$O(\alpha^n \alpha_s^m)$ Corrections in $e^+e^-$ Annihilation and $\tau$ Decay

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Abstract

The results of evaluation of mixed QED×QCD corrections to $R(s)$ in $e^+e^-$ annihilation and $R_\tau$ in hadronic decay of the $\tau$ lepton to $O(\alpha^n \alpha_s^m)$, $m+n \leq 3$, are presented. The net effect on $\alpha_s(M_Z)$ from the $Z$ decay is only about 0.1% and in the $\tau$ decay case the net effect is negligible, as expected.

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Inclusive processes of the hadronic decays of the Z boson and the τ lepton are in the center of experimental and theoretical considerations. These processes are best suitable for, e.g., high and low energy determinations of the strong coupling \( \alpha_s(s) \), thus providing one of the crucial tests of the Standard Model (SM). High experimental and theoretical precisions are important to to spot any deviations from the SM and thus to find a room for a possible new physics. The precise \( \alpha_s(s) \) is also crucial for checking of fundamentals of GUT. The up to date results for \( \alpha_s \) extracted at different energies, using various observables are in satisfactory agreement with the SM. However, the problem of unification of couplings at the GUT scale remains. There has been much attention drown to the so called \( R_b - R_c \) problem. In the above examples, high experimental and theoretical precisions are necessary. A significant progress has been made in the last decade in theoretical evaluation of the above quantities, using perturbation theory methods. To resolve the observed discrepancy between the experiment and the SM predictions, numerous calculations of contributions beyond the SM has been done recently (see, e.g., [8]).

In the present work we evaluate corrections due to multiple photon exchange between quarks and between leptons taking place along the gluon exchange between “final state” quarks. The calculation has been done to four-loop level corresponding to \( O(\alpha^n\alpha^m_s) \ (m+n \leq 3) \) corrections. The objective of this work is to check explicitly that there are no anomalously large perturbative coefficients in front of a small \( \alpha^n\alpha^m_s \) expansion parameter, that could make the correction significant. Such a possibility is not excluded (see, e.g., [9]). Indeed, individual diagram (or even gauge invariant sets of diagrams) contributions are often large and only thanks to some delicate cancellations, they add up to small numbers.

We found that all calculated new corrections are indeed small and, as expected, they can be neglected at the relevant level of precision of \( O(1\%) \).

The method of calculation is similar to the one used in earlier works and described in ref. [11]. Here we discuss issues specific mainly for the present calculation. Known electroweak and QCD corrections to the Z decay rates are summarised in refs. [3] and [7], correspondingly.

The decay rate of the Z boson is usually calculated as the imaginary part of the correlation function of two neutral weak currents of quarks coupled to Z boson. The decay rate has vector and axial-vector parts, which generally are different. In the limit of massless quarks - the approximation used in this work, the vector and axial parts are identical up to the known factors. Therefore, here we calculate only a nonsinglet vector part, or the quantity \( R(s) \). The total hadronic decay rate \( \Gamma_Z \) can then be obtained straightforwardly, by multiplying the result on the sum of the squares of vector and axial couplings and by adding other known corrections from refs. [3,7].

The four-loop \( R(s) \), including the mixed QCD×QED corrections of \( O(\alpha^n\alpha^m_s) \ (m+n \leq 3) \) has the following form

\[
R(s) = \ldots
\]
\[ R(s) = R_0 + \frac{\alpha(s)}{\pi} R_1^{(a)} + \left( \frac{\alpha(s)}{\pi} \right)^2 R_2^{(a^2)} + \left( \frac{\alpha(s)}{\pi} \right)^3 R_3^{(a^3)} + \frac{\alpha_s(s)}{\pi} R_1^{(a_s)} + \frac{\alpha(s)}{\pi} R_2^{(a_s a)} + \left( \frac{\alpha(s)}{\pi} \right)^2 R_3^{(a^2 a_s)} + \left( \frac{\alpha(s)}{\pi} \right)^3 R_3^{(a^3)} \] (1)

The powers of electromagnetic coupling \( \alpha \) and the strong coupling \( \alpha_s \) in the superscripts of \( R_i \) indicate correspondingly the number of photon and the number of gluon exchanges involved in the corresponding multiloop Feynman graphs. The couplings are renormalized according to the MS prescription \[12\]

\[ \alpha_s^B = \mu^{2\varepsilon} \alpha_s \left[ 1 - \frac{\alpha_s}{\pi} \left( \beta_0^{(a_s)} + \frac{\alpha_s}{\pi} \beta_1^{(a_s a)} \right) + \left( \frac{\alpha_s}{\pi} \right)^2 \frac{(\beta_0^{(a_s)})^2}{\varepsilon^2} - \frac{\beta_1^{(a^2)}}{2\varepsilon} \right], \] (2)

\[ \alpha^B = \mu^{2\varepsilon} \alpha \left[ 1 - \frac{\alpha}{\pi} \left( \beta_0^{(a)} + \frac{\alpha}{\pi} \beta_1^{(a a)} \right) + \left( \frac{\alpha}{\pi} \right)^2 \frac{(\beta_0^{(a)})^2}{\varepsilon^2} - \frac{\beta_1^{(a^2)}}{2\varepsilon} \right], \] (3)

where the couplings run according to the renormalization group equations

\[ \alpha_s \beta^{QCD}(\alpha, \alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2}, \quad \alpha \beta^{QED}(\alpha, \alpha_s) = \mu^2 \frac{d\alpha}{d\mu^2} \] (4)

and the two-loop renormalization group \( \beta \) functions now include \( O(\alpha \alpha_s) \) perturbative terms

\[ \beta^{QCD}(\alpha, \alpha_s) = -\frac{\alpha_s}{\pi} \left( \beta_0^{(a_s)} + \frac{\alpha_s}{\pi} \beta_1^{(a_s a)} \right) - \left( \frac{\alpha_s}{\pi} \right)^2 \beta_1^{(a^2)} + \cdots, \] (5)

\[ \beta^{QED}(\alpha, \alpha_s) = -\frac{\alpha}{\pi} \left( \beta_0^{(a)} + \frac{\alpha}{\pi} \beta_1^{(a a)} \right) - \left( \frac{\alpha}{\pi} \right)^2 \beta_1^{(a^2)} + \cdots. \] (6)

The known values for one- and two-loop \( \beta \) function coefficients can be found, e.g., in ref. \[11\]

\[ \beta_0^{(a_s)} = \frac{1}{4} \left( \frac{11}{3} C_A - \frac{4}{3} T N_f \right), \quad \beta_0^{(a)} = -\frac{1}{3} (N_l + N_F) \sum_{f=u,d,...} Q_f^2, \]

\[ \beta_1^{(a^2)} = \frac{1}{16} \left( \frac{34}{3} C_A^2 - \frac{20}{3} C_A T N_f - 4 C_F T N_f \right), \quad \beta_1^{(a^2)} = -\frac{1}{4} (N_l + N_F) \sum_{f=u,d,...} Q_f^2. \]

The \( O(\alpha \alpha_s) \) coefficients can be found from the analysis of two-loop graphs and the known two-loop results
\[ \beta_1^{(\alpha_1)} = -\frac{1}{4} T \sum_{f=u,d,...} Q_f^2, \quad \beta_1^{(\alpha_2)} = -\frac{1}{4} C_F N_F \sum_{f=u,d,...} Q_f^2. \]

The eigenvalues of the Casimir operators for the adjoint \( (N_A = 8) \) and the fundamental \( (N_F = 3) \) representations of SU\(_c\)(3) gauge group are \( C_A = 3 \) and \( C_F = 4/3 \). The Dynkin index \( T \) for the fundamental representation is usually chosen \( T = 1/2 \). \( N_l \) is the number of leptons appearing in closed fermion loops not attached to the gluon propagators. \( Q_f \) denotes electric charge of the quark of flavor \( f \) in the units of the electron charge.

In order to calculate the unknown perturbative terms in Eq. (1), we have rederived Eq. (6.5) of ref. [11] - the four-loop expression for \( R(s) \) in terms of perturbative coefficients of the correlation function, its renormalization constant and the \( \beta \) function, taking into account \( O(\alpha_m \alpha_n s) \) terms. We have also found the renormalization group constraints, similar to the ones in Eqs. (2.29), (2.30) and (2.32) of ref. [11]. These relations are very helpful in testing the intermediate results. Contributions from two-, three- and four-loop graphs were found using the graph-by-graph results of the work [10]. This required a careful counting of symmetric and statistical factors for each of the contributing 98 four-loop, 14 three-loop and 2 two-loop Feynman graphs and calculation of the gauge group weights.

We obtain the following analytical results for the QED and QED×QCD terms in Eq. (1) for the standard U(1) and SU\(_c\)(3) gauge groups within the \( \overline{\text{MS}} \) framework

\[ R_1^{(\alpha)} = \frac{9}{4} \sum f Q_f^4, \]

\[ R_2^{(\alpha^2)} = -\frac{9}{32} \sum f Q_f^6 - \left(\frac{33}{8} - 3\zeta(3)\right) (N_l + 3 \sum j Q_j^2) \sum f Q_f^4, \]

\[ R_3^{(\alpha^3)} = -\frac{207}{128} \sum f Q_f^8 + \left(\frac{27}{8} + \frac{39}{4} \zeta(3) - 15\zeta(5)\right) (N_l + 3 \sum j Q_j^2) \sum f Q_f^6 \]

\[ + \left(\frac{151}{18} - \frac{1}{2} \zeta(2) - \frac{19}{3} \zeta(3)\right) (N_l + 3 \sum j Q_j^2)^2 \sum f Q_f^4 \]

\[ - \left(\frac{303}{64} - \frac{9}{2} \zeta(3)\right) (N_l + 3 \sum j Q_j^4) \sum f Q_f^4 \]

\[ + \left(\frac{11}{4} - 6\zeta(3)\right) (N_l + 3 \sum i Q_i^4) \sum f Q_f^4, \]

\[ R_2^{(\alpha \alpha_2)} = -\frac{3}{4} \sum f Q_f^4, \]

\[ R_3^{(\alpha^2 \alpha_2)} = -\frac{207}{32} \sum f Q_f^8 - \left(\frac{303}{16} - 18\zeta(3)\right) \sum f Q_j^2 \sum f Q_f^4 \]

\[ + \left(\frac{9}{2} + 13\zeta(3) - 20\zeta(5)\right) (N_l + 3 \sum j Q_j^2) \sum f Q_f^4, \]
\( R_3^{(\alpha s^2)} = -\left[ \left( \frac{519}{16} + \frac{429}{4} \zeta(3) - 165\zeta(5) \right) - N_f \left( \frac{9}{4} + \frac{13}{2}\zeta(3) - 10\zeta(5) \right) \right] \sum_f Q_f^4 \)

\[ -\left( \frac{101}{32} - 3\zeta(3) \right) \sum_j Q_j^2 \sum_f Q_f^2 \]

\[ + \left( \frac{11}{2} - 12\zeta(3) \right) \sum_i Q_i^2 \sum_j Q_j^2 \]  

where summations run over participating quark flavors, normally - u, d, s, c, b. We keep summation indices different in order to identify its source topology. For instance, the summation over \( f \) is due to the outer quark loops - the “final state” quarks. The summations over \( i \) and \( j \) correspond to the inner quark loops where virtual quarks and leptons can appear. Note that, of course, leptons can appear only in the case when there is no gluon line attached to the fermion loop. The last terms in Eqs. (10) and (13) are due to the specific four-loop topology where current operators are inserted in the separate fermion loops. The two-, three- and four-loop QCD terms are known and can be found in the original work [10]. Corrections due to non-vanishing quark masses to these terms are also known and can be found, for instance, in the review paper [7]. For simplicity, in the present work we neglect all mass corrections which can trivially be added to our results.

For five massless quarks and infinitely heavy top quark we obtain the following numerical result

\[ R(s) = 3 \sum_{u,d,s,c,b} Q_f^2 \left\{ 1 + 0.2652 \frac{\alpha(s)}{\pi} - 3.2747 \left( \frac{\alpha(s)}{\pi} \right)^2 - 2.3539 \left( \frac{\alpha(s)}{\pi} \right)^3 \right. \]

\[ + \frac{\alpha_s(s)}{\pi} \left[ 1 - 0.0884 \frac{\alpha(s)}{\pi} - 0.4089 \left( \frac{\alpha(s)}{\pi} \right)^2 \right] \]

\[ + \left( \frac{\alpha_s(s)}{\pi} \right)^2 \left[ 1.409 - 2.4857 \frac{\alpha(s)}{\pi} \right] \]

\[ -12.805 \left( \frac{\alpha(s)}{\pi} \right)^3 \}. \]  

One can see that the calculated corrections are very small. The largest corrections beyond the leading QED term - \( \sim \alpha^2 \), \( \sim \alpha_s \alpha \) and \( \sim \alpha_s^2 \alpha \) have a same sign and they add up to the total effect of about 0.1 MeV on the Z width and only 0.1% on \( \alpha_s \). The other calculated corrections are clearly negligible for any phenomenological applications. The calculated terms of \( O(\alpha^2) \) and \( O(\alpha \alpha_s) \) confirm known results [13].

Let us use the calculated corrections on the Z width and obtain the similar terms for the quantity

\[ R_\tau = \frac{\Gamma(\tau^- \to \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \to \nu_\tau e^-\nu_e)} \]  

in the \( \tau \) lepton hadronic decay process. Following the well known procedure [14], we calculate the perturbative part as an integral (see, e.g., [11] for details)

\[ R_{\tau}^{\text{pert}} = \frac{3i}{8\pi} \int_C \frac{ds}{M_\tau^2} \left( 1 - \frac{s}{M_\tau^2} \right)^2 \left( 1 + 2 \frac{s}{M_\tau^2} \right) \Pi(s), \]  

(16)
where $\Pi(s)$ is the transverse part of the correlation function of weak currents of quarks coupled to W boson. The countour $C$ is the circle of radius $|s| = M^2$. In the integrand we substitute the $\alpha_s(s)$ and $\alpha(s)$ by their expansion in terms of $\alpha_s(M)$ and $\alpha(M)$ using the equations

$$
\frac{\alpha_s(s)}{\pi} = \frac{\alpha_s(M)}{\pi} + \left( \frac{\alpha_s(M)}{\pi} \right)^2 \left[ \beta_0^{(a_s)} \log \frac{M^2}{s} + \left( \frac{\alpha_s(M)}{\pi} \right) \beta_1^{(a_s)} \log \frac{M^2}{s} \right] \\
+ \left( \frac{\alpha_s(M)}{\pi} \right)^3 \left[ \beta_1^{(a_s)} \log \frac{M^2}{s} + (\beta_0^{(a_s)})^2 \log^2 \frac{M^2}{s} \right],
$$

(17)

$$
\frac{\alpha(s)}{\pi} = \frac{\alpha(M)}{\pi} + \left( \frac{\alpha(M)}{\pi} \right)^2 \left[ \beta_0^{(a)} \log \frac{M^2}{s} + \left( \frac{\alpha_s(M)}{\pi} \right) \beta_1^{(a_s)} \log \frac{M^2}{s} \right] \\
+ \left( \frac{\alpha_s(M)}{\pi} \right)^3 \left[ \beta_1^{(a)} \log \frac{M^2}{s} + (\beta_0^{(a)})^2 \log^2 \frac{M^2}{s} \right].
$$

(18)

Then we evaluate contour integral and express $R_\tau$ in terms of the perturbative coefficients $R_i$ in Eq. (4), making obvious substitution $\sum_f Q^2_f \to |V_{ud}|^2 + |V_{us}|^2$ and omitting terms coming from the four-loop topologies, where current insertions are placed in the separate fermion loops. Thus for the perturbative part of the $R_\tau$ we obtain the following $\overline{\text{MS}}$ analytical result

$$
R_\tau^\text{pert} = 3(|V_{ud}|^2 + |V_{us}|^2) \left\{ 1 + \left( \frac{\alpha(M)}{\pi} \right) r_1^{(a)} + \left( \frac{\alpha_s(M)}{\pi} \right)^2 r_2^{(a^2)} + \left( \frac{\alpha(M)}{\pi} \right)^3 r_3^{(a^3)} \\
+ \frac{\alpha_s(M)}{\pi} \left( r_1^{(a_s)} + \frac{\alpha(M)}{\pi} r_2^{(a_a)} + \left( \frac{\alpha_s(M)}{\pi} \right)^3 r_3^{(a^2_a)} \right) \\
+ \left( \frac{\alpha_s(M)}{\pi} \right)^2 \left( r_2^{(a^2)} + \frac{\alpha_s(M)}{\pi} r_3^{(a^2_a)} \right) \\
+ \left( \frac{\alpha_s(M)}{\pi} \right)^3 \right\},
$$

(19)

where for the standard U(1) and SU_c(3) gauge groups within the $\overline{\text{MS}}$ framework we obtain

$$
r_1^{(a)} = \frac{3}{4} \sum_f Q^2_f,
$$

(20)

$$
r_2^{(a^2)} = -\frac{3}{32} \sum_f Q^4 - \left( \frac{85}{48} - \zeta(3) \right) (N_l + 3 \sum_j Q^2_j) \sum_f Q^2_f,
$$

(21)

$$
r_3^{(a^3)} = -\frac{69}{128} \sum_f Q^6 + \left( \frac{235}{192} + \frac{13}{4} \zeta(3) - 5 \zeta(5) \right) (N_l + 3 \sum_j Q^2_j) \sum_f Q^4_f \\
+ \left( \frac{3935}{864} - \frac{1}{3} \zeta(2) - \frac{19}{6} \zeta(3) \right) (N_l + 3 \sum_j Q^2_j)^2 \sum_f Q^2_f \\
- \left( \frac{15}{8} - \frac{3}{2} \zeta(3) \right) (N_l + 3 \sum_j Q^4_j) \sum_f Q^2_f,
$$

(22)
\[ r_2^{(\alpha \alpha_s)} = -\frac{1}{4} \sum_f Q_f^2, \quad (23) \]

\[ r_3^{(\alpha^2 \alpha_s)} = -\frac{69}{32} \sum_f Q_f^4 + \left( \frac{235}{144} + \frac{13}{3} \zeta(3) - \frac{20}{3} \zeta(5) \right) (N_i + 3 \sum_j Q_j^2) \sum_f Q_f^2 
\quad - \left( \frac{15}{2} - 6 \zeta(3) \right) \sum_j Q_j^2 \sum_f Q_f^2, \quad (24) \]

\[ r_3^{(\alpha \alpha_s)} = -\left( \frac{605}{64} + \frac{117}{4} \zeta(3) - 45 \zeta(5) \right) - \left( \frac{5}{4} - \zeta(3) \right) \sum_j Q_j^2. \quad (25) \]

The summation index \( f \) runs over \( u, d, s \) “final state” quark flavors appearing in the outer fermionic loops and the summation over \( j \) corresponds inner (virtual) fermionic loops. We take \( j = u, d, s \). For the number of virtual leptons in the inner fermionic loops, we will take \( N_l = 2 \) (\( e, \mu \)). In fact, the \( \tau \)-lepton can also appear in the virtual loops. However, within our approximation, masses of all quarks and leptons, including virtual ones are neglected. On the other hand, the \( \tau \) mass cannot be neglected at this scale. Therefore, the effect of the virtual \( \tau \) lepton is left out of our consideration. This effect can be evaluated via exact evaluation of Feynman graphs with massive loops. The corresponding numerical effect is expected to be small. The two-, three- and four-loop QCD terms are known and can be found in [15]. We obtain the following numerical expression

\[ R_{\tau}^{\text{pert}} = 3 \left( | V_{ud} |^2 + | V_{us} |^2 \right) \left\{ 1 + 0.5 \frac{\alpha(s)}{\pi} - 1.5376 \left( \frac{\alpha(s)}{\pi} \right)^2 + 1.9041 \left( \frac{\alpha(s)}{\pi} \right)^3 + \frac{\alpha_s(s)}{\pi} \left[ 1 - 0.166 \frac{\alpha(s)}{\pi} - 0.799 \left( \frac{\alpha(s)}{\pi} \right)^2 \right] \right. 
\quad \left. + \left( \frac{\alpha_s(s)}{\pi} \right)^2 \left[ 5.202 + 1.2778 \frac{\alpha(s)}{\pi} \right] \right. 
\quad + 26.366 \left( \frac{\alpha_s(s)}{\pi} \right)^3 \right\}. \quad (26) \]

As one can see, the higher order QED and QED×QCD corrections are very small. Even the largest ones of \( O(\alpha \alpha_s) \) and \( O(\alpha^2 \alpha_s) \) have similar size but opposite signs and they cancel each other along the \( O(\alpha^2) \) correction. Thus, what remains is the leading \( O(\alpha) \) correction that has some numerical relevance. All other corrections are clearly negligible.

Summarising, we note that the calculated higher order QED and mixed QED×QCD corrections to the hadronic decay rates of the Z boson and the \( \tau \) lepton can be safely neglected in present phenomenological applications.

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