Homotopy and q-homotopy skein modules of 3-manifolds: an example in Algebra Situs.

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Dedicated to my teacher Joan Birman on her 70’th birthday.

Abstract

Algebra Situs is a branch of mathematics which has its roots in Jones’ construction of his polynomial invariant of links and Drinfeld’s work on quantum groups. It encompasses the theory of quantum invariants of knots and 3-manifolds, algebraic topology based on knots, operads, planar algebras, q-deformations, quantum groups, and overlaps with algebraic geometry, non-commutative geometry and statistical mechanics.

Algebraic topology based on knots may be characterized as a study of properties of manifolds by considering links (submanifolds) in a manifold and their algebraic structure. The main objects of the discipline are skein modules, which are quotients of free modules over ambient isotopy classes of links in a manifold by properly chosen local (skein) relations.

We concentrate, in this lecture, on one relatively simple example of a skein module of 3-manifolds – the q-homotopy skein module. This skein module already has many ingredients of the theory: algebra structure, associated Lie algebra, quantization, state models...

1 Introduction

Algebra Situs\textsuperscript{2} is a branch of mathematics which has its roots in Jones’ construction of his polynomial invariant of links, Jones polynomial, and Drinfeld’s work on quantum groups. It encompasses theory of quantum invariants of knots and 3-manifolds, algebraic topology based on knots, operads, \textsuperscript{1}Supported by NSF-DMS-98089555.

\textsuperscript{2}This part of the paper is based on the talk Algebraic topology based on knots: a case study in the history of ideas given at a Conference in Low-Dimensional Topology in Honor of Joan Birman’s 70th Birthday; Columbia University, March 14–15, 1998.
$q$-deformations, quantum groups, and overlaps with algebraic geometry, non-commutative geometry and statistical mechanics.

Algebraic topology based on knots may be characterized as a study of the properties of manifolds by considering the space of links (submanifolds) in a manifold and its algebraic structure. The main objects of the discipline are skein modules, which are quotients of free modules over ambient isotopy classes of links in a manifold by properly chosen local (skein) relations. Of course, this is not a complete definition of the field, which has its purely algebraic component (skein algebras of groups), higher manifold generalization and rich internal structure, but at least it gives the idea of our subject.

In searching for a starting point of the theory one should consider Listing book (1847), Dedekind and Weber’s paper (1882), and Poincaré’s paper “Analysis Situs” (1895). In knot theory, skein modules (building blocks of algebraic topology based on knots) have their origin in the observation by Alexander (1928) that his polynomials of three links $L_+, L_-$ and $L_0$ in $S^3$ (see Fig.2.1) are linearly related. This line of research was continued by Conway (linear skein, 1969). In graph theory the idea of forming a ring of graphs and dividing it by an ideal generated by local relations was developed by Tutte in his 1946 PhD thesis. The theory of Hecke algebras, as introduced by Iwahori (1964), is closely connected to the theory of skein modules. Another connection can be found in the Temperley-Lieb algebra (1971). The main motivation for skein modules was the discovery/construction of the Jones polynomial (1984). Skein algebras of groups use rich ideas of Poincaré (1884), Vogt (1889), Fricke and Klein (1897) and the school of Magnus (e.g. “Rings of Fricke characters”, 1980).

Joan Birman introduced me to the world of knots and braids, to the work of her advisor W.Magnus, and her grand-advisor M.Dehn. Her work is continued by her students and grand-students (one of the best recent results related to algebraic topology based on knots was obtained by A.Sikora [Si-2]).

We concentrate, here, on one relatively simple example of a skein module of 3-manifolds – the $q$-homotopy skein module. This skein module already has many ingredients of the rich theory: algebra structure, associated Lie algebra, Hopf algebra, quantization, state models, relation to graph theory...

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3See Fig.1.1 for genealogical table.
Jean Le Rond d’Alembert (1717-1783)

Joseph Louis Lagrange (1736-1813)  Pierre-Simon Laplace (1749-1827)

Jean-Baptiste Joseph Fourier (1768-1830)  Simeon Denis Poisson (1781-1840)

Gustav Peter Lejeune Dirichlet (1805-1859)

Julius Plucker (1801-1868)  Rudolf Otto Sigismund Lipschitz (1832-1903)

Felix Klein (1849-1925)

Ferdinand Lindemann (1852-1939)

David Hilbert (1862-1943)

Max Dehn (1878-1952)

Wilhelm Magnus (1907-1990)

Joan Sylvia Lyttle Birman

Students of J.Birman:
Tara Brendle, Nathan Broaddus, Abhijit Champanerkar, Zung He Chen, Richard Fein, Liz Finkelstein, Tat San Fun, Matt Greenwood, Hessam Hamidi-Tehrani, Efstratia Kalfagianni, Marcello Kupferwasser, John D. McCarthy, Ka Yi Ng, Radu Popescu, Jerry Powell, Józef H. Przytycki, Ted Stanford, Rolland J. Trapp, Nancy Wrinkle, Peijun Xu, Matt Zinno,..., Keiko Kawamura.

Figure 1.1. Teachers and students of Joan Birman.
2 Definitions

One of the earliest skein modules considered\(^4\) was the homotopy skein module, where self-crossings were ignored and mixed crossings were resolved using the Alexander-Conway skein relation \([H-P-1]\). In the fall of 1991 the \(q\)-analogue version of the homotopy skein module was first considered \([Pr-3]\) and showed to distinguish some links with the same Jones-Conway (Hom-flypt) polynomial. In the case of \(M = F \times I\) we gave in \([H-P-1]\) the precise description of the homotopy skein module and showed that as an algebra it is isomorphic to the universal enveloping algebra of the Goldman-Wolpert Lie algebra of curves on the surface \(F\), \([Gol]\). For the \(q\)-analogue version of the homotopy skein module we prove that for \(M = F \times I\) and \(\pi_1(F)\) abelian, the skein module is free. We define a \(q\)-version of the Goldman-Wolpert Lie algebra. We also give the formula for an element of the \(q\) homotopy skein module represented by a link in \(S^3\) in terms of linking numbers of the components of the link. We show that the module (algebra), for \(M = S^3\), is equivalent to the dichromatic polynomial of the graph associated to the link. We generalize these to surfaces \(F\) with an abelian fundamental group. We show that for \(\pi_1(F)\) nonabelian the \(q\)-homotopy skein module has torsion.

The \(q\)-homotopy skein module \(\mathcal{HS}^q(M)\) is a \(\mathbb{Z}[z, q^{\pm 1}]\)-module associated to an oriented 3-manifold, \(M\), according to the general scheme described in \([Pr-2, H-P-2]\). It generalizes the homotopy skein module \(\mathcal{HS}(M)\) considered in \([H-P-1]\) and it can be thought of as a \(q\)-analogue of \(\mathcal{HS}(M)\). We will follow \([H-P-1]\) closely in our description. Let \(M\) be an oriented 3-manifold. The \(q\)-homotopy skein module \(\mathcal{HS}^q(M)\) is defined as follows.

**Definition 2.1** Let \(\mathcal{L}^h\) be the set of all oriented links in \(M\) modulo link homotopy (i.e. we ignore self-crossings), including the empty link, \(T_0\). Let \(R = \mathbb{Z}[q^{\pm 1}, z]\) and \(RL^h\) be the free \(R\) module generated by \(\mathcal{L}^h\). Let \(S\) be a submodule of \(RL^h\) generated by the homotopy skein expressions \(q^{-1}L_+ - qL_- - zL_0\), where \(L_+\), \(L_-\) and \(L_0\) are three oriented links in \(M\), which are identical except inside a small ball where they appear instead as shown in Figure 2.1.
Additionally we assume that the two strings of \(L_+\) (or \(L_-\)) involved in the crossing in Fig.2.1 belong to different components of \(L_+\) (or \(L_-\)), that is, we deal with a mixed crossing. We define the \(q\)-homotopy skein module to be the quotient \(\mathcal{HS}_q^\mathcal{L}(M) = R\mathcal{L}^h / S\).

If we do not allow the empty link, we get the reduced \(q\)-homotopy skein module \(\mathcal{H}\tilde{S}_q^\mathcal{L}(M)\) (and we have \(\mathcal{HS}_q^\mathcal{L}(M) = \mathcal{H}\tilde{S}_q^\mathcal{L}(M) \oplus R\)).

Remark 2.2 It may also be convenient to define \(\mathcal{HS}_q^\mathcal{L}(M)\), equivalently, starting from the set of oriented links, \(\mathcal{L}\), in \(M\) and quotient \(R\mathcal{L}\) by the submodule generated by skein relations \(L_+ - L_-\) in the case of a self-crossing and \(q^{-1}L_+ - qL_- - zL_0\) in the case of a mixed crossing.

The \(q\)-homotopy skein module shares with other skein modules several useful elementary properties, like the Universal Coefficients property, several functorial properties, etc. We will discuss these in Section 5. In the next two sections we analyze the \(q\)-homotopy skein module of classical links (that is \(M = S^3\)) and show that it can be interpreted as a polynomial, which we call the homotopy polynomial. The homotopy polynomial corresponds to (a variant of) the dichromatic polynomial of an associated graph, and depends only on the linking numbers between components of the link.

3 \(\mathcal{HS}_q^3(S^3)\) and a homotopy polynomial of classical links

It is relatively easy to show that \(\mathcal{HS}_q^3(S^3)\) is freely generated by unlinks \(T_0, T_1, T_2, T_3, \ldots\), where \(T_i\) denotes the unlink of \(i\) components, and that for a given \(n\)-component link \(L \subset S^3\), its presentation as \(L = w_0(q)T_n + w_1(q)zT_{n-1} + \ldots + w_{n-1}(q)z^{n-1}T_1\) depends only on the linking numbers between components of \(L\) (compare [H-P-1]; Theorem 1.2). In Sections 6
we give detailed proof of much more general result. However, the case of $M = S^3$ is of special interest as $\mathcal{HS}^q(S^3)$ is equivalent to a classical object—the dichromatic polynomial, properties of which are very well understood. More precisely, we can put $T_i = t^i$ and interpret $L \in \mathcal{HS}^q(S^3)$ as $L = HP_L(q, t, z)T_0$, where $HP_L(q, t, z) \in Z[q^{±1}, t, z]$; in the previous notation $HP_L(q, t, z) = w_0(q)t^n + w_1(q)zt^{n-1} + ... + w_{n-1}(q)z^{n-1}t$. We call the polynomial $HP_L(q, t, z)$ the homotopy polynomial of a link $L$. It can be interpreted as a dichromatic polynomial of the weighted graph associated to $L$, and the set of linking numbers between components of $L$ can be recovered from the coefficient $w_1(q)$.

To formulate the main result of this section we need some preliminary definitions. For a link $L = K_1 \cup ... \cup K_n$ in $S^3$ we denote by $[l_{i,j}]$ its linking matrix where $l_{i,j} = \text{lk}(K_i, K_j)$. Let $E$ be the set of all pairs $(i, j)$, $i \neq j$, $1 \leq i, j \leq n$. We consider the notion of a cycle in $E$. The meaning of a cycle can be best explained by considering the complete graph $F_n$ of $n$ vertices, $1, 2, 3, ..., n$, and edges $(i, j)$ ($i \neq j$). Cycles (i.e. simple closed edge-paths) in $F_n$ determine cycles in $E$. For $S \subset E$ where $S$ does not contain a cycle, let $A_S$ denote the subset of $E - S$ such that $(i, j) \in A_S$ if and only if either $S \cup \{(i, j)\}$ has no cycle or, otherwise, if $C$ is the unique cycle in $S \cup \{(i, j)\}$ containing $(i, j)$ then $(i, j)$ is not the first element of $C$ with respect to lexicographical order of pairs $(i, j)$. $\ |S| \ $denotes the cardinality of $S$.

**Theorem 3.1** $T_0, T_1, T_2, T_3, ...$ form a free basis of $\mathcal{HS}^q(S^3)$ and for $L = K_1 \cup ... \cup K_n$ one has the formula:

$$L = \sum_S q^{\Sigma_{(i,j) \in A_S} 2l_{i,j} |S|} T_{n-|S|} \prod_{(i,j) \in S} \frac{q^{2l_{i,j}} - 1}{q - q^{-1}}$$

where the sum is taken over all subsets $S$ of $E$ which do not contain a cycle.

Equivalently we can write:

$$HP_L(q, t, z) = \sum_S q^{\Sigma_{(i,j) \in A_S} 2l_{i,j} |S|} T_{n-|S|} \prod_{(i,j) \in S} \frac{q^{2l_{i,j}} - 1}{q - q^{-1}}$$

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$^5$In [H-P-1] we didn’t write the closed formulas for $w_k(1)$. I noticed Formula 3.4 shortly after [H-P-1] was published and generalized to $w_k(q)$ [Pr-3]. Formula for $w_k(1)$ was also independently discovered by A.Sikora in his Master Degree Thesis [Si-1].

$^6$As correspond to the set of externally inactive elements in the sense of Tutte; compare [Int-4, Tra].
where as before the sum is taken over all subsets $S$ of $E$ which do not contain a cycle.

**Corollary 3.3** If $q = 1$, then the formula in 3.1 reduces to:

$$L = \sum_{S} z^{|S|} T_{n-|S|} \prod_{(i,j) \in S} l_{i,j}$$

In particular $w_k(1)$ is equal to $\sum_{S_k} \prod_{(i,j) \in S_k} l_{i,j}$, where $S_k$ is the set of all $k$-element subsets of $E$ which do not contain a cycle. Compare [H-P-1], formulas of Part 1.

The proof of Theorem 3.1 is not very difficult, but we can omit it totally by showing that Formula 3.2 can be interpreted as a formula for dichromatic polynomial of the signed (or weighted) graph associated to $L$.

We will consider three, closely related, graphs: $G(L), G_1(L)$ and $G_2(L)$ and their dichromatic polynomials. In all of them vertices correspond to components of $L$.

Consider the following signed graph $G(L)$: vertices of $G(L)$ correspond to components of $L$ and vertices $v_i, v_j (i \neq j)$ are joined by $|lk(L_i, L_j)|$ edges of sign equal to $\text{sign}(lk(L_i, L_j))$.

Let $R(G)$ be a dichromatic polynomial of a given signed graph $G$ defined recursively by the rules:

(i) $R(\bullet \cdots \bullet) = t^n$,

(ii) if $e_{\pm}$ is not a loop then:

$$R(G) = q^2 R(G - e_+) + qz R(G/e_+)$$

$$R(G) = q^{-2} R(G - e_-) - q^{-1} z R(G/e_-),$$

(iii) If $e$ is a loop then $R(G) = R(G - e)$.

Remark.
Notice that if $e_+$ and $e_-$ are two edges joining the same endpoints, then $R(G - e_+ - e_-) = R(G)$ because, either $e_+$ and $e_-$ are loops and the equality follows from (iii), or we have:

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7In Section 6 we prove a generalization of the first part of Theorem 3.1.
\[ R(G) = q^2 R(G - e_+) + qz R(G/e_+) = q^2(q^{-2} R(G - e_+ - e_-) - q^{-1} z R((G - e_+)/e_-)) + qz R(G/e_+) = R(G - e_+ - e_-) \] as \( e_- \) is a loop in \( G/e_+ \) so \( R(G/e_+) = R((G - e_+)/e_-) \).

It is a standard fact that \( R(G) \) is well defined, in particular one has the state model formula (compare [F-K, Tra, P-P]). One can immediately see a validity of Formula 3.5 when one notices that (iii) can be rewritten as:

\[
R(G) = \left\{ \begin{array}{l}
q^2 R(G - e_+) + qz \frac{q^{-1} - q}{z} R(G/e_+) \\
q^{-2} R(G - e_-) - q^{-1} z^{\frac{q^{-1} - q}{z}} R(G/e_-)
\end{array} \right.
\]

3.5

\[
R(G) = \sum_{S \in E(G)} t^{p_0(G;S)}(\frac{q^{-1} - q}{z})^{p_1(G;S)} (-1)^{|S| - z}^{\frac{q^2}{|E - S|} + |E - S_-| + |S| - |S_-|} q^{|E|} + p_0(G),
\]

where \((G;S)\) is the subgraph of \( G \) which includes all vertices of \( G \) but only edges of \( S \). \( p_0(G) \) is the number of components of \( G \), and \( p_1(G) \) is the chromatic number of \( G \) (i.e. the first Betti number). \( V = V(G) \) denotes the set of vertices of \( G \) and \(|V|\) the cardinality of \( V \). \( E = E(G) \) denotes the set of edges of \( G \) and \(|E|\) (resp. \(|E|_+\) or \(|E|_-\)) denotes the cardinality of \( E \) (resp. the number of positive or negative edges in \( E \)). In particular \( p_1(G) = |E| - |V| + p_0(G) \).

Remark 3.6  
(a) If \( Q(G; t, z') \) is Traldi’s version of the dichromatic polynomial ([Tra]), then

\[
R(G) = q^{2|E|_+ - |E|_-} Q(G; t, z'),
\]

for \( z' = \frac{2^{-1} - q}{z} \) and Traldi’s weight \( w(e) \) of \( e \in E \) is defined by \( w(e_+) = q^{-1} z \) and \( w(e_-) = -qz \).

(b) If \( < G >_{\mu, A, B} \) is the Kauffman bracket of \( G \) then (compare [P-P], Lemma 5.2):

\[
< G >_{\mu, A, B} = \mu^{-1} (-A^3)^{|E|_+ - |E|_-} R(G)_{z = -i}
\]

for \( \mu = t, A = (iq)^{-\frac{1}{2}}, B = (iq)^{\frac{1}{2}} \) and if \( R(G)_{z = -i} = \sum b_j(q) t^j \) then

\[
R(G) = \sum b_j(q) t^j (iz)^{|V| - j}.
\]

8
Theorem 3.7 Let \( \hat{w} : \mathcal{L} \rightarrow \mathcal{G} \) be a map from the set of links in \( S^3 \) to the set of signed graphs, given by \( \hat{w}(L) = G(L) \). Then \( \hat{w} \) yields an algebra isomorphism \( w : \mathcal{HS}^q(S^3) \rightarrow \mathbb{Z}[q^{\pm 1}, z, t] \), where \( w(T_i) = t^i \) and \( w(L) = R(G(L)) \). The product in \( \mathcal{HS}^q(S^3) \) is given by the disjoint sum of links. Furthermore \( L = R(G(L))T_0 \) in \( \mathcal{HS}^q(S^3) \).

Proof: Because \( T_i \)'s clearly generate \( \mathcal{HS}^q(S^3) \), it suffices to compare defining properties (i)-(iii) of \( R(G) \) with the definition of \( \mathcal{HS}^q(S^3) \). The condition (ii) corresponds to the homotopy skein relation. \( \square \)

Notice that we have proven the first part of Theorem 3.1. Formula 3.2 can be deducted from 3.5, but we will not present it here. Instead we will consider a graph \( G_1(L) \) yielded by \( L \) and from its dichromatic polynomial we will derive 3.2.

We can construct a graph \( G_1 \) out of any signed graph \( G \).

Definition 3.8 Let \( G \) be a signed graph, then \( G_1 \) is a weighted graph (i.e. a graph with a function \( f : E(G_1) \rightarrow \mathbb{Z} \)), with no loops and no multi-edges, obtained from \( G \) by deleting its loops and replacing a multi-edge by a single edge with the weight equal to the sum of signs of edged in the multi-edge. Deleting, \( G_1 - e \), and contracting, \( G_1/e \), are operations on \( G_1 \) with the usual meaning with the convention that whenever a multi-edge is created in \( G_1/e \) then it is replaced by a single edge with a weight being the sum of weights of components of the multi-edge.

Now we define the dichromatic polynomial \( R_1(G_1) \) by the rules:

1. \( R_1(\bullet \cdots \bullet) = t^n \),

2. \( R_1(G_1) = q^{2f(e)}R_1(G_1 - e) + \frac{q^{2f(e)} - 1}{q - q^{-1}}zR_1(G_1/e) \), where \( f(e) \) is the weight of \( e \). Notice that \( e \) is never a loop (as we delete edges of weight zero).

Observe also that for \( q = 1 \), (ii) reduces to:

\( R_1(G_1)_{q=1} = R_1(G_1 - e)_{q=1} + f(e)zR_1(G_1/e)_{q=1} \).

It is easy to check that if \( e \) is an edge of \( G_1 \) with weight \( f(e) \) and \( G' \) is obtained from \( G_1 \) by changing the weight of \( e \) to \( f(e) - 1 \) then \( R_1(G_1) = q^2R_1(G') + qzR_1(G_1/e) \). From these we get:

Lemma 3.9 \( R(G) = R_1(G_1) \).
The following theorem of Tutte [Tut-4] (compare [Tra] or [Za]) yields Theorem 3.1.

**Theorem 3.10** Let $G_1$ be a connected weighted graph. Given an arbitrary linear ordering of edges in $E(G_1)$ and a spanning tree $T$ of $G_1$, an edge $e$ in $T$ is called internally active with respect to $T$ if it precedes all other edges of $G$ whose end vertices lie in different components of $T - e$. An edge $e$ not in $T$ is called externally active with respect to $T$ if it precedes all other edges of $T$ that lie in the unique cycle determined by $T$ and $e$. Then:

$$R_1(G_1) = t \sum_{T} \prod_{e \in EI} q^{2f(e)} \prod_{e \in II} (\frac{q^{2f(e)} - 1}{q - q^{-1}}) \prod_{e \in IA} (tq^{2f(e)} + z \frac{q^{2f(e)} - 1}{q - q^{-1}})$$

where the sum is taken over the set of all spanning trees $T$ of $G_1$ and the three products are taken, respectively, over the set of edges which are externally inactive (EI), internally inactive (II) and internally active (IA) with respect to $T$.

Let $|S|$ denote the number of edges of $S \subset E(G_1)$, and $n$ number of vertices of $G_1$. If we multiply out the last product of Formula 3.11 (having in mind the defining relations (i)-(iii) for $R_1(G_1)$) we get:

$$R_1(G_1) = \sum_{S} t^{n-|S|} z^{|S|} \prod_{e \in A_S} q^{2f(e)} \prod_{e \in S} (q^{2f(e)} - 1) (q - q^{-1})$$

where the sum is taken over all forests $S$ of $G_1$ (that is, subgraphs without cycles), $A_S$ denotes the set of externally inactive edges with respect to $S$ (i.e. $e$ in $E - S$ is inactive if $S \cup e$ is a forest or $S \cup e$ contains a (unique) cycle and $e$ does not precedes all other edges of $C$). Formula 3.2 now follows from 3.12 if edges of $G_1$ are lexicographically ordered. The proof of Theorem 3.1 is complete.

We will end this section by introducing another related graph, $G_2$, and showing that we can interpret Jones-Conway (Homflypt) polynomial of links in $S^3$ as a dichromatic polynomial of a related graph.

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8We can visualize this by applying recursive relations to the edges of the trees.
**Definition 3.13** Let $G_2$ be a signed graph obtained from $G$ by doubling each (signed) edge of $G$; that is, $\varepsilon$ $\varepsilon$ $\varepsilon$. Let $R_2(G_2)$ be a polynomial of $G_2$ given by the rules:

(i) $R_2(\bullet \bullet \ldots \bullet) = t^n$,

(ii) if $e$ is not a loop and $\epsilon(e)$ denote a sign of the edge $e$, then:

$$R_2(G) = q^{2\epsilon(e)}R_2(G_2(\epsilon(e))) + \epsilon(e)q^{\epsilon(e)}zR_2(G_2/e)$$

where $G_2(\epsilon(e))$ denotes the graph obtained from $G_2$ by changing the sign of the edge $e$.

(iii) If $e$ is a loop then $R_2(G_2) = R_2(G_2 - e)$.

We do not claim that $R_2(\ )$ is defined for every signed graph, but it is defined for $G_2$ constructed from $G$, as above. In fact we have:

**Lemma 3.14** $R(G) = R_2(G_2)$.

For any plane signed graph $G$, we can associate the link diagram $D(G)$ (matched diagram) as shown in Fig. 3.1 below:

```
+ + \\
\| /  \\
|  |
- - \\
\| /  \\
|  |
```

Figure 3.1

Then the dichromatic polynomial of $G$ and the Jones-Conway (Homflypt) polynomial of $D(G)$, $P_{D(G)}(v, z)$, are related as follows:

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9We can also say that $D(G)$ is obtained from $G_2$ by the standard (from P.G.Tait times) construction – median graph diagram, with the properly chosen orientation and crossing resolution.
Lemma 3.15  \( P_{D(G)}(v, z) = t^{-1}R(G) \) for \( v = q \) and \( t = \frac{q^3 - 1}{z} \). Thus we can recover \( P_{D(G)}(v, z) \) out of \( R(G) \). Conversely if \( P_{D(G)}(v, z) = \sum_{i=0}^{m} a_{-m+2i}(v)z^{-m+2i} \), where \( m = \text{com}(D(G)) - 1 = |V(G)| - 1 \), then \( R(G) = t \sum_{i=0}^{m} \frac{a_{-m+2i}(q)}{(q^{-1}-q)^{m-i}}z^{m-i} \). Here \( \text{com}(D(G)) \) denotes the number of components of the link diagram \( D(G) \).

4 Examples

As noted in Section 3, the \( q \)-homotopy skein module, \( \mathcal{HS}^q(S^3) \), can be identified with the ring \( \mathbb{Z}[q^{\pm 1}, t, z] \). Thus the class of a link \( L \) in \( \mathcal{HS}^q(S^3) \) defines the homotopy polynomial, \( HP_L(q, t, z) \in \mathbb{Z}[q^{\pm 1}, t, z] \), which depends exclusively on the linking matrix \( [\ell_{i,j}] \) of \( L \). It is worth comparing \( HP_L(q, t, z) \) with the Jones-Conway (Homflypt) polynomial \( P_L(v, z) \). It is well known (compare [L-M, Pr-1, Si-1]) that \( HP_L(1, t, z) \) (i.e. \( q = 1 \)) is determined by \( P_L(v, z) \). This is not true, however, for the more general polynomial \( HP_L(q, t, z) \). In the following example, we use links described by J.Birman [Bir].

Example 4.1  The 3-component links from Fig. 4.1 share the same Jones-Conway polynomial [Bir] but they have different homotopy polynomials. Namely, \( HP_{L_1}(q, t, z) = q^6 t^3 + (q^{-1} + q + q^3 + q^5 - q^7)zt^2 - (1 + q^2 + q^4 + q^6)z^2t \) and \( HP_{L_2}(q, t, z) = q^6 t^3 + (q + 2q^3 + 2q^5 - q^7 - q^9)zt^2 - (q^4 + 2q^6 + q^8)z^2t \).

Another interesting example is also due to Joan Birman, this time it deals with two different closed three braids (one being the mirror image of the other).
**Example 4.2** The 3-component links from Fig. 4.2 share the same Jones-Conway polynomial \([\text{Bin}]\) but they have different homotopy polynomials. Namely, the coefficient, \(w_1(q)\), of \(zt^2\) in \(HP_L(q, t, z)\) is equal to \(-q^3 - q + 2q^{-1}\), and in \(HP_{\bar{L}}(q, t, z)\) is equal to \(q^{-3} + q^{-1} - 2q\)\(^{10}\).

\[
\begin{array}{c}
\includegraphics[width=0.4\textwidth]{example4.2.png}
\end{array}
\]

Fig. 4.2.

The polynomial \(HP_L(q, t, z)\) allows us to recover all linking numbers of \(L\) (as a set with multiplicities); see Theorem 4.3, but is not sufficient to recover the linking matrix. The simplest example is shown in Fig. 4.3.

\[
\begin{array}{c}
\includegraphics[width=0.4\textwidth]{example4.3.png}
\end{array}
\]

Fig. 4.3.

Here \(HP_{L_1}(q, t, z) = HP_{L_2}(q, t, z) = t(q^2t + qz)^3\).

This example is, in a sense, trivial because the graphs \(G(L_1)\) and \(G(L_2)\) are 2-isomorphic. The first nontrivial example is based on the example of M.C.Gray obtained around 1933 (see \([\text{Tut-2}]\)). The graphs \(G(L_1)\) and \(G(L_2)\) are 2-isomorphic.

\(^{10}\)Generally for a link \(L\) and its mirror image \(\bar{L}\) one has: \(HP_{\bar{L}}(q, t, z) = HP_L(-q^{-1}, t, z)\).
of Fig. 4.4 are that of Gray. It is an open question how much of the graph can be recovered from its dichromatic polynomial and whether there are some “elementary moves” on graphs which link different graphs with the same dichromatic polynomial. Most of examples known today are based on the “rotors” idea of Tutte ([BSST Tut-2 Tut-3 APR T Jo Pr-5]).

We will now show that $HP_L(q, t, z)$ allows us to recover the set of linking numbers $\{\ell_{i,j}\}$.

**Theorem 4.3** Let $L$ be an oriented link on $n$ components in $S^3$ and

$$HP_L(q, t, z) = \sum_{i=0}^{n-1} w_i(q) z^i t^{n-i}.$$  

Then the set $\ell_{i,j}$ and the multiplicity with which each given non-zero number appears among linking numbers, can be recovered from $w_1(q)$. That is, $w_1(q)$ determines the polynomial $\prod_{(i,j)} (x - \ell_{i,j})$ where the product is taken over pairs $(i, j)$ with non-zero $\ell_{i,j}$.

**Proof:** The formula 3.2 (where $t^i = T_i$), or straightforward computation using homotopy skein relations, gives us:
4.4 \( w_1(q) = \sum_{(i,j)} q^{\sum_{(k,l) \neq (i,j)} \ell_{k,l}} q^{2\ell_{i,j} - 1} = -q^{2lk(L)} \sum_{(i,j)} q^{2\ell_{i,j} - 1} \)

where \( lk(L) = \sum_{(i,j)} \ell_{i,j} \). It is now an easy exercise to see that \( \sum_{(i,j)} q^{2\ell_{i,j} - 1} \) determines the (unordered) sequence \( \{ \ell_{i,j} \} \). Notice that \( w_1(1) = lk(L) \) and \( w_0(q) = q^{2lk(L)} \). \( \square \)

Let us take for a moment a slightly more general point of view. Assume that \( a_1, a_2, ..., a_k \) is a sequence of integers. Form from the sequence the “Young diagram” (positive numbers “build” the part of the diagram in the first quadrant and negative in the third), so that rows of the diagram corresponds to numbers \( \{a_i\} \), compare Fig.4.5.

![The Young diagram of the sequence (1,2,2,3,-1,-1,-2,-3,-3)](image)

**Fig. 4.5.**

**Lemma 4.5** The columns of the “Young diagram” of the sequence \( a_1, a_2, ..., a_k \) correspond to the coefficients of \( \Sigma(q) = \sum_{i=1}^{k} \frac{q^{2a_i - 1}}{q - q^{-1}} \), that is, if \( \Sigma(q) = \sum_{j \neq 0} b_j q^{2j - \text{sgn}(j)} \) then \( b_j \) is the number of elements of the \( j \)’th column of the “Young diagram” of the sequence \( \{a_i\} \), with appropriate signs. We can say shortly that sequences \( \{a_i\} \) and \( \{b_j\} \) are dual one to another (i.e. they represent dual “Young diagrams”).

**Proof:** It follows immediately from the identities:

\[
\frac{q^{2a} - 1}{q - q^{-1}} = q^{2a-1} + q^{2a-3} + \cdots + q \quad \text{for} \quad a > 0,
\]

and

\[
\frac{q^{2a} - 1}{q - q^{-1}} = -q^{2a+1} - q^{2a+3} - \cdots - q^{-1} \quad \text{for} \quad a < 0,
\]
which describe a positive (resp. negative) row of length \(a\) of the “Young diagram”. □

**Corollary 4.6** The polynomial \(w_1(q)\) has the form:

\[
 w_1(q) = -q^{2k(L)} \sum_{j \neq 0} b_j q^{2j - \text{sgn}(j)}
\]

where \(b_j\)'s are heights of the columns of the “Young diagram” corresponding to the sequence \(\{-\ell_{i,j}\}\).

**Corollary 4.7** The polynomial \(w_1(q)\) is unimodal, that is, if \(w_L(q) = \sum c_i q^{2i+1}\) then there is \(j\) such that \(\ldots \leq |c_{j-1}| \leq |c_j| \geq |c_{j+1}| \geq \ldots\).

**Example 4.8** For the link \(L_2\) of Fig. 4.1 one has linking numbers \((2, 3, -2)\). Corollary 4.6 allows us to find immediately \(w_1(q)\) by building the Young diagram for numbers \((-2, -3, 2)\) as in Fig. 4.6.

![The Young diagram of the sequence (2,-2,-3)](image)

Thus, according to Corollary 4.6 one gets:

\[
 w_1(q) = -q^6(-2q^{-1} - 2q^{-3} - q^{-5} + q + q^3) = q + 2q^3 + 2q^5 - q^7 - q^9
\]

as in Example 4.1.

**Remark 4.9** It follows from Corollary 4.6 that non-zero linking numbers, \(\ell_{i,j}\) can be recovered from \(w_1(q)\) and, vice versa, \(w_1(q)\) can be recovered from the non-zero linking numbers. Furthermore, the coefficient of \(z\) in \(HP_L(q,t,z)\) is equal to \(w_1(q)t^{n-1}\), so from this the number of components of \(L\) can be recovered (it is \(n\)), as well as the number of \(\ell_{i,j}\) which are equal to 0.
5 Elementary properties of homotopy skein modules

It is convenient to consider homotopy and q-homotopy skein modules, as special cases of homotopy skein modules with general coefficients.

Definition 5.1 We define $\mathcal{HS}(M; R, q, z) = R\mathcal{L}^h / S$, where $S$ is the submodule generated by the expressions $q^{-1}L_+ - qL_- - zL_0$ arising from mixed crossings, and $R$ is a commutative ring with identity, $q$ an invertible element and $z$ any element of $R$.

One has of course $\mathcal{HS}(M; \mathbb{Z}[z], 1, z) = \mathcal{HS}(M)$ and $\mathcal{HS}(M; \mathbb{Z}[q^{\pm 1}, z], q, z) = \mathcal{HS}^q(M)$.

We list below a few useful elementary properties of homotopy skein modules. Proofs of these properties are analogous to that of other skein modules, [P-S, Pr-7, Pr-8] and we omit them.

Theorem 5.2

(1) An orientation preserving embedding of 3-manifolds $i : M \to N$ yields a homomorphism of skein modules $i_* : \mathcal{HS}(M; R, q, z) \to \mathcal{HS}(N; R, q, z)$. The above correspondence leads to a functor from the category of 3-manifolds and orientation preserving embeddings (up to ambient isotopy) to the category of $R$-modules (with a specified element $z \in R$, and an invertible element $q \in R$).

(2) (Universal Coefficient Property)

Let $r : R \to R'$ be a homomorphism of rings (commutative with 1). We can think of $R'$ as an $R$ module. Then the identity map on $\mathcal{L}$ induces the isomorphism of $R'$ (and $R$) modules:

$$\bar{r} : \mathcal{HS}(M; R, q, z) \otimes_R R' \to \mathcal{HS}(M; R', r(q), r(z))$$

In particular $\mathcal{HS}(M; R, q, z) = \mathcal{HS}^q \otimes_{\mathbb{Z}[q^{\pm 1}, z]} R$.

(3) Let $M = F \times I$ where $F$ is an oriented surface. Then $\mathcal{HS}(M; R, q, z)$ is an algebra, where $L_1 \cdot L_2$ is obtained by placing $L_1$ above $L_2$ with respect to the product structure. The empty link $T_0$ is the neutral element of the multiplication. Every embedding $i : F' \to F$ yields an algebra homomorphism $i_* : \mathcal{HS}(F' \times I; R, q, z) \to \mathcal{HS}(F \times I; R, q, z)$. 

17
6 The case of $M = F \times I$

Let $\mathcal{L}^h$ denote the set of homotopy links in $M$, that is, $\mathcal{L}^h = \mathcal{L}/(L_+ - L_-)$ where relations are yielded by self-crossings. Let $\hat{\pi}$ denote the set of conjugacy classes in $\pi_1(M)$, or equivalently the set of homotopy knots in $M$. Choose some linear ordering, denoted by $\leq$, of elements of $\hat{\pi}$. Given a homotopy link $L = \{K_1, K_2, \ldots, K_n\}$ in $F \times I$, we shall say that $L$ is a layered homotopy link with respect to the ordering of $\hat{\pi}$ if each $K_i$ is above $K_{i+1}$ in $F \times I$ and $K_i \leq K_{i+1}$. Let $\mathcal{B}$ be the set of all layered homotopy links with respect to the ordering of $\hat{\pi}$, including the empty link.

**Theorem 6.1**  
(i) The $q$-homotopy skein module $\mathcal{H}S^q(F \times I)$ is generated by $\mathcal{B}$.

(ii) The homotopy skein module $\mathcal{H}S(F \times I)$ is freely generated by $\mathcal{B}$; [H-P-2].

(iii) If $\pi_1(F)$ is abelian then the $q$-homotopy skein module $\mathcal{H}S^q(F \times I)$ is freely generated by $\mathcal{B}$.

**Proof:** We will use the following notation: If $D$ is a link diagram in $F \times I$ and $p_1, \ldots, p_s$ are some of its crossings then $D_{\epsilon_1, \ldots, \epsilon_s}$ denotes the link diagram obtained from $D$ by choosing at $p_i$ positive or negative crossing, or smoothing depending on whether $\epsilon_i$ is equal to $+,-$ or $0$. Consider a map $\alpha: R\mathcal{B}$ to $\mathcal{H}S^q(F \times I)$ given by $\alpha(L) = L$. We will prove that $\alpha$ is an isomorphism if either $q = \pm 1$ or $\pi_1(F)$ is abelian.

We follow closely the proof of Theorem 2.1 in [H-P-2] and the proof of Theorem 1.6 in [P-T].

We will construct the inverse map $W$, to $\alpha$. The plan for constructing $W$ is as follows:

(i) We define $W$ on the diagrams by inducting on the number of components and number of “bad” crossings.

(ii) The initial definition, in the inductive step, depends on the ordering of components of a diagram and on the order of eliminating bad crossings. We prove that our choices do not give different results if either $q = \pm 1$ or $\pi_1(F)$ is abelian.

(iii) We show that $W$ is invariant under Reidemeister moves, and satisfies the homotopy skein relations.
To define $W$ we first we use induction on the number of components, $c(D)$, of the link diagram $D$. For each $n \geq 0$ we define a function $W_n$ defined on the set of oriented link diagrams with no more than $n$ components. Then $W$ will be defined for every diagram by $W(D) = W_n(D)$ where $n \geq c(D)$. Of course the functions $W_n$ must satisfy certain coherence conditions for this to work. First we put $W_0(\emptyset) = \emptyset$ and $W_1(D_K) = K$ where $D_K$ is a diagram of a knot $K$.

To define $W_{n+1}$ and prove its properties we will use induction several times. The following will be called the “Main Inductive Hypothesis”: M.I.H. We assume that we have already defined a function $W_n$ for each diagram $D$ with no more than $n$ components ($c(D) \leq n$). We assume that $W_n$ has the following properties:

(1) $W_n(D) = L_D$ if $D$ is a layered diagram of $n$ or less components representing a layered link ($L_D$) respecting the $\leq$ ordering of $\hat{\pi}$.

(2) $W_n(D_+) = W_n(D_-)$ for a self-crossing ($c(D_+) \leq n$),

(3) $q^{-1}W_n(D_+) - qW_n(D_-) = zW_n(D_0)$ for a mixed crossing ($c(D_+) \leq n$).

(4) $W_n(D^R) = W_n(D)$ where $D^R$ is the result of a Reidemeister move on $D$ ($c(D) \leq n$).

Then we want to make the Main Inductive Step, M.I.S., to obtain the existence of a function $W_{n+1}$ with analogous properties defined on diagrams with at most $n + 1$ components.

Before dealing with the task of making the M.I.S. let us explain that it will end the proof of the theorem. It is clear that the function $W_n$ satisfying M.I.H. is uniquely determined by properties (1)-(3) and the fact that any diagram can be changed to a layered diagram (respecting the ordering $\leq$ on $\hat{\pi}$) by changing some crossings (and observing that smoothing is lowering the number of components). Thus the compatibility of the functions $W_n$ is obvious and they define a function $W$ on diagrams. The function satisfies skein relations by (2)-(3) and Reidemeister moves by (4). By property (1) it is the inverse function to $\alpha$.

The rest of the section will be occupied by M.I.S.

First we define a function $W_d$ on diagrams having $n + 1$ components ($n \geq 1$) where components are ordered according to their homotopy type and the chosen ordering of $\hat{\pi}$. If no two of the components are of the same homotopy
type then their ordering is uniquely determined. However, if some of the components have the same homotopy type then any choice when ordering them is allowed. We proceed by induction on the number, \( b(D) \), of bad crossings of an ordered diagram \( D \), that is, crossings that have to be changed in order to obtain a diagram of a layered link (in the chosen ordering of components). If \( b(D) = 0 \) then we put \( W_d(D) = L_D \) where \( L_D \) is a layered link represented by \( D \).

Assume that \( W_d \) is defined for all \( D \) with \( b(D) < t \), \( t \geq 1 \). Let \( D \) be an ordered \( n + 1 \)-component link diagram with \( b(D) = t \). In the next steps, (a)-(c), we define \( W_d(D) \) and prove its properties.

(a) Let \( p \) be a bad crossing of \( D \). We define a function \( W_p(D) \) by the formula (depending on \( p \) being positive or negative we have \( D = D'_p \) or \( D = D''_p \)):

\[
W_p(D) = \begin{cases} 
q^2 W_d(D'_p) + qz W_n(D''_0) & \text{if } \operatorname{sgn}(p) = + \\
q^{-2} W_d(D''_p) - q^{-1} z W_n(D'_0) & \text{if } \operatorname{sgn}(p) = - 
\end{cases}
\]

The right-hand side of the equation is defined by inductive assumptions.

We show that \( W_d \) is independent on the choice of a bad crossing of \( D \).

Let \( s \) be another bad crossing and \( W_s(D) \) associated with \( s \) value. We show that \( W_s(D) = W_p(D) \).

If \( p \) and \( s \) are mixed crossings between different pairs of components then we get \( W_s(D) = W_p(D) \) without restricting \( q \) or the fundamental group. The computation is the same as for the Jones-Conway polynomial (see [P-T]), so we perform this only in the case of \( \operatorname{sgn}(p) = \operatorname{sgn}(s) = 1 \):

\[
W_p(D) = q^2 W_d(D'_p) + qz W_n(D''_0) = q^4 W_d(D''_{p,-}) + q^3 z W_n(D''_{0,-}) + q^3 z W_n(D''_{0,s}) + q^2 z^2 W_n(D''_{0,0}).
\]

The result is symmetric with respect to \( p \) and \( s \), thus \( W_s(D) = W_p(D) \).

Now assume that \( p \) and \( s \) are mixed crossings between the same pair of components of \( D \).

For \( q = \pm 1 \) we will check all sign cases at once. Let \( \epsilon(x) \) be the sign of a crossing \( x \), and \( \epsilon(q) \) the sign of \( q \). Then we have:

\[
W_p(D) = W_d(D_{-\epsilon(p)}) + \epsilon(p) \epsilon(q) z W_n(D''_0) = \]

20
\[ W_d(D_{-\epsilon(p),-\epsilon(s)}) + \epsilon(s)\epsilon(q)zW_n(D_{-\epsilon(p),0}) + \epsilon(p)\epsilon(q)zW_n(D_0^p), \]
\[ W_n(D) = W_d(D_{-\epsilon(s),0}) + \epsilon(s)\epsilon(q)zW_n(D_0^s) = \]
\[ W_d(D_{-\epsilon(s),-\epsilon(p)}) + \epsilon(p)\epsilon(q)zW_n(D_{-\epsilon(s),0}) + \epsilon(s)\epsilon(q)zW_n(D_0^s). \]

By (2) of the M.I.H. we have \( W_n(D_{-\epsilon(p),0}) = W_n(D_0^s), \) and \( W_n(D_0^p) = W_n(D_{-\epsilon(s),0}). \) Thus \( W_n(D) = W_n(D_{\pi_1(F)}) \) and we put \( W_d(D) = W_n(D_{\pi_1(F)}), \) independent of the choice of a bad crossing.

In the case when \( \pi_1(F) \) is abelian but there are no restrictions on \( q \) we will check positive, negative and mixed cases of signs of \( p \) and \( s \) separately.

\((++)\) If \( \text{sgn}(p) = \text{sgn}(s) = +1 \) we get:
\[ W_n(D_{\pi_1(F)}) = q^2W_d(D_+^p) + qzW_n(D_0^p) = q^4W_d(D_+^p_+, -) + q^3zW_n(D_0^s_+, -) + qzW_n(D_0^p), \]
\[ W_n(D) = q^2W_d(D^s) + qzW_n(D_0^s) = q^4W_d(D^s_+, -) + q^3zW_n(D_0^p_+, -) + qzW_n(D_0^s), \]

\((-\)) If \( \text{sgn}(p) = \text{sgn}(s) = -1 \) we get:
\[ W_n(D_{\pi_1(F)}) = q^{-2}W_d(D_+^p) - q^{-1}zW_n(D_0^p) = \]
\[ q^{-4}W_d(D_+^p_+, -) - q^{-3}zW_n(D_0^p_+, -) - q^{-1}zW_n(D_0^p), \]
\[ W_n(D_{\pi_1(F)}) = q^{-2}W_d(D_0^s) - q^{-1}zW_n(D_0^s) = \]
\[ q^{-4}W_d(D_0^s_+, -) - q^{-3}zW_n(D_0^p_+, -) - q^{-1}zW_n(D_0^s), \]

\((+-)\) If \( \text{sgn}(p) = +1 \) and \( \text{sgn}(s) = -1 \) we get:
\[ W_n(D_{\pi_1(F)}) = q^2W_d(D_+^p) + qzW_n(D_0^p) = W_d(D_+^p, -) - qzW_n(D_0^s_+, -) + qzW_n(D_0^p), \]
\[ W_n(D_{\pi_1(F)}) = q^{-2}W_d(D_+^p) - q^{-1}zW_n(D_0^p) = \]
\[ W_d(D_+^p_+, -) + q^{-1}zW_n(D_0^p_+, -) - q^{-1}zW_n(D_0^p), \]

\((-+)\) The same as \((+-)\) with the role of \( p \) and \( s \) switched.
By (2) of the M.I.H. we have $W_n(D_{-\epsilon(p),0}^{p,s}) = W_n(D_0^{p,s})$, and $W_n(D_0^{p,s}) = W_n(D_{-\epsilon(s),0}^{s,p})$. Furthermore, for $\pi_1(F)$ abelian we have $W_n(D_0^p) = W_n(D_0^s)$ (for $n = 2$ it is immediate as $D_0^p$ and $D_0^s$ are homotopic knots; in general one can use induction on the number of crossing at which $D_0$ is below the rest of the diagram). Thus $W_s(D) = W_p(D)$ and we put $W_d(D) = W_p(D)$, independent of the choice of a bad crossing. We should stress here that in cases (+–) and (−+) we have got the equality $W_d(D_{\epsilon(p),\epsilon(s)}^{p,s}) = W_d(D_{-\epsilon(p),-\epsilon(s)}^{p,s})$.

(b) Homotopy skein relations.

The fact that $W_d$ satisfies homotopy skein relation follows from the construction for the mixed crossing and by an easy induction on $b(D)$ for a self-crossing:

(m.c.) Let $p$ be a mixed crossing of $D$. Then $p$ is a bad crossing of $D_+^p$ or $D_-^p$. Using the defining relation for $W_p$ for the diagram in which $p$ is a bad crossing we get the required skein relation: $q^{-1}W_d(D_+^p) - qW_d(D_-^p) = zW_n(D_0^p)$.

(s.c.) Let $p$ be a self-crossing of $D$. We proceed by induction on $b(D)$ of the number of bad crossings of an $n + 1$ component ordered link diagram $D$. If $b(D) = 0$ then $b(D_{-\text{sgn}(p)}) = 0$ as well and $W_d(D_{-\text{sgn}(p)}) = W_d(D_{p+}) = L_D$ the layered link represented by $D$. Assume the skein relation holds for $b(D) < t$, $(t > 0)$. Assume $b(D) = t$ and let $s$ be its bad crossing. Then $W_d(D_{s+}^p) = W_d(D_{s+}^{p,s}) = W_d(D_{s-}^{p,s}) = W_d(D_{s+}^p) = W_d(D_{s-}^p)$, by inductive assumptions.

(c) Independence of $W_d$ on Reidemeister moves.

(Ω1) Let $\Omega_1(D)$ be obtained from $D$ by the first Reidemeister move, where $D$ is an ordered link diagram of $n + 1$ components. We proceed by induction on $b(D)$. We have $b(\Omega_1(D)) = b(D)$ and if $b(D) = 0$ then $W_d(D) = L_D = W_d(\Omega_1(D))$. Assume that equality holds for $b(D) < t$, $(t > 0)$, and let $b(D) = t$. Finally let $p$ be a bad crossing of $D$ (and $\Omega_1(D)$). Then $W_d(D) = q^{2\epsilon(p)}W_d(D_{-\epsilon(p)}^p) + \epsilon(p)q^\epsilon(p)W_n(D_0^p) = q^{2\epsilon(p)}W_d(\Omega_1(D)_0^{p}) + \epsilon(p)q^\epsilon(p)W_n(\Omega_1(D)_0^{p}) = W_d(\Omega_1(D)))$, by inductive assumptions.
Let \( \Omega_2(D) \) be obtained from \( D \) by a second Reidemeister move, where \( D \) is an ordered link diagram of \( n+1 \) components. We have to consider two cases: Reidemeister move does not create any new bad crossings or it creates two new bad crossings.

(i) In the first case we proceed by induction on \( b(D) \), exactly as in the case of \( \Omega_1(D) \).

(ii) \( \Omega_2(D) \) is introducing two new bad crossings of opposing signs \( p \) and \( s \) (we can assume \( sgn(p) = + \) and \( sgn(s) = - \)) (see Fig. 6.1). In particular \( p \) and \( s \) are mixed crossings. Then:

\[
\begin{align*}
W_d(\Omega_2(D)) &= W_d(\Omega_2(D)_{p,s}^+,\Omega_2(D)_{-0}^+,\Omega_2(D)_{0-}^+,\Omega_2(D)_{-p}^+), \\
W_d(D) &= W_d(D)_{p,s}^+,W_d(D)_{-0}^+,W_d(D)_{0-}^+,W_d(D)_{-p}^+, \\
W_d(\Omega_3(D)) &= W_d(\Omega_3(D)_{p,s}^+,\Omega_3(D)_{-0}^+,\Omega_3(D)_{0-}^+,\Omega_3(D)_{-p}^+).
\end{align*}
\]

\( \Omega_3 \) Let \( p_1 \) be the top crossing of the third Reidemeister move, \( p_3 \) be a bottom crossing and \( p_2 \) be a crossing between the top and bottom arcs of the move (see Fig.6.2). Notice that \( b(\Omega_3(D)) = b(D) \). To show that \( W_d(\Omega_3(D)) = W_b(D) \), we proceed by induction on \( b(D) \). If \( b(D) = 0 \) then \( W_d(D) = L_D = W_d(\Omega_3(D)) \). Assume that the equality \( W_d(D) = W_d(\Omega_3(D)) \) holds for \( b(D) < t \ (t > 0) \), and
let $b(D) = t$. Then either:

(i) there is a bad crossing $p$ different than $p_i$ in $D$. Then:

$$W_d(D) = q^{2x(p)}W_d(D) + q^e(p)W_n(D^p) = q^{2x(p)}W_d(\Omega_3(D)) + q^e(p)W_n(\Omega_3(D)^p) = W_d(\Omega_3(D)),$$

by inductive assumptions.

(ii) Assume that $p = p_1$ or $p_3$ is a bad crossing (assume for simplicity of notation that $sgn(p) = +$). Then:

$$W_d(\Omega_3(D)) = q^{2x(p)}W_d(\Omega_3(D))^p + q^e(p)W_n(\Omega_3(D)^p) = q^{2x(p)}W_d(\Omega_3(D))^p + q^e(p)W_n(\Omega_3(D)^p) = W_d(\Omega_3(D)),$$

by inductive assumptions (compare Fig. 6.2). More precisely: because $b(D^p) < b(D)$ therefore by the inductive assumption $W_d(\Omega_3(D))^p = W_d(\Omega_3(D)^p)$.

Furthermore, by M.I.H. we have $W_n(\Omega_3(D)^p) = W_n(\Omega(D^p)) = W_n(D^p)$, where $\Omega$ can be a composition of
By construction \( W \) is the two-sided inverse of \( \alpha \) so the proof of Theorem 6.1 is complete. \( \Box \)
Remark 6.2 Our assumption that $q = \pm 1$ or $\pi_1(F)$ is abelian was used only in the step (a), where we proved independence of $W_d$ on the ordering of “bad” crossings. Careful analysis of this step would allow us to identify the kernel of $\alpha : R\mathcal{B} \to \mathcal{H}\mathcal{S}^q(F \times I)$, however I am not sure how to find good description of it (one approach, reduction to homologies, is sketched in Section 8). We will show, in the next section, that $\ker(\alpha)$ is not trivial for any $F$ with negative Euler characteristic.

7 Torsion

If the Euler characteristic, $\chi(F)$, is negative then the $q$-homotopy skein module has torsion and $\ker(\alpha) \neq \{0\}$. This is described in Theorem 7.1. On the other hand we have proven in Theorem 6.1(ii) that for $q = \pm 1$ the module is free. We have another very interesting case of coefficients, $\mathcal{F}(q)[z]$, where $\mathcal{F}(q)$ is the field of rational functions in variable $q$. We discuss this in Problem 7.3.

Theorem 7.1 Let $F$ be a surface (not necessary compact) which contains a disc with 2 holes or a torus with a hole embedded $\pi_1$-injectively; equivalently, $\pi_1(F_0)$ is not abelian for a connected component $F_0$ of $F$ (in the compact connected case this means that $\chi(F) < 0$). Then
(a) $\mathcal{HS}^q(F \times I)$ has torsion.

(b) Let $\alpha : R\mathcal{B} \to \mathcal{HS}^q(F \times I)$ be an $R$-homomorphism introduced in the previous section. Then $\ker \alpha \neq \{0\}$.

Proof: Let $i : F' \to F$ be an embedding of surfaces which is $\pi_1$-injective (that is, $i_* : \pi_1(F') \to \pi_1(F)$ is a monomorphism), then by Theorem 6.1 (ii), the embedding yields a monomorphism of homotopy skein modules, $i_* : \mathcal{HS}(F' \times I) \to \mathcal{HS}(F \times I)$. Now the theorem follows from the following lemma, which describes torsion elements in a disc with two holes and a torus with a hole, as torsion elements go to torsion elements under the homomorphism $i_\# : \mathcal{HS}^q(F' \times I) \to \mathcal{HS}^q(F \times I)$. Furthermore, torsion elements constructed in Lemma 7.2 are also nontrivial for $q = 1$ so they “survive” the homomorphism $i_\#$ by Theorem 6.1 (ii) and Theorem 5.2(2). □

Lemma 7.2  (1) Let $F$ be a disk with two holes with $\pi_1(F) = \{x, y\}$. Consider the word $w = xy^{-1}$ and its inverse $\bar{w} = yx^{-1}$. Let $\gamma$ and $\bar{\gamma}$ be two knots realizing these words and $L = \gamma \cdot \bar{\gamma}$ be a two component link in $F \times I$, being a product of these knots and $D_L$ the diagram of $L$ with two mixed crossings, $p$ and $s$, as shown in Fig. 7.1. Then we can “compute” $D_L$ in $\mathcal{HS}^q(F \times I)$ resolving first $p$ or $s$:

$$(p,s) \quad D_L = q^2(D_L)_p - qz(D_L)_0 = \begin{cases} (D_L)_{p,s}^{+,-} - qz((D_L)_0^{p,s} + qz(D_L)_0^-) \quad \text{if} \quad p = 1 \\ (D_L)_{p,s}^{+,-} + qz((D_L)_0^{p,s} - (D_L)_0^-) \quad \text{if} \quad p = 0 \end{cases}.$$

$$(s,p) \quad D_L = q^{-2}(D_L)_s - q^{-1}z(D_L)_0 = \begin{cases} (D_L)_{s,p}^{+,-} - q^{-1}z((D_L)_0^{s,p} + q^{-1}z(D_L)_0^s) \quad \text{if} \quad s = 1 \\ (D_L)_{s,p}^{+,-} + q^{-1}z((D_L)_0^{s,p} - (D_L)_0^s) \quad \text{if} \quad s = 0 \end{cases}.$$

Thus in $\mathcal{HS}^q(F \times I)$ one has:

$$(q - q^{-1})z((D_L)_0^p - (D_L)_0^s) = 0.$$

To see that we really identified a nontrivial torsion element one should show that $(D_L)_0^p - (D_L)_0^s \neq 0$, but this is the case even for $q = 1$, as $(D_L)_0^p$ is a knot representing $xy^{-1}x^{-1}y$ in $\pi_1(F)$ and $(D_L)_0^s$ is a knot representing $xyx^{-1}y^{-1}$. These two elements are not conjugate in $\pi_1(F)$ so, by Theorem 6.1, are different in the homotopy skein module of $F \times I$. 27
(2) Let $F$ be a torus with a hole with $\pi_1(F) = \{x, y\}$. Consider the word $w = xy$ and the word $w' = xy^{-1}$. Let $\gamma$ and $\gamma'$ be two knots realizing these words and $L = \gamma \cdot \gamma'$ be a two component link in $F \times I$, being a product of these knots. Let $D_L$ be the diagram of $L$ with two mixed crossings, $p$ and $s$, as shown in Fig. 7.2. Then we can “compute” $D_L$ in $\mathcal{H}S^q(F \times I)$ resolving first $p$ or $s$ and getting (similarly as in (1)): 

$$(q - q^{-1})z((D_L)_0^p - (D_L)_0^s) = 0.$$ 

Again we deal with a nontrivial torsion element because elements of $\pi_1(F)$ represented by $(D_L)_0^p$ and $(D_L)_0^s$ are not conjugate.
If we allow polynomials $q^n - 1$ to be invertible in the ring of coefficients, then our examples are not producing torsion but instead reduce the number of generators in the skein module.

**Problem 7.3** If two knots are homologous in $F \times I$, are they equal in the homotopy skein module with coefficients in $F(q)[z]$?

If the answer is yes then $\mathcal{HS}(F \times I; F(q)[z], q, z)$ is algebra isomorphic to the quantization of a $q$-symmetric Poisson algebra described in Section 8 (see Theorem 8.11).
8 Lie algebras, Poisson algebras, Universal enveloping algebras, $q$-algebras, quantizations and relations to homotopy skein modules.

We base, in part, our discussion of the general concept of quantization and its application to homotopy of skein modules on [Tu-3, H-P-1].

**Definition 8.1**  
(a) A Lie ring is a $Z$ module $B$, with a map $[ , ] : B \times B \to B$ satisfying the condition:

(i) $[ x + y, z ] = [ x, z ] + [ y, z ]$ and $[ x, y + z ] = [ x, y ] + [ x, z ].$

(ii) $[ x, x ] = 0$, in particular $[,]$ is anti-symmetric ($[ x, y ] = -[ y, x ]$).

(iii) $[,]$ satisfies the Jacobi identity:  
$[ x, [ y, z ] ] + [ y, [ z, x ] ] + [ z, [ x, y ] ] = 0.$

(b) Let $R$ be a commutative ring and $R$-module $B$ a Lie ring, then $B$ is called an $R$-Lie algebra if $a[ x, y ] = [ ax, y ] = a[ x, y ]$, that is, $[,]$ is an $R$ bilinear map.

**Example 8.2** Let $B$ be a ring and define $[ x, y ] = xy - yx$. Then $B$ becomes a Lie ring. If $B$ is an $R$-algebra then $(B, [ , ])$ becomes an $R$-Lie algebra.

**Example 8.3** Consider the homotopy skein algebra $\mathcal{HS}(F \times I)$. As in Example 8.2 it is a Lie algebra with a bracket defined by $[ L_1, L_2 ] = L_1 \cdot L_2 - L_2 \cdot L_1$. The Lie bracket of knots is a linear combination of knots, thus the submodule of $\mathcal{HS}(F \times I)$ generated by knots, $Z[z]\hat{\pi}$, is a Lie subalgebra of $\mathcal{HS}(F \times I)$. The bracket $[K_1, K_2]$ can be written now as

$$[K_1, K_2] = z \sum_{p \in K_1 \cap K_2} sgn(p)(K_1 \cdot K_2)_p.$$  

As it is equal to $K_1 \cdot K_2 - K_2 \cdot K_1$, the formula does not depend on the choice of diagrams for $K_1$ and $K_2$. The bracket was first considered by Goldman [Gol] (for $z = 1$). The Lie algebra $Z\hat{\pi}$ is called the Goldman-Wolpert Lie algebra of curves on $F$, and its relation to knot theory was first noticed by Turaev [Tu-3].
Definition 8.6
(a) Let $A$ be a Poisson algebra (e.g. the first homology group of a surface) with addition denoted by $\oplus$, and consider a bilinear antisymmetric form$^{11}$ on $H$, $f : H \times H \to Z$ (e.g. the homology intersection form). We have the structure of a Lie algebra on $Z[\{-1\}]H$ with the bracket $[\ , \ ]_q : Z[\{-1\}]H \times Z[\{-1\}]H \to Z[\{-1\}]H$ defined on elements of $H$ by $[g, h]_q = f(g, h)_q(q \oplus h)$, where the $q$-integer $[n]_q$ is given by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, and extended bilinearly to $Z[\{-1\}]H$.\footnote{A form $f_n : H \times H \to Z_n$ and the bracket $[\ , \ ]_q : Z[\{-1\}]H \times Z[\{-1\}]H \to Z[\{-1\}]H$ can be considered analogously.}

First notice that $[-n]_q = -[n]_q$, so $f(g, h)_q = -f(h, g)_q$, and $[g, h]_q = f(g, h)_q(q \oplus h) = -f(h, g)_q(q \oplus h) = -[h, g]_q$. We will check the Jacobi identity on elements of $H$ using the identity $[m + n]_q = q^n[m]_q + q^{-m}[n]_q$.

Definition 8.5 A Poisson algebra is a commutative algebra equipped with a Lie bracket which satisfies the following Leibniz rule:
$[ab, c] = a[b, c] + [a, c]b$.

Definition 8.6 (a) Let $B$ be an $R$-module and $B^\otimes m$ the tensor product of $m$ copies of $B$ (with $B^\otimes 0 = R$). Then the tensor algebra $T_B$ is an $R$-module $\bigotimes_{i=0}^n B$ with the algebra multiplication defined by the rule:
$\langle a_1 \otimes \ldots \otimes a_n \rangle \langle b_1 \otimes \ldots \otimes b_m \rangle = \langle a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m \rangle$. If $B$ is a free module (with basis $E = \{e_i\}$) we can identify $T_B$ with the algebra of noncommutative polynomials in variables $\{e_i\}$, denoted by $R\{E\}$.

\footnote{It may be more "orthodox" to define a bracket by the formula $[g, h]'_q = q^f(g, h)_q(g \oplus h)$ but this leads to a $q$-Lie algebra. In particular $[g, h]'_q = -q^2f(g, h)[h, g]_q$.}
(b) The symmetric tensor algebra $SB$ is the quotient of $TB$ by the ideal generated by commutators $a \otimes b - b \otimes a$ where $a, b \in B$ (it suffices to consider a generating set of $B$). If $B$ is a free module (with basis $E = \{e_i\}$) then $SB$ is an algebra of (symmetric) polynomials in variables $\{e_i\}, R[E]$.

(c) If $B$ is a Lie algebra then the universal enveloping algebra $UB$ is the quotient of $TB$ by the ideal generated by expressions $a \otimes b - b \otimes a - [a, b]$ $(UB = TB/(a \otimes b - b \otimes a - [a, b])).$

Example 8.7 A symmetric tensor algebra $SB$, with $B$ being an $R$-Lie algebra is a Poisson algebra with the bracket extended from $B$ by the Leibniz rule. We can write the global formula as follows: $[a_1 \otimes a_2 \ldots \otimes a_m, b_1 \otimes b_2 \otimes \ldots \otimes b_n] = \sum_{i,j} a_1 \otimes a_2 \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_m \otimes [a_i, b_j] \otimes b_1 \otimes b_2 \otimes \ldots \otimes b_{j-1} \otimes b_{j+1} \otimes \ldots \otimes b_n,$ where $a_i, b_j \in B$.

Example 8.4 and topological motivation suggest the following deformation of the Poisson algebra which we call a $q$-Poisson algebra.

Example 8.8 Let $Z[q^{\pm 1}]$ with $[,]_q$ be the Lie algebra of Example 8.4. We define the $q$-symmetric tensor algebra as the quotient of the tensor algebra by the ideal generated by relations (q-commutators) $q^{-f(g,h)}g \otimes h - q^{f(g,h)}h \otimes g,$ for $g, h \in H$. That is, $S_q Z[q^{\pm 1}]H = T Z[q^{\pm 1}]H/(q^{-f(g,h)}g \otimes h - q^{f(g,h)}h \otimes g)$.

Let $a = g_1 \otimes \ldots \otimes g_k,$ $b = h_1 \otimes \ldots \otimes h_l,$ and $c = d_1 \otimes \ldots \otimes d_m,$ where $g_i, h_j, d_i \in H.$ Define $f(a, b) = \sum_{i,j} f(g_i, h_j).$ We can extend our bracket $[,]_q$ to $S_q Z[q^{\pm 1}]H$ by a $q$-version of the Leibniz rule:

$$[a \otimes b, c]_q = q^{-f(a, c)}a \otimes [b, c]_q + q^{f(b, c)}[a, c]_q \otimes b.$$

Our Lie bracket $[,]_q$ on $Z[q^{\pm 1}]H$ leads also to a "$q$-deformation" of an universal enveloping algebra:

$$U(q) Z[q^{\pm 1}]H = T Z[q^{\pm 1}]H/(q^{-f(g,h)}g \otimes h - q^{f(g,h)}h \otimes g - [g, h]_q).$$

Our $q$-deformation of the universal enveloping algebra satisfies the $q$-version of the Poincaré-Birkhoff-Witt theorem, that is, $U(q) Z[q^{\pm 1}]H$ is $Z[q^{\pm 1}]$-module isomorphic to $S_q Z[q^{\pm 1}]H.$\footnote{Compute, in $U(q) Z[q^{\pm 1}]H,$ the balanced difference $q^{-f(a,b)-f(a,c)+f(b,c)-c \otimes b \otimes a} - q^{-f(a,b)+f(a,c)} - f(b,c) \otimes b \otimes a,$ for $a, b, c \in H,$ using two different methods (reflecting the symmetric group relation $s_1 s_2 s_1 = s_2 s_1 s_2$). The difference of results: $\Delta_1 - \Delta_2 = [[a, b]_q, c]_q + [[b, c]_q, a]_q + [[c, a]_q, b]_q$ is exactly the Jacobi expression which is equal to 0 as $Z[q^{\pm 1}]H$ is a Lie algebra (Example 8.4). This observation, $\Delta_1 - \Delta_2 = 0,$ is the key in proving the Poincaré-Birkhoff-Witt Theorem.

32
The relation between the Poisson algebras defined above and homotopy skein modules (of $F \times I$) is best formulated in the language of “quantizations”. In fact knot theory leads to several nontrivial quantizations.

**Definition 8.9** (a) Let $P$ be a Poisson algebra over $Z$ and let $A$ be an algebra over $Z[z]$ which is free as a $Z[z]$-module. A $Z$-module epimorphism $\phi : A \to P$ is called a Drinfeld-Turaev quantization of $P$ if

- $\phi(p(z)a) = p(0)\phi(a)$ for all $a \in A$ and all $p(z) \in Z[z]$, and
- $ab - ba \in z\phi^{-1}([\phi(a), \phi(b)])$ for all $a, b \in P$.

(b) If we do not require, as in (a), that $A$ is free as a $Z[z]$-module, we call this a weak Drinfeld-Turaev quantization.

**Example 8.10** Let $A$ be an algebra over the polynomial ring $R[z]$ which is free as an $R[z]$-module. Assume that the quotient algebra $A/\pi A$ is abelian so that $ab - ba \in \pi A$. Then the formula $[a \mod \pi A, b \mod \pi A] = z^{-1}(ab - ba) \mod \pi A$ equips $A/\pi A$ with a Lie bracket which satisfies the Leibniz rule. Thus $A \to A/\pi A$ is a quantization.

**Theorem 8.11** (1.) [H-P-1, Tu-3]

Let $Q : HS(F \times I) \to SZ\tilde{\pi}$ be the $Z$-module homomorphism from the homotopy skein module of $F \times I$ to the Poisson (symmetric tensor) algebra of the Lie algebra $Z\tilde{\pi}$, defined by $Q(\sum p_i(z)L_i) = \sum p_i(0)Q(L_i)$ where for $L_i = K_1 \cdots K_m \in B(F), Q(L_i) = K_1 \otimes \cdots \otimes K_m$. Then $Q$ is a quantization of $SZ\tilde{\pi}$.

(2.) Let $V(M)$ be a submodule of $HS^q(M)$ generated by relations $K_1 - K_2$, where $K_1$ and $K_2$ are homologous knots in $M$ (that is, $K_1 = K_2$ in $H_1(M)$). Then $HS^q(F \times I)/V(F \times I)$ is $Z[q^{\pm 1}, z]$-algebra isomorphic to $U^{q,z}Z[q^{\pm 1}, z]H_1(F \times I) = TZ[q^{\pm 1}, z]H_1(F \times I)/q^{-f(g,h)}g \otimes h - q^{f(g,h)}h \otimes g - z[g,h]_q$, where $g, h \in H_1(F \times I)$.

(3.) If we relax the definition of quantization to allow deformation of $q$-Poisson algebras (like in Example 8.8), then the map $Q : U^{q,z}Z[q^{\pm 1}, z]H_1(F \times I) \to SZ[q^{\pm 1}]H_1(F \times I)$ is a quantization.

(4.) If we modify the multiplication of links in $F \times I$ by putting $L_1 \cdot L_2 = q^{-f(L_1, L_2)}L_1 \cdot L_2$ where $f(L_1, L_2) = \Sigma_{p \in L_1 \cap L_2} \text{sgn}(p)$, and the homomorphism $Q$ so that $\hat{Q}$ is a $\hat{\cdot}$ algebra homomorphism from $HS^q(F \times I)$}

33
I)\(V(F \times I)\) to the symmetric tensor algebra \(SZ[q^{\pm 1}]H_1(F \times I)\), then \(\hat{Q}\) is a quantization. Furthermore, \(\mathcal{H}S^q(F \times I)/V(F \times I)\) with the product \(\hat{\cdot}\) is a \(\mathbb{Z}[q^{\pm 1}, z]\)-algebra isomorphic to \(TZ[q^{\pm 1}, z]H_1(F \times I)/g \otimes h - h \otimes g - z[gh]_q\), where \(g, h \in H_1(F \times I)\).

Proof:

1. It follows from Theorem 6.1(ii) that \(\mathcal{H}S(F \times I)\) is a \(\mathbb{Z}[z]\)-algebra isomorphic to \(TZ[z\tilde{\pi}]/\langle K_1 \otimes K_2 - K_2 \otimes K_1 - z[K_1, K_2] \rangle\) where \([K_1, K_2]\) is the Goldman-Wolpert Lie bracket (Example 8.3). This algebra is in turn a Drinfeld-Turaev quantization of the Poisson algebra \(SZ\tilde{\pi}\) (see Example 8.10).

2. One should carefully follow the proof of Theorem 6.1. Notice that (2.) reduces to Theorem 6.1(iii) for \(\pi_1(F)\) abelian (\(\pi_1(F) = H_1(F)\)).

3. For \(z = 0\), \(U^{q,z}Z[q^{\pm 1}, z]H_1(F \times I)\) reduces to \(SZ[q^{\pm 1}]H_1(F \times I)\).

4. For \(z = 0\), \(U^Z[q^{\pm 1}, z]H_1(F \times I)\) reduces to \(SZ[q^{\pm 1}]H_1(F \times I)\). The advantage of our modified product (\(\hat{\cdot}\)) is that it allows us to equip the quotient \(\mathcal{H}S^q(F \times I)/V(F \times I)\) with the Hopf algebra structure (as \(U^Z[q^{\pm 1}, z]H_1(F \times I)\) is a Hopf algebra).

\[\square\]

9 Speculation

The only previous work on homotopy skein modules beyond \[H-P-1, Tn-3\] was the work by U.Kaiser \[Kai\]. Kaiser showed that \(\mathcal{H}S(M)\) is free for lens spaces \(L(p, q)\) \((p \neq 0)\). One should be able to generalize his result to the \(q\)-homotopy skein modules.

Of great interest is to produce invariants of 3-manifolds from homotopy skein modules, say following the method of Lickorish (compare also \[BHMV\]). The first step is to extend homotopy skein modules to framed links. The most natural solution is, following Kauffman, to consider the skein relations of unframed links as relations among framed links with zero framing (at least in a homology spheres). Thus adding framing relation \(L^{(1)} = sL\), where \(L^{(1)}\)

\[\text{There is related work by Andersen, Mattes and Reshetikhin \[AMR-1, AMR-2\] but it does not touch homotopy skein modules.}\]
denotes the framed link obtained from a framed link $L$ by adding one positive twist to the framing of a component of $L$. Our skein expressions then have the form:

$$s^{-1}L_{+}^{fr} - sL_{-}^{fr}$$ in the case of a self-crossing, and

$$s^{-1}q^{-1}L_{+}^{fr} - sqL_{-}^{fr} - zL_{0}^{fr}$$ in the case of a mixed crossing. The resulting skein module is $\mathcal{H}S_{q,s}^{fr}(M) = Z[q^{\pm 1}, s^{\pm 1}, z]L^{fr}/S_{q,s}$ where $S_{q,s}$ is a submodule of $Z[q^{\pm 1}, s^{\pm 1}, z]L^{fr}$ generated by framing relations and homotopy skein relation as above. One can hope that this skein module should produce several interesting invariants of 3-manifolds including Dijkgraaf-Witten invariants [De-Wi]. One can get at least the Murakami-Ohtsuki-Okada invariant [M-O-O], as $\mathcal{H}S_{q,s}^{fr}(M)$ dominates the second skein module $S_{2}(M;q)$ which in turn can be used to construct the Murakami-Ohtsuki-Okada invariant [Pr-7].

We can also work with homotopy skein modules of unoriented links. In this case we can follow fruitfully the idea of D.Johnson ([Go], [Tu-2], [H-P-1]) of embedding an unoriented link into the sum of all its orientations. This can be used to analyze the Kauffman homotopy skein modules (compare [Tu-2], [H-P-1]). These modules, as well as related Vassiliev-Gusarov homotopy skein modules (compare [Pr-4]), are worthy of detailed consideration.

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