Qualifying ringdown and shadow of black holes under general parametrized metrics with photon orbits

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Abstract The motion of photons around black holes determines the shape of shadow and match the ringdown properties of a perturbed black hole. Observations of shadows and ringdown waveforms will reveal the nature of black holes. In this paper, we study the motion of photons in a general parametrized metric beyond the Kerr hypothesis. We investigated the radius and frequency of the photon circular orbits on the equatorial plane and obtained fitted formula with varied parameters. The Lyapunov exponent which connects to the decay rate of the ringdown amplitude is also calculated. We also analyzed the shape of shadow with full parameters of the generally axisymmetric metric. Our results imply the potential constraint on black hole parameters by combining the Event Horizon Telescope and gravitational wave observations in the future.

1 Introduction

One of the main problems of gravity theories is to test the theory in strong field regime with high accuracy. However, the LIGO-Virgo experiments on gravitational wave observation [1–3] and observation of black hole shadow image at the center of M87 elliptic galaxy [4] give us the opportunity to develop new tests of general relativity and modified/alternative theories of gravity in the strong field regime. Further improvements of the experiments and observations will give more precise results and opportunity to obtain constraints on different theories of gravity.

Up to now the general relativity proposed by Albert Einstein in 1916 is considered as the main theory of gravity which is justified by the different tests in the weak (e.g. Solar system tests [5, 6]) and strong field (e.g. gravitational waves [7, 8], shadow of M87 [9–12, 114, 116] regimes with high accuracy. However, general relativity meets number of problems connected with the spacetime singularity at the origin of the solutions of the field equations, renormalization, compatibility with quantum field theory and etc. Despite the attempts to resolve the singularity problem introducing conformal transformation [13–17], coupling with nonlinear electrodynamics [18–20], higher dimensional corrections [21] and etc, no single unique theory has been found to resolve all fundamental problems of general theory of relativity. Thus, we need to deal with the big number of the modified/alternative theories of gravity to resolve the fundamental problems of general relativity and describe the current observational and experimental data.

There is a belief that astrophysical black holes are described by the Kerr spacetime (at least by Schwarzschild spacetime when the effect of rotation is negligible). However, the so-called Kerr hypothesis has not been yet well confirmed by current experimental and observational data. The new parameters of solutions within modified or alternative theories of gravity representing the deflections from Kerr spacetime may mimic the effects of the spin parameter of the Kerr black holes [22–24]. Due to this fact one needs to develop more independent tests of the gravity theory [25, 26] together with further improvements of the astrophysical instrumentation [27, 28].

On the other hand big number of alternative and modified theories of gravity and corresponding solutions of field equation describing the compact gravitating object creates the difficulties associated with resolving the cross mimicking of the parameters of theories. One of the attempt to resolve this issue is to use the parametrization of the spacetime metric. The parametrization of the spacetime around rotating black hole may help to cover big number of the solutions of different gravity models. Several ways of parametrization of the spacetime describing the rotating black hole have been...
proposed by different authors [29–31]. So-called KRZ metric was proposed by Konoplya, Rezzolla, and Zhidenko [30] where they have suggested parametrization of the spacetime of rotating black hole. They have proposed a parametric frame to describe the spacetime of the axisymmetric black holes which have the Killing vector \( \eta = (0, 0, 0, 1) \). The most interesting point of the KRZ metric is that it contains many significant parameters. These parameters associated with physical properties of the black hole and we briefly discuss them in the text of the paper. Particularly, when all the parameters of KRZ parametrization are equal to zero, the spacetime metric reduces to Kerr one.

The most distinguishable feature of the metric theories of gravity is the light deflection and other effects connected with photon motion in the curved spacetime. Some authors studied the photon surfaces [32] through the photon motion. The gravitational lensing was the one of the first consequences of the general relativity and has been discovered by Einstein. The study of the light deflection can be used to study either gravitational object or distant source. The review of the gravitational lensing effect can be found in Refs. [33–36]. The effects of the plasma on gravitational lensing in different spacetimes have been studied in [37–56]. The most basic gravitational lensing is the Schwarzschild lensing and it has been studied in [57, 58], some papers also studied the cosmic censorship hypothesis (CCH) with the gravitational lensing [59, 60].

Additionally, the properties of circular orbits of photons around black holes reflect on the ringdown signal from the merger of binary black holes [61, 62]. The latter are quasinormal modes (QNMs) which describe the end state of a black hole-black hole merger. Therefore the gravitational-wave emission at late times can be well described by the properties of null geodesics on unstable circular orbits at the black hole’s light ring [63, 64]. The direct calculation of QNMs of the KRZ metric is difficult, however, one may use the alternative method to calculate the frequency and Lyapunov exponent of unstable circular orbits leading to the features of ringdown of waveforms [61–64]. Some authors work on the spherical and static parametrized RZ metric and proposed the higher order WKB method to reduce the difficulty of computing the QNMs [65]. On the other hand the standard metric perturbations of the Schwarzschild black hole has been studied in the pioneer work of Regge and Wheeler [66] and Zerilli [67]. Later the QNM of the black holes have been calculated using the perturbative method in different works (see, e.g., [68–73] and reference therein).

The recent [74–77] and future observation of the black hole shadow by Event Horizon Telescope (EHT) using very long baseline interferometry (VLBI) technique can be used to explore the the gravity in the strong field regime around supermassive black hole (SMBH). At the same time one may test the gravity theories using the observational data from the black hole shadow. The shadow of SMBH has been theoretically studied in Refs. [43, 52, 78–104] within the different gravity models. Here we plan to study shadow of the black holes described by the parametric spacetime metric proposed in [30].

The paper is organized as follows: Sect. 2 is devoted to briefly review of the motion of massive and massless particles in the KRZ space-time and construction of the ray tracing algorithm necessary to investigate the shadow. We also study the frequency of photon orbits (Sect. 3) and the Lyapunov Exponent (Sect. 4) of light ring. Sect. 5 describes the ray-tracing code used to construct the shadow of the KRZ metric. In Sect. 6, we consider the shadow cast by the KRZ space-time for observer at infinity. Finally, in Sect. 7 we summarize the obtained results. Throughout the paper we use a space-like signature \((-, +, +, +)\), a system of units in which \(G = c = 1\). Greek indices run from 0 to 3, Latin indices from 1 to 3.

## 2 Photon motion

In this section we explore the parametrized KRZ metric proposed in [30] and investigate the photon motion around compact object described by KRZ metric. The lowest-order metric expression of the KRZ parametrization has the following form:

\[
\begin{align*}
\Sigma &= 1 + a^2 \cos^2 \theta / \bar{r}^2 , \\
N^2 &= (1 - r_0 / \bar{r}) \\
&\left[ 1 - \epsilon_0 r_0 / \bar{r} + (k_0 - \epsilon_0) r_0^2 / \bar{r}^3 + \delta_1 r_0^3 / \bar{r}^5 \right] \\
B &= 1 + \delta_1 r_0^2 / \bar{r}^2 + \delta_3 r_0^4 \cos^2 \theta / \bar{r}^2 , \\
W &= \left[ w_0 r_0^2 / \bar{r}^2 + \delta_2 r_0^3 / \bar{r}^3 + \delta_4 r_0^5 / \bar{r}^5 \cos^2 \theta \right] / \Sigma , \\
K^2 &= 1 + a W / \bar{r} \\
&+ \left\{ k_0 r_0^2 / \bar{r}^2 + k_1 r_0^3 / \bar{r}^3 \cos^2 \theta \right\} / \Sigma , \\
L &= \left[ 1 + \kappa_2 (1 - r_0 / \bar{r}) \right]^{-1} ,
\end{align*}
\]

In this paper we use the following parameters defined as [105]:

\[
\begin{align*}
\Sigma &= 1 + a^2 \cos^2 \theta / \bar{r}^2 , \\
N^2 &= (1 - r_0 / \bar{r}) \\
B &= 1 + \delta_1 r_0^2 / \bar{r}^2 + \delta_3 r_0^4 \cos^2 \theta / \bar{r}^2 , \\
W &= \left[ w_0 r_0^2 / \bar{r}^2 + \delta_2 r_0^3 / \bar{r}^3 + \delta_4 r_0^5 / \bar{r}^5 \cos^2 \theta \right] / \Sigma , \\
K^2 &= 1 + a W / \bar{r} \\
L &= \left[ 1 + \kappa_2 (1 - r_0 / \bar{r}) \right]^{-1} .
\end{align*}
\]
$$r_0 = 1 + \sqrt{1 - \delta^2},$$ (8)  
$$a_{20} = 2\delta^2/r_0^3,$$ (9)  
$$a_{21} = -\delta^2/r_0^4 + \delta_0,$$ (10)  
$$\varepsilon_0 = (2 - r_0)/r_0,$$ (11)  
$$k_{00} = \frac{2\delta}{r_0^2},$$ (12)  
$$k_{21} = \frac{a^2}{r_0^4} - 2\delta^2/r_0^3 - \delta_0,$$ (13)  
$$w_{00} = \frac{2\delta}{r_0^2},$$ (14)  
$$k_{22} = -\frac{a^2}{r_0^4} + \delta_0,$$ (15)  
$$k_{23} = \frac{a^2}{r_0^4} + \delta_0,$$ (16)  

where \(r_0\) is the radius of the event horizon in the equatorial plane and \(\delta_i\) (\(i = 1, 2, 3, 4, 5, 6, 7, 8\)) is the dimensionless parameter describing the corresponding deformation of the parameter in the metric (1). Particularly, \(\delta_1\) corresponds to the deformation of \(g_{tt}\), \(\delta_2\) and \(\delta_3\) correspond to the deformations of spin, \(\delta_4\) and \(\delta_5\) correspond to the deformations of \(g_{rr}\), \(\delta_6\) corresponds to the deformation of the event horizon. In the case when \(\delta = 0\) the KRZ one (1) reduces to Kerr metric and \(a = 0\) reduces the Kerr metric to Schwarzschild one.

The stationary and axisymmetric KRZ metric is independent of \(t\) and \(\phi\) coordinates which leads to existence of timelike and spacelike Killing vectors. Consequently, these two Killing vectors correspond to two conserved quantities: the energy \(E\) and the z-component of the angular momentum \(L_z\) of test particle. The conserved energy and angular momentum of the test particle can be expressed as:

\[-E = g_{tt} \dot{t} + g_{\phi\phi} \dot{\phi}, \quad \Phi = g_{\phi\phi} \dot{\phi} + g_{\phi\phi} \dot{\phi},\]  
(17)  
(18)

where the overhead dot represents the derivative with respect to the affine parameter (proper time for a massive particle). One can thus express the equation of motion of test particles with these two conserved quantities. Substituting Eqs. (17)-(18) into the normalization condition of the four-velocity \(u^a u_a = -1\) for a massive particle, where \(u^a = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})\) is the 4-velocity, one may obtain the following equation for the motion in the equatorial plane \((\theta = \pi/2)\):

\[g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 + 2g_{\phi\phi} \dot{\phi} = -1 .\]  
(19)

Similarly one can consider the orbits of photons around black hole. For the photon orbits the normalization condition of the four-velocity take the form \(u^a u_a = 0\). Considering the orbit in the equatorial one may get the following expression:

\[g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 + 2g_{\phi\phi} \dot{\phi} = 0 .\]  
(20)

1. We expand the normalization equation \(u^a u_a = -1\) or \(u^a u_a = 0\).
2. We substitute the equations for the conserved quantities \(E\) and \(\Phi\) (17)-(18) into the normalization equation.
3. We rewrite the normalization equation by the two conserved quantities in a form similar to the equation in Newtonian mechanics.

The equations of radial motion containing the effective potential for particle and photon have the following form:

\[
\frac{E^2 - 1}{2} = \frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r),
\]  
(21)

\[
\frac{E^2}{\Phi^2} = \frac{1}{\Phi^2} \dot{\phi}^2 + W_{\text{eff}}(r),
\]  
(22)

where \(V_{\text{eff}}\) and \(W_{\text{eff}}\) are the effective potential for particle and photon orbits, respectively. The expressions for the \(V_{\text{eff}}\) and \(W_{\text{eff}}\) depend on parameters of the chosen spacetime metric. However, application of these steps to the KRZ metric becomes problematic due to its complicate form. Thus here we use a method described in Ref. [106].

For the light ring (LR) we have the following condition

\[V_{\text{eff}} = \nabla V_{\text{eff}} = 0 ,\]  
(23)

where the \(V_{\text{eff}}\) has the following form

\[V_{\text{eff}} = -\frac{1}{D} \left( E^2 g_{\phi\phi} + 2E \Phi g_{\phi\theta} + \Phi^2 g_{tt} \right) ,\]  
(24)

One may now easily introduce new potential functions rewriting the Eq. (24) in the following form

\[V_{\text{eff}} = -\Phi^2 g_{\phi\phi} \left( \sigma - H_+ \right) \left( \sigma - H_- \right) / D ,\]  
(25)

where the \(\sigma = 1/b, b = |\Phi|/E\) is the inverse impact parameter. Newly introduced effective potential \(H_+\) of photon orbits on the orthogonal 2-space has the following form

\[H_+(r, \theta) = -g_{\phi\phi} \pm \sqrt{D} g_{\phi\phi} ,\]  
(26)

where \(D \equiv \left( g_{\phi\phi}^2 - g_{\phi\theta} g_{\theta\phi} \right)\). The circular orbits of photon obey the condition \(\partial_r H_+ = 0\), where the \(\pm\) sign is associated to the two direction of the rotation. In this paper for our analysis we consider the case corresponding to the sign “+”. Now we can calculate the circular orbits of photon in the equatorial plane i.e. \(\theta = \pi/2\) through the method described in [106]. Since we consider the motion in the equatorial plane and use the condition \(\partial_r H_+ = 0\), one can find out that the circular orbits of photon depend on \(\delta_1, \delta_2\) and \(a\) only. We cannot get the exact analytical relationship of these three variables \(\delta_1, \delta_2\) and \(a\) because the relationship of the three variables \(\delta_1, \delta_2\) and \(a\) in the equation \(\partial_r H_+ = 0\) are complicated. So we try to perform a numerical fit to get the fitting equation \(F_1(\delta_1, \delta_2, a)\). After a series of fittings, we get the equation \(F_1(\delta_1, \delta_2, a)\) (see Appendix A). Fig. 1 shows the
numerical results and fitting results with different variables. From Fig. 1 we can see that the fitting results fits well with the numerical results. With the increase of spin $a$ and $\delta_2$, the radius of photon orbits decrease.

After obtaining the radius of the photon circular orbit, one can explore the photon motion. The geodesic equations for null geodesics has the following form:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\tau} \frac{dx^\nu}{d\lambda} \frac{dx^\tau}{d\lambda} = 0,$$

(27)

where $\lambda$ is the affine parameter, $\Gamma^\mu_{\nu\tau}$ are Christoffel Symbols defined as

$$\Gamma^\mu_{\nu\tau} = \frac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\tau} + g_{\sigma\tau,\nu} - g_{\nu\tau,\sigma}),$$

(28)

From Eqs. (27)-(28), one can easily get a differential equations including the terms $d^2t/d\lambda^2$, $d^2r/d\lambda^2$, $d^2\theta/d\lambda^2$, $d^2\phi/d\lambda^2$. Using the relation for radius of photon orbits and the Eqs. (17), (18) and (20), one can determine the whole set of initial conditions.

3 Frequency of the photon orbits

As we have mentioned before, the QNM of a perturbative black hole can be determined by the unstable circular orbit of photons around the black hole. The frequency of QNM is related to the orbital frequency of the light ring [61]. Following the procedure described in Ref. [64], we calculate the frequency of light ring in order to quantify the frequency of ringdown waveforms from KRZ black holes.

In this subsection, we will explore the frequency of the photon orbits in the equatorial plane around black hole described the KRZ spacetime metric. One can calculate the frequency of photon orbits around Kerr black hole in the following way. Using the normalization condition of the four-velocity $u^\mu u_\mu = 0$ one may easily write the radial equation of motion in the form:

$$\frac{1}{\Phi^2} \left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - W_{\text{eff}}(r, b, \sigma_m),$$

(29)

where $\sigma_m \equiv \text{sign}(\Phi)$ and photon effective potential $W_{\text{eff}}(r, b, \sigma_m)$ has the following form:

$$W_{\text{eff}}(r, b, \sigma_m) = \frac{1}{r^2} \left[ 1 - \frac{a}{b} \right]^2 - \frac{2M}{r} \left( 1 - \sigma_m \frac{a}{b} \right)^2.$$

(30)

Solving the equation $\frac{dW_{\text{eff}}}{dr}|_{r_{\text{cir}} = 0} = 0$ and using the value for $r_{\text{cir}}$ one can easily get the the angular velocity $\Omega = d\phi/dt$ and consequently get information about the frequency. We may apply the same method to calculate the frequency for the KRZ spacetime.

From the Eqs. (18) and (20) one may get the following equation

$$\frac{1}{\Phi^2} \left(\frac{dr}{d\lambda}\right)^2 = -\frac{g_{tt} + g_{t\phi} \Omega^2 + 2g_{t\phi}}{g_{rr} \left(g_{t\phi}^2 + g_{t\phi}^2 \Omega^2 + 2g_{t\phi} g_{\phi\phi} \Omega^2 \right)}.$$

(31)

Comparing the Eq. (29) with Eq. (31), one can see that the Eq. (31) does not contain the term $1/b^2$. This is due to fact that the parameter $b$ is independent of the radial coordinate $r$. Consequently, one can include the parameter $b$ into the expression for $W_{\text{eff}}$ in order to calculate the frequency. Now we solve the equation

$$\frac{dW_{\text{eff}}}{dr} \Bigg|_{r_{\text{cir}} = 0} = 0,$$

using the value $r_{\text{cir}}$ obtained using the fitting and get the frequency of the circular photon orbits in KRZ spacetime in the equatorial plane. Since we consider the equatorial plane the frequency of photon orbits depend only on $\delta_1$, $\delta_2$ and $\tilde{a}$ under the condition $\delta_3 = 0$. It is easy to find the frequency of the photon orbits for the different fixed values of $\delta_1$, $\delta_2$ and $\tilde{a}$. However, the analytical expression describing the relation of radius with $\delta_1$, $\delta_2$ and $\tilde{a}$ cannot be found explicitly due to complicated view of the metric functions. Here we tried to perform a numerical fit to obtain the equation $F_\delta(\delta_1, \delta_2, \tilde{a})$ for the frequency of photon orbits. After a number of fittings, we have obtained the expressions $F_\omega(\delta_1, \delta_2, \tilde{a})$ (see Appendix B). Fig. 2 shows the numerical and fitting results for the frequency of photon orbits. Although the data shows some differences on the graphs, the relative error between the data is found to be less than one percent after calculation. From Fig. 2 we can find that when $\delta_{1,2}$ is as large as 0.5, the frequency changes less than 10%. Therefore, only when these non-Kerr parameters are large enough, we can read the derivation from the frequency of ringdown waveform.

4 The Lyapunov Exponent of Light Ring

The Lyapunov Exponent (LE) is the key indicator of chaos in dynamical systems. Interestingly, the LE of the unstable circular orbit of photon corresponds to the decay rate of the ringdown amplitude from a perturbed black hole [62]. The Lyapunov coefficient characterizes the rate of divergence of nearby null geodesics. Based on the orbital frequency and LE of light-ring, analytical black hole binary merger waveforms are constructed in [64], and the waveform amplitude decays as $|\mathcal{A}_0| = \Lambda_p \text{sech}[\gamma(t - t_p)]$, where $\Lambda_p$ is the amplitude when the congruence converges. In this section we compute the LE of the photon circular motion around an generally axisymmetric black hole which is described by the KRZ metric.
\begin{align*}
\delta_1 &= 0.48, \quad \delta_2 = 0.48 \\
\delta_1 &= 0.07, \quad \delta_2 = 0.07
\end{align*}

Numerical results

Fig. 1 Comparison of numerical and fitting results for the radius of photon orbit depending on the spin $a$ (left panel) and the $\delta_2$ (right panel) parameters for different values of $a$, $\delta_1$ and $\delta_2$. Note the plane we choose is the equatorial one i.e. $\theta = \pi/2$.

\begin{align*}
\omega &= \frac{\omega_{KRZ} - \omega_{Kerr}}{\omega_{Kerr}} \\
\delta_1 &= 0.10, \quad \delta_2 = 0.00 \\
\delta_1 &= 0.50, \quad \delta_2 = 0.50
\end{align*}

Numerical results

Fig. 2 Comparison of numerical and fitting results for the frequency depending on the spin $a$, $\delta_1$ and $\delta_2$ parameters. The corresponding parameters have been given in the figures. In all calculations we use the equatorial plane i.e. $\theta = \pi/2$. 
Authors of Ref. [107] have considered two particle’s orbital motions with small differences in their initial conditions, then using the metric functions they have calculated LE. However, in this paper we use another approach proposed by McWilliams [64]. We consider the perturbation in the radial direction and use the following expression for LR posed by McWilliams [64]. We consider the perturbation in the radial direction and use the following expression for LR posed by McWilliams [64].

\[ r = r_t [1 + \varepsilon \gamma (t - t_p)] , \]  

(32)

where \( t_p \) is the time when the congruence converges, \( \varepsilon \) is a small dimensionless order-counting parameter and \( r_t \) is the light ring radius. \( \gamma \) in Eq. (32) is a function defined as:

\[ \gamma = \sinh [\gamma (t - t_p)] , \]

(33)

where \( \gamma \) is the Lyapunov exponent. Since we consider the motion in the equatorial plane we can consider the effect of the parameters \( \delta_1 \) and \( \delta_2 \) in the KRZ metric Eq. (1).

Using the Eqs. (32)-(33) one may explore the dependence of LE from the metric parameters. For the appropriate time we plan to plot three-dimensional graphs and explore the influence of the parameters \( \delta_1 \) and \( \delta_2 \) of the metric Eq. (1) on LE.

We have chosen the different orbits having separation in the \( \varepsilon \) of the order \( 10^{-5} \) with the other initial conditions to be the same. In Fig. 3 we present the 3-D dependence of LE from the parameters \( \delta_1 \) and \( \delta_2 \). From the Fig. 3 one can easily see that when \( \delta_1 \) increases from 0 to 1, LE will have two maximum values and there is no obvious downward trend in the overall dependence. When \( \delta_2 \) increases from 0 to 1 LE have a single maximum value and the overall trend of the graph is down. From the results of Fig. 3 one may speculate that this is related to the initial conditions: the \( l_0 \) and \( \phi_0 \) decrease with increasing \( \delta_1 \), the \( l_0 \) and \( \phi_0 \) increase with increasing \( \delta_2 \). Moreover, the angular momentum \( l \) increases with the decrease of \( \delta_1 \) and \( \delta_2 \). However, the angular momentum \( l \) decreases faster with the increase of \( \delta_2 \). Finally, we conclude that the results show in Fig. 3 is related to the interaction of these initial conditions affected by the parameters of KRZ metric.

From Fig. 3, we can find that the magnitude of LE is very sensitive to the KRZ parameters \( \delta_1 \) and \( \delta_2 \). Comparing with the frequency changes due to non-Kerr parameters, the decay rate of ringdown can constrain \( \delta_1 \) and \( \delta_2 \) better. However, from the figure, the variation of LE is very complicated and hard to be fitted. This creates a problem to construct a parameterized ringdown waveform.

5 Ray-tracing code for photons

The information from the distant source in the KRZ spacetime comes through the study and analyze of the light ray from them to the observer. Through the light ray one can get an image of the source and consequently get some information. In this work we study the trajectories of photons in the KRZ spacetime using the ray-tracing code described in the Ref. [108]. The code describes the trajectories of photons near the black hole.

The evolution of the photon’s position with different components: the \( t \)- and \( \phi \)-components can be obtained by the first-order differential equations (17)-(18). Then one can rewrite \( p_t \) and \( p_\phi \) in terms of two parameters: the normalized affine parameter \( \lambda' = E/\lambda \)

\[ \frac{dt}{d\lambda'} = -\frac{g_{\theta\theta} - b g_{\phi\phi}}{g_{\phi\phi} g_{\alpha\alpha} - g_{\phi\phi}^2} , \]

(34)

\[ \frac{d\phi}{d\lambda'} = \frac{b g_{\alpha\alpha} + b g_{\phi\phi}}{g_{\phi\phi} g_{\alpha\alpha} - g_{\phi\phi}^2} . \]

(35)

For the remaining \( r \)- and \( \phi \)-components of the photon’s position in the KRZ space-time, we can use the second-order geodesic equations with the normalized affine parameter and the Christoffel symbols \( \Gamma^\sigma_{\mu\nu} \) as:

\[ \frac{d^2x^\sigma}{d\lambda'^2} + \Gamma^\sigma_{\mu\nu} dx^\mu d\lambda' dx^\nu = 0. \]

(36)

In this way we can get the system of equations that the ray-tracing code can be used for KRZ spacetime.

We suppose that the massive source described by the KRZ spacetime is located at the origin of the reference frame and coordinate system. We choose the mass of the object \( M = 1 \) since it does not affect the shape of the shadow. We assume that the observer’s screen is located at a distance away the source of \( d = 1000 \), the azimuthal and polar angles are \( \gamma_t \) and 0, respectively. The celestial coordinates (\( \alpha, \beta \)) on the observer’s sky are related to polar coordinates \( r_{scr} \) and \( \phi_{scr} \) on the screen by \( \alpha = r_{scr} \cos(\phi_{scr}) \) and \( \beta = r_{scr} \sin(\phi_{scr}) \). Since we only know the positions and momenta of the photon in the screen, we should solve the geodesic equations from the screen to the source. The photons depart from the screen with a four-momentum perpendicular to the screen and other initial conditions. The method assumes that the screen at spatial infinity, only the photons which moving perpendicular to the screen at a distance \( d \) could influence the infinite screen.

The initial position and four-momentum of each photon in the KRZ spacetime are given as [109]

\[ r_i = (d^2 + \alpha^2 + \beta^2)^{1/2} , \]

(37)

\[ \theta_i = \arccos \left( \frac{d \cos \gamma_t + \beta \sin \gamma_t}{r_i} \right) . \]

(38)
\[ \phi_i = \arctan \left( \frac{\alpha}{d \sin \gamma_{fr} - \beta \cos \gamma_{fr}} \right), \]  
\[ \left( \frac{dr}{d\lambda^i} \right)_i = \frac{d}{r_i}, \]  
\[ \left( \frac{d\theta}{d\lambda^i} \right)_i = -\frac{\cos \gamma_{fr} + \frac{d}{r_i}(d \cos \gamma_{fr} + \beta \sin \gamma_{fr})}{\sqrt{r_i^2 - (d \cos \gamma_{fr} + \beta \sin \gamma_{fr})^2}}, \]  
\[ \left( \frac{d\phi}{d\lambda^i} \right)_i = -\frac{\alpha \sin \gamma_{fr}}{\alpha^2 + (d \cos \gamma_{fr} + \beta \sin \gamma_{fr})^2}. \]  

Using of the equation (20) of the photon four-momentum to be zero one can find the component \((dt/d\lambda^i)\). The conserved quantity \(b\), which is involved in Eqs. (34) and (35), is calculated from the initial conditions of \(E\) and \(L_z\).

The initial conditions of the code on the screen is defined in the following way. The confines of the location of the compact source is found inside \(0 \leq r_{\text{surf}} \leq 20\), and the value of the \(\phi_{\text{scr}}\), in the range \(0 \leq \phi_{\text{scr}} \leq 2\pi\) with step of \(\pi/180\). The confines is the border between the photons that are captured by the compact source and the photons that are able to escape to infinity. The photons are considered as captured by the compact source if they cross the surface \(t = r_{\text{surf}} + \delta r\) with \(\delta r = 10^{-3}\), where \(r_{\text{surf}}\) is the radius of the horizon. Then the confines is amplified in to an accuracy of \(\delta r_{\text{scr}} = 10^{-3}\) to accurately determine the shadow boundary with the value of \(r_{\text{scr}}\) for the corresponding value of \(\phi_{\text{scr}}\). This method allows one to accurately calculate the shadow produced by light ray in the KRZ parametrized metric with high accuracy with respect to sampling the entire screen.

6 The shadow of the KRZ metric

In this section, we plan to study the apparent shape of the compact object shadow under the KRZ spacetime. We can use the celestial coordinates \(\alpha\) and \(\beta\) [17] to describe the shadow of the compact object described by the KRZ spacetime

\[ \alpha = \lim_{r_0 \to \infty} \left( -r_0^2 \sin \theta_0 \frac{d\phi}{dr} \right), \]  
\[ \beta = \lim_{r_0 \to \infty} \left( r_0^2 \frac{d\theta}{dr} \right), \]  

where \(r_0\) is the distance between the massive source and observer and \(\theta_0\) is the inclination angle between the observer lens axis and the normal of observer’s sky plane (see Fig. 4).

In order to describe the dependence of the shadow’s shape with different deformation parameters, we will use the coor-
The shape of the shadow is described by the horizontal displacement from the center of the image $D$, the average radius of the sphere $\langle R \rangle$, and the asymmetry parameter $A$. Since the KRZ spacetime is axially symmetric, the parameter $D$ is always identically equal to zero. In the papers Ref. [111, 112] authors proposed different ways to describe the shape of the shadow. However, the results of the different approaches are similar to each other. The average radius $\langle R \rangle$ is the average distance of the boundary of the shadow from its center, which is defined by

$$\langle R \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} R(\theta) d\theta,$$

(45)

where $R(\theta) \equiv \left[ (\alpha - D)^2 + \beta (\alpha)^2 \right]^{1/2}$. $D = 0$ and $\theta \equiv \tan^{-1} [\beta (\alpha)]$. The asymmetry parameter $A$ is the distortion of the shadow from a circle and defined as

$$A \equiv 2 \left[ \frac{1}{2\pi} \int_0^{2\pi} \left( R - \langle R \rangle \right)^2 d\theta \right]^{1/2}.$$  

(46)

The shadow of the compact object in KRZ spacetime for the different values of metric parameters is shown in Figs. 5-8. From the Figs. 5-8 one may come to the following conclusions:

- From Fig. 5 (a) one can see that with the increase of rotation parameter $a$ one side of the shadow goes away from the center while other one comes closer.
- From Fig. 5 (b) one can observe that the presence of the parameters $\delta_1$ and $\delta_2$ force the shape of shadow to be more flatter.
- The Fig. 5 (c) shows the same effect similar to one caused by the rotation parameter $a$: the result is obvious because...

**Fig. 5** Ray-traced shadow images in the Kerr and KRZ spacetime. The (a) is for the shadow with different spin $a$ in the Kerr spacetime, the (b) is for the shadow with different spin $a$ in the KRZ spacetime, the (c) is for the shadow with different inclination in the Kerr spacetime, the (d) is for the shadow with different inclination in the KRZ spacetime.
Fig. 6 Ray-traced shadow images in the KRZ spacetime. The (a)-(f) are for the shadow corresponding different values of $\delta$. And the text $\delta_i = 0$ in the graph means that other parameters except one varying in the plot equal to zero.
The shadow of the black hole for the different values of parameter $\delta_7$ in the KRZ spacetime.

From Fig. 7 we can see when $\delta_7$ increases, the change of the shadow is small, the small picture in the upper right corner of the picture is the enlarged view of the graph. It shows the minor difference corresponding to the different values of parameter $\delta_7$. From Fig. 8 one can observe similar results corresponding to the parameter $\delta_8$. Since the effect of the $\delta_7$ and $\delta_8$ is very small, in some places (see, e.g. [31]) authors neglect the parameters $\delta_7$ and $\delta_8$ and put the Eq. (7) to be 1. Although the simplification has very weak impact on the shadow, it is mathematically misleading, because the metric used in the Ref. [31] does not reduce to the Kerr metric when $\delta_7 = 0$.

As for sensitivity, the shadow is sensitive to parameters $\delta_1, \delta_2, \delta_4$. The shadow is less sensitive to parameters $\delta_3, \delta_5, \delta_8$. The shadow is almost not sensitive to parameters $\delta_7, \delta_8$.

We also want to know how the KRZ metric can reproduce the exact metric, so we use the Kerr-Sen(KS) metric and Einstein Dilaton Gauss-Bonnet(EDGB) metric[115]. Fig. 9 shows the difference between the KS/EDGB metric and KRZ metric. Through this picture we can know KRZ metric has a good approximation to KS and EDGB metric.

Fig. 10 shows the dependence of $(R)$ and $A$ as a function of the spin parameter (Kerr metric) and $\delta_1$ (KRZ metric) for the fixed values of inclination angle $\theta_0 = \pi/4$ and the spin parameter $a = 0.5$. From Fig. 10 (b) one can easily see that the $\delta_1$ and $\delta_8$ have a greater impact on $(R)$, but with different trend: when $\delta_1$ increases the value of $(R)$ decreases and...
Fig. 9 The shadow of the black hole for the different metric. In figure (a) we compare the KS metric and KRZ metric then in figure (b) we compare the EDGB metric and KRZ metric (a is the value of spin, b is the value of scalar (dilaton) field and ζ is the value of deformation).

Fig. 10 Average radius $\langle R \rangle$ (top row) and asymmetry parameter $A$ (bottom row). The first column corresponds to the Kerr metric with the values of spin $0 \leq a \leq 1$, the second column corresponds to the KRZ metric with different values of $\delta_i$ and the spin $a = 0.5$, when one of $\delta_i$ is confirmed, the other $\delta_i$ is set to be zero.
when $\delta_1$ increases the value of $\langle R \rangle$ increases. This is due to the fact that $\delta_1$ corresponds the deformation of $g_{tt}$, while $\delta_1$ corresponds the deformation of $g_{rr}$. On the other hand $\delta_2$ and $\delta_1$ have the same trend: when their values increase the $\langle R \rangle$ decreases. From Fig. 10 (d), we may also see that the $\delta_2$ have a greater impact on $A$. Other $\delta_j$’s have small influence on $A$. From Fig. 10 (b)-(d), one may see the trend of $\langle R \rangle$ and $A$ with $\delta_1$, $\delta_5$ always opposite to effect of $\delta_2$, $\delta_3$ and $\delta_6$. Finally, with the increase of $\delta_7$ and $\delta_8$, the value of $\langle R \rangle$ and $A$ change very slowly, which also shows that $\delta_7$ and $\delta_8$ are adjustment parameters and have weak effect on the KRZ metric.

7 Conclusion

In the present work we have studied the photon motion in the equatorial plane around the generally axisymmetric black hole which is described by the parametrized KRZ metric. From the properties of the circular orbits of photon, we can quantify the frequency and decay rate of ringdown wave signals by the orbital frequency and Lyapunov exponent of the light-ring. At the same time, the shape of the shadow which be measured by distant observers is also gotten from the photons motion around the parametrized black hole. We have calculated the frequency and LE of the unstable circular orbits of photon in order to get the information on QNMs. We have also obtained the frequency and decay rate of ringdown which can be used to construct a waveform model for the KRZ black hole merger.

In the special case when the photon orbits in the equatorial plane, we have found that only two primary parameters $\delta_1$ and $\delta_2$ of the KRZ metric affect on photon trajectory. It has been shown that with the increase of the spin parameter $a$, $\delta_1$ and $\delta_2$ the radius of photon circular orbits decreases. However, the change of the radius of photon circular orbits are more sensitive to the change of spin parameter $a$, especially when the spin parameter exceeds the value 0.4. It has been found out that effects of spin parameter $a$ and $\delta_2$ on the frequency of photon orbits is the same order while the effect of $\delta_1$ is weaker. With these fitted formula, one can in principle get the frequency of ringdown waveforms from a perturbed black hole described by the KRZ metric.

The decay rate of QNM amplitude can be obtained from the LE characterizing the rate of divergence of nearby null geodesics. In this work we have shown that when $\delta_1$ increases from 0 to 1, LE will have two maximum values and there is no obvious downward trend in the overall trend of the dependence. When $\delta_2$ increases from 0 to 1 LE will have one maximum value and the overall trend of the graph is down. This is related to the initial conditions of the equation of motion. Though the angular momentum $I$ increases with the decrease of $\delta_1$ and $\delta_2$. We have shown that the decay of ringdown is very sensitive to the KRZ parameters.

We have also studied the shadow of the black hole described by the parametrized KRZ spacetime. It has been shown that the parameters $\delta_1$ and $\delta_3$ can change the size of the shadow but with opposite effect. The parameter $\delta_2$ makes the shadow deviate from the center, the parameter $\delta_5$ also can change the size of the shadow but effect is relatively weak. The parameters $\delta_1$ and $\delta_6$ can change the contour shape. The shape of the shadow has different sensitivity to different parameters: the shadow is more sensitive to parameters $\delta_1$, $\delta_2$, $\delta_4$ and less sensitive to parameters $\delta_5$, $\delta_6$. The effects of the parameters $\delta_7$ and $\delta_8$ to the shape of shadow is very weak and almost negligible (compare with [31] where authors have neglected the effects of $\delta_7$ and $\delta_8$ on the iron line in the X-ray spectrum of black holes).

One of the main results of this paper is the analysis of the dependence of the average radius of the shadow and asymmetry (distortion) parameter from the spin parameter and KRZ parameters. Among the effects of other parameters the effect of parameter $\delta_1$ is dominant in changing the average radius of the shadow. On the other hand the main contribution to the change of the asymmetry parameters comes due to the presence of the parameter $\delta_2$. In principle, one may see that the two observable quantities (radius of shadow and asymmetry parameter) could provide the rough estimation of the parameters $\delta_1$ and $\delta_2$. Further analysis of the KRZ parameters and comparison with particular black hole solutions may provide a useful tool to probe the gravity models.

The ringdown and shadow reflect the dynamical process and geometric properties in the strong field of the black hole, respectively. The former can be observed by the ground and space-borne GW detectors and the later can be observed by the EHT. From the GWs and image of EHT, one can reveal the nature of the black holes and test if they are described by the Kerr spacetime which is an exact solution of Einstein field equation in general relativity and assumed to describe the astrophysical black holes. Fortunately, both of these two phenomenon are related to the photon orbits around the black hole at the light-ring. In the present work, by calculating the photon’s motion at the light-ring, we qualify the QNMs and shadows of generally axisymmetric black holes under general parametrized metrics. Perturbing the supermassive black hole and radiating the ringdown signals can be expected in our Galaxy [113], and the imaging of this nearest supermassive black hole is a target of the EHT project. Our results may play a role to construct a joint constraint of dynamical and stationary spacetime of black hole with both LISA and EHT observations.

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Appendix subsection

Appendix A:

The function describing the radius of the photon orbits in equatorial plane and under parametrized KRZ metric has the following form

\[ F_r(\delta_1, \delta_2, \tilde{a}) = k + F_1(\tilde{a}) + F_2(\delta_1) + F_3(\delta_2) + F_4(\delta_1, \tilde{a}) + F_5(\delta_2, \tilde{a}) + F_6(\delta_1, \delta_2) + F_7(\delta_1, \delta_2, \tilde{a}) \]

\[ K = 3.00209 \]

\[ F_1(\tilde{a}) = -1.28767\tilde{a} + 0.34098\tilde{a}^2 - 0.76082\tilde{a}^3 \]

\[ F_2(\delta_1) = -0.444022\delta_1 + 0.13889\delta_1^2 \]

\[ F_3(\delta_2) = -1.40618\delta_2 - 9.94242\delta_2^2 + 37.6563\delta_2^3 - 53.23631\delta_2^4 + 34.56977\delta_2^5 - 8.59813\delta_2^6 \]

\[ F_4(\delta_1, \tilde{a}) = 0.0612867\delta_1\tilde{a} + 1.14277\delta_1a^2 - 0.618507\delta_1a^3 + 0.6384\delta_1^2\tilde{a} - 1.498999\delta_1^2a^2 + 0.59456\delta_1^2a^3 + 0.339307\delta_1^2\tilde{a}^2 - 0.573049\delta_1^2\tilde{a}^3 - 0.174293\delta_1^2a^3 \]

\[ F_5(\delta_2, \tilde{a}) = -1.45769\delta_2\tilde{a} + 9.1916\delta_2a^2 - 5.08388\delta_2a^3 + 18.3726\delta_2^2\tilde{a} - 40.509\delta_2^2a^2 + 20.9555\delta_2^2a^3 + 29.5518\delta_2^3\tilde{a} + 55.2847\delta_2^3a^2 - 27.9031\delta_2^3a^3 + 13.799\delta_2^4\tilde{a} - 24.2201\delta_2^4a^2 + 12.0752\delta_2^4a^3 \]

\[ F_6(\delta_1, \delta_2) = -1.40231\delta_1\delta_2 + 21.6537\delta_1\delta_2 - 64.9458\delta_1\delta_2^3 + 85.1184\delta_1^2\delta_2 - 52.1275\delta_1^2\delta_2^2 + 12.1391\delta_1^2\delta_2^6 + 1.25238\delta_1^2\delta_2 - 11.7479\delta_1^2\delta_2^2 + 32.1809\delta_1^2\delta_2^3 - 39.8198\delta_1^2\delta_2^4 + 22.9467\delta_1^2\delta_2^5 - 4.93156\delta_1^2\delta_2^6 \]

\[ F_7(\delta_1, \delta_2, \tilde{a}) = -4.30625\delta_1\delta_2\tilde{a} + 27.7323\delta_1^2\delta_2\tilde{a} - 20.6891\delta_1^2\delta_2^2 \]

\[ + 49.0293\delta_1^2\delta_2^2\tilde{a} - 212.63\delta_1^2\delta_2^2a^2 + 147.897\delta_1^3\delta_2^2a^2 \]

\[ - 103.967\delta_1^3\delta_2^2a^3 - 396.656\delta_1^3\delta_2^2a^4 + 269.764\delta_1^3\delta_2^3a^5 \]

\[ + 59.1911\delta_1^4\delta_2^2a^5 - 212.401\delta_1^4\delta_2^2a^6 + 142.897\delta_1^4\delta_2^3a^7 \]

\[ - 29.2685\delta_1^4\delta_2^3a^8 + 43.0102\delta_1^4\delta_2^3a^9 - 22.4658\delta_1^4\delta_2^4a^{10} + 83.9201\delta_1^4\delta_2^4a^{11} - 93.1165\delta_1^4\delta_2^4a^{12} + 43.369\delta_1^3\delta_2^2a^2 \]

\[ - 75.7107\delta_1^3\delta_2^2a^2 - 31.539\delta_1^3\delta_2^2a^3 - 0.681802\delta_1^3\delta_2^3a^2 \]

\[ + 19.9327\delta_1^3\delta_2^3a^2 + 20.2448\delta_1^3\delta_2^3a^3 - 20.796\delta_1^3\delta_2^4a^4 + 25.6721\delta_1^3\delta_2^4a^5 + 31.4421\delta_1^3\delta_2^5a^6 \]

\[ - 97.0986\delta_1^3\delta_2^5a^7 + 204.307\delta_1^3\delta_2^5a^8 - 125.38\delta_1^3\delta_2^6a^9 + 122.861\delta_1^3\delta_2^6a^{10} - 260.441\delta_1^3\delta_2^6a^{11} + 161.188\delta_1^3\delta_2^7a^{12} \]

\[ - 50.8252\delta_1^3\delta_2^7a^{12} + 107.8\delta_1^4\delta_2^4a^3 - 67.0748\delta_1^4\delta_2^4a^4 \]

Appendix B:

The function describing the frequency of the photon orbits in equatorial plane under parametrized KRZ metric has the form

\[ F_\omega(\delta_1, \delta_2, \tilde{a}) = k + F_1(\tilde{a}) + F_2(\delta_1) + F_3(\delta_2) + F_4(\delta_1, \tilde{a}) + F_5(\delta_2, \tilde{a}) + F_6(\delta_1, \delta_2) + F_7(\delta_1, \delta_2, \tilde{a}) \]

\[ K = 0.193 \]

\[ F_1(\tilde{a}) = 0.04952\tilde{a} + 0.21633\tilde{a}^2 - 0.39405\tilde{a}^3 + 0.34292\tilde{a}^4 \]

\[ F_2(\delta_1) = 0.029013\delta_1 + 0.0038367\delta_1^2 - 0.0014046\delta_1^3 \]

\[ F_3(\delta_2) = 0.04095\delta_2 + 0.47366\delta_2^2 - 0.17638\delta_2^3 \]

\[ F_4(\delta_1, \tilde{a}) = + 0.0257472\delta_1\tilde{a} + 0.0434687\delta_1a^2 - 0.0517708\delta_1a^3 + 0.00400781\delta_1\tilde{a}^2 - 0.0577969\delta_1a^2^3 + 0.0404687\delta_1^2a^3 \]

\[ - 0.00567736\delta_1^2\tilde{a} + 0.0235175\delta_1^2a^2 - 0.0140544\delta_1^3a^3 \]

\[ F_5(\delta_2, \tilde{a}) = + 0.048684\delta_2\tilde{a} + 0.0161719\delta_2a^2 - 0.229635\delta_2a^3 \]

\[ - 0.413807\delta_2^2\tilde{a} - 0.611906\delta_2^2a^2 + 0.508229\delta_2^2a^3 \]

\[ - 0.0418396\delta_2^3\tilde{a} + 0.550562\delta_2^3a^2 - 0.326042\delta_2^3a^3 \]
where:

\[ F_5(\delta_1, \delta_2) = +0.7407 \delta_1 \delta_2 - 1.5587 \delta_1 \delta_2^2 0.820459 \delta_1 \delta_2^3 - 1.6814 \delta_1 \delta_2 + 3.60301 \delta_1 \delta_2^2 - 1.93261 \delta_1 \delta_2^3 + 1.04502 \delta_1 \delta_2 - 2.501 \delta_1 \delta_2^2 + 1.21019 \delta_1 \delta_2^3 \]

One can get the expression for

\[ g_{rr} = \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2F + a^2} \tag{C.5} \]

which coincides the metric component \(g_{rr}\) in Kerr metric with the unit mass \(M = 1\).

However in [31] authors have written \(N^2\) in the form:

\[ N^2 = \left( 1 - \frac{r_0}{\bar{r}} \right) \left[ 1 - \epsilon_0 \left( \frac{\bar{r}}{r_0} \right) + (k_{00} - \epsilon_0) \right] \frac{r_0^2}{\bar{r}} + \delta r_0 \frac{\bar{r}}{r^3} \]

\[ + \left[ (k_{21} + a_{20}) \frac{r_0^2}{\bar{r}} + a_{21} r_0^4 \right] \cos^2 \theta \tag{C.6} \]

From (C.1) and (C.6), one can see the difference between the parameter \(N^2\) is the function in front of the \(\cos^2 \theta\).

One can simplify the expression (C.6) and after simple calculation one may find the function in front of the \(\cos^2 \theta\) in the form:

\[ N^2 = (a^2 / (r^3 ((1 - a^2) / (1/2) + 1) - a^4 / r^4) \]

so (C.6) takes the form

\[ N^2 = (a^2 + 2a^2 - 2r)^2 / r^2 + (a^2 / (r^3 (1 - a^2) / (1/2) + 1) - a^4 / r^4) \cos^2 \theta \tag{C.7} \]

When we put the function (C.7) into the function \(g_{rr}\), we find that the \(g_{rr}\) can not reduce to the \(g_{rr}\) for Kerr metric. However, since the parameters \(\delta r\) and \(a\) have a small impact to the spacetime, one may neglect these parameter while using the KRZ parametrization.

**Appendix C:**

When \(\delta_i = 0\) the KRZ metric (1) reduces to the Kerr one. One can get the expression for \(g_{rr}\) as

\[ g_{rr} = \Sigma \frac{\bar{r}^2}{N^2} \]

where:

\[ \Sigma = 1 + \frac{\bar{r}^2}{r^2} \cos^2 \theta \]

\[ B = 1 + \delta_i \frac{r_0^2}{\bar{r}^2} + \delta_j \frac{r_0^2}{\bar{r}^2} \cos^2 \theta / r^2 \]

\[ N^2 = \left( 1 - \frac{r_0}{\bar{r}} \right) \left[ 1 - \epsilon_0 \left( \frac{\bar{r}}{r_0} \right) + (k_{00} - \epsilon_0) \right] \frac{r_0^2}{\bar{r}} + \delta r_0 \frac{\bar{r}}{r^3} \]

\[ + \left[ (k_{21} + a_{20}) \frac{r_0^2}{\bar{r}} + a_{21} r_0^4 \right] \cos^2 \theta \tag{C.1} \]

when \(\delta_i = 0\), then we get

\[ N^2 = (a^2 + 2a^2 - 2r)^2 / r^2 \]

\[ B = 1 \]

\[ \Sigma(r, \theta) = 1 + \frac{\bar{r}^2}{r^2} \cos^2 \theta \tag{C.4} \]

**References**

1. T. Damour and G. Esposito-Farèse, Phys. Rev. D **58**, 042001 (1998).

2. M. Agathos, W. Del Pozzo, T. G. F. Li, C. Van Den Broeck, J. Veitch, and S. Vitale, Phys. Rev. D **89**, 082001 (2014).

3. A. Cárdenas-Avendaño, S. Nampalliwar, and N. Yunes, Classical and Quantum Gravity **37**, 135008 (2020).

4. K. Akiyama et al. (Event Horizon Telescope), Astrophys. J. **875**, L1 (2019), arXiv:1906.11238 [astro-ph.GA].

5. M. L. Ruggiero and N. Radicella, Phys. Rev. D **91**, 104014 (2015).

6. S. Bahamonde, J. L. Said, and M. Zubair, Journal of Cosmology and Astroparticle Physics **2020**, 024 (2020).

7. E. Berti, K. Yagi, and N. Yunes, General Relativity and Gravitation **50**, 46 (2018).

8. E. Berti, K. Yagi, H. Yang, and N. Yunes, General Relativity and Gravitation **49** (2018).
9. S. Vagnozzi and L. Visinelli, Phys. Rev. D 100, 024020 (2019).
10. C. Bambi, K. Freese, S. Vagnozzi, and L. Visinelli, Phys. Rev. D 100, 044057 (2019).
11. I. Banerjee, S. Sau, and S. SenGupta, Phys. Rev. D 101, 104057 (2020).
12. M. Khodadi, A. Allahyari, S. Vagnozzi, and D. F. Mota, Journal of Cosmology and Astroparticle Physics 2020, 026 (2020).
13. C. Bambi, L. Modesto, S. Porey, and L. Rachwał, Journal of Cosmology and Astroparticle Physics 2017, 033–033 (2017).
14. M. Zhou, Z. Cao, A. Abdikamalov, D. Ayzenberg, C. Bambi, L. Modesto, and S. Nampalliwar, Physical Review D 98 (2018), 10.1103/physrevd.98.024007.
15. B. Toshmatov, C. Bambi, B. Ahmedov, Z. Stuchlík, and J. Schee, Physical Review D 96 (2017), 10.1103/physrevd.96.064028.
16. B. Toshmatov, C. Bambi, B. Ahmedov, Z. Stuchlík, and J. Schee, Physical Review D 96 (2017), 10.1103/physrevd.96.064028.
17. V. Perlick, Ray optics, Fermat’s principle, and applications to general relativity, Vol. 61 (Springer Science & Business Media, 2000).
18. A. Rogers, Living Reviews in Relativity 7, 9 (2004).
19. A. Rogers, Monthly Notices of the Royal Astronomical Society 451, 17 (2015), https://academic.oup.com/mnras/article-pdf/451/1/17410178/stv903.pdf .
20. A. Rogers, Universe 3 (2017), 10.3390/universe3010003.
21. X. Er and A. Rogers, Monthly Notices of the Royal Astronomical Society 475, 867 (2017), https://academic.oup.com/mnras/article-pdf/475/1/867/23565410/stx3290.pdf.
22. A. Rogers, Monthly Notices of the Royal Astronomical Society 342, 1280 (2003), https://academic.oup.com/mnras/article-pdf/342/4/1280/2826492/342-4-1280.pdf.
23. J. Bicak and P. Hadrava, Astronomy and Astrophysics 44, 389 (1975).
24. S. Kichenassamy and R. A. Krikorian, Phys. Rev. D 93, 104004 (2016).
25. E. Ayón-Beato and A. García, Phys. Rev. Lett. 80, 5056 (1998).
26. E. Ayón-Beato and A. García, Physics Letters B 464, 25 (1999).
27. L. Hollenstein and F. S. N. Lobo, Phys. Rev. D 78, 124007 (2008).
28. V. P. Frolov and I. L. Shapiro, Phys. Rev. D 80, 044034 (2009).
29. P. Pani, C. F. Macedo, L. C. Crispino, and V. Cardoso, Physical Review D 84, 087501 (2011).
30. N. Yunes and L. C. Stein, Physical Review D 83, 104002 (2011).
31. F. Filipponi and G. Tasinato, Journal of Cosmology and Astroparticle Physics 2018, 033 (2018).
32. T. Damour and J. H. Taylor, Phys. Rev. D 45, 1840 (1992).
33. P. G. Ferreira, Annual Review of Astronomy and Astrophysics 57, 335 (2019), https://doi.org/10.1146/annurev-astro-091918-104423 .
34. W.-R. Hu and Y.-L. Wu, National Science Review 4, 685 (2017), https://academic.oup.com/nst/article-pdf/4/5/685/31566708/nwx116.pdf .
35. J. Luo, L.-S. Chen, H.-Z. Duan, Y.-G. Gong, S. Hu, J. Ji, Q. Liu, J. Mei, V. Milyukov, M. Sazhin, C.-G. Shao, V. T. Toth, H.-B. Tu, Y. Wang, Y. Wang, H.-C. Yeh, M.-S. Zhan, Y. Zhang, V. Zharov, and Z.-B. Zhou, Classical and Quantum Gravity 33, 035010 (2016).
Z. Shen, D. Small, B. W. Sohn, J. SooHoo, H. Sun, F. Tazaki, A. J. Tetarenko, P. Tiede, R. P. J. Tilanus, M. Titus, K. Toma, P. Torne, T. Trent, E. Traianou, S. Trippe, I. van Bemmel, H. J. van Langevelde, D. R. van Rossum, J. Wagner, D. Ward-Thompson, J. Wardle, J. Weintraub, N. Wex, R. Wharton, M. Wielgus, G. N. Wong, Q. Wu, D. Yoon, A. Young, K. Young, F. Yuan, Y.-F. Yuan, J. A. Zensus, G.-Y. Zhao, and S.-S. Zhao (EHT Collaboration), Phys. Rev. D 103, 104047 (2021).