A new recurrence relation for the truncated very-well-poised 6ψ6 series and Bailey’s summation formula

Jin Wang¹,², Xinrong Ma³,⁴

¹Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P. R. China
²Department of Mathematics, Soochow University, Suzhou 215006, P.R.China

Abstract

In this paper we introduce the truncated very-well-poised 6ψ6 series and set up an explicit recurrence relation for it by means of the classical Abel lemma on summation by parts. This new recurrence relation implies an elementary proof of Bailey’s well-known 6ψ6 summation formula.

Keywords: Basic hypergeometric series; truncated; very-well-poised; Bailey’s 6ψ6 summation formula; Abel’s lemma; Weierstrass’ theta identity

AMS subject classification 2000: 05A10; 33D15

1. Introduction

As is well known, Bailey’s bilateral very-well-poised (in short, VWP) 6ψ6 summation formula is one of the deepest results in the theory of basic hypergeometric series, which can be recorded as follows.

Theorem 1 (Bailey’s VWP 6ψ6 summation formula: [9, (II.33)]). Let a, b, c, d, e be five nonzero complex parameters subject to |a²q/(bcde)| < 1. Then there holds the summation formula

\[
6\psi_6 \left( a; b, c, d, e; q, \frac{a^2 q}{bcde} \right) = \frac{q, aq, q/a, aq/(be), aq/(ce), aq/(de), aq/(bc), aq/(bd), aq/(cd); q}_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a²q/(bcde); q)_\infty}
\]
We would like to refer the reader to [1] by G. E. Andrews for some applications of the $6\psi_6$ summation to partitions and number theory.

To the best of our knowledge, finding simple and elementary proof of Bailey’s VWP $6\psi_6$ summation formula is still one of attractive problems in basic hypergeometric series. As a supporting evidence, we would like to readdress the comment of R. Askey in his paper [2, p. 575] “...However, it is annoying that a sum that is this important has not been obtained from a more elementary special case.” Up to now, many different proofs of Bailey’s VWP $6\psi_6$ summation formula have been found, among are the method of integral and functional equations by R. Askey [2], the method of Liouville’s analytic continuation by R. Askey and M. E. H. Ismail [3], the $q$-difference method with series expansion by G. E. Andrews [1], the method of $q$-Gosper algorithm by V. Y. B. Chen, W. Y. C. Chen, and N. S. S. Gu [5], the difference method together Abel’s lemma by W. Chu [6], the method of Cauchy’s residue by F. Jouhet and M. Schlosser [11, 16], and elementary manipulations of series by M. Schlosser alone [16], L. J. Slater and A. Lakin [17]. It should be mentioned that in his paper [4], Bailey described how to deduce Bailey’s VWP $6\psi_6$ summation formula [4, (4.7)] and Weierstrass’ theta identity [4, (5.2)] from some three-term relations for VWP $8\phi_7$ series, provided that Rogers’ VWP $6\phi_5$ summation formula is given. However, he did not give any direct connection between these summation formulas. In our paper [19], we have revealed certain relation among Bailey’s VWP $6\psi_6$ and Rogers’ $6\phi_5$ summation formulas, as well as Weierstrass’ theta identity.

For purpose of comparison, we especially point out that it is just Abel’s lemma on summation by parts with which W. Chu rediscovered in a series of papers such as [6, 7] many important results for basic hypergeometric series. Among these, there are the $q$-binomial theorem, $q$-Gauss theorem, Ramamujan’s bilateral $1\psi_1$, the $q$-Pfaff-Saalschütz sum, and Jackson’s VWP $8\phi_7$ sum, and Bailey’s VWP $6\psi_6$ summation formula. In our view, both Chu’s proof in [6] and Chen-Chen-Gu’s proof in [5] use more than four recurrence relations and require the Jacobi triple product identity.

In this short article, as a possibly desired way by R. Askey, we will introduce

**Definition 2.** For any integer $N \geq 0$ and nonzero complex parameters $A, B, C, D, E$, define the truncated very-well-poised $6\psi_6$ series $S_N(A, B, C, D, E)$ to be the following finite sum

$$
\sum_{n=-N}^{N} \frac{\nabla (BDEq^{2n+1}/A)}{\nabla (BDEq/A)} \frac{(Bq, Dq, Eq, BCDEq^2/A^2; q)_n}{(DEq/A, BEq/A, BDq/A, A/C; q)_n} \left(\frac{1}{Cq^2}\right)^n,
$$

where the notation $\nabla(x) := 1 - x$.

In our working below we often write $S_N(A; C)$ for $S_N(A, B, C, D, E)$ by suppressing the dependence of the various summations on the complex parameters $B, D, E$ for easy of notation. As one of the main results, we will present a somewhat more “unexpected” recurrence relation of (1.2) underlying Bailey’s VWP $6\psi_6$ summation.
formula. In other word, Bailey’s \( 6 \psi_6 \) summation formula is just a limitation of this new recurrence relation. Our argument, apart from Abel’s lemma on summation by parts, only depends on the following self-evident identity.

**Lemma 3.** For any complex parameters \( b, c, x, z \) with \( bc \neq 0 \), we have

\[
\nabla \left( cx, \frac{x}{c}, bz, \frac{z}{b} \right) - \nabla \left( bx, \frac{x}{b}, cz, \frac{z}{c} \right) = \frac{z}{c} \nabla \left( bc, \frac{c}{b}, xz, \frac{x}{z} \right).
\]

(1.3)

Hereafter, for brevity, we employ the notation

\[
\nabla(x_1, x_2, \ldots, x_n) := \prod_{i=1}^{n} \nabla(x_i).
\]

(1.4)

Some remarks on notation are necessary. Throughout this paper, we will adopt the standard notation and terminology for basic hypergeometric series (or \( q \)-series) from the book \([9]\) (Gasper and Rahman, 2004). For instance, the \( q \)-shifted factorial with \( 0 < |q| < 1 \) is defined by

\[
(a; q)_n := \begin{cases} 
(1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n = 1, 2, \ldots, \\
1, & n = 0, \\
((1 - aq^{-1})(1 - aq^{-2}) \cdots (1 - aq^{n})))^{-1}, & n = -1, -2, \ldots 
\end{cases}
\]

(1.5)

Its multi-parameter form is compactly abbreviated to

\[
(x_1, x_2, \cdots, x_m; q)_n := (x_1; q)_n(x_2; q)_n \cdots (x_m; q)_n.
\]

The basic and bilateral hypergeometric series with the base \( q \) and the argument \( z \) are defined, respectively, by

\[
\begin{align*}
\phi_{r-1} \left[ \begin{array}{c}
{a_1, \ldots, a_r} \\
{b_1, \ldots, b_{r-1}}
\end{array} ; q, z \right] &:= \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_{r-1}; q)_n} z^n, \\
\psi_r \left[ \begin{array}{c}
{a_1, \ldots, a_r} \\
{b_1, \ldots, b_r}
\end{array} ; q, z \right] &:= \sum_{n=-\infty}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_r; q)_n} z^n.
\end{align*}
\]

(1.6, 1.7)

In particular, the compact notation \( \psi_r(a; a_3, \cdots, a_r; q, z) \) denotes the special case of \( \psi_r \) series above, called **very-well-poised** (VWP), in which all parameters satisfy the relations

\[
b_1 a_1 = b_2 a_2 = \cdots = b_r a_r = aq; \quad a_1 = a\sqrt{q}, a_2 = -a\sqrt{q}
\]

(1.8)

and \( \phi_{r-1}(a; a_4, \cdots, a_r; q, z) \) if there exists certain parameter \( a_i \) (say \( a_3 \)) = \( a \) in (1.8).

2. A recurrence relation for \( S_N(A; C) \)

Let us begin with Abel’s lemma on summation by parts.
Lemma 4 (Abel’s lemma). For any two sequences \( \{U_n, V_n\mid n = 0, \pm 1, \pm 2, \cdots \} \) and integers \( M, N \geq 0 \), it always holds

\[
\sum_{n=-M}^{N} V_n(U_n - U_{n+1}) = V_M U_{-M} - V_N U_{N+1} + \sum_{n=-M+1}^{N} U_n(V_n - V_{n-1}). \tag{2.1}
\]

As one of our main results, the following recurrence relation for \( S_N(A; C) \) may serve as an essential characteristic for Bailey’s VWP \( _6\psi_6 \) summation formula. It is a direct application of Lemmas 3 and 4.

Theorem 5. Let \( S_N(A; C) \) be defined by (1.2). Then \( S_N(A; C) \) satisfies the following recurrence relation

\[
S_{N+1}(A; C) = K_N(A; B, C, D, E) \left( Cq^3 \right)^N
+ \frac{A^2 q}{BDE} \nabla \left( BDEq/A, BCDEq/A^2, Aq/B, Aq/D, Aq/E \right) S_N(Aq; Cq),
\]

where

\[
K_N(A; B, C, D, E) := \frac{q^{N-2} \nabla \left( BDEq^{2N+3}/A \right) (Bq, Dq, Eq, BCDEq^{2}/A^2; q)_{N+1}}{C \nabla \left( BDEq/A \right) (A/C, BDq/A, BEq/A, DEq/A; q)_{N+1}}
+ \frac{BDE \nabla \left( q^{N-1}/A \right) (A/BD, A/BE, A/DE, C/A; q)_{N+2}}{C \nabla \left( C/A, Aq/B, Aq/D, Aq/E, BDEq/A \right) (1/B, 1/D, 1/E, A^2/(BCDEq); q)_{N+1}}
- \frac{A^2 q}{BDE} \nabla \left( Bq, Dq, Eq, BDEq^{N-1}/A^2 \right) (BCDEq^2/A^2, Bq^2, Dq^2, Eq^2; q)_N
\]

\[
\nabla \left( Aq / D, BDE / Aq^2, BDEq / A, Aq / E \right). \tag{2.3}
\]

Proof. To establish (2.2), we start with two sequences

\[
U_n := \frac{(Bq, Dq, Eq, BDE/A^2 q; q)_n}{(BD/A, BE/A, DE/A, Aq^2; q)_n}, \tag{2.4a}
\]

\[
V_n := \frac{(Aq^2, BCDEq/A^2; q)_{n+1}}{(A/Cq, BDE/A^2 q^2; q)_{n+1}} \left( \frac{1}{Cq^3} \right)^n. \tag{2.4b}
\]

In view of Lemma 4, we need only to calculate the differences \( U_n - U_{n+1} \) and \( V_n - V_{n-1} \). To do this, we start with (1.3) and make the parameter replacement

\[
(b, c, x, z) \rightarrow \left( \frac{Aq}{B \sqrt{DE} / A}, \sqrt{B} \sqrt{D}, \sqrt{B} \sqrt{Dq}, \sqrt{B \sqrt{DE} / A} \right). \tag{2.5}
\]

As a result, it follows immediately

\[
\nabla \left( cx / c, xz / z, b, b \frac{z}{z} \right) = \nabla \left( Bq, Dq, Eq, BDE / A^2 \right),
\]

\[
\nabla \left( bx / b, cz / z, b \frac{z}{z} \right) = \nabla \left( Aq^2, BD / A, BE / A, DE / A \right),
\]

\[
\nabla \left( bc / b, xz / z, b \frac{z}{z} \right) = \nabla \left( BDEq / A, Aq / E \right),
\]

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specializing (1.3) to the form
\[
\nabla \left( Bq, Dq, Eq, \frac{BDE}{A^2 q} \right) - \nabla \left( \frac{Aq^2}{A}, \frac{BD}{A}, \frac{BE}{A}, \frac{DE}{A} \right) = \frac{DE}{A} \nabla \left( \frac{Aq}{D}, \frac{B}{Aq}, \frac{BDEq}{A^2 q}, \frac{Aq}{E} \right). \tag{2.6}
\]

In such case, it is easy to find
\[
U_n - U_{n+1} = U_n \times \left( 1 - \frac{U_{n+1}}{U_n} \right)
= U_n \times \left( 1 - \nabla \left( \frac{Bq^{1+n}, Dq^{1+n}, Eq^{1+n}, \frac{BDEq^n}{Aq}}{Aq^{n+2}, \frac{BD}{A}, \frac{BE}{A}, \frac{DE}{A}} \right) \right).
\]

Herein and in what follows, the notation \( \sigma \) denotes the parameter replacement
\[
\sigma : (A, B, D, E) \rightarrow (Aq^n, Bq^n, Dq^n, Eq^n)
\]
and \( F|_\sigma \) means applying \( \sigma \) to the function \( F \). Applying (2.6) to the right-hand side of the last identity gives rise to
\[
U_n - U_{n+1} = U_n \times \left( -\frac{DEq^n}{A} \nabla \left( \frac{Bq/Aq, Aq/D, Aq/E, BDEq^{2n+1}/A}{Aq^n, \frac{BD}{A}, \frac{BE}{A}, \frac{DE}{A}} \right) \right)|_{\sigma}
= -\frac{DEq^n}{A} \nabla \left( \frac{Bq}{Aq}, Aq/D, Aq/E \right)
\times \frac{(BDEq^{2n+1}/A)(Bq, Dq, Eq, BDE/A^2 q; q)_n}{(BD/A, BE/A, DE/A, Aq^2;q)_{n+1}}. \tag{2.7}
\]

On the same lines, we can compute the difference \( V_n - V_{n-1} \). We still appeal to the basic identity (1.3) and make, instead of (2.5), the following parameter replacement
\[
(b, c, x, z) \rightarrow \left( \frac{A^{3/2}q^{1/2}}{\sqrt{B\sqrt{C}\sqrt{D}\sqrt{E}}}, \frac{\sqrt{B\sqrt{D}\sqrt{E}}q^{-3/2}}{\sqrt{A\sqrt{C}}}, 0, \frac{\sqrt{B\sqrt{C}\sqrt{D}\sqrt{E}}q^{3/2}}{\sqrt{A}} \right).
\]

In the sequel, we have
\[
\nabla \left( \frac{Aq^2}{A^2}, \frac{BCDEq}{A^2} \right) - Cq^3 \nabla \left( \frac{Aq}{Cq}, \frac{BDE}{A^2 q^2} \right) = \nabla \left( \frac{BDE}{A}, Cq^3 \right). \tag{2.8}
\]
Now we proceed to calculate the difference

\[ V_n - V_{n-1} = -V_{n-1} \times \left( 1 - \frac{V_n}{V_{n-1}} \right) \]

\[ = -V_{n-1} \times \left( 1 - \frac{\nabla(\frac{Aq^{n+2}, BCDEq^{n+1}}{A^2})}{\nabla(\frac{Aq^n/Cq, BCDEq^n}{A^2q^2}) Cq^3} \right) \]

This identity together with (2.7) specifies (2.1) to the form

\[ \tau : (A, B, C, D, E) \rightarrow (Aq^n, Bq^n, C, Dq^n, Eq^n). \]

At this stage, by applying (2.8) to the last identity, we arrive at

\[ V_n - V_{n-1} = V_{n-1} \times \left( \frac{\nabla(BDE/A, Cq^3)}{\nabla(A/Cq, BDE/A^2q^2) Cq^3} \right)_\tau \]

This identity specifies (2.7) to the form

\[ -\frac{DE}{A} \sum_{n=-M}^{N} \frac{(Aq^2, BCDEq^2/A^2; q)_{n+1}}{(A/Cq, BDE/A^2q^2; q)_{n+1}} \nabla(B/Aq, Aq/D, Aq/E, BCDEq^{2n+1}/A) \]

\[ \times \frac{(Bq, Dq, Eq, BDE/A^2q^3; q)_{n}}{(BD/A, BE/A, DE/A, Aq^2; q)_{n+1}} \]

\[ = V_{-M}U_{-M} - V_NU_{N+1} + \sum_{n=-M}^{N} \frac{(Aq^2, BCDEq^2/A^2; q)_{n+1}}{(A/Cq, BDE/A^2q^2; q)_{n+1}} \nabla(BDEq^{2n}/A, Cq^3) \]

Simplifying the last identity by the relation \( \nabla(x) = -x\nabla(1/x) \), we have

\[ \frac{BDE}{A^2q} \frac{\nabla(B/Aq, Aq/D, Aq/E)}{\nabla(B, D, E, BDE/A^2q^2)} \]

\[ \times \sum_{n=-M}^{N} \frac{\nabla(BDEq^{2n+1}/A)}{(Cq^3)^n} \frac{(B, D, E, BCDEq^2/A^2; q)_{n+1}}{(A/Cq, BD/A, BE/A, DE/A; q)_{n+1}} \]

\[ = V_{-M}U_{-M} - V_NU_{N+1} + \nabla(Cq^3) \]

\[ \times \sum_{n=-M}^{N} \frac{\nabla(BDEq^{2n}/A)}{(Cq^3)^n} \frac{(BCDEq^2/A^2, Bq, Dq, Eq; q)_{n}}{(A/C, BD/A, BE/A, DE/A; q)_{n}}. \]
By dividing both sides by
\[ \frac{BDE \nabla (Aq/B, Aq/D, Aq/E)}{A^2q^2 \nabla (B, D, E, BDE/A^2q^2)} \]
we obtain
\[
\sum_{n=-M}^{N} \nabla \left( \frac{BDEq^{2n+1}/A}{(Cq)^n} \right) \frac{(B, D, E, BCDEq/A^2; q)_{n+1}}{(A/Cq, BD/A, BE/A, DE/A; q)_{n+1}}
= \frac{A^2q^2}{BDE \nabla (Aq/B, Aq/D, Aq/E)}
\]
\[
\times \sum_{n=-M+1}^{N} \nabla \left( \frac{BDEq^{2n}/A}{(Cq)^n} \right) \frac{(BCDEq/A^2, Bq, Dq, Eq; q)_{n}}{(A/C, BD/A, BE/A, DE/A; q)_{n+1}}
\]
Furthermore, on multiplying both sides of (2.10) with
\[
\frac{\nabla (A/Cq, BD/A, BE/A, DE/A)}{\nabla (B, D, E, BDEq/A^2, BDEq/A)}
\]
we obtain
\[
\sum_{n=-M}^{N} \nabla \left( \frac{BDEq^{2n+1}/A}{(Cq)^n} \right) \frac{(Bq, Dq, Eq, BCDEq^2/A^2; q)_{n}}{(A/C, BD/A, BE/A, DE/A, BDEq/A)}
= \frac{A^2q^2}{BDE \nabla (Aq/B, Aq/D, Aq/E)}
\]
\[
\times \sum_{n=-M+1}^{N} \nabla \left( \frac{BDEq^{2n}/A}{(Cq)^n} \right) \frac{(BCDEq/A^2, Bq, Dq, Eq; q)_{n}}{(A/C, BD/A, BE/A, DE/A; q)_{n+1}}
\]
The final step is to calculate two terms \( V_{-M}U_{-M} \) and \( V_{N}U_{N+1} \). For this, we easily find
\[
U_{N+1}V_{N} = \frac{(BCDEq/A^2, Bq, Dq, Eq, BDEq/A^2; q)_{N+1}}{(A/Cq, BD/A, A^2q^2, BD/A, BE/A, DE/A; q)_{N+1}} \left( \frac{1}{Cq^3} \right)^N
\]
while, according to the basic relation (see [9, (I.11)])
\[
\frac{(x; q)_m}{(y; q)_m} = \frac{(q/y; q)_m}{(q/x; q)_m} \left( \frac{y}{x} \right)^m,
\]
we easily check
\[
V_{-M}U_{-M}
= \frac{(Bq, Dq, Eq, BDEq/A^2; q)_{-M}}{(BD/A, BE/A, DE/A, Aq^2, q)_{-M}} \times \frac{(A^2q^2, BCDEq^2/A^2; q)_{-M+1}}{(A/Cq, BDEq^2/A^2; q)_{-M+1}} \left( \frac{1}{Cq^3} \right)^{-M}
\]
\[
= \frac{(Aq/BD, Aq/BE, Aq/DE, 1/Aq; q)_{M}}{(1/B, 1/D, 1/E, A^2q^2/BDEq)_{M}} \times \frac{(Cq^2/A, A^2q^3/BDEq; q)_{M-1}}{(1/Aq, A^2/BCDEq; q)_{M-1}} \left( \frac{1}{Cq^3} \right)^{M-2}.
\]
For our purpose, here we need only to consider the case $M = N + 1$. As such, (2.11) can be recast into the form

$$ \sum_{n=-N}^{N+1} \frac{\nabla (BDEq^{2n+1}/A)}{\nabla (BDEq/A)} (Bq, Dq, Eq, BCDEq^{2}/A^2; q)_n \left( \frac{1}{Cq^2} \right)^n $$

$$ - \frac{A^2 q}{BDE} \nabla (Cq^3, BD/A, BE/A, DE/A, BDE/A) $$

$$ \times \sum_{n=-N}^{N} \frac{\nabla (BDEq^{2n}/A)}{\nabla (BDE/A)} (BDEq/A^2, Bq, Dq, Eq; q)_n \left( \frac{1}{Cq^3} \right)^n $$

$$ = K_N^{(1)} (A; B, C, D, E) (Cq^3)^{-N} - K_N^{(2)} (A; B, C, D, E) (Cq^3)^{-N} $$

$$ + K_N^{(3)} (A; B, C, D, E) (Cq^3)^{-N-1}, $$

(2.13)

where for $i = 1, 2, 3$, $K_N^{(i)} (A; B, C, D, E)$ are defined, respectively, by

$$ K_N^{(1)} (A; B, C, D, E) := V_{-N-1} U_{N-1} A^2 q \frac{\nabla (A/Cq, BD/A, BE/A, DE/A, BDE/A^2q^2)}{BDE \nabla (Aq/B, Aq/D, Aq/E, BCDEq/A^2, BDEq/A)} (Cq^3)^N; $$

$$ K_N^{(2)} (A; B, C, D, E) := V_N U_{N+1} A^2 q \frac{\nabla (A/Cq, BD/A, BE/A, DE/A, BDE/A^2q^2)}{BDE \nabla (Aq/B, Aq/D, Aq/E, BCDEq/A^2, BDEq/A)} (Cq^3)^N; $$

$$ K_N^{(3)} (A; B, C, D, E) := \frac{\nabla (BDEq^{2N+3}/A)}{\nabla (BDEq/A)} (Bq, Dq, Eq, BCDEq^{2}/A^2; q)_{N+1} \left( \frac{1}{Cq^3} \right)_{N+1}. $$

For more clarity, let us write $K_N (A; B, C, D, E)$ for the sum

$$ K_N^{(1)} (A; B, C, D, E) - K_N^{(2)} (A; B, C, D, E) + K_N^{(3)} (A; B, C, D, E) q^{N-2}/C, $$

which, after some routine computations, is the same as given by (2.3). In conclusion, we are able to reformulate (2.13) in terms of the truncated VWP series $S_N (A; C)$ and $K_N (A; B, C, D, E)$ as follows:

$$ S_{N+1} (A; C) = A^2 q \frac{\nabla (BDE/A, Cq^3, BD/A, BE/A, DE/A)}{BDE \nabla (BDEq/A, BCDEq/A^2, Aq/B, Aq/D, Aq/E)} S_N (Aq; Cq) $$

$$ + K_N (A; B, C, D, E) (Cq^3)^{-N}. $$

This gives the complete proof of Theorem 5.
3. A new proof of Bailey’s VWP $\psi_6$ summation formula

Having established Theorem 5, we now turn to show that Bailey’s VWP $\psi_6$ summation formula can be derived from the limit of the truncated VWP series $S_N(A;C)$ as $N \to \infty$. Actually, from Theorem 5, we may derive without any difficulty that

**Lemma 6.** For any five nonzero complex parameters $B, C, D, E, X$ subject to $|C/q^3| < 1$, define

$$T(X; C) := \psi_6(BCDEX^2; BCDEXq, BXq, DXq, EXq; q, C/q^3). \quad (3.1)$$

Then it holds

$$T(X; C) = \nabla \left( \frac{1/(BCDEXq) , BCDEX^2q, BC/q, CD/q, CE/q}{(C/q^3, BCq, BCEq, CDEq; q)} \right) T(Xq; Cq). \quad (3.2)$$

In particular,

$$T(q; C) = \frac{(BCDEq^3, BC/q, CD/q, CE/q; q)}{(C/q^3, BCq, BCEq, CDEq; q)}. \quad (3.3)$$

**Proof.** Starting from Theorem 5 with the tentative assumption $|Cq^2| > 1$, we can take the limit of (2.2) as $N \to +\infty$. It is easy to check

$$\lim_{N \to +\infty} \frac{K_N(A; B, C, D, E)}{(Cq^3)^N} = 0,$$

which results from the fact

$$\lim_{N \to +\infty} K_N(A; B, C, D, E) = \frac{BDE (A/BD, A/BE, A/DE, Cq/A; q)}{C \nabla (Aq/B, Aq/D, Aq/E, BDEq/A)(1/B, 1/D, 1/E, A^2/(BCDEq); q)}$$

$$= \frac{BDE \nabla (Aq/B, Aq/D, Aq/E, BDEq/A)(A/C, BDq/A, BEq/A, DEq/A; q)}{C \nabla (Aq/B, Aq/D, Aq/E, BDEq/A)}$$

$$\times \frac{\theta(A/BD, A/BE, A/DE, A/C; q)}{(1/B, 1/D, 1/E, A^2/(BCDEq), A/C, BDq/A, BEq/A, DEq/A; q)}.$$
which, written out in full form, amounts to

\[
\sum_{n=-\infty}^{\infty} \frac{\nabla (BDEq^{2n+1}/A)}{\nabla (BDEq/A)} (Bq, Dq, Eq, BCDEq^2/A^2; q_n) \frac{1}{Cq^2} \]

\[
= \frac{A^2q}{BDE} \sum_{n=-\infty}^{\infty} \frac{\nabla (BDEq^2/A, q_n) (BCDEq^2/q, BCDEq, q/BC, q/CD, q/CE)}{\nabla (BDEq/A, BCDEq^2/A, Aq/B, Aq/D, Aq/E)} \] (3.4)

Next, we further make the simultaneous substitution

\[
(A, B, D, E) \rightarrow (CX, BX, DX, EX)
\]
in (3.4) and then replace \(C\) with \(1/C\). Consequently, we have

\[
\sum_{n=-\infty}^{\infty} \frac{\nabla (BCDEX^2q^{2n+1})}{\nabla (BCDEX^2q)} (BCDEXq^2, BCDEXq, BCDEXq; q_n) \frac{C}{q^2}
\]

\[
= \frac{q}{C^2BDEX} \sum_{n=-\infty}^{\infty} \frac{\nabla (1/(BCDEXq), BCDEX^2q, BCdex, BCDEXq, q/BC, q/CD, q/CE)}{\nabla (1/(BCDEX^2), C/q^3, BCDX, BCEX, CDEX)} \] (3.5)

It is of importance to realize that the infinite sum on the far right-hand side of (3.5) is just \(T(X; C)\) while the left-hand side of (3.5) is nothing but \(T(X; Cq)\). Then we obtain the following recursive relation

\[
T(X; C) = X \frac{\nabla (BCDEX^2q, BCDEXq, BCDEXq, q/BC, q/CD, q/CE)}{\nabla (BCDEX^2, C/q^3, BCDX, BCEX, CDEX)} T(X; Cq)
\]

\[
= \frac{\nabla (1/(BCDEXq), BCDEX^2q, BCdex, BCDEXq, q/BC, q/CD, q/CE)}{\nabla (1/(BCDEX^2), C/q^3, BCDX, BCEX, CDEX)} T(X; Cq).
\]

It gives the complete proof of (3.2).

Obviously, when \(X = q\), (3.2) reduces to

\[
T(q; C) = \frac{\nabla (BCDEq^3, BC/q, CD/q, CE/q)}{\nabla (C/q^3, BCDq, BCEq, CDEq)} T(q; Cq).
\] (3.6)

By iterating (3.6) \(m\) times, we obtain

\[
T(q; C) = \frac{(BCDEq^3, BC/q, CD/q, CE/q; q)_m T(q; Cq^m)}{(C/q^3, BCDq, BCEq, CDEq; q)_m}.
\] (3.7)

Since \(T(q; C)\) is analytic at \(C = 0\) and

\[
\lim_{m \to +\infty} T(q; Cq^m) = T(q; 0) = 1,
\]
(3.7) reduces to
\[
T(q; C) = \frac{(BCDEq^3, BC/q, CD/q, CE/q; q)_\infty}{(C/q^3, BCDq, BCEq, CDEq; q)_\infty}.
\]

Thus we have (3.3). The lemma is proved. \(\blacksquare\)

We remark here that (3.3) is just Rogers’ \(_6\phi_5\) summation formula [9, (II. 20)]. Even more, from Lemma 6 we obtain the following result.

**Lemma 7.** Let \(T(X; C)\) be the same as in Lemma 6. Then there exists certain function \(Q(X; B, D, E)\) being independent of \(C\), such that
\[
T(X; C) = Q(X; B, D, E) \frac{(BCDEX^2q, q/(BCDEX^2), BC/q, CD/q, CE/q; q)_\infty}{(1/(BCDEX), C/q^3, BCDX, BCEX, CDEX; q)_\infty}.
\]

**(3.8)**

**Proof.** It only needs to consider the function
\[
F(C) := \frac{(BCDEX^2q, q/(BCDEX^2), BC/q, CD/q, CE/q; q)_\infty}{(1/(BCDEX), C/q^3, BCDX, BCEX, CDEX; q)_\infty}
\]

and to check that
\[
F(C) = \frac{\nabla (1/(BCDEXq), BCDEX^2q, BC/q, CD/q, CE/q)}{\nabla(1/(BCDEX^2), C/q^3, BCDX, BCEX, CDEX)} F(Cq).
\]

A direct comparison with (3.2) of Lemma 6 yields
\[
\frac{T(X; C)}{F(C)} = \frac{T(X; Cq)}{F(Cq)} = \ldots = \frac{T(X; Cq^m)}{F(Cq^m)}, m \geq 0
\]

**(3.10)**

Next, we appeal to the uniqueness of Laurent series expansion, which states that if certain function \(G(x)\) satisfies
\[
G(x) := \sum_{n=-\infty}^{\infty} a_n x^n = \sum_{n=-\infty}^{\infty} a_n (xq)^n = G(xq),
\]
then there must hold that \(a_n = 0\) for \(n \neq 0\). As such, we now define such constant \(a_0\) by
\[
Q(X; B, D, E) := \frac{T(X; C)}{F(C)}.
\]

**(3.11)**

Apparently, it is independent of \(C\). The lemma is thereby proved. \(\blacksquare\)

We are now prepared to show an equivalent variant of Theorem 1.
Theorem 8 (Bailey’s VWP 6ψ₆ summation formula). Let B, C, D, E, X be arbitrary nonzero complex parameters subject to |C/q³| < 1. Then there holds

\[ 6ψ₆ \left( BCDEX²; BCDEXq, BXq, DXq, EXq; q/q³ \right) \]

\[ = \left( q, 1/Bq, 1/Dq, 1/Eq, BCDEX²q, q/(BCDEX²), BC/q, CD/q, CE/q; q \right)_∞ \]

\[ \times (X, 1/BX, 1/DX, 1/EX, 1/(BCDEX), C/q³, BCDX, BCEX, CDEX; q)∞. \]

Proof. Obviously, by Lemma 7 we need only to find \( Q(X; B, D, E) \). On account of its being independent of \( C \), we now set in (3.8) \( BCDEXq = 1 \), namely, \( C = 1/BDEXq \).

As is expected, we find

\[ Q(X; B, D, E) = \left( T(X; C) \left( 1/(BCDEX), C/q³, BCDX, BCEX, CDEX; q \right)_∞ \right) \bigg|_{C = 1/BDEXq}. \]

In this case, it is easy to check by the definitions of \( rφ_r-1 \) and \( rψ_r \) series (see (1.6) and (1.7)) that

\[ T \left( X; \frac{1}{BDEXq} \right) = 6ψ₆ \left( X/q; 1, BXq, DXq, EXq; q, \frac{1}{BDEXq} \right) \]

\[ = 6φ₅ \left( q/X; Bq², Dq², Eq²; q, \frac{1}{BDEXq} \right). \] (3.13)

As already proved, (3.3) asserts that for any \( B, C, D, E \), it holds

\[ 6φ₅(BCDEq²; Bq², Dq², Eq²; q, C/q³) = \frac{(BCDEq³, BC/q, CD/q, CE/q; q)_∞}{(C/q³, BCDq, BCEq, CDEq; q)_∞}. \] (3.14)

Therefore, by replacing \( C \) in (3.14) with \( 1/(BDEXq) \) and substituting the result back to (3.13), we obtain

\[ T \left( X; \frac{1}{BDEXq} \right) = \frac{(q²/X, 1/(BDEXq²), 1/(BEXq²), 1/(DEXq²); q)_∞}{(1/(BDEXq⁴), 1/BX, 1/DX, 1/EX; q)_∞}. \]

Finally, we arrive at

\[ Q(X; B, D, E) = \frac{(q²/X, 1/(BDEXq²), 1/(BEXq²), 1/(DEXq²); q)_∞}{(1/(BDEXq³), 1/BX, 1/DX, 1/EX; q)_∞} \]

\[ \times \frac{(q, 1/(BDEXq⁴), 1/Bq, 1/Dq, 1/Eq; q)_∞}{(X, q²/X, 1/(BDEXq²), 1/(BEXq²), 1/(DEXq²); q)_∞} \]

\[ = \frac{(q, 1/Bq, 1/Dq, 1/Eq; q)_∞}{(X, 1/BX, 1/DX, 1/EX; q)_∞}. \]
Upon substituting this back to (3.8), we obtain (3.12) immediately. This completes our proof.

We conclude our paper with the following comments.

**Remark 9.** It is easy to verify that (3.12) reduces to (1.1) at once by making the replacement

\[(B, C, D, E, X) \to \left( \frac{bc}{aq^2}, \frac{a^2q^4}{bcde}, \frac{bd}{aq^2}, \frac{be}{aq^2}, \frac{aq}{b} \right).\]

**Remark 10.** We may view the recurrence relation (2.2) as saying that it is a common source and finite version for both Rogers’ 6\(\phi_5\) and Bailey’s VWP 6\(\psi_6\) summation formula.

**Remark 11.** It is clear that (1.3) of Lemma 3 is in fact the special case \(q = 0\) of the famous Weierstrass theta identity (see [9, Exercise 2.16(i)] or [12])

\[
\theta \left( cx, \frac{x}{c}, bz, \frac{z}{b}; q \right) - \theta \left( bx, \frac{x}{b}, cz, \frac{z}{c}; q \right) = \frac{z}{c} \theta \left( bc, \frac{c}{b}, xz, \frac{x}{z}; q \right).
\]

(3.15)

Note that \(\theta(x; q)\) denotes the Jacobi modified theta function given by

\[(x, q/x; q)_\infty\]

and its multi-parameter form

\[\theta(x_1, x_2, \ldots, x_m; q) := \theta(x_1; q)\theta(x_2; q) \cdots \theta(x_m; q).\]

The reader may consult [12] for a full history and further applications concerning Weierstrass’ theta identity. It is worth mentioning that in the recent paper [18, Theorem 1.7], the author showed that (3.15) is equivalent to (1.3).

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