Linear transfer guidance based on Lyapunov method

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Abstract. Low-thrust propulsion technology is widely used in space missions. However, traditional methods are not appropriate for the general case. In this study, a linearization Lyapunov method for constant thrust control is proposed. Based on relative orbit elements (ROEs), analytical constant-thrust linear equations are introduced. Moreover, via Lyapunov method, the linear constant-thrust control strategy is obtained. Numerical simulations are conducted to demonstrate the validation of the proposed method. Given different thrust magnitudes, the accuracy meets the requirement in these cases. Thus, it proves to be a practical choice for engineering applications.

1. Introduction

Low propulsion technology plays important roles in space missions [1-3], such as orbital transfer [4], formation maintenance and geosynchronous orbit keeping [5,6], due to its efficient fuel consumption. Several analytical or semi-methods for approximating a low-thrust transfer have been proposed. The Lambert method, as a traditional method, is applied for rapidly evaluating the low-thrust transfer in some fields [7,8]. Based on sensitive matrix, Han [9,10] proposed an approximating low-thrust solution for two-point boundary value problems (TPBVP). Furthermore, via first-order extension method, Avanzini [11] obtained analytical algebraic equations, which is suitable for different types of orbital transfer. However, as a drawback of these methods, few of them are appropriate for the general case such as arbitrary eccentricity or $J_2$ perturbation force.

Lyapunov function techniques, as a common and stable strategy for low-thrust guidance, have gained wide attention in recent years. In Early years, Ilgen [12] applied the Lyapunov function to control the Earth orbital transfers. Based on Lyapunov-feedback strategy, Petropoulos [13] proposed the Q-law to evaluate the propellant mass and flight time for low-thrust transfer. Moreover, Yuan et al. [14] developed his work via introducing modified equinoctial elements. As one of advantage, Lyapunov guidance method are usually simple and analytical.

The main contribution of this paper is to propose a linearized method for solving the transfer problems, which could reduce the demand for the engine in the space missions. Firstly, based on relative orbit elements (ROEs), analytical constant-thrust linear equations are introduced. Then, via Lyapunov method, the time-varying linear constant-thrust control strategy is obtained. Finally, numerical simulations are conducted to validate the proposed method.

This paper is organized as follows. In Sec. II, the relative orbit elements (ROEs) and analytical constant-thrust linear equations are introduced. In Sec. III, the linear constant-thrust control strategy is derived based on Lyapunov method. In Sec. IV, the effectiveness of the methods above is examined through several numerical simulations. Finally, Sec. V summarizes the paper.
2. The constant-thrust linear equations based on ROEs
To describe the relative motion, Han and Yin [15] presented a set of parameters named the relative orbital elements (ROEs). The set of ROEs comprises the relative average drift rate $\Delta e = \{\Delta e_x, \Delta e_y\}$, the relative eccentricity vector $\Delta e = \{\Delta e_x, \Delta e_y\}$, the relative inclination vector $\Delta i = \{\Delta i_x, \Delta i_y\}$, and the difference of the mean argument of latitude $\Delta M'$. Based on proximity hypothesis, the ROEs can be simply expressed by the orbital elements of the chief and chasing spacecrafts:

$$\Delta a = a_b - a_a$$
$$\Delta e_x = \Delta e \cos \omega_a - e_a \sin \omega_a (\Delta \omega + \Delta \Omega \cos i_a)$$
$$\Delta e_y = \Delta e \sin \omega_a + e_a \cos \omega_a (\Delta \omega + \Delta \Omega \cos i_a)$$
$$\Delta i_x = \Delta \Omega \sin i_a$$
$$\Delta i_y = -\Delta i$$
$$\Delta M' = \Delta \omega + (M_b - M_a) + \Delta \Omega \cos i_a$$

where $\Delta \Omega = \Omega_b - \Omega_a, \Delta e = e_b - e_a, \Delta i = i_b - i_a, \Delta \omega = \omega_b - \omega_a$. Subscripts $a$ and $b$ represent the chief and chasing spacecrafts respectively. Obviously, if the ROEs become zeros, the orbital elements of both spacecraft are equal. To further simplify the equations (1), Li [16] defined another ROEs by frame transformation:

$$\Delta a = a_b - a_a$$
$$\Delta e_x = \Delta e$$
$$\Delta e_y = e_a (\Delta \omega + \Delta \Omega \cos i_a)$$
$$\Delta i_x = \Delta \Omega \sin i_a$$
$$\Delta i_y = -\Delta i$$
$$\Delta M' = \Delta \omega + (M_b - M_a) + \Delta \Omega \cos i_a$$

Combining classical perturbation equations, the perturbation equations regarding ROEs can be expressed as:

$$\frac{d}{dt} \Delta a = \frac{2a^2_v f_b}{\mu}$$
$$\frac{d}{dt} \Delta e_x = \frac{1}{v_b} [2(e_b + \cos \theta_b) f_b - \frac{r_b}{a_b} \sin \theta_b f_b]$$
$$\frac{d}{dt} \Delta e_y = \frac{e_b}{v_b} [\frac{2 \sin \theta_b}{v_b} f_b + (\frac{2e_b}{v_b} + \frac{r_b \cos \theta_b}{a_b v_b}) f_b]$$
$$\frac{d}{dt} \Delta i_x = \frac{r_b \sin i_b}{h_b} \sin (\omega_b + \theta_b) f_b$$
$$\frac{d}{dt} \Delta i_y = -\frac{r_b}{h_b} \cos (\omega_b + \theta_b) f_b$$
$$\frac{d}{dt} \Delta M' = \left[ \frac{2 \sin \theta_b}{e_b v_b} - \frac{2b}{e_b a_b v_b} (1 + \frac{e_b^2}{p_b}) \sin \theta_b \right] f_b + \left[ \frac{2}{v_b} + \frac{r_b \cos \theta_b}{a_b v_b} - \frac{b_f r_f}{e_b a_b v_b} \cos \theta \right] f_b$$

where $p = a(1-e^2), h = \sqrt{\mu p}, b = a\sqrt{1-e^2}, v = \sqrt{\mu (2/r - 1/a)}$, and $f_b, f_s, f_n$ denote the components of the perturbation acceleration in the TNH frame [16]. Equations (4) can be denoted by
In addition, if \( \Delta X = 0 \), it means the orbital elements of two spacecraft coincide. Moreover, for orbital transfer mission, the target orbital elements can be regard as the “chief spacecraft” and the mission spacecraft at initial orbital elements can be regard as the “chasing spacecraft”. Thus, the orbital transfer can be achieved by controlling the \( \Delta X \rightarrow 0 \).

### 3. The linear transfer guidance strategy via Lyapunov method

In this section, the linear constant-thrust control strategy is obtained via Lyapunov method. Assume that the constant thrust magnitude is \( |u| = \sqrt{f_e^2 + f_s^2 + f_h^2} = c \). Based on equations (4), the derivatives of Lyapunov function \( \Delta X^T \Delta X \) regarding ROEs can be expressed as:

\[
\frac{d}{dt} (\Delta X^T \Delta X) = 2\Delta X^T A(X_s, X_w) u
\]

(5)

where \( u \) is instantaneous thrust acceleration vector. If \( \Delta X^T \Delta X = 0 \), the mission spacecraft transfers to the target orbit. For reducing the \( \Delta X^T \Delta X \), the vector \( u \) needs be optimized

\[
\min_{|u|=c} |A(X_s, X_u) u| = \min_{|u|=c} \left| A(X_s, X_u)^T \Delta X \right| u
\]

(6)

Which can be solved analytically by Lyapunov method:

\[
u = -c \left( A(X_s, X_u)^T \Delta X \right) / \left| A(X_s, X_u)^T \Delta X \right|
\]

(7)

Therefore,

\[
\frac{d}{dt} (\Delta X^T \Delta X) = -2c A(X_s, X_u) \left( A(X_s, X_u)^T \Delta X \right) / \left| A(X_s, X_u)^T \Delta X \right| = -2c \left| A(X_s, X_u)^T \Delta X \right| < 0
\]

(8)

Thus, for instantaneous ROEs \( \Delta X \), the thrust vector \( u \) can be obtained rapidly by equations (7). Because the Lyapunov function \( \Delta X^T \Delta X \) is decreasing with the thrust vector \( u \), it can be considered that the spacecraft transfers close to the target orbit by the linear constant thrust control and the orbital transfer can be achieved when \( \Delta X \rightarrow 0 \).

### 4. Simulation results

To demonstrate the accuracy of the linear transfer guidance strategy proposed in this paper, three separate numerical simulations are carried out with different initial orbital elements of mission spacecraft and relative position and velocity between mission spacecraft and target orbital elements. Table 1 shows the initial parameters of simulations:

| Parameters       | Simulation 1 | Simulation 2 | Simulation 3 |
|------------------|--------------|--------------|--------------|
| Semi-major axis, km | 8555         | 7555         | 8555         |
| Eccentricity     | 0.15         | 0.2          | 0.2          |
| Inclination, deg | 45           | 25           | 25           |
| Ascending node, deg | 10           | 30           | 30           |
| Argument of Perigee, deg | 10           | 20           | 20           |
| Mean anomaly, deg | 20           | 13           | 30           |
| Relative position, m | [440, 0, 1593] | [1211, 61.7, 706.6] | [1318.6, 103.7, 271.4] |
| Relative velocity, m/s | [1.32, 0, -2.55] | [-2.48, 0.09, -2.71] | [-4.63, 0.07, -2.11] |
| Thrust magnitude, m/s² | 0.001        | 0.005        | 0.01         |
Table 2. The results of three simulations.

|                          | Simulation 1 | Simulation 2 | Simulation 3 |
|--------------------------|--------------|--------------|--------------|
| Relative position error, m | 0.18         | 3.19         | 4.71         |
| Relative velocity error, m/s | $2 \times 10^{-4}$ | 0.11         | 0.09         |
| Transfer time, s         | 2000         | 2700         | 2700         |

Table 2 illustrates the results of the three simulations. The relative position and velocity errors denote the norms of relative position and velocity between mission spacecraft and target orbital elements. With the proposed linear transfer guidance strategy, the relative position and velocity norms become small values in each simulation, which proves that the mission spacecraft successfully transfers to the target orbital elements.

5. Conclusions
In this study, based on relative orbit elements (ROEs), a linearization Lyapunov transfer guidance strategy is proposed. Numerical simulations demonstrate the validation of the proposed method. The proposed constant control strategy is much convenient because the thrust vector can be obtained rapidly from instantaneous ROEs. Thus, the proposed guidance strategy can provide practical choice for engineering applications.

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