Intrinsic Angular Momentum and Intrinsic Magnetic Moment of Chiral Superconductor on Two-Dimensional Square Lattice

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The intrinsic magnetic moment (IMM) and intrinsic angular momentum (IAM) of a chiral superconductor with $p$-wave symmetry on a two-dimensional square lattice are discussed on the basis of the Bogoliubov-de Gennes equation. The IMM and IAM are shown to be on the order of $\mu_B N$ and $\hbar N$, respectively, where $N$ is the total number of particles, without an extra factor $(T_c/T_F)^\gamma$ ($\gamma = 1, 2$), and parallel to the pair angular momentum. They arise from the current in the surface layer with a width on the order of the coherence length $\xi_0$, the size of Cooper pairs. However, in a single-band model, they are considerably canceled by the contribution from the Meissner surface current in a layer with the width of the penetration depth $\lambda$, making it difficult to observe them experimentally. In the case of multi-band metals with both electron-like and hole-like bands, however, considerable cancellation still occurs for the IMM but not for the IAM, making it possible to observe the IAM selectively because the effect of the Meissner current becomes less important. As an example of a multi-band metal, the case of the spin-triplet chiral superconductor Sr$_2$RuO$_4$ is discussed and experiments for observing the IAM are proposed.

1. Introduction

1.1 Issues concerning intrinsic angular momentum in $^3$He A-phase

The size of the intrinsic angular momentum (IAM) $L_{in}$ has been a major issue since the middle of the ’70s, when the size of the IAM in the A-phase of superfluid $^3$He was keenly discussed. Namely, the issue is the value of the exponent $\gamma$ when the IAM in the ground state is expressed as $L_{in} = (Nh/2)(T_c/T_F)^\gamma$ in the case where the $\hat{\ell}$-vector is uniform in space. This problem was first addressed by Anderson and Morel in their seminal paper$^1$ discussing a possible state of superfluidity of liquid $^3$He, who stated that $\gamma = 1$. This estimation is based on the idea that the IAM is sustained by a pair condensate with angular momentum $\hbar$ so that the IAM is proportional to the pair amplitude ($\propto T_c/T_F$), as is clearly elucidated by Leggett in a review article on superfluid $^3$He.$^2$ After the discovery of the superfluidity of $^3$He, a bunch of theoretical works were performed on the size of the IAM by microscopic calculations based on the $k$-space approach, which predicted $\gamma = 2.3$–7) This result seems to have been interpreted
such that the IAM is formed by two particles inside the thin surface layer in a $k$-space near the Fermi surface whose width is on the order of the superfluid gap $\Delta$, implying that the number of particles forming the IAM is of the order of $N \times (\Delta/E_F)^2$ or $N \times (T_c/T_F)^2$.

On the other hand, in 1976, Ishikawa proposed an alternative idea\(^8\) that the IAM in the A-phase is $L = \left( \frac{N\hbar}{2} \right)$ in the ground state if we adopt the wave function proposed by BCS, whose property was discussed by Ambegaokar in the review article appearing in the textbook on superconductivity edited by Parks.\(^9\) The point is that the orbital-part of the many particle ground state $\Psi(r_1, r_2, r_3, r_4, \cdots, r_{N-1}, r_N)$ ($N$ being an even number) is given by the product of the wave function of the Cooper pair $\varphi(r, r')$ which is the eigenstate of the relative angular momentum $\ell_z = \hbar$ concerning $(r - r')$. Namely,

$$\Psi(r_1, r_2, r_3, r_4, \cdots, r_{N-1}, r_N) = A [\varphi(r_1, r_2)\varphi(r_3, r_4)\cdots\varphi(r_{N-1}, r_N)], \quad (1)$$

where $A$ indicates the anti-symmetrization with respect to all the spatial coordinates, and the spin coordinates are discarded as irrelevant. Indeed, Ambegaokar showed explicitly that this form of the wave function can be transformed into the wave function proposed by BCS if the Fourier component of the wave function of the Cooper pair $\varphi(r, r')$ is well defined with respect to the relative coordinate $(r - r')$. The issue at that stage was whether the last condition is satisfied or not. Indeed, the Fourier component of $\varphi(r - r')$, $\varphi_k$, is given by

$$\varphi_k = \frac{u_k}{\bar{u}_k}, \quad (2)$$

where the coefficient $u_k$ expressing the coherence of the BCS state vanishes outside the thin region around the Fermi surface of the width of cut-off $\varepsilon_c$ so long as the weak-coupling theory is applied. However, the weak-coupling theory might discard a crucial effect on the coherence between the region near the Fermi surface and the core region of the Fermi sphere. Namely, the issue is to what extent the coherence of the Cooper pair condensate is maintained into the core of the Fermi sphere in a real situation. This seems to be a fundamental question regarding how to understand the Fermi superfluid state, including superconductivity.

It is instructive to remember the discussions of Bogoliubov et al.\(^10\) and Anderson and Morel,\(^1\) who estimated the effect of the Coulomb repulsion on the Cooper pair formation. They solved the gap equation with the phonon-mediated attractive interaction near the Fermi surface with a width on the order of the Debye energy $\omega_D$ and the Coulomb repulsive interaction acting in the whole $k$-space, and they obtained a superconducting gap that is finite not only in the thin surface around the Fermi surface but also in the whole $k$-space including the core of the Fermi sphere. This was crucial to understanding the systematic deviation of the index $\alpha$ for the isotope effect from the canonical value $\alpha = 1/2$.\(^{11}\) Therefore, it is a real effect that the coherence of the Cooper pair extends to the core of the Fermi sphere. In this sense, a difficulty posed for the idea of Ishikawa was safely avoided.\(^{12,13}\) Adopting the wave function in Eq. (1), McClure and Takagi showed\(^{14}\) that the result $L_{in} = N\hbar/2$ holds more
generally within the manifold of the ground state with an axially symmetric configuration of the $\hat{\ell}$-vector, such as the Mermin-Ho\textsuperscript{15}) and Anderson-Toulouse\textsuperscript{16}) textures. Namely, the state with a uniform configuration of the $\hat{\ell}$-vector is adiabatically continued from such axially symmetric states.

A related issue concerned the structure of the supercurrent $j_s$ in the A-phase of superfluid $^3$He. By the symmetry argument, $j_s$ is expressed as

$$j_s = \tilde{\rho}_s v_s + \frac{1}{2} \nabla \times \left( \frac{1}{2} \hbar n_s \hat{\ell} \right),$$

where $\tilde{\rho}_s$ is the superfluid density tensor and the tensor $\tilde{C}$ has the following form in general:

$$C_{ij} = C \delta_{ij} - C_0 \hat{\ell}_i \hat{\ell}_j.$$  

The issue concerned the form of $\tilde{C}$ together with the size of $L_{\text{in}}$. According to microscopic calculations based on the $k$-space representation, $C_{ij}$ is given as $C_{ij} = C(T)(\delta_{ij} - 2 \hat{\ell}_i \hat{\ell}_j)$ and $C(T = 0) = \hbar n/4$, where $n$ is the number density, and $L_{\text{in}} \sim \hbar n_s(T_c/T_F)^{24,17}$ or $L_{\text{in}} = 0$,\textsuperscript{3}) where $n_s$ is the superfluid number density. On the other hand, the supercurrent density at $T = 0$ is given by

$$j_s = \rho_s v_s + \frac{1}{4} \hbar (\nabla n_s) \times \hat{\ell},$$

according to a modified wave function based on Eq. (1), which is generalized so as to take into account a gradual variation of the center of mass coordinate in $\varphi(r, r')$.\textsuperscript{12}) This supercurrent is in a form parallel to that appearing in the electromagnetism in materials,\textsuperscript{18}) in which the electric current $j_M$ induced by a variation of the magnetization density $\mathbf{M}$ is given by

$$j_M = \frac{1}{\mu_0} \nabla \times \mathbf{M}.$$  

Indeed, if one remembers the relation $\mathbf{M} = (e\hbar/2m) \mathbf{L}$, $\mathbf{L}$ being the orbital angular momentum, and that $j_s$ corresponds to $(m/e)j_M$, the second term in Eq. (5) is precisely Eq. (6). On the other hand, Mermin and Muzikar\textsuperscript{19}) claimed that the calculation in Ref. 12 misses a subtle singularity of the wave function $\varphi(r, r')$ when it is calculated in the $k$-space representation, leading to the expression for the supercurrent

$$j_s = \rho v_s + \frac{1}{4} \hbar (\nabla n_s) \times \hat{\ell}.$$  

This form was also derived by Nagai on the basis of kinetic theory.\textsuperscript{20}) The physical meaning of the difference between Eqs. (5) and (7) has been discussed by Volovik from a general viewpoint.\textsuperscript{21}) In any case, it is crucial that both expressions, Eqs. (5) and (7), include the term proportional to $(\nabla n_s) \times \hat{\ell}$ that makes an essential contribution to the IAM through the surface current when $\hat{\ell}$ is parallel to the surface where the number density $n$ vanishes abruptly.

This fact leads to the following physical picture of the IAM in a uniform configuration of the $\hat{\ell}$-vector. In the region at a distance from the system boundary of more than the coherence length $\xi_0$ of the Cooper pairs, the relative motions of the Cooper pairs around the $\hat{\ell}$-vector...
cancel with each other, resulting in no contribution to the IAM. On the other hand, in a thin region with a width on the order of $\xi_0$ near the system boundary, its cancellation is incomplete, giving rise to the surface current that is the main origin of the IAM. This physical picture is exactly the same as the picture for an electric current induced by the (classical) orbital magnetism in materials. Precisely speaking, it is not self-evident that Eq. (5) or (7) cannot be applied near a system boundary where the number density changes abruptly, because they were derived on the assumption that the spatial variations of physical quantities are gradual compared with the length scale characterizing the physics, i.e., the coherence length $\xi_0$.

Recently, this problem has been revived in the context of the topological effect associated with the surface state of a chiral superfluid or superconductivity. According to detailed calculations that take into account the microscopic structure of the surface state with spatial variation on the order of $\xi_0$, the result derived from the supercurrent, given by Eq. (5) or (7), is essentially correct.\textsuperscript{22,23} On the other hand, these calculations show that the IAM at finite temperatures is given by

$$L_{in}(T) = \frac{N^\parallel_s(T)\hbar}{2},$$

where $N^\parallel_s(T)$ is the superfluid number parallel to the $\hat{\ell}$-vector.\textsuperscript{22–24} This result cannot be easily understood intuitively, but seems to be much more involved than a naive physical picture. In particular, it is difficult to find the physical reason why $N^\parallel$ appears in a two-dimensional system when the $\hat{\ell}$-vector is perpendicular to the two-dimensional plane. Indeed, it should be compared with the result obtained by a calculation using a cylindrical representation for one-particle states, in which $L_{in}(T)$ is given by

$$L_{in}(T) = \frac{N^\perp_s(T)\hbar}{2},$$

where $N^\perp_s(T)$ is the superfluid number perpendicular to the $\hat{\ell}$-vector.\textsuperscript{25,26}

Regarding an experiment for detecting the IAM in the $^3$He-A phase, NMR measurement of the structural change in the Mermin-Ho texture in a rotating cryostat was proposed by Takagi.\textsuperscript{27} This experiment has been performed at the Institute for Solid State Physics (ISSP) of the University of Tokyo and suggested that the size of the IAM is on the order of $N_s\hbar/2$.\textsuperscript{28}

1.2 Issues concerning intrinsic magnetic moment in Sr$_2$RuO$_4$

In the past decade, the problem concerning the IAM has been revived as that of the intrinsic magnetic moment (IMM) in the spin-triplet chiral superconductor Sr$_2$RuO$_4$,\textsuperscript{29} in which the orbital part of the superconducting gap has been identified as $\Delta_k = \Delta(\sin k_x a + i \sin k_y a)$, $a$ being the lattice constant of the two-dimensional lattice. This is also supported by a muon-spin-resonance ($\mu$SR) measurement showing the breaking of time-reversal symmetry.\textsuperscript{30} Information on the $k$-representation of $\Delta_k$ is obtained from the temperature dependence of the specific heat (under a magnetic field) and theoretical investigations that suggest the impor-
stance of short-range ferromagnetic correlations among quasiparticles.\textsuperscript{31–33} If the IAM \( L_{\text{in}} \) is on the order of \( N_s \hbar /2 \) and the gyromagnetic ratio is given by \((-e/2m)\), with \( e > 0 \) being the elementary charge, as in the classical case, the IMM density \( M_{\text{in}} \) is estimated as

\[
M_{\text{in}} \simeq -\frac{n_s}{2} \mu_0 \frac{m}{m_{\text{occ band}}} \mu_B,
\]

where \( n_s \equiv N_s / V, \mu_0 = 4\pi \times 10^{-7} \text{ H} \cdot \text{m}^{-1} \) is the magnetic permeability of vacuum, \( \mu_B = e\hbar /2m \) is the Bohr magneton, and \( m_{\text{occ band}} \) is the harmonic average of the band mass of electrons over the occupied state in the Brillouin zone discussed in Appendix A and should be distinguished from the effective mass averaged over the Fermi surface discussed in Appendix B. Then, the magnetic flux density \( B_{\text{in}} \) without the external magnetic field \( H \) is given by \( M_{\text{in}} \) because the relation \( B = M + \mu_0 H \) holds by definition.\textsuperscript{18}\) The electron number density \( n \) of the \( \gamma \)-band, which is the electron-like band, in \( \text{Sr}_2\text{RuO}_4 \), is estimated as

\[
n = \frac{1}{abc},
\]

where \( a = b = 3.9 \times 10^{-10} \text{ m} \) and \( c = (12.7/2) \times 10^{-10} \text{ m} \) are the lengths of an edge of the primitive cell of \( \text{Sr}_2\text{RuO}_4 \) along the \( a, b \) and \( c \) directions, respectively.\textsuperscript{34}\) The magnetization density \( M_{\text{in}} \) is given by the relation

\[
M_{\text{in}} = -\mu_0 \frac{e}{2m_{\text{occ band}}} L_{\text{in}} = -\frac{\hbar}{2} n \mu_0 \frac{e}{2m_{\text{occ band}}},
\]

where \( m_{\text{occ band}} \approx 2.9m \) is the effective mass of the \( \gamma \)-band of \( \text{Sr}_2\text{RuO}_4 \).\textsuperscript{34}\) Therefore, the intrinsic magnetic flux density \( B_{\text{in}} \) is estimated as

\[
B_{\text{in}} = \frac{10^{-30}}{abc} \frac{m}{m_{\text{occ band}}} \times 5.8 \text{ T} \approx -2.1 \times 10^{-2} \text{ T} = -2.1 \times 10^2 \text{ G}.
\]

This value is larger than the “observed” lower critical field \( B_{\text{cl1}}^{\text{obs}} = 5.0 \times 10^{-3} \text{ T} \) for \( \text{Sr}_2\text{RuO}_4 \).\textsuperscript{35}\) Therefore, at first sight, this intrinsic magnetic flux density \( B_{\text{in}} \) will not be completely screened by the Meissner current. However, since \( \text{Sr}_2\text{RuO}_4 \) has two other bands, a hole-like \( \alpha \)-band and an electron-like \( \beta \)-band, considerable cancellation in the IMM is expected among the electron-like \( \beta \)- and the \( \gamma \)-bands and the hole-like \( \alpha \)-band, as discussed in Sect. 5.

Another issue is whether the IAM is also cancelled by the orbital angular momentum arising from the Meissner current moving through a thin surface with the width of the penetration depth \( \lambda \) near the boundary of the system. If this cancellation is incomplete, the remnant IAM may be detected by the Richardson-Einstein-de Haas effect.\textsuperscript{36,37}\)

1.3 Purpose and organization of the present paper

In the “rotationally symmetric” system, in which the angular momentum is a conserved quantity, the result for the IAM, \( L_{\text{in}} = (N\hbar /2)\hat{\ell} \), is almost self-evident from the viewpoint of the BEC-BCS crossover or of the adiabatic continuation. In the BEC limit, each diatomic
molecule has angular momentum $\hbar \ell$ as shown in Fig. 1(a). Therefore, the IAM is given by $L_{\text{in}} = (N\hbar/2)\hat{\ell}$, where the number of diatomic molecules is $N/2$. So long as the pairing interaction has rotationally symmetry, the value of $L_{\text{in}}$, which is a conserved quantity, should not change even if the pairing interaction is weakened to approach the BCS limit in which ”molecules” overlap each other as shown in Fig. 1(b). A subtlety is that the gap amplitude $\Delta$ of a Cooper pair has a very weak singularity. Namely, $d^2\Delta/d\mu^2$ has a discontinuity as the chemical potential $\mu$ passes through the bottom of the quasiparticle band.$^{38}$

On the other hand, in a lattice system in which the rotation symmetry is broken, the problem is not so trivial. Therefore, it is necessary to explicitly investigate the problem for a specific lattice model. One of the purposes of this paper is to investigate how the results for the IAM of the $^3$He-A phase in three-dimensional free space are modified in the case of a chiral superconductor on a two-dimensional square lattice, which simulates Sr$_2$RuO$_4$. Another purpose is to investigate, using this two-dimensional model, to what extent the IMM is screened by the Meissner effect and the IAM is lessened by the Meissner current. On the basis of our results, it is discussed how the IMM and IAM are observed in Sr$_2$RuO$_4$.

Fig. 1. (a) Picture of strong-coupling (BEC) limit, where diatomic molecules with angular momentum $\hbar \ell$ are in the state of Bose-Einstein condensation. (b) Picture of weak coupling (BCS) limit, where the angular momentum is distributed on the Cooper pair condensation.
The organization of the present paper is as follows. In Sect. 2, we introduce the model on a square lattice with an attractive interaction between nearest-neighbor sites and discuss a formulation for explicit calculations. In Sect. 3, the results for the IAM and IMM are shown. In Sect. 4, the effect of the Meissner current on the size of the IMM and IAM is discussed. Finally, in Sect. 5, we propose how to observe the IMM and IAM of the two dimensional chiral superconductor Sr$_2$RuO$_4$.

2. Chiral Superconductor on Square Lattice

2.1 Model Hamiltonian

In order to study the problem of the IAM and IMM of a chiral superconductor on a two-dimensional lattice, a model of Sr$_2$RuO$_4$, we start with the following Hamiltonian:

$$\mathcal{H} = -\mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} - t \sum_{\langle i,j \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - \frac{V}{2} \sum_{\langle i,j \rangle \sigma} c_{i\sigma}^\dagger c_{i\sigma} c_{i\sigma}^\dagger c_{j\sigma},$$

(14)

where $\mu$, $t$, and $V$ are the chemical potential, the transfer integral between nearest-neighbor (n.n.) sites of the square lattice, and the attractive interaction between electrons at n.n. sites, respectively, and $c_{i\sigma}^\dagger$ ($c_{i\sigma}$) is the creation (annihilation) operator of an electron at the $i$-th site with spin component $\sigma$ ($=\uparrow$ or $\downarrow$). The symbol $\langle i,j \rangle$ indicates that the summation is taken over the n.n. sites. Here, we consider the spin-triplet pairing with $S_z = 0$ and introduce a superconducting gap $\Delta_{ij}$ in the spin-triplet manifold as

$$\Delta_{ij} = \frac{V}{2} (c_{i\uparrow}^\dagger c_{j\downarrow} + c_{i\downarrow}^\dagger c_{j\uparrow}),$$

(15)

where $\langle \cdots \rangle$ means the average by the mean-field Hamiltonian $\mathcal{H}_{mf}$ given as

$$\mathcal{H}_{mf} = -\mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} - t \sum_{\langle i,j \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + \sum_{\langle i,j \rangle} \left\{ \Delta_{ij} (c_{j\uparrow}^\dagger c_{i\uparrow} + c_{j\downarrow}^\dagger c_{i\downarrow}) + \text{h.c.} \right\} - \frac{|\Delta_{ij}|^2}{V}. $$

(16)

Here the gap $\Delta_{ij}$ depends on lattice sites $i$ and $j$ in general, and its dependence is determined self-consistently by solving the (lattice version of the) Bogoliubov-de Gennes equation together with Eq. (15).$^{39}$ The gap $\Delta_{ij}$ is odd with respect to the interchange of $i \equiv j$:

$$\Delta_{ij} = -\Delta_{ji},$$

(17)

which results in the odd-parity pairing. Note that, in the case of a uniform system without a boundary, the most stable gap among those given by Eq. (15) is expressed in a wave-vector representation as$^{31}$

$$\Delta_k = \Delta (\sin k_x a + i \sin k_y a),$$

(18)

where $a$ is the lattice constant.

2.2 Orbital magnetization and angular momentum in band picture

In order to take into account the effect of the magnetic field $B$, we adopt the following way of giving the Peierls phase the transfer integral $t_{ij}$ between electrons at the $i$-th and $j$-th
sites
\[ \tilde{t}_{ij}(B) = t_{ij} \exp \left[ -\frac{ie}{\hbar} \int_{r_i}^{r_j} A(r) \cdot dr \right], \tag{19} \]
where \( A \) is the vector potential giving the magnetic field as \( B = \nabla \times A \) and the contour integral is performed along the line connecting the two sites. Then, the band-energy part \( \mathcal{H}_{\text{band}} \) of the Hamiltonian is expressed as
\[ \mathcal{H}_{\text{band}} = - \sum_{(i,j)} \tilde{t}_{ij}(B) c_{i\sigma}^\dagger c_{j\sigma}. \tag{20} \]
Adopting the gauge of the vector potential such that \( A = (B \times r)/2 \), \( \mathcal{H}_{\text{band}} \) is reduced to the following form:
\[
\mathcal{H}_{\text{band}} = -t \sum_i \left[ \exp \left( \frac{ieB}{2\hbar} y_i a \right) c_{(i_x+1,i_y)}^\dagger c_{(i_x,i_y)} + \exp \left( -\frac{ieB}{2\hbar} y_i a \right) c_{(i_x,i_y)}^\dagger c_{(i_x+1,i_y)} \right] \\
- t \sum_i \left[ \exp \left( \frac{ieB}{2\hbar} y_i a \right) c_{(i_x,i_y+1)}^\dagger c_{(i_x,i_y)} + \exp \left( -\frac{ieB}{2\hbar} y_i a \right) c_{(i_x,i_y+1)}^\dagger c_{(i_x,i_y)} \right] \\
+ t \sum_i \left[ \exp \left( \frac{ieB}{2\hbar} x_i a \right) c_{(i_x,i_y+1)}^\dagger c_{(i_x,i_y+1)} + \exp \left( -\frac{ieB}{2\hbar} x_i a \right) c_{(i_x,i_y+1)}^\dagger c_{(i_x,i_y+1)} \right], \tag{21} \]
where the spin coordinate is abbreviated for concise presentation and the \( i \)-th position of the lattice is designated by the two-dimensional representation \((i_x,i_y)\). In deriving Eq. (21), the contour integral Eq. (19) along the \( x \)-direction has been approximated by the trapezoidal rule as
\[
\int_{x_i-a}^{x_i} A_x(x,y_i) dx \simeq A_x \left( x_i - \frac{a}{2}, y_i \right) a \tag{22} \]
and
\[
\int_{x_i}^{x_i+a} A_x(x,y_i) dx \simeq A_x \left( x_i + \frac{a}{2}, y_i \right) a. \tag{23} \]
A similar approximation is adopted for the integral along the \( y \)-direction.

Then, the operator of the magnetization \( M \) of the system (parallel to \( B \), \( z \)-component) is given, in the limit \( B \to 0 \), as
\[
\frac{M_z}{\mu_0} = - \left( \frac{\partial \mathcal{H}}{\partial B} \right)_{B=0} = - \left( \frac{\partial \mathcal{H}_{\text{band}}}{\partial B} \right)_{B=0} = \frac{ta^2}{\hbar^2} (-e) \sum_i \langle r_i \times p_i \rangle_z, \tag{24} \]
where the “momentum” operator \( p_i \) at the \( i \)-th site is defined by
\[
p_{xi} = \frac{-i}{\hbar} \sum_{\sigma} \left[ c_{(i_x+1,i_y)}^\dagger c_{(i_x,i_y)} - c_{(i_x-1,i_y)}^\dagger c_{(i_x,i_y)} - c_{(i_x,i_y+1)}^\dagger c_{(i_x,i_y)} - c_{(i_x,i_y)}^\dagger c_{(i_x,i_y+1)} \right],
\]
\[
p_{yi} = \frac{-i}{\hbar} \sum_{\sigma} \left[ c_{(i_x,i_y+1)}^\dagger c_{(i_x,i_y)} - c_{(i_x,i_y-1)}^\dagger c_{(i_x,i_y)} - c_{(i_x,i_y)}^\dagger c_{(i_x,i_y+1)} - c_{(i_x,i_y)}^\dagger c_{(i_x,i_y-1)} \right]. \tag{25} \]
If we introduce the band mass \( m_b \) at the zone boundary, say at the \( \Gamma \)-point, as
\[
\frac{ta^2}{\hbar^2} \equiv \frac{1}{2m_b}, \tag{26} \]
the magnetization operator \( M_z \) [Eq. (24)] is reduced to a band version of the conventional
form with the gyromagnetic ratio \((-e/2m_b)\):
\[ M_z = \mu_0 \frac{-e}{2m_b} \sum_i (\mathbf{r}_i \times \mathbf{p}_i)_z. \] (27)

This implies that the definition of the orbital angular momentum \(\mathbf{L}\),
\[ \mathbf{L} \equiv \sum_i (\mathbf{r}_i \times \mathbf{p}_i), \] (28)
is a valid and natural one. The definition of \(m_b\) [Eq. (26)] corresponds to the free-electron-like dispersion of tight-binding dispersion around the \(\Gamma\)-point, \((k_x, k_y) = (0, 0)\). Namely,
\[ \epsilon_k = -2t(\cos k_x a + \cos k_y a) \sim -4t + ta^2(k_x^2 + k_y^2) + \cdots. \] (29)
Using this dispersion, \(m_{\text{band}}^{\text{occ}}\) in Eqs. (10), (12), and (13) is estimated as \(m_{\text{band}}^{\text{occ}} = (\pi^2/4)m_b\) in the half-filled case, as shown in Appendix A.

3. Results for IAM and IMM

An explicit form of the Bogoliubov-de Gennes equation for the mean-field Hamiltonian [Eq. (15)] with the superconducting gap of \(S_z = 0\) [Eq. (18)] is given by
\[ \epsilon u_i = -\mu u_i - t u_j + \sum_{\langle j, i \rangle} \Delta_{ij} v_j, \] (30)
\[ \epsilon v_i = \mu v_i + t v_j + \sum_{\langle j, i \rangle} \Delta_{ij}^* u_j, \] (31)
where \(\langle j, i \rangle\) means that the summation is taken over the nearest-neighbor sites. An actual calculation is performed as follows. Hereafter, we mainly focus our discussion on the half-filled case, unless otherwise stated. Equations (30) and (31) are diagonalized by means of a unitary transformation \(U\) to give the mean-field Hamiltonian
\[ H_{\text{mf}} = \sum_{m=1}^{N_L} \epsilon_m \gamma_m^\dagger \gamma_m + \sum_{m=1}^{N_L} (-\epsilon_m) \gamma_m^\dagger \gamma_m, \] (32)
where \(N_L\) is the number of lattice sites, \(0 \leq \epsilon_1 \leq \epsilon_2 \cdots \leq \epsilon_{N_L}\), and the fermion operators \(\gamma\) describing the quasiparticles are related to the electron operators \(c\) by
\[ \begin{bmatrix} c_{1\uparrow}^\dagger, \cdots, c_{N_L\uparrow}^\dagger, c_{1\downarrow}, \cdots, c_{N_L\downarrow} \end{bmatrix} = \begin{bmatrix} \gamma_{1\uparrow}^\dagger, \cdots, \gamma_{N_L\uparrow}^\dagger, \gamma_{1\downarrow}, \cdots, \gamma_{N_L\downarrow} \end{bmatrix} U^\dagger. \] (33)
Substituting Eq. (33) into Eq. (15), we obtain the self-consistent equation for the gap \(\Delta_{ij}\) as
\[
\Delta_{ij} = \frac{V}{2} \sum_{m=1}^{N_L} \left[ (U)_{j+N_L,i,m}^\ast (U)_{i,m} - (U)_{i+N_L,m}^\ast (U)_{j,m} \right] [1 - f(\epsilon_m)]
\]
\[
+ \frac{V}{2} \sum_{m=1}^{N_L} \left[ (U)_{j+N_L,i,m+N_L}^\ast (U)_{i,m+N_L} - (U)_{i+N_L,m+N_L}^\ast (U)_{j,m+N_L} \right] f(\epsilon_m),
\] (34)
where \( U \) depends on \( \Delta_{ij} \) and \( \varepsilon_m \) \( (m = 1, \cdots, N_L) \), and \( f(x) \) is the Fermi distribution function \( f(x) \equiv (e^{x} + 1) \).

We have solved Eqs. (15), (16), and (32)-(34) self-consistently using the numerical diagonalization method and obtained the gap \( \Delta_{ij} \) and the energy levels \( \varepsilon_m \) \( (m = 1, \cdots, N_L) \). Numerical calculations have been performed for lattice sizes of up to \( N_L = 30 \times 30 \) squares with both open and periodic boundary conditions. Throughout the present paper, the phase of the superconducting gap \( \Delta_{ij} \) is chosen as shown in Fig. 2, while \( \Delta_i \) \( (i = 1-4) \) are determined self-consistently.

![Phase of superconducting gap](image)

Fig. 2. Phase of superconducting gap \( \Delta_{ij} \) with \( i \)-th site chosen as a center.

First of all, the IMM [Eq. (27)] and IAM [Eq. (28)] are shown to be zero for the periodic boundary condition for which \( \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 \), resulting in the recovery of the chiral gap given by Eq. (18) in the \( k \)-space representation. This is because the orbital currents due to the relative motion of the Cooper pairs cancel with each other. On the other hand, if we use the open boundary condition, we obtain finite values of the IMM and IAM because the cancellation of the relative motion of the Cooper pairs is incomplete near the boundary of the system within the coherence length \( \xi_0 \) of the Cooper pairs, as in the case of the classical theory of magnetic materials where the current \( j_M \) given by Eq. (6) exists only in a thin surface layer of the system if the magnetization \( M \) is uniform in the bulk of the system.

Figure 3 shows the phase diagram in the \( T/t-V/t \) plane for the system size \( N_L = 9 \times 9 \) in the half-filled case. Three different superconducting states exist, which we call SI, SII, and SIII, as shown in the figure. The SI phase appears around the phase boundary between the SII phase and the normal phase in the intermediate-coupling region \( V/t < 4.5 \). On the other hand, the SIII phase appears not only around the phase boundary between the SII phase and the normal phase in the strong-coupling region \( V/t > 4.5 \), but also as a low-temperature phase in the strong-coupling region \( V/t > 8.5 \).

Figure 4 shows the pattern of current in the SI phase for the parameter set, \( V/t = 3.25 \).
and $T/t = 0.564$, shown by a closed square in Fig. 3. The system size is taken as $N_L = 9 \times 9$. The phases of the superconducting gap $\Delta_{ij}$ shown in Fig. 2 are given by $\Delta_1 = \tilde{\Delta}_1$, $\Delta_2 = -i\tilde{\Delta}_2$, $\Delta_3 = -\tilde{\Delta}_3$, and $\Delta_4 = i\tilde{\Delta}_4$, where $\tilde{\Delta}_1 \sim \tilde{\Delta}_4$ are real and have the same sign. Note that the sign of $\tilde{\Delta}_1 \sim \tilde{\Delta}_4$ is determined so as to satisfy Eq. (17). In this phase, the current is induced around the center of the bulk and its pattern can be seen as a deformed vortex pair lattice as shown by symbols $\otimes$ and $\odot$. The physical reason why this pattern is realized is unclear for the moment.\textsuperscript{41)

Figure 5 shows the pattern of current in the SII phase for the parameter set, $V/t = 8.00$ and $T/t = 1.482$, shown by a closed circle in Fig. 3. The system size is taken as $N_L = 9 \times 9$. This phase is the bulk phase in the intermediate-coupling region as shown in Fig. 3. The phases of the superconducting gap $\Delta_{ij}$ shown in Fig. 2 are given by $\Delta_1 \sim \Delta_4$, which are real and have the same sign. The pattern of the current distribution is that expected physically. Namely, the current is induced near the boundary of the system owing to incomplete cancellation of the relative angular momentum of the Cooper pairs. The reason why the current exists at the center of the system is that the system size $N_L = 9 \times 9$ is comparable to the extent of the Cooper pair $\xi(T) \equiv \pi \xi(T)$, $\xi(T)$ being the coherence length at finite $T$.

If we follow the BCS theory for an s-wave weak-coupling superconductor, the coherence length in the ground state (at $T = 0$) is given by $\xi_0 = \hbar v_F / \pi \Delta$. Then, the ratio of $\xi_0$ to the lattice constant $a$ is estimated as

$$\frac{\xi_0}{a} = \frac{\gamma \hbar v_F}{a \pi^2 T_c} \approx \frac{2\gamma E_F}{\pi^4 T_c} \approx 0.11 \frac{E_F}{T_c},$$

(35)

where we have used the BCS relation $\Delta = \pi T_c / \gamma$, where $\log \gamma$ is the Euler constant $C = 0.57721 \cdots$, and assumed the free dispersion for the quasiparticles and that the Fermi momentum is given by $p_F \approx h \pi / a$. As shown later in Fig. 8, $T_c/t \approx 1.8$ for the attractive interaction $V/t = 8.00$. The Fermi energy $E_F$ is estimated as $E_F \approx 4t$ in the case of half-filling.
Fig. 4. Pattern of current in the SI phase for $V/t = 3.25$ and $T/t = 0.564$ as indicated by a closed square in Fig. 3. The system size is taken as $N_L = 9 \times 9$. The current at each site is shown by an arrow whose length represents the relative size of the lattice momentum defined by Eq. (25). Symbols $\otimes$ and $\odot$ indicate localized vortices with opposite circulations.

Therefore, with the use of Eq. (35), $\xi_0/a$ is estimated as $\xi_0/a \simeq 0.24$, giving the estimated extent of the Cooper pair $\xi^* = \pi \xi_0$ as $\xi^*/a \simeq 0.75$. On the other hand, the temperature $T/t = 1.482$ for Fig. 5 is about 90% of the transition temperature $T_c$, i.e., $(T_c - T)/T_c \simeq 0.1$, as seen in Fig. 3. Then, with the use of the correlation length $\xi(T) \simeq 0.74 \xi_0 \sqrt{T_c/(T_c - T)}$ at a finite temperature $T$, the extent of the Cooper pair at $T/t = 1.482$ is estimated as $\xi^*(T)/a \simeq 0.75 \times 0.74 \times \sqrt{10} \simeq 1.76$, which is not negligible compared with the system size $N_L = 9 \times 9$.

Figure 6 shows the pattern of current in the SII phase for the same parameter set as above, $V/t = 8.00$ and $T/t = 1.482$, but for a much larger system size, $N_L = 30 \times 30$. We can see that the current is essentially confined near the system boundary with width $\xi^* \simeq 1.76a$. The current along and near the boundary of the system can be regarded as a lattice version of the surface current of the A-phase of $^3$He, which is a topological superfluid characterized by the topological class D defined in Ref. 43.

Figure 7 shows the pattern of current in the SIII phase for the parameter set, $V/t = 9.00$ and $T/t = 0.644$, shown by a closed triangle in Fig. 3. The system size is taken as $N_L = 9 \times 9$. The phases of the superconducting gap $\Delta_{ij}$ shown in Fig. 2 are given by $\Delta_1 \sim \Delta_4$ which are real and have the same sign. In this phase, the induced current forms concentric layers of flows, with adjacent layers having opposite signs. The physical reason why this pattern is realized is unclear for the moment.
Fig. 5. Pattern of current in the phase SII for $V/t = 8.00$ and $T/t = 1.482$ as indicated by a closed circle in Fig. 3. The system size is taken as $N_L = 9 \times 9$. Current at each site is shown by an arrow whose length represents the relative size of lattice momentum defined by Eq. (25).

In Fig. 8, $m_z \equiv M_z/\mu_0 N_L$, which is the IMM per site divided by $\mu_0$, and $T_c/t$, the superconducting transition temperature, are shown as functions of $V/t$, which is the strength of the attractive interaction, at $T/t = 0.01$ for the system size $N_L = 13 \times 13$ at half-filling ($\mu/t = 0$). Note that, in order to avoid the effect of the boundary, $T_c/t$ is calculated with the periodic boundary condition. It is noteworthy that $m_z$ does not decrease even though $T_c$ decreases and that a negative correlation exists between $m_z$ and $T_c/t$ except in the SI phase where the coherence length becomes comparable to the system size and the superconducting state is greatly suppressed over the whole system. This suggests that the IAM, connected to the IMM by Eqs. (27) and (28), is on the order of $\hbar N/2$ without the extra factor $(T_c/T_F)^\gamma$ ($\gamma = 1$ or 2).

Indeed, according to Eqs. (27) and (28), the IAM, $L_{in}$, is expressed in terms of $\hbar N$ and $m_z/\alpha \mu_B$ as follows:

$$L_{in} = \frac{2m_b M_z}{(-e) \mu_0} = -\frac{\alpha m_b \hbar N_L m_z}{\alpha \mu_B},$$

(36)

where $\alpha \equiv t m a^2/\hbar^2$ and we have used the definition of the Bohr magneton $\mu_B = e\hbar/2m$. Since it is easily derived that $\alpha m_b/m = 1/2$ if we use the definition of $\alpha$ given above and Eq. (26), the IAM at half-filling (i.e., $N_L = N$) is given by

$$L_{in} = -\frac{\hbar}{2} N \frac{m_z}{\alpha \mu_B}.$$  

(37)

Considering the size of $m_z/\alpha \mu_B$ in the SII phase shown in Fig. 8, we can see that the IAM in
the SII phase is given by $L_{\text{in}} \sim \hbar N/2$. This implies that $\gamma = 0$ for the exponent of the extra factor $(T_c/T_F)^\gamma$, verifying the validity of the original theory by Ishikawa for the IAM in the $^3$He-A phase.\(^8\)

Figure 9 shows the system-size dependence of $m_z/\alpha \mu_B$ for the case of $V/t = 3.0$ and $T/t = 0.01$ at half-filling, i.e., $\mu/t = 0$, up to $N_L = 19 \times 19$. We can see that the system-size scaling works reasonably well, giving $m_z/\alpha \mu_B \simeq 0.302$ in the limit $N_L \to \infty$.

Figure 10 shows how $\text{IMM}/\mu_0$ per site, $m_z/\alpha \mu_B$, changes depending on the strength of the attractive interaction $V/t$, together with the behavior of $\Delta/t$, with $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4$ defined in Fig. 2. The system size is taken as $N_L = 30 \times 30$, the maximum size adopted in the present paper. The reason why $m_z/\alpha \mu_B$ approaches zero at $V/t \simeq 1$ can be understood as follows. At $V/t \simeq 1$, the transition temperature is $T_c/t \simeq 0.08$ as shown in Fig. 8. Therefore, according to $\xi_0/a$ given by Eq. (35) and $\xi^* = \pi \xi_0$, the extent of the Cooper pair $\xi^*/a$ for $T_c/t \simeq 0.08$ is estimated as

$$\frac{\xi^*}{a} = \frac{\pi \xi_0}{a} \simeq \pi \times 0.11 \times \frac{4t}{0.08t} \simeq 17.$$  \hspace{1cm} (38)

This implies that the superconducting order for the system size $N_L = 30 \times 30$ is almost
Fig. 7. Pattern of current in the SIII phase for $V/t = 9.00$ and $T/t = 0.644$ as indicated by a closed triangle in Fig. 3. The system size is taken as $N_L = 9 \times 9$. The current at each site is shown by an arrow whose length represents the relative size of the lattice momentum defined by Eq. (25).

Fig. 8. IMM/$\mu_0$ per site $m_z$ as a function of the attractive interaction $V/t$ for the system size $N_L = 13 \times 13$ at $T/t = 0.01$ in the case of half-filling, i.e., $\mu/t = 0$. SI, SII, and SIII indicate the phases shown in Fig. 3. The dimensionless parameter $\alpha$ is defined by $\alpha \equiv tma^2/\hbar^2$.

destroyed by the effect of the boundary of the system.

We can see that $m_z/\alpha\mu_B$ decreases as $V/t$ increases. The $V/t$ dependence of $m_z/\alpha\mu_B$ in the region $V/t > 10$ can be fitted by const.\,/($V/t$), although we do not show this explicitly. In this region, the pattern of the current corresponds to the lattice of vortices and antivortices with domain walls as shown in Fig. 11 for the system size $N_L = 15 \times 15$ with $V/t = 50$ and $T/t = 0$. Namely, the tendency that the magnetic moments of the vortex and antivortex cancel
Fig. 9. System-size scaling for IMM/μ per site \(m_z\) for \(V/t = 3.0\) and \(T/t = 0.01\) at half-filling, i.e., \(μ/t = 0\). The dimensionless parameter \(α\) is defined by \(α \equiv t^2/v^2\).

Fig. 10. IMM/\(μ_0\) per site \(m_z\) and magnitude of the superconducting gap \(Δ\) as functions of the attractive interaction \(V/t\) for the system size \(N_L = 30 \times 30\) at \(T/t = 0\) at half-filling, i.e., \(μ/t = 0\). The dimensionless parameter \(α\) is defined by \(α \equiv t^2/v^2\). The dashed curve is that expected in the limit \(N_L \to \infty\).

with each other becomes prominent in this region. However, this extremely strong coupling region does not seem to be realized in actual systems.

The dashed curve in Fig. 10 is a smooth extrapolation to \(V/t \to 0\) where \(m_z/αμ_B\) is expected to be \(m_z/αμ_B = 4/π^2\), as argued below. Indeed, according to Eq. (10) and Eq. (A-4), which is valid at half-filling, and the definition of \(α\), \(α \equiv t^2/v^2\), \(m_z/αμ_B\) is transformed as follows:

\[
\frac{m_z}{μ_B} = \frac{1}{2} \frac{m}{m_{\text{occ}}^2} \frac{μ_B}{αμ_B} = \frac{1}{2α} \frac{4}{π^2} \frac{m}{m_b}
\]
Fig. 11. Pattern of current in the SIII phase for $V/t = 50$ and $T/t = 0$. The system size is taken as $N_L = 15 \times 15$. The current at each site is shown by an arrow whose length represents the relative size of the lattice momentum defined by Eq. (25). Symbols $\otimes$ and $\odot$ indicate localized vortices with opposite circulations.

\[
\frac{4}{\pi^2} = 0.40528 \cdots \simeq 0.41, \tag{39}
\]

where, in deriving the last equality, we have used the relation $\alpha m_b/m = 1/2$, which is derived from Eq. (26).

Figure 12 shows the filling ($n_c$) dependence of IMM/$\mu_0$, $m_z/\alpha \mu_B n_c$, for the attractive interactions $V/t = 2.0$ and 3.0 at $T/t = 0$. The reason why $m_z/\alpha \mu_B n_c$ approaches zero at $n_c \simeq 0.15$ in the case of $V/t = 3.0$ can be understood by the effect of competition between the system size and the extent of the Cooper pairs as in the case in Fig. 10. The extent of the Cooper pairs $\xi^*$ is estimated by Eq. (35) but with $a$ replaced by $a/\sqrt{n_c}$ because the average distance between electrons increases in inverse proportion to the square root of the filling $n_c$.

Then, the following relation holds:

\[
\frac{\xi^*}{a} \simeq \pi \times 0.11 \frac{E_F}{T_c \sqrt{n_c}}, \tag{40}
\]

The transition temperature $T_c$ is calculated as $T_c \simeq 0.071t$. Therefore, $\xi^*$ for $V/t = 3.0$ and the filling $n_c = 0.15$, which gives $E_F \simeq 0.887t$, is estimated as

\[
\frac{\xi^*}{a} \simeq 11, \tag{41}
\]

which is approximately half of the size of the system of $30a$. This explains why $m_z$ becomes
zero at approximately $n_c = 0.15$.

The dashed curve in Fig. 12 is a smooth extrapolation to $n_c \to 0$, where $m_z/\alpha \mu_B n_c$ is expected to be equal to 1. Indeed, in the dilute limit ($n_c \to 0$), the effect of the lattice fades away so that the result in free space should be recovered. The IAM $L_{\text{in}}$ in the free space is expected to be given by $L_{\text{in}} = hN/2$. On the other hand, extending Eq. (37), $L_{\text{in}}$ is given by

$$L_{\text{in}} = \frac{h}{2} N \frac{m_z}{\alpha \mu_B n_c},$$

(42)

where $n_c = N/N_L$. Therefore, in the limit $n_c \to 0$, $m_z/\alpha \mu_B n_c$ is expected to approach 1.

Fig. 12. (Color online) IMM/\mu_0 per particle $m_z/\alpha \mu_B n_c$ as a function of the filling $n_c$ for the attractive interactions, $V/t = 2.0$ and 3.0. The system size is taken as $N_L = 30 \times 30$. The dimensionless parameter $\alpha$ is defined by $\alpha \equiv tma^2/h^2$. The dashed curve is that expected in the limit $n_c \to 0$. 
4. Effect of Meissner Current on IAM

In this section, we discuss the effect of the Meissner current on the IMM and IAM induced in a chiral superconductor as discussed in the previous section. Here, we estimate the orbital angular momentum due to the Meissner current flowing in a thin surface layer within penetration depth $\lambda$. First of all, let us consider the situation shown in Fig. 13(a), where the flat boundary surface of the superconductor is the $xy$-plane and the magnetic field $B_x(z)$ is parallel to the $x$-direction and decreases in the superconductor ($z > 0$). Then, by the Ampère law, the current density $j_y(z)$ is given by

$$j_y(z) = \frac{1}{\mu_0} \frac{dB_x(z)}{dz}. \quad (43)$$

Therefore, the Meissner current $J_M$ (per unit length), flowing in the $y$-direction in a surface layer in the $x$-direction, is given by

$$J_M = \int_0^\infty dz \frac{1}{\mu_0} \frac{dB_x(z)}{dz} = \frac{B_0}{\mu_0}, \quad (44)$$

where $B_0 \equiv B_x(0)$. The corresponding mass current $P_M$ is estimated as

$$P_M \simeq \frac{m_{\text{band}}}{e} J_M, \quad (45)$$

where $m_{\text{band}}$ is the band mass averaged near the Fermi level. The reasons why the mass $m_{\text{band}}$ appears are that the actual current is caused by a deformation of the Fermi surface, and that the Fermi liquid correction by “$F_1$” due to the back flow effect does not cancel the dynamical mass enhancement in the case where the Galilean invariance is broken in lattice systems such as Sr$_2$RuO$_4$.\(^{44-46}\) Then, in the situation shown in Fig. 13(b), the orbital angular momentum $L_M$ due to this $P_M$ is given by

$$L_M = 2\pi R \times R \times P_M. \quad (46)$$

With the use of Eqs. (44) and (45), $L_M$ is reduced to

$$L_M = 2\pi R^2 \times \frac{m_{\text{band}}}{e} \times \left( -\frac{B_0}{\mu_0} \right). \quad (47)$$

If we equate $B_{\text{in}} = M_{\text{in}}$ in Eq. (12) to $B_0$ at the surface, $L_M$ is finally given by

$$L_M = -\frac{n(\pi R^2)h}{2} \frac{m_{\text{band}}}{m_{\text{occ}} \text{band}} = -\frac{Nh}{2} \frac{m_{\text{band}}}{m_{\text{occ}} \text{band}}. \quad (48)$$

The first equality of Eq. (48) is also valid for columnar systems with a general cross-section shape if the factor $\pi R^2$ is replaced by the cross-section area $S$ because the angular momentum is related to the areal velocity. In this sense, the expression for $L_M$, Eq. (48), is valid for columnar systems with an arbitrary shape of the cross section. This result implies that $L_M$ cannot cancel the IAM $L_{\text{in}} = Nh/2$ in general as far as $m_{\text{band}} \neq m_{\text{occ}} \text{band}$, which is usually the case including the case of free space, as for liquid $^3$He, if the many-body effect near the Fermi surface is taken into account. Namely, the total angular momentum $L$ in the ground state,
given by

\[ L = L_{\text{in}} + L_{M} = \frac{Nh}{2} \left( 1 - \frac{m_{\text{band}}^{\text{FS}}}{m_{\text{band}}^{\text{occ}}} \right), \tag{49} \]

is not vanishing but is on the order of \( O(N\hbar) \) and has the opposite sign to the \( \hat{\ell} \) vector because \( m_{\text{band}}^{\text{FS}} > m_{\text{band}}^{\text{occ}} \) in correlated systems in general. We note, however, that the total angular momentum \( L \), given by Eq. (49), vanishes in the hypothetical system with a free electron dispersion because \( m_{\text{band}}^{\text{FS}} / m_{\text{band}}^{\text{occ}} = 1 \) in such a case.

Fig. 13. (Color online) (a) Magnetic field near the boundary between a superconductor and a vacuum. (b) Distribution of the magnetic field (indicated by arrows) in a cylindrical superconductor sample.

5. Multi-band Effect

The discussions so far have been for a single-band model. However, multiple bands exist in general. In particular, \( \text{Sr}_2\text{RuO}_4 \), which is a promising candidate for exhibiting the IAM and IMM, has three bands, one hole-like band (\( \alpha \)) and two electron-like bands (\( \beta \) and \( \gamma \)) (see Fig. 14). Therefore, the IMMs of the electron-like and hole-like bands partially cancel with each other because they have opposite signs, reflecting the opposite signs of the gyromagnetic ratio. The characters of the bands are summarized in Table I. \(^{47}\) According to Eq. (10), the IMM is inversely proportional to the band mass. Therefore, the relative size of the IMM is also inversely proportional to the band mass. The relative value of the IMM is given in the last row in Table I for the case that the IAMs of each band have the same magnitude. Then, total IMM density, \( M_{\text{in}}^{\text{tot}} \), is approximately given by

\[ M_{\text{in}}^{\text{tot}} \simeq -\frac{n_s}{2} \mu_0 \mu_B \left( -\frac{1}{1.1} + \frac{1}{2.2} + \frac{1}{2.9} \right), \tag{50} \]

if Eq. (10) is valid for the three bands shown in Table I. The factor in parenthesis in Eq. (50) gives a small value of \(-0.110\); thus, the magnitude of \( M_{\text{in}}^{\text{tot}} \) is one order smaller than that
Table I. Band-dependent $m_{\text{band}}/m$ according to Ref. 47 and the IAM and IMM ratios.

| Band | $\alpha$ | $\beta$ | $\gamma$ |
|------|----------|---------|----------|
| Character | Hole    | Electron | Electron |
| $m_{\text{band}}/m$ | 1.1     | 2.2     | 2.9     |
| IAM ratio  | 1       | 1       | 1       |
| IMM ratio  | 1       | −0.5    | −0.38   |

expected in the single-band model:

$$M_{\text{in}}^{\text{tot}} \simeq \frac{n_s}{2} \mu_0 \mu_B \times 0.110.$$  \hspace{1cm} (51)

Therefore, $M_{\text{in}}^{\text{tot}}$ induces the magnetic field $B_{\text{in}} \simeq 2.3 \times 10^{-3} \text{T} = 23 \text{G}$, which is smaller than the lower critical field $B_{\text{cl}}^{\text{obs}} \simeq 5.0 \times 10^{-3} \text{T}$ and is easily screened out by the Meissner effect, in contrast to the case of the single-band model as discussed in Sect 1.2.

Fig. 14. (Color online) Fermi surfaces of Sr$_2$RuO$_4$. The $\alpha$-band is hole-like, and the $\beta$- and $\gamma$-bands are electron-like.

On the other hand, this cancellation does not occur in the total IAM, $I_{\text{in}}^{\text{tot}}$; $I_{\text{in}}^{\text{tot}}$ is given by the sum of the IAM of three bands, which are all expected to be on the order of $\mathcal{O}(N_s \hbar/2)$. Therefore, the effect of the Meissner current on the total IAM is much less important than in the case of the single-band IAM given by Eq. (49). Thus, the total IAM remains to be technically unscreened by the Meissner current. This multiband effect is crucial to distinguish the IAM from the IMM.
6. How to Observe IMM and IAM in Sr$_2$RuO$_4$

In this section we discuss how to detect the IMM and IAM by carrying out experiments that are possible to perform on Sr$_2$RuO$_4$.

As discussed in the previous sections, it will be difficult to detect the magnetic field induced by the IMM owing to the Meissner screening in the bulk sample. However, it may be possible to detect it for a small sample with size on the order of the penetration depth $\lambda \sim 1300 \, \text{Å}$. On the other hand, the current pattern near the boundary of a small sample will be rather complicated in the sense that the directions of the Meissner current and the current inducing the IMM are opposite and the ranges of both currents are different, leading to considerable cancellation between the two currents up to the penetration depth $\lambda$. Therefore, in order to detect the IMM, the experiments should be carefully designed.

Another possibility exists of observing the IMM by performing $\mu$SR experiment, which can detect the IMM induced around the stopping site of $\mu^+$ in principle. This is because $\mu^+$ attracts electrons from adjacent Ru sites and acts as a nonmagnetic impurity that destroys the chiral superconducting order at sites surrounding $\mu^+$, resulting in a local circulating current around the $\mu^+$ site because the cancellation of the chiral current of Cooper pairs becomes incomplete there. If the effective impurity potential is sufficiently strong to completely suppress the chiral order at surrounding sites,$^{31}$ the magnetic field induced by this circulating current will be on the order of $B_{\text{in}}$ [Eq. (13)], which can be easily shown by standard calculations in classical electrodynamics based on the expressions for the surface current given by Eqs. (5) and (7). A crucial point here is that $B_{\text{in}}$ at the $\mu^+$ site is free from the Meissner screening effect. On the other hand, if the effective impurity potential is not so strong, the induced magnetic field at the $\mu^+$ site is decreased considerably depending on the strength of the potential. Indeed, the magnetic field observed by $\mu$SR, $B_{\mu\text{SR}}^{\text{obs}} \simeq 0.5 \, \text{G}$, is rather small compared with $B_{\text{in}} \simeq 2.3 \times 10^{-3} \, \text{T} = 23 \, \text{G}$ given by Eq. (51). This fact can be understood by assuming that the effective impurity potential from $\mu^+$ is only moderate. However, this should be verified by an explicit model calculation, which is left for a future study.

On the other hand, the IAM of each band gives additive contributions to the total IAM on the order of $\mathcal{O}(N_s \hbar)$ which can be probed by the so-called Richardson-Einstein-de Haas effect, which has been used to detect the macroscopic spin angular momentum in the ferromagnetic state.$^{36, 37}$ Since the size of the IAM is on the same order as the spin angular momentum of ferromagnetic compounds, it is expected that the IAM can be observed by this effect in practice, although a sufficiently low temperature will be required.

7. Conclusion

On the basis of the tight-binding model with the nearest-neighbor attraction on a square lattice, the IMM and IAM have been calculated by solving the Bogoliubov-de Gennes equation. It turned out that, in the ground state, the IMM $m_z$ per site divided by $\mu_0$ is on the order
of $\mu_B$ and the IAM is on the order of $L_{\text{in}} \sim hN$. In particular, in the dilute limit, the result strongly indicates that the IAM approaches $L_{\text{in}} = hN/2$, which is the value first predicted by Ishikawa.\textsuperscript{8) The IAM and IMM are induced by the surface current flowing in a thin layer with a width on the order of the coherence length $\xi_0$.}

It has been shown that thus created IAM is partially screened by the Meissner current if the particles have an electric charge. The extent of the cancellation depends on the ratio of the effective mass $m_{\text{band}}^{\text{FS}}$ of quasiparticles near the Fermi level to that of the harmonic average $m_{\text{band}}^{\text{occ}}$ over the occupied state: if they were equal, the cancellation would become perfect. This is in marked contrast to the IMM, which is almost completely screened by the Meissner effect if the spontaneous magnetic field created by the IMM is smaller than the lower critical field $H_{c1}$, as in the case of Sr$_2$RuO$_4$.

On the other hand, it turned out that the multiband effect is important, as in the case of Sr$_2$RuO$_4$. Namely, a certain amount of cancellation of the IMM occurs among particle and hole bands because they have charges with different signs, while such a cancellation does not occur for the IAM.

An interpretation of the spontaneous magnetic field observed in Sr$_2$RuO$_4$ by $\mu$SR, and a possible means of probing the IAM have also been proposed.

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Appendix A: Harmonic Average of Band Mass of Electrons on Square Lattice

In this appendix, we derive the harmonic average of the band mass of electrons with dispersion, Eq. (29). The inverse mass tensor $\hat{m}^{-1}$ of a band electron is given by

$$\hat{m}^{-1} = \frac{1}{\hbar^2} \begin{pmatrix} \frac{\partial^2 \epsilon_k}{\partial k_x^2} & \frac{\partial^2 \epsilon_k}{\partial k_y \partial k_x} \\ \frac{\partial^2 \epsilon_k}{\partial k_y \partial k_x} & \frac{\partial^2 \epsilon_k}{\partial k_y^2} \end{pmatrix}.$$  \hfill (A-1)

Substituting the dispersion of an electron, Eq. (29), into this expression, it is easily seen that the matrix in Eq. (A-1) is diagonal. Indeed, its explicit form is

$$\hat{m}^{-1} = \frac{2ta^2}{\hbar^2} \begin{pmatrix} \cos k_x a & 0 \\ 0 & \cos k_y a \end{pmatrix}. $$  \hfill (A-2)

The averaging of $\hat{m}^{-1}_{xx}$ over the occupied states at half-filling is performed as follows:

$$\langle \hat{m}^{-1}_{xx} \rangle = \frac{2ta^2}{\hbar^2} \int_0^{\pi/a} dk_y \int_0^{\pi/a-k_y} dk_x \cos k_x a = \frac{2ta^2}{\hbar^2} \times \frac{4}{\pi^2},$$  \hfill (A-3)

where the area of integration with respect to $k_x$ and $k_y$ is restricted to the part surrounded by the dashed line (the Fermi surface) of the first quadrant in Fig. A-1. The averaging of $\hat{m}^{-1}_{yy}$ is performed in a similar way, giving the same value. Then, using the definition of the band mass (near the Γ-point) $m_b$ [Eq. (26)], the harmonic average of the band mass $m_{\text{band}}^{\text{occ}}$ over the occupied states is given by

$$m_{\text{band}}^{\text{occ}} = \frac{1}{\langle \hat{m}^{-1}_{xx} \rangle} = \frac{\pi^2}{4} m_b. $$  \hfill (A-4)

Fig. A-1. Brillouin zone of electrons on the square lattice. Dashed lines indicate the Fermi surface at half-filling and dotted lines indicate the order of two-dimensional integration.
Appendix B: Effective Mass Averaged over the Fermi Surface

In this appendix, we derive the effective mass averaged over the Fermi surface of electrons with dispersion, Eq. (29). The inverse mass tensor $\hat{m}^{-1}$ of this band electron is given by Eq. (A-2). Therefore, the average of $\hat{m}^{-1}_{xx}$ over the Fermi surface is calculated as follows:

\[
\langle \hat{m}^{-1}_{xx} \rangle = \frac{2ta^2}{\hbar^2} \frac{\int d\mathbf{k} \cos k_x a \delta (\cos k_x a + \cos k_y a + \mu/2t)}{\int d\mathbf{k} \delta (\cos k_x a + \cos k_y a + \mu/2t)}
\]

\[
= \frac{1}{2} \frac{2ta^2}{\hbar^2} \frac{\int d\mathbf{k} (\cos k_x a + \cos k_y a) \delta (\cos k_x a + \cos k_y a + \mu/2t)}{\int d\mathbf{k} \delta (\cos k_x a + \cos k_y a + \mu/2t)}
\]

\[
= -\frac{1}{2} \frac{\mu a^2}{\hbar^2}.
\]  

(B-1)

The expression for $\langle \hat{m}^{-1}_{yy} \rangle$ is also given by Eq. (B-1). Therefore, the effective mass averaged over the Fermi surface $m_{\text{band}}^{\text{FS}}$ is given by

\[
m_{\text{band}}^{\text{FS}} = -\frac{2\hbar^2}{\mu a^2}.
\]

(B-2)

It is remarked that $m_{\text{band}}^{\text{FS}}$ diverges toward half-filling, i.e., $\mu \to 0$, which is consistent with the existence of the van Hove singularity in the density of states at $\mu = 0$. On the other hand, in the dilute limit, i.e., $\mu \to -4t$, $m_{\text{band}}^{\text{FS}}$ approaches $\hbar^2/2ta^2$, which is the same as the band mass $m_b$ near the $\Gamma$-point defined by Eq. (26). This guarantees the validity of the definition in Eq. (B-2). In the almost-filled case, $\mu \simeq 4t$, $m_{\text{band}}^{\text{FS}}$ approaches $-\hbar^2/2ta^2 = -m_b$, the hole band mass near the top of the band.
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