A NOTE ON ANDERSON’S THEOREM
IN THE INFINITE-DIMENSIONAL SETTING

RIDDHICK BIRBONSHI, ILYA M. SPITKOVSKY, AND P. D. SRIVASTAVA

Abstract. Anderson’s theorem states that if the numerical range $W(A)$ of
an $n$-by-$n$ matrix $A$ is contained in the unit disk $\mathbb{D}$ and intersects with
the unit circle at more than $n$ points, then $W(A) = \mathbb{D}$. An analogue of this result
for compact $A$ in an infinite dimensional setting was established by Gau and
Wu. We consider here the case of $A$ being the sum of a normal and compact
operator.

1. Introduction

The numerical range (also known as the field of values, or the Hausdorff set) of
a bounded linear operator $A$ acting on a Hilbert space $\mathcal{H}$ is defined as

$$W(A) = \{ \langle Ax, x \rangle : \|x\| = 1 \}.$$ 

Here $\langle , \rangle$ and $\| . \|$ stand for the scalar product on $\mathcal{H}$ and the norm generated by it, respectively.

The set $W(A)$ is a convex (Toeplitz-Hausdorff theorem), bounded, and in the
case $\dim \mathcal{H} < \infty$ also closed subset of the complex plane $\mathbb{C}$.

We will use the standard notation $\overline{X}, X^o, \partial X, X'$ for the closure, interior, the boundary, and the set of the limit points, respectively, of subsets $X \subset \mathbb{C}$. In particular, $\mathbb{D} = \{ z : |z| < 1 \}$ is the open unit disk, $\partial \mathbb{D} = \mathbb{T}$ is the unit circle, and $\overline{\mathbb{D}} = \mathbb{D} \cup \partial \mathbb{D}$ is the closed unit disk.

The closure $\overline{W(A)}$ of the numerical range of $A$ contains the spectrum $\sigma(A)$, and thus the convex hull $\text{conv} \sigma(A)$ of the latter. For normal $A$, $\overline{W(A)} = \text{conv} \sigma(A)$. We refer to [4] for these and other well known properties of the numerical range.

Anderson’s theorem (unpublished by the author but discussed e.g. in [2]) states that if $W(A)$ is contained in $\overline{\mathbb{D}}$ and the intersection of $W(A)$ with $\mathbb{T}$ consists of more than $n = \dim \mathcal{H}$ points, then in fact $W(A) = \mathbb{D}$. This result is sharp in a sense that for a unitary operator $U$ with a simple spectrum acting on an $n$-dimensional $\mathcal{H}$, $W(U)$ is a polygon with $n$ vertices on $\mathbb{T}$ and thus different from $\overline{\mathbb{D}}$.

Unitary diagonal operators also deliver easy examples showing that Anderson’s
theorem does not generalize to the infinite-dimensional setting. Indeed, if $A$ is a
diagonal operator with the point spectrum $\sigma_p(U) = \{ \lambda_j, \ j = 1, 2, \ldots \} \subset \mathbb{T}$, then $\overline{W(A)} = \text{conv} \sigma_p(A) \subset \overline{\mathbb{D}}$ while $W(A) \cap \mathbb{T} = \sigma_p(A)$ is infinite.

2010 Mathematics Subject Classification. Primary 47A12; Secondary 47B07, 47B15, 47B37.

Key words and phrases. Numerical range, Normal operator, compact operator, Weighted shift.

Supported in part by Faculty Research funding from the Division of Science and Mathematics, New York University Abu Dhabi.
Moreover, according to [7] every bounded convex set $G$ for which $G \setminus G^o$ is the union of countably many singletons and conic arcs is the numerical range of some operator acting on a separable $H$.

On the positive side, Anderson’s theorem generalizes quite naturally to the infinite dimensional case under some restrictions on the operators involved. As was shown more recently in [3], the following result holds:

**Theorem 1.** If $A$ is a compact operator on a Hilbert space with $W(A)$ contained in $\mathbb{D}$ and $W(A)$ intersecting $T$ at infinitely many points, then $W(A) = \mathbb{D}$.

In this paper, we single out a wider class of operators for which analogs of Anderson’s theorem are valid in an infinite dimensional setting.

2. Main results

We start with a lemma.

**Lemma 2.** Let $A = N + K$, where $N$ is normal and $K$ is a compact operator on a Hilbert space $H$. If $W(A) \subset \mathbb{D}$ and $\gamma$ is a closed arc of $T$ such that the intersection $\gamma \cap W(A)$ is infinite while $\gamma \cap \sigma_{\text{ess}}(A) = \emptyset$, then $\gamma \subset W(A)$.

Recall that the essential spectrum $\sigma_{\text{ess}}(A)$ of an operator $A$ is the set of $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not Fredholm. Equivalently, $\sigma_{\text{ess}}(A)$ is the spectrum of the equivalence class of $A$ in the Calkin algebra of the algebra of bounded linear operators by the ideal of compact operators.

The proof of this lemma is delegated to the next section; we will discuss here some of its consequences.

**Theorem 3.** Let $A = N + K$, where $N$ is normal and $K$ is a compact operator on a Hilbert space $H$. Let also $W(A) \subset \mathbb{D}$ and $\Gamma$ be a (relatively) open subset of $T$ disjoint with $\sigma_{\text{ess}}(A)$. If every connected component of $\Gamma$ contains limit points of its intersection with $W(A)$, then $\Gamma \subset W(A)$.

**Proof.** Connected components of $\Gamma$ are open arcs $\Gamma_j$. Writing $\Gamma_j$ as $\bigcup_{k=1}^\infty \gamma_{jk}$, where

$$\gamma_{j1} \subset \gamma_{j2} \subset \cdots \subset \gamma_{jk} \subset \cdots$$

is an expanding family of closed arcs, we see that $\gamma = \gamma_{jk}$ satisfy the conditions of Lemma 2 and thus $\gamma_{jk} \subset W(A)$, for $k$ large enough. Consequently,

$$\Gamma = \bigcup_{j,k=1}^\infty \gamma_{jk} \subset W(A).$$

**Corollary 1.** Let $A$ and $\Gamma$ satisfy the conditions of Theorem 3 and in addition $\Gamma$ is dense in $T$. Then

$$\mathbb{D} \cup \Gamma \subset W(A) \subset \overline{W(A)} = \mathbb{D}.$$  

**Proof.** By Theorem 3 we have $\Gamma \subset W(A)$, and so $\text{conv} \Gamma \subset W(A)$ due to the convexity of the numerical range. But $\Gamma$ being dense in $T$ implies that $\text{conv} \Gamma \supset \mathbb{D}$. This proves the left inclusion in (2.1). The right equality then follows by combining $\mathbb{D} \subset W(A)$ with the given $W(A) \subset \mathbb{D}$. □

If the normal component $N$ of $A$ is in fact hermitian, then $\sigma_{\text{ess}}(A) \subset \mathbb{R}$. Choosing $\Gamma = T \setminus \{1, -1\}$ immediately yields
Corollary 2. Let $A = H + K$, where $H$ is hermitian and $K$ is a compact operator on a Hilbert space $\mathcal{H}$. If $W(A) \subset \mathbb{D}$ and the set $\overline{W(A)} \cap T$ has limit points both in the upper and lower open half plane, then $\overline{W(A)} = \mathbb{D}$ and $\overline{W(A)} \setminus W(A) \subset \{1, -1\}$.

The next statement also is an immediate consequence of Corollary 1 we nevertheless state it as a theorem.

Theorem 4. Let $A = N + K$, where $N$ is normal and $K$ is a compact operator on a Hilbert space $\mathcal{H}$. If $W(A) \subset \mathbb{D}$ and the intersection $T \cap \overline{W(A)}$ is infinite while $\sigma_{\text{ess}}(A) \subset \mathbb{D}$, then $W(A) = \mathbb{D}$.

Proof. Indeed, $A$ satisfies the conditions of Corollary 1 with $\Gamma = T$, and so the inclusions in (2.1) turn into the equalities. □

3. Proof of Lemma 2

Note that the essential spectrum is invariant under addition of compact summands, and so $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(N)$. The latter coincides with $\sigma(N)$ from which the isolated eigenvalues of finite multiplicity were removed. If $A$ is compact, that is, $N = 0$, then of course $\sigma_{\text{ess}}(A) = \{0\}$, and condition $\sigma_{\text{ess}}(A) \subset \mathbb{D}$ holds. So, Theorem 1 is a particular case of Theorem 3 which was derived in the previous section from Lemma 2. On the other hand, our proof of Lemma 2 below follows the lines of Gau-Wu’s proof of Theorem 3.

Let $d_A(\theta) = \sup W(\text{Re}(e^{-i\theta}A))$, $\theta \in \mathbb{R}$, where as usual $\text{Re} X$ denotes the hermitian part $(X + X^*)/2$ of the operator $X$. Since $d_A$ is the support function of the convex set $\overline{W(A)}$, condition $W(A) \subset \mathbb{D}$ is equivalent to

$$d_A(\theta) \leq 1, \quad \theta \in \mathbb{R},$$

while the condition imposed on $\gamma \cap \overline{W(A)}$ means that the set

$$\alpha := \{e^{i\theta} \in \gamma: d_A(\theta) = 1\}$$

is infinite. Consequently, $\alpha' \neq \emptyset$.

Observe now that for operators $A$ of the form $N + K$ the essential spectrum coincides with their Weyl spectrum $\omega(A)$, that is, the set of $\lambda$ for which $A - \lambda I$ is not a Fredholm operator with index zero. By Berberian’s spectral mapping theorem [1, Theorem 3.1], for any normal operator $T$ and a function $f$ continuous on $\sigma(T)$, $\omega(f(T)) = f(\omega(T))$. Since $e^{-i\theta}A = e^{-i\theta}N + e^{-i\theta}K$ is the sum of a normal and compact operator along with $A$, we have

$$\sigma_{\text{ess}}(\text{Re}(e^{-i\theta}A)) = \omega(\text{Re}(e^{-i\theta}A)) = \text{Re} (\omega(e^{-i\theta}A))$$

$$= \text{Re} (\sigma_{\text{ess}}(e^{-i\theta}A)) = \text{Re} (e^{-i\theta}\sigma_{\text{ess}}(A)).$$

So, the condition $\gamma \cap \sigma_{\text{ess}}(A) = \emptyset$ implies that

$$1 \in \sigma(\text{Re}(z^{-1}A)) \setminus \sigma_{\text{ess}}(\text{Re}(z^{-1}A))$$

for all $z \in \alpha$.

In other words, 1 is an isolated eigenvalue of $\text{Re}(z^{-1}A)$ of finite multiplicity whenever $z \in \alpha$.

As in [3], we now invoke [6, Theorem 3.3] according to which the points $z \in \alpha$ possess the following property: there exists a neighborhood $U_z$ of such $z$ and two (possibly coinciding) open analytic arcs $\gamma_j(z) \ni z$, $j = 1, 2$ satisfying

$$\partial W(A) \cap U_z \subset \gamma_1(z) \cup \gamma_2(z) \subset W(A).$$
For \( z \in \alpha' \) we have in addition that at least one of the arcs \( \gamma_j(z) \) contains infinitely many points of the unit circle and thus lie in \( T \). Say for definiteness, \( \gamma_1(z) \subset T \). Since \( \mathbb{D} \supset W(A) \supset \gamma_1(z) \), in fact the whole arc \( \gamma_1(z) \) is a subset of \( \alpha \), implying that \( z \) is an interior point of \( \alpha' \). So, \( \alpha' \) is not only closed but also open in \( \gamma \), and thus \( \alpha' = \gamma \). So, \( \alpha = \gamma \) as well. Inclusions (3.3) imply in particular that \( \alpha \subset W(A) \), thus completing the proof. \( \square \)

4. Additional Observations

1. As in [3], the results of Section 2 remain valid with \( \mathbb{D} \) and \( T \) replaced by an arbitrary elliptical disk and its boundary, respectively. In order to see that, it suffices to consider a suitable affine transformation \( \alpha A + \beta A^* + \gamma I \) of \( A \) in place of \( A \) itself.

2. Recall that Theorem [4] is a generalization of Theorem [1] from the case of compact \( A \) to \( A \) being the sum of a normal and compact summands under the additional condition \( \sigma_{ess}(A) \cap \mathbb{T} = \emptyset \). The following examples show that merely the condition on \( \sigma_{ess}(A) \) would not suffice.

Example 1. Consider the 2-by-2 matrix \( C = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix} \) for which \( \sigma(C) = \{1/2\} \) and \( W(C) \) is the closed disk \( E \) centered at \( 1/2 \) with the radius also equal \( 1/2 \). In particular, \( 1 \in E \subset \mathbb{D} \).

Let now \( Z \) be a countable subset of \( T \), and \( A = \bigoplus_{z \in Z} zC \). For any \( \lambda \not\in \frac{1}{2}Z \) we then have

\[
(A - \lambda I)^{-1} = \bigoplus_{z \in Z} \begin{bmatrix} \left(\frac{1}{2} - \lambda\right)^{-1} & -z \left(\frac{1}{2} - \lambda\right)^{-2} \\ 0 & \left(\frac{1}{2} - \lambda\right)^{-1} \end{bmatrix},
\]

and so

\[
\sigma_p(A) = \frac{1}{2}Z \subset \sigma(A) = \frac{1}{2}\overline{Z} \subset \frac{1}{2}T,
\]

implying that \( \sigma(A) \) is disjoint with \( T \). At the same time

\[
Z \subset W(A) = \text{conv}\{zE : z \in Z\} \subsetneq \mathbb{D}.
\]

Moreover, by choosing \( Z \) located on a sufficiently small arc it is possible to arrange for a sector in \( \mathbb{D} \) disjoint with \( W(A) \) and having an opening arbitrarily close to \( \pi \).

Example 2. Let now \( S \) be a weighted shift, that is, \( S e_j = s_j e_{j+1} \), where \( \{e_j\}_{j=1}^{\infty} \) is an orthonormal basis of \( \mathcal{H} \), and \( \{s_j\} \) is a bounded sequence. It is well known (and easy to see) that both the numerical range \( W(S) \) and the spectrum \( \sigma(S) \) are invariant under rotation, and depend only on the absolute values of \( s_j \) and not their arguments. So, without loss of generality let us suppose that \( s_j \geq 0 \). Being convex, \( W(S) \) is then either an open or a closed circular disk, while \( \sigma(S) \) is a (naturally, closed) circular disk according to e.g. [5] Problem 93.

Suppose in addition that the sequence \( \{s_j\} \) is periodic, say with the period \( r \). Then \( W(S) \) is open [10] Proposition 6, while its radius (coinciding in this case with the numerical radius \( w(S) \) of the operator \( S \)) is given by

\[
w(S) = \max\{\sum_{j=1}^{r} s_j x_j x_{j+1} : x_j \in \mathbb{R}, \sum_{j=1}^{r} x_j^2 = 1, x_{r+1} = x_1\}.
\]
In particular, $w(S) \geq (s_1 + \cdots + s_r)/r$. On the other hand, the spectral radius $r(S)$ of $S$ is the geometric mean $\sqrt[s_1\cdots s_r]{s_1\cdots s_r}$ of the weights $s_1, \ldots, s_r$.

Corollary 2]. So, $r(S) < w(S)$, unless all the weights $s_j$ are the same.

By an appropriate scaling, we may arrange for $w(S) = 1$ and thus $W(S) = \mathbb{D} \neq \overline{\mathbb{D}}$, in spite of $\sigma(S) \subset \mathbb{D}$ being disjoint with $\mathbb{T}$.

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