Nonexistence of solutions for Dirichlet problems with supercritical growth in tubular domains

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Abstract. - We deal with Dirichlet problems of the form
\[\Delta u + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega\]
where \(\Omega\) is a bounded domain of \(\mathbb{R}^n\), \(n \geq 3\), and \(f\) has supercritical growth from the viewpoint of Sobolev embedding. In particular, we consider the case where \(\Omega\) is a tubular domain \(T_\varepsilon(\Gamma_k)\) with thickness \(\varepsilon > 0\) and centre \(\Gamma_k\), a \(k\)-dimensional, smooth, compact submanifold of \(\mathbb{R}^n\). Our main result concerns the case where \(k = 1\) and \(\Gamma_k\) is contractible in itself. In this case we prove that the problem does not have nontrivial solutions for \(\varepsilon > 0\) small enough. When \(k \geq 2\) or \(\Gamma_k\) is noncontractible in itself we obtain weaker nonexistence results. Some examples show that all these results are sharp for what concerns the assumptions on \(k\) and \(f\).

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1 Introduction

The results we present in this paper are concerned with existence or nonexistence of nontrivial solutions for Dirichlet problems of the form

$$\Delta u + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$, $n \geq 3$ and $f$ has supercritical growth from the viewpoint of the Sobolev embedding.

Let us consider, for example, the case where $f(t) = |t|^{p-2}t \forall t \in \mathbb{R}$ (this function obviously satisfies the condition (2.4) we use in this paper). In this case, a well known nonexistence result of Pohozaev (see [23]) says that the Dirichlet problem

$$\Delta u + |u|^{p-2}u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has only the trivial solution $u \equiv 0$ when $\Omega$ is starshaped and $p \geq \frac{2n}{n-2}$ (the critical Sobolev exponent).

On the other hand, if $\Omega$ is an annulus it is easy to find infinitely many radial solutions for all $p > 1$ (as pointed out by Kazdan and Werner in [6]). Thus, it is natural to ask whether or not the nonexistence result of Pohozaev can be extended to non starshaped domains and the existence result in the annulus can be extended, for example, to all noncontractible domains of $\mathbb{R}^n$.

Following some stimulated questions pointed out by Brezis, Nirenberg, Rabinowitz, etc. (see [2, 3]) many results have been obtained, relating nonexistence, existence and multiplicity of nontrivial solutions to the shape of $\Omega$ (see [4, 5, 7, 8, 10, 11, 14–17, 19, 21, 22], etc.).

In the present paper our aim is to show that, even if the Pohozaev nonexistence result cannot be extended to all the contractible domains of $\mathbb{R}^n$, one can prove that there exist contractible non starshaped domains $\Omega$, which may be very different from the starshaped ones and even arbitrarily close to noncontractible domains, such that the Dirichlet problem (1.2) has only the trivial solution $u \equiv 0$ for all $p > \frac{2n}{n-2}$.

In order to construct such domains, we use suitable Pohozaev type integral identities in tubular domains $\Omega = T_\varepsilon(\Gamma_k)$ with thickness $\varepsilon > 0$ and centre $\Gamma_k$, where $\Gamma_k$ is a $k$-dimensional, compact, smooth submanifold of $\mathbb{R}^n$.

If $k = 1$, $\Gamma_k$ is contractible in itself and $p > \frac{2n}{n-2}$, we prove that there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, the Dirichlet problem (1.2) with $\Omega = T_\varepsilon(\Gamma_k)$ does not have any nontrivial solution (this nonexistence result follows, as a particular case, from Theorem 2.2).

Let us point out that, if $k = 1$ but $\Gamma_k$ is noncontractible in itself or if $k > 1$, a nonexistence result analogous to Theorem 2.2 cannot hold under the assumption $p > \frac{2n}{n-2}$.

In fact, the method we use in Theorem 2.2 fails when $k = 1$ and $\Gamma_k$ is noncontractible because the multipliers to be used in the Pohozaev type integral identity are not well
defined. Using other multipliers, we obtain a weaker nonexistence result which holds only when \( n \geq 4 \) and \( p > \frac{2(n-1)}{n-3} \) (it follows from Theorem 2.2). On the other hand, this weaker result is sharp because, if \( \Gamma_k \) is for example a circle of radius \( R \) (that is \( T_\varepsilon(\Gamma_k) \) is a solid torus), one can easily obtain infinitely many solutions for all \( \varepsilon \in (0, R) \) when \( n = 3 \) and \( p > 1 \) or \( n \geq 4 \) and \( p \in \left( 1, \frac{2(n-1)}{n-3} \right) \).

Propositions 3.2, 3.3 and 3.4 give examples of existence and multiplicity results of positive and sign changing solutions for some \( p \geq \frac{2n}{n-2} \) in tubular domains \( T_\varepsilon(\Gamma_k) \) with \( k \geq 2 \) and \( \Gamma_k \) contractible in itself. This examples explain why Theorem 2.2 cannot be extended to the case \( k > 1 \) under the assumption \( p > \frac{2n}{n-2} \).

However, in the case \( k > 1 \), with \( \Gamma_k \) contractible or not, we prove a weaker nonexistence result (given by Theorem 3.5) which holds only when \( n > k + 2 \) and \( p > \frac{2(n-k)}{n-k-2} \).

Some existence and multiplicity results, when \( n \leq k + 2 \) or \( n > k + 2 \) and \( p < \frac{2(n-k)}{n-k-2} \), in tubular domains \( T_\varepsilon(\Gamma_k) \) with \( k \geq 2 \) and \( \varepsilon \) non necessarily small, show that also the nonexistence result given by Theorem 3.5 is sharp.

Finally, let us point out that if in the equation \( \Delta u + f(u) = 0 \) we replace the Laplace operator \( \Delta \) by the operator \( \text{div}(|Du|^{q-2}Du) \) with \( 1 < q < 2 \), then critical and supercritical nonlinearities arise also for \( n = 2 \) and produce analogous nonexistence results (see [12, 13]). These results suggest that if \( n = 2, 1 < q < 2 \) and \( p > \frac{2q}{2-q} \), the Pohozaev nonexistence result for starshaped domains can be extended to all the contractible domains of \( \mathbb{R}^2 \) while it is not possible for example if \( n \geq 3, q = 2 \) and \( p \geq \frac{2n}{n-2} \) because of Propositions 3.2, 3.3 and 3.4 (see Remark 3.7).

## 2 Integral identities and nonexistence results

In order to obtain nonexistence results for nontrivial solutions of problem (1.1), we use the Pohozaev type integral identity given in the following Lemma.

**Lemma 2.1** Let \( \Omega \) be a piecewise smooth bounded domain of \( \mathbb{R}^n \), \( n \geq 3 \), \( v = (v_1, \ldots, v_n) \in C^1(\overline{\Omega}, \mathbb{R}^n) \) a vector field in \( \overline{\Omega} \) and \( f \) a continuous function in \( \mathbb{R} \). Then every solution of problem (1.1) satisfies the integral identity

\[
\frac{1}{2} \int_{\partial\Omega} |Du|^2 v \cdot \nu \, d\sigma - \int_\Omega dv[Du] \cdot Du \, dx + \int_\Omega \text{div} v \left( F(u) - \frac{1}{2} |Du|^2 \right) \, dx,
\]

where \( \nu \) denotes the outward normal to \( \partial\Omega \), \( dv[\xi] = \sum_{i=1}^n D_i v \xi_i \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and \( F(t) = \int_0^t f(\tau) \, d\tau \forall t \in \mathbb{R} \).

For the proof it is sufficient to apply the Gauss-Green formula to the function \( v \cdot Du Du \) and argue as in [23]. Notice that the Pohozaev identity is obtained for \( v(x) = x \).
Now our aim is to find suitable domains $\Omega$ and vector fields $v \in C^1(\overline{\Omega}, \mathbb{R}^n)$ such that the identity (2.1) can be satisfied only by a trivial solution of problem (1.1).

In order to construct $\Omega$ and $v$ with this property, let us consider a curve $\gamma \in C^3([a, b], \mathbb{R}^n)$ such that $\gamma'(t) \neq 0$ for all $t \in [a, b]$ and $\gamma(t_1) \neq \gamma(t_2)$ if $t_1 \neq t_2$, $t_1, t_2 \in [a, b]$.

For all $t \in [a, b]$ and $r > 0$, let us set $N(t) = \{ \xi \in \mathbb{R}^n : \xi \cdot \gamma'(t) = 0 \}$ and $N_r(t) = \{ \xi \in N(t) : ||\xi|| \leq r \}$.

Notice that there exists $\varepsilon > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$,

$$[\gamma(t_1) + N_\varepsilon(t_1)] \cap [\gamma(t_2) + N_\varepsilon(t_2)] = \emptyset \quad \text{if} \quad t_1 \neq t_2, \quad t_1, t_2 \in [a, b]. \quad (2.2)$$

For all $\varepsilon \in (0, \varepsilon_1)$ let us consider the open, piecewise smooth, bounded domain $T_\varepsilon^\gamma$ defined by

$$T_\varepsilon^\gamma = \bigcup_{t \in (a, b)} [\gamma(t) + N_\varepsilon(t)]. \quad (2.3)$$

Then, the following nonexistence result holds for the nontrivial solutions in the domain $\Omega = T_\varepsilon^\gamma$.

**Theorem 2.2** Assume the continuous function $f$ satisfies the condition

$$tf(t) \geq p \int_0^t f(\tau) d\tau \geq 0 \quad \forall t \in \mathbb{R} \quad (2.4)$$

for a suitable $p > \frac{2n}{n+2}$. Then, there exists $\varepsilon > 0$ such that for all $\varepsilon \in (0, \varepsilon)$ the Dirichlet problem (1.1) has only the trivial solution $u \equiv 0$ in the domain $\Omega = T_\varepsilon^\gamma$.

It is clear that condition (2.4) implies $f(0) = 0$, so the function $u \equiv 0$ in $T_\varepsilon^\gamma$ is a trivial solution $\forall \varepsilon \in (0, \varepsilon_1)$.

In order to prove that it is the unique solution for $\varepsilon$ small enough, we need some preliminary results.

Notice that if $\varepsilon \in (0, \varepsilon_1)$, the following property holds: for all $x \in T_\varepsilon^\gamma$ there exists a unique $t(x) \in (a, b)$ such that $\text{dist}(x, \Gamma) = |x - \gamma(t(x))|$, where

$$\Gamma = \{ \gamma(t) : t \in [a, b] \}. \quad (2.5)$$

If we set $\xi(x) = x - \gamma(t(x))$, we have $\xi(x) \cdot \gamma'(t(x)) = 0 \forall x \in T_\varepsilon^\gamma$. Therefore, for all $y \in T_\varepsilon^\gamma$ there exists a unique pair $(t(x), \xi(x))$ such that $t(x) \in [a, b]$, $\xi(x) \in N_\varepsilon(t(x))$ and $x = \gamma(t(x)) + \xi(x)$.

Without any loss of generality, we can assume in addition that $a \leq 0 \leq b$ and $|\gamma'(t)| = 1 \forall t \in [a, b]$.

For all $\xi \in N_\varepsilon(0)$ let us consider the function $\tau \mapsto x(\xi, \tau)$ which solves the Cauchy problem

$$\begin{align*}
\frac{dx}{d\tau} &= \gamma'(t(x)) \\
x(\xi, 0) &= \gamma(0) + \xi.
\end{align*} \quad (2.6)$$
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Notice that $\text{dist}(x(\xi, \tau), \gamma) = |\xi| \forall \tau \in [a, b]$. Moreover, for all $\xi \in N_{\varepsilon}(0)$, the function $\tau \mapsto t(x(\xi, \tau))$ is increasing. As a consequence, we can consider the inverse function $t \mapsto \tau(\xi, t)$ which satisfies $t(x(\xi, \tau(\xi, t))) = t \forall t \in [a, b]$. Notice that $\tau(\xi, 0) = 0 \forall \xi \in N_{\varepsilon}(0)$ because $t(x(\xi, 0)) = 0$. For all $\xi \in N_{\varepsilon}(0)$, let us set $\psi(\xi, t) = x(\xi, \tau(\xi, t)) - \gamma(t)$. Then, $\psi(\xi, t) \in N_{\varepsilon}(t)$ and $|\psi(\xi, t)| = |\xi| \forall \xi \in N_{\varepsilon}(0) \forall t \in [a, b]$. Moreover, for all $x \in T_{\varepsilon}^\gamma$ there exists a unique $\xi \in N_{\varepsilon}(0)$ such that $\xi(x) = \psi(\xi, t(x))$ and the function $\xi \mapsto \psi(\xi, t)$ is a one to one function between $N_{\varepsilon}(0)$ and $N_{\varepsilon}(t)$, satisfying $|\psi(\xi_1, t) - \psi(\xi_2, t)| = |\xi_1 - \xi_2| \forall \xi_1, \xi_2 \in N_{\varepsilon}(0), \forall t \in [a, b]$. Now, let us consider the vector field $v$ defined by

$$v(\gamma(t) + \psi(\xi, t)) = t\gamma'(t)[1 - \psi(\xi, t) \cdot \gamma''(t)] + \psi(\xi, t) \quad \forall t \in (a, b), \forall \xi \in N_{\varepsilon}(0).$$

(2.7)

Since $\gamma \in C^3([a, b], \mathbb{R}^n)$, we have $v \in C^1(T_{\varepsilon}^\gamma, \mathbb{R}^n)$, so the integral identity (2.1) holds. In the following lemma we establish some properties of the vector field $v$.

**Lemma 2.3** In the domain $T_{\varepsilon}^\gamma$, let us consider the vector field $v \in C^1(T_{\varepsilon}^\gamma, \mathbb{R}^n)$ defined in (2.7). Then we have

(a) $v \cdot v > 0$ on $\partial T_{\varepsilon}^\gamma \forall \varepsilon \in (0, \varepsilon_1),$

(b) $\lim_{\varepsilon \to 0} \sup \{|n - \text{div } v(x)| : x \in T_{\varepsilon}^\gamma\} = 0,$

(c) $\lim_{\varepsilon \to 0} \sup \{|1 - dv(x)[\eta] \cdot \eta| : x \in T_{\varepsilon}^\gamma, \eta \in \mathbb{R}^n, |\eta| = 1\} = 0.$

**Proof** Taking into account the choice of $\varepsilon_1$, since we are assuming $|\gamma'(t)| = 1 \forall t \in [a, b]$, we have $|1 - \psi(\xi, t) \cdot \gamma''(t)| \geq 0 \forall t \in [a, b]$. Therefore, since we are also assuming $a \leq 0 \leq b$, property (a) is a direct consequence of the definition of $T_{\varepsilon}^\gamma$ and $v$. In order to prove (b), notice that, since $v \in C^1(T_{\varepsilon}^\gamma, \mathbb{R}^n) \forall \varepsilon \in (0, \varepsilon_1)$, there exist $t_\varepsilon \in [a, b]$ and $\xi_\varepsilon \in N_{\varepsilon}(0)$ such that

$$|n - \text{div } v(\gamma(t_\varepsilon) + \psi(\xi_\varepsilon, t_\varepsilon))| = \max\{|n - \text{div } v(x)| : x \in T_{\varepsilon}^\gamma\} \forall \varepsilon \in (0, \varepsilon_1).$$

(2.8)

When $\varepsilon \to 0$, we obtain (up to a subsequence) $t_\varepsilon \to t_0$ for a suitable $t_0 \in [a, b]$ while $\xi_\varepsilon \to 0$ (because $|\xi_\varepsilon| \leq \varepsilon$) and, as a consequence, also $\psi(\xi_\varepsilon, t_\varepsilon) \to 0$ (because $|\psi(\xi_\varepsilon, t_\varepsilon)| = |\xi_\varepsilon|$). Therefore we get

$$\lim_{\varepsilon \to 0} \max\{|n - \text{div } v(x)| : x \in T_{\varepsilon}^\gamma\} = |n - \text{div } v(\gamma(t_0))|.$$

(2.9)

Now, notice that

$$dv(\gamma(t_0))[\gamma'(t_0)] = \gamma'(t_0) + t_0 \gamma''(t_0)$$

(2.10)

and

$$dv(\gamma(t_0))[\psi] = -t_0[\psi \cdot \gamma''(t_0)\gamma'(t_0)] + \psi \quad \forall \psi \in N(t_0).$$

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It follows that \( \operatorname{div} v(\gamma(t_0)) = n \), so property (b) holds.
In a similar way we can prove property (c). In fact, since \( v \in C^1(\mathbb{T}_\varepsilon, \mathbb{R}^n) \) \( \forall \varepsilon \in (0, \xi_1) \), there exist \( \bar{t}_\varepsilon \in [a, b] \), \( \xi_\varepsilon \in N_\varepsilon(0) \) and \( \bar{\eta}_\varepsilon \in \mathbb{R}^n \) such that \( |\bar{\eta}_\varepsilon| = 1 \) and

\[
|1 - dv(\gamma(\bar{t}_\varepsilon) + \psi(\xi_\varepsilon, \bar{t}_\varepsilon))[\bar{\eta}_\varepsilon] \cdot \bar{\eta}_\varepsilon| = \max\{|1 - dv(x)[\eta] \cdot \eta| : x \in \mathbb{T}_\varepsilon, \eta \in \mathbb{R}^n, |\eta| = 1\}. \tag{2.12}
\]

Since \( |\psi(\xi_\varepsilon, \bar{t}_\varepsilon)| = |\xi_\varepsilon| \leq \varepsilon \) \( \forall \varepsilon \in (0, \xi_1) \), we have \( \lim_{\varepsilon \to 0} \psi(\xi_\varepsilon, \bar{t}_\varepsilon) = 0 \). Moreover, there exist \( \bar{t}_0 \in [a, b] \) and \( \bar{\eta}_0 \in \mathbb{R}^n \) such that (up to a subsequence) \( \bar{t}_\varepsilon \to \bar{t}_0 \) and \( \bar{\eta}_\varepsilon \to \bar{\eta}_0 \) as \( \varepsilon \to 0 \).

It follows that

\[
\lim_{\varepsilon \to 0} \max\{|1 - dv(x)[\eta] \cdot \eta| : x \in \mathbb{T}_\varepsilon, \eta \in \mathbb{R}^n, |\eta| = 1\} = |1 - dv(\gamma(\bar{t}_0))[\bar{\eta}_0] \cdot \bar{\eta}_0| \tag{2.13}
\]

Now, let us set \( \bar{\psi}_0 = \bar{\eta}_0 - \bar{\eta}_0 \cdot \gamma'(\bar{t}_0) \gamma'(\bar{t}_0) \) and notice that \( \bar{\psi}_0 \in N(\bar{t}_0) \). Therefore we have

\[
dv(\gamma(\bar{t}_0))[\bar{\psi}_0] = \bar{\psi}_0 - \bar{t}_0 \bar{\psi}_0 \cdot \gamma''(\bar{t}_0) \gamma'(\bar{t}_0). \tag{2.14}
\]

Thus, since

\[
dv(\gamma(\bar{t}_0))[\gamma'(\bar{t}_0)] = \gamma'(\bar{t}_0) + \bar{t}_0 \gamma''(\bar{t}_0) \tag{2.15}
\]

and \( \gamma'(\bar{t}_0) \cdot \gamma''(\bar{t}_0) = 0 \), we obtain

\[
dv(\gamma(\bar{t}_0))[\bar{\eta}_0] \cdot \bar{\eta}_0 = dv(\gamma(\bar{t}_0))[\bar{\eta}_0 \cdot \gamma'(\bar{t}_0) \gamma'(\bar{t}_0) + \bar{\psi}_0] \cdot (\bar{\eta}_0 \cdot \gamma'(\bar{t}_0) \gamma'(\bar{t}_0) + \bar{\psi}_0)
= \{\bar{\eta}_0 \cdot \gamma'(\bar{t}_0) [\gamma'(\bar{t}_0) + \bar{t}_0 \gamma''(\bar{t}_0)] + \bar{\psi}_0 - \bar{t}_0 \bar{\psi}_0 \cdot \gamma''(\bar{t}_0) \gamma'(\bar{t}_0)\}
\cdot (\bar{\eta}_0 \cdot \gamma'(\bar{t}_0) \gamma'(\bar{t}_0) + \bar{\psi}_0)
= |\bar{\eta}_0 \cdot \gamma'(\bar{t}_0)|^2 + |\bar{\psi}_0|^2 = |\bar{\eta}_0|^2 = 1,
\]

which implies property (c).

q.e.d.

**Corollary 2.4** Let \( f \) and \( F \) be as in Lemma 2.1. Let \( T_\varepsilon^\gamma \) and \( v \in C^1(\mathbb{T}_\varepsilon, \mathbb{R}^n) \) be as in Lemma 2.3. Then, every solution \( u_\varepsilon \) of the Dirichlet problem (1.1) in \( \Omega = T_\varepsilon^\gamma \) satisfies the inequality

\[
0 \leq \left[ 1 - \frac{n}{2} + \mu(\varepsilon) \right] \int_{T_\varepsilon^\gamma} |Du_\varepsilon|^2 dx + \int_{T_\varepsilon^\gamma} \operatorname{div} v(u_\varepsilon) F(u_\varepsilon) dx, \tag{2.17}
\]

where \( \mu(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

The proof follows directly from Lemmas 2.1 and 2.3.

**Proof of Theorem 2.2** In order to prove that the trivial solution \( u \equiv 0 \) in \( T_\varepsilon^\gamma \) is the unique solution for \( \varepsilon \) small enough, for every \( \varepsilon \in (0, \xi_1] \), let us consider a solution \( u_\varepsilon \)
of problem (1.1) in $\Omega = T^\epsilon_\gamma$. Taking into account Lemma 2.1 and condition (2.4), from Lemma 2.3 and Corollary 2.4 we obtain

$$0 \leq \left[ 1 - \frac{n}{2} + \mu(\epsilon) \right] \int_{T^\epsilon_\gamma} |Du_\epsilon|^2 dx + \left[ n + \bar{\mu}(\epsilon) \right] \frac{1}{p} \int_{T^\epsilon_\gamma} u_\epsilon f(u_\epsilon) dx,$$

(2.18)

where $\bar{\mu}(\epsilon) \to 0$ as $\epsilon \to 0$. On the other hand, since $u_\epsilon$ is a solution of problem (1.1) in $\Omega = T^\epsilon_\gamma$, we have

$$\int_{T^\epsilon_\gamma} u_\epsilon f(u_\epsilon) dx = \int_{T^\epsilon_\gamma} |Du_\epsilon|^2 dx.$$

(2.19)

Therefore we obtain

$$0 \leq \left[ 1 - \frac{n}{2} + \frac{n}{p} + \mu(\epsilon) + \bar{\mu}(\epsilon) \right] \int_{T^\epsilon_\gamma} |Du_\epsilon|^2 dx.$$

(2.20)

Since $1 - \frac{n}{2} + \frac{n}{p} < 0$ for $p > \frac{2n}{n-2}$, there exists $\bar{\epsilon} \in (0, \epsilon_1)$ such that $1 - \frac{n}{2} + \frac{n}{p} + \mu(\epsilon) + \bar{\mu}(\epsilon) < 0 \forall \epsilon \in (0, \bar{\epsilon})$. Therefore, for all $\epsilon \in (0, \bar{\epsilon})$, we must have $\int_{T^\epsilon_\gamma} |Du_\epsilon|^2 dx = 0$ which implies $u_\epsilon \equiv 0$ in $T^\epsilon_\gamma$ and completes the proof.

$q.e.d.$

Notice that if, instead of the vector field $v$ defined in (2.7), we consider the vector field $\tilde{v}$ defined by

$$\tilde{v}(\gamma(t) + \psi(\xi, t)) = \psi(\xi, t) \quad \forall t \in (a, b), \quad \forall \xi \in \overline{N(0)},$$

(2.21)

we obtain a nonexistence result for $n \geq 4$ and $p > \frac{2(n-1)}{n-3}$ (the critical Sobolev exponent in dimension $n - 1$, which is greater than $\frac{2n}{n-2}$).

Let us point out that the vector field $\tilde{v}$ is well defined also when $\gamma$ is a smooth circuit, that is $\gamma(a) = \gamma(b)$ and $\Omega$ is the interior of $T^\gamma$. Therefore, also in these domains we can prove nonexistence results for $n \geq 4$ and $p > \frac{2(n-1)}{n-3}$, see Theorem 2.5. On the contrary, in these domains the vector field $v$ could not be well defined because

$$v(\gamma(a) + \psi(\xi, a)) \neq v(\gamma(b) + \psi(\xi, b)) \quad \forall \xi \in N_\epsilon(0),$$

(2.22)

while $\gamma(a) + \psi(\xi, a) = \gamma(b) + \psi(\xi, b)$ when $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$.

On the other hand, in these domains one cannot expect to obtain nonexistence results for $p > \frac{2n}{n-2}$ since it is possible that there exist nontrivial solutions when $n \geq 4$ and $\frac{2n}{n-2} < p < \frac{2(n-1)}{n-3}$, while they do not exist for $p \geq \frac{2(n-1)}{n-3}$, which happens for example in the case of a solid torus (see [7, 18, 20]).

In next theorem we consider the case where $\Omega$ is a tubular domain near a circuit, $n \geq 4$ and condition (2.4) holds with $p > \frac{2(n-1)}{n-3}$ (see Theorem 3.5 for an extension to more general tubular domains).
Theorem 2.5 Assume that \( \tilde{\gamma} : [a, b] \to \mathbb{R}^n \) is a smooth curve which satisfies \( \tilde{\gamma}'(t) \neq 0 \) \( \forall t \in [a, b] \), \( \tilde{\gamma}(a) = \tilde{\gamma}(b) \), \( \tilde{\gamma}'(a) = \tilde{\gamma}'(b) \), \( \tilde{\gamma}(t_1) \neq \tilde{\gamma}(t_2) \) if \( t_1, t_2 \in (a, b) \) and \( t_1 \neq t_2 \). Let us set
\[
\tilde{T} = \{ \tilde{\gamma}(t) : t \in [a, b] \} \quad \text{and} \quad \tilde{T}_\varepsilon(\tilde{T}) = \{ x \in \mathbb{R}^n : \text{dist}(x, \tilde{T}) < \varepsilon \} \quad \forall \varepsilon > 0. \tag{2.23}
\]
Moreover assume that \( n \geq 4 \) and condition (2.4) holds with \( p > \frac{2(n-1)}{n-3} \).
Then there exists \( \varepsilon > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_1) \), the Dirichlet problem (1.1) has only the trivial solution \( u \equiv 0 \) in the smooth bounded domain \( \Omega = \tilde{T}_\varepsilon(\tilde{T}) \).

Proof First notice that there exists \( \varepsilon_1 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_1) \) and \( x \in \tilde{T}_\varepsilon(\tilde{T}) \) there exists a unique \( y \in \tilde{\gamma} \) such that \( \text{dist}(x, \tilde{T}) = |x - y| \). Let us denote this \( y \) by \( p(x) \) and consider in \( \tilde{T}_\varepsilon(\tilde{T}) \) the vector field \( \tilde{\nu} \) defined by \( \tilde{\nu}(x) = x - p(x) \).
One can verify by direct computation that
\[
d \tilde{\nu}(\tilde{\gamma}(t))[\tilde{\gamma}'(t)] = 0, \quad d \tilde{\nu}(\tilde{\gamma}(t))[\psi] = \psi \quad \forall t \in [a, b], \forall \psi \in \mathbb{R}^n \quad \text{such that} \quad \psi \cdot \tilde{\gamma}'(t) = 0 \tag{2.24}
\]
and, as a consequence,
\[
d \tilde{\nu}(\tilde{\gamma}(t)) = n - 1 \quad \forall t \in [a, b] \tag{2.25}
\]
\[
d \tilde{\nu}(\tilde{\gamma}(t))[\eta] \cdot \eta = |\eta|^2 - \frac{[\eta \cdot \tilde{\gamma}'(t)]^2}{|\tilde{\gamma}'(t)|^2} \quad \forall t \in [a, b], \forall \eta \in \mathbb{R}^n. \tag{2.26}
\]
It follows that
\[
\limsup_{\varepsilon \to 0} \{|n - 1 - d \tilde{\nu}(x)| : x \in \tilde{T}_\varepsilon(\tilde{T})\} = 0 \tag{2.27}
\]
as one can easily obtain from (2.25) arguing as in the proof of assertion (b) of Lemma 2.3. Moreover, from (2.26) we obtain
\[
\limsup_{\varepsilon \to 0} \{|d \tilde{\nu}(x)[\eta] \cdot \eta : x \in \tilde{T}_\varepsilon(\tilde{T}), \eta \in \mathbb{R}^n, |\eta| = 1\} = 1. \tag{2.28}
\]
In fact, for all \( \varepsilon \in (0, \varepsilon_1) \), choose \( x_\varepsilon \in \tilde{T}_\varepsilon(\tilde{T}) \) and \( \eta_\varepsilon \in \mathbb{R}^n \) such that \( |\eta_\varepsilon| = 1 \) and \( s_\varepsilon - \varepsilon \leq d \tilde{\nu}(x_\varepsilon)[\eta_\varepsilon] \cdot \eta_\varepsilon \) where
\[
s_\varepsilon = \sup\{d \tilde{\nu}(x)[\eta] \cdot \eta : x \in \tilde{T}_\varepsilon(\tilde{T}), \eta \in \mathbb{R}^n, |\eta| = 1\}. \tag{2.29}
\]
Since \( \text{dist}(x_\varepsilon, \tilde{T}) \to 0 \) as \( \varepsilon \to 0 \), and \( \tilde{T} \) is a compact manifold, from (2.26) we infer that \( \limsup_{\varepsilon \to 0} s_\varepsilon \leq 1 \). On the other hand, (2.26) implies \( s_\varepsilon \geq 1 \ \forall \varepsilon \in (0, \varepsilon_1) \), so (2.28) is proved.
Furthermore, one can easily verify that \( \tilde{\nu} \cdot \nu > 0 \) on \( \partial \tilde{T}_\varepsilon(\tilde{T}) \ \forall \varepsilon \in (0, \varepsilon_1) \). Thus, taking also into account condition (2.4), from Lemma 2.1 we infer that every solution \( \tilde{u}_\varepsilon \) of problem (1.1) in the domain \( \tilde{T}_\varepsilon(\tilde{T}) \) satisfies
\[
0 \leq \left[ 1 - \frac{n - 1}{2} + \tilde{\mu}(\varepsilon) \right] \int_{\tilde{T}_\varepsilon(\tilde{T})} |D \tilde{u}_\varepsilon|^2 dx + \left[ \frac{n - 1}{p} + \tilde{\mu}(\varepsilon) \right] \int_{\tilde{T}_\varepsilon(\tilde{T})} \tilde{u}_\varepsilon f(\tilde{u}_\varepsilon) dx, \tag{2.30}
\]
where $\tilde{\mu}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since
\[
\int_{\tilde{T}_\varepsilon(\tilde{\Gamma})} \tilde{u}_\varepsilon f(\tilde{u}_\varepsilon) \, dx = \int_{\tilde{T}_\varepsilon(\tilde{\Gamma})} |D\tilde{u}_\varepsilon|^2 \, dx
\]
(2.31)
(because $\tilde{u}_\varepsilon$ solves problem (1.1) in $\tilde{T}_\varepsilon(\tilde{\Gamma})$) we obtain
\[
0 \leq \left[ 1 - \frac{n-1}{2} + \frac{n-1}{p} + 2\tilde{\mu}(\varepsilon) \right] \int_{\tilde{T}_\varepsilon(\tilde{\Gamma})} |D\tilde{u}_\varepsilon|^2 \, dx
\]
(2.32)
where $1 - \frac{n-1}{2} + \frac{n-1}{p} < 0$ because $n \geq 4$ and $p > \frac{2(n-1)}{n-3}$. Therefore, there exists $\varepsilon \in (0, \bar{\varepsilon}_1)$ such that for all $\varepsilon \in (0, \varepsilon)$ (2.32) implies $\tilde{u}_\varepsilon \equiv 0$ in $\tilde{T}_\varepsilon(\tilde{\Gamma})$. So the proof is complete.

q.e.d.

3 Tubular domains of higher dimension and final remarks

The nonexistence results presented in Section 2 are concerned with domains $\Omega$ which are thin neighbourhoods of 1-dimensional manifolds (with boundary and contractible in Theorem 2.2, without boundary and noncontractible in Theorem 2.5). In this section we consider the case where $\Omega$ is a thin neighbourhood of $k$-dimensional smooth, compact manifold $\Gamma_k$ with $k > 1$.

If $\Gamma_k$ is a submanifold of $\mathbb{R}^n$ with $n > k$, for all $x \in \Gamma_k$ we set $N(x) = T^\perp(x)$ and $N_\varepsilon(x) = \{ x \in N(x) : |x| < \varepsilon \}$, where $T(x)$ is the tangent space to $\Gamma_k$ in $x$ and $N(x)$ is the normal space. Since $\Gamma_k$ is a compact smooth submanifold, there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, we have $[x_1 + N_\varepsilon(x_1)] \cap [x_2 + N_\varepsilon(x_2)] = \emptyset$ for all $x_1$ and $x_2$ in $\Gamma_k$ such that $x_1 \neq x_2$. Then, for all $\varepsilon \in (0, \varepsilon_1)$, we consider the piecewise smooth, bounded domain $T_\varepsilon(\Gamma_k)$ defined as the interior of the set $\cup_{x \in \Gamma_k} [x + N_\varepsilon(x)]$ (we say that $T_\varepsilon(\Gamma_k)$ is the tubular domain with thickness $\varepsilon$ and center $\Gamma_k$). Our aim is to study existence and nonexistence of nontrivial solutions of problem (1.1) in the domain $\Omega = T_\varepsilon(\Gamma_k)$.

Let us point out that when $k > 1$ one cannot prove a theorem analogous to Theorem 2.2. In fact, if $\Gamma_k$ is a $k$-dimensional manifold contractible in itself and $k > 1$, one cannot obtain nonexistence results for nontrivial solutions of problem (1.1) in the domain $\Omega = T_\varepsilon(\Gamma_k)$ under the assumption that condition (2.4) holds with $p > \frac{2n}{n-2}$ as in Theorem 2.2. The reason is explained by the following examples where existence results hold.
Example 3.1 For all $n \geq k + 1$, let us consider the function $\gamma_k : \mathbb{R}^k \to \mathbb{R}^n$ defined as follows:

$$
\gamma_k(x_1, \ldots, x_k) = \begin{cases} 
\frac{2x_i}{|x|^2 + 1} & \text{for } i = 1, \ldots, k \\
\frac{|x|^2 + 1}{2} & \text{for } i = k + 1 \\
0 & \text{for } i = k + 2, \ldots, n
\end{cases}
$$

(3.1)

($\gamma_k$ is the stereographic projection of $\mathbb{R}^k$ on a $k$-dimensional sphere of $\mathbb{R}^n$).

Moreover, for all $r > 0$, let us set $\Gamma^r_k = \{ \gamma_k(x) : x \in \mathbb{R}^k, |x| < r \}$.

Then one can easily verify that the domain $T_\varepsilon(\Gamma^r_k)$ is contractible in itself for all $r > 0$ and $\varepsilon \in (0, 1)$. Moreover, the following propositions hold.

Proposition 3.2 Let $k \geq 2$ and $n \geq k + 1$. Assume that $f(t) = |t|^{p-2}t$ with $p \geq \frac{2n}{n-2}$ and that $p < \frac{2(n-k+1)}{n-k-1}$ if $n > k + 1$.

Then, there exists $r > 0$ such that if $r > \bar{r}$ and $\varepsilon \in (0, 1)$, problem (1.1) in the domain $\Omega = T_\varepsilon(\Gamma^r_k)$ has positive and sign changing solutions; moreover, under the additional assumption $p > \frac{2n}{n-2}$, for all $\varepsilon \in (0, 1)$ the number of solutions tend to infinity as $r \to \infty$.

For the proof it suffices to look for solutions having radial symmetry with respect to the first $k$ variables and argue as in [9, 11, 14, 16, 19, 21, 22].

Proposition 3.3 Let $k \geq 2$, $n \geq k + 1$, $r > 1$, $\varepsilon \in (0, 1)$. Moreover, assume that $f(t) = |t|^{p-2}t \forall t \in \mathbb{R}$. Then, there exists $\bar{p} > \frac{2n}{n-2}$ such that, if $n = k + 1$ and $p \geq \bar{p}$ or if $n > k + 1$ and $p \in \left(\bar{p}, \frac{2(n-k+1)}{n-k-1}\right)$, problem (1.1) with $\Omega = T_\varepsilon(\gamma_k^r)$ has solution.

The proof can be carried out arguing for example as in [10] in order to obtain solutions having radial symmetry with respect to the first $k$ variables.

Proposition 3.4 Let $k \geq 2$, $n \geq k + 1$, $r > 1$, $\varepsilon \in (0, 1)$ and assume that $f(t) = |t|^{p-2}t \forall t \in \mathbb{R}$. Then, there exists $\bar{p} > \frac{2n}{n-2}$ such that problem (1.1) with $\Omega = T_\varepsilon(\gamma_k^r)$ has positive solutions for all $p \in \left(\frac{2n}{n-2}, \bar{p}\right)$. Moreover, the number of solutions tends to infinity as $p \to \frac{2n}{n-2}$.

The proof is based on a Lyapunov-Schmidt type finite dimensional reduction method as in [1, 9], etc.

Thus, while Theorem 2.2 gives a nonexistence result for all $p > \frac{2n}{n-2}$ when $k = 1$, $\Gamma_k$ is contractible in itself and $\Omega$ is a thin tubular domain centered in $\Gamma_k$, Propositions 3.2, 3.3 and 3.4 give examples of existence results for some $p > \frac{2n}{n-2}$ when $\Omega$ is a tubular domain centered in a suitable $k$-dimensional manifold $\Gamma_k^r$, contractible in itself but with $k \geq 2$. In this sense we mean that Theorem 2.2 cannot be extended to the case $k \geq 2$.
Taking into account the definition of the tubular domain noncontractible in itself) when $n > k + 2$ and $p > \frac{2(n-k)}{n-k-2}$, as we prove in the following Theorem 3.5.

If $n \leq k + 2$ or $n > k + 2$ and $p < \frac{2(n-k)}{n-k-2}$, the existence of nontrivial solutions can be proved even if $\Omega$ is a tubular domain $T_\varepsilon(\Gamma_k)$ with $\varepsilon$ not necessarily small; for example, if $\Gamma_k$ is a $k$-dimensional sphere, we can look for solutions with radial symmetry with respect to $k+1$ variables, so we obtain infinitely many solutions for all $\varepsilon \in (0, R)$ where $R$ is the radius of the sphere.

**Theorem 3.5** Let $k \geq 1$, $n > k + 2$ and assume that $\Gamma_k$ is a $k$-dimensional, compact, smooth submanifold of $\mathbb{R}^n$. Moreover, assume that condition (2.4) holds with $p > \frac{2(n-k)}{n-k-2}$.

Then, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, the Dirichlet problem (1.1) has only the trivial solution $u \equiv 0$ on the tubular domain $\Omega = T_\varepsilon(\Gamma_k)$.

**Proof** Taking into account the definition of the tubular domain $T_\varepsilon(\Gamma_k)$, for all $\varepsilon \in (0, \bar{\varepsilon})$ and $x \in T_\varepsilon(\Gamma_k)$ there exists a unique $y \in \Gamma_k$ such that $x \in y + N_\varepsilon(y)$. Then, denote this $y$ by $p_\varepsilon(x)$ and set $v_\varepsilon(x) = x - p_\varepsilon(x)$ $\forall x \in T_\varepsilon(\Gamma_k)$. One can easily verify that the vector field $v_\varepsilon$ satisfies $v_\varepsilon \cdot \nu \geq 0$ on $\partial T_\varepsilon(\Gamma_k) \forall \varepsilon \in (0, \bar{\varepsilon})$.

Therefore, from Lemma 2.1 we infer that every solution $u_\varepsilon$ of problem (1.1) in $T_\varepsilon(\Gamma_k)$ satisfies

$$0 \leq \int_{T_\varepsilon(\Gamma_k)} dv_\varepsilon[Du_\varepsilon] \cdot Du_\varepsilon \, dx + \int_{T_\varepsilon(\Gamma_k)} \text{div} \, v_\varepsilon \left( F(u_\varepsilon) - \frac{1}{2} |Du_\varepsilon|^2 \right) \, dx. \tag{3.2}$$

Notice that

$$dv_\varepsilon(x)[\phi] = 0, \quad dv_\varepsilon(x)[\psi] = \psi \quad \forall x \in \Gamma_k, \ \forall \phi \in T(x), \ \forall \psi \in N(x) \tag{3.3}$$

as one can verify by direct computation.

As a consequence we obtain

$$\text{div} \, v_\varepsilon(x) = n-k, \quad dv_\varepsilon(x)[\phi + \psi] \cdot (\phi + \psi) = |\psi|^2 \quad \forall x \in \Gamma_k, \ \forall \phi \in T(x), \ \forall \psi \in N(x). \tag{3.4}$$

Since $\Gamma_k$ is a compact manifold, it follows that

$$\lim_{\varepsilon \to 0} \sup \{|n-k - \text{div} \, v_\varepsilon(x)| : x \in T_\varepsilon(\Gamma_k)\} = 0 \tag{3.5}$$

and

$$\lim_{\varepsilon \to 0} \sup \{|dv_\varepsilon(x)[\eta] \cdot \eta : x \in T_\varepsilon(\Gamma_k), \ \eta \in \mathbb{R}^n, \ |\eta| = 1\} = 1 \tag{3.6}$$
as one can infer arguing as in the proof of Theorem \(2.5\).

Thus, taking also into account that
\[
\int_{T_\varepsilon(\Gamma_k)} u_\varepsilon f(u_\varepsilon) \, dx = \int_{T_\varepsilon(\Gamma_k)} |Du_\varepsilon|^2 \, dx,
\]
from condition \((2.4)\) we infer that
\[
0 \leq \left[ 1 - \frac{n-k}{2} + \frac{n-k}{p} \right] \int_{T_\varepsilon(\Gamma_k)} |Du_\varepsilon|^2 \, dx
\]
where \(\mu_k(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Since \(1 - \frac{n-k}{2} + \frac{n-k}{p} < 0\) (because \(n > k + 2\) and \(p > \frac{2(n-k)}{n-k-2}\)), it follows that there exists \(\varepsilon \in (0, \varepsilon_1)\) such that, for all \(\varepsilon \in (0, \varepsilon)\) we have \(u_\varepsilon \equiv 0\) in \(T_\varepsilon(\Gamma_k)\), so the problem has only the trivial solution \(u \equiv 0\).

\(q.e.d.\)

**Remark 3.6** Proposition \(3.2\), as well as the results reported in \([9, 11, 14, 16, 13, 21, 22]\), suggest that the existence of nontrivial solutions is related to the property that the domain \(\Omega\) is obtained by removing a subset of small capacity from a domain having a different \(k\)-dimensional homology group with \(k \geq 2\).

For example, in the case of domains with small holes, every hole has small capacity and changes the \((n-1)\)-dimensional homology group.

In the case of tubular domains \(T_\varepsilon(\Gamma_k^r)\), the existence results for \(k \geq 2\) and \(r\) large enough given by Proposition \(3.2\) is related to the fact that \(\Gamma_k^r\) tends to a \(k\)-dimensional sphere \(S_k\) as \(r \to \infty\), the capacity of \(T_\varepsilon(S_k) \setminus T_\varepsilon(\Gamma_k^r)\) tends to 0 as \(r \to \infty\) and the domains \(T_\varepsilon(S_k)\) and \(T_\varepsilon(\Gamma_k^r)\) have different \(k\)-dimensional homology group.

On the contrary, when \(k = 1\), the capacity of \(T_\varepsilon(S_1) \setminus T_\varepsilon(\Gamma_1^r)\) does not tend to 0 as \(r \to \infty\). This fact explains the nonexistence result given by Theorem \(2.2\) in the case of the domains \(T_\varepsilon(\Gamma_1^r)\), when \(\varepsilon\) is small enough, for all \(r > 0\).

**Remark 3.7** If \(n = 2\) we do not have critical or supercritical phenomena for the Laplace operator. But, if we replace it by the \(q\)-Laplace operator, this phenomena arise and may produce nonexistence results for nontrivial solutions. For example, if we consider the Dirichlet problem
\[
\text{div}(|Du|^{p-2}Du) + |u|^{p-2}u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]
where \(\Omega\) is a bounded domain of \(\mathbb{R}^2\), \(1 < q < 2\), \(p \geq \frac{2q}{2-q}\), then one can prove nonexistence results in some bounded contractible domains which can be non starshaped and even arbitrarily close to noncontractible domains (see \([12, 13]\)). For example, if \(\Omega = T_\varepsilon(\Gamma_1^r)\), there exists \(\varepsilon > 0\) such that problem \((3.9)\) has only the trivial solution \(u \equiv 0\) for all \(r > 0\) and \(\varepsilon \in (0, \varepsilon)\).
The results obtained in [12, 13] suggest that the nonexistence of nontrivial solutions for Dirichlet problem (3.9) might be proved in all the contractible domains of $\mathbb{R}^2$ (while it is not possible for problem (1.2) when $n \geq 3$ and $p \geq \frac{2n}{n-2}$ because of Proposition 3.2).

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