Path integral solution by fractional calculus

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Abstract. In this paper, the Path Integral solution is developed in terms of complex moments. The method is applied to nonlinear systems excited by normal white noise. Crucial point of the proposed procedure is the representation of the probability density of a random variable in terms of complex moments, recently proposed by the first two authors. Advantage of this procedure is that complex moments do not exhibit hierarchy. Extension of the proposed method to the study of multi degree of freedom systems is also discussed.

1. Introduction

One of the most challenging problems in the field of stochastic calculus is to find approximated statistics of the solution of linear and non-linear stochastic differential equations that arise in many problems of physical interest. Indeed, beside few particular cases, solutions of the partial differential equation that rules the time evolution of the probability density function is not known in analytical form. Many approximated solution techniques have been proposed, for a comprehensive progress report on this topic readers are referred to [1] and to the book of Grigoriu [2]. Among the approximated procedures proposed in literature, the Path Integral (PI) method [3] is a useful tool to evaluate the response of non-linear systems in terms of probability density. It mainly consists in a step-by-step procedure in which, at each time instant, the probability density function (PDF) must be approximated in the entire (often unbounded) domain of definition. Proposed approximating schemes rely on polynomial or cubic spline interpolation scheme [4-7]. The most important limits of these procedures are (i) the great numerical effort needed to handle Multi-Degree of Freedom Systems which dramatically increases with the number of dimensions, (ii) tackling the loss of accuracy in the evolution of the solution due to the interpolation schemes.

In this paper a new perspective is given to the problem, starting from a new way to represent the PDF by means of complex order moments, proposed in [9] and further developed in [10]. The paper is organized as follows: first, the relevant representation of the statistics of a random variable in terms of complex moments is found, by means of the fractional calculus. Both the probability density function and the characteristic function (CF) show to have a dual representation depending on the complex moments. The extension to multivariate random variables is also shown. That is, the joint PDF and the joint CF of multivariate random variables is expressed by means of complex moments. Once this result is achieved, the path integral method is easily rewritten in terms of the new PDF representation.

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In the appendix, the method is extended to the solution of multidimensional stochastic differential equations. Numerical examples support the proposed procedure.

2. Complex moment representation of the random variable statistics

In [9-10], the first two authors gave a new insight on the probabilistic characterization of random variables by using complex moments that is the key issue for the PI method proposed in this paper. Fractional calculus, although it was born contextually with the ordinary differentiation theory, has received a growing interest only during the last decades of the last century. Applications in physics, in quantum mechanics, in the study of porous systems [11], in fracture mechanics [12] and so on, are available in literature. In stochastic dynamics, the PDF of the response to differential equations driven by Lévy $\alpha$-stable white noise processes is ruled by a fractional differential equation, involving fractional derivative in the diffusive term [13]. Fractional derivatives are encountered also in random vibration with frequency dependent parameter [14] and in the analysis of linear or non-linear systems driven by fractional Brownian motion [15-16]. In this section, we will report just some definitions of fractional operators needed in developing the proposed method. For further insight readers are referred to the excellent monograph of Samko et Al. [18]. Now, let us introduce the problem we solve by means of fractional calculus.

Let $X \in \mathbb{R}$ be a real random variable. The probabilistic characterization of $X$ may be given both by the PDF $p_X(x)$ and by its Fourier transform, namely the CF $\phi_X(\vartheta)$, that is

$$\phi_X(\vartheta) = E[\exp(i \vartheta X)] = \int_{-\infty}^{\infty} \exp(i \vartheta x) p_X(x) dx$$

(1)

where $\vartheta \in \mathbb{R}$, $i = \sqrt{-1}$ is the imaginary unit and $E[\cdot]$ means stochastic average. A Taylor expansion of $\phi_X(\vartheta)$ gives the well-known formula

$$\phi_X(\vartheta) = \sum_{j=0}^{\infty} (i \vartheta)^j/E[X^j]$$

(2)

where $E[X^j]$ are the so called moments of $X$, which are given in the two alternative forms

$$E[X^j] = \int_{-\infty}^{\infty} p_X(x)x^j dx = \frac{1}{i^j} \frac{d^j \phi_X(\vartheta)}{d \vartheta^j} \bigg|_{\vartheta = 0}; j = 1, 2, \ldots$$

(3)

From equation (3), it is apparent that $E[X^j]$ exists if derivatives in zero exist at least up to order $j$. This condition is not in general satisfied as in the case of $\alpha$-stable random variables that exhibit a slope discontinuity of the CF in zero, consequently they have infinite variance, and the Taylor expansion (2) cannot be applied. Another problem connected with (2) is that its Fourier transform gives the expression of the PDF in the form

$$p_X(x) = \sum_{j=0}^{\infty} (-1)^j \frac{E[X^j]}{j!} \frac{d^j \delta(x)}{dx^j}$$

(4)

that is a sum of derivatives of Dirac’s delta $\delta(x)$. This relation, although formally correct, is useless in practical applications. We have shown in [9-10] that the fractional calculus is useful to extend equation (2), (3) and (4) in a more general context, and some important result is in the following summarized.

Using the notation of Samko [18], the Riemann–Liouville (RL) fractional integral of complex order $\gamma = \rho + i \eta$ with $\rho \in \mathbb{R}$, $\eta \in \mathbb{R}$, denoted as $\left( I_{\vartheta}^{\gamma} f \right)(x)$ is defined as
\[ (I_x f)(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \xi^{\gamma-1} f(x + \xi) d\xi \]  

(5)

where \( \Gamma(\cdot) \) is the Euler gamma function. The inverse operator is the Riemann–Liouville (RL) fractional derivative of complex order \( \gamma = \rho + i \eta \) with \( \rho \in \mathbb{R}, \eta \in \mathbb{R} \), denoted as \( (D^\gamma_x f)(x) \) and expressed as

\[ (D^\gamma_x f)(x) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_0^\infty \xi^{n-\gamma-1} f(x + \xi) d\xi \]  

(6)

with \( n = \lfloor \rho \rfloor + 1 \) and \( \lfloor \cdot \rfloor \) means the integer part of the real number within brackets. The first step, shown in [9], is that the fractional derivative of the characteristic function in zero is related to fractional moments as follows

\[ (I_x \phi_x)(0) = E[(\bar{\tau} i \lambda)^{-\gamma}]; \quad \text{Re} \, \gamma > 0 \]  

(7)

\[ (D^\gamma_x \phi_x)(0) = E[(\bar{\tau} i \lambda)^{-\gamma}], \quad \text{Re} \, \gamma > 0 \]  

(8)

These equations are the generalizations of equation (3). Similar relations, extended to other kind of fractional integro-differential, like Marchaud, Grünwald-Letnikov and Riesz operators are reported in [8]. It has to be recalled that an attempt in this direction has been made by Wolfe [17], using fractional Marchaud derivatives of real order, attaining to a relation in terms of absolute moments in a quite involved form.

The second important result in [10] follows from the fact that the terms \( \Gamma(\gamma)(I_x \phi_x)(\omega) \) is the Mellin transform of the function \( \phi_x(\omega \pm \vartheta) \), and therefore, using the inversion theorem of the Mellin transform ([18], pp. 144-5) and taking into account equation (7) the CF can be represented in the form

\[ \phi_x(\pm \vartheta) = \frac{1}{2 \pi i} \int_{\rho-i \infty}^{\rho+i \infty} \Gamma(\gamma) E[(\bar{\tau} i \lambda)^{-\gamma}] |\vartheta|^{-\gamma} d\gamma \]  

(9)

with \( 0 < \rho = \text{Re}(\gamma) < 1 \). This is an integral in the complex plane along the imaginary axis with real abscissa \( \rho \), similar to the Bromwich contour, appearing in the inverse Laplace transform. In [9-10] it is demonstrated that, in order to use equation (9), it suffices to choose a value of \( \rho \) in the interval \( 0 < \rho < 1 \). Making a Fourier transform and recalling that \( \phi_x(\vartheta) \) is the complex conjugate of \( \phi_x(-\vartheta) \), then also the PDF \( p_x(x) \) can be written in terms of complex moments

\[ p_x(x) = \frac{1}{2 \pi i} \text{Re} \left\{ \int_{\rho-i \infty}^{\rho+i \infty} \Gamma(\gamma)(1-\gamma) E[(-i \lambda)^{-\gamma}][(ix)^{-\gamma} d\gamma] \right\} \]  

(10)

Equations (9) and (10) are the generalizations of (2) and (4). For clarity’s sake, we develop an easy example by considering the CF of a Gaussian distribution showing the relation between the generalized integral representation (9) proposed and the Taylor series (2). We consider the characteristic function of a standard Gaussian random variable \( X \), that is \( \phi_X(\vartheta) = \exp(-\vartheta^2/2) \). In order to apply equation (9) the value of \( \rho = \text{Re}(\gamma) \) must be properly selected in the range \( 0 < \rho < 1 \), that will be called fundamental strip in the complex plane. Further, it can be shown that the integrand function is holomorph inside the fundamental strip, ensuring that one can choose every value inside the fundamental strip. Outside the fundamental strip the integrand is not necessarily holomorph and shows isolated singularities in the no-positive part of the real axes. This is highlighted in Figure 1,
where just the real part of the integrand is plotted choosing a particular value of $\theta = \tilde{\theta} > 0$ and $\rho = 0.5$.

It has to be noted that, in this particular case, the function has poles at $0, -2, -4, -6, \ldots$ etc. Now, due to the residue theorem, integrating along the line at $\rho = 0.5$ corresponds to integrate in the whole half plane at the left of the line that is to the sum of the residues at the poles of the integrand obtaining:

$$
\phi_x(\theta) = \frac{1}{2\pi i} \int_{\rho=0.5}^{\rho+i\infty} \Gamma(\gamma) E\left[(-iX)^{-\gamma}\right] \theta^{-\gamma} d\gamma = \sum_{k=0,2,\ldots}^{\infty} \text{Res} \left( \Gamma(k) E\left[(-iX)^{-k}\right] \theta^{-k} \right) = \\
\sum_{k=0,2,\ldots}^{\infty} \frac{(i\theta)^k}{k!} E[X^k] (11)
$$

The former expression is the connection between the Taylor series representation of the characteristic function and the integral representation proposed.

![Figure 1: Real part of the integrand in equation (9)](image)

Now, it is well-known that using the Taylor series in terms of integer moments one needs to properly truncate the sum to a finite number of terms and, moreover, it is not possible to know a priori how the moments of lower order are related to moments of higher order. This does not happen in the integral representation by means of fractional moments. Indeed, as Figure 1 highlights, the integral along the Bromwich contour may be easily truncated in the range $[-\eta, \eta]$ and numerically evaluated. Of course, different numerical integration schemes can be used to this purpose, and different transformations of the contour integral by means of conformal mapping can also be done. In the following, performing an easy rectangle integration scheme for simplicity’s sake, it is shown that with a finite number of fractional moments one can represent both the CF and the PDF of every distribution. Indeed, by an opportune discretization, the equation (9) has been rewritten as

$$
\phi_x(\pm \theta) = \frac{\Delta}{2\pi} \left[ \sum_{k=-\infty}^{\infty} \Gamma(\gamma_k) E\left[\left(\mp iX\right)^{-\gamma_k}\right] |\gamma|^{-\gamma_k} \right] ; (12)
$$

where $\gamma_k = \rho + ik\Delta$, and $\Delta$ is the sampling step. It has to be emphasized that the summation involves only the imaginary part of $\gamma$, while the real part remains fixed. Without going on further, it is worth
to stress that the series (12) is applicable to every distribution. Further, the sum can be truncated in correspondence of \( r \) terms, such that \( \bar{r} = r \Delta \), i.e.

\[
\phi_{\gamma}(\pm \vartheta) = \frac{\Delta}{2 \pi} \left\{ \sum_{k=-\infty}^{\infty} \Gamma(\gamma_k) \Gamma(1-\gamma_k) E\left[ (-\vartheta i X)^{\gamma_k} \right] i \vartheta^{\gamma_k} \right\};
\]

(13)

The most interesting application, to the author’s opinion, is the case of \( \alpha \)-stable random variables (for exhaustive treatment on \( \alpha \)-stable random variables readers are referred to [19] and [20]). Indeed, although such distributions have convergent moments just in the open set \( ]-1, \alpha[ \), with \( 0 < \alpha < 2 \), it suffices to select \( \rho < \alpha \) to ensure the convergence of the series (12).

A Fourier transform of equation (12) gives the PDF of the random variable \( X \) in the form

\[
p_X(x) = \frac{\Delta}{2 \pi} \text{Re} \left\{ \sum_{k=-\infty}^{\infty} \Gamma(\gamma_k) \Gamma(1-\gamma_k) E\left[ (-iX)^{\gamma_k} \right] (ix)^{\gamma_k-1} \right\};
\]

(14)

that, in the truncated form reads

\[
p_X(x) = \frac{\Delta}{2 \pi} \text{Re} \left\{ \sum_{k=-r}^{r} \Gamma(\gamma_k) \Gamma(1-\gamma_k) E\left[ (-iX)^{\gamma_k} \right] (ix)^{\gamma_k-1} \right\};
\]

(15)

In Figure 2, and Figure 3 the CF and the PDF, respectively, of an \( \alpha \)-stable random variable \( (\alpha = 1/2) \) are plotted for different number of terms involved in the sum, that is varying \( r \). For such results we selected \( \rho = 0.4 \) and \( \Delta = 0.4 \). From this figure it can be evicted that the series convergences uniformly in the whole domain. Conditions of applicability and further details on derivation of (12) and (14) are reported in [10].

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure2.png}
\caption{Real part of the CF of a ½-stable random variable}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure3.png}
\caption{PDF of a ½-stable random variable}
\end{figure}

The extension to multivariate distributions of dimension \( d \) is straightforward. Indeed, indicate by \( p_X(x) \) and \( \phi_X(\vartheta) \) the multidimensional joint probability density function (JPDF) and the joint characteristic function (JCF) of the random vector \( X = \{X_1, \ldots, X_d \} \), with \( \vartheta = \{\vartheta_1, \ldots, \vartheta_d \} \) and \( x = \{x_1, \ldots, x_d \} \). Starting from the definition of the multidimensional RL fractional integral of the function \( f \),
\[
(I_{x_{i,\ldots,n}}')f(x) = \frac{1}{\Gamma(\gamma_i)\ldots\Gamma(\gamma_d)} \int_0^\infty \cdots \int_0^\infty x_1^{\gamma_1-1} \cdots x_d^{\gamma_d-1} f(x_1 \mp \xi_1, \ldots, x_d \mp \xi_d) \, dx_1 \cdots dx_d
\]  
(16)

following the exposed idea [9-10], it is not hard to prove that the extension of equation (7),

\[
(I_{x_{i,\ldots,n}}')\phi_\chi(\theta) = E\left[(\mp iX_1)^{\gamma_1} \cdots (\mp iX_d)^{\gamma_d}\right], \quad \text{with} \quad \gamma = \gamma_1 + \ldots + \gamma_d
\]  
(17)

holds true.

Then, interpreting \(\Gamma(\gamma_1)\ldots\Gamma(\gamma_d)(I_{x_{i,\ldots,n}}')\phi_\chi(\theta)\) as the multidimensional Mellin transform of the function \(\phi_\chi(\omega_1 \mp \delta_1, \ldots, \omega_d \mp \delta_d)\), the JCF can be restored by the inversion theorem of the Mellin transform, in the form

\[
\phi_\chi(\pm\delta_1, \pm\delta_2, \ldots, \pm\delta_d) = \frac{1}{(2\pi i)^\gamma} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \cdots \int_{\rho_d - i\infty}^{\rho_d + i\infty} \Gamma(\gamma_1)\ldots\Gamma(\gamma_d) E\left[(\mp iX_1)^{\gamma_1} \cdots (\mp iX_d)^{\gamma_d}\right].
\]  
(18)

By considering the symmetry properties of the JCF, the multidimensional JPDF is obtained by the Fourier transform of equation (18). In particular, in the case of bivariate random variable it assumes the form

\[
\rho_\chi(x_1, x_2) = \frac{2}{(2\pi i)^2} \text{Re}\left[ \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} \Gamma(\gamma_1)\Gamma(1-\gamma_1) \int_{\rho_2 - i\infty}^{\rho_2 + i\infty} \Gamma(\gamma_2)\Gamma(1-\gamma_2) \left\{ \Gamma(\gamma_1)^{\gamma_1} \Gamma(\gamma_2)^{\gamma_2} E\left[(\mp iX_1)^{\gamma_1} \cdots (\mp iX_d)^{\gamma_d}\right] \right\} x_1^{\gamma_1-1} x_2^{\gamma_2-1} \, d\gamma_1 \, d\gamma_2 \right].
\]  
(19)

As numerical example, consider a bivariate normal random variable with null mean vector and covariance matrix \(\Sigma_\chi\). JPDF and JCF given in by

\[
\Sigma_\chi = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 2 \end{pmatrix}
\]  
(20)

\[
\rho_\chi(x) = \left[2\pi \det(\Sigma_\chi)\right]^{1/2} \exp\left[-\frac{1}{2} x^t (\Sigma_\chi)^{-1} x\right], \quad \phi_\theta(\theta) = \exp\left[-\frac{1}{2} \theta^t \Sigma_\chi \theta\right]
\]  
(21)

respectively.

Equations (18) and (19) have been evaluated by means of the simple rectangle integration scheme already shown in the scalar case, with step in both directions of \(\Delta = 0.6\), \(\rho_1 = \rho_2 = 0.5\) and the sums have been truncated with \(r = 30\) terms. In Figure 4 and Figure 5, contour plots show the results obtained for the approximated JCF and JPDF (continuous line) contrasted with the exact functions (dashed).
3. Path integral method in terms of moments

In this section we briefly recall some basics on the use of PI method. Main issue of this section is to show that it is not possible to work in terms of moments in the framework of the PI method, unless closure schemes are adopted. For simplicity’s sake, we consider first a scalar nonlinear stochastic differential equation excited by an external normal white noise \( W(t) \),

\[
X(t) = f(X,t) + W(t); \quad X(0) = X_0
\]

where \( f(X,t) \) is a nonlinear function in the drift term and \( X_0 \) is a random variable with assigned density \( p_\lambda(X,0) \). The Path Integral method consists in the step-by-step time integration of the Chapman-Kolmogorov (CK) equation, which rules the evolution of the probability density function of the response, in the form

\[
p_\lambda(x,t+\tau) = \int_{-\infty}^{\infty} p_\lambda(x,t+\tau|\overline{x},t) p_\lambda(\overline{x},t) d\overline{x}
\]

once the conditional probability \( p_\lambda(x,t+\tau|\overline{x},t) \) is known. If the time increment \( \tau \) is small enough, the short time Gaussian approximation (STGA) [3] can be used

\[
p_\lambda(x,t+\tau|\overline{x},t) = \frac{1}{\sqrt{2\pi q\tau}} \exp \left( -\frac{(x-y(\overline{x},\tau))^2}{2q\tau} \right)
\]

where \( q \) is the intensity of \( W(t) \) and \( y(\overline{x},\tau) = \overline{x} - f(\overline{x},t)\tau \). Further details on this approach are reported in [3-8]. Multiplying equation (23) by \( x^k \), and taking into account the STGA, the starting CK equation is recast in terms of moments as

\[
E[X^k(t+\tau)] = \int_{-\infty}^{\infty} g_k(\overline{x},t) p_\lambda(\overline{x},t) d\overline{x} = E[g_k(X,t)]
\]

having indicated with \( g_k(X,t) \) the moments of order \( k \) of the random variable with density given by (24), namely, with mean value \( y(\overline{x},\tau) \) and variance \( q\tau \). Some of the terms involved in (25) are restored as function of \( y(X,\tau) \) by means of simple calculations, i.e. \( g_1(X,t) = y(X,\tau) \), \( g_2(X,t) = q\tau + y^2(X,\tau) \), and so on. From close inspection to such expression, it
can be highlighted that in the general case of non-linear drift term, equation (25) is a recursive relation which exhibits hierarchy. For example, for a cubic half oscillator with \( f(X) = \varepsilon X^3 \), the second moment of the response at the time instant \( t + \tau \), i.e. \( E[X(t+\tau)^2] \), is known only if \( E[q\tau + (X(t) - \varepsilon X^3(t)\tau)^2] \) is known, which includes moments of higher order at the previous time instant. For such a reason, the PI method based on moments is not feasible. On other hand, a PI method based on numbers (like moments), and not on functions (like PDFs), is extremely attractive for applications. In the following, we show how to develop a new procedure that relies on the complex moments representation of the PDF.

4. Path Integral Method by means of complex moments

Multiply equation (23) by \((-ix)^{\gamma_{\tau}}\) and, after some algebra, it leads to the relation ruling the time evolution of the complex moment of order \( \gamma_{\tau} \), in the form

\[
E\left[ (-iX(t+\tau))^{\gamma_{\tau}} \right] = \int_{-\infty}^{\infty} g_{\gamma_{\tau}}(\overline{x}, \tau) p_{\chi}(\overline{x}, t) d\overline{x} \tag{26}
\]

where

\[
g_{\gamma_{\tau}}(\overline{x}, \tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi q\tau}} \exp\left(-\frac{(x - y(\overline{x}))^2}{2q\tau}\right)(-ix)^{\gamma_{\tau}} dx \tag{27}
\]

are complex moments of a Gaussian distribution, known in closed form. The CK equation is therefore rewritten introducing equation (14) into equation (26).

\[
E\left[ (-iX(t+\tau))^{\gamma_{\tau}} \right] = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} g_{\gamma_{\tau}}(\overline{x}, \tau) \text{Re}\left\{ \sum_{1<\gamma_{\chi}} \Gamma(\gamma_{\chi}) \Gamma(1-\gamma_{\chi}) E\left[ (-iX(t))^{\gamma_{\chi}} \right] (i\overline{x})^{\gamma_{\chi}-1} \right\} d\overline{x} \tag{28}
\]

Then, at each time step, firstly, the necessary moments are calculated by equation (28), and then the PDF at that instant is reconstructed by the series (14). In this formulation, the hierarchy of moments does not appear, and this relies on the fact that, at each time step, the PDF can be approximated in the whole domain by equation (14). Evidence of this is that both sides of equation (28) have the same real part in the order of moments.

4.1. Numerical Examples

We report as numerical application a cubic oscillator of the form \( \dot{X}(t) = X - X^3 + W(t) \) under a standard gaussian initial condition, solved with the proposed procedure. We used \( q = 1 \), a time step of \( \tau = 0.05 \) seconds and at each steps 100 complex moments with real part \( \rho = 1/2 \) have been calculated. In Figure 6, the \( p_{\chi}(x,t) \) and in Figure 7, the stationary PDF is reported and compared with the exact solution, showing a very good agreement.

As further application we consider a quadratic oscillator \( \dot{X}(t) = X - X^2 + W(t) \) under a gaussian initial condition with mean \( \mu = 3 \) and standard deviation \( \sigma = 2 \). Also in this case the strength of the gaussian white noise used was \( q = 1 \). The time step was of \( \tau = 0.025 \) seconds and, at each time step, 60 complex moments with real part \( \rho = 1/2 \) have been calculated. In Figure 8 and Figure 9 the evolutionary and the stationary PDF are plotted, respectively.
Figure 6: Evolutionary PDF of the response to a cubic oscillator

Figure 7: Stationary PDF of the response to a cubic oscillator

Figure 8: Evolutionary PDF of the response to a quadratic oscillator

Figure 9: Stationary PDF of the response to a quadratic oscillator

5. Conclusions
In this paper a new form of Path Integral Solution has been proposed involving fractional moments. The appeal of the proposed procedure lies in the fact that, with a limited number of complex moments, the PDF is well approximated. It has been shown that, with such a representation, the problem of hierarchy of moments disappear, since the fractional moments necessary to restore the PDF are those having fixed real exponent and the only variable remains the imaginary part of the exponent of the fractional moments. Applications show that the complex moment representation proposed is a significant improvement for the implementation of the Path Integral Solution.

Appendix A. Extension to Multi-Degree of Freedom Systems (MDOFS)
Let us consider in this appendix the MDOFS ruled by the stochastic differential system of equations

\[ \dot{X}(t) = f(X(t),t) + W(t); \quad X(0) = X_0 \]  

(29)
where $f(\mathbf{X}(t), t)$ is a $d \times 1$ vector of nonlinear function, $W(t)$ is a zero mean $d \times 1$ vector of white noise processes, $X_0$ is the $d \times 1$ vector of initial conditions with assigned joint probability density function (JPDF). The CK equation reads

$$p_X(x; t + \tau) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_X(x, t + \tau|\mathbf{X}, t) p_X(\mathbf{X}, t) \, dx_1 \cdots dx_d$$  \hspace{1cm} (30)

where the conditional JPDF $p_X(x; t + \tau|\mathbf{X}, t)$ can be evaluated as the JPDF of the solution to the system

$$\mathbf{X}(\nu) = f(\mathbf{X}(\nu), \nu) + W(\nu + t); \quad \mathbf{X}(0) = \mathbf{X} \quad 0 \leq \nu < \tau$$  \hspace{1cm} (31)

that will be indicated as $p_X(x, \tau)$. Under the SGTA (for small $\tau$), the former is approximated by a multi-normal distribution, with mean vector given by $y(\mathbf{X}) = E[\mathbf{X}(\tau)] = \mathbf{X} + f(\mathbf{X}, t) \tau$ and covariance matrix $\Sigma_{XX}(\tau) = \mathbf{Q} \tau$, being $\mathbf{Q}$ a $d \times d$ diagonal matrix whose diagonal elements are the strengths $q_j$ of the $j$th element of $W(t)$.

Finally, multiplying eq.(30) by $(-ix_1)^{-\gamma_1} \cdots (-ix_d)^{-\gamma_d}$ and integrating in the whole domain $\mathbb{R}^d$, the following relation

$$E[(-iX_1)^{-\gamma_1}(t+\tau), \ldots, -iX_d)^{-\gamma_d}(t+\tau)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_r(\mathbf{X}, \tau) \, p_X(\mathbf{X}, t) \, dx_1 \cdots dx_d$$  \hspace{1cm} (32)

holds true. The function $g_r(\mathbf{X}, \tau)$, provided $\mathbf{Q}^{-1}$ exists, represents the complex moments of the multi-normal Gaussian distribution $p_X(x, \tau)$, explicitly

$$g_r(\mathbf{X}, \tau) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{\det(\mathbf{Q} \tau)}} \exp\left\{ -\frac{(\mathbf{x} - y(\mathbf{X}))^T \mathbf{Q}^{-1} (\mathbf{x} - y(\mathbf{X}))}{2 \tau} \right\} (-ix_1)^{-\gamma_1} \cdots (-ix_d)^{-\gamma_d} \, dx$$  \hspace{1cm} (33)

and $p_X(\mathbf{X}, t)$ must be evaluated by means of the equation (19).

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