BIREGULAR GRAPHS WITH THREE EIGENVALUES

XI-MING CHENG, ALEXANDER L. GAVRILYUK♦, GARY R. W. GREAVES♣, AND JACK H. KOOLEN♠

Abstract. We consider nonregular graphs having precisely three distinct eigenvalues. The focus is mainly on the case of graphs having two distinct valencies and our results include the construction of new examples, structure theorems, parameter constraints, and a classification of certain special families of such graphs. We also present a new example of a graph with three valencies and three eigenvalues of which there are currently only finitely many known examples.

1. Introduction

In the late 1990s, as a generalisation of strongly regular graphs, attention was brought to the study of nonregular graphs whose adjacency matrices have precisely three distinct eigenvalues. We continue this investigation focussing mainly on graphs having precisely two distinct valencies, so-called biregular graphs. Muzychuk and Klin [17] called such graphs ‘strongly biregular graphs’.

An n-vertex graph with a vertex of valency n−1 is called a cone. Given a graph Γ, the cone over Γ is the graph formed by adjoining a vertex adjacent to every vertex of Γ. Examples of families of strongly biregular graphs are complete bipartite graphs and cones over strongly regular graphs. Indeed, a complete bipartite graph $K_{n,m}$ (for $n \geq m \geq 1$) has spectrum $\{[\sqrt{nm}]^1, [0]^{n+m-2}, [-\sqrt{nm}]^1\}$. The following result due to Muzychuk and Klin offers a method for finding strongly biregular cones.

Proposition 1.1 (See [17]). Let $\Gamma$ be a strongly regular graph with $n$ vertices, valency $k$, and smallest eigenvalue $\theta_2$. Then the cone over $\Gamma$ has three distinct eigenvalues if and only if $\theta_2(k-\theta_2) = -n$.

There are infinitely many strongly regular graphs satisfying the assumption of the proposition and so there are infinitely many cones over strongly regular graphs having three distinct eigenvalues [3,17]. As well as giving some sporadic examples, using symmetric and affine designs, Van Dam [10] exhibited a couple of infinite families of strongly biregular graphs that are neither cones nor complete bipartite graphs.

2010 Mathematics Subject Classification. 05E30, 05C50.

Key words and phrases. three distinct eigenvalues, strongly regular graphs, biregular graphs, nonregular.

♦Part of the work was done while A.L.G. was visiting Tohoku University as a JSPS Postdoctoral Fellow.

♣G.R.W.G. was supported by JSPS KAKENHI; grant number: 24-02789.

♠J.H.K. is partially supported by the ‘100 talents’ program of the Chinese Academy of Sciences, and by the National Natural Science Foundation of China (No. 11471009).
So far we only have a finite list of graphs with three valencies and three distinct eigenvalues and no examples of graphs with precisely three distinct eigenvalues with more than three valencies. Below we contribute a new graph to the list of graphs with three eigenvalues and three valencies whilst also showing the nonexistence of an, a priori, putative graph with three eigenvalues and four valencies (see Section 5).

There exist some partial classifications of graphs having three distinct eigenvalues in the following senses. Van Dam [10] classified all such graphs having smallest eigenvalues at least \(-2\) and also classified all such graphs on at most 29 vertices. Chuang and Omidi [8] classified those graphs whose spectral radius is less than 8. By showing the existence of some previously unknown graphs on 30 vertices (see Theorem 5.2), we extend Van Dam’s classification up to 30 vertices. We also contribute a classification of strongly biregular graphs whose second largest eigenvalue is at most 1 (see Section 3.3).

In this paper we further develop the theory of graphs with precisely three distinct eigenvalues. We begin with Section 2 where we present our preliminaries, classify graphs that have three distinct eigenvalues and a disconnected complement, and give some bounds that are used in later sections. In Section 3 we focus on strongly biregular graphs and, in particular, we show the finitude of such graphs when the second largest eigenvalue is bounded in the biregular non-bipartite case. We also study the structure of strongly biregular graphs whose two smaller eigenvalues sum to \(-1\). Section 4 is concerned with graphs with three distinct eigenvalues whose complements also have precisely three distinct eigenvalues. In Section 5 using the star complement method, we show the existence of new graphs having precisely three distinct eigenvalues. Throughout the paper some nonexistence results are also established. Furthermore, as an appendix, we provide a table of feasible parameters for biregular graphs having precisely three distinct eigenvalues.

2. Graphs with three distinct eigenvalues

In this section we develop some basic theory for graphs with three distinct eigenvalues. We begin with some notation.

Let \(\Gamma = (V, E)\) be an \(n\)-vertex connected graph. Recall that the adjacency matrix \(A\) of \(\Gamma\) is an \(n \times n\) matrix whose \((i, j)\)th entry, \(A_{i,j}\), is 1 if the \(i\)th vertex of \(\Gamma\) is adjacent to the \(j\)th vertex of \(\Gamma\) and 0 otherwise. By the eigenvalues of \(\Gamma\) we mean the eigenvalues of \(A\). Assume that \(\Gamma\) has precisely three distinct eigenvalues \(\theta_0 > \theta_1 > \theta_2\). Then \(\Gamma\) has diameter two and since such a graph cannot be complete, it follows by interlacing that \(\theta_1 \geq 0\) and \(\theta_2 \leq -\sqrt{2}\).

We write \(m_i\) for the multiplicity of eigenvalue \(\theta_i\) of \(\Gamma\). If \(\Gamma\) has \(n\) vertices then, since \(1 + m_1 + m_2 = n\) and \(\theta_0 + m_1\theta_1 + m_2\theta_2 = 0\), we have

\[
(1) \quad m_1 = \frac{(n-1)\theta_2 + \theta_0}{\theta_1 - \theta_2} \quad \text{and} \quad m_2 = \frac{(n-1)\theta_1 + \theta_0}{\theta_1 - \theta_2}.
\]

By the Perron-Frobenius Theorem (see, for example, [13]), \(\theta_0\) has multiplicity 1 and for any eigenvector for \(\theta_0\) all entries have the same sign. This implies that there exists a positive eigenvector \(\alpha\) for the eigenvalue \(\theta_0\) such that

\[
(A - \theta_1 I)(A - \theta_2 I) = \alpha \alpha^t.
\]

For a vertex \(x\), denote the entry of \(\alpha\) corresponding to \(x\) by \(\alpha_x\). This implies that if a vertex \(x\) has valency \(d_x\), then \(d_x = \alpha_x^2 - \theta_1\theta_2\). Let \(x\) and \(y\) be vertices of \(\Gamma\). We
write \( \nu_{xy} \) for the number of common neighbours of \( x \) and \( y \). By the above formulae we have

\[ \nu_{xy} = (\theta_1 + \theta_2)A_{x,y} + \alpha_x \alpha_y. \]

We will often abuse our above notation by writing \( \alpha_i \) to mean \( \alpha_x \) for some vertex \( x \) having valency \( k_i \). We may also write \( \nu_{ij} \) to mean \( \nu_{xy} \) where \( d_x = k_i \) and \( d_y = k_j \).

Throughout the paper, we will assume this notation to be standard.

\( \Gamma \) denoted constants \( \lambda \) and \( \mu \) such that every pair of vertices has \( \lambda \) or \( \mu \) common neighbours if they are respectively adjacent or non-adjacent. If \( \Gamma \) is regular, then it is well-known that \( \Gamma \) must be strongly regular [10]. In this paper we focus on the less well studied case of when \( \Gamma \) is nonregular.

Let \( \Gamma \) have \( s \) distinct valencies \( k_1, \ldots, k_s \). We write \( V_i := \{ v \in V(\Gamma) \mid \deg_v = k_i \} \) and \( n_i := |V_i| \) for \( i \in \{1, \ldots, s\} \). Clearly the subsets \( V_i \) partition the vertex set of \( \Gamma \). We call this partition the **valency partition** of \( \Gamma \). Let \( \pi = \{\pi_1, \ldots, \pi_s\} \) be a partition of the vertices of \( \Gamma \). For each vertex \( x \) in \( \pi_i \), write \( d^{(x)}_{ij} \) for the number of neighbours of \( x \) in \( \pi_j \). Then we write \( b_{ij} = 1/|\pi_i| \sum_{x \in \pi_i} d^{(x)}_{ij} \) for the average number of neighbours in \( \pi_j \) of vertices in \( \pi_i \). The matrix \( B_{\pi} := (b_{ij}) \) is called the **quotient matrix** of \( \pi \) and \( \pi \) is called **equitable** if for all \( i \) and \( j \), we have \( d^{(x)}_{ij} = b_{ij} \) for each \( x \in \pi_i \). We will use repeatedly properties of the quotient matrices of partitions of the vertex set of a graph and we refer the reader to Godsil and Royles’ book [13], Chapter 9] for the necessary background on equitable partitions and interlacing.

For fixed \( \theta_0 > \theta_1 > \theta_2 \), define the set \( G(\theta_0, \theta_1, \theta_2) \) of connected nonregular graphs having precisely three distinct eigenvalues \( \theta_0, \theta_1, \) and \( \theta_2 \).

Among graphs with three eigenvalues, complete bipartite graphs are distinguished in the following way.

**Theorem 2.1** (Proposition 2 [10]). Let \( \Gamma \) be a graph in \( G(\theta_0, \theta_1, \theta_2) \) where \( \theta_0 \) is not an integer. Then \( \Gamma \) is a complete bipartite graph.

It was shown by Smith [21] that if the second largest eigenvalue of a connected graph \( \Gamma \) is at most 0 then \( \Gamma \) is a complete \( r \)-partite graph with parts of size \( p_1, \ldots, p_r \), denoted \( K_{p_1, \ldots, p_r} \). We will see below that complete bipartite graphs are the only nonregular multipartite graphs with precisely three distinct eigenvalues.

**Theorem 2.2.** Let \( \Gamma \) be a graph in \( G(\theta_0, \theta_1, \theta_2) \). If the complement of \( \Gamma \) is disconnected, then \( \Gamma \) is a cone or \( \Gamma \) is complete bipartite.

**Proof.** Let \( V \) be the vertex set of \( \Gamma \) and suppose that the complement \( \overline{\Gamma} \) has at least 2 connected components.

**Claim 1.** \( \Gamma \) has at most three valencies.

Suppose that \( x \) and \( y \) are vertices in different components of \( \overline{\Gamma} \). Then we must have \( x \sim y \) (in \( \Gamma \)) and, since the other \( n - 2 \) vertices must be adjacent to \( x \) or \( y \), we have \( \nu_{xy} = d_x + d_y - n \). Hence we can write

\[ \alpha_x^2 - \alpha_x \alpha_y + \alpha_y^2 = n + \theta_1 + \theta_2 + 2\theta_1 \theta_2. \] (2)

If \( d_x = d_y \) then, by Eq. (3), \( \alpha_x = \alpha_y = \sqrt{n + \theta_1 + \theta_2 + 2\theta_1 \theta_2} \). Furthermore, since any other vertex \( z \) of \( \Gamma \) must be adjacent to \( x \) or \( y \), by Eq. (2), we would have \( \alpha_z = \alpha_x = \alpha_y \). But this cannot happen since \( \Gamma \) is not regular. Hence, vertices in different components cannot have the same valency.
From Eq. (2) observe that if $\alpha_x$ is fixed then, there are only two possible values for $\alpha_y$, say $\alpha$ and $\alpha'$, satisfying $\alpha + \alpha' = \alpha_x$. Thus, for any vertex $x$ in one connected component $C$ of $\Gamma$, there can be at most two vertices outside of $C$ having distinct valencies. Moreover, in the case where there are vertices with two distinct valencies $y$ and $z$ outside of $C$ then we have $\alpha_x = \alpha_y + \alpha_z$, and hence the valency of any other vertex of $\Gamma$ must be $d_x$, $d_y$, or $d_z$, and the claim is established. One can also see that, since for each vertex $v$ each $\alpha_v$ is positive, $x$ has the largest of the valencies. Therefore we have also established the following claim (the claim is trivial if $\Gamma$ has only two valencies).

**Claim 2.** The vertices of the largest (or larger) valency in $\Gamma$ induce a regular connected component of $\Gamma$.

If $y$ and $z$ were in different connected components, then we could simultaneously write $\alpha_x = \alpha_y + \alpha_z$, $\alpha_y = \alpha_z + \alpha_x$, and $\alpha_z = \alpha_x + \alpha_y$ which is clearly impossible. Thus we can also deduce the following.

**Claim 3.** $\Gamma$ has precisely two connected components.

Let $C$ be a regular connected component of $\Gamma$ and let $v$ be an eigenvector of $C$ with non-trivial eigenvalue $\theta$ (i.e., $v$ is orthogonal to the ‘all ones’ vector). Let $w \in \mathbb{R}^V$ be defined by $w_x = v_x$ if $x \in V(C)$ and 0 otherwise. Then $w$ is an eigenvector of $\Gamma$ with eigenvalue $-\theta - 1$. This means that $\Gamma$ has at most three distinct eigenvalues and is either a complete graph $K_t$ with $t \geq 1$ or a strongly regular graph. If $C = K_1$, then either $t = 1$, in which case $\Gamma$ has a vertex of valency $n - 1$, i.e., $\Gamma$ is a cone or $t \geq 2$, in which case $\Gamma$ has an eigenvalue 0, hence $\Gamma$ is complete multipartite and, since $\Gamma$ has only two connected components, $\Gamma$ must be complete bipartite.

It remains to consider the case when each regular connected component of $\Gamma$ is a strongly regular graph. Let $k_1 > k_2$ be the two largest valencies of $\Gamma$ and let $\Lambda$ be the regular subgraph induced on $V_1$ with $n_1$ vertices and valency $k_{11}$. By above, we assume that $\Lambda$ is strongly regular and, by interlacing, it must have eigenvalues, $k_{11}$, $\theta_1$, and $\theta_2$. Let vertices $x$ and $y$ have respective valencies $d_x = k_1$ and $d_y = k_2$. Note that $x \sim y$ since they must be in different connect components of $\Gamma$. Hence we have

\begin{equation}
(3) \quad k_2 - \nu_{xy} - 1 = \alpha_2^2 - \theta_1 \theta_2 - \alpha_1 \alpha_2 - \theta_1 - \theta_2 - 1 = \alpha_2(\alpha_2 - \alpha_1) - (\theta_1 + 1)(\theta_2 + 1).
\end{equation}

On the other hand, since the complement $\overline{\Lambda}$ of $\Lambda$ is strongly regular with valency $n_1 - k_{11} - 1$ and non-trivial eigenvalues $-\theta_1 - 1$ and $-\theta_2 - 1$, we can also write

\begin{equation}
(4) \quad k_2 - \nu_{xy} - 1 = n_1 - k_{11} - 1 = \overline{\nu} - (\theta_1 + 1)(\theta_2 + 1),
\end{equation}

where $\overline{\nu}$ is the number of common neighbours of two non-adjacent vertices in the component $\overline{\Lambda}$ of $\Gamma$. The second equality follows from a well-known equality \[4\] Thm 1.3.1]. By comparing Eqs. (3) and (4), since $\alpha_1 > \alpha_2$, we have $\overline{\nu} < 0$, which is impossible.

Note that if a bipartite graph is not complete bipartite then its diameter must be at least 3 and hence it cannot have fewer than 4 distinct eigenvalues. The next result follows from this observation and from the above proof.

**Corollary 2.3.** Let $\Gamma$ be a graph in $G(\theta_0, \theta_1, \theta_2)$. Then the following are equivalent.

1. $\Gamma$ is bipartite;
2. $\Gamma$ is complete bipartite;
3. $\theta_1 = 0$. 

\textit{Proof.} 

(1) $\Gamma$ is bipartite;

(2) $\Gamma$ is complete bipartite;

(3) $\theta_1 = 0$. 

\[\square\]
In the proof of Theorem 2.2 we also saw that the disconnected complement \( \Gamma \) must have at most three valencies. Hence we have the following corollary.

**Corollary 2.4** (cf. [10]). Let \( \Gamma \) be a cone in \( G(\theta_0, \theta_1, \theta_2) \). Then \( \Gamma \) has at most three valencies.

**Remark 2.5.** The above corollary generalises a result of Bridges and Mena [4] who studied cones having distinct eigenvalues \( \theta_0, \theta_1, \) and \( -\theta_1 \). They proved that, except for at most three cones having three valencies, such graphs are cones over strongly regular graphs with parameters \((v, k, \lambda, \lambda)\). (Only two of these three exceptional cones have been constructed, it is still an open problem to decide the existence of the largest cone.)

In the remainder of this section we present a series of bounds for graphs having precisely three distinct eigenvalues.

**Lemma 2.6.** Let \( \Gamma \) be a graph in \( G(\theta_0, \theta_1, \theta_2) \) and let \( x \) and \( y \) be vertices with respective valencies \( d_x > d_y \). Then the following hold:

1. if \( x \sim y \), then \( \alpha_x - 1 \leq (\alpha_x - \alpha_y)\alpha_y \leq -(\theta_1 + 1)(\theta_2 + 1) \) and \( \alpha_x \alpha_y \geq -\theta_1 - \theta_2 \);
2. if \( x \not\sim y \), then \( \alpha_x - 1 \leq (\alpha_x - \alpha_y)\alpha_y \leq -\theta_1 \theta_2 \).

**Proof.** Since \( \alpha^2, \alpha_x \alpha_y, \) and \( \alpha^2 \) are all integers and \( \alpha_x > \alpha_y \geq 1 \), we have \( \alpha_x \geq \alpha_y + 1 \) and hence \( \alpha_x - 1 \leq (\alpha_x - \alpha_y)\alpha_y \). The rest follows from the fact that \( 0 \leq \nu_{x,y} \leq d_y - 1 \) when \( x \sim y \) and \( \nu_{x,y} \leq d_y \) when \( x \not\sim y \). \( \square \)

Van Dam and Kooij [11] showed that the number \( n \) of vertices of a connected graph \( \Gamma \) with diameter 2 with spectral radius \( \rho \) satisfies \( n \leq \rho^2 + 1 \) with equality if and only if \( \Gamma \) is a Moore graph of diameter 2 or \( \Gamma \) is the \( K_{1,n-1} \). As a consequence we have the following lemma.

**Lemma 2.7.** Let \( \Gamma \) be an \( n \)-vertex graph in \( G(\theta_0, \theta_1, \theta_2) \). Let \( \delta, \delta_{\text{avg}}, \) and \( \Delta \) respectively denote the smallest, average, and largest valency of \( \Gamma \). Then \( \delta < \delta_{\text{avg}} < \theta_0 < \Delta \) and \( n \leq \theta_0^2 + 1 \) with equality if and only if \( \Gamma \) is \( K_{1,n-1} \).

**Proposition 2.8.** Let \( \Gamma \) be a non-bipartite graph in \( G(\theta_0, \theta_1, \theta_2) \). Let \( \Delta \) be the maximal valency in \( \Gamma \) and let \( \ell := \min\{1 - (\theta_1 + 1)(\theta_2 + 1), -\theta_1 \theta_2 + 1\} \). Then the following hold:

1. \( \Delta \leq (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1 \theta_2 \);
2. if the complement of \( \Gamma \) is connected, then \( \Delta \leq \ell^2 - \theta_1 \theta_2 \);
3. \( n \leq \max\{(\ell^2 - \theta_1 \theta_2 - 1)^2 + 1, (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1 \theta_2 + 1\} \).

**Proof.** Let \( x \) be a vertex with valency \( \Delta \), having a neighbour \( y \) with \( d_y < \Delta \). Now (1) follows from Lemma 2.6. If all vertices \( x \) with valency \( \Delta \) do not have a non-neighbour \( y \) with \( d_y < \Delta \), then the complement of \( \Gamma \) is not connected. So we may assume that there is a vertex \( x \) with valency \( \Delta \) having a non-neighbour \( y \) with \( d_y < \Delta \). Hence (2) follows from Lemma 2.6. For (3), we have by Lemma 2.7 that \( n \leq \theta_0^2 + 1 \leq (\Delta - 1)^2 + 1 \). So the result follows by Theorem 2.2 and (2) if \( \Delta \neq n - 1 \). Otherwise, if \( \Delta = n - 1 \) then \( n = \Delta + 1 \leq (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1 \theta_2 + 1 \). \( \square \)

**Lemma 2.9.** Let \( \Gamma \) be an \( n \)-vertex non-bipartite graph in \( G(\theta_0, \theta_1, \theta_2) \). Assume that \( \theta_1 \) and \( \theta_2 \) have the same multiplicity, \( m = (n - 1)/2 \). Then there exists a positive integer \( t \) such that \( \theta_1 = (-1 + \sqrt{4t + 1})/2, \theta_2 = (-1 - \sqrt{4t + 1})/2, \) and \( \theta_0 = (n - 1)/2 \) and \( 4t + 3 \leq n \).
Proof. Essentially the same as the proof of Proposition 3 in [10]. □

Proposition 2.10. Let \( \Gamma \) be an \( n \)-vertex non-bipartite graph in \( G(\theta_0, \theta_1, \theta_2) \). Assume that \( \theta_1 \) and \( \theta_2 \) have the same multiplicity. Let \( t \) be defined as is Lemma 2.9. Then the following hold:

1. For the maximal valency \( \Delta \) in \( \Gamma \) we have \( \Delta \leq t^2 + 3t + 1 \);
2. For the number of vertices we have \( 4t + 3 \leq n \leq 2\Delta - 1 \leq 2t^2 + 6t + 1 \).

Proof. Let \( x \) be a vertex with valency \( \Delta \). For the first part, we have \( \alpha_x = \sqrt{d_x + \theta_1 \theta_2} = \sqrt{d_x - t} \). Then \( \alpha_x \leq -(\theta_1 + 1)(\theta_2 + 1) + 1 = t + 1 \).

For the second part, we have \( \Delta \geq (n+1)/2 \) since \( \Delta > \theta_0 = (n-1)/2 \) is an integer. The upper bound follows easily. The lower bound follows from Lemma 2.9. □

To prove our next result we will need a theorem of Bell and Rowlinson which enables us to bound the number of vertices of a graph in terms of the multiplicity of one of its eigenvalues.

Theorem 2.11 (See [1]). Let \( \Gamma \) be a graph on \( n \) vertices with an eigenvalue \( \theta \) with multiplicity \( n - t \) for some positive integer \( t \). Then either \( \theta \in \{0, -1\} \) or \( n \leq \frac{t(t+1)}{2} \).

We note that, using the classification of Van Dam [10] Section 7] of graphs with three distinct eigenvalues with at most 29 vertices and the above result of Bell and Rowlinson, we obtain readily the graphs in \( G(\theta_0, \theta_1, \theta_2) \) where the multiplicity of \( \theta_1 \) or \( \theta_2 \) is at most 6.

Lemma 2.12. Let \( \Gamma \) be an \( n \)-vertex graph in \( G(\theta_0, \theta_1, \theta_2) \). For \( \{\theta_1, \theta_s\} \in \{\theta_1, \theta_2\} \) where the multiplicity of \( \theta_1 \) is at least that of \( \theta_s \), we have the following inequalities.

\[
\begin{align*}
\theta_1^2 & \leq 2(n - (1 - 1/n)); \\
\theta_s^2 & \leq n\sqrt{(n-1)/2 + 1/(2(n-1))}.
\end{align*}
\]

Proof. Lemma 2.9 deals with the case when the multiplicities are equal. We can therefore assume that the multiplicity \( m_1 \) of \( \theta_1 \) is at least \( n/2 \). Let \( A \) be the adjacency matrix of \( \Gamma \) and let \( k_1 \) be the largest valency of the vertices of \( \Gamma \). Then since \( nk_1 \geq \text{tr}(A^2) \) we have

\[
n^2 \geq nk_1 \geq \theta_0^2 + m_1 \theta_1^2 + m_2 \theta_2^2 \geq n - 1 + n/2\theta_1^2.
\]

This gives the first inequality. By Theorem 2.11 the multiplicity \( m_s \) of \( \theta_s \) is at least \( \sqrt{2(n - 1)} \). Then the second inequality follows in the same way as above. □

Lemma 2.12 gives us a crude bound on the size of the two smaller eigenvalues of a graph with precisely three distinct eigenvalues.

We call the 11-vertex cone over the Petersen graph, the Petersen cone (see [10] Fig. 1]) and the graph derived from the complement of the Fano plane, the Fano graph (see [10] Fig. 2]).

Lemma 2.13. Let \( \Gamma \) be an \( n \)-vertex non-bipartite graph in \( G(\theta_0, \theta_1, \theta_2) \). Then \( \theta_0 \leq n - 6 \) with equality if and only if \( \Gamma \) is the Petersen cone or the Fano graph.

Proof. By Theorem 2.11 we know that \( \theta_0 \) is rational. First suppose that \( \theta_1 \) and \( \theta_2 \) are irrational. Then they must have the same multiplicities, hence we can apply Lemma 2.12 to obtain the equality \( \theta_0 = (n - 1)/2 \). If \( n - 6 \leq (n - 1)/2 \) then \( n \leq 11 \), but by computation we see that no such graph exists.
Now suppose all eigenvalues are rational. If \( \theta_2 = -2 \) then the lemma holds by the classification theorem of Van Dam [10, Theorem 7]. Otherwise \( \theta_2 \leq -3 \) and we assume that \( n - 6 \leq \theta_0 \leq n - 2 \). Then, by Lemma 2.4, we can write 
\[
\theta_0^2 + m_1 \theta_1^2 + m_2 \theta_2^2 = \sum_{v \in \Gamma} d_v < n \theta_0.
\]
The expressions for the multiplicities \( m_1 \) and \( m_2 \) from [10] yield \( \theta_1 \leq 2 \) and 
\[
3 \leq -\theta_2 \leq \frac{\theta_0(n - \theta_0 + \theta_1)}{\theta_0 + (n - 1)\theta_1}.
\]
It reduces to checking the cases when \( \theta_1 = 1 \) and \( \theta_2 = -3 \). But then there exist no possible pair \( \theta_0 \) and \( n \) such that the multiplicities \( m_1 \) and \( m_2 \) are integral. \( \square \)

3. Biregular graphs with three eigenvalues

3.1. Computing feasible parameters. Let \( \Gamma \) be a graph having \( r \) distinct valencies \( k_1, \ldots, k_r \) with multiplicities \( n_1, \ldots, n_r \), i.e., \( n_i := |\{v \in V(\Gamma) : d_v = k_i\}| \). We refer to the valencies \( k_i \) and their multiplicities \( n_i \) as the \textbf{parameters} (or \textbf{parameter set}) of \( \Gamma \). The main result of this section gives us strong restrictions on the parameters of biregular graphs with three eigenvalues.

We begin with a result about biregular cones.

**Proposition 3.1** (See [10]). Let \( \Gamma \) be a biregular cone in \( G(\theta_0, \theta_1, \theta_2) \). Then \( \Gamma \) is a cone over a strongly regular graph.

**Remark 3.2.** Our Theorem 2.2 is reminiscent of Proposition 6.1 (a) given by Muzychuk and Klin [17]. Let \( \Gamma \in G(\theta_0, \theta_1, \theta_2) \) and let \( W(\Gamma) \) denote its Weisfeiler-Leman closure (see [17, Section 6]). We also remark that, by Theorem 2.2 and Proposition 3.1, we see that [17, Proposition 6.1 (a)] says that if \( \dim(W(\Gamma)) = 6 \) then \( \Gamma \) is biregular with a disconnected complement. Muzychuk and Klin [17] suggest classifying all graphs \( \Gamma \in G(\theta_0, \theta_1, \theta_2) \) satisfying \( \dim(W(\Gamma)) = 9 \), which is the next interesting case after \( \dim(W(\Gamma)) = 6 \).

Van Dam [10] showed that if a graph \( \Gamma \) has precisely three distinct eigenvalues and at most three distinct valencies then the valency partition is equitable. We show a slightly refined version of this result where we assume that \( \Gamma \) has precisely two distinct valencies.

**Theorem 3.3.** Let \( \Gamma \) be an \( n \)-vertex non-bipartite biregular graph in \( G(\theta_0, \theta_1, \theta_2) \) with valencies \( k_1 > k_2 \). Then the following conditions hold:

(i) The partition \( \{V_1, V_2\} \) is an equitable partition of \( \Gamma \) with quotient matrix 
\[
Q = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},
\]
where
\[
k_{11} = \frac{\alpha_1 \theta_0 - \alpha_2 k_1}{\alpha_1 - \alpha_2}, \quad k_{12} = \frac{k_1 - \theta_1}{\alpha_1 - \alpha_2}, \quad k_{21} = \frac{\theta_0 - k_2}{\alpha_1 - \alpha_2}, \quad k_{22} = \frac{\alpha_1 k_2 - \alpha_2 \theta_0}{\alpha_1 - \alpha_2}.
\]

(ii) All eigenvalues of \( \Gamma \) are integers.

(iii) If the matrix \( Q \) has eigenvalues \( \theta_0 \) and \( \theta \), then \( \alpha_1 \alpha_2 = -\theta(\theta' + 1) \) where \( \{\theta, \theta'\} = \{\theta_1, \theta_2\} \). In particular, if \( k_{11} = 0 \) or \( k_{22} = 0 \) then \( \alpha_1 \alpha_2 = -\theta_2 / (\theta_1 + 1) \).

(iv) We have 
\[
n = \frac{(\alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2^2 - \theta_0 - \theta_1 \theta_2)(\theta_0 - \theta_1)(\theta_0 - \theta_2)}{(\theta_0 + \alpha_1 \alpha_2 + \theta_1 \theta_2)\alpha_1 \alpha_2}.
\]
appendix.

We will prove each part of the theorem in turn. 

**Proof.**

(v) The following conditions are equivalent:

- (a) $k_{21} = n_1$;
- (b) $k_{12} = n_2$;
- (c) $n_1 = 1$;
- (d) $\Gamma$ is a cone over a strongly regular graph.

(vi) If $n$ is a prime at least 3, then $\Gamma$ is a cone over a strongly regular graph.

(vii) $\alpha_1 - 1 \leq (\alpha_1 - \alpha_2)\alpha_2 \leq \min\{-(\theta_1 + 1)(\theta_2 + 1), -\theta_1 \theta_2\}$, unless $\Gamma$ is a cone over a strongly regular graph.

**Proof.** We will prove each part of the theorem in turn.

(i) Let a vertex $x$ of valency $k_1$ have $k_{11}$ neighbours in $V_1$ and $k_{12} := k_1 - k_{11}$ neighbours in $V_2$. The vector $\alpha$ is the $\theta_0$-eigenvector of $\Gamma$, therefore $k_{11}\alpha_1 + k_{12}\alpha_2 = \theta_0\alpha$. Since $\alpha_1 > \alpha_2$, it follows that

$$k_{11} = \frac{\alpha_1 \theta_0 - \alpha_2 k_1}{\alpha_1 - \alpha_2} \quad \text{and} \quad k_{12} = \frac{\alpha_1 k_1 - \theta_0}{\alpha_1 - \alpha_2}.$$ 

Applying this idea again to a vertex of valency $k_2$ gives the first part of the theorem.

(ii) By Theorem 2.11 $\theta_0$ is an integer. Since $\theta_0$ and $\theta$ are eigenvalues of $Q$, we have $\theta_0 + \theta = k_{11} + k_{22} \in \mathbb{Z}$ and hence $\theta \in \mathbb{Z}$. The trace of the adjacency matrix of $\Gamma$ is zero, whence the remaining eigenvalue of $\Gamma$ is integral.

(iii) Clearly $Q$ has $\theta_0$ as an eigenvalue (with eigenvector $(\alpha_1, \alpha_2)$). The eigenvalues of $Q$ are a subset (with multiplicity) of the eigenvalues of $\Gamma$, hence the other eigenvalue $\theta$ of $Q$ is in $\{\theta_1, \theta_2\}$. Note that if $k_{11} = 0$ or $k_{22} = 0$ then the determinant of $Q$ is negative, hence $Q$ has a negative eigenvalue, namely, $\theta_2$. Taking the determinant of $Q$, det $Q = k_{11}k_{22} - k_{12}k_{21}$ and using the expressions for the $k_{ij}$ in (i), one obtains the expression for $\alpha_1\alpha_2$.

(iv) Since the valency partition is equitable we have $k_{12}n_1 = k_{21}n_2$. Moreover, from $n_1 + n_2 = n$, we obtain that

$$n_1 = \frac{k_{21}}{k_{12} + k_{21}}n \quad \text{and} \quad n_2 = \frac{k_{12}}{k_{12} + k_{21}}n.$$ 

From, $\theta_0^2 + m_1\theta_1^2 + m_2\theta_2^2 = n_1k_1 + n_2k_2$, using Eq. (1) for the multiplicities and the formulae for the $k_{ij}$’s in (i), one readily obtains the formula for $n$. The formulae for $n_1$ and $n_2$ follow easily.

(v) That (a) is equivalent to (b) follows from (iv). Both (a) and (b) imply that the complement of $\Gamma$ is disconnected, which implies (c) by Theorem 2.2.

By Proposition 3.1 (c) implies (d) and clearly (d) implies (a).

(vi) If $n$ is a prime then by Eqs. (4) one sees that $n_1$ and $n_2$ must be equal to $k_{21}$ and $k_{12}$. Then use (v).

(vii) This follows from Proposition 2.8. 

Using Theorem 3.3 and Lemma 2.12 we have compiled a table of the feasible parameters for biregular graphs with precisely three distinct eigenvalues, see the appendix.
Remark 3.4. Let $\Gamma$ be a biregular graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ with the spectrum of $\Gamma$ fixed. Using Theorem 3.3, one can see that $\Gamma$ can have at most two possible parameter sets. The problem of determining if the parameters of $\Gamma$ are determined by its spectrum comes down to Diophantine analysis. So far we do not have any examples of a pair of parameter sets corresponding to graphs with the same spectrum.

3.2. Bounding the second largest eigenvalue. Neumaier [13] showed that for a fixed $m$, all but finitely many primitive strongly regular graphs with smallest eigenvalue at least $-m$ fall into two infinite families.

Theorem 3.5 (See [13]). Let $m \geq 0$ be a fixed integer. Then there exists a constant $C(m)$ such that any connected and coconnected strongly regular graph $\Gamma$ with smallest eigenvalue $-m$ having more than $C(m)$ vertices has the following parameters (given in the form $\text{sg}(n, k, \lambda, \mu)$):

\[ (1) \text{ sg}(n, sm, s-1+(m-1)^2, m^2) \text{ where } s \in \mathbb{N} \text{ and } n = (s+1)(s(m-1)+m)/m; \]

or

\[ (2) \text{ sg}((s+1)^2, sm, s-1+(m-2)(m-1), m(m-1)) \text{ where } s \in \mathbb{N}. \]

In the next result we show that for fixed $\theta \neq 0$ there are only finitely many cones over strongly regular graphs with exactly three distinct eigenvalues and one of them equal to $\theta$.

Lemma 3.6. Let $\theta \neq 0$ be a fixed algebraic integer and let $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ be a cone over a strongly regular graph with $\theta \in \{\theta_0, \theta_1, \theta_2\}$. Then $\Gamma$ is one of a finite number of graphs.

Proof. If $\theta = \theta_0$, then it follows from Lemma 2.1. Let $\Lambda$ be an $n$-vertex strongly $k$-regular graph with smallest eigenvalue $\theta = \theta_2$. By Theorem 3.5, there exists a constant $C(-\theta)$ such that $n \leq C(-\theta)$, otherwise, for a positive integer $s$, either $\Lambda$ has parameters $\text{sg}(n, s(-\theta), s-1+(-\theta-1)^2, \theta^2)$ and $n = (s+1)(s(-\theta-1)-\theta)/(-\theta)$ or $\Lambda$ has parameters $\text{sg}((s+1)^2, -s\theta, s-1+(\theta+2)(\theta+1), \theta(\theta+1))$. If the cone over $\Lambda$ has precisely three distinct eigenvalues then, by Proposition 1.1, the equation $\theta(k-\theta) = -n$ must be satisfied. It is easily checked that $s$ must equal either $\theta^2 - \theta$ or $\theta^2 - 1$ and that only finitely many graphs from these parameter sets satisfy this condition.

Finally, suppose instead that the second largest eigenvalue of $\Lambda$ is $\theta = \theta_1$. Then, we can apply the same argument as above where we consider $\Lambda$ to be the complement of a strongly regular graph.

Now we show that, there are only finitely many biregular connected graphs with three distinct eigenvalues and bounded second largest eigenvalue.

Theorem 3.7. Let $\Gamma$ be an $n$-vertex biregular graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ with valencies $k_1 > k_2$ and let $t$ be a positive integer. Then there exists a constant $C(t)$ such that if $0 < \theta_1 \leq t$, then $n \leq C(t)$.

Proof. If $\theta_2 \geq -2t$, then the existence of $C(t)$ follows from Proposition 2.8. So from now on we will assume $\theta_2 < -2t$.

From Theorem 3.3, the eigenvalue $\theta_1$ is an integer and without loss of generality we may assume that $\theta_1 = t$. If $\Gamma$ is a cone over a connected strongly regular graph, then the existence of $C(t)$ follows from Lemma 3.6, hence from now on we assume that $\Gamma$ is not a cone over a strongly regular graph.
By Theorem 3.3 we have two cases to consider, namely the case \(\alpha_1\alpha_2 = -\theta_2(\theta_1 + 1)\) and the case \(\alpha_1\alpha_2 = -\theta_1(\theta_2 + 1)\). Let us first consider the case \(\alpha_1\alpha_2 = -\theta_2(\theta_1 + 1)\). Since \(\Gamma\) is not a cone, there exist a vertices \(x\) and \(y\) with respective valencies \(k_1\) and \(k_2\) such that \(x \not\sim y\). Therefore \(\nu_{xy} = \alpha_1\alpha_2 \leq k_y = \alpha_2^2 - \theta_1\theta_2\). Using \(\alpha_1\alpha_2 = -\theta_2(\theta_1 + 1)\), we see that \(\alpha_2^2 \geq -\theta_2\) and hence \(\alpha_1^2 \leq -(\theta_1 + 1)^2\theta_2\). Thus any two non-adjacent vertices have at least \(\alpha_2^2 \geq -\theta_2\) common neighbours. Hence, since \(\Gamma\) has diameter 2,

\[n \leq 1 + k_2 + k_3(k_1 - 1)/\alpha_2^2 = 1 + k_2 + k_1 - 1 - t\theta_2(k_1 - 1)/\alpha_2^2 < (2 + t)k_1.\]

We also have \(k_1 = \alpha_1^2 - \theta_1\theta_2 \leq -\theta_2(t^2 + 3t + 1)\).

Now we find that for the multiplicity \(m_2\) the following holds:

\[m_2 = \frac{(n - 1)\theta_1 + \theta_2}{\theta_1 - \theta_2} < \frac{(2 + t)k_1 t + k_1}{-\theta_2} = \frac{(t + 1)^2 k_1}{-\theta_2} \leq (t + 1)^2(t^2 + 3t + 1).
\]

Then Theorem 3.11 yields \(n \leq (t + 1)^2(t^2 + 3t + 1 + 1)((t + 1)^2(t^2 + 3t + 1 + 2)/2).\) This shows the existence of \(C(t)\) in this case.

It remains to consider the case \(\alpha_1\alpha_2 = -\theta_1(\theta_2 + 1)\). If \(k_2 = 0\) then, by Theorem 3.3, \(\alpha_1\alpha_2 = -\theta_2(\theta_1 + 1)\) which we have dealt with above. We therefore can assume that there exist \(x, y \in V_2\) with \(x \sim y\). Hence \(\nu_{xy} = \alpha_2^2 + \theta_1 + \theta_2 > 0\). We conclude that \(\alpha_2^2 \geq -\theta_2/2\). Now the bound follows in similar fashion as for the case \(\alpha_1\alpha_2 = -\theta_2(\theta_1 + 1)\). \(\square\)

Note that the above result is not true for connected graphs with exactly 4 distinct eigenvalues and exactly two distinct valencies. Indeed, the friendship graphs, i.e., cones over a disjoint union of copies of \(K_2\), can have unbounded number of vertices and all but two of the eigenvalues are equal to \(\pm 1\).

### 3.3. Second largest eigenvalue 1

In this section we will determine the connected biregular graphs with three distinct eigenvalues and second largest eigenvalue 1. First we determine the cones of strongly regular graphs with second largest eigenvalue 1. Seidel [19] (see also [5, Thm 3.12.4 (i)]) classified the strongly regular graphs with smallest eigenvalue 1.

**Theorem 3.8 ([19]).** Let \(\Gamma\) be a connected strongly regular graph with smallest eigenvalue \(-2\). Then \(\Gamma\) is either a triangular graph \(T(m)\) for \(m \geq 5\); an \((m \times m)\) grid for \(m \geq 3\); the Petersen graph; the Shrikhande graph; the Clebsch graph; the Schl"afli graph; or one of the three Chang graphs.

**Lemma 3.9.** Let \(\Gamma\) be a cone over a strongly regular graph in \(\mathcal{G}(\theta_0, 1, \theta_2)\). Then \(\Gamma\) is the Petersen cone.

**Proof.** The strongly regular graphs with second largest eigenvalue 1 are exactly the complements of the graphs in Theorem 3.8. Checking whether each graph satisfies the condition of Proposition 1.1 gives the lemma. \(\square\)

**Lemma 3.10.** There do not exist graphs with the following parameters.

(a) \(n = 30; n_1 = 15; n_2 = 15; k_1 = 20; k_2 = 8;\) spectrum \([16]^1, [1]^{20}, [-4]^9\);
(b) \(n = 51; n_1 = 12; n_2 = 39; k_1 = 35; k_2 = 14;\) spectrum \([22]^1, [1]^{41}, [-7]^9\).

**Proof.** The valency partitions of the parameter sets (a) and (b) have quotient matrices

\[
\begin{pmatrix}
12 & 8 \\
8 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
9 & 26 \\
8 & 6
\end{pmatrix}.
\]
respectively. The first case was shown to be impossible by Van Dam \[10\]. In the second case, for any two vertices \( x \sim y \) in \( V_2 \), we have \( \nu_{xy} = 1 \), but this is impossible as both are adjacent to 8 out of the 12 vertices in \( V_1 \).

**Theorem 3.12.** Let \( \mathbb{G} \) be a two parameter sets from Lemma 3.10. the cone over the Petersen graph, the graph associated to the Fano plane, and the parameters for graphs satisfying these conditions. We obtain 4 parameter sets including \( \alpha \) respectively. The first case was shown to be impossible by Van Dam \[10\]. In the holds.

**Proof.** Assume (a) holds. Let \( V \) as both are adjacent to 8 out of the 12 vertices in \( \mathbb{G} \).

Then we have \( \nu_{vw} = 1 - t + \alpha_2^2 < 1 - t + \alpha_1 \alpha_2 = 0 \). Hence we must have \( k_{22} = 0 \), but then, by Theorem 3.3, we must have \( \alpha_1 \alpha_2 = 2t \), contradicting that we are in case \( \alpha_1 \alpha_2 = t - 1 \).

Now let us consider the other case \( \alpha_1 \alpha_2 = 2t \). If \( k_{12} = n_2 \) or \( k_{21} = n_1 \), then by Theorem 3.3 and Lemma 3.9 the graph, \( \mathbb{G} \), is the cone over the Petersen graph. Hence we can assume that there are vertices \( x \in V_1 \) and \( y \in V_2 \) such that \( x \sim y \). By Lemma 2.6 it follows that \( (\alpha_1 - \alpha_2)\alpha_2 \leq t \), and this implies that \( \alpha_2^2 \geq t \), as \( \alpha_1 \alpha_2 = 2t \).

Using the fact that \( \alpha_1 \alpha_2 = 2t \), it follows from Theorem 3.3 that

\[
n = \frac{(\theta_0 - 1)(\alpha_1^2 + \alpha_2^2 + 3t - \theta_0)}{2t}
\]

holds.

Since \( \alpha_1 \alpha_2 = 2t \), and \( \alpha_2^2 \geq t \), it follows that \( 4t + 1 \leq \alpha_1^2 + \alpha_2^2 \leq 5t \) and hence, by \[0\], we have \( n < 8t \). By Theorem 3.3 we have \( \theta_0 \leq \frac{\alpha_1}{\alpha_2}k_2 = \alpha_1 \alpha_2 + \frac{\alpha_2}{\alpha_1}k_2 \leq 2t + t \times 2 = 4t \). Therefore \( m_2 = n - \frac{1 + \theta_0}{1 + t} < \frac{8t - 1 + 4t}{1 + t} < 12 \) and by Theorem 2.11 we deduce \( n \leq \frac{12 \times 13}{2} = 78 \).

Now, using Theorem 3.3 and Lemma 2.12 we can compute all the feasible parameters for graphs satisfying these conditions. We obtain 4 parameter sets including the cone over the Petersen graph, the graph associated to the Fano plane, and the two parameter sets from Lemma 3.11.

Now we can strengthen Theorem 3.3 further.

**Theorem 3.12.** Let \( \mathbb{G} \) be a non-bipartite biregular graph in \( \mathbb{G}(\theta_0, \theta_1, \theta_2) \) with valencies \( k_1 > k_2 \) whose valency partition has quotient matrix \( (\begin{array}{cc} k_{11} & k_{12} \\ k_{21} & k_{22} \end{array}) \). The following are equivalent:

(a) \( k_{21} = 1 \);
(b) \( k_{21} = n_1 \);
(c) \( k_{12} = n_2 \);
(d) \( \mathbb{G} \) is a cone over a strongly regular graph.

**Proof.** Assume (a) holds. Let \( x \) be a vertex of valency \( k_2 \). Then \( k_2 + k_{11} = 1 + \sum_{y \in V_2} \nu_{yx} = n_1 \alpha_1 \alpha_2 + \theta_1 + \theta_2 + 1 \). We also have that \( k_2 = \alpha_2^2 - \theta_1 \theta_2 \). It follows that \( n_1 = \frac{\alpha_2}{\alpha_1} - \frac{(\theta_1 + 1)(\theta_2 + 1) - k_{11}}{\alpha_1 \alpha_2} \). By Proposition 3.11 we can assume that \( \theta_1 \geq 2 \) and since Van Dam has classified such graphs with \( \theta_2 \geq 2 \), we can assume that \( \theta_2 \leq -3 \).

Using these assumptions and that \( k_{11} \leq n_1 - 1 \), we can write \( \alpha_1 \alpha_2 \) in terms of \( \theta_1 \) and \( \theta_2 \) as in (iii) of Theorem 3.3 to find that \( n_1 \leq 2 \). If \( n_1 = 2 \) then \( \alpha_1 \alpha_2 = -\theta_1 (\theta_2 + 1) \) and det \( Q = \theta_0 \theta_1 > 0 \). But in this case \( k_{11} \in \{0, 1\} \) and for either value of \( k_{11} \) we obtain det \( Q < 0 \), a contradiction. Therefore \( n_1 = 1 \), as in (b). The rest follows from the proof of Theorem 3.3.
3.4. Graphs in $G(\theta_0, \theta_1, \theta_2)$ with $\theta_1 + \theta_2 = -1$. In this section we examine graphs in $G(\theta_0, \theta_1, \theta_2)$ where $\theta_1 + \theta_2 = -1$. We find that such graphs have properties which make it convenient to use the so-called ‘star complement method’ to attempt to construct them. Compare Eq. (1) to the equation in Theorem 5.1. Indeed, in Section 5 we use the star complement method to construct such graphs (see Theorem 5.2). We denote by $j$ the ‘all ones’ (column) vector and we define the matrix $J := 1^T$. First we will need a lemma from linear algebra.

**Lemma 3.13** (See [6, 14]). Let $M$ be a symmetric $n \times n$ matrix with a symmetric partition

$$M = \begin{pmatrix} M_1 & N \\ N^\top & M_2 \end{pmatrix},$$

where $M_1$ has order, say, $n_1$. Suppose that $M$ has just two distinct eigenvalues $r > s$, with multiplicities $f$ and $n - f$. Let $\lambda_1 \geq \cdots \geq \lambda_{n_1}$ be the eigenvalues of $M_1$ and let $\mu_1 \geq \cdots \geq \mu_{n-n_1}$ be the eigenvalues of $M_2$. Then $r \geq \lambda_i \geq s$ for $i = 1, \ldots, n_1$, and

$$\mu_i = \begin{cases} r & \text{if } 1 \leq i \leq f - n_1, \\ s & \text{if } f + 1 \leq i \leq n - n_1, \\ r + s - \lambda_{f-i+1} & \text{otherwise.} \end{cases}$$

Now we give a structural result for graphs with a certain spectrum.

**Lemma 3.14.** Let $\Gamma$ be a connected graph with spectrum $\{[\theta_0]^1, [\theta]^n, [-\theta - 1]^{n-1}\}$, so that its adjacency matrix $A$ under the valency partition has block form:

$$A = \begin{pmatrix} A_1 & B^\top \\ B & A_2 \end{pmatrix}$$

with quotient matrix

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

where $A_1$ and $A_2$ are both $n \times n$ matrices, i.e., $n = n_1 = n_2$, where $n_i$ is the number of vertices with valency $k_i$.

Then the matrices $A_1$ and $J - I - A_2$ are cospectral.

**Proof.** We first recall that if $\Lambda$ is a regular graph with eigenvalues $\eta_0 = k$ (its valency), $\eta_1, \eta_2, \ldots$, then the complement of $\Lambda$ has eigenvalues $|\Lambda| - 1 - k$ and $-1 - \eta_i$ for $i \geq 1$.

For given number $c$, the matrix $M' := A + c\alpha\top$ has eigenvalues $\theta_0 + c|\alpha|^2$ with multiplicity 1, and $-\theta$ and $-\theta - 1$ with multiplicities $n$ and $n - 1$ respectively. Choose $c$ equal to $-(\theta + 1 + \theta_0)/|\alpha|^2$. Then the spectrum of $M'$ is $\{[\theta]^n, [-\theta - 1]^n\}$. Moreover, we can write $M'$ as follows:

$$M' = \begin{pmatrix} A_1 + c\alpha_1^2 J & B_1^\top + c\alpha_1\alpha_2 J \\ B + c\alpha_1\alpha_2 J & A_2 + c\alpha_2^2 J \end{pmatrix},$$

In the notation of Lemma 3.13 applied for $M'$, we have $v = 2n, v_1 = n, \{r, s\} = \{\theta, -\theta - 1\},$ and $f = v - f = n$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $M_3 := A_1 + c\alpha_1^2 J$ and let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of $M_2 := A_2 + c\alpha_2^2 J$.

Since $A_1$ and $A_2$ are the adjacency matrices of regular graphs, the matrices $M_1$ and $M_2$ have the same eigenvalues as the matrices $A_1$ and $A_2$, respectively, except for the eigenvalues with eigenvector $j$.

By the conclusion of Lemma 3.13, we see that

$$\mu_i = r + s - \lambda_{J-i+1} = -1 - \lambda_{J-i+1}, \text{ for } i = 1, \ldots, n.$$
Lemma 3.15. Let $\Gamma$ be a graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$, so that its adjacency matrix $A$ under the valency partition has block form:

$$A = \begin{pmatrix} A_1 & B^\top \\ B & A_2 \end{pmatrix}$$

with quotient matrix $\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, where $A_1$ and $J - I - A_2$ are cospectral. Assume $n = n_1 = n_2$, where $n_i$ is the number of vertices with valency $k_i$.

Then $k_{12} = k_{21}$ and $\Gamma$ has spectrum $\{[\theta_0]_1^1, [\theta]_1^n, [\theta - 1]_1^{n-1}\}$, where $\theta \in \{\theta_1, \theta_2\}$ is an eigenvalue of the quotient matrix of the valency partition of $\Gamma$. Moreover, $BA_1 = (J - I - A_2)B$, and

$$B^\top B = (\theta I - A_1)(\theta I - (J - I - A_1)), \quad \text{and} \quad BB^\top = (\theta I - A_2)(\theta I - (J - I - A_2)).$$

Proof. First, in order to simplify the exposition below, we write $A_2 = J - I - A_1$. Observe that $\tilde{A}_1$ is cospectral to $A_1$. Also note that $k_{12} = k_{21}$, since $n_1 = n_2$ (see the proof of Theorem 3.14). In particular, $Bj = B^\top j = k_{12}j$. Also note that $k_{11} + k_{22} = n - 1 = \theta_0 + \theta$ and we can write

$$A - \theta_1 I)(A - \theta_2 I) = A^2 - (\theta_1 + \theta_2)A + \theta_1 \theta_2 I = \alpha \alpha^\top.$$

For a matrix $X$ whose rows and columns are indexed by the vertex set of $\Gamma$, we will denote by $X_{i,j}$ a submatrix of $X$ whose rows (columns) correspond to the vertices of $\Gamma$ of valency $k_i$ ($k_j$, respectively). We have

$$A^2 = \begin{pmatrix} A_1 + B^\top B & A_1 B^\top - B^\top \tilde{A}_1 + B^\top (J - I) \\ BA_1 - A_1 B + (J - I)B & A_2^2 + BB^\top \end{pmatrix}.$$}

On the other hand, by Eq. (8), we have

$$(A^2)_{1,1} = A_1(\theta_1 + \theta_2) + \alpha_1^2 J - \theta_1 \theta_2 I,$$

$$(A^2)_{2,2} = (J - I - \tilde{A}_1)(\theta_1 + \theta_2) + \alpha_2^2 J - \theta_1 \theta_2 I,$$

so that

$$B^\top B = -A_1^2 + A_1(\theta_1 + \theta_2) + \alpha_1^2 J - \theta_1 \theta_2 I,$$

$$BB^\top = -\tilde{A}_1^2 - \tilde{A}_1(2 + \theta_1 + \theta_2) + (\theta_1 + \theta_2 - n + 2 + 2k_{11} + \alpha_2^2)J - (\theta_1 \theta_2 + \theta_1 + \theta_2 + 1)I,$$

Note that $BB^\top j = B^\top B j = k_{12}k_{21}j$. Multiplying Eqs. (9) and (10) by $j$, and then comparing their right hand sides, we obtain that

$$(2k_{11} + 1)(\theta_1 + \theta_2 + 1) = n(\theta_1 + \theta_2 + 1).$$

Now if $(2k_{11} + 1) = n$ holds then $k_{22} = n - 1 - k_{11} = k_{11}$ and, thus, $\Gamma$ is a regular graph, a contradiction. Therefore $\theta_1 + \theta_2 = -1$.

Suppose that $\Gamma$ had spectrum $\{[\theta_0]_1^1, [\theta]_1^{m_1}, [\theta - 1]_1^{m_2}\}$, where $1 + m_1 + m_2 = |V(\Gamma)| = 2n$. Then

$$\theta_0 + (m_1 - m_2)\theta - m_2 = \text{tr}(M) = 0.$$
This, together with the equation \( \theta_0 + \theta = n - 1 \), gives \( n - m_1 = 2(m_1 - n)\theta \). Since \( \theta \) is an integer, we see that \( n = m_1 \) and hence \( \Gamma \) has spectrum \( \{\theta_0, 1, \theta, \theta^2 - \theta - 1\} \).

Substituting \( \theta_1 + \theta_2 = -1 \) into Eqs. 9 and 10, we see that
\[
B^\top B = -A_1^2 - A_1 + \alpha_2^2 J - \theta_1 \theta_2 I,
\]
\[
BB^\top = -\tilde{A}_1^2 - \tilde{A}_1 + \alpha_2^2 J - \theta_1 \theta_2 I.
\]

Further, let us show that \( BA_1 = \tilde{A}_1 B \) holds. From Eq. 8, we can write
\[
(A^2)_{2,1} - (\theta_1 + \theta_2)M_{2,1} = BA_1 - \tilde{A}_1 B + JB - (1 + \theta_1 + \theta_2)B = \alpha_1 \alpha_2 J.
\]

Taking into account \( \theta_1 + \theta_2 = -1 \) and \( JB = k_{12} J \), we have that
\[
(11) \quad BA_1 - \tilde{A}_1 B = (\alpha_1 \alpha_2 - k_{12}) J.
\]

Multiply Eq. (11) by \( J \) to obtain the equality
\[
(12) \quad \alpha_1 \alpha_2 - k_{12} = 0,
\]
hence the required equality holds.

Finally, we shall show Eq. 11 (we prove only the first equation, the second one follows similarly). Start with the equation
\[
(\theta I - A_1)(\theta + 1)I - (J - A_1)) = -A_2^2 - A_1 + (k_{11} - \theta)J + \theta(\theta + 1)I.
\]

Since \( \theta \in \{\theta_1, \theta_2\} \) and \( \theta_1 + \theta_2 = -1 \), we see that \( \theta(\theta + 1) = -\theta_1 \theta_2 \). Further, \( k_{11} - \theta = \theta_0 - k_{22} \), and by Eq. (12), we have \( (\theta_0 - k_{22})\alpha_2 = k_{21} \alpha_1 = \alpha_2^2 \). Thus
\[
(\theta I - A_1)(\theta + 1)I - (J - A_1)) = -A_2^2 - A_1 + \alpha_2^2 J - \theta_1 \theta_2 I = B^\top B,
\]
and the lemma follows. \( \square \)

In Table 1 we observe that, apart from the Petersen cone, all feasible parameter sets for graphs in \( \mathcal{G}(\theta_0, \theta_1, \theta_2) \) with \( \theta_1 + \theta_2 = -1 \) have \( n_1 = n_2 \). It is an interesting problem to decide whether this property follows from the spectrum in general except from the Petersen cone.

4. Complements and switchings

In this section we will discuss connected graphs \( \Gamma \) that, together with their complements \( \overline{\Gamma} \), both have precisely 3 distinct eigenvalues. Clearly, strongly regular graphs that are both connected and cocomplement satisfy this property. We will show that, for nonregular graphs, such a graph must be biregular. First we describe the eigenvalues of the complement of a biregular graph with three eigenvalues.

**Proposition 4.1.** Let \( \Gamma \) be a biregular \( n \)-vertex graph in \( \mathcal{G}(\theta_0, \theta_1, \theta_2) \) and let \( A \) be its adjacency matrix, and let \( \theta_0 + \theta \theta \) be the trace of the quotient matrix of the valency partition of \( \Gamma \). Then the complement \( \overline{\Gamma} \) of \( \Gamma \) has at most 4 distinct eigenvalues. For \( n > 6 \) the eigenvalues of \( \overline{\Gamma} \) are \( -1 - \theta_1, -1 - \theta_2, \) and
\[
\frac{n - 2 - (\theta_0 + \theta)}{2} \pm \frac{\sqrt{(n + \theta_0 + \theta)^2 - 4(\theta_0 \theta + tr A^2)}}{2}.
\]
Moreover, for \( i \in \{1, 2\} \), the multiplicity of \( -1 - \theta_i \) in \( \overline{\Gamma} \) is at least \( m_i - 1 \) if \( \theta_i = \theta \) or at least \( m_i \) if \( \theta_i \not= \theta \).
Proof. By Theorem 3.3 the valency partition of $\Gamma$ is equitable with quotient matrix $Q$. Therefore the valency partition of $\overline{\Gamma}$ is also equitable with quotient matrix $Q$. For $n > 6$, by Theorem 2.11 both the multiplicities $m_1$ and $m_2$ are at least 2. Now $\overline{\Gamma}$ has eigenvalues $-1 - \theta_1$ and $-1 - \theta_2$ with multiplicities at least $m_1 - 1$ and $m_2 - 1$ respectively. The two eigenvalues of $\overline{Q}$ are determined from the trace and determinant of $Q$. These eigenvalues have the form

$$\frac{n - 2 - (\theta_0 + \theta)}{2} \pm \sqrt{(n + tr Q)^2 - 4(det Q + tr A^2)}$$

Since the sum of the eigenvalues of $\overline{\Gamma}$ is zero, we obtain that the remaining eigenvalue is $tr Q - (\theta_0 + \theta_1 + \theta_2) - 1$. By Theorem 3.3 we have either $tr Q = \theta_0 + \theta_1$ or $tr Q = \theta_0 + \theta_2$, whence the remaining eigenvalue is equal to $-1 - \theta_2$ or $-1 - \theta_1$ respectively.

Theorem 4.2. Let $\Gamma$ be an $n$-vertex graph in $G(\theta_0, \theta_1, \theta_2)$ such that its complement $\overline{\Gamma}$ is in $G(\theta'_0, \theta'_1, \theta'_2)$. Then $\Gamma$ is biregular with valencies $k_1$ and $k_2$ satisfying

$$k_1, k_2 = \frac{n + \theta_1 + \theta_2 \pm \sqrt{(n + \theta_1 + \theta_2 + 2\theta_1\theta_2)^2 - 4\theta_2^2(\theta_1 + 1)^2}}{2}.$$  

Moreover,

$$n = \frac{(\theta_0 - \theta_1)^2}{\theta_0 - \theta_1 + \theta_2 \theta_2}; \quad \theta_0 = \frac{n}{2} + \theta_1 \pm \sqrt{n(n + 4\theta_2(\theta_1 + 1))/2};$$

$$\theta'_0 = n - 1 - \theta_0 + \theta_1 - \theta_2; \quad \theta'_1 = -1 - \theta_2, \quad \theta'_2 = -1 - \theta_1.$$

Proof. We first show that $\Gamma$ is biregular. To distinguish between $\Gamma$ and $\overline{\Gamma}$ we denote elements of $\overline{\Gamma}$ with an apostrophe, e.g., we denote the degree of a vertex $x$ in $\overline{\Gamma}$ by $d'_x$ and its eigenvalues $\theta'_i$ have multiplicity $m'_i$.

Firstly, if either $m_1$ or $m_2$ are in $\{1, 2\}$, then, by Theorem 2.11 either $\theta_1 = 0$ or $n \leq 3 \times 4/2 = 6$. The only graphs satisfying this condition are the complete bipartite graphs (see Corollary 2.3), and these graphs do not satisfy the assumption of the theorem. Therefore we can assume that both $m_1 \geq 3$ and $m_2 \geq 3$, and hence, for each $i \in \{1, 2\}$, the complement $\overline{\Gamma}$ has eigenvalue $-1 - \theta_i$ with multiplicity at least $m_i - 1$. Thus the two smaller eigenvalues of $\overline{\Gamma}$ are $\theta'_i = -1 - \theta_2$ with multiplicity $m'_i \geq m_2 - 1$ and $\theta'_2 = -1 - \theta_1$ with multiplicity $m'_2 \geq m_1 - 1$.

Since $d_x + d'_x = n - 1$, by Lemma 2.7 for all $x \in V(\Gamma)$ we have

$$\tag{13} \theta_0 \geq \frac{\sum d_x}{n} \quad \text{and} \quad \theta'_0 \geq \frac{\sum d'_x}{n}.$$  

It follows that $\theta_0 + \theta'_0 \geq \frac{n(n - 2)}{n} = n - 1$. Further, we have $m_1 + m_2 = m'_1 + m'_2 = n - 1$ and

$$\tag{14} \theta_0 + m_1 \theta_1 + m_2 \theta_2 = 0 \quad \text{and} \quad \theta'_0 + m'_1 \theta'_1 + m'_2 \theta'_2 = 0.$$  

Hence $\theta_1(m_1 - m'_1) + \theta_2(m_2 - m'_2) = n - 1 - (\theta_0 + \theta'_0)$. Thus $\theta_1(m_1 - m'_1) + \theta_2(m_2 - m'_2) \leq 0$ which implies either $m_1 = m'_1$ and $m_2 = m'_2$, or $m_1 = m'_2 - 1$ and $m_2 = m'_1 + 1$.

We will first consider the case $m_1 = m'_1$, and $m_2 = m'_2$. In this case we have equality in (13), which implies that $\Gamma$ is regular (in fact strongly regular), but $\Gamma$ is nonregular so we must have that $m_1 = m'_2 - 1$ and $m_2 = m'_1 + 1$. 

Take two vertices \(x, y\). Since \(\Gamma\) and \(\overline{\Gamma}\) are both connected, we can assume \(x \sim y\) in \(\Gamma\). Denote by \(N_x\) the set of neighbours of \(x\) that are not adjacent to \(y\) in \(\Gamma\). Then we can write

\[
|N_x| = d_x - \nu_{xy} - 1 = \alpha_x^2 - \theta_1 \theta_2 - \alpha_x \alpha_y - \theta_1 - \theta_2 - 1 = \alpha_x (\alpha_x - \alpha_y) - (\theta_1 + 1)(\theta_2 + 1).
\]

On the other hand, all vertices in \(N_x\) are the neighbours of \(y\) that are not adjacent to \(x\) in \(\overline{\Gamma}\), and hence

\[
|N_x| = d'_y - \nu'_{xy} = \alpha_y^2 - (\theta_1 + 1)(\theta_2 + 1) - \alpha'_x \alpha'_y
= \alpha'_y (\alpha'_x - \alpha'_x) - (\theta_1 + 1)(\theta_2 + 1).
\]

Then for all pairs of vertices \(x, y\), we have the following equations

\[
\begin{align*}
\alpha_x (\alpha_x - \alpha_y) &= \alpha'_y (\alpha'_x - \alpha'_x), \\
\alpha_y (\alpha_y - \alpha_x) &= \alpha'_x (\alpha'_y - \alpha'_y). 
\end{align*}
\]

(15)

(16)

Now, taking the sum and difference of the equations (15) and (16) implies that \((\alpha_x - \alpha_y)^2 = (\alpha'_y - \alpha'_x)^2\) and \(\alpha_x^2 - \alpha_y^2 = \alpha_x^2 - \alpha_x^2\). Whence we have \(\alpha_x = \alpha'_y\) and \(\alpha_y = \alpha'_x\) for all pairs of vertices \(x\) and \(y\) with \(d_x \neq d_y\). Therefore \(\Gamma\) is biregular.

Since \(m_1 = m'_2 - 1\) and \(m_2 = m'_1 + 1\), we see from Eq. (14) that \(\theta_0 + \theta'_0 = n - 1 + \theta_1 - \theta_2\).

By Proposition 4.1, the quotient matrix of the valency partition of \(Q\) has eigenvalues \(\theta_1\) and \(\theta_2\). Applying Theorem 3.3 gives \(\alpha_1 \alpha_2 = -\theta_2 (\theta_1 + 1)\), and hence \(\alpha_1^2 + \alpha_2^2 = n + 2\theta_1 \theta_2 + \theta_1 + \theta_2\). Therefore we have

\[
k_1, k_2 = \frac{n + \theta_1 + \theta_2}{2} \pm \sqrt{\left(\frac{n + \theta_1 + \theta_2}{2} + \theta_1 \theta_2\right)^2 - \theta_2^2 (\theta_1 + 1)^2}.
\]

Moreover, again by Theorem 3.3, we can write \(n = \frac{(\theta_0 - \theta_1)^2}{\theta_0 - \theta_1 \theta_2 + \theta_2}\), and hence \(\theta_0 = \frac{\theta_1}{n} + \theta_1 \pm \sqrt{n(n + 4\theta_2(\theta_1 + 1))/2}\).

We describe an infinite family of feasible parameters for biregular graphs with three distinct eigenvalues whose complements also have three distinct eigenvalues.

**Proposition 4.3.** For \(t \geq 1\), set \(\theta_1 = -\theta_2 = 2t^2 + 2t - 1\), \(\theta_0 = \theta_1 - 2\theta_2(\theta_1 + 1)\), and \(k_1, k_2 = 2(\theta_1 + 1)\theta_1 (\theta_1 + (2t + 1)\theta_1\). Suppose \(\Gamma\) is in \(\mathcal{G}(\theta_0, \theta_1, \theta_2)\) with two valencies \(k_1\) and \(k_2\). Then the complement \(\overline{\Gamma}\) is in \(\mathcal{G}(\theta_0 - 1, -1, -1, -\theta_2, -\theta_1 + 1)\).

**Proof.** Let \(\theta_0 + \theta\) be the trace of the quotient matrix of the valency partition of \(\Gamma\). Using Theorem 3.3 one can check that \(\theta = \theta_2\) and that the other parameters \(k_{11}\), \(k_{12}\), etc. are all integral. By Proposition 4.1, it suffices to show that

\[
-1 - \theta_1 = \frac{n - 2 - (\theta_0 + \theta)}{2} \pm \sqrt{(n + \theta_0 + \theta)^2 - 4(\theta_0 \theta + \text{tr} A^2)/2}.
\]

We leave it to the reader to check this equality.

**Remark 4.4.** In Proposition 4.3 the case having smallest \(n\) is obtained when \(t = 1\). This gives \(n = 48\) with spectrum \(\{[27]^1, [3]^{19}, [-3]^{28}\}\), and its complement has spectrum \(\{26]^{1}, [2]^{27}, [-4]^{20}\}\). We now show that there exists no graph having these parameters.
Theorem 4.5. There do not exist any graphs having either of the following parameters: \( n = 48; n_1 = 24; n_2 = 24; k_1 = 33; k_2 = 15; \) spectrum \([27]^1, [3]^{19}, [-3]^{28}\); \( n = 48; n_1 = 24; n_2 = 24; k_1 = 32; k_2 = 14; \) spectrum \([27]^1, [2]^{27}, [-4]^{20}\).

Proof. Suppose such a graph \( \Gamma \) exist having the first set of parameters. The subgraph \( \Lambda \) induced on the vertices having valency 15 is regular with valency 3. Moreover, by interlacing, this subgraph has an eigenvalue of \(-3\) with multiplicity at least 4. Therefore \( \Lambda \) is four copies of \( K_{3,3} \). Partition \( \Gamma \) into five parts consisting of the vertex sets of each \( K_{3,3} \) and \( V_1 \). The quotient matrix

\[
Q = \begin{pmatrix}
3 & 0 & 0 & 0 & 12 \\
0 & 3 & 0 & 0 & 12 \\
0 & 0 & 3 & 0 & 12 \\
0 & 0 & 0 & 3 & 12 \\
3 & 3 & 3 & 3 & 21
\end{pmatrix}
\]

has eigenvalues 27, 3 (with multiplicity 3), and \(-3\). Since the interlacing of the eigenvalues of \( Q \) with the eigenvalues of \( \Gamma \) is tight, we know that the partition must be equitable. Therefore, each vertex of \( V_1 \) is adjacent to precisely three vertices of each \( K_{3,3} \). Let \( X \) be an induced subgraph of \( \Gamma \) consisting of a vertex adjacent to three vertices of a \( K_{3,3} \). Then \( X \) can be one of two graphs both of which have smallest eigenvalue less than \(-3\). This gives a contradiction. \( \Box \)

So far we know of no graphs that satisfy Theorem 4.2 it is an open problem to determine their existence.

Let \( \Gamma \) be a graph in \( G(\theta_0, \theta_1, \theta_2) \) with two valencies (say \( k_1, k_2 \)). We can also consider the graph \( \Gamma' \) obtained by switching with respect to \( V_1 \) in \( \Gamma \). Let \( Q \) and \( Q' \) be the quotient matrices of the valency partitions of \( \Gamma \) and \( \Gamma' \) respectively, \( Q = \begin{pmatrix} k_{11} & k_{12} \\
k_{21} & k_{22} \end{pmatrix} \) and \( Q' = \begin{pmatrix} k_{11} & n_2 - k_{12} \\
n_1 - k_{21} & k_{22} \end{pmatrix} \). In the proposition below, we show that \( \Gamma' \) has at most 4 distinct eigenvalues.

Proposition 4.6. Let \( \Gamma \) be a graph in \( G(\theta_0, \theta_1, \theta_2) \), so that its adjacency matrix \( A \) under the valency partition has block form:

\[
A = \begin{pmatrix}
A_1 & B \\
B^\top & A_2
\end{pmatrix}
\]

with quotient matrix \( Q = \begin{pmatrix} k_{11} & k_{12} \\
k_{21} & k_{22} \end{pmatrix} \), and let \( \Gamma' \) be obtained by switching the vertices with respect to \( V_1 \). Then \( \Gamma' \) has at most 4 distinct eigenvalues. Namely, \( \theta_1, \theta_2, \) and

\[
\frac{k_{11} + k_{22}}{2} \pm \sqrt{(k_{11} - k_{22})^2 + 4(n_1 - k_{21})(n_2 - k_{12})}.
\]

Moreover, for \( k_{11} + k_{22} = \theta_0 + \theta \) and \( i \in \{1, 2\} \), the multiplicity of \( \theta_i \) in \( \Gamma' \) is at least \( n_i - 1 \) if \( \theta_i = \theta \) or at least \( n_i \) if \( \theta_i \neq \theta \).

Proof. By Theorem 3.3 the valency partition is equitable. Apply Corollary 3.2 of Muzychuk and Klin \[17\] to find that \( \Gamma' \) has eigenvalues \( \theta_1 \) and \( \theta_2 \) where the multiplicity of \( \theta_i \) is at least \( n_i - 1 \) if \( \theta_i = \theta \) or at least \( n_i \) if \( \theta_i \neq \theta \). Moreover, the remaining two eigenvalues are the eigenvalues of the quotient matrix \( Q' \) of the corresponding equitable partition of \( \Gamma' \), \( \begin{pmatrix} k_{11} & n_2 - k_{12} \\
n_1 - k_{21} & k_{22} \end{pmatrix} \). \( \Box \)
Suppose that $\Gamma'$ has precisely three distinct eigenvalues. By Proposition 4.6, $\Gamma'$ is in $G(\theta'_0, \theta_1, \theta_2)$. On the one hand, if $\theta'_0 = \theta_0$, then $Q$ and $Q'$ have same eigenvalues. Hence we have the equality $\det Q = \det Q' = k_{11}k_{22} - k_{12}k_{21} = k_{11}k_{22} - (n_1 - k_{21})(n_2 - k_{12})$ and thus, $n_2 = 2k_{12}$ and $n_1 = 2k_{21}$.

On the other hand, if $\theta'_0 \neq \theta_0$ then, without loss generality, we can assume that $Q$ has eigenvalues $\theta_0$ and $\theta_2$, and that $Q'$ has eigenvalues $\theta'_0$ and $\theta_1$.

Consider the special case $k_{11} = k_{22}, \alpha_1 = so_2$, and $\theta_2 = -(s - 1)\theta_1 - s$. We describe an infinite family of feasible parameters of biregular graphs with three eigenvalues such that switching with respect to $V_1$ gives another graph having three distinct eigenvalues.

\textbf{Proposition 4.7.} For $s \geq 2$ and $t \geq 1$, set $\theta_1 = st$, $\theta_2 = -(s - 1)\theta_1 - s$, $\theta_0 = s(2st + 1)(st + 1 - t)$, $k_1 = (s^2(st + 1) + st)(st - t + 1)$, and $k_2 = (s^2t + st + 1)(st - t + 1)$.

Suppose $\Gamma$ is in $G(\theta_0, \theta_1, \theta_2)$ with two valencies $k_1$ and $k_2$ and let $\Gamma'$ be the graph obtained by switching the vertices in $V_1$ of $\Gamma$. Then $\Gamma'$ is in $G(\theta_0 - s(st + 1), \theta_1, \theta_2)$.

\textbf{Proof.} By Proposition 4.6, it suffices to show that the largest eigenvalue of $\Gamma'$ is $\theta_0 - s(st + 1)$, that is, we want to show

$$\theta_0 - s(st + 1) = \frac{k_{11} + k_{22}}{2} + \frac{\sqrt{(k_{11} - k_{22})^2 + 4(n_1 - k_{21})(n_2 - k_{12})}}{2}.$$

Using Theorem 5.1, we can write each of the terms $k_{11}, k_{12}, k_{21}, k_{22}, n_1$, and $n_2$ in terms of $s$ and $t$. We leave it to the reader to check the above equality. \(\square\)

\textbf{Remark 4.8.} In Proposition 4.7, the case having smallest $n$ is obtained when $s = 2$ and $t = 1$. This gives $n = 45$ with spectrum $\{[20]^1, [2]^26, [-4]^{18}\}$, and $\Gamma'$ has spectrum $\{[14]^1, [2]^{27}, [-4]^{17}\}$. (See Table 4 for this case and the case on 80 vertices.)

5. Constructions and Nonexistence

In this section we describe constructions for certain graphs with three eigenvalues. In particular we show the existence of a case which was an open problem from Van Dam’s paper [11]. We also construct a new graph having three valencies and three eigenvalues and we show that there is no 44-vertex biregular graph with three eigenvalues.

Let $\theta$ be an eigenvalue of an $n$-vertex graph $\Gamma$ and suppose that $\theta$ has multiplicity $m$. Define a star set for $\theta$ to be a subset $X \subset V(\Gamma)$ such that $|X| = m$ and $\theta$ is not an eigenvalue of $\Gamma - X$. Now we can state the Reconstruction Theorem (See [9 Theorems 7.4.1 and 7.4.4]).

\textbf{Theorem 5.1.} Let $X$ be a subset of vertices of a graph $\Gamma$ and suppose that $\Gamma$ has adjacency matrix

$$
\begin{pmatrix}
A_X & B^\top \\
B & C
\end{pmatrix},
$$

where $A_X$ is the adjacency matrix of the subgraph induced by $X$. Then $X$ is a star set for $\theta$ if and only if $\theta$ is not an eigenvalue of $C$ and $\theta I - A_X = B^\top (\theta I - C)^{-1}B$.

The graph $\Lambda$ induced by $\Gamma - X$ (having adjacency matrix $C$ in Theorem 5.1) is called the star complement of $\theta$. Star sets and star complements exist for any eigenvalue and any graph and moreover, for $\theta \notin \{0, 1\}$, it can be shown that $\Lambda$-neighbourhoods of the vertices of $X$ are non-empty and distinct [9 Chapter 7]. For vectors in $v, w \in \mathbb{Z}^{n-m}$, define the bilinear map $\langle v, w \rangle := v^\top(\theta I - C)^{-1}w$. Let
V be the set of vectors \( v \in \{0,1\}^{n-m} \) satisfying \( \langle v, v \rangle = 0 \). Form a graph having V as its vertex set where two vectors \( v \) and \( w \) are adjacent if \( \langle v, w \rangle \in \{0, -1\} \). This graph is known as the compatibility graph for \( \Lambda \). Cliques in the compatibility graph then give the columns of the matrix \( B \) as in Theorem 5.1.

**Theorem 5.2.** There exist at least 21 graphs having parameters \( n = 30; n_1 = 15; n_2 = 15; k_1 = 14; k_2 = 8; \) spectrum \( \{[12]^1,[2]^{15},[-3]^{14}\} \).

**Proof.** Let \( \Gamma \) be a graph having the assumed parameters so that its valency partition has the quotient matrix \( \begin{pmatrix} 10 & 4 \\ 4 & 4 \end{pmatrix} \). Now we assume that the star complement of \( \Gamma \) for the eigenvalue 2 is the subgraph induced on either \( V_1 \) or \( V_2 \). Next we check all possibilities for these subgraphs using Magma [3] and nauty [11]. Such graphs are regular with valency equal to either 4 or 10. By interlacing, any such star complement must have smallest eigenvalue at least \(-3\) and second largest eigenvalue less than 2. There are 94 (resp. 43) regular graphs on 15 vertices with valency 10 (resp. 4), smallest eigenvalue at least \(-3\), and second largest eigenvalue less than 2. For each of these potential star complements, we construct the compatibility graph and search for cliques of size 15. This process produced a list of 21 non-isomorphic graphs with the assumed parameters.

We remark that, by combining the above theorem with Van Dam’s classification [10] Section 7, we now have a complete understanding of graphs with three distinct eigenvalues on at most 30 vertices.

A graph in \( G(20, 2, -3) \) having 3 distinct valencies was constructed in [7]. We show that (see Theorem 5.3) there exists precisely one more graph having the same parameters. First we describe the graphs \( \Gamma_1 \) and \( \Gamma_2 \). Form a set \( P \) of five ordered partitions of \( \{0, \ldots, 5\} \) whose parts are pairs of elements of \( \{0, \ldots, 5\} \) such that, for any two partitions \( p \) and \( q \) in \( P \), we have

\[
\sum_{i=0}^{2} |p(i) \cap q(i)| = 2.
\]

Note that each of the 15 pairs of elements of \( \{0, \ldots, 5\} \) appears in precisely one of the ordered partitions in \( P \) and hence each pair appears precisely once. In the following descriptions, the subscripts should be reduced modulo 3.

Now we describe the graph \( \Gamma_1 \), which is isomorphic to the one given in [7]. For each \( i \in \{1, \ldots, 5\} \), assign to \( p^{(i)} \) a unique partition \( p_i \in P \). Let \( X \) be a graph whose vertices are the disjoint union of the triples in \( T \) where each triple forms a triangle and let \( Y \) be the complement of \( X \). For the sake of clarity we denote by \( s^{(i)} \) the corresponding (complement) triangles in \( Y \). Then \( \Gamma_1 \) is equal to \( X \cup Y \) where the edges between \( X \) and \( Y \) are given as follows. For each \( j \in \{0, 1, 2\} \), the vertex \( t_j^{(0)} \) is adjacent to the vertices \( s_i^{(j)} \) for all \( i \in \{0, \ldots, 5\} \). Now partition the set \( \{1, \ldots, 5\} \setminus \{p\} \) into two sets \( L \) and \( R \) of size 2 and 3 respectively. For \( i \in L \), the vertex \( t_i^{(j)} \) is adjacent to \( s_j^{(i)} \) for \( j \in \{0, 1, 2\} \) and \( l \in p_i(k) \). Otherwise, for \( i \in R \), \( t_i^{(j)} \) is adjacent to all vertices of \( Y \) except \( s_{j+k}^{(i)} \) for \( j, k \in \{0, 1, 2\} \) and \( l \in p_i(k) \).

Next we describe a new graph, \( \Gamma_2 \), having the same parameters as \( \Gamma_1 \). Fix \( r \in \{1, \ldots, 5\} \). Let \( p \in P \) be the partition that contains the part \( \{0, r\} \) and let \( f : T \to T \) be the involution that interchanges the elements of the other two parts of \( p \). Assign the partitions to the triples as above such that \( p_r = p \). Let \( X \) be
a graph whose vertices are the disjoint union of the triples in \( T \) and for each for 
\( i \in \{0, \ldots, 5\} \) join each vertex \( t_{ij}^{(i)} \) to \( t_{ij}^{f(i)} \) and \( t_{ij}^{f(i)} \) such that \( \{j, k, l\} = \{0, 1, 2\} \).
Let \( Y \) be the complement of \( X \). Set \( \Gamma_2 \) equal to \( X \cup Y \) where the edges between \( X \) and \( Y \) are given as follows. As before, \( t_{ij}^{(0)} \) is adjacent to the vertices \( s_{j}^{(i)} \) for all \( i \in \{0, \ldots, 5\} \) and \( t_{ij}^{(r)} \) is adjacent to all vertices of \( Y \) except \( s_{j+k}^{(l)} \) for \( k \in \{0, 1, 2\} \) and \( l \in p_i(k) \). Now partition the set \( \{1, \ldots, 5\} \setminus \{r\} \) into two sets \( L \) and \( R \) of size 2. For \( i \in R, k \in \{0, 1, 2\} \), and \( l \in p_i(k) \),
\[
\begin{align*}
t_{ij}^{(i)} &\sim \begin{cases} s_{j+k}^{(l)}, & l \neq i; \\ s_{j}^{(f(i))}, & \text{otherwise}. \end{cases} \\
t_{ij}^{(i)} &\nsim \begin{cases} s_{j+k}^{(l)}, & l \neq i; \\ s_{j}^{(f(i))}, & \text{otherwise}, \end{cases}
\end{align*}
\]
and \( t_{ij}^{(i)} \) is adjacent to all other vertices of \( Y \).

**Theorem 5.3.** The graphs \( \Gamma_1 \) and \( \Gamma_2 \) are the only graphs having parameters \( n = 36; k_1 = 24; k_2 = 14; k_3 = 8; \) spectrum \( \{[20]^1, [2]^17, [-3]^18\} \).

**Proof.** Let \( \Gamma \) be a graph having the assumed parameters so that its valency partition has quotient matrix
\[
\begin{pmatrix}
15 & 6 & 3 \\
12 & 2 & 0 \\
6 & 0 & 2
\end{pmatrix}.
\]
Hence, the graphs induced on the subsets \( V_2 \) and \( V_3 \) are unions of cycles. On 9 vertices, there are only 4 graphs that are unions of cycles. Therefore, there are 16 possible graphs for the subgraph \( \Lambda \) induced on the set \( V_2 \cup V_3 \). Each of these 16 graphs is a potential star complement for the eigenvalue \(-3\). For each of these potential star complements, we construct the compatibility graph and search for cliques of size 18. This process produced a list of two non-isomorphic graphs with the assumed parameters.

Now, unlike in the proof of Theorem 5.2, the graph induced on the set \( V_2 \cup V_3 \) must be a star complement for the eigenvalues \(-3\) (in the proof Theorem 5.2 we assumed this was the case for the graph induced on \( V_1 \) or \( V_2 \)). This means that there can exist no other graphs on the assumed parameters.

**Theorem 5.4.** There do not exist any graphs having parameters \( n = 44; n_1 = 22; n_2 = 22; k_1 = 22; k_2 = 7; \) spectrum \( \{[19]^1, [2]^22, [-3]^21\} \).

**Proof.** Let \( \Gamma \) be a graph having the assumed parameters so that its valency partition has quotient matrix \( \begin{pmatrix} 18 & 4 \\ 4 & 3 \end{pmatrix} \). The subgraph \( \Lambda \) induced on the set \( V_2 \) is cubic. For vertices \( x \) and \( y \) in \( V_2 \) we have \( \nu_{xy} = 0 \) if \( x \sim y \) or \( \nu_{xy} = 1 \) if \( x \not\sim y \). Therefore, \( \Lambda \) cannot contain any triangles or any four-cycles. Moreover, \( \Lambda \) is a 22 vertex cubic graph which, by interlacing, has second largest eigenvalue at most 2. Using **Magma** [3] and **nauty** [16], we find that \( \Lambda \) must be one of four possible graphs. After adjoining 5 vertices in all possible ways to each of the four possible graphs we find that none of the resulting graphs has both smallest eigenvalue at least \(-3\) and second largest eigenvalue at most 2. \( \square \)
In [12], the authors gave a set of feasible parameters for a graph in $G(30, 3, -3)$ having four valencies. We show that no such graph exists.

**Theorem 5.5.** There do not exist any graphs having parameters

$n = 51; n_1 = 1; n_2 = 30; n_3 = 5; n_4 = 15; k_1 = 45; k_2 = 34; k_3 = 18; k_4 = 13; spectrum \{[30]^1, [3]^20, [-3]^30\}.$

The technique of the proof of this result is similar to the techniques used above. We therefore merely give a sketch of the proof.

**Sketch of proof.** Assume there exists a graph $\Gamma$ with the assumed parameters. Van Dam et al. [12] determined much of the structure of $\Gamma$. In particular, it is shown that the valency partition of such a graph $\Gamma$ would be equitable with quotient matrix

$$
\begin{pmatrix}
0 & 30 & 0 & 15 \\
1 & 25 & 3 & 5 \\
0 & 18 & 0 & 0 \\
1 & 10 & 0 & 2
\end{pmatrix}.
$$

Starting with the valency 2 subgraph, one can apply similar techniques as given in the proofs above to determine the nonexistence of $\Gamma$. □

The following lemma is a simple application of the Cauchy-Schwarz inequality.

**Lemma 5.6.** Let $\Gamma$ be a connected $n$-vertex $k$-regular graph having a non-trivial eigenvalue $\theta$ with multiplicity $m$. Then $(k + m\theta)^2 \leq (n - 1 - m)(nk - k^2 - m\theta^2)$.

**Theorem 5.7.** There do not exist any graphs having either of the following parameters

$n = 100; n_1 = 50; n_2 = 50; k_1 = 69; k_2 = 33; spectrum \{[57]^1, [7]^24, [-3]^75\}; n = 100; n_1 = 50; n_2 = 50; k_1 = 64; k_2 = 28; spectrum \{[52]^1, [2]^74, [-8]^25\}.$

**Proof.** The technique is the same for both sets of parameters; we will deal only with first. Consider the 50-vertex 9-regular subgraph $\Lambda$ induced by the subset of vertices $V_2$. By interlacing, $\Lambda$ has eigenvalues 9 with multiplicity 1 and $-3$ with multiplicity 25. Now apply Lemma 5.6 to deduce its nonexistence. □

Our final results link some highly structured graphs to certain designs. A set $V$ of cardinality $v$ together with a collection of $k$-subsets $B_1, \ldots, B_h$ (called blocks) of $V$ is called a $(v, b, r, k)$-configuration if each point of $V$ occurs in precisely $r$ blocks. A $(v, b, r, k)$-configuration $D$ is called a group divisible design (GDD) if its $v$ points can be partitioned into $m$ sets $T_1, \ldots, T_m$ each of size $n \geq 2$ such that any two points $x \in T_i$ and $y \in T_j$ occur together in $\lambda_1$ blocks if $i = j$ and $\lambda_2$ blocks otherwise. We say that $D$ is a GDD with parameters $(v, b; r, k; \lambda_1, \lambda_2; m, n)$ and in Table 1 we write gdd$(v, b; r, k; \lambda_1, \lambda_2; m, n)$. (We refer to Bose [2] for more details about GDDs.) Note that if $n = v$ then $D$ is a 2-design with parameters $(v, b, r, k, \lambda_1)$ or, for short, a $(v, k, \lambda_1)$-design.

**Theorem 5.8.** Let $\Gamma$ be a graph in $G(\theta_0, \theta_1, \theta_2)$ such that its adjacency matrix $A$ under the valency partition has block form:

$$
A = \begin{pmatrix}
A_1 & B_1 \\
B_2 & A_2
\end{pmatrix}
$$

with quotient matrix

$$
\begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix}
$$

and set $n_i = |V_i|$. 
(1) Suppose that, for some $i \in \{1, 2\}$, the induced subgraph on $V_i$ is a complete $m$-partite graph for some $m$. Set $\lambda_1 = \alpha_i^2 - k_{ii}$ and $\lambda_2 = \alpha_i^2 + \theta_1 + \theta_2 - k_{ii} + n_i/m$.

(2) Suppose that, for some $i \in \{1, 2\}$, the induced subgraph on $V_i$ is the disjoint union of cliques $K_{n_i/m}$ for some $m$. Set $\lambda_1 = \alpha_i^2 + \theta_1 + \theta_2 + 1 - k_{ii}$ and $\lambda_2 = \alpha_i^2$.

In either of the above cases, set $j$ such that $\{i, j\} = \{1, 2\}$, then the matrix $B_i$ is an incidence matrix of a GDD with parameters $(n_i, n_j; k_{ij}, k_{ji}; \lambda_1, \lambda_2; m, n_i/m)$.

Proof. Suppose we are in the first case of the theorem, so that the induced subgraph, $K$, on $V_i$ is a complete $m$-partite graph with parts $T_1, \ldots, T_m$. For any two vertices $x$ and $y$ in $V_i$, we have $\nu_{xy} = (\theta_1 + \theta_2)A_{xy} + \alpha_i^2$. Now, $x$ is adjacent to $y$ if and only if they are both in the same part $T_l$ for some $l$. Moreover, the number of common neighbours of $x$ and $y$ in $K$ is $k_{ii} - n_i/m$ if they are adjacent and $k_{ii}$ otherwise. Hence, taking $V_i$ as the set of points and $V_j$ as the set of blocks, we have that $B_i$ is the incidence matrix the required GDD. The other case follows similarly. \qed

Remark 5.9. We can apply Theorem 5.8 to some of the parameter sets given in Table I. In the comment column of the table, we write the parameter $s$ of the quotient matrix $A$.

Van Dam [10, p. 104] showed that, given any affine resolvable $(q^3, q^2, q+1)$-design, the total graph is nonregular with spectrum $\{[q^3 + q^2 + q]^{-1}, [q]^q, [-q]^q\}$. In our final result we show that graphs having that parameters of the total graph $\Gamma$ of such a design are necessarily isomorphic to $\Gamma$.

**Theorem 5.11.** Let $q$ be a positive integer. Let $\Gamma$ be a bipartite graph with parameters $n = 2q^3 + q^2 + q; n_1 = q^3 + q^2 + q; n_2 = q^3; k_1 = q^3 + 2q^2; k_2 = q^2 + q + 1$; spectrum $\{[q^3 + q^2 + q]^{-1}, [q]^q, [-q]^q\}$. Then $\Gamma$ is the total graph of some affine resolvable $(q^3, q^2, q+1)$-design.

Proof. The adjacency matrix of $\Gamma$, under the valency partition, has block form:

$$
\begin{pmatrix}
A_1 & B^T \\
B & A_2
\end{pmatrix}
$$

with quotient matrix $\begin{pmatrix}
q^3 + q^2 & q^2 \\
q^2 + q + 1 & 0
\end{pmatrix}$. By Theorem 5.8, $B$ is the incidence matrix of a $(q^3, q^2, q+1)$-design. Take $V_2$ as the set of points and $V_1$ as the set of blocks. The induced subgraph $\Lambda$ on the vertex set $V_1$ is a $q^3 + q^2$-regular graph with $n_1 = q^3 + q^2 + q$ vertices and, by interlacing, this subgraph has eigenvalue $-q$ with multiplicity at least $q^2 + q$. Hence the complement
of $\Lambda$ is a $q - 1$-regular graph, which has eigenvalue $q - 1$ with multiplicity at least $q^2 + q + 1$. It follows that $\Lambda$ is a complete $q^2 + q + 1$-partite graph $K_{q,q,q}$. So any blocks in the same part are disjoint and two blocks in different part intersect in $q$ points. Therefore, our $(q^3, q^2, q + 1)$-design is affine resolvable.

Remark 5.12. In Table I the feasible parameter set with 66 vertices and largest eigenvalue 39, satisfies the assumptions of Theorem 5.11. Indeed, it corresponds to an affine resolvable $(27, 9, 4)$-design of which there are known [15] to exist precisely 68.

APPENDIX A. FEASIBLE PARAMETERS

In this section we give a table of feasible parameters for biregular graphs having precisely three eigenvalues. Since we have a complete understanding in these cases, we omit parameters that correspond to complete bipartite graphs (second largest eigenvalue equal to 0) and parameters with second largest eigenvalue equal to 1 that correspond to nonexistent graphs.

In the comment column of Table I we give some information about some of the parameter sets. If a graph corresponding to the parameter set exists, we try to give some indication of how it can be constructed. If no such graph exists then we give a reference to a proof of its nonexistence. Otherwise, if the existence of a graph is unknown, we may refer to a design related to the parameters such as a group divisible design (see Remark 5.9).

Table 1: Feasible parameters for biregular graphs with three eigenvalues

| $v$ | Spectrum | $k_1$ | $k_2$ | $n_1$ | $n_2$ | $k_{12}$ | Existence | Comment |
|-----|----------|-------|-------|-------|-------|---------|-----------|---------|
| 11  | $[5]^1, [1]^9, [-2]^9$ | 10 | 4 | 1 | 10 | 10 | Petersen cone |
| 14  | $[8]^1, [1]^6, [-2]^7$ | 10 | 4 | 7 | 7 | 4 | 1 | Fano graph |
| 17  | $[8]^1, [2]^6, [-2]^10$ | 16 | 7 | 1 | 16 | 16 | 2 | $\text{sr}(16, 6, 2, 2)$ cone |
| 22  | $[14]^1, [2]^7, [-2]^14$ | 16 | 7 | 14 | 18 | 4 | 1 | [10] |
| 29  | $[14]^1, [4]^7, [-2]^21$ | 28 | 13 | 1 | 28 | 28 | 4 | $\text{sr}(28, 12, 6, 4)$ cone |
| 30  | $[13]^1, [3]^9, [-2]^20$ | 15 | 7 | 15 | 15 | 3 | 0 | [10] Theorem 7 |
| 30  | $[18]^1, [3]^8, [-2]^21$ | 22 | 10 | 15 | 15 | 8 | 0 | [10] Theorem 7 |
| 30  | $[12]^1, [2]^15, [-3]^14$ | 14 | 8 | 15 | 15 | 4 | $\geq 21$ | Theorem 5.2 |
| 32  | $[14]^1, [4]^8, [-2]^23$ | 16 | 10 | 16 | 16 | 4 | 0 | [10] Theorem 7 |
| 36  | $[21]^1, [5]^7, [-2]^28$ | 28 | 18 | 8 | 28 | 21 | 1 | [10] Theorem 7 |
| 39  | $[14]^1, [2]^23, [-4]^15$ | 17 | 12 | 12 | 27 | 9 | $\geq 120$ | Proposition 4.6 [10] |
| 29  | $[20]^1, [2]^22, [-4]^16$ | 26 | 16 | 12 | 27 | 18 | $\geq 120$ | Proposition 4.6 [10] |
| 44  | $[22]^1, [6]^8, [-2]^35$ | 24 | 15 | 28 | 16 | 4 | 0 | [10] Theorem 7 |
| 44  | $[19]^1, [2]^22, [-3]^21$ | 23 | 7 | 22 | 22 | 4 | 0 | Theorem 5.3 |
| 45  | $[32]^1, [5]^8, [-2]^36$ | 34 | 16 | 36 | 9 | 4 | 0 | [10] Theorem 7 |
| 45  | $[20]^1, [2]^26, [-4]^18$ | 32 | 14 | 9 | 36 | 24 | $\geq 9$ | Proposition 4.4 [10] |
| 45  | $[14]^1, [2]^27, [-4]^17$ | 20 | 11 | 9 | 36 | 12 | $\geq 9$ | Proposition 4.4 [10] |
| 46  | $[15]^1, [3]^20, [-3]^25$ | 45 | 13 | 1 | 45 | 45 | 78 | $\text{sr}(45, 12, 3, 3)$ cone |
| 48  | $[27]^1, [3]^19, [-3]^28$ | 33 | 15 | 24 | 24 | 12 | 0 | Theorem 4.5 |
| 48  | $[20]^1, [2]^27, [-4]^20$ | 32 | 14 | 24 | 24 | 12 | 0 | Theorem 4.5 |
| 50  | $[27]^1, [2]^24, [-3]^25$ | 33 | 9 | 25 | 25 | 9 | 78 | Corollary 5.10 |
### Table 1 – continued from previous page

| $v$ | Spectrum | $k_1$ | $k_2$ | $n_1$ | $n_2$ | $k_{12}$ | Existence | Comment |
|-----|----------|-------|-------|-------|-------|--------|-----------|---------|
| 54  | $[34]^1, [6]^9, [-2]^{44}$ | 40 | 19 | 30 | 24 | 12 | 0 | [10] Theorem 7 |
| 56  | $[21]^1, [3]^24, [-3]^{31}$ | 27 | 11 | 20 | 36 | 9 | ? | gdd(20, 36; 9, 5; 0, 2; 10, 2) |
| 57  | $[21]^1, [4]^21, [-3]^{38}$ | 28 | 16 | 15 | 42 | 14 | ? | (15, 5, 4)-design |
| 57  | $[14]^1, [2]^35, [-4]^{21}$ | 56 | 11 | 1 | 56 | 56 | 1 | srg(56, 10, 0, 2) cone |
| 66  | $[39]^1, [3]^26, [-3]^{39}$ | 45 | 13 | 39 | 27 | 9 | 68 | Remark 5.17 |
| 66  | $[33]^1, [6]^18, [-3]^{47}$ | 34 | 27 | 54 | 12 | 4 | ? |
| 66  | $[36]^1, [3]^32, [-4]^{33}$ | 44 | 20 | 33 | 33 | 16 | ? |
| 68  | $[34]^1, [2]^46, [-6]^{21}$ | 37 | 16 | 48 | 20 | 5 | ? |
| 69  | $[33]^1, [6]^19, [-3]^{49}$ | 36 | 26 | 42 | 27 | 9 | ? |
| 69  | $[24]^1, [2]^48, [-6]^{20}$ | 32 | 17 | 21 | 48 | 16 | ? |
| 70  | $[23]^1, [5]^23, [-3]^{46}$ | 69 | 21 | 1 | 69 | 69 | ? | srg(69, 20, 7, 5) cone |
| 70  | $[25]^1, [3]^40, [-5]^{29}$ | 33 | 23 | 10 | 60 | 24 | ? | (10, 4, 8)-design |
| 70  | $[30]^1, [2]^48, [-6]^{21}$ | 48 | 21 | 14 | 56 | 36 | ? | gdd(14, 56; 36, 9; 24, 22; 7, 2) |
| 70  | $[22]^1, [2]^49, [-6]^{20}$ | 32 | 17 | 14 | 56 | 20 | ? |
| 74  | $[33]^1, [3]^37, [-4]^{36}$ | 39 | 15 | 37 | 37 | 9 | ? |
| 74  | $[38]^1, [2]^50, [-6]^{23}$ | 48 | 21 | 34 | 40 | 20 | ? |
| 78  | $[33]^1, [3]^44, [-5]^{33}$ | 55 | 25 | 12 | 66 | 44 | ? |
| 80  | $[47]^1, [7]^19, [-3]^{60}$ | 53 | 39 | 40 | 40 | 24 | ? |
| 80  | $[27]^1, [3]^46, [-5]^{33}$ | 39 | 21 | 16 | 64 | 24 | $\geq 1$ | Proposition 4.7, [10] |
| 80  | $[35]^1, [3]^45, [-5]^{34}$ | 55 | 25 | 16 | 64 | 40 | $\geq 1$ | Proposition 4.7, [10] |
| 80  | $[42]^1, [2]^59, [-8]^{20}$ | 48 | 34 | 40 | 40 | 24 | ? |
| 81  | $[33]^1, [6]^23, [-3]^{57}$ | 42 | 24 | 27 | 54 | 18 | ? |
| 81  | $[29]^1, [2]^59, [-7]^21$ | 38 | 20 | 27 | 54 | 18 | ? |
| 82  | $[27]^1, [6]^24, [-3]^{57}$ | 81 | 25 | 1 | 81 | 81 | $\geq 1$ | srg(81, 24, 9, 6) cone |
| 84  | $[49]^1, [7]^20, [-3]^{63}$ | 57 | 37 | 42 | 42 | 24 | ? |
| 84  | $[44]^1, [4]^36, [-4]^{47}$ | 56 | 26 | 36 | 48 | 24 | ? |
| 84  | $[39]^1, [3]^51, [-6]^{32}$ | 43 | 27 | 54 | 30 | 10 | ? |
| 84  | $[44]^1, [2]^62, [-8]^{21}$ | 52 | 32 | 42 | 42 | 24 | ? |
| 85  | $[32]^1, [2]^64, [-8]^{20}$ | 64 | 28 | 5 | 80 | 64 | 0 | (5, 4, 48)-design |
| 86  | $[51]^1, [3]^51, [-6]^{34}$ | 54 | 34 | 68 | 18 | 9 | ? | (18, 9, 16)-design |
| 96  | $[44]^1, [4]^42, [-4]^{53}$ | 52 | 20 | 48 | 48 | 12 | ? |
| 96  | $[43]^1, [3]^54, [-5]^{41}$ | 51 | 19 | 48 | 48 | 12 | ? |
| 97  | $[24]^1, [4]^45, [-4]^{51}$ | 96 | 21 | 1 | 96 | 96 | $\geq 1$ | srg(96, 20, 4, 4) cone |
| 98  | $[44]^1, [4]^49, [-5]^{48}$ | 52 | 28 | 49 | 49 | 16 | ? |
| 99  | $[44]^1, [2]^78, [-10]^{20}$ | 47 | 32 | 72 | 27 | 9 | ? |
| 100 | $[57]^1, [7]^24, [-3]^{75}$ | 69 | 33 | 50 | 50 | 24 | 0 | Theorem 5.7 |
| 100 | $[52]^1, [2]^74, [-8]^{25}$ | 64 | 28 | 50 | 50 | 24 | 0 | Theorem 5.7 |

### References

[1] F. K. Bell and P. Rowlinson. On the multiplicities of graph eigenvalues. *Bull. London Math. Soc.*, 35(3):401–408, 2003.

[2] R. C. Bose. Symmetric group divisible designs with the dual property. *J. Stat. Plann. Inferenc*, 1(1):87–101, 1977.
[3] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[4] W. G. Bridges and R. A. Mena. Multiplicative cones—a family of three eigenvalue graphs. *Aequationes Math.*, 22(2-3):208–214, 1981.

[5] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs*, volume 18 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1989.

[6] F. C. Bussemaker, W. H. Haemers, and E. Spence. The search for pseudo orthogonal Latin squares of order six. *Des. Codes Cryptogr.*, 21(1-3):77–82, 2000. Special issue dedicated to Dr. Jaap Seidel on the occasion of his 80th birthday (Oisterwijk, 1999).

[7] D. de Caen, E. R. van Dam, and E. Spence. A nonregular analogue of conference graphs. *J. Combin. Theory Ser. A*, 88(1):194–204, 1999.

[8] H. Chuang and G. R. Omid. Graphs with three distinct eigenvalues and largest eigenvalues less than 8. *Linear Algebra Appl.*, 430(8-9):2053–2062, 2009.

[9] D. Cvetković, P. Rowlinson, and S. Simić. *Eigenspaces of graphs*, volume 66 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1997.

[10] E. R. van Dam. Nonregular graphs with three eigenvalues. *J. Combin. Theory Ser. B*, 73(2):101–118, 1998.

[11] E. R. van Dam and R. E. Kooij. The minimal spectral radius of graphs with a given diameter. *Linear Algebra Appl.*, 443(2-3):408–419, 2007.

[12] E. R. van Dam, J. H. Koolen, and Z.-J. Xia. Graphs with many valencies and few eigenvalues. arXiv:1405.3383.

[13] C. Godsil and G. Royle. *Algebraic graph theory*, volume 207 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[14] W. H. Haemers. *Eigenvector techniques in design and graph theory*, volume 121 of Mathematical Centre Tracts. Mathematisch Centrum, Amsterdam, 1980. Dissertation, Technische Hogeschool Eindhoven, Eindhoven, 1979.

[15] C. Lam and V. D. Tonchev. Classification of affine resolvable 2-(27, 9, 4) designs. *J. Statist. Plann. Inference*, 56(2):187–202, 1996. Special issue on orthogonal arrays and affine designs, Part II.

[16] B. D. McKay and A. Piperno. Practical graph isomorphism, II. *J. Symbolic Comput.*, 60:94–112, 2014.

[17] M. Muzychuk and M. Klin. On graphs with three eigenvalues. *Discrete Math.*, 189(1-3):191–207, 1998.

[18] A. Neumaier. Strongly regular graphs with smallest eigenvalue $-m$. *Arch. Math. (Basel)*, 33(4):392–400, 1979/80.

[19] J. J. Seidel. Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3. *Linear Algebra and Appl.*, 1:281–298, 1968.

[20] S. S. Shrikhande. Affine resolvable balanced incomplete block designs: a survey. *Aequationes Math.*, 14(3):251–269, 1976.

[21] J. H. Smith. Some properties of the spectrum of a graph. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 403–406. Gordon and Breach, New York, 1970.

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China

E-mail address: xmcheng@mail.ustc.edu.cn

Department of Algebra and Topology, N.N. Krasovsky Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia

E-mail address: alexander.gavriliouk@gmail.com

Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

E-mail address: grvgrv@gmail.com
Wen-Tsun Wu Key Laboratory of CAS, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China

E-mail address: koolen@ustc.edu.cn