Gapless superconductivity and the Fermi arc in the cuprates

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We argue that the Fermi arc observed in angle resolved photoemission measurements in underdoped cuprates can be understood as a consequence of inelastic scattering in a d-wave superconductor. We analyze this phenomenon in the context of strong coupling Eliashberg theory, deriving a ‘single lifetime’ model for describing the temperature evolution of the spectral gap as measured by single particle probes such as photoemission and tunneling.

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Underdoped cuprates are characterized by a gap which exists in the excitation spectrum even above the superconducting transition temperature \( T_c \), and is detectable up to a temperature \( T^\ast \); the “pseudogap temperature”. The origin of this pseudogap has been a source of much debate. One of the most intriguing experimental characteristics of this pseudogap regime is the presence of two gaps related to the pairing. The proposals include orbital currents\textsuperscript{6,7}, SDW\textsuperscript{8,9}, CDW\textsuperscript{9}, and dDW\textsuperscript{10,11} ordering. A generic consequence of these theories is the presence of two gaps – a smaller superconducting gap in the nodal regions, and a larger gap of different origin in the antinodal regions. Above \( T_c \), the superconducting gap vanishes leaving only the other order, which leads to Fermi surface pockets near the zone diagonals\textsuperscript{10}. The ARPES intensity on one side of these pockets is expected to be weaker than on the other side (as observed in experiments on materials with density wave ordering) so that the pockets look like arcs in ARPES measurements. The notion of two gaps is also implicit in scenarios relating the arc to Mott physics\textsuperscript{11,12}.

An alternative ‘one-gap’ scenario interprets the pseudogap as a remnant of the superconducting gap, with thermal fluctuations destroying long-range order at \( T \geq T_c \), but allowing gap-like features to remain in the spectra for a range of higher temperatures\textsuperscript{13,14,15}. The effect of phase disordering on the spectral function has been extensively discussed in Refs.\textsuperscript{2,16}. A phenomenological Green’s function has the form

\[
G(k_F, \omega) = \frac{1}{\omega + i\gamma_2 - \Phi^2(\phi)} = \frac{\omega + i\gamma_1}{(\omega + i\gamma_1)(\omega + i\gamma_2) - \Phi^2(\phi)}
\]

(1)

Here \( \Phi(\phi) \) is the d-wave anomalous self energy, and \( \phi \) is an angle denoting the position along the Fermi surface. The scattering \( \gamma_1 \) comes from pairing (or vortex) fluctuations, and \( \gamma_2 \) is the fermionic scattering rate.

A simplified model with \( \gamma_1 = \gamma_2 = \gamma \) is sufficient to understand how the Fermi arc, as experimentally defined, emerges due to scattering. At small \( \gamma \), Im\( G(k_F, \omega) \) has sharp peaks at \( \omega = \pm \Phi(\phi) \). However, as \( \gamma \) increases, the peaks broaden and slowly shift to smaller absolute frequencies (Fig. 1a). For \( \gamma \leq \sqrt{3} |\Phi(\phi)| \), the maxima

![Image](http://example.com/figure1.png)

FIG. 1: (a) Im \( G \) around the Fermi surface from Eq. (1) with \( \Phi = \Phi_0 \cos(2\phi) \), where \( \Phi_0 = 40 \text{ meV} \) and \( \gamma = 30 \text{ meV} \). The top curve is at the node (\( \phi = 45^\circ \)), the bottom curve at the antinode (\( \phi = 0^\circ \)), with \( \phi \) increments of 5\(^{\circ}\). (b) Spectral gap (half the peak to peak separation) versus \( \phi \) for \( \gamma \) ranging from 0 (dashed curve) to 60 \text{ meV} (bottom curve) with \( \gamma \) increments of 10 \text{ meV}. 

\[\text{Im G}(k_F, \omega) = \frac{1}{\omega + i\gamma_2 - \Phi^2(\phi)} = \frac{\omega + i\gamma_1}{(\omega + i\gamma_1)(\omega + i\gamma_2) - \Phi^2(\phi)}\]
are located at \( \omega = \pm |\Phi| \sqrt{\gamma^2 + \Phi^2} \) for \( \gamma > |\Phi| \), a single maximum occurs at \( \omega = 0 \). The boundary value \(|\Phi(\phi)| = \gamma/\sqrt{3}\) sets the length of the Fermi arc (Fig. 1b). Because \( \Phi(\phi) \) vanishes at the node, the arc has a non-zero length for arbitrarily small \( \gamma \).

Our goal is to provide a microscopic justification for Eq. (\textbf{1}) (with \( \gamma_1 = \gamma_2 = \gamma \)) and to show that this is equivalent to gapless superconductivity. The latter naturally emerges at non-zero \( T \) in an Eliashberg theory, independent of the nature of the electron-boson interaction and the symmetry of the pairing gap. Obtaining a two-lifetime model with \( \gamma_1 \neq \gamma_2 \) would require additional physics\textsuperscript{16}. We show that even in the absence of impurities, \( \gamma \) is always non-zero for \( T > 0 \). Both \( \gamma \) and the length of the arc scale linearly with \( T \) at high temperatures, when the strongest interaction is between fermions and classical (thermal) bosons. The linear \( T \) behavior of \( \gamma \) sets in well before the temperature exceeds the “effective Debye frequency” of the bosons for the same reason that linear \( T \) resistivity becomes apparent for \( T > \Omega_{\text{Debye}}/3 \). The linear \( T \) dependence of \( \gamma \) in the pseudogap phase could also arise from the same physics that gives rise to the marginal Fermi liquid behavior of the normal state\textsuperscript{14}.

This scenario is consistent with the ARPES experiments. The key evidence is the vanishing of the extrapolated arc length at \( T = 0 \), and the linear \( T \) dependence of the arc length for \( T > T_c \). Consistent with this is the “closing” of the gap in the nodal region, i.e. the reduction of the spectral peak position to zero for small \( \Phi \) relative to \( \gamma \), and the “filling in” of the gap in the antinodal region, i.e. the spectral peaks broadening without moving much for large \( \Phi \) relative to \( \gamma \). This behavior is evident from Eq. (\textbf{1}) (see Fig. 1a). For a two-gap scenario, the arc length above \( T_c \) should have only a weak dependence on \( T \), and would extrapolate to a non-zero value in the \( T=0 \) limit, in contradiction to the ARPES data\textsuperscript{10}.

The rapid shrinking of the arc length below \( T_c \) found in Ref. \textsuperscript{3} is a much subtler issue. Our analysis shows that a Fermi arc with \( T \)-dependent length is a generic property of a \( d \)-wave superconductor at non-zero \( T \), independent of issues associated with superconducting coherence. In other words, long-range superconducting order does not have to be destroyed by fluctuations for the arc to be present. This generally implies that the arcs must survive below \( T_c \). We find, however, that the length of the arc is very small at the lowest \( T \) (exponentially small if the spectrum of the scattering bosons is completely gapped well below \( T_c \)), but becomes both sizable and linear in \( T \) at temperatures above the fluctuation-driven \( T_c \). This would be consistent with the drop of the scattering rate below \( T_c \) that has been inferred from a variety of techniques, including ARPES and various conductivity measurements\textsuperscript{19}. Another possibility, which we don’t explore here, is that the rapid variation of the arc length near \( T_c \) is due to a strong temperature variation of the damping originating from the interaction between electrons and pairing and/or vortex fluctuations\textsuperscript{2,16,20,21}. Phenomenologically this can be described as the vanishing of \( \gamma_1 \) in the two-lifetime model of Eq. (\textbf{1}). Once \( \gamma_1 = 0 \) at \( T_c \), \( A(k_F, \omega) \) vanishes at \( \omega = 0 \), and the spectral peak is located at a non-zero frequency for all \( k_F \) points except the \( d \)-wave node\textsuperscript{6}.

Gapless superconductivity was first discussed in the context of a BCS \( s \)-wave superconductor with magnetic impurities\textsuperscript{22}. Although not widely appreciated, even in a clean \( s \)-wave system, the Eliashberg equations yield gapless superconductivity\textsuperscript{24,25,26,27}. The reason is that at non-zero \( T \), the density of the bosons is non-zero, and a fermion can undergo scattering by on-shell thermal bosons\textsuperscript{26}. This leads to a non-zero damping term in the fermionic self-energy \( \Sigma(\omega) \), just as in a superconductor with magnetic impurities.

To understand gapless superconductivity for the clean case and to discuss the Fermi arcs, we consider a \( d \)-wave Eliashberg theory with an effective (dimensionless) interaction \( \chi(k - k', \Omega) \) between the electrons at the Fermi surface \(|k| = |k'| = k_F \). We assume \( \chi(k - k', \Omega) \) can be decoupled into \( s \)-wave and \( d \)-wave harmonics, i.e.,

\[
\chi(k - k', \Omega) = \lambda [\chi_s(\Omega) d_k d_k^\ast + \chi_d(\Omega) s_k s_k^\ast]
\]

where \( \lambda \) is a dimensionless coupling, \( s_k = 1 \), and \( d_k = \frac{1}{2} (\cos k_x - \cos k_y) \). We choose a normalization such that \( \chi_s(0) = 1 \). We assume that \( \chi_d \) is attractive, and \( \chi_s \) is repulsive. We also assume for simplicity that there is no feedback effect from the pairing on \( \chi(k - k', \Omega) \) – this last simplification may be partly justified by the fact that, experimentally, feedback effects are relatively weak in the pseudogap phase.

Eliashberg theory provides a mechanism for the opening of a gap in the excitation spectrum, without giving information on phase coherence. Therefore, in what follows, we shall use it to describe the pseudogap behavior for \( T < T_c \). In a superconductor, the self energy has a normal part expressed in terms of a renormalization factor \( Z_k(\omega) \) (which is \( k \) independent if \( s_k = 1 \), and an anomalous (pairing) part expressed in terms of the amplitude \( \Phi_k(\omega) = d_k \Phi(\omega) \) (the equality follows from our choice of interaction, Eq. (\textbf{2})). The superconducting gap is \( \Delta_k(\omega) = \Phi_k(\omega)/Z_k(\omega) = d_k \Delta(\omega) \). The quantities \( \Delta(\omega) \) and \( Z(\omega) \) are given by the solution of two coupled integral equations\textsuperscript{23}, of which the equation for \( \Delta \) involves only \( \Delta \) while the other expresses \( Z \) in terms of \( \Delta \). As long as the interaction is finite at zero frequency, \( \Phi(\omega_m) \) tends to a non-zero value at the smallest \( \omega_m \), and for the purposes of our small \( \omega \) analysis can be safely replaced by a frequency-independent value \( \Phi \). The low-frequency behavior of \( Z(\omega_m) \) is more involved, and the conversion of \( Z \) to real frequencies at finite \( T \) is nontrivial\textsuperscript{25}.

Note that Eq. (\textbf{2}) is an approximation – there is no guarantee that for a realistic \( \chi(k - k', \Omega) \) the pairing amplitude \( \Phi \) will have the simple \( d \)-wave form \( \Phi(k) = d_k \Phi \). In particular, near the onset of pairing, the form of the gap will be such as to minimize the pairbreaking effects specific to systems with a non-\( s \)-wave gap symmetry\textsuperscript{25}. This by itself may give rise to a suppressed gap around
the node, even without the contribution from $Z_k(\omega)$. We, however, will not explore this possibility here and simply assume $\Phi_k \propto d_k$.

It is more convenient to solve the gap equation directly along the real frequency axis and to cast the Eliashberg equations in terms of $D(\omega) = \Delta(\omega)/\omega$. For an electron-phonon superconductor, this procedure has been exploited in Ref. [24] to obtain an equation for $\Delta(\omega)$ along the real axis. The extension to a $d$-wave superconductor, and to arbitrary $\chi_s(\Omega)$ and $\chi_d(\Omega)$, is straightforward and yields

$$D(\omega)B(\omega) = A(\omega) + C(\omega)$$

(3)

where

$$A(\omega) = \frac{\lambda}{2} \int_{-\infty}^{\infty} d\Omega \frac{\Omega}{2T} \text{Re} \left[ \frac{D(\Omega)\chi_d(\Omega - \omega)}{\sqrt{1 - D^2(\Omega)}} \right]$$

$$B(\omega) = \omega + \frac{\lambda}{2} \int_{-\infty}^{\infty} d\Omega \frac{\Omega}{2T} \text{Re} \left[ \frac{\chi_s(\Omega - \omega)}{\sqrt{1 - D^2(\Omega)}} \right]$$

(4)

All integrals should be understood as principal parts. The terms $A(\omega)$ and $B(\omega)$ are non-singular at small frequencies and can be safely approximated by $A(\omega) = A(0) = \text{const}$, and $B(\omega) = B_1 \omega (B_1 = 1 + \lambda)$ for vanishing $D(0))$. The non-trivial physics is due to the presence of the $C(\omega)$ term in Eq. (3). This term originates from the imaginary part of the interaction $\chi(k - k', \Omega)$ along the real frequency axis and describes scattering by on-shell thermal bosons. Neglecting $C(\omega)$ would lead to a conventional BCS-type result $\Delta(\omega) = A(\omega)\omega/B(\omega) = A(0)/B_1 = \text{const}$. However, the second term in the integral for $C(\omega)$ is proportional to $D(\omega) = \Delta(\omega)/\omega$, and diverges at zero frequency if $\Delta(0)$ is nonzero. Keeping only this term and approximating the remainder of the integral by its $\omega \to 0$ limit, we find from Eq. (4) $C(\omega) = -iD(\omega)Q(T)$, where

$$Q(T) = \lambda \int_{-\infty}^{\infty} \frac{d\Omega}{\sinh \frac{\Omega}{2T}} \frac{\text{Im} \chi_s(\Omega)}{\sqrt{1 - D^2(\Omega)}}$$

(5)

Upon substituting into Eq. (3) we get

$$\Delta(\omega) = \frac{A(0)}{B_1} \frac{\omega}{\omega + i\gamma}, \quad \gamma = \frac{Q(T)}{B_1}$$

(6)

This is a gapless superconductor – the gap $\Delta$ is imaginary and linear in frequency at small frequencies, but recovers to a real value, equal to the pairing amplitude $\Phi$, at frequencies larger than $\gamma$. We then obtain at small frequencies $Z(\omega) = Z_0(1 + i\gamma/\omega)$, where $Z_0 = B_1\Phi/A_0$ is a mass-renormalization factor. For the Green’s function we then get

$$G(\omega, k_F) = \frac{1}{Z_0} \frac{\omega + i\gamma}{(\omega + i\gamma)^2 - (\Phi/Z)^2 d_k^2}$$

(7)

This is precisely the $d$-wave single-lifetime model of Eq. (1) ($\gamma_1 = \gamma_2$). Note that the gap amplitude $\Phi/Z_0$ and the scattering rate $\gamma$ are determined by different interactions. In particular, $\gamma Z_0/\Phi = \gamma B_1/A_0$ is strongly enhanced if the repulsive $s$-wave component of the interaction is much larger than the $d$-wave component.

As stated earlier, the model of Eq. (7) gives rise to arcs of the Fermi surface simply because the condition for the arc, $\gamma > \sqrt{3}T(\Phi/Z_0)|d_k|$, is always satisfied sufficiently close to the nodal direction, as long as $\gamma$ is non-zero, as it is for any non-zero $T$.

To understand the temperature dependence of the arc length, we note that below the onset of pairing, $\Phi$ and $Z_0$ are weakly $T$-dependent, so that the $T$ dependence of the arc length comes primarily from $\gamma(T)$. This $T$ dependence is given by Eq. (6), and the functional form of $\gamma(T)$ depends on the form of $\chi_s(\Omega)$. As an example, consider the situation where well below $T^*$, $\text{Im} \chi_s(\Omega)$ can be approximated by an Einstein mode at some frequency $\Omega_0$, i.e. $\text{Im} \chi_s(\Omega) = \pi\Omega_0^2 \delta(\Omega^2 - \Omega_0^2)$. This can be a phonon or collective spin excitations – the latter form a gapped continuum in the normal state, but become a mode (plus a gapped continuum) once $\Phi$ is nonzero.

Substituting $\text{Im} \chi_s(\Omega)$ into (1) and (5), we obtain that the boundary condition for the arc reduces to

$$\pi\eta \frac{\lambda}{1 + \lambda} \frac{\Omega_0}{\sinh \frac{\Omega_0}{2T}} = \sqrt{3} d_k$$

(8)

where $\eta$ is the normalized density of states at $\omega = \Omega_0$. For a circular Fermi surface, this is $\eta = (2/\pi)\text{Re} \int_0^{\pi/2} d\phi / \sqrt{1 - D^2(\phi, \Omega_0)}$ ($\eta > 1$ if the largest contribution to $\eta$ comes from antinodal fermions). The quantity $\Phi/Z_0$ is the measured gap at frequencies well above $\gamma$, and it is well approximated by the maximum value of the ARPES gap at the antinode. We see from Eq. (8) that the arc length is exponentially small at small $T \ll \Omega_0$, but becomes linear in $T$ at high temperatures. This by itself is not a surprising result as at large $T$, the interaction is dominated by the thermal (zero-Matsubara frequency) term in which case the damping scales as $T$ (the zero-Matsubara frequency contribution corresponds to approximating $\sinh(\Omega/T)$ in Eq. (5) by $\Omega/T$). A more
interesting result, well known from the electron-phonon literature\cite{17}, is that, numerically, the r.h.s. of Eq. (8) is essentially linear in $T$ already from $T \sim 0.3\hbar\Omega_0$.

Experimentally, the arc length is linear in $T$ for $T_c < T < T^*$. This is roughly consistent with Eq. (8) if we associate $T_e$ with the mode energy (e.g. for the neutron resonance mode\cite{21}, $T_e \sim 0.2\hbar\Omega_0$, in which case a linear in $T$ behavior sets in at $T \geq 1.5T_c$). Moreover, this behavior will also exhibit $T/T^*$ scaling as in Ref. 3 if $\Phi/\Omega_0$ scales with $T^*$, as would be expected if $T^*$ were associated with pair formation. To look into this further, we show in Fig. 2a a plot of $\gamma(T)$ as given by the l.h.s. of Eq. (8) versus the experimental $\gamma(T)$ values extracted from ARPES data\cite{29} on a slightly underdoped sample of Bi$_2$Sr$_2$CaCu$_2$O$_8$ ($T_c = 90$K). A good fit is obtained with a reasonable value for $\Omega_0$ (28 meV) and a prefactor of one (corresponding to $\hbar\lambda/(1 + \lambda) = 1$). The resulting arc length is shown in Fig. 2b. The agreement is quite reasonable given the number of approximations made in deriving Eq. (8), and the fact that we did not take into account any temperature or doping dependence of $\Phi/\Omega_0$ and $\Omega_0$ when constructing these plots. We note that a dependence of $\gamma$ that is strictly linear in $T$ is more consistent with the data, though a better fit can be obtained if $d_k$ is flattened near the nodes as in Ref. 3. In addition, all the data shown in Fig. 2b were restricted to temperatures above $T_c$ (unlike in Fig. 2a).

In conclusion, we find that the temperature dependent Fermi arcs observed by photoemission in the pseudogap phase can be explained by lifetime broadening of a $d$-wave paired state. We have investigated this in detail by examining the effects of inelastic scattering within the context of strong coupling Eliashberg theory, and find that the results are in accord with the data. This suggests that $T^*$ can be thought of as an energy scale associated with $d$-wave pair formation.

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