Quantum incompressibility and Razumov Stroganov type conjectures

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Abstract

We establish a correspondence between polynomial representations of the Temperley and Lieb algebra and certain deformations of the Quantum Hall Effect wave functions. When the deformation parameter is a third root of unity, the representation degenerates and the wave functions coincide with the domain wall boundary condition partition function appearing in the conjecture of A.V. Razumov and Y.G. Stroganov. In particular, this gives a proof of the identification of the sum of the entries of a $O(n)$ transfer matrix eigenvector and a six vertex-model partition function, alternative to that of P. Di Francesco and P. Zinn-Justin.

1 Introduction

This paper is aimed at establishing a correspondence between the deformation of certain wave functions of the Hall effect and polynomial representations of the Temperley and Lieb (T.L.) algebra.

This work originates from an attempt to understand the conjecture of A.V. Razumov and Y.G. Stroganov [1][2][3], and some partial results towards its proof by P.Di Francesco and P. Zinn-Justin [4].

We consider the analogue of spin singlet wave functions of the Hall effect when one deforms the permutations into the braid group. This amounts to analyze some simple representations of the T.L. algebra on a space of polynomials in $N_e$ variables where $N_e$ is the number of electrons. The relation with the Hall effect arises when we require certain incompressibility properties.

One of the wave functions we consider here is the Halperin wave function [5] for a system of spin one half electrons at filling factor two. When the deformation parameter
$q$ is a third root of unity, the braid group representation degenerates into a trivial representation. In this way, we obtain a proof alternative to, and apparently simpler than, that given in [4] of the equality between the sum of the components of the transfer matrix eigenvector and the six vertex model partition function with domain wall boundary conditions [2][6].

Another wave function we consider is the Haldane Rezayi wave function [7] describing a system of electrons of spin one half at filling factor one. This wave function is a permanent, and its deformation is described in terms of Gaudin’s determinants [8]. When $q$ is a third root of unity, it degenerates to the square of the six vertex model partition function.

In a separate publication [9], we shall consider the Moore Read wave function describing spinless bosons at filling factor one [10]. Its deformation involves an extension of the braid group known as the Birman-Wenzl algebra [11] which can be represented on a polynomial space similarly to the cases presented here. In some appropriate limit, the representation degenerates and the wave function coincides with the transfer matrix eigenvector considered in [12] related to the conjecture of J.De Gier and B. Nienhuis [13].

In general, when a Quantum Hall Effect wave function is discovered, it is soon after observed experimentally. We argue here, that as a bonus, Quantum Hall Effect wave functions and their deformations yield nice mathematical objects. Moreover, all these objects seem to be in relation with striking conjectures emanating from the six vertex model.

Since the permutation group relevant in the quantum Hall effect is technically simpler than the braid group case, let us for pedagogical reasons explain why finding a wave function turns out to be a useful tool to obtain a polynomial representations of the permutation algebra. Essentially, the rest of the paper extends the idea presented here to the braid group case.

We consider electrons in a strong magnetic field projected in the lowest Landau level. In a specific gauge the orbital wave functions are given by:

$$\psi_n(z) = \frac{z^n}{\sqrt{n!}} e^{-\frac{\bar{z}^2}{4l^2}},$$

(1)

where $z = x + iy$ is the coordinate of the electron, and $l$ the magnetic length defines the length scale related to the strength of the magnetic field. These orbitals are shells of radius $\sqrt{2nl}$ occupying an area $2\pi l^2$. Each orbital $n$ is represented by a monomial $z^n$.

The quantum Hall effect [14] ground state $\Psi$ is obtained by combining these individual orbitals into a manybody wave function. A monomial $\lambda_1^{N_1} \cdots \lambda_{N_e}^{N_e}$ describes a configuration

\footnote{More precisely a minor modification of it considered in [15].}
where the electron \( j \) occupies the orbital \( \lambda_j \). The wave function is a linear combinations of such monomials. The effect of the interactions is to impose some vanishing properties when electrons are in contact: \( \Psi \sim (z_i - z_j)^m \) with \( m \) an integer when \( z_i - z_j \to 0 \).

The physical properties are mainly characterized by the filling factor \( \nu \) which is the number of electrons per unit cell of area \( 2\pi l^2 \). When the filling factor is equal to \( \nu \), the accessible orbitals and thus the maximal degree in each variable is bounded by \( \nu^{-1}N_e \). On the other hand, the effect of the interactions \( (m) \) is to force the electrons to occupy more space, thus to occupy higher orbitals and and has the effect of increasing the degree. The problem is thus to obtain wave functions with the maximal possible filling factor (equivalently the lowest degree in each variable) compatible with the vanishing properties imposed by the interactions.

Once such a wave function is obtained, it is the nondegenerate lowest energy state of a Hamiltonian invariant under the permutations, thus we know that it is left invariant under the permutations. By disentangling the coordinate part from the spin part, we obtain an irreducible representation of the permutation algebra acting on polynomials.

Let us illustrate this point in the case of the Halperin wave function \[5\] which describes a system of spin one half electrons at filling factor two. There are no interactions between the electrons, but due to the Pauli principle, the wave function must vanish when two electrons of the same spin come into contact. Each independent orbital can be occupied with two electrons of opposite spin, which is why the maximal filling factor is equal to two.

An equivalent way to impose the constraint is to require that any linear combination of the spin components of the wave function vanishes when three electrons come into contact. The reason for this is that two of the electrons involved will necessary have the same spin. When this constraint is taken into account with the minimal degree hypotheses, one obtains a space of polynomials which can be recombined with the spin components into a wave function changing sign under the permutations. Thus we know a priori that the spatial part of the wave function carries an irreducible representation of the permutation algebra dual to that of the spins. This is precisely by generalizing this argument to the braid group case that we obtain the representations of the T.L. algebra mentioned above.

In the permutation group case case, the components have the simple structure of a product of two Slater determinants grouping together the electrons with the same spin and one does not require to recourse to this machinery.

Let us now briefly indicate why the Halperin wave function may have something to do with the eigenvector of a transfer matrix in the link pattern formulation \[16\]. The wave function is a spin singlet, and the spin components can best be described in a resonating valance bond (RVB) picture as follows: The labels of the electrons are disposed cyclically
around a circle and are connected by a link when two electrons form a spin singlet. Links are not allowed to cross in order to avoid overcounting states. These RVB states coincide with the link patterns of [10]. Thus, the Halperin wave function as the eigenvector of the transfer matrix develops on a basis of link patterns. By deforming the permutation action on link patterns into a T.L algebra action, one is forced to deform accordingly the polynomial representation so as to insure the invariance of the total wave function. When \( q \) is a third root of unity, this property is shared by the transfer matrix eigenvector and allows to identify the two.

In the braid group case, the situation is technically more involved than for the permutations. Nevertheless, the minimal degree hypothesis combined with some annihilation constraint satisfied by linear combination of the spin components yields a wave function with the correct invariance properties. A major difference with the Hall effect is that the cancelation no longer occurs at coincident points, but at points shifted proportionally to the deformation parameter \( q \). Typically, we require that for three arbitrary electron labels \( i < j < k \) ordered cyclically, the wave function vanishes when the corresponding coordinates take the values \( z, q^2 z, q^4 z \).

One is also led to study the affine extension in order to impose cyclic invariance properties which are tautologically satisfied with the permutations. While defined in a natural way on the link patterns, the cyclic properties require to introduce a shift parameter \( s \) when we identify the coordinate \( i + N_e \) with the coordinate \( i \): \( z_{i+N_e} = s z_i \). When this shift parameter is related in a specific way to the braid group deformation parameter, the generalized statistics properties can be established coherently. Here, \( z_{i+N_e} = q^6 z_i \), but the same annihilation property can also be satisfied with \( s \) not related to \( q \), and this can be achieved at the price of doubling the degree and enlarging the algebra [9].

In the the Haldane Rezayi case, [7], the interactions are such that the wave function must vanish as the square of the distance when electrons of the same spin come into contact. For the same reason as before, this amounts to impose that any linear combination of its spin components vanishes as the square of the distance when three electrons come into contact. This wave function is a permanent, and its deformation is described in terms of Gaudin’s determinants [8]. It degenerates to the square of the six vertex model partition function when the deformation parameter is a third root of unity.

The paper is organized as follows. In section 2 we recall some properties about Hecke algebras and their polynomial representations. Section 3 introduces the T.L. algebra representation used here. Section 4 is the core of the paper where we work out the deformed Hall effect wave functions.

We have attempted to be self contained, but in order not to overload the text with technicalities, we have relegated most of the proofs to appendices to which we refer when it is useful.
2 Hecke Algebra.

In this section, we recall some known facts about the Hecke and Temperley and Lieb algebras [17] [18].

The Braid group algebra is generated by the braid group generators \( t_1, t_2, ..., t_{n-1} \), obeying the braid relations:

\[
\begin{align*}
t_{i+1}t_i &= t_it_{i+1} \\
t_i t_j &= t_j t_i \quad \text{if } |i-j| > 1,
\end{align*}
\]

for \( 1 \leq i \leq n - 1 \). It can be convenient to use the notation \( t^{i+1}_i \) instead of \( t_i \), and we will use it when necessary. The Hecke algebra is the quotient of the Braid group algebra by the relations:

\[
(t_i - q)(t_i + \frac{1}{q}) = 0,
\]

It can also be defined using the projectors \( e_i = t_i - q \) obeying the relations:

\[
\begin{align*}
\tau e_i &= e_i^2 \\
e_i e_j &= e_j e_i \quad \text{if } |i-j| > 1 \\
e_i e_{i+1} e_i - e_i &= e_{i+1} e_i - e_{i+1}.
\end{align*}
\]

where we set \( \tau = -(q + q^{-1}) \). The Temperley-Lieb (T.L.) algebra \( A_n \) is the quotient of the Hecke algebra by the relations:

\[
e_i e_{i+1} e_i - e_i = e_{i+1} e_i - e_{i+1} = 0.
\]

In \( A_n \), a trace can be defined [18] as:

\[
\text{tr}(x e_p) = \tau^{-1} \text{tr}(x) \quad \forall x \in A_p.
\]

The affine Hecke algebra, [19] [20] [21], is an extension of the Hecke algebra [3] by generators \( y_i \), \( 1 \leq i \leq n \) obeying the following relations:

\[
\begin{align*}
a) & \quad y_i y_j = y_j y_i \\
b) & \quad t_i y_j = y_j t_i \quad \text{if } j \neq i, i + 1 \\
c) & \quad t_i y_{i+1} = y_{i+1} t_i^{-1} \quad \text{if } i \leq n - 1.
\end{align*}
\]

In (1.2), we indicate why (7c) is natural from the Yang-Baxter algebra point of view.

This algebra can be endowed with two possible involutions: \( e_i^* = e_i \), \( y_i^* = y_i^{\pm 1} \), \( q^* = q^{\pm 1} \).

The symmetric polynomials in the \( y_i \) are central elements.

We define the affine T.L. algebra \( A'_n \) as the extension of the T.L. algebra [5] by the generators \( y_i \).
2.1 Yang’s realization of the Affine relations.

The commutation relations of the affine generators \( y_i \) become simpler to understand if we assume that we have a representation of the permutations \( k_{ij} \) acting in the natural way on the indices. Let us introduce the operators \( x_{ij} = t_{ij}k_{ij} \) for \( i < j \) and \( x_{ji} = x_{ji}^{-1} \). These operators obey the Yang’s relations:

\[
\begin{align*}
    x_{ij}x_{ji} &= 1, \\
    x_{ij}x_{kl} &= x_{kl}x_{ij} \quad \forall i \neq j \neq k \neq l, \\
    x_{ij}x_{ik}x_{jk} &= x_{ik}x_{jk}x_{ij}. \quad (8)
\end{align*}
\]

We also assume that we have commuting operators \( s_i \) such that \( s_is_jx_{ij} = x_{ij}s_is_j \).

Using (8), one verifies that the operators defined as (9) commute. Indeed, they coincide with the scattering matrices of Yang [23]. (7b) follows directly from (8) once we substitute \( t_{i} = x_{ii+1}k_{ii+1} \). (7c) is a direct consequence of the definition (9) of \( y_i \).

Gathering the permutation operators \( k_{ij} \) together, we can obtain another presentation of the \( y_i \). Let us introduce the cyclic operator:

\[
\sigma = k_{n-1}...k_{23}k_{12}s_1. \quad (10)
\]

Then we have:

\[
\begin{align*}
    y_1 &= t_1t_2...t_{n-1}\sigma, \\
    y_2 &= t_1^{-1}y_1t_1^{-1}, \\
    y_n &= t_n^{-1}y_{n-1}t_n^{-1}. \quad (11)
\end{align*}
\]

We can define an additional generator to the \( t_i \): \( t_n = \sigma t_1\sigma^{-1} \), which makes the relations (2) become cyclic. One has:

\[
\sigma t_i = t_{i-1}\sigma. \quad (12)
\]

So that the affine Hecke (or T.L.) algebra is generated by the generators \( t_i \) and the cyclic operator \( \sigma \) obeying (12) and does not require a representation of the permutations. \( \sigma^n \) is a central element which can be set equal to one, and \( \sigma^* = \sigma^{-1} \) if we take \( t^* = t^{-1} \).
Given the Hecke algebra, there is a simple realization of the affine Hecke algebra which consists in taking $y_1 = 1$. Then, $\sigma$ is defined as:

$$\sigma = t_{n-1}^{-1}t_1^{-1}. \quad (13)$$

Using the braid relations, one sees that $\sigma t_i = t_{i-1}\sigma$ for $i > 1$, and one can define $t_n$ by $t_n = \sigma t_1\sigma^{-1}$. Using the braid relations again, one gets $\sigma t_n = t_{n-1}\sigma$. This defines an operator $\sigma$ which allows to construct the affine generators with (11).

2.2 Polynomial representations.

We consider a space of polynomials in $n$ variables $z_i$, constructed as a linear combinations of monomials: $z^\mu = z_1^{\mu_1}z_2^{\mu_2}\ldots z_n^{\mu_n}$ with a total degree $|\mu| = \sum \mu_i$ fixed. There is a natural action of the permutations and of the operators $s_i$ on this space defined by:

$$\bar{\psi}(z_1,\ldots, z_i,\ldots, z_j,\ldots, z_n)_{k_{ij}} = \bar{\psi}(z_1,\ldots, z_j,\ldots, z_i,\ldots, z_n),$$

$$\bar{\psi}(z_1,\ldots, z_i,\ldots, z_n)_{s_i} = c \bar{\psi}(z_1,\ldots, sz_i,\ldots, z_n). \quad (14)$$

It is convenient to consider the polynomials in an infinite set of variables $z_i$, $i \in \mathbb{Z}$, with the identification: $z_{i+n} = sz_i$. The operator $\bar{\sigma}$ (10) takes the form:

$$\bar{\psi}(z_i) = c\bar{\psi}(z_{i+1}). \quad (15)$$

The condition $\sigma^n = 1$ imposes the relation $c^n s^{\mu} = 1$.

As shown in D.1, it is straightforward to derive the following representation of the Hecke relations 2.3:

$$\bar{t}_{ij} = -q^{-1} + (1 - k_{ij})\frac{q z_i - q^{-1}z_j}{z_i - z_j}. \quad (16)$$

In this way we obtain a representation of the affine Hecke algebra acting on homogenous polynomials of a given total degree.

The operators $x_{ij}$ take the form:

$$x_{ij} = -q^{-1} + (q - q^{-1})(1 - k_{ij})\frac{z_j}{z_i - z_j}. \quad (17)$$

In the appendix D.3 we show that there is a natural order on the monomial basis, $z^\mu$, for which the operators $x_{ij}$, and hence the $y_i$ are realized as lower triangular matrices.
The operator $y = y_1 + \ldots + y_n$ can be seen to commute with the Hecke generators. It is therefore equal to a constant in an irreducible representation. Its eigenvalue evaluated on the highest weight polynomial $P_\lambda$ thus characterizes the representation. It is given by:

$$y_\lambda = c(-q)^{1-n}(s^{\lambda_1} + s^{\lambda_2}q^2 + \ldots + s^{\lambda_n}q^{2(n-1)}).$$  \hspace{1cm} (18)

If $\lambda'$ is a permutation of the partition $\lambda$, we say that $z^\lambda$ is of degree $\lambda$. In this paper, we are mainly concerned with the monomials $z^\lambda$, of degree:

$$\lambda = (\frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 2, \frac{n}{2} - 2, \ldots, 0, 0),$$  \hspace{1cm} (19)

and of total degree $|\lambda| = \frac{n}{2}(\frac{n}{2} - 1)$.

We will consider the subclass $\lambda_\pi$ of permutations of $\lambda$ which are smaller than $\lambda$ for the order introduced in the appendix D.3. According to the analysis made there, $\lambda_\pi$ can be obtained from $\lambda$ by a sequence of permutations $(\lambda'_i, \lambda'_{i+1}) \rightarrow (\lambda''_i, \lambda''_{i+1})$ with $\lambda'_i > \lambda'_{i+1}$. The only monomials which can be obtained that way are indexed by the standard Young tableaus with two columns of $\frac{n}{2}$ boxes:

$$z^{\lambda_\pi} = (z_{\mu_1}z_{\nu_1})^{\frac{n}{2}}(z_{\mu_2}z_{\nu_2})^{\frac{n}{2}-1}\ldots(z_{\mu_n}z_{\nu_n})^0,$$  \hspace{1cm} (20)

with $\mu_1 > \mu_2 > \ldots > \mu_{\frac{n}{2}}$, $\nu_1 > \nu_2 > \ldots > \nu_{\frac{n}{2}}$, and $\mu_i > \nu_i$. To simplify notations, we denote these monomials by $z^\pi$ instead of $z^{\lambda_\pi}$.

We identify the standard Young tableaus with the paths $\pi = [h_i]$ introduced in the appendix A. $h_0 = h_n = 0$, $h_i \geq 0$ and $h_{i+1} - h_i = \pm 1$. These paths are obtained using the rule: $h_i - h_{i-1} = 1$ if $i \in \{\mu_j\}$, and $h_i - h_{i-1} = -1$ if $i \in \{\nu_j\}$. For the paths, we use the order $\pi \geq \pi'$, if $[h_i] \geq [h'_i] \forall i$, which coincides with the reverse order for the monomials: $z^\pi \leq z^{\pi'}$.

This identification is illustrated in figure II.

### 3 Representation of the affine T.L. algebra on words.

For $n$ even, there is a simple representation $(H_n)$ of the T.L. algebra $A_n$ obtained as follows. One considers the left action of $A_n$ on the space $A_n\alpha$ where $\alpha$ is the minimal projector $\alpha = e_1e_3\ldots e_{n-1}$. A basis of this space is given by reduced monomial words in the $e_i$. The elements of this basis can be put into correspondence with paths or link patterns. In the appendix A we exhibit a basis of reduced words and we define an order relation on the reduced words.
A scalar product can be defined as:

\[ \pi^* \pi' = \langle \pi | \pi' \rangle \alpha, \tag{21} \]

where \( e_i^* = e_i \) and the involution reverses the order of the letters. In the link-pattern representation, this scalar product is given by: \( \tau \) where \( l \) is the number of loops one gets by stacking the link patterns of \( \pi \) and \( \pi' \) on top of each other. If \( \tau = -(q + q^{-1}) \) with \( q \) not a root of one, this scalar product is positively definite \[18\]. For this scalar product the T.L generators \( e_i \) are by construction hermitian.

To obtain the affine algebra representation, let us define as in \[18\] the cyclic operator:

\[ \sigma = -q^{\frac{n}{2} - 2} t_{n-1}^{-1} \ldots t_1^{-1}, \tag{22} \]

where the normalization is such that in the link-pattern representation, \( \sigma \) acts by cyclicly permuting the indices \( i \rightarrow i - 1 \) (see appendix \[A.4\]). One can define an additional generator, \( e_{n1} = \sigma e_1 \sigma^{-1} \), which acts in the same way as \( e_{ii+1} \) with the two indices \( 1, n \). The affine generators are constructed using \[11\] with \( y_1 = -q^{\frac{n}{2} - 2} \).

In the appendix \[A.3\] we show that the operators \( y_i \) are realized as triangular matrices in \( H_n \), they are hermitian for the choice \( q = q^* \). Their sum \( y = \sum_i y_i \) is constant with a value given by:

\[ y = -(q + q^{-1}) \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q - q^{-1}}. \tag{23} \]

There is an imbedding of \( H_{n-2} \) into \( H_n \) given by \( \pi \rightarrow \pi e_1 \) and a projection \( E \) from \( H_n \) to \( H_{n-2} \) given by:

\[ e_1 \pi = \tau E(\pi) e_1. \tag{24} \]

This projection is both triangular and hermitian.

In \[A.2\] we identify \( H_n \) with \( A_{n+1} \). This allows us to interpret the projection \( E \) as a conditional expectation value of \( A_{n+1} \rightarrow A_{n+1} \) \[18\].

4 q-deformed Quantum Hall Effect wave functions.

4.1 Statement of the Problem.

Let us consider a vector \( \Psi \):

\[ \Psi = \sum_{\pi} \pi F_\pi(z_i), \tag{25} \]
constructed in the following way. The vectors $\pi$ are the basis vectors of $\mathcal{H}_n$ on which the T.L. algebra acts to the left. $F_\pi$ are homogeneous polynomials in the variables $z_1, z_2, \ldots, z_n$ ($n$ is even). The polynomial coefficients of $\Psi$ carry a representation of the affine Hecke algebra generated by the operators $\bar{t}_i$ and $\bar{\sigma}$ acting to the right. The problem is to determine the coefficients $F_\pi$ in such a way that both actions give the same result on the vector $\Psi$:

$$
\Psi \bar{t}_i = t_i \Psi \\
\Psi \bar{\sigma} = \sigma \Psi,
$$

(26)

The first of these relations is equivalent to the more familiar relation (63) derived in D.1.

Said differently, we look for a dual action of the affine T.L. algebra acting on polynomials. Unless we specify it, we address this problem for a generic value of the parameter $q$, not a root of unity, for which the T.L. algebra is semisimple [18].

4.2 Module $\mathcal{M}_n$.

The dual representation of $\mathcal{H}_n$ is obtained by acting with the T.L. generators on the dual $F_\omega$ of the highest vector $\omega \in \mathcal{H}_n$. $\omega$ is given by the sequence ($a_{2p+1} = p + 1$) in the characterization of words we use in the appendix A and is fully characterized by the property that it can be written $\omega = e_i \pi$ only for $i = \frac{n}{2}$. The dual vector $F_\omega$ must therefore be annihilated by all the $e_i$ with $i \neq n/2$. We realize the module $\mathcal{H}_n$ upon acting on $F_\omega$ with the generators $e_i$ for $1 \leq i \leq n - 1$. We define:

$$
\mathcal{M}_n = \text{Vec}\{\bar{\psi} = F_\omega \psi\},
$$

(27)

where we denote with a bar $\bar{\psi}$ the result of the action of the monomial $\psi$ on $F_\omega$. Thus we have $\bar{1} = F_\omega$. In the appendix C.1 we show that $\mathcal{M}_n$ defined in this way is a module over the T.L. algebra as long as the $e_i$ obey the Hecke relations (4). In other words, the projectors $U_{i,i+1}^- = e_i e_{i+1} e_i - e_i$ are null in $\mathcal{M}_n$. This formal module is however not isomorphic to $\mathcal{H}_n$ unless $F_\omega$ obeys some supplementary condition (56). Here, we construct a representation of the T.L. algebra by identifying a state $F_\omega$ dual to $\omega$ and satisfying the condition (56).

The expression of the T.L. generators $e_i = t_i - q$ for $1 \leq i \leq n - 1$ follows from (16):

$$
e_i = -q \frac{z_{i+1} - q^{-1} z_i (1 + k_{ii+1})}{z_{i+1} - z_i}
$$

$$
e_i - \tau = (1 - k_{ii+1}) \frac{q z_i - q^{-1} z_{i+1}}{z_i - z_{i+1}}.
$$

(28)
The effect of $e_i$ and $\tau - e_i$ is to split a polynomial $\bar{\psi}$ into two polynomials belonging to $\mathcal{M}_n$: $\bar{\psi} = S_1 + (qz_i - q^{-1}z_{i+1})S_2$, where both $S_1$ and $S_2$ are symmetrical under the exchange of $z_i$ and $z_{i+1}$. This decomposition is unique and characterizes the projector $e_i$.

It can be convenient to distinguish the representation on $\mathcal{H}_n$ from its dual on $\mathcal{M}_n$. When this is the case, we denote $\bar{e}_i$ the dual projectors which act on polynomials.

One verifies that:

$$\Delta^q_m(z_1,\ldots,z_p) = \prod_{1 \leq i < j \leq m} (qz_i - q^{-1}z_j)$$

(29)

is annihilated by all the $e_i$, $1 \leq i \leq m - 1$, and this defines $\Delta_m$ up to a product by a symmetric polynomial in $z_1,\ldots,z_m$. Therefore, the minimal degree polynomial candidate for $F_\omega$ is:

$$F_\omega = \Delta^q_{\frac{n}{2}}(z_1,\ldots,z_{\frac{n}{2}})\Delta^q_{\frac{n}{2}}(z_{\frac{n}{2}+1},\ldots,z_n).$$

(30)

This polynomial cannot be $q$-antisymmetrized over $\frac{n}{2} + 1$ variables, since the result would have a degree at least $\frac{n}{2}$ in $z_1$, and this is the content of the condition (56). Thus, $\mathcal{M}_n$ is a simple module which can be identified with $\mathcal{H}_n$. This representation is characterized by its Young diagram ($2\frac{n}{2}$) having two columns of length $\frac{n}{2}$. Its dimension is given by the Catalan number $C_n = \left(\begin{array}{c}n \\ \frac{n}{2}\end{array}\right) - \left(\begin{array}{c}n \\ \frac{n}{2} - 1\end{array}\right)$.

If we denote $\pi|\omega$ the coefficient of $\omega$ in the reduced expression of $\pi$, we can identify the polynomials $\bar{\psi} \in \mathcal{M}_n$ with the dual of $\mathcal{H}_n$ through the relation: $\bar{\psi}(\pi) = \psi\pi|\omega$.

We can also introduce the dual basis $F_{\pi}$ defined by its action on reduced words:

$$F_{\pi}(\pi') = \delta_{\pi,\pi'}.$$ 

(31)

Let $\pi_{\psi}$ be the complementary word of $\psi$ (defined in appendix [C]) such that one can write $\psi\pi_{\psi} = \omega$ without reducing the expression. One has $\psi\pi_{\psi}|\omega = 1$ and $\psi\pi_{\psi}|\omega = 0$ if $\pi < \pi_{\psi}$. Expanding $\bar{\psi}$ on the basis $F_{\pi}$, we get: $\bar{\psi} = \sum_{\pi \geq \pi_{\psi}} \psi\pi|\omega F_{\pi}$, and by inverting the triangular system, we can obtain the expression of $F_{\pi}$.

Let us verify that the highest monomial of $\bar{\psi}$, and thus of $F_{\pi_{\psi}}$, as well, is proportional to $z^{\pi_{\psi}}$. We show this by recursion. It is true for $\psi = 1$: $\bar{1} = F_{\omega}$, $\pi_1 = \omega$ and $z^\omega$ is the highest monomial of $F_{\omega}$. We assume that the property is true for $\psi' < \psi$. If $\psi \neq 1$, one can write $\bar{\psi} = \bar{\psi}'e_i$ with $\bar{\psi}' < \bar{\psi}$, and we have $\pi_{\psi'} = e_i\pi_{\psi}$ with $\pi_{\psi} < \pi_{\psi'}$.

Then, according to the recursion hypothesis, the highest monomial of $\bar{\psi}'$ is $z^{\pi_{\psi'}}$ which contains the factor $z^{m^i}z^{n^i}_{i+1}$ with $n > m$. Since $z^{m^i}z^{n^i}_{i+1}e_i = -q^nz^{m^i}_{i+1} + \text{lower monomials}$, the highest monomial of $\psi = \bar{\psi}'e_i$ is $z^{\pi_{\psi}}$. 

11
We also obtain the normalization coefficient of $z^\pi$ up to a global factor: $F_\pi = c_\pi z^\pi +$ lower monomials, with $c_\pi = (-\frac{1}{q})^{l_\pi}$, and $l_\pi$ is the number of letters $e_i$ entering the reduced expression of $\pi$.

4.3 Module $M'_n$.

We now consider a larger module $M'_n \supset M_n$ by letting the operator $\bar{\sigma}$ defined in (15) act on the polynomials. We will put some constraint on the parameter $s$ (which characterizes $\bar{\sigma}$) to have $M'_n = M_n$. We consider the simple case $n = 4$ in the appendix [15] and we obtain $s = q^6$ which is the general case as we show here.

Let us assume that $M'_n = M_n$ and see what constraints $s$ must satisfy to identify $\bar{\sigma}$ defined by its action on polynomials (15) with $\sigma$ defined in terms of generators (13).

We observe that $\sigma^{-1} \omega_n = e_1 \omega_{n-2}$, where $\omega_{n-2}$ is the highest state in $H_{n-2}$. This can easily be verified in the link pattern representation. Thus, we must have:

$$
\sigma E(\pi)e_1|_{\omega_n} = E(\pi)e_1|_{\sigma^{-1}\omega_n} = E(\pi)|_{\omega_{n-2}}. \quad (32)
$$

Let us consider the dual to the projection $E$, $E'$ from $M_n \rightarrow M_{n-2}$ defined as $\bar{\psi} e_1 = \tau E'(\bar{\psi})$. $E'$ needs to satisfy the conditions:

$$
\begin{align*}
 a) & \quad E'(\bar{\psi} e_1) = \tau E'(\bar{\psi}) \\
 b) & \quad E'(\bar{\psi} e_i) = E'(\bar{\psi}) e_i \ \forall i > 2 \\
 c) & \quad E'(\bar{\psi} e_1) = 0 \Rightarrow \bar{\psi} e_1 = 0. \quad (33)
\end{align*}
$$

From (32), in order to identify $\bar{\sigma}$ with $\sigma$, we see that the projection $E'$ must satisfy:

$$
E'(F_{\omega_n} \bar{\sigma}) = F_{\omega_{n-2}}. \quad (34)
$$

$E'$ can be realized as:

$$
E'(\bar{\psi}) = c' \frac{1}{\phi(z, z_i)} \bar{\psi}(z_1 = z, z_2 = q^2 z, z_i), \quad (35)
$$

where $\phi(z, z_i) = \prod_{i=3}^n (z_i - q^4 z)$ and $c'$ is a normalization constant. $E'$ verifies (33a,b) by construction as can be seen from the expression (28) of $e_1 - \tau$.

Using the explicit expression (15) of $\sigma$, we have:

$$
E'(F_{\omega_n} \sigma) = c's^{-\frac{3}{2}} \frac{1}{\phi(z, z_i)} \prod_{i=3}^{i+1} (q^3 z - q^{-1} z_i) \prod_{i=2}^n (q z_i - q^{-1} s z) F_{\omega_{n-2}}(z_3, \ldots, z_n). \quad (36)
$$
which imposes \( s = q^6 \) for the polynomial in the numerator to be proportional to \( \phi(z, z_i) \) and (33) to be satisfied.

To identify \( \mathcal{M}_n \) and \( \mathcal{M}'_n \), we give a more convenient characterization of \( \mathcal{M}'_n \). Consider the space \( \mathcal{M}''_n \) of homogenous polynomials in \( n \) variables, and of the minimal total degree, obeying the property:

\[
(P) : \quad \bar{\psi}(z_i = z, z_j = q^2z, z_k = q^4z) = 0, \quad \text{if } i, j, k, \text{ are cyclically ordered.} \tag{37}
\]

This property is obviously compatible with the cyclic identification \( z_{i+n} = q^6z_i \), it is thus preserved by \( \sigma \) (34). By applying (P) to the triplets \((1, 2, j)\), we see that the projection (35) is well defined from \( \mathcal{M}''_n \) to \( \mathcal{M}''_{n-2} \).

We show that \( \mathcal{M}''_n = \mathcal{M}_n \). For this, we first show that \( \mathcal{M}''_n \) is a module over the T.L. algebra \( A_n \) and that it contains \( \mathcal{M}_n \), then we show that \( \mathcal{M}''_n \) is irreducible over \( A_n \).

To show that \( \mathcal{M}''_n \) is a module over \( A_n \), we verify that the generators \( e_i \) preserve the property (P). Assuming that the polynomial \( \bar{\psi} \) verifies (P) we verify that \( \bar{\psi}e_i \) obeys (P) for a cyclically ordered triplet \( k, l, m \). If \( \{i, i+1\} \cap \{k, l, m\} = \emptyset \), it is obvious. If \( i+1 = k \), it results from the fact that \( \bar{\psi} \) obeys (P) for the triplets \( i, l, m \) and \( i+1, l, m \). The same type of argument applies if \( i = m \). If \( \{i, i+1\} \subset \{k, l, m\} \), \( \bar{\psi}(e_i - \tau) \) is proportional to \((qz_i - q^{-1}z_{i+1})\) and therefore obeys (P).

Let us show that (33) is satisfied in \( \mathcal{M}''_n \). If \( E'(\bar{\psi}e_1) = 0 \), \( \bar{\psi}e_1 \) vanishes when \( z_2 = q^2z_1 \), and from the definition (28) of \( e_1 \), it is symmetric in \( z_1, z_2 \). It is therefore divisible by \((z_1 - q^2z_2)(z_2 - q^2z_1)\). Hence, \( \bar{\psi}e_1/(z_1 - q^2z_2) \) satisfies (P) and has a total degree reduced by one. It is thus equal to zero according to our minimal degree hypothesis.

It is clear that \( F_\omega \) satisfies the property (P). To show that \( \mathcal{M}_n \subset \mathcal{M}''_n \), we need to show that the degree of the polynomials in \( \mathcal{M}''_n \) is the degree \( \frac{n}{2}(\frac{n}{2} - 1) \) of \( F_\omega \). We proceed by recursion on \( n \) and for the moment, we exclude the case where \( e_1 \) is represented as zero in \( \mathcal{M}''_n \). Due to (28) there are polynomials \( \bar{\psi} \) in \( \mathcal{M}''_n \) such that \( E'(\bar{\psi}) \neq 0 \). This implies that \( \bar{\psi} \) has a degree at least \( n - 2 \) in \( z_1, z_2 \). We can apply the recursion hypothesis to \( E'(\bar{\psi}) \in \mathcal{M}''_{n-2} \) to conclude that the minimal degree is \( \frac{n}{2}(\frac{n}{2} - 1) \).

To show that \( \mathcal{M}''_n \) is irreducible as a T.L. module, we use the recursion hypothesis that \( \mathcal{M}''_{n-2} = \mathcal{M}_{n-2} \). Due to (28), \( E' \) is injective from \( \mathcal{M}''_ne_1 \) to \( E'(\mathcal{M}''_n) \subset \mathcal{M}_{n-2} \). Since \( \mathcal{M}_ne_1 = \mathcal{M}_{n-2} \subset \mathcal{M}''_ne_1 \), we have \( \mathcal{M}''e_1 = \mathcal{M}e_1 \). Thus, if \( \mathcal{M}''_n \) contains an irreducible submodule \( R \neq \mathcal{M}_n, Re_1 = 0 \). If \( Re_1 = 0 \), from (14) we see that all the \( e_i \) are represented.

\(^2\)The same argument shows that the maximal degree of the polynomials in \( \mathcal{M}''_n \) is \( \geq \lambda \) for the order defined in (D3).
as 0 in $R$, and therefore, the polynomials in $R$ are proportional to $\Delta_n$ defined in (29) times a symmetric polynomial. Since the total degree of $\Delta_n$ is larger than $\frac{n^2}{2}(\frac{n}{2}-1)$, $R = 0$. We conclude that $\mathcal{M}_n'' = \mathcal{M}_n$ as a T.L. module.

Finally, to identify $\mathcal{M}_n''$ and $\mathcal{M}_n$ as affine modules, we observe that $y_1 = \sigma^{-1}\bar{\sigma}$ commutes with $\mathcal{A}_{n-1}$ generated by $e_2, ..., e_n$. Since $\mathcal{M}_n$ is irreducible over $\mathcal{A}_{n-1}$ [18], $y_1$ is proportional to the identity, thus $\sigma$ and $\bar{\sigma}$ can be identified.

### 4.3.1 Relation with the Macdonald Polynomials and the work of Di Francesco and Zinn-Justin.

As a check of consistency, we must verify that the two expressions of the eigenvalue of the central operators $y$ [18,23] are the same when $s = q^6$. This is indeed the case if we substitute in (18) the degree $\lambda$ [19] of the highest polynomial in $\mathcal{M}_n$ and $c = q^{3(n-1)}$.

For a generic $s$, the operator $y$ [18] can be diagonalized on the basis of symmetric polynomials and its eigenvectors define the Macdonald polynomials [24]. We have seen that when $s = q^6$, the polynomial representation is reducible. As a counterpart, some diagonal elements $y_{\lambda'}$ of $y$ become degenerate with $y_{\lambda}$, for example, $\lambda_2' = \lambda_2 - 1, \lambda_5' = \lambda_5 + 1$. Thus, $y$ cannot be diagonalized. We must use another operator such as $\frac{dy}{ds}$ to define the analogous symmetric polynomial.

In the non semisimple case $q^2 + q + 1 = 0$, $\tau = 1$, the T.L. representation admits a sub-representation given by $\text{Vec}\{\sum x_{\pi} \pi, \text{with } \sum x_{\pi} = 0\}$. The trivial representation $\Omega$ is obtained by equating to zero these vectors. The dual polynomial $F_\Omega = \sum_{\pi} F_{\pi}$ is therefore symmetrical of degree $\lambda$, and obeys the property (P) [37]. This completely determines it to be proportional to the Schur function $s_{\lambda}$ with $\lambda$ given by (19). Indeed, $s_{\lambda}$ has a degree $\lambda$ and satisfies (P) since three columns of the determinant which defines it become linearly dependent when we make the substitution (P). By the same argument as used in 4.3 the degree of a symmetric polynomial satisfying (P) must be at least $\lambda$ (relatively to the order of partitions which follows from [D.3]), and this proves its uniqueness.

In this limit, the $F_{\pi}$ are also the components the ground state of the $O(n = 1)$ transfer matrix, and thus, we have proved that this sum is $s_{\lambda}$, which is the result of [4].

It would be interesting to see if in this limit, $F_\Omega$ can be recovered as the eigenvector of some operator such as $\frac{dy}{ds}$.

### 4.4 Representation on Gaudin’s determinants.

It is well known that the Bethe scalar products [8] can be expressed using a quotient of two determinants. Here, we construct a representation of the T.L. algebra acting on these
quotients. We split the variables $z_i$ into $A = \{z_1, \ldots, z_{\frac{n}{2}}\}$ and $B = \{z_{\frac{n}{2}+1}, \ldots, z_n\}$. We also introduce $p$ a square root of $q$, $p^2 = q$. We define the polynomial $F'_\omega$:

$$F'_\omega(z_1, \ldots, z_n) = \frac{[(p^2 z_i - q^{-2} z_j)^{-1}(pz_i - p^{-1} z_j)^{-1}]}{[(pz_i - p^{-1} z_j)^{-1}]} \Delta_n^q(z_1, \ldots, z_n), \text{ with } i \in A, j \in B. \quad (38)$$

The first factor is the ratio of the Gaudin determinant with the Cauchy determinant \[8\]. It is also related to the domain wall boundary condition partition function \[6\] of a six vertex model with weights: $a = qx - q^{-1} y, b = px - p^{-1} y, c = \sqrt{qy/(p - p^{-1})}$.

The second factor $\Delta_n$ \[29\] transforms the scalar product into a polynomial. This factor has an innocuous effect on the T.L. algebra since:

$$\Delta_n^q(z_1, \ldots, z_n) t_i = \bar{t}_i \Delta_n^q(z_1, \ldots, z_n), \quad (39)$$

where $\bar{t}_i$ is obtained from $t_i$ \[16\] by the substitution $q \rightarrow -1/q$ which preserves the relations \[213\], but exchanges $e_i$ with $e_i - \tau$.

The ratio of the two determinants being symmetrical in the two sets of variables $A$ and $B$, $F'_\omega$ is annihilated by all the $e_i$ with $i \neq \frac{n}{2}$.

To show that the action of the T.L. algebra \[28\] on $F'_\omega$ produces an irreducible module, we proceed as in \[4.3\]. Consider the space $\mathcal{M}_n$ of homogenous polynomials in $n$ variables, and of the minimal total degree, obeying the property:

$$(P') : \quad \bar{\psi}(z_{i_1} = q^{a_1}z, z_{i_2} = q^{a_2}z, z_{i_3} = q^{a_3}z) = 0, \text{ if } i_1, i_2, i_3, \text{ are cyclically ordered, and for : } (a_1, a_2, a_3) = (-1, 0, 1), (-1, 1, 0), (-2, 0, 2), (0, -1, 1). \quad (40)$$

Note that these triplets are stable under the cyclic permutation, $(a_1, a_2, a_3) \rightarrow (a_3 - 2, a_2 + 1, a_1 + 1)$, and the transpositions, $(a_i, a_{i+1}) \rightarrow (a_{i+1}, a_i)$, whenever $|a_{i+1} - a_i| = 1$.

From the cyclic invariance, we deduce that this space is preserved under the action of $\sigma$ \[15\] if we take $s = q^3$.

By applying the property $(P')$ to $z_1, z_2, z_i$ with $(a_1, a_2, a_3) = (-1, 1, 0)$ and $(-2, 0, 2)$, we can define a projection \[33\] from $\mathcal{M}_n \rightarrow \mathcal{M}_{n-2}$. The polynomial $\phi(z, z_i)$ is now a product of two factors $\phi(z, z_i) = \prod_{i=3}^{n}(qz - z_i)(q^{4}z - z_i)$. Arguing as in \[1.3\] we see that this projection satisfies the properties \[33\].

This space is stable under the action of the generators $e_i$. The proof is similar to the one given in \[1.3\] and requires the stability of the triplets $(a_1, a_2, a_3)$ under the transpositions. The minimal degree is now $n(\frac{n}{2} - 1) = 2|\lambda|$ with $|\lambda|$ given by \[19\].

\[3\]Notice that for this six vertex model, $\frac{a^2 + b^2 - c^2}{ab} = p + p^{-1} \neq \tau$. 

15
Let us show that $F'_\omega$ (38) satisfies the property (P') (40). We consider $(i_1, i_2, i_3)$ and $(a_1, a_2, a_3)$. If the variables $z_l, z_m$ with $l < m$, corresponding to two $a_i$ which differ by 2, belong to the same set $A$ or $B$, $F'_\omega(z_m = q^2z_l) = 0$ due to the factor $\Delta_n$. Otherwise, two variables $z_l = z \in A$ and $z_m = q^2z \in B$ differ by a factor $q^2$. By isolating the contribution of the pole $(p^2z_l - p^{-2}z_m)$ in the Gaudin determinant, we factorize a term $\prod_{i}(qz - z_i)$ coming from the Cauchy denominator, and this enables to conclude that $F'_\omega(z_{i_1} = q^{a_1}z, z_{i_2} = q^{a_2}z, z_{i_3} = q^{a_3}z) = 0$ in all the other cases.

Arguing as in 4.3 we conclude that $\mathcal{M}_n$ is an irreducible module over the affine T.L. algebra and that it coincides with the module obtained upon acting with the generators on $F'_\omega$.

We verify again that the eigenvalue of the central operators $y$ (15) is given by (23). Now, $s = q^3$ instead of $q^6$ in 4.3, but the degree $2\lambda$ (19) of the highest polynomial in $\mathcal{M}_n$ is doubled and $c$ keeps the same value $c = q^{3(1 - \frac{\lambda}{2})}$.

In the nonsemisimple case $q^2 + q + 1 = 0$, using the result of [25] we have $F'_\pi = s_\lambda F_\pi$, and therefore, $\sum_\pi F'_\pi = s_\lambda^2$.

5 Conclusion.

Let us conclude with a few comments and questions.

On the mathematical side, this work provides a unification ground around the conjectures relating the eigenvector components of a loop model transfer matrix, the six vertex model domain wall boundary condition partition function and other mathematical objects. It opens the possibility to deform the polynomials underlying these conjectures by presenting them from the algebra representation point of view. We believe that these conjectures are related to incompressibility, and we hope to return to this point in a future publication.

From a technical point of view, it would be interesting to repeat the Jones construction of 4.2 on the polynomials directly. This would allow to recover in a direct way the product structure which they carry since they are associated to elements of the T.L. algebra.

The precise correspondence between the polynomial obtained here and the Macdonald polynomials needs to be clarified.

Finally, do the deformed wave functions considered here have anything to do with physics? At this moment, we have no answer to this question. A step towards a physical interpretation would be to identify a scalar product and a Hermitian Hamiltonian for which these wave functions are the ground states. This could also be useful to access to the excited states which play an important role in the Quantum Hall Effect.
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A Word representation.

A.1 Reduced words.

The module $\mathcal{H}_n$ is obtained by acting with the T.L. generators of $\mathfrak{A}_n$ on the lowest state $\alpha = e_1 e_3 ... e_{n-1}$. Using the relations (5), we obtain a basis of $\mathcal{H}_n$ given by reduced words $\pi$:

$$\pi = (e_{a_{n-1}} e_{a_{n-1}+1} ... e_{a_{2p+1}} e_{a_{2p+1}+1} ... e_{a_n} e_{a_n+1} e_3 e_1), \quad (41)$$

with, $a_{2p+1} \leq 2p+1$, and $1 < a_3 < ... < a_{2p+1} < ... < a_{n-1}$. So, a word is fully characterized by the sequence $(a_{2p+1})$.

On reduced words there is a natural order relation: $\pi > \pi'$ if $\pi$ is written $b\pi'$ with $b$ a monomial. One has $\pi \geq \pi'$ if $a_{2p+1} \leq a'_{2p+1}$ for all $p$.

Another way to represent a reduced word is in terms of paths. Let $m_i$ be the number of times the generator $e_i$ appears in the reduced expression of $\pi$. One has $m_{2i} = m_{2i-1}$ or $m_{2i-1} = m_{2i}$ or $m_{2i} + 1$. We define $h_{2i} = 2m_{2i} - 1$, $h_{2i-1} = 2m_{2i-1} - 2$ and $h_0 = h_n = 0$ by convention. We can describe the words $\pi$ by the paths $\pi = [h_i]$ where $h_0 = h_n = 0$, $h_i \geq 0$ and $h_{i+1} - h_i = \pm 1$. Using the path representation, one has $\pi \geq \pi'$, if $[h_i] \geq [h'_i] \forall i$.

If $\pi'$ is a non reduced word, by reducing it, one decreases the number of times the generator $e_i$ appears in its expression. We thus see that the order relation can be presented in a weaker form valid for non reduced words: If $\pi'$ is a word, not necessarily reduced and $\pi$ is a reduced word, $\pi > \pi'$ if $\pi'$ can be obtained by erasing letters $e_k$ from the (reduced) expression of $\pi$.

Finally, there is way to characterize this representation in terms of link patterns. It is convenient to dispose the $n$ points cyclically around a circle. A link pattern is obtained by pairing all the points in the set $\{1, 2, ..., n\}$: $\pi = \{[i_1, i_2], [i_3, i_4], ..., [i_{n-1}, i_n]\}$, in such a way that two links never cross. In practise, if $[i, j]$ is a link, then the other links $[k, l]$ are either inside, or outside the interval $[i, j]$. The action of $e_{i,i+1}$ is given by: $e_{i,i+1} \{[i, i+1] = [i+1, i].$
$e_{5}^{e_{3}}e^{e_{1}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$e_{5^{2}}e_{3}^{e_{1}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$

$e_{4^{5}}e_{3}^{e_{1}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{bmatrix}$

$e_{5^{2}}e_{3}^{e_{1}} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{bmatrix}$

$e_{3^{4}}e_{5}^{e_{1}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Figure 1: The three different ways to represent a word illustrated in the case of $H_6$.

$1, \ldots, [i_{n-1}, i_n] = \tau\{[i, i+1], \ldots, [i_{n-1}, i_n]\}$, and $e_{i,i+1}\{[i, j], [i+1, k], \ldots, [i_{n-1}, i_n]\} = \{[i, i+1], [j, k], \ldots\}$. In this representation, $\alpha = \{[1, 2], [3, 4], \ldots, [n-1, n]\}$, and $\omega = \{[1, n], [2, n-1], \ldots, [n-1, n]\}$.

These representations are illustrated in figure 1.

A.2 Identifying $H_n$ with $A_{\mathbb{Z}_2}$.

The link pattern representation allows to identify in a natural way $H_n$ with $A_{\mathbb{Z}_2}$. If we split $\{1, 2, \ldots, n\}$ into two subsets : $\{1, 2, \ldots, \frac{n}{2}\}$ and $\{\frac{n}{2}, \ldots, n\}$, the product $\pi \ast \pi'$ is defined on the link patterns by “stacking” the two link patterns and identifying the last $\frac{n}{2}$ points of $\pi$ with the first $\frac{n}{2}$ points of $\pi'$ through $i \equiv n+1-i$. The link pattern $\pi \ast \pi'$ is obtained by removing the loops which appear in this concatenating operation by giving them a weight $\tau$. 

18
Another identification can be achieved on paths by folding a path of length \( n \) into a loop of length \( n \). In this way, we realize \( \mathcal{A}_\frac{n}{2} \) as the algebra of double paths acting on Bratteli diagrams \([26][17][18]\).

In this identification, \( \mathcal{A}_\frac{n}{2} \) is a bimodule over itself. The first \( \frac{n}{2} - 1 \) generators \( e_i \in \mathcal{A}_n \) are identified with the generators of \( \mathcal{A}_\frac{n}{2} \) acting to the left, while the last \( \frac{n}{2} - 1 \) generators are identified with \( e_{n+i} \) acting to the right.

The state \( \omega \) is the identity in \( \mathcal{A}_\frac{n}{2} \), and the trace in \( \mathcal{A}_\frac{n}{2} \) coincides with the scalar product with \( \omega \) in \( \mathcal{A}_n \):

\[
\text{tr}(x) = \tau^{-\frac{n}{2}}(\omega|x).
\] (42)

The projection: \( E_\frac{n}{2} = \sigma^{-\frac{n}{2}+1}\sigma \sigma^{-\frac{n}{2}-1} \), with \( E \) given by \((24)\) can be reinterpreted as a conditional expectation value \([18]\). \( E_\frac{n}{2} : \mathcal{A}_\frac{n}{2} \to \mathcal{A}_{\frac{n}{2}-1} \). Jones construction enables then to construct \( e_\frac{n}{2} \in \mathcal{A}_{\frac{n}{2}+1} \) algebraically from the knowledge of \( E_\frac{n}{2} \).

### A.3 Triangularity of \( y_m \).

Let us show that the affine generators \( y_{m+1} = t_m^{-1}t_{m-1}^{-1}...t_1^{-1} \) are triangular in the word representation. It is obvious for \( y_1 = 1 \) and \( y_2 = t_1^{-2} \) since \( e_1 \) is triangular. We proceed by recursion and assume that \( y_k \) are triangular for \( k < m + 1 \). Using these hypotheses, we show that \( y_{m+1} \) is also triangular.

First we show that \( y_{m+1} \) acts diagonally on \( \alpha \). To study the action of \( y_{m+1} \) on \( \alpha \), we distinguish the two cases \( m \) odd or even. If \( m \) is odd, then:

\[
y_{m+1}\alpha = t_m^{-1}y_m t_m^{-1}e_m... = -qt_m^{-1}y_m e_m... = -\lambda_m q t_m^{-1} e_m... = q^2 \lambda_m \alpha,
\] (43)

where \( \lambda_m \) is the eigenvalue of \( y_m \) on \( \alpha \). If \( m \) is even, we make use of the fact that \( t_m^{-1}t_m e_{m-1} = \frac{1}{q} e_m e_{m-1} \) and the same relation with the indices \( m \) and \( m - 1 \) exchanged to obtain:

\[
y_{m+1}\alpha = t_m^{-1}t_{m-1}^{-1}y_m t_m^{-1}t_{m-1}^{-1}t^{-1} e_m... = \frac{1}{q} t_m^{-1}y_m e_m e_{m-1}... = \frac{1}{q} \lambda_m^{-1} t^{-1} e_m e_{m-1}... = \frac{1}{q^2} \lambda_m^{-1} \alpha.
\] (44)

We deduce that \( \alpha \) is an eigenstate of \( y_m \) with the eigenvalue \( \lambda_m \) obeying the recursion relations \( \lambda_{2m} = q^2 \lambda_{2m-1} \). \( \lambda_{2m+1} = \frac{1}{q^2} \lambda_{2m-1} \). Together with the fact that \( \lambda_1 = 1 \), we deduce \([18]\).
To show that $y_{m+1}$ is triangular on words $\neq \alpha$. We proceed by recursion and assume that $y_{m+1}$ acts in a triangular way on words $< \pi$ and show that the property is also true for $\pi$.

Let us consider the action of $y_{m+1}$ on a reduced word $\pi \neq \alpha$. This word can be put under the form $\pi = e_i \pi'$ where $\pi' < \pi$. We consider the three cases, $i \neq m, m+1$, $i = m$, $i = m + 1$. In the third case, either the word can be written in the form $e_{m+1}e_m \pi'$ with $\pi'$ reduced, or it can be written $e_p \pi'$ with $p < m$. The second possibility reduces to the first case and we need only consider the first possibility.

We observe that $y_{m+1}$ commutes with $e_i$:

\[y_{m+1}e_i \pi' = e_i(y_{m+1} \pi') \quad \text{for } i \neq m, m+1,
\]
\[y_{m+1}e_m \pi' = t_m^{-1}y_m t_m^{-1}e_m \pi' = -qt_m^{-1}(y_m e_m \pi'),
\]
\[y_{m+1}e_m e_{m+1} \pi' = t_m^{-1}y_m t_m^{-1}e_m e_{m+1} \pi' = t_m^{-1}(y_m e_m \pi' + \frac{1}{q} y_m e_{m+1} e_m \pi').
\]  

(45)

It follows from the hypothesis that the terms in brackets are less than $\pi$. In the first case because $\pi' < \pi$, and in the two others because $y_m$ is assumed to be triangular.

To conclude that $y_{m+1}$ is triangular, we must show that the action of $e_i$ in the first case and $e_m$ in the two other cases preserves the triangularity: If $e_i \pi$ is a reduced word and $\pi' \leq \pi$, then $e_i \pi' \leq e_i \pi$. If $e_m \pi$ is a reduced word and $\pi' \leq e_m \pi$, then $e_m \pi' \leq e_m \pi$. Finally, if $e_{m+1} e_m \pi$ is a reduced word and $\pi' \leq e_{m+1} e_m \pi$, then $e_{m+1} e_m \pi' \leq e_{m+1} e_m \pi$. These properties follow from the weak form of the order relation. This concludes the proof of triangularity of $y_{m+1}$.

### A.4 Action of $\sigma$ on words.

The action of $\sigma = -q^{\frac{\pi}{2}}t_{n-1}^{-1} \cdots t_1^{-1}$ on words can be computed similarly. First, using the braid relation (2), one sees that $\sigma e_i = e_i \sigma$ for $i > 1$. To fully characterize its action, we must compute $(\sigma \alpha)$. Using $t_1^{-1}e_1 = -qe_1$ and $t_m^{-1}t_m^{-1}e_{m+1} = \frac{1}{q} e_m e_{m+1}$ we obtain:

\[\sigma \alpha = \prod_{i=1}^{\frac{\pi}{2} - 1} e_{2i} \alpha.\]  

(46)

Thus, $(\sigma \alpha)$ can be characterized by the property:

\[e_{2i}(\sigma \alpha) = \tau(\sigma \alpha),\]  

(47)

20
for $1 \leq i \leq \frac{n}{2}$. $(\sigma \alpha)$ can then be used as a lowest state to construct a canonical basis by acting on it with the generators $e_2, \ldots, e_n$.

B Explicit construction of $\mathcal{M}_4$.

Let us construct $\mathcal{M}_4$ the dual of $\mathcal{H}_4$. The basis of $\mathcal{H}_4$ is given by the words $e_1e_3 = \alpha$, $e_2e_1e_3 = \omega$. So we search for a vector $\Psi$ of the form:

$$\Psi = F_{\alpha}(z_1, \ldots, z_4)\alpha + F_{\omega}(z_1, \ldots, z_4)\omega,$$

where $F_{\alpha}, F_{\omega}$ are polynomials of degree $(1, 1)$ in the variables $z_i$. The action of the T.L. affine algebra is given by the matrices:

$$e_1 = e_3 = \begin{pmatrix} \tau & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = e_4 = \begin{pmatrix} 0 & 0 \\ 1 & \tau \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (49)$$

We can obtain the dual representation by acting with the generators on $F_{\omega} \equiv (0, 1)$ annihilated by $\bar{e}_1 = e_1, \bar{e}_3 = e_3$. The minimum degree polynomial annihilated by $e_1, e_3$ is given by:

$$F_{\omega} = (qz_1 - q^{-1}z_2)(qz_3 - q^{-1}z_4). \quad (50)$$

Let us take $\bar{\sigma} = \sigma$ of the form:

$$F(z_1, z_2, z_3, z_4)\sigma = cF(z_2, z_3, z_4, sz_1). \quad (51)$$

We obtain two different expression for $F_{\alpha} \equiv (1, 0)$ which we must equate. One results from the cyclic property: $F_{\alpha} = F_{\omega}\sigma$, the other given by: $F_{\alpha} = F_{\omega}(e_2 - \tau)$.

We get the equation:

$$\frac{(qz_1 - q^{-1}z_2)(qz_3 - q^{-1}z_4) - (qz_1 - q^{-1}z_3)(qz_2 - q^{-1}z_4)}{z_2 - z_3} = c(qz_4 - q^{-1}sz_1), \quad (52)$$

which determines $s = q^6$, $c = q^{-3}$, and:

$$F_{\alpha} = (qz_2 - q^{-1}z_3)(q^{-2}z_4 - q^2z_1). \quad (53)$$
Module $F_\omega$.

Let us define a T.L. module $M$ defined in terms of a state $F_\omega$ satisfying $F_\omega e_i = 0$ for $i \neq \frac{n}{2}$. The module is obtained by acting with the T.L. generators and reducing words using the T.L. relations (5). In this module, a canonical basis is:

$$\bar{\psi} = F_\omega (e_{\frac{a+1}{2}}e_{\frac{a+2}{2}}...e_{\frac{n-1}{2}}),$$

where the $p$ take the all the values between $\frac{n}{2}$ and $n - 1$ and the $a_p$ are restricted by the conditions: $a_p \leq p + 1$, $a_{\frac{n}{2}} < a_{\frac{n}{2}+1} < ... < a_{n-1}$. The convention is that if $a_p = p + 1$, the sequence $(e_p...e_{a_p})$ is empty. A word $\bar{\psi}$ is fully characterized by the sequence $(a_p)$.

There is a reflection symmetry, $i \rightarrow n - i$, and an alternative description of the module in terms of reflected words:

$$\bar{\psi'} = F_\omega (e_{\frac{n-1}{2}}e_{\frac{n-2}{2}}...e_{\frac{1}{2}}),$$

1 $\leq p \leq \frac{n}{2}$, $b_p \geq p - 1$, $b_{\frac{n}{2}} > ... > b_1$. It is associated to the dual Young diagram $[\mu_{p+1-\frac{n}{2}}] = [p - a_p + 1]$.

A similar order relation as defined earlier holds for reduced words, $\bar{\psi}' < \bar{\psi}$ if $\bar{\psi}$ can be written $\bar{\psi} = \bar{\psi}'a$. For non reduced words $\bar{\psi}'$, it is sufficient that $\bar{\psi}'$ can be obtained by erasing letters $e_k$ from the (reduced) expression of $\bar{\psi}$.

In general, the module $F_\omega$ is reducible, it will be irreducible if $F_\omega$ satisfies the Fock condition:

$$F_\omega \left(1 + \sum_{m=0}^{\frac{n-1}{2}} q^{m+1} t_{\frac{n}{2}}...t_{\frac{n}{2}-m}\right) = 0.$$  \hspace{1cm} (56)

In this case, The only allowed words $\bar{\psi}$ (54) can be associated to their complementary $\pi_\psi$ in such a way that one can write without reducing the expression:

$$\bar{\psi} \pi_\psi = \omega.$$  \hspace{1cm} (57)

Thus, we get the supplementary constraint $a_p > 2p + 1 - n$, $b_p < 2p - 1$.

C.1 Reducing the Hecke Module to its T.L. form.

Let us consider a module $M'$ over the Hecke algebra defined by acting with the Hecke algebra generators satisfying (4) on the state $F_\omega$ satisfying $F_\omega e_i = 0$ for $i \neq \frac{n}{2}$. We want
to show that the Hecke algebra acts as a T.L. algebra on this module. For this, we first show that the Hecke relations (4) are sufficient to reduce the word basis of $M'$ to be of the T.L. form (54). Thus, $M'$ and $M$ can be identified as vector spaces. From this, we will deduce that $M' = M$ as modules. In other words, the projectors $U_{i,i+1}^- = e_i e_{i+1} - e_i$ are null in $M'$.

Let us assume that it is not true. Since all the basis elements of $M'$ are obtained upon acting on $F_\omega$ with letters $e_k$, there is a basis element $\bar{\psi} e_i$ which cannot be expressed as a linear combination of words of the form (54) although $\bar{\psi}$ is of the form (54). Among all the $\bar{\psi}$ which verify this property, we can take the smallest possible for the order relation, so that $\bar{\psi}' e_i$ is of the form (54) when $\bar{\psi}' < \bar{\psi}$. We show that this leads to a contradiction.

Let us consider the word $\bar{\psi} e_i$. It is a word of the form (54) in the three following cases. When $\bar{\psi} e_i$ is a reduced word $> \bar{\psi}$, for $i = a_p - 1$ if $a_p - 1 > a_{p-1}$. When $\bar{\psi} e_i = \tau \bar{\psi}$ when $i = a_p$ and $a_p > a_{p+1} - 1$. When $\bar{\psi} e_i = 0$ if $i < a_p - 1$ or $i > b_p + 1$.

The two remaining cases to consider are: First, when $a_p < i < a_{p+1} - 1$ for some $p$. Second, when $a_p < i \leq a_p + k$ if $a_{p+k} = a_p + k$ with $k \geq 1$. The second case can be studied similarly to the first one using the reflection symmetry $i \to n - i$ and corresponds to $b_p' > i > b_{p-1}' + 1$.

In the first case, $\bar{\psi} e_i = \bar{\psi}' (e_p ... e_{a_p+1} e_{a_p} e_i (e_{p+1} ... e_{a_{p+1}}) ...$, and using the relation (4), we see that:

$$e_p ... e_{a_p+1} e_{a_p} e_i = e_{i-1} e_p ... e_{a_p+1} e_{a_p} + e_p ... e_{i+1} (e_i - e_{i-1}) e_{i-2} ... e_{a_p+1} e_{a_p}.$$  \hspace{2cm} (58)

The second term is $< \bar{\psi}$ and therefore of the T.L. form by the recursion hypothesis. The first term can be eliminated by repeating this relation $p - \frac{n}{2}$ times to push $e_i$ and then $e_{i-1}, ..., e_{i+\frac{n}{2} - p}$ to the left of the word. The last application of the relation gives a term $F_\omega e_i e_{a_p} = 0$ since $i + \frac{n}{2} - p - 1 < \frac{n}{2}$.

This exhaust all the possibilities and $\bar{\psi} e_i$ can always be expressed as a linear combination of reduced T.L. words (54) in contradiction with the hypothesis. Therefore, the word basis of $M'$ coincides with the word basis (54).

To conclude that $M' = M$, let us consider the projectors $U_{i,i+1}^- = e_i e_{i+1} - e_i$, and the space $M'' \subset M'$ annihilated by all the $U_{i,i+1}^-$. The space $M''$ defines a module for the T.L. algebra. Since $F_\omega \in M''$, this module can be identified with $M$. Therefore, $M$ is a subspace of $M'$ with the same dimension, and thus, $M = M'$. 

23
### D Yang-Baxter Equation and Polynomials:

#### D.1 Polynomial representation of the Hecke generators

In this section, we derive the expression of the Hecke generators \( \bar{t}_i \) \([16]\) from the Yang-Baxter equation.

The Yang-Baxter algebra \([8]\) (also called \( RLL = LLR \) relation) can be expressed as:

\[
R_{12}(z_1, z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R_{12}(z_1, z_2),
\]

where \( R_{12}(z_1, z_2) \) is a solution of the Yang-Baxter equation:

\[
R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2).
\]

If we assume that \( R_{12}(z_1, z_2) = Y_{12}(z_1, z_2)P_{12} \) where \( P_{12} \) acts in the natural way on the spin indices, \([59]\) rewrites as:

\[
Y_{12}(z_1, z_2)L_2(z_1)L_1(z_2) = L_2(z_2)L_1(z_1)Y_{12}(z_1, z_2) = L_2(z_1)L_1(z_2)k_{12},
\]

where \( k_{12} \) acts to the left by permuting the variables \( z_1, z_2 \). If we normalize of \( Y(z_1, z_2) \) so that:

\[
Y_{12}(z_1, z_2)Y_{12}(z_2, z_1) = 1,
\]

it is consistent to demand that the \( Y_{ii+1} \) act as a representation of the permutation algebra on some wave function \( \Psi \):

\[
Y_{12}(z_1, z_2)\Psi(z_1, z_2) = \Psi(z_2, z_1) = \Psi(z_1, z_2)k_{12}.
\]

The \( Y_{ij} \) are are called Yang’s operators in \([8]\).

A well known solution of \((60),(62)\) in terms of the Hecke algebra \([3]\) is:

\[
Y_{12}(z) = \frac{t_{12} - zt_{12}^{-1}}{zq - q^{-1}},
\]

where \( z = z_1z_2 \). Substituting \((64)\) in \((63)\), we can also rewrite this relation as:

\[
t_{12}\Psi(z_1, z_2) = \Psi(z_1, z_2)t_{12},
\]

Where \( t_{12} \) takes the form:

\[
t_{12} = -q^{-1} + (1 - k_{12})\frac{qz_1 - q^{-1}z_2}{z_1 - z_2},
\]

and this coincides with \((16)\).
D.2 Commutation relations of the affine generators $y_i$.

We motivate the commutation relation (7c) from the Yang-Baxter algebra (59) point of view. This also reveals a complete symmetry between the spectral parameters $z_i$ and the generators $y_i$.

Let us substitute the spectral parameters $z_i$ with the affine generators $y_i$ in $L(z_i)$, and we require that the relation (65) are preserved under the action of the algebra $L_i$ on $\Psi$:

$$t_{12}L_1(y_1)L_2(y_2)\Psi = L_1(y_1)L_2(y_2)\Psi \tilde{t}_{12},$$

(67)

assuming that (65) holds for $\Psi$.

To avoid cumbersome expressions, we use from here the transposed notation $\tilde{a}X$ for $X\tilde{a}$. We must then transpose back the final algebraic relations we deduce. In the transposed notations (67) is equivalent to:

$$(t_{12} - \tilde{t}_{12})L_1(y_1)L_2(y_2) = 0,$$

(68)

under the hypothesis that $t_{12} = \tilde{t}_{12}$ to the right of any expression. Let us for the moment assume that $\tilde{t}_{12}$ commutes with the symmetrical expressions in $y_1, y_2$.

After substituting the expression of $L_i(y_i)$ deduced from (64):

$$L_1(y_1) = (yt_{10} - y_1t_{10}^{-1})P_{01},$$

(69)

the term proportional to $y^0$ requires that $\tilde{t}_{12}$ commutes with $y_1y_2$, while the term proportional to $y$ imposes that:

$$(t_{12} - \tilde{t}_{12})(y_2t_{01}t_{12}^{-1} + y_1t_{01}^{-1}t_{12}) = 0,$$

(70)

under the hypothesis that $t_{01} = \tilde{t}_{12}$ to the right of any expression. This gives:

$$y_2\tilde{t}_{12} + (q - q^{-1})y_1 - \tilde{t}_{12}y_1 = 0,$$
$$y_1 - \tilde{t}_{12}y_2\tilde{t}_{12} = 0,$$

(71)

which is equivalent to $y_2\tilde{t}_{12} = \tilde{t}_{12}^{-1}y_1$ and implies in particular that $\tilde{t}_{12}$ commutes with the symmetrical expressions in $y_1, y_2$. After transposition, it yields (7c) back.

Alternatively, we can substitute $z_i$ for $y_i$ in (7c) and verify that the relation is obeyed when we use the expression (66) of $\tilde{t}_i$. 

25
D.3 Eigenvalues of the $y_j$ in the polynomial case.

We show that the operators $y_j$ defined with the polynomial representation\cite{2,2} are triangular matrices. Let us recall the expression of $y_j$:

$$ y_i = x_{i+1}x_{i+2}...x_{i+n-1}s_ix_{i+1}...x_{i-1} $$  \hspace{1cm} (72)

where the operator $x_{i,j}$ takes the form for $i < j$:

$$ x_{ij} = -q^{-1} + (q - q^{-1})(1 - k_{ij}) \frac{z_j}{z_i - z_j}, $$  \hspace{1cm} (73)

and the operators $s_i$ act as:

$$ P(z_1, ..., z_i, ..., z_n)s_i = cP(z_1, sz_i, ..., z_n). $$  \hspace{1cm} (74)

$x_{12}$ commutes with $z_1z_2$ and with $z_k$ for $k \neq 1, 2$. It acts in the following way for on the monomials $z_1^m$, $z_2^m$:

$$ z_1^mx_{12} = -q^{-1}z_1^m + (q - q^{-1})(z_1^{m-1}z_2 + z_1^{m-2}z_2^2 + ... + z_2^m), $$

$$ z_2^mx_{12} = -qz_2^m - (q - q^{-1})(z_1^{m-1}z_2 + z_1^{m-2}z_2^2 + ... + z_1z_2^{m-1}) $$  \hspace{1cm} (75)

Let us consider which new monomials $z_1^\lambda$ can appear when one acts with $x_{12}$ on the monomial $z_\lambda$. First, all the $\lambda_l$ for $l \neq 1, 2$ are equal to $\lambda_i$. Then, if $\{\lambda_i', \lambda_j'\} \neq \{\lambda_i, \lambda_j\}$ with $\{i, j\} = \{1, 2\}$ and $\lambda_j' \leq \lambda_i'$, we must have $\{\lambda_i', \lambda_j'\} = \{\lambda_i - p, \lambda_j + p\}$ for some integer $p$. Finally, if $\{\lambda_i', \lambda_j'\} = \{\lambda_1, \lambda_2\}$, the only possibility is that: $(\lambda_1', \lambda_2') = (\lambda_2, \lambda_1)$ with $\lambda_1 > \lambda_2$.

Let us define an order on the monomials by saying that $z_1^\lambda$ is larger than $z_1^\lambda'$ if either $\lambda'$ is obtained from $\lambda$ by a sequence of squeezing operations $\{\lambda_i, \lambda_j\} \rightarrow \{\lambda_i - 1, \lambda_j + 1\}$ with $\lambda_i > \lambda_j$, or if $\lambda'$ is a permutation of $\lambda$ and can be obtained from $\lambda$ by a sequence of permutations $\{\lambda_i, \lambda_{i+1}\} \rightarrow \{\lambda_{i+1}, \lambda_i\}$ with $\lambda_i > \lambda_{i+1}$. It follows from the above analysis that the action of $y_j$ on a monomial produces only monomials which are smaller with respect to this order. Thus the eigenvalues of the operators $y_j$ are given by the diagonal elements in the monomial basis.

Given the partition $\lambda = (\lambda_1, ..., \lambda_n)$, the eigenvalues corresponding to the monomials associated to it are all obtained by permutations of the multiplet:

$$ (y_j) = c(-q)^{1-n}(t^\lambda, q^{2(j-1)}). $$  \hspace{1cm} (76)

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