Form Factors in Affine Toda Field Theories

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Abstract

I briefly review the properties of classical affine Toda field theories and indicate how some of this features survive in the quantum theory on-shell. I demonstrate how this knowledge can be extended off-shell, i.e. how to compute correlation functions for completely integrable models via the form factor approach. For the latter I present an axiomatic system and two explicit computation (the Sinh-Gordon theory and the Bullough-Dodd model) which provide a consistent solution of it.

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1 Introduction

Completely integrable models have attracted a lot of attention and research activity in recent years, since they offer the possibility to obtain exact, i.e. non-perturbative, results in quantum field theory. Affine Toda Field Theories (ATFTs) represent a very important example of this models, since they are explicit Lagrangian versions of integrable deformations of conformal field theories. The breaking of the conformal symmetry simply corresponds to an affinisation of the Lie algebra $\mathfrak{g}$ underlying the conformally invariant theory. Many other features of ATFT can be expressed as well very neatly in terms of Lie algebraic quantities. Before turning to the main subject of my talk, the computation of form factors for this theories, I shall briefly review some of their properties which will be of relevance. I shall start with the classical theory of which many features survive in the quantum version. It is most conveniently described by the Lagrangian density

$$\mathcal{L} = tr \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - e^{\beta \Phi} E e^{-\beta \Phi} E^\dagger \right) , \quad (1.1)$$

where $\Phi$ is an $x, t$ dependent field contained in the Cartan subalgebra of the simple Lie algebra $\mathfrak{g}$ underlying the theory, $E$ is a cyclic element of $\mathfrak{g}$ and the coupling constant $\beta$ is taken to be real. Developing the above Lagrangian to second order in $\Phi$ it has been proven [1] that the masses are proportional to the right Perron-Frobenius vector $y_i$ of the Cartan matrix

$$m_i = \beta y_i \sin \frac{\pi}{h} , \quad (1.2)$$

with $h$ denoting the Coxeter number of $\mathfrak{g}$. Going to the next order one finds that the three point couplings are proportional to the area bounded by the sides of length equal to the masses of the fusing particles, $\Delta_{ijk}$

$$C_{ijk} = \frac{4i \beta \varepsilon(i, j, q)}{\sqrt{h}} \Delta_{ijk} \quad (1.3)$$

where $\varepsilon(i, j, q)$ denotes a structure constant of the algebra. A selection rule which decides whether the coupling constant is non-vanishing can be cast elegantly in the root space of the Cartan matrix

*Denoting by $\Psi^2$ the length of the highest root, $\sqrt{2/\Psi^2} \varepsilon(i, j, q)$ acquires always the value $\pm 1$, unless all three particles correspond to short roots, in which case it becomes $\pm 1/\sqrt{2}$ for $B, C, F_4$ and $\pm 2/\sqrt{3}$ for $G_2$. 
\( g \), i.e. \( C_{ijk} \neq 0 \) if and only if
\[
\sigma^{\xi(i)} \gamma_i + \sigma^{\xi(j)} \gamma_j + \sigma^{\xi(k)} \gamma_k = 0 .
\] (1.4)

Here \( \sigma \) denotes the Coxeter transformation, the triplet of integers \((\xi(i), \xi(j), \xi(k))\) form an equivalence class in which each integer takes values between 0 and \( h \) and \( \gamma_i \) a simple root \( \alpha_i \) associated to particle \( i \) multiplied by the colour value \( c(i) = \pm 1 \), due to the bicolouration of the Dynkin diagram. Going to higher order one finds that all \( n \)-point couplings with \( n \geq 4 \) can be completely determined in terms of the masses and the three-point couplings \([2]\). A generating function for all couplings is given by
\[
C_{l_1 \ldots l_n} = (-1)^{n+1} \sum_{t=1}^{n-2} \sum_{x} \frac{x_1 \ldots x_t}{N_t} m_{l_{n-1}}^2 \delta_{l_{n-1}l_n} ,
\] (1.5)

where the quantities \( x_i \) for each particular \( t \) are expressible in terms of the three point couplings \( x_1, \ldots, x_{2(t+1)-n} = \sum_k \frac{C_{nuk}}{m_k^2} \) or the masses \( x_{2t+3-n}, \ldots, x_t = m_{l_{t}}^2 \delta_{l_{t}l_n} \). Here \( \sum_x \) denotes the sum over all possible permutations of the \( x_i \). The factor \( N_t \) takes care of the overcounting \( g \) when symmetric terms are permuted. Its explicit value is given by
\[
N_t = (2t + 2 - n)!/(n - t - 2)! .
\]

The central object for the on-shell quantum theory, the two-body scattering matrix, can be expressed very compactly for ATFT related to simply laced algebras
\[
S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q - \frac{c(i) + c(j)}{2} \right\}^{\frac{1}{2} \lambda_i - \gamma^q \gamma_j} .
\] (1.6)

Here \( \theta \) denotes the rapidity, \( \lambda_i \) a fundamental weight of the algebra and \( \{ \} \) is a building block consisting out of sinh-functions, i.e. \( \{ x \} = [x]_{\theta} / [x]_{-\theta} \), \( [x]_{\theta} = < x + 1 >_{\theta} < x - 1 >_{\theta} / < x + 1 - B >_{\theta} < x - 1 + B >_{\theta} \) and \( < x >_{\theta} = \sinh \frac{\theta}{2} \left( \theta + \frac{i\pi x}{h} \right) \). \( B(\beta) \) is a function containing the coupling constant dependence which takes values between 0 and 2. The S-matrix possess the remarkable property to be left invariant when mapped as \( B \rightarrow 2 - B \), i.e. the theory contains a duality between the strong and weak coupling region. Formula (1.6) has been found to satisfy all the consistency requirements demanded from a scattering matrix \([3]\) and is furthermore supported by perturbative checks \([4]\). A particular nice example where some classical features survive in the quantum theory is the fusion rule (1.4) which becomes equivalent to the so-called bootstrap equation.

\[
S_{ii}(\theta + i\eta(i)) \ S_{ij}(\theta + i\eta(j)) \ S_{ik}(\theta + i\eta(k)) = 1
\] (1.7)
with \( \eta(t) = -\frac{2}{\pi} \left( 2\zeta(t) + \frac{1-c(t)}{2} \right) \) for \( t = i, j, k \).

Once the on-shell physics is understood, the question towards the off-shell physics arises naturally and it turns out that the knowledge of the S-matrix is useful to compute off-shell quantities like correlation functions of some operator, say \( \mathcal{O} \)

\[
\mathcal{G}_n(r_1, \ldots, r_n) = \langle \mathcal{O}_1(r_1) \ldots \mathcal{O}_n(r_n) \rangle
\]

where \( r \) is denoting the radial distance \( r = \sqrt{x_0^2 + x_1^2} \). In particular the two point function for an hermitian operator \( \mathcal{O} \) in real Euclidean space is given by

\[
\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} |F_n^\mathcal{O}(\theta_1 \ldots \theta_n)|^2 \exp \left( -mr \sum_{i=1}^{n} \cosh \theta_i \right),
\]

where the so-called form factors of an operator \( \mathcal{O} \) have been introduced

\[
F_n^\mathcal{O}(\theta_1, \ldots, \theta_n) = \langle 0 | \mathcal{O}_1 | \theta \rangle | \ldots | \mathcal{O}_n | \theta \rangle | 0 \rangle.
\]

Thus once one has obtained all \( n \)-particle form factors, one is left with the problem to compute the sum and the integrals in (1.9) in order to obtain two-point correlation functions. In comparison with a perturbative expansion in the coupling constant, this approach has the advantage that it contains the coupling constant dependence to all orders. Furthermore, the integrals and the sum are fast convergent. In the rest of my talk I shall be concerned with the explicit computation of form factors \( F_n^\mathcal{O}(\theta_1, \ldots, \theta_n) \). It is based on a sequence of papers jointly written with G. Mussardo and P. Simonetti \([5, 6]\), to whom I’m grateful for collaboration.

## 2 General Properties of Form Factors

I shall now state in a fairly axiomatic fashion some general properties which are expected to hold for form factors. Due to lack of time and space I shall refer for a proper justifications of them in terms of general principles of quantum field theory to various places in the literature \([7-14]\). In order to keep the notation as simple as possible I shall concentrate
on theories involving solely one type of particle, since it will be for this theories where I present a solution of the axiomatic system in the next section.

i) **Commutation of States**

As a consequence of the commutation of two operators in the Zamolodchikov algebra, one obtains

\[ F_n^O(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = F_n^O(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n) S_{i(i+1)}(\theta_{i(i+1)}) \]  

(2.11)

ii) **Analytic Continuation**

The discontinuity at the cuts at \( \theta_{ij} = 2\pi i \) is fixed by

\[ F_n^O(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = F_n^O(\theta_2, \ldots, \theta_n, \theta_1) = \prod_{l=2}^{n} S_{il}(\theta_l - \theta_1) F_n^O(\theta_1, \ldots, \theta_n) \]  

(2.12)

iii) **Relativistic Invariance**

Since we are describing relativistic invariant theories, we expect for an operator \( O \) with spin \( s \)

\[ F_n^O(\theta_1 + \Delta, \ldots, \theta_n + \Delta) = e^{s \Delta} F_n^O(\theta_1, \ldots, \theta_n) \]  

(2.13)

iv) **Kinematic Residue Equation**

Describing theories with equal masses we have a kinematical pole at \( i\pi \) (for different masses we require \([13]\)) , which leads to a recursive equation connecting the \((n + 2)\) and \(n\)-particle form factor

\[ -i \lim_{\bar{\theta} \to \theta} (\bar{\theta} - \theta) F_{n+2}^O(\bar{\theta} + i\pi, \theta, \theta_1, \ldots, \theta_n) = \left( 1 - \prod_{i=1}^{n} S(\theta - \theta_i) \right) F_n^O(\theta_1, \ldots, \theta_n) \]  

(2.14)

v) **Bound State Residue Equation**

A further pole arises as a consequence of a possible bound state, due to the process \( i + j \to k \), in which case we obtain a recursive equation connecting the \((n + 1)\) and the \(n\)-particle form factors

\[ -i \lim_{\bar{\theta} \to \theta} (\bar{\theta} - \theta) F_{n+1}^O(\bar{\theta} + i\eta(i), \theta + i\eta(j), \theta_1, \ldots, \theta_n) = \Gamma_{ij}^k F_n^O(\theta + i\eta(k), \theta_1, \ldots, \theta_n) \]  

(2.15)

where \( \Gamma_{ij}^k \) denotes the three particle vertex on mass-shell.
vi) Cluster Formula

An interesting equation results if one shifts part of the operators forming the physical state to infinity

\[
\lim_{\Delta \to \infty} F_{k+1}^O(\theta_1, \ldots, \theta_k, \theta_{k+1} + \Delta, \ldots, \theta_{k+l} + \Delta) F_0^O = F_k^O(\theta_1, \ldots, \theta_k) F_l^O(\theta_{k+1}, \ldots, \theta_{k+l}).
\]  
(2.16)

vii) Form Factors of the Energy Momentum Tensor

Once one possesses a conservation law involving different kinds of operators one can always utilize it to relate their form factors. The energy momentum tensor is particular good example for this since it is present in every theory. Denoting the trace of the energy momentum tensor by \( \Theta \) and the parts which correspond to the holomorphic and anti-holomorphic in the limit towards the Euclidean version of a conformal field theory by \( T \) and \( \bar{T} \), respectively, we have

\[
\partial_z T(z, \bar{z}) + \partial_{\bar{z}} \Theta(z, \bar{z}) = 0 \quad \text{and} \quad \partial_z \bar{T}(z, \bar{z}) + \partial_{\bar{z}} \Theta(z, \bar{z}) = 0,
\]  
(2.17)

from which we derive

\[
\sigma_1^{(2n)} \sigma_2^{(2n)} F_{2n}^T(\beta_1, \ldots, \beta_{2n}) = \sigma_1^{(2n)} F_{2n}^\Theta(\beta_1, \ldots, \beta_{2n})
\]  
(2.18)

\[
\sigma_2^{(2n)} F_{2n}^\bar{T}(\beta_1, \ldots, \beta_{2n}) = \sigma_1^{(2n)} \sigma_2^{(2n)} F_{2n}^\Theta(\beta_1, \ldots, \beta_{2n}).
\]  
(2.19)

viii) Form Factors of Descendent Operators

Employing the Lorentz group to decompose the space of form factors into

\[
\mathcal{P} = \bigoplus_s \mathcal{P}_s,
\]  
(2.20)

where \( F_n^O \in \mathcal{P}_s \) when \( O \) has spin \( s \), the previous consistency equations provide a criterion to decide whether a particular subspace \( \mathcal{P}_s \) is empty or not. Assuming that we can express an operator \( O' \in \mathcal{P}_s \) in the form \( O' = [OQ_s] \), where \( O \) is spinless, and employing the eigenvalue equation for the conserved charges

\[
Q_s |\beta_1, \ldots, \beta_n\rangle = \sigma^s_n (x^s_1 x_1, \ldots, x^s_n x_n) |\beta_1, \ldots, \beta_n\rangle
\]  
(2.21)

\[\dag\] The function \( \sigma_k^{(n)}(x_1, \ldots, x_n) \) denotes elementary symmetric polynomials which can be generated by

\[
\prod_{i=1}^{n} (x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k^{(n)}(x_1, x_2, \ldots, x_n). \]
we derive the following relations between the two form factors.

\[ F_n^{\mathcal{O}'}(\beta_1, \ldots, \beta_n) = q_n^s(\chi_1^s x_1, \ldots, \chi_n^s x_n) F_n^\mathcal{O}(\beta_1, \ldots, \beta_n). \]  

(2.22)

The kinematic- and bound state residue equation for the form factor \( F_n^{\mathcal{O}'} \) then lead to the two constraints on the eigenvalues for the conserved charge

\[ q_{n+2}^s(-\chi^s x, \chi^s x, \chi_1^s x_1, \ldots, \chi_n^s x_n) = q_n^s(\chi_1^s x_1, \ldots, \chi_n^s x_n) \]  

(2.23)

\[ q_{n+1}^s(\chi_i^s e^{i\bar{i}i k}, \chi_j^s e^{i\bar{j}j k}, \chi_1^s x_1, \ldots, \chi_{n-1}^s x_{n-1}) = q_n^s(\chi_k^s x, \chi_1^s x_1, \ldots, \chi_{n-1}^s x_{n-1}) \]  

(2.24)

Due to the relativistic invariance the eigenvalues will be of the form

\[ q_n^s(\chi_1^s x_1, \ldots, \chi_n^s x_n) = \sum_{i=1}^n \chi_i^s e^{s\beta_i} = s_k^{(n)}(\chi_1^s x_1, \ldots, \chi_n^s x_n). \]  

(2.25)

The polynomial \( s_k^{(n)} \) can always be expressed entirely in terms of the polynomial \( I_{2s-1}^{(n)} = (-1)^{s+1} \det \mathcal{I} (\mathcal{I} \text{ being an } (s \times s)-\text{matrix}), \) whose entries are \( \mathcal{I}_{1j} = \sigma_{2j-1} \) and \( \mathcal{I}_{ij} = \sigma_{2j-2i+2} \) via equation

\[ s_k^{(n)} = \left( I_1^{(n)} \right)^k + k \sum_{\lambda} \prod_{i} I_{\lambda_i}^{(n)}, \]  

(2.26)

where \( \{\lambda\} \) denotes a partition of the integers \( \lambda \) into odd integers, i.e. \( \sum_{i} \lambda_i = \lambda \) for all \( \lambda_i \) odd. This guarantees that equation (2.23) will always be satisfied due to the invariance property \( I_{2s-1}^{(n+2)} = I_{2s-1}^{(n)} \). On the other hand equation (2.24) leads to the non-trivial condition

\[ (\chi_i^s e^{i\bar{i}i k})^s + (\chi_j^s e^{-i\bar{j}j k})^s = (\chi_k^s)^s, \]  

(2.27)

which is precisely the consistency equation derived by Zamolodchikov [15] in a different context. It selects out a particular set of possible spin values for descendent operators.

ix) Minimal Form Factors

In [7] it was proven that a general form factor can always be decomposed into the form

\[ F_n^\mathcal{O}(\theta_1, \ldots, \theta_n) = K_n^\mathcal{O}(\theta_1, \ldots, \theta_n) \prod_{i<j} F_{\min}(\theta_{ij}) \]  

(2.28)

where \( K_n^\mathcal{O}(\theta_1, \ldots, \theta_n) \) is a totally symmetric function in \( \theta_i, 2\pi i \) periodic, containing the entire pole structure and determines the asymptotic behaviour for large values of the rapidities. On the other hand the minimal form factor contains no poles and zeros in
the physical sheet, converges to a constant for \( \theta_i \to \infty \) and satisfies Watson’s equations (2.11) and (2.12) for \( n = 2 \)

\[
F_{\text{min}}(\theta) = F_{\text{min}}(-\theta)S(\theta) \quad \text{and} \quad F_{\text{min}}(i\pi - \theta) = F_{\text{min}}(i\pi + \theta) .
\] (2.29)

3 Form Factors for one scalar field ATFTs

In this section I shall present a general solution for ATFTs involving one scalar field only, i.e. the Sinh-Gordon theory and the Bullough-Dodd model. For the purpose of this talk it will be sufficient to illustrate the general techniques involved.

3.1 The Sinh-Gordon theory

The Lagrangian density for the Sinh-Gordon theory reads

\[
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{g^2} \cosh g\phi(x) .
\] (3.30)

The theory posses a \( \mathbb{Z}_2 \)-symmetry, that is, it remains invariant under the map \( \phi \to -\phi \).
Its conservation laws are graded by the spin values \( 1, 3, 5, 7, \ldots \) and its S-matrix is given by \( S(\theta) = \{1\}_\theta \), which does not posses a pole in the physical sheet such that a fusing process is not possible. With the scattering matrix an input we can solve (2.29) by the usual technique of using Fourier transform after taking the logarithm of this equation and obtain

\[
F_{\text{min}}(\theta, B) = \prod_{k=0}^{\infty} \left| \frac{\Gamma \left( k + \frac{3}{2} + i\hat{\theta} \frac{2\pi}{B} \right) \Gamma \left( k + \frac{1}{2} + B \frac{4}{2\pi} + i\hat{\theta} \frac{2\pi}{B} \right) \Gamma \left( k + 1 - B \frac{4}{2\pi} + i\hat{\theta} \frac{2\pi}{B} \right) \Gamma \left( k + 1 + B \frac{4}{2\pi} + i\hat{\theta} \frac{2\pi}{B} \right)}{\Gamma \left( k + \frac{1}{2} + i\hat{\theta} \frac{2\pi}{B} \right) \Gamma \left( k + \frac{3}{2} - B \frac{4}{2\pi} + i\hat{\theta} \frac{2\pi}{B} \right) \Gamma \left( k + 1 - B \frac{4}{2\pi} + i\hat{\theta} \frac{2\pi}{B} \right) \Gamma \left( k + 1 + B \frac{4}{2\pi} + i\hat{\theta} \frac{2\pi}{B} \right)} \right|^2,
\] (3.31)

\( \hat{\theta} = i\pi - \theta \) which satisfies the functional equation

\[
F_{\text{min}}(\theta + i\pi, B)F_{\text{min}}(\theta, B) = [1]_\theta .
\] (3.32)

This identity is required in the process of solving the recursive equation resulting from (2.14). It turns out to be useful to parameterize the the non-minimal part of the form factor further and extract the polestructure from it

\[
K_n(\theta_1, \ldots, \theta_n) = \frac{Q_n(\theta_1, \ldots, \theta_n)}{\prod_{i<j} x_i + x_j} ,
\] (3.33)
where \( x_i = e^{\theta_i} \) is introduced. Furthermore, we obtain that the function

\[
Q_n(x_1, \ldots, x_n)
\]

can always be written as

\[
Q_n(x_1, \ldots, x_n) = \begin{cases} 
\sigma_n^{(n)} P_n(x_1, \ldots, x_n) & \text{for } \phi \\
\sigma_1^{(n)} \sigma_{n-1}^{(n)} P_n(x_1, \ldots, x_n) & \text{for } \Theta 
\end{cases}
\tag{3.34}
\]

then the functions \( P_n(x_1, \ldots, x_n) \) obey the recursive equation

\[
(-1)^{n+1} P_{n+2}(-x, x, x_1, \ldots, x_n) = x D_n(x, x_1, \ldots, x_n) P_n(x_1, \ldots, x_n)
\tag{3.35}
\]

with

\[
D_n = \frac{1}{2 \sin(\pi B/2)} \sum_{l,k=0}^{n} (-1)^l \sin \left( (k - l) \frac{\pi B}{2} \right) x^{2n-l-k} \sigma_l^{(n)} \sigma_k^{(n)}
\tag{3.36}
\]

as a result of (2.14). Due to the \( Z_2 \)-symmetry we have \( P_{2n}^\phi = P_{2n+1}^\Theta = 0 \). Fixing the initial condition of the recursive equation to be \( F_1^\phi(\theta_1) = \frac{1}{\sqrt{2}} \) and \( F_2^\Theta = 2\pi m^2 \) we can solve this equation iteratively

\[
\begin{align*}
P_3(x_1, \ldots, x_3) &= 1 \\
P_4(x_1, \ldots, x_4) &= \sigma_2 \\
P_5(x_1, \ldots, x_5) &= \sigma_2 \sigma_3 - 4 \cos^2 \left( \frac{\pi B}{2} \right) \sigma_5 \\
P_6(x_1, \ldots, x_6) &= \sigma_2 \sigma_3 \sigma_4 + \left( 4 \cos^2 \left( \frac{\pi B}{2} \right) - 1 \right) \sigma_3 \sigma_6 - 4 \cos^2 \left( \frac{\pi B}{2} \right) (\sigma_4 \sigma_5 + \sigma_1 \sigma_2 \sigma_6).
\end{align*}
\tag{3.37}
\]

Obviously we can carry on and compute iteratively \( P_n \) for higher values of \( n \). However, if we are solely concerned with the computation of correlation function, the first few terms are usually sufficient to achieve high precision. In [5] we have verified this with the particular application to the correlation function of the energy-momentum tensor, which is expressed via the c-theorem. None-the-less for particular values of the coupling constant we can prove that the \( P_n \) take on relatively simple forms. For instance for the self-dual point, i.e. \( B(\sqrt{8\pi}) = 1 \) we obtain

\[
P_n(x_1, \ldots, x_n) = \det A(n)(x_1, \ldots, x_n)
\tag{3.38}
\]

where \( A(n)(x_1, \ldots, x_n) \) is a \((n-3) \times (n-3)\)-matrix whose entries are given by

\[
A_{ij}^{(n)} = \sigma_{2j-i+1}^{(n)} \cos^2 \left( (i - j) \frac{\pi}{2} \right)
\tag{3.39}
\]

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Whereas for the inverse Yang-Lee point, i.e. \( B(2\sqrt{\pi}) = \frac{2}{3} \), we obtain

\[ P_n(x_1, \ldots, x_n) = \det B^{(n)}(x_1, \ldots, x_n) \]  

(3.40)

where the entries of the matrix are now given by

\[ B_{ij}^{(n)} = \sigma_{3j-2i+1}^{(n)} \]  

(3.41)

It is interesting to note that the recursive equation only possesses two solutions, suggesting that \( \phi \) and \( \Theta \) are the fundamental operator content of the theory. As a non-trivial check of our solution it is instructive to verify that it satisfies the cluster equation (2.16). Hence we have found a solution for all the axioms presented in the previous section.

### 3.2 The Bullough-Dodd model

I shall now present a further example, which involves the additional complication of the involvement of the bound state residue equation, which was absent in the Sinh-Gordon theory due to the lack of fusing process. The Bullough-Dodd Lagrangian density reads

\[ \mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \varphi \right)^2 - \frac{m_0^2}{6g^2} \left( 2e^{g\varphi} + e^{-2g\varphi} \right) . \]  

(3.42)

The model possesses conserved quantities graded by the spins \( 1, 5, 7, 11, 13, \ldots \) and its \( S \)-matrix is given by \( S(\theta, B) = \{1\}_{\theta} \{2\}_{\theta} \quad [10] \), which possess a pole a \( i\pi/3 \) representing the fusing process \( A + A \to A \). In the same fashion as in the previous subsection we solve (2.29) and find

\[ F_{\min}(\theta, B) = \prod_{k=0}^{\infty} \frac{\Gamma \left( k + 3 + \frac{3\theta}{2\pi} \right) \Gamma \left( k + 7 + \frac{3\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{3\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{5\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{7\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{9\theta}{2\pi} \right)}{\Gamma \left( k + \frac{5}{6} + \frac{B + i\theta}{2\pi} \right) \Gamma \left( k + \frac{7}{6} + \frac{B + i\theta}{2\pi} \right) \Gamma \left( k + \frac{5}{6} + \frac{B - i\theta}{2\pi} \right) \Gamma \left( k + \frac{7}{6} + \frac{B - i\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{B}{6} + \frac{i\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{B}{6} - \frac{i\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{B}{6} + \frac{B - i\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{B}{6} - \frac{B - i\theta}{2\pi} \right)} \]  

(3.43)

which now satisfies the functional equation

\[ F_{\min}(\theta + i\pi, B)F_{\min}(\theta, B) = [1]_{\theta}[2]_{\theta} \]  

(3.44)

\[ F_{\min}(\theta + i\pi/3, B)F_{\min}(\theta - i\pi/3, B) = [0]_{\theta}F_{\min}(\theta, B) \]  

(3.45)
Extracting again the polestructure from the minimal part of $F_n$, we obtain

$$F_n(\beta_1, \ldots, \beta_n) = Q_n(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\text{min}}(\beta_{ij})}{(x_i + x_j)(\omega x_i + x_j)(\omega^{-1} x_i + x_j)}. \quad (3.46)$$

( $\omega = e^{i\pi/3}$) which we further factorize as in (3.34). Finally we derive the recursive equation resulting from the kinematic residue equation

$$(-1)^n Q_{n+2}(-x, x, x_1, \ldots, x_n) = \frac{1}{F_{\text{min}}(i\pi)} x^3 U(x, x_1, \ldots, x_n) Q_n(x_1, x_2, \ldots, x_n) \quad (3.47)$$

where

$$U(x, x_1, \ldots, x_n) = 2 \sum_{k_1, \ldots, k_6 = 0}^n (-1)^{k_2 + k_3 + k_5} x^{6n-(k_1 + \cdots + k_6)} \sigma_{k_1}^{(n)} \sigma_{k_2}^{(n)} \cdots \sigma_{k_6}^{(n)} \times \sin \left[ \frac{\pi}{3} \left( 2(k_2 + k_4 - k_1 - k_3) + B(k_3 + k_6 - k_4 - k_5) \right) \right] \quad (3.48)$$

and the bound state residue equation

$$Q_{n+2}(\omega x, \omega^{-1} x, x_1, \ldots, x_n) = -\frac{\sqrt{3}}{F_{\text{min}}(i\pi)} \Gamma(g) x^3 D(x, x_1, \ldots, x_n) Q_{n+1}(x, x_1, \ldots, x_n) \quad (3.49)$$

where

$$D(x, x_1, \ldots, x_n) = \sum_{k_1, k_2, k_3 = 0}^n x^{3n-(k_1 + k_2 + k_3)} \omega^{(2+B)(k_2 - k_3)} \sigma_{k_1}^{(n)} \sigma_{k_2}^{(n)} \sigma_{k_3}^{(n)} \quad (3.50)$$

The three-particle vertex on mass-shell acquires the form

$$\Gamma^2(B) = 2\sqrt{3} \tan \left( \frac{\pi B}{6} \right) \tan \left( \frac{\pi}{3} - \frac{\pi B}{6} \right) \tan \left( \frac{\pi}{3} + \frac{\pi B}{6} \right) \quad (3.51)$$

Apart from vanishing in the free theory for $B = 0, 2$, this function becomes zero as well at the self-dual point for $B = 1$. This is a peculiar feature not known in ATFTs related to simply laced algebras. At this particular point the S-matrix coincides with the one of the Sinh-Gordon theory at the inverse Yang-Lee point. In [6] we showed that this feature extends off-shell, that is

$$F_{n}^{BD}(B = 1, x_1, \ldots, x_n) = F_{n}^{SG} \left( B = \frac{2}{3}, x_1, \ldots, x_n \right) \quad (3.52)$$

for both the elementary field and the energy-momentum tensor. Thus the $\mathbb{Z}_2$-symmetry is realised dynamically in the Bullough-Dodd model. Furthermore, we showed that equations (3.47) and (3.49) indeed posses consistent solutions in accordance with all consistency requirements.
4 Conclusions

We conclude that the form factor approach indeed provides a successful method to compute correlation functions for massive integrable models. For particular values of the coupling constant it has been shown that one can obtain closed formulas for $F_n$. It is desirable to extend this to arbitrary values of $\beta$ and obtain closed expressions for those values too. Despite the fact that high numerical precision can be achieved relatively easy in (1.9), by computing only the first few terms, it is desirable to obtain a closed analytic solution of it. Alternatively it would be illuminating to find out whether it is possible to obtain correlation functions via differential equations satisfied by the form factors.

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