Optimal insurance contract with benefits in kind under adverse selection

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Abstract

A significant loss of income can have a negative impact on households who are forced to reduce their consumption of some particular staple goods. This can lead to health issues and consequently generates significant costs for society. In order to prevent these negative consequences, we suggest that consumers can buy an insurance to have a sufficient amount of staple good in case they lose a part of their income. We develop a two-period/two-good Principal-Agent problem with adverse selection and endogenous reservation utility to model an insurance with in kind benefits. This model allows us to obtain semi-explicit solutions for the insurance contract and is applied to the context of fuel poverty.

Key words: contract theory, adverse selection, in-kind insurance, fuel poverty, calculus of variations.

AMS 2000 subject classifications: 91B30; 91B08; 49K15.

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Introduction

This article examines a short–term solution to protect vulnerable households from the risk of temporary poverty in a particular staple good. Staple goods are essential products that consumers are unable, or unwilling, to cut out of their budgets, such as food, beverage, water, energy... They tend to consume staples goods at a relatively constant level, regardless of price or their financial situation. However, in the event of a substantial loss of income, consumers will reduce the budget allocated to some staple goods, such as fresh food, energy, medicines or feminine hygiene products, which can lead to serious illnesses. Indeed, these particular staple goods may not seem essential to poor consumers, unlike food or water. A poor household, facing the choice of heating, healing or eating, will systematically make the choice to eat. This question has been the subject of many surveys and discussions on this type of goods which, for households in a very precarious situation, can be considered, in a sense, as a luxury. In particular, we can mention the works of Milne and Molana [1991], Freeman [2003] for health care, and Meier et al. [2013], Schulte and Heindl [2017] for energy. We focus in this paper on this particular type of staple goods, whose consumption is strongly impacted by income losses.

Since we are focusing on modest households, who are not used to saving money in anticipation of the future, a loss of income would force them to reduce their consumption of some particular basic commodities, such as residential heating, which could lead to health issues (see Lacroix and Chaton [2015]). A common solution would be to offer an income insurance that will provide money to the household in the event of an income loss. However, in the situation under consideration, the household will probably spend the money received to buy products that are more essential from its point of view. This type of insurance thus appears not to be an appropriate solution to the problem we are confronted with. Therefore, our suggestion is rather an in–kind support than a financial assistance: if the household suffers from a loss of income, it will receive a specified amount of the particular good under consideration, ensuring an adequate consumption in that good. This approach, despite the
fact that it could be perceived as paternalistic, should protect individuals from the negative effects of some of their decisions, and thus prevent the health issues associated with the lack of this good. Indeed, many articles show the effectiveness of in–kind support compared to financial one to fight poverty, such as Blackorby and Donaldson [1988] or Slesnick [1996].

Our objective is therefore to develop an insurance dedicated to a particular good, in order to guarantee the insured household a sufficient consumption of this product, in the event of an income loss. Since the risk of income loss is different among the considered population, there is a need to offer a menu of contracts: the insurer should provide different types of insurance, to let each household choose the best suited for its need and risk. Offering a menu of contracts is relatively traditional in the insurance field (see the survey of Dionne et al. [2013]), it allows the insurer to fight against the asymmetry of information between him and the insured, in particular against adverse selection. In general, in insurance problems, this asymmetry is related to the risk to which the insured is exposed, and it is assumed in most models that the insured knows better his own risk than the insurance company. In our context, this assumption seems reasonable since the insured has a better estimate of his risk of income loss than the insurer.

The field of insurance models with adverse selection can be divided into two categories, depending on the status of the insurer. In the first category, authors are considering insurance as a model of pure competition between insurance companies, which implies that the price of insurance is set such that insurers do not make any profit. One of the leading models in this trend is the RS model of Rothschild and Stiglitz [1976] and its various extensions (see Boone [2015], Chassagnon and Chiappori [1997], De Donder and Hindriks [2009], Cook and Graham [1977], Alary and Bien [2008], Janssen and Karamychev [2005] and the survey of Mimra and Wambach [2014]). Our model will be classified in the second (more technically difficult and unfortunately less developed) category: we will assume that the insurer is a monopoly. In this literature, we can mention among others the extension of the RS model to a monopoly by Stiglitz [1977], or more generally the class of models with
adverse selection (not in the field of insurance) developed by Mirrlees [1971], Spence [1974], Guesnerie and Laffont [1984], Salanié [2005], Laffont and Martin-mort [2009] and their extensions to multi-products by, for example, Armstrong [1996], or in continuous-time by Alasseur et al. [2019]. The most reasonable justification for the choice of a monopoly model is that, for the application we have in mind, namely a fuel-poverty\(^1\) insurance, the best suited insurer is the customer’s current energy supplier, who knows his client better than other companies. Moreover, since the insurer is a monopoly, the household only has the choice between purchasing an insurance contract, among those offered by the monopoly, or not. As a result, we consider that the household refuse any contracts if none provides it more utility than its utility without insurance, defined as the reservation utility, which thus depend on its risk.

Inspired by the literature on standard contract theory with adverse selection, mainly through pioneering works of Baron and Myerson [1982], Guesnerie and Laffont [1984], Maskin and Riley [1984], and the few applications to insurance problems such as Stiglitz [1977], Landsberger and Meilijson [1994, 1996], we model this situation as a Principal–Agent problem with adverse selection. The Principal (She – an insurance company, a supplier...) can offer an insurance to the Agent (he – a household), which allows him to receive a specified amount of the staple good under consideration, in the event of a loss of income. We assume that the adverse selection concerns the Agent’s probability of income loss, defined as his type. This assumption is classical in the literature, particularly in all extensions of Rothschild and Stiglitz [1976] and Stiglitz [1977]. The reservation utility we considered will thus depend on the Agent’s type. The problem of an endogenous reservation utility is studied in some adverse selection models such as Lewis and Sappington [1989], Biglaiser and Mezzetti [1993], Maggi and Rodriguez-Clare [1995], Jullien [2000], and Alasseur et al. [2019] also discuss this issue for an application close to the one we have in mind. However, to the best of our knowledge, this problem is rarely

\(^1\)According to the french law Grenelle 2: 'A person in a fuel poverty situation is a person who has particular difficulties in his/her home in obtaining the necessary supply of energy to meet his/her basic needs because of the inadequacy of his/her resources or living conditions.'
addressed for a continuum of types in insurance models, as in our framework.

One of the cornerstone of our approach is to combine a model of consumption (two goods and a budget constraint) with traditional insurance models under adverse selection. Another particularity of our method lies in the choice of a two-period insurance model, as in Schlesinger and Zhuang [2014, 2019]. Most of the literature on insurance with adverse selection is focusing on only one period, where the Agent pays and receives the insurance at the same time, or repeated versions of this scheme. The closest application where this type of two-period models are being developed is self-prevention. In the works of Eeckhoudt et al. [2012], Wang and Li [2015], Peter [2017] (model with savings) or Menegatti [2009], Courbage and Rey [2012] (without savings), the authors consider a two-period model to account for the delay between the prevention effort and the real benefit of it. In our framework, a two-period model is necessary to model the fact that the insured is not in a precarious situation when he subscribes to the insurance to be covered over the next period. Considering a two-period model also allows us to compare, with and without insurance, the evolution of the Agent’s consumption, when he suffers from a loss of income.

The remainder of this paper is organised as follows. The Principal-Agent model for insurance with two periods, two goods and endogenous reservation utility is detailed in Section 1. Section 2 presents the benchmark case, i.e. the problem without adverse selection. Under adverse selection, the problem is solved in Section 3. Through a simple but not simplistic model, we obtain the most explicit results possible. In particular, we find the optimal design for the menu of contracts, and we study the Agent’s optimal choice of contract and consumption. A remarkable feature of our problem is that the optimal menu of contracts excludes the less risky Agents. We apply our results to the context of fuel poverty. By numerical simulations, we show that our insurance with benefits in kind can be a tool to help the riskiest Agents to consume a sufficient quantity of electricity in the case of an income loss. To our opinion, this type of insurance can therefore protect risky Agents from fuel-poverty. Section 4 concludes by suggesting some policy recommendations.
1 The model

1.1 A Principal–Agent model with adverse selection

We consider a two–period/two–good Principal–Agent model with adverse selection. The Agent, He, represents a household consuming the essential good considered and another representative good. More precisely, at each time $t \in \{0, 1\}$, the Agent has an income $w_t$ which allows him to consume a quantity $e_t$ of the considered staple good and a quantity $y_t$ of another good, with respective unitary constant positive price $p_e$ and $p_y$. However, between the two periods, the Agent is likely to suffer from a loss of income, which will put him in a precarious situation at time $t = 1$: he will be constrained to reduce his consumption. However, if he does not consume a sufficient quantity of the staple good, this can lead to serious issues of which the Agent is not necessarily aware. To prevent him from staple good poverty, the risk–neutral Principal, she, who may be the good producer or supplier, an insurance company, or even the government, offers an insurance. This insurance ensures that the Agent receives a specified quantity of the staple good, denoted $e_{min}$, in the event of an income loss. At time $t = 0$, the Agent thus chooses if he wants to subscribe to the insurance and if so, he pays the insurance premium $T$ associated to a contractible quantity $e_{min}$. At time $t = 1$, if he has purchased the insurance and if his income has decreased sufficiently, the Agent receives the quantity $e_{min}$ of the staple good.

We define the random income of an Agent of type $\varepsilon$ at time $t = 1$ by $w_1 := \omega w_0$ where $w_0$ is the income at time $t = 0$ and $\omega$ is a random variable, defined on the probability space $(\Omega, \mathcal{A}, P^\varepsilon)$, where $\Omega$ is a subset in $\mathbb{R}_+$ and $\mathcal{A}$ its natural $\sigma$–algebra. We assume that the insurance is activated when $\omega \leq \bar{\omega}$, where the income loss barrier $\bar{\omega}$ is set in an exogenous way. In order to obtain closed–form solutions, we will make the following assumption:

**Assumption 1.1.** The random variable $\omega$ takes two values, $\underline{\omega}$ with probability $\varepsilon$ and $\bar{\omega}$ with probability $1 - \varepsilon$, where $\varepsilon \in [0, 1]$ and $\bar{\omega} > \bar{\omega} \geq \underline{\omega} > 0$.

We assume that the constants $\underline{\omega}$ and $\bar{\omega}$ are common knowledge. The in-
equality $\tilde{\omega} > \tilde{\omega} > \omega$ means that the insurance is only activated when $\omega = \tilde{\omega}$.

This model for the distribution of losses is traditional in insurance models based on the pioneer works of Rothschild and Stiglitz [1976] and Stiglitz [1977]. We consider that the Agent is better informed than the Principal about the risk of income loss he is facing, which depends on his work quality, his job insecurity, the relation he has with his supervisor… The Principal has only access to an overview of risks among the population, which leads to an adverse selection problem, precisely defined by the following assumption.

**Assumption 1.2** (Adverse Selection). *The Principal cannot observe the type of an Agent, but knows the distribution of the types of her potential clients.*

As classical in adverse selection problems, the Principal has interest in offering a menu of contracts, i.e. various $e_{\min}$ with associated premium $T$. The agent then chooses the contract that best suits him among all contracts offered by the Principal, depending on his risk. In our study, we look for the best continuous menu of contracts that the insurer can offer.

### 1.2 Agent’s problem

In most insurance models, the utility function of the Agent is not specified. With the aim of obtaining the most explicit results possible, we choose here to represent the preferences of the Agent toward the goods’ consumption, at time $t$, by a separable utility function based on logarithmic felicities:

$$U(e_t, y_t) := \alpha \ln(e_t) + \ln(y_t), \quad \text{for} \quad e_t, y_t > 0,$$

where $\alpha$ parametrises the *longview* elasticity of substitution between the staple good and the composite one.

Our model does not take into account the possibility for the Agent to save between the two periods (contrary to Schlesinger and Zhuang [2014]). This hypothesis may seem restrictive but is consistent with the literature on two–period models (in particular on prevention with Menegatti [2009], Courbage and Rey [2012]) and is justified in our framework in view of the particular
households\(^2\) on which we want to focus our study. Indeed, one can assume that a household already used to saving has built up sufficient funds to pay its bills in the event of a loss of income. This household should thus not be concerned by the insurance we develop throughout this paper, unless it has inadequate savings. In fact, our insurance can precisely be interpreted as a form of incentive to save: it is a way for households, who have no savings, to obtain a quantity of staple good in case of an income loss. The concept of insurance is very effective in this type of situation, and allows risks to be shared among the population. Moreover, this choice of model is also based on the willingness to keep a tractable model with (relatively) explicit solutions, and to focus our study on the design of insurance contracts. In parallel with Menegatti [2009] for prevention, the interaction between insurance and savings in a two–period model is a different problem, but could represent a potentially interesting extension for future work.

1.2.1 Reservation utility

Without insurance, the Agent maximises, independently at each period \(t\), the utility previously defined, under his budget constraint:

\[
V^\varepsilon(w_t) := \max_{(e_t, y_t) \in \mathbb{R}^2_+} \quad U(e_t, y_t), \quad \text{u.c.} \quad e_t p_e + y_t p_y \leq w_t. \tag{1.2}
\]

Given a discount factor \(\beta \in [0, 1]\), we define the intertemporal expected utility without insurance of an Agent of type \(\varepsilon\) as follows:

\[
EU^\varepsilon(\varepsilon) := V^\varepsilon(w_0) + \beta \mathbb{E}^{\mathbb{P}_t}[V^\varepsilon(\omega w_0)]. \tag{1.3}
\]

\(^2\)Households that do not save but want to are widespread, as evidenced by the many mobile applications or services to help them. For example, the mobile application *Birdycent* rounds up each payment made by the consumer to feed a piggy bank, with a zero interest rate, which is equivalent to losing money in relation to inflation. A second application, called *Yeeld*, offers 4% in cash back on Amazon instead of an interest rate. The bank *Crédit Mutuel* proposes the service *Budget +*, which is subject to a fee, to automatically save from a current account to a savings one. In our opinion, this highlights the need to encourage households to save money and that they are willing to pay for these types of services.
In our framework, the Agent is likely to accept the insurance contract only if it provides him a level of utility at least equal to his utility without. Therefore, the reservation utility of an Agent of type $\varepsilon$ will be defined\(^3\) by (1.3).

1.2.2 Expected utility with insurance

Let us now fix an insurance contract $(e_{\text{min}}, T)$. If the Agent decides to subscribe to this contract, we assume that the payment of the insurance premium $T$ only impacts his budget constraint at time $t = 0$, and his maximum utility is thus naturally given by:

$$V_0(w_0, T) := V^\omega(w_0 - T). \quad (1.4)$$

As described in Subsection 1.1, the insurance we consider is an in–kind support: it ensures the Agent a fixed non–negative amount $e_{\text{min}} \geq 0$ of a determined staple good at time $t = 1$, if he suffers from a sufficient loss of income, \textit{i.e.} if $\omega = \omega$. Therefore, his maximisation problem is:

$$V_1(\omega w_0, e_{\text{min}}) := \max_{(e_1, y_1) \in \mathbb{R}^2_+} U(e_1 + e_{\text{min}} 1_{\omega = \omega}, y_1), \quad (1.5)$$

subject to:

$$e_1 p_e + y_1 p_y \leq \omega w_0.$$ 

Similarly to the case without insurance, we define the intertemporal expected utility of an Agent of type $\varepsilon$, with an insurance contract $(e_{\text{min}}, T)$, by:

$$\text{EU}^Q(\varepsilon, e_{\text{min}}, T) := V_0(w_0, T) + \beta \mathbb{E}^{\mathbb{P}_\varepsilon}[V_1(\omega w_0, e_{\text{min}})]. \quad (1.6)$$

1.3 The Principal’s problem

We assume that the Principal is risk–neutral\(^4\) and wants to maximise her profit: she receives at time 0 the earnings from the sale of the insurance

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\(^3\)With the aim of simplifying the notations, only the dependency in the type is highlighted: the reservation utility is stated as a function of $\varepsilon$.

\(^4\)The Principal’s risk–neutrality seems reasonable because shareholders of insurance companies generally have a diversified portfolio.
to Agents of type $\varepsilon \in [0,1]$ who agree to subscribe, but needs to provide
them the quantity $e_{\min}$ they have chosen if they suffer from an income loss
in the next period. We consider in this model that the insurers are not in
perfect competition, so that the price of the insurance is not determined by
the actuarial price. Therefore, in this monopoly situation, the insurer can
choose the range of $e_{\min}$ she wants to offer, but also the price associated to
each quantity. We properly define the notion of admissible contracts and menu
of contracts in our framework:

**Definition 1.3.** An admissible contract $(e_{\min}, T)$ for the insurance is a quan-
tity $e_{\min} \geq 0$ with an associated premium $T < w_0$. An admissible menu is
then defined as a continuum of admissible contracts $(e_{\min}, T)$, i.e. a contin-
um of non-negative quantities and a continuous price function $T$ defined for
all quantities offered. Under Assumption 1.2, the function price $T$ is required
to be independent of the Agent’s type.

In the First–Best case, the Principal knows the type $\varepsilon$ of the Agent, and
can thus offer him a particular contract. Since she has to pay with probability
$\varepsilon$ the quantity $e_{\min}$ at the unitary price $p_e$, her optimisation problem is:

$$
\pi_\varepsilon := \sup_{e_{\min}, T} (T - \varepsilon p_e e_{\min}),
$$

(1.7)

under the constraint that the contract $(e_{\min}, T)$ is an admissible contract and
provides the Agent of type $\varepsilon$ with at least his reservation utility.

In the Third–Best case, *i.e.* with adverse selection, we consider a menu
of revealing contracts, in the sense that an Agent of type $\varepsilon$ will subscribe to the
insurance contract designed for him, *i.e. $(e_{\min}(\varepsilon), T(\varepsilon))$. If the distribution of
the type $\varepsilon$ in the population considered by the Principal has a density function
$f$, the Principal’s problem will be defined as follows:

$$
\sup_{e_{\min}, T} \int_0^1 \left( T(\varepsilon) - \varepsilon p_e e_{\min}(\varepsilon) \right) f(\varepsilon) d\varepsilon,
$$

(1.8)

under the participation constraint, and where, in this case, $e_{\min}$ and $T$ will be
appropriate functions of $\varepsilon$. To solve this case, we assume that the distribution of the type $\varepsilon$ in the population considered by the Principal is uniform on $[0, 1]$, i.e. $f = 1_{[0,1]}$. It is equivalent to consider that, from the Principal’s point of view, an Agent has a probability one half to experience a significant loss of income. This assumption is actually not necessary, computations could easily be made for another distribution, but it allows to simplify the Principal’s problem. This distribution models a Principal who does not really have data on the Agent’s income loss. This will be the case in the application in question, where the insurer is an electricity supplier, who is not intended to have insight on the distribution of the risks of income loss of its customers. Moreover, even if a probability of one half seems high for the population, the Agents likely to subscribe to our insurance are rather risky people and the probability in the population considered by the insurer is necessarily higher than in the global population. This is particularly true given that our study focuses on middle-class households without savings, which naturally have a higher probability of losing income.

1.4 Application to fuel poverty

We apply this model on a particular staple good, the electricity, to develop an insurance against fuel poverty. According to Chaton and Gouraud [2019], the fuel poverty is essentially linked to a temporary loss of income. It particularly affects low-income and vulnerable households, with a low propensity to save, and who already spend a large part of their income on energy. This situation can lead households to adopt risky behaviours, causing health problems and housing deterioration (see Lacroix and Chaton [2015]). For example, to keep heat inside their homes, some obstruct vents, thereby generating moisture and mould. Households in fuel poverty are often forced to make choices with harmful consequences for their health: eating or heating, giving up health care or social interactions. The consequences of fuel poverty are often neglected by households but are highly expensive for the society. To avoid these harmful and costly consequences, mechanisms are being developed to help vulnerable
households. For example, in France, energy vouchers are distributed by the State since 2018. This voucher can be used to pay for energy expenses such as electricity, gas, wood and fuel oil bills, but also for energy renovation. In 2019, it targets 5.8 million of households with modest incomes.

The motivation of our work is to act on the prevention side by proposing a complementary tool, to avoid the number of households in fuel poverty from increasing. The idea is to develop an insurance policy that is activated if the household becomes energy constrained. Two French electricity suppliers propose a slightly different insurance: Assurénnergie proposed by Electricité de France (EDF) and Assurance Facture by ENGIE. These two monthly insurances offer a refund of part of the electricity bill in the event of job loss, sick leave, hospitalisation, disability or death. For the first insurance, the amount refunded depends on the contract chosen from the proposed menu. The second insurance is a unique contract. Our goal is to compute the optimal menu of contracts thanks to contract theory with adverse selection, in order to study the structure of the contracts obtained, and to know what types of Agents will be likely to subscribe to the insurance. One can notice that the monopolistic framework under consideration makes sense in this situation, since the fuel provider of a household has more inside information than other fuel providers or classical insurance companies.

2 Benchmark case: the First–Best problem

In this section, we first start by solving the optimal consumption problem of an Agent of type $\varepsilon$: given an insurance contract $(e_{\text{min}}, T)$, and the utility function specified in (1.1), we compute the Agent’s optimal consumption in both goods at each period. As a result, we can compute the maximum utility the Agent can achieve for a given contract. Comparing this utility with the reservation utility, we can determine the maximum price the Agent is willing to pay for the insurance. This first part will allow us to properly define in our context the participation constraint mentioned in the definition of the Principal’s problem in Subsection 1.3. With this in mind, we can then solve the
problem in the First–Best case, i.e. without adverse selection. In particular, since the Principal knows the type of the Agent, she can offer him a specific contract, with which his participation constraint is binding. In other words, she may charge the insurance at the highest price the Agent is willing to pay, to the point that he is in fact indifferent between subscribing or not to the insurance.

2.1 Solving the Agent’s problem

We first solve the consumption problem of an Agent who has not subscribed to the insurance. Let us define the following constant:

$$C_{\alpha,p_e,p_y} := \alpha \ln(\alpha) - (1 + \alpha) \ln(1 + \alpha) - \alpha \ln(p_e) - \ln(p_y).$$  \hfill (2.1)

Since our framework does not allow the Agent to transfer income from one period to another, the Agent maximises his utility at each period independently by solving (1.2), which leads to the following result.

**Lemma 2.1** (Without insurance). *The optimal consumptions at time $t \in \{0, 1\}$ in each goods of an Agent with income $w_t$ are given by:

$$y^\varnothing_t := \frac{1}{1 + \alpha p_y} w_t \quad \text{and} \quad e^\varnothing_t := \frac{\alpha}{1 + \alpha p_e} w_t,$$

and induce the maximum utility $V^\varnothing(w_t) = (1 + \alpha) \ln(w_t) + C_{\alpha,p_e,p_y}$.  

Then, by a simple computation of the expected utility defined by (1.3), we can explicitly write the reservation utility:

**Proposition 2.2.** *Under Assumption 1.1, the expected utility without insurance of an Agent of type $\varepsilon$ is given by

$$EU^\varnothing(\varepsilon) = (1 + \alpha) \ln \left( \bar{\omega}^{\beta(1-\varepsilon)} w_0^2 \right) + (1 + \beta) C_{\alpha,p_e,p_y}. \hfill (2.2)$$

Usually, in insurance models, the reservation utility is taken to be independent of the Agent’s type. In our framework, we consider that an Agent
will not subscribe to the insurance if his utility without is higher. Therefore, the reservation utility we consider is endogenous, and depends on the probability $\varepsilon$. This problem is addressed in some adverse selection models, for example Lewis and Sappington [1989], Biglaiser and Mezzetti [1993], Maggi and Rodriguez-Clare [1995], Jullien [2000], Alasseur et al. [2019], but is rarely considered in insurance problems. As explained in Laffont and Martimort [2009], in this case, determining which participation and incentive constraints are binding becomes a more difficult task. Nevertheless, Proposition 3.3 will establish that only the most risky Agents will be selected by the Principal, and this type of feature only happens in models of countervailing incentives. The Principal excludes the good types, those with a low probability of losing their income, because the price they are willing to pay is very low, while the riskiest Agents are more profitable since they are easily satisfied and willing to pay much more.

Similarly, we can solve the consumption problem of an Agent who subscribes to a given admissible contract (see Lemmas A.1 and A.2). Without loss of generality, we can assume that any admissible contract $(e_{min}, T)$, in the sense of Definition 1.3, is of the following form:

$$e_{min} := q_0 w_0 / p_e, \quad \text{for } q \in \mathbb{R}_+ \text{ and } T := t_0 w_0, \quad \text{for } t_0 \in [0, 1].$$  \hspace{1cm} (2.3)

The pair $(q, t_0)$ will be referred to as an admissible normalised contract. We then denote by $\bar{U}$ the following function, for $q \in \mathbb{R}_+$:

$$\bar{U}(q) := \begin{cases} (1 + \alpha) \ln(1 + q\alpha) & \text{if } q < 1, \\ \alpha \ln(q) + (1 + \alpha) \ln(1 + \alpha) & \text{if } q \geq 1. \end{cases}$$  \hspace{1cm} (2.4)

The preliminary results in Appendix A allow us to provide an explicit form in the following proposition for the expected utility defined by (1.6).

**Proposition 2.3.** Given an admissible normalised contract $(q, t_0)$, and under
Assumption 1.1, the expected utility of an insured Agent of type $\varepsilon$ is given by:

$$EU^Q(\varepsilon, q, t_0) = EU^\varnothing(\varepsilon) + (1 + \alpha) \ln(1 - t_0) + \beta \varepsilon \bar{U}(q).$$ \hspace{1cm} (2.5)

**Remark 2.4.** The separation of cases between $q$ less or greater than 1 is related to the fact that the Agent is not allowed to resell part of the insured quantity of staple good. If $q < 1$, the quantity insured is not sufficient from the Agent’s point of view. He therefore supplements this quantity by purchasing additional energy at $t = 1$. On the contrary, if $q \geq 1$, the Agent will consume only the corresponding amount $e_{\min}$ of the staple good, his optimal complementary consumption becoming equal to zero. However, in this case, it could have been more optimal from his point of view to resell part of the insured quantity. Nevertheless, it is precisely the purpose of this paper that the household consumes more of this particular good, in order to avoid the problems induced by a decrease in consumption, of which the household is not aware.

It remains to determine when the Agent of type $\varepsilon$ will subscribe the insurance, i.e. when his expected utility with the insurance is greater than his reservation utility. With this in mind, by computing the difference between (1.6) and (1.3), we can state the following proposition.

**Proposition 2.5** (Participation constraint). An admissible normalised contract $(q, t_0)$ satisfies the participation constraint of the Agent of type $\varepsilon$ if and only if

$$t_{\max}(\varepsilon, q) := 1 - \begin{cases} (1 + q\alpha)^{-\beta \varepsilon} & \text{if } q < 1, \\ q^{-\beta \varepsilon} \frac{\alpha}{1 + \alpha} (1 + \alpha)^{-\beta \varepsilon} & \text{if } q \geq 1. \end{cases}$$ \hspace{1cm} (2.6)

Therefore, $w_0 t_{\max}(\varepsilon, q)$ is the maximum price the Agent is willing to pay for a quantity $e_{\min} = \alpha q w_0 / p_e$: as soon as the premium $T$ associated to a quantity $e_{\min} = \alpha q w_0 / p_e$ is below $w_0 t_{\max}(\varepsilon, q)$, an Agent of type $\varepsilon$ is willing to purchase the insurance contract. We say in this case that the admissible

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5If the agent could resell part of the quantity, the insurance would be strictly equivalent to an income insurance, which is why we ignore any reselling possibility.
contract \((e_{\min}, T)\) satisfies the participation constraint for \(\varepsilon\) types. One can remark that, for a given normalised quantity \(q\), this maximum price increases with the type \(\varepsilon\) of the Agents, \textit{i.e.} the probability of income loss. Therefore, the riskier the agent, the more willing he is to pay a high price for the same quantity insured.

### 2.2 Solving the Principal’s problem

As detailed in Subsection 1.3, the problem of the Principal in the First–Best case is defined by (1.7), under the participation constraint of the Agent. Thanks to the reasoning developed in the previous subsection, and denoting by \(\Xi_{\varepsilon} := \{(q, t_0) \in \mathbb{R}_+^2, \text{ s.t. } t_0 \leq t_{\text{max}}(\varepsilon, q)\}\), her problem is equivalent to:

\[
\pi_{\varepsilon} := w_0 \sup_{(q, t_0) \in \Xi_{\varepsilon}} (t_0 - \varepsilon \alpha q \omega). 
\]  

(2.7)

**Proposition 2.6.** If \(\beta > \omega\), the optimal contract \((e_{\min}, T)\) for an Agent of type \(\varepsilon \in [0, 1]\) is given by \(e_{FB}^\varepsilon := \alpha q_{FB}^\varepsilon w_0/p_e\) and \(T_{FB}^\varepsilon := w_0 t_{\text{max}}(\varepsilon, q_{\varepsilon})\) where

\[
q_{\varepsilon} := \begin{cases} 
\left(\beta(1 + \alpha)^{-\frac{\beta - 1}{\omega}}\right)^{\frac{1}{(1 + \alpha) + \alpha \beta}} & \text{if } \varepsilon \leq \varepsilon_{FB}^1, \\
\frac{1}{\alpha} \left(\left(\frac{\beta}{\omega}\right)^{\frac{1}{\beta + 1}} - 1\right) & \text{if } \varepsilon > \varepsilon_{FB}^1,
\end{cases}
\]

(2.8)

for \(\varepsilon_{FB}^1 := \frac{1}{\beta} \left(\frac{\ln(\beta) - \ln(\omega)}{\ln(1 + \alpha)} - 1\right)\).

The previous result solves the Principal’s optimisation problem in the First–Best case, its proofs is reported to Appendix B.1. The assumption \(\beta > \omega\) is made to simplify the result and makes perfect sense in this framework (see Remark B.2). Given this optimal normalised contract \((q_{\varepsilon}, t_{\text{max}}(\varepsilon, q_{\varepsilon}))\), the maximum profit obtained by the Principal for each type \(\varepsilon\) of Agents is computed explicitly in Corollary B.1. One can notice that the quantity chosen by an Agent does not depend on his expected revenue at time \(t = 1\) but is decreasing with only the lower income level \(\omega\).
2.3 First–Best insurance against fuel poverty

With the motivation described in Subsection 1.4, we apply the results of the First–Best case in the context of an insurance against fuel poverty in France. Many French households that belong to the lower class of the poorer 30% are in fuel poverty. Indeed, in 2018, in mainland France, 78% of households in the first income decile are in fuel poverty. This percentage falls to 54% (respectively 26%) for the 2nd (respectively 3rd) income decile. These households are already struggling to have the necessary energy to avoid fuel poverty, and obviously do not have the financial means to subscribe to an insurance against fuel poverty.

Therefore, to perform our numerical simulations, we consider a middle–class household, whose annual disposable income, after taxes and social benefits, is \( w_0 = 35,000 \, \text{€} \). We assume that this household lives in an all–electric house (electric heating and hot water), with an annual electricity consumption\(^6\) of \( e_0^\ominus = 14,403 \, \text{kWh} \). Since in 2018, in France, the average price \( p_e \) per kilowatt hour was 0.18 €, the share of household income spent on household energy expenditure was 7.41%. We deduce from the expression of \( e_0^\ominus \) the value \( \alpha = 8\% \). Moreover, we set \( \omega = 0.4 \): the household thus has a probability \( \varepsilon \) to have an annual disposable income equal to 0.4 \( \times \) 35,000 € = 14,000 € at time \( t = 1 \). To simplify, we assume \( \beta = 1 \).

Thanks to Proposition 2.6, we can compute the optimal quantity insured and insurance premium, as well as the Principal’s profit (blue curves on Figure 1). In this case, a risky Agent will pay a higher insurance premium for a smaller quantity insured than a less risky one, as we can see combining the two upper graphs of Figure 1. More precisely, the insured quantity varies from approximately \( e_{\min}^{FB} = 14,403 \) to 12,653 kWh, while the price, \( T^{FB} \) increases from 0 to 4,252 €. Moreover, the middle graph of Figure 1 shows that the riskier the agent, the greater the difference between the insurance premium and the actuarial price. Since the actuarial price also corresponds to the cost of

\(^6\)Note that this consumption is approximately the average electricity consumption of a french household, which is equal to 14,527 kWh according to Belaïd [2016].
Figure 1: Optimal insurance in the First–Best case. The blue curves represent, from top to bottom, the quantity insured, the insurance premium and the Principal’s profit, with respect to the type $\varepsilon$ of the Agent. On the middle graph, the insurance price is compared to the future price of the quantity (orange curve) and to its actuarial price (green curve), which also correspond to the Principal’s cost. The red dotted line on the bottom graph is her average profit.

The Principal, the greater the difference is, the greater her profit is. Therefore, from the Principal’s point of view, the more efficient Agents are those who are ready to pay more than the actuarial price (bottom graph of Figure 1). Her average profit $\Pi^F B = 1,035 \ €$ is given by the integral of her profit per Agent, assuming that the distribution of type is uniform.
Therefore, in our framework, the efficient Agents are those who are at risk of losing their income, which may seem counter–intuitive. Nevertheless, this result can be explained by the reservation utility we have chosen. More precisely, one can compute the information rent, which is the difference between \( \text{EU}(\varepsilon, q, t_0) \) and \( \text{EU}^0(\varepsilon) \) for an Agent of type \( \varepsilon \). This information rent increases with \( \varepsilon \) for every \( q \) and \( t_0 \), which means exactly that the riskier the agent, the more interesting it is for him to buy the insurance. In the First–Best case, the Principal knows the Agent’s type and can thus reduce to zero his information rent. This explain why a risky Agent is ready to pay more for less quantity.

The intuition for the Third–Best case, i.e. when the Principal cannot differentiate the Agents by their type since they are unknown to her, is the following: if she offers the optimal First–Best contract, an Agent with a positive probability of losing income should lie and pretend to be less risky, in order to pay less for a higher quantity insured. Only non–risky Agents will have no interest in pretending they are more risky, since they will pay more for less quantity. As classical in adverse selection problems, the efficient Agents, which are those who have interest in lying, will receive an information rent, generated by the informational advantage they have over the Principal.

However, one can already notice one limit of our model: Agents with type \( \varepsilon \geq 0.53 \) (red dotted curve, middle graph) are even ready to pay more than the future price of the quantity, i.e. \( p\varepsilon e_{\text{min}}(\varepsilon) \). This fact highlights a significant inconsistency of these Agents, who are willing to pay a high price for an insurance when it would be more efficient for them to save money instead. One way to address this inconsistency would be to offer another option in parallel with the insurance, such as a prepayment option (see Appendix C). Indeed, if insurance represents their only option, the Principal abuses from her monopoly position. In our opinion, this result already highlights the importance of regulating this type of market. If the State’s interest is to fight against fuel poverty, insurance seems to be a good option, but at the same time alternative solutions must also be developed.
3 Third–Best case: under adverse selection

In this section, we focus on finding the optimal menu of contracts for the insurance in the presence of adverse selection. As explained in Subsection 2.3, the intuition is that the First–Best contract given by Proposition 2.6 is no longer optimal if the Principal cannot observe the Agents’ type. Indeed, with this contract, an Agent with a positive probability of losing income should lie and pretend to be a less risky agent in order to pay less for a higher quantity insured. Therefore, the risky Agents should receive an information rent, generated by the informational advantage they have over the Principal. Conversely, the Agent with the smaller type considered should have no information rent since he has no interest in lying: if he pretends he is more risky, he will pay a larger premium for less quantity insured.

Following this reasoning, the Principal has to find a new optimal menu of contracts. The classical scheme to find it in this case is to use the Revelation Principle: the Principal has to design a menu of contracts indexed by ε, such that the Agent of type ε will choose the contract designed for him. As classical in adverse selection problems, the more efficient Agents, i.e. Agents with high type in our framework, will receive an information rent, generated by the informational advantage they have over the Principal. Moreover, the Agent with the higher risk will be insured for the optimal quantity computed in the First–Best case. On the contrary, there will be a distortion on the optimal quantity for other types (ε < 1): they will be insured for a smaller quantity than in the First–Best case.

The Revelation Principle detailed in Subsection 3.1 allows us to write the premium of the insurance as a function of the quantity insured and the type, to within a constant c_q. In fact, the value of this constant will be related in Subsection 3.2 to the participation constraint of the Agents: the Principal can choose the constant depending on the Agents’ type she wants to select. Subsection 3.3 will be dedicated to solve the Principal’s problem, and Subsection 3.4 deals with the application to fuel poverty.
3.1 Revelation Principle

Traditionally in adverse selection models (see Salanié [2005] for the general theory on adverse selection), the contract offered by the Principal has to satisfy the incentive compatible (IC) constraint: the contract has to be such that an Agent of type $\varepsilon$ should subscribe the contract corresponding to him, and thus reveal his type $\varepsilon$, previously unknown to the Principal. Indeed, the well–known revelation principle implies that we can restrict the study to incentive compatible mechanisms. More precisely, the revelation principle stated in Salanié [2005] can be adapted to our framework as follows: If the optimal quantity $e_{\min}$ chosen by an Agent of type $\varepsilon$ can be implemented through some mechanism, then it can also be implemented through a direct and truthful mechanism where the Agent reveals his information $\varepsilon$.

First, we can show that the Spence–Mirrlees condition, also called the constant sign assumption in Guesnerie and Laffont [1984], as defined in Laffont and Martimort [2009] is automatically satisfied in our framework (see Lemma B.3). This property makes the incentive problem well behaved in the sense that only local incentive constraints need to be considered. This condition was introduced by Spence [1973] in his theory of signaling on the labour market, and similarly by Mirrlees [1971] in his theory of optimal income taxation, as the single–crossing assumption: it indeed implies that the indifference curves of two different types of Agents can only cross once. This condition also has an economic content, it implies in our framework that Agents with a higher probability of income loss are willing to pay more for a given increase in $e_{\min}$ than the less risky Agents. This condition ensures that it should be possible to separate the high risks from the low risks by offering them a better coverage in return for a higher premium.

In order to find revealing contracts, we define, for an admissible menu of contracts $(e_{\min}, T)$, an associated pair $(q, t_0)$ of functions of $\varepsilon \in [0, 1]$.

**Definition 3.1.** A mechanism $(q, t_0)$ is said to be admissible if

(i) $q$ and $t_0$ are continuous functions on $[0, 1]$ taking values in an interval respectively contained in $\mathbb{R}_+$ and $[0, 1)$;
(ii) \( q \) and \( t_0 \) have continuous first and second derivatives on \( (0,1) \) except at \( \varepsilon_1 := \min\{\varepsilon \in [0,1], \ s.t. \ q(\varepsilon) = 1\} \).  

More precisely, an admissible menu of contracts \( (e_{\min}, T) \) is associated to an admissible mechanism \( (q, t_0) \) if for all quantities \( e_{\min} \) available with price \( T \), there is an \( \varepsilon \in [0,1] \) such that \( e_{\min} = \alpha q(\varepsilon) w_0 / p_e \) and \( T = w_0 t_0(\varepsilon) \).

The first point of the previous definition is entirely based on assumptions made about an admissible menu of contracts in Definition 1.3. The second point on the regularity of \( q \) and \( t_0 \) is more a technical assumption made to simplify the reasoning: it allows us to use the First and Second Order Conditions to define an incentive compatible contract. Unfortunately, the cases separation between \( q < 1 \) and \( q \geq 1 \) will subsequently imply a loss of \( C^1 \) continuity of the quantity and price at this point.

We thus limit our study to mechanisms smooth enough in the sense of Definition 3.1 (ii). According to the reasoning of Guesnerie and Laffont [1984], our results could be easily extend to piecewise continuously differentiable mechanisms of class \( C^1 \), and even some could be generalised to all mechanisms. Nevertheless, significant additional difficulties can be avoided with this smoothness assumption. Moreover, one can notice that the optimal mechanism in the First–Best case is smooth in the sense of Definition 3.1 (ii), and it therefore makes sense to restrict our study in this way.

The IC constraint says that the utility of an Agent of type \( \varepsilon \in [0,1] \) has to be maximal for the choice of the contract \( (q(\varepsilon), t_0(\varepsilon)) \), i.e.

\[
\text{EU}^Q(\varepsilon, q(\varepsilon), t_0(\varepsilon)) = \max_{\varepsilon' \in [0,1]} \text{EU}^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')).
\]

In other words, if a menu of contracts satisfies the IC constraint, then the Agent has an interest in revealing his type by choosing the contract made for him. We denote by \( C^Q \) the set of admissible mechanism satisfying this constraint. With the aim of lightening the equations, we denote throughout

\footnote{With the convention that \( \varepsilon_1 = 0 \) if \( q(\varepsilon) \geq 1 \) for all \( \epsilon \in [0,1] \).}
this section:

\[
Q_0(\varepsilon) := \begin{cases} 
\int_0^\varepsilon \ln (1 + \alpha q(\varepsilon)) \, d\varepsilon & \text{if } \varepsilon \in [0, \varepsilon_1) \\
\int_0^{\varepsilon_1} \ln (1 + \alpha q(\varepsilon)) \, d\varepsilon + \frac{\alpha}{1 + \alpha} \int_{\varepsilon_1}^\varepsilon \ln (q(\varepsilon)) \, d\varepsilon & \text{if } \varepsilon \in [\varepsilon_1, 1].
\end{cases}
\] (3.1)

**Theorem 3.2.** An admissible mechanism \((q, t_0)\) satisfies the IC constraint for all \(\varepsilon \in [0, 1]\) if and only if the function \(q\) is non-decreasing on \([0, 1]\) and there exists \(c_q \geq 0\) such that the price \(t_0\) satisfies for all \(\varepsilon \in [0, 1],\)

\[
t_0(\varepsilon) = 1 - c_q e^{\beta Q_0(\varepsilon)} \times \begin{cases} 
(1 + \alpha q(\varepsilon))^{-\beta \varepsilon} & \text{if } \varepsilon \in [0, \varepsilon_1), \\
(1 + \alpha)^{-\beta \varepsilon_1} (q(\varepsilon))^{-\beta \varepsilon_1/(1 + \alpha)} & \text{if } \varepsilon \in [\varepsilon_1, 1].
\end{cases}
\] (3.2)

The previous proposition provides a characterisation of an admissible mechanism \((q, t_0)\) satisfying the IC constraint for all type of Agents, its proof is postponed to Appendix B.2. Nevertheless, the concrete menu of contracts proposed by the Principal must be composed of quantities \(e_{\text{min}}\) and a price \(T\) associated with each quantity, regardless of the type of Agent, as specified in Definition 1.3. The form of the practical menu of contracts associated to an admissible mechanism is a consequence of the previous theorem and is given by Corollary B.5. We can summarise this result by saying that considering a sufficiently smooth admissible menu of revealing contracts \((e_{\text{min}}, T)\) is equivalent to considering an admissible mechanism \((q, t_0)\), where \(q\) is non-decreasing and the price \(t_0\) is given by (3.2). It is now necessary to establish conditions implying that such a mechanism satisfies the Agent’s participation constraint.

### 3.2 Adding the participation constraint

Recall that an Agent of type \(\varepsilon \in [0, 1]\) will accept the contract if his utility with it is bigger than his reservation utility, defined in our framework as his utility without insurance. To establish a precise result, we define the function
c for all $\varepsilon \in [0, 1]$ by:

$$
\xi(\varepsilon) := \begin{cases} 
  e^{-\beta Q_0(\varepsilon)} & \text{if } \varepsilon < \varepsilon_1, \\
  (1 + \alpha)^{-\beta(\varepsilon_1-\varepsilon_1)} e^{-\beta Q_0(\varepsilon)} & \text{if } \varepsilon \geq \varepsilon_1. 
\end{cases}
$$

(3.3)

The following proposition states that by controlling the constant $c_q$ in the insurance premium $t_0$ given by (3.2), the Principal can choose to select or not Agents with smaller types. As a result, only the most risky Agents will be selected by the Principal. Indeed, the Agents with a high probability of losing their income are easily satisfied and willing to pay much more than the less risky ones. This result is entirely implied by the fact that the reservation utility of an Agent depends on his type, and only happens in Principal–Agent problems with countervailing incentives. Additional information including the proof of the proposition can be found in Appendix B.3. In particular, Remark B.6 shows that if a constant reservation utility had been chosen, the selected Agents would have been the less risky.

**Proposition 3.3.** If the mechanism $(q,t_0)$ is admissible and incentive compatible, an Agent of type $\varepsilon \in [0, 1]$ subscribes to the insurance if and only if $c_q \geq \xi(\varepsilon)$. Moreover, by defining $\underline{c} := \min\{\varepsilon \in [0, 1], \text{ s.t. } c_q = \xi(\varepsilon)\}$, the participation constraint is satisfied only for Agents of type $\varepsilon \in [\underline{c}, 1]$.

Now that the set of the menu of revealing contracts satisfying the Agents’ participation constraint is well–defined, we can study the Principal’s problem.

### 3.3 The optimal menu of contracts

In the Third–Best case, the Principal’s goal is to find an optimal admissible menu of contracts $(\epsilon_{\min}, T)$, in order to maximise her profit, as defined by (1.8), without knowing the Agent’s type. In fact, instead of maximising the utility of the Principal’s over all possible contracts, we restrict the study to menu of contracts associated to an admissible mechanism $(q,t_0)$ in the sense of Definition 3.1. Then, by the revelation principle, it is sufficient to only consider admissible mechanisms that are revealing. Recalling that an Agent
will subscribe a contract if and only if it satisfies his participation constraint, the Principal’s problem becomes:

\[
\sup_{(q,t_0)\in\mathcal{C}Q} \int_{\varepsilon\in\Xi(q,t_0)} \pi(\varepsilon)d\varepsilon, \quad \text{for} \quad \pi(\varepsilon) := w_0t_0(\varepsilon) - \varepsilon\alpha_q(\varepsilon)\omega w_0, \quad (3.4)
\]

where \(\Xi(q,t_0)\) denotes the set of \(\varepsilon \in [0,1]\) such that the participation constraint \(t_0(\varepsilon) \leq t_{\text{max}}(\varepsilon,q(\varepsilon))\) is satisfied, where \(t_{\text{max}}\) is defined by (2.6).

By Theorem 3.2, we know that an admissible mechanism \((q,t_0)\) satisfies the IC constraint if and only if \(q\) is increasing and the price \(t_0\) is given by (3.2). Moreover, we know by Proposition 3.3 that the participation constraint is satisfied only for Agents of type \(\varepsilon \in [\varepsilon,1]\) if and only if \(c_q = c(\varepsilon)\), where \(c\) is defined by (3.3). For \(\varepsilon \in [0,1]\), we thus denote by \(\mathcal{C}Q(\varepsilon)\) the set of admissible and revealing mechanisms such that the participation constraint is satisfied for all \(\varepsilon \in [\varepsilon,1]\) only. Following the previous reasoning, the Principal’s problem is equivalent to:

\[
\sup_{\varepsilon\in[0,1]} \Pi(\varepsilon), \quad \text{where} \quad \Pi(\varepsilon) := \sup_{(q,t_0)\in\mathcal{C}Q(\varepsilon)} \int_{\varepsilon}^{1} \pi(\varepsilon)d\varepsilon.
\]

To solve the Principal’s problem, we first fix \(\varepsilon \in [0,1]\). We denote by \(\mathcal{Q}(\varepsilon)\) the space of functions \(\mathcal{Q}\) defined on \([\varepsilon,1]\), continuous and piecewise continuously differentiable of class \(\mathcal{C}^3\), satisfying:

(i) \(\mathcal{Q}\) is continuous on \([\varepsilon,1]\) and such that \(\mathcal{Q}(\varepsilon) = 0\);

(ii) \(\mathcal{Q}'\) is positive and continuous except at \(\varepsilon_1\), where \(\mathcal{Q}'(\varepsilon_1^-) = \ln(1+\alpha)\) and \(\mathcal{Q}'(\varepsilon_1^+) = 0\);

(iii) \(\mathcal{Q}''\) is positive and continuous except at \(\varepsilon_1\).

We consider the following second–order non–linear ordinary differential equation (ODE):

\[
\frac{\beta}{\omega}(\beta\varepsilon^2\mathcal{Q}''(\varepsilon) - 2)e^{\beta(\mathcal{Q}(\varepsilon) - \varepsilon\mathcal{Q}'(\varepsilon))} + G(\varepsilon, \mathcal{Q}) = 0, \quad (3.5)
\]
with initial conditions $Q(\varepsilon) = 0$ and $Q'(\varepsilon) = \eta$, for $\eta \in \mathbb{R}_+$, and where the function $G$ is defined for any $(\varepsilon, Q) \in [\bar{\varepsilon}, 1] \times Q(\varepsilon)$ by:

$$G(\varepsilon, Q) := \begin{cases} (1 + \varepsilon Q''(\varepsilon)) e^{Q'(\varepsilon)}, & \text{for } \varepsilon \in [\bar{\varepsilon}, \varepsilon_1 \vee \bar{\varepsilon}), \\ (1 + \alpha)^{\beta(\varepsilon_1 \vee \bar{\varepsilon})+1} \left( 1 + \frac{1 + \alpha}{\alpha} Q''(\varepsilon) \right) e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)}, & \text{for } \varepsilon \in [\varepsilon_1 \vee \bar{\varepsilon}, 1]. \end{cases}$$

This ODE is at the heart of the resolution of the Principal’s problem, since it characterises the optimal admissible mechanism, for $\varepsilon$ and $\eta$ fixed.

**Theorem 3.4.** Given $\varepsilon \in [0, 1]$ and $\eta \in \mathbb{R}_+$, if there exists $Q \in Q(\varepsilon)$ solution to the ODE (3.5), then the optimal admissible mechanism $(q, t_0)$ for the Principal is given by:

$$\begin{align*}
&\left(\frac{1}{\alpha} (e^{Q'(\varepsilon)} - 1) , 1 - e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))}\right) \quad \text{for } \varepsilon \in [\bar{\varepsilon}, \varepsilon_1 \vee \bar{\varepsilon}) \\
&\left( e^{\frac{1+\alpha}{\alpha} Q'(\varepsilon)} , 1 - (1 + \alpha)^{-\beta(\varepsilon_1 \vee \bar{\varepsilon})} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} \right) \quad \text{for } \varepsilon \in [\varepsilon_1 \vee \bar{\varepsilon}, 1].
\end{align*}$$

**Remark 3.5.** Theorem 3.4 only gives a sufficient condition for the Principal’s optimisation problem. In fact, it would be possible to obtain a necessary condition. Nevertheless, in the numerical example we are interested in (detailed in the following subsection), as the solution of the ODE (3.5) naturally satisfies the constraint of being in $Q(\varepsilon)$, we decide to simplify the result by presenting it in this way. For more details, the reader is referred to Remark B.8.

For the sake of clarity, the proof of the theorem is reported in Appendix B.4. However, the ODE (3.5) cannot be solved other than numerically. Therefore, to solve the Principal’s problem, one have to first fix $\varepsilon \in [0, 1]$ and an arbitrary initial value $\eta \in \mathbb{R}_+$ for $Q'(\varepsilon)$. Then, the solution of the previous ODE can be computed. With this solution, one can compute the Principal’s profit in this case, using Corollary B.7. This profit can then be maximised by choosing an optimal initial condition $\eta$ and an optimal $\varepsilon \in [0, 1]$. For the numerical results, readers are referred to the next subsection, which discusses the application of this model to a particular framework: the fuel poverty.
3.4 Third–Best insurance against fuel poverty

We consider the same household as the one studied in Subsection 2.3. By the recursive scheme explained in the previous subsection, we obtain the optimal $\varepsilon^* \approx 0.63$ with $Q' (\varepsilon^*) \approx 0$, and thus, Agents of type $\varepsilon < 0.63$ are not insured. Solving the ODE (3.5) for these parameters, we obtain the optimal function $Q \in Q (\varepsilon^*, 0)$, and in particular $\varepsilon^*_1 \approx 0.66$. Thanks to Theorem 3.4, we can compute the optimal admissible revealing mechanism $(q^*, t^*_0)$, and thus the optimal quantity insured for an Agent of type $\varepsilon \in [\varepsilon^*, 1]$ which is $e_{min}^*(\varepsilon) = a q^*(\varepsilon) \omega w_0 / p_e$, and its corresponding price $T^*(\varepsilon) = w_0 t^*_0 (\varepsilon)$. The optimal quantities and prices are represented by blue curves in Figure 2.

![Figure 2: Optimal insurance contract in the Third–Best case.](image)

On the left graph: the optimal quantity insured (blue) is compared to the quantity of the First–Best case (green). On the right graph: the optimal premium (blue) is compared to the maximum price (green) and to the future price (orange). Dotted black axes: $\varepsilon = \varepsilon_1$.

The left graph shows that the quantity insured in the Third–Best case (blue curve) is smaller than the one in the First–Best case (green curve). Only Agents of type $\varepsilon = 1$ will be insured for the same quantity of the First–Best case, modulo numerical errors, i.e. $e_{min}^*(1) \approx 12,665$ kWh/year. On the right graph, the green curve represents the maximum price an Agent of type $\varepsilon$ will be ready to pay for the quantity $e_{min}^*(\varepsilon)$, given by $T_{max} (\varepsilon) := w_0 t_{max} (\varepsilon; q^*(\varepsilon))$. 

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We thus observe that, as stated before, an Agent of type $\varepsilon^*$ pay his maximum price, and his information rent is thus reduced to zero. On the contrary, an Agent of type $\varepsilon > \varepsilon^*$ obtain an information rent. In particular, the information rent is increasing with $\varepsilon$. Finally, as already noticed in the First–Best case, the insurance premium is higher than the future price of the quantity, $p_\varepsilon e^*_{\min}$ (orange curve). This result highlights a form of inconsistency of the Agents and reflects the need to put in place, in addition to the insurance, other options to encourage the risky Agents to save.

![Figure 3: Principal’s profit in the Third–Best case.](image)

Blue curve: profit per type. Dotted red curve: expected profit. Dotted black axis: $\varepsilon = \varepsilon_1$.

Given the optimal admissible revealing mechanism $(q^*, t^*_0)$, we can compute the profit $\pi^*$ of the Principal for every type of Agents, represented in Figure 3. In particular, her total profit $\Pi^*$ on the considered population, given by the integral of $\pi^*(\varepsilon)$ between $\varepsilon^*$ and 1, is equal to 351 €. To show the benefit of considering a menu of contracts, we can compare our results to the profit induced by a unique contract. In this case, the optimal quantity and price can easily be computed, theoretically and numerically: $e_{min} \approx 10,538$ kWh/year and $T \approx 2,549$ €. This contract induces an average profit of 342 € for the Principal, and only Agents with type $\varepsilon \geq 0.65$ subscribe to the insurance. Therefore, there is a positive gain for the Principal in implementing a menu of contracts instead of a unique contract. More precisely, it represents an average
gain of 2.7% on each contract offered. Moreover, with a menu of contracts, more Agents are insured, since $\varepsilon^* < 0.65$.

Given the menu of contracts $(e_{\text{min}}^*, T^*)$ associated to the optimal admissible revealing mechanism $(q^*, t^*_*)$, we can compute the optimal consumption of the Agents who subscribe to the insurance (see Lemmas 2.1, A.1 and A.2 for the formulas). The left graph of Figure 4 represents the optimal consumption of energy at time $t = 0$, $e_0^Q$ with insurance (blue curve) and $e_0^\emptyset$ without (orange curve). Obviously, with insurance, an Agent will consume less than without insurance, since paying the insurance decreases his effective income to be split between the two goods. At time $t = 1$, if the Agent does not suffer from a loss of income, the insurance is not activated and he will thus consume the same quantity of energy in both cases, with and without insurance. On the contrary, as we can see in Figure 4 (right graph), if he suffers from a loss of income, he will consume more energy with insurance (blue curve) than without (orange curve). More precisely, without insurance, his optimal consumption is $e_1^\emptyset \approx 5,800$ KWh, which is around the level of consumption of an household of four people without electric heating. Therefore, one can consider that the household renounces heating its house, for example, because of its loss of income. Otherwise, if the Agent of type $\varepsilon$ is insured, he will receive the quantity $e_{\text{min}}^* (\varepsilon)$. By Lemma A.2, we can then compute his optimal consumption $e_1^Q (\varepsilon)$ of energy. His effective consumption $e_1^{\text{eff}} (\varepsilon)$ (blue curve) is given by the sum of $e_1^Q (\varepsilon)$ and $e_{\text{min}}^* (\varepsilon)$. It is interesting to note that the more risky the Agent is, the higher the insured quantity is, and it tends towards the quantity $e_0^\emptyset$ consumed with the initial income $w_0$.

By definition of $\varepsilon_1$, an Agent of type $\varepsilon \in [\varepsilon^*, \varepsilon_1)$ receives a quantity $e_{\text{min}}^{*} (\varepsilon) < \bar{e}_{\text{min}}$. In this case, the insurance acts as an earmarked fund, or a liquid asset: the Agent behaves exactly as if his income had been increased by $p_e e_{\text{min}}^{*} (\varepsilon)$. More precisely, a part $\alpha/(1 + \alpha)$ of this supplementary income is dedicated to electricity consumption, and the other part, $1/(1 + \alpha)$, is dedicated to the other good, in the same way that his income $w_t$ at time $t$ is distributed between the two goods. Actually, since the insurance is an in–kind support, the Agent has to decrease his consumption $e_1^Q$ and increases his consumption...
\( y_1^Q \), in order to perfectly split this fictive supplementary income between the two goods. On the contrary, when the insured quantity is higher, precisely if 
\[
\varepsilon^*_{\min}(\varepsilon) > \bar{v}_{\min},
\]
the Agent’s consumption \( e_1^Q \) is reduced to 0, and he cannot decrease it anymore. In this case, his effective consumption is given by \( e_{\min}^* \), and the Agent cannot properly split this fictive supplementary income between the two goods. For these types of Agents, \( i.e. \varepsilon \geq \varepsilon_1 \), the insurance no longer acts as a liquid asset, thus ensuring a higher electricity consumption.

Figure 4: Energy consumption in the Third–Best case. Consumption with the insurance (blue) and without (orange), at time \( t = 0 \) on the left, and at time \( t = 1 \) in the case of an income loss on the right. Dotted black axes: \( \varepsilon = \varepsilon_1 \).

The previous result is very interesting in the situation we considered. Indeed, a household that falls into fuel poverty due to a loss of income will tend to consume less electricity, which can lead to health problems and housing damage. This is exactly the kind of problems we want to avoid by proposing an insurance. Nevertheless, a traditional income insurance will allow the agent to receive money in the event of loss of income. However, an Agent in fuel poverty has other needs to satisfy that he considers more important. He would therefore use the insurance money largely for these expenses, ignoring the significance for his health of heating his home sufficiently, for example\(^8\).

\(^8\)Considering our model for the Agent’s consumption, it can be shown that an income insurance will be less efficient than the insurance with benefits in kind we developed.
The insurance we model prevents this bias, it somehow constrains the Agent to consume enough electricity to live decently in his home.

4 Conclusion

We develop a two-period Principal-Agent problem with adverse selection and endogenous reservation utility to model an insurance with benefits in kind. This model allows us to obtain semi-explicit solutions. Applied to the energy sector, this in-kind support helps to prevent fuel poverty among households. Indeed, when a household suffers from a loss of income, if it has subscribed to the insurance we propose, it will consume more energy than without insurance. In this application, providing support in kind therefore forces the household to consume more energy, and thus avoids risky behaviour that can lead to serious health problems. The insurance thus makes it possible to cover Agents’ risks of fuel poverty, but also to pool costs between the risky Agents.

Following the same approach as developed in this paper, it can be shown that the conclusions on consumption would not be the same in the case of an income insurance, the household would not increase its energy consumption sufficiently. The insurance we propose is also different from those provided by the two French energy suppliers EDF and ENGIE: these insurances offer a reimbursement of part of electricity consumption, which means the household has to pay its bill first. However, if it suffers a loss of income, the household will tend to reduce its consumption for fear of not being able to pay its bill, even if it is reimbursed afterwards. An in-kind support helps to avoid this bias. Moreover, our model can be extended to random prices, which would allow the insured to have a guaranteed quantity even in case of a price increase.

The simplicity of our model makes it easy to extend it to a multi-period model, keeping in mind that only Agents who are not in precarious situations are entitled to subscribe to the insurance. Thus, an Agent who has not suffered from a loss of income can pay again the premium to reinsure himself for the next period. A simple repetition of the model is sufficient to deal with this
case. Moreover, we have chosen to apply it for an insurance against fuel poverty. Nevertheless, this type of model could be used for another staple, whose consumption is affected by the loss of income. More generally, this type of insurance could also be purchased by production firms to ensure that they have sufficient input in the event of a temporary downturn in revenues.

Developing such insurance could therefore make it possible to prevent households from fuel poverty for example, and thus avoid significant societal costs. The effects of fuel poverty on the physical and mental health of individuals are not questionable (see Lacroix and Chaton [2015]): to keep heat inside their homes, some households obstruct vents, thus generating moisture and mould, that can cause respiratory problems such as chronic asthma or rhinitis. Moreover, households in fuel poverty are often forced to make choices with harmful consequences for their health: eating or heating, giving up care or giving up going out. However, the societal cost of fuel poverty is difficult to quantify, and this is why it is not taken into account in our model. A possible extension could therefore be to consider the State’s problem, who is faced with the costs of fuel poverty, and try to encourage insurers to offer this type of insurance, or to persuade households to subscribe to it. Consideration could also be given to making this type of insurance mandatory, either by law or by a contract between a landlord and a tenant for example. Indeed, on the one hand, the State could have an interest in ensuring that the Agent is not in fuel poverty in order to reduce health expenses. On the other hand, a landlord, or even a social landlord, could have the same interest in order to avoid deterioration by the tenant of the housing he owns.

However, before developing extensions to this model, it would be necessary to address the issue of high price. Indeed, since the households considered do not think about saving to have a sufficient quantity of essential goods regardless of their future income, their choice are limited to subscribing to the insurance or doing nothing. Due to the form of the utility chosen, in particular its concavity, Agents who anticipate a loss of income with a high probability are willing to pay a very high price for the insurance we offer. Indeed, they prefer to significantly reduce their disposable income by subscribing to the insurance,
which only slightly reduces their utility, to ensure sufficient consumption the following year, which significantly increase their utility. This results in a very high insurance price, higher than the future price of the quantity subscribed. Therefore, insurance can be a tool to protect at-risk households, but it cannot be set up alone if the welfare of the Agents is desired. Indeed, even if our model shows a gain in utility for the Agents subscribing to the insurance, this can be explained by the lack of option for the Agent.

Some suggestions could be considered to make the insurance premium more realistic. First of all, if a regulator required the insurance to be offered to all types of Agents, i.e. $\epsilon = 0$, the insurance price would be lower. In addition, the regulator could also better control the monopoly position of electricity suppliers in the case of energy insurance. By introducing competition in the market for this type of insurance, the price should fall towards the marginal cost of insurance. Another solution that seems, in our opinion, easy to implement, would be to increase the possibilities of the Agents by offering them an additional option: the prepayment. This option can be a way to encourage high-risk Agents to save money, because he does not voluntarily.

The model with prepayment is described in Appendix C.1. In fact, adding this option only changes the participation constraint of the Agent: if the Agent’s utility with prepayment is higher than with insurance, he will not subscribe to the insurance. In this situation, we can see in Appendix C.2 that, even in the case of First-Best, this option allows a large decrease in the price of the insurance. Unfortunately, this addition implies that the Third-Best case detailed in Appendix C.3 is more complicated to solve, although the techniques developed throughout this paper are a step towards resolution. In our opinion, this case would require further study, since it appears to be a very good way to lower the insurance premium and to insure medium-risk Agents.

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A Optimal consumption with insurance

For a better readability, this appendix regroups the results obtained on optimal consumptions in each good and at each period of time, with insurance. These results are not at the heart of our study, but they represent necessary steps to establish Proposition 2.3 that define the expected utility of an Agent with insurance. Moreover, results on optimal consumption in each case is used in numerical simulation to compare the effect of the insurance on consumption.

Lemma A.1 (With insurance, at time 0). Given an insurance premium \( T \), the optimal Agent’s consumptions in each good at time \( t = 0 \) are given by

\[
y_0^Q := \frac{1}{1 + \alpha} \frac{w_0 - T}{p_y} \quad \text{and} \quad e_0^Q := \frac{\alpha}{1 + \alpha} \frac{w_0 - T}{p_e};
\]

and his corresponding maximum utility is \( V_0(T) = (1 + \alpha) \ln(w_0 - T) + C_{\alpha,p_e,p_y} \).

Without loss of generality, we can assume that \( T \) is of the form \( T := t_0 w_0 \) for \( t_0 \in [0, 1) \). Therefore, by slightly abusing the notations as in Lemma 2.1, we denote by \( V_0(t_0) \) the maximum utility the Agent can achieve by optimally choosing his consumption, which can be written as:

\[
V_0(t_0) = (1 + \alpha) \ln(w_0) + (1 + \alpha) \ln(1 - t_0) + C_{\alpha,p_e,p_y}.
\]

The Agent thus pays a price \( w_0 t_0 \) at time \( t = 0 \), depending on the amount of staple good \( e_{\text{min}} \) he wants to receive at time \( t = 1 \). By setting the quantity \( \bar{e}_{\text{min}} := \omega w_0 / p_e \), we obtain the following result on optimal consumption at time \( t = 1 \).

Lemma A.2 (With insurance, at time 1). Given an insurance contract \((e_{\text{min}}, T)\), the optimal Agent’s consumptions in each good at time \( t = 1 \) are given by

\[
e_1^Q := \left( \frac{\alpha}{1 + \alpha} \frac{\omega w_0}{p_e} - \frac{1}{1 + \alpha} e_{\text{min}} 1_{\omega = \omega} \right)^+ \quad \text{and} \quad y_1^Q := \frac{\omega w_0 - p_e e_1^Q}{p_y},
\]

where \( x^+ := \max\{x, 0\} \) for all \( x \in \mathbb{R} \), and provide the following maximum utility to the Agent:

\[
V_1(\omega w_0, e_{\text{min}}) = \begin{cases} 
(1 + \alpha) \ln(\omega w_0 + p_e e_{\text{min}} 1_{\omega = \omega}) + C_{\alpha,p_e,p_y} & \text{if } e_{\text{min}} 1_{\omega = \omega} < \bar{e}_{\text{min}}, \\
\ln(\omega w_0) + \alpha \ln(e_{\text{min}}) - \ln(p_y) & \text{if } e_{\text{min}} 1_{\omega = \omega} \geq \bar{e}_{\text{min}}.
\end{cases}
\]

The case separation in the previous proposition is needed to ensure that the consumption \( e_1^Q \) at time \( t = 1 \) is non-negative. Indeed, the consumer should not be allowed to sell back the staple good. In the first case, i.e. when \( e_{\text{min}} 1_{\omega = \omega} < \bar{e}_{\text{min}}, \) the Agent’s utility at time \( t = 1 \) depends only on his effective income \( \omega w_0 + p_e e_{\text{min}} 1_{\omega = \omega} \). Assuming that the choice of \( e_{\text{min}} \) is restricted to the interval \([0, \bar{e}_{\text{min}}]\) is equivalent to assuming that the quantity offered is smaller than the optimal quantity consumed in the event of an income loss. Therefore, in this case, the insurance acts as
an earmarked fund, or a liquid asset: the Agent would have spent at least the quantity $e_{\text{min}}$, so he reacts as if his income is increased by this value. On the contrary, in the second case, the Agent will consume only the amount $e_{\text{min}}$ of staple good, the optimal $e_1^Q$ becoming equal to zero. In this case, the insurance is not interpreted as a liquid asset, the Agent consumes all the quantity $e_{\text{min}}$ offered to him, leading to a utility $\alpha \ln(e_{\text{min}})$, and does not consider it as an increase in income. He then spend all his income $\omega w_0$ in the other good.

To simplify the notations, we can assume without loss of generality that $e_{\text{min}}$ is of the form $e_{\text{min}} = q \alpha \omega w_0 / p_e$ for some $q \in \mathbb{R}_+$. The maximum utility obtained by the Agent at time $t = 1$ can thus be written as a function of $\omega$ and $q$ as follows:

$$V_1(\omega, q) = (1 + \alpha) \ln(\omega w_0) + \bar{U}(q) I_{\omega = \omega} + C_{\alpha, p_e, p_e},$$

where $\bar{U}$ is defined by (2.4). Combining (A.1) and (A.2), we can compute explicitly the expected utility of an Agent subscribing to an insurance contract, which allows to state Proposition 2.3. Then, comparing the utility with and without insurance, we can determine when an Agent of type $\varepsilon$ will subscribe to the insurance (see Proposition 2.5). The Agent will thus subscribe the insurance as soon as the premium is below a specific level, given by (2.6). Given the form of the maximum price, it can already be noticed that some Agents show a certain form of irrationality, due to their unwillingness to save money from one period to the next.

**Remark A.3 (Maximum price without uncertainty).** One can notice that, in our framework, the maximal price the consumer is willing to pay in the case without uncertainty is not equal to the actuarial price $p_e e_{\text{min}}$. Indeed, assuming that $q < 1$ and setting $\beta = 1$ for simplicity, we obtain $w_{\text{max}}(1, q) > p_e e_{\text{min}}$ as soon as $w_0 > \omega w_0 + p_e e_{\text{min}}$. Therefore, if the income of the consumer at time $t = 0$ is larger than the effective money he will have at time $t = 1$ with the insurance, he is willing to pay a certain amount of money at time $t = 0$ to obtain less at time $t = 1$. Conversely, if his income $w_0$ is lower than the money he will have at time $t = 1$, he will not be prone to pay the real price of the energy he will get. This result is a little bit counter intuitive, but is totally explained by the choice of concave utilities in a two–period model and the absence of saving. This problem does not occur in single period models. However, a one–period model would not allow to model a household willing to insure against a possible loss of future income. One solution could be to offer the Agent the opportunity to have savings, but this is not consistent with the type of household being considered. Therefore, an alternative approach to address this issue is initiated in Appendix C.
B Technical results and proofs...

B.1 ... for the First–Best Case

The profit of the Principal in the First–Best case, induced by the optimal contract detailed in Proposition 2.6, is given by the following result.

Corollary B.1. Let us assume \( \beta > \omega \). The optimal contract \( (\epsilon_{min}^{FB}, T_{FB}) \) for an Agent of type \( \epsilon \in [0, 1] \) induced the following profit for the Principal:

\[
\pi_{\epsilon}^{FB} := \begin{cases} 
  w_0 \left(1 - (\beta(1 + \alpha)/\omega)^{-\beta\epsilon/(1+\alpha+\alpha\beta\epsilon)} - \epsilon\omega(\beta(1 + \alpha)^{-\beta\epsilon/\omega})^{(1+\alpha)/(1+\alpha+\alpha\beta\epsilon)}\right) & \text{if } \epsilon \leq \epsilon_{1}\,^{FB}, \\
  w_0 \left(1 + \epsilon\omega - (\beta/\omega)^{-\beta\epsilon/(1+\beta\epsilon)} - \epsilon\omega(\beta/\omega)^{1/(1+\beta\epsilon)}\right) & \text{if } \epsilon > \epsilon_{1}\,^{FB}. 
\end{cases}
\]

The proof of the previous corollary results from the proof of the associated proposition, detailed below.

Proof of Proposition 2.6. We fix the Agent’s type \( \epsilon \in [0, 1] \). Since the profit of the Principal is increasing in \( T \), she has interest in setting the price of the insurance equal to the maximum price the Agent is willing to pay, i.e. \( t_0 = t_{\text{max}}(\epsilon, q) \), for \( q \in \mathbb{R}_+ \). The participation constraint of the Agent is thus binding and the maximisation problem of the Principal (2.7) becomes:

\[
\pi_\epsilon = w_0 \max \left\{ \sup_{q \in [0,1]} \left\{ 1 - (1 + \alpha q)^{-\beta \epsilon} - \epsilon \alpha q \omega \right\}, \sup_{q \geq 1} \left\{ 1 - q^{-\beta \epsilon/(1+\beta \epsilon)} - \epsilon \alpha q \omega \right\} \right\}.
\]

Computing the First and Second Order Conditions (FOC and SOC) for each supremum, and since \( \beta > \omega \), we obtain that the two suprema are respectively attained for \( q_1 \) and \( q_2 \) where:

\[
q_1 = \min \left\{ \frac{1}{\alpha} \left( (\beta/\omega)^{1/(1+\beta\epsilon)} - 1 \right), 1 \right\}, \quad \text{and} \quad q_2 = \max \left\{ (\beta(1 + \alpha)^{-\beta\epsilon/\omega})^{(1+\alpha)/(1+\alpha+\alpha\beta\epsilon)}, 1 \right\}.
\]

If \( \epsilon > \epsilon_{1}\,^{FB} \), then \( q_1 < 1 \) and \( q_2 = 1 \), and conversely if \( \epsilon < \epsilon_{1}\,^{FB} \), then \( q_1 = 1 \) and \( q_2 > 1 \). Since the two suprema have the same value for \( q = 1 \), we conclude that \( q_{\epsilon} \) defined by (2.8) is optimal.

Remark B.2. We assume in Proposition 2.6 and Corollary B.1 that \( \beta > \omega \) because it is the most interesting case. Otherwise, we would have \( \epsilon_{1}\,^{FB} < 0 \) and the maximum would be reached for \( q_{1} \) defined in the previous proof. However, in this particular case, \( q_{1} \) is negative for all \( \epsilon \in [0, 1] \). Therefore, the optimal \( q_{\epsilon} \) is zero in this case, for all \( \epsilon \in [0, 1] \). This means that the Principal has no interest in offering the insurance. Indeed, when \( \beta \) is too small, the Agent has very little concern for his future, so he is not willing to pay for an insurance to protect him.
B.2 ...to find a revealing menu of contracts

Before seeking for a revealing menu of contracts, we first prove that the Spence–Mirrlees condition is satisfied in our framework (see Lemma B.3). This condition is important since it makes the incentive problem well behaved in the sense that only local incentive constraints need to be considered. Together with Lemma B.4, this allows us to establish Theorem 3.2, whose proof is reported below, after the two lemmas.

**Lemma B.3.** The marginal rates of substitution between the in–kind support and the insurance price can be ranked in a monotonic way. More precisely,

\[
\frac{\partial}{\partial \varepsilon} \left( \frac{\partial Q'(\varepsilon, q, t_0)}{\partial t_0} \right) \leq 0.
\]

**Proof.** Indeed, recalling that the expected utility of an Agent of type $\varepsilon$ is given by (2.5), we have:

\[
\hat{Q}'_{\varepsilon}(\varepsilon, q, t_0) = \begin{cases} 
\beta \varepsilon \alpha (1 + \alpha)/(1 + q \alpha) & \text{if } q < 1, \\
\beta \varepsilon \alpha/q & \text{if } q \geq 1,
\end{cases}
\]

and \( \hat{Q}'_{t_0}(\varepsilon, q, t_0) = -\frac{1 + \alpha}{1 - t_0} \),

which leads to

\[
\frac{\partial}{\partial \varepsilon} \left( \frac{\hat{Q}'_{\varepsilon}(\varepsilon, q, t_0)}{\hat{Q}'_{t_0}(\varepsilon, q, t_0)} \right) = \begin{cases} 
\frac{\beta \alpha (1 - t_0)}{1 + q \alpha} & \text{if } q < 1, \\
-\frac{\beta \alpha (1 - t_0)}{q(1 + \alpha)} & \text{if } q \geq 1.
\end{cases}
\]

Both quotients are indeed non–positive since the price of the insurance should be at least smaller than the Agents’ income which implies $t_0 < 1$, and all other quantities and prices are positive. \( \square \)

**Lemma B.4.** Let $(q, t_0)$ be an admissible mechanism such that $q$ is non–decreasing and $t_0$ is given by (3.2) for some $c_q \geq 0$. If the function $q$ is constant on some interval contained in $[0, 1]$, then the price $t_0$ is also constant on this interval.

**Proof.** Let us first assume that $q$ is constant on some interval $[x, y]$, where $y < \varepsilon_1$. For all $\varepsilon$ in this interval, we have in particular $q(\varepsilon) = q(x)$ and thus:

\[
t_0(\varepsilon) = 1 - c_q (1 + \alpha q(x))^{-\beta x - \beta (\varepsilon - x)} \exp \left( \beta \int_{0}^{x} \ln (1 + \alpha q(\varepsilon)) \, d\varepsilon + \beta \int_{x}^{\varepsilon} \ln (1 + \alpha q(x)) \, d\varepsilon \right)
\]

\[
= 1 - c_q (1 + \alpha q(x))^{-\beta x} \exp \left( \beta \int_{0}^{x} \ln (1 + \alpha q(\varepsilon)) \, d\varepsilon \right).
\]

Therefore, for all $\varepsilon \in [x, y]$, $t_0(\varepsilon) = t_0(x)$, i.e. $t_0$ is constant on this interval. The proof is highly similar for an interval $[x, y]$ such that $x \geq \varepsilon_1$. Finally, if the interval $[x, y]$ contains $\varepsilon_1$, we necessarily have $q(\varepsilon) = 1$ for all $\varepsilon$ in the interval. By definition of $\varepsilon_1$, we actually have $x \geq \varepsilon_1$, and the problem is reduced to the previous case. \( \square \)
**Proof of Theorem 3.2.** (i) We first prove that \( q \) non-decreasing with respect to \( \varepsilon \in [0, 1] \) and \( t_0 \) satisfying (3.2) are necessary conditions for the admissible mechanism \( (q, t_0) \) to satisfy the IC constraint on \([0, 1] \). To prove this, we first fix \( \varepsilon \in (0, 1) \), such that \( q(\varepsilon) \neq 1 \), and focus the study on an Agent of type \( \varepsilon \). Using (2.5), his expected utility if he choses a contract \( \langle q(\varepsilon'), t_0(\varepsilon') \rangle \), for some \( \varepsilon' \in [0, 1] \), is as follows:

\[
EU^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = EU^o(\varepsilon) + (1 + \alpha) \ln(1 - t_0(\varepsilon')) + \beta \varepsilon \tilde{U}(q(\varepsilon')).
\] (B.1)

The mechanism \( (q, t_0) \) is incentive compatible if the Agent choses the contract designed for him to maximise his utility. Therefore, the utility computed above must attain its maximum on \( \varepsilon' = \varepsilon \). Since the mechanism is assumed to be regular enough, we can compute the first and second derivatives of the previous utility, with respect to \( \varepsilon' \). The First Order Condition (FOC) says that the first derivative has to be equal to zero for \( \varepsilon' = \varepsilon \). Since the derivative with respect to \( \varepsilon' \in (0, 1) \) of \( EU^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) \) is given by:

\[
\partial_{\varepsilon'} EU^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = - (1 + \alpha) \frac{\partial_{\varepsilon'} t_0(\varepsilon')} {1 - t_0(\varepsilon')} + (1 + \alpha) \beta \varepsilon \times \begin{cases} \frac{\alpha \partial_{\varepsilon'} q(\varepsilon')} {1 + \alpha q(\varepsilon')} & \text{if } q(\varepsilon') < 1, \\ \frac{\alpha \partial_{\varepsilon'} q(\varepsilon')} {1 + \alpha q(\varepsilon')} & \text{if } q(\varepsilon') > 1, \end{cases}
\] (B.2)

the FOC for the Agent of type \( \varepsilon \) is as follows:

\[
\partial_{\varepsilon} t_0(\varepsilon) = \begin{cases} \beta \varepsilon \frac{\alpha \partial_{\varepsilon} q(\varepsilon)} {1 + \alpha q(\varepsilon)} (1 - t_0(\varepsilon)) & \text{if } q(\varepsilon) < 1, \\ \beta \varepsilon \frac{\alpha \partial_{\varepsilon} q(\varepsilon)} {1 + \alpha q(\varepsilon)} (1 - t_0(\varepsilon)) & \text{if } q(\varepsilon) > 1. \end{cases}
\] (B.3)

Moreover, to check that \( \varepsilon' = \varepsilon \) attains a local maximum, the second order derivative has to be negative for \( \varepsilon' = \varepsilon \) (Second Order Condition – SOC), which gives:

\[
0 \geq - \frac{\partial_{\varepsilon}^2 t_0(\varepsilon)} {1 - t_0(\varepsilon)} - \left( \frac{\partial_{\varepsilon} t_0(\varepsilon)} {1 - t_0(\varepsilon)} \right)^2 + \beta \varepsilon \times \begin{cases} \frac{\alpha \partial_{\varepsilon}^2 q(\varepsilon)} {1 + \alpha q(\varepsilon)} - \frac{\alpha^2 (\partial_{\varepsilon} q(\varepsilon))^2} {(1 + \alpha q(\varepsilon))^2} & \text{if } q(\varepsilon) < 1, \\ \frac{\alpha \partial_{\varepsilon}^2 q(\varepsilon)} {1 + \alpha q(\varepsilon)} - \frac{\alpha (\partial_{\varepsilon} q(\varepsilon))^2} {(1 + \alpha)(q(\varepsilon))^2} & \text{if } q(\varepsilon) > 1. \end{cases}
\] (B.4)

The mechanism \( (q, t_0) \) must be revealing for every types of Agents, which implies that the previous FOC and SOC has to be true at least for all \( \varepsilon \in (0, 1) \) such that \( q(\varepsilon) \neq 1 \).
On the one hand, by differentiating (B.3), we prove that \( t_0 \) should satisfy:

\[
c^2_\varepsilon t_0(\varepsilon) = \begin{cases} 
\frac{\beta \alpha (1 - t_0(\varepsilon))}{(1 + \alpha q(\varepsilon))^2} \left( (\partial q(\varepsilon) + \varepsilon c^2 q(\varepsilon)) (1 + \alpha q(\varepsilon)) - \alpha \varepsilon (\partial q(\varepsilon))^2 (1 + \beta \varepsilon) \right) & \text{if } q(\varepsilon) < 1, \\
\frac{\beta \alpha (1 - t_0(\varepsilon))}{(1 + \alpha)(q(\varepsilon))^2} \left( (\partial q(\varepsilon) + \varepsilon c^2 q(\varepsilon)) q(\varepsilon) - \varepsilon (\partial q(\varepsilon))^2 \left( 1 + \frac{\alpha}{1 + \alpha \beta \varepsilon} \right) \right) & \text{if } q(\varepsilon) > 1.
\end{cases}
\]

Replacing the first and second derivatives of \( t_0 \) by their values computed above, we obtain that the SOC (B.4) is equivalent in both cases to \( \partial q(\varepsilon) \geq 0 \). Therefore, \( q \) is non-decreasing before attaining 1, and also non-decreasing after. By continuity of the function \( q \), it can either cross the constant line equal to 1 only once, or be equal to 1 over an interval. In both cases, it implies that the function \( q \) is non-decreasing on (0, 1). We can thus denote by \([\varepsilon_1, \varepsilon_2]\) the interval on which \( q \) is constant equal to 1, with the convention that this interval is reduced to \( \{\varepsilon_1\} \) if there exists only one point where \( q \) is equal to 1, \( \varepsilon_1 = \varepsilon_2 = 1 \) if \( q \) is always strictly less than 1, and \( \varepsilon_1 = \varepsilon_2 = 0 \) if \( q \) is always strictly greater than 1.

On the other hand, by solving (B.3) when \( q(\varepsilon) < 1 \), i.e. \( \varepsilon \in (0, \varepsilon_1) \), we obtain

\[
t_0(\varepsilon) = 1 - c_q (1 + \alpha q(\varepsilon))^{-\beta \varepsilon} e^{\beta Q_0(\varepsilon)},
\]

for some constant \( c_q \in \mathbb{R} \), using the notation defined in (3.1). This proves the first form in (3.2). Moreover, solving the second part of (B.3), i.e. for \( \varepsilon \in (\varepsilon_2, 1) \), leads to:

\[
t_0(\varepsilon) = 1 - \tilde{c}_q (q(\varepsilon))^{-\beta \varepsilon \alpha / (1 + \alpha)} \exp \left( \frac{\beta \alpha}{1 + \alpha} \int_{\varepsilon_2}^{\varepsilon} \ln(q(\varepsilon)) d\varepsilon \right),
\]

for some \( \tilde{c}_q \in \mathbb{R} \). The two previous forms are respectively valid on \([0, \varepsilon_1]\) and \([\varepsilon_2, 1]\) by the assumed continuity of \( t_0 \). If \( \varepsilon_1 = \varepsilon_2 \in (0, 1) \), the price is continuous at this point if and only if

\[
\tilde{c}_q = c_q (1 + \alpha)^{-\beta \varepsilon_1} e^{\beta Q_0(\varepsilon_1)}, \tag{B.5}
\]

and we thus obtain the second form in Equation (3.2). With this setting, if \( \varepsilon_1 = \varepsilon_2 = 0 \), we obtain \( \tilde{c}_q = c_q \) and the price is given by (3.2) for all \( \varepsilon \in (0, 1) \), to within the constant \( c_q \). The similar reasoning applies if \( \varepsilon_1 = \varepsilon_2 = 1 \). It remains to deal with the case where \( q \) is constant on the interval \([\varepsilon_1, \varepsilon_2]\), not reduced to a singleton. To address this case, let us consider an Agent of type \( \varepsilon \in (\varepsilon_1, \varepsilon_2) \). For the mechanism to be revealing to him, his expected utility \( EU^Q(\varepsilon, q(\varepsilon), t_0(\varepsilon)) \) must at least reach a local maximum in \( \varepsilon' = \varepsilon \). His utility for any \( \varepsilon' \in (\varepsilon_1, \varepsilon_2) \) is as follows, since \( q \) is constant equal to 1 on this interval:

\[
EU^Q(\varepsilon, q(\varepsilon), t_0(\varepsilon)) = EU^Q(\varepsilon) + (1 + \alpha) \ln (1 - t_0(\varepsilon)) + \beta \varepsilon (1 + \alpha) \ln (1 + \alpha).
\]

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Therefore, $\varepsilon$ is a maximum point on $(\varepsilon_1, \varepsilon_2)$ if for any $\varepsilon'$ on this interval, $t_0(\varepsilon) \leq t_0(\varepsilon')$. Conversely, the mechanism is revealing for the Agent of type $\varepsilon'$ at least if $t_0(\varepsilon) \geq t_0(\varepsilon')$. This naturally implies that $t_0$ is also constant on $(\varepsilon_1, \varepsilon_2)$. In particular, by continuity of the price, we should have $t_0(\varepsilon_1) = t_0(\varepsilon_2)$, which also implies (B.5). Finally, we obtain for all $\varepsilon \in [\varepsilon_2, 1]$:}

$$t_0(\varepsilon) = 1 - c_q(1 + \alpha)^{-\beta\varepsilon_1}(q(\varepsilon))^{\beta\varepsilon_0/1+\alpha}e^{\beta Q_0(\varepsilon_1)} \exp \left( \frac{\beta\alpha}{1+\alpha} \int_{\varepsilon_2}^{\varepsilon} \ln(q(\varepsilon))d\varepsilon \right)$$

where the second equality is implied by $q$ constant equal to 1 on $[\varepsilon_1, \varepsilon_2]$. Therefore, the form (3.2) is proven to be true in any cases. Finally, since the mechanism has to be admissible in the sense of Definition 3.1, we should have $t_0(\varepsilon) < 1$ for all $\varepsilon \in [0, 1]$, which implies $c_q > 0$. We therefore have shown that $q$ non–decreasing with respect to $\varepsilon \in [0, 1]$ and $t_0$ satisfying (3.2) are necessary conditions for the menu of contracts to satisfy the IC constraint on $[0, 1]$.

(ii) It remains to prove that these conditions are sufficient. To this end, we recall that the expected utility of an Agent of type $\varepsilon$ who chooses a contract $(q(\varepsilon'), t_0(\varepsilon'))$ is given by (B.1). In particular, its derivative with respect to $\varepsilon'$ for $\varepsilon' \in (0, 1)$ such that $q(\varepsilon') \neq 1$, is given by (B.2). Since $t_0$ satisfies (B.3) in particular in $\varepsilon'$, we obtain

$$\partial_{\varepsilon'} EU^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = \begin{cases} 
(1 + \alpha)^{\beta\alpha\varepsilon q(\varepsilon')} (\varepsilon - \varepsilon') & \text{if } q(\varepsilon') < 1, \\
\beta\alpha \varepsilon q(\varepsilon') (\varepsilon - \varepsilon') & \text{if } q(\varepsilon') > 1.
\end{cases}$$

Moreover, if we consider without loss of generality that $q$ is constant equal to 1 on some interval $[\varepsilon_1, \varepsilon_2]$, and take $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, then, by Lemma B.4, for any neighbourhood of $\varepsilon$ contained in $[\varepsilon_1, \varepsilon_2]$, the price given by (3.2) is also constant. Therefore the expected utility $EU^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon'))$ is in fact also differentiable on this neighbourhood, and its derivative is equal to zero. In summary, the following values are obtained for the derivative of the expected utility:

$$\partial_{\varepsilon'} EU^Q(\varepsilon, q(\varepsilon'), t_0(\varepsilon')) = \begin{cases} 
\beta(1+\alpha)^{\alpha\varepsilon q(\varepsilon')} (\varepsilon - \varepsilon') & \text{if } 0 < \varepsilon' < \varepsilon_1, \\
0 & \text{if } \varepsilon' \in (\varepsilon_1, \varepsilon_2), \\
\beta\alpha \varepsilon q(\varepsilon') (\varepsilon - \varepsilon') & \text{if } \varepsilon_2 < \varepsilon' < 1.
\end{cases}$$

We first check that the contract is revealing for the interior types of Agents, i.e. where the previous derivative is defined. It suffices to remark that the expected utility of an Agent of type $\varepsilon$ is non–decreasing for $\varepsilon' \leq \varepsilon$ and non–increasing after, which proves, by continuity of the utility, that $\varepsilon' = \varepsilon$ is a maximiser. If the Agent’s type is $\varepsilon = 0$, his continuous utility is non–increasing with
to $ε' \in (0, 1)$, and a maximum is attained for $ε' = 0$. The similar reasoning can be applied if the Agent’s type is $ε = 1$, and therefore, the contract is revealing for the extreme types. For $ε = ε_1$ (resp. $ε = ε_2$), the utility is non-decreasing before $ε$, constant on $(ε_1, ε_2)$, and non-increasing after. Therefore, the (continuous) utility is constant and maximal on the interval $[ε_1, ε_2]$, in particular the maximum is also attained at $ε_1$ (resp. $ε_2$). Therefore, the conditions stated in the proposition are sufficient for the mechanism to satisfy the IC constraint for all $ε \in [0, 1]$.

Theorem 3.2 thus provides a characterisation of a mechanism $(q, t_0)$ satisfying the IC constraint for all type of Agents. However, the real menu of contracts offered by the Principal must be composed of quantities $e_{\text{min}}$ and a price $T$ associated with each quantity, independent of the type of Agent which is not observed by the Principal. So, in the end, we will have to get a price $T$ that is only a function of $e_{\text{min}}$, not also a function of $ε$. Nevertheless, Lemma B.4 states that when the function $q$ is constant, the associated price $t_0$ is necessarily constant too. Together with the fact that the function $q$ is non-decreasing, this naturally implies that if two different types of Agents chooses the same quantity, they will pay the same price. This result therefore prevents the contract resulting from a revealing mechanism from depending on the type of Agents.

To precisely define the menu of contract associated to an admissible revealing mechanism, let us fix an interval $I \subset \mathbb{R}_+$ and define $\tilde{I} := \{k \in \mathbb{R}_+ \text{ s.t. } ak \omega w_0/p_e \in I\}$. For a function $f$ non-decreasing on $[0, 1]$, taking values in $\tilde{I}$, its generalised inverse for all $k \in \tilde{I}$ is defined by:

$$f^{-1}(k) = \inf\{ε \in [0, 1] \text{ such that } f(ε) = k\}. \quad (B.6)$$

The following corollary allows us to give a characterisation of a sufficiently smooth admissible menu of revealing contracts. The proof of this result is highly similar to the one of Theorem 3.2.

**Corollary B.5.** An admissible menu of contracts $(e_{\text{min}}, T)$, for $e_{\text{min}} \in I$, is associated to an admissible revealing mechanism if and only if there exists a non-decreasing continuous function $q$, with values in $\tilde{I}$ and continuous second derivatives except where it is equal to 1, such that the price $T$ for a quantity $e_{\text{min}} = ak \omega w_0/p_e$ is given for some $c_q > 0$ by:

$$T(k) = w_0 - c_q w_0 e^{\beta q q^{-1}(k)} \times \begin{cases} (1 + αk)^{-\beta q q^{-1}(k)}, & \text{if } k < 1, \\ (1 + α)^{-\beta q q^{-1}(1)}k^{-\beta q q^{-1}(1)α/(1+α)}, & \text{if } k \geq 1. \end{cases} \quad (B.7)$$

for $k \in \tilde{I}$ and where $q^{-1}$ is the generalised inverse of $q$, as defined in (B.6).

**Proof.** (i) To prove that it is a necessary condition, let us fix an admissible menu of contracts $(e_{\text{min}}, T)$ and an associated admissible revealing mechanism $(q, t_0)$. Since the mechanism $(q, t_0)$ is admissible and satisfies the IC constraint, by Theorem 3.2 we obtain that $q$ is non-decreasing and the price function $t_0$ is given by (3.2) with a constant $c_q > 0$. Moreover, by the previous discussion
on admissible contracts, $T$ should be independent of $\varepsilon$, and thus constant when $q$ is constant, which is true by Lemma B.4. Hence we can write the price $t_0$ given by (3.2) in $\varepsilon = q^{-1}(k)$, where $k := pe_{\min}/(\alpha \omega w_0)$ and $q^{-1}$ is the generalised inverse of $q$. Moreover, noticing that $\varepsilon < \varepsilon_1$ is equivalent to $k < q^{-1}(\varepsilon_1) = 1$, and conversely if $\varepsilon \geq \varepsilon_1$ then $k \geq 1$, the price of a quantity $e_{\min} := \alpha k \omega w_0$ is given by $T(k) = w_0 t_0(q^{-1}(k))$ which is (B.7).

(ii) To prove the equivalence, let us consider an admissible menu of contracts $(e_{\min}, T)$ where $T$ is given by (B.7) and assume that the function $q$ has the right properties. First, we can show that given this menu of contracts, the optimal quantity chosen by an Agent of type $\varepsilon$ is $e_{\min} = \alpha k \omega w_0 / p_e$ where $k = q(\varepsilon)$. Indeed, by computing the derivative of his utility given by (2.5) with respect to the normalised quantity $k$, we obtain the following FOC for the optimal $k$:

$$0 = -\frac{\partial_k t_0(k)}{1 - t_0(k)} + \beta \varepsilon \times \begin{cases} \frac{\alpha}{1 + \alpha k} & \text{if } k < 1, \\ \frac{\alpha}{(1 + \alpha)k} & \text{if } k > 1. \end{cases}$$

Since the derivative of $t_0$ with respect to $k$ satisfies:

$$\partial_k t_0(k) = \begin{cases} \frac{\beta \alpha q^{-1}(k)}{1 + \alpha k} (1 - t_0(k)) & \text{if } k < 1, \\ \frac{\beta \alpha q^{-1}(k)}{(1 + \alpha)k} (1 - t_0(k)) & \text{if } k > 1, \end{cases}$$

the previous FOC is equivalent to $k = q(\varepsilon)$. By continuity of $q$, the result is extendable to $k = 1$. It remains to check the following SOC:

$$0 \geq -\frac{\partial_k^2 t_0(k)(1 - t_0(k)) + (\partial_k t_0(k))^2}{(1 - t_0(k))^2} - \beta \varepsilon \times \begin{cases} \frac{\alpha^2}{(1 + \alpha k)^2} & \text{if } k < 1, \\ \frac{\alpha}{(1 + \alpha)k} & \text{if } k > 1. \end{cases}$$

The second order derivative of $T$ satisfies:

$$\partial_k^2 t_0(k) = \begin{cases} \frac{\beta \alpha (1 - t_0(k))}{(1 + \alpha k)^2} (\partial_k q^{-1}(k)(1 + \alpha k) - \alpha q^{-1}(k) - \beta \alpha (q^{-1}(k))^2) & \text{if } k < 1, \\ \frac{\beta \alpha (1 - t_0(k))}{(1 + \alpha k)^2} ((\partial_k q^{-1}(k) - q^{-1}(k))(1 + \alpha) - \beta \alpha (q^{-1}(k))^2) & \text{if } k > 1, \end{cases}$$
thus the SOC is equivalent to:

\[
\begin{cases}
\hat{c}_k q^{-1}(k)(1 + \alpha k) + \alpha (\varepsilon - q^{-1}(k)) \geq 0 & \text{if } k < 1, \\
\hat{c}_k q^{-1}(k) k - q^{-1}(k) + \varepsilon \geq 0 & \text{if } k > 1.
\end{cases}
\]

In \( k = q(\varepsilon) \neq 1 \), the SOC becomes in both cases \( \hat{c}_k q^{-1}(k) \geq 0 \) which is true since \( q \) is non-decreasing. By continuity of the utility, this result is also true for \( k = 1 \). Therefore, an Agent of type \( \varepsilon \) will choose the quantity \( e_{\min} = \alpha q(\varepsilon) w_0/p_e \), which is an available quantity because \( q \) takes values in \( \tilde{I} \) and thus \( e_{\min} \in I \). By computing the function \( T(k) \) for \( k = q(\varepsilon) \), and divide it by \( w_0 \), we recover the function \( t_0 \) defined by (3.2), which associates to any \( \varepsilon \) the price \( w_0 t_0(\varepsilon) \) of the normalised quantity \( k = q(\varepsilon) \). Moreover, since the mechanism \((q, t_0)\) satisfies the assumptions to be admissible in the sense of Definition 3.1, by Theorem 3.2, the mechanism associated to the menu \((e_{\min}, T)\) is admissible and satisfies the IC constraint.

\( \Box \)

**B.3 …to select the Agents**

Given an admissible revealing mechanism \((q, t_0)\), we can write the informational rent of an Agent of type \( \varepsilon \), as a function of \( \varepsilon \):

\[
\Delta EU^Q(\varepsilon) = (1 + \alpha) \ln (1 - t_0(\varepsilon)) + \beta \varepsilon \bar{U}(q(\varepsilon)).
\]

Since \((q, t_0)\) is a menu of revealing contract, we can use the FOC (B.3) to compute its derivative:

\[
\hat{c}_\varepsilon \Delta EU^Q(\varepsilon) = \beta \bar{U}(q(\varepsilon)) = \begin{cases}
\beta (1 + \alpha) \ln (1 + \alpha q(\varepsilon)) & \text{if } q(\varepsilon) < 1, \\
\beta \alpha \ln (q(\varepsilon)) + \beta (1 + \alpha) \ln (1 + \alpha) & \text{if } q(\varepsilon) > 1.
\end{cases}
\]

This derivative is non-negative in both cases and implies that the information rent is non-decreasing. Therefore, if there exists \( \varepsilon \in [0, 1] \) such that \( \Delta EU^Q(\varepsilon) \geq 0 \), then for all \( \varepsilon \in [\varepsilon, 1] \), \( \Delta EU^Q(\varepsilon) \geq 0 \), which means that the participation constraint of Agents with type \( \varepsilon \in [\varepsilon, 1] \) is satisfied. A more precise result is established in Proposition 3.3, and its proof is reported below.

**Proof of Proposition 3.3.** We consider an admissible and incentive compatible mechanism \((q, t_0)\). Applying Theorem 3.2, the price \( t_0 \) satisfies (3.2). Therefore, in the one hand, if \( \varepsilon \in [0, 1] \) is such that \( q(\varepsilon) < 1 \), i.e. \( \varepsilon < \varepsilon_1 \), the participation constraint of the Agent of type \( \varepsilon \) becomes:

\[
c_q \geq \exp \left(-\beta \int_0^\varepsilon \ln (1 + \alpha q(\varepsilon)) d\varepsilon\right) = c(\varepsilon).
\]
On the other hand, if $\varepsilon \geq \varepsilon_1$, the participation constraint is equivalent to:

$$c_q \geq (1 + \alpha)^{-\beta_{(\varepsilon - \varepsilon_1)} } e^{\beta Q_0(\varepsilon)} = c(\varepsilon).$$

The participation constraint for an Agent of type $\varepsilon \in [0, 1]$ is thus equivalent in both cases to $c_q \geq c(\varepsilon)$. We can then compute the derivative of $c$ with respect to $\varepsilon$:

$$c'(\varepsilon) = -\beta c(\varepsilon) \times \begin{cases} \ln(1 + \alpha q(\varepsilon)) & \text{if } \varepsilon < \varepsilon_1, \\ \ln(1 + \alpha) + \frac{\alpha}{1 + \alpha} \ln(q(\varepsilon)) & \text{if } \varepsilon > \varepsilon_1. \end{cases}$$

Since $q(\varepsilon) \geq 0$ for all $\varepsilon \in [0, \varepsilon_1)$ and $q(\varepsilon) \geq 1$ for all $\varepsilon \in (\varepsilon_1, 1]$, we obtain that the derivative of $c$ is negative in both cases. Since the function $c$ is continuous on $[0, 1]$ (in particular in $\varepsilon_1$), the function is non-increasing on $[0, 1]$. Moreover, by definition of $\varepsilon$ and continuity of $c$, $c(\varepsilon) = c_q$. Thus, in the one hand, for any $\varepsilon \in [\varepsilon, 1]$, we have: $c(\varepsilon) \leq c(\varepsilon) = c_q$, and thus the participation constraint of the Agent of type $\varepsilon$ is satisfied. Conversely, for any $\varepsilon \in [0, \varepsilon)$, we have $c(\varepsilon) > c(\varepsilon) = c_q$, which means that the participation constraint is not satisfied. \hfill \Box

Proposition 3.3 thus states that only the most risky Agents will be selected by the Principal. This result is entirely implied by the fact that the reservation utility of an Agent depends on his type, and only happens in Principal–Agent problems with countervailing incentives. Indeed, the following remark shows that if a constant reservation utility had been chosen, the selected Agents would have been the less risky.

**Remark B.6.** If the Agents’ reservation utility is assumed to be a constant $R_0$, the participation constraint for an Agent of type $\varepsilon$ becomes $\text{EU}^Q(\varepsilon) \geq R_0$, where $\text{EU}^Q(\varepsilon)$ is defined by (2.5) for a revealing contract $(q(\varepsilon), t_0(\varepsilon))$. By computing the derivative of $\text{EU}^Q(\varepsilon)$ with respect to $\varepsilon$ for a menu of revealing contracts, using FOC (B.3), we obtain:

$$c(\varepsilon) = (1 + \alpha) \beta \ln \left( \frac{\alpha}{\alpha + 1} \right) + \beta U(q(\varepsilon)).$$

Under the assumption\(^9\) that $(1 + \alpha) \omega \leq \bar{\omega}$, the information rent $\text{EU}^Q(\varepsilon) - R_0$ is decreasing for all $\varepsilon \in [0, 1]$ such that $q(\varepsilon) \in \left(1, \left(\frac{\bar{\omega}}{(\alpha + 1)}\right)^{(1+\alpha)/\alpha}\right]$. Thus, in this case, if there exists $\bar{\varepsilon} \in [0, 1]$ such that $\text{EU}^Q(\bar{\varepsilon}) \geq R_0$, then the participation constraint of Agents of type $\varepsilon \in [0, \bar{\varepsilon}]$ is satisfied.

Nevertheless, in our opinion, it makes little sense to consider in our framework that the reservation utility is constant for any Agents, regardless of their type.

\(^9\)This is the case in the application considered throughout this paper, since $\alpha = 0.08$, $\omega = 0.4$ and $\bar{\omega} = 1.$
B.4 ...to solve the Principal’s problem

Corollary B.7. Let \( \xi \in [0, 1] \) and \( q \in \mathbb{R}_+ \). If there is a solution \( Q \in Q(\xi) \) to the ODE (3.5), the Principal’s profit given by (3.4) is equal to:

\[
\Pi(\xi) = w_0(1 - \xi) + \frac{1}{2}w_0((\xi_1 + \xi)_2 - \xi^2) - w_0F_1(Q) - w_0F_2(Q),
\]

where \( F_1 \) and \( F_2 \) are respectively defined as follows:

\[
F_1(Q) := \int_{\xi_1 + \xi}^{\xi_2} \left( e^{\beta(Q(\xi) - \xi F'(\xi))} + \xi wF'(\xi) \right) d\xi
\]

\[
F_2(Q) := \int_{\xi_1 + \xi}^{1} \left( (1 + \alpha)^{-\beta(\xi_1 + \xi)} e^{\beta(Q(\xi) - \xi F'(\xi))} + \xi w\frac{1+\alpha}{\alpha}F'(\xi) \right) d\xi.
\]

Proof of Theorem 3.4 and Corollary B.7. Let us fix a mechanism \((q, t_0) \in C^2(\xi)\). This mechanism satisfies the assumption of Theorem 3.2, the price \( t_0 \) is therefore given by (3.2). Moreover, since this mechanism is assumed to be in \( C^2(\xi) \), the participation constraint has to be satisfied only for all \( \xi \in [\xi_1, 1] \), which implies by Proposition 3.3 that the constant \( c_q \) in the price is given by \( c_q = c(\xi) \). We thus obtain that, if \( \xi \in [0, \xi_1] \), the price for all \( \xi \in [\xi, 1] \) is given by:

\[
t_0(\xi) = 1 - e^{\beta(Q_0(\xi) - Q_0(\xi))} \times \left\{ \begin{array}{ll}
(1 + \alpha q(\xi))^{-\beta\xi} & \text{if } \xi \in [\xi_1, \xi_1), \\
(1 + \alpha)^{-\beta \xi_1 q(\xi)} & \text{if } \xi \in [\xi_1, 1].
\end{array} \right.
\]

Similarly, if \( \xi \in [\xi_1, 1] \), the price for all \( \xi \in [\xi_1, 1] \) is given by:

\[
t_0(\xi) = 1 - e^{\beta(Q_0(\xi) - Q_0(\xi))} (1 + \alpha)^{-\beta \xi} (q(\xi))^{\beta \xi/\alpha}.\]

To reconcile the two cases, we denote by \( Q \) the following function, for all \( \xi \in [\xi, 1] \):

\[
Q(\xi) := \left\{ \begin{array}{ll}
\int_{\xi}^{\xi_1 + \xi} \ln (1 + \alpha q(\xi)) d\xi & \text{if } \xi \in [\xi, \xi_1 + \xi) \\
\int_{\xi}^{\xi_1 + \xi} \ln (1 + \alpha q(\xi)) d\xi + \alpha \int_{\xi_1 + \xi}^{\xi} \ln (q(\xi)) d\xi & \text{if } \xi \in [\xi_1 + \xi, 1].
\end{array} \right.
\]

Since \((q, t_0)\) is an admissible revealing mechanism, \( q \) is continuous on \([0, 1]\) and \( C^2 \) on \((0, 1)\) except where it is equal to 1, and, by Theorem 3.2, \( q \) is a non-decreasing function. This naturally implies that the function \( Q \) satisfies the right properties to be in \( Q(\xi) \).
Thanks to the definition of the function $Q$, we can write $q$ as a function of $Q'$, for all $\varepsilon \in [\xi, 1]$:

$$ q(\varepsilon) = \begin{cases} 
\frac{1}{\alpha} (e^{Q'(\varepsilon)}) - 1 & \text{if } \varepsilon \in [\xi, \xi_1 \vee \xi) \\
\frac{1 + \alpha}{\alpha} Q'(\varepsilon) & \text{if } \varepsilon \in [\xi_1 \lor \xi, 1]. 
\end{cases} $$

Therefore, the price $t_0$ can be written as follows, for all $\varepsilon \in [\xi, 1]$:

$$ t_0(\varepsilon) = \begin{cases} 
1 - e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} & \text{if } \varepsilon \in [\xi, \xi_1 \lor \xi), \\
1 - (1 + \alpha)^{-\beta(\xi_1 \vee \xi)} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} & \text{if } \varepsilon \in [\xi_1 \lor \xi, 1]. 
\end{cases} $$

Moreover, optimising on admissible revealing mechanisms $(q, t_0) \in C^Q(\xi)$ is thus equivalent to optimising on $Q \in Q(\xi)$, and the Principal’s problem for $\varepsilon \in [0, 1]$ fixed is thus given by:

$$ \Pi(\xi) = \sup_{(q, t_0) \in C^Q(\xi)} \left\{ \int_{\xi}^{\xi_1 \vee \xi} (t_0(\varepsilon) - \varepsilon \alpha q(\varepsilon) \omega) \, d\varepsilon + \int_{\xi_1 \lor \xi}^{1} (t_0(\varepsilon) - \varepsilon \alpha q(\varepsilon) \omega) \, d\varepsilon \right\} $$

$$ = w_0 (1 - \xi) + \frac{1}{2} \omega w_0 ((\xi_1 \lor \xi)^2 - \xi^2) - w_0 \inf_{Q \in Q(\xi)} \left\{ \int_{\xi}^{\xi_1 \lor \xi} \left( e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \omega e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)} \right) \, d\varepsilon \right\} $$

$$ + w_0 \int_{\xi_1 \lor \xi}^{1} \left( (1 + \alpha)^{-\beta(\xi_1 \lor \xi)} e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \alpha e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)} \right) \, d\varepsilon \right\}. \quad (B.10) $$

This problem is relatively standard in the field of Calculus of Variation. Given the form of the previous optimisation, we are led to study the optimal function $Q$ separately on $(\xi, \xi_1 \lor \xi)$ and on $(\xi_1 \lor \xi, 1)$.

With this in mind, we first study the problem on $(\xi, \xi_1 \lor \xi)$. Let $R$ be an arbitrary function that has at least one derivative and vanishes at the endpoints $\xi$ and $\xi_1 \lor \xi$. For any $\eta \in \mathbb{R}$, we denote $g_1(\eta) := F_1(Q + \eta R)$, where $F_1$ is defined by (B.8a). We can compute the derivative of $g_1$ with respect to $\eta$:

$$ g_1'(\eta) = \int_{\xi}^{\xi_1 \lor \xi} \left( \beta(R(\varepsilon) - \varepsilon R'(\varepsilon)) e^{\beta(Q(\varepsilon) + \eta R(\varepsilon) - \varepsilon Q'(\varepsilon) - \eta R'(\varepsilon)) + \varepsilon \omega R'(\varepsilon) e^{\beta(Q(\varepsilon) + \eta R'(\varepsilon))} \right) \, d\varepsilon. $$

The Gâteaux differential of $F_1$ with respect to $Q$ in the direction $R$ denoted by $DF_1(Q)(R)$ is given by $g_1'(0)$:

$$ DF_1(Q)(R) = \int_{\xi}^{\xi_1 \lor \xi} \left( \beta(R(\varepsilon) - \varepsilon R'(\varepsilon)) e^{\beta(Q(\varepsilon) - \varepsilon Q'(\varepsilon)) + \varepsilon \omega R'(\varepsilon) e^{\beta(Q(\varepsilon))} \right) \, d\varepsilon. $$

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By computing an integration by parts and since \( R(\bar{\varepsilon}) = R(\varepsilon_1 \lor \bar{\varepsilon}) = 0 \) by assumption, we obtain:

\[
DF_1(Q)(R) = -\int_{\bar{\varepsilon}}^{\varepsilon_1 \lor \bar{\varepsilon}} R(\varepsilon) \left( \beta (\beta \varepsilon^2 Q''(\varepsilon) - 2) e^{\beta (Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \omega (1 + \varepsilon Q''(\varepsilon)) e^{Q'(\varepsilon)} \right) d\varepsilon.
\]

Therefore, the Euler–Lagrange equation associated to the optimisation problem on \([\bar{\varepsilon}, \varepsilon_1 \lor \bar{\varepsilon}]\) is thus equivalent to the following non–linear second–order ODE:

\[
0 = \omega (1 + \varepsilon Q''(\varepsilon)) e^{Q'(\varepsilon)} + \beta (\beta \varepsilon^2 Q''(\varepsilon) - 2) e^{\beta (Q(\varepsilon) - \varepsilon Q'(\varepsilon))}.
\]  

(B.11)

Moreover, we can compute the second derivative of \( g_1 \) with respect to \( \eta \):

\[
g''_1(\eta) = \int_{\bar{\varepsilon}}^{\varepsilon_1 \lor \bar{\varepsilon}} \left( \beta^2 (R(\varepsilon) - \varepsilon R'(\varepsilon))^2 e^{\beta (Q(\varepsilon) + \eta R(\varepsilon) - \varepsilon Q'(\varepsilon) - \varepsilon \eta R'(\varepsilon))} + \varepsilon \omega (R'(\varepsilon))^2 e^{Q'(\varepsilon) + \eta R'(\varepsilon)} \right) d\varepsilon.
\]

This second derivative is therefore positive for any \( \eta \in \mathbb{R} \), and implies that \( F_1 \) attains a minimum for \( Q \) the solution on \([\bar{\varepsilon}, \varepsilon_1 \lor \bar{\varepsilon}]\) of the ODE (B.11), if it exists.

Similarly, to study the problem on \((\varepsilon_1 \lor \bar{\varepsilon}, 1)\), we consider \( F_2 \) defined by (B.8b). Let \( R \) be an arbitrary function that has at least one derivative and now vanishes at the endpoints \( \varepsilon_1 \lor \bar{\varepsilon} \) and 1. For any \( \eta \in \mathbb{R} \), we denote \( g_2(\eta) := F_2(Q + \eta R) \). The Gâteaux differential of \( F_2 \) with respect to \( Q \) in the direction \( R \) denoted by \( DF_2(Q)(R) \) is given by \( g''_2(0) \):

\[
DF_2(Q)(R) = \int_{\varepsilon_1 \lor \bar{\varepsilon}}^{1} \left( (1 + \alpha) -\beta (\varepsilon_1 \lor \bar{\varepsilon}) \beta (R(\varepsilon) - \varepsilon R'(\varepsilon)) e^{\beta (Q(\varepsilon) - \varepsilon Q'(\varepsilon))} + \varepsilon \omega (1 + \alpha) R'(\varepsilon) e^{\frac{1 + \alpha}{\alpha} Q'(\varepsilon)} \right) d\varepsilon.
\]

By integration by parts and since \( R(\varepsilon_1 \lor \bar{\varepsilon}) = R(1) = 0 \) by assumption, we obtain that the Euler–Lagrange equation associated to the optimisation problem on \([\bar{\varepsilon}, \varepsilon_1 \lor \bar{\varepsilon}]\) is equivalent to the following non–linear first–order ODE:

\[
0 = \omega (1 + \alpha)^{\beta (\varepsilon_1 \lor \bar{\varepsilon}) + 1} \left( 1 + \alpha \frac{\varepsilon}{\alpha} \right) Q''(\varepsilon) + \beta (\beta \varepsilon^2 Q''(\varepsilon) - 2) e^{\beta (Q(\varepsilon) - \varepsilon Q'(\varepsilon))}.
\]  

(B.12)

Moreover, the second derivative of \( g_2 \) with respect to \( \eta \) is positive for any \( \eta \in \mathbb{R} \), which implies that \( F_2 \) attains a minimum for \( Q \), the solution on \([\varepsilon_1 \lor \bar{\varepsilon}, 1]\) of the ODE (B.12), if it exists.

We can thus conclude that if there is a function \( Q \in \mathcal{Q}(\bar{\varepsilon}) \) solution to the ODE (B.11) on \([\bar{\varepsilon}, \varepsilon_1 \lor \bar{\varepsilon}]\) and to the ODE (B.12) on \([\varepsilon_1 \lor \bar{\varepsilon}, 1]\), maximises the Principal’s profit for \( \bar{\varepsilon} \in [0, 1] \) and \( q \) fixed. Combining both ODEs leads to the ODE (3.5), which proves the theorem. Moreover, using Equation (B.10), we obtain the form of the Principal’s profit given in Corollary B.7.

\[ \Box \]

**Remark B.8.** As explained in Remark 3.5, Theorem 3.4 only gives a sufficient condition for the Principal optimisation problem. To obtain a necessary condition, one should adapt the previous proof by writing the Euler–Lagrange equation for the problem with constraints. A new ODE would
then be obtained, and the existence of a solution to this ODE would be equivalent to the existence of an optimal contract. Nevertheless, in the application developed in Subsection 3.4, solving the ODE (3.5) is sufficient since its solution has the required regularity. Moreover, one can prove that the ODE has a unique solution for $\varepsilon$ bounded away from 0, which confirms that the numerical scheme converges to the solution of the Principal’s problem in our application. More precisely, on $[\varepsilon_1, \varepsilon_1 \vee \varepsilon])$, the ODE (3.5) can be written as a system of two first-order ODEs as follows:

$$
\begin{align*}
Q'(\varepsilon) &= \frac{\beta Q(\varepsilon) - R(\varepsilon)}{1 + \beta \varepsilon}, \\
R'(\varepsilon) &= \frac{1 + \beta \varepsilon - 2\beta \varepsilon R(\varepsilon)/\omega}{\varepsilon} - \frac{\beta^2 \varepsilon e^{R(\varepsilon)/\omega}}{1 + \beta^2 \varepsilon e^{R(\varepsilon)/\omega}}.
\end{align*}
$$

By Cauchy–Lipschitz Theorem, the second ODE has a unique solution if $\varepsilon \geq c > 0$. This solution is in particular bounded with bounded derivatives on the interval considered, which implies that the first ODE also has a unique solution. The same reasoning can be applied on the interval $[\varepsilon_1 \vee \varepsilon, 1]$.

C Adding the prepayment option

The results we developed throughout this paper highlight a certain form of irrationality of the Agents, due to their unwillingness to save money from one period to the next. Indeed, since the Agent has only the choice between subscribing or not subscribing to the insurance, and does not think about saving from one period to the next, he is ready to pay a very high price for the insurance. This appendix propose a solution to address this problem of irrationality.

We consider that a regulator offers (or forces the insurer to offer) another form of contract: the prepayment option. In this situation, the Agent can: (i) subscribe at time $t = 0$ to an insurance contract; (ii) prepay a quantity $e^P$, i.e. pay at time $t = 0$ the price $p_e e^P$ to receive $e^P$ in $t = 1$; (iii) do nothing. Adding this new option of prepayment is a way to encourage a specific form of savings, and could limit the price of the insurance. Although some theoretical results could be obtain, we choose in this section to only present numerical results. Indeed, the theoretical formulations are in the same spirit as those developed throughout the paper but more complicated, so it seems more relevant and meaningful in our opinion to discuss only the results obtained numerically, with the parameters defined for the application to fuel poverty.

C.1 A new reservation utility

If the Agent decides to prepay at time $t = 0$ a quantity $e^P$, his utility function is defined by (1.4), where $T = p_e e^P$ is the price of the chosen quantity. As previously, we assume without loss of generality that $e^P = \omega q w_0 / p_e$ for some $q \in \mathbb{R}_+$. Through easy optimisation techniques, we obtain the following result:

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Lemma C.1. If the Agent subscribes the prepayment option for a quantity \( q < 1/\alpha \omega \), his optimal consumptions at time \( t = 0 \) in each good are given by

\[
y_0^P := \frac{w_0}{1 + \alpha} - \frac{1 - \alpha q \omega}{p_y} \quad \text{and} \quad e_0^P := \frac{\alpha w_0}{1 + \alpha} - \frac{1 - \alpha q \omega}{p_e};
\]

and give him the following utility: \( V_0^P(q) = (1 + \alpha) \ln(w_0) + (1 + \alpha) \ln(1 - \alpha q \omega) + C_{\alpha,p_e,p_y}. \)

At time \( t = 1 \), he receives the prepaid quantity \( e^P \) in any case, not only in the case of an income loss. His utility to maximise at time \( t = 1 \) is thus naturally defined by:

\[
V_1^P(\omega, q) = \max_{(e_1, y_1) \in \mathbb{R}^2_+} \alpha \ln(e_1 + \alpha q \omega w_0/p_e) + \ln(y_1), \quad \text{u.c.} \quad e_1 p_e + y_1 p_y \leq \omega w_0.
\]

By maximising this utility with respect to \( e_1 \) and \( y_1 \), we obtain the following result:

Lemma C.2. The optimal quantities consumed in each good at time \( t = 1 \) are given by:

\[
e_1^P := \frac{\alpha}{1 + \alpha} \frac{w_0}{p_e} (\omega - q \omega)^+ \quad \text{and} \quad y_1^P := \frac{\omega w_0 - p_e e_1^P}{p_y},
\]

and provide the following utility to the Agent: \( V_1^P(\omega, q) = (1 + \alpha) \ln(\omega w_0) + C_{\alpha,p_e,p_y} + \bar{U}(q \omega/\omega). \)

Therefore, by Lemmas C.1 and C.2, the expected utility of an Agent of type \( \varepsilon \) who chooses to prepay the quantity \( e^P = \alpha q \omega w_0/p_e \) for \( q \in [0, 1/\alpha \omega) \) is given by:

\[
EU^P(\varepsilon, q) = EU^\sigma(\varepsilon) + (1 + \alpha) \ln(1 - \alpha q \omega) + \beta \varepsilon \bar{U}(q) + \beta (1 - \varepsilon) \bar{U}(q \omega/\omega).
\]

The Agent then chooses the optimal amount he wants to prepay by maximising his expected utility over admissible \( q \). The easiest way to solve this optimisation problem is to perform a simple numerical optimisation to find the optimal quantity \( q^*(\varepsilon) \) an Agent of type \( \varepsilon \) should prepay and his associated expected utility, denoted \( EU^P(\varepsilon) \), for every \( \varepsilon \in [0, 1] \). For the parameters previously defined in Subsection 2.3, the results are presented in Figure 5.

Facing this new option, an Agent of type \( \varepsilon \) will subscribe to an insurance contract \((e_{\text{min}}, T)\), with \( e_{\text{min}} = \alpha q \omega w_0/p_e \) and \( T = w_0 t_0 \), if and only if the two following conditions hold:

\[
EU^Q(\varepsilon, q, t_0) \geq EU^P(\varepsilon) \quad \text{and} \quad EU^Q(\varepsilon, q, t_0) \geq EU^\sigma(\varepsilon). \quad (C.1)
\]

By Definition of \( EU^P(\varepsilon) \), for every \( \varepsilon \in [0, 1] \) we have \( EU^P(\varepsilon, q, 0) = EU^\sigma(\varepsilon) \). Therefore, the second inequality in \( (C.1) \) is implied by the first and is thus not necessary. In this framework, the reservation utility of an Agent of type \( \varepsilon \) is therefore defined by the utility he obtained thanks to the prepayment option. To simplify the notation, we denote by \( \Delta EU^P(\varepsilon) \) the difference between
the expected utility with prepayment of an Agent and his utility without:

$$\Delta EU^{F^*,\xi} := EU^{F^*,\xi} - EU^\emptyset(\xi),$$

(C.2)

which corresponds to the information rent in this framework.

![Graph showing the relationship between prepayment and expected utility](image)

Figure 5: Optimal quantity prepaid and associated expected utility.
Top: the optimal quantity prepaid (blue) is compared to the quantity consumed with the initial income (dotted green) and with an income loss (dotted red). Bottom: the maximum expected utility with prepayment (blue) is compared to the previous reservation utility (dotted orange).

### C.2 First–Best case

As detailed in Subsection 2.2, the problem of the Principal in this case is defined by (1.7), under a new participation constraint of the Agent, since his reservation utility is now given by $EU^{F^*,\xi}(\xi)$: an Agent will accept the contract if it provides him at least his utility with prepayment. Similarly to the reasoning developed in Subsection 2.2, when the Principal knows the type of the Agent, she may charge him the highest price he is willing to pay for the insurance. In this case, the Agents’ informational rent are then reduced to zero, for any type $\xi \in [0, 1]$.

Using the notation (C.2), the participation constraint of an Agent of type $\xi$ is equivalent to:

$$t_0 \leq T_{\text{max}}^P(\xi, q) := 1 - \exp\left(\frac{\Delta EU^{F^*,\xi}(\xi)}{1 + \alpha}\right) \times \begin{cases} 
(1 + q\alpha)^{-\beta\xi} & \text{if } q < 1, \\
q^{-\beta\xi \alpha \beta \xi (1 + \alpha)} & \text{if } q \geq 1,
\end{cases}$$

However, contrary to Subsection 2.2, since it is relatively complicated to obtain explicitly the
expected utility with prepayment, we cannot give a more detailed formula for the maximum price that an Agent of type $\varepsilon$ is willing to pay for insurance. Nevertheless, all the results can easily be computed numerically. Figure 6 presents, from top to bottom, the quantity insured, the price paid by the Agents, and the Principal’s profit, in the case of an insurance against fuel poverty, i.e. with the parameters defined in Subsection 2.3. We can compare these graphs with those of Figure 1. The most interesting point is that the price of insurance is significantly lower in this new situation. Indeed, the price paid by the Agents is now barely higher than the actuarial price, whereas without prepayment it was sometimes even higher than $p_e e_{\min}$, which is actually precisely the price of the prepayment.

![Graph 1](image1.png)

![Graph 2](image2.png)

![Graph 3](image3.png)

Figure 6: Optimal insurance in the First–Best case with prepayment.

The blue curves represents, from top to bottom, the quantity insured, the premium and the Principal’s profit, with respect to the probability $\varepsilon$. On the middle graph, the premium is compared to the actuarial price (orange curve), which also correspond to the Principal’s cost. The red dotted line on the bottom graph is her average profit.
C.3 Third–Best case

Since we are only changing the reservation utility, the results of Subsection 3.1 remain true. In particular, a Principal will offer a menu of revealing contracts $(e_{min}, T)$ defined by Corollary B.5, such that an Agent of type $\varepsilon$ will choose the quantity $e_{min} = \alpha q(\varepsilon) w_0 / p_e$, and will pay the price $w_0 t_0(\varepsilon)$ given by (3.2), for a particular function $q$ which will be optimised by the Principal. The only thing that changes is the participation constraint, which is now given by $EU^Q(\varepsilon, q, t_0) \geq EU^{P, *}(\varepsilon)$. Using the form of the price given by (3.2), this constraint is equivalent to

$$c_q \geq \varepsilon^P(\varepsilon) := \exp \left( \frac{\Delta EU^{P, *}(\varepsilon)}{1 + \alpha} \right) \varepsilon(\varepsilon),$$

where $\varepsilon$ is defined by (3.3). In Subsection 3.2, the function $\varepsilon$ being decreasing, the participation constraint was satisfied for Agents of type $\varepsilon$ above a specific level. Unfortunately, in this case, it is not possible to determine precisely the variations of the new function $\varepsilon^P$, since we do not have an explicit form for the expected utility with prepayment. More precisely, the function $\varepsilon$ is decreasing while, as we can see in Figure 5, the difference defined in Equation (C.2) is increasing.

In the case with prepayment, it is therefore difficult to determine a monotonicity or even the variations of the information rent. With the help of the First–Best case, we can still intuit that the information rent in the Third–Best case is increasing up to a specific $\varepsilon \in [0, 1]$ and then decreasing. More precisely, as can be seen on the graph with the Principal’s profit, she earns money on the medium–risky Agents, since Agents of type $\varepsilon = 0$ and $\varepsilon = 1$ have no interest in subscribing to the insurance. Indeed, on the one hand, the problem of non–risky Agents remains the same as in the case without prepayment: the optimal quantity they would like to prepay is zero, which implies that their utility with prepayment is equal to their utility without insurance. Their reservation utility is therefore unchanged from the case studied throughout this paper, and these Agents are not of interest from the insurer point of view. On the other hand, the Agents of type $\varepsilon = 1$ are now indifferent between prepayment and insurance, the two options providing them with a sufficient quantity of energy for the future. The riskiest Agents, who were highly courted by the insurer, are now hard to satisfy and will instead turn to prepayment. Hence, the most interesting Agents in this case for the Principal seem to be the intermediate ones, and she should choose the constant $c_q$ so as to select only those Agents. Therefore, instead of maximising the integral from an $\varepsilon$ to 1 of the benefits, she will maximise the integral of the benefit on some interval contained in $(0, 1)$.

Unfortunately, in order to address the Third-Best case, further study would be required. The reasoning would be similar to the one developed in Section 3, but the non–monotonicity of the information rent makes the problem more difficult. Nevertheless, in our opinion, this study would be interesting since the addition of the prepayment option seems to (i) allow Agents to consume sufficient energy in case of loss of income; (ii) decrease the premium compared to the case without prepayment; (iii) allow medium–risk Agents to be insured.