Generation of tunable entanglement and violation of a Bell’s like inequality between
different degrees of freedom of a single photon

Adam Vallés,1 Vincenzo D’Ambrosio,2 Martin Hendrych,1 Michal Mičuda,3 Lorenzo Marrucci,4 Fabio Sciarrino,2 and Juan P. Torres1,5

1ICFO—Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860, Castelldefels, Barcelona, Spain
2Departamento de Fisica, Sapienza Universita di Roma, Roma 00185, Italy
3Department of Optics, Palacký University, 17. listopadu 12, 77146 Olomouc, Czech Republic
4Dipartimento di Fisica, Universita di Napoli Federico II, Complesso Universitario di Monte Sant’Angelo, Napoli, Italy
5Department of Signal Theory and Communications, Universitat Politecnica Catalunya, Campus Nord D3, 08034 Barcelona, Spain

We demonstrate a scheme to generate non-coherent and coherent correlations, i.e., a tunable degree of entanglement, between degrees of freedom of a single photon. Its nature is analogous to the tuning of the purity (first-order coherence) of a single photon forming part of a two-photon state by tailoring the correlations between the paired photons. Therefore, well-known tools such as the Clauser-Horne-Shimony-Holt (CHSH) Bell-like inequality can also be used to characterize entanglement between degrees of freedom. More specifically, we perform CHSH inequality tests between the polarization and the spatial shape (two modes with orbital angular momentum) of a single photon.

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I. INTRODUCTION

Entanglement, a concept introduced in Quantum Theory nearly eighty years ago by E. Schrödinger [1], is one of the main traits of quantum theory, for some it is even its weirdest feature [2]. Since the publication of the seminal Gedanken EPR experiment by Einstein, Podolsky and Rosen (EPR) in their famous 1935 paper [3], and the appearance of the first comments about it the very same year [4], innumerable theoretical discussions and experiments related to this subject have appeared.

Arguably the most relevant contribution to this discussion has been the introduction, fifty years ago now, of the nowadays well-known Bell’s inequalities [5]. One of these Bell-like inequalities, the Clauser-Horne-Shimony-Holt (CHSH) inequality [6], which we will use here, is surely the most commonly used in experiments [7]. Originally, Bell’s inequalities were considered for composite systems made up of two separate subsystems, i.e., two subsystems propagating along different directions that had interacted in the past. For instance, the two subsystems can be each one of the two photons generated by means of the nonlinear process of spontaneous parametric down-conversion (SPDC), when an intense pump beam interacts with the atoms of a non-centrosymmetric nonlinear crystal [8]. Entanglement can reside in any of the degrees of freedom that characterize each of the photons, being the use of polarization the most common. In this case, the quantum state that allows the maximum violation of the CHSH inequality can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left\{ |1\rangle_{AH}|0\rangle_{AV}|0\rangle_{BH}|1\rangle_{BV} + |0\rangle_{AH}|1\rangle_{AV}|1\rangle_{BH}|0\rangle_{BV} \right\},$$

(1)

where $A_H$, $A_V$, $B_H$ and $B_V$ designate the four modes $A_H \equiv$ propagation along direction $A$ and polarization $H$, $A_V \equiv$ propagation along direction $A$ and polarization $V$, $B_H \equiv$ propagation along direction $B$ and polarization $H$, $B_V \equiv$ propagation along direction $B$ and polarization $V$. $H$ and $V$ refer to horizontal and vertical polarization of the photon, and 0 and 1 refer to the number of photons in these modes.

However, correlations of a similar nature to the ones existing between physically separated photons can also exist considering different degrees of freedom of a single system. Therefore, along the lines of the case mentioned above for two separate subsystems, Bell’s inequalities can also be used as well to characterize these correlations existing between different parts of a single system. In [9], the violation of a Bell-like inequality measuring the correlations in two degrees of freedom (polarization and path) of a single photon was demonstrated. In that case, the quantum state of the photon can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left\{ |1\rangle_{AH}|0\rangle_{B_V} + |0\rangle_{AH}|1\rangle_{B_V} \right\}.$$  

(2)

The key point to consider Bell’s inequalities between different parts of a single system is to make the analogy

$|1\rangle_{AH} \equiv |1\rangle_{a_1}|1\rangle_{a_2}$,
$|0\rangle_{AH} \equiv |0\rangle_{a_1}|0\rangle_{a_2}$,
$|1\rangle_{B_V} \equiv |1\rangle_{b_1}|1\rangle_{b_2}$,
$|0\rangle_{B_V} \equiv |0\rangle_{b_1}|0\rangle_{b_2}$.
where

\[ a_1 \equiv \text{photon propagating along direction } A, \]
\[ a_2 \equiv \text{photon with polarization } H, \]
\[ b_1 \equiv \text{photon propagating along direction } B, \]
\[ b_2 \equiv \text{photon with polarization } V, \]

and \( l \) and \( \theta \) no longer refer to number of photons, but to the fact that the mode where the photon is embedded has, or not, this characteristic. Importantly, one has to be able to perform independent measurements in any of the degrees of freedom involved.

Bell-like inequalities can be also used to characterize beams containing many photons, i.e., intense beams, coherent or not. In [10, 11] the authors make use of coherent beams whose electric field reads

\[ E(r) = \frac{1}{\sqrt{2}} \{ \Psi_H(r)\hat{e}_H + \Psi_V(r)\hat{e}_V \}, \quad (3) \]

and use a CHSH inequality to characterize their coherence properties in one of the two degrees of freedom involved. Non-quantum entanglement, or inseparability of degrees of freedom has also been considered [12, 13] as a fundamental tool to address and shed new light into certain characteristics of classical fields, by applying analysis and techniques usually restricted to entanglement in a quantum scenario.

Here we intend to move further into this analogy and show experimentally that one can generate tunable entanglement between two degrees of freedom of a single photon, going from the generation of coherent correlations to incoherent ones. For the single-photon case, the control of the degree of entanglement between degrees of freedom is fully equivalent to tuning the first-order coherence [14] of one of the degrees of freedom involved, in full analogy with the relationship existing between the degree of entanglement between separate photons and the first-order coherence of one of the photons that forms the pair.

Different types of quantum states provide different results in the measurement of the CHSH inequality. This notwithstanding, for any quantum state with any degree of first-order coherence or purity, we demonstrate that the results of a Bell’s measurement obtained using different degrees of freedom of a single photon, are the same than when using the properties of separate photons.

In our experiments we make use of single-photons where the two degrees of freedom involved are the polarization (horizontal and vertical linear polarizations) and spatial modes (two spatial modes with orbital angular momentum index \( m = \pm 1 \)). The orbital angular momentum (OAM) states allow for a relatively simple experimental generation, filtering, detection, and control [15]. These states are characterized by the index \( m \), which can take any integer number, and determines the azimuthal phase dependence of the mode, which is of the form \( \sim \exp(imm) \). Each mode carries an OAM of \( mh \) per photon. The feasibility to generate entangled
states in the lab using polarization and spatial modes with OAM is greatly facilitated by the use of the so-called q-plates \[^{16}\]: liquid crystal devices which couple together polarization and orbital angular momentum and allow to generate states that have been recently exploited in fundamental quantum mechanics \[^{17, 18}\], quantum communications \[^{19}\] and metrology \[^{20}\]. In \[^{21}\], Nagali et al. generated a single-photon quantum state with high purity of the form given by Eq. \[^{2}\]. Karimi et al. \[^{22}\] used this same state to demonstrate the violation of the CHSH inequality.

**II. EXPERIMENTAL SETUP**

The experimental setup used in our experiments is shown in Fig. \[^{1}\]. Paired photons are generated in a 2 mm long beta-barium borate (BBO) nonlinear crystal by means of spontaneous parametric down-conversion (SPDC). We choose a type II source, where the photons generated have orthogonal (horizontal and vertical) polarizations in order to generate a polarization-entangled photon pair by post-selection with a beam splitter.

The pumping laser is a Mira 900 (Coherent) working in a second-harmonic set-up (Inspire Blue, Radiantis). The output light at 405 nm traverse an optical system with 5 dichroic mirrors and a short-pass filter to filter out the remaining 810 nm light. A spatial filter taoleds the spatial shape of the pump beam to obtain the sought-after Gaussian beam profile. We use a 750 mm focal distance lens to obtain a pump beam with 400 μm beam waist that is focused in the middle of the nonlinear crystal. The presence of spatial walk-off in the BBO crystal impedes the use of a smaller beam waist that would increase the efficiency of the SPDC process, since it would also introduce harmful spatial distinguishability between the generated photons. The down-converted photons are collected with a 400 nm focal distance lens.

Another filtering system, formed by 2 dichroic mirrors, a long-pass filter and a band-pass filter removes the residual pumping light at 405 nm. The spectra of the photons in a pair are different, since they are orthogonal in polarization. This translates into both photons having different group velocities, and thus mixing the polarization and frequency properties of the photons. The use of a filter with 3 nm full-width-half-maximum bandwidth centered at 810 nm helps reducing the spectral distinguishability between the photons.

After the beam splitter, the quantum state of the two photons, considering only the cases where there is one photon in a detector and the remaining photon in the other detector (post-selection by coincidence detections), can be generally written as

\[
\rho = \epsilon |\Psi\rangle \langle \Psi | + \frac{1-\epsilon}{2} \times (|1\rangle_{A_H} |0\rangle_{A_V} |0\rangle_{B_H} |1\rangle_{B_V} + |0\rangle_{A_H} |1\rangle_{A_V} |1\rangle_{B_H} |0\rangle_{B_V} + |0\rangle_{A_H} |1\rangle_{A_V} |1\rangle_{B_H} |0\rangle_{B_V} + |1\rangle_{A_H} |0\rangle_{A_V} |0\rangle_{B_H} |1\rangle_{B_V}) ,
\]

(4)

where $|\Psi\rangle$ is given by Eq. \[^{1}\], and $\epsilon$ depends on the delay ($\tau$) between the two orthogonal photons generated.

A delay line, formed by quartz prisms, can be used to tune its value. If photons could be distinguished by their time of arrival to the detectors, then $\epsilon = 0$ and the purity of the quantum state that describes the two photons generated is minimum ($P = 1/2$). The purity of the quantum state can be increased by adding...
or removing the length of quartz that the photons traverse along its optical path [23], necessary to remove all distinguishing information coming from the temporal/frequency degree of freedom. For a specific arrangement of the quartz prisms, that we define as \( \tau = 0 \), we can have \( \epsilon = 1 \). For the \( L = 2 \) mm long BBO crystal of our experiment, with group velocity mismatch of \( D = 190 \) fs/mm, this requires [24] adding with the tunable delay line \( DL/2 = 190/2 \) fs/mm \( \times 2 \) mm = 190 fs to the delay already introduced due to the SPDC process itself.

To entangle the polarization and the orbital angular momentum (OAM) degrees of freedom in a single photon, the photon reflected from the the beam splitter (photon 1) is projected into the linear diagonal polarization state: \( 1/\sqrt{2} [ |H \rangle \pm |V \rangle ] \), with a half wave plate (HWP1) and a Glan-Thompson polarizer (GT1), coupled into a single mode fibre, to remove the remaining spatial distinguishability introduced by the presence of spatial walk-off in the BBO crystal, and detect it to measure in coincidences. The transmitted photon (photon 2) traverse a quarter wave plate (QWP1) to rotate its polarization from horizontal/vertical to circular right (R)/circular left (L), a q-plate (QP1) correlates polarization with OAM, and another quarter wave plate (QWP2) transform back the polarization from circular right/circular left to horizontal/vertical. In summary,

\[
|H \rangle \Rightarrow |R \rangle \Rightarrow |L, m = -1 \rangle \Rightarrow |H, m = -1 \rangle,
|V \rangle \Rightarrow |L \rangle \Rightarrow |R, m = +1 \rangle \Rightarrow |V, m = +1 \rangle. \tag{5}
\]

After the second quarter-wave plate, the quantum state of photon 2, after projection and detection of photon 1, writes

\[
\rho = \epsilon |\Psi^\pm \rangle \langle \Psi^\pm | + \frac{1 - \epsilon}{2} \\
\times \{ |1\rangle_A |0\rangle_B \langle 1\rangle_A \langle 0\rangle_B + |0\rangle_A |1\rangle_B \langle 0\rangle_A \langle 1\rangle_B \}, \tag{6}
\]

where

\( A \equiv \text{Polarization } H \) and spatial mode \( m = -1 \),
\( B \equiv \text{Polarization } V \) and spatial mode \( m = +1 \),

and

\[
|\Psi^\pm \rangle = \frac{1}{\sqrt{2}} \left( |1\rangle_A |0\rangle_B \pm |0\rangle_A |1\rangle_B \right) \equiv \frac{1}{\sqrt{2}} \left( |H, m = -1 \rangle \pm |V, m = +1 \rangle \right). \tag{7}
\]

The purity of the state is \( P = (1 + \epsilon^2)/2 \). If one would apply the concept of concurrence [25] to this single-photon state, considering as the two subsystems the polarization and OAM degrees of freedom of the photon, one would obtain \( C = \epsilon \).

The measurement stage consist of projecting the quantum state generated into specific polarization and OAM states in two steps. First, the state of polarization is projected into the desired state with a half-wave plate (HWP2) and a polarizing beam splitter (PBS).

![FIG. 4. Value of the parameter S in a CHSH inequality as a function of the angle \( \theta = b_1 - a_1 \). The small blue circles with error bars represent the experimental data with their standard deviations. The blue curves are the theoretical predictions assuming a visibility factor of \( V = 0.92 \). (a) \( \epsilon = 1 \), (b) \( \epsilon = 0.8 \), (c) \( \epsilon = 0.32 \) and (d) \( \epsilon = 0.03 \). The values of \( \epsilon \) correspond to delays (a) 0 fs, (b) 200 fs, (c) 400 fs and (d) 600 fs, as depicted in the HOM dip of Figure 2. The dashed red line (upper) corresponds to the Tsirelson bound, and the dashed green line (lower) is the CHSH inequality limit.](image)

The OAM can be projected into any state using several polarization optic elements, before and after a second q-plate (QP2) [24]. More specifically, the OAM state information is transferred into a polarization state with a half-wave plate (HWP3) and a quarter-wave plate (QWP3) located before the q-plate, to transform horizontal/vertical polarizations to right/left polarizations base, and another Glan-Thompson polarizer (GT2) located after. Finally, the photon is spatially filtered by coupling it to a single-mode fibre and detect it in coincidence with the other photon.

### III. EXPERIMENTAL RESULTS

In order to be able to relate the value of \( \epsilon \) in Eq. (4) to the delay introduced by the delay line, and determine the value of the delay which makes the quantum state pure (\( \epsilon = 1 \)), we construct a Hong-Ou-Mandel interferometer (HOM). If we choose the temporal delay introduced by the delay line so that coincidences are close to zero, the state given by Eq. (10) is pure (\( \epsilon = 1 \)) and corresponds to a Bell state. We choose to generate the quantum state \( |\Psi^- \rangle \) to obtain the HOM dip. Fig. 2(a) shows the coincidence photons measured in detectors 1 and 2, and Fig. 2(b) shows the single photons detected in each detector. We should notice that all the results presented in this paper are shown with no subtraction of the accidental coincidences (\( \sim 4 \) pairs in 10 seconds). Figure 2(c) shows coincidence detections re-normalized using the singles measurements from detector 1.

When we change the projection of photon 1 from the state \( 1/\sqrt{2} [ |H \rangle + |V \rangle ] \) to \( 1/\sqrt{2} [ |H \rangle - |V \rangle ] \) with
HWP$_1$, we change the sign of the corresponding Bell state, from $|\Psi^-\rangle$ to $|\Psi^+\rangle$. By modifying the transformation of photon 2 from $L/R \rightarrow H/V$ to $L/R \rightarrow V/H$ with QWP$_2$, we can go from the generation of $|\Phi^{\pm}\rangle$ to $|\Phi^{\pm}\rangle$, where $\Phi^{\pm}$, which can be written as

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|H, m = +1 \rangle \pm |V, m = -1 \rangle).$$

(8)

With this procedure we are able to create the four Bell states.

Figure 3 shows the coincidences measured for each of the four Bell states. Photon 2 is projected first into the polarization state $\sim \cos \beta_1 |H\rangle + \sin \beta_1 |V\rangle$, with $\beta_1 = 0^\circ, 45^\circ$, and after that a second projection is performed into a set of OAM states of the form $\cos \beta_2 |+1\rangle + \sin \beta_2 |-1\rangle$, with $\beta_2$ spanning from 0 to $2\pi$. Ideally, for the state $|\Psi^-\rangle$, coincidence counts as a function of $\beta_2$ follow the form of $\sin^2(\beta_1 - \beta_2)$, which yield a visibility $V = (\text{Max} - \text{Min})/(\text{Max} + \text{Min})$ of 100%. Therefore, the highest the visibility measured, the highest the quality of the entangled state generated.

Measurements of the CHSH inequality requires choosing two polarizations states and two OAM states where the state of photon 2, given by Eq. (6), is projected. When considering any possible state projection, following $|\Phi^{\pm}\rangle$, one finds that the maximum violation of the CHSH inequality for this state is

$$S_{\text{max}} = 2\sqrt{1 + \epsilon^2}. \quad (9)$$

For $\epsilon = 1$ we reach the Tsirelson bound. We will restrict here only to projections into states of the form

$$|a_i\rangle = \frac{1}{\sqrt{2}}(\cos \alpha_i |H\rangle + \sin \alpha_i |V\rangle),$$

$$|b_i\rangle = \frac{1}{\sqrt{2}}(\cos \beta_i |m = +1\rangle + \sin \beta_i |m = -1\rangle), \quad (10)$$

where states $a_i$ $(i = 1, 2)$ refers to linear polarization states and $b_i$ $(i = 1, 2)$ refer to OAM states which are linear combinations of modes $m = +1$ and $m = -1$. By proper combinations of all of the polarization optical elements of the set-up (half-wave and quarter wave plates), one can project the photon into any combination $(a_i, b_i)$ as required.

For the single photon case, and restricting our attention to state projections of the form given in Eq. (10), the CHSH inequality can be written as

$$S = E(a_1, b_1) - E(a_1, b_2) + E(a_2, b_1) + E(a_2, b_2) \leq 2, \quad (11)$$

where

$$E(a_i, b_i) = \frac{N_{++}(a_i, b_i) + N_{--}(a_i^\dagger, b_i^\dagger) - N_{+-}(a_i, b_i^\dagger) - N_{-+}(a_i^\dagger, b_i)}{N_{++}(a_i, b_i) + N_{--}(a_i^\dagger, b_i^\dagger) + N_{+-}(a_i, b_i^\dagger) + N_{-+}(a_i^\dagger, b_i)} \quad (12)$$

$N_{++}(a_i, b_i)$ is the number of photons detected when its quantum state is projected into a polarization state determined by the angle $a_i$ and an OAM state determined by the angle $b_i$. All other cases follow similarly, taking into account that $a_i^\dagger = a_i + \pi/2$ and $b_i^\dagger = b_i + \pi/2$. One can easily find that for the state given by Eq. (6),

$$E(a_i, b_i) = \cos 2a_i \cos 2b_i + \epsilon \sin 2a_i \sin 2b_i. \quad (13)$$

Figure 6 shows the value of $S$ measured when we go from a pure to a mixed state, i.e., for different values of $\epsilon$ from 0 to 1. It shows the value of $S$ as a function of the angle $\theta$, where $\theta \equiv b_1 - a_1 = b_2 + a_2 = -b_1 - a_2$. For the case of a pure state, one would obtain $S(\theta) = 3 \cos 2\theta - \cos 6\theta$. The experimental values measured decrease from the theoretical (ideal) expected values due to the existence of accidental coincidences or the inevitable misalignment of optical elements, by a factor $V$, the visibility measured in Fig. 5. In our case, the maximum CHSH inequality value measured is $S(\theta = 22.5^\circ) = 2.601 \pm 0.037$ and the visibility is $V = 0.92$.

Figure 4 shows that there is a complete analogy between a Bell-like inequality involving the same degree of freedom of two separate photons and that involving two distinct degrees of freedom of the same single photon, independently of the purity (or first-order coherence) of the quantum state. Figure 5 shows the CHSH violation measured for $\theta = 22.5^\circ$, which gives the maximum vi-
IV. CONCLUSIONS

In conclusion, we have demonstrated experimentally that there is a full analogy between the general quantum state (pure or mixed) that describes two-photon states entangled in its polarization degree of freedom, and the correlations (coherent or non-coherent) existing between the polarization and spatial degrees of freedom of a single photon. Along these lines, concepts such as purity, degree of entanglement or concurrence can be as well used to described coherent and non-coherent correlations between properties of a single system. This fact naturally allows one to use Bell’s inequalities to characterize both types of systems, as we have demonstrated here.

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