Massless particle in 2d spacetime

with constant curvature

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Abstract

We consider dynamics of massless particle in 2d spacetimes with constant curvature. We analyze different examples of spacetime. Dynamical integrals are constructed from spacetime symmetry related to sl(2,R) algebra. Mass-shell condition restricts dynamical integrals to a cone (without vertex) which defines physical-phase space. We parametrize the cone by canonical coordinates. Canonical quantization with definite choice of operator ordering leads to unitary irreducible representations of SO+(2.1) group.

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1 Introduction

We analyze classical and quantum dynamics of a massless particle in two dimensional spacetime with constant curvature $R_0 \neq 0$. We consider the cases when spacetime is: (i) hyperboloid with $R_0 < 0$, (ii) half-plane with $R_0 < 0$ and (iii) stripe with $R_0 > 0$. Presented results complete our analysis of dynamics of a particle with non-zero mass $m_0$ in 2d curved spacetime [1,2]. Taking formally $m_0 \to 0$ leads to some singularities both at classical and quantum levels. Thus, the massless case needs separate treatment. As the result we get clear picture of the role played by topology and global symmetries of spacetime in the procedure of canonical quantization.

Dynamics of a massless particle in gravitational field $g_{\mu\nu}(x^0, x^1)$ is defined by the action [3]

$$S = \int L(\tau) \, d\tau, \quad L(\tau) := -\frac{1}{2\lambda(\tau)} \, g_{\mu\nu}(x^0(\tau), x^1(\tau)) \, \dot{x}^\mu(\tau) \dot{x}^\nu(\tau), \quad (1.1)$$

where $\tau$ is an evolution parameter, $\lambda$ plays the role of Lagrange multiplier and $\dot{x}^\mu := dx^\mu/d\tau$. It is assumed that $\lambda > 0$ and $\dot{x}^0 > 0$.

The action (1.1) is invariant under reparametrization $\tau \to f(\tau), \lambda(\tau) \to \lambda(\tau)/\dot{f}(\tau)$. This gauge symmetry leads to the constrained dynamics in the Hamiltonian formulation [4]. The constraint reads

$$\Phi := g^{\mu\nu} p_\mu p_\nu = 0, \quad (1.2)$$

where $p_\mu := \partial L/\partial \dot{x}^\mu$ are canonical momenta (we use units with $c = 1 = \hbar$).

As in [1,2] we use the gauge invariant description. In the case of massive particle the set of spacetime trajectories can be considered as a physical phase-space of the system. This set has natural symplectic structure, which
can be used for quantization. The spacetime trajectories of a massless particle has no symplectic structure (this set is only one dimensional, since particle velocity is fixed). For the gauge invariant description we use the dynamical integrals constructed from the global symmetries of spacetime.

2 Dynamics on hyperboloid

Let \((y^0, y^1, y^2)\) be the standard coordinates on 3d Minkowski space with the metric tensor \(\eta_{ab} = \text{diag}(+,-,-)\). A one-sheet hyperboloid \(H\) is defined by

\[-(y^0)^2 + (y^1)^2 + (y^2)^2 = m^{-2},\]

where \(m > 0\) is a fixed parameter. \(H\) has a constant curvature \(R = -2m^2\) (see [5]).

Any 2-dimensional Lorentzian manifold with constant curvature \(R_0\) can be described (locally) by the conformal metric tensor [5]

\[g_{\mu\nu}(X) = \exp \varphi(X) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := (x^0, x^1),\]

where the field \(\varphi(X)\) satisfies the Liouville equation [6]

\[(\partial_0^2 - \partial_1^2)\varphi(X) + R_0 \exp \varphi(X) = 0.\]

Making use of the parametrization

\[y^0 = -\frac{\cot m\rho}{m}, \quad y^1 = \frac{\cos m\theta}{m \sin m\rho}, \quad y^2 = \frac{\sin m\theta}{m \sin m\rho},\]

where \(\rho [0, \pi/m], \quad \theta [0, 2\pi/m]\)

we get the conformal form (2.2), with

\[\varphi = -\log \sin^2 m\rho,\]
where the spacetime coordinates \( x^0 \) and \( x^1 \) are identified with the parameters \( \rho \) and \( \theta \), respectively. The function (2.5) satisfies the Liouville equation (2.3) for \( R_0 = -2m^2 \).

The Lagrangian (1.1) in this case reads

\[
L = -\frac{\dot{\rho}^2 - \dot{\theta}^2}{2\lambda \sin^2 m\rho}.
\]

(2.6)

The hyperboloid (2.1) is invariant under the Lorentz transformations, i.e., \( SO_1(2,1) \) is the symmetry group of our system. The corresponding infinitesimal transformations (rotation and two boosts) are

\[
\begin{align*}
(\rho, \theta) & \rightarrow (\rho, \theta + \alpha_0/m), \\
(\rho, \theta) & \rightarrow (\rho - \alpha_1/m \ \sin m\rho \sin m\theta, \theta + \alpha_1/m \ \cos m\rho \cos m\theta), \\
(\rho, \theta) & \rightarrow (\rho + \alpha_2/m \ \sin m\rho \cos m\theta, \theta + \alpha_2/m \ \cos m\rho \sin m\theta).
\end{align*}
\]

(2.7)

The dynamical integrals for (2.7) read

\[
\begin{align*}
J_0 &= \frac{p_\theta}{m}, \\
J_1 &= -\frac{p_\rho}{m} \sin m\rho \sin m\theta + \frac{p_\theta}{m} \cos m\rho \cos m\theta, \\
J_2 &= \frac{p_\rho}{m} \sin m\rho \cos m\theta + \frac{p_\theta}{m} \cos m\rho \sin m\theta,
\end{align*}
\]

(2.8)

where \( p_\theta := \partial L/\partial \dot{\theta} \), \( p_\rho := \partial L/\partial \dot{\rho} \) are canonical momenta.

Since \( J_0 \) is connected with space translations (see (2.7)), it defines particle momentum \( p_\theta = mJ_0 \).

It is clear that the dynamical integrals (2.8) satisfy the commutation relations of \( sl(2,\mathbb{R}) \) algebra

\[
\{J_a, J_b\} = \varepsilon_{abc} \eta^{cd} J_d,
\]

(2.9)

where \( \eta^{cd} \) is the Minkowski metric tensor and \( \varepsilon_{abc} \) is the anti-symmetric tensor with \( \varepsilon_{012} = 1 \).
The mass shell condition (1.2) takes the form \(p^2 - p^2_\theta = 0\) and it leads to the relation

\[J_0^2 - J_1^2 - J_2^2 = 0.\] (2.10)

Eq. (2.10) defines two cones. The singular point of the cones \(J_0 = 0 = J_1 = J_2\) should be removed, since it corresponds to the massless particle with zero momentum \((p_\rho = 0 = p_\theta)\), which does not exist.

Thus, the dynamical integrals (2.8) define the physical phase-space of the system and it consists of two disconnected cones \(C_+\) and \(C_-\), for \(J_0 > 0\) and \(J_0 < 0\), respectively.

According to (2.8) and due to \(p_\rho < 0\) (since \(\dot{\rho} > 0\) and \(\lambda > 0\)) the trajectories satisfy the equations

\[J_a y^a = 0, \quad J_1y_2 - J_2y_1 = \frac{p_\rho}{m^2} = -\frac{|J_0|}{m}.\] (2.11)

Each point \((J_a)\) of the cone \(C_+\) or \(C_-\) defines uniquely the trajectory on the hyperboloid \(H\). The trajectories (2.11) are straight lines in 3d Minkowski space. Hence, the set of trajectories is the set of generatrices of the hyperboloid (2.1). For \(J_0 > 0\) we get the ‘right’ moving particle with \(\dot{\theta} > 0\), while for \(J_0 < 0\) we have the ‘left’ moving one, with \(\dot{\theta} < 0\). Both cones, \(C_+\) and \(C_-\), are invariant under \(SO_T(2.1)\) transformations.

To quantize the system, we consider the cones \(C_+\) and \(C_-\) separately. We parametrize \(C_+\) as follows

\[J_0 = \frac{1}{2}(p^2 + q^2), \quad J_1 = \frac{p}{2} \sqrt{p^2 + q^2}, \quad J_2 = \frac{q}{2} \sqrt{p^2 + q^2}.\] (2.12)

where \((0,0) \neq (p,q) \in \mathbb{R}^2\).

It is easy to see that (2.12) defines the one-to-one map from the plane without
the origin to \( C_+ \). The canonical commutation relation \( \{ p, q \} = 1 \) provides (2.9).

For quantization of \( C_+ \) system we introduce the creation-annihilation operators \( a^\pm := (\hat{p} \pm i\hat{q})/\sqrt{2} \) and choose the definite operator ordering in (2.12). This ordering is defined by the following requirements:

a) the operators \( \hat{J}_a \) are self-adjoint,

b) they generate global \( SO_\uparrow(2.1) \) transformations,

c) the spectrum of \( \hat{J}_0 \) is positive,

d) the Casimir number operator \( \hat{C} := \hat{J}_0^2 - \hat{J}_1^2 - \hat{J}_2^2 \) is zero.

This leads to the following expressions

\[
\hat{J}_0 = a^- a^+, \quad \hat{J}_+ = a^+ \sqrt{a^- a^+ + 1}, \quad \hat{J}_- = \sqrt{a^- a^+ + 1} a^-,
\]

where \( \hat{J}_\pm = \hat{J}_1 \pm \hat{J}_2 \). The creation and annihilation operators are defined in the Fock space in the standard way

\[
a^+ |n\rangle = \sqrt{n + 1} |n + 1\rangle, \quad a^- |n\rangle = \sqrt{n} |n - 1\rangle,
\]

where the vectors \( |n\rangle \), \( n \geq 0 \) form the basis of the corresponding Hilbert space \( \mathcal{H}_+ \).

The states \( |n\rangle \) are the eigenstates of \( \hat{J}_0 \)

\[
\hat{J}_0 |n\rangle = (n + 1) |n\rangle,
\]

and from (2.13) we get

\[
\hat{J}_+ |n\rangle = \sqrt{(n + 1)(n + 2)} |n + 1\rangle, \quad \hat{J}_- |n\rangle = \sqrt{n(n + 1)} |n - 1\rangle.
\]

Eqs.(2.13)-(2.15) define the unitary irreducible representation (UIR) of the group \( SO_\uparrow(2.1) \). We identify this representation with the representation of \( D^+_1 \) of the discrete series of \( SL(2,R) \) group [7].
Quantization of the case $C_-$ can be done in the same way. The corresponding Hilbert space $H_-$ is again the Fock space, but we have to make the following replacements: $\hat{J}_\pm \to \hat{J}_\mp$ and $\hat{J}_0 \to -\hat{J}_0$. As a result we get the representation $D_1^-$. 

The Hilbert space of the whole system is $H = H_+ \oplus H_-$ and the corresponding representation $D_1^+ \oplus D_1^-$ describes the $SO(2,1)$ symmetry of the quantum system.

It is interesting to mention the following:

At the classical level our system can be obtained from the massive case in the limit $m_0 \to 0$ (where $m_0$ is a particle mass) and by removing the singular point of the physical phase-space. The quantum theory of the massive case was considered recently in [1]. The corresponding representation is defined by the operators

$$
\hat{J}_0 \psi_n = n \psi_n, \quad \hat{J}_+ \psi_n = \sqrt{n^2 + n + a^2} \psi_{n+1}, \\
\hat{J}_- \psi_n = \sqrt{n^2 - n + a^2} \psi_{n-1},
$$

where $\psi_n := \exp in\phi$ ($n \in \mathbb{Z}$) form the basis of the Hilbert space $L_2(S^1)$ and $a = m_0/m$. For $a = 0$ ($m_0 = 0$) this representation turns into $D_1^+ \oplus A \oplus D_1^-$, where $A$ is a one dimensional trivial representation on the vector $\psi_0$. By removing $A$ we get the quantum theory of the massless case.

The momentum of a quantum particle $\hat{p}_\theta = m \hat{J}_0$ can take only discrete values $P_n = mn$, where $n$ is a nonzero integer.
3 Dynamics on half-plane

Let us consider the Liouville field \( \varphi = -2 \log m|x^0| \), given on a plane \((x^0, x^1)\). This field defines constant spacetime curvature \( R_0 = -2m^2 \) (see (2.3)). The Lagrangian (1.1) in this case reads

\[
L = -\frac{2\dot{x}^+\dot{x}^-}{\lambda m^2(x^+ + x^-)^2},
\]

where \( x^\pm := x^0 \pm x^1 \). It is assumed that \( \dot{x}^0 > 0 \), which leads to \( p_+ + p_- < 0 \).

Formally, (3.1) is invariant under the fractional-linear transformations

\[
x^+ \to \frac{ax^+ + b}{cx^+ + d}, \quad x^- \to \frac{ax^- - b}{-cx^- + d}, \quad ad - bc = 1.
\]

Thus, formally, \( SL(2, \mathbb{R})/\mathbb{Z}_2 \) (which is isomorphic to \( SO_+^+(2, 1) \)) is the symmetry of our system. The transformations (3.2) are well defined on the plane only for \( c = 0 \). The corresponding transformations with \( c = 0 \) form the group of dilatations and translations (along \( x^1 \)), which is a global symmetry of the considered spacetime.

The infinitesimal transformations for (3.2) are

\[
x^\pm \to x^\pm + \alpha_0, \quad x^\pm \to x^\pm + \alpha_1 x^\pm, \quad x^\pm \to x^\pm + \alpha_2 (x^\pm)^2
\]

and the corresponding dynamical integrals read

\[
P = p_+ - p_-, \quad K = p_+ x^+ + p_- x^-, \quad M = p_+ (x^+)^2 - p_- (x^-)^2,
\]

where \( p_\pm = \partial L/\partial \dot{x}^\pm \).

The dynamical integrals (3.4) satisfy again the commutation relations (2.9) with

\[
J_0 = \frac{1}{2}(P + M), \quad J_1 = \frac{1}{2}(P - M), \quad J_2 = K.
\]
The mass-shell condition (1.2) leads to $p_+ = 0$ for $P > 0$ and $p_- = 0$ for $P < 0$. Due to (3.4) and the mass-shell condition, we have

$$K^2 - PM = 0$$

(3.6)

and the trajectories read

$$x^1 - \epsilon(P)x^0 = \frac{K}{P}, \quad \text{where} \quad \epsilon(P) = \frac{P}{|P|}.$$  

(3.7)

It is clear that $x^0 = 0$ is the singularity line in the spacetime. In the massive case this singularity leads to the dynamical ambiguities [2,8]. However, the dynamics of the massless particle is defined uniquely due to (3.7).

The physical phase-space is defined by two cones (3.6) without the line corresponding to $P = 0$. Thus, we have two disconnected parts: $\mathcal{P}_+$ with $P > 0$ and $\mathcal{P}_-$ with $P < 0$. Both cones ($\mathcal{P}_+$ and $\mathcal{P}_-$) are invariant under dilatations and translations generated by the dynamical integrals $K$ and $P$. But, the transformations generated by $M$ are not defined globally. Thus, the physical phase-space has the same symmetry as the spacetime.

Let us quantize the system corresponding to $\mathcal{P}_+$ case (the case $\mathcal{P}_-$ can be done in the same way).

We parametrize $\mathcal{P}_+$ as follows [9]

$$P = p, \quad K = pq, \quad M = pq^2,$$

(3.8)

where $(p, q)$ are the coordinates on half-plane with $p > 0$. The canonical commutation relation $\{p, q\} = 1$ provides the commutation relations of $sl(2, \mathbb{R})$ algebra

$$\{P, K\} = P, \quad \{P, M\} = 2K, \quad \{K, M\} = M.$$  

(3.9)
In the ‘p-representation’ (\( \hat{q} = i\partial_p \)) we choose the operator ordering for \( \hat{K} \) and \( \hat{M} \) by the following requirements:

a) \( \hat{K} \) and \( \hat{M} \) are self-adjoint,

b) \( \hat{P}, \hat{K} \) and \( \hat{M} \) satisfy the commutation relations corresponding to (3.9),

c) the Casimir number operator \( \hat{C} = \frac{1}{2}(\hat{P}\hat{M} + \hat{M}\hat{P}) - \hat{K}^2 \) equals zero.

As a result we get

\[
\hat{P} = p, \quad \hat{K} = i(p\partial_p + \frac{1}{2}), \quad \hat{M} = -p\partial_p^2 - \partial_p + \frac{1}{4p}.
\] (3.10)

The operators defined by (3.10) have continuous spectrum.

The operator \( \hat{J}_0 = (\hat{P} + \hat{M})/2 \) has the discrete spectrum

\[
\hat{J}_0\psi_n(p) = \lambda_n\psi_n(p),
\] (3.11)

\[
\psi_n(p) = \sqrt{p} \exp(-p)L^1_n(2p), \quad \lambda_n = n + 1, \quad n \geq 0,
\] (3.12)

where \( L^1_n(x) \) are the Laguerre polynomials defined by [10]

\[
L^1_n(x) = \sum_{k=0}^{n-1} (-)^k \frac{n!(n + 1)\cdots(k + 2)}{k!(n-k)!} x^k + (-)^n x^n.
\] (3.13)

Eqs. (3.11)-(3.13) show that our representation is unitarily equivalent to \( D_1^+ \) representation (see (2.14) and (2.15)). The scheme presented here can be generalized to include other representations of the discrete series of \( SL(2, \mathbb{R}) \) group [9].

4 Dynamics on stripe

Let us consider the spacetime to be a stripe

\[
\mathcal{S} := \{(t, x) \mid t \in \mathbb{R}, \quad x \in [0, \pi/m]\},
\] (4.1)
with the conformal metric tensor (2.2) and the Liouville field

$$\varphi(t, x) = -\log \sin^2 mx. \tag{4.2}$$

It defines the spacetime with constant positive curvature $R = 2m^2$.

The Lagrangian (1.1) in this case reads

$$L = -\frac{\dot{t}^2 - \dot{x}^2}{2\lambda \sin^2 mx} \tag{4.3}$$

and it is invariant under the action of the universal covering group $\widetilde{SL}(2, \mathbb{R})$.

The corresponding infinitesimal transformations

$$(t, x) \rightarrow (t - \alpha_0/m, x), \tag{4.4}$$

$$(t, x) \rightarrow (t - \alpha_1/m \cos mx \cos mt, x + \alpha_1/m \sin mx \sin mt),$$

$$(t, x) \rightarrow (t - \alpha_2/m \cos mx \sin mt, x - \alpha_2/m \sin mx \cos mt)$$

lead to the dynamical integrals

$$J_0 = -\frac{p_t}{m}, \quad J_1 = -\frac{p_t}{m} \cos mx \cos mt + \frac{p_x}{m} \sin mx \sin mt,$$

$$J_2 = -\frac{p_t}{m} \cos mx \sin mt - \frac{p_x}{m} \sin mx \cos mt, \tag{4.5}$$

which satisfy the commutation relations (2.9).

The mass-shell condition $p_t^2 = p_x^2$ provides

$$J_0^2 - J_1^2 - J_2^2 = 0, \tag{4.6}$$

which defines two cones.

The physical conditions $\dot{t} > 0$ and $\lambda > 0$ give $p_t < 0$. Thus, the upper-cone ($J_0 > 0$) without the vertex $J_0 = 0 = J_1 = J_2$ is the physical phase-space of our system. By (4.5) we get

$$J_0 \cos mx = J_1 \cos mt - J_2 \sin mt. \tag{4.7}$$
Due to (4.6), the trajectories of massless particle (4.7) are zigzag lines with the slope $\dot{x} = \pm 1$. A particle with the velocity equal to one reaches the ‘edge’ of space where it ‘changes’ the direction. Then, it reaches another ‘edge’, again changes the direction, etc. This motion is periodic with the period $2\pi/m$.

Since the physical phase-space is the upper-cone without the vertex, the system can be quantized as in the case of $C_+^+$ system, presented in Section 2. The corresponding quantum theory is defined by the $D_{1}^+$ representation.

5 Discussion

Any two Lorentzian 2d manifolds with the same constant curvature $R_0$ are (locally) isometric. Therefore, the considered systems give the exhaustive picture for the dynamics of massless particle in 2d spacetime with constant curvature.

It is interesting to note that these mechanical systems are related to the model of 2d gravity with the dilaton field [11]. In the conformal gauge this 2d gravity model is described by the Liouville and free fields, which have equal traceless energy-momentum tensors. One can check that the massless field, which has the same traceless energy-momentum tensor as the Liouville field (4.2), is singular and the singularity lines coincide with the trajectories of the massless particle given by (4.7). A similar picture exists for the hyperboloid (2.1) described by the Liouville field (2.5). The Hamiltonian reduction to the physical variables eliminates all field degrees of freedom of the indicated model of 2d gravity and the physical phase-space of the model
is finite dimensional. The character of the reduced system and its relation to the considered dynamics of massless particle depends on the global properties of spacetime. Details concerning this relation will be discussed elsewhere.

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