An EPTAS for Budgeted Matroid Independent Set

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Abstract

We consider the budgeted matroid independent set problem. The input is a ground set, where each element has a cost and a non-negative profit, along with a matroid over the elements and a budget. The goal is to select a subset of elements which maximizes the total profit subject to the matroid and budget constraints. Several well known special cases, where we have, e.g., a uniform matroid and a budget, or no matroid constraint (i.e., the classic knapsack problem), admit a fully polynomial-time approximation scheme (FPTAS). In contrast, already a slight generalization to the multi-budgeted matroid independent set problem has a PTAS but does not admit an efficient polynomial-time approximation scheme (EPTAS). This implies a PTAS for our problem, which is the best known result prior to this work.

Our main contribution is an EPTAS for the budgeted matroid independent set problem. A key idea of the scheme is to find a representative set for the instance, whose cardinality depends solely on $1/\varepsilon$, where $\varepsilon > 0$ is the accuracy parameter of the scheme. The representative set is identified via matroid basis minimization, which can be solved by a simple greedy algorithm. Our scheme enumerates over subsets of the representative set and extends each subset using a linear program. The notion of representative sets may be useful in solving other variants of the budgeted matroid independent set problem.

1 Introduction

We consider the budgeted matroid independent set (BMI) problem, defined as follows. We are given a set of elements $E$, a membership oracle for a collection of independent sets $\mathcal{I} \subseteq 2^E$, where $(E, \mathcal{I})$ is a matroid, a budget $\beta > 0$, a cost function $c : E \to [0, \beta]$, and a profit function $p : E \to \mathbb{R}_+$. A solution for the problem is an independent set $X \in \mathcal{I}$ of total cost at most $\beta$ (i.e., $c(X) = \sum_{e \in X} c(e) \leq \beta$). The profit of a solution $X$ is $p(X) = \sum_{e \in X} p(e)$, and the objective is to find a solution of maximal profit.

BMI is a generalization of the classic 0/1-knapsack problem, which is NP-hard and therefore unlikely to admit an exact polynomial-time algorithm. Thus, there is a long line of work on finding efficient approximations for knapsack and its variants (for comprehensive surveys see, e.g., [20, 17]).

Let $\text{OPT}(I)$ be the value of an optimal solution for an instance $I$ of a maximization problem $\Pi$. For $\alpha \in (0, 1]$, we say that $A$ is an $\alpha$-approximation algorithm for $\Pi$ if, for any instance $I$ of $\Pi$, $A$ outputs a solution of value at least $\alpha \text{OPT}(I)$. A polynomial-time approximation scheme (PTAS) for a maximization problem $\Pi$ is a family of algorithms $(A_\varepsilon)_{\varepsilon > 0}$ such that, for any $\varepsilon > 0$, $A_\varepsilon$ is a polynomial-time $(1 - \varepsilon)$-approximation algorithm for $\Pi$; $(A_\varepsilon)_{\varepsilon > 0}$ is an EPTAS if the running time of $A_\varepsilon$ is of the form $f\left(\frac{1}{\varepsilon}\right) \cdot n^{O(1)}$, where $f$ is an arbitrary function, and $n$ is the bit-length encoding size of the input instance; $(A_\varepsilon)_{\varepsilon > 0}$ is an FPTAS if the running time of $A_\varepsilon$ is of the form $\left(\frac{n}{\varepsilon}\right)^{O(1)}$.

Polynomial-time approximation schemes allow us to obtain almost optimal solutions for NP-hard optimization problems via a speed-accuracy trade-off. However, the strong dependency of run-times on the error parameter, $\varepsilon > 0$, often renders these schemes highly impractical. Thus, a natural goal is to seek the fastest scheme for a given problem, assuming one exists. Having obtained a PTAS, the next step is to consider possible improvements to an EPTAS, or even to an FPTAS. This is the focus of our work.

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For the classic knapsack problem, a very efficient FPTAS exists since the 1970’s. Lawler’s scheme [19], based on ideas from [12], achieves a running time of $O(n \log 1/\varepsilon + 1/\varepsilon^4)$ for a $(1 - \varepsilon)$-approximation. In contrast, already the two-dimensional knapsack problem has a PTAS but does not admit an EPTAS [18]. For the well known multiple knapsack problem, Chekuri and Khanna [3] derived the first PTAS, which was later improved by Jansen to an EPTAS [13, 14]. The existence of an FPTAS is ruled out by a simple reduction from Partition [3]. More generally, resolving the complexity status of NP-hard optimization problem with respect to approximation schemes has been the focus of much research relating to resource allocation and scheduling (see, e.g., [24, 6, 8, 15, 16] and the surveys [22, 23]).

For BMI, FPTASs are known for instances in which the matroid belongs to a restricted family. One notable example is multiple choice knapsack [19], where the elements are partitioned into groups, and a set is independent if it contains at most one element from each group. Another example is 1.5-dimensional knapsack [2], in which a set is independent if it contains at most $k$ elements for some $k \geq 1$ (i.e., a uniform matroid constraint).

There are known results also for generalizations of BMI which involve multiple budget constraints or an additional matroid constraint. Specifically, Grandoni and Zenklusen developed in [9] a PTAS for multi-budgeted matroid independent set (MBMI), a generalization of BMI in which the costs are $d$-dimensional (for some constant $d \in \mathbb{N}$). The PTAS of [9] is based on integrality properties of a linear programming relaxation of the problem. As MBMI generalizes the two-dimensional knapsack problem, it does not admit an EPTAS unless W[1] = FPT [18].

The budgeted matroid intersection problem is a generalization of BMI in which the input includes membership oracles for independent sets of two different matroids, and the solution must be an independent set of both matroids. A PTAS for budgeted matroid intersection was developed in [1]. The algorithm of [1] uses a Lagrangian relaxation along with some combinatorial properties of the problem to patch two solutions (i.e., a feasible solution with sub-optimal profit, and a non-feasible solution with high profit) into the final solution. The existence of an EPTAS (or an FPTAS) for budgeted matroid intersection is still open.

The multi-budgeted matroid intersection problem is a generalization of both multi-budgeted matroid independent set and budgeted matroid intersection, in which the cost function is $d$-dimensional, and the input contains two matroids. In [4] the authors developed a PTAS for multi-budgeted matroid intersection, based on randomized rounding of a solution for a linear programming (LP) relaxation of the problem.

The budgeted matroid independent set problem is also a special case of multiple knapsack with a matroid, a variant of BMI in which the input contains $m$ budgets $\beta_1, \ldots, \beta_m$. A solution consists of $m$ sets $S_1, \ldots, S_m$, where the cost of the $j$th set is at most the $j$th budget ($c(S_j) \leq \beta_j$), and the union of the sets is an independent set of the matroid. A PTAS for the problem (based on randomized rounding) was given in [7]. The existence of an FPTAS is ruled out, as multiple knapsack is a special case [3].

To the best of our knowledge, BMI is studied here for the first time. A PTAS for the problem follows from known results for any of the above generalizations. In all cases, the running time of the scheme is dominated by an enumeration phase which guesses the most profitable elements in an optimal solution.

Our main result is an EPTAS for BMI, thus substantially improving the running times of existing schemes for the problem. Let $K = (E, I, c, p, \beta)$ denote a BMI instance, $\text{OPT}(K)$ the profit of an optimal solution for $K$, and $|K|$ the bit-length encoding of the instance $K$.

**Theorem 1.1.** Given an instance $K$ of BMI and $0 < \varepsilon < 1/2$, there is an algorithm that outputs a solution of profit at least $(1 - \varepsilon) \cdot \text{OPT}(K)$ in time $2^{O(\varepsilon^{-2} \log \frac{1}{\varepsilon})} \cdot \text{poly}(|K|)$.

**Main Technique.** Our algorithm builds upon a framework of Grandoni and Zenklusen [9] for multi-budgeted matroid independent set. In [9] the authors show that a basic solution for a linear programming relaxation of the problem has only a few non-integral entries. Thus, a solution for MBMI is constructed by solving the LP relaxation and adding all the elements with non-zero integral entries to the MBMI solution. An exhaustive enumeration phase which guesses the $\Theta(\frac{1}{\varepsilon})$ most profitable
elements in an optimal solution is used to mitigate the profit loss caused by discarding the (few) elements with non-integral entries in the solution for the LP. The running time of the algorithm is dominated by the $|E|^{O(\frac{1}{\varepsilon})}$ operations required for exhaustive enumeration.

The improved running time of our algorithm is obtained by reducing the time complexity of the enumeration phase. Consider a BMI instance $K = (E, I, c, p, \beta)$. Given some $0 < \varepsilon < \frac{1}{2}$, we say that an element $e \in E$ is profitable if $p(e) > \varepsilon \cdot \text{OPT}(K)$. Our algorithm identifies a representative set of elements $R \subseteq E$ satisfying $|R| \leq f\left(\frac{1}{2}\right)$, for a computable function $f$. Furthermore, we show that the BMI instance has a solution $S$ where all elements in $S \setminus R$ are non-profitable and $p(S) \geq (1 - O(\varepsilon)) \cdot \text{OPT}(K)$. Thus, we can use enumeration to guess $S \cap R$ and then extend the solution using an LP relaxation similar to [9]. Crucial to our construction of $S$ is the notion of profit gap, used for identifying elements that may be added to $S$ by solving the LP (see Section 3). Since all the elements in $S \setminus R$ are non-profitable, the profit loss caused by the few non-integral entries is negligible. Moreover, since $|R| \leq f\left(\frac{1}{2}\right)$, guessing $S \cap R$ can be done in $2^{f\left(\frac{1}{2}\right)}$ steps (in fact, we obtain a slightly better running time), eliminating the dependency of enumeration on the input size. The representative set is identified via matroid basis minimization, which can be solved by a simple greedy algorithm (see Section 3).

**Organization.** In Section 2 we give some definitions and notation. Section 3 presents our approximation scheme, EPTAS, and its analysis. In Section 4 we give a proof of correctness for algorithm FindRep that is used as a subroutine by the scheme. We conclude in Section 5 with a summary and open problems.

## 2 Preliminaries

For simplicity of the notation, for any set $A$ and an element $e$, we use $A + e$ and $A - e$ to denote $A \cup \{e\}$ and $A \setminus \{e\}$, respectively. Also, for any $k \in \mathbb{R}$ let $[k] = \{1, 2, \ldots, [k]\}$. Finally, for a function $f : A \rightarrow B$ and a subset of elements $C \subseteq A$, we define $f(C) = \sum_{e \in C} f(e)$.

### 2.1 Matroids

Let $E$ be a finite ground set and $\mathcal{I} \subseteq 2^E$ a non-empty set containing subsets of $E$ called the independent sets of $E$. Then, $\mathcal{M} = (E, \mathcal{I})$ is a matroid if the following hold:

1. (Hereditary Property) For all $A \in \mathcal{I}$ and $B \subseteq A$, it holds that $B \in \mathcal{I}$.
2. (Exchange Property) For any $A, B \in \mathcal{I}$ where $|A| > |B|$, there is $e \in A \setminus B$ such that $B + e \in \mathcal{I}$.

The next observation follows by repeatedly applying the exchange property.

**Observation 2.1.** Given a matroid $(E, \mathcal{I})$ and $A, B \in \mathcal{I}$, there is $D \subseteq A \setminus B$, $|D| = \max\{|A| - |B|, 0\}$ such that $B \cup D \in \mathcal{I}$.

A basis of a matroid $\mathcal{M} = (E, \mathcal{I})$ is an independent set $B \in \mathcal{I}$ such that for all $e \in E \setminus B$ it holds that $B + e \notin \mathcal{I}$. Given a cost function $c : E \rightarrow \mathbb{R}^+$, we say that a basis $B$ of $\mathcal{M}$ is a minimum basis of $\mathcal{M}$ w.r.t. $c$ if, for any basis $A$ of $\mathcal{M}$ it holds that $c(B) \leq c(A)$. A minimum basis of $\mathcal{M}$ w.r.t. $c$ can be easily constructed in polynomial-time using the greedy approach (see, e.g., [5]). In the following we define several operations on matroids that will be useful for deriving our results.

**Definition 2.2.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid.

1. (restriction) For every $F \subseteq E$ define $\mathcal{I} \cap F = \{A \in \mathcal{I} \mid A \subseteq F\}$ and $\mathcal{M} \cap F = (F, \mathcal{I} \cap F)$.
2. (contraction) For every $F \in \mathcal{I}$ define $\mathcal{I} / F = \{A \subseteq E \setminus F \mid A \cup F \in \mathcal{I}\}$ and $\mathcal{M} / F = (E \setminus F, \mathcal{I} / F)$.
3. (truncation) For every $q \in \mathbb{N}$ define $\mathcal{I} \leq q = \{A \in \mathcal{I} \mid |A| \leq q\}$ and $\mathcal{M} \leq q = (E, \mathcal{I} \leq q)$.
4. (union) Let $M_1 = (E_1, I_1), \ldots, (E_k, I_k)$ be matroids; define $\bigcup_{i \in [k]} M_i = (E_{\bigcup_{i \in [k]} I_i}, I_{\bigcup_{i \in [k]} I_i})$, where $E_{\bigcup_{i \in [k]} I_i} = \bigcup_{i \in [k]} E_i$ and $I_{\bigcup_{i \in [k]} I_i} = \left\{ \bigcup_{i \in [k]} F_i \mid \forall i \in [k] : F_i \in I_i \right\}$.

The next lemma gathers known results which follow directly from the definition of a matroid (see, e.g., [21]).
Lemma 2.3. Let $M = (E, I)$ be a matroid.

1. For any $F \subseteq E$, the restriction of $M$ to $F$ (i.e., $M \cap F$) is a matroid.  
2. For any $F \in I$, the contraction of $M$ by $F$ (i.e., $M/F$) is a matroid.  
3. For any $q \in \mathbb{N}$, the truncation of $M$ (i.e., $[M]_{\le q}$) is a matroid.  
4. Given matroids $M_1 = (E_1, I_1), \ldots, M_k = (E_k, I_k)$, the union of $M_1, \ldots, M_k$ (i.e., $\bigvee_{i \in [k]} M_i$) is a matroid.

2.2 Matroid polytope

Let $M = (E, I)$ be a matroid. Given $B \in I$, the indicator vector of $B$ is the vector $1^B \in \{0, 1\}^E$, where for all $a \in B$ and $b \in E \setminus B$ we have $1_a^B = 1$ and $1_b^B = 0$, respectively. Then the matroid polytope of $M$ is the convex hull of the set of indicator vectors of all independent sets of $M$: $P_M = \text{conv}\{1^B \mid B \in I\}$.

The next observation will be used in the analysis of our scheme.

Observation 2.4. Let $M = (E, I)$ be a matroid, and $\bar{x} \in P_M$. Then $\{e \in E \mid \bar{x}_e = 1\} \in I$.

3 The Algorithm

In this section we present an EPTAS for BMI. Our scheme initially handles elements of high profits. This is done by finding a small representative set out of which the scheme selects the most profitable elements in the solution. More elements, of lower profits, are then added to the solution using a linear program. For the remainder of this section, fix a BMI instance $K = (E, I, c, p, \beta)$ and a parameter $0 < \varepsilon < 1/2$. W.l.o.g., for all $e \in E$ we assume that $\{e\} \in I$ (otherwise $e$ cannot belong to any solution for $K$).

Let $H(K, \varepsilon) = \{\ell \in E \mid p(\ell) > \varepsilon \cdot \text{OPT}(K)\}$ be the set of profitable elements in $K$, and $E \setminus H(K, \varepsilon)$ the set of non-profitable elements; when understood from the context, we simply use $H = H(K, \varepsilon)$. We can easily obtain a PTAS by enumerating over all subsets of at most $\varepsilon^{-1}$ profitable elements to find the profitable elements in the solution. However, such exhaustive search is done in time $\Omega\left(|K|^{\varepsilon^{-1}}\right)$, which cannot lead to an EPTAS. Thus, we take a different approach.

A key observation is that efficient solution can be obtained without enumerating over all subsets of profitable elements. Instead, we limit our scheme to a smaller search space using the notions of replacements and representative sets. Let $S$ be an independent set having a bounded number of elements. A replacement of $S$ is a subset of elements which can replace the profitable elements in $S$, resulting with an independent set of lower cost and almost the same profit. A representative set $R$ is a subset of elements which contains a replacement within $R$ for any independent set with bounded number of elements. Definitions 3.1 and 3.2 give the formal properties of replacements and representative sets, respectively. Let $q(\varepsilon) = \varepsilon^{-\varepsilon^{-1}}$, and recall that $I_{\le q(\varepsilon)} = \{A \in I \mid |A| \le q(\varepsilon)\}$ (the selection of value for $q(\varepsilon)$ becomes clear in the proof of Lemma 3.3).

Definition 3.1. Given a BMI instance $K = (E, I, c, p, \beta), 0 < \varepsilon < \frac{1}{2}$, $G \in I_{\le q(\varepsilon)}$, and $Z_G \subseteq E$, we say that $Z_G$ is a replacement of $G$ for $K$ and $\varepsilon$ if the following holds:

1. $(G \setminus H) \cup Z_G \in I_{\le q(\varepsilon)}$.
2. $c(Z_G) \le c(G \cap H)$.
3. $p((G \setminus H) \cup Z_G) \ge (1 - \varepsilon) \cdot p(G)$.
4. $|Z_G| \le |G \cap H|$.

Definition 3.2. Given a BMI instance $K = (E, I, c, p, \beta), 0 < \varepsilon < \frac{1}{2}$ and $R \subseteq E$, we say that $R$ is a representative set of $K$ and $\varepsilon$ if, for any $G \in I_{\le q(\varepsilon)}$, there is a replacement $Z_G \subseteq R$ of $G$ for $K$ and $\varepsilon$.

In particular, observe that for any solution $S$ of $K$ we have that $S \cap H$ is a replacement of $S$; also, $E$ is a representative set. In the next lemma we show that there exists an almost optimal solution in which all profitable elements belong to a given representative set. Hence, guessing the profitable elements only requires enumerating over subsets of a representative set.
Lemma 3.3. Let $K = (E, \mathcal{I}, c, p, \beta)$ be a BMI instance and $0 < \varepsilon < \frac{1}{2}$. Also, let $R$ be a representatives set of $K$ and $\varepsilon$. Then, there is a solution $S$ of $K$ such that the following holds.

1. $S \cap H \subseteq R$.
2. $p(S) \geq (1 - 3\varepsilon) \text{OPT}(K)$.

We give a brief outline of the proof of Lemma 3.3. Informally, we consider the elements in some optimal solution, OPT, in non-increasing order by profit. We then partition these elements into three sets: $L, J^*$, and $Q$, such that the maximal profit in $Q$ is at most $\varepsilon$ times the minimum profit in $L$. This is the profit gap of $L$ and $Q$. In addition, $L \in \mathcal{I}_{\leq q(\varepsilon)}$, and $p(J^*) \leq \varepsilon \cdot \text{OPT}(K)$. Thus, we can use $Z_L \subseteq R$ as a replacement of $L$, i.e., $L \cap H$ will be replaced by elements in $Z_L$ (note that $Z_L$ is not necessarily a subset of the profitable elements). We now discard $J^*$, and define $\Delta_L = (L \setminus H) \cup Z_L$. As $\Delta_L \cup Q$ may not be an independent set, we use Observation 2.1 to construct $T \subseteq Q, |T| \geq |Q| - |L|$ such that $S = \Delta_L \cup T \in \mathcal{I}$. An illustration of the proof is given in Figure 1.

![Figure 1: The construction of the solution $S$ in the proof of Lemma 3.3.](image)

Proof of Lemma 3.3: With a slight abuse of notation, we use OPT also to denote the profit of an optimal solution for $K$. Given an optimal solution, we partition a subset of the elements in the solution into $\varepsilon^{-1}$ disjoint sets (some sets may be empty). Specifically, let $N = \lceil\varepsilon^{-1}\rceil$; for all $i \in [N]$ define

$$J_i = \{ e \in \text{OPT} \mid p(e) \in (\varepsilon^i \cdot \text{OPT}(K), \varepsilon^{i-1} \cdot \text{OPT}(K)) \}.$$  \hspace{1cm} (1)

Let $i^* = \arg \min_{i \in [N]} p(J_i)$. By (1) we have at least $\varepsilon^{-1}$ disjoint sets; thus, $p(J_{i^*}) \leq \varepsilon \cdot \text{OPT}(K)$. Now, let $L = \bigcup_{k \in i^*[1]} J_k$ be the subset of all elements in $\text{OPT}$ of profits greater than $\varepsilon^{i^*-1} \cdot \text{OPT}(K)$, and $Q = \text{OPT} \setminus (L \cup J_{i^*})$. To complete the proof of the lemma, we need several claims.

Claim 3.4. $L \in \mathcal{I}_{\leq q(\varepsilon)}$.

Proof. Since $L \subseteq \text{OPT}$, by the hereditary property of $(E, \mathcal{I})$ we have that $L \in \mathcal{I}$. Also,

$$|L| \leq \sum_{e \in L} \frac{p(e)}{\varepsilon^{i^*-1} \cdot \text{OPT}(K)} = \frac{p(L)}{\varepsilon^{i^*-1} \cdot \text{OPT}(K)} \leq \varepsilon^{-i^*-1} \leq \varepsilon^{-N+1} \leq \varepsilon^{-\varepsilon^{-1}} = q(\varepsilon)$$

The first inequality holds since $p(e) \geq \varepsilon^{i^*-1} \cdot \text{OPT}$ for all $e \in L$. For the second inequality, we note that $L$ is a solution for $K$. By the above and Definition 2.2, it follows that $L \in \mathcal{I}_{\leq q(\varepsilon)}$. By Claim 3.4 and as $R$ is a representative set, it follows that $L$ has a replacement $Z_L \subseteq R$. Let $\Delta_L = (L \setminus H) \cup Z_L$. By Property 1 of Definition 3.1, we have that $\Delta_L \in \mathcal{I}_{\leq q(\varepsilon)}$, and by Definition 2.2
it holds that $I_{\leq q(e)} \subseteq I$. Hence, $\Delta_L \in I$. Furthermore, as $Q \subseteq \text{OPT} \in I$, by the hereditary property for $(E, I)$ we have that $Q \in I$. Therefore, by Observation 2.1, there is a subset $T \subseteq Q \setminus \Delta_L$, where $|T| = \max\{|Q| - |\Delta_L|, 0\}$, such that $\Delta_L \cup T \in I$.

Let $S = \Delta_L \cup T$. We show that $S$ satisfies the conditions of the lemma.

**Claim 3.5.** $S$ is a solution for $K$.

*Proof.* By the definition of $T$ it holds that $S = \Delta_L \cup T \in I$. Moreover,

\[
c(S) = c(\Delta_L \cup T) \leq c(Z_L) + c(L \setminus H) + c(T) \leq c(L \cap H) + c(L \setminus H) + c(T) \leq c(L) + c(Q) \leq c(\text{OPT}) \leq \beta.
\]

The second inequality holds since $c(Z_L) \leq c(L \cap H)$ (see Property 2 of Definition 3.1). For the third inequality, recall that $T \subseteq Q$. The last inequality holds since $\text{OPT}$ is a solution for $K$. □

The proof of the next claim relies on the profit gap between the elements in $Q$ and $L$.

**Claim 3.6.** $p(Q \setminus T) \leq \varepsilon \cdot \text{OPT}(K)$.

*Proof.* Observe that

\[
|Q \setminus T| \leq |\Delta_L| \leq |Z_L| + |L \setminus H| \leq |L \cap H| + |L \setminus H| \leq |L|.
\]  

The first inequality follows from the definition of $T$. For the third inequality, we use Property 4 of Definition 3.1. Hence,

\[
p(Q \setminus T) \leq |Q \setminus T| \cdot \varepsilon^* \cdot \text{OPT}(K) \leq |L| \cdot \varepsilon^* \cdot \text{OPT}(K) \leq \varepsilon \cdot p(L) \leq \varepsilon \cdot \text{OPT}(K).
\]

The first inequality holds since $p(\varepsilon) \leq \varepsilon^* \cdot \text{OPT}(K)$ for all $e \in Q$. The second inequality is by (2). The third inequality holds since $p(\varepsilon) > \varepsilon^* - 1 \cdot \text{OPT}(K)$ for all $e \in L$. □

**Claim 3.7.** $p(S) \geq (1 - 3\varepsilon)\text{OPT}(K)$.

*Proof.* By Property 3 of Definition 3.1,

\[
p(\Delta_L) = p((L \setminus H) \cup Z_L) \geq (1 - \varepsilon) \cdot p(L).
\]  

Moreover,

\[
p(T) \geq p(Q) - p(Q \setminus T) \geq p(Q) - \varepsilon \cdot \text{OPT}(K) \geq (1 - \varepsilon) \cdot p(Q) - \varepsilon \cdot \text{OPT}(K).
\]  

The second inequality is by Claim 3.6. Using (3) and (4), we have that

\[
p(\Delta_L) + p(T) \geq (1 - \varepsilon) \cdot p(L \cup Q) - \varepsilon \cdot \text{OPT}(K) = (1 - \varepsilon) \cdot p(\text{OPT} \setminus J^*) - \varepsilon \cdot \text{OPT}(K) \geq (1 - 3\varepsilon) \cdot \text{OPT}(K).
\]

The last inequality holds since $p(J^*) \leq \varepsilon \cdot \text{OPT}(K)$. Observe that $S = \Delta_L \cup T$ and $T \cap \Delta_L = \emptyset$. Therefore,

\[
p(S) = p(\Delta_L) + p(T) \geq (1 - 3\varepsilon) \cdot \text{OPT}(K).
\]

Finally, we note that, by (1) and the definition of $Q$, $Q \cap H = \emptyset$. Consequently, by the definition of $S$, we have that $S \cap H \subseteq Z_L$. As $Z_L \subseteq R$ (by Definition 3.2), it follows that $S \cap H \subseteq R$. Hence, using Claims 3.5 and 3.7, we have the statement of the lemma. □
Our scheme for BMI constructs a representative set whose cardinality depends solely on \( \varepsilon \). To this end, we first partition the profitable elements (and possibly some more elements) into a small number of profit classes, where elements from the same profit class have similar profits. The profit classes are derived from a 2-approximation \( \alpha \) for \( \text{OPT}(K) \), which can be easily computed in polynomial time. Specifically, for all \( r \in [\log_{1-\varepsilon}(\frac{1}{2}) + 1] \) define the \( r \)-profit class as

\[
C_r(\alpha) = \left\{ e \in E \mid \frac{p(e)}{2 \cdot \alpha} \in \left( (1-\varepsilon)^r, (1-\varepsilon)^r - 1 \right] \right\}.
\]

For each \( r \in [\log_{1-\varepsilon}(\frac{1}{2}) + 1] \), we define \( [(E, I) \cap C_r(\alpha)]_{\leq q(\varepsilon)} \), the corresponding matroid for the \( r \)-profit class. We construct a representative set by computing a minimum basis w.r.t. the cost function \( c \), for the matroid defined as the union of the corresponding matroids of all profit classes. Note that by Lemma 2.3 the latter is a matroid. The pseudocode of algorithm \text{FindRep}, which outputs a representative set, is given in Algorithm 1.

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**Algorithm 1: FindRep**

1. Compute a 2-approximation \( S^* \) for \( K \) using a PTAS for BMI with parameter \( \varepsilon' = \frac{1}{2} \).
2. Set \( \alpha = p(S^*) \).
3. Return a minimum basis w.r.t. \( c \) of the matroid

\[
\bigvee_{r \in [\log_{1-\varepsilon}(\frac{1}{2}) + 1]} [(E, I) \cap C_r(\alpha)]_{\leq q(\varepsilon)}.
\]

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**Lemma 3.8.** Given a BMI instance \( K = (E, I, c, p, \beta, \varepsilon) \) and \( 0 < \varepsilon < \frac{1}{2} \), Algorithm 1 returns in time \( \text{poly}(|K|) \) a representative set \( R \) of \( K \) and \( \varepsilon \), such that \( |R| \leq q(\varepsilon) \cdot (\log_{1-\varepsilon}(\frac{1}{2}) + 1) \).

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In the following we outline the main ideas of the proof. Let \( R = \text{FindRep}(K, \varepsilon) \), and consider some \( G \in I_{\leq q(\varepsilon)} \) and \( e \in (G \cap H) \setminus R \). By (5) the element \( e \) belongs to some profit class \( C_r(\alpha), r \in [\log_{1-\varepsilon}(\frac{1}{2}) + 1] \). Since \( R \) is a minimum basis w.r.t. \( c \), we can use matroid properties to show that there is some \( b \in C_r(\alpha) \cap R \) of cost \( c(b) \leq c(e) \) such that \( G - a + b \in I_{\leq q(\varepsilon)} \). We now keep replacing elements in \((G \cap H) \setminus R\) in a similar manner, until no such element exists. Thus, we have a replacement of \( G \) within \( R \), i.e., \( R \) is a representative set. A formal proof of Lemma 3.8 is given in Section 4.

Our scheme uses the representative set to select profitable elements for the solution. Using a linear program, the solution is extended to include also non-profitable elements. As the exact set of non-profitable elements is unknown, we use an approximation for the optimal profit. Specifically, let \( \frac{\text{OPT}(K)}{2} \leq \alpha \leq \text{OPT}(K) \), and denote by \( E(\alpha) = \{ e \in E \mid p(e) \leq 2\varepsilon \cdot \alpha \} \) the set including the non-profitable elements, and possibly also profitable elements \( e \) such that \( p(e) \leq 2\varepsilon \cdot \text{OPT}(K) \).

The LP is based on the matroid polytope of the following matroid. Given a solution \( F \) for \( K \), we define \( \mathcal{M}_F(\alpha) = ((E, I)/F) \cap E(\alpha) \); in other words, \( \mathcal{M}_F(\alpha) = (E(\alpha), I_F(\alpha)) \), where \( I_F(\alpha) = \{ A \subseteq E(\alpha) \mid A \cup F \in I \} \). Note that \( \mathcal{M}_F(\alpha) \) is indeed a matroid, by Properties 1 and 2 of Lemma 2.3. The LP formulation is given by

\[
\begin{align*}
\text{LP}(K, F, \alpha) \quad & \max & \sum_{e \in E(\alpha) \setminus F} \bar{x}_e \cdot p(e) \\
\text{s.t.} & & \sum_{e \in E(\alpha) \setminus F} \bar{x}_e \cdot c(e) \leq \beta - c(F) \\
& & \bar{x} \in P_{\mathcal{M}_F(\alpha)}
\end{align*}
\]

The linear program \( \text{LP}(K, F, \alpha) \) maximizes the total profit of a point in the matroid polytope of \( \mathcal{M}_F(\alpha) \) (i.e., \( P_{\mathcal{M}_F(\alpha)} \)), such that the total cost of elements is at most \( \beta - c(F) \); that is, the residual budget after selecting for the solution the elements in \( F \).
Observation 3.9. Let $K = (E, I, c, p, \beta)$ be a BMI instance, $\text{OPT}(K)\leq \alpha \leq \text{OPT}(K)$, $S$ a solution for $K$, and $\bar{x}$ an optimal basic solution for $LP(K, S \cap H, \alpha)$. Then, $\sum_{e \in E(\alpha) \setminus (S \cap H)} \bar{x} e \cdot p(e) \geq p(S \setminus H)$.

It is folklore that a linear program such as $LP(K, F, \alpha)$ can be solved in polynomial-time in $|K|$. As we could not find a proper reference, we include the proof of the next lemma in the appendix.

**Lemma 3.10.** For any BMI instance $K = (E, I, c, p, \beta)$, $\frac{\text{OPT}(K)}{2} \leq \alpha \leq \text{OPT}(K)$, and a solution $F$ of $K$, a basic optimal solution of $LP(K, F, \alpha)$ can be found in time $\text{poly}(|K|)$.

The next lemma will be useful for deriving a solution of high profit using $LP(K, F, \alpha)$. The proof follows as a special case of a result of [9].

**Lemma 3.11.** Let $K = (E, I, c, p, \beta)$ be a BMI instance, $\frac{\text{OPT}(K)}{2} \leq \alpha \leq \text{OPT}(K)$, and $\bar{x}$ a basic solution for $LP(K, F, \alpha)$. Then $\bar{x}$ has at most two non-integral entries.

Using the above, we have the required components for an EPTAS for BMI. Let $R$ be the representative set returned by $\text{FindRep}(K, \varepsilon)$. For all solutions $F \subseteq R$ with $|F| \leq \varepsilon^{-1}$, we find a basic optimal solution $\bar{\lambda}^F$ for $LP(K, F, \alpha)$ and define $C_F = \{(e \in E(\alpha) \setminus F \mid \bar{\lambda}^F_e = 1) \cup F\}$ as the solution of $F$. Our scheme iterates over the solutions $C_F$ for all such subsets $F$ and chooses a solution $C_{F^*}$ of maximal total profit. The pseudocode of the scheme is given in Algorithm 2.

**Algorithm 2: EPTAS($K = (E, I, c, p, \beta), \varepsilon$)**

1. Construct the representative elements $R = \text{FindRep}(K, \varepsilon)$.
2. Compute a 2-approximation $S^*$ for $K$ using a PTAS for BMI with parameter $\varepsilon' = \frac{1}{2}$.
3. Set $\alpha = p(S^*)$.
4. Initialize an empty solution $A \leftarrow \emptyset$.
5. for $F \subseteq R$ s.t. $|F| \leq \varepsilon^{-1}$ and $F$ is a solution of $K$ do
   6. Find a basic optimal solution $\bar{\lambda}^F$ of $LP(K, F, \alpha)$.
   7. Let $C_F = \{(e \in E(\alpha) \setminus F \mid \bar{\lambda}^F_e = 1) \cup F\}$.
   8. if $p(C_F) > p(A)$ then
      9. Update $A \leftarrow C_F$
   10. end
11. end
12. Return $A$.

**Lemma 3.12.** Given a BMI instance $K = (E, I, c, p, \beta)$ and $0 < \varepsilon < \frac{1}{2}$, Algorithm 2 returns a solution for $K$ of profit at least $(1 - 7\varepsilon) \cdot \text{OPT}(K)$.

**Proof.** By Lemma 3.3, there is a solution $S$ for $K$ such that $S \cap H \subseteq R$, and $p(S) \geq (1 - 3\varepsilon) \text{OPT}(K)$. As for all $e \in S \cap H$ we have $p(e) \geq \varepsilon \cdot \text{OPT}(K)$, and $S$ is a solution for $K$, it follows that $|S \cap H| \leq \varepsilon^{-1}$.

We note that there is an iteration of Step 5 in which $F = S \cap H$; thus, in Step 6 we construct a basic optimal solution $\bar{\lambda}^{S \cap H}$ of $LP(K, S \cap H, \alpha)$. We use $X(A) = \{(e \in E(\alpha) \setminus A \mid \bar{\lambda}^A_e = 1\}$ for every basic solution $\bar{\lambda}^A$ computed in Step 6 for $A \subseteq R$. Then,

$$p(X(S \cap H)) = \sum_{e \in E(\alpha) \setminus (S \cap H)} p(e) \sum_{\bar{\lambda}^{S \cap H}_e = 1} \bar{\lambda}^{S \cap H}_e \cdot p(e) - 2 \cdot 2\varepsilon \cdot \text{OPT}(K)$$

$$\geq p(S \setminus H) - 4\varepsilon \cdot \text{OPT}(K).$$

The first inequality holds since, by Lemma 3.11, $|\{e \in E(\alpha) \setminus (S \cap H) \mid \bar{\lambda}^{S \cap H}_e \in (0, 1)\}| \leq 2$, and for all $e \in E(\alpha)$, $p(e) \leq 2\varepsilon \cdot \alpha \leq 2\varepsilon \cdot \text{OPT}(K)$. The second inequality follows from Observation 3.9. Now,

$$p(C_{S \cap H}) = p(S \cap H) + p(X(S \cap H)) \geq p(S) - 4\varepsilon \cdot \text{OPT}(K) \geq (1 - 7\varepsilon) \text{OPT}(K).$$

The first inequality uses (7). The last inequality is by Lemma 3.3.
Claim 3.13. $A = \text{EPTAS}(\mathcal{K}, \varepsilon)$ is a solution of $\mathcal{K}$.

Proof. If $A = \emptyset$ the claim trivially follows since $\emptyset$ is a solution of $\mathcal{K}$. Otherwise, by Step 9 of Algorithm 2, there is a solution $F$ of $\mathcal{K}$ such that $A = C_F$. By Observation 2.4, $X(F)$ is an independent set in the matroid $M_F(\alpha)$. Also, recall that $\mathcal{M}_F(\alpha) = ((E, \mathcal{I})/F) \cap E(\alpha)$. Hence, by Definition 2.2, we have that $X(F) \cup F = C_F \in \mathcal{I}$. Additionally,

$$c(C_F) = c(F) + \sum_{e \in X(F)} c(e) \leq c(F) + \beta - c(F) = \beta.$$ 

The inequality follows from (6). By the above, we conclude that $A$ is a solution for $\mathcal{K}$. \hfill \Box

By Claim 3.13, Steps 5, 9 and 12 of Algorithm 2 and (8), we have that $A = \text{EPTAS}(\mathcal{K}, \varepsilon)$ is a solution for $\mathcal{K}$ satisfying $p(A) \geq p(C_{S \cap H}) \geq (1 - 7\varepsilon)\text{OPT}(\mathcal{K})$. This completes the proof. \hfill \blacksquare

Lemma 3.14. Given a BMI instance $\mathcal{K} = (E, \mathcal{I}, c, p, \beta)$ and $0 < \varepsilon < \frac{1}{2}$, Algorithm 2 runs in time $2^O(\varepsilon^{-2} \log \frac{1}{\varepsilon}) \cdot \text{poly}(|\mathcal{K}|)$.

Proof. By Lemma 3.8, the time complexity of Step 1 is $\text{poly}(|\mathcal{K}|)$. Step 2 can be computed in time $\text{poly}(|\mathcal{K}|)$, by using a PTAS for BMI taking $\varepsilon = \frac{1}{2}$ (see, e.g., [9]). Now, using logarithm rules we have

$$\log_{1-\varepsilon} \left( \frac{\varepsilon}{2} \right) + 1 \leq \frac{\ln \left( \frac{\varepsilon}{2} \right)}{\ln (1 - \varepsilon)} + 1 \leq \frac{2\varepsilon^{-1}}{\varepsilon} + 1 \leq 3\varepsilon^{-2}. \tag{9}$$

The second inequality follows from $x < -\ln(1 - x), \forall x > -1, x \neq 0$, and $\ln y < y, \forall y > 0$. Therefore,

$$|R| \leq \left( \log_{1-\varepsilon} \left( \frac{\varepsilon}{2} \right) + 1 \right) \cdot q(\varepsilon) \leq 3\varepsilon^{-2} \cdot \varepsilon^{-\varepsilon^{-1}} \leq \varepsilon^{-3\varepsilon^{-1}}. \tag{10}$$

The first inequality is by Lemma 3.8. The second inequality is by (9). Let

$$W = \left\{ F \subseteq R \mid |F| \leq \varepsilon^{-1}, F \in \mathcal{I}, c(F) \leq \beta \right\}$$

be the set of solutions considered in Step 5 of Algorithm 2. Then,

$$|W| \leq (|R| + 1)^{\varepsilon^{-1}} \leq (\varepsilon^{-3\varepsilon^{-1}} + 1)^{\varepsilon^{-1}} \leq (\varepsilon^{-4\varepsilon^{-2}}) = 2^O(\varepsilon^{-2} \log \frac{1}{\varepsilon}). \tag{11}$$

The second inequality is by (10). Hence, by (11), the number of iterations of the for loop in Step 5 is bounded by $2^O(\varepsilon^{-2} \log \frac{1}{\varepsilon})$. In addition, by Lemma 3.10, the running time of each iteration is $\text{poly}(|\mathcal{K}|)$. By the above, the running time of Algorithm 2 is $2^O(\varepsilon^{-2} \log \frac{1}{\varepsilon}) \cdot \text{poly}(|\mathcal{K}|)$. \hfill \blacksquare

Proof of Theorem 1.1: Given a BMI instance $\mathcal{K}$ and $0 < \varepsilon < \frac{1}{2}$, using Algorithm 2 for $\mathcal{K}$ with parameter $\frac{1}{2}$ we have the desired approximation guarantee by Lemma 3.12. Also, by Lemma 3.14, the running time is $2^O(\varepsilon^{-2} \log \frac{1}{\varepsilon}) \cdot \text{poly}(|\mathcal{K}|)$.

4 Correctness of FindRep

In this section we give the proof of Lemma 3.8. The proof is based on substitution of subsets by profit classes. A substitution is closely related to replacement (see Definition 3.1); indeed, we can construct replacements for independent sets (and therefore a representative set) by specific substitutions. We start by stating several lemmas that will be used in the proof of Lemma 3.8. The first lemma refers to a generalized exchange property of matroids.

Lemma 4.1. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid, $A, B \in \mathcal{I}$, and $a \in A \setminus B$ such that $B + a \notin \mathcal{I}$. Then there is $b \in B \setminus A$ such that $A - a + b \in \mathcal{I}$.
Lemma 4.2. Given a matroid $M = (E, I)$ and a weight function $w : E \to \mathbb{R}_{\geq 0}$, let $B$ be a minimum basis of $M$ w.r.t. $w$. Then, for any $a \in E \setminus B$ it holds that $\{e \in B \mid w(e) \leq w(a)\} \neq \emptyset$.

Proof. Let $D = \{e \in B \mid w(e) \leq w(a)\}$. Assume towards contradiction that $D + a \in I$. Then, by Observation 2.1 there is $C \subseteq B \setminus (D + a), |C| = \max\{|B| - |D + a|, 0\}$ such that $(D + a) \cup C \in I$; let $F = (D + a) \cup C$. Note that $|F| = |D + a| + |C| \geq |D + a| + |B| - |D + a| = |B|$. Hence, as $|F| \geq |B|$ and $B$ is a basis of $M$, we have that $F$ is a basis of $M$ (since all bases have the same cardinality; see, e.g., (39.2) in [21]).

By (13), we have that $F$ is a basis of $M$ satisfying $w(F) < w(B)$. Contradiction (to the minimality of $B$ w.r.t. $w$).

Lemma 4.3. Let $M_1 = (E_1, I_1), \ldots, M_k = (E_k, I_k)$ be matroids such that $E_i \cap E_j = \emptyset \forall i, j \in [k], i \neq j$; also, let $w : E \to \mathbb{R}_{\geq 0}$ be a weight function, and $R$ a minimum basis of $\bigvee_{i \in [k]} M_i$ w.r.t. $w$. Then, for all $i \in [k]$, $R \cap E_i$ is a minimum basis of $M_i$ w.r.t. $w$.

Proof. Let $i \in [k]$. For all $e \in E_i \setminus R$, it holds that $R \cap E_i + e$ is not an independent set of $M_i$ since otherwise $R + e$ is an independent set of $\bigvee_{i \in [k]} M_i$ by Definition 2.2, contradicting that $R$ is a basis of $\bigvee_{i \in [k]} M_i$; we conclude that $R \cap E_i$ is a basis of $M_i$. Assume towards contradiction that there is a basis $B$ of the matroid $M_i$ such that $w(B) < w(R \cap E_i)$. As $E_i \cap E_j = \emptyset \forall i, j \in [k], i \neq j$, by Definition 2.2 it follows that $(R \setminus E_i) \cup B$ is a basis of the matroid $\bigvee_{i \in [k]} M_i$. In addition,

$$w((R \setminus E_i) \cup B) = w(R) - w(R \cap E_i) + w(B) < w(R).$$

We reach a contradiction since $R$ is a minimum basis w.r.t. $w$ of $\bigvee_{i \in [k]} M_i$.

We now prove Lemma 3.8 using several auxiliary lemmas. For the remainder of this section, let $K = (E, I, c, p, \beta)$ be a BMI instance, $0 < \varepsilon < \frac{1}{2}$, $R = \text{FindRep}(K, \varepsilon)$, and $\alpha$ the value from Step 2 of Algorithm 1. For the next lemma, recall the sets $C_r(\alpha)$ for $r \in [\log_{1-\varepsilon}(\frac{\varepsilon}{2}) + 1]$ were defined in (5).

Lemma 4.4. For all $r \in [\log_{1-\varepsilon}(\frac{\varepsilon}{2}) + 1]$ it holds that $R \cap C_r(\alpha)$ is a minimum basis w.r.t. $c$ of the matroid $M = [(E, I) \cap C_r(\alpha)]_{\leq q(\varepsilon)}$, where $R = \text{FindRep}(K, \varepsilon)$.

Proof. Observe that for all $r, t \in [\log_{1-\varepsilon}(\frac{\varepsilon}{2}) + 1]$ it holds that $C_r(\alpha) \cap C_t(\alpha) = \emptyset$ by (5). Hence, the statement of the lemma follows from Step 3 of Algorithm 1 and Lemma 4.3.

Lemma 4.5. For all $e \in H$ there is exactly one $r \in [\log_{1-\varepsilon}(\frac{\varepsilon}{2}) + 1]$ such that $e \in C_r(\alpha)$.

1With a slight abuse of notation we use $w$ also for the restriction of $w$ on $E_i$. 

10
Proof. Let \( e \in H \). Observe that:
\[
\frac{\varepsilon}{2} \leq \frac{p(e)}{2 \cdot \text{OPT}(K)} \leq \frac{p(e)}{2 \cdot \alpha} \leq \frac{p(e)}{\text{OPT}(K)} \leq 1.
\] (14)
The first inequality holds since \( e \in H \). The second and third inequalities are because \( \alpha \) is the value of a 2-approximation for the optimum of \( K \); thus, \( \frac{\text{OPT}(K)}{2} \leq \alpha \leq \text{OPT}(K) \). The last inequality is because \( \{e\} \) is a solution of \( K \). In addition, observe that for \( r_0 = \log_{1 - \varepsilon} \frac{1}{2} \) it holds that \( (1 - \varepsilon)^{r_0} \leq \frac{1}{2} \) and that for \( r_1 = 1 \) it holds that \((1 - \varepsilon)^{r_1 - 1} = 1 \). Hence, because \( r_0 \leq \log_{1 - \varepsilon} \left( \frac{1}{2} \right) + 1 \), by (14), there is exactly one \( r \in \left[ \log_{1 - \varepsilon} \left( \frac{1}{2} \right) + 1 \right] \) such that \( \frac{p(e)}{2^r} \in \left( (1 - \varepsilon)^r, (1 - \varepsilon)^{r-1} \right] \); thus, by (5) it holds that \( e \in C_r(\alpha) \) and \( e \notin C_r'(\alpha) \) for \( r' \in \left[ \log_{1 - \varepsilon} \left( \frac{1}{2} \right) + 1 \right] \setminus \{r\} \). \( \square \)

For the proof of Lemma 3.8, we define a substitution of some independent set. The first two properties of substitution of some \( G \in \mathcal{I}_{\leq q}(e) \) are identical to the first two properties in the definition of a replacement of \( G \). However, we require that a substitution preserves the number of profitable elements in \( G \) from each profit class, and that a substitution must be disjoint to the set of non-profitable elements in \( G \).

Definition 4.6. For \( G \in \mathcal{I}_{\leq q}(e) \) and \( Z_G \subseteq \bigcup_{r \in \left[ \log_{1 - \varepsilon} \left( \frac{1}{2} \right) + 1 \right]} C_r(\alpha) \), we say that \( Z_G \) is a substitution of \( G \) if the following holds:

1. \( (G \setminus H) \cup Z_G \in \mathcal{I}_{\leq q}(e) \).
2. \( c(Z_G) \leq c(G \cap H) \).
3. For all \( r \in \left[ \log_{1 - \varepsilon} \left( \frac{1}{2} \right) + 1 \right] \) it holds that \( |C_r(\alpha) \cap Z_G| = |C_r(\alpha) \cap G \cap H| \).
4. \( (G \setminus H) \cap Z_G = \emptyset \).

Lemma 4.7. For all \( G \in \mathcal{I}_{\leq q}(e) \) there is a substitution \( Z_G \) of \( G \) such that \( Z_G \subseteq R \).

In the proof of Lemma 4.7, we assume towards a contradiction that there is no substitution for some \( G \in \mathcal{I}_{\leq q}(e) \) which is a subset of \( R \). To reach a contradiction, we take a substitution \( Z_G \) of \( G \) with maximal number of elements from \( R \) and show that using Lemma 4.1 and Lemma 4.2 we can create a substitution with more elements from \( R \) by replacing an element from \( Z_G \setminus R \) by an element from \( R \setminus Z_G \).

Proof of Lemma 4.7. Let \( G \in \mathcal{I}_{\leq q}(e) \) and let \( Z_G \) be a substitution of \( G \) such that \( |Z_G \cap R| \) is maximal among all substitutions of \( G \); formally, let \( S(G) \) be all substitutions of \( G \) and let \( Z_G \in \{ Z \in S(G) \mid |Z \cap R| = \max \{Z' \in S(G) \mid Z' \cap R \} \} \). Since \( G \cap H \) is in particular a substitution of \( G \) it follows that \( S(G) \neq \emptyset \), and thus \( Z_G \) is well defined. Assume towards a contradiction that there is \( a \in Z_G \setminus R \); then, by Definition 4.6 there is \( r \in \left[ \log_{1 - \varepsilon} \left( \frac{1}{2} \right) + 1 \right] \) such that \( a \in C_r(\alpha) \). Let \( \Delta_G = (G \setminus H) \cup Z_G \).

Claim 4.8. There is \( b \in (C_r(\alpha) \cap R) \setminus \Delta_G \) such that \( c(b) \leq c(a) \) and \( \Delta_G - a + b \in \mathcal{I}_{\leq q}(e) \).

Proof. Let \( (C_r(\alpha), T') = \{(E, T) \cap C_r(\alpha)\}_{\leq q}(e) \) and \( M = (\overline{C_r(\alpha)}, T') \). By Lemma 4.4 it holds that \( R \cap C_r(\alpha) \) is a minimum basis w.r.t. \( c \) of \( M \). Define \( D = \{e \in C_r(\alpha) \cap R \mid c(e) \leq c(a) \} \). Then, since \( a \in C_r(\alpha) \setminus R \), it holds that \( D + a \notin T' \) by Lemma 4.2. In addition, by Definition 2.2 it holds that \( T' = \{A \subseteq C_r(\alpha) \mid A \in \mathcal{I}_{\leq q}(e)\} \). Hence, \( D + a \notin \mathcal{I}_{\leq q}(e) \). Therefore, by Lemma 4.1 there is \( b \in D \setminus \Delta_G \) such that \( \Delta_G - a + b \in \mathcal{I}_{\leq q}(e) \). The claim follows since \( c(b) \leq c(a) \) because \( b \in D \). \( \square \)

Using Claim 4.8, let \( b \in C_r(\alpha) \cap R \setminus \Delta_G \) such that \( c(b) \leq c(a) \) and \( \Delta_G - a + b \in \mathcal{I}_{\leq q}(e) \). Then, the properties of Definition 4.6 are satisfied for \( Z_G - a + b \) by the following:

1. \( (G \setminus H) \cup (Z_G - a + b) = \Delta_G - a + b \in \mathcal{I}_{\leq q}(e) \) by the definition of \( b \).
2. \( c(Z_G - a + b) \leq c(Z_G) \leq c(G \cap H) \) because \( c(b) \leq c(a) \).
3. for all $r' \in [\log_{1-\epsilon}(\frac{1}{2}) + 1]$ it holds that $|C_{r'}(\alpha) \cap (Z_G - a + b)| = |C_{r'}(\alpha) \cap Z_G| = |C_{r'}(\alpha) \cap G \cap H|$ because $a, b \in C_{r'}(\alpha)$.

4. $|(G \setminus H) \cap (Z_G - a + b)| \leq |(G \setminus H) \cap (Z_G)| = 0$ where the inequality follows because $b \notin \Delta_G$ and the equality is since $Z_G$ is a substitution of $G$.

By the above and Definition 4.6, it follows that $Z_G + a - b$ is a substitution of $G$; that is, $Z_G + a - b \in S(G)$. Moreover,

$$|R \cap (Z_G - a + b)| > |R \cap Z_G| = \max_{Z \in S(G)} |Z \cap R|. \quad (15)$$

The first inequality is because $a \in Z_G \setminus R$ and $b \in R$. By (15) we reach a contradiction since we found a replacement of $G$ with more elements in $R$ than $Z_G \in S(G)$, which is defined as a replacement of $G$ with maximal number of elements in $R$. Therefore, $Z_G \subseteq R$ as required.

We are now ready to prove Lemma 3.8. The proof follows by showing that for any $G \in I_{\leq q}(\epsilon)$ a substitution of $G$ which is a subset of $R$ is in fact a replacement of $G$.

**Proof of Lemma 3.8:** Let $G \in I_{\leq q}(\epsilon)$. By Lemma 4.7, $G$ has a substitution $Z_G \subseteq R$. Then,

$$p(Z_G) \geq \sum_{r \in [\log_{1-\epsilon}(\frac{1}{2}) + 1]} p(C_{r}(\alpha) \cap Z_G)$$

$$\geq \sum_{r \in [\log_{1-\epsilon}(\frac{1}{2}) + 1] \text{ s.t. } C_r(\alpha) \neq \emptyset} |C_{r}(\alpha) \cap Z_G| \cdot \min_{e \in C_{r}(\alpha)} p(e) \quad (16)$$

$$\geq \sum_{r \in [\log_{1-\epsilon}(\frac{1}{2}) + 1] \text{ s.t. } C_r(\alpha) \neq \emptyset} |C_{r}(\alpha) \cap G \cap H| \cdot (1 - \epsilon) \cdot \max_{e \in C_{r}(\alpha)} p(e)$$

$$\geq (1 - \epsilon) \cdot p(G \cap H).$$

The third inequality is by (5) and Property 3 of Definition 4.6. The last inequality follows from Lemma 4.5. Therefore,

$$p((G \setminus H) \cup Z_G) = p(G \setminus H) + p(Z_G)$$

$$\geq p(G \setminus H) + (1 - \epsilon) \cdot p(G \cap H) \quad (17)$$

$$\geq (1 - \epsilon) \cdot p(G).$$

The first equality follows by Property 4 of Definition 4.6. The first inequality is by (16). Now, $Z_G$ satisfies Property 1 and 2 of Definition 3.1 by Properties 1, 2 of Definition 4.6, respectively. In addition, $Z_G$ satisfies Property 3 of Definition 3.1 by (17). Finally, $Z_G$ satisfies Property 4 of Definition 3.1 by

$$|Z_G| = \sum_{r \in [\log_{1-\epsilon}(\frac{1}{2}) + 1]} |Z_G \cap C_r(\alpha)| \leq \sum_{r \in [\log_{1-\epsilon}(\frac{1}{2}) + 1]} |G \cap H \cap C_r(\alpha)| = |G \cap H|.$$}

The first inequality holds since $Z_G$ is a substitution of $G$. The last equality follows from Lemma 4.5. We conclude that $Z_G$ is a replacement of $G$ such that $Z_G \subseteq R$; thus, $R$ is a representative set by Definition 3.2. By Step 3 of Algorithm FindRep and Definition 2.2, it holds that $|R \cap C_r(\alpha)| \leq q(\epsilon)$; thus, it follows that $|R| \leq (\log_{1-\epsilon}(\frac{1}{2}) + 1) \cdot q(\epsilon)$.

We now bound the time complexity of Algorithm FindRep. Step 1 can be computed in time $\text{poly}(|\mathcal{K}|)$ using a PTAS for BMI with an error parameter $\epsilon = \frac{1}{2}$ (see, e.g., [9]). In addition, a membership oracle for the matroid in Step 3 can be implemented in time $\text{poly}(|\mathcal{K}|)$ using a greedy matroid basis minimization algorithm [5]. Hence, the running time of the algorithm is $\text{poly}(|\mathcal{K}|)$.

## 5 Discussion

In this paper we showed that the budgeted matroid independent set problem admits an EPTAS, thus improving upon the previously known schemes for the problem. The speed-up is achieved by replacing
the exhaustive enumeration used by existing algorithms [9, 1, 4, 7] with efficient enumeration over subsets of a representative set whose size depends only on $1/\varepsilon$.

The representative set found by our algorithm is a minimum cost matroid basis for a union matroid. The union matroid is a union of a matroid for each profit class in the given instance. The basis itself can be easily found using a simple greedy procedure. The correctness relies on matroid exchange properties, optimality properties of minimal cost bases and a “profit gap” obtained by discarding a subset of elements from an optimal solution.

We note that our EPTAS, which directly exploits structural properties of our problem, already achieves substantial improvement over schemes developed for generalizations of BMI. Furthermore, we almost resolve the complexity status of the problem w.r.t approximation schemes. The existence of an FPTAS remains open.

The notion of representative sets can potentially be used to replace exhaustive enumeration in the PTASs for multiple knapsack with matroid constraint [7] and budgeted matroid intersection [1], leading to EPTASs for both problems. It seems that the construction of a representative set, as well as the ideas behind the main lemmas, can be adapted to the setting of the multiple knapsack with matroid problem. However, to derive an EPTAS for budgeted matroid intersection, one needs to generalize first the concept of representative set, so it can be applied to matroid intersection. We leave this generalization as an interesting direction for follow-up work.

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A Solving the Linear Program

In this section we show how LP($\mathcal{K}, F, \alpha$) can be solved in polynomial time, thus proving Lemma 3.10. The main idea is to show that the feasibility domain of LP($\mathcal{K}, F, \alpha$) is a well described polytope, and that a separation oracle for it can be implemented in polynomial time. Thus, using a known connection between separation and optimization, we obtain a polynomial-time algorithm which solves LP($\mathcal{K}, F, \alpha$). Our initial goal, however, is to show that LP($\mathcal{K}, F, \alpha$) is indeed a linear program, for which we need a characterization of the matroid polytope via linear inequalities.

Let $\mathcal{N} = (\Omega, J)$ be a matroid. The matroid rank function of $\mathcal{N}$, $\text{rank}_\mathcal{N} : 2^\Omega \to \mathbb{N}$, is defined by $\text{rank}_\mathcal{N}(S) = \max \{|T| \mid T \in J, T \subseteq S\}$. That is, $\text{rank}_\mathcal{N}(S)$ is the maximal size of an independent set which is also a subset of $S$. The rank function is used in a characterization of a matroid polytope $P_\mathcal{N}$ via a system of linear inequalities.

**Theorem A.1** (Corollary 40.2b in [21]). For any matroid $\mathcal{N} = (\Omega, J)$, it holds that

$$P_\mathcal{N} = \left\{ \bar{x} \in \mathbb{R}_{\geq 0} \mid \forall S \subseteq \Omega : \sum_{\omega \in S} \bar{x}_\omega \leq \text{rank}_\mathcal{N}(S) \right\}.$$
Let \( K = (E, I, c, p, \beta) \) be a BMI instance, \( F \subseteq E \) and \( \frac{\text{OPT}(K)}{2} \leq \alpha \leq \text{OPT}(K) \). By Theorem A.1 it holds that (6) is equivalent to the following linear program.

\[
\text{LP}(K, F, \alpha) \quad \text{max} \quad \sum_{e \in E(\alpha) \setminus F} \bar{x}_e \cdot p(e) \\
\text{s.t.} \quad \sum_{e \in E(\alpha) \setminus F} \bar{x}_e \cdot c(e) \leq \beta - c(F) \\
\sum_{e \in S} \bar{x}_e \leq \text{rank}_{\mathcal{M}_F(\alpha)}(S) \quad \forall S \subseteq E(\alpha) \setminus F \\
\bar{x}_e \geq 0 \quad \forall e \in E(\alpha) \setminus F
\] (18)

We follow the definitions and techniques presented in [10]. We say a polytope \( P \subseteq \mathbb{R}^n \) is of facet complexity \( \varphi \) if it is the solution set of a system of linear inequalities with rational coefficients, and the encoding length of each inequality in the system is at most \( \varphi \). The following is an immediate consequence of the representation of \( \text{LP}(K, F, \alpha) \) as a linear program.

**Observation A.2.** The feasibility region of \( \text{LP}(K, F, \alpha) \) is of facet complexity polynomial in \( |K| \).

A separating hyperplane between a polytope \( P \subseteq \mathbb{R}^n \) and \( \bar{x} \in \mathbb{R}^n \) is vector \( \bar{c} \in \mathbb{R}^n \) such that \( \bar{c} \cdot \bar{x} > \max_{y \in P} \bar{c} \cdot \bar{y} \), where \( \bar{c} \cdot \bar{y} \) is the inner product of \( \bar{c} \) and \( \bar{y} \). With a slight abuse of notation, we say that the constraint \( \bar{c} \cdot \bar{x} \leq L \), where \( \bar{c} \in \mathbb{R}^n \) and \( L \in \mathbb{R} \), is a separation hyperplane between \( P \) and \( \bar{x} \) if \( \bar{c} \cdot \bar{x} > L \) and \( \bar{c} \cdot \bar{y} \leq L \) for every \( \bar{y} \in P \). A separation oracle for a polytope \( P \subseteq \mathbb{R}^n \) is an oracle which receives \( \bar{x} \in \mathbb{R}^n \) as input, and either determines that \( \bar{x} \in P \) or returns a separating hyperplane between \( P \) and \( \bar{x} \). We use the following known connection between separation and optimization.

**Theorem A.3** (Thm 6.4.9 and Remark 6.5.2 in [10]). There is an algorithm which given \( n, \varphi \in \mathbb{N} \), \( \bar{c} \in \mathbb{R}^n \) and a separation oracle for a non-empty polytope \( P \subseteq \mathbb{R}^n \) of facet complexity \( \varphi \), returns a vertex \( \bar{y} \) of \( P \) such that \( \bar{c} \cdot \bar{y} = \max_{\bar{x} \in P} \bar{c} \cdot \bar{x} \) in time polynomial in \( n \) and \( \varphi \).

Thus, to show \( \text{LP}(K, F, \alpha) \) can be solved in polynomial time, we need to show that a separation oracle for \( \text{LP}(K, F, \alpha) \) can be implemented in polynomial time. To this end, we use a known result for matroid polytopes.

**Theorem A.4** (Thm 40.4 in [21]). There is a polynomial time algorithm MatroidSeparator which given a subset of elements \( \Omega \), a membership oracle for a matroid \( \mathcal{N} = (\Omega, \mathcal{J}) \) and \( \bar{x} \in \mathbb{Q}_{\geq 0}^\Omega \) either determines that \( \bar{x} \in P\mathcal{N} \) or returns \( S \subseteq \Omega \) such that \( \sum_{\omega \in S} \bar{x}_\omega > \text{rank}_\mathcal{N}(S) \).

A polynomial-time implementation of a separation oracle for \( \text{LP}(K, F, \alpha) \) is now straightforward.

**Lemma A.5.** There is an algorithm which given a BMI instance \( K = (E, I, c, p, \beta) \), \( F \subseteq E \), \( \frac{\text{OPT}(K)}{2} \leq \alpha \leq \text{OPT}(K) \) and \( \bar{x} \in \mathbb{R}^E \) implements a separation oracle for the feasibility region of \( \text{LP}(K, F, \alpha) \) in polynomial time.

**Proof.** To implement a separation oracle, the algorithm first checks if the input \( \bar{x} \) satisfies constraints (18) and (20). If one of the constraints is violated, the algorithm returns the constraint as a separating hyperplane.

Next, the algorithm invokes MatroidSeparator (Theorem A.4) with the matroid \( \mathcal{M}_F(\alpha) \) and the point \( \bar{x} \). If MatroidSeparator returns that \( \bar{x} \in P_{\mathcal{M}_F(\alpha)} \) then the algorithm returns that \( \bar{x} \) is in the feasibility region. Otherwise, the MatroidSeparator returns \( S \subseteq E(\alpha) \setminus F \) such that \( \sum_{e \in S} \bar{x}_e > \text{rank}_{\mathcal{M}_F(\alpha)}(S) \); that is, a set \( S \) for which (19) is violated. In this case, the algorithm returns \( 1^S \) as a separating hyperplane. Observe that for every \( \bar{y} \) in the feasibility region of \( \text{LP}(K, F, \alpha) \), it holds that

\[
1^S \cdot \bar{y} = \sum_{e \in S} \bar{y}_e \leq \text{rank}_{\mathcal{M}_F(\alpha)}(S) < 1^S \cdot \bar{x},
\]

where the first inequality is by (19). Thus, \( 1^S \) is indeed a separating hyperplane.
Clearly, the separating hyperplanes returned by the algorithm are indeed separating hyperplanes. Furthermore, if the algorithm asserts that $\bar{x}$ is in the feasibility region then constraints (18) and (20) hold as those were explicitly checked, and (19) holds by $\bar{x} \in P_{\mathcal{M}_F(\alpha)}$ (Theorem A.4 and Theorem A.1). That is, $\bar{x}$ is indeed in the feasibility region of $\text{LP}(\mathcal{K}, F, \alpha)$. The algorithm runs in polynomial-time as each of its steps can be implemented in polynomial-time.

Proof of Lemma 3.10. Observe that the vector $(0, \ldots, 0) \in \mathbb{R}^{E(\alpha)}$ is in the feasibility region of $\text{LP}(\mathcal{K}, F, \alpha)$, and thus the feasibility region is not empty. By Theorem A.3 and Observation A.2, there is an algorithm which finds an optimal basic solution for $\text{LP}(\mathcal{K}, F, \alpha)$ using a polynomial number of operations and calls for a separation oracle. By Lemma A.5, the separation oracle can be implemented in polynomial time as well. Thus, an optimal basic solution for $\text{LP}(\mathcal{K}, F, \alpha)$ can be found in time polynomial in $|\mathcal{K}|$. ■