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Cost allocation rule of minimum spanning tree problems without a supplier

Jia-quan Zhan, Hao Cheng and Zhen-sen Zhang

School of Management Engineering, Qingdao University of Technology, Qingdao, People’s Republic of China

ABSTRACT
In this paper, the cost allocation problem of minimum spanning tree without a supplier has been discussed. By assigning nodes to play the role of supplier in turn, we can get n classic m.c.s.t.s. A cooperative cost game model is established whose cost function is defined on cost functions of the n classic m.c.s.t.s. We show that the cooperative game is totally balanced and it has a large core that includes each irreducible core of the classic m.c.s.t. mentioned above. As for a specific core allocation, the arithmetic mean of all Bird imputations is recommended.

1. Introduction
In a typical minimum cost spanning tree (m.c.s.t.) problem, there exists a node named supplier who does not bear any costs. Solving the problem usually involves two steps: first, find an m.c.s.t., and then allocate the costs among all the nodes except the supplier. An m.c.s.t is usually obtained by the circle avoidance method or the circle broken method (Kruskal, 1956). Bird’s irreducible core is the most common method of cost allocation (Bird, 1976). After Bird’s pioneering research, the m.c.s.t. problem has attracted the attention of many scholars. Norde, Moretti, and Tjjs (2004) proposed an axiomatic method for Bird’s imputation. Bergantiños and Vidal-Puga (2015) provided a general framework to identify the family of rules satisfying monotonicity over cost and population, and they proved that the set of allocations induced by the family coincides with the so-called irreducible core. M.c.s.t game models are widely applied for cost sharing and/or earnings sharing in capacity networks (Bogomolnaia, Holzman, & Moulin, 2010), supply chain cooperation (Drechsel, 2010; Granot, Granot, & Sosic, 2014), combinatorial optimization problems (Moulin, 2013) and proceeds distribution in hierarchical venture (Hougaard, Moreno-Ternero, Tvede, & Østerdal, 2017) and many other fields. The model is particularly widely used for cost allocation in transportation networks, i.e. cost allocation in a rapid-transit network (Rosenthal, 2017), route cost assignment problem consisting of both operators’ and users’ decisions (Rasulkhani & Chow, 2019). The classic m.c.s.t. model provides the basic framework for the above studies, and some of them are the natural extensions of the classic model in various fields. However, not in all m.c.s.t. scenarios there exists such a supplier. There does exist a nature supplier in a power transmission system or an oil pipeline network. But in a communication system or traffic network, all nodes may be the same in essence, there does not necessarily exist a supplier in these networks.

This paper intends to modify the cost function of the m.c.s.t. game based on the work of Bird to make it suitable for the network without a supplier. In the next section, we will recall some preliminaries of trees and cooperative games; in the third section we will present the modified characteristic function and prove the existence of the core; in the fourth section an illustrative example will be elaborated; finally, there is a summary and outlook of the future work.

2. Some preliminaries
A graph is a tuple \( G = (V, E) \) where \( V \) denotes the set of vertexes and \( E \) denotes the set of edges. A graph is connected if there exists at least one chain between any two vertexes. Removing some edges and/or some vertexes of graph \( G \), the remaining graph is said to be a subgraph of graph \( G \). If the vertex set of the subgraph is the same as the vertex set of the original graph, then this subgraph is said to be a spanning subgraph of graph \( G \). Sometimes a graph is not enough to accurately describe the relationship between subjects in an actual problem. A network or weighed graph is a graph whose vertexes or edges have some certain indicators. The indicator is usually called ‘weight’ that can represent such things as distance, cost, capacity, and so on.
A tree is a connected and acyclic graph. If a spanning subgraph of graph $G$ is a tree, then this tree is said to be a spanning tree of graph $G$. Given a connected graph $G = (V, E)$ where each edge has a nonnegative weight, a spanning tree of graph $G$ is said to be a minimum spanning tree of graph $G$ if it has the minimum weight among all spanning trees.

**Kruskal algorithm:** Select an edge satisfying the following two conditions: first, it must not form a circle with any edges selected before; second, this edge must be the shortest one in the unselected edges satisfying condition 1.

Note that the minimum spanning tree of a graph is not necessarily unique, but all the minimum spanning trees of a graph have the same weight.

A cooperative game of characteristic function is a pair $(N, v)$ where $N$ is a coalition and $v$ is a function that associates a real number $v(S)$ with each subset $S$ of $N$. We always assume that $v(\emptyset) = 0$. The number $v(S)$ is called the value of $S$. A game $(N, v)$ is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ and $S \cap T = \emptyset$.

Superadditivity is satisfied in most of the applications of cooperative games. The above definition of cooperative game and superadditivity is from the perspective of income. If we define a cooperative from the perspective of cost, then $v(S)$ is replaced by $c(S)$ that represents the least cost of serving the members of $S$ by the most efficient means. Then a cost game $(N, c)$ is subadditive if $c(S \cup T) \leq c(S) + c(T)$ for all $S, T \subseteq N$ and $S \cap T = \emptyset$. That is, the cost of two coalitions cooperating to complete a task is less than the sum of the costs of two coalitions completing tasks independently. In applications, cost games are usually subadditive.

Let $(N, v)$ be a game. We denote $X^*(N, v) = \{ x \in \mathbb{R}^N \mid \forall S \subseteq N, x(S) \geq v(S) \}$. The set $X^*(N, v)$ is the set of feasible payoff vectors for the game $(N, v)$. Core is one of the most important solution concepts in cooperative game theory.

**Definition 2.1:** The core of a game $(N, v)$, denoted by $C(N, v)$, is defined by

$$C(N, v) = \{ x \in X^*(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N \}.$$  

Obviously, if $x \in C(N, v)$, no coalition can improve upon $x$. Thus, each member of the core is a highly stable payoff distribution. Let $S$ be a singleton $\{i\}$, the grand coalition $N$ respectively, it’s easy to find that a core imputation is Pareto optimal and individually rational.

If a game has a non-empty core, we say that the game is balanced. To restrict a game $(N, v)$ on a subset of $N$, we get a subgame of $(N, v)$. If all subgames of $(N, v)$ are balanced, we say that the game $(N, v)$ is totally balanced.

Let $N$ be a nonempty finite set of customers, let $0 \notin N$ be a supplier, let $N^* = N \cup \{0\}$, and let $c_{ij} \in p$ be the cost of connecting $i,j \in N^*$, $i \neq j$, by the edge $e_{ij}$. The cost game $(N^*, c)$ in which $c(S)$ represents the minimum cost of coalition $S$ connecting to the supplier is called an m.c.s.t. game since $c(S)$ is just the weight of the minimum spanning tree connecting $S \cup \{0\}$.

In a minimum spanning tree of the graph $(V, E)$ where $V = N^*$ and $E = \{(i,j) \mid (n+1)x(n+1), i \neq j\}$, there is a unique road $T_i$ from the supplier to a given node (customer) $i$. In a Bird imputation, the cost that the customer $i$ has to bear is the weight of the edge associated with the node in the road $T_i$.

**Theorem 2.1:** (Bird, 1976) Every m.c.s.t. game is totally balanced and each Bird imputation is a core allocation.

The core of an m.c.s.t. game is denoted by $C(G, c)$.

**Example 2.1:** Consider the following m.c.s.t. problem consisting of two nodes and one supplier. The m.c.s.t. is made up of bold lines. Clearly, in this m.c.s.t. game $(G, c)$, $c(\{1\}) = 4$, $c(\{2\}) = 5$, $c(\{1, 2\}) = 7$. By Bird imputation, customer 1 should bear 4 and customer 2 should bear 3. It’s easy to verify this Bird imputation is a core allocation. Note that the core of this cost game is not a.
point. In fact, any imputation like \((4 - \alpha, 3 + \alpha)\) satisfying \(0 \leq \alpha \leq 2\) is a core allocation. The Bird imputation is a vertex of the core.

Usually the core of an m.c.s.t. game is quite large. From an m.c.s.t. problem \((G, c)\), Bird (1976) defined an irreducible m.c.s.t. problem \((G, \hat{c})\) that has exactly the same minimum spanning trees as \((G, c)\). The core of \((G, \hat{c})\), that is, \(C(G, \hat{c})\), is the convex hull of all Bird imputations, and it is much smaller than \(C(G, c)\). Usually \(C(G, \hat{c})\) is called irreducible core of the m.c.s.t. problem \((G, c)\) in Graph by \(c\) function. For a classic or ordinary m.c.s.t. problem, defining the cost function without a supplier, defining an appropriate cost function \(c\) for different minimum spanning trees to determine a same Bird imputation. Take a look at the square, there’re four different m.c.s.t.s, but there is only one unique Bird imputation.

3. A cooperative model for m.c.s.t. problem without a supplier

For a classic or ordinary m.c.s.t. problem, defining the cost function \(c(S)\) is not a problem at all. For an m.c.s.t. problem without a supplier, defining an appropriate cost function \(c(S)\) becomes a critical issue. It is not appropriate to define the cost function as the weight of the minimum spanning tree connecting \(S\). Under this definition, if \(|S| = 1\), then we have \(c(S) = 0\). This eliminates the possibility and necessity of cooperation and it does not meet the actual situation. If we define \(c(S)\) according to its purpose, that is, to connect all nodes (players) in \(N\) (not just in \(S\) together, then coalition \(S\) needs to bear the full cost in order to realize its purpose. Taking this assumption, any \(S \notin \emptyset\), \(c(S)\) equals the weight of m.c.s.t. It’s straight to find the m.c.s.t. game is symmetric in this definition, and then the most reasonable allocation, of course, is that the cost is shared equally by all nodes. The above assumption takes into account the purpose of the nodes, but does not take into account the positions of the nodes. If we designate a node to act as the supplier, then the game becomes an ordinary m.c.s.t. game. Imagine if the node 1 becomes the supplier, then \(c(S)\) is the cost of the m.c.s.t. connecting \(S \cup \{1\}\); if the node 2 becomes the supplier, then \(c(S)\) is the cost of the m.c.s.t. connecting \(S \cup \{2\}\). \(c(S)\) will of course change when the supplier changes.

Definition 3.1: The cost function \(c(S)\) of an m.c.s.t. problem \((G, c)\) without a supplier is defined by

\[
\text{c}(S) = \max\{c_i(S)\} \quad \text{for all } S \subseteq N, \text{ where } i = 1, 2, \ldots, n.
\]

Example 3.1: The costs connecting the four cities are shown in the table below.

|       | A   | B   | C   | D   |
|-------|-----|-----|-----|-----|
| A     | 0   | 2   | 3   | 8   |
| B     |     | 0   | 6   | 7   |
| C     |     |     | 0   | 1   |
| D     |     |     |     | 0   |

Obviously, if \(|S| = 1\), then \(c(S)\) equals the largest number of the row and column in which \(S\) is located. That is, \(c(A) = 8\), \(c(B) = 7\), \(c(C) = 6\) and \(c(D) = 8\).

When \(|S| \geq 2\), it is a little tedious to calculate the cost function strictly according to Definition 3.1, so we need to apply the definition flexibly.

First, we notice that the cost \(c_{ij}\) is just \(c_i(j)\) and \(c_i(j)\) that is the weight of the m.c.s.t. connecting city \(i\) and city \(j\). Then we calculate the weights of the m.c.s.t.s of all third and fourth order subgraphs of the problem. For convenience, in this example a subgraph is denoted by \(G(S)\) where \(S\) is the vertex set of the subgraph, also a coalition and a subset of \(N\). For example, \(G(ABC)\) denotes the graph \((V, E)\) where \(V = \{A, B, C\}\) and \(E\) is the set of edges connecting the vertexes.

The weight of the m.c.s.t. of \(G(ABCD)\) is 5; The weight of the m.c.s.t. of \(G(ABD)\) is 9; The weight of the m.c.s.t. of \(G(BCD)\) is 7; The weight of the m.c.s.t. of \(G(ACD)\) is 4. The weight of the m.c.s.t. of \(G(ABCD)\), i.e. the original graph, is 6.

Now consider coalition \(AB\). Note that \(c_1(AB)\) is the weight of the m.c.s.t. of \(G(ABC)\) and \(c_4(AB)\) is the weight of the m.c.s.t. of \(G(ABD)\). Therefore,

\[
c(AB) = \max\{c_1(AB), c_2(AB), c_3(AB)\}
\]

\[
c_4(AB) = \max\{2, 2, 5, 9\} = 9
\]
Consider coalition ABC. Note that $c_4(ABC)$ is just the weight of the m.c.s.t. of $G(ABC)$ and $c_1(ABC)$, $c_2(ABC)$ and $c_3(ABC)$ are the same, they’re all the weight of the m.c.s.t. of $G(ABC)$. Therefore,

$$c(ABC) = \max(c_1(ABC), c_4(ABC)) = \max(5, 6) = 6.$$ 

By the same way, $c(ABD) = 9$, $c(ACD) = 6$, $c(BCD) = 7$.

As for the grand coalition $N$, apparently, $c(ABCD)$ is the weight of the m.c.s.t. of $G(ABCD)$. Therefore,

$$c(ABCD) = 6.$$

**Theorem 3.1:** The cost game $(G, c)$ where $c(S)$ is defined by Definition 3.1 is subadditive.

**Proof:** for $S, T \subseteq N, S \cap T = \emptyset$, there is

$$c(S) \geq c_i(S), c(T) \geq c_i(T)$$

and

$$c_i(S) + c_i(T) \geq c_i(S \cup T) \text{ where } i = 1, 2, \ldots, n.$$ 

Therefore,

$$c(S) + c(T) \geq c_i(S) + c_i(T) \geq c_i(S \cup T)$$

Note that

$$c(S \cup T) = \max\{c_i(S \cup T)\}$$

Hence,

$$c(S) + c(T) \geq c(S \cup T).$$

That ends proof.

**Theorem 3.2:** The cost game $(G, c)$ is totally balanced and $C(G, c_i) \subseteq C(G, c)$ where the cost game $(G, c_i)$ is obtained by designating the node $i$ as the supplier in the m.c.s.t. problem $(G, c)$.

**Proof:** let an imputation $x \in C(G, c_i)$, there is $x(S) \leq c_i(S)$. Note that $c_i(S) \leq c(S)$, thus, $x(S) \leq c(S)$. That is, $x \in C(G, c)$. Therefore, The cost game $(G, c)$ is balanced and $C(G, c_i) \subseteq C(G, c)$.

By the same manipulation, we can prove that any sub-game of $(G, c)$ is balanced. Hence the cost game $(G, c)$ is totally balanced.

That ends proof.

As mentioned in the previous section, the Bird imputations of game $(G, c_i)$ are vertexes of the set $C(G, c_i)$. Are these Bird imputations also vertexes of the set $C(G, c)$? Unfortunately, unless $n$ is less than or equal to 3, we cannot guarantee that Bird imputations are vertexes of $C(G, c)$.

**4. An illustrate example**

In Section 3, how to calculate the cost function has been explained in detail. In fact, like ordinary m.c.s.t. problems, we don’t need to compute the cost function to allocate the cost, figuring out the minimum spanning tree(s) is enough.

In m.c.s.t. I, designating Node $A$ as the supplier, then we get that $x_A = 0, x_B = 1, x_C = 2$ and $x_D = 1$. By designating each node as the supplier by turns in the two m.c.s.t.s we can get 8 Bird imputations. They are

$$\{(0, 1, 2, 1), (0, 1, 1, 2), (1, 0, 2, 1), (1, 0, 1, 2), (1, 2, 0, 1), (2, 1, 0, 1), (1, 2, 1, 0) \text{ and } (2, 1, 1, 0)\}.$$ 

As proved above, all these Bird imputations are core allocations, that is, no coalition can reject any Bird imputation. However, the costs incurred by the same node vary greatly in different Bird imputations. Like ordinary m.c.s.t. games, the convex hull of these 8 Bird imputation is also a subset of the core $C(G, c)$, so it can be taken as an allocation. Although these Bird imputations may not be vertexes of $C(G, c)$, they are vertexes of the convex hull of themselves. Otherwise, there exists at least one Bird imputation that can be represented as a convex combination of other Bird imputations. But it’s impossible because there is a component equals to 0 in each imputation and there is only one imputation in which the component at the same position is also equal to 0. However, the convex hull is still very large.

Is there a reasonable single valued allocation? Since there isn’t a ‘natural’ supplier and the status of different minimum spanning trees is completely equal, then each Bird imputation should play an equal role in determining the solution. Hence it’s natural to take the arithmetic mean of all Bird imputations as the single valued solution. In our example, the arithmetic mean is $(1, 1, 1, 1)$. It’s not surprising since the cost game is perfectly symmetrical.

**5. Conclusion and discussion**

To solve an m.c.s.t. problem without a supplier, by assigning nodes to play the role of supplier in turn, we have
determined the cost that each coalition is willing to pay at most so that a cooperative game model have been established. The existence of the core of the game has been proven and a reasonable single valued allocation is also proposed. However, the convex hull of Bird imputations is still too large to be applied. Although the arithmetic mean of all Bird imputations seems reasonable, we have not studied it in depth. What other properties does this single valued solution have besides the properties of the core itself? Can it be characterized axiomatically? All these questions remain untouched and they need further study in the future work.

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