I. INTRODUCTION

Randomly branched polymers (RBPs) are a classical topic in statistical physics. Seminal advancements in the theoretical understanding of these polymers have been made not long after the advent of renormalization group theory starting with the seminal work of Lubensky and Issacson (LI). With the surge of biophysics, there recently has been renewed interest in RBPs because RNA in its molten phase belongs to the same universality class as swollen RBPs. However, the current understanding of RBPs is still not quite satisfactory. For example, the topology of their phase diagram is not entirely clear. In particular the part of the phase diagram that contains the so-called θ-transition gives reason for debate. The existing theories for the collapse θ-transition are not entirely correct. As far as we know, there exist no theories for the transport properties and the related fractal dimensions of RBPs such as the dimensions of the backbone, the shortest path and so on.

In this paper we are not interested in chemical or mechanical properties of randomly branched polymers. Rather, we are interested in their structure. More precisely, we are interested in their universal structural properties in the limit where the number of constituent monomers is large. In this limit, an RBP can be regarded as a large cluster, and its structural properties are universal, i.e., common to large RBPs as a class irrespective of their physical or chemical details. Phenomenologically, only their large size and their branching on all length scales are relevant. In the language of critical phenomena – phenomena with large correlation lengths, here the diameters of clusters – all such systems of fractal clusters with different microscopic aspects but with these common relevant properties belong to one universality class, which we denote in the following with the pars pro toto randomly branched polymers. In computer simulations such clusters are usually constructed as so-called lattice animals, i.e., clusters of connected sites (monomers) on a d-dimensional regular lattice. The recent publication of Hsu and Grassberger on the collapse transition of animals and the unresolved issues mentioned above have triggered us to reconsider this classical topic with field theoretic methods.

In the much-studied case of a single large linear polymer in a diluted solvent, the phase diagram is one-dimensional. When the solvent quality is lowered (typically by lowering its temperature) below the so-called θ-point, the polymer undergoes a collapse transition from a swollen coil-like conformation to a compact globule-like conformation. In simple lattice models, the monomer-solvent repulsion that drives the collapse transition is generically implemented via an effective attractive interaction between non-bonded monomers which is equivalent to the monomer-solvent repulsion at least as far as universal properties are concerned. Thus, the fugacity for non-bonded monomer-monomer contacts, let’s call it \( z_{\text{cont}} \), can be chosen as the control variable spanning the phase diagram of a linear polymer in a solvent. Evidently, \( z_{\text{cont}} \) is closely related to temperature.

In the case of a single large RBP in a diluted solvent, the phase diagram is two-dimensional, see Fig. The basic reason for the additional dimension is that one has to deal with an additional fugacity stemming from the fact that the number of bonds \( b \) of an RBP is not uniquely determined by its number of sites \( N \), \( b - N + 1 =: l \geq 0 \), whereas it is uniquely determined for a linear polymer (as well as for a tree-like branched polymer) with \( l = 0 \). The additional fugacity, let’s call it \( z_{\text{cycle}} \), regulates the cyclomatic index (the number of cycles \( l \)) of the polymer in the grand partition sum. For \( z_{\text{cycle}} = 0 \), the RBP has no cycles and the minimal number of bonds, i.e., it is
The phase diagram becomes one-dimensional (it reduces to the vertical axis with $z_{cycle} = 0$ in Fig. 1). Physically, $z_{cycle}$ can be varied, e.g., by adding polyfunctional chemical units to the solution whose insertion into RBPs results in additional bond cycles.

Over the last two decades or so, a number of numerical studies have been undertaken to map out this phase diagram. The picture that arises from these studies can be summarized as follows: There is a swollen phase where the polymer is in a tree-like or sponge-like conformation and a compact phase, where the polymer is in a coil-like or vesicle-like conformation. There is some debate, whether there exists a phase transition between the two compact conformations or not. Between the swollen and the compact phases, there is a line of collapse transitions. One part, called the $\theta$-line (labelled collapse, blue), corresponds to continuous transitions with universal critical exponents from the tree-like conformation to the coil-like confirmation. The other part of the transition line, called the $\theta'$-line (red, in the dashed region), corresponds to the transition between the foam- or sponge-like conformation to the to vesicle-like conformation. Between the $\theta$- and $\theta'$-lines there is a tricritical point. There has been some controversy, if the $\theta'$ transition is continuous or not. With the assumption of it being continuous, computer simulations in 2 dimensions yield nonuniversal critical exponents. As we will explain in detail below, our RG study shows that the collapse transition to the right of the tricritical point is characterized by a runaway of the RG flow. This suggests that the $\theta'$-transition is a fluctuation induced first order transition instead. It could also mean that two of the lines observed in numerical studies of the phase diagram, viz. the lines interpreted as the line of transitions between two compact phases and the $\theta'$-line, respectively, are merely shadows of the spinodals of the discontinuous transition.

The most fruitful theoretical approach to RBPs is based on the asymmetric Potts model, although Flory theory and real space renormalization have also been applied successfully. For the swollen phase, the field theoretic problem was settled by Parisi and Sourlas (PS) via mapping the relevant part of the asymmetric Potts model to the Yang-Lee edge problem using dimensional reduction. Subsequently, this mapping has been applied to further problems such as the exact calculation of universal scaling functions characterizing the behavior in the physical dimension 3. Dimensional reduction was confirmed later with the discovery of an exact relationship between swollen RBPs and repulsive gases at negative activity in two fewer dimensions by Brydges and Imbrie.

The asymmetric Potts model also provides a vantage point for studying the $\theta$-transition and is the basis of the seminal field theoretic work of LI and Harris and Lubensky. Their 1-loop calculation for the $\theta$-transition, however, contains a systematic error in the RG procedure, and as a consequence their long-standing 1-loop results for the collapse transition are strictly speaking not correct although the numerical deviation from the correct results is fortunately small.

Very recently, we developed a new dynamical field theory for RBPs, see for a brief account. In the present paper, we extend our work, and we present it in more detail to make it easier accessible for non-specialist readers. Our theory is based on a stochastic epidemic process which models especially dynamical percolation processes and leads to the well known Parisi–Sourlas dimensional reduction. The collapse transition is characterized by a runaway of the RG flow which suggests that this transition is a fluctuation induced first order transition contrary to what has been assumed in recent numerical studies.

The outline of our paper is as follows: In Sec. I, we derive our dynamical field theoretical model starting from the Langevin equation for a generalization of the so-called general epidemic process (GEP). We discuss different limits of this model and recast it into different forms to reveal the symmetry contents and to establish the connections to previous work in particular that of LI and PS.
In Sec. III we present the core of our RG analysis with focus on the \( \theta \)-transition. We define our RG scheme and we set up RG equations. We analyze the RG flow and its fixed points, and we point out the implication of this flow for the \( \theta' \)-transition. In Sec. IV we extract from our RG results for various observables common in polymer physics. In particular, we calculate scaling forms and critical exponents for the \( \theta \)-transition. We also present results for the fractal dimension of the minimal model in the sense of renormalized field theory. For background on field theory methods in general, we refer to \[26, 27\]. For background on dynamical field theory in the limit \( n \to 0 \), see \[18\].

Typically, one considers the partition sum for large animals: \( N \gg 1 \). The phase diagram in this limit in terms of the fugacities \( z_{cy} \) and \( z_{co} \) is shown in Fig. 4. The special curve \( z_{cy} = (z_{co} - 1) z_{co} \), parametrized by a bond-probability \( p \) as \( z_{cy} = p / (1 - p)^2 \), \( z_{co} = 1 / (1 - p) \) defines a bond-percolation model with the percolation probability \( p = p_c \) depending on the specific type of the lattice. In general, if \( N \gg 1 \), there is a swollen phase for small fugacities, and a compact phase separated by the collapse transition \( z_{co}(z_{cy}) \) which consists of two parts separated by the percolation point as a higher order critical point. Whereas in the swollen phase \( A_N(z_{cy}, z_{co}) \sim \kappa_N(z_{cy}, z_{co}) N^{-\theta} \) with universal \( \theta \) and non-universal \( \kappa_N(z_{cy}, z_{co}) \), one finds at least for the left part of the transition line the scaling law

\[
A_N(z_{cy}, z_{co}(z_{cy})) \sim \kappa(z_{cy}) N^{-\theta} \tag{2.2}
\]

with non-universal \( \kappa(z_{cy}) \), and universal \( \theta \) in general different from \( \theta \). The percolation point as a separating point on the transition line with higher order critical behavior has a \( \theta_{perc} \) which is in general different from \( \theta \) and \( \theta_{perc} \). Only in mean-field theory (Landau approximation) these exponents are equal: \( \theta = \theta_{perc} = 5/2 \).

Other fundamental quantities are given by correlations of sites on the cluster. The correlation function may be defined by

\[
G_N(r, r') = \frac{1}{A_N(z_{cy}, z_{co})} \sum_{l, c} A(N, l, c; r, r') z_{cy}^l z_{co}^c \tag{2.3}
\]

where \( A(N, l, c; r, r') \) is the total number of clusters with \( N \) sites, \( l \) loops, and \( c \) contacts, containing the lattice sites \( r \) and \( r' \). Of course it is

\[
\sum_{r} A(N, l, c; r, r') = N A(N, l, c) \tag{2.4}
\]

The radius of gyration \( R_N \) is then defined by

\[
R_N^2 = \frac{1}{2dN} \sum_{r, r'} (r - r')^2 G_N(r, r') \tag{2.5}
\]

For \( N \gg 1 \), it shows also an universal scaling law

\[
R_N \sim N^{\nu_{L}} \tag{2.6}
\]

The fractal dimension \( d_f = 1 / \nu_L \) is different at the transition line from its value in the swollen phase and at the separating percolation point. However, in mean-field theory it has the uniform value \( d_f = 4 \). Of course, in the compact phase, the fractal dimension is always equal to the lattice dimension \( d \).

In this section we develop our model for RBPs based on the GEP which is perhaps the most widely studied reaction diffusion process in the universality class of dynamical isotropic percolation. To be more specific, we use the Langevin equation for this generalized GEP which we will refine into a minimal model in the sense of renormalized field theory. For background on field theory methods in general, we refer to \[22\].

For background on dynamical field theory in the context of percolation problems, we refer to \[28\]. For a related approach to the somewhat simpler problem of directed randomly branched polymers, see \[29\].

### A. Lattice animals

Usually, one models RBPs by means of so-called lattice animals which are nothing but clusters of connected sites on a regular lattice. One considers as the primary quantity the number \( A(N, l, c) \) of all different configurations (up to translations) of a single cluster (animal) which is a collection of \( N \) sites, connected by \( b \geq N - 1 \) bonds, \( l \) cycles of the bonds, and \( c \) contacts (nearest-neighbor pairs of non-bonded sites). The number of occupied bonds is then given by \( b = l + N - 1 \). There is no need for introducing a separate number \( s \) of nearest-neighbor pairs of occupied and non-occupied sites. This number is given by the relation \( N' = 2b + 2c + s \), where \( N' \) is the lattice coordination number, which is equal to \( 2d \) on a simple hypercubic lattice. The weighted animal number

\[
A_N(z_{cy}, z_{co}) = \sum_{l, c} A(N, l, c) z_{cy}^l z_{co}^c \tag{2.1}
\]

represents a general partition sum for the system. If one sets \( z_{cy} \) to zero, the sum only includes tree configurations. It is well known that this partition function, also known as the generating function of lattice animals, can be obtained from the asymmetric \((n + 1)\)-state Potts model in the limit \( n \to 0 \). Usually, one models RBPs by means of so-called lattice animals which are nothing but clusters of connected sites on a regular lattice. One considers as the primary quantity the number \( A(N, l, c) \) of all different configurations (up to translations) of a single cluster (animal) which is a collection of \( N \) sites, connected by \( b \geq N - 1 \) bonds, \( l \) cycles of the bonds, and \( c \) contacts (nearest-neighbor pairs of non-bonded sites). The number of occupied bonds is then given by \( b = l + N - 1 \). There is no need for introducing a separate number \( s \) of nearest-neighbor pairs of occupied and non-occupied sites. This number is given by the relation \( N' = 2b + 2c + s \), where \( N' \) is the lattice coordination number, which is equal to \( 2d \) on a simple hypercubic lattice. The weighted animal number

\[
A_N(z_{cy}, z_{co}) = \sum_{l, c} A(N, l, c) z_{cy}^l z_{co}^c \tag{2.1}
\]
B. Reactions, Langevin equation, and dynamic response functional

The model that we are about to develop is in the spirit of Landau’s ideas for modeling second-order phase transitions, i.e., it is a mesoscopic model that focuses on general principles unifying processes belonging to the same universality class and is therefore necessarily phenomenological [28]. To set the stage, however, we find it worthwhile to discuss in some detail a specific model belonging to the RBP universality class, viz. a generalization of the GEP. The reaction-diffusion equations defining this process will nurture our intuition and will help us to establish our ideas.

The following generalization of the GEP is a variant of a process that we have introduced for the description of tricritical isotropic percolation [22]. We denote by \( X(\mathbf{r}) \) an agent, i.e., an infected individual, at site \( \mathbf{r} \). An agent can infect a neighboring site \( \mathbf{r} + \delta \) via the percolation step

\[
X(\mathbf{r}) \to X(\mathbf{r}) + X(\mathbf{r} + \delta). \tag{2.7}
\]

This fundamental reaction gives rise to spreading and branching of the epidemic. The agents spontaneously become immune (or decay) and produce spam as a marker of the agent through the reactions

\[
X(\mathbf{r}) \to Z(\mathbf{r}), \tag{2.8a}
\]
\[
X(\mathbf{r}) \to X(\mathbf{r}) + Z(\mathbf{r}), \tag{2.8b}
\]

where \( Z(\mathbf{r}) \) denotes an immune individual or spam at site \( \mathbf{r} \). In the language of forest fires, the \( Z(\mathbf{r}) \) are also referred to as debris. It is the debris left behind by the epidemic that forms the clusters which serve us as prototypes for RBPs. Their self-avoidance or excluded volume interaction is modelled with help of the reaction

\[
X(\mathbf{r}) + kZ(\mathbf{r}) \to (k + 1)Z(\mathbf{r}), \tag{2.9}
\]

where \( k = 1, 2, \ldots \), which damps the epidemic. A mechanism for the RBPs to compactify is introduced into the process through the reaction

\[
X(\mathbf{r} - \delta) + Z(\mathbf{r} + \delta) \to X(\mathbf{r} - \delta) + X(\mathbf{r}) + Z(\mathbf{r} + \delta), \tag{2.10}
\]

which simulates an effective attraction of the agents by the debris.

Having these reactions, one possible way to proceed would be to reformulate the corresponding master-equation in terms of bosonic creation and annihilation operators and then to produce a field theoretic action from these operators via coherent state path integrals [30]. However, we prefer to extract directly the mesoscopic Langevin equations that incorporate the universal features of the above reactions, namely the percolation of agents, their spontaneous decay, their suppression and possible effective attraction by the debris, and the possible existence of vacua without agents as absorbing states of the system.

The primary density-fields describing our generalized GEP are the field of agents \( n(\mathbf{r}, t) \) and the field of the inactive debris \( m(\mathbf{r}, t) = \lambda \int_{-\infty}^{t} dt' n(\mathbf{r}, t') \) which ultimately forms the polymer cluster. A non-Markovian Langevin equation describing such a process, and represents therefore the universality class, is given by

\[
\lambda^{-1} \partial_t n = \nabla^2 n + c \nabla m \cdot \nabla n - \left[ r + g' m + \frac{f'}{2} m^2 \right] n + \zeta. \tag{2.11}
\]

Here, the parameter \( r \) tunes the "distance" to the percolation threshold. Below this threshold, i.e., in the absorbing phase, \( r \) is positive. Throughout this paper, we will assume that the system is deep in the absorbing phase. In this case, a typical final cluster generated from an additional source \( q \delta(\mathbf{r}) \delta(t) \) of agents adding such a source is equivalent to specifying an initial condition for the process) consists of \( N = \langle \int d^d r m(\mathbf{r}, \infty) \rangle \approx q/r \) debris-particles, and has a mean diameter \( 1/\sqrt{r} \). However, we are interested in the large non-typical clusters, the rare events of the stochastic process, with \( N \gg q/r \). We know from percolation theory [24] that these clusters belong to the universality class of lattice animals. Hence, they are the same in a statistical sense as randomly branched polymers as far as their universal properties go. The gradient-term proportional to \( c \) describes the attractive influence of the debris on the agents if \( c \) is negative (as a negative contribution to \( g' \) does). At this point other forms of gradient-terms like \( m \nabla^2 n \) and \( n \nabla^2 m \) are conceivable. However al long as we include any one of these gradient terms into our theory, an omission of the other gradient terms has no effect on the final results, and we choose to work with the term proportional to \( c \) only for simplicity. For usual percolation problems (ordinary or tricritical), these gradient terms are irrelevant. As long as \( g' > 0 \), the second order term \( f' m^2 \) is irrelevant near the transition point and the process models ordinary percolation near \( r = 0 \) [31] or non-typical very large clusters, the swollen RBPs, for \( r > 0 \). We permit both signs of \( g' \) (negative values of \( g' \) correspond to an attraction of the agents by the debris, see above). Hence, our model allows for a tricritical instability (tricritical percolation near \( r = 0 \) [22] or the collapse transition of the RBPs for \( r > 0 \)). Consequently we need the second order term \( f' > 0 \) (which represents the self-avoidance property) to limit the density to finite values. Physically it originates from the suppression of agents by the debris. The Gaussian noise-source \( \zeta(\mathbf{r}, t) \) has correlations

\[
\langle \zeta(\mathbf{r}, t) \zeta(\mathbf{r}', t') \rangle = \left[ \lambda^{-1} g n(\mathbf{r}, t) \delta(t - t') - f n(\mathbf{r}, t) n(\mathbf{r}', t') \right] \times \delta(\mathbf{r} - \mathbf{r}'). \tag{2.12}
\]

The process is assumed to be locally absorbing, and thus all terms in the noise-correlation function contain at least one power of \( n \). The first part of the noise correlation takes into account that the agents decay spontaneously, and thus \( g > 0 \). The non-Markovian term proportional to \( f \) simulates the anticorrelating or, respectively, correlating (from attraction) behavior of the noise in regions
where debris has already been produced with \( f \) being negative if the attraction effects are overwhelming.

Two points are worth mentioning at this stage: (i) For the Langevin-equation with the local noise to be meaningful mathematically, an appropriate cut-off procedure of long wavelengths has to be used. (ii) The stochastic process (2.11) with \( c = r = g' = f' = g = 0 \) but \( f > 0 \) belongs to the universality class of self-avoiding random walks (SAW), and generates therefore the statistics of linear polymers [32].

To proceed towards a field theoretic model, the Langevin equations are now transformed into a stochastic response functional in the Ito-sense [28, 33, 35]

\[
\mathcal{J} = \int d^dx \left\{ \lambda \int dt \dot{\tilde{n}} \left[ \lambda^{-1} \partial_t - \nabla^2 - c \nabla m \cdot \nabla + r + g' m \right] + \frac{f}{2} m^2 - \frac{g}{2} \ddot{n} \right\} + \frac{f}{2} \left[ \lambda \int dt \ddot{n} \right]^2 \right\}.
\]  

(2.13)

With this functional, we now have a vantage point for the calculation of statistical quantities via path-integrals with the exponential weight \( \exp(-\mathcal{J}) \). When a source term \((\tilde{h}, \tilde{n})\) is added, where \( \tilde{h}(r,t) = h_0(r,t) = q \delta(r) \delta(t) \) and \((...,.)\) denotes an integral of a product of two fields over time and space, this functional describes, in particular, the statistics of clusters of debris generated by the stochastic process (2.11) from a source of \( q \) agents at the point \( r = 0 \) at time zero. Denoting by \( \text{Tr}[...] \) the functional integration over the fields, we generally have

\[
\text{Tr}[\exp(-\mathcal{J} + (\tilde{h}, \tilde{n}) + (h, n))] = 1
\]  

(2.14)

if \( h \) or \( \tilde{h} \) are zero. The first property follows from causality whereas the second one originates from the absorptive properties of the process. Note that the role of causality and adsorptivity can be interchanged via the duality transformation \( m(r,t) \leftrightarrow -\ddot{n}(r,-t) \) [28, 31, 36].

C. Branched polymers as rare events

Averaging an observable \( O[n] \) over final clusters of debris (the RBPs) of a given mass \( N \) generated from a source \( h_0(r,t) = q \delta(r) \delta(t) \) at the origin \( r = 0 \) at time \( t = 0 \) leads to the quantity [28, 31, 36]

\[
\langle O \rangle_N \mathcal{P}(N) = \left\langle O[n] \delta(N - M) \exp\left((\tilde{h}, \tilde{n})\right) \right\rangle
\]  

\[
= \text{Tr} \left[ O[n] \delta(N - M) \exp\left(-\mathcal{J} + q \tilde{n}(0,0)\right) \right]
\]  

\[
\approx q \text{Tr} \left[ O[n] \tilde{n}(0,0) \delta(N - M) \exp(-\mathcal{J}) \right],
\]  

(2.15)

where

\[
\mathcal{P}(N) = \langle \delta(N - M) \exp(q \tilde{n}(0,0)) \rangle
\]  

(2.16)

is the probability distribution for finding a cluster of mass \( N \).

\[
\mathcal{M} = \int d^4r dt \lambda n(r,t) = \int d^4r m_{\infty}(r)
\]  

(2.17)

is the total mass of the debris. The field \( m_{\infty}(r) = m(r, t = \infty) \) describes the distribution of the debris after the epidemic has become extinct. Since the probability distribution should be proportional to the number of different configurations, we expect by virtue of universality arguments the following proportionality between the probability distribution \( \mathcal{P}(N) \) and the lattice animal number \( A_N \) for asymptotically large \( N \):

\[
A_N \sim N^{-1} \kappa_0 N \mathcal{P}(N),
\]  

(2.18)

where \( \kappa_0 \) is an effective coordination number of the underlying lattice. The fugacities in \( A_N(z_\text{cy}, z_\text{co}) \) are then considered as analytical functions of the different parameters in the response functional \( \mathcal{J} \) or vice versa. The factor \( N^{-1} \) arises in Eq. (2.18) because the generated clusters are rooted at the source at the point \( r = 0 \), and each site of a given lattice animal may be the root of given cluster. Hence, we expect a scaling

\[
\mathcal{P}(N) \sim N^{1-\theta} \rho_0^N
\]  

(2.19)

with an universal scaling exponent \( \theta \) but non-universal \( \rho_0 \).

In actual calculations, the delta function appearing in averages like in Eq. (2.10) is hard to handle. This problem can be simplified by using Laplace-transformed observables like, e.g., the Laplace transformation of \( \mathcal{P}(N) \), which are functions of a variable conjugate to \( N \), say \( z \),

\[
\mathcal{P}(N) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dz}{2\pi i} e^{zN} \langle \exp(-zM + q \tilde{n}(0,0)) \rangle,
\]  

(2.20)

and applying inverse Laplace transformation (where all the singularities of the integrand lie to the left of the integration path) in the end. Note that the relationship between \( \mathcal{P}(N) \) and \( A_N \) signals the existence of a singularity \( \sim (z - z_c)^{\delta - 2} \) of the integrand in Eq. (2.20) at some critical value \( z_c \). The switch to Laplace-transformed observables can be done in a pragmatic way by augmenting the original \( \mathcal{J} \) with a term \( z \mathcal{M} \) and then working with the new response functional

\[
\mathcal{J}_z = \mathcal{J} + z \mathcal{M}.
\]  

(2.21)

Denoting averages with respect to the new functional by \( \langle ... \rangle_z \), and defining

\[
q \Phi(z) = \ln(\exp(q \tilde{n}))_z \approx q \langle \tilde{n} \rangle_z
\]  

(2.22)

for small \( q \), we get by using Jordans lemma that the asymptotic behavior for large \( N \) is given by

\[
\mathcal{P}(N) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dz}{2\pi i} e^{zN + q \Phi(z)}
\]  

\[
= e^{z_c N + q \Phi(z_c)} \int \frac{dz'}{2\pi i} e^{z'N} + q \Phi(z_c + z') + O(q^2)
\]  

\[
\approx q e^{z_c N + q \Phi(z_c)} \int_0^\infty dx \frac{\text{Disc} \Phi(z_c - x)}{2\pi i} e^{-z_x N},
\]  

(2.23)
where the last row gives the asymptotics for large $N$ and small $q$. Here, $z_c$ is the first singularity of $\Phi(z)$, which as we will show is a branch point on the negative real axis, and the contour of the path integral is deformed into a path above and below the branch cut beginning at the singularity. $\Phi$ denotes the discontinuity of the function $\Phi$ at the branch cut. The non-universal factor \[ q e^{c z_c N + q \Phi(z_c)} \] depending exponentially on $N$ is common to all averages defined by Eq. (2.15), and therefore cancels from all mean values $\langle \mathcal{O} \rangle_N$.

**D. Mean-field theory**

Before we assent to the heights of field theory (or decent to its depths, if the reader prefers, we first apply a mean-field approximation to our theory, i.e., we solve the functional integrals with the weight $\exp(-J_z)$ using a saddle-point approximation. The linear term in $J_z$ that is proportional to the Laplace-variable $z$ leads to a non-zero saddle-point value of the field $\tilde{n}$:

$$\tilde{n}_{SP} = \Phi(z).$$

(2.24)

Therefore, shifting this field, $\tilde{n} \to \tilde{n} + \Phi$, so that

$$\langle \tilde{n} \rangle_z := \text{Tr}[\tilde{n} \exp(-J_z)] = 0$$

(2.25)

the harmonic (Gaussian) part of $J_z$ becomes

$$J_z^{(0)} = \int d^d x \left\{ \lambda \int dt \tilde{n} \{ -\partial^2_n - (r - g \Phi) \} n + \frac{c \Phi}{2} (\nabla m_\infty)^2 + \frac{\Phi}{2} (g' + f \Phi)m_\infty^2 + (z + r \Phi - \frac{g}{2} \Phi^2)m_\infty \right\}.$$  

(2.26)

Here, we have implied that the saddlepoint-value of $m_\infty$ is zero, i.e., we have assumed that $\rho = (g' + f \Phi)\Phi$ is positive. If $\rho = 0$, which is the case near the tricritical instability of our stochastic process, a phase transition to a positive value of $\langle m_\infty \rangle$ sets in. Whether or not this transition is the anticipated collapse transition deserves some further scrutiny. A shift

$$\tilde{n} \to \tilde{n} + \alpha m_\infty$$

(2.27)

(which does not change the condition (2.23)) changes $\Phi(g' + f \Phi)$ to $\Phi(g' + f \Phi) + \alpha \tau$, where $\tau = r - g \Phi$. The special value $\alpha = -c \Phi/2$ eliminates the gradient-term $\sim (\nabla m_\infty)^2$ and hence $\rho = 0$ signals the collapse only if $\tau$ goes to zero which is indeed the critical value corresponding to large clusters with $N \gg 1$. This can be seen from the saddlepoint condition $h = z + r \Phi - g \Phi^2/2 = 0$ that leads to

$$g \Phi(z) = r - \sqrt{r^2 + 2gz}.$$  

(2.28)

Thus, the meanfield-solution shows a branch-point singularity at $z_c = -r^2/2g$, and $\tau(z) = \sqrt{r^2 + 2gz}$ becomes zero at this singularity.

Until now, we have kept the gradient term proportional to $c$ in our theory. The discussion in the last paragraph revealed that this term is redundant in the sense of field theory as it can be eliminated via the shift transformation (2.27). Hence, we will formally set $c = 0$ unless noted otherwise.

Next, let us calculate $P(N)$ from Eq. (2.23). Inserting $\Phi(z)$ from Eq. (2.28), we easily obtain the probability density of branched polymers with size $N$ in mean-field approximation,

$$P(N) = \frac{q}{\sqrt{2\pi g}} N^{-3/2} \exp \left( \frac{rq}{2g} - \frac{r^2}{2} N - \frac{q^2}{2g} N^{-1} \right).$$

(2.29)

The maximum of this distribution is found at $N = N_0 = q/r$. For $N \gg q/r$, the distribution drops down exponentially. However, this is the region of rare events of our stochastic process where the large branched polymers are found. Hence, small $q$ means effectively $q \ll r N$, and $q = 1$ is ‘small’ in this region. Combining Eqs. (2.22) and (2.18), we obtain the asymptotic result

$$P(N) \sim N^{-3/2} \exp(-r^2 N/2g),$$

(2.30)

and the well-known mean-field animal exponent $\theta = 5/2$ common to the swollen phase, the percolation point, as well as the collapse transition-line.

Now, we calculate the monomer distribution (the distribution of the debris-particles) of a single large cluster rooted at the point $r = 0$. We recall from our remarks above that such a root is represented field theoretically by an insertion of the field $\tilde{n}(0,0)$. According to Eq. (2.15), the monomer distribution is given by the inverse Laplace transformation of the correlation function calculated with the harmonic response functional (2.26):

$$\langle m_\infty(r)\tilde{n}(0,0) \rangle_z = G_{1,1}(r;z) = \int \frac{dk}{k^2} \frac{\exp[ik \cdot r]}{\tau(z) + k^2} \frac{dz}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{-(s/4)z} G_{1,1}(r;z) \frac{ds}{s^{d/2-1}}.$$  

(2.31)

It follows that

$$G_N(r) = \frac{1}{P(N)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dz}{2\pi i} e^{sN G_{1,1}(r;z)} \int_0^\infty \frac{ds}{(4\pi s)^{d/2}} \exp(-gs^2/2N - r^2/4s).$$

(2.32)

This function can be written in terms of generalized hypergeometric series $gF_2$, however, we prefer the integral representation shown in Eq. (2.32). Easily, we verify the sum rule

$$\int d^d r G_N(r) = N.$$  

(2.33)

The radius of gyration $R_N$ can be calculated straightforwardly from its definition,

$$R_N^2 = \frac{1}{Nd} \int d^d r G_N(r) r^2 = (2\pi N/g)^{1/2}.$$  

(2.34)
Hence, the gyration exponent is $\nu_A = 1/4$ as anticipated. The integral representation (2.32) yields the asymptotic forms of the monomer distribution for $|r| \ll R_N$

$$G_N(r) \sim \frac{1}{|r|^{d-4}}, \quad (2.35)$$

and

$$G_N(r) \sim \frac{N}{R_N^2} \left(\frac{R_N}{|r|}\right)^{(d-2)/3} \exp \left(-\frac{3\pi^{1/3}}{4} \left(\frac{|r|}{R_N}\right)^{4/3}\right) \quad (2.36)$$

if $|r| \gg R_N$. We see that the monomer-distribution in the fractal interior of the cluster has a fractal dimension $d_f = 4$ independent of $N$. The distribution in the outer region drops down exponentially in $|r|$, however, with an exponent $4/3 = 1/(1 - \nu_A)$. Besides the exponential factor, the distribution decreases algebraically with an exponent $(d-2)/3 = (d/2 - \nu_A + 2 - \theta)/(1 - \nu_A)$. We will show later on that these scaling relations comprising the independent critical exponents $\theta$ and $\nu_A$ hold generally and are not restricted to the mean-field approximation.

Another interesting quantity is the correlation of two roots. Evidently, two roots can either belong to one cluster or they can belong to two separate clusters. Their correlation function is of some value in polymer physics because it determines the second virial coefficient of the equation of state of a dilute solution of branched polymers. The connected part of this correlation function, i.e., the cumulant, is given by

$$\langle \tilde{n}(r,0)\tilde{n}(0,0) \rangle^{(cum)} = C(r;z) = -\rho \int \frac{\exp(ik \cdot r)}{(\tau(z) + k^2)^2} ~ dk$$

$$= -\rho \int_0^\infty \frac{ds}{(4\pi s)^{d/2}} s \exp(-s\tau(z) - r^2/4s). \quad (2.37)$$

Inverse Laplace-transformation leads to

$$C_N(r) \sim \frac{-\rho N^{3/2}}{R_N^2} \left(\frac{R_N}{|r|}\right)^{(d-4)/3} \times \exp \left[-\frac{3\pi^{1/3}}{4} \left(\frac{|r|}{R_N}\right)^{4/3}\right] \quad (2.38)$$

in the region $|r| \gg R_N$, where $N$ should be understood here as the total number of monomers. Since the correlation of roots on the same cluster goes down proportionally to the density of monomers on one single cluster, the increasing behavior of the fraction

$$C_N(r)/G_N(r) \sim -\rho N^{1/2} \left(\frac{|r|}{R_N}\right)^{2/3} \quad (2.39)$$

results mainly from the interaction of two separate clusters. They are repelling one another if $\rho$ is positive, and attracting one another for negative $\rho$. The sharp difference between repelling and attracting is a clear signature of the collapse transition located at $\rho = 0$. Note that a contribution to $\rho$ proportional to $\tau$ as discussed above leads only to a change of the pivotal factor $\rho N^{1/2}$ of order 1 since $\tau(z)$ converts to a term $\sim N^{-1/2}$ through the inverse Laplace-transformation.

E. Dynamical response functional revisited

Now, we return to our response functional to refine it into a form that suits us best for our actual field theoretic analysis. As discussed above, the gradient term proportional to $c$ is redundant. To eliminate this term, we apply to the field $\tilde{n}$ the shift and mixing transformation

$$\tilde{n}(r,t) \rightarrow \tilde{n}(r,t) + \Phi - c\Phi m_\infty(r), \quad (2.40)$$

where $\Phi$ is a free parameter at this stage. Defining in consistency with our mean-field considerations above $\tau = r - g\Phi$, $\rho = (\rho' + f\Phi)\Phi - c\Phi \tau$, $h = z + r\Phi - g\Phi^2/2$, the stochastic functional $\mathcal{J}_z$ [2.31] takes the form

$$\mathcal{J}_z = \int d^dx \left\{ \int dt \left[ \lambda \int \tilde{n} \left[ \rho \frac{\rho'}{2} \dot{\Phi} + \frac{g_1}{6} m_\infty + \frac{g_2}{2} \right] \tilde{n} \right] + \int dt \left[ \tilde{\tau} \right]^{\partial_\tau \tilde{\tau} \Phi} + \int dt \left[ \tilde{\tau} \right]^{\partial_\tau \tilde{\tau} \Phi} \right\}. \quad (2.41)$$

Here, we could have set $\tau$ equal to zero by exploiting that $\Phi$ is a free parameter. Instead of doing so, we rather keep $\tau$ in our theory as a small free parameter. We will see later on that keeping $\tau$ comes in handy for renormalization purposes. In Eq. [2.31], we have eliminated couplings that are of more than third order in the fields because they are irrelevant. We do not write down in detail the relatively uninteresting relations between the new third-order coupling constants and the old ones. Note that $\mathcal{J}_z$ contains two similar couplings: $g_2 m_\infty$ and $g_1 n m_\infty$. Whereas the first coupling respects causal ordering, which means that $\tilde{n}$ is separated by an infinitesimal positive time-element from the $nm$-part resulting from the Ito-calculus [35], the second one respects causal-ity only between $\tilde{n}$ and $n$. In contrast to the $m$-part, the $m_\infty$-part contains all the $n$ with times that lie in the past and in the future of $\tilde{n}$. This property is the heritage of the time-delocalized noise term. Even if we had disregarded the noise term proportional to $f$ in Eq. [2.18], initially, the $nnm_\infty$-coupling would be generated by coarse graining, and hence it must be ultimatively incorporated into the theory to yield a renormalizable theory.

The relevance of the different terms in $\mathcal{J}_z$ follows from their dimensions with respect to an inverse length scale $\mu$ such that time scales as $\mu^{-2}$. Fundamentally, one has to decide which parameters are the critical control-parameters going to zero in mean-field theory. As we have seen, at the collapse transition these are $\tau \sim \rho \sim \mu^2$, and $h \sim \mu^{d+2}/2$. The dimensions of the fields are then given by $\tilde{n} \sim m \sim \mu^{(d-2)/2}$, and $n \sim \mu^{(d+2)/2}$. It follows that all the coupling constant $g_0$, $g_1$, $g_2$, and $g_2'$ have the same dimension $\mu^{(d-4)/2}$. Note that $\tilde{n}$ is tied always to at least one factor of $n$ as a result of absorptivity of the process. Hence, we have retained all the couplings that are
relevant for \( d \leq 6 \) spatial dimensions, and the model is renormalizable below the upper critical dimension \( d_c = 6 \) of the collapse transition. The situation is different if \( \rho \) is a finite positive quantity, that is in the swollen phase. Then \( \rho \) can be absorbed into the fields by a scale transformation which amounts to formally setting \( \rho = 2 \). The field dimensions then become \( m \sim \mu^{d/2} \), \( \nu \sim \mu^{(d+4)/2} \), and \( \chi \sim \mu^{(d-4)/2} \). It follows that \( h \sim \mu^{d/2} \), \( g_0 \sim \mu^{-d/2} \), \( g_1 \sim \mu^{(4-d)/2} \), \( g_2 \sim \mu^{(2-d)/2} \), and \( g_2 \sim \mu^{(8-d)/2} \). Hence, in the swollen phase only \( g_2 = g \) is relevant, now below \( 8 \) spatial dimensions. The other couplings can be safely removed.

F. Quasi-static limit and ghosts

In the following, we focus on the static properties of the generated clusters after the epidemic has become extinct. Here, we are interested only in time-dependent static expectation values of the form \( \langle \prod_i (\tau_i \rho_{\tau_i}(r_i) \prod_j \tilde{n}(r_j, 0) \rangle \). Thus, we take the quasi-static limit \( \tau \to 0 \), see Appendix A, by setting \( \tilde{n}(r, t) \to n_0(r) =: \varphi(r) \) in the dynamic response functional \( J_z \). We rename \( m_\infty(r) =: \tilde{\varphi}(r) \) and get

\[
J_z \to H_{qs} = \int d^d x \left\{ \tilde{\varphi}(\tau - \nabla^2)\varphi + \frac{1}{2} \tilde{\varphi}^2 + h\tilde{\varphi} + \frac{g_0}{6} \tilde{\varphi}^3 + g_1 \tilde{\varphi}^2 \tilde{\varphi} + \frac{1}{2} \tilde{\varphi} (g_2 \tilde{\varphi} - g_2 \varphi) \varphi \right\},
\]

(2.42)

where we have denoted the original time-delocalisation of the \( \tilde{n}m_\infty \)-term by a separating dot in \( \tilde{\varphi}\varphi \cdot \tilde{\varphi} \). Using the quasi-static limit, one has to be careful to account for the former causal ordering of fields in the diagrammatic perturbation expansion. This means that one has to rule out diagrams with closed propagator-loops. But note that only the \( \varphi \varphi \cdot \tilde{\varphi} \)-term can contribute to such a closed loop.

Of course these additional rules make the perturbation expansion very clumsy in higher loop-order calculations. Fortunately there exists an elegant way to overcome these difficulties associated with the additional rules by introducing so-called ghost fields whose sole purpose is to generate additional diagrams that cancel any diagrams with non-causal propagator-loops. Such a procedure does not change the physical content of the theory but simplifies calculations and makes it easier to find higher symmetries. To one-loop order, non-causal loops are easily cancelled by a corresponding loop of contrary sign. The ghost fields for producing such loops that come to mind first are a pair of fermionic fields. Note, however, that \( D \) independent similar bosons can also create a loop with a negative sign in the limit \( D \to -2 \), see Fig. 2.

To be more specific, the ghost fields that we use are \( D \) independent bosonic fields \( \langle \psi_1, \ldots, \psi_D \rangle \), in the limit \( D \to -2 \) which is taken at the end of the calculation. These ghosts are incorporated into our theory by adding the term

\[
\frac{1}{2} \sum_{k=1}^D \psi^* \left[ \tau_0 - \nabla^2 + \left( (g_1 + g_2)\varphi - g_2 \varphi \right) \right] \psi_k
\]

(2.43)
to the integrand of \( J_z \). Note that this term arises formally if one replaces each causal ordered \( \tilde{n}n \)-pair in \( J_z \) by the sum over \( \psi_k \psi_{k/2} \)-pairs. Here comes a new symmetry into play: the additional term (2.43) is trivially invariant under any permutation of the \( D \) ghost fields \( \psi_k \), i.e., we have symmetry under the permutation (or symmetric) group \( S_D \). However, since in general \( \sum_{k=1}^D \psi_k \neq 0 \), this representation is reducible. Hence, it is more useful to introduce new ghost fields \( \chi_1, \ldots, \chi_{D+1} \) with constraint \( \sum_{\alpha=1}^{D+1} \chi_\alpha = 0 \), and \( \sum_{k=1}^D \psi_k^2 = \sum_{\alpha=1}^{D+1} \chi_\alpha^2 = \chi^2 \). This is easily achieved by using \( D + 1 \) Potts–spin-vectors \( \vec{\epsilon}^{(\alpha)} = (\epsilon^{(\alpha)}_k) \) directed to the corners of a \( D \)-dimensional simplex. The spin-vectors have the usual properties: \( \sum_{\alpha=1}^{D+1} \epsilon^{(\alpha)}_k = 0 \), \( \sum_{k=1}^D \epsilon^{(\alpha)}_k \epsilon^{(\beta)}_k = 0 \), \( \sum_{k=1}^D \epsilon^{(\alpha)}_k \epsilon^{(\alpha)}_k = \delta_{\alpha \beta} - 1/(D+1) \). Hence, the relation between the old and the new ghosts are given by \( \chi_\alpha = \sum_{k=1}^D \epsilon^{(\alpha)}_k \psi_k \). Now, we have symmetry under the permutation group \( S_{D+1} \) of permutations of the \( (D + 1) \) ghost fields \( \chi_\alpha \), and this representation is irreducible.

Inspection shows that the ghosts also work in multi-loop diagrams provided that the non-causal loops are separated from each other in these diagrams. However, as long as \( g_2 \) is not zero (note that \( g_2 \) is always greater than zero because otherwise only diagrams without loops are generated), non-separated non-causal loops arise, see the first diagram in Fig. 3. The cancellation requires a permutation-symmetric irreducible interaction \( \chi^3 = \sum_{\alpha=1}^{D+1} \chi_\alpha^3 \) of the \( (D + 1) \) ghosts, see the third diagram in Fig. 3. Using these new ghosts, the quasi static
Hamiltonian becomes
\[
\mathcal{H} = \int d^d x \left\{ \bar{\psi} (\tau - \nabla^2) \psi + \frac{g}{2} \bar{\psi}^2 + h \bar{\psi} \\
+ \frac{1}{2} (\tau \chi^2 + (\nabla \chi)^2) + \frac{g_0}{6} \bar{\psi}^3 + \frac{g_1}{2} \bar{\psi} [2 \bar{\psi} \varphi + \chi^2] \\
+ \frac{1}{6} [3 \bar{\psi} (g_2^\prime \bar{\psi} - g_2 \varphi) \bar{\varphi} + 3 (g_2^\prime \bar{\psi} - g_2 \varphi) \chi^2 \\
+ \sqrt{g_2} g_2 \chi^2] \right\}.
\]
(2.44)

Perturbation theory with this Hamiltonian is no longer burdened with additional rules. It will serve as the vantage point of our RG calculations. As it stands, it is general enough to capture both the swollen phase and the collapse transition. As we have shown in mean field theory, the collapse transition corresponds to vanishing \( \tau, \rho \), and \( h \). Swollen RBPs correspond to vanishing \( \tau \) and \( h \), but positive and finite \( \rho \).

The Hamiltonian (2.44) is form-invariant under three transformations of the fields. Therefore, three parameters of the Hamiltonian are redundant. One of these transformations, the mixing \( \varphi \to \varphi + \kappa \bar{\psi}, \bar{\psi} \to \bar{\psi} \), we have already used to eliminate the gradient term \( (\nabla \bar{\psi})^2 \). The second of these transformations, the rescaling \( \varphi \to \lambda \varphi, \bar{\psi} \to \lambda^{-1} \bar{\psi} \) can be used either to identify coupling-constants \( g_2^\prime = g_2 \), or to transform \( g_2 \) to one and use only scaling-invariant quantities. Via the third transformation, the shift \( \varphi \to \varphi + \gamma, \bar{\psi} \to \bar{\psi} \), either \( \tau \) or \( \rho \) can be transformed away. At this point, a word of caution is in order. Using these transformations to eliminate parameters from the field theoretic functional, one is well advised to make sure that non of the parameters \( \kappa, \lambda, \gamma \) featured in the transformations is singular. Otherwise, parameters eliminated from the unrenormalized theory will have to re-emerge in the renormalization procedure. This is no problem per se, but it is a fact that can be easily overlooked, and if so, will lead to ill-defined renormalization schemes.

Before moving on to our actual RG calculation, we find it worthwhile to comment on the renormalizability of \( \mathcal{H} \). Simple power counting shows that the ghosts have the same dimensionality as the fields \( \varphi \) and \( \bar{\varphi} \), namely \( \lambda_\alpha \sim \mu^{(d-2)/2} \). For the swollen phase, the coupling constants \( g_0, g_1, \) and \( g_2^\prime \) are irrelevant and hence can and should be set equal to zero. Then one can easily ascertain that the remaining \( \mathcal{H} \) contains all the relevant terms generated under renormalization, and hence \( \mathcal{H} \) is renormalizable as far as the swollen phase is concerned. For the collapse transition, the situation is more intricate. Simple inspection by means of power counting lends credence to the renormalizability of \( \mathcal{H} \). However, one has to be more careful here, because the way the various \( g \)'s appear in multiple places, i.e., a given \( g \) may appear as a factor of different monomials of the fields, viz. in couplings amongst the ghost, in couplings amongst the primary fields \( \varphi \) and \( \bar{\varphi} \), and in couplings of the primary fields and the ghosts. Does this spoil renormalizability? The answer is clearly no because we know for certain that \( \mathcal{H} \) is renormalizable by virtue of its equivalence in the quasi-static limit to the renormalizable dynamic functional \( \mathcal{J}_z \) which is renormalizable. Hence, there must exist some hidden symmetry that masks the renormalizability of \( \mathcal{H} \).

Once revealed, this underlying symmetry will provide for relations between different vertex-functions. We will show shortly that this is the symmetric group \( S_{D+2} \) (not only the permutation symmetry \( S_{D+1} \) of the \( D+1 \) ghosts alone) of the permutation of \( (D+2) \) field combinations. First, however, we will look briefly at a 1-loop calculation that underpins and exemplifies the considerations just presented.

G. 1-loop diagrams with ghosts

The elements of our diagrammatic perturbation expansion, the propagators, the correlators, and the vertices, are listed in Fig. (4) and Fig. (5), respectively. For the time being, we focus here just on the decorations of Feynman diagrams, i.e., the combinations of coupling-constants and symmetry-factors of the diagrams without the integrations over loop-momenta. We list the relevant 1-loop diagrams, writing them in a form that makes evident the cancellations in the limit \( D \to -2 \). For the tadpole-diagrams, Fig. (6), we find

1a) \[ g_2 + \frac{D}{2} g_2 \to 0, \] (2.45a)
1b) \[ -\left[ g_1 + (g_1 + g_2^\prime) \right] - \frac{D}{2} (g_1 + g_2^\prime) \to -g_1. \] (2.45b)

The selfenergy-diagrams, Fig. (7), yield

2a) \[ g_2^2 + \frac{D}{2} g_2^2 \to 0, \] (2.46a)
2b) \[ -\frac{1}{2} g_2 (2g_1 + g_2^\prime) - g_2 [g_1 + (g_1 + g_2^\prime)] \\
- \frac{D}{2} g_2 (g_1 + g_2^\prime) \to -2g_1 g_2 - \frac{1}{2} 2g_2 g_2^\prime, \] (2.46b)
2c) \[ -g_0 g_2 + \left[ g_1 + (g_1 + g_2^\prime) \right]^2 \\
+ \frac{D}{2} (g_1 + g_2^\prime)^2 \to -g_0 g_2 + 3g_1^2 + 2g_1 g_2^\prime. \] (2.46c)
In the same way we obtain the decorations of the vertex-diagrams, Figs. (5) to (11).

\[ 3a) \quad 2g_2^3 + Dg_2^3 \to 0, \quad (2.47a) \]
\[ 3b) \quad -2g_2^3(2g_1 + g_2^2 - 2g_2^2[g_1 + (g_1 + g_2)] \]
\[ -Dg_2^3(g_1 + g_2^2) \to -6g_1g_2^2 - 2g_2^2g_2', \quad (2.47b) \]
\[ 3c) \quad 2g_2(2g_1 + g_2^2)^2 - 2g_0g_2^2 + 2g_2[g_1 + (g_1 + g_2)]^2 \]
\[ + Dg_2(g_1 + g_2^2)^2 \to 14g_1^2g_2 + 12g_1g_2g_2' \]
\[ + 2g_2g_2^2 - 2g_0g_2^2, \quad (2.47c) \]
\[ 3d) \quad 6g_0g_2(2g_1 + g_2^2) - 2[g_1 + (g_1 + g_2)^3] \]
\[ + 2D(g_1 + g_2^2)^3 \to 12g_0g_1g_2 + 6g_0g_2g_2' \]
\[ - 14g_1^3 - 18g_1g_2g_2' - 6g_1g_2^2. \quad (2.47d) \]

Of course, the cancellation of non-causal loops, see Fig. (2), should occur also in higher loop-orders. Hence, not only the propagator and the ghost-correlator must be equal but also the full Greens functions \( \langle \phi(\mathbf{r}) \phi(0) \rangle \) and \( \langle \chi(\mathbf{r}) \chi(0) \rangle \). Therefore, the 1-loop self-energy diagrams 2b shown in Fig. (7) must be equal to the 1-loop self-energy diagrams 2d of ghost shown in Fig. (12):

\[ 2d) = -2g_2(g_1 + g_2^2) + \frac{1}{2} \left( \frac{D - 1}{D + 1} \right) g_2^2g_2' \]
\[ \to -2g_1g_2 - \frac{1}{2} g_2^2g_2' = 2b. \quad (2.48) \]

Hence, the \( \sim \chi^3 \) self-interaction of ghosts featured in \( \mathcal{H} \) is needed already at 1-loop order to guarantee the equality of self-energies.

**H. Hidden symmetry and relation to other models**

Now, we come back to the search for the symmetry that ensures the renormalizability of the Hamiltonian \( \mathcal{H} \), Eq. (2.44). At first glance, this Hamiltonian only has the permutation-symmetry \( S_{D+1} \) of the \( (D+1) \) ghost-fields \( \langle \chi(\mathbf{r}) \rangle \). Next we use the form-invariance of the Hamiltonian under a rescaling of the original fields

\[ \varphi \to \lambda \varphi, \quad \varphi \to \lambda^{-1} \tilde{\varphi}, \quad (2.49) \]

which is compensated for and hence becomes a scaling symmetry when augmented by the following redefinition of parameters:

\[ g_0 \to \lambda^3 g_0, \quad g_1 \to \lambda g_1, \quad g_2' \to \lambda g_2', \quad \rho \to \lambda^2 \rho, \quad h \to \lambda h. \quad (2.50) \]

Under the choice \( \lambda = \sqrt{g_2'/g_2} \), which is possible as long as \( g_2' \neq 0 \), we gain the equality \( g_2' = g_2 \). Now, it is easy to show that the Hamiltonian \( \mathcal{H} \) is invariant for each \( \alpha = 1, \ldots, D + 1 \) under the resonator-transformations

\[ \varphi \to \varphi, \quad \varphi \to (\varphi - \tilde{\varphi}) - \chi_\alpha, \quad \chi_\alpha \to -\chi_\alpha - 2\tilde{\varphi}, \quad \chi_\beta \to (\chi_\beta - \chi_\alpha) - \varphi, \quad (2.51) \]

for all \( \beta \neq \alpha \), and always in the limit \( D \to -2 \). This invariance ensures, e.g., the equality of the ghost-correlation functions with the propagator

\[ \langle \chi_\alpha \chi_\beta \rangle = \left( \delta_\alpha \beta - \frac{1}{D + 1} \right) \langle \varphi \tilde{\varphi} \rangle \quad (2.52) \]

FIG. 8. 1-loop vertex diagrams a.
that we have demonstrated explicitly to 1-loop order above. The mirror-transformations which mix original fields with ghosts complete the permutation-symmetries of the ghosts to the full symmetry-group $S_{D+2}$.

To make this hidden symmetry more transparent, we define a new order parameter field with $(D + 3)$ components: $s_0 = \bar{\varphi}$, $s_1 = -\varphi$, and for $\mu \geq 2$: $s_\mu = \chi_{\mu-1} - (\bar{\varphi} - \varphi)/(D + 1)$. With $s^k := \sum_{\mu=0}^n s_\mu$, where $n = D + 2$, we have $s^1 = 0$, and in the limit $n \to 0$

$$s^2 = 2\bar{\varphi}\varphi + \chi^2,$$

$$s^3 = 3\bar{\varphi}(\bar{\varphi} - \varphi)\varphi + 3(\bar{\varphi} - \varphi)\chi^3. \tag{2.53a}$$

$$s^6 = 0, \text{ and in the limit } n \to 0$$

Using this order parameter, it is easy to see that the Hamiltonian can be written as

$$\mathcal{H}_{\text{sp}} = \int d^4x \left\{ \frac{1}{2}(\gamma s^2 + (\nabla s)^2) + \frac{\mu}{2} s_0^2 + h s_0 + \frac{g_0}{6} s_0^6 + \frac{g_1}{2} s_0 s^2 + \frac{g_2}{2} s^3 \right\}, \tag{2.54}$$

which is identical to the Hamiltonian of Eq. (2.44), in the limit $n \to 0$. It is therefore equivalent to our original dynamical model. The Hamiltonian $\mathcal{H}_{\text{sp}}$ describes the field theory of the asymmetric $(n+1)$-state Potts model. The previously hidden symmetry is now the symmetry $S_n$ of permutations of the $n$ fields $(s_1, \ldots, s_n)$. As mentioned in the introduction, the established theories for RBP1, RBP2 and RBP3 are mainly based on the asymmetric Potts model, and the Hamiltonian (2.51) therefore establishes the connection with these theories. Note, furthermore, that for $g_0 = g_1 = 0$ the Hamiltonian (2.51) describes the symmetric $(n+1)$-state Potts model with a linear and quadratic (so-called hard) symmetry breaking. The interaction represented by the third-order terms has $S_{n+1}$-symmetry and yields the field theory of percolation in the limit $n \to 0$. There exists another connection:

$$\mathcal{H}_{\text{sp}} = \int d^4x \left\{ \frac{1}{2}(\gamma s^2 + (\nabla s)^2) + \frac{\mu}{2} s_0^2 + h s_0 + \frac{g_0}{6} s_0^6 + \frac{g_1}{2} s_0 s^2 + \frac{g_2}{2} s^3 \right\}.$$

one can show that for $4g_0 = 2g_1 = -g_2$ the Hamiltonian $\mathcal{H}_{\text{sp}}$ decomposes in a sum of $n$ uncoupled Hamiltonians each describing the Yang-Lee edge problem. The choice of these special combinations of coupling-constants yield important checks for higher order calculations [23].

Now, we turn to the case that $g_2'$ is zero where we cannot rescale the fields to attain $g_2' = g_2$. As we will show, $g_2'$ becomes zero at the fixed point of our model for the collapse, and it is irrelevant for the model in the swollen phase of the RBP. Hence, the case $g_2' = 0$ is important in general for the statistics of branched polymers. Now the third order coupling $\sim \chi^3$ of the ghosts in $\mathcal{H}$, Eq. (2.44), vanishes. The ghosts appear only quadratic, and we can integrate them out formally producing a ghost-determinant raised to the power $(-D/2)$. Taking the limit $D \to -2$, this determinant can be reimported into the Hamiltonian by introducing a pair $(\bar{\psi}, \psi)$ of anti-commuting fermionic ghost-fields. The Hamiltonian becomes

$$\mathcal{H}_{\text{ss}} = \int d^4x \left\{ \bar{\psi}(\tau - \nabla^2)\psi + \frac{\rho}{2} \bar{\psi}^2 + h \bar{\psi} \
\right. \
+ \bar{\psi}(\tau - \nabla^2 + g_1 \bar{\psi} - g_2 \psi) \psi \
+ \frac{g_0}{6} \bar{\psi}^3 + g_1 \bar{\psi}^2 \bar{\psi} - \frac{g_2}{2} \bar{\psi}^3 \right\}. \tag{2.55}$$

Introducing Grassmannian anticommuting super-coordinates $\theta, \bar{\theta}$ with integration rules $\int d\theta = \int d\bar{\theta} = 0, \int d\theta \bar{\theta} = \int d\bar{\theta} d\theta = 1$, and defining a super-field $\Phi(r, \bar{\theta}, \theta) = i \varphi(r) + \bar{\theta} \psi(r) + \psi(r) \theta + i d\bar{\theta} \bar{\psi}(r)$, the Hamiltonian $\mathcal{H}_{\text{ss}}$ takes the form

$$\mathcal{H}_{\text{ss}} = \int d^4x d\bar{\theta} d\theta \left\{ \frac{1}{2} \Phi \left( \tau - \nabla^2 - \rho \partial_\bar{\theta} \partial_\theta \right) \Phi + i h \Phi \
+ i \left( \frac{g_2}{6} \Phi^3 + g_1 \Phi^2 \left( \partial_\bar{\theta} \partial_\theta \Phi \right) - \frac{g_0}{6} \Phi (\partial_\bar{\theta} \partial_\theta \Phi)^2 \right) \right\}. \tag{2.56}$$

This Hamiltonian shows Becchi–Rouet–Stora (BRS)-symmetry [24, 27], i.e., $\mathcal{H}_{\text{ss}}$ is invariant under a super-transformation $\theta \rightarrow \theta + \varepsilon, \bar{\theta} \rightarrow \bar{\theta} + \varepsilon$. Moreover, if the control parameter $\rho$ is positive and finite, i.e., if we consider the problem of swollen RBP’s, $\rho$ can be reset by a scale transformation of the super-coordinates to 2. The super-coordinates get a dimension $\sim \mu^{-1}$ equal to the dimension of the spatial coordinates, and the derivatives combine to a super-Laplace operator $\nabla^2 + \rho \partial_\bar{\theta} \partial_\theta \nabla^2 + 2 \partial_\bar{\theta} \partial_\theta = : \Box :$. As we have shown above, the coupling constants $g_0$ and $g_1$ become irrelevant and hence can be neglected in which case the Hamiltonian takes the super-Yang-Lee form

$$\mathcal{H}_{\text{YLYL}} = \int d^4x d\bar{\theta} d\theta \left\{ \frac{1}{2} \Phi \left( \tau - \Box \right) \Phi + i \frac{g_2}{6} \Phi^3 + i h \Phi \right\}, \tag{2.57}$$

where we have set $g_2 = g$. The Hamiltonian $\mathcal{H}_{\text{YLYL}}$ has, besides the super-translational invariance, super-rotation invariance. Now dimensional reduction can be used to reduce the problem to the normal Yang-Lee problem in two lesser dimensions. This establishes the connection
between our model and the work of Parisi and Sourlas\cite{4} on swollen RBPs.

Before moving on to the core of our RG analysis, we would like to highlight the following implication of our symmetry considerations for the collapse transition. We will see later on that $g'_2$ vanishes at the RG fixed point describing the $\theta$-transition. Thus, this transition is associated with BRS symmetry which is in contrast to the swollen phase which is associated with full supersymmetry. The BRS symmetry indicates that the statistics of the RBPs is dominated by tree configurations. This fact can be understood, for example, by using Cardy’s presentation\cite{21} of the work of Brydges and Imbrie\cite{20}. Cardy reformulates their model of swollen RBPs in $d$ dimensions which is exactly reducible to the problem of the universal repulsive gas singularity in $d - 2$ dimensions which, in turn, belongs to the same universality class as the Yang-Lee problem in a fully supersymmetric way. If one adds an attracting potential between the monomers of the tree-polymers that can lead eventually to the collapse of the RBPs, the rotational supersymmetry is lost, and with it dimensional reduction. However, BRS symmetry is retained, and this symmetry is indeed the vehicle that reduces all configurations to trees. Another route to understand the connection between BRS symmetry and trees lies in a dynamical calculation. At first, this may sound somewhat surprising because BRS symmetry is a feature of the quasi-static Hamiltonian at the collapse fixed point. However, a calculation\cite{37} of the fractal dimension of the minimal path from the origin to swollen RBPs, the rotational supersymmetry indicates that the statistics of the swollen phase which is associated with full supersymmetry. Thus, asymptotically large RBPs at the $\theta$-transition have the topology of trees.

III. RENORMALIZATION AND THE RENORMALIZATION GROUP

Now, we turn to the core of our RG analysis. As announced above, we will base our discussion on the Hamiltonian $H$ of Eq. (2.44). Likewise, we could use $H_{\alpha p}$ with the limit $n \to 0$ which, as we have shown above, is equivalent to $H$. For our discussion here, we choose $H$ over $H_{\alpha p}$ because we feel that the relation of the former to the original GEP is somewhat more intuitive than that of the latter. Actual diagrammatic calculations in higher loop-orders, however, are better to handle when $H_{\alpha p}$ instead of $H$ is used. The renormalization-group functions that feed into our RG analysis for RBPs stem from a renormalized field theoretic calculation for the asymmetric Potts model that we performed recently. Details of this work will be presented elsewhere\cite{37}.

A. The renormalization scheme

Our main focus here lies on the collapse transition, i.e., we are mainly interested in the case that the control parameters $\tau$ and $\rho$ take critical values (zero in mean-field theory) where the correlation length diverges, and correlations between different polymers vanish. Via the equation of state this implies the critical value of $\epsilon$. The principal objects of the perturbation theory are the superficially UV-divergent vertex functions $\Gamma_{kk}$ which consist of irreducible diagrams with $k$ and $k$ amputated legs of $\bar{\varphi}$ and $\varphi$, respectively, as functions of the wave vector $q$. The UV-divergences are then handled via a renormalization scheme that introduces counter terms which absorb said divergences. For our calculations, we use minimal renormalization, i.e., dimensional regularization and minimal subtraction in conjunction with the $\varepsilon$-expansion about $d = 6$ dimensions ($\varepsilon = 6 - d$). Our renormalization scheme leading from bare to renormalized quantities reads

$$G^{1/2}_{\varepsilon}g_{\alpha} \to G^{1/2}_{\varepsilon}\tilde{g}_{\alpha} = Z^{3/2}(u_{\alpha} + B_{\alpha})\mu^{\varepsilon/2},$$

where $G_{\varepsilon}$ is a convenient numerical factor which we chose here to be $G_{\varepsilon} = \Gamma(1 + \varepsilon/2)/(4\pi)^{d/2}$. Note, however, that all choices with $(4\pi)^{3}G_{\varepsilon} = 1 + O(\varepsilon)$ work equally well since their differences only amount to a finite rescaling of the momentum scale $\mu$. We introduce the two-dimensional control-vector $A = (\rho, \tau)$, and $(g_{\alpha}) = (g_{0}, g_{1}, g_{g}, g_{z})$. In a theory regularized by means of a large momentum-cutoff $\Lambda$, the additive non-universal counter terms $\tilde{Z}_{\varepsilon}, \tilde{\sigma}_{\varepsilon}$, and $\tilde{C}_{\varepsilon}$ would diverge $\sim \Lambda^{2}, \Lambda^{4-\varepsilon/2}$, and $\Lambda^{2-\varepsilon/2}$. In our perturbative approach based on dimensional regularisation and minimal subtraction with $\varepsilon$-expansion, they formally vanish. In minimal renormalization, all the other counter-terms are expanded into pure Laurent-series, e.g.,

$$Z = 1 - \frac{Z^{(1)}}{\varepsilon} + O(\varepsilon^{-2}),$$

and so on, where the residue $Z^{(1)}, K^{(1)}, \ldots$ of the $\varepsilon$-poles are pure functions of the dimensionless renormalized coupling-constants $(u_{\alpha}) = (u_{0}, u_{1}, u_{g}, u_{z})$. We present the calculation of all the counter-terms to 1-loop order in Appendix B.

Note that the renormalization scheme\cite{44} introduces a counter-term proportional to $K$ that has no counterpart in the Hamiltonian\cite{24}. This counter-term can be viewed as a remnant of the gradient-term proportional to
the redundant parameter $c$ in the original response functional (2.13) which we removed from our model via the mixing-transformation stated in Eq. (2.40). As a counter term this term is indispensable, however, because the quadratically superficial divergent vertex function
\[
\Gamma_{2,0}(q) = \Gamma_{2,0}'(0) + q^2 \Gamma_{2,0}''(0) + \ldots \tag{3.3}
\]
contains an UV-divergent $\Gamma_{2,0}''(0)$. This fact was overlooked by LI [1] in their calculation, and their long-standing 1-loop results are incorrect although, fortunately, the numeric deviations from the correct 1-loop results are rather small. It must be stressed, however, that the omission of this counter term is not just a technical glitch that affects some numbers. Without this term, renormalization does not cure the theory from non-primitive divergences and is thus not really meaningful. In a 1-loop calculation one does not see these non-primitive divergences explicitly, and hence they are easily overlooked. At higher loop order, however, they inevitably pop up, and one can see explicitly and in detail how the theory fails if not renormalized properly.

The alert reader might ask why the different fields are renormalized with the same renormalization factor $Z$. The fields belong to two different irreducible representations of the symmetry group $S_n$, mathematically denoted by $\{n\}$ and $\{n - 1, 1\}$, the trivial and the fundamental representation, respectively. They should therefore require two independent factors $Z_0$ and $Z_1$. In general, this argumentation is correct, and hence, the primed fields and parameters are renormalized with the same renormalization factor $Z$. In a 1-loop calculation one does not see these non-primitive divergences explicitly, and hence they are easily overlooked. At higher loop order, however, they inevitably pop up, and one can see explicitly and in detail how the theory fails if not renormalized properly.

This discussion sheds another light on what went wrong in the calculation by LI. They overlooked that $Z_0$ and $Z_1$ approach their limit $Z$ differently as manifest in Eq. (3.7a). This difference, when overlooked, leads to erroneous results.

The bare Hamiltonian (2.41) is form-invariant under a rescaling of the fields that makes one of the coupling constants redundant. This rescaling can be chosen in particular so that $\beta_2' = \beta_2$ (see the discussion after Eqs. (2.49) and (2.50)) which leads to the Hamiltonian (2.54) in form of the asymmetric Potts model. Owing to the permutation-symmetry $S_n$ of this Hamiltonian, this relation holds even in renormalized form, $u_2' = u_2$, where $u_2$ is related to the bare $\beta_2$ by the renormalization factor $Z_2$. It follows the relation
\[
\frac{B_2'}{u_2'} = \frac{B_2}{u_2} = Z_2 - 1, \tag{3.8}
\]
where $Z_2$ depends only on scaling invariant combinations of the coupling constants, say
\[
u = u_2 u_2', \tag{3.9a}\]
\[
u = u_1 u_2, \tag{3.9b}\]
\[
u = u_0 u_2', \tag{3.9c}\]

B. Shift-symmetry and Ward identities

The Hamiltonian (2.44) is, as typical for a $\phi^3$-theory, form-invariant under a shift of the order parameter by an arbitrary constant. To be more specific, the Hamiltonian is form-invariant under
\[
\phi \rightarrow \phi' = \phi + \gamma, \tag{3.10}
\]
in conjunction with the parameter-change
\[
\tau \rightarrow \tau' = \tau + g_2 \gamma, \tag{3.11a}\]
\[
\rho \rightarrow \rho' = \rho - (2 g_1 + g_2') \gamma, \tag{3.11b}\]
\[
h \rightarrow h' = h - \tau \gamma - \frac{g_2}{2} \gamma^2. \tag{3.11c}\]

Note that the coupling-constants are not transformed. Hence, the primed fields and parameters are renormalized with the same counter-terms as the original ones. Thus, the transformations represent a scaling symmetry in renormalized as well as in bare form. We introduce the two-dimensional vector $f = (-2 g_1 - g_2', g_2) = G_\xi^2/\mu^2/2 \nu$ with $\nu = (-2 u_1 - u_2', u_2)$ together with its bare form $f_0$, define $\tilde{\gamma} = Z^{1/2} \gamma$, and compare the renormalizations, e.g.,
\[
Z^2 \tilde{\gamma}' = Z \cdot \tilde{\gamma}' = Z \cdot (\tilde{\xi} + \gamma \tilde{f}) = Z(\tilde{\xi} + \tilde{\gamma} \tilde{f}) = Z \cdot \tilde{\xi} + (\gamma G_\xi^1/2 \mu^{3/2} (\nu + V)), \tag{3.12}
\]
where we have defined \( \mathbf{V} = (-2B_1 - B_2, B_2) \). It follows the Ward identity
\[
(\mathbf{Z} - \mathbf{I}) \cdot \mathbf{u} = \mathbf{V}.
\]
In the same way, we derive a second Ward identity
\[
(\mathbf{u} \cdot \mathbf{A})_i = \delta_{2,i} - Z_{2,i}.
\]
In particular, we have \( B_2 = -\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u} \). Both Ward identities are easily verified at 1-loop order with the diagrammatic results given in Appendix B. They reduce higher-order calculations enormously, and lead to important relations between renormalization group functions and critical exponents. Being linear relations between the counter-terms, the Ward identities hold for each term of the Laurent-expansions, in particular for the residua
\[
Z_{2,i}^{(1)} = -\mathbf{u} \cdot \mathbf{A}^{(1)}, \quad (\text{3.15a})
\]
\[
V^{(1)} = Z^{(1)} \cdot \mathbf{u}, \quad (\text{3.15b})
\]
\[
B_2^{(1)} = -\mathbf{u} \cdot \mathbf{A}^{(1)} \cdot \mathbf{u}. \quad (\text{3.15c})
\]
It is of some interest to state the Ward identities also in terms of the Vertex functions. The shift-invariance leads to the following identity for the vertex-function generating functional (remember that no renormalizations are influenced by the shift)
\[
\Gamma[\hat{\phi}, \phi; \mathbf{Z}, h] = \Gamma[\hat{\phi}, \phi; \mathbf{Z}] + (h, \hat{\phi}) = \Gamma[\hat{\phi}, \phi + \gamma; \mathbf{Z} + f \gamma, h - \tau \gamma - \frac{\partial \hat{\gamma}}{2} \gamma^2]. \quad (\text{3.16})
\]
Differentiation with respect to \( \gamma \) leads to the Ward identities
\[
\Gamma_{k,k+1}(\{\mathbf{q} = 0\}) = \tau \delta_{k,1} \delta_{k,0} - f \frac{\partial}{\partial \mathbf{Z}} \Gamma_{k,k}(\{\mathbf{q} = 0\})
\]
\[
(\text{3.17})
\]
between the vertex-functions.

C. RG functions

RG functions express the change of the renormalized quantities under an infinitesimal change of the momentum-scale \( \mu \) (while holding bare quantities constant). They are the essential ingredients of the RG equations. As a scale change between two renormalized and therefore finite theories, the RG functions are themselves finite quantities without \( \varepsilon \)-poles. We define
\[
\beta_\alpha = \mu \partial_\mu u_\alpha|_0 = -\frac{\varepsilon}{2} u_\alpha + \beta^{(0)}_\alpha, \quad (\text{3.18a})
\]
\[
\gamma = \mu \partial_\mu \ln Z|_0, \quad (\text{3.18b})
\]
where \( \beta^{(0)}_\alpha \) and \( \gamma \) are independent of \( \varepsilon \) in minimal renormalization. It follows that
\[
\mu \partial_\mu|_0 (Z, K, \cdots) = -\frac{1}{2} \mathbf{u} \cdot \partial_\mathbf{u} (Z^{(1)}, K^{(1)}, \cdots) + O(\varepsilon^{-1}), \quad (\text{3.19})
\]
where we abbreviate \( \sum u_\alpha \partial u_\alpha =: \mathbf{u} \cdot \partial_\mathbf{u} \). Expanding in the following all expressions in Laurent-series with respect to \( \varepsilon \), and making use of the fact that all renormalized quantities are free of \( \varepsilon \)-poles, we obtain
\[
\beta^{(0)}_\alpha = \frac{3}{2} \gamma u_\alpha - \frac{1}{2} (1 - u_\alpha \partial_\mu) B^{(1)}_\alpha, \quad (\text{3.20a})
\]
\[
\gamma = -\frac{1}{2} u_\alpha \partial_\mu Z^{(1)}, \quad (\text{3.20b})
\]
\[
\gamma' = -\frac{1}{2} u_\alpha \partial_\mu K^{(1)}, \quad (\text{3.20c})
\]
so that
\[
\mu \partial_\mu|_0 \hat{\phi} = -\frac{\gamma}{2} \hat{\phi}, \quad (\text{3.21a})
\]
\[
\mu \partial_\mu|_0 \varphi = -\frac{\gamma}{2} \varphi - \gamma' \hat{\phi} \quad (\text{3.21b})
\]
in Greens functions. Similarly, we get
\[
\mu \partial_\mu|_0 Z = \varphi \cdot \hat{\varphi}, \quad (\text{3.22a})
\]
\[
\mu \partial_\mu|_0 h = \frac{\gamma}{2} h + \frac{1}{2} G^{1/2} \mu^{-\varepsilon/2} (\mathbf{Z} \cdot \hat{\mathbf{A}} \cdot \mathbf{Z}), \quad (\text{3.22b})
\]
where we have defined
\[
\hat{\mathbf{A}} = \frac{\gamma}{2} + \frac{1}{2} u_\alpha \partial_\mu (Z^{(1)})^T, \quad (\text{3.23a})
\]
\[
\hat{\mathbf{Z}} = \frac{1}{2} (1 + u_\alpha \partial_\mu) \mathbf{A}^{(1)}. \quad (\text{3.23b})
\]
It is now easy to derive relations between the Gell-Mann–Low functions with help of the Ward identities (3.15). We obtain
\[
\hat{\alpha} = \gamma + \frac{1}{2} u_\alpha \partial_\mu (\mathbf{Z}^{(1)})^T, \quad (\text{3.24a})
\]
\[
\hat{\beta}_1 = \frac{\gamma}{2} \mathbf{u} + \mathbf{v} \cdot \hat{\mathbf{Z}}, \quad (\text{3.24b})
\]
\[
\hat{\beta}_2 = \frac{3}{2} \gamma - \varepsilon u_2 - u_2 \cdot \hat{\mathbf{A}} \cdot \mathbf{u}. \quad (\text{3.24c})
\]
Here we used the two-dimensional vectors \( \mathbf{u} = (-2u_1 - u_2, u_2) \) and \( \hat{\beta} = (-2\beta_1 - \beta_2, \beta_2) \). In Appendix B, we state all the RG functions to 1-loop order. With the results given there, the relations (3.24) are verified easily.

D. RG equations

Now, we derive the RG equations that determine how the quantities featured in our theory transform or flow under variation of the momentum-scale \( \mu \). In order for the RG equations to produce reliable results, we have to remove a this stage any remaining scaling-redundancy that could contaminate the RG-flow. For example, if we continued using the variables of Sec. III C we were at risk to erroneously conclude from Eq. (3.24) that there is an eigenvalue \( \varepsilon - \gamma \alpha \) of the matrix \( \hat{\mathbf{k}} \) at a fixed point \( (u_\alpha)_s \) with \( \sum \hat{\beta} = 0 \).
To remove the one remaining scaling redundancy from our theory, we switch to rescaling invariant fields
\[ \phi = u_2 \varphi, \quad \tilde{\phi} = u_2^{-1} \tilde{\varphi}, \]  
(3.25)
ccontrol parameters \( \underline{\lambda} = (\sigma, \tau) \) with
\[ \sigma = u_2^2 \rho, \]
(3.26a)
\[ H = 2g_2 h, \]
(3.26b)
and the dimensionless coupling constants given by Eqs. (3.29). This procedure yields the new \( \beta \)-functions
\[ \beta_u = u_2 \beta'_2 + u_2^2 \beta_2, \]
(3.27a)
\[ \beta_v = u_2 \beta_1 + u_2^2 \beta_2, \]
(3.27b)
\[ \beta_w = u_2^3 \beta_0 + 3u_0 u_2^2 \beta_2. \]
(3.27c)
The Gell-Mann–Low functions designated with a hat change to
\[ \gamma' = u_2^2 \gamma', \]
(3.28a)
\[ \kappa_{11} = \kappa_{11} + \zeta, \quad \kappa_{12} = u_2^2 \kappa_{12}, \]
(3.28b)
\[ \kappa_{21} = u_2^2 \kappa_{21}, \quad \kappa_{22} = \hat{\kappa}_{22}, \]
(3.28c)
\[ \alpha_{11} = u_2^3 \alpha_{11}, \quad \alpha_{12} = u_2^2 \alpha_{12}, \quad \alpha_{22} = u_2 \alpha_{22}, \]
(3.28d)
where we have defined
\[ \zeta = \frac{\beta_u}{u} = 2 \frac{\beta_2}{u_2} = 2 \frac{\beta'_2}{u_2}. \]
(3.29)
Note that in case of \( u = 0 \), the function \( \zeta \) is in general finite and non-zero.

Now, we are in the position to set up our ultimate RG equations. The generator \( D_\mu \) of the RG, i.e., the derivative \( \mu \partial_{\mu} \) purely expressed in terms of renormalized parameters, is given by
\[ D_\mu = \mu \frac{\partial}{\partial \mu} + \underline{\lambda} \cdot \cdot \beta \cdot \frac{\partial}{\partial \underline{\lambda}} + \beta_u \frac{\partial}{\partial \beta_u} + \beta_v \frac{\partial}{\partial \beta_v} + \beta_w \frac{\partial}{\partial \beta_w}. \]
(3.30)
Its application to the fields in a correlation function produces the RG equations
\[ D_\mu \tilde{\phi} = - \frac{\gamma + \zeta}{2} \tilde{\phi}, \]
\[ D_\mu \phi = - \frac{\gamma - \zeta}{2} \phi - \gamma' \tilde{\phi}. \]
(3.31a)
In addition the RG equation of the external field \( H \), which is linearly related to \( z \) (the integration variable of the inverse Laplace transformation) and the control-parameters \( \underline{\lambda} \) are
\[ D_\mu H = \frac{\gamma + \zeta + \varepsilon}{2} H + \underline{\lambda} \cdot \cdot \beta \cdot \underline{\lambda}, \]
(3.32a)
\[ D_\mu \underline{\lambda} = \underline{\lambda} \cdot \underline{\kappa}, \]
(3.32b)
We introduce the combination
\[ a = u + 2v = -v_1 v_2, \]
(3.33)
with the corresponding Gell-Mann–Low function \( \beta_a = \beta_u + 2\beta_v \), and the 2-dimensional orthogonal vectors
\[ \underline{w} = (a^{-1}, 1), \quad \underline{v} = (-a, 1). \]
(3.34)
The Ward identities (3.24) yield
\[ \kappa_{22} = \gamma \delta_{1,2} - (\underline{w} \cdot \underline{v}), \]
(3.35)
and the important relations between RG functions
\[ (\underline{w} \cdot \underline{a})_2 = (\varepsilon - \gamma + \zeta)/2, \]
(3.36a)
\[ a^{-1} \beta_a = -\underline{w} \cdot \underline{\kappa} \cdot \underline{w}, \]
(3.36b)
The last equation in combination with the orthogonality of \( \underline{w} \) and \( \underline{v} \) shows that these vectors are for \( \beta_a = 0 \) right and left eigen-vectors of \( \underline{\kappa} \), respectively, with eigenvalues
\[ \kappa_1 = (\underline{w} \cdot \underline{a})_2 = a (\underline{w} \cdot \underline{w})_1 - a^{-1} \beta_a, \]
(3.37a)
\[ \kappa_2 = (\underline{w} \cdot \underline{w})_2 = (\varepsilon - \gamma + \zeta)/2. \]
(3.37b)
Note that \( \kappa_2 = (\varepsilon + \zeta - \gamma)/2 \) determines the RG-flow of the order-parameter field \( \phi \), Eq. (3.31a). This shows that each control parameter combination proportional to \( \underline{w} \) is redundant and can be eliminated by an order-parameter shift. Otherwise, the combination
\[ y := \underline{w} \cdot \underline{w} = a^{-1} \sigma + \tau \]
(3.38)
is free of the shift-redundancy and has the the independent scaling exponent \( \kappa_1 \). We expect that \( y \) defines the distance from the collapse transition line in the phase diagram.

To 1-loop order, our diagrammatic calculation leads to
\[ \beta_u = -\varepsilon + \frac{7}{2} u + 10v \]
(3.39a)
\[ \beta_v = -\varepsilon + \frac{25}{6} u + \frac{21}{2} v - \frac{5}{6} w, \]
(3.39b)
\[ \beta_w = -2 \varepsilon + \frac{21}{2} u + 25v \]
\[ - \left( 5u^2 + \frac{29}{2} uv + 11v^2 \right) w, \]
(3.39c)
\[ \gamma = \frac{u + 4v}{6}, \quad \gamma' = \frac{2uv + 3v^2 - w}{6}, \]
(3.39d)
and the matrices
\[ \underline{\kappa} = \begin{pmatrix} 8(2u + 5v)/3 - \varepsilon, & -1 & 5(u - 2w - 3\varepsilon^2)/3, & 5(u + 4v)/6 \end{pmatrix}, \]
(3.40a)
\[ \underline{\alpha} = \begin{pmatrix} 0, & 1, & 1, & -2v \end{pmatrix}. \]
(3.40b)
With these 1-loop results, the general results (3.35), (3.36) and (3.37), which hold to all loop-orders, are easily verified.
TABLE I. RG fixed points to leading order.

| \(u_\ast\) | \(v_\ast\) | \(w_\ast\) | stability |
|---|---|---|---|
| G | 0 | 0 | 0 | -- -- |
| C | \(\frac{\epsilon}{760}\) | \(\frac{\epsilon^2}{5 \times 760^2}\) | ++ | |
| P | \(\frac{\epsilon}{760}\) | 0 | 0 | ++ -- |
| YL | \(\frac{\epsilon}{2}\) | \(-\frac{\epsilon^2}{9}\) | ++ -- |
| In1 | \(\frac{11\epsilon}{40}\) | \(-\frac{517\epsilon^2}{8000}\) | ++ -- |
| In2 | \(\frac{\epsilon}{760}\) | \(-\frac{\epsilon^2}{5 \times 760^2}\) | ++ -- |

E. RG flow and fixed points

The fixed points of our RG are determined by the zeros of the Gell-Mann–Low RG functions for the three coupling-constants as given in Eqs. (3.39a) to (3.39c). The picture of the topology of the fixed points, invariant lines, and separating surfaces resulting from the RG flow that arises from these equations in the three-dimensional space spanned by these coupling-constants is sketched in Fig. 13. The BRS-plane \(u = 0\) (red) is an invariant plane of the flow equations (3.39a) to (3.39c) to all orders and divides the \((u, v, w)\)-space in two parts: the percolation-part with \(u > 0\) (blue, I) and the Yang-Lee-part with \(u < 0\) (green, I and II). The latter part is non-physical for the branched polymer problem. The percolation line \(v = w = 0\) is an invariant line for both signs of \(u\). For \(u > 0\) the flow goes to the percolation fixed point (P) whereas for \(u < 0\) the flow tends to infinity. The Yang-Lee-line (bold green line) with \(a = b = 0\), where \(a = u + 2v\) and \(b = u^2 + 4w\), is also an invariant line for both signs of \(u\). For \(u < 0\) the flow goes to the Yang-Lee fixed point (YL) whereas for \(u > 0\) the flow runs away to infinity. Altogether we have six fixed points which are compiled in Table I to 1-loop order. Besides the trivial Gaussian fixed point (G) we find in the BRS-plane the stable collapse fixed point (C), and an unstable fixed point (In2). This point lies on a separatrix in the BRS-plane (bold red line) and is attracting on it. The flow of the part which contains C is of course attracting to C. The other part shows runaway flow. Turning to the percolation-part of the \((u, v, w)\)-space, there is the aforementioned unstable percolation fixed point P on the percolation line \(v = w = 0\). Because P has two stable directions, it defines a separating invariant surface with P as an attracting fixed point that divides the space in two parts. The flow in one of it goes to C whereas the flow in the other part is again running away. The separating surface, the stability plane of P for \(u > 0\), is a continuation of the separatrix found above on the BRS-plane for \(u = 0\). In the Yang-Lee-part of the \((u, v, w)\)-space, we also find a separating surface which is the continuation of the BRS-separatrix now into the region with \(u < 0\). This invariant surface is separated in two parts by the Yang-Lee-line. One part is attracting to an unstable fixed point (In1), the other part shows runaway flow. Both surfaces divide the \((u, v, w)\)-space in a wedge-shaped part attracting to C, and a part where the flow goes to infinity. The edge of the wedge is the separatrix in the BRS-plane. Note that the two separating surfaces are not smoothly connected at the separatrix since the BRS-plane is itself a separating surface.

FIG. 13. (Color online) Sketch of the invariant manifolds of the RG-flow as explained in the main text.

The line labelled phys. (brown) is closely related to the collapse line in the phase diagram, Fig. 11 and its meaning is as follows. Recall that we focus on asymptotically large RBP’s, and hence the external field \(h\) is near criticality. The control parameter \(y\) and the three coupling constants are thought to be expressed as functions of the two fugacities spanning the phase diagram. At the collapse, i.e., when \(y\) becomes critical, the two fugacities are not independent and hence, the coupling-constants can be parametrized in terms of a single fugacity. Hence the collapse line in the phase diagram corresponds to a line in the flow diagram which we represent by the brown line. As long as this line lies above the percolation surface, the RG flows to C. From the point where the brown line pierces the blue percolation surface, the RG flows to the percolation fixed point P. From any point on the line below the percolation surface, the RG runs off to infinity.

Before returning to the \(\theta\)-transition as our main focus, we would like point out the following lesson regarding the \(\theta\)-transition that our flow diagram teaches. Usually, runaway flows are associated with fluctuation induced first order transitions. Here, the region below the percolation surface where the coupling-constants runs away to ever more positive values indicate that this transition might be discontinuous and not, as previously assumed, a second order transition.

F. Scaling at the collapse-transition

Now, we determine the scaling behavior of the order parameter \(\langle \bar{n} \rangle_z = \langle \bar{\varphi} \rangle_z = \Phi/g_2\) (here we have included a factor \(g_2\) in the definition of \(\Phi\) for convenience), the correlation function of \(\varphi\) and \(\bar{\varphi}\) and the correlation length. The external field \(H = 2g_2h\) (which is a linear function of the Laplace-variable \(z\)) is related to \(\Phi\) via the equation

\[
\Phi = \frac{2g_2h}{1 - \frac{2g_2h}{\epsilon}}
\]
of state
\[ h + \delta \Gamma[\varphi, \varphi; z] \delta \varphi = 0 , \]  
(3.31)

where \( \Gamma[\varphi, \varphi; z] \) is the vertex generating functional. The equation of state guarantees that tadpole insertions in diagrams are cancelled by the external field \( h \), and \( \varphi(\varphi) = 0 \) after the shift \( \varphi \rightarrow \varphi + \Phi/g_2 \). Using again the shift-symmetry of the vertex generating function, Eq. (3.10), the equation of state (3.31) is reduced to
\[ H + \tau^2 = (\tau - \Phi)^2 + T(\sigma + a\Phi, \tau - \Phi) , \]  
(3.32)

where \( T(l) = -2g_2\Gamma_{1,0}(\tau) \) is the sum of the tadpole-diagrams, which we have calculated to 1-loop order. To find \( \Phi \) as a function of \( z \), we invert equation (3.32) and obtain \( (T - \Phi) \) as a function of \( (H + \tau^2) \) and \( y \). The inverse has according to (3.41) a critical point at a value of \( \Phi \) where
\[ g_2 \frac{\partial}{\partial \Phi} \left( \frac{\delta \Gamma[\varphi, \varphi; z]}{\delta \varphi} \right)_{\varphi=0, \varphi=\Phi/g_2} = \Gamma_{1,1}(q = 0, \sigma + a\Phi, \tau - \Phi) = 0 . \]  
(3.33)

This condition determines eventually the critical value \( z_c \) of the inverse Laplace-transformation where the first singularity in the complex \( z \)-plane is positioned. It is therefore the value where the correlation length \( \xi(z) \sim 1/\sqrt{\Gamma_{1,1}(q = 0)} \) tends to infinity.

To find the scaling behavior of \( \Phi \) as a function of \( (z - z_c) \) near this critical point, we examine the RG flow of the shift-invariant combinations of control parameters \( y = (\tau + a^{-1}\sigma), M = (\tau - \Phi), \) and \( L = (\tau^2 + H) \sim (z - z_c) \).

The RG equations for these combinations are easily derived from the equations (3.31a), (3.31b), and (3.31c) using the properties which follow from the Ward-identities. They are given by
\[ D_\mu y = \kappa_1 y , \]  
(3.43a)
\[ D_\mu M = \kappa_2 M + \kappa_{1,2} a y , \]  
(3.43b)
\[ D_\mu L = (\kappa_2 + \gamma) L + \alpha_{1,1} a^2 y^2 . \]  
(3.43c)

The solutions of these flow equations at a fixed point in terms of a flow parameter \( l \) such that \( \mu(l) = \mu l \) are given by
\[ y(l) = l^{\kappa_1} y , \]  
(3.45a)
\[ M(l) + p_1 y(l) = l^{\kappa_2} (M + p_1 y) , \]  
(3.45b)
\[ L(l) + p_2 y(l)^2 = l^{(\kappa_2 + \gamma)} (L + p_2 y^2) , \]  
(3.45c)

where \( p_1 = [\kappa_{1,2} a/(\kappa_2 - \kappa_1)] \), and \( p_2 = [\alpha_{1,1} a^2/(\kappa_2 + \gamma - 2\kappa_1)] \). Taking into account the naive dimensions of \( M, y, \) and \( L \), the relation between these quantities as the inversion of Eq. (3.32) is
\[ (M(l) + p_1 y(l))/\mu(l)^2 = F((L(l) + p_2 y(l)^2)/\mu(l)^4, y(l)/\mu(l)^2) \]  
(3.46)
in dimensionless form. Choosing \( l \) so that \( (L(l) + p_2 y(l)^2)/\mu(l)^4 = 1 \), we obtain the order-parameter equation in scaling form
\[ M + p_1 y = (L + p_2 y^2)^{\beta/\Delta} \]  
(3.47)

and setting \( (L + p_2 y^2) \sim (z - z_c) \) and \( (M + p_1 y) \sim (\Phi_c - \Phi) \), we obtain
\[ \Phi_c - \Phi = (z - z_c)^{\beta/\Delta} \Phi \]  
(3.48)

Here, the scaling function \( \Phi \) is identical to \( \Phi \) up to some non-interesting constant factors, and the critical exponents are given by the fixed point values of the various RG-functions
\[ 1/\nu = 2 - \kappa_{1,1} , \]  
\[ \gamma = \gamma + \zeta , \]  
\[ \gamma = \gamma + \zeta , \]  
\[ \beta/\nu = 2 - \kappa_{2,2} = 1/2(d - 2 + \eta), \]  
\[ \Delta/\nu = 4 - \kappa_{2,2} - \gamma = 1/2(d + 2 - \eta). \]  
(3.49)

If \( \zeta \neq 0 \), which happens if \( u_* \neq 0 \) and thus holds true at the collapse-transition, we find three independent critical exponents \( \gamma, \eta, \) and \( \nu \).

The RG equation for the correlation function \( G_{1,1}(r) = \langle \phi(r)\phi(0) \rangle_{z}^{(cum)} \) follows from Eq. (3.31a) as
\[ (D_\mu + \gamma)G_{1,1}(r) = 0 \]  
(3.50)
at a fixed point. Using again the flow parameter \( l \), we obtain the solution
\[ G_{1,1}(r, y, M + p_1 y, \mu) = \Gamma_{1,1}(r, l^{\kappa_1} y, l^{\kappa_2} (M + p_1 y), \mu l) = l^{d-2+\gamma} G_{1,1}(r, y/\mu^{1/\nu}, (M + p_1 y)/\mu^{1/\nu}). \]  
(3.51)

Taking \( y \) and \( (z - z_c) \) as independent variables, and expressing \( (M + p_1 y) \) through the equation of state (3.31), we find after choosing \( l \) as above the scaling form
\[ G_{1,1}(r, z) = \frac{G_{1,1}(r, z - z_c)^{\beta/\Delta} y/(y - z_c)^{\beta/\Delta}}{|r|^{d-2+(\eta+\beta)/2}} . \]  
(3.52)

The correlation length \( \xi \) is defined by
\[ \xi^2 = \frac{1}{2d} \int d^d r |r|^2 G_{1,1}(r)/ \int d^d r G_{1,1}(r) = \frac{\partial \ln \Gamma_{1,1}(q)}{\partial q^2} \bigg|_{q=0} , \]  
(3.53)

where the vertex function \( \Gamma_{1,1}(r) \) is related to the Fourier-transformed correlation function by \( \tilde{G}_{1,1}(q) = 1/\Gamma_{1,1}(q) \). Hence the correlation length scales as
\[ \xi(z) \sim (z - z_c)^{-\nu/\Delta} . \]  
(3.54)

In terms of \( \xi \), the correlation function is given by
\[ G_{1,1}(r, z) = \frac{G_{1,1}(r, z/\xi)^{\beta/\Delta} y/\xi^{\beta/\Delta}}{|r|^{d-2+(\eta+\beta)/2}} . \]  
(3.55)
with the radius of gyration

\[ R_N = N^{\nu_A} R(yN^\phi). \]  \hfill (4.6)

Its exponent is given by

\[ \nu_A = \nu/\Delta = \frac{2}{d+2-\eta}. \]  \hfill (4.7)

As it should, our result satisfies the sum rule

\[ \int d^dx G(x,yN^\phi) = 1 \]  \hfill (4.8)

Next, we state our \( \varepsilon \)-expansion results for the exponents governing the collapse transition. Thus far, when it came to the diagrammatic part of our theory, we centered our discussion around the 1-loop order of our calculation to keep matters as simple as possible. Our actual calculation, however, went to higher order which allows us to present here results for the critical exponents of the \( \theta \)-transition to second order in \( \varepsilon \). Details of this calculation will be presented elsewhere \[37\]. For completeness, we list in Appendix \( \ref{Appendix} \) our 2-loop results for the RG functions that went into the calculation of the critical exponents. For the three independent exponents defined in Eqs. \((4.3)\) and \((4.4)\), we obtain

\[ \theta = \frac{5}{2} - 0.4925 \varepsilon(6) - 0.5778 \varepsilon(6)^2, \]  \hfill (4.9a)

\[ \phi = \frac{1}{2} + 0.0225 \varepsilon(6) - 0.3580 \varepsilon(6)^2, \]  \hfill (4.9b)

\[ \nu_A = \frac{1}{4} + 0.1915 \varepsilon(6) + 0.0841 \varepsilon(6)^2, \]  \hfill (4.9c)

From these expansions, we derive numerical results of the exponents for dimensions 2 to 5 by performing simple Padé-estimates \[26, 27\]. These results are compiled in Table \( \ref{TableII} \).
Next, we consider corrections to scaling. To determine the leading corrections, it is useful to distinguish between two phenomena. First, there is the irrelevance of cycles near the $\theta$-transition and the associated crossover to tree-behavior with BRS symmetry. For this crossover, the coupling constant $u$ is proportional to the cycle fugacity $z_{cy}$. Using the RG result $u(l) = ul^{\xi}$ and choosing a small parameter $l$ proportional to $R_N^{-1}$, we find that this crossover leads to a correction to all scaling functions proportional to $u/N^x$, where

$$x_u = \nu_A \zeta_u = d\nu_A + 1 - \Theta$$

(4.9d)

is the corresponding crossover exponent. Second, there is the approach of the coupling constants $v$ and $w$ to their fixed-point values. This approach is described by the eigenvalues of the matrix of first derivatives of the functions $\beta_v$ and $\beta_w$, respectively,

$$\omega_1 = \varepsilon - 0.7614 \varepsilon^2, \quad \omega_2 = 1.0344 \varepsilon - 0.6830 \varepsilon^2.$$  

(4.9e)

These so-called Wegner exponents lead to corrections proportional to $N^{-x_i}$ with $x_i = \nu_A \omega_i$.

**B. The shape of the collapsing branched polymer**

Here we will derive the asymptotic forms of the shape function $G(r/R_N, yN^g)$, Eq. (4.3), of the monomer distribution for small and large $|r|/R_N$ at the collapse transition line $y = 0$. We use methods analogous to methods applied in [39–41] to the case of linear polymers.

In a first and somewhat hand-waving approach, we assume that the monomer distribution in the interior of the branched polymer is independent of the size $N$. Hence, for $x \to 0$, we should have

$$G(x, 0) \sim x^{-d+1/\nu_A},$$

(4.10)

leading to the monomer distribution for $|r| \ll R_N$

$$G_N(r) \sim \frac{1}{|r|^{d-1/\nu_A}}.$$ 

(4.11)

Next, we derive this result more rigorously by application of the short distance expansion. The leading terms of the operator product expansions are given by

$$\hat{\phi}(r + x/2)\hat{\phi}(r - x/2) = c_1(x, \mu)\hat{\phi}(r),$$

(4.12a)

$$\phi(r + x/2)\phi(r - x/2) = c_2(x, \mu)\phi(r) + c_3(x, \mu)\hat{\phi}(r),$$

(4.12b)

$$\phi(r + x/2)\phi(r - x/2) = c_4(x, \mu)\phi(r) + c_5(x, \mu)\phi(r).$$

(4.12c)

The form of these expansions is dictated by the symmetry of our model: $\hat{\phi}$ belongs to the trivial representation of the permutation group $S_n\to0$, and $\phi$ has components belonging to the trivial and the fundamental representation. The scaling behavior of the functions $c_i(x, \mu) \sim \mu^{(d-3)/2}$ follows from the RGE. Applying the RG differential $D_u$ operator to both sides of (4.12) and comparing the results, we find

$$D_\mu c_{1,3}(x, \mu) = -\frac{\eta}{2} c_{1,3}(x, \mu)$$

(4.13)

at the collapse fixed point. Hence

$$c_{1,3}(x, \mu) = (l\mu)^{d/2} c_{1,3}(l\mu, x) = (l\mu)^{d/2} c_{1,3}(l, 1) = \frac{c_{1,3}(l, 1)}{\mu^{\nu/2} |x|^{d-1/\nu_A}}.$$ 

(4.14)

Using Eq. (4.12b), we obtain

$$G_{1,1}(r; z) \sim \frac{\Phi(z)}{|r|^{d-1/\nu_A}}.$$  

(4.15)

This argument has to be taken with a grain of salt. Strictly speaking, the operator product expansion has to be inserted in Greens functions that are superficially convergent, otherwise one has to deal with additive renormalizations. Therefore $G_{1,1}(r; z)$ in Eq. (4.15) is determined only up to a polynomial in $z$. However, this polynomial is cancelled by the inverse Laplace transformation as long as $N > 0$. Hence, after the application of the inverse Laplace transformation to Eq. (4.15) and division by $P(N)$, we indeed get the result stated in Eq. (4.11).

Now we turn to the large $|r|$ (or small $|q|$) behavior of the correlation function. In this regime, the appropriate vertex function is well approximated by

$$G_{1,1}(r; z) \sim \xi(z)^{-2+(n+\eta)/2}.$$ 

(4.16a)

$$G_{1,1}(r; z) \sim \xi(z)^{-2+(n+\eta)/2}.$$ 

(4.16b)

The correlation function has the representation

$$G_{1,1}(r; z) \sim \xi(z)^{-(\eta+\tilde{\eta})/2} \int_0^\infty ds \exp \left(-\xi(z)^{-2}s-r^2/4s\right).$$

(4.17)

Taking the condition $r^2/\xi(z)^2 \gg 1$, $N \gg 1$ into consideration, we calculate the monomer distribution employing a double saddle-point approximation of the $s$- and $z$-integral. We find the distribution in the form of Eq. (4.3) with the shape function

$$G(x, 0) \sim x^{-t} \exp \left(-cx^{1/(1-\nu_A)}\right).$$

(4.18)

c is a constant, and the exponent is

$$t = d - \frac{d/2 - 2 + \theta}{1 - \nu_A}.$$ 

(4.19)

**C. Fractal dimensions**

We conclude this section by briefly discussing the fractal dimensions associated with RBPs. As discussed on
several occasions in this paper, collapsing RBP s have a tree-like structure, i.e., they are quasi one dimensional. Thus, the dimension $d_{\text{min}}$ of the shortest path between two points on the polymer, also known as the chemical distance, the backbone dimension $d_b$, and the resistor dimension $d_r$ coincide. The fractal dimension $d_f$ governing the total mass of the RBP is $d_f = 1/\nu_A$, and the exponent for random walks on a RPB is given by $d_w = d_{\text{min}} + d_f$. From what we have presented thus far in this paper, we know $d_f$ to 2-loop order. Knowing the other fractal dimensions requires to calculate $d_{\text{min}}$, which is identical to the dynamical exponent $\varepsilon$ of our model. This calculation is beyond the scope of this paper and will be presented elsewhere [57]. For completeness, however, we find it useful to mention here the results of our dynamical calculation. For the $\theta$-transition, we find

$$d_{\text{min}} = 2 - 0.8756 (\varepsilon/6) - 1.1528 (\varepsilon/6)^2. \tag{4.20}$$

For the swollen RPBs, we obtain

$$d_{\text{min}} = 2 - (\varepsilon/9) - \frac{35}{18} (\varepsilon/9)^2, \tag{4.21}$$

where $\varepsilon = 8 - d$ because $d = 8$ is the upper critical dimension for the swollen phase. Padé-estimates are given in Table III.

| $d$ | $d_{\text{min}}$ (swollen) | $d_{\text{min}}$ (collapse) |
|-----|-----------------------------|-----------------------------|
| 2   | 1.09                        | 1.21                        |
| 3   | 1.22                        | 1.415                       |
| 4   | 1.37                        | 1.624                       |
| 5   | 1.536                       | 1.8277                      |
| 6   | 1.707                       | 2                           |
| 7   | 1.868                       | 2                           |
| 8   | 2                           | 2                           |

TABLE III. Padé-estimates of the minimal dimension.

V. CONCLUDING REMARKS AND OUTLOOK

In summary, we developed a new, dynamical field theory for isotropic randomly branched polymers, and we used this model in conjunction with the RG to take a fresh look at this classical problem of statistical physics. We demonstrated that our model provides an alternative vantage point to understand the swollen phase via dimensional reduction. We corrected and pushed ahead the critical exponents for the $\theta$-transition. We showed that at the stable fixed point the model has BRS symmetry. Hence, asymptotically the RPBs are dominated by tree configurations. Our RG analysis produces evidence for the $\theta'$-transition being a fluctuation induced first order transition and not as previously assumed a second order transition. It would be interesting to see if future experimental or numerical studies can confirm the latter finding.

Complementary to the quasi-static RG analysis presented in this paper, we have also conducted a field theoretic calculation of the dynamical exponent $\varepsilon$ of our dynamical model [35]. This calculation produced the first ever field theoretic results, quoted above, for the fractal dimension $d_{\text{min}}$ of the shortest path and related fractal dimensions for RPBs. We are currently completing a three-loop calculation of the asymmetric Potts-model. This calculation pushes the exponents $\theta$, $\phi$ and $\nu_A$ to third order in $\varepsilon$ [8].

Appendix A: The quasi-static limit

This Appendix provides some background on the quasi-static limit that we invoke in Sec. III in the derivation of our field theoretic Hamiltonian. Let us consider a dynamic response functional of the general form

$$\mathcal{F}[\tilde{n}, n] = \int d^d x d t \lambda \tilde{n} \left[ \lambda^{-1} \partial_t + \tau - \nabla^2 \right] n + \mathcal{W}[\tilde{n}, n], \tag{A1}$$

where the interaction-part $\mathcal{W}$ reduces to a time independent functional $\mathcal{W}[\tilde{n}_0, m_\infty] \sim \tilde{n}_0(\mathbf{r})$ and $m_\infty(\mathbf{r}) = \lambda \int_{-\infty}^{+\infty} d t \tilde{n}(\mathbf{r}, t)$ after setting $\tilde{n}(\mathbf{r}, t) \rightarrow \tilde{n}_0(\mathbf{r}) = \tilde{n}(\mathbf{r}, 0)$. We define

$$\mathcal{H}_{qs}[\tilde{n}_0, m_\infty] := \mathcal{F}[\tilde{n}_0, n] = \int d^d x \tilde{n}_0 \left[ \tau - \nabla^2 \right] m_\infty + \mathcal{W}[\tilde{n}_0, m_\infty], \tag{A2}$$

where $\mathcal{H}_{qs}$ denotes the quasi-static Hamiltonian. The free causal propagator

$$G(\mathbf{r} - \mathbf{r}', t - t') = \langle n(\mathbf{r}, t) \tilde{n}(\mathbf{r}', t') \rangle_0 \sim \theta(t - t') \tag{A3}$$

with $\theta(t) = 1$ if $t > 0$ and $\theta(t) = 0$ if $t \leq 0$ becomes the static propagator of $\mathcal{H}_{qs}$ after time integration

$$\lambda \int_{-\infty}^{\infty} d t \langle n(\mathbf{r}, t) \tilde{n}(\mathbf{r}', t') \rangle_0 = \lambda \int_{0}^{\infty} d t \langle n(\mathbf{r}, t) \tilde{n}(\mathbf{r}', 0) \rangle_0 = \langle m_\infty(\mathbf{r}) \tilde{n}_0(\mathbf{r}') \rangle_0 = G_{st}(\mathbf{r} - \mathbf{r}'). \tag{A4}$$

Now consider a diagram of the graphical perturbation expansion of the connected correlation function $\langle \prod_i m_\infty(\mathbf{r}_i) \prod_j \tilde{n}(\mathbf{r}_j, 0) \rangle$. By causality, the vertices of the diagram are ordered in time from ‘left’ (i.e., the largest time involved) to ‘right’ (the smallest time), $\tilde{n}$-legs are left-going, $n$-legs are right-going. Consider the first vertex which has only propagators (we suppress the space arguments) $\langle m_\infty(\mathbf{r}_{11}) \rangle_0 = \langle m_\infty \tilde{n}_0 \rangle_0$ on its $\tilde{n}$-legs. Hence, the time-dependence of the $\tilde{n}$-legs of this vertex is absorbed by the $m_\infty$, each $\tilde{n}(t_1)$ becomes a time independent $\tilde{n}_0$, and after integration over the vertex-time $t_1$, the integrated vertex becomes a vertex generated by the quasi-static interaction $\mathcal{W}[\tilde{n}_0, m_\infty]$. By induction,
one can prove that this mechanism carries through all the way to and including the last vertex. The full diagram is therefore generated only by static propagators and the interaction-vertices of the quasi-static Hamiltonian \( \mathcal{H}_q [\hat{n}_0, m_{\infty}] \). By itself, however, the quasi-static Hamiltonian is insufficient to describe the static properties of the theory. As a remnant of its dynamical origin, \( \mathcal{H}_q \) must be supplemented with the causality rule that forbids the former time-closed propagator loops. Hence the terminology quasi-static.

**Appendix B: 1-loop perturbation theory**

In this Appendix we assemble and list our results for the superficially diverging vertex functions \( \Gamma_{1,0}, \Gamma_{1,1}, \Gamma_{2,0}, \Gamma_{1,2}, \Gamma_{2,1} \), and \( \Gamma_{3,0} \) in the case \( \rho = 0 \). Recall that we have already calculated the decorations of the diagrams contributing to these vertex functions in Sec. \([II] \). Thus it remains to perform the integrations over the internal momenta of these diagrams. There are three types of integrals appearing:

\[
I_1(\tau) = \int_\mathbb{R} \frac{1}{\tau + \mathbf{p}^2} \frac{G_\epsilon \tau^{2-\epsilon/2}}{(1-\epsilon/4)(1-\epsilon/2)\tau^\epsilon},
\]

\[
I_2(\tau, \mathbf{q}) = \int_\mathbb{R} \frac{1}{(\tau + \mathbf{p}^2)(\tau + (\mathbf{p} + \mathbf{q})^2)} = \frac{-2G_\epsilon \tau^{-\epsilon/2}}{(1-\epsilon/2)\tau^\epsilon} - \frac{(1-\epsilon/4)G_\epsilon \tau^{-\epsilon/2}}{3(1-\epsilon/6)\tau^\epsilon} - \frac{G_\epsilon \tau^{-\epsilon/2}}{\tau^\epsilon} \mathbf{q}^2,
\]

\[
I_3(\tau) = \int_\mathbb{R} \frac{1}{(\tau + \mathbf{p}^2)^3} \frac{G_\epsilon \tau^{-\epsilon/2}}{\tau^\epsilon},
\]

where we have dropped the UV convergent parts of the integrals which are unimportant for our purposes. In addition to the \((\rho = 0)\)-diagrams listed in Sec. \([II] \) we need a few more diagrams that determine the renormalization of \( \rho \). Those are the diagrams with an insertion of a \( \rho \)-vertex, or in other words, diagrams where a propagator is replaced by a correlator, see Fig. \([III] \). These diagrams can be expressed as

\[
1c) = -\frac{g_2}{2} \rho I_2(\tau, 0),
\]

\[
2e) = -g_2^2 \rho I_3(\tau),
\]

\[
2f) = 2g_2(2g_1 + g_2^2)\rho I_3(\tau).\]

Altogether we obtain the \( \epsilon \)-pole contributions

\[
\Gamma_{1,0} = h - (1b) - (1c)
\]

\[
= h + \frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} (g_1 \tau^2 - g_2 \tau \rho),
\]

\[
\Gamma_{1,1} = (\tau + \mathbf{q}^2) - (2b) - (2e)
\]

\[
= \left\{ \tau - \frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} [g_2(4g_1 + g_2^2)\tau - g_2^2 \rho] \right\} + \left\{ 1 - \frac{G_\epsilon \tau^{-\epsilon/2}}{6\epsilon} g_2(4g_1 + g_2^2) \right\} \mathbf{q}^2,
\]

\[
\Gamma_{2,0} = \rho - (2c) - (2f)
\]

\[
= \left\{ \rho - 2\frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} [(g_0g_2 - 3g_1^2 - 2g_1g_2^2)\tau + (2g_1g_2 + g_2^2)\rho] \right\} + \left\{ 1 - \frac{G_\epsilon \tau^{-\epsilon/2}}{3\epsilon} (g_0g_2 - 3g_1^2 - 2g_1g_2^2) \right\} \mathbf{q}^2,
\]

\[
\Gamma_{1,2} = -g_2 - (3b)
\]

\[
= -\left[ 1 - 2\frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} (3g_1g_2 + g_2g_2^2) \right] g_2,
\]

\[
\Gamma_{2,1} = (2g_1 + g_2^2) - (3c)
\]

\[
= \left[ 2g_1 - 2\frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} [(7g_1g_2 + 3g_2^2)g_1 - g_2g_0] + \left[ 1 - 2\frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} (3g_1g_2 + g_2g_2^2) \right] g_2,\right.
\]

\[
\Gamma_{3,0} = g_0 - (3d)
\]

\[
= g_0 - 2\frac{G_\epsilon \tau^{-\epsilon/2}}{\epsilon} [3(2g_1g_2 + g_2g_2^2)g_0 - (7g_1^2 + 9g_1g_2 + 3g_2g_2^2)g_1].
\]

where all quantities, vertex functions, control parameters, and couplings, are bare quantities. Recall from the main text that we switch notation when we apply our renormalization scheme in that we put an overcirc over bare quantities, e.g., \( \Gamma_{1,0} \rightarrow \hat{\Gamma}_{1,0} \), and we understand quantities without an overcirc as renormalized ones once the renormalization scheme has been applied. Keeping this in mind when we compare the vertex generating function in its bare and renormalized forms,

\[
\Gamma = \sum_{k,k} \frac{\hat{\Gamma}_{k,k}}{k!k!} = \sum_{k,k} \frac{\hat{\Gamma}_{k,k} z_k \bar{z}_k}{k!k!},
\]

we obtain the following renormalizations of the vertex functions,

\[
\Gamma_{1,0} = Z^{1/2} \hat{\Gamma}_{1,0},
\]

\[
\Gamma_{1,1} = Z \hat{\Gamma}_{1,1},
\]

\[
\Gamma_{2,0} = Z \left( \hat{\Gamma}_{2,0} + 2K \hat{\Gamma}_{1,1} \right),
\]

\[
\Gamma_{1,2} = Z^{3/2} \hat{\Gamma}_{1,2},
\]

\[
\Gamma_{2,1} = Z^{3/2} \left( \hat{\Gamma}_{2,1} + 2K \hat{\Gamma}_{1,2} \right),
\]

\[
\Gamma_{3,0} = Z^{3/2} \left( \hat{\Gamma}_{3,0} + 3K \hat{\Gamma}_{2,1} + 3K^2 \hat{\Gamma}_{1,2} \right).
\]

Further exploiting the renormalization scheme \([3.1] \) and using the scaling-invariant coupling constants \( u = u_2 u_2' \),
\[ v = u_1 u_2, \quad w = u_0 u_3^2, \] it is simple algebra to find
\[ Z = 1 + \frac{u + 4v}{6\varepsilon}, \quad K = \frac{w - 2uv - 3u^2}{6u_2^2}, \quad (B18) \]
\[ \Xi = 1 + \frac{1}{\varepsilon} \left( 2u + 4v, 5(w - 2uv - 3v^2)/3u_2^2, -u_2^2, u + 4v \right), \quad (B19) \]
\[ \Delta = \frac{1}{\varepsilon u_2} \begin{pmatrix} 0, u_2^3 \\ u_2^2, -2v \end{pmatrix}, \quad (B20) \]
\[ B_0 = \frac{11uw + 22vw - 10u^2v - 29uw^2 - 22v^3}{2\varepsilon}, \quad (B21) \]
\[ B_1 = \frac{16w + 39u^2 - 5w}{6\varepsilon} u_2^{-1}, \quad (B22) \]
\[ B_2 = \frac{2u + 6v}{\varepsilon} u_2 = (Z_2 - 1) u_2, \quad (B23) \]
for the 1-loop renormalizations.

**Appendix C: 2-loop results of the RG functions**

Here, we list our 2-loop results for the RG functions that went into the calculation of the critical exponents for the \( \theta \)-transition. Details of the calculation leading to these results will be presented elsewhere [37].

The 2-loop parts of the \( \alpha \)-matrix are given by
\[ a_{1,1}^{(2)} = 1, \quad (C1) \]
\[ a_{1,2}^{(2)} = -\frac{47}{24} u + \frac{35}{6} v, \quad (C2) \]
\[ a_{2,2}^{(2)} = \frac{23}{4} u + \frac{161}{12} v - \frac{23}{12} w. \quad (C3) \]

The 2-loop parts of the \( \gamma \)- and \( \gamma' \)-function read
\[ \gamma^{(2)} = \left( \frac{37}{216} u^2 + \frac{7}{6} uv + \frac{191}{108} v^2 \right) - \frac{13}{108} w, \quad (C4) \]
\[ \gamma'^{(2)} = \left( \frac{29}{72} u + \frac{25}{27} v \right) w - \left( \frac{7}{12} u^2 + \frac{469}{216} uv + \frac{17}{9} v^2 \right) v. \quad (C5) \]

The parts of the \( \kappa \)-matrix that are not given by the shift-invariance:
\[ \kappa_{1,1}^{(2)} = \frac{611}{108} w - \left( \frac{1519}{108} u^2 + \frac{1403}{18} uv + \frac{10873}{108} v^2 \right), \quad (C6) \]
\[ \kappa_{2,1}^{(2)} = \left( \frac{43}{3} u^2 + \frac{3001}{54} uv + \frac{452}{9} v^2 \right) v - \left( \frac{161}{18} u + \frac{580}{27} v \right) w. \quad (C7) \]

The \( \beta \)-function that is not given by shift-invariance reads
\[ \beta^{(2)} = \left( \frac{55}{2} u^3 + \frac{10727}{72} u^2 v + \frac{4657}{18} uv^2 + \frac{887}{6} v^3 \right) v - \left( \frac{2809}{72} u^2 + \frac{1754}{9} uv + \frac{1391}{6} v^2 - \frac{85}{9} w \right) w. \quad (C8) \]

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