PERIODIC BOUNCING SOLUTIONS FOR HILL’S TYPE SUB-LINEAR OSCILLATORS WITH OBSTACLES

CHAO WANG∗
School of Mathematics and Statistics, Yancheng Teachers University
Yancheng, 224002, China

QIHUAII LIU
School of Mathematics and Computing Science
Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation
Guilin University of Electronic Technology, Guilin, 541004, China

ZHIGUO WANG
School of Mathematical Sciences, Soochow University
Suzhou, 215006, China

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Abstract. In this paper, we prove the existence and multiplicity of subharmonic bouncing motions for a Hill’s type sublinear oscillator with an obstacle. Furthermore, we also consider the existence, multiplicity and dense distribution of symmetric periodic bouncing solutions when the weight function is even. Based on an appropriate coordinate transformation and the method of phase-plane analysis, we can study our main results via Poincaré map by applying some suitable fixed point theorems.

1. Introduction. In this paper, we consider the existence and multiplicity of periodic motions as well as the dense distribution of symmetric periodic bouncing solutions for a Hill’s type sublinear oscillator with an obstacle

\[\begin{cases}
  x'' + q(t)g(x) = p(t, x), \text{for } x(t) > 0, \\
  x'(t_0) \geq 0, \\
  \text{if } x(t_0) = 0, \text{then } x'(t_0+) = -x'(t_0-),
\end{cases}\]

where \(q : [0, 2\pi] \to \mathbb{R}^+ \setminus \{0\}\) is a continuous and 2\(\pi\)-periodic function, \(p : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}\) is continuous, bounded and 2\(\pi\)-periodic in the first variable such that \(p(t, 0) \equiv 0\) for each \(t \in [0, 2\pi]\), and \(g : \mathbb{R} \to \mathbb{R}\) is locally Lipschitzian continuous satisfying \(g(0) = 0\). Here, we denote by \(\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)\). We also need the following hypotheses:

\[
\begin{align*}
  (g_0) \quad & \lim_{x \to +\infty} g(x) = +\infty, \\
  (\tau^+_{\infty}) \quad & \lim_{h \to +\infty} \tau^+_g(h) = \infty,
\end{align*}
\]

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∗ Corresponding author.
where

\[ \tau_g^+(h) = \sqrt{2} \int_0^{G_x'(h)} \frac{1}{\sqrt{h - G(s)}} \, ds, \]

\[ G(x) = \int_0^x g(s) \, ds. \]

Now we first recalling the definition of bouncing solutions with completely elastic impacts at \( x = 0 \) given in [12].

**Definition 1.1.** A continuous function \( x : \mathbb{R} \to \mathbb{R}^+ \) is called a bouncing solution of equation (1.1) if the following conditions hold:

1. \( x(t) \geq 0 \) and \( x(t) + |x'(t)| > 0 \) for all \( t \in \mathbb{R} \);
2. the set \( W = \{ t : t \in \mathbb{R}, x(t) = 0 \} \) is discrete and not empty;
3. \( x'(t_0+) = -x'(t_0-) \) for any \( t_0 \in W \);
4. given an interval \( I \subset \mathbb{R} \), if \( I \cap W = \emptyset \), then \( x \in C^2(I, \mathbb{R}^+) \) and it is a classical solution of the regular equation

\[ x'' + q(t)g(x) = p(t, x). \]

We call \( t_0 \) a bouncing moment of the solution \( x(t) \), and we say \( x(t) \) a \( 2k\pi \)-periodic bouncing solution if \( x(t) \) is also \( 2k\pi \)-periodic (\( k \in \mathbb{Z}^+ \)).

**Remark 1.** Our definition of bouncing solution is slightly different from that in [1, 5], where the bouncing solution admits remaining attached to the obstacle for some time. In fact, the concept of the solutions of impact equations in our context are as the same as the definition in [11, 12].

The resonance problems of oscillators are the classical problems and there are numerous results upon it for the linear, asymptotically linear and linear-like impact equations [1, 5, 7, 9, 12, 14, 13]. In [9, 12, 14], some linear and asymptotically linear Duffing type equations are concerned and the existence of periodic bouncing solutions or the invariant curves have been proved correspondingly by applying different twisted map theorem to an adequate Poincaré section called the successor map. We refer the readers to [10] for details about the successor map, by which Ortega study the boundedness of solutions of an asymmetric nonlinear oscillators; also see [11, 15] for equations with singularity. We point out that the successor map approach is not always available for other kinds of equations. For instance, since lacks of the property of strong oscillation, it is inconvenient to define the successor map for a sub-linear equation. Inspired by [7], an approximating strategy is considered in [5], by a limit procedure, to prove the existence of periodic admissible bouncing solution for a class of nonlinear impact oscillators. In fact, in [5], the restoring forces are asymptotically linear-like functions which are so called avoiding-resonance-points type. The approximating method is also used in [1, 13].

The generalized resonance problems are also considered for some nonlinear impact oscillators, for example, periodic motions of a class of sublinear plane Hamiltonian system with obstacle are studied in [3, 6] by applying calculation of variation; periodic motions of impact oscillators with superlinear restoring force nearby the origin and with bounded restoring force are considered in [2, 16] correspondingly by the phase-plane analysis method; the quasi-periodic motions and Littlewood’s boundedness are considered in [17]. Obviously, it is difficult to obtain the multiplicity of periodic bouncing solutions via only the calculation of variation. Recently,
when \( p(t, x) \equiv 0 \), the problem of the existence and multiplicity of symmetric and periodic solutions are considered to a class of super-linear equations in [18] where \( q(t) \) are periodic and may change sign. By an appropriate coordinate transformation, the problem stated above can be solved by a zero point theorem used via Poincaré maps.

Our aim is to consider the bouncing periodic motions of Hill’s type equation (1.1) with sublinear assumption provided by time-map \((\tau_0)\). The following theorem is the main results.

**Theorem 1.2.** Assume that \((g_0)\) and \((\tau_0^+)\) hold. Then for each \( j \in \mathbb{Z}^+ \), there exists \( m_j \in \mathbb{Z}^+ \) such that for each \( m > m_j \), equation (1.1) has at least two \(2m\pi\)-periodic bouncing solutions which have exactly \( j \) impacts on \([0, 2m\pi]\). Furthermore, if \((m, j) = 1\), then \(2m\pi\) is the least period. Moreover, for each \( m \in \mathbb{Z}^+ \), there exists a positive constant \( L_m \) such that \(|(x(t), x'(t))| \leq L_m\) for each \(2m\pi\)-periodic bouncing solution \( x(t) \).

Furthermore, when \( q(t) \) is an even function, we shall consider the existence, multiplicity of even and periodic bouncing solutions of system (1.1). We have

**Theorem 1.3.** Assume that \((g_0)\) and \((\tau_0^+)\) hold. Then for each \( j \in \mathbb{Z}^+ \), there exists \( m_j \in \mathbb{Z}^+ \), for \( \forall m > m_j \), equation (1.1) has at least \( j \) even and \(4m\pi\)-periodic bouncing solutions which have exactly \( k \) \((k = 1, 2, \ldots, j)\) impacts on \([0, 4m\pi]\). Furthermore, if \((2m, k) = 1\) \((k = 1, 2, \ldots, j)\), then either \(4m\pi\) or \(2m\pi\) is the least period.

Moreover, we also obtain the dense distribution of the bouncing \( \varepsilon^- \) points which correspond to the even and periodic bouncing solutions, see Theorem 5.5.

The rest of this paper is organized as follows. In Section 2, a pair of different polar coordinate transformations are performed alternately after and before bouncing times, by which we obtain a new system which is equivalent to equation (1.1). This new system is composed by two equations that can be used alternately in consecutive adjacent intervals relatively. Furthermore, by the polar coordinate transformations, we can define a continuous lifting of Poincaré map of impact equation. Thus, the traditionary strategy of proof can be used via generalized Poincaré–Birkhoff twisted fixed point theorem by proving the property of twist of Poincaré map on annulus. To this end, the fundamental phase-plane analysis are performed in Section 3. The existence of infinitely many subharmonic solutions of (1.1) will be proved in Section 4. Finally, the existence and multiplicity, as well as the distribution of the even and periodic bouncing solutions will be considered in Section 5.

Along this paper, we always denote by \( \mathbb{R}^+ = [0, +\infty) \), \( \mathbb{R}^+_0 = (0, +\infty) \).

**2. Coordinate transformation.** Write \( x' = y \), and equation (1.1) is equivalent to the following one order system

\[
\begin{cases}
x' = y, & y' = -q(t)g(x) + p(t, x), \quad \text{for } x(t) > 0, \\
x(t) & \geq 0, \\
& \text{if } x(t_0) = 0, \text{ then } x'(t_0^+) = -x'(t_0^-) .
\end{cases}
\]  

(2.1)

We note that the bouncing solutions of system (2.1) are defined on the impact phase-plane (i.e. \( \{(x, y) : (x, y) \in \mathbb{R}^2, x \geq 0\} \)).

Since \( g(0) = 0 \) and \( p(t, 0) \equiv 0 \) for each \( t \in [0, 2\pi] \), the origin is an equilibrium point of system (2.1), and so any other solutions shouldn’t go through the origin by the uniqueness of the solutions.
Notice that at any bouncing moment \( t_0 \), the solution \( x(t) \) of the impact system satisfies that
\[
x(t_0^+) = x(t_0^-) = 0, \quad x'(t_0^+) = -x'(t_0^-) \text{ and } |x(t_0^+)| + |x'(t_0^+)| = |x(t_0^-)| + |x'(t_0^-)|.
\]

Doing as in [18], denote by
\[
\Xi_1 := \bigcup_{k \in \mathbb{Z}} \left\{ \theta : 4k\pi - \frac{3}{2}\pi < \theta < 4k\pi + \frac{\pi}{2} \right\}
\]
and
\[
\Xi_2 := \bigcup_{k \in \mathbb{Z}} \left\{ \theta : 4k\pi + \frac{\pi}{2} < \theta < 4k\pi + \frac{5}{2}\pi \right\}.
\]
Let us define a map
\[
\phi_1 : \mathbb{R}_0^+ \times \Xi_1 \to \mathbb{R}_0^+ \times \mathbb{R}, \\
(r, \theta) \mapsto (x, y) = (x_1(r, \theta), y_1(r, \theta)),
\]
by
\[
\begin{align*}
x_1 &= 2\sqrt{r} \cos \left( \frac{\theta}{2} + \frac{\pi}{4} \right), \\
y_1 &= 2\sqrt{r} \sin \left( \frac{\theta}{2} + \frac{\pi}{4} \right).
\end{align*}
\]
Similarly, let
\[
\phi_2 : \mathbb{R}_0^+ \times \Xi_2 \to \mathbb{R}_0^+ \times \mathbb{R}, \\
(r, \theta) \mapsto (x, y) = (x_2(r, \theta), y_2(r, \theta)),
\]
defined by
\[
\begin{align*}
x_2 &= 2\sqrt{r} \cos \left( \frac{\theta}{2} - \frac{3}{4}\pi \right), \\
y_2 &= 2\sqrt{r} \sin \left( \frac{\theta}{2} - \frac{3}{4}\pi \right).
\end{align*}
\]

We can verify that, for each \( k \in \mathbb{Z} \), \( \phi_1 \) and \( \phi_2 \) are one to one, continuous mappings from
\[
\left\{ (r, \theta) : r > 0, \ 4k\pi - \frac{3}{2}\pi < \theta < 4k\pi + \frac{\pi}{2} \right\}
\]
and
\[
\left\{ (r, \theta) : r > 0, \ 4k\pi + \frac{\pi}{2} < \theta < 4k\pi + \frac{5}{2}\pi \right\}
\]
to the set \( \{(x, y) : (x, y) \in \mathbb{R}^2, x > 0\} \), respectively. Moreover, for each sequence of points \( \{(r_n, \theta_n)\}_n \), \( r_n > 0 \) such that if
\[
r_n \to r_0 > 0, \quad \theta_n \to (4k\pi - \frac{3}{2}\pi)^+ (\text{or} (4k\pi + \frac{\pi}{2})^+),
\]
as \( n \to +\infty \), then
\[
x_n \to 0, \quad y_n \to -2\sqrt{r_0};
\]
if
\[
r_n \to r_0 > 0, \quad \theta_n \to (4k\pi - \frac{3}{2}\pi)^- (\text{or} (4k\pi + \frac{\pi}{2})^-),
\]
then
\[
x_n \to 0, \quad y_n \to 2\sqrt{r_0}.
\]
Therefore, for each solution \((x(t), y(t))\) of (2.1) defined on its maximal interval \(I\) with \(x(t) + |y(t)| > 0\), we can find a continuous function \((r(t), \theta(t))\) \(\subset \mathbb{R}_0^+ \times \mathbb{R}\), \(t \in I\), such that (2.2) is satisfied as \(\theta \in \Xi_1\) and (2.3) is satisfied as \(\theta \in \Xi_2\).

Moreover, for each \(t_0 \in I\), if
\[
\theta(t_0) = 4k'\pi - \frac{3}{2}\pi \quad (\text{resp. } \theta(t_0) = 4k'\pi + \frac{\pi}{2})
\]
for some \(k' \in \mathbb{Z}\), then we have \(x(t_0) = 0\) and
\[
x'(t_0-) = -x'(t_0+) = -2\sqrt{r_0} < 0 \quad (\text{or } x'(t_0-) = x'(t_0+) = 2\sqrt{r_0} > 0),
\]
where \(r(t_0) = r_0\). This corresponds to the bouncing moment.

**Definition 2.1.** We call \((r(t), \theta(t))\) the polar form of \((x(t), y(t))\) in the sense of the variable transformations (2.2) and (2.3).

**Remark.** The use of the variable transformations (2.2) and (2.3) are alternate before and after \(t_0\), where \(x(t_0) = 0\), corresponding to the bouncing moments. We refer the readers to [18] (Remark 2.1) for more detailed stating.

Therefore, by coordinate transformation (2.2)-(2.3), the equation (2.1) are equivalent to the following systems
\[
\begin{aligned}
&\begin{cases}
&\quad r' = 2r \sin(\frac{\theta}{2} + \frac{\pi}{4}) \cos(\frac{\theta}{2} + \frac{\pi}{4}) \left[1 - \frac{q(t)g(x)}{4} + \sqrt{r}p(t, x) \sin(\frac{\theta}{2} + \frac{\pi}{4})\right], \\
&\quad \theta' = -2\sin^2(\frac{\theta}{2} + \frac{\pi}{4}) - \frac{q(t)g(x)}{4} - \frac{p(t, x)}{\sqrt{r}} \cos(\frac{\theta}{2} + \frac{\pi}{4}),
\end{cases} \\
&\text{when } r > 0, 4k\pi - \frac{3}{2}\pi < \theta < 4k\pi + \frac{\pi}{2}; \text{ and}
\end{aligned}
\tag{2.4}
\]
and
\[
\begin{aligned}
&\begin{cases}
&\quad r' = 2r \sin(\frac{\theta}{2} + \frac{5\pi}{4}) \cos(\frac{\theta}{2} + \frac{5\pi}{4}) \left[1 - \frac{q(t)g(x)}{4} + \sqrt{r}p(t, x) \sin(\frac{\theta}{2} + \frac{5\pi}{4})\right], \\
&\quad \theta' = -2\sin^2(\frac{\theta}{2} + \frac{5\pi}{4}) - \frac{q(t)g(x)}{4} - \frac{p(t, x)}{\sqrt{r}} \cos(\frac{\theta}{2} + \frac{5\pi}{4}),
\end{cases} \\
&\text{when } r > 0, 4k\pi + \frac{\pi}{2} < \theta < 4k\pi + \frac{5\pi}{2}; \text{ and}
\end{aligned}
\tag{2.5}
\]
and
\[
\begin{aligned}
&\begin{cases}
&\quad \theta(t_0) = 2k_0\pi - \frac{3\pi}{2}, \\
&\quad r(t_0) = \lim_{t \to t_0} r(t),
\end{cases} \quad \text{when } r(t) > 0, \lim_{t \to t_0} \theta(t) = 2k_0\pi - \frac{3\pi}{2}, \quad k_0 \in \mathbb{Z}. \quad (2.6)
\end{aligned}
\]

**Definition 2.2.** A continuous function \((r(t), \theta(t))\) defined in \(I \subset \mathbb{R}\) is called a solution of systems (2.4)-(2.5)-(2.6) if the following conditions hold:

1. \((r(t)) > 0\) for each \(t \in I\);
2. for each interval \(I_1 \subset I\), if there is a \(k \in \mathbb{Z}\) such that \(4k\pi - \frac{3}{2}\pi < \theta(t) < 4k\pi + \frac{\pi}{2}\) for each \(t \in I_1\), then \((r(t), \theta(t))\) is a solution of (2.4) on \(I_1\);
3. for each interval \(I_2 \subset I\), if there is a \(k \in \mathbb{Z}\) such that \(4k\pi + \frac{\pi}{2} < \theta(t) < 4k\pi + \frac{5\pi}{2}\) for each \(t \in I_2\), then \((r(t), \theta(t))\) is a solution of (2.5) on \(I_2\);
4. (2.6) holds for each \(t_0 \in I\) such that \(\lim_{t \to t_0} \theta(t) = 4k\pi - \frac{3\pi}{2}\) or \(\lim_{t \to t_0} \theta(t) = 4k\pi + \frac{\pi}{2}\).

By the Definition 2.2, we see that the continuous solution switches between (2.4) and (2.5) before and after every bounce moment. In fact, each left limit at a bounce moment for a continuous solution of (2.4) can be regarded as an initial condition of the Cauchy problem of (2.5) and vice versa.
Remark 3. Obviously, if \((x(t), y(t))\) is a solution of (2.1) defined on its maximal interval \(I\) with \(x(t) + |y(t)| > 0\), then, by Definition 2.1, its polar form \((r(t), \theta(t))\) is a solution of systems (2.4)-(2.5)-(2.6) defined on \(I\). Conversely, assume that \((r(t), \theta(t))\) is a continuous solution of systems (2.4)-(2.5)-(2.6) defined on \(I \subset (a, b)\), then, by (2.2)-(2.3), \((x(t), y(t))\) is a solution of (2.1) defined on \(I\) with \(x(t) + |y(t)| > 0\). In particular, if \(I = \mathbb{R}\), \((x(t), y(t))\) is a bouncing solution of (2.1) by Definition 1.1.

Remark 4. We can check that the solution of the problem of initial value of systems (2.4)-(2.5)-(2.6) is unique and continuously dependent on the initial values.

Remark 5. The functions lie in the right side of equation (2.4)-(2.5) are continuously differentiable with respect to the variables \(r\) and \(\theta\) when \(t \neq t_0\). When \(t = t_0\), together with (2.4) and (2.5) we have \(\theta'_-(t_0) = \theta'_+(t_0) = -2\) and \(r'_-(t_0) = r'_+(t_0) = 0\), noting that \(x(t_0) = 0\). So, by the existence of derivatives theorem, we define complementally \(\theta'(t_0) = -2 < 0\) and \(r'(t_0) = 0\). Thus, the functions \(\theta(t)\) and \(r(t)\) are differentiable in a neighbourhood of \(t_0\). And, \(\theta(t)\) is also strictly monotone decreasing in a neighbourhood of \(t_0\).

Lemma 2.3 ([18]Lemma 2.2). The functions \(x(r, \theta)\) and \(y(r, \theta)\) are 2\(\pi\) periodic with respect to the variable \(\theta\), that is, \(x(r, \theta + 2\pi) = x(r, \theta)\), \(y(r, \theta + 2\pi) = y(r, \theta)\).

Now, let \((x(t; x_0, y_0), y(t; x_0, y_0))\) be a solution of (2.1) defined in an interval \(I\) with the initial conditions
\[
x(0; x_0, y_0) = x_0, \quad y(0; x_0, y_0) = y_0
\]
satisfying \(x(t; x_0, y_0) + |y(t; x_0, y_0)| > 0\) for each \(t \in I\). We denote by \((r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))\) the polar form of the solution \((x(t; x_0, y_0), y(t; x_0, y_0))\) according to the coordinate transformation (2.2)-(2.3). By Lemma 2.3, we have that, the solutions \((r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))\) and \((r(t; r_0, \theta_0 + 2k\pi), \theta(t; r_0, \theta_0 + 2k\pi))(k \in \mathbb{Z})\) of the systems (2.4)-(2.5)-(2.6) are corresponding to the same solution \((x(t; x_0, y_0), y(t; x_0, y_0))\) of the equation (2.1) with initial value \((x_0, y_0)\) for \(t = 0\), where \((x_0, y_0)\) is defined by (2.2) when \(4k_1\pi - \frac{3}{2}\pi < \theta_0 \leq 4k_1\pi + \frac{5}{2}\pi\) and defined by (2.3) when \(4k_1\pi + \frac{5}{2}\pi < \theta_0 \leq 4k_1\pi + \frac{9}{2}\pi(k_1 \in \mathbb{Z})\), respectively.

Now, for each \(r > 0, \theta \in \mathbb{R}\), let
\[
u = \sqrt{2r} \cos \theta, \quad v = \sqrt{2r} \sin \theta. \tag{2.7}
\]
It is easy to check that
\[
du \wedge dv = dr \wedge d\theta.
\]

For each fixed \(T \in \mathbb{R}\), let
\[
P_T : \Omega_T \rightarrow \mathbb{R}_0^* \times \mathbb{R}
\]
\[
(r_0, \theta_0) \mapsto (r(T; r_0, \theta_0), \theta(T; r_0, \theta_0)) \tag{2.8}
\]
be the Poincaré map of time \(T\) associated to system (2.4)-(2.5)-(2.6), where \(\hat{\Omega}_T\) is the set that for each \((r_0, \theta_0) \in \hat{\Omega}_T\), \((r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))\) is continuable in \([0, T]\) with the initial value \((r_0, \theta_0)\) at \(t = 0\). Thus,
\[
P_T : \Omega_T := P_T(\hat{\Omega}_T) \subset \mathbb{R}_0^2 \setminus \{O\}
\]
\[
(u_0, v_0) \mapsto (u(T; u_0, v_0), v(T; u_0, v_0)) \tag{2.9}
\]
is a homeomorphism and area-preserving mapping, where
\[
\Omega_T := \{(u_0, v_0) : u_0 = r_0 \cos \theta_0, \quad v_0 = r_0 \sin \theta_0, (r_0, \theta_0) \in \hat{\Omega}_T\} \subset \mathbb{R}_0^2
\]
is an open set.
Lemma 2.4. Assume \((r(t), \theta(t))\) is a continuous solution of equation (2.4)-(2.5)-(2.6) defined on \(\mathbb{R}\). If \(I \subset \mathbb{R}\) is an interval such that \(0 < r_1 \leq r(t) \leq r_2 < +\infty\) for every \(t \in I\) and \((u(t), v(t))\) crosses the positive half \(v\)-axis \(j\) times in \(I\), then, there is a continuous bouncing solution \(x(t)\) of system (2.1) defined on \(\mathbb{R}\) satisfying that there are exactly \(j\) impacts in \(I\).

Remark 6. Particularly, if there is a continuous solution of (2.2)-(2.3)-(2.4) satisfying \(r(2k\pi) = r(0)\) and \(|\theta(2k\pi) - \theta(0)| = 2j\pi, k \in \mathbb{Z}_0^+, j \in \mathbb{Z}_0^+\), then, there is a \(2k\pi\)-periodic bouncing solution of (2.1) with exactly \(j\) impacts in a period.

3. Phase-plane analysis. In the following, we always denote by \((r(t); 0, \theta_0)\), \(\theta(t); 0, \theta_0)\) the solution of (2.4)-(2.5)-(2.6) with the initial value \((r_0, \theta_0) \in \mathbb{R}_0^+ \times \mathbb{R}\) at \(t = 0\). For simplicity, we denote the solution by \((r(t); 0, \theta_0), \theta(t); 0, \theta_0)\) when \(t_0 = 0\).

Let \(p = (r_0, \theta_0) \in \mathbb{R}_0^+ \times \mathbb{R}\) and \((r(t); a, \theta_0), \theta(t); a, \theta_0)\) be a continuous solution of (2.4)-(2.5)-(2.6) defined on certain interval \(I \subset \mathbb{R}\). For each \([a, b] \subset I\), let

\[
rot_{(a, b)} p := \int_a^b \theta' dt = \theta(b) - \theta(a),
\]

where \(\theta(b) - \theta(a)\) is the angle spanned by the vector \((u(t), v(t))\) along the time \(t \in [a, b]\).

Since \(\theta'(t_0)\) exists for the bouncing moment \(t_0\), in the following, when we estimate the value of \(\theta(t)\) on the interval \(I \subset \mathbb{R}\), we will no longer consider whether there is a bounce moment in \(I\).

Lemma 3.1. Assume that \((g_0)\) holds, then for any \((r_0, \theta_0) \in \mathbb{R}_0^+ \times \mathbb{R}\), the solution \((r(t); 0, \theta_0), \theta(t); 0, \theta_0)\) of (2.4)-(2.5)-(2.6) defines on the interval \((-\infty, +\infty)\).

Proof. Let \(T > 0\) be an arbitrary positive real number. We will prove that \((r(t); 0, \theta_0), \theta(t); 0, \theta_0)\) can be continuos on \([-T, T]\). To this aim, in the following, we finish the proof of it with three steps.

First step. By \((g_0)\), we can take a \(d_0 > 1\) sufficiently large such that

\[
q_1 g(x) \geq 2p_0
\]

for each \(x > d_0\), where \(q_1 = \min\{q(t), p(t, x) : (t, x) \in [0, 2\pi] \times \mathbb{R}\}\).

Now, we give a partition of the impact phase plane \(XOY\). Denote

\[
I = \{(x, y) : 0 \leq x \leq d_0, \ y > 0\},
\]

\[
II = \{(x, y) : x \geq d_0, \ y \geq 0\},
\]

\[
III = \{(x, y) : x \geq d_0, \ -d_0 \leq y \leq d_0\},
\]

\[
IV = \{(x, y) : x \geq d_0, \ y \leq -d_0\},
\]

\[
V = \{(x, y) : 0 \leq x \leq d_0, \ y \leq 0\},
\]

we will estimate the size of \(\|(x(t), y(t))\|\) in these domains.

(i) Assume that \((x(t), y(t))\) is a solution of (2.1) defined in \([t_1, t_2]\) (or \((t_2, t_1)\)) such that \((x(t), y(t)) \in I\) for each \(t \in \[t_1, t_2\]\) (or \((t_2, t_1)\)). Let

\[
H(x, y) := \frac{1}{2}(y^2 + 1) + (G(x) - G_{\min}),
\]

where \(H(x, y)\) is a function of \((x, y)\) and \(G(x)\) is the energy function of the system (2.1). By the conservation of energy, we have

\[
H(x, y) = constant = H_0.
\]

Since \(H(x, y)\) is a constant, the trajectory \((x(t), y(t))\) of the system (2.1) is confined to the level set \(H(x, y) = H_0\). By the continuity of \(H(x, y)\), we have

\[
\|(x(t), y(t))\| \leq M
\]

for some constant \(M\) such that \(M > 0\).
where $G_{\min} = \inf_{x \in \mathbb{R}^+} G(x) > -\infty$. Obviously, we have $H(x, y) > 0$ for each $(x, y) \in \mathbb{R}^+ \times \mathbb{R}$ and $H(x, y) \to +\infty$ as $|x, y| \to +\infty$.

Let $h(t) := H(x(t), y(t))$, then
\[ \frac{dh(t)}{dt} = |yy' + g(x)x'| = |y[-q(t)g(x) + p(t, x)] + g(x)x'|. \]

Therefore, we have
\[ |h'(t)| \leq Ky < Kh(t), \quad \forall t \in [t_1, t_2), \]
where $K := (q_0 + 1) \max_{0 \leq x \leq d_0} g(x) + p_0$. Furthermore, we have
\[ h(t) \leq h(t_1)e^{K|t-t_1|}, \quad \forall t \in [t_1, t_2). \]
It means that $(x(t), y(t))$ is continuuble to $t_2$. Similar arguments can be given in domain V, so we omit the proof.

(ii) Assume that $(x(t), y(t))$ is a solution of (2.1) defined in $[t_1, t_2)$ such that $(x(t), y(t)) \in \Pi$ for each $t \in [t_1, t_2)$. Without loss of generality, we set $t_1 < t_2$, the case of $t_2 < t_1$ can be argued similarly.

By (2.1), we have
\[ \frac{dy}{dx} = \frac{-a(t)g(x) + p(t, x)}{y}, \]
\[ ydy = (-a(t)g(x) + p(t, x))dx. \]

Take $t_3 \in [t_1, t_2)$ arbitrarily and suppose that $(x_1, y_1), (x_3, y_3)$ are two points in the trajectory when $t = t_1$ and $t = t_3$ respectively. Note that $x = x(t)$ is monotone increasing on $[t_1, t_2)$, so we can define its inverse function $t = t(x)$ defined on $[x_1, x_2)$.

Integrating both sides of above equation, we get
\[ \int_{y_1}^{y_3} ydy = \int_{x_1}^{x_3} (-a(t(x))g(x) + p(t(x), x))dx. \]

Then it follows that
\[ \frac{y_3^2 - y_1^2}{2} = (-a(t(\xi))g(\xi) + p(t(\xi), \xi))(x_3 - x_1), \]
where $\xi \in [x_1, x_3]$.

Then
\[ x_3 - x_1 = \frac{y_3 + y_1}{2(-a(t(\xi))g(\xi) + p(t(\xi), \xi))}(y_3 - y_1). \tag{3.1} \]

Let $\rho(t) = \sqrt{x^2(t) + y^2(t)}$, then $y_1 < \rho_1 := \rho(t_1)$. It is easy to see that $y(t)$ is monotone decreasing by (2.1) and the choose of $d_0$. So, $y_3 < y_1 < \rho_1$. By (3.1),
\[ 0 < x_3 - x_1 < \frac{2\rho_1}{2(a(t(\xi))g(\xi) - p(t(\xi), \xi))}(y_1 - y_3) < \frac{\rho_1}{p_0}(y_1 - y_3). \]

Thus,
\[ |(x_3 - x_1, y_3 - y_1)| < (y_1 - y_3)\sqrt{1 + \frac{\rho_1^2}{p_0^2}} < \rho_1\sqrt{1 + \frac{\rho_1^2}{p_0^2}}, \]
and then
\[ \rho_3 < \rho_1(1 + \sqrt{1 + \frac{\rho_1^2}{p_0^2}}) := F_1(\rho_1), \]
which means the solution $(x(t), y(t))$ can be continuuble to $t_2$. Similarly, we have the same estimate of the size of $r(t)$ in domain IV.
(iii) Assume that \((x(t), y(t))\) is a solution of (2.1) defined in \([t_1, t_2)\) such that 
\((x(t), y(t)) \in \mathbb{H}\) for each \(t \in [t_1, t_2)\). Take \(t_3 \in [t_1, t_2)\) arbitrarily. Since 
\[
|x(t_3) - x(t_1)| = \left| \int_{t_1}^{t_3} x'(s) \, ds \right| \leq d_0 |t_3 - t_1| < d_0 |t_2 - t_1|,
\]
so 
\[
|\rho(t_3) - \rho(t_1)| \leq \sqrt{(x(t_3) - x(t_1))^2 + (y(t_3) - y(t_1))^2} \leq d_0 \sqrt{4 + (t_2 - t_1)^2}, \quad (3.2)
\]
and so 
\[
\rho_3 \leq \rho_1 + d_0 \sqrt{4 + (t_2 - t_1)^2} := F_2(\rho_1), \quad (3.3)
\]
which means the solution \((x(t), y(t))\) can be continuable to \(t_2\).

Thus, it is easy to see that, there is a continuous and monotonically increasing function \(F_T : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) with 
\[
\lim_{\rho \to +\infty} F_T(\rho) = +\infty,
\]
such that for each \(\rho > 2d_0\) and \(r_0 \leq \rho\) we have 
\[
\rho(t; r_0, \theta_0) \leq F_T(\rho)
\]
for each \(\theta_0 \in \mathbb{R}\) and \(t \in [-T, T]\) that 
\[
|\theta(t; r_0, \theta_0) - \theta_0| < 2\pi.
\]
Furthermore, by Lemma 3.3, there is a \(R_T > 0\), such that for each solution 
\((r(t), \theta(t))\) with
\[
r(t) \geq R_T, \quad \forall t \in [t_1, t_2],
\]
where \(0 < t_2 - t_1 \leq T\), we have 
\[
|\theta(t_2) - \theta(t_1)| < 2\pi.
\]
As a consequence, when \(r_0 \geq R_T\), we have 
\[
r(t; r_0, \theta_0) \leq F_T(r_0)
\]
for each \(t \in [-T, T]\) and \(\theta_0 \in \mathbb{R}\).

Second step. By the definition of \(\eta(x, y)\) and (2.4)-(2.5), we have 
\[
r'(t) \leq 2r(1 + q_0 M_0)
\]
whenever 
\[
0 < r(t) \leq \frac{1}{16},
\]
where \(q_0 = \max_{[0, 2\pi]} q(t)\), \(M_1\) is a positive real number satisfying 
\[
|\frac{q(t)}{x}| \leq M_1\text{ for each }|x| \leq \frac{1}{2}
\]
by the locally Lipschitzian condition on \(g\).

Thus, if \(0 < r_0 \leq \frac{1}{16} e^{-2(1+q_0 M_1)T}\), we have 
\[
0 < r_0 e^{-4(1+q_0 M_1)T} \leq r(t; r_0, \theta_0) \leq r_0 e^{2(1+q_0 M_1)T} \leq \frac{1}{16}
\]
for each \(t \in [-T, T]\).

Third step. We complete the proof by pointing out that \(|\theta'(t; r_0, \theta_0)|\) is bounded uniformly on \([-T, T]\) when
\[
\frac{1}{16} e^{-2(1+q_0 M_1)T} \leq r_0 \leq R_L
\]
according to (2.4)-(2.5), which excludes that the bouncing times accumulate at a finite time. 
\[\Box\]
Remark 7. By Lemma 3.1, it is easy to see that \( \Omega_T := \mathbb{R}^+ \times \mathbb{R} \) in (2.8) for each \( T > 0 \). So, for each \( k \in \mathbb{Z}^+ \),
\[
\tilde{P}_{2k\pi} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}
\]
is an area preserving and homeomorphic mapping. So, by the definition of \( P_{T > 0} \) (see 2.9), it is clear that \( \tilde{P} \) is an area preserving and homeomorphic mapping. Furthermore, \( P_{2k\pi} \) is a homeomorphism which preserves the element of area \( dudv = rdrd\theta \).

By Lemma 3.1 and arguments of compactness, we have the following elastic properties.

Corollary 1. Assume that \((g_0)\) holds, then for any \( T > 0 \) and \( \alpha > 0 \), there exists a constant \( \beta(\alpha, T) > 0 \) satisfying the following:

1. If \((r(t), \theta(t))\) is a solution of (2.4)-(2.5)-(2.6) such that \( 0 < r(0) \leq \alpha \), then \( r(t) \leq \beta(\alpha, T) \) for all \(|t| \leq T\);

2. If \((r(t), \theta(t))\) is a solution of (2.4)-(2.5)-(2.6) such that \( r(0) \geq \beta(\alpha, T) \), then \( r(t) \geq \alpha \) for all \(|t| \leq T\).

Given a sufficiently small positive number \( \delta > 0 \), we divide \( uv\)-phase plane into three domains:

\[
I := \{(u, v) : -\frac{\pi}{2} - 2\delta < \theta < -\frac{\pi}{2} + 2\delta\}, \quad II := \{(u, v) : -\frac{\pi}{2} + 2\delta \leq \theta < \frac{\pi}{2}\}
\]

and

\[
III := \{(u, v) : \frac{\pi}{2} < 2\theta < \frac{3\pi}{2} - 2\delta\}.
\]

In the following lemma, we use the notation

\[
\theta_* = \begin{cases} 
\frac{\pi}{4}, & -\frac{3}{2} \pi < \theta (\mod 4\pi) < -\frac{\pi}{2} \\
-\frac{5}{4} \pi, & \frac{\pi}{2} < \theta (\mod 4\pi) < \frac{5}{2} \pi.
\end{cases}
\]

Lemma 3.2. There exists a constant \( R_0 > 1 \) such that

\[
\theta'(t) < 0
\]

for each solution \((r(t), \theta(t))\) of (2.4)-(2.5)-(2.6) and each \( t \) in the continuable interval satisfying \( r(t) > R_0 \).

Proof. From \((g_0)\) we know that \( g(x) \cdot x > 0 \), for all \(|x| \geq M_0\) with some constant \( M_0 > 0 \).

If \((u, v) \in I\), then \( \cos(\frac{\theta}{2} + \theta_*) \geq \cos \delta \). By \((g_0)\), there exists \( A_1 > 0 \) such that \( q_1g(x) > 2P_0 \) for \( r > A_1 \), where \( q_1 = \min_{[0,2\pi]} \{g(t)\} \). Therefore, we have

\[
-\frac{q(t)g(x)}{x} 2 \cos^2 \theta(t) + \frac{p(t,x)}{\sqrt{r}} \cos(\frac{\theta}{2} + \theta_*) \leq -\frac{P_0}{\sqrt{r}} \cos \delta < 0,
\]

where \( P_0 = \max \{ p(t,x) : (t,x) \in [0,2\pi] \times \mathbb{R} \} \). Thus from (2.4)-(2.5) we have \( \theta'(t) < 0 \).

If \((u, v) \in II\), then \( \sin^2(\frac{\theta}{2} + \theta_*) \geq \sin^2 \delta \). Notice that \(|g(x)|\) is bounded for all \( 0 \leq x \leq M_0 \). Take \( A_2 > 0 \) such that

\[
\frac{P_0}{\sqrt{r}} < \sin^2 \delta \quad \text{and} \quad \frac{q_0g_{11}}{\sqrt{r}} < \frac{\sin^2 \delta}{4}, \quad \text{for} \ r > A_2,
\]
where \( q_0 = \max_{[0,2\pi]} \{ q(t) \} \), \( g_1 := \max_{[0,\infty]} |g(x)| \). Then we have

\[
\theta'(t) < -\sin^2 \delta + \frac{\sin^2 \delta}{2} = -\frac{\sin^2 \delta}{2} < 0.
\]

Similarly, if \((u,v) \in \textbf{III}\), we have \( \theta'(t) < 0 \), for \( r(t) > A_2 \). Taking \( R_0 := \max\{A_1,A_2\} \), we complete the proof. \( \square \)

**Lemma 3.3.** Assume that \((g_0)\) and \((\tau^+_g)\) hold. Then, for each large \( T_0 > 0 \) there is an \( R_1 \geq R_0 \) such that if \((r(t),\theta(t))\) is a continuous solution of (2.4)-(2.5)-(2.6) satisfying

\[
r(t) \geq R_1, \quad t_1 \leq t \leq t_2,
\]

and

\[
\theta(t_2) - \theta(t_1) = -2\pi,
\]

then we have

\[
t_2 - t_1 \geq T_0.
\]

**Proof.** By \((\tau^+_g)\), for each \( T_0 > 0 \) there exists a positive number \( h_0 \) such that

\[
\tau^+_g(h) > 4\sqrt{2} T_0
\]

for each \( h \geq h_0 \). By \((g_0)\), there is a positive number \( R_1 \geq R_0 \) such that \( G(x) \geq h_0 \) for each \( x \geq R_1 \).

By (2.2)-(2.3), there exists \([s_1,s_2] \subset [t_1,t_2]\) such that, the solution \((x(t),y(t))\) of (2.1) goes through \( \Omega_1 := \{(x,y) : x \geq d_0, y \geq 0\} \) or \( \Omega_2 := \{(x,y) : x \geq d_0, y \leq 0\} \) around the origin with the direction of clockwise for \( t \in [s_1,s_2] \).

Let’s assume

\[
x(t) > d_0, \quad x'(t) = y(t) > 0, \quad \forall t \in [s_1,s_2]
\]

and

\[
x(s_1) = d_0, \quad y(s_2) = 0, \quad x(s_2) \geq R_1.
\]

From (2.1) we have

\[
y(t) \frac{dy(t)}{dt} = -q(t)g(x) + p(t,x) \frac{dx(t)}{dt} \geq -(g_0 g(x(t)) + P_0) \frac{dx(t)}{dt}.
\]

Integrating on the interval \([t,s_2] \subset [s_1,s_2]\), we obtain

\[
\frac{1}{2} y^2(t) \leq q_0 [G(x_2) - G(x(t))] + P_0 \frac{x(t)}{q_0} (x_2 - x(t))
\]

\[
= q_0 [G(x_2) - G(x(t))](1 + \frac{1}{q_0 g(\xi)}),
\]

where \( \xi \in (x_1, x_2) \). In view of \((g_0)\), taking \( d_0 \) large enough, we have

\[
1 + \frac{1}{q_0 g(\xi)} < 2.
\]

Thus we have

\[
\frac{dx(t)}{dt} = y(t) \leq 2\sqrt{G(x_2) - G(x(t))}, \quad s_1 \leq t \leq s_2,
\]

which follows that

\[
s_2 - s_1 \geq \int_{d_0}^{x_2} \frac{dx}{2\sqrt{G(x_2) - G(x)}} = \frac{1}{2\sqrt{2}} [\tau^+(x_2) - \tau^+(d_0)].
\]
Lemma 3.4. There exists rotating a round. 

By Lemma 3.3, we know that under the condition \((\tau_+^+\), the larger the norm of the solution of \((2.4)-(2.5)-(2.6)\) is, the more slowly the solution goes through the domain \(\Pi\) on the phase plane, which leads to the more time needed for the solution rotating a round.

Inspired by the method of proof in [4], we have the following lemma.

**Lemma 3.4.** There exists \(\nu > \frac{1}{3}\) satisfying that, for each \(R > R_0\), there is \(L(R) > R\) such that if \((r(t), \theta(t))\) is a solution of \((2.4)-(2.5)-(2.6)\) with \(r(t_1) = L(R), r(t_2) = R\) \((or r(t_1) = R, r(t_2) = L(R))\) and

\[
R \leq r(t) \leq L(R), \quad \forall t \in [t_1, t_2],
\]

then

\[
\theta(t_2) - \theta(t_1) < -\nu \cdot 2\pi.
\]

**Proof.** Let \((x(t), y(t))\) be a solution of \((2.1)\) corresponding to \((r(t), \theta(t))\).

By the condition \((g_0)\), there exists \(M_1 > R_0\) such that \([g_1 g(x) - p(t, x)] > P_0\) for all \(x \geq M_1\), where \(q_1 = \min_{t \in [0, 2\pi]} q(t), P_0 = \max_{\{(t, x) : (t, x) \in [0, 2\pi] \times \mathbb{R}\}} p(t, x)\). Take \(\varepsilon \in (0, \frac{P_0}{2})\) and define the function \(g_1 : \mathbb{R} \to \mathbb{R}\) by

\[
g_1(x) = \min \left\{ \frac{P_0}{2}, \inf \{q_1 g(\xi) - p(t, x) - \varepsilon : t \in \mathbb{R}, \xi \geq x\} \right\}.
\]

Obviously, for \(x\) large enough, \(q_1 g(\xi) - p(t, x) - \varepsilon > \frac{P_0}{2}\), which yields that \(g_1(x) = \frac{P_0}{2}\).

By the definition of \(g_1\), we know that \(g_1\) is a monotonically non-decreasing function satisfying that

\[
g_1(x) \leq q_1 g(x) - p(t, x) - \varepsilon, \forall x \in \mathbb{R}^+, t \in [0, 2\pi].
\]

Let us define a monotonically non-decreasing function \(g_2\) such that \(g_2(x) \leq g_1(x), x \in \mathbb{R}^+\) and \(g_2(x) = \frac{P_0}{2}\), for \(x\) large enough. Similarly, we can define a monotonically non-decreasing function \(h(x)\) such that

\[
h(x) \geq q_0 g_1(x) - p(t, x) + \varepsilon, \forall x \in \mathbb{R}^+, t \in [0, 2\pi],
\]

where \(g_0 = \max_{t \in [0, 2\pi]} q(t)\). Let

\[
G(x) = \int_0^x g_2(s)ds, \quad H(x) = \int_0^x h(s)ds.
\]

Obviously, \(G, H\) are convex and \(G(x) < H(x)\) for \(x > 0\). Moreover, since \(g_2(x) = \frac{P_0}{2}\) for large positive values of \(x\), we have

\[
\lim_{x \to +\infty} G(x) \to +\infty.
\]

Let \(B[R] := \{(x, y) : x^2 + y^2 \leq R^2, x \geq 0\}\) and choose \(k > 0\) such that, for all \((x, y) \in B[R]\),

\[
\frac{y^2}{2} + H(x) < k, \quad \frac{y^2}{2} + G(x) < k.
\]

Let \(\alpha > 0\) be such that \(H(\alpha) = k\). Define a curve in the right half plane \(XOY(x \geq 0)\) by

\[
\Gamma_1 := \{(x, y) : \frac{y^2}{2} + H(x) = H(\alpha), x \geq 0, y \geq 0\}.
\]
We can see that \( \Gamma_1 \) intersects the positive y-axis at the point \((0, \sqrt{2k}) := (0, v_1)\). Taking \( v_2 > v_1 \), we can define another curve on the right half plane \( X_0Y(x \geq 0) \) by

\[
\Gamma_2 := \{(x, y) : \frac{y^2}{2} + G(x) = \frac{v_2^2}{2}, x \geq 0, y \leq 0\}.
\]

Similarly, \( \Gamma_2 \) intersects the negative y-axis at the point \((0, -v_2) \). Assume \( \Gamma_2 \) intersects the positive x-axis at the point \((\beta, 0)\). We have \( \beta > \alpha \), since \( G(x) < H(x) \) for \( x > 0 \).

It is easy to see that if the curve \( t \mapsto (x(t), y(t)) \) crosses \( \Gamma_1 \), the crossing must be from the “inside” towards the ”outside”. In fact, along solutions of (2.1), for \( y(t) > 0 \),

\[
\frac{d}{dt} \left( \frac{y^2}{2} + H(x) \right) = y[h(x) - (q(t)g(x) - p(t, x))] > 0,
\]

showing that the vector field associated to the differential equations (2.1) points downwards along the half-line \( \{(x, 0) : x \geq \beta\} \).

Therefore, if \((x, y) : [t_1, t_2] \to \mathbb{R}^+ \times \mathbb{R} \) is a solution of (2.1) with the initial value

\[
(x(t_1), y(t_1)) \in \{(x, y) : y \leq 0, \frac{y^2(t_1)}{2} + G(x(t_1)) > G(\beta)\}
\]

such that \( |(x(t_2), y(t_2))| = R \) and \( |(x(t), y(t))| \geq R, \forall t \in [t_1, t_2], \) then there exists \( t_1 < \omega \leq t_2 \) such that

(a) \( x(\omega) \geq \alpha, x'(\omega) = 0 \);

(b) \( \forall t \in [t_1, \omega), (x(t), y(t)) \) is disjoint with both \( \Gamma_1 \) and \( \Gamma_2 \), which implies \( |(x(t), y(t))| > R \);

(c) \( x(t) \) has at least one zero point on \((t_1, \omega)\).

One sees that the curve \( t \mapsto (x(t), y(t)) \) must circle at least a quadrant before crossing the segment \( \{(x, 0) : \alpha \leq x \leq \beta\} \) and entering the half ball \( B[R] \), that is, \( \theta(\omega) - \theta(t_1) \leq -\pi \). In view of (b) and \( R > R_0 \), we have \( \theta(t_2) - \theta(t_1) < -\pi \). Let \( \nu = \frac{5}{12} \), then we have \( \theta(t_2) - \theta(t_1) < -\nu \cdot 2\pi \).

We have the same discussion for the solution of (2.1) with the initial value \((x(t_1), y(t_1)) \in \{(x, y) : x \in \mathbb{R}^+ \times \mathbb{R} : y \geq 0\}\). Therefore, we can choose a large enough \( L[R] > R \) so that the conclusion of this theorem holds.

**Corollary 2.** For every \( R > R_0 \) and every \( j > 0 \), there exists \( L(R, j) > R \) such that if \( (r(t), \theta(t)) \) is a solution of (2.4)-(2.5)-(2.6) with \( r(t_1) = L(R, j), r(t_2) = R \) (or \( r(t_1) = R, r(t_2) = L(R, j) \)) and

\[
R \leq r(t) \leq L(R, j), \quad \forall t \in [t_1, t_2],
\]

then

\[
\theta(t_2) - \theta(t_1) < -2j\pi.
\]

**Proof.** Let us write \( \nu = \delta + \frac{1}{3} \), so that \( \delta > 0 \). Fix \( R > R_0 \), by Lemma 3.3, and set \( R_1 := L(R) > R \) and \( R_2 := L(R_1) \). Assume \((r(t), \theta(t))\) is a solution of (2.4)-(2.5)-(2.6) such that \( r(t_1) = R_2, r(t_2) = R \) and \( R \leq r(t) \leq R_2, t \in [t_1, t_2] \). Denote \( s_1, s_2 \in [t_1, t_2] \) the first and the last time such that \( r(t) = R_1 \) and notice that \( \theta(s_2) - \theta(s_1) < 0 \), then we have

\[
\theta(t_2) - \theta(t_1) = (\theta(t_2) - \theta(s_2)) + (\theta(s_2) - \theta(s_1)) + (\theta(s_1) - \theta(t_1))
\]

\[
< -(\delta + \frac{1}{3}) \cdot 2\pi - (\delta + \frac{1}{3}) \cdot 2\pi = -(\delta + \frac{2}{3}) \cdot 2\pi.
\]
Let $R_3 := L(R_2)$. It is similar to prove that, if $(r(t), \theta(t))$ is a solution of (2.4)-(2.6) such that $r(t_1) = R_3, r(t_2) = R$ or $r(t_1) = R, r(t_2) = R_3$ and $R \leq r(t) \leq R_3, \forall t \in [t_1, t_2]$, then

$$\theta(t_2) - \theta(t_1) < -(3\delta + 1) \cdot 2\pi.$$  

Repeating the above discussion, for every $j > 0$, we can find $L(R, j) > R$ satisfying the desired conclusion.

**Lemma 3.5.** For each $R_2 > R_1 > R_0$ and if $(r(t), \theta(t))$ is a solution of (2.4)-(2.5)-(2.6) such that

$$R_1 \leq r(t) \leq R_2, \quad \forall t \geq t_0,$$

then

$$\theta(t) - \theta(t_0) \to -\infty, \quad t \to +\infty,$$

uniformly for $t_0 \in [0, 2\pi]$.

**Proof.** Since $\theta'$ is continuous, there exist $a > b > 0$ such that $-a < \theta' < -b$. Then we have $\theta(t) - \theta(t_0) < -b(t - t_0)$. The we can complete the proof when $t \to +\infty$.  

4. **The existence of infinitely many subharmonic solutions.** In this section, we shall prove the existence of infinitely many subharmonic solutions of (1.1) with some lemmas in the previous sections. The proof of the main result is based on a new generalized version of Poincaré–Birkhoff twist theorem provided in [12]. For the reader’s convenience we repeat it in the following.

Let $\mathcal{A}$ and $\mathcal{B}$ be two annuli

$$\mathcal{A} := S^1 \times [a_1, a_2], \quad \mathcal{B} := S^1 \times [b_1, b_2]$$

with $0 < b_1 < a_1 < a_2 < b_2 + \infty$. A map $f : \mathcal{A} \to \mathcal{B}$ possesses a lift $\tilde{f} : \mathbb{R} \times [a_1, a_2] \to \mathbb{R} \times [b_1, b_2]$ with the form

$$\theta' = \theta + h(\theta, r), \quad r' = g(\theta, r),$$

where $h, g$ are continuous and $2\pi$–periodic in $\theta$. We say that $\tilde{f}$ satisfies the boundary twist condition if

$$h(\theta, a_1) \cdot h(\theta, a_2) < 0, \quad \text{for } \theta \in [0, 2\pi].$$

**Lemma 4.1.** ([12] Theorem 2.1) Assume that $f : \mathcal{A} \to \mathcal{B}$ is an area-preserving homeomorphism homotopic to the inclusion such that $f(\mathcal{A}) \cap \partial \mathcal{B} = \emptyset$. Moreover, $f$ possesses a lift $\tilde{f}$ satisfying the boundary twist condition and the area of the two connected components of the complement of $f(\mathcal{A})$ in $\mathcal{B}$ is the same as the area of the corresponding connected components of the complement of $f(\mathcal{A})$ in $\mathcal{B}$. Then, $f$ has at least two geometrically distinct fixed points $(\theta_i, r_i) (i = 1, 2)$, satisfying $h(\theta_i, r_i) = 0$ for $i = 1, 2$.

Now we give the proof of Theorem 1.2. Fix $j > 0$, and by Corollary 2 we can take $R_1 > R_0, R_2 = L(R_1, j+1)$ and $R_3 = L(R_2, j+1)$. Then each solution crossing either $B[R_2] \setminus B[R_1]$ or $B[R_3] \setminus B[R_2]$ turns at least $j+1$ times around the origin. Let $\mathcal{D} = B[R_3] \setminus B[R_1]$. By Lemma 3.5, there is $m_j^* \in \mathbb{Z}^+$ such that: for $m \geq m_j^*(m \in \mathbb{Z}^+)$, if $R_1 \leq r(t) \leq R_3$, $\forall t \in [0, 2m\pi]$, then

$$\theta(2m\pi) - \theta(0) < -2j\pi.$$  

Consider any solution $(r(t), \theta(t))$ of (2.4)-(2.5)-(2.6) with $r(0) = R_2$.
• either \((u(t), v(t)) \in D, \forall t \in [0, 2m\pi]\), where \((u(t), v(t))\) is given by (2.7);
• or there exist a \(t \in (0, 2m\pi)\) such that \((u(t), v(t))\not\in D\).

In the former case we already know that \(\theta(2m\pi) - \theta(0) < -2j\pi\); in the latter one we can select an interval \([t_1, t_2] \subset [0, 2m\pi]\) such that:

• either \(r(t_1) = R_2, r(t_2) = R_1\) and \(R_1 \leq r(t) \leq R_2\) for all \(t \in [t_1, t_2]\);
• or \(r(t_1) = R_2, r(t_2) = R_3\) and \(R_2 \leq r(t) \leq R_3\) for all \(t \in [t_1, t_2]\).

In both these situations we can conclude that \(\theta(2m\pi) - \theta(t_1) < -2(j + 1)\pi\) by the choice of \(R_1, R_2\) and \(R_3\). Note that \([\theta(2m\pi) - \theta(t_2)] + [\theta(t_1) - \theta(0)] < 2\pi\), then we have \(\theta(2m\pi) - \theta(0) < -2j\pi\).

In conclusion, if \((r(t), \theta(t))\) is a solution of (2.4)-(2.5)-(2.6), then
\[
r(0) = R_2 \Rightarrow \theta(2m\pi) - \theta(0) < -2j\pi.
\]

Now fix \(m \geq n^*_0\). By Lemma 3.3 there exists \(S_1 > R_2\) such that, for any solution of (2.4)-(2.5)-(2.6), if \(r(t) \geq S_1\) for each \(t \in [0, 2m\pi]\), then \(\theta(2m\pi) - \theta(t) > -2\pi\).

Moreover by Corollary 1, there exists \(S_2 > S_1\) such that for any solution of (2.4)-(2.5)-(2.6) with \(r(0) = S_2\),
\[
r(t) \geq S_1, \quad \forall t \in [0, 2m\pi].
\]

Therefore,
\[
r(0) = S_2 \Rightarrow \theta(2m\pi) - \theta(0) > -2\pi.
\]

Now we begin to construct two annuli \(\mathcal{A}\) and \(\mathcal{B}\) as follows in order to apply Lemma 4.1 to prove the existence of periodic bouncing equations. By Corollary 1, on one hand, we can choose \(R_1\) suitably large such that for each solution of (2.4)-(2.5)-(2.6) \((r(t), \theta(t))\) with \(r(0) \geq R_1\), we have that
\[
r(t) > R_0, \quad \forall t \in [0, 2m\pi].
\]

On the other hand, there is a positive real number \(S_3 > S_2\) such that
\[
r(t) > S_2, \quad \forall t \in [0, 2m\pi]
\]

for each solution of (2.4)-(2.5)-(2.6) \((r(t), \theta(t))\) with \(r(0) = S_2\). Hence, let \(\mathcal{A}\) be the annulus bounded by \(S^1 \times \{R_2\}\) and \(S^1 \times \{S_2\}\) and let \(\mathcal{B}\) be the annulus bounded by \(S^1 \times \{R_0\}\) and \(S^1 \times \{S_3\}\).

Consider the \(2m\pi\)-Poincaré map
\[
\mathcal{P}_{2m\pi} : \mathcal{A} \to \mathcal{B}
\]

which is determined by \(\mathcal{P}_{2m\pi}\) :
\[
[R_2, S_2] \times \mathbb{R} \ni (r_0, \theta_0) \mapsto (r(2m\pi; 0, r_0, \theta_0), \theta(2m\pi; 0, r_0, \theta_0))
\]

when \(T := 2m\pi\) in (2.9).

By Remark 7, it is easy to see that \(\mathcal{P}_{2m\pi}\) is an area-preserving homeomorphism homotopic to the inclusion and \(\mathcal{P}_{2m\pi}\) satisfies the boundary twist condition. Then, by Lemma 4.1, we have at least two fixed points \((r_i, \theta_i) \in (R_2, S_2) \times \mathbb{R}\) \((i = 1, 2)\), which are corresponding to the solutions \((r(t; 0, r_i, \theta_i), \theta(t; 0, r_i, \theta_i))\) of (2.4)-(2.5)-(2.6) such that \(r(2m\pi; 0, r_i, \theta_i) = r_i\) and \(\theta(2m\pi; 0, r_i, \theta_i) - \theta_i = -2j\pi, \quad i = 1, 2\). Thus it follows that (2.1) has at least two \(2m\pi\)-periodic bouncing solutions \((x(t; 0, x_i, y_i), y(t; 0, x_i, y_i))\) \((i = 1, 2)\) for each \(m \geq m_j\), when \(j \geq 1\), which have exactly \(j\) impacts on \([0, 2m\pi]\), where \(x_i\) and \(y_i\) are given by (2.2)-(2.3). By the choice of \(R_2\), we know that \(r(t; 0, r_i, \theta_i) > R_1\) for each \(t \in [0, 2m\pi]\). Then we know that \(x(t; 0, x_i, y_i)\) \((i = 1, 2)\) are \(2m\pi\)-periodic bouncing solutions of (1.1).
Furthermore, by Lemma 3.3, for any given $m \in \mathbb{Z}^+$, there exists $L_m > 0$ such that for every $2m\pi$-periodic solution $((x(t), y(t))$ of (2.1), $r(t) \leq L_m$, that is, $||((x(t), y(t))|| \leq L_m$.

**Remark 8.** We point out that one can find twice as much periodic bouncing solutions as in previous literatures (e.g. [12]) by using the half-plane phase space.

5. Periodic motions of symmetric impact oscillators. In this section, in addition to the conditions stated above, we also assume that $p : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $q(t) : \mathbb{R} \to \mathbb{R}^+ \setminus \{ 0 \}$ are even functions in $t$. We shall consider the existence, multiplicity of even and periodic bouncing solutions of system (1.1). We call $(x(0), x'(0))$ an impact $\varepsilon$-point of equation (2.1), if $x(t)$ is an even and periodic bouncing solution of equation (1.1) with $x(0) > 0$ and $x'(0) = 0$. Especially, if $x(t)$ has the least period $2m\pi$ ($m \in \mathbb{Z}^+$), we say the impact $\varepsilon$-point $(x(0), x'(0))$ is of order $m$. Moreover, we are also concerned with the distribution of the impact $\varepsilon$-points of system (2.1).

Let $t_0 \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ such that $|a| + |b| > 0$. Assume that $x(t, t_0; a, b)$ is a solution of equation (1.1) with the initial value

$$x(t_0) = a, \quad x'(t_0) = b.$$  

Particularly, we set $x(t; a, b) = x(t; 0, a, b)$ for $t_0 = 0$.

**Lemma 5.1.** Let $m \in \mathbb{Z}^+_0$ and $a \in \mathbb{R}^+_0 = (0, +\infty)$. Assume that $x(t, a) := x(t, 0; a, 0)$ is a solution of (1.1) with

$$x'(m\pi, a) = 0,$$  

then $(a, 0)$ is an impact $\varepsilon$-point of (2.1). Furthermore, $(a, 0)$ is of order $m$ if and only if $x'(k\pi) \neq 0$, $k = 1, \ldots, m - 1$.

**Proof.** Before proving the main conclusion, we shall prove a proposition first. That is, if $x(t)$ is a bouncing solution of (1.1) such that $\omega'(k\pi) = 0$ for $k \in \mathbb{Z}$ a certain integer, then, $\omega(2k - t) = \omega(t)$. In fact, let $\nu(t) = \omega(2k - t)$, then, on the one hand, for each $t \in \mathbb{I}$ we have

$$\nu''(t) + q(t)g(\nu(t)) = \omega''(2k\pi - t) + q(2k\pi - t)g(\omega(2k\pi - t))$$

$$= p(2k\pi - t, \omega(2k\pi - t)) = p(t, \nu(t)),$$

where $\mathbb{I}$ is any interval such that $\nu(t) > 0$ for each $t \in \mathbb{I}$. On the other hand, when $\nu(t_0) = 0$, it also means that $u(2k\pi - t_0) = 0$, we have that

$$\nu'(t_0) = \lim_{t \to t_0^+} \frac{\nu'(t) - \nu'(t_0)}{t - t_0} = \lim_{t \to t_0^+} \frac{\omega'(2k\pi - t) - \omega'(2k\pi - t_0)}{t - t_0} = \lim_{t \to t_0^-} \frac{-\omega'(2k\pi - t) - \omega'(2k\pi - t_0)}{t - t_0}.$$  

So, $\nu(t)$ is also a bouncing solution of (1.1). When $t = k\pi$, we have $\omega(k\pi) = \omega(k\pi)$ and $\omega(k\pi) \neq 0$ since $\omega'(k\pi) = 0$. So $\omega'(k\pi) = -\omega'(k\pi) = 0$ and so, by the theorem of existence and uniqueness of solution, we have $\omega(2k\pi - t) = \omega(t)$.

Now, if $x'(0) = x'(m\pi) = 0$, then by the above proposition, we have $x(-t) = x(t)$ when we take $k = 0$. Furthermore, take $k = m$ and let $t = -t$, then we have $x(2m\pi + t) = x(-t)$, and so we have $x(t) = x(2m\pi + t)$.

By Lemma 2.4 and Lemma 5.1, we have the following result.

**Lemma 5.2.** Let $m \in \mathbb{Z}^+_0$ and $r_0 > 0$. Assume that $(r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ is a solution of (2.4)-(2.5)-(2.6) with $\theta_0 = \frac{\pi}{2}(\text{mod} 2\pi)$. If

$$\theta(m\pi; r_0, \theta_0) = -\frac{\pi}{2}(\text{mod} 2\pi),$$  

then...
then \((2\sqrt{r_0}, 0)\) is an impact \(\varepsilon\)-point of (2.1). Furthermore, \((a, 0)\) is of order \(m\) if and only if \(\theta(k\pi; r_0, \theta_0) \neq -\frac{\pi}{2}(\text{mod} 2\pi), k = 1, \ldots, m - 1\).

In the following, we denote by \((r(t, a), \theta(t, a))\) a solution of (2.4)-(2.5)-(2.6) such that \(r(0) = a\) and \(\theta(0) = -\frac{\pi}{2}(\text{mod} \pi)\). It is easy to see that \((r(t, a), \theta(t, a))\) is continuous with respect to \(t\). By Corollary 1, for \(\alpha > 1\) and \(m \in \mathbb{Z}^+\), if \(\alpha > \beta(\alpha, 2m\pi)\), then \(r(t, a) > \alpha > 1\) for all \(|t| \leq 2m\pi\). By the coordinate transformation (2.2)-(2.3) and Lemma 5.2, we have

**Lemma 5.3.** For any \(\alpha > \beta(\alpha, 2m\pi)\), if the solution \((r(t, a), \theta(t, a))\) of (2.4)-(2.5)-(2.6) satisfies that
\[
\theta(m\pi, a) = -\frac{\pi}{2}(\text{mod} 2\pi),
\]
then \((2\sqrt{a_1}, 0)\) is an impact \(\varepsilon\)-point of order \(m\), that is, \(x(t; 2\sqrt{a_1}, 0)\) is an even and \(2m\pi\)-periodic bouncing solution of (1.1).

### 5.1. Infinitely many even and periodic bouncing solutions

The following theorem gives the existence of infinitely many even and periodic bouncing solutions.

**Theorem 5.4.** Assume that \((g_0)\) and \((\tau^+_{\infty})\) hold. Then for each \(j \in \mathbb{Z}^+\), there exists \(m_j \in \mathbb{Z}^+\), for \(\forall m > m_j\), equation (1.1) has at least \(j\) even and \(4m\pi\)-periodic bouncing solutions which have exactly \(k(k = 1, 2, \ldots, j)\) impacts on \([0, 4m\pi]\). Furthermore, if \((2m, k) = 1\) \((k = 1, 2, \ldots, j)\), then either \(4m\pi\) or \(2m\pi\) is the least period.

**Proof.** Form the proof of Theorem 1.2, we know that for each \(j \in \mathbb{Z}^+\) and \(R_1 > R_0\), there exists \(m_j \in \mathbb{Z}^+\), \(R_2 = R_2(R_1, j + 1) > R_1\) and \(S_2 > R_2\) such that for all \(m > m_j\),
\[
\theta(2m\pi, R_2) - \theta(0, R_2) < -2j\pi
\]
and
\[
\theta(2m\pi, S_2) - \theta(0, S_2) > -2\pi,
\]
where both \((r(t, R_2), \theta(t, R_2))\) and \((r(t, S_2), \theta(t, S_2))\) are solutions of (2.4)-(2.5)-(2.6). Since \(\{\theta(\pi, a); a > \beta\}\) is a continuum for \(\beta > 0\) large enough, there exist \(R_2 < a_k < S_2\) \((k = 1, 2, \ldots, j)\) such that
\[
\theta(2m\pi, a_k) - \theta(0, a_k) = -2k\pi, \quad k = 1, 2, \ldots, j,
\]
where \((r(t, a_k), \theta(t, a_k))\) is the solution of (2.4)-(2.5)-(2.6). Then we have
\[
\theta(2m\pi, a_k) = -\frac{\pi}{2}(\text{mod} 2\pi).
\]
Therefore, by Lemma 5.3, we know that \(x(t; 2\sqrt{a_k})(k = 1, 2, \ldots, j)\) are even and \(4m\pi\)-periodic bouncing solutions of (1.1) which have exactly \(2k\) impacts on \([0, 4m\pi]\). Obviously, if \((2m, k) = 1\) \((k = 1, 2, \ldots, j)\), then \(x(t; 2\sqrt{a_k})\) has the least period \(4m\pi\) or \(2m\pi\), that is, \((2\sqrt{a_k}, 0)\) is an impact \(\varepsilon\)-point of order \(2m\) or order \(m\).

### 5.2. The distribution of even and periodic bouncing solutions

In this subsection, we shall consider the distribution of impact \(\varepsilon\)-points of (2.1), where \(g(x) : \mathbb{R}^+ \to \mathbb{R}\) is once continuously differentiable satisfying
\[
(g_0) \quad g(x) > 0, \quad x > 0, \quad g(0) = 0,
\]
\[
(\tau^+_{\infty}) \lim_{e \to +\infty} \tau^+_{\infty}(e) = +\infty,
\]
\[
q(t) : [0, 2\pi] \to \mathbb{R}^+ \setminus \{0\} \text{ is once continuously differentiable and } 2\pi\text{-periodic.}
\]
Obviously, by \((\hat{g}_0)\) and the existence and uniqueness of solutions, we know that for any \(r_0\) and \(T > 0\), the solution \((r(t; t_0, r_0, \theta_0), \theta(t; t_0, r_0, \theta_0))\) of (2.4)-(2.5)-(2.6) with the initial value \((r_0, \theta_0)\) defines continuously on \([-T, T]\). Moreover, from Theorem 5.4, equation (2.1) has infinitely many even and periodic bouncing solutions.

Take two impact \(\varepsilon\)-points \(p_1\) and \(p_2\), with \(p_1 = (a_1, 0), p_2 = (a_2, 0) (0 < a_1 < a_2)\) and connect two points \(p_1, p_2\) in positive semi \(x\)-axis. We denote by
\[
\overline{p_1p_2} = \{(a, 0) \in \mathbb{R}^2 : a_1 \leq a \leq a_2\}.
\]
Now we state one of our main result in the following.

**Theorem 5.5.** Assume \((\hat{g}_0)\) and \((r_\infty^+)\) hold. Then the segment \(\overline{p_1p_2}\) contains at least one impact \(\varepsilon\)-point of (2.1) being different from \(p_1\) and \(p_2\), which implies the segment \(\overline{p_1p_2}\) contains infinitely many impact \(\varepsilon\)-points of (2.1).

We remark that the impact \(\varepsilon\)-point \((a, 0)\) in the \(xy\)-plane is corresponding to the impact \(\varepsilon\)-point \((0, -a^2/4)\) in the \(uv\)-plane. Moreover, the segment \(\overline{p_1p_2}\) in positive semi \(x\)-axis is corresponding to the segment \(\overline{p'_1p'_2}\) in negative semi \(v\)-axis. We shall follow the method used in [8] to prove Theorem 5.5. It is easy to see that, if \(\theta(t; t_0, r_0, \theta_0) = -\frac{\pi}{2} r (\text{mod } 2\pi)\), then \(r_+'(t_1) = r_-'(t_1), r_-'(t_1) = r_+(t_1)\). Therefore, we know that \((r(t; t_0, r_0, \theta_0), \theta(t; t_0, r_0, \theta_0))\) is differentiable at \(t_1\).

Assume that \((r(t; 0, r_0, \theta_0), \theta(t; 0, r_0, \theta_0))\) is a solution of (2.4)-(2.5)-(2.6) starting from \((r_0, \theta_0)\) at time \(t = 0\). The continuous function \((u(t, 0; a, b), v(t, 0; a, b))\) is obtained by the covering projection (2.7). We denote by \(\mathcal{P}\) the Poincaré mapping such that
\[
\mathcal{P}q = (u(2m\pi, 0; a, b), v(2m\pi, 0; a, b)), \quad q = (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\},
\]
where \(m\) is an integer. \(\mathcal{P}\) is differentiable in \(q\) by the assumption of the differentiability of \(g(\cdot)\) and is one-to-one from \(\mathbb{R}^2 \setminus \{(0, 0)\}\) to \(\mathbb{R}^2 \setminus \{(0, 0)\}\). If \(C\) is a simply closed curve in \(\mathbb{R}^2 \setminus \{(0, 0)\}\), then the image of \(C\) by \(\mathcal{P}\), \(C' = \mathcal{PC}\), is a simply closed curve. Furthermore, by the definition of \(\mathcal{P}\), and considering that \(\mathcal{P}\) is area preserving on the \((r, \theta)\)-half plane, it is easy to check that \(\mathcal{P}\) is an area-preserving mapping on the \(uv\)-plane.

Let \(q_0 > 0\) be such that \(q(t) \geq q_0\), for all \(t \in \mathbb{R}\).

**Proof of Theorem 5.5.** Step I: Assume \(p_1\) and \(p_2\) are impact \(\varepsilon\)-point of order \(n_1\) and \(n_2\), respectively. Let \(m = n_1n_2\), then we have
\[
\mathcal{P}P_1 = P_1, \quad \mathcal{P}P_2 = P_2.
\]
Let \(l_0\) is an open segment jointing \((0, -a_2^2/4)\) to \((0, -a_1^2/4)\), that is,
\[
l_0 = \overline{p'_1p'_2} - \{P_1, P_2\}.
\]
In the following, we shall prove that there exists at least one point \((0, -a_2^2/4) \in l_0\) such that \((r(0; t_0, -a_2^2/4, -\frac{\pi}{2}), \theta(0; t_0, -a_2^2/4, -\frac{\pi}{2}))\) is corresponding to an impact \(\varepsilon\)-point \((0, a)\) of (2.1). Let
\[
l_k = Pl_{k-1}, \quad k \geq 1.
\]
In view of \(PP_1 = P_1\) and \(PP_2 = P_2\), \(l_k\) is a continuous curve jointing \(P_1\) to \(P_2\).

Denote by \(L\) the negative \(v\)-axis,
\[
L = \{(u, v) \in \mathbb{R}^2 : u = 0, v < 0\}.
\]
To the contrary for our assertion, suppose that \(l_0\) contains no \(\varepsilon\)-point. Then we have \(l^k \cap L = \emptyset\), for \(k \geq 1\).
Step II: For any given \( q = (a, b) \in \mathbb{R}^2 \setminus \{O\} \), let \((r(t), q, \theta(t), q) := (r(t; 0, r_0, \theta_0), \theta(t; 0, r_0, \theta_0))\), where \( r_0 \) and \( \theta_0 \) are given by (2.7), \( \theta_0 \in [0, 2\pi] \). We can see that \( \theta(t; q) \) is continuous on \([0, \pi]\) for \(|q| > 0\). We take \( r_1 > R_0 \) so that the open circle centered at origin with radius \( r_1 \) contains points \( P_1 \) and \( P_2 \). It is similar to the proof of Theorem 1.2 that for any positive integer \( j > 2 \), we can take large enough \( r_2 := r_2(r_1, j + 1) > r_1 \) such that

\[
\theta(2m\pi; 0, r_2, \theta_0) - \theta_0 < -2j\pi, \quad \forall \theta_0 \in \mathbb{R},
\]

where \( m \) is a positive integer suitable large. We denote by \( O_1 \) the circle centered at origin with radius \( r_2 \). By Lemma 3.3 and Corollary 1, there exists \( S_2 \gg 1 \) such that

\[
r(t) \geq r_2, \quad \forall t \in [0, 2m\pi]
\]

and \( \theta(2m\pi) - \theta(0) > -2\pi \), where \((r(t), \theta(t))\) is a solution of (2.4)-(2.5)-(2.6) with \( r(0) = S_2 \).

Let \( O_2 \) be a circle centered at origin with radius \( S_2 \). Note that \( \mathcal{P} \) is area-preserving and direction-preserving. Similarly to [8], we can prove that there exists \( k > 0 \) such that \( l_k \cap O_i \neq \emptyset, i = 1, 2 \). Assume \( q_1 \in l_k \cap O_1 \) and \( q_2 \in l_k \cap O_2 \), then we have \( \theta(2m\pi, q_1) - \theta(0, q_1) < -2j\pi \), and \( \theta(2m\pi, q_2) - \theta(0, q_2) > -2\pi \). Since \( \theta(2m\pi, q) - \theta(0, q) \) is continuous for \(|q| > 0\), there exists \( q_3 \in l_k \) such that \( \theta(2m\pi, q_3) = -\frac{\pi}{2} \text{(mod} 2\pi) \) and \( r_2 \leq |q_3| \leq S_2 \).

Let \( q_3 = (a', b') \in \mathbb{R}^2 \). From (2.7), we obtain

\[
u(2m\pi; 0, a', b') = 0.
\]

Then there exists \( q_0 = (0, -\frac{a_0^2}{4}) \in l_0 \) such that

\[
q_3 = \mathcal{P}^k q_0 = (u(2km\pi; 0, 0, -\frac{a_0^2}{4}), u'(2km\pi; 0, 0, -\frac{a_0^2}{4})),
\]

which yields that

\[
\theta(2km\pi + 2m\pi, q_0) = -\frac{\pi}{2} \text{(mod} 2\pi).
\]

Therefore, By Lemma 5.3, \((a_0, 0)\) is an impact \( \varepsilon \)-point of (2.1). \( \square \)

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E-mail address: wangchaosudamath@163.com (C. Wang)
E-mail address: lqh520@tom.com, qhuailiu@gmail.com (Q. Liu)
E-mail address: zgwang@suda.edu.cn (Z. Wang)