THE FUNDAMENTAL GROUP OF A \( \mathbb{CP}^2 \) COMPLEMENT OF A BRANCH CURVE AS AN EXTENSION OF A SOLVABLE GROUP BY A SYMMETRIC GROUP

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ABSTRACT. The main result in this paper is as follows:

**Theorem.** Let \( S \) be the branch curve in \( \mathbb{CP}^2 \) of a generic projection of a Veronese surface. Then \( \pi_1(\mathbb{CP}^2 - S) \) is an extension of a solvable group by a symmetric group.

A group with the property mentioned in the theorem is “almost solvable” in the sense that it contains a solvable normal subgroup of finite index. We pose the following question.

**Question.** For which families of simply connected algebraic surfaces of general type is the fundamental group of the complement of the branch curve of a generic projection to \( \mathbb{CP}^2 \) an extension of a solvable group by a symmetric group?

Introduction. Our study of fundamental groups of complements of branch curves is part of our plan to use fundamental groups in order to distinguish among different components of moduli spaces of surfaces of general type.

There are not many known computations of fundamental groups of complements of branch curves. The topic started with Zariski who proved in the 30’s that if \( X \) is a cubic surface in \( \mathbb{CP}^3 \) and \( S \) is the branch curve of a generic projection of \( X \) then \( \pi_1(\mathbb{CP}^2 - S) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \) (see [Z]). In the late 70’s Moishezon proved that if \( X \) is a deg \( n \) surface in \( \mathbb{CP}^3 \) then \( \pi_1(\mathbb{CP}^2 - S) \cong B_n/\text{Center} \), where \( B_n \) is the braid group of order \( n \) (see [Mo]. In fact, Moishezon’s result for \( n = 3 \) is the same as Zariski’s.
result since $B_3/\text{Center} \simeq \mathbb{Z}_2 \ast \mathbb{Z}_3$. The next example was Veronese of order 2 (see [MoTe3]). In all the above examples $\pi_1(\mathbb{CP}^2 - S)$ contains a free noncommutative subgroup with 2 generators, so it is “big”.

Unlike in the early results, in this paper we present $\pi_1(\mathbb{CP}^2 - S)$ not “big”. We study here the fundamental group of the complement in $\mathbb{CP}^2$ of the branch curve of a generic projection of a Veronese surface.

Our main result is as follows:

**Theorem.** Let $S$ be the branch curve in $\mathbb{CP}^2$ of a generic projection of a Veronese surface. Then $\pi_1(\mathbb{CP}^2 - S)$ is an extension of a solvable group by a symmetric group.

We believe that the statement of the theorem is valid for many classes of surfaces of general type. A group with the property mentioned in the theorem is “almost solvable” in the sense that it contains a solvable normal subgroup of finite index. We pose the following question.

**Question.** For which families of simply connected algebraic surfaces of general type is the fundamental group of the complement of the branch curve of a generic projection to $\mathbb{CP}^2$ an extension of a solvable group by a symmetric group?

In [MoTe10], Proposition 2.4, we proved an almost solvability theorem for the complement of $S$ in $\mathbb{C}^2$ where $\mathbb{C}^2$ is a generic affine piece of $\mathbb{CP}^2$. In this paper we move from $\mathbb{C}^2$ to $\mathbb{CP}^2$. This situation involves new techniques ($\S$3 and $\S$5), the Van Kampen Theorem for projective curves, quoted in $\S$1, and different results on the structure of $\pi_1(\mathbb{C}^2 - S)$ from [MoTe9] and [MoTe10] quoted in $\S$4. To formulate the results in $\S$3 and $\S$4 we need some information on the braid group $B_n$ and its quotient $\tilde{B}_n$ which we give in $\S$2. The main theorem is proven in $\S$5.

The theorem can be generalized for any Veronese embedding. In this paper we choose to prove it for an embedding of order 3 to simplify the presentation. The result for any Veronese appears in Section 6.
The braid group $B_n$ plays an important role in describing fundamental groups of complements of curves. There is a quotient of $B_n$, namely $\tilde{B}_n$, which acts on our group $\pi_1(\mathbb{C}P^2 - S)$. We believe that $\tilde{B}_n$ acts on fundamental groups of complements of branch curves for many classes of surfaces of general type, and we can characterize such fundamental groups through the classification of $\tilde{B}_n$-groups.

Lately, there is a growing interest in fundamental groups in general, in classical algebraic geometry and in Kähler geometry cf., for example, [L], [Si], [To]. For fundamental groups of complements of curves see also [CT] and [DOZ].

§1. The Van Kampen Theorem.

As we stated in the introduction, our starting point for proving the main theorem is the Van Kampen Theorem for projective complements of curves. The Van Kampen Theorem from the 1930’s deals with fundamental groups of affine and projective complements of curves. Since our main result is a statement on the fundamental group of the complement in $\mathbb{C}P^2$, we shall only quote in this section the Van Kampen Theorem for the projective complement. We shall start with a few definitions, that we need in order to formulate the theorem.

1.1 Definition. $\ell(\gamma)$.

Let $D$ be a disk. Let $p \in \text{Int}(D)$. Let $u \in \partial D$. Let $\gamma$ be a simple path connecting $u$ with $p$. We assign to $\gamma$ a loop as follows: Let $c$ be a small (oriented) circle around $p$. Let $\gamma'$ be the part of $\gamma$ outside of $c$. We define $\ell(\gamma) = \gamma' \cup c \cup \gamma'^{-1}$. We also use the same notation $\ell(\gamma)$ for the element of $\pi_1(D - K, u)$ corresponding to $\ell(\gamma)$. If $p \in K$, $K \subset D$, $K$ finite, and $\gamma$ does not meet any other point of $K$, then $\ell(\gamma)$ can be chosen to be in $\pi_1(D - K, u)$.

1.2 Definition. $g$-basis (good geometric base)

Let $D$ be a disk, $K \subseteq D$, $K = \{a_1, \ldots, a_m\}$. Let $u \in \partial(D) - K$. Let $\{\gamma_i\}_{i=1}^m$ be a bush in $(D, K, u)$, i.e., $\gamma_i$ is a simple path connecting $u$ with $a_i$, $\forall i, j \gamma_i \cap \gamma_j = u$, $\forall i \gamma_i \cap K = \text{one point}$, and $\{\gamma_i\}$ are ordered counterclockwise around $u$. Let $\Gamma_i =$
\( \ell(\gamma_i) \in \pi_1(D - K, u) \) be the loop around \( a_i \) determined by \( \gamma_i \). \( \{\Gamma_i\}_{i=1}^m \) is a \( g \)-base of \( \pi_1(D - K, u) \).

**1.3 Remark.** A \( g \)-base is a free base of \( \pi_1(D - K, \ast) \) which is essential in the formulation of the Van Kampen Theorem.

**1.4.** Consider the following situation: Let \( S \) be a curve in \( \mathbb{CP}^2 \) of deg \( m \), s.t. \( S \) is transversal to the line in infinity. Let \( \pi : \mathbb{C}^2 \to \mathbb{C} \) be a generic projection. Let \( N = \{ x \in \mathbb{C} \mid \# \pi^{-1}(x) \cap S < m \} \). Let \( u \in \mathbb{C} - N \), s.t. \( u \) is real and \( |x| < u \) \( \forall x \in N \). Let \( \mathbb{C}_u = \pi^{-1}(u) \). Let \( \{\Gamma_i\}_{i=1}^m \) be a \( g \)-base of \( \pi_1(\mathbb{C}_u - S \cap \mathbb{C}_u, \ast) \). By abuse of notation we also use the notation \( \Gamma_i \) for the image of \( \Gamma_i \) in \( \pi_1(\mathbb{C}^2 - S, \ast) \).

**1.5 Theorem.** (Projective Van Kampen Theorem) In the situation of 1.4 we have

\[
\pi_1(\mathbb{CP}^2 - S, \ast) \approx \frac{\pi(\mathbb{C}^2 - S, \ast)}{\langle \prod_{i=1}^m \Gamma_i \rangle},
\]

where \( \langle \prod_{i=1}^m \Gamma_i \rangle \) is the subgroup normally generated by \( \prod_{i=1}^m \Gamma_i \).

**Proof.** [VK]

§2. **Introducing \( \tilde{B}_n \), a quotient of \( B_n \).**

In this section we bring the definition of the braid group and we distinguish certain elements, called half-twists. Using half-twists we present Artin’s Structure Theorem for the braid group and the natural homomorphism to the symmetric group. We also define transversal half-twists and the quotient of \( B_n \) called \( \tilde{B}_n \).

**2.1 Definition.** Braid group \( B_n = B_n[D, K] \)

Let \( D \) be a closed disc in \( \mathbb{R}^2 \), \( K \subset D \), \( K \) finite. Let \( B \) be the group of all diffeomorphisms \( \beta \) of \( D \) such that \( \beta(K) = K, \beta|_{\partial D} = \text{Id}_{\partial D} \). For \( \beta_1, \beta_2 \in B \), we say that \( \beta_1 \) is equivalent to \( \beta_2 \) if \( \beta_1 \) and \( \beta_2 \) induce the same automorphism of \( \pi_1(D - K, u) \). The quotient of \( B \) by this equivalence relation is called the braid group \( B_n[D, K] \) \( (n = \# K) \). The elements of \( B_n[D, K] \) are called braids.
2.2 Definition. \( H(\sigma) \), half-twist defined by \( \sigma \)

Let \( D, K \) be as above. Let \( a, b \in K \), \( K_{a,b} = K - a - b \) and \( \sigma \) be a simple path in \( D - \partial D \) connecting \( a \) with \( b \) s.t. \( \sigma \cap K = \{a, b\} \). Choose a small regular neighborhood \( U \) of \( \sigma \) and an orientation preserving diffeomorphism \( f : \mathbb{R}^2 \to \mathbb{C}^1 \) (\( \mathbb{C}^1 \) is taken with the usual “complex” orientation) such that \( f(\sigma) = [-1, 1] \), \( f(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\} \). Let \( \alpha(r), r \geq 0 \), be a real smooth monotone function such that \( \alpha(r) = 1 \) for \( r \in [0, \frac{3}{2}] \) and \( \alpha(r) = 0 \) for \( r \geq 2 \).

Define a diffeomorphism \( h : \mathbb{C}^1 \to \mathbb{C}^1 \) as follows. For \( z \in \mathbb{C}^1 \), \( z = re^{i\varphi} \), let \( h(z) = re^{i(\varphi + \alpha(r))} \). It is clear that on \( \{z \in \mathbb{C}^1 \mid |z| \leq \frac{3}{2}\} \), \( h(z) \) is the positive rotation by \( 180^\circ \) and that \( h(z) = \text{Identity} \) on \( \{z \in \mathbb{C}^1 \mid |z| \geq 2\} \), in particular, on \( \mathbb{C}^1 - f(U) \). Considering \( (f \circ h \circ f^{-1})|_D \) (we always take composition from left to right), we get a diffeomorphism of \( D \) which interchanges \( a \) and \( b \) and is the identity on \( D - U \). Thus it defines an element of \( B_n[D, K] \), called the half-twist defined by \( \sigma \) and denoted \( H(\sigma) \).

Using half-twists we build a set of generators for \( B_n \).

2.3 Definition. \( \text{Frame of } B_n[D, K] \)

Let \( D \) be a disc in \( \mathbb{R}^2 \). Let \( K = \{a_1, \ldots, a_n\}, K \subset D \). Let \( \sigma_1, \ldots, \sigma_{n-1} \) be a system of simple paths in \( D - \partial D \) such that each \( \sigma_i \) connects \( a_i \) with \( a_{i+1} \) and for

\[
    i, j \in \{1, \ldots, n-1\}, \ i < j \quad \sigma_i \cap \sigma_j = \begin{cases} 
    \emptyset & \text{if } |i - j| \geq 2 \\
    a_{i+1} & \text{if } j = i + 1.
    \end{cases}
\]

Let \( H_i = H(\sigma_i) \). We call the ordered system of half-twists \( (H_1, \ldots, H_{n-1}) \) a frame of \( B_n[D, K] \) defined by \( (\sigma_1, \ldots, \sigma_{n-1}) \), or a frame of \( B_n[D, K] \) for short.

2.4 Notation.

\[
    [A, B] = ABA^{-1}B^{-1}.
\]

\[
    \langle A, B \rangle = ABAB^{-1}A^{-1}B^{-1}.
\]

\[
    (A)_{B} = B^{-1}AB.
\]
2.5 Theorem. (E. Artin’s braid group presentation) Let \( \{H_i\} \) be a frame of \( B_n \). Then \( B_n \) is generated by the half-twists \( H_i \) and all the relations between \( H_1, \ldots, H_{n-1} \) follow from

\[
[H_i, H_j] = 1 \quad \text{if} \quad |i - j| > 1, \\
\langle H_i, H_j \rangle = 1 \quad \text{if} \quad |i - j| = 1, \\
1 \leq i, j \leq n - 1.
\]

Proof. [A] (or [MoTe4], Chapter 5).

2.6 Theorem. Let \( \{H_i\} \) be a frame of \( B_n \). Then

(i) for \( n \geq 2 \), Center \( B_n \) is isomorphic to \( \mathbb{Z} \) with a generator \( \Delta^2_n = (H_1 \cdot \ldots \cdot H_{n-1})^n \).

(ii) \( B_2 \simeq \mathbb{Z} \) with a generator \( H_1 \).

Proof. [MoTe4], Corollary V.2.3.

2.7 Proposition. There is a natural defined homomorphism \( B_n \rightarrow S_n \) (symmetric group on \( n \) elements) defined by \( H_i \rightarrow (i \ i + 1) \).

Proof. Since the transpositions \( \alpha_i = (i \ i + 1) \) satisfy the relations from Artin’s theorem (2.5), the above homomorphism is well defined.

2.8 Definition. \( P_n \).

The kernel of the above homomorphism is denoted by \( P_n \).

2.9 Remark. The transpositions \( \alpha_i \) satisfy a relation that \( H_i \) do not satisfy, which is \( \alpha_i^2 = 1 \). In fact it is true for any transposition. Under the above homomorphism the image of any half-twist is a transposition and thus any square of a half-twist belongs to \( \ker(B_n \rightarrow S_n) \) which is \( P_n \).

2.10 Definition. Transversal half-twists, adjacent half-twist, disjoint half-twist.

Let \( \sigma_1 \) and \( \sigma_2 \) be 2 paths in \( D \) with endpoints in \( K \) which do not intersect \( K \) otherwise (like in 2.2). The half-twists \( H(\sigma_1) \) and \( H(\sigma_2) \) will be called transversal
if \( \sigma_1 \) and \( \sigma_2 \) intersect transversally in one point which is not an end point of either of the \( \sigma_i \)'s.

The half-twists \( H(\sigma_1) \) and \( H(\sigma_2) \) will be called *adjacent* if \( \sigma_1 \) and \( \sigma_2 \) have one endpoint in common.

The half-twists \( H(\sigma_1) \) and \( H(\sigma_2) \) will be called *disjoint* if \( \sigma_1 \) and \( \sigma_2 \) do not intersect.

2.12 Claim. Disjoint half-twists commute and adjacent half-twists satisfy the triple relation \( ABA = BAB \).

*Proof.* By Proposition 2.7 and the fact that every 2 half-twists are conjugated to each other.

2.12 Definition. \( \tilde{B}_n \).

Let \( Q_n \) be the subgroup of \( B_n \) normally generated by \([X, Y]\) for \( X, Y \) transversal half-twists. \( \tilde{B}_n \) is the quotient of \( B_n \) modulo \( Q_n \). For \( X \in B_n \) we denote by \( \tilde{X} \) the image of \( X \) in \( \tilde{B}_n \). \( \{\tilde{H}_i\} \) is a frame of \( \tilde{B}_n \) if \( \{H_i\} \) is a frame of \( B_n \).

Later we shall need some basic relations satisfied in \( \tilde{B}_n \) (and not in \( B_n \)). We formulate this in the following claim.

2.13 Claim. Let \( \tilde{P}_n \) be the image of \( P_n \) (from 2.7) in \( \tilde{B}_n \). Then \( \tilde{P}_n' = \{1, c\} \) where \( c^2 = 1 \), \( c \in \text{Center } \tilde{B}_n \). In particular, if \( \tilde{X} \) and \( \tilde{Y} \) are 2 adjacent half-twists \([\tilde{X} \pm 2, \tilde{Y} \pm 2]\) = \( c \).

*Proof.* [MoTe9], Proposition II.5.2.

§3. General results on fundamental groups of complements of curves.

In this section we prove two general results based on the situation described in 1.4. The first one concerns the action of the braid group on the fundamental group \( \pi_1(\mathbb{C}^2 - S) \) and the second one is a corollary on the structure of \( \pi_1(\mathbb{C}^2 - S) \).

3.1 Proposition. Consider the situation of 1.4. Let \( \Delta^2_m \) be the generator of the center of \( B_m[C_u, C_u \cap S] \). Then when considered as elements of \( \pi_1(\mathbb{C}^2 - S) \),
\[ \Delta^2_m(\Gamma_k) = \Gamma_k \quad \forall \Gamma_k. \]

**Proof.** Let \( \varphi_u \) be the naturally defined homomorphism from \( \pi_1(\mathbb{C} - N, u) \to B_m[\mathbb{C}_u, \mathbb{C}_u \cap S] \). This homomorphism is called the braid monodromy and it factors through the classical monodromy from \( \pi_1 \) to \( S_m, \pi_1 \varphi_u \rightarrow B_m \rightarrow S_m \). Since \( B_m \) acts on \( \pi_1(\mathbb{C}_u - S, \ast) \), so does \( \varphi_u(\pi_1) \). Moreover, \( \varphi_u(\pi_1) \) acts on the elements of \( \pi_1(\mathbb{C}_u - S, \ast) \) when considered as elements of \( \pi_1(\mathbb{C}^2 - S, \ast) \). By the affine Van Kampen theorem (see [RoTe]), for \( \gamma \in \pi_1(\mathbb{C} - N, u), \varphi_u(\gamma)(\Gamma_k) = \Gamma_k \) when considered as elements of \( \pi_1(\mathbb{C}^2 - S) \). By [MoTe4], Lemma VI.2.1, \( \Delta^2_m \) is a product of elements of the form \( \varphi_u(\gamma) \) for \( \gamma \in \pi_1(\mathbb{C} - N, u) \) and thus \( \Delta^2_m(\Gamma_k) = \Gamma_k \forall \Gamma_k. \)

### 3.2 Proposition

**Consider the situation of 1.4.** Let \( \Gamma = \prod_{i=1}^m \Gamma_i \). Then when \( \Gamma \) is considered as an element of \( \pi_1(\mathbb{C}^2 - S, \ast) \), it is a central element and \( \langle \Gamma \rangle \) is an infinite cyclic group.

**Proof.** We first consider \( \Gamma = \prod_{i=1}^m \Gamma_i \) as an element of \( \pi_1(\mathbb{C}_u - S, \ast) \). Clearly, \( \prod \Gamma_i \) is homotopic to a loop \( \partial D \) around all the points of \( \mathbb{C}_u \cap S \). By Proposition V.2.1 of [MoTe5] in \( \pi_1(\mathbb{C}_u - S, \ast) \) the conjugation of \( \Gamma_k \) by \( \partial D \) is equal to the action of \( \Delta^2_m \) (which is defined in 2.6) on \( \Gamma_k \). But in \( \pi_1(\mathbb{C}^2 - S, \ast) \), \( \Delta^2_m \) acts trivially on \( \Gamma_k \), (by 3.1) so conjugation of \( \Gamma_k \) by \( \partial D \) in \( \pi_1(\mathbb{C}^2 - S) \) is stable and thus \( \Gamma = \partial D \) is in the center of \( \pi_1(\mathbb{C}^2 - S, u) \). Moreover, since \( \langle \Delta^2_m \rangle \) is an infinite cyclic group (see ([MoTe4], V.2.1)), so is \( \langle \Gamma \rangle \).

### §4. Results on \( \pi_1(\mathbb{C}^2 - S, \ast) \) for a Veronese branch curve.

In this section we restrict ourselves to a curve which is the branch curve of a Veronese generic projection. We will quote results concerning its complement in \( \mathbb{C}^2 \) (cf. [MoTe9] and [MoTe10]) which will be used later in the proof of the main result, concerning its complement in \( \mathbb{C}\mathbb{P}^2 \).

The fundamental group of the complement in \( \mathbb{C}^2 \) turned out to be a quotient of a semidirect product of \( \tilde{B}_n^2 \) (for a Veronese embedding of \( \deg n \)) and \( G_0(n^2) \) which is a \( \mathbb{Z}_2 \) extension of a free group on \( n^2 - 1 \) elements (see [MoTe9]).
From now on we will restrict ourselves to a Veronese embedding of deg 3. Let $S$ be the branch curve of a generic projection to $\mathbb{C}P^2$ of a Veronese surface of deg 3. The degree of the projection is 9, and the degree of the branch curve is 18. Let $\mathbb{C}^2$ be a big affine piece of $\mathbb{C}P^2$ s.t. $S$ is transversal to the line in infinity.

Consider $\tilde{B}_9$ as defined in §2. Instead of working with a frame of $\tilde{B}_9$ we will work with $\{\tilde{T}_i\}$, a set of generators for $\tilde{B}_9$ as follows:

**4.1 Definition.** Let $\{T_i\}_{i=1}^9$ be s.t. $\tilde{T}_i$ is a half-twist w.r.t. $t_i$ where $t_i$ are arranged as in Figure 4.1

**Figure 4.1**

**4.2 Remark.** The choice of the base originates from a configuration of planes in the degeneration of $V_3$ to a union of planes. We constructed this degeneration in [MoTe7], but we do not use it directly in this paper. It was used in [MoTe9] to prove the results which are quoted here.

**4.3.** It is easy to see that $T_i$ and $T_j$ are adjacent for $(i, j)$ as follows:

- $i, j \in \{1, 2, 3\}$
- $i = 5: j = 3, 8, 9$
- $i = 6, 7, 8: j = i + 1$.

$T_i$ and $T_j$ are disjoint for $(i, j)$ as follows:

- $i \in \{1, 2, 3\}: j \in \{6, 7, 8, 9\}$
\[ i = 5 \quad j = 1, 2, 6, 7 \]
\[ i = 6 \quad j = 8, 9 \]
\[ i = 7 \quad j = 9. \]

4.4 Claim. The set \( \{T_i\} \) satisfies the following relations:

\[
\langle T_i, T_j \rangle = 1 \quad \text{if } T_i \text{ and } T_j \text{ are adjacent} \\
[T_i, T_j] = 1 \quad \text{if } T_i \text{ and } T_j \text{ are disjoint} \\
[T_1, T_2^{-1}T_3T_2] = 1 \\
[T_5, T_8^{-1}T_9T_8] = 1
\]

Proof. Since the sequence of half-twist \( \{T_1, T_2, T_2^{-1}T_3T_2, T_5, T_9, T_9^{-1} T_9 T_8, T_7, T_6\} \) is represented by a consecutive sequence of paths (see Fig. 4.2), it is a frame. By E. Artin’s Theorem, they satisfy the relations that a frame satisfies (Theorem 2.5).

When writing down the triple relations for the above frame, we get

\[
\langle T_1, T_2 \rangle = 1 \\
\langle T_2, T_2^{-1}T_3T_2 \rangle = 1 \\
\langle T_2^{-1}T_3T_2, T_5 \rangle = 1 \\
\langle T_5, T_9 \rangle = 1 \\
\langle T_9, T_8^{-1}T_9T_8 \rangle = 1 \\
\langle T_8^{-1}T_9T_8, T_7 \rangle = 1 \\
\langle T_7, T_6 \rangle = 1
\]

When writing down the commutative relations, we get:

\[
[T_i, T_j] = 1 \text{ for } T_i \text{ or } T_j \text{ disjoint plus} \\
[T_1, T_2^{-1}T_3T_2] = 1 \\
[T_5, T_8^{-1}T_9T_8] = 1.
\]
We just need to show $\langle T_8, T_9 \rangle = 1$, $\langle T_2, T_3 \rangle = 1$. Since $T_8$ and $T_9$ are adjacent, by Claim 2.11, $T_8^{-1}T_9T_8 = T_9T_8T_9^{-1}$. Now from $\langle T_9, T_8^{-1}T_9T_8 \rangle = 1$, we get

$$1 = T_9T_8^{-1}T_9T_8T_9T_8^{-1}T_9T_9^{-1}T_8^{-1}T_9^{-1} =$$
$$= T_9T_9T_8^{-1}T_9T_9^{-1}T_8^{-1}T_9^{-1} =$$
$$= T_9T_8T_8^{-1}T_9^{-1}T_8^{-1}T_9^{-1}$$

Thus also $T_9T_8T_9T_8^{-1}T_9^{-1} = 1$. Thus $\langle T_9, T_8 \rangle = 1$. Similarly, $\langle T_2, T_3 \rangle = 1$, and we get the claim.

**Figure 4.2**

4.5 Definition. Polarization, orderly adjacent, non orderly adjacent.

We choose an orientation on each $T_i$ with compatibility with its “bigger” neighbor. We call it a polarization. See Figure 4.3.

Most of the adjacent $T_i$’s are orderly adjacent (compatible polarization) apart from $\{T_1, T_2\}$ and $\{T_5, T_8\}$ which are non orderly adjacent.

4.6 Definition. $G_0(9)$.

$G_0(9)$ is a $\mathbb{Z}_2$ extension of a free group on 8 elements. We take the following model for $G_0(9)$:

Let $G_0(9)$ be generated by $\{g_i\}_{i=1}^9$ s.t.

$$[g_i, g_j] = \begin{cases} 
1 & T_i, T_j \text{ are disjoint} \\
\tau & \text{otherwise} 
\end{cases}$$
where $\tau^2 = 1$, $\tau \in \text{Center} \, G_0(9)$.

We take the following action of $\tilde{B}_9$ on $G_0(9)$

$$(gi)\tilde{T}_k = \begin{cases} 
    g_i^{-1}\tau & k = i \\
    g_i & T_i, T_k \text{ are disjoint} \\
    g_i g_k^{-1} & T_i, T_k \text{ are not orderly adjacent} \\
    g_k g_i & \text{otherwise}
\end{cases}$$

4.7 Definition. $\tilde{G}_9$, $c$.

Consider the semidirect product $\tilde{B}_9 \ltimes G_0(9)$ w.r.t. the chosen action.

Let $c = [\tilde{T}_1^2, \tilde{T}_2^2]$.

Let $\xi_1 = (\tilde{T}_2 \tilde{T}_1 \tilde{T}_2^{-1})^2 \tilde{T}_2^{-2}$.

Let $N_9 \triangleleft \tilde{B}_9 \ltimes G_0(9)$ be normally generated by $c\tau^{-1}$ and $(g_1 \xi_1^{-1})^3$.

Let $G_9 = \frac{\tilde{B}_9 \ltimes G_0(9)}{N_9}$.

4.8 Definition. $\hat{\psi}_9$.

Let $\tilde{\psi}_9$ be the homomorphism $\tilde{B}_9 \to S_9$ induced from the standard homomorphism $B_9 \to S_9$ (see 2.7). $\tilde{\psi}_9$ exists since $[X, Y] \to 1$ under the standard homomorphism. Let $\hat{\psi}_9 : G_9 \to S_9$ be defined by the first coordinate $\hat{\psi}_9(\alpha, \beta) = \tilde{\psi}_9(\alpha)$.

4.9 Definition. $\psi$.

The projection $V_3 \to \mathbb{C}^2$, of degree 9, induces a standard monodromy homomorphism $\pi_1(\mathbb{C}^2 - S, \ast) \to S_9$ which we denote by $\psi$. 
4.10 Proposition. $\pi_1(\mathbb{C}^2 - S, \ast) \simeq G_9$ s.t. $\psi$ is compatible with $\hat{\psi}_9$.

Proof. [MoTe9], VI.1.

4.11 Definition. $H_9, H_{9,0}, H'_9, H'_{9,0}$.

Let $Ab : B_9 \to \mathbb{Z}$ be the abelianization of $B_9$ and $B_9$ over its commutator subgroup.

Let $\tilde{A}_b : \tilde{B}_9 \to \mathbb{Z}$ be a homomorphism induced from $Ab$ (which exists since $Ab([X,Y]) = 1$).

Let $\tilde{A}_b : G_9 \to \mathbb{Z}$ be defined by the first coordinate $\tilde{A}_b(\alpha, \beta) = \tilde{A}_b(\alpha)$.

Let $H_9 = \ker \hat{\psi}_9$.

Let $H_{9,0} = \ker \hat{\psi}_9 \cap \ker \tilde{A}_b$.

Let $H'_{9,0}$ be the commutant subgroup of $H_9$ and $H_{9,0}$ respectively.

4.12 Proposition. There exists a series $1 \trianglelefteq H'_9 \trianglelefteq H_{9,0} \trianglelefteq H_9 \trianglelefteq G_9$, where $G_9 / H_9 \simeq S_9$, $H_9 / H_{9,0} \simeq \mathbb{Z}$, $H_{9,0} / H'_9 \simeq (\mathbb{Z} \oplus \mathbb{Z} / 3\mathbb{Z})^8$, $H'_9 = H'_9 \simeq \mathbb{Z} / 2\mathbb{Z}$.

Proof. [MoTe10], Proposition 2.4.

Our main result is a result of type 4.12.

To this end we need to get into the proofs of the structure theorems for $G_9$, which are quoted in 4.10 and 4.12. We need this for the proof of the main result

4.13. $H_{9,0}$ is generated by $\{g_i\}_{i=1}^9_{i \neq 4}$, $\{\xi_i\}_{i=1}^9_{i \neq 4}$, $c$ where

$$[g_i, g_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ c & \text{otherwise.} \end{cases}$$

$$[\xi_i, \xi_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ c & \text{otherwise.} \end{cases}$$

$$[\xi_i, g_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ c & \text{otherwise.} \end{cases}$$

$g_i, \xi_i$ of infinite order.

$g_i^3 = \xi_i^3$. $c^2 = 1$. 

\( c \in \text{Center}(G_9). \)

\( H_9 \) is generated by \( H_{9,0} \) and \( \bar{T}_1^2 \) where \( \bar{T}_1^2 \) is of infinite order.

\( H'_9 = H'_{9,0} \) is generated by \( c \). (\( c \) is the image of the generator of \( \bar{P}'_1 \) from 2.12).

For simplicity we also denote \( \zeta_i = g_\xi \xi_i^{-1}, (\zeta_i^3 = 1). \)

From 4.13 we get the following:

4.14. \( H'_{9,0} = H'_9 \simeq \mathbb{Z}_2 (\subseteq \text{Center}(G_9)). \)

\( H_{9,0}/H'_{9,0} \) is generated by \( \{\xi_i\}_{i=1, i \neq 4}^9 \) and \( \{\zeta_i\}_{i=1, i \neq 4}^9 \), when the only relations are the commutativity relation and \( \zeta_i^3 = 1 \). Thus \( \frac{H_{9,0}}{H'_{9,0}} \simeq (\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^8. \)

\( H_9/H_{9,0} \) is generated by \( \bar{T}_1^2 \) and thus is isomorphic to \( \mathbb{Z}. \)

\( H_9 \) is the kernel of \( G_9 \to S_9 \), and thus \( G_9/H_9 \simeq S_9. \)

\section{§5. The Main Result.}

Our main result is the following theorem.

5.0 Theorem. Let \( S \) be the branch curve of a generic projection to \( \mathbb{CP}^2 \) of a Veronese embedding of deg 3. Then \( \pi_1(\mathbb{CP}^2 - S, *) \) is an extension of a solvable group by the symmetric group of 9 elements. In fact, we have \( 1 \left< \mathcal{P}_9 \right>< \mathcal{P}_9 \left< \mathcal{P}_9 \right< H_9 \right< \mathcal{G}_9 \)

where \( \mathcal{G}_9/\mathcal{H}_9 \simeq S_9, \mathcal{H}_9/\mathcal{H}_{9,0} \simeq Z_9, \mathcal{H}_{9,0}/\mathcal{H}_{9,0}' \simeq (\mathbb{Z} \oplus \mathbb{Z}_3)^8, \mathcal{H}_{9,0}' \simeq \mathbb{Z}_2. \)

Proof. It is easy to calculate \( \text{deg } S \) (see [MoTe3]) and it is 18.

We consider the situation of 1.4 for the branch curve from our Theorem. By [MoTe9], Lemma 2.3, there is a possibility to choose a \( g \)-base \( \{\Gamma_i, \Gamma_i'\} \) such that \( \psi(\Gamma_i) = \psi(\Gamma_i') = \text{transposition} \). (This choice is a consequence of the degeneration of the surface to a union of 9 planes.)

By 1.5, \( \pi_1(\mathbb{CP}^2 - S, *) = \frac{\pi_1(\mathbb{CP}^2 - S, *)}{\langle \prod_{i=9}^1 \Gamma_i, \Gamma_i' \rangle} \). Denote \( \hat{\beta} : \pi_1(\mathbb{CP}^2 - S, *) \to G_9 \) to be \( \hat{\beta} = \frac{G_9}{\langle \delta \rangle} \) the isomorphism from 4.10 and \( \delta = \hat{\beta} \left( \prod_{i=9}^1 \Gamma_i \right) \). Clearly, \( \pi_1(\mathbb{CP}^2 - S, *) \simeq G_9/\langle \delta \rangle \) which we denote by \( \mathcal{G}_9 \). To prove the theorem we shall prove that \( \mathcal{G}_9 = G_9/\langle \delta \rangle \) is an extension of a solvable group by a symmetric group.
In 4.12 we introduced a sequence \( 1 < H'_9,0 < H_9,0 < H_9 < G_9 \) and the appropriate quotients. Let \( \overline{H}_9,0, \overline{H}_9,0, \overline{H}_9 \) be the images of \( H'_9,0, H_9,0, H_9 \) in \( G_9 \) respectively. To prove the theorem we shall compute \( G_9/\overline{H}_9,0, \overline{H}_9/\overline{H}_9,0, \overline{H}_9,0/\overline{H}_9,0 \) and \( \overline{H}_9,0 \).

We first need to prove some results on \( G_9 \) in general and on \( \delta \) in particular. This is done in Claims 5.1–5.10. From general arguments we already know that \( \delta \in \text{Center}(G_9) \) (cf. Proposition 3.2).

\[ 5.1 \text{ Claim.} \quad \delta \in H_9, \hat{\beta}(\Gamma_i \Gamma_i') \in H_9. \]

**Proof of Claim 5.1.** By 4.10, \( \hat{\psi}_9 \hat{\beta}(\Gamma_i \Gamma_i') = \psi_i \Gamma_i \Gamma_i' \). Since \( \psi(\Gamma_i \Gamma_i') = 1 \), and thus \( \hat{\beta}(\Gamma_i \Gamma_i') \in \ker \hat{\psi}_9 = H_9 \). Since \( \delta = \prod \hat{\beta}(\Gamma_i \Gamma_i') \), it is also in \( H_9 \). \[ \square \] for Claim 5.1

The new quotients will be determined by an expression of \( \delta \) as a product of elements in \( H_9 \) and elements which are in \( H_9 \) but not in \( H_9,0 \).

\[ 5.2 \text{ Definition.} \]

By abuse of notation the images in \( G_9 = \tilde{B}_9 \rtimes G_0(9) / N_9 \), of \( \tilde{T}_i \) from \( \tilde{B}_9 \) (see 4.2) are also denoted by \( \tilde{T}_i \). We also define:

\[ \tilde{T}_4 = (\tilde{T}_5)_{\tilde{t}_3^{-1}} \tilde{T}_7 \tilde{T}_3^{-1} \tilde{T}_2. \]

\[ g_4 = (g_5)_{\tilde{t}_3^{-1}} \tilde{T}_7 \tilde{T}_3^{-1} \tilde{T}_2 \] for \( g_5 \) from 4.13.

\[ \xi_4 = (\xi_5)_{\tilde{t}_3^{-1}} \tilde{T}_7 \tilde{T}_3^{-1} \tilde{T}_2 \] for \( \xi_5 \) from 4.13.

To work in \( G_9 \) we need some commutativity relations:

\[ 5.3 \text{ Claim.} \quad \text{In } G_9: \]

(i) \( \tilde{T}_i^2, \xi_i, g_i \in H_9 \) \( i = 1, \ldots, 9 \).

(ii) \([\tilde{T}_i^2, g_i], [\xi_i g_i] = 1 \) or \( c \).

(iii) If \( X \) and \( Y \) are 2 adjacent half-twists, then \([\tilde{X}^2, \tilde{Y}^2] = c \).

**Proof.**

(i) Since \( T_i \) is a half-twist \( i = 1 \ldots 9 \), thus \( \hat{\psi}_9(\tilde{T}_i) \) is a transposition and \( \hat{\psi}_9(\tilde{T}_i^2) = 1 \). Thus \( \tilde{T}_i^2 \in H_9 \). The elements \( \{g_i, \xi_i \}_{i=1}^9 \) are in \( H_9 \) by 4.13. Since
$g_5$ is in $H_9$ and $H_9$ is a normal subgroup ($= \ker \hat{\psi}_9$), $g_4$ is also in $H_9$. The same applies for $\xi_4$.

(ii) Since $H'_9 = \{1, c\}$, $c^2 = 1$.

(iii) Since it is true in $\tilde{B}_9$ by Claim 2.13. \hfill \Box$ for Claim 5.3

In order to compute the corresponding quotients in $\overline{G}_9$, we need to express $\delta$ which is in $H_9$ (see 5.1) in terms of the following generators of $H_9$: $\{\zeta_i\}_{i=1, i \neq 4}$, $\{\xi_i\}_{i=1, i \neq 4}$, $c$ and $\hat{T}_i^2$ (see 4.13). Recall that $\delta = \prod_{i=9} \hat{\beta}(\Gamma_i'\Gamma_i)$ where $\forall i = 1 \ldots 9 \hat{\beta}(\Gamma_i'\Gamma_i) \in H_9$. Thus, we shall first express $\hat{\beta}(\Gamma_i'\Gamma_i)$ for $i = 1 \ldots 9$ in terms of $\{\zeta_i\}_{i=1, i \neq 4}$, $\{\xi_i\}_{i=1, i \neq 4}$, $c$ and $\tilde{T}_i^2$, and then we multiply these expressions to get an expression for $\delta$ in these generators (see 5.9). In 5.10 we replace $g_i$ by $\zeta_i\xi_i$.

5.4 Claim.

\[
\begin{align*}
\hat{\beta}(\Gamma_1'\Gamma_1) &= g_1\tilde{T}_1^2 \\
\hat{\beta}(\Gamma_2'\Gamma_2) &= g_2^{-1}\xi_2\tilde{T}_2^2 \\
\hat{\beta}(\Gamma_3'\Gamma_3) &= g_3\xi_3^{-1}\tilde{T}_3^2 \\
\hat{\beta}(\Gamma_4'\Gamma_4) &= g_4^{-1}\xi_4\tilde{T}_4^2 \\
\hat{\beta}(\Gamma_5'\Gamma_5) &= g_5^{-1}\xi_5\tilde{T}_5^2 \\
\hat{\beta}(\Gamma_6'\Gamma_6) &= g_6\tilde{T}_6^2 \\
\hat{\beta}(\Gamma_7'\Gamma_7) &= g_7\xi_7^{-1}\tilde{T}_7^2 \\
\hat{\beta}(\Gamma_8'\Gamma_8) &= g_8^{-1}\xi_8\tilde{T}_8^2 \\
\hat{\beta}(\Gamma_9'\Gamma_9) &= cg_g^{-1}\tilde{T}_9^2
\end{align*}
\]

Proof. We take a new set of generators for $G$:

$E_i = \begin{cases} 
\Gamma_i & i \neq 2, 7 \\
\Gamma_{i'} & i = 2, 7 
\end{cases}$  \quad $E_i' = \begin{cases} 
\Gamma_{i'} & i \neq 2, 7 \\
\Gamma_{i'}\Gamma_i\Gamma_{i'}^{-1} & i = 2, 7 
\end{cases}$

(This choice which was made in [MoTe9] originated from a certain relation in $G$ induced by the affine Van Kampen Theorem.) Clearly $\Gamma_{i'}\Gamma_i = E_{i'}E_i$. Let $A_i =$
$E_i E_i^{-1}$. Clearly, $\Gamma_i \Gamma_i = E_i E_i = A_i E_i^2$. By the construction of $\hat{\beta}$ (see [MoTe9], Ch.V), $\hat{\beta}(E_i^2) = \hat{T}_i^2$ and $\hat{\beta}(A_i)$ is as follows:

$$
\begin{align*}
\beta(A_1) &= g_1 \\
\beta(A_2) &= g_2^{-1}\xi_2 \\
\beta(A_3) &= g_3\xi_3^{-1} \\
\beta(A_4) &= g_4^{-1}\xi_4 \\
\beta(A_5) &= g_5^{-1}\xi_5 \\
\beta(A_6) &= g_6 \\
\beta(A_7) &= g_7\xi_7^{-1} \\
\beta(A_8) &= g_8^{-1}\xi_8 \\
\beta(A_9) &= cg_9^{-1}
\end{align*}
$$

□ for Claim 5.6

In the next step we express $\hat{T}_i^2$ in terms of $\{\xi_i\}_{i=1}^9$ and $\hat{T}_i^2$. The main point in the proof of the next claim is that for 3 half-twists which form a triangle where one of the edges is $T_i$, the product $\hat{X}^2\hat{Y}^{-2}$ of the other 2 half-twists can be expressed in terms of $\xi_i, \xi_i^{-1}$ and $c$. The exact statement is as follows:

**5.5 Claim.** Let $X, Y$ be 2 half-twists, $X = H(x), Y = H(y), T_i = H(t_i)$ s.t. $x, y, t_i$ make a triangle. Assume that $x$ and $y$ meet in $\nu$, and a counterclockwise rotation around $\nu$ inside the triangle meets $x$ before it meets $y$.

(i) If the polarization of $T_i$ goes from $x$ to $y$, then $\xi_i = \hat{X}^2\hat{Y}^{-2}$.

(ii) If the polarization of $T_i$ goes from $y$ to $x$, then $\xi_i = \hat{X}^{-2}\hat{Y}^2$.

**Proof.** Claim IV.4.1 of [MoTe9].
Using this claim we prove the following

5.6 Lemma.

(i) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = c \xi_1^{-1} \xi_2 \)
(ii) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = \xi_1^{-1} \xi_3^{-1} \)
(iii) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = \xi_1^{-1} \xi_3^{-2} \xi_5^{-1} \)
(iv) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = c \xi_1^{-1} \xi_3^{-2} \xi_5^{-2} \xi_9^{-1} \)
(v) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = \xi_1^{-1} \xi_3^{-2} \xi_5^{-2} \xi_8 \)
(vi) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = c \xi_1^{-1} \xi_3^{-2} \xi_5^{-2} \xi_7 \xi_8 \)
(vii) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = \xi_1^{-1} \xi_3^{-2} \xi_5^{-2} \xi_6 \xi_7 \xi_8 \)
(viii) \( \tilde{T}_2^2 \tilde{T}_1^{-2} = \xi_1^{-1} \xi_3^{-2} \xi_4 \xi_5^{-2} \xi_7 \xi_8 \)

Proof. The proof is based on Claim 5.5.

Moreover, we interchange between the \( \xi_i \)'s using the commutator from 4.13

\[ [\xi_i^{\pm 1}, \xi_j^{\pm 1}] = \begin{cases} 1 & T_i \text{ and } T_j \text{ are disjoint} \\ c & \text{otherwise} \end{cases} \]

(i) We write \( \tilde{T}_2^2 \tilde{T}_1^{-2} = \tilde{T}_2^2 \tilde{W}^{-2} \tilde{T}_1^{-2} \) for \( W = (T_1)_{T_2^{-1}} \) which creates a triangle with \( \tilde{T}_1 \) and \( \tilde{T}_2 \) (see Fig. 5.1) We use Claim 5.5 twice – first when we take \( W, T_2, T_1 \) instead of \( X, Y, T \) from Claim 5.5(i), and second when we take \( T_1, W, T_2 \) instead of \( X, Y, T_2 \) from Claim 5.5(ii). By Claim 5.5(i) \( \tilde{W}^{-2} \tilde{T}_2^{-2} = \xi_1 \), and thus \( \tilde{T}_2^2 \tilde{W}^{-2} = \xi_1^{-1} \). By Claim 5.5(ii) \( \tilde{T}_1^{-2} \tilde{W}^2 = \xi_2 \), and since by Claim 5.3(iii) \( [\tilde{T}_1^{-2}, \tilde{W}^2] = c \), we get \( \tilde{W}^2 \tilde{T}_1^{-2} = c \xi_2 \). Together we get (i).

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw (0,0) -- (2,2) node[midway, above right] {T_2} -- (4,0) node[midway, below] {T_1} -- cycle;
\node at (2,2) [circle,fill,inner sep=1.5pt] {};
\node at (0,0) [circle,fill,inner sep=1.5pt] {};
\node at (4,0) [circle,fill,inner sep=1.5pt] {};
\draw[->] (0,0) -- (2,2);
\draw[->] (2,2) -- (4,0);
\draw[->] (0,0) -- (4,0);
\end{tikzpicture}
\caption{Figure 5.1}
\end{figure}

(ii) We write \( \tilde{T}_2 \tilde{T}_1^{-2} = (\tilde{T}_3 \tilde{Z}^{-2}) (\tilde{Z}^2 \tilde{T}_1^{-2}) \) for \( Z = (\tilde{T}_1)_{\tilde{T}_3} \), which creates a triangle with \( T_1 \) and \( T_3 \). (See Fig. 5.2) By Claim 5.5 applied twice, \( \tilde{T}_3^{-2} \tilde{Z}^2 = \xi_1 \)
and \( \tilde{Z}^{-2}T_1^{-2} = \xi_3 \). Thus \( \tilde{Z}^{-2}T_3^{-2} = \xi_1^{-1} \) and \( \tilde{T}_1^{-2} \tilde{Z}^2 = \xi_3^{-1} \). By 5.3(iii) we get \( \tilde{T}_3^2 \tilde{Z}^{-2} = c \xi_1^{-1} \), \( \tilde{Z}^2 \tilde{T}_1^{-2} = c \xi_3^{-1} \). Since \( c \in \text{Center}(G_9) \) and \( c^2 = 1 \), then \( \tilde{T}_3^2 \tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-1} \).

\[ \text{Figure 5.2} \]

(iii) We install \( \tilde{T}_3^{-2}T_3^2 \) in the middle. Since \( \tilde{T}_3 \) relates to \( \tilde{T}_5 \) as \( \tilde{T}_1 \) relates to \( \tilde{T}_3 \) we have \( \tilde{T}_3^2 \tilde{T}_3^{-2} = \xi_3^{-1}\xi_1^{-1} \). Then \( \tilde{T}_3^2 \tilde{T}_3^{-2} \tilde{T}_3^2 \tilde{T}_3^{-1} = \xi_3^{-1}\xi_5^{-1}\xi_1^{-1}\xi_3^{-1} \). Since \( [\xi_3,\xi_5] = [\xi_3,\xi_1] = c, \ [\xi_1,\xi_5] = 1, \ c^2 = 1 \), and \( c \in \text{Center}(G_9) \), we get \( \tilde{T}_3^2 \tilde{T}_1^{-2} = \xi_1^{-1}\xi_3^{-2}\xi_5^{-1} \).

(iv) We install \( \tilde{T}_5^{-2}T_5^2 \) in the middle. Since \( \tilde{T}_9 \) relates to \( \tilde{T}_3 \) as \( \tilde{T}_1 \) relates to \( \tilde{T}_9 \), then \( \tilde{T}_9 \tilde{T}_9^{-2} = \xi_5^{-1}\xi_9^{-1} \). Thus \( \tilde{T}_9 \tilde{T}_9^{-2} \tilde{T}_9 \tilde{T}_1^{-1} = \xi_5^{-1}\xi_9^{-1}\xi_1^{-1}\xi_3^{-2}\xi_5^{-1} = c \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_9^{-1} \). The last equation is based on the commutators of \( \xi_i \) and the fact that \( c \in \text{Center} G_9 \), \( c^2 = 1 \).

(v) \( \tilde{T}_5^2 \tilde{T}_1^{-2} = \tilde{T}_5^2 \tilde{T}_5^{-2} \tilde{T}_5^2 \tilde{T}_1^{-2} \). \( T_8 \) relates to \( T_5 \) as \( T_2 \) relates to \( T_1 \) and thus \( \tilde{T}_8^2 \tilde{T}_1^{-2} = c \xi_5^{-1}\xi_8 \). Thus \( \tilde{T}_8^2 \tilde{T}_1^{-2} = c \xi_5^{-1}\xi_8 \xi_1^{-1}\xi_3^{-2}\xi_5^{-1} \). Since \( \xi_8 \) commutes with \( \xi_1, \xi_3 \) and \( \xi_5 \) commutes with \( \xi_1 \) and \( [\xi_5,\xi_8] = [\xi_5,\xi_3] = c \), we have \( \tilde{T}_8^2 \tilde{T}_1^{-2} = c^3 \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8 \) which equals \( \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8 \).

(vi) \( \tilde{T}_7^2 \tilde{T}_1^{-2} = \tilde{T}_7^2 \tilde{T}_8^{-2} \tilde{T}_8^2 \tilde{T}_1^{-2} \). Since \( T_7 \) relates to \( T_8 \) as \( T_1 \) relates to \( T_3 \), then \( \tilde{T}_7^2 \tilde{T}_7^{-2} = \xi_7^{-1}\xi_8^{-1} \) and \( \tilde{T}_7^2 \tilde{T}_7^{-2} = (\xi_7^{-1}\xi_8^{-1})^{-1} = \xi_7 \xi_7 \) and thus \( \tilde{T}_7^2 \tilde{T}_1^{-2} = \tilde{T}_7^2 \tilde{T}_8^{-2} \tilde{T}_8^2 \tilde{T}_1^{-2} = (\xi_7 \xi_7) \cdot \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_8 \). As before, \( \tilde{T}_7^2 \tilde{T}_1^{-2} = c^3 \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7 \tilde{\xi}_8^2 \) which equals \( c \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7 \tilde{\xi}_8^2 \).

(vii) \( \tilde{T}_6^2 \tilde{T}_1^{-2} = \tilde{T}_6^2 \tilde{T}_7^{-2} \tilde{T}_7^2 \tilde{T}_1^{-2} \). \( T_6 \) relates to \( T_7 \) as \( T_7 \) relates to \( T_8 \) and thus \( \tilde{T}_6^2 \tilde{T}_7^{-2} = \xi_7 \xi_6 \) (see (vi)). Therefore \( \tilde{T}_6^2 \tilde{T}_1^{-2} = c \xi_7 \xi_6 \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_7 \tilde{\xi}_8^2 \) which equals as before \( c^2 \xi_1^{-1}\xi_3^{-2}\xi_5^{-2}\xi_6 \xi_7 \tilde{\xi}_8^2 \).

(viii) \( T_4 = (T_5)_{T_8^{-1}} ) \). By Claim II.1.0 of [MoTe9], if \( X = H(x) \) is represented by a diffeomorphism \( \beta \) and \( Y = H(y) \), then \( Y_X = H((y)\beta) \). Therefore
$T_4$ is a half-twist and we write $T_4 = H(t_4)$. Moreover, to describe $T_4$ we must apply $T_8^{-1}, T_7, T_3^{-1}, T_2$ on $t_5$ and we get $t_4$ as in Figure 5.3:

![Figure 5.3](image)

Since $T_6$ relates to $T_4$ as $T_2$ relates to $T_1$, then $\tilde{T}_6^2 \tilde{T}_4^{-2} = c \xi_4^{-1} \xi_6$. Now $\tilde{T}_4^2 \tilde{T}_1^{-2} = \tilde{T}_4^2 \tilde{T}_6^{-2} \tilde{T}_6^2 \tilde{T}_1^{-2} = (c \xi_4^{-1} \xi_6)^{-1} \xi_1^{-1} \xi_3^{-1} \xi_5^{-2} \xi_6 \xi_2 \xi_2^2 \xi_2^2$ which equals as before to $\xi_1^{-1} \xi_3^{-2} \xi_4 \xi_5^{-2} \xi_7^2 \xi_8^2$.

□ for Lemma 5.6

In order to express $g_4$ and $\xi_4$ in terms of $\{g_i\}_{i=1, i \neq 4}^9$ and $\{\xi_i\}_{i=1, i \neq 4}^9$, we need the following claim from [MoTe9]

**5.7 Claim.** For $f_i = g_i$ or $\xi_i$,

$$
(f_i)_{T_k} = \begin{cases}
    f_i^{-1} \nu & k = i \\
    f_i & T_i, T_k \text{ weakly disjoint} \\
    f_k f_i & T_i, T_k \text{ orderly adjacent} \\
    f_i f_k^{-1} & T_i, T_k \text{ are not orderly adjacent.}
\end{cases}
$$

$$
(f_i)_{T_k^{-1}} = \begin{cases}
    f_i^{-1} \nu & k = i \\
    f_i & T_i, T_k \text{ weakly disjoint} \\
    f_i f_k & T_i, T_k \text{ orderly adjacent} \\
    f_k^{-1} f_i & T_i, T_k \text{ are not orderly adjacent.}
\end{cases}
$$

**Proof.** [MoTe9], Lemma IV.6.3. The conjugations for $g_i$ are part of the definition of $G_0(9)$ (see 4.6) and remains when moving to $G_9 = \frac{\tilde{B}_9 \times G_0(9)}{N_9}$. □ for Claim 5.7

Now we can express $g_4, \xi_4$ in terms of $\{g_i\}_{i=1, i \neq 4}^9$. 
5.8 Claim.

(i) \( \xi_4 = c \xi_2 \xi_3 \xi_5 \xi_7^{-1} \xi_8^{-1} \)

(ii) \( g_4 = c g_2 g_3 g_5 g_7^{-1} g_8^{-1} \)

Proof. The proof is similar for (i) and (ii), and is based on 5.9. We shall only prove (i). By 5.2, \( \xi_4 = (\xi_5)^{\tilde{T}_{8^{-1}} \tilde{T}_{7^{-1}} \tilde{T}_{5^{-1}} \tilde{T}_2} \). Since \( T_5 \) and \( T_8 \) are not orderly adjacent by 5.7, \( (\xi_5)^{\tilde{T}_{8^{-1}}} = \xi_8^{-1} \xi_5 \). Since \( T_5 \) and \( T_7 \) are disjoint, \( (\xi_5)^{\tilde{T}_7} = \xi_5 \). Since \( T_7 \) and \( T_8 \) are orderly adjacent, \( (\xi_5)^{\tilde{T}_7} = \xi_7 \xi_8 \) and thus \( (\xi_8^{-1})^{\tilde{T}_7} = \xi_8^{-1} \xi_7^{-1} \). Together we have \( (\xi_5)^{\tilde{T}_{8^{-1}} \tilde{T}_7} = \xi_8^{-1} \xi_7^{-1} \xi_5 \). We now apply \( \tilde{T}_3^{-1} \). Since \( T_3 \) is disjoint from \( T_7 \) and \( T_8 \), then \( (\xi_8^{-1} \xi_7^{-1})^{\tilde{T}_3^{-1}} = \xi_8^{-1} \xi_7^{-1} \). On the other hand, \( T_5 \) and \( T_3 \) are orderly adjacent and thus \( (\xi_5)^{\tilde{T}_3^{-1}} = \xi_5 \xi_3 \) and \( (\xi_5)^{\tilde{T}_{8^{-1}} \tilde{T}_7 \tilde{T}_3^{-1}} = \xi_8^{-1} \xi_7^{-1} \xi_5 \xi_3 \). Now \( \tilde{T}_2 \) acts on \( \xi_3 \) to get \( \xi_2 \xi_3 \) and does not move the other factors. Thus \( \xi_4 = \xi_8^{-1} \xi_7^{-1} \xi_5 \xi_2 \xi_3 \). We rearrange the factors using the commutators of 4.13 to get

\[
\xi_4 = c \xi_2 \xi_3 \xi_5 \xi_7^{-1} \xi_8^{-1}.
\]

\( \square \) for Claim 5.8

In fact we are interested in \( \delta \) up to a product with \( c \) and thus we formulate the following:

5.9 Claim. Up to multiplication by \( c \)

\[
\delta = g_1 g_2^{-2} g_5^{-2} g_6 g_7 g_9^{-1} \xi_1^{-8} \xi_2 \xi_3^{-12} \xi_5^{-8} \xi_6 \xi_7 \xi_8 \xi_9^{-1} \tilde{T}_1^{18}.
\]

Proof of Claim 5.9. By definition, \( \delta = \hat{\beta} \left( \prod_{i=9}^{i=9} \Gamma_i \Gamma_i \right) \) which equals \( \prod_{i=9}^{i=9} \hat{\beta}(\Gamma_i, \Gamma_i) \). We substitute in the product the values of \( \hat{\beta}(\Gamma_i, \Gamma_i) \), \( i = 1, \ldots, 9 \) from Claim 5.4. In the resulting formula, we replace \( \tilde{T}_1^2 \) by \( \left( \tilde{T}_1^2 \tilde{T}_1^{-2} \right) \tilde{T}_1^2 \) for each \( i = 1, \ldots, 9 \). We then substitute the formula for \( \tilde{T}_1^2 \tilde{T}_1^{-2} \) from Claim 5.6. We also substitute the values of \( g_4 \) and \( \xi_4 \) from Claim 5.8. We then get a formula for \( \delta \) as a product of \( \{\xi_i, g_i\}_{i=1}^{9} \) \( i \neq 4 \) and \( \tilde{T}_1^2 \). Since we are not interested in the appropriate power of \( c \), we can use Claim 5.3(ii) by “pushing” all the powers of \( \tilde{T}_1^2 \) to the right end of the last formula and rearrange the \( \xi_i \)’s and the \( g_i \)’s to the get the claim.

\( \square \) for Claim 5.9
5.10 Claim. Up to multiplication by \( c \), for \( \zeta_i \) and \( \xi \) from 4.13, we have \( \delta = \zeta_1 \zeta_2 \zeta_3 \zeta_6 (\zeta \zeta_9)^{-1} \zeta_1^{-7} \xi_2 \xi_3^{-12} x i_5 i_5 = 10 \xi_6 \xi_7 \xi_8 \xi_9^{-2} \bar{T} \).

Proof of Claim 5.10. In the formula from Claim 5.9 we replace \( g_i \) by \( \zeta_i \). Since \( H'_9 = \{ 1, c \} \) where \( c \in \text{Center}(G_9) \), we can rearrange the terms of the formula. Also using \( \xi_i^3 = 1 \) we get the claim. \( \square \) for Claim 5.10

We go back to the proof the theorem.

To prove that \( \overline{G}_9 \) is an extension of a solvable group by a symmetric group, it is enough to find a normal subgroup whose quotient is \( S_9 \). The subgroup will be \( \overline{H}_9 \).

Recall (3.12) that there exists a series \( 1 \triangleleft H_{9,0} \triangleleft H_9 \triangleleft G_9 \). We defined \( \overline{G}_9 = G_9/\langle \delta \rangle \), and \( \overline{H}_9, \overline{H}_{9,0}, \overline{H}'_{9,0} \) to be the images of \( H_9, H_{9,0}, H'_{9,0} \) in \( \overline{G}_9 \) respectively, and we have a series \( 1 \triangleleft \overline{H}'_{9,0} \triangleleft \overline{H}_{9,0} \triangleleft \overline{H}_9 \triangleleft \overline{G}_9 \). We shall compute the quotients. Since \( \delta \in H_9 \) (Claim 5.1), \( \overline{G}_9/\overline{H}_9 \simeq G_9/\delta \simeq S_9 \). Since \( H_9/H_{9,0} \) is generated by \( \bar{T}^2 \) (see 4.14), \( \overline{H}_9/\overline{H}_{9,0} \) is also generated by \( \bar{T}^2 \). By Claim 5.12, \( (\bar{T}^2)^9 \delta^{-1} \in H_{9,0} \). So when considered as elements of \( \overline{H}_9 \), \( (\bar{T}^2)^9 \in \overline{H}_{9,0} \), and thus as elements of \( \overline{H}_9/\overline{H}_{9,0} \), \( \bar{T}^2 \) is of order 9.

Thus \( \overline{H}_9/\overline{H}_{9,0} \simeq \mathbb{Z}_9 \). Now let \( Y_1 = \xi_1^{-7} \xi_2 \xi_3^{-12} \xi_5^{-10} \xi_6 \xi_7 \xi_8 \xi_9^{-2} (\bar{T}^2)^{18} \). We complete \( Y_1 \) to a base \( Y_1, \ldots, Y_9 \) of \( \langle \bar{T}_1^2, \{ \xi_i \}_{i=1, i \neq 4} \rangle \). \( H_9/H_9' \) is generated by \( Y_1, \ldots, Y_9, \{ \xi_i \}_{i=1, i \neq 4} \).

Modulo \( \langle \delta \rangle \), \( Y_1 \in \{ \xi_i \}_{i=1, i \neq 4} \) and \( Y_2, \ldots, Y_8 \) are of infinite order. Thus \( \overline{H}_9/\overline{H}_{9,0} \simeq (\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^8 \).

By 3.2, \( \langle \delta \rangle \) is an infinite cyclic group where \( \delta \in \text{Center}(G_9) \). Since \( c^2 = 1 \), then \( \langle c \rangle \cap \langle \delta \rangle = 1 \) and \( \overline{H}'_{9,0} \simeq \overline{H}_{9,0}/\langle \delta \rangle \cap \overline{H}'_{9,0} \simeq \overline{H}_{9,0} \simeq \mathbb{Z}_2 \). Thus we have a series \( 1 \triangleleft \overline{H}'_{9,0} \triangleleft \overline{H}_{9,0} \triangleleft \overline{G}_9 \) s.t. \( \overline{H}_9 \) is a solvable group, and \( \overline{G}_9/\overline{H}_9 \simeq \text{symmetric group of 9 elements} \). \( \square \) for Theorem 5.0
§6. The result of Veronese of order $p$.

The result for any Veronese is similar (see below), but the representation of the proof for $p = 3$ is much more “reader friendly”. The result for any $p$ is as follows:

There exist 2 series $A \triangleleft B \triangleleft C \triangleleft G$ and $A \triangleleft \overline{B} \triangleleft \overline{C} \triangleleft G$ s.t.

$$G/C \simeq \overline{G}/\overline{C} \simeq S_{p^2}$$

$$C/B \simeq \mathbb{Z}, \overline{C}/\overline{B} \simeq \mathbb{Z}_q$$

$$B/A \simeq \overline{B}/\overline{A} \simeq \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}_3)^{p^2 - 1} & p = O(3) \\ \mathbb{Z}^{p^2 - 1} & p \neq O(3) \end{cases}$$

$$A \simeq \overline{A} \simeq \begin{cases} \mathbb{Z}_2 & p \text{ odd} \\ 0 & p \text{ even} \end{cases}$$

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