Homological Stability Among Moduli Spaces of Holomorphic Curves in $\mathbb{C}P^n$

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Abstract

The primary goal of this paper is to find a homotopy theoretic approximation to $M_{\text{deg}}^d(\mathbb{C}P^n)$, the moduli space of degree $d$ holomorphic maps of genus $g$ Riemann surfaces into $\mathbb{C}P^n$. There is a similar treatment of a partial compactification of $M_{\text{deg}}^d(\mathbb{C}P^n)$ of irreducible stable maps in the sense of Gromov-Witten theory. The arguments follow those from a paper of G. Segal ([Seg79]) on the topology of the space of rational functions.

1 Introduction

Throughout this paper, all maps between topological spaces will be taken to be continuous and all diagrams commutative unless otherwise stated. The symbol $H_q(-)$ will be used to denote the $q^{\text{th}}$ rational homology group of $-$. 

1.1 Background and Statement of Main Result

A Riemann surface is a pair $(F_g, J)$ where $F_g \subset \mathbb{R}^\infty$ is a smooth oriented surface of genus $g$ embedded in $\mathbb{R}^\infty$, and $J$ is a complex structure on $F_g$ which agrees with given the orientation on $F_g$. Write $J_g$ for the space of such complex structures on $F_g$; a description of the topology of $J_g$ is postponed to §2.2. Also shown in §2.2 is the contractibility of $J_g$.

Let $Map^d(F_g, \mathbb{C}P^n)$ denote the space of continuous maps from $F_g$ to $\mathbb{C}P^n$ having homological degree $d$ and endowed with the compact open topology. Let $Hol^d((F_g, J), \mathbb{C}P^n)$ denote the subspace of $Map^d(F_g, \mathbb{C}P^n)$ consisting of holomorphic maps from the Riemann surface $(F_g, J)$ to $\mathbb{C}P^n$.

Theorem 1.1.1 (Segal [Seg79]). For $g > 0$, the inclusion $Hol^d((F_g, J), \mathbb{C}P^n) \hookrightarrow Map^d(F_g, \mathbb{C}P^n)$ induces an isomorphism in $H_q(-; \mathbb{Z})$ for $q < (d-2g)(2n-1)$. 

This paper is devoted to proving such a theorem as the complex structure $J$ is allowed to vary. This will amount to a statement about the moduli space of Riemann surfaces which will be defined presently.

For a fixed surface $F_g$, consider the set of pairs

$$J^d_g(CP^n) := \{(J, h) \mid J \in J(F_g) \text{ and } h \in Hol^d((F_g, J), CP^n)\}.$$  

This set $J^d_g(CP^n)$ is a subset of the product space $J_g \times Map^d(F_g, CP^n)$ and inherits a subspace topology. Let $Diff^+_g$ be the topological group of orientation preserving diffeomorphisms of $F_g$ endowed with the Whitney $C^\infty$ topology. There is a continuous action of $Diff^+_g$ on $J^d_g(CP^n)$ given by pulling back the complex structure and precomposing maps. Define the *moduli space of degree $d$ genus $g$ holomorphic curves in $CP^n$* as the resulting quotient

$$\mathcal{M}^d_g(CP^n) := J^d_g(CP^n)/Diff^+_g.$$  

Similarly, consider the orbit space

$$\mathcal{M}T^d_g(CP^n) := (J_g \times Map^d(F_g, CP^n))/Diff^+_g;$$

referred to as the *topological moduli space of degree $d$ genus $g$ curves in $CP^n$*. The distinction between $\mathcal{M}^d_g(CP^n)$ and $\mathcal{M}T^d_g(CP^n)$ is that the first involves holomorphic maps while the latter merely continuous maps.

More well-behaved are the homotopy orbit spaces

$$\mathcal{M}^d_g(CP^n)^h := J^d_g(CP^n)/\!\!/Diff^+_g$$

and

$$\mathcal{M}T^d_g(CP^n)^h := J_g \times Map^d(F_g, CP^n)/\!\!/Diff^+_g.$$  

**Theorem 1.1.2** (Main Theorem). The map

$$\mathcal{M}^d_g(CP^n)^h \rightarrow \mathcal{M}T^d_g(CP^n)^h$$

induced by the natural inclusion induces an isomorphism in $H_q(\cdot)$ for $q < (d-2g)(2n-1)$.

**Remark.** Theorem 1.1.2 can be viewed as a statement about families. Indeed, in the appropriate sense, it says that any $q$-dimensional family of curves in $CP^n$ is concordant to a family of holomorphic curves provided $q < (d-2g)(2n-1)$.  

\[\square\]
Recall the action of $\text{Diff}_g^+$ on $\mathcal{J}_g$. The isotropy subgroups $(\text{Diff}_g^+)_J$ are finite for each $J \in \mathcal{J}_g$. Moreover, the diagonal action of $\text{Diff}_g^+$ on $E\text{Diff}_g^+ \times \mathcal{J}_g$ is by slices. The contractability of $\mathcal{J}_g$ mentioned above then results in the product $E\text{Diff}_g^+ \times \mathcal{J}_g$ rationally being a model for $E\text{Diff}_g^+$.

It follows that the projection maps

$$\text{Map}^d(F_g, \mathbb{C}P^n)//\text{Diff}_g^+ \simeq \mathcal{MT}_g^d(\mathbb{C}P^n)^h \to \mathcal{MT}_g^d(\mathbb{C}P^n)$$

and

$$\mathcal{M}_g^d(\mathbb{C}P^n)^h \to \mathcal{M}_g^d(\mathbb{C}P^n)$$

induce rational homotopy equivalences and thus equivalences in rational singular homology. There is an immediate corollary.

**Corollary 1.1.3.** Induced by the obvious inclusion, the map

$$\mathcal{M}_g^d(\mathbb{C}P^n) \to \mathcal{MT}_g^d(\mathbb{C}P^n)$$

induces an isomorphism in $H_q(\_)$ for $q < (d - 2g)(2n - 1)$.

**Remark.** As described in §2.2, the space $\mathcal{J}_g$ of complex structures on $F_g$ is canonically homeomorphic to the space of almost-complex structures on $F_g$. It follows that the topological moduli space $\mathcal{MT}_g^d(\mathbb{C}P^n)$, and better yet $\mathcal{MT}_g^d(\mathbb{C}P^n)^h$, is entirely homotopy-theoretic.

**Corollary 1.1.4.** Induced by a zig-zag of maps of spaces is an isomorphism

$$H_q(\mathcal{M}_g^d(\mathbb{C}P^n)) \cong H_q(\text{Map}^d(F_g, \mathbb{C}P^n)//\text{Diff}_g^+)$$

for $q < (d - 2g)(2n - 1)$.

There is the following useful group completion argument due to Cohen and Madsen.

**Theorem 1.1.5 (CM06).** For $X$ simply connected and for $h_*$ any generalized homology theory, there is a map

$$\text{Map}^d(F_g, \mathbb{C}P^n)//\text{Diff}_g^+(F_g) \to \Omega^\infty(\mathbb{C}P_1^\infty \wedge X_+)$$

which induces an isomorphism in $h_q$ for $q > (g - 5)/2$.

**Corollary 1.1.6.** Induced by a zig-zag of maps of spaces is an isomorphism

$$H_q(\mathcal{M}_g^d(\mathbb{C}P^n)) \cong H_q(\Omega^\infty(\mathbb{C}P_1^\infty \wedge \mathbb{C}P^n))$$

for $q < (d - 2g)(2n - 1)$ and $q < (g - 5)/2$. 

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The rational (co)homology of $\Omega^\infty$-spaces being reasonably well-understood from [DL62], there is the following corollary. To state it takes some setting up. For $V$ a graded vector space over $\mathbb{Q}$ denote by $A(V)$ the free $\mathbb{Q}$-algebra generated by $V$. The $\mathbb{K}$ be the graded vector space over $\mathbb{Q}$ generated by the set $\{k_i\}_{i \geq -1}$ where $k_i$ has degree $2i$. Let $W$ another graded vector space over $\mathbb{Q}$. Denote by $(\mathbb{K} \otimes W)_+$ the part of the vector space $\mathbb{K} \otimes W$ that has positive grading. Recall that $H^q(\mathbb{C}P^n) \cong \mathbb{Q}[c]/c^{n+1}$.

**Corollary 1.1.7.** There is an isomorphism of graded rings through a range

$$H^q(\mathcal{M}_g^d(\mathbb{C}P^n), \mathbb{Q}) \cong A((\mathbb{K} \otimes \mathbb{Q}[c])_+).$$

for $q < (d - 2g)(2n - 1)$ and $q < (g - 5)/2$.

### 1.2 Statement of a Result for the Partial Compactification $\overline{\mathcal{M}}_g^d(\mathbb{C}P^n)$

A more detailed discussion of the relevant terms in this section will be postponed to §3.

In [Seg79], Segal proves a version of Theorem 1.1.1 for irreducible singular Riemann surfaces. His idea was to regard a map from such a singular Riemann surface as a map from the normalization of the surface, with marked points, subject to conditions on the marked points. Such a result is sufficient for the program of the main theorem 1.1.2 to hold and result in a statement about the moduli space $\mathcal{M}_g^d(\mathbb{C}P^n)$ of degree $d$ holomorphic maps from singular surfaces of topological type $[F]$ to $\mathbb{C}P^n$. Denote by $\mathcal{M}_g^d(F)(\mathbb{C}P^n)$ the analogous topological moduli space and similarly for the superscript $h$. In §3 there is the following theorem.

**Theorem 1.2.1.** Let $F$ be an irreducible singular surface. The standard map

$$\mathcal{M}_g^d(F)(\mathbb{C}P^n)^h \to \mathcal{M}_g^d(F)(\mathbb{C}P^n)^h \simeq \text{Map}(F, \mathbb{C}P^n)//\text{Diff}^+(F)$$

induces an isomorphism in $H_q(-)$ for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$.

One might expect from this a statement about a partial compactification of the moduli space $\mathcal{M}_g^d(\mathbb{C}P^n)$ consisting of irreducible stable maps in the sense of Gromov-Witten theory (see [KV99] for example). Write this partial compactification by $\overline{\mathcal{M}}_g^d(\mathbb{C}P^n)$. There is a topological analogue introduced in §3.4 which is denoted $\overline{\mathcal{M}}_g^d(\mathbb{C}P^n)$. There is the following theorem.
Theorem 1.2.2. There is a map
\[ \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \to \overline{\mathcal{M}}_d^t(\mathbb{C}P^n) \]
which induces an isomorphism in \( H_q(-) \) for \( q < (d - 2g)(2n - 1) \).

The right hand term in the above theorem is far more homotopy theoretic. Indeed, it captures the (somewhat simple) combinatorics of the boundary of the partial compactification \( \overline{\mathcal{M}}_d^t(*) \) but retains no holomorphic data.

2 Proof of Theorem 1.1.2

This section is devoted to a proof of Theorem 1.1.2. The argument will follow that of Segal's ([Seg79]) for when the complex structure is fixed. Details will be supplied for \( n = 1 \) in which case the target complex manifold is \( \mathbb{C}P^1 \approx S^2 \). The general situation is not much more difficult as will be outlined later.

2.1 Strategy

The idea is to regard a holomorphic map \( (F_g, J) \to \mathbb{C}P^1 \) as a rational function on \( (F_g, J) \), then to regard a rational function as a pair \( (\eta, \xi) \) of divisors given by its zeros and poles. The degree to which such a pair of divisors is realized in this way from rational function is described by a theorem of Abel’s. Abel’s theorem results in a map from the space of divisors on \( (F_g, J) \) to the Jacobian of \( (F_g, J) \) whose fiber is the space of rational functions on \( (F_g, J) \). This Jacobian is then identified with a standard torus which is independent of \( J \). The resulting sequence is a homology fibration through a range.

There is a comparison fibration to this homology fibration using scanning maps. These scanning maps are shown to be equivalences from which it follows that the space of pairs \( (h, J) \), where \( h \) is a rational function on \( (F_g, J) \), is homology equivalent to \( Map(F_g, S^2) \) through a range. A spectral sequence argument is then in place to have a similar comparison on homotopy quotients by \( Diff^+_g \) and the result follows.

2.2 The Topology of \( J_g \)

On a surface, the Nijenhuis tensor will always vanish simply for dimensional reasons (see [NN57]). Therefore every almost-complex structure on a surface
is integrable and thus comes from an honest complex structure. The space of almost-complex structures on $F_g \subset \mathbb{R}^\infty$ is in bijection with the set of lifts

$$\begin{array}{c}
BU(1) \\
\downarrow \theta \\
F_g \longrightarrow BSO(2).
\end{array}$$

Endow $J_g$ with the topology coming from the subspace topology of the compact open topology on the mapping space $\text{Map}(F_g, BU(1))$. This works well provided we choose a good model for $BSO(2)$ and $BU(1)$ which we will do presently.

Let $\text{Gr}_d^+(\mathbb{R}^N)$ denote the oriented grassmann manifold of $d$-planes in $\mathbb{R}^N$ and similarly for the complex case. Choose $\text{Gr}_d(\mathbb{R}^\infty) := \text{colim}_N \text{Gr}_d^+(\mathbb{R}^N)$ as a model for $BSO(2)$ and similarly $\text{Gr}_1(\mathbb{C}^\infty)$ as a model for $BU(1)$. There is a forgetful map $\theta : BU(1) \to BSO(2)$ given by forgetting the complex structure of $\mathbb{C}^N$. Because $F_g$ is embedded in $\mathbb{R}^\infty$, there is a canonical map $F_g \to BSO(2)$.

Note that the homotopy equivalence $\text{GL}_d^+(\mathbb{R}) \simeq \text{SO}(2) = \text{U}(1) \simeq \text{GL}_1(\mathbb{C})$ yields the space $J_g$ as contractible.

### 2.3 The Homology Fibration

#### Using Abel’s Theorem

Let $V_J$ be the space of holomorphic 1-forms on the Riemann surface $(F_g, J)$. There is the natural inclusion $H_1(F_g; \mathbb{Z}) \hookrightarrow V_J^*$ as a non-degenerate lattice yielding the $g$-torus

$$T_J := V_J^*/H_1(F_g; \mathbb{Z})$$

known as the Jacobian variety of the Riemann surface $(F_g, J)$.

Consider the space of pairs

$$\mathcal{J}_g \times T_J := \{(J, v) \mid v \in T_J\}.$$

Projection onto the first coordinate makes $\mathcal{J}_g \times T_J \to \mathcal{J}_g$ into a fiber bundle with fiber over any $J \in \mathcal{J}_g$ is the Jacobian variety $T_J$.

Once and for all, choose a base point $x_0 \in F_g$. Let $\text{Div}_d$ be the space of pairs $(\eta, \zeta)$ of disjoint positive divisors on $F_g - \{x_0\}$ of bi-degree $(d, d)$.

For each $p \in F_g$ choose a path $\gamma : I \to F_g$ from $p$ to $x_0$. Integration along such $\gamma$ gives a map $I : \text{Div}_d \to T_J$. More explicitly, given a positive
divisor $\eta = \sum m_i p_i$ of $F_g$ and paths $\gamma_i$ from $p_i$ to $x_0$, write $\int_{\gamma_i}$ for $\sum m_i \int_{\gamma_i}$. Define

$$I(\eta, \zeta) = (\alpha \mapsto (\int_{\gamma_\eta} - \int_{\gamma_\zeta})\alpha).$$

Interpreting Abel’s theorem (\cite{GH94}, p. 231, for a general reference), the fiber of $I$ over $0 \in T_J$ is the space of degree $d$ meromorphic functions on $(F_g, J)$, that is, $\text{Hot}^d((F_g, J), \mathbb{C}P^1)$. Here, the $*$ denotes based maps.

**Definition 2.4.** A map $p : E \to B$ is a homology fibration up to degree $k$ if the inclusion $\text{fiber}_b(p) \to \text{hofiber}_b(P)$ induces an isomorphism in $H_q(\cdot)$ for $q < k$.

**Remark.** This definition is similar to that in \cite{MS76} where they introduced the notion of a homology fibration.

Segal shows that $I : \text{Div}_d \to T_J$ is a homology fibration up to degree $d - 2g$. In this paper, we are interested in a similar statement as the complex structure $J$ is allowed to vary. This is accomplished as follows.

There is a natural inclusion $V_J^* \hookrightarrow H_1(F_g; \mathbb{C})$. This along with the projection $\mathbb{C} \to \mathbb{R}$ yields the canonical equivalence

$$V_J^* \cong H_1(F_g; \mathbb{R})$$

which is $H_1(F_g; \mathbb{Z})$-equivariant. We get

$$T_J \cong H_1(F_g; \mathbb{R})/H_1(F_g; \mathbb{Z}).$$

Denote this standard $g$-torus $H_1(F_g; \mathbb{R})/H_1(F_g; \mathbb{Z})$ by $T_0$. There results an equivalence of fiber bundles

$$\begin{array}{ccc}
\mathcal{J}_g \times T_J & \cong & \mathcal{J}_g \times T_0 \\
\downarrow & & \downarrow \\
\mathcal{J}_g & \xrightarrow{id} & \mathcal{J}_g,
\end{array}$$

Along with Abel’s theorem, this provides the sequence

$$\mathcal{J}_g^d(\mathbb{C}P^1)_* \to \mathcal{J}_g \times \text{Div}_d \xrightarrow{I} T_0$$

where the left hand term is the space of pairs $(J, h)$ with $h$ a **based** holomorphic map and as such is the fiber of $I$. 

7
Nesting Spaces of Divisors

We wish to prove a statement about the map $I$ above. To do so we would like to use induction on $d$. For this we will alter the spaces $\text{Div}_d$ so that we can regard $\text{Div}_d \subset \text{Div}_{d+1}$.

Choose a nested sequence of closed disk-neighborhoods $\{U_d\}$ of $x_0$ in $F_g$. Define

$$\text{Div}_d(F_g - U_d) := \{ (\eta, \zeta) \in \text{Div}_d \mid (\eta \cup \zeta) \cap U_d = \emptyset \}.$$

Clearly, $\text{Div}_d(F_g - U_d) \approx \text{Div}_d$. Fixing sequences $\{y_d\}$ and $\{z_d\}$ with $y_d, z_d \in U_d - U_{d+1}$ allows for an embedding $\text{Div}_d(F_g - U_d) \hookrightarrow \text{Div}_d(F_g - U_{d+1})$ given by $(\eta, \zeta) \mapsto (\eta + y_d, \zeta + z_d)$. Moreover, this embedding has trivial normal bundle since it extends to an open embedding of $\text{Div}_d(F_g - U_d) \times V_{y_d} \times V_{z_d}$ where $V_p$ is a small open ball around $p \in F_g$.

In this way, regard $\text{Div}_d$ as sitting inside $\text{Div}_{d+1}$ (up to homotopy). Label the inclusion as $\iota: \text{Div}_d \to \text{Div}_{d+1}$.

The Homology Fibration

The following theorem is the crux of the paper and is the most technical argument presented.

**Theorem 2.4.1.** The map $I: J_g \times \text{Div}_d \to T_0$ is a homology fibration up to degree $d - 2g$.

**Proof.** Throughout this proof, to avoid too many subscripts we will suppress the subscript $g$ in $J_g$.

For $Sp^n := Sp^d(F_g, x_0)$ the $n$-ford symmetric product, interpolate from $Sp^d$ to $P_d := Sp^d \times Sp^d$ by the spaces $P_{d,k} := \{ (\eta, \zeta) \mid \deg(\eta \cap \zeta) \geq k \}$:

$$Sp^d \subset P_{d,d-1} \subset \ldots \subset P_{d,1} \subset P_d.$$

Notice that the compliment $P_{d,k} \setminus P_{d,k+1} \cong \text{Div}_{d-k} \times Sp^k$.

Extend the map $I$ to each $J \times P_{d,k}$. For $v \in \sigma \subset T_0$ a contractible neighborhood, denote $(J \times P_{d,k})^\sigma := I^{-1}(\sigma) \subset J \times P_{d,k}$. Do similarly for $(J \times P_{d,k})^v$. There is still a sequence of inclusions

$$(J \times Sp^d)^\sigma \subset (J \times P_{d,d-1})^\sigma \subset \cdots \subset (J \times P_{d,1})^\sigma \subset (J \times P_d)^\sigma,$$

and $(J \times P_{d,k})^\sigma \setminus (J \times P_{d,k+1})^\sigma = (J \times (\text{Div}_{d-k} \times Sp^k))^\sigma$. Similarly for $v$ in place of $\sigma$. Let $j_\sigma$ denote any such inclusion $(-)^\sigma \hookrightarrow (-)^\sigma$. The proof of the lemma amounts to showing $H_r(j_{J \times \text{Div}_d}) = 0$ when $r < d - 2g$. 

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The spaces $P_{d,k}$ and $Div_d$ are smooth manifolds. The inclusion $Div_{d-k} \times Sp^k \hookrightarrow P_{n,k}$ is an open embedding. The inclusion $P_{d,k} \hookrightarrow P_{d,k-1}$ has normal bundle denoted $\nu_{P_{d,k}}$. It is thus possible to form the well-behaved normal bundle $\nu_{J \times P_{d,k}}$ of the embedding $J \times P_{d,k} \hookrightarrow J \times P_{d,k-1}$, namely,

$$\nu_{J \times P_{d,k}} := pr^* \nu_{P_{d,k}}$$

where $pr : J \times P_{d,k} \to P_{d,k}$ is projection onto the second factor. There is an analogous normal bundle for the two embeddings $(J \times Div_m \times Sp_{d-k}) \hookrightarrow (J \times P_{d,k} \setminus P_{d,k+1})$.

Consider the diagram of exact sequences of homology associated to the pair

$$(J \times P_{d,k}, J \times (P_{d,k} \setminus P_{d,k+1})) \cong (J \times P_{d,k}, J \times Div_{d-k} \times Sp^k).$$

From the above discussion, the relative term is homotopy equivalent to the Thom space $Th(\nu_{J \times P_{d,k+1}})$.

\[
\begin{array}{cccc}
\cdots & \longrightarrow & H_{r-1}Th(\nu_{J \times P_{d,k+1}}^v) & \longrightarrow \\
& \downarrow & & \downarrow \\
\cdots & \longrightarrow & H_r(\nu_{J \times P_{d,k+1}}^v) & \longrightarrow \\
& & & \downarrow \\
\cdots & \longrightarrow & H_r(\nu_{J \times P_{d,k+1}}^\sigma) & \longrightarrow \\
& & & \downarrow \\
& & & H_r(\nu_{J \times P_{d,k}}^\sigma) & \longrightarrow \\
& & & \cdots \\
\end{array}
\]

(1)

Our inductive hypothesis, the Thom isomorphism for rational homology, and the 5-lemma then yield $H_r(\nu_{J \times P_{d,k}}) = 0$ for $r \leq d - k$ ($k > 0$) by using downward induction on $k$ provided it is true for $k = d$. The case $k = d$ is easy enough as outlined by the following two facts.

Firstly, the Riemann-Roch formula and Abel’s theorem tells us that, for $d > 2g$, $I : \{J\} \times Sp^d \to T_0$ is a fiber bundle having fiber the $d-g$ dimensional complex vector space of (based) meromorphic functions on $(F_g, J)$. Secondly, as we allow variation in $J \in \mathcal{J} \simeq \ast$, the map $I : J \times Sp^d \to T_0$ is a fiber bundle with fiber identified with the product $J \times \mathbb{C}^{d-g}$.

Using these same two facts above, it follows that $H_r(\nu_{J \times P_{d}}) = 0$ since $P_d = Sp^d \times Sp^d$. Considering the diagram (1) for $k = 0$, the 5-lemma then tells us that $H_r(\nu_{J \times Div_d}) = 0$ provided $d \geq 2g$ and $r \leq d - 2g$. Rephrasing, we have shown for $m < d + 1$, $H_r(\nu_{J \times Div_m}) = 0$ when $r \leq m - 2g$ and the inductive step is finished. 

\[\square\]
Remark. It should be pointed out that the argument above is sourced at idea from Arnol’d ([Arn70]) where he proves a homological stability result for braid groups. Segal in [Seg79] promotes Arnold’s argument to configuration spaces of the sort above.

2.5 Comparing Fibration Sequences

The Comparison Fibration

Theorem 2.4.1 thus gives the sequence

\[ J_g(\mathbb{C}P^n)_* \to J_g \times \text{Div}_d \to T_0 \]

as a homology fibration up to degree \( d - 2g \). The idea now is to compare this sequence with a homotopy theoretic sequence whose fiber is \( J_g \times \text{Map}_*(F_g, S^2) \). For this, consider the (homotopy) fibration sequence

\[ S^2 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to \mathbb{C}P^\infty \]

which is defined as follows.

The topological group \( S^1 \) act on \( S^2 \) by rotating \( S^2 \) fixing the north and south poles. Construct the fibration

\[ ES^1 \times_{S^1} S^2 \to BS^1. \]

Regard the total space of this fiber bundle as a union of two disk bundles, one corresponding to the northern hemisphere of \( S^2 \), the other to the southern; the two disk bundles are glued together along their fiber-wise boundary \( S^1 \)-bundle. The total space of each disk-bundle is homotopy equivalent to \( BS^1 \). The equitorial \( S^1 \)-bundle is a model for \( ES^1 \) and is thus contractible. When all is said and done, we are left with the (homotopy) fibration sequence

\[ S^1 \to BS^1 \vee BS^1 \to BS^1. \]

Lastly, recall that \( \mathbb{C}P^\infty \) is a model for \( BS^1 \).

The Scanning Map

Our goal now is to define a ‘scanning map’

\[ S : \text{Div}_d \to \text{Map}_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty). \]

Regard \( \mathbb{C}P^\infty \) as the projectivization of the vector space \( \mathbb{C}[z] \). Write \( Sp^\infty(S^2) \) for the infinite symmetric product of \( S^2 \) with base point the north
pole, ∞. There is a map $\mathbb{C}[z] \to Sp^\infty(S^2)$ which sends a polynomial to its roots. This map descends to a homeomorphism $\mathbb{C}P^\infty \simeq Sp^\infty(S^2)$.

Choose a riemannian metric on $F_g$ and a parallelization of $F_g \setminus \{x_0\}$. Let $B_\varepsilon(0) \subset \mathbb{C}$ be the ball of radius $\varepsilon > 0$ about $0 \in \mathbb{C}$. There is an $\varepsilon > 0$ small enough so that for each $p \in F_g$ there is an exponential map

$$exp_p : B_\varepsilon(0) \to F_g - \{x_0\}.$$ 

Let $(\eta, \zeta) \in Div_d$ and choose $\varepsilon > 0$ small enough so that $\eta, \zeta,$ and $x_0$ are all distance at least $\varepsilon$ from each other. For possibly smaller $\varepsilon > 0$ still, there is a map $F_g \to \mathbb{C}P^\infty$ given by

$$p \mapsto exp^{-1}_p(\eta)$$

and $x_0 \mapsto \infty$. This procedure results in a homotopy class of a map

$$S : Div_d \to Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$$

(that the map is only well-defined up to homotopy is due to the choices of $\varepsilon > 0$). This map $S$ is the scanning map. Think of $S$ as scanning $F_g$ for either $\eta$ or $\zeta$ with a very zoomed microscope through which $F_g$ looks like $\mathbb{C}$.

**Comparing Sequences**

Observe that the connected component $Map_0^d(F_g, \mathbb{C}P^\infty)$ is a model for $K(\mathbb{Z}, 1)$ as is any $g$-torus, namely $T_0$. So there are homotopy equivalences $D : T_0 \to Map_0^d(F_g, \mathbb{C}P^\infty)$. It is then reasonable to expect a homotopy equivalence of fibrations

$$\xymatrix{hofiber_0(I) \ar[r] \ar[d] & J_g \times Map_*(F_g, S^2) \ar[d] \ar[r] & J_g \times Div_d \ar[r]^S \ar[d]^I & J_g \times Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \ar[d] \ar[r]^D & Map_0^d(F_g, \mathbb{C}P^\infty) \ar[d] \ar[r] & T_0 \ar[r]^\iota & Map_0^d(F_g, S^2)}$$

where the right fibration is that induced from the sequence in $\mathbb{3}$. We would also hope that the map on fibers restricts to the standard inclusion $J_g(\mathbb{C}P^1)_* \hookrightarrow Map_*(F_g, S^2)$.

For this to work we will need to consider the scanning map as being defined on $Div_d$ for all $d$ at once. Recall our inclusion (up to homotopy) $\iota :$
$\text{Div}_d \to \text{Div}_{d+1}$. Let $\hat{\text{Div}}$ denote the space of disjoint pairs of infinite divisors which almost agree with the fixed pair $(\sum y_i, \sum z_j)$ mentioned earlier. A pair $(\eta, \zeta)$ is said to almost agree with $(\sum y_i, \sum z_j)$ means $\eta - \sum y_i$ and $\zeta - \sum z_j$ are finite sums. Clearly, $\bigcup \text{Div}_d \hookrightarrow \hat{\text{Div}}$.

There are homotopy commutative diagrams

\[
\begin{array}{ccc}
\text{Div}_d & \xrightarrow{S} & \text{Map}_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \\
\downarrow & & \downarrow \\
\text{Div}_{d+1} & \xrightarrow{S} & \text{Map}_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)
\end{array}
\]

from which we get a well-defined (up to homotopy) scanning map

$S : \hat{\text{Div}} \xrightarrow{\simeq} \text{Map}_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$.

**Theorem 2.5.1** (Segal ([Seg79] §4)). The scanning map $S$ is a a homotopy equivalence.

The idea for introducing $\hat{\text{Div}}$ rather than being satisfied with $\bigcup \text{Div}_d$ is that $\hat{\text{Div}}(-)$ tends to behave better than $\bigcup \text{Div}(-)$ on non-compact arguments. Namely, the usual maps among $\hat{\text{Div}}(-)$ which one expects to be quasifibrations from Dold-Thom theory are indeed quasifibrations.

Given $d$, it is possible to choose the sequences $\{y_i\}, \{z_i\}$ so that the diagram

\[
\begin{array}{ccc}
\text{Div}_d & \xrightarrow{\iota} & \text{Div}_{d+N} \\
\downarrow \quad \iota & & \downarrow \quad \iota \\
T_0 & \xrightarrow{id} & T_0 \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Div}_{d+1} & \xrightarrow{\iota} & \text{Div}_{d+N} \\
\downarrow \quad \iota & & \downarrow \quad \iota \\
T_0 & \xrightarrow{id} & T_0 \\
\end{array}
\]

is homotopy commutative and restricts to a map between fibers. To do this, simply ensure that the pair of divisors $(\sum_{k=1}^N y_{d+k}, \sum_{k=1}^N z_{d+k})$ comes from the zeros and poles for a meromorphic function on $F_g$. The homotopy colimit of the top row is homotopy equivalent to the connected component $\hat{\text{Div}}_d$ of $\hat{\text{Div}}$ corresponding the component $\text{Map}^d_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$ consisting of homological degree $(d,d)$ maps.

The next task is to prove the following

**Lemma 2.5.2.** The induced map on fibers in the above diagram 4 induces an isomorphism in $H_q(-)$ for $q < d - 2g$. 

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Postponing the proof of lemma 2.5.2 for now, it follows from Theorem 2.4.1 and Proposition 5 from [MS76] (with “homology fibration” replaced by “homology fibration up to degree $d - 2g$”) that the resulting map

$$I : \text{hocolim}_N \text{Div}_{d+N} \to T_0$$

is then a homology fibration up to degree $d - 2g$. It follows further that

$$I : \widehat{\text{Div}}_d \to T_0$$

is a homology fibration up to degree $d - 2g$. As in ([Seg79] §4), it is possible to choose the homotopy equivalence $D$ via Poincare’s duality such that there results a homotopy commutative diagram

$$\begin{array}{ccc}
\mathcal{J}_g^d(\mathbb{C}P^1) & \longrightarrow & \mathcal{J}_g \times \text{Map}^d(F_g, S^2) \\
\downarrow & & \downarrow \\
\mathcal{J}_g \times \widehat{\text{Div}}_d & \longrightarrow & \mathcal{J}_g \times \text{Map}^d(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \\
\downarrow & & \downarrow \\
T_0 & \longrightarrow & \text{Map}^0(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \\
\end{array}$$

where the right vertical map is induced by subtraction in the H-space $\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$.

Moreover, immitating Segal’s lines ([Seg79], Lemma 4.7), one can verify that the resulting map on fibers is the standard inclusion

$$\mathcal{J}_g^d(\mathbb{C}P^1) \hookrightarrow \mathcal{J}_g \times \text{Map}^d(F_g, S^2).$$

Because the scanning map $S$ is an equivalence (Theorem 2.5.1), we conclude the following lemma.

**Lemma 2.5.3.** The inclusion

$$\mathcal{J}_g^d(\mathbb{C}P^1) \hookrightarrow \mathcal{J}_g \times \text{Map}^d(F_g, S^2)$$

induces an isomorphism in $H_q(-)$ for $q < d - 2g$.

**Proof of Lemma 2.5.2.** To prove Lemma 2.5.2 it will suffice to show $\iota$ induces an isomorphism in $H_q(-)$ for $q < d - 2g$. This is indeed sufficient using the Zeeman comparison theorem ([Zee57]) for the apparent Leray-Serre spectral sequences (see [McC01], §5).
It turns out that most of the work has already been done. In the proof of Theorem 2.4.1 we compared the $H_\ast$-long exact sequences for the pairs $((P_{d,k})^v, (Div_{d-k} \times Sp^k)^v)$ and $((P_{d,k})^v, (Div_{d-k} \times Sp^k)^v)$. Here we compare the $H_\ast$-long exact sequences of the pairs $((P_{d,k})^v, (Div_{d-k} \times Sp^k)^v)$ and $((P_{d+1,k})^v, (Div_{d+1-k} \times Sp^k)^v)$. The appropriate argument here is nearly identical to that of Theorem 2.4.1. Details are left to the reader. \[\square\]

### Unbased Mapping spaces

We are interested in the unbased mapping spaces $\mathcal{J}^d_g(\mathbb{C}P^1)$ and $Map^d(F_g, S^2)$.

**Lemma 2.5.4.** The inclusion $\mathcal{J}^d_g(\mathbb{C}P^1) \hookrightarrow Map^d(F_g, S^2)$ induces an isomorphism in $H_q(-)$ for $q < d - 2g$.

**Proof.** Consider the base-point evaluation fibrations

\[
\begin{array}{ccc}
\mathcal{J}^d_g(\mathbb{C}P^1) & \longrightarrow & \mathcal{J}_g \times Map^d(F_g, S^2) \\
\downarrow & & \downarrow \\
\mathcal{J}^d_g(\mathbb{C}P^1) & \longrightarrow & \mathcal{J}_g \times Map^d(F_g, S^2) \\
\downarrow & & \downarrow \\
\mathbb{C}P^1 & \longrightarrow & S^2.
\end{array}
\]

There results a morphism of Leray-Serre $H_\ast$-spectral sequences. Lemma 2.5.3 shows that this functor induces an isomorphism on the $E_{p,q}^r$ page for $q < d - 2g$. We thus have an isomorphism on the $E_{p,q}^r$ pages for all $r$ when $p + q < d - 2g$. In particular, there is an isomorphism on the $E_{p,q}^\infty$ page for $p + q < d - 2g$.

We are now confronted with the common issue of concluding the middle horizontal map in diagram 5 induces an isomorphism in $H_\ast$, for $\ast = p + q < d - 2g$, knowing it is an isomorphism on the filtration quotients $E_{p,q}^\infty$. For this one uses induction on the filtration degree; the inductive step is clear in light of the 5-lemma. Alternatively, because we are working over the rationals, all extension problems can be solved. Details are left to the reader. \[\square\]

### 2.6 The Action of $Diff^+_g$

Our ultimate goal being to understand moduli space, we now want a version of Lemma 2.5.4 which is quotiented by the action of $Diff^+_g$. 

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**Theorem 2.6.1** (Main theorem \((n = 1)\)). The map \(\mathcal{M}_g^d(\mathbb{C}P^1)^h \to \mathcal{M}_g^d(\mathbb{C}P^1)^h\) induced by the natural inclusion induces an isomorphism in \(H_q(-)\) for \(q < d - 2g\).

**Proof.** There is the following diagram

\[
\begin{array}{ccc}
\mathcal{J}_g^d(\mathbb{C}P^1) & \cong_{H_{<d-2g}} & \mathcal{J}_g \times \text{Map}^d(F, S^2) \\
\downarrow & & \downarrow \\
\text{Diff}^+_g \times_{\text{Diff}^+_g} \mathcal{J}_g^d(\mathbb{C}P^1) & \to & \text{Diff}^+_g \times_{\text{Diff}^+_g} (\mathcal{J}_g \times \text{Map}^d(F, S^2)) \\
\downarrow & & \downarrow \\
\text{BDiff}^+_g & \xrightarrow{id} & \text{BDiff}^+_g
\end{array}
\]

Lemma 2.5.4 shows that the top horizontal map is an isomorphism in \(H_q(-)\) for \(q < d - 2g\). Recognizing the total spaces in the above diagram as the homotopy moduli spaces, the same spectral sequence argument proving Lemma 2.5.4 finishes the proof.

\[\square\]

### 2.7 Maps to \(\mathbb{C}P^n\) for \(n \geq 1\)

In this subsection we will sketch the idea for how to deal with maps to \(\mathbb{C}P^n\) for \(n > 1\).

**Theorem 2.7.1** (Main Theorem). The map \(\mathcal{M}_g^d(\mathbb{C}P^n)^h \to \mathcal{M}_g^d(\mathbb{C}P^n)^h\) induces an isomorphism in \(H_q(-)\) for \(q < (d - 2g)(2n - 1)\).

**Proof (sketch).** The proof is analogous to the proof of Theorem 2.6.1. We will sketch the appropriate modifications. Details will be left to the interested reader. Some helpful details can be found in [Seg79].

Think of a holomorphic map \(h \in \mathcal{J}_g^d(\mathbb{C}P^1)\) as a meromorphic function. Such a meromorphic function is determined by its poles and zeros. Equivalently, we could regard \(h\) as a pair \((r_0, r_1)\) of degree \(d\) polynomial functions on \((F_g, J)\) whose zeros are disjoint. This is the essence of Abel’s theorem that we used earlier to regard \(\mathcal{J}_g^d(\mathbb{C}P^1)_*\) as a subset of \(\text{Div}_d\). Now, think of \(h \in \mathcal{J}_g^d(\mathbb{C}P^n)_*\) similarly as an \((n + 1)\)-tuple \((r_0, ..., r_n)\) of degree \(d\) polynomials with \(\bigcap_{i=0}^n \{r_i = 0\} = \emptyset\). One would thus generalize \(\text{Div}_d^n\) to \(\text{Div}_d^{(n)}\) consisting of \((n+1)\)-tuples of positive divisors which are \((n+1)\)-wise disjoint.
It is a fact that such an \((n+1)\)-tuple comes from a map into \(\mathbb{CP}^n\) if and only if each divisor has the same image in the Jacobian variety. Abel’s theorem can then be modified to reveal that the fiber of the similarly defined integration map \(I : \text{Div}^n_d \to T^*_n\) is the space of based holomorphic maps \(\mathcal{J}^d_g(\mathbb{CP}^n)_*\).

On the homotopy theoretic side, one constructs the space

\[
W_{n+1}(\mathbb{CP}^\infty) \subset \prod_{i=1}^{n+1} \mathbb{CP}^\infty
\]

consisting of \((n+1)\)-tuples with at least one entry the base point of \(\mathbb{CP}^\infty\). Indeed, \(W_2(\mathbb{CP}^\infty) = \mathbb{CP}^\infty \vee \mathbb{CP}^\infty\). Similar to sequence 3 is the (homotopy) fibration sequence

\[
\mathbb{CP}^n \to W_{n+1}(\mathbb{CP}^\infty) \to \prod_{i=1}^{n} \mathbb{CP}^\infty.
\]

There is a similar scanning map used to compare \(\text{Div}^n_d\) to \(\text{Map}^d_*(F_g, W_{n+1}(\mathbb{CP}^\infty))\).

\[\square\]

2.8 Marked Points

Choose \(k\) points \(\{r_i\} \subset F_g\). Let \(\text{Diff}^+_g\) denote the subgroup of \(\text{Diff}^+_g\) consisting of those diffeomorphisms of \(F_g\) which fix \(r_i\) for each \(1 \leq i \leq k\). Define the moduli space of degree \(d\) marked holomorphic curves of genus \(g\) in \(\mathbb{CP}^n\) as the orbit space

\[
\mathcal{M}^d_{g,k}(\mathbb{CP}^n) := \mathcal{J}^d_g(\mathbb{CP}^n)/\text{Diff}^+_g.\]

Do similarly for the topological moduli space \(\mathcal{MT}^d_{g,k}(\mathbb{CP}^n)\). Denote by a superscript \(h\), the analogous homotopy orbit spaces.

**Theorem 2.8.1.** The standard map

\[
\mathcal{M}^d_{g,k}(\mathbb{CP}^n) \to \mathcal{MT}^d_{g,k}(\mathbb{CP}^n)
\]

induces an isomorphism in \(H_q(-)\) for \(q < (d - 2g)(2n - 1)\).

**Proof.** The proof is identical to that of [1.1.2] \[\square\]
3 Irreducible Stable Maps to $\mathbb{C}P^n$

The ultimate goal of the remainder of this paper is to have a statement about the homology of a moduli space $\overline{\mathcal{M}}^d_g(\mathbb{C}P^n)$ of irreducible stable maps to $\mathbb{C}P^n$. This is similar to the Deligne-Mumford compactification of the moduli space curves in $\mathbb{C}P^n$ in sense of Gromov-Witten theory ([DM69], [HM98], [MS04]).

The rough idea is that the boundary of such moduli spaces is built from such moduli spaces of smaller indices (see [KV99] for example). In this way our moduli space of irreducible stable maps to $\mathbb{C}P^n$ can be realized as a colimit of moduli spaces of singular surfaces as they include into each other via resolutions of singularities. Segal’s result, which was the basis of the main theorem above, holds for singular surfaces as well and the arguments above hold for singular surfaces and neighborhoods of their resolutions. Because homology behaves well with respect to homotopy colimits, a Meyer-Vietoris argument can be invoked to conclude a statement about compactified moduli space.

3.1 Preliminaries

For our purposes, we will only consider singular surfaces $F$ which are realized as a disjoint union of oriented surfaces $F_i$, each having a collection of marked points $\{p_{i_k}\}^n_{i=1} \subset F_i$, which are pairwise identified $p_{i_k} \sim p_{j_l}$ in some way. So

$$F = (\coprod_i F_i)/\sim .$$

From here on out, the phrase “singular surface” will refer to one of this form. The data $\{(F_i, \{p_{i_k}\})\}$ is called a normalization of $F$ and it is unique up to homeomorphism. The image of the marked points are the nodes of $F$. The number of nodes of $F$ is well-defined and denoted by $n(F)$. The genus of $F$ is the number $g(F)$ given by

$$\chi(F) = 2 - 2g(F) + n(F)$$

A singular surface is said to be irreducible if its normalization consists of a single connected surface.

Remark/Definition. An alternative description of such a singular description is as follows. Let $\{\alpha_r\}_1^N$ be a collection of disjoint closed curves in the oriented closed smooth surface $\tilde{F}$. Declare $p \sim p'$ if $p$ and $p'$ both lie on some $\alpha$ in the chosen collection. We could alternatively think of a singular surface $F$ as the quotient space $\tilde{F}/\sim$. For $F$ a singular surface and $p \in F$
a node, a singular surface $\tilde{F}$ will be called a smoothing of $F$ at $p$ if there is a closed curve $\alpha \subset \tilde{F}$ with $F \cong \tilde{F}/\alpha$. Up to homeomorphism, a smoothing of $F$ at a node $p \in F$ is unique.

3.2 Singular Riemann Surfaces

A complex structure $J$ on such a singular surface $F$ is the data of a complex structure $J_i$ on each component $F_i$ of its normalization. As before, write $\mathcal{J}(F)$ for the space of such complex structures. A pair $(F, J)$ is called a singular Riemann surface. More classically, a singular Riemann surface of this form is locally modeled on fibers of the holomorphic map $\mathbb{C}^2 \to \mathbb{C}$ given by $(z, w) \mapsto zw$.

A holomorphic map from $(F, J)$ to a complex manifold $Y$ is the data of holomorphic maps $(F_i, J_i) \xrightarrow{h_i} Y$ with $h_i(p_{i_k}) = h_j(p_{j_l})$ whenever $p_{i_k} \sim p_{j_l}$ in $F$. The degree of such a map is the sum $\Sigma_i \deg(h_i)$. Let $\text{Hol}^d((F, J), Y)$ be the space of degree $d$ holomorphic maps from $(F, J)$ into $Y$, topologized in the obvious way.

Remark. Most of the statements about singular surfaces are stated only for irreducible singular surfaces. One could make statements about not-irreducible surfaces provided one specifies the degree of maps on each component of a normalization.

In [Seg79], Segal proves the following result.

**Theorem 3.2.1** (Segal). Let $F$ be an irreducible singular surface. The standard inclusion

$$\text{Hol}^d((F, J), \mathbb{C}P^n) \hookrightarrow \text{Map}^d(F, \mathbb{C}P^n)$$

induces an equivalence in $H_q(\cdot)$ for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$.

The idea of the proof is that a holomorphic map $(F, J) \to \mathbb{C}P^n$ from a singular surface is a holomorphic map $(F_1, J_1) \to \mathbb{C}P^n$ with conditions on the marked points $\{p_{i_k}\}$. The theorem follows from Segal’s previous work when $F$ is not singular.

For $F$ a singular surface, write $\text{Diff}^+(F)$ as the topological group of homeomorphisms of $F$ which restrict to each $F_i$ as a diffeomorphism. Note that it is possible for an element $\phi \in \text{Diff}^+(F)$ to permute the collection $\{F_i\}$. Note also that any such $\phi$ necessarily permutes the marked points $p_{i_k}$. The group $\text{Diff}^+(X)$ acts on the space of pairs

$$\mathcal{J}^d_{\{F\}}(\mathbb{C}P^n) := \{(J, h) \mid J \in \mathcal{J}(F) \text{ and } h \in \text{Hol}^d((F, J), Y)\}$$
by pulling back complex structures and precomposing with holomorphic maps.

Let $F$ be a singular surface. Define the moduli space of type $[F]$ to be the orbit space

$$ \mathcal{M}_{[F]}^d(Y) := \mathcal{J}_{[F]}^d(\mathbb{C}P^n)/Diff^+(F). $$

Define similarly the topological moduli space of type $[F]$ as

$$ \mathcal{M}_{T[2]}(Y) := (\mathcal{J}(F) \times Map(F,Y))/Diff^+(F). $$

As before, write a superscript $h$ to denote the respective homotopy orbit spaces rather than orbit spaces.

**Theorem 3.2.2.** Let $F$ be an irreducible singular surface. The standard map

$$ \mathcal{M}_{[F]}^d(\mathbb{C}P^n)^h \to \mathcal{M}_{T[2]}(\mathbb{C}P^n)^h \simeq Map(F,\mathbb{C}P^n)//Diff^+(F) $$

induces an isomorphism in $H_q(\cdot)$ for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$.

**Proof.** The proof will merely be outlined. As demonstrated with the proof of Theorem 1.1.2, it is possible to follow the lines of Segal for singular Riemann surfaces while keeping track of the complex structure on such Riemann surfaces as a variable. This leads to a statement analogous to Lemma 2.5.4 for singular surfaces. The same spectral sequence argument from the proof of Theorem 2.6.1 is in place to account for quotienting by the action of $Diff^+(F)$.

Corollary 3.2.3. Let $F$ be an irreducible singular surface. The standard map

$$ \mathcal{M}_{[F]}^d(\mathbb{C}P^n) \to \mathcal{M}_{T[2]}(\mathbb{C}P^n) $$

induces an isomorphism in $H_q(\cdot)$ for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$.

**Proof.** The proof is merely the observation that, as before, $Diff^+(F)$ acts on $\mathcal{J}(F)$ with finite stabilizers. It follows that the projections from homotopy quotients to quotients,

$$ \mathcal{M}_{[F]}^d(\mathbb{C}P^n)^h \to \mathcal{M}_{[F]}^d(\mathbb{C}P^n) \quad \text{and} \quad \mathcal{M}_{T[2]}(\mathbb{C}P^n)^h \to \mathcal{M}_{T[2]}(\mathbb{C}P^n), $$

are rational homology equivalences.
3.3 Smoothings of Surfaces

Recall our model of $EDiff^+_F$ as $Emb(F, \mathbb{R}^\infty)$ so that $BDiff^+_F$ consists of embedded singular surfaces in $\mathbb{R}^\infty$ which are diffeomorphic to $F$. Such an embedded singular surface $F' \in BDiff^+_F$ admits a canonical normalization

$$\{F'_i\} \subset \bigsqcup_g BDiff^+_g.$$

Moreover, a point in the homotopy orbit space $\mathcal{M}^{d}_{[F]}(\mathbb{C}P^n)^h$ is a triple $(F', J, h)$ where $F' \subset \mathbb{R}^\infty$ is a singular surface, $J$ is a complex structure on $F$, and $h$ is a holomorphic map $(F', J) \rightarrow \mathbb{C}P^n$.

With the above model for $EDiff^+ (F)$, each embedding $e \in EDiff^+ (F)$ determines a metric on $F$ from that on $\mathbb{R}^N$ ($N$ large). The conformal class of such a metric determines a point in $\mathcal{J}(F)$ in light of §2.2. In this way, there is a distinguished map

$$EDiff^+ (F) \rightarrow \mathcal{J}(F)$$

which is apparently $Diff^+ (F)$-equivariant. It follows that the data $(F', J, h)$ of a point in $\mathcal{M}^{d}_{[F]}(\mathbb{C}P^n)$ is redundant and is determined by the data $(F', h)$.

Similarly for $\mathcal{M}^{Td}_{[F]}(\mathbb{C}P^n)$.

**Remark.** Take $p = p_{i_k} = p_{j_l}$ to be a node of the singular Riemann surface $F \subset \mathbb{R}^\infty$. For $\hat{F}$ a smoothing of $F$ at $p$, the fiber of the projection $\hat{F} \rightarrow F$ over $p$ is a circle $\alpha \subset \hat{F}$. The data of such a smoothing $\hat{F}$ of $F$ is a point in the unit sphere bundle of the vector space $T_{p_{i_k}}F_i \otimes T_{p_{j_l}}F_j$. Better still, given a nonzero point in this vector space, there is a contractible choice of embedded resulting smoothings $(\hat{F}') \subset \mathbb{R}^\infty$ which agree with the embedded $F$ outside a small neighborhood of $p \in F$.

Consider then the vector bundle $\nu_{[F],p}$ over $\mathcal{M}^{d}_{[F]}(\mathbb{C}P^n)^h$ whose fiber over a point $(F', J, h)$ is the complex line

$$(T_{p_{i_k}}F_i \otimes T_{p_{j_l}}F_j, J_i \otimes J_j).$$

Denote by $S(\nu_{[F],p})$ the associated principle $U(1)$-bundle with total space also denoted as $S(\nu_{[F],p})$. In the same way, there is a principle $U(1)$-bundle $S(\nu_{[F],p}^T)$ over the homotopy orbit space $\mathcal{M}^{Td}_{[F]}(\mathbb{C}P^n)^h$. There is an obvious
morphism of principle $U(1)$-bundles

$$
\begin{array}{rcl}
S(\nu_{[F],p}) & \longrightarrow & S(\nu_{[F],p})^T \\
\downarrow & & \downarrow \\
\mathcal{M}^d_{[F]}(\mathbb{C}P^n)^h & \longrightarrow & \mathcal{M}^d_T([F](\mathbb{C}P^n)^h). \\
\end{array}
$$

\begin{equation}
(6)
\end{equation}

**Lemma 3.3.1.** Let $F$ be an irreducible surface. The map

$$
S(\nu_{[F],p}) \rightarrow S(\nu_{[F],p})^T
$$

from diagram (6) induces an isomorphism in $H^q(-)$ for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$.

**Proof.** Diagram (6) induces a morphism of Leray-Serre spectral sequences. This morphism is an isomorphism on $E^2_{p,q}$-pages for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$. It follows that this morphism is an isomorphism on $E^\infty_{p,q}$-pages for $p + q < (d - 2(g(F) - n(F) + 1))(2n - 1)$. Because we are working over the rationals, all extensions are possible and we conclude the desired result.

From the above remark, up to a contractible choice, there is a map $S(\nu_{[F],p})^T \rightarrow \mathcal{M}^d_T([F](\mathbb{C}P^n))$ described by $\tilde{F} \rightarrow F \rightarrow \mathbb{C}P^n$. Similarly, provided $d > 2g$ there is a distinguished homotopy class of a map $S(\nu_{[F],p}) \rightarrow \mathcal{M}^d_{[F]}(\mathbb{C}P^n)$. Briefly, the reason for this is as follows. For $p : \tilde{F} \rightarrow F$ the projection and $h : (F,J) \rightarrow \mathbb{C}P^n$ a holomorphic map, apply the Riemann-Roch formula to the sheaf $(p \circ h)^*\mathcal{O}(1)^n$ knowing that a map to $\mathbb{C}P^n$ is the same as a global section of this sheaf.

There results a homotopy commutative diagram

$$
\begin{array}{rcl}
S(\nu_{[F],p}) & \longrightarrow & \mathcal{M}^d_{[F]}(\mathbb{C}P^n) \\
\downarrow & & \downarrow \\
S(\nu_{[F],p})^T & \longrightarrow & \mathcal{M}^d_T([F](\mathbb{C}P^n)) \\
\end{array}
$$

\begin{equation}
(7)
\end{equation}

which can be made into a homotopy coherent diagram.

Denote by $\mathcal{H}_{[F],p}$ the homotopy colimit of

$$
\mathcal{M}^d_{[F]}(\mathbb{C}P^n) \leftarrow S(\nu_{[F],p}) \rightarrow \mathcal{M}^d_T([F](\mathbb{C}P^n)).
$$

\begin{equation}
(8)
\end{equation}

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Similarly, denote by $H_{[F],p}$ the homotopy colimit of
\[
\mathcal{M}^d_{[F]}(\mathbb{C}P^n) \leftarrow S(\nu^T_{[F],p}) \to \mathcal{M}^d_{[\tilde{F}]}(\mathbb{C}P^n).
\] (9)

Diagrams 6 and 7 together yield a map
\[
H_{[F],p} \to H_{[F],p}'
\] (10)

Lemma 3.3.2. The map in (10) induces an equivalence in $H_q(\cdot)$ for $q < (d - 2(g(F) - n(F) + 1))(2n - 1)$.

Proof. The proof is a simple application of the Mayer-Vietoris sequence in homology along with an application of the five-lemma.

\[\square\]

3.4 A Moduli Space $\overline{\mathcal{M}}^d_{g}(\mathbb{C}P^n)$

From Lemma 3.3.2 one might expect a statement about the moduli space of stable maps in the sense of Gromov-Witten theory. However, such a stable map of (total) degree $d$ can restrict to a map of small degree on irreducible components. For this reason, we will denote by $\overline{\mathcal{M}}^d_{g}(\mathbb{C}P^n)$ moduli space of those stable maps consisting of irreducible surfaces. With the geometric picture of the boundary of the moduli space of stable maps being described by pinching curves on surfaces, the space $\overline{\mathcal{M}}^d_{g}(\mathbb{C}P^n)$ parameterizes maps of surfaces which are allowed to be pinched only along non-separating curves.

One can define the homotopy type of $\overline{\mathcal{M}}^d_{g}(\mathbb{C}P^n)$ as a homotopy colimit. This will be done as follows. Fix $g$ and $d$ and consider a diagram whose entries are
\[
\bigsqcup_{[F]} \{\mathcal{M}^d_{[F]}(\mathbb{C}P^n)\} \amalg \bigsqcup_{[F],p} \{S(\nu_{[F],p})\}
\]
where the first disjoint union is over irreducible singular surfaces $F$ with $g(F) = g$, one in each diffeomorphism class, and the second disjoint union is over such $F$ and nodes on $F$. The maps in this diagram are either $S(\nu_{[F],p}) \to \mathcal{M}^d_{[F]}(\mathbb{C}P^n)$ or $S(\nu_{[F],p}) \to \mathcal{M}^d_{[\tilde{F}]}(\mathbb{C}P^n)$ if $\tilde{F}$ is a smoothing of $F$. Denote this indexing diagram by $\mathcal{D}$ and this diagram by $\mathcal{M} : \mathcal{D} \to \text{Top}$. Define
\[
\overline{\mathcal{M}}^d_{g}(\mathbb{C}P^n) := \text{hocolim}\mathcal{M}.
\]

There is a similar functor $\mathcal{M}^T : \mathcal{D} \to \text{Top}$ whose entries are the topological moduli spaces $\mathcal{M}^d_{[F]}(\mathbb{C}P^n)$ and their $U(1)$-bundles $S(\nu^T_{[F],p})$. Define
\[
\overline{\mathcal{M}}^d_{g}(\mathbb{C}P^n) := \text{hocolim}\mathcal{M}^T
\]

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From §3.3 there is a map of homotopy colimits
\[ \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \to \overline{\mathcal{M}T}_g^d(\mathbb{C}P^n). \] (11)

**Theorem 3.4.1.** The map in diagram (11) induces an isomorphism in \( H_q(\cdot) \) for \( q < (d - 2g)(2n - 1) \).

**Proof.** The work has already been done. The space \( \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \) is a union of the spaces
\[ \partial_k \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \]
which are defined as a similar homotopy colimit as \( \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \) but over a *truncated* diagram consisting of the appropriate spaces indexed by singular surfaces \( F \) which have at least \( k \) nodes. The same is true for \( \overline{\mathcal{M}T}_g^d(\mathbb{C}P^n) \).

Proceed inductively on \( k = n(F) \) by applying Mayer-Vietoris as in Lemma 3.3.2. The induction begins at \( k = g/2 \) which is the maximal number of nodes of an irreducible singular surface of genus \( g \). Observe that the homotopy colimit of \( \mathcal{M} \) is a homotopy colimit of the homotopy colimits \( \mathcal{H}[F] \).

\[ \square \]

**Remark.** This later homotopy colimit \( \overline{\mathcal{M}T}_g^d(\mathbb{C}P^n) \) is still moderately complicated. It is stratified in the same combinatorial way as the partial compactification \( \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \). In fact, when \( n = 0 \) so that \( \mathbb{C}P^n \) is a point, \( \overline{\mathcal{M}}_g^d(*) = \overline{\mathcal{M}T}_g^d(*) \). The benefit of \( \overline{\mathcal{M}T}_g^d(\mathbb{C}P^n) \) is that it is far more homotopy theoretic. Indeed, there is no holomorphic data retained by \( \overline{\mathcal{M}T}_g^d(\mathbb{C}P^n) \), only the combinatorics of the strata of \( \overline{\mathcal{M}}_g^d(*) \) and continuous maps from (singular) genus \( g \) surfaces to \( \mathbb{C}P^n \).

The author has heard in conversation that this partial compactification \( \overline{\mathcal{M}}_g^d(\mathbb{C}P^n) \) of irreducible stable maps is much better behaved in a homotopy theoretic sense than that of all stable maps. For one, the boundary stratum is considerably simpler than that of the full compactification of all stable maps. But more, it is said that this moduli space of irreducible stable maps satisfies a version of Harer stability ([Har85]) when \( \mathbb{C}P^n \) is replaced by a point.

\[ \square \]

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