Asymptotic quasinormal modes of a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime

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Abstract

The analytic forms of the asymptotic quasinormal frequencies of a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime are investigated by using the monodromy technique proposed by Motl and Neitzke. It is found that the asymptotic quasinormal frequencies depend not only on the structure parameters of the background spacetime, but also on the coupling between the scalar fields and gravitational field. Moreover, our results show that only in the minimally coupled case, i.e., $\xi$ tends to zero, the real parts of the asymptotic quasinormal frequencies agree with Hod’s conjecture, $T_H \ln 3$.

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1. Introduction

It is well known that quasinormal modes possess a discrete spectra of complex characteristic frequencies which are entirely fixed by the structure of the background spacetime and are independent of the initial perturbations [1]. Thus, one can directly identify a black-hole existence through comparing the quasinormal modes with the gravitational waves observed in the universe. Meanwhile, it is generally believed that the study of quasinormal modes may lead to a deeper understanding of black holes and quantum gravity because the quasinormal frequency spectra are related to the AdS/CFT correspondence, string theory and loop quantum gravity [2–8]. Therefore, much attention has been devoted to the study of quasinormal modes in the last 30 years [9–25]. In the Schwarzschild spacetime, one found that the asymptotic quasinormal frequencies of high overtones are described by

$$\frac{2\pi \omega}{\kappa} = \ln 3 + i(2n + 1)\pi, \quad n \to \infty, \quad (1)$$
where $\kappa$ is the surface gravity constant of the black hole. Formula (1) was derived numerically [12] and subsequently confirmed analytically [13, 14].

Hod [2] first conjectured that the real parts of the asymptotic quasinormal frequencies of a Schwarzschild black hole can be expressed as $\omega_R = TH \ln 3$. Together with Bohr’s correspondence principle, the first law of black-hole thermodynamics and the asymptotic quasinormal modes, he also obtained some new information about the quantization of area at a black-hole event horizon. Using Hod’s conjecture, Dreyer [3] found that the quasinormal modes can entirely fix the Barbero–Immirzi parameter [4], which was introduced as an indefinite factor by Immirzi to obtain the right form of the black-hole entropy in the loop quantum gravity. Most significantly, the presence of $\ln 3$ also means that the gauge group in the loop quantum gravity should be $SO(3)$ rather than $SU(2)$. Thus, one suggested that Hod’s conjecture may create a new way to probe the quantum properties of black hole.

However, the question whether Hod’s conjecture applies to more general black holes still remains open. Recently, we probed the asymptotic quasinormal modes of a massless scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime [26] and found that the frequency spectra formula satisfies Hod’s conjecture. Cardoso [27] found that the asymptotic quasinormal frequencies in the Schwarzschild de Sitter and anti-de Sitter spacetimes depend on the cosmological constant. Only under the condition that the cosmological constant vanishes, the real parts of the asymptotic quasinormal frequencies return to $TH \ln 3$. For the Reissner–Nordström black hole, Motl and Neitzke [14] obtained the asymptotic quasinormal frequencies which are relevant to the electric charge $Q$. It is unfortunate that the asymptotic quasinormal frequencies do not return to $TH \ln 3$ as the black-hole charge $Q$ tends to zero. Thus, some authors [28] suggested that Hod’s conjecture should be modified in some way. However, what does the correct modification look like? It is an interesting subject that needs to be studied more deeply in the future. At present, it is necessary and important to study the asymptotic quasinormal modes in the more general background spacetimes.

In this paper, our main purpose is to investigate the asymptotic quasinormal modes of a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime. We find that besides dependence on the structure parameters of the background spacetime, the asymptotic quasinormal frequencies are also relevant to the coupling constant $\xi$. The plan of the paper is as follows. In section 2, we derive analytically the asymptotic quasinormal frequency formula of a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime by making use of the monodromy method [14]. At last, a summary and some discussions are presented.

### 2. The asymptotic quasinormal frequencies formula of a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime

In standard coordinates, the metric for the Garfinkle–Horowitz–Strominger dilaton black-hole spacetime can be expressed as [29]

$$ds^2 = -\left(1 - \frac{2M}{r}ight)dt^2 + \left(1 - \frac{2M}{r}ight)^{-1}dr^2 + r'(r' - 2a)\,d\Omega^2,$$

$$e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{2a}{r}\right),$$

where $M$ represents the black-hole mass and $a$ is a parameter related to the dilaton field. The dilaton field is given by $e^{-2\phi} = e^{-2\phi_0}(1 - \frac{2a}{r})$, where $\phi_0$ is the dilaton value at $r' \to \infty$ and $Q$ is the electric charge carried by this black hole. The relationship between mass $M$, the charge $Q$ and $a$ is described as $a = \frac{Q^2}{2Mr}$. This black hole has an event horizon at $r' = 2M$ and two
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singular points at \( r' = 0 \) and \( r' = 2a \). The Hawking temperature \( T_H = \frac{1}{8\pi M} \) is the same as that of the Schwarzschild spacetime.

In order to simplify the calculation, we introduce a coordinate change

\[
r = \sqrt{r'(r' - 2a)}.
\]

Then metric (2) can be rewritten as

\[
ds^2 = - \left( 1 - \frac{2M}{r + \sqrt{a^2 + r^2}} \right) dt^2 + \left( 1 - \frac{2M}{r + \sqrt{a^2 + r^2}} \right)^{-1} \frac{r^2}{r^2 + a^2} dr^2 + r^2 d\Omega^2.
\]

The event horizon of the black hole is now located at \( r = 2\sqrt{M(M - a)} \) and the Hawking temperature is still described by \( T_H = \frac{1}{8\pi M} \). By means of the quantity \( R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \), we find that the point \( r = 0 \) is a curvature singular point.

The general perturbation equation for a coupled massless scalar field in the dilaton spacetime is given by [30]

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \psi - \xi R \psi = 0,
\]

where \( \psi \) is the scalar field and \( R \) is the Ricci scalar curvature. The coupling between the scalar field and the gravitational field is represented by the term \( \xi R \psi \), where \( \xi \) is a numerical coupling factor.

After adopting WKB approximation \( \psi = e^{i\omega t} Y(\theta, \phi) \), introducing a tortoise coordinate

\[
x = \sqrt{a^2 + r^2} - a + 2M \ln \left( \frac{\sqrt{a^2 + r^2} - (2M - a)}{2(M - a)} \right),
\]

and substituting equations (4) and (6) into equation (5), we know that the radial perturbation equation for a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime can be expressed as

\[
\frac{d^2 \phi}{dx^2} + (\omega^2 - V[r(x)]) \phi = 0,
\]

where

\[
V[r(x)] = \left( 1 - \frac{2M}{a + \sqrt{a^2 + r^2}} \right) \times \left[ \frac{l(l+1)}{r^2} + \frac{2M(a^2 + r^2)^{3/2} - 2\sqrt{a^2 + r^2}a^3 + 2Ma^3 - 2a^2 - r^2a^2}{r^4(a + \sqrt{a^2 + r^2})^2} + \xi R \right],
\]

and

\[
R = \frac{2a^2(r^2 + 2aM - 2M\sqrt{a^2 + r^2})}{r^6}.
\]

It is well known that the quasinormal modes consist of the solutions of the perturbation equation (7) with the boundary conditions appropriate for purely ingoing waves at the event horizon and purely outgoing waves at infinity, namely,

\[
\phi \to e^{i\omega t}, \quad x \to -\infty,
\]

\[
\phi \to e^{-i\omega t}, \quad x \to +\infty.
\]

In general, we just consider the perturbation equation (7) in the physical region \( r \geq 2\sqrt{M(M - a)} \) in the Garfinkle–Horowitz–Strominger dilaton black hole. However, in the
monodromy method, it is fundamental to extend analytically equation (7) to the whole complex r-plane. In the process of analytical extension, we find that both the tortoise coordinate $x(r)$ and the wavefunction $\phi(r)$ are multivalued around the singular points $r = 0$ and $r = 2\sqrt{M(M-a)}$. This multivaluedness plays an important and essential role in our analysis. As in [14], we can put branch cuts in the complex r-plane from $r = 0$ to $r = 2\sqrt{M(M-a)}$ in order to avoid dealing with multivalued functions. The monodromy of $\phi(r)$ can be defined by the discontinuity across the cut. Finally, by comparing the local and global monodromy of $\phi(r)$ along the selected contour $L$ around the point $r = 2\sqrt{M(M-a)}$, we can obtain the asymptotic quasinormal frequency spectra in the Garfinkle–Horowitz–Strominger dilaton black-hole spacetime.

From equation (6), we find that $x$ is not uniquely defined as a function of $r$. However, it is very fortunate that we can determine the sign of $\text{Re}(x)$ in the complex r-plane. The regions for the different sign of $\text{Re}(x)$ are shown in figure 1. As in [14], in order to compute conveniently, we may introduce the variable $z = x - \frac{\pi}{2\kappa}$. For $r = 0$, we have $z = 0$. To fix the angle of the variable $z$ at the point $r = 0$, we define the branch $n = 0$ for $\ln(-1)$.

Now, we must define the boundary condition at $r = \infty$. Similar to [14], we can analytically continue $\phi(r)$ via ‘Wick rotation’ to the line $\text{Im}(\omega x) = 0$. For the highly damped modes, i.e., $\omega$ are almost purely imaginary, the line $\text{Im}(\omega x) = 0$ is just slightly sloped off the line $\text{Re}(x) = 0$. Assuming initially that $\text{Re}(\omega) > 0$ and then making use of the condition $\text{Im}(\omega x) = 0$, we find $x = +\infty$ is rotated to $\omega x = +\infty$. Thus, on the line $\text{Re}(x) = 0$, the boundary condition at $r = +\infty$ actually becomes

$$\phi(r) \sim e^{-i\omega x}, \quad \omega x \rightarrow +\infty. \quad (11)$$

Let us now compute the local monodromy around the singular point $r = 2\sqrt{M(M-a)}$. This can be done by matching the asymptotic along the line $\text{Re}(x) = 0$, i.e., the contour $L$ shown in figure 1. When we start at point $A$ and move along the contour $L$ towards interior, the $\phi(x)$ can be looked as the plane waves because the term $\omega^2$ dominates the potential in equation (7) away from the origin point. At the vicinity of the point $r = 0$, we have

$$z \sim -\frac{r^2}{2(M-a)}, \quad (12)$$

and the behaviours of the Ricci scalar curvature and the potential are

$$R \sim \frac{2a(a-M)}{r^4}, \quad (13)$$
and
\[ V[r(z)] \sim -\frac{1 - 2\xi}{4z^2}. \] (14)

We make the identification \( j = \sqrt{2} \xi \), and then the perturbation equation (7) can be rewritten as
\[ \left( \frac{d^2}{dz^2} + \omega^2 + 1 - \frac{j^2}{4z^2} \right) \phi(z) = 0. \] (15)

From [31], we find it can be exactly solved in terms of the Bessel function and the general solution near the origin point can be expressed as
\[ \phi(z) = A_+ c_+ \sqrt{\omega z} J_{j/2}(\omega z) + A_- c_- \sqrt{\omega z} J_{-j/2}(\omega z). \] (16)

Now, let us look for the asymptotic forms of solution (16) away from the origin. After considering the asymptotic behaviour of \( J_{j/2}(\omega z) \) as \( \omega z \to \infty \), we can select the normalization factors \( c_\pm \) in (16) so that we can write the asymptotic forms as
\[ c_\pm \sqrt{\omega z} J_{\pm j/2}(\omega z) \sim 2 \cos(\omega z - \alpha_\pm), \] (17)
as \( \omega z \to \infty \), where \( \alpha_\pm = \frac{\pi}{4}(1 \pm j) \). From equations (16), (17) and the boundary condition (11), we have
\[ A_+ e^{-i\alpha} + A_- e^{i\alpha} = 0, \] (18)

and
\[ \phi(z) \sim (A_+ e^{i\alpha} + A_- e^{-i\alpha}) e^{-i\omega z}. \] (19)

To follow the contour \( L \) and approach the point \( B \), we have to turn an angle \( \frac{3\pi}{2} \) around the origin \( r = 0 \), corresponding to \( 3\pi \) around \( z = 0 \). From the Bessel function behaviour near the origin point
\[ J_{\pm j/2}(\omega z) = z^{\pm j/2} \varphi(z), \] (20)
where \( \varphi(z) \) is an even holomorphic function, we find that after the \( 3\pi \) rotation the asymptotic is
\[ c_\pm \sqrt{\omega z} J_{\pm j/2}(\omega z) \sim e^{i\alpha_\pm} 2 \cos(-\omega z - \alpha_\pm), \] (21)
as \( \omega z \to -\infty \). Thus the asymptotic at the point \( B \)
\[ \phi(z) \sim (A_+ e^{i\alpha} + A_- e^{-i\alpha}) e^{-i\omega z} + (A_+ e^{\alpha} + A_- e^{-\alpha}) e^{i\omega z}, \quad \omega z \to -\infty. \] (22)

Finally, we can come back from point \( B \) to point \( A \) along the large semicircle in the right half plane. In this region, because the term \( \omega^3 \) dominates the potential \( V[r(z)] \), we can approximate the solutions of the perturbation equation as plane waves. When we return to point \( A \), the coefficient of \( e^{-i\omega z} \) remains unchanged while the coefficient of \( e^{i\omega z} \) makes only an exponentially small contribution to \( \phi(z) \) in the right plane. Finally, we find that the monodromy around the contour \( L \) must multiply the coefficient of \( e^{-i\omega z} \) by a factor
\[ \frac{A_+ e^{i\alpha} + A_- e^{-i\alpha}}{A_+ e^{\alpha} + A_- e^{-\alpha}} = \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} - e^{-i\alpha}} = -(1 + 2 \cos \pi j). \] (23)

Now, let us calculate the global monodromy around the contour \( L \). Since the only singularity of \( \phi(r) \) or \( e^{-i\omega z} \) inside the contour occurs at the point \( r = 2\sqrt{M(M-a)} \), according to the boundary condition of the quasinormal modes, we can obtain the monodromy of \( \phi(r) \) or \( e^{-i\omega z} \) at this point. After a full clockwise round trip, \( \phi(r) \) acquires a phase \( e^{\mp \pi j} \), while \( e^{-i\omega z} \) acquires a phase \( e^{-\mp \alpha} \). So the coefficient of \( e^{-i\omega z} \) in the asymptotic of \( \phi(r) \) must be multiplied...
by $e^{\frac{2\pi}{\kappa}}$. Substituting $j = \sqrt{2\xi}$ into equation (23) and comparing the local monodromy with the global one, we find that the asymptotic quasinormal frequencies of a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton black-hole spacetime satisfy

$$e^{\frac{2\pi}{\kappa}} = -[1 + 2\cos(\sqrt{2\xi}\pi)].$$

(24)

Making a simple operation on equation (24), we obtain easily the formula

$$\frac{2\pi \omega}{\kappa} = \ln[1 + 2\cos(\sqrt{2\xi}\pi)] + i(2n + 1)\pi, \quad n \rightarrow \infty.$$  

(25)

It is interesting to note that the right-hand side of formula (25) contains the coupling factor $\xi$. This means that the asymptotic quasinormal modes depend not only on the structure parameters of the background spacetimes, but also on the coupling between the matter fields and gravitational field. Furthermore, we find only when $\xi = 0$, namely, in the minimally coupled case, the real part of the right-hand side of formula (25) becomes $\ln 3$, which is consistent with Hod’s conjecture.

3. Summary and discussion

We have investigated the analytical forms of the asymptotic quasinormal frequencies for a coupled scalar field in the Garfinkle–Horowitz–Strominger dilaton spacetime by adopting the monodromy technique. It is shown that the asymptotic quasinormal frequencies depend not only on the structure parameters of the background spacetime, but also on a coupling constant $\xi$. The fact tells us that the interaction between the matter fields and gravitational field will affect the frequencies spectra formula of the asymptotic quasinormal modes. It is a novel property of the quasinormal modes in Garfinkle–Horowitz–Strominger dilaton spacetime. In the Schwarzschild and Reissner–Nordström spacetimes, we find that the asymptotic quasinormal frequencies do not possess this behaviour because the curvature scalar $R$ in both the spacetimes is equal to zero and then the coupled term in equation (5) vanishes. Moreover, we find that Hod’s conjecture, the real parts of the asymptotic quasinormal frequencies equals to $T_H \ln 3$, is valid only for the minimally coupled case in the Garfinkle–Horowitz–Strominger dilaton spacetime, i.e., $\xi$ becomes zero. It implies that maybe there exists a more general form of the Hod’s conjecture.

It should be pointed out that although formula (25) is not related to the dilaton field parameter $a$ obviously, it does not mean that the quasinormal modes are independent of the dilaton. The reason is that we just consider the contribution of the leading term $r^{-4}$ in the potential $V$ to the local monodromy of the wavefunction $\phi(z)$. If we consider the contribution of the lower order terms in the potential $V(r)$, the corrected term to the asymptotic quasinormal frequencies will depend on the parameter $a$ of the dilaton field. For example, as $a$ is very small, the main corrected term is roughly in proportion to $\frac{(1-i)}{\sqrt{4\pi^2}} \left[ I(l+1) + 1 - \frac{2M}{\lambda M_{\text{max}}} \right]$, which shows that the quasinormal frequencies increase as $a$ increases in the Garfinkle–Horowitz–Strominger dilaton spacetime.

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