Correlations equalities and some upper bounds for the critical temperature for spin one systems.

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Abstract

Starting from correlation identities for the Blume-Capel spin 1 systems and using correlation inequalities, we obtain rigorous upper bounds for the critical temperature. The obtained results improve over effective field type results.

1. Introduction

Correlation inequalities combined with exact identities are useful in obtaining rigorous results in statistical mechanics. Among the various questions that are resolved by them one is the decay of the correlation functions. The decay of the correlation functions give information about the critical couplings of statistical mechanics models. In this work, the method will be applied to study systems described by the spin one Blume-Capel model \cite{1, 2}. Firstly, we present the derivation of an exact relation for the two spin correlation function, valid in any dimension, which is an extension of Callen’s identity for spin 1/2 Ising model \cite{3}. Starting from these identities we will then make use of the first and second Griffiths inequalities and Newman’s inequalities to obtain the exponential decay of the two spin correlation function. The coupling constant which are the upper bounds for the critical temperature are obtained for d=2 and d=3 dimensions. In this study the coupling parameters obtained improve effective field results. Upper bounds for the critical temperature \( T_c \) for Ising and multi-component spin systems have been obtained by showing (for \( T > T_c \)) the exponential decay of the two-point function \cite{4, 5, 6}. Spin correlation inequalities and their iteration are used by Brydges et al \cite{6}, Lieb \cite{7} and Simon \cite{5}. The procedure to improve the bound for the critical temperature over the effective field result for the classical \( S = 1 \) model is as follows: starting from a two-point correlation function identity, a generalization of Callen’s identity \cite{3} for this model \cite{8} and using Griffiths 1st and 2nd inequalities (Griffith I, II) (see \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}) and Newman’s inequalities \cite{10, 15} we establish the inequality for the two-point function, \( \langle S_0 S_l \rangle \), as

\[
\langle S_0 S_l \rangle \leq \sum_j a_j \langle S_j S_l \rangle, 0 \leq a_j \leq 1
\]  

\( 1 \)

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which when iterated (see [5]) implies exponential decay for \(T > T_c\). In section 2 we present the derivation of the correlation identities for the Blume-Capel model [8]. In section 3, we apply these identities to the \(d = 2\) and \(d = 3\) lattices. Next, in section 4, we apply the correlation inequalities to obtain the upper bounds for \(T_c\). Numerical results can be found in section 5, and in section 6 we present our concluding remarks.

We write the Hamiltonian for the classical spin one system, known as the Blume-Capel model, as

\[
H = -J \sum_{i,j} S_i S_j - D \sum_i S_i^2,
\]

where \(J > 0\), \(D\) is the single ion anisotropy and the first sum is over the nearest neighbours spins on the lattice. We define the thermal average \(\langle ... \rangle\) by

\[
\langle ... \rangle = \frac{\text{Tr}(\{S_i\} e^{-\beta H})}{\text{Tr}(e^{-\beta H})}
\]

where each \(S_i\) is restricted by \(S_i = -1, 0, +1\).

2. Correlation identity for the spin one model

We reproduce the generalization of Callen’s identity for the spin 1 Blume-Capel model which has been obtained previously by Siqueira and Fittipaldi [8]. Let

\[
\langle F(S) S_i \rangle = \frac{\text{Tr}(F(S) S_i e^{-\beta H})}{\text{Tr}(e^{-\beta H})},
\]

where \(F(S)\) is any function of \(S\) different from \(S_i\). We can write \(H = H_i + H'\), where

\[
H_i = -\left( \sum_{|j|=1} J_{ij} S_j \right) S_i - DS_i^2,
\]

is the Hamiltonian describing site \(i\) and its neighbours, and \(H'\) corresponds to the Hamiltonian of the rest of the lattice. Consequently \([H_i, H'] = 0\). From Eq. (2) and Eq. (3), we get,

\[
\langle F(S) S_i \rangle = \frac{\text{Tr}'(F(S) e^{-\beta H}) S_i}{\text{Tr}'(e^{-\beta H})} = \frac{\text{Tr}'(F(S) e^{-\beta H}) S_i}{\text{Tr}'(e^{-\beta H})} e^{-\beta H'}
\]

or

\[
\langle F(S) S_i \rangle = \frac{\text{Tr}'(F(S) e^{-\beta H}) S_i}{\text{Tr}'(e^{-\beta H})}
\]

where \(\text{Tr}' \text{Tr} = \text{Tr}\). Finally, we obtain,

\[
\langle F(S) S_i \rangle = \left( F(S) \frac{\text{Tr}' e^{-\beta H_i} S_i}{\text{Tr}' e^{-\beta H_i}} \right).
\]

Explicitly operating the trace \(\text{Tr}_i\), we get,

\[
\langle F(S) S_i \rangle = \left< F(S) \left( \frac{2 e^{\beta D} \sinh(\sum_j \beta J_{ij} S_j)}{2 e^{\beta D} \cosh(\sum_j \beta J_{ij} S_j) + 1} \right) \right>
\]

\[
= \left< F(S) \prod_{|j|=1} e^{\beta J_{ij} S_j} f(x) \right|_{x = 0},
\]

where
with \( \nabla \equiv \frac{\partial}{\partial x} \), such that
\[
e^{\alpha \nabla} f(x) = f(x + \alpha), \quad \text{and}
\]
\[
f(x) = \frac{2e^{\beta D} \sinh(x)}{2e^{\beta D} \cosh(x) + 1}. \quad (10)
\]
As \( S_j^{2n} = S_j^2 \) and \( S_j^{2n+1} = S_j \) for \( n = 0, 1, 2, 3, \ldots \), we obtain,
\[
e^{S_j A} = S_j^2 \cosh(A) + S_j \sinh(A) + 1 - S_j^2, \quad (11)
\]
and, applying Eq. (10) and Eq. (11) in Eq. (9), we get
\[
< F(Si)S_i > = \left\{ F(S) \prod_{j \neq i, |j| = 1} (S_j^2 \cosh(\beta J_{ij} \nabla) + S_j \sinh(\beta J_{ij} \nabla) + 1 - S_j^2) \right\} f(x)|_{x=0} \quad (12)
\]
Similarly for the correlation function involving the square of the spin function \( S_i^2 \), we obtain,
\[
< G(Si)S_i^2 > = \left\{ G(S) \prod_{j \neq i, |j| = 1} e^{\beta J_{ij} S_j \nabla} \right\} g(x)|_{x=0}, \quad (13)
\]
with,
\[
g(x) = \frac{2e^{\beta D} \cosh(x)}{2e^{\beta D} \cosh(x) + 1}. \quad (14)
\]
resulting in,
\[
< G(S)S_i^2 > = \left\{ G(S) \prod_{j \neq i, |j| = 1} (S_j^2 \cosh(\beta J_{ij} \nabla) + S_j \sinh(\beta J_{ij} \nabla) + 1 - S_j^2) \right\} g(x)|_{x=0}. \quad (15)
\]
The function \( G(S) \) is any function of \( S \), except \( S_i^2 \). The equations (12) and (15) are exact and generalize Callen’s identity which was obtained for the \( S = 1/2 \) Ising model [3].

3. Exact correlation identities applied to the \( d=2 \) and \( d=3 \) lattices

Let us apply the previous results for \( < F(S)S_i > \) and \( < G(S)S_i^2 > \) given by equations (12) and (15) for specific lattices in two- and three-dimensions. The two spins correlation functions, \( < S_0 S_i > \), are obtained from equations (12) and (15) by defining \( F(S) = S_i \).

3.1. For the \( d=2 \) and \( z=3 \), the honeycomb lattice

We obtain from Eq. (12)
\[
< S_0 S_i > = A_1 \sum_i < S_i S_i > + A_2 \sum_{i<j} < S_i S_j^2 S_i > + A_3 \sum_{i<j<k} < S_i S_j S_k S_i > + A_4 \sum_{i<j<k} < S_i S_j^2 S_k^2 S_i >, \quad (16)
\]
where the A coefficients are given in appendix Appendix A.1. We also obtain, from Eq. (15),

\[
\langle S_0^2 S_l \rangle = B_0 + B_1 \sum_i \langle S_i^2 S_l \rangle + B_2 \sum_{i<j} \langle S_i S_j S_l \rangle \\
+ B_3 \sum_{i<j} \langle S_i S_j^2 S_l \rangle + B_4 \sum_{i<j<k} \langle S_i S_j S_k^2 S_l \rangle \\
+ B_5 \sum_{i<j<k} \langle S_i^2 S_j^2 S_k^2 S_l \rangle ,
\]

(17)

where the B coefficients are given in appendix Appendix A.1.

3.2. For the \( d = 2 \) and \( z = 4 \), the square lattice

We obtain from Eq. (12) for the two spin correlation functions \( \langle S_0 S_l \rangle \) the expression,

\[
\langle S_0 S_l \rangle = A_1 \sum_i \langle S_i S_l \rangle + A_2 \sum_{i<j} \langle S_i S_j S_l \rangle + A_3 \sum_{i<j<k} \langle S_i S_j S_k S_l \rangle \\
+ A_4 \sum_{i<j<k} \langle S_i S_j^2 S_k S_l \rangle + A_5 \sum_{i<j<k<m} \langle S_i S_j S_k S_m S_l \rangle \\
+ A_6 \sum_{i<j<k<m} \langle S_i S_j^2 S_k^2 S_m S_l \rangle ,
\]

(18)

where the A coefficients are given in appendix Appendix A.2. We also obtain, for the function \( \langle S_0^2 S_l \rangle \),

\[
\langle S_0^2 S_l \rangle = B_0 + B_1 \sum_i \langle S_i^2 S_l \rangle + B_2 \sum_{i<j} \langle S_i S_j S_l \rangle + B_3 \sum_{i<j<k} \langle S_i S_j S_k S_l \rangle \\
+ B_4 \sum_{i<j<k} \langle S_i S_j^2 S_k S_l \rangle + B_5 \sum_{i<j<k<m} \langle S_i S_j S_k S_m S_l \rangle \\
+ B_6 \sum_{i<j<k<m} \langle S_i S_j^2 S_k^2 S_m S_l \rangle + B_7 \sum_{i<j<k<m} \langle S_i S_j S_k S_m^2 S_l \rangle \\
+ B_8 \sum_{i<j<k<m} \langle S_i^2 S_j^2 S_k^2 S_m S_l \rangle ,
\]

(19)

where the B coefficients are given in appendix Appendix A.2.

3.3. For the \( d = 3 \) and \( z = 6 \), the cubic lattice

We obtain from Eq. (12)

\[
\langle S_0 S_l \rangle = A_1 \sum_i \langle S_i S_l \rangle + A_2 \sum_{i<j} \langle S_i S_j S_l \rangle + A_3 \sum_{i<j<k} \langle S_i S_j S_k S_l \rangle \\
+ A_4 \sum_{i<j<k} \langle S_i S_j^2 S_k S_l \rangle + A_5 \sum_{i<j<k<m} \langle S_i S_j S_k S_m S_l \rangle \\
+ A_6 \sum_{i<j<k<m} \langle S_i S_j^2 S_k^2 S_m S_l \rangle + A_7 \sum_{i<j<k<m} \langle S_i S_j S_k S_m^2 S_l \rangle \\
+ A_8 \sum_{i<j<k<m} \langle S_i^2 S_j^2 S_k^2 S_m S_l \rangle ,
\]

(20)
\[ + A_6 \sum_{i<j<k<m} < S_i S_j S_m S_l > + A_7 \sum_{i<j<k<m<n} < S_i S_j S_m S_n S_l > + A_8 \sum_{i<j<k<m} < S_i S_j S_m S_n S_l > + A_9 \sum_{i<j<k<m<n<p} < S_i S_j S_m S_n S_p S_l > + A_{10} \sum_{i<j<k<m} < S_i S_j S_m S_n S_l > + A_{11} \sum_{i<j<k<m<n<p} < S_i S_j S_m S_n S_p S_l >, \]                      

\[ + A_{12} \sum_{i<j<k<m<n} < S_i S_j S_m S_n S_l > + A_{13} \sum_{i<j<k<m<n} < S_i S_j S_m S_n S_p S_l > + A_{14} \sum_{i<j<k<m<n} < S_i S_j S_m S_n S_l > + A_{15} \sum_{i<j<k<m<n} < S_i S_j S_m S_n S_p S_l >, \]                      

where the \( A \) coefficients are given in appendix \[ \text{Appendix A.3} \] We also obtain, for the function \( < S_0^2 S_l > \) ,

\[ < S_0^2 S_l > = B_0 + B_1 \sum_i < S_i S_l > + B_2 \sum_{i<j} < S_i S_j S_l > + B_3 \sum_{i<j} < S_i S_j^2 S_l > + B_4 \sum_{i<j<k} < S_i S_j S_k S_l > + B_5 \sum_{i<j<k} < S_i S_j S_k S_l > + B_6 \sum_{i<j<k<m} < S_i S_j S_m S_l > + B_7 \sum_{i<j<k<m} < S_i S_j S_m S_l > + B_8 \sum_{i<j<k<m} < S_i S_j^2 S_k S_l > + B_9 \sum_{i<j<k<m} < S_i S_j S_k S_m S_l > + B_{10} \sum_{i<j<k} < S_i S_j S_k S_m S_l > + B_{11} \sum_{i<j<k} < S_i S_j S_k S_m S_l > + B_{12} \sum_{i<j<k} < S_i S_j S_k S_m S_l > + B_{13} \sum_{i<j<k} < S_i S_j S_k S_m S_l > + B_{14} \sum_{i<j<k} < S_i S_j S_k S_m S_l > + B_{15} \sum_{i<j<k} < S_i S_j S_k S_m S_l >, \]                      

where the \( B \) coefficients are given in appendix \[ \text{Appendix A.3} \] The sums over \( i, j, k, m, n \) and \( p \) are over the nearest neighbors of 0 to which we have given a numerical ordering. The proof of results (16) for the case (3.1), the honeycomb lattice, is presented in the appendix \[ \text{Appendix B} \] as an example for the other cases.
4. Application of the correlation inequalities

In the following results we will made use of the following inequalities: \( < S_A > \geq 0 \) (Griffiths I), \( < S_A S_B > - < S_A > < S_B > \geq 0 \) (Griffiths II) (see [3], [10], [11], [14]), \( < S_i F > \leq \sum_j < S_i S_j > dF/dS_j > \) (Newman’s) ([13], [10]) and \( < S_i^2 S_A > \leq < S_A > \) ([12], [12]), where \( < S_A > = \prod_i S_i \), \( < S_B > = \prod_i S_i \) and \( F \) is a polynomial function of variables \( S_i \).

From the equations for the two spin correlation functions obtained in subsections (3.1), (3.2) and (3.3) and applying the Griffith’s and Newman’s inequalities we obtain an inequality of the form

\[
< S_0 S_l > \leq \sum_{|i|=1} a_i < S_i S_l >,
\]

where \( a_i \) is a sum of products of two-point functions.

(a) Case \( d=2 \), \( z=3 \), honeycomb lattice.

Using

\[
< S^2 S_i S_l > \leq < S_i S_l >
\]

in equation (19), in the \( A_2 \) term and Griffiths II, i.e.,

\[
< S_i S_j S_k S_l > \geq < S_i S_j > < S_k S_l >
\]

in the \( A_3 \) term, and noticing that \( A_2 \) and \( A_3 \) are negative, we get for \( d=2 \), \( z=3 \),

\[
< S_0 S_l > \leq (A_1 - |A_2| - |A_3| < S_0 S_1 >_{1D} + A_4) \sum_{|i|=1} < S_i S_l >,
\]

(b) Case \( d=2 \), \( z=4 \), square lattice.

Using inequality (23) in equation (20), in the \( A_2 \) term (\( A_2 < 0 \)) term, Griffiths II in the \( A_3 \) term (\( A_3 < 0 \)), the inequalities

\[
< S_j^2 S^2 S_i S_l > \leq < S_i S_l >
\]

in the \( A_4 \) term (\( A_4 > 0 \)) and

\[
< S_j^2 S^2 S^2 S^2 S_i S_l > \leq < S_i S_l >
\]

in the \( A_6 \) term (\( A_6 > 0 \)) and in the \( A_5 \) term using Griffiths II, we get for \( d=2 \), \( z=4 \),

\[
< S_0 S_l > \leq (A_1 - |A_2| - < S_i S_j >_{1D} |A_3| + A_4 + < S_1 S_2 >_{1D} A_5 + A_6) \sum_{|i|=1} < S_i S_l >,
\]
(c) Case \( d = 3, \ z = 6, \) cubic lattice.

As before, we use in equation (20), inequality (23) in the \( A_2 \) term \( (A_2 < 0) \) term, Griffiths II in the \( A_3 \) term \( (A_3 < 0) \), the inequalities (26) in the \( A_4 \) term \( (A_4 > 0) \), inequality (24) in the \( A_6 \) term \( (A_6 > 0) \), and Griffiths II in the \( A_5 \) term. For the term \( A_7(> 0) \) we use Newman’s inequality and for the terms \( A_8(> 0), A_9(> 0), A_{10}(> 0) \) and \( A_{11}(> 0) \), we use inequality (22). Then, we get for \( d=3, z=6, \)

\[
< S_0 S_1 > \leq (A_1 - |A_2| - |S_1 S_2 >_{1D}| A_3 | + A_4 \\
+ S_1 S_2 >_{1D} A_5 \\
+ A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} \sum_{i=1} < S_i S_i >
\]  

(29)

The two-spin correlation function \( < S_1 S_2 >_{1D} \) is the one-dimension model two spin correlation separated by a distance of two lattice sites. By bounding the resulting two-point function occurring in the previous results from below with the two-point function of the one-dimensional infinite chain (see Appendix B), we get:

\[
< S_0 S_1 > \leq \sum_{i=1} a_i < S_i S_i > ,
\]  

(30)

where,

(a) For \( d = 2, \ z = 3, \) honeycomb lattice.

\[
a_j = A_1 - |A_2| - |S_1 S_1 >_{1D}| A_3 | + A_4;
\]  

(31)

(b) For \( d = 2, \ z = 4, \) square lattice.

\[
a_j = A_1 - |A_2| - |S_1 S_2 >_{1D}| A_3 | + A_4 \\
+ S_1 S_2 >_{1D} A_5 + A_6;
\]  

(32)

(c) For \( d = 3, \ z = 6, \) cubic lattice.

\[
a_j = A_1 - |A_2| - |S_1 S_2 >_{1D}| A_3 | + A_4 \\
+ S_1 S_2 >_{1D} A_5 + A_6 \\
+ A_7 + A_8 + A_9 + A_{10} + A_{11}.
\]  

(33)

The one-dimensional correlation function is given by (see Appendix C):

\[
< S_1 S_2 >_{1D} = \frac{1 + \sqrt{1 - 2f(2\beta J)}}{f(2\beta J)}
\]  

(34)

and \( f(2\beta J) \) is given by (10).

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Table 1: Estimatives for $kT_C/J$ for $D = 0$ in previous and in the present work.

|       | $d = 2, z = 3$ | $d = 2, z = 4$ | $d = 3, z = 6$ |
|-------|---------------|---------------|---------------|
| MFA   | 2             | 2.667         | 4             |
| Siqueira | 1.518        | 2.188         | 3.516         |
| Yuksel| -             | 1.964         | -             |
| CVM   | -             | -             | 2.886         |
| Series| -             | 1.688         | 3.192         |
| RG    | -             | 2.128         | 3.474         |
| Monte Carlo [30] | -    | 1.695         | -             |
| Monte Carlo [29] | -    | 1.681         | -             |
| Wang Landau [28] | -    | 1.714         | -             |
| Present work | 1.591 | 2.322         | 3.678         |

Table 2: Estimatives for $kT_C/J$ for $D = ∞$ in previous and in the present work.

|       | $d = 2, z = 3$ | $d = 2, z = 4$ | $d = 3, z = 6$ |
|-------|---------------|---------------|---------------|
| MFA   | 3             | 4             | 6             |
| Siqueira | 2.103        | 3.088         | 5.076         |
| CVM   | -             | -             | 3.876         |
| Series| -             | -             | 4.482         |
| RG    | -             | 2.884         | 4.932         |
| Present work | 1.999 | 3.070         | 5.084         |

5. Numerical results

Evaluating numerically the value of $T$ such that $\sum a_j \leq 1$, $a_j > 0$, we obtain, by sufficient condition, upper bounds for $T_c$, which are shown in tables 1 and 2 in comparison with results obtained by other methods.

For the evaluation of the self-correlation terms ($< S_i^2 >$) that emerge from the application of the Griffith’s and Newman’s inequalities, we use, for the $D = 0$ case, $< S_i^2 > \leq 2/3$, correct for a spin 1 ferromagnetic system, and, for the $D = ∞$ case, $< S_i^2 > = 1$, since in this limit the $S_i = 0$ spin value is suppressed.

For the honeycomb lattice our result has to be compared with the mean field and the effective field calculations. Those results are not rigorous, as ours, and the numerical values we obtain improve those mean field type results and therefore represent the upper bounds, for the limits $D = 0$ and $D = ∞$. For the square and cubic lattices besides the mean field type results, for which the previous comments apply, there are other results, better than mean field type, obtained by series and renormalization group calculations, which can be used as a comparison. The importance of the present numerical results lies in the fact that they were obtained using an identity and rigorous inequalities for the two-spin correlation function. For this reason they represent rigorous upper bounds for the critical temperature.
6. Final Comments

We have presented the derivation of correlation identities for the Blume-Capel spin-1 model which are exact in all dimensions, and we have made use of correlation inequalities to obtain the upper bounds for the transition temperature. The coupling constants obtained for those bounds are calculated for $d=2$ (honeycomb and square lattices) and $d=3$ (cubic lattice). We obtain rigorous results that improve mean field type calculations.

Appendix A. Coefficients of the Spin Correlation Identities for $d=2$, $z=3$ and $z=4$.

With $k = \beta J$ and $f(x)$ given by relation (10), we have for

**Appendix A.1. $d=2$, $z=3$**

\[ A_1 = 3f(k) > 0, \quad (A.1) \]
\[ A_2 = (3f(2k) - 6f(k)) < 0, \quad (A.2) \]
\[ A_3 = \frac{1}{4}(f(3k) - 3f(k)) < 0 \quad (A.3) \]
\[ A_4 = \frac{3}{4}(5f(k) + f(3k) - 4f(2k)) > 0 \quad (A.4) \]

and
\[ B_0 = g(0), \quad (A.5) \]
\[ B_1 = 3(g(k) - g(0)), \quad (A.6) \]
\[ B_2 = \frac{3}{2}(g(2k) - g(0)), \quad (A.7) \]
\[ B_3 = \frac{3}{2}g(2k) + -6g(k) + \frac{9}{2}g(0), \quad (A.8) \]
\[ B_4 = \frac{3}{4}(g(3k) - g(k) - 2g(2k) + 2g(0)), \quad (A.9) \]
\[ B_5 = \frac{1}{4}g(3k) - \frac{3}{2}g(2k) + \frac{15}{4}g(k) - \frac{5}{2}g(0). \quad (A.10) \]

**Appendix A.2. $d=2$, $z=4$**

\[ A_1 = 4f(k) > 0, \quad (A.11) \]
\[ A_2 = 6f(2k) - 12f(k) < 0, \quad (A.12) \]
\[ A_3 = f(3k) - 3f(k) < 0, \quad (A.13) \]
\[ A_4 = 15f(k) - 12f(2k) + 3f(3k)) > 0, \quad (A.14) \]
\[ A_5 = \frac{1}{2}f(4k) - f(3k) - f(2k) + 3f(k) > 0 \quad (A.15) \]
\[ A_6 = \frac{1}{2} f(4k) - 3f(3k) + 7f(2k) - 7f(k) < 0 \quad \text{(A.16)} \]

and

\[ B_0 = g(0), \quad \text{(A.17)} \]
\[ B_1 = 4(g(k) - g(0)), \quad \text{(A.18)} \]
\[ B_2 = 3(g(2k) - g(0)), \quad \text{(A.19)} \]
\[ B_3 = 3(g(2k) - 4g(k) + 3g(0)), \quad \text{(A.20)} \]
\[ B_4 = 3(g(3k) - 2g(2k) - g(k) + 2g(0)), \quad \text{(A.21)} \]
\[ B_5 = g(3k) - 6g(2k) + 15g(k) - 10g(0), \quad \text{(A.22)} \]
\[ B_6 = \frac{1}{8} (g(4k) - 4g(2k) + 3g(0)), \quad \text{(A.23)} \]
\[ B_7 = \frac{3}{4} g(4k) - 3g(3k) + 3g(2k) + 3g(k) - \frac{15}{4} g(0), \quad \text{(A.24)} \]
\[ B_8 = \frac{1}{8} g(4k) - g(3k) + \frac{7}{2} g(2k) - 7g(k) + \frac{35}{8} g(0). \quad \text{(A.25)} \]

Appendix A.3. \(d=3, \ z=6\)

\[ A_1 = 6f(k) > 0, \quad \text{(A.26)} \]
\[ A_2 = -30f(k) + 15f(2k) < 0, \quad \text{(A.27)} \]
\[ A_3 = 5f(3k) - 15f(k) < 0, \quad \text{(A.28)} \]
\[ A_4 = 75f(k) + 15f(3k) - 60f(2k) > 0, \quad \text{(A.29)} \]
\[ A_5 = -15f(3k) + 45f(k) + \frac{15}{2} f(4k) - 15f(2k) > 0, \quad \text{(A.30)} \]
\[ A_6 = -45f(3k) - 105f(k) - f(2k) + \frac{15}{2} f(4k) < 0, \quad \text{(A.31)} \]
\[ A_7 = \frac{3}{8} f(5k) - \frac{15}{8} f(3k) + \frac{15}{4} f(k) > 0, \quad \text{(A.32)} \]
\[ A_8 = \frac{45}{4} f(3k) - \frac{105}{2} f(k) + \frac{15}{4} f(5k) - 15f(4k) + 30f(2k) < 0, \quad \text{(A.33)} \]
\[ A_9 = \frac{3}{8} f(5k) + \frac{15}{8} f(3k) - \frac{15}{4} f(k) + \frac{3}{16} f(6k) - \frac{3}{4} f(4k) + \frac{15}{16} f(2k) < 0, \quad \text{(A.34)} \]
\[ A_{10} = \frac{405}{8} f(3k) + \frac{315}{4} f(k) + \frac{15}{8} f(5k) - 15f(4k) - 90f(2k) > 0, \quad \text{(A.35)} \]
\[ A_{11} = -\frac{5}{4} f(3k) + \frac{45}{2} f(k) + \frac{15}{2} f(4k) - \frac{135}{8} f(2k) - \frac{15}{4} f(5k) + \frac{5}{8} f(6k) > 0 \quad \text{(A.36)} \]

and

\[ B_0 = g(0), \quad \text{(A.37)} \]
\[ B_1 = 6(g(k) - g(0)), \quad \text{(A.38)} \]
\[ B_2 = \frac{15}{2} (g(2k) - g(0)), \quad (A.39) \]
\[ B_3 = \frac{15}{2} (g(2k) - 4g(k) + 3g(0)), \quad (A.40) \]
\[ B_4 = 15 (g(3k) - 2g(2k) - g(k) + 2g(0)), \quad (A.41) \]
\[ B_5 = 5 (g(3k) - 6g(2k) + 15g(k) - 10g(0)), \quad (A.42) \]
\[ B_6 = \frac{15}{8} (g(4k) - 4g(2k) + 3g(0)), \quad (A.43) \]
\[ B_7 = 45 \left( \frac{1}{4} g(4k) - g(3k) + g(2k) + g(k) - \frac{5}{4} g(0) \right), \quad (A.44) \]
\[ B_8 = 15 \left( \frac{1}{8} g(4k) - g(3k) + \frac{7}{2} g(2k) - 7g(k) + \frac{35}{8} g(0) \right), \quad (A.45) \]
\[ B_9 = \frac{1}{32} (g(6k) - 6g(4k) + 15g(2k) - 10g(0)), \quad (A.46) \]
\[ B_{10} = \frac{3}{8} (-126g(0) + 45g(3k) + 210g(k) - 120g(2k) - 10g(4k) + g(5k)), \quad (A.47) \]
\[ B_{11} = \frac{3}{8} \left( -\frac{55}{3} g(3k) - 66g(k) - \frac{165}{4} g(2k) + \frac{77}{2} g(0) + \frac{1}{12} g(6k) + \frac{11}{2} g(4k) - g(5k) \right), \quad (A.48) \]
\[ B_{12} = \frac{15}{4} (-8g(2k) + 14g(0) + g(5k) - 14g(k) + 13g(3k) - 6g(4k)), \quad (A.49) \]
\[ B_{13} = \frac{15}{32} (-40g(3k) + 48g(k) + 15g(2k) - 42g(0) + 26g(4k) + g(6k) - 8g(5k)), \quad (A.50) \]
\[ B_{14} = \frac{15}{8} (-2g(4k) + 8g(2k) - 6g(0) + g(5k) - 3g(3k) + 2g(k)), \quad (A.51) \]
\[ B_{15} = \frac{15}{32} (2g(4k) - 17g(2k) + 14g(0) - 4g(5k) + 12g(3k) - 8g(5k) + g(6k)). \quad (A.52) \]

Appendix B. Proof of the correlation identity for the honeycomb lattice

From equation (12)

\[ < F(S)S_i > = \left< F(S) \prod_{j \neq i} (S_j^z \cosh(\beta J_{ij} \nabla) + S_j \sinh(\beta J_{ij} \nabla) + 1 - S_j^z) \right> f(x)|_{x = 0} \quad (B.1) \]

where,

\[ f(x) = \frac{2e^{\beta D \sinh(x)}}{2e^{\beta D \cosh(x)} + 1}, \quad (B.2) \]

we obtain \( < S_0 S_i > \), for the honeycomb lattice,

\[ < S_0 S_i > = < S_i (1 + S_1 \sinh J \nabla + S_1^2 [\cosh J \nabla - 1]) \times (1 + S_2 \sinh J \nabla + S_2^2 [\cosh J \nabla - 1]) \times (1 + S_3 \sinh J \nabla + S_3^2 [\cosh J \nabla - 1]) > \quad (B.3) \]
where $S_1$, $S_2$ and $S_3$ are the neighbours of $S_0$.

Or,

$$< S_0 S_1 > = 3a_1 < S_1 S_1 > + 6(a_2 - a_1) < S_1 S_2^2 > + a_3 < S_1 S_2 S_3 > + (a_1 - 2a_2 + a_4) < S_1 S_2^2 S_3^2 >,$$

(B.4)

where,

$$a_1 = \sinh J \nabla \cdot f(x) \mid_{x=0} = f(\beta J)$$
$$a_2 = \sinh J \nabla \cosh J \nabla \cdot f(x) \mid_{x=0} = 1/2 f(2\beta J)$$
$$a_3 = \sinh J^3 \nabla \cdot f(x) \mid_{x=0} = 1/4 f(3\beta J) - 3 f(\beta J)$$
$$a_4 = \sinh J \nabla \cosh^2 J \nabla \cdot f(x) \mid_{x=0} = 1/4 [f(3\beta J) + f(\beta J)]$$

(B.5)

From those results we obtain equations (16) and (17) of section 3.1.

Appendix C. Spin Correlation for the One-Dimensional $S=1$ Blume-Capel Model

For the linear chain, we have,

$$< S_0 > = (1 + S_1 \sinh J \nabla + S_1^2 [\cosh J \nabla - 1]) (1 + S_{-1} \sinh J \nabla + S_{-1}^2 [\cosh J \nabla - 1]) > f(x) \mid_{x=0}$$

(C.1)

with $f(x)$ given by expression (10) and $S_1$ and $S_{-1}$ are neighbors of $S_0$. We obtain for the two-spin correlation function

$$< S_0 S_R > = (S_1 S_R + S_{-1} S_R) > f(k)$$
$$+ < (S_1 S_{-1} S_1 S_R + S_{-1} S_1 S_{-1} S_R) > (1/2 f(2k) - f(k))$$

(C.2)

where $k = \beta J$. Applying the inequalities (12, 13)

$$< S_1^2 S_{-1} S_R > \leq < S_{-1} S_R >$$
$$< S_{-1} S_1 S_R > \leq < S_1 S_R >$$

(C.3)

we get,

$$< S_0 S_R > \leq ( < S_1 S_R > + < S_{-1} S_R > ) f(k)$$
$$+ ( < S_{-1} S_R > + < S_1 S_R > ) [1/2 f(2k) - f(k)]$$

(C.4)

Defining $C(R) = < S_0 S_R >$ we get

$$C(R) = A(k)(C(R + 1) + C(R - 1)),$$

(C.5)

where $A(k) = f(2k)/2$.

If $\gamma(R) = C(R + 1)/C(R)$ is inserted in the previous equation we get

$$1 = A(k)(\gamma(R) + \gamma(R)^{-1}).$$

(C.6)

So, $C(R) = \gamma^R$ and

$$\gamma = \frac{1 + \sqrt{1 - 2 f(2\beta J)}}{f(2\beta J)}.$$
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