Monomial operators

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Abstract. We study monomial operators on \( L^2[0, 1] \), that is bounded linear operators that map each monomial \( x^n \) to a multiple of \( x^{p_n} \) for some \( p_n \). We show that they are all unitarily equivalent to weighted composition operators on a Hardy space. We characterize what sequences \( p_n \) can arise. In the case that \( p_n \) is a fixed translation of \( n \), we give a criterion for boundedness of the operator.

1. Introduction

When studying polynomial approximation in \( L^2[0, 1] \), the following class of operators arises naturally.

Definition 1.1. A monomial operator is a bounded linear operator \( T : L^2[0, 1] \to L^2[0, 1] \) with the property that there exist constants \( c_n \) and \( p_n \) so that

\[
T : x^n \mapsto c_n x^{p_n}, \quad \forall n \in \mathbb{N}.
\] (1.2)

If, in addition, there is some \( \tau \) so that \( p_n = n + \tau \) for every \( n \), we call it a flat monomial operator.

The powers \( p_n \) may be complex, but must lie in the half plane

\[
\mathbb{S} := \{ s \in \mathbb{C} : \text{Re}(s) > -\frac{1}{2} \},
\]

in order for \( x^{p_n} \) to lie in \( L^2 \). Well-known examples of monomial operators include the Hardy operator \( H \)

\[
H : f \mapsto \frac{1}{x} \int_0^x f(t) dt \\
H x^n = \frac{1}{n+1} x^n;
\]
the operator $M_x$ of multiplication by $x$; and the Volterra operator

$$V = M_x H : f \mapsto \int_0^x f(t) dt$$

$$V x^n = \frac{1}{n+1} x^{n+1}. $$

In [2] the authors studied flat monomial operators, and showed that they all leave invariant every subspace of the form \( \{ f \in L^2[0,1] : f = 0 \text{ on } [0,t] \} \). It was shown independently by Brodskii [6] and Donoghue [10] that these subspaces are exactly the invariant subspaces for the Volterra operator.

It is the purpose of this note to examine general monomial operators of the form (1.2). To describe them, it is convenient to introduce a Hardy space associated with $S$. The Hardy space of the unit disk, which we denote $H^2(D)$, is the Hilbert space of holomorphic functions on $D$ with finite norm, where the norm is given by

$$\| \phi \|^2_{H^2(D)} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |\phi(re^{i\theta})|^2 d\theta.$$ 

There are two distinct definitions for the Hardy space of a half-plane. Let $\lambda$ be the linear fractional transformation give by

$$\lambda(s) = \frac{s}{s+1},$$

that takes $S$ onto $D$.

**Definition 1.4.** By $H^2(S)$ we mean \{ $\phi \circ \lambda : \phi \in H^2(D)$ \}, and the norm is defined so that $\phi \mapsto \phi \circ \lambda$ is unitary.

An equivalent definition of $H^2(S)$ is the set of functions $f$ that are holomorphic in $S$ and such that $|f|^2$ has a harmonic majorant there. The norm squared is equal to the value of this harmonic majorant at 0. If $f \in H^2(S)$, then it has boundary values a.e. (see e.g. [11]), and the norm is given by

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|f(-\frac{1}{2} + it)|^2}{t^2 + \frac{1}{4}} dt.$$ 

The reproducing kernel for $H^2(S)$ is given by

$$k(s, u) = \langle k_u, k_s \rangle = \frac{(1 + s)(1 + \bar{u})}{1 + s + \bar{u}}.$$ 

In light of (1.6), the map

$$U : L^2[0,1] \to H^2(S)$$

$$x^s \mapsto \frac{1}{1 + s} k_s$$
extends to a unitary from $L^2[0,1]$ onto $H^2(\mathbb{S})$. In [1] we show that for any $f \in L^2$, we have

$$Uf(s) = (1 + s) \int_0^1 f(x)x^s dx.$$ 

If $T$ is a monomial operator, we shall define

$$\hat{T} = UTU^*: H^2(\mathbb{S}) \to H^2(\mathbb{S}). \quad (1.7)$$

One might wonder whether the definition of monomial operator should require that it take $x^s$ to some multiple of a monomial for every $s \in \mathbb{S}$; our first theorem asserts that this always happens, just by assuming it on the natural numbers. It also shows that, after moving to the Hardy space as in (1.7), monomial operators correspond to the adjoints of weighted composition operators.

If $g$ is a holomorphic function on some domain, let us define

$$g^\cup(s) = \overline{g(s)}.$$ 

By (1.5) we see that if $f \in H^2(\mathbb{S})$, then $\|f^\cup\|_{H^2(\mathbb{S})} = \|f\|_{H^2(\mathbb{S})}$.

**Theorem 1.8.** Let $T: L^2[0,1] \to L^2[0,1]$ be a monomial operator given by (1.2). Then there exists a holomorphic map $\beta: \mathbb{S} \to \mathbb{S}$ and a function $g \in H^2(\mathbb{S})$ so that, for every $s \in \mathbb{S}$, we have

$$T(x^s) = \frac{1 + \beta(s)}{1 + s} g(s) x^{\beta(s)}. \quad (1.9)$$

Moreover, we have

$$\hat{T}^* f(s) = g^\cup(s) f(\beta^\cup(s)) \quad \forall f \in H^2(\mathbb{S}). \quad (1.10)$$

Equation (1.10) says

$$\hat{T}^* = M_g \circ C_{\beta^\cup},$$

where $C_\beta$ denotes the composition operator $f \mapsto f \circ \beta$, and $M_g$ denotes the multiplication operator $f \mapsto gf$.

Weighted composition operators have been studied for some time. See e.g. [4,5,7–9,12,15]. By Littlewood’s subordination principle, the operator $C_\beta$ is always bounded whenever $\beta$ is a holomorphic self-map of $\mathbb{S}$ [16, Thm. 10.4.2]. A multiplication operator $M_h$ is bounded if and only if $h \in H^\infty(\mathbb{S})$. It was observed in [14] that it is possible for the product $M_h C_\beta$ to be bounded even when $M_h$ is not.

A consequence of Theorem 1.8 is that it allows us to specify what sequences $p_n$ can occur in (1.2).
Corollary 1.11. Let \((p_n)\) be a sequence in \(S\). Then there exists some choice of scalars \(c_n\), not all zero, so that \(T: x^n \mapsto c_n x^{p_n}\) extends to be a bounded linear operator on \(L^2[0,1]\) if and only if

\[
\left[ \frac{p_m + p_n + 1}{m + n + 1} \right] \geq 0.
\]

In Section 3 we study which pairs of functions \(\beta\) and \(g\) give rise to a bounded operator in (1.9). We answer this question only for flat monomial operators, i.e. when \(\beta(s) = s + \tau\) for some constant \(\tau\). If \(\text{Re}(\tau) < 0\), it follows from Corollary 1.11 that such a \(T\) can never be bounded.

If \(\text{Re}(\tau) \geq 0\), we have:

**Theorem 1.12.** Let \(\text{Re}(\tau) \geq 0\), and let \(T\) be defined by

\[
T(x^n) = \frac{1 + n + \tau}{1 + n} g(n) x^{n+\tau}
\]

for some function \(g \in H^2(S)\).

(i) If \(\text{Re}(\tau) > 0\), then \(T\) extends to be a bounded linear operator from \(L^2[0,1]\) to \(L^2[0,1]\) if and only if the Poisson integral of \(|g|^2\) is bounded on all half-planes that are strictly contained in \(S\).

(ii) If \(\text{Re}(\tau) = 0\), then \(T\) is bounded if and only if \(g\) is bounded on \(S\).

This theorem is proved in Theorems 3.3 and 3.6.

2. Proof of Theorem 1.8

**Proof of Theorem 1.8.** Step 1. Let \(\hat{T}\) be given by (1.7). Define \(g\) by \(g^\cup = \hat{T}^*1\). We have

\[
\hat{T}k_n = UT(1 + n)x^n = (1 + n)Uc_n x^{p_n} = c_n \frac{1 + n}{1 + p_n} k_{p_n}.
\]

Since \(k_0 = 1\), we have

\[
\langle g^\cup, k_n \rangle = \langle 1, c_n \frac{1 + n}{1 + p_n} k_{p_n} \rangle = c_n \frac{1 + n}{1 + p_n}.
\]

This gives \(g(n) = c_n \frac{1 + n}{1 + p_n}\), and so we can write

\[
\hat{T}k_n = g(n)k_{p_n}.
\]

(2.1)

For any \(u \in S\), let \(h_u = \hat{T}^*k_u\). We get from (2.1)

\[
\langle h_u, k_n \rangle = g^\cup(n) \langle k_u, k_{p_n} \rangle
\]

\[
\Rightarrow h_u(n) = g^\cup(n) \frac{(1 + \bar{u})(1 + p_n)}{1 + \bar{u} + p_n}
\]

(2.2)

\[
\Rightarrow h_u^\cup(n) = g(n) \frac{(1 + u)(1 + p_n)}{1 + u + p_n}.
\]

(2.3)
Define $\beta$ by
\[
\beta(s) = \frac{(1 + u)(h_u(s) - g(s))}{(1 + u)g(s) - h_u(s)}.
\]

(2.4)

As $g$ and $h_u$ are both in the Hardy space $H^2(\mathbb{S})$, a priori we know that $\beta$ is in the Nevanlinna class of meromorphic functions on $\mathbb{S}$, the class of quotients of $H^2$ functions. Moreover, it follows from (2.3) that
\[
p_n = \frac{(1 + u)(h_u(n) - g(n))}{(1 + u)g(n) - h_u(n)} = \beta(n).
\]

Observe that $N$ is not a zero set for $H^2(\mathbb{S})$—indeed, $(\lambda(n) = \frac{n}{n+1})$ is not a Blaschke sequence for $H^2(\mathbb{D})$. Therefore it is a set of uniqueness for the Nevanlinna class, and hence $\beta$ is the unique function in the class that satisfies $\beta(n) = p_n$. In particular, the definition (2.4) is actually independent of $u$.

We can write (2.2) as
\[
(\hat{T}^*k_u)(n) = g^\cup(n)k_u(\beta^\cup(n)) \quad \forall n \in N.
\]

(2.5)

Since $N$ is a set of uniqueness, (2.5) holds everywhere
\[
(\hat{T}^*k_u)(s) = g^\cup(s)k_u(\beta^\cup(s)) \quad \forall s \in \mathbb{S}.
\]

(2.6)

Step 2. We must show that $\beta : \mathbb{S} \to \mathbb{S}$. Suppose there is some point $s \in \mathbb{S}$ where $g(s) \neq 0$ and $\beta^\cup(s) = w$ is in $\mathbb{C} \setminus \mathbb{S}$. Then there is a sequence $q_n$ such that each $q_n$ is a finite linear combination of kernel functions, $\|q_n\| \leq 1$, and $q_n(w) \to \infty$. By (2.6), we have
\[
(\hat{T}^*q_n)(s) = g^\cup(s)q_n(w).
\]

(2.7)

The right-hand side of (2.7) tends to infinity, the left-hand side is bounded by $\|T\||k_u||$, a contradiction. Since $\beta^\cup$ is meromorphic, we conclude that whenever $g^\cup(s) \neq 0$ then $\beta(s) \in \mathbb{S}$. Therefore any singularities of $\beta^\cup$ on the zero set of $g^\cup$ are removable, so we conclude that $\beta^\cup$, and hence also $\beta$, is a self-map of $\mathbb{S}$.

Thus we have proved (1.10) for any finite linear combination of kernel functions, and so, by a limiting argument, it is true for all $f \in H^2(\mathbb{S})$.

Step 3. From (2.6), we have
\[
(\hat{T}k_s, k_u) = (\hat{T}^*k_u, k_s) = g(\tilde{s})k_u(\beta^\cup(\tilde{s})) = g(\tilde{s})\frac{(1 + u)(1 + \beta(\tilde{s}))}{1 + u + \beta(\tilde{s})} = g(\tilde{s})k_{\beta^\cup(\tilde{s})}(u).
\]

So
\[
\hat{T}k_s = g(\tilde{s})k_{\beta^\cup(\tilde{s})}.
\]

(2.8)

Therefore
\[
Tx^* = U^*\hat{T}U[x^*] = U^*[\frac{1}{1 + s}k_s] = U^*[\frac{1}{1 + s}g(s)k_{\beta(s)}] = \frac{1 + \beta(s)}{1 + s}g(s)x^{\beta(s)}.
\]

This proves (1.9).
Proof of Corollary 1.11. By Theorem 1.8, a necessary condition for the existence of a non-zero bounded $T$ that maps each $x^n$ to a multiple of $x^{p_n}$ is that there be some holomorphic self-map $\beta$ of $S$ that maps $n$ to $p_n$. This condition is also sufficient, since choosing \[ c_n = \frac{1 + p_n}{1 + n} \] gives $UTU^*$ is the adjoint of $C_{\beta^0}$, which is bounded.

When is there a map $\beta: S \to S$ that interpolates $n$ to $p_n$? Composing with the Riemann map $\lambda$ from (1.3), this is equivalent to asking when there exists $\phi = \lambda \beta \lambda^{-1}$ from $D$ to $D$ that maps $\frac{n}{n+1}$ to $\frac{p_n}{p_n+1}$. By Pick’s theorem [3, Thm. 1.81], this occurs if and only if
\[
\begin{bmatrix}
1 - \frac{p_m \overline{p_n}}{p_{m+1} \overline{p_{n+1}}}
\hline
1 - \frac{m \overline{n}}{m+1 \overline{n+1}}
\end{bmatrix} \geq 0.
\] (2.9)

Rearranging (2.9), we get
\[
\begin{bmatrix}
m + 1 \overline{n} + 1 & p_m + \overline{p_n} + 1
\hline
p_m + 1 & m + \overline{n} + 1
\end{bmatrix} \geq 0.
\]

As conjugating by the rank one operator
\[
\begin{bmatrix}
m + 1 \overline{n} + 1
\hline
p_m + 1 \overline{p_n} + 1
\end{bmatrix}
\]
does not affect positivity, and $n = \bar{n}$, we get that an interpolating $\beta$ exists if and only if
\[
\begin{bmatrix}
p_m + \overline{p_n} + 1
\hline
m + n + 1
\end{bmatrix} \geq 0.
\]

3. Flat monomial operators

In this section, we shall consider operators of the form
\[
T: x^n \mapsto c_n x^{n+\tau},
\] (3.1)
and in particular when they are bounded. Note first that if $\tau = \tau_0 + i\tau_1$, with $\tau_0$ and $\tau_1$ real, then the effect of $\tau_1$ is to multiply everything in the range by $x^{i\tau_1}$, which is unimodular. So without loss of generality we can assume that $\tau$ is real. Moreover, by Corollary 1.11, $T$ can only be bounded if $\tau \geq 0$.

Let us first handle the case $\tau = 0$. In the terminology of Theorem 1.8, $\beta(s) = s$, so $T$ is bounded if and only if there is a bounded function $g$ on $S$ that satisfies $g(n) = c_n$. By Pick’s theorem, this happens if and only if
\[
\begin{bmatrix}
1 - \frac{cm \overline{c_n}}{1 + m + n}
\hline
1 + m + n
\end{bmatrix} \geq 0.
\] (3.2)

So we get:

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Theorem 3.3. The map $T: x^n \mapsto c_n x^n$ extends to be a bounded linear map if and only if (3.2) holds. Moreover, if it is non-zero, it is never compact.

When $\tau > 0$, how do we determine whether (3.1) extends to be bounded? Let

$$\gamma(s) = s + \tau.$$  \hspace{1cm} (3.4)

We must find $g \in H^2(S)$ that satisfies

$$g(n) = \frac{1 + n}{c_n 1 + n + \tau}.$$  

Then $T$ is bounded if and only if $M_g C_\gamma$ is bounded. (We have changed notation slightly from Theorem 1.8 to avoid the use of $g^\cup$ and to emphasize that we have a fixed choice of $\beta$). To investigate $M_g C_\gamma$, we turn to the Poisson kernel.

The Poisson kernel for $S$ at a point $s = -\frac{1}{2} + \sigma + it$ is given by

$$P_{-\frac{1}{2} + \sigma + it} \left( -\frac{1}{2} + iy \right) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (y-t)^2}.$$  

The Poisson integral of some function $f$ defined on the line $\{\text{Re}(s) = -\frac{1}{2}\}$ is given by

$$P[f] \left( -\frac{1}{2} + \sigma + it \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma}{\sigma^2 + (y-t)^2} f \left( -\frac{1}{2} + iy \right) dy.$$  

It is convenient to introduce another Hardy space, $H^2(S)$.

Definition 3.5. The space $H^2(S)$ consists of all functions $f$ that are holomorphic in $S$ and satisfy

$$\sup_{x > -\frac{1}{2}} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy < \infty.$$  

Every function $f \in H^2(S)$ has non-tangential boundary limits almost everywhere, and its norm is then given by (see [13]):

$$\|f\|_{H^2(S)}^2 = \int_{-\infty}^{\infty} \left| f \left( -\frac{1}{2} + iy \right) \right|^2 dy.$$  

Let us write $H_\rho := \{ s \in \mathbb{C} : \text{Re}(s) > \rho \}$. So $S = H_{\frac{1}{2}}$ in this notation.

Theorem 3.6. The operator $M_g C_\gamma$ is bounded if and only if the Poisson integral of $|g|^2$ is bounded on some (and hence every) half-plane $H_\rho$ for $\rho > -\frac{1}{2}$.

Proof of Thm. 3.6. The map

$$W: f(s) \mapsto \frac{1}{\sqrt{2\pi}} \frac{1}{s + 1} f(s)$$  

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is a unitary from $H^2(S)$ onto $\mathcal{H}^2(S)$. One checks

$$WM_gC_\gamma W^* = M_{s+\tau+1}g(s)C_\gamma.$$ \hspace{1cm} (3.7)

For $s$ in $S$ we have

$$1 \leq \left| \frac{s + \tau + 1}{s + 1} \right| \leq 1 + 2\tau,$$

so for boundedness purposes we can drop this factor in (3.7) and conclude that $M_gC_\gamma$ is bounded on $H^2(S)$ if and only if it is bounded on $\mathcal{H}^2(S)$. Thus we wish to know for which $g$ does

$$\int_{-\infty}^{\infty} \left| g\left( -\frac{1}{2} + iy \right) \right|^2 \left| f\left( -\frac{1}{2} + \tau + iy \right) \right|^2 dy \lesssim \int_{-\infty}^{\infty} \left| f\left( -\frac{1}{2} + iy \right) \right|^2 dy$$

hold? This is the same as asking when integrating along the vertical line $\{\text{Re}(s) = -\frac{1}{2} + \tau\}$ with weight $|g(-\frac{1}{2} + iy)|^2$ is a Carleson measure for $H^2(S)$. By [13, VI.3], this happens if and only if

$$\sup_{\sigma > 0, t \in \mathbb{R}} \int_{-\infty}^{\infty} \frac{\sigma}{(\tau + \sigma)^2 + (y - t)^2} \left| g\left( -\frac{1}{2} + iy \right) \right|^2 dy = C < \infty. \hspace{1cm} (3.8)$$

Taking $\sigma = \tau$ in (3.8), we get that $M_gC_\gamma$ is bounded implies

$$\sup_{t \in \mathbb{R}} \int_{-\infty}^{\infty} \frac{\tau}{(2\tau)^2 + (y - t)^2} \left| g\left( -\frac{1}{2} + iy \right) \right|^2 dy \leq K,$$

for some constant $K$, so

$$\sup_{t \in \mathbb{R}} \int_{-\infty}^{\infty} \frac{\tau}{(\tau)^2 + (y - t)^2} \left| g\left( -\frac{1}{2} + iy \right) \right|^2 dy \leq 4K.$$
Example 3.9. Let $\tau > 0$, and take $g(s) = \frac{1}{(1+s)^{\frac{\tau}{2}}} c$ for some $0 < c < \frac{1}{2}$. Then $g$ is not bounded on $S$, so $M_g$ is not bounded. But

$$P[|g|^2] \left( -\frac{1}{2} + \sigma + it \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma}{\sigma^2 + (y-t)^2} \left| y \right|^2 \frac{1}{4} + y^2 \, dy$$

$$\leq \frac{1}{\pi\sigma} \int_{-\infty}^{\infty} \frac{1}{y^2} \frac{1}{4} + y^2 \, dy$$

$$\leq \frac{1}{\pi\sigma} \left[ \int_{-1}^{1} \frac{4}{|y|^2} \, dy + \int_{-\infty}^{\infty} \frac{1}{\frac{1}{4} + y^2} \, dy \right]$$

$$= \frac{1}{\pi\sigma} \left[ \frac{8}{1 - 2c} + 2\pi \right]$$

This is bounded in each half-plane $H_{\rho}$, so by Theorem 3.6 $M_g C_{\gamma}$ is bounded.

4. Unitary monomial operators

Bourdon and Narayan characterized unitary weighted composition operators [5]. Their theorem (translated to $S$) is:

**Theorem 4.1.** (Bourdon–Narayan) The operator $M_g C_{\beta}$ is unitary on $H^2(S)$ if and only if $\beta$ is an automorphism of $S$ and $g(s) = e^{i\theta} k_{s_0}/\|k_{s_0}\|$, where $s_0 = \beta^{-1}(0)$.

We shall give a proof of their theorem directly in the context of monomial operators. First, let us describe the automorphisms of $S$ in a convenient way.

**Lemma 4.2.** The function $\beta : S \to \mathbb{C}$ is an automorphism of $S$ if and only if there is a function $\phi : S \to \mathbb{C}$ so that

$$\frac{1 + \beta(s) + \beta(t)}{1 + s + t} = \phi(s) \phi(t). \quad (4.3)$$

**Proof.** Notice that $\beta$ is an automorphism of $S$ if and only if $b := \lambda \circ \beta \circ \lambda^{-1}$ is an automorphism of $D$. By Pick’s theorem (see [3, Sec. 2.6]) the latter occurs if and only if

$$\frac{1 - b(w)b(z)}{1 - \overline{w}z} = \psi(w) \psi(z) \quad (4.4)$$

for some function $\psi$ on $D$. Doing some algebra, (4.4) becomes (4.3) with

$$\phi(s) = \frac{1 + \beta(s)}{1 + s} \psi \left( \frac{s}{1 + s} \right).$$

Consequently, we have the following characterization of unitary monomial operators (which can also be derived from Bourdon-Narayan and Theorem 1.8).
Theorem 4.5. The operator $T : x^s \mapsto c(s)x^{\beta(s)}$ is unitary on $L^2[0, 1]$ if and only if $\beta$ is an automorphism of $\mathbb{S}$ and $c(s)$ is defined by

$$c(s) = \frac{e^{i\theta}}{\sqrt{1 + 2 \text{Re} \beta(0)}} \frac{1 + \beta(0) + \beta(s)}{1 + s}. \quad (4.6)$$

Proof. Since $\beta$ is holomorphic by Theorem 1.8 and has to be non-constant for $T$ to be bounded, the range of $T$ must be dense. Therefore it is unitary if and only if it is isometric. It is isometric if and only if for every $s, t$ we have

$$\langle x^s, x^t \rangle = \langle c(s)x^{\beta(s)}, c(t)x^{\beta(t)} \rangle \iff \frac{1}{1 + s + t} = \frac{c(s)c(t)}{1 + \beta(s) + \beta(t)}.$$

That means (4.4) holds, so $\beta$ is an automorphism. Letting $t = 0$ we get

$$c(0) = e^{i\theta} \sqrt{1 + 2 \text{Re} \beta(0)},$$

and solving for $c(s)$ we get (4.6).

Conversely, suppose (4.6) holds and $\beta$ is an automorphism. By Lemma 4.2, we have (4.3) for some $\phi$. Letting $t = 0$ and solving, we get that $\phi$ is given by (4.6).

5. Questions

Question 5.1. If $M_g C_\gamma$ is bounded, with $\gamma$ as in (3.4), can one approximate it in norm by operators of the form $M_{g_n} C_\gamma$ where $g_n \in H^\infty(\mathbb{S})$?

Question 5.2. A generalization of the previous question is for non-flat monomial operators. Can every bounded operator of the form $M_g C_\beta$ be approximated by operators $M_{g_n} C_\beta$ with bounded $g_n$?

Question 5.3. What are necessary and sufficient conditions for the functions $\beta$ and $g$ so that the operator $T$ defined by (1.9) is bounded? Compact?

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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