Algebraically grid-like graphs have large tree-width

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Abstract
By the Grid Minor Theorem of Robertson and Seymour, every graph of sufficiently large tree-width contains a large grid as a minor. Tree-width may therefore be regarded as a measure of 'grid-likeness' of a graph.

The grid contains a long cycle on the perimeter, which is the $\mathbb{F}_2$-sum of the rectangles inside. Moreover, the grid distorts the metric of the cycle only by a factor of two. We prove that every graph that resembles the grid in this algebraic sense has large tree-width:

Let $k, p$ be integers, $\gamma$ a real number and $G$ a graph. Suppose that $G$ contains a cycle of length at least $2\gamma pk$ which is the $\mathbb{F}_2$-sum of cycles of length at most $p$ and whose metric is distorted by a factor of at most $\gamma$. Then $G$ has tree-width at least $k$.

1 Introduction
For a positive integer $n$, the $(n \times n)$-grid is the graph $G_n$ whose vertices are all pairs $(i, j)$ with $1 \leq i, j \leq n$, where two points are adjacent when they are at Euclidean distance 1. The cycle $C_n$, which bounds the outer face in the natural drawing of $G_n$ in the plane, has length $4(n - 1)$ and is the $\mathbb{F}_2$-sum of the rectangles bounding the inner faces. This is by itself not a distinctive feature of graphs with large tree-width: The situation is similar for the $n$-wheel $W_n$, the graph consisting of a cycle $D_n$ of length $n$ and a vertex $x \notin D_n$ which is adjacent to every vertex of $D_n$. There, $D_n$ is the $\mathbb{F}_2$-sum of all triangles $xyz$ for $yz \in E(D_n)$. Still, $W_n$ only has tree-width 3.

The key difference is the fact that in the wheel, the metric of the cycle is heavily distorted: any two vertices of $D_n$ are at distance at most 2 within $W_n$, even if they are far apart within $D_n$. In the grid, however, the distance between two vertices of $C_n$ within $G_n$ is at least half of their distance within $C_n$.

In order to incorporate this factor of two and to allow for more flexibility, we equip the edges of our graphs with lengths. For a graph $G$, a length-function on $G$ is simply a map $\ell : E(G) \to \mathbb{R}_{>0}$. We then define the $\ell$-length $\ell(H)$ of a subgraph $H \subseteq G$ as the sum of the lengths of all edges of $H$. This naturally...
induces a notion of distance between two vertices of $G$, where we define $d^\ell_G$ as the minimum $\ell$-length of a path containing both. A subgraph $H \subseteq G$ is $\ell$-geodesic if it contains a path of length $d^\ell_G(a, b)$ between any two vertices $a, b \in V(H)$.

When no length-function is specified, the notions of length, distance and geodecity are to be read with respect to $\ell \equiv 1$ constant.

On the grid-graph $G_n$, consider the length-function $\ell$ which is equal to 1 on $E(C_n)$ and assumes the value 2 elsewhere. Then $C_n$ is $\ell$-geodesic of length $\ell(C_n) = 4(n-1)$ and the sum of cycles of $\ell$-length at most 8. We show that any graph which shares this algebraic feature has large tree-width.

**Theorem 1.** Let $k$ be a positive integer and $r > 0$. Let $G$ be a graph with rational-valued length-function $\ell$. Suppose $G$ contains an $\ell$-geodesic cycle $C$ with $\ell(C) \geq 2rk$, which is the $\mathbb{F}_2$-sum of cycles of $\ell$-length at most $r$. Then the tree-width of $G$ is at least $k$.

The starting point of Theorem 1 was a similar result of Matthias Hamann and the author [2]. There, it is assumed that not only the fixed cycle $C$, but the whole cycle space of $G$ is generated by short cycles.

**Theorem 2** ([2, Corollary 3]). Let $k, p$ be positive integers. Let $G$ be a graph whose cycle space is generated by cycles of length at most $p$. If $G$ contains a geodesic cycle of length at least $kp$, then the tree-width of $G$ is at least $k$.

It should be noted that Theorem 2 is not implied by Theorem 1 as the constant factors are different. In fact, the proofs are also quite different, although Lemma 5 below was inspired by a similar parity-argument in [2].

It is tempting to think that, conversely, Theorem 1 could be deduced from Theorem 2 by adequate manipulation of the graph $G$, but we have not been successful with such attempts.

## 2 Proof of Theorem 1

The relation to tree-width is established via a well-known separation property of graphs of bounded tree-width, due to Robertson and Seymour [3].

**Lemma 3** ([3]). Let $k$ be a positive integer, $G$ a graph and $A \subseteq V(G)$. If the tree-width of $G$ is less than $k$, then there exists $X \subseteq V(G)$ with $|X| \leq k$ such that every component of $G - X$ contains at most $|A \setminus X|/2$ vertices of $A$.

It is not hard to see that Theorem 1 can be reduced to the case where $\ell \equiv 1$. This case is treated in the next theorem.

**Theorem 4.** Let $k, p$ be positive integers. Let $G$ be a graph containing a geodesic cycle $C$ of length at least $4|p/2|k$, which is the $\mathbb{F}_2$-sum of cycles of length at most $p$. Then for every $X \subseteq V(G)$ of order at most $k$, some component of $G - X$ contains at least half the vertices of $C$.
Proof of Theorem 1 assuming Theorem 4. Let $\mathcal{D}$ be a set of cycles of length at most $r$ with $C = \bigoplus \mathcal{D}$.

Since $\ell$ is rational-valued, we may assume that $r \in \mathbb{Q}$, as the premise also holds for $r'$ the maximum $\ell$-length of a cycle in $\mathcal{D}$. Take an integer $M$ so that $rM$ and $\ell'(e) := M\ell(e)$ are natural numbers for every $e \in E(G)$.

Obtain the subdivision $G'$ of $G$ by replacing every $e \in E(G)$ by a path of length $\ell'(e)$. Denote by $C', D'$ the subdivisions of $C$ and $D \in \mathcal{D}$, respectively. Then $C' = \bigoplus_{D \in \mathcal{D}} D'$ and $|C'| = M\ell(C) \geq 2(Mr)k$, while $|D'| = M\ell(D) \leq Mr$ for every $D \in \mathcal{D}$. By Theorem 4 for every $X \subseteq V(G')$ with $|X| \leq k$ there exists a component of $G' - X$ that contains at least half the vertices of $C'$.

By Lemma 3, $G'$ has tree-width at least $k$. Since tree-width is invariant under subdivision, the tree-width of $G$ is also at least $k$.

Our goal is now to prove Theorem 4. The proof consists of two separate lemmas. The first lemma involves separators and $\mathbb{F}_2$-sums of cycles.

Lemma 5. Let $G$ be a graph, $C \subseteq G$ a cycle and $\mathcal{D}$ a set of cycles in $G$ such that $C = \bigoplus \mathcal{D}$. Let $\mathcal{R}$ be a set of disjoint vertex-sets of $G$ such that for every $R \in \mathcal{R}$, $R \cap V(C)$ is either empty or induces a connected subgraph of $C$. Then either some $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$ or there is a component $Q$ of $G - \bigcup \mathcal{R}$ with $V(C) \subseteq V(Q) \cup \bigcup \mathcal{R}$.

Proof. Suppose that no $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$. Then $C$ has no edges between the sets in $\mathcal{R}$: Any such edge would have to lie in at least one $D \in \mathcal{D}$. Let $Y := \bigcup \mathcal{R}$ and let $Q$ be the set of components of $G - Y$.

Let $Q \in \mathcal{Q}$, $R \in \mathcal{R}$ and $D \in \mathcal{D}$ arbitrary. If $D$ has an edge between $Q$ and $R$, then $D$ cannot meet $Y \setminus R$. Therefore, all edges of $D$ between $Q$ and $V(G) \setminus Q$ must join $Q$ to $R$. As $D$ is a cycle, it has an even number of edges between $Q$ and $V(G) \setminus Q$ and thus between $Q$ and $R$. As $C = \bigoplus \mathcal{D}$, we find

$$e_C(Q, R) \equiv \sum_{D \in \mathcal{D}} e_D(Q, R) \equiv 0 \mod 2.$$ 

For every $R \in \mathcal{R}$ which intersects $C$, there are precisely two edges of $C$ between $R$ and $V(C) \setminus R$, because $R \cap C$ is connected. As mentioned above, $C$ contains no edges between $R$ and $Y \setminus R$, so both edges join $R$ to $V(G) \setminus Y$. But $C$ has an even number of edges between $R$ and each component of $V(G) \setminus Y$, so it follows that both edges join $R$ to the same $Q(R) \in \mathcal{Q}$.

Since every component of $C - (C \cap Y)$ is contained in a component of $G - Y$, it follows that there is a $Q \in \mathcal{Q}$ containing all vertices of $C$ not contained in $Y$.

To deduce Theorem 4, we want to apply Lemma 5 to a suitable family $\mathcal{R}$ with $\bigcup \mathcal{R} \supseteq X$ to deduce that some component of $G - X$ contains many vertices of $C$. Here, $\mathcal{D}$ consists of cycles of length at most $\ell$, so if the sets in $\mathcal{R}$ are at pairwise distance $> \lfloor \ell/2 \rfloor$, then no $D \in \mathcal{D}$ can pass through two of them. The next lemma ensures that we can find such a family $\mathcal{R}$ with a bound on $|\bigcup \mathcal{R}|$, when the cycle $C$ is geodesic.
Lemma 6. Let $d$ be a positive integer, $G$ a graph, $X \subseteq V(G)$ and $C \subseteq G$ a geodesic cycle. Then there exists a family $\mathcal{R}$ of disjoint sets of vertices of $G$ with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|\bigcup \mathcal{R} \cap V(C)| \leq 2d|X|$ such that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of $C$ and the distance between any two sets in $\mathcal{R}$ is greater than $d$.

Proof. Let $Y \subseteq V(G)$ and $y \in Y$. For $j \geq 0$, let $B^j_{t}(y)$ be the set of all $z \in Y$ at distance at most $jd$ from $y$. Since $|B^1_{t}(y)| = 1$, there is a maximum number $j$ for which $|B^j_{t}(y)| \geq 1 + j$, and we call this $j = j_Y(y)$ the range of $y$ in $Y$. Observe that every $z \in Y \setminus B^{j_Y(y)}(y)$ has distance greater than $(j_Y(y) + 1)d$ from $y$.

Starting with $X_1 := X$, repeat the following procedure for $k \geq 1$. If $X_k \cap V(C)$ is empty, terminate the process. Otherwise, pick an $x_k \in X_k \cap V(C)$ of maximum range in $X_k$. Let $j_k := j_{x_k}(x_k)$ and $B_k := B^{j_k}_{x_k}(x_k)$. Let $X_{k+1} := X_k \setminus B_k$ and repeat.

Since the size of $X_k$ decreases in each step, there is a smallest integer $m$ for which $X_{m+1} \cap V(C)$ is empty, at which point the process terminates. By construction, the distance between $B_k$ and $X_{k+1}$ is greater than $d$ for each $k \leq m$. For each $1 \leq k \leq m$, there are two edge-disjoint paths $P^1_k, P^2_k \subseteq C$, starting at $x_k$, each of length at most $j_kd$, so that $B_k \cap V(C) \subseteq S_k := P^1_k \cup P^2_k$. Choose these paths minimal, so that the endvertices of $S_k$ lie in $B_k$. Note that every vertex of $S_k$ has distance at most $j_kd$ from $x_k$. Therefore, the distance between $R_k := B_k \cup S_k$ and $X_{k+1}$ is greater than $d$.

We claim that the distance between $R_k$ and $R_{k'}$ is greater than $d$ for any $k < k'$. Since $B_{k'} \subseteq X_{k+1}$, it is clear that every vertex of $B_{k'}$ has distance greater than $d$ from $R_k$. Take a vertex $q \in S_{k'} \setminus R_{k'}$ and assume for a contradiction that its distance to $R_k$ was at most $d$. Then the distance between $x_k$ and $q$ is at most $(j_k + 1)d$. Let $a, b \in B_{k'}$ be the endvertices of $S_{k'}$. If $x_k \notin S_{k'}$, then one of $a$ and $b$ lies on the shortest path from $x_k$ to $q$ within $C$ and therefore has distance at most $(j_k + 1)d$ from $x_k$. But then, since $j_k$ is the range of $x_k$ in $X_k$, that vertex would already lie in $B_k$, a contradiction. Suppose now that $x_k \in S_{k'}$. Then $x_k$ lies on the path in $S_{k'}$ from $x_k$ to one of $a$ or $b$, so the distance between $x_k$ and $x_{k'}$ is at most $j_{k'}d$. Since $x_{k'} \in X_k \cap V(C)$, it follows from our choice of $x_k$ that

$$j_k = j_{x_k}(x_k) \geq j_{x_k}(x_{k'}) \geq j_{x_{k'}}(x_{k'}) = j_{k'},$$

where the second inequality follows from the fact that $X_{k'} \subseteq X_k$ and $j_Y(y) \geq j_{Y'}(y)$ whenever $Y \supseteq Y'$. But then $x_{k'} \in B_{k'}$, a contradiction. This finishes the proof of the claim.

Finally, let $\mathcal{R} := \{R_k : 1 \leq k \leq m\} \cup \{X_{m+1}\}$. The distance between any two sets in $\mathcal{R}$ is greater than $d$. For $k \leq m$, $R_k \cap V(C) = S_k$ is a connected
subgraph of $C$, while $X_{m+1} \cap V(C)$ is empty. Moreover,

$$| \bigcup \mathcal{R} \cap V(C) | = \sum_{k=1}^{m} |S_k| \leq \sum_{k=1}^{m} (1 + 2j_k d)$$

$$\leq \sum_{k=1}^{m} (1 + 2|B_k|-1)d$$

$$\leq \sum_{k=1}^{m} 2|B_k|d \leq 2d|X|.$$ 

\[ \Box \]

**Proof of Theorem 4.** Let $X \subseteq V(G)$ of order at most $k$ and let $d := \lfloor p/2 \rfloor$. By Lemma 6, there exists a family $\mathcal{R}$ of disjoint sets of vertices of $G$ with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|\bigcup \mathcal{R} \cap V(C)| \leq 2dk$ so that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of $C$ and the distance between any two sets in $\mathcal{R}$ is greater than $d$.

Let $\mathcal{D}$ be a set of cycles of length at most $p$ with $C = \bigoplus \mathcal{D}$. Then no $D \in \mathcal{D}$ can meet two distinct $R, R' \in \mathcal{R}$, since the diameter of $D$ is at most $d$. By Lemma 5, there is a component $Q$ of $G - \bigcup \mathcal{R}$ which contains every vertex of $C \setminus \bigcup \mathcal{R}$. This component is connected in $G - X$ and therefore contained in some component $Q'$ of $G - X$, which then satisfies

$$|Q' \cap V(C)| \geq |C| - |\bigcup \mathcal{R} \cap V(C)| \geq |C| - 2dk.$$ 

Since $|C| \geq 4dk$, the claim follows. \[ \Box \]

### 3 Remarks

We have described the content of Theorem 4 as an *algebraic* criterion for a graph to have large tree-width. The reader might object that the cycle $C$ being $\ell$-geodesic is a metric property and not an algebraic one. Karl Heuer has pointed out to us, however, that geodecity of a cycle can be expressed as an algebraic property after all. This is a consequence of a more general lemma of Gollin and Heuer [1], which allowed them to introduce a meaningful notion of geodecity for cuts.

**Proposition 7 (1).** Let $G$ be a graph with length-function $\ell$ and $C \subseteq G$ a cycle. Then $C$ is $\ell$-geodesic if and only if there do not exist cycles $D_1, D_2$ with $\ell(D_1), \ell(D_2) < \ell(C)$ such that $C = D_1 \oplus D_2$.

Finally, we’d like to point out that Theorem 4 does not only offer a ‘one-way criterion’ for large tree-width, but that it has a qualitative converse. First, we recall the Grid Minor Theorem of Robertson and Seymour [4], phrased in terms of walls. For a positive integer $t$, an *elementary* $t$-wall is the graph obtained from the $2t \times t$-grid as follows. Delete all edges with endpoints $(i, j), (i, j+1)$.
when \(i\) and \(j\) have the same parity. Delete the two resulting vertices of degree one. A \(t\)-wall is any subdivision of an elementary \(t\)-wall. Note that the \((2t \times 2t)\)-grid has a subgraph isomorphic to a \(t\)-wall.

**Theorem 8** (Grid Minor Theorem [4]). **For every** \(t\) **there exists a** \(k\) **such that every graph of tree-width at least** \(k\) **contains a** \(t\)-wall.

Here, then, is our qualitative converse to Theorem 1, showing that the algebraic condition in the premise of Theorem 1 in fact captures tree-width.

**Corollary 9.** **For every** \(L\) **there exists a** \(k\) **such that for every graph** \(G\) **the following holds. If** \(G\) **has tree-width at least** \(k\) **then there exists a rational length-function on** \(G\) **so that** \(G\) **contains a** \(\ell\)-geodesic cycle \(C\) **with** \(\ell(C) \geq L\) **which is the** \(\mathbb{F}_2\)-sum of cycles of \(\ell\)-length at most 1.

**Proof.** Let \(s := 3L\). By the Grid Minor Theorem, there exists an integer \(k\) so that every graph of tree-width at least \(k\) contains an \(s\)-wall. Suppose \(G\) is a graph of tree-width at least \(k\). Let \(W\) be an elementary \(s\)-wall so that \(G\) contains some subdivision \(W'\) of \(W\), where \(e \in E(W)\) has been replaced by some path \(P_e \subseteq G\) of length \(m(e)\).

The outer cycle \(C\) of \(W\) satisfies \(d_C(u, v) \leq 3d_W(u, v)\) for all \(u, v \in V(C)\). Moreover, \(C\) is the \(\mathbb{F}_2\)-sum of cycles of length at most six.

Define a length-function \(\ell\) on \(G\) as follows. Let \(e \in E(G)\). If \(e \in P_f\) for \(f \in E(C)\), let \(\ell(e) := 1/m(f)\). Then \(\ell(P_f) = 1\) for every \(f \in E(C)\). If \(e \in P_f\) for \(f \in E(W) \setminus E(C)\), let \(\ell(e) := 3/m(f)\). Then \(\ell(P_f) = 3\) for every \(f \in E(W) \setminus E(C)\). If \(e \notin E(W')\), let \(\ell(e) := 10s^3\), so that \(\ell(e) > \ell(W')\).

It is easy to see that the subdivision \(C' \subseteq G\) of \(C\) is \(\ell\)-geodesic in \(G\). It has length \(\ell(C') = |C| \geq 6s\) and is the \(\mathbb{F}_2\)-sum of the subdivisions of 6-cycles of \(W\). Each of these satisfies \(\ell(D) \leq 18\). Rescaling all lengths by a factor of 1/18 yields the desired result.

\[\square\]

**References**

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