THE SECOND BOGOLYUBOV THEOREM AND GLOBAL AVERAGING PRINCIPLE FOR SPDES WITH MONOTONE COEFFICIENTS

MENGYU CHENG AND ZHENXIN LIU

Abstract. In this paper, we establish the second Bogolyubov theorem and global averaging principle for stochastic partial differential equations (in short, SPDEs) with monotone coefficients. Firstly, we prove that there exists a unique $L^2$-bounded solution to SPDEs with monotone coefficients and this bounded solution is globally asymptotically stable in square-mean sense. Then we show that the $L^2$-bounded solution possesses the same recurrent properties (e.g. periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, etc.) in distribution sense as the coefficients. Thirdly, we prove that the recurrent solution of the original equation converges to the stationary solution of averaged equation under the compact-open topology as the time scale goes to zero — in other words, there exists a unique recurrent solution to the original equation in a neighborhood of the stationary solution of averaged equation when the time scale is small. Finally, we establish the global averaging principle in weak sense, i.e. we show that the attractor of original system tends to that of the averaged equation in probability measure space as the time scale goes to zero. For illustration of our results, we give two applications, including stochastic reaction diffusion equations and stochastic generalized porous media equations.

1. Introduction

Averaging principle is an effective method for studying dynamical systems with highly oscillating components. Under suitable conditions, the highly oscillating components can be “averaged out” to produce an averaged system. The averaged system is easier for analysis and governs the evolution of the original system over long time scales.

Consider the following deterministic systems in $\mathbb{R}^n, n \in \mathbb{N}$:

\begin{equation}
\dot{X}^\varepsilon = F\left(\frac{t}{\varepsilon}, X^\varepsilon\right)
\end{equation}

and

\begin{equation}
\dot{X} = \bar{F}(X)
\end{equation}

for small parameter $0 < \varepsilon \ll 1$, where $F \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $\bar{F}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t,x)dt$.

It is a basic problem of averaging principle to determine in what sense the behavior of solutions to the averaged system (1.2) approximates the behavior of solutions to the non-autonomous system (1.1) as the time scale $\varepsilon$ goes to zero. For the connotation of approximation, there are three natural types of interpretation. One is the so-called first Bogolyubov theorem, i.e. the convergence of the solution of the original Cauchy problem (1.1) to that of the averaged equation (1.2) on a finite interval $[0,T]$ when the initial data are such that $X^\varepsilon(0) = X(0)$. And another one is to request that the approximation be valid on the entire real axis, which is the so-called second Bogolyubov theorem (sometimes called “ theorem
for periodic solution by averaging”). In addition, it is meaningful to determine whether the attractor of the averaged equation (1.2) approximates the attractor of the original equation (1.1). One calls this result the **global averaging principle**.

The idea of averaging dates back to the perturbation theory which was proposed by Clairaut, Laplace and Lagrange in the 18th century. Then fairly rigorous averaging method for nonlinear oscillations was presented by Krylov, Bogolyubov and Mitropolsky [28, 2], which is called the Krylov-Bogolyubov method nowadays. After that, there is a lot of works on averaging for deterministic finite and infinite dimensional systems, which we will not mention here.

Meanwhile, Stratonovich firstly proposed the stochastic averaging method on the basis of physical considerations, which was later proved mathematically by Khasminskii. Then extensive investigations concerning averaging principle for stochastic differential equations were conducted, following Khasminskii’s mathematically pioneering work [25]; see, e.g. [1, 4, 5, 6, 7, 14, 16, 17, 19, 20, 26, 30, 32, 34, 41, 42, 43, 44, 45] and the references therein. Note that the above existing results are concerned with the first Bogolyubov theorem.

Despite considerable advances in this direction, there are few works on stochastic averaging concerning with the second Bogolyubov theorem, which states: there exists a unique periodic solution to the original equation in a neighborhood of the stationary solution of averaged equation. Consider the following SPDEs on a separable Hilbert space \((H, \langle \cdot, \cdot \rangle)\)

\[
dX_\varepsilon(t) = \left( A(X_\varepsilon(t)) + F\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right) \right) dt + G\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right) dW(t),
\]

where \(A\) satisfies some monotone condition, \(F\) and \(G\) are Lipschitz in second variable. Here \(W\) is a two-sided cylindrical Wiener process defined on another separable Hilbert space \(U\) and \(0 < \varepsilon \leq 1\). Recall that the second Bogolyubov theorem for stochastic differential equations with almost periodic coefficients was studied in [24], and [10] investigated the averaging principle for stochastic ordinary differential equations with general recurrent coefficients. As discussed in [24] and [10], equations are semilinear with globally Lipschitz and linear growth nonlinear terms, which cannot cover the monotone case. However, the coefficients of many interesting practical models just satisfy monotone conditions. Some typical examples are reaction diffusion equations and porous media equations.

Generally, reaction diffusion equations can be used to describe the growth of biological population and the spatial spread of epidemic diseases, which are largely affected by time-varying environment. In particular, the recurrent phenomenon has been found in the growth of population and the spread of diseases, since some regular environmental changes such as seasonal changes. And porous media equations appear in the description of different natural phenomenon related to diffusion, filtration or heat propagation. Since the noise models the small irregular fluctuations generated by microscopic effects, it is more practical to consider the above systems perturbed by white noise.

From the perspective of theoretical and practical value, we establish the second Bogolyubov theorem for SPDEs with monotone coefficients in this paper. More specifically, consider equation (1.3), we assume that \(A\) is strongly monotone. Compared with the assumption that \(A\) is a linear bounded operator in [10], this condition admits wider applications. It includes unbounded linear operators and quasi-linear operators.

Denoting by \(F_\varepsilon(t, x) := F\left(\frac{t}{\varepsilon}, x\right)\) and \(G_\varepsilon(t, x) := G\left(\frac{t}{\varepsilon}, x\right)\), we transform equation (1.3) to

\[
dX_\varepsilon(t) = \left( A(X_\varepsilon(t)) + F_\varepsilon(t, X_\varepsilon(t)) \right) dt + G_\varepsilon(t, X_\varepsilon(t)) dW(t).
\]

In this paper, we firstly show that there exists a unique \(L^2\)-bounded solution \(X_\varepsilon(t), t \in \mathbb{R}\) of (1.4) which shares the same recurrent properties (in particular, periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, Poisson stable) in distribution sense as the
coefficients for each $0 < \varepsilon \leq 1$. And we prove that this $L^2$-bounded solution $X_\varepsilon(t), t \in \mathbb{R}$ to (1.4) is globally asymptotically stable in square-mean sense. Without loss of generality, we assume $\varepsilon = 1$ in this part, then the $L^2$-bounded solution is denoted by $X(t), t \in \mathbb{R}$. Note that coefficient $F$ in (1.4) need not to be Lipschitz in this part. This result is interesting on its own. To our knowledge, there are only a few works on general recurrent solutions to SPDEs in a unified framework, see [9, 31]. As discussed in [9] and [31], they dealt with recurrent solutions to semilinear SPDEs with Lipschitz continuous and globally linear growth nonlinearities according to Shcherbakov’s comparability method by character of recurrence. B. A. Shcherbakov gave the existence condition of at least one (or exactly one) solution to deterministic equation with the same character of recurrence as the coefficient. This solution is said to be compatible (respectively, uniformly compatible). Comparing to [9] and [31], we consider SPDEs with monotone coefficients.

Let $X_\varepsilon$ be the recurrent solution to equation (1.4). Then one of the major aims of this paper is to prove that
\begin{equation}
\lim_{\varepsilon \to 0} d_{BL}(\mathcal{L}(X_\varepsilon), \mathcal{L}(\bar{X})) = 0 \quad \text{in } Pr(C(\mathbb{R}, H))
\end{equation}
(see Theorem 4.7 and Corollary 4.8), where $d_{BL}$ is the bounded Lipschitz distance (also called Fortet-Mourier distance); see Subsection 2.4 for details. And $\bar{X}$ is the unique stationary solution of the following averaged equation
\begin{equation}
d\bar{X}(t) = (A(X(t)) + \bar{F}(X(t))) \, dt + \bar{G}(X(t))dW(t).
\end{equation}
Here $\bar{F} \in C(H, H), \bar{G} \in C(H, L_2(U, H))$, $\bar{F}$ and $\bar{G}$ satisfy
\begin{equation*}
\lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} F(s, x) \, ds = \bar{F}(x), \quad \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} \|G(s, x) - \bar{G}(x)\|_{L_2(U, H)}^2 \, ds = 0
\end{equation*}
uniformly with respect to $t \in \mathbb{R}$.

This averaging principle is also applicable to the following system
\begin{equation}
d\bar{X}_\varepsilon(t) = \varepsilon (A(X(t)) + F(t, X(t))) \, dt + \sqrt{\varepsilon} G(t, X(t))dW(t),
\end{equation}
where $0 < \varepsilon \leq 1$. With the time scaling $t \mapsto \frac{t}{\varepsilon}$, denote by $\Phi_\varepsilon(t) := X(\frac{t}{\varepsilon})$ and $W_\varepsilon(t) := \sqrt{\varepsilon} W(\frac{t}{\varepsilon})$ for all $t \in \mathbb{R}$, we transform equation (1.7) to
\begin{equation}
d\Phi_\varepsilon(t) = (A(\Phi_\varepsilon(t)) + F_\varepsilon(t, \Phi_\varepsilon(t))) \, dt + G_\varepsilon(t, \Phi_\varepsilon(t))dW_\varepsilon(t).
\end{equation}
Then we can consider the following equation
\begin{equation}
d\bar{X}_\varepsilon(t) = \left( A(\bar{X}_\varepsilon(t)) + F_\varepsilon(t, \bar{X}_\varepsilon(t)) \right) \, dt + G_\varepsilon(t, \bar{X}_\varepsilon(t))dW(t).
\end{equation}
It is obvious that $\mathcal{L}(\bar{X}_\varepsilon(t)) = \mathcal{L}(\Phi_\varepsilon(t))$ for any $t \in \mathbb{R}$.

In contrast to the first Bogolyubov averaging principle for Cauchy problem of stochastic differential equations on finite intervals, we prove that there exists a unique recurrent solution in a small neighborhood of the stationary solution to the averaged equation when the time scale is small. Note that it is non-initial value problem, and this recurrent solution is more general than the classical second Bogolyubov theorem which only treats the periodic case.

Since the SPDEs we concern in this paper are not semilinear, the semigroup framework in [24] and [10] is not applicable to our problem. Therefore, a difficulty that we face is how to deal with the monotone SPDEs. Firstly, under some suitable conditions, employing the technique of truncation which is used in [4, 6, 7, 30], we show that
\begin{equation*}
\lim_{\varepsilon \to 0} E \sup_{s \leq t \leq s + T} \|X_\varepsilon(t, s, \zeta_s^\varepsilon) - \bar{X}(t, s, \zeta_s)\|^2 = 0
\end{equation*}
for all $s \in \mathbb{R}$ and $T > 0$ provided $\lim_{\varepsilon \to 0} E\|\zeta_s^\varepsilon - \zeta_s\|^2 = 0$, where $X_\varepsilon(t, s, \zeta_s^\varepsilon)$ is the solution of (1.4) with the initial condition $X_\varepsilon(s, s, \zeta_s^\varepsilon) = \zeta_s^\varepsilon$ and $\bar{X}(t, s, \zeta_s)$ is the solution of (1.6) with
the initial condition $\bar{X}(s,s,\zeta_s) = \zeta_s$ (see Theorem 4.5). In fact, this is the first Bogolyubov theorem, which is new despite that there have already been many results in this direction mentioned above.

In view of Theorem 4.5, tightness of family of measures $\{\mathbb{P} \circ [X_\varepsilon(t)]^{-1}\}_{\varepsilon \in (0,1]}$ for any $t \in \mathbb{R}$ plays an important role in establishing the second Bogolyubov theorem. Although the tightness of $\{\mathbb{P} \circ [X_\varepsilon(t)]^{-1}\}_{\varepsilon \in (0,1]}$ on $C([0,T];H)$ was proved by using Ascoli-Arzelà theorem and the Garcia-Rademich-Rumsey theorem in [4, 6, 7], it is different from the technique used in our paper. We find that

$$\lim_{\varepsilon \to 0} \text{dist}_{\mathbb{P} \circ [X_\varepsilon(t)]^{-1}}(\mathbb{P} \circ [X_\varepsilon(t)]^{-1}) = 0$$

(see Theorem 5.14), where $\text{dist}_{\mathbb{P} \circ [X_\varepsilon(t)]^{-1}}$ is the Hausdorff semi-metric and $\mathcal{A} := \{\mathcal{L}(\bar{X}(0))\}$ is the attractor of $\mathbb{P}$ to the averaged equation (1.6). Note that $H(\mathbb{F})$ is compact provided $\mathbb{F}$ is Birkhoff recurrent.

The remainder of this paper is organized as follows. In the next section, we recall some definitions and facts concerning dynamical systems, Poisson stable (or recurrent) functions, Shcherbakov’s comparability method by character of recurrence and variational approach. In the third section, we show that there exists a unique $L^2$-bounded solution which possesses the same recurrent properties in distribution sense as the coefficients and this bounded solution is globally asymptotically stable in square-mean sense. In section 4, we establish the second Bogolyubov theorem for SPDEs with monotone coefficients. In section 5, we prove the global averaging principle for these SPDEs. In the last section, we illustrate our theoretical results by stochastic reaction diffusion equations and stochastic generalized porous media equations.
2. Preliminaries

In this section, we introduce some useful preliminaries, including dynamical systems, poisson stable functions, Shcherbakov’s comparability method by character of recurrence, variational approach.

2.1. Shift dynamical systems. In this subsection, let \((\mathcal{X}, \rho)\) be a complete metric space and \((\mathcal{X}, \mathbb{R}, \pi)\) be a dynamical system (flow) on \(\mathcal{X}\), i.e. the mapping \(\pi : \mathbb{R} \times \mathcal{X} \to \mathcal{X}\) is continuous, \(\pi(0, x) = x\) and \(\pi(t + s, x) = \pi(t, \pi(s, x))\) for any \(x \in \mathcal{X}\) and \(t, s \in \mathbb{R}\). We write \(C(\mathcal{X}, \mathcal{X})\) to mean the space of all continuous functions \(\varphi : \mathbb{R} \to \mathcal{X}\) equipped with the distance

\[
d(\varphi_1, \varphi_2) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(\varphi_1, \varphi_2)}{1 + d_k(\varphi_1, \varphi_2)},
\]

where

\[
d_k(\varphi_1, \varphi_2) := \sup_{|t| \leq k} \rho(\varphi_1(t), \varphi_2(t)),
\]

which generates the compact-open topology on \(C(\mathbb{R}, \mathcal{X})\). The space \((C(\mathbb{R}, \mathcal{X}), d)\) is a complete metric space (see, e.g. [35, 37, 39, 40]).

**Remark 2.1.** Let \(\{\varphi_n\}_{n=1}^{\infty}, \varphi \in C(\mathbb{R}, \mathcal{X})\). Then the following statements are equivalent.

(i) \(\lim_{n \to \infty} d(\varphi_n, \varphi) = 0\).

(ii) \(\lim_{n \to \infty} \max_{|t| \leq t} \rho(\varphi_n(t), \varphi(t)) = 0\) for any \(t > 0\).

(iii) There exists a sequence \(t_n \to +\infty\) such that \(\lim_{n \to \infty} \max_{|t| \leq t_n} \rho(\varphi_n(t), \varphi(t)) = 0\).

Let us now consider two examples of shift dynamical systems which we will use in this paper.

**Example 2.2.** We say \(\varphi^\tau\) is the \(\tau\)-translation of \(\varphi\) if \(\varphi^\tau(t) := \varphi(t + \tau)\) for any \(t \in \mathbb{R}\) and \(\varphi \in C(\mathbb{R}, \mathcal{X})\). For any \((\tau, \varphi) \in \mathbb{R} \times C(\mathbb{R}, \mathcal{X})\), the mapping \(\sigma : \mathbb{R} \times C(\mathbb{R}, \mathcal{X}) \to C(\mathbb{R}, \mathcal{X})\) is defined by \(\sigma(\tau, \varphi) := \varphi^\tau\). Then the triplet \((C(\mathbb{R}, \mathcal{X}), \mathbb{R}, \sigma)\) is a dynamical system which is called shift dynamical system or Bebutov’s dynamical system. Indeed, it is easy to check that \(\sigma(0, \varphi) = \varphi\) and \(\sigma(\tau_1 + \tau_2, \varphi) = \sigma(\tau_2, \sigma(\tau_1, \varphi))\) for any \(\varphi \in C(\mathbb{R}, \mathcal{X})\) and \(\tau_1, \tau_2 \in \mathbb{R}\). And it can be proved that the mapping \(\sigma : \mathbb{R} \times C(\mathbb{R}, \mathcal{X}) \to C(\mathbb{R}, \mathcal{X})\) is continuous, see, e.g. [8, 35, 37, 40].

In what follows, let \((\mathcal{Y}, \rho_1)\) be a complete metric space. We employ \(H(\varphi)\) to denote the hull of \(\varphi\), which is the set of all the limits of \(\varphi^n\) in \(C(\mathbb{R}, \mathcal{X})\), i.e.

\[
H(\varphi) := \{\psi \in C(\mathbb{R}, \mathcal{X}) : \psi = \lim_{n \to \infty} \varphi^n \text{ for some sequence } \{\tau_n\} \subset \mathbb{R}\}.
\]

Notice that the set \(H(\varphi) \subset C(\mathbb{R}, \mathcal{X})\) is closed and translation invariant. Consequently, it naturally defines on \(H(\varphi)\) a shift dynamical system \((H(\varphi), \mathbb{R}, \sigma)\). Now we give the second example, which is similar to Section 2.4 in [9].

**Example 2.3.** We write \(\text{BUC}(\mathbb{R} \times \mathcal{X}, \mathcal{Y})\) to mean the space of all continuous functions \(f : \mathbb{R} \times \mathcal{X} \to \mathcal{Y}\) which satisfy the following conditions:

(i) \(f\) is bounded on every bounded subset from \(\mathbb{R} \times \mathcal{X}\);

(ii) \(f\) is continuous in \(t \in \mathbb{R}\) uniformly with respect to \(x\) on each bounded subset \(Q \subset \mathcal{X}\).

We endow \(\text{BUC}(\mathbb{R} \times \mathcal{X}, \mathcal{Y})\) with the following \(d\) metric

\[
d(f, g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f, g)}{1 + d_k(f, g)},
\]

where

\[
d_k(f, g) := \sup_{|t| \leq k} \rho(f(t), g(t)).
\]
where \( d_k(f,g) := \sup_{|t| \leq k,x \in Q_k} |f(t,x) - g(t,x)|. \) Here \( Q_k \subset X \) is bounded, \( Q_k \subset Q_{k+1} \) and \( \cup_{k \in \mathbb{N}} Q_k = X. \) Note that \( d \) generates the topology of uniform convergence on bounded subsets on \( BUC(\mathbb{R} \times X, \mathcal{Y}) \) and \( (BUC(\mathbb{R} \times X, \mathcal{Y}), d) \) is a complete metric space.

Given \( f \in BUC(\mathbb{R} \times X, \mathcal{Y}) \) and \( \tau \in \mathbb{R}. \) We write \( f^\tau \) to mean the \( \tau \)-translation of \( f \) if \( f^\tau(t,x) := f(t+\tau,x) \) for all \( (t,x) \in \mathbb{R} \times X. \) It is proved that \( BUC(\mathbb{R} \times X, \mathcal{Y}) \) is invariant with respect to translations. Let \( h \) be a mapping \( \sigma : \mathbb{R} \times BUC(\mathbb{R} \times X, \mathcal{Y}) \to BUC(\mathbb{R} \times X, \mathcal{Y}), (\tau,f) \mapsto f^\tau. \) Then it can be proved that the triplet \( (BUC(\mathbb{R} \times X, \mathcal{Y}),\mathbb{R},\sigma) \) is a dynamical system. See Chapter I in \([8]\) for details. Given \( f \in BUC(\mathbb{R} \times X, \mathcal{Y}), H(f) \subset BUC(\mathbb{R} \times X, \mathcal{Y}) \) is closed and translation invariant. Consequently, it naturally defines on \( H(f) \) a shift dynamical system \( (H(f),\mathbb{R},\sigma). \)

We employ \( BC(\mathcal{X},\mathcal{Y}) \) to denote the space of all continuous functions \( f : \mathcal{X} \to \mathcal{Y} \) which are bounded on every bounded subset of \( \mathcal{X} \) and equipped with the distance

\[
d(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f,g)}{1 + d_k(f,g)},
\]

where \( d_k(f,g) := \sup_{x \in Q_k} |f(x) - g(x)|, \) \( Q_k \) is similar to that in Example 2.3. Note that \( (BC(\mathcal{X},\mathcal{Y}),d) \) is a complete metric space. For any \( F \in BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), \) the mapping \( F : \mathbb{R} \to BC(\mathcal{X},\mathcal{Y}) \) is defined by \( F(t) := F(t,\cdot) : \mathcal{X} \to \mathcal{Y} \). Clearly, \( F \in C(\mathbb{R},BC(\mathcal{X},\mathcal{Y})). \)

**Remark 2.4.** It can be proved that the following statements are true.

(i) The mapping \( h : BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}) \to C(\mathbb{R},BC(\mathcal{X},\mathcal{Y})) \) defined by equality \( h(F) := F \) establishes an isometry between \( BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}) \) and \( (C(\mathbb{R},BC(\mathcal{X},\mathcal{Y}))). \)

(ii) \( h(F^\tau) = F^\tau \) for any \( \tau \in \mathbb{R} \) and \( F \in BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), \) i.e. the shift dynamical systems \( (BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}),\mathbb{R},\sigma) \) and \( (C(\mathbb{R},BC(\mathcal{X},\mathcal{Y})),\mathbb{R},\sigma) \) are (dynamically) homeomorphic.

**2.2. Poisson stable functions.** Let us recall the types of Poisson stable (or recurrent) functions to be studied in this paper; For further details and the relations among these types of functions, see \([35, 37, 39, 40]\).

**Definition 2.5.** We say that a function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is \textit{T-periodic}, if there exists a constant \( T \in \mathbb{R} \) such that \( \varphi(t+T) = \varphi(t) \) for all \( t \in \mathbb{R}. \) In particular, \( \varphi \) is called \textit{stationary} provided \( \varphi(t) = \varphi(0) \) for all \( t \in \mathbb{R}. \)

**Definition 2.6.** A function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is called \textit{Bohr almost periodic} if the set \( \mathcal{T}(\varphi, \varepsilon) \) of \( \varepsilon \)-almost periods of \( \varphi \) is relatively dense for each \( \varepsilon > 0, \) i.e. for each \( \varepsilon > 0 \) there exists a constant \( l = l(\varepsilon) > 0 \) such that \( \mathcal{T}(\varphi, \varepsilon) \cap [a,a+l] \neq \emptyset \) for all \( a \in \mathbb{R}, \) where

\[
\mathcal{T}(\varphi, \varepsilon) := \left\{ \tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} \rho(\varphi(t+\tau),\varphi(t)) < \varepsilon \right\},
\]

and \( \tau \in \mathcal{T}(\varphi, \varepsilon) \) is called \( \varepsilon \)-\textit{almost period} of \( \varphi. \)

**Definition 2.7.** We say that \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is \textit{pseudo-periodic in the positive (respectively, negative) direction} if for each \( \varepsilon > 0 \) and \( l > 0 \) there exists a \( \varepsilon \)-almost period \( \tau > l \) (respectively, \( \tau < -l \)) of the function \( \varphi. \) The function \( \varphi \) is called \textit{pseudo-periodic} if it is pseudo-periodic in both directions.

**Definition 2.8.** A function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is called \textit{almost recurrent (in the sense of Bebutov)} if the set \( \mathcal{I}(\varphi, \varepsilon) \) is relatively dense for every \( \varepsilon > 0, \) where \( \mathcal{I}(\varphi, \varepsilon) := \{ \tau \in \mathbb{R} : d(\varphi^\tau, \varphi) < \varepsilon \}. \) And \( \tau \in \mathcal{I}(\varphi, \varepsilon) \) is said to be \( \varepsilon \)-\textit{shift} for \( \varphi. \)

**Definition 2.9.** (i) We say that a function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is \textit{Lagrange stable} provided \( \{ \varphi^h : h \in \mathbb{R} \} \) is a relatively compact subset of \( C(\mathbb{R}, \mathcal{X}). \)
Definition 2.10. We say that \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is Poisson stable in the positive (respectively, negative) direction if for every \( \varepsilon > 0 \) and \( l > 0 \) there exists \( \tau > l \) (respectively, \( \tau < -l \)) such that \( d(\varphi^\tau, \varphi) < \varepsilon \). The function \( \varphi \) is called Poisson stable if it is Poisson stable in both directions.

Definition 2.11. We say that \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is Levitan almost periodic if there exists an almost periodic function \( \psi \in C(\mathbb{R}, \mathcal{Y}) \) such that for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( T(\psi, \delta) \subseteq \mathcal{T}(\varphi, \varepsilon) \).

Definition 2.12. We say that a function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is almost automorphic if it is Levitan almost periodic and Lagrange stable.

Definition 2.13. We say that \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is quasi-periodic with the spectrum of frequencies \( \nu_1, \nu_2, \ldots, \nu_k \) if it satisfies the following conditions:

(i) The numbers \( \nu_1, \nu_2, \ldots, \nu_k \) are rationally independent;
(ii) There exists a continuous function \( \Phi : \mathbb{R}^k \to \mathcal{X} \) such that \( \Phi(t_1 + 2\pi, t_2 + 2\pi, \ldots, t_k + 2\pi) = \Phi(t_1, t_2, \ldots, t_k) \) for all \( (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \);
(iii) \( \varphi(t) = \Phi(\nu_1 t, \nu_2 t, \ldots, \nu_k t) \) for \( t \in \mathbb{R} \).

Let \( \varphi \in C(\mathbb{R}, \mathcal{X}) \). We employ \( \mathcal{N}_\varphi \) (respectively, \( \mathcal{M}_\varphi \)) to denote the family of all sequences \( \{t_n\} \subset \mathbb{R} \) such that \( \varphi^{t_n} \to \varphi \) (respectively, \( \{\varphi^{t_n}\} \) converges) in \( C(\mathbb{R}, \mathcal{X}) \) as \( n \to \infty \).

Definition 2.14. ([36, 37, 39]) A function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is called pseudo-recurrent if for any \( \varepsilon > 0 \) and \( l \in \mathbb{R} \) there exists \( L \geq l \) such that for any \( \tau_0 \in \mathbb{R} \) we can find a number \( \tau \in [l, L] \) satisfying

\[
\sup_{|t| \leq 1/\varepsilon} \rho(\varphi(t + \tau_0 + \tau), \varphi(t + \tau_0)) \leq \varepsilon.
\]

Remark 2.15. ([36, 37, 39, 40])

(i) Every Birkhoff recurrent function is pseudo-recurrent, but not vice versa.
(ii) Suppose that \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is pseudo-recurrent, then every function \( \psi \in H(\varphi) \) is pseudo-recurrent.
(iii) Suppose that \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is Lagrange stable and every function \( \psi \in H(\varphi) \) is Poisson stable, then \( \varphi \) is pseudo-recurrent.

Finally, we remark that a Lagrange stable function is not Poisson stable in general, but all other types of functions introduced above are Poisson stable.

Definition 2.16. (i) We say that a function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) possesses the property A if the motion \( \sigma(\cdot, \varphi) \) through \( \varphi \) with respect to the Bebutov dynamical system \( (C(\mathbb{R} \times \mathcal{X}), \mathbb{R}, \sigma) \) possesses the property A.
(ii) Similarly, we say that \( F \in \text{BUC}(\mathbb{R} \times \mathcal{X}, \mathcal{Y}) \) possesses the property A in \( t \in \mathbb{R} \) uniformly with respect to \( x \) on each bounded subset \( Q \subset \mathcal{X} \), if the motion \( \sigma(\cdot, F) : \mathbb{R} \to \text{BUC}(\mathbb{R} \times \mathcal{X}, \mathcal{Y}) \) through \( F \) with respect to the Bebutov dynamical system \( (\text{BUC}(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), \mathbb{R}, \sigma) \) possesses the property A.

Here the property A may be stationary, periodic, Bohr/Levitian almost periodic, etc.

2.3. Shcherbakov's comparability method by character of recurrence.

Definition 2.17. A function \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is called comparable (respectively, strongly comparable) by character of recurrence with \( \psi \in C(\mathbb{R}, \mathcal{Y}) \) provided \( \mathcal{N}_\psi \subseteq \mathcal{N}_\varphi \) (respectively, \( \mathcal{M}_\psi \subseteq \mathcal{M}_\varphi \)).

Theorem 2.18. ([37, ChII], [38])
(i) $\mathcal{M}_\psi \subseteq \mathcal{M}_\varphi$ implies $\mathcal{N}_\psi \subseteq \mathcal{N}_\varphi$, and hence strong comparability implies comparability.

(ii) Assume that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is comparable by character of recurrence with $\psi \in C(\mathbb{R}, \mathcal{Y})$. If the function $\psi$ is stationary (respectively, $T$-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is $\varphi$.

(iii) Assume that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is strongly comparable by character of recurrence with $\psi \in C(\mathbb{R}, \mathcal{Y})$. If the function $\psi$ is quasi-periodic with the spectrum of frequencies $\nu_1, \nu_2, \ldots, \nu_k$ (respectively, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable), then so is $\varphi$.

(iv) Assume that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is strongly comparable by character of recurrence with $\psi \in C(\mathbb{R}, \mathcal{Y})$. And suppose further that $\psi$ is Lagrange stable. If $\psi$ is pseudo-periodic (respectively, pseudo-recurrent), then so is $\varphi$.

2.4. Variational approach. Recall that $H$ is a separable Hilbert space with norm $\| \cdot \|_H$ and inner product $\langle \cdot, \cdot \rangle_H$, and that $H^*$ is the dual space of $H$. Let $(V, \| \cdot \|_V)$ be a reflexive Banach space such that $V \subset H$ continuously and densely. So we have $H^* \subset V^*$ continuously and densely. Identifying $H$ with its dual $H^*$ via the Riesz isomorphism, then we have $V \subset H \subset V^*$ continuously and densely. We write $\langle \cdot, \cdot \rangle_V$ to denote the pairing between $V^*$ and $V$. It follows that

$$\langle \cdot, \cdot \rangle_V = \langle h, v \rangle_H$$

for all $h \in H$, $v \in V$. $(V, H, V^*)$ is called Gelfand triple. Since $H \subset V^*$ continuously and densely, we deduce that $V^*$ is separable, hence so is $V$.

Assume that $(V_1, \| \cdot \|_{V_1})$ and $(V_2, \| \cdot \|_{V_2})$ are reflexive Banach spaces and embedded in $H$ continuously and densely. Then we get two triples:

$$V_1 \subset H \subset V_1^* \quad \text{and} \quad V_2 \subset H \subset V_2^*.$$  

We define the norm $\| v \|_V := \| v \|_{V_1} + \| v \|_{V_2}$ on the space $V := V_1 \cap V_2$. Note that $(V, \| \cdot \|_V)$ is also a Banach space. Since $V_1^*$ and $V_2^*$ can be thought as subspaces of $V^*$, we get a Banach space $W := V_1^* + V_2^* \subset V^*$ with norm

$$\| f \|_W := \inf \{ \| f_i \|_{V_1} + \| f_2 \|_{V_2} : f = f_1 + f_2, \ f_i \in V_i^*, \ i = 1, 2 \}.$$  

Similarly, we write $\langle \cdot, \cdot \rangle_{V_i}$ to denote the pairing between $V_i^*$ and $V_i$, $i = 1, 2$. Then, for all $v \in V$ and $f = f_1 + f_2 \in W \subset V^*$ we have

$$\langle f, v \rangle_V = \langle f_1, v \rangle_{V_1} + \langle f_2, v \rangle_{V_2}.$$  

Note carefully that if $f \in H$ and $v \in V$, then we obtain

$$\langle f, v \rangle_V = \langle f_1, v \rangle_{V_1} + \langle f_2, v \rangle_{V_2} = \langle f, v \rangle_H.$$  

We write $Pr(H)$ to mean the set of all Borel probability measures on $H$. Denote by $C_b(H)$ the space of all continuous functions $\varphi : H \to \mathbb{R}$ for which the norm $\| \varphi \|_\infty := \sup_{x \in H} |\varphi(x)|$ is finite. Let $\{ \mu_n \} := \{ \mu_n \}_{n=1}^\infty \subset Pr(H)$ and $\mu \in Pr(H)$. We say $\mu_n$ converges weakly to $\mu$ in $Pr(H)$ provided $\int \varphi d\mu_n$ converges to $\int \varphi d\mu$ for all $\varphi \in C_b(H)$. Let $\varphi \in C_b(H)$ be Lipschitz continuous, we define

$$\| \varphi \|_{BL} := Lip(\varphi) + \| \varphi \|_\infty,$$

where $Lip(\varphi) = \sup_{x \neq y} \frac{\varphi(x) - \varphi(y)}{\| x - y \|_H}$. Then $Pr(H)$ is a separable complete metric space with the following bounded Lipschitz distance (also called Fortet-Mourier distance)

$$d_{BL}(\mu, \nu) := \sup \left\{ \left| \int \varphi d\mu - \int \varphi d\nu \right| : \| \varphi \|_{BL} \leq 1 \right\}.$$
for all \( \mu, \nu \in \text{Pr}(H) \). It is well known that \( dB_L \) generates the weak topology on \( \text{Pr}(H) \), i.e. \( \mu_n \to \mu \) weakly in \( \text{Pr}(H) \) if and only if \( dB_L(\mu_n, \mu) \to 0 \) as \( n \to \infty \). See Chapter 11 in [15] for this metric \( dB_L \) (denoted by \( \beta \) there) and its related properties.

We assume in the following exposition that \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a complete probability space. The space \( L^2(\Omega, \mathbb{P}; H) \) consists of all \( H \)-valued random variables \( \zeta \) such that \( E\|\zeta\|_H^2 = \int_{\Omega} \|\zeta\|_H^2 \, d\mathbb{P} < \infty \). An \( H \)-valued stochastic process \( X = X(t) \), \( t \in \mathbb{R} \) is called \( L^2 \)-bounded provided \( \sup_{t \in \mathbb{R}} E\|X(t)\|_H^2 < \infty \). Throughout the paper, we denote by \( \mathcal{L}(\zeta) \in \text{Pr}(H) \) the law or distribution of \( H \)-valued random variable \( \zeta \). A sequence of \( H \)-valued continuous stochastic processes \( \{X_n\} \) is said to converge in distribution to \( X \) (on \( C(\mathbb{R}, H) \)) provided \( \mathcal{L}(X_n) \) weakly converges to \( \mathcal{L}(X) \) in \( \text{Pr}(C(\mathbb{R}, H)) \), where \( \mathcal{L}(X) \) is the law or distribution of \( X \) on \( C(\mathbb{R}, H) \).

If \( dB_L(\mathcal{L}(X_n(t)), \mathcal{L}(X(t))) \to 0 \) as \( n \to \infty \) for each \( t \in \mathbb{R} \), we simply say that \( X_n \) converges in distribution to \( X \) on \( H \).

3. Compatible solutions

Let \( W(t), t \in \mathbb{R} \) be a two-sided cylindrical \( Q \)-Wiener process with \( Q = I \) on a separable Hilbert space \( (U, \langle \cdot, \cdot \rangle_U) \) with respect to a complete filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \). Denote by \( L_2(U, H) \) the space of all Hilbert-Schmidt operators from \( U \) into \( H \).

In this section, coefficient \( F \) in (1.3) need not to be Lipschitz. Therefore, instead of explicitly writing \( F \) in (1.3), we consider the following stochastic partial differential equation on \( H \)

\[
(3.1) \quad dX(t) = A(t, X(t))dt + G(t, X(t))dW(t),
\]

where \( A(t, x) = A_1(x) + A_2(t, x) \), \( A_i : \mathbb{R} \times V_i^* \to V_i^* \), \( i = 1, 2 \) and \( G : \mathbb{R} \times V \to L_2(U, H) \).

Consider equation (3.1). Let us introduce the following conditions.

**H1** (Continuity) For all \( u, v, w \in V \) and \( t \in \mathbb{R} \) the map

\[
R \ni \theta \mapsto V^*_1 \langle A_1(u + \theta v), w \rangle_{V_i}
\]

is continuous. \( A_2 : \mathbb{R} \times V_2 \to V_2^* \) and \( G : \mathbb{R} \times V \to L_2(U, H) \) are continuous. Here \( A_1 \) is called hemicontinuity provided (3.2) hold.

**H2** (Strong monotonicity) There exist constants \( \lambda \geq 0 \), \( r > 2 \) and \( \lambda' \geq 0 \) such that for all \( u, v, w \in V, t \in \mathbb{R} \)

\[
\langle A(t, u) - A(t, v), u - v \rangle_V \leq -\lambda \|u - v\|_H^2 - \lambda' \|u - v\|_H^2,
\]

and

\[
\|G(t, u) - G(t, v)\|_{L_2(U, H)}^2 \leq L^2_G \|u - v\|_H^2.
\]

**H3** (Coercivity) There exist constants \( \alpha_1, \alpha_2 \in (1, \infty), c_1 \in \mathbb{R}, c_2, c_2' \in (0, \infty) \) and \( M_0 \in (0, \infty) \) such that for all \( v \in V, t \in \mathbb{R} \)

\[
\langle A(t, v), v \rangle_V \leq c_1 \|v\|_H^2 - c_2 \|v\|_{V_1}^{\alpha_1} - c_2' \|v\|_{V_2}^{\alpha_2} + M_0.
\]

**H4** (Boundedness) There exist constants \( c_3, c_3' \in (0, \infty) \) such that for all \( v \in V, t \in \mathbb{R} \)

\[
\|A_1(v)\|_{V_1} \leq c_3 \|v\|_{V_1}^{\alpha_1 - 1} + M_0, \quad \|A_2(t, v)\|_{V_1} \leq c_3' \|v\|_{V_2}^{\alpha_2 - 1} + M_0
\]

and

\[
\|G(t, 0)\|_{L_2(U, H)} \leq M_0,
\]

where \( \alpha_i \) and \( M_0 \) are as in (H3).

**H5** \( A_2 \) and \( G \) are continuous in \( t \in \mathbb{R} \) uniformly with respect to \( v \) on each bounded subset \( Q \subset V \).
Remark 3.1. Since we consider compatible solutions (see Definition 3.1) by the method of dynamical systems, we assume that $A_2$ and $G$ satisfy (H1) and (H5) that are different from the usual situation (i.e. we request stronger continuity conditions here). Under conditions of (H1)–(H2) and (H4)–(H5), $(H(A_2), \mathbb{R}, \sigma)$ and $(H(G), \mathbb{R}, \sigma)$ are dynamical systems, where $\sigma : \mathbb{R} \times H(A_2) \to H(A_2)$, $(\tau, A_2) \to A_2^\tau$ and similarly for the action $\sigma$ on $H(G)$. Note that we only need hemisensitivity of $A_2$ and do not need (H5), as usual, when we consider estimates of solutions, such as Lemmas 3.3–3.5, Theorems 3.6 and 3.9, Proposition 3.12, Lemma 3.13.

Definition 3.2 (see, e.g. [33, 47]). We say continuous $H$-valued $(\mathcal{F}_t)$-adapted process $X(t)$, $t \in [0, T]$ is a solution to equation (3.1), if $X \in \cap_{i=1,2}^\infty L_{\alpha_i}^\infty(\mathbb{R} \times \Omega, dt \otimes \mathbb{P}) \cap L^2([0, T] \times \Omega, dt \otimes \mathbb{P}; H)$ with $\alpha_i$ as in (H3) and $\mathbb{P}$-a.s.

\begin{equation}
X(t) = X(s) + \int_s^t A(\sigma, X(\sigma)) d\sigma + \int_s^t G(\sigma, X(\sigma)) dW(\sigma), \quad 0 \leq s \leq t \leq T.
\end{equation}

Moreover, we say $X(t), t \in \mathbb{R}$ is a solution to equation (3.1) provided (3.3) holds for all $t \geq s$ and each $s \in \mathbb{R}$.

Fix $s \in \mathbb{R}$. Under conditions (H1)–(H4), for any $\zeta \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}; H)$ and $T > 0$ there exists a unique solution $X(t, s, \zeta), s \leq t \leq s + T$ to (3.1) with initial condition $X(s, s, \zeta) = \zeta$ (see, e.g. [33]). In this paper, we write $C_\alpha$ to mean some positive constant which depends on $\alpha$. Here $\alpha$ is one or more than one parameter and $C_\alpha$ may change from line to line. Now we discuss the $L^2$-bounded solution to equation (3.1) by employing the classical pullback attraction method in random and non-autonomous dynamics (see, e.g. [11, 13] etc). For this we need three lemmas.

Lemma 3.3. Assume that (H1)–(H4) hold. Let $\zeta_s \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}; H)$ and $X(t, s, \zeta_s)$, $t \geq s$ be the solution to the following Cauchy problem

\begin{equation}
\begin{cases}
  dX(t) = A(t, X(t)) dt + G(t, X(t)) dW(t) \\
  X(s) = \zeta_s.
\end{cases}
\end{equation}

(i) If $2\lambda > L^2_G$, let $\eta \in (0, 2\lambda - L^2_G)$. Then there exist constants $1 \leq p < \frac{2}{2L^2_G} + 1$ and $\kappa, M_1 > 0$ such that

\begin{equation}
E\|X(t, s, \zeta_s)\|_H^{2p} \leq e^{-\kappa(t-s)} E\|\zeta_s\|_H^{2p} + M_1,
\end{equation}

where $M_1$ depends only on $\eta, c_2, c_3, c_2', c_3', \alpha_1, \alpha_2, \kappa, p, r$.

(ii) If $\lambda' > 0$ then estimate (3.4) hold for any $p \in [1, +\infty)$ and $\kappa > 0$.

Proof. By (H2)–(H4) and Young’s inequality, we have

\begin{equation}
2V \langle A(t, u), u \rangle_V + \|G(t, u)\|_{L^2(u, H)}^2 \leq \begin{cases} 
  -\eta\|u\|_H^2 + C_{\alpha_1, \alpha_2, c_2, c_2', M_0}, & \text{if } 2\lambda > L^2_G \\
  -\lambda'\|u\|_H^2 + (c_1 + 2L^2_G - \lambda)\|u\|_H^2 + C_{\alpha_1, \alpha_2, c_2, c_2', M_0}, & \text{if } \lambda' > 0.
\end{cases}
\end{equation}

Given $\kappa > 0$ and $p \geq 1$, in view of Itô’s formula (see, e.g. [33, Theorem 4.2.5]), we get

\begin{equation}
E\left(e^{\kappa(t-s)}\|X(t, s, \zeta_s)\|_H^{2p}\right)
\end{equation}

\begin{align*}
&= E\|\zeta_s\|_H^{2p} + \int_s^t \kappa e^{\kappa(\sigma-s)} E\|X(\sigma, s, \zeta_s)\|_H^{2p} d\sigma \\
&\quad + pE \int_s^t \|X(\sigma, s, \zeta_s)\|_H^{2p-2} e^{\kappa(\sigma-s)} \left(2V \langle A(\sigma, X(\sigma, s, \zeta_s)), X(\sigma, s, \zeta_s) \rangle_V \\
&\quad + \|G(\sigma, X(\sigma, s, \zeta_s))\|_{L^2(u, H)}^2 \right) d\sigma.
\end{align*}
Lemma 3.4. Consider equation (3.1). Assume that $2\lambda - L_G^2 \geq 0$ and (H1)–(H4) hold. Let $X$ and $Y$ be two solutions of equation (3.1). If $\lambda' > 0$ or $2\lambda > L_G^2$, then for any $s \leq t$ we have

$$E\|X(t, s, X(s)) - Y(t, s, Y(s))\|_H^2 \leq \begin{cases} E\|X(s) - Y(s)\|_H^2 \wedge \{\lambda'(r - 2)(t - s)\} \cdot \frac{1}{r - 2}, & \text{if } \lambda' > 0 \\
e^{-2\lambda(2-L_G^2)(t-s)} E\|X(s) - Y(s)\|_H^2, & \text{if } 2\lambda > L_G^2. \end{cases}$$

Therefore, for any $p \in [1, +\infty)$ and $\kappa > 0$

$$E\|X(t, s, \zeta_0)\|_H^p \leq e^{-\kappa(t-s)} E\|\zeta_0\|_H^p + M_1.$$
In particular, for any \( t \in \mathbb{R} \) there exists some random variable \( X(t) \) such that
\begin{equation}
X(t, -n, 0) \rightarrow X(t) \quad \text{in } L^2(\Omega, \mathbb{P}; H) \quad \text{as } n \rightarrow \infty.
\end{equation}

**Proof.** If \( 2\lambda > L_G^2 \), by Itô’s formula and (H2) we get
\begin{align*}
E\|X(t, s, X(s)) - Y(t, s, Y(s))\|^2_H &\leq E\|X(s) - Y(s)\|^2_H + E \int_s^t (-2\lambda + L_G^2) \|X(\sigma, s, X(s)) - Y(\sigma, s, Y(s))\|^2_H d\sigma.
\end{align*}
It follows from Gronwall’s lemma that
\begin{align*}
E\|X(t, s, X(s)) - Y(t, s, Y(s))\|^2_H &\leq e^{-2\lambda(t-s)} E\|X(s) - Y(s)\|^2_H.
\end{align*}
If \( \lambda' > 0 \) and \( 2\lambda \geq L_G^2 \), in view of Itô’s formula and (H2), we have
\begin{align*}
E\|X(t, s, X(s)) - Y(t, s, Y(s))\|^2_H &\leq E\|X(s) - Y(s)\|^2_H + E \int_s^t -2\lambda' \|X(\sigma, s, X(s)) - Y(\sigma, s, Y(s))\|^2_H d\sigma
\end{align*}
\begin{align*}
&\leq E\|X(s) - Y(s)\|^2_H - 2\lambda' \int_s^t (E\|X(\sigma, s, X(s)) - Y(\sigma, s, Y(s))\|^2_H)^{\frac{r-2}{2}} d\sigma.
\end{align*}
Employing comparison theorem, we obtain
\begin{align*}
E\|X(t, s, X(s)) - Y(t, s, Y(s))\|^2_H &\leq E\|X(s) - Y(s)\|^2_H \wedge \{\lambda'(r-2)(t-s)\}^{-\frac{2}{r-2}}.
\end{align*}
\[ \Box \]

**Lemma 3.5.** Suppose that (H1)–(H4) hold. Let \( X(t, s, \zeta_s) \) be a solution to equation (3.1) with initial value \( X(s, s, \zeta_s) = \zeta_s \). We have
\begin{align}
E \left( \sup_{t \in [s, s+T]} \|X(t, s, \zeta_s)\|^2_H \right) &= E \int_s^{s+T} \left( \|X(t, s, \zeta_s)\|_{V_1}^{\alpha_1} + \|X(t, s, \zeta_s)\|_{V_2}^{\alpha_2} \right) dt
\end{align}
\begin{align*}
&\quad + E \int_s^{s+T} \left( \|A_1(t, X(t, s, \zeta_s))\|_{V_1}^{\alpha_1-1} + \|A_2(t, X(t, s, \zeta_s))\|_{V_2}^{\alpha_2-1} \right) dt
\end{align*}
\begin{align*}
&\quad \leq C_{c_1, L_G, M_0, T} \left( 1 + E\|\zeta_s\|^2_H \right)
\end{align*}
for any \( s \in \mathbb{R}, T > 0 \).

**Proof.** By Itô’s formula, (H2) and (H3), we have
\begin{align}
\|X(t, s, \zeta_s)\|^2_H &= \|\zeta_s\|^2_H + \int_s^t \left( 2V^* \langle A(\sigma, X(\sigma, s, \zeta_s)), X(\sigma, s, \zeta_s) \rangle_V + \|G(\sigma, X(\sigma, s, \zeta_s))\|^2_{L^2(U, H)} \right) d\sigma
\end{align}
\begin{align*}
&\quad + 2 \int_s^t \langle X(\sigma, s, \zeta_s), G(\sigma, X(\sigma, s, \zeta_s)) \rangle_H dW(\sigma)
\end{align*}
\begin{align*}
&\quad \leq \|\zeta_s\|^2_H + \int_s^t \left( 2c_1 \|X(\sigma, s, \zeta_s)\|^2_H - 2c_2 \|X(\sigma, s, \zeta_s)\|_{V_1}^{\alpha_1} - 2c^2 \|X(\sigma, s, \zeta_s)\|_{V_2}^{\alpha_2} + 2M_0
\end{align*}
\begin{align*}
&\quad + 2L_G^2 \|X(\sigma, s, \zeta_s)\|^2_H + 2M_0^2 \right) d\sigma + 2 \int_s^t \langle X(\sigma, s, \zeta_s), G(\sigma, X(\sigma, s, \zeta_s)) \rangle_H dW(\sigma)
\end{align*}
Dropping negative terms on the right of the above inequality, according to Burkholder-Davis-Gundy inequality (see, e.g. [33]) and Young’s inequality, we get
\begin{align}
E \sup_{t \in [s, s+T]} \|X(t, s, \zeta_s)\|^2_H
\end{align}
Proof. For any fixed interval \([s, T]\) we have
\[
\leq E\|\zeta_s\|_H^2 + E \int_s^{s+T} \left( (2c_1 + 2L_G^2) \|X(\sigma, s, \zeta_s)\|_H^2 + 2M_0^2 + 2M_0 \right) d\sigma
\]
\[
+ 6E \left( \int_s^{s+T} |G(\sigma, X(\sigma, s, \zeta_s))|^2_{L_2(U, H)} \|X(\sigma, s, \zeta_s)\|_H^2 d\sigma \right)^{\frac{1}{2}}
\]
\[
\leq E\|\zeta_s\|_H^2 + E \int_s^{s+T} (C_{1, L_G} \|X(\sigma, s, \zeta_s)\|_H^2 + C_{M_0}) d\sigma
\]
\[
+ \frac{1}{2} E \sup_{t \in [s, s+T]} \|X(t, s, \zeta_s)\|_H^2.
\]

By Gronwall’s lemma, we obtain
\[E \sup_{t \in [s, s+T]} \|X(t, s, \zeta_s)\|_H^2 \leq C_{1, L_G, T, M_0} \left(1 + E\|\zeta_s\|_H^2\right).\] (3.12)

Take expectations on both sides of (3.10) and let \(t = s + T\), then by (3.12) we have
\[E \int_s^{s+T} \left( \|X(t, s, \zeta_s)\|_{V_1}^\alpha + \|X(t, s, \zeta_s)\|_{V_2}^\alpha \right) dt \leq C_{1, L_G, T, M_0} \left(1 + E\|\zeta_s\|_H^2\right).\]

In view of (H4), we complete the proof. \(\square\)

**Theorem 3.6.** Consider equation (3.1). Suppose that \(2\lambda - L_G^2 \geq 0\) and (H1)–(H4) hold. If \(\lambda' > 0\) or \(2\lambda > L_G^2\), then there exists a unique \(L^2\)-bounded continuous \(H\)-valued solution \(X(t)\), \(t \in \mathbb{R}\) to equation (3.1). Moreover, the mapping \(\hat{\mu} : \mathbb{R} \to \text{Pr}(H)\), defined by \(\hat{\mu}(t) := \mathbb{P} \circ X(t)^{-1}\), is unique with the following properties:

(i) \(L^2\)-boundedness: \(\sup_{t \in \mathbb{R}} \|x\|_H^2 \hat{\mu}(t)(dx) < +\infty\);

(ii) Flow property: \(\mu(t, s, \hat{\mu}(s)) = \hat{\mu}(t)\) for all \(t \geq s\).

Here \(\mu(t, s, \mu_0)\) denotes the distribution of \(X(t, s, \zeta_s)\) on \(H\), with \(\mu_0 = \mathbb{P} \circ \zeta_s^{-1}\).

**Proof.** For any fixed interval \([a, b] \subset \mathbb{R}\), we denote
\[J := L^2([a, b] \times \Omega, dt \otimes \mathbb{P}; L_2(U, H)), \quad K_i := L^{\alpha_i}([a, b] \times \Omega, dt \otimes \mathbb{P}; V_i),\]
\[K_i^* := L^{\beta_i}([a, b] \times \Omega, dt \otimes \mathbb{P}; V_i^*), \quad i = 1, 2.\]

According to the reflexivity of \(K_i\), \(i = 1, 2\), (3.7) and (3.9), we may assume, going if necessary to a subsequence, that

1. \(X(\cdot, -n, 0) \to X(\cdot)\) in \(L^2([a, b] \times \Omega, dt \otimes \mathbb{P}; H)\) and \(X(\cdot, -n, 0) \to X(\cdot)\) weakly in \(K_1\) and \(K_2\);
2. \(A_i(\cdot, X(\cdot, -n, 0)) \to Y_i(\cdot)\) weakly in \(K_i^*, \quad i = 1, 2\);
3. \(G(\cdot, X(\cdot, -n, 0)) \to Z(\cdot)\) weakly in \(J\) and hence
\[
\int_a^t G(\sigma, X(\sigma, -n, 0))dW(\sigma) \to \int_a^t Z(\sigma)dW(\sigma)
\]
weakly* in \(L^\infty([a, b], dt; L^2(\Omega, \mathbb{P}; H))\).

Thus for all \(v \in V, \varphi \in L^\infty([a, b] \times \Omega)\) by Fubini’s theorem we get
\[
E \int_a^b \langle X(t), \varphi(t)v \rangle_V dt
\]
\[
= \lim_{n \to \infty} E \int_a^b \langle X(t, -n, 0), \varphi(t)v \rangle_V dt
\]
\[
= \lim_{n \to \infty} E \int_a^b \langle X(a, -n, 0) + \int_a^t A(\sigma, X(\sigma, -n, 0))d\sigma, \varphi(t)v \rangle_V dt
\]
+ \lim_{n \to \infty} E \left( \int_a^b \int_a^t B(\sigma, X(\sigma, -n, 0))dW(\sigma), \varphi(t)v)_{H} dt \right) \\
= E \int_a^b \psi(t) \left( \left\| X(t) \right\|_H^2 - \left\| X(a) \right\|_H^2 \right) dt \\
\leq E \left( \int_a^b \psi(t) \left( \int_a^t \left( 2V_1 \cdot (Y_1(\sigma) - A_1(\phi(\sigma)), \phi(\sigma))_{V_1} + 2V_1 \cdot (A_1(\phi(\sigma)), X(\sigma))_{V_1} \\
+ 2V_2 \cdot (Y_2(\sigma) - A_2(\sigma, \phi(\sigma)), \phi(\sigma))_{V_2} + 2V_2 \cdot (A_2(\sigma, \phi(\sigma)), X(\sigma))_{V_2} \\
+ 2(Z(\sigma), G(\sigma, \phi(\sigma)))_{L_2(U, H)} - \|G(\sigma, \phi(\sigma))\|_{L_2(U, H)}^2 \right) d\sigma dt \right).\]
Thus (3.7) yields that $\mu$ and $\hat{\phi}$ are not necessarily unique. But in Theorem 3.6, we prove that there exists a unique bounded solution $X(t)$ to (3.1) for any given initial data because backward orbits through the initial data are random variables with the distributions $\mu_{1}(-n)$ and $\mu_{2}(-n)$ respectively. Then consider the solutions $X(t,-n,\zeta_{n,1})$ and $X(t,-n,\zeta_{n,2})$ on $[-n,\infty)$, we have

$$
\text{d}_{BL}(\mu_{1}(t),\mu_{2}(t)) = \sup_{\|f\|_{L^{1}} \leq 1} \left| \int_{H} f(x) \text{d} \left( (\mu(t,-n,\mu_{1}(-n)) \right) - \mu(t,-n,\mu_{2}(-n)) \right) \right|

\leq \left( E\|X(t,-n,\zeta_{n,1}) - X(t,-n,\zeta_{n,2})\|_{H}^{2} \right)^{1/2}.
$$

Thus (3.7) yields that $\mu_{1}(t) = \mu_{2}(t)$ for all $t \in \mathbb{R}$. \hfill \Box

**Remark 3.7.** Note that we call $X(t), t \in \mathbb{R}$ a solution to (3.1) if for any $[s,r] \subset \mathbb{R}, X(t), t \in [s,r]$ is a solution to (3.1). Here we cannot obtain the existence and uniqueness of solutions to (3.1) for $t \in \mathbb{R}$ for any given initial data because backward orbits through the initial data are not necessarily unique. But in Theorem 3.6, we prove that there exists a unique $L^{2}$-bounded solution $X(t), t \in \mathbb{R}$ by the pullback attraction method. And we will also show that this bounded solution $X$ is globally asymptotically stable in square-mean sense below (see Theorem 3.9). Therefore, if $Y(t), t \in \mathbb{R}$ is another solution to (3.1) and there exists $s \in \mathbb{R}$ such that $E\|Y(s)\|_{H}^{2} < \infty$, then we have

$$
\sup_{t \geq s} E\|Y(t)\|_{H}^{2} < \infty.
$$

But on the other hand, we necessarily have

$$
\lim_{t \to -\infty} \sup_{t \geq s} E\|Y(t)\|_{H}^{2} = +\infty.
$$
Indeed, if this is false, then $Y$ is also an $L^2$-bounded solution to (3.1), which contradicts the uniqueness of $L^2$-bounded solution.

**Definition 3.8** (See [18]). We say that a solution $X(\cdot)$ of equation (3.1) is **stable in square-mean sense**, if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \geq 0$
\[
E\|X(t, 0, \zeta_0) - X(t)\|_H^2 < \epsilon,
\]
whenever $E\|\zeta_0 - X(0)\|_H^2 < \delta$. The solution $X(\cdot)$ is said to be **asymptotically stable in square-mean sense** if it is stable in square-mean sense and
\[
\lim_{t \to \infty} E\|X(t, 0, \zeta_0) - X(t)\|_H^2 = 0.
\]
We say $X(\cdot)$ is **globally asymptotically stable in square-mean sense** provided (3.16) holds for any $\zeta_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.

Applying Lemma 3.4 we obtain the following result:

**Theorem 3.9.** Consider equation (3.1). Suppose that $2\lambda - L_G^2 \geq 0$ and (H1)-(H4) hold. If $\lambda' > 0$ or $2\lambda > L_G^2$, then the unique $L^2$-bounded solution of equation (3.1) is globally asymptotically stable in square-mean sense. Moreover,
\[
(3.17) \quad E\|X(t, s, \zeta_s) - X(t)\|_H^2 \leq \begin{cases} \frac{E\|\zeta_s - X(s)\|_H^2 \wedge \{\lambda'((r-2)(t-s))\}}{e^{-\lambda t} E\|\zeta_s - X(s)\|_H^2}, & \text{if } \lambda' > 0 \\ e^{-((2\lambda - L_G^2)(t-s))} E\|\zeta_s - X(s)\|_H^2, & \text{if } 2\lambda > L_G^2 \end{cases}
\]
for any $t \geq s$ and $\zeta_s \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}; H)$.

**Remark 3.10.**
(i) If $A_2$ and $G$ satisfy (H2) and (H3), then every pair of functions
\[
(A_2, \tilde{G}) \in H(A_2, G)
\]
possess the same property with the same constants, where
\[
H(A_2, G) := \{(A_2^\tau, G^\tau) : \tau \in \mathbb{R}\}.
\]
Here $\{(A_2^\tau, G^\tau) : \tau \in \mathbb{R}\}$ means the closure of $\{(A_2^\tau, G^\tau) : \tau \in \mathbb{R}\}$.
(ii) If $A_2$ and $G$ satisfy the conditions (H1), (H2), (H4) and (H5), then $A_2 \in \text{BUC}(\mathbb{R} \times V, V_0^2)$, $G \in \text{BUC}(\mathbb{R} \times V, L_2(U, H))$ and $H(A_2, G) \subset \text{BUC}(\mathbb{R} \times V, V_0^2) \times \text{BUC}(\mathbb{R} \times V, L_2(U, H))$.

**Definition 3.11.** Let $\{\varphi(t)\}_{t \in \mathbb{R}}$ be a solution of equation (3.1). Then $\varphi$ is called **compatible (respectively, strongly compatible) in distribution** if the following conditions are fulfilled:

(i) there exists a bounded closed subset $Q \subset L^2(\Omega, \mathcal{F}; H)$ such that $\varphi(\mathbb{R}) \subseteq Q$;
(ii) $\mathfrak{M}(F, G) \subseteq \mathfrak{M}_\varphi$ (respectively, $\mathfrak{M}(F, G) \subseteq \mathfrak{M}_\varphi$), where $\mathfrak{M}_\varphi$ (respectively, $\mathfrak{M}_\varphi$) means the set of all sequences $\{t_n\} \subseteq \mathbb{R}$ such that the sequence $\varphi(\cdot + t_n)$ converges in distribution.

Now we show that the $L^2$-bounded solution $X(t), t \in \mathbb{R}$ for equation (3.1) is strongly compatible in distribution. To this end, we need the tightness of the family of distributions $\{\mathbb{P} \circ [X(t)]^{-1}\}_{t \in \mathbb{R}}$. Therefore, we need the following condition (H6) which is used by many works (see, e.g. [29]).

**H6** Assume that there exists a closed subset $S \subset H$ equipped with the norm $\| \cdot \|_S$ such that $V \subset S$ is continuous and $S \subset H$ is compact. Let $T_n$ be a sequence of positive definite self-adjoint operators on $H$ such that for each $n \geq 1$,
\[
\langle x, y \rangle_n := \langle x, T_n y \rangle_H, \quad x, y \in H,
\]
defines a new inner product on $H$. Assume further that the norms $\| \cdot \|_n$ generated by $\langle \cdot, \cdot \rangle_n$ are all equivalent to $\| \cdot \|_H$ and for all $x \in S$ we have
\[
\|x\|_n \uparrow \|x\|_S \quad \text{as } n \to \infty.
\]
Furthermore, we suppose that for each $n \geq 1$, $T_n : V \to V$ is continuous and there exist constants $c_4 > 0$, $M_0 > 0$ such that for all $v \in V$, $t \in \mathbb{R}$
\[ 2v' \langle A(t, v), T_nv \rangle_V + \|G(t, v)\|_{L_2(t;H_n)}^2 \leq -c_4\|v\|_n^2 + M_0. \]

**Proposition 3.12.** Consider equation (3.1). Suppose that conditions of Theorem 3.9 hold. If (H6) hold then the $L^2$-bounded solution $X(\cdot)$ satisfies
\[ (3.18) \quad \sup_{t \in \mathbb{R}} E\|X(t)\|_2^2 < \infty. \]

In particular, the family of distributions $\{P \circ [X(t)]^{-1}\}_{t \in \mathbb{R}}$ is tight.

**Proof.** Similar to the proof of Proposition 1 in [11], (3.18) can be obtained by Itô’s formula, (H6) and Gronwall’s lemma. \(\square\)

The following lemma is a direct corollary of Theorem 3.1 in [11].

**Lemma 3.13.** Suppose that $A_n$, $A$, $G_n$, $G$ satisfy (H1)–(H4) with the same constants $c$, $c_1$, $c_2$, $c_3$, $c'_3$, $M_0$, $\alpha_i$, $i = 1, 2$ and $L_G$. Let $X_n$ be the solution of the Cauchy problem
\[ (3.19) \quad \begin{cases} dX(t) = A_n(t, X(t))dt + G_n(t, X(t))dW(t) \\ X(s) = \zeta_n^s \end{cases} \]
and $X$ be the solution to the Cauchy problem
\[ (3.20) \quad \begin{cases} dX(t) = A(t, X(t))dt + G(t, X(t))dW(t) \\ X(s) = \zeta^s. \end{cases} \]

Assume further that
1. $\lim_{n \to \infty} A_i(t, x) = A_i(t, x)$ in $V_i^*$ for all $t \in \mathbb{R}$, $x \in V$, $i = 1, 2$;
2. $\lim_{n \to \infty} G_n(t, x) = G(t, x)$ in $L_2(U, H)$ for all $t \in \mathbb{R}$, $x \in V$.

Then we have the following conclusions:
1. If $\lim_{n \to \infty} E\|\zeta_n^s - \zeta^s\|^2_H = 0$, then $\lim_{n \to \infty} E\sup_{s \leq \tau \leq t} \|X_n(\tau) - X(\tau)\|^2_H = 0$ for any $t > s$;
2. If $\lim_{n \to \infty} \zeta_n^s = \zeta^s$ in probability, then $\lim_{n \to \infty} \sup_{t \geq s} \|X_n(\tau) - X(\tau)\|_H = 0$ in probability;
3. If $\lim_{n \to \infty} d_{BL}(L(\zeta_n^s), L(\zeta^s)) = 0$ in $\Pr(H)$, then
\[ \lim_{n \to \infty} d_{BL}(L(X_n), L(X)) = 0 \quad \text{in } \Pr(C([s, \infty), H)). \]

**Theorem 3.14.** Consider equation (3.1). Suppose that $2\lambda - L_G^2 \geq 0$ and (H1)–(H6) hold. If $\lambda' > 0$ or $2\lambda > L_G^2$, then the unique $L^2$-bounded solution is strongly compatible in distribution.

**Proof.** It follows from Remark 3.10 that $H(A_2, G) \subset BUC(\mathbb{R} \times V, V_2^*) \times BUC(\mathbb{R} \times V, L_2(U, H))$. Let $\{t_n\} \in \mathcal{M}_{(A_2, G)}$, then there exists $(\tilde{A}_2, \tilde{G}) \in H(A_2, G)$ such that
\[ \lim_{n \to \infty} \sup_{|t| \leq t_n, |x| \leq r} \|A_2(t + t_n, x) - \tilde{A}_2(t, x)\|_{V_2^*} = 0, \]
\[ \lim_{n \to \infty} \sup_{|t| \leq t_n, |x| \leq r} \|G(t + t_n, x) - \tilde{G}(t, x)\|_{L_2(U, H)} = 0, \]
for any $l > 0$ and $r > 0$. Let $X_n$ be the unique $L^2$-bounded solution of
\[ dX(t) = (A_1(X(t)) + A_2(t + t_n, X(t)))dt + G(t + t_n, X(t))dW(t) \]
and $\tilde{X}$ be the unique $L^2$-bounded solution of
\[ (3.21) \quad dX(t) = (A_1(X(t)) + \tilde{A}_2(t, X(t)))dt + \tilde{G}(t, X(t))dW(t). \]
We now prove that for any \([a, b] \subset \mathbb{R}\), \(\lim_{n \to \infty} \sup_{t \in [a, b]} d_{BL}(\mathcal{L}(X_n(t)), \mathcal{L}(\tilde{X}(t))) = 0\). According to Lemma 3.13, we only need to prove that \(\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \mathcal{L}(\tilde{X}(t))) = 0\) in \(Pr(H)\) for every \(t \in \mathbb{R}\). To this end, it suffices to show that for every sequence \(\{\gamma_k\} := \{\gamma_k^{(i)}\}_{k=1}^\infty \subset \mathbb{N}\), there exists a subsequence \(\{\gamma_k\}\) of \(\{\gamma_k^{(i)}\}\) such that \(\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \mathcal{L}(\tilde{X}(t))) = 0\) in \(Pr(H)\) for every \(t \in \mathbb{R}\).

Given \(r \geq 1\), according to the tightness of \(\{\mathcal{L}(X_{\gamma_k}(-r))\}\), there exists a subsequence \(\{\gamma_k\} \subset \{\gamma_k^{(i)}\}\) such that \(\mathcal{L}(X_{\gamma_k}(-r))\) converges weakly to some probability measure \(\mu_r\) in \(Pr(H)\). Let \(\xi_r\) be a random variable with distribution \(\mu_r\). Define \(Y_r(t) := X(t, -r, \xi_r)\), where \(X(t, -r, \xi_r), t \in [-r, +\infty)\) is a solution to the following Cauchy problem:

\[
\begin{align*}
&dX(t) = \left( A_1(X(t)) + \hat{A}_2(t, X(t)) \right) dt + \hat{G}(t, X(t)) dW(t) \\
&X(-r) = \xi_r.
\end{align*}
\]

In view of Lemma 3.13, we have

\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}), \mathcal{L}(Y_r)) = 0 \quad \text{in} \quad Pr(C([-r, +\infty), H)).
\]

Since \(\{\mathcal{L}(X_{\gamma_k}(-r - 1))\}\) is tight, going if necessary to a subsequence, we can assume that \(\mathcal{L}(X_{\gamma_k}(-r - 1))\) converges weakly to some probability measure \(\mu_{r+1}\) in \(Pr(H)\). Let \(\xi_{r+1}\) be a random variable with distribution \(\mu_{r+1}\). In light of Lemma 3.13, we have

\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}), \mathcal{L}(Y_{r+1})) = 0 \quad \text{in} \quad Pr(C([-r - 1, +\infty), H)),
\]

where \(Y_{r+1}(t) := X(t, -r-1, \xi_{r+1}), t \in [-r-1, +\infty)\). Therefore, we have \(d_{BL}(\mathcal{L}(Y_r), \mathcal{L}(Y_{r+1})) = 0\) in \(Pr(C([-r, +\infty), H))\). In particular, \(\mathcal{L}(Y_r(t)) = \mathcal{L}(Y_{r+1}(t))\) for all \(t \geq -r\).

Define \(\nu(t) := \mathcal{L}(Y_r(t)), t \geq -r\). We use a standard diagonal argument to extract a subsequence which we still denote by \(\{X_{\gamma_k}\}\) satisfying

\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \nu(t)) = 0 \quad \text{in} \quad Pr(H)
\]

for every \(t \in \mathbb{R}\). Note that \(\sup_{t \in \mathbb{R}} \int_{\mathbb{R}} \|x\|^2 \nu(t)(dx) < +\infty\). And we have \(\mathbb{P}\)-a.s.

\[
Y_r(t) = Y_r(s) + \int_s^t \left( A_1(Y_r(\sigma)) + \hat{A}_2(\sigma, Y_r(\sigma)) \right) d\sigma + \int_s^t \hat{G}(\sigma, Y_r(\sigma)) dW(\sigma),
\]

where \(t \geq s \geq -r\). By the uniqueness in law of the solutions for equation (3.21), we have \(\mathcal{L}(Y_r(t)) = \mu(t, s, \mathcal{L}(Y_r(s))), t \geq s \geq -r\), i.e. \(\nu(t) = \mu(t, s, \nu(s)), t \geq s\). In view of Theorem 3.6, we obtain \(\nu = \mathcal{L}(\tilde{X})\). Therefore, we have

\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \mathcal{L}(\tilde{X}(t))) = 0 \quad \text{in} \quad Pr(H)
\]

for every \(t \in \mathbb{R}\).

Note that \(X(\cdot + t_n)\) and \(X_n(\cdot)\) share the same distribution. It follows from Definition 3.11 and Lemma 3.13 that \(X(\cdot)\) is strongly compatible in distribution. \(\square\)

By Theorems 2.18 and 3.14, we have the following result.

**Corollary 3.15.** Under the conditions of Theorem 3.14 the following statements hold:

(i) If \(A_2 \in C(\mathbb{R} \times V, V^*_2)\) and \(G \in C(\mathbb{R} \times V, L_2(U, H))\) are jointly stationary (respectively, \(T\)-periodic, quasi-periodic with the spectrum of frequencies \(\nu_1, \ldots, \nu_k\), almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in \(t \in \mathbb{R}\) uniformly with respect to \(x\) on each bounded subset, then so is the unique solution \(\varphi \in C_0(\mathbb{R}, L^2(\Omega, \mathbb{P}; H))\) of equation (3.1) in distribution;
(ii) If $A_2 \in C(\mathbb{R} \times V, V_2^*)$ and $G \in C(\mathbb{R} \times V, L_2(U, H))$ are Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent) in $t \in \mathbb{R}$ uniformly with respect to $x$ on each bounded subset, then equation (3.1) has a unique solution $\varphi \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; H))$ which is pseudo-periodic (respectively, pseudo-recurrent) in distribution.

4. THE SECOND BOGOLYUBOV THEOREM

Consider the following stochastic partial differential equation

$$dX_\varepsilon(t) = \left(A(X_\varepsilon(t)) + F\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right)\right) dt + G\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right) dW(t),$$

where $A(x) = A_1(x) + A_2(x)$, $A_i : V_i \to V_i^*$, $i = 1, 2$, $F \in C(\mathbb{R} \times H, H)$, $G \in C(\mathbb{R} \times H, L_2(U, H))$ and $0 < \varepsilon \leq 1$. Here $W$ is a $U$-valued two-sided cylindrical Wiener process with the identity covariance operator with respect to a filtered probability space $(\Omega, F, \mathbb{P}, F_t)$, where $F_t := \sigma\{W(u) - W(v) : u, v \leq t\}$. In this section, we will omit the index $H$ of $\| \cdot \|_H$ and $\langle \cdot, \cdot \rangle_H$ if it does not cause confusion.

We employ $\Psi$ to denote the space of all decreasing, positive bounded functions $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to \infty} \omega(t) = 0$. Below we need additional conditions.

(H2') There exist constants $\lambda, \lambda' \geq 0$, $L_F, L_G, M_0 > 0$, $r > 2$ and $\lambda_F \in \mathbb{R}$ such that

$$\langle A(u) - A(v), u - v \rangle_V \leq -\lambda\| u - v \|^r - \lambda'\| u - v \|^2,$$

$$\langle F(t, x) - F(t, y), x - y \rangle \leq \lambda_F\| x - y \|^2, \quad \| F(t, 0) \| \leq M_0$$

$$\| F(t, x) - F(t, y) \| \leq L_F\| x - y \|, \quad \| G(t, x) - G(t, y) \|_{L_2(U, H)} \leq L_G\| x - y \|$$

for any $t \in \mathbb{R}$, $u, v \in V$ and $x, y \in H$;

(G1) There exist functions $\omega_1 \in \Psi$ and $\tilde{F} \in C(H, H)$ such that

$$\frac{1}{T} \left\| \int_t^{t+T} [F(s, x) - \tilde{F}(x)] ds \right\| \leq \omega_1(T)(1 + \| x \|)$$

for any $T > 0$, $x \in H$ and $t \in \mathbb{R}$;

(G2) There exist functions $\omega_2 \in \Psi$ and $\tilde{G} \in C(H, L_2(U, H))$ such that

$$\frac{1}{T} \int_t^{t+T} \| G(s, x) - \tilde{G}(x) \|_{L_2(U, H)}^2 ds \leq \omega_2(T)(1 + \| x \|^2)$$

for any $T > 0$, $x \in H$ and $t \in \mathbb{R}$.

Remark 4.1. (i) Note that the estimate of solutions to (4.1) for the integral of time increment on $H$ in Lemma 4.4 is weaker than the Hölder continuity but helpful to establish the first Bogolyubov theorem. If $F$ is just monotone instead of Lipschitz, we need this estimate on $V$, which is crucial to establishing the first Bogolyubov theorem based on the technique of time discretization. But we cannot obtain this estimate on $V$ unless there are additional assumptions on higher regularity of initial data and coefficients. However, this higher regularity condition is too strong to apply to porous media equations, which is one of our examples. On the other hand, the higher regularity of initial data is too strong to establish the second Bogolyubov theorem and global averaging principle, which are our main results in this paper. Indeed, the first Bogolyubov theorem and the existence and uniqueness of $L^2$-bounded solutions play important roles in establishing the second Bogolyubov theorem and global averaging principle (see Theorems 4.7 and 5.14). But we only establish that this bounded solution belongs to $L^\infty(\mathbb{R}; L^2(\Omega, \mathbb{P}; H)) \cap L^\infty(\mathbb{R}; L^2(\Omega, \mathbb{P}; S))$ (see Theorem 3.6 and Proposition 3.12), whose regularity is not higher enough for our purpose.
(ii) In order to obtain recurrent solutions, the systems (4.1) need to be dissipative; that is, $2\lambda - 2\lambda F - L_2^2 \geq 0$. Since the condition $2\lambda - 2LF - L_2^2 \geq 0$ is stronger than $2\lambda - 2\lambda F - L_2^2 \geq 0$ when $\lambda F$ is negative, we introduce the conditions of monotonicity and Lipschitz continuity of $F$ in $\left( H2' \right)$ simultaneously.

(iii) Notice that we can just assume $A_2$ is hemicontinuous when $A_2$ is independent of time $t$. In the following, we still say that equation (4.1) satisfies $\left( H5 \right)$ (respectively, $\left( H6 \right)$), if $F$ and $G$ are continuous in $t \in \mathbb{R}$ uniformly with respect to $u$ on each bounded $Q \subset H$ (respectively, there exist constants $c_4, M_0 > 0$ such that $2V, \left( A(v), T_n v \right)_V + 2(F(t,v), T_n v) + \left\| G(t,v) \right\|_{L_2(t,H)} \leq c_1 \left\| v \right\|_n^2 + M_0$ for all $v \in V$ and $t \in \mathbb{R}$).

(iv) It can be verified that $\left( G1 \right)$ (respectively, $\left( G2 \right)$) implies

$$
\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} F(s, x) ds = \bar{F}(x)
$$

(respectively, $\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \left\| G(s, x) - \bar{G}(x) \right\|_{L_2(t,H)} ds = 0$) uniformly with respect to $t \in \mathbb{R}$ and $x$ in any bounded subset on $H$.

Denote by $F_\varepsilon(t, x) := F(t, \frac{x}{\varepsilon}, x)$ and $G_\varepsilon(t, x) := G(t, \frac{x}{\varepsilon}, x)$ for any $t \in \mathbb{R}$, $x \in H$ and $\varepsilon \in (0, 1]$. Equation (4.1) can be written as

$$
dX_\varepsilon(t) = (A(X_\varepsilon(t)) + F_\varepsilon(t, X_\varepsilon(t))) dt + G_\varepsilon(t, X_\varepsilon(t)) dW(t).
$$

Along with equations (4.1)–(4.2) we consider the following averaged equation

$$
dX(t) = (A(X(t)) + \bar{F}(X(t))) dt + \bar{G}(X(t)) dW(t).
$$

**Lemma 4.2.** If $F$ and $G$ satisfy $\left( H2' \right)$ and $\left( G1 \right)$–$\left( G2 \right)$, then $\bar{F}$ and $\bar{G}$ in $\left( G1 \right)$–$\left( G2 \right)$ also satisfy $\left( H2' \right)$ with the same constants.

**Proof.** We only need to prove the conclusion for $\bar{F}$; the case of $\bar{G}$ is similar. It follows from Remark 4.1 (iv) that

$$
\left\| \bar{F}(0) \right\| = \left\| \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t, 0) dt \right\| \leq \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\| F(t, 0) \right\| dt \leq M_0.
$$

For any $x, y \in H$ one sees that

$$
\langle \bar{F}(x) - \bar{F}(y), x - y \rangle
$$

$$
= \langle \bar{F}(x) - \frac{1}{T} \int_0^T F(t,x) dt, x - y \rangle + \langle \frac{1}{T} \int_0^T (F(t,x) - F(t,y)) dt, x - y \rangle
$$

$$
+ \langle \frac{1}{T} \int_0^T F(t,y) dt - \bar{F}(y), x - y \rangle
$$

$$
\leq \frac{1}{T} \left\| \int_0^T (\bar{F}(x) - F(t,x)) dt \right\| \cdot \left\| x - y \right\| + \frac{1}{T} \left\| \int_0^T (F(t,x) - F(t,y), x - y) dt \right\|
$$

$$
+ \frac{1}{T} \left\| \int_0^T (F(t,y) - \bar{F}(y)) dt \right\| \cdot \left\| x - y \right\|
$$

$$
\leq \omega_1(T) (1 + \left\| x \right\|) \left\| x - y \right\| + \lambda_F \left\| x - y \right\|^2 + \omega_1(T) (1 + \left\| y \right\|) \left\| x - y \right\|
$$

and

$$
\left\| \bar{F}(x) - \bar{F}(y) \right\| \leq \left\| \bar{F}(x) - \frac{1}{T} \int_0^T F(t,x) dt \right\| + \left\| \frac{1}{T} \int_0^T (F(t,x) - F(t,y)) dt \right\|
$$

$$
+ \left\| \frac{1}{T} \int_0^T F(t,y) dt - \bar{F}(y) \right\|
$$

$$
\leq \omega_1(T) (1 + \left\| x \right\|) + L_F \left\| x - y \right\| + \omega_1(T) (1 + \left\| y \right\|).$$

Letting $T \to \infty$ in the above two inequalities, we have
\[ \langle \bar{F}(x) - \bar{F}(y), x - y \rangle \leq \lambda_F \|x - y\|^2, \quad \|\bar{F}(x) - \bar{F}(y)\| \leq L_F \|x - y\| \]
for all $x, y \in H$ provided $\lim_{T \to \infty} \omega_1(T) = 0$. \hfill \square

**Remark 4.3.** It follows from Lemma 4.2 that estimates (3.4), (3.9), (3.17), (3.18) uniformly hold for $\varepsilon \in (0, 1)$, and $\bar{F}$ and $\bar{G}$.

Recall that $C_\alpha$ mean some positive constant which depends on $\alpha$. For simplicity, we just write $C$ when $C$ depends on some parameters of $\lambda, \lambda', r, \lambda_F, L_F, L_G, c_1, c_2, \varepsilon_2, \alpha_2, M_0, c_3, \varepsilon_3$ in (H1), (H2') and (H3). Let $\delta > 0$ be a fixed constant depending on $\varepsilon$. For any given stochastic process $\phi$, define a step process $\bar{\phi}$ such that $\bar{\phi}(\sigma) = \phi(s + k\delta)$ for any $\sigma \in [s + k\delta, s + (k + 1)\delta)$. Employing the technique of time discretization, we have the following estimates.

**Lemma 4.4.** Assume (H1), (H2'), (H3)--(H4) and (G1)--(G2) hold. Let $X_\varepsilon(t, s, \zeta_\varepsilon), t \geq s$ be the solution of (4.2) with the initial value $X_\varepsilon(s, s, \zeta_\varepsilon) = \zeta_\varepsilon$ and $\bar{X}(t, s, \bar{\zeta}_s), t \geq s$ be the solution of (4.3) with the initial value $\bar{X}(s, s, \bar{\zeta}_s) = \bar{\zeta}_s$. Then we have
\[ E \int_s^{s+T} \|X_\varepsilon(\sigma, s, \zeta_\varepsilon) - \bar{X}_\varepsilon(\sigma, s, \zeta_\varepsilon)\|^2 \, d\sigma \leq C_T (1 + E \|\zeta_\varepsilon\|^2) \delta^{\frac{1}{2}} \] (4.4)
and
\[ E \int_s^{s+T} \|\bar{X}(\sigma, s, \bar{\zeta}_s) - \bar{X}(\sigma, s, \bar{\zeta}_s)\|^2 \, d\sigma \leq C_T (1 + E \|\bar{\zeta}_s\|^2) \delta^{\frac{1}{2}} \] (4.5)
for any $s \in \mathbb{R}$ and $T > 0$, where $\bar{X} := \bar{X}$.

**Proof.** For simplicity, let $X_\varepsilon(\sigma) := X_\varepsilon(\sigma, s, \zeta_\varepsilon)$ and $\bar{X}(\sigma) := \bar{X}(\sigma, s, \bar{\zeta}_s)$. By Lemma 3.5 we have
\[ E \int_s^{s+T} \|X_\varepsilon(\sigma) - \bar{X}_\varepsilon(\sigma)\|^2 \, d\sigma \]
(4.6)
\[ = E \int_s^{s+T} \|X_\varepsilon(\sigma) - \zeta_\varepsilon\|^2 \, d\sigma + E \sum_{k=1}^{T(\delta)-1} \int_{s+k\delta}^{s+(k+1)\delta} \|X_\varepsilon(\sigma) - X_\varepsilon(s+k\delta)\|^2 \, d\sigma \]
\[ + E \int_{s+T(\delta)\delta}^{s+T} \|X_\varepsilon(\sigma) - X_\varepsilon(s+T(\delta)\delta)\|^2 \, d\sigma \]
\[ \leq C_T (1 + E \|\zeta_\varepsilon\|^2) \delta + 2E \sum_{k=1}^{T(\delta)-1} \int_{s+k\delta}^{s+(k+1)\delta} \|X_\varepsilon(\sigma) - X_\varepsilon(s+T(\delta)\delta)\|^2 \, d\sigma \]
\[ \quad + 2E \sum_{k=1}^{T(\delta)-1} \int_{s+k\delta}^{s+(k+1)\delta} \|X_\varepsilon(\sigma - \delta) - X_\varepsilon(s+k\delta)\|^2 \, d\sigma \]
\[ =: C_T (1 + E \|\zeta_\varepsilon\|^2) \delta + 2 \sum_{k=1}^{T(\delta)-1} I_k + 2 \sum_{k=1}^{T(\delta)-1} J_k. \]

Given $k \in [1, T(\delta) - 1)$, for any $\sigma \in [s+k\delta, s+(k+1)\delta)$, by Itô’s formula, (H2'), (H3)--(H4) and Young’s inequality we get
\[ \|X_\varepsilon(\sigma) - X_\varepsilon(\sigma - \delta)\|^2 \] (4.7)
\[
\begin{align*}
&= \int_{\sigma-\delta}^{\sigma} (2V\cdot \langle A(X_\varepsilon(u)), X_\varepsilon(u) - X_\varepsilon(\sigma - \delta) \rangle_V + 2\langle F_\varepsilon(u), X_\varepsilon(u) - X_\varepsilon(\sigma - \delta) \rangle) \, du \\
&\quad + \int_{\sigma-\delta}^{\sigma} \|G_\varepsilon(u, X_\varepsilon(u))\|_{L_2(v, H)}^2 \, du + 2 \int_{\sigma-\delta}^{\sigma} \langle X_\varepsilon(u) - X_\varepsilon(\sigma - \delta), G_\varepsilon(u, X_\varepsilon(u)) \rangle dW(u) \\
&\leq \int_{\sigma-\delta}^{\sigma} \left(2\|A_1(X_\varepsilon(u))\|_{V_1^*} \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\|_{V_1} + 2\|A_2(X_\varepsilon(u))\|_{V_2^*} \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\|_{V_2} \\
&\quad + 2\|F_\varepsilon(u, X_\varepsilon(u))\| \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\| + 2L_G^2 \|X_\varepsilon(u)\|^2 + 2M_0^2 \right) \, du \\
&\quad + 2 \int_{\sigma-\delta}^{\sigma} \langle X_\varepsilon(u) - X_\varepsilon(\sigma - \delta), G_\varepsilon(u, X_\varepsilon(u)) \rangle dW(u) \\
&\leq \int_{\sigma-\delta}^{\sigma} \left[ 2 \left(\|X_\varepsilon(u)\|_{V_1^*}^{\alpha_1} + M_0 \right) \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\|_{V_1} + 2 \left(\|X_\varepsilon(u)\|_{V_2^*}^{\alpha_2} + M_0 \right) \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\|_{V_2} \\
&\quad + 2 \left(2L_F \|X_\varepsilon(u)\| + M_0 \right) \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\| + 2L_G^2 \|X_\varepsilon(u)\|^2 + 2M_0^2 \right] \, du \\
&\quad + 2 \int_{\sigma-\delta}^{\sigma} \langle X_\varepsilon(u) - X_\varepsilon(\sigma - \delta), G_\varepsilon(u, X_\varepsilon(u)) \rangle dW(u) \\
&\leq \int_{\sigma-\delta}^{\sigma} \left[ 4\|X_\varepsilon(u)\|_{V_1^*}^{\alpha_1} + \frac{4}{\alpha_1} \|X_\varepsilon(\sigma - \delta)\|_{V_1}^{\alpha_1} + 4\|X_\varepsilon(u)\|_{V_2^*}^{\alpha_2} + \frac{4}{\alpha_2} \|X_\varepsilon(\sigma - \delta)\|_{V_2}^{\alpha_2} + \frac{4(\alpha_1 - 1)}{\alpha_1} M_0^{\alpha_1 - 1} \\
&\quad + \frac{4(\alpha_2 - 1)}{\alpha_2} M_0^{\alpha_2 - 1} + (2L_F + L_F^2 + 2L_G^2 + 1) \|X_\varepsilon(u)\|^2 + 2\|X_\varepsilon(\sigma - \delta)\|^2 + 4M_0^2 \right] \, du \\
&\quad + 2 \int_{\sigma-\delta}^{\sigma} \langle X_\varepsilon(u) - X_\varepsilon(\sigma - \delta), G_\varepsilon(u, X_\varepsilon(u)) \rangle dW(u).
\end{align*}
\]

Set \( s^{k\delta} := s + k\delta \) for all \( s \in \mathbb{R} \), \( k \geq 0 \). Then we have

(4.8)

\[
I_k := \mathbb{E} \int_{s^{k\delta}}^{s^{(k+1)\delta}} \|X_\varepsilon(\sigma) - X_\varepsilon(\sigma - \delta)\|^2 \, d\sigma
\]

\[
\leq \mathbb{E} \int_{s^{k\delta}}^{s^{(k+1)\delta}} \left\{ \int_{\sigma-\delta}^{\sigma} \left[ 4\|X_\varepsilon(u)\|_{V_1^*}^{\alpha_1} + \frac{4}{\alpha_1} \|X_\varepsilon(\sigma - \delta)\|_{V_1}^{\alpha_1} + 4\|X_\varepsilon(u)\|_{V_2^*}^{\alpha_2} + \frac{4}{\alpha_2} \|X_\varepsilon(\sigma - \delta)\|_{V_2}^{\alpha_2} \\
&\quad + \frac{4(\alpha_1 - 1)}{\alpha_1} M_0^{\alpha_1 - 1} + \frac{4(\alpha_2 - 1)}{\alpha_2} M_0^{\alpha_2 - 1} + (2L_F + L_F^2 + 2L_G^2 + 1) \|X_\varepsilon(u)\|^2 \\
&\quad + 2\|X_\varepsilon(\sigma - \delta)\|^2 + 4M_0^2 \right] \, du + 2 \int_{\sigma-\delta}^{\sigma} \langle X_\varepsilon(u) - X_\varepsilon(\sigma - \delta), G_\varepsilon(u, X_\varepsilon(u)) \rangle dW(u) \right\} \, d\sigma
\]
\[
\begin{aligned}
& \leq E \int_{s-k+1}^{s-k}\int_{u}^{u+\delta} \left[ 4\|X_\varepsilon(u)\|_{V_1}^{\alpha_1} + 4\|X_\varepsilon(u)\|_{V_2}^{\alpha_2} + C + C\|X_\varepsilon(u)\|^2 \right] d\sigma d\nu
+ E \int_{s-k+1}^{s-k}\delta \left( \frac{4}{\alpha_1}\|X_\varepsilon(\sigma - \delta)\|_{V_1}^{\alpha_1} + \frac{4}{\alpha_2}\|X_\varepsilon(\sigma - \delta)\|_{V_2}^{\alpha_2} + 2\|X_\varepsilon(\sigma - \delta)\|^2 \right) d\sigma + I_k^2 \\
& \leq E \int_{s-k+1}^{s-k}\delta \left[ 4\|X_\varepsilon(u)\|_{V_1}^{\alpha_1} + 4\|X_\varepsilon(u)\|_{V_2}^{\alpha_2} + C + C\|X_\varepsilon(u)\|^2 \right] du
+ E \int_{s-k+1}^{s-k}\delta \left( \frac{4}{\alpha_1}\|X_\varepsilon(\sigma - \delta)\|_{V_1}^{\alpha_1} + \frac{4}{\alpha_2}\|X_\varepsilon(\sigma - \delta)\|_{V_2}^{\alpha_2} + 2\|X_\varepsilon(\sigma - \delta)\|^2 \right) d\sigma + I_k^2 \\
& \leq \delta CE \int_{s-k+1}^{s-k}\left( \|X_\varepsilon(u)\|_{V_1}^{\alpha_1} + \|X_\varepsilon(u)\|_{V_2}^{\alpha_2} + \|X_\varepsilon(u)\|^2 + 1 \right) du + I_k^2.
\end{aligned}
\]

Now we estimate \( I_k^2 \). In view of Burkholder-Davis-Gundy inequality, (H2'), (H4) and Young’s inequality, we obtain

(4.9)

\[
I_k^2 := 2E \int_{s-k}^{s} \int_{\sigma-\delta}^{\sigma} \langle X_\varepsilon(u) - X_\varepsilon(\sigma - \delta), G_\varepsilon(u, X_\varepsilon(u))dW(u) \rangle d\sigma
\]

\[
\leq 6 \int_{s-k}^{s} \left( \frac{2L_\varepsilon^2}{E} \|X_\varepsilon(u)\|^2 + 2M_\varepsilon^2 \right) \|X_\varepsilon(\sigma - \delta)\|^2 d\sigma
\]

\[
\leq C \int_{s-k}^{s} \left( \left( \sup_{s \leq t \leq s+T} \|X_\varepsilon(t)\|^2 + 1 \right) \left( \int_{\sigma-\delta}^{\sigma} \|X_\varepsilon(u) - X_\varepsilon(\sigma - \delta)\|^2 du \right)^{\frac{1}{2}} \right) d\sigma
\]

\[
\leq \delta \frac{1}{2} C \left( E \sup_{s \leq t \leq s+T} \|X_\varepsilon(u)\|^2 + 1 \right) \left( \int_{s-k}^{s} \int_{\sigma-\delta}^{\sigma} \left( \|X_\varepsilon(u)\|^2 + \|X_\varepsilon(\sigma - \delta)\|^2 \right) dud\sigma \right)^{\frac{1}{2}}
\]

\[
\leq \delta \frac{1}{2} C \left( E \|\zeta_\varepsilon^s\|^2 + 1 \right) \left( \int_{s-k}^{s} \int_{\sigma-\delta}^{\sigma} \left( \|X_\varepsilon(u)\|^2 + \|X_\varepsilon(\sigma - \delta)\|^2 \right) dud\sigma \right)^{\frac{1}{2}}
\]

\[
\leq \delta C \left( E \|\zeta_\varepsilon^s\|^2 + 1 \right) \left( E \int_{s-k}^{s} \|X_\varepsilon(u)\|^2 du \right)^{\frac{1}{2}}.
\]

Therefore (4.8) and (4.9) yield

(4.10)

\[
I_k \leq \delta CE \int_{s-k+1}^{s-k} \left( \|X_\varepsilon(u)\|_{V_1}^{\alpha_1} + \|X_\varepsilon(u)\|_{V_2}^{\alpha_2} + \|X_\varepsilon(u)\|^2 + 1 \right) du
+ \delta C \left( E \|\zeta_\varepsilon^s\|^2 + 1 \right) \left( E \int_{s-k+1}^{s-k} \|X_\varepsilon(u)\|^2 du \right)^{\frac{1}{2}}.
\]

By (3.9) and Remark 4.3 we get

(4.11)

\[
2 \sum_{k=1}^{T(\delta)-1} I_k \leq \delta CE \int_{s+T}^{s+T} \left( \|X_\varepsilon(u)\|_{V_1}^{\alpha_1} + \|X_\varepsilon(u)\|_{V_2}^{\alpha_2} + \|X_\varepsilon(u)\|^2 + 1 \right) du
\]
\[ + \delta C \left( E \| \zeta_s^\varepsilon \|^2 + 1 \right) \sum_{k=1}^{T(\delta)-1} \left( E \int_{s(k-1)\delta}^{s(k+1)\delta} \| X_\varepsilon(u) \|^2 du \right) \frac{1}{2} \]

\[ \leq \delta C E \int_s^{s+T} \left( \| X_\varepsilon(u) \|_{V_1}^2 + \| X_\varepsilon(u) \|_{V_2}^2 + \| X_\varepsilon(u) \|^2 + 1 \right) du \]

\[ + \delta^2 C \left( E \| \zeta_s^\varepsilon \|^2 + 1 \right) \left( \int_s^{s+T} E \| X_\varepsilon(t) \|^2 dt \right) \frac{1}{2} \]

\[ \leq C_T \left( 1 + E \| \zeta^\varepsilon_s \|^2 \right) \delta^2. \]

Similarly, we have

\[ 2 \sum_{k=1}^{T(\delta)-1} J_k \leq C_T \left( 1 + E \| \zeta^\varepsilon_s \|^2 \right) \delta^2. \]

Combining (4.6), (4.11) and (4.12), we obtain

\[ E \int_s^{s+T} \| X_\varepsilon(\sigma) - \bar{X}_\varepsilon(\sigma) \|^2 d\sigma \leq C_T \left( 1 + E \| \zeta^\varepsilon_s \|^2 \right) \delta^2. \]

It follows from the same steps as in the proof of (4.4) that

\[ E \int_s^{s+T} \| \bar{X}(\sigma) - \bar{X}(\sigma) \|^2 d\sigma \leq C_T \left( 1 + E \| \zeta^\varepsilon_s \|^2 \right) \delta^2. \]

Now we establish the following first Bogolyubov theorem.

**Theorem 4.5.** Suppose that (G1)–(G2), (H1), (H2') and (H3)–(H4) hold. For any \( s \in \mathbb{R} \), let \( X_\varepsilon(t, s, \zeta_s^\varepsilon), t \geq s \) be the solution of the following Cauchy problem

\[
\begin{align*}
\frac{dX(t)}{dt} &= (A(X(t)) + F_\varepsilon(t, X(t))) dt + G_\varepsilon(t, X(t))dW(t) \\
X(s) &= \zeta_s^\varepsilon
\end{align*}
\]

and \( \bar{X}(t, s, \zeta_s), t \geq s \) be the solution of the following Cauchy problem

\[
\begin{align*}
\frac{dX(t)}{dt} &= (\bar{A}(X(t)) + \bar{F}(X(t))) dt + \bar{G}(X(t))dW(t) \\
X(s) &= \zeta_s
\end{align*}
\]

Assume further that \( \lim_{\varepsilon \to 0} E \| \zeta_s^\varepsilon - \zeta_s \|^2 = 0 \). Then

\[ \lim_{\varepsilon \to 0} E \sup_{s \leq t \leq s+T} \| X_\varepsilon(t, s, \zeta_s^\varepsilon) - \bar{X}(t, s, \zeta_s) \|^2 = 0 \]

for any \( T > 0 \).

**Proof.** Set \( X_\varepsilon(\sigma) := X_\varepsilon(\sigma, s, \zeta_s^\varepsilon) \) and \( \bar{X}(\sigma) := \bar{X}(\sigma, s, \zeta_s) \) for all \( \sigma \geq s \). In view of Itô’s formula and (H2'), we have

\[ \| X_\varepsilon(t) - \bar{X}(t) \|^2 \]

\[ = \| \zeta_s^\varepsilon - \zeta_s \|^2 + \int_s^t \left( 2V^* (A(X_\varepsilon(\sigma)) - A(\bar{X}(\sigma)), X_\varepsilon(\sigma) - \bar{X}(\sigma)) \right) \]

\[ + 2 \langle F_\varepsilon(\sigma, X_\varepsilon(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle + \| G_\varepsilon(\sigma, X_\varepsilon(\sigma)) - \bar{G}(\bar{X}(\sigma)) \|_{L_2(U, H)}^2 \] \]
\[ + 2 \int_s^t \langle X_\xi(\sigma) - \bar{X}(\sigma), (G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma))) \rangle \text{dW}(\sigma) \]
\[ \leq \| \zeta_\xi - \zeta_s \|^2 + 2 \int_s^t \langle X_\xi(\sigma) - \bar{X}(\sigma), (G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma))) \rangle \text{dW}(\sigma) \]
\[ + \int_s^t \left( 2\langle F_\xi(\sigma, X_\xi(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\xi(\sigma) - \bar{X}(\sigma) \rangle + \| G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma)) \|^2_{L_2(U,H)} \right) \text{d}\sigma. \]

Therefore, by Burkholder-Davis-Gundy inequality and Young’s inequality we get
\[ (4.14) \quad E \sup_{s \leq t \leq s+T} \| X_\xi(t) - \bar{X}(t) \|^2 \]
\[ \leq E \| \zeta_\xi - \zeta_s \|^2 + E \sup_{s \leq t \leq s+T} \int_s^t 2\langle F_\xi(\sigma, X_\xi(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\[ + E \int_s^{s+T} \| G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma)) \|^2_{L_2(U,H)} \text{d}\sigma \]
\[ + 6E \left( \int_s^{s+T} \| G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma)) \|^2_{L_2(U,H)} \| X_\xi(\sigma) - \bar{X}(\sigma) \|^2 \text{d}\sigma \right)^{\frac{1}{2}} \]
\[ \leq E \| \zeta_\xi - \zeta_s \|^2 + E \sup_{s \leq t \leq s+T} \int_s^t 2\langle F_\xi(\sigma, X_\xi(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\[ + \frac{1}{2} E \sup_{s \leq t \leq s+T} \| X_\xi(t) - \bar{X}(t) \|^2 + 19E \int_s^{s+T} \| G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma)) \|^2_{L_2(U,H)} \text{d}\sigma. \]

Then we obtain
\[ (4.15) \] \[ E \sup_{s \leq t \leq s+T} \| X_\xi(t) - \bar{X}(t) \|^2 \]
\[ \leq 2E \| \zeta_\xi - \zeta_s \|^2 + 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\xi(\sigma, X_\xi(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\[ + 38E \int_s^{s+T} \| G_\xi(\sigma, X_\xi(\sigma)) - \bar{G}(\bar{X}(\sigma)) \|^2_{L_2(U,H)} \text{d}\sigma \]
\[ =: 2E \| \zeta_\xi - \zeta_s \|^2 + I_1 + I_2. \]

For \( I_1 \),
\[ (4.16) \] \[ I_1 := 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\xi(\sigma, X_\xi(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\[ \leq 4E \int_s^{s+T} \| F_\xi(\sigma, X_\xi(\sigma)) - F_\xi(\sigma, \bar{X}(\sigma)) \| \| X_\xi(\sigma) - \bar{X}(\sigma) \| \text{d}\sigma \]
\[ + 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\xi(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\[ \leq 4E \int_s^{s+T} L_F \| X_\xi(\sigma) - \bar{X}(\sigma) \|^2 \text{d}\sigma \]
\[ + 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\xi(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\[ + 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\xi(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\xi(\sigma) - \bar{X}(\sigma) \rangle \text{d}\sigma \]
\begin{align*}
&+ 4E \sup_{s \leq t \leq s+T} \int_s^t (F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}(\sigma) - \bar{X}(\sigma))d\sigma \\
&\leq 4E \int_s^{s+T} L_F \|X_\varepsilon(\sigma) - \bar{X}(\sigma)\|^2 d\sigma + \mathbb{I}_1^2 + \mathbb{I}_2^3 + \mathbb{I}_3^4.
\end{align*}

For \( \mathbb{I}_2^1 \), by (H2'), Hölder’s inequality, (3.9), Remark 4.3 and (4.4) we have

\begin{equation}
\mathbb{I}_2^1 := 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), X_\varepsilon(\sigma) - \bar{X}_\varepsilon(\sigma) \rangle d\sigma
\end{equation}

\begin{align*}
&\leq 4E \int_s^{s+T} \|F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma))\| \|X_\varepsilon(\sigma) - \bar{X}_\varepsilon(\sigma)\| d\sigma \\
&\leq 4 \left( E \int_s^{s+T} (2L_F \|\bar{X}(\sigma)\| + 2M_0)^2 d\sigma \right)^{\frac{1}{2}} \left( E \int_s^{s+T} \|X_\varepsilon(\sigma) - \bar{X}_\varepsilon(\sigma)\|^2 d\sigma \right)^{\frac{1}{2}} \\
&\leq C_T \left( 1 + E \|\zeta_\delta\|^2 \right) \delta^4.
\end{align*}

For \( \mathbb{I}_3^1 \), similar to \( \mathbb{I}_1^2 \), employing (H2'), Hölder’s inequality, (3.9), Remark 4.3 and (4.5) we get

\begin{equation}
\mathbb{I}_3^1 := 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma
\end{equation}

\begin{align*}
&\leq C_T \left( 1 + E \|\zeta_\delta\|^2 \right) \delta^4.
\end{align*}

For \( \mathbb{I}_1^3 \), in view of (H2'), Hölder’s inequality, (3.9) and Remark 4.3, we have

\begin{equation}
\mathbb{I}_1^3 := 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma
\end{equation}

\begin{align*}
&= 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - F_\varepsilon(\sigma, \bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma \\
&+ 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma \\
&+ 4E \sup_{s \leq t \leq s+T} \int_s^t \langle \bar{F}(\bar{X}(\sigma) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma \\
&\leq 4E \int_s^{s+T} \|F_\varepsilon(\sigma, \bar{X}(\sigma)) - F_\varepsilon(\sigma, \bar{X}(\sigma))\| \|\bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma)\| d\sigma \\
&+ 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma \\
&+ 4E \int_s^{s+T} \|\bar{F}(\bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma))\| \|\bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma)\| d\sigma \\
&\leq 8E \int_s^{s+T} L_F \|\bar{X}(\sigma) - \bar{X}(\sigma)\| \|\bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma)\| d\sigma \\
&+ 4E \sup_{s \leq t \leq s+T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma) \rangle d\sigma \\
&\leq 8L_F \left( E \int_s^{s+T} \|\bar{X}(\sigma) - \bar{X}(\sigma)\|^2 d\sigma \right)^{\frac{1}{2}} \left( E \int_s^{s+T} \|\bar{X}_\varepsilon(\sigma) - \bar{X}(\sigma)\|^2 d\sigma \right)^{\frac{1}{2}}.
\end{align*}


\[ + 4E \sup_{s \leq t \leq s + T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\bar{X}(\sigma)), \bar{\bar{X}}(\sigma) - \bar{\bar{X}}(\sigma))d\sigma \]

\[ \leq C_T (1 + E\|\zeta_s\|^2)^{\frac{1}{2}} + \Pi_1^{3,2}. \]

Define \( t(s, \delta) := s + \left\lfloor \frac{t}{\delta} \right\rfloor \delta \), where \( \left\lfloor \frac{t}{\delta} \right\rfloor \) is the integer part of \( \frac{t}{\delta} \). Now we estimate \( \Pi_1^{3,2} := 4E \sup_{s \leq t \leq s + T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{\bar{X}}(\sigma) - \bar{\bar{X}}(\sigma) \rangle d\sigma \) by (H2'), (G1), (3.9) and Remark 4.3:

(4.20)

\[ 4E \sup_{s \leq t \leq s + T} \int_s^t \langle F_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{F}(\bar{X}(\sigma)), \bar{\bar{X}}(\sigma) - \bar{\bar{X}}(\sigma) \rangle d\sigma \]

\[ = 4E \sup_{s \leq t \leq s + T} \left\{ \frac{1}{\delta}(k+1)^\delta \sum_{k=0}^{\left\lfloor \frac{t}{\delta} \right\rfloor - 1} \int_{s(k\delta)}^{s(k+1)\delta} \langle F_\varepsilon(\sigma, \bar{X}(s(k\delta))) - \bar{F}(\bar{X}(s(k\delta))), \bar{\bar{X}}(s(k\delta)) - \bar{\bar{X}}(s(k\delta)) \rangle d\sigma \right\} \]

\[ + \int_{t(s, \delta)}^{t} \langle F_\varepsilon(\sigma, \bar{X}(t(s, \delta))) - \bar{F}(\bar{X}(t(s, \delta))), \bar{\bar{X}}(t(s, \delta)) - \bar{\bar{X}}(t(s, \delta)) \rangle d\sigma \]

\[ \leq 4E \sup_{s \leq t \leq s + T} \left\{ \frac{1}{\delta}(k+1)^\delta \sum_{k=0}^{\left\lfloor \frac{t}{\delta} \right\rfloor - 1} \left\| \int_{s(k\delta)}^{s(k+1)\delta} \langle F_\varepsilon(\sigma, \bar{X}(s(k\delta))) - \bar{F}(\bar{X}(s(k\delta))), \bar{\bar{X}}(s(k\delta)) - \bar{\bar{X}}(s(k\delta)) \rangle d\sigma \right\| \right\}

\[ \times \|X_\varepsilon(s(k\delta)) - \bar{\bar{X}}(s(k\delta))\| \} + C_T (1 + E\|\zeta_s\|^2)^{\frac{1}{2}} \delta \]

\[ \leq \frac{4T}{\delta} \max_{0 \leq k \leq T(\delta) - 1} \left( E \left\| \int_{s(k\delta)}^{s(k+1)\delta} \langle F_\varepsilon(\sigma, \bar{X}(s(k\delta))) - \bar{F}(\bar{X}(s(k\delta))), \bar{\bar{X}}(s(k\delta)) - \bar{\bar{X}}(s(k\delta)) \rangle d\sigma \right\|^2 \right)^{\frac{1}{2}} C_T (1 + E\|\zeta_s\|^2) \]

\[ + C_T (1 + E\|\zeta_s\|^2) \delta \]

\[ \leq \frac{4T}{\delta} \max_{0 \leq k \leq T(\delta) - 1} \delta \omega_1 \left( \frac{\delta}{\varepsilon} \right) C_T (1 + E\|\zeta_s\|^2) \left( 1 + E\|\zeta_s\|^2 \right) \delta \]

\[ \leq C_T (1 + E\|\zeta_s\|^2) \left( \omega_1 \left( \frac{\delta}{\varepsilon} \right) + \delta \right). \]

Combining (4.19) and (4.20), we deduce

(4.21)

\[ \Pi_1^{3} \leq C_T (1 + E\|\zeta_s\|^2) \left( \delta^{\frac{1}{2}} + \delta + \omega_1 \left( \frac{\delta}{\varepsilon} \right) \right). \]

Therefore, (4.16)–(4.18) and (4.21) yield

(4.22)

\[ \Pi_1 \leq 4L_E \int_s^{s + T} E \sup_{s \leq u \leq \sigma} \|X_\varepsilon(u) - \bar{\bar{X}}(u)\|^2 d\sigma + C_T (1 + E\|\zeta_s\|^2) \left( \delta^{\frac{1}{2}} + \delta + \omega_1 \left( \frac{\delta}{\varepsilon} \right) \right). \]
Now we estimate $I_2$, 

\[
I_2 := 38E \int_s^{s+T} \|G_\varepsilon(\sigma, X_\varepsilon(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
\leq 76E \int_s^{s+T} \|G_\varepsilon(\sigma, X_\varepsilon(\sigma)) - G_\varepsilon(\sigma, \bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
+ 76E \int_s^{s+T} \|G_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
\leq 76L_G^2E \int_s^{s+T} \|X_\varepsilon(\sigma) - \bar{X}(\sigma)\|^2 d\sigma \\
+ 76E \int_s^{s+T} \|G_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
\leq 76L_G^2 \sup_{s \leq u \leq T} \|X_\varepsilon(u) - \bar{X}(u)\|^2 d\sigma + I_2^2.
\]

Denote by $T(\delta) := \left[ \frac{T}{\delta} \right]$. For $I_2^2$, it follows from (H2'), (G1), (4.5), (3.9) and Remark 4.3 that (4.24)

\[
I_2^2 := 76E \int_s^{s+T} \|G_\varepsilon(\sigma, X_\varepsilon(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
\leq 228E \int_s^{s+T} \|G_\varepsilon(\sigma, \bar{X}(\sigma)) - G_\varepsilon(\sigma, \bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
+ 228E \int_s^{s+T} \|G_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
+ 228E \int_s^{s+T} \|\bar{G}(\bar{X}(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
\leq 456E \int_s^{s+T} L_G^2 \|\bar{X}(\sigma) - \bar{X}(\sigma)\|^2 d\sigma + 228E \int_s^{s+T} \|G_\varepsilon(\sigma, \bar{X}(\sigma)) - \bar{G}(\bar{X}(\sigma))\|^2_{L^2(U,H)} d\sigma \\
\leq C_T \left(1 + E\|\varsigma_s\|^2\right) \frac{T(\delta)-1}{\delta} + 228E \sum_{k=0}^{T(\delta)-1} \int_{s+T(\delta)\delta}^{s+k\delta} \|G_\varepsilon(\sigma, \bar{X}(s+k\delta)) - \bar{G}(\bar{X}(s+k\delta))\|^2_{L^2(U,H)} d\sigma \\
+ 228E \int_s^{s+T} \|G_\varepsilon(\sigma, \bar{X}(s + T(\delta)\delta)) - \bar{G}(\bar{X}(s + T(\delta)\delta))\|^2_{L^2(U,H)} d\sigma \\
\leq C_T \left(1 + E\|\varsigma_s\|^2\right) \sum_{k=0}^{T(\delta)-1} \delta \omega_2 \left(\frac{\delta}{\varepsilon}\right) E \left(1 + \|\bar{X}(s + k\delta)\|^2\right) \\
+ CE \int_s^{s+T} \left(\|\bar{X}(s + T(\delta)\delta)\|^2 + 1\right) d\sigma \\
\leq C_T \left(1 + E\|\varsigma_s\|^2\right) \left(\frac{T}{\delta} + \delta + \omega_2 \left(\frac{\delta}{\varepsilon}\right)\right).
\]

Therefore, 

\[
I_2 \leq 76L_G^2 \int_s^{s+T} E \sup_{s \leq u \leq T} \|X_\varepsilon(u) - \bar{X}(u)\|^2 d\sigma + C_T \left(1 + E\|\varsigma_s\|^2\right) \left(\frac{T}{\delta} + \delta + \omega_2 \left(\frac{\delta}{\varepsilon}\right)\right).
\]
Combining (4.15), (4.22) and (4.25), we get
\begin{align}
E \sup_{s \leq t \leq s + T} \| X_\varepsilon(t) - \bar{X}(t) \|^2 \\
& \leq 2E\| \zeta^\varepsilon_s - \zeta_s \|^2 + (4L_F + 76L_G^2) \int_s^{s+T} E \sup_{s \leq u \leq \sigma} \| X_\varepsilon(u) - \bar{X}(u) \|^2 d\sigma \\
& \quad + C_T (1 + E\| \zeta_s \|^2) \left( \delta^{\frac{1}{4}} + \omega_1 \left( \frac{\delta}{\varepsilon} \right) + \omega_2 \left( \frac{\delta}{\varepsilon} \right) \right).
\end{align}

It follows from Gronwall’s lemma that
\begin{align}
E \sup_{s \leq t \leq s + T} \| X_\varepsilon(t) - \bar{X}(t) \|^2 \leq \left[ 2E\| \zeta^\varepsilon_s - \zeta_s \|^2 + C_T (1 + E\| \zeta_s \|^2) \left( \delta^{\frac{1}{4}} + \omega_1 \left( \frac{\delta}{\varepsilon} \right) \right) \right. \\
& \quad \left. + \omega_2 \left( \frac{\delta}{\varepsilon} \right) \right] \exp \{ (4L_F + 76L_G^2) T \},
\end{align}

which implies
\begin{align}
E \sup_{s \leq t \leq s + T} \| X_\varepsilon(t) - \bar{X}(t) \|^2 \leq C_T \left( 2E\| \zeta^\varepsilon_s - \zeta_s \|^2 + \varepsilon^{\frac{1}{4}} + \omega_1 \left( \frac{1}{\sqrt{\varepsilon}} \right) + \omega_2 \left( \frac{1}{\sqrt{\varepsilon}} \right) \right)
\end{align}

provided \( \delta = \sqrt{\varepsilon} \). Letting \( \varepsilon \to 0 \), we obtain
\[ \lim_{\varepsilon \to 0} E \sup_{s \leq t \leq s + T} \| X_\varepsilon(t) - \bar{X}(t) \|^2 = 0. \]

\[ \square \]

**Remark 4.6.**
(i) For simplicity, we take \( \delta = \sqrt{\varepsilon} \) in (4.27) to obtain (4.28), which gives a convergence rate for the first Bogolyubov theorem. If we take \( \delta = \psi(\varepsilon) \) satisfying \( \psi(\varepsilon) \to 0 \) and \( \frac{\psi(\varepsilon)}{\varepsilon} \to \infty \) as \( \varepsilon \to 0 \) such that \( \delta^{\frac{1}{4}} = \omega_1 \left( \frac{\delta}{\varepsilon} \right) = \omega_2 \left( \frac{\delta}{\varepsilon} \right) \), then we obtain a better convergence rate. But we are not sure if our method can give the optimal rate.

(ii) As mentioned in Introduction, there are three types of averaging principle and most existing works (except for [10, 24]) on stochastic averaging focus on the first Bogolyubov theorem. But to the best of our knowledge, the above result is new and hence interesting on its own rights. Meanwhile, it is helpful for us to establish the second Bogolyubov theorem and global averaging principle in what follows.

With the help of Theorem 4.5, we can now establish the second Bogolyubov theorem.

**Theorem 4.7.** Suppose that conditions (G1)–(G2), (H1), (H2′) and (H3)–(H6) hold. Assume further that \( 2\lambda - 2\lambda_F - L_G^2 \geq 0 \). If \( \lambda' > 0 \) or \( 2\lambda - 2\lambda_F - L_G^2 > 0 \) then for any \( 0 < \varepsilon \leq 1 \)
(i) equation (4.2) has a unique solution \( X_\varepsilon \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; H)) \);
(ii) the \( L^2 \)-bounded solution \( X_\varepsilon \) of (4.2) is strongly compatible in distribution, i.e. \( \mathcal{M}_{(F_\varepsilon,G_\varepsilon)} \subseteq \mathcal{M}_{X_\varepsilon} \), and
\[ \lim_{\varepsilon \to 0} d_{BL}(\mathcal{L}(X_\varepsilon), \mathcal{L}(\bar{X})) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)), \]
where \( \bar{X} \) is the unique stationary solution of averaged equation (4.3).

**Proof.** (i) follows from Theorem 3.6.

(ii) By Theorem 3.14 the bounded solution \( X_\varepsilon \) of equation (4.2) is strongly compatible in distribution, i.e. \( \mathcal{M}_{(F_\varepsilon,G_\varepsilon)} \subseteq \mathcal{M}_{X_\varepsilon} \), for any \( 0 < \varepsilon \leq 1 \).

Now we prove that \( \lim_{\varepsilon \to 0} d_{BL}(\mathcal{L}(X_\varepsilon(t)), \mathcal{L}(\bar{X}(t))) = 0 \) in \( Pr(H) \) for any \( t \in \mathbb{R} \). Take a sequence \( \{ \varepsilon_n \}_{n=1}^\infty \subseteq (0, 1] \) such that \( \varepsilon_n \to 0 \) as \( n \to \infty \). Similar to Proposition 3.12, we have
Therefore, we have $dY$ random variable (4.29) \[ \lim_{r \to \infty} \Psi^k_r(-r) \] Then we have solution of \[ \{L(Y_{\varepsilon_n}(t)) \}_{n=1}^{\infty} \] with initial value \[ \text{equation (4.3)) is unique, } L \] \[ \epsilon(0) \ni \in \mathbb{R} \] converges to \[ \|A\| \leq \frac{\|x\|^2}{\epsilon(0)} \|x\| \] \[ E \|X_r(t)\|_2^2 < \infty. \] It follows from Chebychev's inequality and the compact imbedding $S \subset H$ that $\{L(X_{\varepsilon_n}(t))\}_{n=1}^{\infty}$ is tight for all $t \in \mathbb{R}$. For every $r \geq 1$, according to the tightness of $\{L(X_{\varepsilon_n}(t))\}_{n=1}^{\infty}$, there exists a subsequence $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$ such that $L(X_{\varepsilon_{n_k}}(-r))$ weakly converges to $\mu_r$ in $Pr(H)$. Due to the Skorohod representation theorem, there exists a sequence of random variables $\Psi^k_r(-r)$ and $\zeta_r$ with laws of $L(X_{\varepsilon_{n_k}}(-r))$ and $\mu_r$, respectively, defined on another probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P})}$, such that $\Psi^k_r(-r) \to \zeta_r \bar{\mathbb{P}}$-a.s. It follows from (3.4) and Remark 4.3 that there exists $p > 1$ such that \[ \hat{E}\|\Psi^k_r(-r)\|_{2p}^2 = \int_H \|x\|^{2p} L(\Psi^k_r(-r))(dx) = \int_H \|x\|^{2p} L(X_{\varepsilon_{n_k}}(-r))(dx) < \infty. \] By the Vitali $L^p$ convergence criterion, we have $\lim_{k \to \infty} \hat{E}\|\Psi^k_r(-r) - \zeta_r\|^2 = 0.$

Let $\Psi^k_r$ be the solution of the following Cauchy problem \[ \begin{cases} dX(t) = \left( A(X(t)) + F_{\varepsilon_{n_k}}(t, X(t)) \right) dt + G_{\varepsilon_{n_k}}(t, X(t))dW(t) \\ X(-r) = \Psi^k_r(-r) \end{cases} \] and $\bar{Y}_r$ be the solution of the following Cauchy problem \[ \begin{cases} dX(t) = \left( A(X(t)) + \bar{F}(X(t)) \right) dt + \bar{G}(X(t))d\bar{W}(t) \\ X(-r) = \zeta_r, \end{cases} \] where $\bar{W}$ is a cylindrical Wiener process with the identity covariance operator on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P})}$. In view of Theorem 4.5, we get \[ \lim_{k \to \infty} \hat{E}\sup_{-r \leq t \leq t} \|\Psi^k_r(s) - \bar{Y}_r(s)\|^2 = 0 \] for any $t \geq -r$.

Let $\zeta_r$ be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $L(\zeta_r) = \mu_r$, and $Y_r$ be the solution of \[ dX(t) = \left( A(X(t)) + \bar{F}(X(t)) \right) dt + \bar{G}(X(t))dW(t) \] with initial value $Y_r(-r) = \zeta_r$. Since the law of the solutions for equation (4.2) (respectively, equation (4.3)) is unique, $L(\Psi^k_r) = L\left( X_{\varepsilon_{n_k}} \right)$ and $L(\bar{Y}_r) = L(Y_r)$ in $Pr(C([-r, +\infty), H))$. Then we have \[ (4.29) \lim_{k \to \infty} d_{BL}\left( L(X_{\varepsilon_{n_k}}), L(Y_r) \right) = 0 \quad \text{in } Pr(C([-r, +\infty), H)). \]

It follows from the tightness of $\left\{ L\left( X_{\varepsilon_{n_k}}(-r-1) \right) \right\}$ that there exists a subsequence $\{\varepsilon_{n_k}\} \subset \{\varepsilon_{n_k}\}$ such that $L\left( X_{\varepsilon_{n_k}}(-r-1) \right)$ weakly converges to $\mu_{r+1}$. We can find a random variable $\zeta_{r+1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $L(\zeta_{r+1}) = \mu_{r+1}$. Let $Y_{r+1}$ be the solution of \[ dX(t) = \left( A(X(t)) + \bar{F}(X(t)) \right) dt + \bar{G}(X(t))dW(t) \] with initial value $Y_{r+1}(-r-1) = \zeta_{r+1}$. Similar to the procedure of calculating (4.29), we get \[ \lim_{j \to \infty} d_{BL}\left( L(X_{\varepsilon_{n_k}}), L(Y_{r+1}) \right) = 0 \quad \text{in } Pr(C([-r-1, +\infty), H)). \]

Therefore, we have $d_{BL}(L(Y_r), L(Y_{r+1})) = 0$ in $Pr(C([-r, +\infty), H))$. In particular, $L(Y_r(t)) = L(Y_{r+1}(t))$ for all $t \geq -r$. 

Define \( \nu(t) := \mathcal{L}(Y_t(t)), t \geq -r. \) We can extract a subsequence which we still denote by \( \{X_{\varepsilon n_k j}\} \) satisfying \( \lim_{j \to \infty} d_{BL} \left( \mathcal{L}(X_{\varepsilon n_k j}(t)), \nu(t) \right) = 0 \) in \( Pr(H) \) for every \( t \in \mathbb{R} \). In view of Theorem 3.6, we obtain that \( \nu \) is the law of the \( L^2 \)-bounded solution of (4.3). Therefore we have

\[
\lim_{j \to \infty} d_{BL} \left( \mathcal{L}(X_{\varepsilon n_k j}(t)), \mathcal{L}(\bar{X}(t)) \right) = 0 \quad \text{in} \quad Pr(H)
\]

for every \( t \in \mathbb{R} \). By the arbitrariness of \( \{\varepsilon_n\}_{n=1}^{\infty} \subset (0, 1] \), we have

\[
\lim_{\varepsilon \to 0} d_{BL} \left( \mathcal{L}(X_{\varepsilon}(t)), \mathcal{L}(\bar{X}(t)) \right) = 0 \quad \text{in} \quad Pr(H).
\]

Now we show that \( \lim_{\varepsilon \to 0} d_{BL} \left( \mathcal{L}(X_{\varepsilon}), \mathcal{L}(\bar{X}) \right) = 0 \) in \( Pr(C(\mathbb{R}, H)) \). For any \([a, b] \subset \mathbb{R} \), we have \( \mathcal{L}(X_{\varepsilon}(a)) \) converges weakly to \( \mathcal{L}(X(a)) \) in \( Pr(H) \). In light of Skorohod representation theorem, there exist random variables \( \hat{\psi}_{\varepsilon}(a) \) and \( \hat{\psi}(a) \) defined on another probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) satisfying \( \lim_{\varepsilon \to 0} \hat{\psi}_{\varepsilon}(a) = \hat{\psi}(a) \hat{\mathbb{P}}\text{-a.s.}, \) where \( \mathcal{L} \left( \hat{\psi}_{\varepsilon}(a) \right) = \mathcal{L} \left( X_{\varepsilon}(a) \right) \) and \( \mathcal{L} \left( \hat{\psi}(a) \right) = \mathcal{L} \left( \bar{X}(a) \right) \). Similar to the procedure of calculating (4.29), we have

\[
\lim_{\varepsilon \to 0} d_{BL} \left( \mathcal{L}(X_{\varepsilon}), \mathcal{L}(\bar{X}) \right) = 0 \quad \text{in} \quad Pr(C([a, b], H)).
\]

The proof is complete. \( \square \)

**Corollary 4.8.** Under the conditions of Theorem 4.7 the following statements hold:

(i) If \( F \in C(\mathbb{R} \times H, H) \) and \( G \in C(\mathbb{R} \times H, L_2(U, H)) \) are jointly stationary (respectively, \( T \)-periodic, quasi-periodic with the spectrum of frequencies \( \nu_1, \ldots, \nu_k \), almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in \( t \) uniformly with respect to \( x \) on each bounded subset, then equation (4.2) has a unique solution \( X_{\varepsilon} \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; H)) \) which is stationary (respectively, \( T \)-periodic, quasi-periodic with the spectrum of frequencies \( \nu_1, \ldots, \nu_k \), almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in distribution;

(ii) If \( F \in C(\mathbb{R} \times H, H) \) and \( G \in C(\mathbb{R} \times H, L_2(U, H)) \) are Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent) in \( t \) uniformly with respect to \( x \) on each bounded subset, then equation (4.2) has a unique solution \( X_{\varepsilon} \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; H)) \) which is pseudo-periodic (respectively, pseudo-recurrent) in distribution;

(iii)

\[
\lim_{\varepsilon \to 0} d_{BL} \left( \mathcal{L}(X_{\varepsilon}), \mathcal{L}(\bar{X}) \right) = 0 \quad \text{in} \quad Pr(C(\mathbb{R}, H)),
\]

where \( \bar{X} \) is the unique stationary solution of averaged equation (4.3).

5. Global averaging principle in weak sense

We recall firstly that some known definitions and lemma in dynamical systems (see, e.g. [3, 8, 12, 27, 35] for more details). Let \((X, d_X)\) and \((P, d_P)\) be metric spaces.

**Definition 5.1.** A nonautonomous dynamical system \((\sigma, \varphi)\) (in short, \( \varphi \)) consists of two ingredients:

(i) A dynamical system \( \sigma \) on \( P \) with time set \( \mathbb{T} = \mathbb{Z} \) or \( \mathbb{R} \), i.e.

1. \( \sigma_0(\cdot) = Id_P, \)
2. \( \sigma_{t+s}(p) = \sigma_t(\sigma_s(p)) \) for all \( t, s \in \mathbb{T} \) and \( p \in P, \)
3. the mapping \( (t, p) \mapsto \sigma_t(p) \) is continuous.

If \( \mathbb{T} = \mathbb{R} \), \( \sigma \) is called flow on \( P \); if \( \mathbb{T} = \mathbb{R}^+, \sigma \) is called semiflow on \( P \).

(ii) A cocycle \( \varphi : \mathbb{T}^+ \times P \times X \rightarrow X \) satisfies

1. \( \varphi(0, p, x) = x \) for all \( (p, x) \in P \times X, \)
(2) $\varphi(t + s, p, x) = \varphi(t, \sigma_s(p), \varphi(s, p, x))$ for all $s, t \in \mathbb{T}^+$ and $(p, x) \in P \times X$, 

(3) the mapping $(t, p, x) \mapsto \varphi(t, p, x)$ is continuous.

$P$ is called the base or parameter space and $X$ is the fiber or state space. For convenience, we also write $\sigma_t(p)$ as $\sigma tp$.

**Definition 5.2.** Let $(\sigma, \varphi)$ be a nonautonomous dynamical system with base space $P$ and state space $X$. The skew product semiflow $\Pi : \mathbb{T}^+ \times P \times X \to P \times X$ is a semiflow of the form:

$$
\Pi(t, (p, x)) := (\sigma tp, \varphi(t, p, x)).
$$

**Definition 5.3.** Define $\mathcal{X} := P \times X$. A nonempty compact subset $\mathcal{A}$ of $\mathcal{X}$ is called global attractor for skew product semiflow $\Pi$, if

(i) $\Pi(t, \mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{T}^+$,

(ii) $\lim_{t \to +\infty} \operatorname{dist}_X(\Pi(t, D), \mathcal{A}) = 0$ for every nonempty bounded subset $D$ of $\mathcal{X}$,

where $\operatorname{dist}_X(A, B)$ is the Hausdorff semi-metric between sets $A$ and $B$, i.e. $\operatorname{dist}_X(A, B) := \sup_{x \in A} d_x(A, B) = \inf_{y \in B} d_x(x, y)$. Here $d_x((p_1, x_1), (p_2, x_2)) = d_P(p_1, p_2) + d_X(x_1, x_2)$ for all $(p_1, x_1), (p_2, x_2) \in P \times X$.

**Lemma 5.4** (see, e.g. [12]). Let $\{S(t)\}_{t \geq 0}$ be a semiflow in a complete metric space $\mathcal{X}$ having a compact attracting set $K \subset \mathcal{X}$, i.e.

$$
\lim_{t \to +\infty} \operatorname{dist}_X(S(t)B, K) = 0
$$

for all bounded set $B \subset \mathcal{X}$. Then $\{S(t)\}_{t \geq 0}$ has a global attractor $\mathcal{A} := \omega(K)$. Where $\omega(K)$ is the $\omega$-limit set of $K$, i.e. $\omega(K) := \cap_{t \geq 0} U_{s \geq t} S(s)K$.

**Definition 5.5.** A family $D := \{D_p : p \in P\}$ of subsets of $X$ is called a non-autonomous set. If every fiber $D_p$ is compact, then $D := \{D_p : p \in P\}$ is called non-autonomous compact set.

**Definition 5.6** (see, e.g. [12]). A compact set $A \subset X$ is called the uniform attractor (with respect to $p \in P$) of cocycle $\varphi$ if the following conditions are fulfilled:

(i) The set $A$ is uniformly attracting, i.e.

$$
\lim_{t \to +\infty} \sup_{p \in P} \operatorname{dist}_X(\varphi(t, p, B), A) = 0
$$

for every bounded set $B \subset X$.

(ii) If $A_1$ is another closed uniformly attracting set, then $A \subset A_1$.

**Remark 5.7.** It follows from Definition 5.6 (ii) that the uniform attractor is unique.

Denote by $F := (F, G) \in \operatorname{BUC}(\mathbb{R} \times H, H) \times \operatorname{BUC}(\mathbb{R} \times H, L_2(U, H))$. Recall that $F^\tau(t, x) = F(t + \tau, x)$ for all $(t, x) \in \mathbb{R} \times H$.

$$
H(F) = \{F^\tau = (F^\tau, G^\tau) : \tau \in \mathbb{R}\} \subset \operatorname{BUC}(\mathbb{R} \times H, H) \times \operatorname{BUC}(\mathbb{R} \times H, L_2(U, H)),
$$

and $(H(F), \sigma)$ is a shift dynamical system. Here $\sigma : \mathbb{R} \times H(F) \to H(F)$, $(\tau, F) \mapsto F^\tau$.

Let $X(t, s, x, t \geq s$ be the solution of equation

$$
\frac{dX(t)}{dt} = (A(X(t)) + F(t, X(t))) \, dt + G(t, X(t)) \, dW(t)
$$

with initial condition $X(s, s, x) = x$. Define $P_t(s, x, t, dy) := \mathbb{P} \circ (X(t, s, x))^{-1} (dy)$. Then we can associate a mapping $P^*(t, F, \cdot) : \mathbb{P} \to \mathbb{P}$ defined by

$$
P^*(t, F, \mu)(B) := \int_{H} P_t(0, x, t, B) \mu(dx)
$$

for all $\mu \in \mathbb{P}$ and $B \in \mathcal{B}(H)$. We write $\mathbb{P}_2(H)$ to mean the space of probability measures $\mu \in \mathbb{P}$ such that $\int_H \|z\|^2 \mu(\text{d}z) < \infty$. We say that $B \subset \mathbb{P}_2(H)$ is bounded if
there exists a constant \( r > 0 \) such that \( \int_H \|z\|^2 \mu(\text{d}z) \leq r^2 \) for all \( \mu \in B \). In the following, we define
\[
B_r := \left\{ \mu \in Pr_2(H) : \int_H \|z\|^2 \mu(\text{d}z) \leq r^2 \right\}
\]
and
\[
\mathcal{O}_\rho(B) := \{ \mu \in Pr_2(H) : d(\mu, B) < \rho \}
\]
for all \( r, \rho > 0 \), where \( d(\mu, B) := \inf_{\nu \in B} d_{BL}(\mu, \nu) \).

**Lemma 5.8.** Consider equation (5.1). Assume that conditions (H1), (H2') and (H3)–(H5) hold. Then \( P^* \) is a cocycle on \( (H(\mathbb{F}), \mathbb{R}, \sigma) \) with fiber \( Pr_2(H) \).

**Proof.** It follows from Lemma 3.13 that \( P^* \) is a continuous mapping from \( \mathbb{R}^+ \times H(\mathbb{F}) \times Pr_2(H) \) into \( Pr_2(H) \). For any \( \mu \in Pr_2(H) \), \( t, \tau \in \mathbb{R}^+ \) and \( \tilde{F} \in H(\mathbb{F}) \), according to the uniqueness in law of the solutions for equation (5.1), we have
\[
P^*(t + \tau, \tilde{F}, \mu) = P^* \left( t, \sigma_\tau \tilde{F}, P^*(\tau, \tilde{F}, \mu) \right).
\]
And by the definition of \( P^* \) we have \( P^*(0, \tilde{F}, \cdot) = Id_{Pr_2(H)} \) for all \( \tilde{F} \in H(\mathbb{F}) \).

**Corollary 5.9.** Under conditions of Lemma 5.8, the mapping given by
\[
\Pi : \mathbb{R}^+ \times H(\mathbb{F}) \times Pr_2(H) \to H(\mathbb{F}) \times Pr_2(H), \\
\Pi(t, (\tilde{F}, \mu)) := \left( \sigma_t \tilde{F}, P^*(t, \tilde{F}, \mu) \right)
\]
is a continuous skew-product semiflow.

For any given \( \tilde{F} \in H(\mathbb{F}) \), suppose that (H1), (H2'), (H3)–(H4) hold and \( 2\lambda - 2\lambda_F - L_G^2 \geq 0 \). If \( \lambda' > 0 \) or \( 2\lambda - 2\lambda_F - L_G^2 > 0 \), then equation (5.1) has a unique \( L^2 \)-bounded solution \( X_{\tilde{F}} \) with the distribution \( L(X_{\tilde{F}}(t)) =: \mu_{\tilde{F}}(t), t \in \mathbb{R} \). In the following, we denote by \( \mathfrak{X} := H(\mathbb{F}) \times Pr_2(H) \).

**Lemma 5.10.** Let
\[
\bar{F}(x) := \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} F(s, x) \text{d}s \
\text{and} \\
\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \|G(s, x) - \bar{G}(x)\|_{L_2(U,H)}^2 \text{d}s = 0
\]
uniformly with respect to \( t \in \mathbb{R} \). Assume that \( F \) and \( G \) satisfy (G1)–(G2). If \( H(\mathbb{F}) \) is compact, then for any \( \tilde{F} = (\tilde{F}, \tilde{G}) \in H(\mathbb{F}) \) we have
\[
\frac{1}{T} \int_t^{t+T} \left\| \tilde{F}(s, x) - \bar{F}(x) \right\| \text{d}s \leq \omega_1(T)(1 + \|x\|)
\]
and
\[
\frac{1}{T} \int_t^{t+T} \|\tilde{G}(s, x) - \bar{G}(x)\|_{L_2(U,H)}^2 \text{d}s \leq \omega_2(T)(1 + \|x\|^2)
\]
for all \( T > 0, x \in H \) and \( t \in \mathbb{R} \).

**Proof.** Given \( \tilde{F} \in H(F) \), there exists \( \{t_n\} \subset \mathbb{R} \) such that
\[
\lim_{n \to \infty} \sup_{|t| \leq l, \|x\| \leq r} \|\tilde{F}(t, x) - F(t + t_n, x)\| = 0
\]
for all \( l, r > 0 \). Then we have
\[
\frac{1}{T} \int_t^{t+T} \left\| \tilde{F}(s, x) - \bar{F}(x) \right\| \text{d}s
\]
\[
\leq \frac{1}{T} \int_t^{t+T} \left\| \tilde{F}(s, x) - F(s + t_n, x) \right\| \text{d}s + \frac{1}{T} \int_t^{t+T} \|F(s + t_n, x) - \bar{F}(x)\| \text{d}s
\]
\[
\leq \frac{1}{T} \int_t^{t+T} \left\| \tilde{F}(s, x) - F(s + t_n, x) \right\| \text{d}s + \omega_1(T)(1 + \|x\|).
\]
Letting \( n \to \infty \) in (5.4), by Lebesgue dominated convergence theorem, we get

\[
\frac{1}{T} \left\| \int_t^{t+T} \left( \tilde{F}(s,x) - \tilde{F}(x) \right) \, ds \right\| \leq \omega_1(T)(1 + \|x\|).
\]

The proof of (5.3) is similar. \( \square \)

**Remark 5.11.** It follows from Remark 3.10 and Lemma 5.10 that estimates (3.4), (3.7), (3.9), (3.17), (3.18) and (4.28) hold uniformly for all \( \tilde{F} \in H(\mathcal{F}) \) and \( \varepsilon \in (0,1) \).

**Proposition 5.12.** Consider equation (5.1). Assume that conditions (H1), (H2'), (H3)–(H6) hold, and \( 2\lambda - 2\lambda_F - L^2_G \geq 0 \). Suppose further that \( \lambda' > 0 \) or \( 2\lambda - 2\lambda_F - L^2_G > 0 \). Then we have the following results.

(i) Define \( \mathfrak{A}_{\tilde{F}} := \{ \mu_{\tilde{F}}(t) \in Pr_2(H) : t \in \mathbb{R} \} \). Then

\[
P^*(t,\tilde{F},\mathfrak{A}_{\tilde{F}}) = \mathfrak{A}_{\sigma t \tilde{F}}
\]

for all \( t \in \mathbb{R}^+ \) and \( \tilde{F} \in H(\mathcal{F}) \).

(ii) If \( H(\mathcal{F}) \) is compact, then the skew product semiflow \( \Pi \) admits a global attractor \( \mathfrak{A} := \omega \left( H(\mathcal{F}) \times \bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \right) \). Moreover, \( \Pi_2 \mathfrak{A} \) is the uniform attractor of cocycle \( P^* \).

Here \( \Pi_2(\tilde{F},\mu) := \mu \) for all \( (\tilde{F},\mu) \in H(\mathcal{F}) \times Pr_2(H) \).

**Proof.** (i) Given \( t \in \mathbb{R}^+ \) and \( \tilde{F} \in H(\mathcal{F}) \), let \( X_{\sigma t \tilde{F}} \) be the unique \( L^2 \)-bounded solution of equation

\[
dX(s) = \left( A(X(s)) + \tilde{F}(s+t,X(s)) \right) \, ds + \hat{G}(s+t,X(s)) \, dW(s).
\]

Note that \( \mathcal{L}(X_{\tilde{F}}(s+t)) = \mathcal{L}(X_{\sigma t \tilde{F}}(s)) \) for all \( s \in \mathbb{R} \). Consequently, \( P^*(t,\tilde{F},\mathfrak{A}_{\tilde{F}}) = \mathfrak{A}_{\sigma t \tilde{F}} \).

(ii) As mentioned in Remark 5.11, (3.18) in Proposition 3.12 holds uniformly for all \( \tilde{F} \in H(\mathcal{F}) \). Namely,

\[
\sup_{\tilde{F} \in H(\mathcal{F})} \sup_{t \in \mathbb{R}} \int_S \|z\|^2 \mu_{\tilde{F}}(t)(dz) < \infty.
\]

Then there exists a constant \( R > 0 \) such that

\[
\bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \subset \left\{ \mu \in Pr_2(H) : \int_S \|z\|^2 \mu(dz) < R^2 \right\}.
\]

According to the Chebychev’s inequality and the compactness of the inclusion \( S \subset H \), \( \bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \) is compact in \( Pr(H) \).

Let \( r > 0 \) be an arbitrary constant. For any \( \mu \in B_r \), take a random variable \( \xi \) such that \( \mathcal{L}(\xi) = \mu \). Let \( Y(t,\xi), t \geq 0 \) be the solution to

\[
Y(t,\xi) = \xi + \int_0^t \left( A(Y(s,\xi)) + \tilde{F}(s,Y(s,\xi)) \right) \, ds + \int_0^t \hat{G}(s,Y(s,\xi)) \, dW(s).
\]

In view of Theorem 3.9, we have

\[
E\|Y(t,\xi) - X_{\tilde{F}}(t)\|^2 \leq \begin{cases} E\|\xi - X_{\tilde{F}}(s)\|^2_H \land \{ \lambda'(r-2)(t-s) \}^{-\frac{1}{2}}, & \text{if } \lambda' > 0 \\ e^{-(2\lambda - 2\lambda_F - L^2_G)(t-s)}E\|\xi - X_{\tilde{F}}(s)\|^2_H, & \text{if } 2\lambda - 2\lambda_F - L^2_G > 0. \end{cases}
\]

Therefore, \( \lim_{t \to +\infty} \sup_{\tilde{F} \in H(\mathcal{F})} \left( P^*(t,\tilde{F},\mu), \bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \right) = 0 \) uniformly with respect to \( \mu \in B_r \), i.e. \( \bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \) is a compact uniformly attracting set. Obviously, \( H(\mathcal{F}) \times \bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \) is a compact attracting set for \( \Pi \). By Lemma 5.4, \( \Pi \) admits a global attractor \( \mathfrak{A} := \omega \left( H(\mathcal{F}) \times \bigcup_{\tilde{F} \in H(\mathcal{F})} \mathfrak{A}_{\tilde{F}} \right) \).
Let us now prove that \( \Pi_2\mathfrak{A} \) is the uniform attractor of cocycle \( P^* \). Let \( B \subset \Pr_2(H) \) be bounded, then \( H(\mathbb{F}) \times B \) is bounded in \( H(\mathbb{F}) \times \Pr_2(H) \). Therefore,

\[
\text{dist}_{\Pr_2(H)}(P^*(t, \hat{\mathbb{F}}, B), \Pi_2\mathfrak{A}) \leq \text{dist}_X(H(\mathbb{F}) \times P^*(t, \hat{\mathbb{F}}, B), \mathfrak{A}) = \text{dist}_X(\Pi(t, H(\mathbb{F}) \times B), \mathfrak{A}) \rightarrow 0 \text{ as } t \rightarrow +\infty.
\]

Next we verify the minimality property. Denote by \( \omega \) bounded, then \( \omega \) equation (4.3).

\[
(5.8)
\]

\[
(5.9)
\]

for all \( \bar{\lambda} \) and \( \mu \). Then it follows from Proposition 5.12 that \( \mu \) is attractor, we can choose \( \bar{\lambda} \) as special cases.

Theorem 5.14. It is known that \( H(\mathbb{F}) \) is compact provided \( \mathbb{F} \) is Birkhoff recurrent, which includes periodic, quasi-periodic, almost periodic, almost automorphic as special cases.

Next we prove the global averaging principle for strongly monotone SPDEs.

**Theorem 5.14.** Suppose that \( 2\lambda - 2\lambda F - L^2_G \geq 0 \), (G1)–(G2), (H1), (H2') and (H3)–(H6) hold. Assume further that \( \lambda > 0 \) or \( 2\lambda - 2\lambda F - L^2_G > 0 \). If \( H(\mathbb{F}) \) is compact, then

(i) the cocycle \( P^* \) associated with SPDE (4.2) has a uniform attractor \( \mathcal{A}^\varepsilon \) for any \( 0 < \varepsilon \leq 1 \);

(ii) the cocycle \( \hat{P}^* \) associated with SPDE (4.3) has a uniform attractor \( \hat{A} \), which is a singleton set;

(iii) for arbitrary large \( R_1 \) and small \( \rho > 0 \) there exist \( \varepsilon_0 = \varepsilon_0(R_1, \rho) \) and \( T = T(R_1, \rho) \) such that for all \( \varepsilon \leq \varepsilon_0 \), \( t \geq T \) and \( \mathbb{F} \in H(\mathbb{F}) \)

\[
(5.5)
\]

\[
(5.6)
\]

In particular,

\[
(5.6)
\]

\[
\lim_{\varepsilon \rightarrow 0} \text{dist}_{\Pr_2(H)}(\mathcal{A}^\varepsilon, \hat{A}) = 0.
\]

**Proof.** (i)–(ii) It follows from Proposition 5.12 that \( P^*_\varepsilon \) and \( \hat{P}^* \) admit uniform attractors, and \( \mathcal{A} = \{ \mathcal{L}(X_0(0)) \} \in \Pr_2(H) \). Here \( X(t), t \in \mathbb{R} \) is the unique stationary solution to averaged equation (4.3).

(iii) It follows from Theorem 3.9 that there exists \( \delta > 0 \) and \( \varepsilon_0 = \varepsilon_0(R_1, \rho) \) such that

\[
(5.7)
\]

for all \( t \geq 0 \). Fix \( R_1 \) large enough. In view of (3.4), there exists \( T_0 > 0 \) such that

\[
(5.8)
\]

for all \( t \geq T_0 \). Since \( \hat{A} \) is attractor, we can choose \( T_1 = T_1(R_1, \rho) \) so large such that

\[
(5.9)
\]
for all $t \geq T_1$. Denote by $T := \max\{T_0, T_1\}$. Employing (4.28), we have
\begin{equation}
\sup_{0 \leq t \leq T} d\left( P^*_\varepsilon(t, \tilde{F}, \mu), \tilde{P}^*_\varepsilon(t, \mu) \right) < \eta(T, R_1)(\varepsilon)
\end{equation}
for all $\mu \in B_{R_1}$ and $\tilde{F} \in H(\mathbb{F})$, where $\eta(T, R_1)(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, there exists $\varepsilon_0 = \varepsilon_0(T, R_1)$ such that $\eta(T, R_1)(\varepsilon) < \frac{\delta}{2}$ for all $\varepsilon \leq \varepsilon_0$.

For any $\mu \in B_{R_1}$, in view of (5.8)–(5.10), we have
\begin{equation}
P^*_\varepsilon(T, \tilde{F}, \mu) \in \mathcal{O}_\delta(\tilde{A}) \cap B_{R_1}
\end{equation}
for all $\varepsilon \leq \varepsilon_0$. It can be verified that $P^*_\varepsilon(t, \tilde{F}, \mu) \in \mathcal{O}_\rho(\tilde{A})$ for all $t \geq T$ and $\varepsilon \leq \varepsilon_0$. To this end, define $\mu^*_1 := P^*_\varepsilon(T, \tilde{F}, \mu)$. Then $\tilde{P}^*_\varepsilon(t, \mu^*_1) \in \mathcal{O}_\frac{\delta}{2}(\tilde{A})$ and $P^*_\varepsilon(t + T, \tilde{F}, \mu) = P^*_\varepsilon(t, \sigma_T \tilde{F}, \mu_1)$ for all $t \geq 0$. Therefore, according to (5.9)–(5.10), we get
\begin{equation}
P^*_\varepsilon(2T, \tilde{F}, \mu) \in \mathcal{O}_\delta(\tilde{A}) \cap B_{R_1}
\end{equation}
and
\begin{equation}
P^*_\varepsilon(t + T, \tilde{F}, \mu) \in \mathcal{O}_{\frac{\delta}{2} + \frac{\varepsilon}{2}}(\tilde{A}) \subset \mathcal{O}_\rho(\tilde{A})
\end{equation}
for all $t \in [0, T]$. Repeating the above procedure, we have
\begin{equation}
P^*_\varepsilon(t, \tilde{F}, \mu) \in \mathcal{O}_\rho(\tilde{A})
\end{equation}
for all $t \geq T$ and $\varepsilon \leq \varepsilon_0$.

Take $R_1$ large enough so that $\mathcal{A}^c \subset B_{R_1}$, then (5.6) follows from (5.5) and Definition 5.6. \hfill \Box

6. Applications

In this section, we illustrate our theoretical results by two examples. We mainly consider the additive or linear multiplicative noise in these examples for brevity. Let $\Lambda \subset \mathbb{R}^n, n \in \mathbb{N}$ be an open bounded subset and $0 < \varepsilon \leq 1$. Denote by $f^+(t) := \max\{f(t), 0\}$ for all $t \in \mathbb{R}$ and $\lambda_*$ the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition.

6.1. Stochastic reaction diffusion equations. Consider the equation
\begin{equation}
du = (\Delta u - au|u|^{p-2} + \phi(t/\varepsilon)u + g(t/\varepsilon)) \, dt + \kappa dW(t),
\end{equation}
where $W(\cdot)$ is a two-sided standard real-valued Wiener process, $p \in [2, +\infty)$ and $g \in C_b(\mathbb{R}, \mathcal{H}_0^{1,2}(\Lambda))$. Here $a > 0$ and $\kappa \in \mathbb{R}$ are constants. We define $V_1 := H_0^{1,2}(\Lambda), V_2 := L^p(\Lambda), H := L^2(\Lambda), V := V_1 \cap V_2$ and
\begin{align*}
A_1(u) & := \Delta u, \quad A_2(u) := -au |u|^{p-2}, \quad F(t,u) := \phi(t)u + g(t), \quad G(t,u) = \kappa u.
\end{align*}
Assume that $\lambda_* - |\phi^+|_{\infty} - \frac{\kappa^2}{2} > 0$, then we have the following theorem.

**Theorem 6.1.**

1. There exists a unique $L^2$-bounded solution $X_\varepsilon(\cdot)$ to equation (6.1) which is globally asymptotically stable in square-mean sense for any $0 < \varepsilon \leq 1$.

2. If $\phi$ is almost automorphic and $g$ is almost periodic, then the $L^2$-bounded solution $X_\varepsilon(\cdot)$ is almost automorphic in distribution.

3. Let $\bar{X}$ be the unique stationary solution of the following averaged equation
\begin{equation}
du = (\Delta u - a|u|^{p-2}u + \tilde{\phi}u + \tilde{g}) \, dt + \kappa dW(t),
\end{equation}
where $\tilde{\phi} = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \phi(s) \, ds$ and $\tilde{g} = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} g(s) \, ds$ uniformly for all $t \in \mathbb{R}$.

Then
\begin{equation}
\lim_{\varepsilon \to 0} d_{BL}(\mathcal{L}(X_\varepsilon), \mathcal{L}(\bar{X})) = 0 \quad \text{in } Pr(C(\mathbb{R}, L^2(\Lambda))).
\end{equation}
(4) The cocycle \( P^*_\xi \) generated by equation (6.1) has a uniform attractor \( \mathcal{A}^\varepsilon \), and
\[
\lim_{\varepsilon \to 0} \text{dist}_{P^*_{\varepsilon}(H)} (\mathcal{A}^\varepsilon, \mathcal{A}) = 0.
\]

Here \( \mathcal{A} := \mathcal{L}(\mathcal{X}(0)) \) is the attractor for \( \bar{P}^* \) and \( H := L^2(\Lambda) \).

Proof. (1)–(2) It suffices to show that conditions of Theorem 3.9 and Corollary 3.15 hold.

(H1) \( A_1 \) is obviously hemi-continuous. We now prove that \( A_2 \) is hemi-continuous. Let \( u, v, w \in V \). For \( \theta \in \mathbb{R} \), without loss of generality, we assume \( |\theta| \leq 1 \), then we have
\[
\begin{align*}
&\langle v^*_2 \Delta (u + \theta v) - A_2(u), w \rangle_{V^2} \\
&\quad = \int_{\Lambda} \bigg( - (u(\xi) + \theta v(\xi)) |u(\xi) + \theta v(\xi)|^{p-2} w(\xi) + u(\xi)|u(\xi)|^{p-2} w(\xi) \bigg) \, d\xi \\
&\quad \leq \int_{\Lambda} \bigg( 4 |u(\xi)|^{p-1} + |v(\xi)|^{p-1} \bigg) |w(\xi)| + |u(\xi)|^{p-1} |w(\xi)| \, d\xi < \infty.
\end{align*}
\]
The last inequality holds since \( u, v, w \in L^p(\Lambda) \). Then \( v^*_2 \Delta (u + \theta v) - A_2(u), w \rangle_{V^2} \) converges to zero as \( \theta \to 0 \) by Lebesgue’s dominated convergence theorem. So, (H1) holds.

(H2') For all \( u, v, w \in V \) and \( t \in \mathbb{R} \)
\[
\begin{align*}
&\langle v^*_2 A_1(u) - A_1(v), (u - v) \rangle_{V_1} \leq -\lambda_* \|u - v\|_{H}^2, \\
&\langle v^*_2 A_2(u) - A_2(v), (u - v) \rangle_{V_2} = -a \int_{\Lambda} \bigg( u(\xi)|u(\xi)|^{p-2} - v(\xi)|v(\xi)|^{p-2} \bigg) (u(\xi) - v(\xi)) \, d\xi \leq 0,
\end{align*}
\]
\[
\langle F(t, u) - F(t, v), u - v \rangle \leq |\phi^+|_{\infty} \|u - v\|_{H}^2, \quad \|F(t, 0)\|_{H} \leq \sup_{t \in \mathbb{R}} \|g(t)\|_{H^{0,2}(\Lambda)},
\]
\[
\|F(t, u) - F(t, v)\|_{H} \leq |\phi|_{\infty} \|u - v\|_{H} \quad \text{and} \quad \|\kappa u - \kappa v\|_{H}^2 \leq \kappa^2 \|u - v\|_{H}^2.
\]
So (H2') holds with \( \lambda = \lambda_* \), \( \lambda = 0 \), \( \lambda_F = |\phi^+|_{\infty} \), \( L_F = |\phi|_{\infty} \) and \( L_G = |\kappa| \).

(H3) For all \( u, v \in V \), \( t \in \mathbb{R} \) we have
\[
\begin{align*}
&\langle v^*_1 A_1(u), (u - v) \rangle_{V_1} = \int_{\Lambda} |\nabla v(\xi)|^2 \, d\xi = \|v\|_{H}^2 - \|v\|_{V_1}^2, \\
&\langle v^*_2 A_2(u), (u - v) \rangle_{V_2} = -a \int_{\Lambda} |v(\xi)|^p \, d\xi = -a \|v\|_{V_2}^p.
\end{align*}
\]
Then (H3) holds with \( \alpha_1 = 2 \), \( \alpha_2 = p \).

(H4) For all \( u, v \in V \), \( t \in \mathbb{R} \) we have
\[
\begin{align*}
\| v^*_1 A_1(u), (u - v) \|_{V_1} &\leq \|\nabla u\|_{H} \|\nabla v\|_{H} \leq \|u\|_{V_1} \|v\|_{V_1}, \\
\| v^*_2 A_2(u), (u - v) \|_{V_2} &\leq a \int_{\Lambda} |u(\xi)|^p |u(\xi)|^{p-2} v(\xi) |d\xi| \leq a \|u\|_{V_2}^{p-1} \|v\|_{V_2}.
\end{align*}
\]
Therefore, we get \( \|A_1(u)\|_{V_1} \leq \|u\|_{V_1} \) and \( \|A_2(u)\|_{V_2} \leq a \|u\|_{V_2}^{p-1} \).

(2) In order to prove the almost automorphic property of the \( L^2 \)-bounded solution, it suffices to show that \( (H5)-(H6) \) holds. To this end, let \( S := H^{0,2}_0(\Lambda) \), we define \( T_n = -\Delta \left( I - \frac{\Delta}{n} \right)^{-1} = \kappa(I - (I - \frac{\Delta}{n})^{-1}) \). Note that \( T_n \) are continuous on \( W^{0,2}_0(\Lambda) \). Since the heat semigroup \( \{P_t\}_{t \geq 0} \) (generated by \( \Delta \)) is contractive on \( L^p(\Lambda), p > 1 \) and \( (I - \frac{\Delta}{n})^{-1} u = \int_0^\infty e^{-t} P_n u \, dt \), \( T_n \) are continuous on \( L^p(\Lambda) \).

For all \( u \in \mathbb{R} \) we have
\[
\begin{align*}
\langle v^*_1 \Delta (u), T_n u \rangle_{V_1} &\leq -\lambda_* \|u\|_{n}^2 \quad \text{and} \quad \phi(t) \langle u, T_n u \rangle_{H} = \phi(t) \|u\|_{n}^2 \leq |\phi^+|_{\infty} \|u\|_{n}^2,
\end{align*}
\]
In view of the contractivity of \( \{P_t\}_{t \geq 0} \) on \( L^p(\Lambda) \), we have
\[
\begin{align*}
\langle v^*_2 A_2(u), T_n u \rangle_{V_2} &\leq -a \|u\|^{p-2} u, nu - n \left( I - \frac{\Delta}{n} \right)^{n-1} u.
\end{align*}
\]
Young's inequality, we get

\[ n \int_0^\infty e^{-t} \left( \int_\Lambda -au(\xi)|u(\xi)|^{p-2} \left( u(\xi) - P_{\frac{1}{n}} u(\xi) \right) d\xi \right) dt \leq 0. \]

Then we obtain

\[ 2V\ast (A(t,u), T_n u)_V + 2 \langle F(t,u), T_n u \rangle \leq -2 \left( \lambda_a - |\phi^+|_\infty - \varepsilon \right) \|u\|_{n-1}^2 + C \sup_{t \in \mathbb{R}} \|g(t)\|_{L^2}^2. \]

That is, (H6) holds. And (H5) is obvious.

Remark 6.2. (i) We mention that the first Bogolyubov theorem was also studied for reaction-diffusion equations with polynomial nonlinearities by Cerrai [5] and Gao [19].

(ii) Note that equation (6.1) is the real Ginzburg-Landau equation when \( p = 4 \).

6.2. Stochastic generalized porous media equations. Consider the equation

\[ \dot{u} = \left( \Delta (|u|^{p-2} u + au) + \phi \left( \frac{t}{\varepsilon} \right) u \right) dt + GdW(t), \]

where \( W(\cdot) \) is a two-sided cylindrical \( Q \)-Wiener process with \( Q = I \) on \( L^p(\Lambda) \), \( p > 2 \) and \( a \geq 0 \), \( G \in L_2(L^p(\Lambda)) \). And there exists a constant \( C_1 > 0 \) such that \( \phi(t) \leq -C_1 \) for all \( t \in \mathbb{R} \). We define

\[ V := L^p(\Lambda) \subset H := W_0^{-1,2}(\Lambda) \subset V^*. \]

Theorem 6.3. (1) There exists a unique \( L^2 \)-bounded solution \( X_\varepsilon(\cdot) \) to equation (6.4), which is globally asymptotically stable in square-mean sense.

(2) The \( L^2 \)-bounded solution \( X_\varepsilon(\cdot) \) is almost periodic in distribution provided \( \phi \) is almost periodic.

(3) Let \( X \) be the unique stationary solution of averaged equation

\[ \dot{u} = \left( -\Delta (|u|^{p-2} u + au) + \bar{\phi} u \right) dt + GdW(t), \]

where \( \bar{\phi} := \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(s) ds \) uniformly for all \( t \in \mathbb{R} \). Then

\[ \lim_{\varepsilon \to 0} d_{BL}(\mathcal{L}(X_\varepsilon), \mathcal{L}(\bar{X})) = 0 \quad \text{in} \quad Pr(C(\mathbb{R}^1, H)). \]

(4) The cocycle \( P_\varepsilon^t \) generated by equation (6.4) has a uniform attractor \( \mathcal{A}^\varepsilon \), and

\[ \lim_{\varepsilon \to 0} \text{dist}_{Pr^2(H)}(\mathcal{A}^\varepsilon, \mathcal{A}) = 0. \]

Here \( \mathcal{A} := \mathcal{L}(\bar{X}(0)) \) is the attractor for \( P^t \).

Proof. Let \( A(u) := \Delta (|u|^{p-2} u + au), F(t,u) := \phi(t)u \) for all \( u \in V \) and \( t \in \mathbb{R} \). Fix \( u \in V \), for all \( v \in V \) we denote

\[ \langle V \ast (A(u), v)_V := - \int_\Lambda u(\xi)|u(\xi)|^{p-2}v(\xi)d\xi - a \int_\Lambda u(\xi)v(\xi)d\xi \rangle. \]

We first show that \( A : V \to V^* \) is well-defined. Indeed, employing Hölder’s inequality and Young’s inequality, we get

\[ |V \ast (A(u), v)_V| \leq \|u\|_{L^p}^{p-1} \|v\|_{L^p} + a \left( C_p \|u\|_{L^p}^{p-1} + C_p (|\Lambda|)^{\frac{p-1}{p}} \right) \|v\|_{L^p}, \]

for all \( u, v \in V \), where \( C_p \) is a constant depending only on \( p \). Therefore, \( A : V \to V^* \) is well-defined and

\[ \|A(u)\|_{V^*} \leq (1 + aC_p) \|u\|_{L^p}^{p-1} + C_p (|\Lambda|)^{\frac{p-1}{p}} a. \]

Similar to the proof of Theorem 6.1, it suffices to show that (H1), (H2\') and (H3)–(H6) hold. Note that (H1), (H5) hold and \( \lambda_F = 0, L_F = |\phi|_\infty \) in (H2\').
(H2') For all \( u, v \in V \), \( t \in \mathbb{R} \) we have
\[
\langle \partial_t A(u) - A(v), u - v \rangle_V = -\langle u|u|^{p-2} - v|v|^{p-2}, u - v \rangle_{L^2} - a\|u - v\|_{L^2}^2
\]
\[
\leq -2^{2-p}\|u - v\|_{L^2}^p - a\|u - v\|_{L^2}^2
\]
\[
\leq -2^{2-p}\|u - v\|_{H^1}^p - a\|u - v\|_{H^1}^2.
\]
Therefore, (H2) holds with \( r = p \), \( \lambda' = 2^{2-p} \) and \( \lambda = a \).

(H3) Note that for all \( u \in V \), \( t \in \mathbb{R} \)
\[
\langle \partial_t A(u), u \rangle_V = -\int \Lambda u(\xi) |u(\xi)|^{p-2} u(\xi) d\xi - a\int \Lambda u(\xi) u(\xi) d\xi \leq -\|u\|_V^p.
\]
That is, (H3) holds with \( \alpha = p \).

(H4) holds by (6.6) with \( \alpha = p \).

(H6) Let \( S = L^2(\Lambda) \) and \( \Delta \) be the Laplace operator on \( L^2(\Lambda) \) with the Dirichlet boundary condition. Define \( T_n = -\Delta \left(I - \frac{\Delta}{n}\right)^{-1} = n \left(I - (I - \frac{\Delta}{n})^{-1}\right) \). Then we obtain
\[
\langle \partial_t \Delta(u) |u|^{p-2} + au + \phi(t)u, -\Delta(I - \frac{\Delta}{n})^{-1} u \rangle_V
\]
\[
= -\langle u|u|^{p-2} + au, nu - n \int_0^\infty e^{-t} P_n^t u dt \rangle_{L^2} + \phi(t)\|u\|_n^2
\]
\[
\leq -C_1\|u\|_n^2.
\]
That is, (H6) holds.

\[ \square \]

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M. Cheng: School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China

*Email address*: mengyucheng@mail.dlut.edu.cn; mengyu.cheng@hotmail.com

Z. Liu (Corresponding author): School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China

*Email address*: zxliu@dlut.edu.cn