GROUND STATES IN SPATIALLY DISCRETE NON-LINEAR SCHRODINGER LATTICES

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ABSTRACT. In the seminal work [1], Weinstein considered the question of the ground states for discrete Schrödinger equations with power law nonlinearities, posed on \( \mathbb{Z}^d \). More specifically, he constructed the so-called normalized waves, by minimizing the Hamiltonian functional, for fixed power \( P \) (i.e. \( l^2 \) mass). This type of variational method allows one to claim, in a straightforward manner, set stability for such waves.

In this work, we revisit these questions and build upon Weinstein’s work in several directions. First, for the normalized waves, we show that they are in fact spectrally stable as solutions of the corresponding discrete NLS evolution equation. Next, we construct the so-called homogeneous waves, by using a different constrained optimization problem. Importantly, this construction works for all values of the parameters, e.g. \( l^2 \) supercritical problems. We establish a rigorous criterion for stability, which decides the stability on the homogeneous waves, based on the classical Grillakis-Shatah-Strauss/Vakhitov-Kolokolov quantity \( \partial_\omega \| \phi_\omega \|_{l^2}^2 \). In addition, we provide some symmetry results for the solitons. Finally, we complement our results with numerical computations, which showcase the full agreement between the conclusion from the GSS/VK criterion vis-à-vis with the linearized problem. In particular, one observes that it is possible for the stability of the wave to change as the spectral parameter \( \omega \) varies, in contrast with the corresponding continuous NLS model.

1. INTRODUCTION AND MOTIVATION

The discrete nonlinear Schrödinger model [2] has been one of the workhorses within the realm of nonlinear dynamical lattice models that has enabled the identification of numerous nonlinear waveforms, their stability analysis, instability manifestations and complex dynamics and thermodynamics. Arguably, central to its popularity can be thought of as being the prototypical inclusion of the main ingredients for such phenomena, namely the interplay between nonlinearity and lattice dispersion. Another key of its features is its generic nature and multi-fold physical motivation stemming, originally, from the example of optical waveguide arrays [3–4], but also extending nowadays to quite different fields, such as the atomic realm of Bose-Einstein condensates in optical lattices [5]. As some among the numerous notable features that the theoretical analysis and experimental investigations of the mode have enabled to explore, we mention the manifestation of discrete diffraction [6] and diffraction managed solitons [7–8], the illustration of lattice solitary waves [9–10], the emergence of discrete vortices [11–12], the Talbot revivals [13], the examination of \( \mathcal{PT} \)-symmetric lattices [14].

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At the same time, the DNLS has been a rich source of interesting problems in the applied mathematics literature and community. The early work of [11] (that will be central herein) set the stage for some of the important variational considerations that later extended to both periodic and decaying solutions in similar models [15], but also for both focusing and defocusing nonlinearities [16, 17] and continues to be an inspiration for further variational work on the subject more recently [18,19]. On the other hand, a considerable effort has been expended to obtain information about the stability of different waveforms in one- [20,21], two [22] and even three [23] spatial dimensions. Additional efforts have been made to address the spectral theory and dispersive estimates for this model [24], the asymptotic stability of small solutions [25], the distinction between on- and off-site solutions [26], as well as extensions more recently going beyond nearest-neighbor interactions [27,28]. While, admittedly, we cannot offer an exhaustive list, we believe that the above yields a representative flavor of the range of contributions and interest in the subject.

Here, we revisit the widely studied topic of the stability of fundamental waveforms in the model, providing in the spirit of [1] a novel constrained optimization perspective for the so-called normalized waves pertaining to the ground state of the system. On the one hand, the relevant formulation works in the entire parametric regime of the system, while on the other hand, it offers an alternative yet rigorous criterion for the stability of the solutions, which is tantamount to the famous Vakhitov-Kolokolov criterion [29], subsequently made rigorous for continuous systems in the work of Grillakis-Shatah-Strauss [30] (see also the recent exposition of [31]). Our presentation of the relevant results is structured as follows. In section 2, we provide the model formulation and the results/theorems stemming from the above reformulation. After the preliminaries of Section 3, we turn to the construction of the normalized waves in Section 4. The resulting existence and stability properties are presented in Section 5, while Section 6 briefly summarizes our findings and presents our conclusions.

2. THE MODEL AND MAIN RESULTS

We consider the focusing DNLS equation [2] on $\mathbb{Z}^d$, $d \geq 1$ of the form:

$$
i \partial_t u_n + \Delta_{disc} u(n) + |u_n|^{2\sigma} u_n = 0, u : \mathbb{Z}^d \to \mathbb{C}$$

where

$$\Delta_{disc} u(n) = \sum_{j \in \mathbb{Z}^d : |j-n|=1} u_j - 2d u_n.$$  

The equation is well-known to conserve the Hamiltonian

$$H = \sum_{j \in \mathbb{Z}^d : |j-n|=1} |u_j - u_n|^2 - \frac{1}{\sigma+1} \sum_{n \in \mathbb{Z}^d} |u_j|^{2\sigma+2}.$$  

Substituting the standard standing wave ansatz $u_n = e^{i\omega t} \varphi_n$, we obtain the following difference equation, posed on $\mathbb{Z}^d$

$$-\Delta_{disc} \varphi_n + \omega \varphi_n - |\varphi_n|^{2\sigma} \varphi_n = 0.$$  

Relevant quantities, which will be helpful in the sequel are the following

\[ P = \sum_{n \in \mathbb{Z}^{d}} |\psi_n|^2 \]
\[ V = \sum_{n \in \mathbb{Z}^{d}} |\psi_n|^{2\sigma+2} \]

It is also relevant to note in passing here that the squared \( l^2 \) norm is also a conserved quantity (mass) for the dynamics of Eq. (1). Our aim in what follows will be, in line with many of the above discussed works, to explore localized solutions of (2) and more specifically the fundamental discrete solitons with \( \psi_n \geq 0 \) that numerous earlier studies touched upon theoretically \([1, 26]\) and experimentally \([9]\). There is a number of (non-equivalent) ways in which one can introduce such objects, but a quite natural approach, that we adopt in this work, is to consider such waves as appropriate (multiples of) minimizers of appropriate variational problems. Even within this framework, there is a number of ways one can do this, which affects the stability of such waves significantly. We analyze herein two constrained variational problems, which will provide us each with a one-parameter family of solutions.

We start by introducing the notion of normalized waves. That is, for any fixed \( l^2 \) norm of the wave, we minimize the Hamiltonian \( H \), subject to this constraint. This approach has been worked out, in some detail, in the work of Weinsein \([1]\) - here we consider it again, as we need more specific properties regarding the stability of these waves. More precisely, for a fixed \( \lambda > 0 \), we solve the following variational problem:

\[ \begin{align*}
\min & \quad H[u] = \sum_{n \in \mathbb{Z}^{d}} \sum_{j \in \mathbb{Z}^{d} : |j-n|=1} |u_j - u_n|^2 - \frac{\sigma}{\sigma+1} \sum_{j \in \mathbb{Z}^{d}} |u_j|^{2\sigma+2} \\
\text{subject to} & \quad \|u\|_{l^2}^2 = \sum_{n \in \mathbb{Z}^{d}} |u_n|^2 = \lambda.
\end{align*} \]

Let us emphasize right away that we do not expect to be able to solve (3) for all values of \( d, \sigma \). In fact, the variational problem (3) turns out to be ill-posed, i.e., \( \inf_{\|u\|_{l^2}^2=\lambda} H[u] = -\infty \), when \( \sigma \geq \frac{2}{d} \), for all values of \( \lambda \) small enough. This is, in fact, illustrated in the original work of Weinsein \([1]\). More precisely, he shows that for any \( \sigma \geq \frac{2}{d} \), there is \( \lambda^* > 0 \), so that for all \( 0 < \lambda < \lambda^* \),

\[ \inf_{\|u\|_{l^2}^2=\lambda} H[u] = 0, \]

and as it turns out, no constrained minimizers for (3) exist in this case.

A different approach will be to construct the waves as (multiples of) constrained minimizers of a variational problem, with fixed potential energy. That is, letting \( \omega > 0 \) be a fixed parameter (compare with (2)), we solve

\[ \begin{align*}
\min & \quad J[u] := \sum_{n \in \mathbb{Z}^{d}} \sum_{j \in \mathbb{Z}^{d} : |j-n|=1} |u_j - u_n|^2 + \omega \sum_{n \in \mathbb{Z}^{d}} |u_j|^2 \\
\text{subject to} & \quad \|u\|_{l^{2\sigma+2}}^2 = \sum_{n \in \mathbb{Z}^{d}} |u_n|^{2\sigma+2} = 1.
\end{align*} \]

It is rather straightforward to see that the problem (4) is equivalent to the problem for the minimization of the following un-constrained, but homogeneous functional

\[ \bar{J}[u] = \inf_{u \neq 0} \frac{J[u]}{\left( \sum_{n \in \mathbb{Z}^{d}} |u_n|^{2\sigma+2} \right)^{\frac{1}{\sigma+1}}} \rightarrow \min. \]
It is worth noting that, as we shall establish later, the problem (4) is more flexible, in the sense that it allows a larger set of parameters, for which it produces non-trivial waves. Indeed, the requirement $\sigma < \frac{2}{d}$ is no longer necessary and the waves exist for all values of $0 < \sigma < \infty, \omega > 0$.

Next, we discuss the stability of these waves, as solutions to the DNLS (1). More specifically, for solution $\varphi$, consider a perturbation in the form $u_n(t) = e^{it\lambda}(\varphi_n + e^{i\theta}\varphi_n)$.

Plugging this in (1) and ignoring terms like $O(t^2)$, we obtain the linearized problem

$$i\lambda \varphi_n + \Delta_{disc} \varphi_n - \omega \varphi_n + \varphi_n^{2\sigma} \varphi_n + 2\sigma \varphi_n^{2\sigma} \varphi_n = 0.$$ 

Taking $v_n = (\Re \varphi_n, \Im \varphi_n)$, we obtain the following autonomous problem for the perturbation $(\Re \varphi_n, \Im \varphi_n)$,

$$(5) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix} \begin{pmatrix} \Re \varphi_n \\ \Im \varphi_n \end{pmatrix} = \lambda \begin{pmatrix} \Re \varphi_n \\ \Im \varphi_n \end{pmatrix},$$

where

$$\mathcal{L}_+ = -\Delta_{disc} + \omega - (2\sigma + 1)\varphi_n^{2\sigma},$$

$$\mathcal{L}_- = -\Delta_{disc} + \omega - \varphi_n^{2\sigma}.$$ 

Accordingly, we say that the wave $e^{it\lambda}\varphi_n$ is spectrally stable, if the eigenvalue problem (5) does not have a non-trivial solution $(\lambda, \varphi): \Re \lambda > 0, \varphi \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$. The notion of orbital stability concerns the solution to the original non-linear problem, and it is defined as follows - for every $\epsilon > 0$, there exists $\delta > 0$, so that whenever the initial data $u(0) : \|u(0) - \varphi\| < \delta$, then

$$\sup_{t > 0} \inf_{\theta \in \mathbb{R}} \sum_{n} |u_n(t) - e^{i\theta}\varphi_n|^2 < \epsilon.$$ 

Note that the Cauchy problem for the full non-linear evolution, (1) is not a concern in the discrete setting. More specifically, standard energy estimates show that the $l^2$ norm of the solutions of (1) is conserved, so that global solutions of (1) exist whenever the initial data $u(0) \in l^2(\mathbb{Z})$.

2.1. **Existence and stability for the normalized waves.** We start with the general existence results for $d$ dimensional lattices and then we state more precise results for the case $d = 1$.

Our first result is about the normalized waves.

**Theorem 1.** (Construction of the normalized waves for $d$ dimensional lattices)

Let $d \geq 1$ and $0 < \sigma < \frac{2}{d}$. For each $\lambda > 0$, the constrained variational problem (3) has a solution $u$. Any solution $u$ is with non-negative entries $u_j$. Moreover, for every $j_0 \in \{1, \ldots, d\}$ and fixed indices $k_1, \ldots, k_{j_0-1}, k_{j_0+1}, \ldots, k_d$, we have that there exist integers $l_0, l_0$ depending on $k_1, \ldots, k_{j_0-1}, k_{j_0+1}, \ldots, k_d$, so that

(6) \quad $u_{k_1, \ldots, k_{j_0-1}, l_0, k_{j_0+1}, \ldots, k_d} = \ldots = u_{k_1, \ldots, k_{j_0-1}, l_0 + 1, k_{j_0+1}, \ldots, k_d} > u_{k_1, \ldots, k_{j_0-1}, l_0, k_{j_0+1}, \ldots, k_d} \ldots$

(7) \quad $u_{k_1, \ldots, k_{j_0-1}, l_0, k_{j_0+1}, \ldots, k_d} > u_{k_1, \ldots, k_{j_0-1}, l_0 - 1, k_{j_0+1}, \ldots, k_d} \geq u_{k_1, \ldots, k_{j_0-1}, l_0 - 2, k_{j_0+1}, \ldots, k_d} \ldots$

In the case $d = 1$, and in addition to the (6) and (7), we claim that the minimizer is symmetric in the following sense

$$u_0 = u_1 = \ldots = u_{i_0} > u_{i_0+1} = u_{i_0+2} = u_{i_0+3} \ldots$$

Next, the minimizer $u$ satisfies the Euler-Lagrange equation

(8) \quad $-\Delta_{disc} u_n + c(\lambda) u_n - u_n^{2\sigma+1} = 0, n \in \mathbb{Z}^d$. 
for some Lagrange multiplier $c(\lambda) > 0$. Finally, the linearized discrete Schrödinger operator

$$\mathcal{L}_+ f = -\Delta_{\text{disc}} f_n + c(\lambda) f_n - (2\sigma + 1) u_n^{2\sigma} f_n$$

enjoys the property $\mathcal{L}_+ |_{u} \geq 0$. In particular,

**Remark:**

- As a straightforward consequence of the property $\mathcal{L}_+ |_{u} \geq 0$, we see that the Morse index $n(\mathcal{L}_+) \leq 1$. On the other hand by Euler-Lagrange, (8), we obtain
  $$\langle \mathcal{L}_+ u, u \rangle = -2\sigma \sum_{n \in \mathbb{Z}^d} u_n^{2\sigma + 2} < 0,$$

  it follows that $n(\mathcal{L}_+) \geq 1$, so $n(\mathcal{L}_+) = 1$. That is $\mathcal{L}_+$ has exactly one negative eigenvalue.

- It is possible, and this has in fact been done in [1], to construct the normalized waves in the $l^2$ supercritical regime $\sigma \geq \frac{2}{d}$, but only for large enough $\lambda$.

Our next result concerns the spectral stability of the normalized waves constructed in Theorem 1.

**Theorem 2.** Let $d \geq 1$, $0 < \sigma < \frac{2}{d}$ and $\lambda > 0$. Then all normalized waves $e^{itc(\lambda)} u_\lambda$ constructed in Theorem 1 satisfy $u \perp \text{Ker}(\mathcal{L}_+)$. As such the quantity $\mathcal{L}_+^{-1} u$ is uniquely defined in $\text{Ker}(\mathcal{L}_+)^\perp$. In addition, $\langle \mathcal{L}_+^{-1} u, u \rangle \leq 0$.

Assuming the non-degeneracy condition, $\langle \mathcal{L}_+^{-1} u, u \rangle \neq 0$, this implies $\langle \mathcal{L}_+^{-1} u, u \rangle < 0$, hence spectral stability for $e^{itc(\lambda)} u_\lambda$.

**Remarks:**

1. The condition $\langle \mathcal{L}_+^{-1} u, u \rangle \neq 0$ is equivalent to the lack of crossing of a pair of eigenvalues (either purely imaginary or a pair of positive and a negative one) through the origin. Crossing is not expected to happen.

2. With the same proof, it is easy to see that the normalized waves in the $l^2$ supercritical regime $\sigma \geq \frac{2}{d}$, $e^{itc(\lambda)} u_\lambda$, which exist only for large $\lambda > 0$, are also spectrally stable solutions of (1).

3. It is well-known [2] that the DNLS solutions come into onsite and offsite symmetric varieties, i.e., centered on or between adjacent sites. Per our numerical computations, It is only the former that satisfy the condition of $n(\mathcal{L}_+) = 1$ and hence correspond to the constrained minimizers of interest herein.

### 2.2. The existence and stability of the homogeneous waves

Our next results concern the constrained minimization problem (4). We state the existence result first.

**Theorem 3.** (Existence for minimizers of homogeneous functionals) Let $\omega > 0$. Then, the constrained minimization problem (4) has a solution $u_\omega$. The solution has non-negative entries only, which in addition obey (5), (7). The minimizer $u_\omega$ satisfies the Euler-Lagrange equation

$$-\Delta_{\text{disc}} u_n + \omega u_n - j(\omega) u_n^{2\sigma + 1} = 0,$$

where

$$j(\omega) = \inf_{\|u\|_{l^{2\sigma+2}} = 1} J[u] > 0.$$
Moreover, every minimizing sequence for (4) has a convergent subsequence, in the $l^2$ norm, to a constrained minimizer of (4).

The linearized Schrödinger operator

$$\mathcal{L}_+ f_n = -\Delta_{disc} f_n + \omega f_n - (2\sigma + 1) j(\omega) u_n^{2\sigma} f_n$$

has the property $n(\mathcal{L}_+) = 1$.

The minimizers constructed in Theorem 3 do not satisfy the required profile equation (2), but rather the Euler-Lagrange equation (9). It is clear however, that we can construct solutions of (2) starting from solutions $u$ as in Theorem 3. Indeed, setting $\varphi_n := \left( j(\omega) \right)^{1/\sigma} u_n$ produces solutions of (2).

Our next result concerns the stability of these waves. Unfortunately, we need to put forward some technical constructions before we state a rigorous result.

**Definition 1.** Let $\omega > 0$. We say that $u_\omega$ is a limit wave, if there exists a sequence $\delta_j \to 0$, so that $u_{\omega + \delta_j}$ are minimizers for (4) and $\lim_j \| u_{\omega + \delta_j} - u_\omega \|_{l^2} = 0$.

Since $\{u_{\omega + \delta_j}\}_j$ is a minimizing sequence for (4), and since by the results of Theorem 3 we can select a further $l^2$ convergent subsequence, it is clear that limit waves are constrained minimizers of (4). As such, they do satisfy the Euler-Lagrange equation (9). Starting with $u$, we can construct waves of the original DNLS via $\varphi_n := \left( j(\omega) \right)^{1/\sigma} u_n$. We call these also limit waves.

Our next result concerns the stability of these waves.

**Theorem 4.** Let $\omega > 0$ and assume $\varphi_\omega$ is a limit wave, in the sense of Definition 1. Assume also that $\varphi_\omega$ is non-degenerate, in the sense that $\text{Ker}(-\Delta_{disc} + \omega - (2\sigma + 1)\varphi^{2\sigma}) = \{0\}$.

Then, the solution $e^{i\omega t} \varphi_\omega$ is spectrally stable in the context of DNLS, if

$$\lim_{\delta_j \to 0} \frac{\| \varphi_{\omega + \delta_j} \|_{l^2}^2 - \| \varphi_\omega \|_{l^2}^2}{\delta_j} \geq 0.$$}

and it is unstable if

$$\lim_{\delta_j \to 0} \frac{\| \varphi_{\omega + \delta_j} \|_{l^2}^2 - \| \varphi_\omega \|_{l^2}^2}{\delta_j} < 0.$$

**Remarks:**

1. Under the assumptions made in Theorem 4, the limit

$$\lim_{\delta_j \to 0} \frac{\| \varphi_{\omega + \delta_j} \|_{l^2}^2 - \| \varphi_\omega \|_{l^2}^2}{\delta_j}$$

always exists.

2. The non-degeneracy property of the wave,

$$\text{Ker}(-\Delta_{disc} + \omega - (2\sigma + 1)\varphi^{2\sigma}) = \{0\},$$

is a technical, but useful statement, which is expected to hold, at least for a generic subset of the parameter $\omega$.

We have the following corollary.
Corollary 1. Assume that for an interval \((a, b) \subset \mathbb{R}_1^+,\) we have that \(\omega \to \phi_\omega\) is continuous in the \(l^2\) norm and all the waves \(\phi_\omega\) are non-degenerate. Then, the scalar function \(\omega \to \|\phi_\omega\|_{l^2}\) is differentiable, and the wave \(e^{i\omega t}\phi\) is stable if and only if

\[ \partial_\omega \|\phi_\omega\|_{l^2}^2 \geq 0. \]

Remark: This result is of course identical of the Grillakis-Shatah-Strauss stability condition for stability of waves in continuous models. This is, as far as we know, the first such rigorous result in the setting of discrete NLS.

Next, we express everything in terms of the function \(j(\omega)\).

Proposition 1. The function \(\omega \to j(\omega)\) is concave down. As such, it has a second derivative a.e. on \(0, \infty)\). Moreover, at such points of differentiability,

\[ \|\phi_\omega\|_{l^2}^2 = j'(\omega) j''(\omega). \]
Thus, the stability condition is exactly that $j^{1+\frac{1}{\sigma}}(\omega)$ is convex or
\begin{equation}
(12) \quad s(\omega) := j''(\omega) + \frac{(j'(\omega))^2}{\sigma j(\omega)} > 0.
\end{equation}

Figures 1-2 show the diagrams of $P$ and of $V$, as well as the Hamiltonian $H$ vs. $P$ for different values of $\sigma$. It is important to highlight that in line with the earlier works of, e.g., \cite{32, 33}, there exist values of $\sigma$ (e.g., for $1.34 < \sigma < 2$ for $d = 1$ \cite{33}) for which there is multi-stability and the graph of $P$ vs. $\omega$ is non-monotonic; see also Figure 1 for a case with $d = 2$ (bottom panels).

Figures 3-6 (left panels) show $s(\omega)$ compared to the quantity $\partial_\omega \|\varphi_\omega\|_{L^2}^2$ from Corollary 1. According to Proposition 1 and Corollary 1, both functions can be used as indicators of spectral stability. We observe their zero crossings are identical and correspond to stability switches at the power extrema displayed in Figure 1 (top panels).

It is relevant to note here that for ease of visualization in these figures, we multiply $s(\omega)$ by a constant pre-factor so that it is on the same order as $\partial_\omega \|\varphi_\omega\|_{L^2}^2$. Of course, this has no effect on the zero crossings and the overall sign which are central for our stability conclusions.

To corroborate the analytical results further, the middle and right panels of Figs. 3-6 illustrate stability switching via numerical computation of spectral stability eigenvalues in Eq. (5), for fixed $\omega$ in the regions between the zero crossings of $s(\omega)$. For $\sigma = 2, 2.5$, the zero crossings occur approximately at $\omega = 1, 1.2$, so we check the stability at $\omega = 0.5$ and at $\omega = 1.5$. Our results predict that in one of these cases the wave should be stable, while in the other it should be unstable. For $\sigma = 1.5$, $s(\omega)$ features two zero crossings, near $\omega = 0.4$ and $\omega = 0.8$. Hence it is also relevant to test a value to the left of $\omega = 0.4$, one between the two crossings and one to the right of the rightmost crossing; we choose $\omega = 0.1, 0.5$ and $1.5$, respectively. For consistency, we display the $\sigma = 2$ and $2.5$ cases for the same test values $\omega = 0.5, 1.5$, observing that in the latter too, a zero-crossing occurs. This is, respectively, at $\omega = 1$ and $\omega = 1.2$ for these two cases. In all the cases, our stability conclusions are in line with the standard VK criterion and the newly proposed criterion of Eq. (12).

Moreover, our numerical computations lead us to conjecture here the following conclusion: for the critical index $\sigma = \frac{d}{\sigma}$, the excitation threshold is achieved precisely at $\omega = 1$ and moreover $\varphi_\omega$, for $\omega > 1$ are stable. That is, $\min_{\omega > 0} \|\varphi_\omega\|_2 = \|\varphi_1\|_2$, $\partial_\omega \|\varphi_\omega\|_2^2 > 0$ for $\omega > 1$. The opposite monotonicity is true leading to instability for $\omega < 1$.

**Figure 3.** The functions $s(\omega)$ and $\partial_\omega \|\varphi_\omega\|_{L^2}^2$ vs. $\omega$ in one dimension, for $\sigma = 1$ (left). The middle and right panels show the spectral plane $(\lambda_r, \lambda_i)$ for eigenvalues $\lambda = \lambda_r + i\lambda_i$ of Eq. (5), for $\omega = 0.5$ (middle) and $\omega = 1.5$ (right). As expected, both waves are spectrally stable.
We consider the following spaces

\[ l^p = \{ \{x_n\}_{n=-\infty}^\infty : \|x\|_p = \left( \sum_{n=-\infty}^{\infty} |x_n|^p \right)^{1/p} < \infty \}. \]

In particular the spaces \( l^2 \) can be identified via the isometry map \( \mathcal{F} : l^2 \to L^2[0, 1] \) as follows

\[ x(\xi) = \mathcal{F}[\{x_n\}] := \sum_n x_n e^{2\pi i n \xi}, \quad x_n = \int_0^1 x(\xi) e^{-2\pi i n \xi} d\xi. \]

Note that \( \|x\|_{L^2[0,1]} = \|\{x_n\}\|_{l^2} \). Sometimes, we denote \( u = \{u_n\}_{n=-\infty}^\infty \) and \( u(\xi) := \sum_n u_n e^{2\pi i n \xi} \). We will often tacitly identify the sequence \( u \) and the function \( u(\xi) \). Note the basis vectors \( e_n \), which are defined via the Kronecker \( \delta \)'s: 

\[ e_n(m) = \begin{cases} 
1 & n = m \\
0 & n \neq m 
\end{cases}. \]
3.1. **The discrete Laplacian.** The one dimensional discrete Laplacian is given explicitly by
\[
(\Delta_{disc.} u)_n = u_{n+1} - 2u_n + 2u_{n-1}.
\]
In fact, \(\Delta_{disc.}\) has a nice representation in terms of the shift operators. Indeed, let \(S : l^2 \to l^2\), be defined by \(S[u](n) = u_{n+1}\), while its inverse/adjoint \(S^* = S^{-1}\), is given by \(S^{-1}[u](n) = u_{n-1}\).

In such a case, we can represent \(\Delta_{disc.} = S + S^{-1} - 2Id\). On the level of \(L^2[0,1]\) functions, we have
\[
u_{n+1} - 2u_n + 2u_{n-1} = \int_0^1 u(\xi)|e^{-2\pi i(n+1)\xi} + e^{-2\pi i(n-1)\xi} - 2e^{-2\pi in\xi}|d\xi =
\]
\[
-4\int_0^1 u(\xi)\sin^2(\pi\xi)e^{-2\pi in\xi}d\xi.
\]
That is, on the set of \(L^2[0,1]\) functions, the discrete Laplacian can be realized as the Fourier multiplier \(-4\sin^2(\pi\xi)\).

More generally, for reasonable functions \(f(-\Delta_{disc.})\) acts as follows \(f(-\Delta_{disc.})u\) corresponds to a function \(u(\xi)f(4\sin^2(\pi\xi))\). For example, one can easily see (summation by parts) that
\[
\langle -\Delta_{disc.}u, u \rangle = \sum_n (2u_n - u_{n+1} - u_{n-1})\bar{u}_n = \sum_n |u_{n+1} - u_n|^2 \geq 0,
\]
whence \(-\Delta_{disc.}\) is a positive operator. Equivalently, one could have computed this on the level of functions \(u(\xi)\) as follows
\[
\langle -\Delta_{disc.}u, u \rangle = 4\int_0^1 |u(\xi)|^2 \sin^2(\pi\xi)d\xi.
\]
In particular, since \(\int_0^1 |u(\xi)|^2 \sin^2(\pi\xi)d\xi \leq \|u\|^2\), we conclude that
\[
\langle -\Delta_{disc.}u, u \rangle \leq 4\|u\|^2,
\]
whence \(0 < -\Delta_{disc.} \leq 4\) is a bounded operator, with norm at most \(4\). In higher dimensions, \(d \geq 1\), one might reach similar conclusions about \(\Delta_{disc.}\), for example \(0 < -\Delta_{disc.} \leq 4d\).

3.2. **The heat semigroup \(e^{t\Delta_{disc.}}\) and the Perron-Frobenius property.** We introduce the semigroup generated by \(\Delta_{disc.}\) via the heat equation on \(l^2\), namely
\[
\begin{aligned}
\frac{\partial}{\partial t} u_n(t) &= u_{n+1}(t) - 2u_n(t) + u_{n-1}(t) = (\Delta_{disc.} u)_n, \\
u_n(0) &= u_n
\end{aligned}
\]
Equivalently, the solution is given by \(u(t)\), given by \(u(0,\xi)e^{-4ts\sin^2(\pi\xi)}\). More explicitly
\[
u_n(t) = \sum_m u_m \int_0^1 e^{-4ts\sin^2(\pi\xi)}e^{-2\pi i(n-m)\xi}d\xi.
\]
Introduce the coefficients \(K_n(t) = \int_0^1 e^{-4ts\sin^2(\pi\xi)}e^{-2\pi in\xi}d\xi\) of the heat semigroup. In other words,
\[
u(t) = e^{t\Delta_{disc.}u}(0) = \sum_m K_{n-m}(t)u_m(0) = K(t) \ast u(0).
\]

**Proposition 2.** For every \(t > 0\), the sequence \(\{K_n(t)\}\) is even, positive and bell-shaped. That is, for every \(n \geq 1\), \(K_n(t) = K_{-n}(t) > 0\). For every \(t \in (0,1)\), \(K\) is bell-shaped as well, that is \(K_0(t) > K_1(t) > \ldots\).\footnote{one can easily see that the norm is actually exactly four, by taking test functions \(u\) supported near \(\xi \sim \frac{1}{2}\)}
Theorem 5 in the Appendix), we show that 

\[ \sigma \]

for a self-adjoint operator

\[ \text{on it.} \]

It is well-known, (Theorem XIII.44, p. 204, [34]) that the Perron-Frobenius property

ways equivalent to) the positivity improving property of the associated semigroup

and so we refer to it

In the continuous case, it is well-known that such an eigenvalue is simple, with pointwise

For this step, we use the following explicit formula, which is obtained by the Taylor expansion

}\]

Clearly, at least for \( t \in (0, 1) \), \( j \geq 0 \), we have \( K_j(t) > K_{j+1}(t) \).

We now turn our attention to a general (linear) Schrödinger operator in the form

\[ (\mathcal{L}f)_n = -\Delta_{disc} f_n - V_n f_n = \sum_{j \in \mathbb{Z}^d : |j-n| = 1} (f_j - f_n) - V_n f_n, \]

for bounded potentials \( \mathbb{V} = \{V_n\}_{n \in \mathbb{Z}^d} \). From the general form of Weyl's criteria (see the proof of Theorem 5 in the Appendix), we show that \( \sigma_{sp}(\mathcal{L}) = \sigma_{sp}(-\Delta_{disc}) = \sigma(-\Delta_{disc}) = [0, 4d] \). Our interest is in the lowest eigenvalue of \( \mathcal{L} \), if such an object exists. Assume that it does - by the min-max principle, we assume that

\[ \lambda_0(\mathcal{L}) \]

In the continuous case, it is well-known that such an eigenvalue is simple, with pointwise positive eigenfunction. This is known as the Perron-Frobenius theorem and so we refer to it as the Perron-Frobenius property. We have the same result for discrete Schrödinger operators, which to the best of our knowledge appears to be a new result.

**Theorem 5.** (Discrete Schrödinger operators are Perron-Frobenius)

Let \( d \geq 1 \) and \( \mathbb{V} \in \ell^\infty(\mathbb{Z}^d) \). Assume that \( \inf_{\|\mathbf{f}\|_1} \langle \mathcal{L}\mathbf{f}, \mathbf{f} \rangle < 0 \). Then, \( \lambda_0(\mathcal{L}) = \inf_{\|\mathbf{f}\|_1 = 1} \langle \mathcal{L}\mathbf{f}, \mathbf{f} \rangle \) is a simple eigenvalue for \( \mathcal{L} \), with an eigenfunction (ground state) \( \mathbf{g} : g_n \geq 0 \). In particular, there exists \( \delta > 0 \), so that \( \mathcal{L}_n^\infty \geq \lambda_0(\mathcal{L}) + \delta \).

We provide the proof in the Appendix, as it is a bit technical and the methods are somewhat different from the scope of this paper. Regardless, we would like to provide some comments on it. It is well-known, (Theorem XIII.44, p. 204, [34]) that the Perron-Frobenius property for a self-adjoint operator \( \mathcal{H} \), which is bounded from below, follows from (and it is in many ways equivalent to) the positivity improving property of the associated semigroup \( e^{t\mathcal{H}} \).
is worth observing that for the standard continuous Schrödinger operators \(-\Delta + V\), the positivity improving for the semigroup \(e^{t(\Delta + V)}\) (and hence of the Perron-Frobenius) is a direct consequence of the Feynman-Kac formula, which presents the solution to \(u_t = \mathcal{H}v\) as an expectation, against a positive kernel of the initial data. In fact, this can be extended to fractional Schrödinger operators \(\mathcal{H} = (-\Delta)^s + V, 0 < s < 1\), via a similar representation formula, in terms of Lévy processes (instead of Brownian motion). The same representation formulas, while in principle possible, seem unavailable at the moment, hence our direct proof in the Appendix.

3.3. Discrete Szegö inequality. We start with an intuitively clear combinatorial lemma, which is somewhat cumbersome to write down.

**Lemma 1.** Let \(a_0 \geq a_1 \geq \ldots \geq a_N \geq 0 = a_{N+1}\), be a sequence of non-negative numbers and \(\mu : \{0, \ldots, N + 1\} \to \{0, \ldots, N + 1\}\) be a permutation. Then, for every \(p > 1\),

\[
    (16) \quad \sum_{j=0}^{N} |a_{\mu(j+1)} - a_{\mu(j)}|^p \geq \sum_{j=0}^{N} |a_{j+1} - a_j|^p.
\]

Assuming in addition that the sequence \(|a_j|^\infty_{j=0}\) is strictly decreasing, we have that equality in (16) occurs if and only if \(\mu = \text{id}\). That is, any non-trivial permutation \(\mu\) leads to strict inequality in (16).

**Proof.** Introduce \(\varepsilon_j = a_j - a_{j+1} \geq 0, j = 0, \ldots, N\), so that \(a_j = \varepsilon_j + \ldots + \varepsilon_N\). We need to show

\[
    (17) \quad \sum_{j=0}^{N} |\varepsilon_{\min(\mu(j+1),\mu(j))} + \ldots + \varepsilon_{\max(\mu(j+1),\mu(j))}|^p \geq \sum_{j=0}^{N} |\varepsilon_j|^p.
\]

Since

\[
|\varepsilon_{\min(\mu(j+1),\mu(j))} + \ldots + \varepsilon_{\max(\mu(j+1),\mu(j))}|^p \geq \varepsilon_{\min(\mu(j+1),\mu(j))}^p + \ldots + \varepsilon_{\max(\mu(j+1),\mu(j))}^p,
\]

matters clearly reduce to showing the following statement: for every \(k \in \{0, \ldots, N\}\), there exists \(j = j_k \in \{0, \ldots, N + 1\}\), so that \(k \in \{\min(\mu(j+1),\mu(j))\}, \max(\mu(j+1),\mu(j))\} - 1\), so we are done in this case. Similarly, if \(k \in B\), so \(\mu(k) \geq k + 1\), choose \(\mu(j) \in A\), so that \(\mu(j) \leq k\). Again, for some \(j_0 \in \{\min(\mu(j+1),\max(\mu(j))\}, k \in \{\min(\mu(j_0 - 1),\mu(j_0)), \max(\mu(j_0 - 1),\mu(j_0))\} - 1\), so we are done again.

Regarding the equality in (16), assuming strictly decreasing sequence (and hence \(\varepsilon_j > 0, j = 0, \ldots, N\)), we saw that each \(\varepsilon_j\) appears in the sum (17). We also saw in the analysis, that if \(\max(\mu(j+1),\mu(j)) - \min(\mu(j+1),\mu(j)) \geq 2\), for any \(j\), then at least one of \(\varepsilon_j^p, j = 0, \ldots, N\) will appear at least twice leading to strict inequality. Thus, equality in (16) is possible only if \(\max(\mu(j+1),\mu(j)) - \min(\mu(j+1),\mu(j)) = 1\). This is of course possible, only for \(\mu = \text{id}\). Conversely, there is clearly equality in (16) for \(\mu = \text{id}\), so equality in (16) occurs if and only if \(\mu = \text{id}\). \qed
To set up the function spaces, assume \( \{f_n\}_{n \in \mathbb{Z}} \in L^p(\mathbb{Z}), 1 < p < \infty \). Our next task is to study the minimal possible value of expressions of the form

\[
T(f) = \sum_{n=-\infty}^{\infty} |f_{n+1} - f_n|^p,
\]

where we allow ourselves only to permute and translate the sequence \( \{f_n\} \). The reverse triangle inequality \( |f_{n+1} - f_n| \geq |f_{n+1} - f_n| \), which is strict if \( f_n f_{n+1} < 0 \) shows that the quantity \( T \) is minimized on non-negative sequences. In addition, it is clear that to minimize \( T(f) \) effectively, we need to discard redundancies.

More precisely, we say that that two sequences \( \{f_n\} \) and \( \{h_n\} \) are one step equivalent, only if \( f = (\ldots, f_{j-1}, f_j, f_{j+1}, \ldots), f_j = f_{j-1} \neq 0 \) and \( h = (\ldots, f_{j-1}, f_{j+1}, \ldots) \). That is, \( f, h \) are one step equivalent, if one is obtained from the other by erasing one copy of a non-zero element that repeats itself. Note that the erasure operation only allows one to erase a repeating element, which is equal to an immediate neighbor. We say that \( f, h \), are equivalent, if one is obtained from the other by the erasure operation, possibly countably many times. Clearly, for each sequence \( f \in L^p \), there is a uniquely determined equivalent element, which has all different values - that is \( f_j \neq f_k \) for each \( j \neq k \).

**Proposition 3.** (Discrete Szegő inequality)

Let \( \{f_n\}_{n \in \mathbb{Z}} \in L^p(\mathbb{Z}), 1 < p < \infty \). Then, there exists an element \( \tilde{f} \in L^p \), so that

1. \( \tilde{f} \) is a permutation and translation of \( |\{f_n\}| \).
2. \( \tilde{f}(0) = \max_{n \in \mathbb{Z}} |\tilde{f}(n)| \).
3. \( \tilde{f} : \mathbb{N}_+ \to \mathbb{R}^1_+ \) is non-increasing, while \( \tilde{f} : \mathbb{N}_- \to \mathbb{R}^1_- \) is non-decreasing.

for which, we have

\[
\sum_{n=-\infty}^{\infty} |f_{n+1} - f_n|^p \geq \sum_{n=-\infty}^{\infty} |\tilde{f}_{n+1} - \tilde{f}_n|^p,
\]

Equality in (18) holds if and only if \( f_n \) does not change sign and for some \( j_0 \in \mathbb{Z} \),

\[
|f_{j_0}| \geq |f_{j_0-1}| \geq \ldots |f_{j_0-n}| \ldots; \quad |f_{j_0}| \geq |f_{j_0+1}| \geq |f_{j_0+2}| \geq \ldots |f_{j_0+n}| \geq \ldots
\]

**Proof.** As we have argued above,

\[
\sum_{n=-\infty}^{\infty} |f_{n+1} - f_n|^p \geq \sum_{n=-\infty}^{\infty} |f_{n+1} - f_n|^p,
\]

with strict inequality, if there is a change of sign \( f_{n+1} f_n < 0 \). So, we might reduce to the case \( f_n \geq 0 \). Also, since \( f \in L^p \), there is a maximal element, which we assume, by translational invariance to be at \( n = 0 \). That is, \( f_n(0) = \max_{n \in \mathbb{Z}} f(n) \). Next, it suffices to consider such positive sequences with compact support, say on \([0, N]\), and \( f_n(0) = \max_{n \in \mathbb{Z}} f(n) \), for which we need to show

\[
\sum_{n=0}^{N} |f_n - f_{n+1}|^p \geq \sum_{n=0}^{N} |f_n^* - f_{n+1}^*|^p.
\]

where \( \{f_n^*\} \) is the decreasing rearrangement of \( f_n, 0 \leq n \leq N \). For the proof of (19), we simply refer to Lemma [1]. Indeed, we start with \( a_n = f_n^*, n = 0, \ldots, N, a_{N+1} = 0 \) and a permutation \( \mu : \{0, \ldots, N+1\} \to \{0, \ldots, N+1\} \), defined so that \( f_n = f_{\mu(n)}^* \).

---

*We say that \( \{g_n\} \) is a translate of \( \{f_n\} \) if there exists \( k_0 \), so that \( g_n = f_{n+k_0} \) for all \( n \in \mathbb{Z} \).*
Based on (19), we can also prove

\[
\sum_{n=-N}^{-1} |f_n - f_{n+1}|^p \geq \sum_{n=-N}^{-1} |f_n^* - f_{n+1}^*|^p.
\]

After combining the two and taking the limit \(N \to \infty\), we obtain (18), where \(\tilde{f}\) is the decreasing rearrangement of \(f\).

The equality occurs, if it occurs for both (19) and (20). Applying the criteria for equality in Lemma 1 implies that one must have, say for the case \(f_n \geq 0\) and \(f_0 = \max f(n)\), that \(f_0 > f_1 \ldots f_n \ldots\) and \(f_0 > f_{-1} \ldots\). Clearly, in such a case, equality in (18) holds true.

\[\square\]

4. Construction of the normalized waves

We start with some basic properties of the variational problem (3).

4.1. Properties of the variational problem (3). The following estimate is instrumental to the well-posedness of the variational problem (3). In fact, this is a version of a discrete Sobolev embedding.

**Lemma 2.** Let \(d \geq 1\) and \(2 < p \leq p_d = \begin{cases} +\infty & d = 1, 2 \\ \frac{2d}{d-2} & d \geq 3 \end{cases}\).

Then, for every \(0 < \theta < \min(1, d(\frac{1}{2} - \frac{1}{p}))\), there exists \(C_{\theta, p}\), so that for every sequence \(u \in l^2(\mathbb{Z}^d)\)

\[
\left( \sum_{n \in \mathbb{Z}^d} |u_n|^p \right)^{1/p} \leq C_{\epsilon} \left( \sum_{j \in \mathbb{Z}^d : |j - n| = 1} |u_j - u_n|^2 \right)^{\frac{\theta}{2}} \left( \sum_{n \in \mathbb{Z}^d} |u_n|^2 \right)^{\frac{1}{2} - \frac{\theta}{2}}
\]

For \(p > p_d\), (21) holds with \(\theta = 1\).

**Remark:** Note that since \(\sum_{j \in \mathbb{Z}^d : |j - n| = 1} |u_j - u_n|^2 \leq C_d \sum_n |u_n|^2\), the estimates with higher \(\theta\) imply the ones with smaller \(\theta\).

**Proof.** Per our earlier computations, see Section 3.1 we have that

\[
\sum_{j \in \mathbb{Z}^d : |j - n| = 1} |u_j - u_n|^2 = \langle -\Delta_{\text{disc}} u, u \rangle = 4 \int_{[0,1]^d} |u(\xi)|^2 \left( \sum_{m=1}^d \sin^2(\pi \xi_m) \right) d\xi
\]

Denoting the symbol \(\Delta(\xi) := \sum_{m=1}^d \sin^2(\pi \xi_m)\), we see that (21) is better expressed in the following form

\[
\left( \sum_{n \in \mathbb{Z}^d} |u_n|^p \right)^{1/p} \leq C_{\epsilon} \left( \int_{[0,1]^d} |u(\xi)|^2 \Delta(\xi) d\xi \right)^{\frac{\theta}{2}} \left( \int_{[0,1]^d} |u(\xi)|^2 d\xi \right)^{\frac{1}{2} - \frac{\theta}{2}},
\]

which we now prove. Starting with the Hausdorff-Young’s inequality, we obtain

\[
\left( \sum_{n \in \mathbb{Z}^d} |u_n|^p \right)^{1/p} \leq C\|u\|_{L^p([0,1]^d)}.
\]
Next, observe that since
\[ \Delta(\xi) \geq \kappa \min(\{\xi_1^2, \sum_{m=1}^{d} |\xi_m - 1|^2\}), \]
we have that
\[ \int_{[0,1]^d} \frac{1}{\Delta(\xi)^{\gamma}} d\xi < C_\gamma, \]
for all \( \gamma < \frac{d}{2} \). Thus, we can estimate by Hölder’s as follows
\[
\|u\|_{L^p([0,1]^d)} \leq \left( \int_{[0,1]^d} |u(\xi)|^2 \Delta(\xi)^{\frac{2a}{p}} d\xi \right)^{\frac{1}{2}} \left( \int_{[0,1]^d} |u(\xi)|^2 d\xi \right)^{\frac{1}{2} - \frac{a}{p}},
\]
where \( q : \frac{1}{q} + \frac{p'}{2} = 1, \alpha > 0 \). Note that we select \( \alpha : \alpha q < \frac{d}{2} \), so that the second integral is finite. If we can select \( \alpha : \frac{2a}{p} \geq 1 \) (noting that \( \Delta(\xi) < d \)), then we have shown \([\text{21}]\) with \( \theta = 1 \). This happens in the case \( p > p_d \).

In the case \( p : 2 < p \leq p_d \), the inequality \( \alpha < \frac{d}{2q} \) implies \( \frac{2a}{p} < 1 \). Then, again by Hölder’s,
\[
\left( \int_{[0,1]^d} |u(\xi)|^2 \Delta(\xi)^{\frac{2a}{p}} d\xi \right)^{\frac{1}{2}} \leq \left( \int_{[0,1]^d} |u(\xi)|^2 \Delta(\xi) d\xi \right)^{\frac{1}{2}} \left( \int_{[0,1]^d} |u(\xi)|^2 d\xi \right)^{\frac{1}{2} - \frac{\alpha}{p}}.
\]
Putting all the inequalities together yields for \( \theta = \frac{2a}{p'} < 1 \),
\[
(\sum_{n \in \mathbb{Z}^d} |u_n|^p)^{\frac{1}{p}} \leq C_\theta \left( \int_{[0,1]^d} |u(\xi)|^2 \Delta(\xi) d\xi \right)^{\frac{\theta}{q}} \left( \int_{[0,1]^d} |u(\xi)|^2 d\xi \right)^{\frac{1}{2} - \frac{\theta}{2}}.
\]
As \( \alpha \) can assume all values \( \alpha < \frac{d}{2q} = \frac{d}{2}(1 - \frac{p'}{2}) \), then \( \theta \) can be taken in \([\text{23}]\) to be \( \theta = d(\frac{1}{2} - \frac{1}{p}) \). □

Based on Lemma\([\text{2}]\) it can be shown that the variational problem \([\text{3}]\) has solutions, for all values of \( \lambda > 0 \), if \( \sigma < \frac{2}{d} \). On the other hand, it can be shown that for \( \sigma \geq \frac{2}{d} \), solutions exist only for large enough values of \( \lambda > 0 \). This has been shown by Weinstein, see Propositions 4.1 and 4.2, \([\text{1}]\). His proof is based on the availability of the estimate \([\text{21}]\), in the case \( p = 2\sigma + 2 \), with \( \theta : (2\sigma + 2) \frac{1 - \theta}{q} < \sigma \), see the question (4.1) in \([\text{1}]\). This amounts to exactly \( \sigma < \frac{2}{d} \).

In addition, Weinstein, \([\text{1}]\) has shown that for \( \sigma > \frac{2}{d} \), the problem \([\text{3}]\) is not well-posed, in the sense that for small enough \( \lambda > 0 \), \( \inf_{\sum_n |u_n|^2 = \lambda} H[u] = -\infty \).

Our next lemma concerns \( h(\lambda) \).

**Lemma 3.** Let \( d \geq 1, \sigma < \frac{2}{d} \) and \( \lambda > 0 \). Then, \( h(\lambda) := \inf_{\sum_n |u_n|^2 = \lambda} H[u] < 0 \).

**Remark:** This has already been shown by Weinstein, \([\text{1}]\), but we offer alternative proof.

**Proof.** We need to produce an example, for which \( \sum_n |u_n|^2 = \lambda \), but \( H[u] < 0 \). For large \( N >> 1 \), consider
\[
\begin{align*}
 u_{k_1, \ldots, k_d} := \begin{cases}
 c_{d, \lambda, N} \left[ \frac{1}{N^\frac{d}{2}} - \frac{|k_1| + \cdots + |k_d|}{N^{1 + \frac{d}{2}}} \right] & |k_1| + |k_2| + \cdots + |k_d| \leq N - 1 \\
 0 & |k_1| + |k_2| + \cdots + |k_d| \geq N.
\end{cases}
\end{align*}
\]
Figure 7. The functions $c(\lambda)$ and $h(\lambda)$ in dimension $d = 1$.

where $c_{d,\lambda,N}$ so that $\|u\|_{L^2}^2 = \lambda$. Note that for $|u_{k_1,...,k_d}| \leq \text{const.}N^{-d/2}$, while $|u_{k_1,...,k_d}| \sim N^{-d/2}$ for $|k_1| + ... + |k_d| < N/2$, we have that

$$\sum_{k_1,...,k_d} |u_{k_1,...,k_d}|^2 \sim c_{d,\lambda,N}^2,$$

whence $c_{d,\lambda,N} \sim c_{d,\lambda}$. That is, it is bounded above and below by constants independent of $N$.

Next, for $|k_1| + |k_2| + ... + |k_d| \leq N - 2$, we have that for all $j : |j - k| = 1$, $|u_k - u_j| = c_{d,\lambda,N}N^{-1-d/2}$, while for the boundary $|k_1| + |k_2| + ... + |k_d| = N - 1$ and $|k_1| + |k_2| + ... + |k_d| = N$, we also have that $|u_k - u_j| \leq c_{d,\lambda,N}N^{-1-d/2}$ for all $j : |j - k| = 1$. Thus,

$$\sum_{j \in \mathbb{Z}^d : |j - n| = 1} |u_j - u_n|^2 \leq c_{d,\lambda,N}N^dN^{-2-d} \leq c_{d,\lambda,N}N^{-2}$$

Also, it is clear

$$\sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma + 2} \sim N^dN^{-\frac{d}{2}(2\sigma + 2)} \sim N^{-d\sigma}.$$

All in all, we conclude

$$H[u] \leq C_{\lambda,d}N^{-2} - \tilde{C}_{\lambda,d}N^{-d\sigma}$$

Clearly, if $\sigma < \frac{2}{d}$, $H[u]$ can be made negative, if $N = N_{\lambda,d}$ is selected large enough.

The next lemma shows some further useful properties of $h$, in particular the sublinearity of $h$, the concavity of $h$ and the Lipschitz properties of $h$.

**Lemma 4.** The function $h : \mathbb{R}_+^1 \to \mathbb{R}^1$ is locally Lipschitz continuous and concave.

In addition, it is sub-linear. That is for $\lambda > 0$ and $\alpha \in (0, \lambda)$, we have

$$h(\lambda) < h(\alpha) + h(\lambda - \alpha).$$

**Proof.** For the Lipschitz property, we write

$$h(\lambda) = \lambda \inf_{\|u\| = 1} \left[ \sum_{j \in \mathbb{Z}^d : |j - n| = 1} |u_j - u_n|^2 - \frac{\lambda^\sigma}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma + 2} \right] =: \lambda \inf_{\|u\| = 1} \tilde{H}_\lambda[u].$$
It clearly suffices to show that \( \tilde{h} := \inf_{|u|=1} \tilde{H}_\lambda[u] \) is Lipschitz. We have

\[
|\tilde{H}_{\lambda+\delta}[u] - \tilde{H}_\lambda[u]| \leq \frac{1}{\sigma + 1} ((\lambda + \delta)\sigma - \lambda\sigma) \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \leq C_\lambda \delta,
\]

for all \( u : \|u\|_2 = 1 \). It follows that for all \( u : \|u\|_2 = 1 \), there is

\[
\tilde{H}_\lambda[u] - C_\lambda \delta \leq \tilde{H}_{\lambda+\delta}[u] \leq \tilde{H}_\lambda[u] + C_\lambda \delta
\]

Taking \( \inf_{\|u\|_2 = 1} \) implies \( \tilde{h}(\lambda + \delta) - \tilde{h}(\lambda) \leq C_\lambda \delta \), whence the Lipschitz property of \( h \) follows.

For the concavity, recall the formula

\[
h(\lambda) = \inf_{\sum_n |u_n|^2 = 1} \left[ \lambda \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j - u_n|^2 - \frac{\lambda^{\sigma+1}}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \right].
\]

For \( a \in (0, 1), \lambda_1, \lambda_2 > 0 \), we have due to the convexity of the function \( \lambda \rightarrow \lambda^{\sigma+1} \),

\[
(a\lambda_1 + (1-a)\lambda_2) \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j - u_n|^2 - \frac{(a\lambda_1 + (1-a)\lambda_2)^{\sigma+1}}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \geq
\]

\[
\geq a \left[ \lambda_1 \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j - u_n|^2 - \frac{\lambda_1^{\sigma+1}}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \right] +
\]

\[
+ (1-a) \left[ \lambda_2 \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j - u_n|^2 - \frac{\lambda_2^{\sigma+1}}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \right].
\]

Taking infimum on \( u : \|u\|_2 = 1 \), we obtain the inequality

\[
h(a\lambda_1 + (1-a)\lambda_2) \geq ah(\lambda_1) + (1-a)h(\lambda_2),
\]

which is the desired concavity.

Finally, regarding the sub-additivity, observe that

\[
h(\lambda) = \inf_{\sum_n |u_n|^2 = \lambda} \left[ \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j - u_n|^2 - \frac{1}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \right] =
\]

\[
= \frac{\lambda}{\alpha} \inf_{\sum_n |u_n|^2 = \alpha} \left[ \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j - u_n|^2 - \frac{(\lambda/\alpha)^\sigma}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} |u_n|^{2\sigma+2} \right] < \frac{\lambda}{\alpha} h(\alpha),
\]

where the last inequality is strict, because \( \alpha < \lambda \) and there is a minimizing sequence \( u^m \) for (3), with the property \( \lim \sup_m \|u^m\|_2^{\sigma+2} > 0 \). Indeed, otherwise, if there is a minimizing sequence \( u^m : \lim_m \|u^m\|_2^{\sigma+2} = 0 \), then, we would have had

\[
h(\lambda) = \lim_m \sum_{j \in \mathbb{Z}^d : |j-n| = 1} |u_j^m - u_n^m|^2 \geq 0,
\]

in contradiction with Lemma 3.
Now that we have established $h(\lambda) < \frac{\lambda}{\alpha} h(\alpha)$, note that this means that $\lambda \to \frac{h(\lambda)}{\lambda}$ is strictly decreasing.\(^3\) Assume that $\alpha \in (\frac{1}{2}, \lambda)$ (and otherwise, work with $\lambda - \alpha$ instead of $\alpha$). We get

$$h(\lambda) < \frac{\lambda}{\alpha} h(\alpha) = h(\alpha) + \frac{\lambda - \alpha}{\alpha} h(\alpha) \leq h(\alpha) + h(\lambda - \alpha),$$

where in the last inequality, we have used that since $\lambda - \alpha \leq \frac{1}{2} \leq \alpha$, one has $\frac{h(\alpha)}{\alpha} \leq \frac{h(\lambda - \alpha)}{\lambda - \alpha}$. \(\square\)

We are now ready for the existence result.

4.2. **Conclusion of the proof of Theorem**\(^1\) We show that if $d \geq 1$, $0 < \sigma < \frac{2}{d}$ and $\lambda > 0$, then the variational problem \(^3\) has a solution as described in Theorem\(^1\).

The proof consists of applying the method of compensated compactness. Indeed, take a minimizing sequence $\{u^k\}$ for \(^3\). That is, $\|u^k\|_{L^2}^2 = \lambda$ and $H[u^k] \to h(\lambda)$. For every sequence with the property, $\|u^k\|_{L^2}^2 = \lambda$, standard arguments show that one of three alternatives takes place:

1. **(Tightness)** There exists a sequence $n_k \in \mathbb{Z}^d$, so that for any $\varepsilon > 0$, there exists $R = R(\varepsilon)$, so that
   $$\sum_{n \in \mathbb{Z}^d : |n - n_k| < R} |u^k_n|^2 > \lambda - \varepsilon.$$  

2. **(Vanishing)** For every $R > 0$,
   $$\lim_{k \to \infty} \sup_{y \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d : |n - y| < R} |u^k_n|^2 = 0.$$

3. **(Splitting)** There exists $\alpha \in (0, \lambda)$, so that for any $\varepsilon > 0$, there is $R$ and $R_k \to \infty$ and $n_k$, so that
   $$\left| \sum_{n \in \mathbb{Z}^d : |n - n_k| < R} |u^k_n|^2 - \alpha \right| \leq \varepsilon, \quad \sum_{n \in \mathbb{Z}^d : |n - n_k| > R_k} |u^k_n|^2 - (\lambda - \alpha) \leq \varepsilon$$

We will show that the minimizing sequence introduced above, cannot be anything but tight. Indeed, assume vanishing. In particular, this means that $\lim_k \|u^k\|_{L^\infty} = 0$. Then,

$$\sum_n |u^k_n|^{2\sigma + 2} \leq \|u^k\|_{L^\infty}^{2\sigma} \|u^k\|_{L^2}^2,$$

whence $\lim_k \|u^k\|_{L^{2\sigma + 2}} = 0$. Then,

$$h(\lambda) = \lim_k H[u^k] = \lim_k \sum_{j \in \mathbb{Z}^d : |j - n| = 1} |u^k_j - u^k_n|^2 \geq 0,$$

a contradiction with Lemma\(^3\)

Next, assume splitting. Denote

$$v^k = u^k \chi_{|n - n_k| < R}, w^k = u^k \chi_{|n - n_k| > R_k}, z^k = u^k \chi_{|n - n_k| > R_k}$$

\(^3\)This conclusion can also be easily reached for differentiable functions $h$, which are concave, as was established earlier. Unfortunately, the differentiability of $h$ is only valid a.e., and we prefer to give the direct argument instead.
Clearly, we have that \( \|v^k\|_{P^2}^2 \in (\alpha - \epsilon, \alpha + \epsilon), \|w^k\|_{P^2}^2 \in (\lambda - \alpha - \epsilon, \lambda - \alpha + \epsilon), \) while \( \|z^k\|_{P^2}^2 \leq \epsilon. \) Also, by the definition of the functional \( H, \) there exists an absolute constant \( C > 0, \) so that

\[
H[u^k] = H[v^k] + H[w^k] + O(\epsilon) \geq h(\alpha + \epsilon) + h(\lambda - \alpha + \epsilon) - C\epsilon.
\]

Taking limits in \( k \to \infty \) and using the fact that \( \lambda \to h(\lambda) \) is decreasing, we obtain the inequality

\[
h(\lambda) \geq h(\alpha + \epsilon) + h(\lambda - \alpha + \epsilon) - C\epsilon,
\]

which is true for arbitrary \( \epsilon > 0. \) By the continuity of \( h, \) it follows that \( h(\lambda) \geq h(\alpha) + h(\lambda - \alpha), \) which is in contradiction with Lemma 4. Thus, it remains that \( u^k \) is tight, whence, it has a convergent subsequence, which is clearly a solution to (3). The inequalities (6), (7) follow by an application of the discrete Szegö inequality Proposition 3 for each fixed \((k_1, \ldots, k_{j_0-1}, k_{j_0+1}, \ldots, k_d).\)

The Euler-Lagrange equation (8) is derived in the usual fashion. Namely, starting with a constrained minimizer \( u^0 = (u_n)_{n \in \mathbb{Z}^d} : \|u^0\|_{P^2}^2 = \lambda, \) we consider a perturbation \( u^0 + \epsilon q, q \in l^2 \) and then the scalar function

\[
f(\epsilon) := H \left[ \sqrt{\frac{\lambda}{\|u^0 + \epsilon q\|^2}} (u^0 + \epsilon q) \right] = \frac{\lambda}{\|u^0 + \epsilon q\|^2} \langle -\Delta_{disc}(u^0 + \epsilon q), u^0 + \epsilon q \rangle - \frac{1}{\sigma + 1} \left( \frac{\lambda}{\|u^0 + \epsilon q\|^2} \right)^{\sigma+1} \sum_{n \in \mathbb{Z}^d} |u_n + \epsilon q|^2.
\]

Clearly \( f(0) = h(\lambda) = H[u^0] = \inf_{\|u\|_{P^2}^2 = \lambda} H[u], \) whence 0 is a local minimum for the function \( f \) and so \( f'(0) = 0. \) Writing the expansion in powers of \( \epsilon \) and isolating the coefficient in front of \( \epsilon \) (which must be zero), brings about the equation

\[
(24) \quad \sum_n (-\Delta_{disc} u_n + c(\lambda) u_n - u_n^{2\sigma+1}) q_n = 0,
\]

where \( \lambda c(\lambda) = \sum_{n \in \mathbb{Z}^d} u_n^{2\sigma+2} - \sum_{j,l \in \mathbb{Z}^d, j \neq l = 1} \|u_j - u_l\|^2. \) Noting that (24) must be valid for arbitrary \( q, \) we conclude that (3) holds. Regarding the sign of \( c(\lambda), \) note that since \( h(\lambda) = \sum_{j,l \in \mathbb{Z}^d, j \neq l = 1} \|u_j - u_l\|^2 \leq \frac{1}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} u_n^{2\sigma+2} < 0, \) it follows that

\[
\lambda c(\lambda) = \frac{\sigma}{\sigma + 1} \sum_{n \in \mathbb{Z}^d} u_n^{2\sigma+2} - h(\lambda) > 0.
\]

For the second variation claim, consider again the function \( f, \) with \( q \perp u^0, \|q\|_{P^2} = 1. \) That is \( \langle u^0, q \rangle = 0. \) In such case, \( \|u^0 + \epsilon q\|^2 = \|u^0\|^2 + \epsilon^2 = \lambda + \epsilon^2. \) We take advantage of the fact that 0 is a minimum for \( f. \) It now implies \( f''(0) \geq 0. \) The coefficient in front of \( \epsilon^2 \) in the Taylor expansion of \( f(\epsilon) \) around zero is

\[
\langle -\Delta_{disc} q, q \rangle + c(\lambda) \langle q, q \rangle - (2\sigma + 1) \sum_n u_n^{2\sigma} |q_n|^2.
\]

Hence, we conclude that for every \( q : \|q\| = 1, q \perp u^0, \) one has

\[
\langle \mathcal{L}_+ q, q \rangle = \langle -\Delta_{disc} q, q \rangle + c(\lambda) \langle q, q \rangle - (2\sigma + 1) \sum_n u_n^{2\sigma} |q_n|^2 > 0
\]

This means that \( \mathcal{L}_+ |u^0|^2 \geq 0. \)
4.3. **Proof of Theorem** We have already established that (see Theorem 1), that $n(\mathcal{L}_+)$ = 1, and in fact $\mathcal{L}_+|_{|u|} \geq 0$. The next task to is to discuss the spectral picture for the other linearized operator $\mathcal{L}_-$. We show that $\mathcal{L}_- \geq 0$, with a simple eigenvalue at zero, with $\text{Ker}(\mathcal{L}_-) = \text{span}[u]$.

To this end, and by a direct verification shows that $\mathcal{L}_-[u] = 0$, this is simply the profile equation (8). By construction, $u \geq 0$. Note that $\mathcal{L}_-c(\lambda)$ satisfies the requirements of Theorem 5, so we know that the lowest eigenvalue of $\mathcal{L}_-$ is a simple eigenvalue, with an eigenfunction with positive entries. We claim that this eigenvalue is zero. Indeed, zero is an eigenvalue, and assume that it is not the smallest one. Then, there is a negative eigenvalue for $\mathcal{L}_-$. But then, according to Theorem 5, this eigenvalue will have an eigenfunction (ground state), with nonnegative entries. This causes a contradiction as the ground state needs to be orthogonal to $u$, as eigenfunctions corresponding to different eigenvalues. This shows that zero is at the bottom of the spectrum for $\mathcal{L}_-$, so $\mathcal{L}_- \geq 0$, and there is $\delta > 0$, so that $\mathcal{L}_-|_{|u|} \geq \delta$.

Now that we know that $n(\mathcal{L}_+) = 1$, $n(\mathcal{L}_-) = 0$, the general instability index theory, as developed in [35, 36, 37, 38], stability will be established as follows. First, we shall verify that $u \perp \text{Ker}(\mathcal{L}_+)$ and then, as the element $\mathcal{L}_+^{-1}u$ is defined uniquely in $\text{Ker}(\mathcal{L}_+)\perp$, the following quantity $\langle \mathcal{L}_+^{-1}u, u \rangle$ is uniquely defined as well. According to the instability index theory, [35, 36, 37, 38], spectral stability is a consequence of $\langle \mathcal{L}_+^{-1}u, u \rangle < 0$.

To this end, we show that $u \perp \text{Ker}(\mathcal{L}_+)$. Take an element $\psi \in \text{Ker}(\mathcal{L}_+), \|\psi\| = 1$. Let

$$\eta := \psi - \|\psi\|^{-2}\langle \psi, u \rangle u \perp u.$$ 

As $\mathcal{L}_+|_{|u|} \geq 0$. it must be that

$$0 \leq \langle \mathcal{L}_+\eta, \eta \rangle = \|\psi\|^{-4}\langle \psi, u \rangle^2\langle \mathcal{L}_+u, u \rangle. \tag{25}$$

Recall however that $\langle \mathcal{L}_+u, u \rangle = -2\sigma \sum_n |u_n|^{2\alpha + 2} < 0$, whence (25) is contradictory, unless $\langle \psi, u \rangle = 0$. So, we have established that $u \perp \text{Ker}(\mathcal{L}_+)$. As we have argued already, it follows that $\mathcal{L}_+^{-1}u$ is defined uniquely in $\text{Ker}(\mathcal{L}_+)\perp$. Introduce

$$h := \mathcal{L}_+^{-1}u - \|u\|^{-2}\langle \mathcal{L}_+^{-1}u, u \rangle u \perp u.$$ 

Again, it must be that $\langle \mathcal{L}_+h, h \rangle \geq 0$. But, by simple computation,

$$0 \leq \langle \mathcal{L}_+h, h \rangle = -\langle \mathcal{L}_+^{-1}u, u \rangle + \|u\|^{-4}\langle \mathcal{L}_+^{-1}u, u \rangle^2\langle \mathcal{L}_+u, u \rangle = -\langle \mathcal{L}_+^{-1}u, u \rangle,$$

since as established already $\langle \mathcal{L}_+u, u \rangle < 0$. It follows that $\langle \mathcal{L}_+^{-1}u, u \rangle \leq 0$. Under the extra assumption $\langle \mathcal{L}_+^{-1}u, u \rangle \neq 0$, it follows that $\langle \mathcal{L}_+^{-1}u, u \rangle < 0$, and the spectral stability follows.

5. Existence and Stability Properties of the Minimizers of Homogeneous Functionals

We first provide the proof of Theorem 3.

5.1. **Proof of Theorem** We apply the compensation compactness arguments again, but this time for sequences in the $l^{2\alpha + 2}$ norms. As before, pick a minimizing sequence for [41], $\|u^0\|^{2\alpha + 2} = 1, j(u^n) \to j(\omega)$. We have three alternatives

(1) (Tightness) There exists a sequence $n_k \in \mathbb{Z}^d$, so that for any $\epsilon > 0$, there exists $R = R(\epsilon)$, so that

$$\sum_{n \in \mathbb{Z}^d: |n-n_k| < R} |u_n^{k,2\alpha + 2} > 1 - \epsilon.$$
(2) (Vanishing) For every $R > 0$,
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d : |n-y| < R} |u_n^k|^{2\sigma+2} = 0.
\]

(3) (Splitting) There exists $\alpha \in (0, \lambda)$, so that for any $\epsilon > 0$, there is $R$ and $R_k \to \infty$ and $n_k$, so that
\[
\left| \sum_{n \in \mathbb{Z}^d : |n-n_k| < R} |u_n^k|^{2\sigma+2} - \alpha \right| \leq \epsilon, \quad \left| \sum_{n \in \mathbb{Z}^d : |n-n_k| > R_k} |u_n^k|^{2\sigma+2} - (1-\alpha) \right| \leq \epsilon
\]

The vanishing is handled similarly as before. More concretely, the vanishing implies $\lim_k \|u_k\|_{\ell^\infty} \to 0$. But then, since $J[u_k^\lambda] \geq \omega \|u_k\|_{\ell^2}^2$, we have that $\sup_k \|u_k\|_{\ell^2} < \infty$, whence
\[
1 - \|u_k\|_{\ell^{2\sigma+2}}^2 \leq \|u_k\|_{\ell^{2\sigma}} \|u_k\|_{\ell^2}^2
\]
is contradictory for large $k$.

For splitting, we take again
\[
v^k = u^k \chi_{|n-n_k| < R}, \quad w^k = u^k \chi_{|n-n_k| > R_k}, \quad z^k = u^k \chi_{R < |n-n_k| < R_k}
\]
so that
\[
\|v^k\|_{2\sigma+2}^2 - \alpha \leq \epsilon, \quad \|w^k\|_{2\sigma+2}^2 - (1-\alpha) \leq \epsilon, \quad \|z^k\|_{2\sigma+2}^2 - (1-\alpha) \leq \epsilon
\]
and we have the estimate $J[u_k^\lambda] \geq J[v^k] + J[w^k] - C\epsilon$. Observe that a simple rescaling argument shows that for all $\alpha > 0$, we have
\[
J_\alpha(\omega) := \inf_{\|v\|_{2\sigma+2}^2 = \alpha} J[v] = \alpha^{\frac{1}{1+\sigma}} \inf_{\|v\|_{2\sigma+2}^2 = 1} J[v] = \alpha^{\frac{1}{1+\sigma}} j(\omega).
\]
Thus, $J[v^k] \geq (\alpha - \epsilon)^{\frac{1}{1+\sigma}} j(\omega)$, $J[w^k] \geq (1-\alpha - \epsilon)^{\frac{1}{1+\sigma}} j(\omega)$. It follows that
\[
J[u_k^\lambda] \geq \left( (\alpha - \epsilon)^{\frac{1}{1+\sigma}} + (1-\alpha - \epsilon)^{\frac{1}{1+\sigma}} \right) j(\omega) - C\epsilon.
\]
Taking limits $\epsilon \to 0$, and then $k \to \infty$, we conclude
\[
\liminf_k j[u_k] \geq J[u],
\]
which is also a contradiction. Thus, it remains that the sequence is tight, which implies that there is a convergent, in $l^{2\sigma+2}$ subsequence. We take it to be $u^k$. Therefore there is $u : \|u\|_{l^{2\sigma+2}} = 1$, $\lim_k \|u^k - u\|_{l^{2\sigma+2}} = 0$. Since $\sup_k \|u_k\|_{l^2} < \infty$, it follows that $u^k \rightharpoonup u$. It follows that $\{u_{n+e_j}^k - u_n^k\}_{n \in \mathbb{Z}^d} \rightharpoonup \{u_{n+e_j} - u_n\}_{n \in \mathbb{Z}^d}$ for every $j \in \{1, \ldots, d\}$. All in all, by the lower semi-continuity of the $l^2$ norms with respect to weak convergence,
\[
\liminf_k J[u_k] \geq J[u].
\]

On the other hand, $J[u] \geq j(\omega)$, whence $J[u] = \lim_k J[u_k] = j(\omega)$ and $u$ is a minimizer for $\mathcal{P}$. Note that we have in fact proved more - since $u^k \rightharpoonup u$ and $\|u^k\|_{l^2} \to \|u\|_{l^2}$, it follows from the parallelogram identity that in fact the stronger convergence $\lim_k \|u^k - u\|_{l^2} = 0$ holds. In essence, this means that for all minimizing sequences for $\mathcal{P}$, we can always find an $l^2$ (and hence $l^{2\sigma+2}$) convergent subsequence, which converges to a minimizer.
Finally, the derivation of the Euler-Lagrange equation (9) and the property \( \langle \mathcal{L}_+ f, f \rangle \geq 0 \) for all \( f: \sum_{n \in \mathbb{Z}^d} f_n u_n^{2\sigma+1} = 0 \), are proved in exactly the same fashion, as in the corresponding constrained problem for the normalized waves. Specifically, consider

\[
g(\varepsilon) := J \left[ \frac{u^0 + \varepsilon f}{\|u^0 + \varepsilon f\|^{2\sigma+2}} \right]
\]

The critical point condition \( g'(0) = 0 \), which is necessary for the minimizer, implies the Euler-Lagrange equation (9). Taking the increments \( f: \sum_{n \in \mathbb{Z}^d} f_n u_n^{2\sigma+1} = 0 \), we express the necessary condition for the minimizer, namely

\[
g''(0) = \frac{\langle \mathcal{L}_+ f, f \rangle}{2} \geq 0
\]

which implies that \( \langle \mathcal{L}_+ f, f \rangle \geq 0 \) for all \( f: \sum_{n \in \mathbb{Z}^d} f_n u_n^{2\sigma+1} = 0 \). Thus, \( n(\mathcal{L}_+) \leq 1 \), as the operator is non-negative on a co-dimension one subspace, namely \( \{ f: f \perp \{ u_n^{2\sigma+1} \} \} \). On the other hand, we still have \( \langle \mathcal{L}_+ u, u \rangle = -2\sigma \sum_n u_n^{2\sigma+1} < 0 \), whence \( n(\mathcal{L}_+) \geq 1 \). Altogether, \( n(\mathcal{L}_+) = 1 \).

5.2. Proof of Theorem 4. We start with a technical lemma regarding the function \( j(\omega) \).

**Lemma 5.** The function \( j(\omega) \) satisfies the bounds \( \omega < j(\omega) < \omega + 2 \). Moreover, it is uniformly Lipschitz in the intervals \((a, \infty), a > 0\). That is, for each \( a > 0 \), there is \( C_a \), so that for all \( a < \omega_1 < \omega_2 \), there is \( |j(\omega_1) - j(\omega_2)| \leq C_a|\omega_1 - \omega_2| \).

**Remark:** From the proof, one can take \( C_a = 1 + \frac{2}{a} \), which clearly blows up as \( a \to 0^+ \).

**Proof:** Testing with the element \( e_0 \) yields the bound \( j(\omega) < J[e_0] = \omega + 2 \). On the other hand, since \( \|u\|_{l^2} \geq \|u\|_{\ell^{2\sigma+2}} = 1 \), we have that \( J[u] > \omega \) for each \( u: \|u\|_{\ell^{2\sigma+2}} = 1 \), whence \( j(\omega) > \omega \).

The bound \( j(\omega) < \omega + 2 \) implies that

\[
j(\omega) = \inf_{\|u\|_{\ell^{2\sigma+2}} = 1} J[u] = \inf_{\|u\|_{\ell^{2\sigma+2}} = 1, \|u\|_{l^2} < 1 + \frac{2}{\omega}} J[u].
\]

Indeed, testing with \( u: \|u\|_{l^2} > 1 + \frac{2}{\omega} \) leads to \( J[u] > \omega + 2 \), which will not be optimal. Having this in mind, let \( 0 < a < \omega_1 < \omega_2 \). In order to test for both values \( \omega_1, \omega_2 \), it suffices to take \( u: \|u\|_{\ell^{2\sigma+2}} = 1, \|u\|_{l^2}^2 < 1 + \frac{2}{\omega_1} \). Since,

\[
J_{\omega_1}[u] - J_{\omega_2}[u] = (\omega_1 - \omega_2)\|u\|_{l^2}^2,
\]

it follows that \( |j(\omega_1) - j(\omega_2)| \leq \left(1 + \frac{2}{\omega_1}\right)|\omega_1 - \omega_2| \leq \left(1 + \frac{2}{\omega}\right)|\omega_1 - \omega_2| \). \( \square \)

Having Lemma 5 at hand, we can finish off the proof of Theorem 4. Fix \( \omega > 0 \). Recall that the limit waves \( \lim_j \|u_{\omega + \delta_j} - u_\omega\|_{l^2} = 0 \). By the continuity of \( j(\omega) \), we have that \( \varphi_n = j(\omega)^{\frac{1}{\omega}} u_n \) also satisfies \( \lim_j \|\varphi_{\omega + \delta_j} - \varphi_\omega\|_{l^2} = 0 \). Write

\[
\varphi_{\omega + \delta_j} = \varphi_\omega + \delta_j z^j,
\]

so that \( \lim_j \|\delta_j z^j\|_{l^2} = 0 \). Next, subtract the profile equations

\[
-\Delta_{disc}(\varphi_\omega + \delta_j z^j) + (\omega + \delta_j)(\varphi_\omega + \delta_j z^j) - (\varphi_\omega + \delta_j z^j)^{2\sigma+1} = 0,
\]

\[
-\Delta_{disc}\varphi_\omega + \omega \varphi_\omega - \varphi_\omega^{2\sigma+1} = 0.
\]
and divide by $\delta_j$. We obtain

$$
-\Delta_{disc} z_n^j + \omega z_n^j - (2\sigma + 1)\varphi_n^{2\sigma} z_n^j = -\varphi_\omega - \delta_j z_j + E_n^j,
$$

where

$$
E_n^j = \delta_j^{-1} \left( (\varphi_\omega + \delta_j z_j)^{2\sigma + 1} - \varphi_\omega^{2\sigma + 1} - (2\sigma + 1)\varphi_n^{2\sigma} \delta_j z_n^j \right)
$$

By Taylor expansions (separately in the cases $\varphi_n < |\delta_j||z_n^j|$ and otherwise), it is clear that there exists a constant $C = C_\sigma$, so that

$$
|E_n^j| \leq C_\sigma \varphi_n^{2\sigma}(|\delta_j||z_n^j|^2) |z_n^j|.
$$

Thus, we can interpret the equation (26) in the form $L_+[z^j] = -\varphi_\omega - \delta_j z_j + E^j$. This is a non-linear relation for $z^j$, which we use for a posteriori estimates. Indeed, resolving (26) leads to

$$
z^j = -L_+^{-1}[\varphi_\omega] - L_+^{-1}(-\delta_j z_j + E^j),
$$

which leads to the estimate

$$
\|z^j\|_{l^2} \leq C \|\varphi_\omega\|_{l^2} + C|\delta_j|\|z_j\|_{l^2} + C\|E^j\|_{l^2} \leq C \|\varphi_\omega\|_{l^2} + C|\delta_j|\|z_j\|_{l^2} + C\|z^j\|_{l^2} (|\delta_j|\|z^j\|_{l^2})^\sigma
$$

Since $|\delta_j|\|z^j\|_{l^2} \leq |\delta_j|\|z^j\|_{l^2} \to 0$, we can hide the term $\|z^j\|_{l^2} (|\delta_j|\|z^j\|_{l^2})^\sigma$ behind the left-hand side. This implies the estimate

$$
\limsup_j \|z^j\|_{l^2} \leq C \|\varphi_\omega\|_{l^2}.
$$

Going back to the relation (27), this yields that $z^j$ converges strongly in $l^2$ to $-L_+^{-1}[\varphi_\omega]$. That is

$$
\lim_j \|z^j + L_+^{-1}[\varphi_\omega]\|_{l^2} = 0.
$$

Now, consider

$$
\lim_j \frac{\|\varphi_\omega + \delta_j\|_{l^2}^2 - \|\varphi_\omega\|_{l^2}^2}{\delta_j} = \lim_j \frac{\|\varphi_\omega + \delta_j z^j\|_{l^2}^2 - \|\varphi_\omega\|_{l^2}^2}{\delta_j} = \lim_j \left( 2\langle z^j, \varphi_\omega \rangle + \delta_j \|z^j\|_{l^2}^2 \right) = -2 \langle L_+^{-1}[\varphi_\omega], \varphi_\omega \rangle.
$$

By the Vakhitov-Kolokolov criteria, since $n(L) = n(L_1) + n(L_+) = 1$, stability of $e^{i\omega t}\varphi_\omega$ is equivalent to $\langle L_+^{-1}[\varphi_\omega], \varphi_\omega \rangle < 0$. Hence, we have proved that the wave is stable if and only if the following limit (which exists)

$$
\lim \frac{\|\varphi_\omega + \delta_j\|_{l^2}^2 - \|\varphi_\omega\|_{l^2}^2}{\delta_j} > 0.
$$

5.3. **Proof of Proposition**. The fact that $j(\omega)$ is concave follows from the definition. Thus, it is twice differentiable a.e. (and we conjecture that it is smooth everywhere). Taking the profile equation for $u^\omega$, and differentiating in $\omega$, we conclude

$$
L_+ (\partial_\omega u_\omega) = -u_\omega + j'(\omega)\varphi^{2\sigma + 1} = -u_\omega - \frac{j'(\omega)}{\sigma j(\omega)} L_+ u_\omega.
$$

This implies

$$
L_+ (\partial_\omega u_\omega + \frac{j'(\omega)}{2\sigma j(\omega)} u_\omega) = -u_\omega
$$
Taking dot product of the last equation with $\mathbf{u}^{2\sigma+1}$ yields
$$\|\mathbf{u}_\omega\|_2^2 = j'(\omega).$$
It follows that $\|\varphi_\omega\|_2^2 = j'(\omega) j^{\frac{1}{\sigma}}(\omega)$.

6. Conclusions and Future Work

In the present work we have revisited the original formulation of [1] regarding the identification of suitable minimizers in the form of normalized waves that are associated with standing waves of the discrete nonlinear Schrödinger equation. We have provided an alternative formulation of the relevant variational problem, and we have illustrated that the zeros of a suitably defined function can offer definitive information about the stability of the relevant states. We have showcased the equivalence of this distinct formulation’s stability conclusions in connection with the more standard VK (or GSS) stability criteria. However, a key advantage of the formulation is that it works for all values of the system parameters. In addition to establishing the relevant existence theorems for such minimizers, and identifying the associated stability conditions, we have also showcased the results via select numerical computations for different system parameters (including the frequency $\omega$, or the nonlinearity exponent $\sigma$, as well as for different spatial dimensionalities of the problem).

Even within the space of even spatial solutions, however, there are the classes of so-called onsite and offsite solutions [26] and the energetic difference between them represents the celebrated Peierls-Nabarro barrier [2]. Understanding further the differences between such states at the level of the variational formulation would be something of interest for future considerations. On the other hand, it is well-known within this DNLS model that various excited state solutions exist, including the so-called twisted localized modes [39, 40] belong to a different class of solutions that are anti-symmetric. Yet, these waveforms are robust and can be observed to be very long-lived [2], hence potentially developing a variational formulation within such a class of functions (or over more complex waveforms involving a nontrivial phase distribution) could be a challenging, yet interesting direction.

Appendix A. Proof of Theorem 5

First, we observe that $\mathcal{L} = -\Delta_{disc} + \mathbb{V}$ is a bounded self-adjoint operator on $l^2(\mathbb{Z}^d)$. It is clear that $\mathcal{L}$ is relatively compact with respect to the constant coefficient operator $-\Delta_{disc}$, whence by Weyl's theorem (see for example Theorem 14.6, p. 142, [41]), their essential spectra coincide
$$\sigma_{ess}(\mathcal{L}) = \sigma_{ess}(-\Delta_{disc}) = \sigma(-\Delta_{disc}) = [0, 4d].$$
Since by assumption $\lambda_0(\mathcal{L}) = \inf(\mathcal{L}, \mathbf{u}, \mathbf{u}) < 0$, it follows that $\lambda_0(\mathcal{L})$ is an eigenvalue. It remains to show that it is a simple eigenvalue and the corresponding eigenfunction consists of positive entries only.

According to Theorem XIII.44, p. 204, [34], it is enough to show that the the semigroup $e^{t\mathcal{L}}$, is positivity improving. That is, for every $\mathbf{f} \in l^2(\mathbb{Z}^d)$: $\mathbf{f}_n \geq 0, \mathbf{f} \neq 0$, we need to show that $(e^{t\mathcal{L}} \mathbf{f})_n > 0, n \in \mathbb{Z}^d$. Clearly, $\mathbf{u} = e^{t\mathcal{L}} \mathbf{f}$ solves the following discrete parabolic initial value problem
(28)$$\partial_t \mathbf{u}_n(t) - \Delta_{disc} \mathbf{u}_n(t) = V_n \mathbf{u}_n(t), \quad \mathbf{u}(0) = \mathbf{f}, \quad n \in \mathbb{Z}^d.$$
By the Duhamel's formula

\[ u(t) = e^{t \Delta_{\text{disc}}} f + \int_0^t e^{(t-s) \Delta_{\text{disc}}} [\nabla u(s)] ds. \]  

(29)

Recall, that since \( e^{t \Delta_{\text{disc}}} \) is given by a convolution with a positive kernel\(^4\) so we have the pointwise estimate

\[ |e^{t \Delta_{\text{disc}}} h(n)| \leq (e^{t \Delta_{\text{disc}}}|h|)(n), \quad n \in \mathbb{Z}^d, \]

(30)

where we have used the notation \( |h|(n) = |h(n)| \).

We can now develop a Born series type expansion for \( u \) in (29). Namely, we have

\[ u(t) = \sum_{j=0}^{\infty} I_j(f), \]

(31)

where \( I_0(f) = e^{t \Delta_{\text{disc}}} f, I_1(f) = \int_0^t e^{(t-s) \Delta_{\text{disc}}} [\nabla e^{s \Delta_{\text{disc}}} f] ds_1 \), and for \( j \geq 2, \)

\[ I_j(f) = \int_0^t e^{(t-s) \Delta_{\text{disc}}} \left[ \nabla \int_0^{s_1} \cdots \int_0^{s_{j-1}} e^{(s_{j-1}-s_j) \Delta_{\text{disc}}} \left[ \nabla e^{s_j \Delta_{\text{disc}}} f \right] ds_j \right] ds_1 \cdots ds_{j-1}. \]

For each fixed \( n \in \mathbb{Z}^d \), and by taking absolute values in the previous formula, and applying (30) judiciously, we obtain

\[ |I_j(f)(n)| \leq \| \nabla \|_{\infty, \mathbb{Z}^d} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{j-1}} e^{(s_{j-1}-s_j) \Delta_{\text{disc}}} e^{(s_{j-1}-s_j) \Delta_{\text{disc}}} \cdots e^{(s_{j-1}-s_j) \Delta_{\text{disc}}} e^{s_j \Delta_{\text{disc}}} f ds_1 \cdots ds_j \]

\[ = \frac{1}{j!} \| \nabla \|_{\infty, \mathbb{Z}^d} e^{t \Delta_{\text{disc}}} |f|(n). \]

Summing up in \( j \) yields the bound, for each \( t > 0, n \in \mathbb{Z}^d \)

\[ |u_n(t)| \leq \| \nabla \|_{\infty, \mathbb{Z}^d} e^{t \Delta_{\text{disc}}} |f|(n). \]

(32)

This also confirms that the expansion (31) is valid, at least in \( l^2 \) sense.

Having this a priori bound on \( u(t) \) allows us to finish the proof of the positivity improving property of the semigroup. Indeed, by virtue of (30)

\[ |\int_0^t e^{(t-s) \Delta_{\text{disc}}} [\nabla u(s)] ds| \leq \int_0^t e^{(t-s) \Delta_{\text{disc}}} [\nabla e^{\|\nabla\|_{\infty, \mathbb{Z}^d} t \Delta_{\text{disc}}} \nabla e^{t \Delta_{\text{disc}}} f] ds| \leq t \| \nabla \|_{\infty, \mathbb{Z}^d} e^{\|\nabla\|_{\infty, \mathbb{Z}^d} t \Delta_{\text{disc}}} e^{t \Delta_{\text{disc}}} |f|. \]

Suppose now that \( f: \mathbb{Z}^d \to \mathbb{R} \), \( f \geq 0, f \neq 0 \). From (29), we have

\[ u(t) = e^{t \Delta_{\text{disc}}} f + \int_0^t e^{(t-s) \Delta_{\text{disc}}} [\nabla u(s)] ds \geq (1 - t \| \nabla \|_{\infty, \mathbb{Z}^d} e^{\|\nabla\|_{\infty, \mathbb{Z}^d} t \Delta_{\text{disc}}} f) e^{t \Delta_{\text{disc}}} f. \]

Recall that \( e^{t \Delta} > 0 \), as it is given by a convolution with a strictly positive sequence \( K_j(t) \), see (15). Clearly, for all \( t: t \| \nabla \|_{\infty, \mathbb{Z}^d} e^{\|\nabla\|_{\infty, \mathbb{Z}^d} t \Delta_{\text{disc}}} < 1 \), we have that \( u(t) > 0 \). That is, for all small

\[ ^{4}\text{see formula (15) for the } \Delta_{\text{disc}} \text{ for } d = 1, \text{ the general case follows from } e^{t \Delta_d} = e^{t \Delta_1} \cdots e^{t \Delta_1}. \]
enough $t$, in fact for $t < \frac{1}{2\|V\|_{\ell^\infty(Z^d)}}$, we have that $u(t) > 0$. Note that the required smallness of $t$ is only in terms of the potential $V$, and importantly, it is independent on the initial data $f$.

Since one can write, for each $t$,

$$e^{t\mathcal{L}}f = e^{\frac{t}{N}\mathcal{L}} \ldots e^{\frac{t}{N}\mathcal{L}}f$$

with some large $N : \frac{t}{N} < \frac{1}{2\|V\|_{\ell^\infty(Z^d)}}$, we conclude that $e^{t\mathcal{L}}f > 0$, which completes the proof of Theorem 5.

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