Initiality for Typed Syntax and Semantics

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Abstract. We give an algebraic characterization of the syntax and semantics of a class of simply–typed languages, such as the language PCF: we characterize simply–typed binding syntax equipped with reduction rules via a universal property, namely as the initial object of some category. For this purpose, we employ techniques developed in two previous works: in [2], we model syntactic translations between languages over different sets of types as initial morphisms in a category of models. In [1], we characterize untyped syntax with reduction rules as initial object in a category of models. In the present work, we show that those techniques are modular enough to be combined: we thus characterize simply–typed syntax with reduction rules as initial object in a category. The universal property yields an operator which allows to specify translations — that are semantically faithful by construction — between languages over possibly different sets of types.

We specify a language by a 2–signature, that is, a signature on two levels: the syntactic level specifies the types and terms of the language, and associates a type to each term. The semantic level specifies, through inequations, reduction rules on the terms of the language. To any given 2–signature we associate a category of models. We prove that this category has an initial object, which integrates the types and terms freely generated by the 2–signature, and the reduction relation on those terms generated by the given inequations. We call this object the (programming) language generated by the 2–signature.

1 Introduction

We give a characterization, via a universal property, of the syntax and semantics of simply–typed languages with variable binding. More precisely, we characterize the terms and sorts associated to a signature equipped with reduction rules as the initial object in a category of models. Initiality in this category gives rise to an iteration principle (cf. Rem. 45) which allows to specify translations between languages in a convenient way as initial morphisms. The category of models is sufficiently large — and thus the iteration principle stemming from initiality is sufficiently general — to account for translations between languages over different sets of sorts. Furthermore, translations specified via this principle are ensured to be faithful with respect to reduction in the source and target languages, as well as compatible in a suitable sense with substitution on either side.

To illustrate the iteration operator stemming from initiality, we use it to specify a translation from PCF to the untyped lambda calculus ULC. We do
so in the proof assistant Coq \[5\]; for this purpose, we prove formally, in Coq, an instance of our main theorem for the 2–signature of PCF: the types and terms of PCF, equipped with their usual reductions, form an initial object in the category of models of PCF. We then use the iteration principle to obtain an initial morphism — a translation, faithful with respect to reductions — to ULC, as an executable Coq function. The Coq theory files as well as documentation are available online at [http://math.unice.fr/laboratoire/logiciels](http://math.unice.fr/laboratoire/logiciels).

**Summary.** We define a notion of 2–signature which allows the specification of the types and terms of a language — via an underlying 1–signature — as well as its semantics in form of reduction rules. A 1–signature \((S, \Sigma)\) is given by a pair of a signature \(S\) for types and a binding signature \(\Sigma\) for terms typed over the set of types associated to \(S\). Reduction rules for terms generated by \(\Sigma\) are specified via a set \(A\) of inequations over \((S, \Sigma)\). A 2–signature \(((S, \Sigma), A)\) is a pair of a 1–signature \((S, \Sigma)\) and a set \(A\) of inequations over \((S, \Sigma)\). To such a 2–signature we associate a category of representations, for which the types and terms generated by \((S, \Sigma)\), equipped with reductions according to \(A\), forms an initial object.

1–signatures are defined in [2]. There, we associate a category \(\text{Rep}(S, \Sigma)\) of representations to any 1–signature \((S, \Sigma)\), and show that the types and terms freely generated by \((S, \Sigma)\) form an initial object in this category. Representations there are built from monads on families of sets. In the present work, we build a different category \(\text{Rep}^\Delta(S, \Sigma)\) of representations using relative monads from sets to preordered sets, which allows — in a second step, cf. below — the integration of reduction rules to account for semantic aspects. The two categories of representations, \(\text{Rep}(S, \Sigma)\) and \(\text{Rep}^\Delta(S, \Sigma)\), are connected through an adjunction which transports the initial object of the former to the latter category (cf. [Lem. 34]).

Inequations over untyped 1–signatures are considered in [1]. There, we define a notion of 2–signature for untyped syntax with semantics in form of reduction rules and show that its associated category of representations has an initial object. In the present work, we define inequations over typed 1–signatures as defined in [2]. Given a set \(A\) of inequations over a 1–signature \((S, \Sigma)\), the representations of \((S, \Sigma)\) that satisfy each inequation of \(A\), form a full subcategory of \(\text{Rep}^\Delta(S, \Sigma)\), which we call the category of representations of \((S, \Sigma, A)\). Our main theorem (cf. [Thm. 44]) states that this category has an initial object, which integrates the types and terms freely generated by \((S, \Sigma)\), equipped with reduction rules generated by the inequations of \(A\).

**Related Work.** Related work is reviewed extensively in [12], as well as in the author’s PhD thesis [3]. We give a brief overview: rewriting in nominal settings is examined by Férnandez and Gabbay [6]. Ghani and Lüth [8] present rewriting for algebraic theories without variable binding; they characterize equational theories resp. rewrite systems as coequalizers resp. coinserters in a category of monads on the categories Set resp. Pre. Fiore and Hur [7] have extended Fiore’s work to integrate semantic aspects into initiality results. In particular, Hur’s thesis