Topological generation results for free unitary and orthogonal groups

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Abstract

We show that for every $N \geq 3$ the free unitary group $U^+_N$ is topologically generated by its classical counterpart $U_N$ and the lower-rank $U^+_{N-1}$. This allows for a uniform inductive proof that a number of finiteness properties, known to hold for all $N \neq 3$, also hold at $N = 3$. Specifically, all discrete quantum duals $\hat{U}^+_N$ and $\hat{O}^+_N$ are residually finite, and hence also have the Kirchberg factorization property and are hyperlinear. As another consequence, $U^+_N$ are topologically generated by $U_N$ and their maximal tori $\hat{\mathbb{Z}}^* \mathbb{N}$ (dual to the free groups on $N$ generators) and similarly, $O^+_N$ are topologically generated by $O_N$ and their tori $\hat{\mathbb{Z}}^* \mathbb{N}$.

Key words: compact quantum group, free unitary group, free orthogonal group, torus, topological generation, residually finite, hyperlinear, Kirchberg factorization property

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Introduction

The compact quantum group literature has recently seen considerable interest in the notion of topological generation (e.g. [9, 4, 2, 3]). The term was coined in [9], and the concept naturally extends its classical counterpart, applicable to ordinary compact groups:

Let $H, K \leq G$ be two closed subgroups of a compact group. Let also $O(\cdot)$ denote the algebra of representative functions on a compact group (i.e. the algebra generated by matrix entries of finite-dimensional representations on Hilbert spaces). Then, $G$ is the closure of the subgroup generated by $H$ and $K$, written

$$G = \langle H, K \rangle,$$

if and only if there is no proper quotient $\ast$-Hopf algebra $O(G) \to A$ through which both

$$O(G) \to O(H) \quad \text{and} \quad O(G) \to O(K)$$

factor.

Formally, the compact quantum groups in this paper are objects dual to the CQG algebras of [12]: cosemisimple complex Hopf $\ast$-algebras $A$ with positive unital integral $h : A \to \mathbb{C}$. These are to be thought of as algebras $\hat{O}(G)$ of representative functions on the corresponding compact quantum group. Given that classically topological generation can be cast in function-algebra terms, the notion carries over to the quantum setting (this is a paraphrase of [9, Definition 4]):

**Definition 0.1** Let $G_i \leq G$, $i \in I$ be a family of quantum subgroups of a compact quantum group. The family topologically generates $G$, written as

$$G = \langle G_i, \ i \in I \rangle$$
if the quotients
\[ O(G) \to O(G_i) \]
do not factor through any proper quotient Hopf \( * \)-algebra of \( O(G) \).

Here (and throughout the paper) ‘quantum subgroups’ means quotient Hopf \( * \)-algebra.

The notion of topological generation appears under different terminology in \([10]\). Specifically, \([10, \text{Definition 2.15}]\) introduces the concept of a jointly full family of functors \( \mathcal{C} \to \mathcal{C}_i \). In that language, \( G_i \) topologically generate \( G \) precisely when the scalar corestriction functors
\[ \mathcal{M}^{O(G)} \to \mathcal{M}^{O(G_i)} \]
between the respective categories of comodules form a jointly full family.

As yet another characterization, the condition is equivalent to the requirement that the morphism
\[ O(G) \to \prod O(G_i) \]
to the product in the category of CQG algebras (not the product of underlying algebras!) resulting from the maps (1) is an embedding.

In the language of \([7, \text{Definition 1.12}], [10, \text{Theorem 3.1}]\) shows that the Pontryagin duals to the free unitary and orthogonal groups \( U_N^+ \) and \( O_N^+ \), \( N \geq 2 \) are residually finite provided \( N \neq 3 \) (see Section 1 below for a reminder on these objects). The reason for the gap is that the argument proceeds inductively, using the following topological generation result (\([10, \text{Lemma 3.13 and Remark 3.14}]\):

**Theorem 0.2** For all \( N \geq 4 \) we have
\[ U_N^+ = \langle U_N, U_{N-1}^+ \rangle. \]

Similarly, the case \( N = 3 \) is problematic in \([9, 8]\) for reasons ultimately traceable to the same phenomena (see e.g. \([9, \text{Remark 5}]\)).

One of the main goals of the present note is to extend Theorem 0.2 to \( N = 3 \) in Proposition 2.1 below. This will then have a number of consequences:

- Firstly, the results of \([9, 10, 8]\) extend to \( N = 3 \).
- Secondly, we obtain in Section 3 results to the effect that for all \( N \geq 2 \) the free unitary and orthogonal quantum groups \( U_N^+ \) and \( O_N^+ \) are topologically generated by their “maximal tori”.

The underlying Hopf algebra \( O(U_N^+) \) is generated as a \( * \)-algebra by generators \( u_{ij}, 1 \leq i, j \leq N \). Annihilating the off-diagonal generators
\[ u_{ij}, \ 1 \leq i \neq j \leq N \]
produces a quotient Hopf algebra isomorphic to the group algebra \( O(T_N^+) \) of the free group on \( N \) generators (the images of the diagonal \( u_{ii} \)). This is the non-commutative analogue of the function algebra on the “standard torus” embedded diagonally in the unitary group \( U_N \), and the second bullet point above is a paraphrase of Theorem 3.1, that \( U_N^+ \) is generated by all of the conjugates of \( T_N^+ \) by the elements of the classical subgroup \( U_N < U_N^+ \).

The orthogonal picture provided by Theorem 3.3 is similar, the analogue of the maximal torus \( T_N^+ \) this time being the Pontryagin dual of the free product \( \mathbb{Z}_2^{*N} \) of \( N \) copies of \( \mathbb{Z}_2 \).
The study of maximal tori in compact quantum groups $G$ (meaning maximal cocommutative Hopf $*$-algebra quotients of $O(G)$) was initiated in [6] and [3] treats torus-generation themes similar to some the present paper’s contents.

Section 1 briefly recalls some of the relevant background.

In Section 2 we prove Proposition 2.1, extending the topological generation result in Theorem 0.2 to the case $N = 3$.

Finally, Section 3 records the torus-generation consequences alluded to above, in Theorems 3.1 and 3.3.

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1 Preliminaries

We assume some Hopf algebra background; [17, 1, 15, 16] are all good sources. For the purposes of this paper compact quantum groups appear in their CQG algebra guise, as in [12, Definition 2.2]. An equivalent characterization of those objects reads as follows.

Definition 1.1 A CQG algebra is a complex Hopf $*$-algebra $A$ with the following properties:

- $A$ is cosemisimple, i.e. its category of (either left or right) comodules is semisimple.
- the unique unital left and right integral ([15, Definition 2.4.4 and Theorem 2.4.6]) $h : A \to \mathbb{C}$ is positive, in the sense that $h(x^*x) \geq 0$ for all $x \in A$.

The category of compact quantum groups is the category dual to that of CQG algebras. We write $O(G)$ for the Hopf algebra attached to a compact quantum group $G$.

We occasionally refer to compact quantum groups as ‘quantum groups’, the phrase being unambiguous throughout this note.

Example 1.2 The preeminent examples in our context are algebras $O(G)$ of representative functions on compact groups $G$. $h$ is then simply integration against the Haar measure on $G$, justifying the terminology of Haar state for the functional $h$ from Definition 1.1.

Example 1.3 For $N \geq 2$ the free unitary group $U^+_N$ is the compact quantum group whose underlying CQG algebra $O(U^+_N)$ is generated by $u_{ij}$, $1 \leq i, j \leq N$ as a $*$-algebra, with relations demanding that both

$u := (u_{ij})_{i,j}$ and $\pi := (u^*_{ij})_{i,j} \in M_N(A)$

are unitary.

The Hopf algebra structure is given by comultiplication and counit

$\Delta : u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ and $\varepsilon : u_{ij} \mapsto \delta_{ij}$

respectively, where $\delta_{ij}$ denotes the Kronecker delta.
Example 1.4 Let $N \geq 2$ again. The free orthogonal group $O^+_N$ has associated Hopf algebra $O(O^+_N)$ as in Example 1.3, with the additional constraints that the generators $u_{ij}$ be self-adjoint.

Examples 1.3 and 1.4 are central to the discussion below. The objects were introduced in [18], there the respective Hopf algebras were denoted by $A_u(N)$ and $A_o(N)$. Similarly, we have the following construction introduced in [19].

Example 1.5 For $N \geq 2$ the free symmetric group $S^+_N$ has CQG algebra $O(S^+_N)$ generated by the self-adjoint idempotent elements $u_{ij}$ (i.e. projections) such that the sums across the rows and columns of the matrix $u = (u_{ij})_{i,j}$ in $M_N(O(S^+_N))$ are all equal to $1 \in O(S^+_N)$.

Example 1.6 For every discrete group $\Gamma$ the group algebra $\mathbb{C} \Gamma$ is a CQG algebra when equipped with the $\ast$-structure making all $\gamma \in \Gamma$ unitary and with the comultiplication and counit

$$\Delta : \gamma \mapsto \gamma \otimes \gamma \quad \text{and} \quad \varepsilon : \gamma \mapsto 1.$$  

Conceptually, the compact quantum group attached to this CQG algebra should be thought of as the Pontryagin dual of $\Gamma$, for which reason we write $\hat{\Gamma} = O(\hat{\Gamma})$.

Motivated by Example 1.6 we write, as is customary in the field, $\hat{G}$ for the discrete quantum group" dual to $G$. For our purposes, no separate definition of a discrete quantum group is necessary: $\hat{G}$ is to be thought of simply as a virtual object whose underlying “group algebra” $O(G)$.

We will also refer frequently to representations of a compact quantum group:

Definition 1.7 Let $G$ be a compact quantum group. Its category of representations is the category $\mathcal{M}^{O(G)}$ of $O(G)$-comodules.

Just as is the case for an ordinary compact group, representations form a monoidal category. Furthermore, the finite-dimensional representations form a rigid monoidal category: every object $V$ has a dual $V^*$ and the canonical evaluation map $V^* \otimes V \to \mathbb{C}$ is a morphism in $\mathcal{M}^{O(G)}$ to the trivial object $\mathbb{C}$.

2 Free unitary groups in small degree

As recalled in the introduction, one of the results of [10] is the fact that for all $N \geq 4$ we have

$$U^+_N = \langle U_N, U^+_N \rangle :$$

see [10, Lemma 3.13 and Remark 3.14].

The first aim of the present note is to observe that in fact the proof of [10, Lemma 3.13] can be slightly altered so as to also allow (2) for $N = 3$:

Proposition 2.1 For all $N \geq 3$ we have

$$U^+_N = \langle U_N, U^+_N \rangle.$$  

Let $V$ be the fundamental representation of $U^+_N$. Reprising the notation of [10], for a tuple $(\varepsilon_i)_{i=1}^k$ consisting of blanks and ‘$\ast$’ we set

$$V^{(\varepsilon_i)} := V^{\varepsilon_1} \otimes \cdots \otimes V^{\varepsilon_k}$$
(i.e. a tensor product of copies of $V$ and its dual $V^*$).

The representation theory of $U^+_N$ is well understood, having been worked out in [5]. In particular, every $U^+_N$-intertwiner

$$V^{(\varepsilon_i)} \to \mathbb{C}$$

is a linear combination of non-crossing partitions: evaluation of each copy of $V^*$ against one copy of $V$ in such a manner that strings connecting the $(V^*, V)$ pairs can be drawn in the plane so as not to intersect (and all $V$ and $V^*$ get paired off).

**Proof of Proposition 2.1** As noted above, [10, Lemma 3.13] already takes care of the case $N \geq 4$ so we assume $N = 3$. We will examine the proof of that result and amplify it appropriately.

In the proof in question, one considers a tensor product $V^{(\varepsilon_i)}$ of the fundamental representation of $U^+_N$ and seeks to argue that a linear map $f : V^{(\varepsilon_i)} \to \mathbb{C}$ is a $U^+_N$-intertwiner provided it is an $U^+_N$-intertwiner over both $U_N$ and $U^+_N - 1$.

In turn, the hypothesis on $f$ is reinterpreted as follows: for every decomposition $V = W \oplus U$ with

$$\dim W = N - 1, \quad \dim U = 1$$

the restriction of $f$ to every summand of the form

$$Z^{(\varepsilon_i)} \subset V^{(\varepsilon_i)}, \quad Z \in \{W, U\}$$

is a linear combination of non-crossing pairings between $W$, $W^*$ and $U$, $U^*$ tensorands. The conclusion would then have to be that $f$ is a span of non-crossing pairings $V^* \otimes V \to \mathbb{C}$.

By moving the complement $U$ of $W$ continuously, we can furthermore assume that $W$ and $U$ are in arbitrary position: $U$ can be contained in $W$ as well as complementary to it. Furthermore, by subtracting an appropriate span of non-crossing pairings from $f$ we can assume that

$$f|_{W^{(\varepsilon_i)}} \cong 0;$$

the desired conclusion would then be that $f$ vanishes identically on $V^{(\varepsilon_i)}$.

We now proceed essentially as in the proof of [10, Lemma 3.13]. Consider the restriction of $f$ to a subspace of the form (3) for a one-dimensional complement $U$ of $W$. Let $(\omega_j)$ be the sub-tuple of $(\varepsilon_i)$ consisting of those $i$ for which the $i^{th}$ tensorand $Z$ in $V^{(\varepsilon_i)}$ is $W$ (i.e. the $(N - 1)$-dimensional subspace rather than the 1-dimensional one). If $e$ and $e^*$ are mutually dual bases of $U$ and $U^*$ respectively, let

$$\psi : W^{(\delta_j)} \to Z^{(\varepsilon_i)}$$

be the map obtained by inserting $e$ and $e^*$ in the spots deleted in passing from $(\varepsilon_i)$ to $(\delta_j)$ (see Example 2.2). Then, the restriction of $f$ to $Z^{(\varepsilon_i)}$ is uniquely determined by $f \circ \psi$, and the latter is a linear combination

$$\sum_{\pi} c_\pi \pi$$

of non-crossing pairings $\pi$ of $W$ and $W^*$ tensorands in $W^{(\delta_j)}$. Because $f$ is a $U_N$-intertwiner, the same linear combination (4) is valid for any choice of $W$ and $U$, including, as observed above, the non-complementary case of $U \subseteq W$. But in that case $f = 0$ because $f$ vanishes on $W^{(\varepsilon_i)}$, meaning that (4) is identically zero.

The linear independence of the non-crossing pairings on an $(N - 1)$-dimensional space $W$ for $N - 1 = 2$ now finishes the proof.
Example 2.2 If, say, 
\[ Z(\varepsilon_i) = W \otimes U \otimes U^* \otimes W^* \]
then 
\[ W(\delta_j) = W \otimes W^* \] and \( \psi : W \otimes W^* \to W \otimes U \otimes U^* \otimes W^* \) is the map 
\[ W \otimes W^* \ni w \otimes w^* \mapsto w \otimes e \otimes e^* \otimes w^* \in W \otimes U \otimes U^* \otimes W^*. \]

2.1 Residual finiteness, hyperlinearity and factorization

As noted briefly in the introductory discussion above, Proposition 2.1 allows us to extend some of the finiteness results in the literature to \( U_3^+ \) and \( O_3^+ \). This requires that we recall some terminology. The following notion appears in [7, Definition 1.12].

Definition 2.3 Let \( G \) be a compact quantum group. The corresponding discrete quantum group \( \hat{G} \) is residually finite if \( O(G) \) is finitely generated and embeds as a \( \ast \)-algebra in a product of matrix algebras.

With this in hand, we have

Theorem 2.4 One can simply reprise the proof of [10, Theorem 3.1], in light of the new information provided by Proposition 2.1 at \( N = 3 \).

For all \( N \geq 2 \) the discrete quantum groups \( \widehat{U}_N^+ \) and \( \widehat{O}_N^+ \) are residually finite.

Proof [10, Corollary 2.16] says that residual finiteness for discrete quantum groups \( \hat{G} \) is inherited from families \( \hat{G}_i \) if 
\[ G = \langle G_i \rangle \]
is a topologically generating family. [10, Lemma 3.9] proves the claim for \( \hat{U}_2^+ \), and since residual finiteness also holds for duals \( \hat{U}_N \) of classical unitary groups the conclusion in the unitary case follows inductively from Proposition 2.1.

As for the orthogonal claim, it is equivalent to the unitary counterpart by [10, Proposition 3.8].

Residual finiteness, in turn, entails other properties of interest in the literature. The Connes embedding problem (CEP for short) raised in [11] has driven much of the development in operator algebras. It asks whether every finite von Neumann algebra \( N \) with separable predual \( N^* \) equipped with a trace \( \tau \) embeds in a trace-preserving fashion in an ultrapower \( R^\omega \) of the hyperfinite II\(_1\) factor \( R \) with respect to some ultrafilter \( \omega \) on \( \mathbb{N} \).

A discrete group is hyperlinear when its GNS von Neumann algebra with respect to the standard trace satisfies CEP. Motivated by this, we have [9, §3.2].

Definition 2.5 A discrete quantum group \( \hat{G} \) is hyperlinear if the GNS von Neumann algebra associated to \( O(G) \) equipped with the state \( h \) satisfies CEP.

Remark 2.6 Note that the discussion in Definition 2.5 is only meaningful for Kac type quantum groups, i.e. those for which the Haar state \( h \) is tracial: \( h(xy) = h(yx) \) for all \( x, y \in O(G) \).
Corollary 2.7 The discrete duals $\hat{U}_N^+$ and $\hat{O}_N^+$ are hyperlinear for all $N \geq 2$.

**Proof** This follows from Theorem 2.4: as in the case of ordinary discrete groups, residual finiteness entails hyperlinearity (see Remark 2.10 below).

We next we turn to the Kirchberg factorization property, introduced in [14] for discrete groups and studied more generally in the context of discrete quantum groups in [8, 7] (see [8, Theorem 28 and Definitions 2.9 and 2.10]).

**Definition 2.8** A discrete quantum group $\hat{G}$ has the Kirchberg factorization property if the natural action of the algebraic tensor product $O(G) \otimes O(G)^{op}$ on the GNS Hilbert space $L^2(G,h)$ with respect to the Haar stat extends to the minimal $C^*$-tensor product of the two enveloping $C^*$-algebras.

Once more, the bulk of the following result known: all cases $N \neq 3$ are covered by [8, Theorems 4.3 and 4.4].

**Corollary 2.9** All $\hat{U}_N^+$ and $\hat{O}_N^+$, $N \geq 2$ have the Kirchberg factorization property.

**Proof** This is again a consequence of Proposition 2.1: by [7, Theorem 2] residual finiteness implies the Kirchberg factorization property.

**Remark 2.10** The proof of Corollary 2.9 also sheds some light on that of Corollary 2.7: as observed in [8, Remark 2.11], the factorization property implies hyperlinearity. Together with [7, Theorem 2] this justifies the claim made in the proof of Corollary 2.7 that residual finiteness does too, and shows that the three properties discussed above are ordered by strength as follows:

residual finiteness $\Rightarrow$ factorization property $\Rightarrow$ hyperlinearity.

### 3 An application to generation by tori

Assuming Proposition 2.1, we propose to address the following problem. Denote

$$T_N^+ = \mathbb{Z}_2^{sN}, \quad T_N^+ = \mathbb{Z}^{sN}$$

Then, we have

**Theorem 3.1** For all $N \geq 2$,

$$U_N^+ = \langle U_N, T_N^+ \rangle.$$

**Proof** We do this by induction on $N$.

**Induction step**: $N \geq 3$. We know from Proposition 2.1 that

$$U_N^+ = \langle U_N, U_N^+ \rangle$$

and by the induction hypothesis

$$U_{N-1}^+ = \langle U_{N-1}, T_{N-1}^+ \rangle.$$

The conclusion now follows from $T_{N-1}^+ < T_N^+$.  


**Base case:** $N = 2$. According to [5, Lemme 7] we have a surjection

$$\mathbb{T}^* SU_2 \to U^+_2$$

(in the sense that the opposite morphism of Hopf algebras is an inclusion). Since the left hand side is generated by $\mathbb{T}^* \mathbb{T}$ and $SU_2$ (by Theorem 3.4 applied to $G = SU_2$), the right hand side will be generated by the images $T^+_2$ and $SU_2 \subset U_2$ of these two quantum groups through (5).

**Corollary 3.2** For all $N \geq 2$,

$$U^+_N = \langle O_N, T^+_N \rangle.$$  

**Proof** This follows from Theorem 3.1 and the fact that $U_N$ is topologically generated by $O_N$ and $T_N < T^+_N$.

As a consequence, we have the analogous orthogonal result:

**Theorem 3.3** For all $N \geq 2$,

$$O^+_N = \langle O_N, T^+_N \rangle.$$  

**Proof** First, consider the projectivization $PO^+_N$ whose underlying $*$-algebra is generated by

$$u^*_ij u_{kl} = u_{ij}u_{kl}$$

and the unitary analogue $PU^+_N$. The inclusion $O^+_N \to U^+_N$ induces an isomorphism

$$PO^+_N \cong PU^+_N$$

(e.g. by [5, Théorème 1] or [10, Proposition 3.3]). Since Corollary 3.2 ensures that

$$PO^+_N = PU^+_N = \langle PO_N, PT^+_N \rangle = \langle PO_N, PT^+_N \rangle,$$

we at least know that (6) holds “projectively”.

Now let

$$G = \langle O_N, T^+_N \rangle \leq O^+_N.$$  

Since the center $Z_2 < O^+_N$ is contained in $O_N$ and $T^+_N$, we have a commutative diagram

$$\begin{array}{cccccc}
\mathbb{C} & \xrightarrow{\mathcal{O}(PO^+_N)} & \mathcal{O}(O^+_N) & \xrightarrow{\downarrow} & \mathcal{O}(Z_2) & \xrightarrow{\mathbb{C}} \\
\downarrow & & \downarrow & & & \\
\mathcal{O}(H) & \xrightarrow{\mathcal{O}(G)} & \mathcal{O}(G) & \xrightarrow{\downarrow} & \mathbb{C}
\end{array}$$

of Hopf algebras with exact rows, surjective columns, and

$$H = \langle PO_N, PT^+_N \rangle \leq PO^+_N.$$  

Because $H = PO^+_N$ by (7) the left hand vertical arrow is an isomorphism, and hence so is the right hand vertical arrow. But this means precisely that we have (6), as desired.
3.1 Dual free products by the circle

The following result, used above in the proof of \textbf{Theorem 3.1}, might be of some independent interest.

\textbf{Theorem 3.4} Let $G$ be a compact connected Lie group and $T \leq G$ a maximal torus. We then have

$$\mathbb{T} \ast G = \langle \mathbb{T} \ast T, G \rangle.$$ 

\textbf{Proof} We have to argue that the Hopf algebra surjections

$$\mathcal{O}(\mathbb{T}) \ast \mathcal{O}(G) \to \mathcal{O}(\mathbb{T}) \ast \mathcal{O}(T), \quad \mathcal{O}(\mathbb{T}) \ast \mathcal{O}(G) \to \mathcal{O}(G)$$

do not factor through any proper Hopf quotient of $\mathcal{O}(\mathbb{T}) \ast \mathcal{O}(G)$. To that end, let

$$\mathcal{O}(\mathbb{T}) \to \mathcal{O}(G) \to H$$

be the smallest Hopf quotient factoring the two maps. Since all maximal tori of $G$ are mutually conjugate, we have factorizations

$$\mathcal{O}(\mathbb{T}) \ast \mathcal{O}(G) \to H \to \mathcal{O}(\mathbb{T}) \ast \mathcal{O}(T_i)$$

for every maximal torus $T_i \leq G$.

Consider a finite set of maximal tori $T_i \leq G$, $1 \leq i \leq k$ such that the product

$$T_1 \times \cdots \times T_k \to G$$

is onto. At the level of function algebras, this means that the iterated coproduct

$$\Delta^{(k-1)} : \mathcal{O}(G) \to \mathcal{O}(T_1) \otimes \cdots \otimes \mathcal{O}(T_k) \quad (8)$$

is an embedding.

The analogous iterated coproduct

$$\Delta^{(k-1)} : \mathcal{O}(\mathbb{T}) \ast \mathcal{O}(G) \to (\mathcal{O}(\mathbb{T}) \ast \mathcal{O}(T_1)) \otimes \cdots \otimes (\mathcal{O}(\mathbb{T}) \ast \mathcal{O}(T_k)) \quad (9)$$

lands inside the algebra $\mathcal{A}$ generated by the diagonal subalgebra

$$\mathcal{O}(\mathbb{T}) \subset \mathcal{O}(\mathbb{T})^\otimes k$$

and

$$\mathcal{O}(T_1) \otimes \cdots \otimes \mathcal{O}(T_k).$$

Now note that

$$\mathcal{A} \cong \mathcal{O}(\mathbb{T}) \ast (\mathcal{O}(T_1) \otimes \cdots \otimes \mathcal{O}(T_k))$$

and hence (9) must be one-to-one because (8) is. Since on the other hand (9) factors through the quotient $\mathcal{O}(\mathbb{T}) \ast \mathcal{O}(G) \to H$, it follows, as desired, that this quotient is the identity. \hfill \blacksquare
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