Finding singularly cospectral graphs

Cristian Condea,b, Ezequiel Dratmana,c and Luciano N. Grippoa,c

aInstituto de Ciencias, Universidad Nacional de General Sarmiento, Los Polvorines Argentina; bInstituto Argentina de Matemática “Alberto Calderón” – Consejo Nacional de Investigaciones Científicas y Tecnicas, Los Polvorines Argentina; cConsejo Nacional de Investigaciones Científicas y Tecnicas, Los Polvorines Argentina

ABSTRACT
Two graphs having the same spectrum are said to be cospectral. A pair of singularly cospectral graphs is formed by two graphs such that the absolute values of their nonzero eigenvalues coincide. Clearly, a pair of cospectral graphs is also singularly cospectral but the converse may not be true. Two graphs are almost cospectral if their nonzero eigenvalues and their multiplicities coincide. In this paper, we present necessary and sufficient conditions for a pair of graphs to be singularly cospectral, giving an answer to a problem posted by Nikiforov. In addition, we construct an infinite family of pairs of noncospectral singularly cospectral graphs with unbounded number of vertices. It is clear that almost cospectral graphs are also singularly cospectral but the converse is not necessarily true, we present families of graphs where both concepts: almost cospectrality and singularly cospectrality agree.

ARTICLE HISTORY
Received 18 December 2020
Accepted 3 December 2021

COMMUNICATED BY
R. B. Bapat

KEYWORDS
Almost cospectral graphs; cospectral graphs; singularly cospectral graphs

1. Introduction
Two graphs are said to be cospectral if they have the same spectrum. The problem of finding families of pairs of nonisomorphic cospectral graphs has attracted the attention of many researchers. Probably, the first relevant result on this subject is due to Schwenk, who proves that almost all trees are cospectral [1]. Since then, many articles, presenting constructions to either generate pairs of cospectral graphs or finding families of graphs that do not have a mate, known as defined by their spectrum, have been published. See, for instance [2–4].

The energy of a graph was defined by Gutman in 1978 as the sum of the absolute values of its eigenvalues, counted with their multiplicity [5]. Two graphs, with the same number of vertices, are said to be equienergetic if they have the same energy. Clearly, two cospectral graphs are also equienergetic. Nevertheless, there are examples of pairs of noncospectral equienergetic graphs [6,7]. Indeed, finding noncospectral equienergetic pairs of graphs is a very active and actual topic of research. The energy of a graph is nothing but the trace norm of its adjacency matrix; i.e. the sum of the singular values of its adjacency matrix. Notice that, when a matrix is symmetric, its singular values are precisely the absolute values of its eigenvalues. In [8], Nikiforov defines two graphs as singularly cospectral if their nonzero singular values, counted with their multiplicity, coincide. Hence any two
singularly cospectral graphs, having the same number of vertices, are equienergetic. In that article, he also posted the following problem.

**Problem 1.1 ([8]):** Find necessary and sufficient conditions for two graphs to be singularly cospectral.

In addition, we formulate the following two natural problems, in connection with the notion of singularly cospectral.

**Problem 1.2:** Find infinite pairs of noncospectral singularly cospectral graphs.

**Problem 1.3:** Find families of graphs where singularly cospectrality implies cospectrality.

The remainder of the paper is organized as follows. In the next section we recall the definition and basic properties of graphs, and review known results about the eigenvalues of a graph. In Section 3 we solve the Problem 1.1. In Section 4 we deal with Problem 1.2 by presenting a construction based on the spectral decomposition of a symmetric matrix. Finally, in Section 5, we define families of graphs by imposing certain constraints on the spectrum of their graphs that makes equivalent the notions of cospectrality and singularly cospectrality, answering the Problem 1.3.

### 2. Preliminaries

All graphs considered in this article are finite, and without loops or multiple edges. Concepts about graphs used here are defined in the book [9].

Let $G$ be a graph. We use $V(G)$ to denote the set of $n = |V(G)|$ vertices and $E(G)$ the set of edges of $G$, respectively. The adjacency matrix associated with the graph $G$ is defined by $A_G = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if $v_i \sim v_j$ i.e. the vertex $v_i$ is adjacent to the vertex $v_j$ and $a_{ij} = 0$, otherwise. We often denote $A_G$ simply by $A$. We use $d_G(v)$ to denote the degree of $v$ in $G$ (the number of edges incident to $v$), or $d(v)$ provided the context is clear. A $d$-regular graph is a graph such that $d_G(v) = d$ for every $v \in V(G)$.

For a symmetric matrix $M \in \mathbb{R}^{n \times n}$, the spectrum, $\sigma(M)$ is the multiset $\sigma(M) = \{[\lambda_1(M)]^{m_1}, \ldots, [\lambda_r(M)]^{m_r}\}$, where $\lambda_1(M) > \lambda_2(M) > \cdots > \lambda_r(M)$ are the distinct eigenvalues of $M$ and $m_i$ is the multiplicity of $\lambda_i(M)$ for any $i \in \{1, \ldots, r\}$ and the spectral radius, $\rho(M)$, is the maximum among $|\lambda_1(M)|, |\lambda_2(M)|, \ldots, |\lambda_r(M)|$, where $\lambda_i(M) \in \sigma(M)$. To denote the spectrum of $A_G$ we use $\sigma(G)$. We use $P_G(x)$ to denote the characteristic polynomial of $A_G$; i.e. $P_G(x) = \det(xI_n - A_G)$.

The inertia of a graph $G$ is the triple $In(G) = (p(G), z(G), n(G))$, in which $p(G), z(G), n(G)$ stand for the number of positive, zero, and negative eigenvalues of $G$, respectively. The energy of $G$, denoted by $E = E(G)$, is the sum of the absolute values of the eigenvalues of $G$.

The next proposition collects several results about the eigenvalues of a graph. Most of these statements are well-known.

**Proposition 2.1 ([9, Chapter 6]):** Let $G$ be a graph with $n$ vertices and $m$ edges. Then,

1. $|\lambda_i(G)| \leq \lambda_1(G)$ for all $i \geq 2$. 
(2) \( \lambda_1(G) \geq \frac{2m}{n} \). Equality holds if and only if \( G \) is a \( \frac{2m}{n} \)-regular.

(3) If \( G \) is connected, the equality \( \lambda_r(G) = -\lambda_1(G) \) holds if and only if \( G \) is bipartite.

(4) If \( G \) is bipartite, then \( \sigma(G) \) is symmetric respect to zero.

Let \( G \) and \( H \) be two graphs with vertex sets \( V(G) \) and \( V(H) \), respectively. The tensor product of \( G \) and \( H \) is the graph \( G \times H \) such that the vertex set is \( V(G) \times V(H) \) and in which two vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent if and only if \( g_1 \) is adjacent to \( g_2 \) and \( h_1 \) is adjacent to \( h_2 \). The cartesian product of \( G \) and \( H \) is the graph \( G \square H \) such that the vertex set is \( V(G) \times V(H) \) and in which two vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent if and only if either \( g_1 = g_2 \) and \( h_1 \) is adjacent to \( h_2 \) in \( H \) or \( g_1 \) is adjacent to \( g_2 \) in \( G \) and \( h_1 = h_2 \).

We denote by \( C_n, P_n \) and \( K_n \) the cycle, the path and the complete graph on \( n \) vertices. By \( K_{r,s} \) we denote the complete bipartite graph with a bipartition on \( r \) and \( s \) vertices respectively. It is easy to prove that \( \sigma(K_n)(x) = (x - n + 1)(x + 1)^{n-1} \) and in consequence \( \sigma(K_n) = \left\{ [n-1]^1, [-1]^{n-1} \right\} \).

### 3. Characterizing singularly cospectral graphs

Consider a matrix \( M \in \mathbb{R}^{n \times n} \). Recall that the singular values of a matrix \( M \) are the square roots of the eigenvalues of \( M^*M \), where \( M^* \) is the conjugate transpose of \( M \). We denote by \( s_1(M), s_2(M), \ldots, s_r(M) \) the singular values of \( M \) arranged in descending order. Let \( r = \text{rank}(M) \) then \( s_{r+1}(M) = \cdots = s_n(M) = 0 \).

Define for \( p > 0 \),

\[
\|M\|_p = \left( \sum_{i=1}^{n} s_i(M)^p \right)^{\frac{1}{p}}.
\]

For \( p \geq 1 \), it is a norm over \( \mathbb{R}^{n \times n} \) called the \( p \)-Schatten norm. When \( p = 1 \), it is also called the trace norm or nuclear norm. When \( p = 2 \), it is exactly the Frobenius norm \( \|M\|_2 \).

**Remark 3.1:** As \( M^*M \) is a positive semi-definite matrix, so the eigenvalues of \( (M^*M)^{1/2} \) coincide with the singular values of \( M \). It holds that

\[
\lim_{p \to \infty} \|M\|_p = s_1(M).
\]

Given a graph \( G \), following Nikiforov’s notation \([8]\), consider the function \( f_G(p) \) for any \( p \geq 1 \) as:

\[
f_G(p) := \|G\|_p = \left( \sum_{i=1}^{n} s_i(G)^p \right)^{\frac{1}{p}},
\]

where \( \|G\|_p \) and \( s_i(G) \) stands for \( \|A_G\|_p \) and \( s_i(A_G) \), respectively.

The following statement collects some known properties of \( f_G \).

**Lemma 3.1 ([8]):** The following statements hold for a graph \( G \).

1. \( f_G(p) \) is differentiable in \( p \).
(2) \( f_G(p) \) is decreasing in \( p \).
(3) If \( G \) is a graph with \( m \) edges, then \( f_G(2) = \sqrt{2m} \). Furthermore, for any \( k > 1 \), the number of closed walks of length \( 2k \) of a graph \( G \) is equal to
\[
\frac{(f_G(2k))^{2k}}{4k}.
\]

Following [8] we recall the next definition.

**Definition 3.2:** Two graphs \( G \) and \( H \) are called **singly cospectral** if they have the same nonzero singular values with the same multiplicities. We denote a pair of noncospectral singularly cospectral graphs by \( \text{NCSC} \).

We use rank\((G)\) and nullity\((G)\) to denote rank\((A_G)\) and nullity\((A_G)\) for a graph \( G \), respectively. Let us start stating some relevant properties satisfied by singularly cospectral graphs.

**Lemma 3.3:** Let \( G \) and \( H \) be singularly cospectral graphs. Then,
(1) \( G \) and \( H \) have the same number of edges.
(2) \( \text{rank}(G) = \text{rank}(H) \), \( |\text{nullity}(G) - \text{nullity}(H)| = ||V(G)| - |V(H)|| \) and \( p(G) - p(H) = n(H) - n(G) \).
(3) \( E(G) = E(H) \).

**Proof:** We denoted by \( m_G \) and \( m_H \) the number of edges of \( G \) and \( H \), respectively. By item (3) in Lemma 3.1 we have that \( \sqrt{2m_G} = f_G(2) = f_H(2) = \sqrt{2m_H} \). Then, \( m_G = m_H \).

Recall that if \( A \in \mathbb{R}^{n \times n} \), the number of nonzero singular values of \( A \) is equal to the rank\((A)\). Then \( \text{rank}(G) = \text{rank}(H) \). The rank of any square matrix equals the number of nonzero eigenvalues (with repetitions), so the number of nonzero singular values of \( A \) equals the rank of \( A^T A \) (notice that \( A^T = A^* \)). As \( A^T A \) and \( A \) have the same kernel then it follows from the Rank-Nullity Theorem that \( A^T A \) and \( A \) have the same rank. On the other hand, from the Rank-Nullity Theorem we have
\[
|\text{nullity}(G) - \text{nullity}(H)| = |(|V(G)| - \text{rank}(G)) - (|V(H)| - \text{rank}(H))|
= ||V(G)| - |V(H)||.
\]
Therefore, since \( p(G) + n(G) = \text{rank}(G) = \text{rank}(H) = p(H) + n(H) \) then \( p(G) - p(H) = n(H) - n(G) \).

Finally, two singularly cospectral graphs have the same energy since the singular values of any graph coincide with the absolute value of its eigenvalues.

**Example 3.4 (Coenergetic does not imply singularly cospectral):** We present a pair of coenergetic nonsingularly cospectral graphs. For this purpose, we consider the tensor product and the cartesian product of two complete graphs. Let \( r, s \) two integer number with \( r > 3 \) and \( s > 3 \). Let \( G = K_r \times K_s \) and \( H = K_r \square K_s \). Then \( E(G) = E(H) \); i.e. \( G \) and \( H \) are coenergetic [7] but we have that such graphs are no singularly cospectral since \( |E(G)| = \frac{r^2}{2}(r - 1)(s - 1) \) and \( |E(H)| = \frac{r^2}{2}(s + r - 2) \).
In the sequel, the following lemma is useful to obtain a characterization for a pair of singularly cospectral graphs.

**Lemma 3.5:** Let \( x_1, x_2, \ldots, x_r \) and \( y_1, y_2, \ldots, y_s \) be non-increasing sequences of positive real numbers. If

\[
\sum_{i=1}^{r} x_i^{k_n} = \sum_{j=1}^{s} y_j^{k_n} \quad \text{for all } n \in \mathbb{N},
\]

where \( \{k_n\}_{n \in \mathbb{N}} \) is a sequence of real numbers such that \( k_n \geq 1 \) for all \( n \in \mathbb{N} \) and \( k_n \to \infty \), then \( r = s \) and \( x_i = y_i \) for \( i = 1, \ldots, r \).

**Proof:** We denote by \( x = (x_1, x_2, \ldots, x_r, 0, 0, \ldots) \) and \( y = (y_1, y_2, \ldots, y_s, 0, 0, \ldots) \). We have, by (2), that \( \|x\|_{k_n} = \|y\|_{k_n} \) for all \( k_n \), then from (1) we have that

\[
x_1 = \lim_{n \to \infty} \|x\|_{k_n} = \lim_{n \to \infty} \|y\|_{k_n} = \|y\|_\infty = y_1.
\]

By repeating this procedure we can prove that

\[
x_i = y_i \quad \forall i = 1, 2, \ldots, \min\{r, s\}.
\]

Now we assume that \( r < s \). Then by (2) and (3) we obtain that \( \sum_{j=r+1}^{s} y_j^{k_n} = 0 \), as \( y_j \) is a positive real number for any \( j = r + 1, \ldots, s \) we conclude that \( y_j = 0 \) for all \( j \geq r + 1 \). This shows that \( r = s \) and \( x_i = y_i \) for all \( i = 1, \ldots, r \).

Notice that if we replace ‘non-increasing of positive real numbers’ with ‘non-increasing of non-negative real numbers’ in Lemma 3.5, the number of positive terms in both sequences are the same.

Now, we obtain the following theorem for characterizing two graphs \( G \) and \( H \) which are singularly cospectral in terms of their functions \( f_G(x) \) and \( f_H(x) \).

**Theorem 3.6:** The following conditions are equivalent:

1. \( G \) and \( H \) are singularly cospectral.
2. \( f_G(p) = f_H(p) \) for all \( p \geq 1 \).
3. \( f_G(x_n) = f_H(x_n) \) for any sequence \( \{x_n\}_{n \in \mathbb{N}} \) such that \( x_n \geq 1 \) for all \( n \in \mathbb{N} \) and \( x_n \to \infty \).

**Proof:** The implications \( 1 \Rightarrow 2 \) and \( 2 \Rightarrow 3 \) are trivial. Now, suppose that 3 holds. Then

\[
\sum_{i=1}^{r} s_i(G)^{x_n} = \sum_{j=1}^{s} s_j(H)^{x_n} \quad \text{for all } n \in \mathbb{N},
\]

hence by Lemma 3.5 we have that \( G \) and \( H \) are singularly cospectral.

We are ready to prove the main result of this section, giving an answer to Problem 1.1. Such theorem provides a natural and intrinsic characterization of singularly cospectral graphs.
**Theorem 3.7:** Two graphs are singularly cospectral if and only if, for each \( k \in \mathbb{N} \), they have the same number of closed walks of length \( 2k \).

**Proof:** Assume that \( G \) and \( H \) are singularly cospectral, then \( s_i(G) = s_i(H) \) for all \( i = 1, \ldots, \) rank\((G) = \) rank\((H) \) and \( f_G(2k) = f_H(2k) \) for all \( k \in \mathbb{N} \). This implies, by Lemma 3.1, that the number of closed walks of length \( 2k \) of \( G \) and \( H \) coincides.

Conversely, assume that \( G \) and \( H \) have the same number of closed walks of length \( 2k \) for all \( k \in \mathbb{N} \). This implies, by Lemma 3.1, that \( f_G(2k) = f_H(2k) \) for any \( k \in \mathbb{N} \). Then by Theorem 3.6 we conclude that \( G \) and \( H \) are singularly cospectral. \( \blacksquare \)

The following example is an immediate consequence of previous Theorem. We exhibit two pairs of singularly cospectral graphs, in the first pair, one of them is connected and the other one is disconnected, and, in the second pair, both of them are connected. Let \( G = C_{2j} \) and \( F \) be the disjoint union of two copies of the cycle \( C_j \) with an odd integer \( \geq 3 \). We use \( V(F) = \{v_0, v_1, \ldots, v_{2j-1}\} \), and \( V(G) = \{w_0, w_1, \ldots, w_{2j-1}\} \) to denote the set of vertices of \( F \) and \( G \), respectively. Vertices \( v_0, \ldots, v_{j-1} \) and \( v_j, \ldots, v_{2j-1} \) are ordered according to a cyclic order of \( C_j \) and \( w_0, \ldots, w_{2j-1} \) follows a cyclic order of \( C_{2j} \). Given a graph \( H \) such that \( V(H) = \{h_1, \ldots, h_n\} \), for each \( k \in \mathbb{N} \), we denote by \( W_k^H \) a walk of length \( k \), \( h_{n_1}, h_{n_2}, \ldots, h_{n_k} \), when \( h_{n_1} = h_{n_k} \), \( W_k^H \) is said to be a closed walk. Denote by \( CW^H \) the set of closed walks of even length in \( H \). Set \( g \) a function on \( \{0, \ldots, 2j - 1\} \) onto \( \{0, \ldots, 2j - 1\} \) for each \( i = 1, 2 \), defined as \( g_1(n) = r_j(n) \) and \( g_2(n) = r_j(n) + j \), \( r_j(n) \) represents the remainder of \( n \) when divided by \( j \). We define the following functions between the walks in \( G(V^G) \) and the walks in \( F(V^F) \) as follows:

\[
\phi_i : V^G \rightarrow V^F, \quad \phi_i(W^G) = \phi_i(w_{n_1}, \ldots, w_{n_r}) = v_{g_i(n_1)}, \ldots, v_{g_i(n_r)} = W^F_i,
\]

for \( i = 1, 2 \), and \( \phi : V^G \rightarrow V^F \) as

\[
\phi(w_{n_1}, \ldots, w_{n_r}) = \begin{cases} 
\phi_1(w_{n_1}, \ldots, w_{n_r}), & \text{if } n_1 \in \{0, \ldots, j - 1\}, \\
\phi_2(w_{n_1}, \ldots, w_{n_r}), & \text{if } n_1 \in \{j, \ldots, 2j - 1\}.
\end{cases}
\]

Clearly, when the domain of \( \phi \) is restricted to the set \( CW^G \), \( \phi|_{CW^G} : CW^G \rightarrow CW^F \) turns on to be a bijection between \( CW^G \) and \( CW^F \) and by Theorem 3.7 we conclude that this pair of graphs are singularly cospectral. Next, we will describe a pair of singularly cospectral connected graphs. By \( F^* \) (resp \( G^* \)) we denote the graph obtained from \( F \) (resp \( G \)) by adding a vertex \( v_{2j} \) (resp \( w_{2j} \)) adjacent to \( v_0 \) and \( v_j \) (resp \( w_0 \) and \( w_j \)). We will denote each walk in \( G^* \) as \( (w_{2j})W_1, w_{2j}, W_2, \ldots, w_{2j}, W_{t}, (w_{2j}) \), where the last vertex of \( W_t \) is distinct of the first vertex of \( W_{t+1} \) and the subwalks \( W_t \)'s has no occurrence of \( w_0w_{2j}w_j \) or \( w_jw_{2j}w_0 \) and \( (w_{2j}) \) indicates the possible absence of \( w_{2j} \). Set \( g_i^* \) a function on \( \{0, \ldots, 2j\} \) onto \( \{0, \ldots, 2j\} \) for each \( i = 1, 2 \), defined as \( g_i^*(n) = g_i(n) \) if \( n \in \{0, \ldots, 2j - 1\} \) and \( g_i^*(2j) = 2j \). For each \( i = 1, 2 \), we define \( \phi_i^* \) as follows \( \phi_i^*(w_{n_1}, \ldots, w_{n_r}) = v_{g_i^*(n_1)}, \ldots, v_{g_i^*(n_r)} \). Using these two functions we define the function \( \phi^* \) as below:

\[
\phi^* \left( W \right) = \begin{cases} 
(v_{2j})\phi_i^*(W_1), v_{2j}, \ldots, v_{2j}, \phi_{2j}^*(t+1)(W_t), (v_{2j}) & \text{if } w^* \in \{w_0, \ldots, w_{j-1}\}, \\
(v_{2j})\phi_2^*(W_1), v_{2j}, \ldots, v_{2j}, \phi_{2j}^*(t+1)(W_t), (v_{2j}) & \text{if } w^* \in \{w_j, \ldots, w_{2j-1}\}.
\end{cases}
\]
where \( W = (w_{j})_1, w_{2j}, W_2, \ldots, w_{2j}, W_i, (w_{2j}) \) and \( w^* \) stands for the first vertex of \( W_1 \). Analogously to the previous example, the singularly cospectrality, between \( G^* \) and \( F^* \), follows by restricting the domain of \( \phi^* \) to the set \( CW_{G^*} \).

Even though Theorem 3.7 gives a characterization that may be useful from a theoretic perspective, it does not seem to be helpful from a practical point of view when we try to construct pairs of singularly cospectral graphs, unless these graphs are well-structured as in the last example.

4. Constructing pairs of NCSC

Recall that two graphs are called cospectral if they have the same spectrum. Obviously, cospectral graphs are singularly cospectral, but the converse may not be true. See Figure 1 for an example. In order to construct infinite pairs of NCSC graphs, it will be useful the spectral decomposition of a symmetric matrix.

As the adjacency matrix of a graph \( G \) is symmetric with real entries, then it has a spectral decomposition \( A = \sum_{i=1}^{m} \mu_i P_i \) where \( \mu_1, \mu_2, \ldots, \mu_m \) are the distinct eigenvalues of \( G \) and \( P_i \) represents the orthogonal projection onto the eigenspace \( E_{\mu_i} \). The next result is an immediate consequence of [10, Theorem 2.5.4] and it allows us to construct a pair of singularly cospectral graphs but not cospectral.

**Theorem 4.1:** Let \( F \) be a nonbipartite graph with \( |V(F)| = n \) and \( n \geq 3 \). Then, the bipartite graph \( G_F = F \times K_2 \) and \( H_F = 2F \) (the union disjoint of two copies of \( F \)) are NCSC.

**Remark 4.1:** Notice that if \( \sigma(F) = \{[\lambda_1]^{m_1}, \ldots, [\lambda_k]^{m_k} \} \) with spectral decomposition \( A_F = \sum_{i=1}^{k} \lambda_i P_i \), then

\[
A_{H_F} = \begin{pmatrix} A_F & 0 \\ 0 & A_F \end{pmatrix} = \sum_{i=1}^{k} \lambda_i \begin{pmatrix} P_i & 0 \\ 0 & P_i \end{pmatrix} = \sum_{i=1}^{k} \lambda_i \hat{P}_i, \quad (4)
\]

and

\[
A_{G_F} = \begin{pmatrix} 0 & A_F \\ A_F & 0 \end{pmatrix} = \sum_{i=1}^{k} \lambda_i \begin{pmatrix} 0 & P_i \\ P_i & 0 \end{pmatrix}
\]
\[
= \sum_{i=1}^{k} \lambda_i \left[ \frac{1}{2} \begin{pmatrix} P_i & P_i \\ P_i & P_i \end{pmatrix} - \frac{1}{2} \begin{pmatrix} P_i & -P_i \\ -P_i & P_i \end{pmatrix} \right]
= \sum_{i=1}^{k} \lambda_i \left[ \frac{1}{2} \begin{pmatrix} P_i & P_i \\ P_i & P_i \end{pmatrix} - \sum_{i=1}^{k} \lambda_i \left( \frac{1}{2} \begin{pmatrix} P_i & -P_i \\ -P_i & P_i \end{pmatrix} \right) \right]
= \sum_{i=1}^{k} \lambda_i \tilde{P}_i^+ + \sum_{i=1}^{k} (-\lambda_i) \tilde{P}_i^-. 
\] (5)

**Remark 4.2:** If \( F = K_n \) with \( n \geq 3 \), then the bipartite graph \( G_F = F \times K_2 \) and the disconnected graph \( H_F = 2K_n \) are \((n - 1)\)-regular and NCSC.

Let \( G \) be a graph. In [11], Rowlinson obtained from the spectral decomposition of \( A_G \) the relationship between the characteristic polynomials \( P_G \) and \( P_{G^*} \), where \( G^* \) is a graph constructed by adding a new vertex to \( G \). In order to prove our next result, we need to recall such a statement.

**Theorem 4.2 ([11, Theorem 2.1]):** Let \( G \) be a finite graph whose adjacency matrix \( A \) has spectral decomposition \( A = \sum_{i=1}^{m} \mu_i P_i \). Let \( \emptyset \neq S \subseteq V(G) = \{1, 2, \ldots, n\} \) and let \( G^* \) be the graph obtained from \( G \) by adding one new vertex whose neighbours are the vertices in \( S \). Then

\[
P_{G^*}(x) = P_G(x) \left( x - \sum_{i=1}^{m} \rho_i^2 \frac{\mu_i}{x - \mu_i} \right), 
\] (6)

where \( \rho_i = \| \sum_{k \in S} P_i e_k \| \) and \( \{e_1, \ldots, e_n\} \) is the standard orthonormal basis of \( \mathbb{R}^n \).

Combining Theorems 4.1 and 4.2, we obtain a construction that leads to a pair of NCSC graphs by properly adding a vertex to the original pair of graphs.

![Figure 2.](image)

**Figure 2.** In this example \( F = K_6 \), the vertices of the graphs \( G_F \) and \( H_F \) are labelled from 1 to 10 consecutive in each connected component of \( G_F \) and each set of the bipartition of \( H_F \) respectively, and \( S = \{3, 4, 5, 8, 9, 10\} \) is the neighbourhood of the new added vertex. Both graphs are NCSC graphs.
We will use the following notation for the below statement. Let \( F, G_F \) and \( H_F \) as in Theorem 4.1, where \( V(H_F) = \{ w_1, \ldots, w_{2n} \} \) and \( V(G_F) = \{ z_1, \ldots, z_{2n} \} \) such that the first \( n \) vertices belong to the first of \( F \) and the others to the second copy. We label the vertices \( w_1, \ldots, w_{2n} \) (respectively \( z_1, \ldots, z_{2n} \)) by \( 1, \ldots, 2n \). For any \( 1 \leq j \leq n \), let \( G_{F,j}^v \) be the graph obtained from \( G_F \) by adding one new vertex \( v \) whose neighbours are the vertices labelled by \( S_j = \{ 1, \ldots, j \} \cup \{ n + 1, \ldots, n + j \} \). Analogously, we define \( H_{F,j}^w \) by adding a vertex \( w \) and connecting it to the same set \( S_j \) (Figure 2).

**Theorem 4.3:** Let \( \sigma(F) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_k]^{m_k} \} \) and \( \rho_i = \| \sum_{l \in S_j} P_i e_l \| \). Then

\[
P_{G_{F,j}^v}(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i-1} \prod_{i=1}^{k} (x + \lambda_i)^{m_i} \left( x \prod_{i=1}^{k} (x - \lambda_i) - \sum_{i=1}^{k} \rho_i^2 \prod_{j=1, j \neq i}^{k} (x - \lambda_j) \right)
\]

and

\[
P_{H_{F,j}^w}(x) = \prod_{i=1}^{k} (x - \lambda_i)^{2m_i-1} \left( x \prod_{i=1}^{k} (x - \lambda_i) - \sum_{i=1}^{k} \rho_i^2 \prod_{j=1, j \neq i}^{k} (x - \lambda_j) \right).
\]

In particular, \( G_{F,j}^v \) and \( H_{F,j}^w \) are NCSC.

**Proof:** By Theorems 4.1 and 4.2, we have that

\[
P_{G_{F,j}^v}(x) = P_{G_F}(x) \left( x - \sum_{i=1}^{k} \frac{\rho_i^2}{x - \lambda_i} \right),
\]

and

\[
P_{H_{F,j}^w}(x) = P_{H_F}(x) \left( x - \sum_{i=1}^{k} \frac{\sigma_i^2}{x - \lambda_i} - \sum_{i=1}^{k} \frac{\tau_i^2}{x + \lambda_i} \right),
\]

where \( \rho_i = \| \sum_{l \in S_j} P_i e_l \|, \sigma_i = \| \sum_{l \in S_j} P_+^i e_l \| \) and \( \tau_i = \| \sum_{l \in S_j} P_-^i e_l \| \) (see Remark 4.1). It follows that \( \rho_i = \sigma_i \) and \( \tau_i = 0 \) for any \( 1 \leq i \leq k \). Since \( P_{G_F}(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i} \prod_{i=1}^{k} (x + \lambda_i)^{m_i} \) and \( P_{H_F}(x) = \prod_{i=1}^{k} (x - \lambda_i)^{2m_i} \), we conclude that

\[
P_{G_{F,j}^v}(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i-1} \prod_{i=1}^{k} (x + \lambda_i)^{m_i} \left( x \prod_{i=1}^{k} (x - \lambda_i) - \sum_{i=1}^{k} \rho_i^2 \prod_{j=1, j \neq i}^{k} (x - \lambda_j) \right)
\]

and

\[
P_{H_{F,j}^w}(x) = \prod_{i=1}^{k} (x - \lambda_i)^{2m_i-1} \left( x \prod_{i=1}^{k} (x - \lambda_i) - \sum_{i=1}^{k} \rho_i^2 \prod_{j=1, j \neq i}^{k} (x - \lambda_j) \right).
\]
Corollary 4.4: It holds that
\[ \sigma(G_{K_{n,j}}) = \{-(n-1)^1, -(1)^{n-2}, [1]^{n-1}, \lambda_1, \lambda_2, \lambda_3 \} \]
and
\[ \sigma(H_{K_{n,j}}^w) = \{-(1)^{2n-3}, [n-1]^1, \lambda_1, \lambda_2, \lambda_3 \}, \]
with \( \lambda_1 > 0, \lambda_3 < 0 \) and \( \lambda_2 > 0 \) if \( 1 \leq j \leq n - 2 \), or \( \lambda_2 = 0 \) if \( j = n - 1 \), or \( \lambda_2 < 0 \) if \( j = n \). In particular, \( G_{K_{n,j}}^w \) and \( H_{K_{n,j}}^w \) are NCSC.

Proof: Let us suppose that the eigenvalues of \( G_K \) and \( H_K \) are denoted by \{\( \mu_1, \mu_2, \mu_3, \mu_4 \)\} and \{\( v_1, v_2 \)\}, respectively, in increasing order and

\[ A_{G_K} = \sum_{r=1}^{4} \mu_r P_r^{G_K} \quad \text{and} \quad A_{H_K} = \sum_{s=1}^{2} v_s P_s^{H_K}, \quad (9) \]

where \( P_r^{G_K} \) and \( P_s^{H_K} \) represent the orthogonal projection onto the eigenspace \( E_{\mu_r} \) and \( E_{v_s} \), respectively.

Forsake of simplicity, we denote by \( G_{K_{n,j}} \) and \( H_{K_{n,j}} \) the graphs omitting in the notation the added vertex in each one of them. From Theorem 4.3, for each \( j \in \{1, 2, \ldots, n\} \) we obtain the characteristic polynomials of \( G_{K_{n,j}} \) and \( H_{K_{n,j}} \), respectively:

\[ P_{G_{K_{n,j}}}(x) = P_{G_K}(x) \left( x - \sum_{r=1}^{4} \frac{\rho_r^2}{x - \mu_r} \right), \]
\[ P_{H_{K_{n,j}}}(x) = P_{H_K}(x) \left( x - \sum_{s=1}^{2} \frac{\sigma_s^2}{x - v_s} \right), \quad (10) \]

where \( \rho_r = \| \sum_{k \in S_j} P_r^{G_K} e_k \|, \sigma_j = \| \sum_{k \in S_j} P_s^{H_K} e_k \| \) and \( S_j = \{1, \ldots, j, n + 1, \ldots, n + j\} \).

For the sake of clarity, given a matrix \( B \in \mathbb{R}^{n \times n} \) and \( k \in \{1, \ldots, n\} \) we denote by \( c_k(B) \in \mathbb{R}^n \) the \( k \)th column of \( B \). Then,

1. \( \rho_1 = \| \sum_{k \in S_j} P_1^{G_K} e_k \| = \| \sum_{k \in S_j} c_k(P_1^{G_K}) \| = 0. \)
2. \( \rho_2^2 = \| \sum_{k \in S_j} P_2^{G_K} e_k \|^2 = \| \sum_{k \in S_j} c_k(P_2^{G_K}) \|^2 = \frac{2j(n-j)}{n}. \)
3. \( \rho_3 = \| \sum_{k \in S_j} P_3^{G_K} e_k \| = \| \sum_{k \in S_j} c_k(P_3^{G_K}) \| = 0. \)
4. \( \rho_4 = \| \sum_{k \in S_j} P_4^{G_K} e_k \| = \| \sum_{k \in S_j} c_k(P_4^{G_K}) \| = \frac{j}{n} \| c_1(J_{2n}) \| = \frac{j}{n} (2n)^{1/2}. \)
5. \( \sigma_1^2 = \| \sum_{k \in S_j} P_1^{H_K} e_k \|^2 = \| \sum_{k \in S_j} c_k(P_1^{H_K}) \|^2 = \frac{2j(n-j)}{n}. \)
6. \( \sigma_2^2 = \| \sum_{k \in S_j} P_2^{H_K} e_k \|^2 = \| \sum_{k \in S_j} c_k(P_2^{H_K}) \|^2 = \frac{j}{n} \| c_1(J_{2n}) \| = \frac{j}{n} (2n)^{1/2}. \)

Now we have to consider three possible cases.

1. Case 1: \( j = n \).
Then from (10) we can assert that:

\[
P_{GKn,j}(x) = P_{GKn}(x) \left( x - \frac{2n}{x - (n - 1)} \right)
\]

\[
= \frac{P_{GKn}(x)}{(x - (n - 1))} (x^2 - (n - 1)x - 2n),
\] (11)

and

\[
P_{HKn,j}(x) = \frac{P_{HKn}(x)}{(x - (n - 1))} (x^2 - (n - 1)x - 2n).
\] (12)

Let \(Q_1(x) = (x^2 - (n - 1)x - 2n)\) and \(\lambda_1, \lambda_3\) its roots, it is easily seen that \(\lambda_1 \lambda_3 < 0\). Finally, letting \(\lambda_2 = -1 < 0\) we conclude this part of the proof.

(2) **Case 2:** \(j = n - 1\).

In this case from (10) we obtain that the characteristic polynomials have the following expressions:

\[
P_{GKn,j}(x) = \frac{P_{GKn}(x)}{(x - (n - 1))(x + 1)} x \left( x^2 + (2 - n)x - 3n + 3 \right)
\]

and

\[
P_{HKn,j}(x) = \frac{P_{HKn}(x)}{(x - (n - 1))(x + 1)} x \left( x^2 + (2 - n)x - 3n + 3 \right).
\]

Let \(Q_2(x) = x(x^2 + (2 - n)x - 3n + 3)\) and \(\lambda_1, \lambda_2, \lambda_3\) its roots with \(\lambda_2 = 0\). It is easily seen that \(\lambda_1 \lambda_3 < 0\) and this concludes the proof in this case.

(3) **Case 3:** \(1 \leq j \leq n - 2\).

We have that

\[
P_{GKn,j}(x) = \frac{P_{GKn}(x)}{(x - (n - 1))(x + 1)} Q_3(x),
\]

and

\[
P_{HKn,j}(x) = \frac{P_{HKn}(x)}{(x - (n - 1))(x + 1)} Q_3(x),
\]

where \(Q_3(x) = x^3 + (2 - n)x^2 - (n - 1 + 2j)x + 2j((n - 1) - j)\).

We denote by \(\lambda_1, \lambda_2, \lambda_3\) the real roots of \(Q_3\) such that \(\lambda_3 \leq \lambda_2 \leq \lambda_1\). We know that \(\lambda_3 < 0\), since \(\lim_{x \to -\infty} Q_3(x) = -\infty\) and \(Q_3(0) = 2j((n - 1) - j) > 0\). On the other hand, by Descartes’ rule of signs, we conclude that \(\lambda_2\) and \(\lambda_1\) are positive real numbers and this finishes the proof.

\[\blacksquare\]

**Remark 4.3:** From the previous proof we obtain that

\[
P_{GKn,j}(x) = (x - 1)^{n-1} (x + 1)^{n-2} (x + (n - 1)) Q_3(x)
\] (13)
and

\[ P_{H_{Kn,j}}(x) = (x + 1)^{2n-3}(x - (n - 1))Q_3(x), \quad (14) \]

with \( Q_3(x) = x(x + 1)(x - (n - 1)) - 2j(x - (n - 1 - j)). \)

Next, we present a generalization of the previous construction via coalescence operation between two graphs. If \( G \) and \( H \) are two graphs, \( g \in V(G) \) and \( h \in V(H) \), the coalescence between \( G \) and \( H \) at \( g \) and \( h \), denoted \( G \cdot H(g, h : v_{g,h}) \), is the graph obtained from \( G \) and \( H \), by identifying vertices \( g \) and \( h \) (see Figure 3). We denote the coalescence between two copies of a graph \( G \) at a vertex \( g \in V(G) \) and its corresponding copy by \( G(g : v_g) \).

In the 70s Schwenk published an article containing a useful formula for the characteristic polynomial of a graph [12] by linking the characteristic polynomial of two graphs and the coalescence between them.

**Lemma 4.5 ([12]):** Let \( G \) and \( H \) be two graphs. If \( g \in V(G) \), \( h \in V(H) \), and \( F = G \cdot H \), then

\[ PF(x) = PG(x)P_{H-h}(x) + PG_{-g}(x)PH(x) - xPG_{-g}(x)P_{H-h}(x). \]

More details on the above result can be found in [13].

Let \( G_{Kn,j}^v \) and \( H_{Kn,j}^w \) with \( n \geq 3 \) and \( 1 \leq j \leq n \) as above. Let us denote them \( G_{n,j}^v \) and \( H_{n,j}^w \), respectively, for the sake of shortness. For each \( k \geq 1 \), the graph \( G_{n,j,k}^v \) is defined inductively as follows: \( G_{n,j,1}^v := G_{Kn,j}^v \) and \( G_{n,j,k}^v = G_{n,j,k-1}^v(v_{k-1} : v_k) \) for \( k \geq 2 \). Analogously, we consider the graphs \( H_{n,j,1}^w := H_{n,j}^w \) and \( H_{n,j,k}^w = H_{n,j,k-1}^w(w_{k-1} : w_k) \) for \( k \geq 2 \).

Combining Lemma 4.5 and Theorem 4.2, we obtain a pair of graphs singularly cospectral but not (almost) cospectral via coalescence.

**Remark 4.4:** The graphs \( G_{n,j,k}^v \) and \( H_{n,j,k}^w \) are NCSC for every \( k \geq 1 \).

In Remark 4.2 we presented families of pairs of regular graphs that are singularly cospectral. To finish the section we will show that if we have a pair of singularly cospectral graphs, on the same number of vertices, and one of them is regular, then the other one is also regular.
Remark 4.5: Let $G, H$ be two graphs with the same numbers of vertices, such that $G$ is a $d$-regular graph and $H$ is a non-regular one. Then $G$ cannot be singularly cospectral with $H$.

**Proof:** Let $n = |V(G)| = |V(H)|$ and let $m = |E(G)|$. Suppose that $G$ and $H$ are singularly cospectral, then from Proposition 2.1 and Lemma 3.3 we have $m = |E(H)|$ and

$$d = \frac{2|E(H)|}{|V(H)|} = \lambda_1(G) = s_1(G) = s_1(H) = \lambda_1(H),$$

thus $H$ is a $d$-regular graph as consequence of Proposition 2.1. But this contradict our hypothesis and conclude the proof. □

5. Families of graphs where singularly cospectral implies almost cospectral

Since two graphs to be singularly cospectral is not necessary to have the same number of vertices, the notion of almost cospectral is nearer to singularly cospectral than cospectral. Two graphs are *almost cospectral* if their nonzero eigenvalues (and their multiplicities) coincide. The connected components of $P_3 \times P_3$, $C_4$ and $K_{1,4}$ are almost cospectral, see [14, Theorem 3.16]. For more details and results in connection with almost cospectral graphs, we referred to the reader to [15] and the references therein. Notice that if two almost cospectral graphs have the same number of vertices, then they are cospectral. In Section 4, we have presented constructions of singularly cospectral graphs which are not almost cospectral. This section is devoted to present families of graphs where the notion of singularly cospectral and cospectral are equivalent, namely, bipartite graphs (Theorem 5.1), connected graphs having maximum singular value with multiplicity at least two (Theorem 5.2), and connected graphs having the same inertia (Theorem 5.4).

**Theorem 5.1:** Let $G, H$ be two singularly cospectral bipartite graphs then they are almost cospectral.

**Proof:** Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the non-zero eigenvalues of $G$ and $H$, respectively. As $G$ and $H$ are singularly cospectral then $n = m$ and the spectrum of a bipartite graph is symmetric about 0 then $n$ is even. It follows that, for any $1 \leq i \leq \frac{n}{2}$, it holds

$$\lambda_i = -\lambda_{n+1-i} \quad \text{and} \quad \mu_i = -\mu_{n+1-i}.$$ 

Since the singular values of a symmetric matrix are the absolute values of its nonzero eigenvalues, then

$$\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_{\frac{n}{2}}, \lambda_{\frac{n}{2}}$$

and

$$\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_{\frac{n}{2}}, \mu_{\frac{n}{2}},$$

are the singular values of $G$ and $H$, respectively. This shows that the graphs are almost cospectral. □
Figure 4. $\sigma(H_1) = \{[-1]^6, [3]^2\}$ and $\sigma(H_2) = \{[-3]^3, [-1]^3, [1]^3, [3]^1\}$.

Notice that, by Proposition 2.1, if $G$ is a connected graph then the multiplicity of the greater singular value is at most two.

**Theorem 5.2:** Let $G$, $H$ be two singularly cospectral connected graphs such that its largest singular value has multiplicity equal to 2. Then, $G$ and $H$ are almost cospectral.

**Proof:** Let $s_1 > s_2 > \cdots > s_k$ be the non-zero singular values of $G$ with multiplicity $m_1, m_2, \ldots, m_k$, respectively. If we denote by $\lambda_1$ and $\mu_1$ the largest eigenvalue of $G$ and $H$, respectively, then $s_1 = \lambda_1 = \mu_1$. On the other hand, by the connectivity of both graphs such eigenvalues are simple. As by hypothesis, $s_1$ has multiplicity equal to 2, then the smallest eigenvalue of $G$ and $H$ is equal to $-\lambda_1$. So by Proposition 2.1 and Theorem 5.1, we conclude that both graphs are almost cospectral. ■

**Remark 5.1:** The connectivity condition can not be dropped as a hypothesis from Theorem 5.2. See Figure 4 for an example of two graphs, one of them disconnected which are singularly cospectral but not almost cospectral.

**Proposition 5.3:** Let $G$, $H$ be two singularly cospectral connected graphs with exactly three different singular values, without considering their multiplicities, such that $n(G) = n(H)$ or $p(G) = p(H)$. Then they are almost cospectral.

**Proof:** Let $s_1 > s_2 > s_3$ be the non-zero singular values of $G$ with multiplicity $m_1, m_2, m_3$, respectively. If we denote by $\lambda_1$ and $\mu_1$ the largest eigenvalue of $G$ and $H$, respectively, then $s_1 = \lambda_1 = \mu_1$. On the other hand, by the connectivity of both graphs such eigenvalues are simple.

The non-zero eigenvalues of $G$ and $H$ are included in the set

$$\{\pm s_1, \pm s_2, \pm s_3\}.$$

Even more, for any $i \in \{1, 2, 3\}$, the non-zero eigenvalues of $G$ are exactly $s_i$ and $-s_i$ with multiplicity $m_i^+$ and $m_i^-$ respectively. Analogously, the non-zero eigenvalues of $H$ are $s_i$ and $-s_i$ with multiplicity $\hat{m_i}^+$ and $\hat{m_i}^-$, respectively. Obviously, it holds that $m_i^+ + m_i^- = m_i = \hat{m_i}^+ + \hat{m_i}^-$ with $m_i^+, m_i^-, \hat{m_i}^+, \hat{m_i}^- \geq 0$. 
This gives two cases to consider:

(1) **Case 1**: $m_1 = 2$.

From Theorem 5.2 we have that $G$ and $H$ are almost cospectral.

(2) **Case 2**: $m_1 = 1$.

As $m_1 = 1$ thus $m_1^+ = \hat{m}_1^+ = 1$ and $m_1^- = \hat{m}_1^- = 0$. Using the well-known fact that the sum of all eigenvalues of a graph is always zero, we have the following equalities

\[
\begin{align*}
\begin{cases}
    s_1 + (m_2^+ - m_2^-)s_2 + (m_3^+ - m_3^-)s_3 = 0, \\
    s_1 + (\hat{m}_2^+ - \hat{m}_2^-)s_2 + (\hat{m}_3^+ - \hat{m}_3^-)s_3 = 0.
\end{cases}
\end{align*}
\]

From the hypothesis about the positive or negative inertia one of the following identity holds

\[
1 + m_2^+ + m_3^+ = 1 + \hat{m}_2^+ + \hat{m}_3^+, \tag{16}
\]

or

\[
m_2^- + m_3^- = \hat{m}_2^- + \hat{m}_3^- . \tag{17}
\]

From now on, without loss of generality, we assume that $p(G) = p(H)$. Then, by (15) and (16), we have that

\[
(m_2^+ - \hat{m}_2^+)(s_2 - s_3) = 0.
\]

As $s_2 - s_3 > 0$, then we conclude that $m_2^+ - \hat{m}_2^+ = 0 = m_3^+ - \hat{m}_3^+$. This shows that $G$ and $H$ are almost cospectral graphs and it concludes the proof.

**Remark 5.2:** The condition about the negative or positive inertia cannot be dropped from Proposition 5.3, as can be seen in Figure 5.
As a consequence of Proposition 5.3, the following result holds.

**Theorem 5.4:** Let $G$, $H$ be two singularly cospectral connected graphs with exactly three different singular values, without counting its multiplicities, such that $\text{In}(G) = \text{In}(H)$. Then, $G$ and $H$ are almost cospectral.

Since $|V(G)| = |V(H)|$, in the above Theorem almost cospectral can be replaced by cospectral.

**Acknowledgments**

The authors would like to thank the anonymous referee for their generous observations and comments that we have made ample use of in the paper.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

Cristian M. Conde acknowledges partial support from ANPCyT PICT 2017-2522. Ezequiel Dratman and Luciano N. Grippo acknowledge partial support from ANPCyT PICT 2017-1315. Luciano N. Grippo acknowledges partial support from Programa Regional MATHAMSUD MATH190013.

**References**

[1] Schwenk AJ. Almost all trees are cospectral. New directions in the theory of graphs (Proceedings of Third Ann Arbor Conference, University of Michigan, Ann Arbor, MI, 1971); 1973. p. 275–307.

[2] Godsil CD, McKay BD. Constructing cospectral graphs. Aequationes Math. 1982;25(1):257–268.

[3] van Dam ER, Haemers WH. Developments on spectral characterizations of graphs. Discrete Math. 2009;309(3):576–586.

[4] Qiu L, Ji Y, Wang W. On a theorem of Godsil and McKay concerning the construction of cospectral graphs. Linear Algebra Appl. 2020;603:265–274.

[5] Gutman I. The energy of a graph: old and new results. Algebraic combinatorics and applications (Gößweinstein, 1999). Berlin: Springer; 2001. p. 196–211.

[6] Stevanović D. Energy and NEPS of graphs. Linear Multilinear Algebra. 2005;53(1):67–74.

[7] Bonifácio AS, Vinagre CTM, de Abreu NMM. Constructing pairs of equienergetic and non-cospectral graphs. Appl Math Lett. 2008;21(4):338–341.

[8] Nikiforov V. Beyond graph energy: norms of graphs and matrices. Linear Algebra Appl. 2016;506:82–138.

[9] Bapat RB. Graphs and matrices. 2nd ed. London: Springer; 2014. (Universitext). New Delhi: Hindustan Book Agency.

[10] Cvetković D, Rowlinson P, Simić S. An introduction to the theory of graph spectra. Cambridge: Cambridge University Press; 2010. (London Mathematical Society student texts; Vol. 75).

[11] Rowlinson P. The spectrum of a graph modified by the addition of a vertex. Univ Beograd Publ Elektrotehn Fak Ser Mat. 1992;3:67–70.

[12] Schwenk AJ. Computing the characteristic polynomial of a graph. Graphs and combinatorics (Proceedings of the Capital Conference, George Washington University, Washington, D.C., 1973). 1974. p. 153–172. (Lecture notes in math.; Vol. 406).

[13] Cvetković D, Rowlinson P, Simić S. Eigenspaces of graphs. Cambridge: Cambridge University Press; 1997. (Encyclopedia of mathematics and its applications; Vol. 66).
[14] Cvetković DM, Doob M, Gutman I, et al. Recent results in the theory of graph spectra. Amsterdam: North-Holland Publishing Co.; 1988. (Annals of Discrete Mathematics; Vol. 36).

[15] Beineke LW, Wilson RJ, editors. Topics in algebraic graph theory. Cambridge: Cambridge University Press; 2004. (Encyclopedia of mathematics and its applications; Vol. 102).