MATHEMATICAL COMPUTABILITY QUESTIONS FOR SOME CLASSES OF LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS ORIGINATED FROM HILBERT’S TENTH PROBLEM

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Abstract. Inspired by Quantum Mechanics, we reformulate Hilbert’s tenth problem in the domain of integer arithmetics into problems involving either a set of infinitely-coupled non-linear differential equations or a class of linear Schrödinger equations with some appropriate time-dependent Hamiltonians. We then raise the questions whether these two classes of differential equations are computable or not in some computation models of computable analysis. These are non-trivial and important questions given that: (i) not all computation models of computable analysis are equivalent, unlike the case with classical recursion theory; (ii) and not all models necessarily and inevitably reduce computability of real functions to discrete computations on Turing machines. However unlikely the positive answers to our computability questions, their existence should deserve special attention and be satisfactorily settled since such positive answers may also have interesting logical consequence back in the classical recursion theory for the Church-Turing thesis.

Monday, July 4, 2005

Introduction

Hilbert’s tenth problem [2, 11] is concerned with the availability of a universal procedure to determine whether an arbitrarily given Diophantine equation

\[ D(x_1, \ldots, x_K) = 0 \]

has any positive integer solution or not. After more than 70 years since its inception, it has finally been shown that the problem is recursively noncomputable since no such universal procedure which is also classically recursive can exist. The problem is equivalent to the Turing halting problem and intimately links to the concept of effective computability as defined by the Church-Turing thesis in classical recursion theory.

Nevertheless, we have established elsewhere, through an inspiration provided by quantum mechanics [4, 5, 6, 7, 8, 9], some surprising connections of the above problem in number theory to problems over the continuous variables. In particular, we have been able to reformulate Hilbert’s tenth problem in terms of a set of infinitely coupled non-linear differential equations for any given Diophantine equation [8]. Also, through the framework of quantum adiabatic computation [3], we have also associated Hilbert’s tenth problem with a class of linear Schrödinger equations with appropriate time-dependent Hamiltonians. It has been proposed then that a physical implementation of quantum mechanical processes for these Schrödinger equations could provide the physical means to solve Hilbert’s tenth problem [9]. Here, in this paper we will only concern ourselves with the differential equations as mathematical objects, however, and will not appeal to any real physical processes. The mathematical objects so derived are extremely valuable as they provide us a direct link between the well-established classical recursion theory and the infantile subject of computable analysis, through a known noncomputable problem in the former theory.
We especially want to raise in this paper the computability questions for these differential equations in the domain of computable analysis, outside and encompassing the classical recursion theory where Hilbert’s tenth problem was originally formulated.

From the recursive noncomputability of Hilbert’s tenth problem, one might conclude that these differential equations should also be noncomputable in the wider framework. Such hasty conclusion, however, is not warranted because of several reasons. Firstly, however likely the case one might expect, it would need to be established rigorously as a mathematical truth –because the unsolvability of Hilbert’s tenth problem is only established in the framework of Turing computability, not necessarily in mathematics in general.

Secondly, there are many computation models in computable analysis but they are not all equivalent. And it is known that computability in one model may not be the same in some other model, see, for example, for a brief discussion comparison of the various models. This situation is in stark contrast to the classical recursion theory of functions from \( \mathbb{N} \) to \( \mathbb{N} \). There, many different formulations have been given (notably by Kleene, Turing, Post, Herbrand/Gödel, Markov) but in the end these all lead to the same notion of computability with the same class of computable functions. Such an equivalence has led to the postulation and support of the Church-Turing thesis in that theory.

The computability definitions for a single real number in most, if not all, different computation models of computable analysis are equivalent. However, for sequences of reals the definitions diverge and are not equivalent. (The discussions on sequences of real numbers are necessary because of the necessity of topological notions in analysis.) This divergence results in the dependence of the notion of computability on the different computation models, or even on different choices within a single model.\(^1\)

In view of such an inequivalence of computability in different approaches, it is not at all a forgone conclusion that the noncomputability of the differential equations mentioned above is trivially the only, inevitable possibility.

In the next Section we briefly present the observation inspired by quantum mechanics that leads to the connections between Hilbert’s tenth problem with unbounded self-adjoint operators acting on some infinite-dimensional Hilbert space. From this we then reformulate Hilbert’s tenth problem in terms of a set of infinitely coupled non-linear differential equations (eqs. (15, 16) below). We also propose a so-called \textit{continuation procedure} to approximate some relevant part of its solution; the computability of the continuation procedure is left as an unanswered question for the time being.

We then present a class of linear Schrödinger equations (eq. (18) below) with a special class of time-dependent Hamiltonians that are intimately connected to Hilbert’s tenth problem. We call this the \textit{dynamical approach}, because of its use of the Schrödinger equations, to distinguish it from the \textit{kinematic approach} above from which we derive the set of non-linear differential equations. We conclude the paper with some remarks and a discussion on the possible implications, including those on the Church-Turing thesis, if the differential equations are indeed computable in some computation model of computable analysis.

\[\text{\textsuperscript{1}}\text{For example, the so-called real-RAM approach \cite{1,10,13,14,15} introduces computable real functions directly and borrows only the concept of control structure from Turing computation, without any further referencing to the latter. The computability notion in this model, as a result, is different from that of other real computation models that are based and built from the Turing computation.} \]

\[\text{\textsuperscript{2}}\text{One famous example is that the choice of different norms in a Banach space can lead to opposing conclusions about the computability of solutions of the same wave equation in three spatial dimensions with computable initial functions \cite{13}.} \]
Hilbert’s tenth problem and unbounded operators in Hilbert spaces

Given a Diophantine equation with $K$ unknowns $x$’s as in eq. (1), we can make a connection, following [4, 5, 6, 7, 8, 9], with the following self-adjoint operator acting on some appropriate Fock space (a special type of Hilbert space)

$$H_P = \left( D(a_1a_1^\dagger, \cdots, a_Ka_K^\dagger) \right)^2,$$

where

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_k, a_j] = 0,$$

which are usually termed the creation and annihilation operators, and most commonly seen in text-book treatment of the quantum simple harmonic oscillators. The Fock space is built out of the “vacuum” $\bigotimes_{j=1}^K |0_j\rangle$ by repeating applications of the creation operators $a_j^\dagger$.

The operator (2) has non-negative and discrete eigenvalues $(D(n_1, \cdots, n_K))^2$, with natural numbers $n_1, \cdots, n_K$. There is an eigenstate $|E_g\rangle$ corresponding to the smallest eigenvalue $E_g$. If the self-adjoint operator is considered as a Hamiltonian for some dynamical process then these are respectively the ground state and its energy.

It is then clear that the Diophantine equation [4] has at least one integer solution if and only if $E_g = (D(n_1^{(0)}, \cdots, n_K^{(0)}))^2 = 0$, for some $K$-tuple of natural numbers $(n_1^{(0)}, \cdots, n_K^{(0)})$.

To sort out this $E_g$ among the infinitely many eigenvalues is almost an impossible task. The strategy we will employ, as inspired by quantum adiabatic processes, is to tag the state $|E_g\rangle$ by some other known state $|E_I\rangle$ which is the ground state of some other self-adjoint operator $H_I$, which can be smoothly deformed to $H_P$ through some continuous parameter $s \in [0, 1]$. To that end, we consider the interpolating operator

$$\mathcal{H}(s) = H_I + f(s)(H_P - H_I),$$

which has an eigenproblem at each instant $s$,

$$[\mathcal{H}(s) - E_q(s)]|E_q(s)\rangle = 0, \quad q = 0, 1, \cdots$$

with the subscript ordering according to the sizes of the eigenvalues, and $f(s)$ some continuous and monotonically increasing function in $[0, 1]$

$$f(0) = 0; \quad f(1) = 1.$$

Clearly, $E_0(0) = E_I$ and $E_0(1) = E_g$.

A suitable $H_I$ is, where $\alpha$’s $\in \mathbb{C}$,

$$H_I = \sum_{i=1}^K \lambda_i(a_i^\dagger - \alpha_i^\ast)(a_i - \alpha_i),$$

which we will employ from now on. Here, $E_I = 0$ and $|E_I\rangle = |\alpha_1 \cdots \alpha_K\rangle$ is the Cartesian product of the coherent states

$$|\alpha_i\rangle = e^{-|\alpha_i|^2} \sum_{n_i=0}^{\infty} \frac{\alpha_i^n}{\sqrt{n_i!}} |n_i\rangle.$$
where \( |n_i\rangle \) are the eigenstates of \( a_0^\dagger a_i \) with eigenvalues \( n_i \). The \( \lambda_i \) can be chosen to be rational or even irrational numbers such that the first order equation

\[
\sum_{i=1}^{K} \lambda_i p_i = 0,
\]

has no integer solutions in \( p_i \). This condition is to ensure that all the eigenvalues of \( H_I \) are non-degenerate, since the eigenvalues of \( H_I \), which are of the form \( \sum_{i=1}^{K} \lambda_i n_i \), are then easily seen to be unique for different \( K \)-tuples of natural numbers \( n_i \).

**The spectral flow – The “kinematic” approach**

We now derive the differential equations for the tagging connection between the instantaneous eigenvalues and eigenvectors at different instants in (5).

Note firstly that, from the normalisation condition \( \langle E_q | E_q \rangle = 1 \), we can write

\[
\langle E_q | \partial_s | E_q \rangle = -i \partial_s \phi_q,
\]

for some real \( \phi_q \). This can be absorbed away with the redefinition

\[
e^{i\phi_q(s)} | E_q(s) \rangle \rightarrow | E_q(s) \rangle,
\]

upon which

\[
\langle E_q | \partial_s | E_q \rangle = 0.
\]

(10)

(11)

Differentiating (5) with respect to \( s \) yields

\[
[f'(s)W - \partial_s E_q]|E_q\rangle + [\mathbf{J} - E_q] \partial_s |E_q\rangle = 0.
\]

(12)

We next insert the resolution of unity at each instant \( s \),

\[
1 = \sum_{m=0}^{\infty} |E_m(s)\rangle \langle E_m(s)|,
\]

just after \( \mathbf{J} \) in (12) to get, by virtue of (11),

\[
E_q \partial_s |E_q\rangle = [f'(s)W - \partial_s E_q]|E_q\rangle + \sum_{m \neq q} \infty \sum_{m=0}^{\infty} E_m \langle E_m | \partial_s | E_q \rangle |E_m\rangle.
\]

(13)

The inner product of the last equation with \( |E_l\rangle \) gives

\[
( E_q - E_l ) \langle E_l | \partial_s | E_q \rangle = f'(s) \langle E_l | W | E_q \rangle - \partial_s E_q \delta_{ql}.
\]

Thus, for \( q \neq l \) this gives the components of \( \partial_s |E_q\rangle \) in \( |E_l\rangle \), provided \( E_q(s) \neq E_l(s) \) at any \( s \in (0, 1) \), a condition whose proof has been outlined in [6]. Consequently, together with (11),

\[
\partial_s |E_q\rangle = f'(s) \sum_{l \neq q} \infty \sum_{m=0}^{\infty} \langle E_l | W | E_q \rangle |E_l\rangle.
\]

(15)

Also, putting \( q = l \) in (14) we have

\[
\partial_s E_q(s) = f'(s) \langle E_q(s) | W | E_q(s) \rangle.
\]

(16)

Equations (15) and (16) form the set of infinitely coupled differential equations providing the tagging linkage we have been looking for.

In this reformulation, the Diophantine equation (1) has an integer solution if and only if

\[
\lim_{s \to 1} E_0(s) = 0,
\]

(17)
from the constructively known eigenvalues and eigenstates of $H_I$ as the initial conditions. The limiting process might be necessary since $H_P$, i.e. $S_1(1)$, may have a degenerate ground state eigenvalue in general.

“Kinematic” continuation procedure?

The non-linear differential equations of the last Section are infinitely coupled and may not be solved explicitly or computably in general. However, we are only interested in the ground state eigenvalue $E_0(1)$ being zero or not. And since the influence on the ground state by states having larger and larger indices diminishes more and more thanks to the denominators in (15), this information might be derived in some approximation scheme in which the number of states involved is truncated to a finite number. The size of the truncation cannot be universal and must of course depend on the particular Diophantine equation under consideration.

In the below we speculate on an analytic approximation under the name of continuation procedure, and we make no claim about its computability here but leave it as a challenge for the future.

- Starting from the initial condition comprising of the constructively known eigenvalues and eigenvectors of $H_I$ at $s = 0$, the differential equations (15, 16) give us the series expansions

$$|E_q(\epsilon_1)\rangle = |E_q(0)\rangle + \epsilon_1 f'(0) \sum_{l \neq q} \frac{(E_l(0))[W|E_q(0)\rangle}{E_q(0) - E_l(0)}|E_l(0)\rangle + \mathcal{R}_1,$$

$$E_q(\epsilon_1) = E_q(0) + \epsilon_1 f'(0)|E_q(0)\rangle|W|E_q(0)\rangle + \Omega_1.$$

- If the remainders $\mathcal{R}_1$ and $\Omega_1$ above are computable, we would be able to evaluate the radii of convergence in $s$, which contain $s = \epsilon_1$, and also the truncation size $N_1$ which determines the accuracy of the expansions.

- We then proceed to evaluate new series approximations similar to the ones above but this time centred at $s = \epsilon_1$. The new series have new radii of convergence and a new truncation of $N_2$ eigenvectors previously approximated at $s = \epsilon_1$. After this step we have then covered a finite domain in $s$ away from zero, with some computable degree of accuracy.

- We keep reiterating this procedure, if possible, to obtain new remainders and radii of convergence and thus extend the covered domain in $s$, until we could evaluate the limit $\lim_{s \to 0} E_0(s)$.

This is reminiscent of the procedure of analytic continuation of functions in complex analysis.

Note that at each step of the above procedure we only require some finite truncation $N_i$ for a given accuracy of the series approximations. (That accuracy would also “recursively” determine the truncations $N_i$ at all previous steps, $j < i$.) Note also that, in general, the condition $x = 0$ for a computable real number $x$ may not be effectively decidable in some computation model. But here we have the imposed condition that the eigenvalues at $s = 1$ must be integer-valued. This additional condition might help making the equality condition effectively decidable at $s = 1$. That is, we would only need to approximate $E_0(1)$ by the procedure above up to some accuracy, say 0.3, which sufficiently enables us to distinguish different integers, and we would not require infinite precision. The imposed condition of integer-valued eigenvalues for $H_P$, by construction, would provide us the built-in infinite precision at no extra cost!

Hilbert’s tenth and the Schrödinger equation – The “dynamical” approach

The decision result for Hilbert’s tenth problem can also be encoded in yet another class of linear differential equations, apart from the class of nonlinear equations (15, 16) above. The linear
Let $|\psi(t)\rangle$ be the quantum state at time $t$ (of some quantum system), its time evolution is given by the Schrödinger equation, for $0 < t < \tau$,

$$\partial_t |\psi(t)\rangle = -i\mathcal{H}(t/\tau)|\psi(t)\rangle,$$

(18)

$$|\psi(0)\rangle = |\alpha_1 \cdots \alpha_K\rangle,$$

where we have chosen the initial state at time $t = 0$ to be the non-degenerate ground state of $H_I$. Once again we are only interested in the ground state of $H(1) = \mathcal{H}_P$. The quantum adiabatic theorem [12] asserts that as $\tau \to \infty$ (that is, when the Hamiltonian $\mathcal{H}(t/\tau)$ in eq. (18) varies sufficiently slowly in the time $t$) the state of the above system at $t = \tau$ would be in the desired ground state, $|\psi(\tau)\rangle \approx |E_g\rangle$, to any arbitrary degree of precision! For a given precision, various versions of the theorem dictate different conditions on $\tau$ in terms of some intrinsic properties of the eigenvalues and eigenfunctions of $\mathcal{H}(s)$. Those conditions thus are highly dependent on the individual Diophantine equation, and hence are not suitable to the spirit of a universal procedure required by Hilbert’s tenth problem.

We have proposed a different and universal criterion for the identification of the ground state $|E_g\rangle$ from $|\psi(\tau)\rangle$: the Fock state $|n_1^{(0)} , \cdots , n_K^{(0)}\rangle$ is the ground state $|E_g\rangle$ if it has an occupation probability greater than one-half. That is,

$$\left|\left\langle\psi(\tau)|n_1^{(0)} , \cdots , n_K^{(0)}\right\rangle\right|^2 > 1/2 \Rightarrow |n_1^{(0)} , \cdots , n_K^{(0)}\rangle = |E_g\rangle.$$

(19)

It should be emphasised here that such a criterion should be taken at the present time as a postulate as it has only been proved in some limited settings [9]. With this criterion, we only need to repeatedly solve the Schrödinger (18) for larger and larger $\tau$ each time until the probability condition is satisfied so that the ground state can be identified. Such a time $\tau$ can be shown to exist and be finite by the quantum adiabatic theorem. More details of these can be found in [6, 9].

Note also that while the criterion (19) may not be the only one suitable for a physical implementation of the Schrödinger equation, it is the only one that we could yet find suitable for the mathematical discussion of this paper.

Unlike the case of nonlinear differential equations of a previous Section, here we could try to make use of a powerful computability result in a computation model which is known as the First Main Theorem by Pour-El & Richards [13]. Essentially, the theorem asserts that a bounded linear operator from a Banach space to a Banach space which maps a computable sequence of spanning vectors into another computable sequence will also map any computable element into another computable element. For the case at hand, our Schrödinger equation defines a linear operator,

$$U(\tau) = \mathcal{T}\exp\left\{-i\tau\int_0^1 \mathcal{H}(s)ds\right\},$$

(20)

where $\mathcal{T}$ is the time-ordering symbol, which maps the initial state to the final state in the same separable Hilbert space. Now, our initial state $|\alpha_1 \cdots \alpha_K\rangle$ is computable by construction. On the other hand, the linear operator (20) coming out of the Schrödinger equation must be unitary and thus be bounded. Hence, the conditions of the theorem remained to be checked for fulfillment are: (i) which mathematical norm should be chosen to enable the identification criterion (19), or some other equivalent criterion, for the ground state at $t = \tau$; and (ii) whether the image of a particular computable basis is computable or not with this suitably chosen norm.
Concluding remarks

Inspired by quantum mechanics, we have reformulated the question of solution existence of a Diophantine equation into the question of certain properties conceived in an infinitely coupled set of nonlinear differential equations. In words, we encode the answer of the former question into the smallest eigenvalue and corresponding eigenvector of a self-adjoint operator whose integer-valued eigenvalues are bounded from below. To find these eigen-properties we next deform the operator continuously to another self-adjoint operator whose spectrum is known. Once the deformation is also expressible in the form of a set of nonlinearly coupled differential equations, we could now start from the constructive knowns as a handle to study the desired unknowns.

In addition, we also explicitly present a class of linear Schrödinger equations whose solutions at some time $\tau$ from appropriate initial conditions contain the decision results for any given Diophantine equation.

These reformulations map a noncomputable problem in the domain of integer arithmetics into the wider framework of computable analysis. We have given the names, as explained in the Introduction, “dynamical” and “kinematic” respectively for the resulting linear and non-linear equations. In particular, our set of nonlinear equations would be an important topic for the largely untouched subject of nonlinear computable analysis.

For the various reasons given in the Introduction Section, the questions of computability for these differential equations are non-trivial and important. Towards some answers for these questions, we have advocated and speculated on an approach based on the First Main Theorem by Pour-El & Richards [13] for the Schrödinger equations, and some other approximation procedure for the set of nonlinear differential equations. However, for now, these computability questions will have to be left as open problems and challenges.

If in the (admittedly unlikely) event that there exists computation model (with suitably chosen norm) in which the differential equations above are computable and that this model could be restricted and applied to functions from $\mathbb{N}$ to $\mathbb{N}$ then Hilbert’s tenth problem would seem to be solvable in integer arithmetics (as opposed to be solvable in, for example, some physical quantum adiabatic computation)! And this would entail a logical breakdown of the Church-Turing thesis. Because the restriction of such a computable analysis model to the domain of integers would provide a new notion of effective computability different from that of the class of Turing computation.

To be sure that this is not the case, further investigations are urgently needed to find out (i) whether the above equations are noncomputable in all computation models in computable analysis; or (ii) whether the model that admits such computability, if exists, cannot be restricted to integer arithmetics.

Acknowledgements

I am indebted to Peter Hannaford and Alan Head for discussions and support. I would also like to thank Marian Pour-El for a discussion on the computability of quantum mechanics and for her gift of the out-of-print book [13]. This work has been supported by the Swinburne University Strategic Initiatives.

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