Tetrahedral Coxeter groups, large group-actions on 3-manifolds and equivariant Heegaard splittings

Bruno P. Zimmermann

Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
34127 Trieste, Italy

Abstract. We consider finite group-actions on closed, orientable and nonorientable 3-manifolds $M$ which preserve the two handlebodies of a Heegaard splitting of $M$ of some genus $g > 1$ (maybe interchanging the two handlebodies). The maximal possible order of a finite group-action on a handlebody of genus $g > 1$ is $12(g - 1)$ in the orientation-preserving case and $24(g - 1)$ in general, and the maximal order of a finite group preserving the Heegaard surface of a Heegaard splitting of genus $g$ is $48(g - 1)$. This defines a hierarchy for finite group-actions on 3-manifolds which we discuss in the present paper; we present various manifolds with an action of type $48(g - 1)$ for small values of $g$, and in particular the unique hyperbolic 3-manifold with such an action of smallest possible genus $g = 6$ (in strong analogy with the Euclidean case of the 3-torus which has such actions for $g = 3$).

1. A hierarchy for large finite group-actions on 3-manifolds

The maximal possible order of a finite group $G$ of orientation-preserving diffeomorphisms of an orientable handlebody of genus $g > 1$ is $12(g - 1)$ ([Z1], [MMZ]); for orientation-reversing finite group-actions on an orientable handlebody and for actions on a nonorientable handlebody, the maximal possible order is $24(g - 1)$; we will always assume $g > 1$ in the present paper.

Let $G$ be a finite group of diffeomorphisms of a closed, orientable or nonorientable 3-manifold $M$. We consider Heegaard splittings of $M$ into two handlebodies of genus $g$ (nonorientable if $M$ is nonorientable) such that each element of $G$ either preserves both handlebodies or interchanges them. The maximal possible orders are $12(g - 1)$, $24(g - 1)$ or $48(g - 1)$, and we distinguish four types of $G$-actions:

Definition 1.1

1.1.1 The $G$-action is of type $12(g - 1)$:

$M$ is orientable, $G$ is orientation-preserving and non-interchanging (i.e., preserves both handlebodies of the Heegaard splitting), and $G$ is of maximal possible order $12(g - 1)$.
for such a situation (these actions are called *maximally symmetric* in various papers, see \([Z3,Z5,Z6]\)).

1.1.2 *The G-action is of non-interchanging type 24\((g - 1)\):*

either \(M\) is orientable and \(G\) orientation-preserving or \(M\) is non-orientable, and \(G\) is of maximal possible order 24\((g - 1)\) for such a situation (these actions are called *strong maximally symmetric* in \([Z8]\)).

1.1.3 *The G-action is of interchanging type 24\((g - 1)\):*

\(M\) is orientable, the subgroup \(G_0\) of index 2 preserving both handlebodies is orientation-preserving, and \(G\) is of maximal possible order 24\((g - 1)\) for such a situation.

1.1.4 *The G-action is of type 48\((g - 1)\):*

\(G\) is interchanging, either \(M\) is orientable and the subgroup \(G_0\) preserving both handlebodies is orientation-reversing, or \(M\) is non-orientable, and \(G\) is of maximal possible order 48\((g - 1)\) for such a situation.

The second largest orders in the four cases are 8\((g - 1)\), 16\((g - 1)\) and 32\((g - 1)\), then 20\((g - 1)\)/3, 40\((g - 1)\)/3 and 80\((g - 1)\), next 6\((g - 1)\), 12\((g - 1)\) and 24\((g - 1)\) etc.; in the present paper, we consider only the cases of largest possible orders 12\((g - 1)\), 24\((g - 1)\) and 48\((g - 1)\).

In section 2, we present examples of 3-manifolds for various of these types, and in particular the unique hyperbolic 3-manifold of type 48\((g - 1)\) of smallest possible genus \(g = 6\). The situation for the orientation-preserving actions of types 1.1.1 and 1.1.3 is quite flexible and has been considered in various papers (see \([Z3,Z5,Z6]\)), so in the present paper we concentrate mainly on the orientation-reversing actions and actions on non-orientable manifolds of cases 1.1.2 and 1.1.4 where the situation is more rigid. We finish this section with a short discussion of 3-manifolds of type 12\((g - 1)\), for small values of \(g\).

**Theorem 1.2.** ([Z7]) *The closed orientable 3-manifolds with a G-action of type 12\((g - 1)\) and of genus \(g = 2\) are exactly the 3-fold cyclic branched coverings of the 2-bridge links, the group \(G\) is isomorphic to the dihedral group \(D_6\) of order 12 and obtained as the lift of a symmetry group \(\mathbb{Z}_2 \times \mathbb{Z}_2\) of each such 2-bridge link.*

Examples of such 3-manifolds are the spherical Poincaré homology sphere (the 3-fold branched covering of the torus knot of type \((2,5)\)), the Euclidean Hantzsche-Wendt manifold (the 3-fold branched covering of the figure-8 knot, see \([Z2]\)) and the hyperbolic Matveev-Fomenko-Weeks manifold of smallest volume among all closed hyperbolic 3-manifolds (the 3-fold branched covering of the 2-bridge knot \(5_2\)).
Examples of 3-manifolds of type $12(g - 1)$ and genus 3 are the 3-torus and again the Euclidean Hantzsche-Wendt manifold, of genus 6 the spherical Poincaré homology 3-sphere and the hyperbolic Seifert-Weber dodecahedral space, see Corollaries 2.3, 2.4 and 2.5.

The finite groups $G$ which admit an action of type $12(g - 1)$ and genus $g \leq 6$ are $D_6$, $S_4$, $D_3 \times D_3$, $S_4 \times \mathbb{Z}_2$ and $A_5$ ([Z6]).

2. Tetrahedral Coxeter groups and large group-actions

2.1 Non-interchanging actions of type $24(g - 1)$ and actions of type $48(g - 1)$

The following is proved in [Z8].

Theorem 2.1. i) Let $M$ be a closed, irreducible 3-manifold with a non-interchanging $G$-action of type $24(g - 1)$. Then $M$ is spherical, Euclidean or hyperbolic and obtained as a quotient of the 3-sphere, Euclidean or hyperbolic 3-space by a normal subgroup $K$ of finite index, acting freely, in a spherical, Euclidean or hyperbolic Coxeter group $C(n, m; 2, 2; 2, 3)$ or in a twisted Coxeter group $C_\tau(n, m; 2, 2; 3, 3)$; the $G$-action is the projection of the Coxeter or twisted Coxeter group to $M$. Conversely, each such subgroup $K$ gives a $G$-action of type $24(g - 1)$ on the 3-manifold $M = \mathbb{H}^3/K$.

ii) The $G$-action of type $24(g - 1)$ on $M$ extends to a $G$-action of type $48(g - 1)$ if and only $n = m$ and the universal covering group $K$ of $M$ is a normal subgroup also of the twisted Coxeter group $C_\mu(n, n; 2, 2; 2, 3)$ or in the doubly-twisted Coxeter group $C_{\tau \mu}(n, n; 2, 2; 3, 3)$.

In Theorem 2.1, we use the notation in [Z8] which we now explain. A Coxeter tetrahedron is a tetrahedron in the 3-sphere, Euclidean or hyperbolic 3-space all of whose dihedral angles are of the form $\pi/n$ (denoted by a label $n$ of the edge, for some integer $n \geq 2$) and, moreover, such that at each of the four vertices the three angles of the adjacent edges define a spherical triangle (i.e., $1/n_1 + 1/n_2 + 1/n_3 > 1$). We will denote such a Coxeter tetrahedron by $C(n, m; a, b; c, d)$ where $(n, m)$, $(a, b)$ and $(c, d)$ are the labels of pairs of opposite edges, and we denote by $C(n, m; a, b; c, d)$ the Coxeter group generated by the reflections in the four faces of the tetrahedron $C(n, m; a, b; c, d)$, a properly discontinuous group of isometries of one of the three geometries. In the following, we list the Coxeter groups of the various types occuring in Theorem 2.1.

2.1.1 The Coxeter groups $C(n, m; 2, 2; 2, 3)$:

spherical: \[C(2, 2; 2, 2; 2, 3), \ C(2, 3; 2, 2; 2, 3), \ C(2, 4; 2, 2; 2, 3), \ C(2, 5; 2, 2; 2, 3), \]
\[C(3, 3; 2, 2; 2, 3), \ C(3, 4; 2, 2; 2, 3), \ C(3, 5; 2, 2; 2, 3);\]
Euclidean: \[C(4, 4; 2, 2; 2, 3);\]
hyperbolic: \[C(4, 5; 2, 2; 2, 3), \ C(5, 5; 2, 2; 2, 3).\]
A Coxeter tetrahedron $C(n, m; 2, 2; 3, 3)$ has a rotational symmetry $\tau$ of order two (an isometric involution) which exchanges the opposite edges with labels 2 and 3 and inverts the two edges with labels $n$ and $m$. The involution $\tau$ can be realized by an isometry and defines a twisted Coxeter group $C_\tau(n, m; 2, 2; 3, 3)$, a group of isometries containing the Coxeter group $C(n, m; 2, 2; 3, 3)$ as a subgroup of index two.

### 2.1.2 The twisted Coxeter groups $C_\tau(n, m; 2, 2; 3, 3)$:

- **spherical:** $C_\tau(2, 2; 2, 2; 3, 3)$, $C_\tau(2, 3; 2, 2; 3, 3)$, $C_\tau(2, 4; 2, 2; 3, 3)$;
- **Euclidean:** $C_\tau(3, 3; 2, 2; 3, 3)$;
- **hyperbolic:** $C_\tau(2, 5; 2, 2; 3, 3)$, $C_\tau(3, 4; 2, 2; 3, 3)$, $C_\tau(3, 5; 2, 2; 3, 3)$, $C_\tau(4, 4; 2, 2; 3, 3)$, $C_\tau(4, 5; 2, 2; 3, 3)$, $C_\tau(5, 5; 2, 2; 3, 3)$.

A Coxeter tetrahedron $C(n, n; 2, 2; 2, 3)$ has a rotational symmetry $\mu$ of order two (an isometric involution) which exchanges the opposite edges with labels $n$ and 2 and inverts the two remaining edges with labels 2 and 3. As before, this defines a twisted Coxeter group $C_\mu(n, n; 2, 2; 2, 3)$ containing $C(n, n; 2, 2; 2, 3)$ as a subgroup of index 2.

### 2.1.3 The twisted Coxeter groups $C_\mu(n, n; 2, 2; 2, 3)$:

- **spherical:** $C_\mu(2, 2; 2, 2; 2, 3)$, $C_\mu(3, 3; 2, 2; 2, 3)$;
- **Euclidean:** $C_\mu(4, 4; 2, 2; 2, 3)$;
- **hyperbolic:** $C_\mu(5, 5; 2, 2; 2, 3)$.

Finally, a Coxeter tetrahedron $C(n, n; 2, 2; 2, 3)$ has a group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of rotational isometries generated by involutions $\tau$ and $\mu$ as before, and hence defines a doubly-twisted Coxeter group $C_{\tau\mu}(n, n; 2, 2; 3, 3)$ containing $C(n, n; 2, 2; 2, 3)$ as a subgroup of index 4 (and both $C_\tau(n, n; 2, 2; 3, 3)$ and $C_\mu(n, n; 2, 2; 3, 3)$ as subgroups of index 2).

### 2.1.4 The doubly-twisted Coxeter groups $C_{\tau\mu}(n, n; 2, 2; 3, 3)$:

- **spherical:** $C_{\tau\mu}(2, 2; 2, 2; 3, 3)$;
- **Euclidean:** $C_{\tau\mu}(3, 3; 2, 2; 3, 3)$;
- **hyperbolic:** $C_{\tau\mu}(4, 4; 2, 2; 3, 3)$, $C_{\tau\mu}(5, 5; 2, 2; 3, 3)$.

In the following, we will discuss finite-index normal subgroups of small index, acting freely (i.e., torsion-free in the Euclidean and hyperbolic cases) of various Coxeter and tetrahedral groups (their orientation-preserving subgroups). Since this requires computational methods, we need presentations of the various groups.
2.2 Presentations of Coxeter and tetrahedral groups

Denoting by \( f_1, f_2, f_3 \) and \( f_4 \) the reflections in the four faces of a Coxeter polyhedron \( C(n, m; 2, 2; c, 3) \), the Coxeter group \( C(n, m; 2, 2; c, 3) \) has a presentation

\[
< f_1, f_2, f_3, f_4 \mid f_1^2 = f_2^2 = f_3^2 = f_4^2 = 1, \]

\[
(f_1 f_2)^c = (f_2 f_3)^2 = (f_3 f_4)^3 = (f_4 f_1)^2 = (f_1 f_3)^n = (f_2 f_4)^m = 1 > .
\]

A presentation of the twisted group \( C_\tau(n, m; 2, 2; 3, 3) \) is obtained by adding to this presentation a generator \( \tau \) and the relations

\[
\tau^2 = 1, \quad \tau f_1 \tau^{-1} = f_3, \quad \tau f_2 \tau^{-1} = f_4.
\]

If \( n = m \), for a presentation of \( C_\mu(n, n; 2, 2; c, 3) \) one adds a generator \( \mu \) and the relations

\[
\mu^2 = 1, \quad \mu f_1 \mu^{-1} = f_2, \quad \mu f_3 \mu^{-1} = f_4,
\]

and for a presentation of \( C_{\tau\mu}(n, n; 2, 2; 3, 3) \) both generators \( \tau \) and \( \mu \) with their relations, and also the relation \((\tau \mu)^2 = 1\).

We consider also the orientation-preserving subgroups of index 2 of the Coxeter groups. The generators \( f_i \) in their presentations denote rotations now, see [Z5] or [Z6] for such computations of the orbifold fundamental groups in some of these cases. Representing the 1-skeleton of a tetrahedron by a square with its two diagonals, a Wirtinger-type representation of the orbifold-fundamental group is obtained; here the two horizontal edges of the square have labels \( n \) and \( m \), the two vertical edges labels \( c \) and \( 3 \) (generators \( f_1 \) and \( f_4 \)), the two diagonals labels \( 2 \) (generators \( f_2 \) and \( f_3 \)), and similarly for the quotients of the 1-skeleton of the tetrahedron by the involutions \( \tau \) and \( \mu \) (represented by rotations around a vertical and a horizontal axis, so one easily depicts the singular sets of the quotient orbifolds). In this way on obtains the following presentations:

the tetrahedral group \( T(n, m; 2, 2; c, 3) \) of index 2 in \( C(n, m; 2, 2; c, 3) \):

\[
< f_1, f_2, f_3, f_4 \mid f_1^c = f_2^2 = f_3^3 = f_4^3 = 1, \quad f_1 f_2 f_3 f_4 = (f_1 f_2)^n = (f_2 f_4)^m = 1 > ;
\]

the twisted tetrahedral group \( T_\tau(n, m; 2, 2; 3, 3) \) of index 2 in \( C_\tau(n, m; 2, 2; 3, 3) \):

\[
< f_1, f_2, f_3, f_4 \mid f_1^2 = f_2^2 = f_3^3 = f_4^3 = 1, \quad f_1 f_2 f_3 f_4 = (f_1 f_2)^n = (f_2 f_3 f_2 f_4)^m = 1 > ;
\]

the twisted tetrahedral group \( T_\mu(n, n; 2, 2; c, 3) \) of index 2 in \( C_\mu(n, n; 2, 2; c, 3) \):

\[
< f_1, f_2, f_3, f_4, f_5 \mid f_1^c = f_2^2 = f_3^2 = f_4^3 = 1, \quad f_1 f_2 f_3 f_4 = (f_1 f_2)^n = 1, \quad f_5^2 = 1, \quad (f_1 f_5)^n = (f_4 f_5)^n = f_3(f_4 f_5) f_2(f_4 f_5) = 1 > ;
\]
the doubly-twisted tetrahedral group $T_{\tau\mu}(n, n; 2, 2; 3, 3)$ of index 2 in $C_{\tau\mu}(n, n; 2, 2; 3, 3)$:

$$< f_1, f_2, f_3, f_4, f_5 \mid f_1^2 = f_2^2 = f_3^3 = f_4^4 = f_5^5 = 1, \quad f_1 f_2 f_3 f_4 = (f_1 f_2)^n = 1,$$

$$f_5^5 = 1, \quad (f_1 f_5)^2 = (f_4 f_5)^2 = (f_2 f_4 f_5)^2 = 1 >.$$ 

As a typical example, we consider the doubly-twisted Coxeter group $C_{\tau\mu}(5, 5; 2, 2; 3, 3)$ in the next section. In the hyperbolic and Euclidean cases, we call a surjection of a Coxeter group or tetrahedral group admissible if its kernel is torsionfree.

**2.3 Manifolds of type $48(g-1)$ and $24(g-1)$**

By Theorem 2.1, we are interested in torsionfree normal subgroups of finite index of the Coxeter groups 2.1.1 - 2.1.4. Using the presentations in the previous section, all computations in the following are easily verified by GAP (which classifies surjections onto finite groups up to isomorphisms of the image). As a typical example, we consider the twisted Coxeter group $C_{\tau}(5, 5; 2, 2; 3, 3)$.

**Theorem 2.2** i) There is a unique torsionfree normal subgroup $K_0$ of smallest possible index 120 in the twisted Coxeter group $C_{\tau}(5, 5; 2, 2; 3, 3)$. The quotient manifold $M_0 = \mathbb{H}^3/K_0$ is an orientable hyperbolic 3-manifold of type $48(g-1)$ and genus $g = 6$, for an action of $S_5 \times \mathbb{Z}_2$. Since $H_1(M_0) \cong \mathbb{Z}^6$, also the ordinary Heegaard genus of $M_0$ is equal to 6.

ii) The manifold $M_0$ is the unique hyperbolic 3-manifold with an action of type $48(g-1)$, and also of non-interchanging type $24(g-1)$, for genera $g \leq 6$.

iii) There is a unique torsionfree normal subgroup $K_1$ of smallest possible index 120 in the Coxeter group $C(5, 5; 2, 2; 3, 3)$. The quotient $M_1 = \mathbb{H}^3/K_1$ is an orientable hyperbolic 3-manifold of type $48(g-1)$ and genus $g = 11$. Since $H_1(M_1) \cong \mathbb{Z}^{11}$, also the ordinary Heegaard genus of $M_1$ is equal to 11 ($M_1$ is a 2-fold covering of $M_0$).

**Proof.** i) Since $C_{\tau}(5, 5; 2, 2; 3, 3)$ has a finite subgroup (a vertex group) isomorphic to the extended dodecahedral group $\hat{A}_5 \cong A_5 \times \mathbb{Z}_2$ of order 120 (isomorphic to the extended triangle group $[2, 3, 5]$ generated by the reflections in the sides of a hyperbolic triangle with angles $\pi/2, \pi/3$ and $\pi/5$), a torsionfree subgroup of $C_{\tau}(5, 5; 2, 2; 3, 3)$ has index at least 120. Similarly, $T_{\tau}(5, 5; 2, 2; 3, 3)$ has a vertex group $A_5$ and a torsionfree subgroup has index at least 60.

Up to isomorphism of the image (this will always be the convention in the following), there are exactly three surjections of the twisted tetrahedral group $T_{\tau}(5, 5; 2, 2; 3, 3)$ to its vertex group $A_5$; the abelianizations of the kernels of the three surjections are $\mathbb{Z}^6$ and two times $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5^2$. The three surjections are admissible (have torsionfree kernel) since, when killing an element of finite order in a vertex group of the twisted tetrahedron, the twisted tetrahedral group becomes trivial or of order two. The kernel $K_0$
of the unique surjection with abelianized kernel $\mathbb{Z}^6$ is normal also in the twisted Coxeter group $C_\tau(5, 5; 2, 2; 3, 3)$, the doubly-twisted tetrahedral group $T_{\tau\mu}(5, 5; 2, 2; 3, 3)$, and hence also in the doubly-twisted Coxeter group $C_{\tau\mu}(5, 5; 2, 2; 3, 3)$. By Theorem 2.1, the quotient manifold $M_0 = \mathbb{H}^3/K_0$ is a closed orientable hyperbolic 3-manifold of type $48(g - 1)$ and genus $g = 6$. Since $C_\tau(5, 5; 2, 2; 3, 3)/K_0 \cong A_5 \times \mathbb{Z}_2$ and there is no surjection of $C_{\tau\mu}(5, 5; 2, 2; 3, 3)$ to $A_5$, the only remaining possibility is $C_{\tau\mu}(5, 5; 2, 2; 3, 3)/K_0 \cong S_5 \times \mathbb{Z}_2$.

There are three surjections of $C_\tau(5, 5; 2, 2; 3, 3)$ to its vertex group $A_5 \times \mathbb{Z}_2$, with abelianizations $\mathbb{Z}^6$, $\mathbb{Z}^{12}$ and $\mathbb{Z}^5 \times \mathbb{Z}_2^2$, but only the surjection with kernel is admissible: since the rank of the other two kernels is larger than 6, they cannot uniformize a 3-manifold with a Heegaard splitting of genus 6. Hence $K_1$ is the unique torsionfree subgroup of index 120 in $C_\tau(5, 5; 2, 2; 3, 3)$.

ii) By the lists 2.1.3 and 2.1.4 and Theorem 2.1 ii), apart from $C_{\tau\mu}(5, 5; 2, 2; 3, 3)$ the other two Coxeter groups to obtain a hyperbolic 3-manifold of type $48(g - 1)$ are $C_{\tau\mu}(4, 4; 2, 2; 3, 3)$ and $C_\mu(5, 5; 2, 2; 3, 3)$.

Let $K$ be a torsionfree normal subgroup of $C_\tau(4, 4; 2, 2; 3, 3)$, with factor group of order $24(g - 1)$. Since the extended octahedral group $S_4 \times \mathbb{Z}_2$ of order 48 is a vertex group of $C_\tau(4, 4; 2, 2; 3, 3)$, the cases $g = 2, 4$ and 6 are not possible. Also the case $g = 3$ is not possible since there is no surjection of $C_\tau(4, 4; 2, 2; 3, 3)$ to $S_4 \times \mathbb{Z}_2$.

Considering $g = 5$, suppose that there is an admissible surjection of $C_\tau(4, 4; 2, 2; 3, 3)$ to a group $G$ of order 96; since there are no surjections of $C(4, 4; 2, 2; 3, 3)$ onto its vertex group $S_4 \times \mathbb{Z}_2$, its restriction to $C(4, 4; 2, 2; 3, 3)$ also surjects onto $G$. Now $G$ has $S_4 \times \mathbb{Z}_2$ as a subgroup of index 2; dividing out $\mathbb{Z}_2$, it surjects onto $S_4 \times \mathbb{Z}_2$, and then also $C(4, 4; 2, 2; 3, 3)$ surjects onto $S_4 \times \mathbb{Z}_2$. Since no such surjection exists, this excludes also the case $g = 5$, and hence $g \leq 6$ is not possible.

Considering the case of $C_\mu(5, 5; 2, 2; 2, 3)$, there is no surjection of $C(5, 5; 2, 2; 2, 3)$ to its vertex group $A_5 \times \mathbb{Z}_2$, and again $g \leq 6$ is not possible.

This completes the proof of ii) for the case of actions of type $48(g - 1)$; for the case of actions of non-interchanging type $24(g - 1)$ one excludes all other Coxeter groups in a similar way.

iii) There are exactly three surjections of the tetrahedral group $T(5, 5; 2, 2; 3, 3)$ to $A_5$, with abelianized kernels $\mathbb{Z}^{11}$ and two times $\mathbb{Z}_2 \times \mathbb{Z}_3^4 \times \mathbb{Z}_4^4 \times \mathbb{Z}_6^3$. The kernel $K_1$ of the unique surjection with abelianized kernel $\mathbb{Z}^{11}$ is normal also in $C(5, 5; 2, 2; 3, 3)$, $T_\tau(5, 5; 2, 2; 3, 3)$, $T_\mu(5, 5; 2, 2; 3, 3)$ and hence also in $C_{\tau\mu}(5, 5; 2, 2; 3, 3)$. By Theorem 2.1 ii), $M_1 = \mathbb{H}^3/K_1$ is an orientable 3-manifold to type $48(g - 1)$.

Since there is just one surjection of $C(5, 5; 2, 2; 3, 3)$ to its vertex group $A_5 \times \mathbb{Z}_2$, the kernel $K_1$ is the unique normal subgroup of index 120 in $C(5, 5; 2, 2; 3, 3)$.
This completes the proof of Theorem 2.2.

The manifold $M_1$ appeared first in [Z4] where an explicit geometric description is given.

Next we consider the Coxeter group $C(4, 4; 2, 2; 3, 3)$. Its subgroup $T(4, 4; 2, 2; 3, 3)$ has a unique surjection to $\text{PSL}(2, 7)$, its kernel $K_2$ has abelianization $\mathbb{Z}^{13}$ and is normal also in $C(4, 4; 2, 2; 3, 3)$. There are three surjections of $C(4, 4; 2, 2; 3, 3)$ to $\text{PSL}(2, 7) \times \mathbb{Z}_2$, exactly one with abelianization $\mathbb{Z}^{13}$, hence its kernel $K_2$ is normal also in the twisted groups $C_\tau(4, 4; 2, 2; 3, 3)$, $C_\mu(4, 4; 2, 2; 3, 3)$ and $C_{\tau \mu}(4, 4; 2, 2; 3, 3)$ and Theorem 2.1 implies:

**Corollary 2.3** The quotient manifold $M_2 = \mathbb{H}^3/K_2$ is an orientable hyperbolic 3-manifold of type $48(g - 1)$ and genus $g = 29$.

As noted in the proof of Theorem 2.2 ii), the Coxeter group $C(5, 5; 2, 2; 2, 3)$ admits no surjection onto its vertex group $A_5 \times \mathbb{Z}_2$. On the other hand, the tetrahedral group $T(5, 5; 2, 2; 2, 3)$ has exactly two admissible surjections onto its vertex group $A_5$, their kernels are conjugate in $C(5, 5; 2, 2; 2, 3)$, normal in $T_\mu(5, 5; 2, 2; 2, 3)$ with factor group $S_5$, and uniformize the Seifert-Weber hyperbolic dodecahedral 3-manifold ([WS]). By [M], $S_5$ is in fact the full isometry group of the Seifert-Weber manifold which has no orientation-reversing isometries.

**Corollary 2.4** The Seifert-Weber hyperbolic dodecahedral 3-manifold is a closed orientable 3-manifold of interchanging type $24(g - 1)$ and genus $g = 6$, for the action of its isometry group $S_5$.

There are three surjections of the tetrahedral group $T(5, 5; 2, 2; 2, 3)$ to $\text{PSL}(2, 19)$, all admissible, and exactly one kernel $K_3$ has infinite abelianization $\mathbb{Z}^{56}$ and is normal also in the Coxeter group $C(5, 5; 2, 2; 2, 3)$. There is exactly one surjection of $C(5, 5; 2, 2; 2, 3)$ to $\text{PSL}(2, 11) \times \mathbb{Z}_2$, with kernel $K_3$, hence $K_3$ is normal also in the twisted Coxeter group $C_\mu(5, 5; 2, 2; 2, 3)$ and Theorem 2.2 implies:

**Corollary 2.5** The quotient manifold $M_3 = \mathbb{H}^3/K_3$ is an orientable hyperbolic 3-manifold of type $48(g - 1)$ and genus $g = 286$.

Infinite series of finite quotients of the hyperbolic Coxeter group $C(5, 5; 2, 2; 2, 3)$ are considered in the papers [JL] and [P].

It is shown in [JM] that the twisted Coxeter group $C_\tau(5, 2; 2, 2; 3, 3)$ has a unique torsion-free normal subgroup of smallest possible index 2640, with factor group $\text{PGL}(2, 11) \times \mathbb{Z}_2$, which uniformizes an orientable hyperbolic 3-manifold of non-interchanging type $24(g - 1)$ and genus $g = 111$, for an action of $\text{PGL}(2, 11) \times \mathbb{Z}_2$. 

8
Next we discuss the case of the Euclidean Coxeter group $C_\tau(3,3;2,2;3,3)$ (with strong analogies with the hyperbolic case of $C_\tau(5,5;2,2;3,3)$ in Theorem 2.2).

There are five surjections of the tetrahedral group $T(3,3;2,2;3,3)$ to its vertex group $A_4$, the abelianizations of the kernels are $\mathbb{Z}^3$, two times $\mathbb{Z}_4^2$ and two times $\mathbb{Z}_2^3$. The kernel with abelianization $\mathbb{Z}^3$ uniformizes the 3-torus and is normal also in $C(3,3;2,2;3,3)$, $T_\tau(3,3;2,2;3,3)$ and hence $C_\tau(3,3;2,2;3,3)$. The two surjections with abelianized kernel $\mathbb{Z}_4^2$ are conjugate in $C(3,3;2,2;3,3)$ and uniformize the Hantzsche-Wendt manifold (the only of the six orientable Euclidean 3-manifolds with homology $\mathbb{Z}_2^2$, see [W]), and the remaining two surjections are not admissible.

**Corollary 2.6** i) The 3-torus is a 3-manifold of type $48(g−1)$ and genus $g = 3$ which is also its ordinary Heegaard genus.

ii) The Euclidean Hantzsche-Wendt manifold is of interchanging type $24(g−1)$ and genus $g = 3$, for the action of its orientation-preserving isometry group $S_4 \times \mathbb{Z}_2$ (by section 1, it is also a 3-manifold of type $12(g−1)$ and genus $g = 2$, for an action of the dihedral group $D_6$ of order 12).

By [Z2], the full isometry group of the Hantzsche-Wendt manifold has order 96 but the orientation-reversing elements do not preserve the Heegaard splitting of genus 3 of Theorem 2.4.

The 3-torus is a manifold of type $48(g−1)$ and genus $g = 3$ in still a different way. The Euclidean tetrahedral group $T(4,4;2,2;3,3)$ has three surjections to its vertex group $S_4$, with abelianized kernels $\mathbb{Z}^3$ and two times $\mathbb{Z}_2^3 \times \mathbb{Z}_4$. The kernel with abelianization $\mathbb{Z}^3$ is normal also in the Coxeter group $C(4,4;2,2;3,3)$, the twisted tetrahedral group $T_\mu(4,4;2,2;3,3)$ and hence in the twisted Coxeter group $C_\mu(4,4;2,2;3,3)$, it uniformizes the 3-torus which is again a 3-manifold of type $48(g−1)$ (but for an action not equivalent to the action arising from $T(3,3;2,2;3,3)$).

As a spherical case, we consider the Coxeter group $C(5,3;2,2;2,3)$. There is no surjection of $C(5,3;2,2;2,3)$ to its vertex group $A_5 \times \mathbb{Z}_2$, and there are two surjections of the tetrahedral group $T(5,3;2,2;2,3)$ to its vertex group $A_5$; both kernels have trivial abelianization, are conjugate in $C(5,3;2,2;2,3)$ and uniformize the spherical Poincaré homology 3-sphere.

**Corollary 2.7** The spherical Poincaré homology 3-sphere is a 3-manifold of type $12(g−1)$ and genus $g = 6$, for an action of $A_5$.

The case of non-orientable manifolds is more elusive. It is shown in [CMT] that, for sufficiently large $n$, the alternating group $A_n$ is a quotient of $C_\tau(5,2;2,2;3,3)$ by a torsionfree normal subgroup, and such a subgroup uniformizes a non-orientable 3-manifold:
since $A_n$ is simple, the orientation-preserving subgroup $T_\tau(5, 2; 2, 2; 3, 3)$ of index 2 surjects onto $A_n$, and hence the kernel contains an orientation-reversing element.

**Corollary 2.8** For all sufficiently large $n$, there is a non-orientable hyperbolic 3-manifold of non-interchanging type $24(g - 1)$, for an action of the alternating group $A_n$.

Some other finite simple quotients of the nine hyperbolic Coxeter groups are listed in [H], and for the Coxeter groups of type $C(n, m; 2, 2; 2, 3)$ these define non-orientable hyperbolic 3-manifolds of non-interchanging type $24(g - 1)$. At present, we don’t know explicit examples of small genus of non-orientable manifolds of non-interchanging type $24(g - 1)$, and no example of type $48(g - 1)$.

**References**

[CMT] M. Conder, G. Martin, A. Torstensson, *Maximal symmetry groups of hyperbolic 3-manifolds*. New Zealand J. Math. 35 (2006), 37-62

[H] S.P. Humphries, *Quotients of Coxeter complexes, fundamental groupoids and regular graphs*. Math. Z. (1994), 247-273

[JL] G.A. Jones, C.D. Long, *Epimorphic images of the [5,3,5] Coxeter group*. Math. Z. 275 (2013), 167-183

[JM] G.A. Jones, A.D. Mednykh, *Three-dimensional hyperbolic manifolds with large isometry groups*. Preprint 2003

[M] A.D. Mednykh, *On the isometry group of the Seifert-Weber dodecahedron*. Siberian Math. J. 28 (1987), 798-806

[MMZ] D. McCullough, A. Miller, B. Zimmermann, *Group actions on handlebodies*. Proc. London Math. Soc. 59 (1989), 373-415

[P] L. Paoluzzi, *PSL(2,q) quotients of some hyperbolic tetrahedral and Coxeter groups*. Comm. Algebra 26 (1998), 759-778

[WS] C. Weber, H. Seifert, *Die beiden Dodekaederräume*. Math. Z. 37, 237-253 (1933)

[W] J. Wolf, *Spaces of Constant Curvature*. Publish or Perish, Boston 1974

[Z1] B. Zimmermann, *Über Abbildungsklassen von Henkelkörpern*. Arch. Math. 33 (1979), 379-382

[Z2] B. Zimmermann, *On the Hantzsche-Wendt manifold*. Monatsh. Math. 110 (1990), 321-327

[Z3] B. Zimmermann, *Finite group actions on handlebodies and equivariant Heegaard genus for 3-manifold*. Top. Appl. 43 (1992), 263-274
[Z4] B. Zimmermann, *On a hyperbolic 3-manifold with some special properties.* Math. Proc. Camb. Phil. Soc. 113 (1993), 87-90

[Z5] B. Zimmermann, *Hurwitz groups and finite group actions on hyperbolic 3-manifolds.* J. London Math. Soc. 52 (1995), 199-208

[Z6] B. Zimmermann, *Genus actions of finite groups on 3-manifolds.* Mich. Math. J. 43 (1996), 593-610

[Z7] B. Zimmermann, *Determining knots and links by cyclic branched coverings.* Geom. Ded. 66 (1997), 149-157

[Z8] B. Zimmermann, *On large orientation-reversing group-actions on 3-manifolds and equivariant Heegaard decompositions.* Monatsh. Math. 191 (2020), 437-444