Explicit Formulae for Rank Zero DT Invariants and the OSV Conjecture

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Abstract. Fix a Calabi-Yau 3-fold $X$ satisfying the Bogomolov-Gieseker conjecture of Bayer-Macrì-Toda, such as the quintic 3-fold.

By two different wall-crossing arguments we prove two different explicit formulae relating rank 0 Donaldson-Thomas invariants (counting torsion sheaves on $X$ supported on ample divisors) in terms of rank 1 Donaldson-Thomas invariants (counting ideal sheaves of curves) and Pandharipande-Thomas invariants. In particular, we prove a slight modification of Toda’s formulation of OSV conjecture for $X$.

When $X$ is of Picard rank one, we also give an explicit formula for rank two DT invariants in terms of rank zero and rank one DT invariants.

1. Introduction

Let $(X,\mathcal{O}_X(1))$ be a smooth polarised Calabi-Yau threefold satisfying the Bogomolov-Gieseker conjecture of [BMT14, BMS16] and let $H := c_1(\mathcal{O}(1))$. Let $J(\alpha) \in \mathbb{Q}$ be the Joyce-Song’s generalised Donaldson-Thomas invariant [JS12] counting $H$-Gieseker semistable sheaves of numerical K-theory class $\alpha$ on $X$. Fix a rank zero, dimension 2 class

$$ v = (0, D, \beta, m) \in K(X) $$

with $D \neq 0$. In this paper, we relate $J(v)$ to Donaldson-Thomas (DT) type invariants on $X$ counting ideal sheaves of curves and Pandharipande-Thomas (PT) stable pairs via two different procedures.

Method I: We do wall-crossing in the space of weak stability condition for the class $v$ as in [Tod13]. Then we find an explicit expression of $J(v)$ in terms of rank 1 DT invariants and PT invariant. However, the construction works only for specific classes $v$.

The PT stable pair invariant $P_{m_1, \beta_1}$ counts pairs $(F, s)$ consisting of a 1-dimensional pure sheaf $F$ with $(\text{ch}_2(F), \text{ch}_3(F)) = (\beta_1, m_1)$ and a section

$$ s: \mathcal{O}_X \to F $$

with zero-dimensional cokernel. Also $I_{m_2, \beta_2}$ is the rank 1 DT invariant counting ideal sheaves of subschemes $C \subset X$ satisfying

$$ [C] = \beta_2 \quad \text{and} \quad \chi(\mathcal{O}_C) = m_2. $$

For the rank zero class $v$ (1), we define

$$ Q(v) := \frac{1}{2} \left( \frac{D.H^2}{H^3} \right)^2 + 6 \left( \frac{\beta.H}{D.H^2} \right)^2 - \frac{12m}{D.H^2}. $$
Theorem 1.1. (i) If $Q(\nu) < 0$, there is no slope-semistable sheaf of class $\nu$.

(ii) If
\[(H^3)^2 Q(\nu) < D.H^2 + \frac{2}{D.H^2} \cdot \frac{5}{2} \cdot \frac{2}{(D.H^2)^2},\]
any slope-semistable sheaf of class $\nu$ is slope-stable and
\[J(\nu) = \left( \# H^2(X,\mathbb{Z})_{\text{tors}} \right)^2 \sum_{v_1 = -e^{D_1}(1,0,-\beta_1,-m_1)} (-1)^{\chi(v_2,v_1)} \chi(v_2,v_1) \cdot P_{-m_1,\beta_1} \cdot I_{m_2,\beta_2}.\]
Here $M(\nu)$ is a subset of $H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \oplus H^6(X,\mathbb{Z})$ described in Definition 3.4 which only depends on $\text{ch}_1(\nu).H^2 = D.H^2$.

In [Tod13, Theorem 3.18] Toda proved Theorem 1.1 for Calabi-Yau 3-folds $X$ with $\text{Pic}(X) = \mathbb{Z}.H$. In this paper, we apply a different wall-crossing argument to manage all Calabi-Yau 3-folds with arbitrary Picard rank.

Toda’s work in [Tod13] is motivated by Denef–Moore’s approach [DM11] toward Ooguri–Strominger–Vafa (OSV) conjecture [OSV04] relating black hole entropy and topological string on Calabi-Yau 3-folds. In [Tod13, Conjecture 1.1], he gives a mathematical formulation of OSV conjecture and proves the conjecture when $X$ is of Picard rank one. In Section 4, as a result of Theorem 1.1, we prove a slight modification\(^1\) of his conjecture for arbitrary Calabi-Yau 3-folds, see Theorem 4.1.

Method II. To find an explicit expression for rank 0 DT invariants in terms of rank 1 DT and PT invariants, we may apply the same technique as [FT21c]. Fix $n > 0$ and consider a non-zero map $s: O_X(-n) \to F$ for an $H$-Gieseker semistable sheaf $F$ of class $\nu$. Then we do wall-crossing on the space of weak stability conditions for class $\nu_0 := \text{ch}(\text{cone}(s))$ instead of class $\nu$. By [FT21c, Theorem 2], this gives a universal formula expressing $J(\nu)$ in terms of PT invariants. In this paper we show that the formula can be made explicit when $\text{Pic}(X) = \mathbb{Z}.H$. Note that against Method I, this approach works for any arbitrary class $\nu$ of rank zero, but the final relation is not as direct as what we get via Method I. We set
\[\text{PT}^{\nu,n}(x,y,z) := \sum_{\alpha_1 = -e^{H}(1,0,-\beta_1,m_1) \in M_{\nu,n}} \text{P}_{m_1,\beta_1} x^{k_1} y^{\beta_1 - \frac{1}{2}k_1^2 H^2} z^{-m_1 - k_1 \beta_1 H + \frac{1}{2}k_1^3 H^3},\]
where $M_{\nu,n}$ is a set of rank $-1$ classes in $K(X)$ depending on $\nu$ and $n$, see Section 5. For any $\mu \in \mathbb{R}$, we consider the generating series
\[A(\nu_n,\mu) := \text{PT}^{\nu,n}(x,y,z) : \prod_{\alpha' = (0,k'H,\beta',m') \in K(X)} \exp \left( (-1)^{\chi_{\alpha'}} \chi_{\alpha'} \cdot J(\alpha') \cdot x^{k'} y^{\beta'} z^{m'} \right) \]
where $\chi_{\alpha'}$ is a real number described in Definition 5.5.

\(^1\)See Remark 4.2 for more details.
Theorem 1.2. Let \( X \) be a smooth Calabi-Yau 3-fold with \( \text{Pic}(X) = \mathbb{Z}.H \), and let \( \text{ch}_1(v) = D = kH \). There is \( \mu_{v,n} \in \mathbb{R} \) such that the coefficient of \( x^{n+k}y^\beta z^m + \frac{n^2H^2}{2}z^m + \frac{n^3H^3}{6} \) in the series

\[
\chi(O_X(-n), v) \frac{(-1)^{\chi(O_X(-n), v)+1}}{\chi(O_X(-n), v)} A(v_n, \mu_{v,n})
\]

is equal to \( J(v) \). This expresses \( J(v) \) in terms of PT invariants and DT invariants for rank zero classes with lower \( \text{ch}_1 H^2 \), so an inductive argument gives \( J(v) \) in terms of PT invariants.

**Rank 2 case.** Fix a rank 2 class \( w \in K(X) \) and \( n \gg 0 \). It has been shown in [FT21b] that there exists a universal formula expressing \( J(w) \) in terms of rank 1 DT invariant and rank zero DT invariants. By applying a similar argument as in Method II above, we prove in Section 6 that the formula can be made explicit when \( \text{Pic}(X) = \mathbb{Z}.H \), see Theorem 6.1. Combining this with Theorem 1.2 gives an expression of rank 2 DT invariants in terms of rank 1 DT invariants and PT invariants.

**Outlook.** The main wall-crossing arguments in this paper have been done in Section 3 which are valid for any smooth projective 3-fold satisfying the Bogomolov-Gieseker conjecture of [BMT14]. Applying Joyce’s new wall-crossing formula for Fano 3-folds [Joy21, Theorem 7.69] should hopefully give a Fano version of Theorem 1.1 and Theorem 1.2 with insertions.

In this paper, we consider arbitrary Calabi-Yau 3-folds and arbitrary rank 0 or 2 classes in \( K(X) \). However, to get a more explicit formula we need to either

(a) restrict ourselves to special types of Calabi-Yau 3-folds like \( \mathbb{C}^3 \times E \) [Obe22] or a smooth intersection of quadratic and quartic hypersurfaces in \( \mathbb{P}^5 \) [LQ13], or

(b) consider special rank zero or 2 classes \( \alpha \in K(X) \) like \( \text{ch}_0(\alpha) = 2 \) and \( \text{ch}_1(\alpha) = \text{ch}_2(\alpha) = 0 \) as discussed in [Sto12, Tod10b].

Our main technique for controlling walls of instability is the Bogomolov-Gieseker conjecture of [BMT14]. This is now proved for many 3-folds, including some Calabi-Yau 3-folds [BMS16, MP16]. It is proved for a restricted set of weak stability conditions on quintic threefold [Li19] and a complete intersection of quadratic and quartic hypersurfaces in \( \mathbb{P}^5 \) [Liu21] which are sufficient for our purposes, see Lemma 3.7 and Remark 6.7. A weaker version is also proved for double or triple cover CY3 [Kos20] which is enough for Theorem 1.2 and Theorem 6.1 but leads to a weaker version of Theorem 1.1.

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2. Weak stability conditions & wall-crossing formula

Let \((X, \mathcal{O}_X(1))\) be a smooth polarised complex projective threefold with bounded derived category of coherent sheaves \(\mathcal{D}(X)\) and Grothendieck group \(K(\text{Coh}(X))\). Dividing by the kernel of the Mukai pairing gives the numerical Grothendieck group

\[
K(X) := \frac{K(\text{Coh}(X))}{\ker \chi(\ ,\ )}.
\]

Notice \(K(X)\) is torsion-free, isomorphic to its image in \(H^\ast (X, \mathbb{Q})\) under the Chern character. Denoting \(H = c_1(\mathcal{O}_X(1))\), for any \(v \in K(X)\) we set

\[
\begin{align*}
\text{ch}_H(v) &:= \left( \text{ch}_0(v), \frac{1}{H^2} \text{ch}_1(v).H^2, \frac{1}{H^2} \text{ch}_2(v).H, \frac{1}{H^2} \text{ch}_3(v) \right) \in \mathbb{Q}^4, \\
\text{ch}^{\leq 2}_H(v) &:= \left( \text{ch}_0(v), \frac{1}{H^2} \text{ch}_1(v).H^2, \frac{1}{H^2} \text{ch}_2(v).H \right) \in \mathbb{Q}^3.
\end{align*}
\]

We define the \(\mu_H\)-slope of a coherent sheaf \(E\) to be

\[
\mu_H(E) := \begin{cases} 
\frac{\text{ch}_1(E).H^2}{\text{ch}_0(E).H} & \text{if } \text{ch}_0(E) \neq 0, \\
+\infty & \text{if } \text{ch}_0(E) = 0.
\end{cases}
\]

Associated to this slope every sheaf \(E\) has a Harder-Narasimhan filtration. Its graded pieces have slopes whose maximum we denote by \(\mu_H^+(E)\) and minimum by \(\mu_H^-(E)\).

For any \(b \in \mathbb{R}\), let \(\mathcal{A}_b \subset \mathcal{D}(X)\) denote the abelian category of complexes

\[
\mathcal{A}_b = \{ E^{-1} \xrightarrow{d} E^0 : \mu_H^+(\ker d) \leq b, \mu_H^-(\text{coker } d) > b \}.
\]

In particular, setting \(\text{ch}^b_H(E) := \text{ch}(E)e^{-bH}\), each \(E \in \mathcal{A}_b\) satisfies

\[
\text{ch}_1^b(H)(E).H^2 = \text{ch}_1(E).H^2 - bH^3 \text{ch}_0(E) \geq 0.
\]

By [Bri08, Lemma 6.1] \(\mathcal{A}_b\) is the heart of a bounded t-structure on \(\mathcal{D}(X)\). We denote its positive cone by

\[
C(\mathcal{A}_b) := \left\{ \sum_i a_i[E_i] : a_i \in \mathbb{N}, E_i \in \mathcal{A}_b \right\} \subset K(X).
\]

For any \(w > \frac{1}{2}b^2\), we have on \(\mathcal{A}_b\) the slope function

\[
\nu_{b,w}(E) = \begin{cases} 
\frac{\text{ch}_2(E).H - w \text{ch}_0(E).H^3}{\text{ch}_1^b(H)(E).H^2} & \text{if } \text{ch}_1^b(H)(E).H^2 \neq 0, \\
+\infty & \text{if } \text{ch}_1^b(H)(E).H^2 = 0.
\end{cases}
\]
By \([BMT14]^{2}\) \(\nu_{b,w}\) defines a Harder–Narasimhan filtration on \(\mathcal{A}_b\), and so a \textit{weak stability condition} on \(\mathcal{D}(X)\).

**Definition 2.1.** Fix \(w > \frac{1}{2}b^2\). Given an injection \(F \hookrightarrow E\) in \(\mathcal{A}_b\) we call \(F\) a \textit{destabilising subobject} of \(E\) if and only if
\[
\nu_{b,w}(F) \geq \nu_{b,w}(E/F),
\]
and \textit{strictly destabilising} if \(>\) holds. We say \(E \in \mathcal{D}(X)\) is \(\nu_{b,w}\)-(semi)stable if and only if
1. \(E[k] \in \mathcal{A}_b\) for some \(k \in \mathbb{Z}\), and
2. \(E[k]\) contains no (strictly) destabilising subobjects.

By \([BMS16]\), Theorem 3.5\] any \(\nu_{b,w}\)-semistable object \(E \in \mathcal{D}(X)\) satisfies
\[
\Delta_H(E) := (\text{ch}_1(E).H^2)^2 - 2(\text{ch}_2(E).H)\text{ch}_0(E)H^3 \geq 0.
\]
Therefore, if we plot the \((b, w)\)-plane simultaneously with the image of the projection map
\[
\Pi: K(X) \setminus \{E: \text{ch}_0(E) = 0\} \longrightarrow \mathbb{R}^2,
\]
\[
E \longmapsto \left(\frac{\text{ch}_1(E).H^2}{\text{ch}_0(E)H^3}, \frac{\text{ch}_2(E).H}{\text{ch}_0(E)H^3}\right),
\]
as in Figure 1, then \(\nu_{b,w}\)-semistable objects \(E\) lie outside the open set
\[
U := \left\{(b, w) \in \mathbb{R}^2: w > \frac{1}{2}b^2\right\}
\]
while \((b, w)\) lies inside \(U\). The slope \(\nu_{b,w}(E)\) of \(E\) is the gradient of the line connecting \((b, w)\) to \(\Pi(E)\) (or \(\text{ch}_2(E).H/\text{ch}_1(E).H^2\) if \(\text{rank}(E) = 0\)).

\[
\begin{tikzpicture}
    % TikZ code for the diagram
    % ... (the actual code is not shown here)
\end{tikzpicture}
\]

**Figure 1.** \((b, w)\)-plane and the projection \(\Pi(E)\) when \(\text{ch}_0(E) < 0\)

Objects in \(\mathcal{D}(X)\) give the space of weak stability conditions a wall and chamber structure by \([BMS16],\) Proposition 12.5], as rephrased in \([FT21a],\) Proposition 4.1] for instance.

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2We use notation from \([FT20]\); in particular the rescaling \([FT20,\) Equation 6] of \([BMT14]\)'s slope function.
Proposition 2.2 (Wall and chamber structure). Fix \( v \in K(X) \) with \( \Delta_H(v) \geq 0 \) and \( \text{ch}^\leq_2(v) \neq 0 \). There exists a set of lines \( \{ \ell_i \}_{i \in I} \) in \( \mathbb{R}^2 \) such that the segments \( \ell_i \cap U \) (called “walls of instability”) are locally finite and satisfy

(a) If \( \text{ch}_0(v) \neq 0 \) then all lines \( \ell_i \) pass through \( \Pi(v) \).
(b) If \( \text{ch}_0(v) = 0 \) then all lines \( \ell_i \) are parallel of slope \( \frac{\text{ch}_2(v).H}{\text{ch}_1(v).H^2} \).
(c) The \( \nu_{b,w} \)-stability of any \( E \in \mathcal{D}(X) \) of class \( v \) is unchanged as \( (b, w) \) varies within any connected component (called a “chamber”) of \( U \setminus \bigcup_{i \in I} \ell_i \).
(d) For any wall \( \ell_i \cap U \) there is a map \( f : F \to E \) in \( \mathcal{D}(X) \) such that
   - for any \( (b, w) \in \ell_i \cap U \), the objects \( E, F \) lie in the heart \( \mathcal{A}_b \),
   - \( E \) is \( \nu_{b,w} \)-semistable of class \( v \) with \( \nu_{b,w}(E) = \nu_{b,w}(F) = \text{slope} (\ell_i) \) constant on the wall \( \ell_i \cap U \), and
   - \( f \) is an injection \( F \to E \) in \( \mathcal{A}_b \) which strictly destabilises \( E \) for \( (b, w) \) in one of the two chambers adjacent to the wall \( \ell_i \). □

In this paper, we always assume \( X \) satisfies the conjectural Bogomolov-Gieseker inequality of Bayer-Macrì-Toda [BMT14]. In the form of [BMS16, Conjecture 4.1], rephrased in terms of the rescaling [FT20, Equation 6], it is the following.

Conjecture 2.3 (Bogomolov-Gieseker inequality). For any \( (b, w) \in U \) and \( \nu_{b,w} \)-semistable \( E \in \mathcal{D}(X) \), we have the inequality

\[
B_{b,w}(E) := (2w - b^2)\Delta_H(E) + 4(\text{ch}_2^H(E).H)^2 - 6(\text{ch}_1^H(E).H^2)\text{ch}_3^H(E) \geq 0.
\]

Multiplying out and cancelling we find that \( B_{b,w} \) is actually linear in \( (b, w) \):

\[
\frac{1}{2}B_{b,w}(E) = (C_1^2 - 2C_0C_2)w + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3),
\]

where \( C_i := \text{ch}_i(E).H^{3-i} \). The coefficient of \( w \) is \( \geq 0 \) by (9). When it is \( > 0 \) the Bogomolov-Gieseker inequality (11) says that \( E \) can be \( \nu_{b,w} \)-semistable only above the line \( \ell_f(E) \) defined by the equation \( B_{b,w}(E) = 0 \). When \( \text{ch}_0(E) \neq 0 \neq \text{ch}_1(E).H^2 \) we can rearrange to see \( \ell_f(E) \) is the line through the points \( \Pi(E) \) and

\[
\Pi'(E) := \left( \frac{2\text{ch}_2(E).H}{\text{ch}_1(E).H^2}, \frac{3\text{ch}_3(E)}{\text{ch}_1(E).H^2} \right).
\]

Tilt (Gieseker) stability. Given a sheaf of class \( \alpha \in K(X) \) with Hilbert polynomial

\[
P_\alpha(t) := \chi(\alpha(t)) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0,
\]

where \( d \leq 3 \) and \( a_d \neq 0 \). Its reduced Hilbert polynomial is

\[
p_\alpha(t) := \frac{P_\alpha(t)}{a_d}.
\]

Following Joyce [Joy07, Section 4.4] we introduce a total order \( \prec \) on \( \{ \text{monic polynomials}\} \cup \{0\} \) by saying \( p \prec q \) if and only if

(i) \( \deg p > \deg q \), or
(ii) \( \deg p = \deg q \) and \( p(t) < q(t) \) for \( t \gg 0 \).
Our convention is that $\deg 0 = 0$. Then $E \in \text{Coh}(X)$ is called Gieseker (semi)stable if for all exact sequences $0 \to A \to E \to B \to 0$ in $\text{Coh}(X)$ we have

$$p_{[A]} (\preceq) p_{[B]}.$$ 

Here $(\preceq)$ means $\prec$ for stability and $\preceq$ (which is $\prec$ or $=$) for semistability. In particular (i) ensures that Gieseker semistable sheaves are pure.

Discarding the constant term of the Hilbert polynomial before dividing by its top coefficient as before gives

$$(14) \quad \tilde{p}_\alpha(t) := p_\alpha(t) - \frac{a_0}{a_d} = t^d + \cdots + \frac{a_1}{a_d} t.$$ 

It depends only on $\text{ch} \leq 2 \text{H}(\alpha)$. Then tilt (semi)stability on $\text{Coh}(X)$ is defined by the inequalities

$$\tilde{p}_{[A]} (\preceq) \tilde{p}_{[B]}$$ 

for all exact sequences of sheaves $0 \to A \to E \to B \to 0$.

**Large volume limit.** We can improve on the local finiteness of walls by showing we have finiteness as $w \to \infty$. This gives, for each fixed $v \in K(X)$, a large volume chamber $\subseteq U$ in which there are no walls for $v$, so the $\nu_{b,w}$-(semi)stability of objects of class $v$ is independent of $w \gg 0$, see for instance [FT21c, Proposition 1.3]. Moreover, large volume stability coincides with classical stability.

**Lemma 2.4.** Take an object $E \in A_b$ for some $b \in \mathbb{R}$.

(a) Suppose $\text{ch}_0(E) \geq 0$ and $b < \mu_H(E)$. Then $E$ is $\nu_{b,w}$-(semi)stable for $w \gg 0$ if and only if $E$ is a tilt-(semi)stable sheaf.

(b) If $\text{ch}_0(E) = -1$ and $\mu_H(E) < b$, then $E$ is $\nu_{b,w}$-semistable for $w \gg 0$ if and only if $E' \otimes (\det(E))^{-1} [1]$ is a stable pair, i.e. isomorphic to a 2-term complex $\mathcal{O}_X \to F$ in $\mathcal{D}(X)$ (with $\mathcal{O}_X$ in degree 0) such that

- $F$ is a pure 1-dimensional sheaf, and
- $s: \mathcal{O}_X \to F$ has zero-dimensional cokernel.

In particular, in this case $E$ is $\nu_{b,w}$-semistable for $w \gg 0$ if and only if it is $\nu_{b,w}$-stable.

**Proof.** Part (a) follows by the same argument as in [Bri08, Proposition 14.2]. Part (b) is proved in [Tod13, Section 3] and [FT21c, Lemma A.2 and Lemma A.3].

**Wall-crossing formula.** Now assume $X$ is a Calabi-Yau 3-fold, i.e. $K_X \cong \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$. For any two objects $E, F \in \mathcal{D}(X)$, the Hirzebruch–Riemann–Roch Theorem implies that the Euler form $\chi([E], [F]) = \sum_i (-1)^i \dim \text{Ext}^i(E, F)$ is given in terms of their Chern characters by

$$
\chi([E_1], [E_2]) = \text{ch}_0(E_1) \text{ch}_3(E_2) - \text{ch}_0(E_2) \text{ch}_3(E_1) + \text{ch}_1(E_2) \text{ch}_2(E_1) - \text{ch}_1(E_1) \text{ch}_2(E_2) + \frac{1}{12} c_2(X)(\text{ch}_0(E_1) \text{ch}_1(E_2) - \text{ch}_0(E_2) \text{ch}_1(E_1)).
$$
For any \((b, w) \in U\) and any class \(v \in K(X)\) with \(\nu_{b,w}(v) < +\infty\), we can apply the work of Joyce-Song [JS12] to define generalised DT invariants

\[ J_{b,w}(v) \in \mathbb{Q} \]

counting \(\nu_{b,w}\)-semistable objects \(E \in \mathcal{A}_b\) of class \(v\). In [FT21b, Section 4 and Appendix C], these invariants have been described and shown that they are independent of the choice of \((b, w)\) inside each chamber described in Proposition 2.2 for class \(v\). Also the Joyce-Song wall crossing formula applies to the \(J_{b,w}(v)\) under the same \(\nu_{b,w}(v) < +\infty\) condition.

Suppose \(\ell\) is a non-vertical wall for a class \(v \in K(X)\) with \(\text{ch}_1(v)H^2 > 0\) if \(\text{ch}_0(v) = 0\). Thus \(\ell\) passes through \(\Pi(v)\) if \(\text{ch}_0(v) \neq 0\); otherwise is of gradient \(\text{ch}_2(v).H/\text{ch}_1(v).H^2\). Let \((b, w_0)\) be a point on the line segment \(\ell \cap U\). By the local finiteness of walls of Proposition 2.2 we may choose

\[(b, w^\pm) \in U \text{ just above and below the wall } \ell,\]
in the sense that \((b, w^\pm) \notin \ell\) and between \((b, w_-)\) and \((b, w_+)\) there are no walls for \(v\), nor any of its finitely many semistable factors, except for \(\ell\). Then the wall crossing formula [JS12, Equation (5.13)] gives

\[
J_{b,w^-}(v) = \sum_{q \geq 1, \alpha_1, \ldots, \alpha_q \in C(\mathcal{A}_b): \alpha_1 + \cdots + \alpha_q = v} \sum_{\text{connected, simply-connected digraphs } \Gamma: \text{vertices } \{1, \ldots, q\}, \text{edge } i \to j \text{ implies } i < j} (-1)^{q-1+\sum_{1 \leq i < j \leq q} \chi(\alpha_i, \alpha_j)} \frac{\chi(\alpha_1, \ldots, \alpha_q; (b, w^+), (b, w^-))}{2q-1} \prod_{\text{edges } i \to j \text{ in } \Gamma} \chi(\alpha_i, \alpha_j) \prod_{i=1}^q J_{b,w^+}(\alpha_i).
\]

Here \(C(\mathcal{A}_b)\) is the positive cone (6) and the coefficients \(U(\alpha_1, \ldots, \alpha_q; (b, w^+), (b, w^-))\) are defined as follows.

**Definition 2.5.** [JS12, Definition 3.12] Take two points \((b, w_1), (b, w_2) \in U\) and let \(q \geq 1\) and \(\alpha_1, \ldots, \alpha_q \in C(\mathcal{A}_b)\). If for all \(i = 1, \ldots, q - 1\) we have either

(a) \(\nu_{b,w_1}(\alpha_i) \leq \nu_{b,w_1}(\alpha_{i+1})\) and \(\nu_{b,w_2}(\alpha_1 + \cdots + \alpha_i) > \nu_{b,w_2}(\alpha_{i+1} + \cdots + \alpha_q)\) or

(b) \(\nu_{b,w_1}(\alpha_i) > \nu_{b,w_1}(\alpha_{i+1})\) and \(\nu_{b,w_2}(\alpha_1 + \cdots + \alpha_i) \leq \nu_{b,w_2}(\alpha_{i+1} + \cdots + \alpha_q)\),

then define \(S(\alpha_1, \ldots, \alpha_q; (b, w_1), (b, w_2)) = (-1)^r\) where \(r\) is the number of \(i = 1, \ldots, q - 1\) satisfying (a). Otherwise define \(S(\alpha_1, \ldots, \alpha_q; (b, w_1), (b, w_2)) = 0\). Now define

\[
U(\alpha_1, \ldots, \alpha_q; (b, w_1), (b, w_2)) = \sum_{1 \leq p \leq q} \frac{(-1)^{p-1}}{p!} \prod_{i=1}^p S(\mathcal{B}_{b_{i-1}+1}, \mathcal{B}_{b_{i-1}+2}, \ldots, \mathcal{B}_{b_i}; (b, w_1), (b, w_2)) \cdot \prod_{i=1}^t \frac{1}{(a_i - a_{i-1})!}.
\]
One can easily check that the term \( U(\alpha_1, \ldots, \alpha_q; (b, w^+), (b, w^-)) \) is zero unless there is a \( \nu_{b,w_0} \)-semistable object \( E \) of class \( v \) with \( \nu_{b,w_0} \)-semistable factors of classes \( \alpha_1, \ldots, \alpha_q \).

The formula reflects the different Harder-Narasimhan filtrations of \( E \) on the two sides of the wall, and then further filtrations of the semistable Harder-Narasimhan factors by semi-stabilising subobjects.

The complicated formula (16) can be simplified when \( q = 2 \). Suppose \( \nu_{b,w_0}(\alpha_1) = \nu_{b,w_0}(\alpha_2) \),

\[
\nu_{b,w^+}(\alpha_1) > \nu_{b,w^+}(\alpha_2) \quad \text{and} \quad \nu_{b,w^-}(\alpha_1) < \nu_{b,w^-}(\alpha_2).
\]

In the definition of \( U(\alpha_1, \alpha_2; (b, w^+), (b, w^-)) \), we must have \( p = 1 \) and \( t = 2 \), thus

\[
U(\alpha_1, \alpha_2; (b, w^+), (b, w^-)) = S(\alpha_1, \alpha_2; (b, w^+), (b, w^-)) = 1
\]

and

\[
U(\alpha_2, \alpha_1; (b, w^+), (b, w^-)) = S(\alpha_2, \alpha_1; (b, w^+), (b, w^-)) = -1
\]

Thus the summation of coefficient for these two factors in (16) is

\[
(-1)^{\chi(\alpha_1, \alpha_2)+1}\chi(\alpha_1, \alpha_2).
\]

Let

\[
J_{\text{ti}}(\alpha) \in \mathbb{Q} \quad \text{and} \quad J(\alpha) \in \mathbb{Q}
\]

count tilt (Gieseker)-semistable sheaves of class \( \alpha \), see [FT21b, Section 4] for more details.

If all Gieseker semistable sheaves of class \( \alpha \in K(X) \) are Gieseker stable then \( J(\alpha) \in \mathbb{Z} \) is the original DT invariant defined in [Tho00]. We know tilt stability dominates Gieseker stability in the sense of [Joy07, Definition 4.10]. Thus we may apply Joyce-Song wall-crossing formula:

\[
J_{\text{ti}}(v) = \sum_{q \geq 1, \alpha_1, \ldots, \alpha_q \in C(\text{Coh}(X)):\ \text{connected, simply-connected digraphs } \Gamma:\ \text{vertices } \{1, \ldots, q\}, \text{edge } i \to j \text{ implies } i < j}
\]

\[
(-1)^{q-1+\sum_{1 \leq i < j \leq q} \chi(\alpha_i, \alpha_j)} \sum_{\alpha_1 + \cdots + \alpha_q = v} U(\alpha_1, \ldots, \alpha_q; \text{Gi, ti}) \prod_{\text{edges } i \to j \text{ in } \Gamma} \chi(\alpha_i, \alpha_j) \prod_{i=1}^{q} J(\alpha_i).
\]

Here \( U(\alpha_1, \ldots, \alpha_k; \text{Gi, ti}) \) is defined as in Definition 2.5 by replacing \( \nu_{b,w_1}(\alpha) \) by \( p_{[\alpha]} \) and \( \nu_{b,w_2}(\alpha) \) by \( \bar{p}_{[\alpha]} \) for any \( \alpha \in K(X) \).

### 3. Walls for rank-zero classes

Let \( (X, \mathcal{O}_X(1)) \) be a smooth polarised complex projective threefold with \( H = c_1(\mathcal{O}_X(1)) \). Fix a rank zero class \( v = (0, D, \beta, m) \in K(X) \) with

\[
\chi_H(v) = (0, k, s, d)
\]

satisfying inequality (2), i.e.

\[
Q(v) < \frac{1}{2}k^2 - \frac{1}{2} \left( k - \frac{1}{H^3} + \frac{2}{k(H^3)^2} \right)^2
\]
where
\[ Q(v) = \frac{1}{2} k^2 + 6 s^2 \frac{k^2}{k^2} - 12 \frac{d}{k}. \]

In this section, we analyse walls in the space of weak stability conditions \( U \) for class \( v \). By Proposition 2.2, walls for class \( v \) are parallel lines of slope \( \frac{b}{k} = \frac{3 H^3}{2 k}. \)

**Proposition 3.1.** Any wall \( \ell \) for class \( v \) lies above or on the line \( \ell_f \) with equation
\[
(22) \quad w = \frac{s}{k} b + \frac{k^2}{8} - \frac{s^2}{2k^2} - \frac{1}{4} Q(v).
\]

Let \( E' \to E \to E'' \) be a destabilising sequence for an object \( E \) of class \( v \) along a wall \( \ell \), then

- one of the destabilising factors \( E_1 \) (which is either \( E' \) or \( E'' \)) is of Chern character \( v_1 = -e^{D_1}(1,0,-\beta_1,-m_1) \) such that \( (E_1 \otimes \det E_1)^{\vee}[1] \) is a stable pair,
- the other factor \( E_2 \) is a torsion-free sheaf of Chern character \( v_2 = e^{D_2}(1,0,-\beta_2,-m_2). \)

Moreover, both \( E_1 \) and \( E_2 \) are \( b, w \)-semistable for any \((b, w) \in \ell \cap U\).

**Proof.** By (12), if there is a \( b, w \)-semistable object of class \( v \) for some \((b, w) \in U\), then
\[
(23) \quad 0 \leq \frac{1}{2(kH^3)^2} B_{b,w}(v) = w - b \frac{s}{k} + 2 \frac{s^2}{k^2} - \frac{3}{k} \frac{d}{k} = w - b \frac{s}{k} - \frac{k^2}{8} + \frac{s^2}{2k^2} + \frac{1}{4} Q(v).
\]

Thus \( \ell_f \) is the line given by \( B_{b,w}(v) = 0 \) and so the first claim follows. The line \( \ell_f \) intersects \( \partial U \) at two points with \( b \)-values \( b_2 < b_1 \),
\[
b_1, b_2 = \frac{s}{k} \pm \sqrt{\frac{k^2}{4} - \frac{1}{2} Q(v)}.
\]

Then (2) or equivalently (21) gives
\[
(24) \quad b_1 - b_2 = \sqrt{k^2 - 2Q(v)} > k - \frac{1}{H^3} + \frac{2}{k(H^3)^2}.
\]

Since the wall \( \ell \) lies above \( \ell_f \), the destabilising factors \( E_1 \) and \( E_2 \) satisfy
\[
(25) \quad \mu^+_H(H^{-1}(E_i)) \leq b_2 \quad \text{and} \quad b_1 \leq \mu^-_H(H^0(E_i)).
\]

Therefore
\[
kH^3 = ch_1(H^0(E_1))H^2 + ch_1(H^0(E_1))H^2 - ch_1(H^{-1}(E_1))H^2 - ch_1(H^{-1}(E_1))H^2 \geq b_1 H^3 (ch_0(H^0(E_1)) + ch_0(H^0(E_2))) - b_2 H^3 (ch_0(H^{-1}(E_1)) + ch_0(H^{-1}(E_2)))
\]
\[
= (b_1 - b_2) H^3 (ch_0(H^0(E_1)) + ch_0(H^0(E_2))).
\]

The last inequality comes from \( \text{rank}(E) = \text{rank}(E_1) + \text{rank}(E_2) = 0 \). Combining this with (24) implies that
\[
ch_0(H^{-1}(E_1)) + ch_0(H^{-1}(E_2)) = ch_0(H^0(E_1)) + ch_0(H^0(E_2)) \leq 1
\]
Therefore, one of the factors $E_1$ is of rank $-1$ with $\mathcal{H}^{-1}(E_1)$ of rank one and $\mathcal{H}^0(E_1)$ of rank zero; and the other factor $E_2$ is a sheaf of rank one.

We claim

(27) \[ \mu_H(E_2) - b_1 < \frac{1}{H^3}. \]

Otherwise, (25) gives

\[ b_1 - b_2 \leq \mu_H(E_2) - \frac{1}{H^3} - \mu_H(\mathcal{H}^{-1}(E_1)). \]

Since $\mathcal{H}^0(E_1)$ is of rank zero, we have $\mu_H(E_1) \leq \mu_H(\mathcal{H}^{-1}(E_1))$, thus

\[ b_1 - b_2 \leq \mu_H(E_2) - \frac{1}{H^3} - \mu_H(E_1) = k - \frac{1}{H^3}. \]

The last equality comes from $\text{rank}(E_2) = - \text{rank}(E_1) = 1$. But the above is not possible by (24).

Since $k > 0$ and $k \in \frac{1}{H^2} \mathbb{Z}$, (24) gives $b_1 - b_2 > \frac{1}{H^2}$. Therefore (27) implies that the line $\ell_f$ intersects the vertical line $b = \mu_H(E_2) - \frac{1}{H^2}$ at a point inside $U$. Since the wall $\ell$ lies above $\ell_f$, the same holds for $\ell$. Thus [FT21a, Lemma 8.1] implies that $E_2$ is $\nu_{b_0,w}$-stable for $b_0 = \mu_H(E_2) - \frac{1}{H^2}$ and all $w > \frac{k^2}{2}$, hence by Lemma 2.4, $E_2$ is tilt-stable and so torsion-free.

By applying a similar argument as above, one can show

(28) \[ b_2 - \mu_H(E_1) < \frac{1}{H^3}. \]

Thus [FT21a, Lemma 8.2] implies that $E_1$ is stable along the wall $\ell$ and in the large volume limit, so the claim follows from Lemma 2.4. \[ \square \]

For any sheaf $E$ of rank zero, we define $\nu_H$-slope as

(29) \[ \nu_H(E) := \begin{cases} \frac{\text{ch}_2(E).H}{\text{ch}_1(E).H^2} & \text{if } \text{ch}_1(E).H^2 \neq 0, \\ +\infty & \text{if } \text{ch}_1(E).H^2 = 0. \end{cases} \]

We say that a sheaf $E$ of rank zero is slope (semi)stable if for all non-trivial quotient sheaves $E \twoheadrightarrow E'$ one has $\nu_H(E) (\leq) \nu_H(E')$.

Hence a sheaf $E$ of class $v$ is slope(semi)-stable if and only if it is tilt-(semi)stable. Thus Lemma 2.4 implies that $E$ is $\nu_{0,w}$-(semi)stable for $w \gg 0$ if and only if it is slope-(semi)stable. The following proves the first part of Theorem 1.1.

**Lemma 3.2.** If $Q(v) < 0$, there is no slope-semistable sheaf of class $v$.

**Proof.** Let $\ell_v$ be the line of slope $\frac{s}{k}$ which intersects $\partial U$ at two points with $b$-values $b_v < a_v$ such that $a_v - b_v = k$. It is of equation

(30) \[ w = \frac{s}{k} b + \frac{k^2}{8} - \frac{s^2}{2k^2}. \]
The same argument as in the proof of [FT21b, Lemma B.3] implies that there is no wall for class \( v \) above \( \ell'_v \). Thus if there is a slope-semistable sheaf of class \( v \), then it is \( \nu_{b,w} \)-semistable for any \((b, w) \in U\) above \( \ell'_v \). Hence the final line \( \ell_f \) of equation (22) must lie on or below \( \ell'_v \). Comparing its equation (22) with (30) gives \( Q(v) \geq 0 \). □

Applying a similar argument as in [FT20, Section 2] implies the next lemma.

**Lemma 3.3.** Any slope-semistable sheaf of class \( v \) is slope-stable.

**Proof.** Suppose there is a strictly slope semistable sheaf \( E \) of class \( v \). We know \( E \) is \( \nu_{b,w} \)-semistable for \( w \gg 0 \). When we move down it hits a wall \( \ell \) which lies above \( \ell_f \). by Proposition 3.1, the destabilising sequence is of the form \( E_2 \hookrightarrow E \twoheadrightarrow E_1 \) such that both \( E_1 \) is a rank one torsion-free sheaf. Note that since \( \nu_{b,w} \) is a rank one torsion-free sheaf. The same argument as in the proof of [12 SOHEYLA FEYZBAKHSH] implies the next lemma.

Suppose \( E \rightarrow E' \) is a proper quotient sheaf with \( \nu_H(E') = \nu_H(E) \). Since rank \( E' = 0 = \) rank \( E \) the formula (7) gives

\[
\nu_{b,w}(E') = \frac{\text{ch}_2(E')H}{\text{ch}_1(E')H^2} = \frac{s}{k} = \nu_{b,w}(E)
\]

for all \((b, w) \in U\). Since all torsion sheaves are in \( \mathcal{A}_{b_0} \), \( E' \) is a quotient of \( E \) in the abelian category \( \mathcal{A}_{b_0} \), and any quotient of \( E' \) in \( \mathcal{A}_{b_0} \) is also a quotient of \( E \). Therefore \( E' \) is also \( \nu_{b_0,w_0} \)-semistable for \((b_0, w_0) \in \ell \cap U\).

Suppose \( E_2 \) is \( \nu_{b_0,w_0} \)-splitstable, the composition

\[
E_2 \hookrightarrow E \twoheadrightarrow E'
\]

in \( \mathcal{A}_{b_0} \) must either be zero or injective. And it cannot be zero, because this would give a surjection \( E_1 \rightarrow E' \) in \( \mathcal{A}_{b_0} \), contradicting the \( \nu_{b_0,w_0} \)-stability of \( E_1 \). So it is injective. Let \( C \) denote its cokernel in \( \mathcal{A}_{b_0} \), sitting in a commutative diagram

\[
\begin{array}{ccc}
E_2 & \hookrightarrow & E \\
\ | & & \downarrow \\
E_2 & \hookrightarrow & E' \twoheadrightarrow C.
\end{array}
\]

Since \( E' \) and \( E_2 \) are \( \nu_{b_0,w_0} \)-semistable of the same phase, \( C \) is also \( \nu_{b_0,w_0} \)-semistable. Therefore the right hand surjection contradicts the \( \nu_{b_0,w_0} \)-stability of \( E_1 \). □

The next step is to analyse Chern character of the destabilising factors along a wall for class \( v \).

**Definition 3.4.** Let \( M(v) \) be the set of all classes \((D', \beta', m') \in H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \oplus H^6(X, \mathbb{Z})\) such that

\[
\frac{1}{2} \left( \frac{D'H^2}{H^3} \right)^2 - \frac{D'H^2}{2H^3} + \frac{\beta'.H}{H^3} \leq \frac{DH^2}{2(H^3)^2} - \frac{1}{(H^3)^2} \quad \text{and} \quad m' \leq \frac{DH^2(DH^2 + H^3)}{6(H^3)^2}.
\]
Proposition 3.5. The destabilising classes \( v_i = (-1)^i e^{D_i}(1, 0, -\beta_i, -m_i) \) for \( i = 1, 2 \) in Proposition 3.1 satisfy

\[
\frac{1}{2} \left( \frac{D_i H^2}{H^3} \right)^2 - \frac{D_i^2 H}{2 H^3} + \frac{\beta_i H}{H^3} \leq \frac{1}{2} \left( k^2 - k \sqrt{k^2 - 2Q(v)} \right),
\]

and

\[
(-1)^i+1 m_i \leq \frac{2}{3} \beta_i H \left( \beta_i H + \frac{1}{2H^3} \right).
\]

In particular \( (D_i, \beta_i, (-1)^{i+1}m_i) \in M(v) \).

Proof. As in the proof of Proposition 3.1, we assume \( b_2 < b_1 \) are the \( b \)-values of the intersection points of \( \ell_f \) with \( \partial U \). By (25),

\[
b_1 \leq \frac{D_2 H^2}{H^3} \leq b_2 + k.
\]

We know the wall \( \ell \) is of slope \( \frac{s}{k} \) and passes through

\[
\Pi(v_2) = \left( \frac{D_2 H^2}{H^3}, \frac{D_2 H}{2H^3} - \frac{\beta_2 H}{H^3} \right).
\]

Since \( \ell \) lies above or on the parallel line \( \ell_f \), \( \Pi(v_2) \) lies above or on \( \ell_f \). By the classical Bogomolov inequality (9), \( \Pi(v_2) \) lies outside \( U \). Therefore (33) implies that the vertical distance from \( \Pi(v_2) \) to \( \partial U \) which is equal to \( \frac{1}{2} \left( \frac{D_2 H^2}{H^3} \right)^2 - \left( \frac{D_2^2 H}{2 H^3} - \frac{\beta_2 H}{H^3} \right) \) is maximum when \( \frac{D_2 H^2}{H^3} \) is maximum and \( \Pi(v_2) \) lies on \( \ell_f \), see Figure 2.

Therefore

\[
\frac{1}{2} \left( \frac{D_2 H^2}{H^3} \right)^2 - \frac{D_2^2 H}{2 H^3} + \frac{\beta_2 H}{H^3} \leq \frac{1}{2} \left( b_2 + k \right)^2 - \left( \frac{s}{k} (b_2 + k) + \frac{k^2}{8} - \frac{s^2}{2k^2} - \frac{1}{4} Q(v) \right)
\]

\[
= \frac{1}{2} \left( b_2 + k - \frac{s}{k} \right)^2 - \frac{k^2}{8} + \frac{1}{4} Q(v)
\]

\[
= \frac{1}{2} \left( k - \sqrt{k^2 - \frac{1}{2} Q(v)} \right)^2 - \frac{k^2}{8} + \frac{1}{4} Q(v)
\]

\[
= \frac{1}{2} \left( k^2 - k \sqrt{k^2 - 2Q(v)} \right).
\]

This proves (31) for class \( v_2 \). Applying a similar argument proves it for class \( v_1 \).

We know \( E_2 \otimes D_2^{-1} \) is a torsion-free sheaf of class \( (1, 0, -\beta_2, -m_2) \), thus \([FT21a, Proposition 8.3]\) implies

\[
-m_2 \leq \frac{2}{3} \beta_2 H \left( \beta_2 H + \frac{1}{2H^3} \right).
\]

We know \( (E_1 \otimes \det(E_1))^{\vee}[1] \) is a stable pair, so it lies in an exact triangle

\[
E' \longrightarrow (E_1 \otimes D_1^{-1})^{\vee}[1] \longrightarrow Q[-1],
\]
with $Q$ a zero-dimensional sheaf and $E'$ a torsion-free sheaf of class $(1,0,-\beta_1,m_1+\ell(Q))$ where $\ell(Q)$ is the length of $Q$. Thus [FT21a, Proposition 8.3] implies

$$m_1 \leq m_1 + \ell(Q) \leq \frac{2}{3} \beta_1.H \left( \beta_1.H + \frac{1}{2H^3} \right).$$

This completes the proof of (32).

Finally we show that $(D_i,\beta_i,(-1)^{i+1}m_i) \in M(v)$. The right hand side in (31) is maximum when $Q(v)$ is maximum, so our assumption (2) implies that

$$1/2 \left( \frac{D_i.H^2}{H^3} \right)^2 - \frac{D_i^2.H}{2H^3} + \frac{\beta_i.H}{H^3} \leq \frac{1}{2} \left( k^2 - k\sqrt{k^2 - 2Q(v)} \right) \leq \frac{1}{2} \left( k^2 - k(k - \frac{1}{H^3} + \frac{2}{k(H^3)^2}) \right) = \frac{k}{2H^3} - \frac{1}{(H^3)^2}.$$

By Hodge index theorem $0 \leq 1/2 \left( \frac{D_i.H^2}{H^3} \right)^2 - \frac{D_i^2.H}{2H^3}$, hence

$$\beta_i.H \leq H^3 \left( \frac{1}{2} \left( \frac{D_i.H^2}{H^3} \right)^2 - \frac{D_i^2.H}{2H^3} \right) + \beta_i.H \leq \frac{k}{2} - \frac{1}{H^3} < \frac{k}{2}. \tag{34}$$
Thus (32) implies

\[ (-1)^{i+1} m_i \leq \frac{2}{3} \cdot \left( \frac{k}{2} + \frac{1}{2H^3} \right) \leq \frac{k(k+1)}{6} \]

as claimed. \qed

The final step to prove Theorem 1.1 is to show the converse of Proposition 3.1.

**Proposition 3.6.** Take objects \( E_1, E_2 \in \mathcal{D}(X) \) of classes \( v_1 = -e^{D_1}(1,0,-\beta_1,-m_1) \) and \( v_2 = e^{D_2}(1,0,-\beta_2,-m_2) \) such that \( v_1 + v_2 = v \), and

(a) \((E_1 \otimes \det E_1)^{\vee} [1]\) is a stable pair,

(b) \( E_2 \) is a torsion-free sheaf,

(c) \( \frac{1}{2} \left( \frac{D_i H^2}{H^3} \right)^2 - \frac{D_i^2 H}{2H^3} + \frac{\beta_i H}{H^3} \leq \frac{k}{2H^3} - \frac{1}{H^3} \) for \( i = 1, 2 \).

Then there is a point \((b, w) \in U\) such that \( E_1 \) and \( E_2 \) are \( \nu_{b,w} \)-stable of the same slope. In particular, their extensions are strictly \( \nu_{b,w} \)-semistable objects of class \( v \).

**Proof.** Taking \( \text{ch}_2 \) from \( v_1 + v_2 = v \) and applying Hodge index Theorem give

\[ \frac{\beta_1 H}{H^3} \cdot D_1 H^2 - \frac{D_1^2 H}{2H^3} + \frac{\beta_1 H}{H^3} \geq -\frac{k^2}{2H^3} - k \frac{D_1 H^2}{H^3} + \frac{\beta_1 H}{H^3} + s. \]

The assumption in part (c) gives

\[ \frac{k}{2H^3} > \frac{1}{(H^3)^2} \geq \frac{1}{2} \left( \frac{D_1 H^2}{H^3} \right)^2 - \frac{D_1^2 H}{2H^3} + \frac{\beta_1 H}{H^3}. \]

Combining this with (35) implies

\[ \frac{s}{k} - \frac{k}{2H^3} < \frac{D_1 H^2}{H^3} - \frac{D_2 H^2}{2H^3} = \frac{D_2 H^2}{H^3} - k. \]

By applying a similar argument for class \( v_2 \) and using part (c), one can show

\[ \frac{D_1 H^2}{H^3} < \frac{s}{k} - \frac{k}{2H^3} + \frac{1}{2H^3}. \]

Let \( \ell \) be the line passing through \( \Pi(v_2) \) of slope \( \frac{s}{k} \) (which also passes through \( \Pi(v_1) \)). We show that both \( E_1 \) and \( E_2 \) are \( \nu_{b,w} \)-stable for \((b, w) \in \ell \cap U\). Let \( \ell_{v_2} \) be the line passing through

\[ \Pi(v_2) = \left( \frac{D_2 H^2}{H^3}, \frac{D_2 H^2}{2H^3} - \frac{\beta_2 H}{H^3} \right) \]
and the point \( \left( \frac{D_2 H^2}{H^3} - \frac{1}{H^3}, \frac{1}{2} \left( \frac{D_2 H^2}{H^3} - \frac{1}{H^3} \right)^2 \right) \) on \( \partial U \). By [FT21a, Lemma 8.1], there is no wall for class \( v_2 \) passing through the vertical line \( b = \frac{D_2 H^2}{H^3} - \frac{1}{H^3} \). Thus \( E_2 \) is \( \nu_{b,w} \)-stable for any \( (b, w) \in U \) above \( \ell_{v_2} \) when \( b < \frac{D_2 H^2}{H^3} \).

We claim the line segment \( \ell \cap U \) lies above \( \ell_{v_2} \). Otherwise, slope of \( \ell_{v_2} \) is smaller than or equal to \( \frac{s}{k} \), i.e.

\[
\frac{s}{k} \geq \frac{\frac{D_2^2 H}{2H^3} - \frac{\beta_2 H}{H^3} - \frac{1}{2} \left( \frac{D_2 H^2}{H^3} - \frac{1}{H^3} \right)^2}{\frac{1}{H^3}}
\]

This implies

\[
\frac{1}{2} \left( \frac{D_2 H^2}{H^3} \right)^2 - \frac{D_2^2 H}{2H^3} + \frac{\beta_2 H}{H^3} \geq - \frac{s}{k H^3} - \frac{1}{2(2^{\frac{1}{3}})^2} + \frac{D_2 H^2}{(H^3)^2}
\]

which is not possible by part (c). Therefore \( E_2 \) is \( \nu_{b,w} \)-stable for any \( (b, w) \in \ell \cap U \). By applying a similar argument, one can show the same holds for \( E_1 \), so any extension of \( E_1 \) and \( E_2 \) is strictly \( \nu_{b,w} \)-semistable for any \( (b, w) \in \ell \cap U \), as claimed.

We now apply the wall-crossing results in this section to prove Theorem 1.1. So assume that \( X \) is a Calabi-Yau 3-fold: \( K_X \cong \mathcal{O}_X \) and \( H^1(\mathcal{O}_X) = 0 \).

**Proof of Theorem 1.1.** Part (i) follows from Lemma 3.2. For part (ii), we know in the large volume limit when \( w \gg 0 \), we have \( J_{b,w}(v) = J_{f}(v) \). By Lemma 3.3, there is no strictly-tilt semistable sheaf of class \( v \), so

\[ J_{b,w \gg 0}(v) = J_{f}(v) = J(v). \]

On the other hand, we know \( J_{b,w}(v) = 0 \) for \( (b, w) \in U \) below \( \ell_f \). Let \( \ell \) be a wall for class \( v \) between large volume limit and \( \ell_f \). Let \( (b, w^+) \) be points just above and below the wall \( \ell \) and \( (b, w_0) \in \ell \). Proposition 3.1 implies that there are precisely two destabilising factors \( v_1, v_2 \) along the wall \( \ell \). We know the slope function \( \nu_{b,w}(v_1) \) is an increasing linear function with respect to \( w \) and \( \nu_{b,w}(v_2) \) is a decreasing linear function with respect to \( w \). Also \( \nu_{b,w_0}(v_1) = \nu_{b,w_0}(v_2) \), thus

\[ \nu_{b,w_+}(v_1) > \nu_{b,w+}(v_2) \quad \text{and} \quad \nu_{b,w_-}(v_1) < \nu_{b,w-}(v_2). \]

This implies

\[ U(v_1, v_2, (b, w_+), (b, w_-)) = 1 \quad \text{and} \quad U(v_2, v_1, (b, w_+), (b, w_-)) = -1. \]
The wall-crossing formula (16) gives
\[ J_{b,w_\pm}(v) = J_{b,w_+}(v) + \sum_{\substack{v_1 = v_1(b,1,0,\beta_1,m_1) \\ v_2 = v_2(b,1,0,\beta_2,m_2) \\ v_1 + v_2 = v}} (-1)^{v_1} \chi(v_1,v_2) J_{b,w_+}(v_1) J_{b,w_+}(v_2). \]

By Proposition 3.1, \( J_{b,w_+}(v_i) = J_{b,w\geq 0}(v_i) \). We may assume \( \mu_H(E_1) < b < \mu_H(E_2) \). Thus Lemma 2.4 shows that
\[ J_{b,w\geq 0}(v_2) = J(1,0,-\beta_2,-m_2) \]

The latter counts torsion-free sheaves of class \((1,0,-\beta_1,-m_1)\) which are ideal sheaves of 1-dimensional subscheme after tensioning by a line bundle \(L\) with torsion \(c_1\). Thus
\[ J_{b,w_+}(v_1) = \binom{\#H^2(X,\mathbb{Z})_{tors}}{1} I_{m_2,\beta_2}. \]

Similarly, we know \( J_{b,w_+}(v_1) = J_{\infty}(-1,0,\beta_1,m_1) \) where the latter counts dual of stable pairs of class \((-1,0,\beta_1,-m_1)\) up to tensoring by a line bundle with torsion \(c_1\)-class, thus
\[ J_{b,w_+}(v_1) = \binom{\#H^2(X,\mathbb{Z})_{tors}}{1} P_{-m_1,\beta_1}. \]

Summing up over all walls for class \(v\) between the large volume limit and \(\ell_f\), and applying Proposition 3.5 and Proposition 3.6 imply the final wall-crossing formula in Theorem 1.1.

Lemma 3.7. Theorem 1.1 holds if \(X\) is a smooth projective quintic threefold or a smooth projective threefold of complete intersection of quadratic and quartic hypersurfaces in \(\mathbb{P}^5\).

Proof. By [Li19, Liu21], Conjecture 2.3 holds on \(X\) for \((b,w)\) satisfying
\[ w > \frac{1}{2} b^2 + \frac{1}{2} (b - |b|)(|b| - b + 1). \]

Hence if \(b \in \mathbb{Z}\), it holds for any \(w > \frac{1}{2} b^2\). To prove Theorem 1.1, we applied Conjecture 2.3 in two places: (1) to find the line \(\ell_f\) in (23) and (2) in the proof of [FT21a, Proposition 8.3]. It has been shown in [FT21a, Theorem 3.2] that (2) holds true if Conjecture 2.3 is true for \((b,w)\) satisfying (37).

For case (1), since \(\text{Pic}(X) = \mathbb{Z}.H\), we know \(k \in \mathbb{Z}\). As shown in (24), the line \(\ell_f\) intersects \(\partial U\) at two points with \(b_1 - b_2 \geq 1\) if \(k \geq 2\), so \(\ell_f\) intersects a vertical line \(b = b_0 \in \mathbb{Z}\) at a point in the closure \(\overline{U}\) as we required. If \(k = 1\), then \(Q(v) \in \frac{1}{2} \mathbb{Z}\), so the bound (21) implies that \(Q(v) \leq 0\). Thus (24) implies again \(b_1 - b_2 \geq 1\).

4. OSV CONJECTURE

In this section, we prove a slight modification of Toda’s formulation of OSV conjecture [Tod13, Conjecture 1.1]. Part (i) of [Tod13, Conjecture 1.1] follows from Theorem 1.1(i). Thus we only consider the second part of [Tod13, Conjecture 1.1].

Fix \(k > 0\). We consider the invariants \(J(0,kH,\beta,m)\) which counts \(H\)-Gieseker semistable sheaves of Chern character \((0,kH,\beta,m)\). The generating series of these invariants is

\[ \sum_{v} Q(v) = \sum_{v} J(0,kH,\beta,m). \]

This is a deep result.

\footnote{We replaced the class \((0,nH,-\beta,-m)\) in Toda’s notations [Tod13] by \((0,kH,\beta,m)\).}
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defined by

\[ Z_{D_4}^k(x, y) := \sum_{\beta, m} J(0, kH, \beta, m) x^{-m} y^{-\beta}. \]

Define the subset

\[ N(k) := \left\{ D \in H^2(X, \mathbb{Z}) : \left( \frac{D \cdot H}{H^3} \right)^2 - \frac{D^2 \cdot H}{H^3} < \frac{k}{2H^3} \right\} \subset H^2(X, \mathbb{Z}). \]

If \( \text{Pic}(X) = \mathbb{Z} \cdot H \), for instance, then \( N(k) = H^2(X, \mathbb{Z}) \). For any \( \epsilon > 0 \), consider the generating series

\[ Z_{D_6-D_6}^k(x, y, z) := \sum_{D_1, D_2 \in \overline{N}(k) \atop D_2-D_1=kH} \frac{d^3}{dx^3} \frac{d^3}{dy^3} \frac{d^2}{dz^2} + \frac{d^3 H \cdot c(X)}{12} I^{k, \epsilon}(xz^{-1}, xD_2 yz^{-kH}) P^{k, \epsilon}(xz^{-1}, x-D_1 y z^{-kH}) \]

where

\[ P^{k, \epsilon}(x, y) := \sum_{(\beta, -m) \in C(k, \epsilon)} P_{m, \beta} x^m y^{-\beta}, \quad I^{k, \epsilon}(x, y) := \sum_{(\beta, -m) \in C(k, \epsilon)} I_{m, \beta} x^m y^{-\beta}. \]

and

\[ (38) \quad C(k, \epsilon) := \left\{ (\beta, m) \in H^4(X, \mathbb{Z}) \oplus H^6(X, \mathbb{Z}) : \beta \cdot H < \epsilon k^2, \ m < \epsilon k^3 \right\}. \]

A slight modification of [Tod13, Conjecture 1.1(ii)] says the following:

**Theorem 4.1.** Let \( X \) be a smooth projective Calabi-Yau 3-fold. For any \( \xi \geq 1 \), there are \( \mu > 0, \delta > 0 \) and a constant \( k(\xi, \mu) > 0 \) which depends only on \( \xi, \mu \) such that for any \( k > k(\xi, \mu) \), we have the equality of the generating series,

\[ (39) \quad Z_{D_4}^k(x, y) = \left( \#H^2(X, \mathbb{Z})_{\text{tors}} \right)^2 \frac{\partial}{\partial z} Z_{D_6-D_6}^{k, \epsilon}(x, y, z) \bigg|_{z=-1} \]

modulo terms of \( x^m y^\beta \) with

\[ (40) \quad -\frac{H^3}{24} k^3 \left( 1 - \frac{\mu}{k^2} \right) \leq m + \frac{(\beta \cdot H)^2}{2kH^3}. \]

**Remark 4.2.** There are only three places in the above set-up that are different from the Toda ones [Tod13]:

(i) In the definition of generating series \( Z_{D_6-D_6}^{k, \epsilon}(x, y, z) \), we added the extra condition that \( D_1, D_2 \in \overline{N}(k) \).

(ii) In the definition of \( C(k, \epsilon) \), Toda restricts the second factor \( m \) to satisfy \( |m| < \epsilon k^3 \).

(iii) We added the extra factor \( \left( \#H^2(X, \mathbb{Z})_{\text{tors}} \right)^2 \) in (39).
**Proof of Theorem 4.1.** By applying a similar argument as in [Tod13, Section 3.8], one can obtain the coefficient of \(x^{-m}y^{-\beta}\) in the right hand side of (39) and show that the equality (39) is equivalent to

\[
J(0, kH, \beta, m) = \sum_{\substack{v_1 = -e^{D_1}(1,0,-\beta_1,-m_1) \\ v_2 = e^{D_2}(1,0,-\beta_2,-m_2) \\ v_1 + v_2 = (0, kH, \beta, m) \in C(k, \epsilon) \quad D_1, D_2 \in N(k)}} (-1)^{\chi(v_2, v_1)} - 1 \chi(v_2, v_1) I_{m_2, \beta_2} P_{-m_1, \beta_1}.
\]

By our assumption (40), we only need to consider the terms \(x^{-m}y^{-\beta}\) with

\[
\frac{H^3}{24} k^3 \left(\frac{\mu}{k^4}\right) > -m + \frac{(\beta.H)^2}{2kH^3} + \frac{k^3 H^3}{24} = \frac{kH^3}{12} Q(v)
\]

for \(v := (0, kH, \beta, m)\), i.e.

\[
Q(v) < \frac{k^2}{2} \frac{\mu}{k^4}.
\]

For any \(\xi \geq 1\), there are \(\mu > 0, \delta > 0\) and \(k(\xi, \mu) > 0\) such that for \(k > k(\xi, \mu)\) the following three conditions are satisfied:

\[
(i) \quad \mu < \frac{2}{H^3} \delta, \quad (ii) \quad \delta < \frac{1}{4} - \frac{1}{kH^3}, \quad (iii) \quad \frac{\mu}{k^4} \leq 1 - \left(1 - \frac{1}{kH^3} + \frac{2}{k^2(H^3)^2}\right)^2.
\]

We also set \(\epsilon := \frac{\delta}{k}\). Combining (42) and (43)(iii) imply that the condition (2) holds, so we may apply Theorem 1.1 and the results in Section 3 for class \(v\). To prove (41), we need to show the following two claims:

(a) Take \(E_1, E_2 \in D(X)\) of classes \(v_1, v_2\) as in (41) such that \(E_1 \otimes D_1^{-1}\) is a torsion-free sheaf and \((E_2 \otimes D_2^{-1})^v\) is a stable pair, then there is a point \((b, w) \in U\) where \(E_1\) and \(E_2\) are both \(v_{b,w}\)-stable of the same slope, and so they make a wall for the class \(v = (0, kH, \beta, m)\).

(b) All classes \(v_1, v_2 \in K(X)\) which give a non-zero term in the wall-crossing formula for \(J(v)\) in Theorem 1.1 are included in (41).

In (41), we know \(D_i \in N(k)\) for \(i = 1, 2\), so

\[
\frac{1}{2} \left(\frac{D_i.H^2}{H^3}\right)^2 - \frac{D_i^2.H}{2H^3} < \frac{k}{4H^3}.
\]

Since \((\beta_i, (-1)^{i+1}m_i) \in C(k, \epsilon)\), we get

\[
\frac{1}{2} \left(\frac{D_i.H^2}{H^3}\right)^2 - \frac{D_i^2.H}{2H^3} + \frac{\beta_i.H}{H^3} < \frac{k}{4H^3} + \frac{1}{H^3} \epsilon k^2 < \frac{k}{2H^3} - \frac{1}{(H^3)^2}.
\]
Here the last inequality follows from the assumptions \( \xi \geq 1 \) and (43)(ii), i.e.

\[
\frac{1}{H^3} \epsilon k^2 = \frac{k^2 \delta}{H^3 k^\xi} < \frac{k}{H^3} \delta < \frac{k}{4H^3} - \frac{1}{(H^3)^2}
\]

Thus Proposition 3.6 implies claim (a).

To prove claim (b), take two classes \((D_i, \beta_i, (-1)^{i+1} m_i) \in M(v)\) for \( i = 1, 2 \) satisfying inequalities (31) and (32) in Proposition 3.5, then we only need to show \( D_i \in N(k) \) and \((\beta_i, (-1)^{i+1} m_i) \in C(k, \epsilon)\). We know

\[
\frac{k^2}{2} \left( 1 - \sqrt{1 - 2 \frac{Q(v)}{k^2}} \right) = \frac{k^2}{2} \frac{2 \frac{Q(v)}{k^2}}{1 + \sqrt{1 - 2 \frac{Q(v)}{k^2}}} \leq Q(v) \left( \frac{42}{2} \frac{k^2}{k^\xi} \right) < \frac{k^2}{H^3} \delta \xi < k^{(43)(ii)}(ii).
\]

Combining this with (31) implies

\[
\frac{1}{2} \left( \frac{D_i H^2}{H^3} \right)^2 - \frac{D_i^2 H}{2H^3} + \frac{\beta_i H}{H^3} < \frac{k^2}{H^3} \epsilon.
\]

By Hodge index Theorem and (9), both terms \( \frac{1}{2} \left( \frac{D_i H^2}{H^3} \right)^2 - \frac{D_i^2 H}{2H^3} \) and \( \beta_i H \) are non-negative, so the above gives \( \beta_i H \leq \epsilon k^2 \) and

\[
\left( \frac{D_i H^2}{H^3} \right)^2 - \frac{D_i^2 H}{H^3} < 2 \frac{k^2}{H^3} \epsilon = \frac{2 k^2 \delta}{H^3 k^\xi} \leq \frac{2 k}{H^3} \left( \frac{1}{4} - \frac{1}{k H^3} \right) < \frac{k}{2H^3}
\]

which proves \( D_i \in N(k) \). Moreover (32) gives

\[
(-1)^{i+1} m_i \leq \frac{2}{3} \beta_i H \left( \beta_i H + \frac{1}{2H^3} \right) \leq \frac{2}{3} \epsilon k^2 \left( \epsilon k^2 + \frac{1}{2H^3} \right).
\]

To finish the proof of claim (b), it suffices to show

\[
\frac{2}{3} \epsilon k^2 \left( \epsilon k^2 + \frac{1}{2H^3} \right) \leq \epsilon k^3.
\]

This is equivalent to \( \epsilon = \frac{\delta}{k^\xi} \leq \frac{3}{2k} - \frac{1}{2k H^3} \), so it is enough to show

\[
\frac{\delta}{k^\xi} \leq \frac{3}{2k} - \frac{1}{2k} = \frac{1}{k}.
\]

which clearly holds by (43)(ii).

\[\square\]

5. Joyce–Song pair

As before, we fix a rank 0 class \( v = (0, D, \beta, m) \in K(X) \) with

\[
\text{ch}_H(v) = (0, k, s, d),
\]

where \( k > 0 \). Then we pick \( n \gg 0 \) and set \( v_n := v - [O_X(-n)] \), so

\[
\text{ch}_H(v_n) = \left( -1, k + n, s - \frac{1}{2} n^2, d + \frac{1}{6} n^3 \right).
\]
In this section, we do wall-crossing for class $v_n$ instead of class $v$, and capture the information of slope-semistable sheaves of class $v$ along the Joyce-Song wall where $v$ and $\mathcal{O}_X(-n)[1]$ have the same slope.

We know any wall for class $v_n$ is a line segment $\ell \cap U$ where $\ell$ is a line passes through $\Pi(v_n) = \left(-k - n, -s + \frac{1}{2} n^2\right)$.

If $b > -k - n$ and $w \gg 0$, any $v_{b,w}$-semistable object $E \in \mathcal{A}(b)$ of class $v_n$ is isomorphic to the derived dual of a stable pair up to twisting by a line bundle, see Lemma 2.4. On the other hand, conjectural Bogomolov inequality (11) implies that there is a line $\ell_f$ described in [FT21c, Equation (23)] such that there is no $v_{b,w}$-semistable object of class $v_n$ for $(b,w)$ below $\ell_f$, see Figure 3. Hence there are finitely many walls for class $v_n$ between the large volume limit and $\ell_f$, see [FT21c, Proposition 1.4] for more details.

Let $M_{v,n}$ be the set of all rank $-1$ classes $\alpha = (-1, \text{ch}_1, \text{ch}_2, \text{ch}_3) \in K(X)$ with non-negative discriminant $\Delta_H(\alpha) \geq 0$ such that

$$ n + k < \frac{\text{ch}_1 H^2}{H^3} \leq n - \frac{k}{3} $$

and $\Pi(\alpha)$ lies above or on $\ell_f$.

For any class $\alpha \in M_{v,n}$ we define a line $L_\alpha \subset \mathbb{R}^2$ by using induction on $\Delta_H(\alpha)$:

- If $\Delta_H(\alpha) = 0$, then $L_\alpha$ is the vertical line passing through $\Pi(\alpha)$ (which is of gradient $-\infty$).
- If $\Delta_H(\alpha) > 0$, then $L_\alpha$ is the line of smallest gradient passing through $\Pi(\alpha)$ which satisfies the following condition:

  If an object $E$ of class $\alpha$ gets destabilised along a wall $\ell$ on the right of $\Pi(\alpha)$ and above $L_\alpha$ with the destabilising sequence $E_1 \to E \to E_2$, then $\ell \cap U$ lies to the right of the vertical line passing through $\Pi(E)$.

\[ \text{Figure 3. The line } \ell_f \]
Theorem 5.1. \cite[Section 2]{FT21c} Suppose an object $E$ of class $\nu_n$ gets destabilised along a wall $\ell$ with a destabilising sequence $E_1 \to E \to E_2$, then

- one of the factors $E_i$ is of rank $-1$ with $\text{ch}(E_i) \in M_{v,n}$ and the wall $\ell$ lies above $L_{[E_i]}$, \\
- the other factor $E_j$ is of rank zero with $\frac{\text{ch}_1(E_j)H^2}{H^2} \leq k$ such that there is no wall for class $[E_j]$ above or on $\ell$. \\

Moreover $\text{ch}_1(E_j)H = kH^3$ if and only if the wall $\ell$ is the Joyce–Song wall, i.e. $E_1$ is a slope-semistable sheaf of class $\nu$ and $E_2 = T(-nH)[1]$ for a line bundle $T$ with torsion first Chern class.

Proof. By \cite[Lemma 2.1 & Lemma 2.2]{FT21c},\footnote{Here we replaced the parameters $n_0, c$ and $s_0$ in \cite[Section 2]{FT21c} by $n, k$ and $s$, respectively.} one of the factors $E_j$ is of rank zero and there is no wall for $[E_j]$ above or on $\ell$. The other factor $E_i$ is of rank $-1$ and $\text{ch}(E_i) \in M_{v,n}$. \cite[Proposition 2.6]{FT21c} implies that $E_i$ is either

- close to $v_n$ (see \cite[Definition 2.3]{FT21c}), or \\
- the wall $\ell$ lies in the safe area of $[E_i]$ (see \cite[Definition 2.7]{FT21c}).

In case (b), by induction on $\Delta_H(E_i)$, one sees $L_{[E_i]}$ lies below the safe area of $[E_i]$, see \cite[Definition 2.7 and Lemma 2.8]{FT21c}. Thus the wall $\ell$ lies above $L_{[E_i]}$ as claimed.

In case (a), for any class $\alpha$ close to $v_n$, the line $\ell(\alpha)$ is defined to be the line parallel to $\ell$ through the point $\Pi(\alpha)$. We can again proceed by induction on $\Delta_H(E_i)$ and apply \cite[Proposition 2.6]{FT21c} to prove the line $\ell([E_i])$ is above $L_{[E_i]}$. Since the wall $\ell$ lies above $\ell([E_i])$, we get $\ell$ lies above $L_{[E_i]}$ as well.

When $\text{ch}_1(E_j)H = kH^3$, the final claim follows from \cite[Proposition 2.5]{FT21c}. \hfill \Box

Wall-crossing formula. From now on, we assume $X$ is a Calabi-Yau 3-fold with $\text{Pic}(X) = \mathbb{Z}H$. In this case, we can simplify the wall-crossing formula \eqref{WallCrossingFormula} for any class in $M_{v,n}$ and so for $v_n$.

Lemma 5.2. For any rank zero class $\alpha \in K(X)$ with $\text{ch}_1(\alpha) \neq 0$, we have

$$J(\alpha) = J_{b,w}(\alpha) \quad \text{for any } b \in \mathbb{R} \text{ and } w \gg 0.$$
Proof. By Lemma 2.4, we know $J_{b,w>0}(\alpha) = J_{\text{ui}}(\alpha)$. So we only need to analyse the wall-crossing formula (20) for class $\alpha$. Consider the semistable factors $\alpha_1, \ldots, \alpha_q$. Then $U(\alpha_1, \ldots, \alpha_k; G_i, t_i) = 0$ unless $\bar{p}(\alpha_i)(t) = \bar{p}(\alpha_i)(t)$, i.e. $\nu_H(\alpha_i) = \nu_H(\alpha)$. But then $\chi(\alpha, \alpha_j) = \chi_1(\alpha_i) \chi_1(\alpha_j) - \chi_1(\alpha_i) \chi_2(\alpha_j) = 0$

as $\chi_1(\alpha_i) = k_i H$ for some $k_i \in \mathbb{Z}$. Thus the only non-zero term in (20) is for $q = 1$, so $J_{\text{ui}}(\alpha) = J(\alpha)$ as claimed.

Proposition 5.3. Fix $\alpha \in M_{\nu,n}$ and let $\ell$ be a wall for class $\alpha$ which lies above $L_\alpha$. Suppose $(b_0, w^-_0)$ and $(b_0, w^+_0)$ are points just above and below the wall $\ell$, then

$$
(48) \ J_{b_0,w^-_0}(\alpha) = \sum_{q \geq 1, \alpha_1, \ldots, \alpha_q \in \ell} \frac{1}{(q - 1)!} J_{b_0,w^+_0}(\alpha_1) \prod_{i=2}^q (-1)^{\chi(\alpha, \alpha_i)} \chi(\alpha_i, \alpha) J(\alpha_i).
$$

Here $\alpha_i \in \ell$ means $\Pi(\alpha_i)$ lies on the line $\ell$ if $i = 1$, and if $2 \leq i \leq q$, the slope $\frac{\beta_H \nu_H}{k_i H^3}$ is equal to the gradient of $\ell$.

Proof. The argument is similar to [Tod10a, Theorem 5.8]. In the wall-crossing formula (16), we know the coefficient $U(\alpha_1, \ldots, \alpha_q; (b_0, w^+_0), (b_0, w^-_0))$ is zero unless $\nu_{b_0,w^-_0}(\alpha_1) = \cdots = \nu_{b_0,w^-_0}(\alpha_q)$

where $(b_0, w_0)$ lies on the wall $\ell$. Since $\ell$ lies above $L_\alpha$, by definition one of the factors $\alpha_e$ (lying in position $e$) is of rank $-1$ and $\alpha_e \in M_{\nu,n}$. The other factors $\alpha_i$ for $i \neq e$ are of rank zero with $\chi_1(\alpha_i) H^2 < k_i H^3$. The factors of rank zero have the same $\nu_H$-slope (29), so they have the same $\nu_{b_0,w}$-slope with respect to any $(b, w) \in U$.

Step 1. We claim

$$
(49) \ S(\alpha_1, \ldots, \alpha_q; (b_0, w^+_0), (b_0, w^-_0)) = \begin{cases} 
(-1)^{e-1} & \text{if } e = q - 1 \text{ or } q, \\
0 & \text{otherwise}.
\end{cases}
$$

First assume $q > 2$ and $e \leq q - 2$. If $S(\alpha_1, \ldots, \alpha_q; (b_0, w^+_0), (b_0, w^-_0)) \neq 0$, then for $i = q-1$, we have $\nu_{b_0,w^-_0}(\alpha_1) = \cdots = \nu_{b_0,w^-_0}(\alpha_q)$, so we must have

$$
(50) \ \nu_{b_0,w^-_0}(\alpha_1 + \cdots + \alpha_{q-1}) > \nu_{b_0,w^-_0}(\alpha_q).
$$

Let $\alpha_e = (-1, k_e H, \beta_e, m_e)$ and $\alpha_i = (0, k_i H, \beta_i, m_i)$ for $i \neq e$, then

$$
\frac{\beta_e H}{k_i H^3} = \nu_{b_0,w^-_0}(\alpha_i) = \nu_{b_0,w^-_0}(\alpha_e) = \frac{\beta_e H + w_0 H^3}{k_e H^3 + b_0 H^3}.
$$

To have a $\nu_{b_0,w^-_0}$-semistable object of class $\alpha_e$ in $A_{b_0}$, we must have $k_e H^3 + b_0 H^3 > 0$. Thus for $w^-_0 < w_0$, we get $\nu_{b_0,w^-_0}(\alpha_e) < \nu_{b_0,w^-_0}(\alpha_i)$ for $i \neq e$, so

$$
\nu_{b_0,w^-_0}(\alpha_e) = \frac{\beta_e H + w^-_0 H^3}{k_e H^3 + b_0 H^3} < \frac{\beta_e H + w_0 H^3 + \sum_{i \neq e,q} \beta_i H}{k_e H^3 + b_0 H^3 + \sum_{i \neq e,q} k_i H^3} < \frac{\sum_{i \neq e,q} \beta_i H}{\sum_{i \neq e,q} k_i H^3} = \nu_{b_0,w^-_0}(\alpha_i).
$$
The middle term is equal to \( \nu_{b_0,w_0^-}(\alpha_1 + \ldots + \alpha_{q-1}) \) and the right hand one is equal to 
\( \nu_{b_0,w_0^-}(\alpha_q) \), so this is in contradiction to (50). Thus 
\( S(\alpha_1, \ldots, \alpha_q, (b_0, w_0^+), (b_0, w_0^-)) = 0 \).

The same argument as above shows that for \( i = 1, \ldots, e - 1 \),
\[ \nu_{b_0,w_0^+}(\alpha_i) = \nu_{b_0,w_0^-}(\alpha_{i+1}) \quad \text{and} \quad \nu_{b_0,w_0^-}(\alpha_1 + \ldots + \alpha_i) > \nu_{b_0,w_0^-}(\alpha_{i+1} + \ldots + \alpha_q), \]
and if \( e = q - 1 \), we have
\[ \nu_{b_0,w_0^+}(\alpha_e) > \nu_{b_0,w_0^-}(\alpha_{e+1}) \quad \text{and} \quad \nu_{b_0,w_0^-}(\alpha_1 + \ldots + \alpha_e) < \nu_{b_0,w_0^-}(\alpha_e). \]
Thus \( S(\alpha_1, \ldots, \alpha_q; (b_0, w_0^+), (b, w_0^-)) = (-1)^{e-1} \) if \( e = q \), or \( q - 1 \).

**Step 2.** The next claim is
\[ U(\alpha_1, \ldots, \alpha_q; (b_0, w_0^+), (b_0, w_0^-)) = \frac{(-1)^{e-1}}{(e-1)!(q-e)!}. \]

Here \( e \) is again the position of the factor of rank \(-1\). Note that the slope function \( \nu_{b_0,w} \) of any class is a linear function of \( w \). The same computations as in Step 1 implies that if for some \( i < j \),
\[ \nu_{b_0,w_0^-}(\alpha) = \nu_{b_0,w_0^-}(\alpha_i + \alpha_{i+1} + \ldots + \alpha_j) \]
then \( i = 1 \) and \( j = q \). Thus the parameter \( p \) in the definition (17) must be equal to 1 and 
we only need to pick \( t \in [1, q] \) and \( 0 = a_0 < a_1 < \ldots < a_t = q \). Define
\[ B_i := \alpha_{a_i-1+1} + \ldots + \alpha_{a_i} \quad \text{for } i \in [1, t]. \]
We require \( \nu_{b_0,w_0^+}(B_i) = \nu_{b_0,w_0^-}(\alpha_j) \) for \( a_{i-1} < j \leq a_i \). Thus for any \( i \in [1, t] \), either
(a) \( a_i - a_{i-1} = 1 \), or 
(b) \( \alpha_j \)'s are all of rank zero when \( a_{i-1} < j \leq a_i \).

Suppose \( e \leq q - 1 \). By (49) in Step 1, if
\[ S(B_1, \ldots, B_t; (b_0, w_0^+), (b_0, w_0^-)) \neq 0, \]
then all classes \( \alpha_i \) for \( i > e \) lie in one bunch, i.e. \( B_t = \alpha_{e+1} + \ldots + \alpha_q \) and \( B_{t-1} = \alpha_e \).

Also the division of \( \alpha_1, \ldots, \alpha_{e-1} \) into \( B_1, \ldots, B_{t-2} \) can be parametrised by non-decreasing surjective maps
\[ \psi: \{1, \ldots, e-1\} \to \{1, \ldots, t-2\}. \]
Thus
\[ U(\alpha_1, \ldots, \alpha_e; (b_0, w_0^+), (b_0, w_0^-)) = \sum_{e \in [1, q], t' \in [1, e-1]} (-1)^{t'} \frac{1}{(q-e)!} \prod_{i=1}^{t'} \frac{1}{|\psi^{-1}(i)|!} \],
If \( e = 1 \), then \( t' = 0 \) and \( \psi = 0 \) in the above sum. Finally, the claim (51) follows by [Tod10a, Equation (72)].

**Step 3.** Now we apply the wall-crossing formula (16). We know for \( i, j \neq e \),
\[ \chi(\alpha_i, \alpha_j) = k_i H \beta_i - k_i H \beta_j = 0 = \chi(\alpha_j, \alpha_i). \]
and so
\begin{equation}
\chi(\alpha_i, \alpha_e) = \chi(\alpha_i, \alpha).
\end{equation}
Thus there exists only one connected digraph \( \Gamma \) which gives a non-zero term in the wall-crossing formula (16): it is made of edges \( i \to e \) for \( i < e \) and \( e \to j \) for \( j > e \). The corresponding coefficient in (16) is
\begin{align*}
&\frac{(-1)^{q-1+\sum_{1 \leq i < j \leq q} \chi(\alpha_i, \alpha_j)}}{2^{q-1}} U(\alpha_1, \ldots, \alpha_q; (b_0, w_0^+), (b_0, w_0^-)) (-1)^{q-e} \prod_{i \neq e} \chi(\alpha_i, \alpha_e) \\
&= \frac{(-1)^{\sum_{i < e} \chi(\alpha_i, \alpha_e) + \sum_{e < j} \chi(\alpha_j, \alpha_e)}}{2^{q-1}(e-1)!(q-e)!} \prod_{i \neq e} \chi(\alpha_i, \alpha_e) \\
&= \frac{1}{2^{q-1}(e-1)!(q-e)!} \prod_{i \neq e} (-1)^{\chi(\alpha_i, \alpha_e)} \chi(\alpha_i, \alpha_e)
\end{align*}
For any fixed list of classes \( (\alpha_1, \ldots, \alpha_{e-1}, \alpha_{e+1}, \ldots, \alpha_q) \), we sum up the above coefficients over the position \( e \) of the rank \(-1\) class. Since
\begin{equation*}
\sum_{1 \leq e \leq q} \frac{1}{2^{q-1}(e-1)!(q-e)!} = \frac{1}{(q-1)!},
\end{equation*}
we obtain
\begin{equation*}
J_{b_0, w_0^{-}}(\alpha) = \sum_{q \geq 1, \alpha_1, \ldots, \alpha_q \in \ell \atop \alpha_1 \in M_{v,n}} \frac{1}{(k-1)!} J_{b_0, w_0^{+}}(\alpha_1) \prod_{i=2}^{q} (-1)^{\chi(\alpha_i, \alpha_e)} \chi(\alpha_i, \alpha_e) J_{b_0, w_0^{+}}(\alpha_i).
\end{equation*}
Finally, by definition of the line \( L_{\alpha} \), there is no wall for rank zero classes \( \alpha_i \) between \( \ell \) and the large volume limit. Thus for \( i \geq 2 \), we have \( J_{b_0, w_0^{+}}(\alpha_i) = J_{b_0, \alpha}(\alpha_i) = J(\alpha_i) \) by Lemma 5.2. Moreover by (52), \( \chi(\alpha_i, \alpha_e) = \chi(\alpha_i, \alpha) \), so the claim (48) follows.

For any point \( (b, w) \in U \) with \( b > -n + \frac{k}{3} \), we define
\begin{equation}
PT_{b, w}^{\ell, n}(x, y, z) := \sum_{\alpha_1 = (-1, k_1H, \beta_1, m_1) \in M_{v,n} \atop 0 < k_1 < k, \alpha_1 + \alpha_2 + \cdots + \alpha_q = \alpha} J_{b, w}(\alpha_1) x^{k_1} y^{\beta_1} z^{m_1}.
\end{equation}
As a result of Proposition 5.3, we get the following.

**Corollary 5.4.** Let \( \ell \) be a wall for a class \( \alpha = (-1, \tilde{k}H, \tilde{\beta}, \tilde{m}) \in M_{v,n} \) which lies above \( L_{\alpha} \), and let \( (b_0, w_0^+) \) be points just above and below the wall \( \ell \). Then \( J_{b_0, w_0^{-}}(\alpha) \) is equal to the coefficient of \( x^k y^{\beta} z^{m} \) in the series
\begin{equation*}
PT_{b_0, w_0^{+}}^{\ell, n}(x, y, z) \cdot \prod_{\alpha' = (0, k' H, \beta', m') \in \ell \atop 0 < k' < k} \exp \left( -1 \chi(\alpha', \alpha) \chi(\alpha', \alpha) J(\alpha') x^{k'} y^{\beta'} z^{m'} \right). 
\end{equation*}
For any rank $-1$ class $\alpha_1 = (-1, k_1 H, \beta_1, m_1) \in K(X)$, we define
\[ J_\infty(\alpha) := J_{b, w > 0} (\alpha) \quad \text{where } b > \mu_H(\alpha_1) = -k_1. \]
We know $(e^{k_1 H} \alpha_1)^\vee [1] = (1, 0, -\beta_1 - \frac{1}{2} k_1^2 H^2, m_1 + k_1 \beta_1 H + \frac{1}{3} k_1^3 H^3)$ and $\det(\alpha) = \mathcal{O}_X(k)$ because $\mathrm{Pic}(X) = \mathbb{Z} H$. Thus Lemma 2.4 implies
\[ J_\infty(\alpha_1) = P_{m'_1, \beta'_1} \]
where $\beta'_1 := \beta_1 + \frac{1}{2} k_1^2 H^2$ and $m'_1 := -m_1 - k_1 \beta_1 H - \frac{1}{3} k_1^3 H^3$.

Take $b > -n + \frac{b}{3}$ and let $w \to +\infty$, then the limit of the generating series (53) is
\[ \exp \left( \frac{1}{-1} \chi_{\alpha'} \ J(\alpha') \ x^{k'y^z} \right) \]
where $\chi_{\alpha'}$ is defined via the following procedure: any non-zero term of $A(\alpha, \mu)$ is of the form
\[ \chi_{\alpha_1} := \chi \left( \alpha_1, \alpha - \sum_{2 \leq j < i} q_j \alpha_j \right). \]

We know the rank $-1$ destabilising factor $\alpha_1$ in (48) also lies in $M_{v,n}$, so we can apply Proposition 5.3 to this factor as well. Then finiteness of the number of walls when we move to the large volume limit implies the following.

**Proposition 5.6.** Take a class $\alpha = (-1, kH, \beta, m) \in M_{v,n}$ and a point $(b, w) \in U$ above $L_\alpha$ which does not lie on a wall for class $\alpha$. Then $J_{b, \alpha}(\alpha)$ is the coefficient of $x^k y^z m$ in the series $A(\alpha, \mu(\ell))$ where $\mu(\ell)$ is the gradient of the line $\ell$ passing through $\Pi(\alpha)$ and $(b, w)$.
Proof. We know any non-trivial sentence in $A(\alpha, \mu(\ell))$ which is $\mathbb{Q}$-multiple of $x^k y^\beta z^m$ is of the form (55) satisfying (56) and

$$\alpha = (-1, \tilde{k} H, \tilde{\beta}, \tilde{m}) = (-1, k_1 H, \beta_1, m_1) + \sum_{i=2}^{p} q_i(0, k_i H, \beta_i, m_i).$$

Therefore there is a wall $\ell_1$ for class $\alpha$ which is made by classes

$$v_2 := \alpha - q_2(0, k_2 H, \beta_2, m_2) \quad \text{and} \quad w_2 := q_2(0, k_2 H, \beta_2, m_2).$$

The wall $\ell_1 \cap U$ lies above $\ell \cap U$ as it is of slope $\frac{\beta_2 H}{k_2 H^*} > \mu(\ell)$ and both lines $\ell, \ell_1$ pass through $\Pi(\alpha)$. Moreover [FT21b, Lemma 3.2] implies that

$$\Delta_H(v_2) < \Delta_H(\alpha).$$

Similarly, there is a numerical wall $\ell_i$ for class

$$v_i := \alpha - \sum_{j=2}^{i} q_j(0, k_j H, \beta_j, m_j)$$

for any $i \in [2, p - 1]$ which is made by the destabilising factors

$$v_{i+1} = \alpha - \sum_{j=2}^{i+1} q_j(0, k_j H, \beta_j, m_j) \quad \text{and} \quad w_{i+1} := q_i(0, k_{i+1} H, \beta_{i+1}, m_{i+1}).$$

Note that the ordering (56) implies that for any $i \in [1, p - 1]$ the wall $\ell_i \cap U$ lies above or on $\ell_{i-1} \cap U$ where $\ell_0 = \ell$. Also [FT21b, Lemma 3.2] gives

$$0 \leq \Delta_H(-1, k_1 H, \beta_1, m_1) = \Delta_H(v_p) < \Delta_H(v_{p-1}) < \cdots < \Delta_H(v_2) < \Delta_H(\alpha).$$

To prove the main statement, we proceed by induction on $\Delta_H(\alpha)$. If $\Delta_H(\alpha) = 0$, then there is no wall for $\alpha$ by [FT21b, Lemma 3.2], thus $J_{b,w}(\alpha) = J_\infty(\alpha)$. The above argument also shows that the coefficient of $x^k y^\beta z^m$ in the series $A(\alpha, \mu(\ell))$ is also $J_\infty(\alpha)$, so the claim follows.

Now assume the claim holds for all classes $\alpha \in M_{v,n}$ with discriminant less than $\Delta_H(\alpha)$. Since there are only finitely many walls for class $\alpha$ above $L_\alpha$, we can prove the claim for $\alpha$ by induction on walls.

If $(b, w)$ lies in the large volume limit for $\alpha$, the claim is trivial. So assume the claim holds for $(b, w_+)$ just above a wall $\ell_\alpha$ for class $\alpha$. Finiteness of the number of walls for each class implies that we may choose $(b, w_\pm)$ just above and below the wall $\ell_\alpha$ so that they are not on walls for the destabilising classes of $\alpha$ along the wall $\ell_\alpha$. Corollary 5.4 implies that $J_{b,w}(\alpha)$ is the coefficient of $x^k y^\beta z^m$ in the series

$$M := PT_{b,w}(x, y, z) \cdot \prod_{\alpha' = (0, k't H, \beta', m') \in \ell_\alpha \atop 0 < k' < k} \exp \left( (-1)^{\chi(\alpha', \alpha)} \chi(\alpha', \alpha) J(\alpha') x^{e'} y^{\beta'} z^{m'} \right).$$
We claim it is equal to the coefficient of $x^k y^\beta z^n$ in the series $A(\alpha, \mu(\ell^-))$ where $\ell^-$ passes through $\Pi(\alpha)$ and $(b, w^-)$.

Any non-zero $Q$-valued multiple of $x^k y^\beta z^n$ in $M$ is of the form

$$J_{b,w^+}(-1, k_1 H, \beta_1, m_1) x^{k_1} y^{\beta_1} z^{m_1}.$$ 

Let $\alpha = (0, k_1 H, \beta_1, m_1)$ then $\alpha + \beta H = \mu(\ell^-)$.

If $p \geq 2$, we know $\Delta_H(-1, k_1 H, \beta_1, m_1) < \Delta_H(\alpha)$. Thus applying the induction on discriminant and on walls implies that $J_{b,w^-}(\alpha)$ is the coefficient of $x^k y^\beta z^n$ in the series

$$\left( \sum_{\alpha_1 \in M_{\nu, n}} A\left(\alpha_1, \mu(\ell^+_{\alpha_1})\right) \right) \cdot \prod_{\alpha_2 = (0, k_2 H, \beta_2, m_2) \atop 0 < k_2 < k \atop \beta_2 H = \mu(\ell_\alpha)} \exp\left( (-1)^{\chi(\alpha_2, \alpha)} \chi(\alpha_2, \alpha) J(\alpha_2) x^{2} y^{\beta_2} z^{m_2} \right)$$

where $\ell^+_{\alpha_1}$ is the line passing through $\Pi(\alpha_1)$ and $(b, w^+)$. Then the coefficient of $x^k y^\beta z^n$ in the above series is the same as the one in $A(\alpha, \mu(\ell^-))$, so the claim follows.

Finally Theorem 1.2 follows from the following.

**Theorem 5.7.** The coefficient of $x^{n+k} y^\beta x^{2} y^{\beta_2} z^{m+n} w^3 u^3 / 6$ in the series

$$\frac{(-1)^{\chi(\mathcal{O}_{X}(-n), v)+1}}{\chi(\mathcal{O}_{X}(-n), v)} A(v_{n_0}, \mu(\ell_f))$$

is equal to $J(v)$. Here $\ell_f$ is a line passing through $\Pi(v_{n_0})$ which lies just below $\ell_f$.

**Proof.** First consider the Joyce-Song wall $\ell_{JS}$ for class $v$, that $\mathcal{O}_{X}(-n)[1]$ is making. We know the destabilising factors are either of the form

$v_1 = [\mathcal{O}_{X}(-n)][1]$ and $v_2 = v$

or

$$v_1 = M_{\nu, n} \text{ and } v_2 = (0, k_2 H, \beta_2, m_2) \text{ with } 0 < k_2 < k.$$

Let $(b, w \pm)$ be point just above (below) the Joyce-Song wall. Then applying wall-crossing formula (16) and the same argument as in Proposition 5.3 imply

$$J_{b,w^-}(v) = (-1)^{\chi(\mathcal{O}_{X}(-n), v)} \chi(\mathcal{O}_{X}(-n), v) J_{b,w^+}([\mathcal{O}_{X}(-n)[1]]) J_{b,w^+}(v)$$

$$+ \sum_{q \geq 1, \alpha_1, \ldots, \alpha_q \in \ell_{JS}, \alpha_q \in M_{\nu, n}, \alpha_i = (0, k_i H, \beta_i, m_i) \text{ for } i \in [2, q]} \frac{1}{(q - 1)!} J_{b,w^+}(\alpha_1) \prod_{i=2}^q (-1)^{\chi(\alpha_i, \alpha)} \chi(\alpha_i, \alpha) J(\alpha_i).$$
There is no wall for class \( \nu \) up to the large volume limit, so \( J_{b,w^+}(\nu) = J_{b,\infty}(\nu) \) which is equal to \( J(\nu) \) by Lemma 5.2. Moreover \( \mathcal{O}_X(-n) \) is a rigid objects, so \( J_{b,w^+}(\mathcal{O}_X(-n)) = 1 \) as \( \text{Pic}(X) = \mathbb{Z}.H \). We know the destabilising classes along all walls above the Joyce-Song wall are of the form (58). Thus applying a similar argument as in Proposition 5.6 shows \( J_{b,w^-}(\nu_n) \) is the coefficient of \( x^{n+k}y^{\beta-\frac{a_3^2}{2}z^m+\frac{a_3^3}{6}} \) in the series
\[
(-1)^{\chi(\mathcal{O}_X(-n),\nu)} \chi(\mathcal{O}_X(-n),\nu) J(\nu) x^{n+k} y^{\beta-\frac{a_3^2}{2}z^m+\frac{a_3^3}{6}} + A(\nu_n, \mu(\ell^-_{JS})).
\]
For walls below \( \ell_{JS} \), the destabilising factors are all again of the form (58), so \( J_{b,w^f}(\nu_n) \) for \((b, w^-)\) just below \( \ell_f \) is equal to the coefficient of \( x^{n+k}y^{\beta-\frac{a_3^2}{2}z^m+\frac{a_3^3}{6}} \) in the series
\[
(-1)^{\chi(\mathcal{O}_X(-n),\nu)} \chi(\mathcal{O}_X(-n),\nu) J(\nu) x^{n+k} y^{\beta-\frac{a_3^2}{2}z^m+\frac{a_3^3}{6}} + A(\nu_n, \mu(\ell^-_f)).
\]
But \( J_{b,w^f}(\nu_n) = 0 \) as there is no \( \nu_{b,w^-} \)-semistable object of class \( \nu_n \) for \((b, w)\) below \( \ell_f \) and so the claim follows. \( \square \)

6. Rank 2 DT theory

The goal of this section is to prove the following.

**Theorem 6.1.** Let \( X \) be a smooth Calabi-Yau 3-fold with \( \text{Pic}(X) = \mathbb{Z}.H \). For any rank 2 class \( \alpha \in K(X) \), there is an explicit formula expressing \( J(\alpha) \) in terms of rank zero and one DT invariants.

We consider two different cases depending on \( \text{ch}_1(\alpha) \) whether it is even or odd multiple of \( H \).

**Case (i)** Fix a rank 2 class \( \nu = (2, H, \beta, m) \in K(X) \) with
\[(59) \quad \text{ch}_1(\nu) = (2, 1, s, d). \]
Then we pick \( n \gg 0 \) and set \( w_n := \nu - \mathcal{O}_X(-n) \), so
\[
\text{ch}_1(w_n) = \left( 1, n+1, s - \frac{1}{2}n^2, d + \frac{1}{6}n^3 \right).
\]
Consider the line \( \ell_f \) given by \( B_{b,w}(w_n) = 0 \) (11). One can easily show that for \( n \gg 0 \), \( \ell_f \) intersects \( \partial U \) at two points with \( b_1^f \leq b_1' \) such that \( b_1' - b_1^f > n \).

Let \( \ell \) be a wall for an object \( E \) of class \( w_n \) with the destabilising sequence \( E' \rightarrow E \rightarrow E'' \). Then by [FT21b, Theorem 3.3], one of the factors \( E_1 \) is a rank one sheaf and the other factor \( E_0 \) is a rank zero sheaf. We know \( \ell \) lies above or on \( \ell_f \) and passes through \( \Pi(E_1) \), see Figure 4. Thus \( 0 < b_1' \leq \mu(E_1) \) and so \( \text{ch}_1(E_0).H^2 \leq nH^3 \). Then the same argument as in [FT21b, Lemma B.3] implies that there is no wall for \( E_0 \) above \( \ell_f \), so we only need to analyse the destabilising factor \( E_1 \).

Define \( \tilde{M}_{w,n} \subset K(X) \) to be the set of all rank one classes \( \alpha = (1, \text{ch}_1, \text{ch}_2, \text{ch}_3) \in K(X) \) with \( \Delta_H(\alpha) \geq 0 \) such that
\[
0 < \frac{\text{ch}_1 H^2}{H^3} < n + 1 \quad \text{and} \quad \Pi(\alpha) \text{ lies above or on } \ell_f.
\]
As explained in [FT21b, Section 3], for any class $\alpha \in \widetilde{M}_{w,n}$, there is a unique line $\ell_\alpha$ going through $\Pi(\alpha)$ which intersects $\partial U$ at two points with $b$-values $a_v < b_v \leq \mu_H(\alpha)$ satisfying

$$H^3(b_v - a_v) = \text{ch}_1(\alpha)H^2 - b_vH^3 \geq 0.$$ 

The area above the line $\ell_\alpha$ is called the safe area for $\alpha$ and denoted by

$$U_{\alpha} := \left\{(b,w) \in U : b < \frac{\text{ch}_1(\alpha)H^2}{H^3} \text{ and } (b,w) \text{ lies above } \ell_{\alpha}\right\}.$$

Take an object $E \in \mathcal{A}_b$ of class $\alpha \in \widetilde{M}_{w,n}$ which is $\nu_{b,w}$-semistable for some $(b,w) \in U_{\alpha}$. Then [FT21b, Proposition 3.1] implies that

(a) $E$ is a sheaf,
(b) if $E_1 \hookrightarrow E \twoheadrightarrow E_2$ is a short exact sequence in $\mathcal{A}_b$ with $\nu_{b,w}(E_1) = \nu_{b,w}(E_2)$ then one of $E_i$’s is a rank zero tilt-semistable sheaf, and the other factor $E_j$ is a rank one sheaf such that $\text{ch}(E_j) \in \widetilde{M}_{w,n}$ and $(b,w) \in U_{E_j}$.

Let $\ell$ be a wall for class $w_n$, then [FT21b, Theorem 3.3] implies that the destabilising factors are either of the form

(a) $v_1 = [\mathcal{O}_X(-n)][1], v_2 = w$ and there is no wall for class $w$ between $\ell$ and the large volume limit,
(b) $v_1 \in \widetilde{M}_{w,n}, v_2 = (0,k_2H,\beta_2,m_2)$ with $\ell \in U_{v_1}$ and there is no wall for class $v_2$ between $\ell_f$ and the large volume limit.

Figure 4. The final wall $\ell_f$ for class $w_n$
Consider the generating series

\[
\text{DT}^{w,n}(x, y, z) := \sum_{\alpha_1 = (1, k_1 H, \beta_1, m_1) \in \tilde{M}_{w,n}} J(\alpha_1) \cdot x^{k_1} y^{\beta_1} z^{m_1} = \sum_{\alpha_1 = e^{k_1 H} (1, 0, -\beta'_1, -m'_1) \in \tilde{M}_{w,n}} \text{I}_{m'_1, \beta'_1} \cdot x^{k_1} y^{-\beta'_1 + \frac{1}{2} k_2^2 H^2} z^{-m'_1 - k \beta'_1 H + \frac{1}{4} k^3 H^3}.
\]

**Definition 6.2.** Let \( \alpha \in K(X) \) be either equal to \( w_n \) or \( \alpha \in \tilde{M}_{w,n} \). For any real number \( \mu \in \mathbb{R} \), we define

\[
\tilde{A}(\alpha, \mu) := \text{DT}^{w,n}(x, y, z) \cdot \prod_{\alpha' = (0, k'H, \beta', m') \in K(X)} \exp \left( (-1)^{\chi_{\alpha'}} \chi_{\alpha'} J_{\infty}(\alpha') x^{\alpha'} y^{\beta'} z^{m'} \right)
\]

where \( \chi_{\alpha'} \) is defined via the following procedure: any non-zero term of \( \tilde{A}(\alpha, \mu) \) is of the form

\[
(61) \quad J(1, k_1 H, \beta_1, m_1) x^{k_1} y^{\beta_1} z^{m_1} \cdot \prod_{i \in [2, p]} \left( (-1)^{\chi_{\alpha_i}} \chi_{\alpha_i} J(\alpha_i) x^{k_i} y^{\beta_i} z^{m_i} \right)^{q_i} q_i!
\]

where \( q_i \neq 0 \) for \( i \in [2, p] \). We may assume

\[
(62) \quad \frac{\beta_p H}{k_p H^3} \leq \frac{\beta_{p-1} H}{k_{p-1} H^3} \leq \cdots \leq \frac{\beta_2 H}{k_2 H^3},
\]

then for any \( i \in [2, p] \) we define

\[
(63) \quad \chi_{\alpha_i} := \chi(\alpha_i, \alpha - \sum_{2 \leq j < i} q_j \alpha_j).
\]

Applying the same argument as in Proposition 5.6 implies the following.

**Lemma 6.3.** Take a class \( \alpha = (1, k H, \beta, \bar{m}) \in \tilde{M}_{w,n} \) and a point \((b, w) \in U_\alpha \) which does not lie on a wall for class \( \alpha \). Then \( J_{b,w}(\alpha) \) is the coefficient of \( x^k y^\bar{m} \) in the series \( \tilde{A}(\alpha, \mu(\ell)) \) where \( \mu(\ell) \) is the gradient of the line \( \ell \) passing through \( \Pi(\alpha) \) and \((b, w) \).

As a result, we can prove Theorem 6.1 in odd cases:

**Proposition 6.4.** Consider a rank 2-class \( \alpha \in K(X) \) with \( \text{ch}_1(\alpha) = k H \) where \( \gcd(2, k) = 1 \). Let \( w = e^{-\frac{k+1}{2} H} \alpha = (2, H, \beta, m) \) and \( w_n = w - [O_X(-n)] \) for \( n \gg 0 \). Then the coefficient of \( x^{n+1} y^\beta \frac{H^2}{2} z^{m+n\frac{3}{2} H^3} \) in the series

\[
\frac{(-1)^{\chi(\mathcal{O}_X(-n), w)+1}}{\chi(\mathcal{O}_X(-n), w)} \tilde{A}(w_n, \mu(\ell_f))
\]

is equal to \( J(\alpha) \). Here \( \ell_f \) is a line passing through \( \Pi(w_n) \) which lies just below \( \ell_f \).
Proof. We know any tilt-semistable sheaf of class \( \alpha \) or \( w \) is slope-stable, so it is in particular H-Gieseker stable. Therefore \( J(\alpha) = J(w) = J_{t_i}(w) \). Since all walls for class \( w_n \) lie above or on \( \ell_f \), the same argument as in Theorem 5.7 implies the claim. \( \square \)

Case (ii) Now fix a rank 2-class

\[
w = (2, 0, \beta, m).
\]

Then pick \( n \gg 0 \) and let \( w_n = w - [O_X(-n)] \) as before. [FT21b, Theorem B.2] implies that there is no \( \nu_{b,w} \)-semistable object of class \( w_n \) below the Joyce-Song wall \( \ell_{JS} \) which \( O_X(-n) \) is making.

We first examine walls for class \( w_n \) above \( \ell_{JS} \). Define \( \widetilde{M}_{w,n} \subset K(X) \) to be the set of all rank one classes \( \alpha = (1, \chi_1, \chi_2, \chi_3) \in K(X) \) with \( \Delta_H(\alpha) \geq 0 \) such that

\[
0 \leq \frac{\text{ch}_1 H^2}{H^3} < n \quad \text{and} \quad \Pi(\alpha) \text{ lies above or on } \ell_{JS}.
\]

For any class \( \alpha \in \widetilde{M}_{w,n} \), we consider the open subset \( U_{\alpha} \) as in (60). Let \( \ell \) be a wall for class \( w_n \) above \( \ell_{JS} \), then [FT21b, Theorem 3.3] implies that the destabilising factors are of the form

\[
v_1 \in \widetilde{M}_{w,n} \quad \text{and} \quad v_2 = (0, k_2 H, \beta_2, m_2)
\]

such that \( \ell \in U_{v_1} \) and there is no wall for class \( v_2 \) between \( \ell_{JS} \) and the large volume limit\(^6\). Thus we get the following.

Lemma 6.5. Let \((b, w^+) \in U \) be a point just above \( \ell_{JS} \). Then \( J_{b,w^+}(w_n) \) is the coefficient of \( x^n y^\beta - \frac{n^2 \mu^2}{2} z^m + \frac{n^3 \mu^3}{6} \) in the series \( A(\alpha, \mu(\ell_{JS}^+)) \) where \( \ell_{JS}^+ \) is a line passing through \( \Pi(w_n) \) which lies just above \( \ell_{JS} \)

Along the Joyce-Song wall, there are three possibilities for the destabilising factors:

(a) \( v_1 = w \) and \( v_2 = [O_X(-n)][1] \),

(b) \( v_1 = (1, 0, \beta', m') \) and \( v_2 = (0, nH, \beta - \beta' - \frac{n^2 H^2}{2}, m - m' + \frac{n^3 H^3}{6}) \) such that \( \beta'. H = \frac{1}{2} \beta. H \),

(c) \( v_1 = (1, 0, \beta', m'), v_1' = (1, 0, \beta'', m'') \) and \( v_2 = [O_X(-n)][1] \) such that \( v_1 + v_1' + v_2 = w_n \) and \( \beta'. H = \beta''. H = \frac{1}{2} \beta. H \).

Let \((b, w^+) \) be points just above (below) \( \ell_{JS} \). We know there is no wall for all the above destabilising factors between \((b, w^+) \) and the large volume limit. Applying the wall-crossing formula (16) gives

\[
J_{b,w^-}(w_n) = J_{b,w^+}(w_n) + (-1)^{\chi(O_X(-n), w)} \chi(O_X(-n), w) J_{t_i}(w) + A_{w_n}
\]

\(^6\)Note that in case (2)(b) of [FT21b, Theorem 3.3], we apply [FT21b, Theorem B.1] which says there is no wall for our rank zero class \( v_2 \) above the wall that \( O_X(-n) \) is making.
where

\[ A_{v_n} := \sum_{v_1 = (1, 0, s', m')} (-1)^{\chi(v_2, v_1) + 1} \chi(v_2, v_1) J(v_1) J(v_2) \]

\[ v_2 = \left( 0, nH, \beta' - \frac{n^2 H^2}{2}, m - m' + \frac{n^3 H^3}{6} \right) \]

\[ + \sum_{v_1 = (1, 0, s', m')} C_{v_1, v'_1} J(v_1) J(v'_1). \]

Here \( C_{v, v'} \in \mathbb{Q} \) depends on \( v, v' \) and can be explicitly determined by (16).

**Proposition 6.6.** Consider a rank 2-class \( \alpha \in K(X) \) with \( \text{ch}_1(\alpha) = kH \) where \( \gcd(2, k) = 2 \). Let \( w = e^{-\frac{H}{2}} \alpha = (2, 0, \beta, m) \) and \( w_n = w - [O_X(-n)] \) for \( n \gg 0 \). Then \( J_{\parallel}(\alpha) \) is equal to

\[ \frac{(-1)^{\chi(O_X(-n), w) + 1}}{\chi(O_X(-n), w)} (J_{b, w}(w_n) + A_{w_n}) \]

where \( J_{b, w}(w_n) \) is the coefficient of \( x^n y^\beta \frac{x^2 y^2}{2} z^m + x^3 y^3 \) in the series

\[ \frac{(-1)^{\chi(O_X(-n), w) + 1}}{\chi(O_X(-n), w)} \tilde{A}(w_n, \mu(t_{12}^+)). \]

The final step is to apply the wall-crossing formula (20) between tilt-stability and Gieseker-stability. Since \( \alpha \) is of rank 2, there are at most two factors \( \alpha_1, \alpha_2 \) and we know they have the same tilt-slope, so (18) implies

\[ J_{\parallel}(\alpha) = J(\alpha) + \sum_{\alpha_1 = e^H (1, 0, -\beta_1, -m_1) \in C(\text{Coh}(X))} (-1)^{m_1 - m_2} (m_1 - m_2) \chi_{m_1, \beta_1} \chi_{m_2, \beta_2}. \]

Combining this with Proposition 6.6 completes the proof of Theorem 6.1.

**Remark 6.7.** The results in Section 5 and Section 6 are based on the wall-crossing arguments in [FT21b, FT21c]. As discussed in these papers, the restricted set of weak stability conditions handled in [Li19, Kos20, Liu21] are sufficient for our purposes, thus Theorem 1.2 and Theorem 6.1 are valid in these cases.

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