Simultaneous Input and State Interval Observers for Nonlinear Systems

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Abstract—We address the problem of designing simultaneous input and state interval observers for Lipschitz continuous nonlinear systems with unknown inputs and bounded noise signals. Benefiting from the existence of nonlinear decomposition functions and affine abstractions, our proposed observer recursively computes the maximal and minimal elements of the estimate intervals that are proven to contain the true states and unknown inputs, and leverages the output/measurement signals to shrink the intervals by eliminating estimates that are incompatible with the measurements. Moreover, we provide sufficient conditions for the existence and stability (i.e., uniform boundedness of the sequence of estimate interval widths) of the designed observer, and show that the input interval estimates are tight, given the state intervals and decomposition functions.

I. INTRODUCTION

Motivation. In several engineering applications such as aircraft tracking, fault detection, attack (unknown input) detection and mitigation in cyber-physical systems and urban transportation [1]–[3], algorithms for unknown input reconstruction and state estimation have become increasingly indispensable and crucial to ensure their smooth and safe operation. Specifically, in safety-critical bounded-error systems, set/interval membership methods have been applied to guarantee hard accuracy bounds. Further, in adversarial settings with potentially strategic unknown inputs, it is critical and desirable to simultaneously derive compatible estimates of states and unknown inputs, without assuming any a priori known bounds/intervals for the input signals.

Literature review. Interval observer design has been extensively studied in the literature [4]–[14]. However, relatively restrictive assumptions about the existence of certain system properties were imposed to guarantee the applicability of the proposed approaches, such as cooperativeness [8], linear time-invariant (LTI) dynamics [10], linear parameter-varying (LPV) dynamics that admits a diagonal Lyapunov function [12], monotone dynamics [6], [7], and Metzler and/or Hurwitz partial linearization of nonlinearities [9], [11].

The problem of designing an $L_2/L_\infty$ unknown input interval observer for continuous-time LPV systems is studied in [15], where the required Metzler property is formulated as a part of a semi-definite program. However, this approach is not directly applicable for general discrete-time nonlinear systems. Moreover, in their setting, the unknown inputs do not affect the output (measurement) equation.

Leveraging bounding functions, the design of interval observers for a class of continuous-time nonlinear systems without unknown inputs has been addressed in [13]. However, no necessary and/or sufficient conditions for the existence of bounding functions or how to compute them have been discussed. Moreover, to conclude stability, somewhat restrictive assumptions on the nonlinear dynamics have been imposed. On the other hand, the authors in [14] studied interval state estimation for a class of uncertain nonlinear systems, by extracting a known nominal observable subsystem from the plant equations and designing the observer for the transformed system, but without providing guarantees that the derived functional bounds have finite values, i.e., are bounded sequences. Moreover, the derived conditions for the existence and stability of the observer are not constructive. More importantly, none of the aforementioned works consider unknown inputs (without known bounds/intervals) nor the reconstruction/estimation of the uncertain inputs.

For systems with linear output equations and where both the state and output equations are compromised by unknown inputs, the problem of simultaneously designing state and unknown input set-valued observers has been studied in our prior works for LTI [3], LPV [16], switched linear [17] and nonlinear [18] systems with bounded-norm noise. Further, our recent work [19] considered the design of state and unknown input interval observers for nonlinear systems but with the assumption of a full-rank direct feedthrough matrix.

Contributions. By leveraging a combination of nonlinear decomposition mappings [20], [21] and affine abstraction (bounding) functions [22], we design an observer that simultaneously returns interval-valued estimates of states and unknown inputs for a broad range of nonlinear systems [23], in contrast to existing interval observers in the literature that to the best of our knowledge, only return either state [4]–[14] or input [15] estimates. Moreover, we consider arbitrary unknown input signals with no assumptions of a priori known bounds/intervals, being stochastic with zero mean (as is often assumed for noise) or bounded. Further, we relax the assumption of a full-rank feedthrough matrix in [19], and extend the observer design by including a crucial update step, where starting from the intervals from the propagation step, the framers are iteratively updated by intersecting it with the state and input intervals that are compatible with the observations. As a result, the updated framers have decreased widths, i.e., tighter intervals can be obtained.

In addition, we derive sufficient conditions for the existence of our observer that can be viewed as structural properties of the nonlinear systems, as an extension of the rank condition that is typically assumed in linear state and input estimation, e.g., [1]–[3]. We also provide several sufficient

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This work is partially supported by NSF grant CNS-1932066.
conditions in the form of Linear Matrix Inequalities (LMI) for the stability of our designed observer (i.e., the uniform boundedness of the sequence of estimate interval widths). In addition, we show that given the state intervals and specific decomposition functions, our input interval estimates are tight and further provide upper bound sequences for the interval widths and derive sufficient conditions for their convergence and their corresponding steady-state values.

II. PRELIMINARIES

Notation. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space and $\mathbb{R}_{++}$ positive real numbers. For vectors $v, w \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{p \times q}$, $\|v\| \triangleq \sqrt{v^T v}$ and $\|M\|$ denote their (induced) 2-norm, and $v \leq w$ is an element-wise inequality. Moreover, the transpose, Moore-Penrose pseudoinverse, $(i, j)$-th element and rank of $M$ are given by $M^T, M^+, M_{i,j}$ and $\text{rank}(M)$. $M_{(i,j)}$ is a sub-matrix of $M$, consisting of its $i$-th through $s$-th rows and $j$-th through $t$-th columns, and $M$ is a non-negative matrix, i.e., $M \geq 0$, if $M_{i,j} \geq 0, \forall i \in \{1, \ldots, p\}, \forall j \in \{1, \ldots, q\}$. We also define $M^+, M^{++} \in \mathbb{R}^{p \times q}$ as $M_{i,j}^+ = M_{i,j}$ if $M_{i,j} \leq 0$, $M_{i,j}^{++} = 0$ if $M_{i,j} < 0$, $M^{++} = M^+ - M$ and $|M| \triangleq M^+ + M^++$. Furthermore, $r = \text{row}(M) \in \mathbb{R}^p$, where $r(i) = 0$ if the $i$-th row of $A$ is zero and $r(i) = 1$ otherwise, $\forall i \in \{1, \ldots, p\}$. For a symmetric matrix $S$, $S \succeq 0$ and $S \preceq 0$ are positive and negative (semi-)definite, respectively. Next, we introduce some definitions and related results that will be useful throughout the paper. The proofs for the lemmas will be provided in the appendix.

Definition 1 (Interval, Maximal and Minimal Elements, Interval Width). An (multi-dimensional) interval $\mathcal{I} \subset \mathbb{R}^n$ is the set of all real vectors $x \in \mathbb{R}^n$ that satisfies $\underline{x} \leq x \leq \overline{x}$, where $\underline{x}$ and $|\overline{x} - \underline{x}|$ are called minimal vector, maximal vector and width of $\mathcal{I}$, respectively.

Next, we will briefly restate our previous result in [22], tailoring it specifically for intervals to help with computing affine bounding functions for our vector fields.

Proposition 1. [22, Affine Abstraction] Consider the vector field $f(.) : \mathcal{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, where $\mathcal{B}$ is an interval with $\overline{x}, \underline{x}, \mathcal{V}_B$ being its maximal, minimal and set of vertices, respectively. Suppose $\overline{A}_B, \underline{A}_B, \overrightarrow{A}_B, \underline{g}_B, \overrightarrow{g}_B$ is a solution of the following linear program (LP):

$$\min_{\theta} \quad \theta$$

s.t. $$(\overline{A} - \underline{A})x + \underline{g} + \sigma \leq f(x) \leq (\overline{A} - \underline{A})x + \overline{g} - \sigma,$$

$$s = 1_m \in \mathbb{R}^m$$

where $1_m \in \mathbb{R}^m$ is a vector of ones and $\sigma$ can be computed via [22, Proposition 1] for different function classes. Then, $\overline{A}x + \underline{g} \leq f(x) \leq (\overline{A} - \underline{A})x + \overline{g}, \forall x \in \mathcal{V}_B$. We call $\overline{A}, \underline{A}$ upper and lower affine abstraction slopes of function $f(.)$ on $\mathcal{B}$.

Corollary 1. By taking the average of upper and lower affine abstractions and adding/subtracting half of the maximum distance, it is straightforward to parallelize the upper and lower abstractions as $Ax + (1/2)(\overline{A}x + \underline{g} - \theta 1_m) \leq f(x) \leq Ax + (1/2)(\overline{A}x + \underline{g} + 1_m)$, where $A = (1/2)(\overline{A} + \underline{A})$. Proposition 2. [13, Lemma 1] Let $A \in \mathbb{R}^{m \times n}$ and $\underline{g} \leq x \leq \overline{x} \in \mathbb{R}^n$. Then, $A^{++} - A^{++}x \leq Ax \leq A^{++} - A^{++}x$. As a corollary, if $A$ is non-negative, $A\underline{g} \leq Ax \leq A\overline{g}$.

Lemma 1. Suppose the assumptions in Proposition 2 hold. Then, the returned bounds for $Ax$ is tight, in the sense that if $\sup Ax = A^{++} - A^{++}x$ and $\inf Ax = A^{++} - A^{++}x$, $\underline{g} \leq x \leq \overline{x}$, where $\sup$ and $\inf$ are considered element-wise.

Definition 2 (Lipschitz Continuity). function $f(.) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is $L_f$-Lipschitz continuous on $\mathbb{R}^n$, if $\exists L_f \in \mathbb{R}_{++}$, such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|, \forall x_1, x_2 \in \mathbb{R}^n$.

Definition 3 (Mixed-Monotone Mappings and Decomposition Functions). [20, Definition 4] A mapping $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathcal{T} \subset \mathbb{R}^m$ is mixed monotone if there exists a decomposition function $f_d : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{T}$ satisfying:

1) $f_d(x, x) = f(x)$,

2) $x_1 \geq x_2 \Rightarrow f_d(x_1, y) \geq f_d(x_2, y)$,

3) $y_1 \geq y_2 \Rightarrow f_d(x_1, y_1) \leq f_d(x_2, y_2)$.

Proposition 3. [21, Theorem 1] Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathcal{T} \subset \mathbb{R}^m$ be a mixed monotone mapping with decomposition function $f_d : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{T}$ where $x, \underline{x}, \overline{x} \in \mathbb{R}^n$, then $f_d(\underline{x}, \overline{x}) \leq f(x) \leq f_d(\overline{x}, \underline{x})$.

Due to non-uniqueness of the decomposition function of a function, a specific one is given in [20, Theorem 2]: If a vector field $q = [q_1 \ldots q_n]^T : X \rightarrow \mathbb{R}^n$ is differentiable and its partial derivatives are bounded with known bounds, i.e., $\frac{\partial q}{\partial x_i}, b_i \in \mathbb{R}^n$, where $a_i^q, b_i^q \in \mathbb{R}$ can be computed in terms of $x, y, a_i^q, b_i^q$ as given in [20, (10)–(13)]. Consequently, for $x = [x_1 \ldots x_n]^T, y = [y_1 \ldots y_n]^T$, we have $q_d(x, y) = q(z) + C^q(z, y)$, where

$C^q \triangleq [(a_1^q - b_1^q) \ldots (a_n^q - b_n^q)] \in \mathbb{R}^{m \times n}$

with $a_i^q, b_i^q$ given in [20, (10)–(13)], $z = [z_1 \ldots z_m]^T$ and $z_j = x_j + y_j$ (dependent on the case, cf. [20, Theorem 1 and (10)–(13)] for details). Moreover, if exact values of $a_i, b_i$ are unknown, their approximations can be obtained using Proposition 1 with the slopes set to $0$.

Corollary 2. As a direct implication of Propositions 1 and 2 for any Lipschitz mixed-monotone vector-field $q(.) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, with a decomposition function $q_d(\underline{x}, \overline{x})$, we can find upper and lower vectors $\overline{q}, \underline{q}$ such that $\underline{q} \leq q(x) \leq \overline{q}, \forall x \in \mathbb{R}^n$. We call $\overline{A}, \underline{A}$ upper and lower affine abstraction slopes of function $f(.)$ on $\mathcal{B}$.
Lemma 2. Let \( q(\cdot) : [\mathcal{I}, \mathcal{T}] \subset \mathbb{R}^n \to \mathbb{R}^m \) be the Lipschitz mixed-monotone vector-field in Corollary \( \mathcal{C} \) with its decomposition function \( q_d(\cdot, \cdot) \) constructed using \( \mathcal{C} \). Then, 
\[
|q - q_0| \leq \|q_d(\mathcal{I}, \mathcal{T}) - q_d(\mathcal{I}, \mathcal{T})\| \leq L_{q_d}\|\mathcal{I} - \mathcal{T}\|, 
\]
where \( L_{q_d} \equiv L_q + 2\|C_q\| \), with \( C_q \) given in \( \mathcal{C} \).

III. Problem Formulation

System Assumptions. Consider the nonlinear discrete-time system with unknown inputs and bounded noise
\[
\begin{align*}
x_{k+1} &= f(x_k) + Bu_k + Gd_k + w_k, \\
y_k &= g(x_k) + Du_k + Hd_k + v_k,
\end{align*}
\]
where at time \( k \in \mathbb{N} \), \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \), \( d_k \in \mathbb{R}^p \) and \( y_k \in \mathbb{R}^l \) are the state vector, a known input vector, an unknown input vector, and the measurement vector, correspondingly. The process and measurement noise signals \( w_k \in \mathbb{R}^n \) and \( v_k \in \mathbb{R}^l \) are assumed to be bounded, with \( w \leq w_k \leq \overline{w}, \) \( \underline{v} \leq v_k \leq \overline{v}, \) and the known lower and upper bounds, \( \underline{w} \), \( \overline{w} \) and \( \underline{v} \), \( \overline{v} \), respectively. We also assume that lower and upper bounds for the initial state, \( \mathcal{I}_0 \) and \( \mathcal{T}_0 \), are available, i.e., \( \underline{x}_0 \leq x_k \leq \overline{x}_0 \). The vector fields \( f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \) and \( g(\cdot) : \mathbb{R}^n \to \mathbb{R}^l \) are mixed-monotone with decomposition functions \( f_d(\cdot, \cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}^n \) and \( g_d(\cdot, \cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}^l \) and Lipschitz continuous, respectively.

Unknown Input (or Attack) Signal Assumptions. The unknown inputs \( d_k \) are not constrained to follow any model nor to be a signal of any type (random or strategic), hence no prior ‘useful’ knowledge of the dynamics of \( d_k \) is available (independent of \( \{d_i\} \forall i \neq k \) and \( \{w_k\} \forall k \)). We also do not assume that \( d_k \) is bounded or has known bounds and thus, \( d_k \) is suitable for representing adversarial attack signals.

Next, we briefly introduce a similar system transformation as in [3], which will be used later in our observer structure.

System Transformation. Let \( \rho_H \equiv \text{rk}(H) \). Similar to [3], by applying singular value decomposition, we have \( H = [U_1 \ U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & V_2^T \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^T \) with \( V_1 \in \mathbb{R}^{n \times n}, \ V_2 \in \mathbb{R}^{n \times (p-n)}, \) \( \Sigma \in \mathbb{R}^{n \times n} \) (a diagonal matrix of full rank), \( U_1 \in \mathbb{R}^{n \times n} \) and \( U_2 \in \mathbb{R}^{n \times (p-n)} \). Then, since \( V \equiv [V_1 \ V_2] \) is unitary:
\[
d_k = V_1 d_{1,k} + V_2 d_{2,k}, \quad d_{1,k} = V_1^T d_k, \quad d_{2,k} = V_2^T d_k. \quad (4)
\]

Finally, by defining \( T_{1,k} \equiv U_1^T, \ T_{2,k} \equiv U_2^T \), the output equation can be decoupled as:
\[
\begin{align*}
z_{1,k} &= g_1(x_k) + D_{1,k} u_k + v_{1,k} + \Sigma d_{1,k}, \\
z_{2,k} &= g_2(x_k) + D_{2,k} u_k + v_{2,k}, \\
g(x, k) &\equiv T_{1,k} g(x_k), g_2(x_k) \equiv T_{2,k} g(x_k).
\end{align*}
\]
The observer design problem can be stated as follows:

**Problem 1.** Given a nonlinear discrete-time system with unknown inputs and bounded noise \( (3) \), design a stable observer that simultaneously finds bounded intervals of compatible states and unknown inputs.

IV. General Simultaneous Input and State Interval Observers (GSISO)

A. Interval Observer Design

We consider a recursive three-step interval-valued observer design, composed of a state propagation (SP) step, which propagates the previous time state estimates through the state equation to find propagated intervals, a measurement update (MU) step, which iteratively updates the state intervals using the observation, and an unknown input estimation (UIE) step, which computes the input intervals using state intervals and observation. We design the observer in the following form:

**State Propagation:** \( \mathcal{I}^{\text{SP}}_k = \mathcal{F}^p_x(\mathcal{I}^{\text{SP}}_{k-1}, y_{k-1}, u_{k-1}) \)

**Measurement Update:** \( \mathcal{I}^{\text{MU}}_k = \mathcal{F}^p_x(\mathcal{I}^{\text{MU}}_k, y_k, u_k) \)

**Unknown Input Estimation:** \( \mathcal{I}^{\text{UIE}}_{k-1} = \mathcal{F}^d_x(\mathcal{I}^{\text{UIE}}_{k-1}, y_{k-1}, u_{k-1}) \)

where \( \mathcal{F}^p_x, \mathcal{F}^p_x, \mathcal{F}^d_x \) are to-be-designed interval mappings, while \( \mathcal{I}^{\text{SP}}_k, \mathcal{I}^{\text{SP}}_{k-1} \) and \( \mathcal{I}^{\text{UIE}}_{k-1} \) are intervals of compatible states, updated states and unknown inputs at time steps \( k, k \) and \( k-1 \), respectively. Note that we are constrained with obtaining a one-step delayed estimate of \( \mathcal{I}^{d}_{k-1} \), because in contrast with [19], the matrix \( H \) is not necessarily full-rank, and hence \( d_k \) cannot be estimated from the current measurement, \( y_k \). However, in Lemma \( \mathcal{L} \) and Remark \( \mathcal{R} \) we will discuss a way of obtaining the current estimate of a component of the input signal, i.e., \( d_{1,k} \).

Considering the computational complexity of optimal observers [24], as well as nice properties of interval sets [15], we consider set estimates of the form:
\[
\begin{align*}
\mathcal{I}^{\text{SP}}_k &= \{ x \in \mathbb{R}^n : x_{\underline{p}} \leq x \leq x_{\overline{p}} \}, \\
\mathcal{I}^{\text{MU}}_k &= \{ x \in \mathbb{R}^n : x_{\underline{c}} \leq x \leq x_{\overline{c}} \}, \\
\mathcal{I}^{\text{UIE}}_{k-1} &= \{ d \in \mathbb{R}^p : d_{\underline{k}} \leq d \leq d_{\overline{k}} \},
\end{align*}
\]
i.e., we restrict the estimation errors to be closed intervals. In this case, the observer design problem boils down to finding \( x_k, x_{\overline{k}}, x_{\underline{k}}, \mathcal{I}_k, \mathcal{I}_{k-1}, d_{k} \). Our interval observer can be defined at each time step \( k \geq 1 \) as follows (with known \( \mathcal{I}_0 \) and \( \mathcal{T}_0 \) such that \( \underline{x}_0 \leq x_k \leq \overline{x}_0 \)):

**State Propagation (SP):**
\[
\begin{align*}
[x_k^T, \zeta_k^T] &\leq M_f [\mathcal{I}_k \ z_k^T]^T + M_g [y_k^T \ \gamma_k^T]^T + \omega^p + \omega^c, \\
M_v [x_k^T \ \zeta_k^T] &+ M_w [\mathcal{T}_k^T \ \zeta_k^T] + M_y y_{k-1} + M_u u_{k-1}.
\end{align*}
\]

**Measurement Update (MU):**
\[
\mathcal{T}_k = \lim_{\tau \to \infty} \mathcal{T}_k, \quad \zeta_k = \lim_{\tau \to \infty} \zeta_k.
\]

**Unknown Input Estimation (UIE):**
\[
\overline{d}_{k-1} = N_{11} \overline{h}_{k-1} + N_{12} \overline{h}_{k-1}, \quad d_{k-1} = N_{21} \overline{h}_{k-1} + N_{22} \overline{h}_{k-1},
\]
where \( \forall q \in \{ f, g \}, \gamma_k, \zeta_k, \mathcal{T}_k, d_{k-1} \) are upper and lower vector values for the function \( g(\cdot) \) on the interval \( [\mathcal{I}_{k-1}, \mathcal{I}_{k-1}] \), which can be recursively computed using Corollary \( \mathcal{C} \). Moreover,
Algorithm 1 GSISIO

1: Initialize: maximal($I_0^u$) = τ₀; minimal($I_0^l$) = z₀;
   ▶ Observer Gains Computation
Compute $M_k, N_k, g_k, u_k, v_k, w_k$, $i, j \in \{1, 2\}$ via Theorem [1]
2: for $k = 1$ to $n$ do
   ▶ Estimation of $x_k$
Compute $\hat{x}_k$ via (6).
Compute $(\hat{x}_k^i, \hat{x}_k^s)^{\infty}_{i=0}$ via (13)-(14);
3: $(\overline{x}_k, \underline{x}_k) = (\hat{x}_k^i, \hat{x}_k^s)^{\infty}_{i=0}$; $\overline{x}_k \in \mathbb{R}^p: \underline{x}_k \leq x_k \leq \overline{x}_k$;
   ▶ Compute $\delta_k$ through Lemma [8]
   ▶ Estimation of $d_{k-1}$
Compute $d_{k-1}, d_{k-1}, \delta_{k-1}^d$ via (10)-(12) and Lemma [5];
4: $I_{k-1}^d = \{d \in \mathbb{R}^p: \underline{d}_{k-1} \leq d \leq \overline{d}_{k-1}\}$;
5: end for

Definitions

Definition 4 (Correctness (Framer Property [11]). Given an initial interval $z_0 \leq x_0 \leq z_0$, the GSISIO observer returns correct interval estimates, if the true states and unknown inputs of the system (3) are within the estimated intervals (8)-(10) for all times. If the observer is correct, we call $(\overline{x}_k^d, \underline{x}_k^d)^{\infty}_{k=0}$, $(\overline{x}_k, \underline{x}_k)^{\infty}_{k=0}$ and $(\overline{d}_k, \underline{d}_k)^{\infty}_{k=1}$ the propagated state, updated state and input framers, respectively.

Definition 5 (Tightness of Input Estimates). The input interval estimates $(\overline{x}_k^d, \underline{x}_k^d)^{\infty}_{k=1}$ are tight, if at each time step $k$, given the state estimate $\overline{x}_k$, the input framers $\overline{d}_k, \underline{d}_k$, coincide with supremum and infimum values of the set of compatible inputs.

We begin by using the result in Lemma [1] to conclude the correctness and tightness of the input estimates, assuming that the state estimates are given. To increase readability, all proofs will be provided in the appendix.

Lemma 3 (Correctness and Tightness of Input Estimates). Consider the system (3) along with the GSISIO in (8)-(10), let $J \triangleq ([G^T, H])$ and suppose that Assumption [7] holds, $N_{11} = N_{22} = J^T$, and $N_{12} = N_{21} = -J^T$. Then, given any pair of state framer sequences $(\overline{\tau}_{k}, \underline{\tau}_{k})^{\infty}_{k=0}$, the input interval estimates given in (10), are correct and tight.

Next, we state our main result on the existence of the GSISIO and correctness of the state estimates.

Theorem 1 (Existence of Correct Framers). Consider the system (3), the transformed output equations (5)-(7) and the GSISIO introduced in (8)-(10). Suppose all the assumptions in Lemma [3] hold and there exists a pair of slope matrices $(\overline{A}, \underline{A})$, which construct affine upper and lower abstractions for the vector field $g_2(.)$ on the entire state space (cf. Proposition [1]). Suppose that the observer gains are chosen as given in Appendix [A]. Then, at each time step $k$, the GSISIO returns finite and correct framers, i.e., finite correct interval estimates for the system (3), if

$$r^T((A_1 + A_2)r + \hat{r}) = 0,$$

with $A_1 \triangleq A^1 + A^1 + A^1 + A^1 + A^1$, $A_2 \triangleq A^1 + A^1 + A^1 + A^1 + A^1$, $A = (1/2)(\overline{A} + \underline{A})$, $\hat{r} \triangleq$ roundsup$(I - A^1)$, $r \triangleq$ roundsup$(I - A^1A_2(1:\nu))$ and $\nu$ given in Appendix [A].

Corollary 3. In the case that only the state propagation step is considered, the existence conditions boil down to rk$I - K_1 - L_1 = rk(I - K_1 - L_1) = n$.

Note that we can only obtain a one-step delayed estimate of $d_k$ in (10), since we can find an estimate for $d_{k,1}$ at current time $k$, but not $d_{k,2}$. We formalize this as follows.

Lemma 4. Suppose all the assumptions in Theorem [7] hold. Then, at time step $k$, $d_{k,1} \leq d_{k,1} \leq d_{k,1}$, where $d_{k,1} = \Sigma^{-1}(z_{k,1} - T_kD_{uk})$, $d_{k,1} = \Sigma^{-1}(z_{k,1} - T_kD_{uk}) + \bar{f}_k$, with $f_k \triangleq (\Sigma^{-1}T_k)^{+}(g(\overline{x}_k, \underline{x}_k) + \overline{y}) - (\Sigma^{-1}T_k)^{+}(g(\overline{x}_k, \underline{x}_k) + \overline{y})$.

Remark 1. The result in Lemma [2] is particularly helpful in the special case when the feedthrough matrix has full rank. In this case, $d_k = d_{k,1}$ and hence, $d_k$ can be estimated at current time $k$. Thus, this can be considered as an alternative approach to the one in [19] for the full-rank H case.
C. Uniform Boundedness of Estimates (Observer Stability)

In this section, we derive several sufficient conditions for the stability of GSISIO via Theorem 2.

**Theorem 2** (Observer Stability). Consider the system \( (5) \) and the GSISIO (5–10). Suppose all the assumptions in Theorem 7 hold, the decomposition functions \( f_2, g_2 \) are constructed using (2) and \( \tilde{f}, \tilde{g} \) are the upper and lower affine abstraction slopes for \( g_2(x) \) on the entire state space. Then, the observer is stable, in the sense that interval widths and compute their steady-state values, if they exist.

Lemma 5 (Uniform Boundedness of Estimates (Observer Stability))

Let \( \Delta_k \) be the width of the interval, and \( \bar{\Delta}_k \) be the upper bound of the interval width. Then, \( \bar{\Delta}_k \) is uniformly bounded, and consequently, interval width and state estimation errors \( \|\Delta_k\|_\infty \leq \max(\|\Delta_k-\Delta_{k-1}\|, \|\Delta_k\|_\infty) \) are also uniformly bounded.

(ii) \( \min_{P \in \mathbb{D}_P} \lambda_{\max}(\mathcal{P}) \geq 0 \), where \( \mathbb{D}_P = \{D^* \in \mathbb{D}^* \mid \mathcal{P}_D \preceq 0 \} \).

(iii) \( \mathcal{P}_D = \{P \mid \mathcal{D}(I-P) = 0 \} \), where \( \mathcal{D}(I-P) = 0 \).

\[ \mathcal{D} = \left[ \begin{array}{ccc} Q & 0 & 0 \\ 0 & 0 & 0 \\ * & T_f & T_f \\ * & T_f & T_f \\ * & T_f & T_f \\ * & T_f & T_f \\
\end{array} \right] \], \( \mathcal{P}_D = \left[ \begin{array}{ccc} P + \mathcal{D} & 0 \\ 0 & P \\ 0 & P \\ 0 & P \\
\end{array} \right] \).

Remark 2. The optimization and feasibility problems in (i–iii) are all (mixed)-integer programs with finitely countable feasible sets \( \{D^* \mid D^* \leq 2^n \} \), which can be easily solved by enumerating all possible solutions and comparing the values.

Finally, we will provide upper bounds for the interval widths and compute their steady-state values, if they exist.

**Lemma 5** (Upper Bounds of the Interval Widths and their Convergence). Consider the system (5) and the GSISIO observer (5–10). Suppose all the assumptions in Theorem 7 hold with strict inequality.

Then, the interval width sequences \( \{\|\Delta_k\|_\infty \}, \{\|\Delta_{k-1}\|_\infty \} \) for \( k=0 \) are uniformly upper bounded by the convex sequences \( \{\delta_k, \delta_{k-1}\} \), as follows:

\[ \|\Delta_k\|_\infty \leq \delta_k = \mathcal{L}^k \delta_0 + \|\Delta_0\|_\infty, \quad \|\Delta_{k-1}\|_\infty \leq \delta_{k-1} = \mathcal{G}(\delta_k(k)) \]

where \( \mathcal{D} \) is a solution to \( \min_{DP^*} \|\Delta_2\|_\infty \), \( P^* \) is the solution of the optimization problem in (6).

**V. ILLUSTRATIVE EXAMPLE**

We consider a slightly modified version of a nonlinear system in [25], without the uncertain matrices, with the inclusion of unknown inputs, and with the following parameters (cf. (3)): \( n = l = p = 2, m = 1, f(x) = [f_1(x), f_2(x)] \), \( g(x) = [g_1(x), g_2(x)] \), \( B = D = 1 \), \( H = 0 \), \( \mathbf{v} = 0 \).

The unknown input signals are depicted in Figure 1 where \( \mathbf{r}(t) \) is not full rank and hence, the approach in [19] is not applicable. Moreover, applying [22, Theorem 1], we can compute finite-valued upper and lower bounds for partial derivatives of \( f(\cdot) \) and \( g(\cdot) \) as:

\[ \frac{\partial g}{\partial a_1} = \left[ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right], \quad \frac{\partial g}{\partial b_1} = \left[ \begin{array}{c} a_5 \\ a_6 \\ a_7 \\ a_8 \end{array} \right] \]

where \( \mathbf{v} \) is a very small positive value, ensuring that the partial derivatives are in open intervals (cf. [20, Theorem 1]). Moreover, \( L_f = 0.35 \) and \( L_g = 0.74 \) and Assumption 1 holds by [20, Theorem 1]. Furthermore, computing \( K = [K_1 K_2] = \left[ \begin{array}{cc} 0.0267 & 0 \\ 0.4177 & 1.2103 \end{array} \right] \) and \( L = [L_1 L_2] = \left[ \begin{array}{cc} 0 & 0.1017 \\ 0 & 0.5194 \end{array} \right] \), we obtain \( \mathbf{v} = 0.0009 \) and \( L_g = 1.19 \). Consequently, \( \mathcal{L} = 0.643 \) is the smallest one that satisfies Condition (i) in Theorem 2 with \( \mathbf{D} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \). So, we expect to obtain uniformly bounded estimate errors with convergent upper bounds. This is shown in Figure 2 where at
conditions for the existence and stability of the observer. With an example. For future work, we seek to find tighter boundedness of the interval widths were derived. Finally, a specific pair of decomposition functions. Further, several input interval estimates, given the state intervals and unknown input framers and proved the tightness of proved that the observer recursively outputs the correct state derived sufficient conditions for the existence of our observer, nonlinear systems with unknown inputs was proposed. We valued observer for bounded-error mixed monotone Lipschitz best efforts, we were unable to find interval-valued observers upper bound for the interval width and the upper bounds width, which in turn is less than or equal to the predicted unknown input estimates for comparison with our results.

VI. CONCLUSION

In this paper, a simultaneous input and state interval-valued observer for bounded-error mixed monotone Lipschitz nonlinear systems with unknown inputs was proposed. We derived sufficient conditions for the existence of our observer, proved that the observer recursively outputs the correct state and unknown input framers and proved the tightness of the input interval estimates, given the state intervals and a specific pair of decomposition functions. Further, several conditions for the stability of the observer, i.e., the uniform boundedness of the interval widths were derived. Finally, we demonstrated the effectiveness of the proposed approach with an example. For future work, we seek to find tighter decomposition (bounding) functions and to provide necessary conditions for the existence and stability of the observer.

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APPENDIX: OBSERVER GAIN DEFINITIONS AND PROOFS

A. GSISIO OBSERVER GAIN DEFINITIONS

∀s ∈ \{f, g, u, w, v, y\}: A_s = A_s^f A_s, A_w = A_w^f A_w, A_{\bar{\mu}} = \left[ I - K_{\bar{\mu}} L_{\bar{\mu}} \right], A_{\bar{\mu}}^f = \left[ I + L_{\bar{\mu}} - K_{\bar{\mu}} L_{\bar{\mu}} \right], A_{\bar{\mu}} = A_{\bar{\mu}}^f L_{\bar{\mu}} - K_{\bar{\mu}} L_{\bar{\mu}}^2, L_{\bar{\mu}} = L_{\bar{\mu}} (1:\bar{\mu}), L_{\bar{\mu}}^2 = L_{(\bar{\mu}+1:\bar{\mu}+1)}, F = (I + L_{\bar{\mu}} - K_{\bar{\mu}}) \mathbf{B}^\top L_{\bar{\mu}} - (L_{\bar{\mu}} - K_{\bar{\mu}}) D_{\bar{\mu}}, D_{\bar{\mu}} = (1/2) (A_{\bar{\mu}} + A_{\bar{\mu}}^f). Further, \( \omega^f_\mu = \mu [r^f - r^f]^T, g_\mu(x_{\bar{\mu}}) \) is a decomposition function of \( g(x) \) and \( \mu \) is a very large positive real number (infinity), while \( \omega^f_\mu(x_{\bar{\mu}}) = \mu \text{ rowsupp}(I - A_{\bar{\mu}}^f A_{\bar{\mu}}), \) where \( \text{rowsupp}(A_{\bar{\mu}}) \) is a solution of the LP (1) for the corresponding vector field \( g(x) \) on the interval \( \mathbb{R}^{i-1} \), with the following extra constraints:

\[
\begin{align*}
(\bar{A}_i - \bar{A}) x_{i,k} - \bar{\pi} &\leq 0 \quad (\bar{A}_i - \bar{A}) x_{i,k} + \bar{\pi} - \bar{\pi} \, (18) \\
\text{for all } x_{i,k} \in \mathcal{V}_{\bar{A}_{\bar{\mu}}}, \text{ at time } k \text{ and at iteration } i \in \{1, \ldots, n\}. 
\end{align*}
\]

B. PROOF OF LEMMA 1

For \( j \in \{1, \ldots, m\} \), consider the problem of \( \overline{\pi}_j = \max_{\bar{\pi} \leq \bar{\pi}} [Ax]_j \), where \( [Ax]_j = \sum_{i=1}^{n} A_i x_i \) is the \( j \)-th component of the vector \( Ax \). It is easy to verify that the solutions of this linear program are \( \bar{\pi}_j = \pi_i \) if \( A_i \geq 0 \), and \( \pi_i = -\pi_i \) if \( A_i < 0 \), for \( i \in \{1, \ldots, n\} \). Consequently, \( \overline{\pi}_j = [Ax]_j \) or \( [Ax]_j^\top \bar{\pi} \), where \( [Ax]_j \) is \( \pi \)-th row of \( Ax \). By a similar reasoning of \( \bar{\pi} \), \( \pi \leq \bar{\pi} \), \( \text{inf}_{\bar{\pi} \leq \bar{\pi}} [Ax]_j \geq [Ax]_j \), which is a solution of (22). Thus, considering that \( \pi \leq \bar{\pi} - \bar{\pi} \text{ and } \text{inf}_{\bar{\pi} \leq \bar{\pi}} [Ax]_j = \pi = [Ax]_j \), the proof is complete.

C. PROOF OF LEMMA 2

Starting from (2), we obtain \( \overline{f}(\bar{\pi}, \bar{\pi}) = f(x_1) + C_f(\bar{\pi} - \bar{\pi}) \) and \( \overline{f}(\bar{\pi}, \bar{\pi}) = f(x_2) + C_f(\bar{\pi} - \bar{\pi}) \), which together imply

\[
\overline{f}(\bar{\pi}, \bar{\pi}) - f(x, \bar{\pi}) = f(x_1) - f(x_2) + 2C_f(\bar{\pi} - \bar{\pi}), \quad (19)
\]

where \( \forall i \in \{1, \ldots, n\} \). Combining (19) with the Lipschitz continuity of \( f \) we obtain

\[
\|f(x_1, \bar{\pi}) - f(x_2, \bar{\pi})\| \leq L_f \|x_1 - x_2\| + 2\|C_f\| \|\bar{\pi}\| \quad (20)
\]

Combining (20) and (21) yields the result.

D. PROOF OF LEMMA 3

Augmenting the state and output equations in (3) and from Corollary 2 we obtain \( \bar{\mu} \leq [G^\top H^\top]_{\bar{\mu}} \bar{\mu} \leq \bar{\mu} \bar{\mu} \leq \bar{\mu} \bar{\mu} \leq \bar{\mu} \bar{\mu} \), defined in (11) and (12). Then, the input framers in (10) can be obtained by using Propositions 1 and 3 and considering the fact that \( J \) is full rank. Finally, tightness is implied by Lemma 1 (where the Matrix Equations J).
equivalent to (17). Moreover, since \( \{x_k^{* \dagger}\} \) and \( \{x_k^{* \dagger}\} \) for all \( r \) are, by construction, computed with upper-approximations of the observation function \( g \), hence correctness follows for the state frame, while correctness for the input frame holds by Lemma 3. Finally, without the update step in (9), (17) reduces to \( r = r_{\text{rowsupp}}(I - A_1^2 A_2) = 0 \), which is equivalent to the rank condition in Corollary 3 by [27].

**F. Proof of Lemma 2**

The bounds for \( d_{1,k} \) can be obtained by applying Propositions 2 and 5 to 5. Moreover, since \( d_{2,k} \) does not appear in 5 and 6, it cannot be estimated at the current time.

**G. Proof of Theorem 2**

Let \( \Delta_k^* = \|x_k - x_k^*\| \) (similarly for \( \Delta_{k,f}^* \)). Then, by (26),
\[
\Delta_k^* = \min(\Delta(x_k^{\dagger,f} + 2 \mu_r (x_k^{1/2} + 2 + (\Delta^{1/2} + \Delta^{1/2})))).
\]
From this and using the fact that \( \min(a, b) \leq (I - D)b \) for all \( D \in \mathbb{R}^n \), we obtain
\[
\Delta_k^* \leq \|D + I - D\| (A_1 + A_2) \|\Delta_{k,f}^* + 2 \mu_r (D + I - D)\| r',
\]
where \( r' \triangleq \|A_1 + A_2\| r + \tilde{r} \). Since (17) holds (equivalently \( r'(j) - r'(j) = 0, \forall j \in \{1, \ldots, n\} \)), choosing any \( D \in \mathbb{R}^n \) with \( \begin{bmatrix} D & 0 \end{bmatrix} \), we eliminate the second term on the right-hand side of the above inequality and returns
\[
\Delta_k^* \leq \|D + I - D\| (A_1 + A_2) \|\Delta_{k,f}^* + 2 \mu_r D\| r', \quad \forall D \in \mathbb{R}^n. \tag{27}
\]
On the other hand, from (23), (23) and Corollary 2, we obtain
\[
\Delta_{k,f}^* \leq \|D + I - D\| (A_1 + A_2) \|\Delta_{k,f}^* + 2 \mu_r (D + I - D)\| r', \quad \forall D \in \mathbb{R}^n. \tag{28}
\]
where \( \Delta_{k,f}^* \triangleq T_f \Delta_{k,f}^* + T_g \Delta_{k,f}^* \), \( \Delta_{k,f}^* \triangleq f_d(x_k, \Phi_k) - f_d(x_k, \Phi_k) \), \( \Delta_{k,f}^* \triangleq g(w_k, \Theta_k) - g(w_k, \Theta_k) \), \( \Delta_{k,f}^* \triangleq T_f \Delta_{k,f}^* + T_g \Delta_{k,f}^* \), \( \Delta_{k,f}^* \triangleq \|D + I - D\| (A_1 + A_2) \|\Delta_{k,f}^* + 2 \mu_r (D + I - D)\| r', \quad \forall D \in \mathbb{R}^n. \tag{29}
\]
Below, we will show that either of the three conditions in the theorem implies uniform boundedness of \( \{\Delta_k^*\}_{k=0}^{\infty} \).

**Condition (i):** Since Assumption 1 holds, the application of triangle inequality to (29) yields
\[
\|\Delta_k^*\| \leq L_D \|\Delta_{k-1}^*\| + \|\tilde{D} \Delta z\|, \quad \forall D \in \mathbb{R}^n, \tag{30}
\]
with \( L_D \triangleq L_{f_d} \|D + I - D\| \|T_f\| + L_{g_d} \|D + I - D\| \|T_g\| \) and \( L_{f_d}, L_{g_d} \) obtained from Lemma 3. Since \( L_D^* \leq 1 \) (by Condition (i)), the sequence \( \{\|\Delta_k^*\|\}_{k=0}^{\infty} \) is uniformly bounded. Therefore, the interval width dynamics is stable.

**Condition (ii):** To show that Condition (ii) implies stability, with slightly abuse of notation, let \( D \) be a specific member of \( \mathbb{R}^n \) and suppose we show the stability of the dynamical system \( \Delta_{k+1}^* = D \Delta_{k}^* + D \Delta z \), where \( D \triangleq \|D + I - D\| (A_1 + A_2) \). Then, by Comparison Lemma [28], the dynamical system \( \Delta_{k+1}^* = D \Delta_{k}^* + D \Delta z \) is stable. To do so, consider a candidate Lyapunov function \( \Delta_k^* = \Delta_k^* \Delta_k^* \) and let \( \tilde{T} \) denote the update step in (17).

**H. Proof of Lemma 3**

Applying (30) repeatedly, for all \( D \in \mathbb{R}^n \), we have
\[
\|\Delta_k^*\| \leq \|\Delta_0^*\| + \sum_{k=0}^{n-1} \|\tilde{T} \Delta z\| = \|\tilde{T} \Delta z\| g_k^{\dagger} + \|\tilde{T} \Delta z\| g_k^{\dagger} / |T_k - 1|.
\]
Further, from (10)–(12) we obtain \( \Delta_{k-1}^* \leq \Delta_{k-1}^* + \Delta_{k-1}^* + \Delta_{k-1}^* + \Delta_{k-1}^* \Delta_{k-1}^* \), where \( J_k \triangleq \|J_k \| J + J_k^T \). Applying Lemma 2 and triangle inequality returns the upper bound for \( \|\Delta_{k-1}^*\| \), while taking the limit of \( k \to \infty \) results in the steady-state values. The rest of the results follow from the non-increasing Lyapunov functions defined in the proof of Theorem 2 and the use of the Rayleigh Quotient.