1. Introduction

Two years ago we gave a talk on abelian 2-categories in the Max-Planck Institute of Mathematics in Bonn. Our approach was motivated by the theory of categorical modules over a categorical rings [2]. Details of the theory of categorical modules will soon appear in our joint work with Vincent Schmitt.

In the recent preprint Mathieu Dupont [1] rediscovered this notion. Our original axioms were equivalent but not the same as one given in [1]. However we did not used (co)pips and (co)roots. Our approach in the subject will appear elsewhere.

The paper [1] contains several interesting results unknown to us, however some of the results of Dupont were known to us including Corollary 192 [1], which claims that the category of discrete and codiscrete (or connected) objects are equivalent abelian categories. Dupont poses also a question whether any abelian category comes in this way. We will give a rather trivial solution of this problem in the case when a given abelian category has enough projective or injective objects, which we have known for several years.

In this note we follow [1] with few exceptions. We use the term 2-kernel and 2-cokernel for what Dupont calls kernel and cokernel and keep the terms kernel and cokernel in the classical meaning. Groupoids arising in this note are in fact Picard categories. In particular $\pi_1$ did not depend on the base point and therefore we omitted it.

2. The 2-category of arrows $\mathcal{A}^{[1]}$

Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}^{[1]}$ be the 2-category of arrows of $\mathcal{A}$. Recall that objects of $\mathcal{A}^{[1]}$ are arrows $a : A_1 \to A_0$ of $\mathcal{A}$, a morphism from $a : A_1 \to A_0$ to $b : B_1 \to B_0$ is a pair $(f_0, f_1)$ of morphisms in $\mathcal{A}$ such that the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 \\
\downarrow{a} & & \downarrow{b} \\
A_0 & \xrightarrow{f_0} & B_0,
\end{array}
\]

commutes, while a 2-arrow $(f_0, f_1) \Rightarrow (g_0, g_1)$ is an arrow $\alpha : A_0 \to B_1$ in $\mathcal{A}$ such that

\[
f_1 - g_1 = \alpha a \\
f_0 - g_0 = b \alpha
\]

with obvious compositions. In this case we also say that $(f_0, f_1)$ is homotopic to $(g_0, g_1)$ and write $(f_0, f_1) \sim (g_0, g_1)$.

For objects $a : A_1 \to A_0$ and $b : B_1 \to B_0$ we let $\text{Hom}_{\mathcal{A}^{[1]}}(a, b)$ be the corresponding hom-groupoid. It is clear that

$\pi_1(\text{Hom}_{\mathcal{A}^{[1]}}(a, b)) = \text{Hom}_\mathcal{A}(\text{Coker}(a), \text{Ker}(b))$

However, in general we do not have a nice description of $\pi_0(\text{Hom}(a, b))$ in terms of $\text{Ker}(a), \cdots, \text{Coker}(b)$. The situation can be improved in some particular cases. To state the corresponding result we need an additional category $\mathcal{E}$. Objects of $\mathcal{E}$ are triples $(M, N, x)$, where $M$ and $N$ are objects of the category $\mathcal{A}$, while $x \in \text{Ext}^2_\mathcal{A}(M, N)$. A morphism from $(M, N, x)$ to $(M', N', x')$ is a pair $(f, g)$. 
where $f : M \to M'$ and $g : N \to N'$ are morphisms in $\mathbb{A}$ such that the equality $f^*(x') = g_*(x)$ holds in $\text{Ext}^2(M, N')$. For an object $a : A_1 \to A_0$ we let $Ch(a)$ be the triple $(\text{Coker}(a), \text{Ker}(a), ch(a))$, where $ch(a)$ is the class of the 2-fold extension

$$0 \to \text{Ker}(a) \to A_1 \xrightarrow{a} A_0 \to \text{Coker}(a) \to 0$$

in $\text{Ext}^2_\mathbb{A}((\text{Coker}(a), \text{Ker}(a))$. In this way one gets a functor $Ch : \mathbb{A}^{[1]} \to \mathcal{E}$. We recall the following well-known result.

**Lemma 2.1.** Let $a : A_1 \to A_0$ and $b : B_1 \to B_0$ be two objects of $\mathbb{A}^{[1]}$. If $A_0$ is a projective object in $\mathbb{A}$, then one has an exact sequence

$$0 \to \text{Ext}^1_\mathbb{A}(\text{Coker}(a), \text{Ker}(b)) \to \pi_0(\text{Hom}_{\mathbb{A}^{[1]}}(a, b)) \to \text{Hom}_\mathcal{E}(Ch(a), Ch(b)) \to 0$$

**Corollary 2.2.**

(i) Let $a : A_1 \to A_0$, $b : B_1 \to B_0$ and $b' : B'_1 \to B'_0$ be objects of $\mathbb{A}^{[1]}$ and let $b \to b'$ be a morphism in $\mathbb{A}^{[1]}$, such that the induced morphisms $\text{Ker}(b) \to \text{Ker}(b')$ and $\text{Coker}(b) \to \text{Coker}(b')$ are isomorphisms. If $A_0$ is a projective object in $\mathbb{A}$, then the induced morphism of groupoids $\text{Hom}_{\mathbb{A}^{[1]}}(a, b) \to \text{Hom}_{\mathbb{A}^{[1]}}(a, b')$ is an equivalence of categories.

(ii) Let $a : A_1 \to A_0$ and $b : B_1 \to B_0$ be objects of $\mathbb{A}^{[1]}$ such that $A_0$ and $B_0$ are projective objects in $\mathbb{A}$. If $a \to b$ is a morphism in $\mathbb{A}^{[1]}$, such that the induced morphisms $\text{Ker}(a) \to \text{Ker}(b)$ and $\text{Coker}(a) \to \text{Coker}(b)$ are isomorphisms, then $a$ and $b$ are equivalent.

**Proof.** i) In this case we have a nice description for $\pi_1(\text{Hom}_{\mathbb{A}^{[1]}}(a, -))$, which shows that the functor $\text{Hom}_{\mathbb{A}^{[1]}}(a, b) \to \text{Hom}_{\mathbb{A}^{[1]}}(a, b')$ yields an isomorphism on $\pi_0$ and $\pi_1$ and hence is an equivalence of categories. ii) By the same reason the induced functor $\text{Hom}_{\mathbb{A}^{[1]}_{\mathcal{E}}}(x, a) \to \text{Hom}_{\mathbb{A}^{[1]}_{\mathcal{E}}}(x, b)$ is an equivalence of categories for all $x \in \mathbb{A}^{[1]}_{\mathcal{E}}$ and hence we can use the Yoneda lemma for 2-categories.

3. The 2-category $\mathbb{A}^{[1]}_{\mathcal{E}}$

In this section we will assume that $\mathbb{A}$ is an abelian category with enough projective objects.

We let $\mathbb{A}^{[1]}_{\mathcal{E}}$ be the full 2-subcategory of the 2-category $\mathbb{A}^{[1]}$ consisting of objects $a : A_1 \to A_0$ such that $A_0$ is a projective object in $\mathbb{A}$.

**Theorem 3.1.** If $\mathbb{A}$ is an abelian category with enough projective objects, then $\mathbb{A}^{[1]}_{\mathcal{E}}$ is a 2-abelian $\text{Gpd}$-category. The subcategory of discrete and codiscrete objects are equivalent to $\mathbb{A}$.

The rest of this work is devoted to the proof. The first observation is that the direct sum in $\mathbb{A}$ yields an additive $\text{Gpd}^{\ast}$-category structure. The next task is to characterize faithful, fully faithful, cofaithful and fully cofaithful morphisms.

**Lemma 3.2.**

(i) A morphism $(f_0, f_1) : a \to b$ is faithful in $\mathbb{A}^{[1]}_{\mathcal{E}}$ iff the morphism

$$
\begin{pmatrix}
-a \\
\begin{array}{c}
f_0 \\
1
\end{array}
\end{pmatrix} : A_1 \to A_0 \oplus B_1
$$

is a monomorphism in $\mathbb{A}$.

(ii) Let $(f_0, f_1) : a \to b$ be a morphism in $\mathbb{A}^{[1]}_{\mathcal{E}}$ and let $g : \text{Ker}(a) \to \text{Ker}(b)$, $h : \text{Coker}(a) \to \text{Coker}(b)$ be induced morphisms. Then $(f_0, f_1)$ is fully faithful in $\mathbb{A}^{[1]}_{\mathcal{E}}$ iff $g$ is an isomorphism and $h$ is a monomorphism in $\mathbb{A}$.

**Proof.** i) By definition $(f_0, f_1) : a \to b$ is faithful iff the induced homomorphism

$$\pi_1(\text{Hom}_{\mathbb{A}^{[1]}}(x, a)) \to \pi_1(\text{Hom}_{\mathbb{A}^{[1]}}(x, b))$$

is a monomorphism for all $x \in \mathbb{A}^{[1]}_{\mathcal{E}}$. But the homomorphism in the question is the same as

$$\text{Hom}_{\mathbb{A}}(\text{Coker}(x), \text{Ker}(a)) \to \text{Hom}_{\mathbb{A}}(\text{Coker}(x), \text{Ker}(b))$$
Since $\mathcal{A}$ has enough projective objects, any object in $\mathcal{A}$ is isomorphic to an object of the form $\text{Coker}(x)$ for a suitable $x \in \mathcal{A}_c^{[1]}$. Thus $(f_0, f_1) : a \to b$ is faithful iff the induced homomorphism $g : \text{Ker}(a) \to \text{Ker}(b)$ is a monomorphism and the i) follows. ii) By definition $(f_0, f_1) : a \to b$ is fully faithful iff the induced functor
\[(f_0, f_1)^x : \text{Hom}_{\mathcal{A}^{[1]}}(x, a) \to \text{Hom}_{\mathcal{A}^{[1]}}(x, b)\]
is full and faithful. This happens iff the functor $(f_0, f_1)^x$ yields an isomorphism on $\pi_1$ and a monomorphism on $\pi_0$. Thus for all $x \in \mathcal{A}_c^{[1]}$ the induced homomorphism
\[\text{Hom}_{\mathcal{A}}(\text{Coker}(x), \text{Ker}(a)) \to \text{Hom}_{\mathcal{A}}(\text{Coker}(x), \text{Ker}(b))\]
is an isomorphism and the induced map
\[\pi_0(\text{Hom}_{\mathcal{A}^{[1]}}(x, a)) \to \pi_1(\text{Hom}_{\mathcal{A}^{[1]}}(x, b))\]
is a monomorphism. From the first condition follows that $g : \text{Ker}(a) \to \text{Ker}(b)$ is an isomorphism. This fact and Lemma 2.1 yields that the induced map $\text{Ch}(a) \to \text{Ch}(b)$ is a monomorphism in $\mathcal{E}$. Since $g$ is an isomorphism without loss of generality we can identify $\text{Ker}(a)$ and $\text{Ker}(b)$. Then we have $\text{ch}(a) = h^*\text{ch}(b)$. We set $K = \text{Ker}(h)$ and let $i : K \to \text{Coker}(a)$ be an inclusion. We have $i^*\text{ch}(a)) = (hi)^*\text{ch}(b)) = 0$. Hence $(i, 0) : (K, 0, 0) \to \text{Ch}(a)$ is a well defined morphism in $\mathcal{E}$ which is annulled by the monomorphism $\text{Ch}(a) \to \text{Ch}(b)$. Hence $K = 0$ and the result follows.

Recall that $a$ is discrete if $a \to 0$ is faithful. Hence an object $a : A_1 \to A_0$ in $\mathcal{A}_c^{[1]}$ is discrete iff $a$ is a monomorphism in $\mathcal{A}$.

**Corollary 3.3.** The functor $\text{Dis}(\mathcal{A}_c^{[1]}) \to \mathcal{A}$ given by $a \mapsto \text{Coker}(a)$ is an equivalence of categories.

**Lemma 3.4.**

(i) A morphism $(f_0, f_1) : a \to b$ is cofaithful iff the morphism
\[(f_0, b) : A_0 \oplus B_1 \to B_0\]
is an epimorphism in $\mathcal{A}$.

(ii) Let $(f_0, f_1) : a \to b$ be a morphism in $\mathcal{A}_c^{[1]}$ and let
\[g : \text{Ker}(a) \to \text{Ker}(b), \quad h : \text{Coker}(a) \to \text{Coker}(b)\]
be induced morphisms. Then $(f_0, f_1)$ is fully cofaithful in $\mathcal{A}_c^{[1]}$ iff $h$ is an isomorphism and $g$ is an epimorphism in $\mathcal{A}$.

**Proof.** By definition $(f_0, f_1) : a \to b$ is cofaithful (resp. fully cofaithful) iff the induced functor
\[(f_0, f_1)_x : \text{Hom}_{\mathcal{A}^{[1]}}(b, x) \to \text{Hom}_{\mathcal{A}^{[1]}}(a, x))\]
is faithful (resp. full and faithful). Since any object of $\mathcal{A}$ is of the form $\text{Ker}(a)$ for a suitable $a \in \mathcal{A}_c^{[1]}$, it follows from the description of $\pi_i(\text{Hom}_{\mathcal{A}^{[1]}})$ given in Section 2 essentially by the same argument as in Lemma 3.2 that this happens iff the map $h$ is a monomorphism (resp. $h$ is an isomorphism and the map $\text{Ch}(a) \to \text{Ch}(b)$ is an epimorphism in $\mathcal{E}$). This already proves i) and in ii) it remains to show $g$ is an epimorphism. We set $C = \text{Coker}(g)$ with canonical morphism $q : \text{Ker}(b) \to C$. Since $h$ is an isomorphism we can and we will identify $\text{Coker}(a)$ and $\text{Coker}(b)$. Then we will have $\text{ch}(b) = g_*(\text{ch}(a))$. Hence $q_*(\text{ch}(b)) = q_*(g_*(\text{ch}(a))) = 0$. Thus $(q, 0, 0) : \text{Ch}(b) \to (C, 0, 0)$ is a well-defined morphism which is annulled by the epimorphism $\text{Ch}(a) \to \text{Ch}(b)$. Thus $C = 0$.

For an object $x : X_1 \to X_0$ of $\mathcal{A}^{[1]}$ we choose a projective object $P_0$ and an epimorphism $\epsilon : P_0 \to X_0$ and consider the pull-back diagram
\[
\begin{array}{ccc}
P_1 & \xrightarrow{c_1} & X_1 \\
\downarrow p & & \downarrow x \\
P_0 & \xrightarrow{\epsilon_0} & X_0
\end{array}
\]
It is clear that the induced morphisms $\text{Ker}(p) \to \text{Ker}(x)$ and $\text{Coker}(p) \to \text{Coker}(x)$ are isomorphisms and $p \in \mathcal{A}_c^{[1]}$. We call $p$ a replacement of $x$. Sometimes it is denoted by $x^{\text{rep}}$. The following easy Lemma shows that this is well-defined.

**Lemma 3.5.** Let $f : a \to x$ be a morphism in $\mathcal{A}_c^{[1]}$ which induce isomorphism on $\text{Ker}$ and $\text{Coker}$. If $a \in \mathcal{A}_c^{[1]}$ then $f$ has the lifting to $x^{\text{rep}}$, which is unique up to unique homotopy.

Now we discuss 2-kernels and 2-cokernels in $\mathcal{A}_c^{[1]}$. Let $(f_0, f_1) : a \to b$ be a morphism in $\mathcal{A}_c^{[1]}$. According to [1] the 2-cokernel of $(f_0, f_1)$ in $\mathcal{A}_c^{[1]}$ is $(q : Q \to B_0, (\text{id}, q), \xi)$:

![Diagram](image)

where

![Diagram](image)

is given by the classical push-out construction. It follows that if $b \in \mathcal{A}_c^{[1]}$, then the 2-cokernel is also in $\mathcal{A}_c^{[1]}$. Hence the 2-category $\mathcal{A}_c^{[1]}$ has 2-cokernels and the inclusion $\mathcal{A}_c^{[1]} \to \mathcal{A}^{[1]}$ preserves 2-cokernels.

Thanks to [1] the 2-kernel of $(f_0, f_1) : a \to b$ in $\mathcal{A}^{[1]}$ is $(k' : A_1 \to K, (k, \text{id}), \kappa)$:

![Diagram](image)

where

![Diagram](image)

is given by the classical pull-back construction. In general $K$ is not a projective object even if $A_0$ and $B_0$ are projective objects in $\mathcal{A}$. Hence $k' : A_1 \to K$ does not belongs to $\mathcal{A}_c^{[1]}$. Let $c : C_1 \to C_0$ be a replacement of $k'$. Thus we have an epimorphism $\epsilon : C_0 \to K$ with projective $C_0$ and the pull-back diagram

![Diagram](image)

We claim that $(c : C_1 \to C_0, (\kappa \epsilon, \epsilon'), (\kappa \epsilon))$ is a 2-kernel of $(f_0, f_1)$. Indeed, we have to show that for any object $x$ in $\mathcal{A}_c^{[1]}$ the groupoids $\text{Hom}_{\mathcal{A}_c^{[1]}}(x, x')$ and the 2-kernel of

$\text{Hom}_{\mathcal{A}_c^{[1]}}(x, a) \to \text{Hom}_{\mathcal{A}_c^{[1]}}(x, b)$

are equivalent. But we know that the last groupoid is equivalent to $\text{Hom}_{\mathcal{A}_c^{[1]}}(x, k')$. Since the morphism $(\epsilon, \epsilon') : c \to k'$ satisfies the conditions of Corollary 2.2 the claim follows.
Now is easy to see that the 2-cokernel of \((k, \epsilon, \epsilon') : c \to a\) is \(k : K \to A_0\) and the 2-kernel of \((\text{id}, q) : b \to q'\) is \(e : E_1 \to E_0\), where \(E_0 \to Q\) is an epimorphism with \(E_0\) projective and \(E_1\) is the pull-back

\[
\begin{array}{ccc}
E_1 & \longrightarrow & B_1 \\
\downarrow & & \downarrow q \\
E_0 & \longrightarrow & Q
\end{array}
\]

It follows from the description of 2-kernels and 2-cokernels that

\[
\Sigma(a : A_1 \to A_0) = (\text{Coker}(a) \to 0)
\]

and

\[
\Omega(a : A_1 \to A_0) = (F_1 \to F_0),
\]

where

\[0 \to F_1 \to F_0 \to A_1 \xrightarrow{\alpha} A_0 \to 0\]

is an exact sequence with projective \(F_0\).

Thus \(\text{Pip}(f) = (r : R_1 \to R_0)\), where

\[0 \to R_1 \to R_0 \xrightarrow{\pi} A_1 \to K\]

is an exact sequence with projective \(R_0\) and \(\pi\) can be considered as a homotopy

\[
\begin{array}{ccc}
R_1 & \longrightarrow & A_1 \\
\downarrow r & & \downarrow a \\
R_0 & \longrightarrow & A_0
\end{array}
\]

Since \(\Sigma(r) = (\text{Ker}(k') \to 0)\) we see that the morphism \(\omega_f : \text{Coroot}(\pi) \to k'\) defined in \([1]\) is an equivalence. A similar argument works also for \(\text{Copip}(f)\). This finishes the proof of Theorem 3.1.

**Remarks.**

1) If \(\mathcal{A}\) has enough injective objects, then we can consider the dual construction. Namely, we let \(\mathcal{A}^{[1]}\) be the full 2-subcategory of the 2-category \(\mathcal{A}^{[1]}\) consisting of objects \(a : A_1 \to A_0\) such that \(A_1\) is an injective object of \(\mathcal{A}\). Then \(\mathcal{A}^{[1]}\) is a 2-abelian \(\text{Gpd}\)-category and the category of discrete and codiscrete objects of \(\mathcal{A}^{[1]}\) are equivalent to \(\mathcal{A}\).

2) The same method can be used to construct another abelian 2-categories. Recall that a symmetric categorical group consists of the following data

\[
C = (\partial : C_{ee} \to C_e, \{-,-\} : C_e \times C_e \to C_{ee})
\]

where \(C_e\) and \(C_{ee}\) are groups and \(\partial\) is a homomorphism, while \(\{-,-\}\) is a map such that the following equalities hold for \(x, y, z \in C_e\) and \(a, b \in C_{ee}\).

\[
\begin{align*}
\partial\{x, y\} &= x^{-1}y^{-1}xy \\
\{\partial a, \partial b\} &= a^{-1}b^{-1}ab \\
\{\partial a, x\}\{x, \partial a\} &= 1 \\
\{x, yz\} &= \{x, z\}\{x, y\}\{y^{-1}x^{-1}yx, z\} \\
\{xy, z\} &= \{y^{-1}xy^{-1}, y^{-1}zy\}\{y, z\}. \\
\{x, y\}\{y, x\} &= 1.
\end{align*}
\]

Symmetric categorical groups obviously form a 2-category \(\text{SCG}\). We let \(\text{SCG}_e\) be the full 2-subcategory formed by objects \(C = (\partial : C_{ee} \to C_e)\) with free \(C_e\). In a similar manner one can prove that \(\text{SCG}_e\) is a 2-abelian \(\text{Gpd}\)-category.

3) All these examples are particular cases of the following general construction. Let \(\mathcal{M}\) be a closed pointed simplicial model category. \(\mathcal{M}\) is called *additive* if the natural maps from the coproduct to the products

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}: X \vee Y \to X \times Y
\]
are weak equivalences. Moreover, $\mathcal{M}$ is called two-stage if $\Sigma^2(X) \to 0$ is a weak equivalence. If $\mathcal{M}$ is a two-stage additive pointed model simplicial category then under some hypothesis $\mathcal{M}_{cf}$ is a 2-abelian Gpd-category, where objects of $\mathcal{M}_{cf}$ are fibrant and cofibrant objects, morphisms in $\mathcal{M}_{cf}$ are usual morphisms, while 2-arrows are homotopy classes of homotopies. The details will be given elsewhere.

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