FLAT 2-ORBIFOLD GROUPS

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Abstract. This is a summary of some of the basic facts about flat 2-orbifold groups, otherwise known as 2-dimensional crystallographic groups. We relate the geometric and topological presentations of these groups, and consider structures corresponding to decompositions of the orbifolds as fibrations or as unions. We also consider covering relations, and record the bases of Seifert fibrations of 4-manifolds with geometries of solvable Lie type.

An $n$-dimensional crystallographic group is a discrete subgroup $\pi$ of the group $E(n) = \mathbb{R}^n \rtimes O(n)$ of isometries of euclidean $n$-space $\mathbb{R}^n$ which acts properly discontinuously on $\mathbb{R}^n$. The translation subgroup $\pi \cap \mathbb{R}^n$ is a lattice of rank $n$, with quotient a finite subgroup of $O(n)$, called the holonomy group of $\pi$. Since conjugation by $\pi$ preserves the lattice, the holonomy group is conjugate in $GL(n, \mathbb{R})$ to a subgroup of $GL(n, \mathbb{Z})$. The quotient $B$ of $\mathbb{R}^n$ by the action has a natural orbifold structure, recording the images of points with nontrivial stabilizers. The group $\pi$ is then the orbifold fundamental group $\pi^{orb}(B)$.

It is well known that when $n = 2$ there are just 17 possibilities. We shall relate the presentations of the groups deriving from their structure as an extension of a finite group by a lattice to those deriving from the orbifold structure. We give explicit embeddings of each group in $E(2)$, where there is not an obvious choice. (However, we do not consider the issue of moduli, i.e., the parametrization of all such embeddings of a given flat 2-orbifold group.) The orbifold fibres over a 1-orbifold if and only if the group is an extension of $\mathbb{Z}$ or the infinite dihedral group $D_\infty$. In the latter case the group is also a generalized free product with amalgamation (GFPA), corresponding to a decomposition of the orbifold along a codimension-1 suborbifold. In §4 we describe the minimal proper covering relations between these orbifolds. Finally we consider which flat 2-orbifolds can be bases of Seifert fibrations of 3- or 4-manifolds. (This was the instigation for this work.) Most of this material is well known; the only novelty here is perhaps in bringing this material together. We include several relevant expository articles in the bibliography, although these are not explicitly invoked in the paper.
1. THE HOLONY GROUP

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $G$ is a nontrivial finite subgroup of $GL(2, \mathbb{Z})$ it is conjugate to one of the cyclic groups generated by $A$, $-I$, $B$, $B^2$, $R$ or $AR$, or to a dihedral subgroup generated by $\{A, R\}$, $\{-I, R\}$, $\{B, R\}$, $\{B^2, R\}$ or $\{B^2, RB\}$. Let $\mathbb{Z}^2$ denote $\mathbb{Z}$, considered as a $G$-module via the inclusion $G < GL(2; \mathbb{Z})$. The flat 2-orbifold groups are the extensions of such groups $G$ by $\mathbb{Z}^2$, and so are determined by the cohomology group $H^2(G; \mathbb{Z}^2)$.

In 10 of these 13 cases $H^2(G; \mathbb{Z}^2) = 0$, and so the semidirect product $\mathbb{Z}^2 \rtimes G$ is the unique extension of $G$ by $\mathbb{Z}^2$ (up to isomorphism). The eight semidirect products with $G \leq \langle A, R \rangle = GL(2, \mathbb{Z}) \cap O(2)$ embed as discrete cocompact subgroups of $E(2) = \mathbb{R}^2 \rtimes O(2)$ in the obvious way. The five other semidirect products embed in $Aff(2) = \mathbb{R}^2 \rtimes GL(2, \mathbb{R})$, and these embeddings may be conjugated into $E(2)$. We shall give explicit embeddings for each of the four non-split extensions.

2. PRESENTATIONS

Each group is identified by the traditional crystallographic symbol (in square brackets) and by the now-standard orbifold symbol $[\mathbb{Z}]$. In each case we give first a presentation arising from the extension and then one deriving from the corresponding flat orbifold. Epimorphisms to $D_\infty$ correspond to GFPA structures, arising naturally from Van Kampen’s Theorem. These are essentially unique for $A$, $Mb$ and $Kb$, since the kernel must contain the centre $\zeta_\pi$.

Let $I = [[[0, 1]$, $J = [[[0, 1]$ be the reflector interval and the interval with one reflector endpoint and one ordinary endpoint. Let $Mb$ and $D(2, 2)$ be the Möbius band and the disc with two cone points, but with ordinary boundaries. Then $I$ is the one-point union of two copies of $J$, and so $\pi_{1 orb}(J) = Z/2Z$ and $\pi_{1 orb}(I) \cong \pi_{1 orb}(D(2, 2)) \cong D_\infty$.

The first four orbifolds $(T, A, Kb$ and $Mb$) fibre over $S^1$.

**Holonomy** $G = 1$.

$p1 = T$. $Z^2 = \langle x, y \mid xy = yx \rangle$.

In the subsequent presentations the generators $a, b, c, d, j, n$ and $r$ shall represent elements whose images in the holonomy group have matrices $A, B, B^2, AR, -I, BR$ and $R$, respectively, with respect to the basis $\{x, y\}$ for the translation subgroup $\mathbb{Z}^2$. (The other generators $m, p, s, t, u, v, w, z$ do not have such fixed interpretations.)

**Holonomy** $Z/2Z = \langle AR \rangle$. In this case $H^2(G; \mathbb{Z}^2) = Z/2Z$.

$pm = A = S^1 \times I$. $\pi = \mathbb{Z} \times D_\infty \cong (\mathbb{Z} \oplus Z/2Z) \ast_\mathbb{Z} (\mathbb{Z} \oplus Z/2Z)$. 


This is the split extension.
\[ \langle \mathbb{Z}^2, d \mid dx = xd, dyd = y^{-1}, d^2 = 1 \rangle. \]

Let \( u = dy \). Then also
\[ \langle d, u, x \mid dx = xd, ux = xu, d^2 = u^2 = 1 \rangle. \]

The subgroups \( \langle x + y, d \rangle \) and \( \langle dx, y \rangle \) are isomorphic to \( \pi_{\text{orb}}(Mb) \) and \( \pi_1(Kb) \), respectively.

\([pg] = Kb = Mb \cup Mb. \quad \pi = \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z} \ast \mathbb{Z}. \]

This is the non-split extension.
\[ \langle \mathbb{Z}^2, d \mid d^2 = x, dyd = y^{-1}, \rangle = \langle \mathbb{Z}^2, d \mid d^2 = x, dyd = y^{-1} \rangle. \]

Let \( u = dy \). Then also
\[ \langle d, u \mid d^2 = u^2 \rangle. \]

The quotient of \( \pi \) by its centre \( \langle x \rangle \cong \mathbb{Z} \) is \( D_\infty \), but this extension does not split.

We may embed \( \pi \) in \( E(2) \) via \( y \mapsto (j, I_2) \) and \( z \mapsto \left(\frac{1}{2}i, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \)

**Holonomy** \( Z/2\mathbb{Z} = \langle R \rangle: \)
\[ [cm] = Mb = Mb \cup S^1 \times \mathbb{J}. \quad \pi \cong \mathbb{Z} \ast \mathbb{Z} (\mathbb{Z} \oplus Z/2\mathbb{Z}). \]
\[ \langle \mathbb{Z}^2, r \mid rxr = y, r^2 = 1 \rangle. \]

Let \( z = xr \). Then also
\[ \langle r, z \mid rz^2 = z^2r, r^2 = 1 \rangle. \]

Let \( \tau \) be the involution of the normal subgroup \( \langle xy^{-1}, r \rangle = \langle r, zrz^{-1} \rangle \cong D_\infty \) which swaps \( r \) and \( zrz^{-1} \). Thus \( \pi \cong D_\infty \ast_\tau \mathbb{Z} \). The centre is \( \langle xy \rangle \cong \mathbb{Z} \) and \( \pi/\zeta \pi \cong D_\infty \), but this extension does not split.

The subgroup \( \langle xr, rx \rangle = \langle z, [r, z] \rangle \) is isomorphic to \( \pi_1(Kb) \).

The next five orbifolds fibre over \( \mathbb{I} \).

**Holonomy** \( Z/2\mathbb{Z} = \langle -I \rangle: \)
\[ [p2] = S(2, 2, 2, 2) = D(2, 2) \cup D(2, 2). \quad \pi \cong D_\infty \ast \mathbb{Z} D_\infty. \]
\[ \langle \mathbb{Z}^2, j \mid jxj = x^{-1}, jyj = y^{-1}, j^2 = 1 \rangle. \]

Let \( u = jx \) and \( v = jy \). Then also
\[ \langle j, u, v \mid j^2 = u^2 = v^2 = (juv)^2 = 1 \rangle. \]

This group is a semidirect product of the normal subgroup \( \langle ju \rangle \) with \( \langle j, v \rangle \). Thus \( \pi \cong \mathbb{Z} \ast D_\infty \).

**Holonomy** \( D_4 = \langle -I, R \rangle:\)
$[cmm] = \mathbb{D}(2, \overline{2}, \overline{2})$. \( \pi \cong \langle S(2, 2, 2, 2), r \mid rxy = y, r^2 = (jr)^2 = 1 \rangle. \)

Let \( u = jx, v = jy \) and \( z = jr \). Then \( v = rur \) and \( juv = rzurur \).

Then also

\[ \langle r, u, z \mid r^2 = u^2 = z^2 = (rz)^2 = (zuru)^2 = 1 \rangle. \]

This group is the semidirect product of the normal subgroup \( \langle j, x \rangle \) with \( \langle j, x \rangle \), and so \( \pi \cong D_\infty \rtimes D_\infty \). The corresponding GFPA structure \( \pi \cong (D_\infty \times Z/2Z) *_{D_\infty} D_\infty \) derives from the decomposition of the disc along a chord which separates the cone point from the corner points.

**Holonomy** \( D_4 = \langle -I, AR \rangle \). In this case \( H^2(G; \tilde{\mathbb{Z}}^2) = (Z/2Z)^2 \).

The group \( \pi \) has a presentation

\[ \langle \mathbb{Z}^2, d, j \mid dx = xd, dyd^{-1} = y^{-1}, jxj = x^{-1}, jyj = y^{-1}, (jd)^2 = y^e, d^2 = x^f, j^2 = 1 \rangle. \]

We may assume that \( 0 \leq e, f \leq 1 \). In all cases, \( \langle x \rangle \) and \( \langle y \rangle \) are the maximal infinite cyclic normal subgroups.

Two extension classes give isomorphic groups, and so there are three possibilities:

$[pmm] = \mathbb{D}(\overline{2}, \overline{2}, \overline{2}, \overline{2})$. \( \pi \cong (D_\infty \times Z/2Z) *_{D_\infty} (D_\infty \times Z/2Z) \).

This is the split extension (with \( e = f = 0 \)). It is also \( D_\infty \times D_\infty \):

\[ \langle \mathbb{Z}^2, d, j \mid dx = xd, dyd = y^{-1}, jxj = x^{-1}, jyj = y^{-1}, d^2 = j^2 = (dj)^2 = 1 \rangle. \]

Let \( s = jdx \) and \( t = dy \). Then also

\[ \langle d, j, s, t \mid d^2 = j^2 = s^2 = t^2 = (st)^2 = (tj)^2 = (jd)^2 = (ds)^2 = 1 \rangle, \]

or \( \langle d, j, s, t \mid d, t = j, s, d^2 = j^2 = s^2 = t^2 = 1 \rangle. \)

The GFPA structure derives from the decomposition of the disc along a chord which separates one pair of adjacent corner points from the others.

$[pmg] = \mathbb{D}(2, 2) = S^1 \times \mathbb{J} \cup D(2, 2)$. \( \pi \cong (\mathbb{Z} \oplus Z/2Z) *_{\mathbb{Z}} D_\infty \).

This corresponds to \( (e, f) = (1, 0) \). (The choice \( (0, 1) \) gives an isomorphic group.)

\[ \langle \mathbb{Z}^2, d, j \mid jxj = x^{-1}, y = (jd)^2, dx = xd, d^2 = j^2 = 1 \rangle. \]

Let \( v = jx \). Then also

\[ \langle d, j, v \mid djv = jvd, d^2 = j^2 = v^2 = 1 \rangle. \]

The relation \( djv = jvd \) is equivalent to \( jdj = vdv \), since \( j^2 = v^2 = 1 \). Hence we also have \( \pi \cong D_\infty *_{D_\infty} D_\infty \). This GFPA structure derives from the decomposition of the disc along a chord which separates the two cone points.
The subgroups $\langle jv \rangle$ and $\langle d, jdv \rangle$ are normal, and so $\pi \cong \mathbb{Z} \times D_\infty$ and $\pi \cong D_\infty \rtimes D_\infty$.

We may embed $\pi$ in $E(2)$ via $d \mapsto (-\frac{1}{2}j, (\frac{1}{3} \ 0)), \ j \mapsto (0, -I_2)$ and $v \mapsto (-i, -I_2)$.

The index-2 subgroups of the group of $\mathbb{D}(2,2)$ corresponding to $\langle AR \rangle < D_4$ and $\langle -AR \rangle < D_4$ are isomorphic to $\pi_1(Kb) = \mathbb{Z} \times -1 \mathbb{Z}$ and $\pi_{orb}(A) = \mathbb{Z} \times D_\infty$, respectively.

$[pgg] = P(2,2) = Mb \cup D(2,2)$. $\pi \cong \mathbb{Z} *_\mathbb{Z} D_\infty$.

This corresponds to $e = f = 1$.

$\langle \mathbb{Z}^2, \ d, \ j \ | \ d^2 = x, \ (jd)^2 = y, \ jd^2j = d^{-2}, \ d^2(jd)^2 = (jd)^2d^2, \ j^2 = 1 \rangle$.

Let $v = jd^2$. Then this reduces to

$\langle d, \ j, \ v \ | \ d^2 = jv, \ j^2 = v^2 = 1 \rangle$ or just $\langle d, j \ | \ (jd)^2 = j^2 = 1 \rangle$.

There is an automorphism which fixes $j$ and swaps $d$ and $jd$. We have $\pi/\langle (jd)^2 \rangle \cong \pi/\langle d^2 \rangle \cong D_\infty$, but these extensions do not split.

We may embed $\pi$ in $E(2)$ via $d \mapsto (\frac{1}{2}i, (\frac{1}{3} \ 0)), \ j \mapsto (\frac{1}{2}(i+j), -I_2)$.

The index-2 subgroups of the group of $P(2,2)$ corresponding to $\langle AR \rangle < D_4$ and $\langle -AR \rangle < D_4$ are both isomorphic to $\pi_1(Kb)$.

The remaining eight orbifolds do not fibre over $S^1$ or $\mathbb{I}$.

**Holonomy** $Z/4Z = \langle A \rangle$:

$[p4] = S(2,4,4)$.

$\langle \mathbb{Z}^2, \ a \ | \ axa^{-1} = y^{-1}, \ aya^{-1} = x, \ a^4 = 1 \rangle$.

Let $j = a^2x$. Then also $\langle a, j \ | \ a^4 = j^2 = (aj)^4 = 1 \rangle$.

**Holonomy** $D_8 = \langle A, R \rangle$. In this case $H^2(G; \mathbb{Z}) = (Z/2Z)$.

The group $\pi$ has a presentation

$\langle \mathbb{Z}^2, \ a, \ r \ | \ axa^{-1} = y^{-1}, \ aya^{-1} = x, \ rxr = y, \ a^4 = r^2 = 1, \ (ar)^2 = y^e \rangle,$

where $e = 0$ or $1$. Let $j = a^2x$, so $x = a^2j$ and $y = aja$. Then this reduces to

$\langle a, j, r \ | \ a^4 = j^2 = (aj)^4 = r^2 = 1, \ ra^2jr = aja, \ (ar)^2 = (aja)^e \rangle$.

$[p4m] = D(\mathbb{Z}, \mathbb{I}, \mathbb{I})$.

This is the split extension (with $e = 0$). It reduces to

$\langle a, j, r \ | \ a^4 = j^2 = (aj)^4 = r^2 = (ar)^2 = 1, \ arj = jar \rangle$,

since $a^2r = ra^2$ when $e = 0$. Let $w = ar$ and $z = jw = jar$. Then also

$\langle r, w, z \ | \ r^2 = w^2 = z^2 = (rz)^4 = (zw)^2 = (wr)^4 = 1 \rangle$.

$[p4g] = D(\mathbb{Z}, 4)$. 
This is the non-split extension (with \( e = 1 \)).
\[
\langle a, j, r \mid a^4 = j^2 = (aj)^4 = r^2 = 1, ra^2jr = aja, (ar)^2 = aja \rangle.
\]
Hence \( aj = a^{-1} r a r = (ra)^{-1} ara \) and \( j = r a r^{-1} \), so this reduces to
\[
\langle a, r \mid a^4 = r^2 = (r a r^{-1})^2 = 1 \rangle.
\]
We may embed \( \pi \) in \( E(2) \) via \( a \mapsto (0, (0 - 1 \ 0)), j \mapsto (-i, -I) \) and \( r \mapsto (\frac{1}{2}i, (0 \ 1 \ 0)) \).
The index-2 subgroup of this group corresponding to \( \langle -I, AR \rangle \) < \( D_8 \) is the split extension. In particular, the groups with holonomy \( \mathbb{Z}/4\mathbb{Z} \) or \( D_8 \) do not contain the groups of \( Kb, \mathbb{D}(2, 2) \) or \( P(2, 2) \).

**Holonomy** \( Z/3Z = \langle B^2 \rangle \):
\[
[p3] = S(3, 3, 3).
\]
\[
\langle Z^2, c \mid cxc^{-1} = x^{-1} y, cyc^{-1} = x^{-1}, c^3 = 1 \rangle.
\]
Let \( u = cx \). Then also
\[
\langle c, u \mid c^3 = u^3 = (cu)^3 = 1 \rangle.
\]

**Holonomy** \( Z/6Z = \langle B \rangle \):
\[
[p6] = S(2, 3, 6).
\]
\[
\langle Z^2, b \mid bxb^{-1} = y, byb^{-1} = x^{-1} y, b^6 = 1 \rangle.
\]
Let \( v = b^2 x \). Then also
\[
\langle b, v \mid b^6 = v^3 = (bv)^2 = 1 \rangle.
\]

**Holonomy** \( D_6 = \langle B^2, R \rangle \):
\[
[p3m1] = \mathbb{D}(3, 3, 3).
\]
\[
\langle S(3, 3, 3), r \mid r x r = y, r^2 = (rc)^2 = 1 \rangle.
\]
Let \( s = rc \) and \( t = crx \). Then also
\[
\langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^3 = (st)^3 = (tr)^3 = 1 \rangle.
\]

**Holonomy** \( D_6 = \langle B^2, BR \rangle \):
\[
[p31m] = \mathbb{D}(3, 3).
\]
\[
\langle S(3, 3, 3), n \mid nxn = x^{-1} y, nyn = y n, n^2 = (nc)^2 = 1 \rangle.
\]
Let \( v = nx^{-1}y^{-1} \) and \( w = nx^2y^{-1} \). Then also
\[
\langle c, v, w \mid c^3 = v^2 = w^2 = 1, w = cvc^{-1} \rangle
\]
or
\[
\langle c, w \mid w^2 = c^3 = (c^{-1} wc)^3 = 1 \rangle.
\]

**Holonomy** \( D_{12} = \langle B, R \rangle \):
\[
[p6m] = \mathbb{D}(2, 3, 6).
\]
\[
\langle S(2, 3, 6), r \mid r x r = y, r^2 = (rb)^2 = 1 \rangle.
\]
Let \( m = brb^{-1} = b^2 r \), \( n = br \) and \( p = r b x y^{-2} \). Then also

\[
\langle m, n, p \mid m^2 = n^2 = p^2 = (mn)^6 = (np)^3 = (pm)^2 = 1 \rangle.
\]

The matrix \( B \) is not orthogonal. However conjugation by \( \left( \begin{array}{cc} -2 & 1 \\ 0 & \sqrt{3} \end{array} \right) \) carries \( \langle B, R \rangle \) into \( O(2) \), and thus carries each of the subgroups \( \mathbb{Z}^2 \rtimes G \) of \( Aff(2) \) determined by the five groups with 3-torsion into \( E(2) \).

In practice it can be useful to first sort these groups by their abelianizations, which are:

- \( \mathbb{Z}^2 \) for \( T \),
- \( \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \) for \( A \),
- \( \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) for \( Kb \) and \( Mb \),
- \( (\mathbb{Z}/2\mathbb{Z})^4 \) for \( \mathbb{D}(2, 2, 2, 2) \), \( \mathbb{D}(2, \bar{2}, \bar{2}) \) and \( \mathbb{D}(\bar{2}, 4, 4) \),
- \( (\mathbb{Z}/2\mathbb{Z})^2 \) for \( \mathbb{D}(2, 3, 6) \),
- \( \mathbb{Z}/2\mathbb{Z} \) for \( \mathbb{D}(3, 3, \bar{3}) \),
- \( Z/4\mathbb{Z} \oplus Z/2\mathbb{Z} \) for \( S(2, 4, 4) \), \( P(2, 2) \) and \( \mathbb{D}(\bar{2}, 4) \),
- \( Z/3\mathbb{Z} \) for \( S(3, 3, 3) \) and
- \( Z/6\mathbb{Z} \) for \( S(2, 3, 6) \) and \( \mathbb{D}(3, \bar{3}) \).

Only the groups of the pairs \( \{ Kb, Mb \} \) and \( \{ \mathbb{D}(2, 2), \mathbb{D}(2, \bar{2}, \bar{2}) \} \) have both isomorphic holonomy groups and isomorphic abelianizations, and in each case one of the pair is a semidirect product.

### 3. Fibrations over \( \mathbb{I} \)

We shall say that two epimorphisms \( \lambda, \lambda' : \pi \to D_\infty = \pi_{\text{orb}}(\mathbb{I}) \) are equivalent if \( \lambda' = \delta \lambda \alpha \) for some \( \alpha \in Aut(\pi) \) and \( \delta \in Aut(D_\infty) \). Such epis are most easily found by considering the possible kernels, which are maximal normal virtually-\( \mathbb{Z} \) subgroups. (Note that since \( Out(D_\infty) = \mathbb{Z}/2\mathbb{Z} \) and we are working with epimorphisms, it is sufficient to take \( \delta \) to be either the identity or the automorphism which swaps the generators \( \{ u, v \} \) of \( D_\infty = \langle u, v \mid u^2 = v^2 = 1 \rangle \).)

All epimorphisms from \( \pi_{\text{orb}}(S(2, 2, 2, 2)) \) to \( D_\infty \) are equivalent. These correspond to fibrations over \( \mathbb{I} \) with general fibre \( S^1 \) and two singular fibres (reflector intervals connecting pairs of cone points).

All epimorphisms from \( \pi_{\text{orb}}(\mathbb{D}(2, 2, 2, 2)) \) to \( D_\infty \) are equivalent. (There are two maximal normal virtually-\( \mathbb{Z} \) subgroups.) These correspond to the projections of \( \mathbb{D}(2, 2, 2, 2) = \mathbb{I} \times \mathbb{I} \) onto its factors.

All epimorphisms from \( \pi_{\text{orb}}(P(2, 2)) \) to \( D_\infty \) are equivalent. (There are two maximal normal virtually-\( \mathbb{Z} \) subgroups.) These correspond to a fibration over \( \mathbb{I} \) with general fibre \( S^1 \) and two singular fibres (the centreline of \( Mb \), and one reflector interval connecting the cone points).
There are two equivalence classes of epimorphisms from $\pi^{orb}(\mathbb{D}(2, 2))$ to $D_\infty$. (There are two maximal normal virtually-$\mathbb{Z}$ subgroups.) One corresponds to a fibration over $\mathbb{I}$ with general fibre $S^1$ and two singular fibres (one reflector interval connecting the cone points and the reflector curve). The other corresponds to a fibration with general fibre $\mathbb{I}$ and two singular fibres (two reflector intervals, each connecting a cone point to a reflector curve).

All epimorphisms from $\pi^{orb}(\mathbb{D}(2, 2))$ to $D_\infty$ are equivalent. (There are two maximal normal virtually-$\mathbb{Z}$ subgroups.) These correspond to a fibration over $\mathbb{I}$ with general fibre $\mathbb{I}$ and one exceptional fibre (a reflector interval connecting the cone point to a reflector curve).

4. COVERINGS

If $\alpha$ and $\beta$ are two flat orbifold groups and there is a monomorphism $\alpha \to \beta$ which is an isomorphism on the translation subgroups then we shall say that the corresponding orbifold cover is equitranslational. In this case $\beta_1(\alpha) \geq \beta_1(\beta)$, the holonomy group $G_\alpha$ is conjugate in $GL(2, \mathbb{Z})$ to a subgroup of $G_\beta$, and the extension class $e_\beta \in H^2(G_\beta; \mathbb{Z}^2)$ must restrict to $e_\alpha$.

Such inclusions are easily determined from the lattices of subgroups of the two maximal subgroups $\langle A, R \rangle$ and $\langle B, R \rangle$ of $GL(2, \mathbb{Z})$ (modulo conjugacy). Determining the lattices of equitranslational covers of $\mathbb{D}(2, 3, 4)$ or of $\mathbb{D}(2, 3, 6)$ is straightforward, since in these cases the orbifold groups arising are split extensions. The only subtle point is how
the nonsplit extensions (with holonomy a 2-group) restrict over sub-

The orientable orbifold $S(2, 2, 2, 2)$ covers all flat orbifolds with ho-

Similarly, $S(2, 4, 4)$ covers each of $D(2, 4)$ and $D(2, 4, 4)$, and $S(3, 3, 3)$ covers all with holonomy divisible by 3.

If we drop the requirement that the translation subgroups coincide, there also less obvious inclusions. It remains necessary that $G_\alpha$ be conjugate in $GL(2, \mathbb{Q})$ to a subgroup of $G_\beta$. We may also use the (non)existence of reflector curves and/or corner points in testing whether one orbifold covers another. In all cases the subgroup generated by $2x$ and $2y$ is normal and of index 4 in the translation subgroup, and so the orbifolds have degree-4 self-coverings. (If $G \leq \langle A, R \rangle$ there is a degree-2 self-covering, since the subgroup generated by $x + y$ and $x - y$ is normal and of index 2 in the translation subgroup.)

For example, although $AR$ and $R$ are not conjugate in $GL(2, \mathbb{Z})$, they are conjugate in $GL(2, \mathbb{Z}[\frac{1}{2}])$. The inclusion $\mathbb{Z} \times D_\infty < D_\infty \rtimes_T \mathbb{Z}$ corresponds to the geometric fact that $A$ covers $Mb$. Folding $Mb$ across its centerline gives $A$, while the quotient of $Kb$ by fibrewise reflection is $Mb$. However $P(2, 2)$ is covered only by $T$, $Kb$, $S(2, 2, 2, 2)$ and itself.

The involution $[x : y : z] \mapsto [x : -y : -z]$ of $RP^2$ has one fixed point and one fixed circle. Hence $P(2, 2)$ covers $D(2, 2)$.

Rotating $D(\overline{2}, \overline{2}, \overline{2})$ about its centre gives $D(\overline{2}, \overline{2}, \overline{2})$. Conversely, folding $D(2, \overline{2}, \overline{2})$ across a diameter through the cone point and separating the corner points gives $D(\overline{2}, \overline{2}, \overline{2})$. Folding $D(2, 2)$ across a diameter separating the cone points gives $D(2, \overline{2}, \overline{2})$.

Since $D(2, 2)$ has no corner points, it is not covered by $D(2, \overline{2}, \overline{2})$. Since $D(\overline{2}, 4)$ has no corner points $\overline{T}$ with stabilizer $D_8$, it is not covered by $D(\overline{2}, \overline{4}, \overline{T})$. Nor does it cover $D(\overline{2}, \overline{4}, \overline{T})$.

The non-orientable orbifold $D(\overline{2}, \overline{3}, \overline{6})$ is covered by all with holonomy divisible by 3. Rotating $D(\overline{3}, \overline{3}, \overline{3})$ about its centre gives $D(3, 3)$. However $D(3, 3)$ does not cover $D(\overline{3}, \overline{3}, \overline{3})$. 

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5. FLAT 2-ORBIFOLDS AS BASES OF SEIFERT FIBRATIONS

A Seifert fibred 3-manifold $M$ is flat or is a $Nil^3$-manifold if and only if it is Seifert fibred over a flat 2-orbifold $B$. The fibration derives from an $S^1$-action (i.e., the image of the general fibre in $\pi_1(M)$ is central) if and only if the orientation character $w = w_1(M)$ factors through $\pi^{\text{orb}}(B)$. Every flat 3-manifold and every $Nil^3$-manifold is Seifert fibred, but the fibration may not be unique. However, not all flat 2-orbifolds arise as bases of such fibrations. In the $Nil^3$ case, the images of elements of finite order in $\pi^{\text{orb}}(B)$ in the holonomy group have determinant $+1$, by part (3) of Theorem 8.6 of [5]. Hence $B$ can have no reflector curves, and so must be one of $T$, $Kb$, $S(2,2,2,2)$, $P(2,2)$, $S(2,4,4)$, $S(3,3,3)$ or $S(2,3,6)$. Each of these is the base of a Seifert fibration of a $Nil^3$-manifold. (See [2].) Since $Nil^3$-manifolds are orientable, the fibration derives from an $S^1$-action if and only if the base is orientable, and so is one of $T$, $S(2,2,2,2)$, $S(2,4,4)$, $S(3,3,3)$ or $S(2,3,6)$.

There are just ten flat 3-manifolds, and it is easy to check the possibilities by hand. Let $M$ be a flat 3-manifold which is Seifert fibred over $B$, and let $h \in \pi = \pi_1(M)$ generate the image of the fundamental group of the general fibre. Then $\pi^{\text{orb}}(B) \cong \pi/\langle h \rangle$. The image of the translation subgroup of $\pi$ in $\pi^{\text{orb}}(B)$ is an abelian normal subgroup. It follows that the holonomy of $B$ is a quotient of the holonomy of $M$, and thus is cyclic or $(Z/2Z)^2$ (by the classification of flat 3-manifold groups). On considering the possible infinite cyclic normal subgroups we find that the possible bases are the seven flat 2-orbifolds with no reflector curves, together with $A$, $Mb$ and $D(2,2)$. (We may exclude $D(2,2,2,2)$, since every flat 3-manifold group can be generated by at most 3 elements, whereas $\beta_1(\pi^{\text{orb}}(D(2,2,2,2)); \mathbb{F}_2) = 4$. Is there a simpler reason to exclude $D(2,2,2,2)$?)

In a little more detail: the 3-torus $G_1$ has base $T$, the half-turn 3-manifold $G_2$ has bases $Kb$ and $S(2,2,2,2)$, $G_3$ has base $S(3,3,3)$, $G_4$ has base $S(2,4,4)$, $G_5$ has base $S(2,3,6)$ and $G_6$ has base $P(2,2)$. The fibrations of $G_3, \ldots, G_6$ are unique. The non-orientable flat 3-manifolds all have Seifert fibrations with base $Kb$. In addition, $B_1 = Kb \times S^1$ also has bases $T$ and $A$, $B_2$ also has bases $T$ and $Mb$, $B_3$ also has bases $A$ and $D(2,2)$, and $B_4$ also has base $Mb$.

In dimension 4, manifolds which are Seifert fibred over a flat 2-orbifold with general fibre a flat 2-manifold are also geometric. (However there are non-geometric 4-manifolds which are Seifert fibred over hyperbolic base 2-orbifolds.) The relevant geometries are $\mathbb{E}^4$, $Nil^3 \times \mathbb{E}^1$, $Nil^4$ and $Sol^3 \times \mathbb{E}^1$. 
All but three of the 74 flat 4-manifolds have Seifert fibrations. (See Chapter 8 of [5].) If the holonomy $G$ of $\pi_{orb}(B)$ is cyclic then $B$ is the base of a Seifert fibration of a flat 3-manifold $N$, and so $N \times S^1$ is a flat 4-manifold which is Seifert fibred with general fibre $T$ and base $B$. These products are also total spaces of $T$-bundles over $T$. If the holonomy $G$ is dihedral it is a quotient of $\pi_1(Kb)$, and pulling back the extension of $G$ by $\mathbb{Z}^2$ over $\pi_1(Kb)$ gives the group of a flat 4-manifold which is Seifert fibred over $B$. These 4-manifolds are total spaces of $T$-bundles over $Kb$.

Manifolds with one of the other geometries have canonical Seifert fibrations, and the fibration is unique for $\text{Nil}^4$-manifolds and $\text{Sol}^3 \times \mathbb{E}^1$-manifolds. If the geometry is $\text{Nil}^3 \times \mathbb{E}^1$ or $\text{Nil}^4$ the manifold is an infranilmanifold, and these are treated at length in [2].

The general fibre $F$ of a Seifert fibration of a $\text{Nil}^3 \times \mathbb{E}^1$-manifold can be either $T$ or $Kb$, and inspection of the lists in [2] shows that every flat 2-orbifold is the base of some such fibration with $F = T$. However, it is not clear from a cursory inspection whether this is always so for the canonical Seifert fibration. If $x, y$ in $\pi$ represent a basis of the translation subgroup of $\pi_{orb}(B)$ and $t \in \pi$ then $[x, y]$ is a nontrivial element of $\pi_1(F)$, and $t[x, y]t^{-1} = [x, y]^{\det(t)}$, where $\det(t)$ is the determinant of the image of $t$ in the holonomy of $\pi_{orb}(B)$. Therefore if the image of $\pi_1(F)$ in $\pi$ is central then $B$ must be orientable. Products of $\text{Nil}^3$-manifolds with $S^1$ give examples realizing the five possibilities.

The simplest examples of $\text{Nil}^3 \times \mathbb{E}^1$-manifolds with general fibre $Kb$ are $Kb$-bundles over $T$, with groups having presentations

$$\langle a, b, c, x \mid [a, b] = x^{2q}, ac = ca, bc = cb, ax = xa, bx = xb, xcx^{-1} = c^{-1} \rangle.$$ 

We do not know whether every flat 2-orbifold is the base of such a fibration.

In the other two cases the general fibre is $T$ and the action of $\pi_{orb}(B)$ on the fundamental group of the fibre has infinite image in $\text{Aut}(\pi_1(T)) \cong \text{GL}(2, \mathbb{Z})$, since $M$ is neither flat nor a $\text{Nil}^3 \times \mathbb{E}^1$-manifold. Since $\pi_{orb}(B)$ is solvable and $\text{GL}(2, \mathbb{Z})$ is virtually free this image is virtually $\mathbb{Z}$, and so $B$ must fibre over $S^1$ or $I$. If the geometry is $\text{Nil}^4$ then $B$ is one of $T$, $A$, $Mb$, $P(2, 2)$ or $D(2, 2)$ [2]. Finally, if the geometry is $\text{Sol}^3 \times \mathbb{E}^1$ then $B$ is one of $T$, $Kb$, $A$, $Mb$, $S(2, 2, 2, 2)$, $P(2, 2)$ or $D(2, 2)$ [6].
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