On the existence of nontrivial solutions for a nonlinear equation relative to a measure-valued Lagrangian on homogeneous spaces

Abstract. We prove the existence of a non-trivial solution for a nonlinear equation related to a measure-valued Lagrangian. The result is based on a compact embedding theorem of the Lagrangian domain and on the application of the Mountain Pass Theorem joined to a Palais-Smale condition.

1. Introduction and Result

We consider a locally compact separable Hausdorff topological space $X$ endowed with a measure $m$ and a quasidistance $d$. A quasidistance $d$ on $X$ is a function on $X \times X$ with the usual properties of a metric and a weaker version of the triangle inequality

$$d(x, y) \leq c_T(d((x, z) + d(z, y))), \quad c_T \geq 1.$$  

The set

$$B(x, R) = \{y \in X : d(x, y) < R\}$$

will be called a quasi-ball. The triple $(X, d, m)$ is assumed to satisfy the following property: for every $R_0 > 0$ there exists a constant $c_0 > 0$, dependent on $R_0$, such that for $r \leq \frac{R}{2} \leq R \leq R_0$

$$0 < c_0 \left(\frac{r}{R}\right)^\nu m(B(x, R)) \leq m(B(x, r))$$  

(1.1)

for every $x \in X$, where $\nu$ is a positive real number independent of $r, R, R_0$. Such a triple $(X, d, m)$ will be called a homogeneous space of dimension $\nu$. We point out, however, that a given exponent $\nu$ occurring in (1.1) should be considered, more precisely, as an upper bound of the “homogeneous dimension”, hence we should better call $(X, d, m)$ a homogeneous space of dimension less or equal than $\nu$. Our setting is given by a couple $(X, \mathcal{L})$, “a homogeneous space $X$ with a Lagrangian $\mathcal{L}$”, with the following properties

$$(L1): \mathcal{L} : C \mapsto \mathcal{M}(X)$$ is a map which associates with each function $u$ from a given subspace $\mathcal{C}$ of $C(X)$ a measure $\mathcal{L}[u] \in \mathcal{M}^+(X)$, where $C(X)$ denotes the space of all continuous functions on $X$ and $\mathcal{M}^+(X)$ the space of all nonnegative Radon measures on $X$.  

Universit`a degli Studi di Bergamo, Facolt`a di Ingegneria, Viale Marconi, 5, 24044 Dalmine (Bergamo) Italy.
E-mail: Garattini@mi.infn.it.
We assume that there exists \( k \geq 1 \) such that for a given \( p \geq 1 \), the following family of Poincaré-like inequalities holds on the metric quasi-balls \( B(x,r) \subset X \) [2]:

\[
\int_{B(x,r)} |u - u_{x,r}|^p \, dm \leq c_{p,r}^p \int_{B(x,kr)} d\mathcal{L}[u],
\]

where \( u_{x,r} \) is the average of \( u \) on \( B(x,r) \), for every \( u \in C \) and \( B(x,r) \subset X \).

If \( u \in C \) and \( g \in C^1(\mathbb{R}) \) with \( g' \) bounded on \( \mathbb{R} \), then \( g(u) : x \mapsto g(u(x)) \) belong also to \( C \) and

\[
\mathcal{L}[g(u)] = |g'(u)|^p \mathcal{L}[u]
\]

We are interested in nontrivial solution of the following problem

\[
\int_X d\mathcal{L}[u] v(x) + \int_X V(x) u^p(x) v(x) m(dx) = \int_X f(u(x)) v(x) m(dx)
\]

for every \( v \in C \cap L^p(X, Vm) \) where \( u \in C \cap L^p(X, Vm) \) ( \( Vm \) is the Radon measure with density \( V \) with respect to \( m \)). Eq. (1.4) is a generalization of the problem of searching for nontrivial solution for a semilinear equation in the framework of Dirichlet forms as studied in Ref. [2] and in the framework of semilinear equations of the form

\[
\triangle u + u^p = 0
\]

considered in Ref. [3]. Further developments on semilinear equations for Dirichlet forms can be found in Ref. [3] for problems of the type

\[
\int_{\Omega} \alpha(u,v)(dx) - \lambda \int_{\Omega} a(x) u(x) v(x) m(dx) = \int_{\Omega} f(u(x)) v(x) m(dx),
\]

where \( \Omega \) is an open bounded subset of \( X \), \( \alpha(u,v) \) is a uniquely defined signed Radon measure on \( X \), \( \lambda \) is an arbitrary nonvanishing number and \( a \in Lip(\Omega) \) with \( a(x) > 0 \). To analyze Eq. (1.4), we assume that

\[
W = \left\{ u : \int_X d\mathcal{L}[u] + \int_X Vu^p m(dx) < +\infty \right\}
\]

and that

\[
\|u\|_W = \left[ \int_X d\mathcal{L}[u] + \int_X Vu^p m(dx) \right]^{\frac{1}{p}}
\]

be a norm in \( W \). Moreover let us assume that \( V \in C(X, \mathbb{R}) \) and

\[
V(x) > 0, \quad \forall x \in X
\]

\[
V(x) \to +\infty, \quad \text{as} \quad d(0, x) \to +\infty
\]

where \( 0 \) is an arbitrarily fixed point in \( X \). We assume also that \( f(t) \in C(X, \mathbb{R}) \) satisfies the following conditions

\[
f(0) = 0, \quad f(t) = o(t), \quad \text{as} \quad t \to 0
\]
\[(1.11)\quad f(t) = o\left(|t|^\frac{\nu + \mu}{\nu - p}\right), \quad \text{as } |t| \to +\infty\]

if \(\nu > p\)
or

\[(1.12)\quad f(t) = o\left(|t|^\sigma\right), \quad \text{as } |t| \to +\infty\]

\(\sigma > p + 1\), if \(\nu \leq p\). Finally we assume that

\[(1.13)\quad 0 < \mu F(t) = \mu \int_0^t f(s) ds \leq tf(t)\]

where \(p < \frac{\nu \mu}{\nu - p}\) if \(\nu > p\) or \(p < \mu\) if \(\nu \leq p\). We observe that from the assumption \((1.13)\) it follows that there exists \(m > 0\) such that

\[(1.14)\quad F(t) \geq m |t|^\mu\]

for \(|t| \geq 1\). The result we will prove in the next Section is the following:

**Theorem 1.** Let the assumptions \((1.8), (1.9), (1.10), (1.13)\) hold together with \((1.11)\) if \(\nu > 2\) or with \((1.12)\) if \(\nu = 2\). Then the problem \((1.4)\) has a nontrivial solution.

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### 2. Preliminary results

We begin the section with a covering Lemma and its Corollary.

**Lemma 1.** A ball \(B(x, R)\) can be covered by a finite number \(n(r, R)\) of balls \(B(x_i, r)\), \(r \leq R\), such that \(x_i \in B(x, R)\) and \(B\left(x_i, \frac{\pi}{2}\right) \cap B\left(x_j, \frac{\pi}{2}\right) = \emptyset\) for \(i \neq j\). Moreover every point of \(B(x, R)\) is covered by at most \(M\) balls \(B(x_i, R)\) where \(M\) depends on \(r\).

**Proof.** The first part of the result follows immediately from assumption \((1.1)\). For the second part we observe that if a point \(x\) in \(B(x, R)\) is covered by the ball \(B(x_i, r)\), then \(x_i \in B(x, r)\); so the number \(M\) of the balls \(B(x_i, r)\), that cover \(x\), is estimated by the greatest number \(Q\) of points \(y_k\) in \(B(x, r)\) with \(d(y_k, y_{k+1}) \geq \frac{\pi}{2}\) and we observe that, by \((1.1)\), \(Q\) is estimated by a number \(M\) depending only on \(r\).

From Lemma 1, we obtain the following

**Corollary 1.** The space \(X\) can be covered by a countable union of balls \(B(x_i, r)\), such that \(B\left(x_i, \frac{\pi}{2}\right) \cap B\left(x_j, \frac{\pi}{2}\right) = \emptyset\) for \(i \neq j\). Moreover every point of \(X\) is covered by at most \(M\) balls, where \(M\) depends only on \(r\).

We prove now a compact embedding result

**Lemma 2.** Let the assumption related to inequality \((1.2)\) holds. Then every sequence \(\{u_n\}\) in \(C\left[B(x, (k + 1)R)\right]\) such that

\[(2.1)\quad \int_{B(x, kr)} dL[u] \leq C\]

is relatively compact in \(L^p\left(B(x, R)\right)\).
Proof. We have to prove that there is a subsequence of \( \{u_n\} \) convergent in \( L^p (B(x, R), m) \). Taking into account assumption (1.1), the ball \( B(x, R) \) can be covered by a finite number of balls \( B(x_j, r_j), r_j \leq \frac{4}{Q}, j = 1, \ldots, Q \) where \( Q \) depends on \( r, R \), such that every point of \( B(x, R) \) belongs at most to \( M \) balls, where \( M \) does not depend on \( r \). Let \( w_{n,m} = u_n - u_m \) and \( \bar{w}_{n,m} = \int_{B(x_j,r)} w_{n,m} \, m(dx) \). Then

\[
\int_{B(x,R)} w_{n,m}^p m(dx) \leq \sum_{j=1}^{Q} \int_{B(x_j,r)} w_{n,m}^p m(dx) = \sum_{j=1}^{Q} \int_{B(x_j,r)} |w_{n,m} - \bar{w}_{n,m} + \bar{w}_{n,m}|^p m(dx)
\]

(2.2) \[
\leq 2^{p-1} \sum_{j=1}^{Q} \int_{B(x_j,r)} |w_{n,m} - \bar{w}_{n,m}|^p m(dx) + 2^{p-1} \sum_{j=1}^{Q} \int_{B(x_j,r)} (\bar{w}_{n,m})^p m(dx).
\]

Since

\[
\int_{B(x_j,r)} (\bar{w}_{n,m})^p m(dx) = \int_{B(x_j,r)} m(dx) \left( \int_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p
\]

(2.3) \[
= \frac{1}{m^{p-1}(B(x_j,r))} \left( \int_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p,
\]

then inequality (2.2) becomes

\[
2^{p-1} \sum_{j=1}^{Q} \int_{B(x_j,r)} |w_{n,m} - \bar{w}_{n,m}|^p m(dx) + 2^{p-1} \sum_{j=1}^{Q} \int_{B(x_j,r)} (\bar{w}_{n,m})^p m(dx)
\]

\[
\leq 2^{p-1} c_{p^{\nu}} \sum_{j=1}^{Q} \int_{B(x_j,kr)} d\mathcal{L}[u] + 2^{p-1} \sum_{j=1}^{Q} \frac{1}{m^{p-1}(B(x_j,r))} \left( \int_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p.
\]

(2.4) \[
\leq 2^{p-1} c_{p^{\nu}} MCK^\nu + \left( \frac{R}{r} \right)^{\nu(p-1)} \frac{2^{p-1}}{m^{p-1}(B(x,R)) c_0} \sum_{j=1}^{Q} \left( \int_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p.
\]

Choose \( r = r_x \) and \( \varepsilon > 0 \) such that \( 2^{p-1} c_{p^{\nu}} MCK^\nu \leq \varepsilon \). Suppose \( \{u_n\} \) is weakly convergent in \( L^p (B(x, (k+1) R), m) \) then

\[
\left( \frac{R}{r_x} \right)^{\nu(p-1)} \frac{2^{p-1}}{m^{p-1}(B(x,R)) c_0} \sum_{j=1}^{Q} \left( \int_{B(x_j,r)} (w_{n,m}) m(dx) \right)^p \leq \frac{\varepsilon}{2}
\]

(2.5)
for \( n, m \geq n_\varepsilon \). This implies
\[
\int_{B(x,R)} w_{n,m}^p \, dx \leq \varepsilon
\]
and \( \{u_n\} \) is a Cauchy sequence in the space \( L^p(B(x,R),m) \) then \( \{u_n\} \) is convergent in \( L^p(B(x,R),m) \).

**Lemma 3.** Let \( W \subset C \) be the space defined in Eq. (1.6) and let us assume that \( W \) be a Banach space w.r.t. \( \| \cdot \|_W \), then the embedding of \( W \) in \( L^p(X,m) \) is compact.

**Proof.** Let \( \|u_k\|_W \leq C \). After extraction of a subsequence, we have that \( \{u_k\} \) is weakly convergent in \( W \) to \( u \). We suppose, without loss of generality that \( u = 0 \) and prove
\[
\int_X u_k^p \, dx \to 0 \quad (2.7)
\]
when \( k \to +\infty \). Let \( \varepsilon > 0 \), \( \exists R > 0 \) such that \( V(x) \geq 1 + \frac{Cp}{\varepsilon} \) when \( d(x,0) \geq R \). Since \( \int_{B(0,R)} u_k^p \, dx \to 0 \) when \( k \to +\infty \), then \( \exists k \) such that for \( k \geq k_\varepsilon \)
\[
\int_{B(0,R)} u_k^p \, dx \leq \frac{\varepsilon}{1 + Cp} \quad (2.8)
\]
Then for \( k \geq k_\varepsilon \)
\[
\int_X u_k^p \, dx = \int_{B(0,R)} u_k^p \, dx + \int_{X \setminus B(0,R)} u_k^p \, dx \leq \frac{\varepsilon}{1 + Cp} \left[ 1 + \int_{X \setminus B(0,R)} V u_k^p \, dx \right] \leq \frac{\varepsilon}{1 + Cp} [1 + \|u_k\|_W^p] \leq \varepsilon. \quad (2.9)
\]

**3. Proof of Theorem 1**

The function on \( W \) associated to our problem can be written as
\[
\varphi(u) = \frac{1}{2} \|u\|_W^p - \int_X F(u(x)) \, m(dx). \quad (3.1)
\]
It can be proved that \( \varphi \in C^1(W,\mathbb{R}) \) and
\[
\langle \varphi'(u), v \rangle = (u,v)_W - \int_X f(x,u(x)) \, v(x) \, m(dx). \quad (3.2)
\]
The critical points of \( \varphi \) are weak solution of our problem, then to prove Theorem 1 it is enough to prove the existence of nontrivial points for \( \varphi \).

**Proposition 1.** The functional \( \varphi \) satisfies the Palais-Smale condition under assumption of Theorem 1.
Proof. Let \( \{u_k\} \) be a sequence in \( W \) such that
\[
|\varphi'(u_k)| \leq C, \quad \varphi'(u_k) \to 0,
\]
in \( W^* \) as \( k \to +\infty \), where \( W^* \) denotes the dual space of \( W \). From (3.3) we obtain that there exists \( k_0 \) such that for \( k \geq k_0 \),
\[
\|W^* u\| \leq \mu \|u_k\|_W.
\]
Then
\[
C + \|u\|_W^p \geq \varphi(u_k) - \frac{1}{\mu} \langle \varphi'(u_k), u \rangle
\]
\[
= \frac{1}{2} \|u_k\|_W^p - \int_0^\infty F(u_k(x)) m(dx) - \frac{1}{\mu} \left( \|u_k\|_W^p - \int_0^\infty f(u_k(x)) u_k m(dx) \right)
\]
\[
= \left( \frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|_W^p - \int_0^\infty F(u_k(x)) m(dx) - \frac{1}{\mu} \int_0^\infty f(u_k(x)) u_k m(dx) \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|_W^p.
\]
Since \( \{u_k\} \) is bounded in \( W \) and from Lemma 3, we know that there exists a subsequence strongly convergent in \( L^p(X, m) \) and weakly to \( u \in W \). We apply now the Lemma 5 if \( \nu \geq p \) or the Lemma 6 if \( \nu < p \) of Ref. [4] to the function \( g(t) = f(t) \) and to the sequence \( \{u_k\} \) and we obtain
\[
\lim_{k \to +\infty} \int_0^{\infty} f(u_k)(u_k - u) m(dx) = 0.
\]
From the assumption we have that
\[
|\langle \varphi'(u), v \rangle| \leq \varepsilon_k \|v\|_W
\]
where \( \varepsilon_k \to 0 \) as \( k \to +\infty \). Then from (3.7) we have
\[
\langle \varphi'(u_k), u_k - u \rangle = \langle u_k, u_k - u \rangle_W - \int_0^{\infty} f(x, u_k(x)) (u_k - u)(x) m(dx)
\]
\[
= \|u_k\|_W^p - (u_k, u)_W - \int_0^{\infty} f(x, u_k(x)) (u_k - u)(x) m(dx).
\]
From (3.6) and (3.7) we obtain
\[
\langle \varphi'(u_k), u_k - u \rangle \to \|u_k\|_W^p - (u_k, u)_W \to 0,
\]
when \( k \to +\infty \). This implies that \( \{u_k\} \) converges to \( u \) strongly in \( W \).

Proof of Theorem 4. First we prove that for \( \rho \leq \min \left( \frac{\mu}{\nu} m(B(0, 1)), \frac{1}{\mu} \right) \) small enough \( \varphi(u) \geq \gamma > 0 \) for \( \|u_k\|_W = \rho \). Consider the case \( \nu \geq p \). As in Lemma 5 of Ref. [4] we obtain that for every \( \varepsilon > 0 \) there exists a constant \( C \) such that
\[
0 \leq F(t) \leq \varepsilon \left( |t|^p + |t|^\beta \right) + C |t|^\beta
\]
where \( \beta = \frac{\nu}{\nu - p} \) if \( \nu > p \) or \( \beta = \sigma + 1 \) if \( \nu = p \). There exists \( C \) such that
\[
\|u\|_{L^p(X, m)} \leq C \|u\|_W, \quad \|u\|_{L^\infty(X, m)} \leq C \|u\|_W.
\]
Choose $\varepsilon < \frac{1}{2\rho}$; then
\[
\int_X F(u) m(dx) \leq \varepsilon \left[ \int_X |u|^p m(dx) + \int_X |u|^\beta m(dx) \right] + C_\varepsilon \int_X |u|^\beta m(dx)
\]
(3.12)
\[
eq \varepsilon \left( \|u\|_{L^p(X,m)} + \|u\|_{L^\beta(X,m)} \right) + C_\varepsilon \|u\|_{L^\beta(X,m)} \leq \varepsilon \left( C^p \|u\|_W + C^\beta \|u\|_W \right) + C_\varepsilon C^\beta \|u\|_W
\]
and
\[
\varphi(u) = \frac{1}{2} \|u\|_W^p - \int_X F(u) m(dx) \geq \left( \frac{1}{2} - \varepsilon C^p \right) \|u\|_W^p - C^\beta (\varepsilon + C_\varepsilon) \|u\|_W^\beta
\]
(3.13)
\[
\geq \rho^p - C^\beta (\varepsilon + C_\varepsilon) \rho^\beta
\]
and the result follows from the last inequality. We consider now the case $\nu < 2$. From the assumption we obtain that for every $\varepsilon > 0$ there exists a constant $\delta > 0$ such that
(3.14)
\[
F(t) \leq \varepsilon |t|^p
\]
for $|t| \leq \delta$. We observe that there exists $C$ such that
(3.15)
\[
\|u\|_{L^p(X,m)} \leq C \|u\|_W, \quad \|u\|_{L^\infty(X,m)} \leq C \|u\|_W.
\]
Choosing $\|u\|_W = \rho = \frac{\delta}{C}$, we have $\|u\|_{L^\infty(X,m)} \leq \delta$; then
(3.16)
\[
\int_X F(u) m(dx) \leq \varepsilon \int_X |u|^p m(dx) = \varepsilon \|u\|_{L^p(X,m)}^p \leq \varepsilon C^p \|u\|_W^p
\]
and
(3.17)
\[
\varphi(u) = \frac{1}{2} \|u\|_W^p - \int_X F(u) m(dx) \geq \left( \frac{1}{2} - \varepsilon C^p \right) \|u\|_W^p \geq \rho^p.
\]
The result follows from the last inequality. Let us prove the existence of $u_0 \in X \setminus B_{\rho}$ such that $\varphi(u) \leq 0$. Let $u_0 \in D[a]$ be the potential of the ball $B(0,1)$ with respect to the ball $B(0,2)$. Then $u_0$ is in $W$ and $\|u_0\|_W \geq \text{am}(B(0,1)) > \rho$; we recall that
(3.18)
\[
F(u_0(x)) \geq m |u_0(x)|^\mu
\]
for $x \in B(0,1)$. Let $\gamma > 1$; we have $u_0(x) = 1$ on $B(0,1)$, so
\[
\varphi(\gamma u_0) = \frac{1}{2} \gamma^p \|u_0\|_W^p - \int_X F(\gamma u_0) m(dx) \leq \frac{1}{2} \gamma^p \|u_0\|_W^p - \int_{B(0,1)} F(\gamma u_0) m(dx)
\]
(3.19)
\[
\leq \frac{1}{2} \gamma^p \|u_0\|_W^p - m\gamma^\mu \int_{B(0,1)} |u_0|^\mu m(dx) \leq \frac{1}{2} \gamma^p \|u_0\|_W^p - m\gamma^\mu \mu (B(0,1)).
\]
Since $\mu > p$ we have for $\gamma > \gamma_0$, $\gamma_0$ suitable, we have $\varphi(\gamma u_0) < 0$. The proof is completed with the application of the Mountain Pass Theorem.\]
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