ON A FINITE RANGE DECOMPOSITION OF THE RESOLVENT
OF A FRACTIONAL POWER OF THE LAPLACIAN
II. THE TORUS

P. K. Mitter

Laboratoire Charles Coulomb
CNRS-Université Montpellier- UMR5221
Place E. Bataillon, Case 070, 34095 Montpellier Cedex 05 France

e-mail: Pronob.Mitter@umontpellier.fr

Abstract: In previous papers, [1], [2], we proved the existence as well as regularity of a
finite range decomposition for the resolvent $G_\alpha(x - y, m^2) = ((-\Delta)^{\frac{\alpha}{2}} + m^2)^{-1}(x - y)$, for
$0 < \alpha < 2$ and all real $m$, in the lattice $\mathbb{Z}^d$ for dimension $d \geq 2$. In this paper, which is a
continuation of the previous one, we extend those results by proving the existence as well
as regularity of a finite range decomposition for the same resolvent but now on the lattice
torus $\mathbb{Z}^d/LN+1\mathbb{Z}^d$ for $d \geq 2$ provided $m \neq 0$ and $0 < \alpha < 2$. We also prove differentiability
and uniform continuity properties with respect to the resolvent parameter $m^2$. Here $L$ is
any odd positive integer and $N \geq 2$ is any positive integer.

1. Introduction

In previous papers [1] and [2] we proved the existence as well as regularity of a finite
range decomposition for the resolvent

$$G_\alpha(x - y, m^2) = ((-\Delta)^{\frac{\alpha}{2}} + m^2)^{-1}(x - y) \quad (1.1)$$

for $0 < \alpha < 2$ and all real $m$, in the lattice $\mathbb{Z}^d$ for dimension $d \geq 2$. The definition
and properties of a finite range decomposition were given in [1, 2]. The reference [2]
incorporates the content of the published version [1] together with its erratum, and will
thus be convenient to refer to. The main result is Theorem 1.1 of [1], restated in [2]. In
this paper, we will prove for all $\alpha$ in the interval $0 < \alpha < 2$ the existence and regularity of
a finite range decomposition of a periodic version of (1.1) on the torus $\mathbb{Z}^d/LN+1\mathbb{Z}^d$. This
is the content of Theorem 1.1 below. Continuity and differentiability properties in $m^2$ are
given in Theorem 1.2 below for $\alpha$ in the interval $1 < \alpha < 2$. We emphasise that Theorems
1.1 and 1.2 are valid only when $m \neq 0$. Results for finite range decompositions of general families of massless models on the discrete torus are given in [9].

The resolvent (1.1) arises as the covariance of the Gaussian measure underlying various statistical/field theoretic systems with long range interactions (see [2]). For $d = 2, 3$ the upper critical dimension for those systems is $d_c = 2\alpha$. Thus for $\alpha$ in the above interval we can arrange for the system to be below the upper critical dimension, and this is where non-trivial critical phenomena for long range systems are expected.

For $\alpha = 2$ the resolvent in (1.1) is that of a standard massive Laplacian. A finite range decomposition on the lattice $\mathbb{Z}^d$ was obtained in [3]. In this case also the methods of this paper can be applied for obtaining a finite range decomposition on the lattice torus starting from the work in [3]. A finite range decomposition for the resolvent of a massive Laplacian on the lattice torus was obtained earlier in [10] using the results of [4].

This paper is a companion to the earlier papers [1, 2]. We will use freely the notations and results, especially Theorem 1.1, Corollary 1.2 and Proposition 2.1, of these references. However, for the convenience of the reader, before embarking on the proofs of Theorems 1.1 and Theorem 1.2 we will give some indications of the strategy to be followed making use of results from earlier references.

A survey of earlier results and references together with motivation was given in [1] and we will not repeat them here. We simply remind the reader that one of the main applications of finite range decompositions is in rigorous Renormalisation Group analysis of statistical/field theoretic systems near and at the critical point of second order phase transitions. The lattice acts as an ultraviolet cutoff but we also need a finite volume cutoff and then later take the infinite volume limit. In the finite volume theory it is desirable to preserve translation invariance. One convenient way of doing that is putting the theory on a torus of finite period which is the edge length of the fundamental domain in the shape of a square, cube or hypercube. The goal of extending the finite range decomposition of the resolvent (1.1) given in [1, 2] (together with regularity properties) to the torus is achieved in this paper.

We should point out that a different way of achieving the same goal has been given in [6] using estimates from [5]. This is an essential ingredient in [6] where critical exponents below the critical dimension $d_c = 2\alpha$ have been studied for the $n$-component $\varphi^4$ model with long range interactions in the regime where $\varepsilon = d_c - d = 2\alpha - d > 0$ is held sufficiently small. Our method and results for extending the finite range decomposition to the torus however differ from that in [6] and therefore we are providing them. It relies on the bounds of Theorem 1.1 and Corollary 1.2 of [1] and [2] together with Fourier analysis on the discretized torus using the discrete Fourier transform and estimates on Fourier coefficients. These are obtained using estimates in [3] and the spectral decomposition given in Proposition 2.1 of [1, 2]. The mass derivatives of the functions appearing in the finite range decomposition are estimated very simply using the spectral decomposition and estimates given in [1], and [2].
Continuity results are consequences. Mass derivative/continuity bounds are known to be useful in the study of critical exponents [6]. A comparison with the bounds given in [6] is given later in Remarks 3 and 5 below.

Definitions

Let \( L = 3^p, p \geq 2, \quad \varepsilon_j = L^{-j}, j \geq 0, \quad 0 < \alpha < 2 \) and \( d \geq 2 \). Let \( N \geq 2 \) be any positive integer. Let \( T_{N+1} = T_{N+1}^d \) denote the torus \( \mathbb{Z}^d / N+1 \mathbb{Z}^d \) of edge length \( L^{N+1} \).

The fundamental cube \( Q_{N+1} = [-\frac{L^{N+1}}{2}, \frac{L^{N+1}}{2}]^d \cap \mathbb{Z}^d \) has the property that every point of \( \mathbb{Z}^d \) has a unique translate with respect to \( Q_{N+1} \). The volume of the fundamental cube is \(|Q_{N+1}| = L^{(N+1)d} \). Functions on the torus are periodic functions. Integration (summation) on the torus is defined as usual as integration over the fundamental cube. Moreover we define \( L^1(T_{N+1}) = L^1(Q_{N+1}) \). If \( X \subset \mathbb{Z}^d \) then \( L^1(X) \) is the space of summable functions on \( X \).

In the following we often speak of periodizing a function. Let \( f : \mathbb{Z}^d \to \mathbb{R} \). We say that \( f \) has a periodization \( f_{T_{N+1}} \) with period \( L^{N+1} \) with \( N \) any positive integer if for \( \forall x \in \mathbb{Z}^d \) the sum

\[
f_{T_{N+1}}(x) = \sum_{y \in L^{N+1} \mathbb{Z}^d} f(x + y)
\]

exists. If \( f \in L^1(\mathbb{Z}^d) \) then the sum converges absolutely in \( L^1(Q_{N+1}) \) and defines \( f_{T_{N+1}} \) as a function in \( L^1(T_{N+1}) \). For analogous considerations in the continuum see e.g. Stein and Weisz, Chapter 7, in [7].

Finally we note that we shall often employ continuum integral notations for lattice sums. The Lebesgue measure in \((\varepsilon_n \mathbb{Z})^d\) is the counting measure times \( \varepsilon_n^d \).

All objects in the following Theorem 1.1 will be defined and introduced below immediately after the statement of the theorem.

**Theorem 1.1**

Let \( 0 < \alpha < 2, \quad d \geq 2 \) and \( N \geq 2 \). Let \( \varepsilon_j = L^{-j}, \forall j \geq 0 \) and let \( m \neq 0 \). Then the positive definite function \( G_\alpha(x - y, m^2) \) on \( \mathbb{Z}^d \) has a periodized version \( G_{\alpha, T_{N+1}}(x - y, m^2) \) which is a function in \( L^1(T_{N+1}) \). Moreover for all \( m \neq 0 \) we have the following finite range decomposition:

\[
G_{\alpha, T_{N+1}}(x - y, m^2) = \sum_{j=0}^{N-1} L^{-2j[\varphi]} \Gamma_{j, \alpha}(\frac{x - y}{L^j}, L^j m^2) + L^{-2N[\varphi]} G_{N, \alpha, T_{N+1}}(\frac{x - y}{L^N}, L^N m^2)
\]

where

\[3\]
\[
[\varphi] = \frac{d - \alpha}{2}
\]  
(1.4)

and the positive definite functions \( \Gamma_{j,\alpha}(\cdot, m^2) \), defined on \((\varepsilon_j \mathbb{Z})^d\), which appear in the sum are those in Theorem 1.1 of [1] and [2]. The function \( G_{N,\alpha,T_{N+1}} \) which did not appear in [1], [2] will be defined later at the end of this theorem. The functions \( \Gamma_{j,\alpha}(\cdot, m^2) \) have finite range \( L \) and satisfy the bounds stated in [1] and [2]:

For all \( j \geq 2 \) and \( 0 \leq q \leq j \), and all \( p \geq 0 \),

\[
||\partial^p \varepsilon_j \Gamma_{j,\alpha}(\cdot, m^2)||_{L^\infty((\varepsilon_j \mathbb{Z})^d)} \leq c_{L,p,\alpha}(1 + m^2)^{-2}. 
\]  
(1.5)

For \( j = 0, 1 \) and \( 0 \leq q \leq j \) we have the bound

\[
||\partial^p \varepsilon_j \Gamma_{j,\alpha}(\cdot, m^2)||_{L^\infty((\varepsilon_j \mathbb{Z})^d)} \leq c_{L,p,\alpha}(1 + m^2)^{-1}. 
\]  
(1.6)

In the above \( \partial \varepsilon_j = \partial \varepsilon_{j,e_k}, k = 1, \ldots, d \) is a forward lattice partial derivative with increment \( \varepsilon_j \) and in any particular direction \( e_k \) in the lattice \((\varepsilon_j \mathbb{Z})^d\). Moreover \( \partial^p \varepsilon_j \) is a multi-derivative of order \( p \) defined as in the continuum but now with lattice forward derivatives. \( e_1, \ldots, e_d \) are unit vectors which give the orientation of \( \mathbb{R}^d \) as well as the orientation of all embedded lattices \((\varepsilon_j \mathbb{Z})^d \subset \mathbb{R}^d\). By construction the lattices are nested in an obvious way. The constant \( c_{L,p,\alpha} \) depends on \( L, p, \alpha \). It depends implicitly on the dimension \( d \).

The functions on \( \mathbb{Z}^d \)

\[
\hat{\Gamma}_{j,\alpha}(x, m^2) = L^{-2j[\varphi]} \Gamma_{j,\alpha}(\frac{x}{L^j}, L^j \alpha m^2) 
\]  
(1.7)

have finite range \( L^{j+1} \)

\[
\hat{\Gamma}_{j,\alpha}(x, m^2) = 0 : |x| \geq L^{j+1} 
\]  
(1.8)

and therefore for \( 0 \leq j \leq N - 1 \) are functions on \( T_{N+1} \). Their periodization give back the functions themselves. They satisfy the regularity bounds of Corollary 1.2 of [1] and [2]: for \( j \geq 2 \),

\[
||\partial^p \hat{\Gamma}_{j,\alpha}(\cdot, m^2)||_{L^\infty(\mathbb{Z}^d)} \leq c_{L,p,\alpha}(1 + L^j \alpha m^2)^{-2} L^{-(2j[\varphi] + pj)} 
\]  
(1.9)

and for \( j = 0, 1 \)

\[
||\partial^p \hat{\Gamma}_{j,\alpha}(\cdot, m^2)||_{L^\infty(\mathbb{Z}^d)} \leq c_{L,p,\alpha}(1 + L^j \alpha m^2)^{-1} L^{-(2j[\varphi] + pj)}. 
\]  
(1.10)

For all \( N \geq 2 \) the function

\[
\hat{G}_{N,\alpha,T_{N+1}}(x, m^2) = L^{-2N[\varphi]} G_{N,\alpha,T_{N+1}}(\frac{x}{L^N}, L^N \alpha m^2) 
\]  
(1.11)
is in $L^1(T_{N+1})$ and satisfies for all $m \neq 0$

$$
|\partial^p_{\mathbb{Z}^d} \tilde{G}_{N,\alpha,T_{N+1}}(x, m^2)| \leq c_{L,\alpha,p} L^{-2N\alpha} m^{-4} L^{-(2N[\varphi]+pN)}. \quad (1.12)
$$

For $m^2 \geq \frac{1}{\sqrt{C}} L^{-N\alpha}$ where $C$ is any positive constant independent of $N$, and all integers $p \geq 0$, we therefore get the bound

$$
||\partial^p_{\mathbb{Z}^d} \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2)||_{L^\infty(T_{N+1})} \leq c_{L,p,\alpha} L^{-(2N[\varphi]+pN)} \quad (1.13)
$$

where the constant $c_{L,p,\alpha}$ depends on $C, L, p, \alpha$ but is independent of $N, m$.

**A guide to Theorem 1.1**

We recall for the benefit of the reader the basic objects introduced above. The functions

$$
\Gamma_{j,\alpha}(\cdot, m^2) : (\varepsilon_j \mathbb{Z})^d \to \mathbb{R}
$$

are defined by

$$
\Gamma_{j,\alpha}(\cdot, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2) \Gamma_j(\cdot, s). \quad (1.14)
$$

where $\Gamma_j(\cdot, s)$ is the rescaled fluctuation covariance in the finite range decomposition of the resolvent of the standard Laplacian (see [3]) and $\rho_\alpha(s, m^2)$ is the spectral function given by Proposition 2.1 in [1, 2]:

$$
\rho_\alpha(s, m^2) = \frac{\sin \pi \alpha/2}{\pi} \frac{s^{\alpha/2}}{s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2}. \quad (1.15)
$$

This latter function has bounds given in [1, 2] and these can be found again in Section 3 of the present paper. These bounds together with the bounds on $\Gamma_j(\cdot, s)$ of [3] were used in [1, 2] to provide the bounds on the fluctuation covariances $\tilde{\Gamma}_{j,\alpha}(\cdot, m^2)$ in Theorem 1.1 above.

The function $\tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2)$ is the periodization of a function $\tilde{G}_{N,\alpha}(\cdot, m^2)$ on $\mathbb{Z}^d$ which is shown to be in $L^1(\mathbb{Z}^d)$. The latter function is the unrescaled version of the function $\mathcal{G}_{N,\alpha}(\cdot, m^2)$ on $(\varepsilon_N \mathbb{Z})^d$ given by

$$
\mathcal{G}_{N,\alpha}(\cdot, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2) \mathcal{G}_N(\cdot, s). \quad (1.16)
$$

In [3] the finite range decomposition for the standard Laplacian on $\mathbb{Z}^d$ was given for an arbitrary but finite number of terms together with an explicit formula for the remainder which is $\mathcal{G}_N(\cdot, s)$. This formula for the remainder is given and used later in Section 2 in the
course of proving the statements about \( \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2) \) in Theorem 1.1 above. Rescalings were performed in [3] which is why \( G_N(\cdot, s) \) is a function on the lattice \( (\varepsilon_N \mathbb{Z})^d \).

**Remark 1: Scale independence of constants**

As in [1] (erratum) and [3], one can get rid of the scale dependence of constants by coarse graining on a larger scale \( L' = L^r \) with \( r \) a large positive integer and holding \( L \) fixed. The finite range expansion can be rewritten by summing the fluctuation covariances and the remainder over the intermediate scales. The fluctuation covariances on the coarser scale \( L' \) are defined by

\[
\tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) = \sum_{l=0}^{r-1} \tilde{\Gamma}_{l+jr,\alpha}(\cdot, m^2).
\]

We now get the coarse scale finite range decomposition

\[
G_{\alpha}(\cdot, m^2) = \sum_{j \geq 0} \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2)
\]

with

\[
\tilde{\Gamma}'_{j,\alpha}(x - y, m^2) = 0, \quad |x - y| \geq (L')^{j+1}.
\]

The bounds on the coarse scale fluctuation covariances and the coarse scale remainder remain unchanged with new constants which are independent of the coarse scale \( L' \). This is explained in [1] as well as in [2] in the paragraph on coarse graining which follows Corollary 1.2 and is proved in Appendix A of [2].

**Remark 2:**

The function \( \tilde{G}_{N,\alpha,T_{N+1}} \) on the torus, introduced earlier, can also be viewed as the sum of all the functions \( \tilde{\Gamma}_{j,\alpha} \) for \( j \geq N \). In [6] the function was estimated as the the sum of estimates of the summands. Instead we estimate this function directly using its explicit representation together with Fourier analysis and estimates on the discrete Fourier transform.

**Remark 3:** The bounds (1.9) and (1.10) on the fluctuation covariances differ from those given in Proposition 10.1 of [7] as was noted earlier in [2] and [3]. In particular the \( (1 + L^{j\alpha} m^2)^{-1} \) term in the bounds occur only for \( j = 0, 1 \) terms and not for \( j \geq 2 \) in contrast to that in Proposition 10.1 of [6] where it occurs in the bounds for all \( j \). However the bound (1.12) agrees with the relevant bound in Proposition 10.1 of [6] once one takes account of the scale dimension \( [\varphi] \) of the Gaussian field \( \varphi \) (which is \( 2[\varphi] = d - \alpha \)).

We have the following continuity and differentiability properties in \( m^2 \) of the functions appearing in the finite range decomposition (1.3) of Theorem 1.1. They are given in

[6] 1:6
Theorem 1.2 for \( \alpha \) restricted to the interval \( 1 < \alpha < 2 \) and \( d \geq 2 \) for reasons explained in the introduction.

**Theorem 1.2**

1. **Differentiability of fluctuation covariances**: Let \( 1 < \alpha < 2 \) and \( d \geq 2 \). For all \( m^2 > 0 \) and all \( j \geq 1 \), the functions \( \tilde{\Gamma}_{j,\alpha}(\cdot, m^2) \) are differentiable functions of \( m^2 \) and the derivatives satisfy the bounds:

\[
\left\| \frac{\partial}{\partial m^2} \partial^p_{Z^d} \tilde{\Gamma}_{j,\alpha}(\cdot, m^2) \right\|_{L^\infty(Z^d)} \leq c_{L,\alpha,p} L^{-pj} L^{-(d-2)} (m^2)^{-2(1-\frac{\alpha}{d})}.
\]

**(1.20)**

**Uniform Continuity**: As a consequence for all \( m^2 > 0 \), \( \tilde{\Gamma}_{j,\alpha}(\cdot, m^2) \) is a uniformly continuous function of \( m^2 \). For all \( m^2_i > 0 \), \( i = 1, 2 \) we have the following uniform bounds:

\[
\left\| \partial^p_{Z^d} \tilde{\Gamma}_{j,\alpha}(\cdot, m^2_1) - \partial^p_{Z^d} \tilde{\Gamma}_{j,\alpha}(\cdot, m^2_2) \right\|_{L^\infty(Z^d)} \leq c_{L,\alpha,p} L^{-j(d-2)} L^{-pj} \left| (m^2_1)^{(\frac{2-\alpha}{d})} - (m^2_2)^{(\frac{2-\alpha}{d})} \right|.
\]

**(1.21)**

The constants \( c_{L,\alpha,p} \) in (1.22) and (1.21) are independent of \( j, m^2_1, m^2_2 \).

2. **Differentiability of** \( \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2) \): For all \( m \neq 0 \) and all integers \( p \geq 0 \)

\[
\left\| \frac{\partial}{\partial m^2} \partial^p_{Z^d} \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2) \right\|_{L^\infty(Z^d)} \leq c_{L,\alpha,p} L^{-pN} L^{-(N+1)d} (m^2)^{-2}
\]

where the constant \( c_{L,\alpha,p} \) is independent of \( N \). As a consequence we have the following

**Uniform continuity of** \( \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2) \)

For all \( m_i \neq 0, i = 1, 2 \) and all integers \( p \geq 0 \)

\[
\left\| \partial^p_{Z^d} \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2_1) - \partial^p_{Z^d} \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2_2) \right\|_{L^\infty(Z^d)} \leq c_{L,\alpha,p} L^{-pN} L^{-(N+1)d} m_1^{-2} m_2^{-2} |m_1^2 - m_2^2|.
\]

**(1.24)**

**Remark 4**: Scale independence of constants in mass derivative estimates:

For \( d \geq 3 \), \( \forall p \geq 0 \) as well as for \( d = 2 \), \( \forall p \geq 1 \) we can get rid of the scale dependence of the constants in the bounds (1.20) by passing to a coarser scale \( L' \) as in Remark 1 above with \( L' = L^r \) with \( L \geq 2 \) fixed and \( r \) a large positive integer. The mass differentiability bound on the coarse scale fluctuation covariance \( \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) \) is then obtained following Appendix

[7]
A of [2] but now using the bound (1.20) in intermediate steps. Then the bound (1.20) continues to hold for \( \tilde{\Gamma}' \) with a new constant \( c'_{L,\alpha,p} \) independent of \( L' \). These statements are proved in Appendix A of the present paper.

For \( d = 2 \) with \( p = 0 \) the bound (1.20) cannot be employed directly and we have to proceed otherwise. We coarse grain the fluctuation covariances in [3] thus producing a \( \log L' \) dependence in bounds as was first done in [4]. This produces \( \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) \) by the steps in Section 3 of [1, 2]. The bound (1.6) now holds for \( \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) \) with a new constant \( c'_{L,\alpha,p} \log L' \) where \( c'_{L,\alpha,p} \) is independent of \( L' \).

**Remark 5:**

Note that the mass derivative bound given in (1.20) agrees after coarse graining (see Remark 4 above) with that given in Proposition 10.1 of [6] for \( d = 3 \) (once one has taken account of the definition of \( \varepsilon = 2\alpha - d \) which figures in the bounds in [6]). For \( d = 2 \) with \( p \geq 1 \) we have no logarithmic dependence either in the scale or the mass (see Remark 4 above) in contrast to Proposition 10.1 of [6]. For \( d = 2 \) with \( p = 0 \) we have a logarithmic scale dependence (as in Remark 4 above) but no logarithmic dependence on \( m^2 \). This too is in contrast to the bound in [6]. These bounds for \( d = 2 \) are thus stronger than the estimate in Proposition 10.1 of [6].

In the next two sections we will give proofs of the above theorems. Before embarking on the proofs we indicate the strategy. First we note that provided \( m \neq 0 \) the resolvent \( G_{\alpha}(x - y, m^2) \) is in \( L^1(\mathbb{Z}^d) \) and therefore periodizable (see e.g. Theorem 2.4 of Stein and Weiss [8], Ch. 7, page 251) and the periodized version exists as an \( L^1(T_{N+1}) \) function on the torus. Provided \( m \neq 0 \), the function \( \tilde{G}_{N,\alpha,T_{N+1}}(\cdot, m^2) \) is the periodized version of an \( L^1(\mathbb{Z}^d) \) function \( \tilde{G}_{N,\alpha} \) which we identified earlier. The periodized function is in \( L^1(T_{N+1}) \) and thus has a multiple Fourier series. This is obtained by a Poisson summation formula for the discrete torus. Fourier analysis for finite Abelian groups is discussed in [8]. We will prove that the Fourier coefficients, supplied by the discrete Fourier transform, have rapid decay which leads not only to the existence but also to very good uniform differentiability properties of the periodic function. This is at the heart of Theorem 1.1. The continuity results of Theorem 1.2 will turn out to be relatively easy consequences.

### 2. Proof of Theorem 1.1

The function \( G_{\alpha}(x - y, m^2) \) on \( \mathbb{Z}^d \) is pointwise positive. This follows from the fact that it is the resolvent of an \( \alpha \)-stable continuous time Lévy walk \( x_t^{(\alpha)} \in \mathbb{Z}^d \):

\[
G_{\alpha}(x - y, m^2) = \int_0^\infty dt \, e^{-m^2 t} \, E_x(x_t^{(\alpha)} = y). \tag{2.1}
\]

Therefore for \( m \neq 0 \)

[8] 2:8
\[ \|G_\alpha(\cdot, m^2)\|_{L^1(\mathbb{Z}^d)} = \int_{\mathbb{Z}^d} dx \left| G_\alpha(x, m^2) \right| = \int_{\mathbb{Z}^d} dx \, G_\alpha(x, m^2) \]
\[ = \hat{G}_\alpha(0, m^2) \]
\[ = \frac{1}{m^2} < \infty. \]  

(2.2)

Hence for \( m \neq 0 \) the function \( G_\alpha(\cdot, m^2) \) is in \( L^1(\mathbb{Z}^d) \) as claimed and the series

\[ G_{\alpha,T_{N+1}}(x, m^2) = \sum_{n \in \mathbb{Z}^d} G_\alpha(x + n L^{N+1}, m^2) \]  

(2.3)

converges absolutely in the norm of \( L^1(Q_{N+1}) \) and defines a function in \( L^1(T_{N+1}) \) (see [8], Ch.7, Theorem 2.4).

Remark: Since \( G_\alpha(\cdot, m^2) \) is in \( L^1(\mathbb{Z}^d) \) for \( m \neq 0 \) it follows that there exists a \( \delta > 0 \) such that as \( |x| \to \infty \), \( G_\alpha(x, m^2) \sim O(|x|^{-(d+\delta)}) \). In fact a precise estimate shows that \( \delta = \alpha \) where \( 0 < \alpha < 2 \). However this fact will play no role in the rest of this paper.

We now proceed to the proof of the finite range decomposition (1.3) and the bounds stated in Theorem 1.1. We only need to prove the existence of the function \( \tilde{G}_{N,\alpha,T_{N+1}}(x, m^2) \) of (1.11) in \( L^1(T_{N+1}) \) and the bound (1.12). As in the proof of Theorem 1.1 of [1, 2] given in section 3 of [1, 2], we will start with the finite range decomposition of the resolvent of the Laplacian in \( \mathbb{Z}^d \) given in [3]. We will stop after the first \( N - 1 \) terms and then use the formula for the remainder. From equation (3.31) of [3] we have

\[ G(x - y, s) = \sum_{j=0}^{N-1} L^{-j(d-2)} \Gamma_j \left( \frac{x-y}{L^j}, L^{2j} s \right) + L^{-N(d-2)} \mathcal{G}_N \left( \frac{x-y}{L^N}, L^{2N} s \right) \]  

(2.4)

\( \Gamma_j \) and \( \mathcal{G}_N \) are defined through equations (3.28), (3.29) and (3.30) of [3] (see equations (2.11), (2.12) and (2.13) below). The products in these equations are convolution products.

We now proceed as in Section 3 of [1], [2]. We insert the above finite range decomposition with remainder in the Fourier transform of equation (2.2) of Proposition 2.1 of [1] to get (see equations (3.8)- (3.12) of [1, 2]):

\[ G_\alpha(x - y, m^2) = \sum_{j=0}^{N-1} L^{-2j[\varphi]} \Gamma_{j,\alpha} \left( \frac{x-y}{L^j}, L^{j\alpha} m^2 \right) + L^{-2N[\varphi]} \mathcal{G}_{N,\alpha} \left( \frac{x-y}{L^N}, L^{N\alpha} m^2 \right) \]  

(2.5)

Because of their support properties (see (1.8)), the periodization (with period \( L^{N+1} \)) of the functions
\[ \tilde{\Gamma}_{j,\alpha}(x-y, m^2) = L^{-2j[\varphi]} \Gamma_{j,\alpha} \left( \frac{x-y}{L^j}, L^j \alpha m^2 \right) \] (2.6)

for \( 0 \leq j \leq N - 1 \) gives back the same functions which are therefore also defined on the torus \( T_{N+1} \). Moreover Theorem 1.1 and Corollary 1.2 of [1] (corrected in the erratum) and [2] gives the bounds (1.9) and (1.10) on these functions. It therefore remains to study the torus boundary function

\[
\tilde{G}_{N,\alpha}(x-y) = L^{-2N[\varphi]} \ G_{N,\alpha} \left( \frac{x-y}{L^N}, L^N \alpha m^2 \right)
\] (2.7)

and its periodization. Now

\[
\hat{G}_{N,\alpha}(\cdot, m^2) : (\mathbb{Z})^d \to \mathbb{R}
\] (2.8)
is given by

\[
\hat{G}_{N,\alpha}(\cdot, m^2) = \int_0^\infty ds \rho_\alpha(s, m^2) \ G_{N}(\cdot, s)
\] (2.9)

where \( \rho_\alpha \) is the spectral function of Proposition 2.1 of [1]. Let us introduce the notation

\[
G_N(s)(x-y) = G_N(x-y, s).
\] (2.10)

Claim:

\[
G_N(s)(x-y) \geq 0
\] (2.11)

Proof

From equations (3.28) and (3.30) of [4] we have

\[
G_N(s) = A_N(s)G_{\varepsilon_N}(s)A_N(s)^*.
\] (2.12)

\( G_{\varepsilon_N}(s)(u-v) \) is the resolvent of the laplacian on the lattice \( (\varepsilon_N \mathbb{Z})^d \) and the products in (2.12) are convolution products with (defective) probability measures:

\[
G_N(s)(x-y) = \int_{(\varepsilon_N \mathbb{Z})^d} \int_{(\varepsilon_N \mathbb{Z})^d} A_N(s)(x, du)G_{\varepsilon_N}(s)(u-v)A_N(s)(y, dv)
\] (2.13)

and \( A_N(s) \) is given by a convolution product of averaging operators:

\[
A_N(s) = \prod_{m=1}^{N} A_{\varepsilon_j, m}(L^{-(m-1)})(s)
\] (2.14)

which themselves are (defective) probability measures. A probability measure is called defective if its total mass is less than 1, which is the case if \( s > 0 \). Now the action of
each averaging operator on a function $f$ is given by equation (3.23) of [4]. It is composed of a non-negative constraining function and the action of a Poisson kernel measure whose action is positivity preserving. Therefore the action of each averaging operator is positivity preserving and hence their convolution product $A_N(s)$ is positivity preserving. Finally $G_{\varepsilon N}(s)(u-v)$ being the resolvent of a random walk in $(\varepsilon_N \mathbb{Z})^d$ is pointwise positive. Therefore $G_N(s)(x-y) \geq 0$.  

Therefore

$$||G_N(s)||_{L^1((\varepsilon_N \mathbb{Z})^d)} = \int_{(\varepsilon_N \mathbb{Z})^d} dx |G_N(x,s)| = \int_{(\varepsilon_N \mathbb{Z})^d} dx G_N(x,s)$$

$$= \hat{G}_N(0,s)$$

(2.15)

Now taking the Fourier transform of (2.13) and (2.14) we get

$$\hat{G}_N(p, s) = \left|\hat{A}_N(p, s)\right|^2 \frac{s}{s-\hat{\Delta}_{\varepsilon N}(p)}$$

(2.16)

where $p \in B_{\varepsilon N} = [-\frac{\pi}{\varepsilon N}, \frac{\pi}{\varepsilon N}]$ and $\hat{\Delta}_{\varepsilon N}(p)$ is the Fourier transform of the $\varepsilon_N$-lattice Laplacian. From Appendix B of [2] we have for every integer $k \geq 0$ and $N \geq 2$ the bound

$$|\hat{A}_N(p, s)|^2 \leq c_{L,k}(1+s)^{-2}(p^2+1)^{-k}.$$  

(2.17)

From (2.16) and (2.17) we get

$$|\hat{G}_N(p, s)| \leq c_{L,k}(1+s)^{-2}(p^2+1)^{-k}(s-\hat{\Delta}_{\varepsilon N}(p))^{-1}$$

(2.18)

where $c_{L,k}$ is independent of $N$. Therefore for $s > 0$ we have from (2.15), (2.16) and (2.18)

$$||G_N(s)||_{L^1((\varepsilon_N \mathbb{Z})^d)} \leq c_{L,k}(1+s)^{-2} \frac{1}{s}.$$  

(2.19)

From (2.9), (2.18) and the bound on the spectral function $\rho(s, m^2)$ in Proposition 2.1, equation (2.4) of [1], [2]

$$0 \leq \rho_\alpha(s, m^2) \leq c_\alpha \frac{s^{\alpha/2}}{s^{\alpha} + m^4}$$

(2.20)

and hence we get for $m \neq 0$

$$||G_{N,\alpha}(\cdot, m^2)||_{L^1((\varepsilon_N \mathbb{Z})^d)} \leq c_\alpha \int_0^\infty ds \frac{s^{\alpha/2}}{s^{\alpha} + m^4} ||G_N(\cdot, s)||_{L^1((\varepsilon_N \mathbb{Z})^d)}$$

$$\leq c_{\alpha,L,k} \int_0^\infty ds \frac{s^{\alpha/2-1}}{s^{\alpha} + m^4}(1+s)^{-2}$$

(2.21)

$$\leq c_{\alpha,L,k} \frac{1}{m^4}.$$
where the constant
\[
c_\alpha = \int_0^\infty ds \frac{s^{\alpha/2-1}(1+s)^{-2}}{2} < \infty
\] (2.22)
is \(O(1)\) since \(0 < \alpha < 2\). It follows from (2.7) and the bound in (2.21) that for \(m \neq 0\)
\[
\|\hat{\mathcal{G}}_{N,\alpha}(\cdot, m^2)\|_{L^1(\mathbb{Z}^d)} = L^{N\alpha}\|\mathcal{G}_{N,\alpha}(\cdot, L^{N\alpha}m^2)\|_{L^1(\mathbb{Z}^d)}
\leq c_{L,\alpha,k}L^{N\alpha} \frac{1}{L^{2N\alpha}m^4}.
\] (2.23)
Therefore for \(m \neq 0\), \(\hat{\mathcal{G}}_{N,\alpha}(\cdot, m^2)\) is in \(L^1(\mathbb{Z}^d)\) and
\[
\tilde{\mathcal{G}}_{N,\alpha,T_{N+1}}(x, m^2) = \sum_{y \in \mathbb{Z}^d} \tilde{\mathcal{G}}_{N,\alpha}(x + y, m^2)
\] (2.24)
converges absolutely in the norm of \(L^1(Q_{N+1})\) and hence defines a function in \(L^1(T_{N+1})\). As a \(L^1(T_{N+1})\) periodic function \(\tilde{\mathcal{G}}_{N,\alpha,T_{N+1}}(\cdot, m^2)\) has a Fourier series which is obtained by Poisson summation with discrete Fourier transform for the discretized torus:
\[
\hat{\tilde{\mathcal{G}}}_{N,\alpha,T_{N+1}}(x, m^2) = \frac{1}{|Q_{N+1}|} \sum_{p \in \frac{\mathbb{Z}^d}{Q_{N+1}}} \hat{\tilde{\mathcal{G}}}_{N,\alpha}(p, m^2) e^{ip \cdot x}.
\] (2.25)
Recall that \(Q_{N+1} = [-\frac{L_{N+1}}{2}, \frac{L_{N+1}}{2}]^d \cap \mathbb{Z}^d\). Here \(L = 3^p\) where \(p \geq 1\) is any positive integer (this \(p\) is not be confused with the \(p\) appearing in the sum). Thus the sum is over a discretization of the Brillouin zone \(B_{\varepsilon_0} = [-\pi, \pi]^d\) where the discrete Fourier transform in \(\mathbb{Z}^d\) occurring in (2.25) is defined. We shall now estimate the Fourier coefficients.

From the definition
\[
\tilde{\mathcal{G}}_{N,\alpha}(x, m^2) = L^{-2N[\varphi]} \mathcal{G}_{N,\alpha}(\frac{x}{L^\varphi}, L^{N\alpha}m^2)
\] (2.26)
we obtain for the Fourier transform
\[
\hat{\mathcal{G}}_{N,\alpha}(p, m^2) = L^{N\alpha} \hat{\mathcal{G}}_{N,\alpha}(L^N p, L^{N\alpha}m^2)
\leq L^{N\alpha} \int_0^\infty ds \rho_\alpha(s, L^{N\alpha}m^2) \hat{\mathcal{G}}_N(L^N p, s)
\] (2.27)
where \(p \in [-\pi, \pi]^d\) and \(\rho_\alpha\) is that of (1.16). Using the bounds supplied in (2.18) and (2.20) we obtain for \(m \neq 0\)
\[ |\tilde{G}_{N,\alpha}(p, m^2)| \leq c_\alpha c_{L,k} L^{N\alpha} \int_0^{\infty} ds \frac{s^{\alpha/2-1}}{s^{\alpha} + L^{2N\alpha} m^4} (1 + s)^{-2} ((L^N p)^2 + 1)^{-k} \]
\[ \leq c_{\alpha,L,k} \frac{L^{N\alpha}}{(L^{N\alpha} m^2)^2} ((L^N p)^2 + 1)^{-k}. \]  
(2.28)

From (2.25) we get
\[ \tilde{G}_{N,\alpha,T_{N+1}}(x, m^2) = \frac{1}{|Q_{N+1}|} \sum_{p \in Q_{N+1}} \tilde{G}_{N,\alpha}(\frac{2\pi}{L^{N+1}} p, m^2) e^{i \frac{2\pi}{L^{N+1}} p \cdot x}. \]  
(2.29)

Taking partial derivatives of order \( l \) with respect to \( x \in \mathbb{Z}^d \) we get
\[ \partial_{\mathbb{Z}^d}^l \tilde{G}_{N,\alpha,T_{N+1}}(x, m^2) = \frac{1}{|Q_{N+1}|} \sum_{p \in Q_{N+1}} \left( \frac{2\pi p}{L^{N+1}} \right)^l \tilde{G}_{N,\alpha}(\frac{2\pi}{L^{N+1}} p, m^2) e^{i \frac{2\pi}{L^{N+1}} p \cdot x} \]  
(2.30)

where the partial derivative \( \partial_{\mathbb{Z}^d}^l \) is in multi-index notation, \( p^l = \prod_{i=1}^d p_i^l, \) \( l_i \geq 0 \) are non-negative integers and \( l = \sum_{i=1}^d l_i \geq 0 \). We now use the bound (2.28) and extend the sum to \( \mathbb{Z}^d \) to get

\[ |\partial_{\mathbb{Z}^d}^l \tilde{G}_{N,\alpha,T_{N+1}}(x, m^2)| \leq c_{\alpha,L,k} L^{-(N+1)d} \frac{L^{N\alpha}}{(L^{N\alpha} m^2)^2} \sum_{p \in \mathbb{Z}^d} \left( \frac{2\pi |p|}{L} \right)^l \left( \frac{(2\pi p)^2}{L} + 1 \right)^{-k}. \]  
(2.31)

The non-negative integer \( k \) is at our disposal. We choose \( 2k > d + l + 1 \). Then the series converges and we get the bound for all \( m \neq 0 \) and all integers \( l \geq 0 \)

\[ |\partial_{\mathbb{Z}^d}^l \tilde{G}_{N,\alpha,T_{N+1}}(x, m^2)| \leq c_{L,\alpha,l} L^{-2N\alpha} m^{-4} L^{-(2N|\varphi|+lN)}. \]  
(2.32)

This proves (1.12) and thus the proof of Theorem 1.1 is complete. \( \blacksquare \)

The next section is devoted to the proof of Theorem 1.2

3. Proof of Theorem 1.2

Throughout the proof we restrict \( \alpha \) to the range \( 1 < \alpha < 2 \).

First, we will prove the differentiability bound. Recall the rescaled fluctuation covariances

\[ \Gamma_{j,\alpha}(\cdot, m^2) : (\varepsilon_j \mathbb{Z})^d \rightarrow \mathbb{R} \]

defined in equation (3.11) in Section 3 of [1, 2].
\[ \Gamma_{j, \alpha} (\cdot, m^2) = \int_0^\infty ds \rho_{\alpha} (s, m^2) \Gamma_{j, \alpha} (\cdot, s) \]

where \( \Gamma_{j, \alpha} (\cdot, s) \) is the rescaled fluctuation covariance in the finite range decomposition of the resolvent of the standard Laplacian (see Section 3 of [1, 3]). Then we have for all \( 0 \leq j \leq N - 1 \), on using the uniform bound in Theorem 5.5 of [4] together with Sobolev embedding:

\[
\left\| \frac{\partial}{\partial m} \partial^p \varepsilon \Gamma_{j, \alpha} (\cdot, m^2) \right\|_{L^\infty ((\varepsilon_q \mathbb{Z})^d)} \leq \int_0^\infty ds \left| \frac{\partial}{\partial m^2} \rho_{\alpha} (s, m^2) \right| \left\| \partial^p \varepsilon \Gamma_{j, \alpha} (\cdot, s) \right\|_{L^\infty ((\varepsilon_q \mathbb{Z})^d)}
\]

\[
\leq c_{L,p} \int_0^\infty ds \left| \frac{\partial}{\partial m^2} \rho_{\alpha} (s, m^2) \right| (1 + s)^{-1}
\]

(3.1)

where \( \rho_{\alpha} (s, m^2) \) is the spectral function of Proposition 2.1 of [1]:

\[
\rho_{\alpha} (s, m^2) = \frac{\sin \pi \alpha/2}{\pi} \frac{s^{\alpha/2}}{s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2}.
\]

Hence

\[
\left| \frac{\partial}{\partial m^2} \rho_{\alpha} (s, m^2) \right| \leq c_{\alpha} \frac{s^{\alpha/2} (m^2 + s^{\alpha/2})}{(s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2)^2}
\]

\[
\leq c_{\alpha} \frac{s^{\alpha/2} (m^2 + s^{\alpha/2})}{(s^\alpha + m^4)^2}
\]

(3.3)

where we have used from the proof of Proposition 2.1 of [1] the bound

\[
d_{\alpha} (s, m^2) = s^\alpha + m^4 + 2m^2 s^{\alpha/2} \cos \pi \alpha/2
\]

\[
\geq c'_{\alpha} (m^4 + s^\alpha).
\]

Therefore

\[
\left\| \frac{\partial}{\partial m^2} \partial^p \varepsilon \Gamma_{j, \alpha} (\cdot, m^2) \right\|_{L^\infty ((\varepsilon_q \mathbb{Z})^d)} \leq c_{L, \alpha, p} \int_0^\infty ds \frac{s^{\alpha/2} (m^2 + s^{\alpha/2})}{(s^\alpha + m^4)^2} (1 + s)^{-1}.
\]

(3.4)

After a change of variables \( s^{\alpha/2} = m^2 \sigma \) we get with a different constant \( c_{L, \alpha, p} \)

\[
\left\| \frac{\partial}{\partial m^2} \partial^p \varepsilon \Gamma_{j, \alpha} (\cdot, m^2) \right\|_{L^\infty ((\varepsilon_q \mathbb{Z})^d)} \leq c_{L, \alpha, p} (m^2)^{\frac{2}{\alpha}} H_{\alpha} (\mu)
\]

(3.5)

where

\[
\mu = (m^2)^{\frac{2}{\alpha}}
\]

(3.6)
\[ H_\alpha(\mu) = \int_0^\infty d\sigma \frac{\sigma^{\frac{2}{\alpha}}(1 + \sigma)}{(1 + \sigma^2)^2} (1 + \mu \sigma^{\frac{2}{\alpha}})^{-1}. \]  

(3.7)

For \(1 < \alpha < 2\) we have the obvious bound

\[ H_\alpha(\mu) \leq H_\alpha(0) \]  

(3.8)

where

\[ H_\alpha(0) = \int_0^\infty d\sigma \frac{\sigma^{\frac{2}{\alpha}}(1 + \sigma)}{(1 + \sigma^2)^2} < \infty \]  

(3.9)

since \(1 < \alpha < 2\) and thus \(H_\alpha(0)\) is a constant \(c_\alpha\) of \(O(1)\).

From (3.5), (3.6), (3.7), (3.8) and (3.9) we get

\[ \| \frac{\partial}{\partial m^2} \partial_{\varepsilon_j} \Gamma_{j, \alpha}(\cdot, m^2) \|_{L^\infty((\varepsilon\mathbb{Z})^d)} \leq c_{L, \alpha, p} (m^2)^{\frac{2}{\alpha} - 2} \]  

(3.10)

and hence

\[ \| \frac{\partial}{\partial m^2} \partial_{\varepsilon_j} \tilde{\Gamma}_{j, \alpha}(\cdot, m^2) \|_{L^\infty(\mathbb{Z}^d)} \leq c_{L, \alpha, p} L^{-pj} L^{-2j[\varphi]} L^{j\alpha} (L^{j\alpha} m^2)^{-2(1 - \frac{1}{\alpha})} = c_{L, \alpha, p} L^{-j(d - 2)} L^{-pj} (m^2)^{-2(1 - \frac{1}{\alpha})} \]  

(3.11)

which proves the differentiability bound (1.6) of Theorem 1.2. The uniform continuity bound (1.21) now follows by integrating the bound (3.11) above.

It remains now to prove the uniform Lipshitz continuity bound (1.20) of \(\tilde{\Gamma}_{N, \alpha, T_{N+1}}(\cdot, m^2)\).

Recall that in the uniform continuity statement \(m^2 > 0\). We will first give an uniform upper bound for its derivative with respect to \(m^2\) from which the uniform Lipshitz continuity bound will follow. To this end we start from the Fourier series representation (2.30) where the Fourier coefficients decay rapidly as in (2.28). We shall show presently that their derivatives with respect to \(m^2\) also decay rapidly. We have

\[ \| \frac{\partial}{\partial m^2} \partial_{\varepsilon_j} \tilde{\Gamma}_{N, \alpha, T_{N+1}}(\cdot, m^2) \|_{L^\infty(\mathbb{Z}^d)} \leq \frac{1}{|Q_{N+1}|} \sum_{p \in Q_{N+1}} \frac{2\pi p}{L N + 1} \| \frac{\partial}{\partial m^2} \tilde{\Gamma}_{N, \alpha}(\cdot, m^2) \|_{L^{\infty}(\mathbb{Z}^d, \mathbb{R})}. \]  

(3.12)

Now using the representation (2.9), the equality (2.27), and the bound (2.18) we obtain for every integer \(k \geq 0\)
\[
\left| \frac{\partial}{\partial m^2} \hat{G}_{N,\alpha}(\frac{2\pi}{L}t, m^2) \right| \leq c_{L,k} L^{2N}\int_0^\infty \, ds \left| \frac{\partial}{\partial m^2} \rho_\alpha(s, m^2) \right|_{m^2 \to L_N^\alpha m^2} \times (1 + s)^{-2} \left( (\frac{2\pi p}{L})^2 + 1 \right)^{-k} s^{-1}. \tag{3.13}
\]

We have the bound (this was obtained in going from (3.1) to (3.4))

\[
\left| \frac{\partial}{\partial m^2} \rho_\alpha(s, m^2) \right| \leq c_\alpha \frac{s^{\alpha/2}(m^2 + s^{\alpha/2})}{(s^\alpha + m^4)^2}. \tag{3.14}
\]

From (3.12), (3.13) and (3.14) and then extending the sum to that over \( \mathbb{Z}^d \) we obtain by choosing \( k \) sufficiently large so that the series converges,

\[
\left| \left\langle \frac{\partial}{\partial m^2} \partial_m \tilde{G}_{N,\alpha,T} \right\rangle_{m^2 \to L_N^\alpha m^2} \right|_{L^\infty(\mathbb{Z}^d)} \leq c_{L,\alpha,l} L^{- (N + 1)d} L^{- Nl} L^{2N} \times \int_0^\infty \, ds \frac{s^{\alpha/2-1}(m^2 + s^{\alpha/2})}{(s^\alpha + (m^2)^2)^2} (1 + s)^{-2} \left| m^2 \to L_N^\alpha m^2 \right| \tag{3.15}
\]

where the integral

\[
F_\alpha(m^2) = \int_0^\infty \, ds \frac{s^{\alpha/2-1}(m^2 + s^{\alpha/2})}{(s^\alpha + (m^2)^2)^2} (1 + s)^{-2} \tag{3.16}
\]

converges since \( \alpha > 0 \) and \( m \neq 0 \). We now change variables as in the line before (3.5):

\[
s^{\alpha/2} = m^2 \sigma \to \mu = (m^2)^{\frac{2}{\alpha}}
\]

\[
F_\alpha(m^2) = (m^2)^{-2} \frac{2}{\alpha} \int_0^\infty \, d\sigma \frac{(1 + \sigma)}{(1 + \sigma^2)^2} (1 + \mu \sigma^{2/\alpha})^{-2} \leq (m^2)^{-2} \frac{2}{\alpha} \int_0^\infty \, d\sigma \frac{(1 + \sigma)}{(1 + \sigma^2)^2} \leq c_\alpha (m^2)^{-2} \tag{3.17}
\]

since the last integral on the right converges to a constant of \( O(1) \). Therefore

\[
F_\alpha(L^\alpha m^2) \leq L^{-2N} c_\alpha (m^2)^{-2}. \tag{3.18}
\]

From (3.15), (3.16) and (3.18) we get for all integers \( l \geq 0 \)

\[
\left| \left\langle \frac{\partial}{\partial m^2} \partial_m \tilde{G}_{N,\alpha,T} \right\rangle_{m^2 \to L_N^\alpha m^2} \right|_{L^\infty(\mathbb{Z}^d)} \leq c_{L,\alpha,l} L^{- (N + 1)d} L^{- Nl} (m^2)^{-2}. \tag{3.19}
\]
The uniform continuity bound for non-zero mass is now obtained by integration. The proof of Theorem 1.2 is complete.

**Appendix A**

In this Appendix we prove the statements in the first paragraph of Remark 4. By definition the fluctuation covariances on the coarser scale $L' = L^r$ with $L \geq 2$ fixed and $r$ a large positive integer is given by (1.17):

$$\tilde{\Gamma}'_{j,\alpha}(\cdot, m^2) = \Gamma_{l+jr,\alpha}(\cdot, m^2). \quad (3.20)$$

Therefore we get

$$||\frac{\partial}{\partial m^2} \prod_{d} \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2)||_{L^\infty(\mathbb{Z}^d)} \leq \sum_{l=0}^{r-1} ||\frac{\partial}{\partial m^2} \prod_{d} \Gamma_{l+jr,\alpha}(\cdot, m^2)||_{L^\infty(\mathbb{Z}^d)}$$

$$\leq c_{L,\alpha,p}(m^2)^{-2(1-\frac{d}{2})} \sum_{l=0}^{r-1} L^{-p(l+jr)} L^{-(l+jr)(d-2)}$$

$$\leq c_{L,\alpha,p}(m^2)^{-2(1-\frac{d}{2})} (L')^{-p_j(L')^{(d-2)}} \sum_{l=0}^{\infty} L^{-(p+(d-2))}.$$

For $d \geq 3$, $\forall p \geq 0$ and $d = 2$, $\forall p \geq 1$ we can bound the sum on the right hand side by

$$\sum_{l=0}^{\infty} L^{-l} = (1 - \frac{1}{L})^{-1}$$

and hence

$$||\frac{\partial}{\partial m^2} \prod_{d} \tilde{\Gamma}'_{j,\alpha}(\cdot, m^2)||_{L^\infty(\mathbb{Z}^d)} \leq c'_{L,\alpha,p}(m^2)^{-2(1-\frac{d}{2})} (L')^{-p_j(L')^{(d-2)}}.$$

which is (1.20) with a new constant independent of $L'$ as claimed.

**Acknowledgements** I wish to thank David Brydges for many helpful conversations and for setting me right on Poisson summation for a discrete torus. I also thank the diligent reviewers for their remarks, questions and suggestions.

**References**

[1] P. K. Mitter: On a finite range decomposition of the resolvent of a fractional power of the laplacian, J Stat Phys (2016) **163**:1235-1246, Erratum: J Stat Phys (2017) **166**.
[2] P. K. Mitter: On A Finite Range Decomposition of the Resolvent of a Fractional Power of the Laplacian: http://arxiv.org/abs/1512.02877

[3] D. Brydges, G. Guadagni and P. K. Mitter: Finite range Decomposition of Gaussian Processes, J. Stat. Phys. (2004) **115**: 415–449

[4] Roland Bauerschmidt: A simple method for finite range decomposition of quadratic forms and Gaussian fields, Probab. Theory Relat. Fields (2013) **157**: 817-845

[5] Roland Bauerschmidt (unpublished)

[6] G. Slade: Critical exponents for long range $O(n)$ models below the upper critical dimension, https://arxiv.org/1611.06169

[7] Elias M Stein and Guido Weiss: Introduction to Fourier Anaysis on Euclidean Spaces, Princeton University Press, second printing (1975), Princeton, New Jersey

[8] Audrey Terras: Fourier Analysis on Finite Groups and Applications, London Mathematical Society Student Texts 43, Cambridge University Press, second printing (2001), Cambridge UK.

[9] Stefan Adams, Roman Kotecký, Stefan Muller: Finite range decomposition for families of gradient Gaussian measures, J. Funct. Anal. (2013) **264**: 169-206

[10] R. Bauerschmidt, D. C. Brydges, G. Slade: A renormalisation group method. III. Perturbative analysis, J Stat Phys (2015) **159**: 492-529