Koszul equivalences in $A_\infty$-algebras

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Abstract. We prove a version of Koszul duality and the induced derived equivalence for Adams connected $A_\infty$-algebras that generalizes the classical Beilinson–Ginzburg–Soergel Koszul duality. As an immediate consequence, we give a version of the Bernstein–Gelfand–Gelfand correspondence for Adams connected $A_\infty$-algebras.

We give various applications. For example, a connected graded algebra $A$ is Artin–Schelter regular if and only if its Ext-algebra $\text{Ext}_A^*(k,k)$ is Frobenius. This generalizes a result of Smith in the Koszul case. If $A$ is Koszul and if both $A$ and its Koszul dual $A'$ are noetherian satisfying a polynomial identity, then $A$ is Gorenstein if and only if $A'$ is. The last statement implies that a certain Calabi–Yau property is preserved under Koszul duality.

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Introduction

Koszul duality is an incredibly powerful tool used in many areas of mathematics. One aim of this paper is to unify some generalizations by using $A_\infty$-algebras. Our version is comprehensive enough to recover the original version of Koszul duality and the induced derived equivalences due to Beilinson–Ginzburg–Soergel [7] and most of the generalizations in ring theory and algebraic geometry. Although we will restrict ourselves to Adams connected $A_\infty$-algebras (a natural extension of a connected graded algebras — see Definition 2.1), we have set up a framework that will work for other classes of algebras arising from representation theory and algebraic geometry.

We fix a commutative field $k$ and work throughout with vector spaces over $k$. We define $A_\infty$-algebras over $k$ in Definition 1.1.

Similar to [24, Section 11] we define the Koszul dual of an $A_\infty$-algebra $A$ to be the vector space dual of the bar construction of $A$ — see Section 2 for details. This idea is not new and dates back at least to Beilinson–Ginzburg–Schechtman [6] for graded algebras. Keller also took this approach in [19] for differential graded algebras. Our first result is a generalization of [7, Theorem 2.10.2].

**Theorem A.** Let $A$ be an augmented $A_\infty$-algebra. Suppose that the Koszul dual of $A$ is locally finite. Then the double Koszul dual of $A$ is $A_\infty$-isomorphic to $A$.

This is proved as Theorem 2.4. A special case of the above theorem was proved in [24, Theorem 11.2].

As in [7] we prove several versions of equivalences of derived categories induced by the Koszul duality. Let $D(A)$ be the derived category of right $A_\infty$-modules over $A$. Let $D_{\text{per}}(A)$ (respectively, $D_{\text{fd}}(A)$) denote the full triangulated subcategory of $D(A)$ generated by all perfect complexes (respectively, all right $A_\infty$-modules whose homology is finite-dimensional) over $A$. The next result is a generalization of [7, Theorem 2.12.6].
Theorem B. Let $A$ be an Adams connected $A_\infty$-algebra and $E$ its Koszul dual. If $HE$ is finite-dimensional, then there is an equivalence of triangulated categories $D^\infty_{\text{per}}(A) \cong D^\infty_\text{fd}(E)$.

This is proved as Corollary 7.2(b).

Other equivalences of triangulated categories can be found in Sections 4 and 5. If $A$ is either Artin–Schelter regular (Definitions 9.1(c) and 9.2(c)) or right noetherian with finite global dimension, then $HE$ is finite-dimensional and hence Theorem B applies.

Koszul duality has many applications in ring theory, representation theory, algebraic geometry, and other areas. The next result is a generalization of the Bernstein–Gelfand–Gelfand correspondence that follows from Theorem B.

Theorem C. Let $A$ be an Adams connected $A_\infty$-algebra that is noetherian Artin–Schelter regular. Let $E$ be the Koszul dual of $A$. Then $HE$ is finite-dimensional and there is an equivalence of triangulated categories $D^\infty(\text{proj } A) \cong D^\infty_{\text{fg}}(E)$.

Applications of Koszul duality in ring theory are surprising and useful. We will mention a few results that are related to the Gorenstein property. In the rest of this introduction we let $R$ be a connected graded associative algebra over a base field $k$.

Corollary D. Let $R$ be a connected graded algebra. Then $R$ is Artin–Schelter regular if and only if the Ext-algebra $\bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_R(k_R, k_R)$ is Frobenius.

This result generalizes a theorem of Smith [32, Theorem 4.3 and Proposition 5.10] that was proved for Koszul algebras. It is proved in Section 9.3. Corollary D is a fundamental result and the project [25] was based on it.

The Gorenstein property plays an important role in commutative algebra and algebraic geometry. We prove that the Gorenstein property is preserved under Koszul duality; see Section 9.4 for details.

Corollary E. Let $R$ be a Koszul algebra and let $R^!$ be the Koszul dual of $R$ in the sense of Beilinson–Ginzburg–Soergel [7]. If $R$ and $R^!$ are both noetherian having balanced dualizing complexes, then $R$ is Gorenstein if and only if $R^!$ is.

The technical hypothesis about the existence of balanced dualizing complexes can be checked when the rings are close to being commutative. For example, Corollary E holds when $R$ and $R^!$ are noetherian and satisfy a polynomial identity. This technical hypothesis is presented because we do
not understand noncommutative rings well enough. We do not know any example in which the technical hypothesis is necessary; however, Corollary E does fail for nonnoetherian rings — for example, the free algebra \( R = k\langle x, y \rangle \) is Gorenstein, but \( R^1 \cong k\langle x, y \rangle / (x^2, xy, yx, y^2) \) is not.

Note that Koszul duality preserves the Artin–Schelter condition (Proposition 9.3). Under the technical hypothesis of Corollary E the Artin–Schelter condition is equivalent to the Gorenstein property. Therefore Corollary E follows. We can restate Corollary E for \( A_\infty \)-algebras in a way that may be useful for studying the Calabi–Yau property of the derived category \( D^\infty(A) \) (see the discussion in Section 9.4).

The following is proved in Section 9.4.

**Corollary F.** Let \( A \) be an Adams connected commutative differential graded algebra such that \( \text{RHom}_A(k, A) \) is not quasi-isomorphic to zero. If the Ext-algebra \( \bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_A(k_A, k_A) \) is noetherian, then \( A \) satisfies the Artin–Schelter condition.

The hypothesis on \( \text{RHom}_A(k, A) \) is a version of finite depth condition which is very mild in commutative ring theory and can be checked under some finiteness conditions. This is automatic if \( A \) is a finitely generated associative commutative algebra. As said before, the Artin–Schelter condition is equivalent to the Gorenstein property under appropriate hypotheses. Hence Corollary F relates the Gorenstein property of \( R \) with the noetherian property of \( R^1 \) and partially explains why Corollary E holds, and at the same time it suggests that Corollary E should be a consequence of a more basic statement that relates the noetherian property of \( R \) with the Gorenstein property of \( R^1 \). On the other hand, we believe that there should be a version of Corollary E without using the noetherian property. As we commented for Corollary E, Corollary F should hold in a class of noncommutative rings \( R \). Corollary F is also a variation of a result of Bøgvad and Halperin about commutative complete intersection rings [9].

This paper is part four of our \( A_\infty \)-algebra project and is a sequel to [24, 25, 26]. Some results were announced in [24]. For example, Theorem B and Corollary D were stated in [24] without proof. We also give a proof of [24, Theorem 11.4] in Section 5.

The paper is divided into three parts: Koszul duality for algebras, Koszul duality for modules, and applications in ring theory.

Part I consists of Sections 1 and 2. Section 1 gives background material on \( A_\infty \)-algebras and their morphisms. The reader may wish to skim it to see the conventions and notation used throughout the paper. Theorem A is proved in Section 2, and we use it to recover the classical Koszul duality of Beilinson, Ginzburg, and Soergel. We also discuss a few examples.

Part II consists of Sections 3–8. Section 3 gives background material on \( A_\infty \)-modules; most of this is standard, but the results on opposites are new. Section 4 sets up a framework for proving equivalences of various derived
categories of DG modules over a DG algebra. Section 5 uses this framework to prove DG and $A_\infty$ versions of the results of Beilinson, Ginzburg, and Soergel which establish equivalences between certain derived categories of modules over a Koszul algebra and over its Koszul dual. The point of Section 6 is a technical theorem which allows us, in Section 7, to rederive the classical results from the $A_\infty$-algebra results. Theorem B is proved in Section 7, also. We discuss a couple of examples in Section 8.

Part III consists of Sections 9–10. Section 9 discusses Artin–Schelter regular algebras and Frobenius algebras, from the $A_\infty$-algebra point of view, and includes proofs of Corollaries D, E, and F. Section 10 gives an $A_\infty$-version of the BGG correspondence; Theorem C is proved there.

Part I. Koszul duality for algebras

1. Background on $A_\infty$-algebras

In this section, we describe background material necessary for the rest of the paper. There are several subsections: grading conventions and related issues; $A_\infty$-algebras and morphisms between them; the bar construction; and homotopy for morphisms of $A_\infty$-algebras.

1.1. Conventions. Throughout we fix a commutative base field $k$. Unless otherwise stated, every chain complex, vector space, or algebra will be over $k$. The unadorned tensor product $\otimes$ is over $k$ also.

Vector spaces (and the like) under consideration in this paper are bi-graded, and for any bihomogeneous element $a$, we write

$$\deg a = (\deg_1(a), \deg_2(a)) \in \mathbb{Z} \times G$$

for some abelian group $G$. The second grading is called the Adams grading. In the classical setting $G$ is trivial, but in this paper we have $G = \mathbb{Z}$; many of the abstract assertions in this paper hold for any abelian group $G$. If $V$ is a bigraded vector space, then the degree $(i, j)$ component of $V$ is denoted by $V^i_j$. Usually we work with bihomogeneous elements, with the possibility of ignoring the second grading. All chain complexes will have a differential of degree $(1, 0)$. The Koszul sign convention is in force throughout the paper, but one should ignore the second grading when using it: when interchanging elements of degree $(i, s)$ and $(j, t)$, multiply by $(-1)^{ij}$.

Given a bigraded vector space $V$, we write $V^s$ for its graded dual. Its suspension $SV$ is the bigraded space with $(SV)^i_j = V^{i+1}_j$: suspension shifts the first grading down by one, and ignores the second grading. Write $s : V \to SV$ for the obvious map of degree $-1$. If $V$ has a differential $d_V$, then define a differential $d_{SV}$ on $SV$ by $d_{SV}(sv) = -sd_V(v)$. The Adams shift of $V$ is $\Sigma V$ with $(\Sigma V)^i_j = V^{i+1}_j$. If $V$ has a differential, then there is no sign in the differential for $\Sigma V$: $d_{\Sigma V}$ is just the shift of $d_V$. 
If \((M,d)\) and \((N,d)\) are complexes, so are \(\text{Hom}_k(M,N)\) and \(M \otimes N(= M \otimes_k N)\), with differentials given by
\[
d(f) = df - (-1)^{\deg_1(f)} fd, \quad \forall f \in \text{Hom}_k(M,N),
\]
\[
d(m \otimes n) = dm \otimes n + (-1)^{\deg_1(m)} m \otimes dn, \quad \forall m \otimes n \in M \otimes N,
\]
respectively.

If \(C\) is a category, we write \(C(X,Y)\) for morphisms in \(C\) from \(X\) to \(Y\). We reserve \(\text{Hom}\) to denote the chain complex with differential as in the previous paragraph.

1.2. \(A_\infty\)-algebras and morphisms. In this paper, we will frequently work in the category of augmented \(A_\infty\)-algebras; in this subsection and the next, we define the objects and morphisms of this category. Keller’s paper [20] provides a nice introduction to \(A_\infty\)-algebras; it also has references for many of the results which we cite here and in later subsections. Lefèvre-Hasegawa’s thesis [23] provides more details for a lot of this; although it has not been published, it is available online. Another reference is [24] which contains some easy examples coming from ring theory. The following definition is originally due to Stasheff [34].

Definition 1.1. An \(A_\infty\)-algebra over \(k\) is a \(\mathbb{Z} \times \mathbb{Z}\)-graded vector space \(A\) endowed with a family of graded \(k\)-linear maps
\[
m_n : A^\otimes n \rightarrow A, \quad n \geq 1,
\]
of degree \((2 - n, 0)\) satisfying the following Stasheff identities: for all \(n \geq 1\),
\[
\text{SI}(n) \quad \sum (-1)^{r+s+t} m_u(1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0,
\]
where the sum runs over all decompositions \(n = r + s + t\), with \(r, t \geq 0\) and \(s \geq 1\), and where \(u = r + 1 + t\). Here \(1\) denotes the identity map of \(A\). Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule.

A \(DG\) (differential graded) algebra is an \(A_\infty\)-algebra with \(m_n = 0\) for all \(n \geq 3\).

The reader should perhaps be warned that there are several different sign conventions in the \(A_\infty\)-algebra literature. We have chosen to follow Keller [20], who is following Getzler and Jones [14]. Stasheff [34] and Lefèvre-Hasegawa [23] use different signs: they have the sign \((-1)^{r+s+t}\) in \(\text{SI}(n)\), and this requires sign changes in other formulas (such as \(\text{MI}(n)\) below).

As remarked above, we work with bigraded spaces throughout, and this requires a (very mild) modification of the standard definitions: ordinarily, an \(A_\infty\)-algebra is singly graded and \(\deg m_n = 2 - n\); in our bigraded case, we have put \(\deg m_n = (2 - n, 0)\). Thus if one wants to work in the singly graded setting, one can just work with objects concentrated in degrees \((*, 0) = \mathbb{Z} \times \{0\}\).
Definition 1.2. An \(A_{\infty}\)-algebra \(A\) is strictly unital if \(A\) contains an element 1 which acts as a two-sided identity with respect to \(m_2\), and for \(n \neq 2\), \(m_n(a_1 \otimes \cdots \otimes a_n) = 0\) if \(a_i = 1\) for some \(i\).

In this paper we assume that \(A_{\infty}\)-algebras (including DG algebras) are strictly unital.

Definition 1.3. A morphism of \(A_{\infty}\)-algebras \(f : A \rightarrow B\) is a family of \(k\)-linear graded maps

\[ f_n : A^\otimes n \rightarrow B, \quad n \geq 1, \]

of degree \((1 - n, 0)\) satisfying the following Stasheff morphism identities: for all \(n \geq 1\),

\[
\text{MI}(n) \sum (-1)^{r+s} f_u(1^\otimes r \otimes m_s \otimes 1^\otimes t) = \sum (-1)^w m_q(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_q}),
\]

where the first sum runs over all decompositions \(n = r + s + t\) with \(r, t \geq 0\), \(s \geq 1\), and where we put \(u = r + 1 + t\); and the second sum runs over all \(1 \leq q \leq n\) and all decompositions \(n = i_1 + \cdots + i_q\) with all \(i_s \geq 1\). The sign on the right-hand side is given by

\[
w = (q - 1)(i_1 - 1) + (q - 2)(i_2 - 1) + \cdots + 2(i_{q-1} - 1) + (i_{q-1} - 1).
\]

When \(A_{\infty}\)-algebras have a strict unit (as we usually assume), an \(A_{\infty}\)-morphism between them is also required to be strictly unital, which means that it must satisfy these unital morphism conditions: \(f_1(1_A) = 1_B\) where \(1_A\) and \(1_B\) are strict units of \(A\) and \(B\) respectively, and \(f_n(a_1 \otimes \cdots \otimes a_n) = 0\) if some \(a_i = 1_A\) and \(n \geq 2\) (see \([20, 3.5], [24, \text{Section 4}]\)).

As with \(A_{\infty}\)-algebras, we have modified the grading on morphisms: we have changed the usual grading of \(\deg f_n = 1 - n\) to \(\deg f_n = (1 - n, 0)\). The composite of two morphisms is given by a formula similar to the morphism identities \(\text{MI}(n)\); see \([20]\) or \([23]\) for details.

Definition 1.4. A morphism \(f : A \rightarrow B\) of \(A_{\infty}\)-algebras is strict if \(f_n = 0\) for \(n \neq 1\). The identity morphism is the strict morphism \(f = 1\). A morphism \(f\) is a quasi-isomorphism or an \(A_{\infty}\)-isomorphism if \(f_1\) is a quasi-isomorphism of chain complexes.

Note that quasi-isomorphisms of \(A_{\infty}\)-algebras have inverses: a morphism is a quasi-isomorphism if and only if it is a homotopy equivalence — see Theorem 1.16 below.

We write \(\text{Alg}\) for the category of associative \(\mathbb{Z} \times \mathbb{Z}\)-graded algebras with morphisms being the usual graded algebra morphisms, and we write \(\text{Alg}^\infty\) for the category of \(A_{\infty}\)-algebras with \(A_{\infty}\)-morphisms.

Let \(A\) and \(B\) be associative algebras, and view them as \(A_{\infty}\)-algebras with \(m_n = 0\) when \(n \neq 2\). We point out that there may be nonstrict \(A_{\infty}\)-algebra morphisms between them. That is, the function

\[
\text{Alg}(A, B) \rightarrow \text{Alg}^\infty(A, B),
\]
sending an algebra map to the corresponding strict $A_\infty$-morphism, need not be a bijection. See Example 2.8 for an illustration of this.

The following theorem is important and useful.

**Theorem 1.5 ([17]).** Let $A$ be an $A_\infty$-algebra and let $HA$ be its cohomology ring. There is an $A_\infty$-algebra structure on $HA$ with $m_1 = 0$ and $m_2$ equal to its usual associative product, and with the higher multiplications constructed from the $A_\infty$-structure of $A$, such that there is a quasi-isomorphism of $A_\infty$-algebras $HA \to A$ lifting the identity of $HA$.

This theorem was originally proved for $\mathbb{Z}$-graded $A_\infty$-algebras, and holds true in our $\mathbb{Z} \times \mathbb{Z}$-setting.

### 1.3. Augmented $A_\infty$-algebras

A strictly unital $A_\infty$-algebra $A$ comes equipped with a strict, strictly unital morphism $\eta : k \to A$.

**Definition 1.6.** (a) A strictly unital $A_\infty$-algebra $A$ is **augmented** if there is a strictly unital $A_\infty$-algebra morphism $\varepsilon : A \to k$ so that $\varepsilon \circ \eta = 1_k$.

(b) If $A$ is an augmented $A_\infty$-algebra with augmentation $\varepsilon : A \to k$, its augmentation ideal is defined to be $\ker(\varepsilon_1)$.

(c) A morphism of augmented $A_\infty$-algebras $f : A \to B$ must be strictly unital and must respect the augmentations: $\varepsilon_A = \varepsilon_B \circ f$.

We write $\text{Alg}_{\text{aug}}^\infty$ for the resulting category of augmented $A_\infty$-algebras.

**Proposition 1.7** (Section 3.5 in [20]). The functor $\text{Alg}_{\text{aug}}^\infty \to \text{Alg}^\infty$ sending an augmented $A_\infty$-algebra to its augmentation ideal is an equivalence of categories. The quasi-inverse sends an $A_\infty$-algebra $A$ to $k \oplus A$ with the apparent augmentation.

Using this equivalence, one can translate results and constructions for $A_\infty$-algebras to the augmented case. The bar construction is an application of this.

### 1.4. The bar construction

The bar construction $B(-)$ is of central importance in this paper, since we define the Koszul dual of $A$ to be the vector space dual of its bar construction. In this subsection, we describe it. We also discuss the cobar construction $\Omega(-)$, the composite $\Omega(B(-))$, and other related issues.

The following definition is a slight variant on that in [20, Section 3.6].

**Definition 1.8.** Let $A$ be an augmented $A_\infty$-algebra and let $I$ be its augmentation ideal. The **bar construction** $B_{\text{aug}} \circ A$ on $A$ is a coaugmented differential graded (DG) coalgebra defined as follows: as a coaugmented coalgebra, it is the tensor coalgebra $T(SI)$ on $SI$:

$$T(SI) = k \oplus SI \oplus (SI)^{\otimes 2} \oplus (SI)^{\otimes 3} \oplus \cdots .$$

As is standard, we use bars rather than tensors, and we also conceal the suspensions $s$, writing $[a_1| \cdots |a_m]$ for the element $sa_1 \otimes \cdots \otimes sa_m$, where
$a_i \in I$ for each $i$. The degree of this element is
\[
\deg[a_1 \cdots a_m] = \left( \sum (-1 + \deg_1 a_i), \sum \deg_2 a_i \right).
\]

The differential $b$ on $B_{\text{aug}}^\infty A$ is the degree $(1, 0)$ map given as follows: its component $b_m : (SI)^\otimes m \to T(SI)$ is given by
\[ b_m([a_1|\cdots|a_m]) = \sum_{j,n} (-1)^{w_{j,n}} [a_1|\cdots|a_j|m_n(a_{j+1}, \cdots, a_{j+n})|a_{j+n+1}|\cdots|a_m], \]
where $m_n = (-1)^n m_n$ and
\[ w_{j,n} = \sum_{1\leq s \leq j} (-1 + \deg_1 a_s) + \sum_{1\leq t < n} (n - t)(-1 + \deg_1 a_{j+t}). \]
That is, its component mapping $(SI)^\otimes m$ to $(SI)^\otimes u$ is
\[ \sum 1^\otimes j \otimes (s_m \circ (s_1)^\otimes n) \otimes 1^\otimes m-j-n, \]
where the sum is over pairs $(j, n)$ with $m \geq j + n$, and where $u = m - n + 1$. See [24, p. 115] for a discussion of the sign involved in $m_n$.

If $A$ is an augmented DG algebra, then the above bar construction is the original bar construction and it is also denoted by $BA$.

Remark 1.10. In [20, 3.6], Keller describes the bar construction in the nonaugmented situation. Aside from grading issues, the relation between his version and ours is as follows: if we write $B^\infty$ for Keller’s version, then $B_{\text{aug}}^\infty$ is the composite
\[ \text{Alg}_{\text{aug}}^{\infty} \to \text{Alg}^{\infty} \xrightarrow{B^\infty} \text{DGC} \to \text{DGC}_{\text{coaug}}, \]
where the first arrow is the equivalence from Proposition 1.7, and the last arrow takes a coalgebra $C$ to $k \oplus C$, with the apparent coaugmentation.

The coderivation $b$ encodes all of the higher multiplications of $A$ into a single operation. Keller [20, 3.6] notes that if $A$ and $A'$ are augmented $A_\infty$-algebras, then there is a bijection between Hom sets
\[ \text{Alg}_{\text{aug}}^{\infty}(A, A') \leftrightarrow \text{DGC}_{\text{coaug}}(B_{\text{aug}}^\infty A, B_{\text{aug}}^\infty A'). \]
(Again, he is working with nonaugmented $A_\infty$-algebras, but Proposition 1.7 allows us to translate his result to this setting.)

We briefly mention the cobar construction. In full generality, this would probably take a coaugmented $A_\infty$-coalgebra as input, and produce an augmented DG algebra. We have no interest in working with $A_\infty$-coalgebras, though, and we do not need this generality.
Definition 1.12. Given a coaugmented DG coalgebra $C$ with coproduct $\Delta$ and differential $b_C$, the cobar construction $\Omega C$ on $C$ is the augmented DG algebra which as an augmented algebra is the tensor algebra $T(S^{-1}J)$ on the desuspension of the coaugmentation coideal $J = \text{cok}(k \rightarrow C)$. It is graded by putting 
\[
\deg[x_1|\cdots|x_m] = \left(\sum (1 + \deg_1 x_i), \sum \deg_2 x_i\right).
\]
Its differential is the sum $d = d_0 + d_1$ of the differentials
\[
d_0[x_1|\cdots|x_m] = -\sum_{i=1}^m (-1)^{n_i}[x_1|\cdots|b_C(x_i)|\cdots|x_m],
\]
and
\[
d_1[x_1|\cdots|x_m] = \sum_{i=1}^m \sum_{j=1}^{k_i} (-1)^{n_i + \deg_1 a_{ij}}[x_1|\cdots|x_{i-1}|a_{ij}|b_{ij}|\cdots|x_m]
\]
where $n_i = \sum_{j<i}(1 + \deg_1 x_j) \text{ and } \sum_{j=1}^{k_i} a_{ij} \otimes b_{ij} = \Delta(x_i)$. Here $\Delta$ is the induced coproduct on $J$.

Definition 1.13 ([23, Section 2.3.4]). If $A$ is an augmented $A_\infty$-algebra, then its enveloping algebra $UA$ is defined to be the DG algebra $UA := \Omega(B^\infty_{\text{aug}}A)$.

Thus the enveloping algebra of an augmented $A_\infty$-algebra is an augmented DG algebra.

Proposition 1.14 ([23, 1.3.3.6 and 2.3.4.3]). There is a natural quasi-isomorphism of $A_\infty$-algebras $A \rightarrow UA$.

The map $A \rightarrow UA$ arises as follows: between the categories of DG coalgebras and DG algebras, the bar $B$ and cobar $\Omega$ constructions are adjoint, with $\Omega$ the left adjoint, and thus for any DG coalgebra $C$, there is a map $C \rightarrow B(\Omega C)$. In the case where $C = B^\infty_{\text{aug}}A$, we get a map
\[
B^\infty_{\text{aug}}A \rightarrow B(\Omega(B^\infty_{\text{aug}}A)) = B^\infty_{\text{aug}}(\Omega(B^\infty_{\text{aug}}A)).
\]
(One can view an augmented DG algebra $R$ as an $A_\infty$-algebra with all higher multiplications equal to zero. In this situation, the $A_\infty$-bar construction $B^\infty_{\text{aug}}R$ reduces to the standard DG algebra bar construction $B(R)$.) The bijection (1.11) says that this corresponds to a map
\[
A \rightarrow \Omega(B^\infty_{\text{aug}}A).
\]
This is the map in Proposition 1.14. This proposition says that every augmented $A_\infty$-algebra is quasi-isomorphic to an augmented DG algebra. A similar result is also true in the nonaugmented case, although we will not need this. The quasi-isomorphism between $A$ and $\Omega(BA)$ is also a standard result in the case when $A$ itself is an augmented DG algebra, although the natural map goes the other way in that setting; indeed, there is a chain
homotopy equivalence $\Omega B(A) \to A$ which is a map of DG algebras, but its inverse need not be an algebra map. See [13, Section 19], for example. One application of Proposition 1.14 is that in the DG case, there is a quasi-inverse in the category $\text{Alg}_{\text{aug}}^\infty$.

Also from [13, Section 19], we have the following result.

**Lemma 1.15** ([13, Section 19]). Let $R$ be an augmented DG algebra. Assume that $R$ is locally finite. Then there is a natural isomorphism $\Omega(R^\sharp) \cong (B^\infty_{\text{aug}} R)^\sharp = B^\sharp R$.

In light of the lemma and Remark 1.10, we point out that if $R$ is a locally finite augmented associative algebra, then the homology of the dual of its bar construction is isomorphic to $\text{Ext}^*_R(k,k) := \bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_R(k,k)$. This is also true when $R$ is an $A_\infty$-algebra; see [24, Lemma 11.1] and its proof.

Thus by Theorem 1.5, there is a quasi-isomorphism of $A_\infty$-algebras

$$\text{Ext}^*_R(k,k) \to B^\sharp R.$$ 

The $A_\infty$-structure on $\text{Ext}^*_R(k,k)$ is studied in [26].

1.5. Homotopy. Earlier, we said that we work in the category of augmented $A_\infty$-algebras. We also need the homotopy category of such algebras, and so we need to discuss the notion of homotopy between $A_\infty$-algebra morphisms. See [20, 3.7] and [23, 1.2.1.7] for the following.

Let $A$ and $A'$ be augmented $A_\infty$-algebras, and suppose that $f, g : A \to A'$ are morphisms of augmented $A_\infty$-algebras. Let $F$ and $G$ denote the corresponding maps $B^\infty_{\text{aug}} A \to B^\infty_{\text{aug}} A'$. Write $b$ and $b'$ for the differentials on $B^\infty_{\text{aug}} A$ and $B^\infty_{\text{aug}} A'$, respectively. Then $f$ and $g$ are homotopic, written $f \simeq g$, if there is a map $H : B^\infty_{\text{aug}} A \to B^\infty_{\text{aug}} A'$ of degree $-1$ such that

$$\Delta H = (F \otimes H + H \otimes G)\Delta \quad \text{and} \quad F - G = b' \circ H + H \circ b.$$

One can also express this in terms of a sequence of maps $h_n : A^\otimes n \to A'$ satisfying some identities, but we will not need this formulation. See [23, 1.2.1.7] for details (but note that he uses different sign conventions).

Two $A_\infty$-algebras $A$ and $A'$ are homotopy equivalent if there are morphisms $f : A \to A'$ and $g : A' \to A$ such that $f \circ g \simeq 1_{A'}$ and $g \circ f \simeq 1_A$.

We will use the following theorem.

**Theorem 1.16** ([18], [29], [23, 1.3.1.3]). (a) Homotopy is an equivalence relation on the set of morphisms of $A_\infty$-algebras $A \to A'$.

(b) An $A_\infty$-algebra morphism is a quasi-isomorphism if and only if it is a homotopy equivalence.

By part (a), we can define the homotopy category $\text{HoAlg}_{\text{aug}}^\infty$ to be the category of augmented $A_\infty$-algebras in which the morphisms are homotopy classes of maps: that is,

$$\text{HoAlg}_{\text{aug}}^\infty(A, A') := (\text{Alg}_{\text{aug}}^\infty(A, A')/\simeq).$$
By part (b), in this homotopy category, quasi-isomorphisms are isomorphisms.

2. The Koszul dual of an $A_\infty$-algebra

Let $A$ be an augmented $A_\infty$-algebra. Its $A_\infty$-Koszul dual, or Koszul dual for short, is defined to be $E(A) := (B_{\text{aug}}^\infty A)^\sharp$. By [24, Section 11 and Lemma 11.1], $E(A)$ is a DG algebra model of the $A_\infty$-Ext-algebra $\bigoplus_{i \in \mathbb{Z}} \text{Ext}_A^i(k_A,k_A)$ where $k_A$ is the trivial right $A_\infty$-module over $A$, and where, by definition, $\text{Ext}_A^i(M,N) = \bigoplus_{j \in \mathbb{Z}} D_\infty^- (A)(M, S^i \Sigma^j(N))$ for any right $A_\infty$-modules $M$ and $N$. See Section 3 for the definitions of $A_\infty$-modules and their derived categories.

In this section, we study some of the basic properties of $A_\infty$-Koszul duality, we connect it to “classical” Koszul duality, and we discuss a few simple examples. The main result is Theorem A, restated as Theorem 2.4.

2.1. Finiteness and connectedness conditions. In this subsection, we introduce some technical conditions related to finite-dimensionality and connectivity of bigraded objects.

Definition 2.1. Let $A$ be an augmented $A_\infty$-algebra and let $I$ be its augmentation ideal. We write $I_i^s$ for the direct sum $I_i^s = \bigoplus_j I_i^j$, and similarly $I_i^k = \bigoplus_j I_i^j$. We say that $A$ is locally finite if each bihomogeneous piece $A_i^j$ of $A$ is finite-dimensional. We say that $A$ is strongly locally finite if $I$ satisfies the following:

1. Each bihomogeneous piece $I_i^j$ of $I$ is finite-dimensional (i.e., $A$ is locally finite).
2. Either for all $j \leq 0$, $I_i^s = 0$; or for all $j \geq 0$, $I_i^k = 0$.
3. Either for all $j$, there exists an $m = m(j)$ so that for all $i > m(j)$, $I_i^j = 0$; or for all $j$, there exists an $m' = m'(j)$ so that for all $i < m'(j)$, $I_i^j = 0$.

We say that $A$ is Adams connected if, with respect to the Adams grading, $A$ is (either positively or negatively) connected graded and locally finite. That is:

- $I_i^s$ is finite-dimensional for all $j$.
- Either for all $j \leq 0$, $I_i^s = 0$, or for all $j \geq 0$, $I_i^k = 0$.

We say that a DG algebra $A$ is weakly Adams connected if:

- The DG bar construction $B(A; A) \cong B(A) \otimes A$ is locally finite.
- The only simple DG $A$-modules are $k$ and its shifts.
- $A$ is an inverse limit of a family of finite-dimensional left DG $A$-bimodules.

Lemma 2.2. We have the following implications.

(a) Adams connected $\Rightarrow$ strongly locally finite $\Rightarrow$ weakly Adams connected.
(b) A weakly Adams connected $\Rightarrow E(A)$ locally finite.
(c) A strongly locally finite $\Rightarrow E(A)$ strongly locally finite, and hence every iterated Koszul dual of $A$ is strongly locally finite.
(d) A Adams connected $\Rightarrow E(A)$ Adams connected, and hence every iterated Koszul dual of $A$ is Adams connected.

Proof. The first implication in part (a) is clear. For the second implication, if $A$ is strongly locally finite, then for connectivity reasons, $k$ and its shifts will be the only simple modules. We defer the proof that $B(A;A)$ is locally finite until after the proof of part (c).

For the inverse limit condition, we assume that $I^*_j = 0$ when $j \leq 0$, and that for each $j$, there is an $m'(j)$ such that $I^*_j = 0$ when $i < m'(j)$. The other cases are similar. We will describe a sequence of two-sided ideals $J_n$ in $A$, $n \geq 1$, so that $A/J_n$ is finite-dimensional, and $A = \varprojlim A/J_n$. Define $J_n$ to be

$$J_n = I^*_n \oplus I^*_j \oplus I^*_m \oplus \cdots$$

$$\oplus I^*_j \oplus I^*_m \oplus \cdots$$

The notation “$\sigma \vdash n - 1$” means that $\sigma$ partitions $n - 1$. The idea here is that, for example, if $J$ contains all of the elements in bidegrees $(1, \geq n)$, and since $A$ has elements in bidegrees $(1, \geq m'(1))$, then for $J$ to be an ideal, it should contain all of the elements in bidegrees $(2, \geq n + m'(1))$.

Part (b) is clear: as graded vector spaces, $B(A;A)$ and $B(A) \otimes_k A$ are isomorphic; thus if $B(A;A)$ is locally finite, so is $B(A)$.

(c) Since $E(A)$ is dual to the tensor coalgebra $T(SI)$, we focus on $T(SI)$. If we suppose that $I^*_j = 0$ when $j \leq 0$, then for all $j < n$, we have $(I^{\otimes n})^*_j = 0$. Shifting $I$ by $S$ does not change this: $((SI)^{\otimes n})^*_j$ will be zero if $j < n$. Therefore, $T(SI)$ satisfies condition (2) of Definition 2.1. Dualizing, we see that $E(A)$ satisfies the other version of condition (2): if $J$ is its augmentation ideal, then $J^*_j = 0$ when $j \geq 0$. Similarly, if $I^*_j = 0$ when $j \geq 0$, then $J^*_j = 0$ when $j \leq 0$.

So if $I$ satisfies condition (2), then so does $J$.

Now suppose that $I$ satisfies (1) and this version of condition (3): for each $j$, there is an $m'(j)$ such that $I^*_j = 0$ when $i < m'(j)$. Then for fixed $n$ and $j$:

- $(I^{\otimes n})^*_j$ is zero if $i$ is small enough.
- $(I^{\otimes n})^*_j$ is finite-dimensional for all $i$.

Therefore $T(SI)$ satisfies condition (3). Furthermore, since for fixed $j$, $(I^{\otimes n})^*_j$ is zero for all but finitely many values of $n$, we see that $T(SI)$ satisfies condition (1). Dualizing, we see that the augmentation ideal $J$ of $E(A)$ satisfies (1) and (3), also (although $J$ satisfies the “other version” of (3)). This completes the proof of part (c).
We return to part (a): if $A$ is strongly locally finite, then by part (c), so is the bar construction $B(A)$; more precisely, $A$ and $B(A)$ will satisfy the same version of condition (3). Hence it is easy to verify that their tensor product will be locally finite. This completes the proof of (a).

(d) $A$ being Adams connected is equivalent to $I$ satisfying (1), (2), and both versions of (3): for each $j$, there are numbers $m(j)$ and $m'(j)$ so that $I_i^j = 0$ unless $m'(j) \leq i \leq m(j)$. By the proof of (c), this implies that $E(A)$ satisfies the same conditions.

**Remark 2.3.** One may interchange the roles of $i$ and $j$ in the definition of strong local finiteness, but the presence of the shift $S$ in $E(A) = T(SI)$ makes the situation asymmetric. Suppose that $I$ satisfies the following:

1. Each bihomogeneous piece $I_i^j$ of $I$ is finite-dimensional.
2. Either for all $i$ there exists an $m = m(i)$ so that for all $j \geq m(i)$, $I_i^j = 0$; or for all $i$ there exists an $m' = m'(i)$ so that for all $j \leq m'$, $I_i^j = 0$.
3. Either for all $i \geq 1$, $I_i^i = 0$; or for all $i \leq 1$, $I_i^i = 0$.

Then by imitating the proof of part (c) of the lemma, one can show that $E(A)$ is locally finite; however, it may not satisfy (2').

Suppose that $I$ satisfies (1'), (2'), and the following:

3'. Either for all $i \geq 0$, $I_i^i = 0$; or for all $i \leq 1$, $I_i^i = 0$.

Then the same proof shows that the augmentation ideal of $E(A)$ satisfies (1'), (2'), and (3'') as well, and hence the same holds for every iterated Koszul dual of $A$.

### 2.2. $A_\infty$-Koszul duality.

Here is the main theorem of this section, which is a slight generalization of [24, Theorem 11.2]. This is Theorem A from the introduction.

**Theorem 2.4.** Suppose that $A$ is an augmented $A_\infty$-algebra with $E(A)$ locally finite. Then there is a natural quasi-isomorphism of $A_\infty$-algebras $A \cong E(E(A))$.

If $A$ is weakly Adams connected, then essentially by definition (or see Lemma 2.2), $E(A)$ is locally finite. Hence by Lemma 2.2, $E(A)$ is locally finite if $A$ is Adams connected or strongly locally finite. The summary of the theorem’s proof is that the double Koszul dual is the enveloping algebra $UA$ of $A$ (see Definition 1.13 and Proposition 1.14).

**Proof.** By definition, the double Koszul dual $E(E(A))$ is $(B_{aug}^\infty((B_{aug}^\infty A)^\sharp))^\sharp$. Apply Lemma 1.15 to the DG algebra $E(A) = (B_{aug}^\infty A)^\sharp$, which is locally finite. Then there are natural DG algebra isomorphisms

$$E(E(A)) = (B_{aug}^\infty((B_{aug}^\infty A)^\sharp))^\sharp \cong \Omega((((B_{aug}^\infty A)^\sharp)^\sharp) \cong \Omega(B_{aug}^\infty A).$$

Proposition 1.14 gives a natural $A_\infty$-isomorphism $A \overset{\cong}{\to} \Omega(B_{aug}^\infty A)$. □
Koszul duality $E(-)$ is a contravariant functor from $A_\infty$-algebras to DG algebras, and since one can view a DG algebra as being an $A_\infty$-algebra, there are functions

$$\text{Alg}_{\text{aug}}^\infty(A, A') \xrightarrow{E(-)} \text{DGA}_{\text{aug}}(E(A'), E(A)) \rightarrow \text{Alg}_{\text{aug}}^\infty(E(A'), E(A)),$$

for augmented $A_\infty$-algebras $A$ and $A'$.

**Corollary 2.5.** Suppose that $A$ and $A'$ are augmented $A_\infty$-algebras with $E(A)$, $E(E(A))$, $E(A')$, and $E(E(A'))$ locally finite.

(a) Then $E(-)$ gives a bijection

$$\text{Alg}_{\text{aug}}^\infty(A, A') \xrightarrow{\sim} \text{DGA}_{\text{aug}}(E(A'), E(A)).$$

(b) The composite

$$\text{Alg}_{\text{aug}}^\infty(A, A') \xrightarrow{\sim} \text{DGA}_{\text{aug}}(E(A'), E(A)) \rightarrow \text{Alg}_{\text{aug}}^\infty(E(A'), E(A))$$

induces a bijection

$$\text{HoAlg}_{\text{aug}}^\infty(A, A') \xrightarrow{\sim} \text{HoAlg}_{\text{aug}}^\infty(E(A'), E(A)).$$

(c) Hence every $A_\infty$-algebra map $f : E(A') \rightarrow E(A)$ is homotopic to a DG algebra map.

**Proof.** (a) From (1.11) we have a bijection

$$\text{Alg}_{\text{aug}}^\infty(A, A') \xrightarrow{\sim} \text{DGC}_{\text{coaug}}(B_{\text{aug}}^\infty A, B_{\text{aug}}^\infty A').$$

We are assuming that $B_{\text{aug}}^\infty A$ and $B_{\text{aug}}^\infty A'$ are locally finite, so the vector space duality maps

$$\text{DGC}_{\text{coaug}}(B_{\text{aug}}^\infty A, B_{\text{aug}}^\infty A') \rightarrow \text{DGA}_{\text{aug}}((B_{\text{aug}}^\infty A')^\sharp, (B_{\text{aug}}^\infty A)^\sharp)$$

$$\rightarrow \text{DGC}_{\text{coaug}}(((B_{\text{aug}}^\infty A)^\sharp)^\sharp, ((B_{\text{aug}}^\infty A')^\sharp)^\sharp)$$

are bijections. Therefore so is

$$\text{Alg}_{\text{aug}}^\infty(A, A') \xrightarrow{\sim} \text{DGA}_{\text{aug}}(E(A'), E(A)).$$

(b, c) The naturality of the quasi-isomorphism (= homotopy equivalence) in Theorem 2.4 says that the function

$$\text{HoAlg}_{\text{aug}}^\infty(A, A') \xrightarrow{E(E(-))} \text{HoAlg}_{\text{aug}}^\infty(E(E(A)), E(E(A'))),$$

$$f \mapsto E(E(f))$$

is a bijection. That is, the composite

$$\text{HoAlg}_{\text{aug}}^\infty(A, A') \rightarrow \text{HoAlg}_{\text{aug}}^\infty(E(A'), E(A))$$

$$\rightarrow \text{HoAlg}_{\text{aug}}^\infty(E(E(A)), E(E(A'))))$$

is a bijection. The first map here is induced by

$$\text{Alg}_{\text{aug}}^\infty(A, A') \xrightarrow{\sim} \text{DGA}_{\text{aug}}(E(A'), E(A)) \xrightarrow{i} \text{Alg}_{\text{aug}}^\infty(E(A'), E(A)),$$
and the second by
\[ \text{Alg}_{\text{aug}}^\infty(E(A'), E(A)) \xrightarrow{j} \text{DGA}_{\text{aug}}(E(E(A)), E(E(A'))) \]
Since both of the functions \( i \) and \( j \) are inclusions, the functions \( \text{Ho} i \) and \( \text{Ho} j \) must be bijections. This proves (b) and (c).

\[ \square \]

2.3. Classical Koszul duality. Classically, a Koszul algebra is a connected graded associative algebra \( R \) which is locally finite, is generated in degree 1, has quadratic relations, and has its \( i \)th graded Ext-group \( \text{Ext}_R^i(k, k) \) concentrated in degree \( -i \) for each \( i \); see [7, Theorem 2.10.1] or [32, Theorem 5.9(6)], for example. (In those papers, \( \text{Ext}_R^i(k, k) \) is actually required to be concentrated in degree \( i \), but that is the result of different grading conventions.) Its (classical) Koszul dual, also denoted by \( R^! \), is \( \text{Ext}^*_R(k, k) \). One can show that if \( R \) is a Koszul algebra, then so is \( R^! \)—see [7, 2.9.1], for example.

A standard example is an exterior algebra \( R = \Lambda(x_1, \ldots, x_n) \) on generators \( x_i \) each in degree 1; then its Koszul dual \( R^! \) is the polynomial algebra \( k[y_1, \ldots, y_n] \), with each \( y_i \) in degree \((1, -1)\).

We want all of our algebras to be bigraded, though, and we want the double Koszul dual to be isomorphic, as a bigraded algebra, to the original algebra. Thus we might grade \( \Lambda(x_1, \ldots, x_n) \) by putting each \( x_i \) in degree \((0, 1)\), in which case \( R^! = k[y_1, \ldots, y_n] \). The grading for \( R^! \) is given as follows: \( y_i \) is represented in the dual of the bar construction for \( R \) by the dual of \([x_i]\), and since \( \deg[x_i] = (-1, 1) \), \( y_i \) has degree \((1, -1)\). The double Koszul dual is exterior on classes dual to \([y_i]\) in the bar construction on \( R^! \), each of which therefore has degree \((0, 1)\).

Note that these are graded in such a way that there are no possible nonzero higher multiplications \( m_n \) on them. This absence of higher multiplications is typical for a Koszul algebra, as Keller [20, 3.3] and the authors [24, Section 11] point out. Conversely, if we grade our algebras in such a way that there are no possible higher multiplications, we can recover classical Koszul duality.

Definition 2.6. Fix a pair of integers \((a, b)\) with \( b \neq 0 \). A bigraded associative algebra \( A \) is an \((a, b)\)-generated Koszul algebra if it satisfies these conditions:

(a) \( A_{0,0} = k \).
(b) \( A \) is locally finite.
(c) \( A \) is generated in bidegree \((a, b)\).
(d) The relations in \( A \) are generated in bidegree \((2a, 2b)\).
(e) For each \( i \), the graded vector space \( \text{Ext}^i_A(k, k) \) is concentrated in degree \((i(a-1), -ib)\).
Koszul equivalences in $A_\infty$-algebras

In fact, conditions (c) and (d) should follow from condition (e): one should be able to imitate the proofs of [7, 2.3.1 and 2.3.2].

If $A$ is a bigraded associative algebra, the classical Koszul dual of $A$, denoted by $A^!$, is defined to be $HE(A)$ — the homology of the $A_\infty$-Koszul dual $E(A) = (B^\infty_{\text{aug}} A)^\natural$. Forgetting grading issues, $A^!$ is isomorphic to $\text{Ext}_A^*(k, k)$.

In classical ring theory, we often consider the classical Koszul dual as an associative algebra — an $A_\infty$-algebra with $m_n = 0$ if $n \neq 2$.

**Corollary 2.7.** Fix a pair of integers $(a, b)$ with $b \neq 0$. If $A$ is an $(a, b)$-generated Koszul algebra, then $E(A)$ and $E(HE(A))$ are quasi-isomorphic to associative algebras $A^!$ and $(A^!)^!$, respectively, and there is an isomorphism of bigraded algebras $A \cong (A^!)^!$.

This result is known [7] so we only give a sketch of proof.

**Sketch of proof of Corollary 2.7.** By Theorem 1.5, the $A_\infty$-Koszul dual $E(A)$ is quasi-isomorphic to $A^!$ with some possible higher multiplications. We need to show, among other things, that in this case, the higher multiplications on $A^!$ are zero.

Since $b \neq 0$, both $E(A)$ and $E(HE(A))$ will be locally finite.

The bidegree $(\deg_1 x, \deg_2 x)$ of a nonzero nonunit element $x \in \text{Ext}_A^*(k, k)$ satisfies

\[
\frac{\deg_1 x}{\deg_2 x} = \frac{a - 1}{b},
\]

and this fraction makes sense since $b \neq 0$. The same is true for any tensor product of such elements. Since the higher multiplication $m_n$ has degree $(2 - n, 0)$, one can see that if $n \neq 2$, the bidegree of $m_n(x_1 \otimes \cdots \otimes x_n)$ will not satisfy (*), and so will be zero. Thus there is no nonzero higher multiplications on $A^!$ which is compatible with the bigrading. This implies that the $A_\infty$-algebra $E(A)$ is quasi-isomorphic to the associative algebra $A^!$.

Now we claim that $A^!$ is $(1 - a, -b)$-generated Koszul. There is an obvious equivalence of categories between $\mathbb{Z}$-graded algebras and $\mathbb{Z} \times \mathbb{Z}$-graded algebras concentrated in degrees $(na, nb)$ for $n \in \mathbb{Z}$; under that equivalence, $(a, b)$-generated Koszul algebras correspond to Koszul algebras in the sense of [7]. Koszul duality takes $\mathbb{Z}$-graded algebras generated in degree 1 to $\mathbb{Z} \times \mathbb{Z}$-graded algebras generated in degree $(1, -1)$. It takes $\mathbb{Z} \times \mathbb{Z}$-graded algebras generated in degree $(a, b)$ to $\mathbb{Z} \times \mathbb{Z}$-graded algebras generated in degree $(1 - a, -b)$. The proof of [7, 2.9.1] carries over to show that since $A$ is $(a, b)$-generated Koszul, its dual $A^!$ is $(1 - a, -b)$-generated Koszul.

Since $A^!$ is Koszul, its Koszul dual $(A^!)^!$ is associative (or there is no nonzero higher multiplications on $(A^!)^!$, by the first part of the proof). A similar grading argument shows that any morphism $f : A \to (A^!)^!$ of $A_\infty$-algebras must be strict; thus the isomorphism of $A_\infty$-algebras $A \to (A^!)^!$ is just an isomorphism of associative algebras. □
2.4. Examples: exterior and polynomial algebras. In this subsection, we consider some simple examples involving exterior algebras and polynomial algebras. The first example shows that in the classical setting, it is crucial that a Koszul algebra be generated in a single degree.

Example 2.8. Assume that the ground field $k$ has characteristic 2, and consider the exterior algebra $\Lambda = \Lambda(x_1, x_2)$ with $\deg x_i = (0, i)$: this is not a classical Koszul algebra, nor is it an $(a, b)$-generated Koszul algebra, since there are generators in multiple degrees. The same goes for $\Lambda \otimes \Lambda$. The Ext-algebra for $\Lambda$ is the polynomial algebra $\Lambda^! = k[y_1, y_2]$ with $\deg y_i = (1, -i)$. Similarly, the Ext-algebra for $\Lambda \otimes \Lambda$ is isomorphic to $\Lambda^! \otimes \Lambda^!$. Although it is true that $(\Lambda^!)^! \cong \Lambda$, there are naturality problems. In particular, the map $\text{Alg}(\Lambda, \Lambda \otimes \Lambda) \to \text{Alg}(\Lambda^! \otimes \Lambda^!, \Lambda^!)$
is not injective: one can show that the following two maps induce the same map on Ext:

$$f : x_1 \mapsto x_1 \otimes 1 + 1 \otimes x_1, \quad g : x_1 \mapsto x_1 \otimes 1 + 1 \otimes x_1,$$

$$f : x_2 \mapsto x_2 \otimes 1 + 1 \otimes x_2, \quad g : x_2 \mapsto x_2 \otimes 1 + 1 \otimes x_2 + x_1 \otimes x_1.$$  

(Indeed, any coassociative coproduct on $\Lambda$ induces the Yoneda product on $\text{Ext}^\ast_\Lambda(k, k)$.) This shows the importance of the requirement that Koszul algebras be generated in a single degree.

Now, Theorem 2.4 and Corollary 2.5 apply here. The $A_\infty$-version of the Koszul dual of $\Lambda$ is quasi-isomorphic to $\Lambda^! = k[y_1, y_2]$, where the $A_\infty$-structure on $\Lambda^!$ is given by $m_n = 0$ when $n \neq 2$. Corollary 2.5 says that there is a bijection $\text{HoAlg}_\infty^\ast(\Lambda, \Lambda \otimes \Lambda) \xrightarrow{\sim} \text{HoAlg}_\infty^\ast(\Lambda^! \otimes \Lambda^!, \Lambda^!)$. This fixes the flaw above; the two (strict) maps $f$ and $g$ correspond to $A_\infty$-algebra morphisms $E(f)$ and $E(g)$, and while $E(f)_1 = E(g)_1$, the morphisms must differ in some higher component. (Even though the algebras involved here have $A_\infty$-structures with zero higher multiplications, there are nonstrict $A_\infty$-algebra morphisms between them. Also, $k[y_1, y_2]$ is quasi-isomorphic, not equal, to the $A_\infty$-Koszul dual of $\Lambda$, so Corollary 2.5(c), which says that every nonstrict map on Koszul duals is homotopic to a strict one, does not apply here.)

Example 2.9. Let $A = \Lambda(x)$ with $\deg x = (a, 0)$; Corollary 2.7 does not apply in this case. The Koszul dual is $E(A) = k[y]$ with $\deg y = (1 - a, 0)$ and with $m_n = 0$ for $n \neq 2$. Assume that $a \neq 0, 1$; then $E(A)$ is locally finite, as is $E(E(A))$ by Remark 2.3. Thus Corollary 2.5 says that there is a bijection $\text{HoAlg}_\infty^\ast(\Lambda(x), \Lambda(x)) \xrightarrow{\sim} \text{HoAlg}_\infty^\ast(k[y], k[y])$. For degree reasons, every map $\Lambda(x) \to \Lambda(x)$ must be strict, so each map is given by the image of $x$: $x \mapsto cx$ for any scalar $c \in k$. 
On the other hand, degree reasons do not rule out nonstrict maps $k[y] \to k[y]$. Strict maps will correspond to those from $\Lambda(x)$ to itself, with the map given by the scalar $c$ corresponding to the map $y \mapsto cy$. Thus, as pointed out in Corollary 2.5(c), if there are any nonstrict maps, then they are homotopic to strict ones. In particular, one can see that if two $A_\infty$-algebra maps $f, g : k[y] \to k[y]$ are homotopic, then $f_1 = g_1$. So if $f = (f_1, f_2, \ldots)$ is such a map, then it will be homotopic to the strict map $(f_1, 0, 0, \ldots)$. None of this is immediately clear from the morphism and homotopy identities, so Koszul duality, in the form of Corollary 2.5, gives some insight into $A_\infty$-maps from $k[y]$ to itself.

**Example 2.10.** Now consider $A = \Lambda(x)$ with $\deg x = (1, 0)$. As in the previous example, every map $\Lambda(x) \to \Lambda(x)$ must be strict. In this case, $B^\infty_{\text{aug}}A$ is the vector space spanned by the classes $[x] \cdots [x]$, all of which are in bidegree $(0, 0)$. Since $B^\infty_{\text{aug}}A$ is not locally finite, Theorem 2.4 does not apply, and the Koszul dual $E(A)$ ends up being the power series ring $k[[y]]$ instead of the polynomial ring $k[y]$. Consider the composite

$$\text{Alg}_{\text{aug}}^\infty(A, A) \xrightarrow{\sim} \text{DGC}_{\text{coaug}}^\infty(B^\infty_{\text{aug}}A, B^\infty_{\text{aug}}A) \to \text{DGA}_{\text{aug}}^\infty(E(A), E(A)).$$

The first map is a bijection by (1.11), but the second map is not, essentially since the map is given by vector space duality and the vector spaces involved are not finite-dimensional. Since strict $A_\infty$-maps are homotopic if and only if they are equal, we get a proper inclusion

$$\text{HoAlg}_{\text{aug}}^\infty(A, A) \ncong \text{HoAlg}_{\text{aug}}^\infty(E(A), E(A)).$$

Thus Corollary 2.5 fails here.

**Example 2.11.** We also mention the case when $A = \Lambda(x)$ with $\deg x = (a, b)$ with $b \neq 0$. In this case, $E(A) = k[y]$ with $\deg y = (1 - a, -b)$. Also, $A_\infty$-morphisms $\Lambda(x) \to \Lambda(x)$ and $k[y] \to k[y]$ must be strict, and it is easy to show that the strict maps are in bijection, as Corollary 2.5 says they should be. Indeed in this case, classical Koszul duality (Corollary 2.7) applies, since $\Lambda(x)$ is $(a, b)$-generated Koszul with $b \neq 0$.

**Part II. Koszul duality for modules**

**3. Background on $A_\infty$-modules**

Koszul duality relates not just to algebras, but also to modules over them. In this section, we briefly review the relevant categories of modules over an $A_\infty$-algebra. See Keller [20, 4.2] for a few more details, keeping in mind that since he is not working in the augmented setting, a little translation is required, especially in regards to the bar construction. The paper [24, Section 6] also has some relevant information, as does Lefèvre-Hasegawa’s thesis [23].
There are several subsections here: the definition of $A_\infty$-module; the bar construction; derived categories; and several sections about “opposites.”

3.1. $A_\infty$-modules. Let $A$ be an $A_\infty$-algebra. A bigraded vector space $M$ is a right $A_\infty$-module over $A$ if there are graded maps

$$m_n : M \otimes A^\otimes n \rightarrow M, \quad n \geq 1,$$

degree $(2 - n, 0)$ satisfying the Stasheff identities $SI(n)$, interpreted appropriately. Similarly, a bigraded vector space $N$ is a left $A_\infty$-module over $A$ if there are graded maps

$$m_n : A^\otimes n \otimes N \rightarrow N, \quad n \geq 1,$$

degree $(2 - n, 0)$ satisfying the Stasheff identities $SI(n)$, interpreted appropriately.

Morphisms of right $A_\infty$-modules are defined in a similar way: a morphism $f : M \rightarrow M'$ of right $A_\infty$-modules over $A$ is a sequence of graded maps

$$f_n : M \otimes A^\otimes n \rightarrow M', \quad n \geq 1,$$

degree $(1 - n, 0)$ satisfying the Stasheff morphism identities $MI(n)$. Morphisms of left $A_\infty$-modules are defined analogously, and so are homotopies in both the right and left module settings.

Now suppose that $A$ is an augmented $A_\infty$-algebra. A right $A_\infty$-module $M$ over $A$ is strictly unital if for all $x \in M$ and for all $a_i \in A$, $m_2(x \otimes 1) = x$ and

$$m_n(x \otimes a_2 \otimes \cdots \otimes a_n) = 0, \quad \text{for all } n \geq 3$$

if $a_i = 1$ for some $i$. A morphism $f$ of such is strictly unital if for all $n \geq 2$, we have $f_n(x \otimes a_2 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some $i$.

Given an augmented $A_\infty$-algebra $A$, let $\text{Mod}^\infty(A)$ denote the category of strictly unital right $A_\infty$-modules with strictly unital morphisms over $A$.

Suppose $A$ is an augmented $A_\infty$-algebra. The morphism $\varepsilon : A \rightarrow k$ makes the vector space $k$ into a left $A_\infty$-module over $A$. It is called the trivial left $A_\infty$-module over $A$ and is denoted by $Ak$. The trivial right $A_\infty$-module over $A$ is defined similarly, and is denoted by $k_A$.

3.2. The bar construction for modules. The bar construction is as useful for $A_\infty$-modules as it is for $A_\infty$-algebras: recall that the bar construction on $A$ is $B^\infty_{\text{aug}} A = T(SI)$. A strictly unital right $A_\infty$-module structure on a bigraded vector space $M$ gives a comodule differential on the right $B^\infty_{\text{aug}} A$-comodule

$$B^\infty_{\text{aug}} (M; A) := SM \otimes T(SI),$$

as in Definition 1.8. Also, morphisms of right modules $M \rightarrow M'$ are in bijection with morphisms of right DG comodules $B^\infty_{\text{aug}} (M; A) \rightarrow B^\infty_{\text{aug}} (M'; A)$ as in (1.11), and the notion of homotopy translates as well.

Similarly, one has a bar construction for left $A_\infty$-modules, defined by

$$B^\infty_{\text{aug}} (A; N) := T(SI) \otimes SN,$$
with the same formula for the differential.

3.3. Derived categories. Let $D^\infty(A)$ be the derived category associated to the module category $\text{Mod}^\infty(A)$. By [20, 4.2] (see also [23, 2.4.2]), this derived category is the same as the homotopy category of $\text{Mod}^\infty(A)$: in the homotopy category for $A_\infty$-modules, quasi-isomorphisms have already been inverted. This statement is the module version of Theorem 1.16.

Given a DG algebra $R$, write $\text{Mod}_R$ for the category of unital DG right $R$-modules, and write $\mathcal{D}(R)$ for its derived category. A good reference for the derived category $\mathcal{D}(R)$ is [19]; also see [21, 22]. See [23, 2.4.3 and 4.1.3] and [24, 7.2 and 7.3] for the following.

**Proposition 3.1.** (a) Suppose that $A$ and $B$ are augmented $A_\infty$-algebras.

If $f : A \to B$ is an $A_\infty$-isomorphism, then the induced functor

$$f^* : D^\infty(B) \to D^\infty(A)$$

is a triangulated equivalence.

(b) Suppose that $R$ is an augmented DG algebra. Then the inclusion $\text{Mod}_R \hookrightarrow \text{Mod}^\infty(R)$ induces a triangulated equivalence

$$\mathcal{D}(R) \to \mathcal{D}^\infty(R).$$

(c) Hence if $A$ is an augmented $A_\infty$-algebra and $R$ is an augmented DG algebra with an $A_\infty$-isomorphism $A \to R$, there is a triangulated equivalence $F : \mathcal{D}(R) \to \mathcal{D}^\infty(A)$. Under this equivalence, $F(k_R) \cong k_A$ and $F(R) \cong A$.

Hence in the category $\mathcal{D}^\infty(A)$ one can perform many of the usual constructions, by first working in the derived category $\mathcal{D}(UA)$ of its enveloping algebra and then applying the equivalence of categories $\mathcal{D}(UA) \to \mathcal{D}^\infty(A)$. See Section 5 for an application of this idea. We note that the Adams shift is an automorphism of $\mathcal{D}^\infty(A)$.

3.4. The opposite of a DG algebra. If $A = (A, m_1, m_2)$ is a DG algebra with differential $m_1$ and multiplication $m_2$, we define the opposite algebra of $A$ to be $(A^{op}, m_1^{op}, m_2^{op})$, where $A^{op} = A$, $m_1^{op} = -m_1$, and

$$m_2^{op}(a \otimes b) = (-1)^{(\deg a)(\deg b)}m_2(b \otimes a).$$

That is, $m_2^{op} = m_2 \circ \tau$, where $\tau$ is the twist function, which interchanges tensor factors at the expense of the appropriate Koszul sign. One can verify that this is a DG algebra: $m_1^{op}$ and $m_2^{op}$ satisfy a Leibniz formula. (Choosing $m_1^{op} = m_1$ also works, but is not compatible with the bar construction: see below.)

If $f : A \to A'$ is a map of DG algebras, define $f^{op} : A^{op} \to (A')^{op}$ by $f^{op} = f$. Then $f^{op}$ is also a DG algebra map, so $\text{op}$ defines an automorphism of the category of DG algebras, and it is clearly its own inverse.
Dually, given a DG coalgebra $C = (C, d, \Delta)$, we define its opposite coalgebra to be $(C^{\text{op}}, d^{\text{op}}, \Delta^{\text{op}})$, where $C^{\text{op}} = C$, $d^{\text{op}} = -d$, and $\Delta^{\text{op}} = \tau \circ \Delta$.

With these definitions, for any DG algebra $A$ there is an isomorphism

$$B(A^{\text{op}}) \xrightarrow{\Phi} B(A)^{\text{op}},$$

$[a_1] \ldots [a_m] \mapsto (-1)^{\sum_{i<j} (-1+\deg a_i)(-1+\deg a_j)} [a_m] \ldots [a_1],$

of DG coalgebras. Note that, had we defined the differential $m_1^{\text{op}}$ in $A^{\text{op}}$ by $m_1^{\text{op}} = m_1$, this map $\Phi$ would not be compatible with the differentials, and so would not be a map of DG coalgebras.

If $F : (C, d, \Delta) \to (C', d', \Delta')$ is a morphism of DG coalgebras, then define $F^{\text{op}}$ to be equal to $F$. One can check that $F^{\text{op}} : C^{\text{op}} \to (C')^{\text{op}}$ is also a map of DG coalgebras, making $\text{op}$ into a functor $\text{op} : \text{DGC} \to \text{DGC}$. As above, it is an automorphism which is its own inverse.

The definition of homotopy in Subsection 1.5 works in general for DG coalgebra morphisms: if $F$ and $G$ are DG coalgebra morphisms $C \to C'$, then a homotopy from $F$ to $G$ is a map $H : C \to C'$ of degree $-1$ such that

$$\Delta H = (F \otimes H + H \otimes G)\Delta, \quad F - G = d' \circ H + H \circ d.$$

We write $H = H(F \to G)$ to indicate the “direction” of the homotopy. One can check that, in this situation, the map $H$ also defines a homotopy from $C^{\text{op}}$ to $F^{\text{op}}$, as maps $C^{\text{op}} \to (C')^{\text{op}}$. Therefore, we may define $H^{\text{op}}(G^{\text{op}} \to F^{\text{op}})$ to be $H(F \to G)$. As a consequence, $\text{op}$ induces an automorphism on the homotopy category of DG coalgebras.

### 3.5. The opposite of an $A_{\infty}$-algebra.

Now we define a functor

$$\text{op} : \text{Alg}^{\infty} \to \text{Alg}^{\infty}$$

which generalizes the opposite functor on the category of DG algebras. Given an $A_{\infty}$-algebra $(A, m_1, m_2, m_3, \ldots)$, define $(A^{\text{op}}, m_1^{\text{op}}, m_2^{\text{op}}, m_3^{\text{op}}, \ldots)$ as follows: as a bigraded vector space, $A^{\text{op}}$ is the same as $A$. The map $m_n^{\text{op}} : (A^{\text{op}})^{\otimes n} \to A^{\text{op}}$ is defined by $m_n^{\text{op}} = (-1)^{\varepsilon(n)} m_n \circ (\text{twist})$, where “twist” is the map which reverses the factors in a tensor product, with the appropriate Koszul sign, and

$$\varepsilon(n) = \begin{cases} 1, & \text{if } n \equiv 0, 1 \pmod{4}, \\ 0, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Equivalently, since only the parity of $\varepsilon(n)$ is important, $\varepsilon(n) = \binom{n+2}{2}$, or $\varepsilon(n) = \binom{n}{2} + 1$. Thus when applied to elements,

$$m_n^{\text{op}}(a_1 \otimes \cdots \otimes a_n) = (-1)^{\varepsilon(n) + \sum_{i<j} (\deg a_i)(\deg a_j)} m_n(a_n \otimes \cdots \otimes a_1).$$
Lemma 3.2. The function $\varepsilon$ satisfies the following additivity formula: for any $q \geq 1$ and any $i_s \geq 1$, $s = 1, 2, \ldots, q$,

$$\sum_{1 \leq s \leq q} \varepsilon(i_s) + \varepsilon\left(\sum_{1 \leq s \leq q} i_s - q + 1\right) + \sum_{1 \leq s < t \leq q} (i_s - 1)(i_t - 1) \equiv q + 1 \pmod{2}.$$

Proof. The $q = 1$ case is trivial, the $q = 2$ case may be established by (for example) considering the different congruence classes of $i_1 \bmod 4$, and for larger $q$, one can use a simple induction argument. □

Lemma 3.3. $(A^{\text{op}}, m_1^{\text{op}}, m_2^{\text{op}}, m_3^{\text{op}}, \ldots)$ is an $A_\infty$-algebra.

Proof. We need to check that $(A^{\text{op}}, m_1^{\text{op}}, m_2^{\text{op}}, m_3^{\text{op}}, \ldots)$ satisfies the Stasheff identities. This is a tedious, but straightforward, verification, which we leave to the reader. The $q = 2$ case of Lemma 3.2 is useful. □

We also need to specify what happens to morphisms. Given a morphism $f : A \to B$, we define $f^{\text{op}} : A^{\text{op}} \to B^{\text{op}}$ by defining

$$f_n^{\text{op}} : (A^{\text{op}})^{\otimes n} \to B^{\text{op}}$$

to be $f_n^{\text{op}} = (-1)^{1+\varepsilon(n)} f_n \circ \text{(twist)}$; that is,

$$f_n^{\text{op}}(a_1 \otimes \cdots \otimes a_n) = (-1)^{1+\varepsilon(n)+\sum_{i<j} \deg a_i \deg a_j} f_n(a_n \otimes \cdots \otimes a_1).$$

Lemma 3.4. The family $f^{\text{op}} = (f_n^{\text{op}})$ is a morphism of $A_\infty$-algebras.

Proof. This is another tedious verification. Lemma 3.2 is used here. □

To complete this circle of ideas, we should consider the bar construction. That is, consider the following diagram of functors:

$$\begin{array}{ccc}
\text{Alg}_\infty & \text{op} & \text{Alg}_\infty \\
\sim & \sim & \sim \\
\text{DGC} & \text{op} & \text{DGC} \\
\sim & \sim & \sim \\
\text{DGC} & \rightarrow & \text{DGC}
\end{array}$$

The horizontal arrows are equivalences of categories. The vertical arrows are fully faithful embeddings. Starting with an $A_\infty$-algebra $A$ in the upper left corner, mapping down and then to the right gives $B(A^{\text{op}})$, while mapping to the right and then down gives $B(A^{\text{op}})$.

It would be nice if these two DG coalgebras agreed, and indeed they do. The proof of the following is left to the reader.

Lemma 3.5. For any $A_\infty$-algebra $A$, the map

$$\Phi : B(A^{\text{op}}) \to B(A)^{\text{op}},$$

$$[a_1] \cdots [a_m] \mapsto (-1)^{\sum_{i<j} (-1+\deg a_i)(-1+\deg a_j)} [a_m] \cdots [a_1]$$

is an isomorphism of DG coalgebras.
Note that $B(A)^{op}$ is the opposite coalgebra to $B(A)$, as defined in Subsection 3.4.

This result gives us a second way to prove Lemma 3.4, that the definition $f_n^{op} = (-1)^{1+\varepsilon(n)}f_n \circ (\text{twist})$ defines a morphism of $A_\infty$-algebras: one just has to check that if the $A_\infty$-algebra morphism $f : A \to A'$ corresponds to the DG coalgebra morphism $B(f) : B(A) \to B(A')$, then the composite

$$B(A)^{op} \xrightarrow{\Phi} B(A)^{op} \xrightarrow{B(f^{op})} B((A')^{op}) \xrightarrow{\Phi} B(A')^{op}$$

is equal to $B(f)^{op} = B(f)$. This is straightforward.

Once we know that the bar construction works well with opposites, we can define the opposite of a homotopy between $A_\infty$-algebra maps in terms of the bar construction. Thus $\text{op}$ defines an automorphism on the homotopy category of $A_\infty$-algebras.

### 3.6. The opposite of an $A_\infty$-module.

Since modules over an $A_\infty$-algebra are defined using exactly the same identities $\text{SI}(n)$ as for $A_\infty$-algebras, and since morphisms between modules satisfy only slight variants on the identities $\text{MI}(n)$, essentially the same proofs show that the opposite of a right $A$-module is a left $A^{op}$-module, etc. That is, there are equivalences of categories

$$(\text{left } A_\infty\text{-modules over } A)^{op} \xrightarrow{\text{op}} \text{Mod}^{\infty}(A^{op}),$$

$$(\text{D}^{\infty}(\text{left } A_\infty\text{-modules over } A))^{op} \xrightarrow{\text{op}} \text{D}^{\infty}(A^{op}).$$

So whenever left $A_\infty$-modules arise, we may easily convert them to right $A_\infty$-modules, and vice versa.

### 4. Adjunctions and equivalences

This section lays more groundwork: generalities for establishing equivalences between categories via Auslander and Bass classes, results about derived functors for DG modules, and $\otimes$-$\text{Hom}$ adjointness. Two of the main results of the section are Propositions 4.10 and 4.11, which describe when certain subcategories of derived categories of DG-modules are equivalent.

#### 4.1. Auslander and Bass classes.

Let $C$ and $D$ be two categories. Let $F : C \to D$ be left adjoint to a functor $G : D \to C$. Then there are natural transformations

$$\eta : 1_C \to GF,$$

$$\varepsilon : FG \to 1_D.$$

We define two full subcategories as follows. The *Auslander class* associated to $(F,G)$ is the subcategory of $C$ whose objects are

$$\{M \mid \eta_M : M \to GF(M) \text{ is an isomorphism}\}.$$
The Auslander class is denoted by $A$. The Bass class associated to $(F, G)$ is the subcategory of $D$ whose objects are

$$\{ N \mid \varepsilon_N : FG(N) \to N \text{ is an isomorphism} \}.$$ 

The Bass class is denoted by $B$. These definitions are abstractions of ideas of Avramov and Foxby [2, Section 3]. The following lemma is proved by imitating [2, Theorem 3.2].

**Lemma 4.1.** Let $(F, G)$ be a pair of adjoint functors between $C$ and $D$.

(a) The functors $F$ and $G$ restrict to an equivalence of categories between $A$ and $B$.

(b) If $C$ and $D$ are additive and $F$ and $G$ are additive functors, then $A$ and $B$ are additive subcategories. If $C$ and $D$ are triangulated and $F$ and $G$ are triangulated functors, then $A$ and $B$ are triangulated subcategories.

### 4.2. Derived functors over a DG algebra.

The derived category and derived functors over a DG algebra are well-understood constructions nowadays. See [33], [19], and [13], for example. We review some details in this subsection. As with $A_\infty$-algebras and modules, every DG module in this paper is $\mathbb{Z} \times \mathbb{Z}$-graded.

Let $R$ be a DG algebra and let $M$ be a DG $R$-module. Then $M$ is called **acyclic** if $HM = 0$; it is called **free** if it is isomorphic to a direct sum of shifts of $R$; and it is called **semifree** if there is a sequence of DG submodules $0 = M_{-1} \subset M_0 \subset \cdots \subset M_n \subset \cdots$ such that $M = \bigcup_n M_n$ and that each $M_n/M_{n-1}$ is free on a basis of cycles. Semifree modules are a replacement for free complexes over an associative algebra.

**Notation.** If $R$ is a DG algebra and $M$ and $N$ are DG $R$-modules, we write $\text{Hom}_R(M, N)$ for the DG $k$-module whose degree $n$ elements are degree $n$ $R$-module maps $M \to N$, ignoring the differential; see Subsection 1.1 for the formula for the differential in $\text{Hom}_R(M, N)$. Similarly, $\text{End}_R(M)$ means the complex $\text{Hom}_R(M, M)$.

In DG homological algebra, $K$-projective and $K$-injective DG modules are used to define derived functors. A DG $R$-module $M$ is called **$K$-projective** if the functor $\text{Hom}_R(M, -)$ preserves quasi-isomorphisms, or equivalently, $\text{Hom}_R(M, -)$ maps acyclic DG $R$-modules to acyclic DG $k$-modules. For example, a semifree DG $R$-module is always $K$-projective. A DG $R$-module $M$ is called **$K$-injective** if the functor $\text{Hom}_R( -, N)$ preserves quasi-isomorphisms, or equivalently, $\text{Hom}_R( -, N)$ maps acyclic DG $R$-modules to acyclic DG $k$-modules.

Given a DG $R$-module $M$, a map $f : L \to M$ is called a **semifree** (or $K$-projective or $K$-flat, respectively) **resolution** of $M$ if $f : L \to M$ is a quasi-isomorphism and $L$ is semifree (or $K$-projective or $K$-flat, respectively).
Similarly, a $K$-injective resolution of $M$ is a quasi-isomorphism $M \to L$ where $L$ is $K$-injective. In all of these cases, we will also abuse notation slightly and refer to $L$ itself as the resolution, omitting mention of the map $f$.

The right derived functor of $\Hom_R(-,-)$ is $\RHom_R(-,-)$, defined by

$$\RHom_R(M,N) := \Hom_R(P,N) \quad \text{or} \quad \RHom_R(M,N) := \Hom_R(M,I)$$

where $P$ is a $K$-projective resolution of $M$ and $I$ is a $K$-injective resolution of $N$. The left derived functor of $- \otimes_R -$ is $- \otimes_R L$, defined by

$$M \otimes_R L := M \otimes_R Q \quad \text{or} \quad M \otimes_R L := S \otimes_R N$$

where $S$ is a $K$-flat resolution of $M$ and $Q$ is a $K$-flat resolution of $N$.

4.3. Tensor-Hom and Hom-Hom adjunctions. We discuss $\otimes$-Hom and Hom-Hom adjointness, both basic and derived. These are well-known, at least in the case of modules over an associative algebra. The DG case may not be as familiar, so we provide some details. Here is the basic version.

**Lemma 4.2.** Let $A$, $B$, $C$, and $D$ be DG algebras.

(a) [Hom-Hom adjointness] Let $A L_C$, $D M_B$ and $A N_B$ be DG bimodules. Then

$$\Hom_A\Hom_{\text{op}}(A L_C, \Hom_B(D M_B, A N_B)) \cong \Hom_B(D M_B, \Hom_A\Hom_{\text{op}}(A L_C, A N_B))$$

as DG $(C,D)$-bimodules.

(b) [$\otimes$-Hom adjointness] Let $D L_B$, $B M_A$ and $C N_A$ be DG bimodules. Then

$$\Hom_A(D L_B \otimes_B M_A, C N_A) \cong \Hom_B(D L_B, \Hom_A(B M_A, C N_A))$$

as DG $(C,D)$-bimodules.

The isomorphism in part (a) gives a pair of adjoint functors

$$A\text{-Mod} \rightleftarrows (\text{Mod } B)^{\text{op}},$$

namely $\Hom_A\Hom_{\text{op}}(-, A N_B)$ (left adjoint) and $\Hom_B(-, A N_B)$ (right adjoint). This explains the label, “Hom-Hom adjointness.”

**Proof.** (a) The desired isomorphism

$$\phi : \Hom_A\Hom_{\text{op}}(A L_C, \Hom_B(D M_B, A N_B)) \to \Hom_B(D M_B, \Hom_A\Hom_{\text{op}}(A L_C, A N_B))$$

is defined by the following rule. Let $f \in \Hom_A\Hom_{\text{op}}(A L_C, \Hom_B(D M_B, A N_B))$, and $l \in L$ and $m \in M$, write $f(l) \in \Hom_B(D M_B, A N_B)$ and $f(l)(m) \in N$; then $\phi(f) : M \to \Hom_A\Hom_{\text{op}}(A L_C, A N_B)$ is determined by

$$\phi(f)(m)(l) = (-1)^{|l||m|} f(l)(m)$$

for all $l \in L, m \in M$. It is straightforward to check that $\phi$ is an isomorphism of DG bimodules.
The above construction is given in the unpublished manuscript [3, (3.4.2), p. 27]. The isomorphism \( \phi \) is called the *swap isomorphism* [3, Sect. 3.4].

(b) This is standard and a proof is given in [3, (3.4.3), p. 28]. A non-DG version is in [31, Theorem 2.11, p. 37]. \( \square \)

To get derived versions of these, we need information about bimodules, semifree resolutions, \( K \)-projectives, etc.

**Lemma 4.3.** Let \( A \) and \( B \) be DG algebras. Let \( M \) and \( L \) be DG \((B,A)\)-bimodules, or equivalently, DG \( B^{op} \otimes A \)-modules.

(a) Let \( N \) be a DG \( A \)-module. If there is a sequence of DG submodules

\[
0 = N_{-1} \subset N_0 \subset \cdots \subset N_n \subset \cdots
\]

such that \( N = \bigcup_n N_n \), \( C_n := N_n/N_{n-1} \) is \( K \)-projective, and the underlying graded module \( C_n \) is projective, then \( N \) is \( K \)-projective.

(b) If \( M \) is semifree \((B,A)\)-bimodule, then it is \( K \)-projective over \( A \). As a consequence, if \( M \to L \) is a semifree resolution of \( L \), then restricted to the right-hand side, it is a \( K \)-projective resolution of \( L_A \).

(c) If \( M \) is \( K \)-injective \((B,A)\)-bimodule, then it is \( K \)-injective over \( A \). As a consequence, if \( L \to M \) is a \( K \)-injective resolution of \( L \), then restricted to the right-hand side, it is a \( K \)-injective resolution of \( L_A \).

**Proof.** (a) First consider the sequence

\[
0 \to N_0 \to N_1 \to C_1 \to 0
\]

of DG \( A \)-modules. This is a split (hence exact) sequence after omitting the differentials, as the underlying graded module \( C_1 \) is projective. Let \( X \) be an acyclic DG \( A \)-module. Then we have an exact sequence

\[
0 \to \text{Hom}_A(C_1,X) \to \text{Hom}_A(N_1,X) \to \text{Hom}_A(N_0,X) \to 0.
\]

If the two ends \( \text{Hom}_A(C_1,X) \) and \( \text{Hom}_A(N_0,X) \) are acyclic, so is the middle term \( \text{Hom}_A(N_1,X) \). This shows that if \( C_0(=N_0) \) and \( C_1 \) are \( K \)-projective, so is \( N_1 \). By induction on \( n \) we see that \( N_n \) is \( K \)-projective and projective for all \( n \). Since every sequence

\[
0 \to N_{n-1} \to N_n \to C_n \to 0
\]

splits, the map \( \text{Hom}_A(N_n,X) \to \text{Hom}_A(N_{n-1},X) \) is surjective. This means that the inverse system \( \{\text{Hom}_A(N_n,X)\}_n \) satisfies Mittag-Leffler condition. Since each \( \text{Hom}_A(N_n,X) \) is acyclic,

\[
\text{Hom}_A(N,X) = \varprojlim \text{Hom}_A(N_n,X)
\]

is acyclic by [37, Theorem 3.5.8].

(b) The second assertion follows from the first one.

By part (a) and the definition of a semifree module, we may assume \( M \) is free. Since a free module is a direct sum of shifts of \( B^{op} \otimes A \), we may assume \( M \) is a copy of \( B^{op} \otimes A \). Therefore \( M \) is free over \( A \), and hence \( K \)-projective over \( A \).
The second assertion follows from the first one. Let $N_A$ be an acyclic DG $A$-module. By $\otimes$-Hom adjointness (Lemma 4.2(b)), we have
\[
\text{Hom}_A(N, B_M A) = \text{Hom}_B(A, B_M A) \\
\cong \text{Hom}_{B^\text{op}}(N \otimes_A (B^\text{op} \otimes A), B_M A) \\
\cong \text{Hom}_{B^\text{op}}(N \otimes B^\text{op}, B_M A).
\]

Since $N$ is acyclic, so is $N \otimes B^\text{op}$. Since $B_M A$ is $K$-injective, the above formula implies that $\text{Hom}_A(N, M_A)$ is acyclic. Hence $M_A$ is $K$-injective. □

We can combine the previous two lemmas to get derived Hom-Hom and $\otimes$-Hom adjointness.

**Lemma 4.4** (Derived Hom-Hom adjointness). Let $A, B, C, and D$ be DG algebras. Let $A L, B M C$ and $A N_D$ be DG bimodules.

(a) There is an isomorphism of complexes
\[
\text{RHom}_{A^\text{op}}(AL, \text{RHom}_B(DM_B, AN_B)) \\
\cong \text{RHom}_B(DM_B, \text{RHom}_{A^\text{op}}(AL, AN_B))
\]
in $D(C^\text{op} \otimes D)$.

(b) There is an isomorphism of $k$-vector spaces
\[
D(A^\text{op})(AL, \text{RHom}_B(M_B, AN_B)) = D(B)(BM_B, \text{RHom}_{A^\text{op}}(AL, AN_B)).
\]

**Proof.** (a) This follows from Lemmas 4.2 and 4.3, and by taking semifree resolutions of the DG bimodules $M$ and $L$.

(b) This follows from (a) by taking $H^0$. □

**Lemma 4.5** (Derived $\otimes$-Hom adjointness). Let $A, B, C$ and $D$ be DG algebras. Let $D L_B, B M_A$ and $C N_A$ be DG bimodules.

(a) There is an isomorphism of complexes
\[
\text{RHom}_A(DL \otimes_B MA, CN_A) \cong \text{RHom}_B(DL_B, \text{RHom}_A(BM_A, CN_A))
\]
in $D(C^\text{op} \otimes D)$.

(b) There is an isomorphism of $k$-vector spaces
\[
D(A)(L \otimes_B MA, N_A) \cong D(B)(LB, \text{RHom}_A(BM_A, N_A)).
\]

**Proof.** (a) This follows from Lemmas 4.2 and 4.3, and by taking semifree resolutions of the DG bimodules $M$ and $L$ and a $K$-injective resolution of the bimodule $N$.

(b) This follows from (a) by taking $H^0$. □
4.4. Balanced bimodules and equivalences. In this subsection we prove Propositions 4.10 and 4.11, the main results in this section; these establish a framework for proving the derived equivalences in Section 5.

Let $A$ and $E$ be two DG algebras. A DG $A$-module $M$ is a DG $(E,A)$-bimodule if and only if there is a map of DG algebras $E \to \text{Hom}_A(M,M)$.

**Definition 4.6.** Let $B$ be a DG $(E,A)$-bimodule. We call $B$ left balanced if and only if there is a quasi-isomorphism $B \to N$ of DG $(E,A)$-bimodules such that $N$ is $K$-injective over $E^{op} \otimes A$ and the canonical map $E \to \text{Hom}_A(N,N)$ is a quasi-isomorphism of DG algebras. The right balanced property is defined in a similar way, in terms of the map $A^{op} \to \text{Hom}_{E^{op}}(N,N)$.

**Lemma 4.7.** Let $B$ be a DG $(E,A)$-bimodule. Then the following conditions are equivalent:

(i) $B$ is left balanced.
(ii) If $P \to B$ is a quasi-isomorphism of DG $(E,A)$-bimodules with $P_A$ being $K$-projective, then the canonical map $E \to \text{Hom}_A(P,P)$ is a quasi-isomorphism of DG algebras.
(iii) If $B \to I$ is a quasi-isomorphism of DG $(E,A)$-bimodules with $I_A$ being $K$-injective, then the canonical map $E \to \text{Hom}_A(I,I)$ is a quasi-isomorphism of DG algebras.

**Proof.** (i) $\iff$ (ii). Suppose that there is a quasi-isomorphism $B \to N$ where $N$ is $K$-injective over $E^{op} \otimes A$. (By a result of Spaltenstein [33, Corollary 3.9], there always is such a map.) By Lemma 4.3(c), $N$ is $K$-injective over $A$. The quasi-isomorphism $f : P \to B \to N$ induces two maps

$$E \xrightarrow{i_N} \text{Hom}_A(N,N) \xrightarrow{f^*} \text{Hom}_A(P,N)$$

and

$$E \xrightarrow{i_P} \text{Hom}_A(P,P) \xrightarrow{f^*} \text{Hom}_A(P,N)$$

of $(E,E)$-bimodules. Since $N$ is $K$-injective over $A$ and $P$ is $K$-projective over $A$, both $g$ and $h$ are quasi-isomorphisms. It is easy to see that $f^*i_N = f^*i_P$: they both map $e \in E$ to $ef = fe \in \text{Hom}_A(P,N)$. Therefore $i_N$ is a quasi-isomorphism if and only if $i_P$ is.

(ii) $\iff$ (iii). This proof is similar. 

By the above lemma, we can construct plenty of left balanced bimodules. For example, let $M$ be a $K$-injective (or $K$-projective) DG $A$-module and let $E = \text{Hom}_A(M,M)$. It follows from the lemma that $M$ becomes a left balanced DG $(E,A)$-bimodule with its natural left $E$-module structure.

We now recall a few definitions.

**Definition 4.8.** An object $M$ in an additive category $C$ with infinite direct sums is called small if $C(M, -)$ commutes with arbitrary direct sums.

Let $A$ be an $A_\infty$-algebra and $M$ be a right $A_\infty$-module over $A$. Let $\text{triang}_A^\infty(M)$ denote the triangulated subcategory of $D^\infty(A)$ generated by
$M$ and its Adams shifts. (Recall that every subcategory in this paper is full.) Let $\text{triang}_A^\infty(M)$ be the thick subcategory generated by $\text{triang}_A^\infty(M)$: the smallest triangulated subcategory, closed under summands, which contains $\text{triang}_A^\infty(M)$. Similarly, if $R$ is a DG algebra and $N$ is a DG $R$-module, let $\text{triang}_R(N)$ denote the triangulated subcategory of $\text{D}(R)$ generated by $N$ and its shifts, and let $\text{thick}_R(N)$ be the thick subcategory generated by $\text{triang}_R(N)$. Let $\text{D}_{\text{per}}(A) = \text{thick}_A^\infty(A)$, and let $\text{D}_{\text{per}}(R) = \text{thick}_R(R)$. We call objects in $\text{D}_{\text{per}}(A)$ and $\text{D}_{\text{per}}(R)$ perfect complexes. Let $\text{loc}_A^\infty(M)$ denote the localizing subcategory (= triangulated and closed under arbitrary direct sums) generated by $\text{triang}_A^\infty(M)$, and similarly for $\text{loc}_R(N)$. We can also define each of these with $M$ or $N$ replaced by a collection of modules. It is well-known that $\text{loc}_R(R) = \text{D}(R)$.

**Lemma 4.9.**

(a) If $R$ is a DG algebra and $N$ is a right DG $R$-module, then $N$ is small in $\text{D}(R)$ if and only if $\text{RHom}_R(N, -)$ commutes with arbitrary colimits, and if and only if $N \in \text{D}_{\text{per}}(R)$.

(b) If $A$ is an $A_{\infty}$-algebra and $M$ is a right $A_{\infty}$-module over $A$, then $M$ is small in $\text{D}_{\text{per}}^\infty(A)$ if and only if $M \in \text{D}_{\text{per}}^\infty(A)$.

**Proof.** (a) The equivalence that $N$ is small if and only if $N$ is in $\text{D}_{\text{per}}(R)$ is somewhat standard; see Keller [19, 5.3], for example. If $\text{RHom}_R(N, -)$ commutes with arbitrary colimits, then since homology also commutes with colimits, $\text{D}(R)(N, -)$ is $\text{H} \text{RHom}_R(N, -)$ does as well, and so $N$ is small. Finally, $\text{RHom}_R(R, -)$ commutes with arbitrary colimits, and hence so does $\text{RHom}_R(N, -)$ for any object $N$ in $\text{D}_{\text{per}}(R) = \text{thick}_R(R)$.

(b) This follows from part (a) and Propositions 1.14 and 3.1(c). \qed

Let $B$ be a DG $(E, A)$-bimodule. By Lemma 4.5(b),

$$F_B := - \otimes_E^L B : \text{D}(E) \to \text{D}(A) \quad \text{and}$$

$$G_B := \text{RHom}_A(B, -) : \text{D}(A) \to \text{D}(E)$$

form a pair of adjoint functors. Let $A_B$ and $B_B$ be the Auslander and Bass classes associated to the pair $(F_B, G_B)$. By Lemma 4.1, $(F_B, G_B)$ induces a triangulated equivalence between $A_B$ and $B_B$.

The next two results are precursors of the derived equivalences in the next section.

**Proposition 4.10.** Let $A$ and $E$ be DG algebras, and suppose that $B$ is a left balanced DG $(E, A)$-module. Define adjoint functors $F = F_B = - \otimes_E^L B$ and $G = G_B = \text{RHom}_A(B, -)$, as above.

(a) Then $F_B$ and $G_B$ induce an equivalence of categories $A_B \cong B_B$. Furthermore, $E_E \in A_B$, $B_A \in B_B$, and $F_B(E_E) = B_A$.

(b) There is an equivalence of triangulated categories

$$\text{triang}_E(E) \cong \text{triang}_A(B).$$
(c) There is an equivalence of triangulated categories
\[ D_{\text{per}}(E) = \text{thick}_E(E) \cong \text{thick}_A(B). \]

(d) If \( B_A \) is small, then there is an equivalence of triangulated categories
\[ D(E) \cong \text{loc}_A(B). \]

**Proof.** (a) The first assertion is Lemma 4.1. Without loss of generality, we assume that \( B \) is \( K \)-injective over \( E^{\text{op}} \otimes A \). Since \( B \) is left balanced,
\[ E \rightarrow \text{Hom}_A(B, B) \cong \text{RHom}_A(B, E \otimes_E^L B) = GF(E) \]
is a quasi-isomorphism of DG \( A \)-modules. Hence \( E \in \mathcal{A}_B \). Clearly \( F(E) = B \) and \( G(B) = E \). Consequently, \( B \in \mathcal{B}_B \).

(b,c) These follow from (a).

(d) By definition, \( F \) commutes with arbitrary colimits. If \( B \) is small, \( G \) commutes with arbitrary colimits. In this case, \( \mathcal{A}_B \) and \( \mathcal{B}_B \) have arbitrary colimits. Since \( E \in \mathcal{A}_B \), \( \mathcal{A}_B = D(E) \). Since \( F \) is an equivalence, \( \mathcal{B}_B = \text{loc}_A(B) \). \( \square \)

Now we consider two other functors \( \tilde{F}^B = \text{RHom}_{E^{\text{op}}}( -, B) : D(E^{\text{op}}) \rightarrow D(A) \) and \( \tilde{G}^B = \text{RHom}_A( -, B) : D(A) \rightarrow D(E^{\text{op}}) \). Both of them are contravariant; however, if we view them as \( F^B : D(E^{\text{op}}) \rightarrow D(A^{\text{op}}) \) and \( G^B : D(A)^{\text{op}} \rightarrow D(E^{\text{op}}) \), then they become covariant. By Lemma 4.4(a), \( (F^B, G^B) \) is an adjoint pair. Let \( \mathcal{A}^B \) and \( \mathcal{B}^B \) be the Auslander and Bass classes associated to the pair \( (F^B, G^B) \).

**Proposition 4.11.** Let \( A \) and \( E \) be DG algebras, and suppose that \( B \) is a left balanced DG \( (E, A) \)-module. Define adjoint functors \( F = F^B \) and \( G = G^B \), as above.

(a) Then \( F^B \) and \( G^B \) induce an equivalence of categories \( \mathcal{A}^B \cong \mathcal{B}^B \). Furthermore, \( E \in \mathcal{B}^B, \mathcal{B}_A \subseteq \mathcal{A}^B, \) and \( F^B(EE) = B_A \).

(b) If \( B \) is also right balanced, then \( A_A \subseteq \mathcal{A}^B, E \mathcal{B} \subseteq \mathcal{B}^B, \) and \( F^B(E \mathcal{B}) = A_A \).

(c) There is an equivalence of triangulated categories
\[ \text{triang}_{E^{\text{op}}}(E) \cong \text{triang}_{A}(B)^{\text{op}}. \]

If \( B \) is also right balanced, then
\[ \text{triang}_{E^{\text{op}}}(E, B) \cong \text{triang}_{A}(A, B)^{\text{op}}. \]

(d) There is an equivalence of triangulated categories
\[ D_{\text{per}}(E^{\text{op}}) = \text{thick}_{E^{\text{op}}}(E) \cong \text{thick}_A(B)^{\text{op}}. \]

If \( B \) is also right balanced, then
\[ \text{thick}_{E^{\text{op}}}(E, B) \cong \text{thick}_A(A, B)^{\text{op}}. \]
(e) If $E\mathcal{B}$ is small and $\mathcal{B}$ is right balanced, then there is an equivalence of triangulated categories
\[ \text{D}_{\text{per}}(E^{\text{op}}) \cong \text{thick}_A(A, \mathcal{B})^{\text{op}} = \text{thick}_A(\mathcal{B})^{\text{op}}. \]
As a consequence, $A \in \text{thick}_A(\mathcal{B})^{\text{op}}$.

**Proof.** (a) The first assertion follows from Lemma 4.1. We may assume that $\mathcal{B}$ is $K$-injective over $E^{\text{op}} \otimes A$. Since $\mathcal{B}$ is left balanced,
\[ G^B F^B(E) = \text{RHom}_A(\text{RHom}_{E^{\text{op}}}(E, \mathcal{B}), \mathcal{B}) = \text{RHom}_A(\mathcal{B}, \mathcal{B}) \leftarrow E \]
is a quasi-isomorphism. This shows that $E \in B^\mathcal{B}$. Since $F^B(E) = \mathcal{B}$, we have $B \in A^\mathcal{B}$.

(b) This is the right-hand version of (a).
(c,d,e) These follow from (a,b). □

Proposition 4.10 also implies the following easy fact.

**Corollary 4.12.** Let $M$ be an object in $D^\infty(A)$ for some $A_\infty$-algebra $A$. Then $\text{thick}_A(M)$ is triangulated equivalent to $\text{D}_{\text{per}}(E)$ for some DG algebra $E$.

**Proof.** By Proposition 3.1 we may assume that $A$ is a DG algebra, and then we may replace $D^\infty(A)$ by $D(A)$. Hence we may assume that $M$ is a right DG $A$-module.

Let $B_A$ be a $K$-projective resolution of $M$ and let $E = \text{End}_A(B_A)$. Then $B$ is a left balanced $(E, A)$-bimodule. Note that $M \cong B$ in $D(A)$. The assertion follows from Proposition 4.10(c). □

## 5. Koszul equivalences and dualities

In the setting of classical Koszul duality [7], there is an equivalence between certain subcategories of the derived categories of a Koszul algebra $A$ and of its Koszul dual; the subcategories consist of objects satisfying certain finiteness conditions. In this section, we explore the analogous results for non-Koszul algebras, DG algebras, and $A_\infty$-algebras. The main results are Theorems 5.4 and 5.5 in the DG setting, and Theorems 5.7 and 5.8 in the $A_\infty$ setting.

### 5.1. Koszul equivalence and duality in the DG case

Let $A$ be an augmented DG algebra and let $E(A) = (B^\text{ang}_A)^2$ be its Koszul dual, as defined in Section 2.2. The usual bar construction $B(A; A)$ [13, p. 269], where the second $A$ is viewed as a DG $A$-bimodule, agrees with the $A_\infty$-module version $B^\text{ang}_A(A; A)$ from Section 3.2. By [13, Proposition 19.2(b)], $B(A; A)$ is a semifree resolution of the right $A$-module $k$. Thus to define derived functors we may replace $k_A$ with $B(A; A)_A$.

The following lemma can be viewed as a dual version of [13, Proposition 19.2].
Lemma 5.1. Let $\mathcal{B} = B(A; A)$ and let $E = E(A)$.

(a) The natural embedding $i : E k \to E \mathcal{B}$ is a quasi-isomorphism of left DG $E$-modules.

(b) If $A$ is weakly Adams connected (Definition 2.1), then $\mathcal{B}$ is a $K$-injective DG left $E$-module.

A left DG $E$-module is called semiinjective if it is an injective left graded $E$-module and a $K$-injective left DG $E$-module.

Proof. (a) By [13, Proposition 19.2(a)], the augmentations in $BA$ and $A$ define a quasi-isomorphism $\varepsilon \otimes \varepsilon : \mathcal{B}_A \to k_A$ of right DG $A$-modules. The map $i$ is a quasi-isomorphism because the composition $k \xrightarrow{i} \mathcal{B} \xrightarrow{\varepsilon \otimes \varepsilon} k$ is the identity. It is easy to see that the map $i$ is a right DG $E$-module homomorphism.

(b) Let $E^\sharp$ be the $E$-bimodule $\text{Hom}_k(E, k)$. It follows from the adjunction formula in Lemma 4.2(b) that $E^\sharp$ is semiinjective as a left and a right DG $E$-module. If $V$ is a finite-dimensional $k$-vector space, then $E^\sharp \otimes V \cong \text{Hom}_k(E, V)$ and this is also semiinjective as a left and a right DG $E$-module.

Since $B(A; A)$ is locally finite, $E = (BA)^\sharp$ is locally finite and $E^\sharp = (BA^\sharp)^\sharp \cong BA$. Hence $B(A; A(k)) \cong BA$ is a semiinjective left DG $E$-module.

By induction one can easily show that if $M$ is a finite-dimensional left DG $A$-module, then the bar construction $B(A; M)$ is a semiinjective left DG $E$-module. Since $A$ is weakly Adams connected, $A = \lim N_n$ where the $\{N_n\}_{n \geq 0}$ are finite-dimensional left DG $A$-modules. Since each $N_n$ is finite-dimensional, we may further assume that the map $N_n \to N_{n-1}$ is surjective for all $n$. By the assertion just proved, $B(A; N_n)$ is a semiinjective left DG $E$-module for each $n$, as is $B(A; N_n/N_{n-1})$.

Since $A = \lim N_n$ and since $B(A; A)$ is locally finite, $B(A; A) = \lim B(A; N_n)$.

A result of Spaltenstein [33, Corollary 2.5] says that such an inverse limit of $K$-injectives is again $K$-injective, and this finishes the proof. (Spaltenstein’s result is for inverse limits of $K$-injectives in the category of chain complexes over an abelian category, but the proof is formal enough that it extends to the category of DG modules over a DG algebra.)

Remark 5.2. By the above lemma, $E \mathcal{B}$ is isomorphic to $E k$ in $D(E^{op})$. By [13, Proposition 19.2(a)], $\mathcal{B}_A$ is isomorphic to $k_A$ in $D(A)$. However, $\mathcal{B}$ is not isomorphic to $k$ in $D(E^{op} \otimes A)$ in general.

Lemma 5.3. Let $\mathcal{B}$ be the right DG $A$-module $B(A; A)$ and let $C = \text{End}_A(\mathcal{B})$.

(a) $\mathcal{B}$ is a left balanced $(C, A)$-bimodule.

(b) If $E := E(A)$ is locally finite, then $\mathcal{B}$ is a left balanced $(E, A)$-bimodule via the natural isomorphism $E \to C$. As a consequence, $HE \cong H \mathrm{RHom}_A(k, k)$.
(c) If $A$ is weakly Adams connected, then $B$ is a right balanced $(E,A)$-bimodule.

**Proof.** (a) Since $B$ is semifree over $A$, the assertion follows from Lemma 4.7.
(b) The first assertion follows from Lemma 4.7 and the fact that $E \to \text{End}_A(B)$ is a quasi-isomorphism [13, Ex 4, p. 272]. Since $B_A$ is a $K$-projective resolution of $k_A$,
$$HE \cong H \text{End}_A(B) \cong H \text{RHom}_A(k,k),$$
where the last isomorphism follows from Theorem 2.4. Therefore, since $HA$ is locally finite, it suffices to show that $H\phi$ is injective. Let $a \in A^\text{op}$ be a cocycle such that $\phi(a) = 0$ in $H \text{End}_{E^\text{op}}(B)$. Then $a \neq 1$ and there is an $f \in \text{End}_{E^\text{op}}(B)$ such that $\phi(a) = d(f)$. Applying this equation to $x = [] \otimes 1 \in B = B(A;A)$, we obtain
$$[] \otimes a = \phi(a)(x) = d(f)(x)$$
$$= (d \circ f - (-1)^{\text{deg}_1 f} f \circ d)(x)$$
$$= d \circ f([] \otimes 1) \pm f \circ d([] \otimes 1).$$
Since $f$ is a left $E$-module homomorphism, $f([] \otimes 1) = [ ] \otimes w$ for some $w \in A$. By definition, $d([] \otimes 1) = 0$. Therefore
$$[] \otimes a = d \circ f([] \otimes 1) = d([] \otimes w) = [ ] \otimes dw,$$
and hence $a = dw$ as required. □

Here is a version of [7, 1.2.6] for DG algebras.

**Theorem 5.4.** Let $A$ be an augmented DG algebra and let $E = E(A)$ be the Koszul dual of $A$. Assume that $E$ is locally finite. The functors $\text{RHom}_A(k,-)$ and $- \otimes_{E^\text{op}}^L k$ induce the following equivalences.
(a) The category $\text{triang}_A(k)$ is triangulated equivalent to $\text{triang}_{E^\text{op}}(E)$.
(b) The category $\text{thick}_A(k)$ is triangulated equivalent to $\text{D}_{\text{per}}(E)$.
(c) Suppose that $k_A$ is small in $D(A)$. Then $\text{loc}_A(k)$ is triangulated equivalent to $D(E)$.

**Proof.** Note that $B = B(A;A) \cong k_A$ as a right DG $A$-module. Then the assertions follow from Proposition 4.10 and Lemma 5.3(b). □

**Theorem 5.5.** Let $A$ be an augmented DG algebra and let $E = E(A)$ be the Koszul dual of $A$. Assume that $A$ is weakly Adams connected. The functors $\text{RHom}_A(-,k)$ and $\text{RHom}_{E^\text{op}}(-,k)$ induce the following equivalences.
Koszul equivalences in $A_\infty$-algebras

(a) $\text{triang}_A(k, A)^{\text{op}}$ is triangulated equivalent to $\text{triang}_{E^{\text{op}}}(E, k)$.
(b) $\text{thick}_A(k, A)^{\text{op}}$ is triangulated equivalent to $\text{thick}_{E^{\text{op}}}(E, k)$.
(c) Suppose that $k_A$ is small in $D(A)$. Then
$$D_{\text{per}}(A)^{\text{op}} = \text{thick}_A(k, A)^{\text{op}} \cong \text{thick}_{E^{\text{op}}}(E, k) = \text{thick}_{E^{\text{op}}}(k).$$

**Proof.** This follows from Proposition 4.11 and Lemmas 5.1 and 5.3. □

**Corollary 5.6.** Let $A$ be a weakly Adams connected augmented DG algebra and let $E = E(A)$ be its Koszul dual. If $k_A$ is small in $D(A)$, then $H_E$ is finite-dimensional.

**Proof.** By Theorem 5.5(c), $E$ is in the thick subcategory generated by $k$, and every object in $\text{thick}_{E^{\text{op}}}(k)$ has finite-dimensional homology. □

See Corollary 6.2 for a related result for DG algebras, and Corollaries 5.9 and 7.2 for similar results about $A_\infty$-algebras.

5.2. Koszul equivalence and duality in the $A_\infty$ case. Now suppose that $A$ is an $A_\infty$-algebra. By Proposition 1.14, $A$ is quasi-isomorphic to the DG algebra $U_A$, so by Proposition 3.1, the derived category $D^\infty(A)$ is equivalent to $D(UA)$. We can use this to prove the following, which is a version of [7, 1.2.6] for $A_\infty$-algebras.

**Theorem 5.7.** Let $A$ be an augmented $A_\infty$-algebra and let $E = E(A)$ be the Koszul dual of $A$. Assume that $A$ is strongly locally finite (Definition 2.1).

(a) The category $\text{triang}_A^\infty(k)$ is triangulated equivalent to $\text{triang}_{E^\infty}(E)$.
(b) The category $\text{thick}_A^\infty(k)$ is triangulated equivalent to $D_{\text{per}}^\infty(E)$.
(c) Suppose that $k_A$ is small in $D^\infty(A)$. Then $\text{loc}_A^\infty(k)$ is triangulated equivalent to $D^\infty(E)$.

**Proof.** We can replace $A$ by $U_A$ and $E$ by $E(UA) = E(E(A))$. The assertions follow from Proposition 3.1 and Theorem 5.4. □

Similarly, Proposition 3.1 combined with 5.5 give the following.

**Theorem 5.8.** Let $A$ be an augmented $A_\infty$-algebra and let $E = E(A)$ be the Koszul dual of $A$. Assume that $A$ is strongly locally finite.

(a) $\text{triang}_A^\infty(k, A)^{\text{op}}$ is triangulated equivalent to $\text{triang}_{E^{\text{op}}^\infty}(E, k)$.
(b) $\text{thick}_A^\infty(k, A)^{\text{op}}$ is triangulated equivalent to $\text{thick}_{E^{\text{op}}^\infty}(E, k)$.
(c) Suppose that $k_A$ is small in $D^\infty(A)$. Then
$$D_{\text{per}}^\infty(A)^{\text{op}} = \text{thick}_A^\infty(k, A)^{\text{op}} \cong \text{thick}_{E^{\text{op}}^\infty}(E, k) = \text{thick}_{E^{\text{op}}^\infty}(k).$$

Just as Theorem 5.5 implied Corollary 5.6, this result implies the following.

**Corollary 5.9.** Let $A$ and $E$ be as in Theorem 5.8. If $k_A$ is small in $D^\infty(A)$, then $H_E$ is finite-dimensional.

See Corollary 7.2 for the converse of Corollary 5.9.
Corollary 5.10. Via the equivalence in Theorem 5.8, there is an isomorphism of $k$-vector spaces
\[ H^i_j(R\Hom_A(k, A)) \cong H^{-i}_{-j}(R\Hom_{E^{op}}(k, E)) \]
for all $i, j$.

Proof. Again we may assume that $A$ and $E$ are DG algebras. The functor $G^B$ in Proposition 4.11 is defined as $G^B = R\Hom_A(-, B) \cong R\Hom_A(-, k)$, and changes $S$ to $S^{-1}$ and $\Sigma$ to $\Sigma^{-1}$; hence the assertion follows from the fact $G^B(k_A) = E^E$ and $G^B(A_A) = Ek$. □

Remark 5.11. One might hope to prove Theorems 5.7 and 5.8 directly, working in the category $D^\infty(A)$ rather than $D(UA)$. Keller [20, 6.3] and Lefèvre-Hasegawa [23, 4.1.1] have described the appropriate functors: $A^\infty$-versions of $R\Hom_A(-, -)$ and $- \otimes^L_E -$, which they write as $\text{Hom}_A^\bullet(-, -)$ and $- \otimes^E -$. Although we fully expect these to satisfy all of the required properties (such as adjointness), it was easier to use the more standard results in the DG setting.

6. Minimal semifree resolutions

In this short section we consider the existence of a minimal semifree resolution of a DG module over a DG algebra. The main result is Theorem 6.1; this result is needed in several places. There are similar results in the literature — see, for example [4, Section 1.11] or [11, Lemma A.3] — but they require that $A$ be connected graded with respect to the first (non-Adams) grading. We need to use this in other situations, though, so we include a detailed statement and proof.

We say that $A$ is positively connected graded in the second (Adams) grading if $A^*_<0 = 0$ and $A^*_0 = k$; negatively connected graded in the Adams grading is defined similarly. Write $m$ for the augmentation ideal of $A$; then a semifree resolution $F \to M$ of a module $M$ is called minimal if $d_F(F) \subset Fm$.

Theorem 6.1. Let $A$ be a DG algebra and let $M$ be a right DG $A$-module.

(a) Assume that $A$ is positively connected graded in the second grading and that $HM^*_<0 = 0$ for some $n$, or that $A$ is negatively connected graded in the second grading and that $HM^*_<0 = 0$ for some $n$. Then $M$ has a minimal semifree resolution $L \to M$ with $L^*_<0 = 0$ (respectively $L^*_>0 = 0$).

(b) Assume further that $HA$ and $HM$ are both bounded on the same side with respect to the first grading: assume that for each $j$, there is an $m$ so that $(HA)^{<m}_j = 0$ and $(HM)^{<m}_j = 0$, or $(HA)^{<m}_j = 0$ and $(HM)^{<m}_j = 0$. If $HA$ and $HM$ are locally finite (respectively, locally finite with respect to the second grading), then so is $L$. 
Proof. Without loss of generality, we assume that $A^*_{<0} = 0$ and $HM^*_{<n} = 0$ for some $n$. After an Adams shift we may further assume that $n = -1$, that is, $HM = HM^*_{<0}$. We will construct a sequence of right DG $A$-modules $\{L_u\}_{u \geq 0}$ with the following properties:

1. $0 = L_{-1} \subset L_0 \subset \cdots \subset L_u \subset \cdots$.
2. $L_u/L_{u-1}$ is a free DG $A$-module generated by a cycles of Adams-degree $u$.
3. $L_u \otimes_A k$ has a trivial differential; that is, $d_{L_u}(L_u) \subset L_u \mathfrak{m}$.
4. There is a morphism of DG $A$-modules $\varepsilon_u : L_u \to M$ such that the kernel and coker of $H(\varepsilon_u)$ have Adams degree at least $u+1$.

If $A$ and $M$ satisfy the hypotheses in part (b), then each $L_u$ will also satisfy

5. $L_u$ is locally finite (respectively, locally finite with respect to the second grading).

Let $L_{-1} = 0$. We proceed to construct $L_u$ inductively for $u \geq 0$, so suppose that $\{L_{-1}, L_0, \ldots, L_u\}$ have been constructed and satisfy (1)–(4), and if relevant, (5).

Consider the map $H(\varepsilon_u) : HL_u \to HM$; let $C$ be its cokernel and let $K$ be its kernel. We will focus on the parts of these in Adams degree $u+1$. Choose an embedding $i$ of $C_{u+1}$ into the cycles in $M$, and let $P_u$ be the free DG $A$-module $C_{u+1} \otimes_A A$ on $C_{u+1}$, equipped with a map $f : P_u \to M$, sending $x \otimes 1$ to $i(x)$ for each $x \in C_{u+1}$. Since $A$ is positively connected graded in the Adams grading, the map $f$ induces an isomorphism in homology in Adams degrees up to and including $u+1$. Similarly, let $Q_u$ be the free DG $A$-module on $K_{u+1}$, mapping to $L_u$ by a map $\bar{g}$ inducing a homology isomorphism in degrees less than or equal to $u+1$. Then let $L_{u+1}$ be the mapping cone of

$$Q_u \to L_u \oplus P_u,$$

where $g$ maps $Q_u$ to the first summand by the map $\bar{g}$. Since $Q_u$ is free and since the composite $(\varepsilon_u + f)g$ induces the zero map on homology, this composite is null-homotopic. Therefore there is a map $\varepsilon_{u+1} : L_{u+1} \to M$.

In more detail, since $L_{u+1}$ is the mapping cone of $g : Q_u \to J_u := L_u \oplus P_u$, it may be written as $L_{u+1} = S(Q_u) \oplus J_u$, with differential given by

$$d(q, l) = (-d_{Q_u}(q), g(q) + d_{J_u}(l)).$$

The null-homotopy gives an $A$-module homomorphism $\theta : Q_u \to M$ of degree $(-1, 0)$ such that

$$\theta d_{Q_u} + d_M \theta = \delta_u g$$

where $\delta_u = \varepsilon_u + f$. The $A$-module homomorphism $\varepsilon_{u+1} : L_{u+1} \to M$ is defined by

$$\varepsilon_{u+1}(q, l) = \theta(q) + \delta_u(l) \quad \forall q \in Q_u \text{ and } l \in J_u.$$

One can check that $\varepsilon_{u+1}$ commutes with the differentials and hence is a morphism of DG $A$-modules. The morphism $\varepsilon_{u+1}$ is an extension of $\varepsilon_u + f$, hence $\varepsilon_u$ is the restriction of $\varepsilon_{u+1}$ to $L_u$. 


Now we claim that the kernel and cokernel of $H(ε_{u+1})$ are in Adams degree at least $u + 2$. There is a long exact sequence in homology

$$\cdots \rightarrow H^j(Q_u) \xrightarrow{H(\varphi)} H^j(L_u \oplus P_u) \rightarrow H^j(L_{u+1}) \xrightarrow{\delta} H^j_{u+1}(Q_u) \rightarrow \cdots.$$  

In Adams degrees less than $u + 1$, $Q_u$ and $P_u$ are zero, so $L_u$ and $L_{u+1}$ are isomorphic. In Adams degree $u + 1$, $H^*(Q_u)$ maps isomorphically to $K_u$, a vector subspace of $H(L_u \oplus P_u)$, so the boundary map $δ$ is zero, and the above long exact sequence becomes short exact. Indeed, there is a commutative diagram, where the rows are short exact:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^i(Q_u)_{\leq u+1} & \rightarrow & H^i(L_u \oplus P_u)_{\leq u+1} & \rightarrow & H^i(L_{u+1})_{\leq u+1} & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & H^i(M)_{\leq u+1} & \rightarrow & H^i(M)_{\leq u+1} & \rightarrow & 0
\end{array}
$$

The snake lemma immediately shows that $H(ε_{u+1})$ has zero kernel and zero cokernel in Adams degree $\leq u + 1$, as desired. This verifies property (4) for $L_{u+1}$.

With property (4), and the fact that $L_i/L_{i-1}$ has a basis of cycles in Adams degree $i$, (1) and (2) are easy to see. To see (3) we use induction on $u$. It follows from the construction and induction that $d_{L_u}(L_u) \subset mL_u + L_{u-1}$.

Since the semibasis of $L_{u-1}$ has Adams degree no more than $u - 1$ and the semibasis of $P_u \oplus Q_u$ has Adams degree $u$, we see that $d_{L_u}(L_u) \subset mL_u + mL_{u-1} = mL_u$.

Let $L$ be the direct limit $\lim L_u$. Then $L$ is semifree and there is a map $ϕ : L \rightarrow M$ such that the kernel and cokernel of $H(ϕ)$ are zero. Such an $L$ is a semifree resolution of $M$. Property (3) implies that $L$ is minimal.

If the hypotheses of part (2) are satisfied, then the construction of $L_u$ shows that (5) holds. Since $L_u^j = (L_u)^j$ for $u \gg 0$, $L$ is also locally finite (respectively, locally finite with respect to the Adams grading).

We are often interested in the complex $R\text{Hom}_A(k, k)$ or in its homology, namely, the Ext-algebra, $\text{Ext}^*_A(k, k)$. As noted in Section 4.2, to compute this, we replace $k$ by a $K$-projective resolution $P$, and then $R\text{Hom}_A(k, k) = \text{Hom}_A(P, k)$. Since semifree implies $K$-projective, we can use a minimal semifree resolution, as in the theorem. The construction of $L$ gives the following: for each $u$, there is a short exact sequence of DG $A$-modules

$$0 \rightarrow L_{u-1} \rightarrow L_u \rightarrow L_u/L_{u-1} \rightarrow 0,$$

where $L_u/L_{u-1}$ is a free DG $A$-module, and this leads to a short exact sequence

$$0 \rightarrow \text{Hom}_A(L_u/L_{u-1}, k) \rightarrow \text{Hom}_A(L_u, k) \rightarrow \text{Hom}_A(L_{u-1}, k) \rightarrow 0.$$  

We see that $\text{Hom}_A(L_u/L_{u-1}, k)_i = 0$ when $i < u$, and therefore

$$\text{Hom}_A(L_u, k)_u \cong \text{Hom}_A(L_u/L_{u-1}, k)_u \cong (B^5)_{-u},$$
where $B$ is a graded basis for the free DG module $L_u/L_{u-1}$. Note that $B$ is concentrated in degrees $(\ast, u)$, so its graded dual $B^\sharp$ is in degrees $(\ast, -u)$. Again by the short exact sequence, by induction on $u$, $\text{Hom}_A(L_{u-1}, k)_i = 0$ when $i \geq u$, so we see that

$$\text{RHom}_A(k, k)_u \cong \text{Hom}_A(L, k)_u \cong \text{Hom}_A(L_u, k)_u \cong (B^\sharp)_{-u}.$$ 

Furthermore, since $L_u/L_{u-1}$ is free on a basis of cycles, or alternatively because the resolution $L$ is minimal, we see that

$$\text{Ext}_A(k, k)_u \cong (B^\sharp)_{-u}.$$ 

This leads to the following corollary; see Corollary 5.6 for a related result.

**Corollary 6.2.** Let $A$ be a DG algebra which is connected graded, either positively or negatively, in the second grading, and let $E = E(A)$ be its Koszul dual.

(a) Then $HE = \text{Ext}_A^*(k, k)$ is finite-dimensional if and only if $k_A$ is small in $D(A)$.

(b) If $HE$ is finite-dimensional (or if $k_A$ is small in $D(A)$), then $HA$ is Adams connected.

**Proof.** (a) If $HE = \text{Ext}_A^*(k, k)$ is finite-dimensional, then by the above computation, the minimal semifree resolution $L$ of $k$ is built from finitely many free pieces, and so $L$ is a perfect complex: it is in $\text{thick}_A(k)$. Therefore $k_A$ is small in $D(A)$.

Conversely, if $k_A$ is small, then it is isomorphic in $D(A)$ to a perfect complex, and we claim that if $X$ is a perfect complex, then $H \text{RHom}_A(X, k)$ is finite-dimensional as a vector space: this is true if $X = A$, and therefore it is true for every object in the thick subcategory generated by $A$. Therefore $k_A$ small implies that $HE$ is finite-dimensional.

(b) Now suppose that $HE$ is finite-dimensional. Without loss of generality, suppose that $A$ is positively graded connected in the second grading. We claim that for each $i$, $(HA)_i^*$ is finite-dimensional.

Note that in the construction of the minimal semifree resolution for $k$, the first term $L_0$ is equal to $A$, and the map $L_0 \to k$ is the augmentation. Consider the short exact sequence

$$0 \to L_{u-1} \to L_u \to L_u/L_{u-1} \to 0,$$

for $u \geq 1$. Since $L_u/L_{u-1}$ is free on finitely many classes in Adams degree $u$, then in Adams degree $i$, $H(L_u/L_{u-1})_i$ is isomorphic to a finite sum of copies of $HA_{i-u}$. Therefore if $i \leq u$, then this is finite-dimensional. Therefore when $i \leq u$, $H(L_u/L_{u-1})_i$ is finite-dimensional if and only if $H(L_u)_i$ is. For $i$ fixed and $u$ sufficiently large, $H(L_u)_i$ stabilizes and gives $H(L)_i$. But $HL \cong k$, since $L$ is a semifree resolution of $k$. Thus $H(L_0)_i = (HA)_i$ is finite-dimensional for each $i$, as desired. □
7. Towards classical Koszul duality

In this section we recover the classical version of the Koszul duality given by Beilinson–Ginzburg–Soergel [7]. First we give some useful results about $A_\infty$-algebras with finite-dimensional cohomology, and then we use these to recover classical Koszul duality, in Theorem 7.5.

7.1. Finite-dimensional $A_\infty$-algebras. Let $A$ be an $A_\infty$-algebra. Let $D^\infty_{fd}(A)$ denote the thick subcategory of $D^\infty(A)$ generated by all $A_\infty$-modules $M$ over $A$ such that $HM$ is finite-dimensional.

Lemma 7.1. Let $A$ be a strongly locally finite $A_\infty$-algebra. For parts (b) and (c), assume that $A$ is Adams connected and that $HA$ is finite-dimensional.

(a) $\text{thick}_{A}^\infty(k) = D^\infty_{fd}(A)$.
(b) $A$ is quasi-isomorphic to a finite-dimensional Adams connected DG algebra.
(c) $\text{thick}_{A}^\infty(k) = D^\infty_{fd}(A) \supseteq D^\infty_{per}(A)$ and $\text{loc}_{A}^\infty(k) = D^\infty(A)$.

Proof. (a) Clearly $\text{thick}_{A}^\infty(k) \subset D^\infty_{fd}(A)$. To show the converse, we may replace $A$ by the DG algebra $UA \cong E(E(A))$ (Proposition 1.14). Since $A$ is strongly locally finite, so is $E(E(A))$, by Lemma 2.2. So we may assume that $A$ is a strongly locally finite DG algebra. In this case every 1-dimensional right DG $A$-module $M$ is isomorphic to a shift of the trivial module $k$. As a consequence, $M \in \text{thick}_{A}^\infty(k)$. Induction shows that $M \in \text{thick}_{A}^\infty(k)$ if $M$ is finite-dimensional. If $M$ is a right DG $A$-module with $HM$ being finite-dimensional, then the minimal semifree resolution $L$ of $M$ is Adams locally finite by Theorem 6.1. Thus $M$ is quasi-isomorphic to a finite-dimensional right DG $A$-module by truncation: replace $L$ by $\bigoplus_{N \leq s \leq N} L_s^*$ for $N$ sufficiently large. Therefore $M$ is in $\text{thick}_{A}^\infty(k)$. This shows that $\text{thick}_{A}^\infty(k) = D^\infty_{fd}(A)$.

(b) By Theorem 2.4, $A$ is quasi-isomorphic to $B := E(E(A))$. Since $A$ is Adams connected, so is $B$. Since $HA \cong HB$, $H(B_{\geq n}) = 0$ for some $n$. Hence $B$ is quasi-isomorphic to $C := B/B_{\geq n}$. Therefore $A$ is quasi-isomorphic to the Adams connected finite-dimensional DG algebra $C$.

(c) By part (b) we may assume that $A$ is finite-dimensional, which implies that $A$ is in $D^\infty_{fd}(A) = \text{thick}_{A}^\infty(k) \subseteq \text{loc}_{A}^\infty(k)$. Therefore

$$\text{loc}_{A}^\infty(A) = D^\infty(A) \subseteq \text{loc}_{A}^\infty(k) \subseteq D^\infty(A).$$

This proves the last statement. \hfill \qed

Corollary 7.2. Let $A$ be an augmented $A_\infty$-algebra and let $E = E(A)$.

(a) Suppose that $A$ is strongly locally finite. $k_A$ is small in $D^\infty(A)$ if and only if $HE$ is finite-dimensional.

(b) Suppose that $A$ is Adams connected. If $k_A$ is small in $D^\infty(A)$ (or if $HE$ is finite-dimensional), then $D^\infty_{per}(A) \cong D^\infty_{fd}(E)$. 

Proof. (a) If $k_A$ is small, then by Corollary 5.9, $HE$ is finite-dimensional. Conversely, suppose that $HE$ is finite-dimensional. By Proposition 1.14, $A$ is quasi-isomorphic to the augmented DG algebra $UA \cong E(E(A))$, so by Proposition 3.1, $k_A$ is small in $D^\infty(A)$ if and only if $k_{UA}$ is small in $D(UA)$. According to Corollary 6.2, $k_{UA}$ is small if and only if $HE(UA)$ is finite-dimensional. Since $A$ is strongly locally finite, so is $E$, and we have a quasi-isomorphism $E(E(E)) \cong E$. This means that $HE(UA) \cong HE(A) = HE$.

(b) This follows from Theorem 5.7(b) (switching $A$ and $E$) and Lemma 7.1(a) (for $E$). (Since $A$ is Adams connected, then by Lemma 2.2, $E$ is Adams connected and strongly locally finite, so these results hold for $E$.)

Part (b) is Theorem B from the introduction.

7.2. Twisting the grading. To recover the classical derived equivalences induced by Koszul duality, we need to twist the grading of $E(A)$. Recall that if $A$ is Adams connected and is concentrated in degrees $\{0\} \times \mathbb{N}$, then $E(A)$ is in degrees $\{(n,-n)|n \geq 0\}$. The aim of this section is to change the grading of $E(A)$, to put it in degrees $\{0\} \times \mathbb{N}$.

Let $A$ be a DG algebra with zero differential (there are some problems if $d \neq 0$). Let $\overline{A}$ be the DG algebra with zero differential such that:

(a) $\overline{A} = A$ as ungraded associative algebras.

(b) $\overline{A}_j = A^i_{-j}$ (so $A^i_{-j} = \overline{A}^{i+j}$).

Note that the sign before $j$ is not essential; for example, we can define a different $\overline{A}$ by $\overline{A}_j = A^i_{-j}$ or $\overline{A}_j = A^i_{-j}$ or $\overline{A}_j = A^i_{i+j}$. Since $A$ is a DG algebra with zero differential, so is $\overline{A}$. If $A$ is Koszul, then $A^i = HE(A) = HE(A)$ which lives in degrees $\{0\} \times \mathbb{Z}$.

Remark 7.3. If $A$ is a DG algebra with nonzero differential $d$, then we don’t know how to make $\overline{A}$ into a DG algebra naturally. That is, we don’t see a good way of defining a differential for $\overline{A}$.

For any right DG $A$-module $M$, we define a corresponding right DG $\overline{A}$-module $\overline{M}$ by:

(a) $\overline{M} = M$ as ungraded right $A$-modules.

(b) $\overline{M}_j = M^i_{i+j}$.

One can easily check that $\overline{M}$ with the differential of $M$ is a right DG $\overline{A}$-module.

The following is routine.

Lemma 7.4. The assignment $M \mapsto \overline{M}$ defines an equivalence between $\text{Mod} \overline{A}$ and $\text{Mod} \overline{A}$ which induces a triangulated equivalence between $D(A)$ and $D(\overline{A})$. If further $A$ is finite-dimensional, then $D_{fd}(A) \cong D_{fd}(\overline{A})$.

We now reprove [7, Theorem 2.12.6] (in a slightly more general setting). Let $D^\infty_{\text{fg}}(A)$ denote the thick subcategory of $D^\infty(A)$ generated by all $A_\infty$-modules $M$ over $A$ such that $HM$ is finitely generated over $HA$. If $A$ is a
DG algebra, $D_{fg}(A)$ is defined in a similar way. Note that $D_{fg}^\infty(A) = D_{fd}^\infty(A)$ if and only if $HA$ is finite-dimensional.

**Theorem 7.5.** Suppose $A$ is a connected graded finite-dimensional Koszul algebra. Let $A^! = HE(A)$.

(a) $D(A) \cong \text{loc}_A(k)$.

(b) If $A^!$ is right noetherian, then $D_{fg}(A) = D_{fd}(A) \cong D_{per}(A^!) = D_{fg}(A^!)$.

**Proof.** Note that $A^!$ is quasi-isomorphic to $E(A)$. It follows from Lemma 7.4 that we can replace $A^!$ by $E := E(A)$.

(a) This follows from Theorem 5.4(b) (switching $A$ and $E$).

(b) By Corollary 7.2(b) (again switching $A$ and $E$), $D_{per}(E) \cong D_{fd}(A)$. Since $A$ is finite-dimensional, $D_{fd}(A) = D_{fg}(A)$. Note that $A^!$ is concentrated in degrees $\{0\} \times \mathbb{N}$. Since $A^!$ is right noetherian and has finite global dimension (because $A$ is finite-dimensional Koszul), it is clear that $D_{per}(A^!) = D_{fg}(A^!)$. \hfill \Box

8. Some examples

**Example 8.1.** Let $\Lambda = \Lambda(x_1, x_2)$ with $\text{deg} x_i = (0, i)$. As noted in Example 2.8, this is not a classical Koszul algebra; nonetheless, it satisfies many of the same derived equivalences as Koszul algebras. Let $\Lambda^!$ denote the Ext algebra $\text{Ext}_\Lambda^*(k, k) \cong k[y_1, y_2]$, where $\text{deg} y_i = (1, -i)$; this is an $A_\infty$-algebra with no higher multiplications. Then $E(\Lambda)$ is quasi-isomorphic to the associative algebra $\Lambda^!$, so there is a triangulated equivalence $D^\infty(E(\Lambda)) \cong D(\Lambda^!)$. Similarly, we have $D^\infty(\Lambda) \cong D(\Lambda)$. Since the homology of $E(\Lambda^!) \cong E(E(\Lambda))$ is isomorphic to $\Lambda$, which is finite-dimensional, the trivial module $k^!_{\Lambda^!}$ is small in $D(\Lambda^!)$. So Theorems 5.7 and 5.8 give triangulated equivalences

\[
\begin{align*}
\text{thick}_\Lambda(k) & \cong D_{per}(\Lambda^!), & \text{thick}_\Lambda(k, \Lambda)^{op} & \cong \text{thick}_{\Lambda^!}(\Lambda^!, k), \\
\text{thick}_{\Lambda^!}(k) & \cong D_{per}(\Lambda), & D_{per}(\Lambda)^{op} & \cong \text{thick}_\Lambda(k), \\
\text{loc}_{\Lambda^!}(k) & \cong D(\Lambda).
\end{align*}
\]

Compare to the classical Koszul equivalences of Theorem 7.5: part (b) of that theorem is the first of these equivalences, while part (a) of the theorem is the last of these.

Slightly more generally, the results of Theorem 7.5 hold for exterior algebras on finitely many generators, as long as they are graded so as to be Adams connected: $\Lambda(x_1, \ldots, x_n)$, graded by setting $\text{deg} x_i = (a_i, b_i)$ with each $b_i$ positive, or each $b_i$ negative.

**Example 8.2.** We fix an integer $p \geq 3$ and define two $A_\infty$-algebras, $B(0)$ and $B(p)$, each with $m_1 = 0$. As associative algebras, they are both isomorphic to $\Lambda(y) \otimes k[z]$ with $\text{deg} y = (1, -1)$ and $\text{deg} z = (2, -p)$. The algebra $B(0)$ has no higher multiplications, while $B(p)$ has a single nonzero higher multiplication $m_p$. This map $m_p$ satisfies $m_p(y^{\otimes p}) = z$; more generally,
$m_p(a_1 \otimes \cdots \otimes a_p)$ is zero unless each $a_i$ has the form $a_i = yz^{j_i}$ for some $j_i \geq 0$, and

$$m_p(yz^{j_1} \otimes \cdots \otimes yz^{j_p}) = z^{1+\sum{j_i}}.$$

See [24, Example 3.5] for more on $B(p)$.

We claim that $k_{B(0)}$ is not small in $D^\infty(B(0)) \cong D(B(0))$, while $k_{B(p)}$ is small in $D^\infty(B(p))$. The Ext algebra $\text{Ext}^*_B(k, k)$ is isomorphic to the associative algebra $k[u] \otimes \Lambda(v)$ with $\deg u = (0,1)$ and $\deg v = (1,p)$. In particular, this algebra is not finite-dimensional, so by Corollary 7.2, $k_{B(0)}$ is not small in $D(B(0))$. In contrast, by [24, Proposition 12.6], the Koszul dual of $B(p)$ is $A_\infty$-isomorphic to the associative algebra $A(p) = k[x]/(xp)$, where $\deg x = (0,1)$. Since $A(p)$ is finite-dimensional, $k_{B(p)}$ is small in $D^\infty(B(p))$. This verifies the claim.

**Part III. Applications in ring theory**

### 9. The Artin–Schelter condition

In this section we prove Corollaries D, E and F. We start by discussing Artin–Schelter regularity, for both associative algebras and $A_\infty$-algebras. The Eilenberg–Moore spectral sequence is a useful tool for connecting results about modules over $HA$ to modules over $A$, if $A$ is a DG algebra or an $A_\infty$-algebra. Then we discuss Frobenius algebras and prove Corollary D, and we discuss dualizing complexes and prove Corollary E. At the end of the section, we prove Corollary F.

#### 9.1. Artin–Schelter regularity.

**Definition 9.1.** Let $R$ be a connected graded algebra.

(a) $R$ is called Gorenstein if $\text{injdim} \, R = \text{injdim} \, R < \infty$.

(b) $R$ is called Artin–Schelter Gorenstein if $R$ is Gorenstein of injective dimension $d$ and there is an integer $l$ such that

$$\text{Ext}^i_R(k, R) \cong \text{Ext}^i_{R^e}(k, R) \cong \begin{cases} 0 & i \neq d, \\ \Sigma^l(k) & i = d. \end{cases}$$

(c) $R$ is called Artin–Schelter regular if $R$ is Artin–Schelter Gorenstein and has global dimension $d$.

Artin–Schelter regular algebras have been used in many ways in noncommutative algebraic geometry.

Now we consider analogues for $A_\infty$-algebras. When $A$ is an unbounded $A_\infty$-algebra, there is no good definition of global or injective dimension, so we only consider a version of condition (b) in Definition 9.1.

**Definition 9.2.** Let $A$ be an augmented $A_\infty$-algebra and let $k$ be the trivial module.
(a) We say $A$ satisfies the right Artin–Schelter condition if there are integers $l$ and $d$ such that
\[
\text{Ext}^i_A(k, A) \cong \begin{cases} 
0 & i \neq d, \\
\Sigma^l(k) & i = d.
\end{cases}
\]
If further $k_A$ is small in $D^\infty(A)$, then $A$ is called right $A_\infty$-Artin–Schelter regular, or just right Artin–Schelter regular, if the context is clear.

(b) We say $A$ satisfies the left Artin–Schelter condition if there are integers $l$ and $d$ such that
\[
\text{Ext}^i_{A^{\text{op}}}(k, A) \cong \begin{cases} 
0 & i \neq d, \\
\Sigma^l(k) & i = d.
\end{cases}
\]
If further $Ak$ is small in $D^\infty(A^{\text{op}})$, then $A$ is called left $A_\infty$-Artin–Schelter regular (or just left Artin–Schelter regular).

(c) We say $A$ satisfies the Artin–Schelter condition if the conditions in parts (a) and (b) hold for the same pair of integers $(l, d)$. If further $k_A$ is small in $D^\infty(A)$, then $A$ is called $A_\infty$-Artin–Schelter regular (or just Artin–Schelter regular).

(d) Finally, if $A$ is $A_\infty$-Artin–Schelter regular, we say that $A$ is noetherian if $D^\infty_{fg}(A) = D^\infty_{\text{per}}(A)$.

Suppose that $R$ is a connected graded algebra. If $R$ is Artin–Schelter regular as an associative algebra, then $R$ is $A_\infty$-Artin–Schelter regular. Conversely, if $R$ is $A_\infty$-Artin–Schelter regular, then one can use Corollary 6.2 to show that $R$ is Artin–Schelter regular.

Also, if $R$ is a connected graded algebra, then the cohomology of every object in $D_{fg}(R)$ is bounded (with respect to the homological grading), and we see that $D_{fg}(R) = D_{\text{per}}(R)$ if and only if $R$ is noetherian of finite global dimension; this motivates our definition of “noetherian” for $A_\infty$-Artin–Schelter regular algebras.

It is easy to see that $A$ satisfies the left $A_\infty$-Artin–Schelter condition if and only if $A^{\text{op}}$ satisfies the right $A_\infty$-Artin–Schelter condition. We conjecture that the left $A_\infty$-Artin–Schelter condition is equivalent to the right $A_\infty$-Artin–Schelter condition. We verify this, with some connectedness and finiteness assumptions, in Theorems 9.8 and 9.11.

**Proposition 9.3.** Let $A$ be an augmented $A_\infty$-algebra and let $E = E(A)$ be the Koszul dual of $A$.

(a) Assume that $A$ is strongly locally finite. Then $A$ satisfies the right Artin–Schelter condition if and only if $E$ satisfies the right Artin–Schelter condition.

(b) Assume that $E$ is strongly locally finite. Then $A$ satisfies the left Artin–Schelter condition if and only if $E$ satisfies the right Artin–Schelter condition.
Proof. (a) By Corollary 5.10 \( A \) satisfies the right Artin–Schelter condition if and only if \( E \) satisfies the left Artin–Schelter condition.

(b) Switch \( A \) and \( E \) and use the fact that \( A \to E(E(A)) \) is a quasi-isomorphism: the assertion follows from part (a).

\[\square\]

9.2. The Eilenberg–Moore spectral sequence. In this subsection we recall the Eilenberg–Moore spectral sequence [22, Theorem III.4.7] and [13, p. 280]. This helps to translate homological results for modules over \( HA \) to similar results for modules over \( A \).

Suppose that \( A \) is a DG algebra and that \( M \) and \( N \) are right DG \( A \)-modules. Because of the \( \mathbb{Z} \times \mathbb{Z} \)-grading on \( A \), \( M \), and \( N \), \( \text{Ext}_{HA}^*(HM,HN) \) is \( \mathbb{Z}^3 \)-graded, and we incorporate the gradings into the notation as follows:

\[
\text{Ext}_{HA}^p(HM,HN)_s^q.
\]

On the other hand, \( \text{Ext}_{A}^*(M,N) \) is \( \mathbb{Z}^2 \)-graded, since it is defined to be the homology of \( \text{RHom}_A(M,N) \). That is,

\[
\text{Ext}_{A}^i(M,N)_j := D(A)(M,S^i \Sigma^j N) \approx H(\text{RHom}_A(M,S^i \Sigma^j N)).
\]

Because of this, each \( E_r \)-page of the Eilenberg–Moore spectral sequence is \( \mathbb{Z}^3 \)-graded, while the abutment is \( \mathbb{Z}^2 \)-graded.

**Theorem 9.4** ([22, Theorem III.4.7]). Let \( A \) be a DG algebra, and let \( M \) and \( N \) be right DG \( A \)-modules. Then there is a spectral sequence of the form

\[
(E_2)^p,q_s \Rightarrow \text{Ext}_{A}^{p+q}(M,N)_s,
\]

natural in \( M \) and \( N \). All differentials preserve the lower (Adams) grading \( s \).

This is a spectral sequence of cohomological type, with differential in the \( E_r \)-page as follows:

\[
d_r : (E_r)^p,q_s \to (E_r)^{p+r,q-r+1}_s.
\]

Ignoring the Adams grading, the \( E_2 \)-term is concentrated in the right half-plane (i.e., \( p \geq 0 \)), and it converges strongly if for each \( (p,q) \), \( d_r : (E^p,q_s) \to (E^{p+r,q-r+1}_r)_s \) is nonzero for only finitely many values of \( r \).

There is also a Tor version of this spectral sequence, which we do not use. See [22, Theorem III.4.7] and [13, p. 280] for more details.

Note that the above theorem also holds for \( A_\infty \)-algebras; see [22, Theorem V.7.3]. Another way of obtaining an \( A_\infty \)-version of this spectral sequence is to use the derived equivalence between \( D^\infty(A) \) and \( D(UA) \).

**Corollary 9.5.** Let \( A \) be an \( A_\infty \)-algebra. If \( HA \) satisfies the left (respectively, right) Artin–Schelter condition, then so does \( A \).

**Proof.** We may assume that \( A \) is a DG algebra. Let \( M = k \) and \( N = A \). Note that \( HK = k \). Since \( HA \) satisfies the left Artin–Schelter condition, \( \bigoplus_{p,q} \text{Ext}_{HA}^p(HM,HN)_s^q \) is 1-dimensional. By Theorem 9.4, \( \bigoplus_{n} \text{Ext}_{A}^n(k,A) \) is 1-dimensional. Therefore \( A \) satisfies the left Artin–Schelter condition. \[\square\]
One naive question is if the converse of Corollary 9.5 holds.

9.3. Frobenius $A_\infty$-algebras. In this subsection we define Frobenius DG algebras and Frobenius $A_\infty$-algebras and then prove Corollary D.

Definition 9.6. An augmented DG algebra $A$ is called left Frobenius (respectively, right Frobenius) if:

(a) $HA$ is finite-dimensional.
(b) There is a quasi-isomorphism of left (respectively, right) DG $A$-modules $\alpha : S^l \Sigma^d(A) \to A^\sharp$ for some integers $l$ and $d$.

An augmented DG algebra $A$ is called Frobenius if it is both left and right Frobenius.

Lemma 9.7. Suppose $A$ is a DG algebra such that there is a quasi-isomorphism of left DG $A$-modules $\alpha : S^l \Sigma^d(A) \to A^\sharp$ for some integers $l$ and $d$.

(a) There is a quasi-isomorphism of right DG $A$-modules $\beta : S^l \Sigma^d(A) \to A^\sharp$ for the same integers $l$ and $d$.
(b) If $A$ is connected graded with respect to some grading which is compatible with the $\mathbb{Z}^2$-grading — see below — then $HA$ is finite-dimensional.
(c) If $HA$ is finite-dimensional, then $HA$ is Frobenius as an associative algebra.
(d) $A$ satisfies the left Artin–Schelter condition and $\text{RHom}_A(k, A)$ is quasi-isomorphic to $S^{-l} \Sigma^{-d}(k)$.

The compatibility requirement for the grading in part (b) means that there should be numbers $a$ and $b$ so that the $n$th graded piece is equal to $\bigoplus_{ai+bj=n} A_i^j$.

Proof. (a) Let $\beta = S^l \Sigma^d(\alpha^\sharp)$.
(b) Since $S^l \Sigma^d(A) \to A^\sharp$ is an quasi-isomorphism,
$$(HA)^{n-l}_{m-d} \cong (HA^\sharp)^{n}_{m} \cong ((HA)_m^n)^\sharp$$
for all $n, m$. This implies that $HA$ is locally finite. If $A$ is connected graded with respect to some compatible grading, then so is $HA$. Then the above formula implies that $HA$ is finite-dimensional.
(c) The quasi-isomorphism $\alpha$ gives rise to an isomorphism $H(\alpha) : S^l \Sigma^d(HA) \to (HA)^\sharp$.
If $HA$ is finite-dimensional, then $HA$ is Frobenius.
(d) Since $A \to S^{-l} \Sigma^{-d}A^\sharp$ is a $K$-injective resolution of $A$, we can compute $\text{RHom}_A(k, A)$ by $\text{Hom}_A(k, S^{-l} \Sigma^{-d}(A^\sharp))$, which is $S^{-l} \Sigma^{-d}(k)$.

By Lemma 9.7 above, $A$ is left Frobenius if and only if it is right Frobenius. So we can omit both “left” and “right” before Frobenius. It is therefore easy to see that $A$ is Frobenius if and only if $A^{\text{op}}$ is. We show that the Frobenius property is a homological property.
Theorem 9.8. Let $A$ be an Adams connected DG algebra such that $HA$ is finite-dimensional. Then the following are equivalent.

(a) $A$ is Frobenius.
(b) $HA$ is Frobenius.
(c) $A$ satisfies the left Artin–Schelter condition.
(d) $A$ satisfies the right Artin–Schelter condition.

Proof. By Lemma 7.1, we may assume that $A$ is finite-dimensional.

(a) $\Rightarrow$ (b): Use Lemma 9.7(c).

(b) $\Rightarrow$ (a): Since $HA$ is Frobenius, there is an isomorphism of right $HA$-modules $f : S^d \Sigma^d(HA) \to HA^2$. Pick $x \in ZA^2$ so that the class of $x$ generates a submodule of $HA^2$ that is isomorphic to $S^d \Sigma^d(HA)$. Hence the map $a \to xa$ is a quasi-isomorphism $A \to A^2$.

(a) $\Rightarrow$ (d): By vector space duality,

$$R\text{Hom}_{A^\op}(k, A) \cong R\text{Hom}_A(A^\#, k) \cong R\text{Hom}_A(S^d \Sigma^d(A), k) = S^{-l}\Sigma^{-d}(k).$$

Hence $A$ satisfies the right Artin–Schelter condition.

(d) $\Rightarrow$ (a): Suppose $R\text{Hom}_{A^\op}(k, A) \cong S^{-l}\Sigma^{-d}(k)$ by the right Artin–Schelter condition. Since $HA$ is locally finite, by vector space duality, we obtain that $R\text{Hom}_A(A^\#, k)$ is quasi-isomorphic to $S^{-l}\Sigma^{-d}(k)$. By Theorem 6.1 $A^2$ has a minimal semifree resolution, say $P \to A^2$. Since $P$ is minimal, it has a semifree basis equal to $(\bigoplus \text{Ext}^i_A(A^\#, k))^\# = \Sigma^d(k)$. Hence $P \cong S^d \Sigma^d(A)$ for some $l$ and $d$. Thus $S^d \Sigma^d(A) \to A^2$ is a quasi-isomorphism and $A$ is left Frobenius. By Lemma 9.7, $A$ is Frobenius.

Thus we have proved that (a), (b) and (d) are equivalent. By left-right symmetry, (a), (b) and (c) are equivalent. $\square$

We obtain an immediate consequence.

Corollary 9.9. Let $f : A \to B$ be a quasi-isomorphism of Adams connected DG algebras. Assume that $HA \cong HB$ is finite-dimensional. Then $A$ is Frobenius if and only if $B$ is.

Since finite-dimensional Hopf algebras are Frobenius, every finite-dimensional DG Hopf algebra is Frobenius.

Suggested by the DG case, we make the following definition.

Definition 9.10. An $A_\infty$-algebra $A$ is called Frobenius if $HA$ is finite-dimensional and there is a quasi-isomorphism of right $A_\infty$-modules $S^d \Sigma^d(A) \to A^2$ for some $l$ and $d$.

The following is similar to Theorem 9.8 and the proof is omitted.

Theorem 9.11. Let $A$ be an Adams connected $A_\infty$-algebra such that $HA$ is finite-dimensional. The following are equivalent.

(a) $A$ is Frobenius.
(b) $HA$ is Frobenius.
(c) $A$ satisfies the left Artin–Schelter condition.

(d) $A$ satisfies the right Artin–Schelter condition.

Now we are ready to prove Corollary D from the introduction. We restate it for the reader’s convenience.

**Corollary D.** Let $R$ be a connected graded algebra. Then $R$ is Artin–Schelter regular if and only if the Ext-algebra $\bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_R(k_R, k_R)$ is Frobenius.

**Proof.** Note that $R$ is right Artin–Schelter regular if and only if $k_R$ is small and $R$ satisfies the right $A_\infty$-Artin–Schelter condition [35, Proposition 3.1]. Now we assume, temporarily, that $R$ is Adams connected; by Lemma 2.2, this means that $\mathcal{E}(R)$ is, as well. Then the above conditions are in turn equivalent to: $HE(R) = \bigoplus_i \text{Ext}^i(k_R, k_R)$ is finite-dimensional and $E(R)$ satisfies the left Artin–Schelter condition. By Theorem 9.11, this is equivalent to $HE(R)$ being Frobenius.

Now, we use Corollary 6.2 to justify the Adams connected assumption: if $R$ is Artin–Schelter regular, then $k_R$ is small, which implies that $R = HR$ is Adams connected. Conversely, if $HE(R)$ is Frobenius, then it is finite-dimensional, which also implies that $R$ is Adams connected. \hfill \Box

This proof in fact shows that (for associative algebras) left Artin–Schelter regularity is equivalent to right Artin–Schelter regularity.

**9.4. Dualizing complexes and the Gorenstein property.** The balanced dualizing complex over a graded ring $B$ was introduced by Yekutieli [38]. We refer to [38] for the definition and basic properties. Various noetherian graded rings have balanced dualizing complexes; see [36, 39].

**Lemma 9.12.** Suppose $R$ is a noetherian connected graded ring with a balanced dualizing complex. Then $R$ satisfies the right Artin–Schelter condition if and only if $R$ is Gorenstein. In this case, $R$ will be Artin–Schelter Gorenstein.

**Proof.** Let $B$ be the balanced dualizing complex over $R$. Then the functor $F := \text{RHom}_R(-, B)$ induces an equivalence $\mathcal{D}_{\text{lg}}(R) \cong \mathcal{D}_{\text{lg}}(R^{\text{op}})^{\text{op}}$ and satisfies $F(k_R) = Rk$. By the right Artin–Schelter condition,

$$\text{RHom}_R(k, R) \cong S^l \Sigma^d(k)$$

for some $l$ and $d$. Applying the duality functor $F$, we have

$$\text{RHom}_{R^{\text{op}}}(B, k) = \text{RHom}_{R^{\text{op}}}(F(R), F(k)) \cong \text{RHom}_R(k, R) \cong S^l \Sigma^d(k).$$

Therefore $RB$ is quasi-isomorphic to $S^{-l} \Sigma^{-d}(R)$. Since $RB$ has finite injective dimension by definition, $R$ has finite injective dimension. Also it follows from $RB \cong S^{-l} \Sigma^{-d}(R)$ that $B_R \cong S^{-l} \Sigma^{-d}(R)$ by the fact that $R^{\text{op}} = \text{RHom}_{R^{\text{op}}}(B, B)$. So since $B_R$ has finite injective dimension, so does $R_R$. 


For the converse, note that both $R \text{RHom}_R(k, R)$ and $R \text{RHom}_{R^{op}}(k, R)$ are finite-dimensional since the existence of $B$ implies that $R$ satisfies the $\chi$-condition. The assertion follows from the proof of [40, Lemma 1.1]. □

Now we restate and prove Corollary E.

**Corollary E.** Let $R$ be a Koszul algebra and let $R^{!}$ be the Koszul dual of $R$. If $R$ and $R^{!}$ are both noetherian having balanced dualizing complexes, then $R$ is Gorenstein if and only if $R^{!}$ is.

**Proof.** By Lemma 9.12 the Artin–Schelter condition is equivalent to the Gorenstein property. The assertion follows from Proposition 9.3. □

We say a connected graded algebra $A$ has enough normal elements if every nonsimple graded prime factor ring $A/P$ contains a homogeneous normal element of positive degree. A noetherian graded ring satisfying a polynomial identity has enough normal elements.

**Corollary 9.13.** Let $R$ be a Koszul algebra and let $R^{!}$ the Koszul dual of $R$. If $R$ and $R^{!}$ are both noetherian having enough normal elements, then $R$ is (Artin–Schelter) Gorenstein if and only if $R^{!}$ is.

**Proof.** By [39, Theorem 5.13], $R$ and $R^{!}$ have balanced dualizing complexes. By [40, Proposition 2.3(2)], under the hypothesis, the Artin–Schelter Gorenstein property is equivalent to the Gorenstein property. The assertion follows from Corollary E. □

Let $R^{e} = R \otimes R^{op}$. Following the work of Van den Bergh [36], Ginzburg [15] and Etingof–Ginzburg [10], an associative algebra $R$ is called twisted Calabi–Yau if

$$\text{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} \phi R^1 & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$$

for some $d$ (note that we do not require $R$ to have finite Hochschild dimension). If the above equation holds for $\phi = \text{Id}_R$, then $A$ is called Calabi–Yau. If $R$ is connected graded, then $\phi R^1$ should be replaced by $\Sigma^l(\phi R^1)$ for some integer $l$. It follows from Van den Bergh’s result [36, Proposition 8.2] that if $R$ is connected graded noetherian and Artin–Schelter Gorenstein, then $R$ is twisted Calabi–Yau. It is easy to see that if $R$ has finite global dimension and $R^e$ is noetherian, then Artin–Schelter regularity is equivalent to the twisted Calabi–Yau property. It is conjectured that the Artin–Schelter Gorenstein property is equivalent to the twisted Calabi–Yau property for all connected graded noetherian rings.

We end this section with a proof of Corollary F.

**Corollary F.** Let $A$ be an Adams connected commutative differential graded algebra such that $R \text{RHom}_A(k, A)$ is not quasi-isomorphic to zero. If the Ext-algebra $\bigoplus_{i \in \mathbb{Z}} \text{Ext}^i_A(k_A, k_A)$ is noetherian, then $A$ satisfies the Artin–Schelter condition.
Proof. Since $A$ is commutative, its Ext algebra $H = HE(A) = \text{Ext}_A^*(k, k)$ is a graded Hopf algebra which is graded cocommutative [12, p. 545]. By hypothesis, $H$ is noetherian, so it satisfies [12, (1.1)]. Since $\text{RHom}_A(k, A) \neq 0$, Corollary 5.10 implies that $\text{RHom}_{E^{\text{op}}}(k, E) \neq 0$ where $E = E(A)$. By Theorem 9.4 $\text{RHom}_{H^{\text{op}}}(k, H) \neq 0$. Since $H^{\text{op}} \cong H$, we have $\text{RHom}_H(k, H) \neq 0$ which says that $H$ has finite depth. By [12, Theorem C], the noetherian property of $H$ implies that $H$ is elliptic, and elliptic Hopf algebras are classified in [12, Theorem B]. It is well-known that the Hopf algebras in [12, Theorem B] are Artin–Schelter Gorenstein. By Corollary 9.5, $E(A)$ satisfies the Artin–Schelter condition, and therefore by Proposition 9.3, $A$ does as well. □

10. The BGG correspondence

The classical Bernstein–Gel’fand–Gel’fand (BGG) correspondence states that the derived category of coherent sheaves over $\mathbb{P}^n$ is equivalent to the stable derived category over the exterior algebra of $(n + 1)$-variables [8, Theorem 2]. Some generalizations of this were obtained by Baranovsky [5], He–Wu [16], Mori [27] and so on. In this section we prove a version of the BGG correspondence in the $A_\infty$-algebra setting, as a simple application of Koszul duality.

If $R$ is a right noetherian ring, then the stable bounded derived category over $R$, denoted by $D^b_{\text{fg}}(R)$, is defined to be the Verdier quotient $D^b_{\text{fg}}(R)/D^b_{\text{per}}(R)$. With $R$ concentrated in degrees $\{0\} \times \mathbb{Z}$, every complex in $D^b_{\text{fg}}(R)$ is bounded. When $R$ is a finite-dimensional Frobenius algebra, then the stable module category over $R$ is equivalent to the stable bounded derived category over $R$ [30].

Recall from Sections 7.1 and 7.2, respectively, that

$$D^\infty_{\text{fd}}(A) = \text{thick}^\infty_{A}(M \mid M \in \text{Mod} A, \dim_k HM < \infty),$$

$$D^\infty_{\text{fg}}(A) = \text{thick}^\infty_{A}(M \mid M \in \text{Mod} A, HM \text{ finitely generated over } HA).$$

If $HA$ is finite-dimensional, then by Lemma 7.1(a), $D^\infty_{\text{fd}}(A) = \text{thick}^\infty_{A}(k)$, which is also equal to $\text{thick}^\infty_{A}(k, A)$. Modelled on the definition of $D^b_{\text{fg}}(R)$, we define the stable derived category of an $A_\infty$-algebra $A$ to be

$$D^\infty_{\text{fg}}(A) = D^\infty_{\text{fg}}(A)/D^\infty_{\text{per}}(A)$$

where the right-hand side of the equation is a Verdier quotient. Boundedness of complexes does not make sense here since $A$ itself may not be bounded, but for a weakly Adams connected $A_\infty$-algebra we have the following variation. The small stable derived category of a weakly Adams connected $A_\infty$-algebra $A$ is defined to be

$$D^\infty_{\text{sm}}(A) = \text{thick}^\infty_{A}(k, A)/D^\infty_{\text{per}}(A)$$
where the right-hand side of the equation is a Verdier quotient. If $HA$ is finite-dimensional, then

$$D_{\text{sm}}^\infty(A) = D_{\text{fd}}^\infty(A)/D_{\text{per}}^\infty(A) = D_{\text{fg}}^\infty(A).$$

In general, $D_{\text{sm}}^\infty(A) \subset D_{\text{fg}}^\infty(A)$. It is arguable whether or not $D_{\text{sm}}^\infty(A)$ is a good definition. One reason we use the above definition is to make the BGG correspondence easy to prove.

It is easy to see that $D_{\text{sm}}^\infty(A) = 0$ if and only if $k_A$ is small (see Definition 4.8 and Lemma 4.9(b)). In this case we call $A$ regular. This is consistent with terminology for associative algebras: Orlov called the triangulated category $D_b(R)$ the derived category of the singularity of $R$ [28].

Given any connected graded ring $R$, we define the projective scheme over $R$ to be the quotient category

$$\text{Proj } R := \text{Mod } R/\text{Tor } R$$

where $\text{Tor } R$ is the Serre localizing subcategory generated by all finite-dimensional graded right $R$-modules [1]. When $R$ is right noetherian, we denote its noetherian subcategory by $\text{proj } R$. The bounded derived category of $\text{proj } R$ is $D^b(\text{proj } A)$ which is modelled by the derived category of coherence sheaves over a projective scheme. When $A$ is an $A_\infty$-algebra, we can define the derived category directly without using $\text{Proj } A$. The derived category of projective schemes over $A$ is defined to be

$$D^\infty(\text{Proj } A) = D^\infty(A)/\text{loc}^\infty_A(k);$$

the derived category of finite projective schemes over $A$ is defined to be

$$D^\infty(\text{proj } A) = D_{\text{fg}}^\infty(A)/\text{thick}_{\text{fg}}^\infty(k);$$

and the derived category of small projective schemes over $A$ is defined to be

$$D_{\text{sm}}^\infty(\text{proj } A) = \text{thick}_{\text{sm}}^\infty(k, A)/\text{thick}_{\text{sm}}^\infty(k).$$

If $A$ is right noetherian and regular (e.g., $A$ is a commutative polynomial ring), then $D^\infty(\text{proj } A) = D_{\text{sm}}^\infty(\text{proj } A)$ and this is equivalent to the derived category of the (noncommutative) projective scheme $\text{proj } A$ [1].

**Lemma 10.1.** Let $A$ be an $A_\infty$-algebra satisfying the Artin–Schelter condition. Then $D_{\text{sm}}^\infty(A)^{\text{op}} \cong D_{\text{sm}}^\infty(A^{\text{op}})$.

**Proof.** First of all we may assume $A$ is a DG algebra. Let $B$ be the $(A, A)$-bimodule $A$. Clearly $B$ is a balanced $(A, A)$-bimodule. The equivalences given in Proposition 4.10 are trivial, but Proposition 4.11 is not trivial. By Proposition 4.11(a), $A^B \cong B^B$ and $F^B(A_A) = A_A \in k^B$. By the Artin–Schelter condition, $F^B(k_A) = \text{RHom}_{A^{\text{op}}}(k_A, A) = S^l\Sigma^d(k)$ and $G^B(S^l\Sigma^d(k)) = S^{-l}\Sigma^{-d}(G^B(k)) = k$. So $k_A \in A^B$ and $F^B(A_A) = S^l\Sigma^d(k)$. This implies that $F^B$ induces an equivalence between $\text{thick}_{\text{op}}^\infty(k, A)^{\text{op}}$ and $\text{thick}_{\text{sm}}^\infty(k, A)$ which sends $A_A$ to $A_A$. The assertion follows from the definition of $D_{\text{sm}}^\infty(A)$. \qed
The next theorem is a version of the BGG correspondence.

**Theorem 10.2.** Let $A$ be a strongly locally finite $A_{\infty}$-algebra and let $E$ be its Koszul dual.

(a) There is an equivalence of triangulated categories

$$D_{\text{sm}}(\text{proj } A)^{\text{op}} \cong D_{\text{sm}}(E^{\text{op}}).$$

(b) If $A$ is Adams connected graded noetherian right Artin–Schelter regular, then there is an equivalence of triangulated categories

$$D^\infty(\text{proj } A) \cong D_{\text{fg}}^\infty(E).$$

**Proof.** (a) By Theorem 5.8(b) there is an equivalence of triangulated categories

$$\text{thick}_{\infty}^A(k,A)^{\text{op}} \cong \text{thick}_{E^{\text{op}}}^\infty(k,E)$$

which maps $k_A$ to $E$. Therefore we have

$$D_{\text{sm}}(\text{proj } A)^{\text{op}} = (\text{thick}_{\infty}^A(k,A)/\text{thick}_{A}^\infty(k))^{\text{op}}$$

$$\cong \text{thick}_{E^{\text{op}}}^\infty(k,E)/\text{thick}_{E^{\text{op}}}^\infty(E) = D_{\text{sm}}^\infty(E^{\text{op}}).$$

(b) Since $A$ is AS regular, Corollary D says that $E$ is Frobenius and hence $HE$ is finite-dimensional. By Lemma 10.1, $D_{\text{sm}}^\infty(E^{\text{op}}) \cong D_{\text{sm}}^\infty(E)^{\text{op}}$. Since $HE$ is finite-dimensional, $D_{\text{sm}}^\infty(E) = D_{\text{fg}}^\infty(E)$. Since $A$ is connected graded Artin–Schelter regular, $k_A$ is small, so

$$\text{thick}_{\infty}^A(k,A) = \text{thick}_{A}^\infty(A) = D_{\text{per}}^\infty(A).$$

By the definition of noetherian (Definition 9.2), $D_{\text{fg}}^\infty(A) = D_{\text{per}}^\infty(A)$. Therefore $D^\infty(\text{proj } A) = D_{\text{sm}}^\infty(\text{proj } A)$. The assertion follows from part (a). \qed

Theorem C is part (b) of the above theorem.

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