Initial and Final Characterized Fuzzy $R_{T_{1}}$-Spaces  

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Abstract

Basic notions related to the characterized fuzzy $R_{T_{1}}$ and characterized fuzzy $T_{1}$-spaces are introduced and studied. The metrizable characterized fuzzy spaces are classified by the characterized fuzzy $R_{T_{1}}$ and the characterized fuzzy $T_{1}$-spaces in our sense. The induced characterized fuzzy space is characterized by the characterized fuzzy $R_{T_{1}}$ and characterized fuzzy $T_{1}$-space if and only if the related ordinary topological space is $\phi_{T_{1}}R_{T_{1}}$-space and $\phi_{T_{1}}T_{1}$-space, respectively. Moreover, the $\alpha$-level and the initial characterized spaces are characterized $R_{T_{1}}$ and characterized $T_{1}$-spaces if the related characterized fuzzy space is characterized fuzzy $R_{T_{1}}$ and characterized fuzzy $T_{1}$, respectively. The categories of all characterized fuzzy $R_{T_{1}}$ and of all characterized fuzzy $T_{1}$-spaces will be denoted by CFR-Space and CRF-Tych and they are concrete categories. These categories are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets and hence all the initial and final lifts exist uniquely in CFR-Space and CRF-Tych. That is, all the initial and final characterized fuzzy $R_{T_{1}}$-spaces and all the initial and final characterized fuzzy $T_{1}$-spaces exist in CFR-Space and in CRF-Tych. The initial and final characterized fuzzy spaces of a characterized fuzzy $R_{T_{1}}$-space and of a characterized fuzzy $T_{1}$-space are characterized fuzzy $R_{T_{1}}$ and characterized fuzzy $T_{1}$-spaces, respectively. As special cases, the characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space of a characterized fuzzy $R_{T_{1}}$ and of a characterized fuzzy $T_{1}$-space are also characterized fuzzy $R_{T_{1}}$ and characterized fuzzy $T_{1}$-spaces, respectively. Finally, three finer characterized fuzzy $R_{T_{1}}$-spaces and three finer characterized fuzzy $T_{1}$-spaces are introduced and studied.

Keywords: Fuzzy filter; Fuzzy topological space; Operation; Characterized fuzzy space; Metriz-able characterized fuzzy space; Induced characterized fuzzy space; $\alpha$-Level characterized space; $\phi_{T_{1}}$, $\phi_{T_{1}}$-interior, $\phi_{T_{1}}$-continuous; Initial and final characterized fuzzy spaces; Characterized fuzzy $T_{1}$-space; Characterized fuzzy $T_{1}$-space; AMS classification; Primary 54E35, 54E52; Secondary 54A40, 03E72

Introduction

Eklund and Gähler [1] introduced the notion of fuzzy filter and by means of this notion the point-based approach to the fuzzy topology related to usual points has been developed. The more general concept for the fuzzy filter introduced by Gähler [2] and fuzzy filters are classified by types. Because of the specific type of the L-filter however the approach of Eklund and Gähler [1] is related only to the L-topologies which are stratified, that is, all constant L-sets are open. The more specific fuzzy filters considered in the former papers are now called homogeneous. The notion of fuzzy real numbers is introduced by Gähler and Gähler [3], as a convex, normal, compactly supported and upper semi-continuous fuzzy subset of the set of all real numbers R. The set of all fuzzy real numbers is called the fuzzy real line and will be denoted by $R_{\infty}$ where L is complete chain.

The operation on the ordinary topological space $(X,T)$ has been defined by Kasahara [4] as a mapping $\varphi$ from T into $2^X$ such that $A \subseteq T$, for all $A \subseteq T$. Abd El-Monsef et al. [5], extend Kasahara [4] operation to the power set $P(X)$ of the set X Kandil et al. [6] extended Kasahara’s and Abd El-Monsef’s operations by introducing operation on the class of all fuzzy sets endowed with an fuzzy topology $\tau$ as a mapping $L^{2} \rightarrow L^{3}$ such that $\mu \leq \mu^{o}$ for all $\mu \in L^{2}$, where $\mu^{o}$ denotes the value of $\varphi$ at $\mu$. The notions of fuzzy filters and the operations on the class of all fuzzy sets on X endowed with an fuzzy topology $\tau$ are applied in ref. [7] to introduce a more general theory including all the weaker and stronger forms of the fuzzy topology. By means of these notions the notion of $\varphi_{T_{1}}$-interior of the fuzzy set, $\varphi_{T_{1}}$-convergence and $\varphi_{T_{1}}$-fuzzy

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neighborhood filters are defined. The notion of $\varphi_{\text{int}}$-interior operator for the fuzzy sets is also defined as a mapping $\varphi_{\text{int}}: L \to L$ which fulfill (11) to (15). Since there is a one-to-one correspondence between the class of all $\varphi_{\text{int}}$-open fuzzy subsets of $X$ and these operators, then the class $\varphi_{\text{int}}(X)$ of all $\varphi_{\text{int}}$-open fuzzy subsets of $X$ is characterized by these operators. Hence, the triple $(X, \varphi_{\text{int}}, \text{int})$ as well as the triple $(X, \varphi_{\text{OF}}(X))$ will be called the characterized fuzzy space of $\varphi_{\text{int}}$-open fuzzy subsets. For each characterized fuzzy space $(X, \varphi_{\text{int}}, \text{int})$ the mapping which assigns to each point $x$ of $X$ the $\varphi_{\text{int}}$-neighborhood filter at $x$ is said to be $\varphi_{\text{int}}$-fuzzy filter pre topology [7]. It can be identified itself with the characterized fuzzy space $(X, \varphi_{\text{int}})$. The characterized fuzzy spaces are characterized by many of characterizing notions, for example by: $\varphi_{\text{int}}$-fuzzy neighborhood filters, $\varphi_{\text{int}}$-fuzzy interior of the fuzzy filters and by the set of all $\varphi_{\text{int}}$-inner points of the fuzzy filters. Moreover, the notions of closeness and compactness in characterized fuzzy spaces are introduced and studied in ref. [8]. For an fuzzy topological space $(X, \tau)$, the operations on $(X, \tau)$ and on the fuzzy topological space $(I, \tau)$, where $I=[0,1]$ is the closed unit interval and $I$ is the fuzzy topology defined on the left unit interval $I$ are applied to introduced and studied the notions of characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces or (characterized Tychonoff spaces) [9]. In this paper, Basic notions related to the characterized fuzzy $R_{\tau}$ and the characterized fuzzy $T_{1}$-spaces are introduced and studied. Some of this the metrizable characterized fuzzy spaces, initial and final characterized fuzzy spaces and three finer characterized fuzzy $R_{\tau}$-spaces are introduced and classified by the characterized fuzzy $R_{\tau}$ and characterized fuzzy $T_{1}$-spaces. The metrizable characterized fuzzy space is introduce as a generalization of the weaker and stronger forms of the fuzzy metric space introduced by Gahler and Gahler [3]. For every stratified fuzzy topological space $(X, \tau)$ generated canonically by a fuzzy metric $d$ on $X$, the metrizable characterized fuzzy space $(X, \varphi_{\text{int}, \tau})$ is characterized fuzzy $T_{1}$-spaces in sense of Abd-Allah [10] and therefore it is characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-L-space.

The induced characterized fuzzy space $(X, \varphi_{\text{int}}, \text{int})$ is characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces if and only if the related ordinary topological space $(X, \tau)$ is $\varphi_{\text{int}}$-space and $\varphi_{\text{int}}$-space, respectively, that is, the notions of characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces are good extension as in sense of Lowen [11]. Moreover, the $\alpha$-level characterized space $(X, \varphi_{\text{int}, \tau})$ and the initial characterized space $(X, \varphi_{\text{int}, \text{int}})$ are characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces if the related characterized fuzzy space $(X, \varphi_{\text{int}, \text{int}})$ is characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces, respectively. We show that the finer characterized fuzzy space of the characterized fuzzy $R_{\tau}$-spaces and of the characterized fuzzy $T_{1}$-spaces is also characterized fuzzy $R_{\tau}$ and characterized fuzzy $T_{1}$-spaces, respectively. The categories of all characterized fuzzy $R_{\tau}$-spaces and of all characterized fuzzy $T_{1}$-spaces will be denoted by CFR-Space and CRF-Tych, respectively. We show that these categories are concrete categories and they are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets and hence all the initial and final lifts exist uniquely in CFR-Space and CRF-Tych, respectively. That is, all the initial and final characterized fuzzy $T_{1}$-spaces and all the initial and final characterized fuzzy $R_{\tau}$-spaces are exist in the categories CFR-Space and CRF-Tych. Moreover, we show that the initial and final characterized fuzzy spaces of the characterized fuzzy $R_{\tau}$-space and of the characterized fuzzy $T_{1}$-space are characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces, respectively. As an special cases, the characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space of the characterized fuzzy $R_{\tau}$-spaces and of the characterized fuzzy $T_{1}$-spaces are also characterized fuzzy $R_{\tau}$-spaces and characterized fuzzy $T_{1}$-spaces, respectively. Finally, in section 5, we introduce and study three finer characterized fuzzy $R_{\tau}$-spaces and three finer characterized fuzzy $T_{1}$-spaces as a generalization of the weaker and stronger forms of the completely regular and fuzzy $T_{1}$-spaces introduced [1,12,13]. The relations between such new characterized fuzzy $R_{\tau}$-spaces and our characterized fuzzy $T_{1}$-spaces are introduced. More general the relations between such new characterized fuzzy $T_{1}$-spaces and our characterized fuzzy $T_{1}$-spaces are also introduced. Meany special cases from these finer characterized fuzzy $R_{\tau}$-spaces and from finer characterized fuzzy $T_{1}$-spaces are listed in Table 1.

Preliminaries

We begin by recalling some facts on fuzzy sets and fuzzy filters. Let $L$ be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Consider $L=L(0)$ and $L=L(1)$. Recall that the complete distributivity of $L$ means that the distributive law $\mu \wedge (\alpha \vee \beta) = (\mu \wedge \alpha) \vee (\mu \wedge \beta)$. Sometimes we will assume more specially that $L$ is a complete chain, that is, $L$ is a complete lattice whose partial ordering is a linear one. The standard example of $L$ is the real closed unit interval $I=[0,1]$. For a set $X$, let $L^{X}$ be the set of all fuzzy subsets of $X$, that is, of all mappings $\mu: X \to L$. Assume that an order-reversing involution $\tau: L^{X} \to L^{X}$ is fixed. For each fuzzy set $\mu$, let $\mu^{\tau}$ denote the complement of $\mu$ defined by: $(\mu \tau)(x)=\mu(x)$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_{\mu}$. $\mu^{\tau}$ means the supremum of the set of values of $\mu$. The fuzzy sets on $X$ will be denoted by Greek letters as $\mu, \eta, \rho, \ldots$ etc. Denote by $\tau$ the constant fuzzy subset of $X$ with value $\mu \in L$. The fuzzy singleton $\{x\}$, is an fuzzy set in $X$ defined by $\mu(x)=\alpha$ and $\mu(y)=0$ for all $y \neq x, \alpha \in L_{\mu}$. The class of all fuzzy singletons in $X$ will be denoted by $S(X)$. For every $x \in S(X)$ and $\mu \in L_{\mu}$, we write $\mu \subseteq \mu$ if and only if $\mu \leq \mu(x)$. The fuzzy set $\mu$ is said to be quasi-coincident with the fuzzy set $\rho$ and written $\mu \equiv \rho$ or if and only if there exists $x \in X$ such that $\mu(x) \equiv \rho(x)-1$. If $\mu$ not quasi-coincident with the fuzzy set $\rho$, then we write $\mu \not\equiv \rho$. The fuzzy filter on $X$ [14] is the mapping $M: L \to L$ such that the following conditions are fulfilled:

\begin{align}
(\text{F1}) \quad M(\mu) & \leq \alpha \quad \text{for all } \alpha \in L \text{ and } M(1)=1.
\end{align}
The fuzzy filter \( M \) is said to be homogeneous \([14]\) if \( M(\sigma) = \sigma \) for all \( \alpha \in L \). For each \( x \in X \), the mapping \( x \cdot : L^2 \rightarrow L \) defined by \( x(\alpha) = \mu(x) \leq \alpha \) for all \( \mu \in L^3 \) is a homogeneous fuzzy filter on \( X \).

The homogeneous fuzzy filter at the fuzzy set is defined by the same way as follows, for each \( \mu \in L^3 \), the mapping \( \mu \cdot : L^2 \rightarrow L \) defined by \( \mu(\alpha) = \sigma(x) \leq \alpha \) for all \( \alpha \in L \) is also a homogeneous fuzzy filter on \( X \), called homogenous fuzzy filter at \( \mu \in L \).

Obviously, the relation between homogenous fuzzy filter \( \mu \cdot \) at \( \mu \in L \) and the homogenous fuzzy filter \( x \cdot \) at \( x \in X \) is given by:

\[
\mu(\eta) = \bigwedge_{\alpha \leq \eta} \sigma(x),
\]

for all \( \eta \in L \). As shown in ref. \([15]\), \( \mu \leq \eta \) if and only if \( \mu \leq \eta \) holds for all \( \mu \in L \). Let \( F_\eta \) and \( F_\mu \) denote the sets of all fuzzy filters and all homogenous fuzzy filters on \( X \), respectively. If \( M, N \) are fuzzy filters on the set \( X \), then \( M \leq N \) provided \( M(\eta) \leq N(\eta) \) for all \( \mu \in L^3 \). Noting that if \( L \) is a complete chain then \( M \) is not finer than \( N \), denoted by \( M \leq N \), provided there exists \( \mu \in L \) such that \( M(\mu) < N(\mu) \). As shown in ref. \([4]\), if \( M, N \) and \( L \) are three fuzzy filters on a set \( X \), then we have:

\[
M \neq L \Rightarrow N \text{ implies } M \neq N \text{ and } M \geq L \Rightarrow N \text{ implies } M \neq N .
\]

The coarsest fuzzy filter \( M \) on \( X \) is the fuzzy filter has the value 1 at 1 and 0 otherwise. Suprema and infimum of sets of fuzzy filters are meant with respect to the finer relation. An fuzzy filter \( M \) on \( X \) is said to be ultra \([2]\) fuzzy filter if it does not have a proper finer fuzzy filter. For each fuzzy filter \( \mu \in \mathcal{F}_X \) there exists a finer ultra fuzzy filter \( U \in \mathcal{F}_X \) such that \( \mu \leq U \). Consider \( A \) as a non-empty set of ultra fuzzy filters of \( X \), then the supremum \( \bigvee_{\mu \in A} M(\mu) = \bigvee_{\mu \in A} M(\mu) \) for all \( L \) but the infimum \( \bigwedge_{\mu \in A} M(\mu) \) does not exists, in general. As shown in ref. \([16]\), the infimum \( \bigwedge_{\mu \in A} M(\mu) \) of \( A \) with respect to the finer relation for fuzzy filters exists if and only if \( M(\mu) \leq \mu \cdot M(\mu) \) holds for all \( \mu \in L^3 \), the strong \( \alpha \)-cut and the weak \( \alpha \)-cut of \( \mu \) are the ordinary subsets \( S_\alpha(\mu) = \{ x \in X | \mu(x) > \alpha \} \) and \( W_\alpha(\mu) = \{ x \in X | \mu(x) \geq \alpha \} \) of \( X \) respectively. For each complete chain \( L \), the \( \alpha \)-level topology and the initial topology \([19]\) of an ultra fuzzy filter \( M \) on the set \( X \) are defined as follows:

\[
\tau_\alpha = \{ S_\alpha(\mu) = \{ x \in X | \mu(x) > \alpha \} \text{ and } W_\alpha(\mu) = \{ x \in X | \mu(x) \geq \alpha \} \},
\]

respectively, where \( \mu \) is the infimum with respect to the finer relation for topologies. On other hand if \( X, (X, \tau) \) is an ordinal topological space, then the induced ultra fuzzy topology on \( X \) is given by Lowen \([17]\) as the following:

\[
\tau(\tau) = \{ \mu \in L^3 | S_\alpha(\mu) = \{ x \in X | \mu(x) > \alpha \} \text{ for all } \alpha \in L \} .
\]

The fuzzy topology on \( X, (X, \tau) \) also \( \tau \) are said to be stratified provided \( \alpha \in \tau \) holds for all \( \alpha \in L \), that is, all constant fuzzy sets are open \([19]\).

The fuzzy unit interval

The fuzzy unit interval will be denoted by \( I \) and it is defined in \([3]\) as the fuzzy subset:

\[
I = \{ x \in R^+_1 | x \leq 1 \},
\]

where \( I = [0,1] \) is the real unit interval and \( x \) is the set of all positive fuzzy real numbers. Note that, the binary relation \( \leq \) is defined on \( R^+_1 \) as follows:

\[
x \leq y \Leftrightarrow x_\alpha \leq y_\alpha \text{ and } x_\alpha \leq y_\alpha ,
\]

for all \( x, y \in R_+^1 \), where \( x_\alpha = \inf_{z \in R^+_1} \{ z \geq x \} \) and \( x_\alpha = \sup_{z \in R^+_1} \{ z(x) \geq \alpha \} \) for all \( \alpha \in L \). Note that the family \( \Omega \), which is defined by:

\[
Omega = \{ R_\alpha(x) : \delta \in R \} \cup \{ R_\alpha(x) : \delta \in R \} \cup \{ 0 \} \}
\]

is a base for an fuzzy topology \( I \) on \( R^+_1 \), where \( R_\alpha \) and \( R^+ \) are the fuzzy subsets of \( R^+_1 \) defined by

\[
R_\alpha(x) = x_\alpha \text{ and } R^+ = \{ x_\alpha \} \}
\]

for all \( x \).
\[ R^O(x) \land R^O(y) \leq R^O(x \land y), \tag{2.2} \]

where, \( x \land y \) is the fuzzy real number defined by \((x \land y)(\xi) = \min(x(\xi), y(\xi)) \) for all \( \xi \in R \).

### Operation on fuzzy sets

In the sequel, let a fuzzy topological space \((X, \tau)\) be fixed. By the operation [6] on the set \(X\) we mean the mapping \(\phi: L^X \to L^X\) such that \(\int(\mu) \leq \phi(\mu)\) holds for all \(\mu \in L^X\), where, \(\phi(\mu)\) denotes the value of \(\phi\) at \(\mu\). The class of all operations on \(X\) will be denoted by \(O_X\). By the identity operation on \(X\), we mean the operation \(1^X\), \(\tau\) such that \(1^X(\mu) = \mu\) for all \(\mu \in L^X\). The constant operation on \(X\), denoted by \(c_X\), is defined as follows: \(\phi \leq c_X \Leftrightarrow \phi(\mu) \leq \mu\) for all \(\mu \in L^X\). If \(\phi\) is a partially ordered relation on \(X\), defined as follows: \(\phi \leq \phi_1 \Leftrightarrow \phi(\mu) \leq \phi_1(\mu)\) for all \(\mu \in L^X\), then \((O_X, \leq)\) is a partially distributive lattice. The operation \(\phi: L^X \to L^X\) is called:

(i) Isotone if \(\mu \leq \eta\) implies \(\phi(\mu) \leq \phi(\eta)\), for all \(\mu, \eta \in L^X\).

(ii) Weakly finite intersection preserving (wpfp, for short) with respect to \(A \subseteq L^X\) if \(\eta \land \phi(\mu) \leq \phi(\eta \land \mu)\) holds, for all \(\eta \in A\) and \(\mu \in L^X\).

(iii) Idempotent if \(\phi(\mu) = \phi(\phi(\mu))\), for all \(\mu \in L^X\).

The operations \(\phi, \psi \in O_X\) are said to be dual if \(\psi(\mu) = c_X(\phi(\mu))\) or equivalently \(\phi(\mu) = c_X(\psi(\mu))\) for all \(\mu \in L^X\), where, \(c_X\) denotes the complement of \(\mu\). The dual operation of \(\phi\) is denoted by \(\phi^*\). In the classical case of \(L = \{0, 1\}\), by the operation on a set \(X\) we mean the mapping \(\phi: P(X) \to P(X)\) such that \(A \subseteq A\) for all \(A \subseteq X\) and the identity operation on the class of all ordinary operations \(O_{X \subseteq X}\) on \(X\) will be denoted by \(1_{X \subseteq X}\) and it defined by: \(1_{X \subseteq X}(A) = A\) for all \(A \subseteq X\).

The \(\phi\)-open fuzzy sets

Let a fuzzy topological space \((X, \tau)\) be fixed and \(\phi \in O_X\). The fuzzy set \(\mu: X \to L\) is said to be \(\phi\)-open fuzzy set if \(\mu \leq \psi(\mu)\) holds. We will denote the class of all \(\phi\)-open fuzzy sets on \(X\) by \(\phi OF(X)\). The fuzzy set \(\mu\) is called \(\phi\)-closed if its complement \(\psi\phi\) is \(\phi\)-open. The operations \(\phi, \psi \in O_X\) are equivalent and written \(\phi \sim \psi\) if \(\phi(\mu) = \psi(\mu)\) for all \(\mu \in L^X\).

The \(\phi\)-interior fuzzy sets

Let a fuzzy topological space \((X, \tau)\) be fixed and \(\phi \in O_X\). Then the \(\phi\)-interior of the fuzzy set \(\mu: X \to L\) is a mapping \(\phi_{\mu} intL: X \to L\) defined by:

\[ \phi_{\mu} intL(\mu) = \vee \{\eta \mid \phi_{\eta} \subseteq \phi_{\mu} \land \eta(\phi_{\mu}) = \eta\}. \tag{2.3} \]

That is, the \(\phi\)-interior is the greatest \(\phi\)-open fuzzy set \(\eta\) such that \(\eta \leq \mu\).

The \(\phi\)-interior of the fuzzy set \(\mu: X \to L\) is defined by:

\[ \phi_{\mu} intL(\mu) = \vee \{\eta \mid \phi_{\eta} \subseteq \phi_{\mu} \land \eta(\phi_{\mu}) = \eta\}. \tag{2.3} \]

### Characterized Fuzzy Spaces

Independently on the fuzzy topologies, the notion of \(\phi\)-interior operator for the fuzzy sets can be defined as a mapping \(\phi intL: L^X \to L^X\) which fulfill (11) to (15). It is well-known that (2.3) and (2.4) give a one-to-one correspondence between the class of all \(\phi\)-open fuzzy sets and these operators, that is, \(\phi of(X)\) can be characterized by the \(\phi\)-interior operators. In this case the triple \((X, \phi OF(X))\) will be called characterized fuzzy space [7] of the \(\phi\)-open fuzzy subsets of \(X\). The characterized fuzzy space \((X, \phi intL)\) is said to be stratified if and only if \(\phi_{\mu} intL = \phi_{\mu} \land \mu\) for all \(\mu \in L^X\). As shown in ref. [7], the characterized fuzzy space \((X, \phi intL)\) is stratified if the related fuzzy topology is stratified. Moreover, the characterized fuzzy space \((X, \phi intL)\) is said to have the weak infimum property [21],...
provided $\varphi_{\alpha,i}(\mu) = \varphi_{\alpha,i}(\mu) \cap \varphi_{\alpha,i}(\mu)$ for all $\mu \in L^\alpha$ and $\alpha \in \Delta$. The characterized fuzzy space $(X, \varphi_{\alpha,i}(\mu))$ is said to be strongly stratified \cite{21}, provided $\varphi_{\alpha,i}(\mu)$ is stratified and have the weak infimum property. If $(X, \varphi_{\alpha,i}(\mu))$ and $(X, \varphi_{\alpha,j}(\mu))$ are two characterized fuzzy spaces, then $(X, \varphi_{\alpha,i}(\mu))$ is said to be finer than $(X, \varphi_{\alpha,j}(\mu))$ and denoted by $\varphi_{\alpha,i}(\mu) \subseteq \varphi_{\alpha,j}(\mu)$, provided $\varphi_{\alpha,i}(\mu) \supseteq \varphi_{\alpha,j}(\mu)$ holds for all $\mu \in L^\alpha$. If $\tau$ is a fuzzy topology on the set $X$ and $\varphi_{\alpha,i} \in O_{X\varphi_{\alpha,i}}$, then by the initial characterized space of $(X, \tau)$ we mean the characterized spaces $(X, \varphi_{\alpha,i}(\mu))$ and $(X, \varphi_{\alpha,j}(\mu))$, respectively, where $(\varphi_{\alpha,i}(\mu))(\alpha)$ and $i(\varphi_{\alpha,i}(\mu))$ are defined as follows:

\[ \{ \alpha, \mu, \nu \in \mathbb{P} \} \mid \varphi_{\alpha,i}(\mu) \wedge \varphi_{\alpha,i}(\nu) \wedge (\alpha \in \Delta) \]

Sometimes we denoted to the $\alpha$-level characterized space and the initial characterized space of $(X, \tau)$ by $(X, \varphi_{\alpha,i}(\mu))$ which is defined in $[21, 22]$. Obviously, $\varphi_{\alpha,i}(\mu)$ is a fuzzy stack with the cutting property, called $\varphi_{\alpha,i}(\mu)$-fuzzy neighborhood for all $\mu \in L^\alpha$. If $\varphi_{\alpha,i}(\mu)$ is characterized fuzzy space $(X, \varphi_{\alpha,i}(\mu))$, then by the induced characterized fuzzy space on $X$ we mean the characterized space $(X, \varphi_{\alpha,i}(\mu))$ which is defined by:

\[ (\varphi_{\alpha,i}(\mu)(\alpha)) \cap \varphi_{\alpha,j}(\mu) \cap \varphi_{\alpha,j}(\mu) \cap \varphi_{\alpha,j}(\mu) \cap \alpha \in \Delta \]

An important notion in the characterized fuzzy space $(X, \varphi_{\alpha,i}(\mu))$ is that of the $\varphi_{\alpha,i}(\mu)$-fuzzy neighborhood filter at the points and at the ordinary subsets of this space. Let $(X, \tau)$ be a fuzzy topological space $(X, \tau)$ by $\varphi_{\alpha,i}(\mu)$ and $(X, \varphi_{\alpha,j}(\mu))$, respectively. If $\tau$ is ordinary topological space on a set $X$ and $\varphi_{\alpha,i} \in O_{X\varphi_{\alpha,i}}$, then by the induced characterized fuzzy space on $X$ we mean the characterized space $(X, \varphi_{\alpha,i}(\mu))$ which is defined by:

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\[ (\varphi_{\alpha,i}(\mu)(\alpha)) \cap \varphi_{\alpha,j}(\mu) \cap \varphi_{\alpha,j}(\mu) \cap \varphi_{\alpha,j}(\mu) \cap \alpha \in \Delta \]
product space [7] of the characterized fuzzy spaces \((X, \delta_{12})\). The \(\varphi_{12} \cdot \text{int}\) will be denoted by \(\bigwedge_{i \in I} \delta_{12} \cdot \text{int}\) and it is initial \(\varphi_{12} \cdot \text{int}\) fuzzy interior operator of \((\delta_{12} \cdot \text{int})_{\alpha i}\) with respect to the family \((\tau_{\alpha})_{\alpha i}\) of projections. The characterized fuzzy product space \((X, \varphi_{12} \cdot \text{int})\) also will be denoted by \(\bigwedge_{i \in I} (X, \delta_{12} \cdot \text{int})\).

### Final characterized fuzzy spaces

It is well-known (cf. e.g., [11,24]) that in the topological category all final lifts uniquely exist and hence also all final structures exist. They are dually defined. In case of the category CF-Space of all characterized fuzzy spaces the final structures can be given as is shown in the following:

Let \(I\) be a class and for each \(i \in I\), let \((X, \delta_{i}) \cdot \text{int}\) be an characterized fuzzy space and \(f_i: X \to X_i\) is the mapping of \(X_i\) into a set \(X_i\). The final \(\varphi_{12} \cdot \text{int}\) fuzzy interior operator of \((\delta_{i} \cdot \text{int})_{\alpha i}\) with respect to \((\tau_{\alpha})_{\alpha i}\) is the finest \(\varphi_{12} \cdot \text{int}\) on \(X\) for which all mappings \(f_i: (X, \delta_{i} \cdot \text{int}) \to (X, \varphi_{12} \cdot \text{int})\) are \(\delta_{i} \cdot \text{int}\) fuzzy continuous [7]. Hence, the triple \((X, \varphi_{12} \cdot \text{int})\) is the final characterized fuzzy space of \(((X, \delta_{i} \cdot \text{int})_{\alpha i})\) with respect to \((\tau_{\alpha})_{\alpha i}\). The final \(\varphi_{12} \cdot \text{int}\) of \((X, \varphi_{12} \cdot \text{int})\) with respect to \((\tau_{\alpha})_{\alpha i}\) exists and is given by

\[
\varphi_{12} \cdot \text{int}_{\alpha i}(\mu(x)) = \bigwedge_{\alpha \in \alpha i} \delta_{12} \cdot \text{int}_{\alpha i}(\mu \circ f_i(x)) \wedge \mu(x)
\]

for all \(x \in X\) and \(\mu \in L^1\).

### Characterized Fuzzy Quotient Spaces

Let \((X, \varphi_{i} \cdot \text{int})\) be a characterized fuzzy space and \(f: X \to A\) is a surjective mapping. Then the mapping \(\varphi_{i} \cdot \text{int}_{\alpha i}: L^1 \to L^1\), defined by:

\[
\varphi_{i} \cdot \text{int}_{\alpha i}(\mu)(x) = \bigwedge_{\alpha \in \alpha i} \varphi_{i} \cdot \text{int}_{\alpha i}(\mu \circ f_i(x))
\]

for all \(\alpha \in A\) and \(\mu \in L^1\), is final \(\varphi_{i} \cdot \text{int}\) fuzzy interior operator of \((\varphi_{i} \cdot \text{int})_{\alpha i}\) with respect to \(f\) which is not idempotent. Then \(\varphi_{i} \cdot \text{int}_{\alpha i}\) will be called quotient \(\varphi_{i} \cdot \text{int}\) fuzzy interior operator and the triple \((A, \varphi_{i} \cdot \text{int}_{\alpha i})\) is said to be characterized fuzzy quotient space [7].

Note that in this case \(\varphi_{12} \cdot \text{int}\) idempotent, \(\varphi_{12} \cdot \text{int}\) need not be. Even in the classical case of \(E=\{0, 1\}\), \(\varphi_{i} \cdot \text{int}\) and \(\varphi_{i} = \{0\} X\) we have the following: If \(\varphi_{i} \cdot \text{int}\) is up to an identification the usual topology, then \(\varphi_{12} \cdot \text{int}\) is a pre topology which need not be idempotent. An example is given [25] (p. 234).

### Characterized Fuzzy Sum Spaces

Assume that \((X, \delta_{i})\) is a characterized fuzzy space for each \(i \in I\), where \(I\) is any class. Let \(X\) be the disjoint union \(\bigsqcup (X, \{i\})\) of the family \((X, \{i\})\) and for each \(i \in I\), let \(\varphi_{i} \cdot \text{int}: L^1 \to L^1\), defined by \(e_i: X \to X\) be the canonical injection from \(X\) into \(X\) given by \(e(x) = (x, i)\). Then the mapping \(\varphi_{i} \cdot \text{int}: L^1 \to L^1\), defined by:

\[
\varphi_{i} \cdot \text{int}(\mu) = \delta_{i} \cdot \text{int}(\mu \circ e_i(x))(x) = \bigwedge_{\alpha \in \alpha i} \delta_{12} \cdot \text{int}_{\alpha i}(\mu \circ f_i(x))
\]

for all \(i \in I\), of \(\mu \in L^1\), is said to be final \(\varphi_{i} \cdot \text{int}\) fuzzy interior operator with respect to \(e_i\).

\((\delta_{i} \cdot \text{int})_{\alpha i}\) \(\varphi_{i} \cdot \text{int}\) will be called sum \(\varphi_{i} \cdot \text{int}\) fuzzy interior operator and \(\varphi_{i} \cdot \text{int}\) is said to be characterized fuzzy sum space [7] and it will be denoted also by \(\bigwedge_{i \in I} (X, \delta_{i} \cdot \text{int})\).
such that \( O_{\varphi}\), and \( \Omega\) is a subbase for the characterized fuzzy space \((X, \varphi_{\text{int}})\).

Then, \((X, \varphi_{\text{int}})\) is characterized fuzzy \( R_{\frac{1}{2}}\)-space if and only if for all \( F \in \Omega\) and \( x \in X\) such that \( x \notin F\), there exists a \( \varphi_{\text{int}}\)-continuous mapping \( f : (X, \varphi_{\text{int}}) \rightarrow (I, \varphi_{\text{int}})\) fuzzy \( T_{\frac{1}{2}}\)-space and characterized fuzzy \( T_{\frac{1}{2}}\)-spaces such that \( f(x) = 0\) and \( f(y) = 1\) for all \( y \in F\).

**Characterized**

Let a fuzzy topological space \((X, \tau)\) be fixed and \( \varphi_1, \varphi_2 \in O_{\varphi}\). Then the characterized fuzzy space \((X, \varphi_{\text{int}})\) is said to be characterized fuzzy \( R_{\frac{1}{2}}\) or characterized Tychonoff fuzzy space \([9]\) (resp. fuzzy \( T_{\frac{1}{2}}\)-space) if and only if it is characterized fuzzy \( R_{\frac{1}{2}}\) (resp. characterized fuzzy \( T_{\frac{1}{2}}\)) and characterized fuzzy \( T_{\frac{1}{2}}\)-space. The related fuzzy topological space \((X, \tau)\) is said to be fuzzy \( \varphi_{\text{int}}\)-\( T_{\frac{1}{2}}\)-space (resp. fuzzy \( \varphi_{\text{int}}\)-\( T_{\frac{1}{2}}\)-space) if and only if it is fuzzy \( \varphi_{\text{int}}\)-\( R_{\frac{1}{2}}\) (resp. fuzzy \( \varphi_{\text{int}}\)-\( T_{\frac{1}{2}}\)) and fuzzy \( \varphi_{\text{int}}\)-\( T_{\frac{1}{2}}\)-space.

**Proposition**

Every characterized fuzzy \( T_{\frac{1}{2}}\)-space is characterized fuzzy \( R_{\frac{1}{2}}\)-space \([9]\).

**Metrizable Characterized Fuzzy Spaces and Characterized \( T_{\frac{1}{2}}\)-Spaces**

By the fuzzy metric on the set \( X \times X \rightarrow R^*_+\) such that the following conditions are fulfilled:

1. \( d(x, y) = 0\) if and only if \( x = y\).
2. \( d(x, y) = d(y, x)\).
3. \( d(x, y) \leq d(x, z) + d(z, y)\) holds for all \( x, y, z \in X\).

Where \( 0^+\) denotes the fuzzy number which has value 1 at 0 and 0 otherwise. The set \( X \times X \rightarrow R^*_+\) will be called fuzzy metric space. Each fuzzy metric on a set \( X \) generated canonically a stratified fuzzy topology \( \tau \) which has the base \( \{ \delta \in d : \delta \in \mu \text{ and } x \in X \} \) as a base, where \( \delta : X \times X \rightarrow R^*_+\) is the mapping defied by: \( d(x, y) = d(x, y)\) and

\[
\mu = \{ \delta \in R^*_+ : \delta > 0 \text{ and } \alpha \in L_1 \} \cup \{ \delta : \alpha \in L, \}
\]

Where \( \alpha \) is the domain of \( R^*_+\). The restriction of \( R^*_+\) on \( R^*_+\). Now, consider \( \varphi_{\text{int}}\) such as \( \varphi_{\text{int}}(x, \alpha)\) as shown in ref. \([20]\), the characterized fuzzy space \((X, \varphi_{\text{int}})\) is said to be metriziable characterized fuzzy space.

In the following proposition we shall prove that every metrizable characterized fuzzy space is characterized fuzzy \( T_{\frac{1}{2}}\)-space in sense of Abd-Allah \([10]\).

**Proposition**

Let \((X, \tau)\) be an stratified fuzzy topological space generated canonically by a fuzzy metric \( d \) on \( X \) and \( \varphi, \varphi_{\text{int}} \in \Omega_{\varphi}\) then the metrizable characterized fuzzy space \((X, \varphi_{\text{int}})\) is characterized fuzzy \( T_{\frac{1}{2}}\)-space.

**Proof**

Let \( F_1, F_2 \in O_{\varphi}(C(X)) \) such that \( F_1 \cap F_2 = \emptyset\). Then for all \( x \in F_1 \) and \( y \in F_2\), we get \( d(x, y) = 0\), that is, there exists \( \delta \) such that \( d(x, y)(\delta) > 0\) and therefore

\[
R^*_+ \bigg|_{\varphi_{\text{int}}} \left( \left( d(x, y) (\delta) \right) \right) = \left( \varphi \right) \left( d(x, y)(\delta) \right) < 1,
\]

holds. Consider \( \mu = R_{\frac{1}{2}} \cdot d \) and \( \eta = R_{\frac{1}{2}} \cdot d \), then

\[
\mu(x) = R_{\frac{1}{2}} (d_x) = R_{\frac{1}{2}} \left( d_x \right) = \left( \varphi \right) (d_x) = 1\]

for all \( x \in F_1 \) and \( \eta \in R_{\frac{1}{2}} \cdot d \), then\( R_{\frac{1}{2}} \cdot (d(x, z) + d(y, z)) \leq R_{\frac{1}{2}} \cdot (d(x, z)) \leq 1 \) and therefore

\[
(\mu \wedge \eta)(z) = R_{\frac{1}{2}} \cdot (d(x, z)) \leq R_{\frac{1}{2}} \cdot d(x, z) \leq 1 \text{ holds for all } z \in X,
\]

that is, \( \sup(\mu \wedge \eta) < 1\). Hence, the infimum \( \text{Np1.2}(F_1 \wedge \text{Np1.2}(F_2)) \) does exist and therefore \((X, \varphi_{\text{int}})\) is characterized fuzzy \( R_{\frac{1}{2}}\)-space. Because of Theorem 3.1 \([27]\), it is clear that \((X, \varphi_{\text{int}})\) is characterized fuzzy \( T_{\frac{1}{2}}\)-space. Consequently, \((X, \varphi_{\text{int}})\) is characterized fuzzy \( T_{\frac{1}{2}}\)-space.

**Example 3.1**

From Propositions 2.9 and 3.1, we get that the metrizable fuzzy space in sense of Gähler and Gähler \([3]\) is an example of a metrizable characterized fuzzy \( T_{\frac{1}{2}}\)-space and that is also example of a metrizable characterized fuzzy \( T_{\frac{1}{2}}\)-space for

**Characterized \( R_{\frac{1}{2}}\) and characterized \( T_{\frac{1}{2}}\)-spaces**

In the following we introduce and study the concepts of characterized \( R_{\frac{1}{2}}\)-space and of characterized \( T_{\frac{1}{2}}\)-spaces in the classical case. Let \((X, T)\) be an arbitrary topological space and \( \varphi_1, \varphi_2 \in O_{\varphi}\). Then the characterized space \((X, \varphi_{\text{int}})\) is said to be characterized \( R_{\frac{1}{2}}\)-space if and only if \((X, \varphi_{\text{int}})\) is characterized \( R_{\frac{1}{2}}\)-space in sense of \( \varphi_{\text{int}}\).

**Proposition**

Let \((X, T)\) be an arbitrary topological space and \( \varphi_1, \varphi_2 \in O_{\varphi}\) such that \( \varphi_{\text{int}}(x, \alpha)\) is isotonous and idempotent. Then \((X, \varphi_{\text{int}})\) is characterized \( R_{\frac{1}{2}}\)-space if and only if the induced characterized fuzzy space \((X, \varphi_{\text{int}})\) is characterized fuzzy \( R_{\frac{1}{2}}\)-space.

**Proof**

Let \((X, \varphi_{\text{int}})\) is characterized \( R_{\frac{1}{2}}\)-space, \( x \in X \) and \( F \in O_{\varphi}(\varphi, \alpha, \Omega(X)) \) such that \( x \notin F \). Then, there exists \( \varphi_{\text{int}}\)-continuous mapping \( g : (X, \varphi_{\text{int}}) \rightarrow (I, \varphi_{\text{int}}) \) such that \( g(x) = \text{int} \Omega_S \rightarrow \emptyset \) for all \( y \in S \subseteq F \) and all \( \alpha \in L_1 \), where \( \delta_1 \delta_2 \in O_{\varphi}(\alpha) \). Hence, the mapping \( g : (X, \varphi_{\text{int}}) \rightarrow (I, \delta_{\varphi_{\text{int}}}\) is \( \varphi_{\text{int}}\)-continuous fuzzy continuous. Consider \( h : (I, \varphi_{\text{int}}) \rightarrow (I, \varphi_{\text{int}}) \) is the mapping defied by \( h(z) = z \) for all \( z \in I_1 \), then \( \delta_{\varphi_{\text{int}}} \) is fuzzy continuous and therefore there exists an \( \varphi_{\text{int}}\)-continuous fuzzy continuous mapping \( f : h \rightarrow (X, \varphi_{\text{int}}) \rightarrow (I, \varphi_{\text{int}}) \) such as

\[
f(x) = \delta_{\varphi_{\text{int}}} (x, \alpha) \text{ for all } x \in X.
\]
that \( f(x) = 1 \) and \( f(y) = 0 \) for all \( y \in F \). Consequently, \((X, \varphi_{12})\) is characterized fuzzy \( R_{\frac{1}{2}} \)-space.

Conversely, let \((X, \varphi_{12})\) is characterized fuzzy \( R_{\frac{1}{2}} \)-space, \( x \in X \) and \( F \in \varphi_{12}(X) \) such that \( x \not\in X_{F} \). Then, \( x \not\in X_{F} \) and \( x \not\in \varphi_{12}(X) \) such that \( f(x) = 1 \) and \( f(y) = 0 \) for all \( y \in X_{F} \). Since \( \varphi_{12} \) is characterized fuzzy continuous mapping \( f: (X, \varphi_{12}) \rightarrow (I_{L}, \psi_{12}) \), then there could be found the mapping \( f_{x} = (X, \varphi_{12}) \rightarrow (I_{L}, \psi_{12}) \) which is \( \psi_{12} \)-continuous with \( f(x) = 1 \) and \( f(y) = 0 \) for all \( y \in X_{F} \). Hence, \((X, \varphi_{12})\) is characterized fuzzy \( R_{\frac{1}{2}} \)-space.

**Corollary 3.1**

Let \((X, T)\) be an ordinary topological space and \( \varphi_{1}, \varphi_{2} \in O_{f}(X, T) \), \( x \in X \) and \( F \) be \( \varphi_{1} \)-open such that \( \varphi_{2} \geq \varphi_{1} \), then \((X, \varphi_{12})\) is characterized fuzzy \( R_{\frac{1}{2}} \)-spaces if and only if the induced characterized fuzzy \( R_{\frac{1}{2}} \)-spaces and the notions \( \varphi_{1}, \varphi_{2} \in \varphi_{12}(X) \) is characterized fuzzy \( R_{\frac{1}{2}} \)-space.

**Proposition 3.3**

Let \((X, \tau)\) be a fuzzy topological space and \( \varphi_{1}, \varphi_{2} \in O_{f}(X, \tau) \), \( x \in X \) and \( F \in \varphi_{12}(X) \) such that \( \varphi_{2} \geq \varphi_{1} \) is isotope and idempotent. Then, \((X, \varphi_{12})\) is characterized fuzzy \( R_{\frac{1}{2}} \)-spaces if and only if the induced characterized fuzzy \( R_{\frac{1}{2}} \)-spaces are finer than \( \varphi_{12} \)-space.

**Proof:** Immediate from Propositions 2.3 and 3.2.

In the following proposition for each fuzzy topological space \((X, \tau)\), we show that the \( \alpha \)-level characterized fuzzy \( R_{\frac{1}{2}} \)-spaces and the initial characterized fuzzy \( R_{\frac{1}{2}} \)-spaces are characterized fuzzy \( R_{\frac{1}{2}} \)-spaces with the \( \alpha \)-level characterized fuzzy \( R_{\frac{1}{2}} \)-spaces.

**Proposition 3.2**

Let \((X, \tau)\) be a fuzzy topological space and \( \varphi_{1}, \varphi_{2} \in O_{f}(X, \tau) \), \( x \in X \) and \( F \in \varphi_{12}(X) \) such that \( \varphi_{2} \geq \varphi_{1} \) is isotope and idempotent. Then, \((X, \varphi_{12})\) is characterized fuzzy \( R_{\frac{1}{2}} \)-spaces if and only if the induced characterized fuzzy \( R_{\frac{1}{2}} \)-spaces are finer than \( \varphi_{12} \)-space.

**Proof:** Immediate from Propositions 2.7 and 3.4.
and CRF-Tych, respectively. Note that the categories CFR-Space and CRF-Tych are concrete categories. The concrete categories CFR-Space and CRF-Tych are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets. Hence, all the initial and final lifts exist uniquely in the categories CFR-Space and CRF-Tych, respectively.

This means that they also topology over the category SET. That is, all the initial and final characterized fuzzy $R_{\frac{1}{2}}$-spaces and all the initial and final characterized fuzzy $T_{\frac{1}{2}}$-spaces exist in CFR-Space and CRF-Tych, respectively.

In the following let $X$ be a set, let $I$ be a class and for each $i \in I$, let the characterized fuzzy space $(X, \delta_{i,\text{int}})$ of all $\delta_{i,\text{open}}$-open fuzzy subsets of $X$ is characterized fuzzy $R_{\frac{1}{2}}$-space. For some $i \in I$, let $f: X \rightarrow X$, is $\varphi_{i}\delta_{i,\text{closed}}$-injective mapping from $X$ into $X$. Then we show in the following that the initial characterized fuzzy space $(X, \varphi_{i,\text{int}})$ of $((X, \delta_{i,\text{int}}))_{a_{i}}$ with respect to $(\varphi_{i})_{a_{i}}$ is also characterized fuzzy $R_{\frac{1}{2}}$-space. More general, we show under the same conditions, that the initial characterized fuzzy space $(X, \varphi_{i,\text{int}})$ of $((X, \delta_{i,\text{int}}))_{a_{i}}$ with respect to $(\varphi_{i})_{a_{i}}$ is characterized fuzzy $T_{\frac{1}{2}}$-space if all the characterized fuzzy spaces $(X, \varphi_{i,\text{int}})$ are characterized fuzzy $T_{\frac{1}{2}}$-spaces for all $i \in I$. Moreover, as special cases we show that the characterized fuzzy subspace, characterized fuzzy product space and characterized fuzzy filter pre topology of a characterized fuzzy $R_{\frac{1}{2}}$-space and of a characterized fuzzy $T_{\frac{1}{2}}$-space are characterized fuzzy $R_{\frac{1}{2}}$-spaces and characterized fuzzy $T_{\frac{1}{2}}$-spaces, respectively.

**Proposition**

Let $X$ be a set and $I$ be a class. For each $i \in I$, let the characterized fuzzy space $(X, \delta_{i,\text{int}})$ of all $\delta_{i,\text{open}}$-open fuzzy subsets of $X$ is characterized fuzzy $R_{\frac{1}{2}}$-space. If $f: X \rightarrow X$ is an $\varphi_{i}\delta_{i,\text{closed}}$-injective mapping from $X$ into $X$, for some $i \in I$, then the initial characterized fuzzy space $(X, \varphi_{i,\text{int}})$ of $((X, \delta_{i,\text{int}}))_{a_{i}}$ with respect to $(\varphi_{i})_{a_{i}}$ is also characterized fuzzy $R_{\frac{1}{2}}$-space.

**Proof:** Let $x \in X$ and $F \in \varphi_{i}\mathbb{C}(X)$ such that $x F$. Since $f: X \rightarrow X$ is $\varphi_{i}\delta_{i,\text{closed}}$-injective for some $i \in I$, then $f(F) \in \delta_{i,\text{open}}(X)$ and $f(x) \notin f(F)$. Because of of $(X, \varphi_{i,\text{int}})$ is characterized fuzzy $R_{\frac{1}{2}}$-space for all $i \in I$, then there exists an $\delta_{i,\varphi_{i,\text{int}}}$-fuzzy continuous mapping $g: (X, \delta_{i,\text{int}}) \rightarrow (U_{i}, \varphi_{i,\text{int}})$ such that $g(f(x)) = \overline{1}$ and $g(f(x)) = \overline{0}$ for all $y \in F$. Therefore the composition $h=g \circ f: (X, \varphi_{i,\text{int}}) \rightarrow (U_{i}, \varphi_{i,\text{int}})$ is $\varphi_{i,\varphi_{i,\text{int}}}$-fuzzy continuous mapping such that $h(x) = (g \circ f)(x)) = \overline{1}$ and $h(y) = (g \circ f)(y)) = \overline{0}$ for all $y \in F$. Consequently, $(X, \varphi_{i,\text{int}})$ is characterized fuzzy $R_{\frac{1}{2}}$-space.

**Corollary 4.1** Let $X$ be a set and $I$ be a class. For each $i \in I$, let the characterized fuzzy space $(X, \delta_{i,\text{int}})$ of all $\delta_{i,\text{open}}$-open fuzzy subsets of $X$ is characterized fuzzy $T_{\frac{1}{2}}$-space. If $f: X \rightarrow X$ is an $\varphi_{i}\delta_{i,\text{closed}}$-injective mapping from $X$ into $X$, for some $i \in I$, then the initial characterized fuzzy space $(X, \varphi_{i,\text{int}})$ of $((X, \delta_{i,\text{int}}))_{a_{i}}$ with respect to $(\varphi_{i})_{a_{i}}$ is also characterized fuzzy $T_{\frac{1}{2}}$-space.

**Proof:** Immediate from Propositions 2.5 and 4.1.

**Corollary 4.2**

The characterized fuzzy subspace $(A, \varphi_{i,\text{int}})$ and the characterized fuzzy product space $\prod_{i \in I}(X_{i}, \varphi_{i,\text{int}})$ of a characterized fuzzy $R_{\frac{1}{2}}$-space (resp. characterized fuzzy $T_{\frac{1}{2}}$-space) are also characterized fuzzy $R_{\frac{1}{2}}$-space (resp. characterized $T_{\frac{1}{2}}$-space).

**Proof:** Follows immediately from Proposition 4.1 and Corollary 4.1.2

As shown in ref. [7], the characterized fuzzy space $(X, \varphi_{i,\text{int}})$ is characterized as a fuzzy filter pre topology, then we have the following result:

**Corollary 4.3**

For each $i \in I$, let $\mathcal{N}_{\varphi_{i}, X} \rightarrow F_{i}X$, is $\delta_{\text{int}}$, as the fuzzy filter pre topology is characterized fuzzy $R_{\frac{1}{2}}$-fuzzy $T_{\frac{1}{2}}$-space. Then, the representation of the initial $\varphi_{i,\text{int}}$-interior operator $\mathcal{N}_{\varphi_{i}, X} \rightarrow F_{i}X$ of the initial characterized fuzzy space $(X, \varphi_{i,\text{int}})$ of $((X, \delta_{i,\text{int}}))_{a_{i}}$ with respect to $(\varphi_{i})_{a_{i}}$ as a fuzzy filter pre topology which is defined by:

$$\mathcal{N}_{\varphi_{i,\text{int}}}((x, \mu)) = \bigvee_{i \in I \cup \{1, 2\}} \mathcal{N}_{\varphi_{i,\text{int}}}((f_{i}(x), \mu))$$

for all $x \in X$ and $\mu \in \mathcal{L}$ is also characterized fuzzy $R_{\frac{1}{2}}$ (resp. characterized fuzzy $T_{\frac{1}{2}}$).

Now, if we consider the case of $I$ being a singleton, then we have the following results as special cases from Proposition 4.1 and Corollary 4.1.

**Proposition**

Let $(X, \tau_{1})$ and $(Y, \tau_{2})$ are two fuzzy topological spaces, $\delta_{1}, \delta_{2} \in O_{\tau_{1},\tau_{2}}$, and $\delta_{1}, \delta_{2} \in O_{\tau_{1},\tau_{2}}$. If the mapping $f: X \rightarrow Y$ is an $\varphi_{i,\delta_{1,\text{closed}}}$-injective mapping from $X$ into $Y$ and $(Y, \delta_{2,\text{int}})$ in characterized fuzzy $R_{\frac{1}{2}}$-terized fuzzy $T_{\frac{1}{2}}$-space, then the initial characterized fuzzy space $(X(Y, \delta_{2,\text{int}})$ with respect to $f$ is also characterized fuzzy $R_{\frac{1}{2}}$ (resp. fuzzy $T_{\frac{1}{2}}$)-space.

**Proof:** Straight forward.

**Corollary 4.4**

Let $(X, \tau_{1})$ be an fuzzy topological spaces and $\delta_{1}, \delta_{2}, f: X \rightarrow Y$ is an $\varphi_{i,\delta_{1,\text{closed}}}$-injective mapping from $X$ into $Y$ fuzzy $\delta_{1, T_{\frac{1}{2}}}$-space), then the initial fuzzy topological space $(X, f^{\tau_{1}}(\tau_{1}))$ of $(Y, \tau_{2})$ with respect to $f$ is fuzzy $\varphi_{i,\tau_{1}}$ $R_{\frac{1}{2}}$-space (resp. fuzzy $\varphi_{i,\tau_{1}}$ $T_{\frac{1}{2}}$-space) for all $\varphi_{i, \varphi_{i,\text{int}}}$ $O_{\tau_{1},\tau_{2}}$.

**Proof:** Follows immediately from Proposition 4.2.2

In the following let $X$ be a set and $I$ be a class. For each $i \in I$, let the characterized fuzzy space $(X, \delta_{i,\text{int}})$ of all $\delta_{i,\text{open}}$-open fuzzy subsets of $X$ is characterized fuzzy $R_{\frac{1}{2}}$-space. For some $i \in I$, let $f_{i}: X \rightarrow X$
is surjective mapping from \(X\) into \(X\) and \(f^{-1}\) is \(\varphi_1, \delta_1\)-closed in the classical sense. Then as in case of the initial characterized fuzzy spaces, we show in the following that the final characterized fuzzy space \((X, \varphi_1, \delta_1, \text{int})\) of \((X, \delta_1, \text{int})\) with respect to \((f)_{\text{int}}\) is also characterized fuzzy \(R_{13}^{T_2}\)-space. More generally, we show under the same conditions that, the final characterized fuzzy space \((X, \varphi_1, \delta_1, \text{int})\) of \((X, \delta_1, \text{int})\) with respect to \((f)_{\text{int}}\) is characterized fuzzy \(T_{13}^{2}\)-space if each of the characterized fuzzy spaces \((X, \delta_1, \text{int})\) is characterized fuzzy \(T_{13}^{2}\)-space and of the characterized fuzzy \(R_{13}^{T_2}\)-space for all \(i \in I\). Moreover, as special cases we show that the characterized fuzzy quotient space and the characterized fuzzy sum space of the characterized fuzzy \(R_{13}^{T_2}\)-space and of the characterized fuzzy \(T_{13}^{2}\)-space are characterized fuzzy \(R_{13}^{T_2}\)-spaces and characterized fuzzy \(T_{13}^{2}\)-spaces, respectively. Proposition 4.3 Let \(X\) be a set and \(I\) be a class. For each \(i \in I\), let the characterized fuzzy space \((X, \delta_1, \text{int})\) of all \(\delta_1\)-open fuzzy subsets of \(X\) is characterized fuzzy \(R_{13}^{T_2}\)-space for all \(i \in I\), then there exists an \(\delta_1, \varphi_1\)-fuzzy continuous mapping \(g: (X, \delta_1, \text{int}) \rightarrow (I, \varphi_1, \text{int})\) such that \(g(x) = 1\) and \(g(z) = 0\) for all \(z \in K\), that is, \(g(f^{-1}(z)) = 1\) and \(g(f^{-1}(z)) = 0\) for all \(z \in F\). Therefore, there exists a mapping \(h = g \circ f^{-1}: (X, \varphi_1, \text{int}) \rightarrow (I, \varphi_1, \text{int})\) such that \(h(x) = 1\) and \(h(z) = 0\) for all \(z \in F\). Since \(f\) is \(\delta_1, \varphi_1\)-fuzzy open, then \(\nu_1, \varphi_2\) \(\mu \subseteq f^{-1}(\nu_1, \varphi_2) \mu \subseteq f^{-1}(\mu) = \delta_1, \varphi_1\) \(\mu \subseteq f^{-1}(\mu) = \delta_1, \varphi_1\) \(\mu \subseteq f^{-1}(\mu) = \delta_1, \varphi_1\)-fuzzy continuous. Hence, the composition \(h = g \circ f^{-1}: (X, \varphi_1, \text{int}) \rightarrow (I, \varphi_1, \text{int})\) is \(\varphi_1, \delta_1\)-fuzzy continuous mapping and therefore the final characterized fuzzy space \((X, \varphi_1, \text{int})\) is characterized fuzzy \(R_{13}^{T_2}\)-space.

**Corollary 4.5**

Let \(X\) be a set and \(I\) be a class. For each \(i \in I\), let the characterized fuzzy space \((X, \delta_1, \text{int})\) of all \(\delta_1\)-open fuzzy subsets of \(X\) is characterized fuzzy \(T_{13}^{2}\)-space. If \(f: X \rightarrow X\) is an surjective \(\delta_1, \varphi_1\)-fuzzy open mapping from \(X\) into \(X\) and \(f^{-1}\) is \(\varphi_1, \delta_1\)-closed, then the final characterized fuzzy space \((X, \varphi_1, \delta_1, \text{int})\) of \((X, \delta_1, \text{int})\) with respect to \((f)_{\text{int}}\) is characterized fuzzy \(R_{13}^{T_2}\) (resp. characterized fuzzy \(T_{13}^{2}\)-space).

**Proof:** Immediate from Propositions 2.6 and 4.3. 2

**Corollary 4.6**

The characterized fuzzy quotient space \((A, \varphi_1, \text{int})\) and the characterized fuzzy \(T_{13}^{2}\)-space are also characterized fuzzy \(R_{13}^{T_2}\) (resp. characterized fuzzy \(T_{13}^{2}\))-L-spaces.

**Proof:** Follows immediately from Proposition 4.3 and Corollary 4.5. 2

Now, if we consider the case of \(I\) being a singleton, then we have the following results as special cases from Proposition 4.3 and Corollary 4.5.

**Proposition 4.4** Let \((X, \tau_1)\) and \((Y, \tau_2)\) are two fuzzy topological spaces, \(\phi_1, \phi_2 \in \mathcal{O}_{\tau_1, \tau_2}(X, Y)\) and \(\delta_1, \delta_2 \in \mathcal{O}_{\tau_1, \tau_2}(X, Y)\). If \(f: Y \rightarrow X\) is an \(\delta_1, \varphi_1\)-fuzzy open surjective mapping from \(Y\) into \(X\) and \(f^{-1}\) is \(\varphi_1, \delta_1\)-closed, then the final characterized fuzzy space \((X, \varphi_1, \delta_1, \text{int})\) of \((Y, \delta_2, \text{int})\) with respect to \(f\) is characterized fuzzy \(R_{13}^{T_2}\) (resp. characterized fuzzy \(T_{13}^{2}\)-space if \((Y, \delta_2, \text{int})\) is characterized fuzzy \(R_{13}^{T_2}\) (resp. characterized fuzzy \(T_{13}^{2}\))-L-spaces.

**Proof:** Straight forward.

**Corollary 4.7**

Let \((X, \tau_1)\) be an fuzzy topological spaces and \(\delta_1, \delta_2 \in \mathcal{O}_{\tau_1, \tau_2}(X, Y)\). If \(f: Y \rightarrow X\) is an \(\delta_1, \varphi_1\)-fuzzy open surjective mapping from \(Y\) into \(X\) and \(f^{-1}\) is \(\varphi_1, \delta_1\)-closed, then the final fuzzy topological space \((X, f(\tau_1))\) of \((Y, \tau_2)\) with respect to \(f\) is fuzzy \(\varphi_1, \delta_1\)-space (resp. fuzzy \(\varphi_2, \delta_2\)-space) if \(f\) is fuzzy \(\varphi_1, \delta_1\)-space (resp. fuzzy \(\varphi_2, \delta_2\)-space).

**Proof:** Follows immediately from Proposition 4.4. 2

**Fine Characterized Fuzzy \(R_{13}^{T_2}\)-spaces and Characterized Fuzzy \(T_{13}^{2}\)-spaces**

In this section we are going to introduce and study some finer characterized fuzzy \(R_{13}^{T_2}\) and finer characterized fuzzy \(T_{13}^{2}\)-spaces as a generalization of the weaker and stronger forms of the completely fuzzy regular and fuzzy \(T_{13}^{2}\)-spaces introduced [28,12,13]. The relations between such characterized fuzzy \(R_{13}^{T_2}\)-spaces and our characterized fuzzy \(R_{13}^{T_2}\)-spaces which presented [9] are introduced. More generally, the relations between such characterized fuzzy \(T_{13}^{2}\)-spaces and our characterized fuzzy \(T_{13}^{2}\)-spaces are also introduced.

Characterized fuzzy \(R_{13}^{T_2}\), \(H\) and characterized fuzzy \(T_{13}^{2}\), \(H\)-spaces. In the following we introduce and study the concept of characterized completely fuzzy regular Hutton and characterized fuzzy \(T_{13}^{2}\), Hutton-spaces as a generalization of the weaker and stronger forms of the completely fuzzy regular and fuzzy \(T_{13}^{2}\)-spaces in sense of Hutton [28], respectively. The relation between characterized completely fuzzy regular Hutton-spaces and the characterized fuzzy \(R_{13}^{T_2}\)-spaces in our sense is introduced. More generally, the relations between characterized fuzzy \(T_{13}^{2}\) Hutton-spaces and the characterized fuzzy \(T_{13}^{2}\)-spaces in our sense is also introduced. Let \((X, \tau)\) be a fuzzy topological space and \(\phi_1, \phi_2 \in \mathcal{O}_{\tau_1, \tau_2}\). Then the characterized fuzzy space \((X, \varphi_1, \text{int})\) is said to be characterized completely fuzzy regular Hutton-space or (characterized fuzzy \(R_{13}^{T_2}\)-space, for short) if for an
\( \mu \in \Phi \text{OF}(X) \), there exists a collection \( \eta \in \mathcal{L} \) and an \( \varphi_1, \varphi_2 \)-fuzzy continuous mapping \( g : (X, \varphi_1, \varphi_2) \to (Y, \psi_1, \psi_2) \) such that \( \mu = \bigvee \eta \) and 
\[ \eta_y(x) \leq g(y)(1) - x \leq g(y)(0) \leq \varphi_1(g)(x) \leq \mu_y(x) \]
holds for all \( y \in X \). Then characterized fuzzy space \( (X, \varphi_1, \varphi_2, \int) \) is said to be characterized fuzzy \( T \), H-space or (characterized fuzzy \( T \), H-space, for short) if and only if it is characterized fuzzy \( R \), H and characterized fuzzy \( T \), -spaces.

In the classical case of \( L = \{0, 1\} \), \( \varphi_1, \varphi_2 \)-int \( \varphi_1 \)-int \( \varphi_2 \)-fuzzy continuity of \( f \) is up to an identification the usual fuzzy continuity of \( f \). Then in this case the notions of characterized fuzzy \( R \), H-spaces and of characterized fuzzy \( T \)-spaces coincide with the notion of fuzzy completely regular spaces and the notion fuzzy \( T \), -spaces defined by Hutton [28], respectively. Another special choices for the operations \( \varphi_1, \varphi_2 \) and \( \varphi_1 \) and \( \varphi_2 \) are obtained (Table 1).

In the following proposition, we show that the characterized fuzzy \( R \), -spaces which are presented [9] are more general than the characterized fuzzy \( T \), H-spaces.

**Proposition 5.1**

Let \( (X, \tau) \) be an fuzzy topological space and \( \varphi_1, \varphi_2 \in \mathcal{O}(L, \tau_1) \).

Then every characterized fuzzy \( R \), H-space \( (X, \varphi_1, \varphi_2, \int) \) is characterized fuzzy \( R \), -space.

**Proof:** Let \( (X, \varphi_1, \varphi_2) \) be characterized fuzzy \( R \), H-space, \( x \in X \) and \( F \in \varphi_2 \text{OF}(X) \) such that \( x \notin F \). Then, \( \chi_{\varphi_1} \cong \Phi \text{OF}(X) \) and \( \chi_{\varphi_2}(x) = 1 \), therefore \( \chi_{\varphi_1} \geq \alpha \) holds for all \( \alpha \in \mathcal{L} \). Hence, \( \chi_{\varphi_1} = \bigvee \alpha_{\varphi_1} \chi_{\varphi_1} \) and therefore for all \( x \in F \), there exists a family \( (x_{\alpha})_{\alpha} \) in \( L \) such that \( \chi_{\varphi_1} = \bigwedge_{\alpha} \chi_{\varphi_1} \) and \( \chi_{\varphi_2}(x) = \chi_{\varphi_1}(x) \leq g(x)(0) < g(x)(0) \leq \chi_{\varphi_2}(y) \) holds for all \( y \in X \). In case of \( y \in F \), we get \( 0 \leq g(x)(1) - g(x)(0) \leq 0 \) for all \( y \in F \) and therefore \( g(x) = 0 \) for all \( y \in F \). In case of \( y = x \), we get \( g(x)(1) - g(x)(0) \leq 0 \) holds for all \( x \in L \) and this means that \( g(x)(s) = 1 \) for all \( s \in 1 \) and therefore \( g(x) = 1 \). Consequently, \( (X, \varphi_1, \varphi_2) \) is characterized fuzzy \( R \), -space in sense [9].

**Corollary 5.1** Let \( (X, \tau) \) be an fuzzy topological space and \( \varphi_1, \varphi_2 \in \mathcal{O}(L, \tau_1) \). Then every characterized fuzzy \( T \), H-space is characterized fuzzy \( T \), -space.

**Proof:** Follows immediately from Proposition 5.1.

The following example shows that the inverse of Proposition 5.1 and of Corollary 5.1 is not true in general.

**Example 5.1.**

Let \( X = \{x, y\} \) with \( x \neq y \) and \( \tau = \{0, 1, 2\} \) be an fuzzy topology on \( X \). Choose \( \varphi_1 = \text{int}, \varphi_2 = \text{cl}, \varphi_1 = \text{int}, \) and \( \varphi_2 = \text{cl} \). Hence, \( \varphi_2 \text{CF}(X) = \{0, 1, 1, 0, 1, 0, 1, 0\} \text{CF}(X) = \{0, 1, 1, 0, 1, 0, 1, 0\} \) and there is the only case of \( x \in X, F = \{y\} \in \varphi_1 \text{OF}(X) \) such that \( x \notin F \). Since the mapping \( f : (X, \varphi_1, \varphi_2, \int) \) -space is characterized fuzzy \( R \), H-space and characterized fuzzy \( T \)-space.

In the classical case of \( L = \{0, 1\} \), \( \varphi_1, \varphi_2 \)-int \( \varphi_1 \)-int \( \varphi_2 \)-fuzzy continuity of \( f \) is up to an identification the usual fuzzy continuity of \( f \). Then in this case the notions of characterized fuzzy \( R \), K-space and of characterized fuzzy \( T \), K-space coincide with the notion of completely fuzzy regular.
spaces and the notion of fuzzy \( T_{13} \)-spaces presented by Katasara [13], respectively. Another special choices for the operations \( \psi_1, \psi_2, \psi_3, \) and \( \psi_4 \) are obtained in Table 1. In the following proposition we show that the notion of characterized fuzzy \( \mathcal{R}_{13} \)-spaces which are presented [9] are more general than the characterized fuzzy \( \mathcal{R}_{13} \)-K.-spaces.

**Proposition**

Let \((X, \tau)\) be an fuzzy topological space and \( \varphi, \varphi_i \in O_{\tau, 1, 2, \ldots} \). Then every characterized fuzzy \( \mathcal{R}_{13} \)-K.-space \((X, \varphi_i, \text{int})\) is characterized fuzzy \( \mathcal{R}_{13} \)-space.

**Proof:** Let \((X, \varphi_i, \text{int})\) is a characterized fuzzy \( \mathcal{R}_{13} \)-K.-space, \( x \in X \) and \( F \in \varphi_i. C(X) \) such that \( x \notin F \). Then, \( \chi_{1, x} \tau(x) = 1 \) and \( \chi_{2, x} \tau(x) = 1 \), therefore \( \chi_{1, x} \tau(x) \geq \alpha \) holds for all \( \alpha \in L \). Because of \((X, \varphi_i, \text{int})\) is characterized fuzzy \( \mathcal{R}_{13} \)-K.-space, then there exists a \( \varphi_i, \psi_i \)-fuzzy continuous mapping \( g_1: (X, \varphi_i, \text{int}) \to (I_1, \psi_i, \text{int}) \) such that \( \chi_{1, x} g_1(x)(t) \geq \tau(x) \) and \( \chi_{2, x} g_1(x)(t) \geq \tau(x) \) hold for all \( y \in X \) and \( \alpha \in L \). In case of \( y \in F \), we have \( \chi_{1, x} g_1(x)(t) \leq 0 \), that is, \( g_1(y)(t) = 0 \) for all \( t > 0 \), \( y \in F \) and therefore \( g_1(x)(t) \geq 0 \) for all \( y \in F \). Consequently, \((X, \varphi_i, \text{int})\) is characterized fuzzy \( \mathcal{R}_{13} \)-space in sense [9].

**Corollary 5.2** Let \((X, \tau)\) be an fuzzy topological space and \( \varphi, \varphi_i \in O_{\tau, 1, 2, \ldots} \). Then every characterized fuzzy \( T_{13} \)-space is characterized fuzzy \( T_{13} \)-space.

**Proof:** Follows immediately from Proposition 5.2.

The following example shows that the inverse of Proposition 5.2 and of Corollary 5.2 is not true in general.

**Example 5.2.**

Consider the characterized fuzzy space \((X, \varphi_i, \text{int})\) which is characterized fuzzy \( \mathcal{R}_{13} \)-space in sense [9] and characterized fuzzy \( T_{13} \)-space, therefore \((X, \varphi_i, \text{int})\) is characterized fuzzy \( T_{13} \)-space in sense [9].

On other hand, for any \( \varphi_i, \psi_i \)-fuzzy continuous mapping \( f_1: (X, \varphi_i, \text{int}) \to (I_1, \psi_i, \text{int}) \) such that \( f_1(x) = 1 \) and \( f_1(x) = 0 \), we shall consider \( x \in \varphi_i. O(F(X)) \) with \( x_1 \tau(x) = \alpha > 0 \), that is, there exists some \( \alpha \in L \) such that \( x_1 \tau(x) = \alpha \). Therefore, \( f_1(x)(t) = \chi_{1, x} f_1(x)(t) \geq \chi_{1, x} \tau(x) \) holds only when \( x \in X \) and it is not fulfilled when \( x \notin X \). Moreover, \( f_1(x)(0) = \chi_{1, x} f_1(x)(0) \leq x_1 \tau(x) \) holds only when \( x \in X \) and it is not fulfilled when \( x \notin X \). Hence, \((X, \varphi_i, \text{int})\) is not characterized fuzzy \( \mathcal{R}_{13} \)-K.-space and therefore it is not characterized fuzzy \( T_{13} \)-K.-space.

**Characterized Fuzzy \( \mathcal{R}_{13} \)-KE and Characterized Fuzzy \( T_{13} \)-KE-Spaces**

In the following we introduce and study the concepts of characterized completely fuzzy regular Kandil and Shaaf spaces and of characterized fuzzy \( T_{13} \)-spaces.

The relation between characterized fuzzy regular Kandil and Shaaf spaces and the characterized fuzzy \( \mathcal{R}_{13} \)-spaces which are presented [9] is also introduced.

Let \((X, \tau)\) be an fuzzy topological space and \( \varphi, \varphi_i \in O_{\tau, 1, 2, \ldots} \). Then the characterized fuzzy space \((X, \varphi_i, \text{int})\) is said to be characterized fuzzy regular Kandil and Shaaf space or (characterized fuzzy \( \mathcal{R}_{13} \)-KE-space, for short) if for every \( x \in X \) and \( \mu \in \varphi_i. CF \) such that \( x \varphi \mu \notin \varphi_i(CF(X)) \) then \( \varphi_i. \psi_i \)-continuous mapping \( f(x, \varphi_i, \text{int}) \to (I_1, \psi_i, \text{int}) \) such that \( f(x)(0) = \mu \) and \( f(x)(1) = \varphi_i \psi_i \) are hold for all \( y \in X \) and \( \alpha \in L \). The characterized fuzzy space \((X, \varphi_i, \text{int})\) is said to characterized quasi fuzzy \( T_{13} \)-space or (characterized QF\( T_{13} \)-space, for short) if for all \( x, y \in X \) such that \( x \neq y \) we have \( x \varphi_i \psi_i \psi_i \beta \) and \( \varphi_i \psi_i \psi_i \beta \) for all \( a, b \in L \). As easily seen that every characterized QF\( T_{13} \)-space is characterized fuzzy \( T_{13} \)-space. The characterized fuzzy space \((X, \varphi_i, \text{int})\) is said to be characterized fuzzy \( T_{13} \)-K.-El-Shafee-space or (characterized fuzzy \( T_{13} \)-KE-space, for short) if and only if it is characterized fuzzy \( T_{13} \)-K.-KE-space and characterized fuzzy \( T_{13} \)-QF\( T_{13} \)-spaces. Obviously, every characterized fuzzy \( T_{13} \)-KE-space is characterized fuzzy \( T_{13} \)-K.-space. In the classical case of \( L = \{0, 1\} \), \( \varphi, \psi_i, \text{int} \), \( \varphi_i \psi_i, \text{int} \), \( \varphi_1 \psi_1 \), and \( \psi_2 \psi_2 \), it is proved that the \( \varphi_i \psi_i \psi_i \)-fuzzy continuity of \( f \) is up to an identification the usual fuzzy continuity of \( f \). Hence, the notions of characterized fuzzy \( \mathcal{R}_{13} \)-KE-spaces and of characterized fuzzy \( T_{13} \)-KE-spaces coincide with the notion of completely fuzzy regular spaces and the notion fuzzy \( T_{13} \)-spaces presented by Kandil and Shaaf [12], respectively.

Another special choices for the operations \( \varphi, \varphi_i, \psi_i, \) and \( \psi_2 \) are obtained in Table 1.

In the following proposition we show that the characterized fuzzy \( \mathcal{R}_{13} \)-spaces which are presented [9] are more general than the characterized fuzzy \( \mathcal{R}_{13} \)-KE-spaces.

**Proposition 5.3.**

Let \((X, \tau)\) be an fuzzy topological space and \( \varphi, \varphi_i \in O_{\tau, 1, 2, \ldots} \). Then every characterized fuzzy \( \mathcal{R}_{13} \)-KE-space \((X, \varphi_i, \text{int})\) is characterized fuzzy \( \mathcal{R}_{13} \)-space.

**Proof:** Let \((X, \varphi_i, \text{int})\) is a characterized fuzzy \( \mathcal{R}_{13} \)-KE-space, \( x \in X \) and \( F \in \varphi_i. C(X) \) such that \( x \notin F \). Then, \( \chi_{1, x} = \varphi_i \psi_i \psi_i(X) \) and \( \chi_{1, x} \tau(x) = 1 \), therefore \( x \varphi_i \psi_i \). Because of \((X, \varphi_i, \text{int})\) is characterized fuzzy \( \mathcal{R}_{13} \)-KE-
space, then there exists a $\varphi_{1,2}$-fuzzy continuous mapping $f_{1,2}(x, \varphi_{1,2})$ such that $f_{1,2}(x)(0)+\leq x_{1,2}(y)$ and $f_{1,2}(x)(1)-\geq x_{1,2}(y)$ are hold for all $y \in X$. In case of $y \in E$, we have $0 \leq f_{1,2}(y)(1)- \leq f_{1,2}(y)(0)+ \leq 0$, that is, $f_{1,2}(y)=0$ for all $y \in E$. In case of $y=x$, we have $1 \leq f_{1,2}(x)(1)- \leq f_{1,2}(x)(0)+ \leq 1$ holds and then $f_{1,2}(x)(s)=1$ for all $s \leq 1$, therefore $f_{1,2}(x)=1$. Hence, there exists a $\varphi_{1,2}$-fuzzy continuous mapping $f_{1,2}(X, \varphi_{1,2}) \rightarrow (I_{1,2}, \psi_{1,2})$ such that $f_{1,2}(x)=1$ and $f_{1,2}(y)=0$ for all $y \in E$. Consequently, $(X, \varphi_{1,2})$ is characterized fuzzy $R_{1,2}$-space in sense [9].

Corollary 5.3

Let $(X, \tau)$ be a fuzzy topological space and $\varphi_{1,2}, \varphi_{3,4} \in O_{1,2}$, then every characterized fuzzy $T_{1,2}$ KE-space is characterized fuzzy $T_{1,2}$-space.

Proof: Follows immediately from Proposition 5.3 and the fact that every characterized space is characterized fuzzy $T_{1,2}$-space.

The following example shows that the inverse of Proposition 5.3 and Corollary 5.3 are not true in general.

Example 5.3.

Consider the characterized fuzzy space $(X, \varphi_{1,2}, \int \_1, \_2)$ which is defined in Example 5.1, then as shown in Example 5.1, $(X, \varphi_{1,2}, \int \_1, \_2)$ is characterized fuzzy $R_{1,2}$-space in sense [9] and characterized fuzzy $T_{1,2}$-space, therefore $(X, \varphi_{1,2}, \int \_1, \_2)$ is characterized fuzzy $T_{1,2}$-space in sense [9].

Now, choose $x_{1} \in S_{1}(X)$ and $\mu_{1}=x_{1} \in \varphi_{1,2}CF_{1}(X)$ then $\mu_{1}=x_{1} \cup y_{1} \in \varphi_{1,2}OF_{1}(X)$ such that $x_{1} \cup y_{1}=\mu_{1}$. Hence, for any $(X, \varphi_{1,2}, \int \_1, \_2)$-fuzzy continuous mapping $f_{1,2}(X, \varphi_{1,2}, \int \_1, \_2)$ such that $f_{1,2}(x)=1$ and $f_{1,2}(y)=0$ for all $y \in E$, we get $x_{1}(z) \leq f_{1,2}(z)(1)- \geq f_{1,2}(z)(1)$ holds for all $z \in X$. But $\mu_{1}(z)=x_{1}(z) \cup y_{1}(z)=f_{1,2}(z)(1)- \geq f_{1,2}(z)(1)$ holds only for $z \neq y$ and it is not fulfilled for $z=x$. Consequently, $(X, \varphi_{1,2}, \int \_1, \_2)$ is not characterized fuzzy $R_{1,2}$ KE-space and therefore it is not characterized fuzzy $T_{1,2}$ KE-space.

Conclusion

In this paper, basic notions related to the characterized fuzzy $R_{1,2}$ and the characterized fuzzy $T_{1,2}$-spaces which are presented [9] are introduced and studied. These notions are named metrizable characterized fuzzy subspaces, initial and final characterized fuzzy subspaces, some finer characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-spaces. The metrizable characterized fuzzy space is introduced as a generalization of the weaker and stronger forms of the fuzzy metric space introduced by Gahler and Gahler [3]. For every stratified fuzzy topological space generated canonically by an fuzzy metric we proved that, the metrizable characterized fuzzy space is characterized fuzzy $T_{1,2}$-space in sense of Abd-Allah [10] and therefore, it is characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-space. The induced characterized fuzzy space is characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-space if and only if the related ordinary topological space is $\varphi_{1,2}$-space and $\varphi_{1,2}$-space, respectively. Hence, the notions of characterized fuzzy $R_{1,2}$ and of characterized fuzzy $T_{1,2}$ are good extension in sense of Lowen [11]. Moreover, the a-level characterized space and the initial characterized space are characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-space if the related characterized fuzzy space is characterized fuzzy $R_{1,2}$-space and characterized fuzzy $T_{1,2}$-space, respectively. We shown that the finer characterized fuzzy space of a characterized fuzzy $R_{1,2}$-space and of a characterized fuzzy $T_{1,2}$-space is also characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-space, respectively. The categories of all characterized fuzzy $R_{1,2}$ and of all characterized fuzzy $T_{1,2}$-spaces will be denoted by CFR-Space and CRF-Tych and they are concrete categories. These categories are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets and hence all the initial and final lifts exist uniquely in CFR-Space and CRF-Tych, respectively. That is, all the initial and final characterized fuzzy $R_{1,2}$-spaces exist in CFR-Space and also all the initial and final characterized fuzzy $T_{1,2}$-spaces exist in CRF-Tych. We shown that the initial and final characterized fuzzy spaces of a characterized fuzzy $R_{1,2}$-space and of characterized fuzzy $T_{1,2}$-space are characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-spaces, respectively. As special cases, the characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space of a characterized fuzzy $R_{1,2}$-space and of a characterized fuzzy $T_{1,2}$-space are also characterized fuzzy $R_{1,2}$ and characterized fuzzy $T_{1,2}$-spaces, respectively. Finally, we introduced and studied three finer characterized fuzzy $R_{1,2}$ and three finer characterized fuzzy $T_{1,2}$ L-spaces as a generalization of the weaker and stronger forms of the completely regular and the fuzzy $T_{1,2}$-spaces introduced [28,12,13].

These fuzzy spaces are named characterized fuzzy $R_{1,2}$ H, characterized fuzzy $R_{1,2}$ K, characterized fuzzy $R_{1,2}$ KE, characterized fuzzy $T_{1,2}$ H, characterized fuzzy $T_{1,2}$ K and characterized fuzzy $T_{1,2}$ KE-spaces.

The relations between characterized fuzzy $R_{1,2}$ H, characterized fuzzy $R_{1,2}$ K, characterized fuzzy $R_{1,2}$ KE-spaces and the characterized fuzzy $R_{1,2}$-space which are presented [9] are introduced. More generally, the relations between characterized fuzzy $T_{1,2}$ H, characterized fuzzy $T_{1,2}$ K, characterized fuzzy $T_{1,2}$ KE-spaces and the characterized fuzzy $T_{1,2}$-spaces are also introduced. Meanly special cases from these finer characterized fuzzy $R_{1,2}$ and finer characterized fuzzy $T_{1,2}$-spaces are listed in Table 1.
| Operations | Char.fuzzy $R_\frac{1}{2}$ | Char.fuzzy $R_\frac{1}{2}$ | Char.fuzzy $R_\frac{1}{2}$ | Char.fuzzy $T_\frac{1}{2}$ | Char.fuzzy $T_\frac{1}{2}$ | Char.fuzzy $T_\frac{1}{2}$ |
|------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1 φ1-intr. $q_2=1$, $X$ | Fuz. $R_\frac{1}{2}$ | Fuz. $R_\frac{1}{2}$ | Fuz. $R_\frac{1}{2}$ | Fuz. $T_\frac{1}{2}$ | Fuz. $T_\frac{1}{2}$ | Fuz. $T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 2 φ1-intr. $q_2=1$, $X$ | Fuz. $\delta R_\frac{1}{2}$ | Fuz. $\delta R_\frac{1}{2}$ | Fuz. $\delta R_\frac{1}{2}$ | Fuz. $\delta T_\frac{1}{2}$ | Fuz. $\delta T_\frac{1}{2}$ | Fuz. $\delta T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 3 φ1-intr. $q_2=1$, $X$ | Fuz. $\lambda R_\frac{1}{2}$ | Fuz. $\lambda R_\frac{1}{2}$ | Fuz. $\lambda R_\frac{1}{2}$ | Fuz. $\lambda T_\frac{1}{2}$ | Fuz. $\lambda T_\frac{1}{2}$ | Fuz. $\lambda T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 4 φ1-intr. $q_2=1$, $X$ | Fuz. $\theta R_\frac{1}{2}$ | Fuz. $\theta R_\frac{1}{2}$ | Fuz. $\theta R_\frac{1}{2}$ | Fuz. $\theta T_\frac{1}{2}$ | Fuz. $\theta T_\frac{1}{2}$ | Fuz. $\theta T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 5 φ1-intr. $q_2=1$, $X$ | Fuz. $\xi R_\frac{1}{2}$ | Fuz. $\xi R_\frac{1}{2}$ | Fuz. $\xi R_\frac{1}{2}$ | Fuz. $\xi T_\frac{1}{2}$ | Fuz. $\xi T_\frac{1}{2}$ | Fuz. $\xi T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 6 φ1-intr. $q_2=1$, $X$ | Fuz. $\alpha R_\frac{1}{2}$ | Fuz. $\alpha R_\frac{1}{2}$ | Fuz. $\alpha R_\frac{1}{2}$ | Fuz. $\alpha T_\frac{1}{2}$ | Fuz. $\alpha T_\frac{1}{2}$ | Fuz. $\alpha T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 7 φ1-intr. $q_2=1$, $X$ | Fuz. $\beta R_\frac{1}{2}$ | Fuz. $\beta R_\frac{1}{2}$ | Fuz. $\beta R_\frac{1}{2}$ | Fuz. $\beta T_\frac{1}{2}$ | Fuz. $\beta T_\frac{1}{2}$ | Fuz. $\beta T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |
| 8 φ1-intr. $q_2=1$, $X$ | Fuz. $\gamma R_\frac{1}{2}$ | Fuz. $\gamma R_\frac{1}{2}$ | Fuz. $\gamma R_\frac{1}{2}$ | Fuz. $\gamma T_\frac{1}{2}$ | Fuz. $\gamma T_\frac{1}{2}$ | Fuz. $\gamma T_\frac{1}{2}$ |
| $\psi_1$-intr. $q_2=1$ | H-space | K-space | H-space | K-space | H-space | K-space |

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Table 1: Some special classes of Char.fuzzy $R_{\alpha}$ H-spaces, Char.fuzzy $R_{\beta}$ K-spaces, Char.fuzzy $R_{\gamma}$ KE-spaces, Char.fuzzy $T_{\alpha}$ H-spaces, Char. fuzzy $T_{\beta}$ K-spaces, $T_{\gamma}$ KE-spaces.

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