Multiple Solutions of a Nonlinear Biharmonic Equation on Graphs

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Abstract
In this paper, we consider a biharmonic equation with respect to the Dirichlet problem on a domain of a locally finite graph. Using the variation method, we prove that the equation has two distinct solutions under certain conditions.

Keywords Locally finite graph · Biharmonic equation · Distinct solutions

Mathematics Subject Classification 35A15 · 35G30

1 Introduction

The existence and non-existence of solutions of boundary value problems for biharmonic equations have been studied by many authors. A lot of results are devoted to the following problem in $H^2_0(\Omega)$,

$$\begin{cases}
\Delta^2 u = \lambda u + |u|^{p-2} u, & x \in \Omega, \\
u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0,
\end{cases} \quad (1.1)$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$; $\lambda$ is a constant; $p = \frac{2N}{N-4} \cdot$ If $\lambda < 0$ and $\Omega$ is star-shaped, then Problem (1.1) has only trivial solution. Denote by $\lambda_1$ the first eigenvalue of the problem.
\[
\begin{aligned}
\Delta^2 u &= \lambda u, \ x \in \Omega, \\
u|_{\partial\Omega} = 0, \ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0.
\end{aligned}
\]

If \(0 < \lambda < \lambda_1\) and \(N \geq 8\), then Problem (1.1) has at least one non-trivial solution \([4]\). For \(5 \leq n \leq 7\), there exists a constant \(\bar{\lambda} > 0\) such that Problem (1.1) has a non-trivial solution for all \(\lambda \in (\bar{\lambda}, \lambda_1)\). Deng and Wang \([3]\) studied the existence and non-existence of multiple solutions of biharmonic equations boundary value problem,

\[
\begin{aligned}
\Delta^2 u &= \lambda u + |u|^{p-2}u + f(x), \ x \in \Omega, \\
u|_{\partial\Omega} = 0, \ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0.
\end{aligned}
\]

We refer the reader to \([1,2,13,14]\) for more related results.

In this paper, we will study the similar problem on a graph. A discrete graph is denoted by \(G = (V, E)\), where \(V\) is the vertex set and \(E\) is the edge set. We say that \(G\) is locally finite if for any \(x \in V\), there is only finite \(y \in V\) such that \(xy \in E\). A graph is called connected if for any \(x, y \in V\), they can be connected via finite edges. Throughout this paper, we assume that \(G\) is locally finite and connected. Let \(\omega_{xy}\) be the weight of an edge \(xy \in E\) such that \(\omega_{xy} > 0\) and \(\omega_{xy} = \omega_{yx}\). We use a positive function \(\mu : V \to \mathbb{R}^+\) to define a measure on \(G\). For a bounded domain \(\Omega \subset V\), the boundary of \(\Omega\) is defined as

\[
\partial\Omega := \{y \notin \Omega : \exists x \in \Omega \text{ such that } xy \in E\}.
\]

We introduce some notations about the partial differential equations on a graph. The \(\mu\)-Laplacian of a function \(u : V \to \mathbb{R}\) is defined by

\[
\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} (u(y) - u(x)) ,
\]

where \(y \sim x\) stands for \(xy \in E\). For any two functions \(u\) and \(v\) on the graph, the gradient form is defined by

\[
\Gamma(u, v) = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy} (u(y) - u(x))(v(y) - v(x)).
\]

If \(u = v\), we write \(\Gamma(u) = \Gamma(u, u)\), which is used to define the length of the gradient for \(u\),

\[
|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy} (u(y) - u(x))^2\right)^{1/2}.
\]
For a function $u$ over $V$, the integral is defined by

$$\int_V u \, d\mu = \sum_{x \in V} \mu(x) u(x).$$

Recently, Grigor’yan et al. [6,7] applied the mountain-pass theorem to establish existence results for some nonlinear equations. Zhang and Zhao [15] studied the convergence of ground state solutions for nonlinear Schrödinger equations via the Nehari method on graphs. Using the similar method, Han et al. [9] studied existence and convergence of solutions for nonlinear biharmonic equations on graphs. One may refer to [10] for the results about the heat equation and refer to [5,8,11] for the results about the Kazdan–Warner equation on a graph.

In this paper, we consider the following equation

$$\begin{cases}
\Delta^2 u = \lambda u + |u|^{p-2}u + \epsilon f, & \text{in } \Omega,
\u = 0, & \text{on } \partial\Omega,
\end{cases} \tag{1.2}$$

where $\Omega$ is a bounded domain of $V$; $\lambda$, $p$ and $\epsilon$ are positive constants, $p > 2$; and $f$ is a given function on $V$.

Let $W^{2,2}(\Omega)$ be the space of functions $u : V \to \mathbb{R}$ under the norm

$$\|u\|_{W^{2,2}(\Omega)} = \left( \int_{\Omega \cup \partial\Omega} (|\Delta u|^2 + |\nabla u|^2) \, d\mu + \int_{\Omega} u^2 \, d\mu \right)^{1/2}. \tag{1.3}$$

Let $H(\Omega) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$, where $W^{1,2}_0(\Omega)$ is the completion of $C_c(\Omega)$ under the norm

$$\|u\|_{W^{1,2}_0(\Omega)} = \left( \int_{\Omega \cup \partial\Omega} |\nabla u|^2 \, d\mu + \int_{\Omega} u^2 \, d\mu \right)^{1/2},$$

and $C_c(\Omega) := \{ u : \Omega \to \mathbb{R} : \text{supp } u \subset \Omega, u|_{\partial\Omega} = 0 \}$.

Define

$$\|u\|_H = \left( \int_{\Omega \cup \partial\Omega} |\Delta u|^2 \, d\mu \right)^{1/2}$$

for any $u \in H$. It is easy to see that $\| \cdot \|_H$ is a norm on $H$. Noting that the dimension of $W^{2,2}(\Omega)$ is finite, we have that $\|u\|_H = \left( \int_{\Omega \cup \partial\Omega} |\Delta u|^2 \, d\mu \right)^{1/2}$ is a norm equivalent to (1.3) on $H(\Omega)$. Lemma 2.6 in [9] implies that $H(\Omega)$ is embedded in $L^q(\Omega)$ for all $1 \leq q < +\infty$ and there is a constant depending only on $q$ and $\Omega$ such that

$$\left( \int_{\Omega} |u|^q \, d\mu \right)^{1/q} \leq C \left( \int_{\Omega \cup \partial\Omega} |\Delta u|^2 \, d\mu \right)^{1/2}. \tag{1.4}$$
**Definition 1.1** For a $u \in H$, if for any $\varphi \in C_c(\Omega)$, there holds that

$$
\int_{\Omega \cup \partial \Omega} \Delta u \varphi \, d\mu = \lambda \int_{\Omega} u \varphi \, d\mu + \int_{\Omega} |u|^{p-2} u \varphi \, d\mu + \epsilon \int_{\Omega} f \varphi \, d\mu,
$$

then $u$ is called a weak solution of (1.2).

Define

$$
J_\epsilon (u) = \frac{1}{2} \int_{\Omega \cup \partial \Omega} |\Delta u|^2 \, d\mu - \frac{1}{2} \int_{\Omega} \lambda u^2 \, d\mu - \frac{1}{p} \int_{\Omega} |u|^p \, d\mu - \epsilon \int_{\Omega} f(x) u \, d\mu,
$$

which will be used in the later variation procedure. Define

$$
\lambda_1 (\Omega) = \inf_{u \neq 0, \, u|_{\partial \Omega} = 0} \frac{\int_{\Omega \cup \partial \Omega} |\Delta u|^2 \, d\mu}{\int_{\Omega} u^2 \, d\mu}.
$$

Using the variation method similar to that in [6,7], we prove the following theorem.

**Theorem 1.2** Let $G = (V, E)$ be a locally finite graph. Suppose that $0 < \lambda < \lambda_1 (\Omega)$, $f \in H' (\Omega)$ where $H'$ is the dual space of $H$. Then, there exists $\epsilon_1 > 0$ such that (1.2) has two distinct solutions if $0 < \epsilon < \epsilon_1$.

## 2 Proof of the Main Results

**Lemma 2.1** There exist positive constants $r_\epsilon$ and $\delta_\epsilon$ such that $J_\epsilon \geq \delta_\epsilon$ for all $u \in H$ with $r_\epsilon \leq \|u\|_H \leq 2r_\epsilon$ if $0 < \epsilon < \epsilon_1$ for a sufficiently small $\epsilon_1$.

**Proof**

$$
J_\epsilon (u) \geq \frac{1}{2} \|u\|^2_H - \frac{\lambda}{2} \int_{\Omega} |u|^2 \, d\mu - \frac{C}{p} \|u\|^p_H - \epsilon \|f\|_{H'} \|u\|_H
$$

$$
\geq \frac{\tau}{2} \|u\|^2_H - \frac{C}{p} \|u\|^p_H - \epsilon \|f\|_{H'} \|u\|_H
$$

$$
\geq \|u\|_H \left( \frac{\tau}{2} \|u\|_H - \frac{C}{2} \|u\|^{p-1} - \epsilon \|f\|_{H'} \right),
$$

where $\tau = \frac{\lambda_1 (\Omega) - \lambda}{\lambda_1 (\Omega)}$. Take $r_\epsilon = \sqrt{\epsilon}$. Since

$$
\lim_{\epsilon \to 0^+} \frac{\frac{\tau}{2} \sqrt{\epsilon} - 2^{p-2} C \epsilon^{(p-1)/2} - \epsilon \|f\|_{H'}}{\frac{\tau}{2} \sqrt{\epsilon}} = 1,
$$

there exists some $\epsilon_1$ such that

$$
\frac{\tau}{2} \sqrt{\epsilon} - 2^{p-2} C \epsilon^{(p-1)/2} - \epsilon \|f\|_{H'} \geq \frac{\tau}{4} \sqrt{\epsilon},
$$
if $0 < \epsilon < \epsilon_1$.

Setting $\delta_\epsilon = \frac{\epsilon}{4}$, we obtain $J_\epsilon(u) \geq \delta_\epsilon$ if $0 < \epsilon < \epsilon_1$.

\end{proof}

\textbf{Lemma 2.2} For any $c \in \mathbb{R}$, $J_\epsilon$ satisfies the \((PS)_c\) condition. If $(u_k) \subset H$ is a sequence such that $J_\epsilon(u_k) \to c$ and $J_\epsilon'(u_k) \to 0$, then up to a subsequence, $u_k$ converges to some $u$ in $H$.

\begin{proof}
If $J_\epsilon(u_k) \to c$ and $J_\epsilon'(u_k) \to 0$, then we have

\begin{equation}
\frac{1}{2} \int_{\Omega \cup \partial \Omega} |\Delta u_k|^2 d\mu - \frac{1}{2} \int_{\Omega} \lambda u_k^2 d\mu - \frac{1}{p} \int_{\Omega} |u_k|^p d\mu - \epsilon \int_{\Omega} f(x) u_k d\mu = c + o_k(1),
\end{equation}

(2.2)

\begin{equation}
|\int_{\Omega \cup \partial \Omega} |\Delta u_k|^2 d\mu - \int_{\Omega} \lambda u_k^2 d\mu - \int_{\Omega} |u_k|^p d\mu - \epsilon \int_{\Omega} f(x) u_k d\mu| = o_k(1) \|u_k\|_H,
\end{equation}

(2.3)

where $o_k(1) \to 0$ as $k \to +\infty$.

From (2.2) and (2.3), we have

\begin{equation}
\left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} |u_k|^p d\mu = c + \frac{\epsilon}{2} \int_{\Omega} f u_k d\mu + o_k(1) \|u_k\|_H + o_k(1).
\end{equation}

(2.4)

Hence,

\begin{equation}
\tau \|u_k\|_H^2 \leq \int_{\Omega \cup \partial \Omega} |\Delta u_k|^2 d\mu - \int_{\Omega} \lambda u_k^2 d\mu
\end{equation}

\begin{equation}
\leq \frac{2pc}{p - 2} + \frac{(2p - 2)\epsilon}{p - 2} \|f\|_{H'} \|u_k\|_H + o_k(1) \|\mu\|_H + o_k(1)
\end{equation}

(2.5)

\begin{equation}
\leq \frac{2pc}{p - 2} + \frac{4(p - 1)^2 \epsilon^2}{(p - 2)^2 \tau} \|f\|_{H'}^2 + \frac{\tau}{4} \|u_k\|_H^2 + \frac{\tau}{4} \|u_k\|_H^2 + o_k(1).
\end{equation}

Thus, we have $(u_k)$ that is bounded in $H$. Since $H$ is pre-compact, it follows that up to a subsequence, $u_k$ converges to some $u$ in $H$.

Now, we arrive at a position to prove the main theorem. For any $u^* \in H$, passing to the limit $t \to +\infty$, we get

\begin{equation}
J_\epsilon(tu^*) = \frac{t^2}{2} \int_{\Omega \cup \partial \Omega} |\Delta u^*|^2 d\mu - \frac{t^2}{2} \int_{\Omega} \lambda u^*^2 d\mu - \frac{t^p}{p} \int_{\Omega} |u^*|^p d\mu
\end{equation}

\begin{equation}
- t\epsilon \int_{\Omega} f(x) u^* d\mu \to -\infty
\end{equation}

as $t \to \infty$. Hence, there exists some $\tilde{u} \in H$ such that $J_\epsilon(\tilde{u}) < 0$ with $\|\tilde{u}\|_H > r_\epsilon$. Combining Lemma 2.1, we see that $J_\epsilon$ satisfies all the hypotheses of the mountain-pass theorem: $J_\epsilon \in C^1(H, \mathbb{R})$; $J_\epsilon(0) = 0$; when $\|u\|_H = r_\epsilon$, $J_\epsilon(u) \geq \delta_\epsilon$; $J_\epsilon(\tilde{u}) < 0$ for some $\tilde{u}$ with $\|\tilde{u}\|_H > r_\epsilon$. 

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Then, we have
\[ c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J_\epsilon(u) \]
that is the critical point of \( J_\epsilon \), where
\[ \Gamma = \{ \gamma \in C([0, 1], H) : \gamma(0) = 0, \gamma(1) = \tilde{u} \}. \]

Thus, there exists a weak solution \( u_\epsilon \in H \) with \( J_\epsilon(u_\epsilon) \geq \delta_\epsilon \).

**Lemma 2.3** There exists \( \tau_0 \) and \( u^* \in H \) with \( \| u^* \|_H = 1 \) such that \( J_\epsilon(tu^*) < 0 \) if \( 0 < t < \tau_0 \).

**Proof** We study the equation
\[ \Delta^2 u = \lambda u + f \]
in \( H(\Omega) \). Define the functional
\[ J_f(u) = \frac{1}{2} \int_{\Omega} |\Delta u|^2 d\mu - \frac{1}{2} \int_{\Omega} \lambda u^2 d\mu - \int_{\Omega} f(x) u d\mu. \]

Noting that
\[ \frac{1}{2} \int_{\Omega} |\Delta u|^2 d\mu - \frac{1}{2} \int_{\Omega} \lambda u^2 d\mu \geq \frac{\tau}{2} \| u \|_H^2 \]
and
\[ \left| \int_{\Omega} f(x) u d\mu \right| \leq \| f \|_{H'} \| u \|_H \leq \frac{\tau}{4} \| u \|_H^2 + \frac{1}{\tau} \| f \|_{H'}^2, \]
we have that \( J_f \) has a lower bound on \( H \).

Set
\[ m_f = \inf_{u \in H} J_f(u). \]

There exists \( u_k \in H \) such that \( J_f(u_k) \to m_f \). From (2.7) and (2.8), we know \( u_k \) is bounded in \( H \).

Hence, there exists \( u^* \in H \) such that \( u_k \rightharpoonup \tilde{u} \) weakly in \( H \). Then,
\[ J_f(\tilde{u}) \leq \liminf_{k \to \infty} J_f(u_k) = m_f \]
and \( \tilde{u} \) is the weak solution of (2.6). It follows that
\[ \int_{\Omega} f \tilde{u} d\mu = \int_{\Omega} |\Delta \tilde{u}|^2 d\mu - \lambda \int_{\Omega} (\tilde{u})^2 d\mu > 0. \]

Now, we compute the derivative of \( J_\epsilon(t\tilde{u}) \):
\[ \frac{d}{dt} J_\epsilon(t\tilde{u}) = t \int_{\Omega} |\Delta \tilde{u}|^2 d\mu - t \int_{\Omega} \lambda (\tilde{u})^2 d\mu - t^{p-1} \int_{\Omega} |\tilde{u}|^p d\mu - \epsilon \int_{\Omega} f(x) \tilde{u} d\mu. \]
By (2.9), we obtain
\[
\frac{d}{dt} J_\epsilon(t\tilde{u}) \bigg|_{t=0} < 0.
\]

Letting \( u^* = \frac{\tilde{u}}{\|\tilde{u}\|_H} \), we finish the proof. \( \Box \)

**Lemma 2.4** Choose \( \epsilon \) such that \( 0 < \epsilon < \epsilon_1 \) where \( \epsilon_1 \) is the same as in Lemma 2.1. Then, there exists a function \( u_0 \in H \) with \( \|u_0\|_H \leq 2r_\epsilon \) such that
\[
J_\epsilon(u_0) = c_\epsilon = \inf_{\|u\| \leq 2r_\epsilon} J_\epsilon(u),
\]
where \( r_\epsilon = \sqrt{\epsilon} \), \( c_\epsilon < 0 \).

**Proof** By (2.1), we see that \( J_\epsilon \) has a lower bound on
\[
B_{2r_\epsilon} = \{ u \in H : \|u\|_H \leq 2r_\epsilon \}.
\]

Combining Lemma 2.1, we get \( c_\epsilon < 0 \).
Let \( (u_k) \subset H \) be a sequence satisfying \( \|u_k\|_H \leq 2r_\epsilon \) and \( J_\epsilon(u_k) \to c_\epsilon \). Then up to a sequence, \( u_k \) converges weakly to \( u_0 \) in \( H \) and converges strongly to \( u_0 \) in \( L^q(V) \) for any \( 1 \leq q \leq +\infty \). It follows that
\[
\lim_{k \to +\infty} \int_{\Omega} fu_k d\mu = \int_{\Omega} fu_0 d\mu,
\]
\[
\|u_0\|_H \leq \limsup_{k \to +\infty} \|u_k\|_H \leq 2r_\epsilon,
\]
\[
\lim_{k \to +\infty} \int_{\Omega} \lambda u_k^2 d\mu = \int_{\Omega} \lambda u_0^2 d\mu,
\]
\[
\lim_{k \to +\infty} \int_{\Omega} u_k^p d\mu = \int_{\Omega} u_0^p d\mu.
\]

Hence,
\[
J_\epsilon(u_0) \leq \limsup_{k \to +\infty} J_\epsilon(u_k) = c_\epsilon
\]
and \( u_0 \) is the minimizer of \( J_\epsilon \) on \( B_{2r_\epsilon} \). Lemma 2.1 implies that \( \|u_0\|_H < r_\epsilon \). For any \( \varphi \in C_c(V) \), let \( \psi(t) = J_\epsilon(u_0 + t\varphi) \). Then, \( \psi \) is a smooth function in \( t \). It is easy to see that there is \( \eta > 0 \) such that \( u_0 + t\varphi \in B_{2r_\epsilon} \) if \( |t| < \eta \). This means that \( \psi(0) \) is the minimum of \( \psi(t) \) on \((-\eta, \eta)\). By \( \psi'(0) = 0 \), we get
\[
\int_{\Omega \cup \partial \Omega} \Delta u_0 \Delta \varphi d\mu - \lambda \int_{\Omega} u_0 \varphi d\mu - \int_{\Omega} |u_0|^{p-1} u_0 \varphi d\mu - \epsilon \int_{\Omega} f \varphi d\mu = 0.
\]

We conclude that \( u_0 \) is a weak solution of (1.2) and complete the proof. \( \Box \)
Clearly, \(u_c\) and \(u_0\) are two distinct solutions of (1.2) since \(J_c(u_c) > 0\) and \(J_c(u_0) < 0\).

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