Hamiltonian Structures on Coadjoint Orbits of Semidirect Product $G = Diff_+(S^1) \ltimes C^\infty(S^1, \mathbb{R})$

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Abstract

We consider the semidirect product of diffeomorphisms of the circle $D = Diff_+(S^1)$ and $C^\infty(S^1, \mathbb{R})$ functions, classify its coadjoint orbits and treat dynamical systems related to corresponding Lie algebra centrally extended by Kac-Moody, Virasoro and semidirect product cocycles with arbitrary coefficients.

The isomorphism (under certain conditions on elements of the coalgebra in the form of a Miura transformation) between Lax $L$-operators and smooth orientation-preserving immersions $S^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is established.

Using the three-hamiltonian structure formalism we prove the integrability of our generalized system in the case of a family of Hamiltonians formed from residues of powers of pseudo-differential operators. Relations between DWW and mKdV hierarchies appear when we introduce an alternative sequence of Hamiltonians.

The construction of commuting flows on coadjoint orbits is generalized for our group and Poisson bracket in the coalgebra of the associated affine loop algebra is presented.

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1 Introduction: Central extensions of semidirect product

\[ \mathcal{G} = Vect(S^1) \ltimes C^\infty(S^1, \mathbb{R}) \]

In the series of papers the Lie algebra of the group of orientation preserving diffeomorphism of the circle \( \mathcal{D} = Diff_+(S^1) \) was considered (see [1], [2], [3]). Authors of [4] and [5] treat the semidirect product of \( \mathcal{D} \) and the space of smooth non-vanishing functions \( \mathcal{V} = C^\infty(S^1, \mathbb{R}) \) on \( S^1 \) with central extensions by three cocycles \( C_{K-M}, C_{SP} \) and \( C_{Vir} \) of corresponding algebra of differential operators introduced in [12].

In [1] and [2] one can find a classification of coadjoint orbits of group \( \mathcal{D} = Diff_+(S^1) \) the algebra of which is centrally extended by the Virasoro cocycle.

The Lax representation for the dynamical system occurred from the coadjoint action of the extended by three cocycles algebra of the above mentioned semidirect product was found in [7]. The theorem in [3] established an isomorphism between the space of Hill equations and the space of smooth orientation-preserving immersions of \( S^1 \to \mathbb{P}^1 \), \( \mathbb{P}^1 \) is the real projective line). Different relations to the KdV hierarchy and procedure of the construction of commuting flows on the phase space are also presented there.

The author of [6] has developed the theory of the non-standard integrable systems and applied to a particular case of the operator \( L = b\partial^{-1} + a + \beta \partial \). The existence of the three-hamiltonian structure form of the DWW (Dispersive Water Waves) system of equations and its integrability were proved due to properties of \( L \) operators. The second hamiltonian structure is a matrix coadjoint representation of the Lie algebra with a particular case of the linear combination of two cocycles with arbitrary coefficients are also integrable.

The main purpose of this paper is to show that the dynamical systems that occur from the coadjoint action of the algebra \( \mathcal{G} = Vect(S^1) \ltimes C^\infty(S^1, \mathbb{R}) \) centrally extended by three cocycles with arbitrary coefficients are also integrable.

The plan of the paper is the following. Section 1 describes our algebra and its central extensions.

In Section 2 we show that the classification of its coadjoint orbits is closely related to the classification in the case of the Virasoro algebra. Namely the space of elements conserving the element of the coalgebra is just the same as for the Virasoro algebra.

In Section 3 the theorem proved in [3] is generalized. It turns out that the space of the Lax operators is isomorphic to the space of immersions \( S^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \) under certain restrictions on elements of the coalgebra. Those restrictions are related to the proof of the integrability of our system.

In Section 4 we present a commuting family of Hamiltonians, corresponding to our algebra which gives us a generalization of the DWW system. The alternative sequence of Hamiltonians starting from the standard quadratic Hamiltonian results in the KdV-like system of equations. In that case we find two reductions to the pair of KdV and mKdV equations.

In Section 5 we generalize the recipe of the construction of commuting flows on the phase space (see [3]) for the centrally extended algebra.

The Poisson bracket can be introduced for the affine loop algebra analog of the Virasoro algebra [10]. In the end of this paper we present the first and the second systems of compatible Poisson brackets for DWW and KdV-like systems corresponding to the extended algebra of semidirect product \( G = \mathcal{D} \ltimes \mathcal{V} \).
As for physical application, we show that the KdV-like pair of equations describes a system of generalized magnetic hydrodynamics on the phase space of our group \([13]\). We also consider small oscillations of our system.

Let \(\mathcal{D} = Diff(S^1)\) be the group of diffeomorphisms of \(S^1\) and \(\mathcal{V} = C^\infty(S^1, R)\) the space of smooth non-vanishing functions on \(S^1\). Consider a semidirect product \(G = \mathcal{D} \ltimes \mathcal{V}\) with the action of \(\mathcal{D}\) on \(\mathcal{V}\) (see \([7]\))

\[
\varrho \nu = \nu \circ \varrho
\]

where \(\varrho \in \mathcal{D}\) and \(\nu \in \mathcal{V}\). The group multiplication in \(G\) is

\[
(\varrho_1, \nu_1)(\varrho_2, \nu_2) = (\varrho_2 \circ \varrho_1, \nu_1 + \nu_2 \circ \varrho_1).
\]

The Lie algebra corresponding to \(G\) is the semidirect product \(\hat{G} = Vect(S^1) \ltimes \mathcal{V}\) of the algebra of vector fields \(Vect(S^1)\) on \(S^1\) and \(\mathcal{V} = C^\infty(S^1, R)\). We can consider a central extension \(\hat{G} = G \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}\) of \(G\) by the following cocycles (\([7]\), \([12]\))

\[
C_{K-M}((f_1, g_1), (f_2, g_2)) = \int_{S^1} g_1 g_2' \, dx
\]

\[
C_{SP}((f_1, g_1), (f_2, g_2)) = \int_{S^1} (f_1'' g_2 - f_2'' g_1) \, dx
\]

\[
C_{Vir}((f_1, g_1), (f_2, g_2)) = \int_{S^1} f_1 f_2''' \, dx
\]

( primes mean \(\partial = \frac{\partial}{\partial x}\) where \(x\) is a coordinate on \(S^1\)).

The Lie algebra structure in \(\hat{G}\) is given by the bracket

\[
[(f_1, g_1, (\alpha_1, \alpha_2, \alpha_3)_1), (f_2, g_2, (\alpha_1, \alpha_2, \alpha_3)_2)] =
(f_1 f_2' - f_2 f_1', f_1 g_2' - f_2 g_1', C_{K-M}, C_{SP}, C_{Vir}).
\]

We associate the dual space \(\hat{G}^*\) (the space of pseudo-differential operators on \(S^1\)) for \(\hat{G}\) by means of the pairing between \((f, g, \alpha_1, \alpha_2, \alpha_3) \in \hat{G}\) and \((b, a, \beta_1, \beta_2, \beta_3) \in \hat{G}^*\)

\[
\langle (f, g, \alpha_1, \alpha_2, \alpha_3), (b, a, \beta_1, \beta_2, \beta_3) \rangle = \int_{S^1} (f b + g a) \, dx + \sum_{i=1}^{3} \alpha_i \beta_i.
\]

The coadjoint action of \(\hat{G}\) on \(\hat{G}^*\) is the generalization of the coadjoint action of \(G\) on \(G^*\)

\[
\hat{ad}^*_{(f,g,\alpha_1,\alpha_2,\alpha_3)}(b, a, \beta_1, \beta_2, \beta_3) = (2 f' b + f b' + a g' + \beta_2 g'' + \beta_3 f''',
(f a)', \beta_1 g' - \beta_2 f'', 0, 0, 0).
\]
We can also consider the extension \( \tilde{G} = G \oplus R \) of \( G \) by a linear combination

\[
Z = l_1 C_{K-M} + l_2 C_{P-P} + l_3 C_{Vir}
\]

(i.e. \( \{ f \partial + g + \alpha Z \} \in \tilde{G} \)) with the appropriate Lie bracket

\[
[(f_1, g_1, \alpha_1), (f_2, g_2, \alpha_2)] = (f_1 f'_2 - f_2 f'_1, f_1 g'_2 - f_2 g'_1, Z).
\]

The pairing between \( (f, g, \alpha) \in \tilde{G}^* \) and \( (b, a, \beta) \in \tilde{G}^{**} \) (i.e. \( \{ b \partial^{-2} + a \partial^{-1} + \beta \log \partial \} \in \tilde{G}^{**} \)) of the form

\[
\langle (f, g, \alpha), (b, a, \beta) \rangle = \int_{S^1} (fb + ag) \, dx + \alpha \beta.
\]

and the coadjoint action

\[
\tilde{ad}^*_{(f, g, \alpha)}(b, a, \beta) = (2f' b + fb' + ag' + \beta (l_2 g'' + l_3 f''''), (fa)' + \beta (l_1 g' - l_2 f''), 0).
\]

The meaning of a special linear combination with

\[
l_1 = 1, l_2 = -\frac{1}{2}, l_3 = \frac{1}{6}
\]

is explained in [12].

### 2 Coadjoint orbits of \( G \)

The orbit method due to Kirillov [1] can be applied to the classification of coadjoint orbits of \( G \). Let us determine the subgroup \( G_{\delta} \) of \( G \) which leaves \( \rho = (b, a, \beta_1, \beta_2, \beta_3) \in \tilde{G}^* \) invariant under the action of an element \( \kappa = (f, g, \alpha_1, \alpha_2, \alpha_3) \in \tilde{G} \). The transformation laws of \( \rho \) under an infinitesimal transformation by \( \kappa \) are

\[
\delta b = 2f' b + fb' + ag' + \beta_2 g'' + \beta_3 f''' = 0,
\]

\[
\delta a = (fa)' + \beta_1 g' - \beta_2 f'' = 0,
\]

\[
\delta \beta_1 = \delta \beta_2 = \delta \beta_3 = 0.
\]

Eliminating \( g \) from (2.14) with the help of (2.15) we get (for \( \beta_1 \neq 0 \)) a system

\[
\begin{align*}
\delta \bar{b} &= \gamma f''' + 2f' \bar{b} + f \bar{b}' = 0 \\
\delta a &= (fa)' + \beta_1 g' - \beta_2 f'' = 0 \\
\delta \beta_1 &= \delta \beta_2 = \delta \gamma = 0
\end{align*}
\]

where
\[ \bar{b} = b - \frac{a^2}{\beta_1} - \frac{2a'}{\beta_1} \]
\[ \gamma = \frac{\beta_2}{\beta_1} + \beta_3. \]  

(2.18)

The first equation in (2.17) is a condition for \( \bar{b} \) to be invariant under the action of the \( f \) which is an element of the Virasoro algebra [1]. Therefore we have just the same classification of \( f \)-elements and \( \bar{b} \) as in [1],[2].

If \( \beta_2^2 + \beta_3 = 0, \beta_1 \neq 0 \) then \( f \) is an element of non-extended algebra of vector fields [2] (see also [3]).

The classification of coadjoint orbits for the \( g \)-element of \( \hat{G} \) can be deduced from the second equation of (2.17). Namely

\[ \beta_1 g = \text{const} - \beta_2 f' - fa. \]  

(2.19)

We see that the \( g \)-element of \( \hat{G} \) is related to the \( f \)-element by means of (2.19) with the functional parameter \( a \).

In principle, one can eliminate from (2.14) the element corresponding to the cocycle \( \text{C}_{\text{Vir}} \). Let us differentiate (2.13) multiplied by \( \frac{\beta_2}{\beta_1} \) \( (\beta_2 \neq 0) \), add (2.13) multiplied by \( a \frac{\beta_3}{\beta_2} \) and subtract the sum from (2.14) then

\[ \begin{align*}
\delta b &= 2f' b_s + fb'_s + a g' + \beta_2 g'' = 0 \\
\delta a &= (fa)' + \beta_1 g' - \beta_2 f'' = 0 \\
\delta \beta_2 &= 0
\end{align*} \]  

(2.20)

where

\[ b_s = \frac{1}{1 + \frac{\beta_3}{\beta_2}} (b + \frac{\beta_3}{\beta_2} a' + \frac{\beta_3}{2\beta_2^2} a^2). \]  

(2.21)

Let us show that we can transform the \( g \)-component of the algebra in such a way that the element of \( \hat{G}^* \) corresponding to \( \text{C}_{PP} \) will not appear explicitly in the coadjoint action. Making the substitution

\[ \tilde{g} = \frac{\beta_2}{2\beta_1} g \]  

(2.22)

we can rewrite the system (2.20) as

\[ \begin{align*}
\delta \tilde{b}_s &= 2f' \tilde{b}_s + fb'_s + a \tilde{g}' + \beta_2 \tilde{g}'' = 0 \\
\delta a &= (af)' + 2\beta_2 \tilde{g}' - \beta_2 f'' = 0 \\
\delta \beta_2 &= 0
\end{align*} \]  

(2.23)

with

\[ \tilde{b}_s = \frac{\beta_1}{2\beta_2} - b_s. \]  

(2.24)

The consequences of such a transformation will be clear in Section 4.
3  Lax operator representation for elements of $\hat{G}$ and $\hat{G}^*$

The element $V \in G$ may be represented as a matrix (see [7])

$$V = \begin{pmatrix} -\frac{1}{2}f' + f \circ \partial & 0 \\ -g' & \frac{1}{2}f' + f \circ \partial \end{pmatrix}$$  \hspace{1cm} (3.25)$$

The set of such differential operators forms a Lie algebra under ordinary commutator of operators. We can introduce the action of $V$ on $L \in \hat{G}^*$

$$ad^*_{\nu} L = V \circ L = [V, L] + AL$$  \hspace{1cm} (3.26)$$

where

$$A = \begin{pmatrix} 2f' & -g' \\ g' & 0 \end{pmatrix}$$  \hspace{1cm} (3.27)$$

and

$$L = \begin{pmatrix} 2b + 4\beta_3 \circ \partial^2 & 2\beta_2 \circ \partial + a \\ 2\beta_2 \circ \partial - a & -\beta_1 \end{pmatrix}$$  \hspace{1cm} (3.28)$$

which is the coadjoint action in that representation. Then the Lax form of the (2.14), (2.15) and (2.16) is

$$\delta L = ad^*_{\nu} L = 0.$$  \hspace{1cm} (3.29)$$

The modification of $L$-operator for the system (2.20) is transparent

$$L_* = \begin{pmatrix} 2b_* & 2\beta_2 \circ \partial + a \\ 2\beta_2 \circ \partial - a & -\beta_1 \end{pmatrix}$$  \hspace{1cm} (3.30)$$

Let $\psi$ be a column vector of two variables

$$\psi = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}.$$  \hspace{1cm} (3.31)$$

Consider the linear problem

$$L\psi = 0$$  \hspace{1cm} (3.32)$$

which is equivalent to

$$\begin{cases} 2bf_1 + 4\beta_3 f_1' + 2\beta_2 g_1' + ag_1 = 0 \\ 2\beta_2 f_1' - af_1 - \beta_1 g_1 = 0. \end{cases}$$  \hspace{1cm} (3.33)$$

Eliminating $g_1$ from the first equation of (3.33) we arrive at Hill equation ([1], [11])
\[ f_1'' + \frac{1}{2\gamma} \bar{b} f_1 = 0. \]  (3.34)

Let us take two independent solution \((f_1, g_1)\) and \((f_2, g_2)\) of (3.33). Then the second equation of it (for \(\beta_1 \neq 0\)) gives

\[ g_1 = \frac{1}{\beta_1} (a f_1 - 2\beta_2 f'_1). \]  (3.35)

It easy to check that the Wronskian

\[ W(g_1, g_2) = \frac{1}{\beta_1} \left( a^2 + 2\beta_2 a' + 2\frac{\beta_3^2}{\gamma} \bar{b} \right) W(f_1, f_2) \]

\[ = \frac{2}{\beta_2} \left( b + \frac{\beta_3^2 b}{\beta_1 \beta_3 + \beta_2^2} \bar{b} \right) W(f_1, f_2). \]  (3.36)

For the operator (3.30) using \(L_*\) we get the following condition

\[ W_*(g_1, g_2) = \frac{1}{\beta_1} b_* W_*(f_1, f_2). \]  (3.37)

Since the Wronskian \(W_*(f_1, f_2) = C_0\) is a constant, therefore the space of \(L_*\) operators is isomorphic to the space of smooth orientation preserving immersions \(S^1 \rightarrow P^1 \times P^1\) when

\[ b_* = \text{const}(x) \]  (3.38)

(compare with [3]). One can mention that (3.38) is some sort of Miura transformation as well as (2.18),(2.21). We can only repeat that the same situation takes place for the space of \(L\)-operators when

\[ b + \frac{\beta_2^2 \bar{b}}{\gamma} = \text{const}(x). \]  (3.39)

4 Hamiltonian structures on coadjoint orbits of \(G\)

On coadjoint orbits of a group we could define as in [3] a classical mechanics system by the following Euler equation for \(\chi \in \hat{G}^*\) (dot means the coadjoint action) (see [3])

\[ \partial_t \chi = (M^{-1} \chi) \cdot \chi \]  (4.40)

where the product in the r.h.s. is the coadjoint action of \(\hat{G}\) on \(\hat{G}^*\). (In the case of \(SO_3\), \(\chi\) and \(M\) are the angular momentum and the inertia tensor of free moving body in the configuration space \(SO_3\) correspondingly).

The Euler equation (4.40) for the case of \(G\) may be written in the form

\[ \begin{pmatrix} a \\ b \end{pmatrix}_t = \left( M^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \right) \cdot \begin{pmatrix} a \\ b \end{pmatrix} \]  (4.41)

or as a coadjoint action
\[
\begin{align*}
\dot{b} &= 2f'b + fb' + ag' + \beta_2g'' + \beta_3f'' \\
\dot{a} &= (fa)' + \beta_1g' - \beta_2f''.
\end{align*}
\]

(dots mean \(\partial_t\)). Using methods of Section 3 (equations (2.17), (2.20)) we obtain two alternative formulations of the dynamical system (4.42)

\[
\begin{align*}
\dot{\tilde{b}} &= \gamma f''' + 2f\tilde{b} + f\tilde{b}' \\
\dot{\tilde{a}} &= (fa)' + \beta_1g' - \beta_2f''.
\end{align*}
\]

(4.43)

and

\[
\begin{align*}
\dot{b^*} &= 2f'b^* + fb^* + ag' + \beta_2g'' \\
\dot{a} &= (fa)' + \beta_1g' - \beta_2f''.
\end{align*}
\]

(4.44)

We may write (4.42) and (4.44) in a hamiltonian form

\[
\begin{pmatrix} a \\ b \end{pmatrix}_t = B \frac{\delta H}{\delta a} \quad \frac{\delta H}{\delta b}
\]

(4.45)

with the column vector of variational derivatives of a Hamiltonian \(H\) in the r.h.s. and \(B\) as a hamiltonian matrix (see [6]). Identifying elements of our algebra with elements of the coalgebra (i.e. linear parts of variations of \(H\) by \(a\) and \(b\)) let us take \(f = a\) and \(g = b\) for (4.42) or \(f = a\) and \(g = b^*\) for (4.44). Then we find that in (4.40)

\[
M^{-1} = M^{-1}_{DW} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(4.46)

and arrive at two generalizations of the DWW system of equations (3)

\[
\begin{align*}
\dot{a} &= (\beta_1b + a^2 - \beta_2a')' \\
\dot{b} &= (2ab + \beta_2b' + \beta_3a'')'.
\end{align*}
\]

(4.47)

and

\[
\begin{align*}
\dot{a} &= (\beta_1b_* + a^2 - \beta_2a')' \\
\dot{b_*} &= (2ab_* + \beta_2b_*').
\end{align*}
\]

(4.48)

A substitution (see (2.24))

\[
\ddot{b}_* = \frac{\beta_1}{2\beta_2}b_*
\]

(4.49)

allows us to rewrite the system (4.48) as

\[
\begin{align*}
\dot{a} &= (2\beta_2b_* + a^2 - \beta_2a')' \\
\dot{b_*} &= (2\dot{a}b_* + \beta_2b_*')'.
\end{align*}
\]

(4.50)

and in the form (4.43) with the matrix \(B = \tilde{B}_*\).
\[ \hat{B}^H = \begin{pmatrix}
2\beta_2 \circ \partial & \partial \circ a - \beta_2 \circ \partial^2 \\
\partial \circ a + \beta_2 \circ \partial^2 & \hat{b}_s \circ \partial + \partial \circ \hat{b}_s
\end{pmatrix} \]  
(4.51)

( substitutions \( b_s \mapsto \hat{b}_s \) should be made in the column vector of variational derivatives in (4.45)).

One can mention that (4.50) is the DWW system of equations [4]. It was proved in [6] that such a system is integrable and there exist the following three hamiltonian structure formulation for it ( \( \hat{B}^H \) coincides with the second hamiltonian structure)

\[ \begin{pmatrix}
\frac{\delta H_{k+1}}{\delta a} \\
\frac{\delta H_k}{\delta b_s}
\end{pmatrix} = \hat{B}^H \begin{pmatrix}
\frac{\delta H_k}{\delta a} \\
\frac{\delta H_k}{\delta b_s}
\end{pmatrix} = \hat{B}^{III} \begin{pmatrix}
\frac{\delta H_{k-1}}{\delta a} \\
\frac{\delta H_{k-1}}{\delta b_s}
\end{pmatrix} \]  
(4.52)

where \( k = 1, 2, \ldots \). The first and third hamiltonian structures are given by

\[ \hat{B}^I = \begin{pmatrix}
0 & \partial \\
\partial & 0
\end{pmatrix} \quad \text{and} \quad \hat{B}^{III} = \begin{pmatrix}
2\beta_2(\partial \circ a + a \circ \partial) & 2\beta_2(\partial \circ \hat{b}_s + \hat{b}_s \circ \partial) + \partial \circ (a - \beta_2 \circ \partial)^2 \\
2\beta_2(\partial \circ \hat{b}_s + \hat{b}_s \circ \partial) - (a + \beta_2 \circ \partial)^2 \circ \partial & (a + \beta_2 \circ \partial)(\hat{b}_s \circ \partial + \partial \hat{b}_s) + (\hat{b}_s \circ \partial + \partial \hat{b}_s)(a - \beta_2 \circ \partial)
\end{pmatrix} \]  
(4.54)

The identity

\[ (\mathbb{L}^\dagger)^{m+1} = (\mathbb{L}^\dagger)^m \mathbb{L}^\dagger = \mathbb{L}^\dagger (\mathbb{L}^\dagger)^m \]  
(4.55)

with the \( \mathbb{L} \)-operator ( \( ^\dagger \) means adjoint)

\[ \mathbb{L} = \hat{b}_s \partial^{-1} + a + \beta_2 \partial \]  
(4.56)

was used in [6] to prove (4.52) for the family of conserved commuting Hamiltonians (see also [4])

\[ \tilde{H}_n = \frac{1}{n} \text{Res} \mathbb{L}^n = \frac{1}{n} \text{Res} (\hat{b}_s \partial^{-1} + a + \beta_2 \partial)^n. \]  
(4.57)

If we expand \( (\mathbb{L})^m \) in \( \xi \) ( \( \xi \) denotes partial derivative)
\((L)^m = \sum s \bar{\phi}_s(m)\xi^s\)  \hspace{1cm} (4.58)

and pick out the \(\xi^i\) terms in \((4.55)\) for \(i = 0, 1, 2\) then we get \(\bar{\phi}\) for all \(m\)

\[
\begin{align*}
\dot{\bar{\phi}}_0(m + 1) &= 2\beta_2\bar{\phi}_{-1}(m) + (a - \beta_2\partial)\bar{\phi}_0(m) \hspace{1cm} (4.59) \\
\partial\bar{\phi}_{-1}(m + 1) &= (a\partial + \partial^2)\bar{\phi}_{-1}(m) + (\bar{b}_s\partial + \partial\bar{b}_s)\bar{\phi}_0(m). \hspace{1cm} (4.60)
\end{align*}
\]

In the theory of non-standard integrable systems \([6]\) the following relations occur

\[
\begin{align*}
\dot{\bar{\phi}}_0(m) &= \delta\dot{H}_{m+1} \hspace{1cm} (4.61) \\
\dot{\bar{\phi}}_{-1}(m) &= \delta\dot{H}_{m+1} \hspace{1cm} (4.62)
\end{align*}
\]

which combined with \((4.59)\) and \((4.60)\) gives the first and second hamiltonian structures. Let us change variables

\[
\begin{align*}
p_0(m) &= \bar{\phi}_0(m) \hspace{1cm} (4.63) \\
p_{-1}(m) &= \frac{2\beta_2}{\beta_1} \bar{\phi}_{-1}(m) \hspace{1cm} (4.64)
\end{align*}
\]

so that

\[
\begin{align*}
p_0(m) &= \frac{\delta H_{s_{m+1}}}{\delta b_*} = \frac{\delta\dot{H}_{m+1}}{\delta b_*} \hspace{1cm} (4.65) \\
p_{-1}(m) &= \frac{\delta H_{s_{m+1}}}{\delta a} = \frac{2\beta_2}{\beta_1} \frac{\delta\dot{H}_{m+1}}{\delta a} \hspace{1cm} (4.66)
\end{align*}
\]

and we can rewrite \((4.59)\) and \((4.60)\) as

\[
\begin{align*}
p_0(m + 1) &= \beta_1 p_{-1}(m) + (a - \beta_2\partial)p_0(m) \hspace{1cm} (4.67) \\
\partial p_{-1}(m + 1) &= (a\partial + \partial^2)p_{-1}(m) + (b_s\partial + \partial b_s)p_0(m) \hspace{1cm} (4.68)
\end{align*}
\]

for all \(m\). \((4.67)\) and \((4.68)\) give us the second hamiltonian structure for systems \((4.48)\)

\[
B_{*II} = \begin{pmatrix}
\beta_1 \circ \partial & \partial \circ a - \beta_2 \circ \partial^2 \\
a \circ \partial + \beta_2 \circ \partial^2 & b_s \circ \partial + \partial \circ b_s
\end{pmatrix}
\hspace{1cm} (4.69)
\]

We derive from \((4.65)\) and \((4.66)\) the family of Hamiltonians

10
\[ H_{*n} = \frac{2\beta_2}{\beta_1^n} \text{Res} \left( \frac{\beta_1}{2\beta_2} b_* \partial^{-1} + a + \beta_2 \partial \right)^n \]  

(4.70)

and calculate the third hamiltonian structure using (4.67) and (4.68) (the first structure is just the same as in (4.52))

\[
B_{III}^* = \begin{pmatrix}
\beta_1 (\partial \circ a + a \circ \partial) & \beta_1 (\partial \circ b_* + b_* \circ \partial) + \partial \circ (a - \beta_2 \circ \partial)^2 \\
\beta_1 (\partial \circ b_* + b_* \circ \partial) & (a + \beta_2 \partial)(b_* \partial + \partial b_*) \\
- (a + \beta_2 \circ \partial)^2 \circ \partial & + (b_* \partial + \partial b_*)(a - \beta_2 \partial)
\end{pmatrix}
\]

(4.71)

It easy to check that \(B_{II}^*\) and \(B_{III}^*\) are hamiltonian. We have to take

\[ H_{*0} = \int_{S^1} b_*(x) \, dx \]  

(4.72)

and change arguments \(\bar{b}_* \mapsto b_*\) in variational derivatives.

Equations of (4.45) with \(B = B_{II}^*\) with \(H_{*1}\) describes a simple dynamical system

\[
\begin{cases}
\dot{a} = a' \\
\dot{b}_* = b_*'
\end{cases}
\]

(4.73)

Now let us get back to the original problem. Systems (4.44) and (4.42) are equivalent up to conditions \(\beta_3 \neq 0, \beta_2 \neq 0\). Therefore we may conclude that (4.47) is also integrable. Leaving the first structure \(\hat{B}^I\) (4.53) unchanged we may present (4.47) in the form of three-hamiltonian structure equation with hamiltonian matrixes

\[
B = \hat{B}^I = \begin{pmatrix}
\beta_1 \circ \partial & \partial \circ a - \beta_2 \circ \partial^2 \\
a \circ \partial + \beta_2 \circ \partial^2 & b \circ \partial + \partial \circ b + \beta_3 \circ \partial^3
\end{pmatrix}
\]

(4.74)

\[
\hat{B}^{III} = \begin{pmatrix}
\beta_1 (\partial \circ a + a \circ \partial) & \beta_1 (\partial \circ b + b \circ \partial) + \partial \circ (a - \beta_2 \circ \partial)^2 + \beta_3 \partial^3 \\
\beta_1 (\partial \circ b + b \circ \partial) & (a + \beta_2 \partial)(b \partial + \partial b) + (b \partial + \partial b)(a - \beta_2 \partial) + \beta_3(a \partial^3 + \partial^3 a)
\end{pmatrix}
\]

(4.75)

With the help of three-hamiltonian structure we can calculate the whole family of Hamiltonians for the systems (4.47). For example the third Hamiltonian is
\[ H_3 = \int_{S^1} \left( a^2 b + \beta_2 (a'b' - a'b) + \frac{1}{2} \beta_1 b^2 - \frac{1}{2} \beta_3 (a')^2 \right) \, dx \]  

(4.76)

while in the first and second coincide with \( H_{*1} \) and \( H_{*2} \) with \( b_* \) substituted by \( b \).

As it is shown in [4] the DWW system of equations has for every odd flow \( n = 1 \) (mod 2) the invariant manifold defined by \( a = 0 \) on which this hierarchy reduces to the KdV hierarchy.

Now let us take a standard quadratic Hamiltonian

\[ \tilde{H}_1 = \int_{S^1} \frac{1}{2} \left( b^2 + a^2 \right) \, dx \]  

(4.77)

or

\[ \tilde{H}_1 = \int_{S^1} \frac{1}{2} \left( \tilde{b}^2 + a^2 \right) \, dx. \]  

(4.78)

Then in (4.41)

\[ M^{-1} = M_{KdV}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(4.79)

We derive \( f = b, g = a \) from the Hamiltonian \( \tilde{H}_1 \) (4.77) and write (4.45) with the same \( B = \tilde{B}^{II} \) as before

\[
\begin{cases}
\dot{a} &= (ba + \beta_1 a - \beta_2 b')' \\
\dot{b} &= \beta_3 b'' + 3bb' + aa' + \beta_2 a''.
\end{cases}
\]  

(4.80)

We have \( f = \bar{b}, a = g \) for the system (4.43) with \( \tilde{H}_1 \)

\[
\begin{cases}
\dot{a} &= (\bar{b}a + \beta_1 a - \beta_2 \bar{b})' \\
\dot{\bar{b}} &= \gamma \bar{b}'' + 3\bar{b} \bar{b}'.
\end{cases}
\]  

(4.81)

One can see, that the second equation in (4.81) is the KdV equation. The infinite number of its conservation laws allows us to integrate the first equation of our system.

Recall that the systems (4.42) and (4.43) are equivalent under conditions \( \beta_1 \neq 0, \beta_2 \neq 0 \). Therefore as in the case of DWW systems we conclude that (4.80) is integrable.

As a generalization of bi-hamiltonian structure of the KdV equation the system (4.80) has a form

\[
\begin{pmatrix} a \\ b \end{pmatrix}_t = \tilde{B}^I \begin{pmatrix} \delta \tilde{H}_{k+1} \\ \delta \tilde{H}_k \end{pmatrix} = \tilde{B}^{II} \begin{pmatrix} \delta \tilde{H}_{k+1} \\ \delta \tilde{H}_k \end{pmatrix}
\]  

(4.82)

where
\[
\hat{B}^I = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}
\]  
(4.83)

\[k = 0, 1\] and

\[\hat{H}_0 = \int_{S^1} (a(x) + b(x)) \, dx.\]  
(4.84)

\[(\text{4.45)} \text{ with } \hat{H}_0 \text{ is again (4.73))}\]

\[\hat{H}_2 = \int_{S^1} \left( \left[ \frac{1}{2} \beta_3 - \frac{\beta_2}{2} \left( \frac{\partial b}{\partial x} \right)^2 \right] + \left[ \frac{1}{2} a^2 b + \beta_2 (a'b - ab') + \beta_1 \frac{1}{2} a^2 \right] \right) \, dx.\]  
(4.85)

It is easy to see that \(\hat{H}_0, \hat{H}_1\) and \(\hat{H}_2\) are integrals of motion. Indeed \([15]\), equations in (4.73) and (4.80) are in the form of the continuity equation. If we denote \(q_0 = a + b\) for \(\hat{H}_0\) and \(q_1 = a^2 + b^2\) for \(\hat{H}_1\) then

\[\partial_t q_k + \partial_j q_k = 0\]  
(4.86)

\[k = 0, 1\] and \(q_0, q_1\) are conserved densities. Using the two-hamiltonian structure we can calculate higher Hamiltonians of the system. Analogous but less interesting dynamical systems appear from (4.44).

The system (4.80) can also be transformed in a way that clarify its coupling with the family of Hamiltonians (4.57). Let

\[\tilde{b} = b + \beta_1\]  
(4.87)

then

\[
\begin{cases}
\dot{a} &= (ba - \beta_2 \tilde{b}'')' \\
\dot{\tilde{b}} &= \beta_3 \tilde{b}''' + 3b \tilde{b}' + aa' + \beta_2 a'' - 3\beta_1 \tilde{b}.
\end{cases}
\]  
(4.88)

(4.88) has the form of (4.82) with the same \(\hat{H}_0, \hat{H}_1, \hat{B}^I\) (b is replaced by \(\tilde{b}\)) and

\[\hat{B}^I_{\tilde{b}} = \begin{pmatrix} 0 & \partial \circ a - \beta_2 \circ \partial^2 \\ a \circ \partial + \beta_2 \circ \partial^2 & \tilde{b} \circ \partial + \partial \circ \tilde{b} + \beta_3 \circ \partial^3 - 3\beta_1 \circ \partial \end{pmatrix}.\]  
(4.89)

The expression in the second square brackets reminds us \(\hat{H}_3\)

\[\hat{H}_2 = \int_{S^1} \left( \left[ \frac{1}{2} \beta_3 - \frac{\beta_2}{2} \left( \frac{\partial \tilde{b}}{\partial x} \right)^2 \right] + \left[ \frac{1}{2} a^2 b + \beta_2 (a'b - ab') \right] \right) \, dx\]  
(4.89)

It is interesting to mention that the condition \(b = 0\) in \(\tilde{b}\) leads to the reduction of (4.81) to
\[
\begin{aligned}
\dot{a} &= \beta_1 a' + \frac{1}{\beta_0} (\beta^2 a''' - \frac{3}{2} a^2 a') \\
\dot{\hat{a}} &= \gamma \hat{a}''' + 3 \hat{a} \hat{a}'
\end{aligned}
\] (4.91)

with

\[
\hat{a} = -\frac{\beta_2}{\beta_1} a' - \frac{1}{2\beta_1} a^2
\] (4.92)

so that we have a mKdV equation for \(a\) and a KdV equation for \(\hat{a}\) related by some kind of Miura transformation.

The system (4.44) with \(f = b_*\) and \(g = a\) gives

\[
\begin{aligned}
\dot{a} &= (b_* a + \beta_1 a - \beta_2 b'_*)' \\
\dot{b}_* &= 3 b_* b'_* + a a' + \beta_2 a''.
\end{aligned}
\] (4.93)

As it was in (4.81) for \(b = 0\) we obtain a pair of mKdV and KdV equations when \(b = 0\)

\[
\begin{aligned}
\dot{a} &= \frac{\beta_3}{\gamma} \left( \frac{3}{2\beta_2} a^2 a' - a''' \right) - \beta_1 a' \\
\dot{a}_* &= \gamma \hat{a}_*''' + 3 \hat{a}_* \hat{a}_*' \\
\end{aligned}
\] (4.94)

where

\[
\tilde{\gamma} = 1 + \frac{\beta_3 \beta_1}{\beta_2^2}
\] (4.95)

\[
\tilde{a}_* = \frac{1}{\gamma} \left( \beta_3 a' + \frac{\beta_3}{2\beta_2} a^2 \right).
\] (4.96)

The Lax representation for the systems (4.42) and (4.44) with two types of Hamiltonians is obvious (3.29)

\[
\partial_t L = ad^*_{v'} L
\] (4.97)

and

\[
\partial_t L_{*} = ad^*_{v'} L_{*}
\] (4.98)

(see (3.25), (3.27), (3.28) and (3.30)) where elements of the algebra should be replaced by coalgebra elements.

(4.42) (with \(\beta_1 = \beta_2 = \beta_3 = 0\)) can also be considered as a system of equation of motion for generalized magnetic hydrodynamics in the configuration space \(G\),

\[
\dot{\mathcal{M}} = -ad^*_{\mathcal{N}} \mathcal{M} + ad^*_{\mathcal{H}} J
\] (4.99)

\[
\dot{\mathcal{H}} = [\mathcal{N}, \mathcal{H}]
\]

where the second equation can be written in the \(ad^*\)-terms in coalgebra so that \(\mathcal{N} = -f, \mathcal{H} = g, \mathcal{M} = b, J = a\) if \((q, J) \in G^*\) corresponding to \((-f, g) \in G\).
It might be interesting to consider small oscillations in the dynamical system described by (4.80). We find

\[-i(\omega - \beta_3 k^3)b = -i\beta_2 k^2 a \] (4.100)

\[-i\omega = \beta_2 k^2 \] (4.101)

which leads us to

\[\omega = \frac{\beta_3 k^3}{2} \pm \sqrt{\frac{\beta_3^2 k^6}{4} + \beta_2 k^4} \] (4.102)

as a \(\omega(k)\)-dependence.

5 Commuting flows and Poisson brackets on \(\hat{G}^*\)

There exists a standard way to obtain commuting flows on coadjoint orbits of \(G\), \([3]\). For two elements \(h_1, h_2 \in \hat{G}\) and \(u \in \hat{G}^*\) the Poisson bracket is given by

\[\{h_1, h_2\}(u) = \langle u, [dh_1, dh_2]\rangle. \] (5.103)

Choose a fixed element \(q = (q_1, q_2, \ldots) \in \hat{G}^*\). Then for each \(G\)-invariant function \(F\) on \(\hat{G}^*\), (i.e.

\[\langle \tau, h, dF(h)\rangle = \langle h, [\tau, dF(h)]\rangle = 0, \] (5.104)

for every \(\tau \in \hat{G}\) and every \(h \in \hat{G}^*\) and for each \(\chi = (\chi_i), \chi_i \in \mathbb{C}\) functions

\[F_\chi(h) = (F_{\chi,1}(u_1 - \chi_1 q_1), F_{\chi,2}(u_2 - \chi_2 q_2), \ldots) \] (5.105)

Poisson-commute \([3]\). The proof is a simple generalization of the lemma in \([3]\) using the skew symmetry of the Poisson bracket (5.103) of multi-component functions (cf. \([3]\)).

Consider now an affine Lie algebra \(L(\hat{G}) = \sum_{i<0} \varsigma_i \lambda^i + \sum_{i\geq0} \varsigma_i \lambda^i\) consisting of Lorant polynomials with coefficients \(\varsigma_i \in G\). Let \(\hat{L}(\hat{G})\) be the loop algebra connected to \(\hat{G}\) and centrally extended by \(R\) \((\lambda, \lambda^{-1})\). Let \(R = P_+ - P_-\) be a standard \(r\)-matrix where \(P_\pm\) are projection operators from \(L\) to corresponding subalgebras. The dual to \(\hat{L}(\hat{G})\) space is introduced by means of the pairing (see \([3], [10]\))

\[\langle (b, a, \beta_1, \beta_2, \beta_3), (f, g, \alpha_1, \alpha_2, \alpha_3)\rangle = \]

\[Res_{\lambda=0} \left( \int (bf + ag) \, dx + \sum_{i=1}^{3} \beta_i \alpha_i \right). \] (5.106)

In the space of polynomials of order \(\leq N\) (with fixed \(\beta_m, m = 1, 2, 3\))

\[\chi(\lambda, x) = \left(\begin{array}{c}a(\lambda, x) \\ b(\lambda, x) \end{array}\right)_{\beta_1, \beta_2, \beta_3} = \sum_{i=0}^{N} \chi_i(x) \lambda^i \] (5.107)
we have three linear systems of $N + 2$ compatible brackets. Two first of them (compare with (4.42)) are

$$\{\chi_i(x), \chi_j(y)\}_{R^\lambda k}^{\text{DW} W_1} = \epsilon \sum_{s=0}^N \left( \begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right) \left( \begin{array}{c} \delta(x-y) \\ \delta(x-y) \end{array} \right)$$  \hspace{1cm} (5.108)

$$\{\chi_i(x), \chi_j(y)\}_{R^\lambda k}^{\text{DW} W_2} =$$

$$\epsilon \sum_{s=0}^N \left( \begin{array}{c} \beta_{1,s} \delta_{l,s} \partial \\ a_l(x) \circ \partial - \beta_{2,s} \delta_{l,s} \partial^2 \\ b_l(x) \circ \partial + \beta_{3,s} \delta_{l,s} \partial^2 \end{array} \right) \left( \begin{array}{c} \delta(x-y) \\ \delta(x-y) \end{array} \right)$$  \hspace{1cm} (5.109)

where $l = i + j + 1 - k$, $k = 0, ..., N + 1$, $\epsilon = 1$ when $i, j \geq k$, $\epsilon = -1$ when $i, j < k$, $\epsilon = 0$ for $i \geq k, j < k$.

The derivation of the third system of brackets from (4.73) is rather tedious but simple exercise. In the KdV-like case the system of second brackets coincides with (5.109) while the first is

$$\{\chi_i(x), \chi_j(y)\}_{R^\lambda k}^{\text{KdV} 1} = \epsilon \sum_{s=0}^N \left( \begin{array}{cc} \partial & 0 \\ 0 & \partial \end{array} \right) \left( \begin{array}{c} \delta(x-y) \\ \delta(x-y) \end{array} \right)$$  \hspace{1cm} (5.110)

As for the third Poisson bracket for (4.80) one could try to generalize the non-local bracket found in [14] for our system of equation.

6 Conclusion

To complete the analysis of hamiltonian strictures considered in this paper, it would be interesting to try to find a whole family of commuting Hamiltonians for the KdV-like system (4.42) using the bi-hamiltonian form of it (4.82) or the resolvent of the inverse to the Lax operator (3.28) (cf. [3]). It is also possible to write alternative Lax representation for our systems using the third hamiltonian structures.

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