ANALOGUES OF SOME TAUBERIAN THEOREMS FOR STRETCHINGS

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Abstract. We investigate the effect of four-dimensional matrix transformation on new classes of double sequences. Stretchings of a double sequence is defined, and this definition is used to present a four-dimensional analogue of D. Dawson’s copy theorem for stretching of a double sequence. In addition, the multidimensional analogue of D. Dawson’s copy theorem is used to characterize convergent double sequences using stretchings.

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1. Introduction. In this paper, RH-regular matrices and the stretching of double sequences are used to characterize P-convergent sequences. To achieve this goal we begin by defining an $\epsilon$-Pringsheim-copy and a stretching of double sequences. In addition, the copy theorem of Dawson in [1] will be extended as follows: if each of $A$ and $T$ is an RH-regular matrix, and $x$ is any bounded double complex sequence with $\epsilon$ being any bounded positive term double sequence with $P\lim_{i,j} \epsilon_{i,j} = 0$, then there exists a stretching $y$ of $x$ such that $T(Ay)$ exists and contains an $\epsilon$-Pringsheim-copy of $x$. By using this extended copy theorem some natural implications and variations of this extended copy theorem will be presented.

2. Definitions, notations, and preliminary results

Definition 2.1 (see [3]). A double sequence $x = [x_{k,l}]$ has Pringsheim limit $L$ (denoted by $P\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k,l > N$. We will describe such an $x$ more briefly as “P-convergent.”

Definition 2.2 (see [3]). A double sequence $x$ is called definite divergent, if for every (arbitrarily large) $G > 0$ there exist two natural numbers $n_1$ and $n_2$ such that $|x_{n,k}| > G$ for $n \geq n_1$, $k \geq n_2$.

Definition 2.3. The double sequence $[y]$ is a double subsequence of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then $y$ is formed by

\[
\begin{array}{cccc}
z_1 & z_2 & z_5 & z_{10} \\
z_4 & z_3 & z_6 & - \\
z_9 & z_8 & z_7 & - \\
- & - & - & -
\end{array}
\]
The double sequence \( x \) is bounded if and only if there exists a positive number \( M \) such that \( |x_{k,l}| < M \) for all \( k \) and \( l \). A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [5, 6] characterizes the regularity of two-dimensional matrix transformations. In [4], Robison presented a four-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is \( P \)-convergent is not necessarily bounded. The definition of regularity for four-dimensional matrices will be stated below along with the Robison-Hamilton characterization of the regularity of four-dimensional matrices.

**Definition 2.4.** The four-dimensional matrix \( A \) is said to be \( RH \)-regular if it maps every bounded \( P \)-convergent sequence into a \( P \)-convergent sequence with the same \( P \)-limit.

**Theorem 2.5** (see [2, 4]). The four-dimensional matrix \( A \) is \( RH \)-regular if and only if

\[
\text{(RH}_1\text{)} \quad P\text{-lim}_{m,n} \sum_{k,l=1}^{\infty} a_{m,n,k,l} = 0 \quad \text{for each } k \text{ and } l;
\]
\[
\text{(RH}_2\text{)} \quad P\text{-lim}_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0 \quad \text{for each } l;
\]
\[
\text{(RH}_3\text{)} \quad P\text{-lim}_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0 \quad \text{for each } k;
\]
\[
\text{(RH}_4\text{)} \quad \sum_{k,l=1}^{\infty} |a_{m,n,k,l}| \text{ is } P\text{-convergent; and}
\]
\[
\text{(RH}_5\text{)} \quad \text{there exist finite positive integers } A \text{ and } B \text{ such that } \sum_{k,l>B} |a_{m,n,k,l}| < A.
\]

**Example 2.6.** The sequences \([y_{n,k}]=1\) and \([y'_{n,k}]=-1\) for each \( n \) and \( k \) are both subsequences of the double sequence whose \( n,k \)th term is \( x_{n,k}=(-1)^n \). In addition to the two subsequences given, every double sequence of 1’s and -1’s is a subsequence of this \( x \).

**Example 2.7.** As another example of a subsequence of a double sequence, we define \( x \) as follows:

\[
x_{n,k} := \begin{cases} 
1, & \text{if } n = k, \\
\frac{1}{n}, & \text{if } n < k, \\
n, & \text{if } n > k.
\end{cases}
\]  

Then the double sequence

\[
y'_{n,k} := \begin{bmatrix}
\frac{1}{2} & 4 & \frac{1}{10} & 20 & \cdots \\
8 & 6 & \frac{1}{12} & 22 & \cdots \\
\frac{1}{18} & \frac{1}{16} & \frac{1}{14} & 24 & \cdots \\
32 & 30 & 28 & 26 & \cdots \\
& & & & \ddots
\end{bmatrix}
\]  

(2.3)

is clearly a subsequence of \( x \).
**Remark 2.8.** Note that if the double sequence $x$ contains at most a finite number of unbounded rows and/or columns, then every subsequence of $x$ is bounded. In addition, the finite number of unbounded rows and/or columns does not affect the $P$-convergence or $P$-divergence of $x$ and its subsequences.

**Definition 2.9.** A number $\beta$ is called a Pringsheim limit point of the double sequence $x = [x_{n,k}]$ provided that there exists a subsequence $y = [y_{n,k}]$ of $[x_{n,k}]$ that has Pringsheim limit $\beta$:

$$P-lim y_{n,k} = \beta.$$ 

**Example 2.10.** Define the double sequence $x$ by

$$x_{n,k} := \begin{cases} (-1)^n, & \text{if } n = k, \\ (-2)^n, & \text{if } n = k + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

This double sequence has five Pringsheim limit points, namely $-2, -1, 0, 1,$ and $2$.

**Remark 2.11.** The definition of a Pringsheim limit point can also be stated as follows: $\beta$ is a Pringsheim limit point of $x$ provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_{i \to \infty} x_{n_i,k_i} = \beta$.

**Definition 2.12.** A double sequence $x$ is divergent in the Pringsheim sense ($P$-divergent) provided that $x$ does not converge in the Pringsheim sense ($P$-convergent).

**Remark 2.13.** Definition 2.12 can also be stated as follows: a double sequence $x$ is $P$-divergent provided that either $x$ contains at least two subsequences with distinct finite Pringsheim limit points or $x$ contains an unbounded subsequence. Also note that, if $x$ contains an unbounded subsequence then $x$ also contains a definite divergent subsequence.

**Example 2.14.** This is an example of a convergent double sequence whose terms form an unbounded set

$$x_{n,k} := \begin{cases} k, & \text{if } n = 1, \\ n, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

**Example 2.15.** This is an example of an unbounded divergent double sequence with three finite Pringsheim limit points, namely $-1, 0,$ and $1$:

$$x_{n,k} := \begin{cases} k + 1, & \text{if } n = 1, \\ (-1)^{n+1}, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

**Example 2.16.** This is an example of a double sequence which contains an unbounded subsequence

$$x_{n,k} := \begin{cases} n, & \text{if } n = k, \\ -n, & \text{if } n = k + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.7)$$
**Example 2.17.** For an example of a definite divergent sequence take \( x_{n,k} = n \) for each \( n \) and \( k \); then it is also clear that \( x \) contains an unbounded subsequence.

The following propositions are easily verified.

**Proposition 2.18.** If \( x = [x_{n,k}] \) is \( P \)-convergent to \( L \) then \( x \) cannot converge to a limit \( M \), where \( M \neq L \).

**Proposition 2.19.** If \( x = [x_{n,k}] \) is \( P \)-convergent to \( L \), then any subsequence of \( x \) is also \( P \)-convergent to \( L \).

**Remark 2.20.** For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the two-dimensional plane, as illustrated by the following example.

**Example 2.21.** The sequence

\[
x_{n,k} := \begin{cases} 
1, & \text{if } n = k = 0, \\
1, & \text{if } n = 0, k = 1, \\
1, & \text{if } n = 1, k = 0, \\
0, & \text{otherwise}
\end{cases} \tag{2.8}
\]

contains only two subsequences, namely, \([y_{n,k}] = 0 \) for each \( n \) and \( k \), and

\[
z_{n,k} := \begin{cases} 
1, & \text{if } n = k = 0, \\
0, & \text{otherwise};
\end{cases} \tag{2.9}
\]

neither subsequences is \( x \).

The following propositions are easily verified.

**Proposition 2.22.** If every subsequence of \( x = [x_{k,l}] \) is \( P \)-convergent, then \( x \) is \( P \)-convergent.

**Proposition 2.23.** The double sequence \( x \) is \( P \)-convergent to \( L \) if and only if every subsequence of \( x \) is \( P \)-convergent to \( L \).

**Definition 2.24.** The double sequence \( y \) contains an \( \epsilon \)-Pringsheim-copy of \( x \) provided that \( y \) contains a subsequence \( y_{n_i,k_j} \) such that \( |y_{n_i,k_j} - x_{i,j}| < \epsilon_{i,j} \), for \( i,j = 1,2,\ldots \).

**Example 2.25.** Let

\[
x_{n,k} := \begin{cases} 
(-1)^n, & \text{if } k = n, \\
0, & \text{otherwise},
\end{cases} \tag{2.10}
\]

and let \( P\)-\( \lim_{n,k} \epsilon_{n,k} = 0 \) with

\[
y_{n,k} := \begin{cases} 
(-1)^n, & \text{if } k = n, \\
\epsilon_{n,k}, & \text{otherwise}. \tag{2.11}
\end{cases}
\]

Observe that, not only does \( y \) contain an \( \epsilon \)-Pringsheim-copy of \( x \), but \( y \) itself is an \( \epsilon \)-Pringsheim-copy of \( x \).
**Definition 2.26.** The double sequence $y$ is a *stretching* of $x$ provided that there exist two increasing index sequences $\{R_i\}_{i=0}^\infty$ and $\{S_j\}_{j=0}^\infty$ of integers such that

$$y_{n,k} := \begin{cases} R_0 = S_0 = 1, & \text{if } R_{i-1} \leq k < R_i, \\ x_{n,i}, & \text{if } S_{j-1} \leq n < S_j, \\ x_{j,k}, & i, j = 1, 2, \ldots. \end{cases} \quad (2.12)$$

**Remark 2.27.** This definition demonstrates the procedure which is used to construct a stretching of a double sequence $x$. This procedure uses a sequence of stages to construct the stretching of $x$. These stages are constructed using a sequence of abutting rows and columns of $x$. These rows and columns are constructed as follows.

**Stage 1.** Begin by repeating the first row of $x$ $R_1$ times and denote the resulting double sequence by $y^{1,0}$ then repeat the first column of $y^{1,0}$ $S_1$ times resulting in $y^{1,1}$.

**Stage 2.** Begin by repeating the $R_1 + 1$ row of $y^{1,1}$, $R_2 - R_1$ times which yields $y^{2,1}$ then repeat the $S_1 + 1$ column of $y^{2,1}$, $S_2 - S_1$ times which yields $y^{2,2}$.

$$\vdots$$

**Stage $i$.** Begin by repeating the $1 + \sum_{p=1}^{i-1} R_p$ row of $y^{i-1,i-1}$, $R_i - R_{i-1}$ times which yields $y^{i,i-1}$ then repeat the $1 + \sum_{q=1}^{i-1} S_q$ column of $y^{i-1,i-1}$, $S_i - S_{i-1}$ times which yields $y^{i,i}$. Note that in each stage we repeat the number of rows and then repeat the number of columns. However the resulting stretching $y$ of $x$ is the same, if we first repeat the number of columns and then repeat the numbers of rows. Also note that every sequence itself is a stretching of itself and the sequences that induce this kind of stretching are $R_i = i$ and $S_j = j$.

**Example 2.28.** The sequence

$$
\begin{array}{cccccccccccc}
& x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\
& x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\
& x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\
& x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\
& x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\
& x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\
& x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\
& x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\
& x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
$$

is a stretching of $x$ induced by $R_i = 3i$ and $S_j = 3j$.

3. **Main results.** The following theorem is given its name because of its similarity to the copy theorem of Dawson in [1].

**Theorem 3.1** (extended copy theorem). If each of $A$ and $T$ is an RH-regular matrix, and $x$ is any bounded double complex sequence with $\epsilon$ being any bounded positive term
double sequence with $P$-$\lim_{i,j} \epsilon_{i,j} = 0$, then there exists a stretching $\gamma$ of $x$ such that $T(Ay)$ exists and contains an $\epsilon$-Pringsheim-copy of $x$.

**Proof.** We begin by introducing a few notations which are used only in this proof. Let

\[ ||A|| := \sup_{m,n>\delta} \left( \sum_{k,l} |a_{m,n,k,l}| \right) < K_A, \quad ||T|| := \sup_{m,n>\delta} \left( \sum_{k,l} |t_{m,n,k,l}| \right) < K_T, \]

\[ M_{i,j} := 1 + \sum_{k,l=1}^{i,j} |x_{k,l}|, \quad \delta_{i,j} := \min_{i,j} \left\{ \frac{\epsilon_{i,j}}{1} \leq k \leq i \cup 1 \leq l \leq j \right\}, \]

\[ K := K_A + K_T + \max_{i,j} \left\{ \frac{\epsilon_{i,j}}{1} \leq k \leq i \cup 1 \leq l \leq j \right\} + 1, \quad Q_{i,j} := K M_{i,j} + 1, \]

\[ c_{i,j}(r,s) := \left\{ \frac{(k,l)}{1} \leq k \leq r_i \cup 1 \leq l \leq s_j \right\}, \]

\[ \tilde{c}_{i,j}(r,s) := \left\{ \frac{(k,l)}{r_i} \leq k < \infty \cup s_j \leq l < \infty \right\}, \quad \tilde{b}_{i,j}(r,s) := c_{i,j}(r,s) \setminus c_{i-1,j-1}(r,s). \]

Then by (RH$_2$) there exist $m_{\alpha_1}$ and $n_{\beta_1}$ such that for $m > m_{\alpha_1} > \tilde{B}$ and $n > n_{\beta_1} > \tilde{B}$, where $\tilde{B}$ is defined by the sixth RH-condition,

\[ \left| \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l} - 1 \right| < \frac{\delta_{\alpha_1,\beta_1}}{16 Q_{\alpha_1,\beta_1}}. \]

(3.2)

Also by (RH$_1$) and (RH$_2$) there exist $a_{\alpha_1}$ and $b_{\beta_1}$ such that

\[ \sum_{(k,l) \in c_{\alpha_1,\beta_1}(m,n)} |t_{a_{\alpha_1},b_{\beta_1},k,l}| < \frac{\delta_{\alpha_1,\beta_1}}{8 Q_{\alpha_1,\beta_1}}, \quad \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l} - 1 \right| < \frac{\delta_{\alpha_1,\beta_1}}{8 Q_{\alpha_1,\beta_1}}. \]

(3.3)

In addition, there exist $m_{\alpha_1}, n_{\beta_1}, \alpha_2$, and $\beta_2$ such that if $1 \leq \psi \leq a_{\alpha_1}$ and $1 \leq \omega \leq b_{\beta_1}$, then

\[ \sum_{(k,l) \in \tilde{c}_{\alpha_1,\beta_1}(m,n)} |t_{\psi,\omega,k,l}| < \frac{\delta_{\alpha_1,\beta_1}}{16 Q_{\alpha_2,\beta_2}}. \]

(3.4)

Also, there exist $r_{\alpha_1} > 1$ and $s_{\beta_1} > 1$ such that if $1 \leq m \leq m_{\alpha_1}$ and $1 \leq n \leq n_{\beta_1}$ then

\[ \sum_{(k,l) \in \tilde{c}_{\alpha_1,\beta_1}(r,s)} |a_{m,n,k,l}| \leq \frac{\delta_{\alpha_1,\beta_1}}{16 Q_{\alpha_2,\beta_2}}. \]

(3.5)

Now, without loss of generality, we set $\alpha_p = p$ and $\beta_q = q$. Having chosen

\[ \left\{ m_p, \tilde{m}_p, a_p, r_p \right\}^{i-1,j-1}_{p=0,q=0}, \quad \left\{ n_q, \tilde{n}_q, b_q, s_q \right\}^{i-1,j-1}_{p=0,q=0} \]

with $m_0 = n_0 = \tilde{m}_0 = \tilde{n}_0 = a_0 = b_0 = r_0 = s_0 = 1$, now choose $m_i > \tilde{m}_{i-1}$ and $n_j > \tilde{n}_{j-1}$ such that if $m > m_i$ and $n > n_j$ then

\[ \sum_{(k,l) \in \tilde{c}_{i-1,j-1}(r,s)} |a_{m,n,k,l} - 1| < \frac{\delta_{i,j}}{16 Q_{i,j} 2^{i+j}}. \]

(3.7)
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\[
\sum_{(k,l) \in c_{i-1,j-1}(r,s)} |a_{m,n,k,l}| \leq \frac{\delta_{i,j}}{8Q_{i-1,j-1}2^{i+j}}. \tag{3.8}
\]

Also choose \(a_i > a_{i-1}\) and \(b_j > b_{j-1}\) such that

\[
\sum_{(k,l) \in c_{i,j}(m,n)} |t_{a_i,b_j,k,l}| \leq \frac{\delta_{i,j}}{8Q_{i,j}}, \quad \sum_{(k,l) \in c_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 < \frac{\delta_{i,j}}{8Q_{i,j}}. \tag{3.9}
\]

Next choose \(\bar{m}_i > m_i\) and \(\bar{n}_j > n_j\) such that if \(1 \leq \psi \leq a_i\) and \(1 \leq \omega \leq b_j\) then

\[
\sum_{(k,l) \in \bar{c}_{i,j}(\bar{m},\bar{n})} |t_{\psi,\omega,k,l}| < \delta_{i,j} 8Q_{i,j}, \quad \left| \sum_{(k,l) \in \bar{c}_{i,j}(\bar{m},\bar{n})} t_{\psi,\omega,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}}. \tag{3.10}
\]

Then choose \(r_i > r_{i-1}\) and \(s_j > s_{j-1}\) such that if \(1 \leq m \leq \bar{m}_i\) and \(1 \leq n \leq \bar{n}_j\) then

\[
\sum_{(k,l) \in \bar{c}_{i,j}(r,s)} \left| a_{m,n,k,l} \right| < \frac{\delta_{i,j}}{2^{2+i+j}Q_{i+1,j+1}}, \tag{3.11}
\]

where \(m_i, n_j, \bar{m}_i, \bar{n}_j, r_i, \) and \(s_j\) are chosen using \((RH_1), (RH_2), (RH_3),\) and \((RH_4)\) such that if \(1 \leq p \leq j-1\) and \(1 \leq q \leq i-1\) the following is obtained:

\[
\left| \sum_{(k,l) \in \bar{c}_{i,j}(r,s)} a_{m,n,k,l} \right| \leq \frac{\delta_{p,j}}{8Q_{p,j}2^{p+j}}, \quad \left| \sum_{(k,l) \in \bar{c}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| \leq \frac{\delta_{i,q}}{8Q_{i,q}2^{i+q}}. \tag{3.12}
\]

Therefore by (3.9) and (3.10) we have

\[
\left| \sum_{(k,l) \in c_{i,j}(\bar{m},\bar{n}) \setminus c_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 \right| \leq \frac{\delta_{i,j}}{4Q_{i,j}}, \tag{3.13}
\]

and by (3.7), (3.8), and (3.11) we also obtain

\[
\left| \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}2^{i+j}}, \tag{3.14}
\]

where \(m_i \leq m \leq \bar{m}_i\) and \(n_j \leq n \leq \bar{n}_j\). Let \(\{y_{k,l}\}\) be the stretching of \(x\) induced by \(\{r_i\}\) and \(\{s_j\}\). Since

\[
(Ay)_{m,n} - x_{i,j} = \sum_{k,l=1}^{r_{i-1},s_{j-1}-1} a_{m,n,k,l}y_{k,l} + \sum_{(k,l) \in \bar{b}_{i,j}(r,s)} a_{m,n,k,l}y_{k,l} - x_{i,j} \tag{3.15}
\]

if \(i,j > 1,\) with \(m_i \leq m \leq \bar{m}_i\) and \(n_j \leq n \leq \bar{n}_j\) the following is obtained:

\[
\left| \sum_{k,l=1}^{r_{i-1},s_{j-1}-1} a_{m,n,k,l}y_{k,l} \right| \leq \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\} \sum_{k,l=1}^{r_{i-1},s_{j-1}-1} |a_{m,n,k,l}y_{k,l}|. \tag{3.16}
\]
By (3.8),
\[
\left| \sum_{k,l=1}^{i-1,j-1} a_{m,n,k,l} y_{k,l} \right| \leq \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\} \frac{\delta_{i,j}}{8 Q_{i-1,j-1}}. \tag{3.17}
\]

Since
\[
Q_{i-1,j-1} = K \left( 1 + \sum_{k,l=1}^{i-1,j-1} |x_{k,l}| \right) + 1 \geq K \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\}, \tag{3.18}
\]
the following holds:
\[
\left| \sum_{k,l=1}^{i-1,j-1} a_{m,n,k,l} y_{k,l} \right| \leq \frac{\delta_{i,j}}{8K}, \tag{3.19}
\]
the following also is obtained:
\[
\left| \sum_{p,q=i+1,j+1}^{\infty} \sum_{(k,l) \in \bar{B}_{p,q}(r,s)} a_{m,n,k,l} y_{k,l} \right| \leq \sum_{p,q=i+1,j+1}^{\infty} |x_{p,q}| \sum_{(k,l) \in \bar{B}_{p,q}(r,s)} |a_{m,n,k,l}| \leq \frac{\delta_{i,j}}{24K} \sum_{p,q=i+1,j+1}^{\infty} \frac{1}{2^{p+q}} \leq \frac{\delta_{i,j}}{8K}, \tag{3.20}
\]
because
\[
\sum_{k,l=r_{p,q}}^{\infty} |a_{m,n,k,l}| \leq \frac{\delta_{p-1,q-1}}{24+p+q Q_{p,q}}, \frac{|x_{p,q}|}{Q_{p,q}} < \frac{1}{K}. \tag{3.21}
\]
Therefore by (3.11),
\[
\left| \sum_{(k,l) \in \bar{B}_{i,j}(r,s)} a_{m,n,k,l} y_{k,l} - x_{i,j} \right| \leq \sum_{q=1}^{i-1} |x_{i,q}| \sum_{(k,l) \in \bar{B}_{i,q}(r,s)} a_{m,n,k,l} + \sum_{p=1}^{j-1} |x_{p,j}| \sum_{(k,l) \in \bar{B}_{p,j}(r,s)} a_{m,n,k,l} + |x_{i,j}| \sum_{(k,l) \in \bar{B}_{i,j}(r,s)} a_{m,n,k,l-1} \leq \sum_{p,q=1}^{i,j} |x_{i,j}| \frac{\delta_{p,q}}{2^{p+q+1}} \leq \frac{\delta_{i,j}}{K8} \sum_{p,q=1}^{i,j} \frac{1}{2^{p+q}} = \frac{\delta_{i,j}}{K^2}. \tag{3.22}
\]
Therefore,
\[
| (Ay)_{m,n} - x_{i,j} | \leq \frac{\delta_{i,j}}{K8} + \frac{\delta_{i,j}}{K4} + \frac{\delta_{i,j}}{K2} < \frac{\delta_{i,j}}{2K}. \tag{3.23}
\]
Note that the inequality (3.23) is true for \( m_1 \leq m \leq \tilde{m}_1 \) and \( n_1 \leq n \leq \tilde{n}_1 \), and also this inequality is true for \( i, j \geq 1 \) with \( m_i \leq m \leq \tilde{m}_i \) and \( n_j \leq n \leq \tilde{n}_j \). Hence

\[
(A \gamma)_{m,n} = x_{i,j} + u_{i,j},
\]

(3.24)

where \( |u_{i,j}| \leq \delta_{i,j}/2K \). Note that if \( \bar{m}_{i-1} \leq m \leq m_i \) and \( \bar{n}_{j-1} \leq n \leq n_j \), then the following is obtained:

\[
\left| (A \gamma)_{m,n} \right| \leq \sum_{k,l=1}^{r_i-1,s_j-1} a_{m,n,k,l} y_{k,l} + \sum_{p,q=i+1,j+1}^{\infty} \sum_{k,l=1}^{\tilde{m}_i,\tilde{n}_j} a_{m,n,k,l} y_{k,l}
\]

\[
\leq \max \left\{ \frac{|x_{i,l}|}{1} \leq k \leq i \leq 1 \leq j \leq f \right\} \sum_{k,l=1}^{r_i-1,s_j-1} |a_{m,n,k,l}|
\]

\[
+ \sum_{p,q=i+1,j+1}^{\infty} |x_{k,l}| \sum_{k,l=1}^{\tilde{m}_i,\tilde{n}_j} |a_{m,n,k,l}|
\]

\[
\leq K m_{i,j} + \sum_{p,q=i+1,j+1}^{\infty} \frac{|x_{k,l}|}{2p+d} Q_{p+1,q+1} \frac{\delta_{p,q}}{4}
\]

\[
\leq K m_{i,j} + \frac{\delta_{i,j}}{4} + K m_{i,j} \leq K m_{i,j} + 1 = Q_{i,j}.
\]

(3.25)

Also, if \( m_{i-1} \leq m \leq m_i \) and \( n_{j-1} \leq n \leq n_j \) then

\[
\left| \sum_{k,l=1}^{\infty} a_{m,n,k,l} y_{k,l} \right| \leq \left| (A \gamma)_{m,n} - x_{i,j} \right| + |x_{i,j}|
\]

\[
\leq \frac{\delta_{i,j}}{2K} + K m_{i,j} \leq K m_{i,j} + 1 = Q_{i,j}.
\]

(3.26)

By using (3.25) we now show the existence of \( T(A \gamma) \). If \( a_{i-1} < m \leq a_i \) and \( b_{j-1} < n \leq b_j \) then

\[
\left| \sum_{k,l=\tilde{m}_i+1,\tilde{n}_j+1}^{\infty} t_{m,n,k,l} (A \gamma)_{k,l} \right| \leq \sum_{r,s=i,j}^{\infty} \sum_{(p,q) \in \bar{F}_{r+1,s+1}(m,n)} |t_{m,n,p,q} (A \gamma)_{p,q}|
\]

\[
\leq \sum_{r,s=i,j}^{\infty} Q_{r+1,s+1} \sum_{(p,q) \in \bar{F}_{r+1,s+1}(m,n)} |t_{m,n,p,q}|
\]

\[
\leq \sum_{r,s=i,j}^{\infty} Q_{r+1,s+1} \sum_{(p,q) \in \bar{F}_{r+1,s+1}(m,n)} \delta_{r,s} \frac{1}{2^{r+s} Q_{r+1,s+1}^{2+r+s}}
\]

\[
\leq \frac{\delta_{i,j}}{4} \sum_{r,s=1}^{\infty} \frac{1}{2^{r+s}} \leq \frac{\delta_{i,j}}{4}.
\]

(3.27)
Therefore $T(Ay)$ exists. Also, by (3.25) we now show that $T(Ay)$ contains an $\epsilon$-Pringsheim-copy of $x$. First note that

$$\left| \sum_{k,l=1}^{\infty,\infty} t_{a_i,b_j,k,l}(Ay)_{k,l} - x_{i,j} \right| \leq \sum_{k,l=1}^{m_i-1,n_j-1} |t_{a_i,b_j,k,l}(Ay)_{k,l}|$$

$$+ \left| \sum_{(k,l) \in \hat{b}_{i,j}(r,s)} t_{a_i,b_j,k,l}(Ay)_{k,l} - x_{i,j} \right|$$

$$+ \sum_{k,l=\hat{m}_i+1,\hat{n}_j+1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l},$$

with

$$\sum_{k,l=1}^{m_i-1,n_j-1} |t_{a_i,b_j,k,l}(Ay)_{k,l}| = \sum_{k,l=1}^{m_i-1,n_j-1} |t_{a_i,b_j,k,l}| Q_{i,j} \leq Q_{i,j} \frac{\delta_{i,j}}{8Q_{i,j}} \leq \frac{\delta_{i,j}}{8}, \quad (3.29)$$

$$\left| \sum_{(k,l) \in \hat{b}_{i,j}(r,s)} t_{a_i,b_j,k,l}(Ay)_{k,l} - x_{i,j} \right| = \left| \sum_{(k,l) \in \hat{b}_{i,j}(r,s)} t_{a_i,b_j,k,l}(x_{i,j} + u_{i,j}) - x_{i,j} \right|$$

$$\leq |x_{i,j}| \sum_{(k,l) \in \hat{b}_{i,j}(r,s)} |t_{a_i,b_j,k,l} - 1|$$

$$+ \sum_{(k,l) \in \hat{b}_{i,j}(r,s)} |t_{a_i,b_j,k,l} u_{i,j}|$$

$$\leq \frac{|x_{i,j}| \delta_{i,j}}{4Q_{i,j}} + \frac{\delta_{i,j}}{4K} \sum_{(k,l) \in \hat{b}_{i,j}(r,s)} |t_{a_i,b_j,k,l}|$$

$$\leq \frac{\delta_{i,j}}{2}, \quad (3.30)$$

$$\left| \sum_{k,l=\hat{m}_i+1,\hat{n}_j+1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l} \right| \leq \sum_{r,s=i,j}^{\infty,\infty} \sum_{(p,q) \in \hat{b}_{r+1,s+1}(\hat{m},\hat{n})} |t_{a_i,b_j,p,q}(Ay)_{p,q}|$$

$$\leq \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1} \sum_{(p,q) \in \hat{b}_{r+1,s+1}(m,n)} |t_{a_i,b_j,p,q}|$$

$$\leq \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1} \frac{\delta_{r,s}}{2^{2s+r+1}Q_{r+1,s+1}} \leq \frac{\delta_{i,j}}{4}, \quad (3.31)$$

Hence,

$$\left| \sum_{k,l=1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l} - x_{i,j} \right| \leq \frac{\delta_{i,j}}{4} + \frac{\delta_{i,j}}{8} < \delta_{i,j} \leq \epsilon_{i,j}, \quad (3.32)$$

This completes the proof of the extended copy theorem.
The next two results are immediate corollaries of the extended copy theorem.

**Corollary 3.2.** If $T$ is any RH-regular matrix summability method and $A$ is an RH-regular matrix such that $Ay$ is $T$-summable for every stretching $y$ of $x$, then $x$ is $P$-convergent.

**Corollary 3.3.** If $T$ is any RH-regular matrix summability method and $A$ is an RH-regular matrix such that $Ay$ is absolutely $T$-summable for every stretching $y$ of $x$, then $x$ is $P$-convergent.

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