ attached to $M$. It follows from [6] that for a Levi generating manifold $M$, there are many non-defective discs through every point of $M$. However, it does not follow that there are non-defective stationary discs. The key new result of this paper is the existence of non-defective stationary discs, which the author conjectured in [8] and reduced it to a linear algebra question. We answer it here for a strongly pseudoconvex Levi generating manifold.

Using the results obtained here, we can improve our earlier result of [7, 8] on the regularity of CR mappings.

**Theorem 1.2.** Let $M_1$ and $M_2$ be $C^\infty$ smooth generic strictly pseudoconvex Levi generating manifolds in $\mathbb{C}^n$, and let $F : M_1 \to M_2$ be a homeomorphism such that both $F$ and $F^{-1}$ are CR and satisfy a Lipschitz condition with some exponent $0 < \alpha < 1$. Then $F$ is $C^\infty$ smooth.
Previously [7, 8], there was an additional condition on the existence of non-defective stationary discs for $M_1$ and $M_2$.

The paper is structured as follows. In Section 2, we recall some basics on CR manifolds. In Section 3, we introduce stationary discs. In Section 4, we discuss parametrization of stationary discs by their jets at a boundary point and prove Theorem 1.1 assuming the needed existence of non-defective stationary discs. In Section 5, we discuss the existence of non-defective stationary discs and reduce it to a result in linear algebra. In Section 5, we prove that result.

2. CR manifolds

Let $M$ be a smooth real submanifold in $\mathbb{C}^n$. Recall that the complex tangent space at $p \in M$ is the maximum complex subspace in $T_p(M)$.

$$T^c_p(M) = T_p(M) \cap JT_p(M), \quad p \in M.$$ 

Here $J : \mathbb{C}^n \to \mathbb{C}^n$ is the operator of multiplication by $i = \sqrt{-1}$. The manifold $M$ is called a CR manifold if $\dim T^c_p(M)$ does not depend on $p \in M$. Then the dimension $\dim C T^c_p(M)$ is called the CR dimension of $M$ and is denoted by $\dim_{CR} M$.

The manifold $M$ is called generic if $T_p(M)$ spans $T_p(\mathbb{C}^n) \simeq \mathbb{C}^n$ over $\mathbb{C}$ for all $p \in M$, that is,

$$T_p(M) + JT_p(M) = \mathbb{C}^n.$$

If $M$ is generic, then $M$ is a CR manifold and

$$\dim_{CR} M + \text{cod} M = n,$$

where $\text{cod} M$ is the real codimension of $M$ in $\mathbb{C}^n$.

Let $M$ be a generic manifold in $\mathbb{C}^n$. Let $T^{*,1,0}(\mathbb{C}^n)$ be the bundle of $(1,0)$ forms in $\mathbb{C}^n$.

$$T^{*,1,0}(\mathbb{C}^n) = \{ \omega = \sum \omega_j dz_j : \omega_j \in \mathbb{C}, 1 \leq j \leq n \}.$$

The conormal bundle $N^*(M)$ of $M$ in $\mathbb{C}^n$ is the real dual bundle to the normal bundle $N(M) = T(\mathbb{C}^n)|_M / T(M)$.

Every real form is a real part of a unique $(1,0)$ form. Then we can view $N^*(M)$ as a real submanifold in $T^{*,1,0}(\mathbb{C}^n)$.

$$N^*_p(M) = \{ \omega \in T^{*,1,0}_p(\mathbb{C}^n) : \text{Re} \langle \omega, X \rangle = 0, X \in T_p(M) \}.$$

Let $M \subset \mathbb{C}^n$ be a generic manifold of codimension $k$. Then $M$ has a local equation $\rho = 0$, where $\rho = (\rho_1, \ldots, \rho_k)$ is a smooth $\mathbb{R}^k$-valued function in a neighborhood of $0 \in \mathbb{C}^n$ such that $\partial \rho_1 \wedge \cdots \wedge \partial \rho_k \neq 0$. 


The spaces $T_p(M)$, $T^*_p(M)$, and $N^*_p(M)$ have the descriptions

$$T_p(M) = \{ X \in T_p(C^n) : \langle d\rho_j(p), X \rangle = 0, 1 \leq j \leq k \},$$

$$T^*_p(M) = \{ X \in T_p(C^n) : \langle \partial\rho_j(p), X \rangle = 0, 1 \leq j \leq k \},$$

$$N^*_p(M) = \{ c\partial\rho = \sum c_j\partial\rho_j(p) : c \in \mathbb{R}^k \}.$$

Here

$$d\rho = \partial\rho + \overline{\partial}\rho, \quad d\rho_j = \partial\rho_j + \overline{\partial}\rho_j, \quad \partial\rho = \sum \frac{\partial\rho}{\partial z_\ell} dz_\ell.$$

3. Stationary discs

A complex disc is a map $\phi : \mathbb{D} \to X$, of the standard unit disc $\mathbb{D} \subset \mathbb{C}$ to a complex manifold $X$, $\phi \in O(\mathbb{D}) \cap C(\mathbb{D})$.

We say a complex disc $\phi$ is attached to a set $M \subset \mathbb{C}^n$ if $\phi(b\mathbb{D}) \subset M$.

**Definition 3.1.** [4, 7] Let $M$ be a generic manifold in $\mathbb{C}^n$. A complex disc $\phi$ attached to $M$ is called stationary if there exists a nonzero continuous holomorphic mapping $\phi^* : \overline{\mathbb{D}} \setminus \{0\} \to T^{*1,0}(\mathbb{C}^n)$, such that $\zeta \mapsto \zeta\phi^*(\zeta)$ is holomorphic in $\mathbb{D}$ and $\phi^*(\zeta) \in N^*_\phi(\zeta)M$ for all $\zeta \in b\mathbb{D}$.

In other words, $\phi^*$ is a punctured complex disc with a pole of order at most one at zero attached to $N^*(M) \subset T^{*1,0}(\mathbb{C}^n)$ such that the natural projection sends $\phi^*$ to $\phi$. We call $\phi^*$ a lift of $\phi$. We will always use the term “lift” in this meaning.

Stationary discs bear this name because they arise from an extremal problem, but we do not need it here.

It can happen that a lift has no pole. Following [6], we call a disc $\phi$ defective if it has a nonzero lift $\phi^*$ holomorphic in the whole unit disc including 0. Defective discs will be the main concern in this paper. For a strictly convex hypersurface, all defective discs are constant. However, this is no longer true for strictly pseudoconvex manifolds of higher codimension, see Remark 5.2.

To state the main result on the existence of stationary discs, we need a general description of their lifts. Let $M$ be again a codimension $k$ generic manifold in $\mathbb{C}^n$ defined by a local equation

$$\rho = 0, \quad \rho(z, w) = x - h(y, w),$$

where $z = x + iy \in \mathbb{C}^k$, $w \in \mathbb{C}^m$, $k + m = n$, $h(0, 0) = 0$, $dh(0, 0) = 0$.

Let $\phi$ be a complex disc attached to $M$. We recall a $k \times k$ matrix function $G$ on $b\mathbb{D}$ such that $G(1) = I$ and $G\rho_z \circ \phi$ extends holomorphically from $b\mathbb{D}$ to $\mathbb{D}$, here $\rho_z$ denotes the matrix of partial derivatives and $I$ is the identity matrix. For a small disc $\phi$, such a matrix $G$ close to the identity always exists and is unique (see [6, 7]). If $h$ does not depend on $y$, then $G \equiv I$. For simplicity we will omit writing “$\circ \phi$”.

**Proposition 3.2.** [7] Let $\phi$ be a small complex disc attached to $M$. 
Every lift $\phi^*$ of $\phi$ holomorphic at 0 has the form $\phi^*|_{bD} = cG\partial\rho$, where $c \in \mathbb{R}^k$.

(ii) $\phi$ is defective if and only if there exists a nonzero $c \in \mathbb{R}^k$ such that $cGh_w$ extends holomorphically to $\mathbb{D}$.

(iii) Every lift $\phi^*$ of $\phi$ has the form $\phi^*|_{bD} = \text{Re} (\lambda\z + c)G\partial\rho$, where $\lambda \in \mathbb{C}^k$, $c \in \mathbb{R}^k$.

(iv) $\phi$ is stationary if and only if there exist $\lambda \in \mathbb{C}^k$ and $c \in \mathbb{R}^k$ such that $\z \text{Re} (\lambda\z + c)Gh_w$ extends holomorphically to $\mathbb{D}$.

We again assume that

\begin{equation}
\label{equation2}
h_j(y, w) = \langle A_j w, \overline{w} \rangle + O(|y|^3 + |w|^3), \quad 1 \leq j \leq k.
\end{equation}

Here is the main result on the existence of stationary discs.

**Theorem 3.3.** [7] Let $M \subset \mathbb{C}^n$ be a $C^4$-smooth strictly pseudoconvex Levi generating manifold defined by [7, 2]. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $\lambda \in \mathbb{C}^k$, $c \in \mathbb{R}^k$, $w_0 \in \mathbb{C}^m$, $y_0 \in \mathbb{R}^k$, $v \in \mathbb{C}^m$ such that

\[ \sum \text{Re} (\lambda_j\z + c_j)A_j > \epsilon(|\lambda| + |c|)I \]

for all $\z \in b\mathbb{D}$ and $|w_0| < \delta, |y_0| < \delta, |v| < \delta$ there exists a unique stationary disc $\z \mapsto \phi(\z) = (z(\z), w(\z))$ with lift $\phi^*$ such that $w(1) = w_0$, $w'(1) = v$, $y(1) = y_0$, and $\phi^*|_{bD} = \text{Re} (\lambda\z + c)G\partial\rho$. The pair $(\phi, \phi^*)$ depends $C^2$-smoothly on $\z \in \partial\mathbb{D}$ and all the parameters $\lambda, c, w_0, y_0, v$.

4. Parametrization by $T(N^*(M))$ and proof of the main result

Let $\mathcal{E}$ be the set of all stationary discs with lifts $(\phi, \phi^*)$ provided by Theorem 3.3. We will call them stationary pairs, or for brevity $s$-pairs. The set $\mathcal{E}$ of $s$-pairs is parametrized by $\lambda, c, w_0, y_0, v$, which add up to $4n$ real parameters independent of the dimension of $M$. Using this parametrization, we can identify $\mathcal{E}$ with an open set in $\mathbb{R}^{4n}$. Note that $\dim T(N^*(M)) = 4n$, which suggests that $\mathcal{E}$ can be parametrized by $T(N^*(M))$ using the values and derivatives of the discs at a boundary point.

We recall the following evaluation maps [5, 7]:

\[ \mathcal{F} : \mathcal{E} \rightarrow T(N^*(M)), \quad \mathcal{F}(\phi, \phi^*) = (\phi(1), \phi^*(1), J\phi'(1), J\phi^*'((1))), \]

\[ \mathcal{G} : \mathcal{E} \rightarrow N^*(M) \times N^*(M), \quad \mathcal{G}(\phi, \phi^*) = (\phi(1), \phi^*(1), \phi(\z_0), \phi^*(\z_0)). \]

Here $\z_0 \in b\mathbb{D}$ is a fixed point, $\z_0 \neq 1$.

**Theorem 4.1.** [5, 7] Let $M$ be $C^4$-smooth Levi generating strongly pseudoconvex manifold in $\mathbb{C}^n$. Let $(\phi, \phi^*) \in \mathcal{E}$. Suppose $\phi$ is not defective. Then $\mathcal{F}$ and $\mathcal{G}$ are diffeomorphisms of a neighborhood of $(\phi, \phi^*)$ onto open sets respectively in $T(N^*(M))$ and $N^*(M) \times N^*(M)$.

For brevity, we call an $s$-pair $(\phi, \phi^*)$ non-defective if $\phi$ is not defective. By Theorem 4.1, every $s$-pair $(\phi, \phi^*)$ close to a given non-defective $s$-pair is uniquely determined by the data $\mathcal{F}(\phi, \phi^*)$. From the properties of the mapping $\mathcal{G}$ it follows that the union of $\phi(b\mathbb{D})$ for all
s-pairs \((\phi, \phi^*)\) close to a given non-defective s-pair with fixed \((\phi, \phi^*)(1)\) covers an open set in \(M\).

Thus the question on the parametrization of \(\mathcal{E}\) by \(T(N^*(M))\) depends on the existence of non-defective stationary discs. We prove the following.

**Theorem 4.2.** Let \(M\) be a strongly pseudoconvex Levi generating manifold in \(\mathbb{C}^n\). Let \(p \in M\). Let \(U\) be an open neighborhood of \(p\) in \(M\). Then there exists a non-defective stationary pair \((\phi, \phi^*) \in \mathcal{E}\) such that \(\phi(1) = p\) and \(\phi(\overline{\mathbb{D}}) \subset U\).

We will discuss this theorem in the next section. We now prove the main result on 2-jet determination assuming the parametrization of \(\mathcal{E}\) by \(T(N^*(M))\) provided by Theorems 4.1 and 4.2.

**Proof of Theorem 4.1.** Let \(f : M_1 \to M_2\) be a germ of a CR diffeomorphism at \(0 \in M_1\), and \(f(0) = 0\). Let \(g\) be a quadratic polynomial map with the same 2-jet at \(0\) as \(f\). Then by replacing \(f\) by \(g^{-1} \circ f\) we can assume that \(f\) has the same 2-jet at \(0\) as the identity. Then we can choose coordinates so that \(M_1\) and \(M_2\) have equations of the same form

\[
x_j = h_j(y, w) = \langle A_j w, \overline{w} \rangle + O(|y|^3 + |w|^3), \quad 1 \leq j \leq k,
\]

that differ only in the order 3 terms.

Let \((\phi_1, \phi_1^*)\) be a non-defective s-pair for \(M_1\), \(\phi_1(1) = 0\). There is a unique s-pair \((\phi_2, \phi_2^*)\) for \(M_2\) such that \((\phi_2, \phi_2^*)\) is close to \((\phi_1, \phi_1^*)\), and \(F_1(\phi_1, \phi_1^*) = F_2(\phi_2, \phi_2^*)\). Here \(F_1\) and \(F_2\) are the parametrization maps for \(M_1\) and \(M_2\).

The pair \((f \circ \phi_1, f_* \circ \phi_1^*)\) is an s-pair for \(M_2\). Since the 2-jet of \(f\) at \(0\) is the identity, we have \(F_2(f \circ \phi_1, f_* \circ \phi_1^*) = F_1(\phi_1, \phi_1^*) = F_2(\phi_2, \phi_2^*)\). Then by the unique parametrization, \(f \circ \phi_1 = \phi_2\).

Thus the map \(f\) is uniquely determined on the set \(\phi_1(b\mathbb{D})\). Since these sets cover an open set in \(M_1\), the mapping \(f\) is uniquely determined. \(\square\)

### 5. Existence of non-defective stationary discs

Let \(M \subset \mathbb{C}^n\) be again a smooth strictly pseudoconvex Levi generating manifold defined by \((\mathbf{1}, \mathbf{2})\). Let \(M_0\) be the corresponding quadric defined by

\[
x_j = \langle A_j w, \overline{w} \rangle, \quad 1 \leq j \leq k.
\]

Let \(\lambda \in \mathbb{C}^k, c \in \mathbb{R}^k\) satisfy

\[
\sum \text{Re} (\lambda_j \zeta + c_j) A_j > 0, \quad \zeta \in b\mathbb{D}.
\]

Let \(P = \sum \lambda_j A_j, Q = \sum c_j A_j\). Then (see [7]) the equation

\[
P^* X^2 + 2QX + P = 0
\]

has a unique solution \(X\) with all eigenvalues in \(\mathbb{D}\). For \(v \in \mathbb{C}^m\) we put \(S(X, v) = \text{Span} \{X^\ell v : \ell = 0, 1, \ldots \}\).
Lemma 5.1. Let \((\phi, \phi^*)\) be a stationary pair constructed for the quadric \(M_0\) by Theorem 3.3 using the data \((\lambda, c, v)\). The disc \(\phi\) is defective iff the linear operators \(S \to \mathbb{C}^m\) defined by the matrices \(A_1, \ldots, A_k\) are linearly dependent over \(\mathbb{R}\). Here \(S = S(X, v)\).

Remark 5.2. In general, defective stationary discs with condition \(\sum \text{Re} (\lambda_j \zeta + c_j) A_j > 0, \zeta \in bD\), do exist for \(M_0\). Say, let \(\lambda = 0\). Then \(X = 0\) and \(S = S(X, v) = \text{Span} \{v\}\). If \(k > 2m\), then the vectors \(A_j v \in \mathbb{C}^m\) are linearly dependent over \(\mathbb{R}\), so all discs with data \(\lambda = 0\) and arbitrary \(c, v\) are defective.

Since every manifold \(M\) is locally approximated by the quadric \(M_0\), it suffices to prove Theorem 4.2 on the existence of non-defective discs for a quadric (see [8], Proposition 8.4).

Theorem 5.3. Let \(M_0\) be a strongly pseudoconvex Levi generating quadric in \(\mathbb{C}^n\). Then there exists a non-defective stationary pair for \(M_0\). Hence, the set of all defective stationary pairs form a proper algebraic set in the space of parameters.

The quadratic equation (3) for \(X\) is hard to solve explicitly, but we can use the approximation\
\[ X = -\frac{1}{2}Q^{-1}P + O(|\lambda|^3). \]

Generically, this matrix \(X\) will have distinct eigenvalues. Then by Vandermonde, \(S(X, v) = \mathbb{C}^m\) for some \(v\), and the conclusion follows.

In general, we can chose \(c \in \mathbb{R}^k\) such that \(Q > 0\). Note that the criterion for defective discs is independent of linear coordinate changes in \(\mathbb{C}^m\). We choose the w-coordinates so that \(Q = I\).

Then Theorem 5.3 reduces to the following.

Theorem 5.4. Let \(A_1, \ldots, A_k\) be linearly independent hermitian \(m \times m\) matrices. Then there exist \(\lambda \in \mathbb{R}^k, v \in \mathbb{C}^m\) such that the matrices \(A_j\) define \(\mathbb{R}\)-linear independent operators \(S \to \mathbb{C}^m\). Here \(S = S(X, v), X = \sum \lambda_j A_j\).

Remark 5.5. We no longer need the strong pseudoconvexity condition. We only needed it for reduction to this theorem. We were able to do the reduction in the strongly pseudoconvex case. This is the main reason we restrict to this case here.

6. Proof of Theorem 5.4

We say that a \(m \times m\) matrix \(A\) is a \(r \times r (m_1, \ldots, m_r)\) block matrix if \(m_1 + \ldots + m_r = m\), \(A = (A_{ij})_{i,j=1}^r\), and each block \(A_{ij}\) is a \(m_i \times m_j\) matrix.

Lemma 6.1. Let \(A\) and \(B\) be hermitian \(2 \times 2 (m_0, m_1)\) block matrices. Suppose \(A_{00} = 0, A_{10} = 0, \det A_{11} \neq 0\). Let \(V_0\) be the space of the first \(m_0\) coordinates. Suppose for every small \(t \in \mathbb{R}\), the matrix \(A - tB\) has the eigenspace \(V(t)\) close to \(V_0\), \(\dim V(t) = m_0\). Then there exist \(\beta, \gamma \in \mathbb{R}\) such that \(B_{00} = \beta I, B_{01}A_{11}^{-1}B_{10} = \gamma I\).
Proof. Let \( \lambda(t) \) be the eigenvalue close to zero of the matrix \( A - tB \). Then \( \lambda(t) \) is an analytic function (see [3]). Since \( \lambda(0) = 0 \), we have \( \lambda(t) = \lambda_1 t + \lambda_2 t^2 + \ldots \). Furthermore, each eigenvector \( u(t) \) of \( A - tB \) with eigenvalue \( \lambda(t) \) can be represented as an analytic function \( u(t) = u_0 + u_1 t + u_2 t^2 + \ldots \), where \( u_0 \in V_0 \) and \( u_1, u_2, \ldots \in V_0^\perp \). We have \( (A - tB)u(t) = \lambda(t)u(t) \), that is,

\[
(A - tB - \lambda_1 t - \lambda_2 t^2 - \ldots)(u_0 + u_1 t + u_2 t^2 + \ldots) = 0.
\]

From vanishing the coefficients of degrees 0, 1, and 2, we obtain

\[
Au_0 = 0, \\
Au_1 = (B + \lambda_1 I)u_0, \\
Au_2 = (B + \lambda_1 I)u_1 + \lambda_2 u_0.
\]

Then (\ref{eq:4}) implies

\[
(B_{00} + \lambda_1 I)u_0 = 0,
\]

\[
A_{11}u_1 = B_{10}u_0,
\]

whereas (\ref{eq:5}) implies

\[
B_{01}u_1 + \lambda_2 u_0 = 0.
\]

From (\ref{eq:7}, \ref{eq:8}) we obtain

\[
(B_{01}A_{11}^{-1}B_{10} + \lambda_2 I)u_0 = 0.
\]

Since \( u_0 \in V_0 \) is arbitrary, the equations (\ref{eq:6}, \ref{eq:9}) imply the desired conclusion with \( \beta = -\lambda_1 \), \( \gamma = -\lambda_2 \).

We say the matrix \( B \) is subordinate to \( A \) if for every small \( t \in \mathbb{R} \) the matrix \( A - tB \) has the same number of distinct eigenvalues as \( A \).

**Lemma 6.2.** Let \( \alpha_1, \ldots, \alpha_r \) be all distinct eigenvalues of a hermitian matrix \( A \). Let \( v_1, \ldots, v_r \) be the corresponding eigenvectors. Let \( B \) be a hermitian matrix subordinate to \( A \). Suppose \( Bv_j = 0 \) for all \( 1 \leq j \leq r \). Then \( B = 0 \).

**Proof.** By applying a unitary transformation, we can assume that \( A \) is a diagonal \((m_1, \ldots, m_r)\) block matrix, where \( m_j \) is the multiplicity of \( \alpha_j \), that is, \( A_{ij} = \delta_{ij}\alpha_i I \).

Let \( V_j = \{ u \in \mathbb{C}^m : Au = \alpha_j u \} \). Then \( A - \alpha_j I \) and \( B \) satisfy the hypotheses of Lemma 6.1. By the latter, \( B_{jj} = \beta_j I \). Since \( v_j \in V_j \) and \( Bv_j = 0 \), we have \( \beta_j = 0 \) and \( B_{jj} = 0 \).

Let \( \tilde{A}_j \) be the matrix obtained from \( A - \alpha_j I \) by deleting all rows and columns contained in the block \( A_{jj} \). Let \( \tilde{B}_j = (B_{j1}, \ldots, \tilde{B}_{jj}, \ldots, B_{jr}) \) be the \( j \)-th row in the block representation of \( B \) with deleted block \( B_{ij} \). Here the hat means that the block \( B_{jj} \) is missing.

By Lemma 6.1, for some \( \gamma_j \in \mathbb{R} \),

\[
\tilde{B}_j \tilde{A}_j^{-1} \tilde{B}_j^* = \gamma_j I.
\]
The latter means
\[
\sum_{\ell \neq j} (\alpha_\ell - \alpha_j)^{-1} B_{\ell j} B_{\ell j} = \gamma_j I.
\]

The hypothesis $B v_j = 0$ means that
\[
B_{\ell j} v_j = 0
\]
for all $\ell$ and $j$. By applying (11) to $v_j$ we get $\gamma_j = 0$.

For definiteness $\alpha_1 < \ldots < \alpha_s$. Let $j = 1$. Then (11) takes the form
\[
\sum_{\ell=2}^{r} (\alpha_\ell - \alpha_1)^{-1} B_{1 \ell} B_{1 \ell}^* = 0,
\]
in which all terms are non-negative semi-definite because $\alpha_\ell - \alpha_1 > 0$. Hence $B_{1 \ell} = 0$ for all $\ell$. Continuing this procedure successively we get $B_{j \ell} = 0$ for all $\ell$ and $j$, that is, $B = 0$ as desired.

\[\square\]

Proof of Theorem 5.4. Put $A(t) = \sum t_j A_j, t \in \mathbb{R}^k$. Let $s(t)$ be the number of distinct eigenvalues of $A(t)$.

Let $r = \max s(t), s(\lambda) = r, X = A(\lambda)$.

Let $\alpha_1, \ldots, \alpha_r$ be distinct eigenvalues of $X$. Let $v_1, \ldots, v_r$ be the corresponding eigenvectors. Let $v = v_1 + \ldots + v_r$. We claim that $\lambda$ and $v$ satisfy the conclusion of Theorem 5.4.

By Vandermonde, $S = S(X, v) = \text{Span} \{v_1, \ldots, v_r\}$.

Arguing by contradiction, assume $A_j$’s are linearly dependent as operators $S \to \mathbb{C}^m$. Then there is $\mu \in \mathbb{R}^k$ such that the matrix $B = \sum \mu_j A_j$ has the property: $B v_j = 0$ for all $1 \leq j \leq r$. Since $r = \max s(t)$, the matrix $B$ is subordinate to $X$. By Lemma 6.2 $B = 0$, which is absurd because $A_j$’s are linearly independent.

\[\square\]

References

[1] V. K. Beloshapka, A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space. Math. Notes 47 (1990), 239–242.
[2] F. Bertrand, L. Blanc-Centi, and F. Meylan, Stationary discs and finite jet determination for non-degenerate generic real submanifolds. Adv. Math. 343 (2019), 910–934.
[3] T. Kato, Perturbation Theory for Linear Operators. Springer, 1980.
[4] L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule. Bull. Soc. Math. France 109 (1981), 427–474.
[5] A. Scalari and A. Tumanov, Extremal discs and analytic continuation of product CR maps. Mich. Math. J. 55 (2007), 25–33.
[6] A. Tumanov, Extending CR functions on a manifold of finite type over a wedge. Mat. Sb. 136 (1988), 129–140.
[7] A. Tumanov, Extremal discs and the regularity of CR mappings in higher codimension. Amer. J. Math. 123 (2001), 445–473.
[8] A. Tumanov, Extremal discs and the geometry of CR manifolds. Lect. Notes in Math. 1848 (2004), 191–212.

Department of Mathematics, University of Illinois, 1409 West Green St., Urbana, IL 61801

E-mail address: tumanov@illinois.edu