Effect of Variable Viscosity and Thermal Conductivity on MHD Natural Convection Flow along a Vertical Flat Plate

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Authors’ contributions

This work was carried out by the author SPKS and supervised by the author MMA. Author SPKS designed the study, performed the mathematical analysis, Numerical Technics, wrote the protocol, did the graphical visualizations, managed the literature searches and wrote the first draft of the manuscript. The draft manuscript was finalized by the supervision of author MMA. The final manuscript was read and approved by both authors.

Abstract

Free convection flow around a heated vertical flat plate in the presence of a magnetic field is very important from the technical standpoint, and several researchers have studied this issue. The effects of variable viscosity and thermal conductivity on Magneto-Hydrodynamics (MHD) free convection flow over an isothermal vertical plate immersed in a fluid with heat conduction will be studied in this study. The two-dimensional, laminar, and unsteady boundary layer equations are considered in this paper. Using relevant variables, simple governing equations are transformed into non-dimensional governing equations. The implicit finite difference scheme, also known as the Crank-Nicolson scheme, is used to solve these equations numerically. This research looks at viscous incompressible fluids with temperature-dependent viscosity and thermal conductivity. The effect of various parameters on velocity, temperature, local skin friction, and local heat transfer coefficient profiles will be shown in this study, and the results will be compared to those of other researchers. The current numerical results will be compared to the results of previously published works. Figures from the current thesis will be compared to those from previously published works. The outcomes result will be shown in graphs for various values of relevant physical parameters.
1 Introduction

Natural convection is a mass and heat transfer process in which fluid motion is caused solely by density variations in the fluid caused by temperature differences. Because of its natural and engineering applications, it attracts a lot of interest from researchers. Furthermore, scientists and researchers are interested in the issue of natural convection flow along a vertical flat plate because of its numerous applications. Convection is often used in the formation of microstructures during the cooling of molten metals and fluid flows around veiled heat dissipation fins, solar ponds, petroleum reservoirs, nuclear energy, fire engineering, and a variety of other engineering applications. The most popular industrial application of natural convection is free air cooling without the use of fans. Aside from that, viscosity is a measure of internal fluid friction caused by fluid flow resistance. The work required to deform a viscous material into energy is referred to as dissipation. Thermal conductivity, on the other hand, refers to the capacity to transmit heat. Many researchers have completed extensive research projects focusing on the significance of viscous dissipation and thermal conductivity.

From various perspectives, the effects of variable viscosity and dependent thermal conductivity on free convection flow along a vertical flat plate with heat conduction are important. For its purposes, researchers are interested in technology and processes. Alam et al. [1] considered the Effect of pressure stress work and viscous dissipation in natural convection flow along a vertical flat plate with heat conduction. Alim et al. [2] investigated the Joule heating effect on the coupling of conduction with MHD free convection flow from a vertical flat plate. Rahman et al. [3] presented the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat conduction. Alim et al. [4] studied the combined effect of viscous dissipation & Joule heating on the coupling of conduction & free convection along a vertical flat plate. Molla et al. [5] considered the natural convection laminar flow with temperature dependent viscosity and thermal conductivity along a vertical wavy surface. Saﬁqul Islam et al. [6] presented the effects of temperature dependent thermal conductivity on natural convection flow along a vertical flat plate with heat generation. Kabir et al. [7] analyzed the effects of viscous dissipation on MHD natural convection flow along a vertical wavy surface. Viscous and Joule heating effects on MHD free convection flow with variable plate temperature is investigated by Hossain [8]. Finite difference analysis of transient free convection on an isothermal flat plate is studied by Soundalgekar et al. [9]. Steady free convection flow with variable viscosity and thermal diffusivity along a vertical plate is analyzes by Elbashbeshy et al. [10]. The numerical study of the combined free and forced convective laminar boundary layer flow past a vertical isothermal flat plate with temperature dependent viscosity is considered by Kafoussius et al. [11]. The effect of radiation on free convection flow of fluid with variable viscosity from a porous vertical plate is presented by Anwar Hossain et al. [12]. Effect of variable viscosity on a MHD free convection flow past a semi-infinite flat plate with an aligned magnetic field in the case of unsteady flow is studied by Seddeek [13]. Gpali.Kwang et al. [14] studied numerical study on vertical plate with variable viscosity and thermal conductivity. Amir Abbas et al. [15] studied the Combined effects of variable viscosity and thermophoretic transportation on mixed convection flow around the surface of a sphere is presented on Thermal Science Vol No. 24(6B). Muhammad Ashraf et al. [16] considered Numerical Simulation of the Combined Effects of Thermophoretic Motion and Variable Thermal Conductivity on Free Convection Heat Transfer. Amir Abbas et al. [17] analyzed the Combined effects of thermal radiation and thermophoretic motion on mixed convection boundary layer flow.

In this research, an analytical solution for the variable viscosity and dependent thermal conductivity in natural convection flow along a vertical flat plate in the presence of magneto-hydrodynamics with heat conduction will be performed based on experimental analysis. The discretization of momentum and energy equations in terms of non-dimensional $\bar{x}$ -axis and $\bar{y}$ -axis in order to express the equations in finite difference form by approximating functions and derivatives in terms of the central differences in both coordinate directions. The numerical calculations of these equations are carried out using an implicit finite difference scheme known as the Crank-Nicolson scheme, which has resulted in the development of the programming code for the current problem. For various parameters such as variable viscosity, dependent thermal conductivity, magneto-hydrodynamics, and Prandtl's number, the outcomes data analysis has evolved for velocity profile, temperature profile, local skin friction, local Nusselt number, average skin friction, and average Nusselt number.

Keywords: Steady state; magneto-hydrodynamics; dependent viscosity; dependent thermal conductivity.
2 Mathematical Analysis

Consider a viscous incompressible fluid flowing past a semi-infinite vertical plate of unsteady flow. As shown in Fig. 1, the $\bar{x}$-axis is taken vertically upward along the plate, while the $\bar{y}$-axis is chosen perpendicular to the plate at the leading edge. The $\bar{x}$-axis is assumed to begin at the plate's leading edge. Except for the fluid viscosity, thermal conductivity, and density, all physical properties of the fluid are considered to be constant. The density differs in the body force term in the momentum equation where the Bossiness approximation is used, and the fluid viscosity varies exponentially and thermal conductivity varies linearly with the fluid temperature.

![Fig. 1. Perpendicular to the plate at the leading edge](image)

We have simplified the mathematical statement of the basic conservation laws of mass, momentum, and energy for a viscous incompressible fluid with an unsteady and electrically conducting flow.

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{v}}{\partial y} &= 0 \\
\frac{\partial \mathbf{u}}{\partial t'} + \mathbf{u} \frac{\partial \mathbf{u}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{u}}{\partial y} &= \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial \mathbf{u}}{\partial y} \right) + g \beta (T' - T_{\infty}') - \sigma_0 \frac{\mathbf{\beta}_0 \mathbf{u}}{\rho} \\
\frac{\partial T'}{\partial t'} + \mathbf{u} \frac{\partial T'}{\partial x} + \mathbf{v} \frac{\partial T'}{\partial y} &= \frac{1}{\rho C_p} \frac{\partial}{\partial y} \left( k \frac{\partial T'}{\partial y} \right)
\end{align*}
\]

Where, $\mathbf{u}$ and $\mathbf{v}$ are the velocity components along the $\bar{x}$ and $\bar{y}$ axis respectively, $t'$ is the time, $T'$ is the temperature of the fluid in the boundary layer, $T_{w}'$ is the temperature at the wall and $T_{\infty}'$ is the fluid temperature far away from the plate, $g$ is the acceleration due to gravity, $\kappa$ is the thermal conductivity of fluid, $\rho$ is the density, $C_p$ is the specific heat at constant pressure and $\mu$ is the variable dynamic co-efficient of viscosity of the fluid, $\sigma_0$ is the electric conduction and $\mathbf{\beta}_0$ is the magnetic field strength.

The initial and boundary conditions are

\[
\begin{align*}
t' \leq 0 : & \quad \mathbf{u} = 0, \mathbf{v} = 0, \quad T' = T_{\infty}' \quad \text{for all} \quad y \\
t' > 0 : & \quad \mathbf{u} = 0, \mathbf{v} = 0, \quad T' = T_{w}' \quad \text{at} \quad y = 0 \\
t' > 0 : & \quad \mathbf{u} = 0, T' = T_{w}' \quad \text{at} \quad x = 0 \\
t' > 0 : & \quad \mathbf{u} \to 0, \quad T' \to T_{\infty}' \quad \text{as} \quad y \to \infty
\end{align*}
\]
On introducing the following non-dimensional quantities in equations (1) to (4), we have

\[ x = \frac{x}{L}, \quad y = \frac{y}{Gr^{1/4}}, \quad u = \frac{u}{v Gr^{-3/2}}, \quad v = \frac{v}{v Gr^{-1/4}}, \quad t = \frac{t'}{E Gr^{1/2}}, \]

\[ T = \frac{T' - T_{\infty}'}{T_w' - T_{\infty}'}, \quad Gr = \frac{g \beta L^2 (T_w' - T_{\infty}')}{\nu^2}, \quad Pr = \frac{\mu_0 C_p}{k_0}, \quad \nu = \frac{\mu_0}{\rho} \]

Here, \( L \) = Length of the plate, \( \nu \) = Kinematic viscosity, \( Gr \) = Grashof number, \( Pr \) = Prandtl number.

Dimensionless temperature \( T \) is available in the literature with the out of many forms of variation of viscosity and thermal conductivity. The following forms are proposed by G. Palani.Kwang-Yong Kim, Stattery [18], Ockendon and Ockendon [19], Elbashbeshy [20], Ockendon and Ockendon [19], and Seddeek and Abdelmeguid [21].

\[ \frac{\mu}{\mu_0} = e^{-\lambda T} \]

\[ \frac{k}{k_0} = 1 + \gamma T \]

Where \( \lambda \) and \( \gamma \) are Viscosity and Thermal conductivity variation parameters which are depended on the nature of the fluid, \( \mu_0 \) and \( k_0 \) are the viscosity and the thermal conductivity at temperature \( T_{w}' \).

In the fluid, the magneto hydrodynamic field is governed by the boundary layer equations, which are in the non-dimensional form and obtained by introducing the dimensionless variables described in (5), may be written as the equation of continuity

\[ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \]

\[ \Rightarrow \frac{\partial}{\partial x} \left( \frac{1}{L^2} \left( \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial x} \right) \right) = 0 \]

\[ \Rightarrow \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \]

Now momentum equation (2) can be reduced by applying the non-dimensional transformation (5) and (6), we have

\[ \frac{\partial \tilde{u}}{\partial t'} + u \frac{\partial \tilde{u}}{\partial x} + v \frac{\partial \tilde{u}}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial \tilde{u}}{\partial y} \right) + g \beta (T' - T_{\infty}') - \sigma_0 \frac{\beta_0^2 \tilde{u}}{\rho} \]

\[ \Rightarrow \frac{\partial \tilde{u}}{\partial t} + u \frac{\partial \tilde{u}}{\partial x} + v \frac{\partial \tilde{u}}{\partial y} = \left[ \epsilon^{-\lambda T} \frac{\partial^2 \tilde{u}}{\partial y^2} - \lambda \epsilon^{-\lambda T} \frac{\partial \tilde{T}}{\partial y} \frac{\partial \tilde{u}}{\partial y} \right] + T - \sigma_0 \frac{\beta_0^2 L^2 Gr^{-3/2}}{\rho v} \frac{1}{\rho} \tilde{u} \]

\[ \Rightarrow \frac{\partial \tilde{u}}{\partial t} + u \frac{\partial \tilde{u}}{\partial x} + v \frac{\partial \tilde{u}}{\partial y} = \left[ \epsilon^{-\lambda T} \frac{\partial^2 \tilde{u}}{\partial y^2} - \lambda \epsilon^{-\lambda T} \frac{\partial \tilde{T}}{\partial y} \frac{\partial \tilde{u}}{\partial y} \right] + T - Mu \]

(9)
Again the energy equation (3) can be reduced by the above similarity transformation (5) and (7), we have

\[
\frac{\partial T'}{\partial t} + u \frac{\partial T'}{\partial x} + v \frac{\partial T'}{\partial y} = \frac{1}{\rho C_p} \frac{\partial}{\partial y} \left( k \frac{\partial T'}{\partial y} \right)
\]

\[
\Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_0}{\mu_0 C_p} \left\{ (1 + \gamma T) \frac{\partial^2 T}{\partial y^2} + \gamma \left( \frac{\partial T}{\partial y} \right)^2 \right\}
\]

\[
\Rightarrow \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{P_r} \left[ \gamma \left( \frac{\partial T}{\partial y} \right)^2 \right] + \frac{1}{P_r L} \left( 1 + \gamma T \right) \frac{\partial^2 T}{\partial y^2}
\]

(10)

The initial and boundary conditions in the dimensionless forms are as follows

\[
t \leq 0: \quad u = 0, \ v = 0, \ T = 0 \quad \text{for all} \ y
\]

\[
t > 0: \quad u = 0, \ v = 0, \ T = 1 \quad \text{at} \ y = 0
\]

\[
u = 0, \quad T = 0 \quad \text{at} \ x = 0
\]

\[
u \to 0, \quad T \to 0 \quad \text{at} \ y \to \infty
\]

(11)

With the boundary condition (11), the equations from (8) to (10) describe the free convective unsteady laminar boundary layer flow with variable viscosity and thermal conductivity along an isothermal semi-infinite vertical plate. Where, \( M = \sigma_0 \beta_0 L^2 Gr \frac{1}{\mu} \) is the magnetic parameter and \( Pr = \mu_0 C_p / k_0 \) is the Prandtl’s number.

The local shear stress in the plate is defined by

\[
\tau_x = \left( \mu \frac{\partial u}{\partial y} \right)_{y=0}
\]

(12)

By introducing the non-dimensional quantities from the equations (5) to (6) in equation (12), we get the non-dimensional form of local skin friction which it is given by

\[
\tau_x = \left( \mu \frac{\partial u}{\partial y} \right)_{y=0}
\]

\[
\Rightarrow \tau_x = Gr^3 e^{-\lambda} \left[ \frac{\partial u}{\partial y} \right]_{y=0}
\]

(13)

Now the integration of the equation (13) from the values \( x = 0 \) to \( x = 1 \) gives the average skin friction and it is given by

\[
\bar{\tau} = e^{-\lambda} Gr^3 \int_0^1 \left[ \frac{\partial u}{\partial y} \right]_{y=0} \ dx
\]

(14)

The local Nusselt number is defined by
\[ \text{Nu}_x = \frac{-L \left( k \frac{\partial T}{\partial y} \right)_{\tau=0} }{k_0 (T_w - T_\infty)} \]

\[ \Rightarrow \text{Gr}^{1/4} \left( T_w - T_\infty \right) \text{Nu}_{(xL)} = -(1 + \gamma T) \left( \frac{\partial T}{\partial y} \right)_{y=0} \]

\[ \Rightarrow \text{Gr}^{1/4} \left( T_w - T_\infty \right) \text{Nu}_{(xL)} = -(1 + \gamma) \left( \frac{\partial T}{\partial y} \right)_{y=0} \because y = 0, T = 1 \]

\[ \Rightarrow \bar{\text{Nu}}_x = -(1 + \gamma) \left( \frac{\partial T}{\partial y} \right)_{y=0} \]

The integration of the equation (15) from the values \( x = 0 \) to \( x = 1 \) gives the average skin friction and it is given by

\[ \therefore \text{Nu}_x = -(1 + \gamma) \int_0^1 \left( \frac{\partial T}{\partial y} \right)_{y=0} \, dx \] (16)

### 2.1 Numerical Techniques

The implicit finite difference scheme of Crank-Nicolson form, which is fast convergent and unconditionally stable, is used to solve the two-dimensional, non-linear, unsteady, and coupled partial differential equations from (8) to (10) under the initial and boundary conditions in equation (11). The finite difference equations that correspond to equations (8) to (10) are as follows:

\[
\begin{align*}
&\left[ u^{k+1}_{i+1,j} - u^k_{i+1,j} + u^k_{i-1,j} - u^k_{i,j-1} + u^{k+1}_{i,j-1} - u^k_{i,j-1} \right] + \left[ v^{k+1}_{i,j+1} - v^k_{i,j+1} + v^k_{i,j} - v^k_{i,j-1} \right] = 0 \\
&\left[ u^{k+1}_{i,j} - u^k_{i,j} \right] + \frac{1}{2} \left[ T^{k+1}_{i,j} + T^k_{i,j} \right] = \frac{1}{2} \left[ T^{k+1}_{i,j} + T^k_{i,j} \right] \\
&= \frac{1}{2 \Delta y} \left[ \lambda v^{k+1}_{i,j} - v^k_{i,j} \right] + \frac{1}{2} \left[ T^{k+1}_{i,j} + T^k_{i,j} \right] \\
&= \frac{1}{4 \Delta y} \left[ T^{k+1}_{i,j-1} - T^{k+1}_{i,j+1} + T^k_{i,j+1} - T^k_{i,j-1} \right] \\
&= \frac{1}{2} \left[ T^{k+1}_{i,j-1} - T^{k+1}_{i,j+1} + T^k_{i,j+1} - T^k_{i,j-1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j-1} - T^{k+1}_{i,j+1} + T^k_{i,j+1} - T^k_{i,j-1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j-1} - 2T^k_{i,j} + T^k_{i,j+1} \right] \\
&\left[ T^{k+1}_{i,j+1} + T^k_{i,j+1} \right] + \frac{1}{2} \left[ T^{k+1}_{i,j+1} - T^{k+1}_{i,j} + T^k_{i,j+1} - T^k_{i,j-1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j+1} - 2T^k_{i,j} + T^k_{i,j+1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j+1} - 2T^k_{i,j} + T^k_{i,j+1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j+1} - 2T^k_{i,j} + T^k_{i,j+1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j+1} - 2T^k_{i,j} + T^k_{i,j+1} \right] \\
&= \frac{1}{2 \Delta y} \left[ T^{k+1}_{i,j+1} - 2T^k_{i,j} + T^k_{i,j+1} \right] \]
\] (17)
3 Results and Discussion

Heat conduction, also called diffusion, is the direct microscopic exchange of kinetic energy of particles through the boundary between two systems. Water has high thermal capacity and low viscosity and also good heat transfer fluid. Oil has a higher liquid temperature than water and has been a preferred choice to get around the high pressure issue. Conduction, radiation and convection all play a role in moving heat between Earth's surface and the atmosphere. Convection is heat transfer by mass motion of a fluid such as air or water when the heated fluid is caused to move away from the source of heat, carrying energy with it. Convection above a hot surface occurs because hot air expands, becomes less dense, and rises. Low Prandtl’s number of liquid metals imply that heat transfer by molecular thermal conduction is significant not only in the near-wall layer, but also in the flow core even in a fully developed turbulent flow.

In the present study, the ranges for λ, γ and Pr are considered:

For air: 0.7 ≤ λ ≤ 0, 0 ≤ γ ≤ 6, \( Pr = 0.733 \)
For water: 0 ≤ λ ≤ 0.6, 0 ≤ γ ≤ 0.12, 2 ≤ Pr ≤ 7.00

We equate our findings to the curves computed by G.palani.Kwang-Yong Kim and Elbashbeshy & Ibrahim to ensure that our computed results are accurate for various values of λ and γ for air (\( Pr = 0.733 \)) which are plotted in Figs. 2(a), 2(b). At the steady state, our findings are very similar to those of G.palani.Kwang-Yong Kim and Elbashbeshy & Ibrahim.
The body force has not had enough time to produce and effective motion in the fluid during the initial phase following step changes in the wall temperature. As a result, for small time $t$, the velocity components $u$ and $v$ are both negligible. Pure heat conduction dominates heat transfer during this initial transient regime, resulting in constant viscosity and thermal conductivity. Now, equation (10) reduces to

$$\frac{\partial T}{\partial t} = \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2}$$

Thus, it is noted that for a given Prandtl's number, magnetic parameter, and normal distance from the wall, the temperature profile is only a function of time and normal distance from the wall for short periods. Now setting $Pr = 1$, the solutions of the Eq. (15) subject to the initial and boundary conditions specified in the local Nusselt number are

$$T = \text{erfc} \left( \frac{y}{2\sqrt{t}} \right)$$

(20)

For different viscosity, thermal conductivity, magnetic variance parameters, and Prandtl's numbers, Figs. 3(a), 3(b), 4(a), 4(b), 5(a), 5(b), 6(a) 6(b), 7(a), 7(b), 8(a), 8(b), 9(a), 9(b), 10(a), 10(b) show the variation of velocity and temperature at their transient, temporal limit, and steady state against the co-ordinate $y$ at the leading edge of the plate, viz., $x = 1.0$. For all time $t$, the fluid velocity increases until it reaches its maximum value very close to the wall (i.e., $0 \leq y \leq 2$) and then decreases monotonically to zero as $y$ increases. It's also worth noting that the velocity and temperature rise with time $t$, reach a temporal limit, and then settle into a steady state.

The variation of transient velocity and temperature profiles with area $A$ for a fixed value of $\gamma = 0.10$ in air ($Pr = 0.733$) and $M = 0.30$ is shown in Figs. 3(a) and 3(b). The fluid's velocity increases over time until it reaches a temporal limit, after which it gradually decreases until it reaches the ultimate steady state. It is observed that as the viscosity variance parameter is increased, the time taken to reach the steady state decreases slightly. The velocity $u$ at any vertical plane near the plate increases as $x$ increases (the viscosity of air decreases), as shown in Fig 3(a). At a certain distance from the wall, however, the opposite pattern is observed. The temperature of the fluid decreases as $\lambda$ increases, as shown in Fig 3(b) (the viscosity of air decreases).

In Figs. 4(a) and 4(b), the numerical values of the variation of transient velocity and temperature profiles with $\gamma$ for a fixed value of $\lambda = 0.30$, $M = 0.50$ in air ($Pr = 0.733$) with the variation of the thermal conductivity parameter are graphically shown. For a fixed value of $\lambda$, $M$ and Prandtl's number, it can be shown that the velocity and temperature distribution in the fluid increases as increases (thermal conductivity of air increases). It's also worth noting that as $\gamma$ increases, the magnitude of the velocity and temperature increases significantly, implying that the volume flow rate increases as well. Even during the initial transient phase, the effect of thermal conductivity variation on velocity and temperature is more important. It is also observed that as the thermal conductivity parameter $\gamma$ is increased, the time to reach the temporal maximum and steady state decreases.

Fig. 5(a) and 5(b) show the numerical values of variance velocity and temperatures determined from Eqs (13) and (14) for different values of $\lambda$ for a fixed value of $M = 0.50$ and $\gamma = 0.40$ in water ($Pr = 3.00$). It is obvious that as the viscosity variance parameter $\lambda$ is increased, the time taken to achieve the temporal maximum and steady state decreases. Since the viscosity of water decreases with an increase in the viscosity variation parameter $\gamma$, as shown in Eq (6), an increase in the viscosity variation parameter, $\lambda$ increases the velocity of the flow near the wall, as seen in Fig. 5(a). For higher $\lambda$ values, the maximum velocity also gets very close to the wall. This qualitative effect occurs because the fluid with variable viscosity ($\lambda > 0$) is able to travel more easily in a region close to the heated surface due to the lower viscosity of the fluid with $\lambda > 0$ compared to the fluid with constant viscosity. As a consequence, the velocity and thermal boundary layers become thinner.
Fig. 2(a) and 2(b), Variation of dimensionless velocity and temperature profiles versus dimensionless $y$ for various values of the variable thermal conductivity parameter $\gamma$ with $Pr = 0.733$, $\lambda = 0.4$ and $M = 0.0$

Fig. 3(a) and 3(b), Variation of dimensionless velocity profiles and temperature profiles against dimensionless $y$ for different values of variable viscosity parameter $\lambda$ and steady state condition with $Pr = 0.733$, $\gamma = 0.10$ and $M = 0.30$

Fig. 4(a) and 4(b), Variation of dimensionless velocity and temperature profiles against dimensionless $y$ for various values of the variable thermal conductivity parameter $\gamma$ and steady state condition with $Pr = 0.733$, $\lambda = 0.30$ and $M = 0.50$
Fig. 5(a) and 5(b), Variation of dimensionless velocity and temperature profiles versus dimensionless $y$ for various values of the vector viscosity parameter $\lambda$ and steady state condition with $Pr = 3.00$, $\gamma = 0.40$ and $M = 0.50$

Fig. 6(a) and 6(b), Variation of dimensionless velocity and temperature profiles versus dimensionless $y$ for various values of the variable thermal conductivity parameter $\gamma$ and steady state condition with $Pr = 3.00$, $\lambda = 0.30$ and $M = 0.20$

Fig. 7(a) and 7(b), Variation of dimensionless velocity profiles and temperature profiles against dimensionless $y$ for different values of Prandtl’s number $Pr$ and steady state condition with $\lambda = 0.30$, $\gamma = 0.40$ and $M = 0.40$
Fig. 8(a) and 8(b), Variation of dimensionless velocity profiles and temperature profiles against dimensionless $y$ for different values of magnetic parameter $M$ and steady state condition with $\lambda = 0.50$, $\gamma = 0.10$ and $Pr = 7.00$

Fig. 9(a) and 9(b), Variation of dimensionless local skin friction and local Nusselt number against dimensionless distance $x$ for different values of $M$, $\lambda$, $\gamma$ and $Pr$ at steady state condition

Fig. 10(a) and 10(b), Variation of dimensionless average skin friction and average Nusselt number against dimensionless distance $x$ for different values of $M$, $\lambda$, $\gamma$ and $Pr$ at steady state condition

It is observed that the velocity of the fluid particle increases only in the region $0 \leq y \leq 2$ as $\lambda$ increases (the viscosity of water decreases). The temperature profiles decrease with increasing $\lambda$, as shown in Fig. 5(b). This is due to the fact that an increase in $\lambda$ results in an increase in peak velocity. The first effect increases the velocity
of the fluid particle due to a decrease in viscosity, while the second effect reduces the velocity of the fluid particle due to a decrease in temperature near the plate, where $T$ is high. As a result, the first force will be dominant, and the velocity $u$ will increase as $\lambda$ increases. When the temperature $T$ is low far away from the plate, on the other side, the second effect takes over, and the velocity decreases as $\lambda$ increases. We can see from the discussion that ignoring the variations in fluid viscosity and thermal conductivity can result in a significant mistake.

The variance of velocity and temperature for different values of $\gamma$ for a fixed value of $\lambda = 0.30$, $M = 0.20$ in water ($Pr = 3.00$) is shown in Figs. 6(a) and 6(b). It is observed that as the value of $\gamma$ increases, the time it takes to reach the steady state decreases. It's also observed that as the value of $\gamma$ rises, the fluid's temperature distribution rises as well. For fixed values of $\lambda = -0.30$, $\gamma = 0.40$ and $M = 0.40$, the difference of transient velocity and temperature with Prandtl's numbers is shown in Figs. 7(a) and 7(b) (b). With increasing Prandtl's number of the fluid, it is observed that the time taken to reach the temporal limit and steady state increases. We can see from the numerical results that the velocity profile decreases as Prandtl's number increases. Since a greater Prandtl's number value indicates that the thermal diffusion from the wall is not prevailing, while the velocity diffusion spreads further from the wall, thinner temperature profiles result.

Figs. 8(a) and 8(b) graphically depict the numerical values of variance of velocity and temperature profiles with $M$ for fixed values of $\lambda = 0.50$, $\gamma = -0.10$ for water ($Pr = 7.00$). It can be seen from these graphs that the time it takes to enter the steady state increases as the magnetic parameter $M$ decreases. In addition, as $M$ decreases near to the vertical plate, the velocity increases. As the magnetic parameter $M$ decreases, the fluid temperature decreases.

Eqs (12), (14), (15), and (16) include the evaluation of derivatives using a five-point approximation formula, followed by the evaluation of integrals using the Newton-Cotes closed integration formula.

Eq. (13) is used to calculate the local skin friction values, which are then plotted in Fig. 9(a) as a function of the axial coordinate $\lambda$, for air and water, as well as selected values of the variance parameters $\lambda$ and $\gamma$. As $\lambda$ increases, the local skin friction increases as well. Local skin friction decreases as the viscous variation parameter $\lambda$, increases. It's also been discovered that as the value of the thermal conductivity parameter $\gamma$ rises, so does the local wall shear stress. Local skin friction is observed to decrease as Prandtl's number increases.

For different values of viscosity and thermal conductivity parameters for air and water, average values of skin friction are determined numerically from Eq. (14) and graphically shown in Fig 10(a). It increases over time and reaches a stable state after a period of time. With increasing values of the viscous parameter $\lambda$ and the magnetic parameter $M$, average skin friction decreases. It is also observed that as the value of the thermal conductivity parameter $\gamma$ increases, the average wall shear stress increases. Average skin friction is observed to decrease as the value of Prandtl's number increases.

Fig. 9(b) depicts the dimensionless steady-state local heat transfer rate for air and water for various variance parameter values. The rate of local heat transfer is found to increase as the viscosity, thermal conductivity, and magnetic parameters increase. The rate of local heat transfer increases as $Pr$ increases. Since a larger $Pr$ results in a thinner thermal boundary layer, a larger wall temperature gradient, and thus a higher heat transfer rate, this pattern is anticipated. The effects of variance parameters and $Pr$ on the average Nusselt number are shown in Fig. 10(b). Since increasing the Prandtl's number speeds up the spatial decay of the temperature in the flow region, increasing the Prandtl's number leads to an increase in the average heat transfer rate, but the magnetic parameter has the opposite effect. It's also observed that as $\lambda$, $\gamma$ and $M$, the average Nusselt's number decreases.

4 Conclusion

The effect of variable viscosity and thermal conductivity on magneto-hydrodynamics laminar natural convection boundary-layer vertical plate with constant surface temperature is investigated in this paper. Thermal conductivity is assumed to be a linear function of temperature, while fluid viscosity is assumed to be an exponential function of temperature. An implicit Crank-Nicolson-type finite difference scheme is used to solve the dimensionless governing equations. Graphically, a distinction is often made between the current numerical
findings and previously published works. The two's agreement has been deemed outstanding. According to the findings of this study:

1. The dimensionless fluid velocity increases as the viscosity parameter $\lambda$ increases and the fluid temperature decreases. Higher velocity is observed in a region near the wall when the viscosity variable parameter $\lambda$ is significant, resulting in a higher Nusselt number and lower skin friction.

2. The fluid velocity, fluid temperature, the dimensionless wall velocity gradient, and the dimensionless rate of heat transfer from the plate to the fluid all increase as the thermal conductivity parameter $\gamma$ increases.

3. It has been discovered that ignoring the viscosity and thermal conductivity variations can result in significant errors. As a result, we conclude that the effects of variable viscosity and thermal conductivity must be considered in order to predict more accurate outcomes.

4. The local skin friction coefficient, the local Nusselt number, and the velocity distribution over the entire boundary layer decrease as a result of the magnetic parameter $M$, but the temperature distribution increases.

**Competing Interests**

Authors have declared that no competing interests exist.

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