CENTRAL BINOMIAL TAIL BOUNDS

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Abstract. An alternate form for the binomial tail is presented, which leads to a variety of bounds for the central tail. A few can be weakened into the corresponding Chernoff and Slud bounds, which not only demonstrates the quality of the presented bounds, but also provides alternate proofs for the classical bounds.

1. Introduction

Let $B(p, n)$ denote a binomial random variable comprising $n$ flips of a bias-$p$ coin, and set $\sigma = \sqrt{p(1-p)}$. The classical form of the central tail, obtained by summing over the possible outcomes, is

$$P[B(p, n) \geq n/2] = \sum_{h=\lceil n/2 \rceil}^{n} \binom{n}{h} p^h (1-p)^{n-h}. \quad (1.1)$$

If instead $B$ is considered from the perspective of random walks on the integer line, another representation is possible by tracking, as $n$ increases, the motion of mass from one side of the origin to the other. This intuition is formalized in section 2, and results in the following statement.

Theorem 1.1. When $n$ is odd and $p < 1/2$,

$$P[B(p, n) \geq n/2] = p - (1/2 - p) \sum_{j=1}^{(n-1)/2} \binom{2j}{j} \sigma^{2j} \quad (1.2)$$

$$= (1/2 - p) \sum_{j \geq (n+1)/2} \binom{2j}{j} \sigma^{2j}. \quad (1.3)$$

To demonstrate the value of this new characterization of the central tail, it is used to derive bounds. In particular, section 3 (“Closed-form Bounds”) approximates the summands of (1.2) and (1.3) in various ways to yield summations with closed-form expressions. On the other hand, section 5 (“Bounding with the Standard Normal”) replaces the summation of (1.3) with an integral, yielding a bound incorporating the distribution function of the standard normal.

A few of these bounds appear in Figure 1. The upper and lower bound pair of (5.1) are the tightest, and their forms are sufficiently similar to allow the gap to be analytically quantified. This comes at the cost of interpretability: they are rather complicated. Contrastingly, the upper bound of (3.7) is simple: when $n$ is odd, it is just $(2\sigma)^{n+1}/2$. Remarkably, the Chernoff bound for $m$ even is $(2\sigma)^m$. (Whenever

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the number of trials is odd, \( n \) is used; when it is even, \( m \) is used.) The relationship between the bounds of section 3 and the Chernoff bound is explored in section 4. The gap between these bounds, seemingly large in Figure 1, is quantified, which furthermore provides an alternate proof of the Chernoff bound.

Among the lower bounds, only that of (5.1) consistently outperforms Slud’s bound. Fortunately, both depend on the standard normal, and thus the comparison can be made precise: section 6 discusses proving Slud’s bound by weakening the bounds of (5.1). The proof is notable because it extends the sufficient conditions of the classical statement of Slud’s bound. Unfortunately, the details of this proof are tedious, and relegated to Appendix B. The task of producing a good, elementary lower bound proved challenging: the lower bound of (3.5) which appears in Figure 1 is only tight for \( p \) away from \( 1/2 \). A comparison of all bounds may be found in Figure 2 of section 7.

To close, section 8 generalizes Theorem 1.1 to arbitrary tails, however no bounds are derived.

2. Central Binomial Tails via Random Walks

Consider random walks on the integer line originating at 0, and at each step incrementing their position with probability \( p \), or decrementing it with probability \( 1 - p \). When \( n \) is odd, \( P[B(p, n) \geq n/2] \) can be interpreted to mean the probability mass of walks terminating with positive coordinate after \( n \) steps. To prove Theorem 1.1, the first step is to quantify the effect which two trials have on this probability mass.
Lemma 2.1. When $n$ is odd,

$$\mathbb{P}[B(p, n + 2) \geq (n + 2)/2] - \mathbb{P}[B(p, n) \geq n/2] = (p - 1/2) \left( \frac{n + 1}{(n + 1)/2} \right) \sigma^{n+1}.$$ 

Proof. For mass to change sign in two steps, it must originate in a path ending at a coordinate adjacent to the origin, and move in the direction of the origin twice. Symbolically,

$$p^2 \left( \frac{n}{(n-1)/2} \right) p^{(n-1)/2} (1-p)^{(n+1)/2} - (1-p)^2 \left( \frac{n}{(n+1)/2} \right) p^{(n+1)/2} (1-p)^{(n-1)/2}$$

\[
\begin{align*}
\text{two increasing steps} & \quad \text{two decreasing steps} \\
= (2p-1) \left( \frac{n}{(n+1)/2} \right) \sigma^{n+1}.
\end{align*}
\]

To finish, note that \(\left( \frac{n}{(n+1)/2} \right) = \left( \frac{n+1}{n+1} \right) \left( \frac{n+1}{(n+1)/2} \right) = \frac{1}{2} \left( \frac{n+1}{(n+1)/2} \right).\)

Although the above proof depends on a random walk interpretation, it also goes through purely algebraically using (1.1).

Accumulating the contribution of such steps up to $n$, it is possible to rewrite the binomial tail.

Lemma 2.2. When $n$ is odd,

$$\mathbb{P}[B(p, n) \geq n/2] = p - (1/2 - p) \sum_{j=1}^{(n-1)/2} \binom{2j}{j} \sigma^{2j}.$$ 

Proof. Invoking Lemma 2.1

$$\mathbb{P}[B(p, n) \geq n/2]$$

\[
\begin{align*}
= \mathbb{P}[B(p, 1) \geq 1/2] + \sum_{j=1}^{n-2} \mathbb{P}[B(p, j + 2) \geq (j + 2)/2] - \mathbb{P}[B(p, j) \geq j/2]) \\
= p + (p - 1/2) \sum_{j=1}^{n-2} \left( \frac{j + 1}{(j + 1)/2} \right) \sigma^{j+1},
\end{align*}
\]

and substituting $2j - 1$ for $j$ yields the lemma.

When $p < 1/2$, as $n \to \infty$, the central tail probability must approach 0 (cf. for instance (3.7)). As such, it should also be possible to compute the tail in a fashion complementary to Lemma 2.2, instead accumulating the contribution of all remaining steps.

Lemma 2.3. When $n$ is odd and $p < 1/2$,

$$\mathbb{P}[B(p, n) \geq n/2] = (1/2 - p) \sum_{j \geq (n+1)/2} \binom{2j}{j} \sigma^{2j}.$$
Proof. When \( p = 0 \), the result is immediate, thus take \( p \in (0, 1/2) \). Combining the Taylor expansion \((1 - 4\sigma^2)^{-1/2} = \sum_{j \geq 0} \binom{2j}{j} \sigma^{2j} \) with Lemma 2.2,

\[
p - \left( \frac{1}{2} - p \right) \sum_{j=1}^{(n-1)/2} \binom{2j}{j} \sigma^{2j} = p - \left( \frac{1}{2} - p \right) \left( \frac{1}{\sqrt{1 - 4\sigma^2}} - \sum_{j \geq (n+1)/2} \binom{2j}{j} \sigma^{2j} \right) = \frac{1}{2} - \frac{1/2 - p}{\sqrt{1 - 4\sigma^2}} + (1/2 - p) \sum_{j \geq (n+1)/2} \binom{2j}{j} \sigma^{2j},
\]
the result following since \( \sqrt{1 - 4\sigma^2} = |1 - 2p| \) and \( p < 1/2 \). □

Proof of Theorem 1.1 (1.2) is handled by Lemma 2.2, whereas Lemma 2.3 takes care of (1.3). □

Remark 2.4. Going forward, the two constraints that \( n \) is odd and \( p < 1/2 \) will frequently appear. First note that \( p < 1/2 \) can be assuaged with

\[
P[B(p,n) \geq n/2 + k] = P[B(1-p,n) \leq n/2 - k].
\]
(And when \( p = 1/2 \), by Lemma 2.2, \( P[B(1/2,n) \geq n/2] = 1/2 \).) Additionally, the tail may be flipped with

\[
P[B(p,n) \geq n/2 + k] = 1 - P[B(p,n) < n/2 + k].
\]
Lastly, using the same random walk reasoning as in the proof of Lemma 2.1 central tail bounds on \( B(p,m) \) where \( m \) is even can be reduced to bounds on \( B(p,m-1) \) via

\[
P[B(p,m) \geq m/2] - P[B(p,m-1) \geq (m-1)/2] = p \left( \frac{m-1}{(m-2)/2} \right) p^{(m-2)/2} (1-p)^{m/2} = \frac{1}{2} \left( \frac{m}{m/2} \right) p^m.
\]

3. Closed-form Bounds

To produce bounds from Theorem 1.1, the first task is to eliminate the binomial coefficient, which the following steps achieve by way of Stirling’s approximation.

**Definition 3.1.** Define
\[
l(n) = -\frac{9n + 1}{3n(12n + 1)},
\]
\[
u(n) = -\frac{18n - 1}{12n(6n + 1)}.
\]

**Remark 3.2.** Note that, for \( n \geq 1 \), both are strictly increasing, and
\[
-\frac{1}{5n} > u(n) > -\frac{1}{4n} > l(n) > -\frac{1}{3n}.
\]

Using the bounded form of Stirling’s approximation (as in (9.15) from chapter 2 of Feller [2])
\[
\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{1/(12n+1)} < n! < \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{1/(12n)},
\]
the central binomial coefficient can be bounded with
\[
\frac{4^j e^{l(2j)}}{\sqrt{\pi j}} < \binom{2j}{j} < \frac{4^j e^{u(2j)}}{\sqrt{\pi j}}.
\]
Notice that combining (3.1) and (for instance) Lemma 2.3 yields the somewhat hopeful relation

\begin{equation}
(1/2 - p) \sum_{j \geq (n+1)/2} \frac{(2\sigma)^{2j} e^{(2j)}}{\sqrt{\pi j}} \leq P[B(p, n) \geq n/2] \leq (1/2 - p) \sum_{j \geq (n+1)/2} \frac{(2\sigma)^{2j} e^{(2j)}}{\sqrt{\pi j}}.
\end{equation}

The remainder of this section starts from (3.2) (or from the analogous formula using the finite summation of Lemma 2.2), and manipulates the summation into one possessing a closed-form expression. The primary difficulty in (3.2) is the term $j - 1/2$, and the derivation of each bound can be characterized by its approach to this term. A sense of the relative performance of the bounds can be gleaned from Figure 2 on page 11.

The first bounds relax $j - 1/2$ trivially; that is, upper bounding it with 1, and lower bounding it with $j - 1$.

**Theorem 3.3.** When $n$ is odd and $p \in (0, 1/2)$,

\begin{equation}
P[B(p, n) \geq n/2] \geq \frac{(1 - 2p)e^{(n+1)(2\sigma)^{n-1}}}{\sqrt{2\pi(n + 1)}} (-\ln(1 - 4\sigma^2)),
\end{equation}

\begin{equation}
P[B(p, n) \geq n/2] \leq \frac{(2\sigma)^{n+1}}{(1 - 2p)^{n+1}2\pi(n + 1)}.
\end{equation}

(Note that the lower and upper bounds may be related using $\ln(x) \leq x - 1$.) Both bounds become poor as $p \to 1/2$; in fact, the upper bound grows unboundedly, and the lower bound goes to zero. The upper bound, however, is sufficiently tight for $p < 1/4$ and odd $n$ to prove the Chernoff bound in section 4.

**Proof.** From (3.2)

\[P[B(p, n) \geq n/2] \geq (1/2 - p) \sum_{j \geq (n+1)/2} \frac{(2\sigma)^{2j} e^{(2j)}}{\sqrt{\pi j}} \geq (1/2 - p) e^{(n+1)2\sigma^{n-1}} \sum_{j \geq 1} \frac{(2\sigma)^{2j+n-1}}{\sqrt{\pi j}(n+1)/2},\]

and the bound follows using the Taylor expansion $-\ln(1 - 4\sigma^2) = \sum_{j \geq 1} (2\sigma)^{2j}/j$.

Similarly for the upper bound,

\[P[B(p, n) \geq n/2] \leq (1/2 - p) \sum_{j \geq (n+1)/2} \frac{(2\sigma)^{2j} e^{(2j)}}{\sqrt{\pi j}} \leq \frac{1 - 2p}{\sqrt{2\pi(n + 1)}} \sum_{j \geq (n+1)/2} (2\sigma)^{2j} = \frac{1 - 2p}{\sqrt{2\pi(n + 1)}} \left( \frac{(2\sigma)^{n+1}}{1 - 4\sigma^2} \right).
\]

To finish, use $1 - 4\sigma^2 = (1 - 2p)^2$. \qed

Another approach to the term $j^{-1/2}$ is to lower bound with an exponential.
Theorem 3.4. When \( p < 1/2 \) and \( n \) is odd,

\[
\Pr[B(p, n) \geq n/2] \geq \frac{(1 - 2p)e^{(n+1)(2\sigma)^{n+1}}}{(1 - 4\sigma^2 \sqrt{(n+1)/(n+3)}) \sqrt{2\pi(n+1)}}.
\]

This lower bound also approaches zero as \( p \to 1/2 \), but is otherwise the tightest elementary lower bound in this paper. It can be seen to dominate the lower bound of \([3.3]\) by taking \( n \) large and using a tangent approximation to \( \ln \).

Proof. Fitting an exponential to \(((n + 1)/2)^{-1/2} \) and \(((n + 3)/2)^{-1/2} \) yields

\[
\frac{1}{\sqrt{(n+1)/2}} \left( \frac{n + 1}{n + 3} \right)^{j-(n+1)/2} = \frac{2}{n + 1} \left( \frac{n + 1}{n + 3} \right)^{j-(n+1)/2}.
\]

Thus, again using \([3.1]\) and Lemma 2.3

\[
\Pr[B(p, n) \geq n/2] \geq \frac{(1/2 - p)e^{(n+1)}}{\sqrt{\pi}} \sum_{j \geq (n+1)/2} (2\sigma)^{2j} \sqrt{2} \left( \frac{n + 1}{n + 3} \right)^{j-(n+1)/2},
\]

with the usual geometric sequence formula giving the statement. \( \square \)

This section’s last method of coping with \( j^{-1/2} \) relies upon the chain of equalities\(^1\)

\[
\sum_{j \geq \eta} \frac{x^j}{\sqrt{j}} = \sqrt{\left( \sum_{j \geq \eta} \frac{x^j}{\sqrt{j}} \right) \left( \sum_{j \geq \eta} \frac{x^j}{\sqrt{j}} \right)} = \sqrt{\sum_{k} x^j \sum_{j=\eta}^{k-\eta} \frac{1}{\sqrt{j(k-j)}}}.
\]

Definition 3.5. For any \( \eta, k \in \mathbb{Z}^+ \) with \( 2\eta \leq k \), define

\[
\psi_{\eta}(k) = \sum_{j=\eta}^{k-\eta} \frac{1}{\sqrt{j(k-j)}}.
\]

As it turns out, \( \psi_{\eta} \) is rather well behaved.

Lemma 3.6. For any \( \eta, k \in \mathbb{Z}^+ \) with \( 2\eta \leq k \), \( \psi_{\eta}(k) \leq \psi_{\eta}(k + 1) \leq \pi \).

Proof. Note that

\[
\psi_{\eta}(k) = \sum_{j=\eta}^{k-\eta} \frac{1}{\sqrt{j(k-j)}} = \sum_{j=\eta}^{k-\eta} \frac{d}{dj} \cos^{-1}(1 - \frac{2j}{k})
\]

\[
= -\frac{2}{k} \sum_{i=1-2j/k}^{j=\eta,\ldots,k-\eta} \frac{d}{di} \cos^{-1}(i) = \frac{2}{k} \sum_{i=1-2j/k}^{j=\eta,\ldots,k-\eta} \frac{1}{\sqrt{1-i^2}}.
\]

As such, \( \psi_{\eta}(k) \) can be interpreted as a Riemann sum lower bounding the function \((1 - x^2)^{-1/2}\) on the interval \((-1, +1)\). Indeed, take \( 2/k \) to be the width of each rectangle, and when \( i \leq 0 \), take \((1 + i^2)^{-1/2}\) to be the height at the right endpoint of a rectangle, otherwise when \( i > 0 \) take it to be the height at the left endpoint. Since \( d(cos^{-1}(x))/dx = -(1 - x^2)^{-1/2} \), the value of the approximated integral is \( \pi \), which gives the upper bound. (To handle the discontinuity, apply the monotone convergence theorem to \( \lim_{n \to \infty} \int_{-1+1/n}^{1-1/n} \frac{dx}{\sqrt{1-x^2}} \).)

\(^1\)The idea for the approach comes rather naturally if attempting to relate \([3.2]\) to the Chernoff bound, as addressed in Section 4.
For the monotonicity statement, note that the Riemann sums of a convex, decreasing function are increasing as the width of the subdivisions decreases (a proof of this fact is in Appendix E). The result follows by applying this to both halves of the function separately. □

With this machinery in place, the final bounds of this section may be established.

**Theorem 3.7.** When \( n \) is odd and \( p < 1/2 \),
\[
\frac{(2\sigma)^{n+1} e^{(n+1)}}{2\pi (n+1)} \leq \mathbb{P}[B(p, n) \geq n/2] \leq \frac{(2\sigma)^{n+1}}{2}.
\]

It is shown in section 4 that, with an even number of trials, the upper bound of (3.7) is tighter than the Chernoff bound for all \( p \in [0, 1/2] \). Furthermore, the ratio of the two approaches 2 as the number of trials grows.

The lower bound of (3.7) has the weakness that, as the number of trials grows, it becomes poor. On the other hand, it has the distinction, among all bounds of this section derived from (3.2), that it does not approach 0 as \( p \to 1/2 \).

**Proof.** Again using (3.2) but now dealing with \( j^{-1/2} \) via (3.6),
\[
\mathbb{P}[B(p, n) \geq n/2] \leq \frac{1 - 2p}{2\sqrt{\pi}} \sqrt{\sum_{k \geq n+1} (2\sigma)^{2k}\psi(n+1/2)(k)} \\
\leq \frac{1 - 2p}{2} \sqrt{(2\sigma)^{2n+2}/1 - 4\sigma^2},
\]
where the conclusion used \( \psi_\eta(k) \leq \pi \); to finish, substitute \( \sqrt{1 - 4\sigma^2} = |1 - 2p| \). The lower bound proceeds analogously, but invoking Lemma 3.6 to grant \( 2/(n+1) = \psi(n+1)/2(n+1) \leq \psi(n+1/2)(k) \) for all \( k \geq n + 1 \). □

Note that all preceding bounds used the infinite summation form as presented in Lemma 2.3. For the last pair of bounds in this section, the finite sum from Lemma 2.2 is used, which predictably leads to a much different bound.

**Theorem 3.8.** When \( n \) is odd and \( p < 1/2 \),
\[
P[B(p, n) \geq n/2] \geq p - \frac{2\sigma^2 e^{(n-1)}\sqrt{(1 - (2\sigma)^{2n-4})\psi_1(n-1)}}{\sqrt{\pi}}.
\]
\[
P[B(p, n) \geq n/2] \leq p - \frac{2\sigma^2 e^{(2)}\sqrt{1 - (2\sigma)^{n-1}}}{\sqrt{\pi}}.
\]

Both (3.8) and (3.9) become exact as \( p \to 1/2 \), but are otherwise inaccurate.

**Proof.** This proof does not differ greatly from the others in this section, with the exception of starting from Lemma 2.2. For the lower bound, the key inequalities are
\[
- \sum_{j=1}^{(n-1)/2} \frac{(2\sigma)^{2j}}{\sqrt{j}} \geq - \sum_{k=2}^{n-1} (2\sigma)^{2k}\psi_1(k) \geq - \sqrt{\frac{((2\sigma)^{4} - (2\sigma)^{2n})\psi_1(n-1)}{1 - 4\sigma^2}},
\]
which makes use of the monotonicity of \( \psi_\eta \). The upper bound is similar, but using the fact that \( \psi_1 \geq 1 \). □
4. Relationship to the Chernoff Bound

Since \([X \geq a] = [e^{tX} \geq e^{ta}]\) for all \(t > 0\), it follows by Markov’s inequality that
\[
P[X \geq a] \leq \inf_{t>0} \frac{\mathbb{E}(e^{tX})}{e^{ta}};
\]
this is a form of the Chernoff bound (see (3.6) in Chernoff [1]). Applying this to the central tail, when \(m\) is even, yields
\[
P[B(p,m) \geq m/2] \leq (2\sigma)^m.
\]
It is no coincidence this bears a striking resemblance to the upper bound in (3.7) that bound was derived with the intent of proving the Chernoff bound. In fact, adjusting (3.7) to even \(m\) as per Remark 2.4,
\[
P[B(p,m) \geq m/2] \leq (2\sigma)^m \left(1 + \frac{1}{\sqrt{2\pi m}} \right)^{n/2}.
\]
This serves to not only prove the Chernoff bound (for this case), it also states that the multiplicative error of the Chernoff bound is at least \(1/(2 + (2\pi m)^{-1/2})^{-1}\).

When \(n\) is odd, the Chernoff bound is the slightly uglier expression
\[
P[B(p,n) \geq (n+1)/2] \leq (2\sigma)^n \sqrt{\frac{1-p}{1-p}} \left(\frac{n^2}{n^2-1}\right)^{n/2} \sqrt{\frac{n-1}{n+1}}.
\]
The ratio of this expression to the upper bound (3.7) is
\[
\sqrt{\frac{1-p}{1-p}} \left(\frac{n^2}{n^2-1}\right)^{n/2} \sqrt{\frac{n-1}{n+1}},
\]
which approaches 2 as \(n \to \infty\). It does not, however, exceed 1 for all \(p\) and all \(n\). On the other hand, dividing the bound by the upper bound in (3.4) yields the ratio
\[
\sqrt{\frac{1-p}{1-p}} \sqrt{\frac{\pi(n-1)}{(n-p)^2}} \left(\frac{n^2}{n^2-1}\right)^{n/2}.
\]
The ratio in (4.2) holds when \(p \geq 1/4\), whereas the latter ratio in (4.3) exceeds 1 when \(p \leq 1/4\). Thus, combining the two via a min yields a better bound. As will be discussed in section 7, a number of the bounds, when paired via min or max, form extremely good bounds.

Lastly, note that the Chernoff bound was chosen because, for an even number of trials, it is it is tighter than the corresponding Hoeffding and Bernstein bounds. (For a proof, see Appendix D.)

5. Bounding with the Standard Normal

The preceding bounds all aimed for a closed-form approximation for either the infinite or finite summation in Theorem 1.1. In this section, however, the strategy is to replace the infinite summation with an integral. As usual, let \(\Phi\) and \(\phi\) be the distribution function and density of the standard normal.

Theorem 5.1. Let \(n\) odd and \(p \in (0,1/2)\) be given, and set
\[
\Upsilon = \frac{2 \left(1 - \Phi(\sqrt{-\ln(4\sigma^2)}\ln(4\sigma^2))\right)}{\sqrt{-\ln(4\sigma^2)}}, \quad \Delta = \frac{(2\sigma)^{n+1}}{\sqrt{2\pi(n+1)}}.
\]
where

\begin{equation}
(5.2) \quad (1/2 - p)e^{l(n+1)}(\Upsilon + \Delta) \leq \mathbb{P}[B(p, n) \geq n/2] \leq (1/2 - p)(\Upsilon + \Delta(1 + R)),
\end{equation}

where

\[
R \leq \min \left\{ 1, \frac{1}{4} \left( \frac{1}{n+1} - \ln(4\sigma^2) \right) \right\}.
\]

The most important property is that the expressions for the upper and lower bounds are nearly the same, providing for easy comparison. Concretely, the additive error of either can be bounded with their difference

\[
(1/2 - p)(\Upsilon + \Delta(1 + R)) - (1/2 - p)e^{l(n+1)}(\Upsilon + \Delta).
\]

Using \(e^{l(n+1)} \geq e^{-1/3(n+1)} \geq 1 - 1/3(n+1)\) and \(R \leq 1\) yields

\[
(5.2) \quad \frac{1}{3(n+1)} \left( \frac{1}{2} - p \right) \Upsilon + \left( 1 + \frac{1}{3(n+1)} \right) \left( \frac{1}{2} - p \right) \Delta.
\]

Since \((1/2 - p)\Upsilon\) is increasing (along \(p \in [0, 1/2]\)), it may be replaced with its limiting value \(1/2\). Substituting the maximizing value for \(p\) into the right summand of \((5.2)\) and simplifying, the error is thus upper bounded by \(2/5(n+1)\). Although coarse, this error bound provides some explanation of the accuracy of \((5.1)\) evidenced in Figure 1 and Figure 2.

The plots use the minimum of the two choices for \(R\) (for every \(p\)). As is discussed in section 6, it is possible to prove Slud’s bound by relaxing these bounds, and when a choice for \(R\) must be made, 1 suffices. Lastly note that the more complicated bound on \(R\), though usually better, is worse for small values of \(p\). For instance, a sufficient condition for the complicated bound to be better (for any \(n \geq 1\)) is \(p \geq 0.0077\).

To prove the theorem, first note that a change of variable suffices to remove \(j^{-1/2}\) from the integral. (Recall that this term was the primary difficulty in section 3.)

**Lemma 5.2.** When \(\sigma \in (0, 1/2)\),

\[
\int_{(n+1)/2}^{\infty} \frac{(2\sigma)^2j}{\sqrt{\pi j}}dj = \frac{2}{\sqrt{\ln(4\sigma^2)}}\Phi\left(-\frac{(n+1)\ln(4\sigma^2)}{\sqrt{\ln(4\sigma^2)}}\right).
\]

**Proof.** To start,

\[
\int_{(n+1)/2}^{\infty} \frac{(2\sigma)^2j}{\sqrt{\pi j}}dj = \int_{(n+1)/2}^{\infty} \frac{e^{j\ln(4\sigma^2)}}{\sqrt{\pi j}}dj.
\]

Applying the map \(j \mapsto -j^2/(2\ln(4\sigma^2))\) yields

\[
\int_{-(n+1)\ln(4\sigma^2)}^{\infty} \frac{e^{-j^2/2}}{-\pi j^2/(2\ln(4\sigma^2))} \left( \frac{-2j}{2\ln(4\sigma^2)} \right) dj,
\]

which gives the statement after some algebra. \(\square\)

**Proof of Theorem 5.1.** By the first order Euler-Maclaurin summation formula,

\[
\sum_{j \geq (n+1)/2} \frac{(2\sigma)^2j}{\sqrt{\pi j}} = \int_{(n+1)/2}^{\infty} \frac{(2\sigma)^2j}{\sqrt{\pi j}}dj + \frac{(2\sigma)^{n+1}}{\sqrt{2\pi(n+1)}} + \int_{(n+1)/2}^{\infty} \left\{ \left\{ x \right\} - 1/2 \right\} \left( \frac{d}{dx} \frac{(2\sigma)^2x}{\sqrt{\pi x}} \right) dx,
\]

where \(\left\{ x \right\}\) is the fractional part of \(x\). (To turn this expression into lower and upper bounds for \(\mathbb{P}[B(p, n) \geq n/2]\), scale by \((1 - 2p)e^{l(n+1)}\) and \(1 - 2p\), which yields both
sides of (3.2). By Lemma 5.2, the first term is \( Y \), and the second is \( \Delta \), so only the last term requires attention. To start, observe that
\[
\frac{d}{dx} \left( \frac{(2\sigma)^{2x}}{\sqrt{\pi x}} \right) = \frac{(2\sigma)^{2x}}{\sqrt{\pi x}} \left( \ln(4\sigma^2) - \frac{1}{2x} \right) =: f(x),
\]
\[
\frac{d^2}{dx^2} \left( \frac{(2\sigma)^{2x}}{\sqrt{\pi x}} \right) = \frac{(2\sigma)^{2x}}{\sqrt{\pi x}} \left( \ln^2(4\sigma^2) - \frac{\ln(4\sigma^2)}{x} + \frac{3}{4x^2} \right).
\]
Since \( \sigma \in (0, 1/2) \), \( f \leq 0 \) and \( f' \geq 0 \), and hence the integral is nonnegative (once again, this follows from exercise 9.16 of Graham et al. [3]), thus establishing the lower bound. Next
\[
\int_{(n+1)/2}^{\infty} \left( \{x\} - 1/2 \right) \left( \frac{d}{dx} \frac{(2\sigma)^{2x}}{\sqrt{\pi x}} \right) \, dx \leq -\frac{1}{2} \int_{(n+1)/2}^{\infty} \left( \frac{d}{dx} \frac{(2\sigma)^{2x}}{\sqrt{\pi x}} \right) \, dx = \Delta,
\]
establishing the first upper bound on \( R \). (The same bound may be derived by starting with the naive integral bound instead of Euler-Maclaurin.) For the second upper bound, write
\[
\int_{(n+1)/2}^{\infty} \left( \{x\} - 1/2 \right) f(x) \, dx = \sum_{j \geq (n+1)/2} \left[ \int_{0}^{1/2} (x - 1/2) f(j + x) \, dx \right] = \sum_{j \geq (n+1)/2} \left( \int_{0}^{1/2} (x - 1/2) f(j + x) \, dx + \int_{1/2}^{1} (x - 1/2) f(j + x) \, dx \right).
\]
Since \( f(x) \) is negative and monotonic increasing, it follows that
\[
\int_{0}^{1/2} (x - 1/2) f(j + x) \, dx \leq \inf_{x \in [0,1/2]} f(j + x) \int_{0}^{1/2} (x - 1/2) \, dx = -f(j)/8, \quad \int_{1/2}^{1} (x - 1/2) f(j + x) \, dx \leq \sup_{x \in [1/2,1]} f(j + x) \int_{1/2}^{1} (x - 1/2) \, dx = f(j)/8.
\]
As such, the sum telescopes, establishing the other bound on \( R \). \( \square \)

6. Relationship to Slud’s Bound

Slud’s bound is the standard tool for lower bounding binomial tails.

**Theorem 6.1** (Slud [3]). Let \( n, k \) be nonnegative integers with \( k \leq n \), and \( p \in [0, 1] \). When either (a) \( p \leq 1/4 \) and \( np \leq k \leq n \), or (b) \( np \leq k \leq n(1 - p) \), then
\[
\Pr[B(p, n) \geq k] \geq \Phi \left( \frac{k - np}{\sigma \sqrt{n}} \right).
\]

**Remark 6.2.** Many presentations omit sufficient condition (a). Many also omit the integrality of \( k \), which can be seen as necessary with the example \( n = 1 \) and \( p = k \in (0, 0.5) \); in this case, Slud’s lower bound is exactly \( 0.5 > p = \Pr[B(p, 1) \geq k] \).

It is possible to start with the bounds in Theorem 5.1 and apply a battery of elementary inequalities (mostly tangent and secant approximations to the relevant functions) to weaken the inequalities into Slud’s bound. The proof is quite tedious, and thus deferred to Appendix B however a few points are worthy of mention.

By the method of proof, it is immediate that the bounds of Theorem 5.1 are tighter than Slud’s inequality. In one case, care is even made to maintain a small separation, however quantifying the exact gap is hard.
Perhaps most importantly, the proof was able to extend the sufficient conditions for Slud’s inequality. For \( n \) odd and \( m \) even, Slud’s bound (for central tails) requires \( p \leq \frac{1}{2} - \frac{1}{2n} \) and \( p \leq \frac{1}{2} \), respectively. The new proof, however, holds for

\[
p \leq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{4(\sqrt{n(n+1)} - 1)}{4n + 2} \right)^{1/2}, \quad p \leq \frac{1}{2} + \frac{1}{6}(e^{\ln(\frac{m}{m/2})})^{1/3}.
\]

This is nice since Slud’s bound is of a significantly simpler (and more interpretable) form than the bounds of Theorem 5.1.

Empirical evidence seems to suggest that Slud’s bound does not hold for all \( p \), and in fact, as \( n \to \infty \), the maximal permissible \( p \) shrinks to \( 1/2 \). Also, the following appears to be true.

**Conjecture 6.3.** When \( p \in (0, 1/2) \) and \( m \) is even,

\[
P[B(p, m) \geq m/2] \geq 1 - \Phi \left( \frac{1/2 - p}{\sqrt{m}} \right) + \frac{1}{2} \left( \frac{m}{m/2} \right)^{1/2} \sigma^m.
\]

Unfortunately, the bounds of Theorem 5.1 are not sufficiently tight to establish this; perhaps if the term \( e^{\ln(n+1)} \) were handled better.

### 7. Summary of Bounds

Figure 2 contains plots of all bounds. The bounds of (5.1) are almost exact, which is in agreement with their error bound. The simple upper bound in (3.7) is
also generally good. On the other hand, most of the other bounds vary performance quite widely with \( p \). This suggests use of pairs of bounds in tandem; for instance, as was mentioned in section 4, the minimum of the upper bounds in (3.7) and (3.4) suffices to prove Chernoff’s bound. Similarly, the max of the lower bounds in (3.5) and (3.8) would work well.

Slud’s bound fares well, being the best lower bound with the exception of the bounds in (5.1) and (3.5) for certain values of \( p \). In contrast, many upper bounds outperform the Chernoff bound.

8. General Binomial Tails via Random Walks

By following steps analogous to those of section 2, Theorem 1.1 readily generalizes. (For details, see Appendix A.)

**Theorem 8.1.** When \( n \) is odd, \( k \in \mathbb{Z} \cap [0, n/2) \), and \( p \in (0, 1/2) \),

\[
\Pr[B(p, n) \geq n/2 + k] = p^{2k+1} - \left( \frac{p}{1-p} \right)^k \sum_{j=k+1}^{(n-1)/2} \binom{k}{2j} \binom{2j}{j+k} \sigma^{2j}
\]

(8.1)

\[
= \left( \frac{p}{1-p} \right)^k \sum_{j \geq (n+1)/2} \binom{k}{2j} \binom{2j}{j+k} \sigma^{2j}.
\]

(8.2)

Furthermore

\[
\Pr[B(1/2, n) \geq n/2 + k] = 2^{-2k-1} + \sum_{j=k+1}^{(n-1)/2} \binom{k}{2j} \binom{2j}{j+k} 4^{-j} - \frac{1}{2} \sum_{j \geq (n+1)/2} \binom{k}{2j} \binom{2j}{j+k} 4^{-j}.
\]

(8.3)

(8.4)

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Appendix A. Proof of Theorem 8.1

Lemma A.1. When \( n \) is odd and \( k \in \mathbb{Z} \cap [-n/2, n/2] \),

\[
P \left[ B(p, n + 2) \geq \frac{n + 2}{2} + k \right] - P \left[ B(p, n) \geq \frac{n}{2} + k \right] = \left( p - \frac{1}{2} + \frac{k}{n + 1} \right) \left( \frac{n + 1}{(n + 1)/2 + k} \right) p^{(n+1)/2+k}(1 - p)^{(n+1)/2-k}.
\]

Proof. As in the proof of Lemma 2.1, the change in probability mass comes entirely from the random walks which after \( n \) steps are at coordinates \((n - 1)/2 + k\) or \((n + 1)/2 + k\). Thus, as before, the mass gained minus the mass lost is

\[
p^2 \left( \frac{n}{(n - 1)/2 + k} \right) p^{(n-1)/2+k}(1 - p)^{(n+1)/2-k} - (1 - p)^2 \left( \frac{n}{(n + 1)/2 + k} \right) p^{(n+1)/2+k}(1 - p)^{(n-1)/2-k} = p^{(n+1)/2+k}(1 - p)^{(n+1)/2-k} \left[ p \left( \frac{n}{(n - 1)/2 + k} \right) - (1 - p) \left( \frac{n}{(n + 1)/2 + k} \right) \right] = \left( \frac{n + 1}{(n + 1)/2 + k} \right) p^{(n+1)/2+k}(1 - p)^{(n+1)/2-k} \left[ p - \frac{1}{2} + \frac{k}{n + 1} \right].
\]

Substituting \( k = 0 \) in the above yields both the statement (and proof) of Lemma 2.1.

Proof of Theorem 8.1. Since the case \( k = 0 \) is Theorem 1.1, take \( k \neq 0 \). To show (8.1), it suffices to rewrite the tail as a telescoping series, and invoke Lemma A.1.

Note that this derivation holds for any \( p \in (0, 1) \).

\[
P[B(p, n) \geq n/2 + k] = P[B(p, 2k + 1) \geq (2k + 1)/2 + k] + \sum_{j=1}^{(n-1)/2-k} \left[ P[B(p, 2k + 1 + 2j) \geq (2k + 1)/2 + k + j] - P[B(p, 2k - 1 + 2j) \geq (2k - 1)/2 + k + j] \right] = p^{2k+1} + \sum_{j=1}^{(n-1)/2-k} \left( p - \frac{1}{2} + \frac{k}{2j + 2k} \right) \left( \frac{2j + 2k}{j + 2k} \right) p^{2k+j}(1 - p)^{j} = p^{2k+1} + \left( \frac{p}{1 - p} \right) k \sum_{j=k+1}^{(n-1)/2} \left( p - \frac{1}{2} + \frac{k}{2j} \right) \left( \frac{2j}{j + k} \right) \sigma^{2j}.
\]
From here, [8.3] may be obtained by substituting \( p = 1/2 \). Note that, with the exception of the last step, \( p \) can happily take on values in \( \{0, 1\} \), and the expression in the penultimate yields probabilities of 0, 1, respectively.

Next, using (2.5.16) from Wilf [5],

\[
\left( \frac{p}{1-p} \right)^k \sum_{j \geq k} \left( \frac{k}{2j} \right) \left( \frac{2j}{j+k} \right) \sigma^{2j} = \frac{p^{2k}}{2} \sum_{j \geq 0} 2^{j + 2k} \left( \frac{2j + 2k}{j} \right) \sigma^{2j} \\
= \frac{p^{2k}}{2} \left( \frac{1 - \sqrt{1 - 4\sigma^2}}{2\sigma^2} \right)^{2k} \\
= \frac{p^{2k}}{2} \left( \frac{1 - |2p - 1|}{2\sigma^2} \right)^{2k}.
\]

(A.1)

When \( p = 1/2 \), this expression is 1/2; thus, starting from [8.3]

\[
2^{-2k-1} + \sum_{j=k+1}^{(n-1)/2} \left( \frac{k}{2j} \right) \left( \frac{2j}{j+k} \right) 4^{-j} \\
= 2^{-2k-1} + \sum_{j \geq k} \left( \frac{k}{2j} \right) \left( \frac{2j}{j+k} \right) 4^{-j} \\
- \left( \frac{k}{2k} \right) \left( \frac{2k}{k+k} \right) 4^{-k} - \sum_{j \geq (n+1)/2} \left( \frac{k}{2j} \right) \left( \frac{2j}{j+k} \right) 4^{-j} \\
= \frac{1}{2} - \sum_{j \geq (n+1)/2} \left( \frac{k}{2j} \right) \left( \frac{2j}{j+k} \right) 4^{-j},
\]

which is exactly (8.4). Finally, to handle (8.2), first use (2.5.15) from Wilf [5] to obtain

\[
\left( \frac{p}{1-p} \right)^k \sum_{j \geq k} \left( p - \frac{1}{2} \right) \left( \frac{2j}{j+k} \right) \sigma^{2j} = p^{2k} \left( p - \frac{1}{2} \right) \sum_{j \geq 0} \left( \frac{2j + 2k}{j} \right) \sigma^{2j} \\
= p^{2k} \left( p - \frac{1}{2} \right) \frac{1}{\sqrt{1 - 4\sigma^2}} \left( \frac{1 - \sqrt{1 - 4\sigma^2}}{2\sigma^2} \right)^{2k} \\
= \left( \frac{p^{2k}}{2} \right) \frac{2p - 1}{|2p - 1|} \left( \frac{1 - |2p - 1|}{2\sigma^2} \right)^{2k}.
\]

(A.2)

Combining [A.1] and [A.2] it follows that

\[
\left( \frac{p}{1-p} \right)^k \sum_{j \geq k} \left( p - \frac{1}{2} + \frac{k}{2j} \right) \left( \frac{2j}{j+k} \right) \sigma^{2j} = \frac{p^{2k}}{2} \left( \frac{1 - |2p - 1|}{2\sigma^2} \right)^{2k} \left( \frac{2p - 1}{|2p - 1|} + 1 \right),
\]
which further reduces to just 0 when \( p < 1/2 \). Thus, from (8.1),
\[
\mathbb{P}[B(p, n) \geq n/2 + k] = p^{2k+1} - \left( \frac{p}{1-p} \right)^k \sum_{j=k+1}^{(n-1)/2} \left( \frac{1}{2} - p - \frac{k}{2j} \right) \left( \frac{2j}{j+k} \sigma^2j \right) \\
= p^{2k+1} - \left( \frac{p}{1-p} \right)^k \sum_{j=k}^{(n-1)/2} \left( \frac{1}{2} - p - \frac{k}{2j} \right) \left( \frac{2j}{j+k} \sigma^2j \right) \\
+ \left( \frac{p}{1-p} \right)^k \sum_{j=(n+1)/2}^{k} \left( \frac{1}{2} - p - \frac{k}{2j} \right) \left( \frac{2j}{j+k} \sigma^2j \right) \\
= p^{2k+1} - 0 - p^{2k+1} \\
+ \left( \frac{p}{1-p} \right)^k \sum_{j=(n+1)/2}^{k} \left( \frac{1}{2} - p - \frac{k}{2j} \right) \left( \frac{2j}{j+k} \sigma^2j \right),
\]
which is (8.2).

\[ \square \]

**APPENDIX B. PROOF OF (CENTRAL) SLUD’S BOUND**

**Theorem B.1.** When \( n \) is odd and \( 0 \leq p \leq \frac{1}{2} + \frac{1}{2 \pi} \left( 1 - \frac{4\sqrt{n(n+1)-1}}{4n+2} \right)^{1/2} \),
\[
\mathbb{P}[B(p, n) \geq n/2] \geq 1 - \Phi \left( \frac{(n+1)/2 - np}{\sigma \sqrt{n}} \right). \tag{B.1}
\]

When \( m \) is even and \( 0 \leq p \leq \frac{1}{2} + \frac{1}{6} (e^{(m)/m})^{1/3} \),
\[
\mathbb{P}[B(p, m/2) \geq m/2] \geq 1 - \Phi \left( \frac{1/2 - p}{\sqrt{m}} \right). \tag{B.2}
\]

When \( p \in \{0, 1/2\} \), the statements are immediate, and thus disregarded. The proof is split into four parts, each reducing to the bounds in Theorem 5.1.

**Proof of (B.1) when \( 0 < p < 1/2 \).** Set \( \alpha = \sqrt{-(n+1) \ln(4\sigma^2)} \) and \( \beta = ((n+1)/2 - np)/(\sigma \sqrt{n}) \); note that \( \alpha < \beta \). Using the lower bound in Theorem 5.1, the statement is implied by
\[
\frac{(1-2p)e^{l(n+1)}(1 - \Phi(\alpha))}{\sqrt{-\ln(4\sigma^2)}} + \frac{(1-2p)e^{l(n+1)}(2\sigma)^{n+1}}{\sqrt{2\pi(n+1)}} \geq 1 - \Phi(\beta).
\]

Dropping the second term and using \( \sqrt{-\ln(4\sigma^2)} \leq \sqrt{(4\sigma^2)^{-1}} - 1 = 2\sigma/(1 - 2p) \), this is a consequence of
\[
2\sigma e^{l(n+1)}(1 - \Phi(\alpha)) \geq 1 - \Phi(\beta).
\]

\( 0 \leq \alpha \leq \beta \) so \( \Phi(\beta) - \Phi(\alpha) \geq \phi(\beta)(\beta - \alpha) \), and \( 1 - \Phi(\beta) < \phi(\beta)/\beta \) (cf. (1.8) from chapter 7 of Feller [2]), so this in turn is implied by either of
\[
2\sigma e^{l(n+1)}\phi(\beta)(\beta - \alpha) \geq (1 - 2\sigma e^{l(n+1)})\frac{\phi'(\beta)}{\beta} \iff \quad 2\sigma e^{l(n+1)}(\beta(\beta - \alpha) + 1) \geq 1.
\]
Establishing the latter inequality is \[ \text{Lemma C.1} \]

Proof of (B.1) when \( \frac{1}{2} < p < \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{4\sqrt{n(n+1)-1}}{4n+2} \right)^{1/2} \). Set \( \beta = (np - (n + 1)/2)/(\sigma \sqrt{n}) \) and \( \alpha = \sqrt{-(n + 1)\ln(4\sigma^2)} \). By Remark 2.4 and symmetry of \( \phi \), the theorem statement may be re-written as

\[ P[B(1 - p, n) \geq n/2] \leq 1 - \Phi(\beta). \]

Discard the case that \( \beta < 0 \), since then \( 1 - \Phi(\beta) > 1/2 \) whereas \( P[B(1 - p, n) \geq n/2] \leq 1/2 \) by Lemma 2.2. Since \( 2p - 1 \leq \sqrt{-\ln(4\sigma^2)} \) this in turn is a consequence of

\[ (B.3) \quad 1 - \Phi(\alpha) + \frac{(1 - 2p)(2\sigma)^{n+1}}{2\pi(n + 1)} \leq 1 - \Phi(\beta). \]

As per Lemma C.2, the conditions on \( p \) imply

\[ \frac{2p - 1}{\sqrt{n + 1}} \leq \alpha - \beta, \]

meaning \( \alpha \geq \beta \). Thus scaling both sides by \( \phi(\alpha) \) yields

\[ \frac{(2p - 1)(2\sigma)^{n+1}}{2\pi(n + 1)} \leq (\alpha - \beta)\phi(\alpha), \]

which implies (B.3) (since an integral can be lower bounded by a rectangle). \( \square \)

Proof of (B.2) when \( 0 < p < 1/2 \). Set \( \alpha = \sqrt{-m \ln(4\sigma^2)} \) and \( \beta = ((1/2 - p)\sqrt{m})/\sigma \); again, \( \alpha \leq \beta \). Invoking Remark 2.4 (3.1) and Theorem 5.1 the theorem statement is implied by

\[ e^{l(m)} \left( \frac{(1 - 2p)(1 - \Phi(\alpha))}{\sqrt{-\ln(4\sigma^2)}} + \frac{(3/2 - p)(2\sigma)^{m}}{\sqrt{2\pi m}} \right) \geq 1 - \Phi(\beta). \]

As in the proof when \( n \) is odd, use \( \sqrt{-\ln(4\sigma^2)} \leq 2\sigma/(1 - 2p) \), but simply drop the term \( \Phi(\beta) - \Phi(\alpha) \), which gives the antecedent statement

\[ (B.4) \quad \frac{e^{l(m)}(3/2 - p)(2\sigma)^{m}}{\sqrt{2\pi m}} \geq (1 - 2e^{l(m)}\sigma)(1 - \Phi(\beta)), \]

which will be established using the fact \( 1 - \Phi(\beta) \leq \min\{\phi(\beta)/\beta, 1/2\} \). Set \( \bar{p} = \sqrt{3/4 - 1/(2e^{l(m)})} \); the statement is established for \( p \in [\bar{p}, 1/2] \) in Lemma C.3 so take \( p \in (0, \bar{p}] \). Rearranging, the condition on \( p \) states

\[ e^{l(m)}\left( \frac{3}{4} - p^2 \right) \geq \frac{1}{2}, \]

and \( \sigma \leq 1/2 \), so this becomes

\[ e^{l(m)}\left( \frac{3}{4} - p^2 \right) \geq 1. \]

Next, \( 3/4 - p^2 = (3/2 - p)(1/2 - p) + 2\sigma^2 \), so the above can be re-arranged into

\[ e^{l(m)}\frac{(1.5 - p)(0.5 - p)}{\sigma} \geq 1 - 2e^{l(m)}\sigma. \]
To finish, the definition of \( \beta \) and scaling by \( \phi(\beta) \) implies

\[
\frac{e^{l(m)(1.5-p)\phi(\beta)}}{\sqrt{m}} \geq (1 - 2e^{l(m)\sigma})\frac{\phi(\beta)}{\beta},
\]

where substituting \( \phi(\beta) \leq \phi(\alpha) = \frac{(2\sigma)^2}{\sqrt{2m}} \) yields (B.4).

\[\square\]

**Proof of (B.2) when** \( \frac{1}{2} < p < 1/2 + (e^{l(m)/m})^{1/3}/6 \). Set \( \alpha = \sqrt{-m\ln(4\sigma^2)} \) and \( \beta = ((p - 1/2)/\sqrt{m})/\sigma \); again using Remark 2.4 and (3.1), the theorem statement may be rewritten

\[
P[B(1 - p, m - 1) \geq m/2] - \frac{e^{l(m)(2\sigma)m/2}}{\sqrt{2\pi m}} \leq 1 - \Phi(\beta).
\]

Invoking Theorem 5.1, \( 2p - 1 \leq \sqrt{-\ln(4\sigma^2)} \), and \( \phi(\alpha) = (2\sigma)^m/\sqrt{2\pi} \), this is implied by

\[
\frac{(2p - 1 - e^{l(m)})\phi(\alpha)}{\sqrt{m}} \leq \Phi(\alpha) - \Phi(\beta),
\]

which by \( \Phi(\alpha) - \Phi(\beta) \geq -(\beta - \alpha)\phi(\alpha) \) is a consequence of

(B.5)

\[
\frac{(2p - 1 - e^{l(m)})}{\sqrt{m}} \leq \alpha - \beta.
\]

Set \( q = 2p - 1 \); the conditions on \( p \) mean \( q \leq (e^{l(m)/m})^{1/3}/3 \). Using algebra and some bounds on \( l(m) \), it follows that

\[
(m - 1)q^3 + e^{l(m)}q^2 + q - e^{l(m)} \leq 0,
\]

which can be rearranged into

\[
e^{l(m)} - q \geq m \left( \frac{q}{1 - q^2} - q \right).
\]

As before, \( q = 2p - 1 \leq \sqrt{-\ln(4\sigma^2)} \), and since \( 1 - q^2 = 2p(2 - p) \leq 2\sigma \), this implies

\[
e^{l(m)} - (2p - 1) \geq m \left( \frac{2p - 1}{2\sigma} - \sqrt{-\ln(4\sigma^2)} \right),
\]

which gives (B.5). \[\square\]

**Appendix C. Supporting Lemmas**

**Lemma C.1.** When \( p \in (0, 1/2) \) and \( n \) odd,

\[
2\sigma e^{(n+1)} \left( \left( \frac{(n+1)/2 - np}{\sigma \sqrt{n}} \right) \left( \frac{(n+1)/2 - np}{\sigma \sqrt{n}} - \sqrt{-((n+1)\ln(4\sigma^2))} + 1 \right) \right) \geq 1.
\]
Proof. Set $r = p(1 - p) = \sigma^2$, and lower bound the left hand side according to

$$2\sqrt{r}e^{(n+1)} \left[ \frac{n\sqrt{1-4r}+1}{2\sqrt{n}r} \left( \frac{n\sqrt{1-4r}+1}{2\sqrt{n}r} - \sqrt{-\ln(4r)} \right) + 1 \right] \geq 2\sqrt{r}e^{(n+1)} \left[ \frac{n\sqrt{1-4r}+1}{2\sqrt{n}r} \left( \frac{n\sqrt{1-4r}+1}{2\sqrt{n}r} - \frac{1}{\sqrt{4r}} \right) + 1 \right]$$

$$= 2\sqrt{r}e^{(n+1)} \left[ \frac{n^2(1-4r)+2n\sqrt{1-4r}+1}{4nr} \right] - \frac{n(1-4r)\sqrt{n+1} + \sqrt{(n+1)(1-4r)}}{4r\sqrt{n}} + 1
\right]$$

\begin{equation}
(C.1) = e^{l(n+1)} \left[ \frac{1-4r}{\sqrt{r}} \left( \frac{n}{2} - \frac{\sqrt{n(n+1)}}{2} \right) + \sqrt{1-4r} \left( \frac{1}{2n} \right) + \sqrt{r}(2) \right].
\end{equation}

The goal will be to show that this final expression is lower bounded by 1. In particular, its derivative with respect to $r$ is

$$r^{-1/2}e^{l(n+1)} \left[ -\frac{1}{2} + \frac{4r}{2r} \left( \frac{n}{2} - \frac{\sqrt{n(n+1)}}{2} \right) - \frac{4r}{2r\sqrt{1-4r}} \left( 1 - \frac{1}{2} \frac{\sqrt{n+1}}{n} \right) - \frac{1}{4nr} + \frac{1}{2} \right];$$

by showing this is negative, it will suffice to show that $\text{(C.1)}$ holds at $p = 1/2$.

First consider the case that $n \in \{1, 2, 3, 4\}$ (and note that $r \in (0, 1/4)$, since $p \in (0, 1/2)$). Each case can be checked manually by plugging in the given $n$, and using $(1-4r)^{-1/2} \geq 1 + 2r$ (the linear approximation at $r = 0$).

So take $n \geq 5$, and it will be shown that

$$* = \frac{1}{2} + \frac{4r}{2r} \left( \frac{n}{2} - \frac{\sqrt{n(n+1)}}{2} \right) - \frac{1}{2r\sqrt{1-4r}} \left( 1 - \frac{1}{2} \frac{\sqrt{n+1}}{n} \right) \leq -1/2.$$  

Since $n \geq 5$, using some calculus, it holds that

$$\frac{n}{2} - \frac{\sqrt{n(n+1)}}{2} \geq \frac{1}{4},$$

$$1 - \frac{1}{2} \frac{\sqrt{n+1}}{n} \geq 0.45.$$  

Therefore

$$* \leq -\frac{1}{2r} \left( -\frac{1}{4} + \frac{0.45}{\sqrt{1-4r}} \right).$$
Consider the case that \( r \in [0, 1/8] \), and replace \( (1 - 4r)^{-1/2} \geq 1 + 2r \) (again, the linear approximation at \( r = 0 \)):

\[
-\frac{1}{2r} \left( -\frac{1 + 4r}{4} + \frac{0.45}{\sqrt{1 - 4r}} \right) \leq -\frac{1}{2r} \left( -\frac{1 + 4r}{4} + (0.45)(1 + 2r) \right) = -\frac{1}{2r} \left( \frac{2}{10} - \frac{r}{10} \right) = \frac{1}{20} - \frac{1}{10r} \leq 120 - \frac{8}{10} < 1/2.
\]

Now take \( r \in [1/8, 1/4] \), and use the linear approximation \( (1 - 4r)^{-1/2} \geq \sqrt{2}(1 + 4r) \):

\[
-\frac{1}{2r} \left( -\frac{1 + 4r}{4} + \frac{0.45}{\sqrt{1 - 4r}} \right) < -1/2.
\]

Since \( \star \) is decreasing, the minimum is obtained when \( r = 1/4 \). Plugging \( r = 1/4 \) into \([C.1]\)

\[
(C.1) \geq e^{l(n+1)}(1 + n^{-1}).
\]

This exceeds 1 for \( n = 1 \); for \( n \geq 3 \), since \( 1 + x \geq \exp(x - x^2) \) when \( x \in [0, 1/2] \), it follows that

\[
e^{l(n+1)}(1 + n^{-1}) \geq \exp(l(n+1) + (n^{-1} - n^{-2})) \geq \exp(-(3n + 1)^{-1} + (n^{-1} - n^{-2})) \geq 1.
\]

**Lemma C.2.** When \( \frac{1}{2} < p < \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{4\sqrt{n(n+1)-1}}{4n+2} \right)^{1/2} \) and \( n \) is odd,

\[
\frac{2p - 1}{\sqrt{n + 1}} \leq \sqrt{\frac{-(n+1)\ln(4\sigma^2) - np - (n+1)/2}{\sigma\sqrt{n}}},
\]

**Proof.** The conditions on \( p \) are equivalent to

\[
0 \geq (4n + 2)p^2 - (4n + 2)p + \sqrt{n(n+1)} - 1 = -4n\sigma^2 - 2\sigma^2 + \sqrt{n(n+1)} - 1.
\]

Note that \( y^{-1/2} \) is lower bounded by its Taylor expansion \( 1 + (1 - y)/2 \), meaning

\[
1 + 2\sigma = 1 + (1 - (1 - 4\sigma^2))/2 \leq (1 - 4\sigma^2)^{-1/2} = (1 - 2p)^{-1}.
\]

Furthermore \( 4\sigma^2 \leq 2\sigma \) (since \( \sqrt{r} \) is nondecreasing on \([0, 1]\)), so the above condition implies

\[
\sqrt{n(n+1)} \leq 2n\sigma + (1 - 2p)^{-1}.
\]

With some rearranging, this becomes

\[
\frac{2p - 1}{\sqrt{n + 1}} \leq \sqrt{\frac{(n+1)(1 - 4\sigma^2)}{\sigma\sqrt{n}}} - \frac{np - (n+1)/2}{\sigma\sqrt{n}},
\]

and since \( -\ln(4\sigma^2) \geq 1 - 4\sigma^2 \), the statement follows. \(\Box\)

**Lemma C.3.** When \( m \) is even and \( p \in [\bar{p}, 1/2] \),

\[
e^{l(m)}(3/2 - p)(2\sigma)^m \sqrt{2\pi m} \geq (1 - 2e^{l(m)}\sigma) \min\left\{ \frac{\phi(\beta)}{\beta}, \frac{1}{2} \right\},
\]

where \( \beta = (1/2 - p)\sqrt{m}/\sigma \) and \( \bar{p} = \sqrt{3/4 - 1/(2e^{l(m)})} \).
Proof. First, notice that the left hand side is decreasing along \( p \in [p, 1/2] \). Starting from the derivative:

\[
\frac{d}{dp} \left( \frac{e^{l(m)}}{\sqrt{2\pi m}} + \frac{3}{2}(p(1-p))^{m/2} - (p(1-p))^{m/2} \right) = \frac{e^{l(m)}}{\sqrt{2\pi m}} \left( \frac{3m}{4}(p(1-p))^{m/2-1}(1-2p) - (p(1-p))^{m/2} - \frac{m}{2}(p(1-p))^{m/2-1}(1-2p) \right)
\]

Setting this to zero and solving the quadratic yields

\[
p = \frac{1 + 2m \pm \sqrt{1 + m + m^2}}{2 + 2m}.
\]

Notice that since \((1 + 2m)/(2 + 2m) \geq (1 + m)/(2 + 2m) = 1/2\), at least one of the solutions exceeds 1/2. Let \( p^* \) denote the solution subtracting the discriminant. Below, it is shown that \( p^* \leq \hat{p} \). But since the derivative evaluated at 1/2 is negative, it must follow that the left hand side is decreasing along \([\hat{p}, 1/2]\).

Since \((1/2 + m)^2 = 1/4 + m + m^2 \leq 1 + m + m^2\), it follows that

\[
p^* \leq \frac{1 + 2m - (1/2 + m)}{2 + 2m} = \frac{1 + 2m}{4 + 4m}.
\]

By inspection, \( p^* < \hat{p} \) for \( m \in \{2, 4, \ldots, 28\} \). Next, since \(-l(m) \leq (3m)^{-1}\),

\[
\hat{p} = \sqrt{3/4 - e^{-l(m)}/2} \geq \sqrt{3/4 - e^{-1/3m}/2}.
\]

Any convex function may be upper bounded by its secant along an interval; thus along \([0, 1/6]\), \( e^x \leq 1 + 6x(e^{1/6} - 1) \). Furthermore, \((3m)^{-1} \in [0, 1/6]\) since \( m \geq 2\), thus

\[
(C.2) \quad \hat{p} \geq \sqrt{\frac{3}{4} - \frac{1}{2} \left( 1 + e^{1/6} - 1 \right) \frac{1}{3m}} = \sqrt{1/4 - e^{1/6} - 1}.
\]

Next, note that

\[
mp^2(4 + 4m)^2 - m(1 + 2m)^2
= 4m(1 + m)^2 - 16(e^{1/6} - 1)(1 + 2m)^2 - m(1 + 2m)^2
= 3m + 4m^2 - 16(e^{1/6} - 1)(1 + 2m)^2 - m(1 + 2m)^2,
\]

which exceeds zero when \( m \geq 30\). Rearranging, this yields \( \hat{p} \geq p^* \).

Now consider the right hand side of the lemma statement, note that it is upper bounded by \( 1/2 - e^{l(m)} \sigma \), which decreases as \( p \to 1/2 \) since \( \sigma \) increases along this interval. Combining all these pieces, to prove the inequality, it suffices to show that the left hand side at 1/2 exceeds the right hand side at \( \hat{p} \).

First upper bound the quantity on the right hand side. Continuing from \((C.2)\) and using the secant approximation \( \sqrt{x} \geq 1/6 + 4x/3 \) along \([1/16, 1/4]\),

\[
\hat{p} \geq \sqrt{1/4 - \frac{e^{1/6} - 1}{m}} \geq \frac{1}{6} + \frac{4}{3} \left( 1/4 - \frac{e^{1/6} - 1}{m} \right) \geq \frac{1}{2} - \frac{1}{4m}.
\]
On the other hand, since $\sqrt{x} \leq x + 1/4$ (first-order Taylor (tangent) at $1/4$),
$e^x \geq 1 + x$, and $(4m)^{-1} \leq -l(m)$,
\[
\sqrt{3/4 - e^{-l(m)/2}} \leq \sqrt{3/4 - e^{-1/4m}/2} \leq \sqrt{3/4 - (1 - 1/4m)/2} = \sqrt{1/4 - 1/8m} \leq 1/2 - 1/8m.
\]
Combining these,
\[
2\sigma = \sqrt{4p(1 - p)} \geq \sqrt{1 - 1/4m - 1/8m^2} \geq 1 - 1/4m - 1/8m^2 \geq 1 - 1/3m
\]
the last two steps following since $\sqrt{x} \geq x$ on $[0, 1]$, and $-1/8m^2 \geq -1/12m$. Inserting this into the right hand side (with $1/2$ in the min) and using $e^{l(m)} \geq 1 - 1/3m$,
\[
\frac{1}{2}(1 - 2e^{l(m)}\sigma) \leq \frac{1}{2}(1 - (1 - 1/3m)^2) \leq 1/3m.
\]
On the other hand, the left hand side (at $p = 1/2$) may be lower bounded as
\[
\frac{e^{l(m)}}{\sqrt{2\pi m}} \geq \frac{1 - 1/3m}{\sqrt{2\pi m}} \geq \frac{5/6}{\sqrt{2\pi m}}.
\]
Comparing the square of this lower bound on the left hand side, and the square on
the upper bound of the right hand side, it is clear the left hand side is greater, thus
completing the proof. \hfill \Box

**Appendix D. Comparison of Existing Central Tail Upper Bounds**

Applying Hoeffding, Bernstein, and Chernoff bounds to $P[B(p, n) \geq m/2]$ with even $m$ and $p < 1/2$ yields
\[
P[B(p, m) \geq m/2] \leq \exp[-m(1 - 2p)^2/2] \quad \text{Hoeffding},
\]
\[
P[B(p, m) \geq m/2] \leq \exp \left[ -\frac{3m(1/2 - p)^2}{(1 + 4p)(1 - p)} \right] \quad \text{Bernstein},
\]
\[
P[B(p, m) \geq m/2] \leq [4p(1 - p)]^{m/2} = \exp(\frac{m}{2} \ln(4p(1 - p))) \quad \text{Chernoff}.
\]

**Theorem D.1.** The following inequalities relate the performance of these bounds:

\[
[4p(1 - p)]^{m/2} \leq \exp \left[ -\frac{3m(1/2 - p)^2}{(1 + 4p)(1 - p)} \right] \leq \exp[-m(1 - 2p)^2/2] \quad \text{if } p \in [0, 1/4];
\]
\[
[4p(1 - p)]^{m/2} \leq \exp[-m(1 - 2p)^2/2] \leq \exp \left[ -\frac{3m(1/2 - p)^2}{(1 + 4p)(1 - p)} \right] \quad \text{if } p \in [1/4, 1/2].
\]

**Proof.** Since $\exp$ is monotonic increasing, bounds will be compared by looking solely at the exponents.

First, note that the Chernoff Bound is always better than the Hoeffding bound:
\[
\frac{m}{2} \ln(4p(1 - p)) \leq \frac{m}{2} (4p(1 - p) - 1) = \frac{m}{2} (1 - 2p)^2.
\]
Next, the ratio of Hoeffding’s bound to Bernstein’s bound is
\[
\frac{2}{3} (1 + 4p)(1 - p).
\]
By setting this quadratic to 1 and solving, the ratio is equal when $p \in \{1/4, 1/2\}$. Furthermore, it is concave, and attains a maximum at $p = 3/8$. Combining these facts, it must be the case that the ratio is less than or equal to 1 along $p \in [0, 1/4]$, and greater than or equal to 1 along $p \in [1/4, 1/2]$. 

To finish, it remains to be shown that the Chernoff bound is less than the Bernstein bound along $p \in [0, 1/4]$. Let $f, g$ denote the Chernoff and Bernstein bounds, respectively. $df/dp \geq 0$ and $dg/dp \geq 0$ along this interval, meaning both are increasing. Thus, the result follows by the fact that $f(1/8) < g(0)$ and $f(1/4) < g(1/8)$. □

**Appendix E. Riemann Sums of Convex, Decreasing Functions**

**Lemma E.1.** Let intervals $(a, b] \subseteq (c, d]$, integers $n < m$ with $(d-c)/m \leq (b-a)/n$, and a function $f$ convex and decreasing on $(c, d]$ be given. Let $R(f; (t, u], v)$ denote the Riemann sum approximation of $\int_t^u f$ consisting of $v$ equal pieces, whose height is $f$ evaluated at their right endpoint. Then $R(f; (a, b], n) \leq R(f; (c, d], m)$.

**Proof.** Let $I$ denote some block of $R(f; (a, b], n)$; that is, there is some integer $1 \leq j \leq n$ such that $I = [a + j(b-a)/m, a + (j+1)(b-a)/m]$. The desired fact will be shown for the restriction to $I$, from which the general statement follows by considering all $n$ pieces of $[a, b]$.

The ingredients of the proof appear in figure 3. In particular, consider the greatest point within $I$ where the right endpoint of a block of $R(f; (c, d], m)$ falls (such a point must exist since $R(f; (c, d], m)$ has narrower blocks). Let $w = f(a + (j+1)(b-a)/m)$ be the height of the block, and let $x$ be the offset at which this point falls. Furthermore, let $y$ be the height above $w$ of the block.

Now consider the immediately next block (in $R(f; (c, d], m)$); its boundary must fall outside $I$. Let $z$ denote this distance beyond the edge of $I$, and let $u$ be the height below $w$ of this point. By assumption, it must be the case that $z \leq x$. Furthermore, since $f$ is convex on $(c, d]$, it must be the case that $y \frac{1}{1/n-x} \geq \frac{w}{w}$. Thus, to compute the area with respect to $R(f; (c, d], m)$, one has the lower bound

$$x(w+y) + \left(\frac{1}{n} - x\right)(w-u) \geq \frac{w}{n} + xy - \left(\frac{1}{n} - x\right) \frac{yz}{1/n-x} \geq \frac{w}{n}.$$
Note that this last quantity is precisely the area of the block with respect to $R(f; [a, b], n)$, which completes the proof.

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