First-order symmetries of the Dirac equation in a curved background: a unified dynamical symmetry condition*

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Abstract

It has been shown that, for all dimensions and signatures, the most general first-order linear symmetry operators for the Dirac equation including interaction with Maxwell field in a curved background are given in terms of Killing–Yano (KY) forms. As a general gauge invariant condition it is found that among all KY forms of the underlying (pseudo) Riemannian manifold, only those which Clifford commute with the Maxwell field take part in the symmetry operator. It is also proved that associated with each KY form taking part in the symmetry operator, one can define a quadratic function of velocities which is a geodesic invariant as well as a constant of motion for the classical trajectory. Some geometrical and physical implications of the existence of KY forms are also elucidated.

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1. Introduction

In many evolutions taking place in a flat or a curved background, isometries of the underlying spacetime metric lead to conservation laws that also have clear geometrical meanings expressed by means of their local generators, Killing vector fields. As spacetime transformations, flows of these fields specify the conserved quantities as their flow invariants. However, since the beginning of the 1970s, it has been recognized that many interesting properties of a given spacetime are intimately related to hidden symmetries of its metric, which make themselves

* Dedicated to Professor T Dereli on the occasion of his 60th birthday.
manifest in higher rank tensorial objects that also provide additional conservation laws [1–3]. The building blocks of the mathematical structure behind these hidden symmetries and associated conserved quantities are the symmetric Killing tensors, KY forms, conformal Killing tensors and conformal KY (CKY)-forms. Their defining relations are natural generalizations of those of Killing vector fields and of conformal Killing vector fields. Although these higher rank objects can completely be determined by the metric itself (see for instance [4, 5]), they are not directly related to isometries or conformal transformations. But they comprise, by definition, their generators in the rank-1 sector of their hierarchy. In most of the cases these additional conserved quantities lead to complete integrability of the considered problem. This important step was initiated with the works of Penrose and his collaborators [1, 2]. They have shown that it is the existence of a second-rank symmetric Killing tensor which can be written as the ‘square’ of a KY 2-form that leads to Carter’s fourth constant of motion, which is responsible for the complete integrability of the geodesic problem in the Kerr geometry [6]. For other research fields utilizing KY and CKY forms and for earlier references for these forms we shall refer to [7–10] and the references therein.

An important place where some or all of these tensorial objects enter the analysis is the study of symmetry operators for the Dirac-type equations describing the motion in curved background with or without additional interactions. By now it has become a well-established fact that while CKY forms take part in symmetry operators, via the $R$-commuting argument for the massless Dirac equation, KY forms are indispensable in constructing first-order symmetries of the massless as well as massive Dirac equation in a curved spacetime. It is a four-dimensional Lorentzian spacetime where most of the applications have taken place. The first seminal studies in this context were carried out by Carter, McLenaghan and Kamran [11–13]. The results of these earlier studies, obtained in four dimensions for the massive or massless Dirac equation in the absence of electromagnetic interactions, were recently extended by Benn and his collaborators, to an arbitrary dimension and signature [14–16].

In the absence of additional interactions, all KY or CKY forms of the spacetime take part in the symmetry operator without any extra restriction. However, when interactions are included, some additional conditions arise which restrict the possible forms that can enter into the symmetry operator. To the best of our knowledge, these restrictions for a four-dimensional curved background were found for the first time by McLenaghan and Spindel [17]. In searching for the most general symmetry operator commuting with the Dirac equation in the presence of an electromagnetic field, they found that the symmetry operator can be constructed from KY forms of the underlying background, provided that they separately fulfil some conditions involving the field itself. These included some conditions found before by Carter and McLenaghan [12]. Some of these conditions were also obtained before by Hughston et al in a slightly different context: in searching quadratic first integrals for the charged particle orbits in the charged Kerr background [3].

In this study we first show that the main results of McLenaghan and Spindel, that is the construction of the symmetry operators out of KY forms, can be extended to an arbitrary dimension and signature. We then obtain a unified condition which allows one to specify which KY forms can take part in the symmetry operator, and hence define hidden dynamical symmetry of the problem. This is an algebraic condition and can be stated as follows; a KY form of the curved background enter the symmetry operator if and only if it Clifford commutes with the force (Maxwell) field. In particular, we solve all the consistency conditions and find a concise way to choose the gauge in order to make 0-form component of the symmetry operator constant. Owing to a non-integrated consistency condition, this point has remained ambiguous in the literature.
Our results include the results of Benn and his collaborators when the electromagnetic field is turned off. Finally, we prove that the quadratic functions of velocities defined in terms of each KY p-form entering the symmetry operator is not only a geodesic invariant but also, for an arbitrary number of dimension and signature, a constant of motion for the classical trajectory.

We shall mainly use the notation of [18] and adopt the following conventions and terminology. The underlying base manifold is supposed to be an n-dimensional pseudo-Riemannian manifold with arbitrary signature. Covariant derivative of spinors, that is of sections of a bundle carrying an irreducible representation of the (real or complexified) Clifford algebra, with respect to the vector field $X$ is denoted by $S_X$. Then the Dirac operator on spinors is $\mathcal{D} = e^\mu S_X^\mu$, where the local co-frame $\{e^\alpha\}$ is the dual to the tangent frame $\{X_a\}$. Summation convention over repeated indices will be used throughout the paper. Juxtaposing $e^\mu$ and $S_X^\mu$, or any other operator, or form will denote the Clifford multiplication. When acting on forms $S_X$ and $\mathcal{D}$ will be denoted, in terms of the pseudo-Riemannian connection $\nabla$, as $\nabla_X$ and $\nabla^\mu = e^\mu \nabla_X$.

$$d = e^\mu \nabla_X \quad \text{and} \quad \delta = -i_X \nabla_X$$

denote the exterior derivative and the co-derivative written in terms of covariant derivative operator $\nabla_X$. $\wedge$ is the exterior product and $i_X$ will represent the interior derivative with respect to $X$ whose action on an arbitrary p-form $\alpha$ is defined, for all vector fields $Y_j$, by

$$(i_X \alpha)(Y_1, \ldots, Y_{p-1}) = p\alpha(X, Y_1, \ldots, Y_{p-1}).$$

For dual basis elements we have $i_X e^\mu = e^\mu (X_b) = \delta_b^\mu$.

The rest of the paper is organized as follows. In section 2, the general form of the first-order symmetry operator of the Dirac equation with a potential term is specified. This is achieved by constructing and then by solving all consistency equations except the equation for the 0-form component of the non-derivative term of the symmetry operator. The unified dynamical symmetry condition announced in the title is established in section 3 by analyzing higher degree components of the consistency equations. Special cases of this condition and the integration of the remaining 0-form component are also presented there. In section 4, the correspondence between KY and closed CKY forms are studied and Yano vectors are introduced. Implications of the existence of Yano vectors related to symmetry analysis and to the global structure of the underlying spacetime are given in the same section. In section 5, the first integrals of geodesic equations, constants of motion of classical trajectories and their connection with the KY forms and the mentioned dynamical symmetry condition are considered. Derivation of consistency equations and the contraction of curvature 2-forms with a Yano vector are given in the two appendices. In appendix B determination of upper bounds for the numbers of linearly independent KY forms is also provided. Section 6 concludes the paper.

2. First order symmetry operators of the Dirac equation

We set out to our analysis by considering the Dirac equation

$$(\mathcal{D} + iA)\psi = m\psi$$

for a complex spinor field $\psi$ and propose the following first-order linear symmetry operator

$$L = 2\omega^\mu S_X^\mu + \Omega,$$

such that $L$ Clifford commutes with $\mathcal{D} + iA$. Equation (1) describes the motion of a massive, charged and spin-1/2 particle, with unit charge and mass $m$, interacting with the curved
background encoded in the Dirac operator $\mathcal{S}$ and the force field represented by the potential form $A$. By fixing the charge from the outset, we assume the effective coupling to the spinor field. In the beginning $A$ is allowed to be an arbitrary inhomogeneous form, and the consistency conditions are derived in this general context. Later on, $A$ will be taken to be a 1-form. In that case, the term involving $A$ in equation (1) describes coupling to the Maxwell field $F = dA$. $A$ may also involve potential terms of non-electromagnetic origin such as those of conservative forces. In the latter case, $F$ will be referred to as a force field. In equation (2) $\omega^a$ and $\Omega$ are $\mathbb{Z}_2$-homogeneous, both even or both odd forms which act, like $A$, on spinor fields by the Clifford multiplication.

In recent studies [14–16] it has been proved that use of the graded Clifford commutator $[,]$ considerably eases the calculations in analyzing the symmetries of the Dirac-type operators. For a $p$-form $\alpha$ and an inhomogeneous Clifford form $\beta = \sum_q \beta(q)$, this commutator is defined as

$$[\alpha, \beta] = \alpha \beta - \sum_q (-1)^{pq} \beta(q) \alpha. \tag{4}$$

If $\alpha$ is a one-form this transforms, in terms of the main involution $\eta$ (which leaves the even forms invariant and changes the sign of the odd forms) of the Clifford algebra, to $[\alpha, \beta] = \alpha \beta - \beta \eta \alpha$. Let us suppose that

$$[\mathcal{S}, L] + i[A, L] = 0, \tag{3}$$

is satisfied and let us call $L$ even (odd) when $\omega^a$ and $\Omega$ are both even (both odd). When $L$ is even its graded commutator with the Dirac equation becomes the usual Clifford commutator $[,]$ and irrespective of $n$, $L$ is a symmetry operator. That is, it Clifford commutes with the Dirac equation and maps a solution to another. When $L$ is odd it anti-commutes with the Dirac equation and fails to be a symmetry operator. However, in such a case if $n$ is even $L_z$ is a symmetry operator since the volume form $z$ anti-commutes with odd forms. (In such a case $A$ must be an odd form.)

2.1. The main consistency equations

To determine $\omega^a$’s and $\Omega$ in equation (2), we equate the symmetrized coefficients of the covariant derivatives of each order to zero in equation (3). The second-order derivatives come only from the first bracket of (3) such that

$$[\mathcal{S}, L] = 2i_X \omega^a [S^2(X_a, X^b) + S^2(X_b, X_a)] + \cdots, \tag{4}$$

where $i_X$ is the interior derivative,

$$S^2(X, Y) = S_X S_Y - S_{[X,Y]},$$

denotes the second covariant derivative and the ellipsis stands for lower order terms. Equating the coefficients of the second-order derivatives to zero in equation (3) yields

$$i_X \omega^a + i_X \omega_b = 0, \tag{5}$$

for all $a, b = 1, 2, \ldots, n$. This is satisfied if and only if $\omega^a = i_X \omega$ where $\omega$ is possibly a $\mathbb{Z}$-inhomogeneous form. It has been shown in appendix A that, in view of (5), by equating the coefficients of equal power of derivatives in equation (3) we obtain

$$\nabla^X \omega = i_X \varphi - i[A, i_X \omega], \tag{6}$$

$$\nabla^2(X_a, X^b)\omega = \varphi + 2i(i_X \omega)^h \nabla^X A - i[A, \Omega]. \tag{7}$$
where $\varphi = d\omega - \Omega$. These constitute two main sets of the consistency conditions that specify the form of possible symmetry operators of the Dirac equation, for all dimensions and signatures, in which $A$ is a general form. Henceforth, we take $A$ to be a 1-form and write, for an arbitrary form $\alpha$,

$$[A, \alpha] = 2i \hat{A}\alpha.$$  

Here $\hat{A}$ is the metric dual of $A$, which in terms of the metric $g$ of the background is defined by $A(X) = g(\hat{A}, X)$ for all $X$.

2.2. Solutions: emergence of KY forms

Let us first concentrate on equation (6), which for 1-form $A$ can be rewritten as

$$\nabla X a \omega = i X a (\varphi + 2i i \hat{A}\omega).$$  

(8)

Applying $i X a$ to both sides of this equation, we first see that $\delta \omega = 0$, that is, $\omega$ must be co-closed. On the other hand, by applying $e_a \wedge$ we obtain

$$d \omega = \pi (\varphi + 2i i \hat{A}\omega),$$

(9)

where the linear map $\pi$ scales each form component by its degree:

$$\pi(\alpha) = e_a \wedge i X a \alpha.$$  

The $(p+1)$-form component of equation (9) reads, for $p = 0, 1, \ldots, n - 2$, as

$$\varphi_{(p+1)} = \frac{1}{p+1} d\omega_{(p)} - 2i i \hat{A}\omega_{(p+2)},$$

(10)

and for $p = n - 1$, as $\varphi_{(n)} = n^{-1} d\omega_{(n-1)}$. In view of the last two equations, equation (8) implies that each $p$-form component of $\omega$ must obey

$$\nabla X a \omega_{(p)} = \frac{1}{p+1} i X a d\omega_{(p)},$$

(11)

which is the well-known KY equation.

From equation (10) and by the fact that $\omega$ is co-closed we also obtain

$$\Omega_{(p+1)} = \frac{p}{p+1} d\omega_{(p)} + 2i i \hat{A}\omega_{(p+2)},$$

(12)

$$\Omega_{(n)} = \left(1 - \frac{1}{n}\right) d\omega_{(n-1)},$$

(13)

for $p = 0, 1, \ldots, n - 2$. Note that for $p = 0, 1, \ldots, n - 1$ we have

$$i \hat{A}\Omega_{(p+1)} = p \nabla i \hat{A}\omega_{(p)}.$$  

(14)

These provide the non-derivative term, except for its 0-form component, of the symmetry operator in terms of KY forms and the potential.

We now turn to the implications of equation (7). Differentiating equation (8) once more, we obtain

$$\nabla^2 (X a, X a) \omega = -\delta (\varphi + 2i i \hat{A}\omega),$$

(15)

and by combining this with equation (7), we arrive at

$$i \hat{A}\Omega - \frac{i}{2} d\Omega = \delta (i \hat{A}\omega) + (i X a \omega) \nabla X a A.$$  

(16)

This can be used to obtain the 0-form component of $\Omega$ and possible relations among its higher degree components. For this purpose we apply the general relation

$$[\delta, i X] = -i X a \hat{A}X a.$$
to KY forms and obtain

$$\delta(i_A \omega) = -i_X^* i_{\nabla X} A \omega. \tag{17}$$

We now make use of

$$(i_X \omega)^\kappa_a = \kappa_a \wedge i_X^* \omega - i_{\kappa_a} i_X^* \omega, \tag{18}$$

where $\kappa_a$‘s are 1-forms. This relation is a direct result of the standard Clifford multiplication rule for a right multiplication of an arbitrary form by a 1-form. Using (18), by taking $\kappa_a = \nabla X_a A$, and (17) in (16) we obtain

$$i_A \Omega - \frac{1}{2} i d\Omega = (\nabla X_a A) \wedge i_X^* \omega. \tag{19}$$

The 1-form component of this relation reads as

$$d\Omega_{(0)} = -2 i(i_A \Omega_{(2)} - i_X^* \omega_{(1)} \nabla X_a A),$$

$$= -2 i(\nabla \omega_{(1)} - \nabla \omega_{a}). \tag{20}$$

In obtaining the second equality we made use of (14). Equation (20) will be integrated in the next section after the higher degree components of equation (19) are further analyzed to uncover the symmetry condition stated in the title.

3. A unified dynamical symmetry condition

In this section we first show that equation (16) contains an important algebraic condition which plays a prominent role in deciding which KY forms can take part in the symmetry operator. To see this, we first recall that $\Omega = d\omega - \varphi$ which implies

$$d\Omega = -d\varphi = 2 i d i_A \omega, \quad i_A \Omega = i_A d\omega - i_A \varphi = \pi i_A \varphi. \tag{21}$$

In the last equality we made use of (9). On substituting these relations into (16), we obtain

$$\pi (i_A \varphi) = -d i_A \omega + (i_X^* \omega)^\kappa_a \nabla X_a A. \tag{22}$$

We now make use of

$$[\nabla, i_X^*]_+ = \nabla X + e^a i_{\nabla X_a X}, \tag{23}$$

where $[\cdot, \cdot]_+$ denotes the Clifford anti-commutator. Using (18), once again by taking $\kappa_a = \nabla X_a A$, and (22) in equation (21) we arrive at

$$\pi (i_A \varphi) - i_A d\omega + \nabla \omega = -\frac{1}{2} [dA, \omega]_-, \tag{24}$$

where we have also used the following relation:

$$\kappa_a \wedge i_X^* - e^a \wedge i_{\kappa_a} \omega = -\frac{1}{2} [dA, \omega]_-. \tag{25}$$

It is now easy to verify that, by the KY-equation (11) and by the equation (10), the left-hand side of (23) vanishes and we obtain an important condition $[dA, \omega]_- = 0$. On the other hand, as is apparent in equation (24), since the Clifford commutator of an arbitrary form by a 2-form does not change the degrees of its components, each p-form component of $\omega$ must Clifford commute with the force field $F = dA$:

$$[F, \omega_{(p)}]_- = 0. \tag{26}$$

This gauge invariant condition means that among the KY forms admitted by the underlying spacetime, only those which satisfy the above condition take part in the symmetry operators. Moreover, it must be emphasized that it is the force-field, not the potential form itself, that plays a selective role in this regard.
At this point we should note that, in view of equations (10), (14) and equation (21) derived above, the higher degree components of equation (19) lead us again to condition (25) and yield nothing new.

In order to integrate equation (20) as well as to obtain more practical statements resulting from equation (25), we rewrite it as

\[ i_X F \wedge i_X \omega_{(p)} = 0. \]  

(26)

This is obviously satisfied for \( p = 0 \) and also for \( p = n \). Hence the extreme cases \( p = 0, n \) do not impose any condition on \( F \) and on possible KY forms that take part in the symmetry operator \( L \). In general, a KY \( p \)-form \( \omega_{(p)} \) takes part in the symmetry operator if and only if \( F \) is in the kernel of the operator \( i_X \omega_{(p)} \wedge i_X \). More practical refinements of this condition are attained for intermediate values of \( p \). We first should note that the 0-form component \( \omega_{(0)} \) of a KY form can be any function and \( \omega_{(1)} \) is the dual of a Killing vector field. Moreover, \( \omega_{(n)} \) is a constant (parallel), that is, it is a constant multiple of the volume form: \( \omega_{(n)} = k \). For \( p = 1 \) we can write \( \omega_{(1)} = \tilde{K} \), where \( K \) is a Killing vector field. In this case, equation (26) reduces to

\[ i_K F = 0, \]  

(27)

which implies that \( F \) remains invariant under the flow generated by \( K \). That is, \( L_K F = 0 \), where \( L_K \) denotes the Lie derivative in the direction of \( K \). In other words, even the generator of the isometries will take part in the symmetry operators if they fulfil condition (27).

In terms of the potential 1-form \( A \) the condition (27) reads as \( i_K (e^a \wedge \nabla_X A) = 0 \) which, in view of \( i_\nabla_X K A = i_\tilde{A} \nabla_X \omega_{(1)} \), \( \nabla \omega_{(1)} \), can be evaluated to obtain

\[ 0 = \nabla K A - di_KA - \nabla_\tilde{A} \omega_{(1)}. \]

By comparing this relation with equation (20) and then by integrating we obtain

\[ \Omega_0 = 2i_K A, \]  

(28)

up to a constant. This relation, which was not recognized in the literature before, provides a concise way of choosing the gauge in order to make \( \Omega_0 \) constant. In such a case, only duals of Killing vector fields whose flows preserve the potential can appear in the symmetry operator. An equivalent but more instructive way of deriving (28) goes as follows. Recalling the fact that for a 1-form \( \beta \) the metric dual of \( \nabla_X \beta \) is \( \nabla_X \tilde{\beta} \), we can rewrite condition (20) as

\[ d\tilde{\omega}_{(0)} = -2i[\tilde{A}, K]_L = 2i L_K \tilde{A}. \]

Here \([,]_L \) denotes the usual Lie bracket of vector fields, and we have also used the zero torsion condition. Since \( K \) is a Killing field, the above relation simply reads as \( d\Omega_{(0)} = 2i L_K A \) and then the usual action of Lie derivative on differential forms produces the desired relation in view of (27).

For a KY 2-form \( \omega_{(2)} = 2^{-1} \omega_{ab} e^a \wedge e^b \) the condition (26) reads as \( \omega_{ab} F^a_{\ c} = 0 \) for all values of \( b \) and \( c \) where the square bracket stands for the anti-symmetrization of the enclosed indices and \( F^a_{\ bc} \) are the components of \( F \) in the same \( \{e^a\} \) basis. This condition is the tensor-language version [12] of the condition first obtained by Hughston et al [3]. Our condition (25), or equivalently (26), are natural generalizations of this and similar conditions. This reflects the efficiency of the Clifford calculus in this context.

To discuss condition (26) for higher forms it is convenient to rewrite it, in terms of \((p + 1)\)-forms defined by

\[ \beta_a = F \wedge i_X \omega_{(p)}, \]

as \( i_X \beta_a = 0 \). This is also satisfied for \( p = 0, n \). In the latter case \( \beta_a = 0 \) are identically satisfied, since all \( \beta_a \)'s are \((n + 1)\)-forms. For \( p = n - 1 \) we have \( \beta_a = f_a \gamma \) for some set of
functions \( f_a \) such that equation (26) amounts to \( i_V z = 0 \) for \( V = f_a X^a \). As the map \( \varphi_z \) defined by \( \varphi_z(V) = i_V z \) between the vector fields and \( (n-1) \)-forms is an isomorphism, \( i_V z \) vanishes if and only if \( V = 0 \), that is if and only if all \( f_a \) vanish. This is equivalent to \( F \wedge i_{X_a} \omega(n-1) = 0 \) for all \( X_a \), or to

\[
i_{X_a} F \wedge \omega(n-1) = 0.
\]

(29)

In the case of \( p = n - 2 \), \( \beta_a \)'s are \( (n-1) \)-forms and each one can be written, in terms of uniquely determined 1-form \( \sigma_a = \sigma_{ab} e^b \), as \( \beta_a = \ast \sigma_a \). In such a case equation (26) amounts to \( \sigma_a \wedge \tilde{X}^a = 0 \), which implies that \( \sigma_{ab} = \sigma_{ba} \). This can succinctly be expressed as

\[
F \wedge (\tilde{X}_a \wedge i_{X_b} \wedge \tilde{X}_b) \omega(n-2) = 0,
\]

for \( a, b = 1, \ldots, n \). Note that the considered cases \( p = 0, 1, n - 2, n - 1, n \) exhaust all possible forms of equation (25) in four dimensions. More appealing versions of some of these conditions will appear in the next section (see equation (34) and the remarks followed).

Finally in this section we should note that, when \( A \) is zero we recover exactly the results of Benn and Kress for all dimensions and signatures, obtained in [14–16]. In such a case, since condition (25) disappears, all KY forms of the underlying spacetime take part in the symmetry operators and all of the non-derivative terms explicitly given by equations (12) and (13), except \( \Omega_0 \) which is constant, are exact.

4. Correspondence between KY forms and closed CKY forms

The appearance of higher rank KY forms in the symmetry operator indicates the presence of dynamical symmetries which are not isometries. We shall now establish a general one-to-one correspondence between the KY forms and closed CKY forms and, as a particular case, we shall show that for each KY \((n-1)\)-form there exists a uniquely determined conformal transformation. This transformation is generated by a locally gradient field whose integral curves are pre-geodesics. In this way we shall see that the higher KY forms are related to special CKY forms and, in the case of \((n-1)\)-forms, to the special conformal transformations rather than isometry transformations.

For the two points mentioned above, let us consider the CKY-equation

\[
\nabla_X \rho(p) = \frac{1}{p+1} i_X d\rho(p) - \frac{1}{n-p+1} \tilde{X} \wedge \delta \rho(p),
\]

(30)

which has the well-established Hodge duality invariance and conformal covariance. A \( p \)-form \( \rho(p) \) is called CKY \( p \)-form if and only if it satisfies equation (30) for all vector fields \( X \). It is obvious that a \( p \)-form is a KY \( p \)-form if and only if it satisfies equation (30) for all vector fields \( X \). It is obvious that a \( p \)-form is a KY \( p \)-form if and only if it is a co-closed CKY \( p \)-form. A less obvious fact is that a \((n-p)\)-form is a KY \((n-p)\)-form if and only if it is the Hodge dual of a closed CKY \( p \)-form. Indeed, the Hodge dual of (30) for closed CKY forms yields

\[
\nabla_{X_a} \ast \rho(p) = \frac{1}{n-p+1} i_{X_a} d^* \rho(p),
\]

where we have made use of the relation \( i_X \ast \phi = \ast (\phi \wedge \tilde{X}) \) which can be considered as the definition of the Hodge map. Since \( \rho(p) \) is closed it is locally exact such that \( \rho(p) = d\alpha \), where the \((p-1)\)-form \( \alpha \) may be termed as the KY-potential for the KY \((n-p)\)-form \( \omega(n-p) = \ast d\alpha \). Note that 0-forms of KY and CKY coincide and that any CKY \( n \)-form is of the form \( \rho(0) = f z \), where \( f \) can be any differentiable function. As a result, the Hodge map establishes a vector space isomorphism between the vector space of all KY \( p \)-forms and that of all closed CKY \((n-p)\)-forms for all values of \( p \).
4.1. KY \((n-1)\)-forms and Yano vectors

In the case of KY \((n-1)\)-forms we can write
\[
\omega_{(n-1)} = i_Y z = \ast \tilde{\omega},
\]
where the vector field \(Y\) is the metric dual of the associated closed CKY 1-form \(\tilde{\omega}\). \(Y\) will be referred to, following McLenaghan and Spindel, as the Yano vector. In that case, connection with the conformal transformations can be established in a more instructive manner. As the space of \((n-1)\)-forms and that of 1-forms are isomorphic, exactly as in (31), any form of the former type can be written as the Hodge dual of a uniquely determined latter type.

Recalling that the divergence of a vector field \(U\) with respect to the volume form \(z\) is defined by
\[
\nabla_X U = (\text{div}_z U) z,
\]
we can write
\[
\nabla_X \omega_{(n-1)} = \frac{1}{n} (\text{div}_z \omega) Y^a.
\]

On the other hand, for an arbitrary vector field \(U\), \(\nabla_X U\) can be decomposed as follows:
\[
\nabla_X U = \frac{1}{2} i_X dU - \frac{1}{n} \tilde{X} \delta \tilde{U} + \Gamma_X (U),
\]
(33)

(see [18] section 6.13 and [7]) where the 1-form \(\Gamma_X (U)\) is defined by
\[
\Gamma_X (U) = \frac{1}{2} \left[ (L_U g) - \frac{1}{n} \text{tr} (L_U g) g \right] (X).
\]
Thus, for each vector field \(V\) corresponding to a KY \((n-1)\)-form we have, in view of (32),
\[
i_X d\tilde{V} = -2 \Gamma_X (V).
\]
That is, \(V\) is conformal if and only if \(\tilde{V}\) is closed.

Let us return to equation (31): \(\tilde{\omega}\) is closed, obeys the relation \(\delta \tilde{\omega} = -\text{div}_z \tilde{\omega}\) and generates conformal transformations \(L_Y g = 2\lambda g\) with the conformal weight
\[
\lambda = \frac{1}{2n} \text{tr} (L_Y g) = \frac{1}{2n} (L_Y g)(X^a, X_a).
\]

\(Y\) is locally a gradient field for \(\tilde{\omega}\) is closed. From (32), it also follows that
\[
\nabla_Y Y = \frac{1}{n} (\text{div}_z Y) Y,
\]
that is, the integral curves of \(Y\) are pre-geodesics and may be reparameterized to become geodesics. If \(Y\) is divergence-free, then it is covariantly constant. We should note that in some literature, a curve whose velocity vector \(\dot{\gamma}\) satisfies \(\nabla \dot{\gamma} = f \dot{\gamma}\), for some function \(f\), is termed geodesic, which we here call pre-geodesic by adhering to the nomenclature of [18, 19]. For the Yano vector we have \(f = \text{div}_z Y / n\).

Substitution of (31) into (29) yields
\[
i_X F \wedge \ast \tilde{\omega} = 0,
\]
which in terms of components can be rewritten as
\[
F_{ab} Y^b = 0.
\]
(34)

Note that condition (27) can also be rewritten as \(F_{ab} K^b = 0\), where \(K^b\)’s are the components of the Killing vector \(K\). Thus, KY \((n-1)\)-forms and 1-forms take part in the symmetry operator if and only if the corresponding Yano and Killing vectors are contained in the kernel of the matrix \(F_{ab}\). These imply that if \(F_{ab}\) is non-singular, which can happen only in even dimensions, no KY \((n-1)\)-form and 1-form can take part in the symmetry operator. Our condition for KY 2-forms can also be stated as follows: a KY 2-form will take part in the symmetry operator if and only if the corresponding anti-symmetric matrix \((\omega_{ab})\) commutes with \(F_{ab}\).
4.2. Contractions with a Yano vector

Contractions of curvature 2-forms \( R_{ab} \), Ricci 1-forms \( P_a \), and conformal (Weyl) 2-forms \( C_{ab} \), defined for \( n > 3 \) by (below \( R \) denotes the scalar curvature)

\[
C_{ab} = R_{ab} - \frac{1}{n-2} \left( P_a \wedge e_b - P_b \wedge e_a - \frac{R}{n-1} e_a \wedge e_b \right),
\]

with a Yano vector enable one to reach decisive statements about the global structures of the underlying spacetimes. The mentioned contractions are found to be as follows:

\[
i_Y R_{ab} = \frac{1}{n-1} \left[ (i_Y P_a) e_b - (i_Y P_b) e_a \right]
\]

\[
Y b i_Y P_a = Y a i_Y P_b,
\]

\[
i_Y C_{ab} = \frac{1}{(n-1)(n-2)} \left[ (R g_{ac} - P_{ac}) e_b - (R g_{bc} - P_{bc}) e_a \right] Y c + \frac{1}{n-2} (P_a Y_b - P_b Y_a),
\]

where \( P_{ac} \) are the components of the Ricci tensor. The second relation easily follows from a second application of \( i_Y \) to the first relation and the last one also follows from the definition of \( C_{ab} \) and the first relation of (35). A derivation of the first relation is given in appendix B.

Equations (35) are generalizations in the language of differential forms to an arbitrary number of dimensions and signature of the tensorial relations first found by McLenaghan and Spindel in the case of four-dimensional Lorentzian spacetimes.

If \( P_{ab} = R g_{ab} / n \) such that \( R \) is constant, that is in Einstein spaces, we have \( i_Y C_{ab} = 0 \). In four dimensions, the existence of a Yano vector on an Einstein space implies that the space is conformally flat or of Petrov type \( N \). In higher dimensions, the equation \( i_Y C_{ab} = 0 \) for nonzero \( C_{ab} \) is a necessary but not a sufficient condition for being Petrov type \( N \). In such a case the spacetime can also be of Petrov type \( III \) [22]. When \( Y \) is non-null we can write the second equation of (35) as \( i_Y P_a = \lambda Y_a \), where \( \lambda = Y^b i_Y P_b / g(Y, Y) \). In such a case the Yano vector is an eigenvector of \( P_{ab} \) with eigenvalue \( \lambda = R / n \). When \( Y \) is null, it is also null with respect to the Ricci tensor, in the sense that \( Y a i_Y P_a = 0 \), in the directions of non-zero components of the Yano vector. In other words, in such directions contraction of Ricci forms with a null Yano vector projects it to an orthogonal direction.

5. Quadratic geodesic invariants and constants of motion

The most important physical implication of the existence of KY forms is the fact that they provide first integrals of the geodesic equation and may lead to quadratic functions of momenta that are invariant also for the classical trajectories. As is shown in the subsection B below, condition (26) plays a crucial and unified role in establishment of the second point.

5.1. First integrals of geodesic equations

The easiest way to see the first point mentioned above is to consider the relation \( [\nabla_X, i_Y] = i_{\nabla_X Y} \) which holds for two arbitrary vector fields \( X, Y \). When \( X \) and \( Y \) are equal to a velocity field \( \dot{\gamma} \) of a geodesic, that is \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \), the covariant and interior derivatives commute and we obtain, in view of the defining relation (11), \( \nabla_{\dot{\gamma}} i_{\dot{\gamma}} \omega = 0 \) for any KY \((p+1)\)-form \( \omega \). Therefore, the \( p \)-form \( \alpha = i_{\dot{\gamma}} \omega \) and hence its ‘length’ \( |\alpha|^2 \) defined by \( |\alpha|^2 = g_p(\alpha, \alpha) \) remain constant along the geodesic \( \dot{\gamma} \). Here \( g_p \) is the compatible metric induced by \( g \) in the space of \( p \)-forms and we assume \( g_0 \) to simply multiply the two 0-forms (see [18], chapter 1).

The constancy of \( |\alpha|^2 \) can be considered as a special case of a more general fact [20]. To see this, let us consider the symmetric bilinear form

\[
K^{(\beta)}(X, Y) = g_p(i_X \beta, i_Y \beta) = \epsilon^*(i_X \beta \wedge ^*i_Y \beta),
\]

(36)
defined, in terms of a \((p + 1)\)-form \(\beta\) on the Cartesian product of the space of vector fields. Here \(\epsilon\) denotes the sign of the determinant of \(g\). In the case of a KY \((p + 1)\)-form \(\omega\), one can easily verify the cyclic identity:

\[
\nabla_X K^{(\omega)}(Y, Z) + \nabla_Y K^{(\omega)}(Z, X) + \nabla_Z K^{(\omega)}(X, Y) = 0.
\]

That is, the symmetrized covariant derivatives of \(K^{(\omega)}\) vanish. This shows that to any KY \((p + 1)\)-form \(\omega\), one can easily verify the cyclic identity:

\[
\nabla_X K^{(\omega)}(Y, Z) + \nabla_Y K^{(\omega)}(Z, X) + \nabla_Z K^{(\omega)}(X, Y) = 0.
\]

That is, the symmetrized covariant derivatives of \(K^{(\omega)}\) vanish. This shows that to any KY \((p + 1)\)-form \(\omega\) is associated a symmetric bilinear form \(K^{(\omega)}\) which is the Killing tensor generalizing the so-called Stackel–Killing tensor that corresponds to a KY 2-form, first recognized by Penrose and Floyd (the second tensor of (38) below). Since

\[
\nabla_X [K^{(\omega)}(X, X)] = 2K^{(\omega)}(\nabla_X X, X),
\]

(37)

\(K^{(\omega)}(\dot{\gamma}, \dot{\gamma})\) is constant along the geodesic \(\gamma\).

For KY 1-form \(\omega(1) = K_{ae}\), 2-form \(\omega(2) = 2^{-1} \omega_{abc} e^{ab}\) and n-form \(\omega(n) = k\), where \(k\) is a constant, the components of the corresponding Killing tensor are found, respectively, to be

\[
K_{ab} = g_{n-2}(i_{X_a} \tilde{\omega}^b, i_{X_b} \tilde{\omega}^a) = \epsilon g_{2}(\tilde{\omega}^a \wedge \tilde{\omega}^b),
\]

(39)

This last Killing tensor, in particular, has remarkable properties worth mentioning. By definition \(K^{(\omega)}(Y, Y) = 0\), that is, Yano vectors are null with respect to the associated Killing tensor. When \(Y\) is non-null (that is, \(g(Y, Y) \neq 0\)), \(K^{(\omega)}\) has a one-dimensional kernel spanned by \(Y\), and therefore it is singular. In that case every vector orthogonal to \(Y\) is an eigenvector with the eigenvalue \(\epsilon g(Y, Y)\) and the trace of \(K^{(\omega)}\) is \(\epsilon(n - 1)g(Y, Y)\).

Moreover, the properly normalized Killing tensor \(K' = K^{(\omega)} / \epsilon g(Y, Y)\) is an idempotent projector \(K_{ab} K'_{bc} = K'_{ac}\) having rank \(n - 1\). When \(Y\) is null we have \(K^{(\omega)}(Y, Y) = -\epsilon Y_a Y_b\) and in such a case, it is a rank-1 nilpotent \((K^{(\omega)}(Y, Y) K^{(\omega)}(Y, b) = 0)\) projector, projecting vector fields to the direction of the Yano vector.

### 5.2. Constants of motion for classical trajectories

It is known that any rank-\(p\) symmetric (covariant) Killing tensor provides a degree-\(p\) polynomial of velocity that is a geodesic invariant [19]. But, the second rank symmetric Killing tensors that can be constructed from KY forms, as in (36), have a distinguished property of providing a quadratic invariant of velocity along classical trajectories. In the remaining part of this section we shall prove this statement in the most general setting of this paper. To be precise, let us consider the quadratic function

\[
f^{(\omega)} = K^{(\omega)}(u, u) = K_{ab} u^a u^b,
\]

(40)

where \(u = \dot{C}\) is the world velocity of a charged (charge assumed to be unit) material particle obeying the classical equation of motion

\[
\nabla_u u = \frac{1}{m} i_u F. \tag{41}
\]

Here, \(u^a = dx^a / d\tau\) such that \(x^a\)’s are the local coordinates of the world curve \(C\) parameterized by the proper time \(\tau\) and \(\nabla_u u\) represents the acceleration of the particle. Hence

\[
\frac{d}{d\tau} (f^{(\omega)} \circ C) = C_\tau (\partial_\tau) f^{(\omega)} = \nabla_u [K^{(\omega)}(u, u)]
\]

\[
= 2K^{(\omega)}(\nabla_u u, u), \tag{42}
\]

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and in view of (41)
\[
\frac{d}{d\tau} (f^{(\omega)} \circ C) = \frac{2}{m} a^a a^b F_{ac} K_b^{(\omega)c}.
\] (43)

What we are going to prove is the constancy of \( f^{(\omega)} \) along the classical trajectory determined by (41). Before doing that, it would be illuminating to examine first some special cases. One can easily verify that the right-hand side of (43) vanishes if the components of the Killing tensor given by (38) are used, provided that KY forms employed in defining \( f^{(\omega)} \) satisfy condition (26). In the case of \( K^{(\ast Y)} \) given by (39), the contraction with the Maxwell field is, in view of (34)
\[
F_{ac} K_b^{(\ast Y)c} = \epsilon g(Y, Y) F_{ab},
\] (44)
which is obviously anti-symmetric and also makes the right-hand side of (43) vanish. So, as a particular result, all quadratic functions constructed from KY \( p \)-forms for \( p = 1, 2, n-1, n \) are constant along the classical trajectory. These exhaust all possible cases in a four-dimensional spacetime and, to the best of our knowledge, these are all that can be found in the related literature.

We shall now prove that \( f^{(\omega)} \) is a constant of motion for the classical trajectories determined by (41) for all dimensions and signatures as well as for all KY \( p \)-forms \( \omega(p) \) obeying the symmetry condition (26). For the proof, we first take the Hodge dual of both sides of equation (43) and write
\[
\frac{d}{d\tau} (f^{(\omega)} \circ C)^* \mathbf{1} = \frac{2}{m} a^a a^b I_{ab},
\] (45)
where the \( n \)-form \( I_{ab} \) is defined as
\[
I_{ab} = \ast \left( F_{ac} K_b^{(\omega)c} \right) = i_{X^a} i_{X^b} F_i X^c \omega \wedge \ast i_{X^b} \omega.
\] (46)
In passing to the second equality of equation (46) we have used (36). Thus, \( f^{(\omega)} \) is constant along the world line \( C(\tau) \) if and only if the right-hand side of equation (45) vanishes. Evidently, the anti-symmetry condition \( I_{ab} = -I_{ba} \) is sufficient (and can also be shown to be necessary) for the right-hand side of (45) to vanish. To show the anti-symmetry of \( I_{ab} \) we rewrite it as follows:
\[
I_{ab} = -\left[ i_{X^a} \left( i_{X^c} F \wedge i_{X^c} \omega \right) + i_{X^c} F \wedge i_{X^a} i_{X^c} \omega \right] \wedge \ast i_{X^b} \omega,
\]
\[
= i_{X^c} F \wedge i_{X^b} \left( i_{X^a} \omega \wedge \ast i_{X^b} \omega \right) - i_{X^a} \omega \wedge i_{X^c} i_{X^b} \omega,
\]
\[
= -i_{X^a} \omega \wedge i_{X^c} F \wedge \ast \left( i_{X^b} \omega \wedge e^c \right).
\] (47)
The first term in the square bracket of the first line vanishes because of condition (26), and the first term of the second line vanishes since it can be written as an interior derivative of a \((n+1)\)-form. In the third line of (47) we have made use of the identity \( i_{X^c} \ast i_{X^b} \omega = \ast \left( i_{X^b} \omega \wedge e^c \right) \) which was also used in the previous section. In view of \( i_{X^c} F = F_{i\alpha} e^\alpha \) and of another Hodge identity \( \alpha \wedge \ast \beta = \beta \wedge \ast \alpha \) which holds for any two \( p \)-forms \( \alpha \) and \( \beta \), we obtain from (47)
\[
I_{ab} = -i_{X^a} \omega \wedge F_{i\alpha} e^\alpha \wedge \ast \left( i_{X^b} \omega \wedge e^\alpha \right)
\]
\[
= i_{X^a} \omega \wedge i_{X^c} F \wedge \ast \left( i_{X^b} \omega \wedge e^c \right).
\] (48)
By comparing this with the third line of (47) we obtain \( I_{ab} = -I_{ab} \) which was what to be demonstrated.

An immediate corollary of the constancy of \( f^{(\omega)} \) is, by virtue of (42), the relation \( K^{(\omega)}(\nabla_u u, u) = 0 \) which can be interpreted as follows. The world velocity and acceleration of a charged material particle are perpendicular to each other, not only with respect to the metric, but also with respect to the symmetric Killing tensors associated with each KY form satisfying condition (26).
6. Summary and conclusion

In this study, the most general first-order linear symmetry operators of the Dirac equation including interaction with the Maxwell field in a curved background of arbitrary $n$-dimension and of signature are specified. We have shown that all coefficients forms $\omega^a$'s of the symmetry operator $L = 2\omega^a S_{X^a} + \Omega$ are given, in terms of an inhomogeneous KY form $\omega$, by $\omega^a = i X^a \omega$. The components of $\Omega$ are explicitly calculated and are given by equations (12), (13) and (28). They depend on the exterior derivative of KY forms and their contraction with the potential field $A$. We have also found a unified, gauge invariant dynamical symmetry condition which states that among all the KY forms of the underlying curved background only those which Clifford commute with the Maxwell field can take part in $L$. When $\omega^a$ and $\Omega$ are even, $L$ itself, but when they are odd and $n$ is even $L_z$ is a first-order symmetry operator which Clifford commutes with the Dirac equation and hence maps a solution to another.

The special cases of the dynamical symmetry condition are also discussed as they may provide valuable insights in applications. In particular, the KY $(n-1)$-forms and 1-forms take part in the symmetry operator if and only if the corresponding Yano and Killing vectors belong to the kernel of the anti-symmetric matrix $F_{ab}$ corresponding to the components of the Maxwell field. These imply that if $F_{ab}$ is non-singular, which can happen only in even dimensions, no KY $(n-1)$-forms and 1-forms can take part in the symmetry operator. Implications of the existence of Yano vectors in specifying the global structure of the curved background are also discussed. Finally it has been proved that, for all KY forms obeying the dynamical symmetry condition, there exists a quadratic function of velocity (defined by the equation (40)) which is a constant of motion for the classical motion in an arbitrary dimension and signature.

All of these results are expected to provide a unified framework for symmetry analysis and to serve as a firm base in studying symmetry algebras and related conserved quantities of the Dirac equation for a given specific curved background and force field.

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Appendix A. Consistency conditions

By direct calculations we obtain

$$[\mathcal{S}, \omega] = 2\omega^a S_{X^a} + d\omega = L + \varphi,$$

where $\omega^a = i X^a \omega$. It proves convenient to take $L$ as $L = [\mathcal{S}, \omega] - \varphi$, where $\Omega = d\omega - \varphi$. Using the fact that

$$[\mathcal{S}, [\mathcal{S}, \omega]] = [\mathcal{S}^2, \omega],$$

the symmetry condition (3) transforms to

$$[\mathcal{S}^2, \omega] = [\mathcal{S}, \varphi] - i[A, L]. \quad (A.1)$$

$\mathcal{R}$ being the scalar curvature of the spinor connection, $\mathcal{S}^2$ acts on the spinor fields as (see [18] chapter 10)

$$\mathcal{S}^2 = S^2(X_a, X^a) - \frac{1}{2} \mathcal{R}. \quad (A.2)$$
Therefore on substituting the relations
\[ [\mathcal{S}^2, \omega] = [\mathcal{S}^2(X_a, X^a), \omega] = 2\nabla_X\omega S_X + \nabla^2(X_a, X^a)\omega, \]
\[ [\mathcal{S}, \varphi] = 2i_X\varphi S_X + \varphi, \]
\[ [A, L] = 2[A, i_X\omega] S_X - 2(i_X\omega)^n \nabla_X\omega + [A, \Omega], \]
into equation \((A.1)\) and then on equating the coefficients of equal power of derivatives, we obtain the consistency conditions \((6)\) and \((7)\) of the main text.

Appendix B. Contraction of curvatures with a Yano vector

By differentiating the KY equation \((11)\), the action of the Hessian (see the remark following equation \((4)\)) on a KY p-form \(\omega(p)\) is found to be
\[ \nabla^2(X_a, X_b)\omega(p) = \frac{1}{p+1} \left( i_X\nabla_X - i_X\nabla_X \right) \omega(p). \]
Since the difference of \(\nabla^2(X_a, X_b)\) and \(\nabla^2(X_b, X_a)\) is the curvature operator \(R(X_a, X_b)\), from \((B.1)\) we obtain
\[ R(X_a, X_b)\omega(p) = \frac{1}{p+1} \left( i_X\nabla_X - i_X\nabla_X \right) \omega(p). \]
On the other hand, the action of the curvature operator on any form \(\alpha\) is known to be (see [18] equation (8.1.11) and [7])
\[ R(X_a, X_b)\alpha = -i_X R_{ab} \wedge i_X\alpha. \]
Note that the action on the 1-form \(\tilde{X}\) simply is \(-i_X R_{ab}\). Using \((B.3)\) in \((B.2)\) and then by multiplying the result with \(\epsilon a\wedge\), we obtain
\[ \nabla_X \omega(p) = \frac{p+1}{p} i_X^b R_a \wedge i_X^a \omega(p). \]
Using this in \((B.2)\) for \(\omega(n-1) = \tilde{Y}\) we find
\[ R(X_a, X_b)^*\tilde{Y} = \frac{(-1)^n}{n-1} [i_X^c \tilde{Y} \wedge P_a - i_X^c \tilde{Y} \wedge P_b] \]
where we have used the contracted Bianchi identity \(i_X P_a = i_X P_b\). Let us first note that
\[ R(X_a, X_b)^*\tilde{Y} = \epsilon(-1)^n i_Y R_{ab}, \]
\[ ^*([\tilde{Y} \wedge e_b] \wedge P_a) = i_{\tilde{Y}}^c \epsilon^*([\tilde{Y} \wedge e_b], \]
\[ = \epsilon\left[ (i_Y P_a) e_b - (i_Y P_b) e_a \right], \]
where \(\epsilon\) is the sign of \(\det g\). In deriving the above relations, we have made use of \((B.3)\) and of the following identities:
\[ ^*\alpha(p) = \epsilon(-1)^{p(n-p)} \alpha(p), \quad i_X^c \tilde{Y} = ^*\tilde{Y} \wedge e_a. \]

If we now take the Hodge dual of both sides of \((B.5)\) and use \((B.6)\), \((B.7)\) we obtain the first relation of \((35)\) of the main text: \(i_Y R_{ab} = (n - 1)^{-1} [i_Y P_a] e_b - (i_Y P_b) e_a\).

As an aside, it is worth mentioning that if \((B.4)\) is substituted into \((B.1)\), we obtain
\[ \nabla^2(X_a, X_b)\omega(p) = \frac{1}{p} i_X^c (R_{ca} \wedge i_X \omega(p)), \]
which shows that the second covariant derivative of any KY \( p \)-form is determined by curvature characteristics and the form itself. This implies that the value of a KY \( p \)-form at any point is entirely determined by the values of the form itself and its first covariant derivatives at the same point. These remarks can be used to determine the upper bound for the numbers of linearly independent KY \( p \)-forms to be the binomial number \( \binom{n+1}{p+1} \) [5, 21].

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