A local lemma via entropy compression

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Abstract

In the framework of the probabilistic method in combinatorics, we revisit the entropy compression method clarifying the setting in which it can be applied and providing a theorem yielding a general constructive criterion. We finally elucidate, through topical examples, the effectiveness of the entropy-compression criterion in comparison with the Lovasz Local Lemma criterion and, in particular, with the improved criterion based on cluster expansion.

Keywords: Probabilistic Method in combinatorics; Lovász Local Lemma; Randomized algorithms.

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1 Introduction

The Lovász Local Lemma (LLL), originally formulated by Erdős and Lovász [14], is a powerful tool in the framework of the probabilistic method in combinatorics to prove the existence of some combinatorial objects with certain desirable properties (such as a proper coloring of the edges of a graph). According to the LLL, the existence of the combinatorial object under analysis is guaranteed if a family of (bad) events in some probability space do not occur. The LLL then provides a criterion providing an upper bound on the probabilities of these bad events which ensures that, with non-zero probability, none of them occur. The popularity of this lemma is due to the fact that it can be implemented for a wide class of problems in combinatorics, in such a way that its sufficient condition, once some few parameters have been suitably tuned, can be easily checked.

1.1 Lovász Local Lemma: the general version

In order to enunciate the most general statement of the LLL, which includes also the lopsided version (see e.g. [33] and references therein), we need to give some basic notations. Let \( G = (V, E) \) be a (simple undirected) graph with vertex set \( V \) and edge set \( E \). Two vertices \( v, v' \in V \) are adjacent if \( \{v, v'\} \in E \). Given
v ∈ V, let \( \Gamma^*_G(v) \) be the set of vertices of \( G \) adjacent to \( v \) (i.e., the neighborhood of \( v \)). We denote \( \Gamma_G(v) = \Gamma^*_G(v) \cup \{ v \} \).

Given a family of events \( \mathcal{F} \) in some probability space \((\Omega, P)\) and given \( e \in \mathcal{F} \), we denote by \( \bar{e} \) the complement event of \( e \in \mathcal{F} \), so that \( \bigcap_{e \in \mathcal{F}} \bar{e} \) is the event that none of the events in the family \( \mathcal{F} \) occur.

Henceforth, the product over the empty set is equal to one, if \( n \in \mathbb{N} \) we set \( [n] = \{1, 2, \ldots, n\} \) and \( [n]_0 = \{0, 1, 2, \ldots, n\} \) and, if \( X \) is a finite set, then \( |X| \) denotes the number of its elements.

**Theorem 1.1 (Lovász local Lemma)** Let \( \mathcal{F} \) be a finite collection of events in a probability space \((\Omega, P)\). Let \( G \) be a graph with vertex set \( \mathcal{F} \). Let \( \mu = \{\mu_e\}_{e \in \mathcal{F}} \) be a collection of real numbers in \((0, +\infty)\). If, for each event \( e \in \mathcal{F} \) and for each \( U \subset \mathcal{F} \setminus \Gamma_G(e) \),

\[
P\left( e \mid \bigcap_{e' \in U} \bar{e}' \right) \leq \frac{\mu_e}{\Phi_e(\mu, G)}
\]

with

\[
\Phi_e(\mu, G) = \prod_{e' \in \Gamma_G(e)} (1 + \mu_{e'})
\]

then

\[
P\left( \bigcap_{e \in \mathcal{F}} \bar{e} \right) > 0
\]

i.e. the probability that none of the events in the family \( \mathcal{F} \) occur is strictly positive.

The sufficient condition given by (1.1) is the LLL criterion.

**Remark.** Theorem 1.1 is generally known in the literature as the “Lopsided” LLL. When the graph \( G \) has edge set such that, each event \( e \in \mathcal{F} \) is independent of the \( \sigma \)-algebra generated by the collection of events \( \mathcal{F} \setminus \Gamma_G(e) \) so that, for any \( e \in \mathcal{F} \) and any \( U \subset \mathcal{F} \setminus \Gamma_G(e) \) we have that \( P\left( e \mid \bigcap_{e' \in U} \bar{e}' \right) = P(e) \), we say that \( G \) is a dependency graph for the family \( \mathcal{F} \). Theorem 1.1 in which \( G \) is a dependency graph for the family \( \mathcal{F} \) is the most common version of the LLL and it is the one used in the vast majority of the applications. An even more simpler version of the LLL is the so-called “symmetric” case. This occurs if it is possible to find \( p > 0 \) such that \( P(e \mid \bigcap_{e' \in U} \bar{e}') \leq p \) for all events in \( \mathcal{F} \) and any \( U \subset \mathcal{F} \setminus \Gamma_G(e) \). In this case it is natural to choose \( \mu_e = \mu \) for all \( e \in \mathcal{F} \) so that condition (1.1) becomes

\[
p \leq \frac{\mu}{(1 + \mu)^{d+1}}
\]

where \( d = \max_{e \in \mathcal{F}} |\Gamma^*_G(e)| \). Optimizing in \( \mu > 0 \) and using that \( \left( 1 + \frac{1}{d} \right)^d < e \), (1.4) is usually written as

\[
ep(d + 1) \leq 1
\]
1.2 The improved version of the LLL via cluster expansion

In recent times a remarkable connection between the LLL and statistical mechanics has been pointed out by Scott and Sokal [33]. More precisely, Scott and Sokal showed that the thesis (1.3) of the Lovász local lemma holds if and only if the probabilities \( \{P(e)\}_{e \in \mathcal{E}} \) are inside the complex polydisc where the partition function of the hard-core self-repulsive lattice gas on \( G \) is nonvanishing and that this necessary and sufficient condition is equivalent to the Shearer “optimal” criterion [35]. Furthermore, they show that the LLL criterion (1.1) providing a sufficient condition for the thesis (1.3) to hold, coincides with the so-called Dobrushin criterion, which is a sufficient condition for the non vanishing of the aforementioned partition function.

Later on, Fernández and Procacci [17], working on the abstract polymer gas model (of which the hard-core, self-repulsive lattice gas on a graph \( G \) is a special case) obtained a sensible improvement of the Dobrushin criterion via cluster expansion methods. Such a result has then been used by Bissacot et al. [7] to obtain an improved version of the LLL. To enunciate this improved version of the LLL we recall that, given a graph \( G = (V,E) \), a set \( Y \subset V \) is independent if no edge \( \{v,v'\} \in E \) is such that \( v \in Y \) and \( v' \in Y \).

**Theorem 1.2 (Cluster Expansion local lemma (CELL))** With the same hypothesis of Theorem 1.1, if, for each event \( e \in \mathcal{E} \) and for each \( U \subset \mathcal{E} \setminus \Gamma(e) \)

\[
P \left( \bigcap_{e \in U} \bar{e} \right) \leq \frac{\mu_e}{\Xi_e(\mu, G)} \tag{1.6}
\]

with

\[
\Xi_e(\mu, G) = \sum_{Y \subseteq \Gamma(e)} \prod_{e' \in Y} \mu_{e'} \tag{1.7}
\]

then

\[
P \left( \bigcap_{e \in \mathcal{E}} \bar{e} \right) \geq 0
\]

**Remark.** The improvement on Theorem 1.1 is immediately recognized by noting that

\[
\Phi_e(\mu, G) = \prod_{e' \in \Gamma(e)} (1 + \mu_{e'}) = \sum_{Y \subseteq \Gamma(e)} \prod_{e' \in Y} \mu_{e'} \geq \sum_{Y \subseteq \Gamma(e)} \prod_{e' \in Y \text{ independent}} \mu_{e'} = \Xi_e(\mu, G)
\]

The improved sufficient condition given by (1.6) is nowadays known as the *cluster expansion criterion* (CE criterion). As remarked in [17], observing that the event \( e \) is connected to any \( e' \in \Gamma^+(e) \) the function \( \Xi_e(\mu, G) \) can be rewritten as

\[
\Xi_e(\mu, G) = \mu_e + \sum_{Y \subseteq \Gamma(e)} \prod_{e' \in Y \text{ independent}} \mu_{e'} \tag{1.8}
\]
1.3 The Moser-Tardos algorithmic version of the local lemma

The LLL provides a sufficient condition for the probability that none of the undesirable events in some (unspecified) probability \((\Omega, P)\) space occur to be strictly positive and this implies the existence of at least one outcome in \((\Omega, P)\) which realizes the occurrence of the “good” event. However, since \((\Omega, P)\) is left completely unspecified, Theorem \ref{thm:lll} as well as Theorem \ref{thm:lll} says nothing about how to find such a configuration. The efforts to devise an algorithmic version of the LLL by trying to say something more precise about the probability space \((\Omega, P)\) go back to the work of Beck \cite{beck}, Alon \cite{alon} and others (see e.g. reference list in \cite{moser-tardos}) and culminated with the breakthrough work by Moser and Tardos \cite{moser-tardos1, moser-tardos2}, who presented a fully algorithmic version of LLL by assuming that the probability space \((\Omega, P)\) is generated by a finite collection of independent random variables. This simple assumption covers nearly all known applications of LLL.

The scheme introduced by Moser and Tardos (called nowadays the \textit{variable setting}) is defined as follows. Given a finite set \(S\), for each \(x \in S\) we have a random variable \(\psi_x\) taking values in some set \(\Psi_x\) according with some given distribution. The collection \(\{\psi_x\}_{x \in S}\) is supposed to form a family of \textit{mutually independent} random variables. Therefore, for any non empty \(U \subset S\), the subfamily \(\{\psi_x\}_{x \in U}\) generates a product probability space \(\Omega_U\) with product probability measure \(P_U\).

Moser and Tardos set \((\Omega, P) = (\Omega_S, P_S)\).

Observe that for any \(U \subset S\), a configuration (or elementary event) \(\omega = \prod_{x \in U} \psi_x\) in \(\Omega_U\) is an element of \(\Psi_U = \prod_{x \in U} \Psi_x\). Given \(\omega \in \Psi_S\) and \(U \subset S\), we let \(\omega|_U\) be the restriction of \(\omega\) to \(U\), i.e. if \(\omega = \prod_{x \in S} \psi_x\) then \(\omega|_U = \prod_{x \in U} \psi_x\).

Let now \(\mathcal{O}\) be a family of subsets of \(S\) (called hereafter “objects”). Given \(A \in \mathcal{O}\), an event \(\bar{\varepsilon}\) in \(\Omega_A\) is a subset of \(\Psi_U = \prod_{x \in U} \Psi_x\). For each \(A \in \mathcal{O}\), we will refer to an event in \(\Omega_A\) as an event associated to \(A\) and write \(\text{obj}(\bar{\varepsilon}) = A\). As usual, \(\bar{\varepsilon}\) denotes the complementary event of \(\varepsilon\). Clearly, since \(\Omega_S\) is a product space, each event \(\varepsilon\) in \(\Omega_A\) can also be viewed as the event in \(\Omega_S\) formed by those \(\omega \in \Psi_S\) such that the restriction \(\omega|_A\) is in \(\varepsilon\) so that \(P_A(\varepsilon) = P_S(\varepsilon)\).

Moser and Tardos considered families \(\mathcal{F}\) of (bad) events such that, for each element of \(\varepsilon \in \mathcal{F}\), \(\text{obj}(\varepsilon) = A\), for some \(A \in \mathcal{O}\).

Remark. Note that by construction an \(\varepsilon \in \mathcal{F}\) depends only on the family of random variables \(\{\psi_x\}_{x \in \text{obj}(\varepsilon)}\) and any two events \(\varepsilon, \varepsilon'\) of \(\mathcal{F}\) such that \(\text{obj}(\varepsilon) \cap \text{obj}(\varepsilon') = \emptyset\) are necessarily independent. This implies that the graph \(\mathcal{G}\) with vertex set \(\mathcal{F}\) and edge-set constituted by the pairs \(\{\varepsilon, \varepsilon'\}\) such that \(\text{obj}(\varepsilon) \cap \text{obj}(\varepsilon') \neq \emptyset\) is a natural dependency graph for the family \(\mathcal{F}\).

In this setting Moser and Tardos defined the following algorithm.

MT-Algorithm.
- Step 0. Sample a random \(\omega \in \Psi_S\) and set \(\omega_0 = \omega\) as the configuration at step 0.
- Step \(i\) (for \(i \geq 1\)). Let \(\omega_{i-1} \in \Psi_S\) be the configuration at step \(i - 1\). If \(\omega_{i-1}\) is such that some event of the family \(\mathcal{F}\) occurs, select one of them (at random or according to some deterministic rule), say \(\varepsilon\), and choose an evaluation (resampling) \(\omega' \in \Psi_{\text{obj}(\varepsilon)}\). Set \(\omega_i\) as the configuration of \(\Psi_S\) such that \(\omega_i|_{S \setminus \text{obj}(\varepsilon)} = \omega_{i-1}|_{S \setminus \text{obj}(\varepsilon)}\) and \(\omega_i|_{\text{obj}(\varepsilon)} = \omega'\)
Moser and Tardos proved [26] that if condition (1.1) of Theorem 1.1 holds, MT-algorithm terminates rapidly finding a configuration in ΨS such that none of the bad events of the family \( \mathcal{F} \) occurs. Later Pegden [29] improved Moser-Tardos result replacing condition (1.1) with condition (1.6). Pegden’s extension of the Moser-Tardos algorithmic local lemma is thus the algorithmic version of Theorem 1.2 and can be stated as follows.

**Theorem 1.3 (Algorithmic CELL)** Given a finite set \( S \) and its associated family of mutually independent random variables \( \psi_S \), let \( \mathcal{O} \) be a collection of subsets of \( S \) and let \( \mathcal{F} \) a family of bad events associated to \( \mathcal{O} \) with natural dependency graph \( G \). Let \( \mu = \{ \mu_e \}_{e \in \mathcal{F}} \) be real numbers in \((0, +\infty)\). If, for each \( e \in \mathcal{F} \), the cluster expansion criterion (1.6) holds, then

\[
\bigcap_{e \in \mathcal{F}} \bar{\mathcal{e}} \neq \emptyset
\]

and MT-algorithm finds \( \omega \in \bigcap_{e \in \mathcal{F}} \bar{\mathcal{e}} \) in an expected total number of steps less than or equal to \( \sum_{e \in \mathcal{F}} O \mu_e \).

**Remark.** In the Moser-Tardos setting above described the function \( \Xi_e(\mu, G) \) defined in (1.7) admits a somehow natural upper bound. Just observe that, for any \( y \in S \), the set \( \mathcal{F}(y) = \{ e \in \mathcal{F} : y \in \text{obj}(e) \} \) is a clique of the natural dependency graph \( G \) of the family \( \mathcal{F} \): all events in \( \mathcal{F}(y) \) contain \( y \) and thus any pair \( \{ e, e' \} \subset \mathcal{F}(y) \) is connected by an edge of \( G \). Thus, the neighbor \( \Gamma_G(e) \) of any event \( e \in \mathcal{F} \) is the (in general not disjoint) union of cliques \( \{ \mathcal{F}(y) \} \) \( y \in \text{obj}(e) \), i.e., we can write \( \Gamma_G(e) = \bigcup_{y \in \text{obj}(e)} \mathcal{F}(y) \). Therefore

\[
\Xi_e(\mu, G) \leq \Xi_e^{\text{clique}}(\mu, G) \equiv \prod_{y \in \text{obj}(e)} \left[ 1 + \sum_{e' \in \mathcal{F}(y)} \mu_{e'} \right]
\]

(1.9)

A slightly better clique estimate is obtained by using the expression (1.8) for \( \Xi_e(\mu, G) \). Namely,

\[
\Xi_e(\mu, G) \leq \tilde{\Xi}_e^{\text{clique}}(\mu, G) \equiv \mu_e + \prod_{y \in \text{obj}(e)} \left[ 1 + \sum_{e' \in \mathcal{F}(y), e' \neq e} \mu_{e'} \right]
\]

(1.10)

Below we will refer to the functions \( \Xi_e^{\text{clique}}(\mu, G) \) and \( \tilde{\Xi}_e^{\text{clique}}(\mu, G) \) as the clique estimates of \( \Xi_e(\mu, G) \) and we will make use of these estimates in Section 4.

Via the upper bound (1.9) for \( \Xi_e(\mu, G) \), an improved estimate for latin transversal has been obtained in [7] and improved bounds for several chromatic indices have been given in [27] and [8]. It is important however to stress that bounds (1.9) or (1.10) are efficient only if the cliques \( \{ \mathcal{F}(y) \} \) \( y \in \text{obj}(e) \) are not too overlapped. When cliques \( \{ \mathcal{F}(y) \} \) \( y \in \text{obj}(e) \) are too overlapped these bounds tends to be too rough. This situation occurs for example in the case of perfect and separating hash families (see [30]).
1.4 Motivation and overview of the present paper

Moser and Tardos brilliant insight has represented a significant advance, but it is not the end of the story. Shortly after their work, the question has been raised as to whether it is possible to design alternative algorithms, eventually depending on the specific problem treated, able to go beyond the bounds given by the LLL or CELL. These ideas have been originally developed in [11, 21] where specific algorithms have been implemented for specific graph coloring problems to obtain bounds which are better than those obtainable by LLL or even CELL. In particular, in the paper [15] Esperet and Parreau devised an algorithm able to obtain a new upper bound for the chromatic index of the acyclic edge coloring of a graph with maximum degree $\Delta$ which sensibly improves on the bound obtained by Ndreca et al. [27] just an year before via the CELL. Esperet and Parreau also suggested that this algorithm yields a general method able to treat most of the applications in graph coloring problems covered by the LLL. Indeed, this was confirmed in several successive papers [22, 31, 28, 32, 10, 9, 10, 18, 34], where the Esperet-Parreau scheme has been applied to several combinatorial problems in graph colorings and beyond, generally improving previous results obtained via the LLL (sometimes the improvement is sensible, sometimes less). This approach, in some sense a variant of the Moser-Tardos algorithmic LLL, has been called entropy compression method (the expression was first used by Tao in reference to the Moser Tardos algorithm [37]). In works mentioned the entropy compression method is used as a set of instructions to devise specific algorithms to face specific combinatorial problems rather than a general theorem which one can use as a possible alternative to the LLL. It is therefore natural to ask whether there is some general entropy-compression criterion, valid in a (possibly restricted) Moser-Tardos setting that can be used as an alternative to the the LLL and cluster expansion criteria. For the specific framework of graph coloring, this question has been addressed in a paper by Gonçalves, Montassier and Pinlou in [22] where a general criterion is somehow outlined (see there Theorem 12). It is also worth to mention a paper by Bernshteyn [6] in which a non algorithmic “Local Cut Lemma”, inspired by the Entropy-compression method, is able to reproduce some of its achievements. A second natural question is to try to understand to what extent the entropy-compression method is more efficient than the local lemma and for what reasons.

In the present paper, basically a survey of entropy compression method, we tried to answer these questions. We first of all of clarify the (as general as possible) framework in which the EC-method can be applied. Namely, the EC-setting coincides with a restricted Moser-Tardos variable setting in which the independent random variables generating the product probability space take values in a common finite set of cardinality $k \in \mathbb{N}$ according to the uniform distribution. We then show that in such a setting, a general constructive criterion (in Theorem 2.2 below) alternative to the LLL and CELL criteria can indeed be formulated. This “Entropy-Compression criterion” (EC criterion), i.e. inequality (2.1) below, is in general easy to be implemented in applications and, once satisfied, furnishes a general random algorithm which efficiently find
a configuration avoiding all bad events.
We also point out that in the specific (and, as far as we know, unique) case of the acyclic edge coloring, the Entropy-Compression setting goes somehow beyond this restricted Moser-Tardos setting in the sense that bad events not sharing variables are not necessarily mutually independent (and this is an extension w.r.t. the Moser-Tardos framework).

We finally show how to implement the EC criterion to several well known (and hopefully pedagogical) problems in combinatorics. These examples will be also used to make a comparison between EC and CE criteria trying to clarify in which conditions the EC criterion beats the CE and LLL criteria and to what extent.

The rest of the paper is organized as follows. In Section 2 we illustrate the EC setting and we state Theorem 2.2 containing the general EC criterion. In Section 3 we give the proof of Theorem 2.2. In section 4 we discuss some applications.

2 The entropy-compression setting

Let $S$ be finite set and let $k \in \mathbb{N}$. We refer to $S$ as the set of “atoms” and to $[k]$ as the set of “colors” We suppose that a total order is fixed in sets $S$ and $[k]$. We let $[k]_0 = [k] \cup \{0\}$. A coloring $\kappa$ of $S$ is a function $\kappa : S \rightarrow [k]$. A partial coloring $\gamma$ of $S$ is a function $\gamma : S \rightarrow [k]_0$ and when $\gamma(x) = 0$ we say that $x$ is uncolored. For any non empty $Y \subseteq S$ we denote by $[k]^Y$ and $[k]_0^Y$ the sets of colorings and partial colorings in $Y$ respectively. For any nonempty set $X$, we denote by $\mathcal{P}(X)$ the set of all nonempty subsets of $X$.

Given the pair $(S,k)$, let $O$ be a family of subsets of $S$ (called objects). Given an object $A \in O$, a standard flaw of $A$ (the bad event in the Moser-Tardos scheme) is a subset $\varepsilon \subset [k]^A$. A a non-standard flaw of $A$ is a function $f : [k]_0^{S\setminus A} \rightarrow \mathcal{P}([k]^A)$ (i.e. for each $\gamma \in [k]_0^{S\setminus A}$, $f(\gamma)$ is a subset $f \subset [k]^A$). If $\varepsilon$ is a flaw of $A \in O$ (either standard or non-standard), we write $\text{obj}(\varepsilon) = A$ and we call $\vert A \vert = \vert \text{obj}(\varepsilon) \vert$ the size of $\varepsilon$ while $\vert \varepsilon \vert$ is the cardinality of $\varepsilon$ (i.e. the number of colorings forming $\varepsilon$). As usual, $\bar{\varepsilon}$ will denote the complement of $\varepsilon$ in $[k]^A$, i.e. $\bar{\varepsilon} = [k]^A \setminus \varepsilon$. A family $\mathfrak{F}$ of flaws is said to be associated to $O$ if each $\varepsilon \in \mathfrak{F}$ is a flaw of $A$ for some $A \in O$. When $y \in \text{obj}(\varepsilon)$ we will (improperly) say that $\varepsilon$ contains $y$ and we denote by $\mathfrak{F}(y) = \{ \varepsilon \in \mathfrak{F} : y \in \text{obj}(\varepsilon) \}$ the sub-family of flaws containing $y$. A a good coloring w.r.t. to $\mathfrak{F}$ is a coloring $c \in [k]^S$ avoiding all flaws in $\mathfrak{F}$, that is to say, $c$ is such that for all $A \in O$ and all $\varepsilon \in \mathfrak{F}$ such that $\text{obj}(\varepsilon) = A$, we have $c|_A \notin \varepsilon$ if $\varepsilon$ is standard and $c|_A \notin \bar{\varepsilon}$ if $\varepsilon$ is non-standard. In other words, the restriction $c|_A$ of the coloring $c$ to the set $A$ belongs to $\bar{\varepsilon}$ for each flaw $\varepsilon$ of $A$. We denote by $\bigcap_{\varepsilon \in \mathfrak{F}} \bar{\varepsilon}$ the set of all good colorings w.r.t. $\mathfrak{F}$ in $[k]^S$.

Remark. When a family $\mathfrak{F}$ associated to $O$ contains only standard flaws (which is what happens in nearly all known applications of the EC method, as far as we know) the EC-setting coincides with a Moser-Tardos variable setting whose probability space is generated by $|S|$ i.i.d. random variables $\{\psi_x\}_{x \in S}$ all taking value in the set $[k]$ according to the uniform distribution. On the other
hand when some of the flaws in the family $\mathcal{F}$ are non-standard (the only example we know is the case of the acyclic edge coloring), the EC-setting has no manifest correspondence with the Moser-tardos setting. We will refer below to first case as the independent EC-setting and to the second case as the dependent EC-setting. We stress once again that virtually all applications of the entropy compression method available in the literature fall in the independent EC-setting, with the sole exception, far as we know, of the the acyclic edge coloring of bounded degree graphs, for which the use the dependent EC-setting is required (see section 4).

**Definition 2.1** A flaw $\epsilon$ is said tidy if, for all $y \in \text{obj}(\epsilon)$, there exists $X \subset \text{obj}(\epsilon) \setminus \{y\}$ such that any coloring $\kappa \in \epsilon$ is uniquely determined by its restriction to $X$ and no $Y \subsetneq X$ has this property. The set $X$ is called a “seed” of $\epsilon$.

**Remark.** By definition a tidy flaw has the empty set as its unique seed if and only if is elementary. Note that if $\epsilon$ is not tidy, then it can be seen as the disjoint union of tidy (in the worst case elementary) flaws. Therefore there is no loss of generality in considering only families in which all flaws are tidy.

Given a family $\mathcal{F}$ of tidy flaws associated to $\mathcal{O}$ and given a flaw $\epsilon \in \mathcal{F}(y)$, we define

$$\kappa_y(\epsilon) = \min\{|X| : X \subset \text{obj}(\epsilon) \setminus \{y\} \text{ and } X \text{ is a seed of } \epsilon\} \quad (2.11)$$

and set

$$\|\epsilon_y\| = |\text{obj}(\epsilon)| - \kappa_y(\epsilon) \quad (2.12)$$

We refer to $\|\epsilon_y\|$ as the power of the event $\epsilon$ (w.r.t. $y$). Note that, for any $\epsilon \in \mathcal{F}(y)$, $\kappa_y(\epsilon) \in \{0, 1, \ldots, |\text{obj}(\epsilon)| - 1\}$ and $\|\epsilon_y\| = |\text{obj}(\epsilon)|$ only if $\epsilon$ is elementary. We set

$$\mathfrak{F}_j(y) = \{\epsilon \in \mathcal{F}(y) \text{ and } \|\epsilon_y\| = j\}, \quad \mathfrak{F}_j = \bigcup_{y \in \mathcal{S}} \mathfrak{F}_j(y)$$

and

$$d_j = \max_{y \in \mathcal{S}, \gamma \in [k]^0} |\mathfrak{F}_j(y)| \quad (2.13)$$

I.e. $d_j$ is an upper bound for the number of events with effective size $j$ associated to objects containing a common atom. We finally define

$$E_\mathcal{O} = \{j \in \mathbb{N} : \exists y \in \mathcal{S} \text{ such that } \mathfrak{F}_j(y) \neq \emptyset\} \quad (2.14)$$

and set, for $x \in (0, +\infty)$,

$$\phi_\mathcal{O}(x) = 1 + \sum_{j \in E_\mathcal{O}} d_j x^j \quad (2.15)$$

**Remark.** Note that a tidy event $\epsilon$ of size 1, i.e. such that $\text{obj}(\epsilon) = x$, is necessarily elementary because its unique seed is the empty set. Therefore, give $A \in \mathcal{O}$ such that $|A| = 1$ and $\gamma \in [k]^0\backslash A$, a tidy flaw of $A$ given $\gamma$ has always the form $\epsilon = \{\text{the atom } x \in \mathcal{S} \text{ has the color } c \text{ and } c \in \Phi(x, \gamma)\}$ where $\Phi(x, \gamma)$ is some subset $\subset [k]$ of forbidden colors for the atom $x$ given the partial coloring $\gamma$ outside $x$. 
2.1 EC-Algorithm and Entropy-Compression Lemma

We remind that a total order is chosen in the sets \( S \) and \([k]\). We also suppose that a total order has been chosen in the sets \( O \) and \( \mathcal{F} \). We choose, for each \( y \in S \) and \( \epsilon \in \mathcal{F}(y) \), a unique subset \( G(\epsilon, y) \subset \text{obj}(\epsilon) \setminus \{y\} \) such that \( G(\epsilon, y) \) is a seed of \( \epsilon \) and \(|G(\epsilon, y)| = \kappa_y(\epsilon)\). We also denote shortly \( G^*(\epsilon, y) = \text{obj}(\epsilon) \setminus G(\epsilon, y)\).

Note that \( y \in G^*(\epsilon, y) \). Given a partial coloring \( \gamma \in [k]_0^S \), given \( x \in S \) and given a color \( s \in [k] \) we denote by \( \gamma_s^i \) the partial coloring which coincides with \( \gamma \) in the set \( S \setminus \{x\} \) and it takes the value \( s \) at \( x \).

**EC-Algorithm.**

Assume that a set of atoms \( S \) is given together with a family of objects \( O \) and a family \( \mathcal{F} \) of flaws associated to \( O \). Let \( t \) be an arbitrary natural number (which can be taken as large as we please). The input of the algorithm is a vector \( F_t \) in the space \([k]^t\) (i.e. a vector with \( t \) entries each of which taking values in the set \([k]\)).

The algorithm performs (at most) \( t \) steps and to each step \( i \in [t] \) a partial coloring \( \gamma_i \) is associated as described below.

- **Step 0.** Set \( \gamma_0 = 0 \) (i.e. in the beginning no atom is colored).
- **Step \( i \) (for \( i \geq 1 \)).**
  - \( i^1 \) If \( \gamma_{i-1}^{-1}(0) \neq \emptyset \), let \( y \) be the smallest atom (in the total order chosen) left uncolored by the partial coloring \( \gamma_{i-1} \). Take \( i^{th} \) entry of the vector \( F_t \) and let \( s \in [k] \) be this entry. Color \( y \) with the color \( s \) and consider the partial coloring \( \gamma_i = \gamma_{i-1}^s \) obtained from \( \gamma_{i-1} \) by coloring \( y \) with the color \( s \).
  - \( i^2 \) Conversely, if some flaw occurs given \( \gamma_{i-1}^s \), set \( \gamma_i = \gamma_{i-1}^s \) and go to the step \( i+1 \).
  - \( i^3 \) If no flaw occurs given \( \gamma_{i-1}^s \), set \( \gamma_i = \gamma_{i-1}^s \) and go to the step \( i+1 \).
  - **Step \( i^* \) If \( \gamma_{i-1}^{-1}(0) = \emptyset \), stops the algorithm discarding all entries \( f_i, f_{i+1}, \ldots, f_t \) of \( F_t \).**

Note that the partial coloring \( \gamma_i \) returned by the algorithm at the end of each step \( i \) necessarily avoids all flaws in \( \mathcal{F} \). The algorithm performs at most \( t \) steps but it can stop earlier, i.e. after having performed \( m < t \) steps and \( \gamma_m^{-1}(0) = \emptyset \), (in other words if at the end of step \( m \) all variables are colored, and no bad event occurs). In this case only the first \( m \) entries of the vector \( F_t \) are used. We say that the algorithm is successful if it stops after \( m < t \) steps, or it lasts \( t \) steps and after the last step \( t \) we have \( \gamma_t^{-1}(0) = \emptyset \). Conversely, we say that the algorithm fails if it performs all \( t \) steps and \( \gamma_t^{-1}(0) \neq \emptyset \). Clearly when the EC-algorithm is successful we have found a coloring \( \kappa \in [k]^S \) which avoid all flaws of the family \( \mathcal{F} \). Note that the algorithm can be either deterministic if \( F_t \) is a given prefixed vector or random if the entries of \( F_t \) are uniformly sampled from the set \([k]^t\) sequentially and independently.

We are now in the position to state the main theorem of this note.
Theorem 2.2 (Entropy-compression Lemma) Assume that a pair \((S, k)\) is given together with a family of objects \(O\) and a family \(\mathcal{F}\) of flaws associated to \(O\). If
\[
k > \inf_{x > 0} \frac{\phi_O(x)}{x}
\] (2.1)
then
\[
\bigcap_{\varepsilon \in \mathcal{F}_O} \bar{\varepsilon} \neq \emptyset
\]
Moreover, if inequality (2.1) holds strictly, the (random) EC-algorithm finds a coloring \(c \in \bigcap_{\varepsilon \in \mathcal{F}_O} \bar{\varepsilon}\) in an expected number of steps linear in \(|S|\).

We will refer below inequality (2.1) as the entropy-compression criterion.

Remark. The random EC-algorithm still finds a coloring avoiding all flaws in the family \(\mathcal{F}_O\) even if we only demand \(k\) to be greater or equal (instead of strictly greater) than the l.h.s. of (2.1), but in this case we have no control on the expected running time.

3 Proof of Theorem 2.2

Let \(\mathcal{F}_t\) be the subset of \([k]^t\) formed by all vectors \(F_t\) such that \(\gamma^{-1}_t(0) \neq \emptyset\). By definition, if \(F_t \in \mathcal{F}_t\), the algorithm performs all the \(t\) steps and fails. Clearly \(|\mathcal{F}_t| \leq k^t\) and if we are able to prove that this inequality is strict, then this means that the set \([k]^t \setminus \mathcal{F}_t\) is non empty and for each vector \(F_t \in [k]^t \setminus \mathcal{F}_t\), the algorithm is successful.

So, let assume from now on that \(F_t \in \mathcal{F}_t\). In this case the algorithm induced by \(F_t\) is such that for any step \(i \in [t]\) there is no case \(i^\bullet\) and therefore we may set \(i^\circ \equiv i\) for all \(i \in [t]\). We start by observing that, if at step \(i\), after coloring the atom \(y\), we fall in the \(i_2\) procedure, some occurring flaw \(\varepsilon\) with effective size \(\|\varepsilon_y\| = j \in E_O\) is selected. Since \(|\mathcal{F}_j(y)| \leq d_j\), this event \(\varepsilon\) can be uniquely determined by a pair \((j, \ell)\) with \(\ell \in [d_j]\).

The EC-algorithm at step \(i\) takes \(i^{th}\) entry of the vector \(F_t\), say \(s \in [k]\), and attributes to the smallest uncolored atom at step \(i\), say \(y\), the color \(s\). The vector \(F_t\) determines thus uniquely all the \(t\) steps performed by the EC-algorithm. We will now show that the converse is also true: any vector \(F_t \in \mathcal{F}_t\) is uniquely determined by a "record" of the algorithm. This record is specified by a pair \((R_t, \gamma_t)\) where \(\gamma_t\) is the partial coloring at step \(t\) and \(R_t\) is a string \(r_1 \cdots r_i \cdots r_t\) with \(r_i\) being defined as follows.

- if at step \(i\), after coloring the atom, \(y\) no bad event occur, set \(r_i = 0\)
- if at step \(i\), after coloring the atom \(y\), the bad event \(\varepsilon \in \mathcal{F}_j(y)\) is selected, then it uniquely determined by the pair \((j, \ell)\) (where \(j \in E_O\) and \(\ell \in [d_j]\)) and set \(r_i = (j, \ell)\).

Thus \(F_t \in \mathcal{F}_t\) determines uniquely the pair \((R_t, \gamma_t)\). We call \(\mathcal{M}\) the map \(F_t \mapsto (R_t, \gamma_t)\). Let us prove that \(\mathcal{M}\) is an injection, which in other words means that
the knowledge of the record \((R_t, \gamma_t)\) of the algorithm permits to reconstruct uniquely the vector \(F_t\) which had produced that record.

**Lemma 3.1** The set \(X_t = \gamma_t^{-1}(0)\) formed by the uncolored atoms at the conclusion of step \(t\) is uniquely determined by the string \(R_t\).

**Proof.** We prove the statement by induction on \(t\). The case \(t = 1\) is easy. If \(R_1 = r_1 = 0\) at the beginning of step 1 the first atom \(x \in S\) is colored, no bad event occur and \(x\) stays colored at the end of step 1. So in this case we have \(X_1 = S \setminus \{x\}\). If \(R_1 \neq 0\), this means that the coloring the first atom \(x \in S\) produces a bad event which is necessarily elementary and of effective size equal to one (recall: we are supposing that all events of the family \(\mathcal{F}\) are tidy). The atom is thus uncolored at the end of step 1 and so in this case \(X_1 = S\). Suppose now \(t \geq 2\). By the induction hypothesis \(X_{t-1}\) is uniquely determined by \(R_{t-1}\). The smallest atom in the set \(X_{t-1}\), call it \(y\), is the one that will be colored at step \(t\). If \(r_t = 0\), this means that the coloring of the atom \(y\) has produced no bad events and so \(X_t = X_{t-1} \setminus \{y\}\). If \(r_t = (j, \ell)\) then the coloring of the atom \(y\) has produced the event \(e\) uniquely determined by the pair \((j, \ell)\) among those of effective size \(j\) containing the atom \(y\) and we know that all atoms in \(\text{obj}(e)\) have been uncolored except those in \(G(e, y) \subset \text{obj}(e)\) so that we have \(X_t = X_{t-1}\}\{\text{obj}(e) \setminus G(e, y)\}. \square

**Lemma 3.2** For any \(t \in \mathbb{N}\), the map \(\mathcal{M}\) assigning to each vector \(F_t \in \mathcal{F}_t\) the pair \((R_t, \gamma_t)\) is injective.

**Proof:** We prove by induction that the pair \((R_t, \gamma_t)\) uniquely determines \(F_t\) (in other words we prove that \(|\mathcal{M}^{-1}(R_t, \gamma_t)| = 1\)). Let us start by considering the case \(t = 1\). Suppose first \(R_1 = r_1 = 0\). Then the first atom \(x \in S\) is colored with the colour say \(s \in [k]\), no bad event occurs and the partial coloring \(\gamma_1\) is such that \(\gamma_1(x) = s\) while \(\gamma_1(y) = 0\) for all \(y \in S \setminus \{x\}\). Then the (one-dimensional) vector \(F_1 = (f_1)\) has entry \(f_1 = s\). On the other hand, if \(r_1 = (1, \ell)\) then necessarily a tidy elementary event \(e\) such that \(|\text{obj}(e)| = \|e_x\| = 1\) is selected with \(e\) being the event that \(x\) receive the color, say \(s\) which is the \(\ell\)th color of the list \(\Phi(x, \gamma_0)\) and thus the (one-dimensional) vector \(F_1 = (f_1)\) has entry \(f_1 = s\). Suppose now the claim true for \(t - 1\). Namely, given the pair \((R_{t-1}, \gamma_{t-1})\), we know the vector \(F_{t-1}\). Therefore we have to show that, given the pair \((R_t, \gamma_t)\) we must be able to find the entry \(f_t\) of the vector \(F_t\) and the function \(\gamma_{t-1}\) (so that, by induction, \((R_{t-1}, \gamma_{t-1})\) gives us all the remaining entries of the vector \(F_t\)). By Lemma 3.1 the knowledge of \(R_{t-1}\) determine uniquely the set \(X_{t-1}\) formed by the uncolored atoms at the conclusion of step \(t - 1\) and hence we know the smallest atom uncolored after \(t - 1\) steps; call it \(y\). We have to consider two cases: a) \((R_t, \gamma_t)\) is such that \(r_t = 0\); b) \((R_t, \gamma_t)\) is such that \(r_t = (j, \ell)\). The case a) is easy. Indeed if \(r_t = 0\) then the variable \(y\) will be colored at step \(t\) and so if \(\gamma_t(y) = s\), then \(f_t = s\) while \(\gamma_{t-1}\) is such that \(\gamma_{t-1}(x) = \gamma_t(x)\) for all \(x \in S \setminus \{y\}\) and \(\gamma_{t-1}(y) = 0\). Let us now consider the case b) i.e. \(r_t = (j, \ell)\) where \(j \in E_\mathcal{O}\) and \(\ell \in [d_j]\). The pair \((j, \ell)\) uniquely determines the event \(e \in \mathcal{F}_j(y)\) that is selected among those occurring at the beginning of step \(t\) after the coloring of
the variable \( y \). Concerning this event \( \epsilon \), we know the subset \( G(\epsilon, y) \subseteq \text{obj}(\epsilon) \) constituted by the atoms that will continue to stay colored after the conclusion of step \( t \) (recall: \( G(\epsilon, y) \) is empty if \( \epsilon \) is elementary). Now we have \( \gamma_t \) so we know the colors of the atoms (eventually) belonging to \( G(\epsilon, y) \). Since \( G(\epsilon, y) \) is a seed of \( \epsilon \), the knowledge of the colors of the atoms of \( G(\epsilon, y) \) (or the fact that \( \epsilon \) is elementary if \( G(\epsilon, y) = \emptyset \)) allows us to deduce which were the colors of all the other atoms in \( G^c(\epsilon, y) = \text{obj}(\epsilon) \setminus G(\epsilon, y) \). Denote by \( c(x) \) the color of the atom \( x \in G^c(\epsilon, y) \) uniquely determined by the coloring of \( G(\epsilon, y) \). Then \( \gamma_{t-1}(x) = c(x) \) for \( x \in G^c(\epsilon, y) \setminus \{y\} \) and \( \gamma_t(y) = 0 \). Finally, since by definition \( y \in G^c(\epsilon, y) \), if \( c(y) \) is, say, equal to \( s \in [k] \) then \( f_t = s \). □

Let us denote, for \( r \in |[S]| \), \( F^r_t \) the set formed by all vectors in \( F_t \) such that \( |\gamma_t^{-1}(r)| = r \). Clearly \( F_t \) is the disjoint union of the family \( \{F^r_t\}_{r \in |[S]|} \) and therefore we have that

\[
|F_t| = \sum_{r=1}^{[S]} |F^r_t| \quad (3.1)
\]

Let \((R_t, G_t) \) (resp. \((R^r_t, G^r_t) \) ) set of all records produced with vectors in \( F_t \) (resp. \( F^r_t \) ), in other words \( (R_t, G_t) = M(F_t) \) (resp. \((R^r_t, G^r_t) = M(F^r_t) \) ).

**Proposition 3.3**

\[
|F_t| \leq ([|k|] + 1)^{|S|} \sum_{r=1}^{N} |R^r_t| \quad (3.2)
\]

**Proof.** From Lemma 3.2 it follows that

\[
|F^r_t| \leq |R^r_t| |G^r_t| \leq |R^r_t| |G_t| \leq |R^r_t| (k + 1)^{|S|} \quad (3.3)
\]

since \( G_t \) is a subset of the set of all partial colorings from \( S \) to \([k] \cup \{0\} \) which has cardinality \((k + 1)^{|S|}\). Putting (3.3) into (3.1) inequality (3.2) follows. □

### 3.1 Upper bound for \( |R^r_t| \)

A string \( w_1 w_2 \ldots w_n \) with \( w_i \in \{0, 1\} \) is usually called a word on the alphabet \( \{0, 1\} \). Give a word \( w \) on the alphabet \( \{0, 1\} \), the mirror image \( \bar{w} \) of \( w \) is the word obtained by reading \( w \) from right to left, i.e if \( w = w_1 w_2 \ldots w_n \) then \( \bar{w} = w_n w_{n-1} \ldots w_1 \). An initial segment of the word \( w_1 \ldots w_n \) is the a sub-string of \( w_1 \ldots w_n \) of the form \( w_1 \ldots w_i \) with \( 1 \leq i \leq n \). A **partial Dyck word** is word on the alphabet \( \{0, 1\} \) such that in any initial segment of the word the number of 0’s is greater or equal than the number of 1’s. A **Dyck word** on the alphabet \( \{0, 1\} \) is a partial Dyck word with equal number of 0’s and 1’s (hence a Dyck word has always an even number of letters). A partial Dyck word can be viewed as a path in \( \mathbb{Z}^2 \) starting at the origin made by steps either \((1, 1)\) or \((1, -1)\) in such a way that the path stays in the first quadrant (i.e. the path never goes below the x axis). A Dyck word (of size \( 2n \)) is then a Dyck path which starts at the origin and ends at the point \((2n, 0)\) of the x axis. A descent (ascent) in a partial Dyck word is a sequence of 1’s (0’s). For \( E \subseteq \mathbb{N} \) let us denote by \( D_{n,E} \) the set of Dyck words with \( n \) 0’s, \( n \) 1’s and with descents having cardinality in \( E \). Moreover,
for \( r < n \) natural number, let us denote by \( \mathcal{D}_{n,r,E} \) the set of partial Dyck words with \( n \) 0’s, \( n - r \) 1’s and with descents having cardinality in \( E \).

**Lemma 3.4** Let \( m_E = \min\{E \setminus \{1\}\} \). Then the following inequality holds.

\[
|\mathcal{D}_{n,r,E}| \leq |\mathcal{D}_{t+r(m_E-1),E}| \quad (3.4)
\]

**Proof.** We can construct an injective function \( f : \mathcal{D}_{n,r,E} \rightarrow \mathcal{D}_{n+r(m_E-1),E} \) by associating to the partial Dick word \( w \in \mathcal{D}_{t,r,E} \) the Dick word \( f(w) \in \mathcal{D}_{t+r(m_E-1),E} \) formed by appending at the end of \( w \) the word \((0^{m_E-1}1^{m_E})^r\). □

Let \( w \in \mathcal{D}_{n,E} \) and let \( j \in E \), we define \( \delta_j(w) \) as the number of descents in \( w \) having cardinality \( j \). We further define \( \alpha_j(w) \) as the number of ascents in \( w \) having cardinality \( j \). Let \( \mathcal{T}_{n+1,E} \) be the set of plane rooted trees with \( n + 1 \) vertices such that the number of children of each internal vertex takes value in \( E \). If \( \tau \in \mathcal{T}_{n+1,E} \) we define \( \delta_j(\tau) \) as the number of vertices in \( \tau \) having \( j \) children.

**Lemma 3.5** There is a one-to-one map \( h : \mathcal{T}_{n+1,E} \rightarrow \mathcal{D}_{n,E} : \tau \mapsto w \) such that for any \( j \in E \)

\[
\delta_j(\tau) = \delta_j(w) \quad (3.5)
\]

**Proof.** We construct the map \( h \) as follows. For a given \( \tau \in \mathcal{T}_{n+1,E} \), consider the string \( \rho_1 = v_1 \ldots v_n \) formed by the first \( n \) vertices of \( \tau \) ordered according to a Deep-first search (these are all vertices of \( \tau \) but a leaf). Now substitute each \( v_i \) in the string \( \rho_i \) with the word

\[
\underbrace{00 \ldots 01}_s
\]

where \( s_i \) is the number of children of the vertex \( v_i \). Let \( u \) the word so obtained. By observing that \( \sum_{i=1}^{n} s_i = n \) we conclude that \( u \) is a Dyck word of size \( 2n \) whose ascents have cardinality in \( E \) and \( \alpha_j(u) = \delta_j(\tau) \) for all \( j \in E \). Moreover the map \( \tau \mapsto u \) is clearly a bijection. Now consider the mirror image \( \tilde{u} \) of \( u \) and interchange the 0’s and the 1’s and let \( w \) the word so obtained. The composition of maps \( \tau \mapsto u \mapsto \tilde{u} \mapsto w \) is clearly a bijection from \( \mathcal{T}_{n+1,E} \) to \( \mathcal{D}_{n,E} \) and the word \( w \) is a Dyck word of size \( 2n \) whose descents have cardinality in \( E \) and, for all \( j \in E \), it holds that \( \delta_j(w) = \delta_j(\tau) \). □

We now construct a (non injective) map \( \mathfrak{M} : \mathcal{R}_t^\ast \rightarrow \mathcal{D}_{t,r,E_O} \).

**Definition 3.6** Let \( R_t = r_1 \ldots r_t \in \mathcal{R}_t^\ast \). We define the map \( \mathfrak{M} \) which associates to \( R_t = r_1 \ldots r_t \) the word \( R_t^\ast = r_1^\ast \ldots r_t^\ast \) on the alphabet \( \{0,1\} \) as follows. If \( r_i = 0 \) then put \( r_i^\ast = 0 \). If \( r_i = (j,\ell) \) then put

\[
\underbrace{011 \ldots 1}_j
\]

**Lemma 3.7** For any \( R_t \in \mathcal{R}_t^\ast \), the word \( R_t^\ast \) defined above is a partial Dyck word with \( t \) 0’s and \( t - r \) 1’s and descents in the set \( E_O \) defined in (2.14). In other words if \( R_t \in \mathcal{R}_t^\ast \), then \( R_t^\ast \in \mathcal{D}_{t,r,E_O} \).
Proof. Reading $R_t^*$ from left to the right, every 0 correspond to a variable that has been colored and every 1 correspond to a variable that has been uncolored if some bad event occurs. So the number of 0’s is $t$. Let $x$ be the number of 1’s. This number represents by construction the total number of variables that have been uncolored during the process of the algorithm. Thus $t - x = r$, i.e. $x = t - r$. The string is by construction a partial Dyck word since we cannot uncolor more variables than the number of colored variable. Finally again by construction we have that the descents are of size $j \in E_O$.

Lemma 3.8 The pre-image of a string $R_t^*$ by the map $\mathfrak{M}$ has cardinality less or equal than

$$\prod_{j \in E_O} (d_j)^{\delta_j(R_t^*)}$$

where recall that $\delta_j(R_t^*)$ is the number of descents in $R_t^*$ of size $j$ and $d_j$ is defined in (2.13).

Proof. For each descent of size $j$ in $R_t^*$, we do not distinguish which flaw of size $j$ that generated this descent. Since there are at most $d_j$ flaws of size $j$ containing a fixed atom, this accounts for the $\prod_{j \in E_O} (d_j)^{\delta_j(R_t^*)}$ factor. \[\square\]

Let us now define

$$\varphi_j = \begin{cases} 
1 & \text{if } j = 0 \\
d_j & \text{if } j \in E_O \\
0 & \text{otherwise}
\end{cases}$$

(3.6)

Note that, for $x > 0$

$$\sum_{j \geq 0} \varphi_j x^j = \phi_O(x)$$

(3.7)

We now can prove the following proposition.

Proposition 3.9 Let $T_n$ denotes the set of all plane rooted trees with $n$ vertices, then we have

$$|R_t^r| \leq \sum_{\tau \in \mathcal{T}_{t+r(m_{E_O} - 1)+1}} \prod_{j \geq 0} (\varphi_j)^{\delta_j(\tau)}$$

(3.8)

where recall that $\delta_j(\tau)$ denotes the number of vertices with $j$ children in $\tau$.

Proof. From Lemmas 3.7 and 3.8 it follows that

$$|R_t^r| \leq \sum_{w \in D_{t,r,E_O}} \prod_{j \in E_O} (d_j)^{\delta_j(w)}$$

Lemma 3.4 then implies

$$\sum_{w \in D_{t,r,E_O}} \prod_{j \in E_O} (d_j)^{\delta_j(w)} \leq \sum_{w \in D_{t+r(m_{E_O} - 1),E_O}} \prod_{j \in E_O} (d_j)^{\delta_j(w)}$$
Lemma 3.5 yields

\[
\sum_{w \in D_{t+r(m_{EO}-1), E_O}} \prod_{j \in E_O} (d_j)^{\delta_j(w)} = \sum_{\tau \in T_{t+r(m_{EO}-1)+1, E_O}} \prod_{j \in E_O} (d_j)^{\delta_j(\tau)}
\]

and finally using definition (3.6) we get

\[
\sum_{\tau \in T_{t+r(m_{EO}-1)+1, E_O}} \prod_{j \in E_O} (d_j)^{\delta_j(\tau)} = \sum_{\tau \in T_{t+r(m_{EO}-1)+1, E_O}} \prod_{j \geq 0} \phi_j^{\delta_j(\tau)}
\]

\[\square\]

We can now use the following well known theorem in analytic combinatorics (see e.g., Theorem 5 in [12]).

**Theorem 3.10** Let \(\{\phi_j\}_{j \geq 0}\) a sequence of nonnegative numbers and let \(\varphi(x) = \sum_{j \geq 0} \phi_j x^j\); let \(R\) be the convergence radius of \(\varphi(x)\) and let \(\xi\) be the first root in \([0, R)\) of the equation \(x\varphi'(x) - \varphi(x) = 0\). Then

\[
\sum_{\tau \in T_n} \prod_{j \geq 0} \phi_j^{\delta_j(\tau)} \leq C_\varphi \left[\frac{\varphi'(\xi)}{\phi(\xi)}\right]^{n/3} \tag{3.9}
\]

with \(C_\varphi\) constant.

Note that

\[\varphi'(\xi) = \min_{x > 0} \frac{\varphi(x)}{x}\]

Inserting now inequality (3.9) into (3.8) and recalling (3.7) we get

\[
|R_t| \leq C_\varphi \left[\frac{\phi_O(\xi)}{t + r(m_{EO} - 1) + 1} \right]^{3/2} \tag{3.10}
\]

**3.2 Conclusion of the proof of Theorem 2.2**

We are now in the position to end the proof of Theorem 2.2. Indeed, inserting (3.10) into (3.2) we get

\[
|F_t| \leq C_\varphi (k + 1)^{|S|} \sum_{r=1}^{|S|} \frac{[\phi_O'(\xi)]^{t+r(m_{EO}-1)+1}}{(t + r(m_{EO} - 1) + 1)^{3/2}} \leq C_{\varphi, S, k}
\]

with

\[
C_{\varphi, S, k} = C_\varphi (k + 1)^{|S|} [\phi_O'(\xi)]^{|S| (m_{EO}-1) + 1}
\]

Hence if

\[
[\phi_O'\xi] = \min_{x > 0} \frac{\phi_O(x)}{x} \leq k \tag{3.11}
\]

we have that for \(t\) larger than \(C_{\varphi, S, k}^{2/3}\)

\[|F_t| < k^t\]
i.e. the algorithm stops. The condition (3.11) which ensures the algorithm to stop is exactly (2.1) with greater or equal replacing strictly greater. Now note that, if each entry of the vector $F_t$ is selected uniformly and independently at random, then $|F_t|/k^t$ is the probability that after $t$ steps the (random) EC-algorithm is still running. Let us suppose that inequality (3.11) holds strictly. I.e., let us suppose that there exists $\varepsilon > 0$ such that $(\phi'_O(\xi)/k) \leq (1 - \varepsilon)$. Then the probability $P(t)$ that the algorithm is still running after $t$ steps can be bounded above as follows

$$P(t) \leq C_\varphi(k+1)^{|S|[\phi'_E(\xi)]}|S|(s-1)+1(1-\varepsilon)^t$$

Let $t_0$ be the solution of the equation

$$C_\varphi(k+1)^{|S|[\phi'_E(\xi)]}|S|(s-1)+1(1-\varepsilon)^t = 1$$

Remark that $t_0$ is linear in $|S|$ and observe that $P(t_0+t) \leq (1-\varepsilon)^t = e^{-t|\ln(1-\varepsilon)|}$. Therefore, if inequality (3.11) holds strictly, the expected running time $T$ of the EC-algorithm is bounded by

$$T \leq t_0 + \frac{1}{|\ln(1-\varepsilon)|}$$

So if the condition (3.11) is replaced by the inequality

$$\phi'_O(\xi) \leq (1 - \varepsilon)k$$

for some $\varepsilon > 0$, then the expected running time of the algorithm is linear in $|S|$. This concludes the proof of Theorem 2.2.

4 Comparison and examples

4.1 Entropy-Compression versus cluster expansion

It is instructive and illuminating, we think, to start by comparing the entropy-compression criterion with the cluster expansion criterion on a general basis. We will assume here that we are in the independent EC-setting. For a comparison between EC and CELL in the dependent EC-setting (i.e. the case of the acyclic edge coloring) we refer the reader to Section 4.2.5.

We have therefore the pair $(S,k)$ together with a family of objects $\mathcal{O}$ and a family $\mathcal{F}$ of standard flaws associated to $\mathcal{O}$. As previously mentioned, this corresponds to a Moser-Tardos setting whose probability space is generated by $|S|$ i.i.d. random variables $\{\psi_x\}_{x \in S}$ all taking value in $[k]$ according to the uniform distribution. In this framework it is natural (and easier) to use the clique estimate (1.9) in order to evaluate $\Xi_{\varepsilon}(\mu,G)$. Therefore the CE criterion (1.6) is satisfied if

$$P(\varepsilon) \leq \frac{\mu_{\varepsilon}}{\Xi_{\varepsilon}(\mu,G)} = \frac{\mu_{\varepsilon}}{\prod_{y \in \text{obj}(t)} \left[ 1 + \sum_{e' \in \mathcal{F}(y)} \mu_{e'} \right]}$$

(4.1)
We will refer below to condition (4.1) as the strong variable setting CE criterion. This condition (4.1), of course, is strongest than the CE criterion (1.6) in the sense that to be fulfilled requires smaller probabilities for the bad events. We stress once again that the strong variable setting CE criterion (4.1) has been the tool used in recent times to improve several bounds previously obtained via the classical version of LLL given by Theorem 1.1.

Now, in the elementary probability space generated by i.i.d. uniformly distributed random variables \( \{ \psi_x \}_{x \in S} \) taking values in \([k]\), given \( y \in S, j \in E_O \), and \( \epsilon \in \mathcal{F}_j(y) \) and recalling definitions (2.11) and (2.12), it is immediate to see that

\[
P(\epsilon) = \frac{k^{\kappa_y(\epsilon)}}{k^{||\text{obj}(\epsilon)||}} = \frac{1}{k^j} \tag{4.2}
\]

Moreover, for any \( \epsilon \in \mathcal{F} \)

\[
\Xi_{\epsilon}^{\text{clique}}(\mu, G) = \prod_{y \in \text{obj}(\epsilon)} \left( 1 + \sum_{e' \in \mathcal{F}_j(y)} \mu_{e'} \right) \leq \prod_{y \in \text{obj}(\epsilon)} \left( 1 + \sum_{j \in E_O} \sum_{e' \in \mathcal{F}_j(y)} \mu_{e'} \right) \tag{4.3}
\]

Given (4.2), it is natural to set \( \mu_{\epsilon} = x^j \) for all \( \epsilon \in \mathcal{F}_j \), where \( x > 0 \), so that, recalling definition (2.13), we get

\[
\Xi_{\epsilon}^{\text{clique}}(\mu) \leq \prod_{y \in \text{obj}(\epsilon)} \left( 1 + \sum_{j \in E_O} d_j x^j \right) = \left[ \phi_O(x) \right]^{||\text{obj}(\epsilon)||} \leq [\phi_O(x)]^{l_j} \tag{4.4}
\]

where

\[ l_j = \max \{ ||\text{obj}(\epsilon)|| : \epsilon \in \mathcal{F}_j \} \]

Note that \( \phi_O(x) \) is the very same function appearing in the entropy-compression criterion (2.1). Using (4.4), condition (4.1) is rewritten as

\[
\frac{1}{k^j} \leq \max_{x > 0} \frac{x^j}{[\phi_O(x)]^{l_j}} \quad \text{for all} \quad j \in E_O
\]

In conclusion, in this restricted Moser-Tardos variable setting in which the random variables \( \{ \psi_x \}_{x \in S} \) are i.i.d. uniformly distributed all taking values in the same set \([k]\), the strong variable setting CE criterion (4.1) can be rewritten as

\[
k \geq \min_{x > 0} \frac{[\phi_O(x)]^{l_j}}{x} \quad \text{for all} \quad j \in E_O \tag{4.5}
\]

It is now crystal clear to compare (4.5) with the entropy-compression criterion (2.1).

Since for all \( j \in E_O \) we have that \( l_j \geq j \) and \( \phi_O(x) > 1 \), it holds

\[
\frac{[\phi_O(x)]^{l_j}}{x} \geq \frac{\phi_O(x)}{x} \quad \text{for all} \quad j \in E_O \quad \text{and for all} \quad x > 0
\]

and the equality only holds if \( l_j = j \) for all \( j \in E_O \) which happens only when all (tidy) events of the family \( \mathcal{F}_j \) are elementary.

The conclusion is that in the independent EC-setting \((S, k)\), the entropy-compression criterion (2.1) is always better than or equal to condition (4.5) and
these two conditions coincide only all events of the family \( \mathcal{F} \) are elementary. However, it is worth to remind that condition (4.5) has been derived from (4.1) where the clique estimate \( \Xi_{\mathcal{G}}^{\text{clique}}(\mu, \mathcal{G}) \) has been used. As said above, condition (4.1) yields bounds poorer than those obtainable by the general CE criterion (1.6) as soon as the cliques composing \( \Gamma_{\mathcal{G}}(\epsilon) \) are too overlapped.

4.2 Some graph coloring examples: when and to what extent EC criterion beats CE criterion

In this subsection \( G = (V, E) \) is a graph with maximum degree \( \Delta \) and \( k \in \mathbb{N} \).

4.2.1 Non repetitive vertex coloring

Let us start with a simple example fitting in the independent EC-setting. A coloring \( c : V \to [k] \) of the vertices of \( G \) is called nonrepetitive if, for any \( n \geq 1 \), no path \( p = \{v_1, v_2, \ldots, v_{2n}\} \subset V \) is repetitive (i.e such that \( c(v_i) = c(v_{i+n}) \) for all \( i = 1, 2, \ldots n \)). Observe that a nonrepetitive vertex coloring is proper.

The EC-setting \( (S, \mathcal{V}) \) for the present case is as follows. The set of atoms is \( S = V \) and the set of colors is \( [k] \). We now determine the families of objects \( \mathcal{O} \) and of flaws \( \mathcal{F} \) associated to \( \mathcal{O} \). Let \( P_n \) be the set of all paths with \( 2n \) vertices in \( G \) and let \( P = \bigcup_{n \geq 1} P_n \). We set \( \mathcal{O} = P \). Let now for \( p \in P \), \( \epsilon_p \) be the event that \( p \) is monochromatic and set \( \mathcal{F}_P = \{\epsilon_p\}_{p \in P} \). Clearly any event of the family \( \mathcal{F}_P \) is standard and therefore here, as in all the examples to follow (except that of Subsection (4.2.5)), we are in the independent EC setting. Moreover observe that any event \( \epsilon \in \mathcal{F}_P \) is tidy and, for any \( v \in \text{obj}(\epsilon) \), we have that \( \|\epsilon_v\| = \frac{1}{2} \text{obj}(\epsilon) \). Therefore in the present case we have

\[
\mathcal{E}_\mathcal{O} = \{j\}_{j \geq 1},
\]

Furthermore, for \( j \in \mathcal{E}_\mathcal{O} \), the number \( d_j \) defined in (2.13) represents here the maximal number of paths of size \( 2j \) in \( G \) containing a fixed vertex, which can be bounded as

\[
d_j \leq j\Delta^{2j-1}
\]

so that

\[
\phi_\mathcal{O}(x) \leq 1 + \sum_{j=1}^{\infty} j\Delta^{2j-1}x^j, \quad \text{with } 0 < \Delta^2x < 1
\]

Hence the EC criterion (2.1) in the present case is fulfilled if

\[
k \geq \min_{0 < \Delta^2 x < 1} \frac{1 + \frac{1}{\Delta} \sum_{j \geq 1} j(\Delta^2x)^j}{x}
\]

Inequality (4.7) can be rewritten as

\[
k \geq \left[ \min_{0 < x < 1} g(x) \right] \Delta^2
\]

where

\[
g(x) = \frac{1 + \frac{1}{\Delta} x}{x} = \frac{1}{x} + \frac{1}{\Delta (1-x)^2}
\]
For $\Delta \geq 3$ we can take $x_1 = 1 - \frac{2^{1/3}}{\Delta^{1/3}}$ (which is such that $x_1 - \min_{0 < x < 1} g(x) = o(\Delta^{-1/3})$) and we can bound

$$g(x_0) \leq g(x_1) = 1 + \frac{3}{2^x \Delta^x} + \frac{2^{2/3}}{\Delta^2} \frac{1}{1 - \frac{2^{x/3}}{\Delta^x}}.$$  

Hence (4.7) is satisfied if

$$k \geq \Delta^2 \left(1 + \frac{3}{2} \Delta^{-\frac{1}{3}} + o(\Delta^{-\frac{1}{3}})\right)$$

which is the bound given in [22].

It is worth to notice that in this first example the strong variable setting CE criterion in the form (4.5) obtains a bound nearly as good as (4.8). Indeed, recalling that in the present case $\|e_v\| = \frac{1}{2} |\text{obj}(e)|$ for any $e \in \mathcal{F}$ and any $v \in \text{obj}(e)$, we have that $\frac{L}{j} = 2$ for all $j \in E_O$. Therefore (4.5) is written as

$$k \geq \left[ \min_{0 < x < 1} \tilde{g}(x) \right] \Delta^2$$

with

$$\tilde{g}(x) = \frac{(1 + \frac{1}{\Delta} \frac{x}{(1-x)^2})^2}{x}$$

and, by numerical computation (4.9) is satisfied if

$$k \geq \Delta^2 \left(1 + \frac{5}{2} \Delta^{-\frac{1}{3}} + o(\Delta^{-\frac{1}{3}})\right)$$

which is worst than the bound (4.8) obtained via EC criterion only at the order $O(\Delta^{-\frac{1}{3}})$.

### 4.2.2 Star vertex coloring

A proper coloring $c : V \to [k]$ of the vertices of $G$ is called a **star coloring** if no path $p = \{v_1, v_2, v_3, v_4\}$ is bichromatic. The minimum numbers of colors required for a graph $G$ to have a vertex star coloring is called the star chromatic index of $G$ and it is denoted by $\pi(G)$.

Let $P_2$ be the set of all paths with 2 vertices and let $P_4$ be the set of all paths with 4 vertices. The EC-setting in the present case is established by taking $S = V$, $O = P_2 \cup P_4$. The family $\mathcal{F}_{P_2 \cup P_4}$ of flaws (a subfamily of the family $\mathcal{F}_P$ of the previous example) is given by $\mathcal{F} = \mathcal{F}_{P_2} \cup \mathcal{F}_{P_4}$ where $\mathcal{F}_{P_2} = \{e_p\}_{p \in P_2}$ and $\mathcal{F}_{P_4} = \{e_p\}_{p \in P_4}$. As in the previous example all flaws of in $\mathcal{F}_{P_2} \cup \mathcal{F}_{P_4}$ are standard, non elementary and tidy. This is once again an independent EC-setting. Clearly, a coloring $c : V \to [k]$ avoiding all flaws in $\mathcal{F}$ is a star coloring.

Moreover $e \in \mathcal{F}_i$ ($i = 2, 4$) is such that for any $v \in \text{obj}(e)$, it holds $\|e_v\| = \frac{i}{2}$. This implies that

$$E_O = \{1, 2\}$$

and, recalling (4.6)

$$\phi_O(x) \leq 1 + \Delta x + 2\Delta^3 x^2, \quad \text{with } x > 0$$
Hence the entropy-compression criterion (2.1) in the actual case is fulfilled if

\[
k \geq \min_{x>0} \frac{1 + \Delta x + 2(\Delta^3 x^2)}{x}
\]  

(4.10)

with \(x > 0\). Setting \(\Delta x = y\), inequality (4.10) can be rewritten as

\[
k \geq \left[\min_{y>0} g(y)\right] \Delta
\]  

(4.11)

where

\[
g(y) = \frac{1 + y + 2\Delta y^2}{y}
\]

The minimum of \(g(y)\) for \(y > 0\) occurs at \(y_0 = (\sqrt{2}\Delta^{\frac{3}{2}})^{-1}\) and \(g(y_0) = \sqrt{2}\Delta^{\frac{3}{2}}(2 + (\sqrt{2}\Delta^{\frac{3}{2}})^{-1})\) and thus

\[
k \geq 2\sqrt{2} \Delta^{\frac{3}{2}} + \Delta
\]  

(4.12)

which is the bound given in [15].

Finally, let us compare the bound (4.11) with that produced by the strong variable setting CE criterion (4.5). As in the previous example, \(\frac{l_j}{j} = 2\) for \(j = 1, 2\), so that (4.5) is

\[
k \geq \left[\min_{y>0} \left(\frac{1 + y + 2\Delta y^2}{y}\right)^2\right] \Delta
\]  

(4.13)

which yields

\[
k \geq \frac{8}{\sqrt{27}} \Delta^{\frac{3}{2}} + \frac{8\Delta}{3} + o(\Delta)
\]

being worst than (4.12) by a factor 1.54 at the lowest order in \(\Delta\).

### 4.2.3 Vertex coloring graphs frugally

Given a natural number \(\beta\), a proper coloring of the vertices of \(G\) is \(\beta\)-frugal if any vertex has at most \(\beta\) members of any color class in its neighborhood.

The minimum number of colors required such that a graph \(G\) has at least one \(\beta\)-frugal proper vertex coloring is called the \(\beta\)-frugal chromatic number of \(G\) and will be denoted by \(\chi_{\beta}(G)\).

Given \(v \in V\), we denote by \(H^\beta_v\) the set of all subsets of the neighborhood \(\Gamma^*_G(v)\) of \(v\) with exactly \(\beta + 1\) vertices. In other words

\[
H^\beta_v = \{Y \subset \Gamma^*_G(v) : |Y| = \beta + 1\}
\]

Moreover, given \(v \in V\) and \(h^\beta_v \in H^\beta_v\), we let \(c_{h^\beta_v}\) to be the event that \(h^\beta_v\) is monochromatic). Let \(\mathcal{H}\beta = \{H^\beta_v\}_v \in V\) and let \(\mathfrak{F}\beta = \{c_{h^\beta_v}\}_{h^\beta_v \in H^\beta_v, v \in V}\). Then the EC-setting in the present case is established by taking \(\mathcal{S} = V\), \(\mathcal{O} = P_2 \cup H\beta\) and \(\mathfrak{F}_{P_2 \cup H\beta} = \mathfrak{F}_{P_2} \cup \mathfrak{F}_{H\beta}\). Note that once again this is an independent EC-setting.
Clearly all flaws in \( \mathcal{F} \) are tidy, with seeds of size 1 and with effective size either 1 (if the flaw belongs to \( \mathcal{F}_{P_2} \)) or \( \beta \) (if the flaw belongs to \( \mathcal{F}_{H_\beta} \)). Thus \( E_0 = \{1, \beta\} \).

As in the previous example \( d_1 \leq \Delta \). Moreover, concerning the estimate of \( d_\beta \), observe that a vertex \( w \in V \) has at most \( \Delta \) neighbors \( \{v_1, \ldots, v_\Delta\} \) and therefore \( w \) belongs to at most \( \Delta \) distinct neighborhoods \( \Gamma^*_G(v_1), \ldots, \Gamma^*_G(v_\Delta) \) and in each \( \Gamma^*_G(v_i) \) of these neighborhoods \( w \) can be contained in at most \( \Delta^\beta \) sets of type \( h^*_\beta = \{v^*_1, v^*_2, \ldots, v^*_\beta+1\} \). Therefore

\[
d_\beta \leq \Delta \left( \frac{\Delta}{\beta} \right) \leq \frac{\Delta^{1+\beta}}{\beta!} \tag{4.14}
\]

Hence we get

\[
\phi_0 = 1 + \Delta x + \frac{\Delta^{1+\beta}}{\beta!} x^\beta
\]

The condition (2.1) is therefore

\[
k \geq \min_{x > 0} \frac{1 + \Delta x + \frac{\Delta^{1+\beta}}{\beta!} x^\beta}{x} \tag{4.15}
\]

i.e, setting \( \Delta x = y \)

\[
k \geq \left[ \min_{y > 0} g_\beta(y) \right] \Delta \tag{4.16}
\]

with

\[
g_\beta(y) = \frac{1 + y + \frac{\Delta}{\beta!} y^\beta}{y}
\]

The minimum occurs in \( y_0 = \left[ \frac{\Delta}{\beta!} (\beta - 1) \right]^{\frac{1}{\beta}} \) and \( g(y_0) = 1 + \left( \frac{\Delta}{\beta!} \right)^{\frac{1}{\beta}} (\beta - 1)^{\frac{1}{\beta}} \frac{\beta}{\beta - 1} \), therefore there is a \( \beta \)-frugal coloring of \( G \) as soon as

\[
k \geq \frac{\Delta^{1+\frac{1}{\beta}}}{(\beta!)^{1/\beta}} \frac{\beta}{(\beta - 1)^{1-\frac{1}{\beta}}} + \Delta
\]

and thus

\[
\chi_\beta(G) \leq \Delta + \frac{\Delta^{1+\frac{1}{\beta}}}{(\beta!)^{1/\beta}} \frac{\beta}{(\beta - 1)^{1-\frac{1}{\beta}}} \tag{4.17}
\]

The bound (4.17) represents an improvement on the previous best bound for the \( \beta \)-frugal chromatic number of a graph with maximum degree \( \Delta \) given in [27]. In particular, for the case of 2-frugal coloring (a.k.a. linear coloring) we get

\[
\chi_2(G) \leq \sqrt{2} \Delta^{\frac{3}{2}} + \Delta \tag{4.18}
\]

In [38] it is proved that \( \Delta^{\frac{3}{2}} \) is the correct order since by an explicit example the author proves that \( \chi_2(G) \geq \Delta^{3/2}/(6\sqrt{3}) \).

To conclude this example let us notice that, for any \( \beta \geq 2 \) we have once again that \( \max_{j=1,\beta} \frac{L_j}{j} = 2 \) and therefore, for the case \( \beta = 2 \), the CE criterion (4.5) is

\[
k \geq \left[ \min_{y > 0} \frac{(1 + y + \frac{\Delta}{2!} y^2)^2}{y} \right] \Delta \tag{4.19}
\]
yielding
\[ k \geq \frac{8}{\sqrt{27}} \sqrt{2\Delta^3} + \frac{8\Delta}{3} + o(\Delta) \]
That is to say in this example as the previous example, EC criterion beats CE criterion by a factor \( \frac{8}{\sqrt{27}} \approx 1.54 \) in the leading order \( \Delta^\frac{3}{2} \).

4.2.4 Acyclic vertex coloring

A proper coloring \( c : V \to [k] \) of the vertices of \( G \) is called an *acyclic vertex coloring* if no cycle of \( G \) is bichromatic. The minimum numbers of colors required for a graph \( G \) to have an acyclic vertex coloring is called the vertex acyclic chromatic index of \( G \) and it is denoted by \( \chi'(G) \).

Following [3], we say that a pair of next nearest neighbor vertices \( u, v \) of \( G \) is a *special pair if they have more than \( \alpha \Delta^{2/3} \) common neighbors* (\( \alpha \) is a positive parameter to be optimized later). We set \( S = V \) as the set of atoms and define the set of objects \( O \) as follows. As before \( P_2 \) is the set of all paths with 2 vertices in \( G \); \( C_4 \) is the set of cycles with four vertices not containing special pairs in \( G \); \( S_2 \) is the set of all special pairs of vertices in \( G \) and finally \( P_6 \) is the set of all paths with 6 vertices in \( G \). We set \( O = P_2 \cup C_4 \cup S_2 \cup P_6 \). Let now, for \( p \in P_2 \) \( e_p \) be the event that \( p \) is monochromatic and let \( \mathcal{F}_2 = \{ e_p \}_{p \in P_2} \). For \( c \in C_4 \) let \( e_c \) be a properly bichromatic 4-cycle not containing special pairs and let \( \mathcal{F}_C = \{ e_c \}_{c \in C_4} \). For \( s \in S_2 \), let \( e_s \) be the event that \( s \) is monochromatic and let \( \mathcal{F}_S = \{ e_s \}_{s \in S_2} \). Finally, for \( q \in P_6 \), let \( e_q \) be the event that \( q \) is properly bichromatic and let \( \mathcal{F}_P = \{ e_q \}_{q \in P_6} \). We set \( \mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_C \cup \mathcal{F}_S \cup \mathcal{F}_P \). Clearly a vertex coloring avoiding all flaws in \( \mathcal{F} \) is acyclic and proper.

Note that all events of \( \mathcal{F} \) are tidy and \( e \in \mathcal{F}_P \) is such that \( ||e|| = 1 \); \( e' \in \mathcal{F}_C \) is such that \( ||e'|| = 2 \); \( e'' \in \mathcal{F}_S \) is such that \( ||e''|| = 1 \) and \( e \in \mathcal{F}_P \) is such that \( ||e|| = 4 \). Therefore in the present case we have \( E_O = \{ 1, 2, 4 \} \). Moreover let \( \tilde{d}_i \) be the number of paths of size 2 containing a fixed vertex, \( \tilde{d}_1 \) the set of special pairs containing a fixed vertex, \( d_i = \tilde{d}_i + \tilde{d}_1 \); let \( d_2 \) be the set of 4-cycles not containing special pairs containing a fixed vertex and let \( d_4 \) be the set of 6-paths containing a fixed vertex, it is easy to check (see e.g. Lemma 2.4 in [3] or Sec. 2 in [34]) that \( \tilde{d}_1 \leq \Delta, \tilde{d}_1 \leq \frac{5}{\alpha} \Delta^{\frac{4}{3}}, d_2 \leq \frac{\beta}{\Delta} \Delta^{\frac{5}{3}} \) and \( d_4 \leq 3 \Delta^5 \). Then we have that
\[ \phi_O(x) = 1 + (\Delta + \frac{1}{\alpha} \Delta^{\frac{4}{3}})x + \frac{\alpha}{2} \Delta^{\frac{8}{3}} x^2 + 3 \Delta^{5} x^3 \]
Hence the EC criterion \((4.21)\) in the actual case is fulfilled if
\[
\min_{x > 0} \frac{1 + \Delta x + \frac{1}{\alpha} \Delta^{\frac{4}{3}} x + \frac{\alpha}{2} \Delta^{\frac{8}{3}} x^2 + 3 \Delta^{5} x^3}{x} \leq k \tag{4.20}
\]
Inequality \((4.20)\) can be rewritten as
\[ k \geq \left[ \min_{x > 0} (g_\alpha(x) + g_1(x)) \right] \Delta^{\frac{3}{2}} \]
where
\[ g_\alpha(x) = \frac{1 + \frac{1}{\alpha} x + \frac{\beta}{\Delta} x^2}{x} \quad \text{and} \quad g_1(x) = \Delta^{-\frac{4}{3}} (1 + x + 3x^3) \]
Taking \( x = \sqrt{\frac{2}{\alpha}} \) (where the minimum of \( g_\alpha(x) \) occurs) we can bound

\[
k \geq (\sqrt{2\alpha} + \frac{1}{\alpha})\Delta^\frac{4}{3} + \left[ 1 + \sqrt{\frac{2}{\alpha}} + 3 \left( \frac{2}{\alpha} \right)^{\frac{3}{2}} \right] \Delta
\]

Optimizing now in \( \alpha \) in the leading term \( \propto \Delta^\frac{4}{3} \) by taking \( \alpha = \sqrt{2} \) we get

\[
k \geq \frac{3}{\sqrt{2}} \Delta^\frac{4}{3} + (7 + \sqrt{2})\Delta
\]

and thus \( \chi(G) \leq (3/\sqrt{2})\Delta^\frac{4}{3} + (7 + \sqrt{2})\Delta \). This bound is asymptotically slightly better than the one obtained in [34].

To compare this bound with the one obtained via CELL, observe that once again \( \max_{j=1,2,4} l_j = 2 \) and therefore the CE criterion (4.5) is

\[
k \geq \left[ \min_{x>0} \left( \frac{1 + \frac{1}{\alpha} x + \frac{\alpha}{2} x^2}{x} \right)^2 \right] \Delta^\frac{4}{3} + o(\Delta^\frac{4}{3})
\]

which, optimizing for \( \alpha \) yields

\[
k \geq \frac{9}{2} \Delta^\frac{4}{3} + o(\Delta^\frac{4}{3})
\]

So in this example EC beats CELL by a factor 1.89 in the leading order \( \Delta^\frac{4}{3} \).

### 4.2.5 A dependent EC-setting example: acyclic edge coloring

A proper edge coloring of \( G \) is acyclic if any cycle of \( G \) is colored with at least three colors. The minimum number of colors required such that \( G \) has at least one acyclic proper edge coloring is called the acyclic edge chromatic number of \( G \) and will be denoted by \( a'(G) \).

If \( G \) has maximum degree \( \Delta \), then, for any fixed order of the edges of \( G \), one can properly colors the edges of \( E \) using \( 2\Delta - 1 \) colors: just color consecutively each edge \( e \) by using the smallest color not already used in the set \( \Gamma(e) \) of the edges adjacent to \( e \). Since \( |\Gamma(e)| \leq 2\Delta - 2 \), at each step this color will be always available. Esperet and Parreau [15] noticed that, still using just \( 2\Delta - 1 \) colors, one can also avoid at each step all bichromatic cycles of length 4. Indeed, when coloring \( e \), if \( f \) and \( f' \) are in \( \Gamma(e) \) and have the same color \( s_1 \) and \( e' \in E \), having color \( s_2 \), is such that \( \{e,f,f',e'\} \) is a 4-cycle, then we must avoid the color \( s_2 \) but at the same time in \( \Gamma(e) \) only at most \( 2\Delta - 3 \) colors are used (since \( f \) and \( f' \) have the same color). In other words the following lemma holds.

**Lemma 4.1** Given a graph \( G = (V,E) \) with maximum degree \( \Delta \), the edges of \( G \) can be colored sequentially using just \( 2\Delta - 1 \) colors, in such a way that the coloring is proper and contains no bichromatic cycle of length 4.

In order to use the EC criterion to estimate \( a'(G) \), let us first of all establish the pair \( (S,k) \). Since here we are coloring the edges of \( G \) we set \( S = E \) and, in view of Lemma 4.1 we choose \( k \) in such a way that \( k \geq 2\Delta - 1 \).
Now, for any partial coloring $\gamma \in [k]^{E}_0$ of the edges of $G$ using $k \geq 2\Delta - 1$ colors and for any $e \in E$, let $\Phi(e, \gamma)$ be the subset of $[k]$ formed by the colors present in $\Gamma(e)$ or such that exists $f, f' \in \Gamma(e)$ and $e' \in E$ such that $\{e, f, f', e'\}$ is a properly bichromatic 4-cycle in $G$. By the discussion above we have that $|\Phi(e, \gamma)| \leq 2(\Delta - 1)$. We now associate to each $e \in E$ and each $s \in \Phi(e, \gamma)$ the non-standard elementary flaw $e = \{e \text{ has the color } s\}$. Let us denote by $F_E$ the set of all these flaws. We stress once again that each flaw $e \in F_E$ is non-standard. Moreover, since all events of $F_E$ are elementary, we have $\|e\| = |\text{obj}(e)| = 1$ for all $e \in F_E$. Clearly a coloring $c : E \rightarrow [k]$ avoiding all flaws in $F_E$ is a proper coloring of the edges of $G$ with no bichromatic 4-cycles.

Let now $C_{2n}$ be the set of all cycles of size $2n$ in $G$ and let $C = \{C_{2n}\}_{n \geq 3}$ the set of all cycles of even length greater than 4 in $G$. Let $F_C$ the family of flaws $F_C = \{\text{the cycle } c \text{ is properly bi-chromatic}\}_{c \in C}$. Any flaw of $e \in F_C$ is standard, non elementary and tidy with seeds of size 2. Thus, for any $e \in F_C$ and for any $e \in \text{obj}(e)$, it holds $\|e\| = |\text{obj}(e)| - 2$.

Clearly a coloring $c : E \rightarrow [k]$ avoiding all flaws in $F_E \cup F_C$ is an acyclic edge coloring of $G$. Therefore the EC-setting in the the present case is established by taking $O = E \cup C$ and $F = F_E \cup F_C$. We stress once again that this is a dependent EC-setting since each event $e$ in the subfamily $F_E$ is non standard (i.e. $e$ depend on the (partial) coloring outside $\text{obj}(e)$). Moreover we have

$$E_O = \{1\} \cup \{2n - 2\}_{n \geq 3}$$

Let us find the numbers $d_j$ defined in (2.13) for $j \in E_O$. Recalling that

$$\max_{v \in V, \gamma \in [k]^{V}_0} |\Phi(v, \gamma)| \leq 2(\Delta - 1)$$

we get

$$d_1 \leq 2(\Delta - 1)$$

Secondly, for $n \geq 3$, the number $d_{2n - 2}$ represents here the maximal number of cycles of size $2n$ in $G$ containing a fixed edge and it can be bounded as

$$d_{2n - 2} \leq (\Delta - 1)^{2n - 2},$$

so that, for $0 < x < \Delta - 1$, we have

$$\phi_O(x) \leq 1 + 2(\Delta - 1)x + \sum_{n \geq 3} (\Delta - 1)^{2n - 2}x^{2n - 2} = 1 + 2(\Delta - 1)x + \frac{[(\Delta - 1)x]^4}{1 - [(\Delta - 1)x]^2}$$

Hence the entropy-compression criterion (2.1) in the actual case is fulfilled if

$$k \geq \min_{x > 0} \frac{1 + 2(\Delta - 1)x + [(\Delta - 1)x]^4}{x}$$

with $x > 0$. Setting $(\Delta - 1)x = y$, inequality (4.23) can be rewritten as

$$k \geq \left(\left[\min_{y > 0} g(y)\right]\right)(\Delta - 1)$$

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where
\[ g(y) = \frac{1 + 2y + \frac{y^4}{1-y^2}}{y} \]
The min\(_{y>0} \) \( g(y) \) is attained at \( x = \frac{1}{2}(\sqrt{5} - 1) \) and \( g\left(\frac{1}{2}(\sqrt{5} - 1)\right) = 4 \) and thus
\[ a'(G) < 4(\Delta - 1) \quad (4.24) \]
which is the bound given by Esperet and Parreau in [15].

**Remark.** As said in the introduction, the constant 4 here above is sensibly better than 9.163 obtained via the cluster expansion criterion in [27]. However this strong improvement is much more due to Lemma 4.1 that to entropy compression. To see this, first observe the cluster expansion criterion applied to the acyclic edge coloring in the ways illustrated in [27] is obtained by assigning to each edge of \( G \) a color chosen uniformly and independently between \( N \) available colors. In view of Lemma 4.1 we can adopt a more efficient strategy. Suppose that our random coloring of the edges of \( G \) is obtained by using \( N = 2(\Delta - 1)+k \) colors (with \( k \geq 1 \)) and by coloring sequentially the edges of \( G \), in such a way that the edge \( e \) colored at step \( t \) is colored by choosing uniformly and independently a number \( s \in [k] \) and then coloring \( e \) using the \( s^{th} \) available color among those avoiding monochromatic cherries and bichromatic 4-cycles (which, at each step, are at least \( k \)). With this kind of random coloring the possible bad events are only those in the family \( \mathcal{F}_C \), i.e. they are the bichromatic cycles of size greater or equal than 6. Due to our random coloring we are not in the Moser-Tardos setting but nothing prevent us to apply the general Theorem 1.2. Let us choose as graph \( G \) the graph with vertex set \( \mathcal{F}_C \) and edge set \( E \) such that \( \{e, e'\} \in E \) if and only if \( \text{obj}(e) \cap \text{obj}(e') \neq \emptyset \). It is not difficult to check that, given the random greedy coloring of the edges of \( E \) described above, for any \( e \in \mathcal{F}_C \) and any \( U \in \mathcal{F} \setminus \Gamma_G(e) \), we have
\[ P\left( e \bigcap_{e' \in U} e' \right) \leq \frac{1}{k^{2|\text{obj}(e)|-2}} \]
Therefore, using bounds (4.3)-(4.5), condition (1.6) can be written as
\[ k \geq \min_{0<x<(\Delta-1)^{-1}} \frac{\left[ 1 + \sum_{k \geq 3}(\Delta - 1)^{2k-2}x^{2k-2} \right]^{\frac{3}{2}}}{x} \quad (4.25) \]
where \( 3/2 = \sup_{e \in \mathcal{F}_C} |\text{obj}(e)|/(|\text{obj}(e)| - 2) \). Condition (4.25) is satisfied as soon as
\[ k \geq \left[ \min_{0<y<1} \frac{\left( 1 + \frac{y^4}{(1-y^2)} \right)^{\frac{3}{2}}}{y} \right] (\Delta - 1) = 2.181(\Delta - 1) \]
Therefore it is possible to color the edges of \( G \) using \( N \geq 2(\Delta - 1) + 2.181(\Delta - 1) = 4.181(\Delta - 1) \) in such a way that the resulting coloring is proper and with no bichromatic cycles of any size, This bound is very close to that obtained
by Esperet and Parreau. In conclusion, bearing in mind Lemma 4.11 as far as
acyclic edge coloring is considered, CELL is beaten by EC only because of the
exponent 3/2 in the numerator of the r.h.s. of (4.25).
We finally want to stress that Giotis et al. [19] have recently improved the
Esperet-Parreau bound proving that \( a'(G) \leq 3.79(\Delta - 1) \) combining Lemma
4.11 with a variant of the Moser-Tardos algorithm specifically designed for the
case of the acyclic edge coloring.

### 4.3 EC and CELL can tie the game. A non graph coloring
example: Independent sets

Let \( G \) be a graph with vertex set \( V \), edge set \( E \) and with maximum degree \( \Delta \).
We want to find the least integer \( k_\Delta \) such that, for any partition \( \{V_1, \ldots, V_n\} \)
of \( V \) with \( |V_i| \geq k \), there exists an independent subset \( I \) of \( V \) with exactly one
vertex in each \( V_i \). This set \( I \) is called an independent transversal of \( G \) w.r.t. the
partition \( \{V_1, \ldots, V_n\} \). It is long known that \( k_\Delta \) is linear in \( \Delta \) [1, 16]. The best
bound for \( k_\Delta \), i.e. \( k_\Delta \leq 2\Delta \) is due to Penny Haxell [23]. The Haxell’s bound is
tight since Szabó and Tardos [36] showed that \( k_\Delta > 2\Delta - 1 \).

Let us see in this section what EC and LLL are able to say about this problem.
Without loss of generality, we can assume that \( G \) is such that \( |V_i| = k \) for all
sets \( V_i \). The general case \( |V_i| \geq k \) follows by considering the graph induced by
\( G \) on a union of \( n \) subsets of cardinality \( k \), each of them a subset of one \( V_i \).

The EC setting in this case can be established as follows. We may assume
that for each \( i \in [n] \) the set \( V_i \) is ordered, namely, \( V_i = \{v_1^{(i)}, \ldots, v_{k}^{(i)}\} \). If now
we identify, for each \( j \in [k] \) and \( i \in [n] \), \( v_j^{(i)} \) with \( j \), we can take as set of
atoms \( S = \{V_1, V_2, \ldots, V_n\} \), and the set of “colors” can be identified with \([k] \);
a “coloring” \( c \) of \( S \) being just the selection of an element in each \( V_i \). Namely,
to choose the color \( s \in [k] \) for the atom \( V_i \) means to select the vertex \( v_s^{(i)} \in V_i \).
Given \( i, j \in [n] \), we say that the pair \( \{V_i, V_j\} \) is connected in \( G \) if there exists
\( v \in V_i \) and \( v' \in V_j \) such that \( \{v, v'\} \in E \). Then the family of objects \( \mathcal{O} \in \) the
present case is constituted by the connected pairs \( \{V_i, V_j\} \) in \( G \). Given
\( A = \{V_i, V_j\} \in \mathcal{O} \) let \( \epsilon \) be the event in \([k]^4 \)
\( \epsilon = \{c(V_i) = v, c(V_j) = v' \text{ and } \{v, v'\} \in E\} \)

Note that \( \epsilon \) is standard and elementary. Let now, for each \( A = \{V_i, V_j\} \in \mathcal{O} \),
\( \mathfrak{F}_A \) be the family of flaws
\( \mathfrak{F}_A = \{\epsilon : c \in [k]^4 : \{c(V_i), c(V_j)\} \in E\} \)

In other words, for each \( A \in \mathcal{O} \), \( \mathfrak{F}_A \) is constituted by the (elementary) events
formed by the colorings that select a vertex \( v \) in \( V_i \) and a vertex \( v' \) in \( V_j \) and
\( \{v, v'\} \) is an edge of \( G \). The family \( \mathfrak{F} = \cup_{A \in \mathcal{O}} \mathfrak{F}_A \), constituted by standard
elementary flaws, is such that a coloring of \( S \) avoiding all the events in the
family \( \mathfrak{F} \) selects an independent set of \( G \).
In the present case, since any \( \epsilon \in \mathfrak{F}_A \) is elementary, we have that \( ||\epsilon|| = |\text{obj}(\epsilon)| = 2 \) for all \( \epsilon \in \mathfrak{F} \). This means that \( E_\mathcal{O} = \{2\} \). Finally we have to estimate \( d_2 \), i.e.,
the maximal number of events containing a fixed \( V_i \in \mathcal{O} \). Since any \( V_i \) has \( k \) vertices and each of these vertices is adjacent to at most \( \Delta \) other vertices of \( G \), we get that \( d_2 \leq k \Delta \) and thus

\[
\phi_{E_\mathcal{O}}(x) = 1 + k \Delta x^2
\]

Therefore condition (2.1) is written as follows

\[
k \geq 2(k \Delta)^{\frac{1}{2}}
\]

which is to say \( k \Delta \leq 4 \Delta \). This bound is equal to that obtained with the CELL (see e.g. [7]). By the discussion made in Section 4.1 this is not a surprise. Indeed, in the present case EC criterion and CELL criterion yield the same bound because any \( e \in \mathcal{F} \) is elementary.

Finally it is worth to mention that very recently Graf and Haxell [20] have designed an efficient algorithm (not related to EC or Moser-Tardos algorithmic LLL!) able to find an independent transversal in a graph \( G = (V, E) \) with maximum degree \( \Delta \) for any partition \( \{V_1, \ldots, V_n\} \) of \( V \) such that \( |V_i| \geq 2\Delta + 1 \) for \( 1 = 1, \ldots, n \).

Declarations of interest: none

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