Almost Engel linear groups

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Abstract. A group $G$ is almost Engel if for every $g \in G$ there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, n g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $n(x, g)$ such that $[x, n g] \in \mathcal{E}(g)$ whenever $n(x, g) \leq n$. A group $G$ is almost nil if it is almost Engel and for every $g \in G$ there is a positive integer $n$ such that $[x, n g] \in \mathcal{E}(g)$ for every $x \in G$. We prove that if a linear group $G$ is almost Engel, then $G$ is finite-by-hypercentral. If $G$ is almost nil, then $G$ is finite-by-nilpotent.

1. Introduction

By a linear group we understand here a subgroup of $GL(m, F)$ for some field $F$ and a positive integer $m$. An element $g$ of a group $G$ is called a (left) Engel element if for any $x \in G$ there exists $n = n(x, g) \geq 1$ such that $[x, n g] = 1$. As usual, the commutator $[x, n g]$ is defined recursively by the rule

$$[x, n g] = [[x, n-1 g], g]$$

assuming $[x, 0 g] = x$. If $n$ can be chosen independently of $x$, then $g$ is a (left) $n$-Engel element. A group $G$ is called Engel if all elements of $G$ are Engel. It is called $n$-Engel if all its elements are $n$-Engel. A group is said to be locally nilpotent if every finite subset generates a nilpotent subgroup. Clearly, any locally nilpotent group is an Engel group. It is a long-standing problem whether any $n$-Engel group is locally nilpotent. Engel linear groups are known to be locally nilpotent (cf. [2, 3]).

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We say that a group \( G \) is almost Engel if for every \( g \in G \) there is a finite set \( \mathcal{E}(g) \) such that for every \( x \in G \) all sufficiently long commutators \( [x, n \cdot g] \) belong to \( \mathcal{E}(g) \), that is, for every \( x \in G \) there is a positive integer \( n(x, g) \) such that \( [x, n \cdot g] \in \mathcal{E}(g) \) whenever \( n(x, g) \leq n \). (Thus, Engel groups are precisely the almost Engel groups for which we can choose \( \mathcal{E}(g) = \{1\} \) for all \( g \in G \).) We say that a group \( G \) is nil if for every \( g \in G \) there is a positive integer \( n \) depending on \( g \) such that \( g \) is \( n \)-Engel. The group \( G \) will be called almost nil if it is almost Engel and for every \( g \in G \) there is a positive integer \( n \) depending on \( g \) such that \( [x, n \cdot g] \in \mathcal{E}(g) \) for every \( x \in G \).

Almost Engel groups were introduced in \([6]\) where it was proved that an almost Engel compact group is necessarily finite-by-(locally nilpotent). The purpose of the present article is to prove the following related result.

**Theorem 1.1.** Let \( G \) be a linear group.

1. If \( G \) is almost Engel, then \( G \) is finite-by-hypercentral.
2. If \( G \) is almost nil, then \( G \) is finite-by-nilpotent.

Recall that the union of all terms of the (transfinite) upper central series of \( G \) is called the hypercenter. The group \( G \) is hypercentral if it coincides with its hypercenter. The hypercentral groups are known to be locally nilpotent (see \([10]\) P. 365)). By well-known results obtained in \([2, 3]\), if under the hypotheses of Theorem 1.1 the group \( G \) is Engel or nil, then \( G \) is hypercentral or nilpotent, respectively.

### 2. Preliminaries

Let \( G \) be a group and \( g \in G \) an almost Engel element so that there is a finite set \( \mathcal{E}(g) \) such that for every \( x \in G \) there is a positive integer \( n(x, g) \) with the property that \( [x, n \cdot g] \) belongs to \( \mathcal{E}(g) \) whenever \( n(x, g) \leq n \). If \( \mathcal{E}'(g) \) is another finite set with the same property for possibly different numbers \( n'(x, g) \), then \( \mathcal{E}(g) \cap \mathcal{E}'(g) \) also satisfies the same condition with the numbers \( n''(x, g) = \max\{n(x, g), n'(x, g)\} \).

Hence there is a minimal set with the above property. The minimal set will again be denoted by \( \mathcal{E}(g) \) and, following \([6]\), called the Engel sink for \( g \), or simply \( g \)-sink for short. From now on we will always use the notation \( \mathcal{E}(g) \) to denote the (minimal) Engel sinks. In particular, it follows that for each \( x \in \mathcal{E}(g) \) there exists \( y \in \mathcal{E}(g) \) such that \( x = [y, g] \).

More generally, given a subset \( K \subseteq G \) and an almost Engel element \( g \in G \), we write \( \mathcal{E}(g, K) \) to denote the minimal subset of \( G \) with the property that for every \( x \in K \) there is a positive integer \( n(x, g) \) such that \( [x, n \cdot g] \) belongs to \( \mathcal{E}(g, K) \) whenever \( n(x, g) \leq n \). Throughout
the article we use the symbols $\langle X \rangle$ and $\langle X^G \rangle$ to denote the subgroup generated by a set $X$ and the minimal normal subgroup of $G$ containing $X$, respectively.

A group is said to virtually have certain property if it contains a subgroup of finite index with that property. The following lemma can be found in [8 Ch. 12, Lemma 1.2] or in [5 Lemma 21.1.4].

**Lemma 2.1.** A virtually abelian group contains a characteristic abelian subgroup of finite index.

As usual, we write $Z_i(G)$ for the $i$th term of the upper central series of $G$ and $\gamma_i(G)$ for the $i$th term of the lower central series. Well-known Schur’s theorem says that if $G$ is central-by-finite, then the commutator subgroup $G'$ is finite (see [10, 10.1.4]). Baer proved that if, for a positive integer $k$, the quotient $G/Z_k(G)$ is finite, then so is $\gamma_k+1(G)$ (see [10, 14.5.1]). Recently, the following related result was obtained in [1] (see also [7]).

**Theorem 2.2.** Let $G$ be a group and let $H$ be the hypercenter of $G$. If $G/H$ is finite, then $G$ has a finite normal subgroup $N$ such that $G/N$ is hypercentral.

We will also require the Dicman Lemma (see [10, 14.5.7]).

**Lemma 2.3.** In any group a normal finite subset consisting of elements of finite order generates a finite subgroup.

In [9] Plotkin proved that if a group $G$ has an ascending series whose quotients locally satisfy the maximal condition, then the Engel elements of $G$ form a locally nilpotent subgroup. In particular we have the following lemma.

**Lemma 2.4.** Let $G$ be a group having an ascending series whose quotients locally satisfy the maximal condition and let $a \in G$ be an Engel element. Then $\langle a^G \rangle$ is locally nilpotent.

Linear groups are naturally equipped with the Zarisski topology. If $G$ is a linear group, the connected component of $G$ containing 1 is denoted by $G^0$. We will use (sometimes implicitly) the following facts on linear groups. All these facts are well-known and are provided here just for the reader’s convenience.

- If $G$ is a linear group and $N$ a normal subgroup which is closed in the Zarissky topology, then $G/N$ is linear (see [12 Theorem 6.4]).
- Since finite subsets of $G$ are closed in the Zarisski topology, it follows that any finite subgroup of a linear group is closed. Hence $G/N$ is linear for any finite normal subgroup $N$. 

• If $G$ is a linear group, the connected component $G^0$ has finite index in $G$ (see [12] Lemma 5.3).
• Each finite conjugacy class in a linear group centralizes $G^0$ (see [12] Lemma 5.5).
• In a linear group any descending chain of centralizers is finite. This follows from [12] Lemma 5.4 and the fact that the Zariski topology satisfies the descending chain condition on closed sets.
• A linear group generated by normal nilpotent subgroups is nilpotent (see Gruenberg [3]).
• Tits alternative: A finitely generated linear group either is virtually soluble or contains a subgroup isomorphic to a non-abelian free group (see [11]).
• The Burnside-Schur theorem: A periodic linear group is locally finite (see [12], 9.1).
• Zassenhaus theorem: A locally soluble linear group is soluble. Every linear group contains a unique maximal soluble normal subgroup (see [12], Corollary 3.8).
• Since the closure in the Zarisski topology of a soluble subgroup is again soluble (see [12], Lemma 5.11), it follows that the unique maximal soluble normal subgroup of a linear group is closed. In particular, if $G$ is linear and $R$ is the unique maximal soluble normal subgroup of $G$, then $G/R$ is linear and has no nontrivial normal soluble subgroups.
• A locally nilpotent linear group is hypercentral (see [2] or [3]).
• Gruenberg: The set of Engel elements in a linear group $G$ coincides with the Hirsch-Plotkin radical of $G$. The set of right Engel elements coincides with the hypercenter of $G$ (see [3]).

Here, as usual, the Hirsch-Plotkin radical of a group is the maximal normal locally nilpotent subgroup. An element $g \in G$ is a right Engel element if for each $x \in G$ there exists a positive integer $n$ such that $[g, n, x] = 1$.

3. Almost Engel elements in virtually soluble groups

In the present section we give certain criteria for a group containing almost Engel elements to be finite-by-nilpotent or finite-by-hypercentral. In particular, we prove that a virtually soluble group generated by finitely many almost Engel elements is finite-by-nilpotent (Theorem 3.3).
Lemma 3.1. Let $G = H\langle a_1, \ldots, a_s \rangle$, where $H$ is a normal subgroup and $a_i$ are almost Engel elements. Assume that $G/H$ is nilpotent. If $N \leq H$ is a finite normal subgroup of $H$, then $\langle N^G \rangle$ is finite.

Proof. Suppose first that $s = 1$ and write $a$ in place of $a_1$. Let $M$ be the subgroup generated by all commutators of the form $[x, j a]$, where $x \in N$ and $j$ is a nonnegative integer. Since both $N$ and $\mathcal{E}(a)$ are finite, it follows that there exists an integer $k$ such that $M$ is contained in the product $\prod_{i=0}^k N^{a^i}$. It is clear that the product $\prod_{i=0}^k N^{a^i}$ is normal in $H$ and $a$ normalizes $M$. Therefore $\langle M^H \rangle$ is normal in $G$ and is contained in $\prod_{i=0}^k N^{a^i}$. Moreover, $\langle N^G \rangle = \langle M^H \rangle$ so in the case where $s = 1$ the lemma follows.

Therefore we will assume that $s \geq 2$ and use induction on $s$. Assume additionally that $G/H$ is abelian. Set $H_0 = H$ and $H_i = H_{i-1}\langle a_i \rangle$ for $i = 1, \ldots, s$. The subgroups $H_i$ are normal in $G$ and $H_s = G$. By induction, $K = \langle N^{H_{s-1}} \rangle$ is finite. Since $G = H_{s-1}\langle a_s \rangle$, the above paragraph shows that $\langle K^G \rangle$ is finite. Obviously, $\langle K^G \rangle = \langle N^G \rangle$ and so in the case where $G/H$ is abelian the lemma follows.

We will now allow $G/H$ to be nonabelian, say of nilpotency class $c$. We will use induction on $c$. Set $B = \langle a_s^G \rangle$ and $G_1 = H B$. Since $G/H$ is a finitely generated nilpotent group, it follows that each subgroup of $G/H$ is finitely generated and so $B$ has finitely many conjugates of $a_s$, say $a_s^{a_1}, \ldots, a_s^{a_p}$ such that $G_1 = H\langle a_s^{a_1}, \ldots, a_s^{a_p} \rangle$. Since $G_1/H$ has nilpotency class at most $c - 1$, by induction $\langle N^{G_1} \rangle$ is finite. We now notice that $G = G_1\langle a_1, \ldots, a_{s-1} \rangle$ so the induction on $s$ completes the proof. \hfill \Box

Lemma 3.2. Let $G = H\langle a \rangle$, where $H$ is a virtually abelian normal subgroup and $a$ is an almost Engel element. Then $\langle a^G \rangle$ is finite-by-(locally nilpotent).

Proof. Assume that $G$ is a counter-example with $|\mathcal{E}(a)|$ as small as possible. In view of Lemma 2.1 we can choose a maximal characteristic abelian subgroup $V$ in $H$. Since $V$ is abelian, we have $[v_1, a][v_2, a] = [v_1 v_2, a]$ for any $v_1, v_2 \in V$. In other words, a product of two commutators of the form $[v, a]$, where $v \in V$, again has the same form. Therefore $\mathcal{E}(a, V)$ is a finite subgroup. Obviously, the normalizer in $G$ of $\mathcal{E}(a, V)$ has finite index. It follows that $\mathcal{E}(a, V)$ is contained in a finite normal subgroup $N$. If $\mathcal{E}(a, V) \neq 1$, we pass to the quotient $G/N$ and use induction on $|\mathcal{E}(a)|$. Therefore without loss of generality we will assume that $\mathcal{E}(a, V) = 1$, that is, $a$ is Engel in $V\langle a \rangle$. Since $\mathcal{E}(a)$ consists of commutators of the form $[x, a]$ with $x \in \mathcal{E}(a)$, it
follows that $E(a) \cap V = \{1\}$. Let $C_0 = 1$ and
$$C_i = \{v \in V \mid [v, a] \in C_{i-1}\}$$
for $i = 1, 2, \ldots$. Since $a$ is Engel in $V$, we have $V = \cup_i C_i$.

Let $T = \langle E(a), a \rangle$ and $U = V \cap T$. We observe that $U$ is a finitely generated abelian subgroup. In view of the fact that $V$ is the union of the $C_i$ we deduce that there exists a positive integer $n$ such that $U = C_n \cap U$.

For $i = 0, \ldots, n$ set $U_i = C_i \cap U$. Thus, $U = U_n$. Observe that $U_1$ centralizes $a$ and therefore $U_1$ normalizes the set $E(a)$. Denote by $W_1$ the intersection $U_1 \cap C_G(E(a))$. Since $E(a)$ is finite, it follows that $W_1$ has finite index in $U_1$. Further, it is clear that $W_1$ is contained in the center $Z(T)$.

The finiteness of the index $[U_1 : W_1]$ implies that $U_2$ contains a normal in $T$ subgroup $W_2$ such that the index $[U_2 : W_2]$ is finite, and $[W_2, T] \leq W_1$. Thus, $W_2$ is contained in $Z_2(T)$, the second term of the upper central series of $T$.

Next, in a similar way we conclude that $U_3 \cap Z_3(T)$ has finite index in $U_3$ and so on. Eventually, we deduce that $U \cap Z_n(T)$ has finite index in $U$. Thus, $T/Z_n(T)$ is finite-by-cyclic and therefore there exists a positive integer $k$ such that $a^k \in Z_{n+1}(T)$. Hence, $T/Z_{n+1}(T)$ is finite and so, in view of Baer’s theorem, we deduce that $T$ is finite-by-nilpotent. In particular, for some positive integer $r$ the subgroup $\gamma_r(T)$ is finite. The observation that for each $x \in E(a)$ there exists $y \in E(a)$ such that $x = [y, g]$ guarantees that $E(a)$ is contained in $\gamma_r(T)$. In particular, we proved that the subgroup $\langle E(a) \rangle$ is finite. Because $V$ is abelian, it is obvious that $V$ normalizes $V \cap \langle E(a) \rangle$. Thus, $V \cap \langle E(a) \rangle$ is a finite subgroup with normalizer of finite index. It follows that $V \cap \langle E(a) \rangle$ is contained in a finite normal subgroup of $G$. We can factor out the latter and without loss of generality assume that $V \cap \langle E(a) \rangle = 1$.

Recall that $C_1 = C_1(a)$. Therefore $C_1$ normalizes $\langle E(a) \rangle$ and in view of the fact that $V \cap \langle E(a) \rangle = 1$ we conclude that $C_1$ centralizes $\langle E(a) \rangle$. So $C_1 \leq Z(VT)$. Same argument shows that $C_2/C_1 \leq Z(VT/C_1)$ and, more generally, $C_{i+1}/C_i \leq Z(VT/C_i)$ for $i = 0, 1, 2, \ldots$. Thus, $V \leq Z_{\infty}(VT)$ where $Z_{\infty}(VT)$ stands for the hypercenter of $T$. Of course, it follows that there exists a positive integer $k$ such that $a^k \in Z_{\infty}(VT)$. We deduce that $Z_{\infty}(VT)$ has finite index in $VT$. Theorem 2.2 now tells us that $VT$ has a finite normal subgroup $N$ such that the quotient group $(VT)/N$ is hypercentral. The hypercentral groups are locally nilpotent and so $VT$ is finite-by-(locally nilpotent).

The observation that for each $x \in E(a)$ there exists $y \in E(a)$ such that $x = [y, g]$ guarantees that $E(a)$ is contained in $N$. 
Since $VT$ has finite index in $G$, Dicman’s lemma tells us that $G$ contains a finite normal subgroup $R$ such that $\mathcal{E}(a) \subseteq N \leq R$. The image of $a$ in $G/R$ is Engel and the required result follows from Lemma 2.4. □

**Theorem 3.3.** A virtually soluble group generated by finitely many almost Engel elements is finite-by-nilpotent.

**Proof.** Let $G$ be a virtually soluble group generated by finitely many almost Engel elements $a_1, \ldots, a_s$ and let $S$ be a normal soluble subgroup of finite index in $G$. We assume that $S \neq 1$ and let $V$ be the last nontrivial term of the derived series of $S$. By induction on the derived length of $S$ we assume that $G/V$ is finite-by-nilpotent. Therefore $G$ contains a normal subgroup $H$ such that $V$ has finite index in $H$ and the quotient $G/H$ is nilpotent. For $i = 1, \ldots, s$ set $G_i = H\langle a_i \rangle$. By Lemma 3.2 each subgroup $\langle a_i^{G_i} \rangle$ has a finite normal subgroup $N_i$ such that $\langle a_i^{G_i} \rangle/N_i$ is locally nilpotent. Since $G_i/H$ are abelian, it is clear that all quotients $G_i/H \cap N_i$ are locally nilpotent and so, replacing if necessary $N_i$ by $H \cap N_i$, without loss of generality we can assume that all subgroups $N_i$ are normal subgroups of $H$. Therefore the product of the subgroups $N_i$ is finite. By Lemma 3.1 the product of $N_1 \cdots N_s$ is contained in a finite subgroup $N$ which is normal in $G$. Obviously the images in $G/N$ of the generators $a_1, \ldots, a_s$ are Engel. Thus, $G/N$ is a virtually soluble group generated by finitely many Engel elements. It follows from Lemma 2.4 that $G/N$ is nilpotent. The proof is complete. □

The next lemma is well-known. For the reader’s convenience we provide the proof.

**Lemma 3.4.** Let $G = H\langle a \rangle$, where $H$ is a nilpotent normal subgroup and $a$ is a nil element. Then $G$ is nilpotent.

**Proof.** Suppose that $a$ is $n$-Engel. Let $K = Z(H)$ and set $K_0 = K$ and $K_{i+1} = [K_i, a]$ for $i = 0, 1, \ldots$. Then $K_{n-1} \leq K \cap C_K(a)$ and so $K_{n-1} \leq Z(G)$. Moreover we observe that $[K_i, G] \leq K_i$ and it follows that $K_{n-1} \leq Z_i(G)$ for $i = 1, 2, \ldots, n$. Therefore $K \leq Z_n(G)$. Passing to the quotient $G/Z_n(G)$ and using induction on the nilpotency class of $H$ we deduce that if $H$ is nilpotent with class $c$, then $G$ is nilpotent with class at most $cn$. □

**Lemma 3.5.** Let $G = H\langle a \rangle$, where $H$ is a hypercentral normal subgroup and $a$ is an Engel element. Then $G$ is hypercentral.
Proof. It is sufficient to show that \( Z(G) \neq 1 \). Let \( Z = Z(H) \). Since \( a \) is an Engel element, \( C_Z(a) \neq 1 \). Obviously, \( C_Z(a) \leq Z(G) \). The proof is complete. \( \square \)

Lemma 3.6. Let \( a \) be an almost Engel element in a group \( G \) and assume that \( E(a) \) is contained in a locally nilpotent subgroup. Then the subgroup \( \langle E(a) \rangle \) is finite.

Proof. Set \( D = \langle E(a) \rangle \). Without loss of generality we can assume that \( G = D \langle a \rangle \). Since \( E(a) \) is finite, \( D \) is nilpotent and we can use induction on the nilpotency class of \( D \). Thus, by induction assume that the quotient of \( D \) over its center is finite. By Schur’s theorem the derived group \( D' \) is finite as well. Factoring out \( D' \) we can assume that \( D \) is abelian. So now \( D \) is abelian and \( D = \langle E(a) \rangle \) and hence \( D \) is finite. \( \square \)

Lemma 3.7. Let \( G = H \langle a \rangle \), where \( H \) is a hypercentral normal subgroup.

1. If \( a \) is almost Engel, then \( G \) is finite-by-hypercentral.
2. If \( H \) is nilpotent and \( a \) is almost nil, then \( G \) is finite-by-nilpotent.

Proof. We will prove Claim 1 first. Assume that \( a \) is almost Engel. Let \( N \) be the product of all normal subgroups of \( G \) whose intersection with \( E(a) \) is \( \{1\} \). It is easy to see that \( N \cap E(a) = \{1\} \) and \( N \) is the unique maximal normal subgroup with that property. Therefore \( K \cap E(a) \neq \{1\} \) whenever \( K \) is a normal subgroup containing \( N \) as a proper subgroup. Since \( E(a) \) is finite, the group \( G \) contains a minimal normal subgroup \( M \) such that \( N < M \). Taking into account that \( H \) is hypercentral, we observe that \( M/N \) is central in \( H/N \).

Let \( D = \langle E(a) \rangle \cap M \). It follows that \( M = ND \). Suppose that \( D \) is not normal in \( M \) and set \( L = N_M(N_M(D)) \). Since \( M \) is hypercentral, it satisfies the normalizer condition and so \( L \neq N_M(D) \). Obviously \( a \) normalizes both \( L \) and \( N_M(D) \). Since \( a \) acts on \( L/N_M(D) \) as an Engel element, the centralizer of \( a \) in \( L/N_M(D) \) is nontrivial. Thus, \( L \) has a subgroup \( C \) such that \( N_M(D) < C \) and \( C \) normalizes \( N_M(D) \langle a \rangle \). Of course, \( D \) is normal in \( N_M(D) \langle a \rangle \). By Lemma 3.5 the quotient of \( N_M(D) \langle a \rangle \) by \( D \) is hypercentral. It is easy to see that \( D \) is a unique minimal normal subgroup of \( N_M(D) \langle a \rangle \) whose quotient is hypercentral. Therefore \( D \) is characteristic in \( N_M(D) \langle a \rangle \) and so \( C \) normalizes \( D \). This is a contradiction since \( N_M(D) < C \).

Hence, \( D \) is normal in \( M \). Again, it is easy to see that \( D \) is a unique minimal normal subgroup of \( M \langle a \rangle \) whose quotient is hypercentral. Therefore \( D \) is characteristic in \( M \) and so it is normal in \( G \).
We pass to the quotient $G/D$ and Claim 1 now follows by straightforward induction on $|\mathcal{E}(a)|$.

We now assume that $H$ is nilpotent and $a$ is almost nil. We already know that $G$ is finite-by-hypercentral. Factoring out a finite normal subgroup we can assume that $G$ is hypercentral. In that case $a$ is actually nil and so by Lemma 3.4 $G$ is nilpotent. The proof of the lemma is complete. \hfill \Box

4. Linear groups

**Lemma 4.1.** A virtually soluble almost Engel linear group is finite-by-hypercentral.

**Proof.** Suppose that $G$ is a virtually soluble almost Engel linear group. Let $S$ be a normal soluble subgroup of finite index in $G$. By induction on the derived length of $S$ we assume that $S'$ is finite-by-hypercentral. Passing to the quotient over a normal finite subgroup without loss of generality we can assume that $S'$ is hypercentral. By Lemma 3.7 the subgroup $\langle S', x \rangle$ is finite-by-hypercentral for each $x \in G$. Thus, for each $x \in G$ there exists a finite characteristic subgroup $R_x \leq \langle S', x \rangle$ such that $\langle S', x \rangle / R_x$ is hypercentral. Since $\langle S', x \rangle$ is normal in $S$, it follows that each element in $R_x$ has centralizer of finite index in $S$, hence centralizer of finite index in $G$. Therefore $G^0$ centralizes $R_x$ and it follows that $\langle S', x \rangle$ is hypercentral for each $x \in G^0$. The subgroup $\prod_{x \in S \cap G^0} \langle S', x \rangle$, where $x$ ranges over $S \cap G^0$, is locally nilpotent and therefore hypercentral. In particular $N = S \cap G^0$ is hypercentral and so $G$ is virtually hypercentral. By Lemma 3.7 the subgroup $\langle N, x \rangle$ is finite-by-hypercentral for each $x \in G$. In other words, for each $x \in G$ there exists a finite characteristic subgroup $Q_x \leq \langle N, x \rangle$ such that the quotient $\langle N, x \rangle / Q_x$ is hypercentral. Since $N$ has finite index in $G$, it follows that $G$ contains only finitely many subgroups of the form $\langle N, x \rangle$. Set $N_0 = \prod_{x \in G} Q_x$. We see that $N_0$ is a finite normal subgroup. Pass to the quotient $G/N_0$. Now the subgroup $\langle N, x \rangle$ is hypercentral for each $x \in G$. It follows that $N$ consists of right Engel elements and so, by the result of Gruenberg, $N$ is contained in the hypercenter of $G$. It follows from Theorem 2.2 that $G$ is finite-by-hypercentral, as required. \hfill \Box

We are now ready to prove Theorem 1.1 in its full generality. For the reader’s convenience we restate it here.

**Theorem 4.2.** Let $G$ be a linear group. If $G$ is almost Engel, then $G$ is finite-by-hypercentral. If $G$ is almost nil, then $G$ is finite-by-nilpotent.
Proof. Assume that $G$ is almost Engel. In view of Lemma 4.1 it is sufficient to show that $G$ is virtually soluble. By the Zassenhaus theorem a linear group is soluble if and only if it is locally soluble. Therefore it is sufficient to show that $G$ is virtually locally soluble. It is clear that $G$ does not contain a subgroup isomorphic to a nonabelian free group. Hence, by Tits alternative, any finitely generated subgroup of $G$ is virtually soluble. Therefore, by Theorem 3.3 any finitely generated subgroup of $G$ is finite-by-nilpotent. It becomes obvious that elements of finite order in $G$ generate a periodic subgroup. Moreover, the quotient of $G$ over the subgroup generated by all elements of finite order is locally nilpotent. Hence, $G$ is virtually locally soluble if and only if so is the subgroup generated by elements of finite order. Therefore without loss of generality we can assume that $G$ is an infinite periodic (and locally finite) group.

Let $R$ be the soluble radical of $G$. We can pass to the quotient and without loss of generality assume that $R = 1$. So in particular $G$ has no nontrivial Engel elements. By the theorem of Hall-Kulatilaka $G$ contains an infinite abelian subgroup. We conclude that some centralizers in $G$ are infinite. Since $G$ satisfies the minimal condition on centralizers, it follows that $G$ has a subgroup $D \neq 1$ such that the centralizer $C = C_G(D)$ is infinite while $C_G(\langle D, x \rangle)$ is finite for each $x \in G \setminus D$. Using that $C$ is infinite we deduce from the Hall-Kulatilaka theorem that $C$ contains an infinite abelian subgroup $A$. Obviously $A \leq C_G(\langle D, A \rangle)$ and it follows that $A \leq D$. Thus, $A \leq Z(C)$.

Now choose $1 \neq a \in A$. The centralizer $C$ normalizes the finite set $\mathcal{E}(a)$ because $a \in Z(C)$. Hence, $C$ contains a subgroup of finite index which centralizes $\mathcal{E}(a)$. It follows that $C_G(\langle D, \mathcal{E}(a) \rangle)$ is infinite and we conclude that $\mathcal{E}(a)$ is contained in $D$ and $C$ centralizes $\mathcal{E}(a)$. In particular, $a$ centralizes $\mathcal{E}(a)$ and so $\mathcal{E}(a) = \{1\}$. Thus, $a$ is an Engel element, a contradiction. This completes the proof of Claim 1.

Suppose now that $G$ is almost nil. We already know that $G$ is finite-by-hypercentral. Passing to a quotient over a finite normal subgroup we can assume that $G$ is hypercentral. Then obviously $G$, being both hypercentral and almost nil, must be nil. By the result of Gruenberg, $G$ is nilpotent.

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