Discreteness corrections and higher spatial derivatives in effective canonical quantum gravity

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Abstract

Canonical quantum theories with discrete space may imply interesting effects. This article presents a general effective description, paying due attention to the role of higher spatial derivatives in a local expansion and differences to higher time derivatives. In a concrete set of models, it is shown that spatial derivatives one order higher than the classical one are strongly restricted in spherically symmetric effective loop quantum gravity. Moreover, radial holonomy corrections cannot be anomaly-free to this order.

1 Introduction

A canonical quantization of gravity implies different types of modifications of the classical space-time continuum, depending on which precise methods are used. In several approaches, discrete structures appear which should modify not only the dynamics of the theory but also its fundamental symmetries. The consistency of such modifications largely remains to be explored, especially regarding conditions for a quantum theory free of gauge anomalies. Several models of consistent [1] or perfect discretizations [2, 3] are available. At the level of effective actions or constraints, however, only pointwise quantum-geometry modifications have been implemented so far, which may indirectly be related to spatial discreteness but remain local and do not give rise to a derivative expansion. In the case of cosmological perturbations in effective models of loop quantum gravity, for instance, current versions either use pointwise exponentials of a connection instead of integrated...
holonomies \[4, 5\], or — so far for vector modes only — perform an expansion but truncate it at one order before higher spatial derivatives would become relevant \[6\]. The form or even the possibility of consistent versions of discreteness in effective theories of loop quantum gravity therefore remains unclear.

Motivated by this crucial gap in the current understanding of canonical quantum gravity, we start in this article a systematic investigation of consistent discreteness corrections using effective methods. These new canonical tools are general, but we aim to provide specific examples in models of loop quantum gravity, the canonical theory in which the most details about discrete representations are available. In particular, in this setting there are not only standard discretization effects of spatial derivatives, but also a new type of modification related to the prominence of holonomies in its kinematical quantum representation.

Loop quantum gravity \[7, 8, 9\] is based on a representation in which holonomies of connections rather than connection components act as operators. This feature, as often emphasized is crucial for the spatial background independence realized in the theory, and for the implication of discrete spatial quantum geometry. Classical expressions that depend on connection components, especially the Hamiltonian constraint of canonical gravity, must then be regularized or modified before they can be turned into operators, reflecting more-indirect discreteness effects in space-time observables. Characteristic corrections to the classical dynamics result, which have been investigated in quite some detail in loop quantum cosmology and also in black-hole models. Most of these investigations, however, have made use either of a complete elimination of local degrees of freedom in exactly homogeneous minisuperspace models, or of gauge fixings or other limitations on space-time structures such as specific choices of time variables (deparameterization). In such models, it is not clear whether the quantum theory analyzed is free of anomalies and whether it respects covariance (or even energy conservation \[10\]). That crucial space-time effects may be missed by such restricted treatments is shown by a recent discovery suggesting that space-time turns Euclidean when holonomy effects are significant \[11\]. It is therefore incorrect to assume classical properties of space-time, as one does when one fixes the gauge before quantization or uses other properties of the classical constraints.

In order to overcome these limitations, one must find a consistent quantum realization of the constraint algebra that remains first class off-shell. For holonomy corrections, such systems have been found only in \(2 + 1\) dimensions \[12\] or partially, and with effective methods, in spherically symmetric models \[13, 14\] and for cosmological perturbations \[5\]. Effective methods, developed for this purpose in \[15, 16\], allow one to compute Poisson brackets of quantum-corrected constraints instead of commutators of constraint operators, implying large simplifications. But still, the status of consistent holonomy-corrected constraints remains incomplete even in symmetric models: So far, only “pointwise” holonomies have been implemented, exponentiating connection components without including spatial integrations along curves. The weak non-locality associated with curve integrations turns out to be difficult to parameterize and implement, even if one approximates it by a derivative expansion. However, it is the most characteristic consequence of spatial discreteness in loop quantum gravity. This non-locality, or higher spatial derivatives, must therefore
be realized in an anomaly-free way before one can be sure that discrete quantum space as envisaged by loop quantum gravity is able to provide consistent space-time models.

In this article, we investigate these issues further. We first provide a systematic treatment of a derivative expansion in effective canonical gravity, taking into account the different appearances of time and space derivatives in a Hamiltonian setting. To facilitate such expansions for first-class constraints, subject to the strong requirement of anomaly-freedom of any correction terms, we will provide several formulas of Poisson brackets applicable generally. Our main examples will be given in spherically symmetric models. So far the results are negative: in the models considered, higher spatial derivatives one order above the classical one are ruled out. But given the complexity of the problem, our analysis of spherically symmetric models of loop quantum gravity remains incomplete, and there is still room for potentially consistent versions based on more general (and more complicated) parameterizations of discreteness effects.

2 Holonomy corrections

Discreteness corrections appear in different forms whenever a Hamiltonian is to be formulated on a discrete structure. In general, the only feature one may assume for an effective formulation is the presence of higher spatial derivatives, resulting from finite differences expanded for a local effective description. But there are also more characteristic consequences of some approaches, most importantly the use of holonomies in loop quantum gravity. Properties of holonomy operators and their conjugate fluxes imply spatial discreteness at a kinematical level [17], and they lead to modifications of the Hamiltonian constraint [18, 19] that have consequences for the dynamics as well as the structure of space-time.

Holonomies $h_s(A^i_a) = \mathcal{P} \exp(\int \tau_i A^i_a \dot{e}^a d\lambda)$ (with Pauli matrices $2i\tau_j$) represent the su(2)-connection $A^i_a$ by SU(2)-elements associated to all (analytic) curves $e$ in space. In loop quantum gravity [7, 8, 9], they act as multiplication operators which, starting with a connection-independent state, construct the whole kinematical Hilbert space [20]. Collecting all curves used in repeated actions of holonomies, one obtains a graph as a model for the underlying discrete space. Spin-network states can be used as a basis of the Hilbert space [20].

On these graphs, holonomies act by generating new edges or by changing the excitation level (the spins) of existing ones. A holonomy around a closed loop can be used to approximate curvature components of $A^i_a$, the better the smaller the loop is. However, the limit of the holonomy around a loop shrinking to a point divided by the coordinate area of the loop, in which case one would obtain exactly the curvature components, does not exist for operators: A holonomy operator maps a state into an orthogonal one, so that the limit state in the attempted construction would be orthogonal to all states in the limiting sequence, a contradiction. One can work only with extended loops, so that classical expressions containing the connection or its curvature must be modified before quantization by reexpressing them in terms of extended holonomies. For low curvature and small loops,
one would still be close to the classical expressions, so that the semiclassical limit would not be in danger.

One can develop a more refined argument if one exploits spatial diffeomorphism invariance of the theory [19]. If one implements the diffeomorphism gauge freedom before one constructs curvature-dependent operators such as the Hamiltonian constraint, a small loop is gauge equivalent to a larger one. The diffeomorphism-invariant state with a loop added does not change as the loop is shrunk, and one can take the limit in a trivial way, the loop remaining attached. Applying this construction to the Hamiltonian constraint, one can argue that the algebra of constraint operators is anomaly-free on the space of states solving the diffeomorphism constraint [21].

However, it is not clear what quantum corrections follow from these operators because the diffeomorphism-invariant level makes it difficult to associate a semiclassical or other geometry with these states. Moreover, these constructions prevent one from addressing the full off-shell constraint algebra, in which the Hamiltonian constraint is paired non-trivially with the diffeomorphism constraint. Even though one ultimately wants to solve all constraints and derive physical observables, the space of diffeomorphism-invariant states is too restricted to analyze full consistency of the theory or to uncover non-observable properties of conceptual interest, such as the form of quantum space-time structure realized. To address these questions, one must look for a consistent realization of all first-class constraints relevant for the space-time gauge without solving any of them. (Canonical formulations that make use of triad variables are subject to a Gauss constraint. This constraint is not relevant for space-time structure and can therefore be treated separately from the diffeomorphism and Hamiltonian constraints. Moreover, it is simple enough to allow explicit solutions and direct implementations in the quantum theory, without giving rise to quantum corrections or deformations of its gauge structure.)

When both the Hamiltonian constraint and the diffeomorphism constraint remain unsolved, looking for an off-shell realization of a first-class constraint algebra becomes very difficult. One must then deal with the issue of “structure functions” in the bracket $\{H[N], H[M]\} = D[h^{ab}(N\partial_b M - M\partial_b N)]$ of two Hamiltonian constraints, where $h^{ab}$ is the inverse spatial metric. Moreover, the argument by which one can eliminate the loop regulator exploiting diffeomorphism invariance no longer applies. Holonomies are not just regulated versions of the connection; they imply modifications of the classical dynamics which may be small at low curvature but can still have important implications. If they break the closure of the constraint algebra, the gauge structure will be violated no matter how small possible modifications are. The theory will be anomalous and inconsistent.

It is not clear at present whether a consistent off-shell realization of loop quantum gravity exists. Models in which this has been achieved, so far in $2 + 1$ dimensions at the operator level [12, 22, 23, 24], show that a consistent first-class algebra can be realized only if it is deformed, with quantum corrections in its structure functions. These results, including the specific form of corrections, are consistent with effective calculations, in which one inserts possible quantum corrections in the classical constraints and computes their Poisson brackets [15, 13, 5]. We will make use of effective constraints in spherically symmetric models, which strike a nice balance between interesting off-shell properties and
rather manageable calculations.

2.1 Spherical symmetry

A spherically symmetric SU(2) connection has the form [25, 26, 27, 28]

\[ A_i^\tau_3 dx^a = A_x \tau_3 dx + A_\varphi \Lambda^A d\vartheta + A_\varphi \bar{\Lambda}^A d\varphi + \cos \vartheta \tau_3 d\varphi \] (1)

with an internal frame such that \( \text{tr}(\Lambda^A \tau_3) = \text{tr}(\bar{\Lambda}^A \tau_3) = \text{tr}(\Lambda^A \bar{\Lambda}^A) = 0 \). The requirement that internal gauge transformations must fix \( \tau_3 \) reduces the internal gauge group to U(1). The gauge is partially fixed at this stage, but the Gauss constraint is so simple that one can easily demonstrate the independence of quantum results of the chosen gauge fixing. Moreover, the constraint is not modified by holonomy corrections: Its quantization makes use only of invariant vector fields on spaces of connections. No anomalies are introduced at this stage. Finally, the part of the gauge freedom fixed here does not refer to space-time structure, and is therefore not crucial in the present context.

A spherically symmetric densitized triad has the dual form

\[ E^a_i \tau^i = E^x \tau_3 \sin \vartheta \frac{\partial}{\partial x} + E^\varphi \Lambda_E \sin \vartheta \frac{\partial}{\partial \vartheta} + E^\varphi \bar{\Lambda}_E \frac{\partial}{\partial \varphi}. \] (2)

Its coefficients determine the spatial metric

\[ ds^2 = \frac{(E^x)^2}{|E^x|} dx^2 + |E^x|(d\vartheta^2 + \sin(\vartheta)^2 d\varphi^2). \] (3)

The internal triad \( (\tau_3, \Lambda_E, \bar{\Lambda}_E) \) is independent of the one in the connection, except that the same \( \tau_3 \) is used in both cases. There is a free angle (denoted by \( \beta \) in [26, 27]) to rotate the internal triads into each other, which together with its momentum is an invariant kinematical degree of freedom, in addition to \( (A_x, E^x) \) and \( (A_\varphi, E^\varphi) \), but is eliminated when the remaining U(1) Gauss constraint is solved. While \( A_x \) is canonically conjugate to \( E^x \), \( E^\varphi \) is not conjugate to \( A_\varphi \) but to the extrinsic-curvature component \( K_\varphi = -2\gamma^{-1}A_\varphi \text{tr}(\Lambda^A \Lambda_E) \). The \( x \)-component of extrinsic curvature is \( K_x = \gamma^{-1}(A_x + \eta) \) with \( \eta = -2\text{tr}(\tau_3 \Lambda_E) \). One can obtain \( K_x \) as a U(1)-gauge invariant combination of \( A_x \) and \( \eta \). In spherical symmetry, one can therefore easily work with extrinsic curvature instead of connections, also in holonomies.

In addition to the appearance of U(1) instead of SU(2), a further simplification of spherical symmetry is the form of graphs. Inhomogeneity is realized only in the radial \( x \)-direction, along which one can align vertices connected by links. Any extended holonomy simply integrates over a piece of the \( x \)-axis, as in \( \bar{h}_e(A_x) = \exp(\tau_3 \int A_x dx) \), or \( \bar{h}_e(A_x) = \exp(i \int A_x dx) \) for matrix elements exhibiting the U(1) nature. (No path ordering is necessary for the Abelian reduced theory.) In any U(1)-gauge invariant state, an \( A_x \)-holonomy is combined with point holonomies for \( \eta \) so that the state depends only on \( A_x + \eta \). It suffices to look at the basis of charge-network states, in which each edge \( e \) carries an integer charge quantum number \( k_e \) for its \( A_x \)-holonomy, and each vertex an integer quantum number \( k_v \) for
a point holonomy \( \exp(i\eta(v)) \) if the \( U(1) \)-field \( \eta \) (in addition to a real quantum number \( \mu_v \) for \( K_x(v) \), taking values in the Bohr compactification of the real line). The conservation of \( U(1) \)-flux at a vertex \( v \) with \( \eta \)-charge \( k_v \) implies the relation \( k_{e_+} - k_{e_-} + k_v = 0 \), where \( e_{\pm} \) are the two edges touching the vertex \( v \) with charge labels \( k_{e_{\pm}} \). The corresponding (point) holonomies then appear in a spin-network state as factors

\[
\ldots e^{ik_{e_-} \int A_x dx} e^{ik_v \int A_x dx} \ldots = \ldots e^{ik_{e_-} \int (A_x + \eta') dx} e^{ik_{e_+} \int (A_x + \eta') dx} \ldots .
\]

Gauge-invariant states therefore depend only on the combination \( A_x + \eta' = \gamma K_x \) of \( A_x \) and \( \eta \). The other connection component \( A_\varphi \) or rather the \( E^\varphi \)-momentum \( K_\varphi \) appears in holonomies along curves in the \( \varphi \)-direction, along which \( K_\varphi \) does not change. Such a holonomy is simply an exponential \( h_{(e,\delta)}(K_\varphi) = \exp(i\delta K_\varphi(v)) \) with \( K_\varphi \) evaluated at a point (or vertex) \( v \), and with a real number \( \delta \) (or possibly a function on phase space) related to the coordinate length of the curve one would integrate over.

With these variables, one can check what kind of holonomy modifications are possible in the Hamiltonian constraint so that a first-class algebra results. The possibility of pointwise modifications in \( K_\varphi \) has been clarified [13], with the result that they can leave the algebra first-class but always deform it. The form of the deformed algebra is very characteristic:

\[
\{H[N], D[N^z]\} = -H[N^z N'] \quad \text{and} \quad \{H[M], H[N]\} = D[\beta|E^\varphi|/(E^\varphi)^{-2}](MN' - M'N)]
\]

with \( N^z \) the only non-vanishing component in the radial direction of the shift vector, and \( \beta \) a correction or deformation function depending on phase space variables. This form also agrees with the one found in 2 + 1-dimensional models and for cosmological perturbations. However, no consistent holonomy modification of the \( K_x \)-dependence has yet been found. These corrections are more difficult to realize, not the least because they require integrations, and therefore lead to either non-locality or higher-derivative theories.

### 2.2 Parameterization

A single gauge-invariant combination of holonomies in spherical symmetry is given by \( h_e = \exp(i\gamma \int_{x_0}^{x_0 + \ell_0} K_x dx) \), where \( x_0 \) is the starting point of the curve and \( \ell_0 \) its coordinate length. As an operator, \( h_e \) will add an edge from \( x_0 \) to \( x_0 + \ell_0 \) to the graph underlying a state it acts on, or increase the quantum numbers on pieces of a spin network overlapping with the curve from \( x_0 \) to \( x_0 + \ell_0 \). Composite operators depending on the connection, such as the Hamiltonian constraint, make use of these basic holonomy operators, but they usually come with a specific description of how the curves for holonomies are chosen with respect to a graph state acted on. They may leave the graph unchanged, making use only of holonomies along curves between vertices already present in the original graph, or they may be graph-changing and create new vertices. In the former case, \( \ell_0 \) for an individual holonomy in a vertex contribution of the operator would be fixed as the coordinate distance to the next vertex.

In the latter case, which is more complicated, but preferred in the full theory for the arguments of anomaly-freedom on diffeomorphism-invariant states to work, \( \ell_0 \) would not
be constant; one would have to find an alternative way to determine its values. One could, for instance, assume that the graph-changing nature of the operator leads to dynamical lattice refinement so that the geometrical length \[ \ell = \int_{x_0}^{x_0 + \ell_0} \sqrt{g_{xx}} \, dx \approx \ell_0 \sqrt{g_{xx}} = \ell_0 E^x / \sqrt{|E^x|}, \] measured with the densitized triad or the metric component \( \sqrt{g_{xx}} = E^\varphi / \sqrt{|E^x|} \), has a certain dependence on geometrical variables such as the orbit area \( |E^x| \). (We assume \( \ell_0 \) to be sufficiently small compared to the scale on which \( g_{xx} \) varies. If this assumption is violated or not precise enough, a derivative expansion of the integral can be used, as described in more detail below.) The simplest possibility in this context would be for \( \ell \) (rather than \( \ell_0 \)) to be some constant, such as the Planck length, but this is not the only choice.

A constant \( \ell \) would be analogous to a certain class of cosmological models \[29\] often studied in loop quantum cosmology, which is also shown by the behavior of holonomies. To see this, we assume that we are close to homogeneous models, so that \( \ell_0 \) for a given holonomy may be very short compared to \( K_x / K'_x \), which is generically large for \( K'_x \) restricted to be small by near homogeneity. The dominant contribution to the argument \( \int_{x_0 + \ell_0}^{x_0 + \ell_0 + \ell} K_x \, dx \) of the exponential in a holonomy is then simply

\[ \ell_0 K_x(x_0) \approx \ell \sqrt{E^x} / E^\varphi K_x. \]  

According to the classical equations of motion, which may be used when the present assumption is satisfied and \( K_x \) is small\(^1\) compared to \( 1/\ell_0 \), we write

\[ \ell \sqrt{E^x} / E^\varphi K_x = -\ell \left( \frac{E^\varphi}{E^\varphi} - \frac{1}{2} \frac{E_x}{E^x} \right) = -\ell \left( \frac{E^\varphi}{E^\varphi} \sqrt{|E^x|} \right)^* = -\ell \frac{\sqrt{g_{xx}}}{\sqrt{|g_{xx}|}}. \]

For a cosmological model, \( \sqrt{g_{xx}} = a \) would be the scale factor, so that the argument of holonomies agrees with \( \ell \mathcal{H} \), using the Hubble parameter \( \mathcal{H} = \dot{a} / a \).

Different parameterizations (or lattice refinement schemes \[30\ \[31\]) are possible in which \( \ell \) is not constant but, for instance, a certain power of \( |E^x| \) or of \( g_{xx} \), or some other function. We will not assume any specific function but simply take into account the fact that the choice of routings of curves may lead to a triad dependence of holonomies in addition to the expected connection or extrinsic-curvature dependence.

\(^1\)Assuming that we are close to a homogeneous model restricts the possible choices of space-time slicings, so that a “low-curvature” regime may be demarcated in terms of the non-invariant curvature component \( K_x \). Subtleties in the general case of inhomogeneous — but still spherically symmetric — geometries will be discussed below.
2.3 Derivative expansion

In the previous subsection, we assumed, restricting ourselves to near-homogeneous low-curvature geometries, that the coordinate length $\ell_0$ is sufficiently small, so that holonomy corrections would be weak. As we approach regimes in which quantum geometry is more pronounced, stronger modifications of the dynamics arise from the use of holonomies and higher-order corrections must be taken into account. In an inhomogeneous model, not just higher powers of $\ell_0 \gamma K_x$ in an expansion of the pointwise

$$\exp(i\ell_0 \gamma K_x) = \sum_{n=0}^{\infty} \frac{1}{n!} (i\ell_0 \gamma K_x)^n$$

will grow, but also higher spatial derivatives of $K_x$ in a derivative expansion of the integrated $\int_{x_0}^{x_0+\ell_0} K_x dx$.

The treatment of expansions now becomes more subtle, related to the interpretation of the non-invariant $K_x$ as some kind of measure for curvature, or at least as a parameter that tells us when holonomy corrections become large. The extrinsic-curvature component $K_x$ depends on the slicing, and one may locally be able to make it small in high-curvature regimes, or to make it large even in flat space-time, just by choosing an appropriate slicing of space-time. It therefore seems inconsistent to use its magnitude to determine the strength of corrections or orders of expansions. This problem, of course, does not arise just in effective descriptions; it plays a role already in constructing a full Hamiltonian constraint [18, 19], in which one replaces connections by holonomies so that the classical expression is obtained in the “classical limit,” or for “small” connections. (In the complete classical limit, loops shrink to points and connections can take arbitrary values. But for correct semiclassical physics, corrections to the classical limit must be small, which can be realized only for sufficiently small connections.) The problem we encounter in formulating a systematic derivative expansion is therefore a more general one: It arises because canonical quantum gravity primarily implies modifications for the Hamiltonian, rather than for space-time covariant expressions such as an action with its coordinate-independent meaning.

On closer inspection, it turns out that the form of discreteness corrections expressed in terms of extrinsic curvature is well-defined. A key role in this argument is played by the fact that, as shown in [4], holonomy modifications cause corrections of the algebra of hypersurface deformations by an additional function $\beta$ depending on extrinsic curvature. The algebra of constraints still closes, and therefore deviations from the classical value $\beta = 1$ have invariant meaning. This statement is clear from the algebra, but it seems surprising given the non-invariant form of extrinsic curvature. Several further implications of modified constraints then come into play: First, modified constraints imply corrections in the classical equations of motion, and canonically, the role of K-components as extrinsic curvature follows only after the Hamiltonian equation of motion for the triad is used. With modified equations, $K$-components no longer are extrinsic curvature in the classical sense, and classical intuition about the values $K$-components may take for different slicings breaks down. Secondly, a deformed algebra means that gauge transformations, although
not violated, no longer generate space-time Lie derivatives or changes of slicings in classical space-time. Therefore, what one may achieve for $K$-components in classical space-time plays no role for possible $K$-values in a deformed system. Two slices of the same classical space-time, one with large $K$ and one with small $K$, do not produce gauge-related solutions of the modified system. The fact that one solution would deviate more strongly from the classical space-time than the other one is not a contradiction. These arguments further highlight the importance of deformed algebras and their derivation (but also the dangers of using too much classical space-time intuition when one interprets canonical quantum gravity). They allow us to quantify the strength of holonomy corrections in terms of “extrinsic-curvature” components, and to organize expansions.

In general, then, one expects that holonomy corrections become strong at high curvature. As one leaves the classical regime, deviations from both the dynamics and the form of space-time will grow. While classical intuition will break down at some point before the Planck regime is reached, effective equations allow one to study the consequences of quantum aspects. For leading orders of an expansion, one may still use classical expectations to estimate what correction terms are relevant, and these should be all terms that contribute to curvature invariance of some order. Relevant terms may equally result from $K^n_x$ (a higher power) or $K^{(n)}_x$ (a higher derivative), or combinations thereof. Riemann curvature is, after all, a sum of powers of the space-time connection and its derivatives. Unless one works in a specific space-time gauge or slicing, which usually is not legitimate before quantization, one cannot assume that only powers $K^n_x$ but no higher spatial derivatives $K^{(n)}_x$ should contribute. (Time derivatives play a special role in a canonical theory. We will discuss them below.)

The classical constraint is therefore modified not only by pointwise holonomy corrections of the form $\exp(i\ell_0 \gamma K_x(x_0))$ but also by higher spatial derivatives of $K_x$. For any explicit local effective constraint, a combined expansion is required, one of the form

$$\exp \left(i\gamma \int_{x_0}^{x_0+\ell_0} K_x dx\right) = \exp \left(i\gamma \int_0^{\ell_0} (K_x(x_0) + hK'_x(x_0) + \frac{1}{2}h^2K''_x(x_0) + \cdots)dh\right)$$

$$= \exp \left(i\gamma (\ell_0 K_x(x_0) + \frac{1}{2}\ell_0^2 K'_x(x_0) + \frac{1}{6}\ell_0^3 K''_x(x_0) + \cdots)\right)$$

$$= 1 + i\ell_0 \gamma K_x(x_0) + \frac{1}{2}\ell_0^2 (i\gamma K'_x(x_0) - \gamma^2 K_x(x_0)^2)$$

$$+ \frac{1}{6}\ell_0^3 (i\gamma K''_x(x_0) - 3\gamma^2 K_x(x_0)K'_x(x_0) - i\gamma^3 K_x(x_0)^3) + \cdots (8)$$

(Note that this derivative expansion, unlike the continuum limit of the difference equation for states [32, 33] in homogeneous models, is not controlled by $\gamma$.). The expansion by powers of $\ell_0$ ensures that all terms with the same number of derivatives are grouped together provided we count the time derivative implicit in $K_x$ once equations of motion are used.

If $\ell_0$ is not fixed but triad-dependent as per (5), it may be evaluated at $x_0$ or at $x_0 + \ell_0$ or some other point, so that a further derivative expansion of $\ell_0$ and the triad variables it may contain would have to be included. (A derivative expansion also contributes higher-order terms to the integral in (5).) The specific evaluation point depends on how lattice
refinement implies a triad-dependent $\ell_0$. Parameterizing the evaluation point as $x_0 + r\ell_0$ with some $0 \leq r \leq 1$, we can write

$$\ell_0(E(x_0 + r\ell_0)) = \ell_0(E(x_0)) + r\ell_0(\frac{d\ell_0}{dE})|_{E(x_0)}E'(x_0) + \cdots .$$  (9)

If $\ell_0$ is small and lattice refinement (or whatever causes the triad dependence of $\ell_0(E)$) is not too violent, $d\ell_0/dE$ is small as well, so that the derivative term in (9) is of second order in our expansion. For instance, for a power law $\ell_0(E) \propto E_y$, as realized in all cases studied so far, including [29], we have $d\ell_0/dE \propto y\ell_0/E$, and we can write the second-order term as $ry\ell_0^2E'/E$. Lattice refinement or the evaluation point of $\ell_0$ therefore determine the coefficients of the derivative expansion, but do not affect its general form. Here, for the general consideration of effective descriptions, it is sufficient to know that we should expect the appearance of further triad derivatives in an expansion whose coefficients, such as $rd\ell_0/dE$ in (9), so far remain undetermined by a derivation from the full theory. The purpose of our constructions is to derive possible restrictions on such coefficients using only the requirement of anomaly-freedom.

### 2.4 Holonomy corrections vs. higher-curvature corrections

Holonomy corrections, according to [5], imply higher powers of the connection or extrinsic curvature in the Hamiltonian constraint, as well as higher spatial derivatives. These features are shared with higher-curvature corrections in an effective action, except that higher time derivatives always come along with higher curvature but are not suggested by holonomy corrections. Holonomy corrections are indeed different from higher-curvature ones; they result from a modification of the Hamiltonian constraint motivated by spatial quantum geometry, not from generic space-time covariant correction terms in the form of higher curvature invariants. Even though holonomy corrections and higher-curvature corrections are expected to be significant in the same regimes — when curvature reaches Planckian values — they must be distinguished from each other both in their formal derivation and in their possible implications.

Holonomy corrections and higher-curvature corrections have very different effects on quantum space-time structure. Generic higher curvature corrections are determined by all possible terms that could modify the classical action by higher derivatives in a covariant way, leaving the classical space-time structure unchanged. Only the dynamics is then modified at high curvature. Generic holonomy corrections, on the other hand, are introduced at the kinematical level in order to quantize the Hamiltonian constraint. The constraints themselves determine what space-time structure is realized, by generating gauge transformations that classically correspond to space-time Lie derivatives of phase-space functions. When quantum corrections are inserted in the constraints, their transformations change. Gauge transformations could be violated, and in general will be unless one is very careful about arranging different correction terms. If this happens, the theory is anomalous and inconsistent because its equations do not have mutually compatible solutions. (One would obtain formally consistent solutions if one solves the constraint as second-class ones. But
then there are neither second-order equations of motion nor gauge transformations that would remove spurious degrees of freedom.) In the consistent case, when all quantized constraints still generate gauge transformations and remain first class, their algebra, not just their functional form, in general carries quantum corrections. Their gauge transformations no longer correspond to Lie derivatives on the constraint surface, which implies that the space-time structure is changed by quantum corrections. This consequence is realized for holonomy modifications in all consistent versions found so far. Unlike higher-curvature corrections, they modify the notion of space-time and general covariance. At high density, modifications can be so strong that space-time turns into a quantum version of 4-dimensional Euclidean space.

Regarding their formal derivation, holonomy corrections and higher-curvature terms make use of different mathematical structures and expansions. They also imply different changes of the number of local degrees of freedom.

### 2.4.1 Derivatives

As already discussed, higher spatial derivatives result from holonomy corrections if a derivative expansion is used to approximate their integrations locally. Spatial derivatives of the connection (or extrinsic curvature) and, if curve lengths are taken as triad-dependent, of the triad result. Since $E^a_i$ and $K^a_{\dot{a}}$, at this stage, both are phase-space functions, one could expect them to appear in integrations $\int_{x_0}^{x_0+\ell_0(E)} K_x dx$ on the same footing, organized in a derivative expansion by increasing orders $n$ of derivatives $K_x^{(n)}$ grouped with $E^{(n)}$. After all, the implicit time derivative contained in $K_x$ can be seen only when equations of motion are used, but the latter are not available before the Hamiltonian constraint is quantized and imposed. They cannot be used for an analysis of the off-shell constraint algebra necessary to study possible anomalies.

Nevertheless, we have already seen in [8] that the expansion is arranged as if the time derivative implicit in $K_x$ were present, if we just expand by powers of the edge length $\ell_0$. Similarly, if we expand a triad-dependent $\ell_0$ as in [9], every factor of $\ell_0$ comes along with an additional spatial derivative of $E^x$. This expansion therefore treats $K_x$ and $(E^x)'$, or in general $K_x^{(n-1)}$ and $(E^x)^{(n)}$ on the same footing. Even if we do not refer to implicit time derivatives or equations of motion, all derivatives are ultimately counted.

For higher-curvature corrections, one considers as being on the same footing all derivatives that would contribute to a curvature invariant of some order. Since equations of motion would classically tell one that $K_x$ is related to a first-order time derivative of the triad, a derivative expansion is automatically organized by increasing orders $n$ of derivatives $K_x^{(n-1)}$ and $E^{(n)}$, taking into account the extra time derivative already present in $K_x$. Derivative expansions for holonomy corrections and higher-curvature corrections therefore combine terms in the same manner.

Still, it is a priori unclear in which way one should organize the derivative expansion of constraints, treating the $K$-components as derivatives or not. The two different types of brackets in the hypersurface-deformation algebra involving the Hamiltonian con-
constraint $H[N]$, $\{H[N], D[N^x]\}$ and $\{H[N], H[M]\}$, seem to require different viewpoints. The bracket $\{H[N], D[N^x]\}$ being first class makes sure that $H[N]$ transforms according to some consistent spatial geometry, which would be just the classical one if we use an unmodified diffeomorphism constraint $D[N^x]$ (as we will do below; see also [11]). A closed bracket then requires all terms in $H[N]$, including its corrections, to combine to scalars of the correct spatial density weight. For the latter, only spatial derivatives count but not the time derivatives implicitly contained in the $K$-components. (Some of the phase-space variables carry intrinsic density weights, with $K_x$ and $E^\varphi$ of density weight one. From the viewpoint of spatial diffeomorphisms, $K_x$ should therefore be on the same footing as $E^\varphi$, not as $K_\varphi$ as the implicit time derivative would suggest.) For $\{H[N], H[M]\}$, on the other hand, closure implies consistent space-time dynamics, expected to be at least partially of higher-curvature type. Here, it would seem more natural to count $K$-components as first-order (time) derivatives.

We will for now avoid making a fixed choice on the order of derivatives and their counting. It turns out that the specific form of variational methods in this context, discussed further below, offers further insights and guidelines. In particular, the order of derivatives of multipliers $N$ and $N^x$, not just those of phase-space variables, plays an important role in organizing expansions of Poisson brackets in the hypersurface-deformation algebra.

2.4.2 Degrees of freedom

Holonomy corrections just modify the dependence of the Hamiltonian constraint (or its expectation values used for effective constraints) on the connection. No additional degrees of freedom are implied since only the connection and the triad are quantized, providing the same number of basic operators that exist as basic classical phase-space variables.

Higher-curvature corrections imply higher time derivatives and therefore new degrees of freedom if initial values for higher-derivative equations are to be imposed. Interpreting higher-derivative equations perturbatively, the number of independent solutions does not change because extra solutions beyond the classical number would be non-analytic in the perturbation parameter and must be discarded for consistency [31]. Nevertheless, higher time derivatives imply corrections which can be understood as coupling terms with these new, virtual degrees of freedom, just as virtual particles imply quantum corrections in perturbative quantum field theory. Canonically, the new degrees of freedom take on a much more explicit form [35, 36]: they arise as fluctuations and higher moments of a quantum state, parameters which are independent of expectation values of basic operators to which the classical phase-space structure can be applied. In certain regimes, these moments, provided they change slowly, can be solved for in terms of expectation values and inserted into expectation-value equations. In this way, the coupling terms implicitly realized in higher-derivative equations become explicit [37].
2.4.3 Algebra

In a complete semiclassical or effective expansion of a loop-quantized theory, both holonomy corrections and higher time derivatives, resulting from couplings to moments of a state, are present. Moreover, because of their relation to curvature they are both expected to be significant in the same regimes and cannot easily be separated from each other. Only a combined treatment including both types of corrections can be fully consistent. Thanks to their different formal and space-time roles, however, one can easily separate these two modifications in formal derivations.

Formally, holonomy corrections modify the dependence of constraints on classical variables, while higher time derivatives come from moments of a state. The gauge transformations they generate (if they indeed do generate gauge) therefore affect different degrees of freedom. While a constraint modified only by holonomy corrections implies modified gauge transformations for expectation values, it leaves moments of canonical basic operators invariant. A constraint modified by moments or higher time derivatives, on the other hand, always generates gauge transformations that change the moments as well. In this way, considering not just the magnitude of typical correction terms but also the form of the modified gauge theory, one can keep holonomy corrections and higher-curvature ones separate from each other.

Gauge transformations of the constraints of gravity encode the form of the space-time structure realized. Since the transformations change in different and distinguishable ways for the two types of curvature-related corrections, taken separately they imply different space-time structures. Higher-curvature terms, by definition, leave the classical space-time structure and the notion of general covariance unchanged. Holonomy corrections, in all consistent versions found so far, modify space-time structure and covariance. These modifications, in general, cannot be canceled by higher time derivatives (or other quantum-geometry corrections such as inverse-triad terms), and therefore the space-time structure following from holonomy corrections alone is a good indication of what a combined system would imply. If an anomaly-free version of holonomy-modified constraints can be found, it will certainly provide a consistent space-time model. For this reason, we focus on holonomy corrections in this paper (but take along inverse-triad corrections), leaving out moment terms which are more difficult to derive.

3 Constraint algebra

As indicated by the prevalence of deformed constraint algebras in loop quantum gravity, we are in a situation much more general than the one of standard higher-curvature effective actions. The latter, even though they may modify the classical dynamics considerably, all have the same classical hypersurface-deformation algebra for their constraints [38]. Models of loop quantum gravity implement quantized space-time structures, while higher-curvature effective actions take into account modified dynamics of a standard space-time. This difference has an influence on the derivation of possible consistent constraint alge-
bras: While higher-curvature actions always produce the classical bracket \( \{H[N], H[M]\} = D[\hbar^{\alpha\beta}(N\partial_\alpha M - M\partial_\alpha N)] \) with only first derivatives of the multipliers, integrations by parts applied to some \( \{H[N], H[M]\} \) with constraints modified by higher spatial derivatives should in general produce terms with as many derivatives of \( N \) and \( M \) as assumed in a derivative expansion. Correspondingly, additional consistency conditions may be obtained by requiring the algebra to close to all orders considered.

### 3.1 General procedure

In the presence of higher spatial derivatives, derivatives of the multipliers \( N \) and \( N' \) may be obtained in Poisson brackets, which raises the question in how far multipliers and their derivatives can be treated as independent. Using integrations by parts, a single constraint such as \( H[N] = 0 \) can be rewritten in such a form, that derivatives of \( N \) appear in the integrand. Such mere rewritings, schematically \( H[N] = H_1[N] + H_2[N'] \), clearly cannot lead to additional constraints because there was just one constraint to begin with. Indeed, one cannot treat \( N \) and \( N' \) as independent and derive two constraints \( H_1 = 0 \) and \( H_2 = 0 \) from the one original \( H = 0 \). The local constraints on phase-space functions are obtained by requiring \( H[N] = 0 \) for all functions \( N \). The function itself and its derivatives (as opposed to their values at a single point) are not independent, and therefore no additional constraints arise by applying integrations by parts.

These circumstances are rather obvious and often used at least implicitly when dealing with smeared constraints \( H[N] \). One may employ them to reduce the freedom in writing the constraints: If we require that only the multiplier \( N \) but none of its spatial derivatives appear in the constraint expression, the freedom of integrations by parts is strongly reduced. This condition could not be used if quantum gravity or some other effects would give rise to corrections with higher spatial derivatives and non-linear functions even of the multipliers. There could then be irreducible higher-derivative terms of multipliers that cannot be rewritten to be proportional to the underived multiplier. However, such corrections could only appear if the multipliers themselves were subject to quantization or other modifications, which never happens in canonical approaches. The multipliers are not turned into operators in canonical quantizations; they remain test functions even for constraint operators. Moreover, they appear in classical constraints without their spatial derivatives, so that they are not subject to discretization modifications. It is therefore safe to assume that all terms in a given effective constraint are proportional to one multiplier function without any one of its derivatives.

### 3.1.1 Derivative expansion of constraint brackets

For Poisson brackets of two smeared constraints, the previous considerations take on a rather different form. As we will see explicitly below, if we assume two constraints, \( C_1[M] \) and \( C_2[N] \), their Poisson bracket \( \{C_1[M], C_2[N]\} = \sum_{i,j} \int M^{(i)} N^{(j)} f_{i,j} \ dx \) may depend on higher spatial derivatives of \( M \) and \( N \), up to some order considered for a derivative expansion of the constraints. The presence of two independent functions \( M \) and \( N \) implies new
features compared to the previous discussion of a single constraint. First, it is, in general,
no longer possible to remove all spatial derivatives of $M$ and $N$ by integrating by parts in
$\sum_{i,j} \int M^{(i)} N^{(j)} f_{i,j} \, dx$. Some higher spatial derivatives of multipliers will therefore remain
in Poisson brackets even if they can always be removed in the constraints themselves. We
may assume a form in which one of the multipliers, say $M$, appears without its derivatives,
$\{C_1[M], C_2[N]\} = \sum_j \int M N^{(j)} g_j \, dx$ with new functions $g_j$, but trying to remove further
the derivatives of $N$ will reinstate those of $M$. As with an individual constraint, the latter
form with underived $M$ may be used to fix some of the freedom of integrating by parts,
but it will not remove all spatial derivatives of multipliers.

Secondly, and more importantly, the presence of two independent multiplier functions
implies that there are several independent terms in the Poisson bracket of two constraints.
If the bracket is required to have a certain form, for instance that it be first class and
therefore vanish on the constraint surface, several independent conditions will result. To
see this, we must consider the freedom contained in a pair of functions, or the set $\{(M, N) : \ M, N \text{ functions on space}\}$. For a first-class algebra $\{C_1[M], C_2[N]\}$, we have the condition
that $\sum_j \int M N^{(j)} g_j \, dx$ be a linear combination of all original constraints. For a single
multiplier in this expression, there would be just one condition. With two multipliers $M$
and $N$, however, a new condition arises for each derivative order $j$.

To show this, we work locally without loss of generality because it is sufficient to vary
functions in a neighborhood $U$ of an arbitrary but fixed point to derive equations of motion.
Furthermore, we may assume the multipliers to be smooth and Taylor-expandable in the
chosen neighborhood. We may then re-organize our set of local multiplier functions as
\[
\{(M, N) : M, N \text{ smooth functions on } U\}
\]
using all monomials of degree $j$ for $N$, and denoting by $\langle \cdot \rangle$ the linear span. We then derive
iteratively that all $g_j$ must independently be a combination of constraints: For $N = c_0$
constant and varying by $c_0$, we have that $\int M g_0 \, dx$ must be a combination of constraints
for all $M$, so that $g_0$ must locally be a combination of constraints. For $N = c_1 x$, varying
by $c_1$ and using the first result on $g_0$, we obtain that $\int M g_1 \, dx$ must be a combination of constraints, still for all $M$ since the $M$-variations of $(M, N)$ with $N = c_1 x$ are independent
of those with $N = c_0$. Proceeding in this way, all $g_j$ must independently be combinations of
the constraints. A first-class algebra of constraints with higher spatial derivatives therefore
requires additional conditions on the possible form of constraints, even if no additional
constraints on phase space are implied.

In the preceding argument on the independence of multiplier functions and independent
conditions $g_j$ it was important that the Poisson bracket $\{C_1[M], C_2[N]\}$ was assumed to be
arranged in the form $\sum_j \int M N^{(j)} g_j \, dx$, using integrations by parts. Sometimes, especially
for the bracket of two Hamiltonian constraints, the series may at first appear in a different
form. In the next subsection, we will see that for two Hamiltonian constraints (or more generally, for the bracket of two copies of the same constraint with different multipliers) it is of-
ten more natural to write the bracket as \( \{H[M], H[N]\} = \sum_{i,j} \int (M^{(i)} N^{(j)} - M^{(j)} N^{(i)}) h_{i,j} dx \) to make the antisymmetry in \( M \) and \( N \) explicit. The sum may be assumed to be such that \( i < j \), with \( j \) ranging from zero to \( n \) at \( n \)-th order. However, it turns out that the antisymmetric combinations \( M^{(i)} N^{(j)} - M^{(j)} N^{(i)} \) cannot all be varied independently of one another, and that the \( h_{i,j} \) in a first-class algebra need not be combinations of constraints independently for all \( i \) and \( j \). To see this, it suffices to rewrite the first few orders of an antisymmetric arrangement in terms of the standard form used before. (Such formulas will be useful for later manipulations in explicit examples. We include the general expressions at \( n \)-th order in an appendix.)

At first order, integrating by parts and ignoring boundary terms, we have

\[
\int dx (MN' - M'N) h_{0,1} = \int dx (MNh'_{0,1} + 2MN'h_{0,1}),
\]

both forms require the same condition for a first-class algebra, namely that \( g_1 = 2h_{0,1} \approx 0 \) vanish on the constraint surface (which implies that \( g_0 = h'_{0,1} \) vanishes on the same surface). At second maximal order, \( j = 2 \), we have two additional terms

\[
\int dx (MN'' - M''N) h_{0,2} = -\int dx (MNh''_{0,2} + 2MN'h'_{0,2}),
\]

and

\[
\int dx (M'N'' - M''N') h_{1,2} = -\int dx (MN' h''_{1,2} + 3MN'' h'_1 + 2MN'' h_{1,2}),
\]

so that adding (11), (12) and (13) results in

\[
\int dx \sum_{j=1}^{j-1} \sum_{i=0}^{j} (M^{(i)} N^{(j)} - M^{(j)} N^{(i)}) h_{i,j} = \int dx (-2MN'' h_{1,2} - 3MN'' h'_1
\]

\[
+ MN' (2h_{0,1} - 2h'_{0,2} - h''_{1,2}) + MN (h'_{0,1} - h''_{0,2})
\]

giving four conditions

\[
g_3 = -2h_{1,2} \approx 0,
\]

\[
g_2 = -3h'_{1,2} \approx 0,
\]

\[
g_1 = 2h_{0,1} - 2h'_{0,2} - h''_{1,2} \approx 0,
\]

\[
g_0 = h'_{0,1} - h''_{0,2} \approx 0,
\]

go of which only two are independent:

\[
h_{1,2} \approx 0,
\]

and

\[
h_{0,1} - h'_{0,2} \approx 0.
\]
The functions $h_{0,1}$ and $h_{0,2}$ in (11) and (12) need not vanish independently.

Similarly at third maximal order there are six conditions

$$
g_5 = 2h_{2,3} \approx 0, \quad g_4 = 5h'_{2,3} \approx 0, \quad g_3 = -2h_{1,2} + 2h_{0,3} + 2h'_{1,3} + 4h''_{2,3} \approx 0, \quad g_2 = -3h'_{1,2} + 3h'_{0,3} + 3h''_{1,3} + h'''_{2,3} \approx 0, \quad g_1 = 2h_{0,1} - 2h_{0,2} - h''_{1,2} + 3h''_{0,3} + h'''_{1,3} \approx 0, \quad g_0 = h'_{0,1} - h''_{0,2} + h'''_{0,3} \approx 0,
$$

but only three of them are independent, implying:

$$
h_{2,3} \approx 0, \quad h_{1,2} - h_{0,3} - h'_{1,3} \approx 0, \quad h_{0,1} - h'_{0,2} + h''_{0,3} \approx 0.
$$

As these examples indicate, even total orders $i + j$ do not lead to conditions independent of those from odd orders because the highest even derivatives $MN^{i+j}$ cancel out after integrating by parts the antisymmetric combinations $M^{i}N^{j} - M^{j}N^{i}$.

### 3.1.2 Functional derivatives

We now turn to explicit formulas to compute Poisson brackets of constraints with derivative corrections. These are again given for fields in one spatial dimension (the case of interest for spherical symmetry), but most of them easily generalize to higher spatial dimensions.

For a functional

$$F[N, q] := \int dx N F (q(x), q'(x), q''(x), \ldots, q^{(n)}(x))$$

depending on some smearing function $N$, and a field $q$ and its spatial derivatives up to order $n$, we compute its functional derivative using

$$\frac{\delta F[N, q]}{\delta q(x)} := \left. \frac{\delta (NF)}{\delta q} \right|_{q=q(x)}
$$

where $\delta (NF)/\delta q$ is the ‘variational derivative’ of $NF$:

$$\frac{\delta (NF)}{\delta q} := \sum_{k=0}^{n} (-1)^k \left( N \frac{\partial F}{\partial q^{(k)}} \right)^{(k)}.
$$

Here and in what follows we use the same letter to denote the smeared and unsmeared (density) functional.
We may expand terms of this form using the binomial identity for the \( k \)-th derivative of a product:
\[
(AB)^{(k)} = \sum_{l=0}^{k} \binom{k}{l} A^{(k-l)} B^{(l)},
\]
and obtain
\[
\frac{\delta(NF)}{\delta q} = \sum_{k=0}^{n} (-1)^k \left( N \frac{\partial F}{\partial q^{(i)}} \right)^{(k)}
\]
\[
= \sum_{k=0}^{n} \sum_{l=0}^{k} (-1)^k \binom{k}{l} N^{(k-l)} \left( \frac{\partial F}{\partial q^{(i)}} \right)^{(l)}
\]
\[
= \sum_{j=0}^{n} N^{(j)} \sum_{k=0}^{n-j} (-1)^{j+k} \binom{j+k}{k} \left( \frac{\partial F}{\partial q^{(i+j)}} \right)^{(k)}.
\]
In the last line, we have used the identity:
\[
\sum_{k=0}^{n} \sum_{l=0}^{k} \binom{k}{l} A_{k,l} = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{j+k}{k} A_{j+k,k},
\]
for arbitrary functions \( A_{k,l} \). The right-hand side of (24) follows from summing ‘diagonally’ as opposed to row by row in the diagram:

\[
\begin{aligned}
0 \quad 0 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 1 & \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \\
0 \quad 2 & \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \\
0 \quad 3 & \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \\
\ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots
\end{aligned}
\]

Making use of (23), we may now write, for general functionals \( F_A[M,q,\ldots,q^{(m)}] \) and \( F_B[N,p,\ldots,p^{(n)}] \), the formula:
\[
\frac{\delta(MF_A)}{\delta q} \frac{\delta(NF_B)}{\delta p} - (M \leftrightarrow N) = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( M^{(i)} N^{(j)} - N^{(i)} M^{(j)} \right) \delta_{q\rightarrow q}^{m,i} F_A \delta_{p\rightarrow p}^{n,j} F_B
\]
with
\[
\delta_{q\rightarrow q}^{m,i} F := \sum_{k=0}^{m-i} (-1)^{i+k} \binom{i+k}{k} \left( \frac{\partial F}{\partial q^{(i+k)}} \right)^{(k)}.
\]

These basic formulas equally apply to the brackets \( \{H[N], H[M]\} \) and \( \{H[N], D[N^x]\} \). However, since we assume the diffeomorphism constraint \( D[N^x] \) to be unaffected by quantum corrections and therefore to remain of first order in spatial derivatives, we may write the bracket containing it in more explicit form.

Returning to general expressions (21), (22), and (24), with the assumption of at most first derivatives in \( N^x \), from the form of (38) for the Poisson bracket of a general functional \( F[N] \) with \( D[N^x] \), for each scalar variable \( q \) there will then be a contribution of the form

\[
\int dx \ N^x \frac{\delta(NF)}{\delta q} q' \quad (27)
\]

and (if \( q \) is a density of weight one) an additional one of the form

\[
\int dx \ (N^x)' \frac{\delta(NF)}{\delta q} q \ . \quad (28)
\]

Using (21), (22), and (24), with added integration by parts, we may write

\[
\int dx \ N^x \frac{\delta(NF)}{\delta q} q' = \int dx \ N^x \sum_{k=0}^{n} (-1)^k \left( N \frac{\partial F}{\partial q^{(k)}} \right)^{(k)} q'
\]

\[
= \int dx \sum_{k=0}^{n} (-1)^{2k} \left( N \frac{\partial F}{\partial q^{(k)}} \right) (N^x q')^{(k)}
\]

\[
= \int dx \sum_{k=0}^{n} \sum_{l=0}^{k} N(N^x)^{(k-l)} \left( \frac{k}{l} \right) \frac{\partial F}{\partial q^{(k)}} q^{(l+1)}
\]

\[
= \int dx \left[ N N^x \sum_{k=0}^{n} \frac{\partial F}{\partial q^{(k)}} q^{(k+1)} + \sum_{i=1}^{n} \sum_{l=0}^{i} N(N^x)^{(i-l)} \left( \frac{k}{l} \right) \frac{\partial F}{\partial q^{(k)}} q^{(l+1)} \right]
\]

\[
= \int dx \left[ N N^x \sum_{k=0}^{n} \frac{\partial F}{\partial q^{(k)}} q^{(k+1)} + \sum_{i=1}^{n} \sum_{k=0}^{n-i} \left( \frac{i+k}{k} \right) \frac{\partial F}{\partial q^{(i+k)}} q^{(k+1)} \right] .
\]  

(29)

Similarly we can write

\[
\int dx \ (N^x)' \frac{\delta(NF)}{\delta q} q = \int dx \sum_{i=1}^{n+1} N(N^x)^{(i)} \left( \frac{\partial F}{\partial q^{(i-1)}} q + \sum_{k=0}^{n-i} \left( \frac{i+k}{k} \right) \frac{\partial F}{\partial q^{(i+k)}} q^{(k+1)} \right) .
\]

(30)
so that the total contribution for a density is
\[
\int dx \left[ N^x \frac{\partial (NF)}{\partial q} q' + (N^x)' \frac{\partial (NF)}{\partial q} q + \sum_{k=0}^{n} \left( i + k + 1 \right) \left( \frac{\partial F}{\partial q^{(i+k)}} \right) q^{(k+1)} \right]
\]
\[
+ \sum_{i=1}^{n} N(N^x)^{(i)} \left( \frac{\partial F}{\partial q^{(i-1)}} q + \sum_{k=0}^{n-i} \left( i + k + 1 \right) \left( \frac{\partial F}{\partial q^{(i+k)}} \right) q^{(k+1)} \right)
\]
\[
+ N(N^x)^{(n+1)} \frac{\partial F}{\partial q^{(n)}} q \right].
\] (31)

These expressions can readily be used to explicitly calculate the bracket of any phase space functional with the diffeomorphism constraint in one dimension.

To compute the Poisson bracket of two Hamiltonian constraints we may use (25), for the special case \( F_A = F_B = H \):
\[
\frac{\delta (MH)}{\delta q} \frac{\delta (NH)}{\delta p} - (M \leftrightarrow N) = \sum_{i=0}^{n} \sum_{j=0}^{n} \left( M^{(i)} N^{(j)} - N^{(i)} M^{(j)} \right) \delta^{n,i}_q H \delta^{n,j}_p H
\]
\[
= \sum_{j=1}^{n} \sum_{i=0}^{j-1} \left( M^{(i)} N^{(j)} - N^{(i)} M^{(j)} \right) \left( \delta^{n,i}_q H \delta^{n,j}_p H - \delta^{n,j}_q H \delta^{n,i}_p H \right),
\] (32)
where now \( n \) is the maximum derivative order considered of the two variables \( q \) and \( p \). (This is true because for \( m > n \), with \( n \) the maximum order of derivatives of \( q \) appearing in \( H \), we have \( \delta^{m,i}_q H = \delta^{n,i}_q H \) and \( \delta^{m,i}_q H = 0 \).) Application of this formula for each canonical pair gives rise to an expression of the form
\[
\{ H[M], H[N] \} = \sum_{j=1}^{n} \sum_{i=0}^{j-1} \int dx \left( M^{(i)} N^{(j)} - N^{(i)} M^{(j)} \right) h_{i,j}
\] (33)
alluded to previously.

The appendix shows how one can use the lower order calculations from the previous subsection to rewrite the general expression (33) in the form \( \sum_{j=0}^{2n-1} \int dx \ N M^{(j)} g_j \).

3.2 Spherical symmetry

We now specialize the previously derived formulas to the spherically symmetric case. The Poisson bracket of functions \( f \) and \( g \) on the phase space of spherically symmetric gravity is
\[
\{ f, g \} = 2G \int dx \left( \gamma \frac{\delta f}{\delta A_x} \frac{\delta g}{\delta E^x} + \frac{1}{2} \frac{\delta f}{\delta K^\varphi} \frac{\delta g}{\delta E^\varphi} + \gamma \frac{\delta f}{\delta \eta} \frac{\delta g}{\delta P_n} \right.
\]
\[
- \gamma \frac{\delta f}{\delta E^x} \frac{\delta g}{\delta A_x} - \frac{1}{2} \frac{\delta f}{\delta K^\varphi} \frac{\delta g}{\delta E^\varphi} - \gamma \frac{\delta f}{\delta P_n} \frac{\delta g}{\delta \eta} \right),
\] (34)
which we apply to $f$ and $g$ being the Hamiltonian or diffeomorphism constraints.

The general modified Hamiltonian constraint we consider is:

$$H[N] = -\frac{1}{2G} \int dx \ N (\alpha |E^x|^{-\frac{1}{2}} E^\varphi f_1 + 2s\bar{\alpha} |E^x|^{\frac{1}{2}} f_2 + \alpha |E^x|^{-\frac{1}{2}} E^\varphi$$

$$- \alpha_{\Gamma} |E^x|^{-\frac{1}{2}} E^\varphi \Gamma^2_{\varphi} + 2s\bar{\alpha}_{\Gamma} |E^x|^{\frac{1}{2}} \Gamma'_{\varphi}),$$

(35)

where $s = \text{sign}E^x$. We use $\Gamma_{\varphi} = -(E^x)'/(2E^\varphi)$ as an abbreviation, which classically would be a component of the spin connection. Classically, $f_1 = K^2_\varphi$ and $f_2 = K_\varphi (A_x + \eta')/\gamma$ for spherically symmetric gravity, and $\alpha = \bar{\alpha} = \alpha_{\Gamma} = \bar{\alpha}_{\Gamma} = 1$. Not all these functions are independent and we could, for instance, absorb $\alpha$ in $f_1$. However, we will keep them separate to indicate their different origins in inverse-triad and holonomy corrections, respectively.

The Gauss and diffeomorphism constraints remain unaltered because their classical action on phase space can directly be lifted to quantum states. The gauge transformations they generate are therefore unmodified, and we have

$$G[\lambda] = \frac{1}{2G\gamma} \int dx \ \lambda ((E^x)' + P^n)$$

(36)

$$D[N^x] = \frac{1}{2G} \int dx \ N^x \left(2E^\varphi K'_\varphi - \frac{1}{\gamma} A_x (E^x)' + \frac{1}{\gamma} \eta' P^n \right)$$

(37)

We keep the full set of constraints, but one can easily solve the Gauss constraint by replacing $A_x/\gamma$ with $K_x$ and eliminating $\eta'$ terms. (The extrinsic-curvature component $K_x = \gamma^{-1}(A_x + \eta')$ is invariant under the action generated by the Gauss constraint, and $P^n$ is expressed in terms of $(E^x)'$ on its constraint surface.)

We first give general formulas to compute the Poisson algebra of constraints when all the correction functions ($f_1$, $f_2$, $\alpha$, $\bar{\alpha}$, $\alpha_{\Gamma}$ and $\bar{\alpha}_{\Gamma}$) are allowed to be arbitrary (smooth) functions of the configuration variables $A_x + \eta'$, $K_\varphi$, triads $E^x$, $E^\varphi$, and their derivatives to some order $n$. For a more detailed analysis, we then specialize to the cases of $n = 0$ and $n = 1$ for holonomy corrections with or without inverse triad corrections. For these explicit considerations of Poisson brackets, we will find it convenient to split the Hamiltonian constraint (35) into its terms

$$H_0[N] := -\frac{1}{2G} \int dx \ N (\alpha |E^x|^{-\frac{1}{2}} E^\varphi (f_1 + 1))$$

$$H_A[N] := -\frac{1}{2G} \int dx \ N (2s\bar{\alpha} |E^x|^{\frac{1}{2}} f_2)$$

$$H_{\Gamma}[N] := H_{\Gamma}^1[N] + H_{\Gamma}^2[N] + H_{\Gamma}^3[N]$$
with
\[ H^1_{\Gamma}[N] := \frac{1}{2G} \int dx N \alpha_T \frac{|E^x|^\frac{1}{2}((E^x)')^2}{4E^\varphi} \]
\[ H^2_{\Gamma}[N] := \frac{1}{2G} \int dx N s\alpha_T \frac{|E^x|^\frac{1}{2}(E^x)''}{E^\varphi} \]
\[ H^3_{\Gamma}[N] := -\frac{1}{2G} \int dx N s\alpha_T \frac{|E^x|^\frac{1}{2}(E^x)'(E^\varphi)'}{E^\varphi^2} \]

### 3.2.1 Diffeomorphism bracket

The bracket \( \{ F, D[N^x] \} \) of some phase-space function \( F \) with the diffeomorphism constraint may be computed explicitly for an arbitrary dependence of \( F \) on the canonical variables and their derivatives up to \( n \)-th order, as indicated in the preceding subsection. Given the form of the diffeomorphism constraint (37), we have
\[
\{ F[N], D[N^x] \} = \int dx N^x \left[ \frac{\delta F[N]}{\delta A_x} A'_x + \frac{\delta F[N]}{\delta \eta} \eta' + \frac{\delta F[N]}{\delta K^\varphi} K'_\varphi \right. \\
+ \left. \frac{\delta F[N]}{\delta E^x} E' + \frac{\delta F[N]}{\delta E^\varphi} E^{\varphi'} + \frac{\delta F[N]}{\delta P^\eta} P'^\eta \right] + \int dx (N^x)' \left[ \frac{\delta F[N]}{\delta A_x} A'_x + \frac{\delta F[N]}{\delta E^\varphi} E^{\varphi'} + \frac{\delta F[N]}{\delta P^\eta} P'^\eta \right]. \tag{38}
\]

Specializing \( F \) to the Hamiltonian constraint \( H \), and using (29) and (31), the functional derivatives take the form
\[
\{ H[N], D[N^x] \} = \int dx NN^x H' \\
+ \sum_{i=1}^n \int dx N(N^x)^{(i)} \left[ \frac{\partial H}{\partial A_x^{(i-1)}} A'_x + \frac{\partial H}{\partial \eta^{(i)}} \eta' + \frac{\partial H}{\partial (E^\varphi)^{(i-1)}} E^\varphi \\
+ \sum_{k=0}^{n-i} \left( \binom{i+k+1}{k+1} \left( \frac{\partial H}{\partial A_x^{(i+k)}} A^{(k+1)}_x + \frac{\partial H}{\partial \eta^{(i+k+1)}} \eta^{(k+2)} \right) \right) + \left( \binom{i+k}{k} \right) \frac{\partial H}{\partial K^{(i+k)}_\varphi} K^{(i+k)}_\varphi \\
+ \left( \binom{i+k}{k} \right) \frac{\partial H}{\partial (E^\varphi)^{(i+k)}} (E^\varphi)^{(k+1)} \right] \\
+ \int dx N(N^x)^{(n+1)} \left[ \frac{\partial H}{\partial A_x^{(n)}} A'_x + \frac{\partial H}{\partial \eta^{(n+1)}} \eta' + \frac{\partial H}{\partial (E^\varphi)^{(n)}} E^\varphi \right], \tag{39}
\]

where \( n \) is the maximum order considered.

The explicit dependence of (33) on \( E^x, E^\varphi, (E^x)', (E^\varphi)' \) and \( (E^x)'' \) gives the contribution term
\[
\int dx N(N^x)' (H_0 + H_\Gamma) = \int dx N(N^x)' (H - H_A),
\]

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so we may also write

\[
\{H[N], D[N^x]\} = \int \text{d}x \, N N^x \, H' + \int \text{d}x \, N(N^x)'(H - H_A)
\]

\[
+ \sum_{i=1}^{n} \int \text{d}x \, N(N^x)^{(i)} \left[ \frac{\partial H}{\partial A_x^{(i-1)}} A_x + \frac{\partial H}{\partial \eta^{(i)}} \eta' + \frac{DH}{D(E^\phi)^{(i-1)}} E^\phi \right]
\]

\[
+ \sum_{k=0}^{n-i} \left( \begin{pmatrix} i + k + 1 \\ k + 1 \end{pmatrix} \frac{\partial H}{\partial A_x^{(i+k)}} A_x^{(k+1)} + \frac{\partial H}{\partial \eta^{(i+k+1)}} \eta^{(k+2)} \right) + \left( \begin{pmatrix} i + k \\ k \end{pmatrix} \frac{\partial H}{\partial K^{(i+k)}} K^{(k+1)} \right)
\]

\[
+ \int \text{d}x \, N(N^x)^{(n+1)} \left[ \frac{\partial H}{\partial A_x^{(n)}} A_x + \frac{\partial H}{\partial \eta^{(n+1)}} \eta' + \frac{DH}{D(E^\phi)^{(n)}} E^\phi \right].
\]

With the \(i\)-sum being zero for \(n = 0\).

Here and in what follows we use the short hand notation

\[
\frac{DH}{Dq}
\]

for the partial derivative of \(H\) with respect to \(q\), acting only on the correction functions. Notice that \(DH/DA_x^{(i)} = \partial H/\partial A_x^{(i)}\) for all \(i \geq 0\), \(DH/D(E^\phi)^{(i)} = \partial H/\partial (E^\phi)^{(i)}\) for \(i > 1\), and \(DH/D(E^x)^{(i)} = \partial H/\partial (E^x)^{(i)}\) for \(i > 2\). We could therefore have written all terms in the previous expression as well as D-derivatives.

If we allow for one higher order of derivatives of the triad, and we group derivatives
\( A^{(k)} \) with derivatives \( E^{(k+1)} \), we rearrange as

\[
\{H[N], D[N^x]\} = \int dx \, NN^x \, H' + \int dx \, N(N^x)'(H - H_A) \\
+ \sum_{i=1}^{n} \int dx \, N(N^x)^{(i)} \left[ \frac{\partial H}{\partial A_x^{(i-1)}} A_x + \frac{\partial H}{\partial \eta^{(i)}} \eta' + \frac{DH}{D(E^x)^{(i-1)}} E^\varphi \\
+ \frac{DH}{D(E^x)^{(i)}} (E^x)' + (i + 1) \frac{DH}{D(E^x)^{(i)}} (E^x)' \right] \\
+ \sum_{k=0}^{n-i} \left( \begin{array}{c} i + k + 1 \\ k + 1 \end{array} \right) \left( \frac{\partial H}{\partial A_x^{(i+k)}} A_x^{(k+1)} + \frac{\partial H}{\partial \eta^{(i+k+1)}} \eta^{(k+2)} \right) + \left( \begin{array}{c} i + k \\ k \end{array} \right) \frac{\partial H}{\partial K\varphi^{(i+k)}} K\varphi^{(k+1)} \\
+ \sum_{k=1}^{n+1-i} \left( \begin{array}{c} i + k \\ k \end{array} \right) \frac{DH}{D(E^x)^{(i+k)}} (E^x)^{(k+1)} + \left( \begin{array}{c} i + k + 1 \\ k + 1 \end{array} \right) \frac{DH}{D(E^x)^{(i+k)}} (E^x)^{(k+1)} \right] \\
+ \int dx \, N(N^x)^{(n+1)} \left[ \frac{\partial H}{\partial A_\varphi^{(n)}} A_x + \frac{\partial H}{\partial \eta^{(n+1)}} \eta' + \frac{DH}{D(E^\varphi)^{(n)}} E^\varphi \\
+ \frac{DH}{D(E^\varphi)^{(n+1)}} (E^\varphi)' + (n + 2) \frac{DH}{D(E^\varphi)^{(n+1)}} (E^\varphi)' \right] \\
+ \int dx \, N(N^x)^{(n+2)} \frac{DH}{D(E^\varphi)^{(n+1)}} E^\varphi. \tag{41}
\]

The last integral immediately shows that the Hamiltonian does not depend on \((E^\varphi)^{(n+1)}\). There can only be non-trivial dependence on \((E^x)^{(n+1)}\).

### 3.2.2 Hamiltonian bracket

For \( n_{A_x}, n_{E^x}, n_{K\varphi}, n_{E^\varphi} \), the maximum order of derivatives of the corresponding variables, we compute the \( \{H[M], H[N]\} \) bracket using formula \([25]\)

\[
\{H[M], H[N]\} = 2G \sum_{i=0}^{n_{A_x}} \sum_{j=0}^{n_{E^x}} \sum_{j\neq i} \int dx \, (M^{(i)} N^{(j)} - N^{(i)} M^{(j)}) \left[ \gamma \delta_{A_x}^{n_{A_x}} \frac{1}{i} H \delta_{E^x}^{n_{E^x}} \frac{j}{H} \right] + \\
+ 2G \sum_{i=0}^{n_{K\varphi}} \sum_{j=0}^{n_{E^\varphi}} \sum_{j\neq i} \int dx \, (M^{(i)} N^{(j)} - N^{(i)} M^{(j)}) \left[ \frac{1}{2} \delta_{K\varphi}^{n_{K\varphi}} \frac{i}{H} \delta_{E^\varphi}^{n_{E^\varphi}} \frac{j}{H} \right],
\]
or more concisely using \([32]\) with \(n = \max(n_{A_x}, n_{E^x}, n_{K^x}, n_{E^\varphi})\):

\[
\{H[M], H[N]\} = 2G \sum_{j=1}^{n} \sum_{i=0}^{j-1} \int dx \left( M^{(i)} N^{(j)} - N^{(i)} M^{(j)} \right) \left[ \gamma (\delta_{A_x}^{n,i} H \delta_{E^x}^{n,j} H - \delta_{A_x}^{n,j} H \delta_{E^x}^{n,i} H) \right. \\
\left. + \frac{1}{2} (\delta_{K^x}^{n,i} H \delta_{E^x}^{n,j} H - \delta_{K^x}^{n,j} H \delta_{E^x}^{n,i} H) \right]. \tag{42}
\]

where again

\[
\delta_{q}^{n,i} H := \sum_{k=0}^{n-i} (-1)^{i+k} \binom{i + k}{k} \left( \frac{\partial H}{\partial q^{(i+k)}} \right)^{(k)}. \tag{43}
\]

For instance, at third maximal derivative order we have

\[
\delta_{q}^{3,0} H = \frac{\partial H}{\partial q} - \left( \frac{\partial H}{\partial q} \right)' - \left( \frac{\partial H}{\partial q} \right)'' - \left( \frac{\partial H}{\partial q} \right)'''. \tag{44}
\]

and \(\delta_{q}^{3,3} H = -\partial H/\partial q'''\).

Taking into account the explicit dependence of \(H\) on \(E^x, E^\varphi, (E^x)', (E^\varphi)', (E^x)''\), and \((E^\varphi)''\), we compute the coefficients in \([43]\). Defining

\[
\Delta_{E^x}^{0} := \frac{1}{2G} \left[ \frac{s \alpha |E^x|^\frac{3}{2} E^\varphi (f_1 + 1) - \bar{\alpha}|E^x|^{-\frac{1}{2}} f_2 - s \bar{\alpha} |E^x|^{-\frac{3}{2}} ((E^x)')^2}{2E^\varphi} \right. \\
\left. + \bar{\alpha} \frac{|E^x|^{-\frac{1}{2}} ((E^x)')^2}{2E^\varphi} - \bar{\alpha} \frac{|E^x|^{-\frac{1}{2}} ((E^\varphi)')}{2E^\varphi^2} \right]
\]

\[
\Delta_{E^x}^{1} := \frac{1}{2G} \left[ \frac{s \alpha \frac{|E^x|^{-\frac{1}{2}} ((E^\varphi)')}{2E^\varphi} + s \bar{\alpha} \frac{|E^x|^\frac{1}{2} (E^\varphi)'}{E^\varphi^2} + 2 \left( s \bar{\alpha} \frac{|E^x|^\frac{1}{2} (E^\varphi)'}{E^\varphi} \right) \right] \tag{45}
\]

\[
\Delta_{E^x}^{2} := \frac{1}{2G} \left( s \bar{\alpha} \frac{|E^x|^\frac{1}{2} (E^\varphi)'}{E^\varphi^2} \right) \tag{47}
\]

\[
\Delta_{E^\varphi}^{0} := \frac{1}{2G} \left[ \frac{s \alpha |E^x|^{-\frac{1}{2}} f_1 + 1 - \bar{\alpha}|E^x|^{-\frac{1}{2}} f_2 - s \bar{\alpha} |E^x|^\frac{1}{2} ((E^\varphi)')^2}{4E^\varphi^2} \right. \\
\left. + \bar{\alpha} \frac{|E^x|^\frac{1}{2} ((E^\varphi)')^2}{4E^\varphi^2} - \bar{\alpha} \frac{|E^x|^\frac{1}{2} ((E^\varphi)')}{4E^\varphi^3} \right]
\]

\[
\Delta_{E^\varphi}^{1} := \frac{1}{2G} \left( s \bar{\alpha} \frac{|E^x|^\frac{1}{2} ((E^\varphi)')}{E^\varphi^2} \right) \tag{49}
\]

we have

\[
\delta_{E^x}^{n,i} H := \Delta_{E^x}^{i} + \sum_{k=0}^{n-i} (-1)^{i+k} \binom{i + k}{k} \left( \frac{DH}{D(E^x)^{(i+k)}} \right)^{(k)} \tag{50}
\]
for $i = 0, 1, 2,$ and
\[
\delta_{E^\varphi}^{n,i} H := \Delta_{E^\varphi}^i + \sum_{k=0}^{n-i} (-1)^{i+k} \binom{i+k}{k} \left( \frac{DH}{D(E^\varphi)^{(i+k)}} \right)^{(k)}
\] (51)
for $i = 0, 1.$

4 Examples

A general treatment of closed constraint algebras in a derivative expansion appears to be complicated, but one can deal with the lowest orders. At first order (without additional derivatives beyond the classical form), we reproduce but also strengthen the results of [13]. At second order, we will obtain the first indications about possible higher-derivative corrections.

4.1 No additional derivatives

The case of an $H[N]$ with a modified dependence on phase-space variables but no additional spatial derivatives has already been studied in [13]. However, starting with more general assumptions on the possible dependence on $A_x$, we will be able to strengthen previous results. It turns out that a consistent deformation is possible with higher powers of $K_\varphi$. According to a derivative expansion, one could expect terms with an additional spatial derivative of $E$ to appear for each new factor of $K_\varphi$ in a series expansion of $f_1$, which will be discussed in the next subsection.

The bracket (40) with $n = 0$ reads
\[
\{H[N], D[N^x] \} = -H[N^x N']
\]
\[
+ \int dx N(N^x) \left[ -H_A + \frac{\partial H}{\partial A_x} A_x + \frac{\partial H}{\partial \eta'} \eta' + \frac{DH}{DE^\varphi} E^\varphi \right]
\] (52)
and, for a first-class algebra, gives the condition
\[
-H_A + \frac{\partial H}{\partial (A_x + \eta')} (A_x + \eta') + \frac{DH}{DE^\varphi} E^\varphi = \mathcal{F}_1 H + \mathcal{F}_2 D
\] (53)
with some functions $\mathcal{F}_1$ and $\mathcal{F}_2$.

Since $H$, by assumption, does not contain derivatives of $K_\varphi$ we must have $\mathcal{F}_2 = 0$. 

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Explicitly, (53) then reads

\[
\begin{align*}
\frac{1}{\alpha(f_1 + 1)} \left( \frac{\partial(\alpha(f_1 + 1))}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial(\alpha(f_1 + 1))}{\partial E^\varphi} E^\varphi \right) & H_0 \\
+ \frac{1}{\alpha f_2} \left( -\bar{\alpha} f_2 + \frac{\partial(\bar{\alpha} f_2)}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial(\bar{\alpha} f_2)}{\partial E^\varphi} E^\varphi \right) & H_A \\
+ \frac{1}{\alpha \Gamma} \left( \frac{\partial \alpha \Gamma}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial \alpha \Gamma}{\partial E^\varphi} E^\varphi \right) & H_1 \\
+ \frac{1}{\alpha \Gamma} \left( \frac{\partial \bar{\alpha} \Gamma}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial \bar{\alpha} \Gamma}{\partial E^\varphi} E^\varphi \right) & H_2 \\
\end{align*}
\]

Only the last two terms contain derivatives \((E^\varphi)', (E^\varphi)''\), and therefore

\[
F_1 = \frac{1}{\alpha \Gamma} \left( \frac{\partial \alpha \Gamma}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial \alpha \Gamma}{\partial E^\varphi} E^\varphi \right)
\]

must be satisfied.

If there are no inverse-triad corrections, that is, \(\alpha = \bar{\alpha} = \alpha \Gamma = \bar{\alpha} \Gamma = 1\), then (55) implies \(F_1 = 0\) and equation (54) reads

\[
\frac{\partial (H_0 + H_A)}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial (H_0 + H_A)}{\partial E^\varphi} E^\varphi = H_0 + H_A
\]

with general solution

\[
H_0 + H_A = c_1 E^\varphi + c_2 (A_x + \eta') + F[(A_x + \eta')/E^\varphi]
\]

for functions \(c_1\) and \(c_2\) independent of \(A_x\) and \(E^\varphi\), and an arbitrary function \(F\). (If \(f_1\) is assumed to be independent of \(A_x + \eta'\), we have the same equation and general solution for \(f_2\).) We may discard the homogeneous solution \(F[(A_x + \eta')/E^\varphi]\) on the basis that \(H_0 + H_A\) has to be of density weight one. Indeed, using the \(\{H, H\}\) bracket we will see explicitly that the dependence of \(H\) on \(A_x + \eta'\) has to be linear in this case.

For general inverse-triad corrections, and if we assume all correction functions except \(f_2\) to be independent of \(A_x + \eta'\), we get, by equating the \(H_A\) terms on the left and right hand side of (54), the slightly more complicated equation

\[
\frac{\partial f_2}{\partial (A_x + \eta')} (A_x + \eta') + \frac{\partial f_2}{\partial E^\varphi} E^\varphi = \left( \frac{1}{\alpha \Gamma} \frac{\partial \alpha \Gamma}{\partial E^\varphi} E^\varphi - \frac{1}{\bar{\alpha} \Gamma} \frac{\partial \bar{\alpha} \Gamma}{\partial E^\varphi} E^\varphi + 1 \right) f_2.
\]

\footnote{Define \(h = \log(H_0 + H_A), y = \log(A_x + \eta')\) and \(z = \log E^\varphi\), and subsequently \(y = Y + Z\) and \(z = Y - Z\). The differential equation then reads \(1 = \partial h/\partial y + \partial h/\partial z = \partial h/\partial Y\), solved by \(h(Y, Z) = Y + g(Z)\) with an arbitrary function \(g(Z)\). In terms of the original variables, \(H_0 + H_A = \exp(g) \exp(Y) = \sqrt{(A_x + \eta')E^\varphi G((A_x + \eta')/E^\varphi)\). In the solution used in the text, we have, without restriction, rewritten \(G(Z) = c_1 \exp(Z) + c_2 \exp(-Z) + F(Z)\) because the first two terms appear in the classical constraint.}
It can be solved as before, with additional factors of derivatives of $\alpha$.

The $\{H[M], H[N]\}$ bracket (52) is

\[
\{H[M], H[N]\} = 2G \int dx (MN' - NM') \\
\times \left[ -\gamma \Delta_{E^x}^2 \left( \frac{\partial H}{\partial A_x} \right)' + \left( \Delta_{E^x}^1 - (\Delta_{E^x}^2)' \right) \gamma \frac{\partial H}{\partial A_x} + \frac{1}{2} \left( \Delta_{E^x}^1 \right) \frac{\partial H}{\partial K_\varphi} \right].
\]

Explicitly, for inverse-triad corrections independent of $A_x$,

\[
\{H[M], H[N]\} =
\]

\[
= \frac{1}{2G} \int dx (MN' - NM') \left[ \alpha \partial f_2 \left| \frac{E^x}{E^\varphi} \right|^2 \left( 2\gamma \left( \frac{\partial f_2}{\partial A_x} \right)' \right) E^\varphi - \frac{\partial f_2}{\partial K_\varphi} (E^\varphi)' \right] \\
+ 2 \left( \alpha \partial f_2 \left| \frac{E^x}{E^\varphi} \right|^2 \frac{\partial f_2}{\partial A_x} \right) \left( \alpha \Gamma \right)' \gamma \partial f_2 \left( \frac{E^x}{E^\varphi} \right)' + \left( \alpha \Gamma \right)' \left( \frac{E^x}{E^\varphi} \right)' \left( \frac{E^x}{E^\varphi} \right)' \right]
\]

Since inverse-triad corrections have already been studied in detail elsewhere, we now consider holonomy corrections only, that is use $\alpha = \bar{\alpha} = \alpha_\Gamma = \bar{\alpha}_\Gamma = 1$. Condition (56) reads explicitly

\[
f_2 - \frac{\partial f_2}{\partial E^\varphi} E^\varphi - \frac{\partial f_2}{\partial A_x} A_x - \frac{\partial f_2}{\partial \eta'} \eta' = \frac{E^\varphi}{2E^x} \left( \frac{\partial f_1}{\partial E^\varphi} E^\varphi + \frac{\partial f_1}{\partial A_x} A_x + \frac{\partial f_1}{\partial \eta'} \eta' \right),
\]

and the bracket (58) gives

\[
\{H[M], H[N]\} =
\]

\[
= \frac{1}{2G} \int dx (MN' - NM') \left[ \frac{\left| E^x \right|}{(E^\varphi)^2} \left( 2\gamma \left( \frac{\partial f_2}{\partial A_x} \right)' \right) E^\varphi - \frac{\partial f_2}{\partial K_\varphi} (E^\varphi)' \right] \\
+ s \left( \gamma \frac{\partial f_2}{\partial A_x} - \frac{1}{2} \frac{\partial f_1}{\partial K_\varphi} \right) (E^\varphi)' + s \gamma \left( \frac{\partial f_1}{\partial A_x} \right)' - s \left( \frac{(E^\varphi)'}{2E^x} - \frac{(E^\varphi)'}{E^\varphi} \right) \frac{\partial f_1}{\partial A_x} \right].
\]

Expanding

\[
\left( \frac{\partial f_1}{\partial A_x} \right)' = \frac{\partial^2 f_1}{\partial A_x^2} A_x + \frac{\partial^2 f_1}{\partial K_\varphi \partial A_x} K_\varphi + \frac{\partial^2 f_1}{\partial \eta' \partial A_x} \eta' + \frac{\partial^2 f_1}{\partial E^\varphi \partial A_x} (E^\varphi)' + \frac{\partial^2 f_1}{\partial E^x \partial A_x} (E^x)' + \frac{\partial^2 f_1}{\partial E^\varphi \partial A_x} (E^\varphi)' \]

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and similarly for $(\partial f_2/\partial A_x)'$, we get

$$\{H[M], H[N]\} =$$

$$\frac{1}{2G} \int dx (MN' - NM') \left[ \gamma \left( \frac{2|E^x|}{E^\varphi} \frac{\partial^2 f_2}{\partial K_\varphi \partial A_x} + s \frac{\partial^2 f_1}{\partial K_\varphi \partial A_x} \right) K_\varphi' + \left( 2 \gamma \frac{|E^x|}{E^\varphi} \frac{\partial^2 f_2}{\partial E^x \partial A_x} + \frac{2|E^x|}{E^\varphi} \frac{\partial f_2}{\partial K_\varphi} - s \gamma \frac{\partial^2 f_1}{\partial E^x \partial A_x} - \frac{\gamma}{2|E^x|} \frac{\partial f_1}{\partial A_x} - \frac{s}{2E^\varphi} \frac{\partial f_1}{\partial K_\varphi} \right) (E^x)' \right. + \left( 2 \gamma \frac{|E^x|}{E^\varphi} \frac{\partial^2 f_2}{\partial A_x^2} + s \gamma \frac{\partial^2 f_1}{\partial A_x^2} \right) A_x' + \left( 2 \gamma \frac{|E^x|}{E^\varphi} \frac{\partial^2 f_2}{\partial \eta' \partial A_x} + s \gamma \frac{\partial^2 f_1}{\partial \eta' \partial A_x} \right) \eta'' + \left. \left( 2 \gamma \frac{|E^x|}{E^\varphi} \frac{\partial^2 f_2}{\partial E^\varphi \partial A_x} + s \gamma \frac{\partial^2 f_1}{\partial E^\varphi \partial A_x} + \frac{s}{2E^\varphi} \frac{\partial f_1}{\partial A_x} \right) (E^\varphi)' \right].$$

If we impose again that the right hand side be a linear combination $F_1 H + F_2 D$, by considering the $H_2^3$ term, we must have $F_1 = 0$ since correction functions do not contain second derivatives of $E^x$. In order to have now a multiple of the diffeomorphism constraint $2E^\varphi K_\varphi' - \frac{1}{\gamma} (A_x + \eta') (E^\varphi)'$, the $A_x'$, $\eta''$ and $(E^\varphi)'$ terms must vanish:

$$\frac{2E^x}{E^\varphi} \frac{\partial^2 f_2}{\partial A_x^2} + \frac{\partial^2 f_1}{\partial A_x^2} = 0 \quad (62)$$

$$\frac{2E^x}{E^\varphi} \frac{\partial^2 f_2}{\partial \eta' \partial A_x} + \frac{\partial^2 f_1}{\partial E^\varphi \partial A_x} = 0 \quad (63)$$

$$\frac{2E^x}{E^\varphi} \frac{\partial^2 f_2}{\partial E^\varphi \partial A_x} + \frac{\partial^2 f_1}{\partial E^\varphi \partial A_x} + \frac{1}{E^\varphi} \frac{\partial f_1}{\partial A_x} = 0 \quad (64)$$

If these conditions are not satisfied, the $A_x'$ and $(E^\varphi)'$ terms in (61) could be part of a first-class algebra only if $F_2$ depends on $A_x'$ and $(E^\varphi)'$, respectively. Then we would need $f_1$ or $f_2$ to depend on $A_x'$ and $(E^\varphi)'$ for the first terms in (61) to be anomaly-free with the same dependence of $F_2$ on derivatives, but such a dependence is assumed to be absent in this subsection.

Equations (62) and (63) imply

$$\frac{2E^x}{E^\varphi} \frac{\partial f_2}{\partial A_x} + \frac{\partial f_1}{\partial A_x} = C[K_\varphi, E^x, E^\varphi] \quad (65)$$

or equivalently

$$\frac{2E^x}{E^\varphi} f_2 + f_1 = C[K_\varphi, E^x, E^\varphi](A_x + \eta') + C_1[K_\varphi, E^x, E^\varphi].$$

for arbitrary functions $C[K_\varphi, E^x, E^\varphi]$ and $C_1[K_\varphi, E^x, E^\varphi]$. Therefore, under the present assumption, the Hamiltonian $H$ can depend only linearly on $A_x + \eta'$. (From the equations, $C_1$ could also depend on $\eta'$, but we know that $H$ must depend on the gauge invariant combination $A_x + \eta'$. Only).
Substituting the derivative of equation (65) with respect to $E^\varphi$ and (64) back in (65) gives the functional dependence of $C$ on $E^\varphi$:

$$\frac{\partial C}{\partial E^\varphi} = -\frac{C}{E^\varphi}$$

so $C[K_\varphi, E^x, E^\varphi] = C_2[K_\varphi, E^x]/E^\varphi$, for some function $C_2[K_\varphi, E^x]$, and

$$\frac{2E^x}{E^\varphi} f_2 + f_1 = C_2[K_\varphi, E^x] \frac{A_x + \eta'}{E^\varphi} + C_1[K_\varphi, E^x, E^\varphi]. \quad (66)$$

Putting these results back in the bracket (61)

$$\{H[M], H[N]\} = \frac{1}{2G} \int dx (MN' - NM') \left[ \frac{s\gamma}{2E^\varphi} \frac{\partial C_2}{\partial K_\varphi} \left( 2E^\varphi K'_\varphi - \frac{1}{\gamma} (A_x + \eta')(E^x)' \right) \right.$$

$$+ s\gamma \left( \frac{\partial C_2}{\partial E^x} - \frac{C_2}{2E^x} - \frac{1}{2\gamma} \frac{\partial C_1}{\partial K_\varphi} \right) \left( \frac{E^x}' \right) \right] \quad (67)$$

gives a condition for functions $C_1$ and $C_2$:

$$\frac{\partial C_2}{\partial E^x} - \frac{C_2}{2E^x} - \frac{1}{2\gamma} \frac{\partial C_1}{\partial K_\varphi} = 0.$$ 

Condition (59) from the $\{H, D\}$ bracket translates into

$$\frac{\partial C_1}{\partial E^\varphi} E^\varphi = 0$$

that is, $C_1$ is independent of $E^\varphi$. Substituting (60) in the Hamiltonian then gives

$$H[N] = -\frac{1}{2G} \int dx N \left( |E^x|^{-\frac{1}{2}} E^\varphi C_1 + |E^x|^{-\frac{1}{2}} C_2 (A_x + \eta') + |E^x|^{-\frac{1}{2}} E^\varphi \right.$$  

$$- |E^x|^{-\frac{1}{2}} E^\varphi \Gamma^2_{\varphi} + 2s |E^x| \frac{1}{2} \Gamma'_\varphi \right)$$

so changing $C \rightarrow C/\gamma$, $C_2 \rightarrow C_2/\gamma$ and renaming variables gives the general solution for $f_1$ and $f_2$ consistent with previous results:

$$f_1 = C_1[K_\varphi, E^x]$$

$$f_2 = \frac{1}{2E^x} C_2[K_\varphi, E^x](A_x + \eta')/\gamma \quad (68)$$

with $C_1$ and $C_2$ satisfying

$$\frac{\partial C_1}{\partial K_\varphi} = 2 \frac{\partial C_2}{\partial E^x} - \frac{C_2}{E^x}.$$ 

(69)
The form \( f_1 = C_1[K, E^x] \) allows holonomy corrections to depend on the triad component \( E^x \) as well as on extrinsic curvature or the connection, which could be used to model lattice refinement by a triad dependent length parameter \( \ell_0 \). However, the relationship (69) rules out this kind of parameterization: In a U(1)-theory as it automatically appears in the reduced setting of spherical symmetry, the \( K \)-dependence of holonomies is almost-periodic. (On the quantum configuration space, \( K \) takes values in the Bohr compactification of the real line.) If we try to model lattice refinement by some form \( \exp(i f(E^x)K) \) of holonomies, only a constant \( f \) is compatible with (69) and an almost-periodic \( C_2 \). Otherwise, the derivative \( \partial \exp(i f(E^x)K)/\partial E^x = i K \phi (df/\partial E^x) \exp(i f(E^x)K) \) is not almost periodic in \( K \). Lattice refinement appears to be incompatible with a consistent algebra if one insists on almost-periodic holonomy modifications. This result shows an interesting relationship with problems of the Bohr compactification as a model for non-Abelian connections, pointed out in [39]: Taking into account the non-Abelian structure leads to a more-complicated representation which automatically incorporates lattice refinement but is not based on almost-periodic functions. (In isotropic models one can formally write lattice-refined holonomies with \( f \) of power-law form as unrefined ones in variables redefined by a canonical transformation. This is not possible here because we are dealing with two variables \( K \) and \( E^x \) that are not part of a canonical pair. The situation is closer to anisotropic models, in which rescalings are not possible in general [40].)

The deformed algebra in this case is

\[
\{H[N], D[N']\} = -H[N'N]
\]

\[
\{H[M], H[N]\} = D \left[ (MN' - NM') s \frac{\partial C_2}{2(E^x)^2 \partial K} \right]
\]

\[
- G \left[ (MN' - NM') \eta' s \frac{\partial C_2}{2(E^x)^2 \partial K} \right].
\]

The deformation function \( \beta = (2E^x)^{-1} \partial C_2/\partial K \), which would equal one classically, is of particular interest. From (69) we obtain

\[
4 \frac{\partial(E^x \beta)}{\partial E^x} - 2\beta = 2\beta - 4E^x \frac{\partial \beta}{\partial E^x} = \frac{\partial^2 C_1}{\partial K^2}.
\] (70)

If \( \beta \) depends only weakly on \( E^x \), which is expected for pure holonomy corrections, it is negative near a maximum of \( C_1 \). As observed in [11], this behavior implies signature change to a quantum version of 4-dimensional Euclidean space whenever holonomy corrections are strong, in a regime where they would bound the curvature dependence of the Hamiltonian constraint.

These results agree with previous constructions, but they are more general because we did not assume but rather derive that the Hamiltonian constraint must depend linearly on \( A_x + \eta' \) to the given order of derivatives.

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4.2 Higher spatial triad derivatives

If we allow for higher spatial derivatives of the triad for extra factors of $K_{\varphi}$ in an expanded $f_1$, as suggested by a derivative expansion, several new terms appear in the equations of the previous subsection. For instance, allowing for one additional order of spatial triad derivatives, using (41), the \{H, D\} bracket and Eq. (53) could include two additional terms $\frac{DH}{D(E^{x})'}(E^{x})'' + 2\frac{DH}{D(E^{\varphi})'}(E^{\varphi})'$. (The last one, however, must be zero from the $N(N^{x})''$-condition found in Sec. 3.2.1.) Such terms would be added to the explicit version (54) as well.

If one considers the implicit time derivative in $K_{\varphi}$ and $A_{x}$ on the same footing as explicit higher spatial derivatives of the triad, a consistent derivatives expansion must limit the polynomial order in $K_{\varphi}$ and $A_{x}$ along with the explicit derivative order. If the next derivative order is considered, with correction functions allowed to depend on $(E^{x})'$ and $(E^{\varphi})'$, the curvature-dependent functions $f_1$ and $f_2$ could be third-order polynomials of $K_{\varphi}$ and $A_{x}$ or quadratic with one spatial derivative. We are no longer allowed to assume a non-polynomial function or a series for the curvature dependence, such as an almost-periodic function. A third-order term in $K_{\varphi}$ or $A_{x}$ would violate time-reversal symmetry, and so the next derivative order amounts to correction functions depending on the first spatial derivative of the phase-space variables. We will discuss such consistent versions in the next subsection, and an explicit derivative expansions in more detail in Sec. 5.

4.3 Dependence on first curvature derivatives

We now consider the case where all correction function may depend on first derivatives of the phase space variables. Bracket (10) reads accordingly

\[
\{H[N], D[N^{x}]\} = -H[N^{x}N']
+ \int dx N(N^{x})' \left[ -HA + \frac{DH}{DA_{x}}A_{x} + \frac{DH}{D\eta'}\eta' + \frac{DH}{D(E^{\varphi})}E^{\varphi}
+ 2\left( \frac{\partial H}{\partial A_{x}}A_{x}' + \frac{\partial H}{\partial \eta'}\eta'' \right) + \frac{\partial H}{\partial K_{\varphi}'}K_{\varphi}' + \frac{DH}{D(E^{x})'}(E^{x})' + 2\frac{DH}{D(E^{\varphi})'}(E^{\varphi})' \right]
+ \int dx N(N^{x}'') \left[ \frac{\partial H}{\partial A_{x}'}A_{x} + \frac{\partial H}{\partial \eta''}\eta' + \frac{DH}{D(E^{\varphi})}E^{\varphi} \right].
\]  

(71)

Requiring the last integral proportional to $N(N^{x})''$ to vanish weakly, gives one condition, analogous to (53)

\[
\frac{\partial H}{\partial (A_{x} + \eta')'}(A_{x} + \eta') + \frac{DH}{D(E^{\varphi})'}E^{\varphi} = \mathcal{F}_1 H + \mathcal{F}_2 D.
\]  

(72)

Again, since we assume unintegrated point holonomies for the $\varphi$-components if we do not consider derivatives of $K_{\varphi}$ (dropping $\partial H/\partial K_{\varphi}'$ in (71)), we must have $\mathcal{F}_2 = 0$. 

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Explicitly,
\[
\frac{1}{\alpha(f_1+1)} \left( \frac{\partial(\alpha(f_1+1))}{\partial(A_x + \eta')} (A_x + \eta') + \frac{\partial(\alpha(f_1+1))}{\partial(E^\varphi)'} E^\varphi \right) H_0 \\
+ \frac{1}{\alpha f_2} \left( \frac{\partial(\alpha f_2)}{\partial(A_x + \eta')} (A_x + \eta') + \frac{\partial(\alpha f_2)}{\partial(E^\varphi)'} E^\varphi \right) H_A \\
+ \frac{1}{\alpha} \left( \frac{\partial\alpha}{\partial(A_x + \eta')} (A_x + \eta') + \frac{\partial\alpha}{\partial(E^\varphi)'} E^\varphi \right) H_1 \\
+ \frac{1}{\alpha} \left( \frac{\partial\alpha}{\partial(A_x + \eta')} (A_x + \eta') + \frac{\partial\alpha}{\partial(E^\varphi)'} E^\varphi \right) H_2 \\
\Gamma = F_1 H. 
\] (73)

Since only the last term contains the derivative \((E^\varphi)''\), one must have
\[
F_1 = \frac{1}{\bar{\alpha}_\Gamma} \left( \frac{\partial\bar{\alpha}_\Gamma}{\partial(A_x + \eta')} (A_x + \eta') + \frac{\partial\bar{\alpha}_\Gamma}{\partial(E^\varphi)'} E^\varphi \right) 
\] (74)

Based on this equation we can already draw one conclusion: If there are no inverse-triad corrections and no further corrections with derivatives of \(E^\varphi\) and \(K^\varphi\), \(F_1 = 0\) and \((72)\) implies that there can be no dependence on \(A'_x\) either.

With or without the former assumptions, the simplest possibility is for each coefficient of \(H_0\), \(H_A\) and \(H_\Gamma\) above to be equal to zero, so the condition for the correction functions \(\alpha(f_1+1), \bar{\alpha} f_2, \alpha_\Gamma\) and \(\bar{\alpha}_\Gamma\) is
\[
\frac{\partial F}{\partial(A_x + \eta')'} (A_x + \eta') + \frac{\partial F}{\partial(E^\varphi)'} E^\varphi = 0 
\]
with general solution
\[
F = F \left[ (A_x + \eta')' - \frac{(A_x + \eta')}{E^\varphi} (E^\varphi)' \right]. 
\]

The \(N(N^\varphi)'\) integral in \((71)\) gives the additional condition (assuming again that there are no derivatives of \(K^\varphi\))
\[
-H_A + \frac{\partial H}{\partial(A_x + \eta')} (A_x + \eta') + \frac{DH}{DE^\varphi} E^\varphi \\
+ 2 \frac{\partial H}{\partial(A_x + \eta')'} (A_x + \eta')' + 2 \frac{DH}{D(E^\varphi)'} (E^\varphi)' + \frac{DH}{D(E^\varphi)'} (E^\varphi)' = F_3 H 
\] (75)

an extension of \((53)\).
On the other hand, the \{H, H\} bracket (12) is

\[
\{H[M], H[N]\} = 2G \int dx (MN' - NM') \left[ \gamma \left( \Delta^1_E^x - \frac{DH}{DE^x} \right) \left( \frac{\partial H}{\partial A'_x} - \left( \frac{\partial H}{\partial A'_x} \right)' \right) + \right.
\]

\[
+ \left. \left( \Delta^0_E^x + \frac{DH}{DE^x} - \left( \frac{DH}{D(E^x)} \right)' \right) \gamma \frac{\partial H}{\partial A'_x} + \right.
\]

\[
+ \frac{1}{2} \left( \Delta^1_{E^x} - \frac{DH}{D(E^x)} \right) \left( \frac{\partial H}{\partial K'_x} - \left( \frac{\partial H}{\partial K'_x} \right)' \right) + \right.
\]

\[
+ \frac{1}{2} \left( \Delta^0_{E^x} + \frac{DH}{D(E^x)} - \left( \frac{DH}{D(E^x)} \right)' \right) \gamma \frac{\partial H}{\partial A'_x} \right]
\]

\[
+ 2G \int dx (MN'' - NM'') \left[ \gamma \left( \Delta^2_{E^x} \right) \left( \frac{\partial H}{\partial A_x} - \left( \frac{\partial H}{\partial A_x} \right)' \right) \right]
\]

\[
- 2G \int dx (M'N'' - N'M'') \gamma \Delta^2_{E^x} \frac{\partial H}{\partial A'_x}. \tag{76}
\]

The last integral, using (15), imposes the condition

\[- 2G \gamma \Delta^2_{E^x} \frac{\partial H}{\partial A_x} = \mathcal{F}_4 H + \mathcal{F}_5 D, \tag{77}\]

or, again explicitly

\[-s\bar{\alpha} \Gamma \frac{|E^x|^4}{E^x} \left[ \frac{1}{\alpha(f_1 + 1)} \left( \frac{\partial (\alpha(f_1 + 1))}{\partial A'_x} \right) H_0 + \frac{1}{\bar{\alpha}f_2} \left( \frac{\partial (\bar{\alpha}f_2)}{\partial A'_x} \right) H_A \right. \right.
\]

\[
\left. + \frac{1}{\bar{\alpha}} \left( \frac{\partial A'_x}{\partial A'_x} \right) H^1_\Gamma + \frac{1}{\alpha} \left( \frac{\partial A'_x}{\partial A'_x} \right) H^{2,3}_\Gamma \right] = \mathcal{F}_4 H + \mathcal{F}_5 D. \tag{78}\]

Once more, if we do not consider derivatives of \(K'_x\), or we take the weaker assumption \(\partial \bar{\alpha} \Gamma / \partial K'_x = 0\), noting that on the left hand side only \(H^2_\Gamma\) contains \((E^x)'\), we have

\[-s |E^x|^{\frac{3}{2}} \frac{\partial \bar{\alpha} \Gamma}{E^x} \frac{\partial \bar{\alpha} \Gamma}{\partial A'_x}. \tag{79}\]

Hence, if we do not consider derivatives of \(K'\) and if \(\bar{\alpha} \Gamma\) is independent of \(A'_x\), we must have \(\partial H / \partial A'_x = 0\) even if we allow for inverse-triad corrections. In particular this shows that radial holonomy corrections alone cannot be anomaly-free at first order. According to our calculations, the Hamiltonian constraint can depend on \(A'_x\) only if it also depends on \(K'_x\) or if correction functions depend on \((E^x)''\). While we have not found such a consistent version (the next section provides further insights), if one exists it would require tightly related radial and angular holonomy corrections for all anomalies to cancel.
5 Extrinsic-curvature expansion

Some properties of consistent curvature constraints can be derived by different, somewhat more condensed methods if one makes use of more explicit expansions of modified constraints. Using such methods, we now consider a derivative order one above the classical one, allowing for derivatives of both $A_x$ (or $K_x$) and $K_{\varphi}$.

In order to make spatial derivatives of extrinsic curvature in the functions $f_1$ and $f_2$ of (35) more explicit, we consider an expansion of the form (8) for these functions. We also assume time reversibility, so that there are no terms with odd powers of the extrinsic curvatures $K_x$ and $K_{\varphi}$. With these assumptions the Hamiltonian density may be expanded as

$$H = H_{00}^{00} + H_{00}^{11}K_xK_{\varphi} + H_{00}^{02}K_x^2 + H_{00}^{10}K_xK_{\varphi}' + H_{01}^{01}K_{\varphi}K_{\varphi}' + H_{10}^{10}K_{\varphi}K_{\varphi}' + H_{10}^{10}K_{\varphi}K_{\varphi}' + H_{02}^{02}K_{\varphi}^2 + H_{01}^{01}K_{\varphi}K_{\varphi}' + H_{10}^{10}K_{\varphi}K_{\varphi}'$$

with ‘coefficients’ $H_{kl}^{ij}$ depending only on triad variables (and their derivatives). For the classical constraint, we have

$$H_{00}^{00} = -\frac{1}{G} \left( \frac{E_x}{\sqrt{|E^x|}} \left( 1 - \Gamma_{\varphi}^2 \right) + 2s \sqrt{|E^x|} \Gamma'_{\varphi} \right)$$

$$H_{00}^{11} = -\frac{1}{G} s \sqrt{|E^x|}$$

$$H_{00}^{02} = -\frac{1}{2G} \frac{E_{\varphi}}{\sqrt{|E^x|}}$$

and the rest zero.

As a short-cut, we will parameterize components and spatial derivatives of the densitized triad by the sequence

$$(E^x, E_{\varphi}, \Gamma_{\varphi}, \Gamma'_{\varphi}, \Gamma''_{\varphi}, \ldots)$$

where, we recall,

$$\Gamma_{\varphi} = -\frac{(E^x)'}{2E_{\varphi}}$$

is a component of the classical spin connection. The sequence (84) is not the most general set of derivatives of both $E^x$ and $E_{\varphi}$. For our specific example, we assume that corrections appear as powers or derivatives of $\Gamma_{\varphi}$, modeled on the form of higher-curvature corrections. We do not attempt a more general analysis in this paper. (If the hypersurface-deformation algebra is deformed, the classical structure of Riemannian space-time does not apply and it is no longer clear what a spin connection is that $\Gamma_{\varphi}$ could be related to by pull-back. But the spatial structure remains unmodified in our setting, and in any case we are still free to use the same $\Gamma_{\varphi}$ as a function of the phase-space degrees of freedom $E^x$ and $E_{\varphi}$.)
We then have
\[
\frac{\delta H(y)}{\delta E^x(z)} = \frac{\partial H(y)}{\partial E^x} \delta(y, z) - \frac{\partial H(y)}{\partial \Gamma_{\varphi}} \frac{\delta(y, z)}{2E^\varphi} + \frac{\partial H(y)}{\partial \Gamma_{\varphi}} \left( - \frac{\delta''(y, z)}{2E^\varphi} + \frac{(E^\varphi)' \delta'(y, z)}{2(E^\varphi)^2} \right)
\]
\[
+ \frac{\partial H(y)}{\partial \Gamma''_{\varphi}} \left( - \frac{\delta''(y, z)}{2E^\varphi} + \frac{(E^\varphi)' \delta'(y, z)}{2(E^\varphi)^2} \right) + \frac{\partial H(y)}{\partial \Gamma'''_{\varphi}} \frac{\delta'(y, z)}{2(E^\varphi)^2} \delta''(y, z) - \frac{(E^\varphi)' \delta'(y, z)}{2E^\varphi} \delta''(y, z)
\]
\[
=: E_1(y) \delta(y, z) + E_2(y) \delta'(y, z) + E_3(y) \delta''(y, z) + E_4(y) \delta'''(y, z)
\]
with
\[
E_1 = \frac{\partial H}{\partial E^x},
\]
\[
E_2 = - \frac{1}{2E^\varphi} \frac{\partial H}{\partial \Gamma_{\varphi}} + \frac{(E^\varphi)' \partial H}{2(E^\varphi)^2 \partial \Gamma_{\varphi}} + \frac{(E^\varphi)''}{2(E^\varphi)^2} \frac{\partial H}{\partial \Gamma''_{\varphi}},
\]
\[
E_3 = - \frac{1}{2E^\varphi} \frac{\partial H}{\partial \Gamma_{\varphi}} + \frac{(E^\varphi)' \partial H}{(E^\varphi)^2 \partial \Gamma_{\varphi}},
\]
\[
E_4 = - \frac{1}{2E^\varphi} \frac{\partial H}{\partial \Gamma''_{\varphi}}.
\]
and
\[
\frac{\delta H(y)}{\delta E^\varphi(z)} = \frac{\partial H(y)}{\partial E^\varphi} \delta(y, z) + \frac{(E^\varphi)' \partial H(y)}{2(E^\varphi)^2} \delta(y, z)
\]
\[
+ \frac{\partial H(y)}{\partial \Gamma_{\varphi}} \left( \frac{(E^\varphi)''}{2(E^\varphi)^2} \delta(y, z) - \frac{(E^\varphi)'(E^\varphi)'}{(E^\varphi)^3} \delta(y, z) + \frac{(E^\varphi)'}{2(E^\varphi)^2} \delta'(y, z) \right)
\]
\[
+ \frac{\partial H(y)}{\partial \Gamma_{\varphi}'} \left( \frac{(E^\varphi)''}{2(E^\varphi)^2} \delta(y, z) + \frac{(E^\varphi)''}{(E^\varphi)^2} \delta'(y, z) - 2 \frac{(E^\varphi)'(E^\varphi)''}{(E^\varphi)^3} \delta'(y, z) + \frac{(E^\varphi)''(E^\varphi)'}{(E^\varphi)^3} \delta(y, z) \right)
\]
\[
+ \frac{(E^\varphi)'}{2(E^\varphi)^2} \delta''(y, z) - \frac{(E^\varphi)'(E^\varphi)''}{(E^\varphi)^3} \delta(y, z) + 3 \frac{(E^\varphi)'(E^\varphi)''}{(E^\varphi)^4} \delta(y, x)
\]
\[
- 2 \frac{(E^\varphi)'(E^\varphi)'}{(E^\varphi)^3} \delta'(y, x)
\]
\[
=: \tilde{E}_1(y) \delta(y, z) + \tilde{E}_2(y) \delta'(y, z) + \tilde{E}_3(y) \delta''(y, z)
\]
with
\[
\tilde{E}_1 = \frac{\partial H}{\partial E^\varphi} + \frac{(E^\varphi)'}{2(E^\varphi)^2} \frac{\partial H}{\partial \Gamma_{\varphi}} + \left( \frac{(E^\varphi)''}{2(E^\varphi)^2} - \frac{(E^\varphi)'(E^\varphi)'}{(E^\varphi)^3} \right) \frac{\partial H}{\partial \Gamma_{\varphi}}
\]
\[
+ \left( \frac{(E^\varphi)''}{2(E^\varphi)^2} - 2 \frac{(E^\varphi)'(E^\varphi)'}{(E^\varphi)^3} \right) \frac{\partial H}{\partial \Gamma_{\varphi}'} + \frac{(E^\varphi)'}{2(E^\varphi)^2} \frac{\partial H}{(E^\varphi)^3} \delta(y, x) + 3 \frac{(E^\varphi)'(E^\varphi)''}{(E^\varphi)^4} \delta(y, x)
\]
\[
\tilde{E}_2 = \frac{(E^\varphi)'}{2(E^\varphi)^2} \frac{\partial H}{\partial \Gamma_{\varphi}'} + \left( \frac{(E^\varphi)''}{(E^\varphi)^3} - 2 \frac{(E^\varphi)'(E^\varphi)'}{(E^\varphi)^3} \right) \frac{\partial H}{\partial \Gamma_{\varphi}'}
\]
\[
\tilde{E}_3 = \frac{(E^\varphi)'}{2(E^\varphi)^2} \frac{\partial H}{\partial \Gamma_{\varphi}}.
\]
Here and in what follows, derivatives of delta functions are taken with respect to the first argument: $y$ for $\delta(y, z)$ above.

In the $\{H, H\}$ bracket, these functional derivatives will appear together with those by $K_x$ and $K_\varphi$, respectively. The latter are

\[
\frac{\delta H(x)}{\delta K_x(z)} = (H_{00}^{11} K_\varphi + 2H_{00}^{20} K_x + H_{10}^{10} K_x' + H_{01}^{10} K_\varphi') \delta(x, z)
\]

\[
+ (H_{10}^{10} K_x + H_{10}^{01} K_\varphi') \delta'(x, z)
\]

\[
=: K_1(x) \delta(x, z) + K_2(x) \delta'(x, z)
\]

\[
\frac{\delta H(x)}{\delta K_\varphi(z)} = (H_{00}^{11} K_x + 2H_{00}^{02} K_\varphi + H_{01}^{01} K_\varphi' + H_{10}^{01} K_\varphi') \delta(x, z)
\]

\[
+ (H_{01}^{10} K_x + H_{01}^{01} K_\varphi) \delta'(x, z)
\]

\[
:= \bar{K}_1(x) \delta(x, z) + \bar{K}_2(x) \delta'(x, z).
\]

With these preparations, integrating out the $x$ and $y$ dependence with smearing functions $M(x)$ and $N(y)$, we write

\[
\frac{1}{G} \{H[M], H[N]\} = 
\]

\[
= \int \int \int dx \, dy \, dz \, M(x) N(y) \left( 2 \frac{\delta H(x)}{\delta K_\varphi(z)} \frac{\delta H(y)}{\delta E^x(z)} + \frac{\delta H(x)}{\delta K_\varphi(z)} \frac{\delta H(y)}{\delta E^z(z)} \right) - (x \leftrightarrow y)
\]

\[
= \int \int \int dx \, dy \, dz \, M(x) N(y) \left( 2 \frac{\delta H(x)}{\delta K_\varphi(z)} \frac{\delta H(y)}{\delta E^x(z)} + \frac{\delta H(x)}{\delta K_\varphi(z)} \frac{\delta H(y)}{\delta E^z(z)} \right) - (M \leftrightarrow N)
\]

\[
= \int dz \left[ 2(MNK_1 E_1 - M(NE_2)'K_1 + M(NE_3)''K_1 - M(NE_4)''''K_1
\]

\[
- (MK_2)' E_1 + (MK_2)'(NE_2)' - (MK_2)'(NE_3)'' + (MK_2)'(NE_4)''''
\]

\[
+ MKNK_1 E_1 - M(NE_2)'K_1 + M(NE_3)''K_1
\]

\[
- (M \bar{K}_2)' E_1 + (M \bar{K}_2)'(NE_2)' - (M \bar{K}_2)'(NE_3)''
\right] - (M \leftrightarrow N)
\]

\[
= \int dz \left( MN' - NM' \right) \left[ 2(-K_1 E_2 + K_2 E_1 + 2K_1 E_3' - K_2 E_2' + K_2 E_2 - 3K_1 E_4''
\]

\[
+ K_2 E_3'' - 2K_2 E_3' + 3K_2 E_4'' - K_2 E_4''') - \bar{K}_1 E_2 + \bar{K}_2 E_1
\]

\[
+ 2K_1 E_3 - K_2 E_2' + K_2 E_2 + K_2 E_3' - 2K_2 E_3]
\]

\[
+ \int dz \left( MN'' - NM'' \right) \left[ 2(K_1 E_3 - 3K_1 E_4' - K_2 E_3 + 3K_2 E_4') + \bar{K}_1 E_3 - \bar{K}_2 E_3
\]

\[
+ \int dz \left( MN''' - NM''' \right) \left[ 2(-K_2 E_3 + 3K_2 E_4') - \bar{K}_2 E_3 \right]
\]

\[
+ \int dz \left( MN'''' - NM'''' \right) (K_2' - K_1) E_4
\]

\[
+ \int dz \left( MN''' - NM''' \right) K_2 E_4
\] (93)
We are now ready to apply (18) and (19) to read off two independent conditions. The second one of these conditions provides a lengthy equation, but coefficients of $K'''$ and $K'''$ on the left-hand side of (19) must vanish since these cannot be matched with a linear combination of constraints. The coefficient of $K'''$ is identically zero, but the requirement of vanishing $K'''$-term gives the equation

$$E_4(H_{10}^{01} - H_{01}^{01}) = 0. \tag{94}$$

We first consider the simplest possibility $E_4 = 0$. With $E_4 = 0$, we have $\partial H/\partial \Gamma_\varphi'' = 0$ and therefore $E_3 = 0$. Condition (18) then has only one non-zero term $K_2 E_3 \approx 0$, which implies $K_2 = 0$. (There is no $K_\varphi$ in the diffeomorphism constraint but only its first derivative, and $E_3$ cannot be zero because it does not vanish classically.) With $K_2 = 0$, we must have $H_{10}^{10} = 0 = H_{01}^{01}$, and there cannot be first-order derivatives of $K_\varphi$ in the Hamiltonian. A consistent version with radial holonomies seems to require higher than next order in derivatives of the triad.

It remains to evaluate all remaining terms of (19):

$$2K_1 E_2 + \bar{K}_1 \bar{E}_2 - \bar{K}_2 \bar{E}_1 + 2K'_1 E_3 - 2K_1 E'_3 + \bar{K}_2 \bar{E}'_2 - \bar{K}_2 \bar{E}'_2 \approx 0. \tag{95}$$

For second derivatives of $K_\varphi$ and $K_\varphi$ to be absent (from $K_1'$), we must have $H_{00}^{01} = 0$ (in addition to $H_{10}^{10}$ which we have already derived). We write out the remaining $K$-terms explicitly:

$$\left(2H_{00}^{11}(E_2 - E'_3) + 2H_{00}^{02} \bar{E}_2 - H_{01}^{01}(E_1 - \bar{E}'_2) + 2(H_{00}^{11})' E_3 - (H_{01}^{01})' \bar{E}_2\right) K_\varphi$$

$$+ 2H_{11}^{11} \bar{E}_3 K_\varphi' + (4H_{00}^{20}(E_2 - E'_3) + H_{00}^{11} \bar{E}_2 + 4(H_{00}^{20})' E_3) K_x + 4H_{00}^{20} \bar{E}_3 K_\varphi' \approx 0.$$ 

Since there are no terms independent of the $K$ components, this expression (assuming $H_{00}^{00} \neq 0$) must be proportional to the diffeomorphism constraint only: $\mathcal{F}(2E_\varphi K'_\varphi - K_x(E^\varphi)')$. For $K'_\varphi$ to be absent, we must have $H_{00}^{20} = 0$, and comparison of the $K_x$- and $K_\varphi'$-terms gives the same consistent relation

$$\mathcal{F} = -\frac{1}{2(E_\varphi)^2} \partial H/\partial \Gamma'_\varphi H_{00}^{11}. \tag{96}$$

This function plays the role of the deformation function in the constraint algebra. It might differ from the previous form of $\beta$ if the correction functions depend non-trivially on $\Gamma_\varphi'$ in the presence of derivative corrections.

The $K_\varphi'$-coefficient must vanish, which provides one further consistency condition

$$2H_{00}^{11}(E_2 - E'_3) + 2H_{00}^{02} \bar{E}_2 - H_{01}^{01}(E_1 - \bar{E}'_2) + 2(H_{00}^{11})' E_3 - (H_{01}^{01})' \bar{E}_2 = 0. \tag{97}$$

This equation can be interpreted as a condition for $H_{01}^{01}$ for given $H_{10}^{11}$ and $H_{02}^{02}$, and the latter two functions remain free. (Seen in this way, (97) is a first-order ordinary differential equation for $H_{01}^{01}(x)$. The integration constant is fixed by the boundary condition that the classical limit $H_{01}^{01} = 0$ should be reached at spatial infinity.) It is then possible to have
a derivative correction proportional to $K_\varphi K'_\varphi$, but the coefficient is strictly related to the correction functions multiplying the classical curvature terms. The derivative term affects the deformation function $\beta$ only indirectly via $H_{00}^{11}$ and (97).

Instead of solving for $H_{01}^{01}$, we may rewrite (97) as

$$H_{02}^{00} = \frac{1}{2} (H_{01}^{01})' - \frac{1}{2G} \frac{E^\varphi}{\sqrt{|E^x|}} \alpha$$

with

$$\alpha = -G \sqrt{|E^x|} \left( \frac{H_{01}^{01} \bar{E}_1 - \bar{E}_2'}{E_2} - 2H_{00}^{11} \frac{E_2 - E_3'}{E_2} - 2(H_{00}^{11})' \frac{E_3'}{E_2} \right).$$

Inserting (98) in (80) shows that $\alpha$ plays the same role as in (35) and, more importantly, that the new derivative term in the Hamiltonian density $H$ can only be a total derivative $1/2 (H_{01}^{01} K'_\varphi)'$. The allowed derivative correction is therefore of a very special form that is not necessarily expected from general holonomy corrections. Nevertheless, the correction would non-trivially affect the solution space of the theory because the Hamiltonian density, upon multiplication with a general lapse function, is not just affected by a total derivative.

As a consistency check, we show that the previously known consistent versions satisfy (97). If there are no derivative corrections, we have the parameterization (35) with coefficients independent of $\Gamma_\varphi$ and its derivatives:

$$H_{00}^{11} = -\frac{1}{G} \sqrt{|E^x|} \bar{\alpha} \quad , \quad H_{00}^{02} = -\frac{1}{2G} \frac{E^\varphi}{\sqrt{|E^x|}} \alpha,$$

$$\frac{\partial H}{\partial \Gamma_\varphi} = -\frac{1}{2G} \frac{(E^\varphi)'}{\sqrt{|E^x|}} \alpha \Gamma \quad , \quad \frac{\partial H}{\partial \Gamma'_\varphi} = -\frac{1}{G} \frac{E^\varphi}{\sqrt{|E^x|}} \bar{\alpha} \Gamma.$$ (101)

With the equations found here, we obtain the deformation function

$$\beta = \bar{\alpha} \bar{\alpha} \Gamma$$ (102)

and one consistency condition

$$\alpha \bar{\alpha} \Gamma - \bar{\alpha} \alpha \Gamma - 2|E^x| (\bar{\alpha} \bar{\alpha} \Gamma - \bar{\alpha} \bar{\alpha} \Gamma) = 0.$$ (103)

This equation reproduces the condition found in [13].

Finally, following similar arguments, it is not difficult to see that the other possibility to fulfill condition (94): $H_{10}^{10} = H_{01}^{01}$ is not consistent. Therefore, $E_4 = 0$ and we cannot have any dependence on $\Gamma''_\varphi$ to this order. Since $\Gamma''_\varphi$ depends on $(E^\varphi)''$, the result that $\partial H/\partial \Gamma''_\varphi = 0$ is related to the condition found in Sec. 3.2.1, but here we are testing for a combination of $(E^\varphi)''$ and $(E^x)'''$. 

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6 Conclusions

The anomaly problem is one of the most crucial issues in canonical and loop quantum gravity. If it cannot be resolved, canonical quantum gravity cannot be shown to be consistent. Unfortunately, the problem is also one of the most complicated ones, and therefore any result in this direction is useful. In some models, one can make progress with commutators of operators [12, 22, 23, 24], but except for the 2+1-dimensional example in [12] it remains difficult to tackle non-local holonomies. Moreover, the step from operator equations to geometrical statements is non-trivial, requiring some handle on semiclassical states which constitutes another difficult and important problem of canonical quantum gravity. Effective methods, as further developed in this article, sidestep many of these complications and can still provide fundamental insights.

If canonical quantum gravity gives rise to a consistent operator version of the constraint algebra, expectation values lead to a consistent effective version of constraints. By ruling out certain terms in effective constraints, one can therefore conclude that the possible form of consistent operators is restricted. We have done just that in the present paper, by limiting (but not completely ruling out) the connection dependence to next-to-leading order in an expansion by spatial derivatives.

We have not attempted to go beyond the next-to-leading order, given the complexity, but our methods are suitable for such a task. We do not speculate on the verdict whether a derivative expansion of holonomies can be implemented consistently. But it is of interest to spell out what implications a negative result would have. (The implications of a positive verdict are obvious.) If higher spatial derivatives can be ruled out, loop quantum gravity as it is commonly understood would be shown to be inconsistent. However, a more careful view could still be possible, a view that relies on a combination of higher spatial with higher time derivatives. Even though this consequence would agree with the expectation from higher-curvature effective actions, it is not necessarily implied by loop quantum gravity, as we have discussed in detail in Sec. 2.3. Holonomy modifications of the Hamiltonian constraint by themselves do not imply a close link with higher time derivatives. If such a link would be required by the condition of anomaly-freedom, it would mean that the construction of Hamiltonian constraint operators must be tightly connected to the form of the allowed dynamical states. After all, higher time derivatives arise in canonical quantum theories from quantum back-reaction of moments of a state. If holonomy modifications can be made consistent only with a careful choice of corrections that contain higher time derivatives, the form of states on which the constraint algebra can be represented must be non-trivially restricted; not all kinematical states could be allowed. One would have to find a domain of states smaller than the kinematical Hilbert space but larger than the physical one (since one aims to represent the off-shell algebra). Such classes of states have not been considered yet in loop quantum gravity, but, given the difficulties in finding consistent effective realizations with non-pointwise holonomies, they might be the only way to make the theory consistent.

Another, perhaps more dramatic consequence would apply to loop quantum cosmology. If consistent holonomy corrections require closely related higher time derivatives, the cur-
rent cosmological models used in this context are wrong: The usual modified Friedmann
equations do not contain higher time derivatives even though such terms would be of a
similar magnitude as the modifications. Results obtained by solving for full wave func-
tions implicitly contain higher time derivatives via quantum back-reaction. However, in
a minisuperspace model these implicit corrections are not tied to holonomy modifications
in a strong-enough way to ensure anomaly-freedom. While higher spatial derivatives do
not matter for minisuperspace equations, higher time derivatives are important terms that
could change the implications claimed for holonomy modifications in models in which a
consistent embedding in inhomogeneous geometries has not been confirmed. In the cos-
mological context, it is also important to mention the mounting evidence for signature
change at high curvature. Our equation (70) confirms and strengthens the phenomenon
in spherically symmetric models, in accordance with a similar result recently obtained for
cosmological perturbations [41].

We repeat that our results do not suffice to rule out consistent holonomy corrections.
Our last remarks are meant to show the importance of the anomaly problem, a question
which is ignored in most of the “physical” results claimed by the theory in cosmological or
black-hole models.

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A Conditions from antisymmetric higher-derivative multipliers

We show here how to rewrite the general expression (33) in the form \( \sum_{j=0}^{2n-1} \int dx \ N^{(j)} g_j \).

First, integrating by parts and using (22) and (24), we have

\[
\sum_{j=1}^{n} \sum_{i=0}^{j-1} \int dx \ M^{(i)} N^{(j)} h_{i,j} = \sum_{j=1}^{n} \sum_{k=0}^{j-1} \int dx \ M N^{(j+k)} \sum_{l=0}^{j-k-1} (-1)^{k+l} \binom{k+l}{l} h_{k+l,j},
\]

and

\[
-\sum_{j=1}^{n} \sum_{i=0}^{j-1} \int dx \ M^{(j)} N^{(i)} h_{i,j} = \sum_{j=1}^{n} \sum_{k=0}^{j-1} \sum_{l=0}^{j} \int dx \ M N^{(j+k-l)} (-1)^{j+l+1} \binom{j}{l} h_{k,j}.
\]
Adding these two expressions results in

\[ \sum_{j=1}^{n} \sum_{i=0}^{j-1} \int dx \left( M^{(i)} N^{(j)} - N^{(i)} M^{(j)} \right) h_{i,j} = \]

\[ = \sum_{j=1}^{n} \sum_{k=0}^{j-1} \int dx \; M^{(j+k)} \left( (-1)^{j+1} h_{k,j} + \sum_{l=0}^{j-k-1} (-1)^{k+l} \binom{k+l}{l} h^{(l)}_{k+l,j} \right) \]

\[ + \sum_{j=1}^{n} \sum_{l=1}^{j-1} \sum_{k=0}^{j-l-1} \int dx \; M^{(j+k-l)} (-1)^{j+1} \binom{j}{l} h^{(l)}_{k,j}. \]  

(104)

Defining \( s := j + k \), the first line of the right hand side of this expression may be written as

\[ \sum_{j=1}^{n} \sum_{k=0}^{j-1} \int dx \; M^{(j+k)} \left( (-1)^{j+1} h_{k,j} + \sum_{l=0}^{j-k-1} (-1)^{k+l} \binom{k+l}{l} h^{(l)}_{k+l,j} \right) = \]

\[ = \sum_{s=1}^{2n-1} \int dx \; M^{(s)} \sum_{j=[s/2]+1}^{\min(s,n)} \left( (-1)^{j+1} h_{s-j,j} + \sum_{l=0}^{2j-s-1} (-1)^{s-j+l} \binom{s-j+l}{l} h^{(l)}_{s-j+l,j} \right), \]  

(105)

where \( \lfloor s/2 \rfloor \) denotes the integer part of \( s/2 \). The second line requires a little more work: defining \( r := k - l + 1 \) and then \( s := j + r - 1 \), we have

\[ \sum_{j=1}^{n} \sum_{k=0}^{j-1} \sum_{l=1}^{j} \int dx \; M^{(j+k-l)} (-1)^{j+1} \binom{j}{l} h^{(l)}_{k,j} = \]

\[ = \sum_{j=1}^{n} \sum_{r=-(j-1)}^{j-1} \int dx \; M^{(j+r-1)} (-1)^{j+1} \sum_{k=\max(r,0)}^{\min(j+r-1,j-1)} \binom{j}{k-r+1} h^{(k-r+1)}_{k,j} \]

\[ = \sum_{s=0}^{2n-2} \int dx \; M^{(s)} \sum_{j=[s/2]+1}^{\min(s,n)} \sum_{k=\max(s-j+1,0)}^{n} (-1)^{j+1} \binom{j}{k+j-s} h^{(k+j-s)}_{k,j}, \]  

(106)

Combining these results we finally get

\[ \sum_{j=1}^{n} \sum_{i=0}^{j-1} \int dx \; M^{(i)} N^{(j)} - N^{(i)} M^{(j)} h_{i,j} = \]

\[ = \sum_{s=1}^{2n-1} \int dx \; M^{(s)} \sum_{j=[s/2]+1}^{\min(s,n)} \left( (-1)^{j+1} h_{s-j,j} + \sum_{l=0}^{2j-s-1} (-1)^{s-j+l} \binom{s-j+l}{l} h^{(l)}_{s-j+l,j} \right) + \]

\[ + \sum_{s=0}^{2n-2} \int dx \; M^{(s)} \sum_{j=[s/2]+1}^{\min(s,n-1)} (-1)^{j+1} \sum_{l=\max(s-j+1,0)}^{n} \binom{j}{l+j-s} h^{(l+j-s)}_{l,j}. \]  

(107)
This gives $2n$ equations from which there will be $n$ independent conditions. We have (for $n \geq 2$):

$$g_0 = \sum_{j=1}^{n} (-1)^{j+1} h_{0,j}^{(j)} \approx 0,$$

$$g_s = \sum_{j=\lceil s/2 \rceil+1}^{\min(s,n)} (-1)^{j+1} h_{s-j,j} + \sum_{l=0}^{2j-s-1} (-1)^{s-j+l} \binom{s-j+l}{l} h_{s-j+l,j}^{(l)} +$$

$$+ \sum_{j=\lceil \frac{s+1}{2} \rceil+1}^{n} (-1)^{j+1} \sum_{l=\max(s-j+1,0)}^{\min(s,j-1)} \binom{j}{l+j-s} h_{l,j}^{(l+j-s)} \approx 0,$$

for $s = 1, \ldots, 2n - 2$, and

$$g_{2n-1} = (-1)^{n-1} 2 h_{n-1,n} \approx 0.$$

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