Estimating the size of the scatterer

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Abstract

Formula for the size of the scatterer is derived explicitly in terms of the scattering amplitude corresponding to this scatterer. By the scatterer either a bounded obstacle $D$ or the support of the compactly supported potential is meant.

1 Introduction

Let $D$ be a bounded $C^2$–smooth connected domain in $\mathbb{R}^3$, $S$ be its boundary, $N$ is the outer unit normal to $S$, $u_N$ is the normal derivative of $u$ on $S$, $S^2$ be the unit sphere in $\mathbb{R}^3$, $D' := \mathbb{R}^3 \setminus D$, $k = const > 0$, $\alpha, \beta \in S^2$, $v$ be the scattered field, $g = g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}; \{e_j\}_{j=1}^3$ be an orthonormal basis in $\mathbb{R}^3$. We drop the $k$–dependence since $k$ is fixed.

Consider the scattering problem

$$\nabla^2 u + k^2 u = 0 \quad \text{in} \quad D', \quad u|_S = 0, \quad u = u_0 + v, \quad (1)$$

where $u_0 := e^{ik\alpha \cdot x}$ is the incident field, and $v$, the scattered field, satisfies the radiation condition

$$v_r - ikv = O(r^{-2}), \quad r = |x| \to \infty, \quad x/r = \beta, \quad (2)$$

uniformly with respect to $\beta$. Problem (1)–(2) is the obstacle scattering problem, $D$ is the obstacle, the scatterer, $u = u(x) := u(x, \alpha)$ is the scattering solution.

It is known (see, for example, [2], [3]) that problem (1)–(2) has a solution and this solution $u$ is unique.

By the Green’s formula one gets

$$u = u_0(x, \alpha) - \int_S g(x,s)h(s)ds, \quad h := u_N(s, \alpha). \quad (3)$$

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Let \( r \to \infty, x/r = \beta \) in (3). Then
\[
  u = u_0 + A(\beta, \alpha) \frac{e^{ikr}}{r} + O(r^{-2}), \quad r \to \infty, \quad \beta = x/r,
\]
where
\[
  A(\beta, \alpha) = -\frac{1}{4\pi} \int_S e^{-ik\beta \cdot s} h(s, \alpha) ds.
\]
The properties of \( A(\beta, \alpha) \) are studied, for example, in [2], [3].

It was proved in [2], p.62, that \( A(\beta, \alpha) \) is an analytic function of \( \beta \) and \( \alpha \) in the algebraic variety \( M \subset \mathbb{C}^3 \) defined by the equation \( z \cdot z = 1, z \in \mathbb{C}^3 \), where \( z \cdot z := \sum_{j=1}^{3} z_j^2, z_j \in \mathbb{C}^1 \).

Indeed, from (5) it follows that \( A(\beta, \alpha) \) is an analytic function of \( \beta \) on the variety \( M \). This function is originally defined on \( S^2 \). For small \( p > 0 \) it is defined for \( |\beta_1| \leq p, |\beta_2| \leq p \) and \( \beta_3 = (1 - \beta_1^2 - \beta_2^2)^{1/2} \). The scattering amplitude \( A(\beta, \alpha) \) admits a unique analytic continuation to the algebraic variety \( M \). Therefore, one can take \( \beta \) in formula (5), for example, as 
\[
  \beta = ae_1 + ibe_2, \quad a, b \in \mathbb{R}, \quad a^2 - b^2 = 1.
\]
This \( \beta \) clearly belongs to \( M \).

Let us call a plane \( P \) supporting \( D \) at a point \( s \) if \( s \in D \cap P \) and \( D \) is contained in one of the two half-spaces bounded by \( P \). A plane is supporting \( D \) if it is supporting \( D \) at some point. The distance \( d \) between two parallel planes supporting \( D \) we call the size of \( D \) in the direction of the normal to these planes.

Our first result is the following theorem.

**Theorem 1.** The size of \( D \) in the direction \( e_2 \) is
\[
  d \leq \lim_{b \to \infty} \frac{\ln |A(\beta_1 + ibe_2, \alpha)|}{bk},
\]
where \( a^2 - b^2 = 1 \).

Consider now the potential scattering:
\[
  [\nabla^2 + k^2 - q(x)] \psi = 0 \quad \text{in} \quad \mathbb{R}^3,
\]
where \( \psi \) satisfies the radiation condition (2), the potential \( q \) is a real-valued compactly supported function with support \( D, q \in L^2(D) \). It is known that \( \psi \) satisfies relation (4) with
\[
  A(\beta, \alpha) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot y} H(y) dy, \quad H(y) = H(y, \alpha) := q(y) \psi(y, \alpha).
\]
The function \( H \) does not depend on \( \beta \). One has
\[
  \max_{y \in D, \alpha \in S^2} |H(y, \alpha)| \leq c,
\]
where \( c > 0 \) is a constant.

Since \( S \) is \( C^2 \)--smooth, any section of \( S \) by a plane is a curve with a bounded length.

Our second result is:

**Theorem 2.** Formula (7) holds for the estimate of the support \( D \) of \( q \) in the direction \( e_2 \).
2 Proofs

**Proof of Theorem 1.** Using formula (5) with $\beta$ from (6), one gets

$$|A(ae_1 + ibe_2, \alpha)| \leq c \int_S e^{bks_2} ds, \quad c = \frac{1}{4\pi} \max_{s \in S} |u_N(s, \alpha)|. \quad (11)$$

The conclusion of Theorem 1 follows from (11) and Lemma 1.

**Lemma 1.** One has

$$\lim_{b \to \infty} \frac{\ln \int_S e^{bks_2} ds}{bk} = d, \quad (12)$$

where $d$ is the size of $D$ in the direction $e_2$.

Theorem 1 is proved. $\square$

**Proof of Lemma 1.** One has

$$\int_S e^{bks_2} ds = \int_{S : s_2 \geq d - \epsilon} e^{bks_2} ds + O(e^{bk(d - \epsilon)}), \quad b \to \infty. \quad (13)$$

We assume that $S$ is $C^2$--smooth. Therefore the section of $S$ by any plane $y = const$ is a curve that has a finite length $\ell(y)$, $\ell(y) = cy^\gamma$, where $\gamma > 0$ depends on the smoothness of $S$ at the supporting point. In the coordinate system that we use one of the planes, supporting $D$, has the equation $y = 0$ and the other plane, supporting $D$, is $y = d$. If one changes variable $y$ setting $d - y = \eta$, then

$$J := \int_{S : s_2 \geq d - \epsilon} e^{bks_2} ds = \int_{d - \epsilon}^d e^{bky} \ell(y) dy = e^{bkd} \int_0^\epsilon \eta e^{-bk\eta} c(d - \eta)^\gamma. \quad (14)$$

Thus

$$\lim_{b \to \infty} \frac{\ln J}{bk} = d. \quad (15)$$

Lemma 1 is proved. $\square$

**Proof of Theorem 2** is essentially the same as the proof of Theorem 1.

3 Remarks

1. We have chosen $\beta$ by formula (6), but we can choose it so that $\text{Im} \beta$ is directed along a given unit vector from $S^2$.

2. The symmetry of the scatterer can influence the functional form of the scattering amplitude.

   In [4] it is proved that if and only if the scatterer is spherically symmetric its scattering amplitude is a function of $\beta \cdot \alpha$, $A = A(\beta \cdot \alpha)$. Therefore, when the scatterer is spherically symmetric $\alpha$ should not be chosen orthogonal to the vector that is parallel to $\text{Im} \beta$.

3. In quantum mechanics the scattering amplitude corresponding to a spherically symmetric compactly supported potential is calculated as a series [1]. This series is less convenient for passing to the limit $b \to \infty$ than formula (9).
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