Variable step block backward differentiation formula with independent parameter for solving stiff ordinary differential equations

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Abstract. Over the last decade, the evolution of block backward differentiation formulas (BBDF) has involved the modifications of the formulation techniques in order to solve ordinary differential equations (ODEs). Better still, if the modified methods have the ability of computing solutions efficiently with any prescribed parameter. Therefore, this research focuses on the derivation of 2-point variable step block backward differentiation formulas (VSBBDF) that possesses independent parameter in the coefficients. In this formula, each block contains two points, which compute two approximate solutions simultaneously. Varying the value of parameter will lead to multiple choice of solutions with different level of accuracy. Since the method is derived using variable step size scheme, the strategy in controlling the step size ratio is also discussed. The capability of the derived method is demonstrated by solving initial value problem of stiff ODEs. A comparison of its performance with several existing methods is made to shed light on the superiority and shortcomings of VSBBDF with respect to independent parameter.

1. Introduction

Many problems in diverse fields of studies, such as physical sciences, engineering and medicine can be modelled by ordinary differential equations (ODEs). The first order ODEs can be written in the form of

\[ y' = f(x, y), \]

with initial condition \( y(a) = y_0 \) in the interval \( a \leq x \leq b \). Stiff ODEs are characterized as those whose exact solution has a term in the form of \( e^{-\lambda t} \), where \( \lambda \) is a large positive constant. The key feature of stiff equations is that the derivative terms may increase rapidly as \( t \) increases [1]. A set of (1) is stiff when an excessively small step is required to produce correct integration [2]. Further occurrences of stiffness can be found in electrical circuits [3] and computational biology [4]. A numerical method with a large region of absolute stability must be used for solving stiff ODEs, or else the step size, \( h \) must be chosen to be very small. Among the numerical methods in literature, the backward
differentiation formula (BDF) is preferred because its stability regions contain all \( \lambda h \) where the eigenvalue, \( \lambda \) is negative or \( \lambda \) is complex with negative real part. A lot of extensions and improvements have been made on the basis of the BDF such as fully implicit block backward differentiation formulas (BBDF) [5–7], diagonally implicit BBDF [8,9], singly diagonally implicit BBDF [10] and BBDF with off-step points [11,12] that have verified the competency of computing simultaneous solutions at different points. These methods are proven to have A-stable properties, which are suitable for solving stiff ODEs. A new development of the BDF method is known as BDF-\( \alpha \) with numerical damping control, which was previously introduced by [13,14]. The advantage of the BDF-\( \alpha \) is that it has larger stability region than the conventional BDF. However, the works presented in [13,14] do not provide the numerical solution of equation (1). In line with that, the similar idea has been applied to BBDF where the authors [15,16] constructed BBDF-\( \alpha \) using constant step size for solving equation (1). The results in their work show a significant accuracy improvement over the BDF, BDF-\( \alpha \) and BBDF in terms of maximum error and average error. This study is primarily concerned with the method pertaining to the BBDF with variable step approach [17]. Throughout this paper, the variable step BBDF with independent parameter \( \alpha \) (VSBBDF- \( \alpha \)) is derived for solving first order stiff ODEs. This method differs from that of [6,15] where the intention to derive a new block method is to obtain better approximations of (1) by using adjustable parameter \( \alpha \) that has been stored in the coefficients of the proposed formulas.

2. Methodology

2.1. Formulation of the method

In this section, the VSBBDF is derived using three previous values, \( y_{n-2}, y_{n-1} \) and \( y_n \), where \( n \) is the grid index, to compute two solutions, \( y_{n+1} \) and \( y_{n+2} \) simultaneously. The step size of the computed block is \( 2h \) and the step size of the previous block is \( 2rh \), where \( r \) is the step size ratio. To begin with, the following Lagrange interpolation polynomial \( P_k(x) \) of degree \( k = 5 \) is used:

\[
P_k(x) = \sum_{j=0}^{k} L_{k,j}(x) y(x_{n+1-j}), \quad \text{where} \quad L_{k,j}(x) = \prod_{i=0}^{k} \left( \frac{x - x_{n+1-j}}{x_{n+1-j} - x_{n+1-i}} \right), \quad j = 0, 1, \ldots, k. \tag{2}
\]

By substituting \( x = x_{n+1} + sh \) into equation (2), we then differentiate the resulting polynomial once with respect to \( s \). Setting \( s = 0 \) and \( s = 1 \), yields the following

\[
hP_s'(x_{n+1}) = \left[ \frac{-3r - 2 - r^2}{4r^2 (2r + 1)(r + 1)(r + 2)} \right] y_{n-2} + \left[ \frac{4 + 16r^2 + 16r}{4r^2 (2r + 1)(r + 1)(r + 2)} \right] y_{n-1} + \left[ \frac{-37r^3 - 20r^4 - 4r^5 - 13r^2 - 2 - 32r^2}{4r^2 (2r + 1)(r + 1)(r + 2)} \right] y_n + \left[ \frac{12r^4 + 32r^3 + 16r^2}{4r^2 (2r + 1)(r + 1)(r + 2)} \right] y_{n+1} + \left[ \frac{r^2 + 4r^5 + 8r^4 + 5r^3}{4r^2 (2r + 1)(r + 1)(r + 2)} \right] y_{n+2}, \tag{3}
\]
\[
hP_3'(x_{n+2}) = \left[\frac{8 + 8r + 2r^2}{4r^2(2r + 1)(r + 1)(r + 2)}\right]y_{n-2} + \left[\frac{-16 - 32r^2 - 48r}{4r^2(2r + 1)(r + 1)(r + 2)}\right]y_{n-1} + \left[\frac{64r^3 + 26r^4 + 4r^5 + 40r + 8 + 74r^2}{4r^2(2r + 1)(r + 1)(r + 2)}\right]y_n + \left[\frac{-16r^5 - 80r^4 - 128r^3 - 64r^2}{4r^2(2r + 1)(r + 1)(r + 2)}\right]y_{n+1} \quad (4)
\]

where \( hP_3'(x_{n+1}) = hy_{n+1}' = hf_{n+1} \) and \( hP_3'(x_{n+2}) = hy_{n+2}' = hf_{n+2} \). Replacing \( r = 1, 2 \) and \( \frac{5}{8} \) into equations (3) and (4), we obtain:

For \( r = 1 \):
\[
y_{n+1} - \frac{1}{10} y_{n-2} + \frac{3}{5} y_{n-1} - \frac{9}{5} y_n + \frac{3}{10} y_{n+2} = \frac{6}{5} hf_{n+1},
\]
\[
y_{n+2} + \frac{3}{25} y_{n-2} - \frac{16}{25} y_{n-1} + \frac{36}{25} y_n - \frac{48}{25} y_{n+1} = \frac{12}{25} hf_{n+2}.
\]

For \( r = 2 \):
\[
y_{n+1} - \frac{3}{128} y_{n-2} + \frac{25}{128} y_{n-1} - \frac{225}{128} y_n + \frac{75}{128} y_{n+2} = \frac{15}{8} hf_{n+1},
\]
\[
y_{n+2} + \frac{2}{115} y_{n-2} - \frac{3}{23} y_{n-1} + \frac{18}{23} y_n - \frac{192}{115} y_{n+1} = \frac{12}{23} hf_{n+2}.
\]

For \( r = \frac{5}{8} \):
\[
y_{n+1} - \frac{208}{775} y_{n-2} + \frac{6912}{5425} y_{n-1} - \frac{13689}{6200} y_n + \frac{351}{1736} y_{n+2} = \frac{117}{124} hf_{n+1},
\]
\[
y_{n+2} + \frac{12544}{29875} y_{n-2} - \frac{53248}{29875} y_{n-1} + \frac{74529}{29875} y_n - \frac{2548}{1195} y_{n+1} = \frac{546}{1195} hf_{n+2}.
\]

2.2. Order condition
Motivated by the work of [14], five parameters, \( \alpha, \beta, \rho, \mu \) and \( \delta \) are inserted in equations (5) – (7) systematically. For the simplicity of the modified expressions, it can be written in the following way, which corresponds to the standard linear multistep method (LMM) of step number \( k = 5 \):

\[
\sum_{j=0}^{k} A_j y_{n+j} = h \sum_{j=0}^{k} B_j f_{n+j},
\]

where \( A_j \) and \( B_j \) are real constants.
For $r = 1$:

$$A_0 = B_0 = B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{10} - \frac{3}{5} \delta \\ \frac{3}{16} + \frac{1}{25} \delta \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{3}{5} + \frac{3}{5} \delta + \frac{9}{5} \mu \\ \frac{16}{25} - \frac{16}{25} \delta - \frac{36}{25} \mu \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{9}{5} - \frac{9}{5} \mu - \rho \\ \frac{36}{25} + \frac{36}{25} \mu + \frac{48}{25} \rho \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 + \rho - \frac{3}{10} \beta \\ -\frac{48}{25} - \frac{48}{25} \rho - \beta \end{bmatrix}, \quad A_5 = \begin{bmatrix} \frac{3}{10} + \frac{3}{10} \beta \\ 1 + \beta \end{bmatrix}, \quad B_3 = \begin{bmatrix} -\frac{6}{5} \alpha \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} \frac{6}{5} + \frac{6}{5} \alpha \\ -\frac{12}{25} \alpha \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 \\ \frac{12}{25} + \frac{12}{25} \alpha \end{bmatrix}.$$}

For $r = 2$:

$$A_0 = B_0 = B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{3}{128} - \frac{25}{128} \delta \\ \frac{2}{115} + \frac{3}{23} \delta \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{25}{128} + \frac{25}{128} \delta + \frac{225}{128} \mu \\ -\frac{3}{23} - \frac{3}{23} \delta - \frac{18}{23} \mu \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{225}{128} - \frac{225}{128} \mu - \rho \\ \frac{18}{23} + \frac{18}{23} \mu + \frac{192}{115} \rho \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 + \rho - \frac{75}{128} \beta \\ -\frac{192}{115} - \frac{192}{115} \rho - \beta \end{bmatrix}, \quad A_5 = \begin{bmatrix} \frac{75}{128} + \frac{75}{128} \beta \\ 1 + \beta \end{bmatrix}, \quad B_3 = \begin{bmatrix} -\frac{15}{8} \alpha \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} \frac{15}{8} + \frac{15}{8} \alpha \\ -\frac{12}{23} \alpha \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 \\ \frac{12}{23} + \frac{12}{23} \alpha \end{bmatrix}.$$}

For $r = \frac{5}{8}$:

$$A_0 = B_0 = B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{208}{775} - \frac{6912}{5425} \delta \\ \frac{12544}{29875} + \frac{53248}{29875} \delta \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{6912}{5425} + \frac{6912}{5425} \delta + \frac{13689}{5425} \mu \\ \frac{53248}{29875} - \frac{53248}{29875} \delta - \frac{74529}{29875} \mu \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \frac{13689}{6200} - \frac{13689}{6200} \mu - \rho \\ \frac{2548}{29875} + \frac{2548}{29875} \rho + \frac{1195}{1195} \mu \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 + \rho - \frac{351}{1736} \beta \\ \frac{2548}{1195} + \frac{2548}{1195} \rho - \beta \end{bmatrix}, \quad A_5 = \begin{bmatrix} \frac{351}{1736} + \frac{351}{1736} \beta \\ 1 + \beta \end{bmatrix},$$

$$B_3 = \begin{bmatrix} \frac{117}{124} \alpha \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} \frac{117}{124} + \frac{117}{124} \alpha \\ \frac{546}{1195} + \frac{546}{1195} \alpha \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 \\ \frac{546}{1195} + \frac{546}{1195} \alpha \end{bmatrix}.$$}

To determine the order of the method, the following definition is referred:
Definition 1

The LMM is said to be of order $q$ if $C_0 = C_1 = \ldots = C_q = 0$ and the error constant, $C_{q+1} \neq 0$, where

$$C_0 = \sum_{j=0}^{k} A_j, \quad C_q = \sum_{j=0}^{k} \left( \frac{1}{q!} j^q A_j - \frac{1}{(q-1)!} j^{q-1} B_j \right), \quad q = 1, 2, \ldots, k.$$  \hfill (9)

For $r = 1$, we may write

$$C_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 3 \delta + 9 \mu + 3 \beta \\ 16 \delta - 48 \mu + 36 \mu + \beta \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 \delta + 5 \mu - 27 \mu + 21 \\ 8 \delta - 24 \mu + 54 \mu - 2 \beta - 25 \alpha \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1 \delta + 19 \mu - 21 \mu + 37 \beta - 3 \alpha \\ 8 \delta - 152 \mu + 42 \mu + 37 \beta - 42 \alpha \\ 75 \delta - 125 \mu + 25 \mu + \beta \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 \delta + 65 \mu - 9 \mu + 35 \\ 40 \delta - 24 \mu + 16 \mu - 5 \beta - 5 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 1 \delta + 19 \mu - 93 \mu + 781 \beta - 8 \mu - 3 \\ 200 \delta - 120 \mu + 400 \mu + 14 \beta - 50 \alpha \end{bmatrix}.$$

For $r = 2$ and $r = \frac{5}{8}$, equations (9) are established in the following forms:

$$C_0 = A_0 + A_1 + A_2 + A_3 + A_4,$$

$$C_q = \frac{1}{q!} \left[ (-2r)^q A_0 + (-r)^q A_1 + (0)^q A_2 + (1)^q A_3 + (2)^q A_4 \right] - \frac{1}{(q-1)!} \left[ (0)^{q-1} B_2 + (1)^{q-1} B_3 + (2)^{q-1} B_4 \right], \quad q = 1, \ldots, 5.$$  \hfill (10)

The coefficients, $A_j$ and $B_j$ are substituted into equation (10) to produce $C_q = \begin{bmatrix} C_{q,1} \\ C_{q,2} \end{bmatrix}$, $q = 0, \ldots, 5$ as follows:

For $r = 2$:

$$C_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 25 \delta - 225 \mu + \rho + 75 \beta \\ 64 \mu - 64 \mu + \rho + 128 \beta \end{bmatrix}, \quad C_2 = \begin{bmatrix} 75 \delta + 225 \mu + 1 \mu + 225 \beta - 15 \\ 64 \delta + 64 \mu + 2 \mu + 256 \beta - 8 \alpha \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 175 \delta - 75 \mu + 1 \mu + 175 \beta - 15 \alpha \\ 96 \delta - 32 \mu + 1 \mu + 256 \beta - 16 \alpha \end{bmatrix}, \quad C_4 = \begin{bmatrix} 125 \delta + 75 \mu + 1 \mu + 375 \beta - 15 \alpha \\ 64 \delta + 64 \mu + 24 \mu + 1024 \beta - 16 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 30 \delta - 12 \mu - 8 \mu + 5 \beta - 14 \alpha \\ 23 \delta - 23 \mu - 8 \mu + \beta - 8 \alpha \end{bmatrix}.$$
\[
C_5 = \begin{bmatrix}
15 + 155 \delta - 15 \mu + \frac{1}{120} \rho + \frac{155}{1024} \beta - \frac{5}{64} \\
24 + 124 \delta + 24 \mu - \frac{8}{575} \rho + 120 \beta - \frac{46}{46}
\end{bmatrix}.
\]

For \( r = \frac{5}{8} \):

\[
C_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix}
864 \delta - \frac{13689}{1085} \mu + \rho + \frac{351}{1736} \beta \\
6656 \delta + \frac{74529}{5975} \mu - \frac{2548}{47800} \rho + \beta
\end{bmatrix}, C_2 = \begin{bmatrix}
-162 \delta + \frac{13689}{217} \mu + \frac{1}{2} \rho + \frac{1053}{3472} \beta - \frac{117}{124} \\
1248 \delta - \frac{74529}{1195} \mu - \frac{1274}{1195} \rho + \frac{3}{2} \beta - \frac{546}{1195}
\end{bmatrix},
\]

\[
C_3 = \begin{bmatrix}
\frac{45}{124} \delta - \frac{22181}{253952} \mu + \frac{1}{6} \rho + \frac{117}{496} \beta - \frac{117}{248} \\
\frac{364}{717} \delta + \frac{24843}{3585} \mu - \frac{1755}{6} \rho + \frac{7170}{1195} \beta - \frac{819}{1195}
\end{bmatrix}, C_4 = \begin{bmatrix}
\frac{3375}{27776} \delta + \frac{114075}{8126464} \mu + \frac{1}{24} \rho + \frac{13888}{248} \beta - \frac{39}{248} \\
\frac{325}{1192} \delta - \frac{124215}{7831552} \mu - \frac{637}{7170} \rho + \frac{5}{8} \beta - \frac{637}{1195}
\end{bmatrix}, C_5 = \begin{bmatrix}
\frac{4536}{158720} \delta - \frac{114075}{6501712} \mu + \frac{1}{120} \rho + \frac{117}{2240} \beta - \frac{273}{992} \\
\frac{24843}{382400} \delta + \frac{124215}{62652416} \mu - \frac{35850}{120} \rho - \frac{956}{120}
\end{bmatrix}.
\]

Hence, \( C_q, q = 0, \ldots, 4 \) are solved simultaneously to determine the order conditions for \( C_{q,1} \) and \( C_{q,2} \) as presented in table 1.

| Table 1. Order conditions for \( C_{q,1} \) and \( C_{q,2} \) for VSBBDF-\( \alpha \). |
|---|
| \( r \) | 1 | 2 | 5 \( \frac{5}{8} \) |
| Condition 1 | \( \mu = \frac{2}{3} \alpha, \beta = \frac{4}{3} \alpha \) | \( \mu = \frac{58}{225} \alpha, \beta = \frac{23}{15} \alpha \) | \( \mu = \frac{96128}{95823} \alpha, \beta = \frac{142}{117} \alpha \) |
| \( \rho = \frac{3}{5} \alpha, \delta = \frac{1}{3} \alpha \) | \( \rho = \frac{13}{128} \alpha, \delta = \frac{7}{25} \alpha \) | \( \rho = \frac{731}{868} \alpha, \delta = \frac{432}{432} \alpha \) |
| Condition 2 | \( \mu = \frac{1}{2} \alpha, \beta = \frac{22}{25} \alpha \) | \( \mu = \frac{37}{180} \alpha, \beta = \frac{77}{92} \alpha \) | \( \mu = \frac{23680}{31941} \alpha, \beta = \frac{1078}{1195} \alpha \) |
| \( \delta = \frac{1}{4} \alpha, \rho = \frac{3}{4} \alpha \) | \( \rho = \frac{511}{768} \alpha, \delta = \frac{11}{60} \alpha \) | \( \rho = \frac{439}{546} \alpha, \delta = \frac{385}{1248} \alpha \) |

Conditions 1 and 2 are substituted into \( C_{q,1} \) and \( C_{q,2} \) respectively to produce

\[
C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

The error constants, \( C_5 \) for \( r = 1, 2 \) and \( r = \frac{5}{8} \) are

\[
\begin{bmatrix}
\frac{1}{10} \alpha + \frac{3}{50} \\
-\frac{3}{25} \alpha - \frac{12}{125}
\end{bmatrix}.
\]
is 4. Subsequently, all conditions from table 1 are substituted into equation (8). From this, the corrector formulas that possess one independent parameter $\alpha$ are obtained as follows:

For $r = 1$:

$$y_{n+1} = \left[ \frac{15}{64} + \frac{31}{64} \alpha \right] y_{n-2} + \left[ \frac{4563}{18720} + \frac{10023}{18720} \alpha \right] y_{n-1} + \left[ \frac{3}{5} \alpha \right] y_n + \left[ \frac{-2}{5} \alpha \right] y_{n+2}$$

$$y_{n+2} = \left[ \frac{3}{25} \alpha \right] y_{n-2} + \left[ \frac{16}{25} \alpha \right] y_{n-1} + \left[ \frac{36}{25} \alpha \right] y_n + \left[ \frac{48}{25} \alpha \right] y_{n+1}$$

$$+ \left[ \frac{12}{25} \alpha \right] hf_{n+2} + \left[ \frac{-12}{25} \alpha \right] hf_{n+1}.$$  

(11)

For $r = 2$:

$$y_{n+1} = \left[ \frac{3}{128} + \frac{7}{128} \alpha \right] y_{n-2} + \left[ \frac{-25}{128} \alpha \right] y_{n-1} + \left[ \frac{225}{128} + \frac{45}{128} \alpha \right] y_n + \left[ \frac{-75}{128} \alpha \right] y_{n+2}$$

$$+ \left[ \frac{15}{8} + \frac{15}{8} \alpha \right] hf_{n+1} + \left[ \frac{-15}{8} \alpha \right] hf_n,$$

$$y_{n+2} = \left[ \frac{-2}{115} \alpha \right] y_{n-2} + \left[ \frac{3}{23} + \frac{17}{23} \alpha \right] y_{n-1} + \left[ \frac{-18}{23} + \frac{117}{23} \alpha \right] y_n + \left[ \frac{192}{23} + \frac{224}{23} \alpha \right] y_{n+1}$$

$$+ \left[ \frac{12}{23} + \frac{23}{23} \alpha \right] hf_{n+2} + \left[ \frac{-23}{23} \alpha \right] hf_{n+1}.$$  

(12)
For $r = \frac{5}{8}$:

\[
y_{n+1} = \left( \begin{array}{c}
\frac{208 + 368 \alpha}{775 + 775} \\
1 + 37 \alpha \\
62 
\end{array} \right) y_{n-2} + \left( \begin{array}{c}
\frac{6912 - 14592 \alpha}{5425 - 5425} \\
1 + 37 \alpha \\
62 
\end{array} \right) y_{n-1} + \left( \begin{array}{c}
\frac{13689 + 9477 \alpha}{6200 + 3100} \\
1 + 37 \alpha \\
62 
\end{array} \right) y_n \\
+ \left( \begin{array}{c}
\frac{351 - 213 \alpha}{1736 - 868} \\
1 + 37 \alpha \\
62 
\end{array} \right) y_{n+2} + \left( \begin{array}{c}
\frac{117 + 117 \alpha}{124 + 124} \\
1 + 37 \alpha \\
62 
\end{array} \right) hf_{n+1} + \left( \begin{array}{c}
\frac{-117 \alpha}{124} \\
1 + 37 \alpha \\
62 
\end{array} \right) hf_n, \\
y_{n+2} = \left( \begin{array}{c}
\frac{12544 + 9856 \alpha}{29875 + 17925} \\
1 + 1078 \alpha \\
1195 
\end{array} \right) y_{n-2} + \left( \begin{array}{c}
\frac{53248 + 14336 \alpha}{29875 + 5975} \\
1 + 1078 \alpha \\
1195 
\end{array} \right) y_{n-1} + \left( \begin{array}{c}
\frac{-74529 + 21294 \alpha}{29875 + 5975} \\
1 + 1078 \alpha \\
1195 
\end{array} \right) y_n \\
+ \left( \begin{array}{c}
\frac{2548 + 1876 \alpha}{1195 + 117} \\
1 + 1078 \alpha \\
1195 
\end{array} \right) y_{n+1} + \left( \begin{array}{c}
\frac{546 + 546 \alpha}{1195 + 1195} \\
1 + 1078 \alpha \\
1195 
\end{array} \right) hf_{n+2} + \left( \begin{array}{c}
\frac{-546 \alpha}{1195} \\
1 + 1078 \alpha \\
1195 
\end{array} \right) hf_{n+1}. 
\]

(13)

The implementation of VSBBDF-\(\alpha\) is done in PECE mode, where \(P\) and \(C\) indicate the application of predictor and corrector respectively, while \(E\) represents the evaluation of (1). The predictor formulas are one order less than the corrector formulas (13), which are derived using the same derivation technique without taking into account the free parameters.

2.3. Convergence properties

The linear multistep method (LMM) is said to be convergent if the computed value at any time step approximates the exact solution as the step size approach to zero. The consistency and zero stability are necessary conditions for the convergence of the LMM [1]. For this purpose, we refer to the following definitions.

Definition 2.
The LMM is consistent if and only if the following conditions are satisfied:

\[
\sum_{j=0}^{k} A_j = 0, \quad \sum_{j=0}^{k} jA_j = \sum_{j=0}^{k} B_j. 
\]

(14)

Definition 3.
The LMM is said to be zero-stable if no root of the first characteristic polynomial, \(p(t)\) has modulus greater than one, and if every root with modulus one is simple.

2.4. Consistency

Definition 2 is established to suit with the step size ratio, \(r = 1, r = 2\) and \(r = \frac{5}{8}\). For the convenience of writing, equations (11) – (13) are written in the form of equation (14) where the coefficients of the VSBBDF-\(\alpha\), \(A_j\) and \(B_j\) are presented in table 2.
| \( r \) | \( A_0 \) | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( B_2 \) | \( B_3 \) | \( B_4 \) |
|------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 1    | \( \frac{1}{10} - \frac{1}{5} \) \( \alpha \) | \( \frac{3}{25} + \frac{4}{25} \) \( \alpha \) | \( \frac{3}{5} + \frac{7}{5} \) \( \alpha \) | \( \frac{9}{25} + \frac{54}{25} \) \( \alpha \) | \( \frac{1}{5} + \frac{1}{5} \) \( \alpha \) | \( -\frac{6}{5} \) \( \alpha \) | \( \frac{6}{5} + \frac{6}{5} \) \( \alpha \) | \( 0 \) |
| 2    | \( \frac{-3}{128} + \frac{7}{128} \alpha \) | \( \frac{2}{115} + \frac{11}{460} \alpha \) | \( \frac{25}{128} + \frac{65}{128} \alpha \) | \( \frac{18}{23} + \frac{117}{92} \alpha \) | \( \frac{1}{192} + \frac{224}{115} \alpha \) | \( -\frac{15}{8} \) \( \alpha \) | \( \frac{15}{8} + \frac{15}{8} \alpha \) | \( \frac{10}{23} + \frac{12}{23} \alpha \) | \( 0 \) |
| 5    | \( \frac{208}{775} + \frac{368}{775} \alpha \) | \( \frac{12544}{29875} + \frac{9856}{17925} \alpha \) | \( \frac{6912}{5425} + \frac{14592}{5425} \alpha \) | \( \frac{74529}{29875} + \frac{21294}{5975} \alpha \) | \( \frac{13689}{6200} + \frac{9477}{3100} \alpha \) | \( -\frac{117}{124} \) \( \alpha \) | \( \frac{117}{124} + \frac{117}{124} \alpha \) | \( \frac{546}{1195} + \frac{546}{1195} \alpha \) | \( 0 \) |
| 8    | \( \frac{5}{5} \) \( \alpha \) | \( \frac{4}{3} \) \( \alpha \) | \( \frac{3}{2} \) \( \alpha \) | \( \frac{1}{1} \) \( \alpha \) | \( \frac{37}{62} \) \( \alpha \) | \( \frac{117}{124} \) \( \alpha \) | \( \frac{117}{124} + \frac{117}{124} \alpha \) | \( \frac{546}{1195} + \frac{546}{1195} \alpha \) | \( 0 \) |

\[
\sum_{j=0}^{4} A_j = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\sum_{j=0}^{4} jA_j = \begin{bmatrix}
\frac{6}{5} & 15 & 117 \\
\frac{12}{25} & 12 & 124 \\
\frac{25}{25} & 12 & 546 \\
\frac{25}{25} & 12 & 115 \\
\frac{25}{25} & 12 & 115 \\
\end{bmatrix}
\]
\[
\sum_{j=0}^{4} B_j = \begin{bmatrix}
\frac{6}{5} & \frac{15}{8} & 117 \\
12 & 12 & 124 \\
25 & 23 & 546 \\
\end{bmatrix}
\]

Since \( \sum_{j=0}^{4} A_j = 0 \) and \( \sum_{j=0}^{4} jA_j = \sum_{j=0}^{4} B_j \), it can be concluded that the VSBBDF-\( \alpha \) is consistent.

2.5. Zero stability
Let us now discuss the zero stability for the proposed method. The stability analysis of the VSBBDF-\( \alpha \) for \( r=1 \), \( r=2 \) and \( r=\frac{5}{8} \) can be conducted through the application of the standard linear test equation, \( f = y' = \lambda y \). The absolute stability region in the \( h\lambda \)-plane is determined by solving \( \det(At^2 - Bt - C) = 0 \) to obtain the stability polynomial as follows:

For \( r=1 \):
\[
p(t, \hat{t}, \alpha) = \frac{12}{25} t^2 \hat{t} \alpha + \frac{1}{25} t - \frac{9}{25} t^2 + \frac{18}{125} t^2 \alpha - \frac{18}{125} t^2 \hat{h} - \frac{372}{125} t^4 \alpha \hat{h} - \frac{168}{125} t^4 \alpha^2 \hat{h} + \frac{144}{125} t^4 \alpha^2 \hat{h} - \frac{264}{125} t^4 \alpha \hat{h}
\]

For \( r=2 \):
\[
p(t, \hat{t}, \alpha) = \frac{63}{368} t^3 \alpha \hat{h} - \frac{1}{2944} t - \frac{873}{736} t^3 \alpha^2 \hat{h} + \frac{45}{46} t^3 \alpha^2 \hat{h} + \frac{279}{184} t^4 \alpha^2 \hat{h} + \frac{243}{736} t^4 \alpha^2 \hat{h} - \frac{33}{184} t^4 \alpha \hat{h} + \frac{15}{4} t^4 \alpha \hat{h}
\]

For \( r=\frac{5}{8} \):
\[
p(t, \hat{t}, \alpha) = -\frac{8348067}{1852250} t^3 \alpha \hat{h} + \frac{126144}{4630625} t^3 \alpha^2 \hat{h} - \frac{126144}{4630625} t^3 \alpha^2 \hat{h} + \frac{2297754}{4630625} t^4 \alpha \hat{h} + \frac{1572864}{4630625} t^4 \alpha \hat{h} + \frac{59750}{4630625} t^4 \alpha \hat{h} + \frac{366912}{4630625} t^4 \alpha \hat{h} + \frac{1155}{4630625} t^4 \alpha \hat{h}
\]
To determine the zero stability, we set $\hat{h} = 0$ and obtain the following equations for each step size selection:

For $r = 1$:

$$p(t, \alpha) = \frac{1}{25} t - \frac{9}{25} t^2 + \frac{18}{125} t^3 + \frac{6}{125} \alpha t + \frac{318}{125} \alpha t^4 - \frac{342}{125} \alpha t^5 + \frac{138}{125} \alpha^2 t^4 - \frac{54}{25} \alpha^2 t^3 + \frac{126}{125} \alpha^2 t^2 + \frac{6}{125} \alpha^2 t + \frac{197}{125} t^4 - \frac{153}{125} t^3.$$ 

For $r = 2$:

$$p(t, \alpha) = -\frac{1}{2944} t + \frac{289}{2944} t^2 - \frac{91}{46} t^3 + \frac{173}{92} \alpha t^4 - \frac{213}{1472} \alpha t^5 - \frac{3}{23} \alpha t^4 - \frac{57}{23} \alpha t^3 - \frac{21}{8} \alpha t^2.$$ 

For $r = \frac{5}{8}$:

$$p(t, \alpha) = \frac{262144}{4630625} t^7 - \frac{4209984}{4630625} t^6 + \frac{106031}{74090} t^5 - \frac{34569}{59750} t^4 - \frac{1897344}{4630625} t^3 + \frac{1572864}{4630625} t^2 \alpha - \frac{207519}{148180} t^4 \hat{h} - \frac{94506}{37045} t^4 \alpha - \frac{2297754}{926125} t^3 \alpha + \frac{43728}{37045} t^4 \alpha^2 - \frac{2645712}{926125} t^3 \alpha^2 + \frac{6189696}{4630625} t^2 \alpha^2 + \frac{1572864}{4630625} t \alpha^2.$$ 

Then, we obtain 4 roots, $t_1, t_2, t_3$ and $t_4$ for each step size ratio, where $t_1 = 0$ and $t_2 = 1$, while $t_3$ and $t_4$ have parameter $\alpha$. Based on Definition 3, it has to be mentioned that the method is zero-stable if all roots have modulus less than or equal to 1. The values of $t_3$ and $t_4$ in relation to some selected values of $\alpha$ must be investigated to ensure the zero stability of the proposed method. Therefore, the graphs of $\alpha$ against $t_3$ and $\alpha$ against $t_4$ are plotted in figures 1 - 6.

**Figure 1.** Graph of $t_3$ against $\alpha$ for VSBBDF-$\alpha$, $r = 1$  

**Figure 2.** Graph of $t_4$ against $\alpha$ for VSBBDF-$\alpha$, $r = 1$
Figure 3. Graph of $t_3$ against $\alpha$ for VSBBDF-$\alpha$, $r = 2$

Figure 4. Graph of $t_4$ against $\alpha$ for VSBBDF-$\alpha$, $r = 2$

Figure 5. Graph of $t_3$ versus $\alpha$ for VSBBDF-$\alpha$, $r = \frac{5}{8}$

Figure 6. Graph of $t_4$ versus $\alpha$ for VSBBDF-$\alpha$, $r = \frac{5}{8}$

Figure 1 shows the graph of $\alpha$ versus $t_3$ while figure 2 shows the graph of $\alpha$ versus $t_4$. The points in the small interval can be seen clearly if the smaller range of $\alpha$ (scale of horizontal axis) is used. In figure 1, $t_3 < 1$ when $\alpha > -0.8$ while figure 2 shows that $t_4 < 1$ for all values of $\alpha$. Thus, it can be said that the VSBBDF-$\alpha$ when $r = 1$ is zero-stable when $\alpha \in [-0.8, \infty)$. In figure 3, $t_3 > 1$ when $\alpha \in (-4.03, -1.14)$ while $t_3 < 1$ when $\alpha \in (-\infty, -4.03]$ and $\alpha \in [-1.14, \infty)$. On the other hand, figure 4 shows that $t_4 < 1$ for all values of $\alpha$. Thus, the VSBBDF-$\alpha$ for $r = 2$ is zero-stable when $\alpha \in (-\infty, -4.03]$ and $\alpha \in [-1.14, \infty)$. In figure 5, $t_3 < 1$ when $\alpha \in [-0.73, 3.47]$. Figure 6 shows that $t_4 < 1$ for all values of $\alpha$. This can be concluded that the VSBBDF-$\alpha$ for $r = \frac{5}{8}$ is zero-stable when $\alpha \in [-0.73, 3.47]$. All computations and graphs in this section are done in Maple™ 18.

2.6. Selection of step size

During the implementation of the method, the choices of the step size ratio are $r = 1$, $2$ and $\frac{5}{8}$, which correspond to maintaining, halving and increasing the step size respectively. Therefore, it is crucial to design the strategy of selecting the step size, thus minimize the number of steps. If the local truncation error, $LTE = \| y_{n+2}^{(k+1)} - y_{n+2}^{(k)} \|$, $k = 2$ is less than or equal to tolerance limit (TOL), the step size is
maintained as constant which equivalent to the formula when \( r = 1 \), thus \( h_{\text{new}} = c h_{\text{old}} \left( \frac{TOL}{LTE} \right)^{1/p} \), is computed, where \( c \) is the safety factor, \( p \) is the order of the method and \( h_{\text{old}} \) is the step size from previous block. The aim of utilizing the safety factor is to reduce the risk of failure step. In our case, we choose \( c = 0.7 \). If \( h_{\text{new}} > 1.6 h_{\text{old}} \) then the new step size becomes \( h_{\text{new}} = 1.6 h_{\text{old}} \) which equivalent to the formula when \( r = \frac{5}{8} \). Otherwise, if the LTE is greater than the accepted TOL, the step failure occurs and \( h_{\text{new}} = \frac{1}{2} h_{\text{old}} \) is computed. In this case, the step is repeated with halving the current step size which is equivalent to the formula when \( r = 2 \).

3. Numerical results
In this section, we will compare the performance of the VSBBDF-\( \alpha \) with the numerical results obtained in [17]. We consider the initial value problem (IVP) of first order stiff ODEs:

\[
y'(x) = -\lambda (y - x) + 1, \quad 0 \leq x \leq 10,
\]

with the initial condition, \( y(0) = 1 \). The exact solution is given by \( y(x) = e^{\lambda x} + x \). Equations with \( \lambda \) negative but large in magnitude are examples of stiff differential equations. The large size of \( |\lambda| \) may force step size, \( h \) to be much smaller in order for \( \lambda h \) to be in the stability region. The computation is done using C programming. The following tables give the numerical results using VSBBDF-\( \alpha \) with tolerance (TOL), \( 10^{-2}, 10^{-4} \) and \( 10^{-6} \). Inherent with this proposed method is the choice of independent parameter \( \alpha \) and how it affects the solution of this problem. Here, the values of parameter, \( \alpha = -0.5 \) and \( \alpha = 0.5 \) are chosen due to the convergence of the method. The results are compared with the results obtained from [17] where the problem is solved using non-block variable step variable order backward differentiation formula (NBDF) and existing VSBBDF of order 4 with no independent parameter. Tables 3-5 show the comparison made in terms of total number of steps (TS), successful steps (SS), failure steps (FS) and maximum error (MAXE). Plots of TOL versus MAXE are illustrated in figures 7 - 9.

| TOL   | Method       | \( \alpha \) | TS | SS | FS | MAXE          |
|-------|--------------|--------------|----|----|----|---------------|
| \( 10^{-2} \) | NBDF         | -            | 39 | 32 | 7  | 4.7188E-02   |
|       | VSBBDF       | -            | 25 | 25 | 0  | 1.2017E-04   |
|       | VSBBDF-\( \alpha \) | -0.5       | 26 | 26 | 0  | 2.0777E-04   |
|       |              | 0.5          | 27 | 27 | 0  | 3.7155E-05   |
| \( 10^{-4} \) | NBDF         | -            | 64 | 53 | 11 | 9.0525E-04   |
|       | VSBBDF       | -            | 39 | 39 | 0  | 2.0600E-06   |
|       | VSBBDF-\( \alpha \) | -0.5       | 41 | 41 | 0  | 2.4545E-06   |
|       |              | 0.5          | 46 | 46 | 0  | 5.9011E-07   |
| \( 10^{-6} \) | NBDF         | -            | 105| 89 | 16 | 1.3568E-05   |
|       | VSBBDF       | -            | 74 | 74 | 0  | 2.5285E-08   |
|       | VSBBDF-\( \alpha \) | -0.5       | 85 | 85 | 0  | 2.6057E-08   |
|       |              | 0.5          | 95 | 95 | 0  | 6.7590E-09   |
Table 4. Numerical results when $\lambda = -50$

| TOL   | Method      | $\alpha$ | TS | SS | FS | MAXE        |
|-------|-------------|----------|----|----|----|-------------|
| $10^{-2}$ | NBDF       | -        | 36 | 30 | 6  | 5.4862E-02 |
|        | VSBBDF     | -        | 26 | 26 | 0  | 2.3300E-04 |
|        | VSBBDF-$\alpha$ | -0.5   | 27 | 27 | 0  | 1.3012E-04 |
|        |            | 0.5      | 28 | 28 | 0  | 8.1878E-05 |
| $10^{-4}$ | NBDF       | -        | 63 | 53 | 10 | 6.1625E-04 |
|        | VSBBDF     | -        | 39 | 39 | 0  | 2.8412E-06 |
|        | VSBBDF-$\alpha$ | -0.5   | 42 | 42 | 0  | 2.1539E-06 |
|        |            | 0.5      | 45 | 45 | 0  | 5.4873E-07 |
| $10^{-6}$ | NBDF       | -        | 102 | 88 | 14 | 7.9504E-06 |
|        | VSBBDF     | -        | 75 | 75 | 0  | 1.9296E-08 |
|        | VSBBDF-$\alpha$ | -0.5   | 85 | 85 | 0  | 2.6648E-08 |
|        |            | 0.5      | 97 | 97 | 0  | 6.9660E-09 |

Table 5. Numerical results when $\lambda = -100$

| TOL   | Method      | $\alpha$ | TS | SS | FS | MAXE        |
|-------|-------------|----------|----|----|----|-------------|
| $10^{-2}$ | NBDF       | -        | 36 | 30 | 6  | 5.4862E-02 |
|        | VSBBDF     | -        | 26 | 26 | 0  | 2.3300E-04 |
|        | VSBBDF-$\alpha$ | -0.5   | 27 | 27 | 0  | 1.2260E-04 |
|        |            | 0.5      | 28 | 28 | 0  | 5.6043E-05 |
| $10^{-4}$ | NBDF       | -        | 65 | 54 | 11 | 6.1625E-04 |
|        | VSBBDF     | -        | 40 | 40 | 0  | 2.8412E-06 |
|        | VSBBDF-$\alpha$ | -0.5   | 43 | 43 | 0  | 2.2360E-06 |
|        |            | 0.5      | 47 | 47 | 0  | 5.5528E-07 |
| $10^{-6}$ | NBDF       | -        | 104 | 88 | 16 | 7.9504E-06 |
|        | VSBBDF     | -        | 76 | 76 | 0  | 1.9296E-08 |
|        | VSBBDF-$\alpha$ | -0.5   | 86 | 86 | 0  | 2.6665E-08 |
|        |            | 0.5      | 99 | 99 | 0  | 6.9895E-09 |

Figure 7. Graph of log(TOL) against log(MAXE) for $\lambda = -30$
Figure 8. Graph of log(TOL) against log(MAXE) for $\lambda = -50$
From tables 3-5, the similar pattern of results has been found for all $\lambda$. This shows the capability of the methods for solving stiff problem. However, there is a couple of interesting results to point out. The results indicate that the VSBBDF and VSBBDF-$\alpha$ reduce the total steps by 5% to 30% from NBDF at most tolerance. It can be seen that the VSBBDF-$\alpha$ requires more steps compared to the VSBBDF. This is attributed to the selection of step size when $\alpha = 0.5$ is used, which requires more steps to integrate the system along the interval. It is also observed clearly that the NBDF takes more steps than the VSBBDF and VSBBDF-$\alpha$. This is expected since both VSBBDF and VSBBDF-$\alpha$ compute the solutions at two points simultaneously. It is apparent from the results that VSBBDF and VSBBDF-$\alpha$ outperform the NBDF in terms of maximum error. We do not notice a significant decrease in the maximum error as we set $\alpha = -0.5$. Although the results of VSBBDF-$\alpha = -0.5$ is almost as accurate as VSBBDF for all tolerance, there is still an effect when the value of $\alpha$ is varied. In contrast, the VSBBDF-$\alpha$ shows more clearly the improvement in accuracy when we set $\alpha = 0.5$.

4. Conclusion

The present study is an effort to show the capability of the VSBBDF-$\alpha$ for solving stiff ODEs. The numerical treatment of VSBBDF-$\alpha$ is complicated in terms of formulation but it turns out to be sufficiently universal and convenient because it has independent parameters to be varied. It is known that the proposed method is sometimes unstable and fail to give better accuracy when inappropriate value of independent parameter is chosen. However, the results allow us to conclude that the influence of appropriate value of $\alpha$ can reduce the errors, hence better accuracy can be achieved. Although the existence of independent parameter in the coefficients of formula does not guarantee the significant improvement of accuracy, further research would be worthwhile and tends to become a realistic criterion in deriving a new numerical method with different order of accuracy.

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