Transient hydraulic calculations with a linear turbine model derived from a nonlinear synthetic model

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Abstract. When doing transient and dynamic analysis of powerplants, mathematical representation of the turbine is needed. Linear models lend themselves easily to be used in combination with control algorithms, block diagrams, as well as for more qualitative analysis and parameter studies. A typical linear model consists of characteristic values representing the gradients. Thus, when these characteristic values are known, a linear turbine model is easy to implement into existing platforms, such as Simulink, Xcos or similar. The characteristic values for the linear model are derived from linearizing a fully synthetic nonlinear model. This nonlinear model uses a first principles approach based on the Euler turbine equations and the operating degree, thus is independent from measured data, and only need input such as nominal head and flow. This paper also presents a new derivation of some of the parts of the non-linear model.

1. Introduction

Accurate transient and stability analysis of hydropower plants is done with specialized software with high complexity and high accuracy. Examples are LVTrans\textsuperscript{[1],[2]} and Alab\textsuperscript{[3]}, amongst others. More general simulation software like Simulink, Xcos, Modelica, are available for a much greater audience and use cases, but they do not lend themselves easily for analysis of hydropower plants. One reason is the difficulty of modelling the plant with sufficient accuracy without using tailor made code and solvers, which will nullify the effort. Another reason is the lack of a turbine model that is physically correct yet is easily modelled without detailed knowledge of the performance chart. The result of this is that oversimplifications are often done.

These highly detailed tailor made software packages, will likely remain with us for the foreseeable future. Still, there are lots of use cases where pinpoint accuracy is not of importance. This does not mean that the simplest possible solution always is adequate. If there are ways to increase the accuracy and the physical correctness, without a huge increase in complexity and detail, suitable for these highly available software packages in widespread use, this surely must be useful.

Typical use cases are; More correct simulations of huge grids. Inclusion of online models in more sophisticated governing systems \textsuperscript{[4],[5],[16]}. Teaching and research. Another use case, which emerges implicitly from the first principles approach, is better understanding of the physical parameters governing turbine transients. This paper presents a linear turbine model derived from a nonlinear analytical model based on Euler turbine equations. This is a first principles approach, although measured charts can readily be included in the non-linear case for added accuracy.
The paper also presents a derivation of the non-linear model from a different angle that is hopefully more intuitive than the original derivation found in [6] or [7].

2. Governing equations
All turbines can be described by the Euler turbine equation. This equation relates hydraulic- and mechanical energy through the efficiency. The variables are as shown in Figure 2.1.

\[ \eta_h = \frac{1}{gH_e} (c_{u1} u_1 - c_{u2} u_2) = \frac{\omega_t}{gH_e} (c_{u1} r_1 - c_{u2} r_2) \]  

(1)

\[ gH_e = g(H_1 - H_2) = \frac{1}{\eta_h} \left[ \frac{1}{2} (c_1^2 - c_2^2) - \frac{1}{2} (v_1^2 - v_2^2) + s_D \omega_t^2 \right] \]  

(3)

\[ s_D = \frac{1}{8} (D_1^2 - D_2^2) \]  

(4)

Equation (1), or equation (3) and (4) general. They are valid for every operating point a turbine may enter. For each operating point there will always exist a numerical value for \( \eta_h \), positive or negative, that makes the relation true, even if those values only can be found accurately by measurements. This is visualized in Figure 2.3.
Figure 2.3. Hill chart with validity of Euler turbine equation. Equation (1), (3) and (5). The efficiency is included, but unknown.

Figure 2.4. Hill chart with validity of equation (9). The efficiency is not included.

\[ gH_e = g(H_1 - H_2) = \frac{1}{2}(c_1^2 - c_2^2) - \frac{1}{2}(v_1^2 - v_2^2) + s_D\omega_t^2 + F_1 \]  \hspace{1cm} (5)

Equation (3) is general, but not very practical. A more practical equation is obtained by rewriting the losses represented by \( \eta_h \) as a general loss term \( F_1 \) without losing any generality. Another important relation for a turbine is the opening degree, \( \kappa \), defined as:

\[ \kappa = \left( \frac{Q}{\sqrt{2gH_e}} \right) / \left( \frac{Q_R}{\sqrt{2gH_{er}}} \right) \text{ yields } gH_e = gH_{er} \left( \frac{Q}{\kappa Q_R} \right)^2 \]  \hspace{1cm} (6)

Where subscript \( R \) denotes "rated" variables. Here the meaning is at “best efficiency point”, (BEP). The opening degree relates the flow, \( Q \), to the total energy, \( gH_e \), with rated values and \( \kappa \). It contains no information of the speed, \( \omega \), or the efficiency \( \eta_h \). It cannot in general be valid for other speeds than rated speed and cannot include any information about the efficiency. It is simply a valve equation as described in [8].

Equation (5) and equation (6) are both expressions of the total energy. Combining them, including the restriction of validity for the opening degree, yields:

\[ gH_R \left( \frac{Q}{\kappa Q_R} \right)^2 = \frac{1}{2}(c_1^2 - c_2^2) - \frac{1}{2}(v_1^2 - v_2^2) + s_D\omega_t^2 + F_1 \]  \hspace{1cm} (7)

Equation (7) relates the opening degree and flow to the velocities and the total loss. The equation is only valid for rated speed, but otherwise also includes the efficiency or losses. The equation is therefore of general validity at rated speed only, but also includes the geometric relation \( s_D \).

\[ \frac{1}{2}(c_1^2 - c_2^2) - \frac{1}{2}(v_1^2 - v_2^2) + F_1 = gH_R \left( \frac{Q}{\kappa Q_R} \right)^2 - s_D\omega_t^2 \]  \hspace{1cm} (8)

Rewriting equation (7) and inserting equation (8) into equation (5):
\[ g(H_1 - H_2) = gH_R \left( \frac{Q}{kQ_R} \right)^2 + s_D (\omega_k^2 - \omega_R^2) \]  

Equation (9) is thus valid in the entire area, as shown in Figure 2.4. Equation (9) is a modified expression of the opening degree that also includes the important term; \( s_D (\omega_k^2 - \omega_R^2) \), making it valid for all speeds. It's an accurate expression of the head-flow-speed relation, except all references to the efficiency is lost. The efficiency will be handled later.

The torque of a turbine is [6]:

\[ T = \rho Q (c_{u1} r_1 - c_{u2} r_2) = \rho Q (t_s - r_2^2 \omega_t) \]

\[ t_s = r_1 c_1 \cos(\alpha_1) + r_2 A_z \cot(\beta_2) c_1 \sin(\alpha_1) \]  

Here \( t_s \) is the starting torque when \( \omega_t = 0 \) [7]. \( A_z \) is the inlet area divided by the outlet area perpendicular to the shaft. It is useful at this point to restrict the derivation and handle other losses separately. The restriction is as shown with the dotted line in Figure 2.5.

**Figure 2.5.** restriction of the derivation in this paper

3. Non-dimensional equations

Equation (9) and equation (10) may be used, but it’s not very practical. The efficiency also needs to be included somehow. Non-dimensional variables are made by dividing by the rated values:

\[ T_R = \frac{\eta_{hr} \rho g Q_R H e_R}{\omega t_R} \]

\[ \eta_h = \frac{T \omega_t}{\rho g Q H_e} = \frac{\eta_{hr} \rho g Q_R H e_R}{\omega t_R^2} \frac{t \omega t R \omega}{q h_e} = \frac{\eta_{hr} t \omega}{q h_e} \]

\[ \tilde{\eta}_h = \frac{\eta_h}{\eta_{hr}} = \frac{t \omega}{q h_e} \]  

The “non-dimensional” or “per unit” efficiency is shown in equation (14). Equation (9) and equation (10) with their geometrical relations [7] become:

\[ h_e = \left( \frac{q}{k} \right)^2 + \sigma (\omega^2 - 1) \Rightarrow q = k \sqrt{h_e - \sigma (\omega^2 - 1)} \]

\[ t = q (m_s - \psi \omega) \quad \text{Pelton} \Rightarrow t = q \left( \frac{2}{k} \frac{q}{k} - 2 \psi \omega \right) \]

\[ \sigma = s_D \frac{\omega_R^2}{g H_R}, \quad m_s = \xi \frac{q}{k} (\cos(\alpha_1) + \tan(\alpha_{3R}) \sin(\alpha_1)) \]
\[ \psi = \frac{u^2_R}{g H_R}, \quad \xi = \frac{1 + \psi}{\cos(\alpha_{1R}) + \tan(\alpha_{1R}) \sin(\alpha_{1R})}, \quad \kappa = \frac{\sin(\alpha_1)}{\sin(\alpha_{1R})} \] (18)

Here \( m_s \) is the non-dimensional version of \( t_s \).

4. The hydraulic efficiency

The hydraulic efficiency is defined by the Euler turbine equation, equation (1). The corresponding per unit version using the derived equations becomes:

\[ \bar{\eta}_h = \frac{t \sigma}{q h_e} = \left( \frac{m_s - \psi \sigma}{h_e} \right) \frac{(m_s - \psi \sigma) \sigma}{\left( \frac{q \sigma}{\kappa} \right)^2 + \sigma(\alpha^2 - 1)} \] (19)

Inspecting equation (19), one can see that for each opening degree, \( \kappa \), the hydraulic efficiency is a function of \( \sigma \) and \( q \). The rest of the variables are geometrical constants suitable for one particular runner design. It is therefore useful to envision the efficiency as a 3D surface with \( \sigma \) and \( q \) as x and y axis, a variant of displaying a so-called hill chart for a turbine. An example is shown in Figure 4.1.

Along constant \( \kappa \) lines the efficiency is predicted well, especially for \( \kappa = 1 \). The reason for this is that along these lines, the efficiency is for the most part, determined by the speed of the turbine, running too fast or too slow compared with optimal inlet and outlet angles [6][7]. The losses are not utilized kinetic energy (the fluid is spinning in the draft tube or exiting from the Pelton buckets with high velocity). Even for \( \kappa \neq 1 \), the relative efficiency compared to the optimum at that specific \( \kappa \) is assumed to be good. However, along the q-axis, keeping \( \sigma \) constant, there’s a different story. This is due to the derivation of equation (9), where all references to the efficiency was lost. The root cause is of course the opening degree itself, where the loss is simply a constant, \( Q_R / \sqrt{H_e R} \). This constant is only correct at rated values. The parameter, \( m_s \), has some effect of loss because it is a function of \( q \), but it is not enough as shown in [6].

Since a good prediction of the efficiency is already achieved along the \( \sigma \)-axis (or along the constant \( \kappa \)-curves), what is needed to complete the diagram, is a good approximation of the efficiency along the \( q \)-axis for \( \sigma = 1 \). That curve is essentially the curve one gets when measurements the efficiency on site. For the remainder of this paper, the approximation and method in [7] is used, radically different and simpler from the method described in [6]. The method is to multiply the efficiency with the incipient efficiency, \( \eta_i \). The incipient efficiency could be any approximation curve, or measured data, but a first order approximation is:

\[ \eta_i = q(2 - q) \] (20)

This curve is 0 at \( q = 0 \) and \( q = 2 \), and 1 at \( q = 1 \). It is a parabola. In this paper, the curve is inserted into the torque equation, but could also be inserted in the head equation instead:

\[ t = \eta_i q (m_s - \psi \sigma) \] (21)

\[ h_e = \frac{\left( \frac{q \sigma}{\kappa} \right)^2 + \sigma(\sigma^2 - 1)}{\eta_i} \] (22)

Equation (22) is perhaps the most intuitive one, because \( \eta_i \) directly represents losses that disappeared in equation (9). Equation (21) is a bit more practical in a fully analytical approach. Even though
mathematically equivalent as far as the efficiency goes, it’s not arbitrary which version is used, as this depends on how the block diagram is set up.

5. Accuracy of the non-linear equations
The accuracy here lies in the physical correctness. All the equations are physical relations that cannot themselves be wrong. Therefore, this model is physically correct, at least as much as one can hope for considering the equations are 1D. The errors stem mainly from the loss functions, and a 1D representation of a 3D object. The incipient efficiency function used here, equation (19), looks perhaps overly simplistic, but it captures all the main physical aspects of the loss along the q-axis.

- At BEP, the efficiency is the rated efficiency
- At \( q \approx 0 \), the efficiency is 0 (usually \( q \) is slightly higher than zero)
- At \( q > 1 \), the efficiency decreases monotonically towards \( q = 2 \).

Further, the equations capture all the characteristics inherent in a turbine, because these characteristics are intrinsic parts of the Euler turbine equation. It is these characteristics that are of interest.

For the sake of completeness, these equations are easily tuned to measurements when the hill charts are known. This is a bit fiddly, but with the loss model presented in [7], and used in this paper, a much more direct approach is possible.

The first step is to take a measured efficiency curve at rated speed, scale it so it has 100% efficiency at BEP, and use that as the incipient efficiency. This will make the diagram highly accurate for all \( q \) and \( \kappa \) at rated speed and for some band along the speed axis. The next step is to take the runaway curve, and use that to fine tune the parameter \( \xi \) and the rotating frictional losses. This can be done easily remembering that the efficiency is zero at runaway.

6. Linearization
Equation (15), (16), (21) and equation (22) are two practical sets of equations on the form (subscript, \( e \), removed for clarity):

\[
h = h[q, \kappa, \varpi, \eta_i(q)] \quad \text{and} \quad t = t(q, \kappa, \varpi)
\]

or

\[
q = q(h, \kappa, \varpi) \quad \text{and} \quad t = t[q, \kappa, \varpi, \eta_i(q)]
\]
Denoting the steady state point with subscript a, and the variation around that point for: \( \Delta x = x - x_a \), the linearized equations take the form:

\[
\Delta q = a_{11} \Delta h + a_{12} \Delta \kappa + a_{13} \Delta \sigma \\
\Delta h = b_{11} \Delta q + b_{12} \Delta \kappa + b_{13} \Delta \sigma \\
\Delta t = a_{21} \Delta q + a_{22} \Delta \kappa + a_{23} \Delta \sigma
\]

or

\[
a_{11} = \left( \frac{\partial q}{\partial h} \right)_a, \quad a_{12} = \left( \frac{\partial q}{\partial \kappa} \right)_a, \quad a_{13} = \left( \frac{\partial q}{\partial \sigma} \right)_a \\
b_{11} = \left( \frac{\partial h}{\partial q} \right)_a, \quad b_{12} = \left( \frac{\partial h}{\partial \kappa} \right)_a, \quad b_{13} = \left( \frac{\partial h}{\partial \sigma} \right)_a
\]

and

\[
a_{21} = \left( \frac{\partial t}{\partial q} \right)_a, \quad a_{22} = \left( \frac{\partial t}{\partial \kappa} \right)_a, \quad a_{23} = \left( \frac{\partial t}{\partial \sigma} \right)_a
\]

A fully analytical derivation of the characteristics can be found in [9]. If an analytical approach is chosen, or an analytical \( \eta_i \) is used, then equation (24) is the most practical one. A numerical calculation of the characteristics is a straight forward numerical Jacobi kind of procedure, using either equation (23) or (24). If one also wants to include a measured \( \eta_i \), then this \( \eta_i \) must be a smoothed continuous function of \( q \), included using the product rule.

7. Obtaining parameters for a turbine

One of the advantages of this approach is that nothing needs to be known about the turbines, except rated flow and rated head. The reason is that the Euler turbine equation is always valid. The main dimensions are the inlet and outlet diagram, diameters and inlet/outlet areas. Simple procedures of obtaining main dimensions can be found in textbooks and lecture notes [10], [11], [12]. Much more refined automated iterative methods are used in [3] where the whole turbine is designed in detail. An example of such a simple procedure is given in [9].

8. Block diagram

A precursor to the method presented here was done by the student and the author in [13]. There a turbine model found in [14], [15] and [16] was used. The aim there was to use LVTrans [1], to find the characteristics used in that former mentioned model. This worked somewhat, but due to different variables used in the equations it simply became extremely convoluted. In that model the torque is a function of \( h, \kappa, \sigma \) instead of \( q, \kappa, \sigma \). It was therefore chosen to follow the Euler turbine equation for later work.

![Figure 8.1. A general block diagram of a hydro power plant](image-url)
Figure 8.1 shows a general block diagram of a powerplant. The block named “Waterway and turbine” is the interest here. The waterway and the power are described as (non-elastic penstock only for clarity):

\[
\Delta h = -\frac{Q_R L_s}{A g H_R} = -T_w s \Delta q, \quad \Delta p = \frac{\omega_R T_a}{P_R} \Delta t + \frac{\omega_R T_a}{P_R} \Delta \sigma \approx \Delta t + \Delta \sigma
\] (31)

![Block Diagram](image)

**Figure 8.2.** Waterway and turbine using the turbine model

9. Simulations

| Table 9.1. 3 different turbines with the same Ta and Tw |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| P_R [MW]        | η_R             | T_a [s]         | T_w [s]         | H_R [m]         | Q_R [m^3/s]    | L [m]           | A_ps [m^2]    | D_ps [m]       | Comment            |
| 90 m Francis    | 100             | 0.96            | 6              | 0.9             | 90             | 118            | 108           | 4.52            | 1.2*Hr             |
| 350 m Francis   | 100             | 0.96            | 6              | 0.9             | 350            | 30.3           | 420           | 2.29            | To get the same T_w |
| 600 m Pelton    | 100             | 0.93            | 6              | 0.9             | 600            | 18.3           | 720           | 1.78            |                     |

a_{11} = 0.50, a_{12} = 1.00, a_{13} = 0.06, a_{21} = 3.19, a_{22} = -2.19, a_{23} = -1.19, K_p = 1, T_i [s] = 10, droop = 0

Table 9.1 shows 3 different turbines, and their characteristics at BEP calculated using the presented model. At BEP, a_{11} and a_{12} becomes the same for all of them. It’s the linear representation of a valve. a_{21} + a_{22} = 1 for all of them at BEP. The two interesting parameters are a_{13} and a_{23}.

\[
a_{13} = \left(\frac{\partial q}{\partial \sigma}\right)_a = 2\sigma = \frac{1}{4} (D_1^2 - D_2^2) \frac{\omega_R^2}{g H_R}
\] (32)

\[
a_{23} = \left(\frac{\partial t}{\partial \sigma}\right)_a = -\psi = -\frac{u_{zR}^2}{g H_R} \quad \text{and for a Pelton} \quad a_{23} = -2 \frac{u_{zR}^2}{g H_R}
\] (33)

Both these two parameters act in a feedback loop from the speed in the block diagram. a_{13} can take both positive and negative values and the effect is the same as opening or closing the turbine. It’s the self-regulating parameter, a positive value causes instability, a negative causes stability. a_{23} is always
negative at BEP, but the magnitude changes according to the turbine. The parameter works directly on the torque and has a stabilizing effect.

Simulations were done using Xcos in Scilab [17] with a model shown in Figure 9.1

![Simulation model in Scilab](image)

**Figure 9.1. Simulation model in Scilab**

![Impulse response calculated with Xcos](image)

**Figure 9.2. Impulse response calculated with Xcos**

![Transient response calculated with Xcos](image)

**Figure 9.3. Transient response calculated with Xcos**

To visualize the different turbines with respect to their self-governing properties, a simulation is done without the PID, shown in Figure 9.2. The guide vanes are opened with an impulse of 10% amplitude and 5s duration. As expected, the low head Francis shows an unstable characteristic due to the positive $a_{13}$. The high head Francis shows a stable characteristic. The Pelton just stay there, because the only thing that can bring it back is increased friction, in particular shaft friction, which is not modelled here.

In Figure 9.3 the PID is connected. A speed step of -1% down is done followed by a step in $p_{grid}$ of -1%. The droop and power setpoint is set to zero. The PID parameters are given in Table 1. The graph shows a natural behaviour.

10. Conclusion

The turbine model was originally presented in [7]. It is used in LVTrans for full transient analysis using MOC. In this paper it is derived from a different angle and derived further as a straight forward linear model. The linear model is such that an arbitrary point of linearization can be done, without resorting to measured data, although measured data can improve the accuracy, and is much easier to include than previously using [6].

The use of this model is in relation to more advanced governors [4], [5], [16], both as a direct implementation in some MPC algorithm, and for studying those algorithms, and take advantage of the high-level modelling of Xcos or Simulink. Other use cases are general transient and stability studies, as part of the grid, or on their own.
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