The Chi Function, Tau Function, and Riemann Zeta Function

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Abstract

The fundamental relationship \( \zeta(s) = \chi(s)\zeta(1-s) \) reveals that no special function is more suitable than the chi function \( \chi(s) \) to study the Riemann zeta function \( \zeta(s) \). When \( t \) is sufficiently large, the modulus and argument of \( \chi(\sigma+it) \) are monotone about \( \sigma \) and \( t \) respectively, which accomodates the construction of a Riemann surface for the multivalued \( z = \chi(s) \). The inverse of \( \chi(s) \), which is branched and much similar to the logarithm function, is introduced as tau function \( s = \tau(z) \). Then \( \zeta(s) = \zeta \circ \tau(z) \) can be studied by its different branches, with a much simpler relationship in single-valued domains, which finally leads to the conclusion about the nontrivial zeros of \( \zeta(s) \).

1. Introduction

The Riemann zeta function \( \zeta(s) \) is one of the most challenging functions, whose zeros have interested many. The analytic function \( \zeta(s) \) satisfies the functional equation

\[
(1) \quad \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \quad s \in \mathbb{C}\setminus\{1\}.
\]

The chi function \( \chi(s) \) is defined as

\[
(2) \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s),
\]

and

\[
(3) \quad \chi(s)\chi(1-s) = 1.
\]

Considering the zeros of \( \sin\left(\frac{\pi s}{2}\right) \) and the poles of \( \Gamma(1-s) \), the chi function \( \chi(s) \) is meromorphic on the entire complex plane, with poles at \( s = 1, 3, 5, \ldots \) and zeros at \( s = 0, -2, -4, \ldots \)

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It has been shown that the only function which satisfies
\[ \zeta(s) = \chi(s)\zeta(1 - s) \]
or
\[ \zeta(1 - s) = \chi(1 - s)\zeta(s), \]
and has the same general characteristics as \( \zeta(s) \), is \( \zeta(s) \) itself\[4\]. The fundamental relationship reveals that no special function is more suitable than \( \chi(s) \) to study \( \zeta(s) \).

2. The modulus and argument of the chi function

Some properties of the chi function \( \chi(s) \) are proposed. One is about its modulus, the other is about its argument, both for the \( s \) far away from the real axis in the complex plane. We only discuss the \( s \) in the upper half-plane for the symmetry of \( t \).

**Lemma 2.1.** Let \( s = \sigma + it \). Then \( |\chi(s)| = 1 \) for \( \sigma = \frac{1}{2} \). There also exists a real number \( M_1 > 0 \), such that the modulus of \( \chi(s) \) is a continuous function of \( \sigma \) and \( t \) when \( t \geq M_1 \), satisfying the following properties:

(i) \(|\chi(s)|\) decreases strictly monotonously with increasing \( \sigma \), and tends to 0 as \( \sigma \to +\infty \).

(ii) \( 0 < |\chi(s)| < 1 \) for \( \frac{1}{2} < \sigma < +\infty \).

(iii) \( 1 < |\chi(s)| < +\infty \) for \(-\infty < \sigma < \frac{1}{2} \).

**Proof.** Taking \( \sigma = \frac{1}{2} \) in (3), then \( |\chi(s)| = 1 \). All the poles and zeros of \( \chi(s) \) are on the real axis, then \( 0 < |\chi(s)| < +\infty \) when \( t \neq 0 \).

The following asymptotic expansion[2] is adopted for real \( x \) and \( y \):

\[ |\Gamma(x + iy)| = \sqrt{2\pi|y|^{x - \frac{1}{2}}e^{-\frac{1}{2}x|y|}}\left\{1 + O\left(\frac{1}{|y|}\right)\right\}; \quad |y| \to \infty. \]

Taking \( x = 1 - \sigma \) and \( y = -t \) in (6), then

\[ |\chi(s)| = |2^{\sigma + it}| \cdot |\pi^{\sigma - 1 + it}| \cdot \left|\sin\left(\frac{\pi}{2}(\sigma + it)\right)\right| \cdot |\Gamma(1 - \sigma - it)| \]
\[ = 2^{\sigma} \pi^{\sigma - 1} \sqrt{\sin^2\left(\frac{\pi}{2}\sigma\right) + \sinh^2\left(\frac{\pi}{2}t\right)} \sqrt{2\pi t^{\frac{1}{2} - \sigma}} e^{-\frac{1}{2}\pi t} \left\{1 + O\left(\frac{1}{t}\right)\right\} \]
\[ = \sqrt{2\pi} 2^{\sigma} \pi^{\sigma - 1} \sqrt{\frac{\sin^2\left(\frac{\pi}{2}\sigma\right) + \sinh^2\left(\frac{\pi}{2}t\right)}{e^{\pi t}}} t^{\frac{1}{2} - \sigma} \left\{1 + O\left(\frac{1}{t}\right)\right\}. \]

The modulus \( |\chi(s)| \) is a continuous function of \( \sigma \) and \( t \) when \( t \) is sufficiently large. It’s obvious that \( 0 < |\chi(s)| < 1 \) for all \( \sigma > \frac{1}{2} \), and \( |\chi(s)| > 1 \) for all \( \sigma < \frac{1}{2} \), both as \( t \to +\infty \).
For any sufficiently large \( t \), we also have \(|\chi(s)| \to 0\) as \( \sigma \to +\infty \), and \(|\chi(s)| \to +\infty\) as \( \sigma \to -\infty \).

The infinitesimal \( \Delta|\chi(s)| \) with respect to \( \Delta\sigma \) is

\[
(8) \quad \frac{\Delta|\chi(s)|}{|\chi(s)|} = \left\{ \log(2\pi) - \log t + \frac{\pi}{2} \sin \pi \sigma + \sinh^2 \left( \frac{\pi}{2} t \right) \right\} \{1 + O\left(\frac{1}{t}\right)\} \Delta\sigma.
\]

It’s obvious that \(|\chi(s)|\) decreases strictly monotonously with increasing \( \sigma \) when \( t \) is sufficiently large.

Lemma 2.1 demonstrates that the critical line \( \{s|\text{Re}(s) = \frac{1}{2}\} \) in the \( s \)-plane is mapped to a unit circle \( S^1 \) by \( \chi(s) \).

**Lemma 2.2.** Let \( s = \sigma + it \). There exists a real number \( M_2 > 0 \), such that the argument of \( \chi(s) \) is a continuous function of \( \sigma \) and \( t \) when \( t \geq M_2 \), satisfying the following properties:

(i) \( \arg(\chi(s)) \) decreases strictly monotonously with increasing \( t \), and tends to \( -\infty \) as \( t \to +\infty \).

(ii) \( \arg(\chi(s)) \) remains almost constant with varying \( \sigma \).

(iii) \( \arg(\chi(s)) = \arg(\chi(1 - \overline{s})) \).

**Proof.** When \( t \neq 0 \), the logarithm of \( \chi(s) \) is obtained as

\[
(9) \quad \log \chi(\sigma + it) = (\sigma + it) \log 2 + (\sigma - 1 + it) \log \pi
\]

\[+ \log \sin \left( \frac{\pi}{2} (\sigma + it) \right) + \log \Gamma(1 - \sigma - it).\]

The third item in (9) is

\[\log \sin \left( \frac{\pi}{2} (\sigma + it) \right) = \log \frac{e^{-\frac{\pi}{2}t} e^{\frac{i\pi}{2} \sigma} - e^{\frac{\pi}{2}t} e^{-i\frac{\pi}{2} \sigma}}{2i} = \log \left( e^{\frac{\pi}{2}t} e^{-i\frac{\pi}{2} \sigma} \right) + \log \left( 1 - \frac{e^{i\pi \sigma}}{e^{\pi t}} \right) = \frac{\pi}{2} t - \log 2 + i \frac{\pi}{2} (1 - \sigma) + O\left(\frac{1}{e^{\pi t}}\right).\]

The last item in (9) can be expressed by the asymptotic expansion[1]

\[
(10) \quad \log \Gamma(z + a) = (z + a - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{z}\right)
\]

where \(|\arg z| < \pi\) and \( a \in \mathbb{C} \). Taking \( z = -it \) and \( a = \sigma \) in (10), then

\[\log \Gamma(\sigma - it) = (-it + \frac{1}{2} - \sigma) \log(-it) + it + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{t}\right).\]
Finally
\[
\arg(\chi(\sigma + it)) = \Im(\log \chi(\sigma + it)) = t + t \log \frac{2\pi}{t} + \frac{\pi}{4} + O\left(\frac{1}{t}\right) + O\left(-\sin \frac{\pi \sigma}{e^{\pi t}}\right).
\]
(11)

The argument \(\arg(\chi(\sigma + it))\), if not restricted in the range of \(2\pi\), is a continuous function of \(\sigma\) and \(t\) when \(t\) is sufficiently large. It’s obvious that
\[
\arg(\chi(\sigma + it)) \rightarrow t + t \log \frac{2\pi}{t} + \frac{\pi}{4}, \quad t \rightarrow +\infty.
\]
(12)

The \(\arg(\chi(\sigma + it))\) decreases strictly monotonously with increasing \(t\) when \(t\) is sufficiently large, and \(\arg(\chi(\sigma + it))\) tends to \(-\infty\) as \(t \rightarrow +\infty\).

It’s also observed that \(\arg(\chi(\sigma + it))\) remains almost constant with varying \(\sigma\) when \(t\) is sufficiently large.

Since \(\chi(s) \neq 0\) when \(t \neq 0\), the argument of (3) is
\[
2\pi m = \arg(\chi(\sigma + it)) + \arg(\chi(1 - \sigma - it)) = \arg(\chi(\sigma + it)) - \arg(\chi(1 - \sigma + it)), \quad m = 0, \pm 1, \pm 2, \ldots.
\]
(13)

And (12) excludes the possibility of any non-zero \(m\) for sufficiently large \(t\), although \(\arg(\chi(s))\) is multivalued. □

**Lemma 2.2** demonstrates that for any real constant \(\sigma\), while \(s = \sigma + it\) moves upwards on the vertical line \(\{s|\Re(s) = \sigma\}\), the image \(\chi(s)\) wraps around the origin clockwise for \(t \geq M_2\). Combined with **Lemma 2.1**, while \(s\) moves upwards on the critical line \(\{s|\Re(s) = \frac{1}{2}\}\), the image \(\chi(s)\) loops on the unit circle \(S^1\) clockwise, passing through one point such as \(\{1\}\) on \(S^1\) infinitely many times.

### 3. The branch and the inverse of the chi function

**Lemma 2.1** and **Lemma 2.2** apply when \(t \geq M_1\) and \(t \geq M_2\) respectively, and the following domain is focused on:

**Definition 3.1.** The domain \(D \subset \mathbb{C}\) is said to be far away from the real axis (FAR for short), if a proper real number \(M \geq \max(M_1, M_2)\) can be chosen, such that \(\arg(\chi(s)) \leq \arg(\chi(\frac{1}{2} + iM))\) for all \(s \in D\).

As a special case, the vertical line \(\{s|\sigma = \frac{1}{2}, t \geq M\}\) is FAR for any \(M \geq \max(M_1, M_2)\), since \(\arg(\chi(\frac{1}{2} + it)) \leq \arg(\chi(\frac{1}{2} + iM))\) for all the points on the line by **Lemma 2.2**. **Definition 3.1** can be extended to the lower half-plane as \(\arg(\chi(\sigma + it)) \geq \arg(\chi(\frac{1}{2} + iM))\) for \(t < 0\).

For all \(s \in D\), we observe that \(0 < |\chi(s)| < +\infty\) by **Lemma 2.1**, and that \(\arg(\chi(s))\) is monotonous decreasing about \(t\) and tends to \(-\infty\) by **Lemma 2.2**. Hence a Riemann surface can be constructed as follows to make the range of \(z = \chi(s)\) single-valued.
The slit $z$-plane $\mathbb{C}\setminus[0, +\infty)$ is designated as the $m$th sheet $S_m$, where $m$ is any integer sufficiently large. And when every two sheets $S_m$ and $S_{m+1}$ are attached along the branch cut $(0, +\infty)$, a Riemann surface $R$ spread over the $z$-plane is constructed.

Then the chi function
\begin{equation}
\label{eq:14}
z = \chi(s), \quad s \in D
\end{equation}
where $z \in R$, is the composition of infinitely many branches
\begin{equation}
\label{eq:15}
z = \chi_m(s) = |\chi(s)|e^{\text{Arg}(\chi(s))}2\pi i m
\end{equation}
where $z \in S_m \subset R$.

**Definition 3.2.** Let $\phi \in \mathbb{R}$ be a constant. The arc $\gamma_\phi$ in the $s$-plane is said to be argument-preserving for the map $\chi: s \to z$, if $\arg(z) = \phi$ for all $s \in \gamma_\phi$.

**Lemma 3.1.** Let $D = \{s|\sigma_1 \leq \text{Re}(s) \leq \sigma_2\}$ be a FAR domain where $\sigma_1 < \frac{1}{2} < \sigma_2$. For any $s_0 \in D$, there exists a unique argument-preserving arc $\gamma_\phi$ for the map $\chi$, such that $s_0 \in \gamma_\phi$ and the arc $\gamma_\phi$ splits the domain $D$ horizontally into two parts.

**Proof.** Let $\chi(\sigma + it) = r(\sigma, t)e^{i\phi(\sigma, t)}$ where $\phi(\sigma,t) = \arg(\chi(\sigma + it))$ and $r(\sigma, t) = |\chi(\sigma + it)|$.

Step 1. Choose $M$ for the domain $D$.

From (11) we know that $\phi(\sigma,t)$ is continuous and bounded for any finite $\sigma$ and $t$. Let
\begin{equation}
\epsilon = \sup_{\sigma \in [\sigma_1, \sigma_2]} |\phi(\sigma, t) - \phi(\frac{1}{2}, t)|.
\end{equation}
There exists a real number $M_3 > 0$ and an arbitrarily small $\epsilon_0 < \pi$, such that $\epsilon < \epsilon_0$ for all $t \geq M_3$, because $\epsilon \to 0$ as $t \to +\infty$ by Lemma 2.2.

Let $M_0 = \max(M_1, M_2, M_3)$. Then $|\phi(\sigma, M_0) - \phi(\frac{1}{2}, M_0)| < \epsilon_0$ for all $\sigma \in [\sigma_1, \sigma_2]$. Take a real number $M > M_0$ satisfying $\phi(\frac{1}{2}, M) = \phi(\frac{1}{2}, M_0) - 2\epsilon_0$. Then for all $\sigma \in [\sigma_1, \sigma_2]$ and $t \geq M$,
\begin{equation}
\label{eq:16}
\phi(\sigma, M) < \phi(\frac{1}{2}, M) + \epsilon_0 = \phi(\frac{1}{2}, M_0) - \epsilon_0 < \phi(\sigma, M_0).
\end{equation}

Choose the domain $D = \{s|\sigma_1 \leq \sigma \leq \sigma_2, t \geq M\}$ as the FAR critical strip.

Step 2. The existence of the unique set $\gamma_\phi$.

If $s_0 = \sigma_0 + it_0 \in D$, then $t_0 \geq M > M_0$. For any constant $\sigma \in [\sigma_1, \sigma_2]$, a continuous real function of $t$ is constructed as
\begin{equation}
\label{eq:17}
F(t) = \phi(\sigma, t) - \phi(\sigma_0, t_0).
\end{equation}
When $t = M_0$, we obtain that $F(M_0) = \phi(\sigma, M_0) - \phi(\sigma_0, t_0) > 0$ by (16). When $t \to \infty$, we obtain that $F(t) = \phi(\sigma, t) - \phi(\sigma_0, t_0)$ decreases strictly monotonously and tends to $-\infty$ by Lemma 2.2.
Therefore there exists a unique and bounded \( \hat{t} \) satisfying \( F(t) = 0 \) for each \( \hat{\sigma} \in [\sigma_1, \sigma_2] \), all of which form the unique set \( \gamma_\phi = \{ \sigma + it | \phi(\sigma, t) = \phi(\sigma_0, t_0), \sigma_1 \leq \sigma \leq \sigma_2 \} \).

Step 3. The set \( \gamma_\phi \) is a continuous and simple arc.

The two-variable function \( H(\sigma, t) = \phi(\sigma, t) - \phi(\sigma_0, t_0) \) is continuous about \( \sigma \) and \( t \) for \( \sigma \in (\sigma_1 - \delta, \sigma_2 + \delta) \) and \( t \geq M \), where \( \delta > 0 \) is small. Because \( H(\sigma, t) \) is monotone decreasing about \( t \), a unique implicit function \( t = h(\sigma) \) can be established from \( H(\sigma, t) = 0 \) for all \( \sigma \in [\sigma_1, \sigma_2] \), which shows that \( \gamma_\phi \) is a continuous and simple arc, splitting the critical strip \( \{ \sigma + it | \sigma_1 \leq \sigma \leq \sigma_2 \} \) horizontally into two parts. \( \square \)

The argument-preserving arc \( \gamma_\phi \subset D \) is mapped to \( \beta_\phi \subset R \) by the map \( (18) \chi : \gamma_\phi \to \beta_\phi \), and \( \beta_\phi \) is a line segment on a ray issuing from the origin.

**Corollary 3.2.** The map \( \chi : \gamma_\phi \to \beta_\phi \) is continuous and one-to-one, satisfying the following properties:

(i) If \( s_0 \in \gamma_\phi \), then \( (1 - \overline{s_0}) \in \gamma_\phi \).

(ii) Two arcs \( \gamma_{\phi_1} \) and \( \gamma_{\phi_2} \) do not intersect in the \( s \)-plane, if \( \phi_1 \neq \phi_2 \).

**Proof.** For any \( s \in \gamma_\phi \), let \( \chi(s) = r(\sigma, t)e^{i\phi} \) where \( \phi \) is a constant real. By Lemma 2.1 the radius \( r \) is continuous and monotone decreasing with respect to \( \sigma \). Then the map \( \chi : \gamma_\phi \to \beta_\phi \) is continuous and one-to-one.

Lemma 2.2 claims that \( \arg(\chi(s)) = \arg(\chi(1 - \overline{s_0})) \), which makes property (i) true.

Suppose \( s_0 \) is the point where \( \gamma_{\phi_1} \) and \( \gamma_{\phi_2} \) intersect. The assumption of \( \gamma_{\phi_1} \neq \gamma_{\phi_2} \) with \( \phi_1 = \phi_2 \) contradicts Lemma 3.1, which makes property (ii) true. \( \square \)

The FAR domain \( D \) is mapped to \( U \) by the map \( (19) \chi : D \to U \)

where \( U \) is the domain in the Riemann surface \( R \).

When the branch cut \([0, +\infty)\) is chosen, the domain \( U \) can be separated into infinitely many sheets as

\( (20) U_m = U(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \beta_\phi \)

where \( m \) is any sufficiently large integer and

\( \phi_1 = -2\pi m, \quad \phi_2 = \phi_1 - 2\pi. \)
Each $U_m$ is a domain in the slit complex plane $S_m$. Correspondingly the domain $D$ can be separated into infinitely many horizontal strips as

$$D_m = D(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \gamma_\phi.$$ (21)

Each $D_m$ is the pre-image of $U_m$.

**Corollary 3.3.** The map $\chi : D_m \to U_m$ is one-to-one, and the inverse map $\chi^{-1} : U_m \to D_m$ can be defined on the whole slit plane $S_m$.

**Proof.** Suppose $s = \sigma + it \in D_m$ and $z = re^{i\phi} \in U_m$.

If $\chi(s_1) = \chi(s_2) = r_0e^{i\phi_0}$ where $s_1, s_2 \in D_m$, then $s_1, s_2 \in \gamma_{\phi_0}$. The Corollary 3.2 requires that $s_1 = s_2$ for they share a single $r_0$. Then the map $\chi$ is one-to-one from $D_m$ to $U_m$.

Let $\chi : \gamma_{\phi} \to \beta_{\phi}$ where $\gamma_{\phi}$ is the arc preserving the argument $\phi$ for the map $\chi$, and let

$$D_0 = D_m \cap \{s|\sigma_1 \leq \text{Re}(s) \leq \sigma_2\}$$

where $\sigma_1 < \frac{1}{2} < \sigma_2$.

Lemma 3.1 claims that $\gamma_{\phi}$ always intersects the vertical lines $\{s|\sigma = \sigma_1\}$ and $\{s|\sigma = \sigma_2\}$ in the FAR domain $D_0$, even as $\sigma_2 \to +\infty$ and $\sigma_1 \to -\infty$. Lemma 2.1 continues to claim that $\beta_{\phi}$ is a line segment with one end tending to the origin, and with the other end tending to $\infty$. As $\phi$ decreases by $2\pi$, the domain $U_m$ covers the whole complex plane $\mathbb{C}$ except the branch cut. \[\square\]

The tau function $\tau(z)$, inverse of the $m$-th branch function (15), can be well defined based on Corollary 3.3 as

$$s = \tau(z) = \chi^{-1}(z), \quad z \in \mathbb{C}\setminus[0, +\infty)$$ (22)

where $-2\pi(m+1) < \arg(z) < -2\pi m$.

**Lemma 3.4.** The tau function $s = \tau(z)$ is a conformal mapping of the slit plane $S_m$ onto the horizontal strip $D_m$.

**Proof.** Suppose $\chi : s \to z$ where $s = \sigma + it \in D_m$ and $z = re^{i\phi} \in S_m$.

The FAR domain $D_m$ contains neither zeros nor poles of $\chi(s)$. Since $\chi(s)$ is analytic for all $s \in D_m$, the function $\chi(s)$ is differentiable for all $s \in D_m$, and the derivative of the nonzero $\chi(s) = re^{i\phi}$ is

$$\chi'(s) = \frac{\partial(r \cos \phi)}{\partial \sigma} + i \frac{\partial(r \sin \phi)}{\partial \sigma}$$ (23)

$$= \frac{\partial r}{\partial \sigma} e^{i\phi} + i \frac{\partial \phi}{\partial \sigma} \chi(s).$$
Both Lemma 2.1 and Lemma 2.2 applies for all \( s \in D_m \), which require \( \frac{\partial r}{\partial \sigma} < 0 \) and \( \frac{\partial \phi}{\partial \sigma} \to 0 \). Therefore \( \chi'(s) \neq 0 \) for all \( s \in D_m \). And the inverse map \( \tau : z \to s \) is analytic on \( S_m \) by implicit function theorem. \( \square \)

The tau function \( \tau(z) \) can also be defined on the negative \( m \)-th sheet.

**Definition 3.3.** The branch

\[
(24) \quad s = \tau_-(z), \quad z \in S_m^*
\]

is said to be the conjugated branch of

\[
(25) \quad s = \tau(z), \quad z \in S_m
\]

if \( S_m^* = \{ \overline{z} | z \in S_m \} \) is the reflection of \( S_m \).

The conjugated branch of (22) is

\[
(26) \quad s = \tau_-(z) = \chi^{-1}(z), \quad z \in \mathbb{C} \backslash [0, +\infty)
\]

where \( 2\pi m < \arg(z) < 2\pi(m + 1) \).

Another pair of conjugated branches are available if \( (-\infty, 0] \) is chosen to be the branch cut. The corresponding tau functions are

\[
(27) \quad s = \tau(z) = \chi^{-1}(z), \quad z \in \mathbb{C} \backslash (-\infty, 0]
\]

where \( -\pi \leq \arg(z) < \pi - 2\pi m \), and

\[
(28) \quad s = \tau_-(z) = \chi^{-1}(z), \quad z \in \mathbb{C} \backslash (-\infty, 0]
\]

where \( -\pi + 2\pi m < \arg(z) < \pi + 2\pi m \).

Generally speaking, the topology of \( \chi(s) \) is similar to the topology of \( e^s \), when \( s \) is far away from the real axis in the complex plane. And the topology of \( \tau(z) \) is similar to the topology of \( \log(z) \).

4. **The logarithmic integral of the branched zeta function**

The composite function is introduced based on (22) or (27) as

\[
(29) \quad w = \zeta(s) = \zeta \circ \chi^{-1}(z) = \zeta \circ \tau(z) = G(z), \quad z \in S_m.
\]

where \( -2\pi(m + 1) < \arg(z) < -2\pi m \) or \( -\pi - 2\pi m < \arg(z) < \pi - 2\pi m \), and \( m \) is sufficiently large.

The function \( G(z) \) is also branched in accordance with the branch of \( \tau(z) \). And \( G(z) \) is analytic on the slit plane \( S_m \) by Lemma 3.4.

By Definition 3.3 the conjugated branch of (29) is

\[
(30) \quad w = \zeta(s) = \zeta \circ \tau_-(z) = G_-(z), \quad z \in S_m^*
\]

where \( S_m^* \) is the reflection of \( S_m \).

Let the point \( z \in S_m \) and

\[
(31) \quad G : z \mapsto s \mapsto \zeta(s).
\]
Supposing $\eta = \chi(1 - s)$, by (3) we have

$$\eta = \chi(1 - s) = \frac{1}{\chi(s)} = \frac{1}{z}.$$  

Lemma 2.2 claims that $\arg(\eta) = \arg(z)$ and $\arg(\eta) = -\arg(z)$, requiring the points $\eta \in S_m$ and $\eta \in S^*_m$ respectively and

$$G : \eta = \left(\frac{1}{z}\right) \tau \to 1 - s \to \zeta(1 - s),$$

$$G_- : \eta = \frac{1}{z} \tau \to 1 - s \to \zeta(1 - s).$$

The fundamental functional equation (4) can be rewritten as

$$(34) \quad G(z) = z G_-(\frac{1}{z})$$

Let $\tau : S_m \to D_m$ and the domain $D_m$ is a horizontal strip

$$(35) \quad D_m = D(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \gamma_{\phi}$$

where $\phi_1 = -2\pi m$ or $\phi_1 = \pi - 2\pi m$, and $\phi_2 = \phi_1 - 2\pi$.

We come to study the logarithmic integral of nonvanishing zeta function

$$(36) \quad \int_{\partial D_s} \frac{\zeta'(s)}{\zeta(s)} ds$$

where the domain $D_s$ is defined as follows.

**Definition 4.1.** A domain $D_s$ is said to be simple, if

(i) the domain $D_s \subset D_m$ is bounded, simple connected, and symmetric with respect to the vertical line $\{ s | \text{Re}(s) = \frac{1}{2} \}$;

(ii) the boundary $\partial D_s$ is piecewise smooth, and $\zeta(s) \neq 0$ for all $s \in \partial D_s$;

(iii) the boundary $\partial D_s$ meets the vertical line $\{ s | \text{Re}(s) = \frac{1}{2} \}$ only twice.

Two types of simple domains $D_s$ are studied.

**Definition 4.2.** A domain $D_s^1$ is said to be type-one, if

(i) the domain $D_s^1 \subset D_m$ is simple and

$$D_m = D(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \gamma_{\phi}$$

where $\phi_1 = -2\pi m$ and $\phi_2 = -2\pi(m + 1)$;

(ii) the boundary $\partial D_s^1$ meets the vertical line $\{ s | \text{Re}(s) = \frac{1}{2} \}$ at two points

$$s_{m+1}^- = \frac{1}{2} + i(t_{m+1} - \epsilon) \quad \text{and} \quad s_m^+ = \frac{1}{2} + i(t_m + \epsilon)$$

where $\arg(\chi(\frac{1}{2} + it_m)) = -2\pi m$, $\arg(\chi(\frac{1}{2} + it_{m+1})) = -2\pi (m + 1)$, and $\epsilon > 0$ is arbitrarily small.
Definition 4.3. A domain $D_s^2$ is said to be type-two, if
(i) the domain $D_s^2 \subset D_m$ is simple and
$$D_m = D(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \gamma_{\phi}$$
where $\phi_1 = \pi - 2\pi m$ and $\phi_2 = -\pi - 2\pi m$;
(ii) the boundary $\partial D_s^2$ meets the vertical line $\{ s \mid \text{Re}(s) = \frac{1}{2} \}$ at two points
$$s_m^+ = \frac{1}{2} + i(t_m + \epsilon) \quad \text{and} \quad s_m^- = \frac{1}{2} + i(t_m - \epsilon)$$
where $\arg(\chi(\frac{1}{2} + it_m)) = -2\pi m$, and $\epsilon > 0$ is arbitrarily small.

Lemma 4.1. The logarithmic integral of nonvanishing zeta function

$$\lim_{\epsilon \to 0} \int_{\partial D_s^1} \frac{\zeta'(s)}{\zeta(s)} \, ds = 2\pi i$$

where $D_s^1$ is type-one simple domain defined in Definition 4.2.

Proof. Step 1. Choose $\epsilon$ for the domain $D_s^1$.

When $s = \frac{1}{2} + it$, we have $1 - s = \frac{1}{2} - it = \overline{\gamma}$. By (4) we obtain
$$\zeta(s) - \overline{\zeta(s)} = (\chi(s) - 1)\zeta(s),$$
which requires $\zeta(s)$ to be real when $\chi(s) = 1$.

Since the zeros of analytic $\zeta(s)$ are isolated, there exists a small $\epsilon_0 > 0$ such that $\zeta(\frac{1}{2} + i(t_m + \epsilon)) \neq 0 \quad \text{and} \quad \zeta(\frac{1}{2} + i(t_m + 1 + \epsilon)) \neq 0 \quad \text{for all} \quad 0 < |\epsilon| < \epsilon_0$.

Choose $0 < \epsilon < \epsilon_0$ for the domain $D_s^1$, where
$$\lim_{\epsilon \to 0} \zeta(s_m^+) = \lim_{\epsilon \to 0} \zeta(s_m^-) = \lim_{\epsilon \to 0} \overline{\zeta(s_m^-)},$$
$$\lim_{\epsilon \to 0} \zeta(s_m+1) = \lim_{\epsilon \to 0} \zeta(s_m+1) = \lim_{\epsilon \to 0} \overline{\zeta(s_m+1)},$$
since $\chi(s) \to 1$ and $\zeta(s) \to \overline{\zeta(s)}$ as $\epsilon \to 0$.

Let $\gamma$ and $\overline{\gamma}$ be any arc and its conjugate respectively, where $\gamma$ starts at $s_m^-$ and ends at $s_m^+$. When $\zeta(s) \neq 0$ on $\gamma$ or $\overline{\gamma}$, the logarithmic integral
$$\lim_{\epsilon \to 0} \left\{ \int_\gamma \frac{\zeta'(s)}{\zeta(s)} \, ds - \int_{\overline{\gamma}} \frac{\zeta'(s)}{\zeta(s)} \, ds \right\} = \lim_{\epsilon \to 0} \left\{ \log \zeta(s) \bigg|_{s_m^+}^{s_m^-} - \log \zeta(s) \bigg|_{s_m^+}^{s_m^-} \right\}$$
$$= 0$$
for $\epsilon \in (0, \epsilon_0)$.

Step 2. Integrate along the boundary $\partial D_s^1$.

Suppose the boundary $\partial D_s^1 \subset D_m$ is separated into two arcs $\gamma^0$ and $\gamma = \{ 1 - \overline{\gamma} \mid s \in \gamma^0 \}$ by the vertical line $\{ s \mid \text{Re}(s) = \frac{1}{2} \}$. And the arc $\overline{\gamma} = \{ \overline{\gamma} \mid s \in \gamma \}$ is the conjugate of the arc $\gamma$. 

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Let 
\[ \chi : \gamma^0 \to \beta^0, \quad \gamma \to \beta, \quad \gamma \to \overline{\beta}. \]
Then the arc \( \overline{\beta} = \{ z \mid \frac{1}{z} \in \beta^0 \} \subset \mathbb{S}^* \) and the arc \( \beta = \{ z \mid \overline{z} \in \beta \} \subset S_m \). And the map \( \chi \) sends both \( \gamma_{\phi_1} \) and \( \gamma_{\phi_2} \) to the branch cut \( (0, +\infty) \).

When the point \( s \in \gamma^0 \) starts at \( s_{m+1}^{-} \) and ends at \( s_m^{+} \) in the counterclockwise direction of \( \partial D_s^1 \), the point \( 1 - \bar{s} \in \gamma \) starts at \( s_{m+1}^{-} \) and ends at \( s_m^{+} \) in the clockwise direction of \( \partial D_s^1 \), and the point \( 1 - s \in \overline{\gamma} \) starts at \( s_{m+1}^{-} \) and ends at \( s_m^{+} \).

As \( s \) describes the arc \( \gamma^0 \), the value \( z = \chi(s) \in \beta^0 \) moves continuously, starting at \( 1 + i0^+ \) on the top edge of the branch cut \( (0, +\infty) \) and ending at \( 1 + i0^- \) on the bottom edge of \( (0, +\infty) \), with \( \arg(z) \) increasing from \( \phi_2 = -2\pi(m+1) \) to \( \phi_1 = -2\pi m \).

For nonvanishing \( \zeta(s) \), taking the logarithm of \((34)\), we obtain
\[
(41) \quad \log G(z) = \log z + \log G_{-\left(\frac{1}{z}\right)}
\]
The derivative of \((41)\) with respect to \( z \) is
\[
(42) \quad \frac{G'(z)}{G(z)} = \frac{1}{z} - \frac{1}{z^2} \frac{G_{-\left(\frac{1}{z}\right)}}{G_{-\left(\frac{1}{z}\right)}}
\]
which can be integrated along the arc \( \beta^0 \subset S_m \) as
\[
(43) \quad \int_{\beta^0} \frac{G'(z)}{G(z)} \, dz = \int_{\beta^0} \frac{dz}{z} - \int_{\beta^0} \frac{1}{z^2} \frac{G_{-\left(\frac{1}{z}\right)}}{G_{-\left(\frac{1}{z}\right)}} \, dz
\]
\[
= \int_{1+i0^-}^{1+i0^+} \frac{dz}{z} + \int_{\beta} \frac{G_{-\left(\eta\right)}}{G_{-\left(\eta\right)}} \, d\eta
\]
\[
\to 2\pi i + \int_{\beta} \frac{G_{-\left(\eta\right)}}{G_{-\left(\eta\right)}} \, d\eta, \quad \epsilon \to 0.
\]

For nonvanishing \( \zeta(s) \), by \((40)\) and \((43)\) we obtain
\[
\lim_{\epsilon \to 0} \int_{\partial D_s^1} \frac{\zeta'(s)}{\zeta(s)} \, ds = \lim_{\epsilon \to 0} \left\{ \int_{\gamma^0} \frac{\zeta'(s)}{\zeta(s)} \, ds - \int_{\gamma} \frac{\zeta'(s)}{\zeta(s)} \, ds \right\}
\]
\[
= \lim_{\epsilon \to 0} \left\{ \int_{\gamma^0} \frac{\zeta'(s)}{\zeta(s)} \, ds - \int_{\gamma} \frac{\zeta'(s)}{\zeta(s)} \, ds \right\}
\]
\[
= \lim_{\epsilon \to 0} \left\{ \int_{\beta^0} \frac{G'(z)}{G(z)} \, dz - \int_{\beta} \frac{G_{-\left(\eta\right)}}{G_{-\left(\eta\right)}} \, d\eta \right\}
\]
\[
= 2\pi i.
\]
Lemma 4.2. The logarithmic integral of nonvanishing zeta function

\[
\lim_{\epsilon \to 0} \int_{\partial D_2^s} \frac{\zeta'(s)}{\zeta(s)} ds = 0
\]

where \( D_2^s \) is type-two simple domain defined in Definition 4.3.

Proof. We only sketch the proof here.

The branch cut is chosen to be \((-\infty, 0]\), such that all the arcs broken from \( \partial D_2^s \) or its conjugate \( \overline{\partial D_2^s} \) stay inside one sheet \( S_m \) or its conjugate \( S_m^* \).

When the point \( s \in \gamma^0 \) starts at \( s^+_m \) and ends at \( s^-_m \) in the counterclockwise direction of \( \partial D_2^s \), the point \( 1 - \frac{s}{z} \in \gamma \) starts at \( s^+_m \) and ends at \( s^-_m \) in the clockwise direction of \( \partial D_2^s \), and the point \( 1 - s \in \gamma \) starts at \( s^+_m \) and ends at \( s^-_m \).

As \( s \) describes the arc \( \gamma^0 \), the value \( z = \chi(s) \in \beta^0 \) moves continuously, starting at \( 1 + i0^+ \) on the top edge of \((0, +\infty)\) and ending at \( 1 + i0^- \) on the bottom edge of \((0, +\infty)\), without cutting through the branch cut \((-\infty, 0]\), and the total increase in \( \arg(z) \) is zero.

That makes the difference in (43), which leads to

\[
\int_{\beta^0} \frac{G'(z)}{G(z)} dz = \int_{\beta^0} \frac{dz}{z} - \int_{\beta^0} \frac{1}{z^2} \frac{G'(\frac{1}{z})}{G(-\frac{1}{z})} dz
\]

\[
= \int_{1+i0^+} \frac{dz}{z} + \int_{\beta} \frac{G'(\eta)}{G(-\eta)} d\eta
\]

\[
\to \int_{\beta} \frac{G'(\eta)}{G(-\eta)} d\eta, \quad \epsilon \to 0
\]

and

\[
\lim_{\epsilon \to 0} \int_{\partial D_2^s} \frac{\zeta'(s)}{\zeta(s)} ds = 0.
\]

\( \square \)

Finally we give some conclusion about the nontrivial zeros of \( \zeta(s) \) when \( s \) is far away from the real axis.

Theorem 4.3. Suppose \( t \) is sufficiently large. All the zeros of \( \zeta(\sigma + it) \) are on the critical line \( \{\sigma + it | \sigma = \frac{1}{2}\} \). And each time \( \chi(\frac{1}{2} + it) \) loops once around the origin from \( \{1\} \) to \( \{1\} \) on the unit circle \( S^1 \) with increasing \( t \), there exists one zero of \( \zeta(\frac{1}{2} + it) \).

Proof. Let \( s = \sigma + it \).

The FAR domain \( D \) in the \( s \)-plane can be separated into infinitely many horizontal strips

\[
D_m = D(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \gamma_\phi
\]
where \( m \) is sufficiently large integer and \( \phi_2 = \phi_1 - 2\pi \), with their boundaries \( \gamma_{\phi_1} \) and \( \gamma_{\phi_2} \).

Step 1. Estimate the number of zeros and poles of \( \zeta(s) \) in each domain \( D_m \) with its boundary.

Let 
\[
\overline{D}_m = \bigcup_{\phi_2 \leq \phi \leq \phi_1} \gamma_{\phi}
\]
and \( D_0 = \overline{D}_m \cap \{ s | 0 < \sigma < 1 \} \) where \( \phi_1 = -2\pi m \). The domain \( D_0 \) is bounded and suppose \( T = \sup_{\sigma + it \in D_0} t \).

By von Mangoldt’s result\[3\] the number of zeros in the domain \( \{ \sigma + it | 0 < \sigma < 1, 0 < t \leq T \} \) is \( \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \), requiring a less number of zeros in each domain \( \overline{D}_m \). The only pole of \( \zeta(s) \) is at \( s = 1 \).

Therefore each domain \( \overline{D}_m \) contains a finite number of zeros and no pole.

Step 2. Count the number of zeros of \( \zeta(s) \) in each domain \( D_m \).

Suppose \( \rho_1, \rho_2, \ldots, \rho_n \) are the finite number of zeros of \( \zeta(s) \) in the domain \( D_m \). And the punctured domain \( D_m \setminus \{ \rho_1, \rho_2, \ldots, \rho_n \} \) is multiply-connected.

Supposing any \( \rho \in \gamma_{\phi} \) where \( \phi = -2\pi m \) such that \( \zeta(\rho) = 0 \), then type-two simple domain in \( D_m^2 \) could be constructed to contain \( \rho \). The argument principle required that the logarithmic integrals along the boundaries of this pair of domains differ by \( 4\pi i \), which made a contradiction with Lemma 4.1.

Therefore all the zeros of \( \zeta(\sigma + it) \) in the domain \( D_m \) are on the critical line, and Lemma 4.1 rules the number to be one. In other words, each time \( \chi(\frac{1}{2} + it) \) loops once around the origin from \( \{ 1 \} \) to \( \{ 1 \} \) on the unit circle \( S^1 \) with increasing \( t \), there exists one zero of \( \zeta(\frac{1}{2} + it) \).

Step 3. Count the number of zeros of \( \zeta(s) \) on the boundary of domain \( D_m \).

Let
\[
D_m^2 = D(\phi_2, \phi_1) = \bigcup_{\phi_2 < \phi < \phi_1} \gamma_{\phi}
\]
where \( \phi_1 = \pi - 2\pi m \) and \( \phi_2 = -\pi - 2\pi m \).

Supposing any \( \rho \in \gamma_{\phi} \) where \( \phi = -2\pi m \) such that \( \zeta(\rho) = 0 \), then type-two simple domain in \( D_m^2 \) could be constructed to contain \( \rho \). The argument principle required that the logarithmic integral along the boundary of the constructed simple domain to be nonzero, which made a contradiction with Lemma 4.2.

Therefore no zero of \( \zeta(s) \) is on the boundary of domain \( D_m \). In other words, \( \zeta(s) \neq 0 \) when \( \arg \chi(s) = -2\pi m \).

\[\square\]
Theorem 4.3 claims that the argument of \( \chi(\frac{1}{2} + it) \) determines the distribution of zeros of \( \zeta(s) \) on the critical line, which can be estimated by (11). Roughly speaking, there exists one nontrivial zero when \( t + t \log \frac{2\pi}{t} \) decreases by \( 2\pi \) for sufficiently large \( t \).

5. Conclusion

The chi function and tau function, as a pair of special functions similar to the exponential function and logarithm function, can be expected to play more roles in the complex analysis than to study Riemann zeta function. And the cross-branch technique is also presented for the analysis of multivalued complex function.

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(Received: May 9, 2020)

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