Generalized holonomy of M-theory vacua$^1$

A. Batrachenko$^2$, M. J. Duff$^3$, James T. Liu$^4$ and W. Y. Wen$^5$

Michigan Center for Theoretical Physics
Randall Laboratory, Department of Physics, University of Michigan
Ann Arbor, MI 48109–1120, USA

Abstract

The number of M-theory vacuum supersymmetries, $0 \leq n \leq 32$, is given by the number of singlets appearing in the decomposition of the 32 of SL(32, $\mathbb{R}$) under $\mathcal{H} \subset$ SL(32, $\mathbb{R}$) where $\mathcal{H}$ is the holonomy group of the generalized connection which incorporates non-vanishing 4-form. Here we compute this generalized holonomy for the $n = 16$ examples of the M2-brane, M5-brane, M-wave, M-monopole, for a variety of their $n = 8$ intersections and also for the $n > 16$ pp waves.

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$^2$abat@umich.edu
$^3$mduff@umich.edu
$^4$jimliu@umich.edu
$^5$wenw@umich.edu
1 Introduction

The equations of M-theory display the maximum number of supersymmetries $N = 32$, and so $n$, the number of supersymmetries preserved by a particular vacuum, must be some integer $0 \leq n \leq 32$. In vacua with vanishing 4-form $F_{(4)}$, it is well known that $n$ is given by the number of singlets appearing in the decomposition of the 32 of SO(1, 10) under $H \subset$ SO(1, 10) where $H$ is the holonomy group of the usual Riemannian connection.

$$D_M = \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB}. \quad (1.1)$$

Here $\Gamma_A$ are the SO(1, 10) Dirac matrices and $\Gamma_{AB} = \Gamma_{[A} \Gamma_{B]}$. This connection can account for vacua with $n = 0, 1, 2, 3, 4, 6, 8, 16, 32$.

Vacua with non-vanishing $F_{(4)}$ allow more exotic fractions of supersymmetry, including $16 < n < 32$. Here, however, it is necessary to generalize the notion of holonomy to accommodate the generalized connection that results from a non-vanishing $F_{(4)}$.

$$D_M = D_M - \frac{1}{288} (\Gamma_M^{NPQR} - 8 \delta_M^N \Gamma^{PQR}) F_{NPQR}. \quad (1.2)$$

As discussed in a previous paper [1], the number of M-theory vacuum supersymmetries is now given by the number of singlets appearing in the decomposition of the 32 of $G$ under $H \subset G$ where $G$ is the generalized structure group and $H$ is the generalized holonomy group. Discussions of generalized holonomy may also be found in [2, 3].

In subsequent papers by Hull [4] and Papadopoulos and Tsimpis [5] it was shown that $G$ may be as large as SL(32, $\mathbb{R}$) and that an M-theory vacuum admits precisely $n$ Killing spinors iff

$$\text{SL}(31 - n, \mathbb{R}) \ltimes (n + 1)\mathbb{R}^{(31-n)} \not\supseteq H \subseteq \text{SL}(32 - n, \mathbb{R}) \ltimes n\mathbb{R}^{(32-n)}, \quad (1.3)$$

i.e. the generalized holonomy is contained in $\text{SL}(32 - n, \mathbb{R}) \ltimes n\mathbb{R}^{(32-n)}$ but is not contained in $\text{SL}(31 - n, \mathbb{R}) \ltimes (n + 1)\mathbb{R}^{(31-n)}$.

In this paper we compute this generalized holonomy for the $n = 16$ examples of the M2-brane, M5-brane, M-wave (MW) and the M-monopole (MK), for a variety of their $n = 8$ intersections: M2/MW, M5/MW, M2/M5, MW/MK and also for the $n > 16$ ppwaves. We begin with a review of generalized holonomy in section 2. Then we turn to $n = 16$ and $n = 8$ solutions in sections 3 and 4. Since pp-waves with exotic fractions of supersymmetry involve a slightly different analysis, they are covered in section 5. Finally, we conclude with some comments in section 6.
2 Holonomy and supersymmetry

The number of supersymmetries preserved by an M-theory background depends on the number of covariantly constant spinors,

\[ D_M \epsilon = 0, \]  
\[ (2.1) \]
called \textit{Killing} spinors. It is the presence of the terms involving the 4-form \( F_{(4)} \) in \((1.2)\) that makes this counting difficult. So let us first examine the simpler vacua for which \( F_{(4)} \) vanishes. Killing spinors then satisfy the integrability condition

\[ [D_M, D_N] \epsilon = \frac{1}{4} R_{MN}^{AB} \Gamma_{AB} \epsilon = 0, \]  
\[ (2.2) \]
where \( R_{MN}^{AB} \) is the Riemann tensor. The subgroup of \( \text{Spin}(10, 1) \) generated by this linear combination of \( \text{Spin}(10, 1) \) generators \( \Gamma_{AB} \) corresponds to the holonomy group \( H \) of the connection \( \omega_M \). We note that the same information is contained in the first order Killing spinor equation \((2.1)\) and second-order integrability condition \((2.2)\). One implies the other, at least locally. The number of supersymmetries, \( n \), is then given by the number of singlets appearing in the decomposition of the 32 of \( \text{Spin}(10, 1) \) under \( H \). In Euclidean signature, connections satisfying \((2.2)\) are automatically Ricci-flat and hence solve the field equations when \( F_{(4)} = 0 \). In Lorentzian signature, however, they need only be Ricci-null so Ricci-flatness has to be imposed as an extra condition. In Euclidean signature, the holonomy groups have been classified \([6]\). In Lorentzian signature, much less is known but the question of which subgroups \( H \) of \( \text{Spin}(10, 1) \) leave a spinor invariant has been answered \([7]\). There are two sequences according as the Killing vector \( v_A = \bar{\epsilon} \Gamma_A \epsilon \) is timelike or null. Since \( v^2 \leq 0 \), the spacelike \( v_A \) case does not arise. The timelike \( v_A \) case corresponds to static vacua, where \( H \subset \text{Spin}(10) \subset \text{Spin}(10, 1) \) while the null case to non-static vacua where \( H \subset \text{ISO}(9) \subset \text{Spin}(10, 1) \). It is then possible to determine the possible \( n \)-values and one finds \( n = 2, 4, 6, 8, 16, 32 \) for static vacua, and \( n = 1, 2, 3, 4, 8, 16, 32 \) for non-static vacua \([8, 9, 10]\).

2.1 Generalized holonomy

In general we want to include vacua with \( F_{(4)} \neq 0 \). Such vacua are physically interesting for a variety of reasons. In particular, they typically have fewer moduli than their zero \( F_{(4)} \) counterparts \([11]\). Now, however, we face the problem that the connection in \((1.2)\) is no
longer the spin connection to which the bulk of the mathematical literature on holonomy groups is devoted. In addition to the Spin(10, 1) generators \( \Gamma_{AB} \), it is apparent from (1.2) that there are terms involving \( \Gamma_{ABC} \) and \( \Gamma_{ABCDE} \). In fact, the generalized connection takes its values in \( \text{SL}(32, \mathbb{R}) \). Note, however, that some generators are missing from the covariant derivative. Denoting the antisymmetric product of \( k \) Dirac matrices by \( \Gamma^{(k)} \), the complete set of \( \text{SL}(32, \mathbb{R}) \) generators involve \( \{ \Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}, \Gamma^{(5)} \} \) whereas only \( \{ \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(5)} \} \) appear in the covariant derivative. Another way in which generalized holonomy differs from the Riemannian case is that, although the vanishing of the covariant derivative of the spinor implies the vanishing of the commutator, the converse is not true, as discussed below in section 2.2.

This generalized connection can preserve exotic fractions of supersymmetry forbidden by the Riemannian connection. For example, M-branes at angles [12] include \( n=5 \), 11-dimensional pp-waves [13, 14, 15, 16] include \( n = 18, 20, 22, 24, 26 \), squashed \( N(1, 1) \) spaces [17] and M5-branes in a pp-wave background [18] include \( n = 12 \) and Gödel universes [19, 20] include \( n = 14, 18, 20, 22, 24 \). However, we can attempt to quantify this in terms of generalized holonomy groups\(^6\). Generalized holonomy means that one can assign a holonomy \( \mathcal{H} \subset \mathcal{G} \) to the generalized connection appearing in the supercovariant derivative \( D \) where \( \mathcal{G} \) is the generalized structure group. The number of unbroken supersymmetries is then given by the number of \( \mathcal{H} \) singlets appearing in the decomposition of the 32 dimensional representation of \( \mathcal{G} \) under \( \mathcal{H} \subset \mathcal{G} \).

For generic backgrounds we require that \( \mathcal{G} \) be the full \( \text{SL}(32, \mathbb{R}) \) while for special backgrounds smaller \( \mathcal{G} \) are sufficient [1]. To see this, let us write the supercovariant derivative as

\[
D_M = \hat{D}_M + X_M, \tag{2.3}
\]

for some other connection \( \hat{D}_M \) and some covariant \( 32 \times 32 \) matrix \( X_M \). If we now specialize to backgrounds satisfying

\[
X_M \epsilon = 0, \tag{2.4}
\]

then the relevant structure group is \( \hat{\mathcal{G}} \subset \mathcal{G} \).

Consider, for example, for the connection \( \hat{D} \) arising in dimensional reduction of \( D = 11 \) supergravity. One can show [1] that the lower dimensional gravitino transformation may be

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\(^6\)In this paper we focus on \( D = 11 \) but similar generalized holonomy can be invoked to count \( n \) in Type IIB vacua [21], which include pp-waves with \( n = 28 \) [16].
written

\[ \delta \psi_\mu = \hat{D}_\mu \epsilon, \]  

(2.5)
in terms of a covariant derivative

\[ \hat{D}_\mu = \partial_\mu + \omega_\mu^{\alpha\beta} \gamma_\alpha \gamma_\beta + Q_\mu^{ab} \Gamma_{ab} + \frac{1}{3!} e^{ia} e^{jb} e^{kc} \partial_\mu \phi_{ijk} \Gamma_{abc}. \]  

(2.6)

Here \( \gamma_\alpha \) are SO\((d - 1, 1)\) Dirac matrices, while \( \Gamma_a \) are SO\((11 - d)\) Dirac matrices. In the above, the lower dimensional quantities are related to their \( D = 11 \) counterparts \((E_M^A, \Psi_M^{(11)}, A_{MNP})\) through

\[ ds^2_{(11)} = \Delta^{1/2} ds^2_4 + g_{ij} dy^i dy^j, \]
\[ \psi_\mu = \Delta^{1/2} \left( \Psi_\mu^{(11)} + \frac{1}{d - 2} \gamma_\mu \Gamma^{i} \Psi_i^{(11)} \right), \quad \lambda_i = \Delta^{1/2} \Psi_i^{(11)}, \]
\[ \epsilon = \Delta^{1/2} \epsilon^{(11)}, \]
\[ Q_\mu^{ab} = e^{ia} \partial_\mu e_i ^{b}, \quad P_{\mu ij} = e_{[i} \partial_\mu e_{j]} a, \quad \phi_{ijk} = A_{ijk}. \]  

(2.7)
The condition \( (2.4) \) is just \( \delta \lambda_i = 0 \) where \( \lambda_i \) are the dilatinos of the dimensionally reduced theory. In this case, the generalized holonomy is given by \( \hat{H} \subseteq \hat{G} \) where the various \( \hat{G} \) arising in spacelike, null and timelike compactifications are tabulated in [1] for different numbers of the compactified dimensions. These smaller structure groups are also the ones appropriate to more general Kaluza-Klein compactifications of the product manifold type, i.e. without a warp factor [4].

This is probably a good time to say a few words about the difference between generalized holonomy and the hidden symmetries conjecture which were both discussed in [1]. There it was argued that the equations of M-theory possess previously unidentified hidden spacetime (timelike and null) symmetries in addition to the well-known hidden internal (spacelike) symmetries. They take the form \( \mathcal{G} = \text{SO}(d - 1, 1) \times G(\text{spacelike}), \mathcal{G} = \text{ISO}(d - 1) \times G(\text{null}) \) and \( \mathcal{G} = \text{SO}(d) \times G(\text{timelike}) \) with \( 1 \leq d < 11 \). For example, \( G(\text{spacelike}) = \text{SO}(16), G(\text{null}) = [\text{SU}(8) \times \text{U}(1)] \rtimes \mathbb{R}^{56} \) and \( G(\text{timelike}) = \text{SO}^*(16) \) when \( d = 3 \). The nomenclature derives from the fact that they coincide with the hidden symmetry groups that appear in the spacelike, null and timelike dimensional reductions of the theory. However, they were proposed as background-independent symmetries of the full unreduced and untruncated \( D = 11 \) equations of motion, not merely their dimensional reduction.

For \( d \geq 3 \), these coincide with the generalized structure groups \( \hat{G} \) discussed above that appear in the dimensionally reduced covariant derivative. A more speculative idea is that
there exists a yet-to-be-discovered version of $D = 11$ supergravity or $M$-theory that displays even bigger hidden symmetries corresponding to $\hat{G}$ with $d < 3$ [11] which could be as large as $\text{SL}(32, \mathbb{R})$ [11].

To avoid possible confusion, we emphasize here that the notion of generalized holonomy $\mathcal{H} \subset \text{SL}(32, \mathbb{R})$ is valid whether or not these hidden symmetry conjectures turn out to be correct, and the misleading phrase ‘generalized holonomy conjecture’ should now be abandoned. This highlights another difference between generalized and Riemannian holonomy, $\mathcal{H}$ need not be a symmetry of the theory, whereas $H \subset \text{SO}(1, 10)$ always is.

Note also that a recent paper [22] calls into question both generalized holonomy and the hidden symmetries conjecture in the presence of fermions because some of the symmetries do not admit spinor representations. While acknowledging that such global considerations are important, we do not take the same pessimistic attitude since the fermions do not generally transform as spinors. In the $d = 3$ spacelike case, for example, the gravitino transforms as a \textit{vector} 16 of $\text{SO}(16)$.

### 2.2 Integrability conditions

Yet another way in which generalized holonomy differs from Riemannian holonomy is that, although the vanishing of the covariant derivative implies the vanishing of the commutator, the converse is not true. Consequently, the second order integrability condition alone may be a misleading guide to the generalized holonomy group $\mathcal{H}$.

To illustrate this, we consider Freund-Rubin [23] vacua with $F_{(4)}$ given by

$$F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma},$$

where $\mu = 0, 1, 2, 3$ and $m$ is a constant with the dimensions of mass. This leads to an $\text{AdS}_4 \times X^7$ geometry. For such a product manifold, the supercovariant derivative splits as

$$\mathcal{D}_\mu = D_\mu + m\gamma_\mu \gamma_5$$

and

$$\mathcal{D}_m = D_m - \frac{1}{2}m\Gamma_m,$$

and the Killing spinor equations reduce to

$$\mathcal{D}_\mu \epsilon(x) = 0$$
and
\[ D_m \eta(y) = 0. \] (2.12)

Here \( \epsilon(x) \) is a 4-component spinor and \( \eta(y) \) is an 8-component spinor, transforming with Dirac matrices \( \gamma_\mu \) and \( \Gamma_m \) respectively. The first equation is satisfied automatically with our choice of \( AdS_4 \) spacetime and hence the number of \( D = 4 \) supersymmetries, \( 0 \leq N \leq 8 \), devolves upon the number of Killing spinors on \( X^7 \) [24]. They satisfy the integrability condition
\[ [D_m, D_n] \eta = -\frac{1}{4} C_{mn}^{ab} \Gamma_{ab} \eta = 0, \] (2.13)
where \( C_{mn}^{ab} \) is the Weyl tensor. Owing to this generalized connection, vacua with \( m \neq 0 \) present subtleties and novelties not present in the \( m = 0 \) case [25], for example the phenomenon of *skew-whiffing* [26, 27]. For each Freund-Rubin compactification, one may obtain another by reversing the orientation of \( X^7 \). The two may be distinguished by the labels *left* and *right*. An equivalent way to obtain such vacua is to keep the orientation fixed but to make the replacement \( m \rightarrow -m \) thus reversing the sign of \( F_4 \). So the covariant derivative (2.10), and hence the condition for a Killing spinor, changes but the integrability condition (2.13) remains the same. With the exception of the round \( S^7 \), where both orientations give \( N = 8 \), at most one orientation can have \( N \geq 0 \). This is the *skew-whiffing theorem*. (Note, however, that skew-whiffed vacua are automatically stable at the classical level since skew-whiffing affects only the spin \( 3/2, 1/2 \) and \( 0^- \) towers in the Kaluza-Klein spectrum, whereas the criterion for classical stability involves only the \( 0^+ \) tower [28, 27].)

The squashed \( S^7 \) provides a non-trivial example [29, 26]: the left squashed \( S^7 \) has \( N = 1 \) but the right squashed \( S^7 \) has \( N = 0 \). Other examples are provided by the left squashed \( N(1, 1) \) spaces [17], one of which has \( N = 3 \) and the other \( N = 1 \), while the right squashed counterparts both have \( N = 0 \). (Note, incidentally, that \( N = 3 \) *i.e.* \( n = 12 \) can never arise in the Riemannian case.)

All this presents a dilemma. If the Killing spinor condition changes but the integrability condition does not, how does one give a holonomic interpretation to the different supersymmetries? We note that in (2.10), the \( SO(7) \) generators \( \Gamma_{ab} \), augmented by presence of \( \Gamma_a \), together close on \( SO(8) \) [30]. Hence the generalized holonomy group satisfies \( \mathcal{H} \subset SO(8) \). We now ask how the 8 of \( SO(8) \) decomposes under \( \mathcal{H} \). In the case of the left squashed \( S^7 \), \( \mathcal{H} = SO(7)^- \), and \( N = 1 \), but for the right squashed \( S^7 \), \( \mathcal{H} = SO(7)^+ \), \( 8 \rightarrow 8 \) and
$N = 0$. From the integrability condition alone, however, we would have concluded naively that $\mathcal{H} = G_2 \subset SO(7)$ for which $8 \rightarrow 1 + 7$ and hence that both orientations give $N = 1$.

2.3 Higher order corrections

Another context in which generalized holonomy may prove important is that of higher loop corrections to the M-theory Killing spinor equations with or without the presence of non-vanishing $F(4)$. As discussed in [31], higher loops yield non-Riemannian corrections to the supercovariant derivative, even for vacua for which $F(4) = 0$, thus rendering the Berger classification inapplicable. Although the Killing spinor equation receives higher order corrections, so does the metric, ensuring, for example, that $H = G_2$ Riemannian holonomy 7-manifolds still yield $N = 1$ in $D = 4$ when the non-Riemannian corrections are taken into account. This would require a generalized holonomy $\mathcal{H}$ for which the decomposition $8 \rightarrow 1 + 7$ continues to hold.

3 Generalized holonomy for $n = 16$

We now turn to a generalized holonomy analysis of some basic supergravity solutions. Starting with the maximally supersymmetric backgrounds ($n = 32$), namely $E^{1,10}$, AdS$_7 \times S^4$, AdS$_4 \times S^7$ and Hpp, it should be clear that they all have trivial generalized holonomy, in accord with (1.3). However, only flat space may be described by (trivial) Riemannian holonomy.

Somewhat more interesting to consider are the four basic objects of M-theory preserving half of the supersymmetries (corresponding to $n = 16$). These are the M5-brane, M2-brane, M-wave (MW) and the Kaluza-Klein monopole (MK). The latter two have $F(4) = 0$ and may be categorized using ordinary Riemannian holonomy, with $H \subset SO(10, 1)$. We now look at these in turn.

3.1 The M5-brane

The familiar supergravity M5-brane solution [32] may be written in isotropic coordinates as

$$ds^2 = H_5^{-1/3} dx^2 + H_5^{2/3} dy^2,$$

$$F_{ijkl} = \epsilon_{ijklm} \partial_m H_5,$$

(3.1)
where $H_5(y)$ is harmonic in the six-dimensional transverse space spanned by $\{y^i\}$, and $\epsilon_{ijklm} = \pm 1$. While the transverse space only needs to be Ricci flat, we take it to be $\mathbb{E}^5$, so as not to further break the supersymmetry.

A simple computation of the generalized covariant derivative on this background yields

\[
\mathcal{D}_\mu = \partial_\mu - \frac{1}{6}\Gamma_{\bar{\mu}}{^j}{^i}P_5^{+}H^{-3/2}\partial_i H,
\]

\[
\mathcal{D}_i = \partial_i + \frac{1}{3}\Gamma_{\bar{i}}{^j}{^i}P_5^{+}\partial_j \ln H - \frac{1}{2}\Gamma{(5)}\partial_i \ln H.
\]

(3.2)

Here, $P_5^{\pm} = \frac{1}{2}(1 \pm \Gamma^{(5)})$ is the standard 1/2-BPS projection for the M5-brane, where $\Gamma^{(5)} = \frac{1}{5!}\epsilon_{ijklm}\Gamma_{\bar{i}}{^j}{^k}{^l}{^m}$. All quantities with bars indicate tangent space indices. To obtain the generalized holonomy of the M5-brane, we examine the commutator of covariant derivatives. Defining

\[
M_{MN} = [\mathcal{D}_M, \mathcal{D}_N],
\]

we find that $M_{\mu\nu} = 0$, so that the holonomy is trivial in the longitudinal directions along the brane. On the other hand, the transverse and mixed commutators are given by

\[
M_{ij} = -\frac{2}{3}\Gamma_{\bar{i}}{^j}{^k}P_5^{+}(\partial_k \ln H)^2 + \frac{2}{3}\Gamma_{\bar{i}}{^j}{^k}P_5^{+}(\partial_i \partial^k \ln H - \frac{2}{3}\partial_i \partial_j \ln H \partial^k \ln H),
\]

\[
M_{\mu i} = H^{-1/2}[\frac{1}{6}\Gamma_{\bar{i}}{^j}{^k}P_5^{+}(\partial_i \partial^j H - \frac{2}{3}\partial_i \partial_j \ln H \partial^k \ln H) + \frac{1}{8}\Gamma_{\bar{\mu}}{^j}{^i}P_5^{+}(\partial_j \ln H)^2].
\]

(3.4)

We first examine the transverse holonomy. Independent of the form of the harmonic function, $H_5$, we see that the only combination of Dirac matrices showing up in $M_{ij}$ are given by $\Gamma_{\bar{i}}{^j}{^k}P_5^{+}$. Defining a set of Hermitian generators $T_{ij} = -\frac{1}{2}\Gamma_{\bar{i}}{^j}{^k}P_5^{+}$, it is easily seen that they generate the $SO(5)$ algebra

\[
[T_{ij}, T_{kl}] = i(\delta_{ik}T_{jl} - \delta_{ij}T_{kl} - \delta_{jk}T_{il} + \delta_{jl}T_{ik}).
\]

(3.5)

As a result, the transverse holonomy is simply $SO(5)_+$, where the + refers to the sign of the M5-projection.

Turning next to the mixed commutator, $M_{\mu i}$, we see that it introduces an additional set of Dirac matrices, $K_{\mu i} = \Gamma_{\bar{i}}{^j}{^k}P_5^{+}$. Since $\Gamma_{\bar{i}}{^j}{^k}P_5^{+} = P_5^{+}\Gamma_{\bar{i}}{^j}{^k}$, it is clear that the $K_{\mu i}$ generators commute among themselves. On the other hand, commuting $K_{\mu i}$ with the $SO(5)_+$ generators $T_{ij}$ yield the additional combinations $K_{\mu} = \Gamma_{\bar{i}}{^j}{^k}P_5^{+}$ and $K_{\mu ij} = \Gamma_{\bar{i}}{^j}{^k}P_5^{+}$. Picking a set of Cartan generators $T_{12}$ and $T_{34}$ for $SO(5)_+$, we may see that the complete set $\{K_{\mu}, K_{\mu i}, K_{\mu ij}\}$ has weights $\pm 1/2$. As a result, they transform as a set of 4-dimensional spinor representations of $SO(5)_+$. We conclude that the generalized holonomy of the M5-brane is

\[
\mathcal{H}_{M5} = SO(5)_+ \rtimes 6\mathbb{R}^{4(4)}.
\]

(3.6)
3.2 The M2-brane

Turning next to the M2-brane, its supergravity solution may be written as [2]

\[ ds^2 = H_2^{-2/3} dx_{\mu}^2 + H_2^{1/3} d\vec{y}^2, \]
\[ F_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} \frac{1}{H_2}. \] (3.7)

A similar examination of the commutator of generalized covariant derivatives, (3.3), for this solution indicates the presence of both compact generators \( T_{ij} = -\frac{i}{2} \Gamma_{\bar{i}j} P^+_2 \) and non-compact ones \( K_{\mu i} = \Gamma_{\bar{\mu} i} P^+_2 \). Here, \( P^+_2 = \frac{1}{2} (1 \pm \Gamma^{(2)}) \) where \( \Gamma^{(2)} = \frac{1}{3!} \epsilon_{\mu\nu\rho} \Gamma^{\bar{\mu} \bar{\nu} \bar{\rho}} \) is the M2-brane projection. Furthermore, the coordinates on (3.7) correspond to a \( \frac{3}{8} \) longitudinal/transverse split. Hence the transverse holonomy in this case is \( \text{SO}(8)_+ \).

To obtain the generalized holonomy group \( \mathcal{H}_{M2} \), we must first close the algebra formed by \( T_{ij} \) and \( K_{\mu i} \). Upon doing so, we find the additional generators \( K_{\mu ijk} = \Gamma_{\bar{\mu} \bar{i} \bar{j} \bar{k}} P^+_2 \). As in the M5 case, we may see that the set \( \{ K_{\mu i}, K_{\mu ijk} \} \) form eight-dimensional representations of \( \text{SO}(8)_+ \). However, some care must be taken in identifying these representations as the \( 8_v, 8_s \) or \( 8_c \) (up to an overall automorphism due to triality).

Since it is instructive for the later intersecting brane examples, we will demonstrate a simple method for investigating the generalized holonomy of this solution. Based on the \( \frac{3}{8} \) split, we may make an explicit decomposition of the 11-dimensional (real) Dirac matrices as follows

\[ \Gamma^0 = 1 \times i\sigma^2 \times 1, \]
\[ \Gamma^1 = 1 \times \sigma^1 \times 1, \]
\[ \Gamma^2 = 1 \times \sigma^3 \times \sigma^3, \]
\[ \Gamma^3 = 1 \times \sigma^3 \times \sigma^1, \]
\[ \Gamma^a = \gamma^a \times \sigma^3 \times \sigma^2. \] (3.8)

Here, the eight-dimensional transverse space is split into \( 7 + 1 \), with \( \gamma^a \) a set of purely imaginary \( 8 \times 8 \) seven-dimensional Dirac matrices. Since \( \Gamma^{(2)} \equiv \Gamma^{012} = 1 \times 1 \times \sigma^3 \), the M2-brane projection is simply

\[ P^+_2 = 1 \times 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \] (3.9)
The explicit $\text{SO}(8)_+$ generators then have the form
\[
T_{ij} \longleftrightarrow \{-\frac{i}{2} \gamma^{ab}, \frac{1}{2} \gamma^a\} \times 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
which highlights the embedding of $\text{SO}(7) \subset \text{SO}(8)_+$. The complete set of mixed generators may be written concisely as
\[
\{K_{\mu i}, K_{\mu ijk}\} \longleftrightarrow \text{Cl}(0,7)_+ \times \{1, \sigma^1, i\sigma^2\} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
where $\text{Cl}(p,q)$ is the real Clifford algebra with signature given by $p$ positive and $q$ negative eigenvalues. In this case, $\text{Cl}(0,7)_+$ is generated by the Dirac matrices $i\gamma^a$, and is isomorphic to $\text{GL}(8, \mathbb{R})$.

Examination of (3.10) and (3.11) demonstrates that the M2 holonomy generators have the schematic form
\[
\begin{pmatrix} \text{SO}(8)_+ \times 1 & 0 \\ \mathbb{R}(8,8) \times \{1, \sigma^1, i\sigma^2\} & 0 \end{pmatrix} \subset \begin{pmatrix} \text{SL}(16, \mathbb{R}) & 0 \\ \mathbb{R}(16,16) & 0 \end{pmatrix},
\]
as appropriate to a solution with $n = 16$. This shows that the M2 generalized holonomy is given by
\[
\mathcal{H}_{M2} = \text{SO}(8)_+ \ltimes 12\mathbb{R}^{2(8_s)}.
\]
This corrects a result obtained in [2] where it was claimed that the generalized holonomy is simply $\mathcal{H}_{M2} = \hat{H}_{M2} = \text{SO}(8)_+$ which also yields $n = 16$.

### 3.3 The M-wave

We now turn to the pure geometry solutions. The wave (MW) is given by
\[
ds^2 = 2 dx^+ dx^- + K dx^+^2 + d\vec{y}^2,
\]
where $K(\vec{y})$ is harmonic on the nine-dimensional Euclidean transverse space $\mathbb{E}^9$. In a vielbein basis $e^+ = dx^+$, $e^- = dx^- + \frac{1}{2}K dx^+$, $e^i = dy^i$, the only non-vanishing component of the spin connection is given by $\omega^{+i} = \frac{1}{2}\partial_i K e^+$. Thus the gravitational covariant derivative acting on $\epsilon$ is given by
\[
D_+ = \partial_+ + \frac{1}{4}\partial_i K \Gamma_{-i}, \quad D_- = \partial_-, \quad D_i = \partial_i.
\]
Note that the metric is given by $ds^2 = 2e^+e^- + e^i e^j$, so that light cone indices are raised and lowered as, e.g., $\Gamma_- = \Gamma_+ \Gamma_i$ in tangent space.

The only non-vanishing commutator of covariant derivatives is given by

$$M_{+i} = -\frac{1}{4}\partial_i \partial_j K \Gamma_- \Gamma_i,$$

so we may identify the generalized holonomy generators as $T^i = \Gamma_- \Gamma_i$. Since $\Gamma^2_2 = 0$, these nine generators are mutually commuting, and the MW generalized holonomy is

$$\mathcal{H}_{\text{MW}} = \mathbb{R}^9.$$

In addition to being a subgroup of $\text{SL}(16, \mathbb{R}) \ltimes 16 \mathbb{R}^{16}$, this may also be viewed as a subgroup of $\text{ISO}(9)$ appropriate to backgrounds with a null Killing vector. We will return to waves in section 5 where we turn on $F^{(4)}$ and consider the generalized holonomy of pp-waves preserving exotic fractions of supersymmetry.

### 3.4 The M-monopole

The final basic M-theory object we consider is the Kaluza-Klein monopole, which is given by the Euclidean Taub-NUT solution

$$ds^2 = dx^2_\mu + H(dr^2 + r^2 d\Omega_2^2) + H^{-1}(dz - q \cos \theta d\phi)^2,$$

where $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $H = 1 + q/r$. As is well known, this space is Ricci flat and hyper-Kähler, and so has $\text{Sp}(1) \simeq \text{SU}(2)$ holonomy. Since this solution does not involve $F^{(4)}$, its generalized holonomy is similarly $\text{SU}(2)$

$$\mathcal{H}_{\text{MK}} = \text{SU}(2).$$

### 4 Some $n = 8$ examples

Having looked at the basic objects of M-theory, we now turn to intersecting configurations preserving fewer supersymmetries. While large classes of intersecting brane solutions and configurations involving to branes at angles have been constructed, we will only examine some of the simple cases of orthogonal intersections yielding $n = 8$. 

11
4.1 Branes with a KK-monopole

It has often been noted that the basic supergravity p-brane solutions are not restricted to having only flat Euclidean transverse spaces. This indicates, in particular, that the M5 and M2 solutions of (3.1) and (3.7) demand only that the transverse space spanned by \( \{ \vec{y} \} \) is Ricci flat. Of course, this Ricci flat manifold must still be supersymmetric in order to preserve some fraction of supersymmetry.

A simple example would be to replace \( E^4 \) with a Taub-NUT configuration in four of the transverse directions to the brane. For the M5 case, the resulting M5/MK solution has the form

\[
d s^2 = H^{-1/3}_5(dx_\mu^2 + H^{-2/3}_5(dy^2 + H_6(2dr^2 + r^2d\Omega^2_2)) + H^{-1}_6(dz - q_6 \cos \theta d\phi)^2).
\] (4.1)

Here, the M5-brane is delocalized along the \( y \) direction, so the harmonic functions have the form \( H_5 = 1 + q_5/r \) and \( H_6 = 1 + q_6/r \). This represents the lifting of a NS5/D6 configuration to eleven dimensions.

Noting that four of the five transverse directions is replaced by a Taub-NUT space, the corresponding Riemannian holonomy is contained in the SO(5) tangent space group in the sense of \( SU(2) \subset SO(4) \subset SO(5) \). The embedding of the self-dual connection in SO(4) leads to explicit \( SU(2) \) generators \( T^{(MK)}_{ab} = -\frac{i}{2} \bar{\Gamma}_{\bar{a} \bar{b}} P^+_K \) where \( P^\pm_K = \frac{1}{2}(1 \pm \Gamma^{1234}) \) and \( a, b, \ldots = 1, \ldots, 4 \). On the other hand, as shown in section 3.1, the \( SO(5)_+ \) generalized holonomy of M5 in the transverse directions involve the \( P^+_5 \) projection, and is generated by \( T^{(M5)}_{ij} = -\frac{i}{2} \bar{\Gamma}_{ij} P^+_5 \), where \( i, j, \ldots = 1, \ldots, 5 \). As a result, the transverse holonomy of this M5/MK configuration arises as the closure of \( T^{(M5)}_{ij} \) and \( T^{(MK)}_{ab} \).

Since \( T^{(MK)}_{ab} \) is comprised of Dirac matrices entirely in the transverse directions, we may perform a trivial decomposition

\[
T^{(MK)}_{ab} = -\frac{i}{2} \bar{\Gamma}_{\bar{a} \bar{b}} P^+_K P^+_5 - \frac{i}{2} \bar{\Gamma}_{\bar{a} \bar{b}} P^+_K P^-_5
\] (4.2)

Because the first term is already contained entirely in \( T^{(M5)}_{ij} \), the resulting algebra is equally well generated by the mutually commuting set

\[
T^{(M5)}_{ij} = -\frac{i}{2} \bar{\Gamma}_{ij} P^+_5, \quad \bar{T}^{(MK)}_{ab} = -\frac{i}{2} \bar{\Gamma}_{\bar{a} \bar{b}} P^+_K P^-_5.
\] (4.3)

This indicates that the transverse holonomy is simply \( SO(5)_+ \times SU(2)_- \) where \( \pm \) refers to the embedding inside the \( \hat{D} \) structure group \( SO(5)_+ \times SO(5)_- \) for a 6/5 split. The additional
M5 mixed commutator generators \( \{K_\mu, K_{\mu i}, K_{\mu ij}\} \) now transform under both \( \text{SO}(5)_+ \) and \( \text{SU}(2)_- \). Working out the weights of these generators under \( \text{SO}(5)_+ \times \text{SU}(2)_- \) demonstrates that the generalized holonomy of this M5/MK configuration is

\[
\mathcal{H}_{\text{M5/MK}} = [\text{SO}(5)_+ \times \text{SU}(2)_-] \ltimes 6^2(4,1)^{(4,2)}. \tag{4.4}
\]

For the M2-brane, the eight-dimensional transverse space may be given a hyper-Kähler metric \([38]\), which is generically of holonomy \( \text{Sp}(2) \). However, we only consider the product of two independent Taub-NUT spaces, with holonomy \( \text{Sp}(1) \times \text{Sp}(1) \). Provided both are oriented properly with the M2, this yields a single additional halving of the supersymmetries, leading to \( n = 8 \). The transverse holonomy of this solution corresponds to the embedding \( \text{SO}(8) \times \text{SU}(2) \times \text{SU}(2) \subset \text{SO}(8) \times \text{SO}(4) \times \text{SO}(4) \subset \text{SO}(8) \times \text{SO}(8) \subset \text{SO}(16) \), where \( \text{SO}(16) \) is the \( \hat{D} \) structure group corresponding to a \( 3/8 \) split. The complete generalized holonomy group is

\[
\mathcal{H}_{\text{M2/MK/MK}} = [\text{SO}(8) \times \text{SU}(2) \times \text{SU}(2) \ltimes 3^2(8,2)] \ltimes 6^2(8,1,1). \tag{4.5}
\]

With only a single Taub-NUT space, the generalized holonomy is instead

\[
\mathcal{H}_{\text{M2/MK}} = [\text{SO}(8) \times \text{SU}(2) \ltimes 3^2(8,2)] \ltimes 6^2(8,1,1). \tag{4.6}
\]

### 4.2 Branes with a wave

For solutions with an extended longitudinal space, it is possible to turn on a wave in a null direction along the brane. We consider the M2/MW and M5/MW combinations, both of which preserve a quarter of the original supersymmetries. For the M2/MW combination, the supergravity solution is given by \([36]\)

\[
\begin{align*}
    ds^2 &= H_2^{-2/3}(2dx^+dx^- + Kdx^+2 + dz^2) + H_2^{1/3}dy^2, \\
    F_{+-zi} &= \partial_i \frac{1}{H_2}.
\end{align*}
\tag{4.7}
\]

Here, both \( K \) and \( H_2 \) are harmonic on the eight-dimensional overall transverse space; the wave is delocalized along the \( z \) direction.

If \( H_2 \) is turned off, the solution reverts to the MW solution of \([3.14]\), however with dependence on only eight of the nine directions transverse to the wave. The resulting holonomy would be \( \mathbb{R}^8 \). Combining this with the M2 generalized holonomy, \([3.13]\), must yield a larger group that is nevertheless contained in \( \text{SL}(24, \mathbb{R}) \ltimes 8^2 \).
To see this explicitly, we first note that the generalized covariant derivative has the form

\[
D_+ = \partial_+ + \frac{1}{4} H_2^{-1/2} \partial_i K \Gamma_i \Gamma_i - \frac{1}{6} H_2^{-3/2} \partial_i H_2 (\Gamma_+ + \frac{1}{2} K \Gamma_-) \Gamma_i P^+_2, \\
D_- = \partial_- - \frac{1}{6} H_2^{-3/2} \partial_i H_2 \Gamma_\mu \Gamma_i P^+_2, \\
D_z = \partial_z - \frac{1}{6} H_2^{-3/2} \partial_i H_2 \Gamma_z \Gamma_i P^+_2, \\
D_i = \partial_i + \frac{1}{12} (\Gamma_i^j - 2 \delta_i^j) P^+_2 \partial_j \ln H_2 - \frac{1}{6} \partial_i \ln H_2,
\]

where all Dirac matrices are written with frame indices. As usual, the M2 projection is defined by \( P_2^{\pm} = \frac{1}{2} (1 \pm \Gamma^{(2)}) \), where \( \Gamma^{(2)} = \Gamma^{++} \).

Taking commutators of the above covariant derivatives, it is clear that the generalized holonomy algebra is formed by the closure of the MW algebra, generated by \( \Gamma_- \Gamma_i \), and the M2 algebra, generated by \(-i \frac{1}{2} \Gamma_{ij} P^+_2\) and \( \Gamma^\mu \Gamma_i P^+_2 \) where \( \mu \) denotes one of the longitudinal coordinates, +, − or \( z \). To be explicit, we may use the Dirac matrix decomposition given by (3.8). The light-cone Dirac matrices are then given by

\[
\Gamma^+ = \frac{1}{\sqrt{2}} (-\Gamma^0 + \Gamma^1), \quad \Gamma^- = \frac{1}{\sqrt{2}} (-\Gamma^0 - \Gamma^1),
\]

so that \( \Gamma^{(2)} = \Gamma^{012} = 1 \times 1 \times \sigma^3 \), and the M2 projection has the identical form as (3.9).

Killing spinors for the wave solution are projected according to \( \Gamma_- \epsilon = 0 \), or equivalently \( \Gamma^+ \epsilon = 0 \). In terms of 0 and 1 components, this corresponds to the condition \( P^+_L \epsilon = 0 \) where the wave projection is given by \( P^{\pm}_L = \frac{1}{2} (1 \pm \Gamma^{01}) \).

The M2/MW Killing spinors satisfy the simultaneous conditions \( P^+_2 \epsilon = 0 \) and \( P^+_L \epsilon = 0 \), where \( P^+_2 \) is given in (3.9) and

\[
P^+_L = 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times 1.
\]

(4.10)

Combining the last two elements in the three-term direct product, the projections may be explicitly written as

\[
P^+_2 = 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^+_L = 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(4.11)
As a result, Killing spinors are given by

\[ \epsilon = \eta \times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (4.12) \]

and a typical generator of the generalized holonomy group must have the form

\[ T = \begin{pmatrix} \cdots & \cdots & \cdots & 0 \\ \cdots & \text{SL}(24, \mathbb{R}) & 0 \\ \cdots & \cdots & \cdots & 0 \\ \cdots & \mathbb{R}(8, 24) & \cdots & 0 \end{pmatrix}. \quad (4.13) \]

In this 4 \times 4 matrix notation, the M2 holonomy generators of (3.12) may be written as

\[ T_{M2} = \begin{pmatrix} \text{SO}(8)_+ & 0 & 0 & 0 \\ A & 0 & B & 0 \\ 0 & 0 & \text{SO}(8)_+ & 0 \\ C & 0 & A & 0 \end{pmatrix}, \quad (4.14) \]

where the single \text{SO}(8)_+ transverse holonomy simultaneously transforms the first and third entries of the four-component vector. Here, \( A, B \) and \( C \) are independent \( \text{GL}(8, \mathbb{R}) \) matrices. In addition, the \( \mathbb{R}^8 \) holonomy of the wave (delocalized along \( z \)) is generated by

\[ T_{MW} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b^0 1 + i b^a \gamma^a & 0 & 0 \\ b^0 1 - i b^a \gamma^a & 0 & 0 & 0 \end{pmatrix}, \quad (4.15) \]

where \( \{b^0, b^a\} \) is an eight-component vector. Closing the algebra generated by \( T_{M2} \) and \( T_{MW} \) results in the M2/MW generators

\[ T_{M2/MW} = \begin{pmatrix} \text{SO}(8) & 0 & 0 & 0 \\ \mathbb{R}(8, 16) & \text{SL}(16, \mathbb{R}) & 0 \\ \vdots & \vdots & 0 \\ \cdots & \mathbb{R}(8, 24) & \cdots & 0 \end{pmatrix}. \quad (4.16) \]

Thus the corresponding generalized holonomy group is

\[ \mathcal{H}_{M2/MW} = [\text{SO}(8) \times \text{SL}(16, \mathbb{R}) \times \mathbb{R}^{(8, 16)}] \times 8 \mathbb{R}^{(8, 1)+(1,16)}. \quad (4.17) \]
The generalized holonomy analysis for the M5/MW solution \[36\]
\[
\begin{align*}
    ds^2 & = H_5^{-1/3}(2dx^+dx^- + K dx^+ + Kdz^2) + H_5^{2/3}d\vec{y}^2, \\
    F_{ijkl} & = \epsilon_{ijklm}\partial_m H_5,
\end{align*}
\]
(4.18)
is similar. Here the functions $H_5$ and $K$ are harmonic on the five-dimensional overall transverse space. This corresponds to a superposition of a M5-brane with a delocalized wave, where the latter has $\mathbb{R}^5$ holonomy. Closing the holonomy algebra over the M5 and MW generators yields the generalized holonomy
\[
    \mathcal{H}_{\text{M5/MW}} = [\text{SO}(5) \times \text{SU}^*(8) \ltimes 4\mathbb{R}^{(4,8)}] \ltimes 8\mathbb{R}^{2(4,1)+2(1,8)}.
\]
(4.19)
Note that $\text{SU}^*(8) \simeq \text{SL}(4,\mathbb{H})$, and the latter is built out of multiple copies of the five-dimensional real Clifford algebra $\text{Cl}(5,0)_+ \simeq \text{GL}(2,\mathbb{H})$.

4.3 Other examples

Additional pure geometry backgrounds may be constructed by combining a wave with a Taub-NUT space. An $n = 8$ example is given by \[36\,39\]
\[
\begin{align*}
    ds^2 & = dx^+ dx^- + K dx^+ + Kdz^2 + H_6(dr^2 + r^2d\Omega_2^2) + H_6^{-1}(dz - q_6 \cos \theta d\phi)^2, \\
    F_{ijk} & = \epsilon_{ijklm}\partial_m H_5,
\end{align*}
\]
(4.20)
where $K = q_0/r + q_0/y^3$ and $H_6 = 1 + q_6/r$. Since the transverse space is a direct product of $\mathbb{E}^5$ with Taub-NUT, the generalized holonomy has the direct product form
\[
    \mathcal{H}_{\text{MW/MK}} = \mathbb{R}^5 \times (\text{SU}(2) \ltimes \mathbb{R}^{2(2)}).
\]
(4.21)
Finally, there are numerous examples of overlapping or intersecting brane configurations involving multiple M2 and/or M5 branes. Various fractions of supersymmetry may be preserved by placing branes at appropriate angles. Here, we only consider the orthogonal intersection of M2 and M5 on a string, given by \[36\,37\]
\[
\begin{align*}
    ds^2 & = H_2^{-2/3}H_5^{-1/3}dx_\mu^2 + H_2^{1/3}H_5^{-1/3}d\bar{y}_4^2 + H_2^{-2/3}H_5^{2/3}dz^2 + H_2^{1/3}H_5^{2/3}d\bar{y}_4^2, \\
    F_{\mu
u zi} & = \epsilon_{\mu\nu}\partial_z \frac{1}{H_2}, \\
    F_{ijkl} & = \epsilon_{ijklm}\partial_m H_5,
\end{align*}
\]
(4.22)
The full holonomy algebra is obtained by the closure of the M5 and M2 holonomies, given by \[3.6\] and \[3.13\], respectively. A slight complication arises in that the individual generators
work on different relative transverse directions for the M5 and M2 branes. By taking the non-compact generators of one of the branes (e.g. the $12 \mathbb{R}^{2(8)}$ for the M2) and commuting with the transverse holonomy generators of the other (in this case the $SO(5)_+$ for the M5) we end up filling up all of $SL(24, \mathbb{R})$. As a result, we find that the M2/M5 generalized holonomy fills all of the maximally allowed case for $n = 8$, namely

$$\mathcal{H}_{M2/M5} = SL(24, \mathbb{R}) \ltimes 8\mathbb{R}^{24}. \quad (4.23)$$

### 5 Waves and supernumerary Killing spinors

In this section, we consider waves with non-vanishing $F_{(4)}$. For a pp-wave with covariantly constant null Killing vector $\partial/\partial x^-$, the metric and four-form take the form

$$\begin{align*}
    ds^2 &= 2 dx^+ dx^- + K dx^+^2 + dy^2, \\
    F_{(4)} &= \mu dx^+ \wedge \Phi_{(3)}, \quad (5.1)
\end{align*}$$

where $\mu$ is a nonzero constant and $\Phi_{(3)}$ is a harmonic three-form on the transverse space. In general, the function $K$ depends on both $x^+$ and $\vec{y}$, while for plane waves, it has the quadratic form $K = K_{ij}(x^+) y^i y^j$.

The metric is identical to that of (3.14), which was considered previously in the pure geometry case. Thus the generalized covariant derivative is given by

$$\begin{align*}
    D_+ &= D_+ - \frac{i}{12} \mu (1 + \Gamma_- \Gamma_+) W, \\
    D_- &= \partial_-, \\
    D_i &= \partial_i + \frac{i}{24} \mu \Gamma_- (\Gamma_i W + 3 W \Gamma_i), \quad (5.2)
\end{align*}$$

where $W = \frac{i}{3!} \Phi_{ijk} \Gamma_{ijk}$, and the gravitational covariant derivative $D_+$ is given in (3.15). With non-vanishing $F_{(4)}$, the integrability condition of (3.16) is modified to become $M_{+i} \epsilon = 0$ where

$$M_{+i} = -\frac{1}{4} [\partial_i \partial_j K \Gamma_j + \frac{\mu^2}{72} (6 \Gamma_+ W + 9 W^2 \Gamma_i + \Gamma_i W^2)] \Gamma_- \equiv -\frac{1}{4} X_i \Gamma_- \Gamma_. \quad (5.3)$$

Note that this integrability condition acting on a spinor $\epsilon$ is in exact agreement with the first order Killing spinor conditions for the pp-wave background [14] [13]. In particular, since $(\Gamma_-)^2 = 0$, half of the original supersymmetries ($n = 16$) are always preserved by spinors
satisfying $\Gamma_\epsilon = 0$. On the other hand, extra supersymmetries (denoted supernumerary supersymmetries in [14]) arise whenever $X_i$ has zero eigenvalues.

If we identify the generalized holonomy generators as $T^i = X_i \Gamma_\epsilon$, and furthermore note that $\{X_i, \Gamma_\epsilon\} = 0$, it is easy to see that $[T^i, T^j] = 0$. Hence the nine generators fill out at most $\mathbb{R}^9$. But the generalized holonomy may be smaller if any of the generators are either degenerate or trivial. In particular, the generalized holonomy group must be trivial for the maximally supersymmetric Hpp-wave [40, 33, 41].

To investigate the generalized holonomy for plane waves with exotic fractions of supersymmetry, consider the ansatz [14, 15]

$$\Phi(3) = m_1 dy_{129} + m_2 dy_{349} + m_3 dy_{569} + m_4 dy_{789},$$

$$K = 1 - \sum_i \mu_i^2 y_i^2,$$  (5.4)

where $dy_{ijk} = dy_i \wedge dy_j \wedge dy_k$, and the equations of motion demand $\sum \mu_i^2 = \frac{1}{12} \mu^2 \Phi^2$. The $\mu_i$ must be chosen appropriately in order to preserve supersymmetry [15]. Since the direction $i = 9$ is singled out, the result is somewhat asymmetrical, with $\mu_9 = \frac{1}{2} (m_1 + 2m_2 + m_3 + m_4)^2$, while $\mu_1 = \mu_2 = \frac{1}{36} (2m_1 - m_2 - m_3 - m_4)^2$ with similar expressions for $\mu_3, \ldots, \mu_8$ (where the factor of 2 is permuted). In this case, we find

$$X_1, 2 = -\frac{1}{18} \mu^2 \Gamma_{1,2}[(2m_1 - m_2 - m_3 - m_4)^2 - (2m_1 - m_2 \Gamma^{1234} - m_3 \Gamma^{1256} - m_4 \Gamma^{1278})^2],$$

$$X_9 = -\frac{2}{3} \mu^2 \Gamma_9[(m_1 + m_2 + m_3 + m_4)^2 - (m_1 + m_2 \Gamma^{1234} + m_3 \Gamma^{1256} + m_4 \Gamma^{1278})^2].$$  (5.5)

The choice of setting all $m_i$ to zero trivially recovers the Minkowski vacuum, with $n = 32$. On the other hand, even when exactly one of the $m_i$ is nonzero, all the $X_i$ still vanish. This case corresponds to the Hpp-wave, which preserves all supersymmetries ($n = 32$), and which has trivial generalized holonomy

$$\mathcal{H}_{\text{Hpp}} = \{1\}.  \quad (5.6)$$

We may see that with each additional non-vanishing $m_i$ turn on, the $X_i$ take on the form of multiple commuting projections, with the projections built from $\Gamma^{1234}$, $\Gamma^{1256}$ and finally $\Gamma^{1278}$. Hence this appropriate connection between the $\mu_i$ (metric) and $m_i$ (four-form) constants allows the addition of 2, 4 or 8 supernumerary supersymmetries. A slightly different ansatz for $\Phi(3)$ also allows for 6 extra supersymmetries. Thus in this manner we obtain plane waves with $n = 18, 20, 22$ and 24 [15].
For all these cases with $n < 32$, none of the $X_i$ in (5.5) vanish. Since each individual $X_i$ is obtained by multiplying $\Gamma_i$ by a suitable projector, they are all linearly independent. Hence the generalized holonomy remains $\mathbb{R}^9$, regardless of the actual number of supersymmetries. Furthermore, even if the $\mu_i$ and $m_i$ were not chosen appropriately (so that there are no extra supersymmetries), the plane wave would still preserve the original $n = 16$. Hence

$$\mathcal{H}_{pp} = \mathbb{R}^9 \quad (n = 16, 18, 20, 22, 24).$$

(5.7)

The $n = 16$ case is essentially that of the MW found before.

As seen in (5.5), the projections which are responsible for the exotic fractions of supersymmetries are hidden inside the $X_i$. Without a detailed examination of the generators $T^i = X_i \Gamma_-$, we cannot tell how many extra supersymmetries there are simply by looking at the generalized holonomy group itself.

6 Discussion

As we have seen in the previous three sections, the generalized holonomy of M-theory solutions takes on a variety of guises. Our results are summarized in table\[1\] We make note of two features exhibited by these solutions. Firstly, it is clear that many generalized holonomy groups give rise to the same number $n$ of supersymmetries. This is a consequence of the fact that while $\mathcal{H}$ must satisfy the condition (1.3), there are nevertheless many possible subgroups of $\text{SL}(32 - n, \mathbb{R}) \ltimes n\mathbb{R}^{(32-n)}$ allowed by generalized holonomy. Secondly, as demonstrated by the plane wave solutions, knowledge of $\mathcal{H}$ by itself is insufficient for determining $n$; here $\mathcal{H} = \mathbb{R}^9$, while $n$ may be any even integer between 16 and 26.

What this indicates is that, at least for counting supersymmetries, it is important to understand the embedding of $\mathcal{H}$ in $\mathcal{G}$. In contrast to the Riemannian case, different embeddings of $\mathcal{H}$ yield different possible values of $n$. Although this appears to pose a difficulty in applying the concept of generalized holonomy towards classifying supergravity solutions, it may be possible that a better understanding of the representations of non-compact groups will nevertheless allow progress to be achieved in this direction.

While the full generalized holonomy involves several factors, the transverse (or $\hat{D}$) holonomy is often simpler, e.g. $SO(5)$ for the M5 and $SO(8)$ for the M2. The results summarized in table\[1\] are suggestive that the maximal compact subgroup of $\mathcal{H}$, which must be contained
Table 1: Generalized holonomies of the objects investigated in the text. For \( n = 16 \), we have \( H \subseteq SL(16, \mathbb{R}) \times 16\mathbb{R}^{16} \), while for \( n = 8 \), it is instead \( H \subseteq SL(24, \mathbb{R}) \times 8\mathbb{R}^{24} \).

For example, the M2/MK/MK solution may be regarded as a 3/8 split, with a hyper-Kähler eight-dimensional transverse space. In this case, the \( \hat{D} \) structure group is SO(16), and the 32-component spinor decomposes under \( SO(32) \supset SO(16) \supset SO(8) \times SU(2) \times SU(2) \) as \( 32 \rightarrow 2(16) \rightarrow 2(8, 1, 1) + 2(1, 2, 2) + 8(1, 1, 1) \) yielding eight singlets. Similarly, for the M5/MW intersection, we consider a 2/9 split, with the wave running along the two-dimensional longitudinal space. Since the \( \hat{D} \) structure group is \( SO(16) \times SO(16) \) and the maximal compact subgroup of \( SU^*(8) \) is \( USp(8) \), we obtain the decomposition \( 32 \rightarrow (16, 1) + (1, 16) \rightarrow 4(4, 1) + (1, 8) + 8(1, 1, 1) \) under \( SO(32) \supset SO(16) \times SO(16) \supset SO(5) \times USp(8) \). This again yields \( n = 8 \). Note, however, that this analysis fails for the plane waves, as \( \mathbb{R}^9 \) has no compact subgroups.

A different approach to supersymmetric vacua in M-theory is through the technique of \( G \)-structures \cite{42}. Hull \cite{41} has suggested that \( G \)-structures may be better suited to finding supersymmetric solutions whereas generalized holonomy may be better suited to classifying
them. In any event, it would be useful to establish a dictionary for translating one technique into the other.

Ultimately, one would hope to achieve a complete classification of vacua for the full M-theory. In this regard, one must at least include the effects of M-theoretic corrections to the supergravity field equations and Killing spinor equations and perhaps even go beyond the geometric picture altogether. It seems likely, however, that counting supersymmetries by the number of singlets appearing in the decomposition 32 of $\text{SL}(32, \mathbb{R})$ under $\mathcal{H} \subset \text{SL}(32, \mathbb{R})$ will continue to be valid.

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