SHORT NOTES ON $L^1(\Omega, X)$ WITH INFINITE MEASURE

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Abstract. This study uses the ideas of [11] to provide the dual of $L^1(\mu, X)$ in the positive and $\sigma$–finite cases. This results in elegant necessary and sufficient criteria for weak compactness in $L^1(S, \mu, X)$ in the $\sigma$–finite case, using the ideas of [5] and [4]. Finally, the result of [10] is extended to compute the sun-dual of $L^1(\mathbb{R}, X)$ with respect to the canonical translation semigroup, dropping the approximation property from $X^{**}$, which is applied to obtain almost periodicity for integrals of non-smooth functions. Moreover, for evolution semigroups, it is shown that weak compactness of the orbits implies strong stability.

1. Introduction

The subject of this study is to extend the results from finite to infinite measure cases and provide some applications to $C_0$-semigroups. To obtain the results, we use a representation of the linear functionals on $L^1(S, X)$. From [2], we know that there are isometries, but the translation of the conditions does not appear obvious. For this, the duality, weak compactness and an application to $C_0$–semigroup theory are given.

To obtain relatively weak compactness, previously, the conditions in [7] and [4] are provided, which are either not necessary or aim at a different topology. The proof in this study follows in the sufficiency part in the idea of [5] to omit condition (2) of [7, Thm. 1] and replace it with the condition on $\text{co} \{f_n : n \geq k\}$, which was used by [5]. In the necessity part, an idea given in [4] is applied. The book [4] is restricted to local compactness, and the dual Banach space was assumed to have the RNP. Further relative $\sigma(L^1(\Omega, X^*), L^\infty(\Omega, X))$–compactness was considered. This is surely weaker than the relative $\sigma(L^1(\Omega, X^*), L^\infty(\Omega, X^{**}))$–compactness if the underlying range space is a dual. Additionally, it is well known that in the general Banach space case, it is not sufficient to consider the functionals coming with $L^\infty(\Omega, X^{**})$. Moreover, we restrict to one arbitrary chosen coverage $\bigcup_{n=1}^\infty A_n = \Omega$. Under these prerequisites, we will provide necessary and sufficient criteria on relatively weak compactness for general Banach spaces $X$, and infinite but $\sigma$–finite and positive measure spaces $(\Omega, \Sigma, \mu)$. The results above are applied with the use of sun-dual semigroups. Recall that for a $C_0$–semigroup,

$$X^\ominus = \left\{ x^* \in X^* : \lim_{h \to 0} T^*(h)x^* = x^* \right\}. $$

The translation semigroup on the bounded uniformly continuous functions is a well-known $C_0$–semigroup. Using the representation of the linear functionals on $L^1(\mathbb{R}, X)$, it was shown that in general, $BUC(\mathbb{R}, X^*) = L^1(\mathbb{R}, X)^\ominus$. This was previously proven under the assumption of $X^*$ having the approximation property [10 Ass. 7.3.3, pp. 131-132]. Additionally, an application of the weak compactness result gives the equality of weak and strong stability for so-called evolution semigroups, compare [4].

2. Representations of linear functionals and consequences

Throughout this study, $\mu : \Sigma \to \mathbb{R}$ is assumed to be a positive and $\sigma$–finite measure.

Key words and phrases. $\sigma$–finite measures, duality, weak compactness, dual semigroups.
Next, we want to consider weak compactness in this measure space, which leads to the following definitions.

**Definition 2.1.** Let $X$ be a Banach space and $\Gamma \subset X^*$ and $(\Omega, \Sigma, \mu)$ be a measure space. Then, $u : \Omega \to X$ is called $\Gamma$-measurable if $x^*(u(\cdot)) : \Omega \to \mathbb{C}$ is $\mu$-measurable. In the case of $X = Y^*$ and $\Gamma = X$, $X$-measurability is called $w^*$-measurability.

We cite the general result for $p \in [1, \infty)$, but we provide a proof for $p = 1$ using [11, Thm. 5.1]. This will provide the main ideas and is sufficient for usage in this study.

**Theorem 2.2.** [2, Thm. 1.5.4, p. 23] Let $X$ be a Banach space and $\Omega$ be a finite measure space. (1 ≤ $p < \infty$), and $q$ is the conjugate of $p$. Then, for each $\varphi \in L^p(\Omega, \Sigma, \mu, X)^*$, we have for some $g : \Omega \to X^*$ with

1. $g$ is $w^*$-measurable.
2. The function $\{\omega \to \|g(\omega)\|\} \in L^q(\mu)$.
3. $\varphi(f) = \int_{\Omega} < g(\omega), f(\omega) >_{X^*, X} \, d\mu(\omega)$.
4. $\|\varphi\| = \|\|g(\cdot)\||_q$.

Using some standard arguments from topology and functional analysis, we obtain the following.

**Corollary 2.3.** Let $X$ be a Banach space and $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. Then, for each $\varphi \in L^1(\Omega, \Sigma, \mu, X)^*$, for some $g : \Omega \to X^*$, we have

1. $g$ is $w^*$-measurable.
2. The function $\{\omega \to \|g(\omega)\|\} \in L^\infty(\mu)$.
3. $\varphi(f) = \int_{\Omega} < g(\omega), f(\omega) >_{X^*, X} \, d\mu(\omega)$.
4. $\|\varphi\| = \|\|g(\cdot)\||_\infty$.

Next, we want to consider weak compactness in this measure space, which leads to the following definitions.

**Definition 2.4.** Let $X$ be a Banach space and $(\Omega, \Sigma, \mu)$ be a finite measure space. Then, $H \subset L^1(\mu, X)$ is called uniformly integrable if

$$\lim_{\mu(E) \to 0} \int_E \|f\| \, d\mu = 0 \text{ uniformly for } f \in H.$$ 

Additionally, we generalize equi-integrability, which is considered in the scalar-valued case [8, IV.15.54, p. 547] and in the infinite-dimensional case [4].

**Definition 2.5.** Let $H \subset L^1(\Omega, X)$.

1. Then, $H$ is $A$-equi-integrable if there exists $\sigma$-finite coverage $A = \{A_n\}_{n \in \mathbb{N}}$, with $A_n \subset A_{n+1}$, $\Omega = \bigcup_{k=1}^\infty A_k$, such that
   (a) $H$ is uniformly integrable,
   (b) there exist $\{a_n\}_{n \in \mathbb{N}} \in l^1(\mathbb{N})^+$, such that for all $f \in H$, and
   $$\int_{A_n \setminus A_{n-1}} \|f\| \, d\mu \leq a_n.$$

2. Then, $H$ is $\Omega$-equi-integrable if for every $\sigma$-finite coverage $A = \{A_n\}_{n \in \mathbb{N}}$, with $A_n \subset A_{n+1}$, $\Omega = \bigcup_{k=1}^\infty A_k$, we have
   (a) $H$ is uniformly integrable,
   (b) there exist $\{a_n\}_{n \in \mathbb{N}} \in l^1(\mathbb{N})^+$, such that for all $f \in H$, and
   $$\int_{A_n \setminus A_{n-1}} \|f\| \, d\mu \leq a_n.$$
Remark 2.6. Let $A = \{A_n\}_{n \in \mathbb{N}}, \sigma$–finite coverage with $A_n \subset A_{n+1}$, $\Omega = \bigcup_{k=1}^{\infty} A_k$. Then,

$$\sum_{n=k}^{\infty} \int_{A_n \setminus A_{n-1}} \|f\|d\mu = \int_{\Omega \setminus A_k} \|f\|d\mu.$$ 

Following the proof of [5, Thm 2.1] and [4] obtains the elegant result for relatively weak compactness in $L^1(\Omega, \Sigma, \mu, X)$.

Theorem 2.7. Let $\mu$ be a positive Borel measure on a $\sigma$–finite measure space and $H$ be a bounded subset of $L^1(S, \mu, X)$. Then, the following are equivalent:

1. $H$ is weakly relatively compact.
2. $H$ is $A$–equi-integrable, and given any sequence $\{f_n\}_{n \in \mathbb{N}} \subset H$, there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ with $g_n \in \text{co} \{f_k : k \geq n\}$ such that $\{g_n(\omega)\}_{n \in \mathbb{N}}$ is norm convergent in $X$ for a.e. $\omega \in \Omega$.
3. $H$ is $A$–equi-integrable, and given any sequence $\{f_n\}_{n \in \mathbb{N}} \subset H$, there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ with $g_n \in \text{co} \{f_k : k \geq n\}$ such that $\{g_n(\omega)\}_{n \in \mathbb{N}}$ is weakly convergent in $X$ for a.e. $\omega \in \Omega$.

Remark 2.8. In the previous Thm [2.7], $A$–equi-integrability can be replaced by $\Omega$–equi-integrability. As the proof of (1) implies (3), the coverage is arbitrary.

Remark 2.9. Surely, the finite measure case is a consequence when choosing $A_n = \Omega$ for all $n \in \mathbb{N}$.

3. Applications

To discuss the translation semigroup, recall the following conclusion from the representation of $L^1(\mathbb{R}, X)^*$.

Remark 3.1. Let $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$ and $\mu$ be Lebesgue measures such that for all $f \in \text{BUC}(\mathbb{J})$, the duality of $(L^1(\mathbb{R}^+, X), \text{BUC}(S, X^*))$ is given by

$$< g, f > = \int_{\mathbb{J}} < g(r), f(r) > d\mu(r).$$

For $t \in \mathbb{J}$, the translation operator

$$T(t) : L^\infty_{w^*}(\mathbb{J}, X^*) \rightarrow L^\infty_{w^*}(\mathbb{J}, X^*)$$

$$f \mapsto \{s \mapsto f(t+s)\},$$

is the dual operator to

$$V(t) : L^1(\mathbb{J}, X) \rightarrow L^1(\mathbb{J}, X)$$

$$f \mapsto \left\{ s \mapsto \begin{cases} f(s-t) : s \in t + \mathbb{J} \\ 0 : \text{otherwise} \end{cases} \right\}.$$ 

As $\{V(t)\}_{t \in \mathbb{J}}$ is a $C_0$–semigroup, it has a generator $B$, and $V^*(t) = T(t)$.

Using the representation of the linear functionals, we extend the result of [10] for the translation semigroup $L^1(\mathbb{R}, X) = \text{BUC}(\mathbb{R}, X^*)$, which was proven under the condition that $X^*$ possesses the a.p. In the following, we drop this condition.

Lemma 3.2. Let $X$ be a Banach space $\{V(t)\}_{t \in \mathbb{R}}$ and the translation semigroup on $L^1(\mathbb{R}, X)$. Then, $L^1(\mathbb{R}, X) = \text{BUC}(\mathbb{R}, X^*)$. 

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Proof of Cor. 3.2. By the previous theorem, we have $L^1(\mathbb{R}, X)^* \subset L^{\infty}(\mathbb{R}, X^*)$. Further, $z \in X^*$, for $g \in L^1(\mathbb{R}, X)^\odot$

$$\|<g(\omega + h), z > -<g(\omega), z >\|_{ess-\infty} = \sup_{h \in B_{L^1(\mu)}} \int_{\Omega} |<g(\omega + h), z > -g(\omega), z >| \, h(\omega) d\omega$$

for every $z \in X^*$. Hence, $<g(\cdot), z >$ fulfills the scalar-valued requirement for $L^1(\mu)^\odot = BUC(\mathbb{R})$, [10] Exa 1.3.9, p.8], which yields $<g(\cdot), z >$ uniformly continuous for all $z \in X^*$

Hence, we found $g_z \in L^{\infty}(\mathbb{R}, X)$ defined on every $t \in \mathbb{R}$, with $<g_z, z > \in BUC(\mathbb{R})$, and $<x, g_z(\cdot) > =<x, g_z(\cdot) > =<x, g(\cdot) >, \text{a.e for all } y, z_1, z_2 \in X$.

We claim that $g \in C(\mathbb{R}, X^*)$. If not, there is a point of discontinuity $t_0$ and null-sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, such that

$$\inf_{n \in \mathbb{N}} \|g(t_0 + t_n) - g(t_0)\| \geq \epsilon > 0.$$ 

For some $\{z_n\}_{n \in \mathbb{N}} \subset B_X$

$$\|g(t_0 + t_n) - g(t_0)\| - \frac{1}{n} = |<g_{z_n}(t_0 + t_n) - g_{z_n}(t_0), z_n>| \leq \sup_{h \in B_{L^1(\mu)}} \int_{\Omega} |<g(\omega + t_n), z_n > -g(\omega), z_n >| \, h(\omega) d\omega \leq \sup_{z \in B_X, h \in B_{L^1(\mu)}} \int_{\Omega} |<g(\omega + h), z > -g(\omega), z >| \, h(\omega) d\omega \leq \sup_{h \in B_{L^1(\mu, X)^*}} \int_{\Omega} |<g(\omega + h), z > -g(\omega), h(\omega) >| \, d\omega \to 0,$$

whereby the convergence comes with the definition of $g \in L^1(\mathbb{R}, X)^\odot$, which yields the contradiction.

Next we present an application of this representation. Using the definitions in Remark 3.2 and applying Corollary 3.2 and [11] Thm. 4.6.11, Lemma 4.6.13], we obtain

Corollary 3.3. Assume $c_0 \not\subset X^*$, let $f \in L^{\infty}(\mathbb{R}, X^*)$, and let $\{V^*(t)\}_{t \in \mathbb{R}} \subset L(L^{\infty}(\mathbb{R}, X^*))$ be the dual translation group. Furthermore, if $V^*(t)f - f \in AP(\mathbb{R}, X^*)$, or $V^*(t)R(\lambda, B)^*f - R(\lambda, B)^*f \in AP(\mathbb{R}, X^*)$ for all $t \in \mathbb{R}$, then $R(\lambda, B)^*f \in AP(\mathbb{R}, X^*)$.

Proof. Let $f \in L^{\infty}(\mathbb{R}, X^*)$ and $V^*(t)f - f \in AP(\mathbb{R}, X^*)$. Then,

$V^*(t)R(\lambda, B)^*f - R(\lambda, B)^*f = V^\odot R(\lambda, B)^*f - R(\lambda, B)^*f \in AP(\mathbb{R}, X^*)$

As $R(\lambda, B)^*f \in D(A^*) \subset L^1(\mathbb{R}, X)^\odot = BUC(\mathbb{R}, X^*)$. Hence, an application of [11] Thm. 4.6.1, p. 298] yields $R(\lambda, B)^*f \in AP(\mathbb{R}, X)$.

In a similar way, the result of Kadets can be extended.

Theorem 3.4. Assume that $c_0 \not\subset X^*$, and let $f \in L^{\infty}(\mathbb{R}, X^*)$, with $R(\lambda, B)^*f \in AP(\mathbb{R}, X^*)$. If for all $\lambda > 0$, the integrals $\int_0^t R(\lambda, B)^*f(s)ds$ are bounded, then

$$w^* - \int_0^t f(s)ds \in AP(\mathbb{R}, X^*).$$
Proof. \( f \in L^\infty_w(\mathbb{R}, X) \). Then, \( R(\lambda, B^*) f \in BUC(\mathbb{R}, X^*) \), and by \cite{[3]} Lemma 4.6.13, p. 300, we have \( \int_0^t R(\lambda, B^*) f(s) ds \in AP(\mathbb{R}, X^*) \) for all \( \lambda > 0 \). Then,
\[
< x, \int_0^t R(\lambda, B) f(r) dr > = \int_0^\infty e^{-\lambda s} < x, f(s + r) > dt dr = \int_0^\infty e^{-\lambda s} \int_0^t < x, f(s + r) > dr ds = \int_0^\infty e^{-\lambda s} \int_s^{s+t} < x, f(r) > dr ds.
\]

As \( \{ s \mapsto w^* - \int_s^{s+t} f(r) dr \} \subseteq BUC(\mathbb{R}, X^*), \) and \( V^*(t) w^* - w^* \)-continuous, we have for the \( w^* \)-integrals
\[
\int_s^{s+t} f(r) dr - \int_s^r f(r) dr = \int_s^r f(r) dr - \int_t^{r+t} f(r) dr = \int_s^r f(r) dr - \int_0^r f_t(r) dr = \delta_0 \left( (I - V^*(t))(\int_s^r f(\cdot + r) dr) \right),
\]
and
\[
\left\| \delta_0 \left( (I - V^*(t))(\int_s^r f(\cdot + r) dr) \right) \right\| \leq \| I - V^*(t) \| \| f \|_\infty |s - t|\]

Consequently, \( \{ w^* - \int_s^{s+t} f(r) dr \} \) is an equi-Lipschitz family, and partial integration gives
\[
AP(\mathbb{R}, X^*) \ni \int_0^\infty \lambda R(\lambda, B) f(r) dr = \int_0^\infty e^{-\lambda s} \int_s^{s+t} f(r) dr ds = w^* - \int_0^\infty f(s) ds,
\]
uniformly for \( \lambda \to \infty \), which concludes the proof. \( \square \)

As an application, so-called evolution semigroups are considered. For a detailed discussion, see \cite{[3]}.

**Theorem 3.5.** Let \( \{ S(t) \}_{t \in \mathbb{R}^+} \subseteq L(X) \) be a bounded \( C_0 \)-semigroup and
\[
T(t) : L^1(\mathbb{R}, X) \longrightarrow L^1(\mathbb{R}, X),
\]
\[
f \longmapsto \{ s \mapsto S(t)f(t - s) \}
\]
If \( \{ T(t) \}_{t \in \mathbb{R}^+} \) is Eberlein weakly almost periodic (i.e., \( \{ T(t)f \}_{t \geq 0} \) is weakly relatively compact for all \( f \in L^1(\mathbb{R}, X) \)), then \( \{ T(t) \}_{t \in \mathbb{R}^+} \) is strongly stable (i.e., \( \lim_{t \to \infty} T(t)f = 0 \) for all \( f \in L^1(\mathbb{R}, X) \)).

**Proof.** Consider the simple function \( f(t) := \chi_{[1,1]}(t)x; \) then, by the assumption that \( \{ T(t)f \}_{t \geq 0} \) is weakly relatively compact, and the equi-integrability given by Remark \( \cite{[2,6]} \)
\[
\int_{|s| > n} \| S(t) \chi_{[-1,1]}(t - s)x \| ds \to 0 \text{ uniformly for } t \geq 0.
\]
The identity
\[
\int_{|s| > n} \| S(t) \chi_{[-1,1]}(t - s)x \| ds = \| S(t)x \| \int_{|t - [-1,1]| \geq n} 1 ds
\]
and using the uniform in \( t \in \mathbb{R}^+ \), we have \( \| S(t)x \| \to 0 \) Hence, \( \{ T(t) \}_{t \in \mathbb{R}^+} \) becomes strongly stable and therefore \( \{ T(t) \}_{t \in \mathbb{R}^+} \) on the simple function and by its boundedness on all of \( L^1(\mathbb{R}, X) \). \( \square \)
4. Proofs

**Proof of Thm. 2.2.** For the case of \( p = 1 \), we present a proof based on [11 Thm. 5.1]. For \( \varphi \in L^1(\Omega, X)^* \), we consider

\[
\begin{align*}
G &: \Sigma \longrightarrow X^* \\
E &\mapsto \{ x \mapsto \varphi(x\chi_E) \}.
\end{align*}
\]

Since

\[
|\sum_{i=1}^n G(E_i)(x_i)| = |\varphi(\sum_{i=1}^n \chi_{E_i} x_i)| \leq |\varphi| \left\| \sum_{i=1}^n \chi_{E_i} x_i \right\|_1
\]

we find that \( G \) has values in \( X^* \) and is countably additive. To verify the bounded variation, let \( \{E_i\}_{i=1}^n \) be a partition and \( \{x_i\}_{i=1}^n \subset B_X \). Then,

\[
|\sum_{i=1}^n G(E_i)(x_i)| = |\varphi(\sum_{i=1}^n \chi_{E_i} x_i)| \leq |\varphi| \left\| \sum_{i=1}^n \chi_{E_i} x_i \right\|_1
\]

\[
= |\varphi| \int_{\Omega} \left\| \sum_{i=1}^n \chi_{E_i} (\omega) x_i \right\| d\mu(\omega) \leq |\varphi| \sum_{i=1}^n |x_i| \mu(E_i)
\]

\[
\leq |\varphi| \mu(\Omega).
\]

Following [11 Thm. 5.1] with \( V = (X^*, w^*) \), it remains to verify in his notation

\[
A_\Omega(G) = \left\{ \frac{G(F)}{\mu(F)} : F \subset \Omega, \mu(F) > 0 \right\}
\]

is bounded. By [11], we have \( A_\Omega(G) \subset ||\varphi||B_X \). Hence, we find by [11 Thm. 5.1] a \( w^* \)-integrable function \( g \), such that \( G(E) = w^* - \int_E g d\mu \), for all \( E \in \Sigma_n \), and the range of \( g \) is a subset of \( A_\Omega(G) \subset ||\varphi||B_{X^*} \), which yields \( ||g||_\infty \leq ||\varphi|| \). Plainly, \( ||g(\cdot)|| \in L^\infty(\Omega) \), and if \( f \in L^1(\Omega, X) \) is a simple function, then \( \varphi(f) = \int_{\Omega} <g, f >_{X^*, X} d\mu \). Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of simple functions with the limit \( f \). Then, for a subsequence \( g(\omega), f(\omega) = \lim_{n \to \infty} <g(\omega), f_n(\omega) > \), the limit is measurable,

\[
|\int_{\Omega} <g, f_n > d\mu - \int_{\Omega} <g, f > d\mu| \leq ||\varphi|| \int_{\Omega} |f_n - f| |d\mu|
\]

and

\[
\varphi(f) = \lim_{k \to \infty} \varphi(f_n) = \lim_{k \to \infty} \int_{\Omega} <g, f_n > d\mu = \int_{\Omega} <g, f > d\mu
\]

concludes the proof.

**Proof of Corollary 2.3.** Let \( \varphi : L^1(\Omega, X) \to \mathbb{C} \) be a linear functional and \( \bigcup A_n = \Omega \) be a \( \sigma \)-finite cover, with \( A_n \subset A_{n-1} \). Then, we find by the previous result, for the restrictions of

\[
\varphi|_{A_n \setminus A_{n-1}} = \varphi_n : L^1(A_n \setminus A_{n-1}, X) \to \mathbb{C},
\]

a sequence \( \{\tilde{g}_n\}_{n \in \mathbb{N}} : A_n \setminus A_{n-1} \to ||\varphi||B_{X^*} \), such that \( \tilde{\varphi}_n(f) = \int_{A_n \setminus A_{n-1}} <\tilde{g}_n(\omega), f(\omega) > d\mu(\omega) \), for all \( f \in L^1(A_n, X) \). Note that with \( A_{-1} = \emptyset \),

\[
\varphi(f\chi_{A_n}) = \sum_{k=1}^n \varphi(f\chi_{A_k \setminus A_{k-1}}) = \sum_{k=1}^n \tilde{\varphi}(f\chi_{A_k \setminus A_{k-1}})
\]

With this observation, we define

\[
g_n(\omega) := \begin{cases} \tilde{g}_k(\omega) & : \omega \in A_n \setminus A_{n-1} \\
0 & : otherwise, \end{cases}
\]
Thus, for given $A \tilde{<} \{ h \}$ the continuous linear functionals $\tilde{g}$ to $g$ so that $g \in L^\infty_\mu(\Omega, X^*)$, and

$$\int \sum_{k=1}^n < g_k, f > d\mu = \varphi(f \chi_{A_n}).$$

For $f \in L^1(\Omega, X)$, we have

$$|\varphi(f) - \int_{\Omega} < g, f > d\mu| \leq |\varphi(f) - \sum_{n=1}^k \varphi_n(f)| + |\sum_{n=1}^k \varphi_n(f) - \int_{\Omega} \sum_{n=1}^k < g_n, f > d\mu| + \int_{\Omega} |f| d\mu \leq C \int_{\Omega} ||f|| d\mu.$$

for some $C > 0$, which proves the representation. For $||\varphi|| = ||g(\cdot)||_{\infty}$, apply the Hölder inequality, which concludes the proof.

Proof of Thm. 2.7. For sufficiency up to the $A-$equi-integrability estimation, we may precisely follow the proof of [5]. We provide the proof (3) that implies (1). Following Eberlein, Smul'yan, and Grothendieck, we have to show that given $\{ f_m \}_{m \in \mathbb{N}} \subset H$ and $\{ \varphi_n \}_{n \in \mathbb{N}} \subset B_{L^1(\mu, X^*)}$, we have

$$\alpha := \lim_{n \to \infty} \lim_{m \to \infty} \varphi_n(f_m) = \lim_{m \to \infty} \lim_{n \to \infty} \varphi_n(f_m) =: \beta,$$

provided that the iterated limits exit [9] Cor. 1 of Thm 7. Given such sequences $\{ f_m \}_{m \in \mathbb{N}}$ and $\{ \varphi_n \}_{n \in \mathbb{N}}$, let $\{ g_n \}_{n \in \mathbb{N}}$ be the sequence associated with $\{ f_m \}_{m \in \mathbb{N}}$ according to (3), $E \in \Sigma, \mu(E) = 0$, the exceptional subset of $\Omega$. Define

$$g(\omega) := \begin{cases} w - \lim_{n \to \infty} g_n(\omega) : & \omega \in \Omega \setminus E \\ 0 : & \text{otherwise} \end{cases},$$

Clearly, $g$ is essentially separably valued and weakly measurable and hence strongly measurable. Moreover, by its very definition and using Fatou's Lemma and boundedness of $H$,

$$\int ||g|| d\mu \leq \int \liminf ||g_n|| d\mu \leq \liminf \int ||g_n|| d\mu < \infty,$$

so that $g \in L^1(\mu, X)$. Let us show that the sequence $\{ g_n \}_{n \in \mathbb{N}}$ converges weakly in $L^1(\mu, X)$ to $g$. First, note that we can assume $L^1(\mu)$ to be separable. Then, according to Thm. 2.2 the continuous linear functionals $\tilde{h}$ on $L^1(\mu, X)$ are represented by $w^*$-measurable functions $h : \Omega \to X^*$ such that $||h(\cdot)|| \in L^\infty(\mu)$, the pairing being given by

$$< \tilde{h}, f > = \int < h, f > d\mu, \quad f \in L^1(\mu, X).$$

Given any such $h$, uniform integrability of the sequence $\{ < h, g_n > \}_{n \in \mathbb{N}}$ and the fact that $< h(\cdot), g_n(\cdot) > \to < h(\cdot), g(\cdot) >$ a.e. $\Omega$, in conjunction with Vitali's convergence theorem imply that for every $A_n$

$$\int_{A_n} |< h(s), g_n(s) > \to < h(s), g(s) >| d\mu(s) \to 0. \quad (2)$$

Thus, for given
\[
\int_S |< h(s), g_n(s) > - < h(s), g(s) >| \, d\mu(s)
\]
\[
\leq \int_{A_n} |< h(s), g_n(s) > - < h(s), g(s) >| \, d\mu(s) + \int_{S \setminus A_n} |< h(s), g_n(s) > - < h(s), g(s) >| \, d\mu(s)
\]
\[
\leq \int_{A_n} |< h(s), g_n(s) > - < h(s), g(s) >| \, d\mu(s) + C \left\{ \int_{S \setminus A_n} \| g_n \| \, d\mu + \int_{S \setminus A_n} \| g \| \, d\mu \right\}.
\]

Again Fatou’s lemma yields
\[
\int_{S \setminus A_n} \| g \| \, d\mu \leq \lim inf \int_{S \setminus A_n} \| g_n \| \, d\mu,
\]
and we find an \( n \in \mathbb{N} \), such that
\[
\begin{bmatrix}
\int_{S \setminus A_n} \| g_n \| \, d\mu + \int_{S \setminus A_n} \| g \| \, d\mu
\end{bmatrix} \leq \frac{\varepsilon}{2} \text{ uniformly for } n \in \mathbb{N}.
\]

An application of (2) leads to weak convergence in \( L^1(S, \mu, X) \). To complete the implication, note that if \( h_m = \sum_1^r \alpha_i f_{k_i} \) is any convex combination of \( \{ f_m \}_{m \in \mathbb{N}} \) and \( \varphi \in L^1(S, \mu, X) \) such that \( \gamma := \lim_{m \to \infty} \varphi(f_m) \) exists, then
\[
| \gamma - \varphi(h_m) | = | \sum_1^r \alpha_i (\gamma - \varphi(f_{k_i})) | \leq \max \{ | \gamma_i - \varphi(f_{k_i}) | : 1 \leq i \leq r \}.
\]

This shows that \( \alpha = \varphi(g) = \beta_i \), where \( \varphi \) is a \( w^* \)-cluster point of \( \{ \varphi_n \}_{n \in \mathbb{N}} \), thus completing that proof (3) implies (1).

Implication (1) implies (3); note that uniform integrability comes with [7, p. 177]. It remains to verify (2) of the definition [2, 3]. Assuming the contrary to part 2 of the definition of \( A \)-equi-integrable, for all positive \( \{ a_n \}_{n \in \mathbb{N}} \in l^1(\mathbb{N}) \), we find an \( f \) such that
\[
\int_{A_n \setminus A_{n-1}} \| f \| \, d\mu \geq a_n.
\]

Choosing \( a_n^k = \varepsilon \) and \( a_k^\epsilon = 0 \), for \( k \neq n \in \mathbb{N} \), we find an \( f_n \in A \), such that
\[
\int_{A_n \setminus A_{n-1}} \| f \| \, d\mu \geq a_n = \varepsilon.
\]

Then using Hahn-Banach, we find \( \psi_k \in L^1(A_k \setminus A_{k-1}, X)^* \) with norm 1,
\[
a_k \leq \int_{A_n \setminus A_{n-1}} \| \tilde{f}_k \| \, d\mu = < \psi_k, f_k >
\]

By Theorem [22] we find \( g_k \in L^\infty(A_k \setminus A_{k-1}, X^*) \), such that \( \| g_k \| \leq 1 \), and
\[
a_k \leq \int_{A_n \setminus A_{n-1}} \| \tilde{f}_k \| \, d\mu = < \psi_k, f_k > = \int_{A_k \setminus A_{k-1}} < g_k, f_k > \, d\mu.
\]

Now, we consider the mapping
\[
T : L^1(S, \mu, X) \longrightarrow l^1(\mathbb{N})
\]
\[
f \longmapsto \left\{ \int_{A_k \setminus A_{k-1}} < g_k, f > \, d\mu \right\}_{k \in \mathbb{N}}
\]
and
\[
T(f_n) = \left\{ \int_{A_k \setminus A_{k-1}} < g_k, f_n > \, d\mu \right\}_{k \in \mathbb{N}}
\]
which fails to be uniformly summable for $\varepsilon_1 = \varepsilon/2$. However, due to the weak conditional compactness of $\{f_n\}_{n \in \mathbb{N}}$, the sequence $\{Tf_n\}_{n \in \mathbb{N}}$ is conditionally compact, a contradiction to [8] [IV.13.3].

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