Classification of \((k;4)\)-arcs up to projective inequivalence, for \(k < 10\)

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Abstract

In this paper, the classification of \((k;4)\)-arcs up to projective inequivalence for \(k < 10\) in \(\text{PG}(2,13)\) is introduced in details according to their inequivalent number, stabilisers, the action of each stabiliser on the associated arc, and the inequivalent classes \(N_c\) of secant distributions of arcs. Here, the strategy is to start from the projective line \(\text{PG}(1,13)\) where there are three projectively inequivalent tetrads.

1 Basic concepts

1.1 Finite fields

A field \(F\) is a set of elements with two operations, addition (\(+\)) and multiplication (\(\times\)), satisfying the following properties:

(a) \((F,+)\) is an abelian group with identity 0;
(b) \((F \setminus \{0\}, \times)\) is an abelian group with identity 1;
(c) \(x(y+z) = xy + xz,\) for all \(x,y,z \in F\).

1.2 Note

A finite field is defined up to an isomorphism by the number \(q\) of its elements. So, \(q\) must be an integer power \(p^h\) of a prime \(p\). Here, \(p\) is the characteristic of the finite field. Then, every
element \( x \in \mathbb{F}_q \) satisfies \( x^q - x = 0 \). When \( q = p \), then \( \mathbb{F}_p = \{0, 1, \ldots, p-1\} \); when \( q = p^h \), \( h > 1 \), then \( \mathbb{F}_q = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{q-2} | \alpha^{q-1} = 1\} \) for some \( \alpha \in \mathbb{F}_q \). The non-zero elements of \( \mathbb{F}_q \) form a group \( \mathbb{F}_q^* \) of order \( q - 1 \) such that \( \mathbb{F}_q^* \cong \mathbb{Z}_{q-1} \).

### 1.3 Finite groups

**Definition 1.** A group is an ordered pair \((G, \ast)\), where \( G \) is a non-empty set and \( \ast \) is a binary operation on \( G \) such that the following properties hold.

1. For all \( a, b, c \in G \), \( a \ast (b \ast c) = (a \ast b) \ast c \).
2. There exists \( e \in G \) such that for all \( a \in G \), \( a \ast e = a = e \ast a \).
3. For all \( a \in G \), there exists \( b \in G \) such that \( a \ast b = e = b \ast a \).

### 1.4 Group action on a set

Let \( G \) be a group acts on a set \( X \) if for each \( g \in G \) and \( x \in X \) an element \( gx \in X \) is defined, such that \( g_2(g_1x) = (g_2g_1)x \) and \( ex = x \) for all \( x \in X \), \( g_1, g_2 \in G \).

The set \( \text{Orb}(x) = \{gx | g \in G\} \),

is called the orbit of the element \( x \). The stabilizer of an element \( x \) of \( X \) is the subgroup

\[ S = \{g \in G | gx = x\} . \]

The fixed points set of an element \( g \) of \( G \) is the set defined as follows:

\[ \text{Fix}(g) = \{x \in X | gx = x\} . \]

### 2 The projective plane \( \text{PG}(2, q) \)

The projective plane \( \text{PG}(2, q) \) over \( \mathbb{F}_q \) contains \( q^2 + q + 1 \) points and lines. There are \( q + 1 \) points on each line and \( q + 1 \) lines passing through each point. The value of \( q \) that has been used in this work is \( q = 13 \). Therefore the projective plane \( \text{PG}(2, 13) \) has 183 points and lines, with 14 points on each line and 14 lines passing through each point. The point \( P(x_0, x_1, x_2) \) in the projective plane, \( \text{PG}(2, q) \), can be represented as a vector of three coordinates over \( \mathbb{F}_q \) as shown in Table 1.
### Table 1: The points in PG(2, q)

| Point format | Number of points |
|--------------|------------------|
| P(x₀, x₁, 1) | q²               |
| P(x₀, 1, 0)  | q                |
| P(1, 0, 0)   | 1                |

A line in PG(2, q) is a set of points P(x₀, x₁, x₂) satisfying the homogeneous linear equation

\[ ax₀ + bx₁ + cx₂ = 0, \]

with \( a, b, c \in \mathbb{F}_q \) not all zero; it is denoted by \( L(a, b, c) \). Thus, a projective plane is an incidence structure of points and lines with the following properties:

(i) every two points are incident with a unique line;
(ii) every two lines are incident with a unique point;
(iii) there are four points, no three collinear.

### 3 General linear group of a vector space

Let \( \mathbb{F}_q \) is a finite field and let \( V(n, q) \) is a vector space of dimension \( n \) over \( \mathbb{F}_q \), then the linear map \( V(n, q) \rightarrow V(n, q) \), such that \( x \rightarrow xA \), for \( x \in V \) a row vector and \( A \) a non-singular \( n \times n \) matrix over \( \mathbb{F}_q \). The group consisting of all linear maps of \( V(n, q) \), that is, the group consisting of all non-singular \( n \times n \) matrices over \( \mathbb{F}_q \), is called the general linear group and is denoted by \( \text{GL}(n, q) \). The order of \( \text{GL}(n, q) \) is as follows:

\[ |\text{GL}(n, q)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}). \]

In addition, the subgroup \( \text{SL}(n, q) \) consisting of all matrices with determinant 1, and it is called the special linear group of degree \( n \) over \( \mathbb{F}_q \). The group \( \text{SL}(n, q) \) contains a subgroup \( \text{UT}(n, q) \) consisting of those matrices with all entries below the main diagonal zero, and with the entries on the main diagonal equal to the identity. This subgroup is called the unitriangular group of degree \( n \) over \( \mathbb{F}_q \).
3.1 The fundamental theorem in $\text{PG}(2,q)$

If $\phi : \mathcal{P} \to \mathcal{P}'$ is a bijective mapping from one projective plane, $\text{PG}(2,q)$, to another, then there is a unique projectivity shifting any quadrangle, that is, a set of four points no three collinear, to another quadrangle.

Definition .2

A $(k;n)$-arc $\mathcal{K}$ in $\text{PG}(2,q)$ is a set of $k$ points such that no $n+1$ of them are collinear but some $n$ are collinear.

3.2 Lexicographically least set

Given the sets $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ of integers, with $a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$. Then $A \leq B$ lexicographically if either $A = B$ or if, for some $i$ with $1 \leq i < r$, we have $a_1 = b_1, \ldots, a_i = b_i$, but $a_{i+1} < b_{i+1}$.

4 Classification of $(k;4)$-arcs up to projective inequivalence, for $k < 10$

The number of projectively inequivalent $(k;4)$-arcs for $k < 10$ is given in the following subsections.

4.1 Projectively inequivalent $(4;4)$-arcs

In this classification, the number of tetrads is constructed by fixing a triad, $\mathcal{U}_1 = \{1,2,9\}$. There are eleven tetrads containing $\mathcal{U}_1$. The lexicographically least sets in the $G$-orbits of tetrads, where $G = PGL(2,13)$ took 2104 msec. Then among these canonical sets there are three projectively inequivalent tetrads; this took 1699 msec. Also, the three tetrads have $sd$-equivalent secant distribution. It took 1734. The statistics are shown in Table 2.

| Number | Tetrad   | $\{t_4,t_3,t_2,t_1,t_0\}$ |
|--------|----------|---------------------------|
| 1      | $\{1,2,9,21\}$ | $\{1,0,0,52,130\}$       |
| 2      | $\{1,2,9,83\}$ | $\{1,0,0,52,130\}$       |
| 3      | $\{1,2,9,115\}$ | $\{1,0,0,52,130\}$       |

Theorem .3 In $\text{PG}(1,13)$, there are exactly three projectively inequivalent tetrads.
4.2 Projectively inequivalent (5;4)-arcs

The (5;4)-arcs are constructed by adding all the points from the plane, PG(2, 13), which are not on the line to each inequivalent tetrad given in Table 2. So, the constructed number of (5;4)-arcs is 507. The lexicographically least set images of the 507 (5;4)-arcs are computed. This shows that the number $\Phi_4$ of projectively inequivalent (5;4)-arcs is three. The three (5;4)-arcs all have the same secant distribution, that is, $\{1, 0, 4, 5, 8, 120\}$. In addition, the stabiliser of each of the three projectively inequivalent (5;4)-arcs is $Z_3 \times ((Z_4 \times Z_4) \rtimes Z_2)$, $Z_3 \times (Z_8 \rtimes Z_2)$, $Z_3 \times (SL(2, 3) \rtimes Z_2)$. The statistics are given in the following tables:

Table 3: Projectively inequivalent (5;4)-arcs

| Number | $\Phi_4$ | Stabiliser | $\{t_4, t_4, t_3, t_3, t_0\}$ |
|--------|---------|------------|-----------------------------|
| 1      | $\{1, 2, 9, 83, 3\}$ | $Z_3 \times ((Z_4 \times Z_4) \rtimes Z_2)$ | $\{1, 0, 4, 5, 8, 120\}$ |
| 2      | $\{1, 2, 9, 21, 3\}$ | $Z_3 \times (Z_8 \rtimes Z_2)$ | $\{1, 0, 4, 5, 8, 120\}$ |
| 3      | $\{1, 2, 9, 115, 3\}$ | $Z_3 \times (SL(2, 3) \rtimes Z_2)$ | $\{1, 0, 4, 5, 8, 120\}$ |

Theorem 4 In PG(2, 13), there are exactly three projectively inequivalent (5;4)-arcs.
Remark

The stabiliser groups in Table 3 split the associated projectively inequivalent (5;4)-arcs into 2 orbits. They are given as follows.

1. The group $Z_3 \times ((Z_4 \times Z_4) \times Z_2)$ partitions the (5;4)-arc \{1,2,9,83,3\} into 2 orbits \{1,9,2,83\}, \{3\}.
2. The group $Z_3 \times (Z_8 \times Z_2)$ splits the (5;4)-arc \{1,2,9,21,3\} into 2 orbits \{1,9,2,21\}, \{3\}.
3. The group $Z_3 \times (\text{SL}(2,3) \times Z_2)$ divides the (5;4)-arc \{1,2,9,115,3\} into 2 orbits \{1,2,115,9\}, \{3\}.

4.3 Projectively inequivalent (6;4)-arcs

In Table 3, for each projectively inequivalent (5;4)-arc the points from the plane which are not on any 4-secant are added to construct the (6;4)-arcs. Therefore, the number of (6;4)-arcs that constructed is 504. Among the 504 (6;4)-arcs the lexicographically least set image and the stabiliser are calculated. So, the number $\Phi_4$ of projectively inequivalent (6;4)-arcs is 10. Also, the secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ for each of the 10 projectively inequivalent (6;4)-arcs is computed. It shows that there are only two \textit{sd}-inequivalent classes $N_c$ of secant distributions. The statistics of the 10 projectively inequivalent (6;4)-arcs are given in the following tables:

\begin{table}[h]
\centering
\caption{Projectively inequivalent (6;4)-arcs}
\begin{tabular}{|c|c|c|}
\hline
Number & $\Phi_4$ & Stabiliser & Orbits \\
\hline
1 & \{1,2,9,83,3,4\} & $Z_3 \times Z_2$ & \{1\}, \{2,\}, \{3,4\}, \{9,83\} \\
2 & \{1,2,9,21,3,4\} & $Z_2$ & \{1\}, \{2\}, \{3,4\}, \{9\}, \{21\} \\
3 & \{1,2,9,115,3,4\} & $Z_3 \times Z_2$ & \{1\}, \{2,115,9\}, \{3,4\} \\
4 & \{1,2,9,83,3,8\} & \{9,2,83\}, \{3,8\} \\
5 & \{1,2,9,21,3,5\} & \{9\}, \{21\} \\
6 & \{1,2,9,21,3,12\} & \{1\}, \{2\}, \{3,12\}, \{9\}, \{21\} \\
7 & \{1,2,9,21,3,14\} & \{1\}, \{2\}, \{3,14\}, \{9,21\} \\
8 & \{1,2,9,83,3,5\} & \{1\}, \{2\}, \{3,5\}, \{9\}, \{83\} \\
9 & \{1,2,9,115,3,7\} & \{1,115\}, \{2\}, \{3,7\} \\
10 & \{1,2,9,115,3,5\} & \{1\}, \{2,115,9\}, \{3,5\} \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{$N_c$ of $\{t_4,t_3,t_2,t_1,t_0\}$}
\begin{tabular}{|c|c|c|}
\hline
Number & $N_c$ & Number of $N_c$ \\
\hline
1 & \{1,1,6,65,110\} & 3 \\
2 & \{1,0,9,62,111\} & 7 \\
\hline
\end{tabular}
\end{table}

\textbf{Theorem 5} In $\text{PG}(2,13)$, there are exactly ten projectively inequivalent (6;4)-arcs.
4.4 Projectively inequivalent $(7;4)$-arcs

In this process, the constructed number of $(7;4)$-arcs is 1670. According to their lexicographically least set images, the number of projectively inequivalent $(7;4)$-arcs is 207. Among the 207 arcs, there are eleven types of the stabiliser groups. In addition, the secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ of each of the $(7;4)$-arcs is also computed. It shows that there are five $sd$-inequivalent classes of secant distributions. The statistics are given in Tables 7 and 8.

| Number | $\Phi_4$     | Stabiliser |
|--------|--------------|------------|
| 1      | $\{1,2,9,83,3,4,57\}$ | $I$       |
| 2      | $\{1,2,9,83,3,4,5\}$      | $I$       |
| 3      | $\{1,2,9,21,3,4,20\}$     | $I$       |
| 4      | $\{1,2,9,21,3,4,5\}$      | $I$       |
| 5      | $\{1,2,9,21,3,4,22\}$     | $I$       |
| 6      | $\{1,2,9,21,3,4,32\}$     | $I$       |
| 7      | $\{1,2,9,21,3,4,37\}$     | $I$       |
| 8      | $\{1,2,9,21,3,4,58\}$     | $I$       |
| 9      | $\{1,2,9,115,3,4,22\}$   | $Z_2$     |
| 10     | $\{1,2,9,115,3,4,5\}$     | $I$       |
| 11     | $\{1,2,9,83,3,4,51\}$     | $I$       |
| 12     | $\{1,2,9,83,3,4,6\}$      | $I$       |
| 13     | $\{1,2,9,83,3,4,19\}$     | $I$       |
| 14     | $\{1,2,9,21,3,4,13\}$     | $I$       |
| 15     | $\{1,2,9,21,3,4,19\}$     | $I$       |
| 16     | $\{1,2,9,21,3,4,96\}$     | $I$       |
| 17     | $\{1,2,9,21,3,4,27\}$     | $I$       |
| 18     | $\{1,2,9,21,3,4,28\}$     | $I$       |
| 19     | $\{1,2,9,21,3,4,56\}$     | $I$       |
| 20     | $\{1,2,9,21,3,4,149\}$    | $I$       |
| 21     | $\{1,2,9,21,3,4,122\}$    | $I$       |
| 22     | $\{1,2,9,21,3,4,6\}$      | $I$       |
| 23     | $\{1,2,9,83,3,4,30\}$     | $I$       |
| 24     | $\{1,2,9,115,3,4,50\}$    | $I$       |
| 25     | $\{1,2,9,115,3,4,15\}$    | $I$       |
| 26     | $\{1,2,9,83,3,4,27\}$     | $I$       |
| 27     | $\{1,2,9,83,3,4,33\}$     | $I$       |
| 28     | $\{1,2,9,115,3,4,10\}$    | $I$       |
| 29     | $\{1,2,9,115,3,4,30\}$    | $I$       |
| 30     | $\{1,2,9,21,3,4,101\}$    | $I$       |
| 31     | $\{1,2,9,21,3,4,30\}$     | $I$       |
| 32     | $\{1,2,9,83,3,4,47\}$     | $I$       |
| 33     | $\{1,2,9,21,3,4,40\}$     | $I$       |
| 34     | $\{1,2,9,21,3,4,75\}$     | $I$       |
|   |  |   |
|---|---|---|
| 35 | \{1, 2, 9, 21, 3, 4, 127\} |   |
| 36 | \{1, 2, 9, 21, 3, 4, 100\} |   |
| 37 | \{1, 2, 9, 21, 3, 4, 14\} |   |
| 38 | \{1, 2, 9, 21, 3, 4, 111\} |   |
| 39 | \{1, 2, 9, 21, 3, 4, 12\} |   |
| 40 | \{1, 2, 9, 83, 3, 4, 15\} |   |
| 41 | \{1, 2, 9, 83, 3, 4, 16\} |   |
| 42 | \{1, 2, 9, 115, 3, 4, 103\} |   |
| 43 | \{1, 2, 9, 115, 3, 4, 6\} |   |
| 44 | \{1, 2, 9, 83, 3, 4, 8\} |   |
| 45 | \{1, 2, 9, 83, 3, 4, 43\} |   |
| 46 | \{1, 2, 9, 115, 3, 4, 7\} |   |
| 47 | \{1, 2, 9, 115, 3, 4, 20\} |   |
| 48 | \{1, 2, 9, 21, 3, 4, 18\} |   |
| 49 | \{1, 2, 9, 21, 3, 4, 65\} |   |
| 50 | \{1, 2, 9, 83, 3, 4, 20\} |   |
| 51 | \{1, 2, 9, 83, 3, 4, 92\} |   |
| 52 | \{1, 2, 9, 21, 3, 4, 136\} |   |
| 53 | \{1, 2, 9, 21, 3, 4, 95\} |   |
| 54 | \{1, 2, 9, 21, 3, 4, 49\} |   |
| 55 | \{1, 2, 9, 21, 3, 4, 44\} |   |
| 56 | \{1, 2, 9, 115, 3, 4, 18\} |   |
| 57 | \{1, 2, 9, 83, 3, 4, 11\} |   |
| 58 | \{1, 2, 9, 83, 3, 4, 31\} |   |
| 59 | \{1, 2, 9, 83, 3, 4, 10\} |   |
| 60 | \{1, 2, 9, 83, 3, 4, 17\} |   |
| 61 | \{1, 2, 9, 83, 3, 4, 49\} |   |
| 62 | \{1, 2, 9, 83, 3, 4, 23\} |   |
| 63 | \{1, 2, 9, 83, 3, 4, 28\} |   |
| 64 | \{1, 2, 9, 83, 3, 4, 54\} |   |
| 65 | \{1, 2, 9, 83, 3, 4, 13\} |   |
| 66 | \{1, 2, 9, 83, 3, 4, 37\} |   |
| 67 | \{1, 2, 9, 83, 3, 4, 40\} |   |
| 68 | \{1, 2, 9, 83, 3, 4, 26\} |   |
| 69 | \{1, 2, 9, 83, 3, 4, 76\} |   |
| 70 | \{1, 2, 9, 83, 3, 4, 25\} |   |
| 71 | \{1, 2, 9, 83, 3, 4, 7\} |   |
| 72 | \{1, 2, 9, 83, 3, 4, 82\} |   |
| 73 | \{1, 2, 9, 83, 3, 4, 71\} |   |
| 74 | \{1, 2, 9, 83, 3, 4, 108\} |   |
| 75 | \{1, 2, 9, 83, 3, 4, 126\} |   |
| 76 | \{1, 2, 9, 83, 3, 4, 14\} |   |
| 77 | \{1, 2, 9, 83, 3, 4, 130\} | \(Z_6\) |
| 78 | \{1, 2, 9, 83, 3, 4, 100\} | \(I\) |
| 79 | \{1, 2, 9, 21, 3, 4, 35\} | \(I\) |
| 80 | \{1, 2, 9, 115, 3, 4, 24\} | \(Z_2\) |
| 81 | \{1, 2, 9, 21, 3, 4, 16\} | \(I\) |
| 82 | \{1, 2, 9, 21, 3, 4, 43\} | \(I\) |
| 83 | \{1, 2, 9, 21, 3, 4, 46\} | \(I\) |
| 84 | \{1, 2, 9, 21, 3, 4, 51\} | \(I\) |
| 85 | \{1, 2, 9, 115, 3, 4, 8\} | \(I\) |
| 86 | \{1, 2, 9, 115, 3, 4, 16\} | \(I\) |
| 87 | \{1, 2, 9, 21, 3, 4, 50\} | \(I\) |
| 88 | \{1, 2, 9, 21, 3, 4, 82\} | \(I\) |
| 89 | \{1, 2, 9, 21, 3, 4, 17\} | \(I\) |
| 90 | \{1, 2, 9, 21, 3, 4, 152\} | \(I\) |
| 91 | \{1, 2, 9, 21, 3, 4, 76\} | \(I\) |
| 92 | \{1, 2, 9, 21, 3, 4, 55\} | \(I\) |
| 93 | \{1, 2, 9, 21, 3, 4, 94\} | \(I\) |
| 94 | \{1, 2, 9, 115, 3, 4, 17\} | \(I\) |
| 95 | \{1, 2, 9, 115, 3, 4, 33\} | \(I\) |
| 96 | \{1, 2, 9, 115, 3, 4, 34\} | \(I\) |
| 97 | \{1, 2, 9, 21, 3, 4, 8\} | \(I\) |
| 98 | \{1, 2, 9, 21, 3, 4, 57\} | \(I\) |
| 99 | \{1, 2, 9, 21, 3, 4, 103\} | \(I\) |
| 100 | \{1, 2, 9, 21, 3, 4, 47\} | \(I\) |
| 101 | \{1, 2, 9, 21, 3, 4, 48\} | \(I\) |
| 102 | \{1, 2, 9, 21, 3, 4, 34\} | \(I\) |
| 103 | \{1, 2, 9, 21, 3, 4, 26\} | \(I\) |
| 104 | \{1, 2, 9, 21, 3, 4, 108\} | \(I\) |
| 105 | \{1, 2, 9, 21, 3, 4, 25\} | \(I\) |
| 106 | \{1, 2, 9, 115, 3, 4, 74\} | \(I\) |
| 107 | \{1, 2, 9, 115, 3, 4, 26\} | \(I\) |
| 108 | \{1, 2, 9, 21, 3, 4, 15\} | \(I\) |
| 109 | \{1, 2, 9, 21, 3, 4, 7\} | \(I\) |
| 110 | \{1, 2, 9, 21, 3, 4, 118\} | \(Z_2\) |
| 111 | \{1, 2, 9, 115, 3, 4, 35\} | \(Z_3\) |
| 112 | \{1, 2, 9, 21, 3, 4, 23\} | \(Z_2\) |
| 113 | \{1, 2, 9, 21, 3, 4, 71\} | \(I\) |
| 114 | \{1, 2, 9, 21, 3, 4, 110\} | \(I\) |
| 115 | \{1, 2, 9, 21, 3, 4, 74\} | \(I\) |
| 116 | \{1, 2, 9, 21, 3, 4, 54\} | \(I\) |
| 117 | \{1, 2, 9, 21, 3, 4, 31\} | \(I\) |
| 118 | \{1, 2, 9, 21, 3, 4, 33\} | \(I\) |
|   |   |   |
|---|---|---|
| 119 | \{1,2,9,21,3,4,66\} | \(I\) |
| 120 | \{1,2,9,21,3,4,130\} | \(Z_3\) |
| 121 | \{1,2,9,21,3,4,77\} | \(I\) |
| 122 | \{1,2,9,115,3,4,32\} | \(I\) |
| 123 | \{1,2,9,115,3,4,14\} | \(I\) |
| 124 | \{1,2,9,115,3,4,130\} | \(Z_3 \times S_3\) |
| 125 | \{1,2,9,83,3,8,17\} | \(Z_2\) |
| 126 | \{1,2,9,21,3,5,56\} | \(I\) |
| 127 | \{1,2,9,21,3,12,18\} | \(I\) |
| 128 | \{1,2,9,83,3,5,7\} | \(I\) |
| 129 | \{1,2,9,115,3,7,12\} | \(Z_2\) |
| 130 | \{1,2,9,115,3,7,6\} | \(Z_2\) |
| 131 | \{1,2,9,21,3,5,31\} | \(I\) |
| 132 | \{1,2,9,83,3,8,60\} | \(Z_4 \times Z_2\) |
| 133 | \{1,2,9,83,3,8,7\} | \(I\) |
| 134 | \{1,2,9,83,3,8,18\} | \(I\) |
| 135 | \{1,2,9,83,3,8,57\} | \(Z_{12}\) |
| 136 | \{1,2,9,83,3,8,40\} | \(Z_2\) |
| 137 | \{1,2,9,83,3,8,24\} | \(I\) |
| 138 | \{1,2,9,83,3,8,62\} | \(I\) |
| 139 | \{1,2,9,83,3,8,26\} | \(I\) |
| 140 | \{1,2,9,83,3,8,5\} | \(I\) |
| 141 | \{1,2,9,83,3,8,19\} | \(Z_4\) |
| 142 | \{1,2,9,83,3,8,12\} | \(I\) |
| 143 | \{1,2,9,21,3,12,17\} | \(I\) |
| 144 | \{1,2,9,21,3,5,13\} | \(Z_2\) |
| 145 | \{1,2,9,115,3,7,16\} | \(Z_2\) |
| 146 | \{1,2,9,83,3,5,6\} | \(I\) |
| 147 | \{1,2,9,21,3,12,68\} | \(Z_2\) |
| 148 | \{1,2,9,21,3,5,6\} | \(I\) |
| 149 | \{1,2,9,21,3,14,52\} | \(Z_2 \times Z_2\) |
| 150 | \{1,2,9,83,3,5,13\} | \(Z_2\) |
| 151 | \{1,2,9,115,3,7,49\} | \(Z_2 \times Z_2\) |
| 152 | \{1,2,9,115,3,5,13\} | \(Z_6\) |
| 153 | \{1,2,9,21,3,5,111\} | \(I\) |
| 154 | \{1,2,9,21,3,5,79\} | \(I\) |
| 155 | \{1,2,9,21,3,5,50\} | \(I\) |
| 156 | \{1,2,9,21,3,12,30\} | \(I\) |
| 157 | \{1,2,9,83,3,5,44\} | \(I\) |
| 158 | \{1,2,9,115,3,7,19\} | \(Z_2\) |
| 159 | \{1,2,9,115,3,7,41\} | \(I\) |
| 160 | \{1,2,9,21,3,5,76\} | \(I\) |
|   |     | \{1,2,9,21,3,5,106\} | \{1,2,9,21,3,5,95\} | \{1,2,9,21,3,12,58\} | \{1,2,9,83,3,5,17\} | \{1,2,9,21,3,5,65\} | \{1,2,9,21,3,5,66\} | \{1,2,9,21,3,5,99\} | \{1,2,9,21,3,5,45\} | \{1,2,9,21,3,5,42\} | \{1,2,9,83,3,5,16\} | \{1,2,9,83,3,5,32\} | \{1,2,9,21,3,5,40\} | \{1,2,9,21,3,14,55\} | \{1,2,9,21,3,12,66\} | \{1,2,9,115,3,5,6\} | \{1,2,9,21,3,14,31\} | \{1,2,9,83,3,5,40\} | \{1,2,9,115,3,7,92\} | \{1,2,9,115,3,5,40\} | Z_3 \times Z_3 | \{1,2,9,21,3,5,28\} | \{1,2,9,21,3,5,26\} | \{1,2,9,21,3,5,8\} | \{1,2,9,21,3,5,41\} | \{1,2,9,21,3,5,27\} | \{1,2,9,21,3,5,20\} | \{1,2,9,21,3,5,126\} | \{1,2,9,21,3,5,17\} | \{1,2,9,21,3,5,100\} | \{1,2,9,21,3,5,29\} | \{1,2,9,21,3,5,43\} | \{1,2,9,21,3,5,167\} | \{1,2,9,21,3,5,15\} | \{1,2,9,115,3,7,8\} | \{1,2,9,115,3,7,26\} | \{1,2,9,115,3,7,15\} | \{1,2,9,115,3,7,42\} | \{1,2,9,115,3,5,42\} | \{1,2,9,115,3,5,13\} | \{1,2,9,21,3,12,14\} | \{1,2,9,115,3,7,45\} | \{1,2,9,83,3,5,27\} | \{1,2,9,115,3,7,25\} | I | I | I | I | I | I | I | Z_3 | Z_2 | Z_3 | Z_6 | Z_3 | Z_6 | I | I | Z_2 | I | I | I | I | I | I | I | I | I | Z_3 | I | Z_2 | Z_3 | I | I | I | I |
In Table 7, there are 11 types of the stabiliser groups as follows:

$$I, \ Z_2, \ Z_3, \ Z_4, \ Z_6, \ D_4, \ Z_3 \times S_3, \ Z_4 \times Z_2, \ Z_{12}, \ Z_2 \times Z_2, \ Z_3 \times Z_3.$$ 

These stabiliser groups of order at least two divide their corresponding projectively inequivalent \((7;4)\)-arcs into a number of orbits. All orbits of these groups are listed in Table 9.

**Theorem .6** In \(PG(2,13)\), there are exactly 207 projectively inequivalent \((7;4)\)-arcs.

**Remark**

In Table 7, there are 11 types of the stabiliser groups as follows:

$$I, \ Z_2, \ Z_3, \ Z_4, \ Z_6, \ D_4, \ Z_3 \times S_3, \ Z_4 \times Z_2, \ Z_{12}, \ Z_2 \times Z_2, \ Z_3 \times Z_3.$$ 

| Number | \(N_c\) | Number of \(N_c\) |
|--------|--------|------------------|
| 1      | \{1, 0, 15, 64, 103\} | 62               |
| 2      | \{1, 1, 12, 67, 102\} | 106              |
| 3      | \{1, 2, 9, 70, 101\}  | 30               |
| 4      | \{1, 3, 6, 73, 100\}  | 3                |
| 5      | \{2, 0, 9, 72, 100\}  | 6                |

| \(\Phi_4\) | Stabiliser | Orbits                      |
|-------------|------------|------------------------------|
| \{1,2,9,115,3,4,22\} | \(Z_2\)   | \{1,2\}, \{3\}, \{4,22\}, \{9,115\} |
| \{1,2,9,21,3,4,136\} | \(Z_2\)   | \{1,2\}, \{3,136\}, \{4\}, \{9,21\} |
| \{1,2,9,83,3,4,11\} | \(D_4\)   | \{1\}, \{2,3\}, \{4,11,83,9\} |
| \{1,2,9,83,3,4,10\} | \(Z_2\)   | \{1,2\}, \{3\}, \{4,10\}, \{9,83\} |
| \{1,2,9,83,3,4,23\} | \(Z_2\)   | \{1\}, \{2\}, \{3\}, \{4\}, \{9,83\}, \{23\} |
| \{1,2,9,83,3,4,37\} | \(Z_2\)   | \{1,2\}, \{3\}, \{4,37\}, \{9\}, \{83\} |
| \{1,2,9,83,3,4,76\} | \(Z_2\)   | \{1\}, \{2\}, \{3\}, \{4\}, \{9,83\}, \{76\} |
| \{1,2,9,83,3,4,82\} | \(Z_2\)   | \{1\}, \{2\}, \{3\}, \{4\}, \{9,83\}, \{82\} |
| \{1,2,9,83,3,4,108\} | \(Z_2\)   | \{1\}, \{2\}, \{3\}, \{4\}, \{9,83\}, \{108\} |
| \{1,2,9,83,3,4,126\} | \(Z_2\)   | \{1\}, \{2\}, \{3\}, \{4\}, \{9,83\}, \{126\} |
| \{1,2,9,83,3,4,14\} | \(Z_2\)   | \{1\}, \{2\}, \{3\}, \{4\}, \{9,83\}, \{14\} |
| \{1,2,9,83,3,4,130\} | \(Z_6\)   | \{1\}, \{2\}, \{3,4\}, \{9,83\}, \{130\} |
| \{1,2,9,115,3,4,24\} | \(Z_2\)   | \{1,2\}, \{3,24\}, \{4\}, \{9,115\} |
\begin{tabular}{|l|l|}
\hline
\{1,2,9,21,3,4,118\} & \(Z_2\) & \{1,2\}, \{3,118\}, \{4\}, \{9,21\} \\
\{1,2,9,115,3,4,35\} & \(Z_3\) & \{1,9,115\}, \{2\}, \{3,35,4\} \\
\{1,2,9,21,3,4,23\} & \(Z_2\) & \{1\}, \{2,3\}, \{4,21\}, \{9,23\} \\
\{1,2,9,21,3,4,130\} & \(Z_3\) & \{1\}, \{2\}, \{3,4,130\}, \{9\}, \{21\} \\
\{1,2,9,115,3,4,130\} & \(Z_3 \times S_3\) & \{1\}, \{2,3,9,115,130,4\} \\
\{1,2,9,83,3,8,17\} & \(Z_2\) & \{1,2\}, \{3,8\}, \{9,83\}, \{17\} \\
\{1,2,9,115,3,7,12\} & \(Z_2\) & \{1,9\}, \{2,115\}, \{3,12\}, \{7\} \\
\{1,2,9,115,3,7,6\} & \(Z_2\) & \{1,115\}, \{2,9\}, \{3,7\}, \{6\} \\
\{1,2,9,83,3,8,60\} & \(Z_4 \times Z_2\) & \{1,9,2,83\}, \{3,8\}, \{60\} \\
\{1,2,9,83,3,8,57\} & \(Z_{12}\) & \{1,9,2,83\}, \{3,8,57\} \\
\{1,2,9,83,3,8,40\} & \(Z_2\) & \{1,2\}, \{3,8\}, \{9,83\}, \{40\} \\
\{1,2,9,83,3,8,19\} & \(Z_4\) & \{1,9,2,83\}, \{3\}, \{8\}, \{19\} \\
\{1,2,9,21,3,5,13\} & \(Z_2\) & \{1\}, \{2\}, \{3,13\}, \{5\}, \{9\}, \{21\} \\
\{1,2,9,115,3,7,16\} & \(Z_2\) & \{1,115\}, \{2,9\}, \{3\}, \{7\}, \{16\} \\
\{1,2,9,21,3,12,68\} & \(Z_2\) & \{1\}, \{2\}, \{3\}, \{9\}, \{12,68\}, \{21\} \\
\{1,2,9,21,3,14,52\} & \(Z_2 \times Z_2\) & \{1,2\}, \{3\}, \{9,21\}, \{14,52\} \\
\{1,2,9,83,3,5,13\} & \(Z_2\) & \{1\}, \{2\}, \{3,13\}, \{5\}, \{9,83\} \\
\{1,2,9,115,3,7,49\} & \(Z_2 \times Z_2\) & \{1,115\}, \{2,9\}, \{3\}, \{7,49\} \\
\{1,2,9,115,3,5,13\} & \(Z_6\) & \{1\}, \{2,115,9\}, \{3,13\}, \{5\} \\
\{1,2,9,115,3,7,19\} & \(Z_2\) & \{1,115\}, \{2,9\}, \{3,7\}, \{19\} \\
\{1,2,9,21,3,5,40\} & \(Z_3\) & \{1\}, \{2,3,5,40\}, \{9\}, \{21\} \\
\{1,2,9,21,3,14,55\} & \(Z_2\) & \{1,2\}, \{3\}, \{9,21\}, \{14,55\} \\
\{1,2,9,21,3,12,66\} & \(Z_3\) & \{1\}, \{2\}, \{3,12,66\}, \{9\}, \{21\} \\
\{1,2,9,115,3,5,6\} & \(Z_2\) & \{1\}, \{2,9,115\}, \{3\}, \{5\}, \{6\} \\
\{1,2,9,21,3,14,31\} & \(Z_6\) & \{1\}, \{2\}, \{3,14,31\}, \{9,21\} \\
\{1,2,9,83,3,5,40\} & \(Z_3\) & \{1\}, \{2\}, \{3,5,40\}, \{9\}, \{83\} \\
\{1,2,9,115,3,7,92\} & \(Z_6\) & \{1,115\}, \{2,9\}, \{3,7,92\} \\
\{1,2,9,115,3,5,40\} & \(Z_3 \times Z_3\) & \{1\}, \{2,9,115\}, \{3,5,40\} \\
\{1,2,9,21,3,5,8\} & \(Z_2\) & \{1,2\}, \{3\}, \{5,8\}, \{9,21\} \\
\{1,2,9,21,3,5,29\} & \(Z_2\) & \{1,2\}, \{3,29\}, \{5\}, \{9,21\} \\
\{1,2,9,115,3,5,42\} & \(Z_3\) & \{1,9,115\}, \{2\}, \{3,5,42\} \\
\{1,2,9,21,3,12,14\} & \(Z_2\) & \{1,2\}, \{3,14\}, \{9,21\}, \{12\} \\
\{1,2,9,115,3,7,45\} & \(Z_3\) & \{1,9,115\}, \{2\}, \{3,45,7\} \\
\{1,2,9,115,3,7,20\} & \(Z_3\) & \{1,2,9\}, \{3,7,20\}, \{115\} \\
\{1,2,9,115,3,7,52\} & \(Z_2\) & \{1,115\}, \{2,9\}, \{3,7\}, \{52\} \\
\{1,2,9,21,3,12,96\} & \(Z_2\) & \{1,2\}, \{3,96\}, \{9,21\}, \{12\} \\
\hline
\end{tabular}

4.5 Projectively inequivalent \((8;4)\)-arcs

In \(\text{PG}(2,13)\), the number of projectively inequivalent \((8;4)\)-arcs is 7399. The stabliser groups of 7399 projectively inequivalent \((8;4)\)-arcs are as follows:
$I, Z_2, Z_3, Z_4, Z_6, Z_{12}, Z_2 \times Z_2, Z_4 \times Z_2, (Z_4 \times Z_4) \rtimes Z_2, Z_3 \times S_3, D_4.$

The number of these groups is listed in Table 10. Also, the 7399 projectively inequivalent $(8;4)$-arcs have eleven $sd$-inequivalent classes of secant distributions as shown in Table 11.

### Table 10: Group statistics of the projectively inequivalent $(8;4)$-arcs

| Number | Stabiliser | Number of stabiliser |
|--------|------------|-----------------------|
| 1      | $I$        | 6895                  |
| 2      | $Z_2$      | 443                   |
| 3      | $Z_3$      | 12                    |
| 4      | $Z_4$      | 15                    |
| 5      | $Z_6$      | 4                     |
| 6      | $Z_{12}$   | 1                     |
| 7      | $Z_2 \times Z_2$ | 18     |
| 8      | $Z_4 \times Z_2$ | 2     |
| 9      | $(Z_4 \times Z_4) \rtimes Z_2$ | 1     |
| 10     | $Z_3 \times S_3$ | 1     |
| 11     | $D_4$      | 7                     |

Note that the groups of order at least eight are as follows:

$Z_4 \times Z_2, Z_{12}, (Z_4 \times Z_4) \rtimes Z_2, Z_3 \times S_3.$

These groups partition the associated projectively inequivalent $(8;4)$-arcs into a number of orbits as shown below.

1. The group $Z_{12}$ splits the $(8;4)$-arc $\{1, 2, 9, 3, 8, 57, 19\}$ into 3 orbits of sizes 4, 3, 1. They are $\{1, 9, 2, 83\}, \{3, 8, 57\}, \{19\}$.

2. The group $Z_4 \times Z_2$ partitions the $(8;4)$-arcs $\{1, 2, 9, 3, 8, 60, 19\}$ and $\{1, 2, 9, 3, 8, 57, 59\}$ into 3 orbits. They are $\{1, 9, 2, 83\}, \{3, 60\}, \{8, 19\}$ and $\{1, 9, 2, 83\}, \{3, 59\}, \{8, 57\}$.

3. The group $(Z_4 \times Z_4) \rtimes Z_2$ divides the $(8;4)$-arc $\{1, 2, 9, 83, 3, 8, 19, 59\}$ into one orbit, that is, $\{1, 2, 3, 19, 8, 83, 59, 9\}$.

4. The group $Z_3 \times S_3$ separates the $(8;4)$-arc $\{1, 2, 9, 115, 3, 5, 6, 132\}$ into two orbits of sizes 2, 6. They are $\{1, 5\}, \{2, 6, 9, 115, 132, 3\}$.
Table 11: \( N_c \) of \( \{t_4, t_3, t_2, t_1, t_0\} \)

| Number | \( N_c \) | Number of \( N_c \) |
|--------|------------|------------------|
| 1      | \{ 1, 0, 22, 64, 96 \} | 534             |
| 2      | \{ 1, 1, 19, 67, 95 \} | 2272            |
| 3      | \{ 1, 2, 16, 70, 94 \} | 2905            |
| 4      | \{ 1, 3, 13, 73, 93 \} | 1188            |
| 5      | \{ 2, 0, 16, 72, 93 \} | 146             |
| 6      | \{ 1, 4, 10, 76, 92 \} | 182             |
| 7      | \{ 2, 1, 13, 75, 92 \} | 128             |
| 8      | \{ 1, 5, 7, 79, 91 \} | 10              |
| 9      | \{ 2, 2, 10, 78, 91 \} | 30              |
| 10     | \{ 1, 6, 4, 82, 90 \} | 1               |
| 11     | \{ 2, 3, 7, 81, 90 \} | 3               |

**Theorem 7** In PG\((2, 13)\), there are exactly 7399 projectively inequivalent \((8; 4)\)-arcs.

### 4.6 Projectively inequivalent \((9; 4)\)-arcs

In PG\((2, 13)\), the number of projectively inequivalent \((9; 4)\)-arcs is 222536 according to the inequivalent lexicographically least set in the G-orbit of each \((9; 4)\)-arc. These arcs have one of the groups \( I, Z_2, Z_3, Z_4, Z_6, Z_2 \times Z_2, Z_4 \times Z_2, D_4, S_3, S_4, A_4 \). In addition, the secant distribution of each of the 222536 projectively inequivalent arcs is calculated. There are 21 \( sd \)-inequivalent classes of secant distributions of the projectively inequivalent \((9; 4)\)-arcs. The statistics are given in Tables 12, 13, and 14.

**Remark**

In Table 12, the large groups of order at least 4 are \( Z_4, Z_6, Z_4 \times Z_2, S_3, S_4, D_4, A_4 \). The action of these groups is shown in the following table:
Table 13: **Group orbits of projectively inequivalent (9;4)-arcs**

| \(\Phi_4\) | Stabiliser | Orbits |
|------------|------------|--------|
| \{1,2,9,83,3,4,5,7,99,105\} | \(Z_4\) | \{1,9,2,83\}, \{3,105,57,4\}, \{99\} |
| \{1,2,9,83,3,4,5,24,135\} | \(Z_4\) | \{1\}, \{2,4\}, \{3,83,135,9\}, \{5,24\} |
| \{1,2,9,21,3,4,22,24,108\} | \(Z_4\) | \{1,2\}, \{3,22,24,4\}, \{9,21\}, \{108\} |
| \{1,2,9,115,3,4,18,151,159\} | \(Z_4\) | \{1,115\}, \{2,9\}, \{3,18,159,4\}, \{151\} |
| \{1,2,9,83,3,4,30,84,124\} | \(Z_4\) | \{1,9,2,83\}, \{3\}, \{4,124,84,30\} |
| \{1,2,9,83,3,4,92,135,118\} | \(Z_4\) | \{1\}, \{2,4\}, \{3,83,135,9\}, \{92,118\} |
| \{1,2,9,83,3,5,13,49,101\} | \(Z_4\) | \{1\}, \{2\}, \{3,49,13,101\}, \{5\}, \{9,83\} |
| \{1,2,9,21,3,12,68,56,151\} | \(Z_4\) | \{1,2\}, \{3\}, \{9,21\}, \{12,56,68,151\} |
| \{1,2,9,83,3,5,13,16,33\} | \(Z_4\) | \{1,2\}, \{3\}, \{16,13,33\}, \{5\}, \{9,83\} |
| \{1,2,9,83,3,5,13,58,97\} | \(Z_4\) | \{1,2\}, \{3,58,13,97\}, \{5\}, \{9\}, \{83\} |
| \{1,2,9,83,3,8,17,32,61\} | \(Z_4\) | \{1,9,2,83\}, \{3,32,8,61\}, \{17\} |
| \{1,2,9,83,3,8,17,79,147\} | \(Z_4\) | \{1,9,2,83\}, \{3,79,8,147\}, \{17\} |
| \{1,2,9,115,3,7,6,154,160\} | \(Z_4\) | \{1,154,115,160\}, \{2,3,9,7\}, \{6\} |
| \{1,2,9,21,3,14,31,8,74\} | \(Z_4\) | \{1,14,2,74\}, \{3,21,31,9\}, \{8\} |
| \{1,2,9,115,3,4,5,25,148\} | \(Z_6\) | \{1,3\}, \{2,25,115,9,148,5\}, \{4\} |
| \{1,2,9,115,3,4,30,43,59\} | \(Z_6\) | \{1,4\}, \{2,43,9,115,59,30\}, \{3\} |
| \{1,2,9,115,3,4,18,35,39\} | \(Z_6\) | \{1,3,2,115,39,35\}, \{4,18\}, \{9\} |
| \{1,2,9,115,3,4,8,51,130\} | \(Z_6\) | \{1\}, \{2,4,9,3,115,130\}, \{8,51\} |
| \{1,2,9,115,3,4,16,37,145\} | \(Z_6\) | \{1,9,3,115,37,145\}, \{2\}, \{4\}, \{16\} |
| \{1,2,9,115,3,4,32,31,130\} | \(Z_6\) | \{1\}, \{2,115,30,9,3,4\}, \{31,32\} |
| \{1,2,9,115,3,4,32,29,130\} | \(Z_6\) | \{1\}, \{2,3,115,4,9,130\}, \{29,32\} |
| \{1,2,9,115,3,4,32,130,149\} | \(Z_6\) | \{1\}, \{2,114,9,130,3\}, \{32,149\} |
| \{1,2,9,83,3,4,57,60,147\} | \(Z_4 \times Z_2\) | \{1,9,3,2,60,83,147,57\}, \{4\} |
| \{1,2,9,21,3,4,58,7,80\} | \(S_3\) | \{1,2,3,7,58,4\}, \{9,21,80\} |
| \{1,2,9,115,3,4,5,130,131\} | \(S_3\) | \{1,2,4\}, \{3,9,131,115,5,130\} |
| \{1,2,9,21,3,4,96,163,166\} | \(S_3\) | \{1,2,163\}, \{3,9,21,163,4,166\} |
| \{1,2,9,115,3,4,15,130,45\} | \(S_3\) | \{1,9,130\}, \{2,3,45,115,15,4\} |
| \{1,2,9,83,3,4,11,10,84\} | \(S_4\) | \{1,2,3\}, \{4,83,11,9,37,129\} |
| \{1,2,9,83,3,4,11,37,129\} | \(D_4\) | \{1,2\}, \{3,86,10,4\}, \{9,83\}, \{82\} |
| \{1,2,9,115,3,7,12,77,76\} | \(D_4\) | \{1,2,9,115\}, \{3,12,7,77\}, \{76\} |
| \{1,2,9,115,3,7,12,70,177\} | \(D_4\) | \{1,2,9,115\}, \{3,70,12,177\}, \{7\} |
| \{1,2,9,83,3,4,5,12,135\} | \(A_4\) | \{1,2,4\}, \{3,83,135,5,9,12\} |
| \{1,2,9,83,3,4,92,135,164\} | \(A_4\) | \{1,2,4\}, \{3,83,135,164,9,92\} |
Table 14: $N_c$ of $\{t_4, t_3, t_2, t_1, t_0\}$

| Number | $N_c$ | Number of $N_c$ |
|--------|-------|-----------------|
| 1      | $\{1, 0, 30, 62, 90\}$ | 1199 |
| 2      | $\{1, 1, 27, 65, 89\}$ | 13688 |
| 3      | $\{1, 2, 24, 68, 88\}$ | 50341 |
| 4      | $\{1, 3, 21, 71, 87\}$ | 74174 |
| 5      | $\{2, 0, 24, 70, 87\}$ | 1776 |
| 6      | $\{1, 4, 18, 74, 86\}$ | 47139 |
| 7      | $\{2, 1, 21, 73, 86\}$ | 7227 |
| 8      | $\{2, 2, 18, 76, 85\}$ | 8259 |
| 9      | $\{1, 5, 15, 77, 85\}$ | 12848 |
| 10     | $\{1, 6, 12, 80, 84\}$ | 1487 |
| 11     | $\{2, 3, 15, 79, 84\}$ | 3388 |
| 12     | $\{3, 0, 18, 78, 84\}$ | 182 |
| 13     | $\{1, 7, 9, 83, 83\}$ | 68 |
| 14     | $\{2, 4, 12, 82, 83\}$ | 518 |
| 15     | $\{3, 1, 15, 81, 83\}$ | 151 |
| 16     | $\{1, 8, 6, 86, 82\}$ | 2 |
| 17     | $\{2, 5, 9, 85, 82\}$ | 39 |
| 18     | $\{3, 2, 12, 84, 82\}$ | 42 |
| 19     | $\{2, 6, 6, 88, 81\}$ | 2 |
| 20     | $\{3, 3, 9, 87, 81\}$ | 5 |
| 21     | $\{3, 4, 6, 90, 80\}$ | 1 |

**Theorem 8** In PG(2, 13), there are exactly 222536 projectively inequivalent (9;4)-arcs.

### 4.7 Projectively inequivalent (10;4)-arcs

The number of (10;4)-arcs is paralleled into 5 processes; each took 6 : 22 : 54 : 11, 4 : 15 : 36 : 77, 5 : 09 : 28 : 12, 5 : 12 : 40 : 46, 3 : 21 : 52 : 13 of CPU time respectively for the construction. Then according to the canonical images of the (10;4)-arcs found from 4 processes, there are at least 5268378 projectively inequivalent (10;4)-arcs. This took 2403232618 msc. The 5268378 arcs have 36 $sd$-inequivalent classes $N_c$ of $i$-secant distributions as listed in Table 15. The total time is 1726578 msc where it was computed in six processes. Then according to the number of $N_c$ there are 36 $sd$-inequivalent (10;4)-arcs, which have five types of stabilisers $I$, $Z_2 \times Z_2$, $Z_2$, $S_3$, $Z_3 \times S_3$. The timing of these groups was 3633 msec. The statistics of the $sd$-inequivalent (10;4)-arcs are given in Table 16.
Table 15: \( N_c \) of \( \{t_4, t_3, t_2, t_1, t_0\} \)

| Number | \( N_c \) | Number of \( N_c \) |
|--------|-----------|---------------------|
| 1      | \{1, 7, 18, 79, 78\} | 192599 |
| 2      | \{1, 0, 39, 58, 85\} | 661 |
| 3      | \{1, 1, 36, 61, 84\} | 15664 |
| 4      | \{1, 2, 33, 64, 83\} | 145027 |
| 5      | \{1, 3, 30, 67, 82\} | 592731 |
| 6      | \{1, 4, 27, 70, 81\} | 1227187 |
| 7      | \{1, 5, 24, 73, 80\} | 1322219 |
| 8      | \{1, 6, 21, 76, 79\} | 719144 |
| 9      | \{1, 8, 15, 82, 77\} | 24434 |
| 10     | \{1, 9, 12, 85, 76\} | 1399 |
| 11     | \{1, 10, 9, 88, 75\} | 31 |
| 12     | \{1, 11, 6, 91, 74\} | 3 |
| 13     | \{2, 0, 33, 66, 82\} | 4572 |
| 14     | \{2, 1, 30, 69, 81\} | 52934 |
| 15     | \{2, 2, 27, 72, 80\} | 207496 |
| 16     | \{2, 3, 24, 75, 79\} | 344994 |
| 17     | \{2, 4, 21, 78, 78\} | 255989 |
| 18     | \{2, 5, 18, 81, 77\} | 87359 |
| 19     | \{2, 6, 15, 84, 76\} | 13784 |
| 20     | \{2, 7, 12, 87, 75\} | 954 |
| 21     | \{2, 8, 9, 90, 74\} | 38 |
| 22     | \{2, 10, 3, 96, 72\} | 1 |
| 23     | \{3, 0, 27, 74, 79\} | 3944 |
| 24     | \{3, 1, 24, 77, 78\} | 17244 |
| 25     | \{3, 2, 21, 80, 77\} | 22990 |
| 26     | \{3, 3, 18, 83, 76\} | 11598 |
| 27     | \{3, 4, 15, 86, 75\} | 2477 |
| 28     | \{3, 5, 12, 89, 74\} | 257 |
| 29     | \{3, 6, 9, 92, 73\} | 12 |
| 30     | \{3, 7, 6, 95, 72\} | 2 |
| 31     | \{4, 0, 21, 82, 76\} | 222 |
| 32     | \{4, 1, 18, 85, 75\} | 297 |
| 33     | \{4, 2, 15, 88, 74\} | 97 |
| 34     | \{4, 3, 12, 91, 73\} | 13 |
| 35     | \{4, 4, 9, 94, 72\} | 2 |
| 36     | \{5, 0, 15, 90, 73\} | 3 |

**Theorem .9** In \( \text{PG}(2, 13) \), there are at least 5268378 projectively inequivalent \((10;4))-\text{arcs}.\)
Table 16: *sd*-inequivalent (10:4)-arcs

| Symbol | (10:4)-arc | \( t_1, t_3, t_2, t_4 \) | Stabiliser |
|--------|------------|-----------------|------------|
| \( K'_1 \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 8\} | \{1, 7, 18, 79, 78\} | \( I \) |
| \( K'_2 \) | \{1, 2, 9, 83, 3, 8, 17, 40, 72, 78\} | \{1, 0, 39, 58, 85\} | \( I \) |
| \( K'_3 \) | \{1, 2, 9, 83, 3, 4, 6, 50, 67, 63\} | \{1, 1, 36, 61, 84\} | \( I \) |
| \( K'_4 \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 99, 40\} | \{1, 2, 33, 64, 83\} | \( I \) |
| \( K'_5 \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 107, 18\} | \{1, 3, 30, 67, 82\} | \( I \) |
| \( K'_6 \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 63\} | \{1, 4, 27, 70, 81\} | \( I \) |
| \( K'_7 \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 16\} | \{1, 5, 24, 73, 80\} | \( I \) |
| \( K'_8 \) | \{1, 2, 9, 115, 3, 4, 5, 6, 7, 8\} | \{1, 6, 21, 76, 79\} | \( I \) |
| \( K'_9 \) | \{1, 2, 9, 115, 3, 4, 5, 6, 7, 90\} | \{1, 8, 15, 82, 77\} | \( I \) |
| \( K'_{10} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 137, 178\} | \{1, 9, 12, 85, 76\} | \( I \) |
| \( K'_{11} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 178, 104\} | \{1, 10, 9, 88, 75\} | \( Z_2 \times Z_2 \) |
| \( K'_{12} \) | \{1, 2, 9, 83, 3, 4, 5, 30, 37, 51\} | \{1, 11, 6, 91, 74\} | \( I \) |
| \( K'_{13} \) | \{1, 2, 9, 83, 3, 4, 6, 11, 167, 33\} | \{2, 0, 33, 66, 82\} | \( I \) |
| \( K'_{14} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 11\} | \{2, 1, 30, 69, 81\} | \( I \) |
| \( K'_{15} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 11, 18\} | \{2, 2, 27, 72, 80\} | \( I \) |
| \( K'_{16} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 7\} | \{2, 3, 24, 75, 79\} | \( I \) |
| \( K'_{17} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 17\} | \{2, 4, 21, 78, 78\} | \( I \) |
| \( K'_{18} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 87\} | \{2, 5, 18, 81, 77\} | \( I \) |
| \( K'_{19} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 166, 163\} | \{2, 6, 15, 84, 76\} | \( I \) |
| \( K'_{20} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 137, 37\} | \{2, 7, 12, 87, 75\} | \( I \) |
| \( K'_{21} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 68, 11\} | \{2, 8, 9, 90, 74\} | \( Z_2 \) |
| \( K'_{22} \) | \{1, 2, 9, 115, 3, 4, 18, 183, 35, 39\} | \{2, 10, 3, 96, 72\} | \( Z_3 \times S_3 \) |
| \( K'_{23} \) | \{1, 2, 9, 83, 3, 4, 5, 57, 166, 11, 51\} | \{3, 0, 27, 74, 79\} | \( I \) |
| \( K'_{24} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 11, 17\} | \{3, 1, 24, 77, 78\} | \( Z_2 \) |
| \( K'_{25} \) | \{1, 2, 9, 83, 3, 4, 5, 7, 6, 113, 77\} | \{3, 2, 21, 80, 77\} | \( I \) |
| \( K'_{26} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 137, 87\} | \{3, 3, 18, 83, 76\} | \( I \) |
| \( K'_{27} \) | \{1, 2, 9, 83, 3, 4, 5, 142, 131, 163\} | \{3, 4, 15, 86, 75\} | \( I \) |
| \( K'_{28} \) | \{1, 2, 9, 83, 3, 4, 5, 6, 95, 163\} | \{3, 5, 12, 89, 74\} | \( I \) |
| \( K'_{29} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 51, 37\} | \{3, 6, 9, 92, 73\} | \( I \) |
| \( K'_{30} \) | \{1, 2, 9, 83, 3, 4, 5, 51, 37, 122\} | \{3, 7, 6, 95, 72\} | \( S_3 \) |
| \( K'_{31} \) | \{1, 2, 9, 83, 3, 4, 5, 166, 38, 160\} | \{4, 0, 21, 82, 76\} | \( I \) |
| \( K'_{32} \) | \{1, 2, 9, 83, 3, 4, 5, 153, 91\} | \{4, 1, 18, 85, 75\} | \( I \) |
| \( K'_{33} \) | \{1, 2, 9, 83, 3, 4, 5, 163, 96\} | \{4, 2, 15, 88, 74\} | \( I \) |
| \( K'_{34} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 112, 39\} | \{4, 3, 12, 91, 73\} | \( I \) |
| \( K'_{35} \) | \{1, 2, 9, 83, 3, 4, 5, 129, 37, 11\} | \{4, 4, 9, 94, 72\} | \( Z_2 \) |
| \( K'_{36} \) | \{1, 2, 9, 21, 3, 4, 37, 91, 90, 178\} | \{5, 0, 15, 90, 73\} | \( Z_2 \) |
Remark

In Table 17, the classification timings of the projectively inequivalent \((k;4)\)-arcs for \(k = 5, \ldots, 9\) are given.

Table 17: **Timing (msec) of projectively inequivalent \((k;4)\)-arcs for \(k = 5, \ldots, 9\)**

| \((k;4)\)-arcs | Construction | Lexicographically least sets | \(\{t_4, t_3, t_2, t_1, t_0\}\) | Stabilisers |
|-----------------|-------------|-----------------------------|-------------------------------|------------|
| \((5;4)\)-arcs  | 2011        | 2134                        | 2193                          | 2181       |
| \((6;4)\)-arcs  | 2138        | 2168                        | 2329                          | 2230       |
| \((7;4)\)-arcs  | 2516        | 2201                        | 2999                          | 3615       |
| \((8;4)\)-arcs  | 26606       | 711630                      | 19554                         | 80338      |
| \((9;4)\)-arcs  | 22729912    | 32126643                    | 176130                        | 3848131    |

5 **Complete \((38;4)\)-arcs from the \(sd\)-inequivalent \((10;4)\)-arcs**

In Table 16, there are 36 \(sd\)-inequivalent \((10;4)\)-arcs together with the corresponding \(sd\)-inequivalent classes of the \(i\)-secant distributions. Therefore, at this stage of the classification the 36-arcs of Table 16 have been extended. The aim of this process is to discover the largest complete \((k;4)\)-arc in \(\text{PG}(2,13)\) that can be established. The result of this method is a complete \((38;4)\)-arc \(K'\). This complete arc is comes from the \(sd\)-inequivalent \((10;4)\)-arc \(K'_8\). The complete \((38;4)\)-arc is as follows: \(K' = \{1, 2, 9, 115, 3, 4, 5, 6, 7, 8, 10, 19, 25, 60, 74, 98, 107, 78, 130, 27, 106, 69, 116, 46, 63, 126, 99, 51, 81, 65, 52, 176, 88, 92, 53, 181, 169, 178\}\). The properties of \(K'\) are given in Table 18.

Table 18: **Complete \((38;4)\)-arc in \(\text{PG}(2,13)\)**

| Symbol | \((38;4)\)-arc | Stabiliser | \(\{t_4, t_3, t_2, t_1, t_0\}\) |
|--------|----------------|------------|---------------------------------|
| \(K'\) | \{1, 2, 9, 115, 3, 4, 5, 6, 7, 8, 10, 19, 25, 60, 74, 98, 107, 78, 130, 27, 106, 69, 116, 46, 63, 126, 99, 51, 81, 65, 52, 176, 88, 92, 53, 181, 169, 178\} | \(D_{12}\) | \{102, 24, 19, 14, 24\} |

Remark

In Table 19, the classification timing of the projectively inequivalent \((k;4)\)-arcs for \(k = 5, \ldots, 9\) are given.

Table 19: **Timing (msec) of projectively inequivalent \((k;4)\)-arcs for \(k = 5, \ldots, 9\)**

| \((k;4)\)-arcs | Construction | Lexicographically least sets | \(\{t_4, t_3, t_2, t_1, t_0\}\) | Stabilisers |
|-----------------|-------------|-----------------------------|-------------------------------|------------|
| \((5;4)\)-arcs  | 2011        | 2134                        | 2193                          | 2181       |
| \((6;4)\)-arcs  | 2138        | 2168                        | 2329                          | 2230       |
| \((7;4)\)-arcs  | 2516        | 2201                        | 2999                          | 3615       |
| \((8;4)\)-arcs  | 26606       | 711630                      | 19554                         | 80338      |
| \((9;4)\)-arcs  | 22729912    | 32126643                    | 176130                        | 3848131    |
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