Novel Multi-Channel Fractional-Order Radial Harmonic Fourier Moments for Color Image Analysis

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ABSTRACT The classical radial harmonic Fourier moments (RHFMs) and the quaternion radial harmonic Fourier moments (QRHFMs) are gray-scale and color image descriptors. The radial harmonic functions with integer orders are not able to extract fine features from the input images. In this paper, the authors derived novel fractional-order radial harmonic functions in polar coordinates. The obtained functions are used to define novel multi-channel fractional-order radial harmonic moments (FrMRHFMs) for color image description and analysis. The invariants to geometric transformations for these new moments are derived. A theoretical comparison between FrMRHFMs and QRHFMs is performed from the aspects of kernel function and the spectrum analysis. Numerical simulation is carried out to test these new moments in terms of image reconstruction capabilities, invariance to the similarity transformations, color image recognition and the CPU computational times. The obtained theoretical and numerical results clearly show that the proposed FrMRHFMs is superior to the QRHFMs and the existing fractional-order orthogonal moments.

INDEX TERMS Color image analysis, fractional-order orthogonal moments, radial harmonic Fourier moments, rotation invariance.

I. INTRODUCTION
The orthogonal moments of gray-scale images were defined in the early 1980s. These moments have three useful characteristics: First, its ability to represent digital images with a minimum redundancy. Second, these moments are robust against different kinds of noise. Third, the values of these moments are almost unchanged with geometric transformations such as rotation, scaling and translation [1]–[4].

Ren et al. [5] derived the RHFMs using triangular functions as radial kernels, which possess the property of being invariant to rotation and can be made invariant to translation and scaling by a set of geometric transformations. Wang et al. [6] utilized the algebra of quaternions and introduced the QRHFMs for color image representation and retrieval. They pointed out that QRHFMs show better characteristics when compared with other orthogonal moments. Due to these characteristics, RHFMs and QRHFMs were utilized with various applications such as: image reconstruction [7], color image retrieval [8], digital watermarking of gray-scale and color images [9], [10] and copy-move forgery detection [11].

However, the RHFMs and QRHFMs and their Rotation, Scaling and Translation (RST) invariants are restricted to being an integer order, where the order of the basis orthogonal polynomials are integers. Mathematicians integrated the principles of the fractional calculus for accurate solution of the fractional differential equations [12]. Inspired by the mathematicians’ success, few researchers derived fractional-order type of few orthogonal moments for image analysis. Xiao et al. [13] derived fractional-order Legendre-Fourier moments. Zhang et al. [14] defined the fractional-order Fourier-Mellin polynomials then derived the FrOFMMs. Benouini et al. [15] presented the orthogonal fractional-order Chebyshev moments (FrOCMs). Benouini et al. [16]
derived the Legendre polynomials of fractional order and defined what is called generalized Legendre moment invariants. These fractional-order moments are devoted to gray-scale images. On the other side, Chen et al. [17] introduced the orthogonal fractional-order quaternion Zernike moments (QFrZMs) for color image processing and utilized them in detection of copy-move forgery in color images. Kaur et al. [18] used the fractional-order Zernike moments with the Support Vector Machine (SVM) classifier in plant disease recognition. Chen et al. [19] used the fractional quaternion cosine transform in image copy-move forgery detection.

Recently, the multi-channel orthogonal moments [20], [21] have been defined as an alternative to the quaternion orthogonal moments for better description of color images. For the same computational framework, multi-channel orthogonal moments are outperforming the corresponding quaternion orthogonal moments.

The attractive characteristics of QRHFMs motivated the authors to generalize the radial harmonic Fourier functions to be defined with a fractional-order, and then, utilizing the successful multi-channel approach to derived a novel set of multi-channel fractional-order radial harmonic Fourier moments (FrMRHFMs) for color image analysis.

In this computational framework, the analytical integration is used in exact computation of the angular-kernel of the FrMRHFMs by integrating the angular function $e^{i\hat{q}\theta}$ over the circular image pixels while the radial-kernel $T_p(r, \alpha)$ could not be evaluated using the same analytical integration. Accurate numerical integration over the same circular image pixels is a proper choice. Moreover, the RST invariants of the novel FrMRHFMs are proved. Experiments have been performed to validate the capability of the proposed FrMRHFMs to reconstruct color images, their RST invariances and computational time efficiency. The results clearly show that proposed FrMRHFMs outperformed the QFrZMs [17] and the conventional QRHFMs [6].

The contributions of this paper can be summarized as follows:
- Novel orthogonal fractional-order radial harmonic functions are defined.
- A new set of fractional-order multi-channel radial harmonic-Fourier moments (FrMRHFMs) for color images analysis is derived.
- The invariance to similarity transformations are derived for FrMRHFMs.
- A theoretical proof for the superiority of FrMRHFMs over the existing QRHFMs and QFrZMs is presented.

The remaining of this paper is organized as follows: The new set of fractional-order radial harmonic Fourier functions, the FrMRHFMs and their RST invariants for RGB color images are explained in Section II. Section III presented the detailed description of the computational framework. Kernel function and spectrum analysis are studied in Section IV. The performed experiments and the discussion are described in Section V. Conclusion and future work are introduced in Section VI.

II. THE PROPOSED FMRHFMS

In this section, we define new fractional-order radial harmonic Fourier functions and derive novel FrMRHFMs for color image analysis. Then, the RST invariants of the FrMRHFMs are proved.

A. FRMRHFMS FOR RGB COLOR IMAGES

Based on the multi-channel approach [20], [21], the RGB representation of the color image, $g_c(r, \theta)$ is represented as:

$$g_c(r, \theta) = (g_R(r, \theta), g_G(r, \theta), g_B(r, \theta))$$

(1)

The three-color channels are $g_R(r, \theta)$, $g_G(r, \theta)$ and $g_B(r, \theta)$ where $c \in \{R, G, B\}$. The FrMRHFMs for the RGB color image $g_c(r, \theta)$ is:

$$\text{FrM}_{pq}(g_c) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c(r, \theta) \left[ W_{pq}(r, \theta) \right]^* r dr d\theta$$

(2)

where $W_{pq}(r, \theta)$ are the basis functions; $p \geq 0$ is the moment order while $|q| \geq 0$ refers to the repetition. The basis functions are:

$$W_{pq}(r, \theta) = T_p(r, \alpha) e^{-i\hat{q}\theta}$$

(3)

where $\hat{\imath} = \sqrt{-1}$, the fractional parameter, $\alpha \in \mathbb{R}^+$ and $T_p(r, \alpha)$ is the radial basis function (RBF):

$$T_p(r, \alpha) = \begin{cases} \sqrt{\rho} r^{\alpha-1} \frac{1}{\rho^2}, & p = 0 \\ \sqrt{\rho} r^{\alpha-1} \frac{2}{\rho^2} \cos (\pi p r^\alpha) , & p \text{ is even} \\ \sqrt{\rho} r^{\alpha-1} \frac{2}{\rho^2} \sin (\pi (p + 1) r^\alpha) , & p \text{ is odd} \end{cases}$$

(4)

The functions $W_{pq}(r, \theta)$ are orthogonal over the interval $0 \leq r \leq 1$ where:

$$\int_0^{2\pi} \int_0^1 W_{pm}(r, \theta) \left[ W_{qm}(r, \theta) \right]^* r dr d\theta = 2\pi \delta_{pm} \delta_{qn}$$

(5)

where $\delta_{pm}$ is the Kronecker function; $[\cdot]^*$ is the complex conjugate and $2\pi$ is the normalization factor. Detailed description of the derivation of (4) and proof of (5) are available in
the appendices A and B. We must note that, (1) is used to compute three multi-channel fractional-order moments, namely, FrM_{pq}(g_{R}), FrM_{pq}(g_{G}), and FrM_{pq}(g_{B}). Based on the orthogonal property of the fractional-order radial harmonic-Fourier functions, W_{pq}(r, \theta), the original input image function g_c(r, \theta) is reconstructed as follows:

\[ g_c^{\text{recons.}}(r, \theta) = \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \text{MFrM}_{pq}(g_c) W_{pq}(r, \theta) \]

Since summation to infinity is impossible in computing environments, a finite summation using pm\text{ax} and qm\text{ax} is used where pm\text{ax} refers to a user predefined maximum order of the computed fractional-order radial harmonic Fourier moments (FrMRHFMs), therefore, (6) is rewritten as follows:

\[ g_c^{\text{recons.}}(r, \theta) \approx \sum_{p=0}^{\text{pm\text{ax}}} \sum_{q=-\text{qm\text{ax}}}^{\text{qm\text{ax}}} \text{MFrM}_{pq}(g_c) W_{pq}(r, \theta) \]

B. GEOMETRIC INVARIANCE OF FRMRHFMS

1) INVARIANCE TO ROTATION

To test the invariance with respect to rotation, the original image function, g_c(r, \theta) is rotated with an angle \beta, hence:

\[ g_c^\beta(r, \theta) = g_c(r, \theta - \beta) \] (8)

According to (1), the FrMRHFMs of the rotated image, g_c^\beta(r, \theta), is:

\[ \text{FrM}_{pq}(g_c^\beta) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c^\beta(r, \theta)^* r dr d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c(r, \theta - \beta) T_p(\alpha, r) e^{-iq\beta} r dr d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c(r, \theta) T_p(\alpha, r) e^{-iq(\theta + \beta)} r dr d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c(r, \theta) T_p(\alpha, r) e^{-iq\beta} e^{-iq\beta} r dr d\theta \]

\[ = \text{FrM}_{pq}(g_c) e^{-iq\beta} \] (9)

Simply, we could write:

\[ \text{FrM}_{pq}(g_c^\beta) = e^{-iq\beta} \text{FrM}_{pq}(g_c), \quad c \in \{R, G, B\} \] (10)

where FrM_{pq}(g_c) are the fractional-order moments, FrMRHFMs, of g_c(r, \theta) whereas FrM_{pq}(g_c^\beta) is the corresponding of the fractional-order moments, FrMRHFMs, of g_c^\beta(r, \theta).

Since |e^{-iq\beta}| = 1 for any value of q and \beta, therefore:

\[ |\text{FrM}_{pq}(g_c^\beta)| = |e^{-iq\beta} \text{FrM}_{pq}(g_c)| \] (11)

Equation (11) clearly proves that the magnitude values of FrMRHFMs are unchanged with rotation.

2) INVARIANCE TO SCALING

Wang et al. [22] showed that circular orthogonal moments are invariant to scaling when these moments are computed over the same unit disk. In the case of FrMRHFMs, these moments are circular orthogonal moments which are defined in polar coordinates over a unit disk, and computed using the circular image pixels as displayed in Fig. 1. Therefore, the proposed FrMRHFMs are scaling invariant.

3) INVARIANCE TO TRANSLATION

According to [23], the invariance to translation is achieved when the coordinates’ origin shifted to coincide with the centroid (x_c, y_c) of the color image which is defined as:

\[ x_c = ((m_{10}(g_{R}) + m_{10}(g_{G}) + m_{10}(g_{B}))/m_{00}, \quad y_c = ((m_{10}(g_{R}) + m_{10}(g_{G}) + m_{10}(g_{B}))/m_{00}) \]

\[ = m_{00}(g_{R}) + m_{00}(g_{G}) + m_{00}(g_{B}) \] (12)

where m_{00}(g_{c}), m_{01}(g_{c}) and m_{10}(g_{c}), are the geometric moments of the RGB color channels. The central FrMRHFMs are:

\[ \text{FrM}_{pq} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c(\bar{r}, \bar{\theta}) T_p(\bar{\alpha}, \bar{r}) e^{-iq\bar{\theta}} \bar{r} d\bar{r} d\bar{\theta} \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g_c(\bar{r}, \bar{\theta}) T_p(\bar{\alpha}, \bar{r}) e^{-iq\bar{\theta}} \bar{r} d\bar{r} d\bar{\theta} \] (13)

where the coordinate origin at coincides with (x_c, y_c).

III. COMPUTATIONAL FRAMEWORK

The FrMRHFMs should be computed in polar coordinates using the circular image pixels. The input image function which is represented using square pixels is converted into circular pixels using the cubic interpolation [24]. Detailed description of the mapping process is available in [25], [26]. The FrMRHFMs for color images as defined by (2) are computed as follows:

\[ \text{FrM}_{pq} = \frac{1}{2\pi} \sum_{i,j} K_{pq}(i, \theta_{ij}) \hat{g}_c(i, \theta_{ij}) \] (14)

with:

\[ K_{pq}(i, \theta_{ij}) = I_p(r_i)J_q(\theta_{ij}) \] (15)
where:
\[
I_q (\theta_{i,j}) = \int_{V_{i,j}} e^{-iq\theta} d\theta
\]
(16)
\[
I_p (r_i) = \int_{U_i} T_p(\alpha, r) rdr = \int_{U_i} R(r)dr
\]
(17)
with:
\[
R(r) = T_p(\alpha, r)r
\]
(18)

The limits of the definite integrals are:
\[
V_{i,j+1} = \theta_{i,j} + \Delta \theta_{i,j}; \quad V_{i,j} = \theta_{i,j} - \Delta \theta_{i,j}/2
\]
(19)
\[
U_{i+1} = R_i + \Delta R_i/2; \quad U_i = R_i - \Delta R_i/2
\]
(20)
The analytical-integration is applied for exact computation of the angular kernel, \(J_q(\theta_{i,j})\):
\[
J_q (\theta_{i,j}) = \begin{cases} 
\frac{1}{q} \left( e^{-iq\theta_{i,j+1}} - e^{-iq\theta_{i,j}} \right), & q \neq 0 \\
V_{i,j+1} - V_{i,j}, & q = 0 
\end{cases}
\]
(21)
The successful utilization of Gaussian numerical integration method [27] in ref. [28], [29] encouraged us to utilize the same method to compute the radial kernel \(I_p(r_i)\):
\[
\int_a^b h(z)dz \approx \frac{(b-a)}{2} \sum_{i=0}^{c-1} w_i h \left( \frac{a+b}{2} - \frac{a-b}{2} t_i \right)
\]
(22)
where \(l = 0, 1, 2, \ldots c - 1\); \(w_i\) and \(t_i\) are weights and locations of the sampling points; the values of \(w_i\) are \(\sum_{i=0}^{c-1} w_i = 2\). The values of \(t_i\) are depending on \(a\) and \(b\); \(c\) is an integer refers to the order of the numerical integration.

Utilizing (22) into (17) yields:
\[
I_p (r_i) = \int_{U_i}^{U_{i+1}} R(r)dr
\]
\[
\approx \frac{(U_{i+1} - U_i)}{2} \sum_{i=0}^{c-1} w_i R \left( \frac{U_{i+1} + U_i}{2} + \frac{U_{i+1} - U_i}{2} t_i \right)
\]
(23)

IV. KERNEL FUNCTION AND SPECTRUM ANALYSIS

Circular orthogonal moments are defined by multiplying two kernel functions. The first one is the radial kernel function while the second is the angular Fourier kernel function, \(e^{-iq\theta}\). Fortunately, the angular Fourier kernel function is common in all kind of circular orthogonal moments. Therefore, the radial kernel function is corner stone and the key success in any successful circular orthogonal moments. In this section, we perform a theoretical analysis of the fractional-order radial function of the proposed FrMRHFMs as defined in equation (4) and compare it with the integer-order radial function of the proposed QRFHMs [6]. This analysis is performed through two steps. The first step discusses how to choose the best value of the fractional parameter \(\alpha\). The second step analyzes the behavior of the radial kernel functions. Theoretically, any positive real value could be assigned to \(\alpha\), \(\alpha\). The authors of the existing methods [13], [14] clearly show that the most suitable values of \(\alpha\) is \(0 < \alpha < 2\). Since the radial kernel function of the FrMRHFMs is dependent on the value of the fractional parameter \(\alpha\), we select four different values (0.7, 1.1, 1.5 and 1.9) with different orders \((p = 0, 1, 2, 3, 4,\) and 5). The fractional-order radial kernel is plotted against the values of \(r\) \((0 \leq r \leq 1)\) as displayed in Figs 2(a) to 2(d). The integer-order radial kernel function of the QRFHMs and the fractional order radial kernel function of the QFrZMs are plotted with the same orders \((p = 0, 1, 2, 3, 4, 5)\) in Fig 2(e) and 2(f), respectively. For more clarity, only one order, \(p = 5\), is considered where the fractional-order radial kernel is re-plotted for \((p = 0.7, 1.1, 1.5\) and 1.9). The plotted curves are displayed in Fig. 3(a) to 3(d). The integer-order radial kernel function of the QRFHMs and The integer-order radial kernel function of the QFrZMs with the same order \(p = 5\) are plotted in Fig. 3(e) and 3(f). The plotted curves clearly show that the value, \(\alpha = 1.9\), results is the best choice where the radial kernel function oscillates at lower frequency and uniformly distributed over the unit disk interval \((0 \leq r \leq 1)\). Generally, the new radial kernel function with fractional-order is much smoother, oscillate at lower frequency and uniformly distributed over the unit disk interval \((0 \leq r \leq 1)\) than the corresponding radial kernel with integer-order and the fractional order of QRFHMs and the fractional order of QFrZMs. These characteristics significantly increase the recognition capabilities of the proposed fractional-order color image descriptors.

V. NUMERICAL SIMULATION

This Section presents a description of the performed experiments and a discussion of the obtained results. Three groups of experiments have been performed to evaluate the performance of the proposed FrMRHFMs in terms of color image reconstruction, RST invariance and computational times. The performance of the proposed method compared with related quaternion moments such as QRFHMs [6] and the recent fractional-order quaternion moments QFrZMs [17]. These experiments have been performed using MATLAB environment.

A. COLOR IMAGE RECONSTRUCTION CAPABILITY

The accuracy and the numerical stability of the proposed fractional-order moments have been assessed through reconstructing different color images. The FrMRHFMs were computed and employed in reconstructing a set of standard color images. The value of the normalized image reconstruction error (NIRE) [29] reflects the quantitative accuracy of the proposed FrMRHFMs. The qualitative accuracy could be determined using the visual inspection of the human eye. The NIRE is defined as:
\[
\text{NIRE} = \frac{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (g_c(i,j) - g_{\text{Recons}}(i,j))^2}{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_c(i,j)^2}
\]
(24)
According to (24), the proposed FrMRHFs are said to be highly accurate and numerically stable when NIRE values are continuously decreasing and approaching zero. In this experiment, the color image, boat, with dimension of $128 \times 128$ was selected from the standard color images shown in Fig. 4.

For the comparison task, the proposed FrMRHFs method is applied against QRHFs [6] and QFrZMs [17] with orders up to 150 and $\alpha = 1.9$ to reconstruct the color image of the boat of size $128 \times 128$. Fig. 5 shows the reconstructed images while Fig. 6 shows the plotted curves of NIRE.
According to Fig. 5, the reconstructed color images using QFrZMs [17] are completely damaged. Whereas, it is partially damaged when reconstructed using QRHFMs [6]. The obtained results ensure that these methods are unsuitable for accurate color image reconstruction. On the other hand, our proposed FrMRHFMs method was able to reconstruct color images with high accuracy rate.

Figure 6 shows that the NIRE values of QFrZMs [17] and QRHFMs [6] are monotonically increased as the moment order increased. On the other side, the proposed method continuously achieves low NIRE values as the moment’s orders increased which ensures the accuracy and stability of the proposed FrMRHFMs method.

For fair comparison, the adopted high-precision computational method is used to FrMRHFMs, QRHFMs, and QFrZMs. Each group of these moments are used to reconstruct a standard color image. The obtained results as displayed in Fig. 7. Despite the significant improvement in the
accuracy of the QRHFMs, the reconstructed color image using the proposed FrMRHFMs is better than the reconstructed color images using QRHFMs or QFrZMs.

**B. INVARIANCES TO SIMILARITY TRANSFORMATIONS**

In this section, three experiments were performed to test the invariance to rotation, scaling and translation. The accuracy of the RST invariance is evaluated using Mean Square Error (MSE):

$$\text{MSE} = \frac{1}{L_{\text{Total}}} \sum_{p=0}^{\max} \sum_{q=0}^{\max} \left( |\text{FrM}_{pq}(g_c)| - \left| \text{FrM}_{pq}(g_{c,\text{Trans}}) \right| \right)^2$$  \hspace{1cm} (25)

where $|\text{FrM}_{pq}(g_c)|$ and $|\text{FrM}_{pq}(g_{c,\text{Trans}})|$ are the magnitudes values of the fractional-order radial harmonic-Fourier moments for original and transformed images respectively; $L_{\text{Total}}$ is total number of both moments. A unified maximum order 20 is used in the performed experiments for RST invariances. The Columbia Object Image Library (COIL-100) [30] as displayed in Fig. 8 is used in the performed experiments. The original color image, $obj_{16,355}$ of size $128 \times 128$ is rotated in the counter-clock wise direction with rotation angles from $0^{\circ}$ to $90^{\circ}$. Fig. 9 shows the rotated color images. The proposed fractional order moment (FrMRHFMs) and the existing QRHFMs [6] and QFrZMs [17] are computed for the un-rotated and rotated color images. The evaluated MSEs are plotted in Fig. 10. The proposed FrMRHFMs show accurate rotation invariance than the existing orthogonal quaternion moments, QRHFMs [6] and fractional-order moments, QFrZMs [17]. The color images of the object, $obj_{81,0}$ of size $128 \times 128$ are reduced with the reduction scaling factors, 0.1, 0.25, 0.5, and 0.75, and magnified with the magnification scaling factors, 1.25, 1.5, 1.75, and 2.0. Fig. 11 show the reduced and magnified color images. The proposed FrMRHFMs, QRHFMs [6] and QFrZMs [17] are computed for original and scaled color images. Fig. 12.a and 12.b show the plotted MSE for reduced and magnified color images. The $obj_{27,0}$ is shifted using different parameters.
in both directions. The coefficients of the translated color images were computed using the proposed FrMRHFs, QRHFs [6] and QFrZMs [17]. Fig. 13 shows the plotted MSE values. The obtained results prove the invariance of the proposed FrMRHFs with respect to rotation, scaling and translation and ensure the outperformance of FrMRHFs over the QRHFs [6] and QFrZMs [17] in terms of RST invariances.

### C. COLOR IMAGE RECOGNITION

The recognition ability of the proposed FrMRHFs moments for color images was tested using the common dataset of color images, soccer team [31]. This dataset contains 280 images from 7 soccer teams taken from the web, containing 40 images per class of different size, all images are resized to unified size $512 \times 512$. Sample images from soccer team dataset are displayed in Fig. 14. The ability of

![Rotated images of the obj16_35.](image)

![Plotted MSE for rotated obj52_0.](image)

![Scaled and original color images of the obj81_0.](image)

the proposed FrMRHFs moments to recognize the similar color images is quantitatively measured by the recognition rate (RT%) which is defined as in [21]:

$$RT(\%) = \frac{(Y \times 100)}{Q_r}$$  \hspace{1cm} (26)

where $Y$ is the total number of correctly recognized images and $Q_r$ is the number of query images which are used for testing. The recognition rates, (RT%) are evaluated using the different distances similarity measures, $L_1$-norm, $L_2$-norm, square-chord, $\chi^2$ and Canberra [21]. The same conditions as described in [21], are used in the performed experiments considering the normal, rotation and scaling cases. In the experimental work, the features of the two images
(query & test) are extracted by the proposed FrMRHFM method and the existing moments, QRHFM [6] and QFrZM [17]. The recognition rate (RT%) of the proposed FrMRHFM is computed for normal, rotated, and the scaled test color images. It is compared with the (RT%) of QRHFM [6] and QFrZM [17]. We have selected maximum moment order where the length of the feature vectors is the same. The maximum moment order, $Z_{max} = 13$ is used with QFrZM [17] where 56 features are obtained. The maximum order $Q_{max} = 8$ is used with the QRHFM [6] that produced 81 features and $C_{max} = 4$. In case of the proposed FrMRHFM, 25 moments is used for each channel to obtain a total number of 75 features. The results for the normal, rotated, and scaled color query images of the Soccer team dataset are shown in the Tables (1)-(3), respectively. The achieved recognition rates are consisting with the theoretical analysis of the real-value radial functions where the proposed FrMRHFM has better ability to recognize the color images due to their oscillation at low frequency, smooth, and uniformly distribution.

Additional experiments are performed to evaluate the recognition rates of noisy images with different noise levels. Finally, an experiment is performed to evaluate the average recognition rates. These rates are show in Table (4). It is clear that the mean recognition rates of the proposed FrMRHFM are much better than its value in case of using QRHFM [6], and QFrZM [17] methods.

### D. COMPUTATIONAL COMPLEXITY

In this subsection, a theoretical proof of the efficiency and superiority of the FrMRHFM over the existing QRHFM [6] and QFrZM [17] is presented. Computational complexity analysis of these three sets of moments is presented. The total number of computational operations required by each method is evaluated. The proposed FrMRHFM depends on the moments’ order, $p_{max}$, and the repetition, $q_{max}$, the order of the numerical integration, $c$, the size of the input image in polar coordinates, and both radial & angular kernels. For a digital color image of size $N \times N$, with $N^2$ pixels, the number of pixels in polar domain is $SM^2$, where $S$ is the number of the inner most circular sectors (i.e., $S = 4$) and $M$ is non-overlapped circular rings (i.e., $N/2 \leq M \leq N$). In this paper, we have selected $M = N/2$. Therefore, the overall computational complexity of
FrMRHFMs is $O(FrMRHFMs) = O(\text{calculation of the radial kernel}) + O(\text{calculation of the angular kernel}) + O(\text{the product of radial kernel, angular kernel and the image function in polar domain})$. The computational complexity of computing the radial kernel, the angular kernel, the product of both radial kernel and angular kernel by the image function in polar domain are $O(c \ pmax \ M)$, $O(qmax \ SM)$, and $O(3 \ pmax \ qmax \ SM^2)$ respectively. Therefore, the computational complexity $O(FrMRHFMs) = O(c \ pmax \ M) + O(qmax \ SM) + O(3 \ pmax \ qmax \ SM^2)$. The computational complexity of QRHFMs [6] is $O(QRHFMs) = O(\text{calculation of the radial function}) + O(\text{the product of radial, the complex Fourier function and the image function in cartesian domain}) + O(\text{Linear combination of one real part and three imaginary parts of RHFMs for three components (R-, G-, B-})$. Therefore, the overall computational complexity is $O(QRHFMs) = O(pmax^2 N^2) + O(qmax N^2) + O(3 \ pmax \ qmax \ N^2) + O(4 \ pmax \ qmax \ N^2) + O(3 \ pmax \ qmax \ SM^2)$. Similarly, the computational complexity of QFrZMs [17] is $O(QFrZMs) = O(\text{calculation of the radial function}) + O(\text{classification of the complex Fourier function}) + O(\text{the product of radial, the complex Fourier function and the image function in cartesian domain}) + O(\text{Linear combination of one real part and three imaginary parts of FrZMs for three components (R-, G-, B-})$. Therefore, the total number of operations required for computing the three sets of orthogonal moments, FrMRHFMs, QRHFMs [6], and QFrZMs [17], for an image of the size $N \times N$ with moment order $\leq pmax = 5$, $qmax = 5$ and $Max = 5$, the explicit total number of operations are shown in Table (6).

### E. COMPUTATIONAL TIME

In this subsection, the quantative measure, elapsed CPU times, are used to evaluate the efficiency of the proposed FrMRHFMs, QRHFMs [6] and QFrZMs [17]. In the experiment, FrMRHFMs, QRHFMs [6] and QFrZMs [17] were used to compute the moments of the different color images
of well-known datasets, COIL-100 [30], the soccer team dataset [31] and OTscene [32]. These datasets have different color image size and numbers. The average elapsed CPU times are calculated. The obtained results are plotted in Fig. 15. It is clear that the proposed FrMRHFMs is faster than QRHFMs [6] and QFrZMs [17]. The average elapsed CPU times clearly show that the FrMRHFMs is much faster than QRHFMs [6] and QFrZMs [17] which is consistent with the results of the complexity analysis as shown in Tables (5) and (6).

VI. CONCLUSION

Novel fractional-order radial harmonic functions were derived. Based on these functions, novel color image descriptors, FrMRHFMs, were introduced. Numerical simulation using various color images were performed. The new set of FrMRHFMs successfully reconstructed different color images where the reconstructed images are almost identical to the original color images. These results reflect the ability of the proposed FrMRHFMs to extract the fine features of color images. The proposed FrMRHFMs show highly accurate invariance to the similarity transformations, rotation, scaling and translation. The fast computation of the proposed FrMRHFMs make suitable for real-time image processing and pattern recognition applications. Generally, the proposed FrMRHFMs show better performance than the QRHFMs and the existing fractional-order moments.

APPENDIXES

APPENDIX A

MATHEMATICAL DERIVATION OF THE FRACTIONAL RADIAL HARMONIC POLYNOMIALS

The radial harmonic polynomials of integer order, \( P_p(r) \), are defined as follows:

\[
P_p(r) = \begin{cases} 
\sqrt{\frac{1}{r^p}}, & p = 0 \\
\sqrt{\frac{2}{r^p}} \cos(\pi pr), & p \text{ is even} \\
\sqrt{\frac{2}{r^p}} \sin(\pi (p + 1)r), & p \text{ is odd}
\end{cases}
\] (A1)

where \( r \in [0, 1] \). The radial harmonic polynomials of integer order are orthogonal in the interval \([0,1]\) where:

\[
\int_0^1 P_p(r)P_q(r)rdr = \delta_{pq}
\] (A2)

The fractional-order radial harmonic polynomials, \( T_p(\alpha, r) \), are derived by modifying the radial harmonic polynomials of integer order, \( P_p(r) \), where the variable \( r \) is replaced by \( r^\alpha \) with \( \alpha \in \mathbb{R}^+ \), which defined as follows:

\[
T_p(\alpha, r) = \sqrt{\alpha} r^{\alpha-1} P_p \left( r^\alpha \right)
\] (A3)

Then:

\[
T_p(\alpha, r) = \begin{cases} 
\sqrt{\alpha} r^{\alpha-1} \frac{1}{r^p}, & p = 0 \\
\sqrt{\alpha} r^{\alpha-1} \frac{2}{r^p} \cos(\pi pr^\alpha), & p \text{ is even} \\
\sqrt{\alpha} r^{\alpha-1} \frac{2}{r^p} \sin(\pi (p + 1)r^\alpha), & p \text{ is odd}
\end{cases}
\] (A4)

The fractional-order radial harmonic polynomials, \( T_p(\alpha, r) \), are orthogonal in the interval \([0,1]\) where:

\[
\int_0^1 T_p(\alpha, r)T_q(\alpha, r)rdr = \delta_{pq}
\] (A5)

APPENDIX B

PROOF OF ORTHOGONALITY

Using equation (A3) in equation (A5) yields:

\[
\int_0^1 T_p(\alpha, r)T_q(\alpha, r)rdr = \int_0^1 \sqrt{\alpha} r^{\alpha-1} P_p \left( r^\alpha \right) \sqrt{\alpha} r^{\alpha-1} P_q \left( r^\alpha \right) rdr
\]

\[
= \int_0^1 \alpha r^{\alpha-1} P_p \left( r^\alpha \right) P_q \left( r^\alpha \right) r^{\alpha-1} dr
\]

\[
= \int_0^1 \alpha r^{\alpha-1} P_q \left( r^\alpha \right) P_q \left( r^\alpha \right) r^{\alpha} dr
\]

\[
= \int_0^1 P_q \left( r^\alpha \right) P_q \left( r^\alpha \right) r^{\alpha} r^{\alpha-1} dr \int_0^1 P_p \left( r^\alpha \right) P_q \left( r^\alpha \right) r^{\alpha+1} dr
\]

\[
= \delta_{pq}
\] (B1)

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