Consider a random walk $S_n = \sum_{i=1}^{n} X_i$ with independent and identically distributed real-valued increments $X_i$ of zero mean and finite variance. Assume that $X_i$ is non-lattice and has a moment of order $2 + \delta$. For any $x \geq 0$, let $\tau_x = \inf \{ k \geq 1 : x + S_k < 0 \}$ be the first time when the random walk $x + S_n$ leaves the half-line $[0, \infty)$. We study the asymptotic behavior of the probability $P(\tau_x > n)$ and that of the expectation $E(f(x + S_n), \tau_x > n)$ for a large class of target function $f$ and various values of $x, y$ possibly depending on $n$. This general setting implies limit theorems for the joint distribution $P(x + S_n \in y + [0, \Delta], \tau_x > n)$ where $\Delta > 0$ may also depend on $n$. In particular, the case of moderate deviations $y = \sigma \sqrt{qn \log n}$ is considered. We also deduce some new asymptotics for random walks with drift and give explicit constants in the asymptotic of the probability $P(\tau_x = n)$. For the proofs we establish new conditioned integral limit theorems with precise error terms.
3.2. Duality identities 21
3.3. Auxiliary statements 22
3.4. Proof of the upper bound 24
3.5. Proof of the lower bound 26
3.6. Proof of Theorems 1.1 and 1.2 32
4. Conditioned local limit theorem near the boundary 35
4.1. A non-asymptotic conditioned local limit theorem 35
4.2. Proof of Theorems 1.3 and 1.4 41
5. Conditioned concentration bounds far from the boundary 43
5.1. Formulation of the result 43
5.2. Proof of the upper bound 43
5.3. Proof of the lower bound 46
5.4. Proof of Theorems 1.5 and 1.6 50
6. Conditioned local limit theorem far from the boundary 53
6.1. A non-asymptotic conditioned local limit theorem 54
6.2. Proof of Theorems 1.7 and 1.8 58
6.3. Proof of Theorems 1.9 and 1.10 59
6.4. Proof of Theorems 1.11 and 1.12 60
7. Appendix: Proof of conditioned integral limit theorems 61
7.1. Auxiliary results 61
7.2. Proof of Theorems 2.7 and 2.8 68
References 79

1. Main results

1.1. Introduction and assumptions. Assume that on the probability
space \((\Omega, \mathcal{F}, \mathbb{P})\) we are given a sequence of independent identically dis-
tributed real-valued random variables \((X_i)_{i \geq 1}\) with \(\mathbb{E}X_1 = 0\) and \(\mathbb{E}X_1^2 = \sigma^2 \in (0, \infty)\). Let \(S_n = \sum_{i=1}^{n} X_i, n \geq 1\). For any starting point \(x \in \mathbb{R}_+ := [0, \infty)\), consider the first moment \(\tau_x\) when the random walk \((x+S_n)_{n \geq 1}\) goes
below the constant boundary 0, which is defined as

\[
\tau_x = \inf \{k \geq 1 : x + S_k < 0\}.
\]

For \(\Delta > 0, x \geq 0\) and \(y \geq 0\) possibly depending on \(n\), consider the proba-
bility

\[
\mathbb{P}(x + S_n \in y + [0, \Delta], \tau_x > n).
\]

The asymptotic behavior of the probability (1.1) has been studied by many
authors, we refer to Lévy [34], Borovkov [8, 10, 11], Feller [21], Spitzer [41],
Bolthausen [7], Iglehart [30], Eppel [20], Bertoin and Doney [6], Caravenna
[13], Vatutin and Wachtel [44], Denisov and Wachtel [15], Kersting and
Vatutin [33], Denisov, Sakhanenko and Wachtel [16] and to the references
therein. In particular, Eppel [20] and later Vatutin and Wachtel [44] have found its asymptotic in the case when $\frac{\tau_n}{\sqrt{n}} \to 0$ and $\frac{y_n}{\sqrt{n}} \to 0$. However, there are very few results dealing with the case when these conditions are not satisfied. We refer to Doney [18] for the case when $\frac{\tau_n}{\sqrt{n}} \sim c_1$ and $\frac{y_n}{\sqrt{n}} \sim c_2$ for some constants $c_1, c_2 > 0$. Precise large deviations for the special class of heavy tailed distributions have been studied in Doney and Jones [19]. Note also that the results in [18] and [44] are stated for the case of random variables in the domain of attraction of stable laws, which will not be considered here.

In this paper, we shall study the asymptotic of the probability (1.1) when $x, y \geq 0$ are moving to $\infty$ under some moment assumptions on the increment $X_1$. Before stating our main results we give the necessary definitions and introduce some notation.

We first recall the definition of non-latticity: a random variable $X$ is said to be non-lattice if for any $h > 0$ and $a \in [0, h)$ it holds that $P(X \in hZ + a) \neq 1$, where $Z$ is the set of integers. We shall make use of the following conditions:

A1. The law of the random variable $X_1$ is non-lattice.

A2. $EX_1 = 0$ and there exists $\delta > 0$ such that $E(|X_1|^{2+\delta}) < \infty$.

The limit behavior of the probability (1.1) depends on two harmonic functions related to the random walk $(x + S_n)_{n \geq 1}$, which we proceed to introduce. It is well known (see [6]) that under condition A2, the function

$$x \in \mathbb{R}_+ \mapsto V(x) = -ES_x$$

is well-defined and strictly positive. Using the results of Tanaka [43, Lemma 1] and the renewal arguments, one can verify that for any $x \geq 0$,

$$E V(x + S_1)1_{\{x+S_1 \geq 0\}} = V(x). \quad (1.2)$$

The function $V$ will be called harmonic function of the random walk $(x + S_n)_{n \geq 1}$ killed at $\tau_1$. The harmonic function $V$ is uniquely defined up to a constant factor. If $U$ is the renewal function in the strict increasing ladder process of $(S_n)_{n \geq 1}$ ([33]), then, for any $x \geq 0$ it holds $U(x) = V(x)/V(0)$. Due to identity (1.2), the function $V$ can be uniquely extended to the whole real line $\mathbb{R}$ by setting $V(x) = EV(x + S_1)1_{\{x+S_1 \geq 0\}}$ for $x < 0$, so that the harmonicity property (1.2) is preserved. Moreover, the support of $V$ is given by ([26, Example 2.10]) $\mathcal{D} = \{x \in \mathbb{R} : P(x + X_1 > 0) > 0\}$.

Let $(S^*_n)_{n \geq 1}$ be the dual random walk: $S^*_n = -\sum_{i=1}^n X_i$. Denote by $\tau^*_x$ the dual exit time: $\tau^*_x = \inf \{k \geq 1 : x + S^*_k < 0\}$. For $x \geq 0$, let $V^*(x) := -EV(S^*_x)$ be the harmonic function of the random walk $(x + S^*_n)_{n \geq 1}$ killed at $\tau^*_x$. The function $V^*$ can be extended to the whole real line in the same way as the function $V$. 
Denote by \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \) \( x \in \mathbb{R} \) the standard normal distribution function. Let \( \phi^+ \) be the Rayleigh density function, i.e.

\[
\phi^+(s) = se^{-s^2/2}1_{\{s \geq 0\}}, \quad s \in \mathbb{R}.
\]

When the starting point \( x \) is far from the boundary we shall make use of the following function:

\[
\psi(s, x) = \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{(s-x)^2}{2}} - e^{-\frac{(s+x)^2}{2}} \right), \quad s, x \in \mathbb{R}.
\]

Note that \( \psi(s, x) > 0 \) for any \( s, x > 0 \), and that \( \psi(s, 0) = 0 \) for any \( s \in \mathbb{R} \). It is useful to note that for any fixed \( x > 0 \), the function \( s \in \mathbb{R}_+ \mapsto \psi(s, x) \) is not a density on \( \mathbb{R}_+ \), but with the normalization factor \( 1/(2\Phi(x) - 1) \) it becomes one. By straightforward calculations, one can verify that uniformly in \( s \) over compact sets of \( \mathbb{R}_+ \),

\[
\lim_{x \to 0} \frac{\psi(s, x)}{2\Phi(x) - 1} = \phi^+(s).
\]

In the sequel, the notation \( f_\alpha(n) \sim g_\alpha(n) \), uniformly in \( \alpha \in A \) as \( n \to \infty \), means that \( \lim_{n \to \infty} \sup_{\alpha \in A} \frac{f_\alpha(n)}{g_\alpha(n)} = 1 \). For short, we will write \( \mathbb{E}(X; B) \) for the expectation \( \mathbb{E}(X \mathbf{1}_B) \).

### 1.2. Starting point near the boundary.

In this section we formulate our results when the starting point \( x \) is near the boundary 0. Below we fix a target function \( f(\cdot) \) on the sum \( x + S_n - y \) where \( y \) is a drift variable. The following result is effective when the drift \( y \) moves to infinity. It will be obtained as a consequence of the more general Theorem 3.1.

**Theorem 1.1.** Assume \( A1 \) and \( A2 \). Let \( f : \mathbb{R} \mapsto \mathbb{R} \) be a directly Riemann integrable function with support in \( \mathbb{R}_+ \) such that \( \int_{\mathbb{R}_+} f(t)(1 + t) dt < \infty \). Then, for any \( \eta \in (0, 1] \) and any sequence of positive numbers \( (\alpha_n)_{n \geq 1} \) satisfying \( \lim_{n \to \infty} \alpha_n = 0 \), we have, as \( n \to \infty \), uniformly in \( x \in [0, \alpha_n \sqrt{n}] \) and \( y \in [\eta \sqrt{n}, \eta^{-1} \sqrt{n}] \),

\[
\mathbb{E}(f(x + S_n - y); \tau_x > n) \sim \frac{2V(x)}{\sqrt{2\pi \sigma^2 n}} \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) \int_{\mathbb{R}_+} f(t) dt.
\]

Moreover, there exist constants \( \varepsilon_0, \delta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), \( q \in (0, \delta_0) \) and \( \eta > 0 \), as \( n \to \infty \), the asymptotic (1.5) holds uniformly in \( x \in [0, n^{1/2-\varepsilon}] \) and \( y \in [\eta \sqrt{n}, \sigma q n \log n] \).

In the particular case when \( f = \mathbf{1}_{[0, \Delta]} \) with \( \Delta > 0 \), one can improve Theorem 1.1 by showing the uniformity in \( \Delta \) in a certain range.

**Theorem 1.2.** Assume \( A1 \) and \( A2 \). Then, for any \( \eta \in (0, 1], \Delta_0 > 0 \) and any sequence of positive numbers \( (\alpha_n)_{n \geq 1} \) satisfying \( \lim_{n \to \infty} \alpha_n = 0 \),
we have, as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$, $y \in [\eta \sqrt{n}, \eta^{-1} \sqrt{n}]$ and $\Delta \in [\Delta_0, \alpha_n \sqrt{n}]$,

$$\mathbb{P}(x + S_n \in [0, \Delta] + y, \tau_x > n) \sim \Delta \frac{2V(x)}{\sqrt{2\pi \sigma^2 n}} \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right). \quad (1.6)$$

Moreover, there exist constants $\varepsilon_0, \delta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $q \in (0, \delta_0)$, $\eta > 0$ and $\Delta_0 > 0$, as $n \to \infty$, the asymptotic (1.6) holds uniformly in $x \in [0, n^{1/2-\varepsilon}]$, $y \in [\eta \sqrt{n}, \sigma \sqrt{q n \log n}]$ and $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$.

In particular, from the second assertion of Theorem 1.2, taking $y = \sigma \sqrt{q n \log n}$ in (1.6), we get, as $n \to \infty$, uniformly in $x \in [0, n^{1/2-\varepsilon}]$ and $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$,

$$\mathbb{P}(x + S_n \in [0, \Delta] + \sigma \sqrt{q n \log n}, \tau_x > n) \sim \frac{2V(x)}{\sqrt{2\pi \sigma^2}} \frac{\Delta \sqrt{q \log n}}{n^{1+q/2}}. \quad (1.7)$$

The asymptotic (1.7) can be compared with the classical local limit theorem with moderate deviations (for non-killed random walks) which can be deduced from the results of Nagaev [35] and Amosova [3]: as $n \to \infty$,

$$\mathbb{P}(S_n \in [0, \Delta] + \sigma \sqrt{q n \log n}) \sim \frac{\Delta}{\sqrt{2\pi \sigma n^{(1+q)/2}}}. \quad (1.8)$$

We refer to Breuillard [12] for similar results for random walks satisfying the diophantine condition, and to Grama [24] for martingales.

In the particular case when $x = 0$, $y = O(\sqrt{n})$ and $\Delta > 0$ is a fixed real number, the asymptotic (1.6) has been established earlier by Caravenna [13] under the non-lattice condition A1 and the optimal second moment assumption A2 with $\delta = 0$. Under the same conditions, Caravenna’s result has been generalized by Doney [18] to the case when $x$ can depend on $n$. Under the stronger moment condition of order $2 + \delta$, our asymptotic (1.6) improves on these results in two aspects. Firstly, $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$ is allowed to depend on $n$; secondly, $y$ can take values in the range $[\eta \sqrt{n}, \sigma \sqrt{q n \log n}]$. Indeed, the case when $y$ goes beyond the range $\sqrt{n}$ has not been considered in the literature so far and our result gives a new asymptotic in this range.

Our second result gives the exact asymptotics for the probability (1.1) when $x$ is near the boundary and $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. It is a consequence of a more general Theorem 4.2.

**Theorem 1.3.** Assume A1 and A2. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \to \infty} \alpha_n = 0$ and $n^{1/4} \alpha_n \geq 1$.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a directly Riemann integrable function with support in $\mathbb{R}_+$ satisfying $\int_{\mathbb{R}_+} f(t)(1 + t)^\gamma \, dt < \infty$ for some constant $\gamma > 1$. Then, for
any $a > 0$, we have, as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$ and $y \in [0, a]$,
\[
\mathbb{E}(f(x + S_n - y); \tau_x > n) \sim \frac{2V(x)}{\sqrt{2\pi\sigma^3n^{3/2}}} \int_{\mathbb{R}_+} f(t - y)V^*(t)dt.
\] (1.9)

2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a directly Riemann integrable function with support in $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} f(t)(1 + t)dt < \infty$. Then, we have, as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$ and $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$,
\[
\mathbb{E}(f(x + S_n - y); \tau_x > n) \sim \frac{2yV(x)}{\sqrt{2\pi\sigma^3n^{3/2}}} \int_{\mathbb{R}_+} f(t)dt.
\] (1.10)

The presence of the target function $f$ in the asymptotics (1.9) and (1.10) is very important since it allows to deal with random walks with drift and to establish local theorems for the time $\tau_x$. Indeed, with a suitable change of measure the drift vanishes so that we can apply (1.9) and (1.10) with an appropriate target function. This will be used in Sections 1.4 and 1.5 to obtain the exact asymptotics for the exit time of random walks with negative drift. The precise asymptotic of the local probability $\mathbb{P}(\tau_x = n)$ is given in Section 1.5.

It is easy to see that in the range $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$, the right hand side of the asymptotic (1.10) is equivalent to that in (1.5). We conjecture that our asymptotics (1.5) and (1.6) should be valid uniformly in the larger interval $y \in [\alpha_n, o(n^{1/6})]$ for any $\alpha_n \to \infty$.

In particular, for any real number $a > 0$, taking $f(t) = e^{-at}$ and $y = 0$ in (1.9), we get that, as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$,
\[
\mathbb{E}(e^{-a(x+S_n)}; \tau_x > n) \sim \frac{2V(x)}{\sqrt{2\pi\sigma^3n^{3/2}}} \int_{\mathbb{R}_+} e^{-at}V^*(t)dt.
\] (1.11)

The result (1.11) holds uniformly in $x \in [0, \alpha_n \sqrt{n}]$, thus improving that of [1, Proposition 2.1] (see also [33, Theorem 4.10]) which was proved for fixed $x \geq 0$. Note that the results in [1, 33] are established using a completely different approach based on the Wiener-Hopf factorization, in particular using the Sparre-Andersen and Spitzer identities. These type of results turn out to be very useful for studying limit theorems for branching processes in random environment.

In the particular case when $f = 1_{[0,\Delta]}$ with $\Delta > 0$, the following result improves the asymptotic (1.10) by showing the uniformly in $\Delta$.

**Theorem 1.4.** Assume $A1$ and $A2$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \to \infty} \alpha_n = 0$ and $n^{1/4} \alpha_n \geq 1$. Then, for any $\Delta_0 > 0$, we have, as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$, $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$ and $\Delta \in [\Delta_0, o(y)]$,
\[
\mathbb{P}(x + S_n \in [0,\Delta] + y, \tau_x > n) \sim \frac{2yV(x)}{\sqrt{2\pi\sigma^3n^{3/2}}}.\] (1.12)
Note that the asymptotic (1.12) does not hold when $\frac{x}{\sqrt{n}} \to \infty$. Therefore Theorem 1.4 corrects a misstatement in [18, Proposition 18], where it is claimed, in particular, that the asymptotic (1.12) holds uniformly in $\Delta > 0$.

1.3. **Starting point far from the boundary.** In this section we formulate our results when the starting point $x$ is far from the boundary, more precisely when $\frac{x}{\sqrt{n}} \in [\eta^{-1}, \eta]$ for any real number $\eta > 1$. The following result is the analog of Theorem 1.1 for large $x$.

**Theorem 1.5.** Assume **A1** and **A2.** Let $\eta \geq 1$ be any fixed real number. Let $f : \mathbb{R} \to \mathbb{R}$ be any directly Riemann integrable function with support in $\mathbb{R}_+$ satisfying $\int_{\mathbb{R}_+} f(t) (1 + t)^{-\gamma} dt < \infty$ for some constant $\gamma > 1$. Then, there exists a constant $q_0 > 0$ such that for any $q \in (0, q_0)$, we have, as $n \to \infty$, uniformly in $x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}]$ and $y \in [\eta^{-1} \sqrt{n}, \sigma \sqrt{qn \log n}]$,

$$
\mathbb{E}(f(x + S_n - y); \tau_x > n) \sim \frac{1}{\sigma \sqrt{n}} \psi \left( \frac{y}{\sigma \sqrt{n}} \frac{x}{\sigma \sqrt{n}} \right) \int_{\mathbb{R}_+} f(t) dt. \quad (1.13)
$$

In the particular case when $f = \mathbb{1}_{[0,\Delta]}$ with $\Delta > 0$, we are able to prove the uniformly in $\Delta$.

**Theorem 1.6.** Under **A1** and **A2**, for any $\varepsilon, \eta, \Delta_0 > 0$, there exists a constant $q_0 > 0$ such that for any $q \in (0, q_0)$, we have, as $n \to \infty$, uniformly in $x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}]$, $y \in [\eta^{-1} \sqrt{n}, \sigma \sqrt{qn \log n}]$ and $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$,

$$
\mathbb{P}(x + S_n \in [0, \Delta] + y; \tau_x > n) \sim \frac{\Delta}{\sigma \sqrt{n}} \psi \left( \frac{y}{\sigma \sqrt{n}} \frac{x}{\sigma \sqrt{n}} \right). \quad (1.14)
$$

In particular, taking $x = \sigma \eta \sqrt{n}$ and $y = \sigma \sqrt{qn \log n}$ in (1.14), we have, as $n \to \infty$, uniformly in $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$,

$$
\mathbb{P} \left( x + S_n \in [0, \Delta] + \sigma \sqrt{qn \log n}; \tau_x > n \right) \sim \frac{\Delta e^{-\frac{q^2}{4} + \eta \sqrt{\log n}}}{\sqrt{2\pi \sigma n (1 + q)/2}}. \quad (1.15)
$$

We continue with an analog of Theorem 1.3 for $x$ large.

**Theorem 1.7.** Assume **A1** and **A2.** Let $\eta \geq 1$ be any fixed real number.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be any directly Riemann integrable function with support in $\mathbb{R}_+$ satisfying $\int_{\mathbb{R}_+} f(t) (1 + t)^{-\gamma} dt < \infty$ for some constant $\gamma > 1$. Then, for any $a > 0$, we have, as $n \to \infty$, uniformly in $x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}]$ and $y \in [0, a]$,

$$
\mathbb{E}(f(x + S_n - y); \tau_x > n) \sim \frac{2}{\sqrt{2\pi \sigma^2 n}} \phi^+ \left( \frac{x}{\sigma \sqrt{n}} \right) \int_{\mathbb{R}_+} f(t - y)V^+(t) dt. \quad (1.16)
$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be any directly Riemann integrable function with support in $\mathbb{R}_+$ satisfying $\int_{\mathbb{R}_+} f(t) (1 + t)^{\gamma} dt < \infty$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of
positive numbers satisfying \( \lim_{n \to \infty} \alpha_n = 0 \) and \( n^{1/4} \alpha_n \geq 1 \). Then we have, as \( n \to \infty \), uniformly in \( x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}] \) and \( y \in [\alpha_n^{-1}, \alpha_n\sqrt{n}] \),
\[
\mathbb{E}(f(x + S_n - y); \tau_x > n) \sim \frac{2y}{\sqrt{2\pi\sigma^2 n}} \phi^+ \left( \frac{x}{\sigma\sqrt{n}} \right) \int_{\mathbb{R}_+} f(t) dt. \tag{1.17}
\]

The following version of (1.17) with \( f = \mathbbm{1}_{[0,\Delta]} \) holds uniformly in \( \Delta \).

**Theorem 1.8.** Assume \( \textbf{A1} \) and \( \textbf{A2} \). Let \( \{\alpha_n\}_{n \geq 1} \) be any sequence of positive numbers satisfying \( \lim_{n \to \infty} \alpha_n = 0 \) and \( n^{1/4} \alpha_n \geq 1 \). Then, for any \( \Delta_0 > 0 \), we have, as \( n \to \infty \), uniformly in \( x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}] \), \( y \in [\alpha_n^{-1}, \alpha_n\sqrt{n}] \) and \( \Delta \in [\Delta_0, o(y)] \),
\[
\mathbb{P}(x + S_n \in [0,\Delta] + y, \tau_x > n) \sim \Delta \frac{2y}{\sqrt{2\pi\sigma^2 n}} \phi^+ \left( \frac{x}{\sigma\sqrt{n}} \right). \tag{1.18}
\]

### 1.4. Random walks with drift.

In this section we consider the case when the random walk has non-zero drift. As before, assume that on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) the sequence \( \{X_i\}_{i \geq 1} \) is independent identically distributed with \( \mathbb{E}X_1 = \mu \in \mathbb{R} \) and \( \mathbb{E}X_1^2 = \sigma^2 \in (0,\infty) \). Let \( S_n = \sum_{i=1}^n X_i, n \geq 1 \). For any starting point \( x \in \mathbb{R}_+ := [0,\infty) \), define
\[
\tau_x = \inf \{k \geq 1 : x + S_k < 0\} \quad \text{with} \quad \inf \emptyset = \infty.
\]

The case when random walk \( \{S_n\}_{n \geq 1} \) has a negative drift, i.e. \( \mu < 0 \), was studied by Iglehart [31], where the exact asymptotic of the probability \( \mathbb{P}(\tau_n > n) \) has been established under an exponential moment condition on \( X_1 \). Further results of the probability (1.1) can be found in Doney [17], Keener [32] and, under the assumption that \( X_1 \) is heavy tailed and has finite second moment condition in Bansaye and Vatutin [5]. When the random variable \( X_1 \) is absolutely continuous with subexponential density, Denisov, Vatutin and Wachtel [14] investigated the asymptotic behavior of (1.1) for fixed \( x \geq 0 \) and various ranges of \( y \).

All the results in Sections 1.2 and 1.3 have analogs for random walks with drift, by using an exponential change of probability measure, under the following additional Cramér-type condition:

**A3.** There exist constants \( \lambda \in \mathbb{R} \) and \( \delta > 0 \) such that
\[
\mathbb{E}X_1 e^{\lambda X_1} = 0 \quad \text{and} \quad \mathbb{E}|X_1|^{2+\delta} e^{\lambda X_1} < \infty.
\]

It is easy to see that \( \mathbb{E}e^{\lambda X_1} \leq \mathbb{E}|X_1|^{2+\delta} 1_{\{|X_1|>1\}} e^{\lambda X_1} < \infty \).

Note that from condition **A3** we have that \( \lambda > 0 \) (\( \lambda = 0 \), \( \lambda < 0 \)) when \( \mu < 0 \) (\( \mu = 0 \), \( \mu > 0 \)). In the case when \( \mu = 0 \) condition **A3** becomes **A2**.

As before, let \( \{S_n^*\}_{n \geq 1} \) be the dual random walk: \( S_n^* = -\sum_{i=1}^n X_i \). Denote by \( \tau_x^* \) the dual exit time: \( \tau_x^* = \inf \{k \geq 1 : x + S_k^* < 0\} \) with \( \inf \emptyset = \infty \). Let \( \lambda \in I_{\mu} \) be from condition **A3**. Denote
\[
\Lambda(\lambda) = \log \mathbb{E}e^{\lambda X_1} \quad \text{and} \quad \sigma_\lambda^2 = \mathbb{E}|X_1|^2 e^{\lambda X_1}.
\]
Define a change of probability measure by setting
\[ \mathbb{P}_\lambda(X_1 \in dx) = e^{\lambda x - \Lambda(\lambda)} \mathbb{P}(X_1 \in dx) \]
and let \( E_\lambda \) be the expectation corresponding to \( \mathbb{P}_\lambda \). From condition \( A3 \) it follows that \( \Lambda(\lambda) < 0 \) whenever \( \lambda \neq 0 \). For \( x \geq 0 \), let \( V_\lambda(x) := -E_\lambda(S_{\tau_x}) \) and \( V^*_\lambda(x) := -E_\lambda(S^*_{\tau^*_x}) \) be the harmonic functions of random walks \( S_n \) and \( S^*_n \) killed at \( \tau_x \) and \( \tau^*_x \) under the probability measure \( \mathbb{P}_\lambda \), respectively. The functions \( V_\lambda \) and \( V^*_\lambda \) can be extended to the whole real line \( \mathbb{R} \) in the same way as the harmonic function \( V \).

The results of this section are obtained from those in Sections 1.2 and 1.3 by using an exponential change of probability measure. The key point in applying this change of measure is the presence of the target functions in Theorems 1.3 and 1.7.

In the particular case when the random walk has negative drift, i.e. \( \mu = \mathbb{E}X_1 < 0 \) (in this case \( \lambda > 0 \) by condition \( A3 \)), we have the following result.

**Theorem 1.9.** Assume \( A1 \) and \( A3 \) for some \( \lambda > 0 \). Let \( (\alpha_n)_{n \geq 1} \) be any sequence of positive numbers satisfying \( \lim_{n \to \infty} \alpha_n = 0 \). Then we have, as \( n \to \infty \), uniformly in \( x \in [0, \alpha_n \sqrt{n}] \),
\[ \mathbb{P}(\tau_x > n) \sim \frac{2V_\lambda(x)e^{n\Lambda(\lambda)+\lambda x}}{\sqrt{2\pi \sigma^3 \lambda n^{3/2}}} \int_{\mathbb{R}^+} e^{-\lambda t} V^*_\lambda(t) dt. \] (1.19)

The asymptotic (1.19) improves the results of Iglehart [31, Theorem 2.1] and Doney [17, Theorem II]. Firstly, our result (1.19) holds uniformly in \( x \in [0, \alpha_n \sqrt{n}] \), while [31] deals with the particular case \( x = 0 \) and [17] deals with fixed \( x \geq 0 \). Secondly, our moment condition \( A3 \) is weaker than that used in [31].

The following theorem complements the results of Iglehart [31] and Doney [17] when \( x \) is large:

**Theorem 1.10.** Assume \( A1 \) and \( A3 \) for some \( \lambda > 0 \). Then, for any \( \eta \geq 1 \), we have, as \( n \to \infty \), uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \),
\[ \mathbb{P}(\tau_x > n) \sim \frac{2e^{n\Lambda(\lambda)+\lambda x}}{\sqrt{2\pi \sigma^2 \lambda n}} \phi^+ \left( \frac{x}{\sigma \lambda \sqrt{n}} \right) \int_{\mathbb{R}^+} e^{-\lambda t} V^*_\lambda(t) dt. \]

1.5. **Local limit theorems for the exit time.** In this section we formulate two local limit theorems for the exit time \( \tau_x \).

Our first result gives the asymptotics of the probability \( \mathbb{P}(\tau_x = n) \) when the random walk is driftless. Set
\[ \kappa := \int_{-\infty}^{t} V^*(t) dt = \int_{\mathbb{R}^+} \mathbb{P}(t + X_1 < 0)V^*(t) dt. \] (1.20)
The second equality follows from the harmonicity property (1.2). By using condition $A_1$ and Markov’s inequality, one can check that the second integral in (1.20) is strictly positive and finite; see Lemma 6.4.

**Theorem 1.11.** Assume $A_1$ and $A_2$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \to \infty} \alpha_n = 0$. Then we have, as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$,

$$P(\tau_x = n) \sim \frac{2\varsigma V(x)}{\sqrt{2\pi \sigma^2 n^{3/2}}}.$$  \hspace{1cm} (1.21)

Moreover, for any $\eta \geq 1$, it holds, as $n \to \infty$, uniformly in $x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}]$,

$$P(\tau_x = n) \sim \frac{2\varsigma}{\sqrt{2\pi \sigma^2 n}} \phi^+ \left( \frac{x}{\sigma \sqrt{n}} \right).$$  \hspace{1cm} (1.22)

For fixed $x \geq 0$, asymptotics of the probability $P(\tau_x = n)$ have been obtained for instance by Eppel [20], Borovkov [9] and Vatutin and Wachtel [44]. A uniform version has been obtained by Doney [18]. The asymptotic (1.21) is in accordance with Theorem 2.8 of [27], which is stated for Markov chains with finite state space and for fixed $x \geq 0$. However, the setting in [27] does not cover the general case of random walks in $\mathbb{R}$.

To the best of our knowledge, the explicit formula (1.20) for the constant $\varsigma$ in the asymptotics (1.21) and (1.22) has not been known in the literature. Our formulas (1.21) and (1.22) (under assumptions $A_1$ and $A_2$) correct some misstatement in the analogous results in papers [44, Theorems 7] and [18, Theorem 2 (A)]. Under the setting of our paper ($X_1$ is in the domain of attraction of the normal law), these results claim that, for any fixed $x \geq 0$, as $n \to \infty$,

$$P(\tau_x = n) \sim \frac{V(x)}{2\sqrt{2\pi \sigma^3 n^{3/2}}}. \hspace{1cm} (1.23)$$

Therefore, compared with (1.21), the factor $\frac{4\varsigma}{\sigma^2}$ is missing in (1.23). The same remark also applies to some other results, in particular to [44, Theorems 10] and [18, Theorem 2 (B)].

Our next result extends Theorem 1.11 to the case of random walks with drift. Let

$$\varsigma_\lambda := E e^{\lambda X_1} \int_{-\infty}^{0} e^{-\lambda t} V^*_\lambda(t)dt = \int_{\mathbb{R}^+} P(t + X_1 < 0) e^{-\lambda t} V^*_\lambda(t)dt.$$ \hspace{1cm} (1.24)

The second equality will be proved in Lemma 6.4 where we also show that the constant $\varsigma_\lambda$ is strictly positive and finite.
Theorem 1.12. Assume $A1$ and $A3$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \to \infty} \alpha_n = 0$. Then we have, as $n \to \infty$, uniformly in $x \in [0, \alpha \sqrt{n}]$,

$$P(\tau_x = n) \sim \frac{2 \sqrt{\lambda} V_\lambda(x)e^{n\Lambda(\lambda)+\lambda x}}{\sqrt{2\pi \sigma^2 \lambda n^{3/2}}}.$$  \hspace{1cm} (1.25)

Moreover, for any $\eta \geq 1$, it holds, as $n \to \infty$, uniformly in $x \in [\eta^{-1}\sqrt{n}, \eta \sqrt{n}]$,

$$P(\tau_x = n) \sim \frac{2 \sqrt{\lambda} e^{n\Lambda(\lambda)+\lambda x}}{\sqrt{2\pi \sigma^2 \lambda n^{3/2}}} \phi^+(\frac{x}{\sigma \sqrt{n}}).$$  \hspace{1cm} (1.26)

The asymptotics (1.25) and (1.26) for random walks with drift are new to our knowledge.

The proof of Theorems 1.11 and 1.12 is given in Section 6.4. Our proof is an elementary application of the local limit theorems with target functions, which is different from the proofs in [20], [9], [44] and [18] based on more involved arguments.

1.6. Method of the proof. Usually to study the asymptotic behavior of the probability (1.1) the Wiener-Hopf factorization is employed. However, for the proof of our results this technique does not seem appropriate. In the present paper we adopt an approach inspired by [15]. To give a sketch of the proofs let us denote $f = \mathbb{1}_{y+[0,\Delta]}$ and let us assume first that in the expectation $E(f(x+S_n);\tau_x > n)$ the starting point $x \geq 0$ is near the boundary (in our setting the level 0). Then splitting the trajectory of the walk $(x+S_n)_{n \geq 0}$ into two parts of length $k$ and $m$ with $k + m = n$ and using the Markov property after some transformations we can approximate the probability (1.1) as follows (see (3.9) and (3.21) for more details):

$$E(f(x+S_n);\tau_x > n) \approx \int_{\mathbb{R}_+} E f(t+S_m) P(x+S_k \in dt, \tau_x > k).$$  \hspace{1cm} (1.27)

The expectation $E f(t+S_m)$ corresponding to the second part of the trajectory is handled by using the ordinary local limit theorem (Theorem 2.5), which gives

$$E f(x+S_n; \tau_x > n) \approx \int_{\mathbb{R}_+} \varphi_n(t) P(x+S_k \sigma \sqrt{k} \in dt, \tau_x > k),$$  \hspace{1cm} (1.28)

where $\varphi_n(t) := \int_{\mathbb{R}} f(\sigma \sqrt{k} s) \phi_{m/k}(t-s) ds$ and $\phi_v$ is the normal density of mean zero and variance $v > 0$. For the integral in (1.28), which corresponds to the first part of the trajectory, we apply the conditioned integral limit theorem (Theorem 2.7). After some elementary calculations, this leads to the approximation

$$E(f(x+S_n); \tau_x > n) \approx \frac{2V(x)}{\sqrt{2\pi k \sigma}} \int_{\mathbb{R}} f(\sigma \sqrt{n} t) \phi_{\delta_n} * \phi_{1-\delta_n}^+(t) dt,$$  \hspace{1cm} (1.29)
where $\delta_n = \frac{m}{n}$ and $\phi^+_v$ is the Rayleigh density with scale parameter $\sqrt{v}$, $v > 0$. Using the convolution Lemma 3.3 as $\delta_n$ becomes small, we obtain, as $n \to \infty$,

$$
E(f(x + S_n); \tau_x > n) \approx \frac{2V(x)}{\sqrt{2\pi \sigma^2 n}} \int_{\mathbb{R}} f(t) \phi^+ \left( \frac{t}{\sigma \sqrt{n}} \right) dt,
$$

(1.30)

from which we can deduce Theorem 1.1. Note that, the main term in the asymptotic (1.30) becomes meaningful only when the support of the function moves to infinity as $n \to \infty$, which is the case for instance when $f = \mathbb{1}_{y+[0,\Delta]}$ and $y$ moves to $\infty$. The use of the convolution step (1.29) is the crucial point of our approach, which makes the difference with that of [15]. It also clarifies the idea of the method and greatly simplifies the computations compared to [15].

The case when $f = \mathbb{1}_{y+[0,\Delta]}$ and both $x$ and $y$ are near the boundary (which is a particular case of Theorem 1.3) is handled in a different way. Using the Markov property we obtain the approximation

$$
E(f(x + S_n); \tau_x > n) \approx \int_{\mathbb{R}_+} H_m(t) \mathbb{P}(x + S_k \in dt, \tau_x > k),
$$

(1.31)

where $H_m(t) = E(f(t + S_m); \tau_t > m)$. First, by the asymptotic (1.30) applied with the function $f = H_m$ and $n = k$, we have

$$
E(f(x + S_n); \tau_x > n) \approx \frac{2V(x)}{\sqrt{2\pi \sigma^2 k}} \int_{\mathbb{R}_+} H_m(t) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt,
$$

from which by the reversibility we get

$$
E(f(x + S_n); \tau_x > n) \approx \frac{2V(x)}{\sqrt{2\pi \sigma^2 k}} \int_{\mathbb{R}_+} f(t) \mathbb{E} \left[ \phi^+ \left( \frac{t + S^*_m}{\sigma \sqrt{m}} \right); \tau^*_t > m \right] dt.
$$

The expectation inside the integral (which corresponds to the second part of the trajectory) is handled using the conditioned integral limit theorem (Theorem 2.7) for the reversed walk $(t + S^*_n)_{n \geq 0}$, which after some elementary transformations leads to the approximation

$$
E(f(x + S_n); \tau_x > n) \approx \frac{2V(x)}{\sqrt{2\pi \sigma^3 n^{3/2}}} \int_{\mathbb{R}} f(t) V^*(t) dt.
$$

(1.32)

This gives the assertion (1.9) of Theorem 1.3. It is worth mentioning that when handling the right hand side of (1.31) there is a problem with the function $H_m$ which does not comply with the assumptions of Theorem 1.1. We overcome this by using a special smoothing technique developed in Section 2.1.

The corresponding assertion of Theorem 1.9 is obtained by applying the approximation (1.32) to the random walk $(x + S_n)_{n \geq 0}$ under an exponential change of measure. At this stage a result with a target function $f$ is necessary.
The case when the starting point \( x \) is of order \( \sqrt{n} \) is treated much in the same way, except that the product \( \phi^+(t)V(x) \) is replaced by the function \( \psi(t,x) \) defined by (1.4). In order to perform this program, we establish effective local limit theorems for \( S_n \) with target functions on \( S_n \) and with explicit remainders expressed in terms of the target function. We will also establish conditioned integral limit theorems with a rate of convergence.

Let us note that the method described above works also for sums of i.i.d. random variables taking values in lattices and can also be applied for random walks in cones. This last point is treated in a subsequent paper. We also would like to point out some other advantages of our method. With our approach we can avoid the use of the reversibility of the random walk \( (x + S_n)_{n \geq 0} \) (this requires a minor modification of the proof of the lower bound in section 3.5). This is in contrast to the previous work in the area \([13, 18, 33, 44]\) where the reversibility is used essentially in the proof. Therefore with our approach we can extend Theorems 1.1, 1.3, 1.5 and 1.7 to the case of Markov random walks. We refer to Grama, Quint and Xiao [29] for a study on conditioned limit theorems for hyperbolic dynamical systems and to Grama, Mentemeier and Xiao [28] where this type of results are applied for the study of branching products of random matrices.

2. Effective limit theorems

2.1. Preliminary results. Let \( f : \mathbb{R} \mapsto \mathbb{R}_+ \) be a non-negative Borel measurable function. Consider the upper and lower ladder functions: for \( I_k = [k\delta, (k+1)\delta) \) with \( \delta > 0 \),

\[
\mathcal{T}_\delta(u) = \sum_{k \in \mathbb{Z}} 1_{I_k}(u) \sup_{u' \in I_k} f(u'), \quad \mathcal{L}_\delta(u) = \sum_{k \in \mathbb{Z}} 1_{I_k}(u) \inf_{u' \in I_k} f(u'), \quad u \in \mathbb{R}.
\]

The function \( f \) is called directly Riemann integrable if \( \int_{\mathbb{R}} \mathcal{T}_\delta(u) du < \infty \) for any \( \delta > 0 \) small enough, and

\[
\lim_{\delta \to 0} \int_{\mathbb{R}} (\mathcal{T}_\delta(u) - \mathcal{L}_\delta(u)) du = 0,
\]

(2.1)

here and below all the integrals are understood in the Lebesgue sense. A Borel measurable function \( f \) with values in \( \mathbb{R} \) is directly Riemann integrable if both its positive and negative parts are directly Riemann integrable. Since \( \mathcal{L}_\delta \leq f \leq \mathcal{T}_\delta \), we have that (2.1) is equivalent to the following property:

\[
\lim_{\delta \to 0} \int_{\mathbb{R}} (\mathcal{T}_\delta(u) - f(u)) du = 0, \quad \lim_{\delta \to 0} \int_{\mathbb{R}} (f(u) - \mathcal{L}_\delta(u)) du = 0.
\]

(2.2)

Every directly Riemann integrable function is necessarily integrable with respect to the Lebesgue measure on \( \mathbb{R} \) and vanishes at infinity. But the converse may not hold because of the possible oscillations at infinity. We refer to [21, §XI.1] and [4, §V.4] for more details.
Let \( f, g : \mathbb{R} \mapsto \mathbb{R}_+ \) be Borel measurable functions and \( \varepsilon > 0 \). We say that 
\( g \) \( \varepsilon \)-dominates \( f \) and we use notation \( f \leq \varepsilon g \) if
\[
    f(u) \leq g(u + v), \quad \forall u \in \mathbb{R}, \quad \forall |v| \leq \varepsilon.
\]  
(2.3)
We say equivalently that \( f \) \( \varepsilon \)-minorates \( g \) and we use notation \( g \geq \varepsilon f \). If \( f \leq \varepsilon g \) we also say that \( g \) is an upper \( \varepsilon \)-envelope of \( f \) or equivalently that \( f \) is an lower \( \varepsilon \)-envelope of \( g \). In the following we are interested in (Borel) measurable \( \varepsilon \)-envelopes. Any function \( f \) has measurable upper and lower \( \varepsilon \)-envelopes for any \( \varepsilon > 0 \) sufficiently small. To see this consider the upper and lower \( \varepsilon \)-envelopes of \( f_\delta \) and \( f_\delta \),
\[
    f_{\delta, u}(u) = \sup_{|v-u| \leq \varepsilon} f_\delta(v), \quad f_{\delta, u}(u) = \inf_{|v-u| \leq \varepsilon} f_\delta(v), \quad u \in \mathbb{R}.
\]  
(2.4)
Then, obviously, it holds that
\[
    f_{\delta, u}(u) \leq \varepsilon f_\delta \leq f \leq f_{\delta, u}(u).
\]  
(2.5)
In addition, both \( f_{\delta, u} \) and \( f_{\delta, u} \) are Borel measurable functions since \( f_\delta \) and \( f_\delta \) take values in a countable set.

The following lemma shows that if \( f \) is directly Riemann integrable then it can be approximated by the help of the functions \( f_{\delta, u} \) and \( f_{\delta, u} \).

**Lemma 2.1.** Let \( f : \mathbb{R} \mapsto \mathbb{R}_+ \) be a directly Riemann integrable function. Then we have
\[
    \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} |f_{\delta, u}(u) - f(u)|du = 0, \quad \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} |f_{\delta, u}(u) - f(u)|du = 0.
\]
Proof. By definition (2.4), it holds that almost surely under the Lebesgue measure on \( \mathbb{R} \),
\[
    \lim_{\varepsilon \to 0} f_{\delta, u} = f_\delta \quad \text{and} \quad \lim_{\varepsilon \to 0} f_{\delta, u} = f_\delta.
\]  
(2.6)
Using the Lebesgue dominated convergence theorem, we conclude that for any \( \delta > 0 \),
\[
    \lim_{\varepsilon \to 0} \int_{\mathbb{R}} |f_{\delta, u}(u) - f_\delta(u)|du = 0, \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}} |f_{\delta, u}(u) - f_\delta(u)|du = 0.
\]  
(2.7)
For this it is enough to prove that there exist \( \varepsilon_0 \in (0, \delta] \) and a Lebesgue integrable function \( g \) on \( \mathbb{R} \) such that
\[
    0 \leq \sup_{\varepsilon \in (0, \varepsilon_0)} f_{\delta, u} \leq \sup_{\varepsilon \in (0, \varepsilon_0)} f_{\delta, u} \leq g.
\]  
(2.8)
Indeed, for \( k \in \mathbb{Z} \), let
\[
    g(u) = f_\delta(u) + f_\delta(u + \delta) + f_\delta(u - \delta), \quad u \in [k\delta, (k + 1)\delta).
\]
We see that \( f_{\delta,-\epsilon}(u) \leq g(u) \) for any \( u \in \mathbb{R} \) and \( \epsilon \in (0, \delta] \). In addition,
\[
\sum_{k=-\infty}^{\infty} \chi_{[k\delta,(k+1)\delta)}(u)g(u) = \sum_{k=-\infty}^{\infty} \chi_{[k\delta,(k+1)\delta)}(u)f_{\delta}(u) \\
+ \sum_{k=-\infty}^{\infty} \chi_{[k\delta,(k+1)\delta)}(u)f_{\delta}(u + \delta) + \sum_{k=-\infty}^{\infty} \chi_{[k\delta,(k+1)\delta)}(u)f_{\delta}(u - \delta).
\]
Since the right hand side is an integrable function, we get that the function \( g \) is Lebesgue integrable, which shows (2.8).

The assertion of the lemma follows from (2.7) and (2.2). \( \square \)

Now we state some smoothing inequalities. For any integrable function \( f : \mathbb{R} \mapsto \mathbb{R} \), its Fourier transform is defined by \( \hat{f}(t) = \int_{\mathbb{R}} e^{-itu} f(u)du, t \in \mathbb{R} \). Note that
\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{-itu} \left( \frac{\sin(u/2)}{u/2} \right)^2 du = (1 - |t|) \chi_{\{|t| \leq 1\}}, \quad t \in \mathbb{R}, \tag{2.9}
\]
where we use the convention that \( \sin 0 = 1 \). We introduce the density function \( \kappa \) by setting
\[
\kappa(u) = \left[ \int_{\mathbb{R}} \left( \frac{\sin(v/4)}{v/4} \right)^4 dv \right]^{-1} \left( \frac{\sin(u/4)}{u/4} \right)^4, \quad u \in \mathbb{R}.
\]
By (2.9), its Fourier transform \( \hat{\kappa} \) is a non-negative even function with support on \([-1, 1]\). For any \( \epsilon > 0 \), we define the rescaled density function \( \kappa_{\epsilon} \) by
\[
\kappa_{\epsilon}(u) = \frac{1}{\epsilon} \kappa \left( \frac{u}{\epsilon} \right), \quad u \in \mathbb{R}. \tag{2.10}
\]
Its Fourier transform \( \hat{\kappa}_{\epsilon} \) is supported on \([-\epsilon^{-1}, \epsilon^{-1}]\). Note also that there exists a constant \( c > 0 \) such that for any \( \epsilon > 0 \),
\[
\int_{|u| > \frac{1}{4}} \kappa(u)du \leq c \int_{\frac{1}{4}}^{\infty} \frac{1}{u}du \leq ce^3. \tag{2.11}
\]

We shall make use of the following smoothing inequalities, whose proofs can be carried out in the same way as that of [27, Lemma 5.2]. Below let
\[
f * g(u) = \int_{\mathbb{R}} f(u-v)g(v)dv, \quad u \in \mathbb{R},
\]
be the convolution of \( f \) and \( g \), whenever the integral makes sense.

**Lemma 2.2.** There exists a constant \( c > 0 \) such that for any \( \epsilon \in (0, 1/2) \) and any integrable functions \( f, g : \mathbb{R} \mapsto \mathbb{R}_+ \) with \( f \leq g \), and any \( u \in \mathbb{R} \),
\[
f(u) \leq (1 + c\epsilon)g * \kappa_{\epsilon^2}(u), \quad g(u) \geq f * \kappa_{\epsilon^2}(u) - \int_{|v| > \epsilon} f(u-v) \kappa_{\epsilon^2}(v)dv.
\]
Remark 2.3. If \( f \leq_{\varepsilon} g \) and the function \( g \) is integrable, then \( f \) is bounded. Indeed, since \( f \leq_{\varepsilon} g \) and \( g \) is an integrable function, by Lemma 2.2 we get
\[
\int f \leq (1 + c\varepsilon)g * \kappa_{\varepsilon^2}.
\]
Since the Fourier transform of \( g * \kappa_{\varepsilon^2} \) is compactly supported on \([-\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^2}]\), by the Fourier inversion formula, the function \( g * \kappa_{\varepsilon^2} \) is bounded on \( \mathbb{R} \), so that \( f \) is bounded on \( \mathbb{R} \).

2.2. Local limit theorems with precise remainders. The local limit theorem is a refinement of the central limit theorem and has been of considerable interest since the groundwork of Gnedenko [23], Shepp [40] and Stone [42] for sums of real-valued random variables. In this section we establish effective local limit theorems for the random walk \( (S_n)_{n \geq 1} \) with target functions and explicit convergence rates, which will be an important ingredient to obtain precise errors terms in the conditioned local limit theorems.

We introduce the necessary notation. The standard normal density is denoted by \( \phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, t \in \mathbb{R} \). Let \( \Phi^+(t) = (1 - e^{-t^2/2})I_{\{t \geq 0\}} \) be the Rayleigh distribution function on \( \mathbb{R} \). By an integrable function \( f : \mathbb{R} \mapsto \mathbb{R} \) we mean a Borel measurable function whose Lebesgue integral \( \|f\|_1 := \int_{\mathbb{R}} |f(x)| \, dx \) exists and is finite. In the sequel \( c \) denotes a positive constant and \( c_{\alpha, \beta} \) denote positive constants depending only on their indices. All these constants are subject to change their values every occurrence.

Theorem 2.4. Assume A1 and A2 for some constant \( \delta \in (0, 1] \). Let \( K \subset \mathbb{R} \) be a compact set. Then there exists a constant \( c_K > 0 \) such that for any integrable function \( f : \mathbb{R} \mapsto \mathbb{R} \) whose Fourier transform has a compact support contained in \( K \) and any \( n \geq 1 \),
\[
\left| \mathbb{E}f(S_n) - \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} f(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) \, dt \right| \leq \frac{c_K}{n^{(1+\delta)/2}} \|f\|_1.
\]

Theorem 2.4 can be established following the standard approach based on the Fourier transform (we refer for example to [42, 12]) and therefore its proof will not be detailed here.

In Theorem 2.4 the target function \( f \) is actually assumed to be infinitely differentiable on \( \mathbb{R} \) since its Fourier transform is compactly supported. Below we shall deduce from Theorem 2.4 the following bounds in the local limit theorem for integrable functions \( f \) which are not necessarily smooth. In the formulation below we assume that the functions \( f, g : \mathbb{R} \mapsto \mathbb{R}_+ \) are integrable and \( f \leq_{\varepsilon} g \). By Remark 2.3, this implies that \( f \) is bounded on \( \mathbb{R} \).

Theorem 2.5. Assume A1 and A2 for some constant \( \delta \in (0, 1] \). Then, for any \( \varepsilon \in (0, \frac{1}{2}) \), there exists a constant \( c > 0 \) not depending on \( \varepsilon \) and a constant \( c_{\varepsilon} > 0 \) such that the following holds:

1. For any integrable functions \( f, g : \mathbb{R} \mapsto \mathbb{R}_+ \) satisfying \( f \leq_{\varepsilon} g \) and \( n \geq 1 \),
\[
\mathbb{E}f(S_n) - \frac{1 + c_{\varepsilon}}{\sigma \sqrt{n}} \int_{\mathbb{R}} g(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) \, dt \leq \frac{c_{\varepsilon}}{n^{1+\delta}/2} \|g\|_1.
\]
2. For any integrable functions \(f, h : \mathbb{R} \mapsto \mathbb{R}_+\) satisfying \(f \geq h\) and \(n \geq 1\),
\[
\mathbb{E} f(S_n) - \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} [h(t) - c \varepsilon f(t)] \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt \geq - \frac{c \varepsilon}{n(1 + \delta/2)} \| f \|_1. \tag{2.13}
\]

In the proof of Theorem 2.5 we shall make use of the following inequality.

**Lemma 2.6.** There exists a constant \(c > 0\) such that for any \(\varepsilon \in (0, 1/2)\), \(n \geq 1\) and any integrable function \(g : \mathbb{R} \mapsto \mathbb{R}_+\),
\[
\int_{\mathbb{R}} g * \kappa_{\varepsilon^2}(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt \leq (1 + c \varepsilon) \int_{\mathbb{R}} g(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt + \frac{c \varepsilon}{\sqrt{n}} \| g \|_1.
\]

**Proof.** Denote by \(\phi_v(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\), \(x \in \mathbb{R}\) the normal density of variance \(v\). By Fubini’s theorem, we have
\[
\int_{\mathbb{R}} g * \kappa_{\varepsilon^2}(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt = \sigma \sqrt{n} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} g(v) \kappa_{\varepsilon^2}(t - v) dv \right) \phi_{\sigma^2 n}(t) dt
\]
\[
= \sigma \sqrt{n} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \kappa_{\varepsilon^2}(t - v) \phi_{\sigma^2 n}(t) dv \right) g(v) dv
\]
\[
= \sigma \sqrt{n} \int_{\mathbb{R}} \phi_{\sigma^2 n} * \kappa_{\varepsilon^2}(t) g(t) dt.
\]

Using the second inequality in Lemma 2.2 gives that for any \(t \in \mathbb{R}\),
\[
\phi_{\sigma^2 n} * \kappa_{\varepsilon^2}(t) \leq \psi_{\sigma^2 n}(t) + \int_{|v| \geq \varepsilon} \phi_{\sigma^2 n}(t - v) \kappa_{\varepsilon^2}(v) dv,
\]
where \(\psi_{\sigma^2 n}(t) = \sup_{|v| \leq \varepsilon} \phi_{\sigma^2 n}(t + v), t \in \mathbb{R}\). Hence, using again Fubini’s theorem, we get
\[
\int_{\mathbb{R}} g * \kappa_{\varepsilon^2}(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt
\]
\[
\leq \sigma \sqrt{n} \int_{\mathbb{R}} \psi_{\sigma^2 n}(t) g(t) dt + \int_{|v| \geq \varepsilon} \left[ \int_{\mathbb{R}} \phi \left( \frac{t - v}{\sigma \sqrt{n}} \right) g(t) dt \right] \kappa_{\varepsilon^2}(v) dv
\]
\[
=: J_1 + J_2.
\]

For \(J_1\), by elementary calculations we see that
\[
J_1 = \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\varepsilon} e^{-\frac{(t+\varepsilon)^2}{2\sigma^4n}} g(t) dt + \int_{-\varepsilon}^{\varepsilon} g(t) dt + \int_{\varepsilon}^{\infty} e^{-\frac{(1-\varepsilon)^2}{2\sigma^4n}} g(t) dt \right]
\]
\[
= \frac{1}{\sqrt{2\pi}} \left[ \int_{\mathbb{R}} e^{-\frac{t^2}{2\sigma^4n}} g(t) dt + \int_{-\varepsilon}^{\varepsilon} \left( e^{-\frac{(t+\varepsilon)^2}{2\sigma^4n}} - e^{-\frac{t^2}{2\sigma^4n}} \right) g(t) dt
\]
\[
+ \int_{\varepsilon}^{\infty} \left( e^{-\frac{(1-\varepsilon)^2}{2\sigma^4n}} - e^{-\frac{t^2}{2\sigma^4n}} \right) g(t) dt + \int_{-\varepsilon}^{\varepsilon} \left( e^{-\frac{t^2}{2\sigma^4n}} - 1 \right) g(t) dt \right]
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2\sigma^4n}} g(t) dt + \frac{c \varepsilon}{\sqrt{n}} \| g \|_1. \tag{2.15}
\]
For $J_2$, since the normal density $\phi$ is Lipschitz continuous on $\mathbb{R}$, we have

$$
\sup_{t \in \mathbb{R}} \left| \phi \left( \frac{t-v}{\sigma \sqrt{n}} \right) - \phi \left( \frac{t}{\sigma \sqrt{n}} \right) \right| \leq c |v| \sqrt{n}.
$$

Since $\int_{|v| \geq \varepsilon} \kappa_{\varepsilon^2}(v) dv \leq c\varepsilon$ and $\int_{|v| \geq \varepsilon} |v| \kappa_{\varepsilon^2}(v) dv \leq c\varepsilon^2$, it follows that

$$
J_2 \leq \int_{|v| \geq \varepsilon} \left[ \int_{\mathbb{R}} \phi \left( \frac{t}{\sigma \sqrt{n}} \right) g(t) dt \right] \kappa_{\varepsilon^2}(v) dv + \frac{c\varepsilon}{\sqrt{n}} \|g\|_1 \int_{|v| \geq \varepsilon} |v| \kappa_{\varepsilon^2}(v) dv
$$

$$
\leq c\varepsilon \int_{\mathbb{R}} \phi \left( \frac{t}{\sigma \sqrt{n}} \right) g(t) dt + \frac{c\varepsilon^2}{\sqrt{n}} \|g\|_1.
$$

(2.16)

Putting together (2.14), (2.15) and (2.16), the desired result follows.

**Proof of Theorem 2.5.** Let $\varepsilon \in (0, \frac{1}{4})$ and $f \leq_{\varepsilon} g$. Note that the function $g * \kappa_{\varepsilon^2}$ is continuous and integrable on $\mathbb{R}$ (the integrability follows from $\|g * \kappa_{\varepsilon^2}\|_1 = \|g\|_1 \|\kappa_{\varepsilon^2}\|_1 = \|g\|_1$). By Lemma 2.2, it holds that $f \leq (1+c\varepsilon) g * \kappa_{\varepsilon^2}$. Moreover, the support of the function $\tilde{\kappa}_{\varepsilon^2}$ is contained in the set $[-\varepsilon^{-2}, \varepsilon^{-2}]$. Therefore, by Theorem 2.4, we have that for any $n \geq 1$,

$$
\mathbb{E} f (S_n) \leq (1+c\varepsilon) \mathbb{E} g * \kappa_{\varepsilon^2} (S_n)
$$

$$
\leq \frac{1+2c\varepsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} g * \kappa_{\varepsilon^2} (t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt + \frac{c\varepsilon}{n(1+\delta)/2} \|g\|_1,
$$

(2.17)

where we used the fact that $\|g * \kappa_{\varepsilon^2}\|_1 = \|g\|_1$. By Lemma 2.6, the upper bound (2.12) follows.

Now we prove the lower bound (2.13). Set $\epsilon = \varepsilon/2$ and let $\overline{h} : \mathbb{R} \mapsto \mathbb{R}_+$ be such that $h \leq_{\varepsilon} \overline{h} \leq_{\varepsilon} f$. By Lemma 2.2, we have

$$
\mathbb{E} f (S_n) \geq \mathbb{E} \overline{h} * \kappa_{\varepsilon^2} (S_n) - \int_{|v| \geq \varepsilon} \mathbb{E} \overline{h} (S_n - v) \kappa_{\varepsilon^2}(v) dv.
$$

(2.18)

For the first term, by Theorem 2.4 and the first inequality in Lemma 2.2,

$$
\mathbb{E} \overline{h} * \kappa_{\varepsilon^2} (S_n) \geq \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} \overline{h} * \kappa_{\varepsilon^2} (t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt - \frac{c\varepsilon}{n(1+\delta)/2} \|\overline{h}\|_1
$$

$$
\geq 1 - \frac{c\varepsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} h (t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt - \frac{c\varepsilon}{n(1+\delta)/2} \|f\|_1.
$$

(2.19)

For the second term on the right hand side of (2.18), applying the upper bound (2.12) gives that for any $v \in \mathbb{R}$,

$$
\mathbb{E} \overline{h} (S_n - v) \leq \frac{1+2c\varepsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} f(t) \phi \left( \frac{t+v}{\sigma \sqrt{n}} \right) dt + \frac{c\varepsilon}{n(1+\delta)/2} \|f\|_1.
$$
In the same way as in the proof of (2.16), one has
\[
\int_{|v|\geq \epsilon} \mathbb{E} \kappa_\epsilon^2(v) dv \\
\leq \frac{1 + c \epsilon}{\sigma \sqrt{n}} \int_{|v|\geq \epsilon} \left( \int_{\mathbb{R}} f(t) \phi \left( \frac{t + v}{\sigma \sqrt{n}} \right) dt \right) \kappa_\epsilon^2(v) dv + \frac{c \epsilon}{n^{(1+\delta)/2}} \|f\|_1 \\
\leq \frac{1 + c \epsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} f(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt + \frac{c \epsilon^2}{n} \|f\|_1 + \frac{c \epsilon}{n^{(1+\delta)/2}} \|f\|_1 \\
\leq \frac{c \epsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} f(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt + \frac{c \epsilon}{n^{(1+\delta)/2}} \|f\|_1, \tag{2.20}
\]
where in the last inequality we used the fact that \( \delta \in (0, 1] \). By collecting the bounds (2.18), (2.19) and (2.20), we get the lower bound (2.13). \( \square \)

2.3. Conditioned integral limit theorems with rate of convergence. The goal of this section is to formulate effective conditioned integral limit theorems for the random walk \( x + S_n \) with explicit rate of convergence and dependence on the starting point \( x \). To the best of our knowledge, these results have not yet been established in the literature and besides the fact that we need them for establishing the main results of the paper, they are of independent interest.

The following result is a conditioned integral limit theorem for small starting point \( x = o(\sqrt{n}) \). Recall that \( \Phi^+(t) = (1 - e^{-t^2/2}) \mathbb{1}_{t \geq 0} \) is the Rayleigh distribution function on \( \mathbb{R} \).

**Theorem 2.7.** Assume A2. Then, there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), it holds uniformly in \( n \geq 1 \), \( t \in \mathbb{R}_+ \) and \( x \in [0, n^{1/2-\varepsilon}] \),
\[
\left| \mathbb{P} \left( \frac{x + S_n}{\sigma \sqrt{n}} \leq t, \tau_x > n \right) - \frac{2V(x)}{\sigma \sqrt{2\pi n}} \Phi^+(t) \right| \leq c \epsilon \frac{1 + x}{n^{1/2+\varepsilon}}. \tag{2.21}
\]
In particular, we have, uniformly in \( n \geq 1 \) and \( x \in [0, n^{1/2-\varepsilon}] \),
\[
\left| \mathbb{P} (\tau_x > n) - \frac{2V(x)}{\sigma \sqrt{2\pi n}} \right| \leq c \epsilon \frac{1 + x}{n^{1/2+\varepsilon}}. \tag{2.22}
\]
Moreover, there exists a constant \( c \) such that for any \( x \geq 0 \),
\[
\mathbb{P} (\tau_x > n) \leq \frac{1 + x}{\sqrt{n}}. \tag{2.23}
\]
For large \( x \), the following result complements Theorem 2.7. Recall that the function \( \psi \) is defined by (1.4).

**Theorem 2.8.** Assume A2 with some \( \delta > 0 \). Then, there exists a constant \( c_\delta > 0 \) such that for any \( n \geq 1 \), \( x \in n^{\frac{\delta}{2}} - n^{\frac{\delta}{2(3+\delta)}} \) and \( t \in \mathbb{R}_+ \),
\[
\left| \mathbb{P} \left( \frac{x + S_n}{\sigma \sqrt{n}} \leq t, \tau_x > n \right) - \int_0^t \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) ds \right| \leq c_\delta n^{-\frac{\delta}{2(3+\delta)}}. \tag{2.24}
\]
In particular, we have, uniformly in \( n \geq 1 \) and \( x > n^{\frac{1}{2} - \frac{3}{2(3+\delta)}} \),

\[
\left| \mathbb{P}(\tau_x > n) - \left( 2\Phi \left( \frac{x}{\sigma \sqrt{n}} \right) - 1 \right) \right| \leq c_\delta n^{-\frac{\delta}{2(3+\delta)}}. \tag{2.25}
\]

Combining the bounds (2.21) and (2.24), one can deduce the following:

**Corollary 2.9.** Assume A2. Then there exists \( \varepsilon > 0 \) such that for any sequence of positive numbers \( (\alpha_n)_{n \geq 1} \) satisfying \( \lim_{n \to \infty} \alpha_n = 0 \), it holds uniformly in \( n \geq 1 \), \( t \in \mathbb{R}_+ \) and \( x \in [0, \alpha_n \sqrt{n}] \),

\[
\left| \mathbb{P}(x + S_n \sigma \sqrt{n} \leq t, \tau_x > n) - \frac{2V(x)}{\sigma \sqrt{2\pi n}} \Phi^+(t) \right| \leq c_\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{1 + x}{n^{1/2}}. \tag{2.26}
\]

In addition, there exists \( \varepsilon \in (0, \frac{1}{2}) \) such that for any \( \beta \in (0, \frac{1}{2} - \varepsilon) \), uniformly in \( n \geq 1 \), \( t \in \mathbb{R}_+ \) and \( x \geq n^\beta \),

\[
\left| \mathbb{P}(x + S_n \sigma \sqrt{n} \leq t, \tau_x > n) - \int_0^t \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) ds \right| \leq c_\varepsilon \frac{1 + \min\{x, n^{1/2-\varepsilon}\}}{n^{1/2+\varepsilon \beta}}. \tag{2.27}
\]

Some comments on the precision of the above results seem to be appropriate. Nagaev [36, 37] showed that, under the assumption that \( \beta_3 = \mathbb{E}(|X_1|^3) < \infty \) (which corresponds to \( \delta = 1 \)), the remainder term in (2.25) can be improved, namely it is of the order \( n^{-1/2} \): uniformly in \( x \geq 0 \),

\[
\left| \mathbb{P}(\tau_x > n) - \left( 2\Phi \left( \frac{x}{\sigma \sqrt{n}} \right) - 1 \right) \right| \leq c(\beta_3)^2 n^{-1/2}. \tag{2.28}
\]

Aleskevicene [2] improved the upper bound in (2.28) to \( c\beta_3 n^{-1/2} \).

Note however that Nagaev’s bound (2.28) makes sense only when \( x = x_n \to \infty \) as \( n \to \infty \). For \( x \) in compact sets it is not precise, since the remainder term \( O(n^{-1/2}) \) is of the same order as the main term \( 2\Phi \left( \frac{x}{\sigma \sqrt{n}} \right) - 1 \).

In fact, for any fixed \( x \geq 0 \) the right asymptotic of the probability \( \mathbb{P}(\tau_x > n) \) is not given by (2.28) but by (2.22). Borovkov [8, Theorem 6] obtained precise asymptotics of the above probabilities under the stronger condition that \( X_1 \) has exponential moment and under additional assumption that the distribution of \( X_1 \) has an absolutely continuous component. The results of Borovkov are stated in terms of some infinite series, which makes the comparison with ours tedious. For instance, we could not deduce (2.27) from [8].

Note that the uniformity in \( t \) and \( x \) is crucial for establishing our main theorems in Section 1.

The proofs of Theorems 2.7, 2.8 and Corollary 2.9 will be given in Appendix 7.
3. Conditioned concentration bounds near the boundary

The goal of this section is to establish Theorems 1.1 and 1.2.

3.1. Formulation of the result. In the proof of Theorems 1.1 and 1.2, we need a refined version of the conditioned local limit theorem formulated below, which is stated as upper and lower bounds.

**Theorem 3.1.** Assume A1 and A2. Then, there exist constants c > 0 and ε0 > 0 such that for any ε ∈ (0, ε0), one can find a constant cε > 0 such that for any sequence of positive numbers (αn)n≥1 satisfying \( \lim_{n \to \infty} \alpha_n = 0 \), the following holds uniformly in \( x \in [0, \alpha_n \sqrt{n}] \) and \( n \geq 1 \):

1. For any integrable functions \( f, g : \mathbb{R} \to \mathbb{R} \) satisfying \( f \leq_{\varepsilon} g \),

\[
\mathbb{E}(f(x + S_n); \tau_x > n) \leq (1 + c \varepsilon) \frac{2V(x)}{\sqrt{2\pi \sigma^2 n}} \int_{\mathbb{R}_+} g(t) \phi^+ \left( \frac{t}{\sigma \sqrt{n}} \right) dt + c \varepsilon^{1/4} \frac{V(x)}{n} \int_{\mathbb{R}} g(t) \phi \left( \frac{t}{\varepsilon^{1/2} \sigma \sqrt{n}} \right) dt + c \varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \|g\|_1. \tag{3.1}
\]

2. For any integrable functions \( f, g, h : \mathbb{R} \to \mathbb{R} \) satisfying \( h \leq_{\varepsilon} f \leq_{\varepsilon} g \),

\[
\mathbb{E}(f(x + S_n); \tau_x > n) \geq \frac{2V(x)}{\sqrt{2\pi \sigma^2 n}} \int_{\mathbb{R}_+} h(t) \phi^+ \left( \frac{t}{\sigma \sqrt{n}} \right) dt - c \varepsilon^{1/4} \frac{V(x)}{n} \int_{\mathbb{R}} g(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) + \phi^+ \left( \frac{t}{\sigma \sqrt{n}} \right) dt - c \varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \|g\|_1. \tag{3.2}
\]

Note that the bounds in Theorem 3.1 become effective when the support of the target function \( f \) moves to infinity with an appropriate rate. As a consequence of this theorem, in Section 3.6 we shall prove Theorems 1.1 and 1.2.

3.2. Duality identities. In this section we establish two duality identities for the random walk \( x + S_n \) jointly with the exit time \( \tau_x \). These identities are simple consequence of the fact that the Lebesgue measure is translation invariant. Recall that \( S_0 = 0 \) and \( S_n = \sum_{i=1}^{n} X_i \) for \( n \geq 1 \), and its dual random walk is given by \( S^*_j = S_{n-j} - S_n \) for \( 0 \leq j \leq n \).

**Lemma 3.2.** For any \( n \geq 1 \) and any bounded measurable functions \( g, h : \mathbb{R} \to \mathbb{R} \), we have

\[
\int_{\mathbb{R}_+} h(x)\mathbb{E}g(x + S_n) \mathbb{I}_{\{\tau_x > n\}} dx = \int_{\mathbb{R}_+} g(y)\mathbb{E}h(y + S^*_n) \mathbb{I}_{\{\tau^*_x > n\}} dy \tag{3.3}
\]

and

\[
\int_{\mathbb{R}} h(x)\mathbb{E}g(x + S_n) \mathbb{I}_{\{\tau_x < n\}} dx = \int_{\mathbb{R}} g(y)\mathbb{E}h(y + S^*_n) \mathbb{I}_{\{\tau^*_x \leq n\}} dy. \tag{3.4}
\]
Proof. By a change of variable $x + S_n = y$, we get
\[
\int_{\mathbb{R}_+} h(x) \mathbb{E} g(x + S_n) \mathbb{1}_{\{r_n > n\}} dx
\]
\[
= \int_{\mathbb{R}} h(x) \mathbb{E} g(x + S_n) \mathbb{1}_{\{x \geq 0, x + S_0 \geq 0, \ldots, x + S_{n-1} \geq 0, x + S_n \geq 0\}} dx
\]
\[
= \int_{\mathbb{R}} g(y) \mathbb{E} h(y - S_n) \mathbb{1}_{\{y - S_n \geq 0, y - S_n + S_0 \geq 0, \ldots, y - S_n + S_{n-1} \geq 0, y \geq 0\}} dy
\]
\[
= \int_{\mathbb{R}} g(y) \mathbb{E} h(y + S_n^*) \mathbb{1}_{\{y + S_n^* \geq 0, y + S_n^* + S_0 \geq 0, \ldots, y + S_n^* + S_{n-1} \geq 0, y \geq 0\}} dy
\]
\[
= \int_{\mathbb{R}_+} g(y) \mathbb{E} h(y + S_n^*) \mathbb{1}_{\{r_n^* > n\}} dy,
\]
which concludes the proof of (3.3).

Using again the change of variable $x + S_n = y$ gives
\[
\int_{\mathbb{R}} h(x) \mathbb{E} g(x + S_n) dx = \int_{\mathbb{R}} g(y) \mathbb{E} h(y + S_n^*) dy. \quad (3.5)
\]
Taking the difference between (3.5) and (3.3), we get (3.4). \qed

3.3. Auxiliary statements. We state several auxiliary statements which will be used in the proofs of the main results. Recall that $\phi_v(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}}$, $x \in \mathbb{R}$ is the normal density of mean 0 and variance $v$ and that $\phi^+_v(x) = \frac{v}{\pi} e^{-\frac{x^2}{2v}} \mathbb{1}_{\{x \geq 0\}}, x \in \mathbb{R}$ is the Rayleigh density with scale parameter $\sqrt{v}$. It holds that $\phi_1(x) = \phi(x)$ and $\phi^+_1(x) = \phi^+(x), x \in \mathbb{R}$. The following lemma shows that when $v$ is small, the convolution $\phi_v * \phi^+_{1-v}$ behaves like the Rayleigh density $\phi^+.

Lemma 3.3. For any $v \in (0, 1/2]$ and $x \geq 0$, it holds
\[
\sqrt{1 - v} \phi^+(x) \leq \phi_v * \phi^+_{1-v}(x) \leq \sqrt{1 - v} \phi^+(x) + \sqrt{v} e^{-\frac{x^2}{2v}}. \quad (3.6)
\]
Proof. By definition, we have that for any $x \geq 0$,
\[
\phi_v * \phi^+_{1-v}(x) = \int_{\mathbb{R}} \phi_v(z) \phi^+_{1-v}(x - z) dz
\]
\[
= \frac{1}{\sqrt{1 - v}} \int_{\mathbb{R}} \frac{x - z}{\sqrt{2\pi v(1 - v)}} e^{-\frac{(x-z)^2}{2\pi v(1 - v)}} \mathbb{1}_{\{x - z \geq 0\}} dz.
\]
Since
\[
\frac{z^2}{2v} + \frac{(x-z)^2}{2(1-v)} = \frac{x^2}{2} + \frac{(z-xv)^2}{2v(1-v)},
\]
we get
\[
\phi_v * \phi_{1-v}^+(x) = \frac{1}{\sqrt{1-v}} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} \frac{x-z}{\sqrt{2\pi v(1-v)}} e^{-\frac{(x-z)^2}{2v(1-v)}} 1_{\{x-z \geq 0\}} dz
\]
\[
=: I_1 + I_2,
\]
where
\[
I_1 = \frac{1}{\sqrt{1-v}} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} \frac{x-z}{\sqrt{2\pi v(1-v)}} e^{-\frac{(x-z)^2}{2v(1-v)}} 1_{\{x-z \geq 0\}} dz = \sqrt{1-v} e^{-\frac{x^2}{2}},
\]
\[
I_2 = \frac{1}{\sqrt{1-v}} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} \frac{z-x}{\sqrt{2\pi v(1-v)}} e^{-\frac{(z-x)^2}{2v(1-v)}} 1_{\{x-z < 0\}} dz.
\]
Hence the first inequality in (3.6) holds since \(I_2 \geq 0\).

To prove the second inequality in (3.6), it remains to give an upper bound for \(I_2\). By a change of variable, we have
\[
I_2 = \sqrt{\frac{v}{2\pi}} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} \frac{w-x}{\sqrt{1-v}} e^{-\frac{w^2}{2}} dw \leq \sqrt{\frac{v}{2\pi}} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} w e^{-\frac{w^2}{2}} dw = \sqrt{\frac{v}{2\pi}} e^{-\frac{x^2}{2}},
\]
which finishes the proof of the second inequality in (3.6).

The following Fuk-Nagaev inequality can be found in [22, 38].

Lemma 3.4. Let \(\mathbb{E}X_1 = 0\) and \(\mathbb{E}(X_1^2) = \sigma^2 < \infty\). Then, for any \(u, v > 0\),
\[
\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| > u\right) \leq 2 \exp\left[\frac{u}{\sigma} \left(1 + \log \frac{n}{uv}\right)\right] + n \mathbb{P}(|X_1| > v).
\]

Let \((B_t)_{t \geq 0}\) be a standard Brownian motion on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For any \(x \geq 0\), define the exit time
\[
\tau^\text{bm}_x = \inf\{t \geq 0 : x + \sigma B_t < 0\}.
\]
The following well known formulas are due to Levy [34] (Theorem 42.1, pp.194-195).

Lemma 3.5. For any \(x \geq 0\), \(n \geq 1\) and \(0 < a < b \leq \infty\),
\[
\mathbb{P}\left(x + \sigma B_n \in [a, b], \tau^\text{bm}_x > n\right) = \frac{1}{\sigma \sqrt{n}} \int_a^b \psi\left(\frac{s}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}}\right) ds. \quad (3.7)
\]
In particular, by taking \(a = 0\) and \(b = \infty\), for any \(x \geq 0\) and \(n \geq 1\),
\[
\mathbb{P}(\tau_x^\text{bm} > n) = \mathbb{P}\left(\sigma \inf_{0 \leq u \leq n} B_u \geq -x\right) = \frac{2}{\sigma \sqrt{2\pi n}} \int_0^x e^{-\frac{s^2}{2\sigma^2 n}} ds. \quad (3.8)
\]

We need the following functional central limit theorem due to Sakhanenko [39], which allows us to couple out the random walk with the Brownian motion.
Lemma 3.6. Assume $A_2$ with some $\delta > 0$. Then there exists a construction of the random walk $(S_n)_{n \geq 0}$ on the initial probability space together with a continuous time Brownian motion $(B_t)_{t \geq 0}$ such that for any $\gamma \in (0, \frac{\delta}{2(2+\delta)})$ and $n \geq 1$,

$$\mathbb{P} \left( \sup_{0 \leq t \leq 1} |S_{nt} - \sigma B_{nt}| > n^{1/2-\gamma} \right) \leq \frac{c_\gamma}{n^{\delta/2-(2+\delta)\gamma}},$$

where $c_\gamma$ is a constant depending only on $\gamma$.

Note that $\delta$ can be greater than 1 in Lemma 3.6.

3.4. Proof of the upper bound. The goal of this section is to prove the upper bound (3.1) of Theorem 3.1.

Let $\varepsilon > 0$ be a sufficiently small constant. With $\delta = \sqrt{\varepsilon}$, set $m = \lfloor \delta n \rfloor$ and $k = n - m$. Note that for $n$ such that $n > n_0(\varepsilon) := \frac{4}{\varepsilon}$, we have $\frac{\delta^{1/2}}{1-\delta} \leq \frac{m}{n} \leq \frac{\delta}{1-\delta}$. It suffices to prove (3.1) only for sufficiently large $n > n_0(\varepsilon)$, where $n_0(\varepsilon)$ depends on $\varepsilon$; otherwise the bound becomes trivial. We start by using the Markov property to get that for any starting point $x \in \mathbb{R}_+$,

$$I_n(x) := \mathbb{E} (f(x + S_n); \tau_x > n)$$

$$= \int_{\mathbb{R}_+} \mathbb{E} (f(t + S_m); \tau_t > m) \mathbb{P} (x + S_k \in dt, \tau_x > k)$$

$$\leq \int_{\mathbb{R}_+} \mathbb{E} f(t + S_m) \mathbb{P} (x + S_k \in dt, \tau_x > k). \quad (3.9)$$

By the local limit theorem (Theorem 2.5), for any integrable function $g : \mathbb{R} \mapsto \mathbb{R}_+$ satisfying $g \geq f$, there exist constants $c, c_\varepsilon > 0$ such that for any $t \in \mathbb{R}$,

$$\mathbb{E} f(t + S_m) \leq (1 + c_\varepsilon) \int_{\mathbb{R}} g(s) \frac{1}{\sigma \sqrt{m}} \phi \left( \frac{t-s}{\sigma \sqrt{m}} \right) ds + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|g\|_1. \quad (3.10)$$

From (3.10) and the bound (2.23), we get that for any $x \in \mathbb{R}_+$,

$$I_n(x) \leq (1 + c_\varepsilon) J_n(x) + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|g\|_1 \mathbb{P} (\tau_x > k)$$

$$\leq (1 + c_\varepsilon) J_n(x) + c_\varepsilon \frac{V(x)}{n^{1+\varepsilon}} \|g\|_1, \quad (3.11)$$

where for brevity we set

$$J_n(x) := \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}} g(s) \frac{1}{\sigma \sqrt{m}} \phi \left( \frac{t-s}{\sigma \sqrt{m}} \right) ds \right] \mathbb{P} (x + S_k \in dt, \tau_x > k). \quad (3.12)$$
By a change of variable, it follows that
\[ J_n(x) = \int_{R_+} \left[ \int_R g(s) \frac{1}{\sigma \sqrt{m}} \phi \left( \frac{t \sigma \sqrt{k} - s}{\sigma \sqrt{m}} \right) ds \right] \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} \in dt, \tau_x > k \right) \]
\[ = \int_{R_+} \varphi_n(t) \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} \in dt, \tau_x > k \right), \]
where
\[ \varphi_n(t) := \int_R g(\sigma \sqrt{k} s) \frac{1}{\sqrt{m/k}} \phi \left( \frac{t - s}{\sqrt{m/k}} \right) ds. \quad (3.13) \]

Using integration by parts and the conditioned integral limit theorem (Corollary 2.9), we claim that uniformly in \( x \in [0, \alpha_n \sqrt{n}] \),
\[ \left| J_n(x) - \frac{2V(x)}{\sigma \sqrt{2\pi k}} \int_{R_+} \varphi_n(t) \phi^+(t) dt \right| \leq c_\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V(x)}{n} \|g\|_1. \quad (3.14) \]

Indeed, since the function \( t \mapsto \varphi_n(t) \) is differentiable on \( \mathbb{R} \) and vanishes as \( t \to \pm \infty \), using integration by parts, we get that for any \( x \in \mathbb{R}_+ \),
\[ J_n(x) = \int_{R_+} \varphi_n'(t) \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} > t, \tau_x > k \right) dt. \quad (3.15) \]

Applying the conditioned integral limit theorem (Corollary 2.9) gives that uniformly in \( x \in [0, \alpha_n \sqrt{n}] \),
\[ \left| \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} > t, \tau_x > k \right) - \frac{2V(x)}{\sigma \sqrt{2\pi k}} (1 - \Phi^+(t)) \right| \leq c_\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V(x)}{n^{1/2}}, \quad (3.16) \]

which, together with (3.15), implies that uniformly in \( x \in [0, \alpha_n \sqrt{n}] \),
\[ \left| J_n(x) - \frac{2V(x)}{\sigma \sqrt{2\pi k}} \int_{R_+} \varphi_n'(t)(1 - \Phi^+(t)) dt \right| \leq c_\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V(x)}{n^{1/2}} \int_{R_+} |\varphi_n'(t)| dt. \quad (3.17) \]

From (3.13) and a change of variable, it is easy to see that
\[ \int_{R_+} |\varphi_n'(t)| dt \leq \int_{R_+} \left[ \int_R g(\sigma \sqrt{k} s) \left| \phi' \left( \frac{t - s}{\sqrt{m/k}} \right) \right| ds \right] \frac{dt}{\sqrt{m/k}} \frac{dt}{\sqrt{m/k}} \]
\[ = \int_{R_+} \left[ \int_R g(\sigma \sqrt{ms}) \left| \phi'(t - s) \right| ds \right] dt \leq \frac{c}{\sqrt{m}} \|g\|_1. \quad (3.18) \]

Using integration by parts, we have
\[ \int_{R_+} \varphi_n'(t)(1 - \Phi^+(t)) dt = \int_{R_+} \varphi_n(t) \phi^+(t) dt. \quad (3.19) \]
Putting together (3.17), (3.18) and (3.19), we obtain (3.14).

Combining (3.14) with (3.11), we get that uniformly in \( x \in [0, \alpha_n \sqrt{n}] \),

\[
I_n(x) \leq (1 + c\varepsilon) \frac{2V(x)}{\sigma \sqrt{2\pi k}} \int_{R^+} \varphi_n(t) \phi^+(t) dt + c\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \|g\|_1.
\]

(3.20)

Denote \( \delta_n = \frac{m}{n} \). By the definition of \( \varphi_n \) (see (3.13)), a change of variable and Fubini’s theorem, we derive that

\[
\int_{R^+} \varphi_n(t) \phi^+(t) dt
= \int_{R^+} \left[ \int_{R} g(\sigma \sqrt{k}s) \frac{1}{\sqrt{m/k}} \phi \left( \frac{s - t}{\sqrt{m/k}} \right) ds \right] \phi^+(t) dt
= \int_{R^+} \left[ \int_{R} g(\sigma \sqrt{n}s') \frac{1}{\sqrt{m/k}} \phi \left( \frac{s' - t'}{\sqrt{m/n}} \right) ds' \right] \phi^+ \left( \frac{t'}{\sqrt{k/n}} \right) \frac{dt'}{\sqrt{k/n}}
= \int_{R} g(\sigma \sqrt{n}s') \phi_{\delta_n} * \phi^+_{1-\delta_n} (s') ds'
= \frac{1}{\sigma \sqrt{n}} \int_{R} g(t) \phi_{\delta_n} * \phi^+_{1-\delta_n} \left( \frac{t}{\sigma \sqrt{n}} \right) dt
\leq \sqrt{k} \frac{1}{\sigma n} \int_{R} g(t) \phi^+ \left( \frac{t}{\sigma \sqrt{n}} \right) dt + \frac{\sqrt{m}}{\sigma n} \int_{R} g(t) e^{-\frac{t^2}{2\sigma^2 m}} dt,
\]

where for the last line we applied the upper bound in Lemma 3.3 with \( v = \delta_n \) and the fact that \( 1 - \delta_n = \frac{k}{n} \). Substituting this bound into (3.20), we get

\[
I_n(x) - (1 + c\varepsilon) \frac{2V(x)}{\sigma \sqrt{2\pi^2 n}} \int_{R} g(t) e^{-\frac{t^2}{2\sigma^2 m}} dt + c\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \|g\|_1,
\]

which concludes the proof of the upper bound (3.1).

3.5. **Proof of the lower bound.** In this section we establish the lower bound (3.2).

Let us keep the notation used in the proof of the upper bound (3.1). By the Markov property we have

\[
I_n(x) = \int_{R^+} \mathbb{E} f(t + S_m) \mathbb{P}(x + S_k \in dt, \tau_x > k)
- \int_{R^+} \mathbb{E} (f(t + S_m); \tau_t \leq m) \mathbb{P}(x + S_k \in dt, \tau_x > k)
=: I_{n,1}(x) - I_{n,2}(x).
\]

(3.21)
Lower bound of $I_{n,1}(x)$. Using the local limit theorem (2.13) leads to
\[
\mathbb{E}f(t + S_m) \geq \frac{1}{\sigma \sqrt{m}} \int_{\mathbb{R}} h(s) \phi \left( \frac{s - t}{\sigma \sqrt{m}} \right) ds - \frac{ce \epsilon}{\sigma \sqrt{m}} \int_{\mathbb{R}} f(s) \phi \left( \frac{s - t}{\sigma \sqrt{m}} \right) ds - \frac{c \epsilon}{n^{1/2+\epsilon}} \|f\|_1. \tag{3.22}
\]
Proceeding in the same way as in the estimate of $J_n(x)$ defined by (3.12), (using the lower bound in (3.6) instead of the upper one, so that the second term on the right hand side of (3.1) does not appear), one has, uniformly in $x \in [0, \alpha_n \sqrt{n}]$,
\[
\frac{1}{\sigma \sqrt{m}} \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}} h(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt - c \epsilon \left( \alpha_n + n^{-\epsilon} \right) \frac{V(x)}{n} \|h\|_1 \right] \mathbb{P}(x + S_k \in dt, \tau_x > k)
\]
and
\[
\frac{1}{\sigma \sqrt{m}} \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}} f(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt - c \epsilon \left( \alpha_n + n^{-\epsilon} \right) \frac{V(x)}{n} \|f\|_1 \right] \mathbb{P}(x + S_k \in dt, \tau_x > k).
\]
By (2.23), these bounds together with (3.22) yield that uniformly in $x \in [0, \alpha_n \sqrt{n}]$,
\[
I_{n,1}(x) \geq \frac{2V(x)}{\sqrt{2\pi\sigma^2 n}} \int_{\mathbb{R}} h(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt - c \epsilon \left( \alpha_n + n^{-\epsilon} \right) \frac{V(x)}{n} \|f\|_1. \tag{3.23}
\]

Upper bound of $I_{n,2}(x)$. Splitting the integral in this term into two parts according to whether the value of $t$ is less or larger than $\epsilon^{1/6} \sqrt{n}$, we have
\[
I_{n,2}(x) = \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}} f(u) \mathbb{P}(t + S_m \in du, \tau_t \leq m) \right] \mathbb{P}(x + S_k \in dt, \tau_x > k)
= K_1 + K_2, \tag{3.24}
\]
where
\[
K_1 = \int_{0}^{\epsilon^{1/6} \sqrt{n}} \left[ \int_{\mathbb{R}} f(u) \mathbb{P}(t + S_m \in du, \tau_t \leq m) \right] \mathbb{P}(x + S_k \in dt, \tau_x > k),
\]
\[
K_2 = \int_{\epsilon^{1/6} \sqrt{n}}^{\infty} \left[ \int_{\mathbb{R}} f(u) \mathbb{P}(t + S_m \in du, \tau_t \leq m) \right] \mathbb{P}(x + S_k \in dt, \tau_x > k).
\]
For $K_1$, we use the local limit theorem (Theorem 2.5) and the bound (2.23) of Theorem 2.7 to get

$$K_1 \leq \int_0^{\varepsilon^{1/6}/\sqrt{n}} \left[ \int_{\mathbb{R}} f(u) \mathbb{P}(t + S_m \in du) \right] \mathbb{P}(x + S_k \in dt, \tau_x > k)
\leq K_{11} + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|g\|_1 \mathbb{P}(\tau_x > k)
\leq K_{11} + \frac{c_\varepsilon}{n^{1+\varepsilon}} \|g\|_1,$$

where

$$K_{11} = \int_0^{\varepsilon^{1/6}/\sqrt{n}} \left[ \int_{\mathbb{R}} g(u) \frac{1}{\sigma \sqrt{m}} \phi \left( \frac{u - t}{\sigma \sqrt{m}} \right) du \right] \mathbb{P}(x + S_k \in dt, \tau_x > k).$$

By Fubini’s theorem, we have $K_{11} = \int_{\mathbb{R}} g(u) J(u) du$, where

$$J(u) = \int_{0}^{\varepsilon^{1/6}/\sqrt{n}} F_u(t) \mathbb{P}(x + S_k \in dt; \tau_x > k), \quad u \in \mathbb{R},$$

and

$$F_u(t) = \frac{1}{\sigma \sqrt{m}} \phi \left( \frac{u - t}{\sigma \sqrt{m}} \right), \quad t \in [0, \varepsilon^{1/6}/\sqrt{n}].$$

Then

$$J(u) = \int_{0}^{\infty} F_u(t) \mathbb{P}(x + S_k \in dt, x + S_k \leq \varepsilon^{1/6}/\sqrt{n}, \tau_x > k)
= \int_{0}^{\infty} F_u'(t) \mathbb{P}(x + S_k \in [0, \varepsilon^{1/6}/\sqrt{n}], x + S_k > t, \tau_x > k) dt
= \int_{0}^{\varepsilon^{1/6}/\sqrt{n}} F_u'(t) \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} \in \left[ \frac{t}{\sigma \sqrt{k}}, \frac{\varepsilon^{1/6}/\sqrt{n}}{\sigma \sqrt{k}} \right], \tau_x > k \right) dt. \quad (3.26)$$

Applying the conditional integral limit theorem (Corollary 2.9), we get that uniformly in $t \in \mathbb{R}_+$ and $x \in [0, \alpha_n \sqrt{n}]$,

$$\left| \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} \in \left[ \frac{t}{\sigma \sqrt{k}}, \frac{\varepsilon^{1/6}/\sqrt{n}}{\sigma \sqrt{k}} \right], \tau_x > k \right) - \frac{2V(x)}{\sigma \sqrt{2\pi k}} \left[ \Phi^+ \left( \frac{\varepsilon^{1/6}/\sqrt{n}}{\sigma \sqrt{k}} \right) - \Phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) \right] \right| \leq c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n^{1/2}}.$$

Hence, using the fact that $F_u(\varepsilon^{1/6}/\sqrt{n}) - F_u(0) \leq \frac{1}{\sqrt{m}} \leq \frac{c_\varepsilon}{\sqrt{n}}$ gives

$$J(u) \leq \frac{2V(x)}{\sigma \sqrt{2\pi k}} \int_{0}^{\varepsilon^{1/6}/\sqrt{n}} F_u'(t) \left[ \Phi^+ \left( \frac{\varepsilon^{1/6}/\sqrt{n}}{\sigma \sqrt{k}} \right) - \Phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) \right] dt
+ c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n}. \quad (3.27)$$
Using integration by parts and the fact that \( F_u(0) \geq 0 \), we have

\[
\int_0^{\varepsilon^{1/6}\sqrt{n}} F_u'(t) \left[ \Phi^+ \left( \frac{\varepsilon^{1/6}}{\sigma \sqrt{k}} \right) - \Phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) \right] dt = -F_u(0) \Phi^+ \left( \frac{\varepsilon^{1/6}}{\sigma \sqrt{k}} \right) + \frac{1}{\sigma \sqrt{k}} \int_0^{\varepsilon^{1/6}\sqrt{n}} F_u(t) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt
\]

\[
\leq \frac{1}{\sigma \sqrt{k}} \int_0^{\varepsilon^{1/6}\sqrt{n}} F_u(t) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt =: H(u). \tag{3.28}
\]

Elementary calculations give

\[
H(u) = \frac{1}{\sigma \sqrt{k}} \int_0^{\varepsilon^{1/6}\sqrt{n}} \frac{1}{\sigma \sqrt{2\pi m}} e^{-\frac{(u-t)^2}{2\sigma^2 m}} t e^{-\frac{t^2}{2\sigma^2 \varepsilon}} dt
\]

\[
= \frac{1}{\sigma \sqrt{k}} e^{-\frac{u^2}{2\sigma^2 n}} \int_0^{\varepsilon^{1/6}\sqrt{n}} \frac{1}{\sigma \sqrt{2\pi m}} e^{-\frac{n}{2\sigma^2 \varepsilon} (t - \frac{u}{\sigma \sqrt{m}})^2} dt.
\]

By a change of variable \( t \sqrt{\frac{n}{\sigma^2 m k}} = z \), it follows that

\[
H(u) = \frac{1}{\sigma \sqrt{k}} e^{-\frac{u^2}{2\sigma^2 n}} \int_0^{\varepsilon^{1/6}\sqrt{n} / \sqrt{\sigma^2 m k}} \frac{z}{\sigma \sqrt{2\pi n}} e^{-\frac{\sqrt{\sigma^2 m k} z}{\sigma \sqrt{n}} \left( \sqrt{\sigma^2 m k} - \frac{u}{\sigma \sqrt{m}} \right)^2} \, dz
\]

\[
\leq \frac{1}{\sigma \sqrt{k}} e^{-\frac{u^2}{2\sigma^2 n}} \int_0^{\varepsilon^{1/6}\sqrt{n} / \sqrt{\sigma^2 m k}} \frac{z}{\sigma \sqrt{2\pi n}} \left( \frac{z}{\sigma \sqrt{n}} \right) ^{\frac{\sigma^2 m k}{\sigma \sqrt{n}} - \frac{u}{\sigma \sqrt{m}}} \, dz
\]

\[
\leq \frac{c \varepsilon^{1/6}}{\sqrt{k} \sqrt{m k}} \frac{n}{\sqrt{\sigma m k}} e^{-\frac{u^2}{2\sigma^2 n}} \leq \frac{c \varepsilon^{1/6}}{\sqrt{k} \sqrt{\sigma m k}} \phi \left( \frac{u}{\sigma \sqrt{n}} \right),
\]

which, together with (3.27) and (3.28), implies that

\[
J(u) \leq c \varepsilon^{1/12} \frac{V(x)}{n} \phi \left( \frac{u}{\sigma \sqrt{n}} \right) + c \varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n}.
\]

Since \( K_{11} = \int_{\mathbb{R}} g(u)J(u) du \), we obtain

\[
K_{11} \leq c \varepsilon^{1/12} \frac{V(x)}{n} \int_{\mathbb{R}} g(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt + c \varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \| g \|_1. \tag{3.29}
\]

We proceed to give an upper bound for \( K_2 \), which can be rewritten as

\[
K_2 = \int_{\mathbb{R}} L(t) \mathbb{P} \left( x + S_k \in dt, \tau_x > k \right), \tag{3.30}
\]

where, for \( t \in \mathbb{R} \),

\[
L(t) := \mathbb{1}_{\{t > \varepsilon^{1/6} \sqrt{m}\}} \mathbb{E}(f(t + S_m); \tau_t \leq m). \tag{3.31}
\]

The function \( t \mapsto L(t) \) is integrable on \( \mathbb{R} \) since \( f \) is integrable on \( \mathbb{R} \). Denote

\[
M(t) := \mathbb{1}_{\{t + \varepsilon > \varepsilon^{1/6} \sqrt{m}\}} \mathbb{E}(g(t + S_m); \tau_{t-\varepsilon} \leq m), \quad t \in \mathbb{R}. \tag{3.32}
\]
Then $L \leq_\varepsilon M$ since $f \leq_\varepsilon g$. Using the upper bound (3.1) of Theorem 3.1 and the fact that $\|M\|_1 \leq \|g\|_1$, we obtain that uniformly in $x \in [0, \alpha_n \sqrt{n}]$,

$$K_2 \leq (1 + c_\varepsilon) \frac{2V(x)}{\sqrt{2\pi \alpha^2 k}} \int_{\mathbb{R}_+} M(t) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt$$

$$+ c_\varepsilon^{1/4} \frac{V(x)}{n} \int_{\mathbb{R}} M(t) e^{-\frac{t^2}{2 \sigma^2 k}} dt + c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \|g\|_1.$$  (3.33)

For the first term, in view of (3.32), we use the duality formula (Lemma 3.2) and the fact that the function $\phi^+$ is bounded on $\mathbb{R}$ to derive that

$$\int_{\mathbb{R}_+} M(t) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt$$

$$= \int_{\mathbb{R}_+} \mathbb{E}(g(t + S_m); \tau_{t-} \leq m) \mathbb{I}_{\{t + \varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \phi^+ \left( \frac{t + \varepsilon}{\sigma \sqrt{k}} \right) dt$$

$$= \int_{\mathbb{R}_+} \mathbb{E}(g(t + \varepsilon + S_m); \tau_{t} \leq m) \mathbb{I}_{\{t + 2\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \phi^+ \left( \frac{t + \varepsilon}{\sigma \sqrt{k}} \right) dt$$

$$= \int_{\mathbb{R}_+} \mathbb{E} g(t + \varepsilon) \mathbb{E} \left[ \phi^+ \left( \frac{t + S_{m} + \varepsilon}{\sigma \sqrt{k}} \right); t + S_{m} + 2\varepsilon > \varepsilon^{1/6} \sqrt{n}, \tau_{t} \leq m \right] dt$$

$$\leq c \int_{\mathbb{R}_+} g(t + \varepsilon) \mathbb{P} \left( t + S_{m} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau_{t} \leq m \right) dt =: c(J_1 + J_2),$$

where

$$J_1 = \int_0^{\varepsilon^{1/4} \sqrt{n}} g(t + \varepsilon) \mathbb{P} \left( t + S_{m} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau_{t} \leq m \right) dt,$$  (3.34)

$$J_2 = \int_{\varepsilon^{1/4} \sqrt{n}}^{\infty} g(t + \varepsilon) \mathbb{P} \left( t + S_{m} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau_{t} \leq m \right) dt.$$  (3.35)

For $J_1$, observe that on the event $\{\tau_{t} \leq m\}$, the inequality $t + S_{m} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}$ implies that there exists $0 \leq j \leq m$ such that $S_{m} - S_{j} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}$. Since $S_{j} = S_{m-j} - S_{m}$, it follows that $-S_{m-j} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}$ for some $0 \leq j \leq m - 1$ and hence

$$\mathbb{P} \left( t + S_{m} > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau_{t} \leq m \right) \leq \mathbb{P} \left( \max_{1 \leq j \leq m} |S_{j}| \geq \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau_{t} \leq m \right).$$  (3.36)

Noting that $m = \lfloor \varepsilon^{1/2} n \rfloor$, we use the Fuk-Nagaev inequality (Lemma 3.4) with $n$ replaced by $m$ and $u = v = \frac{1}{2} \varepsilon^{1/6} \sqrt{n}$, and condition A2 to get

$$\mathbb{P} \left( \max_{1 \leq j \leq m} |S_{j}| \geq \frac{1}{2} \varepsilon^{1/6} \sqrt{n} \right) \leq c_\varepsilon^{1/6} + m \mathbb{P} (|X_1| > v) \leq c_\varepsilon^{1/6} + \frac{c_\varepsilon}{n^{\varepsilon}},$$  (3.37)
which, together with (3.34) and (3.36), implies that

\[ J_1 \leq c \varepsilon^{1/6} \int_0^{\varepsilon^{1/4}/\sqrt{n}} g(t + \varepsilon) dt + \frac{c \varepsilon}{n^{2\varepsilon}} \|g\|_1. \]  

(3.38)

For \( J_2 \), we use (3.36), (3.37) and Hölder’s inequality to get

\[ \mathbb{P} \left( t + S_m^* > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau^*_t \leq m \right) \]
\[ \leq \mathbb{P}^{1/2} \left( \max_{1 \leq j \leq m} |S_j| \geq \frac{1}{2} \varepsilon^{1/6} \sqrt{n} \right) \mathbb{P}^{1/2} (\tau^*_t \leq m) \]
\[ \leq \left( c \varepsilon^{1/12} + \frac{c \varepsilon}{n^{\varepsilon}} \right) \mathbb{P}^{1/2} \left( t + \min_{1 \leq j \leq m} S_j < 0 \right). \]  

(3.39)

Denote

\[ A_m = \left\{ \sup_{0 \leq t \leq 1} |S_t| - \sigma B_t \mid \leq m^{1/2 - 2\varepsilon} \right\} \]  

(3.40)

and by \( A_m^c \) its complement. By the functional central limit theorem (Lemma 3.6) and the identity (3.8) of Lemma 3.5, we obtain that for any \( t \geq \varepsilon^{1/4} / \sqrt{n} \),

\[ \mathbb{P} \left( t + \min_{1 \leq j \leq m} S_j < 0 \right) \leq \mathbb{P} \left( t + \min_{1 \leq j \leq m} S_j < 0, A_m \right) + \frac{c \varepsilon}{n^{2\varepsilon}} \]
\[ \leq \mathbb{P} \left( t - m^{1/2 - 2\varepsilon} + \sigma \min_{1 \leq j \leq m} B_j < 0 \right) + \frac{c \varepsilon}{n^{2\varepsilon}} \]
\[ = \frac{2}{\sigma \sqrt{2\pi m}} \int_{t-m^{1/2 - 2\varepsilon}}^{\infty} e^{-\frac{s^2}{2\sigma^2 m}} ds + \frac{c \varepsilon}{n^{2\varepsilon}} \]
\[ \leq c e^{-\frac{t^2}{4\sigma^2 \sqrt{m}}} + \frac{c \varepsilon}{n^{2\varepsilon}}, \]

where in the last line we used the inequality \( \int_a^{\infty} e^{-\frac{s^2}{2}} ds \leq \frac{1}{a} e^{-\frac{a^2}{2}} \) for \( a > 0 \),

and the fact that \( \sqrt{m} / t = \sqrt{[\varepsilon n]} / t \leq 1 \) for \( t \geq \varepsilon^{1/4} / \sqrt{n} \). Hence,

\[ \mathbb{P}^{1/2} \left( t + \min_{1 \leq j \leq m} S_j < 0 \right) \leq c e^{-\frac{t^2}{16\sigma^2 \sqrt{\varepsilon n}}} + \frac{c \varepsilon}{n^{2\varepsilon}}. \]

This, together with (3.35) and (3.39), implies that

\[ J_2 \leq c \varepsilon^{1/12} \int_{\varepsilon^{1/4} / \sqrt{n}}^{\infty} g(t + \varepsilon)e^{-\frac{t^2}{16\sigma^2 \sqrt{\varepsilon n}}} dt + \frac{c \varepsilon}{n^{2\varepsilon}} \|g\|_1. \]  

(3.41)
Putting (3.38) and (3.41) together and using the fact that \( e^{t^2/16\sigma^2\sqrt{n}} \leq c \) for any \( t \in [0, \varepsilon^{1/4}/\sqrt{n}] \), we have

\[
\int_{\mathbb{R}_+} M(t) \phi^+ \left( \frac{t}{\sigma \sqrt{n}} \right) dt \\
\leq c\varepsilon^{1/4} \sqrt{n} \left[ \int_{\varepsilon^{1/4}/\sqrt{n}}^{\varepsilon^{1/4}/\sqrt{n}} g(t) dt + \int_{\varepsilon^{1/4}/\sqrt{n}}^{\infty} g(t + \varepsilon) e^{-t^2/16\sigma^2\sqrt{n}} dt \right] + \frac{c\varepsilon}{n^\varepsilon} \| g \|_1
\]

\[
\leq c\varepsilon^{1/12} \int_{\mathbb{R}_+} g(t + \varepsilon) e^{-t^2/16\sigma^2\sqrt{n}} dt + \frac{c\varepsilon}{n^\varepsilon} \| g \|_1.
\]

For the second term on the right hand side of (3.33), following the same proof of (3.42) and using the fact that \( e^{-t^2/16\sigma^2\sqrt{n}} \leq 1 \), one has

\[
\int_{\mathbb{R}} M(t) e^{-t^2/2\sigma^2\varepsilon^2/\sqrt{n}} dt \leq c\varepsilon^{1/12} \int_{\mathbb{R}_+} g(t + \varepsilon) e^{-t^2/16\sigma^2\sqrt{n}} dt + \frac{c\varepsilon}{n^\varepsilon} \| g \|_1.
\]

Implementing (3.42) and (3.43) into (3.33) gives

\[
K_2 \leq c\varepsilon^{1/12} \int_{\mathbb{R}_+} g(t + \varepsilon) e^{-t^2/16\sigma^2\sqrt{n}} dt + \frac{c\varepsilon}{n^\varepsilon} \| g \|_1.
\]

From (3.24), (3.29) and (3.44), and using the fact that \( e^{-t^2/16\sigma^2\sqrt{n}} \leq c\phi(t/\sigma\sqrt{n}) \), we obtain

\[
I_{n,2}(x) \leq c\varepsilon^{1/12} \frac{V(x)}{n} \int_{\mathbb{R}} [g(t) + g(t + \varepsilon)] \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt \\
+ c\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \| g \|_1
\]

\[
\leq c\varepsilon^{1/12} \frac{V(x)}{n} \int_{\mathbb{R}} g(t) \phi \left( \frac{t}{\sigma \sqrt{n}} \right) dt + c\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V(x)}{n} \| g \|_1,
\]

where in the last line we used the Lipschitz continuity of \( \phi \). Combining this with (3.21) and (3.23), and using the fact that \( f \leq g \), we conclude the proof of the lower bound (3.2).

### 3.6. Proof of Theorems 1.1 and 1.2.

In this section we show how to deduce Theorems 1.1 and 1.2 from Theorem 3.1.

**Proof of Theorem 1.1.** We first prove that (1.5) holds uniformly in \( x \in [0, n^{1/2-\varepsilon}] \) and \( y \in [\eta\sqrt{n}, \sigma\sqrt{qn\log n}] \). Without loss of generality, we assume that the target function \( f \) is non-negative. Since \( f \equiv_{c} \overline{f}_{\delta, \varepsilon} \) with \( \overline{f}_{\delta, \varepsilon} \) defined...
by (2.4), applying the upper bound (3.1) of Theorem 3.1 with \( g = T_{\delta, \varepsilon} \) and \( \alpha_n = n^{-\varepsilon} \), we derive that for any \( x \in [0, n^{1/2-\varepsilon}] \) and \( y \in \mathbb{R}_+ \),

\[
\mathbb{E}(f(x + S_n - y); \tau_x > n) \leq (1 + c\varepsilon) \frac{2V(x)}{\sqrt{2\pi\sigma^2 n}} J_n,
\]

where

\[
J_n = \int_{-\varepsilon}^{\infty} T_{\delta, \varepsilon}(t) \left[ \phi^+ \left( \frac{t + y}{\sigma \sqrt{n}} \right) + \phi^+ \left( \frac{t + y}{\varepsilon^{1/4} \sigma \sqrt{n}} \right) + \frac{c\varepsilon}{n^{\varepsilon}} \right] dt.
\]

Since the functions \( \phi^+ \) and \( \phi \) are Lipschitz continuous on \( \mathbb{R} \), there exists a constant \( c > 0 \) such that for any \( t \geq -\varepsilon \) and \( y \in \mathbb{R} \),

\[
\left| \phi^+ \left( \frac{t + y}{\sigma \sqrt{n}} \right) - \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) \right| \leq c \frac{|t|}{\sqrt{n}},
\]

\[
\left| \phi \left( \frac{t + y}{\varepsilon^{1/4} \sigma \sqrt{n}} \right) - \phi \left( \frac{y}{\varepsilon^{1/4} \sigma \sqrt{n}} \right) \right| \leq c \frac{|t|}{\varepsilon^{1/4} \sqrt{n}},
\]

which implies that

\[
J_n \leq \left[ \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) + \phi \left( \frac{y}{\varepsilon^{1/4} \sigma \sqrt{n}} \right) + \frac{c\varepsilon}{n^{\varepsilon}} \right] \int_{-\varepsilon}^{\infty} T_{\delta, \varepsilon}(t) dt
\]

\[
+ \frac{c}{\varepsilon^{1/4} \sqrt{n}} \int_{-\varepsilon}^{\infty} T_{\delta, \varepsilon}(t)|t| dt.
\]

Note that for any fixed \( \eta > 0 \), there exists a constant \( c_\eta > 0 \) such that uniformly in \( y \in [\eta \sqrt{n}, \sigma \sqrt{qn \log n}] \),

\[
n^{-q/2} \leq c_\eta \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right), \quad \phi \left( \frac{y}{\varepsilon^{1/4} \sigma \sqrt{n}} \right) \leq c_\eta \exp \left\{ - \frac{\eta^2}{4\varepsilon \sigma^2} \right\} \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right).
\]

(3.45)

It follows that uniformly in \( y \in [\eta \sqrt{n}, \sigma \sqrt{qn \log n}] \),

\[
J_n \leq \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) \left( 1 + c_\eta \exp \left\{ - \frac{\eta^2}{4\varepsilon \sigma^2} \right\} + \frac{c_\eta}{n^{\varepsilon - q/2}} \right) \int_{-\varepsilon}^{\infty} T_{\delta, \varepsilon}(t) dt
\]

\[
+ \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) \frac{c_\eta}{\varepsilon^{1/4} n^{1/2 - q/2}} \int_{-\varepsilon}^{\infty} T_{\delta, \varepsilon}(t)|t| dt.
\]

Therefore, we get that uniformly in \( x \in [0, n^{1/2-\varepsilon}] \) and \( y \in [\eta \sqrt{n}, \sigma \sqrt{qn \log n}] \),

\[
\limsup_{n \to \infty} \frac{\mathbb{E}(f(x + S_n - y); \tau_x > n)}{\sqrt{2\pi\sigma^2 n} \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right)} \leq \left[ 1 + c_\eta \exp \left\{ - \frac{\eta^2}{4\varepsilon \sigma^2} \right\} \right] \int_{-\varepsilon}^{\infty} T_{\delta, \varepsilon}(t) dt.
\]

This proves the upper bound by taking first \( \varepsilon \to 0 \) and then \( \delta \to 0 \), and using Lemma 2.1. The proof of the lower bound can be carried out in the same way using (3.2).
In a similar way, one can also use Theorem 3.1 to prove that (1.5) holds uniformly in \( x \in [0, \alpha_n \sqrt{n}] \) and \( y \in [\eta \sqrt{n}, \eta^{-1} \sqrt{n}] \). The proof of Theorem 1.1 is complete. \( \square \)

We will show in Lemma 7.4 that the harmonic function \( V \) satisfies \( x \leq V(x) \leq c(1 + x) \) for any \( x \geq 0 \). To prove Theorem 1.2 we need the following refinement of this bound.

**Lemma 3.7.** Assume \( A2 \) for some \( \delta > 0 \). Then, for any \( \varepsilon \in (0, \frac{\delta}{2(2+\delta)}) \), there exists a constant \( c_\varepsilon > 0 \) such that for any \( x \geq 0 \) and \( k_0 \geq 1 \),

\[
x \leq V(x) \leq \left(1 + \frac{c_\varepsilon}{k_0}\right)x + c_\varepsilon k_0^{1/2 - \varepsilon}.
\]

**Proof of Theorem 1.2.** We first prove that (1.6) holds uniformly in \( x \in [0, n^{1/2 - \varepsilon}] \), \( y \in [\eta \sqrt{n}, \sigma \sqrt{q n \log n}] \) and \( \Delta \in [\Delta_0, n^{1/2 - \varepsilon}] \). Using the upper bound (3.1) with \( f = I_{[y, y + \Delta]} \), \( g = I_{[y - \varepsilon, y + \Delta + \varepsilon]} \) and \( \alpha_n = n^{-\varepsilon} \), we have, uniformly in \( x \in [0, n^{1/2 - \varepsilon}] \),

\[
P(x + S_n \in [0, \Delta] + y, \tau_x > n) \leq (1 + c\varepsilon) \frac{2V(x)}{\sqrt{2\pi \sigma^2 n}} J_n,
\]

where

\[
J_n = \int_{-\varepsilon}^{\Delta + \varepsilon} \left[ \phi^+ \left( \frac{t + y}{\sigma \sqrt{n}} \right) + \phi \left( \frac{t + y}{\varepsilon 1/4 \sigma \sqrt{n}} \right) + \frac{c_\varepsilon}{n^\varepsilon} \right] dt.
\]

Since the functions \( \phi^+ \) and \( \phi \) are Lipschitz continuous on \( \mathbb{R} \), there exists a constant \( c > 0 \) such that for any \( \Delta \in [\Delta_0, n^{1/2 - \varepsilon}] \), \( t \in [-\varepsilon, \Delta + \varepsilon] \) and \( y \in [\eta \sqrt{n}, \sigma \sqrt{q n \log n}] \),

\[
\left| \phi^+ \left( \frac{t + y}{\sigma \sqrt{n}} \right) - \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) \right| \leq c \frac{|t|}{\sqrt{n}} \leq \frac{c}{n^\varepsilon},
\]

\[
\left| \phi \left( \frac{t + y}{\varepsilon 1/4 \sigma \sqrt{n}} \right) - \phi \left( \frac{y}{\varepsilon 1/4 \sigma \sqrt{n}} \right) \right| \leq c \frac{|t|}{\varepsilon 1/4 \sqrt{n}} \leq \frac{c}{\varepsilon 1/4 n^\varepsilon}.
\]

This implies that there exists a constant \( c_\eta > 0 \) such that uniformly in \( y \in [\eta \sqrt{n}, \sigma \sqrt{q n \log n}] \) and \( \Delta \in [\Delta_0, n^{1/2 - \varepsilon}] \),

\[
J_n \leq (\Delta + 2\varepsilon) \left[ \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) + \phi \left( \frac{y}{\varepsilon 1/4 \sigma \sqrt{n}} \right) + \frac{c}{\varepsilon 1/4 n^\varepsilon} \right]
\]

\[
\leq \Delta (1 + c\varepsilon) \phi^+ \left( \frac{y}{\sigma \sqrt{n}} \right) \left( 1 + c_\eta \exp \left\{ -\frac{\eta^2}{4\sqrt{2}\sigma^2} \right\} + \frac{c_\eta}{\varepsilon 1/4 n^\varepsilon - q/2} \right),
\]

where in the last inequality we used (3.45) and the fact that \( \Delta_0 > 0 \) is a fixed constant. Therefore, we get that uniformly in \( x \in [0, n^{1/2 - \varepsilon}] \), \( y \in
Conditioned local limit theorems

\[ [\eta \sqrt{n}, \sigma \sqrt{qn \log n}] \text{ and } \Delta \in [\Delta_0, n^{1/2-\varepsilon}], \]

\[
\limsup_{n \to \infty} \frac{\mathbb{P}(x + S_n \in [0, \Delta] + y, \tau_x > n)}{\Delta \frac{2V(y)}{\sqrt{2\pi \sigma^2 n}} \phi \left( \frac{y}{\sigma \sqrt{n}} \right)} \leq 1 + c \eta \exp \left\{ - \frac{\eta^2}{4\sqrt{2\varepsilon \sigma^2}} \right\}.
\]

Since \( \varepsilon > 0 \) can be arbitrary small, this proves the upper bound. The proof of the lower bound can be carried out in the same way using (3.2).

Similarly, one can use Theorem 3.1 to prove that the asymptotic (1.6) holds uniformly in \( x \in [0, \alpha \sqrt{n}], y \in [\eta \sqrt{n}, \eta^{-1} \sqrt{n}] \) and \( \Delta \in [\Delta_0, \alpha \sqrt{n}] \). Hence the proof Theorem 1.2 is complete.

4. Conditioned local limit theorem near the boundary

The aim of this section is to establish Theorems 1.3 and 1.4.

4.1. A non-asymptotic conditioned local limit theorem. The following bounds will be used the proofs of Theorems 1.3 and 1.4. It is an easy consequence of the duality formula (Lemma 3.2) and bounds of the exit time for the dual random walk.

Lemma 4.1. Assume A2. Let \( g : \mathbb{R} \to \mathbb{R}_+ \) be a measurable function satisfying \( \int_{\mathbb{R}_+} g(y)(1 + y)dy < \infty \). Then there exists a constant \( c \) such that

\[
\sup_{n \geq 1} \sqrt{n} \int_{\mathbb{R}_+} \mathbb{E}(g(x + S_n); \tau_x > n)dx \leq c \int_{\mathbb{R}_+} g(y)(1 + y)dy.
\]

Moreover,

\[
\lim_{n \to \infty} \sqrt{n} \int_{\mathbb{R}_+} \mathbb{E}(g(x + S_n); \tau_x > n)dx = \frac{2}{\sigma \sqrt{2\pi}} \int_{\mathbb{R}_+} g(y)V^*(y)dy.
\]

Proof. Using Lemma 3.2 with \( h = 1 \), we get that for any \( n \geq 1 \),

\[
\int_{\mathbb{R}_+} \mathbb{E}(g(x + S_n); \tau_x > n)dx = \int_{\mathbb{R}_+} g(y)\mathbb{P}(\tau_x^* > n)dy.
\]

The conclusion follows from (2.22), (2.23) of Theorem 2.7 and the Lebesgue dominated convergence theorem.

Theorems 1.3 and 1.4 will be deduced from the following more general statement in which we establish upper and lower bounds for the joint law of the random walk \( x + S_n \) and of the event \( \{\tau_x > n\} \).

Theorem 4.2. Assume A1 and A2. Then, there exist constants \( c > 0 \) and \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and any sequence of positive numbers \( (\alpha_n)_{n \geq 1} \) satisfying \( \lim_{n \to \infty} \alpha_n = 0 \), one can find a constant \( c_\varepsilon > 0 \) such that uniformly in \( x \in [0, \alpha_n \sqrt{n}] \) and \( n \geq 1 \), the following holds:
1. For any measurable functions \( f, g : \mathbb{R} \to \mathbb{R}_+ \) satisfying \( f \leq \epsilon g \) and \( \int_{\mathbb{R}_+} g(t - \epsilon)(1 + t)dt < \infty \),

\[
\mathbb{E}(f(x + S_n); \tau_x > n) \leq \frac{2V(x)}{\sqrt{2\pi\sigma^3}n^{3/2}} \int_0^{\alpha_n\sqrt{n}} g(t - \epsilon)V^*(t)dt + \left( c\sqrt{\epsilon} + c\epsilon\alpha_n + \frac{c\epsilon}{n^{\epsilon}} \right) \frac{V(x)}{n^{3/2}} \int_{\mathbb{R}_+} g(t - \epsilon)(1 + t)dt + c\frac{V(x)}{n} \int_{\alpha_n\sqrt{n}}^{\infty} g(t - \epsilon)dt.
\]  

(4.1)

2. For any measurable functions \( f, g, h : \mathbb{R} \to \mathbb{R}_+ \) satisfying \( h \leq \epsilon f \leq \epsilon g \) and \( \int_{\mathbb{R}_+} g(t - \epsilon)(1 + t)dt < \infty \),

\[
\mathbb{E}(f(x + S_n); \tau_x > n) \geq \frac{2V(x)}{\sqrt{2\pi\sigma^3}n^{3/2}} \int_0^{\alpha_n\sqrt{n}} h(t + \epsilon)V^*(t)dt - \left( c\epsilon^{1/2} + c\epsilon\alpha_n + \frac{c\epsilon}{n^{\epsilon}} \right) \frac{V(x)}{n^{3/2}} \int_{\mathbb{R}_+} g(t - \epsilon)(1 + t)dt - c\epsilon^{1/2} \frac{V(x)}{n} \int_{\alpha_n\sqrt{n}}^{\infty} h(t - \epsilon)dt.
\]  

(4.2)

**Proof.** We first establish the upper bound (4.1). Set \( m = \lfloor n/2 \rfloor \) and \( k = n - m \). By the Markov property, we have for any \( x \in \mathbb{R}_+ \),

\[
I_n(x) := \mathbb{E}(f(x + S_n); \tau_x > n) = \int_{\mathbb{R}_+} I_m(t) \mathbb{P}(x + S_k \in dt, \tau_x > k). \quad (4.3)
\]

When \( x < 0 \), let \( I_n(x) = 0 \) for any \( n \geq 1 \). Now we are going to find an \( \epsilon \)-domination for the function \( t \mapsto I_m(t) \) given on the right hand side of (4.3). Since \( f \leq \epsilon g \), it holds that for any \( t \in \mathbb{R} \) and \( |v| \leq \epsilon \),

\[
I_m(t) = \mathbb{E}(f(t + S_m); \tau_t > m) \leq H_m(t + v), \quad (4.4)
\]

where

\[
H_m(t) := \mathbb{E}(g(t + S_m); \tau_{t+\epsilon} > m) 1_{\{t+\epsilon\}}, \quad t \in \mathbb{R}. \quad (4.5)
\]

It is easy to see that \( I_m \preceq \epsilon H_m \) and that both \( I_m \) and \( H_m \) are integrable on \( \mathbb{R} \). Thus we can apply the upper bound (3.1) of Theorem 3.1 to obtain

\[
I_n(x) \leq (1 + c\epsilon) \frac{2V(x)}{\sqrt{2\pi\sigma^2k}} \int_{\mathbb{R}_+} H_m(t) \phi^+ \left( \frac{t}{\sigma\sqrt{k}} \right) dt + c\epsilon^{1/4} \frac{V(x)}{n} \int_{\mathbb{R}} H_m(t) \phi \left( \frac{t}{\epsilon^{1/4}\sigma\sqrt{k}} \right) dt + c\epsilon (\alpha_n + n^{-\epsilon}) \frac{V(x)}{n} \|H_m\|_1
\]

\[=: (1 + c\epsilon)J_1 + J_2 + J_3. \quad (4.6)\]
Bound of $J_1$. We use a change of variable and the duality formula (Lemma 3.2) to get

$$J_1 = 2V(x) \int_{\mathbb{R}^+} E(g(t + S_m); \tau_{t+} > m) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt$$

$$= 2V(x) \int_{\mathbb{R}^+} E(g(t + S_m - \varepsilon); \tau_t > m) \phi^+ \left( \frac{t - \varepsilon}{\sigma \sqrt{k}} \right) dt$$

$$= 2V(x) \int_{\mathbb{R}^+} g(t - \varepsilon) E \left[ \phi^+ \left( \frac{t + S_m^* - \varepsilon}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt. \quad (4.7)$$

Then we split this integral into two parts: $J_1 = J_{11} + J_{12}$, where

$$J_{11} = 2V(x) \int_{\alpha n \sqrt{n}}^{\infty} g(t - \varepsilon) E \left[ \phi^+ \left( \frac{t + S_m^* - \varepsilon}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt, \quad (4.8)$$

$$J_{12} = 2V(x) \int_{0}^{\alpha n \sqrt{n}} g(t - \varepsilon) E \left[ \phi^+ \left( \frac{t + S_m^* - \varepsilon}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt. \quad (4.9)$$

Bound of $J_{11}$. Since the function $\phi^+$ is bounded on $\mathbb{R}$, we easily get

$$J_{11} \leq c \frac{V(x)}{n} \int_{\alpha n \sqrt{n}}^{\infty} g(t - \varepsilon) dt. \quad (4.10)$$

Bound of $J_{12}$. Since the function $\phi^+$ is differentiable on $\mathbb{R}^+$, using integration by parts leads to

$$E \left[ \phi^+ \left( \frac{t + S_m^* - \varepsilon}{\sigma \sqrt{k}} \right); \tau^*_t > m \right]$$

$$= \int_{\mathbb{R}^+} (\phi^+)'(u) P \left( \frac{t + S_m^* - \varepsilon}{\sigma \sqrt{k}} > u, \tau^*_t > m \right) du$$

$$= \int_{\mathbb{R}^+} (\phi^+)'(u) P \left( \frac{t + S_m^*}{\sigma \sqrt{m}} > u_{k,m,\varepsilon}, \tau^*_t > m \right) du, \quad (4.11)$$

where $u_{k,m,\varepsilon} = \frac{u \sigma \sqrt{k} + \varepsilon}{\sigma \sqrt{m}}$. Applying the conditioned integral limit theorem (2.26) of Corollary 2.9 (with $S_m$, $\tau_t$ and $V(t)$ replaced by $S_m^*$, $\tau^*_t$ and $V^*(t)$, respectively), we get that uniformly in $t \in [0, \alpha n \sqrt{n}]$ and $u \geq 0$,

$$\left| P \left( \frac{t + S_m^*}{\sigma \sqrt{m}} > u_{k,m,\varepsilon}, \tau^*_t > m \right) - \frac{2V^*(t)}{\sqrt{2\pi m}} \left( 1 - \Phi^+ \left( u_{k,m,\varepsilon} \right) \right) \right|$$

$$\leq c \left( \alpha_n + n^{-\varepsilon} \right) \frac{V^*(t)}{n^1/2}.$$
Substituting this into (4.11) and using the fact that \( \int_{\mathbb{R}_+} |(\phi^+)'(u)| du < \infty \), we obtain that uniformly in \( t \in [0, \alpha_n \sqrt{n}] \),

\[
\left| \mathbb{E} \left[ \phi^+ \left( \frac{t + S_m^* - \varepsilon}{\sigma \sqrt{k}} \right) ; \tau_t^* > m \right] \right| - \frac{2V^*(t)}{\sigma \sqrt{2\pi m}} \int_{\mathbb{R}_+} (\phi^+)'(u) \left[ 1 - \Phi^+ (u_{k,m,\varepsilon}) \right] du \leq c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V^*(t)}{n^{1/2}}.
\]

(4.12)

Since \( m = [n/2] \), \( k = n - m \) and \( u_{k,m,\varepsilon} = \frac{u\sqrt{k} + \varepsilon}{\sigma \sqrt{m}} \), using integration by parts and the fact that \( |(\phi^+)'| \leq c \) for some constant \( c > 0 \), we get

\[
\int_{\mathbb{R}_+} (\phi^+)'(u) \left( 1 - \Phi^+ (u_{k,m,\varepsilon}) \right) du = \sqrt{\frac{k}{m}} \int_{\mathbb{R}_+} \phi^+(u) \phi^+ (u_{k,m,\varepsilon}) du \leq \sqrt{\frac{k}{m}} \int_{\mathbb{R}_+} \phi^+(u) \phi^+ \left( \frac{u\sqrt{k}}{\sqrt{m}} \right) du + \frac{c_\varepsilon}{\sqrt{n}}.
\]

(4.13)

By a change of variable and the fact that \( k + m = n \), we see that

\[
\sqrt{\frac{k}{m}} \int_{\mathbb{R}_+} \phi^+(u) \phi^+ \left( \frac{u\sqrt{k}}{\sqrt{m}} \right) du = \frac{1}{\sqrt{m}} \int_{\mathbb{R}_+} \phi^+ \left( \frac{y}{\sqrt{k}} \right) \phi^+ \left( \frac{y}{\sqrt{m}} \right) dy = \frac{1}{\sqrt{k}} \int_{\mathbb{R}_+} y^2 e^{-\frac{y^2}{2}} dy = \frac{k^{1/2} 2\pi m}{2n^{3/2}}.
\]

Therefore, combining this with (4.9), (4.12) and (4.13), we obtain that uniformly in \( t \in [0, \alpha_n \sqrt{n}] \),

\[
J_{12} \leq \frac{2V(x)}{\sqrt{2\pi \sigma^2 k}} \int_0^{\alpha_n \sqrt{n}} g(t - \varepsilon) \left[ \frac{2V^*(t)}{\sigma^{2\pi m}} \left( \frac{k^{1/2} 2\pi m}{2n^{3/2}} + \left( c_\varepsilon \frac{\varepsilon}{\sqrt{n}} + \varepsilon^2 \alpha_n + \varepsilon^2 n^{-\varepsilon} \right) \frac{V^*(t)}{\sqrt{n}} \right) dt \right] \leq (1 + c_\varepsilon \alpha_n + \varepsilon^2 n^{-\varepsilon}) \frac{2V(x)}{\sqrt{2\pi \sigma^3 n^{3/2}}} \int_0^{\alpha_n \sqrt{n}} g(t - \varepsilon) V^*(t) dt.
\]

(4.14)

**Bound of \( J_2 \).** The estimate of \( J_2 \) (cf. (4.6)) follows the same lines as that of \( J_1 \). Proceeding in the same way as the proof of (4.7), (4.8) and (4.9), one
has $J_2 = J_{21} + J_{22}$, where

\[
J_{21} = c\varepsilon^{1/4} \frac{V(x)}{n} \int_{\alpha_n \sqrt{n}}^{\infty} g(t - \varepsilon)\mathbb{E} \left[ \phi \left( \frac{t + S^*_m - \varepsilon}{\varepsilon^{1/4} \sigma \sqrt{k}} \right) ; \tau^*_t > m \right] dt,
\]

\[
J_{22} = c\varepsilon^{1/4} \frac{V(x)}{n} \int_{0}^{\alpha_n \sqrt{n}} g(t - \varepsilon)\mathbb{E} \left[ \phi \left( \frac{t + S^*_m - \varepsilon}{\varepsilon^{1/4} \sigma \sqrt{k}} \right) ; \tau^*_t > m \right] dt.
\]

**Bound of $J_{21}$**. As in (4.10), since the function $\phi$ is bounded on $\mathbb{R}$, we have

\[
J_{21} \leq c\varepsilon^{1/4} \frac{V(x)}{n} \int_{\alpha_n \sqrt{n}}^{\infty} g(t - \varepsilon)dt.
\]  

(4.15)

**Bound of $J_{22}$**. Similarly to (4.11), it holds that

\[
\mathbb{E} \left[ \phi \left( \frac{t + S^*_m - \varepsilon}{\varepsilon^{1/4} \sigma \sqrt{k}} \right) ; \tau^*_t > m \right] = \int_{\mathbb{R}^+} \phi'(u) \mathbb{P} \left( \frac{t + S^*_m}{\sigma \sqrt{m}} > \tilde{u}_{k,m,\varepsilon} ; \tau^*_t > m \right) du,
\]

where $\tilde{u}_{k,m,\varepsilon} = \frac{e^{1/4}}{\sigma \sqrt{m}} \varepsilon$. Following the proof of (4.12), we derive that uniformly in $t \in [0, \alpha_n \sqrt{n}]$,

\[
\mathbb{E} \left[ \phi \left( \frac{t + S^*_m - \varepsilon}{\varepsilon^{1/4} \sigma \sqrt{k}} \right) ; \tau^*_t > m \right] \leq \frac{2V^*(t)}{\sigma \sqrt{2\pi mn}} \int_{\mathbb{R}^+} \phi'(u) \left[ 1 - \Phi^+ \left( \frac{\tilde{u}_{k,m,\varepsilon}}{\sigma \sqrt{m}} \right) \right] du + c\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V^*(t)}{n^{1/2}}
\]

\[
= \frac{2V^*(t)}{\sigma \sqrt{2\pi mn}} e^{1/4} \sqrt{\frac{k}{m}} \int_{\mathbb{R}^+} \phi(u) \phi^+ \left( \frac{\tilde{u}_{k,m,\varepsilon}}{\sigma \sqrt{m}} \right) du + c\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V^*(t)}{n^{1/2}}
\]

\[
\leq \left( c\varepsilon^{1/4} + c\varepsilon \alpha_n + c\varepsilon n^{-\varepsilon} \right) \frac{V^*(t)}{\sqrt{n}},
\]

which implies that

\[
J_{22} \leq \left( c\varepsilon^{1/4} + c\varepsilon \alpha_n + c\varepsilon n^{-\varepsilon} \right) \frac{V(x)}{n^{3/2}} \int_{0}^{\alpha_n \sqrt{n}} g(t - \varepsilon)V^*(t) dt.
\]  

(4.16)

**Bound of $J_3$**. By (4.5), a change of variable and Lemma 4.1, we have

\[
\|H_m\|_1 = \int_{\mathbb{R}^+} \mathbb{E} \left( g(t - \varepsilon + S_m) ; \tau_t > m \right) dt \leq \frac{c}{\sqrt{n}} \int_{\mathbb{R}^+} g(t - \varepsilon)(1 + t) dt.
\]

(4.17)

Since $m = \lceil n/2 \rceil$, it follows from (4.6) and (4.17) that

\[
J_3 \leq c\varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V(x)}{n^{3/2}} \int_{\mathbb{R}^+} g(t - \varepsilon)(1 + t) dt.
\]

(4.18)

Combining (4.10), (4.14), (4.15), (4.16) and (4.18) concludes the proof of the upper bound (4.1) of the theorem.
We next sketch the proof of the lower bound (4.2). Similarly to (4.6), we use the lower bound (3.2) of Theorem 3.1 to get that uniformly in $x \in [0, \alpha_n \sqrt{n}]$,

$$I_n(x) \geq \frac{2V(x)}{\sqrt{2\pi \sigma^2 k}} \int_{\mathbb{R}_+} L_m(t) \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) dt$$

$$- c \varepsilon^{1/12} \frac{V(x)}{n} \int_{\mathbb{R}} H_m(t) \left[ \phi \left( \frac{t}{\sigma \sqrt{k}} \right) + \phi^+ \left( \frac{t}{\sigma \sqrt{k}} \right) \right] dt$$

$$- c \varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V(x)}{n} ||H_m||_1$$

$$=: K_1 + K_2 + K_3,$$  \hspace{1cm} (4.19)

where $L_m \leq \varepsilon I_m \leq \varepsilon H_m$ with $H_m$ defined by (4.5) and

$$L_m(t) := \mathbb{E} \left( h(t + S_m) ; \tau_{t+} > m \right) \mathbb{1}_{\{t > \varepsilon\}}, \quad t \in \mathbb{R}. \hspace{1cm} (4.20)$$

**Bound of $K_1$.** In the same way as in the proof of (4.7), using the duality formula (Lemma 3.2), one has

$$K_1 = \frac{2V(x)}{\sqrt{2\pi \sigma^2 k}} \int_{\mathbb{R}_+} h(t + \varepsilon) \mathbb{E} \left[ \phi^+ \left( \frac{t + S_m^* + \varepsilon}{\sigma \sqrt{k}} \right) ; \tau^*_t > m \right] dt$$

$$\geq \frac{2V(x)}{\sqrt{2\pi \sigma^2 k}} \int_{0}^{\alpha_n \sqrt{n}} h(t + \varepsilon) \mathbb{E} \left[ \phi^+ \left( \frac{t + S_m^* + \varepsilon}{\sigma \sqrt{k}} \right) ; \tau^*_t > m \right] dt.$$

Following the proof of (4.14), we get that uniformly in $x \in [0, \alpha_n \sqrt{n}]$,

$$K_1 \geq \left( 1 - c \varepsilon \alpha_n - c \varepsilon n^{-\varepsilon} \right) \frac{2V(x)}{\sqrt{2\pi \sigma^2 n^{3/2}}} \int_{0}^{\alpha_n \sqrt{n}} h(t + \varepsilon)V^*(t)dt.$$

**Bound of $K_2$.** Following the proof of (4.15) and (4.16), one can check that

$$K_2 \geq -c \varepsilon^{1/12} \frac{V(x)}{n} \int_{\alpha_n \sqrt{n}}^{\infty} h(t - \varepsilon) dt$$

$$- \left( c \varepsilon^{1/12} + c \varepsilon \alpha_n + c \varepsilon n^{-\varepsilon} \right) \frac{V(x)}{n^{3/2}} \int_{0}^{\alpha_n \sqrt{n}} g(t - \varepsilon)V^*(t)dt.$$

**Bound of $K_3$.** From (4.18) we see that

$$K_3 \geq -c \varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \frac{V(x)}{n^{3/2}} \int_{\mathbb{R}_+} g(t - \varepsilon)(1 + t) dt.$$

Collecting the above bounds for $K_1$, $K_2$ and $K_3$, and using the fact that $h(t + \varepsilon) \leq f(t) \leq g(t - \varepsilon)$ for any $t \in \mathbb{R}$, we get the lower bound (4.2). \qed
4.2. Proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. We first give a proof of (1.9). By the assumption $$\int_{\mathbb{R}^+} f(t)(1+t)^\gamma dt < \infty$$ for some constant $$\gamma > 1$$, the asymptotic (1.9) is a consequence of Theorem 4.2, Lemma 2.1 and the Lebesgue dominated convergence theorem.

We next give a proof of (1.10). Applying the upper bound (4.1) of Theorem 4.2 with $$g = \bar{f}_{\delta, \varepsilon}$$ (cf. (2.4)) and $$\alpha_n$$ replaced by $$2\alpha_n$$, we get that uniformly in $$x \in [0, \alpha_n\sqrt{n}]$$ and $$y \in [\alpha_n^{-1}, \alpha_n\sqrt{n}]$$,

$$\mathbb{E} (f(x + S_n - y); \tau_x > n) \leq \frac{2V(x)}{\sqrt{2\pi \sigma_n^2 n^{3/2}}} \int_0^{\alpha_n\sqrt{n}} \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)V^*(t)dt + \left( c\sqrt{\varepsilon} + c\varepsilon\alpha_n + \frac{c\varepsilon}{n^2} \right) \frac{V(x)}{n^{3/2}} \int_{\mathbb{R}^+} \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)(1 + t)dt + \frac{cV(x)}{n} \int_{2\alpha_n\sqrt{n}}^\infty \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)dt. \quad (4.21)$$

We claim that as $$n \to \infty$$, it holds uniformly in $$y \in [\alpha_n^{-1}, \alpha_n\sqrt{n}]$$ that

$$\int_0^{\alpha_n\sqrt{n}} \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)V^*(t)dt \sim y \int_{-\varepsilon}^\infty \bar{f}_{\delta, \varepsilon}(u)du, \quad (4.22)$$

$$\int_{\mathbb{R}^+} \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)(1 + t)dt \sim y \int_{-\varepsilon}^\infty \bar{f}_{\delta, \varepsilon}(u)du, \quad (4.23)$$

$$\int_{2\alpha_n\sqrt{n}}^\infty \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)dt \sim \frac{y}{\sqrt{n}} o(1). \quad (4.24)$$

Indeed, by a change of variable and the fact that the function $$\bar{f}_{\delta, \varepsilon}$$ is supported on $$[-\varepsilon, \infty)$$, we have

$$\int_0^{\alpha_n\sqrt{n}} \bar{f}_{\delta, \varepsilon}(t - y - \varepsilon)V^*(t)dt = \int_{-\varepsilon}^{\alpha_n\sqrt{n}-y-\varepsilon} \bar{f}_{\delta, \varepsilon}(u)V^*(u + y + \varepsilon)du$$

$$= \int_{-\varepsilon}^\infty \bar{f}_{\delta, \varepsilon}(u)V^*(u + y + \varepsilon)du - \int_{2\alpha_n\sqrt{n}-y-\varepsilon}^\infty \bar{f}_{\delta, \varepsilon}(u)V^*(u + y + \varepsilon)du. \quad (4.25)$$

Since $$V^*(t)/t \to \infty$$ as $$t \to \infty$$ and $$\int_{-\varepsilon}^\infty \bar{f}_{\delta, \varepsilon}(u)du < \infty$$, by the Lebesgue dominated convergence theorem, as $$n \to \infty$$, the first integral is equivalent to $$y \int_{-\varepsilon}^\infty \bar{f}_{\delta, \varepsilon}(u)du$$. The second integral is negligible compared with the first one, using the inequality $$t \leq V^*(t) \leq c(1 + t)$$. Hence (4.22) holds and the relation (4.23) can be proved in the same way. For (4.24), by a change of variable and the fact that $$\int_{\mathbb{R}^+} \bar{f}_{\delta, \varepsilon}(u)(1 + u)du < \infty$$ and $$y \in [\alpha_n^{-1}, \alpha_n\sqrt{n}]$$.
with $\alpha_n \sqrt{n} \to \infty$ as $n \to \infty$,
\[
\int_{2\alpha_n \sqrt{n}}^{\infty} \bar{F}_{\delta,\varepsilon}(t - y - \varepsilon)dt \leq \int_{\alpha_n \sqrt{n} - \varepsilon}^{\infty} \bar{F}_{\delta,\varepsilon}(u)du \\
\leq \frac{1}{\alpha_n \sqrt{n}} \int_{\alpha_n \sqrt{n} - \varepsilon}^{\infty} \bar{F}_{\delta,\varepsilon}(u)(1 + u)du \\
\leq \frac{y}{\sqrt{n}} o(1),
\]
which shows (4.24).

Therefore, from (4.21), (4.22), (4.23) and (4.24), we get that uniformly in $x \in [0, \alpha_n \sqrt{n}]$ and $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$,
\[
\limsup_{n \to \infty} \frac{\mathbb{E}(f(x + S_n - y); \tau_x > n)}{\sqrt{2\pi \sigma^3 n^{3/2}}} \leq (1 + c \sqrt{\varepsilon}) \int_{-\varepsilon}^{\infty} \bar{F}_{\delta,\varepsilon}(u)du,
\]
which implies the desired upper bound by taking first $\varepsilon \to 0$ and then $\delta \to 0$, and using Lemma 2.1. The lower bound can be proved in the same way by using (4.2) of Theorem 4.2. The proof of (1.10) is complete.

Proof of Theorem 1.4. Without loss of generality, we assume that $\alpha_n \geq n^{-\varepsilon}$, so that $\alpha_n \sqrt{n} \geq \Delta$, where $\Delta \in [\Delta_0, o(y)]$. Applying (4.1) with $f = 1_{[y, y + \Delta]}$, $g = 1_{[y - \varepsilon, y + \Delta + \varepsilon]}$ and with $\alpha_n$ replaced by $3\alpha_n$, and noticing that the last integral in (4.1) vanishes, we deduce that uniformly in $x \in [0, \alpha_n \sqrt{n}]$ and $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$,
\[
\mathbb{P}(x + S_n \in [0, \Delta] + y, \tau_x > n) \leq \frac{2V(x)}{\sqrt{2\pi \sigma^3 n^{3/2}}} \int_0^{\Delta + 2\varepsilon} V^*(u + y)du \\
+ \left(c \sqrt{\varepsilon} + c_\varepsilon \alpha_n + \frac{c_\varepsilon}{n^{\varepsilon}}\right) \frac{V(x)}{n^{3/2}} \int_0^{\Delta + 2\varepsilon} (u + y + 1)du. \tag{4.25}
\]

Applying Lemma 3.7 with $k_0 = (\frac{1}{\alpha_n})^{-\varepsilon}$, we get that uniformly in $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$ and $\Delta \in [\Delta_0, o(y)]$,
\[
\int_0^{\Delta + 2\varepsilon} V^*(u + y)du \leq \left(1 + c_\varepsilon \alpha_n^{-\varepsilon} \right) \int_0^{\Delta + 2\varepsilon} (u + y)du + \frac{c_\varepsilon}{\sqrt{\alpha_n}}(\Delta + 2\varepsilon) \\
\leq \left(1 + c_\varepsilon \alpha_n^{-\varepsilon} \right) y(\Delta + 2\varepsilon) + \frac{c_\varepsilon}{\sqrt{\alpha_n}}(\Delta + 2\varepsilon) \\
\sim y(\Delta + 2\varepsilon) \leq (1 + c\varepsilon) y\Delta, \tag{4.26}
\]
where in the last line we used the fact that $\frac{1}{\sqrt{\alpha_n}} = o(y)$ and $\Delta_0 > 0$ is a fixed constant. Similarly, one can check that uniformly in $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$ and $\Delta \in [\Delta_0, o(y)]$,
\[
\int_0^{\Delta + 2\varepsilon} (u + y + 1)du \leq (1 + c\varepsilon) y\Delta. \tag{4.27}
\]
Therefore, substituting (4.26) and (4.27) into (4.25), we obtain that uniformly in \( x \in [0, \alpha_n \sqrt{n}] \), \( y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}] \) and \( \Delta \in [\Delta_0, o(y)] \),

\[
\limsup_{n \to \infty} \mathbb{P} \left( x + S_n \in [0, \Delta] + y, \tau_n > n \right) \leq \left( 1 + c \sqrt{\varepsilon} \right),
\]

which concludes the proof of the upper bound by letting \( \varepsilon \to 0 \). The lower bound can be obtained in a similar way by using (4.2).

\[\Box\]

5. Conditioned concentration bounds far from the boundary

The goal of this section is to establish Theorems 1.5 and 1.6.

5.1. Formulation of the result. Denote

\[
\psi^+(s, x) = \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{(s-x)^2}{2}} - e^{-\frac{(s-x)^2}{2}} \right) \mathbb{I}_{\{s \geq 0\}}, \quad s, x \in \mathbb{R}.
\]

Note that \( \psi^+(s, x) = \psi(s, x) \) for any \( s \geq 0 \) and \( x \in \mathbb{R} \), where \( \psi \) is defined by (1.4). The following result is a conditioned local limit theorem of order \( n^{-1} \) with large starting point \( x \).

**Theorem 5.1.** Assume A1 and A2. Then, for any \( \eta > 0 \), there exist constants \( c, \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), one can find a constant \( c_\varepsilon > 0 \) such that uniformly in \( x \in [n^{1/2 - \varepsilon}, \eta \sqrt{n}] \) and \( n \geq 1 \), the following holds:

1. For any integrable functions \( f, g : \mathbb{R} \to \mathbb{R}_+ \) satisfying \( f \leq \varepsilon g \),

\[
\mathbb{E} \left( f(x + S_n); \tau_n > n \right) \leq \frac{1 + c\varepsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} g(s) \psi \left( \frac{s}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right) ds + \frac{c_\varepsilon}{n^{1/2 + \varepsilon}} \|g\|_1. \tag{5.1}
\]

2. For any integrable functions \( f, g, h : \mathbb{R} \to \mathbb{R}_+ \) satisfying \( h \leq \varepsilon f \leq \varepsilon g \),

\[
\mathbb{E} \left( f(x + S_n); \tau_n > n \right) \geq \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} h(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \sqrt{n}} \right) \right] ds - \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} h(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \sqrt{n}} \right) \right] ds - \frac{c_\varepsilon}{n^{1/2 + \varepsilon}} \|g\|_1. \tag{5.2}
\]

5.2. Proof of the upper bound. In this section we prove the upper bound (5.1) of Theorem 5.1. For any \( v > 0 \), we denote

\[
\psi_v(s, x) = \frac{1}{\sqrt{2\pi v}} \left( e^{-\frac{(s-x)^2}{2v}} - e^{-\frac{(s-x)^2}{2v}} \right), \quad s, x \in \mathbb{R}. \tag{5.3}
\]
Then $\psi_1 = \psi$ with $\psi$ defined by (1.4). We shall make use of the following convolution result. Recall that $\phi_v(z) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{z^2}{2v}}, z \in \mathbb{R}$, and $\Phi$ is the standard normal distribution function on $\mathbb{R}$.

**Lemma 5.2.** For any $v \in (0, 1)$ and $s, x \in \mathbb{R}$, we have

$$
\int_{\mathbb{R}} \phi_v(s - z)\psi_{1-v}(z, x)dz = \psi(s, x). \tag{5.4}
$$

In addition, for any $v \in (0, 1/4]$ and $s, x \in \mathbb{R}$, we have

$$
\left| \int_0^\infty \phi_v(s - z)\psi_{1-v}(z, x)dz - \psi(s, x) \right| \leq 1 - \Phi \left( \frac{s}{\sqrt{v}} \right). \tag{5.5}
$$

**Proof.** Since for any $z, x \in \mathbb{R},$

$$
\psi_{1-v}(z, x) = \frac{1}{\sqrt{2\pi(1-v)}} \int_{-x}^{x} \frac{y - z}{1-v} e^{-\frac{(y-z)^2}{2(1-v)}} dy,
$$

by Fubini’s theorem, we derive that for any $s \in \mathbb{R},$

$$
\int_{\mathbb{R}} \phi_v(s - z)\psi_{1-v}(z, x)dz = \int_{\mathbb{R}} \phi_v(z)\psi_{1-v}(s - z, x)dz
$$

$$
= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi v}} e^{-\frac{z^2}{2v}} \frac{1}{\sqrt{2\pi(1-v)}} \left[ \int_{-x}^{x} \frac{y - s + z}{1-v} e^{-\frac{(y-s+z)^2}{2(1-v)}} dy \right] dz
$$

$$
= \frac{1}{\sqrt{2\pi(1-v)}} \int_{-x}^{x} \left[ \int_{\mathbb{R}} \frac{z + y - s}{\sqrt{2\pi v(1-v)}} e^{-\frac{z^2}{2v}} e^{-\frac{(y-z)^2}{2(1-v)}} dz \right] dy.
$$

Since

$$
\frac{z^2}{2v} + \frac{(z + y - s)^2}{2(1-v)} = \frac{(z + (y-s)v)^2}{2v(1-v)} + \frac{(y-s)^2}{2},
$$

we obtain

$$
\int_{\mathbb{R}} \phi_v(s - z)\psi_{1-v}(z, x)dz
$$

$$
= \frac{1}{\sqrt{2\pi(1-v)}} \int_{-x}^{x} \left[ \int_{\mathbb{R}} \frac{z + y - s}{\sqrt{2\pi v(1-v)}} e^{-\frac{(z+(y-s)v)^2}{2v(1-v)}} dz \right] e^{-\frac{(y-s)^2}{2}} dy
$$

$$
= \frac{1}{\sqrt{2\pi(1-v)}} \int_{-x}^{x} \left[ -(y-s)v + (y-s) \right] e^{-\frac{(y-s)^2}{2}} dy
$$

$$
= \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{(s-x)^2}{2}} - e^{-\frac{(s+x)^2}{2}} \right) = \psi(s, x),
$$

which ends the proof of (5.4).
To prove (5.5), using (5.4) and the fact that $\psi_{1-v}(z, x) = -\psi_{1-v}(-z, x)$, we get that for any $s \in \mathbb{R}$,

$$
\int_0^\infty \phi_v(s - z)\psi_{1-v}(z, x)dz
= \int_\mathbb{R} \phi_v(s - z)\psi_{1-v}(z, x)dz - \int_{-\infty}^0 \phi_v(s - z)\psi_{1-v}(z, x)dz
= \psi(s, x) + \int_{\mathbb{R}^+} \phi_v(s + z)\psi_{1-v}(z, x)dz.
$$

Since $\psi_{1-v}(z, x) < 1$ for any $v \in (0, 1/4]$, and $\int_{\mathbb{R}^+} \phi_v(s + z)dz = 1 - \Phi\left(\frac{s}{\sqrt{v}}\right)$ for $s \in \mathbb{R}$, the inequality (5.5) follows. \hfill \Box

Now we give a proof of the upper bound (5.1) of Theorem 5.1.

Proof of (5.1). As in the proof of (3.1), let $\varepsilon > 0$ be a sufficiently small constant and $\delta = \sqrt{\varepsilon}$, and set $m = [\delta n]$ and $k = n - m$. It suffices to prove (5.1) only for sufficiently large $n$, otherwise the bound becomes trivial. Similarly to the proof of (3.9), (3.10), (3.11), (3.12) and (3.15), one can use the Markov property and the local limit theorem (Theorem 2.5) to get

$$
I_n(x) := \mathbb{E}(f(x + S_n); \tau_x > n) \leq (1 + c\varepsilon)J_n(x) + \frac{c\varepsilon}{n^{1/2+\varepsilon}}\|g\|_1 \mathbb{P}(\tau_x > k)
\leq (1 + c\varepsilon)J_n(x) + \frac{c\varepsilon}{n^{1/2+\varepsilon}}\|g\|_1, \tag{5.6}
$$

where

$$
J_n(x) := \int_{\mathbb{R}^+} \varphi_n(t)\mathbb{P}\left(\frac{x + S_k}{\sigma \sqrt{k}} > t, \tau_x > k\right)dt, \tag{5.7}
$$

$$
\varphi_n(t) := \int_{\mathbb{R}} g(\sigma \sqrt{k}s)\frac{1}{\sqrt{m/k}}\phi\left(\frac{t - s}{\sqrt{m/k}}\right)ds. \tag{5.8}
$$

By the conditioned integral limit theorem (Theorem 2.8), we have that for any $x \geq n^{1/2-\varepsilon}, t \geq 0$ and $n \geq 1$,

$$
\left|\mathbb{P}\left(\frac{x + S_k}{\sigma \sqrt{k}} > t, \tau_x > k\right) - \int_t^\infty \psi\left(s, \frac{x}{\sigma \sqrt{k}}\right)ds\right| \leq \frac{c\varepsilon}{n^{\varepsilon}}. \tag{5.9}
$$

From (5.7) and (5.9), using integration by parts we derive that

$$
J_n(x) - \int_{\mathbb{R}^+} \varphi_n(t)\psi\left(t, \frac{x}{\sigma \sqrt{k}}\right)dt \leq \frac{c\varepsilon}{n^{\varepsilon}} \int_{\mathbb{R}^+} |\varphi_n'(t)|dt \leq \frac{c\varepsilon}{n^{1/2+\varepsilon}}\|g\|_1, \tag{5.10}
$$
where in the last inequality we used the bound (3.18). By the definition of \( \varphi_n \) (cf. (5.8)), Fubini’s theorem and a change of variable, we get

\[
\int_{\mathbb{R}^+} \varphi_n(t) \psi \left( t, \frac{x}{\sigma \sqrt{k}} \right) dt \\
= \int_{\mathbb{R}^+} \left[ \int_{\mathbb{R}} g(\sigma \sqrt{ns}) \frac{1}{\sqrt{m/k}} \phi \left( \frac{s-t}{\sqrt{m/k}} \right) \psi \left( t, \frac{x}{\sigma \sqrt{k}} \right) ds \right] dt \\
= \int_{\mathbb{R}} g(\sigma \sqrt{ns}) \left[ \int_{\mathbb{R}^+} \frac{1}{\sqrt{m/n}} \phi \left( \frac{s-t}{\sqrt{m/n}} \right) \frac{1}{\sqrt{k/n}} \psi \left( t, \frac{x}{\sqrt{k/n}} \right) dt \right] ds \\
= \int_{\mathbb{R}} g(\sigma \sqrt{ns}) \left[ \int_{\mathbb{R}^+} \phi_{\delta_n} (s-t) \psi_{1-\delta_n} \left( t, \frac{x}{\sigma \sqrt{n}} \right) dt \right] ds, \tag{5.11}
\]

where we denoted \( \delta_n = \frac{m}{n} \). By (5.5), we have that for any \( s \in \mathbb{R} \),

\[
\int_{\mathbb{R}^+} \phi_{\delta_n} (s-t) \psi_{1-\delta_n} \left( t, \frac{x}{\sigma \sqrt{n}} \right) dt \leq \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) + 1 - \Phi \left( s \sqrt{\frac{m}{n}} \right).
\]

Thus, substituting this into (5.11), we get that for any \( x \geq n^{1/2-\varepsilon} \),

\[
\int_{\mathbb{R}^+} \varphi_n(t) \psi \left( t, \frac{x}{\sigma \sqrt{k}} \right) dt \leq \int_{\mathbb{R}} g(\sigma \sqrt{ns}) \left[ \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) + 1 - \Phi \left( s \sqrt{\frac{m}{n}} \right) \right] ds \\
= \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} g(s) \psi \left( \frac{s}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right) ds \\
+ \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} g(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \sqrt{m}} \right) \right] ds. \tag{5.12}
\]

Putting together (5.6), (5.10) and (5.12), and recalling that \( m = [\varepsilon^{1/2} n] \), we conclude the proof of the upper bound (5.1) of Theorem 5.1. \( \square \)

5.3. **Proof of the lower bound.** The aim of this section is to establish the lower bound (5.2) of Theorem 5.1.

**Proof of (5.2).** Let us keep the notation used in the proof of the upper bound (5.1). Recall that

\[
I_n(x) := \int_{\mathbb{R}^+} \mathbb{E} f(t + S_m) \mathbb{P} (x + S_k \in dt, \tau_x > k) \\
- \int_{\mathbb{R}^+} \mathbb{E} (f(t + S_m); \tau_t \leq m) \mathbb{P} (x + S_k \in dt, \tau_x > k) \\
=: I_{n,1}(x) - I_{n,2}(x). \tag{5.13}
\]
Lower bound of $I_{n,1}(x)$. By (3.22), we have
\[
\mathbb{E} f(t + S_m) \geq \frac{1}{\sigma \sqrt{m}} \int_{\mathbb{R}} [h(s) - \varepsilon f(s)] \phi \left( \frac{s - t}{\sigma \sqrt{m}} \right) ds - \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|f\|_1.
\] (5.14)

Proceeding in the same way as the proof of the upper bound (5.1) (replacing the function $g$ by $h - \varepsilon f$ and using the fact that $h \leq \varepsilon f$), one has
\[
\int_{\mathbb{R}_+} \left[ \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} [h(s) - \varepsilon f(s)] \phi \left( \frac{s - t}{\sigma \sqrt{n}} \right) ds \right] \mathbb{P}(x + S_k \in dt, \tau_x > k) \geq \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} [h(s) - \varepsilon f(s)] \psi \left( \frac{s}{\sigma \sqrt{n}} \right) ds - \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} h(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \varepsilon^{1/4} \sqrt{n}} \right) \right] ds - \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|f\|_1.
\] (5.15)

Combining (5.14) and (5.15), we derive that for any $x \geq n^{1/2-\varepsilon}$,
\[
I_{n,1}(x) \geq \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} [h(s) - \varepsilon f(s)] \psi \left( \frac{s}{\sigma \sqrt{n}} \right) ds - \frac{1}{\sigma \sqrt{n}} \int_{\mathbb{R}} h(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \varepsilon^{1/4} \sqrt{n}} \right) \right] ds - \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|f\|_1.
\] (5.16)

Upper bound of $I_{n,2}(x)$. It has been shown in (3.24) that
\[
I_{n,2}(x) = K_1 + K_2,
\] (5.17)

where $K_1$ and $K_2$ are given by (3.24). For $K_1$, it is shown in (3.25) and (3.26) that
\[
K_1 \leq \int_{\mathbb{R}} g(u) J(u) du + c_\varepsilon \frac{V(x)}{n^{1+\varepsilon}} \|g\|_1,
\] (5.18)

where, with $F_u(t) = \frac{1}{\sigma \sqrt{m}} \phi \left( \frac{u - t}{\sigma \sqrt{m}} \right)$,
\[
J(u) = \int_0^{\varepsilon^{1/6} \sqrt{n}} F_u(t) \mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} \in \left[ \frac{t}{\sigma \sqrt{k}}, \frac{\varepsilon^{1/6} \sqrt{n}}{\sigma \sqrt{k}} \right], \tau_x > k \right) dt.
\]

Using Theorem 2.8, we get that for any $x \geq n^{1/2-\varepsilon}$,
\[
\mathbb{P} \left( \frac{x + S_k}{\sigma \sqrt{k}} \in \left[ \frac{t}{\sigma \sqrt{k}}, \frac{\varepsilon^{1/6} \sqrt{n}}{\sigma \sqrt{k}} \right], \tau_x > k \right) \leq G(t) + \frac{c_\varepsilon}{n^{\varepsilon}}.
\]

where
\[
G(t) = \int_{\mathbb{R}} \frac{\varepsilon^{1/6}}{\sigma \sqrt{k}} \psi \left( s, \frac{x}{\sigma \sqrt{k}} \right) ds.
\]
Since $G(\varepsilon^{1/6}\sqrt{n}) = 0$ and $F_u(0), G(0) \geq 0$, it follows that

$$J(u) \leq F_u(\varepsilon^{1/6}\sqrt{n})G(\varepsilon^{1/6}\sqrt{n}) - F_u(0)G(0)$$

$$+ \frac{1}{\sigma\sqrt{k}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} F_u(t)\psi\left(\frac{t}{\sigma\sqrt{k}}, \frac{x}{\sigma\sqrt{k}}\right) dt + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} |F'_u(t)| dt$$

$$\leq \frac{1}{\sigma\sqrt{k}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} F_u(t)\psi\left(\frac{t}{\sigma\sqrt{k}}, \frac{x}{\sigma\sqrt{k}}\right) dt + \frac{c_\varepsilon}{n^{1/2+\varepsilon}}, \quad (5.19)$$

where in the last line we used the fact that $\int_{0}^{\varepsilon^{1/6}\sqrt{n}} |F'_u(t)| dt \leq \frac{c}{\sqrt{n}}$. By elementary calculations, we see that

$$J(u) : = \int_{0}^{\varepsilon^{1/6}\sqrt{n}} F_u(t)\psi\left(\frac{t}{\sigma\sqrt{k}}, \frac{x}{\sigma\sqrt{k}}\right) dt$$

$$= \frac{1}{2\pi} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} \frac{1}{\sigma\sqrt{m}} e^{-\frac{(u-t)^2}{2\sigma^2}} \left(e^{-\frac{(t-x)^2}{2\sigma^2}} - e^{-\frac{(t-k)^2}{2\sigma^2}}\right) dt$$

$$= \frac{1}{2\pi} e^{-\frac{u^2}{2\sigma^2}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} \frac{1}{\sigma\sqrt{m}} e^{-\frac{u^2}{2\sigma^2}} e^{\frac{t}{\sigma\sqrt{m}} c \sigma k \varepsilon} \left(1 - e^{-\frac{2u}{\sigma^2}}\right) dt.$$

Since $x \in [n^{1/2-\varepsilon}, \eta\sqrt{n}]$ and $t \in [0, \varepsilon^{1/6}\sqrt{n}]$, we have $e^{\frac{t}{\sigma\sqrt{m}} c \sigma k \varepsilon} \leq c$ for some constant $c > 0$. Using the fact that $e^{-\frac{u^2}{2\sigma^2}} \leq 1$ and the inequality $1 - e^{-t} \leq t$, $t \geq 0$, we get

$$J(u) \leq ce^{-\frac{u^2}{2\sigma^2}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} \frac{t}{\sigma\sqrt{m}} e^{-\frac{u^2}{2\sigma^2} + \frac{1}{\sigma\sqrt{m}} c \sigma k \varepsilon} dt$$

$$= ce^{-\frac{u^2}{2\sigma^2}} \frac{2\sigma^2}{\sigma\sqrt{m}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} \frac{t}{\sqrt{km}} e^{-\frac{n^2}{\sigma^2} + \frac{1}{\sigma\sqrt{m}} c \sigma k \varepsilon} (t-h)^2 dt$$

$$= ce^{-\frac{u^2}{2\sigma^2}} \frac{2\sigma^2}{\sigma\sqrt{m}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}} \frac{t}{\sqrt{km}} e^{-\frac{u^2}{2\sigma^2} + \frac{1}{\sigma\sqrt{m}} c \sigma k \varepsilon} (t-h)^2 dt.$$

By a change of variable $\frac{\sqrt{m}}{\sqrt{km}} t = z$, it follows that

$$J(u) \leq ce^{-\frac{u^2}{2\sigma^2}} \frac{2\sigma^2}{\sigma\sqrt{m}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}/\sqrt{\sigma^2 mk}} ze^{-\frac{z^2}{2\sigma^2} + \frac{1}{\sigma\sqrt{m}} c \sigma k \varepsilon} du$$

$$\leq ce^{-\frac{u^2}{2\sigma^2}} \frac{2\sigma^2}{\sigma\sqrt{m}} \int_{0}^{\varepsilon^{1/6}\sqrt{n}/\sqrt{\sigma^2 mk}} zdz \leq c\varepsilon^{1/3} \frac{n}{\sqrt{km}} e^{-\frac{u^2}{2\sigma^2}} \leq c\varepsilon^{1/12} e^{-\frac{u^2}{2\sigma^2}}.$$

In view of (5.19), this implies that

$$J(u) \leq c\varepsilon^{1/12} \frac{u^2}{2\sigma^2} + \frac{c_\varepsilon}{n^{1/2+\varepsilon}}.$$
Implementing this bound into (5.18), we obtain
\begin{equation}
K_1 \leq c \frac{\varepsilon^{1/12}}{\sqrt{n}} \int_{\mathbb{R}} g(t) e^{-\frac{t^2}{2\sigma^2}} dt + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|g\|_1. \tag{5.20}
\end{equation}

We proceed to give an upper bound for $K_2$ (cf. (3.30)). Applying the upper bound (5.1) of Theorem 5.1, we obtain
\begin{equation}
K_2 \leq \frac{1 + c\varepsilon}{\sigma \sqrt{k}} \int_{\mathbb{R}^+} M(s) \psi \left( \frac{s}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) ds
+ \frac{1 + c\varepsilon}{\sigma \sqrt{k}} \int_{\mathbb{R}^+} M(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \sqrt{1/4} \sqrt{n}} \right) \right] ds + \frac{c\varepsilon}{n^{1/2+\varepsilon}} \|M\|_1, \tag{5.21}
\end{equation}
where $s \mapsto M(s)$ is defined by (3.32). For the first term, applying the duality formula (3.4) of Lemma 3.2, we obtain
\begin{equation}
\int_{\mathbb{R}^+} M(s) \psi \left( \frac{s}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) ds
= \int_{\mathbb{R}^+} \mathbb{E} (g(s + S_m); \tau_{s-\varepsilon} \leq m) \mathbb{I}_{\{s+\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \psi \left( \frac{s}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) ds
= \int_{\mathbb{R}^+} \mathbb{E} (g(t + \varepsilon + S_m); \tau_t \leq m) \mathbb{I}_{\{t+2\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \psi \left( \frac{t + \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) dt
= \int_{\mathbb{R}^+} g(t + \varepsilon) \mathbb{E} \left[ \psi \left( \frac{t + S^*_m + \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); t + S^*_m + 2\varepsilon > \varepsilon^{1/6} \sqrt{n}, \tau^*_t \leq m \right] dt.
\end{equation}
Since the function $(t, x) \mapsto \psi(t, x)$ is bounded on $\mathbb{R} \times \mathbb{R}$, it follows that
\begin{equation}
\int_{\mathbb{R}^+} M(s) \psi \left( \frac{s}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) ds
\leq c \int_{\mathbb{R}^+} g(t + \varepsilon) \mathbb{P} \left( t + S^*_m > \frac{1}{2} \varepsilon^{1/6} \sqrt{n}, \tau^*_t \leq m \right) dt
\leq c\varepsilon^{1/12} \int_{\mathbb{R}^+} g(t + \varepsilon) e^{-\frac{t^2}{16\varepsilon^2 \sqrt{n}}} dt + \frac{c_\varepsilon}{n^{1/2}} \|g\|_1, \tag{5.22}
\end{equation}
where the last inequality holds due to the estimates of (3.34), (3.35) and (3.42). Following the same proof of (5.22) and using the fact that $1 - \Phi(\cdot)$ is bounded on $\mathbb{R}$, one has
\begin{equation}
\int_{\mathbb{R}^+} M(s) \left[ 1 - \Phi \left( \frac{s}{\sigma \sqrt{1/4} \sqrt{n}} \right) \right] ds \leq c\varepsilon^{1/12} \int_{\mathbb{R}^+} g(t + \varepsilon) e^{-\frac{t^2}{16\varepsilon^2 \sqrt{n}}} dt + \frac{c_\varepsilon}{n^{1/2}} \|g\|_1. \tag{5.23}
\end{equation}
Implementing (5.22) and (5.23) into (5.21) and using the fact that $\|M\|_1 \leq \|g\|_1$, we get
\begin{equation}
K_2 \leq c \frac{\varepsilon^{1/12}}{\sqrt{n}} \int_{\mathbb{R}^+} g(t + \varepsilon) e^{-\frac{t^2}{16\varepsilon^2 \sqrt{n}}} dt + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \|g\|_1.
\end{equation}
Combining this with (5.17) and (5.20), we derive that

\[ I_{n,2}(x) \leq c \frac{x^{1/12}}{\sqrt{n}} \int_{\mathbb{R}} \left[ g(t) + g(t + \varepsilon) \right] e^{-\frac{t^2}{2\sigma^2n}} dt + \frac{c\varepsilon}{n^{1/2 + \varepsilon}} \| g \|_1. \]

This, together with (5.16), finishes the proof of the lower bound (5.2). □

5.4. Proof of Theorems 1.5 and 1.6. In this section we prove Theorems 1.5 and 1.6 by making use of Theorem 5.1.

**Proof of Theorem 1.5.** Without loss of generality, we assume that the target function \( f \) is non-negative.

Since \( f \leq \delta, \varepsilon \) with \( \delta, \varepsilon \) defined by (2.4), applying the upper bound (5.1) of Theorem 5.1 with \( g = \delta, \varepsilon \), we derive that

\[ I_n := E(f(x + S_n - y); \tau_x > n) \leq 1 + \frac{c\varepsilon}{\sigma \sqrt{n}} \int_{\mathbb{R}} \delta, \varepsilon (t) \left[ 1 - \Phi \left( \frac{t + y}{\sigma \sqrt{n}} \right) \right] dt + \frac{c\varepsilon}{n^{1/2 + \varepsilon}} \int_{\mathbb{R}} \delta, \varepsilon (t) dt. \]

Since the function \( \psi(\cdot, x) \) is Lipschitz continuous on \( \mathbb{R} \), by elementary calculations, there exists a constant \( c > 0 \) such that for any \( t \in \mathbb{R}, x \in [\eta^{-1/4} \sqrt{n}, \eta \sqrt{n}] \) and \( y \in [\eta \sqrt{n}, \sigma \sqrt{qn \log n}] \),

\[ \left| \psi \left( \frac{t + y}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right) - \psi \left( \frac{y}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right) \right| \leq c \frac{|t|}{\sqrt{n}}, \quad (5.24) \]

\[ 1 - \Phi \left( \frac{t + y}{\sigma \sqrt{n}} \right) \leq e^{-c/\sqrt{\varepsilon}} \psi \left( \frac{y}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right), \quad (5.25) \]

\[ \frac{c\varepsilon}{n^2} \leq \frac{c\varepsilon}{n^{1/2 - q}} \psi \left( \frac{y}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right), \quad (5.26) \]

which implies that

\[ I_n \leq 1 + \frac{c\varepsilon}{\sigma \sqrt{n}} \psi \left( \frac{y}{\sigma \sqrt{n}}, \frac{x}{\sigma \sqrt{n}} \right) \left( 1 + \frac{c\varepsilon}{n^{1/2 - q}} \right) \int_{\mathbb{R}} \delta, \varepsilon (t) dt + \frac{c\varepsilon}{n} \int_{\mathbb{R}} \delta, \varepsilon (t) |t| dt. \]

Therefore, uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \) and \( y \in [\eta \sqrt{n}, \sigma \sqrt{qn \log n}] \),

\[ \limsup_{n \to \infty} \frac{E(f(x + S_n - y); \tau_x > n)}{\sigma \sqrt{n}} \leq (1 + c\varepsilon) \int_{-\varepsilon}^{\infty} \delta, \varepsilon (t) dt, \]

which proves the upper bound by taking first \( \varepsilon \to 0 \) and then \( \delta \to 0 \), and using Lemma 2.1. The proof of the lower bound can be carried out in the same way using the lower bound (5.2) together with the fact that
there exists a constant $c > 0$ such that uniformly in $x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}]$ and $y \in [\eta\sqrt{n}, \sigma\sqrt{qn\log n}]$,
\[
|\phi\left(\frac{t + y}{\sigma\sqrt{n}}\right) - \phi\left(\frac{y}{\sigma\sqrt{n}}\right)| \leq c \frac{|t|}{\sqrt{n}}, \quad \phi\left(\frac{y}{\sigma\sqrt{n}}\right) \leq c\phi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right).
\]
The proof of Theorem 1.5 is complete. \qed

Proof of Theorem 1.6. Using (5.1) of Theorem 5.1 with $f = 1_{[y,y+\Delta]}$ and $g = 1_{[y-\varepsilon, y+\Delta+\varepsilon]}$, we derive that uniformly in $x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}]$,
\[
I_n := \mathbb{P}(x + S_n \in [0,\Delta] + y, \tau_x > n)
\leq 1 + c\varepsilon \int_{-\varepsilon}^{\Delta + \varepsilon} \psi\left(\frac{t + y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) dt
\quad + \frac{1 + c\varepsilon}{\sigma\sqrt{n}} \int_{-\varepsilon}^{\Delta + \varepsilon} \left|1 - \Phi\left(\frac{t + y}{\sigma\varepsilon^{1/4}\sqrt{n}}\right)\right| dt + \frac{c\varepsilon}{n^{1/2+\varepsilon}} (\Delta + 2\varepsilon).
\]
Since the function $(s,x) \mapsto \psi(s,x)$ is Lipschitz continuous on $\mathbb{R} \times \mathbb{R}$, there exists a constant $c > 0$ such that for any $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$, $t \in [-\varepsilon, \Delta + \varepsilon]$, $x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}]$ and $y \in [\eta\sqrt{n}, \sigma\sqrt{qn\log n}]$,
\[
\left|\psi\left(\frac{t + y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) - \psi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right)\right| \leq c \frac{t}{\sigma\sqrt{n}} \leq c \frac{\varepsilon}{n^{\varepsilon}}, \quad (5.27)
\]
which, together with the fact that $\Delta + 2\varepsilon \leq (1 + c\varepsilon)\Delta$, implies that
\[
\int_{-\varepsilon}^{\Delta + \varepsilon} \psi\left(\frac{t + y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) dt \leq \Delta (1 + c\varepsilon) \left[\psi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) + c \frac{\varepsilon}{n^{\varepsilon}}\right].
\]
Similarly, using the inequality (5.25), we get
\[
\int_{-\varepsilon}^{\Delta + \varepsilon} \left|1 - \Phi\left(\frac{t + y}{\sigma\varepsilon^{1/4}\sqrt{n}}\right)\right| dt \leq \Delta (1 + c\varepsilon)e^{-c/\sqrt{\varepsilon}} \psi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right).
\]
Therefore, taking into account (5.26), it follows that uniformly in $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$, $x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}]$ and $y \in [\eta\sqrt{n}, \sigma\sqrt{qn\log n}]$,
\[
I_n \leq \Delta \frac{1 + c\varepsilon}{\sigma\sqrt{n}} \left[\psi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) + c \frac{\varepsilon}{n^{\varepsilon}} + c \frac{\varepsilon}{n^{\varepsilon}}\right] + \Delta \frac{1 + c\varepsilon}{\sigma\sqrt{n}} \psi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) \left(1 + c \frac{\varepsilon}{n^{\varepsilon-q}}\right), \quad (5.28)
\]
which implies that uniformly in $\Delta \in [\Delta_0, n^{1/2-\varepsilon}]$, $x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}]$ and $y \in [\eta\sqrt{n}, \sigma\sqrt{qn\log n}]$,
\[
\limsup_{n \to \infty} \frac{\Delta I_n}{\sigma\sqrt{n}} \psi\left(\frac{y}{\sigma\sqrt{n}}, \frac{x}{\sigma\sqrt{n}}\right) \leq 1 + c\varepsilon.
\]
Since $\varepsilon > 0$ can be arbitrary small, this proves the upper bound.
For the lower bound, using (5.2) with \( f = \mathbb{1}_{[y,y+\Delta]} \), \( g = \mathbb{1}_{[y-\varepsilon,y+\Delta+\varepsilon]} \) and \( h = \mathbb{1}_{[y+\varepsilon,y+\Delta-\varepsilon]} \), we get that uniformly in \( x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}] \),

\[
I_n \geq \frac{1}{\sigma\sqrt{n}} \int_{-\varepsilon}^{\Delta-\varepsilon} \psi \left( \frac{t+y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}} \int_{-\varepsilon}^{\Delta} \psi \left( \frac{t+y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}} \int_{-\varepsilon}^{\Delta} \phi \left( \frac{t+y}{\sigma\sqrt{n}} \right) dt
- c_\varepsilon \int_{-\varepsilon}^{\Delta} \left[ 1 - \Phi \left( \frac{t+y}{\sigma\varepsilon^{1/4}\sqrt{n}} \right) \right] dt
- \frac{c_\varepsilon}{\eta^{1/2+\varepsilon}} (\Delta + 2\varepsilon)
= I_{n,1} - I_{n,2} - I_{n,3} - I_{n,4} - I_{n,5}.
\]

For \( I_{n,1} \) and \( I_{n,2} \), using (5.27) and proceeding in the same way as (5.28), we get that uniformly in \( \Delta \in [\Delta_0, n^{1/2-\varepsilon}] \),

\[
I_{n,1} \geq \Delta \frac{1 - c_\varepsilon}{\sigma\sqrt{n}} \psi \left( \frac{y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}} \left( 1 + \frac{c_\varepsilon}{n^{\varepsilon-q}} \right),
I_{n,2} \leq \Delta \frac{c_\varepsilon}{\sigma\sqrt{n}} \psi \left( \frac{y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}} \left( 1 + \frac{c_\varepsilon}{n^{\varepsilon-q}} \right).
\]

For \( I_{n,3} \), using the inequality (5.25) yields that

\[
I_{n,3} \leq \Delta \frac{c_\varepsilon - c\varepsilon}{\sigma\sqrt{n}} \psi \left( \frac{y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}}.
\]

For \( I_{n,4} \), by (5.26), there exists a constant \( c > 0 \) such that for any \( \Delta \in [\Delta_0, n^{1/2-\varepsilon}] \), \( \Delta \in [-2\varepsilon, \Delta + \varepsilon] \), \( x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}] \) and \( y \in [\eta\sqrt{n}, \sigma\sqrt{qn \log n}] \),

\[
\phi \left( \frac{t+y}{\sigma\sqrt{n}} \right) \leq \phi \left( \frac{y}{\sigma\sqrt{n}} \right) + \frac{c}{n^{\varepsilon}} \leq \psi \left( \frac{y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}} \left( 1 + \frac{c_\varepsilon}{n^{\varepsilon-q}} \right).
\]

This, together with the fact that \( \Delta + 3\varepsilon \leq (1 + c\varepsilon)\Delta \), implies that

\[
I_{n,4} \leq \Delta \frac{c_\varepsilon^{1/2}}{\sqrt{n}} \psi \left( \frac{y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}} \left( 1 + \frac{c_\varepsilon}{n^{\varepsilon-q}} \right).
\]

For \( I_{n,5} \), using again \( \Delta + 2\varepsilon \leq (1 + c\varepsilon)\Delta \) and (5.26), we derive that uniformly in \( \Delta \in [\Delta_0, n^{1/2-\varepsilon}] \), \( x \in [\eta^{-1}\sqrt{n}, \eta\sqrt{n}] \) and \( y \in [\eta\sqrt{n}, \sigma\sqrt{qn \log n}] \),

\[
I_{n,5} \leq \Delta \frac{c_\varepsilon}{n^{1/2+\varepsilon}} \leq \Delta \frac{c_\varepsilon}{n^{1/2+\varepsilon-q}} \psi \left( \frac{y}{\sigma\sqrt{n}} \right) \frac{x}{\sigma\sqrt{n}}.
\]

Putting together the above bounds for \( I_{n,1}, I_{n,2}, I_{n,3}, I_{n,4} \) and \( I_{n,5} \), letting first \( n \to \infty \) and then \( \varepsilon \to 0 \), we obtain the desired lower bound. The proof of Theorem 1.6 is complete. \( \square \)
6. Conditioned local limit theorem far from the boundary

The goal of this section is to establish Theorems 1.7, 1.8, 1.9, 1.10, 1.11 and 1.12. Recall that \( \phi^+ (s) = \frac{s}{v} e^{-\frac{s^2}{4v}} \mathbb{1}_{s \geq 0} \), \( s \in \mathbb{R} \) is the Rayleigh density with scale parameter \( \sqrt{v} \), and \( \psi_v \) is defined by (5.3). The following lemma will be used to prove Theorems 1.7 and 1.8.

**Lemma 6.1.** For any \( x \geq 0 \) and \( v \in (0, 1) \), we have

\[
\int_{\mathbb{R}^+} \phi^+_v (s) \psi_{1-v} (s, x) \, ds = \sqrt{v} \phi^+(x).
\]

**Proof.** Notice that

\[
\int_{\mathbb{R}^+} \phi^+_v (s) \psi_{1-v} (s, x) \, ds
= \frac{1}{\sqrt{2\pi(1-v)}} \int_{\mathbb{R}^+} \frac{s}{v} e^{-\frac{s^2}{2v}} \left( e^{-\frac{(s-x)^2}{2(1-v)}} - e^{-\frac{(s+x)^2}{2(1-v)}} \right) \, ds
= \frac{1}{v\sqrt{2\pi(1-v)}} e^{-\frac{x^2}{2(1-v)}} \int_{\mathbb{R}^+} se^{-\frac{s^2}{2v(1-v)}} \left( e^{\frac{x^2}{2v}} - e^{-\frac{x^2}{2v}} \right) \, ds.
\]

For the first integral, we have that for any \( x \geq 0 \),

\[
\int_{\mathbb{R}^+} se^{-\frac{s^2}{2v(1-v)} + \frac{x^2}{2v}} \, ds = e^{\frac{vx^2}{2(1-v)}} \int_{\mathbb{R}^+} se^{-\frac{x^2}{2v(1-v)}} \, ds
= e^{\frac{vx^2}{2(1-v)}} \int_{-vx}^{\infty} (t + vx) e^{-\frac{t^2}{2v(1-v)}} \, dt
= e^{\frac{vx^2}{2(1-v)}} \left[ v(1-v)e^{-\frac{vx^2}{2v(1-v)}} + vx \int_{-vx}^{\infty} e^{-\frac{t^2}{2v(1-v)}} \, dt \right]. \quad (6.1)
\]

For the second integral, by replacing \( x \) with \( -x \), we get

\[
\int_{\mathbb{R}^+} se^{-\frac{s^2}{2v(1-v)} - \frac{x^2}{2v}} \, ds = e^{\frac{vx^2}{2v(1-v)}} \left[ v(1-v)e^{-\frac{vx^2}{2v(1-v)}} - vx \int_{vx}^{\infty} e^{-\frac{t^2}{2v(1-v)}} \, dt \right]. \quad (6.2)
\]

Therefore, taking the difference between (6.1) and (6.2), we obtain

\[
\int_{\mathbb{R}^+} \phi^+_v (s) \psi_{1-v} (s, x) \, ds
= \frac{1}{v\sqrt{2\pi(1-v)}} e^{-\frac{x^2}{2v}} \left( vx \int_{-vx}^{\infty} e^{-\frac{t^2}{2v(1-v)}} \, dt + vx \int_{vx}^{\infty} e^{-\frac{t^2}{2v(1-v)}} \, dt \right)
= \frac{1}{\sqrt{2\pi(1-v)}} xe^{-\frac{x^2}{2v}} \int_{\mathbb{R}} e^{-\frac{t^2}{2v(1-v)}} \, dt = \sqrt{v} \phi^+(x),
\]

which ends the proof of the lemma. \( \square \)
6.1. A non-asymptotic conditioned local limit theorem.

Theorem 6.2. Assume A1 and A2. Then, for any $\eta > 0$, there exist constants $c > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any sequence of positive numbers $(\alpha_n)_{n \geq 1}$ satisfying $\lim_{n \to \infty} \alpha_n = 0$, one can find a constant $c_{\varepsilon} > 0$ such that uniformly in $x \in [n^{1/2-\varepsilon}, \eta \sqrt{n}]$ and $n \geq 1$, the following holds:

1. For any measurable functions $f, g : \mathbb{R} \mapsto \mathbb{R}_+$ satisfying $f \leq \varepsilon g$ and $\int_{\mathbb{R}_+} g(t - \varepsilon)(1 + t)dt < \infty$,
   \[
   \mathbb{E}(f(x + S_n); \tau_x > n) \leq \frac{2}{\sigma^2 \sqrt{2\pi n}} \left[ \phi^+ \left( \frac{x}{\sigma \sqrt{n}} \right) + \left( c\varepsilon^{1/4} + c_{\varepsilon} \alpha_n \right) \right] \int_{0}^{\alpha_n \sqrt{n}} g(t - \varepsilon)V^*(t)dt 
   + \frac{c}{\sqrt{n}} \int_{\alpha_n \sqrt{n}}^{\infty} g(t - \varepsilon)dt + \frac{c_{\varepsilon}}{n^{1+\varepsilon}} \int_{\mathbb{R}_+} g(t - \varepsilon)(1 + t)dt. \tag{6.3}
   
   2. For any measurable functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}_+$ satisfying $h \leq f \leq \varepsilon g$ and $\int_{\mathbb{R}_+} g(t - \varepsilon)(1 + t)dt < \infty$,
   \[
   \mathbb{E}(f(x + S_n); \tau_x > n) \geq \frac{2}{\sigma^2 \sqrt{2\pi n}} \left[ \phi^+ \left( \frac{x}{\sigma \sqrt{n}} \right) - (c\varepsilon + c_{\varepsilon} \alpha_n) \right] \int_{0}^{\alpha_n \sqrt{n}} h(t + \varepsilon)V^*(t)dt 
   + \frac{c}{\sqrt{n}} \int_{\alpha_n \sqrt{n}}^{\infty} g(t - \varepsilon)dt - \left( \frac{c_{\varepsilon}^{1/2}}{n} + \frac{c_{\varepsilon}}{n^{1+\varepsilon}} \right) \int_{\mathbb{R}_+} g(t - \varepsilon)(1 + t)dt. \tag{6.4}
   
   Proof. We first prove the upper bound (6.3). Set $m = [n/2]$ and $k = n - m$. It is shown in (4.3), (4.4) and (4.5) that
   \[I_n(x) := \mathbb{E}(f(x + S_n); \tau_x > n) = \int_{\mathbb{R}_+} I_m(t) \mathbb{P}(x + S_k \in dt, \tau_x > k)\]
   and $I_m \leq \varepsilon H_m$, where
   \[H_m(t) := \mathbb{E}(g(x + S_m); \tau_{t+\varepsilon} > m) \mathbb{1}_{\{t > -\varepsilon\}}, \quad t \in \mathbb{R}.
   
   By the upper bound (5.1) of Theorem 5.1, we get that uniformly in $x \in [n^{1/2-\varepsilon}, \eta \sqrt{n}]$,
   \[I_n(x) \leq (1 + c\varepsilon)(J_1 + J_2 + J_3), \tag{6.5}\]
   where
   \[J_1 = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}} H_m(t) \psi \left( \frac{t}{\sigma \sqrt{k}} \right) dt, \quad J_2 = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}} H_m(t) \left[ 1 - \Phi \left( \frac{t}{\sigma \varepsilon^{1/4} \sqrt{k}} \right) \right] dt, \quad J_3 = \frac{c_{\varepsilon}}{n^{1+\varepsilon}} \|H_m\|_1.\]
Conditioned Local Limit Theorems

Bound of $J_1$. As in (4.7), (4.8) and (4.9), using a change of variable and the duality formula (Lemma 3.2), we get

$$J_1 = \frac{1}{\sigma \sqrt{k}} \int_{-\varepsilon}^{\infty} E(g(t + S_m); \tau_{t+\varepsilon} > m) \psi \left( \frac{t}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) dt$$

$$\quad = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}_+} E(g(t + S_m - \varepsilon); \tau_t > m) \psi \left( \frac{t - \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right) dt$$

$$\quad = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}_+} (t - \varepsilon) E \left[ \psi \left( \frac{t + S^*_m - \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt$$

$$\quad =: J_{11} + J_{12}, \quad (6.6)$$

where

$$J_{11} = \frac{1}{\sigma \sqrt{k}} \int_{\alpha_n \sqrt{n}}^{\infty} g(t - \varepsilon) E \left[ \psi \left( \frac{t + S^*_m - \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt,$$

$$J_{12} = \frac{1}{\sigma \sqrt{k}} \int_{0}^{\alpha_n \sqrt{n}} g(t - \varepsilon) E \left[ \psi \left( \frac{t + S^*_m - \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt.$$

Bound of $J_{11}$. Since the function $\psi(\cdot, \cdot)$ is bounded on $\mathbb{R} \times \mathbb{R}$, we get

$$J_{11} \leq \frac{c}{\sqrt{n}} \int_{\alpha_n \sqrt{n}}^{\infty} g(t - \varepsilon) dt \quad (6.7)$$

Bound of $J_{12}$. For brevity we denote $\psi'_1(u, x) = \frac{d}{du} \psi(u, x)$. Following the proof of (4.11) and (4.12), from the conditioned integral limit theorem (2.26) of Corollary 2.9 and the fact that $\int_{\mathbb{R}_+} |\psi'_1(u, \frac{x}{\sigma \sqrt{k}})| du < \infty$, one has that uniformly in $t \in [0, \alpha_n \sqrt{n}]$,

$$\left| E \left[ \psi \left( \frac{t + S^*_m - \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] \right| - \frac{2V^*(t)}{\sigma \sqrt{2\pi m}} \int_{\mathbb{R}_+} \psi'_1 \left( u, \frac{x}{\sigma \sqrt{k}} \right) \left( 1 - \Phi^+(u_{k,m,\varepsilon}) \right) du \leq c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V^*(t)}{n^{1/2}},$$

where $u_{k,m,\varepsilon} = \frac{u \sigma \sqrt{k} + \varepsilon}{\sigma \sqrt{m}}$. Using integration by parts and the fact that the function $(\phi^+)'$ is Lipschitz on $\mathbb{R}_+$ and $\int_{\mathbb{R}_+} \psi(u, \frac{x}{\sigma \sqrt{k}}) du < \infty$ uniformly in
where in the last equality we used Lemma 6.1. Thus, we deduce that

\[
J_{12} \leq \frac{2}{\sigma^2 \sqrt{2\pi n}} \left[ \phi^+ \left( \frac{x}{\sigma \sqrt{n}} \right) + c_\varepsilon (\alpha_n + n^{-\varepsilon}) \right] \int_0^{\alpha_n \sqrt{n}} g(t-\varepsilon)V^*(t)dt.
\]

(6.8)

**Bound of** \( J_2 \). In the same way as in (6.6), using a change of variable and the duality formula ((3.3) of Lemma 3.2), one has

\[
J_2 = \frac{1}{\sigma \sqrt{k}} \int_{\text{R}_+} g(t-\varepsilon)\mathbb{E} \left[ 1 - \Phi \left( \frac{t + S_m^* - \varepsilon}{\sigma \varepsilon^{1/4} \sqrt{k}} \right) ; \tau_t^* > m \right] dt.
\]

(6.9)

We decompose \( J_2 \) into two parts: \( J_2 = J_{21} + J_{22} \), where

\[
J_{21} = \frac{1}{\sigma \sqrt{k}} \int_{\alpha_n \sqrt{n}}^{\infty} g(t-\varepsilon)\mathbb{E} \left[ 1 - \Phi \left( \frac{t + S_m^* - \varepsilon}{\sigma \varepsilon^{1/4} \sqrt{k}} \right) ; \tau_t^* > m \right] dt,
\]

\[
J_{22} = \frac{1}{\sigma \sqrt{k}} \int_0^{\alpha_n \sqrt{n}} g(t-\varepsilon)\mathbb{E} \left[ 1 - \Phi \left( \frac{t + S_m^* - \varepsilon}{\sigma \varepsilon^{1/4} \sqrt{k}} \right) ; \tau_t^* > m \right] dt.
\]

**Bound of** \( J_{21} \). Since the function \( 1 - \Phi \) is bounded, we get

\[
J_{21} \leq \frac{c_\varepsilon}{\sqrt{n}} \int_0^{\alpha_n \sqrt{n}} g(t-\varepsilon)dt.
\]

(6.10)

**Bound of** \( J_{22} \). Proceeding in the same way as in (4.11) and (4.12), from the conditioned integral limit theorem (2.26) of Corollary 2.9 and the fact that
1 - \Phi(\infty) = 0$, one has that uniformly in $t \in \left[0, \alpha_n \sqrt{n}\right]$,

$$
\left| \mathbb{E} \left[ 1 - \Phi \left( \frac{t + S_m^* - \varepsilon}{\sigma \varepsilon^{1/4} \sqrt{k}} \right) ; \tau_t^* > m \right] - \frac{2V^*(t)}{\sigma \sqrt{2\pi m}} \int_{\mathbb{R}^+} \phi(v) \Phi^+(v, m, \varepsilon) dv \right| 
\leq c \varepsilon (\alpha_n + n^{-\varepsilon}) \frac{V^*(t)}{n^{1/2}},
$$

(6.11)

where $v, m, \varepsilon = \varepsilon^{1/4} \sqrt{\frac{k}{n}}$. Since the function $\Phi^+$ is Lipschitz continuous, by elementary calculations, we have

$$
\int_{\mathbb{R}^+} \phi(v) \Phi^+(v, m, \varepsilon) dv \leq \int_{\mathbb{R}^+} \phi(v) \Phi^+ \left( \varepsilon^{1/4} \sqrt{\frac{k}{m}} v \right) dv + c \varepsilon \sqrt{\frac{\sigma}{n}}
= \frac{1}{2} - \frac{1}{2} \left( 1 + \frac{\sqrt{\varepsilon k}}{m} \right)^{-1/2} + c \varepsilon \sqrt{\frac{n}{m}} \leq c \varepsilon^{1/4}.
$$

Combining this with (6.11) and (6.10), we derive that

$$
J_2 \leq \frac{1}{n} \left( c \varepsilon^{1/4} + c \varepsilon \alpha_n + c \varepsilon n^{-\varepsilon} \right) \int_0^{\alpha_n \sqrt{n}} g(t - \varepsilon) V^*(t) dt
+ \frac{c \varepsilon}{\sqrt{n}} \int_{\alpha_n \sqrt{n}} g(t - \varepsilon) dt.
$$

(6.12)

Bound of $J_3$. By (4.17), we have

$$
J_3 \leq \frac{c \varepsilon}{n^{1/4}} \int_{\mathbb{R}^+} g(t - \varepsilon)(1 + t) dt.
$$

(6.13)

Putting together (6.7), (6.8), (6.12) and (6.13), we conclude the proof of the upper bound (6.3) of the theorem.

We next sketch the proof of the lower bound (6.4). Similarly to the proof of (6.5), we use the lower bound (5.2) of Theorem 5.1 to get $I_n(x) \geq K_1 - K_2 - K_3$, where

$$
K_1 = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}} L_m(t) \psi \left( \frac{t}{\sigma \sqrt{k}} - \frac{x}{\sigma \sqrt{k}} \right) dt,
$$

$$
K_2 = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}} H_m(t) \left[ 1 - \Phi \left( \frac{t}{\sigma \varepsilon^{1/4} \sqrt{k}} \right) \right] dt
$$

$$
K_3 = \frac{c \varepsilon}{\sigma \sqrt{k}} \int_{\mathbb{R}} H_m(t) \psi \left( \frac{t}{\sigma \sqrt{k}} - \frac{x}{\sigma \sqrt{k}} \right) dt
+ \frac{c \varepsilon^{1/2}}{\sqrt{n}} \int_{\mathbb{R}} \left[ H_m(t) + H_m(t + \varepsilon) \right] \phi \left( \frac{t}{\sigma \sqrt{k}} \right) dt + \frac{c \varepsilon}{n^{1/2 + \varepsilon}} \| H_m \|_1,
$$

with $H_m$ and $L_m$ defined by (4.5) and (4.20), respectively.
Bound of $K_1$. Note that $L_m(t) = 0$ for $t < \varepsilon$. As in the estimate of $J_1$ (cf. (6.6)), using the duality formula (Lemma 3.2), we get

$$K_1 = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}_+} h(t + \varepsilon) \mathbb{E} \left[ \psi \left( \frac{t + S_m^* + \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt \geq \frac{1}{\sigma \sqrt{k}} \int_0^{\alpha \sqrt{n}} h(t + \varepsilon) \mathbb{E} \left[ \psi \left( \frac{t + S_m^* + \varepsilon}{\sigma \sqrt{k}}, \frac{x}{\sigma \sqrt{k}} \right); \tau^*_t > m \right] dt,$$

where in the last inequality we used the fact that $t + S_m^* + \varepsilon \geq \varepsilon$ on the event $\tau^*_t > m$, so that $\psi > 0$. Following the proof of (6.8), one has

$$K_1 \geq \frac{2}{\sigma^2 \sqrt{2\pi n}} \left[ \phi^* \left( \frac{x}{\sigma \sqrt{n}} \right) - c \varepsilon \left( \alpha_n + n^{-\varepsilon} \right) \right] \int_0^{\alpha \sqrt{n}} h(t + \varepsilon) V^*(t) dt.$$

Bound of $K_2$. By (6.9) and (6.12), we have

$$K_2 = \frac{1}{\sigma \sqrt{k}} \int_{\mathbb{R}_+} g(t - \varepsilon) \mathbb{E} \left[ 1 - \Phi \left( \frac{t + S_m^* - \varepsilon}{\sigma \varepsilon^{1/4}\sqrt{k}} \right); \tau^*_t > m \right] dt \leq \frac{1}{n} \left( c^{1/4} + c \alpha_n + c n^{-\varepsilon} \right) \int_0^{\alpha \sqrt{n}} g(t - \varepsilon) V^*(t) dt + \frac{c}{\sqrt{n}} \int_0^{\infty} g(t - \varepsilon) (1 + t) dt.$$

Bound of $K_3$. Since the functions $\psi$ and $\phi$ are bounded, using (4.17) we get

$$K_3 \leq \left( \frac{c^{1/12}}{\sqrt{n}} + \frac{c \varepsilon}{n^{1/2 + \varepsilon}} \right) \| H_m \|_1 \leq \left( \frac{c^{1/12}}{n} + \frac{c \varepsilon}{n^{1 + \varepsilon}} \right) \int_{\mathbb{R}_+} g(t - \varepsilon) (1 + t) dt.$$

Collecting the above bounds for $K_1$, $K_2$ and $K_3$, and using the fact that $V^*(t) \leq c(1 + t)$ for $t \in \mathbb{R}_+$ and $h(t + \varepsilon) \leq f(t) \leq g(t - \varepsilon)$ for any $t \in \mathbb{R}$, we conclude the proof of the lower bound (6.4). \hfill \Box

6.2. Proof of Theorems 1.7 and 1.8.

Proof of Theorem 1.7. We first prove (1.16). Since $\int_{\mathbb{R}_+} f(t) (1 + t)^\gamma dt < \infty$ for some constant $\gamma > 1$, the result (1.16) is a consequence of Theorem 6.2, Lemma 2.1 and the Lebesgue dominated convergence theorem.

We next prove (1.17). Applying the upper bound (6.3) of Theorem 6.2 with $g = T_{\delta, \varepsilon}$ and $\alpha_n$ replaced by $2\alpha_n$, we get that uniformly in $x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}]$ and $y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}]$,

$$\mathbb{E} \left( f(x + S_n - y); \tau_x > n \right) \leq \frac{2}{\sigma^2 \sqrt{2\pi n}} \left[ \phi^* \left( \frac{x}{\sigma \sqrt{n}} \right) + (c^{1/4} + c \varepsilon \alpha_n) \right] \int_0^{\alpha \sqrt{n}} T_{\delta, \varepsilon} (t - y - \varepsilon) V^*(t) dt \leq \frac{c}{\sqrt{n}} \int_{2\alpha \sqrt{n}}^{\infty} T_{\delta, \varepsilon} (t - y - \varepsilon) dt + \frac{c \varepsilon}{n^{1 + \varepsilon}} \int_{\mathbb{R}_+} T_{\delta, \varepsilon} (t - y - \varepsilon) (1 + t) dt.$$
Using (4.22), (4.23) and (4.24), and the fact that \( \phi^+ (\frac{x}{\sigma \sqrt{n}}) \) is bounded from below by a constant \( c_\eta > 0 \) for \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \), we get that uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \) and \( y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}] \),
\[
\limsup_{n \to \infty} \frac{\mathbb{E} \left( f(x + S_n - y); \tau_x > n \right)}{\sqrt{2 \pi \sigma^2 n} \phi^+ (\frac{x}{\sigma \sqrt{n}})} \leq \left( 1 + c \varepsilon^{1/4} \right) \int_{-\varepsilon}^{\infty} F_{\delta, \varepsilon}(u) du.
\]
This yields the desired upper bound by taking first \( \varepsilon \to 0 \) and then \( \delta \to 0 \), and using Lemma 2.1. The lower bound can be proved in the same way by using (6.4) of Theorem 6.2. The proof of (1.17) is complete. \( \square \)

**Proof of Theorem 1.8.** Applying (6.3) of Theorem 6.2 with \( f = \mathbb{1}_{[y, y + \Delta]} \), \( g = \mathbb{1}_{[y - \varepsilon, y + \Delta + \varepsilon]} \) and \( \alpha_n \) replaced by \( 2\alpha_n \) (so that the second integral in (6.3) vanishes), we derive that uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \), \( y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}] \) and \( \Delta \in [\Delta_0, o(y)] \),
\[
\mathbb{P} \left( x + S_n \in [0, \Delta] + y, \tau_x > n \right) 
\]
\[
\leq \frac{2}{\sigma \sqrt{2 \pi n}} \left[ \phi^+ \left( \frac{x}{\sigma \sqrt{n}} \right) + (c \varepsilon^{1/4} + c \varepsilon \alpha_n) \right] \int_{-\varepsilon}^{\Delta + \varepsilon} V^*(t + y + \varepsilon) dt + \frac{c \varepsilon}{n^{1+\varepsilon}} \int_{-\varepsilon}^{\Delta + \varepsilon} \left( 1 + t + y \right) dt.
\]
Since \( 1 \leq c_\eta \phi^+ (\frac{x}{\sigma \sqrt{n}}) \) holds uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \) for some constant \( c_\eta > 0 \), using (4.26) and (4.27), we obtain that uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \), \( y \in [\alpha_n^{-1}, \alpha_n \sqrt{n}] \) and \( \Delta \in [\Delta_0, o(y)] \),
\[
\limsup_{n \to \infty} \frac{\mathbb{P} \left( x + S_n \in [0, \Delta] + y, \tau_x > n \right)}{\frac{2y \Delta}{\sigma^2 \sqrt{2 \pi n}} \phi^+ (\frac{x}{\sigma \sqrt{n}})} \leq \left( 1 + c_\eta \varepsilon^{1/4} \right),
\]
which concludes the proof of the upper bound by letting \( \varepsilon \to 0 \). The lower bound can be obtained in a similar way by using (6.4). The proof of Theorem 1.8 is complete. \( \square \)

6.3. **Proof of Theorems 1.9 and 1.10.** Theorems 1.9 and 1.10 are easy consequences of the following result, which is deduced from Theorems 1.3 and 1.7.

**Theorem 6.3.** Assume \( A1 \) and \( A3 \). Let \( f : \mathbb{R}_+ \mapsto \mathbb{R} \) be a directly Riemann integrable function satisfying \( \int_{\mathbb{R}_+} f(t) e^{-\lambda t} (1 + t)^\gamma dt < \infty \) for some constant \( \gamma > 1 \). Then, for any sequence of positive numbers \( (\alpha_n)_{n \geq 1} \) satisfying \( \lim_{n \to \infty} \alpha_n = 0 \), we have, as \( n \to \infty \), uniformly in \( x \in [0, \alpha_n \sqrt{n}] \),
\[
\mathbb{E} \left( f(x + S_n); \tau_x > n \right) \sim \frac{2 V_\lambda (x) e^{n \lambda (\lambda) + \lambda x}}{\sqrt{2 \pi \sigma^3 n^{3/2}}} \int_{\mathbb{R}_+} f(t) e^{-\lambda t} V_\lambda^*(t) dt. \tag{6.14}
\]
Moreover, for any \( \eta \geq 1 \), we have, as \( n \to \infty \), uniformly in \( x \in [\eta^{-1} \sqrt{n}, \eta \sqrt{n}] \),
\[
E(f(x + S_n); \tau_x > n) \sim \frac{2e^{n\Lambda(\lambda)+\lambda x}}{\sqrt{2\pi \sigma^2_n}} \phi^+(\frac{x}{\sigma \sqrt{n}}) \int_{\mathbb{R}_+} f(t)e^{-\lambda V^*_\lambda(t)} dt.
\]

**Proof.** By a change of measure, we get
\[
E(f(x + S_n); \tau_x > n) = e^{n\Lambda(\lambda)} E_\lambda \left( e^{-\lambda S_n} f(x + S_n); \tau_x > n \right)
\]
\[
= e^{n\Lambda(\lambda)+\lambda x} E_\lambda \left( e^{-\lambda(x+S_n)} f(x + S_n); \tau_x > n \right).
\]

Since \( \int_{\mathbb{R}_+} f(t)e^{-\lambda(1+t)^\gamma} dt < \infty \) for some constant \( \gamma > 1 \), applying (1.9) of Theorem 1.3 we get (6.14), and applying (1.16) of Theorem 1.7 we obtain (6.15).

**Proof of Theorems 1.9 and 1.10.** In the particular case when the random walk \( S_n \) has negative drift, i.e. \( \mu = EX_1 < 0 \) (in this case \( \lambda > 0 \) by condition A3), taking \( f = \mathbb{1}_{[0,\infty)} \) in (6.14) we get Theorem 1.9, and taking \( f = \mathbb{1}_{[0,\infty)} \) in (6.15) we get Theorem 1.10.

**Lemma 6.4.** Assume A3 for some \( \lambda > 0 \). Then,
\[
\int_{\mathbb{R}_+} P(t + X_1 < 0)e^{-\lambda V^*_\lambda(t)} dt = E e^{\lambda X_1} \int_{-\infty}^{0} e^{-\lambda V^*_\lambda(t)} dt \in (0, \infty).
\]

**Proof.** By Fubini’s theorem and a change of variable, we get
\[
I := \int_{\mathbb{R}_+} P(t + X_1 < 0)e^{-\lambda V^*_\lambda(t)} dt
\]
\[
= E \int_{0}^{\infty} \mathbb{1}_{\{t+X_1<0\}} e^{-\lambda V^*_\lambda(t)} dt
\]
\[
= E \int_{X_1}^{0} e^{-\lambda(s-X_1)} V^*_\lambda(s - X_1) ds
\]
\[
= \int_{-\infty}^{0} E \left[ e^{-\lambda(s-X_1)} V^*_\lambda(s - X_1); s - X_1 \geq 0 \right] ds.
\]

By the harmonicity property of the function \( V^*_\lambda \), we have that for any \( s \leq 0 \),
\[
E_\lambda [V^*_\lambda(s - X_1); s - X_1 \geq 0] = V^*_\lambda(s).
\]

Then, by the definition of \( E_\lambda \) it follows that
\[
I = E e^{\lambda X_1} \int_{-\infty}^{0} e^{-\lambda s} E_\lambda [V^*_\lambda(s - X_1); s - X_1 \geq 0] ds
\]
\[
= E e^{\lambda X_1} \int_{-\infty}^{0} e^{-\lambda s} V^*_\lambda(s) ds.
\]
which proves the equality in (6.16). Using Markov’s inequality, condition A3 and the fact that $V^*_a$ is strictly positive and non-decreasing on $\mathbb{R}_+$, it is easy to see that $I \in [0, \infty)$. Suppose that $I = 0$, then $\mathbb{P}(X_1 < 0) = 0$. This contradicts to the first requirement in condition A3, so that $I \in (0, \infty)$. □

Proof of Theorems 1.11 and 1.12. We only give a proof of Theorem 1.12 since Theorem 1.11 is a particular case of Theorem 1.12 by taking $\lambda = 0$.

We first prove (1.25). By the definition of $\tau_x$, we have for any $n \geq 1$,

$$
\mathbb{P}(\tau_x = n + 1) = \mathbb{P}(x + S_{n+1} < 0, \tau_x > n).
$$

From the Markov property, it follows that

$$
\mathbb{P}(x + S_{n+1} < 0, \tau_x > n) = \mathbb{E}(f(x + S_n); \tau_x > n),
$$

where

$$
f(t) = \mathbb{P}(t + X_1 < 0) \mathbb{I}_{(t \geq 0)}, \quad t \in \mathbb{R}.
$$

It is easy to see that $0 \leq f \leq 1$ and $f$ is a non-increasing function on $\mathbb{R}$. In addition, using Markov’s inequality and condition A3, we get that for $t \geq 1$,

$$
f(t) \leq \mathbb{P}(|X_1| > t) \leq \frac{1}{t^{2+\delta} e^{2t}} \mathbb{E}|X_1|^{2+\delta} e^{\Lambda X_1} \leq \frac{c}{t^{2+\delta} e^{2t}},
$$

so that by taking $\gamma = 1 + \frac{\delta}{2}$,

$$
\int_{\mathbb{R}_+} f(t) e^{-\Lambda t} (1 + t)^{\gamma} dt = \int_0^1 f(t) e^{-\Lambda t} (1 + t)^{\gamma} dt + \int_1^\infty f(t) e^{-\Lambda t} (1 + t)^{\gamma} dt \leq c.
$$

Hence the function $f$ satisfies the condition stated in Theorem 6.3. Using (6.14) of Theorem 6.3, we obtain that as $n \to \infty$, uniformly in $x \in [0, \alpha_n \sqrt{n}]$,

$$
\mathbb{P}(\tau_x = n + 1) \sim \frac{2V^*_a(x)e^{\Lambda X_1} + \lambda x}{\sqrt{2\pi} \alpha_n^{3/2}} \int_{\mathbb{R}_+} \mathbb{P}(t + X_1 < 0) e^{-\Lambda t} V^*_a(t) dt,
$$

which, together with Lemma 6.4, concludes the proof of (1.25).

The asymptotic (1.26) can be obtained in the same way by using (6.15) of Theorem 6.3. The proof of Theorems 1.11 and 1.12 is complete. □

7. Appendex: Proof of conditioned integral limit theorems

7.1. Auxiliary results. The goal of this section is to prove some auxiliary results which will be used to establish the conditioned integral limit theorems (Theorems 2.7 and 2.8).

Lemma 7.1. Assume A2 for some $\delta > 0$. Then there exists a constant $c > 0$ such that for any $\varepsilon \in \left[0, \frac{\delta}{2(2+\delta)}\right]$, $n \geq 1$ and $x \geq n^{1/2-\varepsilon}$,

$$
\mathbb{E}(x + S_n; \tau_x > n) \leq x + cn^{1/2-\varepsilon} \leq \left(1 + \frac{c}{n^\varepsilon}\right)x.
$$
Proof. By the optional stopping theorem, we get
\[
\mathbb{E}(x + S_n; \tau_x > n) = x - \mathbb{E}(x + S_n; \tau_x \leq n) \\
= x - \mathbb{E}(x + S_{\tau_x}; \tau_x \leq n).
\]  
(7.1)

By the definition of \(\tau_x\), we have \(X_{\tau_x} \leq x + S_{\tau_x} < 0\) and hence 
\[
\mathbb{E}(x + S_n; \tau_x > n) < x + \mathbb{E}(|X_{\tau_x}|; \tau_x \leq n).
\]

Using Hölder’s inequality, condition A2 and Markov’s inequality gives 
\[
\mathbb{E} \left( |X_k|; |X_k| > n^{1/2-\varepsilon} \right) \leq \mathbb{E}^{\frac{2+\delta}{2+\varepsilon}} \left( |X_k|^{2+\delta} \right) \mathbb{P}^{\frac{\varepsilon}{2+\delta}} \left( |X_k| > n^{1/2-\varepsilon} \right) \\
\leq c n^{-\left(1/2-\varepsilon\right)(1+\delta)},
\]
from which it follows that 
\[
\mathbb{E} \left( |X_{\tau_x}|; \tau_x \leq n \right) \leq n^{1/2-\varepsilon} + \mathbb{E} \left( |X_{\tau_x}|; |X_{\tau_x}| > n^{1/2-\varepsilon}, \tau_x \leq n \right) \\
\leq n^{1/2-\varepsilon} + \sum_{k=1}^{n} \mathbb{E} \left( |X_k|; |X_k| > n^{1/2-\varepsilon} \right) \\
\leq n^{1/2-\varepsilon} + cn^{1-\left(1/2-\varepsilon\right)(1+\delta)} \leq cn^{1/2-\varepsilon},
\]
where in the last inequality we used the fact that \(\varepsilon \in \left[0, \frac{\delta}{2(2+\delta)}\right]\). The desired result follows. \(\square\)

For \(\varepsilon \in (0, 1/2)\) and \(x \geq 0\), consider the first time when the absolute value of the random walk \((x + S_k)_{k \geq 1}\) exceeds the level \(n^{1/2-\varepsilon}\):
\[
\nu_n = \nu_{n,x,\varepsilon} = \inf \left\{ k \geq 1 : |x + S_k| > n^{1/2-\varepsilon} \right\}.
\]  
(7.2)

It is easy to see that the stopping time \(\nu_n\) converges \(\mathbb{P}\)-a.s. to \(\infty\) as \(n \to \infty\). The following result gives the tail behavior of \(\nu_n\).

Lemma 7.2. Assume A2 for some \(\delta > 0\). Then, for any \(\varepsilon \in (0, 1/2)\) and \(\beta > 0\), there exists a constant \(c_{\varepsilon,\beta}\) such that for any \(n \geq 1\) and \(x \geq 0\),
\[
\mathbb{P} \left( \nu_n > \beta n^{1-\varepsilon} \right) \leq c_{\varepsilon,\beta} e^{-c_{\varepsilon,\beta} n^\varepsilon}.
\]

Proof. Set \(K := \left[n^\varepsilon\right]\) and split the interval \([1, \beta n^{1-\varepsilon}]\) into \(K\) subintervals of length \(l := \left[\beta n^{1-2\varepsilon}\right]\), so that \(Kl \leq \beta n^{1-\varepsilon}\). We have for large enough \(n\),
\[
\mathbb{P} \left( \nu_n > \beta n^{1-\varepsilon} \right) \leq \mathbb{P} \left( \max_{1 \leq j \leq Kl} |y + S_j| \leq n^{1/2-\varepsilon} \right).
\]

For each \(k = 1, \ldots, K\), denote \(A_{k,x} := \{ \max_{1 \leq j \leq k} |x + S_k| \leq n^{1/2-\varepsilon} \}\). Using the Markov property, we have 
\[
\mathbb{P}(A_{K,x}) \leq \mathbb{P}(A_{K-1,x}) \sup_{x \in \mathbb{R}} \mathbb{P}(A_{1,x}) \leq \left( \sup_{x \in \mathbb{R}} \mathbb{P}(A_{1,x}) \right)^K.
\]
By the central limit theorem for $S_l$, we get that uniformly in $x \in \mathbb{R}$,
\[ \mathbb{P}(A_{1,x}) \leq \mathbb{P}(|x + S_l| \leq 2n^{1/2 - \varepsilon}) \leq \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\varepsilon} - c_{\varepsilon, \beta}}^{-\frac{x}{\varepsilon} + c_{\varepsilon, \beta}} e^{-\frac{u^2}{2}} du + cr_1 < 1, \]
where $c_{\varepsilon, \beta} = \frac{n^{1/2 - \varepsilon}}{\sqrt{r}}$. Since $K = [n^\varepsilon]$, the result follows. □

The following result shows that the expectation of the random walk $(x + S_n)_{n \geq 1}$ killed at the exit time $\tau_x$ is uniformly bounded with respect to $n$.

**Lemma 7.3.** Assume A2 for some $\delta > 0$. There exists $c > 0$ such that for any $x \geq 0$ and $n \geq 1$,
\[ \mathbb{E}(x + S_n; \tau_x > n) \leq c(1 + x). \]

**Proof.** The proof is a combination of a recursive argument and the Markov property. For brevity, denote $V_n(x) = \mathbb{E}(x + S_n; \tau_x > n)$. For $\varepsilon \in (0, \frac{\delta}{2(2+\delta)})$, we have
\[ V_n(x) = \mathbb{E}
\]
\[ =: J_1 + J_2. \tag{7.3} \]

**Bound of $J_1$.** By Cauchy-Schwarz’s inequality, Minkowski’s inequality and Lemma 7.2, we obtain that for any $\varepsilon \in (0, \frac{\delta}{2(2+\delta)})$,
\[ J_1 \leq \mathbb{E}
\]
\[ \leq \mathbb{E}^{1/2} \left(|x + S_n|^2\right) \mathbb{P}^{1/2} \left(\nu_n > [n^{1-\varepsilon}]\right)
\]
\[ \leq \left(x + cn^{1/2}\right) c_{\varepsilon} e^{-c_{\varepsilon} n^\varepsilon} \leq c_{\varepsilon}(1 + x)e^{-c_{\varepsilon} n^\varepsilon}. \tag{7.4} \]

**Bound of $J_2$.** Using the Markov property leads to
\[ J_2 = \sum_{k=1}^{[n^{1-\varepsilon}]} \mathbb{E}(x + S_n; \tau_x > n, \nu_n = k)
\]
\[ = \sum_{k=1}^{[n^{1-\varepsilon}]} \int_{\mathbb{R}} V_{n-k}(x') \mathbb{P}(x + S_k \in dx'; \tau_x > k, \nu_n = k). \tag{7.5} \]

Note that on the event $\{\tau_x > k, \nu_n = k\}$, we have $x' = x + S_k > n^{1/2 - \varepsilon}$. By Lemma 7.1, it holds that for any $\varepsilon \in (0, \frac{\delta}{2(2+\delta)})$,
\[ V_{n-k}(x') = \mathbb{E}(x' + S_{n-k}; \tau_{x'} > n - k) \leq \left(1 + \frac{c}{(n - k)^{\varepsilon}}\right) x'. \]
Implementing this bound into (7.5), we derive that
\[ J_2 \leq \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \left( 1 + \frac{c \varepsilon}{n^\varepsilon} \right) \mathbb{E}(x + S_k; \tau_x > k, \nu_n = k) \leq \left( 1 + \frac{c \varepsilon}{n^\varepsilon} \right) \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}(x + S_k; \tau_x > k, \nu_n = k). \]  
(7.6)

Since \(((x + S_n) \mathbb{1}_{\{\tau_x > n\}})_{n \geq 1}\) is a submartingale with respect to the natural filtration \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\), we get that for any \(1 \leq k \leq \lfloor n^{1-\varepsilon} \rfloor\),
\[ \mathbb{E}(x + S_k; \tau_x > k, \nu_n = k) \leq \mathbb{E} \left( x + S_{\lfloor n^{1-\varepsilon} \rfloor}; \tau_x > \lfloor n^{1-\varepsilon} \rfloor, \nu_n = k \right), \]
and consequently, for any \(\varepsilon \in (0, \frac{\delta}{2(2+\delta)})\),
\[ J_2 \leq \left( 1 + \frac{c \varepsilon}{n^\varepsilon} \right) V_{\lfloor n^{1-\varepsilon} \rfloor}(x). \]  
(7.7)

Substituting (7.7) and (7.4) into (7.3) gives
\[ V_n(x) \leq \left( 1 + \frac{c \varepsilon}{n^\varepsilon} \right) V_{\lfloor n^{1-\varepsilon} \rfloor}(x) + c \varepsilon (1 + x) e^{-c \varepsilon n^\varepsilon}. \]

Applying [25, Lemma A.1], we get that for any integer \(k_0 \in [1, n]\),
\[ V_n(x) \leq \left( 1 + \frac{c \varepsilon}{k_0^\varepsilon} \right) V_{k_0}(x) + c \varepsilon (1 + x) e^{-c \varepsilon k_0^\varepsilon}. \]

By Lemma 7.1, we have \(V_{k_0}(x) \leq x + c k_0^{1/2-\varepsilon}\) and therefore, for any \(\varepsilon \in (0, \frac{\delta}{2(2+\delta)})\),
\[ V_n(x) \leq \left( 1 + \frac{c \varepsilon}{k_0^\varepsilon} \right) x + c k_0^{1/2-\varepsilon} + c \varepsilon (1 + x) e^{-c \varepsilon k_0^\varepsilon} \leq \left( 1 + \frac{c \varepsilon}{k_0^\varepsilon} \right) x + c' k_0^{1/2-\varepsilon}, \]  
(7.8)
which concludes the proof of the lemma. \(\square\)

The following result proves the existence and gives some properties of the harmonic function \(V\).

**Lemma 7.4.** Assume A2 for some \(\delta > 0\).

1. For any \(x \geq 0\), the limit \(\lim_{n \to \infty} \mathbb{E}(x + S_n; \tau_x > n)\) exists and
\[ V(x) := \lim_{n \to \infty} \mathbb{E}(x + S_n; \tau_x > n). \]  
(7.9)

2. The function \(V\) is increasing on \(\mathbb{R}_+\) and \(x \leq V(x) \leq c(1 + x)\) for any \(x > 0\). Moreover, \(\lim_{x \to \infty} \frac{V(x)}{x} = 1\).
3. The function $V$ is harmonic in the sense that for any $x \geq 0$,
\[ \mathbb{E}(V(x + S_1); \tau_x > 1) = V(x). \] (7.10)

Proof. We proceed with the first assertion. Using (7.1) and Lemma 7.3, we get that for $x \geq 0$,
\[ x \mathbb{P}(\tau_x > n) + \mathbb{E}(-S_{\tau_x}; \tau_x \leq n) = \mathbb{E}(x + S_n; \tau_x > n) \leq c(1 + x). \] (7.11)

Since $x + S_{\tau_x} \leq 0$, we have $S_{\tau_x} \leq 0$ and hence, using (7.11),
\[ \mathbb{E}(-S_{\tau_x}; \tau_x \leq n) \leq c(1 + x). \]

By the Lebesgue monotone convergence theorem, it follows that
\[ \lim_{n \to \infty} \mathbb{E}(-S_{\tau_x}; \tau_x \leq n) = \mathbb{E}(-S_{\tau_x}) \leq c(1 + x). \] (7.12)

This, together with (7.11) and the fact that $\mathbb{P}(\tau_x > n) \to 0$ as $n \to \infty$, proves (7.9).

We now prove the second assertion. Since for any $0 \leq x_1 \leq x_2$,
\[ \mathbb{E}(x_1 + S_n; \tau_{x_1} > n) \leq \mathbb{E}(x_2 + S_n; \tau_{x_2} > n), \]
from (7.9) it follows that $V$ is increasing on $\mathbb{R}_+$.

Using (7.1) and the fact that $x + S_{\tau_x} \leq 0$, we see that $\mathbb{E}(x + S_n; \tau_x > n) \geq x$. Hence, by (7.9), we get $V(x) \geq x$. Since $V(x) = -ES_{\tau_x}$, using (7.12) we get that $V(x) \leq c(1 + x)$ for some constant $c > 0$. Therefore $x \leq V(x) \leq c(1 + x)$.

From (7.8) and (7.9), it follows that
\[ \limsup_{x \to \infty} \frac{V(x)}{x} \leq 1 + \frac{c_x}{k_0}. \]

Taking $k_0 \to \infty$, we get $\limsup_{x \to \infty} V(x)/x \leq 1$. This, together with the inequality $x \leq V(x)$, shows that $\lim_{x \to \infty} V(x)/x = 1$.

For the third assertion, denoting $V_n(x) = \mathbb{E}(x + S_n; \tau_x > n)$, by the Markov property, we have
\[ V_{n+1}(x) = \int_\mathbb{R} V_n(x') \mathbb{P}(x + S_1 \in dx', \tau_x > 1). \]

The identity (7.10) follows from the Lebesgue dominated convergence theorem, the inequality $V_n(x') \leq c(1 + x')$ and (7.9).

Proof of Lemma 3.7. The assertion of the lemma follows from (7.8) and (7.9).

Recall that $\nu_n$ is a stopping time defined by (7.2).
Lemma 7.5. Assume A2 for some $\delta > 0$. Then, for any $\varepsilon \in (0, \frac{\delta}{2(2+\delta)})$, there exists a constant $c_\varepsilon > 0$ such that for any $x \geq 0$ and $n \geq 1$,
\[
E_1 := \mathbb{E} \left( x + S_{\nu_n}; \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}] \right) \leq c_\varepsilon (1 + x), \quad (7.13)
\]
and
\[
E_1 \leq V(x) \leq \left( 1 + \frac{c_\varepsilon}{n^{\varepsilon/2}} \right) E_1 + c_\varepsilon (1 + x) e^{-c_\varepsilon n^\varepsilon}. \quad (7.14)
\]

Proof. We first prove (7.13). Since $x + S_{\nu_n} > 0$, we have
\[
E_1 \leq \mathbb{E} \left( x + S_{[n^{1-\varepsilon}]}; \tau_x > [n^{1-\varepsilon}], \nu_n \leq [n^{1-\varepsilon}] \right)
\leq \mathbb{E} \left( x + S_{[n^{1-\varepsilon}]}; \tau_x > [n^{1-\varepsilon}] \right).
\]
Using Lemma 7.3, we obtain (7.13).

We next prove (7.14). By Lemma 7.4 and the fact that $(V(x+S_n)1_{\{\tau_x > \nu_n\}})_{n \geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, we get
\[
V(x) = \mathbb{E} \left( V(x + S_n); \tau_x > n, \nu_n \leq [n^{1-\varepsilon}] \right)
+ \mathbb{E} \left( V(x + S_n); \tau_x > n, \nu_n > [n^{1-\varepsilon}] \right)
= \mathbb{E} \left( V(x + S_n); \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}] \right)
+ \mathbb{E} \left( V(x + S_n); \tau_x > \nu_n, \nu_n > [n^{1-\varepsilon}] \right). \quad (7.15)
\]
On the one hand, since $V(x) \geq x$, from (7.15) we get
\[
V(x) \geq \mathbb{E} \left( V(x + S_{\nu_n}); \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}] \right) \geq E_1. \quad (7.16)
\]
On the other hand, using Lemma 3.7 and the inequality $V(x) \leq c(1 + x)$, we obtain that for any $\varepsilon \in (0, \frac{\delta}{2(2+\delta)})$,
\[
V(x) \leq \left( 1 + \frac{c_\varepsilon}{n^{\varepsilon/2}} \right) E_1 + c_\varepsilon k_0^{1/2-\varepsilon} \mathbb{P} \left( \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}] \right)
+ c_\varepsilon (1 + x) e^{-c_\varepsilon n^\varepsilon}. \quad (7.17)
\]
By Hölder’s inequality and Lemma 7.2, we get that for any $\varepsilon \in (0, 1/2)$,
\[
\mathbb{E} \left( x + S_n; \tau_x > n, \nu_n > [n^{1-\varepsilon}] \right) \leq c(x + n^{1/2}) \mathbb{P}^{1/2} \left( \nu_n > [n^{1-\varepsilon}] \right)
\leq c_\varepsilon (1 + x) e^{-c_\varepsilon n^\varepsilon}. \quad (7.18)
\]
Since $n^{1/2-\varepsilon} \leq x + S_{\nu_n}$ on the event $\{\tau_x > \nu_n\}$, it holds that
\[
\mathbb{P} \left( \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}] \right)
\leq \frac{1}{n^{1/2-\varepsilon}} \mathbb{E} \left( x + S_{\nu_n}; \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}] \right) = \frac{1}{n^{1/2-\varepsilon}} E_1. \quad (7.19)
\]
Combining (7.17), (7.18) and (7.19) gives that for any \( \varepsilon \in (0, \frac{\delta}{2(2+\delta)}) \),

\[
V(x) \leq \left( 1 + \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}} + \frac{c_{\varepsilon}k_0^{1/2-\varepsilon}}{n^{1/2-\varepsilon}} \right) E_1 + c_{\varepsilon}(1 + x)e^{-c_{\varepsilon}n^\varepsilon}.
\]

By taking \( k_0 = n^{1-2\varepsilon} \), this implies that

\[
V(x) \leq \left( 1 + \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}} \right) E_1 + c_{\varepsilon}(1 + x)e^{-c_{\varepsilon}n^\varepsilon}.
\] (7.20)

Combining this with (7.16), we get (7.14). \( \Box \)

**Lemma 7.6.** Assume \( A_2 \) for some \( \delta > 0 \). Then, for any \( \varepsilon \in (0, \frac{\delta}{2(2+\delta)}) \), there exists a constant \( c_{\varepsilon} > 0 \) such that for any \( x \geq 0 \) and \( n \geq 1 \),

\[
\mathbb{P}(\tau_x > n) \leq \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}}(1 + x).
\]

**Proof.** By Lemmas 7.5 and 7.2, we get

\[
\mathbb{P}(\tau_x > n) = \mathbb{P}(\tau_x > n, \nu_n \leq [n^{1-\varepsilon}]) + \mathbb{P}(\tau_x > n, \nu_n > [n^{1-\varepsilon}])
\leq \frac{1}{n^{1/2-\varepsilon}} \mathbb{E}
\left([X_{\nu_n}; \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}]] + c_{\varepsilon}e^{-c_{\varepsilon}n^\varepsilon}\right).
\]

Using the bound (7.13), the result follows. \( \Box \)

**Lemma 7.7.** Assume \( A_2 \) for some \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), \( x \geq 0 \) and \( n \geq 1 \),

\[
E_2 := \mathbb{E}
\left([X_{\nu_n}; X_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}]] + \frac{c_{\varepsilon}(1 + x)}{n^{\delta/2-\varepsilon(1+\varepsilon+\delta/2)}}\right).
\]

**Proof.** Since \( x + S_{\nu_n} \leq X_{\nu_n} + n^{1/2-\varepsilon} \) on the event \( \{x + S_{\nu_n} > n^{1/2-\varepsilon/2}\} \), we have

\[
E_2 \leq \mathbb{E}
\left([X_{\nu_n}; X_{\nu_n} > n^{1/2-\varepsilon/2} - n^{1/2-\varepsilon}, \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}]\right).
\]
Since \(n^{1/2-\epsilon/2} - n^{1/2-\epsilon} > c_\varepsilon n^{1/2-\epsilon/2}\), it follows that
\[
E_2 \leq \sum_{k=1}^{[n^{1-\epsilon}]} \mathbb{E} \left( x + S_k; X_k > c_\varepsilon n^{1/2-\epsilon/2}, \tau_x > k \right)
\]
\[
\leq \sum_{k=1}^{[n^{1-\epsilon}]} \mathbb{E} \left( x + S_{k-1} + X_k; X_k > c_\varepsilon n^{1/2-\epsilon/2}, \tau_x > k - 1 \right)
\]
\[
= \sum_{k=1}^{[n^{1-\epsilon}]} \mathbb{E} \left( x + S_{k-1}; \tau_x > k - 1 \right) \mathbb{P} \left( X_k > c_\varepsilon n^{1/2-\epsilon/2} \right)
\]
\[
+ \sum_{k=1}^{[n^{1-\epsilon}]} \mathbb{E} \left( X_k; X_k > c_\varepsilon n^{1/2-\epsilon/2} \right) \mathbb{P} \left( \tau_x > k - 1 \right)
\]
\[= E_{21} + E_{22}.\]

For \(E_{21}\), using Lemma 7.3, Markov’s inequality and condition \(\textbf{A2}\), we get
\[
E_{21} \leq c (1 + x) \sum_{k=1}^{[n^{1-\epsilon}]} \mathbb{P} \left( X_k > c_\varepsilon n^{1/2-\epsilon/2} \right) \leq \frac{c(1 + x)}{n^{\delta/2 - \epsilon/2}}.
\]

For \(E_{22}\), using Lemma 7.6 and again Markov’s inequality, we obtain
\[
E_{22} \leq \mathbb{E} \left( X_1; X_1 > c_\varepsilon n^{1/2-\epsilon/2} \right) \sum_{k=1}^{[n^{1-\epsilon}]} \mathbb{P} \left( \tau_x > k - 1 \right)
\]
\[
\leq \frac{c_\varepsilon}{n^{(1/2-\epsilon/2)(1+\delta)}} \sum_{k=1}^{[n^{1-\epsilon}]} \frac{c_\varepsilon}{k^{1/2-\epsilon}} (1 + x)
\]
\[
\leq \frac{c_\varepsilon}{n^{(1/2-\epsilon/2)(1+\delta)}} c_\varepsilon (1 + x) n^{(1-\epsilon)(1/2+\epsilon)} \leq \frac{c_\varepsilon (1 + x)}{n^{\delta/2 - \epsilon(1+\epsilon+\delta/2)}}.
\]
The desired result follows. \(\square\)

### 7.2. Proof of Theorems 2.7 and 2.8

We first give a proof of Theorem 2.7 by using the bounds shown in Section 7.1 and the functional central limit theorem (Lemma 3.6).

**Proof of Theorem 2.7.** As in (3.40), denote
\[
A_k = \left\{ \sup_{0 \leq t \leq 1} \left| S_{tk} - B_{tk} \right| \leq k^{1/2 - 2\epsilon} \right\}
\]
and by \(A_k^c\) its complement. Using the Markov property, we have
\[
\mathbb{P} \left( x + S_n \leq t \sqrt{n}, \tau_x > n \right) =: J_1 + J_2 + J_3,
\]
where

\[ J_1 = \mathbb{P}\left( x + S_n \leq t\sqrt{n}, \tau_x > n, \nu_n > [n^{1-\varepsilon}] \right), \]

\[ J_2 = \sum_{k=1}^{[n^{1-\varepsilon}]} \int_{\mathbb{R}} \mathbb{P}(x' + S_{n-k} \leq t\sqrt{n}, \tau_{x'} > n - k, A_{n-k}^c) \times \mathbb{P}(x + S_k \in dx', \tau_x > k, \nu_n = k), \]

\[ J_3 = \sum_{k=1}^{[n^{1-\varepsilon}]} \int_{\mathbb{R}} \mathbb{P}(x' + S_{n-k} \leq t\sqrt{n}, \tau_{x'} > n - k, A_n - k) \times \mathbb{P}(x + S_k \in dx', \tau_x > k, \nu_n = k). \]

**Bound of \( J_1 \).** By Lemma 7.2, it holds that

\[ J_1 \leq \mathbb{P}(\nu_n > [n^{1-\varepsilon}]) \leq c_\varepsilon e^{-c_\varepsilon n^\varepsilon}. \] (7.23)

**Bound of \( J_2 \).** For \( k \leq [n^{1-\varepsilon}] \), we have \( n - k \geq c_\varepsilon n \). By Lemma 3.6,

\[ \mathbb{P}(x' + S_{n-k} \leq t\sqrt{n}, \tau_{x'} > n - k, A_{n-k}^c) \leq \mathbb{P}(A_{n-k}^c) \leq \frac{c_\varepsilon}{n^{\delta/2 - 2(2+\delta)}}. \] (7.24)

Note that on the event \( \{ \tau_x > \nu_n \} \), we have \( n^{1/2-\varepsilon} \leq x + S_{\nu_n} \). By Lemma 7.5, we get

\[ J_2 \leq \frac{c_\varepsilon}{n^{\delta/2 - 2(2+\delta)}} \mathbb{P}(\tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}]) \leq \frac{c_\varepsilon}{n^{(\delta+1)/2 - 2(2+\delta) - \varepsilon}} \mathbb{E}(x + S_{\nu_n}; \tau_x > \nu_n, \nu_n \leq [n^{1-\varepsilon}]) \leq \frac{c_\varepsilon (1+x)}{n^{(\delta+1)/2 - 2\varepsilon(5+2\delta)}}. \] (7.25)

**Bound of \( J_3 \).** We write \( J_3 = J_{31} + J_{32} \), where

\[ J_{31} = \sum_{k=1}^{[n^{1-\varepsilon}]} \int_{\mathbb{R}} \mathbb{P}(x' + S_{n-k} \leq t\sqrt{n}, \tau_{x'} > n - k, A_{n-k}) \times \mathbb{P}(x + S_k \in dx', x + S_k > n^{1/2-\varepsilon/2}, \tau_x > k, \nu_n = k), \]

\[ J_{32} = \sum_{k=1}^{[n^{1-\varepsilon}]} \int_{\mathbb{R}} \mathbb{P}(x' + S_{n-k} \leq t\sqrt{n}, \tau_{x'} > n - k, A_{n-k}) \times \mathbb{P}(x + S_k \in dx', x + S_k \leq n^{1/2-\varepsilon/2}, \tau_x > k, \nu_n = k). \]

**Bound of \( J_{31} \).** Denote \( x' = x' + (n - k)^{1/2-2\varepsilon} \). Since \( x' > n^{1/2-\varepsilon/2} \), from (7.21) and (3.8), we derive that

\[ \mathbb{P}(\tau_{x'} > n - k, A_{n-k}) \leq \mathbb{P}(\tau_{x'} > n - k) \leq c x'_+ \leq c x'_+ \varepsilon. \] (7.26)
Implementing the bound (7.26) into $J_{31}$, and using Lemma 7.7, we obtain

$$J_{31} \leq \frac{c_x}{\sqrt{n}} E_2 \leq \frac{c_x (1 + x)}{n^{(1+\delta)/2-\varepsilon/(1+\delta/2)}}.$$  (7.27)

**Upper bound of $J_{32}$.** Denote $x' = x' + (n - k)^{1/2-2\varepsilon}$ and $t_+ = t + \frac{1}{n^{\varepsilon/2}}$. By Lemma 3.5, we have

$$\mathcal{K}(x') : = \mathbb{P}(x' + S_{n-k} \leq t \sqrt{n}, \tau_{x'} > n - k, A_{n-k})$$

$$\leq \mathbb{P}(x' + \sigma B_{n-k} \leq t + \sqrt{n}, \tau_{x'} \leq n - k)$$

$$= \frac{1}{\sigma \sqrt{2\pi(n-k)}} \int_0^{\sqrt{n-k}} \left( e^{-\frac{(u - x')^2}{2\sigma^2(n-k)}} - e^{-\frac{(u + x')^2}{2\sigma^2(n-k)}} \right) du$$

$$\leq \frac{1}{\sigma \sqrt{2\pi(n-k)}} \int_0^{\sqrt{n-k}} e^{-\frac{u^2}{2\sigma^2(n-k)}} \left( e^{\frac{u^2}{\sigma^2(n-k)}} - e^{-\frac{u^2}{\sigma^2(n-k)}} \right) du$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{n-k}} e^{-\frac{u^2}{2}} \left( e^{\frac{u^2}{\sigma^2(n-k)}} - e^{-\frac{u^2}{\sigma^2(n-k)}} \right) ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{n-k}} e^{-\frac{u^2}{2}} e^{\frac{x'}{\sqrt{n-k}}} \left( 1 - e^{-\frac{x'}{\sqrt{n-k}}} \right) ds.$$  (7.28)

Using the inequality $1 - e^{-z} \leq z$ for $z \geq 0$, it follows that, with $t_+ = t + \frac{1}{n^{\varepsilon/2}}$,

$$\mathcal{K}(x') \leq \frac{2x_+}{\sqrt{2\pi(n-k)}} \int_0^{\sqrt{n-k}t_+} s e^{-\frac{s^2}{2}} e^{\frac{x'}{\sqrt{n-k}}} ds.$$  (7.29)

Since $k \leq \lfloor n^{1-\varepsilon} \rfloor$ and $x_+ = x' + (n - k)^{1/2-2\varepsilon}$ with $x' \leq n^{1/2-\varepsilon/2}$, we have

$$\frac{x_+}{\sqrt{n-k}} \leq \frac{c_x}{n^{\varepsilon/2}}.$$  

Hence, when $t \in [0, n^{\varepsilon/p}]$ for some constant $p \geq 2$ sufficiently large, we have

$$s \leq \sqrt{n-k} t_+ \leq c_x n^{\varepsilon/p},$$

so that

$$\exp \left( s \frac{x'}{\sqrt{n-k}} \right) \leq \exp \left( \frac{c_x}{n^{\varepsilon/2 - \varepsilon/2}} \right) \leq 1 + \frac{c_x}{n^{\varepsilon/2 - \varepsilon/2}}.$$  

This implies that when $t \in [0, n^{\varepsilon/p}]$,

$$\int_0^{\sqrt{n-k}t_+} s e^{-\frac{s^2}{2}} e^{\frac{x'}{\sqrt{n-k}}} ds \leq \left( 1 + \frac{c_x}{n^{\varepsilon/2 - \varepsilon/2}} \right) \int_0^{\sqrt{n-k}t_+} s e^{-\frac{s^2}{2}} ds$$

$$\leq \left( 1 + \frac{c_x}{n^{\varepsilon/2 - \varepsilon/2}} \right) \left( \Phi^+(t) + \sqrt{n-k} t_+ - t \right)$$

$$\leq \left( 1 + \frac{c_x}{n^{\varepsilon/2 - \varepsilon/2}} \right) \left( \Phi^+(t) + \frac{c_x}{n^{\varepsilon/2}} \right).$$  (7.30)
When \( t > n^{\varepsilon} \), we have
\[
\int_0^{\sqrt{\frac{n^{1/\varepsilon}/n}{n^{1/2}}}} e^{-s^2/n} \, ds = \int_0^{n^{\varepsilon}} e^{-s^2/n} \, ds + \int_{n^{\varepsilon}}^{\sqrt{\frac{n^{1/\varepsilon}}{n^{1/2}}}} e^{-s^2/n} \, ds
\]
\[
\leq \left( 1 + \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}} \right) \int_0^{n^{\varepsilon}} e^{-s^2/n} \, ds + c_{\varepsilon} e^{-c_{\varepsilon} n^{\varepsilon}}
\]
\[
\leq \left( 1 + \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}} \right) \int_0^{t} e^{-s^2/n} \, ds + c_{\varepsilon} e^{-c_{\varepsilon} n^{\varepsilon}}. \quad (7.31)
\]

Combining (7.29), (7.30) and (7.31), we get that uniformly in \( t \in \mathbb{R}_+ \),
\[
K(x') \leq \frac{2x'}{\sqrt{2\pi(n-k)}} \left( 1 + \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}} \right) \left( \Phi^+(t) + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right).
\]

Since \( k \leq \lceil n^{1-\varepsilon} \rceil \), we have \( \frac{1}{\sqrt{n-k}} \leq \frac{1}{\sqrt{n}} (1 + \frac{c_{\varepsilon}}{n^{\varepsilon}}) \) and
\[
\frac{x'}{\sqrt{n-k}} \leq \frac{1}{\sqrt{n}} \left( x' + n^{1/2-2\varepsilon} \right) \left( 1 + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right),
\]
which implies that
\[
K(x') \leq \frac{2}{\sqrt{2\pi n}} \left( x' + n^{1/2-2\varepsilon} \right) \left( 1 + \frac{c_{\varepsilon}}{n^{1/2-\varepsilon}} \right) \left( \Phi^+(t) + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right). \quad (7.32)
\]

Therefore, using the fact that \( n^{1/2-\varepsilon} \leq x + S_{\nu_n} \), we get
\[
J_{32} \leq \frac{2}{\sqrt{2\pi n}} \left( 1 + \frac{c_{\varepsilon,p}}{n^{1/2-\varepsilon}} \right) \left( \Phi^+(t) + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right)
\]
\[
\times \sum_{k=1}^{\lceil n^{1-\varepsilon} \rceil} \mathbb{E} \left( x + S_k + n^{1/2-2\varepsilon}, x + S_k \leq n^{1/2-\varepsilon/2}, \tau_x > k, \nu_n = k \right)
\]
\[
\leq \frac{2}{\sqrt{2\pi n}} \left( 1 + \frac{c_{\varepsilon,p}}{n^{1/2-\varepsilon}} \right) \left( \Phi^+(t) + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right)
\]
\[
\times \mathbb{E} \left( x + S_{\nu_n}, x + S_{\nu_n} \leq n^{1/2-\varepsilon/2}, \tau_x > \nu_n, \nu_n \leq \lceil n^{1-\varepsilon} \rceil \right)
\]
\[
\leq \frac{2}{\sqrt{2\pi n}} \left( 1 + \frac{c_{\varepsilon,p}}{n^{1/2-\varepsilon}} \right) \left( \Phi^+(t) + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right) E_1
\]
\[
\leq \frac{2V(x)}{\sqrt{2\pi n}} \left( 1 + \frac{c_{\varepsilon,p}}{n^{1/2-\varepsilon}} \right) \left( \Phi^+(t) + \frac{c_{\varepsilon}}{n^{\varepsilon}} \right), \quad (7.33)
\]
where in the last inequality we used Lemma 7.5.
Lower bound of $J_{32}$. Denote $x'_- = x' - (n - k)^{1/2} - 2\varepsilon$ and $t_- = t - n^{-2\varepsilon}$.

By Lemma 3.5, we have that for $t \geq n^{-2\varepsilon}$,

$$
\mathcal{K}(x') \geq \mathbb{P} (x'_- + B_{n-k} \leq t_- \sqrt{n}, \tau_{x'_-}^{bn} > n - k) 
$$

$$
= \frac{1}{\sqrt{2\pi(n-k)}} \int_{0}^{\sqrt{t_-}} \left( e^{\frac{(u-x'_-)^2}{2(n-k)}} - e^{\frac{(u+x'_-)^2}{2(n-k)}} \right) du 
$$

$$
= \frac{1}{\sqrt{2\pi(n-k)}} e^{\frac{(x'_-)^2}{2(n-k)}} \int_{0}^{\sqrt{t_-}} e^{-\frac{u^2}{2n-k}} \left( e^{\frac{u}{\sqrt{n-k}}} - e^{-\frac{u}{\sqrt{n-k}}} \right) du 
$$

$$
= \frac{1}{\sqrt{2\pi}} e^{\frac{(x'_-)^2}{2(n-k)}} \int_{0}^{t_-} e^{\frac{s^2}{2}} e^{-\frac{2s}{\sqrt{n-k}}} \left( e^{\frac{s}{\sqrt{n-k}}} - 1 \right) ds. \quad (7.34)
$$

Using the inequality $e^z - 1 \geq z$ for $z \in \mathbb{R}$, we get

$$
\mathcal{K}(x') \geq \frac{2x'_-}{\sqrt{2\pi(n-k)}} e^{\frac{(x'_-)^2}{2(n-k)}} \int_{0}^{t_-} se^{-\frac{s^2}{2}} e^{-\frac{s}{\sqrt{n-k}}} ds. \quad (7.35)
$$

Since $x' \leq n^{1/2-\varepsilon/2}$, $x'_- = x' - (n - k)^{1/2} - 2\varepsilon$ and $k \leq [n^{1-\varepsilon}]$, we have

$$
\left| \frac{x'_-}{\sqrt{n-k}} \right| \leq \frac{c_\varepsilon}{n^{\varepsilon/2}}, \quad e^{-\frac{(x'_-)^2}{2(n-k)}} \geq \left( 1 - \frac{c_\varepsilon}{n^{\varepsilon}} \right). \quad (7.36)
$$

Hence, when $t_- \in [0, n^{\varepsilon p}]$ for some constant $p \geq 2$ sufficiently large, we have $s \leq n^{\varepsilon p}$, so that

$$
\exp \left( -s \frac{x'_-}{\sqrt{n-k}} \right) \geq \exp \left( -\frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon p}} \right) \geq 1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon p}}.
$$

This implies that when $t_- \in [0, n^{\varepsilon p}]$,

$$
\int_{0}^{t_-} se^{-\frac{s^2}{2}} e^{-\frac{s}{\sqrt{n-k}}} ds \geq \left( 1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon p}} \right) \int_{0}^{t_-} se^{-\frac{s^2}{2}} ds. \quad (7.37)
$$

When $t_- > n^{\varepsilon p}$, we have

$$
\int_{0}^{t_-} se^{-\frac{s^2}{2}} e^{-\frac{s}{\sqrt{n-k}}} ds \geq \int_{0}^{n^{\varepsilon p}} se^{-\frac{s^2}{2}} e^{-\frac{s}{\sqrt{n-k}}} ds 
$$

$$
\geq \left( 1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon p}} \right) \int_{0}^{n^{\varepsilon p}} se^{-\frac{s^2}{2}} ds 
$$

$$
\geq \left( 1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon p}} \right) \int_{0}^{t_-} se^{-\frac{s^2}{2}} ds - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon p}}. \quad (7.38)
$$
Combining (7.35), (7.36), (7.37) and (7.38), we get that uniformly in \( t \in \mathbb{R}_+ \),
\[
\mathcal{K}(x') \geq \frac{2x'}{\sqrt{2\pi(n-k)}} \left[ \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \int_0^{t-} se^{-\frac{s^2}{2}} ds - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon'}} \right].
\]
Since \( k \leq [n^{1-\varepsilon}] \) and \( x' = x' - (n-k)^{1/2-2\varepsilon} \), we have
\[
\frac{x'}{\sqrt{n-k}} \geq \frac{1}{\sqrt{n}} \left( x' - c_\varepsilon n^{1/2-2\varepsilon} \right).
\]
It follows that
\[
\mathcal{K}(x') \geq \frac{2}{\sqrt{2\pi n}} \left( x' - c_\varepsilon n^{1/2-2\varepsilon} \right) \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \left[ \int_0^{t-} se^{-\frac{s^2}{2}} ds - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon'}} \right]
\]
\[
= \frac{2}{\sqrt{2\pi n}} \left( x' - c_\varepsilon n^{1/2-2\varepsilon} \right) \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \left[ \Phi^+(t_-) - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon'}} \right].
\]
(7.39)
Following the proof of (7.33), using Lemmas 7.5 and 7.7, we obtain
\[
J_{32} \geq \frac{2}{\sqrt{2\pi n}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \left[ \Phi^+(t_-) - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon'}} \right]
\]
\[
\times \sum_{k=1}^{[n^{1-\varepsilon}]} \mathbb{E} \left( x + S_k - c_\varepsilon n^{1/2-2\varepsilon}, x + S_k \leq n^{1/2-\varepsilon/2}, \tau_x > k, \nu_n = k \right)
\]
\[
\geq \frac{2}{\sqrt{2\pi n}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \left[ \Phi^+(t_-) - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon'}} \right] (E_1 - E_2)
\]
\[
\geq \frac{2}{\sqrt{2\pi n}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \left[ \Phi^+(t_-) - c_\varepsilon e^{-c_\varepsilon n^{\varepsilon'}} \right] \left( V(x) - \frac{c_\varepsilon(1+x)}{n^{\delta/2-\varepsilon(1+\varepsilon+\delta)/2}} \right)
\]
\[
\geq \frac{2V(x)}{\sqrt{2\pi n}} \left(1 - \frac{c_\varepsilon}{n^{\varepsilon/2-\varepsilon'}} \right) \Phi^+(t_-) - \frac{c_\varepsilon(1+x)}{n^{\delta/2-\varepsilon(1+\varepsilon+\delta)/2}}. \quad (7.40)
\]
We conclude the proof of Theorem 2.7 by combining (7.23), (7.25), (7.27), (7.33) and (7.40).

To prove Theorem 2.8, we first show the following result which is based on the coupling method using the functional central limit theorem (Lemma 3.6).

**Theorem 7.8.** Assume \textbf{A2} with some \( \delta > 0 \). For any \( \gamma \in (0, \frac{\delta}{2(2+\delta)}) \), there exists a constant \( c_\gamma > 0 \) such that the following assertions hold:
1. For any \( n \geq 1, x > 0 \) and \( t > 0 \),
\[
\mathbb{P} \left( \frac{x + S_n}{\sigma \sqrt{n}} > t, \tau_x > n \right) \leq \int_t^{\infty} \psi \left( s, \frac{x}{\sigma \sqrt{n}} + \frac{1}{\sigma n^\gamma} \right) ds + \frac{c_\gamma}{n^{\delta/2-\gamma(2+\delta)}}. \quad (7.41)
\]
2. For any $n \geq 1$, $x > n^{1/2-\gamma}$ and $t > 0$,

$$\mathbb{P}\left(\frac{x + S_n}{\sigma \sqrt{n}} > t, \tau_x > n\right) \geq \int_t^\infty \psi\left(s, \frac{x}{\sigma \sqrt{n}} - \frac{1}{\sigma n^{\gamma}}\right) ds - \frac{c_\gamma}{n^{\frac{3}{2} - \gamma(2+\delta)}}. \quad (7.42)$$

Proof. Denote

$$A_n = \left\{ \sup_{0 \leq t \leq 1} \left| S_{[tn]} - \sigma B_{tn} \right| > n^{1/2-\gamma} \right\} \quad (7.43)$$

and by $A_n^c$ the complement of $A_n$. By Lemma 3.6, for any $\gamma \in (0, \frac{\delta}{2(2+\delta)})$, there exists a constant $c_\gamma > 0$ such that

$$\mathbb{P}(A_n) \leq \frac{c_\gamma}{n^{\frac{3}{2} - \gamma(2+\delta)}}. \quad (7.44)$$

Since $x_1^+ = x + n^{\frac{1}{2}-\gamma}$, using Lemma 3.5 and (7.44), we get

$$\mathbb{P}\left(\frac{x + S_n}{\sigma \sqrt{n}} > t, \tau_x > n\right) \leq \mathbb{P}\left(\frac{x_1^+ + \sigma B_n}{\sigma \sqrt{n}} > t, \tau_x > n\right) + \mathbb{P}(A_n)$$

$$\leq \frac{1}{\sigma \sqrt{n}} \int_t^\infty \psi\left(s, \frac{x_1^+}{\sigma \sqrt{n}}\right) ds + \frac{c_\gamma}{n^{\frac{3}{2} - \gamma(2+\delta)}},$$

which ends the proof of the upper bound (7.41).

Since $x_1^- = x - n^{\frac{1}{2}-\gamma}$, using Lemma 3.5 and (7.44), we get that for any $x > n^{\frac{1}{2}-\gamma}$,

$$\mathbb{P}\left(\frac{x + S_n}{\sigma \sqrt{n}} > t, \tau_x > n\right) \geq \mathbb{P}\left(\frac{x_1^- + \sigma B_n}{\sigma \sqrt{n}} > t, \tau_x > n, A_n^c\right)$$

$$\geq \mathbb{P}\left(\frac{x_1^- + \sigma B_n}{\sigma \sqrt{n}} > t, \tau_x > n\right) - \mathbb{P}(A_n)$$

$$\geq \frac{1}{\sigma \sqrt{n}} \int_t^\infty \psi\left(s, \frac{x_1^-}{\sigma \sqrt{n}}\right) ds - \frac{c_\gamma}{n^{\frac{3}{2} - \gamma(2+\delta)}},$$

which concludes the proof of the lower bound (7.42). \qed

Using Theorem 7.8, we now prove Theorem 2.8.
Proof of Theorem 2.8. By the definition of $\psi$ (cf. (1.4)), we have for $t > 0$,

$$
\int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}} + \frac{1}{\sigma \sqrt{n}}\right) ds - \int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}}\right) ds
= \frac{1}{\sqrt{2\pi}} \int_t^\infty \left( e^{-\frac{1}{2}(s-\frac{x}{\sigma \sqrt{n}})^2} - e^{-\frac{1}{2}(s-\frac{x}{\sigma \sqrt{n}})^2} \right) ds
+ \frac{1}{\sqrt{2\pi}} \int_t^\infty \left( e^{-\frac{1}{2}(s+\frac{x}{\sigma \sqrt{n}})^2} - e^{-\frac{1}{2}(s+\frac{x}{\sigma \sqrt{n}})^2} \right) ds
=: I_1 + I_2.
$$

For $I_1$, by a change of variable and elementary calculations, we get

$$
I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma \sqrt{n}}}^{\infty} \left( e^{-\frac{1}{2}(s-\frac{1}{\sigma \sqrt{n}})^2} - e^{-\frac{1}{2}s^2} \right) ds
\leq \int_{\mathbb{R}} \left| e^{-\frac{1}{2}(s-\frac{1}{\sigma \sqrt{n}})^2} - e^{-\frac{1}{2}s^2} \right| ds
= \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} \left| e^{-\frac{1}{2}(s-\frac{1}{\sigma \sqrt{n}})^2} - e^{-\frac{1}{2}s^2} \right| ds + \int_{|s| > \frac{n}{\sigma \sqrt{n}}} \left| e^{-\frac{1}{2}(s-\frac{1}{\sigma \sqrt{n}})^2} - e^{-\frac{1}{2}s^2} \right| ds
=: I_{11} + I_{12}.
$$

For $I_{11}$, using the inequality $e^x - 1 \leq |x| + |x|e^x$, $x \in \mathbb{R}$, we have

$$
I_{11} = \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}s^2} \left| e^{\frac{s}{\sigma \sqrt{n}} - \frac{1}{2\sigma^2 s^2}} - 1 \right| ds
\leq \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}s^2} \left| \frac{s}{\sigma \sqrt{n}} - \frac{1}{2\sigma^2 n^2} \right| ds
+ \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}(s-\frac{1}{\sigma \sqrt{n}})^2} \left| \frac{s}{\sigma \sqrt{n}} - \frac{1}{2\sigma^2 n^2} \right| ds
\leq 2 \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}s^2} \left| \frac{s}{\sigma \sqrt{n}} - \frac{1}{2\sigma^2 n^2} \right| ds
\leq \frac{2}{\sigma \sqrt{n}} \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} |s|e^{-\frac{1}{2}s^2} ds + \frac{1}{\sigma^2 n^2} \int_{|s| \leq \frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}s^2} ds
\leq \frac{c}{n^{\gamma}}.
$$

For $I_{12}$, we have

$$
I_{12} \leq \int_{\frac{n}{\sigma \sqrt{n}}}^{\frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}s^2} ds + \int_{-\frac{n}{\sigma \sqrt{n}}}^{-\frac{n}{\sigma \sqrt{n}}} e^{-\frac{1}{2}s^2} ds \leq \frac{2}{n^{\gamma}}.
$$

Hence, we get $I_1 \leq \frac{c}{n^{\gamma}}$. Similarly, one can also check that $I_2 \leq \frac{c}{n^{\gamma}}$. Therefore,

$$
\int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}} + \frac{1}{\sigma \sqrt{n}}\right) ds - \int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}}\right) ds \leq \frac{c}{n^{\gamma}}.
$$

(7.45)
Using (7.41) and taking \( \gamma = \frac{\delta}{2(3+\delta)} \), we obtain
\[
\mathbb{P}\left( \frac{x + S_n}{\sigma \sqrt{n}} > t, \tau_x > n \right) \leq \int_t^\infty \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) \, ds + \frac{c_\gamma}{n^{\frac{\gamma}{2} - \gamma(2+\delta)}} \\
\leq \int_t^\infty \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) \, ds + \frac{c_\gamma}{n^{\frac{\gamma}{2}}}.
\]  
(7.46)

To show the lower bound, we first write for \( t > 0 \),
\[
\int_t^\infty \psi \left( s, \frac{x}{\sigma \sqrt{n}} - \frac{1}{\sigma n^{\gamma}} \right) \, ds - \int_t^\infty \psi \left( s, \frac{x}{\sigma \sqrt{n}} \right) \, ds
= \frac{1}{\sqrt{2\pi}} \int_1^\infty \left( e^{-\frac{1}{2}(s-\frac{1}{\sigma n^{\gamma}})^2} - e^{-\frac{1}{2}(s-\frac{1}{\sigma n^{\gamma}})^2} \right) \, ds
+ \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \left( e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} - e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} \right) \, ds
=: J_1 + J_2.
\]

For \( J_1 \), by a change of variable and elementary calculations, we get
\[
J_1 = \frac{1}{\sqrt{2\pi}} \int_{\frac{s}{\sigma \sqrt{n}}}^\infty \left( e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} - e^{-\frac{1}{2}s^2} \right) \, ds
\geq - \int_\mathbb{R} e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} - e^{-\frac{1}{2}s^2} \, ds
= - \int_{|s| \leq n^{\gamma}/2} \left| e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} - e^{-\frac{1}{2}s^2} \right| \, ds
- \int_{|s| > n^{\gamma}/2} \left| e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} - e^{-\frac{1}{2}s^2} \right| \, ds
=: J_{11} + J_{12}.
\]

For \( J_{11} \), using again the inequality \( e^x - 1 \leq |x| + |x|e^x, x \in \mathbb{R} \), we have
\[
J_{11} = - \int_{|s| \leq n^{\gamma}/2} e^{-\frac{1}{2}s^2} \left| e^{-\frac{1}{2n^{\gamma}} - \frac{1}{2n^{2\gamma}}} - 1 \right| \, ds
\geq - \int_{|s| \leq n^{\gamma}/2} e^{-\frac{1}{2}s^2} \left| s n^{\gamma} \right| + \frac{1}{n^{2\gamma}} \, ds
- \int_{|s| \leq n^{\gamma}/2} e^{-\frac{1}{2}(s+\frac{1}{\sigma n^{\gamma}})^2} \left| s n^{\gamma} + \frac{1}{2n^{2\gamma}} \right| \, ds
\geq -2 \int_{|s| \leq 2n^{\gamma}/2} e^{-\frac{1}{2}s^2} \left| s n^{\gamma} + \frac{1}{n^{2\gamma}} \right| \, ds
\geq -\frac{2}{n^{\gamma}} \int_{|s| \leq 2n^{\gamma}/2} |s|e^{-\frac{1}{2}s^2} ds - \frac{2}{n^{2\gamma}} \int_{|s| \leq 2n^{\gamma}/2} e^{-\frac{1}{2}s^2} ds \geq -\frac{c}{n^{\gamma}}.
\]

For \( J_{12} \), we have
\[
J_{12} \leq \int_0^{n^{\gamma}/2+\frac{1}{2n^{2\gamma}}} e^{-\frac{1}{2}s^2} ds + \int_{-n^{\gamma}/2}^{-n^{\gamma}/2+\frac{1}{2n^{2\gamma}}} e^{-\frac{1}{2}s^2} ds \leq \frac{2}{n^{\gamma}}.
\]
Hence, we get $I_1 \leq \frac{c}{n^\gamma}$. Similarly, one can also check that $I_2 \leq \frac{c}{n^\gamma}$. Therefore,

$$
\int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}} - \frac{1}{\sigma n^\gamma} \right) ds - \int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}} \right) ds \geq - \frac{c}{n^\gamma}.
$$

Using (7.42) and taking $\gamma = \frac{\delta}{2(3+\delta)}$, we obtain

$$
P \left( \frac{x + S_n}{\sigma \sqrt{n}} > t, \tau_x > n \right) \geq \int_t^\infty \psi \left(s, \frac{x}{\sigma \sqrt{n}} \right) ds - \frac{c}{n^\gamma} - \frac{c\gamma}{n^{\frac{2}{3} - \gamma(2+\delta)}}.
$$

Combining (7.46) and (7.47) finishes the proof of (2.24).

\[ \square \]

**Proof of Corollary 2.9.** Note that

$$
I := \int_0^t \psi \left(s, \frac{x}{\sigma \sqrt{n}} \right) ds = e^{-\frac{x^2}{2\sigma^2 n}} \int_0^t \frac{1}{\sqrt{2\pi}} \int_0^s e^{-\frac{u^2}{2\sigma^2 n}} \left(1 - e^{-\frac{2ux}{\sigma \sqrt{n}}} \right) du ds.
$$

(7.48)

For the upper bound, using the inequality $1 - e^{-u} \leq u, u \geq 0$, we get that for any $x \geq 0$ and $t \geq 0$,

$$
I \leq \frac{2x}{\sigma \sqrt{2\pi n}} e^{-\frac{x^2}{2\sigma^2 n}} \int_0^t se^{-\frac{s^2}{2\sigma^2 n}} e^{\frac{sx}{\sigma \sqrt{n}}} ds
$$

$$
= \frac{2x}{\sigma \sqrt{2\pi n}} \int_0^t se^{-\frac{1}{2}(s - \frac{x}{\sigma \sqrt{n}})^2} ds = \frac{2x}{\sigma \sqrt{2\pi n}} \int_{-\frac{x}{\sigma \sqrt{n}}}^{t - \frac{x}{\sigma \sqrt{n}}} \left(u + \frac{x}{\sigma \sqrt{n}} \right) e^{-\frac{1}{2}u^2} du
$$

$$
\leq \frac{2x}{\sigma \sqrt{2\pi n}} \left( e^{-\frac{x^2}{2\sigma^2 n}} - e^{-\frac{1}{2}t \frac{x^2}{\sigma^2 n}} \right) + \frac{c^2 x^2}{n}
$$

$$
= \frac{2x}{\sigma \sqrt{2\pi n}} e^{-\frac{x^2}{2\sigma^2 n}} \left( 1 - e^{-\frac{t^2}{2} + \frac{1x^2}{\sigma^2 n}} \right) + \frac{c^2 x^2}{n}
$$

$$
\leq \frac{2V(x)}{\sigma \sqrt{2\pi n}} \left( 1 - e^{-\frac{t^2}{2}} \right) + \frac{c^2 x^2}{n} = \frac{2V(x)}{\sigma \sqrt{2\pi n}} \Phi^+(t) + \frac{c^2 x^2}{n},
$$

(7.49)

where in the last inequality we used the fact that $e^{-\frac{x^2}{2\sigma^2 n}} \leq 1, t \geq 0$ and $x \leq V(x)$. 

For the lower bound, from (7.48) and the inequalities $e^{\frac{x^2}{2\sigma^2 n}} \geq 1$ and $1 - e^{-u} \geq u - \frac{u^2}{2}$, $u \geq 0$, we obtain that for any $x \geq 0$ and $t \geq 0$,

$$I \geq e^{-\frac{x^2}{2\sigma^2 n}} \frac{1}{\sqrt{2\pi}} \int_0^t e^{-\frac{s^2}{2\sigma^2 n}} \left(1 - e^{-\frac{2xs}{\sigma^2 n}}\right) ds$$

$$\geq e^{-\frac{x^2}{2\sigma^2 n}} \frac{2x}{\sigma \sqrt{2\pi n}} \left(\int_0^t s e^{-\frac{s^2}{2\sigma^2 n}} ds - \frac{x}{\sqrt{n}} \int_0^t s^2 e^{-\frac{s^2}{2\sigma^2 n}} ds\right)$$

$$\geq \left(1 - \frac{x^2}{n}\right) \frac{2x}{\sigma \sqrt{2\pi n}} \left(\Phi^+(t) - c_t \frac{x}{\sqrt{n}}\right)$$

$$\geq \frac{2x}{\sigma \sqrt{2\pi n}} \left(\Phi^+(t) - c_t \frac{x}{\sqrt{n}}\right),$$

(7.50)

where $c_t = \frac{1}{\sigma} \int_0^t s^2 e^{-\frac{s^2}{2\sigma^2 n}} ds$. Applying Lemma 3.7 with $k_0 = n$, we get that uniformly in $x \in [n^{1/2-\varepsilon}, \alpha_n \sqrt{n}]$,

$$x \geq \frac{1}{1 + c_k k_0^{-\varepsilon}} \left(V(x) - c_k k_0^{1/2-\varepsilon}\right) \geq V(x) - c_k \frac{V(x)}{n^{\varepsilon}} - c_k n^{1/2-\varepsilon}$$

$$\geq V(x) \left(1 - \frac{c_k}{n^{\varepsilon}}\right).$$

Substituting this into (7.49) gives that uniformly in $x \in [n^{1/2-\varepsilon}, \alpha_n \sqrt{n}]$,

$$I \geq \frac{2V(x)}{\sigma \sqrt{2\pi n}} \left(1 - \frac{c_k}{n^{\varepsilon}}\right) \left(\Phi^+(t) - c_t \frac{x}{\sqrt{n}}\right) \geq \frac{2V(x)}{\sigma \sqrt{2\pi n}} \left(\Phi^+(t) - c_t' \frac{x}{\sqrt{n}}\right),$$

(7.51)

where $c_t'$ is bounded uniformly in $t \in \mathbb{R}_+$. Combining (7.49) and (7.51), and taking into account Theorems 2.7 and 2.8, we conclude the proof of (2.26).

For the second assertion (2.27), applying Lemma 3.7 with $k_0 = n^{\beta}$, we get that for any $\varepsilon \in (0, 1/2)$ and $\beta \in (0, 1/2 - \varepsilon)$, uniformly in $x \in [n^{\beta}, \alpha_n \sqrt{n}]$,

$$x \geq \frac{1}{1 + c_k k_0^{-\varepsilon}} \left(V(x) - c_k k_0^{1/2-\varepsilon}\right) \geq V(x) - c_k n^{-\varepsilon \beta} V(x) - c_k n^{\beta/2}$$

$$\geq V(x) - c_k n^{-\varepsilon \beta} V(x) - c_k x n^{-\beta/2}$$

$$\geq V(x) \left(1 - c_k n^{-\varepsilon \beta}\right).$$

Substituting this into (7.50), we get that uniformly in $x \in [n^{\beta}, n^{1/2-\varepsilon}]$,

$$I \geq \frac{2V(x)}{\sigma \sqrt{2\pi n}} \left(1 - c_k n^{-\varepsilon \beta}\right) \left(\Phi^+(t) - c_t \frac{x}{\sqrt{n}}\right)$$

$$\geq \frac{2V(x)}{\sigma \sqrt{2\pi n}} \left(\Phi^+(t) - c_k n^{-\varepsilon \beta}\right).$$
This, together with (7.49) and Theorems 2.7 and 2.8, finishes the proof of (2.27).

REFERENCES

[1] Afanasyev, V.I., Böinghoff, C., Kersting, G., Vatutin, V.A. (2012): Limit theorems for weakly subcritical branching processes in random environment. *Journal of Theoretical Probability*, 25(3), 703-732.

[2] Aleskevičiene, A.K. (1973): Nonuniform estimate of the distribution of the maximum of cumulative sums of independent random variables. *Litov. Mat. Sb.*, 13(2), 15-43. In Russian.

[3] Amosova, N.N. (1972): On limit theorems for probabilities of moderate deviations. *Vestnik Leningrad. Univ*, 13(2), 139-153.

[4] Asmussen, S. (2003): Applied probability and queues. Second Edition, Application of Mathematics 51, *Springer-Verlag*, New York.

[5] Bansaye, V., Vatutin, V.A. (2014): Random walk with heavy tail and negative drift conditioned by its minimum and final values. *Markov Processes and Related Fields*, 20(4), 633-652.

[6] Bertoin, J., Doney, R.A. (1994): On conditioning a random walk to stay nonnegative. *The Annals of Probability*, 22(4), 2152-2167.

[7] Bolthausen, E. (1972): On a functional central limit theorem for random walk conditioned to stay positive. *The Annals of Probability*, 4(3), 480-485.

[8] Borovkov, A.A. (1962): New limit theorems for boundary-valued problems for sums of independent terms. *Sib. Math. J.*, 3(5), 645-694.

[9] Borovkov, A.A. (1970): Factorization identities and properties of the distribution of the supremum of sequential sums. *Theory of Probability and Its Applications*, 15(3), 359-402.

[10] Borovkov, A.A. (2004): On the Asymptotic Behavior of the Distributions of First-Passage Times, I. *Mathematical Notes*, 75(1-2), 23-37. Translated from Matematicheskie Zametki, 75(1), 24-39.

[11] Borovkov, A.A. (2004): On the Asymptotic Behavior of Distributions of First-Passage Times, II. *Mathematical Notes*, 75(3-4), 322-330. Translated from Matematicheskie Zametki, 75(3), 350-359.

[12] Breuillard, E. (2005): Distributions diophantiennes et théorème limite local sur $\mathbb{R}^d$. *Probability Theory and Related Fields*, 132(1), 13-38.

[13] Caravenna, F. (2005): A local limit theorem for random walks conditioned to stay positive. *Probability Theory and Related Fields*, 133(4), 509-530.

[14] Denisov, D., Vatutin, V.A., Wachtel, V. (2014): Local probabilities for random walks with negative drift conditioned to stay nonnegative. *Electronic Journal of Probability*, 88, 1-17.

[15] Denisov, D., Wachtel, V. (2015): Random walks in cones. *The Annals of Probability*, 43(3), 992-1044.

[16] Denisov, D., Sakhanenko, A., Wachtel, V. (2018): First-passage times for random walks with nonidentically distributed increments. *The Annals of Probability*, 46(6), 3313-3350.

[17] Doney, R.A. (1989): On the asymptotic behaviour of first passage times for transient random walk. *Probability Theory and Related Fields*, 81(2): 239-246.

[18] Doney, R.A. (2012): Local behavior of first passage probabilities. *Probability Theory and Related Fields*, 152(3-4): 559-588.
[19] Doney, R.A., Jones, E. (2012): Large deviation results for random walks conditioned to stay positive. *Electronic Communications in Probability*, 38, 1–11.
[20] Eppel, M.S. (1979): A local limit theorem for the first overshoot. *Siberian Math. J.*, 20, 130-138.
[21] Feller, W. (1964): *An Introduction to Probability Theory and Its Applications*. Vol. 2. Wiley, New York.
[22] Fuk, D.K., Nagaev, S.V. (1971): Probability inequalities for sums of independent random variables. *Theory of Probability and Its Applications*, 16(4), 643-660.
[23] Gnedenko, B.V. (1948): On a local limit theorem of the theory of probability. *Uspekhi Mat. Nauk*, 3(25), 187-194.
[24] Grana, I. (1997): On moderate deviations for martingales. *The Annals of Probability*, 25(1), 152-183.
[25] Grana, I., Lauvergnat, R., Le Page, É. (2018): Limit theorems for affine Markov walks conditioned to stay positive. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 54(1), 529-568.
[26] Grana, I., Lauvergnat, R., Le Page, É. (2018): Limit theorems for Markov walks conditioned to stay positive under a spectral gap assumption. *The Annals of Probability*, 46(4), 1807-1877.
[27] Grana, I., Lauvergnat, R., Le Page, É. (2020): Conditioned local limit theorems for random walks defined on finite Markov chains. *Probability Theory and Related Fields*, 176(1-2), 669-735.
[28] Grana, I., Mentemeier, S., Xiao, H. (2021): The extremal position of a branching random walk in the general linear group. *In preparation*.
[29] Grana, I., Quint, J.F., Xiao, H. (2021): Conditioned limit theorems for hyperbolic dynamical systems. *In preparation*.
[30] Iglehart, D.L. (1974): Functional central limit theorems for random walks conditioned to stay positive. *The Annals of Probability*, 2(4), 608-619.
[31] Iglehart, D.L. (1974): Random walks with negative drift conditioned to stay positive. *Journal of Applied Probability*, 11(4), 742-751.
[32] Keener, R. (1992): Limit theorems for random walks conditioned to stay positive. *The Annals of Probability*, 20(2), 801-824.
[33] Kersting, G., Vatutin, V.A. (2017): Discrete time branching processes in random environment. ISTE Limited.
[34] Lévy, P. (1937): Théorie de l’addition des variables aléatoires. Gauthier-Villars.
[35] Nagaev, S.V. (1965): Some limit theorems for large deviations. *Theory of Probability and Its Applications*, 10(2), 214-235.
[36] Nagaev, S.V. (1969): Estimating the rate of convergence for the distribution of the maximum sums of independent random quantities. *Sib. Math. J.*, 10(3), 443-458.
[37] Nagaev, S.V. (1975): On the speed of convergence of the distribution function of the maximum sum of independent identically distributed random quantities. *Theory Probab. Appl.*, 15(2), 309-314.
[38] Nagaev, S.V. (1979): Large deviations of sums of independent random variables. *The Annals of Probability*, 7(5), 745-789.
[39] Sakhnenko, A.I. (2006): Estimates in the invariance principle in terms of truncated power moments. *Siberian Mathematical Journal*, 47(6), 1113-1127.
[40] Sheep, L.A. (1964): A local limit theorem. *The Annals of Mathematical Statistics*, 35(1), 419-423.
[41] Spitzer, F. (1976): *Principles of Random Walk*. Second edition. Springer.
[42] Stone, C. (1965): A local limit theorem for nonlattice multi-dimensional distribution functions. *The Annals of Mathematical Statistics*, 36(2), 546-551.

[43] Tanaka, H. (1989): Time reversal of random walks in one-dimension. *Tokyo J. Math.*, 12(1), 159-174.

[44] Vatutin, V.A., Wachtel, V. (2009): Local probabilities for random walks conditioned to stay positive. *Probability Theory and Related Fields*, 143(1-2), 177-217.

Current address, I. Grama: Université de Bretagne Sud, Laboratoire de Mathématiques de Bretagne Atlantique, UMR CNRS 6205, Centre Yves Coppens, Campus de Tohannic, 56017 Vannes, France

Email address, I. Grama: ion.grama@univ-ubs.fr

Current address, H. Xiao: Universität Hildesheim, Institut für Mathematik und Angewandte Informatik, 31141 Hildesheim, Germany

Email address, H. Xiao: xiao@uni-hildesheim.de