HIGH-FREQUENCY LIMITS AND NULL DUST SHELL SOLUTIONS IN GENERAL RELATIVITY

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Abstract. Consider the characteristic initial value problem for the Einstein vacuum equations without any symmetry assumptions. Impose a sequence of data on two intersecting null hypersurfaces, each of which is foliated by spacelike 2-spheres. Assume that the sequence of data is such that the derivatives of the metrics along null directions are only uniformly bounded in $L^2$ but the derivatives of the metrics along the directions tangential to the 2-spheres obey higher regularity bounds uniformly. By the results in [J. Luk and I. Rodnianski, Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations, Camb. J. Math. 5(4), 2017], it follows that the sequence of characteristic initial value problems gives rise to a sequence of vacuum spacetimes $(\mathcal{M},g_n)$ in a fixed double-null domain $\mathcal{M}$. Since the existence theorem requires only very low regularity, the sequence of solutions may exhibit both oscillations and concentrations, and the limit need not be vacuum. We prove nonetheless that, after passing to a subsequence, the metrics converge in $C^0$ and weakly in $W^{1,2}$ to a solution of the Einstein–null dust system with two families of (potentially measure-valued) null dust.

We show moreover that all sufficiently regular solutions to the Einstein–null dust system (with potentially measure-valued null dust) adapted to a double null coordinate system arise locally as weak limits of solutions to the Einstein vacuum system in the manner described above. As a consequence, we also give the first general local existence and uniqueness result for solutions to the Einstein–null dust system for which the null dusts are only measures. This in particular includes as a special case solutions featuring propagating and interacting shells of null dust.

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1. Introduction

This paper studies two circles of problems in general relativity, namely the problem of high-frequency limits and the problem of null dust shell solutions. We moreover show that there is a close relationship between the two problems. Here is a summary of what we achieve in this paper:

(1) The first circle of problems (see Section 1.1.1) concerns high-frequency limits of vacuum solutions, i.e. we seek to understand “effective matter field” that arises in appropriately defined weak limits of solutions to the Einstein vacuum equations. For “angularly regular” spacetimes adapted to a double null foliation (but without any symmetry assumptions), we give a complete characterization of possible high-frequency limits. Namely we show that all high-frequency limits...
are isometric to solutions to the Einstein–null dust system; and conversely, all solutions to the Einstein–null dust system also arise locally as high-frequency limits of vacuum spacetimes.

(2) The second circle of problems (see Section 1.1.2) concerns null dust shell solutions, i.e. solutions to the Einstein–null dust system with a “shell of null dust” for which the stress-energy-momentum tensor is a delta measure on an embedded null hypersurface. We prove an existence and uniqueness result (again with no symmetry assumptions) for the Einstein–null dust system which describes solutions featuring propagation and interaction of null dust shells (and also more general solutions where the null dust is measure-valued).

(3) We show that the problem of high-frequency limits and the problem of null dust shells are closely related. In fact, they can both be studied and understood from the point of view of low-regularity problems of the Einstein equations. In particular, in this paper we study these problems using the low-regularity local existence and uniqueness result in our previous papers [41, 42] (which was originally developed to understand the propagation and interaction of impulsive gravitational waves, i.e. solutions to the Einstein vacuum equation such that some curvature components admit a delta function singularity on an embedded null hypersurface).

(4) Moreover, our construction of null dust shell solutions is based on studying vacuum solutions and then taking appropriate high-frequency limits. Put differently, using the characterization of high-frequency limit, existence and uniqueness of solutions to the Einstein–null dust can be established by studying vacuum solutions. See Section 1.2.3.

(5) Conversely, we also illustrate through the example of formation of trapped surfaces how the study of the Einstein–null dust system illuminates our understanding of the Einstein vacuum equations. See Section 1.5.

We will explain this further in the remainder of the introduction and give a first descriptions of the main results. In Section 1.1, we will first introduce the two circles of problems regarding high-frequency limits and null dust shell solutions. In Section 1.2, we then give a first description of the main results in this paper, and explain how they relate to [41, 42]. We then discuss some related works in Section 1.3 and the ideas of the proof in Section 1.4.

1.1. The problems.

1.1.1. High-frequency limits. Consider a sequence of (3 +1)-dimensional spacetimes \( \{ (M, g_n) \}_{n=1}^{\infty} \) which solve the Einstein vacuum equations:

\[
Ric_{\mu \nu}(g_n) = 0.
\]

Suppose that \( g_n \to g_{\infty} \) in \( C^0_{loc} \) and the derivatives of \( g_n \) converge weakly\(^1\) in \( L^2_{loc} \). Explicit examples are known (see for instance [8, 18]) in symmetry classes such that the limit \( (M, g_{\infty}) \) may satisfy the Einstein equations

\[
Ric_{\mu \nu}(g_{\infty}) - \frac{1}{2}(g_{\infty})_{\mu \nu} R(g_{\infty}) = T_{\mu \nu},
\]

with a non-vanishing stress-energy-momentum tensor \( T_{\mu \nu} \), where \( R(g_{\infty}) \) is the scalar curvature of the limit metric. Physically, \( T_{\mu \nu} \) can be interpreted as an effective stress-energy-momentum tensor arising from limits of high-frequency gravitational waves. Mathematically, \( T_{\mu \nu} \) can be thought of as a defect measure that arises because taking weak limits do not commute with taking products.

This phenomenon raises the following question:

**Problem 1.1.** Give a description of the non-vanishing stress-energy-momentum tensors that arise in the limiting process as described above.

There are two guiding conjectures concerning Problem 1.1, both of which were introduced by Burnett. In [8], Burnett considered more restrictive assumptions on the convergence of \( g_n \). Namely, he required that for some \( C > 0, \lambda_n \to 0 \),

\[
|g_n - g_{\infty}| \leq \lambda_n, \quad |\partial g_n| \leq C, \quad |\partial^2 g_n| \leq C\lambda_n^{-1}
\]

in some local coordinate system.

Under these assumptions, Burnett made the following conjectures [8]:

**Conjecture 1.2** (Burnett’s conjecture). Any such limit \( (M, g_{\infty}) \) is isometric to a solution to the Einstein–massless Vlasov system for some appropriate choice of Vlasov field.

**Conjecture 1.3** (Reverse Burnett’s conjecture). Any solution to the Einstein–massless Vlasov system arises as a limit of solutions to the Einstein vacuum equations in the sense described above.

\(^1\)Remark that if the convergence of the derivatives of \( g_n \) is strong in \( L^2_{loc} \), then the limit is necessarily vacuum.
Here, “Einstein–massless Vlasov system” is to be interpreted in an appropriate generalized sense which allows the Vlasov field to be a measure on the cotangent bundle. In particular, it includes the known examples for which the limit is described by the Einstein–null dust system.

Conjectures 1.2 and 1.3 remain open in full generality, although there is some recent progress when \((M, g_n)\) is assumed to be \(U(1)\) symmetric [23, 24]; see Section 1.3.2. In between the full problem and the \(U(1)\) symmetric problem, it is of interest to study a setting which is not completely general, but nonetheless does not require any exact symmetries.

**Problem 1.4.** Find a setting in \((3 + 1)\)-dimensions without any exact symmetry such that Conjectures 1.2 and 1.3 can be studied.

Beyond the original conjectures of Burnett, one can try to understand weak limits of vacuum solutions without imposing (1.3), but requiring only the weaker convergence \((g_n \to g_0\) uniformly and \(\partial g_n \to \partial g_\infty\) weakly in \(L^2)\) introduced in the beginning of Section 1.1.1. Notice in particular that while (1.3) allows for oscillations in \(g_n\), it prohibits concentrations. On the other hand, concentrations can in principle occur if we only require \(g_n \to g_\infty\) in \(C^0\) and \(\partial g_n \to \partial g_\infty\) weakly in \(L^2\). This motivates

**Problem 1.5.** Study (appropriate analogues of) Conjectures 1.2 and 1.3 when concentrations are present (in addition to oscillations).

1.1.2. Null shell solutions. In 1957, Synge [51] discovered a solution to the Einstein equations describing a propagating null shell of dust. More precisely, he constructed a spacetime with a distinguished null hypersurface so that the metric is isometric to Schwarzschild to the one side of that null hypersurface, and is isometric to Minkowski to the other side of that null hypersurface. Along the separating null hypersurface, the spacetime is not vacuum. Instead, a component of the Ricci curvature is a delta function supported on this null hypersurface, and the spacetime can be thought of as containing a null shell of dust. Since then, many other explicit solutions have been discovered; see Section 1.3.3 for further discussions.

In view of the explicit solutions, it is desirable to develop a local theory for null dust shells which does not impose any symmetry assumptions. Note that the difficulty in such an endeavor is that a null shell has much lower regularity than that allowed by standard local well-posedness results.

**Problem 1.6.** Prove a local existence and uniqueness theorem for the Einstein–null dust system which incorporates the propagation of null shells of dust.

Once Problem 1.6 is understood, it is natural to further extend the class of initial data for which one can develop a local theory. Motivated by explicit solutions featuring the interaction of two null dust shells (see for instance [13, 14, 50]), it is desirable to understand more generally the interaction of two null shells, described by the transversal intersection of two null hypersurfaces which support the null shells of dust.

**Problem 1.7.** Prove a local existence and uniqueness theorem for the Einstein–null dust system which incorporates the interaction of null shells of dust.

From a PDE point of view, a spacetime containing a null dust shell is a solution to the Einstein–null dust system for which the stress-energy-momentum tensor of the null dust is merely a measure (which is not absolutely continuous with respect to the Lebesgue measure). From this perspective, it is of interest to study more general solutions to the Einstein–null dust system for which the stress-energy-momentum tensor is a measure with singular parts, but not necessarily a measure supported on a single null hypersurface.

**Problem 1.8.** Construct more general solutions to the Einstein–null dust system where the null-dust is measure-valued.

Notice that while null dust shells are particular measure-valued solutions to the Einstein–null dust system, they are very special. Indeed they are so special that often in the physics literature they are constructed (under symmetry assumptions) by considering a “junction condition” across the null hypersurface on which the dust is supported. This is no longer the case for more general measures.

1.2. Main results. In this subsection, we give the informal statements of the main results concerning the problems discussed in Sections 1.1.1 and 1.1.2; see Sections 1.2.2 and 1.2.3. Before that, however, we first discuss our previous low-regularity well-posedness result in Section 1.2.1, which as we will show is closely related to the problems at hand.
A local well-posedness result with \( L^2 \) Christoffel symbols. We recall our earlier local well-posedness result in [41, 42]. The setup of [41, 42] is as follows. We seek a solution \((M = [0, u_*] \times [0, w_*] \times S^2, g)\) to the Einstein vacuum equations in double null coordinates:

\[
g = -2\Omega^2(du \otimes du + dw \otimes dw) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du),
\]

where \( \theta = (\theta^1, \theta^2) \) is a local coordinate system on \( S^2 \), \( \Omega \) is a strictly positive function, \( b \) is a vector field tangent to \( S^2 \), and for every \((u, w)\), \( \gamma \) a Riemannian metric on \( S^2 \). A characteristic initial value problem (for \((\Omega, b, \gamma)\)) is considered in [41, 42], i.e. characteristic data are prescribed on \( H_0 := [0, I] \times \{0\} \times S^2 \) and \( H_0 := \{0\} \times [0, I] \times S^2 \). The function \( \Omega \) can be arbitrarily prescribed on \( H_0 \) and \( H_0 \). The vector field \( b^A \) can be prescribed arbitrarily on \( H_0 \) (but not \( H_0 \)), and additionally \( \delta b^A / \delta u \) can be prescribed arbitrarily on the sphere \( S_{0,0} := \{0\} \times \{0\} \times S^2 \). Finally, the metric \( \gamma \) can be prescribed on \( H_0 \) and \( H_0 \) subject to some constraints equations (see (3.1) in Section 3.1.2).

In [42], we consider data obeying the following estimates\(^2\) (where \( \partial / \partial u \) denotes \( \partial / \partial u \) or \( \partial / \partial w \) derivatives):

\[
\begin{align*}
&g \in \{\gamma, \log \det \gamma, \log \Omega, b\}, \quad t \leq \frac{\epsilon}{C}, \\
&\sum_{g \in \{\gamma, \log \det \gamma, \log \Omega, b\}} \left( \sum_{i \leq 5} \left\| \frac{\partial}{\partial \theta^i} \right\|_{L^2(S)} + \sum_{i \leq 5} \left\| \frac{\partial}{\partial \theta^i} \right\|_{L^2(S)} \right) \\
&\quad + \sum_{g \in \{\gamma, \log \det \gamma, \log \Omega, b\}} \left( \sum_{i \leq 5} \left\| \frac{\partial}{\partial t} \right\|_{L^2(S)} + \sum_{i \leq 5} \left\| \frac{\partial}{\partial t} \right\|_{L^2(S)} \right) \leq C.
\end{align*}
\]

**Theorem 1.9** (L.-R. [42]). Given characteristic initial data satisfying the bounds (1.5), there exists \( \epsilon > 0 \) sufficiently small depending only on \( C \) such that for any \( u_* \in (0, I) \) and \( w_* \in (0, e] \), there exists a unique solution to the Einstein vacuum equations in double null coordinates in \([0, u_*] \times [0, w_*] \times S^2\) which achieves the given data. The solution is \( C^0 \cap W^{1,2} \) with additional regularity in \( \partial / \partial u \) directions, with estimates depending only on \( C \) in (1.5).

A more precise version is given in Theorems 3.2 and 3.3.

The key point here is that when measured in the worst directions (i.e. the \( \partial / \partial u \) and \( \partial / \partial u \) directions), the components of the metric are merely in \( W^{1,2} \).

1.2.2. Results on high-frequency limits. To study high-frequency limits, we consider exactly the setting of the results in Section 1.2.1, where in spite of the very low regularity of the data we still have a well-posedness theory (see Problem 1.4). Our result on high-frequency limits is most easily formulated in the language of Theorem 1.9. As emphasized above, the assumptions of Theorem 1.9 allow the components of the metric to be merely bounded in \( W^{1,2} \). Therefore, given a sequence of characteristic initial data obeying uniformly the estimates in Theorem 1.9, the first derivatives of the metric components do not necessarily have strong limits. In particular, the limits, if they exist, can in principle have non-trivial stress-energy-momentum tensors as discussed in Section 1.1.1. Our first main result shows that non-trivial stress-energy-momentum tensors must correspond to that of null dust. It can be viewed as a resolution of Conjecture 1.2 in our particular setting where the metric is adapted to a double null foliation gauge\(^3\).

**Theorem 1.10.** Take a sequence of characteristic initial data which obey the bounds in Theorem 1.9 uniformly. Then the following holds:

1. There exists a sequence of metric

\[
g_n = -2\Omega_n^2(du \otimes du + dw \otimes dw) + \gamma_{AB}(d\theta^A - b^A_n du) \otimes (d\theta^B - b^B_n du)
\]

in a uniform domain of existence \([0, u_*] \times [0, w_*] \times S^2\).

2. After passing to a subsequence \( g_{n_k} \), there exists a metric

\[
g_\infty = -2\Omega_\infty^2(du \otimes du + dw \otimes dw) + \gamma_\infty_{AB}(d\theta^A - b^A_\infty du) \otimes (d\theta^B - b^B_\infty du)
\]

so that \( g_n \to g_\infty \) in \( C^0 \) and weakly in \( W^{1,2} \) in \([0, u_*] \times [0, w_*] \times S^2\).

3. Moreover \( g_\infty \) satisfies (weakly) the Einstein–null dust system with two families of null dusts which are potentially measure-valued.

In establishing Theorem 1.10, the existence theorem (Theorem 1.9) plays two important roles:

- Theorem 1.9 gives a uniform region of existence of solutions, which allows us to study the limit in this setting.

\(^2\)In fact we only needed slightly weaker estimates, but (1.5) is slightly more concise to state. We remark also that the constraints equations together with (1.5) imply bounds for the Ricci coefficients

\(^3\)Note that in our setting, due to the angular regularity given by Theorem 1.9, we only obtain null dust in the limit, as opposed to more general Vlasov field as in the case of Conjecture 1.2 in general.
• The regularity properties of the solutions proven in Theorem 1.9 allows us to treat the limits of the nonlinear terms. In particular, the regularity properties dictate that only for specific nonlinear terms could the product of the weak limits be different from the weak limit of the product. This implies that nothing “worse” than two families of null dusts can arise in the limit. Understanding the second point above in particular requires studying the effects of compensated compactness.

Moreover, the setup in Theorem 1.10 allows concentrations (in addition to oscillations) to occur in the limiting process. These seem to be first known examples of limiting effective stress-energy-momentum tensors being created by concentrations; see Problem 1.5.

One consequence of Theorem 1.10 is that we know when the limiting spacetime metric is vacuum: since the null dust satisfies a transport equation, it is vanishing if and only if it has vanishing data.

Corollary 1.11. Let the sequence $g_n$, the subsequence $g_{n_k}$ and the limit $g_\infty$ be as in Theorem 1.10. Then the limiting spacetime metric $g_\infty$ is a (weak) solution to the Einstein vacuum equations if and only if the initial data sets for $g_{n_k}$ converge to a limiting initial data set to the Einstein vacuum equations.

In fact, slightly more can be said, and that the solution is determined only by the limit of the initial data. In other words, we have the following uniqueness theorem:

**Theorem 1.12.** Given two sequences of characteristic initial data satisfying the assumptions of Theorem 1.10 which moreover have the same limit on the initial characteristic hypersurfaces, then in fact the limiting spacetime metrics given by Theorem 1.10 also coincide.

1.2.3. **Results on null shells.** Once we have obtained the existence and uniqueness of the limits as solutions to the Einstein–null dust system (see Theorems 1.10 and 1.12), we can use this to obtain an existence and uniqueness theory for the Einstein–null dust system where the null dust is merely a measure (with sufficient angular regularity). More precisely, given an initial data set to the Einstein–null dust system with a potentially measure-valued null dust, we approximate it by a sequence of initial data sets to the Einstein vacuum equations, and then use Theorem 1.10 (!) to construct a solution to the Einstein–null dust system. Uniqueness of solutions constructed in this manner is then given by Theorem 1.12.

Our main result on null shells is the following existence and uniqueness theorem for the characteristic initial value problem for the Einstein–null dust system with measure-valued null dust:

**Theorem 1.13.** Consider a characteristic initial value problem with the Einstein–null dust system with strongly angularly regular initial data with a measure-valued null dust.

Then, in an appropriate local double null domain, there exists a unique angularly regular weak solution to the Einstein–null dust system.

Theorem 1.13 provides an existence and uniqueness result for a large class of data with measure-valued null dust. These include in particular data for which the solutions feature the propagation and interaction of null dust shells. In other words, it simultaneously addresses Problems 1.6, 1.7 and 1.8. We emphasize that Theorem 1.13 imposes no symmetry assumptions.

We remark that in Theorem 1.13 one also gets a stability statement, which follows from the proof of Theorem 1.12. We will however not formulate this precisely for the sake of brevity.

Finally, as explained above, the proof of Theorem 1.13 not only gives existence and uniqueness of measure-valued solutions to the Einstein–null dust system, but it also shows that any such solution is an appropriate limit of vacuum solutions. As a result, we also resolve Conjecture 1.3 in our setting.

**Corollary 1.14.** Let $(\mathcal{M} = [0, u_+) \times [0, u_+] \times S^2, g_\infty, \{d\nu_{u_+}\}_{u_+ \in [0, u_+]}, \{d\nu_{u_-}\}_{u_- \in [0, u_-])}$ be an angularly regular solution to the Einstein–null dust system (potentially with measure-valued dusts) with strongly angularly regular data. Then for any $p \in \mathcal{M}$, there exist $p \in \mathcal{M}' \subseteq \mathcal{M}$ and a sequence of smooth angularly regular vacuum solutions $(\mathcal{M}', g_0)$ such that $g_n \to g_\infty$ in $C^0$ and weakly in $W^{1,2}$ in $\mathcal{M}'$.

Combining Theorem 1.10 and Corollary 1.14, we have thus answered Problem 1.4 in the class of angularly regular solutions with strongly angularly regular data.

1.3. **Related works.**

4We refer the reader to Definition 4.5 for the precise regularity assumptions. For now we just emphasize that the angular regularity that we require for these characteristic initial data is stronger than the angular regularity for the solutions. We thus distinguish then with the terms “angularly regular spacetimes” and “strongly angularly regular data”. This is related to a well-known loss of derivative associate to the characteristic initial value problem for second order hyperbolic system [46].

5Here, $d\nu$ and $d\bar{\nu}$ here denote the measure-valued null dust; see Definition 2.36.
1.3.1. High-frequency limits in general relativity. The study of limits of high-frequency spacetimes has a long tradition in the physics literature, beginning with the pioneering works of Isaacson [25, 26] and Choquet-Bruhat [9], who already observed using some form of “averaging” or “expansion” that high-frequency limits of gravitational waves could lead to an effective stress-energy-momentum tensor mimicking that of null dust. (See also [47].) This was further discussed and explored by MacCallum–Taub [45].

More relevant to our paper is the work of Burnett [8], in which he formulated high-frequency limits of gravitational waves in the language of weak limits. In the same paper, he introduced Conjectures 1.2 and 1.3. Within Burnett’s framework of weak limits, various examples have been constructed, see for instance [22, 18, 53, 54, 23, 39].

Finally, we note interesting connections between high-frequency limits and inhomogeneities in cosmology [17, 18, 19, 20], as well as late-time asymptotics in cosmological spacetimes [37, 38, 39]. See also [49, 52] for other applications.

1.3.2. Burnett’s conjecture in $U(1)$ symmetry. As mentioned earlier, Burnett’s conjecture (Conjecture 1.2) in its full generality remains open. Nevertheless, imposing an $U(1)$ symmetry and an elliptic gauge condition, Burnett’s conjecture has been solved recently in [24].

In fact, under a slightly more restrictive symmetry and gauge assumption, there is a partial result for the reverse Burnett conjecture (Conjecture 1.3) [23]. It was shown that all generic, smooth, small data solutions to the Einstein–null dust system arise as suitable weak limit of solutions to the Einstein vacuum equations.

1.3.3. Null dust shell solutions in the physics literature. As described earlier, to the best of our knowledge the first null dust shell solution was constructed in [51]. This was later generalized by [13]. The interaction of two null dust shells under symmetry assumptions has been studied in [13, 14, 50].

Due to its simplicity, null dust shell solutions are also used as a simplified model to study gravitational collapse; see for instance [48, 21, 55, 4, 7, 6].

For further references, see the book [5].

1.3.4. Low-regularity solutions to the Einstein equations. Our result in this paper heavily relies on the low-regularity existence and uniqueness result in Theorem 1.9. Low-regularity results in general relativity are themselves of independent interest. Perhaps the most celebrated such result is the bounded $L^2$ curvature theorem:

**Theorem 1.15** (Klainerman–R.–Szeftel [31]). The time of existence (with respect to a maximal foliation) of a classical solution to the Einstein vacuum equations depends only on the $L^2$-norm of the curvature and a lower bound of the volume radius of the corresponding initial data set.

Theorem 1.15 handles a very general class of data. This is in contrast to Theorem 1.9, which although allows for lower regularity when measured in the worst directions, the theorem also requires the data to be of a more specific form.

Very recently, in an ongoing work, L.–Van de Moortel [43, 44] study the problem of transversal interaction of three impulsive gravitational waves under polarized $U(1)$ symmetry. While [43, 44] relies heavily on the symmetry assumptions, in view of the presence of three impulsive gravitational waves, the problem requires geometric construction beyond the double null foliation used in [41, 42].

1.4. Brief discussion of the proof.

1.4.1. Examples in symmetries. Before we discuss the proof, it is illuminating to look at a few very simple examples in symmetry. The first example is given already in [8], which shows in the plane wave setting how oscillations give rise to a null dust. This is the basic example of the phenomenon that we explore in our paper (where we consider the much more general case with no symmetry assumptions).

**Example 1.16** (The Burnett example [8]). Consider the following metric on $\mathbb{R}^4$:

$$g = -2du du + H(u)^2(e^{G(u)}dX^2 + e^{-G(u)}dY^2),$$

where $H(u)$ and $G(u)$ are real-valued functions of $u$. This defines a Lorentzian metric as long as $H > 0$.

The Ricci curvature tensor is given by

$$\text{Ric}(g) = \left\{-\frac{1}{2} (G'(u))^2 - \frac{2H''(u)}{H(u)}\right\} du \otimes du.$$

---

$^6$We remark that the exact conditions in [24] are slightly stronger that in Conjecture 1.2; see [24] for details.
Burnett considered a one-parameter family of solutions to the vacuum Einstein equations which take the form (1.6). The family of solutions is parametrized by $\lambda$. For $\lambda > 0$, define $G_\lambda$ by

$$G_\lambda(u) = \lambda k(u) \sin \left( \frac{u}{\lambda} \right),$$

where $k(u)$ is some fixed smooth function. Also, define $H_\lambda$ by the following ordinary differential equation

$$
\begin{cases}
- \frac{1}{2} (G'_\lambda(u))^2 - \frac{2H'_\lambda(u)}{H_\lambda(u)} = 0, \\
H_\lambda(0) = 1, \quad H'_\lambda(0) = 0,
\end{cases}
$$

so that by (1.7) the metric $g_\lambda = -2dud\bar{u} + H_\lambda(u)^2(e^{G_\lambda(u)}dX^2 + e^{-G_\lambda(u)}dY^2)$ is vacuum.

We now consider the limit $\lambda \to 0$. Clearly,

$$G_0(u) := \lim_{\lambda \to 0} G_\lambda(u) = 0.$$

By standard theory of ordinary differential equations, there exists $\epsilon > 0$ such that (1.8) can be solved for $u \in [0, \epsilon]$. It is easy moreover to show that $H_\lambda$ has a limit in $C^1([0, \epsilon])$ after taking $\epsilon$ to be smaller if necessary. We define $H_0(u) := \lim_{\lambda \to 0} H_\lambda(u)$.

For $u \in [0, \epsilon]$, the spacetime metric given by

$$g_0 = -2dud\bar{u} + H_0(u)^2(e^{G_0(u)}dX^2 + e^{-G_0(u)}dY^2)$$

satisfies (by (1.7))

$$\text{Ric}(g_0) = \frac{1}{4} (k(u))^2 du \otimes du = \frac{1}{2} \lim_{\lambda \to 0} (G'_\lambda(u))^2 du \otimes du.$$

In particular, if $k \neq 0$, then $g_0$ is not a solution to the Einstein vacuum equation, but instead solves the Einstein null dust system.

Still within the category of explicit examples in symmetry class, one can also go beyond one family of null dust and get a limit with two families of null dust. We refer the reader to [18] for details.

**Example 1.17** (Green–Wald example in Gowdy symmetry [18]). Green and Wald gave an example of a sequence of vacuum polarized Gowdy spacetimes whose limit is non-vacuum and in fact can be thought of as having two families of null dust. The spacetimes they constructed have topology $^7\mathbb{R} \times \mathbb{T}^3$ so that in a coordinate system $(\tau, \theta, \sigma, \delta) \in \mathbb{R} \times \mathbb{T}^3$, the sequence of metrics are given by

$$g_n = e^{\frac{(\tau - \alpha_n)}{4\pi}}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}(e^{P_n}d\sigma^2 + e^{-P_n}d\delta^2),$$

where $P_n$ and $\alpha_n$ take the following form

$$P_n := \frac{A}{\sqrt{n}} J_0(ne^{-\tau}) \sin(n\theta),$$

$$\alpha_n := -\frac{A^2 e^{-\tau}}{2} J_1(ne^{-\tau}) J_0(ne^{-\tau}) \cos(2n\theta) - \frac{A^2 ne^{-2\tau}}{4} \left( J_0(ne^{-\tau})^2 + 2 (J_1(ne^{-\tau}))^2 - J_0(ne^{-\tau}) J_2(ne^{-\tau}) \right).$$

Here, $J_k$ and $Y_k$ denote the standard Bessel functions of first and second kind respectively, and $A$ is some fixed real valued constant.

As is shown in [18], these metrics are all vacuum and the sequence has a uniform limit on compact subsets of $\mathbb{R} \times \mathbb{T}^3$ to the limiting spacetime

$$g_\infty = e^{\frac{(\tau + \frac{A^2 e^{-\tau}}{2})}{4\pi}}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}(d\sigma^2 + d\delta^2),$$

which has a non-trivial Einstein tensor with the following non-vanishing components

$$G_{\tau\tau} = \frac{A^2 e^{-\tau}}{4\pi}, \quad G_{\theta\theta} = \frac{A^2 e^{-\tau}}{4\pi}.$$  

This corresponds to a solution to the Einstein equations with two families of null dust$^8$.

---

$^7$We remark that as we are only interested in the local behavior of high-frequency, the topology plays no role here.

$^8$It can be easily checked after introducing the null variables $\bar{u} := -e^{-\tau} + \theta$ and $u := -e^{-\tau} - \theta$, the non-vanishing components of the Einstein tensor in the $(\bar{u}, \bar{u}, \sigma, \delta)$ coordinates are

$$G_{\bar{u}\bar{u}} = G_{\bar{u}\bar{u}} = \frac{A^2}{8\pi H^2}.$$
While the explicit examples feature only oscillations and have a limit which is smooth, it is not difficult to modify (1.16) so that the limiting stress-energy-momentum tensor still corresponds to null dust, but is only a measure-valued null dust shell. Our main result will in particular generalize this simple example to general measure-valued null dust without symmetry assumptions.

**Example 1.18** (Null shell in plane symmetry). Let \( k(u) \in C^\infty_c \), \( \supp(k) \subseteq [-\frac{1}{2}, \frac{1}{2}] \), \( f \geq 0 \). Moreover, assume

\[
\int_{-\infty}^{\infty} (k'(u))^2 \, du = 1. \tag{1.11}
\]

Now, we construct a one-parameter family of solutions of the form (1.6) to the Einstein vacuum equation by setting

\[
G_\lambda(u) = \lambda^4 k\left(\frac{u}{\lambda}\right)
\]

for \( \lambda > 0 \). \( H_\lambda \) is then defined to be solutions to (1.8) so that we obtain a 1-parameter family of vacuum solutions. Notice that \( G_\lambda \) is much more singular than that in (1.16) as \( \lambda \to 0 \). In particular, \( G_\lambda \) is not uniformly bounded in \( \lambda \). It is easy to see that \( G_\lambda(u) \) converges uniformly to 0. We thus define

\[
G_0(u) = \lim_{\lambda \to 0} G_\lambda(u) = 0. \tag{1.12}
\]

An ODE argument shows that there is an interval \( u \in [-\epsilon, \epsilon] \) such that \( H_\lambda(u) \) admits a \( C^0 \cap W^{1,p} \) limit (for \( p \in [1, +\infty) \)).

For \( u \in [-\epsilon, \epsilon] \), the spacetime metric given by

\[
g_0 = -2dud\bar{u} + H_0(u)^2(e^{G_0(u)}dX^2 + e^{-G_0(u)}dY^2)
\]

satisfies (by (1.7))

\[
\text{Ric}(g_0) = \frac{1}{2} \delta(u) \, du \otimes du,
\]

which corresponds exactly to a null dust shell.

1.4.2. Characterization of the high-frequency limit. The above examples, while extremely specific, already illustrate the basic phenomenon. In the more general case in Theorem 1.10, we will prove that there are (at most) two non-trivial components of the Einstein tensor that can be generated in the weak limit. In particular, as asserted in Theorem 1.10, the limiting spacetime is isometric to a solution to the Einstein–null dust system. Unlike in the above examples, however, no explicit computations will be available and we will rely on compactness, particularly compensated compactness, arguments.

Our starting point is Theorem 1.9, which states that given a sequence of initial data obeying the estimates in Theorem 1.9 uniformly, there is a uniform region of existence. Moreover, it follows from the estimates established in the proof of Theorem 1.9 that the sequence of spacetime metrics are uniformly bounded in \( C^0 \cap W^{1,2} \). Using standard compactness results, this is sufficient to extract a subsequence \((\mathcal{M}, g_n)\) and a limiting spacetime so that the metrics converge strongly in \( C^0 \) and weakly in \( W^{1,2} \).

To obtain more information about the limiting spacetime, and to show that it indeed satisfies the Einstein–null dust system, we need a more precise understanding of the convergence. Introducing the Ricci coefficients \( \eta, \eta, \chi, \chi, \bar{\chi}, \bar{\chi}, \omega \) and \( \bar{\omega} \) with respect to a null frame adapted to a double null coordinate system (see Section 2.3), to understand whether the limiting Ricci coefficients amounts to checking whether quadratic products of these Ricci coefficients converge weakly to the products of the weak limits. The following are the main observations:

1. The Ricci coefficients \( \eta, \bar{\eta}, \chi, \bar{\chi} \) converge (up to a subsequence) strongly in the (say, spacetime\(^{10}\)) \( L^2 \) norm. In particular, in all the quadratic terms where \( \eta, \bar{\eta}, \chi, \bar{\chi} \) is one of the Ricci coefficients, the weak limit of the product coincides with the product of the weak limits.

2. Notice that even though \( \eta, \bar{\eta}, \chi, \bar{\chi} \) all have strong \( L^2 \) limits, the precise sense of limit (and the proof) is different. The components \( \eta \) and \( \bar{\eta} \) admit (subsequential) uniform pointwise limit, and can be proven by an Arzela–Ascoli type argument. However, \( \chi, \bar{\chi} \) only converge in \( L^p \) (for \( p \neq +\infty \)) (up to a subsequence) and to prove this we rely on the compactness of \( BV \) and the Aubin–Lions lemma.

\(^{9}\)Note however that \( H'_\lambda \) does not have a uniform limit.

\(^{10}\)In fact, stronger convergence holds (and the sense of convergence is different for different Ricci coefficients), but strong spacetime \( L^2 \) convergence is sufficient to ensure that they do not contribution to convergence defect for the Ricci curvature.
(3) The Ricci coefficients which only have weak $L^2$ limits, i.e. $\hat{\chi}, \hat{\chi}, \hat{\omega}$ and $\hat{\omega}$, exhibit some compensated compactness. For instance, since $\hat{\chi}$ is more regular along constant-$\nu$ hypersurfaces and $\hat{\chi}$ is more regular along constant-$\alpha$ hypersurfaces, we have $w^\ast \lim_{k \to +\infty} (\hat{\chi} \otimes \hat{\chi}) = (w^\ast \lim_{k \to +\infty} \hat{\chi}) \otimes (w^\ast \lim_{k \to +\infty} \hat{\chi})$, even though both $\hat{\chi}_n$ and $\hat{\chi}_n$ only admit weak limits.

(4) Finally, the only quadratic terms of the Ricci coefficients in the definition of the Ricci curvature such that the weak limit of the product differs from the product of the weak limits are $|\hat{\chi}|^2$ and $|\hat{\chi}|^2$. In particular, weak-* $\lim_{k \to +\infty} |\hat{\chi}_n|^{2}_{\gamma_k} - |\hat{\chi}|$ and weak-* $\lim_{k \to +\infty} \hat{\chi}_n - |\hat{\chi}|$ are in general non-trivial non-negative measures corresponding to the two families of null dust.

Using the above three observations, it already follows that with respect to the null frame $\{e_1, e_2, e_3, e_4\}$, the only potentially non-vanishing components of the Ricci curvature of the limiting spacetime are $\text{Ric}(e_3, e_3)$ and $\text{Ric}(e_4, e_4)$.

To prove that the limit solves the Einstein–null dust system, we also need to show the propagation equation of the null dust\footnote{In fact we will also show some higher order equations such as transport equations for some first angular derivatives of the Ricci coefficients and a hyperbolic system for the renormalized curvature components. They are strictly speaking not necessary for Theorem 1.10, but are used to prove the uniqueness of the limit.}. For this we need to understand convergence properties of some higher derivative terms and also cubic terms. These turn out to follow from strong convergence statements and compensated compactness statements which are similar to those above but for higher order derivatives.

1.4.3. Proof of the uniqueness theorem. For the uniqueness theorem (Theorem 1.12), first note that since the limiting spacetime metrics are obtained as limits of the metrics from Theorem 1.9, the metric components are mostly in similar function spaces as in [41, 42] (with the notable exception that $\nabla X, \nabla X$ are only BV instead of $W^{1,1}$, and that there are in addition two families of null dust, which are in general only measure-valued; more on this later). This suggests uniqueness to be proven in function spaces that are used in [42].

In order to obtain the uniqueness result, the renormalization introduced in [41, 42] plays an important role. (In fact, in this paper, the presence of the measure-valued null dust makes the renormalization in [40] more convenient to use than that in our original [41, 42]). The main point of the renormalization in [41, 42] is to identify some quantities we called the renormalized curvature components, which are more regular than the spacetime curvature components themselves. Introducing the renormalization moreover allowed us to obtain a closed system of equations which does not involve the spacetime curvature components $R(e_4, e_4, e_4, e_4)$ and $R(e_3, e_3, e_3, e_3)$, which in general only distribution-valued.

After introducing the renormalization, we can recast the Einstein–null dust system in double null coordinates as a coupled quasilinear system of hyperbolic, elliptic and transport equations for the metric components, the Ricci coefficients, the renormalized curvature components and the null dust. We then use this system to prove estimates and deduce the uniqueness statement.

One crucial difference of our uniqueness argument with the proof of a priori estimates in [42] is that a priori we only know that the solutions obey the Einstein–null dust system in an appropriate weak sense. Another (perhaps more important) difference is the presence of the null dust, which is moreover only measure-valued. To prove our uniqueness result will involve estimating differences of the null dust and the null mean curvatures $\hat{\chi}$ and $\hat{\chi}$ using the transport equations they satisfy, and this needs to be carried out in appropriate function spaces which avoid a potential loss of derivative; see Sections 8.4 and 8.7.

1.4.4. Approximating the initial data. The final result that we prove is Theorem 1.13 (from which Corollary 1.14 also follows), which asserts the local existence and uniqueness of solutions to the Einstein–null dust system with measure-valued null dust. Given that all limits satisfy the Einstein–null dust system (Theorem 1.10) and that the limiting spacetime depends only on the limiting data (Theorem 1.12), the final necessary ingredient is a statement that for every given data set to the Einstein–null dust system (with potentially a measure-valued null dust), we can find a sequence of smooth vacuum data which

(1) obey uniformly the estimates required in Theorem 1.10, and

(2) moreover limits in an appropriate weak sense to the given data.

Once this approximation result is achieved, we prove the existence part of Theorem 1.13 using Theorem 1.10 to extract a limit which is a weak solution to the Einstein–null dust system. We then prove the uniqueness part of Theorem 1.13 by Theorem 1.12.

To obtain the approximation result requires solving on the initial hypersurfaces the null constraint equations. For this we rely on the fact, elucidated in [10], that the null constraint equations can be
solved by first prescribing appropriate “free data”, related to the conformal class of $\gamma$, and then solving transport equations. We carry out the proof of the approximation result in two steps\footnote{In particular, even to generate the null dust shell, we first regularize the initial data for the null dust, and then approximate it by high-frequency oscillation in the metric, in contrast to Example 1.18.}, which we now describe.

In the first step, we show that (up to technical assumptions\footnote{We need a technical assumption that require the null dust to vanish near some angular direction; see Section 9.1.}) any smooth null dust data can be approximated by highly oscillatory but smooth vacuum data. We prescribe highly oscillatory data for $\frac{\partial u}{\partial \hat{n}}$ where $\gamma_n$ is an appropriate representation of the conformal class of $\gamma_n$. We choose a sequence $\gamma_n$ carefully so that $|\frac{\partial u}{\partial \hat{n}}|^2_{\gamma_n} - |\frac{\partial u}{\partial \hat{n}}|^2_{\gamma_n}$ converges weakly to the prescribed function corresponding to the null dust. Moreover, using the high-frequency parameter in $\gamma_n$ as a smallness parameter, we solve the vacuum null constraints equations, prove necessary estimates, and show that the limit indeed solves the null constraint equations for the Einstein–null dust system.

In the second step, we show that any measure-valued null dust data (with additional angular regularity) can be approximated by smooth null dust data (and obtain the desired result after combining with Step 1). To obtain this approximation result, we smooth out the given data for the null dust and the metric and prove a stability-type result for the null constraint equations in a low-regularity class consistent with the null dust only being a measure.

We emphasize that such an approximation result is possible precisely because we only need uniform bounds for $\chi$ and $\hat{\chi}$ in $L^2$ (with only additional angular regularity but not regularity in the null directions).

1.5. Formation of trapped surfaces. A main theme of this paper and the discussion thus far is the connection between the Einstein vacuum equations and the Einstein–null dust system via weak limits. In particular, our construction of solutions to the Einstein–null dust system with measure-valued null dust explicitly exploits this connection.

The connection between the Einstein vacuum equations and the Einstein–null dust system can also shed light on the problem of formation of trapped surfaces. Because of the complexity of the formation of trapped surfaces in vacuum, the problem was first studied in simplified settings, such as the collapse of a null dust shell [16, 48, 51]. We will show that the solutions in [16, 48, 51] in fact arise as suitable weak limits of vacuum solutions. These vacuum solutions are moreover exactly those constructed by Christodoulou in his groundbreaking work [10]. In other words, at least for a specific solution regime, the collapse of a null dust shell captures the dynamics of gravitational collapse in vacuum. See further discussions in Section 10.

1.6. Outline of the paper. We end the introduction with an outline of the remainder of the paper. In Section 2, we introduce the geometric setup in this paper. In particular, we will describe the double null foliation gauge. We will also define the precise notions of weak solutions that will be relevant in this paper. After these preliminary discussions, we recall the existence theorem in [42] in Section 3. We then state the precise version of the main theorems (Theorems 4.1, 4.4, 4.6 and 4.7) and the result on approximation by vacuum spacetimes (Theorem 4.7) follow.

The remainder of the paper is then devoted to the proofs of these four theorems.

- Sections 5–7 are devoted to proving that a limit exists and solves the Einstein–null dust system (Theorem 4.1). In Section 5, we begin with some general compactness results. In Section 6, the compactness results will be applied to extract a limiting spacetime with regularity properties. In Section 7, we then show that the limit satisfies the Einstein–null dust system.
- Section 8 will be devoted to proving the uniqueness theorem (Theorem 4.4).
- Section 9 will be devoted to proving an approximation theorem for the characteristic initial data, from which the local existence and uniqueness result for the Einstein–null dust system (Theorem 4.6) and the result on approximation by vacuum spacetimes (Theorem 4.7) follow.

In Section 10, we end the paper with a discussion regarding the relation between null dust shell solutions and the formation of trapped surfaces result of Christodoulou [10].

Finally, we include two appendices. In Appendix A, we derive from [42] some additional estimates that are used in this paper. In Appendix B, we prove our main compensated compactness lemma.

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2. Geometric preliminaries

2.1. Weak solutions to the Einstein equations. In this subsection, we give some definitions of weak solutions to the Einstein vacuum equations and the Einstein–null dust system which make sense under very weak regularity assumptions. We will very soon further restrict the class of solutions that we consider, but it is useful to keep these more general definitions in mind.

**Definition 2.1.** Let \((M, g)\) be a \(C^0 \cap W^{1,2}_{\text{loc}}\) time-oriented Lorentzian manifold. We say that \((M, g)\) is a weak solution to the **Einstein vacuum equations** if for all smooth and compactly supported vector fields \(X\) and \(Y\),

\[
\int_M ((D_\mu X^\nu)(D_\nu Y^\mu) - D_\mu X^{\nu\rho} D_\rho Y^\mu) \, \text{dVol}_g = 0,
\]

where \(D\) and \(\text{dVol}_g\) respectively denote the Levi–Civita connection and the volume form associated to \(g\).

**Definition 2.2.** Let \((M, g)\) be a \(C^0 \cap W^{1,2}_{\text{loc}}\) time-oriented Lorentzian manifold and \(du, dv\) be non-negative Radon measures on \(M\). Let \(u, \varphi : M \to \mathbb{R}\) be two \(C^1\) functions satisfying

\[
g^{-1}(du, du) = g^{-1}(dv, dv) = 0.
\]

We say that \((M, g, dv, du)\) is a **weak solution to the Einstein–null dust system** if the following holds:

1. For all smooth and compactly supported vector fields \(X\) and \(Y\),

\[
\int_M ((D_\mu X^\nu)(D_\nu Y^\mu) - D_\mu X^{\nu\rho} D_\rho Y^\mu) \, \text{dVol}_g = \int_M (Xu)(Yu) \, dv + \int_M (X\varphi)(Y\varphi) \, dv.
\]

2. For every smooth and compactly supported real-valued function \(\varphi\),

\[
\int_M g^{-1}(du, d\varphi) \, dv = 0 = \int_M g^{-1}(dv, d\varphi) \, dv.
\]

**Remark 2.3.** Consider the particular case where \(g\) is \(C^2\) and there exist \(C^1\) functions \(f_\mu\) and \(f_\nu\) such that \(dp = f_\mu^2 \text{dVol}_g\) and \(dv = f_\nu^2 \text{dVol}_g\). Then the two conditions in Definition 2.2 are equivalent to

1. \(\text{Ric}(g) = f_\mu^2 \, du \otimes du + f_\nu^2 \, dv \otimes dv\),

2. \(2g^{-1}(du, df_\mu) + (\Box_g u)f_\mu = 0 = 2g^{-1}(dv, df_\nu) + (\Box_g \varphi)f_\nu\),

where \(\Box_g\) denotes the Laplace–Beltrami operator associated to \(g\).

**Remark 2.4.** Definition 2.2 does not give the most general form of the Einstein–null dust system. Even when the density of the null dust is given by an \(L^1\) function, one in general allows, for null 1-forms \(\xi\) and \(\xi^\ast\)

1. \(\text{Ric} = f_\xi^2 \, \xi \otimes \xi + f_\xi^2 \, \xi^\ast \otimes \xi^\ast\),

2. \(2g^{-1}(\xi, df_\xi) + (D^\alpha \xi_\alpha)f_\xi = 0 = 2g^{-1}(\xi^\ast, df_\xi^\ast) + (D^\alpha \xi^\ast_\alpha)f_\xi^\ast\),

3. \(D_\xi \xi = 0 = D_{\xi^\ast} \xi^\ast\),

where \(\xi\) and \(\xi^\ast\) are not necessarily exact. We restrict however our attention to Definition 2.2 since this is precisely what arises in the limit in our setting.

2.2. Double null foliation and double null coordinates. In this subsection, we define spacetimes with double null foliation and double null coordinates. From now on we consider \(\mathcal{M}\) being a manifold with corners with\(^{14}\)

\[
\mathcal{M} = [0, u_\ast] \times [0, \underline{u}_\ast] \times \mathbb{S}^2,
\]

where \(u_\ast, \underline{u}_\ast > 0\).

**Definition 2.5.** We introduce the following notations for subsets of \(\mathcal{M}\) satisfying (2.1):

1. \(H_u := \{(u', u''), \vartheta) \in [0, u_\ast] \times [0, \underline{u}_\ast] \times \mathbb{S}^2 : u' = u\}\),

\(^{14}\)In fact, all of our results apply if we replace \(\mathbb{S}^2\) by any compact 2-surface \(S\). The choice of \(S = \mathbb{S}^2\) is so that we have consistent notation with [41, 42].
(2) \[ H_u := \{(u', u', \vartheta) \in [0, u_u] \times [0, u_u] \times S^2 : u' = u \}, \]
(3) \[ S_{u,u} := H_u \cap H_u = \{(u', u', \vartheta) \in [0, u_u] \times [0, u_u] \times S^2 : u' = u, u' = u \}. \]

**Definition 2.6.** For a type \( T_{\alpha}^\beta \) tensor field \( \xi \) on \( M \), we say that it is \( S \)-tangent if for every vector fields \( X_1, ..., X_p \in T_x M \), \( \langle X_1, ..., X_p \rangle \) is in \( \otimes^p T_x S_{u,u} \) and \( \xi(X_1, ..., X_p) = 0 \) if any one of \( X_1, ..., X_p \) equals to \( e_3 \) or \( e_4 \).

We introduce the following convention for indices for the remainder of the paper. We will use the convention that the lower case Greek indices run through the spacetime indices \( \mu, \nu = 1, 2, 3, 4 \) while the upper case Latin indices run through the indices on the 2-surfaces \( A, B = 1, 2 \). We will use Einstein’s summation convention unless otherwise stated.

**Definition 2.7** (\( C^0 \cap W^{1,2}_{\text{loc}} \) metrics in double null coordinates). A Lorentzian metric \( g \) on \( M \) satisfying (2.1) is said to be a \( C^0 \cap W^{1,2}_{\text{loc}} \) **metric in double null coordinates** if the following hold:

- There exists an atlas \( \{ U_i \}_{i=1}^N \subset \mathbb{S}^2 \) such that given coordinates \( (\theta^1, \theta^2) \) in each coordinate chart \( U_i \), the metric takes the form\(^\text{15}\)

\[ g = -2\Omega^2 (du \otimes du + dg \otimes du) + \gamma_{AB} (d\theta^A - b^A du) \otimes (d\theta^B - b^B du), \]  

where

- \( \Omega \) is a real-valued function in \( C^0 \cap W^{1,2}_{\text{loc}} \),
- \( b = b^A \frac{\partial}{\partial \theta^A} \) is a \( C^0 \cap W^{1,2}_{\text{loc}} \) \( S \)-tangent vector field tangential to \( S \),
- \( \gamma = \gamma_{AB} d\theta^A d\theta^B \) is a \( C^0 \cap W^{1,2}_{\text{loc}} \) \( S \)-tangent symmetric covariant 2-tensor, which restricts to a positive definite metric on \( TS \).

**Remark 2.8** (Regularity assumptions). In Definition 2.7, the metric components are required only to be in \( C^0 \cap W^{1,2}_{\text{loc}} \). This is the minimal assumption that we need for many of the definitions below. Nevertheless, the spacetime metrics that we actually construct will have higher regularity, at least when viewed in some directions (see already Definition 2.26).

**Remark 2.9** (Geometric significance of the double null coordinate system). At least when the metric is sufficiently regular, (2.2) has the following geometric interpretation:

1. \( u \) and \( u \) are null variables. In particular, \( H_u \) and \( H_u \) (recall (2.5)) are null hypersurfaces.
2. \( (du) \) and \( (du) \) are null geodesic vector fields (where \( \sharp \) denotes the metric dual).
3. The coordinate functions \( \theta^1 \) and \( \theta^2 \) are constant along the integral curves of \( (du) \).

2.3. **Ricci coefficients and the Gauss curvature of the 2-spheres.** We will define the Ricci coefficients for a metric satisfying Definition 2.7. For the rest of this subsection, we assume that \( (M, g) \) satisfying Definition 2.7 is given and fixed.

**Definition 2.10.** The normalized null pair are defined as follows:

\[ e_3 = \Omega^{-1} \left( \frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A} \right), \quad e_4 = \Omega^{-1} \frac{\partial}{\partial u}. \]

We will also write \( \{e_A\}_{A=1,2} \) to denote an arbitrary local frame tangent to \( S_{u,u} \).

We now define the Ricci coefficients as the following \( S \)-tangent tensors, where \( D \) is the Levi–Civita connection with respect to the spacetime metric \( g \):

**Definition 2.11** (Ricci coefficients). (1) Define the following Ricci coefficients\(^\text{16}\) such that for vector fields \( X, Y \) tangential to \( S \):

\[ \chi(X, Y) = g(D_X e_4, Y), \quad \chi(X, Y) = g(D_X e_4, Y), \]
\[ \eta(X) = -\frac{1}{2} g(D_3 X, e_4), \quad \eta(X) = -\frac{1}{2} g(D_3 X, e_4), \]
\[ \omega = -\frac{1}{4} g(D_1 e_3, e_4), \quad \omega = -\frac{1}{4} g(D_1 e_3, e_4). \]

\(^{15}\)Recall our notation that capital Latin letters are summed from 1 to 2.
\(^{16}\)Note that this is well-defined with \( C^0 \cap W^{1,2}_{\text{loc}} \) regularity of the metric coefficients.
(2) Define also
\[ \hat{\chi} = \chi - \frac{1}{2} \nabla \chi, \quad \hat{\chi} = \chi - \frac{1}{2} \nabla \chi, \]
where \( \hat{\chi} \) (resp. \( \hat{\chi} \)) is the traceless part of \( \chi \) (resp. \( \chi \)) (with the trace taken with respect to the metric \( \gamma \) on \( S_{u,u} \) and \( \nabla \chi \) (resp. \( \nabla \chi \)) is the trace of \( \chi \) (resp. \( \chi \)).

(3) We will use \( \chi_{AB}, \chi_{AB}, \eta_{A}, \) etc. to denote the components of the Ricci coefficients with respect to a local coordinate system on \( S \) in a double null coordinate system.

We define the following quantities which depend only on the intrinsic geometry of a Riemannian 2-manifold. Note that in application \( S = S_{u,u} \) for some \( u, u \).

**Definition 2.12.** Let \( (S, \gamma) \) be a closed Riemannian 2-manifold.

1. Define \( \text{Area}(S, \gamma) \) to be the total area of \( S \) with respect to the metric \( \gamma \).
2. Define the isoperimetric constant by
\[ I(S, \gamma) = \sup_{\partial U \subset C^1} \frac{\min\{\text{Area}(U), \text{Area}(U^c)\}}{(\text{Perimeter}(\partial U))^2}. \]

3. Define the Gauss curvature \( K \) by
\[ \gamma_{BC} K = \frac{\partial}{\partial \theta^A} \Gamma_{BA}^A - \frac{\partial}{\partial \theta^C} \Gamma_{BA}^C + \frac{\partial}{\partial \theta^D} \Gamma_{BA}^D, \]
where
\[ \Gamma_{BA}^C = \frac{1}{2} \sum_{i} (\frac{\partial}{\partial \theta^A} \gamma_{BD} + \frac{\partial}{\partial \theta^D} \gamma_{AB} - \frac{\partial}{\partial \theta^B} \gamma_{AD}) \chi_{ABC}. \]

When there is no danger of confusion, we write \( \text{Area}(S) = \text{Area}(S, \gamma) \) and \( I(S) = I(S, \gamma) \).

Note that our regularity assumptions are sufficient to make sense of \( \text{Area}(S_{u,u}, \gamma) \) and \( I(S_{u,u}, \gamma) \). However, for \( (M, g) \) satisfying Definition 2.7, the Gauss curvature \( K \) of the surfaces \( S_{u,u} \) is only to be understood as a distribution.

### 2.4. Differential operators on S-tangent tensor fields.

**Definition 2.13** (Covariant derivatives for \( S \)-tangent tensor fields). Define \( \hat{\nabla}_3 \) and \( \hat{\nabla}_4 \) to be the projections to \( S_{u,u} \) of the covariant derivatives \( D_3 = D_{e_3} \) and \( D_4 = D_{e_4} \) respectively. Define \( \hat{\nabla}_A \) to be the Levi–Civita connection with respect to the metric \( \gamma \).

The following identities hold for the connections \( \hat{\nabla}_3, \hat{\nabla}_4 \) and \( \hat{\nabla} \). The proofs are straightforward and omitted.

**Proposition 2.14.** In the double null coordinate system, for every covariant tensor field \( \phi \) of rank \( r \) tangential to the spheres \( S_{u,u} \), we have
\[ (\hat{\nabla}_3 \phi)_{A_1 A_2 \ldots A_r} = \Omega^{-1} (\hat{\nabla}_u + bC \hat{\nabla}_u) \phi_{A_1 A_2 \ldots A_r} - \sum_{i=1}^{r} (\chi_{A_i} \phi_{A_1 A_2 \ldots A_r} - \Omega^{-1} \frac{\partial b}{\partial \theta^A} \phi_{A_1 A_2 \ldots A_r}), \]
where \( \hat{A}_i \) denotes that the \( A_i \) which was originally present is removed. Similarly, we have
\[ (\hat{\nabla}_4 \phi)_{A_1 A_2 \ldots A_r} = \Omega^{-1} \hat{\nabla}_u \phi_{A_1 A_2 \ldots A_r} - \sum_{i=1}^{r} \chi_{A_i} \phi_{A_1 A_2 \ldots A_r}. \]

Finally, \( \hat{\nabla} \) is given by the Levi–Civita connection associated to the metric \( \gamma \), i.e.,
\[ \hat{\nabla}_{BA} \phi_{A_1 A_2 \ldots A_r} = \frac{\partial}{\partial \theta^B} \phi_{A_1 A_2 \ldots A_r} - \sum_{i=1}^{r} \Gamma_{BA_i} \phi_{A_1 A_2 \ldots A_r}. \]

We introduce the following differential operators:

**Definition 2.15.** Define \( \mathcal{L} \) to be the projection of the Lie derivative to the tangent space of \( S_{u,u} \).

**Definition 2.16.** For a totally symmetric tensor field \( \phi \) of rank \( r + 1 \), define
\[ (\hat{\text{div}} \phi)_{A_1 \ldots A_r} := (\gamma^{-1})_{BC} \hat{\nabla} B \phi_{CA_1 \ldots A_r}, \quad (\text{curl} \phi)_{A_1 \ldots A_r} := \varepsilon_{BC} \hat{\nabla} B \phi_{CA_1 \ldots A_r}, \]
where \( \varepsilon \) denotes the volume form with respect to \( \gamma \).

Finally, define the following operator on \( S \)-tangent 1-forms:
\[ (\hat{\nabla} \otimes \phi)_{AB} := \hat{\nabla} A \phi_B - \hat{\nabla} B \phi_A - \gamma_{AB} \hat{\text{div}} \phi. \]
2.5. Some basic identities.

**Proposition 2.17.** The Ricci coefficients can be expressed as derivatives of the metric components in the double null coordinate system (recall (2.2)). More precisely, we have\(^{17}\)

\[
\mathcal{L}_{\frac{\partial}{\partial u}} \gamma = 2\Omega \chi, \quad \mathcal{L}_{\frac{\partial}{\partial u} + b^A \frac{\partial}{\partial u^A}} \gamma = 2\Omega \chi, \quad \mathcal{L}_{\frac{\partial}{\partial u}} b^A = -2\Omega^2 (\eta^A - \eta^A),
\]

where \(\mathcal{L}\) is as in Definition 2.15 and \(^{\dagger}\) denotes the metric dual with respect to \(\gamma\).

**Proposition 2.18.** The following relation holds:

\[
\omega = -\frac{1}{2} (e^4 \log \Omega), \quad \varpi = -\frac{1}{2} (e^3 \log \Omega), \quad \frac{1}{2} (\eta A + \eta A) = \nabla_A \log \Omega.
\]

2.6. Function spaces and angularly regular metrics. In this paper, we will mostly consider a slightly more restricted class of spacetimes in double null coordinates; see already Definition 2.26. The main feature of this class is that the metric is more regular along the directions tangential to \(S_{u,w}\) (property that it is not in a better isotropic Sobolev space than \(W^{1,2}\)). Moreover, the precise regularity is different for different Ricci coefficients, which can be seen as capturing the null structure of the Einstein equations in a double null coordinate system. The regularity that we impose is consistent with the regularity of spacetimes obtained in [42].

**Definition 2.19.** Denote by \(dA_\gamma\) the standard volume form induced by the (Riemannian) metric \(\gamma\) on \(S_{u,w}\), i.e. in local coordinates, \(dA_\gamma := \sqrt{\det \gamma} \, d\theta^1 \, d\theta^2\).

**Definition 2.20** (Definition of \(L^p\) spaces). In this definition, let \(\phi\) be an \(S\)-tangent rank \(r\) tensor field on \([0,u_\ast] \times [0,\bar{u}_\ast] \times \mathbb{S}^2\).

(i) For every \((u,\bar{u})\), define, for \(p \in [1, +\infty)\)

\[
\|\phi\|_{L^p_u(S_{u,\bar{u}};\gamma)} := \left( \int_{S_{u,\bar{u}}} |\phi|^p \, dA_\gamma \right)^{\frac{1}{p}}
\]

and for \(p = +\infty\), define

\[
\|\phi\|_{L^\infty_u(S_{u,\bar{u}};\gamma)} := \text{ess sup}_{S_{u,\bar{u}}} |\phi| \equiv \text{ess sup}_{S_{u,\bar{u}}} ((\gamma^{-1})^{A_1 A_1'} \cdots (\gamma^{-1})^{A_r A_r'} \phi_{A_1 \cdots A_r} \phi_{A_1' \cdots A_r'})^{\frac{1}{2}}.
\]

We will often view \(\|\phi\|_{L^p_u(S_{u,\bar{u}};\gamma)}\) as a function of \(u\) and \(\bar{u}\).

(ii) For \(q \in [1, +\infty)\), \(p \in [1, +\infty]\), define

\[
\|\phi\|_{L^q_u L^p(S_{u,\bar{u}};\gamma)} := \left( \int_{S_{u,\bar{u}}} \|\phi\|_{L^p_u(S_{u,\bar{u}};\gamma)}^q \, du \right)^{\frac{1}{q}}.
\]

These two terms will be viewed as functions of \(u\) and \(\bar{u}\) respectively. Define also \(L^\infty_u L^p(S_{u,\bar{u}};\gamma)\) and \(L^\infty_u L^p(S_{u,\bar{u}};\gamma)\) after the obvious modifications.

(iii) For \(r \in [1, +\infty)\), \(p, q \in [1, +\infty]\), define

\[
\|\phi\|_{L^r_u L^q_u L^p_u(S_{u,\bar{u}};\gamma)} := \left( \int_0^u \|\phi\|_{L^p_u(S_{u,\bar{u}};\gamma)}^r \, du \right)^{\frac{1}{r}},
\]

\[
\|\phi\|_{L^r_u L^q_u L^p_u(S_{u,\bar{u}};\gamma)} := \left( \int_0^u \|\phi\|_{L^p_u(S_{u,\bar{u}};\gamma)}^r \, du \right)^{\frac{1}{r}}.
\]

In a similar manner as (2), we also allow \(r = +\infty\) after the obvious modifications.

**Definition 2.21** (Definition of Sobolev spaces). Let \(\phi\) be an \(S\)-tangent tensor field.

\[
\frac{\partial}{\partial u} \gamma_{AB} = 2\Omega X_{AB}, \quad \frac{\partial}{\partial u} b^A = \frac{\partial}{\partial u} b^A = -2\Omega^2 (\eta^A - \eta^A),
\]

\[
\frac{\partial}{\partial u} b^A = -2\Omega^2 (\eta^A - \eta^A).
\]

\(^{17}\)In coordinates, the relations in (2.10) above read as follows:

\[
\frac{\partial}{\partial u} \gamma_{AB} = 2\Omega X_{AB}, \quad \frac{\partial}{\partial u} b^A = \frac{\partial}{\partial u} b^A = -2\Omega^2 (\eta^A - \eta^A),
\]

\[
\frac{\partial}{\partial u} b^A = -2\Omega^2 (\eta^A - \eta^A).
\]
For every $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, +\infty]$, define
\[
\|\phi\|_{W^m,p(S, \gamma)} := \|\nabla^m \phi\|_{L^p(S, \gamma)},
\]
where $\nabla$ is the Levi-Civita connection associated to $\gamma$.

(2) Define also $L^m_w S(S, \gamma), L^m w L^m S(S, \gamma), \ldots, \text{etc. in a similar manner as Definition 2.2.2 and 2.20.3 after replacing } L^p(S, \gamma)$ by $W^m(S, \gamma)$.

**Definition 2.22** (Definition of the BV spaces). Let $\phi$ be an $S$-tangent rank $r$ tensor field on $[0, u] \times [0, v] \times S^2$. Define
\[
\|\phi\|_{BV(H, \gamma)} := \int_0^u \|\phi\|_{L^1(S, \gamma)} \, du + \sup \left\{ \left| \int_0^u \int_{S^{u, \gamma}} \frac{\partial}{\partial u} \phi \, dA, \, du \right| \mid \phi \in C^1, \|\phi\|_{L^1(S, \gamma)} \leq 1 \right\}.
\]

Similarly, we define
\[
\|\phi\|_{BV(H, \gamma)} := \int_0^u \|\phi\|_{L^1(S, \gamma)} \, du + \sup \left\{ \left| \int_0^u \int_{S^{u, \gamma}} \frac{\partial}{\partial u} \phi \, dA, \, du \right| \mid \phi \in C^1, \|\phi\|_{L^1(S, \gamma)} \leq 1 \right\}.
\]

**Definition 2.23** (Continuity in $u$ and/or $v$). Define the space $C^0(S, \gamma) W^m_p(S, \gamma)$ as the completion of smooth tensor fields under the $L^m_w S(S, \gamma)$ in a similar manner.

At this point, let us recall that BV functions, even though they are defined only a.e., have well-defined traces on Lipschitz hypersurfaces. We will in particular need the following statement (whose proof can be found for instance in [15, Theorem 5.6]):

**Lemma 2.24.** Let $f : [0, u] \times [0, v] \times S^2 \to \mathbb{R}$ be such that $f \in C^0(S, \gamma)$. Then the following holds.

(1) For every $(u, u) \in [0, u] \times (0, u)\), there is an $L^1(S, \gamma)$ function $f^-(u, v, \theta)$ such that
\[
\lim_{\epsilon \to 0^+} \int_0^u \frac{1}{\epsilon} \int_{S^{u, \gamma}} \left| f^-(u, u, \theta) - f(u, u, \theta) \right| \, dA, \, du = 0.
\]

(2) For every $(u, u) \in [0, u] \times (0, u)$, there is an $L^1(S, \gamma)$ function $f^+(u, v, \theta)$ such that
\[
\lim_{\epsilon \to 0^+} \int_0^u \frac{1}{\epsilon} \int_{S^{u, \gamma}} \left| f^+(u, u, \theta) - f(u, u, \theta) \right| \, dA, \, du = 0.
\]

Similar statements hold for $f \in C^0(S, \gamma)$ after swapping $u$ and $v$; we omit the details.

We need one more definition before we introduce the class of spacetime we consider. We introduce an auxiliary metric in all of $\mathcal{M} = [0, u] \times [0, v] \times S^2$ to measure the regularity of $\gamma$. We define this to be the Lie-transported $\gamma$ $\gamma_{s,\gamma}$.

**Definition 2.25.** Define $\gamma_{s,\gamma}$ on $\mathcal{M} = [0, u] \times [0, v] \times S^2$ so that on the initial 2-sphere $S_{s,\gamma}$, $\gamma_{s,\gamma} = \gamma$ $\gamma_{s,\gamma}$, and that$^\text{18}$
\[
\ell_{s,\gamma} \gamma_{s,\gamma} = 0, \quad \ell_{s,\gamma} \gamma_{s,\gamma} = 0
\]
everywhere else in $\mathcal{M}$.

We are now ready to define the class of spacetimes that we study for the remainder of the paper.

**Definition 2.26** (Angularly regular double null metrics). Let $(\mathcal{M} = [0, u] \times [0, v] \times S, g)$ be a spacetime in double null coordinates (see Definition 2.2). We say that $(\mathcal{M}, g)$ is angularly regular if

(1) $\gamma - \gamma_{s,\gamma}$, $\beta$, $\log \Omega \in C^0(S, \gamma) W^2,4(S, \gamma) \cap L^\infty(S, \gamma)$, $\log \det \gamma_{s,\gamma} \in C^0(S, \gamma) C^0(S, \gamma)$,

(2) $\sup_{u, v} (\text{Area}(S, \gamma) + (\text{Area}(S, \gamma))^{-1}) < +\infty$, $\log \det \gamma_{s,\gamma} \in C^0(S, \gamma) C^0(S, \gamma)$,

(3) $K \in C^0(S, \gamma) L^2(S, \gamma) \cap L^\infty(S, \gamma) W^2,2(S, \gamma) \cap L^\infty(S, \gamma) W^2,2(S, \gamma)$.

$^\text{18}$Note that this is possible since $\frac{\partial}{\partial s} \gamma_{s,\gamma} = 0$. 
(4) \( \chi, \omega \in L^2_u L^\infty_u W^{2, 2}(S_{u, \omega}) \cap C^0_u L^2_u W^{2, 2}(S_{u, \omega}) \cap L^\infty_u L^2_u W^{3, 2}(S_{u, \omega}) \), \( \chi, \omega \in L^2_u L^\infty_u W^{2, 2}(S_{u, \omega}) \cap C^0_u L^2_u W^{2, 2}(S_{u, \omega}) \cap L^\infty_u L^2_u W^{3, 2}(S_{u, \omega}) \).

(5) \( \eta, \eta \in C^0_u L^1_u W^{1, 4}(S_{u, \omega}) \cap L^\infty_u L^2_u W^{2, 2}(S_{u, \omega}) \cap L^\infty_u L^2_u W^{3, 2}(S_{u, \omega}) \cap L^\infty_u L^2_u W^{3, 2}(S_{u, \omega}) \).

(6) \( \psi \chi \in C^0_u L^2_u W^{2, 1}(S_{u, \omega}) \cap L^\infty_u \text{BV}(H_u) \cap L^\infty_u L^2_u W^{3, 2}(S_{u, \omega}) \), \( \psi \chi \in C^0_u L^1_u W^{2, 1}(S_{u, \omega}) \cap L^\infty_u \text{BV}(H_u) \cap L^\infty_u L^2_u W^{3, 2}(S_{u, \omega}) \).

In the above, we have written \( S_{u, \omega}, H_u \) and \( H_u \) instead of \( (S_{u, \omega}, \gamma), (H_u, \gamma) \) and \( (H_u, \gamma) \) to simplify the notations.

Remark 2.27. Note that because of the presence of \( C^0_u \) and \( C^0_u \) in the above spaces, we can talk about \( \gamma, b, \log \Omega, \eta \) and \( \eta \) for every \( u \) and \( y \) (and not just for almost every \( u \) and \( y \)). We can also talk about \( \chi \) and \( \omega \) as an \( L^2_u W^{2, 2}(S_{u, \omega}) \) function and \( \psi \chi \) as a \( L^1_u W^{2, 1}(S_{u, \omega}) \) function for every \( u \) (as opposed to just for almost every \( u \)). Similarly, we can talk about \( \chi \) and \( \omega \) as a \( L^2_u W^{2, 2}(S_{u, \omega}) \) function and \( \psi \chi \) as a \( L^1_u W^{2, 1}(S_{u, \omega}) \) function for every \( y \) (as opposed to just for almost every \( y \)).

2.7. Null structure equations. We introduce the null structure equations, which are equations for the Ricci coefficients which hold on solutions to the Einstein vacuum equations. To state these equations, we need the following definitions (in addition to Definition 2.16):

Definition 2.28. Define the following contractions

\[ \phi^{(1)} \cdot \phi^{(2)} := (\gamma^{-1})^{AC} (\gamma^{-1})^{BD} \phi^{(1)}_{AB} \phi^{(2)}_{CD} \text{ for symmetric 2-tensors } \phi^{(1)}_{AB}, \phi^{(2)}_{AB}, \]

\[ \phi^{(1)} \cdot \phi^{(2)} := (\gamma^{-1})^{AB} \phi^{(1)}_{A} \phi^{(2)}_{B} \text{ for 1-forms } \phi^{(1)}_{A}, \phi^{(2)}_{A}. \]

\[ (\phi^{(1)} \cdot \phi^{(2)})_A := (\gamma^{-1})^{BC} \phi^{(1)}_{AB} \phi^{(2)}_{BC} \text{ for a symmetric 2-tensor } \phi^{(1)}_{AB} \text{ and a 1-form } \phi^{(2)}_{A}. \]

\[ (\phi^{(1)} \otimes \phi^{(2)})_{AB} := \phi^{(1)}_{A} \phi^{(2)}_{B} + \phi^{(1)}_{A} \phi^{(2)}_{B} - \gamma_{AB} (\gamma^{-1})^{CD} \phi^{(1)}_{CD} \phi^{(2)}_{AB} \text{ for one forms } \phi^{(1)}_{A}, \phi^{(2)}_{A}. \]

\[ \phi^{(1)} \land \phi^{(2)} := \varepsilon^{AB} (\gamma^{-1})^{CD} \phi^{(1)}_{AC} \phi^{(2)}_{BD} \text{ for symmetric two tensors } \phi^{(1)}_{AB}, \phi^{(2)}_{AB}. \]

Define * of 1-forms and symmetric 2-tensors respectively as follows (note that on 1-forms this is the Hodge dual on \( S_{u, \omega} \)):

\[ \ast \phi_A := \gamma_{AC} \in CB \quad \ast \phi_{AB} := \gamma_{BC} \in DC \phi_{AC}. \]

Define also the trace for totally symmetric tensors of rank r to be

\[ \left( \ast \phi \right)_{A_1 \ldots A_r} := (\gamma^{-1})^{BC} \phi_{BC A_1 \ldots A_r}. \]

We now give a list of the null structure equations.

Proposition 2.29. Let \((M, [0, u] \times [0, y] \times S, g)\) be a \( C^2 \) spacetime in double null coordinates. If \((M, g)\) solves the Einstein vacuum equations, then the following null structure equations hold:

\[ \nabla_4 \psi \chi + \frac{1}{2} (\psi \chi)^2 = -|\chi|^2 - 2\omega \psi \chi, \]

\[ \nabla_3 \psi \chi - \frac{1}{2} (\psi \chi)^2 = -|\chi|^2 - 2\omega \psi \chi, \]

\[ \nabla_4 \eta + \frac{3}{4} \psi \chi (\eta - \eta) = \nabla_4 \chi - \frac{1}{2} \nabla_4 \psi \chi - \frac{1}{2} (\eta - \eta) \cdot \hat{\chi}, \]

\[ \nabla_3 \eta + \frac{3}{4} \psi \chi (\eta - \eta) = \nabla_3 \chi - \frac{1}{2} \nabla_3 \psi \chi - \frac{1}{2} (\eta - \eta) \cdot \hat{\chi}, \]

\[ \nabla_4 \psi \chi + \psi \chi \psi \chi = 2\omega \psi \chi - 2K + 2\nabla_4 \eta + 2|\eta|^2, \]

\[ \nabla_3 \psi \chi + \psi \chi \psi \chi = 2\omega \psi \chi - 2K + 2\nabla_3 \eta + 2|\eta|^2, \]

\[ \nabla_4 \chi + \frac{1}{3} \psi \chi = \nabla_4 \eta + 2\omega \psi \chi - \frac{1}{2} \psi \chi \hat{\chi} + \eta \cdot \hat{\eta}, \]

\[ \nabla_3 \chi + \frac{1}{3} \psi \chi = \nabla_3 \eta + 2\omega \psi \chi - \frac{1}{2} \psi \chi \hat{\chi} + \eta \cdot \hat{\eta}, \]

\[ \nabla_4 \omega - 2\omega \omega + \eta \cdot \eta - \frac{1}{4} |\eta|^2 = - \frac{1}{2} (K - \frac{1}{2} \chi \cdot \hat{\chi} + \frac{1}{4} \psi \chi \psi \chi), \]

\[ \nabla_3 \omega - 2\omega \omega + \eta \cdot \eta - \frac{1}{4} |\eta|^2 = - \frac{1}{2} (K - \frac{1}{2} \chi \cdot \hat{\chi} + \frac{1}{4} \psi \chi \psi \chi). \]

**Proof.** See the derivation for instance in [10]. \( \square \)
2.8. Weak formulation of transport equations.

**Definition 2.30.** Let \((\mathcal{M}, g)\) be an angularly regular metric in double null coordinates. Consider the transport equations

\[
\begin{align*}
\nabla_3 \phi &= F, \quad (2.23) \\
\nabla_4 \psi &= G, \quad (2.24)
\end{align*}
\]

where \(\phi, F, \psi, G\) are \(S\)-tangent covariant tensor fields of rank \(r\) in \(C^0_uC^1_uL^p(S)\) for some \(p \in [1, +\infty]\).

We say that (2.23) (resp. (2.24)) is satisfied in the integrated sense if for every \(C^1\) contravariant \(S\)-tangent tensor \(\varphi\) of rank \(r\), the following holds:

\[
\int_{S_{\nu,2}} \langle \varphi, \phi \rangle \Omega dA_\gamma - \int_{S_{\nu,1}} \langle \varphi, \phi \rangle \Omega dA_\gamma - \int_{u_1}^{u_2} \int_{S_{\nu,2}} \left( (\varphi, F + (\dot{\psi} \chi - 2\omega) \phi) + \langle \nabla_3 \varphi, \phi \rangle \right) \Omega^2 dA_\gamma du' = 0,
\]

(resp. \(\forall 0 \leq u_1 < u_2 \leq u_*\)).

**Definition 2.31.** Let \((\mathcal{M}, g)\) be an angularly regular metric in double null coordinates. (2.23) and (2.24) where \(\phi, F\) are \(S\)-tangent covariant tensor fields of rank \(r\) in \(C^0_uC^1_uL^p(S)\) for some \(p \in [1, +\infty]\); and \(\psi, G\) are \(S\)-tangent covariant tensor fields of rank \(r\) in \(C^0_uC^1_uL^p(S)\) for some \(p \in [1, +\infty]\).

We say that (2.23) (resp. (2.24)) is satisfied in the weak integrated sense if for every \(C^1\) contravariant \(S\)-tangent tensor \(\varphi\) of rank \(r\), the following holds:

\[
\int_{0}^{u_2} \int_{S_{\nu,2}} \langle \varphi, \phi \rangle \Omega dA_\gamma du - \int_{0}^{u_2} \int_{S_{\nu,1}} \langle \varphi, \phi \rangle \Omega dA_\gamma du = \int_{0}^{u_2} \int_{S_{\nu,2}} \left( (\varphi, F + (\dot{\psi} \chi - 2\omega) \phi) + \langle \nabla_3 \varphi, \phi \rangle \right) \Omega^2 dA_\gamma du' du = 0,
\]

(resp. \(\forall 0 \leq u_1 < u_2 \leq u_*\)).

**Remark 2.32.** It is easy to see by integration by parts, (2.10) and (2.11) that

\[
\text{classical sense} \implies \text{integrated sense} \implies \text{weak integrated sense}.
\]

2.9. Weak formulation of Einstein vacuum equations in the double null gauge. In this subsection, we give a weak formulation of the Einstein vacuum equations in the double null gauge, which is slightly stronger than that in Definition 2.1 and takes advantage of angular regularity. Our formulation relies on notions introduced in Sections 2.7 and 2.8.

**Definition 2.33.** Let \((\mathcal{M} = [0, u_*] \times [0, u_*] \times S^2, g)\) be an angularly regular spacetime in double null coordinates.

We say that \((\mathcal{M}, g)\) is an **angularly regular weak solution to the Einstein vacuum equations** if the following holds:

1. (2.13)–(2.16) are satisfied in the integrated sense (Definition 2.30).
2. (2.17)–(2.22) are satisfied in the weak integrated sense (Definition 2.31).

**Remark 2.34.** We remark that in order to make sense of Definition 2.33, we do not need to full regularity assumptions in Definition 2.26. We make the stronger assumptions in Definition 2.26 because that will be the relevant class of spacetimes in the later parts of the paper.

The following proposition clarifies the relation between the notions of solutions Definitions 2.1 and 2.26, as well as their relation to classical solutions. Part (1) is an immediate consequence of Proposition 2.29; part (2) is a direct computation. We omit the details.

**Proposition 2.35.**

1. Suppose \((\mathcal{M} = [0, u_*] \times [0, u_*] \times S^2, g)\) is a \(C^2\) (classical) solutions to the Einstein vacuum equations in double null coordinates (see Definition 2.7), then \((\mathcal{M}, g)\) angularly regular weak solution to the Einstein vacuum equations in the sense of Definition 2.33.
(2) Suppose \((M = [0, u_1] \times [0, u_2] \times \mathbb{S}^2, g)\) be an angularly regular weak solution to the Einstein vacuum equations in the sense of Definition 2.23, then \((M, g)\) is a weak solution to the Einstein vacuum equations in the sense of Definition 2.1.

2.10. Weak formulation of the Einstein–null dust system in the double null gauge. In analogy to Definition 2.33, we introduce in this subsection a weak formulation of the Einstein–null dust system in the double null gauge that uses angular regularity; see already Definition 2.37.

We begin with defining the class of measures (which will represent the null dusts) which we will consider.

Definition 2.36. Let \((M = [0, u_1] \times [0, u_2] \times \mathbb{S}^2, g)\) be an angularly regular spacetime in double null coordinates. Let \(\{\text{d}v_u\}_{u \in [0, u_1]}\) be a 1-parameter family of measures such that for every \(u \in [0, u_1]\), \(\text{d}v_u\) is a non-negative Radon measure on \([u, 0, 0, u_2, 0, \mathbb{S}^2]\). Similarly, let \(\{\text{d}v_u\}_{u \in [0, u_2]}\) be a 1-parameter family of measures such that for every \(u \in [0, u_2]\), \(\text{d}v_u\) is a non-negative Radon measure on \((0, u_2, 0, 0, u_1, 0, \mathbb{S}^2)\).

We say that \(\{\text{d}v_u\}_{u \in [0, u_1]}\) and \(\{\text{d}v_u\}_{u \in [0, u_2]}\) are \textbf{angularly regular} if the following holds:

1. \(\text{d}v_u\) is continuous in \(u\) and \(\text{d}v_u\) is continuous in \(u\) with respect to the weak-* topology, i.e.

\[
\lim_{u \to u'} \int_{[u, 0, 0, u_2, 0, \mathbb{S}^2]} \varphi \, dv_u = \int_{[u', 0, 0, u_2, 0, \mathbb{S}^2]} \varphi \, dv_u, \quad \forall \varphi \in C_c^\infty([0, u_1] \times (0, u_2)) \times \mathbb{S}^2),
\]

\[
\lim_{u \to u'} \int_{(0, u_1) \times [u, 0, u_2, 0, \mathbb{S}^2]} \varphi \, dv_u = \int_{(0, u_1) \times [u', 0, u_2, 0, \mathbb{S}^2]} \varphi \, dv_u, \quad \forall \varphi \in C_c^\infty((0, u_1) \times [0, u_2, 0, \mathbb{S}^2]).
\]

(2) Angular regularity holds in the following sense. For every \(u\), let

\[
\mathcal{X}_u := \{ \varphi \text{ real valued function : } \varphi \in C^0_\mathcal{L}^1(S_{0, u}), \|\varphi\|_{L^\infty_{\mathcal{L}} L^1(S_{0, u})} \leq 1 \},
\]

\[
\mathcal{Y}_u := \{ \mathcal{X}_u, S\text{-tangent vector field : } \mathcal{X}_u, \text{div} \mathcal{X} \in C^0_\mathcal{L}^1(S_{0, u}), \|\mathcal{X}\|_{L^\infty_{\mathcal{L}} L^1(S_{0, u})} \leq 1 \}
\]

\[
\mathcal{Z}_u := \{ (\mathcal{X}, \mathcal{Y}) : \mathcal{X}, \mathcal{Y} S\text{-tangent vector fields, } \mathcal{X}, \mathcal{Y}, \text{div} \mathcal{X} \otimes \mathcal{Y}, \text{div} \text{div} \mathcal{X} \otimes \mathcal{Y} \in C^0_\mathcal{L}^1(S_{0, u}), \|\mathcal{X} \otimes \mathcal{Y}\|_{L^\infty_{\mathcal{L}} L^4(S_{0, u})} \leq 1 \}
\]

Similarly, for every \(u\), define

\[
\mathcal{X}_u := \{ \varphi \text{ real valued function : } \varphi \in C^0_\mathcal{L}^1(S_{0, u}), \|\varphi\|_{L^\infty_{\mathcal{L}} L^1(S_{0, u})} \leq 1 \},
\]

\[
\mathcal{Y}_u := \{ \mathcal{X}_u, S\text{-tangent vector field : } \mathcal{X}_u, \text{div} \mathcal{X} \in C^0_\mathcal{L}^1(S_{0, u}), \|\mathcal{X}\|_{L^\infty_{\mathcal{L}} L^1(S_{0, u})} \leq 1 \}
\]

\[
\mathcal{Z}_u := \{ (\mathcal{X}, \mathcal{Y}) : \mathcal{X}, \mathcal{Y} S\text{-tangent vector fields, } \mathcal{X}, \mathcal{Y}, \text{div} \mathcal{X} \otimes \mathcal{Y}, \text{div} \text{div} \mathcal{X} \otimes \mathcal{Y} \in C^0_\mathcal{L}^1(S_{0, u}), \|\mathcal{X} \otimes \mathcal{Y}\|_{L^\infty_{\mathcal{L}} L^4(S_{0, u})} \leq 1 \}
\]

Then there exists \(C > 0\) such that

\[
\sup_{u \in [0, u_1]} \left( \sup_{\varphi \in \mathcal{X}_u} \left| \int_{[u, 0, 0, u_2, 0, \mathbb{S}^2]} \varphi \, dv_u \right| + \sup_{\mathcal{X}_u} \left| \int_{[u, 0, 0, u_2, 0, \mathbb{S}^2]} \text{div} \mathcal{X} \, dv_u \right| \right) \leq C,
\]

and

\[
\sup_{u \in [0, u_2]} \left( \sup_{\varphi \in \mathcal{X}_u} \left| \int_{[0, u_1] \times [u, 0, u_2, 0, \mathbb{S}^2]} \varphi \, dv_u \right| + \sup_{\mathcal{X}_u} \left| \int_{[0, u_1] \times [u, 0, u_2, 0, \mathbb{S}^2]} \text{div} \mathcal{X} \, dv_u \right| \right) \leq C,
\]

where we used the convention \(\text{div} \mathcal{X} = \nabla_A \nabla_B (X^A Y^B).

Definition 2.37. Let \((M = [0, u_1] \times [0, u_2] \times \mathbb{S}^2, g)\) be an angularly regular spacetime in double null coordinates, and let \(\{\text{d}v_u\}_{u \in [0, u_1]}, \{\text{d}v_u\}_{u \in [0, u_2]}\) be angularly regular non-negative Radon measures (see Definition 2.36).

We say that \((M, g, \{\text{d}v_u\}_{u \in [0, u_1]}, \{\text{d}v_u\}_{u \in [0, u_2]}\) is an angularly regular weak solution to the Einstein–null dust system if the following holds:
Lemma 2.38. Suppose \((M, g, \{dv_u\}_{u \in [0, u_*]}, \{dv_\omega\}_{\omega \in [0, \omega_*]})\) is an angularly regular weak solution to the Einstein–null dust system in the sense of Definition 2.37. Then, for \(dv := \Omega^2 dv_u du\) and \(dv := \Omega^2 dv_\omega d\omega\), \((M, g, dv, dv)\) is a weak solution to the Einstein–null dust system in the sense of Definition 2.2.

2.11. Renormalized curvature components and the renormalized Bianchi equations. Given an angularly regular spacetime \((M, g)\) (see Definition 2.26), define the following \(S\)-tangent tensor fields:

**Definition 2.39.** Define

\[
\beta := -\hat{\nabla} \hat{\chi} + \frac{1}{2} \hat{\nabla} \hat{\gamma} - \frac{1}{2} (\eta - \frac{2}{3}) \cdot \left( \hat{\chi} - \hat{\gamma} \right), \quad \hat{\beta} := \hat{\nabla} \hat{\chi} - \frac{1}{2} \left( \eta - \frac{2}{3} \right) \cdot \left( \hat{\chi} - \hat{\gamma} \right), \quad \hat{\sigma} := \text{curl} \hat{\eta}.
\]

We will call \((\beta, \hat{\beta}, \hat{\sigma})\) and \(K\) the renormalized curvature components. The relation of \((\beta, \hat{\beta}, \hat{\sigma})\) to the spacetime curvature components is given by the following:

**Lemma 2.40.** If \((M, g)\) is a \(C^2\) (classical) solution to the Einstein vacuum equations, then

\[
\beta_A = \frac{1}{2} R(e_A, e_4, e_3, e_4), \quad \hat{\sigma} = \frac{1}{4} \hat{R}(e_4, e_3, e_4, e_3) + \frac{1}{2} e^{AB} (\gamma^{-1})^{CD} \hat{\chi}_{AC} \hat{\chi}_{BD},
\]

where \(R\) is the Riemann curvature tensor of \(g\) and \(\hat{R}\) denotes the Hodge dual of \(R\).

For sufficiently regular spacetimes in double null coordinates, the renormalized curvature components obey the following renormalized Bianchi equations (recall definitions from Definitions 2.16 and 2.28):
Proposition 2.41. If \((M, g)\) is a \(C^3\) (classical) solution to the Einstein vacuum equations, then the following system of equations holds:

\[
\nabla_3 \beta + \psi \chi \beta = - \nabla K + \nabla \sigma + 2 \omega \beta + 2 \chi \cdot \beta - 3(\eta K - \eta \sigma) + \frac{1}{2}(\nabla (\chi \cdot \chi) + \nabla (\chi \wedge \chi)) + \frac{3}{4}(\eta \chi + \eta \chi \wedge \chi) + \frac{3}{4}(\nabla \psi \chi \psi K + \psi \chi \nabla \psi K) - \frac{3}{4} \eta \psi \chi \psi K,
\]

(2.31)

\[
\nabla_4 \sigma + \frac{3}{2} \psi \chi \sigma = - \nabla \nu \sigma - \frac{1}{2}(\eta - \eta \chi + \eta \chi \wedge \chi) - 2 \eta \wedge \beta - \frac{1}{2} \chi \wedge (\nabla \otimes \eta) - \frac{1}{2} \chi \wedge (\eta \otimes \eta),
\]

(2.32)

\[
\nabla_4 K + \psi \chi K = \nabla \nu \beta - \frac{1}{2}(\eta - \eta \chi) \cdot \beta - 2 \eta \cdot \beta + \frac{1}{2} \chi \cdot \eta \otimes \eta + \frac{1}{2} \chi \cdot (\eta \otimes \eta) - \frac{1}{2} \psi \chi \nabla \nu \eta - \frac{1}{2} \psi \chi |\eta|^2,
\]

(2.33)

\[
\nabla_4 \beta + \psi \chi \beta = \nabla K + \nabla \sigma + 2 \omega \beta + 2 \chi \cdot \beta + 3(\eta K + \eta \sigma) - \frac{1}{2}(\nabla (\chi \cdot \chi) + \nabla (\chi \wedge \chi)) + \frac{1}{2}(\nabla \psi \chi \psi K + \psi \chi \nabla \psi K) - \frac{3}{2} \eta \chi - \eta \chi \wedge \chi + \frac{3}{4} \eta \psi \chi \psi K,
\]

(2.36)

Proof. These equations can be derived starting the decomposing the Bianchi equations \(\nabla^\alpha R_{\alpha \beta \gamma \delta} = 0\) with respect to the null frame and then performing algebraic manipulations; they can be found for instance in [12].

Definition 2.42. Let \((M = [0, u_1] \times [0, u_2] \times S^2, g)\) be an angularly regular spacetime in double null coordinates. We say that the renormalized Bianchi equations are satisfied if

1. the equations (2.32)-(2.35) are satisfied in the integrated sense (Definition 2.30);
2. the equations (2.31) and (2.36) are satisfied in the weak integrated sense (Definition 2.31).

2.12. Auxiliary equations. In this subsection, we discuss a few auxiliary equations. We will show that when the spacetime is sufficiently regular, they hold as a consequence of the null structure equations (recall Section 2.7). We will then introduce an appropriate (weak) notion for these solutions in the setting of angularly regular spacetimes; see definition 2.46.

The equations that we will be interested in are those for the higher derivatives of the metric components, those for the mass aspect functions, and those for the derivatives of \(\psi \chi\) and \(\psi \chi\). These are not necessary to make sense of the Einstein equations weakly, but will be useful for the proof of our uniqueness theorem (Theorem 1.12).

2.12.1. Higher order transport equation for the metric components.

Proposition 2.43. The following holds for a \(C^2\) metric in double null coordinates (see (2.2)):

\[
\nabla_4 \nabla \chi \gamma = 0,
\]

(2.37)

\[
\nabla_4 \nabla \log \Omega = -(\eta + \eta \chi) + \chi \cdot \nabla \log \Omega.
\]

(2.38)

Under the same assumptions, the following equation for \(\nabla b\), which we write in index notations, also holds

\[
\nabla_4 \nabla A b^A = - (\eta + \eta \chi) A^A + \frac{1}{2} (\eta + \eta \chi) B_{AC} b^C - (\gamma - 1)^A B_{BD} \nabla_C b^A + \sum_{i=1}^r (\chi A^A \eta_{AC} - \chi B_{AC} A^A) + \epsilon_{AC} \psi \chi \psi b^A.
\]

(2.39)

Proof. By Lemma 7.3.3 in [11], the following commutation formula holds:

\[
[\nabla_4, \nabla B]_{\phi A_1, \ldots, A_r} = \frac{1}{2} (\eta + \eta \chi) \nabla_4 \phi_{A_1, \ldots, A_r} - (\gamma - 1)^A \chi B_{AC} \phi_{A_1, \ldots, A_r} + \sum_{i=1}^r (\gamma - 1)^A \chi A_{BD} \phi_{A_1, \ldots, A_r} + \epsilon_{AC} \psi \chi \psi b^A.
\]

(2.40)

where \(\ast\) denotes the Hodge dual on the 2-sphere, and \(\hat{A}\) in the indices means that the original \(A\) is removed.
The conclusion then follows from applying the commutation formula to
\[ \nabla_4 \gamma = 0, \quad \nabla_4 b = -2 \Omega (\eta - \eta) + \chi \cdot b, \quad \nabla_4 \log \Omega = -2 \omega, \] (2.41)
noting that the regularity assumptions are sufficient to justify the use of the commutation formula. □

2.12.2. Transport equations for the mass aspect functions. Define the mass aspect functions \( \mu \) and \( \mu \) by
\[ \mu := -d \nabla \eta + K, \quad \mu := -d \nabla \eta + K. \] (2.42)

We then have the following equations\(^{19}\) for \( \mu \) and \( \mu \).

Proposition 2.44. The following holds for a \( C^3 \) solution to the Einstein vacuum equations in double null coordinates (see (2.2)):}

\[
\nabla_4 \mu = d \nabla (\chi \cdot (\eta - \eta)) + \frac{1}{2} (\eta + \eta) \cdot (\chi \cdot (\eta - \eta) + \beta) + \chi \cdot \nabla \eta - \Psi \chi \eta \cdot \eta + \chi \cdot \eta \cdot \eta + \beta \cdot \eta - \Psi \chi K - \frac{1}{2} (\eta - \eta) \cdot \beta - 2 \eta \cdot \beta + \frac{1}{2} \chi \cdot \nabla \eta + \frac{1}{2} \chi \cdot (\eta \nabla \eta) - \frac{1}{2} \Psi \chi d \nabla \eta - \frac{1}{2} \Psi \chi |\eta|^2, \]
(2.43)

and

\[
\nabla_3 \mu = d \nabla (\chi \cdot (\eta - \eta)) + \frac{1}{2} (\eta + \eta) \cdot (\chi \cdot (\eta - \eta) - \beta) + \chi \cdot \nabla \eta - \Psi \chi \eta \cdot \eta + \chi \cdot \eta \cdot \eta - \beta \cdot \eta - \Psi \chi K - \frac{1}{2} (\eta - \eta) \cdot \beta + 2 \eta \cdot \beta + \frac{1}{2} \chi \cdot \nabla \eta + \frac{1}{2} \chi \cdot (\eta \nabla \eta) - \frac{1}{2} \Psi \chi d \nabla \eta - \frac{1}{2} \Psi \chi |\eta|^2. \]
(2.44)

Proof. The following commutation formulae hold for any \( C^2 \) \( S \)-tangent 1-form \( \xi \) on a \( C^2 \) metric in a
double null coordinate system (the equation (2.45) can be derived from (2.40); (2.46) can be achieved similarly starting from
Lemma 7.3.3 in [11]):

\[
[\nabla_4, d \nabla] \xi = \frac{1}{2} (\eta + \eta) \cdot \nabla_4 \xi - \chi \cdot \nabla \xi + \Psi \chi \eta \xi - \chi \cdot \eta \cdot \xi - \beta \cdot \xi, \]
(2.45)

\[
[\nabla_3, d \nabla] \xi = \frac{1}{2} (\eta + \eta) \cdot \nabla_3 \xi - \chi \cdot \nabla \xi + \Psi \chi \eta \xi - \chi \cdot \eta \cdot \xi + \beta \cdot \xi. \]
(2.46)

Now, by Definition 2.39, the equations (2.15) and (2.16) can be rephrased as
\[
\nabla_4 \eta = -\chi \cdot (\eta - \eta) - \beta, \quad \nabla_3 \eta = -\chi \cdot (\eta - \eta) + \beta, \]
(2.47)
i.e. the top order derivatives can be grouped in terms of \( \beta \) and \( \beta \).

Applying (2.45) and (2.46) to (2.47), we thus obtain
\[
\nabla_4 d \nabla \eta = -d \nabla \{ \chi \cdot (\eta - \eta) + \beta \} - \frac{1}{2} (\eta + \eta) \cdot \{ \chi \cdot (\eta - \eta) + \beta \} + \chi \cdot \nabla \eta + \Psi \chi \eta \cdot \eta - \chi \cdot \eta \cdot \eta - \beta \cdot \eta, \]
\[
\nabla_3 d \nabla \eta = -d \nabla \{ \chi \cdot (\eta - \eta) - \beta \} - \frac{1}{2} (\eta + \eta) \cdot \{ \chi \cdot (\eta - \eta) - \beta \} - \chi \cdot \nabla \eta + \Psi \chi \eta \cdot \eta - \chi \cdot \eta \cdot \eta + \beta \cdot \eta. \]

Recalling the definition of \( \mu \) and \( \mu \) in (2.42), we can combine the above equations with (2.33) and (2.35) to obtain the desired conclusion. □

2.12.3. Weak formulation of transport equations for derivatives of \( \Psi \chi \) and \( \Psi \chi \). As in Sections 2.9 and 2.10, the transport equations for \( \Psi \chi \) in the \( e_4 \) direction and for \( \Psi \chi \) in the \( e_3 \) direction are different depending on whether the spacetime satisfies the Einstein vacuum equations or the Einstein–null dust system. We first consider the vacuum case.

Proposition 2.45. Let \((\mathcal{M}, g)\) be a \( C^3 \) solution to the Einstein vacuum equations in double null coordinates. Assume \( \chi \) is a \( C^1 \) \( S \)-tangent vector field. Then the following holds:
\[
\frac{\partial}{\partial u} \chi (\Omega^{-1} \Psi \chi) + \frac{1}{2} \chi (\Psi \chi)^2 = -\chi |\chi|^2 + \left[ \frac{\partial}{\partial u}, \chi \right] (\Omega^{-1} \Psi \chi). \]
(2.48)

\[
\left( \frac{\partial}{\partial u} + \nabla_0 \right) \chi (\Omega^{-1} \Psi \chi) + \frac{1}{2} \chi (\Psi \chi)^2 = -\chi |\chi|^2 + \left[ \frac{\partial}{\partial u} + \nabla_0, \chi \right] (\Omega^{-1} \Psi \chi). \]
(2.49)

Proof. This follows from differentiating (2.13) and (2.14) by \( \chi \) and using (2.11). □

\(^{19}\) Remark that the key point for introducing \( \mu \) and \( \mu \) (instead of just considering \( d \nabla \eta \) and \( d \nabla \eta \)) is that in the transport equations they satisfy, no terms of first derivative of curvature appear on the RHS.
In the presence of dust, (2.48) and (2.49) do not hold. Instead, the dust term acts as a source in these transport equations (cf. Section 2.10). We introduce a weak formulation of these equations, which can be considered as the higher derivative version of (2.27) and (2.28).

Let $\mathcal{X}$ be a $C^1$ $S$-tangent vector field. Consider the following equation for $\mathcal{X}(\Omega^{-1} \hat{\psi} \chi)$ (for all $u \in [0, u_+)$ and all $0 \leq u_1 < u_2 < u_+$)

$$
\int_{S_{u_2}} \Omega^2(\mathcal{X}(\Omega^{-1} \hat{\psi} \chi))^\dagger \, dA_\gamma - \int_{S_{u_1}} \Omega^2(\mathcal{X}(\Omega^{-1} \hat{\psi} \chi))^\dagger \, dA_\gamma,
$$

$$
= \int_{u_1}^{u_2} \int_{S_{u_2}} \left[ \left( \frac{\partial}{\partial u} + \nabla_u \mathcal{X} \right) (\Omega^{-1} \hat{\psi} \chi) - 4\omega \Omega \mathcal{X}(\Omega^{-1} \hat{\psi} \chi) \right] \Omega^2 \, dA_\gamma \, du + \int_\Omega \left( 2 \mathcal{X}(\log \Omega) + \nabla \mathcal{X} \right) \nu_u,
$$

and the following equation for $\mathcal{X}(\Omega^{-1} \hat{\psi} \chi)$ (for all $u \in [0, u_+]$ and all $0 \leq u_1 < u_2 < u_+$)

$$
\int_{S_{u_2}} \Omega^2(\mathcal{X}(\Omega^{-1} \hat{\psi} \chi))^{-} \, dA_\gamma - \int_{S_{u_1}} \Omega^2(\mathcal{X}(\Omega^{-1} \hat{\psi} \chi))^{-} \, dA_\gamma,
$$

$$
= \int_{u_1}^{u_2} \int_{S_{u_2}} \left[ \left( \frac{\partial}{\partial u} + \nabla_u \mathcal{X} \right) (\Omega^{-1} \hat{\psi} \chi) - 4\omega \Omega \mathcal{X}(\Omega^{-1} \hat{\psi} \chi) \right] \Omega^2 \, dA_\gamma \, du + \int_\Omega \left( 2 \mathcal{X}(\log \Omega) + \nabla \mathcal{X} \right) \nu_u.
$$

Recall here that $\pm$ denotes the trace (see Lemma 2.24), which is well defined in an angularly regular double null spacetime (Definition 2.26) since $\mathcal{X}(\Omega^{-1} \hat{\psi} \chi)$ is in $BV(H_u)$ for all $u$ and $\mathcal{X}(\Omega^{-1} \hat{\psi} \chi)$ is in $BV(H_{\Omega u})$ for all $u$.

2.12.4. Weak formulation of all the auxiliary equations. We are now ready to define an appropriate weak formulation of the equations that we have considered in this section.

**Definition 2.46.** Let $(\mathcal{M} = [0, u_+] \times [0, u_+] \times S^2, y)$ be an angularly regular spacetime in double null coordinates. We say that the equations (2.37), (2.38), (2.39), (2.43), (2.44), (2.50) and (2.51) are satisfied if

1. the equations (2.37)–(2.39) and (2.43)–(2.44) are satisfied in the integrated sense (Definition 2.30);
2. the equation (2.50) is satisfied for all $C^1 S$-tangent vector field $\mathcal{X}$, for all $u \in [0, u_+]$ and all $0 \leq u_1 < u_2 < u_+$;
3. the equation (2.51) is satisfied for all $C^1 S$-tangent vector field $\mathcal{X}$, for all $u \in [0, u_+]$ and all $0 \leq u_1 < u_2 < u_+$.

3. Existence theorems

In this section, we recall the existence and uniqueness theorem in [42]. We will in particular need to use the estimates derived in [42] to prove our main theorems.

To simplify the exposition, we restrict our attention to smooth initial data. This will already be sufficient for our purposes. Even though the data are smooth, the key point here is that the result in [42] guarantees that the region of existence and the estimates that the solutions obey depend only on low-regularity norms of the data.

In Section 3.1, we first give some remarks regarding the characteristic initial value problem in the double null foliation gauge. We then give the statement of the main result of [42] in Section 3.2.

3.1. The characteristic initial value problem. We will consider a characteristic initial value problem as follows. We impose characteristic initial data on two transversally intersecting null hypersurfaces $H_0 = [0, I] \times S^2$ and $H_0 = [0, I] \times S^2$, where $\{0\} \times S^2 \subset H_0$ and $\{0\} \times S^2 \subset H_0$ are identified. See Figure 3.1.

We will provide two ways of thinking about the characteristic initial data, which we call the full characteristic initial data and the reduced characteristic initial data; see Sections 3.1.1 and 3.1.2 respectively.  

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20 The original theorems in [41, 42] indeed allow the initial data to be non-smooth. In particular, they allow the initial $\hat{\chi}$ and $\hat{\chi}$ to be discontinuous, which corresponds to impulsive gravitational waves.
3.1.1. The full characteristic initial data. The first way to prescribe characteristic initial data is to prescribe all of the metric components \((\gamma, \Omega, b)\) and all of the Ricci coefficients \((\chi, \chi, \eta, \eta, \omega, \omega)\) on both \(H_0\) and \(\overline{H}_0\). All of these objects are required to be smooth, \(\gamma\) is required to be positive definite, and \(\Omega\) is required to be positive. Moreover, the metric components and the Ricci coefficients are required to satisfy the following:

- The metric components and the Ricci coefficients relate to each other via (2.10) and (2.11).
- The null structure equations (2.13)–(2.22) all hold, where it is understood that the \(\nabla_3\) equations are required to hold on \(H_0\) and the \(\nabla_4\) equations are required to hold on \(H_0\).

3.1.2. The reduced characteristic initial data. As is discussed in [10], there is another way to think about the characteristic initial data. Essentially, this allows one to identify some freely prescribable data and then impose the other constraints by solving appropriate transport equations. From this point of view, the initial data consist of \((\Omega, \Phi, \hat{\gamma}, b \mid_{\mathcal{L}_0}, \hat{\omega} \mid_{\mathcal{L}_0})\), where \(\Phi\) and \(\hat{\gamma}\) are such that \(\gamma = \Phi^2 \hat{\gamma}\), and \(\hat{\gamma}\) is normalized by the condition \(\frac{\det \hat{\gamma}}{\det \gamma} = 1\) for some \(\hat{\gamma}\) satisfying \(\mathcal{L} \hat{\gamma} \mid_{H_0} = 0 = \mathcal{L} \hat{\omega} \mid_{\overline{H}_0}\).

Here, \(\Phi\) and \(\hat{\gamma}\) are required to satisfy the constraint equation:

\[
\frac{\partial^2 \Phi}{\partial u^2} = 2 \frac{\partial \log \Omega}{\partial u} \frac{\partial \Phi}{\partial u} - \frac{1}{s} \frac{\partial}{\partial u} \hat{\gamma} \frac{\partial \Phi}{\partial u} \quad \text{on } H_0, \\
\frac{\partial^2 \hat{\gamma}}{\partial u^2} = 2 \frac{\partial \log \Omega}{\partial u} \frac{\partial \hat{\gamma}}{\partial u} - \frac{1}{s} \frac{\partial}{\partial u} \hat{\gamma} \frac{\partial \hat{\gamma}}{\partial u} \quad \text{on } H_0,
\]

where

\[
\frac{\partial^2 \hat{\gamma}}{\partial u^2} := (\gamma^{-1})^{AC} (\hat{\gamma}^{-1})^{BD} \left( \frac{\partial}{\partial u} \hat{\gamma}_{AB} \right) \left( \frac{\partial}{\partial u} \hat{\gamma}_{CD} \right), \quad \frac{\partial^2 \hat{\gamma}}{\partial u^2} := (\gamma^{-1})^{AC} (\hat{\gamma}^{-1})^{BD} \left( \frac{\partial}{\partial u} \hat{\gamma}_{AB} \right) \left( \frac{\partial}{\partial u} \hat{\gamma}_{CD} \right).
\]

To simplify the exposition, we assume from now on \(b \mid_{\mathcal{L}_0} \equiv 0\).

Once \(\Omega, \Phi, \hat{\gamma}\) and \(\hat{\omega} \mid_{\mathcal{L}_0}\) are prescribed, we can derive the full characteristic initial data set as in Section 3.1.1 by the following procedure (see [10]):

- Given \(\Omega\), we can obtain \(\omega \mid_{H_0}\) and \(\omega \mid_{\overline{H}_0}\) by (see (2.11))

\[
\omega \mid_{H_0} = -2 \Omega^{-1} \frac{\partial}{\partial u} \log \Omega, \quad \omega \mid_{\overline{H}_0} = -2 \Omega^{-1} \frac{\partial}{\partial u} \log \Omega.
\]

- Given \(\Omega, \hat{\gamma}\) and \(\Phi\), we can obtain \(\chi \mid_{H_0}\) and \(\chi \mid_{\overline{H}_0}\) by

\[
\chi_{AB} \mid_{H_0} = \frac{1}{2} \Omega^{-1} \Phi^2 \frac{\partial}{\partial u} \chi_{AB}, \quad \chi \mid_{H_0} = \frac{1}{2} \Omega^{-1} \Phi^2 \frac{\partial}{\partial u} \chi_{AB},
\]

\[
\chi_{AB} \mid_{\overline{H}_0} = \frac{1}{2} \Omega^{-1} \Phi^2 \frac{\partial}{\partial u} \chi_{AB}, \quad \chi \mid_{\overline{H}_0} = \frac{1}{2} \Omega^{-1} \Phi^2 \frac{\partial}{\partial u} \chi_{AB}.
\]

- \(\eta\) and \(\eta\) can be obtained on \(H_0\) by solving (2.15) together with the condition \(\frac{1}{2} (\eta + \eta) = \mathcal{V} \log \Omega\) (see (2.11)); \(\eta\) and \(\eta\) can be obtained on \(\overline{H}_0\) by solving (2.16) together with the condition \(\frac{1}{2} (\eta + \eta) = \mathcal{V} \log \Omega\). The initial condition for both of these ODEs are given in view of \(\mathcal{L} \frac{\partial}{\partial u} b = -2 \Omega^2 (\eta - \eta)\) (see (2.9)) and the fact that \(\mathcal{L} \frac{\partial}{\partial u} b \mid_{S_{0,0}}\) is prescribed.

- \(b\) on \(H_0\) can be obtained by \(\mathcal{L} \frac{\partial}{\partial u} b = -2 \Omega^2 (\eta - \eta)\) and the condition that \(b \mid_{S_{0,0}} = 0\) (which follows from \(b \mid_{\overline{H}_0} = 0\)).
\( \omega \) on \( H_0 \) can be obtained solving the transport equations (2.22); \( \omega \) on \( H_0 \) can be obtained solving the transport equation (2.21). Note that the initial data for \( \omega \) for this transport equation is determined by the value of \( \omega \) on \( H_0 \) (obtained above); similarly for \( \omega \).

- \( \chi \) on \( H_0 \) can be obtained solving the transport equations (2.18) and (2.20); \( \chi \) on \( H_0 \) can be obtained solving the transport equations (2.17) and (2.19). Note that the initial data for \( \chi \) for these transport equations is determined by the value of \( \chi \) on \( H_0 \); similarly for \( \chi \).

The important point for the above procedure is that all the terms on the RHS of the transport equations are given in the previous steps. The transport equations can therefore all be solved (despite the fact that the procedure involves a loss of derivatives in the sense that to obtain estimates for (say) three derivatives for all the initial Ricci coefficients require prescribing more derivatives on \( (\Omega, \Phi, \gamma, L_{\frac{a}{n}} b \mid S_0) \)).

Tracing through the above procedure, we also obtain quantitative estimates for \((b, \chi, \hat{\chi}, \eta, \hat{\eta}, \omega, \hat{\omega})\). We give the result in the following lemma. The proof is straightforward and omitted (see again [10] for details).

**Lemma 3.1.** Suppose \((\Omega, \Phi, \gamma, L_{\frac{a}{n}} b \mid S_0)\) is a reduced characteristic initial data set with the estimates

\[
\| (\log \Omega, \hat{\gamma}, \log \Phi, \frac{\partial \Phi}{\partial u}) \|_{H_0} \lesssim \| c_{0} W^{\infty,2} (s_{0,\gamma}, b) \| + \| (\log \Omega, \hat{\gamma}, \log \Phi, \frac{\partial \Phi}{\partial u}) \|_{H_0} \| c_{0} W^{\infty,2} (s_{0,\gamma}, \chi) \| \leq C_0,
\]

\[
\| (\hat{\gamma}, \hat{\chi}, \hat{b}, \hat{\omega}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} + \| (\hat{\gamma}, \hat{\chi}, \hat{b}, \hat{\omega}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} \leq C_0.
\]

Then, for the metric components and Ricci coefficients derived with the procedure outlined above, there exists \( C_0 > 0 \) depending on \( C_0 \) such that

\[
\| (\gamma, \log \frac{\partial \gamma}{\partial \gamma}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} + \| (\gamma, \log \frac{\partial \gamma}{\partial \gamma}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} \leq \tilde{C}_0,
\]

\[
\| (\hat{\gamma}, \hat{\chi}, \hat{\omega}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} + \| (\hat{\gamma}, \hat{\chi}, \hat{\omega}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} \leq \tilde{C}_0.
\]

### 3.2. The statements of the existence theorems.

We now state the main result in [42]. Recall again that we only focus on the case of the result in [42] where the initial data are smooth. We will first give the existence statement (Theorem 3.2) and then give the estimates (Theorem 3.3).

As above, we consider a characteristic initial value problem for the Einstein vacuum equations with smooth initial data given on \( H_0 = [0, 1] \times S^2 \) and \( H_0 = [0, 1] \times S^2 \) where \( \{0\} \times S^2 \subset H_0 \) and \( \{0\} \times S^2 \subset H_0 \) are identified. Suppose we are given a full characteristic initial data set as in Section 3.1.1 with \( b \mid H_0 = 0 \).

In addition, fix a smooth metric \( \hat{\gamma} \) on \( S^2 \). Extend \( \hat{\gamma} \) to \( H_0 \) by requiring \( L_{\frac{a}{n}} b \mid H_0 = 0 = L_{\frac{a}{n}} b \mid H_0 \).

The following is the main existence theorem \(^{21}\) from [42]:

**Theorem 3.2 (L.-R., Theorem 3 in [42]).** Consider the characteristic initial value problem for the Einstein vacuum equations with smooth full characteristic initial data as described above. Suppose there exists a constant \( D_0 > 0 \) such that the prescribed data satisfy

\[
\| (\gamma, \log \frac{\partial \gamma}{\partial \gamma}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} + \| (\gamma, \log \frac{\partial \gamma}{\partial \gamma}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} \leq D_0,
\]

\[
\| (\hat{\gamma}, \hat{\chi}, \hat{\omega}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} + \| (\hat{\gamma}, \hat{\chi}, \hat{\omega}) \|_{L_{a} W^{\infty,2} (s_{0,\gamma}, \chi)} \leq D_0.
\]

Then there exists \( \varepsilon > 0 \) (sufficiently small) depending only on \( D_0 \) and \( \varepsilon \) such that the following holds:

Given \( u_* \in (0, 1] \) and \( u_* \in (0, \varepsilon] \), there exists a unique smooth solution to the Einstein vacuum equations in double null coordinates in the domain \( [0, u_*] \times [0, u_*] \times S^2 \) which achieves the given initial data.

\(^{21}\)Note that though equivalent, Theorem 3.2 is slightly differently worded as [42, Theorem 3]. In particular, instead of (3.3), the estimates in [42] are stated in terms of local coordinates instead of with respect to a reference metric \( \hat{\gamma} \).
We now turn to the estimates for the Ricci coefficients which are obtained in the proof of Theorem 3.2. We record them in Theorem 3.3 below. Most of the following estimates can be directly read off from [42]. For the convenience of the reader, we include in Appendix A a derivation of these estimates from [42]. (For the statement of Theorem 3.3, recall again the definition of \( \gamma_0, A \) in Definition 2.25.)

**Theorem 3.3.** Consider a characteristic initial value problem as in Theorem 3.2, as well as a spacetime solution given in the conclusion Theorem 3.2.

After choosing \( \epsilon > 0 \) sufficiently small if necessary, there exists \( \tilde{C} > 0 \) depending only on \( D_0 \) and \( I \) (in Theorem 3.2) such that the following estimates hold in \( [0, u_*] \times [0, u_*] \times S^2 \):

\[
I(S_{u, \gamma}, \gamma) \leq \tilde{C}, \quad \tilde{C}^{-1} \leq \text{Area}(S_{u, \gamma}, \gamma) \leq \tilde{C},
\]

\[
\sum_{\psi \in \{v, \xi\}} \left( \| \nabla^3 \psi \|_{C^0 L^2} + \| \nabla^4 \psi \|_{C^0 L^2} \right) \leq \tilde{C},
\]

\[
\sum_{\psi \in \{v, \xi\}} \left( \| \nabla^3 \psi \|_{C^0 L^2} + \| \nabla^4 \psi \|_{C^0 L^2} \right) \leq \tilde{C},
\]

\[
\sum_{\psi \in \{v, \xi\}} \left( \| \nabla^3 \psi \|_{C^0 L^2} + \| \nabla^4 \psi \|_{C^0 L^2} \right) \leq \tilde{C},
\]

\[
\sum_{\psi \in \{v, \xi\}} \left( \| \nabla^3 \psi \|_{C^0 L^2} + \| \nabla^4 \psi \|_{C^0 L^2} \right) \leq \tilde{C},
\]

\[
\sum_{\psi \in \{v, \xi\}} \left( \| \nabla^3 \psi \|_{C^0 L^2} + \| \nabla^4 \psi \|_{C^0 L^2} \right) \leq \tilde{C},
\]

where we have used the shorthand above that \( W^{k,2}(S) = W^{k,2}(S_{u, \gamma}, \gamma) \).

**Remark 3.4.** Notice that the conditions in Lemma 3.1 guarantee that the assumptions (3.3)–(3.6) of Theorem 3.2 hold.

### 4. Main results

#### 4.1. Existence and characterization of the limiting spacetime

**Theorem 4.1.** Consider a sequence of smooth characteristic initial data for the vacuum Einstein equations

\[
\text{Ric} = 0.
\]

Assume that all the geometric quantities obey the bounds in Theorem 3.2 uniformly. Denoting \( (\gamma_0, A)_{n} = \gamma_n \big|_{S_0, 0} \), assume that there is a \( C^3 \) limit \( (\gamma_0, A)_\infty \), i.e.

\[
(\gamma_0, A)_n \to (\gamma_0, A)_\infty \text{ in } C^3.
\]

Then, there exists \( \epsilon > 0 \) sufficiently small (independent of \( n \)) such that the following hold for \( \mathcal{M} := [0, u_*] \times [0, u_*] \times S^2 \) with \( u_* \in (0, \epsilon) \):

1. There exists a sequence of spacetimes \( (\mathcal{M}, g_n) \) in double null coordinates

\[
g_n = -2\Omega_n^2 (du \otimes du + du \otimes du) + (\gamma_n)_{AB} (d\theta^A - b_n^A du) \otimes (d\theta^B - b_n^B du)
\]

    corresponding to the sequence of initial data, which solve the Einstein vacuum equations.
(2) A subsequence of spacetime metrics \((M, g_{n_k})\) converges in \(C^0\) in the null coordinate system to a limiting spacetime \((M, g_\infty)\)
\[
g_\infty = -2\Omega_\infty^2 (du \otimes du + du \otimes du) + (\gamma_\infty)_{AB}(d\theta^A - b^A_\infty du) \otimes (d\theta^B - b^B_\infty du).
\]
In addition, the weak derivatives of the components of \(g_{n_k}\) converge weakly in \((\text{spacetime})\) \(L^2\) to the weak derivatives of the components of \(g_\infty\); and the Ricci coefficients corresponding to \(g_{n_k}\) converge weak in \((\text{spacetime})\) \(L^2\) to the Ricci coefficients corresponding to \(g_\infty\).
(3) After passing to a further subsequence (not relabeled), the following weak-* limits exist:
\[
d_{u} := \text{weak-*} \lim_{k \to +\infty} \left( \Omega_{n_k}^2 \right)^{\frac{1}{2}} d_{\gamma_{n_k}} du - \Omega_\infty^2 \left( \gamma_\infty \right)^{\frac{1}{2}} \Omega_\infty d\gamma_\infty du,
\]
\[
d_{\nu} := \text{weak-*} \lim_{k \to +\infty} \left( \Omega_{n_k}^2 \right)^{\frac{1}{2}} d_{\gamma_{n_k}} \Omega_{n_k}^2 d\gamma_{n_k} du - \Omega_\infty^2 \left( \gamma_\infty \right)^{\frac{1}{2}} \Omega_\infty d\gamma_\infty du.
\]
Moreover, \((M, g_\infty, \{d_{u}\}_{u \in [0, u_\ast]}, \{d_{\nu}\}_{\nu \in [0, \nu_\ast]})\) is an angularly regular weak solution to the Einstein-null dust system in the sense of Definition 2.37.
(4) For \((M, g_\infty, \{d_{u}\}_{u \in [0, u_\ast]}, \{d_{\nu}\}_{\nu \in [0, \nu_\ast]}),\) the renormalized Bianchi equations are satisfied in the sense of Definition 2.42.
(5) For \((M, g_\infty, \{d_{u}\}_{u \in [0, u_\ast]}, \{d_{\nu}\}_{\nu \in [0, \nu_\ast]}),\) the equations (2.37), (2.38), (2.39), (2.43), (2.44), (2.50) and (2.51) are satisfied in the sense of Definition 2.46.

Theorem 4.1 will be proven in Sections 6 and 7. See the conclusion of the proof in Section 7.7. Some remarks are in order.

Remark 4.2. To simplify the statements, we only asserted that the limit is achieved in the \(C^0\) and the \(W^{1,2}\)-weak sense. Nevertheless, in fact, various different Ricci coefficients have better convergence properties; see more precise convergence statements in Section 6.

Remark 4.3. As we explained in the introduction, Theorem 4.1 implies a fortiori that the \((M_\infty, g_\infty)\) is vacuum if and only if both \(d_0 = 0\) and \(d_\nu = 0\) (Corollary 1.11).

4.2. Uniqueness of the limiting spacetime.

Theorem 4.4. Let \(M = [0, u_\ast] \times [0, u_\ast] \times S^2\). Suppose \((M, g^{(1)}, \{d_{u}^{(1)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(1)}\}_{\nu \in [0, \nu_\ast]}),\) and \((M, g^{(2)}, \{d_{u}^{(2)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(2)}\}_{\nu \in [0, \nu_\ast]}),\) are such that the following holds:
(1) \((M, g^{(1)}, \{d_{u}^{(1)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(1)}\}_{\nu \in [0, \nu_\ast]}),\) and \((M, g^{(2)}, \{d_{u}^{(2)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(2)}\}_{\nu \in [0, \nu_\ast]}),\) both are angularly regular weak solutions to the Einstein-null dust system in the sense of Definition 2.37.
(2) \((M, g^{(1)}, \{d_{u}^{(1)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(1)}\}_{\nu \in [0, \nu_\ast]}),\) and \((M, g^{(2)}, \{d_{u}^{(2)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(2)}\}_{\nu \in [0, \nu_\ast]}),\) both satisfy the renormalized Bianchi equations in the sense of Definition 2.42.
(3) \((M, g^{(1)}, \{d_{u}^{(1)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(1)}\}_{\nu \in [0, \nu_\ast]}),\) and \((M, g^{(2)}, \{d_{u}^{(2)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(2)}\}_{\nu \in [0, \nu_\ast]}),\) both satisfy the equations (2.37), (2.38), (2.39), (2.43), (2.44), (2.50) and (2.51) in the sense of Definition 2.46.
(4) \((M, g^{(1)}, \{d_{u}^{(1)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(1)}\}_{\nu \in [0, \nu_\ast]}),\) and \((M, g^{(2)}, \{d_{u}^{(2)}\}_{u \in [0, u_\ast]}, \{d_{\nu}^{(2)}\}_{\nu \in [0, \nu_\ast]}),\) have the same initial data in the sense that\(^{22}\)
\[
(\gamma^{(1)} - \gamma^{(2)}, \tilde{b}^{(1)} - \tilde{b}^{(2)}, \log \frac{\Omega^{(1)}}{\Omega^{(2)}}) \big|_{S_{n,a}} = 0, \forall u \in [0, u_\ast],
\]
\[
(\gamma^{(1)} - \gamma^{(2)}, \tilde{b}^{(1)} - \tilde{b}^{(2)}, \log \frac{\Omega^{(1)}}{\Omega^{(2)}}) \big|_{S_{n,a}} = 0, \forall u \in [0, u_\ast],
\]
\[
((\tilde{\chi}^{(1)} - \chi^{(1)}), \tilde{n}^{(1)} - n^{(1)}, \tilde{n}^{(2)} - n^{(2)}) \big|_{S_{n,a}} = 0, \forall u \in [0, u_\ast],
\]
\[
d_{u}^{(1)} - d_{u}^{(2)} = 0, \quad d_{u}^{(1)} - d_{u}^{(2)} = 0.
\]

Then the following holds:
(1) \(\gamma^{(1)} = \gamma^{(2)}, \tilde{b}^{(1)} = \tilde{b}^{(2)}\) and \(\Omega^{(1)} = \Omega^{(2)}\) everywhere on \(M\).
(2) \(d_{u}^{(1)} = d_{u}^{(2)}\) for all \(u \in [0, u_\ast]\).
(3) \(d_{\nu}^{(1)} = d_{\nu}^{(2)}\) for all \(u \in [0, u_\ast]\).

The proof of Theorem 4.4 will be carried out in Section 8.

\(^{22}\)Note that while we only explicitly assumed that the differences of very specific Ricci coefficients vanish on the initial hypersurfaces, in fact it holds that the all Ricci coefficients coincide on the initial hypersurfaces. This is because the remaining Ricci coefficients can be written as tangential (along the initial hypersurfaces) derivatives of the metric components.
4.3. Characteristic initial value problem for the Einstein–null dust system with angularly regular measure-valued null dust. We now turn to our main results on the characteristic initial value problem for the Einstein–null dust system with angularly regular measure-valued null dust. We first introduce the class of initial data that we consider. For simplicity, we will only consider characteristic initial data for which $b^A \equiv 0$ on $\mathcal{H}_0$.

**Definition 4.5** (Strongly angularly regular reduced characteristic initial data). Impose characteristic initial data on $H_0 = [0, I] \times S^2$ and $\mathcal{H}_0 = [0, I] \times S^2$, where $\{0\} \times S^2 \subset H_0$ and $\{0\} \times S^2 \subset \mathcal{H}_0$ are identified (see Figure 3.3).

Let $\gamma$ be an (arbitrary) auxiliary smooth metric on $S^2$. Define $\dot{\gamma}$ on $H_0 \cup \mathcal{H}_0$ by imposing

$$\mathcal{L}_{\frac{\partial}{\partial u}} \dot{\gamma} = 0 = \mathcal{L}_{\frac{\partial}{\partial \nu}} \dot{\gamma}.$$ 

A **strongly angularly regular reduced characteristic initial data set** to the Einstein–null dust system consists of a hextuple $(\Omega, \Phi, \dot{\gamma}, \mathcal{L}_{\frac{\partial}{\partial u}} b |_{S_{0, \omega}}, \nu_{\text{init}}, d \nu_{\text{init}})$ with the following properties:

1. $\Omega > 0$ is a smooth function on $H_0 \cup \mathcal{H}_0$.
2. $\Phi > 0$ is a Lipschitz function on $H_0 \cup \mathcal{H}_0$ such that on $\frac{\partial \phi}{\partial u} |_{H_0} \in \text{BV}(H_0, \gamma)$ and $\frac{\partial \phi}{\partial u} |_{\mathcal{H}_0} \in \text{BV}(\mathcal{H}_0, \gamma)$ (see Definition 2.22). Moreover,

$$\|\Phi\|_{L^\infty \gamma(S_0, \omega, \gamma)} + \|\Phi^{-1}\|_{L^\infty \gamma(S_0, \omega, \gamma)} + \|\frac{\partial \Phi}{\partial u}\|_{L^\infty \gamma(S_0, \omega, \gamma)} < +\infty,$$

$$\|\Phi\|_{L^\infty \gamma(S_0, 0, \gamma)} + \|\Phi^{-1}\|_{L^\infty \gamma(S_0, 0, \gamma)} + \|\frac{\partial \Phi}{\partial u}\|_{L^\infty \gamma(S_0, 0, \gamma)} < +\infty.$$

3. $\dot{\gamma}$ is a continuous covariant $S$-tangent 2-tensor which restricts to a positive definite metric on each $S_{0, \omega}$ or $S_{0, 0}$. Moreover $\dot{\gamma}$ satisfies the following properties:

(a) \[ \det \dot{\gamma} = 1. \]

(b) \[ \|\dot{\gamma}\|_{L^\infty \gamma(S_0, \omega, \gamma)} + \|\frac{\partial \dot{\gamma}}{\partial u}\|_{L^\infty \gamma(S_0, \omega, \gamma)} < +\infty, \]

\[ \|\dot{\gamma}\|_{L^\infty \gamma(S_0, 0, \gamma)} + \|\frac{\partial \dot{\gamma}}{\partial u}\|_{L^\infty \gamma(S_0, 0, \gamma)} < +\infty. \]

4. $\mathcal{L}_{\frac{\partial}{\partial u}} b |_{S_{0, \omega}}$ is a $W^{5,2}(S_{0, \omega}, \dot{\gamma})$ vector field.

5. $d \nu_{\text{init}}$ is a non-negative Radon measure on $(0, I) \times S^2$ satisfying the following regularity estimate:

$$\sup \left\{ \sum_{0 \leq k \leq 6} \int_{(0, I) \times S^2} (\mathcal{L}^k \Phi(k)(u, \vartheta) \, d \nu_{\text{init}}) : \Phi(k) \in C^\infty, \|\Phi(k)\|_{L^\infty \gamma(S_{0, \omega}, \gamma)} \leq 1 \right\} < +\infty, \quad (4.4)$$

where $\Phi(k)$ is a rank-$k$ tensor field.

6. $d \nu_{\text{init}}$ is a non-negative Radon measure on $(0, I) \times S^2$ satisfying the following regularity estimate:

$$\sup \left\{ \sum_{0 \leq k \leq 6} \int_{(0, I) \times S^2} (\mathcal{L}^k \Phi(k)(u, \vartheta) \, d \nu_{\text{init}}) : \Phi(k) \in C^\infty, \|\Phi(k)\|_{L^\infty \gamma(S_{0, 0}, \gamma)} \leq 1 \right\} < +\infty, \quad (4.5)$$

where $\Phi(k)$ is a rank-$k$ tensor field.

(6) $(\Omega, \Phi, \dot{\gamma}, d \nu_{\text{init}})$ together satisfy the constraint equations

$$-\frac{1}{2} \int_{S^2} (\frac{\partial}{\partial u} \Omega^{-2} \frac{\partial \Phi}{\partial u})(u, \vartheta) \, d \Lambda_{\gamma} \, d u$$

$$= -\frac{1}{8} \int_{S^2} (\Omega^{-2} \frac{\partial \Phi}{\partial u}^{2} \Phi(u, \vartheta) \, d \Lambda_{\gamma} \, d u - \frac{1}{2} \Omega^{-1} \Phi(u, \vartheta) \, d \nu_{\text{init}} \quad (4.6)$$

**23** Smoothness of $\Omega$ can also be dropped and replaced by

$$\|\log \Omega\|_{L^\infty \gamma(S_{0, \omega}, \gamma)} + \|\frac{\partial \log \Omega}{\partial u}\|_{L^\infty \gamma(S_{0, \omega}, \gamma)} < +\infty,$$

$$\|\log \Omega\|_{L^\infty \gamma(S_{0, 0}, \gamma)} + \|\frac{\partial \log \Omega}{\partial u}\|_{L^\infty \gamma(S_{0, 0}, \gamma)} < +\infty.$$
for any \( \varphi \in C_0^\infty((0,1] \times S^2) \).

Similarly, on \( H_0 \), \((\Omega, \Phi, \tilde{\gamma}, d\nu_{\text{init}}) \) together satisfy the constraint equations
\[
- \int_0^t \int_{S^2} \left( \frac{\partial \Phi}{\partial u} \Omega^{-2} \frac{\partial \Phi}{\partial u} \right) (u', \theta) \, dA_\gamma \, du' \\
= - \frac{1}{2} \int_0^t \int_{S^2} \Phi^{-1} \varphi(u', \theta) \, d\nu_{\text{init}}
\]
for any \( \varphi \in C_0^\infty((0,1) \times S^2) \).

We emphasize that the initial data in Definition 4.5 are consistent with the initial null dust being merely a measure, and initial metric being merely \( C^0 \cap W^{1,2} \).

The following is our local existence and uniqueness theorem for the Einstein–null dust system with measure-valued null shells (cf. Theorem 1.13):

**Theorem 4.6.** Given a strongly angularly regular reduced characteristic initial data set \((\Omega, \Phi, \tilde{\gamma}, \mathcal{E} \triangleright b \mid_{\mathcal{S}_0}, d\nu_{\text{init}}, d\nu_{\text{init}}^*) \) as in Definition 4.5, there exists \( \epsilon > 0 \) sufficiently small such that whenever \( u_* \in [0,1] \) and \( u_* \in [0,\epsilon] \), there exists a unique angularly regular weak solution to the Einstein–null dust system in the sense of Definition 2.37 in the domain \([0,u_*] \times [0,\infty] \times S^2 \) which achieves the prescribed initial data.

The proof of Theorem 4.6 will be completed in Section 9.

4.4. Approximating solutions to the Einstein–null dust system by solutions to the Einstein vacuum equations. Finally, we prove that (up to some technical assumptions) any angularly regular weak solution to the Einstein–null dust system can be weakly approximately by smooth solutions to the Einstein vacuum equations (cf. Corollary 1.14):

**Theorem 4.7.** Let \( \mathcal{M} = [0, u_1] \times [0, u_*] \times S^2 \). Suppose \((\mathcal{M}, g_\nu, \{d\nu_\alpha\}_{\alpha \in [0, u_*]}, \{d\nu_\alpha^*\}_{\alpha \in [0, u_*]}) \) is an angularly regular weak solution (see Definition 2.37) to the Einstein–null dust system with strongly angularly regular characteristic initial data (see Definition 4.5) and \( b \mid_{\nu_0} = 0 \). Assume moreover that the strongly angularly regular characteristic initial data set satisfies
\[
\text{supp}(d\nu_{\text{init}}) \subset [0,1] \times U^c, \quad \text{supp}(d\nu_{\text{init}}^*) \subset [0,1] \times U^c
\]
for some non-empty open set \( U \subset S^2 \).

Let \( \tilde{\mathcal{M}} = [0, u_1] \times [0, u_*] \times S^2 \subset \mathcal{M} \). Then, for \( u_* \in [0, u_*] \) sufficiently small, there exists a sequence of smooth solutions to the Einstein vacuum equations \((\tilde{\mathcal{M}}, g_{\nu_*}) \) in double null coordinates such that \((\tilde{\mathcal{M}}, g_{\nu_*}) \) converges to \((\mathcal{M}, g \mid_{\tilde{\mathcal{M}}}) \) in the sense described in Theorem 4.1.

The proof of Theorem 4.7 will be completed in Section 9.

We explain some of the assumptions in the following remarks.

**Remark 4.8.** Theorem 4.7 requires the initial data to satisfy \( b \mid_{\nu_0} = 0 \). This is however not a serious restriction as given any \((\mathcal{M}, g_\nu, \{d\nu_\alpha\}_{\alpha \in [0, u_*]}, \{d\nu_\alpha^*\}_{\alpha \in [0, u_*]}) \), one can always change coordinates on the 2-spheres to achieve the vanishing of \( b \) on \( H_0 \).

**Remark 4.9.** Theorem 4.7 requires the vanishing of the (initial) null dust in some angular directions. This restriction is due to a hairy ball theorem-type obstruction in prescribing a symmetric traceless tensor on the 2-sphere.

A consequence of this assumption that if we consider solutions on \( \tilde{\mathcal{M}} = [0, u_*] \times [0, u_*] \times S^2 \), Theorem 4.7 only concerns a restrictive class of solutions. Nevertheless, by the finite speed of propagation, it does imply that (sufficiently\(^{24}\)) angularly regular weak solutions can always be \( \text{locally weakly approximated by vacuum solutions.} \)

5. General compactness results

5.1. Preliminaries. The following Sobolev embedding result is standard (recall Definition 2.12 for notations).

**Proposition 5.1** (Sobolev embedding). Let \( 2 < p < \infty \) and \( r \in \mathbb{N} \cup \{0\} \).

There exists a constant \( C_{p,r} > 0 \), depending only on \( p \) and \( r \), such that for any closed Riemannian 2-manifold \((S, \gamma)\) with a \( C^2 \) metric\(^{25}\),
\[
(Area(S))^{-\frac{1}{2}} \| \xi \|_{L^p(S)} \leq C_{p,r} \sqrt{\max\{1(S),1\}} \| \nabla \xi \|_{L^2(S)} + (Area(S))^{-\frac{1}{2}} \| \xi \|_{L^2(S)}
\]

\(^{24}\)Note that we still need assumptions more than that in Definition 2.37 as the data are required to be more regular.

\(^{25}\)While the lemma is stated in [10] for smooth metrics, the \( C^2 \) case follows from an easy approximation argument.
for any covariant tensor \(\xi\) of rank \(r\).

(2) ([10, Lemma 5.2]) Let \(2 < p < \infty\) and \(r \in \mathbb{N} \cup \{0\}\). There exists a constant \(C_{p,r} > 0\), depending only on \(p\) and \(r\), such that for any closed Riemannian 2-manifold \((S, \gamma)\) with a \(C^2\) metric,

\[
\|\xi\|_{L^p(S)} \leq C_{p,r} \sqrt{\max\{1, 1\}} (\text{Area}(S))^{\frac{1}{2} - \frac{1}{p}} (\|\nabla \xi\|_{L^p(S)} + (\text{Area}(S))^{-\frac{1}{2}} \|\xi\|_{L^p(S)})
\]

for any covariant tensor \(\xi\) of rank \(r\).

**Proposition 5.2.** The following identities hold for any \(C^1\) function \(f\):

\[
\frac{\partial}{\partial u} \int_{S_{u,u}} f \, dA_\gamma = \int_{S_{u,u}} \Omega (\nabla_3 f + \psi \chi_f) \, dA_\gamma = \int_{S_{u,u}} \left( \frac{\partial}{\partial u} f + \Omega \psi \chi_f \right) \, dA_\gamma,
\]

\[
\frac{\partial}{\partial u} \int_{S_{u,u}} f \, dA_\gamma = \int_{S_{u,u}} \Omega (\nabla_4 f + \psi \chi_f) \, dA_\gamma.
\]

5.2. **Compactness theorems.** Starting from this subsection, we prove various general compactness results. We will consider the following setup for the remaining subsections in this section:

- We will consider as our domain the manifold (with corners) \(\mathcal{M} = [0, u_\ast] \times [0, \underline{u}_\ast] \times \mathbb{S}^2\).
- On \(\mathcal{M}\), we fix a \(C^3\) metric \(\hat{g}\) (independent of \(n\)) such that

\[
\hat{L}_{\hat{g}} \hat{\gamma} = \hat{L}_{\hat{g}} \hat{\gamma} = 0. \tag{5.1}
\]

This will be used for defining the norms.

Our first compactness result is the following simple variation of the Arzelà–Ascoli theorem.

**Proposition 5.3.** Consider either the case \((p, q) = (+\infty, 4)\) or \((p, q) = (4, 2)\).

Let \(\{\psi_n\}_{n=1}^{\infty}\) be a sequence of covariant rank \(r\) smooth \(S\)-tangent tensors satisfying the following uniform bounds:

\[
\sup_n (\|\psi_n\|_{L^\infty_u L^q_u L^p(S_{u,u}, \hat{\gamma}))} + \|\hat{L}_{\hat{g}} \psi_n\|_{L^\infty_u L^q_u L^p(S_{u,u}, \hat{\gamma}))} + \|\hat{L}_{\hat{g}} \hat{\psi_n}\|_{L^\infty_u L^q_u L^p(S_{u,u}, \hat{\gamma}))} < +\infty. \tag{5.2}
\]

Then, there exists a subsequence \(\{\psi_{n_k}\}_{k=1}^{\infty}\) and a \(\psi_\infty \in C^0_u C^0_u L^p(S_{u,u}, \hat{\gamma})) \cap L^\infty_u L^\infty_u W^{1,q}(S_{u,u}, \hat{\gamma}))\) such that

\[
\lim_{k \to +\infty} \|\psi_{n_k} - \psi_\infty\|_{L^\infty_u L^p(S_{u,u}, \hat{\gamma}))} = 0. \tag{5.3}
\]

Moreover, \(\hat{L}_{\hat{g}} \psi_\infty \in L^\infty_u L^2_u L^p(S_{u,u}, \hat{\gamma}))\) and \(\hat{L}_{\hat{g}} \hat{\psi_\infty} \in L^\infty_u L^2_u L^p(S_{u,u}, \hat{\gamma}))\).

**Proof.**

**Step 1:** Existence of \(\psi_\infty \in L^\infty_u L^\infty_u L^p(S_{u,u}, \hat{\gamma}))\) and proof of (5.3). First, since \(W^{1,q}(S_{u,u}, \hat{\gamma})) \subseteq L^p(S_{u,u}, \hat{\gamma}))\) is compact for all \((u, \underline{u})\), we know that for every \((u, \underline{u})\), there is a subsequence \(n_k\) and a \(\psi_\infty \in L^p(S_{u,u}, \hat{\gamma}))\) such that

\[
\lim_{k \to +\infty} \|\psi_{n_k} - \psi_\infty\|_{L^p(S_{u,u}, \hat{\gamma}))} = 0. \tag{5.4}
\]

By a standard argument extracting a diagonal subsequence, we thus obtain that for a fixed subsequence \(n_k\), the (5.4) holds for all \((u, \underline{u})\) rational.

We next show that for this fixed subsequence, \(\psi_{n_k}\) is uniformly (in \((u, \underline{u})\)) Cauchy in \(L^p(S_{u,u}, \hat{\gamma}))\).

Let \(\epsilon > 0\). For \((u_0, \underline{u}_0) \in ([0, u_\ast] \times [0, \underline{u}_\ast]) \cap \mathbb{Q}^2\), let \(\mathcal{R}(u_0, \underline{u}_0, \epsilon) := \{(u, \underline{u}) \in [0, u_\ast] \times [0, \underline{u}_\ast] : |u - u_0| + |\underline{u} - \underline{u}_0| \leq \epsilon\} \). Clearly, we can find a finite set of \(\{(u_i, \underline{u}_i)\}_{i=1}^{m} \subset ([0, u_\ast] \times [0, \underline{u}_\ast]) \cap \mathbb{Q}^2\) (depending on \(\epsilon\)) such that \(\mathcal{R}(u_0, \underline{u}_0, \epsilon) \supseteq [0, u_\ast] \times [0, \underline{u}_\ast]\).

Since (5.4) holds for \((u, \underline{u}) = (u_i, \underline{u}_i)\) for \(i = 1, \ldots, m\), we can find \(K\) sufficiently large such that whenever \(1 \leq i \leq m\) and \(k, k' \geq K\), we have

\[
\|\psi_{n_k} - \psi_{n_{k'}}\|_{L^p(S_{u,u}, \hat{\gamma}))} \leq \epsilon.
\]

Now for every \((u, \underline{u}) \in [0, u_\ast] \times [0, \underline{u}_\ast]\), we can find the closest \((u_i, \underline{u}_i)\) so that we obtain

\[
\|\psi_{n_k} - \psi_{n_{k'}}\|_{L^p(S_{u,u}, \hat{\gamma}))} \leq \|\psi_{n_k} - \psi_{n_{k'}}\|_{L^p(S_{u,u}, \hat{\gamma}))} + \int_{u_i}^{u} (\|\hat{L}_{\hat{g}} \psi_{n_k}\|_{L^p(S_{u,u}, \hat{\gamma}))} + \|\hat{L}_{\hat{g}} \hat{\psi_{n_k}}\|_{L^p(S_{u,u}, \hat{\gamma}))}) \, du' \leq \epsilon,
\]

whenever \(k, k' \geq K\). This proves that there exists \(\psi_\infty \in L^\infty_u L^\infty_u L^p(S_{u,u}, \hat{\gamma}))\) and that (5.3) holds.

**Step 2:** \(\psi_\infty \in C^0_u C^0_u L^p(S_{u,u}, \hat{\gamma})) \cap L^\infty_u L^\infty_u W^{1,q}(S_{u,u}, \hat{\gamma}))\). Since \(\psi_{n_k}\) are smooth and \(\psi_\infty\) is their limit in \(L^\infty_u L^\infty_u L^p(S_{u,u}, \hat{\gamma}))\), it immediate follows that \(\psi_\infty \in C^0_u C^0_u L^p(S_{u,u}, \hat{\gamma}))\).
Next, notice that for every fixed \((u, \psi)\), the given uniform \(W^{1,q}(S_{u,\psi}, \gamma)\) estimate in (5.2) and the Banach–Alaoglu theorem imply that there exists \(\phi_\infty\) such that for a further subsequence, \(\nabla \psi_{n_k} \rightharpoonup \phi_\infty\) weakly in \(L^q(S_{u,\psi}, \gamma)\). Since the \(W^{1,q}(S_{u,\psi}, \gamma)\) estimate is uniform in \((u, \psi)\), it follows that \(\phi_\infty \in L^\infty_0 L^\infty_0 L^q(S_{u,\psi}, \gamma)\). It is easy to check that \(\phi_\infty = \nabla \psi_\infty\), proving that \(\psi_\infty \in L^\infty_0 L^\infty_0 W^{1,q}(S_{u,\psi}, \gamma)\).

Step 3: \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty \in L^\infty_0 L^p_0(L^p_0(S_{u,\psi}, \gamma))\) and \(\mathcal{L}_{\frac{\partial}{\partial n}} \phi_\infty \in L^\infty_0 L^p_0 L^p(S_{u,\psi}, \gamma)\). For every fixed \(u\), the estimate (5.2) and Banach–Alaoglu implies that after passing to a further subsequence, \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_{n_k}\) has a weak limit in \(L^p_0 L^p(S_{u,\psi}, \gamma)\). It is easy to check that the limit coincides with \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty\), and therefore \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty \in L^\infty_0 L^p_0(L^p_0(S_{u,\psi}, \gamma))\).

The proof of \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty \in L^\infty_0 L^p_0 L^p_0(L^p_0(S_{u,\psi}, \gamma))\) is similar; we omit the details. \(\square\)

**Proposition 5.4.** Consider one of the following cases: (1) \(m \in \{0, 1\}, (p, q) = (+\infty, 4); (2) \(m \in \{0, 1, 2\}, (p, q) = (4, 2)\).

Let \(\{\psi_n\}_{n=1}^{\infty}\) be a sequence of covariant rank \(r\) smooth \(S\)-tangent tensors satisfying the following uniform bounds:

\[
\sup_n \|\psi_n\|_{L^\infty_0 L^2_0 W^{m+1,q}(S_{u,\psi}, \gamma)} + \|\mathcal{L}_{\frac{\partial}{\partial n}} \psi_n\|_{L^\infty_0 L^2_0 W^{m+1,q}(S_{u,\psi}, \gamma)} + \|\mathcal{L}_{\frac{\partial}{\partial n}} \psi_n\|_{L^\infty_0 L^2_0 W^{m+1,q}(S_{u,\psi}, \gamma)} < +\infty.
\]

Then, there exists a subsequence \(\{\psi_{n_k}\}_{k=1}^{\infty}\) and a \(\psi_\infty \in C^0_0 C^0_0 W^{m,p}(S_{u,\psi}, \gamma) \cap L^\infty_0 L^\infty_0 W^{m+1,q}(S_{u,\psi}, \gamma)\) such that

\[
\lim_{k \to +\infty} \|\psi_{n_k} - \psi_\infty\|_{L^\infty_0 L^2_0 W^{m,p}(S_{u,\psi}, \gamma)} = 0. \tag{5.6}
\]

Moreover, \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty \in L^\infty_0 L^2_0 W^{m,p}(S_{u,\psi}, \gamma)\) and \(\mathcal{L}_{\frac{\partial}{\partial n}} \phi_\infty \in L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)\).

**Proof.** This is straightforward from Proposition 5.3. For \(m = 0\), this is exactly Proposition 5.3.

We consider the case \(m = 1\). By Proposition 5.3, there exists a subsequence \(\psi_{n_k}\) and \(\psi_\infty \in C^0_0 C^0_0 W^{m+1,p}(S_{u,\psi}, \gamma) \cap L^\infty_0 L^\infty_0 W^{m+1,q}(S_{u,\psi}, \gamma)\) such that \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty \in L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)\), \(\mathcal{L}_{\frac{\partial}{\partial n}} \psi_\infty \in L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)\)

\[
\lim_{k \to +\infty} \|\psi_{n_k} - \psi_\infty\|_{L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)} = 0.
\]

Moreover, using Proposition 5.3 for \(\nabla \psi_{n_k}\), we see that after passing to a further subsequence (not relabeled), there exists \(\phi_\infty \in C^0_0 C^0_0 L^p(S_{u,\psi}, \gamma) \cap L^\infty_0 L^\infty_0 W^{m+1,q}(S_{u,\psi}, \gamma)\) such that \(\mathcal{L}_{\frac{\partial}{\partial n}} \phi_\infty \in L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)\), \(\mathcal{L}_{\frac{\partial}{\partial n}} \phi_\infty \in L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)\) and

\[
\lim_{k \to +\infty} \|\nabla \psi_{n_k} - \phi_\infty\|_{L^\infty_0 L^2_0 L^p(S_{u,\psi}, \gamma)} = 0.
\]

The conclusion of the proposition in the \(m = 1\) case then follows after checking that \(\phi_\infty = \nabla \psi_\infty\) and using that \([\mathcal{L}_{\frac{\partial}{\partial n}} \nabla] = [\mathcal{L}_{\frac{\partial}{\partial n}} \nabla] = 0\) (due to (5.1)).

The \(m = 2\) case is similar to the \(m = 1\) case; we omit the details. \(\square\)

### 5.3. Compactness in BV.

**Lemma 5.5 (Aubin–Lions lemma).** Let \(X_0 \subseteq X \subseteq X_1\) be three Banach spaces such that \(X_0 \subseteq X \subseteq X_1\) is continuous. For \(T > 0\) and \(q > 1\), let

\[
W := \{v \in L^\infty([0,T]; X_0) : \dot{v} \in L^q([0,T]; X_1)\},
\]

where \(^-\) denotes the (weak) derivative in the variable on \([0,T]\).

Then \(W\) embeds compactly into \(C^0([0,T]; X)\).

**Proposition 5.6 (Compactness of BV functions (Theorems 5.2 and 5.5 in [15])).** Let \(U \subseteq \mathbb{R}^k\) be open and bounded, with Lipschitz boundary \(\partial U\). Assume \(\{\psi_{n_k}\}_{k=1}^{\infty}\) is a sequence in (Euclidean) \(BV(U)\) satisfying

\[
\sup_n \|\psi_n\|_{BV(U)} := \sup_n \left(\int_U |\psi_n| \, dx + \sup \left\{ \int_U |\psi_n \text{div} \phi| \, dx : \phi \in C^1_c(U; \mathbb{R}^k), \|\phi\|_{L^\infty(U)} \leq 1 \right\} \right) < +\infty.
\]

Then there exists a subsequence \(\{\psi_{n_k}\}_{k=1}^{\infty}\) and a function \(\psi_\infty \in BV(U)\) such that

\[
\int_U |\psi_{n_k} - \psi_\infty| \, dx \to 0 \quad \text{as } k \to +\infty.
\]

Moreover,

\[
\|\psi_\infty\|_{BV(U)} \leq \liminf_{k \to +\infty} \|\psi_{n_k}\|_{BV(U)}.
\]
We now consider the application of Proposition 5.6 to our setting. In particular, in the proposition below, we continue to work in the setting described in the beginning of Section 5.2.

**Proposition 5.7.** (1) Let \( \{\psi_n\}_{n=1}^{+\infty} \) be a sequence of smooth functions such that
\[
\sup_n(\|\psi_n\|_{C^0_uBV(H_u,\gamma)} + \|\nabla\psi_n\|_{L^2_uL^1(S_u,\gamma)}) < +\infty.
\]

Then there exists a subsequence \( \{\psi_{n_k}\}_{k=1}^{+\infty} \) and a \( \psi_\infty \in C^0_uL^1(S_u,\gamma) \cap L^\infty_uBV(H_u,\gamma) \) such that
\[
\|\psi_{n_k} - \psi_\infty\|_{C^0_uL^1(S_u,\gamma)} \to 0
\]
as \( k \to +\infty \).

(2) An entirely symmetric statement holds after swapping \( u \) and \( \bar{u} \).

**Proof.** We will only consider case (1); case (2) can be treated by swapping \( u \) and \( \bar{u} \).

First, by Lemma 5.5 (with \( X_0 = BV(H_u,\gamma) \), \( X = \mathbb{X} = L^1_uL^1(S_u,\gamma) \), \( T = u \), \( q = 2 \)) and Proposition 5.6 (which gives the compactness of \( X_0 \subseteq X \)), it follows that there exists \( \psi_\infty \in C^0_uL^1(S_u,\gamma) \) such that (5.8) holds.

The fact that \( \psi_\infty \in L^\infty_uBV(H_u) \) then follows from (5.7) in Proposition 5.6.

\[\square\]

### 5.4. Weak compactness theorems

We continue to work in the setting described in the beginning of Section 5.2.

**Proposition 5.8.** (1) Let \( \{\psi_n\}_{n=1}^{+\infty} \) be a sequence of rank-\( r \) S-tangent covariant smooth tensor fields such that
\[
\sup_n(\|\psi_n\|_{\mathcal{C}^0_uL^1(S_u,\gamma)} + \|\nabla\psi_n\|_{L^2_uL^1(S_u,\gamma)}) < +\infty.
\]

Then there exists a subsequence \( \{\psi_{n_k}\}_{k=1}^{+\infty} \) and a rank-\( r \) S-tangent covariant tensor field-valued Radon measure \( \{d\mu_{\psi,n}\}_{n \in [0,u]} \) such that the following hold:

(a) For every \( u \in [0,u] \) and every rank-\( r \) S-tangent bounded contravariant tensor field \( \varphi \in C^0 \) on \( (0,\bar{u}) \times \mathbb{S}^2 \),
\[
\int_{\{u\} \times (0,\bar{u}) \times \mathbb{S}^2} \langle \varphi(u,\vartheta), \psi_{n_k} \rangle \, dA_3 \, du - \int_{\{u\} \times (0,\bar{u}) \times \mathbb{S}^2} \varphi(u,\vartheta) \cdot d\mu_{\psi,u} \to 0
\]
as \( k \to +\infty \).

(b) \( d\mu_{\psi,u} \) is continuous in \( u \) and the following holds:
\[
\left| \int_{\{u\} \times (0,\bar{u}) \times \mathbb{S}^2} \langle \varphi(u,\vartheta), \psi_{n_k} \rangle \, dA_3 \, du - \int_{\{u\} \times (0,\bar{u}) \times \mathbb{S}^2} \langle \varphi(u,\vartheta), \psi_{n_k} \rangle \, dA_3 \, du \right|
\leq \limsup_{k \to +\infty} \|u - u'\|^r \|\nabla\psi_{n_k}\|_{L^2_uL^1(S_u,\gamma)} \|\psi_{n_k}\|_{L^\infty_uL^2(S_u,\gamma)}
\]
\[
\leq \limsup_{k \to +\infty} \sup \left\{ \sup_{u \in [0,u]} \sup_{\varphi \in C^0} \|\psi_{n_k}\|_{\mathcal{C}^0_uL^1(S_u,\gamma)} \leq 1 \left| \int_{\{u\} \times (0,\bar{u}) \times \mathbb{S}^2} \varphi(u,\vartheta) \cdot d\mu_{\psi,u} \right| \right\}
\]
\[
\leq \limsup_{k \to +\infty} \sup \|\psi_{n_k}\|_{\mathcal{C}^0_uL^1(S_u,\gamma)}.
\]

(2) An entirely symmetric statement holds after swapping \( u \) and \( \bar{u} \).

**Proof.** We only consider case (1); case (2) is similar.

**Step 1:** Proof of (5.10) for rational \( u \). For every fixed \( u \in [0,u] \), by the Banach–Alaoglu theorem, there exists a subsequence \( n_k \) and a rank-\( r \) S-tangent covariant tensor field-valued Radon measure \( d\mu_{\psi,u} \) such that (5.10) holds.

By considering \( u \in [0,u] \cap \mathbb{Q} \) and using a standard trick of picking a diagonal subsequence, we can therefore find a fixed subsequence \( n_k \) such that (5.10) holds for all rational \( u \in [0,u] \).

We now fix the subsequence \( n_k \).

**Step 2:** Proof of (5.10) for all \( u \in [0,u] \). We now show that for the subsequence \( n_k \) fixed in Step 1, (5.10) in fact holds for all \( u \in [0,u] \).

First, we note that for every fixed \( u \in [0,u] \) (not necessarily rational), we can take a further subsequence \( n_k \) so that (5.10) holds (for some \( d\mu_{\psi,u} \)). Thus, in order to obtain weak-* convergence of that
full subsequence \( n_k \), it suffices to show that for all \( u \in [0, u_*] \), and all bounded \( C^0 \) \( u \)-independent, rank-\( r \) contravariant, \( S \)-tangent tensor field \( \varphi \),

\[
\left\{ \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_k} \rangle \, dA_\gamma \, du \right\}_{k=1}^{+\infty}
\]

is a Cauchy sequence.

Fix \( u \in [0, u_*] \) (not necessarily rational) and \( \varphi \in C^0 \) for the remainder of this step. Since \( \mathcal{E}_{\frac{\partial}{\partial u}} \hat{\gamma} = 0 \), we can use the fundamental theorem of calculus, Hölder’s inequality and (5.9) to obtain

\[
\begin{align*}
&\left| \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_k} \rangle \, dA_\gamma \, du - \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_{k'}} \rangle \, dA_\gamma \, du \right| \\
\leq & \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \left| \langle \varphi(u, \vartheta), \mathcal{E}_{\frac{\partial}{\partial u}} \psi_{n_k} \rangle \right| \, dA_\gamma \, du \\
\leq & \| \varphi \|_{L^2 L^\infty(S_{\vartheta, \varphi})} \| \mathcal{E}_{\frac{\partial}{\partial u}} \psi_{n_k} \|_{L^2 L^\infty(S_{\vartheta, \varphi})} \lesssim |u - u'|^{\frac{1}{2}} \| \varphi \|_{L^\infty(S)}.
\end{align*}
\]

Let \( \epsilon > 0 \). There exists \( u' \) rational such that

\[
|u - u'|^{\frac{1}{2}} \| \varphi \|_{L^\infty(S)} < \epsilon.
\]

By Step 1, for this rational \( u' \), there exists \( K > 0 \) such that whenever \( k, k' \geq K \), we have

\[
\begin{align*}
&\left| \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_k} \rangle \, dA_\gamma \, du - \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_{k'}} \rangle \, dA_\gamma \, du \right| < \epsilon.
\end{align*}
\]

Hence, by (5.13) (for both \( k \) and \( k' \)), (5.14), (5.15), and the triangle inequality,

\[
\begin{align*}
&\left| \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_k} \rangle \, dA_\gamma \, du - \int_{\{u\} \times (0, u_*] \times \mathbb{S}^2} \langle \varphi(u, \vartheta), \psi_{n_{k'}} \rangle \, dA_\gamma \, du \right| \lesssim \epsilon
\end{align*}
\]

for \( k, k' \geq K \), which is what we wanted to prove.

Step 3: Proof of (5.11) and (5.12). The estimate (5.11) follows from (5.13) after taking \( \limsup_{k \to +\infty} \).

Finally, (5.12) follows from (5.10) and Hölder’s inequality.

**Proposition 5.9.** Suppose, in addition to the assumptions of Proposition 5.8, there exists \( q \in [2, +\infty) \) such that

\[
\sup_n \left( \| \varphi_n \|_{L^q L^\infty L^2(S_{\vartheta, \varphi})} + \| \mathcal{E}_{\frac{\partial}{\partial u}} \varphi_n \|_{L^2 L^q L^2(S_{\vartheta, \varphi})} \right) < +\infty.
\]

Then there is a rank-\( r \) \( S \)-tangent contravariant tensor field \( \psi_\infty \in C^0 \cap L^2 L^2(S_{\vartheta, \varphi}) \) such that \( d\mu_{\psi, u} = \psi_\infty \, dA_\gamma \, du \) (with \( d\mu_{\psi, u} \) as in Proposition 5.8). Moreover, \( \mathcal{E}_{\frac{\partial}{\partial u}} \psi_\infty \in L^2 L^q L^2(S_{\vartheta, \varphi}) \).

A symmetric statement also holds after swapping \( u \) and \( \vartheta \).

**Proof.** Step 1: \( d\mu_{\psi, u} \) is absolutely continuous with respect to \( dA_\gamma \, du \). Fix \( u \in [0, u_*] \). Suppose \( U \subset [0, u_*] \times \mathbb{S}^2 \) is an open subset such that \( \int_U \, dA_\gamma \, du < \epsilon \). Then, using Hölder's inequality and (5.16),

\[
\begin{align*}
&\sup_{\varphi \in C^0 \cap L^\infty L^\infty(S_{\vartheta, \varphi})} \left| \int_U \varphi \cdot d\mu_{\psi, u} \right| \\
\leq & \sup_{\varphi \in C^0 \cap L^\infty L^\infty(S_{\vartheta, \varphi})} \limsup_{k \to +\infty} \int_U |\langle \varphi, \psi_{n_k} \rangle| \, dA_\gamma \, du \\
\leq & \left( \sup_{\varphi \in C^0 \cap L^\infty L^\infty(S_{\vartheta, \varphi})} \| \varphi \|_{L^\infty(S_{\vartheta, \varphi})} \right) \left( \sup_k \| \psi_{n_k} \|_{L^2 L^\infty(S_{\vartheta, \varphi})} \right) \left( \int_0^{u_*} \, du \right)^{\frac{1}{2}} \left( \int_U \, dA_\gamma \, du \right)^{\frac{1}{2}} \\
\leq & \frac{u_*^{\frac{1}{2}}}{\epsilon}.
\end{align*}
\]

It therefore follows that \( d\mu_{\psi, u} = \psi_\infty \, dA_\gamma \, du \) for some rank-\( r \) tensor field \( \psi_\infty \in L^\infty L^1 L^1(S_{\vartheta, \varphi}) \).

**Step 2:** Regularity of \( \psi_\infty \). In view of Step 1, it remains to prove the regularity statement for \( \psi_\infty \).
First, by duality, Fatou’s lemma, Hölder’s inequality and (5.16),
\[
\|\psi_\infty\|_{L^q_u L^\infty_v L^2(S_{u,v}^2)} = \sup_{\phi \in C^0} \|\phi\|_{L^q_u L^\infty_v L^2(S_{u,v}^2)} \int_{[0,u_{\ast}] \times [0,v_{\ast}]} \langle \phi, \psi_\infty \rangle \, dA \, du \, dv \\
\leq \sup_{\phi \in C^0} \|\phi\|_{L^q_u L^\infty_v L^2(S_{u,v}^2)} \int_{[0,u_{\ast}] \times [0,v_{\ast}]} \langle \phi, \psi_n \rangle \, dA \, du \, dv < +\infty,
\]
which proves that \(\psi_\infty \in L^q_u L^\infty_v L^2(S_{u,v}^2)\).

Next, we show that \(E_n \psi_\infty \in L^q_u L^\infty_v L^2(S_{u,v}^2)\). Note that by (5.16) and the Banach–Alaoglu theorem, after passing to a further subsequence (not relabelled), \(E_n \psi_n\) has a weak limit in \(L^q_u L^\infty_v L^2(S_{u,v}^2)\).

This limit coincides with \(E_n \psi_\infty\), proving that \(E_n \psi_\infty \in L^q_u L^\infty_v L^2(S_{u,v}^2)\).

Finally, we show that \(\psi_\infty \in C^0_u L^q_v L^2(S_{u,v}^2)\). By (5.17), we already know \(\psi_\infty \in L^\infty_u L^q_v L^2(S_{u,v}^2)\).

To prove continuity in \(u\), we use the fundamental theorem of calculus and the fact (established just above) that \(E_n \psi_\infty \in L^q_u L^\infty_v L^2(S_{u,v}^2)\).

\[\square\]

6. Existence of a limiting spacetime

From now on until the end of Section 7, we work under the assumptions of Theorem 4.1.

In this section, we prove the existence of a limiting spacetime (recall part (2) of Theorem 4.1). We will use the convention that all constants \(C\) or implicit constants in \(\lesssim\) will depend only on quantities in the assumptions of Theorem 4.1.

To prove convergence we will continually extract subsequences from \((M, g_u)\). Phrases such as “extracting a further subsequence \(n_k\)” will mean that we extract a subsequence from that in the previous lemma, proposition, etc. To simplify notations, we will never relabel the further subsequence.

We begin in Section 6.1 a preliminary step showing that the norms with respect to different metrics are comparable. We then proceed to show that the limit exists and proving the corresponding regularity statement:

1. The existence of uniform limit of the metric components will be proven in Sections 6.2, 6.3.
2. The existence of uniform limit of \(\eta\) and \(\bar{\eta}\) will be proven in Section 6.4.
3. The existence of weak limit of \(\bar{\chi}, \bar{\chi}, \bar{\gamma}, \bar{\chi}, \omega\) and \(\bar{\omega}\) will be proven in Section 6.5.
4. The existence of BV limit of \(\bar{\psi}\chi\) and \(\bar{\psi}\bar{\chi}\) will be proven in Section 6.6.
5. Finally, the existence of limits in (4.2) and (4.3) will be proven in Section 6.8.

It will also be important to prove a compensated compactness result, related to some convergence properties of the Ricci coefficients which have the weakest convergence. This will be treated in Section 6.7.

6.1. Comparability of norms.

**Proposition 6.1** (Comparability of norms). For every \(r \in \mathbb{N} \cup \{0\}\), there exists a constant \(C > 0\) (independent of \(n\) and \((u, v))\) such that for any rank-\(r\) \(S\)-tangent covariant tensor \(\xi\),
\[
C^{-1}\|\xi\|_{L^p(S_{u,v}^2(\gamma_{0,0}, n))} \leq \|\xi\|_{L^p(S_{u,v}^2(\gamma_{0,0}, n))} \leq C\|\xi\|_{L^p(S_{u,v}^2(\gamma_{0,0}, n))} \leq +\infty,
\]
and
\[
C^{-1}\|\xi\|_{W^1,p(S_{u,v}^2(\gamma_{0,0}, n))} \leq \|\xi\|_{W^1,p(S_{u,v}^2(\gamma_{0,0}, n))} \leq C\|\xi\|_{W^1,p(S_{u,v}^2(\gamma_{0,0}, n))} \leq +\infty,
\]

Using in addition that \((\gamma_{0,0}n) \to (\gamma_{0,0})_\infty\) in \(C^0(S)\) (see (4.1)), it follows that
\[
C^{-1}\|\xi\|_{L^p(S_{u,v}^2(\gamma_{0,0}, \infty))} \leq \|\xi\|_{L^p(S_{u,v}^2(\gamma_{0,0}, \infty))} \leq C\|\xi\|_{L^p(S_{u,v}^2(\gamma_{0,0}, \infty))} \leq +\infty,
\]
and
\[
C^{-1}\|\xi\|_{W^1,p(S_{u,v}^2(\gamma_{0,0}, \infty))} \leq \|\xi\|_{W^1,p(S_{u,v}^2(\gamma_{0,0}, \infty))} \leq C\|\xi\|_{W^1,p(S_{u,v}^2(\gamma_{0,0}, \infty))} \leq +\infty.
\]

**Proof.** Step 1: Proof of (6.1). Given \(\xi \in S_{u,v}^2\), extend \(\xi\) to a \(S\)-tangent tensor on \([0,u_{\ast}] \times [0,v_{\ast}] \times S^2\) (still denoted by \(\xi\)) by stipulating that
\[
E_n \xi = E_n \xi = 0
\]
(possible since $\frac{\partial}{\partial u}, \frac{\partial}{\partial u'} = 0$).

From this (and the fact that $\mathcal{L}_{u}(\gamma_{0,0})_{n} = \mathcal{L}_{u'}(\gamma_{0,0})_{n} = 0$) it follows that

$$
\|\xi\|_{L^{p}(S_{n,\pm}(\gamma_{0,0})_{n})} = \|\xi\|_{L^{p}(S_{u',\pm}(\gamma_{0,0})_{n})}, \quad \forall (u', u').
$$

On the other hand, by Proposition 2.17 we have

$$
\frac{\partial}{\partial u}(\gamma_{n})_{AB} = 2\Omega_{n}(\chi_{n})_{AB}, \quad \frac{\partial}{\partial u}(\gamma_{n})_{AB} = 2\Omega_{n}(\chi_{n})_{AB} - (\gamma_{n})_{CA}(\nabla_{n})_{B}b_{C}^{0} - (\gamma_{n})_{CB}(\nabla_{n})_{A}b_{C}^{0}.
$$

Therefore, using also (6.5) and Proposition 5.2, we have

$$
\frac{\partial}{\partial u}\|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} = \frac{\partial}{\partial u} \int_{S_{n,\pm}} |\xi|^{p}_{\gamma_{n}} \, dA_{\gamma_{n}}
$$

$$
= -p \int_{S_{n,\pm}} |\xi|^{p-2}_{\gamma_{n}} \sum_{s=1}^{r} \Omega_{n}(\chi_{n})_{A, B}^{s} \Pi_{s=1}^{(\gamma_{n})_{A, B}}(\gamma_{n})_{A, B}^{s} \xi_{A_{1}...A_{i}...B_{r}} \, dA_{\gamma_{n}} + \int_{S_{n,\pm}} \Omega_{n}\nabla_{n}|\xi|^{p}_{\gamma_{n}} \, dA_{\gamma_{n}},
$$

and similarly,

$$
\frac{\partial}{\partial u}\|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} = \frac{\partial}{\partial u} \int_{S_{n,\pm}} |\xi|^{p}_{\gamma_{n}} \, dA_{\gamma_{n}}
$$

$$
= -p \int_{S_{n,\pm}} |\xi|^{p-2}_{\gamma_{n}} \sum_{s=1}^{r} \Omega_{n}(\chi_{n})_{A, B}^{s} \Pi_{s=1}^{(\gamma_{n})_{A, B}}(\gamma_{n})_{A, B}^{s} \xi_{A_{1}...A_{i}...B_{r}} \, dA_{\gamma_{n}}
$$

$$
+ \int_{S_{n,\pm}} \Omega_{n}\nabla_{n}|\xi|^{p}_{\gamma_{n}} \, dA_{\gamma_{n}}.
$$

Now using the uniform boundedness for the terms $\|\Omega_{n}\chi_{n}\|_{L^{2}}^{\infty} L^{\infty}(S_{n,\pm}, \gamma_{n})$, $\|\Omega_{n}\chi_{n}\|_{L^{2}}^{\infty} L^{\infty}(S_{n,\pm}, \gamma_{n})$, $\|\nabla_{n}b_{n}\|_{L^{\infty}} L^{\infty}(S_{n,\pm}, \gamma_{n})$ (by (3.9), (3.12) and (3.13)) and Grönwall’s inequality, we obtain (for all $(u, u') \in [0, u_{*}] \times [0, u_{*}])$

$$
C^{-1} \|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} \leq \|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} \leq C \|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})}.
$$

The estimate (6.1) therefore follows from (6.6), (6.9) and the fact that $(\gamma_{0,0})_{n} |_{S_{0,\pm}} = \gamma_{0} |_{S_{0,\pm}}$.

**Step 2: Proof of (6.2).** By (3.9), Sobolev embedding, (6.1) and the computation

$$
(\mathcal{F}_{n} - \mathcal{F}_{0,0,0})_{AB} = \frac{1}{2} ((\gamma_{0,0})_{n}^{-1})^{AD}(2(\nabla_{n})_{B}(\gamma_{n} - (\gamma_{0,0})_{n}C)_{D} - (\nabla_{n})_{D}(\gamma_{n} - (\gamma_{0,0})_{n}C))_{D},
$$

we have $\|\mathcal{F}_{n} - \mathcal{F}_{0,0,0}\|_{C_{0} C_{0} L^{2}(S_{n,\pm}, \gamma_{n})} \lesssim 1$. Using again (6.1), we then obtain

$$
\|\nabla_{n}\xi - (\nabla_{0,0})\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} \lesssim \|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})}.
$$

As a result, by the triangle inequality, for any $1 \leq p \leq +\infty$,

$$
\|\nabla_{n}\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} \leq C(\|\nabla_{0,0}\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} + \|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})}),
$$

and

$$
\|\nabla_{n}\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} \leq C(\|\nabla_{0,0}\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})} + \|\xi\|_{L^{p}(S_{n,\pm}, \gamma_{n})}).
$$

The estimate (6.2) then follows from (6.1).

**Step 3: Proof of (6.3) and (6.4).** The proof of (6.3) and (6.4) is similar to that of (6.2). The only difference is that by (3.9) and Sobolev embedding using Proposition 5.1 and (3.7), we only have the estimates $\|\nabla_{n}(\mathcal{F}_{n} - \mathcal{F}_{0,0,0})\|_{C_{0} C_{0} L^{4}(S_{n,\pm}, \gamma_{n})} \lesssim 1$ and $\|\nabla_{n}^{2}(\mathcal{F}_{n} - \mathcal{F}_{0,0,0})\|_{C_{0} C_{0} L^{2}(S_{n,\pm}, \gamma_{n})} \lesssim 1$, which restrict the range of allowable $p$.

6.2. Limit of $\gamma$ and its angular derivatives.

**Proposition 6.2.** There exists a subsequence $n_{k}$ and a limiting metric $\gamma_{\infty} \in C_{0} C_{0} W^{2,4}(S_{n,\pm}, (\gamma_{0,0})_{\infty}) \cap L^{\infty} L^{\infty} W^{2,4}(S_{n,\pm}, (\gamma_{0,0})_{\infty})$ such that

$$
\|\gamma_{n_{k}} - \gamma_{\infty}\|_{C_{0} C_{0} W^{2,4}(S_{n,\pm}, (\gamma_{0,0})_{\infty})} + \|(\nabla_{0,0})\gamma_{n_{k}} - \gamma_{\infty}\|_{C_{0} C_{0} W^{2,4}(S_{n,\pm}, (\gamma_{0,0})_{\infty})} \rightarrow 0.
$$

Moreover, $\gamma_{\infty}$ also satisfies

$$
\|\mathcal{L}_{\nabla_{n}}\gamma_{\infty}\|_{L^{\infty} L^{2} W^{2,4}(S_{n,\pm}, (\gamma_{0,0})_{\infty})} + \|\mathcal{L}_{\nabla_{n}}\gamma_{\infty}\|_{L^{\infty} L^{2} W^{2,4}(S_{n,\pm}, (\gamma_{0,0})_{\infty})} \lesssim 1.
$$
Proposition 6.3. That norm \( (\gamma_n - (\gamma_0,0),n) \|C_u^0 C_{u}^0 W^{1,2}(S_{u,u},(\gamma_0,0)) \) + \( \|\gamma_n - (\gamma_0,0)n\|C_u^0 C_{u}^0 W^{1,2}(S_{u,u},(\gamma_0,0)) \) + \( \|\gamma_n - (\gamma_0,0)n\|C_u^0 C_{u}^0 W^{1,2}(S_{u,u},(\gamma_0,0)) \) is independent of \( \gamma_0 \). Therefore, by Proposition 5.4 (with \( \gamma = (\gamma_0,0) \)), there exists \( \gamma_\infty \) such that

\( \gamma_n - (\gamma_0,0)n \rightarrow \gamma_\infty - (\gamma_0,0) \) in \( C_u^0 C_{u}^0 W^{2,4}(S_{u,u},(\gamma_0,0)) \).

This implies (6.12) and (6.13) (using Sobolev embedding). Moreover, by Proposition 5.4, \( \gamma_\infty - (\gamma_0,0) \) is in \( L_u^\infty L_u^\infty W^{3,2}(S_{u,u},(\gamma_0,0)) \), which implies \( \gamma_\infty \in L_u^\infty L_u^\infty W^{3,2}(S_{u,u},(\gamma_0,0)) \). Finally, Proposition 5.4 gives that \( \|\gamma_n - (\gamma_0,0)\| \leq \|\gamma_\infty - (\gamma_0,0)\| \leq \|\gamma_\infty - (\gamma_0,0)\| \leq L_u^\infty L_u^\infty W^{2,4}(S_{u,u},(\gamma_0,0)) \), which imply (6.14) since \( \|\gamma_n - (\gamma_0,0)\| = 0 \). □

One immediate consequence of Proposition 6.2 is the uniform bound of the isoperimetric constants and the area of each of the 2-sphere \( S_{u,u} \) with respect to the limiting metric \( \gamma_\infty \):

**Proposition 6.3.**

\[ \sup_{n} I(S_{u,u}, \gamma_\infty) \lesssim 1, \]

\[ 1 \lesssim \inf_{n} \text{Area}(S_{u,u}, \gamma_\infty) \leq \sup_{n} \text{Area}(S_{u,u}, \gamma_\infty) \lesssim 1, \]

and

\[ \| \log \frac{\det \gamma_\infty}{\det(\gamma_0,0)} \|C_u^0 C_{u}^0 \leq 1. \]

**Proof.** By the \( C^0 \) convergence statement (6.12) in Proposition 6.2, it follows that for every \( (u, \gamma) \),

\[ \text{Area}(S_{u,u}, \gamma_\infty) \leq \text{Area}(S_{u,u}, \gamma_\infty) \leq \text{Area}(S_{u,u}, \gamma_\infty), \]

and (using also (4.1))

\[ \| \log \frac{\det \gamma_\infty}{\det(\gamma_0,0)} \|C_u^0 C_{u}^0 \leq \sup_{k \rightarrow \infty} \| \log \frac{\det \gamma_\infty}{\det(\gamma_0,0)} \|C_u^0 C_{u}^0 \].

The desired conclusions then follow from (3.7). □

Another immediate consequence of Proposition 6.2 is the following estimates for the angular connections:

**Proposition 6.4.** The following hold for \( \Psi \) being the Christoffel symbols associated to \( \gamma_\infty \):

\[ \|\Psi_n - \Psi\|C_u^0 C_{u}^0 \lesssim 1, \]

\[ \|\Psi_n - \Psi\|C_u^0 C_{u}^0 \lesssim 1, \]

Moreover, \( K_\infty \) (the Gauss curvature of \( (S_{u,u}, \gamma_\infty) \)) is a well-defined \( C_u^0 C_{u}^0 L^4(S_{u,u}, (\gamma_0,0)) \) function which satisfies

\[ \|K_n - K\|C_u^0 C_{u}^0 \lesssim 1. \]

**Proof.** The estimates (6.15) and (6.16) follow from Proposition 8.7 and the fact

\[ \Gamma_n - \Gamma_\infty = \frac{1}{2} (\gamma^{-1})^{AD} (2(V_n)D(\gamma_n - \gamma_\infty)) - (V_n)D(\gamma_n - \gamma_\infty)BC. \]

The statements about the Gauss curvature in (6.17) follow immediate from (6.15), (6.16) and (2.4). □

Given Propositions 8.7 and 6.4, we have the following equivalence of norms:

**Lemma 6.5.** Let \( 0 \leq m \leq 3 \) be an integer and \( p \in [1, p_m] \), where \( p_m = \begin{cases} +\infty & m = 0, 1 \\ 4 & m = 2 \\ 2 & m = 3 \end{cases} \). Then all of

\[ W^{m,p}(S_{u,u}, \gamma_n), W^{m,p}(S_{u,u}, \gamma_\infty), W^{m,p}(S_{u,u}, (\gamma_0,0)n), W^{m,p}(S_{u,u}, (\gamma_0,0)) \]
Proof. The equivalence of $W^{m,p}(S_{u,\psi}, \gamma_{n_k})$, $W^{m,p}(S_{u,\psi}, (\gamma_0,0)_{n_k})$ and $W^{m,p}(S_{u,\psi}, (\gamma_0,0)_{\infty})$ has been proven in Proposition 6.1. That $W^{m,p}(S_{u,\psi}, \gamma_{n_k})$ and $W^{m,p}(S_{u,\psi}, \gamma_{\infty})$ are equivalent is a consequence of Propositions 8.7 and 6.4. \qed

In view of Lemma 6.5, from now on, we will write $L^p(S_{u,\psi})$, $W^{1,p}(S_{u,\psi})$, etc. without specifying the metric with respect to which the norms are defined.

Recall that in the definition of angular regularity (Definition 2.26), we need second derivative estimates for $K_{\infty}$, which does not follow from the estimates for $\gamma_{\infty}$. We derive them in the following proposition:

Proposition 6.6. The limit $K_{\infty}$ from Proposition 6.4 satisfies

$$K_{\infty} \in L^2_u L^2_u W^{2,2}(S_{u,\psi}) \cap L^\infty_u L^2_u W^{2,2}(S_{u,\psi}).$$

Proof. We will only prove the $L^\infty_u L^2_u W^{2,2}(S_{u,\psi})$ estimate; the $L^\infty_u L^2_u W^{2,2}(S_{u,\psi})$ bound can be treated in a completely identical manner after switching $u$ and $w$.

By (3.11),

$$\|K_{n_k}\|_{L^\infty_u L^2_u W^{2,2}(S_{u,\psi})} \lesssim 1.$$

It follows from the Banach–Alaoglu theorem that for every $u \in [0, u_0)$, there exists a further subsequence of $\nabla^2 K_{n_k}$ which admits a weak $L^2_u L^2(S_{u,\psi})$ limit $\psi_{\infty}$ satisfying the estimate

$$\|\psi_{\infty}\|_{L^2_u L^2(S_{u,\psi})} \lesssim 1. \quad (6.18)$$

The weak convergence (together with Proposition 6.4) implies that $\psi_{\infty} = \nabla^2 K$. Since (6.18) moreover holds independently of $u$, this proves that $K_{\infty} \in L^\infty_u L^2_u W^{2,2}(S_{u,\psi})$. \qed

Finally, before we end this subsection, it will be convenient to use the equivalence of norms that we have established above to rephrase the compactness theorems:

Lemma 6.7. Proposition 5.4, 5.7, 5.8 and 5.9 all apply in the setting of this section with $W^{m+1,2}(S_{u,\psi}, \gamma)$, $W^{m,2}(S_{u,\psi}, \gamma)$, etc. replaced by $W^{m+1,2}(S_{u,\psi}, \gamma)$, $W^{m,2}(S_{u,\psi}, \gamma)$, etc.\qed

6.3. Limits of $b$ and $\Omega$.

Proposition 6.8. There exist $b_{\infty}$ and $\Omega_{\infty}$ such that the following hold after passing to a further subsequence $n_k$:

$$\|b_{n_k} - b_{\infty}\|_{C^1_u(C^2_u(S_{u,\psi}))} + \|b_{n_k} - b_{\infty}\|_{C^1_u(C^2_u W^{2,2}(S_{u,\psi}))} \to 0,$$

$$\|\log \frac{\Omega_{n_k}}{\Omega_{\infty}}\|_{C^1_u(C^2_u(S_{u,\psi}))} + \|\log \frac{\Omega_{n_k}}{\Omega_{\infty}}\|_{C^1_u(C^2_u W^{2,2}(S_{u,\psi}))} \to 0.$$  

Moreover, for $\theta_\infty \in \{b_{\infty}, \log \Omega_{\infty}\}$,

$$\|\theta_\infty\|_{L^\infty_u L^2_u W^{1,2}(S_{u,\psi})} + \|\mathcal{L}_{\frac{\partial}{\partial u}} \theta_\infty\|_{L^\infty_u L^2_u W^{2,2}(S_{u,\psi})} + \|\mathcal{L}_{\theta_\infty} \theta_\infty\|_{L^\infty_u L^2_u W^{2,2}(S_{u,\psi})} \lesssim 1.$$

Proof. This is an immediate consequence of Proposition 5.4 and Lemma 6.5. \qed

Remark 6.9. In particular, combining Propositions 6.2 and 6.8, it follows that $(\mathcal{M}, g_{n_k})$ has a $C^0$ limit $(\mathcal{M}, g_\infty)$ given by

$$g_\infty = -2\Omega_{\infty}(d\theta^A - b_{\infty}^A d\theta) \otimes (d\theta^B - b_{\infty}^A d\theta).$$

6.4. Limits of $\eta$ and $\eta$. We next consider the limits of $\eta$ and $\eta$. They in particular have uniform limits. More precisely,

Proposition 6.10. There exist $\eta_{\infty}$ and $\eta_{\infty}$ such that the following hold after passing to a further subsequence $n_k$:

$$\|\eta_{n_k} - \eta_{\infty}\|_{C^1_u(C^2_u(S_{u,\psi}))} + \|\eta_{n_k} - \eta_{\infty}\|_{C^1_u(C^2_u W^{1,4}(S_{u,\psi}))} \to 0,$$

$$\|\eta_{n_k} - \eta_{\infty}\|_{C^1_u(C^2_u(S_{u,\psi}))} + \|\eta_{n_k} - \eta_{\infty}\|_{C^1_u(C^2_u W^{1,4}(S_{u,\psi}))} \to 0.$$  

Moreover, for $\psi_{\infty} \in \{\eta_{\infty}, \eta_{\infty}\}$,

$$\|\psi_{\infty}\|_{L^\infty_u L^2_u W^{1,2}(S_{u,\psi})} + \|\mathcal{L}_{\frac{\partial}{\partial u}} \psi_{\infty}\|_{L^\infty_u L^2_u W^{2,2}(S_{u,\psi})} + \|\mathcal{L}_{\psi_{\infty}} \psi_{\infty}\|_{L^\infty_u L^2_u W^{2,2}(S_{u,\psi})} \lesssim 1.$$

Proof. This is an immediate consequence of the bounds (3.10), Sobolev embedding (Proposition 5.1) and Proposition 5.4 (and Lemma 6.7). \qed
Proposition 6.11. For $\psi_{\infty} \in \{\eta_{\infty}, \eta_{\infty}\}$, where $\eta_{\infty}$ and $\eta_{\infty}$ are as in Proposition 6.10, it holds that

$$\psi_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty}) \cap L^\infty u L^2 u W^{3,2}(S_{u,\infty}) \cap L^\infty u L^\infty u W^{1,2}(S_{u,\infty}).$$

Proof. Step 1: The $W^{2,2}$ estimate. By (3.10), (for $\psi_{n,k} \in \{\eta_{n,k}, \eta_{n,k}\}$) $\psi_{n,k} \in W^{2,2}(S_{u,\infty})$ uniformly in $k$, $u$ and $u$, it follows that for every $u, u$, after passing to a further subsequence, $\psi_{n,k}$ converges so some limit in $W^{2,2}(S_{u,\infty})$. It is easy to check (using (6.10) and a density argument) that this limit coincides with $\psi_{\infty}$ almost everywhere. It follows that $\psi_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty})$.

Step 2: The $W^{3,2}$ estimate. The proof is similar to that in Proposition 6.6; we omit the details. □

6.5. Weak limits of $\hat{\chi}, \omega, \hat{\chi}, \hat{\chi}, \omega$ and $\hat{\chi}$. In this subsection, we now discuss the weak limits of the Ricci coefficients $\hat{\chi}, \omega, \hat{\chi}, \hat{\chi}, \omega$ and $\hat{\chi}$. The Ricci coefficients $\hat{\chi}, \omega, \hat{\chi}, \hat{\chi}, \omega$ and $\hat{\chi}$ indeed admit weak limits. This is related to the fact that they have the weakest regularity estimates. It is also the reason for which the limit spacetime $(\mathcal{M}, g_{\infty})$ is not necessarily vacuum. On the other hand, $\hat{\chi}$ and $\hat{\chi}$ in fact have stronger convergence properties than those proven in Proposition 6.13. We will return to this in Section 6.6.

Proposition 6.12. There exist a further subsequence $n_k$ and $S$-tangent tensor fields $\hat{\chi}_{\infty}, \omega_{\infty}, \hat{\chi}_{\infty}, \hat{\chi}_{\infty}, \omega_{\infty}, \hat{\chi}_{\infty}$ such that for every $u \in [0, u]$, 

$$\nabla^i n_k \hat{\chi}_{n_k} \to W^\infty, \hat{\chi}_{\infty}, \hat{\omega} \to W^\infty \hat{\omega}, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to W^\infty W^\infty, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to W^\infty W^\infty$$

weakly in $L^2 u L^2(S_{u,\infty})$ for $i = 0, 1, 2$, and for every $u \in [0, u]$, 

$$\nabla^i n_k \hat{\chi}_{n_k} \to W^\infty, \hat{\chi}_{\infty}, \hat{\omega} \to W^\infty \hat{\omega}, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to W^\infty W^\infty, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to W^\infty W^\infty$$

weakly in $L^2 u L^2(S_{u,\infty})$ for $i = 0, 1, 2$.

Moreover, the limits satisfy $\hat{\chi}_{\infty}, \omega_{\infty}, \hat{\chi}_{\infty}, \hat{\chi}_{\infty}, \omega_{\infty}, \hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty}) \cap L^\infty u L^2 u W^{3,2}(S_{u,\infty}), \hat{\omega}_{\infty}, \hat{\omega}_{\infty}, \hat{\omega}_{\infty}, \hat{\omega}_{\infty} \in L^2 u L^2 u W^{2,2}(S_{u,\infty}) \cap C^\infty u L^2 u W^{2,2}(S_{u,\infty}), \hat{\omega}_{\infty}, \hat{\omega}_{\infty}, \hat{\omega}_{\infty}, \hat{\omega}_{\infty} \in L^2 u L^2 u W^{2,2}(S_{u,\infty}).$

Proof. We will only discuss the theorem for $\hat{\chi}$. It is easy to see that $\omega$ and $\hat{\chi}$ can be treated in exactly the same way (since $\omega$ satisfies similar estimates, and $\hat{\chi}$ satisfies even stronger bounds); while $\hat{\chi}$ and $\hat{\chi}$ can be handled similarly after changing $u$ and $u$.

Step 1: Existence of weak limit. By Proposition 5.9 with $q = 2$ (and Proposition 5.8, Lemma 6.7) and the estimates in (3.12) and (3.17), for $i = 0, 1, 2$, after passing to a subsequence $n_k$, there exists $\hat{\chi}_{\infty} \in L^\infty u L^\infty u L^\infty u L^2(S_{u,\infty}) \cap L^\infty u L^2 u L^\infty u L^2(S_{u,\infty})$ such that $\nabla^i n_k \hat{\chi}_{n_k} \to \hat{\chi}_{\infty} \in L^\infty u L^\infty u L^2(S_{u,\infty})$ for every $u$. By Proposition 6.4 and the uniqueness of distribution limits it follows that $\hat{\chi}_{\infty} = \nabla^i \hat{\chi}_{\infty}$. This shows that $\hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty}) \cap C^\infty u L^\infty u L^2(S_{u,\infty})$.

Step 2: Higher regularity of $\hat{\chi}_{\infty}$ and $\mathcal{L}_{\infty} u \hat{\chi}_{\infty}$. Step 1 in particular shows that $\hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty}) \cap C^\infty u L^2 u W^{2,2}(S_{u,\infty})$. By Proposition 5.9, we also have $\mathcal{L}_{\infty} u \hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty})$.

It therefore remains to show that $\hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{3,2}(S_{u,\infty})$. To see this, note that for every $u \in [0, u]$, the estimate (3.12) and the Banach–Alaoglu theorem imply that after passing to a further subsequence, $\nabla^2 n_k \hat{\chi}_{n_k}$ converges weakly in $L^2 u W^{3,2}(S_{u,\infty})$ to a limit $\hat{\chi}_{\infty}$ satisfying the bound (independently of $u$)

$$\hat{\chi}_{\infty} \in L^\infty u L^2(S_{u,\infty}).$$

(6.19)

The weak limit implies that $\hat{\chi}_{\infty} = \nabla^i \hat{\chi}_{\infty}$. Thus (6.19) and the fact (established above) that $\hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{2,2}(S_{u,\infty}) \cap C^\infty u L^\infty u W^{2,2}(S_{u,\infty})$ imply $\hat{\chi}_{\infty} \in L^\infty u L^\infty u W^{3,2}(S_{u,\infty})$.

Proof. Let $q \in [2, +\infty)$. There exist a further subsequence $n_k$ and functions $\hat{\chi}_{\infty}$ and $\hat{\chi}_{\infty}$ such that for every $u \in [0, u]$, 

$$\nabla^i n_k \hat{\chi}_{n_k} \to \nabla^i \hat{\chi}_{\infty}, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to \nabla^i \hat{\chi}_{\infty}, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to \nabla^i \hat{\chi}_{\infty}$$

weakly in $L^2 u L^2(S_{u,\infty})$ for $i = 0, 1, 2$, and for every $u \in [0, u]$, 

$$\nabla^i n_k \hat{\chi}_{n_k} \to \nabla^i \hat{\chi}_{\infty}, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to \nabla^i \hat{\chi}_{\infty}, \hat{\nabla}^i n_k \hat{\chi}_{n_k} \to \nabla^i \hat{\chi}_{\infty}$$

weakly in $L^2 u L^2(S_{u,\infty})$ for $i = 0, 1, 2$. 


Moreover, $\Psi \chi_\infty \in L^2_u L^\infty_u W^{2,2}(S_u, \omega) \cap C^0_u L^1_u W^{2,2}(S_u, \omega)$, $\mathcal{L}_u \Psi \chi_\infty \in L^2_u L^\infty_u W^{2,2}(S_u, \omega)$; and similarly $\Psi \chi_\infty \in L^2_u L^\infty_u W^{2,2}(S_u, \omega) \cap C^0_u L^1_u W^{2,2}(S_u, \omega) \cap L^\infty_u L^\infty_u W^{3,2}(S_u, \omega)$, $\mathcal{L}_u \Psi \chi_\infty \in L^2_u L^\infty_u W^{2,2}(S_u, \omega)$.

Proof. This can be proven in exactly the same way as Proposition 6.12, except for using the better bounds that $\Psi \chi$ and $\Psi \chi$ obey ((3.10), (3.15) and (3.16)), and applying Proposition 5.9 with a general $q$ (as opposed to only $q = 2$).

6.6. Strong limits of $\Psi \chi$ and $\Psi \chi$. In this subsection we further show strong convergence for $\Psi \chi$ and $\Psi \chi$ (in addition to Proposition 6.13 in Section 6.5); see the main results in Proposition 6.15. For this we rely on compactness of BV. First, we need the following

**Proposition 6.14** (Comparability of the BV norms). There exists $C > 0$ independent of $n$ such that the following holds for all continuous $\phi : [0, u_n] \times [0, u_n] \times S^2 \to \mathbb{R}$:

$$C^{-1} \| \phi \|_{BV(H_n, (\gamma_n)_{0,0})} \leq \| \phi \|_{BV(H_n, (\gamma_n)_{0,0})} \leq C \| \phi \|_{BV(H_n, (\gamma_n)_{0,0})},$$

and

$$C^{-1} \| \phi \|_{BV(H_n, (\gamma_n)_{0,0})} \leq \| \phi \|_{BV(H_n, (\gamma_n)_{0,0})} \leq C \| \phi \|_{BV(H_n, (\gamma_n)_{0,0})}.$$  

Proof. After recalling Definition 2.22, this is immediate from Proposition 6.1 and (6.11).

In view of Proposition 6.14, from now on we will simply write $BV(H_n)$ and $BV(H_n)$ for either of the equivalent norms.

**Proposition 6.15.** The functions $\Psi \chi_\infty$ and $\Psi \chi_\infty$ in Proposition 6.13 satisfy in addition

$$\Psi \chi_\infty \in C^0_u L^1_u W^{2,1}(S_u, \omega) \cap L^\infty_u L^\infty_u W^{3,2}(S_u, \omega),$$

$$\Psi \chi_\infty \in C^0_u L^1_u W^{2,1}(S_u, \omega) \cap L^\infty_u L^\infty_u W^{3,2}(S_u, \omega).$$

Moreover, after passing to a further subsequence $n_k$,

$$\lim_{k \to +\infty} \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^1_u W^{2,1}(S_u, \omega)} + \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^1_u W^{2,1}(S_u, \omega)} = 0. \quad (6.21)$$

and for every $p \in [1, +\infty)$,

$$\lim_{k \to +\infty} \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^1_u W^{1,p}(S_u, \omega)} \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^1_u W^{1,p}(S_u, \omega)} = 0. \quad (6.22)$$

Proof. In this proof, we are concerned with $\Psi \chi$; the proofs for statements regarding $\Psi \chi$ are similar.

Step 1: BV compactness.

Step 1(a): $C^0_u L^1_u L^1_u(S_u, \omega) \cap L^\infty_u BV(H_u)$ estimates. The bounds (3.10), (3.15), (3.16) give uniform estimates for $\Psi \chi_{n_k}$ which are sufficient to apply Proposition 5.7. Thus, by Proposition 6.14, the BV compactness theorem in Proposition 5.7 (and Lemma 6.7) and the uniqueness of (distributional) limits, we have, after passing to a further subsequence,

$$\lim_{k \to +\infty} \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^1_u L^1_u(S_u, \omega)} + \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^1_u L^1_u(S_u, \omega)} = 0. \quad (6.23)$$

and that $\Psi \chi_\infty \in C^0_u L^1_u L^1_u(S_u, \omega) \cap L^\infty_u BV(H_u)$.

Step 1(b): $C^0_u L^1_u W^{2,1}(S_u, \omega)$ estimates and proof of (6.21). We now apply the same argument as in Step 1(a) but to higher derivatives.

By (3.10), (3.15) and Proposition 6.5, we have that $(\nabla u, 0, 0, r)^2 \Psi \chi_{n_k}$ and $(\nabla u, 0, 0, r)^2 \Psi \chi_{n_k}$ are uniformly bounded in $BV(H_u)$ for all $u$. Using Propositions 6.14 and 5.7 (and Lemma 6.7), it follows that after passing to a further subsequence, $(\nabla u, 0, 0, r)^2 \Psi \chi_{n_k}$ and $(\nabla u, 0, 0, r)^2 \Psi \chi_{n_k}$ both converge in $C^0_u L^1_u L^1_u(S_u, \omega)$ to some limits. It is easy to check that these limits coincide with $(\nabla u, 0, 0, r)^2 \Psi \chi_{\infty}$ and $(\nabla u, 0, 0, r)^2 \Psi \chi_{\infty}$, which proves (6.21).

Step 2: Completion of the proof of (6.20). The only estimate not already established in Step 1 is that $\Psi \chi_\infty \in L^\infty_u L^\infty_u W^{3,2}(S_u, \omega)$. This can be proven in a similar manner as Step 1 in the proof of Proposition 6.11. We omit the details.

Step 3: Proof of (6.22). Let $q \in [2, +\infty)$. By Proposition 6.13, $\Psi \chi_\infty \in C^0_u L^q_u W^{2,2}(S_u, \omega)$. Hence by Sobolev embedding (using Propositions 5.1 and 6.3), $\Psi \chi_\infty \in C^0_u L^q_u W^{1,q}(S_u, \omega)$. Combining with the estimate (3.10), it follows that

$$\sup_k \| \Psi \chi_{n_k} - \Psi \chi_\infty \|_{C^0_u L^q_u W^{1,q}(S_u, \omega)} \leq 1. \quad (6.24)$$
By Hölder’s inequality, for any \( p \in [1, +\infty) \), \( q \in [2, +\infty) \) with \( p < q \).

\[
\|\nabla \chi_{n_k} - \nabla \chi_{\infty}\|_{C_0^0 L^2_x L^{1,p}(S_{n_k})} \leq \|\nabla \chi_{n_k} - \nabla \chi_{\infty}\|_{C_0^0 L^2_x L^{1,p}(S_{n_k})} \|\nabla \chi_{n_k} - \nabla \chi_{\infty}\|_{C_0^0 L^2_x L^{1,p}(S_{n_k})}.
\]

Given any \( p \in [1, +\infty) \), we can choose \( q \in [2, +\infty) \) with \( p < q \) so that (6.21) and (6.24) imply \( \|\nabla \chi_{n_k} - \nabla \chi_{\infty}\|_{C_0^0 L^2_x L^{1,p}(S_{n_k})} \to 0 \). We have thus obtained (6.22). \( \square \)

**Remark 6.16.** Notice that while \( \nabla \chi_{\infty} \) and \( \nabla \chi_{\Sigma} \) are a.e. bounded (by (6.20) and Sobolev embedding), they are not necessarily continuous. Nonetheless, they have well-defined traces in the sense of Lemma 2.24.

6.7. **Compensated compactness.** While \( \check{\chi}, \omega, \check{\chi} \) and \( \omega \) only admit weak limits (see Section 6.5), it is important that there is a compensated compactness phenomenon for some quadratic products of them. Our main result of this subsection in in Proposition 6.18, after we state a more general compensated compactness lemma (Lemma 6.17). The proof of Lemma 6.17 is relegated to Appendix B.

**Lemma 6.17.** Let \( B_{\mathbb{R}^2}(0, R) \subset \mathbb{R}^2 \) be the ball of radius \( R \) in \( \mathbb{R}^2 \). Suppose there are two sequences of functions \( \{f_n\}_{n=1}^{\infty}, \{h_n\}_{n=1}^{\infty} \subset L^2([0, u_*] \times [0, \underline{u}], \times B_{\mathbb{R}^2}(0, R)) \) such that the following hold:

1. There exist \( L^2([0, u_*] \times [0, \underline{u}], \times B_{\mathbb{R}^2}(0, R)) \) functions \( f_\infty \) and \( h_\infty \) such that \( f_n \to f_\infty \) and \( h_n \to h_\infty \) weakly in \( L^2([0, u_*] \times [0, \underline{u}], \times B_{\mathbb{R}^2}(0, R)) \).

2. There exists \( C_0 > 0 \) such that

\[
\sup_n \sum_{i \leq j \leq k \leq 0} \int_{0}^{u_*} \int_{0}^{u_*} \int_{B_{\mathbb{R}^2}(0, R)} \left( \frac{\partial}{\partial u} \right)^i \left( \frac{\partial}{\partial y} \right)^j \left( \frac{\partial}{\partial y^2} \right)^k f_n \, dy \, du \, dw \leq C_0 \tag{6.25}
\]

and

\[
\sup_n \sum_{i \leq j \leq k \leq 0} \int_{0}^{u_*} \int_{0}^{u_*} \int_{B_{\mathbb{R}^2}(0, R)} \left( \frac{\partial}{\partial u} \right)^i \left( \frac{\partial}{\partial y} \right)^j \left( \frac{\partial}{\partial y^2} \right)^k h_n \, dy \, du \, dw \leq C_0 \tag{6.26}
\]

Then, after passing to subsequences \( \{f_{n_k}\}_{k=1}^{\infty} \) and \( \{h_{n_k}\}_{k=1}^{\infty} \), \( f_{n_k}h_{n_k} \to f_\infty h_\infty \) weakly in \( L^2([0, u_*] \times [0, \underline{u}], \times B_{\mathbb{R}^2}(0, R)) \).

**Proposition 6.18.** After passing to a subsequence \( \{n_k\}_{k=1}^{\infty} \), \( (\check{\chi}_{n_k})_{AB}(\check{\Sigma}_{n_k})_{CD} \) converges weakly in \( L^2_x L^2_x L^2(S) \) to \( (\check{\chi}_{\infty})_{AB}(\check{\Sigma}_{\infty})_{CD} \), i.e. for any contravariant \( S \)-tangent 4-tensor \( \varphi_{ABCD} \in L^2_x L^2_x L^2(S) \),

\[
\int_{[0, u_*] \times [0, \underline{u}], \times B_{\mathbb{R}^2}(0, R)} \varphi_{ABCD}(\check{\chi}_{n_k})_{AB}(\check{\Sigma}_{n_k})_{CD} \, dA_{\infty} \, du \, dw \to \int_{[0, u_*] \times [0, \underline{u}], \times B_{\mathbb{R}^2}(0, R)} \varphi_{ABCD}(\check{\chi}_{\infty})_{AB}(\check{\Sigma}_{\infty})_{CD} \, dA_{\infty} \, du \, dw.
\]

Similarly, after passing to a subsequence \( \{n_k\}_{k=1}^{\infty} \), \( (\check{\Sigma}_{n_k})_{AB}(\check{\chi}_{n_k})_{CD} \) converges weakly in \( L^2_x L^2_x L^2(S) \) to \( (\check{\Sigma}_{\infty})_{AB}(\check{\chi}_{\infty})_{CD} \) and \( (\check{\chi}_{n_k})_{AB}(\check{\Sigma}_{n_k})_{CD} \) also respectively converge weakly in \( L^2_x L^2_x L^2(S) \) to \( (\check{\chi}_{\infty})_{AB}(\check{\Sigma}_{\infty})_{CD} \) and \( (\check{\Sigma}_{\infty})_{AB}(\check{\chi}_{\infty})_{CD} \).

**Proof.** Suffices to work component-wise and in a local coordinate chart \( U \). The bounds (3.12) and (3.13) (together with (3.9)) imply that in local coordinates, the following estimates hold:

\[
\sup_n \sum_{i \leq j \leq k \leq 2} \int_{U} \left( \frac{\partial}{\partial u} \right)^i \left( \frac{\partial}{\partial \theta} \right)^j \left( \frac{\partial}{\partial \theta^2} \right)^k (\check{\chi}_{n_k})_{AB} \, dx \, dy \, dz < +\infty
\]

and

\[
\sup_n \sum_{i \leq j \leq k \leq 2} \int_{U} \left( \frac{\partial}{\partial u} \right)^i \left( \frac{\partial}{\partial \theta} \right)^j \left( \frac{\partial}{\partial \theta^2} \right)^k (\check{\Sigma}_{n_k})_{AB} \, dx \, dy \, dz < +\infty
\]

The assertion of the proposition therefore follows from Lemma 6.17. \( \square \)

6.8. **Weak limits of \( \Omega^2 \chi_n^2 \) and \( \Omega^2 \chi_{\Sigma}^2 \).** In this subsection, we discuss the weak limits of \( |\chi_n|^2 \) and \( |\chi_{\Sigma}|^2 \). Notice that unlike \( \chi_n \) and \( \chi_{\Sigma} \) themselves, \( |\chi_n|^2 \) and \( |\chi_{\Sigma}|^2 \) are only in \( L^1 \) (and not in \( L^2 \)). We therefore can only hope to obtain weak limits as measures. For this purpose, our main tool will be Proposition 5.8.
Proposition 6.19. For every \( u \in [0, u_*] \), there exists a non-negative Radon measure \( d\nu_u \) on \( (0, u_*) \times S^2 \), which is uniformly bounded and continuous in \( u \) (see (5.11) and (5.12)), such that after passing to a subsequence \( n_k \), the following convergences hold for every bounded \( \varphi \in C^0(\{u\} \times (0, u_*) \times S^2) \):

\[
\int_{\{u\} \times (0, u_*) \times S^2} \varphi \, d\nu_u = \lim_{k \to +\infty} \left( \int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{n_k}^2 |\hat{\chi}_{n_k}|^2 \gamma_{n_k} \, dA_{\gamma_{n_k}} \, du \right) - \int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{\infty}^2 |\hat{\chi}_{\infty}|^2 \gamma_{\infty} \, dA_{\gamma_{\infty}} \, du.
\]

(6.27)

Similarly, there exists a non-negative Radon measure \( d\nu_u \), which is uniformly bounded and continuous in \( u \), such that after passing to a further subsequence, the following convergences hold for every bounded \( \varphi \in C^0((0, u_*) \times \{0\} \times S^2) \):

\[
\int_{\{u\} \times (0, u_*) \times S^2} \varphi \, d\nu_u = \lim_{k \to +\infty} \left( \int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{n_k}^2 |\hat{\chi}_{n_k}|^2 \gamma_{n_k} \, dA_{\gamma_{n_k}} \, du \right) - \int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{\infty}^2 |\hat{\chi}_{\infty}|^2 \gamma_{\infty} \, dA_{\gamma_{\infty}} \, du.
\]

(6.28)

Proof. We will only prove the statements concerning \( d\nu_u \); \( d
\nu_u \) can be handled similarly.

To show (6.27), we first use Proposition 5.8 (and Lemma 6.7) with \( \psi_{n_k} = \Omega_{n_k}^2 |\hat{\chi}_{n_k} - \hat{\chi}_{\infty}|^2 \gamma_{n_k} \sqrt{\det \gamma_{n_k}} \) \( \sqrt{\det \gamma_{0,0}_{\infty}} \) Note that by (3.9) and (3.12), \( \{\psi_{n_k}\}_{k=1}^{\infty} \) indeed obeys the assumptions of Proposition 5.8. Hence we conclude that there is a further subsequence and a (scalar-valued) Radon measure \( d\nu_u \) which is uniformly bounded and continuous in \( u \) such that for every real-valued function bounded \( \varphi \in C^0((0, u_*) \times \{0\} \times S^2) \),

\[
\int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{n_k}^2 |\hat{\chi}_{n_k} - \hat{\chi}_{\infty}|^2 \gamma_{n_k} \sqrt{\det \gamma_{n_k}} \sqrt{\det \gamma_{0,0}_{\infty}} \, dA_{\gamma_{n_k}} \, du \to \int_{\{u\} \times (0, u_*) \times S^2} \varphi \, d\nu_u.
\]

Note that \( d\nu_u \) as defined is manifestly non-negative. Then noticing that \( \sqrt{\det \gamma_{n_k}} \sqrt{\det \gamma_{0,0}_{\infty}} \, dA_{\gamma_{n_k}} = dA_{\gamma_{\infty}} \), and using the convergence statements in Propositions 6.8 and 6.12, it follows that \( d\nu_u \) indeed satisfies (6.27).

\[\square\]

Proposition 6.20. \( \{d\nu_u\}_{u \in [0, u_]} \), \( \{d\nu_u\}_{u \in \{0\} \times S^2} \) is angularly regular in the sense of Definition 2.36.

Proof. As in the previous proposition, we consider only \( d\nu_u \); the case for \( d
\nu_u \) is similar.

That \( d\nu_u \) is continuous in \( u \) has already been established in Proposition 6.19. It thus remains to bound each of the terms in (2.25), which will be carried out in the three steps below.

**Step 1: Estimating the first term in (2.25).** Let \( \varphi \in \mathcal{X}_u \). By density we can assume that \( \varphi \) is smooth. Hence, by (6.27), for every \( u \),

\[
\left| \int_{\{u\} \times (0, u_*) \times S^2} \varphi \, d\nu_u \right| \\
\leq \limsup_{k \to +\infty} \left( \int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{n_k}^2 |\hat{\chi}_{n_k}|^2 \gamma_{n_k} \, dA_{\gamma_{n_k}} \, du \right) + \int_{\{u\} \times (0, u_*) \times S^2} \varphi \Omega_{\infty}^2 |\hat{\chi}_{\infty}|^2 \gamma_{\infty} \, dA_{\gamma_{\infty}} \, du
\]

(6.29)

\[\leq \|\varphi\|_{L^2(S_{\infty}^+)} \left( \limsup_{k \to +\infty} \|\Omega_{n_k} \|_{C^0(L^\infty(S_{\infty}^+))} + \|\Omega_{\infty} \|_{C^0(L^\infty(S_{\infty}^+))} \right)
\times \left( \limsup_{k \to +\infty} \|\hat{\chi}_{n_k} \|_{L^2(S_{\infty}^+)} + \|\hat{\chi}_{\infty} \|_{L^2(L^\infty(S_{\infty}^+))} \right) \lesssim \|\varphi\|_{L^2(S_{\infty}^+)} \lesssim 1,\]

where in the last line we used the estimates in Propositions 6.8 and 6.12.

**Step 2: Estimating the second term in (2.25).** Let \( \hat{\chi} \in \mathcal{X}_u \). As in Step 1, we assume by density that \( \hat{\chi} \) is smooth. The main difference with Step 1 is that we need to integrate by parts in the angular direction.
to handle the additional derivative. By (6.27), for every $u$,

$$\left| \int \left. d\nu_\infty X \right|_{\{u\} \times S^2} \right|$$

$$\leq \limsup_{k \to +\infty} \left| \int \left. (d\nu_\infty X) \Omega^2_{nk} |\chi_{nk}|^2_{L^2} dA_{\gamma_{nk}} \right| + \int \left. (d\nu_\infty X) \Omega^2_{\infty} |\bar{\chi}|^2_{L^\infty} dA_{\gamma_{\infty}} \right|$$

$$\leq \limsup_{k \to +\infty} \int \{u\} \times S^2 \left| (d\nu_\infty X) \Omega^2_{nk} |\chi_{nk}|^2_{L^2} dA_{\gamma_{nk}} \right| + \int \left. (d\nu_\infty X) \Omega^2_{\infty} |\bar{\chi}|^2_{L^\infty} dA_{\gamma_{\infty}} \right|$$

$$\leq \|X\|_{L^\infty L^1(S_\omega)} \times (1 + \| (\nabla_{\infty})_A - (\nabla_{nk})_A \|_{L^2 L^1(S_\omega)})$$

$$\times \limsup_{k \to +\infty} \| \Omega_{nk} \|_{L^\infty L^1(S_\omega)} \times \| (\nabla_{\infty})_{AB} \|_{L^2 L^1(S_\omega)}$$

$$\times \limsup_{k \to +\infty} \| \bar{\chi}_{nk} \|_{L^\infty L^2 W^{1,\infty}(S_\omega)} + \| \bar{\chi}_\infty \|_{L^\infty L^2 W^{1,\infty}(S_\omega)} \lesssim 1,$$

where in the last line we have used the estimates established in Propositions 6.4, 6.8 and 6.12.

**Step 3: Estimating the third term in (2.25).** To estimate the third term, we carry out an integration by parts argument as in Step 2. The only difference is that we have higher derivatives, e.g. a term $(\nabla_{\infty})^2 \bar{\chi}_{\infty}$, which can only be controlled in $L^\infty L^2 L^1(S_\omega)$ by Proposition 6.12 (but not $L^\infty L^2 L^\infty(S_\omega)$). It is for this reason that we need to assume control of $\|X \otimes Y\|_{L^\infty L^2 L^2(S_\omega)}$. We omit the straightforward details, but will just demonstrate this with one of the most difficult terms:

$$\left| \int \left. X^A Y^B (\nabla_{\infty})_{BA}(\Omega^2_{\infty} |\bar{\chi}|^2_{L^\infty}) dA_{\gamma_{\infty}} \right|$$

$$\lesssim \|X \otimes Y\|_{L^\infty L^2 L^1(S_\omega)} \| (\nabla_{\infty})_{BA} \|_{L^\infty L^2 L^1(S_\omega)} \lesssim 1,$$

where we have used Propositions 6.8 and 6.12.

**At this point, we fix the subsequence $n_k$ such that Propositions 6.2, 6.4, 6.8, 6.10, 6.12, 6.13, 6.15, 6.18 and 6.19 hold.** Along this subsequence, the spacetime metrics converge uniformly to a limiting spacetime $(\mathcal{M}, g_\infty)$, with additional refined convergence for the Ricci coefficients as described by the propositions above. Moreover, combining the above propositions with Propositions 6.6 and 6.11, it also follows that the limit $(\mathcal{M}, g_\infty)$ is angularly regular (see Definition 2.26).

7. **The equations satisfied by the limit spacetime and the proof of Theorem 4.1**

We continue to work under the assumptions of Theorem 4.1, and take $n_k$ as in the end of the last section.

Using the properties of the limits that we showed in the previous section, we now derive the equations satisfied by the various limiting quantities.

- In **Section 7.1**, we derive the transport equations for the metric components.
- In **Section 7.2** and **Section 7.3**, we derive the transport equations for the Ricci coefficients. These equations correspond exactly to a description of the (weak) Ricci curvature.
- In **Section 7.4**, we prove the propagation equation for the null dust.
- Finally, in **Section 7.5** and **Section 7.6**, we derive the higher order equations for the Ricci coefficients and renormalized Bianchi equations respectively.

Combined with the results in the previous section, we will then complete the proof of Theorem 4.1 in **Section 7.7**.
7.1. Equations for the metric components. In Section 6, we defined

- \((\gamma_\infty, b_\infty, \Omega_\infty)\), which are understood as subsequential uniform limits of the metric components, and
- \((\chi_\infty, \chi_\infty, \eta_\infty, \eta_\infty, \omega_\infty, \omega_\infty)\), which are understood as subsequential weak limits of the Ricci coefficients.

It is easy to check that they are related in the expected manner, i.e. that \((\chi_\infty, \chi_\infty, \eta_\infty, \eta_\infty, \omega_\infty, \omega_\infty)\) are indeed the Ricci coefficients associated to the limiting metric, i.e.

**Proposition 7.1.** The equations (2.10) and (2.11) hold when we

- take the metric components to be \((\gamma_\infty, b_\infty, \Omega_\infty)\) (given by Propositions 6.2 and 6.8) and
- take the Ricci coefficients to be \((\chi_\infty, \chi_\infty, \eta_\infty, \eta_\infty, \omega_\infty, \omega_\infty)\) (given by Propositions 6.10, 6.13 and 6.12),
- where all derivatives are to be understood as weak derivatives.

**Proof.** This follows easily from the uniqueness of limits in the sense of distributions. \(\square\)

Proposition 7.1 immediately implies the following transport equations for the metric components.

**Proposition 7.2.** The equations

\[
\nabla_4 \gamma = 0, \quad \nabla_4 b = -2\Omega(\eta - \bar{\eta}) + \chi \cdot b, \quad \nabla_4 \log \Omega = -2\omega
\]

hold in the integrated sense (see Definition 2.30).

**Proof.** The last equation here is just the first equation in (2.11). The other two equations follow from (2.10) and the expression of \(\nabla_4\) in (2.7).

Using Proposition 7.1, we have thus shown that all these equations hold when the derivatives are understood as weak derivatives. Together with the regularity of the metric components and the Ricci coefficients we have derived in the previous section, it follows that these equations are also satisfied in the integrated sense. \(\square\)

In fact, we can also derive transport equations for the derivatives of the metric components.

**Proposition 7.3.** The equations (2.37)–(2.39) hold for \((\mathcal{M}, g_\infty)\) in the integrated sense (see Definition 2.30).

**Proof.** We start from the fact that (by Proposition 2.43), the equations (2.37)–(2.39) hold classically, and hence also in the integrated sense, for \((\mathcal{M}, g_{n_k})\) for all \(n_k \in \mathbb{N}\). Taking (2.37) as an example, we have that for any rank-3 \(C^1\) tensor \(\varphi\),

\[
\int_{S_{u, w}} \langle \varphi, (\nabla_4 \gamma)_{n_k} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}} - \int_{S_{u, w}} \langle \varphi, (\nabla_4 \gamma)_{n_k} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}}
\]

\[
= \int_{S_{u, \omega}} \left( \langle \varphi, (\nabla_4 \gamma)_{n_k} \rangle \Omega_{n_k} \right) dA_{\gamma_{n_k}} + \int_{S_{u, \omega}} \langle \varphi, (\nabla_4 \gamma)_{n_k} \rangle \Omega_{n_k}^2 \, dA_{\gamma_{n_k}}
\]

(7.1)

Now we want to pass to the \(k \to +\infty\) limit to obtain (2.37) in the integrated sense. By Propositions 6.2 and 6.8, \(\gamma_{n_k}, (\nabla_4 \gamma)_{n_k}, \Omega_{n_k}\) and \(dA_{\gamma_{n_k}}\) have uniform limits which allow us to take \(k \to +\infty\). Similarly, \(\nabla_4 \gamma_{n_k}\) also has a strong limit \(C^0 L^2_\infty L^\infty(S_{u, w})\) by (6.22), allowing us to pass to \(k \to +\infty\). The only term without a strong limit is therefore \(\omega_{n_k}\), which, by Proposition 6.12, has a weak \(L^2_\infty L^2(S_{u, w})\) limit (for every fixed \(u\)). Nevertheless, \(\omega_{n_k}\) is multiplied by \((\nabla_4 \gamma)_{n_k} \Omega_{n_k}^2 \, dA_{\gamma_{n_k}}\) which has a uniform limit so that

\[
\int_{S_{u, \omega}} \langle \varphi, \omega_{n_k} (\nabla_4 \gamma)_{n_k} \rangle \, dA_{\gamma_{n_k}} \, dA' \to \int_{S_{u, \omega}} \langle \varphi, \omega_\infty (\nabla_4 \gamma)_{\infty} \rangle \, dA_{\gamma_{\infty}} \, dA'.
\]

Combining all these observations, we can pass (7.1) to the \(k \to +\infty\) limit to obtain (2.37) in the integrated sense.

The equations (2.38) and (2.39) can be proven similarly, noting that in both cases the only terms without a strong limit involve \(\chi_{n_k}\) or \(\omega_{n_k}\), but they are multiplied by terms which have uniform limits. We omit the details. \(\square\)
7.2. The vanishing (weak) Ricci curvature components of the limit spacetime.

Proposition 7.4. The equations (2.15)–(2.16) hold for \((M, g_{\infty})\) in the integrated sense (see Definition 2.30).

Proof. Let us only consider (2.15). The equation (2.16) can be treated in essentially the same manner. According to Definition 2.30, we need to show, for all \(S\)-tangent vector field \(\varphi \in C^1\),
\[
\int_{S_{u,21}} \langle \varphi, \eta_n \rangle \Omega_n \ dA_{\gamma_n} - \int_{S_{u,22}} \langle \varphi, \eta_\infty \rangle \Omega_\infty \ dA_{\gamma_\infty}
\]
\[
= \int_{S_{u,21}} \langle \varphi, \nabla_{\infty} \chi_\infty \rangle - \frac{1}{2} \nabla_{\infty} \cdot \chi_\infty + \frac{1}{4} \nabla_{\infty} \chi_\infty \cdot \nabla_{\infty} \chi_\infty + \frac{1}{4} \chi_\infty \eta_\infty \rangle \Omega^2_\infty \ dA_{\gamma_\infty} \ d\nu' + \int_{S_{u,22}} \langle (\nabla_4)_{\infty} \varphi, \eta_\infty \rangle \Omega^2_\infty \ dA_{\gamma_\infty} \ d\nu'.
\]

(7.2)

Since \((M, g_{n_k})\) is a smooth solution to the Einstein vacuum equations, by Proposition 2.29,
\[
\int_{S_{u,21}} \langle \varphi, \eta_{n_k} \rangle \Omega_{n_k} \ dA_{\gamma_{n_k}} - \int_{S_{u,22}} \langle \varphi, \eta_{n_k} \rangle \Omega_{n_k} \ dA_{\gamma_{n_k}}
\]
\[
= \int_{S_{u,21}} \langle \varphi, \nabla_{n_k} \chi_{n_k} \rangle - \frac{1}{2} \nabla_{n_k} \cdot \chi_{n_k} - \frac{1}{2} (\eta - \eta_{n_k}) \cdot \nabla_{n_k} \chi_{n_k} + \frac{1}{4} \chi_{n_k} \eta_{n_k} \rangle \Omega^2_{n_k} \ dA_{\gamma_{n_k}} \ d\nu' + \int_{S_{u,22}} \langle (\nabla_4)_{n_k} \varphi, \eta_{n_k} \rangle \Omega^2_{n_k} \ dA_{\gamma_{n_k}} \ d\nu'.
\]

(7.3)

Our goal now is to pass to the \(k \to +\infty\) limit in (7.3) to obtain (7.2). The terms on the LHS of (7.3) are easy, since by Proposition 6.2, 6.8 and 6.10, all of \(\gamma_{n_k}, \Omega_{n_k}\) and \(\eta_{n_k}\) converge uniformly to their limits. We therefore have
\[
\int_{S_{u,21}} \langle \varphi, \eta_{n_k} \rangle \Omega_{n_k} \ dA_{\gamma_{n_k}} - \int_{S_{u,22}} \langle \varphi, \eta_{n_k} \rangle \Omega_{n_k} \ dA_{\gamma_{n_k}}
\]
\[
\to \int_{S_{u,21}} \langle \varphi, \eta_\infty \rangle \Omega_\infty \ dA_{\gamma_\infty} - \int_{S_{u,22}} \langle \varphi, \eta_\infty \rangle \Omega_\infty \ dA_{\gamma_\infty}.
\]

(7.4)

The terms on the RHS of (7.3) are not much more difficult. To proceed, let us first recall from Propositions 6.2 and 6.8 that the metric components converge uniformly to their limit so that we can focus on taking the limits of the Ricci coefficients. There are now three types of terms to understand:

(1) Terms in which \(\varphi\) is contracted with \(\nabla_{n_k} \chi_{n_k}\). For these terms, we use that \(\nabla_{n_k} \chi_{n_k}\) converges weakly in \(L^2(\gamma_{n_k})\) for all fixed \(u\) (Propositions 6.12 and 6.13). Hence,
\[
\int_{S_{u,21}} \langle \varphi, \nabla_{n_k} \chi_{n_k} \rangle - \frac{1}{2} \nabla_{n_k} \cdot \chi_{n_k} + \frac{1}{4} \chi_{n_k} \eta_{n_k} \rangle \Omega^2_{n_k} \ dA_{\gamma_{n_k}} \ d\nu'
\]
\[
\to \int_{S_{u,21}} \langle \varphi, \nabla_{\infty} \chi_\infty \rangle - \frac{1}{2} \nabla_{\infty} \cdot \chi_\infty + \frac{1}{4} \chi_{\infty} \eta_\infty \rangle \Omega^2_\infty \ dA_{\gamma_\infty} \ d\nu'.
\]

(7.5)

(2) Terms in which \(\varphi\) is contracted with a product of two Ricci coefficients. There are in turn two types of terms: (a) a product of two Ricci coefficients, each of which converges in the \(C^0L^2(\gamma_{n_k})\) norm, e.g. \(\chi_{n_k} \eta_{n_k}\); (b) a product of two Ricci coefficients, one of which converges weakly in \(L^2(\gamma_{n_k})\) for every \(u\) and the other converges in the \(C^0L^2(\gamma_{n_k})\) norm, e.g. \(\eta_{n_k} \cdot \chi_{n_k}\). In either case, we know that the product converges weakly in \(L^2(\gamma_{n_k})\) to the product of the limits for every \(u \in [0, u_*]\). This implies
\[
\int_{S_{u,21}} \langle \varphi, \frac{1}{2} (\eta - \eta_{n_k}) \cdot \nabla_{n_k} \chi_{n_k} + \frac{1}{4} \chi_{n_k} \eta_{n_k} + \frac{3}{4} \chi_{n_k} \eta_{n_k} - 2 \omega_{n_k} \eta_{n_k} \rangle \Omega^2_{n_k} \ dA_{\gamma_{n_k}} \ d\nu'
\]
\[
\to \int_{S_{u,21}} \langle \varphi, \frac{1}{2} (\eta - \eta_{\infty}) \cdot \nabla_{\infty} \chi_{\infty} + \frac{1}{4} \chi_{\infty} \eta_{\infty} + \frac{3}{4} \chi_{\infty} \eta_{\infty} - 2 \omega_{\infty} \eta_{\infty} \rangle \Omega^2_\infty \ dA_{\gamma_\infty} \ d\nu'.
\]

(7.6)

(3) Terms in which \((\nabla_4)_{n_k} \varphi\) is contracted with \(\eta_{n_k}\). Expanding \((\nabla_4)_{n_k} \varphi\) using (2.7), we see from Propositions 6.2, 6.8, 6.13 and 6.12 that \((\nabla_4)_{n_k} \varphi \to (\nabla_4)_{\infty} \varphi\) weakly in \(L^2(\gamma_{n_k})\) for every
$u \in [0, u_*]$. Now this is contracted with $\eta_{n_k}$, which tends to $\eta_\infty$ in the $C^0_uC^0_\Omega L^2(S_{u_*})$ norm by Proposition 6.10. It follows that
\begin{equation}
\int_{S_{u_*}} \int_{S_{u'_*}} \langle (\nabla_4)_{n_k} \nu, \eta_{n_k} \rangle \Omega^2_{n_k} \, dA_{\gamma_{n_k}} \, du' \to \int_{S_{u_*}} \int_{S_{u'_*}} \langle (\nabla_4)_{\infty} \nu, \eta_\infty \rangle \Omega^2_{\infty} \, dA_{\gamma_\infty} \, du' \tag{7.7}
\end{equation}
Combining (7.4)–(7.7), we therefore deduce (7.2) from (7.3) after taking $k \to +\infty$, as desired. □

**Proposition 7.5.** The equations (2.17)–(2.22) hold for $(\mathcal{M}, g_\infty)$ in the weak integrated sense (see Definition 2.31).

**Proof.** The proof is in fact quite similar to that of Proposition 7.4, with two exceptions:

- The equations are now in weak integrated (as opposed to integrated) form.
- We also use the compensated compactness property in Proposition 6.18.

Since the equations (2.17)–(2.22) can in fact all be treated in a similar fashion, we will only discuss (2.20) in detail.

Recalling Definition 2.31, our goal will be to show that for all contravariant $S$-tangent 2-tensor field $\varphi \in C^1$, 
\begin{equation}
\int_{0}^{u_*} \int_{S_{u'_*}} \langle \varphi, \tilde{\chi}_{n_k} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}} \, du' - \int_{0}^{u_*} \int_{S_{u'_*}} \langle \varphi, \tilde{\chi}_{\infty} \rangle \Omega \, dA_{\gamma_\infty} \, du' \tag{7.8}
\end{equation}
In a similar spirit as the proof of Proposition 7.4, we will derive (7.8) by taking the $k \to +\infty$ limit of the following equation, which holds thanks to Proposition 2.29:
\begin{equation}
\int_{0}^{u_*} \int_{S_{u'_*}} \langle \varphi, \tilde{\chi}_{n_k} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}} \, du' - \int_{0}^{u_*} \int_{S_{u'_*}} \langle \varphi, \tilde{\chi}_{\infty} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}} \, du' \tag{7.9}
\end{equation}
We proceed, again, as in the proof of Proposition 7.4. First, by Propositions 6.2, 6.8 and 6.12,
\begin{equation}
\int_{0}^{u_*} \int_{S_{u'_*}} \langle \varphi, \tilde{\chi}_{n_k} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}} \, du' - \int_{0}^{u_*} \int_{S_{u'_*}} \langle \varphi, \tilde{\chi}_{\infty} \rangle \Omega_{n_k} \, dA_{\gamma_{n_k}} \, du' \tag{7.10}
\end{equation}
For the terms on the RHS of (7.9), we split into three types of terms as in Proposition 7.4. Again, we note that we can focus on the limits of the Ricci coefficients in view of Propositions 6.2 and 6.8.

(1) Terms in which $\varphi$ is contracted with $\nabla_{n_k} \eta_{n_k}$. Using Proposition 6.10 (in addition to Propositions 6.2 and 6.8),
\begin{equation}
\int_{0}^{u_*} \int_{u_*}^{u'_*} \int_{S_{u'_*}} \langle \varphi, \nabla_{n_k} \tilde{\chi}_{n_k} \eta_{n_k} \rangle \Omega^2_{n_k} \, dA_{\gamma_{n_k}} \, du' \tag{7.11}
\end{equation}
(2) Terms in which $\varphi$ is contracted with a product of two Ricci coefficients. As in the proof of Proposition 7.4, it can be checked that in any such products, at least one term has a strong $L^2_uL^2_\nu L^2(S_{u_*})$ limit while the other term has at least a weak $L^2_uL^2_\nu L^2(S_{u_*})$ limit. Thus, using...
and Propositions 6.2, 6.8, 6.10, 6.12, 6.13 and 6.15, we obtain
\[
\int_{0}^{u_{2}} \int_{u_{1}}^{u_{2}} \int_{S_{u_{1}} \setminus U} (\langle \frac{1}{2} \tilde{\nabla}_{n} \tilde{\chi}_{n}, \tilde{\chi}_{n} \rangle_{\ast} - \frac{1}{2} \tilde{\nabla}_{n} \tilde{\chi}_{n} + \gamma_{n} \tilde{\nabla}_{n} \tilde{\gamma}_{n}, \Omega_{n}^{2} \ast dA_{n} \ast dA_{\gamma_{n}} \ast d\mu) du_{1} du_{2} du
\]
which is defined to be the following (Lipschitz) cutoff function:

If \( 0 \leq u_{1} < u_{2} \leq u_{*} \).

We begin with the fact that (2.14) holds for all \((M, g_{n})\). Take a \(C^{1}\) function \(\varphi : [0, u_{*}] \times S^{2} \rightarrow \mathbb{R}\). Multiply (2.14) by \(\varphi(u, \theta)\xi_{\ell}(u)\), where for every \(\ell \in \mathbb{N}\) with \(\ell^{-1} \leq u_{*} - 2u_{1}\), \(\xi_{\ell} : [0, u_{*}] \rightarrow \mathbb{R}\) is defined to be the following (Lipschitz) cutoff function:

\[
\xi_{\ell}(u) := \begin{cases}
0 & \text{if } u \in [0, u_{1}) \\
\ell(u - u_{1}) & \text{if } u \in [u_{1}, u_{1} + \ell^{-1}) \\
1 & \text{if } u \in [u_{1} + \ell^{-1}, u_{2} - \ell^{-1}) \\
-\ell(u - u_{2}) & \text{if } u \in [u_{2} - \ell^{-1}, u_{2}) \\
0 & \text{if } u \in [u_{2}, u_{*})
\end{cases}
\] (7.14)

Therefore, the following holds for every \(u \in [0, u_{*}]\):

\[
\ell \int_{u_{2}}^{u_{2} - \ell^{-1}} \int_{S_{u_{1}} \setminus U} \varphi \Omega_{n} \tilde{\nabla}_{n} \tilde{\chi}_{n} dA_{n} d\mu - \ell \int_{u_{1}}^{u_{1} + \ell^{-1}} \int_{S_{u_{1}} \setminus U} \varphi \Omega_{n} \tilde{\nabla}_{n} \tilde{\chi}_{n} dA_{n} d\mu
\]

We now take \(k \rightarrow +\infty\). Note that we cannot just replace the \(n_{k}\)'s in (7.15) by \(\infty\) because of the term \(|\tilde{\nabla}_{n_{k}}^{2} |_{\gamma_{n_{k}}} \Omega_{n_{k}}^{2} \) (see Proposition 6.19). On the other hand, in all the other terms which are quadratic in the Ricci coefficients, there must be at least one fact which has a strong \(L^{2}_{2} L_{2}(U_{n_{k}})\) limit (for every

7.3. The non-vanishing (weak) Ricci curvature components of the limit spacetime.

**Proposition 7.6.** \(\psi \hat{\chi}_{\infty}^{\pm}\) and \(\psi \tilde{\chi}_{\infty}^{\pm}\) satisfy (2.27) and (2.28) respectively.

**Proof.** We will only derive equation (2.27) for \(\psi \hat{\chi}_{\infty}^{\pm}\); the equation (2.28) for \(\psi \tilde{\chi}_{\infty}^{\pm}\) is similar (and simpler).

Fix \(0 \leq u_{1} < u_{2} \leq u_{*}\).

We begin with the fact that (2.14) holds for all \((M, g_{n})\). Take a \(C^{1}\) function \(\varphi : [0, u_{*}] \times S^{2} \rightarrow \mathbb{R}\). Multiply (2.14) by \(\varphi(u, \theta)\xi_{\ell}(u)\), where for every \(\ell \in \mathbb{N}\) with \(\ell^{-1} \leq u_{*} - 2u_{1}\), \(\xi_{\ell} : [0, u_{*}] \rightarrow \mathbb{R}\) is defined to be the following (Lipschitz) cutoff function:

\[
\xi_{\ell}(u) := \begin{cases}
0 & \text{if } u \in [0, u_{1}) \\
\ell(u - u_{1}) & \text{if } u \in [u_{1}, u_{1} + \ell^{-1}) \\
1 & \text{if } u \in [u_{1} + \ell^{-1}, u_{2} - \ell^{-1}) \\
-\ell(u - u_{2}) & \text{if } u \in [u_{2} - \ell^{-1}, u_{2}) \\
0 & \text{if } u \in [u_{2}, u_{*})
\end{cases}
\] (7.14)

Therefore, the following holds for every \(u \in [0, u_{*}]\):

\[
\ell \int_{u_{2}}^{u_{2} - \ell^{-1}} \int_{S_{u_{1}} \setminus U} \varphi \Omega_{n} \tilde{\nabla}_{n} \tilde{\chi}_{n} dA_{n} d\mu - \ell \int_{u_{1}}^{u_{1} + \ell^{-1}} \int_{S_{u_{1}} \setminus U} \varphi \Omega_{n} \tilde{\nabla}_{n} \tilde{\chi}_{n} dA_{n} d\mu
\]

We now take \(k \rightarrow +\infty\). Note that we cannot just replace the \(n_{k}\)'s in (7.15) by \(\infty\) because of the term \(|\tilde{\nabla}_{n_{k}}^{2} |_{\gamma_{n_{k}}} \Omega_{n_{k}}^{2} \) (see Proposition 6.19). On the other hand, in all the other terms which are quadratic in the Ricci coefficients, there must be at least one fact which has a strong \(L^{2}_{2} L_{2}(U_{n_{k}})\) limit (for every
Indeed, using Propositions 6.2, 6.8, 6.12, 6.13, 6.15 and 6.19, we obtain
\[
\ell \int_{u_2 - \ell^{-1}}^{u_2} \int_{S_{u_2}} \varphi \Omega^\gamma u \hat{\chi}^{\gamma} \, dA_{\gamma} \, du - \ell \int_{u_1}^{u_1 + \ell^{-1}} \int_{S_{u_1}} \varphi \Omega^\gamma u \hat{\chi}^{\gamma} \, dA_{\gamma} \, du \\
= \int_{u_2}^{u_1} \int_{S_{u_2}} \xi_\ell ((\zeta_3)_{\infty} \varphi) \hat{\chi}^{\gamma} - 4 \varphi \Omega^\gamma u \hat{\chi}^{\gamma} + \frac{1}{2} \varphi (\hat{\chi}^{\gamma})^2 - \varphi \frac{1}{2} \partial_u \hat{\chi}^{\gamma} = 0.
\]

Finally, we take \( \ell \to +\infty \) in (7.16). For the LHS, we use Lemma 2.24; for the RHS, we use the dominated convergence theorem. We then obtain
\[
\int_{S_{u_2}} \varphi \Omega^\gamma u \hat{\chi}^{\gamma} \, dA_{\gamma} = \int_{S_{u_1}} \varphi \Omega^\gamma u \hat{\chi}^{\gamma} \, dA_{\gamma} \\
= \int_{u_2}^{u_1} \int_{S_{u_2}} ((\zeta_3)_{\infty} \varphi) \hat{\chi}^{\gamma} - 4 \varphi \Omega^\gamma u \hat{\chi}^{\gamma} + \frac{1}{2} \varphi (\hat{\chi}^{\gamma})^2 - \varphi \frac{1}{2} \partial_u \hat{\chi}^{\gamma} \, dA_{\gamma} \, du
\]
which is exactly the equation (2.27).

7.4. Propagation equation for the null dust. In this subsection, we prove transport equations for \( d\nu_u \) and \( d\nu_u^\gamma \) (recall Proposition 6.19). The main result is the following proposition:

**Proposition 7.7.** For every \( 0 \leq u_1 < u_2 \leq u_+ \), and every \( C^1 \) function \( \varphi : [0, u_+] \times (0, u_+) \times S^2 \to \mathbb{R} \),
\[
\int_{(u_2) \times (0, u_+) \times S^2} \varphi \, d\nu_u = \int_{(u_1) \times (0, u_+) \times S^2} \varphi \, d\nu_u + \int_{u_1}^{u_2} \int_{(u_1, u) \times S^2} (d\varphi / d\eta + \nabla_{b, \infty} \varphi) \, d\nu_u \, du.
\]

Similarly, for every \( 0 \leq u_1 < u_2 \leq u_+ \), and every \( C^1 \) function \( \varphi : [0, u_+] \times (0, u_+) \times S^2 \to \mathbb{R} \),
\[
\int_{(u_2) \times (0, u_+) \times S^2} \varphi \, d\nu_u^\gamma = \int_{(u_1) \times (0, u_+) \times S^2} \varphi \, d\nu_u^\gamma + \int_{u_1}^{u_2} \int_{(u_1, u) \times S^2} (d\varphi / d\eta) \, d\nu_u^\gamma \, du.
\]

The proof of Proposition 7.7 relies on the next two propositions. We refer the reader to the end of this subsection for the conclusion of the proof of Proposition 7.7.

**Proposition 7.8.** The following identity holds for \((M, g_n)\) for all \( n \in \mathbb{N} \) and for \((M, g_\infty)\):
\[
\int_0^{u_1} \int_{S_{u_2}} \psi \Omega^\gamma u \hat{\chi}^{\gamma} \, dA_{\gamma} \, du - \int_0^{u_1} \int_{S_{u_2}} \psi \Omega^\gamma u \hat{\chi}^{\gamma} \, dA_{\gamma} \, du
\]
\[
= \int_0^{u_1} \int_{S_{u_2}} \hat{\chi} \cdot (2 \hat{\nabla} \hat{\eta} - \psi \hat{\chi} + 2 \hat{\eta}) \Omega^\gamma dA_{\gamma} \, du \\
+ \int_0^{u_1} \int_{S_{u_2}} (d\psi / d\eta + \nabla_b \chi) (\Omega^\gamma \hat{\chi}^\gamma) \, dA_{\gamma} \, du.
\]

Similarly, for \((M, g_n)\) for all \( n \in \mathbb{N} \) and for \((M, g_\infty)\)
\[
\int_0^{u_2} \int_{S_{u_2}} \psi \Omega^\gamma \hat{\chi}^{\gamma} \, dA_{\gamma} \, du - \int_0^{u_2} \int_{S_{u_2}} \psi \Omega^\gamma \hat{\chi}^{\gamma} \, dA_{\gamma} \, du
\]
\[
= \int_0^{u_2} \int_{S_{u_2}} \hat{\chi} \cdot (2 \hat{\nabla} \hat{\eta} - \psi \hat{\chi} + 2 \hat{\eta}) \Omega^\gamma dA_{\gamma} \, du \\
+ \int_0^{u_2} \int_{S_{u_2}} (d\psi / d\eta) (\Omega^\gamma \hat{\chi}^\gamma) \, dA_{\gamma} \, du.
\]

**Proof.** We will only prove (7.17); the proof of (7.18) is slightly simpler.

Note that by Theorem 3.3 and the results in Section 6, \((M, g_n)\) and \((M, g_\infty)\) are angularly regular. Moreover, Propositions 2.29 and 7.5 shows that in these spacetimes,
\[
\nabla_3 \hat{\chi} + \frac{1}{2} \psi \hat{\chi} \hat{\chi} = \nabla \hat{\eta} - 2 \hat{\omega} \hat{\chi} - \frac{1}{2} \psi \hat{\chi} \hat{\chi} + \frac{1}{2} \psi \hat{\chi} \hat{\chi} + \hat{\eta} \hat{\eta}
\]
is satisfied in the weak integrated sense of Definition 2.31.

It therefore suffices to show that if the equation (7.19) is satisfied in the weak integrated sense of Definition 2.31 in an angularly regular spacetime, then in fact (7.17) holds.
According to Definition 2.31, that (7.19) is satisfied in the weak integrated sense means

\[
\int_0^\infty \int_{S_{u,v_2}} \langle \varphi, \hat{\chi} \rangle \Omega \, dA, \, du - \int_0^\infty \int_{S_{u,v_2}} \langle \varphi, \hat{\chi} \rangle \Omega \, dA, \, du \\
= \int_0^\infty \int_{u_1}^{v_2} \int_{S_{u,v}} \langle \varphi, (\nabla \hat{\chi} + \frac{1}{2} \hat{\chi} \chi - \frac{1}{2} \hat{\chi} \chi + \eta \hat{\chi} \Omega) \rangle \Omega^2 \, dA, \, du \, d\mu \\
+ \int_0^\infty \int_{u_1}^{v_2} \int_{S_{u,v}} \langle \nabla \chi \varphi, \hat{\chi} \rangle \Omega^2 \, dA, \, du \, d\mu \tag{7.20}
\]

for every smooth and compactly supported S-tangent 2-tensor \( \varphi^{AB} \). By angular regularity and Hölder's inequality, one verifies that

\[
\|\hat{\chi} \chi\|_{L^2_{\Omega} L^2(S)} + \|\nabla \hat{\chi} \chi - \frac{1}{2} \hat{\chi} \chi + \eta \hat{\chi} \Omega\|_{L^2_{\Omega} L^2(S)} < +\infty, \quad \|\hat{\chi}\|_{L^2_{\Omega} L^2(S)} < +\infty.
\]

It therefore follows from a density argument that (7.20) holds for all \( \varphi \) such that \( \varphi \in C^0_u L^2_{\Omega} L^2(S) \) and \( \nabla_3 \varphi \in L^2_{\Omega} L^2_{\Omega} L^2(S) \). In particular, we can choose \( \varphi^{AB} = \psi \Omega \chi^{AB} \), where \( \psi \) is a \( C^1 \) function to obtain (7.17).

\[\square\]

**Proposition 7.9.** Taking the subsequence \( n_k \) as in the end of Section 6, the following terms in (7.17) obey

\[
\begin{align*}
&\int_{[0,u_1] \times [0,u_2] \times \mathbb{R}^3} \psi \left( 2 \hat{\chi} n_k \cdot \hat{\Omega} \nabla \hat{\chi} \nabla \eta n_k + 2 \hat{\chi} n_k \cdot \hat{\Omega} \nabla \hat{\chi} \nabla \eta n_k + 2 \hat{\chi} n_k \cdot \hat{\Omega} \nabla \hat{\chi} \nabla \eta n_k \right) \, dA, \, du \, d\mu \\
&\rightarrow \int_{[0,u_1] \times [0,u_2] \times \mathbb{R}^3} \psi \left( 2 \hat{\chi} n \cdot \hat{\Omega} \nabla \hat{\chi} \nabla \eta n + 2 \hat{\chi} n \cdot \hat{\Omega} \nabla \hat{\chi} \nabla \eta n + 2 \hat{\chi} n \cdot \hat{\Omega} \nabla \hat{\chi} \nabla \eta n \right) \, dA, \, du \, d\mu \tag{7.21}
\end{align*}
\]

A similar convergence statement holds for the corresponding terms in (7.18).

**Proof.** We will only prove (7.21); the terms in (7.18) can be treated in the same way.

We will use the following two facts. First of all, \( g_{n_k} \to g_{\infty} \) uniformly, and since the integrand in every term is at least in \( L^2_{\Omega} L^1(S_{u,w}) \), we can easily pass to the limit \( k \to +\infty \) in all occurrences of \( n_k \), \( \Omega_{n_k} \) and \( dA_{n_k} \). Second, we use the standard fact that whenever there is a product of two quantities, say \( f_{n_k} \) and \( h_{n_k} \), such that \( f_{n_k} \) converges weakly to \( f_{\infty} \) in \( L^2_{\Omega} L^2(S) \) and \( h_{n_k} \) converges to \( h_{\infty} \) in the \( L^2_{\Omega} L^2(S) \) norm, then \( f_{n_k} h_{n_k} \) converges to \( f_{\infty} h_{\infty} \) in the sense of distribution.

**Step 1:** Term I. Note that \( \nabla \hat{\chi} n_k \cdot \hat{\Omega} \nabla \eta n_k \) converges in the \( C^0_u C^0_u L^4(S_{u,w}) \) norm (by Proposition 6.10), and thus in particular converges in the \( L^2_{\Omega} L^2_{\Omega} L^2(S_{u,w}) \) norm. On the other hand, by Proposition 6.12, \( \hat{\chi} n_k \) converges weakly in \( L^2_{\Omega} L^2(S_{u,w}) \) for all \( u \), and thus in particular converges weakly in \( L^2_{\Omega} L^2_{\Omega} L^2(S_{u,w}) \). By the remark in the beginning of the proof, we deduce that I attains the limit as indicated in (7.21) as \( k \to +\infty \).

**Step 2:** Term II. For this term we use compensated compactness. Indeed, by Proposition 6.18, \( \hat{\chi} n_k \cdot n_k \) converges weakly in \( L^2_{\Omega} L^2 L^2(S) \). On the other hand, Proposition 6.15 in particular implies that \( \hat{\chi} n_k \to \hat{\chi} \hat{\chi} \) in the \( L^2_{\Omega} L^2_{\Omega} L^2(S) \) norm. As in Step 1, we then conclude using the remark in the beginning of the proof.

**Step 3:** Term III. By Proposition 6.10, \( \eta n_k \cdot \hat{\Omega} \) converges in the \( C^0_u C^0_u C^0(S_{u,w}) \) norm, and thus also converges in the \( L^2_{\Omega} L^2_{\Omega} L^2(S_{u,w}) \) norm. On the other hand, as argued in Step 1, \( \hat{\chi} n_k \) converges weakly in \( L^2_{\Omega} L^2_{\Omega} L^2(S_{u,w}) \). As before, we then conclude using the remark in the beginning of the proof. \[\square\]
Proof of Proposition 7.7. We start with the identity (7.17) for \((M, g_n)\). Now take \(k \to +\infty\) and use Propositions 6.19 and 7.9 to obtain
\[
\int_0^{u_2} \int_{S_u \gamma_n} \psi \Omega^2_\infty \left| \bar{\chi} \right|^2 \gamma_n \, d\gamma_n \, du - \int_0^{u_1} \int_{S_u \gamma_n} \psi \Omega^2_\infty \left| \bar{\chi} \right|^2 \gamma_n \, d\gamma_n \, du
\]
\[+ \int_{\{u_2\} \times (0, u_2) \times S^2} \psi \, d
u_{u_2} - \int_{\{u_1\} \times (0, u_2) \times S^2} \psi \, d\nu_{u_1}
\]
\[= \int_0^{u_2} \int_{S_u \gamma_n} \bar{\chi} \cdot \left( 2 \bar{\nabla} \otimes \bar{\nabla} \right) \eta_n = \psi \bar{\chi} \otimes \bar{\chi} + 2 \eta_n \bar{\nabla} \eta_n \right) \Omega^3_\infty \, d\gamma_n \, du \text{ du} \quad (7.22)
\]
\[+ \int_0^{u_1} \int_{S_u \gamma_n} \left( \frac{\partial \psi}{\partial u} + \bar{\nabla} \psi \right) (\Omega^2_\infty \left| \bar{\chi} \right|^2) \gamma_n \, d\gamma_n \, du \]
\[+ \int_{u_1}^{u_2} \int_{\{u \} \times (0, u_2) \times S^2} \left( \frac{\partial \psi}{\partial u} + \bar{\nabla} \psi \right) \, d\nu_{u} \, du.
\]
Finally, subtract from (7.22) the equation (7.17) for \((M, g)\). We then obtain the desired transport equation for \(d\nu_u\). The transport equation for \(d\nu_{u_2}\) can be derived analogously. \(\square\)

7.5. Higher order transport equations for the Ricci coefficients. We next derive the equations (2.43), (2.44), (2.50) and (2.51) for the derivatives of the Ricci coefficients.

Proposition 7.10. \(\mu_\infty\) and \(\mu_\infty\) respectively obeys (2.43) and (2.44) in the integrated sense (Definition 2.30).

Proof. By Proposition 2.44, we know that (2.43) and (2.44) are satisfied for \((M, g_n)\) for all \(k\). It therefore suffices to check that we can take the limit \(k \to +\infty\). Except for having some cubic terms, this is similar to Proposition 7.4.

Now, (2.43) and (2.44) can be schematically written as
\[
\bar{\nabla}^2 \phi = \bar{\nabla} \chi \ast (\eta, \eta) \ast \chi + K \ast \chi \ast (\eta, \eta) \ast (\eta, \eta),
\]
\[\text{and similarly for}\ n.
\]
Consider now (7.23) since (7.24) is similar. For \(i = 1, 2\), the terms
\[
\int_{S_{u_1}} \langle \varphi, \mu_{n_k} \rangle \Omega_{n_k} \, dA_{n_k} \to \int_{S_{u_2}} \langle \varphi, \mu_\infty \rangle \Omega_{\infty} \, dA_{\infty}
\]
due to (2.42), Propositions 6.8, 6.4, 6.8 and 6.10.

It thus remains to pass to the \(k \to +\infty\) limit for the terms which are integrated in \(u\):

1. Term with \(\nabla^2 \varphi\). In a completely analogous manner as the proof of (7.7), we have
\[
\int_{u_1}^{u_2} \int_{S_{u_2}} \left( \langle \nabla^2 \varphi \rangle \right)_{n_k} \, \eta_{n_k} \Omega_{n_k} \, dA_{n_k} \right. \to \left. \int_{u_1}^{u_2} \int_{S_{u_2}} \langle \nabla^2 \varphi \rangle \right. \to \left. \Omega_{\infty} \, dA_{\infty} \right. \, d\nu_{u_2}.
\]

2. Quadratic terms\(^{26}\): \(\eta \bar{\nabla} \chi, \eta \bar{\nabla} \eta, \chi \bar{\nabla} \eta, \chi \bar{\nabla} \eta, \chi K, \omega \bar{\nabla} \eta, \omega K\). By Propositions 6.12 and 6.13, all of \(\bar{\chi}_{n_k}, \bar{\eta}_{n_k}, \bar{\omega}_{n_k}, \bar{\nabla}_{n_k} \bar{\chi}_{n_k}, \bar{\nabla}_{n_k} \bar{\eta}_{n_k}, \bar{\eta}_{n_k} \chi_{n_k}\) converge weakly to their (weak) limits in \(L^2_u L^2(S_{u_2})\) for all \(u\). By Propositions 6.4 and 6.10, \(\eta_{n_k}, \bar{\eta}_{n_k}, \bar{\omega}_{n_k}\) all converge to their limits in the \(L^\infty_u L^2(S_{u_2})\) norm. Hence, all the quadratic terms converge to the appropriate limit in the sense of distribution.

3. Cubic terms: \(\eta \ast \eta \ast \chi, \eta \ast \eta \ast \chi, \eta \ast \bar{\chi} \ast \chi\). By Proposition 6.10, \(\eta_{n_k}\) and \(\bar{\eta}_{n_k}\) have pointwise uniform limits. By Proposition 6.12, \(\chi_{n_k}\) (i.e. both \(\bar{\chi}_{n_k}\) and \(\bar{\chi}_{n_k}\)) has a weak \(L^2_u L^2(S_{u_2})\) limit for every \(u\). It therefore follows that all the cubic terms have the desired limits. \(\square\)

Proposition 7.11. (1) The equation (2.50) is satisfied for all \(C^1\) \(S\)-tangent vector field \(X\), for all \(u \in [0, u_2]\) and all \(0 \leq u_1 < u_2 \leq u_*\).

(2) The equation (2.51) is satisfied for all \(C^1\) \(S\)-tangent vector field \(X\), for all \(u \in [0, u_2]\) and all \(0 \leq u_1 < u_2 \leq u_*\).

\(^{26}\)Note that in addition to the inhomogeneous terms in (7.23), the quadratic term includes terms \(\bar{\chi} \bar{\chi} + \bar{\omega} \bar{\omega}\) in Definition 2.30.
Proof. In view of their similarities, we will only prove (the slightly harder) (2.51).

By (2.49) in Proposition 2.45,

$$
\left( \frac{\partial}{\partial u} + \nabla_{\nu_n} \chi \right) (\Omega_n^{-1} \psi \chi_{\nu_n}) + \frac{1}{2} \chi (\psi \chi_{\nu_n})^2 = -\chi [\nabla_{\nu_n} |^2 + | \frac{\partial}{\partial u} + \nabla_{\nu_n} \chi | (\Omega_n^{-1} \psi \chi_{\nu_n})].
$$

(7.25)

Fix $0 \leq u_1 < u_2 \leq u$, and let $\xi_\ell$ be a cutoff as in (7.14) when $\ell^{-1} \leq u_2 - u_1$. Multiplying (7.25) by $\xi_\ell$, integrating with respect to $\Omega_n$ d$\Lambda_{\gamma_n} du$, and integrating by parts, we obtain that for every $u \in [0, u_2]$: \[
\ell \int_{u_2 - \ell^{-1}}^{u_2} \int_{S_{u, \nu_n}} \Omega_n \chi (\Omega_n^{-1} \psi \chi_{\nu_n}) \, d\Lambda_{\gamma_n} du - \ell \int_{u_1}^{u_1 + \ell^{-1}} \int_{S_{u, \nu_n}} \Omega_n \chi (\Omega_n^{-1} \psi \chi_{\nu_n}) \, d\Lambda_{\gamma_n} du 
= \int_{u_1}^{u_2} \int_{S_{u, \nu_n}} \xi_\ell \left( \frac{\partial}{\partial u} + \nabla_{\nu_n} \chi \right) (\Omega_n^{-1} \psi \chi_{\nu_n}) - 4\omega_n \Omega_n \chi (\Omega_n^{-1} \psi \chi_{\nu_n}) - \chi [\nabla_{\nu_n} |^2 + \chi | (\Omega_n^{-1} \psi \chi_{\nu_n})] \, d\Lambda_{\gamma_n} du.
\]

In order to be able to pass to the $k \to +\infty$ limit, we integrate by parts the last term to obtain
\[
\ell \int_{u_2 - \ell^{-1}}^{u_2} \int_{S_{u, \nu_n}} \Omega_n \chi (\Omega_n^{-1} \psi \chi_{\nu_n}) \, d\Lambda_{\gamma_n} du - \ell \int_{u_1}^{u_1 + \ell^{-1}} \int_{S_{u, \nu_n}} \Omega_n \chi (\Omega_n^{-1} \psi \chi_{\nu_n}) \, d\Lambda_{\gamma_n} du 
= \int_{u_1}^{u_2} \int_{S_{u, \nu_n}} \xi_\ell \left( \frac{\partial}{\partial u} + \nabla_{\nu_n} \chi \right) (\Omega_n^{-1} \psi \chi_{\nu_n}) - 4\omega_n \Omega_n \chi (\Omega_n^{-1} \psi \chi_{\nu_n}) \, d\Lambda_{\gamma_n} du + \int_{u_1}^{u_2} \int_{S_{u, \nu_n}} \xi_\ell (2\chi (\log \Omega_n) + d\nu_n \chi) \, d\Lambda_{\gamma_n} du.
\]

We now argue in a similar manner as Proposition 7.6. First we pass to the $k \to +\infty$ limit using Propositions 6.2, 6.4, 6.8, 6.12, 6.13, 6.15, 6.19. Note that except for the terms in the last line of (7.26), all the other terms have a most one factor which does not admit a strong limit so that we can replace $n_k$ by $+\infty$ in the limit. In particular, because $[\frac{\partial}{\partial u} + \nabla_{\nu_n} \chi]$ is an $S$-tangent vector field, $[\frac{\partial}{\partial u} + \nabla_{\nu_n} \chi] (\Omega_n^{-1} \psi \chi_{\nu_n}) \to [\frac{\partial}{\partial u} + \nabla_{\nu_n} \chi] (\Omega_\infty^{-1} \psi \chi_{\infty})$ in the $L^2 \mu(S_{u, \nu_n})$ norm for every $u \in [0, u_2]$. For the terms on the last line of (7.26), we obtain an extra term involving $d\nu_{\gamma_n}$ in the limit.

After taking the $k \to +\infty$ limit we then take $\ell \to +\infty$ (using Lemma 2.24 on the LHS and the dominated convergence theorem on the RHS), we obtain that
\[
\int_{S_{u_2, \nu_n}} \Omega_\infty^2 (\chi (\Omega_\infty^{-1} \psi \chi_{\infty}))^- \, d\Lambda_{\gamma_n} - \int_{S_{u_1, \nu_n}} \Omega_\infty^2 (\chi (\Omega_\infty^{-1} \psi \chi_{\infty}))^+ \, d\Lambda_{\gamma_n} 
= \int_{u_1}^{u_2} \int_{S_{u, \nu_n}} (\chi (\log \Omega_\infty) + d\nu_\infty \chi) \, d\Lambda_{\gamma_n} du 
+ \int_{(u_1, u_2) \times \{\Omega\} \times S^2} \left( \chi (\log \Omega_\infty) + d\nu_\infty \chi \right) \, d\nu_\gamma du \leq \frac{\|\chi |\|_{L^2(\Omega)} \|\psi \|_{L^2(\Omega)} \|\chi \|_{L^2(\Omega)} \|\chi \|_{L^2(\Omega)}}{\lambda} \chi \text{ as desired.}
\]

\[\square\]

7.6. Renormalized Bianchi equations. Recall the definition of the curvature components in Definition 2.39. The set of renormalized Bianchi equations Proposition 2.41 are satisfied by the limit spacetime in an appropriate sense:

**Proposition 7.12.** In the limiting spacetime $(\mathcal{M}, g_\infty)$, the renormalized Bianchi equations are satisfied in the sense of Definition 2.42.

**Proof.** The proof is similar to various previous propositions; we will only indicate the main points. As in Propositions 7.4, 7.5 and 7.10, the main goal will be to make use of the fact that all the equations are satisfied for $(\mathcal{M}, g_{\nu_k})$ according to Proposition 2.41 and then take limits.

Now taking the limit of the (integrated form of) (2.32)–(2.35) can be done in exactly the same way as in the proof of Proposition 7.10. Indeed, it can be easily checked that except for the top derivative terms $d\nu_\beta, d\nu_\beta$, etc., (2.32)–(2.35) have schematically the same type of terms as (2.43) and (2.44). The top derivative terms can be handled using Propositions 6.12 and 6.13 (which guarantees the weak convergence of up to second derivatives of $\chi, \psi \chi, \bar{\chi}$ and $\bar{\psi} \chi$) together with Propositions 6.2, 6.8 and 6.10.
Finally, in order to take the limit of (the weak integrated form of) (2.31) and (2.36), we need in addition to use the compensated compactness result in Proposition 6.18 to handle the terms $\chi\nabla X^i, \chi\nabla X^j, \eta\chi\chi, \eta\chi\chi$, etc. (cf. the difference between Propositions 7.4 and 7.5). We omit the details.

7.7. Proof of Theorem 4.1. We now have all the ingredients to complete the proof of Theorem 4.1:

Proof of Theorem 4.1. (1) This assertion follows directly from Theorem 3.2.

(2) The existence of a $C^0$ limit of the metric components in double null coordinates is a consequence of Propositions 6.2 and 6.8. The same propositions also give the weak $L^2$ convergence statements for the first derivatives of the metric components.

The weak $L^2$ convergence of the Ricci coefficient follow as a consequence of Propositions 6.19 and 6.20.

Finally, Proposition 7.1 shows that the weak limit of the Ricci coefficients coincide with the Ricci coefficients associated with the limit metric.

(3) The weak-* limits (4.2) and (4.3) exist as a consequence of Proposition 6.19. The angular regularity of $(\mathcal{M}, g_{\infty})$ (see Definition 2.26) follows from Propositions 6.2, 6.3, 6.4, 6.6, 6.8, 6.10, 6.11, 6.12, 6.13 and 6.15. The angular regularity of $\{\{d\nu\}_{u\in[0, u_s]}; \{d\omega\}_{u\in[0, u_s]})$ (see Definition 2.36) follows from Propositions 6.19 and 6.20.

Finally, that $(\mathcal{M}, g_{\infty}, \{d\nu\}_{u\in[0, u_s]}, \{d\omega\}_{u\in[0, u_s]})$ is an angularly regular weak solution to the Einstein–null dust system (see Definition 2.37) follows from Propositions 7.4, 7.5, 7.6 and 7.7.

(4) The renormalized Bianchi equations follow from Proposition 7.12.

(5) The auxiliary equations hold because of Propositions 7.3, 7.10 and 7.11.

8. Uniqueness of the Limit

In this section, we prove the uniqueness theorem (Theorem 4.4). For the whole section, we work under the assumptions of Theorem 4.4. In particular, we are given two angularly regular weak solutions $(\mathcal{M}, g^{(1)}, \{d\nu^{(1)}\}_{u\in[0, u_s]}, \{d\omega^{(1)}\}_{u\in[0, u_s]})$ and $(\mathcal{M}, g^{(2)}, \{d\nu^{(2)}\}_{u\in[0, u_s]}, \{d\omega^{(2)}\}_{u\in[0, u_s]})$ to the Einstein–null dust system. We will first define in Section 8.1 a distance function (see (8.3)) that controls the difference of the two solutions. The remaining subsections are devoted to controlling this distance function.

- In Section 8.2, we prove some easy preliminary estimates.
- In Section 8.3, we prove general estimates for differences of transport equations, and apply them to control the differences of metric components and the Ricci coefficients $\eta, \chi\chi, \chi\chi, \omega$ and $\omega$. The transport equations for $\chi\chi$ and $\chi\chi$ (and their angular derivatives) will be treated separately in Section 8.4, because they involve the measure-valued $d\nu$ and $d\omega$ on the RHSs.
- We then treat the top-order estimates. This is dealt with by a combination of energy estimates for the renormalized curvature components (Section 8.5) and elliptic estimates to handle the top-order derivatives of the Ricci coefficients (Section 8.6).
- In Section 8.7, we estimate the difference of the (measure-valued) null dust.

Putting all these together in Section 8.8, we obtain Theorem 4.4.

8.1. Distance function. To proceed, we first introduce a reduction so that the analysis is carried out in a small rectangle. We partition the set $[0, u_s] \times [0, u_s]$ into $N^2$ rectangles; namely we write $[0, u_s] \times [0, u_s] = \cup_{i=0}^{N-1} \cup_{j=0}^{N-1} [u_i, u_{i+1}] \times [u_j, u_{j+1}]$ where $u_i = \frac{i}{N}u_s$ and $u_j = \frac{j}{N}u_s$. It suffices to show that for $N$ sufficiently large (depending on the size of $(\mathcal{M}, g^{(1)}, \{d\nu^{(1)}\}_{u\in[0, u_s]}, \{d\omega^{(1)}\}_{u\in[0, u_s]})$ and $(\mathcal{M}, g^{(2)}, \{d\nu^{(2)}\}_{u\in[0, u_s]}, \{d\omega^{(2)}\}_{u\in[0, u_s]})$, if the two sets of data agree on $[u_i] \times [u_j, u_{j+1}] \times S^2$ and $[u_i, u_{i+1}] \times [u_j, u_{j+1}] \times S^2$, then in fact the two solutions agree in $[u_i, u_{i+1}] \times [u_j, u_{j+1}] \times S^2$.

The parameter $N$ will be chosen later. For the remainder of the section, we fix some $0 \leq i, j \leq N - 1$, and will only concern ourselves with the region $[u_i, u_{i+1}] \times [u_j, u_{j+1}] \times S^2$. In particular, when applying the definitions or equations from the previous sections, we will replace $[0, u_s]$ by $[u_i, u_{i+1}]$ (and respectively $[0, u_s]$ by $[u_j, u_{j+1}]$).

---

27Of course we have in fact proven much stronger convergence statements.
In order to define a distance function between two solutions to the Einstein–null dust system: dist := sup \( \| \psi^1 - \psi^2 \|_{L^\infty_u L^2_x W^{1,2}(S_u, \mathbb{R}^3)} + \sum \| \psi^1 - \psi^2 \|_{L^\infty_u L^2_x L^2(S_u, \mathbb{R}^3)} + \sum \| \psi^1 - \psi^2 \|_{L^\infty_u L^2_x W^{1,2}(S_u, \mathbb{R}^3)} \)

We then define a distance function between two solutions to the Einstein–null dust system:

\[
\operatorname{dist}_\mu(\mu^1, \mu^2) := \sup_{\mu \in \mathcal{M}} \sup_{\mu \in \mathcal{M}} \left| \int_{\mathcal{H}_u} \mu^2 - \mu^1 \right| + \sup_{\mu \in \mathcal{M}} \sup_{\mu \in \mathcal{M}} \left| \int_{\mathcal{H}_u} \bar{\mu}^1 - \bar{\mu}^2 \right|.
\]

In the following subsections of the section, we will control each piece in (8.3) and prove an estimate \( \lesssim \frac{\mu}{N^2} \). We will use the following convention for constants. The angular regularity assumption of Theorem 4.4 (see Definitions 2.26 and 2.36) gives control of the geometric quantities associated to \((M, g^1, \mu^1, \nu^1, \mu^2, \nu^2)\) and \((M, g^2, \mu^1, \nu^1, \mu^2, \nu^2)\) (in the full region \([0, u_0] \times [0, u_2] \times \mathbb{S}^2\)). All implicit constants in \( \lesssim \) in this section will depend only on the estimates for the geometric quantities given in Definitions 2.26 and 2.36. Importantly, they are independent of \( N \). (Moreover, for instance when we say we use Definition 2.26, we mean that we use the corresponding quantitative estimates.)

8.2. Some auxiliary estimates.

**Proposition 8.1.** For every \( u \), \( \nu \),

\[
\| \gamma^1 \|_{L^2(S_{\nu}, \gamma^1)} \lesssim \| \gamma^1 \|_{L^2(S_{\nu}, \gamma^2)} \lesssim \| \gamma^1 \|_{L^2(S_{\nu}, \gamma^1)}.
\]

**Proof.** As in the proof of Proposition 6.1, for \( i = 1, 2 \), \( L^2(S_{\nu}, \gamma^1) \) is comparable to \( L^2(S_{\nu}, \gamma^1) \), where \( \gamma^1(0, 0) \) is the metric that agrees with \( \gamma^1(0, 0) = \gamma^1(0, 0) = 0 \).

Therefore, to establish the proposition, it suffices to show that \( L^2(S_{\nu}, \gamma^1) \) and \( L^2(S_{\nu}, \gamma^2) \) are comparable, which is obviously the case since by assumption \( \gamma^1(0, 0) = \gamma^2(0, 0) \).

**Proposition 8.2.**

\[
\| \gamma^1 \|_{L^2(S_{\nu}, \gamma^1)} \lesssim \| \gamma^2 \|_{L^2(S_{\nu}, \gamma^1)},
\]

and

\[
\| \gamma^2 \|_{L^2(S_{\nu}, \gamma^1)} \lesssim \| \gamma^1 \|_{L^2(S_{\nu}, \gamma^1)}.
\]
In particular, by (8.3), we also have

\[ \text{LHS of (8.4) + LHS of (8.5)} \lesssim \text{dist.} \]

**Proof.** The estimate (8.4) is a standard statement regarding the continuity of inverses. We omit the details; see for instance [12, Proposition 9.2] for the relevant calculations.

To prove (8.5), first note that for every \( x \in (0, +\infty) \), we have the calculus inequality \( |x - 1| \leq \max\{x, 1/x\} \log x \). It follows (by setting \( x = \frac{\sqrt{\det \gamma(1)} \sqrt{\det \gamma(2)}}{\sqrt{\det \gamma(2)}} \)) that

\[ |\frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1| \leq \max\{|\frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1|, |\frac{\sqrt{\det \gamma(2)}}{\sqrt{\det \gamma(1)}} - 1|\} \log \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}}. \]

By Definition 2.26, we have uniform \( L^\infty \) bounds for \( \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} \) and \( \frac{\sqrt{\det \gamma(2)}}{\sqrt{\det \gamma(1)}} \). Therefore, taking the \( L^\infty L^2(S_{u,x}, \gamma(1)) \) norm of the above inequality yields

\[ \| \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1 \|_{L^\infty L^2(S_{u,x}, \gamma(1))} \lesssim \| \log \det \gamma(1) - \log \det \gamma(2) \|_{L^\infty L^2(S_{u,x}, \gamma(1))}. \]

(8.6)

For the first derivative, we compute

\[ \nabla \left( \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1 \right) = \frac{1}{2} \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} \nabla (\log \det \gamma(1) - \log \det \gamma(2)). \]

Using the bounds in Definition 2.26, we obtain

\[ \| \nabla \left( \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1 \right) \|_{L^\infty L^2(S_{u,x}, \gamma(1))} \lesssim \| \nabla (\log \det \gamma(1) - \log \det \gamma(2)) \|_{L^\infty L^2(S_{u,x}, \gamma(1))}. \]

(8.7)

Combining (8.6) and (8.7) yields the bound for \( \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1 \) in (8.5); the bound for \( \frac{\sqrt{\det \gamma(1)}}{\sqrt{\det \gamma(2)}} - 1 \) can be proven in an entirely analogous manner. \( \square \)

**Proposition 8.3.**

\[ \| \mathcal{F}(1) - \mathcal{F}(2) \|_{L^\infty L^2(S_{u,x}, \gamma(1))} \lesssim \| \gamma(1) - \gamma(2) \|_{L^\infty L^2(S_{u,x}, \gamma(1))}, \]

and

\[ \| \mathcal{F}(1) - \mathcal{F}(2) \|_{L^\infty L^2(S_{u,x}, \gamma(1))} \lesssim \text{dist.} \]

**Proof.** The first statement is immediate from (2.5) and Proposition 8.2. The second statement then follows after applying also (8.3). \( \square \)

**Proposition 8.4.**

\[ \| 1 - \frac{\Omega(1)}{\Omega(2)} \|_{L^\infty L^2 W^{1,2}(S_{u,x}, \gamma(1))} + \| 1 - \frac{\Omega(2)}{\Omega(1)} \|_{L^\infty L^2 W^{1,2}(S_{u,x}, \gamma(1))} \lesssim \text{dist.} \]

**Proof.** It suffices to control \( 1 - \frac{\Omega(1)}{\Omega(2)}, 1 - \frac{\Omega(2)}{\Omega(1)} \) and their derivatives by \( \log \frac{\Omega(1)}{\Omega(2)} \) and its derivatives. This is a computation almost exactly the same as the proof of (8.5); we omit the details. \( \square \)

8.3. **Transport estimates for the metric coefficients and the Ricci coefficients.** In this subsection, we prove some estimates which are derivable using transport equations. We first prove some general estimates regarding general transport equations in Propositions 8.5 and 8.6. These will then be applied in Propositions 8.7–8.10 to control the differences of the metric coefficients and the Ricci coefficients.

**Proposition 8.5.** Suppose\(^{28}\) \( \bar{\nabla}^{(1)}_3 \phi = F \) holds in the integrated sense (Definition 2.30) such that \( \phi_{|_{u = u_0}} = 0 \). Then

\[ \| \phi \|_{L^\infty L^2(S_{u,x}, \gamma(1))} \lesssim \| F \|_{L^\infty L^1 L^2(S_{u,x}, \gamma(1))}. \]

\[ (8.8) \]

Suppose \( \bar{\nabla}^{(1)}_4 \phi = F \) holds in the integrated sense (Definition 2.30) such that \( \phi_{|_{u = u_0}} = 0 \). Then

\[ \| \phi \|_{L^\infty L^2(S_{u,x}, \gamma(1))} \lesssim \| F \|_{L^\infty L^1 L^2(S_{u,x}, \gamma(1))}. \]

\[ (8.9) \]
Suppose $V_3^{(1)} \phi = F$ holds in the weak integrated sense (Definition 2.31) such that $\phi |_{\{u=a_i\}} = 0$. Then
\[ \|\phi\|_{L^\infty_t L^2_x(S_{a_i,\infty}^{(1)}))} \lesssim \|F\|_{L^1_t L^2_x(S_{a_i,\infty}^{(1)})). \tag{8.10} \]

Suppose $V_4^{(1)} \phi = F$ holds in the weak integrated sense (Definition 2.31) such that $\phi |_{\{u=a_i\}} = 0$. Then
\[ \|\phi\|_{L^\infty_t L^2_x(S_{a_i,\infty}^{(1)}))} \lesssim \|F\|_{L^1_t L^2_x(S_{a_i,\infty}^{(1)}))}. \tag{8.11} \]

Proof. We will prove (8.8) and (8.10). The proofs for (8.9) and (8.11) are similar and omitted.

Step 1: Proof of (8.8). Fix $(U, U) \in [u_i, u_j+1] \times [u_j, u_{j+1}]$. Let $\varphi \in C^1$ satisfy
\[ V_3 \varphi = 0 \tag{8.12} \]
and
\[ \|\varphi\|_{L^2_x(S_{a_i,\infty}^{(1)}))} \leq 1. \tag{8.13} \]

By Proposition 5.2 and (8.12),
\[ \frac{\partial}{\partial u} \int_{S_{u,\infty}^1} |\varphi|^2 \gamma \, d\Lambda_0 = \int_{S_{u,\infty}^1} \Omega_1 \left( V_3 \varphi + \chi \right)^2 \, d\Lambda_0 = \int_{S_{u,\infty}^1} \Omega_1 \, d\Lambda_0 = 0. \tag{8.15} \]

Applying H"{o}lder’s inequality and (8.14) to (8.15), and using the estimates in Definition 2.26, we see that for every $u \in [u_i, u_{i+1}]$,
\[ \left| \int_{S_{u,\infty}^1} \langle \varphi, \phi \rangle \Omega_1 \, d\Lambda_0 \right| \lesssim \|F\|_{L^1_t L^2_x(S_{a_i,\infty}^{(1)}))} + \|\Omega_1 \chi - 2\gamma \Omega_1\|_{L^1_t L^2_x(S_{a_i,\infty}^{(1)})} \|\phi\|_{L^\infty_t L^2_x(S_{a_i,\infty}^{(1)})). \tag{8.16} \]

In particular, it follows from (8.16) by duality and the boundedness of $\log \Omega_1$ that
\[ \|\phi\|_{L^2_x(S_{a_i,\infty}^{(1)}))} \lesssim \sup_{\|\varphi\|_{L^2_x(S_{a_i,\infty}^{(1)}))} \leq 1 \left| \int_{S_{u,\infty}^1} \langle \varphi, \phi \rangle \Omega_1 \, d\Lambda_0 \right| \lesssim \|F\|_{L^1_t L^2_x(S_{a_i,\infty}^{(1)}))} + \frac{1}{N^2} \|\phi\|_{L^\infty_t L^2_x(S_{a_i,\infty}^{(1)})). \tag{8.17} \]

In view of the arbitrariness of $(U, U)$, we then obtain
\[ \|\phi\|_{L^\infty_t L^\infty_x(S_{a_i,\infty}^{(1)}))} \lesssim \|F\|_{L^1_t L^2_x(S_{a_i,\infty}^{(1)}))} + \frac{1}{N^2} \|\phi\|_{L^\infty_t L^\infty_x(S_{a_i,\infty}^{(1)})), \]
which, after choosing $N$ sufficiently large, implies (8.8).

Step 2: Proof of (8.10). Fix $U \in [u_i, u_{i+1}]$. Pick $\varphi \in C^1$ satisfying (8.12), but instead of (8.13), assume
\[ \|\varphi\|_{L^2_x(S_{a_i,\infty}^{(1)}))} \leq 1. \tag{8.18} \]

Integrating (8.14) in $u$ and applying Gr"{o}nwall’s inequality, we obtain
\[ \|\varphi\|_{L^\infty_t L^\infty_x(S_{a_i,\infty}^{(1)}))} \lesssim 1. \tag{8.19} \]

By Definition 2.31 and (8.12), we have, for all $u \in [u_i, u_{i+1}]$,
\[ \int_{u_i}^{u_{i+1}} \int_{S_{u,\infty}^1} \langle \varphi, \phi \rangle \Omega_1 \, dA_0 \, du + \int_{u_i}^{u_{i+1}} \int_{S_{u,\infty}^1} \langle \varphi, \phi \rangle \Omega_1 \, dA_0 \, du' \, du = 0. \tag{8.20} \]
Applying (8.20) when \( u = U \) and using Hölder’s inequality together with (8.19), we obtain
\[
\|\phi\|_{L^2_u L^2_u(S_{\omega, \gamma}^{(1)})} \lesssim \sup_{\|\phi\|_{L^2_u L^2_u(S_{\omega, \gamma}^{(1)})} \leq 1} \left| \int_{S_{\omega, \gamma}^{(1)}} \langle \varphi, \phi \rangle \Omega \, dA, \, du \right|
\]
\[
\lesssim \|F\|_{L^2_u L^2_u(S_{\omega, \gamma}^{(1)})} + \frac{1}{N^2} \|\|\hat{U}\|_{L^2_u L^2(S_{\omega, \gamma}^{(1)})} - 2\|u\|_{L^2_u L^2(S_{\omega, \gamma}^{(1)})}\|\phi\|_{L^\infty_u L^2_u(S_{\omega, \gamma}^{(1)})}
\]
\[
\lesssim \|F\|_{L^2_u L^2_u(S_{\omega, \gamma}^{(1)})} + \frac{1}{N^2} \|\phi\|_{L^\infty_u L^2_u(S_{\omega, \gamma}^{(1)})}.
\]
(8.21)

Since \( U \) is arbitrary, it follows from (8.21) that
\[
\|\phi\|_{L^2_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} \lesssim \|F\|_{L^2_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} + \frac{1}{N^2} \|\phi\|_{L^\infty_u L^2_u L^2(S_{\omega, \gamma}^{(1)})},
\]
which implies (8.10) after taking \( N \) sufficiently large.

**Proposition 8.6.** The following holds for every \( \phi \) an \( S \)-tangent tensorfield of arbitrary rank (with the implicit constant depending on the rank):
\[
\|(\mathbf{\nabla}^4 - \mathbf{\nabla}_0^2)\phi\|_{L^2_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} \lesssim \frac{\text{dist}}{N^2} \left( \|\phi\|_{L^\infty_u L^2_u W^{1,4}(S_{\omega, \gamma}^{(1)})} + \|(\mathbf{\nabla}^4 - \mathbf{\nabla}_0^2)\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} \right),
\]
(8.22)
\[
\|(\mathbf{\nabla}^4 - \mathbf{\nabla}_0^2)\phi\|_{L^2_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} \lesssim \frac{\text{dist}}{N^2} \left( \|\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} + \|\phi\|_{L^\infty_u L^2_u W^{1,4}(S_{\omega, \gamma}^{(1)})} + \|(\mathbf{\nabla}^4 - \mathbf{\nabla}_0^2)\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} \right),
\]
(8.23)
\[
\|(\mathbf{\nabla}_3^3 - \mathbf{\nabla}_0^2)\phi\|_{L^2_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} \lesssim \frac{\text{dist}}{N^2} \left( \|\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} + \|\phi\|_{L^\infty_u L^2_u W^{1,4}(S_{\omega, \gamma}^{(1)})} + \|\mathbf{\nabla}_3^3\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} \right).
\]
(8.24)

and
\[
\|(\mathbf{\nabla}_3^3 - \mathbf{\nabla}_0^2)\phi\|_{L^2_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} \lesssim \frac{\text{dist}}{N^2} \left( \|\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} + \|\phi\|_{L^\infty_u L^2_u W^{1,4}(S_{\omega, \gamma}^{(1)})} + \|\mathbf{\nabla}_3^3\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} \right).
\]
(8.25)

**Proof.** We will only prove (8.24) and (8.25); the estimates (8.22) and (8.23) are slightly easier.

Before we proceed, first note that by Hölder’s inequality and Fubini’s theorem, it suffices to prove that for \( p \in [2, +\infty] \),
\[
\|(\nabla^4 - \nabla_0^2)\phi\|_{L^p_u L^2_u L^2(S_{\omega, \gamma}^{(1)})} \lesssim \text{dist} \times \left( \|\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} + \|\phi\|_{L^\infty_u L^2_u W^{1,4}(S_{\omega, \gamma}^{(1)})} + \|\nabla_3^3\phi\|_{L^\infty_u L^2_u L^4(S_{\omega, \gamma}^{(1)})} \right).
\]
(8.26)

From now on, we take \( p \in [2, +\infty] \) and our goal will be to prove (8.26).

By (2.6),
\[
\|(\nabla^4 - \nabla_0^2)\phi\|_{A_1, A_2, \ldots, A_r} = \frac{(\nabla^4 - \nabla_0^2)\phi}{\Omega^{(1)}} A_1, A_2, \ldots, A_r + (\Omega^{(1)})^{-1} (b^{(1)} - b^{(2)}) C^{(1)} \phi_{A_1, A_2, \ldots, A_r} \]
(8.27)
\[
+ \frac{(b^{(2)} - b^{(1)})}{\Omega^{(1)}} \sum_{i=1}^r (\gamma^{-1} B C A_i)^{(2)} \phi_{A_1, A_2, \ldots, A_r} \]
(8.28)
\[
- \frac{(b^{(1)} - b^{(2)})}{\Omega^{(1)}} \sum_{i=1}^r (\gamma^{-1} B C A_i)^{(1)} \phi_{A_1, A_2, \ldots, A_r}
\]
(8.29)
\[
+ \frac{(\Omega^{(1)})^{-1} (b^{(1)} - b^{(2)}) B \phi_{A_1, A_2, \ldots, A_r}}{\Omega^{(1)}} \]
(8.30)

For the first term in (8.27), we first note that by Proposition 8.4 and Sobolev embedding (Proposition 5.1),
\[
\frac{(b^{(2)} - b^{(1)})}{\Omega^{(1)}} \sum_{i=1}^r (\gamma^{-1} B C A_i)^{(2)} \phi_{A_1, A_2, \ldots, A_r} \lesssim \frac{(b^{(2)} - b^{(1)})}{\Omega^{(1)}} \sum_{i=1}^r (\gamma^{-1} B C A_i)^{(2)} \phi_{A_1, A_2, \ldots, A_r} \]
(8.31)

Note in particular that a smallness factor \( N^{-\frac{1}{2}} \) arises from the difference between \( L^4_u \) on the LHS of (8.24), (8.25) and \( L^2_u \) on the LHS of (8.26).
Therefore, using the fact $p \geq 2$ and Hölder’s inequality, we obtain
\[
\frac{\Omega^{(2)} - \Omega^{(1)}}{\Omega^{(1)}} \leq L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \frac{\Omega^{(2)} - \Omega^{(1)}}{\Omega^{(1)}} \leq \frac{\Omega^{(2)} - \Omega^{(1)}}{\Omega^{(1)}} \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

The second term in (8.27) can be treated in a similar manner. We note that by (8.3) and Proposition 5.1, $\|b^{(1)} - b^{(2)}\|L^p_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist}$ and so using also the bounds provided by Definition 2.26, we have
\[
\|\Omega^{(2)} - \Omega^{(1)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

For (8.28), we use (8.31), bounds in Definition 2.26 and Hölder’s inequality to obtain
\[
\|\Omega^{(2)} - \Omega^{(1)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

We next consider (8.29). For this, we note that by (8.3) and Sobolev embedding (Proposition 5.1) in $L^2_\Omega L^2(S_{u,u}\gamma^{(1)})$, we have
\[
\|\Omega^{(1)} - \Omega^{(2)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

Thus, Hölder’s inequality implies that
\[
\|\Omega^{(1)} - \Omega^{(2)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

Finally, we consider (8.30). By (8.3),
\[
\|\Omega^{(1)} - \Omega^{(2)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

Therefore, by Hölder’s inequality, Sobolev embedding (Proposition 5.1) and the fact that $p \geq 2$,
\[
\|\Omega^{(1)} - \Omega^{(2)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3} \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

Combining the above estimates, we have thus achieved (8.26). This concludes the argument. \(\square\)

**Proposition 8.7.**
\[
\|\gamma^{(1)} - \gamma^{(2)}\|L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)}) + \|\log \text{det} \gamma^{(1)}\|L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)}) + \|\log \text{det} \gamma^{(2)}\|L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)}) + \|\gamma^{(2)}\|L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

**Proof.** Step 1: Proof of $L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)})$ estimates. By Propositions 8.5 and 8.6, it suffices to bound $(\nabla^4_1 - \nabla^4_2)^{(1)} - (\nabla^4_1 - \nabla^4_2)^{(2)}$, $(\nabla^4_1 \log \text{det} \gamma^{(1)}) - (\nabla^4_1 \log \text{det} \gamma^{(2)})$ and $(\nabla^4_1 \omega^{(1)}) - (\nabla^4_1 \omega^{(2)})$ in $L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)})$.

By (2.10), we have
\[
(\nabla^4_1 \gamma^{(1)}) - (\nabla^4_2 \gamma^{(2)}) = 0,
\]
\[
(\nabla^4_1 \log \text{det} \gamma^{(1)}) - (\nabla^4_2 \log \text{det} \gamma^{(2)}) = 2\nabla^4_1 \chi^{(1)} - 2\nabla^4_2 \chi^{(2)},
\]
\[
(\nabla^4_1 \omega^{(1)}) - (\nabla^4_2 \omega^{(2)}) = -2(\nabla^4_1 \gamma^{(1)}) \cdot \omega^{(1)} + (\nabla^4_2 \gamma^{(2)}) \cdot \omega^{(2)} - 2\nabla^4_1 \chi^{(1)} \cdot \omega^{(1)}.
\]

Now by the estimates in Definition 2.26 and (8.3), the RHS of each of these equations is bounded above in $L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)})$ by dist. In particular, using the Cauchy–Schwarz inequality, we obtain
\[
\|\nabla^4_1 - \nabla^4_2\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) + \|\nabla^4_1 \log \text{det} \gamma^{(1)} - (\nabla^4_2 \log \text{det} \gamma^{(2)})\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) + \|\nabla^4_1 \omega^{(1)} - (\nabla^4_2 \omega^{(2)})\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]

Therefore, we obtain the desired $L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)})$ estimates.

**Step 2:** Proof of $L^\infty_\Omega L^2(S_{u,u}\gamma^{(1)})$ estimates. This is similar to Step 1, except that we use the equations (2.37)–(2.39) instead; we omit the details. \(\square\)

**Proposition 8.8.**
\[
\|\eta^{(1)} - \eta^{(2)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) + \|\eta^{(1)} - \eta^{(2)}\|L^2_\Omega L^2(S_{u,u}\gamma^{(1)}) \leq \text{dist} \frac{N^2}{\mathcal{V}_3}.
\]
Proof. We will prove the estimate for $\eta^{(1)} - \eta^{(2)}$; the estimate for $\frac{1}{2} \mu^{(1)} - \frac{1}{2} \mu^{(2)}$ is similar.

By Propositions 8.5, we need to bound $\nabla_4^{(1)} (\eta^{(1)} - \eta^{(2)})$. We write

$$\nabla_4^{(1)} (\eta^{(1)} - \eta^{(2)}) = - (\nabla_4^{(1)} - \nabla_4^{(2)}) \eta^{(2)} + (\nabla_4 \eta^{(1)}) - (\nabla_4 \eta^{(2)}).$$

The term $(\nabla_4^{(1)} - \nabla_4^{(2)}) \eta^{(2)}$ can be estimated using (8.22) in Proposition 8.6 and the bounds for $\eta^{(2)}$ given by Definition 2.26, the equation (2.15) satisfied by $\eta^{(2)}$, together with Propositions 8.1 and 8.3 so that we have

$$\| (\nabla_4^{(1)} - \nabla_4^{(2)}) \eta^{(2)} \|_{L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist} \frac{1}{N^\frac{3}{4}}.$$

Therefore, according to Proposition 8.5, it suffices to bound the terms in $(\nabla_4 \eta^{(1)}) - (\nabla_4 \eta^{(2)})$. By (2.15), these terms are either of the form $(\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)})$ or $(\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)})$ or $(\chi \eta^{(1)}) - (\chi \eta^{(2)})$ (where * denotes some contraction with respect to $\gamma$).

We first handle the term $(\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)})$. A direct computation gives

$$(\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)})$$

$$= \{ (\gamma^{(1)})^{-1} - (\gamma^{(2)})^{-1} \} AB \nabla_4 \chi^{(1)} + \{ (\gamma^{(2)})^{-1} \} AB \nabla_4 \chi^{(2)} + \{ (\gamma^{(1)})^{-1} \} AB (\nabla_4 \chi^{(1)} - \nabla_4 \chi^{(2)}).$$

To proceed, note that by Definition 2.26 and Sobolev embedding (Proposition 5.1), $\| \chi^{(1)} \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim 1$. Therefore, using (2.6), (8.3) and Propositions 8.2 and 8.3, we obtain

$$\| (\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)}) \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist} \frac{1}{N^\frac{3}{4}}.$$

We next consider $(\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)})$. Again, we use Definition 2.26 and (8.3) to obtain

$$\| (\nabla_4 \chi^{(1)}) - (\nabla_4 \chi^{(2)}) \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist} \frac{1}{N^\frac{3}{4}}.$$

Finally, the terms not involving derivatives of $\chi$. We will look at $(\chi \eta^{(1)}) - (\chi \eta^{(2)})$; the term $(\chi \eta^{(1)}) - (\chi \eta^{(2)})$ is similar. By Definition 2.26 and (8.3), we have

$$\| (\chi \eta^{(1)}) - (\chi \eta^{(2)}) \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist} \frac{1}{N^\frac{3}{4}}.$$

Combining all the above estimates, we thus obtain

$$\| (\nabla_4 \eta^{(1)}) - (\nabla_4 \eta^{(2)}) \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist} \frac{1}{N^\frac{3}{4}}.$$

As argued above, this estimate concludes the proof. \(\square\)

Proposition 8.9. The following estimate holds:

$$\| \mu^{(1)} - \mu^{(2)} \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} + \| \mu^{(2)} - \mu^{(1)} \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist}.$$

Proof. We will only prove the estimate for $\mu^{(1)} - \mu^{(2)}$; the other term can be treated similarly. Arguing as in Proposition 8.8, we first write

$$\nabla_4^{(1)} (\mu^{(1)} - \mu^{(2)}) = - (\nabla_4^{(1)} - \nabla_4^{(2)}) \mu^{(2)} + (\nabla_4 \mu^{(1)}) - (\nabla_4 \mu^{(2)}).$$

Again as in Proposition 8.8, we use (8.22) in Proposition 8.6 to obtain

$$\| (\nabla_4^{(1)} - \nabla_4^{(2)}) \mu^{(2)} \|_{L^\infty L^2 L^2 (S_{\infty} \gamma)} \lesssim \text{dist} \frac{1}{N^\frac{3}{4}}.$$

It thus remains to bound $(\nabla_4 \mu^{(1)}) - (\nabla_4 \mu^{(2)})$ in $L^\infty L^2 L^2 (S_{\infty} \gamma)$. To this end, we consider the terms in the equation (2.43). Note that schematically, we need to consider the following terms: $\nabla \chi \star (\eta, \eta)$, $\nabla (\eta, \eta) \star \chi$, $\chi \star (\eta, \eta)$, where $(\eta, \eta)$ means we take either $\eta$ or $\eta$, and * denotes some contraction with respect to $\gamma$.

We now consider each of these types of terms. For simplicity of the exposition, we will take $\eta$ as a representative of $(\eta, \eta)$. 

}\end{proof}
We begin with the term $\nabla \chi \ast \eta$. This can be treated as the terms in Proposition 8.8.

$$\left\| (\nabla \chi \ast \eta) - (\nabla \chi \ast \eta) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})}$$

$$\lesssim N^{-\frac{1}{2}} \left\| (\nabla \chi) (\chi) - (\nabla \chi) (\chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + N^{-\frac{1}{2}} \left\| (\nabla \chi) - (\nabla \chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})}$$

$$+ N^{-\frac{1}{2}} \left\| (\nabla \chi) - (\nabla \chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + \left\| \eta - \eta \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

We next consider $\chi \ast \nabla \eta$. Note that there is a contribution $\chi (\nabla \chi) (\eta - \eta)$ for which we need to put $\chi$ in $L^2_t L^\infty_x(S_{u, \gamma})$ and put $\nabla \chi$ in $L^2_t L^2_x(S_{u, \gamma})$. Therefore we will not be able to obtain a smallness factor of $N^{-\frac{1}{2}}$.

$$\left\| (\chi \ast \nabla \eta) - (\chi \ast \nabla \eta) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})}$$

$$\lesssim N^{-\frac{1}{2}} \left\| (\chi) - (\chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + N^{-\frac{1}{2}} \left\| (\nabla \chi) - (\nabla \chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + \left\| \eta - \eta \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

Finally, we handle $\chi \ast \eta \ast \eta$. This can again be treated as the terms in Proposition 8.8.

$$\left\| (\chi \ast \eta \ast \eta) - (\chi \ast \eta \ast \eta) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})}$$

$$\lesssim N^{-\frac{1}{2}} \left\| (\chi) - (\chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + N^{-\frac{1}{2}} \left\| (\nabla \chi) - (\nabla \chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + \left\| \eta - \eta \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

This concludes the proof. □

**Proposition 8.10.**

$$\left\| (\chi) - (\chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + \left\| (\nabla \chi) - (\nabla \chi) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

$$\left\| (\omega) - (\omega) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} + \left\| (\nabla \omega) - (\nabla \omega) \right\|_{L^\infty_t L^2_x(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

**Proof.** All of the estimates can be obtained in a similar way, we will consider only $\chi$ in the proof.

By (8.10) in Proposition 8.5, it suffices to bound $\nabla \chi (\chi) - (\chi)$ in $L^1_t L^2_x L^2(S_{u, \gamma})$. We write

$$\nabla \chi (\chi) = -\nabla \chi (\chi) + \nabla \chi (\chi) - \nabla \chi (\chi).$$

By Proposition 8.6, the estimates in Definition 2.26, Propositions 8.1 and 8.3, we have

$$\left\| (\chi) - (\chi) \right\|_{L^1_t L^2_x L^2(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

It therefore suffices to bound $\nabla \chi (\chi)$ in $L^1_t L^2_x L^2(S_{u, \gamma})$. We now look at the corresponding terms in (2.20).

We begin with the $\nabla \eta$ term:

$$\left\| (\nabla \eta) - (\nabla \eta) \right\|_{L^1_t L^2_x L^2(S_{u, \gamma})}$$

$$\lesssim \left\| (\eta) - (\eta) \right\|_{L^1_t L^2_x L^2(S_{u, \gamma})} + \left\| (\nabla \eta) - (\nabla \eta) \right\|_{L^1_t L^2_x L^2(S_{u, \gamma})} \lesssim \frac{\text{dist}}{N^{\frac{3}{2}}}.$$

Next, we consider the quadratic term $\eta$. We need to be more careful since some of the terms involved cannot be bounded in $L^\infty_t L^\infty_x$ type norms. We consider $\nabla \chi$ as an example (the other terms are...
similar). By Hölder’s inequality,
\[
\|\varphi \chi^{(1)} - \varphi \chi^{(2)}\|_{L^2_u L^2(S, \mathbb{R})}^2 \lesssim N^{-\frac{3}{2}} \|\varphi^{(1)} - \varphi^{(2)}\|_{L^\infty_u L^2(S, \mathbb{R})} \|\chi^{(1)}\|_{L^\infty_u L^2(S, \mathbb{R})}^2 + N^{-\frac{3}{2}} \|\varphi^{(2)}\|_{L^\infty_u L^2(S, \mathbb{R})} \|\chi^{(1)} - \chi^{(2)}\|_{L^\infty_u L^2(S, \mathbb{R})} \lesssim \frac{\text{dist}}{N^\frac{1}{2}}. \tag{8.32}
\]

We note explicitly that in the above estimate, while the differences $\varphi^{(1)} - \varphi^{(2)}$ has to be controlled by taking $L_u^\infty$ first before taking $L_u^\infty$, it is important that according to Definition 2.26, $\chi^{(1)}$ can be bounded by taking $L_u^\infty$ first before taking $L_u^2$. A similar comment applies to the product $\varphi^{(2)}(\chi^{(1)} - \chi^{(2)})$.

Putting all these together and using (8.10) in Proposition 8.5, we obtain the desired estimate. \qed

### 8.4. Estimates for $\psi \chi$, $\psi \chi$ and their derivatives.

**Proposition 8.11.**
\[
\|\psi \chi^{(1)} - \psi \chi^{(2)}\|_{L^2_u L^2(S, \mathbb{R})} + \|\psi \chi^{(1)} - \psi \chi^{(2)}\|_{L^2_u L^2(S, \mathbb{R})} \lesssim \text{dist}. \tag{8.33}
\]

In particular,
\[
\|\psi \chi^{(1)} - \psi \chi^{(2)}\|_{L^2_u L^2(S, \mathbb{R})} + \|\psi \chi^{(1)} - \psi \chi^{(2)}\|_{L^2_u L^2(S, \mathbb{R})} \lesssim \frac{\text{dist}}{N^\frac{1}{2}}. \tag{8.34}
\]

**Proof.** We will only prove the estimate for $\psi \chi$; the estimate for $\psi \chi$ is similar.

Fix $\varphi \in [u_j, u_{j+1}]$ and $U \in [u_i, u_{i+1}]$ for the remainder of the proof.

Let $\varphi$ be a function on $S_U$ satisfying
\[
\|\varphi\|_{L^2(S_U)} \lesssim 1. \tag{8.35}
\]

Extend $\varphi$ on $[u_i, u_{i+1}] \times S^2$ by $e_3^{(1)} = 0$. Proposition 5.2, Grönwall’s inequality, and the estimates for $\chi^{(1)}$ and $\Omega^{(1)}$ together imply that
\[
\|\varphi\|_{L^2_u L^2(S, \mathbb{R})} \lesssim 1. \tag{8.36}
\]

Using $e_3^{(2)} = 0$, we also obtain
\[
\varphi = - (\varphi^{(1)} - \varphi^{(2)}) = -(\varphi^{(1)} - \varphi^{(2)})^2 \varphi. \tag{8.37}
\]

To proceed, we use the equation (2.27) for both $\langle \mathcal{M}, g^{(1)} \rangle$ and $\langle \mathcal{M}, g^{(2)} \rangle$. For $\langle \mathcal{M}, g^{(1)} \rangle$ we will use $\varphi$ as the test function; while for $\langle \mathcal{M}, g^{(2)} \rangle$ we will use $\varphi^{(1)} \varphi^{(2)} \sqrt{\det \gamma^{(1)}} \sqrt{\det \gamma^{(2)}}$ (instead of $\varphi$) as the test function.

Taking the difference between the two identities, using the fact that the initial data coincide, and applying (8.35), we then obtain
\[
\int_{S_{U, \varphi}} \varphi \Omega^{(1)}((\psi \chi^{(1)})^{-1} - \Omega^{(2)}((\psi \chi^{(2)})^{-1}) dA_{\gamma^{(1)}}
\]
\[
= \int_{U, \varphi} \int_{S_{U, \varphi}} (\nabla S_{(1)} - S_{(2)}) \psi \chi^{(2)} \Omega^{(1)}(\nabla S_{(1)}^2) dA_{\gamma^{(1)}} du \tag{8.38}
\]
\[
+ \int_{U, \varphi} \int_{S_{U, \varphi}} \varphi \chi^{(1)} - \chi^{(2)} \frac{\Omega^{(1)}(\nabla S_{(1)}^2)}{\Omega^{(2)}} (\nabla S_{(2)}^2) dA_{\gamma^{(1)}} du \tag{8.39}
\]
\[
+ \int_{U, \varphi} \int_{S_{U, \varphi}} (\nabla S_{(1)} - S_{(2)}) \frac{\varphi^{(1)}(\nabla S_{(1)}^2)}{\sqrt{\det \gamma^{(1)}}} dA_{\gamma^{(1)}} du \tag{8.40}
\]
\[
- \int_{U, \varphi} \int_{S_{U, \varphi}} \varphi (\nabla S_{(1)} - S_{(2)}) \frac{\varphi^{(1)}(\nabla S_{(1)}^2)}{\sqrt{\det \gamma^{(1)}}} dA_{\gamma^{(1)}} du \tag{8.41}
\]

We now estimate each of the terms (8.36)–(8.40).
For \((8.36)\), we integrate by parts, use Hölder’s inequality and use the estimates in Definition 2.26 and (8.3) and (8.34) to obtain

\[
\|(8.36)\| \leq \left| \int_{U} \int_{S_{\omega}} \varphi \nabla \chi_{(1)}(\partial \chi_{(2)} \frac{(\Omega_{(1)})^{2}}{\Omega_{(2)}}) \, dA_{\gamma(1)} \, du \right| \\
+ \left| \int_{U} \int_{S_{\omega}} \varphi \left| \partial \psi \frac{1}{\Omega_{(2)}} \right| (\partial \chi_{(2)} \frac{(\Omega_{(1)})^{2}}{\Omega_{(2)}}) \, dA_{\gamma(1)} \, du \right| \\
\lesssim \|\varphi\|_{L^{\infty}(S_{\omega}, \gamma(1))} \|\chi_{(1)} - \psi\|_{L^{\infty}(S_{\omega}, \gamma(1))} \|\partial \chi_{(2)} \frac{(\Omega_{(1)})^{2}}{\Omega_{(2)}}\|_{L^{\infty}(\omega, \gamma(1))} \lesssim \text{dist.} N^\frac{1}{2}.
\]  
(8.41)

For \((8.37)\), we compute

\[
\epsilon_{3}^{(2)} \left\{ \Omega_{(1)}^{2} \right\} \sqrt{\det \gamma(1)} \\
= \left\{ \Omega_{(1)}^{2} \right\} \sqrt{\det \gamma(1)} \left\{ -2\epsilon_{1} + 2\epsilon_{2} + \Omega_{(2)}^{-1} \right\} \\
- \Omega_{(1)} \chi_{(1)} + \Psi_{(2)} \\
- \Omega_{(1)}^{-1} \epsilon_{1} - \Omega_{(2)}^{-1} \epsilon_{2} \\
+ \sqrt{\frac{\det \gamma(1)}{\det \gamma(2)}} \log \frac{\Omega_{(1)}}{\Omega_{(2)}}.
\]

Therefore, using Definition 2.26, (2.3), Hölder’s inequality and (8.34),

\[
\|(8.37)\| \lesssim \text{dist.}
\]  
(8.42)

Using Definition 2.26, (8.3), Hölder’s inequality and (8.34),

\[
\|(8.38)\| + \|(8.39)\| \lesssim \text{dist.}
\]  
(8.43)

Finally, \((8.40)\) can be directly estimate using the definition (8.1) and (8.34):

\[
\|(8.40)\| \lesssim \text{dist}(d\nu(1), d\nu(2)) \lesssim \text{dist.}
\]  
(8.44)

Combining (8.41), (8.42), (8.43) and (8.44) and plugging into the identity preceding these estimates, we obtain

\[
\left| \int_{S_{\omega}} \varphi \Omega_{(1)}(\partial \chi_{(2)} - \frac{\Omega_{(1)}}{\Omega_{(2)}}) \, dA_{\gamma(1)} \right| \lesssim \text{dist.}
\]

By duality and the bounds of \(\Omega_{(1)}\) in Definition 2.26, it follows that

\[
\|(\partial \chi_{(2)} - \frac{\Omega_{(1)}}{\Omega_{(2)}}) \|_{L^{2}(S_{\omega}, \gamma(1))} \lesssim \text{dist.}
\]  
(8.45)

Since \((U, \varphi)\) is arbitrary, by (8.45), we have

\[
\|(\partial \chi_{(1)} - \frac{\Omega_{(1)}}{\Omega_{(2)}}) \|_{L^{\infty}(S_{\omega}, \gamma(1))} \lesssim \text{dist.}
\]  
(8.46)

Finally, we compute

\[
(\partial \chi_{(1)} - \frac{\Omega_{(1)}}{\Omega_{(2)}}) - (\partial \chi_{(2)} - \frac{\Omega_{(1)}}{\Omega_{(2)}}) = (\partial \chi_{(1)} - \frac{\Omega_{(1)}}{\Omega_{(2)}}) - (1 - \frac{\Omega_{(1)}}{\Omega_{(2)}}) (\partial \chi_{(2)} - \frac{\Omega_{(1)}}{\Omega_{(2)}})
\]

and observe that the desired conclusion follows from (8.46), the bounds for \(1 - \frac{\Omega_{(1)}}{\Omega_{(2)}}\) in Proposition 8.4 and the bounds for \((\partial \chi_{(2)} - \frac{\Omega_{(1)}}{\Omega_{(2)}})\) in Definition 2.26.

\[\square\]

**Proposition 8.12.**

\[
\|(\nabla^{(1)}(\partial \chi_{(1)} - \partial \chi_{(2)})\|_{L^{\infty}(\omega, \gamma(1))} + \|(\nabla^{(1)}(\partial \chi_{(1)} - \partial \chi_{(2)})\|_{L^{\infty}(\omega, \gamma(1))} \lesssim \text{dist.}
\]  
(8.47)

In particular,

\[
\|(\nabla^{(1)}(\partial \chi_{(1)} - \partial \chi_{(2)})\|_{L^{\infty}(\omega, \gamma(1))} + \|(\nabla^{(1)}(\partial \chi_{(1)} - \partial \chi_{(2)})\|_{L^{\infty}(\omega, \gamma(1))} \lesssim \text{dist.}
\]  
(8.48)
Proof. It clearly suffices to prove (8.47); as (8.48) follows from (8.47) and Hölder’s inequality. We will only prove the estimate for $\hat{\mathbf{X}}$; the estimate for $\hat{\mathbf{Y}}$ is similar. To this end, we will use the equation for $\hat{\mathbf{X}}(\Omega^{-1}\hat{\chi}_1)$ in (2.51).

Fix $\mathbf{u} \in [u_j, u_{j+1}]$ and $U \in [u_i, u_{i+1}]$ for the remainder of the proof.

Step 1: Definition of $\mathbf{X}$. Let $\mathbf{X}$ be a smooth vector field on $S_{\hat{\mathbf{u}}}$ satisfying

$$
\|\mathbf{X}\|_{L^2(S_{\hat{\mathbf{u}}}, \gamma^{(1)})} \leq 1. 
$$  
(8.49)

Extend $\mathbf{X}$ to an $S$-tangent vector field $\mathbf{X}$ on $[u_i, U] \times S^2$ by stipulating to be the unique solution to

$$
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial}{\partial r} + \nabla_{\mathbf{b}^{(2)}} \mathbf{X} = 0 \\
\mathbf{X}(U, \theta) = \mathbf{X}(\theta).
\end{array} \right.
\end{aligned}
$$  
(8.50)

It is easy to check using (8.49), (8.50) and estimates in Definition 2.26 that for every $u \in [u_i, U]$,

$$
\|\mathbf{X}\|_{L^\infty(L^2(S_{\hat{\mathbf{u}}}, \gamma^{(1)})))} \lesssim 1.
$$  
(8.51)

Step 2: Preliminary computations. Before we proceed, we first carry out a couple of computations for terms that will arise later in Step 3 when we apply the equation (2.51) for $\hat{\mathbf{X}}(\Omega^{-1}\hat{\chi}_1)$. The main point of these computations is to rewrite the expression in a form so that any term involving derivatives of $\mathbf{X}$ can be regrouped as a total divergence.

First, by (8.50), for any $f \in L^\infty_u L^\infty_w W^{2,2}(S_{\hat{\mathbf{u}}}, \gamma^{(1)})$,

$$
\frac{\partial}{\partial r} + \nabla_{\mathbf{b}^{(2)}} \mathbf{X} f = \nabla_{\mathbf{b}^{(2)} - \mathbf{b}^{(1)}} \mathbf{X} f
$$  

$$
div^{(2)}((\nabla_{\mathbf{X}} f)(b^{(2)} - b^{(1)})) = (\nabla_{\mathbf{X}} f)(div^{(2)}(b^{(2)} - b^{(1)})) = - \nabla_{\mathbf{X}}(\nabla_{\mathbf{b}^{(2)} - \mathbf{b}^{(1)}} f - \text{Hess}(f)(\mathbf{X}, b^{(2)} - b^{(1)})).
$$  
(8.52)

Notice that, after integrating by parts and using (2.11), the first term also satisfies

$$
\int_{S_{\hat{\mathbf{u}}}} div^{(2)}((\nabla_{\mathbf{X}} f)(b^{(2)} - b^{(1)}))(\Omega^{(2)})^2 dA_{\gamma^{(2)}} = -2 \int_{S_{\hat{\mathbf{u}}}} (\nabla_{\mathbf{X}} f)(\nabla_{\mathbf{b}^{(2)} - \mathbf{b}^{(1)}} \log \Omega^{(2)})(\Omega^{(2)})^2 dA_{\gamma^{(2)}}.
$$  
(8.53)

On the other hand, we also have

$$
(div^{(1)} \mathbf{X}) \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} - div^{(2)} \mathbf{X}
$$  

$$
= (div^{(2)} \mathbf{X}) \left( \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} - 1 \right) + (div^{(1)} \mathbf{X} - div^{(2)} \mathbf{X}) \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}}
$$  
(8.54)

$$
= div^{(2)} \left((\frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} - 1) \mathbf{X} - \nabla_{\mathbf{X}} (\frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} - 1) \right) + \nabla_{\mathbf{X}} \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}}.
$$

Step 3: Application of (2.51). We now apply (2.51) to both $(M, g^{(1)})$ and $(M, g^{(2)})$, using the fact that $\mathbf{X}$ satisfies (8.50) and that $\Omega^{(1)} = \Omega^{(2)}$, $\gamma^{(1)} = \gamma^{(2)}$ and $(t\hat{\chi}_1^{(1)} + t\hat{\chi}_1^{(2)})^+ = (t\hat{\chi}_1^{(2)})^+$ when $t = 0$. Using also
the computations (8.52)–(8.54), we obtain
\begin{align*}
&\int_{S_{\psi,\gamma}} \left( (\Omega^{(1)})^2 (X ((\Omega^{(1)})^{-1} \psi^{(1)})) - (\Omega^{(2)})^2 \sqrt{\frac{\det \gamma^{(2)}}{\det \gamma^{(1)}}} (X ((\Omega^{(2)})^{-1} \psi^{(2)})) \right) \, dA_{\gamma}, \\
&= \int_{u_i} \int_{S_{\psi,\gamma}} \left\{ 2 \nabla \psi_{b^{(2)}b^{(1)}} \log \Omega^{(2)} + d\psi^{(2)}(b^{(2)} - b^{(1)}) \right\} ((\Omega^{(2)})^{-1} \psi^{(2)})(\Omega^{(2)})^2 \, dA_{\gamma}, \\
&\quad + \int_{u_i} \int_{S_{\psi,\gamma}} \left( \psi_{X}((\Omega^{(2)})^{-1} \psi^{(2)})(\Omega^{(2)})^2 + (\Omega^{(2)})^2 H_{\text{ess}}((\Omega^{(2)})^{-1} \psi^{(2)})(X, b^{(2)} - b^{(1)}) \right) \, dA_{\gamma}. \\
&\quad - 4 \int_{u_i} \int_{S_{\psi,\gamma}} \left\{ \psi_{(1)}((\Omega^{(1)})^3 X ((\Omega^{(1)})^{-1} \psi^{(1)})) - \psi_{(2)}(\Omega^{(2)})^3 X ((\Omega^{(2)})^{-1} \psi^{(2)})) \right\} \sqrt{\frac{\det \gamma^{(2)}}{\det \gamma^{(1)}}} \, dA_{\gamma}, \\
&\quad + \int_{u_i} \int_{S_{\psi,\gamma}} \left( \psi_{X}((\Omega^{(2)})^{-1} \psi^{(2)})(\Omega^{(2)})^2 \right) \sqrt{\frac{\det \gamma^{(2)}}{\det \gamma^{(1)}}} \, dA_{\gamma}, \\
&\quad + \int_{u_i} \int_{S_{\psi,\gamma}} \left( \psi_{X}((\Omega^{(2)})^{-1} \psi^{(2)})(\Omega^{(2)})^2 \right) \sqrt{\frac{\det \gamma^{(2)}}{\det \gamma^{(1)}}} \, dA_{\gamma}, \\
&\quad + \int_{u_i} \int_{S_{\psi,\gamma}} \left( \psi_{X}((\Omega^{(2)})^{-1} \psi^{(2)})(\Omega^{(2)})^2 \right) \sqrt{\frac{\det \gamma^{(2)}}{\det \gamma^{(1)}}} \, dA_{\gamma}, \\
&\quad + \int_{u_i} \int_{S_{\psi,\gamma}} \left( \psi_{X}((\Omega^{(2)})^{-1} \psi^{(2)})(\Omega^{(2)})^2 \right) \sqrt{\frac{\det \gamma^{(2)}}{\det \gamma^{(1)}}} \, dA_{\gamma}.
\end{align*}

Now (8.56)–(8.59) do not involve derivatives of $X$ and we can directly estimate them when using Definition 2.26, (8.3), (8.51), Propositions 8.2 and 8.4, and Hölder’s inequality to get
\begin{align*}
|\langle \psi \rangle| |\langle \psi \rangle| |\langle \psi \rangle| |\langle \psi \rangle| \lesssim \text{dist}.
\end{align*}

For (8.66), we first integrate by parts away the derivative on $X$ and then argue as above to obtain
\begin{align*}
|\langle \psi \rangle| |\langle \psi \rangle| |\langle \psi \rangle| |\langle \psi \rangle| 
&= \int_{u_i} \int_{S_{\psi,\gamma}} \nabla X \left\{ ((\Omega^{(1)})^2) X ((\Omega^{(1)})^{-1} \psi^{(1)})) \right\} \, dA_{\gamma} \, du - \int_{u_i} \int_{S_{\psi,\gamma}} \nabla X \left\{ ((\Omega^{(2)})^2) X ((\Omega^{(2)})^{-1} \psi^{(2)})) \right\} \, dA_{\gamma}, \\
&\lesssim \text{dist}.
\end{align*}

We then consider the terms involving the measure $d\nu^{(2)}$. To handle the terms (8.61)–(8.63), we first use the regularity of $d\nu^{(2)}$ given in (2.26), and then argue as above (with Definition 2.26, (8.3), (8.5), (8.51), Propositions 8.2 and 8.4, and Hölder’s inequality) to obtain
\begin{align*}
|\langle \psi \rangle| |\langle \psi \rangle| |\langle \psi \rangle| |\langle \psi \rangle| &\lesssim \|2X (\log \Omega^{(1)}) \sqrt{\frac{\det \gamma^{(1)}}{\det \gamma^{(2)}}} - 2X (\log \Omega^{(2)}) \|_{L^\infty L^1(S_{\psi,\gamma}, \gamma^{(2)})} + \|((\sqrt{\frac{\det \gamma^{(1)}}{\det \gamma^{(2)}}} - 1)X) \|_{L^\infty L^1(S_{\psi,\gamma}, \gamma^{(2)})} \\
&\quad + \|(-\nabla X (\frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} - 1) + \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} \nabla X \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}}) \|_{L^\infty L^1(S_{\psi,\gamma}, \gamma^{(2)})} \\
&\lesssim \text{dist}.
\end{align*}
Finally, we bound the term (8.64). Using (8.1), (8.3), Hölder’s inequality, (8.51) and Definition 2.26, we obtain
\[
|8.64| \lesssim \text{dist}(\|X\log^2\|_{L^\infty_x L^2_y(S_u, u, \gamma)} + \|X\|_{L^\infty_x L^2_y(S_u, u, \gamma)}) 
\lesssim \text{dist}(1 + \|\nabla \log^2\|_{L^\infty_x L^2_y(S_u, u, \gamma)}) \lesssim \text{dist}.
\] (8.68)

**Step 4: Conclusion.** Plugging the estimates (8.65)–(8.68) back into (8.56)–(8.64), we obtain
\[
\|X\|_{L^2_y(S_u, u, \gamma)} \lesssim \text{dist}.
\] (8.69)
By duality, Definition 2.26, (8.3), Propositions 8.2 and 8.4, and (8.69) that we have just established,
\[
\|\nabla (\tilde{\chi}^{(1)}(1) - \tilde{\chi}^{(2)}(1))\|_{L^2(S_u, u, \gamma)} 
= \sup_{\|X\|_{L^2_y(S_u, u, \gamma)} \leq 1} \int_{S_u} \left( (\nabla (\tilde{\chi}^{(1)}(1))) - (\nabla (\tilde{\chi}^{(2)}(1))) \right) \, dA(y)
\lesssim \sup_{\|X\|_{L^2_y(S_u, u, \gamma)} \leq 1} \int_{S_u} \Omega^{(1)} \left( (\nabla (\tilde{\chi}^{(1)}(1))) - (\nabla (\tilde{\chi}^{(2)}(1))) \right) \, dA(y)
\leq \sup_{\|X\|_{L^2_y(S_u, u, \gamma)} \leq 1} (8.55) + \text{dist} \lesssim \text{dist}.
\] (8.70)
Since \(U \in [u_i, u_{i+1}]\) and \(\underline{u} \in [\underline{u}_j, \underline{u}_{j+1}]\) are both arbitrary, the desired conclusion follows from (8.70).

8.5. **Energy estimates for the renormalized curvature components.** In this subsection, we control the differences of the renormalized curvature components using energy estimates. We first bound the terms appearing in the equations for the difference of the renormalized curvature components.

**Proposition 8.13.** The following equations, when viewed as transport equations, hold in the weak integrated sense\(^{30}\) of Definition 2.30:
\[
(\tilde{\chi}^{(1)}(1)(\beta^{(1)} - \beta^{(2)}) + \nabla (K^{(1)} - K^{(2)}) - (\sigma^{(1)} \nabla (\sigma^{(1)} - \sigma^{(2)})) = \text{error}_{\beta, K, \sigma},
\] (8.71)
\[
(\tilde{\chi}^{(1)}(1)(\beta^{(1)} - \beta^{(2)}) + (\nabla \tilde{\nu}^{(1)}(1)(\beta^{(1)} - \beta^{(2)})) = \text{error}_{\sigma, \beta},
\] (8.72)
\[
(\tilde{\chi}^{(1)}(1)(\beta^{(1)} - \beta^{(2)}) + (\nabla \tilde{\nu}^{(1)}(1)(\beta^{(1)} - \beta^{(2)})) = \text{error}_{\beta, K, \sigma},
\] (8.73)
\[
(\tilde{\chi}^{(1)}(1)(\beta^{(1)} - \beta^{(2)}) - \nabla (K^{(1)} - K^{(2)}) - (\sigma^{(1)} \nabla (\sigma^{(1)} - \sigma^{(2)})) = \text{error}_{\beta, \sigma},
\] (8.74)
where
\[
\|\text{error}_{\beta, K, \sigma}\|_{L^2_y L^2_x L^2(S_u, u, \gamma)} \leq \frac{\text{dist}}{N^2},
\] (8.77)
\[
\|\text{error}_{\sigma, \beta}\|_{L^2_y L^2_x L^2(S_u, u, \gamma)} \leq \frac{\text{dist}}{N^2},
\] (8.78)
\[
\|\text{error}_{\beta, K, \sigma}\|_{L^2_y L^2_x L^2(S_u, u, \gamma)} \leq \frac{\text{dist}}{N^2}.
\] (8.79)

**Proof.** We only consider a subset of the equations in view of their similarities. More precisely, we will consider (8.71) and (8.72). The equation (8.73) is similar to (8.72). On the other hand, the equations (8.74)–(8.76) are similar to (8.71)–(8.73) after changing \(u, \underline{u}, c_3, c_4\) etc. appropriately.

**Step 1: The equation** (8.71). We compute
\[
\text{LHS of (8.71)} = (\tilde{\chi}^{(1)}(1)(\beta^{(1)} - \beta^{(2)}) - \nabla (K^{(1)} - K^{(2)}) - (\sigma^{(1)} \nabla (\sigma^{(1)} - \sigma^{(2)}))^{(1)}
- (\tilde{\chi}^{(2)}(1)(\beta^{(1)} - \beta^{(2)}) + (\nabla \tilde{\nu}^{(2)}(1)(\beta^{(1)} - \beta^{(2)})) = \text{error}_{\beta, \sigma},
\] (8.76)

We now control the RHS in \(L^1_y L^2_x L^2(S_u, u, \gamma))\).

To bound the term I we use the equation (2.31). Schematically, there are three types of terms:

\(^{30}\)We remark that (8.72), (8.73), (8.75) and (8.76) in fact hold in the integrated sense, and therefore a fortiori hold in the weak integrated sense.
where $\ast$ denotes some arbitrary contraction with respect to the metric.

We will take one example from each group of terms above; the other terms can be treated in exactly the same manner.

For the first group, we consider $\nabla \chi \ast \chi$, which can be treated as (8.32). More precisely, we use Hölder’s inequality, Definition 2.26, (8.3), Propositions 8.2 and 8.3 to obtain

$$
\| (\nabla \chi \ast \chi) \|_{L^1_t L^2_x(S,u,w)} \lesssim N^{-\frac{1}{4}} \| (\nabla \chi \ast \chi) \|_{L^1_t L^2_x(S,u,w)} \left( \sum_{i=1,2} \| \nabla \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \right) + N^{-\frac{1}{4}} \| \nabla \chi \|_{L^\infty_t L^2_x(S,u,w)} \| \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \| \ast \|_{L^2_x(S,u,w)}.
$$

For the second group, we consider $\eta \ast \chi \ast \omega$. We use Hölder’s inequality, Definition 2.26, (8.3), Proposition 8.2 and (8.32) to obtain

$$
\| (\eta \ast \chi \ast \omega) \|_{L^1_t L^2_x(S,u,w)} \lesssim N^{-\frac{1}{4}} \| (\eta \ast \chi \ast \omega) \|_{L^1_t L^2_x(S,u,w)} \left( \sum_{i=1,2} \| \nabla \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \right) + N^{-\frac{1}{4}} \| \nabla \chi \|_{L^\infty_t L^2_x(S,u,w)} \| \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \| \ast \|_{L^2_x(S,u,w)}.
$$

For the third group, we consider $\eta K$. Using Hölder’s inequality, Definition 2.26 and (8.3), we obtain

$$
\| (\eta K) \|_{L^1_t L^2_x(S,u,w)} \lesssim N^{-\frac{1}{4}} \| (\eta K) \|_{L^1_t L^2_x(S,u,w)} \left( \sum_{i=1,2} \| \nabla \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \right) + N^{-\frac{1}{4}} \| \nabla \chi \|_{L^\infty_t L^2_x(S,u,w)} \| \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \| \ast \|_{L^2_x(S,u,w)}.
$$

To handle $\Pi$ we use (8.25) in Proposition 8.6. Using the estimates for $\beta(2)$ given by Definition 2.26 and the estimate for $\nabla(2)/\beta(2)$ obtained after using (2.31), we obtain

$$
\| \Pi \|_{L^1_t L^2_x(S,u,w)} \lesssim \frac{\text{dist}}{N^{\frac{1}{4}}}
$$

Finally, the term $\Pi$ can be bounded by Hölder’s inequality, Sobolev embedding (Proposition 5.1), Definition 2.26 and (8.3),

$$
\| \Pi \|_{L^1_t L^2_x(S,u,w)} \lesssim N^{-\frac{1}{4}} \| \chi \|_{L^\infty_t L^\infty_x(S,u,w)} \| \ast \|_{L^2_x(S,u,w)}.
$$

Combining the above considerations, we thus obtain (8.71).

**Step 2: The equation (8.72).** We compute as in Step 1 to obtain

$$
\text{LHS of (8.72)} = \left( \nabla \hat{\sigma} + \partial \hat{\sigma} \right) - \left( \nabla \hat{\sigma} + \partial \hat{\sigma} \right)^{(2)}
$$

$\ast =$

$$
= \left( \nabla \hat{\sigma} - \nabla \hat{\sigma} \right)^{(2)} - \left( \partial \hat{\sigma} - \partial \hat{\sigma} \right)^{(2)}.
$$

According to (2.32), $(\nabla \hat{\sigma} + \partial \hat{\sigma}) - (\nabla \hat{\sigma} + \partial \hat{\sigma})^{(2)}$ consists of terms similar to those in $(\nabla \hat{\sigma}) - (\nabla \mu)$ and therefore I can be treated in a similar manner as in Proposition 8.9. We thus obtain

$$
\| I \|_{L^1_t L^2_x(S,u,w)} \lesssim N^{-\frac{1}{4}} \| I \|_{L^1_t L^2_x(S,u,w)} \lesssim \frac{\text{dist}}{N^{\frac{1}{4}}}.
$$
We omit the details.

The term II can be controlled using (8.23) in Proposition 8.6 after using the estimates for $K$ given by combining Definition 2.26 and the equation (2.32) so that

$$\|\Pi\|_{L^1 \rightarrow L^q(S_u, \gamma)} \lesssim \frac{\text{dist}}{N^\frac{1}{2}}.$$  

Finally, the term III can be bounded by Hölder's inequality, Sobolev embedding (Proposition 5.1), Definition 2.26, (8.3), Propositions 8.2 and 8.3 so that

$$\|\Pi\|_{L^1 \rightarrow L^q(S_u, \gamma)} \lesssim N^{-1} \|\gamma(1) - \gamma(2)\|_{L^q(S_u, \gamma)} \|\beta(1) - \beta(2)\|_{L^q(S_u, \gamma)} + N^{-1} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^q(S_u, \gamma)} \|\beta(1) - \beta(2)\|_{L^q(S_u, \gamma)}$$

$$\lesssim N^{-1} \|\gamma(1) - \gamma(2)\|_{L^q(S_u, \gamma)} \|\beta(1) - \beta(2)\|_{L^q(S_u, \gamma)} \lesssim \frac{\text{dist}}{N}.$$  

This concludes the proof of (8.72). \hfill \Box

**Proposition 8.14.**

$$\|\beta(1) - \beta(2)\|_{L^q(S_u, \gamma)} + \|\mathbf{v}(1) - \mathbf{v}(2)\|_{L^q(S_u, \gamma)} \lesssim \frac{\text{dist}}{N^\frac{1}{2}},$$  

(8.79)

$$\|\beta(1) - \beta(2)\|_{L^q(S_u, \gamma)} + \|\mathbf{v}(1) - \mathbf{v}(2)\|_{L^q(S_u, \gamma)} \lesssim \frac{\text{dist}}{N^\frac{1}{2}}.$$  

(8.80)

**Proof.** The proofs of (8.79) and (8.80) are similar; we will only treat (8.79). This will be achieved by considering (8.71)–(8.73).

**Step 1: Derivation of the main energy identities.** We begin with (8.71), which is satisfied in the weak integrated sense (see Definition 2.31). Using Definition 2.31 the fact that $\beta(1) = \beta(2)$ a.e. on $\{u = u_i\}$, this means that

$$\int_{u_i}^{u_{i+1}} \int_{S_u} \langle \varphi, \beta(1) - \beta(2) \rangle \Omega(1) \, dA_{\gamma(1)} \, du$$

$$+ \int_{u_i}^{u_{i+1}} \int_{S_u} \langle \varphi, - \mathbf{v}(K(1) - K(2)) + (\gamma(1)) \mathbf{v}(\mathbf{v}(1) - \mathbf{v}(2)) + \text{error}_\beta, K, \sigma \rangle \Omega(1) \, dA_{\gamma(1)} \, du \, du'$$

$$+ \int_{u_i}^{u_{i+1}} \int_{S_u} \langle (\varphi, \mathbf{v}(1) - 2\mathbf{v}(2)) \Omega(1) \rangle \, dA_{\gamma(1)} \, du = 0.$$  

A priori this holds for $\varphi \in C^1$, but using the bounds in Definition 2.26 and (8.77), we can apply a density argument to show that in fact it suffices to have $\varphi \in C^0 L^1 L^2(S_u, \gamma)$ and $\mathbf{v}_\beta \in L^1 L^2(S_u, \gamma)$. Therefore, for every fixed $u \in [u_i, u_{i+1}]$, we can choose

$$\mathcal{A}(u, u', \theta) := ((\gamma(1))^{-1}) \mathcal{A}(\beta(1) - \beta(2))((u, u', \theta) \mathcal{A}(u, u', \theta),$$

where $\mathcal{A}$ denotes the indicator function. We then obtain

$$\int_{u_i}^{u} \int_{S_{u'}} \langle \beta(1) - \beta(2) \rangle_\gamma(1) \Omega(1) \, dA_{\gamma(1)} \, du'$$

$$+ 2 \int_{u_i}^{u} \int_{S_{u'}} \langle \beta(1) - \beta(2), - \mathbf{v}(K(1) - K(2)) + (\gamma(1)) \mathbf{v}(\mathbf{v}(1) - \mathbf{v}(2)) \rangle_\gamma(1) \Omega(1)^2 \, dA_{\gamma(1)} \, du' \, du'$$

$$+ \int_{u_i}^{u} \int_{S_{u'}} \langle (\beta(1) - \beta(2), \text{error}_\beta, K, \sigma) \rangle_\gamma(1) \Omega(1)^2 \, dA_{\gamma(1)} \, du' \, du' = 0.$$  

(8.81)

In a completely similar manner, but using (8.72) and (8.73) instead of (8.71), we obtain

$$\int_{u_i}^{u} \int_{S_{u'}} |K(1) - K(2)|^2 \Omega(1) \, dA_{\gamma(1)} \, du'$$

$$- 2 \int_{u_i}^{u} \int_{S_{u'}} \langle K(1) - K(2) \rangle \Omega(1) \, dA_{\gamma(1)} \, du' \, du'$$

$$+ \int_{u_i}^{u} \int_{S_{u'}} \langle (2(K(1) - K(2)) \text{error}_K, \beta + \mathbf{v}(1) - 2\mathbf{v}(2)) \rangle_\gamma(1) \Omega(1)^2 \, dA_{\gamma(1)} \, du' \, du' = 0.$$  

(8.82)
and
\[
\int_{u_i}^{u} \int_{S_{u',u}} |\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}|^2 \Omega^{(1)} \, dA_{\gamma_i} \, du' \\
- 2 \int_{u_i}^{u} \int_{S_{u',u}} (\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}) \, (d\nu^{*})(\beta^{(1)} - \beta^{(2)})(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
+ \int_{u_i}^{u} \int_{S_{u',u}} (2(\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}) \text{error}_{\sigma,\beta} + (\nu \chi^{(1)} - 2 \omega^{(1)})|\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}|^2)(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \, du' = 0.
\]
(8.83)

Our goal will be to derive an estimate after summing (8.81), (8.82) and (8.83). In the next two steps, we will estimate terms in (8.81)–(8.83). We will then return to summing (8.81), (8.82) and (8.83) in Step 4.

**Step 2: Handling the main angular terms.** We note that the highest order terms in (8.81)–(8.83) involving angular derivatives of $\beta^{(1)} - \beta^{(2)}$, $K^{(1)} - K^{(2)}$ and $\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}$ cannot be directly controlled by dist. Instead, we need to integrate by parts. Using also (2.11) and Hölder’s inequality, we obtain
\[
\begin{align*}
2 \int_{u_i}^{u} \int_{S_{u',u}} (\beta^{(1)} - \beta^{(2)}, -\nabla (K^{(1)} - K^{(2)}) + (\nu^{*})(\sigma^{(1)} - \sigma^{(2)}))_{\gamma_i}(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
- 2 \int_{u_i}^{u} \int_{S_{u',u}} (K^{(1)} - K^{(2)}) \, d\nu^{*}(\beta^{(1)} - \beta^{(2)})(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
- 2 \int_{u_i}^{u} \int_{S_{u',u}} (\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}) \, (d\nu^{*})(\beta^{(1)} - \beta^{(2)})(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
= 2 \int_{u_i}^{u} \int_{S_{u',u}} (K^{(1)} - K^{(2)})(\beta^{(1)} - \beta^{(2)}) : \nabla (\eta + \eta)(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
+ 2 \int_{u_i}^{u} \int_{S_{u',u}} (\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)})(\beta^{(1)} - \beta^{(2)}) : \nabla (\eta + \eta)(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
\lesssim \frac{1}{N} \|\beta^{(1)} - \beta^{(2)}\|_{L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|\Omega^{(1)} - K^{(2)}, \sigma^{(1)} - \sigma^{(2)})\|_{(L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|^2) \lesssim \text{dist}^2 \frac{1}{N}.
\]
(8.84)

**Step 3: Estimating the error terms.** We now handle the remaining terms in (8.81), (8.82) and (8.83):
\[
\begin{align*}
\int_{u_i}^{u} \int_{S_{u',u}} (2(\beta^{(1)} - \beta^{(2)}, \text{error}_{\beta}, K, \sigma)_{\gamma_i} + (\nu \chi^{(1)} - 2 \omega^{(1)})(\beta^{(1)} - \beta^{(2)})(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
+ \int_{u_i}^{u} \int_{S_{u',u}} (2(K^{(1)} - K^{(2)}) \text{error}_{K, \beta} + (\nu \chi^{(1)} - 2 \omega^{(1)})(K^{(1)} - K^{(2)})(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
+ \int_{u_i}^{u} \int_{S_{u',u}} (2(\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)}) \text{error}_{\sigma, \beta} + (\nu \chi^{(1)} - 2 \omega^{(1)})(\dot{\sigma}^{(1)} - \dot{\sigma}^{(2)})(\Omega^{(1)})^2 \, dA_{\gamma_i} \, du' \\
\lesssim \|\beta^{(1)} - \beta^{(2)}\|_{L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|\text{error}_{\beta}, K, \sigma)_{L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|2} \\
+ \|\beta^{(1)} - \beta^{(2)}\|_{L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|2} \|\nu \chi^{(1)} - 2 \omega^{(1)}\|_{L^\infty \|L^2 \|L^\infty((S_{u',u},\gamma_i)\|2} \\
+ \|(K^{(1)} - K^{(2)}, \dot{\sigma}^{(1)} - \dot{\sigma}^{(2)})\|_{(L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|^2} \|\text{error}_{K, \beta, \sigma, \beta}\|_{(L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|^2} \\
+ \|(K^{(1)} - K^{(2)}, \dot{\sigma}^{(1)} - \dot{\sigma}^{(2)})\|_{(L^\infty \|L^2 \|L^2((S_{u',u},\gamma_i)\|^2} \|\nu \chi^{(1)} - 2 \omega^{(1)}\|_{L^\infty \|L^2 \|L^\infty((S_{u',u},\gamma_i)\|2} \lesssim \text{dist}^2 \frac{1}{N^2},
\]
(8.85)

where we have used (8.77) and (8.78), as well as the estimates
\[
\|\nu \chi^{(1)} - 2 \omega^{(1)}\|_{L^\infty \|L^2 \|L^\infty((S_{u',u},\gamma_i)\|2} + \|\nu \chi^{(1)} - 2 \omega^{(1)}\|_{L^\infty \|L^2 \|L^\infty((S_{u',u},\gamma_i)\|2} \lesssim N^{-\frac{1}{2}},
\]
which in turn follow from the bounds in Definition 2.26 and Hölder’s inequality.
Step 4: Putting everything together. Summing (8.81), (8.82), (8.83), and using (8.84) and (8.85), we obtain that for every \((u, \tilde{u}) \in [u_1, u_{n+1}] \times [\tilde{u}_j, \tilde{u}_{j+1}]\),

\[
\int_{u_1}^{u} \int_{S_{u', \eta}} \left( |K^{(1)}(u) - K^{(2)}| + |\beta^{(1)}(u) - \beta^{(2)}(u)| \right) dA_{\gamma}(u) du' + \int_{\tilde{u}_j}^{\tilde{u}} \int_{S_{\tilde{u}' \eta, \tilde{\gamma}}} |\beta^{(1)}(\tilde{u}) - \beta^{(2)}(\tilde{u})| \Omega^{(1)}(\tilde{u}) dA_{\tilde{\gamma}}(\tilde{u}) du' \lesssim \frac{\text{dist}^2}{N^2}.
\]

Since \((u, \tilde{u})\) is arbitrary, we obtain (8.79) after taking square roots. \(\square\)

8.6. Elliptic estimates for the Ricci coefficients. We recall the following standard elliptic estimate for div-curv systems:

**Proposition 8.15.** Let \((S^2, \gamma)\) be a Riemannian manifold such that \(\gamma \in C^1\) and the Gauss curvature \(K_\gamma \in L^2(S^2, \gamma)\). Then the following holds for all covariant symmetric tensor \(\xi\) of rank \((r + 1)\) belonging to \(W^{1,2}(S^2, \gamma)\):

\[
\int_{S^2} \left( |\nabla \xi|^2 + (r + 1) K_\gamma |\xi|^2 + r K_\gamma |\text{tr}_\gamma \xi|^2 \right) dA_\gamma = \int_{S^2} \left( |\text{div}_\gamma \xi|^2 + |\text{curl}_\gamma \xi|^2 + r K_\gamma |\text{tr}_\gamma \xi|^2 \right) dA_\gamma. \tag{8.86}
\]

In particular, there exists a constant \(C > 0\) depending only on \(r\), \(\|K_\gamma\|_{L^2(S^2, \gamma)}\), the area \(\text{Area}(S^2, \gamma)\), and the isoperimetric constant \(I(S^2, \gamma)\) such that

\[
\|\nabla \xi\|_{L^2(S^2, \gamma)} \leq C(\|\text{div}_\gamma \xi\|_{L^2(S^2, \gamma)} + \|\text{curl}_\gamma \xi\|_{L^2(S^2, \gamma)} + \|\xi\|_{L^2(S^2, \gamma)} + \|\text{tr}_\gamma \xi\|_{L^2(S^2, \gamma)}). \tag{8.87}
\]

**Proof.** A proof of (8.86) can be found in Lemma 7.1 in [10].

In order to prove (8.87), we first note that by Hölder’s inequality and the Sobolev inequality (Proposition 5.1),

\[
\int_{S^2} ((r + 1) K_\gamma |\xi|^2 + r K_\gamma |\text{tr}_\gamma \xi|^2) dA_\gamma \lesssim \|K_\gamma\|_{L^2(S^2, \gamma)} (\|\xi\|_{L^2(S^2, \gamma)} + \|\text{tr}_\gamma \xi\|_{L^2(S^2, \gamma)}) \lesssim \|\xi\|_{L^2(S^2, \gamma)}^2 + \|\text{tr}_\gamma \xi\|_{L^2(S^2, \gamma)}^2.
\]

Plugging this into (8.86) and applying Young’s inequality to absorb \(\|\nabla \xi\|_{L^2(S^2, \gamma)}\), we obtain (8.87). \(\square\)

**Proposition 8.16.**

\[
\|\nabla (\eta^{(1)} - \eta^{(2)})\|_{L^2(S^2, \gamma^{(1)}')} + \|\nabla (\eta^{(1)} - \eta^{(2)})\|_{L^2(S^2, \gamma^{(2)}')} \lesssim \frac{\text{dist}}{N^4},
\]

\[
\|\nabla (\eta^{(1)} - \eta^{(2)})\|_{L^2(S^2, \gamma^{(1)}')} + \|\nabla (\eta^{(1)} - \eta^{(2)})\|_{L^2(S^2, \gamma^{(2)}')} \lesssim \frac{\text{dist}}{N^4}.
\]

**Proof.** By Definition 2.39 and (2.42),

\[
\text{div} (\eta^{(1)} - \eta^{(2)}) = -(\mu^{(1)} - \mu^{(2)}) + K^{(1)} - K^{(2)} - (\text{div}^{(1)} - \text{div}^{(2)})\eta^{(2)},
\]

\[
\text{curl} (\eta^{(1)} - \eta^{(2)}) = \tilde{\sigma}^{(1)} - \tilde{\sigma}^{(2)} - (\text{curl}^{(1)} - \text{curl}^{(2)})\eta^{(2)}.
\]

Applying Proposition 8.15 with \((S^2, \gamma) = (S_{u, \omega}, \gamma^{(1)}')\) for every \((u, \omega)\), and using Hölder’s inequality, we then obtain

\[
\|\nabla (\eta^{(1)} - \eta^{(2)})\|_{L^2(S_{u, \omega}, \gamma^{(1)})} + \|\nabla (\eta^{(1)} - \eta^{(2)})\|_{L^2(S_{u, \omega}, \gamma^{(2)})} \lesssim \frac{\text{dist}}{N^4}.
\]

The first four terms in (8.88) can be bounded above by \(\lesssim \frac{\text{dist}}{N^4}\) using Propositions 8.8, 8.9 and 8.14.

For the last two terms in (8.88), we use Proposition 8.2 to bound the difference of the inverse metrics and use Proposition 8.3 to control the difference of the connections, and combine them with the estimate.
in Proposition 8.7. Using also the bound for $\eta^{(2)}$ given by Definition 2.26, Hölder’s inequality and Sobolev embedding (Proposition 5.1), we obtain
\[
\|(d\hat{\nabla}^{(1)} - d\hat{\nabla}^{(2)})\eta^{(2)}\|_{L^2(S_{u,u}, \gamma^{(1)})} + \|(c\hat{\nabla}^{(1)} - c\hat{\nabla}^{(2)})\eta^{(2)}\|_{L^2(S_{u,u}, \gamma^{(1)})} \\
\lesssim \|\gamma^{(1)} - \gamma^{(2)}\|_{L^4(S_{u,u}, \gamma^{(1)})} \|\nabla\eta^{(2)}\|_{L^4(S_{u,u}, \gamma^{(1)})} + \|\hat{\nabla}^{(1)} - \hat{\nabla}^{(2)}\|_{L^2(S_{u,u}, \gamma^{(1)})} \|\eta^{(2)}\|_{L^\infty(S_{u,u}, \gamma^{(1)})} \\
\lesssim \text{dist} \frac{N}{\xi^2}.
\]

Combining all the above bounds gives the desired estimates for $\hat{\nabla}^{(1)}(\eta^{(1)} - \eta^{(2)})$. The estimate for $\eta^{(1)} - \eta^{(2)}$ can be derived in a similar manner but using instead the following equations\footnote{To derive the equation for $c\hat{\nabla}^{(1)}(\eta^{(1)} - \eta^{(2)})$, we use (2.11), and $c\hat{\nabla}^{(1)}\nabla \Omega^{(1)} = c\hat{\nabla}^{(2)}\nabla \Omega^{(2)} = 0$, in addition to (2.42).}:
\[
d\hat{\nabla}^{(1)}(\eta^{(1)} - \eta^{(2)}) = -(\hat{\eta}^{(1)} - \hat{\eta}^{(2)}) + K^{(1)} - K^{(2)} - (d\hat{\nabla}^{(1)} - d\hat{\nabla}^{(2)})\eta^{(2)},
\]
\[
c\hat{\nabla}^{(1)}(\eta^{(1)} - \eta^{(2)}) = -(\hat{\sigma}^{(1)} - \hat{\sigma}^{(2)}) - (c\hat{\nabla}^{(1)} - c\hat{\nabla}^{(2)})\eta^{(2)}.
\]
We omit the details. \(\square\)

**Proposition 8.17.**
\[
\|\hat{\nabla}^{(1)}(\chi^{(1)} - \chi^{(2)})\|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})} + \|\nabla^{(1)}(\hat{\chi}^{(1)} - \hat{\chi}^{(2)})\|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})} \lesssim \text{dist} \frac{N}{\xi^2}.
\]

**Proof.** We will only handle the estimate for $\hat{\chi}^{(1)} - \hat{\chi}^{(2)}$; that for $\chi^{(1)} - \chi^{(2)}$ can be treated similarly.

**Step 1:** Estimates for $d\hat{\nabla}^{(1)}(\chi^{(1)} - \chi^{(2)})$. By Definition 2.39,
\[
d\hat{\nabla}^{(1)}(\chi^{(1)} - \chi^{(2)}) = -(d\hat{\nabla}^{(1)} - d\hat{\nabla}^{(2)})\chi^{(2)} - (\beta^{(1)} - \beta^{(2)}) + \frac{1}{2} (\nabla^{(1)} \chi^{(1)} - \nabla^{(2)} \chi^{(2)})
\]
\[
- \frac{1}{2} \left[ \left(\eta^{(1)} - \eta^{(2)}\right) \cdot \left(\chi^{(1)} - \chi^{(2)}\right) \right] - \frac{1}{2} \left(\eta^{(1)} - \eta^{(2)}\right) \cdot \left(\chi^{(1)} - \chi^{(2)}\right) \right)^2.
\]  
(8.89)

We now estimate each term in (8.89). By Propositions 8.2, 8.3 and 8.7
\[
\|(d\hat{\nabla}^{(1)} - d\hat{\nabla}^{(2)})\chi^{(2)}\|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})} \lesssim \text{dist} \frac{N}{\xi^2}.
\]
The $\beta^{(1)} - \beta^{(2)}$ term can be estimated using Proposition 8.14 by
\[
\|\beta^{(1)} - \beta^{(2)}\|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})} \lesssim \text{dist} \frac{N}{\xi^2}.
\]
The remaining terms can be controlled using the bounds in Definition 2.26, the estimates obtained in Propositions 8.8, 8.11, 8.12 and Hölder’s inequality as follows:
\[
\|\frac{1}{2} (\nabla^{(1)} \chi^{(1)} - \nabla^{(2)} \chi^{(2)}) \cdot \left(\eta^{(1)} - \eta^{(2)}\right) \|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})}
\]
\[
\lesssim \text{dist} \frac{N}{\xi^2}.
\]

Combining all the above estimates, we thus obtain
\[
\|d\hat{\nabla}^{(1)}(\chi^{(1)} - \chi^{(2)})\|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})} \lesssim \text{dist} \frac{N}{\xi^2}.
\]  
(8.90)

**Step 2:** Estimates for $\nabla^{(1)}(\chi^{(1)} - \chi^{(2)})$. Next, we compute, using $\nabla^{(1)}(\chi^{(1)} - \chi^{(2)}) = 0$, that
\[
\nabla^{(1)}(\chi^{(1)} - \chi^{(2)}) = -((\gamma^{(1)})^{-1} - (\gamma^{(2)})^{-1}) \cdot \hat{\chi}^{(2)}.
\]  
(8.91)

Combining (8.91) with Propositions 8.2 and 8.7, we obtain
\[
\|\nabla^{(1)}(\chi^{(1)} - \chi^{(2)})\|_{L^\infty_t L^2_x L^2(S_{u,u}, \gamma^{(1)})} \lesssim \text{dist} \frac{N}{\xi^2}.
\]  
(8.92)

**Step 3:** Estimates for $c\hat{\nabla}^{(1)}(\chi^{(1)} - \chi^{(2)})$ and conclusion of argument. To proceed, note that for any symmetric rank-2 tensor $\xi$, we have (see for instance computations leading up to (7.63) in [10])
\[
c\hat{\nabla}^{(1)}\xi = -(*\nabla)^{(1)}\text{tr}_{(1)}\xi + (*d\nabla)^{(1)}\xi.
\]
The following estimates hold:

8.7. Estimates for the measure-valued null dusts.

**Proposition 8.18.** The following estimates hold:

\[
\sup_{u' \in [u_i, u_{i+1}]} \sup_{\varphi \in \mathcal{C}_c^\infty} \left| \int_{\Omega_{u'}} \varphi \left( \nu'_{u'} \right) \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \, du' \right| \lesssim \frac{\text{dist}}{N},
\]

(8.94)

\[
\sup_{u' \in [u_i, u_{i+1}]} \sup_{\varphi \in \mathcal{C}_c^\infty} \left| \int_{\Omega_{u'}} \varphi \left( \nu'_{u'} \right) \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \, du' \right| \lesssim \frac{\text{dist}}{N}.
\]

(8.95)

**Proof.** In view of their similarities we will only prove (8.94).

Let \( \varphi(u, \theta) \) be a smooth function compactly supported in \((u_j, u_{j+1}) \times S^2\) which satisfies \(\|\varphi\|_{L^\infty} \lesssim 1\).

For every fixed \(U \in [0, u_i]\), define \(\varphi_U(u, \theta)\) to be the unique solution to the following initial value problem:

\[
\begin{aligned}
\partial_u \varphi_U + \nabla_{b^{(1)}} \varphi_U &= 0 \\
\varphi_U(U, u, \theta) &= \varphi(u, \theta).
\end{aligned}
\]

(8.96)

It then follows from integrating (8.96) and using the estimates in Definition 2.26 that

\[
\|\varphi_U\|_{L^\infty_t L^2_x(S^2, \mu^{\gamma(1)})} \lesssim \|\varphi\|_{L^\infty_t L^2_x(S^2, \mu^{\gamma(1)})} \lesssim 1.
\]

(8.97)

We compute

\[
(\partial_u + \nabla_{b^{(2)}}) \left( \varphi_U \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \right) \\
= \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \{ \nabla_{b^{(2)}} - b^{(1)} \} \varphi_U + \varphi_U (\partial_u + \nabla_{b^{(2)}}) \left( \log \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \right)
\]

\[
= \left[ \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \left( b^{(2)} - b^{(1)} \right) \right] \varphi_U - \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \left[ \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \left( b^{(2)} - b^{(1)} \right) \right] \varphi_U
\]

(8.98)

\[
= \left[ \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \left( b^{(2)} - b^{(1)} \right) \right] \varphi_U + \left[ \Omega \chi \right]^{(2)} - \left( \Omega \chi \right)^{(1)} \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \varphi_U,
\]

where in the last line we have used \((by (2.9))\)

\[
(\partial_u + \nabla_{b^{(2)}}) \left( \log \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \right) - \left[ \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \right] \left( b^{(2)} - b^{(1)} \right) = (\Omega \chi)^{(2)} - (\Omega \chi)^{(1)}.
\]
Proof. We will only prove (8.99); the estimate (8.100) is similar.

We have

\[ \frac{1}{N} \| b^{(2)} - b^{(1)} \|_{L^\infty_\nu L^2(S_u \pm \gamma(2))} + \frac{1}{N} \| b^{(2)} - b^{(1)} \|_{L^\infty_\nu L^2(S_u \pm \gamma(2))} \leq \frac{\text{dist} N}{N}, \]

where in the second to last inequality we have used (8.97) and the Cauchy–Schwarz inequality; and in the last inequality we have used (8.3) and Proposition 8.11.

**Proposition 8.19.** The following estimates hold:

\[ \| X \|_{L^\infty_\nu L^2(S_u \pm \gamma(1))} \leq 1. \]  

(8.101)

Proof. We will only prove (8.99); the estimate (8.100) is similar.

Fix \( U \in [u_i, u_{i+1}] \).

**Step 1: Choice of \( \hat{X} \).** Let \( \hat{X}(u, \nu, \vartheta) \) be a \( C^\infty \) \( S \)-tangent vector field on \( (u_i, u_{i+1}) \). Extend \( \hat{X} \) to \( \hat{X}(u, \nu) \), which is defined to be the unique solution to the following initial value problem:

\[ \begin{aligned}
    \hat{X}(1) &= 0 \\
    \partial_u \hat{X}(u, \nu) &= \hat{X}(u, \nu)
\end{aligned} \]

so that we in particular have

\[ \| \hat{X} \|_{L^\infty_\nu L^2(S_u \pm \gamma(1))} \leq 1. \]

(8.101)

We now compute (using\(^{32}\) (2.46)) that

\[ \begin{aligned}
    (\partial_u + \hat{y}_{b^{(1)}})(d\hat{X}^{(1)}) &= \Omega^{(1)} \left( -\frac{1}{2} \hat{b}^{(1)} \hat{X}^{(1)} \hat{X} - \hat{X}^{(1)} \hat{X}^{(1)} \right) \\
    &= F_{\hat{X}}.
\end{aligned} \]

(8.102)

On the other hand, computing as in (8.98) and using (8.102), we obtain

\[ \begin{aligned}
    (\partial_u + \hat{y}_{b^{(2)}})(d\hat{X}^{(1)}) &= \Omega^{(1)} \left( -\frac{1}{2} \hat{b}^{(2)} \hat{X}^{(1)} \hat{X} - \hat{X}^{(1)} \hat{X}^{(1)} \right) + [\Omega \hat{X}^{(2)}] - \Omega \hat{X}^{(1)} \right) \frac{\text{det} \gamma^{(1)}}{\text{det} \gamma^{(2)}} (d\hat{X}^{(1)}) + F_{\hat{X}} \frac{\text{det} \gamma^{(1)}}{\text{det} \gamma^{(2)}} \\
    &= G_{\hat{X}} + F_{\hat{X}} \frac{\text{det} \gamma^{(1)}}{\text{det} \gamma^{(2)}},
\end{aligned} \]

(8.103)

where \( F_{\hat{X}} \) is as in (8.102).

\(^{32}\) Notice that we have enough regularity to justify this commutation, where the derivatives on the LHS of (8.102) are understood as weak derivatives.
Step 2: Application of (2.29). We now apply the transport equation\(^\text{33}\) for \(\nu_u\) (in (2.29)) with (8.102) and (8.103), and the fact that \(\nu_u^{(1)} = \nu_u^{(2)}\) to obtain

\[
\left| \int_{H_u} (d\nu_u^{(1)} \mathcal{X}) (d\nu_u^{(1)} - \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} d\nu_u^{(2)}) \right|
\]

\[
= - \int_{u_1}^{U} \int_{H_u} (\partial_u + \nabla \psi(\nu_u^{(1)})) d\nu_u^{(1)} \, du
\]

\[
+ \int_{u_1}^{U} \int_{H_u} (\partial_u + \nabla \psi(\nu_u^{(1)})) (d\nu_u^{(1)} \mathcal{X}) \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} \, d\nu_u^{(2)} \, du
\]

\[
\leq \left| \int_{u_1}^{U} \int_{H_u} F_X (d\nu_u^{(1)} - \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} d\nu_u^{(2)}) \, du \right| \tag{8.104}
\]

\[
+ \int_{u_1}^{U} \int_{H_u} G_X d\nu_u^{(2)} \, du \tag{8.105}
\]

For the remainder of the proof we will estimate (8.104) and (8.105).

Step 3: Estimating (8.104). For the term (8.104), we recall the definition of \(F_X\) in (8.102) and compute

\[
F_X = - \frac{1}{2} d\nu_u^{(1)} (\Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X})) + \frac{1}{2} (\nabla \psi(\Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}))) - d\nu_u^{(1)} (\Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X})) \mathcal{X} + \mathcal{X} \cdot d\nu_u^{(1)} (\Omega^{(1)} \mathcal{X})
\]

\[
+ \Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}) - \Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}) \eta^{(1)} \mathcal{X}
\]

\[
= - d\nu_u^{(1)} (F_{X,1}) + F_{X,2},
\]

where \(F_{X,1}\) and \(F_{X,2}\), given by

\[
F_{X,1} := \frac{1}{2} \Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}) + \Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}) \mathcal{X},
\]

\[
F_{X,2} := \frac{1}{2} (\nabla \psi(\Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}))) \mathcal{X} + \mathcal{X} \cdot d\nu_u^{(1)} (\Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X})) + \Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}) - \Omega^{(1)} \psi(\nu_u^{(1)} \mathcal{X}) \eta^{(1)} \mathcal{X},
\]

obey the estimates

\[
\|F_{X,1}\|_{L^2_x L^\infty_y L^2(S)} + \|F_{X,2}\|_{L^2_x L^\infty_y L^2(S)} \lesssim 1.
\]

(To see that, we use Definition 2.26, (8.101) and Hölder’s inequality.)

Recalling then the definition of \(\text{dist}_U(\nu_u^{(1)}, \nu_u^{(2)})\) (see (8.1)), we obtain

\[
(8.104) \lesssim \text{dist}_U(\nu_u^{(1)}, \nu_u^{(2)}) \int_{u_1}^{U} \left[ \|F_{X,1}\|_{L^2_x L^\infty_y L^2(S)} + \|F_{X,2}\|_{L^2_x L^\infty_y L^2(S)} \right] \, du
\]

\[
\lesssim \frac{1}{N^\frac{1}{4}} \text{dist}_U(\nu_u^{(1)}, \nu_u^{(2)}) \lesssim \frac{\text{dist}}{N^\frac{1}{4}}. \tag{8.107}
\]

Step 4: Estimating (8.105). We further compute the \(G_X\) terms (recall (8.103)). Using \((d\nu_u^{(1)} - d\nu_u^{(2)})\mathcal{Y} = \mathcal{Y} \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}}\), we obtain

\[
d\nu_u^{(2)} \left[ \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) (d\nu_u^{(1)} \mathcal{X}) \right]
\]

\[
= d\nu_u^{(2)} \left[ \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) \otimes \mathcal{X} \right] - d\nu_u^{(2)} \left[ \mathcal{X} d\nu_u^{(2)} \left( \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) \right) \right]
\]

\[
+ d\nu_u^{(2)} \left[ \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) (\mathcal{X} \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}}) \right].
\]

\(^{33}\)Note that a priori, to apply (2.29) requires the test function to be \(C_1^\infty\), but it can easily be checked by an approximation argument that \(d\nu_u^{(1)} \mathcal{X}\) and \((d\nu_u^{(1)} \mathcal{X}) \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}}\) are also admissible test functions.
and
\[
\begin{align*}
[(\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1})] & \left( \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (d\nu^{(1)} X) \right) \\
= d\nu^{(2)} \left\{ [ (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1})] \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} X \right\} - \nabla_X \left\{ [ (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1})] \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} \right\} \] & \quad (8.109) \\
+ \left[ (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1}) \right] \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (X \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} ) .
\end{align*}
\]

Therefore, (8.103), (8.108) and (8.109) together imply that we have the following decomposition\footnote{Note that $G_X$ is a tensor of rank $r$.}
\[
G_X = d\nu^{(2)} d\nu^{(2)} G_{X,2} + d\nu^{(2)} G_{X,1} + G_{X,0},
\]
where
\[
G_{X,2} = \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) \otimes X,
\]
\[
G_{X,1} = -X d\nu^{(2)} \left\{ \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) \right\} + \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (b^{(2)} - b^{(1)}) (X \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} )
\]
\[
+ \left[ (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1}) \right] \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} X,
\]
\[
G_{X,0} = -\nabla_X \left\{ [ (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1})] \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} \right\} + \left[ (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1}) \right] \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} (X \log \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} ).
\]

By Definition 2.26 (and the definition of $X$), $G_{X,2} \in C_0^0 L^2(S_{u,u}, \gamma^{(1)})$, $G_{X,1}$, $G_{X,0} \in C_0^0 L^2(S_{u,u}, \gamma^{(1)})$. Moreover, the following estimates are satisfied (using Hölder’s inequality, Definition 2.26, (8.101) and Sobolev embedding (Proposition 5.1)):
\[
\| G_{X,2} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} \lesssim \| b^{(1)} - b^{(2)} \|_{L^2_u L^1(S_{u,u}, \gamma^{(1)})}
\]
\[
\lesssim \| \nabla (b^{(1)} - b^{(2)}) \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} + \| b^{(1)} - b^{(2)} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})},
\]
\[
\| G_{X,1} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} \lesssim \| \nabla (b^{(1)} - b^{(2)}) \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} + \| b^{(1)} - b^{(2)} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})}
\]
\[
+ \| (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1}) \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})},
\]
\[
\| G_{X,0} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} \lesssim \| \nabla (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1}) \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})}
\]
\[
+ \| (\Omega \delta \chi)^{(2)} - (\Omega \delta \chi)^{(1}) \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})}.
\]

Using Definition 2.26, (8.3), Proposition 8.11 and the Cauchy–Schwarz inequality, this implies that
\[
\int_{u_i}^{u_{i+1}} (\| G_{X,2} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} + \| G_{X,1} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} + \| G_{X,0} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})}) \, du \lesssim \frac{\text{dist}}{N^2} .
\]

Hence, using the regularity estimate for $du_u$ in Definition 2.36 together with (8.101), we obtain
\[
(8.105) \lesssim \int_{u_i}^{U} (\| G_{X,2} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} + \| G_{X,1} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} + \| G_{X,0} \|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})}) \, du \lesssim \frac{\text{dist}}{N^2} .
\]

**Step 5: Putting everything together.** Using the estimates (8.107) and (8.111) for (8.104) and (8.105), and returning to Step 2, we obtain
\[
\left| \int_{\mathcal{H}_u} (d\nu^{(1)} X) (d\nu^{(1)} U - \frac{\sqrt{\det \gamma^{(1)}}}{\sqrt{\det \gamma^{(2)}}} d\nu^{(2)} U) \right| \lesssim \frac{\text{dist}}{N^2}.
\]

In view of (1) the arbitrariness of the prescription of $X$ at $u = U$ (subject to $\|X\|_{L^2_u L^2(S_{u,u}, \gamma^{(1)})} \leq 1$) and (2) the arbitrariness of $U \in [u_i, u_{i+1}]$, we have obtain (8.99).
8.8. Putting everything together: Proof of Theorem 4.4.

Proof of Theorem 4.4. Recalling the definition of the distance function in (8.3), we see that by Propositions 8.7, 8.8, 8.10, 8.11, 8.12, 8.14, 8.16, 8.17, 8.18 and 8.19,
\[ \text{dist} \lesssim \frac{\text{dist}}{N^4}. \]
Choosing \( N \) large enough gives \( \text{dist} = 0 \), which implies the desired uniqueness result. \( \square \)

9. Weak approximation theorem

In this section we prove our final two theorems, Theorem 4.6 and Theorem 4.7. Given what we have obtained so far (Theorems 4.1 and 4.4), the key step to both Theorem 4.6 and Theorem 4.7 is an approximation result which allows us to approximate any null dust initial data set (with merely measure-valued null dust) by a smooth vacuum initial data set; see already Proposition 9.4. This result will be carried out in the following steps:

1. We first show that all smooth null dust initial data sets with two families of null dusts in double null foliation can be approximated by smooth vacuum initial data sets (Proposition 9.1 in Section 9.1).
2. Our next step is to approximate measure-valued null dust initial data by smooth null dust initial data (Proposition 9.2 in Section 9.2).
3. Combining the previous steps, we then achieve an approximation of measure-valued null dust initial data by smooth null dust initial data sets (Proposition 9.4 in Section 9.3).

Once we obtain the main approximation result, we conclude the proofs of Theorem 4.6 and Theorem 4.7. This will be carried out in Section 9.4.

Since the approximation results are the same on \( H_0 \) and \( H_0 \), in what follows we will focus on \( H_0 \). The case for \( H_0 \) is the same; see Proposition 9.5.

9.1. Approximating smooth null dust data by vacuum data. Before we proceed, recall Definition 4.5 only gives a notion of the null dust where the null dust is a measure. We now introduce a convention for smooth null dust data in order to carry out the approximation procedure. For this we will stipulate that \( \text{d} \nu_{\text{init}} \) is absolutely continuous with respect to \( \text{d} A, \text{d} u \) and for a smooth function \( f \),
\[ \text{d} \nu_{\text{init}} = \Phi^{-2} \Omega^{-2} f \text{d} A, \text{d} u. \]
In this smooth setting, the constraint equation (3.1) is replaced by
\[ \frac{\partial^2 \Phi}{\partial u^2} = 2 \frac{\partial \log \Omega}{\partial u} \frac{\partial \Phi}{\partial u} - \frac{1}{8} \left( \frac{\partial^2 \Phi}{\partial u^2} \right)^2 \Phi - \frac{1}{2} \Phi^{-1} f \text{ on } H_0, \] (9.1)
where \( f \) is non-negative and \( \frac{\partial^2 \Phi}{\partial u^2} \) is defined as in (3.2), (with obviously modifications on \( H_0 \)).

Proposition 9.1. Let \( K \in \mathbb{N} \). Fix an arbitrary smooth metric \( \hat{g} \) on \( S_{0,0} \). Extend \( \hat{g} \) to \([0, u_\ast]\) \( \times \mathbb{S}^2 \) by
\[ \hat{\mathcal{L}}_{\hat{g}} \hat{g} = 0. \] (9.2)
Assume the following:

- Let \( \Omega \) be a given smooth positive function on \([0, u_\ast]\) \( \times \mathbb{S}^2 \).
- Suppose \( \hat{\gamma}^{(\text{dust})} \) is a smooth \( S \)-tangent covariant 2-tensor on \([0, u_\ast]\) \( \times \mathbb{S}^2 \), which is a Riemannian metric on \( S_{0, u} \) for every \( u \in [0, u_\ast]\).
- Suppose \( \Phi^{(\text{dust})} \) is a positive smooth function on \([0, u_\ast]\) \( \times \mathbb{S}^2 \).
- Assume that the following constraint equation is satisfied
\[ \frac{\partial^2 \Phi^{(\text{dust})}}{\partial u^2} = 2 \frac{\partial \log \Omega^{(\text{dust})}}{\partial u} \frac{\partial \Phi^{(\text{dust})}}{\partial u} - \frac{1}{8} \left( \frac{\partial^2 \Phi^{(\text{dust})}}{\partial u^2} \right)^2 \Phi^{(\text{dust})} - \frac{1}{2} \Phi^{-1} f \text{ for a non-negative smooth function } f(u, \vartheta). \] (9.3)

Then there exists a sequence of smooth \( \{ (\hat{\gamma}_n, \Phi_n) \}^{+\infty}_{n=1} \) on \([0, u_\ast]\) \( \times \mathbb{S}^2 \) such that the following holds:

1. (Positivity of \( \hat{\gamma}_n \) and \( \Phi_n \)) For every \( n \) sufficiently large, \( \hat{\gamma}_n \) is a smooth \( S \)-tangent tensor on \([0, u_\ast]\) \( \times \mathbb{S}^2 \) which is a Riemannian metric on \( S_{0, u} \) satisfying \( \frac{\text{dist} \hat{\gamma}_n}{\text{dist} \hat{g}} = 1 \), and \( \Phi_n \) is a positive smooth function on \([0, u_\ast]\) \( \times \mathbb{S}^2 \).
(2) (Vacuum constraint) For every \( n \geq 1 \),
\[
\begin{cases}
\frac{\partial^2 \Phi_n}{\partial u^2} = 2 \frac{\partial \log \Omega}{\partial u} \frac{\partial \Phi_n}{\partial u} - \frac{1}{2} \frac{\partial^2 \Phi_n}{\partial u^2}, \\
\Phi_n(u = 0) = \Phi^{(dust)}(u = 0), \quad \frac{\partial \Phi_n}{\partial u}(u = 0) = \frac{\partial \Phi^{(dust)}}{\partial u}(u = 0).
\end{cases}
\] (9.4)

(3) (Uniform estimates and convergence) The following hold for some implicit constants depending only on \( f, \gamma, \gamma^{(dust)}, \Phi^{(dust)} \) and \( \Omega \) but independent of \( n \):
\[
\| \gamma_n - \gamma^{(dust)} \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} \lesssim n^{-1}, \quad \| \frac{\partial \gamma_n}{\partial u} \|_{L^\infty W^{1,\infty}(S_0, u^\gamma)} \lesssim 1,
\] (9.5)
\[
\| \Phi_n - \Phi^{(dust)} \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} \lesssim n^{-1}, \quad \| \frac{\partial (\Phi_n - \Phi^{(dust)})}{\partial u} \|_{L^\infty W^{1,\infty}(S_0, u^\gamma)} \lesssim n^{-1}.
\] (9.6)

(4) (Refined uniform estimates for \( \frac{\partial \gamma_n}{\partial u} \)) For any fixed \( \gamma \) (satisfying (9.2)) and \( \gamma_0 \) (satisfying \( \frac{\det \gamma_0}{\det \gamma} = 1 \)), there exist \( C > 0 \) and \( \epsilon > 0 \) (both depending only on \( \gamma \) and \( \gamma_0 \)) such that
\[
\| \gamma^{(dust)} - \gamma_0 \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} < \epsilon,
\] (9.7)
then for all \( n \) sufficiently large
\[
\| \frac{\partial \gamma_n}{\partial u} \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} \leq C(1 + \| \frac{\partial \gamma^{(dust)}}{\partial u} \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} + \| f \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)})).
\] (9.8)

(5) (Oscillatory estimates) There exist smooth functions \( \{ F_n \}_{n=1}^\infty \) with uniform in \( n \) estimates
\[
\| F_n \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} \lesssim 1
\] (9.9)
such that
\[
\| \frac{\partial^2 \gamma_n}{\partial u^2} - \frac{\partial \gamma^{(dust)}}{\partial u}(\Phi^{(dust)})^2 - 4f - \frac{1}{n} \frac{\partial F_n}{\partial u} \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} \lesssim n^{-1},
\] (9.10)
In both (9.9) and (9.10), the implicit constants depend only on \( f, \gamma, \gamma^{(dust)} \) and \( \Phi^{(dust)} \) but are independent of \( n \).

Proof. Since there exists \( \theta \neq U \subset S^2 \) such that \( f(u, \theta) = 0 \) for \( \theta \in U \) and \( u \in [0, u_*] \), we can work with a local coordinate system on \( S^2 \).

Step 1: Construction of \( \gamma_n \). Suppose that in local coordinates \( \gamma \) be given by
\[
\gamma^{(dust)} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}
\]
where \( a, b \) and \( d \) are smooth functions of \( u \) and \( \theta \). By assumption \( \gamma^{(dust)} = \gamma \) and thus we have
\[
ad - b^2 = \det \gamma.
\] (9.11)
Notice moreover that since \( \gamma \) is positive definite, we must have \( a > 0 \) and \( d > 0 \).

We now define the sequence \( \gamma_n \) by
\[
\gamma_n := \begin{pmatrix} a + \frac{(ad - b^2)}{d} & \frac{2f}{\Phi^{(dust)}} \frac{1}{kn} \sin(knu) \\ \frac{2f}{\Phi^{(dust)}} \frac{1}{kn} \sin(knu) & d - (ad - b^2) \end{pmatrix},
\] (9.12)
where \( k \) is some large but fixed real parameter chosen so that \( \gamma_n \) is positive definite for all \( n \geq 1 \). Note that (9.12) is well-defined since\( a > 0, d > 0, \Phi^{(dust)} \geq 0 \) and \( f \geq 0 \). Furthermore, (9.12) is chosen so that indeed by (9.11),
\[
det(\gamma_n) = (ad - b^2)[1 + \frac{2f}{\Phi^{(dust)}} \frac{1}{kn} \sin(knu) - \frac{2af}{\Phi^{(dust)}} \frac{1}{kn} \sin(knu)]
\]
\[
= (ad - b^2) \left[ 1 + \frac{2f}{\Phi^{(dust)}} \frac{1}{kn} \sin(knu) \right] = ad - b^2 = \det(\gamma^{(dust)}) = \det(\gamma).
\]

\( ^{35} \)Note that Part (3) already contains the statement \( \| \frac{\partial \gamma_n}{\partial u} \|_{L^\infty W^{k,\infty}(S_0, u^\gamma)} \lesssim 1 \) (after using Hölder’s inequality). The key point here is that we further analyze the dependence of the implicit constant. In particular, the constant \( C > 0 \) in the estimate can be chosen uniformly for all \( \gamma^{(dust)} \) satisfying (9.7).

\( ^{36} \)It is for this step that we have used the condition of non-negativity of \( \Phi^{(dust)} \) and \( f \).
From the formula (9.12), the smoothness of \((a, b, d, f)\), and the positivity of \(a\) and \(d\), it immediately follows that (9.5) holds.

To obtain (9.8) and (9.10), we compute

\[
\frac{\partial}{\partial u} \tilde{\gamma}_n = \left( \frac{\partial a}{\partial u} + \frac{(a-b)^2}{\Phi^{(\text{dust})}} \frac{2}{\Phi^{(\text{dust})}} \right) \frac{1}{\Phi^{(\text{dust})}} \cos(knu) \left( \frac{\partial}{\partial u} \tilde{\gamma}_n \right)_{AB} \left( \frac{\partial}{\partial u} \tilde{\gamma}_n \right)_{CD} + O_K \left( \frac{1}{n} \right),
\]

where \(O_K \left( \frac{1}{n} \right)\) denotes terms whose \(L^\infty W^{K,\infty}(S_\infty, \gamma)\) norm is bounded above up to a constant by \(\frac{1}{n}\) (where the constant depends on \(f, \tilde{\gamma}, \tilde{\gamma}^{(\text{dust})}\), and \(\Phi^{(\text{dust})}\)).

From this formula we obtain (9.8). Indeed,

- the \(\frac{\partial a}{\partial u}, \frac{\partial b}{\partial u}, \frac{\partial d}{\partial u}\) terms in the \(L^2 W^{K,\infty}(S_0, \gamma)\) norm can be controlled by \(\|\frac{\partial a}{\partial u}\|_{L^2 W^{K,\infty}(S_0, \gamma)}\); and
- the \(O_K \left( \frac{1}{n} \right)\) terms in the \(L^2 W^{K,\infty}(S_0, \gamma)\) norm can be bounded by 1 after choosing \(n\) to be sufficiently large;

- the \(\frac{(a-b)^2}{\Phi^{(\text{dust})}}\) terms in the \(L^2 W^{K,\infty}(S_0, \gamma)\) norm can be controlled by \(\|f\|_{L^2 W^{K,\infty}(S_0, \gamma)}\) norm. Moreover, the constant of the estimate can be chosen to depend uniformly on \(\tilde{\gamma}_n\). In particular, the constant can be chosen to be uniform under the assumption (9.7).

To establish (9.10), we compute using (9.12) and (9.13) that

\[
\frac{1}{(a-b)^2} \left[ \left(\frac{\partial a}{\partial u} \right)^2 - 2(b \frac{\partial b}{\partial u})^2 + a \frac{\partial d}{\partial u} \right] = \left( \frac{\partial \tilde{\gamma}^{(\text{dust})}}{\partial u} \right)^2.
\]

Next, for the highly oscillatory terms, we can write

\[
\left(\Phi^{(\text{dust})}\right)^2 \times \left\{ \frac{4f}{\Phi^{(\text{dust})}} \cos(2knu) + \frac{2}{a-b^2} \left( \frac{\partial a}{\partial u} - d \frac{\partial d}{\partial u} \right) \right\} \cos(knu)
\]

\[
= \frac{1}{n} \frac{\partial a}{\partial u} - \frac{2f}{k} \sin(2knu) + \frac{4}{k(a-b^2)} \left( d \frac{\partial d}{\partial u} - \frac{\partial a}{\partial u} \right) \Phi^{(\text{dust})} \sin(knu)
\]

\[
= \frac{1}{n} \frac{\partial a}{\partial u} - \frac{2f}{k} \sin(2knu) + \frac{4}{k(a-b^2)} \left( d \frac{\partial d}{\partial u} - \frac{\partial a}{\partial u} \right) \Phi^{(\text{dust})} \sin(knu)
\]

Thus, defining

\[
F_n = \frac{2f}{k} \sin(2knu) + \frac{4}{k(a-b^2)} \left( d \frac{\partial d}{\partial u} - \frac{\partial a}{\partial u} \right) \Phi^{(\text{dust})} \sin(knu),
\]

which clearly satisfies (9.9), it follows that (9.10) holds.

**Step 2:** Estimates for \(\Phi_n\). Define now \(\Phi_n\) by the initial value problem (9.4). Since \(\Omega\) and \(\tilde{\gamma}_n\) are smooth, this linear ODE has a unique solution. Our goal now is to prove (9.6), from which the positivity of \(\Phi_n\) (for \(n\) large) would also follow.
By (9.3) and (9.4), \((\Phi_n - \Phi^{(\text{dust})})\) satisfies the ODE
\[
\frac{\partial^2 (\Phi_n - \Phi^{(\text{dust})})}{\partial u^2} = 2 \frac{\partial \log \Omega}{\partial u} \frac{\partial (\Phi_n - \Phi)}{\partial u} - \frac{1}{8} \left( \frac{\partial^2 \gamma_n}{\partial u^2} \right)^2 |\gamma_n|^2 - \frac{1}{8} \left( \frac{\partial^2 \gamma_n}{\partial u^2} \right)^2 |\gamma_n|^2 \Phi^{(\text{dust})} + \frac{1}{2} f \Phi^{(\text{dust})},
\]
(9.15)
where the initial data for both \(\Phi_n - \Phi^{(\text{data})}\) and \(\frac{\partial}{\partial u} (\Phi_n - \Phi^{(\text{data})})\) vanish.

Using (9.10) and then integrating by parts and using (9.9),
\[
\int_0^u \left[ - \frac{1}{8} \left( \frac{\partial^2 \gamma_n}{\partial u^2} \right)^2 |\gamma_n|^2 - \frac{1}{8} \left( \frac{\partial^2 \gamma_n}{\partial u^2} \right)^2 |\gamma_n|^2 \Phi^{(\text{dust})} + \frac{1}{2} f \Phi^{(\text{dust})} \right] \Phi^{(\text{dust})} + O_K \left( \frac{1}{n} \right) = O_K \left( \frac{1}{n} \right).
\]
(9.16)
Integrating (9.15) and using (9.16), we obtain
\[
\frac{\partial (\Phi_n - \Phi^{(\text{dust})})}{\partial u} (u, \vartheta) = \int_0^u \left[ \frac{\partial \log \Omega}{\partial u} \frac{\partial (\Phi_n - \Phi)}{\partial u} - \frac{1}{8} \left( \frac{\partial^2 \gamma_n}{\partial u^2} \right)^2 \Phi^{(\text{dust})} \right] \Phi^{(\text{dust})} + O_K \left( \frac{1}{n} \right),
\]
(9.17)
Taking the \(W^{K, \infty}\) norm along the 2-spheres, (9.17) implies
\[
sup_{u' \in [0, u]} \left\| \frac{\partial (\Phi_n - \Phi^{(\text{dust})})}{\partial u} \right\|_{W^{K, \infty}(S_0, \vartheta, \hat{\gamma})} \lesssim \int_0^u \left[ \sup_{u'' \in [0, u']} \left\| \frac{\partial (\Phi_n - \Phi^{(\text{dust})})}{\partial u} \right\|_{W^{K, \infty}(S_0, \vartheta', \hat{\gamma})} + \sup_{u'' \in [0, u']} \left\| \Phi_n - \Phi^{(\text{dust})} \right\|_{W^{K, \infty}(S_0, \vartheta', \hat{\gamma})} \right] \Phi^{(\text{dust})} \Phi^{(\text{dust})} d\mu' + n^{-1},
\]
(9.18)
where in the last line we used the fundamental theorem of calculus and the initial vanishing of \(\Phi_n - \Phi^{(\text{data})}\) to control sup_{u'' \in [0, u']} \left\| \Phi_n - \Phi^{(\text{dust})} \right\|_{W^{K, \infty}(S_0, \vartheta', \hat{\gamma})}.

The estimate (9.6) therefore follows from first applying Grönwall’s inequality to (9.18), and then using the fundamental theorem of calculus to estimate \(\left\| \Phi_n - \Phi^{(\text{dust})} \right\|_{L^\infty W^{K, \infty}(S_0, \vartheta, \hat{\gamma})}\).
(3) For $0 \leq k \leq K$, $f_m$ satisfies the quantitative convergence estimate

\[
\left| \int_0^{2\gamma} \int_{S_{0,\omega}} (\partial\bar{\nabla}^k \varphi(k)) \Omega^{-2} f_m \, dA_3 \, du - \int_{(0,\omega) \times S^2} \partial\bar{\nabla}^k \varphi(k) \, d\nu_{\text{init}} \right| \leq 2^{-m} \left\| \frac{\partial \varphi(k)}{\partial u} \right\|_{L^2 \Omega^1(S_{0,\omega}^3)} + 2^{-m} \left\| \varphi(k) \right\|_{L^\infty \Omega^1(S_{0,\omega}^3)},
\]

(9.21)

for every continuous rank-$k$ S-tangent tensor field $\varphi(k)$ such that $\frac{\partial \varphi(k)}{\partial u} \in L^2 \Omega^1(S_{0,\omega}^3)$.

Here, the implicit constants in (9.20) and (9.21) depend only on $\omega$, $\gamma$, $\Omega$ and $\Lambda$.

Finally, if $d\nu_{\text{init}}$ is supported on $[0, \omega] \times U^c$ for some $U \subset S^2$, then for any open $V \subseteq U$ with $\nabla \subset U$, $f_m$ can be chosen so that $\supp(f_m) \subset [0, \omega] \times V^c$.

Proof. In this proof, we will suppress the explicit dependence on $\gamma$ in the norms if there is no risk of confusion.

Step 1: Definition of $\{\tilde{f}_c\}_{c \in (0,10^{-10}\omega]}$. We first define an auxiliary one parameter family of (not necessarily smooth) functions $\{\tilde{f}_c\}_{c \in (0,10^{-10}\omega]}$. The functions $\tilde{f}_c$ are obtained by mollifying $d\nu_{\text{init}}$ in the $u$-direction and then taking the density with respect to $\Omega^{-2} \, dA_3 \, du$.

Let

- $\varphi : [0, \omega] \times S^2 \to \mathbb{R}$ be a smooth tensor field; and
- $\{\zeta_i\}_{i=1}^3$ be a smooth partition of unity of $[0, \omega]$ such that $\supp(\zeta_1) \subset [0, \frac{2\omega}{3}]$, $\supp(\zeta_2) \subset [\frac{2\omega}{3}, \frac{3\omega}{4}]$, $\supp(\zeta_3) \subset [\frac{3\omega}{4}, \omega]$ and $\sum_{i=1}^3 \zeta_i \equiv 1$; and
- $\varrho : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth even cutoff function with $\varrho \varrho \leq [-1,1]$ and $\int_{\mathbb{R}} \varrho = 1$.

Given $\epsilon \in (0,10^{-10}\omega]$, define\(^{37}\) $\varphi_{c} : [0, \omega] \to \mathbb{R}$ by

\[
\varphi_{c}(u', \theta) := \int_{\mathbb{R}} \zeta_1(\varrho)(u + \epsilon, \theta) \frac{1}{\epsilon} \varrho\left(\frac{u - u'}{\epsilon}\right) \, du + \int_{\mathbb{R}} \zeta_2(\varrho)(u, \theta) \frac{1}{\epsilon} \varrho\left(\frac{u - u'}{\epsilon}\right) \, du + \int_{\mathbb{R}} \zeta_3(\varrho)(u - \epsilon, \theta) \frac{1}{\epsilon} \varrho\left(\frac{u - u'}{\epsilon}\right) \, du
\]

\[
= \sum_{i=1}^3 \int_{\mathbb{R}} \zeta_i(\varrho)(u + \alpha_i \epsilon, \theta) \frac{1}{\epsilon} \varrho\left(\frac{u - u'}{\epsilon}\right) \, du,
\]

where $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -1$. It is easy to check that $\varphi_{c} \to \varphi$ uniformly.

Note that (for an implicit constant independent of $\varphi$) $\sup_{\omega \in [0, \omega]} \| \varphi_{c}(u', \cdot) \|_{L^1(\mathbb{R})} \leq \epsilon^{-\frac{1}{2}}$. Therefore, using (9.19), the map

\[
\varphi \mapsto \int_{[0,\omega] \times S^2} \varphi_{c}(u', \theta) \, d\nu_{\text{init}}(u', \theta)
\]

extends to a bounded linear map: $L^2 \Omega^1(S_{0,\omega}) \to \mathbb{R}$. It follows by duality that there exists $\tilde{f}_c \in L^2 \Omega^1(S_{0,\omega})$ such that

\[
\int_{[0,\omega] \times S^2} \varphi_{c}(u', \theta) \, d\nu_{\text{init}}(u', \theta) = \int_{[0,\omega] \times S^2} \varphi(u, \theta) \Omega^{-2} \tilde{f}_c(u, \theta) \, dA_3 \, du.
\]

(9.23)

By (9.23), it follows that $\Omega^{-2} \tilde{f}_c \, dA_3 \, du \to d\nu_{\text{init}}$ in the weak-$^*$ topology as $c \to 0$.

Step 2: Uniform estimates for $\{\tilde{f}_c\}_{c \in (0,10^{-10}\omega]}$. Consider the class of rank-$k$ tensor fields

\[
D^{(k)} = \{ \varphi(k) \in \Gamma(T^k([0, \omega] \times S^2)) : \varphi \in L^\infty C^\infty(S_{0,\omega}) \}.
\]

Since $D^{(k)}$ is dense in $L^\infty \Omega^1(S_{0,\omega})$, we can compute using duality (9.23) and (9.19) that

\[
\left\| \tilde{f}_c \right\|_{L^1\Omega^1(S_{0,\omega})} \leq \sup_{\{\varphi(k) \in D^{(k)} : \|\varphi(k)\|_{L^\infty \Omega^1(S_{0,\omega})} = 1}\} \int_{S_{0,\omega}} (\partial\bar{\nabla}^K \varphi(k))(u, \theta) \tilde{f}_c(u, \theta) \, dA_3 \, du
\]

\[
+ \sup_{\{\varphi(k) \in D^{(k)} : \|\varphi(k)\|_{L^\infty \Omega^1(S_{0,\omega})} = 1\}} \int_{S_{0,\omega}} (\tilde{f}_c(u, \theta) \partial\bar{\nabla}^K \varphi(k))(u, \theta) \, dA_3 \, du \leq 1.
\]

\(^{37}\)Note that $\varphi_{c}$ is defined on all of $[0, \omega]$ because we have shifted $\varphi$ on the support of $\zeta_1$ and $\zeta_3$ near the boundary.
Step 3: Speed of convergence. Using additional regularity of the test function $\varphi$, we show a quantitative speed for the weak-* convergence of $\Omega^{-2}\tilde{f}_e dA_3 \, du \to d\nu_{init}$. We first compute, for $i = 1, 2, 3$,

$$\sup_{\Omega} \int_{\Omega} (\zeta_i \varphi)(u + \alpha_i \epsilon, \theta) \frac{1}{\epsilon} g\left(\frac{u - u'}{\epsilon}\right) du = (\zeta_i \varphi)(u', \theta) \|_{L^1(S^2)} \leq \sup_{\Omega} \int_{\Omega} (\zeta_i \varphi)(u + \alpha_i \epsilon, \theta) \frac{1}{\epsilon} g\left(\frac{u - u'}{\epsilon}\right) du = (\zeta_i \varphi)(u', \theta) \|_{L^1(S^2)} \leq \sup_{\Omega \in [0, u]} \int_{\Omega} (\zeta_i \varphi)(u + \alpha_i \epsilon, \theta) \frac{1}{\epsilon} g\left(\frac{u - u'}{\epsilon}\right) du = (\zeta_i \varphi)(u', \theta) \|_{L^1(S^2)} \leq \epsilon \frac{\partial \varphi}{\partial u} \|_{L^2(S^2)} + \epsilon \| \varphi \|_{L^\infty(S^2)}.$$

(9.25)

Let $0 \leq k \leq K$. Using (9.25), (9.19) and the definition of $\tilde{f}_e$ in (9.22) and (9.23), we obtain

$$\left| \int_0^{\frac{1}{2}} \int_{S_0} (\tilde{u}^k \varphi(k)) \Omega^{-2} \tilde{f}_e dA_3 du - \int_{[0, u] \times S^2} \tilde{u}^k \varphi(k)(u', \theta) d\nu_{init}(u', \theta) \right| \leq \epsilon \| \varphi \|_{L^2(S^2)} + \epsilon \| \varphi \|_{L^\infty(S^2)}.$$

(9.26)

Step 4: Definition of $\{f_m\}_{m=1}^{+\infty}$ and conclusion of proof. Finally, let $\{f_m\}_{m=1}^{+\infty}$ be smooth and such that

$$\| f_m - \tilde{f}_{2-2m} \|_{L^2(S_0) \times S^2} \leq 2^{-2m},$$

(9.27)

(where by $\tilde{f}_{2-2m}$ we mean $\tilde{f}_e$ with $\epsilon = 2^{-2m}$). Since $\tilde{f}_e$ is non-negative by (9.23), it is easy to see that $f_m$ can be arranged to be non-negative. In the case $d\nu_{init}$ is supported on $[0, u] \times U$ for some $U \subset S^2$, then for any open $V \subseteq U$, we can moreover impose $\text{supp}(f_m) \subseteq [0, u] \times V$.

We need to check that the three properties asserted in the statement of the proposition hold.

1. Since $\Omega^{-2} \tilde{f}_e dA_3 du \to d\nu_{init}$ in the weak-* topology, it follows from (9.27) that $\Omega^{-2} f_m dA_3 du \to d\nu_{init}$ in the weak-* topology.

2. The estimate (9.20) follows from (9.24), (9.27) and the triangle inequality.

3. Integrating by parts and using (9.27), smoothness of $\Omega$ and H"older's inequality, we obtain

$$\left| \int_0^{\frac{1}{2}} \int_{S_0} (\tilde{u}^k \varphi(k)) \Omega^{-2} (f_m - \tilde{f}_{2-2m}) dA_3 du \right| \leq 2^{-2m} \| \varphi \|_{L^2(S_0) \times S^2}.$$

Combining this with (9.26), we then obtain (9.21) using the triangle inequality. 

\[ \square \]

**Proposition 9.3.** Let $d\nu_{init}$, $\{f_m\}_{m=1}^{+\infty}$ and $\gamma$ be as in Proposition 9.2. Suppose there exist continuous functions $(\Phi, \log \Omega)$ and a continuous $S$-tangent 2-tensor $\gamma$ such that the following holds for some $K \in \mathbb{N}$:

- $\Phi$ is positive, Lipschitz and $L^\infty(S_0) \times S^2$, $\frac{\partial \Phi}{\partial u}$ is BV and $L^\infty(S_0) \times S^2$, and the following estimates hold:

$$\| \Phi \|_{L^\infty(S_0) \times S^2} + \| \Phi^{-1} \|_{L^\infty(S_0) \times S^2} + \| \frac{\partial \Phi}{\partial u} \|_{L^\infty(S_0) \times S^2} \leq 1.$$

(9.28)

- $\Omega$ is smooth, positive and satisfy

$$\| \log \Omega \|_{L^\infty(S_0) \times S^2} + \| \frac{\partial \log \Omega}{\partial u} \|_{L^2(S_0) \times S^2} \leq 1.$$

(9.29)

- $\gamma$ satisfies $\frac{\text{det}(\gamma)}{\text{det}(\gamma)} = 1$ and obeys the following bounds:

$$\| \gamma \|_{L^\infty(S_0) \times S^2} + \| \frac{\partial \gamma}{\partial u} \|_{L^2(S_0) \times S^2} \leq 1.$$

(9.30)
Assume also that \((d
u_\text{init}, \Phi, \log \Omega, \hat{\gamma})\) satisfies the equation (4.6) for any \(\varphi \in C^\infty_c((0, u_*) \times \mathbb{S}^2)\).

Consider the initial value problem for \(\Phi_m\) with smooth data:

\[
\begin{aligned}
\frac{\partial^2 \Phi_m}{\partial t^2} - \frac{2}{\partial \log \Omega} \frac{\partial \Phi_m}{\partial \Omega} + \frac{1}{\hat{\gamma}_m} \frac{\partial^2 \gamma_m}{\partial u^2} \Phi_m + \frac{1}{2} f_m \Phi_m^{-1} & = 0, \\
\Phi_m(0, \vartheta) & = \tilde{\Phi}_m(\vartheta), \\
\frac{\partial \Phi_m}{\partial u}(0, \vartheta) & = \tilde{\Psi}_m(\vartheta),
\end{aligned}
\tag{9.31}
\]

where \((\tilde{\gamma}_m, \tilde{\Phi}_m, \tilde{\Psi}_m)\) are such that the following holds for all \(m \in \mathbb{N}^\times:\)

- The tensor \(\tilde{\gamma}_m\) is smooth and such that
  \[
  \frac{\det \tilde{\gamma}_m}{\det \tilde{\gamma}} = 1, \quad \| \tilde{\gamma}_m - \tilde{\gamma} \|_{L^\infty_2 W^{k, \infty}(S_0, \tilde{\gamma})} + \| \frac{\partial}{\partial u} (\tilde{\gamma}_m - \tilde{\gamma}) \|_{L^2_2 W^{k, \infty}(S_0, \tilde{\gamma})} \leq 2^{-m}. \tag{9.32}
  \]

- The data \((\tilde{\Phi}_m, \tilde{\Psi}_m)\) of (9.31) are smooth and such that
  \[
  \| \tilde{\Phi}_m - \Phi(0, \cdot) \|_{W^{k, \infty}(S_0, \tilde{\gamma})} + \| \tilde{\Psi}_m - \left( \frac{\partial \Phi}{\partial u} \right)^{-1}(0, \cdot) \|_{W^{k, \infty}(S_0, \tilde{\gamma})} \leq 2^{-m}. \tag{9.33}
  \]

Then the following holds for \(m \in \mathbb{N}\) sufficiently large:

1. The solution \(\Phi_m\) to (9.31) is defined on all of \([0, u_*] \times \mathbb{S}^2\).
2. \(\Phi_m\) converges to \(\Phi\) in the following sense\(^{39}\):
   \[
   \| \Phi_m - \Phi \|_{L^\infty_2 W^{k, \infty}(S_0, \tilde{\gamma})} + \| \frac{\partial}{\partial u} (\Phi_m - \Phi) \|_{L^2_2 W^{k, \infty}(S_0, \tilde{\gamma})} \leq 2^{-m}. \tag{9.34}
   \]
3. \(\frac{\partial \Phi_m}{\partial u}\) satisfies uniformly the estimate
   \[
   \sup_m \| \frac{\partial \Phi_m}{\partial u} \|_{L^\infty_2 W^{k, \infty}(S_0, \tilde{\gamma})} \lesssim 1. \tag{9.35}
   \]

Here, the implicit constants depend only on \(u_*, \tilde{\gamma}, \Omega, \Lambda\) in Proposition 9.2.

Proof. We will only prove the estimates which will then in particular imply existence of solution on all of \([0, u_*] \times \mathbb{S}^2\).

Since all norms on the 2-spheres in this proof will be taken with respect to \(\tilde{\gamma}\), when there is no risk of confusion we will suppress explicit references to \(\tilde{\gamma}\).

For Steps 1 and 2 of the argument, let us fix \(U \in [0, u_*]\). We will first be deriving estimates for \(\Phi_m - \Phi\) and \(\frac{\partial}{\partial u} (\Phi_m - \Phi)\) on the interval \([0, U]\). Thus, in Steps 1 and 2, \(L^\infty_2\) and \(L^\infty_2\) mean \(L^\infty_2([0, U])\) and \(L^\infty_\Lambda([0, U])\). Moreover, all the implicit constants in the estimates will be independent of \(U\).

Step 1: Equation for \(\Phi_m - \Phi\). We first derive an equation for \(\Phi_m - \Phi\) using (4.6) and (9.31). Let \(k \in \{0, \ldots, K\}\). We will consider smooth rank-\(k\) covariant tensor \(\tilde{\varphi} = \tilde{\varphi}_{A_1 \cdots A_k}\) such that

\[
\| e^{A_u} \frac{\partial \tilde{\varphi}}{\partial u} \|_{L^2_2 L^1(S_0 \tilde{\omega})} = 1, \quad \tilde{\varphi} \big|_{u = \tilde{U}} = 0, \tag{9.36}
\]

for some large \(A > 1\) to be determined\(^{40}\).

For every \(\ell \in \mathbb{N}\) with \(\ell^{-1} \leq \tilde{U}/2\), define \(\xi_\ell : [0, u_*] \to \mathbb{R}_{\geq 0}\) by

\[
\xi_\ell(u) := \begin{cases} 
\ell u & \text{if } u \in [0, \ell^{-1}) \\
1 & \text{if } u \in [\ell^{-1}, \ell U) \\
0 & \text{if } u \in (\ell U, u_*]
\end{cases} \tag{9.37}
\]

Note that given \(\tilde{\varphi}\) as above and \(\xi_\ell\) as in (9.37), \(\xi_\ell \tilde{\varphi} e^{k\xi_\ell \tilde{\varphi}} \in C^\infty_c((0, u_*) \times \mathbb{S}^2)\) for all \(\ell \in \mathbb{N}\). After an easy limiting argument, we can thus apply (4.6) with \(\varphi = \xi_\ell \tilde{\varphi} e^{k\xi_\ell \tilde{\varphi}}\). Together with (9.31), we thus obtain

---

\(^{38}\)Here \(\cdot\) denote the trace of BV functions, see Lemma 2.24.

\(^{39}\)We emphasize that while we have uniform \(L^\infty_2\) bounds for \(\frac{\partial \Phi_m}{\partial u}\) in (9.35), the convergence estimate in (9.34) for \(\frac{\partial \Phi_m}{\partial u}\) is only in \(L^2_2\). In fact, it can be easily checked that the convergence in general does not hold in \(L^\infty_2\).

\(^{40}\)The largeness of \(A\) will be used to facilitate the proof of the estimates in Step 2. One could think of the weight \(e^{A_u}\) as a device to prove a Grönwall-like estimate in the setting involving the measure \(d\nu_\text{init}\).
that, for every \( \ell \in \mathbb{N} \) with \( \ell^{-1} \leq \frac{2}{\mathcal{A}} \),

\[
- \int_{0}^{\infty} \int_{S^{2}} \xi(U')( ((\mathcal{A}^{k}) \partial \varphi_{\ell} \partial \Phi_{m} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma)(u', \vartheta) \, dA_{\ell} \, du' \\
\leq \mathcal{A}^{-1} \int_{0}^{\infty} \int_{S^{2}} ((\mathcal{A}^{k}) \partial \varphi_{\ell} \partial \Phi_{m} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma)(u', \vartheta) \, dA_{\ell} \, du'.
\]  

Taking \( \ell \to +\infty \), applying the dominated convergence theorem for \( \mathcal{O}' \), \( \mathcal{O}'_{A} \), \( \mathcal{O}'_{B} \) and using the fact that \( \frac{\partial \Phi_{m}}{\partial \mathcal{A}_{\ell}} \) is smooth and \( \lim_{\Phi_{m} \to 0} \frac{\partial \Phi_{m}}{\partial \mathcal{A}_{\ell}} \) is well-defined in the trace sense for \( \mathcal{I}' \), we obtain

\[
\| \varphi \|_{L^{1}(S_{\vartheta}, d\vartheta)} \leq \mathcal{A}^{-1} \int_{0}^{\infty} \int_{S^{2}} ((\mathcal{A}^{k}) \partial \varphi_{\ell} \partial \Phi_{m} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma)(u', \vartheta) \, dA_{\ell} \, du' \\
\leq \mathcal{A}^{-1} \int_{0}^{\infty} \int_{S^{2}} ((\mathcal{A}^{k}) \partial \varphi_{\ell} \partial \Phi_{m} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma)(u', \vartheta) \, dA_{\ell} \, du'.
\]  

Step 2: Estimating the terms in (9.39). We now estimate the terms from (9.39). Before we proceed, note that it follows easily from (9.36) that for every \( U \in [0, \mathcal{L}] \),

\[
\| \varphi \|_{L^{1}(S_{\vartheta}, d\vartheta)} \leq \mathcal{A}^{-1} \int_{0}^{\infty} \int_{S^{2}} ((\mathcal{A}^{k}) \partial \varphi_{\ell} \partial \Phi_{m} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma)(u', \vartheta) \, dA_{\ell} \, du'.
\]  

Now for all of the terms in (9.39), we first integrate by parts in the angular variables and then control the resulting terms with (9.40) and the bounds in the assumption of the proposition.

By Hölder’s inequality, (9.40), (9.29), (9.31) and (9.33),

\[
|I| \leq \sum_{k_{1} + k_{2} = k} \| \varphi \|_{L^{1}(S_{\vartheta}, d\vartheta)} \| \mathbf{v}^{k_{1}} \|_{L^{2}(S_{\vartheta}, d\vartheta)} \| \mathbf{v}^{k_{2}} \|_{L^{2}(S_{\vartheta}, d\vartheta)} \| \mathbf{v}^{k_{3}} \|_{L^{2}(S_{\vartheta}, d\vartheta)} \| (\mathcal{A}^{k_{1}} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma) - (0, \vartheta) \|_{L^{2}(S_{\vartheta}, d\vartheta)} \leq \frac{2^{-m}}{\sqrt{A}}.
\]  

By Hölder’s inequality, (9.40), (9.28), (9.29), (9.30) and (9.32),

\[
|I| \leq \mathcal{A}^{-1} \int_{0}^{\infty} \int_{S^{2}} ((\mathcal{A}^{k}) \partial \varphi_{\ell} \partial \Phi_{m} \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma \partial \mathcal{A}_{\ell} \gamma)(u', \vartheta) \, dA_{\ell} \, du'.
\]  

(9.41)
We now turn to term III, which we will split into two terms (see (9.44) and (9.45) below). First we observe that by (9.28),
\[\|\Phi_m^1 - \Phi^{-1}\|_{L^\infty(S_0, \mathbb{R})} = \|\Phi^{-1} \Phi^{-1}_m(\Phi - \Phi_m)\|_{L^\infty(S_0, \mathbb{R})} \lesssim \|\Phi_m^1\|_{L^\infty(S_0, \mathbb{R})} \|\Phi - \Phi_m\|_{L^\infty(S_0, \mathbb{R})}.\]  
(9.43)

Therefore, integrating by parts the $d\hat{\nu}^k$ and using the estimates (9.20), (9.29), (9.43) and (9.40), we obtain
\[\frac{1}{2} \left| \int_{\Omega} (\phi^k(\Phi)^{-1}_m(u', \theta) \phi|u|) \right|_{\Omega} \nu_m \int_{\Omega} \left( \frac{\hat{\nu}^k}{\Omega} \right) \Phi^{-1}_m(u', \theta) + \frac{1}{2} \int_{\Omega} \int_{\Omega} (\phi^k(\Phi)^{-1}_m(u', \theta) \phi|u|) \right|_{\Omega} \nu_m \int_{\Omega} \left( \frac{\hat{\nu}^k}{\Omega} \right) \Phi^{-1}_m(u', \theta) \sum_{0 \leq k' \leq k} \|e^{-A u \phi^k} \frac{\partial \Phi}{\partial u} \|_{L^\infty(S_0, \mathbb{R})}.\]  
(9.44)

On the other hand, notice that we can write $d\hat{\nu}^k(\phi^k)\Phi^{-1}$ as a linear combination of terms of the form
\[d\hat{\nu}^k(\phi^k)\Phi^{-1}\]
for $k_1 + k_2 = k$. Therefore, using (9.21) in Proposition 9.2, and then using (9.36), (9.40) (and using $A \phi \geq 0$) and (9.28), we obtain
\[\frac{1}{2} \int_{(0, \mathbb{R}) \times \mathbb{R}^2} (d\hat{\nu}^k(\phi^k)\Phi^{-1}_m(u', \theta) \phi|u|) \right|_{\Omega} \nu_m \int_{\Omega} \left( \frac{\hat{\nu}^k}{\Omega} \right) \Phi^{-1}_m(u', \theta) \sum_{0 \leq k' \leq k} \|e^{-A u \phi^k} \frac{\partial \Phi}{\partial u} \|_{L^\infty(S_0, \mathbb{R})}.\]  
(9.45)

Combining (9.44) and (9.45) and using the triangle inequality, we thus obtain
\[\|\Phi_m^1\|_{L^\infty(W^{k, \infty}(S_0, \mathbb{R}))} \sum_{0 \leq k' \leq k} \|e^{-A u \phi^k} \frac{\partial \Phi}{\partial u} \|_{L^\infty(S_0, \mathbb{R})} + 2^{-m}.\]  
(9.46)

Starting with duality and using (9.39), (9.41), (9.42) and (9.46), we obtain
\[\sum_{0 \leq k \leq K} \|e^{-A u \phi^k} \frac{\partial \Phi}{\partial u} \|_{L^\infty(S_0, \mathbb{R})} \sup_{\Omega} \int_{\Omega} \left( \frac{\hat{\nu}^k}{\Omega} \right) \Phi^{-1}_m(u', \theta) \sum_{0 \leq k' \leq k} \|e^{-A u \phi^k} \frac{\partial \Phi}{\partial u} \|_{L^\infty(S_0, \mathbb{R})}.\]  
(9.47)

We then complement the estimate (9.47) of $\frac{\partial (\phi^k - \Phi_m)}{\partial u}$ with the following estimate on $\Phi - \Phi_m$ for $u \in [0, U]$, which is derived using the fundamental theorem of calculus, (9.31), (9.33) and Hölder’s inequality:
\[\|\hat{\nu}^k(\Phi - \Phi_m)\|_{L^\infty(S_0, \mathbb{R})} \leq \|\hat{\nu}^k(\Phi)\|_{L^\infty(S_0, \mathbb{R})} + \int_{\Omega} \|\hat{\nu}^k(\Phi - \Phi_m)\|_{L^\infty(S_0, \mathbb{R})} d\nu_m + 2^{-m} \left( \int_{\Omega} \|\hat{\nu}^k(\Phi - \Phi_m)\|_{L^\infty(S_0, \mathbb{R})} d\nu_m \right)^{1/2} \]  
(9.48)
The estimate (9.48) implies (using $A u \geq 0$) that for every $u \in [0, U]$, 
\[
\sum_{0 \leq k \leq K} \| e^{-A u \tilde{\Psi}^k} (\Phi - \Phi_m) \|_{L^\infty_0 L^\infty_0(S_{0, \omega})} \lesssim 2^{-m} + \sum_{0 \leq k \leq K} \frac{1}{\sqrt{A}} \| e^{-A u \tilde{\Psi}^k} (\Omega^{-2} \frac{\partial (\Phi - \Phi_m)}{\partial u}) \|_{L^\infty_0 L^\infty_0(S_{0, \omega})}.
\]
Adding (9.47) and (9.49), we obtain
\[
\sum_{0 \leq k \leq K} (\| e^{-A u \tilde{\Psi}^k} (\Omega^{-2} \frac{\partial (\Phi - \Phi_m)}{\partial u}) \|_{L^\infty_0 L^\infty_0(S_{0, \omega})} + \| e^{-A u \tilde{\Psi}^k} (\Phi - \Phi_m) \|_{L^\infty_0 L^\infty_0(S_{0, \omega})}) \lesssim 2^{-m} + \sum_{0 \leq k \leq K} \frac{1}{\sqrt{A}} \| e^{-A u \tilde{\Psi}^k} (\Phi_m - \Phi) \|_{L^\infty_0 L^\infty_0(S_{0, \omega})}
\]
uniformly for all subintervals $[0, U] \subseteq [0, \omega]$ and for all $m \in N$.

Using (9.50), we first claim that for $m \in N$ sufficiently large, $\| \Phi_m^{-1} \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})} \lesssim 2 \| \Phi_m^{-1} \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})}$.

If not, then by continuity there exists $\tilde{U}$ such that
\[
\| \Phi_m^{-1} \|_{W^{K, \infty}(S_{0, \omega})} = 2 \| \Phi_1^{-1} \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})}, \quad \sup_{m \in [0, \tilde{U}]} \| \Phi_m^{-1} \|_{W^{K, \infty}(S_{0, \omega})} \lesssim 2 \| \Phi_1^{-1} \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})}.
\]

However, for $A$ sufficiently large (independent of $\tilde{U}$ or $m$), (9.50) and (9.51) imply that on the interval $[0, \tilde{U}]$, 
\[
\sum_{0 \leq k \leq K} (\| e^{-A u \tilde{\Psi}^k} \frac{\partial (\Phi - \Phi_m)}{\partial u} \|_{L^\infty_0 L^\infty_0(S_{0, \omega})} + \| e^{-A u \tilde{\Psi}^k} (\Phi - \Phi_m) \|_{L^\infty_0 L^\infty_0(S_{0, \omega})}) \lesssim 2^{-m}.
\]

For $m \in N$ sufficiently large, (9.52) contradicts (9.51). Hence
\[
\| \Phi_m^{-1} \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})} \lesssim 2 \| \Phi_1^{-1} \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})}.
\]

Now using (9.53), we can repeat the above argument to derive (9.52) from (9.50) as long as $A$ and $m$ are chosen to be sufficiently large. Finally, we fix $A$ in (9.52) and absorb it into the implicit constants. Since $\tilde{U}$ is arbitrary, we have thus obtained (9.34).

**Step 3: Proof of (9.35).** The proof of (9.35) is much more straightforward since we are only have to deal with the *smooth* ODE (9.31).

Using the uniform estimates given by (9.20), (9.28), (9.30) and (9.34), we know that
\[
\sup_m \| e^{-A u \tilde{\Psi}^k} \frac{\partial \gamma_m}{\partial \tilde{u}} \|_{L^\infty_0 L^\infty_0(S_{0, \omega})} \lesssim 2 \frac{\partial \gamma_m}{\partial \tilde{u}} \| \Phi_m \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})} < +\infty.
\]

We now let $\Psi_m = \frac{\partial \gamma_m}{\partial \tilde{u}}$, $H_m = 2 \frac{\partial \log M}{\partial \tilde{u}}$ and $R_m = \frac{1}{2} \frac{\partial \gamma_m}{\partial \tilde{u}} (\Phi_m + \frac{1}{2} f_m \Phi_m^{-1})$. Then by (9.31), (9.29) and (9.54), it suffices to show that for $\tilde{\Psi}_m$, solving the ODE
\[
\frac{\partial \Psi_m}{\partial \tilde{u}} + H_m \Psi_m + R_m = 0,
\]
with the estimates
\[
\sup_m \| \Psi_m(0, \tilde{u}) \|_{W^{K, \infty}(S_{0, \omega})} + \| H_m \|_{L^2_0 W^{K, \infty}(S_{0, \omega})} + \| R_m \|_{L^2_0 W^{K, \infty}(S_{0, \omega})} \lesssim D,
\]
we can obtain uniform (in $m$) estimates for the $L^\infty_0 W^{K, \infty}(S_{0, \omega})$ norm of $\Psi_m$.

Assume as a bootstrap assumption that $\| \Psi_m \|_{W^{K, \infty}(S_{0, \omega})} \lesssim \sqrt{B} D e^{B \tilde{u}}$ (which is satisfied initially as long as $B \geq 1$). Differentiating (9.55) by $\tilde{\Psi}_m^k$ (for $0 \leq k \leq K$), integrating in $\tilde{u}$, and then using the bootstrap assumption and the Cauchy–Schwarz inequality, we obtain
\[
\| \tilde{\Psi}_m^k \|_{L^\infty_0(S_{0, \omega})} \lesssim D + \sum_{k_1 + k_2 = k} \| \tilde{\Psi}_m^k : H_m \|_{L^2_0 L^\infty_0(S_{0, \omega})} \| \tilde{\Psi}_m^k \|_{L^2_0 L^\infty_0(S_{0, \omega})} + \| \tilde{\Psi}_m^k \|_{L^2_0 L^\infty_0(S_{0, \omega})} \lesssim D + D^2 D e^{B \tilde{u}} \lesssim (D + 1) e^{B \tilde{u}}.
\]

In particular, we obtain $\| \Psi_m \|_{W^{K, \infty}(S_{0, \omega})} \lesssim (D + 1) e^{B \tilde{u}}$. If we choose $B$ such that $(D + 1) \ll \sqrt{B}$, we have improved the bootstrap assumption. Finally, after fixing $B$ and allowing the implicit constant to depend on $B$, we obtain that $\| \Psi_m \|_{L^\infty_0 W^{K, \infty}(S_{0, \omega})} \lesssim (D + 1)$. As argued above, this implies (9.35).
9.3. Putting everything together: Approximating measure-valued null dust data by vacuum data.

**Proposition 9.4.** Let \( K \in \mathbb{N} \). Suppose we are given \( \nu_{\text{init}} \) and \( \gamma \) as in the assumptions of Proposition 9.2, and \( \Phi, \log \Omega, \tilde{\gamma} \) as in Proposition 9.3. Assume moreover that \( \nu_{\text{init}} \) is supported on \([0, u_\gamma] \times U^c\) for some \( U \subset S^2\).

Then there exist smooth \( \{ (\tilde{\gamma}_m^{(\text{vac})}, \Phi_m^{(\text{vac})}) \}_{m=m_0}^{+\infty} \) on \([0, u_\gamma] \times S^2\) such that the following holds:

1. **(Basic properties)** \( \Phi_m^{(\text{vac})} \) is strictly positive. \( \tilde{\gamma}_m^{(\text{vac})} \) is a positive definite metric on each \( S_0 \).

Moreover,

\[
\frac{\text{det} \tilde{\gamma}_m^{(\text{vac})}}{\text{det} \gamma} = 1 \tag{9.57}
\]

2. **(Vacuum constraint)** For every \( m \in \mathbb{N} \) with \( m \geq m_0 \),

\[
\begin{align*}
\frac{\partial^2 \Phi_m^{(\text{vac})}}{\partial u_2^2} &= 2 \log \Omega \frac{\partial \Phi_m^{(\text{vac})}}{\partial u_2} - \frac{1}{\sqrt{\nu}} \frac{\partial \Phi_m^{(\text{vac})}}{\partial u_2} \left( \frac{1}{\gamma_m^{(\text{vac})}} \right) \Phi_m^{(\text{vac})}, \\
\Phi_m^{(\text{vac})}(u = 0) &= \Phi(u), \quad \frac{\partial \Phi_m^{(\text{vac})}}{\partial u_2}(u = 0) = \frac{\partial \Phi}{\partial u}(u = 0).
\end{align*}
\]

3. **(Uniform estimates)**

\[
\|\tilde{\gamma}_m^{(\text{vac})}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} + \|\frac{\partial \tilde{\gamma}_m^{(\text{vac})}}{\partial u}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim 1, \tag{9.58}
\]

\[
\|\Phi_m^{(\text{vac})}\|_{L^\infty_w W^{K,\infty}} + \|\frac{\partial \Phi_m^{(\text{vac})}}{\partial u}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim 1, \tag{9.59}
\]

\[
\Phi_m^{(\text{vac})} \gtrsim 1. \tag{9.60}
\]

4. **(Convergence)** The following convergences hold:

\[
\tilde{\gamma}_m^{(\text{vac})} \to \tilde{\gamma}, \quad \Phi_m^{(\text{vac})} \to \Phi \text{ uniformly}, \tag{9.61}
\]

and for every \( \varphi \in C_0^0([0, u_\gamma] \times S^2) \),

\[
\lim_{m \to +\infty} \int_{[0, u_\gamma] \times S^2} \varphi \left( \tilde{\Omega}^{-2} \frac{\partial \tilde{\gamma}_m^{(\text{vac})}}{\partial u} \right)^2 \left( \gamma_m^{(\text{vac})} \right)^{-1} d\nu_{\text{init}} - \frac{1}{4} \int_{[0, u_\gamma] \times S^2} \varphi \left( \Omega^{-2} \frac{\partial \gamma}{\partial u} \right)^2 d\nu_{\text{init}}. \tag{9.62}
\]

**Proof. Step 1: Approximate with a sequence of smooth null dust data.** Let \( \{ f_m \}_{m=1}^{+\infty} \) be as in the conclusion of Proposition 9.2. Since \( \nu_{\text{init}} \) is supported on \([0, u_\gamma] \times U^c\), by Proposition 9.2, we choose \( f_m \) so that \( \text{supp}(f_m) \subset [0, u_\gamma] \times U^c \) for some fixed non-empty open \( V \subset \mathbb{R} \times U \) with \( V \subset \mathbb{R} \times U \).

Define a smooth sequence \( \{ (\gamma_m^{(\text{dust})}, \Phi_m^{(\text{dust})}, \tilde{\Phi}_m^{(\text{dust})}), \tilde{\Psi}_m^{(\text{dust})}) \}_{m=1}^{+\infty} \) so that the estimates (9.32) and (9.33) hold (with \( (\gamma_m, \Phi_m, \tilde{\Phi}_m) = (\gamma_m^{(\text{dust})}, \Phi_m^{(\text{dust})}, \tilde{\Phi}_m^{(\text{dust})}) \)). We in addition choose

\[
\|\gamma_m^{(\text{dust})} - \tilde{\gamma}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim \min\{\epsilon, 2^{-m}\}, \tag{9.63}
\]

where \( \epsilon > 0 \) is as in Proposition 9.1.4 (see in particular (9.7)) with \( \tilde{\gamma}_0 = \tilde{\gamma} \).

We then apply Proposition 9.3 so that by (9.34) and (9.35) we have

\[
\|\Phi_m^{(\text{dust})} - \Phi\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim 2^{-m}, \quad \sup_m \|\frac{\partial \Phi_m^{(\text{dust})}}{\partial u}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim 1. \tag{9.64}
\]

**Step 2: Approximate with a sequence of vacuum data.** From Step 1, we have obtained a sequence of smooth data \( \{ (f_m, \gamma_m^{(\text{dust})}, \Phi_m^{(\text{dust})}) \}_{m=1}^{+\infty} \) to the Einstein–null dust system. Now for each \( m \in \mathbb{N} \), we apply Proposition 9.1. We choose \( n(m) \) sufficiently large depending on \( m \) so that

\[
n^{-1} \leq 2^{-m} \tag{9.65}
\]

and that all the \( O(\frac{1}{n}) \) error terms in Proposition 9.1 are made \( \lesssim 2^{-m} \). Since we have already required (9.63), we thus obtain a sequence \( \{ (\tilde{\gamma}_m^{(\text{vac})}, \Phi_m^{(\text{vac})}) \}_{m=1}^{+\infty} \) so that

\[
\|\tilde{\gamma}_m^{(\text{vac})} - \gamma_m^{(\text{dust})}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim 2^{-m} \tag{9.66}
\]

\[
\left\| \frac{\partial \tilde{\gamma}_m^{(\text{vac})}}{\partial u} \right\|_{L^\infty_w (\gamma_m^{(\text{vac})})}^2 \left( \Phi_m^{(\text{dust})} \right)^2 - \left\| \frac{\partial \gamma_m^{(\text{dust})}}{\partial u} \right\|_{L^\infty_w (\gamma_m^{(\text{vac})})}^2 \left( \Phi_m^{(\text{dust})} \right)^2 - 4 f_m - \frac{1}{n(m)} \frac{\partial F_m}{\partial u}\|_{L^\infty_w W^{K,\infty}(S_0, \tilde{\gamma})} \lesssim 2^{-m}, \tag{9.67}
\]

\[
\frac{\text{det} \tilde{\gamma}_m^{(\text{vac})}}{\text{det} \gamma} = 1 \tag{9.57}
\]
\[ \| \Phi^{(\text{vac})}_m - \Phi^{(\text{dust})}_m \|_{L^2_\infty W^{K,\infty}} \lesssim 2^{-m}, \quad \left\| \frac{\partial (\Phi^{(\text{vac})}_m - \Phi^{(\text{dust})}_m)}{\partial u} \right\|_{L^2_\infty W^{K,\infty}(S_0, \Sigma)} \lesssim 2^{-m}, \tag{9.68} \]

and
\[ \left\| \frac{\partial \gamma^{(\text{vac})}_m}{\partial u} \right\|_{L^2_\infty W^{K,\infty}(S_0, \Sigma)} \lesssim 1. \tag{9.69} \]

Here, in (9.67), \( F_m \) are smooth functions which according to (9.9) satisfy
\[ \| F_m \|_{L^2_\infty W^{K,\infty}} \lesssim 1. \tag{9.70} \]

Note also that in deriving (9.69), we have also used the estimate (9.20) for \( f_m \).

**Step 3: Putting everything together.** In this last step, we check that for sufficiently large \( m_0 \), the sequence \( \{ (\gamma^{(\text{vac})}_m, \Phi^{(\text{vac})}_m) \}_{m=m_0}^\infty \) constructed in Step 2 indeed satisfies the necessary requirements.

First, the constructions in Propositions 9.1 guarantees (9.57). After choosing \( m_0 \) to be sufficiently large, the positivity follows from (9.63), (9.64), (9.66), (9.68) and the triangle inequality.

Second, since \( \{ (\gamma^{(\text{vac})}_m, \Phi^{(\text{vac})}_m) \}_{m=1}^\infty \) are constructed by Proposition 9.1, the vacuum constraints are satisfied by definition.

Third, the uniform estimates (9.58) and (9.59) follow from (9.63), (9.64), (9.66), (9.68) and the triangle inequality. The lower bound (9.60) follows from the lower bound of \( \Phi \), the estimates (9.64), (9.68) and the triangle inequality.

Fourth, the convergence statements in (9.61) follow from (9.63), (9.64), (9.66), (9.68) and the triangle inequality.

Finally, we prove the convergence statements in (9.62). By density we assume that \( \varphi \) is \( C^1 \).

Note that \( dA_\gamma = \varphi^2 \, dA_\gamma \). Also, by (9.57), \( dA_{\gamma^{(\text{vac})}_m} = \left( \Phi^{(\text{vac})}_m \right)^2 \, dA_\gamma \). Then, using (9.28), (9.30), (9.64), (9.67), (9.68) and (9.69), we obtain that for each fixed \( m \in \mathbb{N} \) with \( m \geq m_0 \),
\[
\frac{1}{4} \int_{[0, u_m] \times S^2} \varphi \Omega^{-2} \left| \frac{\partial \gamma^{(\text{vac})}_m}{\partial u} \right|^2 \, dA_{\gamma^{(\text{vac})}_m} \, du \quad = \quad \frac{1}{4} \int_{[u, u_m] \times S^2} \varphi \Omega^{-2} \left| \frac{\partial \gamma^{(\text{vac})}_m}{\partial u} \right|^2 \, dA_\gamma \, du \\
= \quad \frac{1}{4} \int_{[0, u_m] \times S^2} \varphi \Omega^{-2} \left( \left| \frac{\partial \gamma^{(\text{vac})}_m}{\partial u} \right|^2 \Phi^{(\text{vac})}_m \right)^2 - \left| \frac{\partial \gamma^{(\text{vac})}_m}{\partial u} \right|^2 \Phi^2 \, dA_\gamma \, du \\
= \quad \int_{[0, u_m] \times S^2} \varphi \Omega^{-2} \left( f_m + \frac{1}{4n(m)} \frac{\partial F_m}{\partial u} \right) \, dA_\gamma \, du + O(2^{-m}). \tag{9.71} \]

We then integrate by parts and use (9.65) and (9.70) to obtain
\[
\left| \int_{[0, u_m] \times S^2} \varphi \Omega^{-2} \frac{1}{4n(m)} \frac{\partial F_m}{\partial u} \, dA_\gamma \, du \right| \lesssim 2^{-m}. \tag{9.72} \]

Plugging (9.72) into (9.71), taking the \( m \to +\infty \) limit, and using the first conclusion in Proposition 9.2, we obtain
\[
\text{RHS of (9.62)} = \lim_{m \to +\infty} \int_{[0, u_m] \times S^2} \varphi \Omega^{-2} f_m \, dA_\gamma \, du = \text{LHS of (9.62)}. \]

The same construction as Proposition 9.4 can be carried out on the \( \nu = 0 \) hypersurface in a completely analogous manner. We state this as a proposition:

**Proposition 9.5.** Proposition 9.4 holds on \( H_0 \) after replacing \( u \mapsto u \), \( \nu \mapsto \nu_\text{init} \).

**9.4. Proofs of Theorem 4.6 and Theorem 4.7.**

**Proof of Theorem 4.6.** We first consider the case that the strongly angularly regular reduced characteristic initial data set satisfies the additional assumption that \( \text{supp}(d\nu_\text{init}) \subset [0, I] \times U^c \) and \( \text{supp}(d\nu_\text{init}) \subset [0, I] \times U^c \) for some non-empty open \( U \subset \mathbb{S}^2 \).

We first apply Propositions 9.4 and 9.5 in the previous subsection so as to obtain a sequence of reduced characteristic initial data sets (see Section 3.1.2) for the Einstein vacuum equations.

By Lemma 3.1, the sequence of vacuum characteristic initial data sets corresponding to this sequence of vacuum reduced characteristic initial data set satisfies the assumption of Theorem 4.1. Therefore, Theorem 4.1 shows that there exists \( \epsilon > 0 \) so that the sequence of vacuum solutions arising from this sequence of data admits a subsequence that converges to an angularly regular weak solution to the Einstein–null dust system in \( [0, u_*] \times [0, \nu_*] \times \mathbb{S}^2 \) whenever \( u_* \in (0, I] \) and \( \nu_* \in (0, \epsilon) \). Because of
the convergence statements (9.61) and (9.62), it follows moreover that this solution indeed achieve the prescribed initial data. We have thus proven the existence part of the theorem.

Uniqueness then follows from Theorem 4.4.

Finally, in the general case where \( \partial r_{\text{init}} \) or \( \partial u_{\text{limit}} \) is not supported away from some angular direction, we can cut off and use the finite speed of propagation to reduce to the previous case. \( \square \)

**Proof of Theorem 4.7.** This is similar to the proof of Theorem 4.6 so we will be brief. Given an angularly regular weak solution to the Einstein–null dust system as in Theorem 4.6, we first approximate the data using Propositions 9.4 and 9.5 and then use Lemma 3.1 and Theorem 4.1 to obtain a limiting angularly regular weak solution \((\bar{\mathcal{M}}, g_\infty)\) (for \( u_\ast \), sufficiently small) to the Einstein–null dust system.

By definition, \((\bar{\mathcal{M}}, g_\infty)\) is a limit of a sequence of smooth solutions \((\mathcal{M}_n, g_n)\) to the Einstein vacuum equations in the sense of Theorem 4.1. On the other hand, by Theorem 4.4, \((\mathcal{M}, g) = (\bar{\mathcal{M}}, g_\mid_{\bar{\mathcal{M}}})\). Combining these two facts gives the conclusion of the theorem. \( \square \)

# 10. Relation with the formation of trapped surfaces

In this final section, we discuss a connection of high-frequency limits and null dust shells with Christodoulou’s work [10] on the formation of trapped surfaces in vacuum. As is well-known, Penrose proved that a vacuum spacetime (or more generally a spacetime obeying the null energy condition) with a non-compact Cauchy hypersurface and a trapped surface must be future causally geodesically incomplete. In a monumental work [10], Christodoulou showed moreover that trapped surfaces could form dynamically in vacuum spacetimes from characteristic initial data which are arbitrarily dispersed. In particular, he found open sets of initial data such that there are no trapped surfaces initially, but a trapped surface is formed in the dynamical evolution. We will recall the results of [10] in Section 10.1; see also [1, 2, 3, 27, 28, 30, 32, 33, 34, 35, 36, 42, 56, 57] and references therein for various extensions.

Christodoulou’s construction is based on what he called the short pulse method, for which the large initial data is concentrated on a short length scale of size \( \delta \). It is precisely the short length scale that allowed Christodoulou to propagate a hierarchy of \( \delta \)-dependent estimates for the geometric quantities so that the estimates can be closed despite being a large data problem.

We will show below (see Section 10.3) that if we take the \( \delta \to 0^+ \) limit in Christodoulou’s short pulse ansatz, then one obtains a limiting spacetime which is not vacuum, but solves the Einstein–null dust system with a null dust shell (in a similar manner as the main results of this paper). The limiting solution in fact coincides with the Synge–Gibbons–Penrose construction (see Section 10.2) of collapsing null dust shell solutions in which trapped surfaces form dynamically. In particular, one could think of Christodoulou’s solutions as “approximating” the spacetimes with collapsing null shells.

## 10.1. Christodoulou’s trapped surface formation result

In [10], Christodoulou proved the dynamical formation of trapped surfaces by considering a characteristic initial value problem, where the initial data are prescribed on two intersecting null hypersurfaces.

Before describing Christodoulou’s data, first define \( \mathcal{M}^- := \{(u, \bar{y}, \vartheta) : 0 < u < u + 1 < 1, \vartheta \in S^2\} \) to be the past light cone of a point in Minkowski spacetime\(^{41}\), i.e. consider the metric \( g \) on \( \mathcal{M}^- \) taking the form

\[
g = -2\Omega^2 (du \otimes du + du \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du)
\]

with \( \Omega^2 \mid_{\mathcal{M}^-} = 1, b \mid_{\mathcal{M}^-} = 0 \) and \( \gamma \mid_{\mathcal{M}^-} = (u - u + 1)^2 \gamma_{S^2(1)}, \) where \( \gamma_{S^2(1)} \) is the round metric on the 2-sphere with radius 1.

For the characteristic initial value problem that Christodoulou considered, one initial characteristic hypersurface is \( H_0 := \{(u, \bar{y}, \vartheta) \in \mathcal{M}^- : u = 0\} \) with the induced geometry. The other initial characteristic surface is given by \( H_0 = \{u = 0\} \times [0, \delta] \times S^2 \), and the data consist of a “short pulse” mentioned above, where \( \delta > 0 \) is a small parameter. The following is a version\(^{42}\) of the main theorem in [10], which gives a condition on the initial data on \( H_{-1} \) which guarantees the dynamical formation of trapped surfaces.

\(^{41}\)In polar coordinates, the Minkowski metric is given by \( m = -dt^2 + dr^2 + r^2 \gamma_{S^2(1)}. \) Here, the null coordinates correspond to \( u = \frac{1}{2}(t - r) + 1, \bar{y} = \frac{1}{2}(t + r). \) Thus in the \((t, r, \vartheta)\) coordinate system, we have \( \mathcal{M}^- = \{(t, r, \vartheta) : -2 < t + r < 0, -2 < t - r < 0\}, \) which is a truncated subset of the past light cone of the origin in Minkowski spacetime.

\(^{42}\)The original theorem in fact applies when the data are posed on past null infinity to obtain a semi-global spacetime; see details in [10].
Theorem 10.1 (Christodoulou [10]). For every $B > 0$ and $u_* < 1$, there exists $\delta = \delta(B, u_*) > 0$ sufficiently small such that if the initial $\chi$ (denoted $\chi(0)$, prescribed on $H_0 := \{(u, u, \vartheta) : u = 0, \vartheta \in [0, \delta], \vartheta \in \mathbb{S}^2\}$ satisfy
\[
\sum_{i \leq 5, j \leq 3} \delta^{\frac{1}{2} + j} \| W_i W_j \chi(0) \|_{L^2(S_{-1 + u_*})} \leq B, \tag{10.1}
\]
then there is a unique solution to the Einstein vacuum equation in double null coordinates in $\{(u, u) : u \in [0, u_*], u \in [0, \delta]\} \times \mathbb{S}^2$ with the prescribed data.

Moreover, if the initial data also verify the lower bound
\[
\inf_{\vartheta \in \mathbb{S}^2} \int_0^\delta \chi(0)^2 (u', \vartheta) \, du' \geq M_* > 2(1 - u_*), \tag{10.2}
\]
then, after choosing $\delta$ to be smaller (depending on $B$, $u_*$ and $M_*$) if necessary, the sphere $S_{-1 + u_*}$ is a trapped surface.

We remark that by definition, the sphere $S_{-1 + u_*}$ is a trapped surface exactly when the inequalities $\xi \chi > 0$ and $\xi \chi < 0$ hold pointwise on $S_{-1 + u_*}$.

As already noted in [42], the scaling of the initial data in Theorem 10.1 is such that the $L^2(H_0)$ norm of the initial data for $\chi$ and its angular derivatives are uniformly bounded as $\delta \to 0$. In particular, suppose we extend the data in a "regular" way to $S_{-1 + u_*}$.

Define, in exactly the same manner as in Section 10.1, $M_{-} := \{(u, u, \vartheta) : 0 < u < u_* + 1, \vartheta \in \mathbb{S}^2\}$ and impose that the Ricci coefficient $\Omega_{\vartheta \vartheta} > 0$ on $S_{-1 + u_*}$ and $\delta_{\vartheta \vartheta} = 0$, $\delta_{\vartheta \vartheta} > 0$ on $S_{-1 + u_*}$.

As a result, it follows that in $M_{-}$, $\xi \chi = -\frac{2}{u + 1}$ and $\xi \chi = -\frac{2}{u + 1}$ and all the other Ricci coefficients vanish.

Define $M_{+} := \{(u, u, \vartheta) : 0 < u < 1, 0 \leq u \leq f(u)\}$ for some decreasing function $f : [0, 1] \to \mathbb{R}$ with $\lim_{u \to 1^-} f(u) = 0$. The choice of the metric in $M_{+}$ does not matter so much, let us just assume that it is a vacuum metric in a double null coordinate system (10.3). Assume also that the metric coefficients $\Omega, \chi$ and $\omega$ are continuous up to and across $N$ and that all the Ricci coefficients except for $\xi \chi$, $\hat{\chi}$ and $\omega$ are also continuous up to and across $N$.

Impose that the Ricci coefficient $\xi \chi$ has a jump discontinuity across $N$ so that $\xi \chi = m(\vartheta) = \frac{m(\vartheta)}{(1 - u)^2}$, for some smooth function $m : \mathbb{S}^2 \to \mathbb{R}_{> 0}$. By (2.28) (and taking appropriate limit), this means that the spacetime has a null shell data given by
\[
\delta u = m(\vartheta) \delta(u),
\]
where $\delta(u)$ denotes the delta measure at $u = 0$ and $m(\vartheta)$ is as before. Note also that with this definition of $\delta u$, it is easy to check that the propagation equation (2.29) holds.

Suppose now that there are $M_* > 0$ and $u_* \in (0, 1)$ such that
\[
\inf_{\vartheta \in \mathbb{S}^2} m(\vartheta) \geq M_* > 2(1 - u_*) > 0. \tag{10.4}
\]
We claim that in fact the sphere \( S_{u, \epsilon} \) is trapped for \( \epsilon > 0 \) sufficiently small. In order to prove that, it suffices to show that \( \Psi^\chi(u = u_*, \ u = 0, \ \sigma) < 0 \) and that\(^{45} \) \( \Psi^\chi^{-1}(u = u_*, \ u = \epsilon, \ \sigma) < 0 \) for all \( \sigma \in S^2 \)

These are very easy to check: since \( \Psi^\chi \) is continuous across the null dust shell, it takes the Minkowskian value \( \Psi^\chi(u = u_*, \ u = 0, \ \sigma) = -\frac{2}{1 - u_*^2}; \) on the other hand, since \( \Psi^\chi^{-1}(u = u_*, \ u = 0, \ \sigma) = \frac{2}{1 - u_*^2} \) (taking Minkowskian value), the jump condition above gives

\[
\Psi^\chi^+(u = u_*, \ u = 0, \ \sigma) = \Psi^\chi^-(u = u_*, \ u = 0, \ \sigma) - \frac{m(\sigma)}{(1 - u_*^2)^2} = \frac{2}{1 - u_*^2} - \frac{m(\sigma)}{(1 - u_*^2)^2}.
\]

In particular, this means that whenever \( m(\sigma) \) obeys the lower bound (10.4), we have \( \Psi^\chi^+ < 0 \) everywhere on \( S_{u, 0} \). Now since the metric is smooth in \( \mathcal{M}^+ \cap \{ u > 0 \} \), it follows that \( S_{u, u} \) is trapped for some \( u > 0 \) sufficiently small.

This simple construction shows that a trapped surface is formed dynamically from the collapse of a null dust shell.

10.3. Connection between the Synge–Gibbons–Penrose construction and trapped surface formation in vacuum. We now observe that by taking the \( \delta \to 0^+ \) limit in Christodoulou’s construction in Theorem 10.1, one obtains a solution to the Einstein–null dust system with a null dust shell as in Section 10.2.

To make this precise, we need slightly more assumptions than that in Theorem 10.1. In order to streamline the exposition, let us first state a propagation of regularity result, before turning to the precise setup relating Christodoulou’s result to Section 10.2. The following propagation of regularity result is a small modification of \([42, \text{Proposition 52}]\), and can be proven in exactly the same manner.

**Lemma 10.2.** Consider a characteristic initial value problem with initial data satisfying the assumptions of Theorem 3.2.

(1) (Propagation of angular regularity) If \( \forall i \in \mathbb{N} \cup \{0\} \), \( \exists \mathcal{C}_i > 0 \) such that the initial data satisfy (in addition to the assumptions of Theorem 3.2)

\[
\sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \| \Psi_i^\psi \|_{L^2(S_{u, \omega})} + \sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \| \Psi_i^\psi \|_{L^\infty(S_{u, 0})} \\
+ \sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \| \Psi_i^\psi \|_{L^2(S_{u, \omega})} + \sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \| \Psi_i^\psi \|_{L^\infty(S_{u, 0})} \leq C_i.
\]

Then \( \forall i' \in \mathbb{N} \cup \{0\} \), \( \exists \mathcal{C}'_i > 0 \) (where for each \( i' \), \( \mathcal{C}'_i \) depends only on the constants in Theorem 3.2 and finitely many \( C_i \)'s) such that the following bounds hold in \( [0, u_*] \times [0, u_*] \times S^2 \):

\[
\sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \| \Psi_i^\psi \|_{L^2(S_{u, \omega})} + \sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \| \Psi_i^\psi \|_{L^\infty(S_{u, 0})} \leq C'_i.
\]

(2) (Propagation of higher regularity in the “regular region”) Suppose the assumptions of part (1) hold, and

- \( \exists u_1, u_2, \ u_1, \ u_2 \) with \( 0 \leq u_1 < u_2 \leq u_* \) and \( 0 \leq u_1 < u_2 \leq \omega \),
- \( \exists J, \ L \in \mathbb{N} \cup \{0\}, \) and
- \( \forall i \in \mathbb{N} \cup \{0\}, \exists \mathcal{C}^{(J, L)}_i > 0 \)

such that the initial data satisfy

\[
\sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \sum_{\ell \leq L} \| \Psi_i^\psi \|_{L^2((u_1, u_2), L^\infty(S_{u, \omega}))} + \sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \sum_{j \leq J} \| \Psi_j^\psi \|_{L^2((u_1, u_2), L^\infty(S_{u, 0}))} \leq \mathcal{C}^{(J, L)}_i.
\]

Then \( \forall i' \in \mathbb{N} \cup \{0\} \), \( \exists \mathcal{C}'^{(J, L)}_i > 0 \) (where for each \( i' \), \( \mathcal{C}'^{(J, L)}_i \) depends only on the constants in Theorem 3.2 and finitely many \( C_i \)'s and \( \mathcal{C}^{(J, L)}_i \)'s) such that the following estimates hold:

\[
\sum_{\psi \in \{ \eta, \hat{\eta}, \psi, \hat{\psi}, \chi, K \}} \sum_{\ell \leq L} \sum_{j \leq J} \| \Psi_i^\psi \|_{L^2(L^\infty(S_{u, \omega}))} \leq \mathcal{C}'^{(J, L)}_i.
\]

\(^{45}\) Recall again that \( \Psi^\chi \) is not continuous across the hypersurface \( \{ u = 0 \} \)!
We remark that an important point of part (2) Lemma 10.2 is that the higher regularity estimates hold even when the data are singular for $u < u_1$ or $\mathbf{y} < \mathbf{y}_1$.

We now return to the discussion of the relation between Theorem 10.1 and null dust shells.

Consider a sequence of characteristic initial data on $H_0$ that is the backward Minkowskian light cone as in Theorem 10.1 and $H_0 = \{\emptyset\} \times [0, \epsilon] \times S^2$. Fix a decreasing sequence $\delta_n \to 0$. On $H_0$, require the (smooth) characteristic initial data to obey the following:

1. (Christodoulou’s conditions) Fix $M_\ast$, $B > 0$ and $\mathbf{u}_\ast \in (0, 1)$. When restricted to $\mathbf{y} \in [0, \delta_n]$, (10.1) and (10.2) both hold (with $\delta$ replaced by $\delta_n$).
2. (Additional angular regularity) Assume pointwise estimates for all higher angular derivatives, i.e. assume (10.5) holds.
3. (Additional regularity beyond short pulse) Impose that when restricted to $\mathbf{y} \in (\delta_n, I)$, the metric components $(\gamma_{\alpha\beta}, \log \Omega_{\alpha\beta}, b_{\alpha\beta})$ are uniformly bounded in the $C^k$ norm (with respect to derivatives tangential to $H_0$) for every $n \geq m$ and for every $k \in \mathbb{N} \cup \{0\}$.

By Theorem 4.1, there exists an $\epsilon > 0$ such that if $\mathbf{u}_\ast \leq \epsilon$, then for every $n \in \mathbb{N}$ there is a solution to the Einstein–null dust equations in the region $[0, \mathbf{u}_\ast] \times [0, \mathbf{y}_\ast] \times S^2$ with the prescribed initial data. Moreover, there exists a limiting solution to the Einstein–null dust system in $[0, \mathbf{u}_\ast] \times [0, \mathbf{y}_\ast] \times S^2$.

Given the conditions (1)–(3) that we imposed above for the sequence of vacuum initial data, we can in fact conclude that the limiting solution to the Einstein–null dust system has the following features:

1. (Propagation of angular regularity) By part (1) of Lemma 10.2, the improved angular regularity estimates (10.6) hold for the full sequence of vacuum solutions, and hence also hold for the limiting solution.
2. (Regularity in the $\nabla_3$ direction) Since the data on $H_0$ are smooth, by part (2) of Lemma 10.2, the estimates (10.8) hold for $L = 0$ and for all $J \geq 0$ for the full sequence of vacuum solutions. In particular, the limiting spacetime metric is smooth away from $\{\mathbf{y} = 0\}$.
3. (Improved regularity away from $\mathbf{y} = 0$) For every $\delta > 0$, (since $\delta_n \to 0^+$), there exists $N \in \mathbb{N}$ such that for every $n \geq N$, the assumption (10.7) holds for $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1, \mathbf{u}_2) := (0, \mathbf{u}_\ast, \delta, \mathbf{y}_\ast)$ and any $J, L \in \mathbb{N} \cup \{0\}$. It follows that for the limiting spacetime, (10.5) holds away from $\{\mathbf{y} = 0\}$.
4. (Continuity of $\eta, \mathbf{y}, \nabla \chi, \nabla \chi$ and $\nabla \omega$) By Theorem 4.1 and the definition of angular regularity (Definition 2.26), the limiting spacetime has continuous $\eta, \mathbf{y}, \nabla \chi, \nabla \chi$ and $\nabla \omega$. Using angular regularity and the improved $\nabla_3$ regularity established above, it follows that $\nabla \chi, \nabla \chi$ and $\nabla \omega$ are also continuous.
5. (Presence of a null shell) The Christodoulou conditions (10.1) and (10.2) exactly imply that in the limit there is a null dust shell — supported exactly on the $\{\mathbf{y} = 0\}$-hypersurface — given by the measure $d\nu_u = m(\mathbf{y})d\nu_0$, where $m:S^2 \to \mathbb{R}_{>0}$ which is both bounded above and bounded below away from 0. (Correspondingly, this gives a jump of $\nabla \chi$ across the $\{\mathbf{y} = 0\}$ hypersurface.)

With this it is easy to conclude that the limiting spacetime is exactly one as in Section 10.2, i.e. there is a propagating null dust shell which drives the dynamical formation of trapped surfaces. This demonstrates a connection between Christodoulou’s construction in [10] and collapsing null dust shells.

**Appendix A. Derivation of the estimates in Theorem 3.3**

In this appendix, we derive the estimates in Theorem 3.3. The most difficult bounds are already in [42]. Here we indicate how to obtain the remaining estimates.

**Proof of Theorem 3.3.** In this proof, all constants $C > 0$ and implicit constants in $\lesssim$ depend only on the constants in the assumptions of Theorems 3.2 and 3.3.

**Step 1:** Proof of (3.10)–(3.13): Ricci coefficient estimates from [42]. These estimates follow directly from [42, Theorem 4]. (Note that the norms $\mathcal{O}, \mathcal{O}_{3,2}$ and $\mathcal{R}$ defined in [42] controls all the norms in (3.10)–(3.13).)

**Step 2:** Proof of (3.9) I: Estimates for the metric components. In this step we prove the $C^0_u C^0_{\mathbf{w}} W^{3,2}(S_{\mathbf{w}, \gamma})$ estimates for $\mathbf{f}$ in (3.9).

**Step 2(a): Preliminaries.** We first argue as in Proposition 6.1 to obtain that for any $p \in \{1, +\infty\}$,

$$C^{-1} \|\xi\|_{L^p(S_{\mathbf{w}, \gamma}; \mathbf{u}, \mathbf{y}, \mathbf{z})} \leq \|\xi\|_{L^p(S_{\mathbf{w}, \gamma})} \leq C \|\xi\|_{L^p(S_{\mathbf{w}, \gamma}; \mathbf{u}, \mathbf{y}, \mathbf{z})}. \quad (A.1)$$

(For this we first use the transport equation (6.9) in the $\mathbf{u}$ direction on $H_{b_0}$, where $b \mid_{H_{b_0}} = 0$. We then use the transport equation (6.6) in the $\mathbf{u}$ direction for the rest of the spacetime.)
Next we derive some commutation formulas. By (2.10),
\[
|\mathcal{L}_{\frac{\partial}{\partial u}} \nabla B|\psi_{A_1 \cdots A_r} = 2 \sum_{i=1}^r (\gamma^{-1})^{C_D} \{ \nabla_D (\chi) A_i B - 2 \nabla_D (A_i \Omega) B \} \psi_{A_1 \cdots A_i \cdots A_r},
\]
(A.2)
where \( \hat{A} \) means that \( A_i \) is removed.

Similarly, since \( b |_{\mathcal{H}_0} = 0 \), we have by (2.10)
\[
|\mathcal{L}_{\frac{\partial}{\partial u}} \nabla B|\psi_{A_1 \cdots A_r} |_{\mathcal{H}_0} = 2 \sum_{i=1}^r (\gamma^{-1})^{C_D} \{ \nabla_D (\chi) A_i B - 2 \nabla_D (A_i \Omega) B \} \psi_{A_1 \cdots A_i \cdots A_r} |_{\mathcal{H}_0}.
\]
(A.3)

**Step 2(b): Estimates for \( \Omega \).** The zeroth estimates for \( \Omega \) in (3.9) can be obtained from [42, Propositions 1, 3]. The higher order derivatives for \( \Omega \) follows from the third equation in (2.11) together with the estimates for \( \eta \) and \( \tilde{\eta} \) in (3.10).

**Step 2(c): Estimates** for \( \gamma \). Using (2.10) and \( \mathcal{L}_{\frac{\partial}{\partial u}} \gamma_{0,0} = 0 \), we have
\[
\frac{\partial}{\partial u} |\gamma - \gamma_{0,0}|^2 |_{\gamma_{0,0}} = 4 \Omega (\chi - \gamma - \gamma_{0,0}) |_{\gamma_{0,0}} \frac{\partial}{\partial u} |\gamma - \gamma_{0,0}|^2 |_{\gamma_{0,0}} |_{\mathcal{H}_0},
\]
Integrating (A.4) (first in \( u \) along \( \mathcal{H}_0 \), then in \( \gamma \)), and using (A.1) together with the estimates for \( \chi, \tilde{\chi} \), \( \Omega \) established above, we obtain the \( C^0_u C^0_w L^2(S_{u, \tilde{u}, \gamma}) \) bounds for \( \gamma - \gamma_{0,0} \) in (3.9).

To obtain the (first to third) derivative estimates for \( \gamma - \gamma_{0,0} \), we first derive the transport equations for \( \nabla^i (\gamma - \gamma_{0,0}) \) using (A.2) and (A.3), and then argue similarly as above.

**Step 2(d): Estimates for \( b \).** The \( C^0_u C^0_w W^{3,2}(S_{u, \tilde{u}, \gamma}) \) estimates for \( b \) can be proven in a similar way as those for \( \gamma - \gamma_{0,0} \), except that we instead use as transport equation the third equation in (2.10) and also the fact that \( b |_{\mathcal{H}_0} = 0 \).

**Step 3: Proof of (3.7) and (3.8): area density, isoperimetric constant and area estimates.** For the area density estimate in (3.8), note that by (2.10) and \( b |_{\mathcal{H}_0} = 0 \),
\[
\frac{\partial}{\partial u} (\log \frac{\det \gamma}{\det \gamma_{0,0}}) = 2 \Omega \frac{\det \gamma}{\det \gamma_{0,0}} \frac{\partial}{\partial u} (\log \frac{\det \gamma}{\det \gamma_{0,0}}) |_{\mathcal{H}_0} = 0 \Omega \frac{\det \gamma}{\det \gamma_{0,0}} |_{\mathcal{H}_0}.
\]
Integrating (first in \( u \) along \( \mathcal{H}_0 \) and then in \( \gamma \)) and using the already established estimates with (A.1), we obtain (3.8).

Next, consider the isoperimetric constant bound in (3.7). Take \( (u, \gamma) \in [0, u_*] \times [0, \tilde{u}_*] \). To bound \( I(S_{u, \tilde{u}}, \gamma) \), we first bound \( I(S_{u, 0}, \gamma) \) in terms of \( I(S_{u, 0}, 0) \), and then bound \( I(S_{u, \tilde{u}}, \gamma) \) in terms of \( I(S_{u, 0}, 0) \).

Let \( U \subset S_{u, \tilde{u}} \) be a domain as in the definition in (2.3). We first map \( U \) to \( S_{u, 0} \) via the flow of \( \frac{\partial}{\partial t} \), and then map it to \( S_{0, 0} \) via the flow of \( \frac{\partial}{\partial u} \). According to (2.3), it then suffices to control the change of \( \text{Area}(U), \text{Area}(U^*) \) and \( \text{Perimeter}(\partial U) \) under these maps. The changes of \( \text{Area}(U) \) and \( \text{Area}(U^*) \) have already been bounded in (3.8). The change in \( \text{Perimeter}(\partial U) \) can be controlled in a similar manner as [10, Lemmas 5.3, 5.4]. This gives a uniform bound on \( \text{f}(S_{u, \tilde{u}}, \gamma) \).

Finally, the area estimate in (3.7) is an immediate consequence of (3.8).

**Step 4: Proof of (3.14)−(3.17): Using the equations for the Ricci coefficients.** For these estimates involving \( \mathcal{L}_{\frac{\partial}{\partial u}} \) and \( \mathcal{L}_{\frac{\partial}{\partial u}} \), we use the null structure equations.

**Step 4(a): (3.14) for \( \eta \) and \( \tilde{\eta} \).** For \( \psi \in \{ \eta, \tilde{\eta} \} \), by Proposition 2.14,
\[
\mathcal{L}_{\frac{\partial}{\partial u}} \psi = \Omega \nabla_3 \psi + \Omega \chi \cdot \psi, \quad \mathcal{L}_{\frac{\partial}{\partial u}} \psi = \Omega \nabla_3 \psi + \Omega \chi \cdot \psi - \nabla_b \psi - \nabla_{b^*} b.
\]
(A.5)
Let us consider the case \( \psi = \eta \) (\( \tilde{\eta} \) is similar). First, using (A.5) and (2.15),
\[
\mathcal{L}_{\frac{\partial}{\partial u}} \eta = \Omega \chi \cdot \eta + \Omega \{ \frac{3}{4} \frac{\partial}{\partial u} \eta - \nabla_b \chi - \frac{1}{2} \nabla_{b^*} \chi - \frac{1}{2} (\eta - \tilde{\eta}) \cdot \tilde{\chi} \}.
\]
(A.6)
It suffices to bound each term on the RHS of (A.6) in \( L^2_u C^0_w W^{3,2}(S_{u, \tilde{u}}, \gamma) \). Since we have obtained a uniform bound on the isoperimetric constant in Step 3, we can apply Sobolev embedding in Proposition 5.1 (together with Hölder’s inequality) so that given any two tensor fields \( \phi^{(1)} \) and \( \phi^{(2)} \)
\[
\| \phi^{(1)} \otimes \phi^{(2)} \|_{W^{3,2}(S_{u, \tilde{u}}, \gamma)} \leq \| \phi^{(1)} \|_{W^{3,2}(S_{u, \tilde{u}}, \gamma)} \| \phi^{(2)} \|_{W^{3,2}(S_{u, \tilde{u}}, \gamma)}.
\]
(A.7)

Note that the estimates we need for \( \gamma \) here are slightly different from those in [42]. In [42], we gave the bounds for \( \gamma \) in local coordinates.

Note that while (3.14) only requires the slightly weaker \( C^0_u L^2_u W^{3,2}(S_{u, \tilde{u}}, \gamma) \) estimates, the stronger estimate holds.
With the product estimate (A.7), it is not hard to control terms on the RHS of (A.6). To consider a couple of representative terms, we have
\[
\|\Omega \partial u \partial \psi \chi \|^2 S^2 \| \Omega \|^2 \| \nabla \partial u \partial \psi \chi \|^2 \lesssim \| \Omega \|^2 \| \nabla \partial u \partial \psi \chi \|^2 \lesssim 1,
\]
which is (A.8).
\[
\|\Omega \cdot \chi \| L^2 \| \chi \|^2 \lesssim \| \Omega \|^2 \| \nabla \partial u \partial \psi \chi \|^2 \lesssim \| \Omega \|^2 \| \nabla \partial u \partial \psi \chi \|^2 \lesssim 1.
\]
For \( L^\perp \), note that even though our list of null structure equations do not explicitly contain a \( \nabla_\perp \eta \), combining (2.16) with the second and third equations in (2.11), we have
\[
\mathcal{L} \nabla_\perp \eta = \Omega \mathcal{L} \eta - \nabla_\perp \eta - \nabla_\perp b - \Omega \left[ -\frac{3}{4} \partial \chi (\eta - \eta) + \partial \psi \chi - \frac{1}{2} \nabla \psi \chi - \frac{1}{2} (\eta - \eta) \cdot \chi \right] - 4 \nabla (\Omega \partial \eta) - 2 \nabla (\nabla_\perp \log \Omega).
\]
Using equation (A.10), we can then bound the terms on its RHS in \( C^0_{\perp} L^2 W^{2,2}(S_{u, \gamma}) \) in a similar way as (A.8) and (A.9). Notice that because of the vector \( b \), we need the third order angular derivative estimates for \( b \) and \( \eta \), which are both established above. (On the other hand, it is precisely because of the third order angular derivatives for \( \eta \) that we only have \( C^0_{\perp} L^2 W^{2,2}(S_{u, \gamma}) \), instead of \( L^2 C^0_{\perp} W^{2,2}(S_{u, \gamma}) \), estimates.)

**Step 4(b):** (3.17) for \( \chi, \omega, \hat{\chi} \) and \( \omega \). As in (A.5) we rewrite \( \mathcal{L} \nabla \hat{\psi} \) and \( \mathcal{L} \nabla \psi \) as \( \nabla_4 \psi_4 \) and \( \nabla_3 \psi_3 \) (plus lower order terms) and then use the relevant equations.

Consider \( \psi_H \in \{ \hat{\psi}, \psi \} \), which obeys \( \nabla_4 \) equations (2.19) and (2.21). (Note that \( \psi_H \in \{ \hat{\psi}, \psi \} \) does not obey \( \nabla_3 \) equations. We thus only have estimates for \( \mathcal{L} \nabla \psi_H \).)

Rewriting the \( \nabla_4 \) equations for \( \psi_H \in \{ \hat{\psi}, \psi \} \) in terms of \( \mathcal{L} \nabla \hat{\psi} \) and \( \mathcal{L} \nabla \psi \), it can be checked that all the terms on the RHS can be bounded in \( L^2 S \) using the estimates we have derived.

Consider for instance the following term on the RHS of (2.19), which can be bounded using (A.7) and estimates we have established:
\[
\| \Omega \omega \chi \| L^2 S^2 \| \partial u \partial \psi \chi \| \| \hat{\chi} \| \| \nabla \partial u \partial \psi \chi \| \lesssim 1.
\]
This term in particular limits the regularity in \( u \) and \( \psi \) that can be proven for \( \mathcal{L} \nabla \psi_H \). Other terms are either similar or easier; we omit the details.

The terms \( \mathcal{L} \nabla \psi_H \) for \( \psi_H \in \{ \hat{\psi}, \psi \} \) are similar except for having to handle terms related to \( b \).

**Step 4(c):** (3.15) and (3.16) for \( \nabla \chi \) and \( \nabla \hat{\chi} \). Finally, we argue similarly for \( \nabla \chi \) and \( \nabla \hat{\chi} \). Note that both \( \nabla \chi \) and \( \nabla \hat{\chi} \) obey both \( \nabla_4 \) and \( \nabla_3 \) equations; see (2.13), (2.14), (2.17) and (2.18). As above, we rewrite these equations in terms of \( \mathcal{L} \nabla \chi \) and \( \mathcal{L} \nabla \hat{\chi} \) and then estimate the RHS.

Let us just point out the terms that limit the regularity. For \( \mathcal{L} \nabla \chi \), we have the term
\[
\| \chi \partial u \partial \psi \chi \|^2 S^2 \| \chi \| \| \nabla \partial u \partial \psi \chi \|^2 \lesssim 1;
\]
and for \( \mathcal{L} \nabla \hat{\chi} \), we have the term
\[
\| \nabla \partial u \partial \psi \chi \|^2 S^2 \| \nabla \partial u \partial \psi \chi \|^2 \lesssim 1.
\]
These terms are responsible for the choice of function spaces for \( \mathcal{L} \nabla \chi \) and \( \mathcal{L} \nabla \hat{\chi} \) in (3.15). The estimates (3.16) are similar.

**Step 5: Proof of (3.9) II: Estimates for \( u, \psi \) derivatives of the metric components.** We first consider \( \gamma \). The first equation in (2.10) together with estimates we have established above gives the necessary bound for \( \mathcal{L} \nabla \gamma \). For \( \mathcal{L} \nabla \gamma \), note that \( \mathcal{L} \nabla \gamma - \mathcal{L} \frac{b}{b} \gamma \) can be expressed as angular covariant derivatives of \( b \). Thus, after using the estimates that we have established above, the desired estimate in (3.9) for \( \mathcal{L} \nabla \gamma \) follows from (second equation in) (2.10).

The estimates for \( \mathcal{L} \nabla \log \Omega \) and \( \mathcal{L} \nabla \log \Omega \) in (3.9) are similar (as those for \( \gamma \)) except that we use instead the first two equations of (2.11).

Finally, we turn to \( b \). The estimates for \( \mathcal{L} \nabla b \) in (3.9) follow directly from (2.10). For \( \mathcal{L} \nabla b \), we derive from (2.10) that
\[
\mathcal{L} \nabla (\mathcal{L} \nabla b) = -2 \mathcal{L} \nabla \{ \Omega^2 (\eta^2 - \eta^2) \}.
\]
Using (2.10), (2.11), (A.10) (and analogous equation for \( \mathcal{L} \nabla \eta \)), and the established estimates, it follows that
\[
\| \nabla \partial u \partial \psi \chi \|^2 S^2 \| \nabla \partial u \partial \psi \chi \|^2 \lesssim 1.
\]
Finally, integrating (A.11) using the bound (A.12) yields the estimate for $\mathcal{L}^b$ in (3.9).

\section*{Appendix B. Proof of Lemma 6.17}

In this appendix we prove Lemma 6.17, which is a compensated compactness lemma. We will first give a high-level proof, assuming a few results that we will later prove in Propositions B.1–B.4.

Proof of Lemma 6.17. It will be convenient to introduce the following notations. Let $(x^1, x^2, x^3, x^4) = (u, y, y^1, y^2)$. Denote also $dx = dx^1 dx^2 dx^3 dx^4$. Denote the corresponding Fourier variables by $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $d\xi = d\xi_1 d\xi_2 d\xi_3 d\xi_4$.

Step 1: A simple reduction. We first extend $f_n$ and $h_n$ so that they are compactly supported in a fixed ball $U \subset \mathbb{R}^4$ and satisfy the estimates

\[ \sup_{n} \sum_{i \leq 1, j \leq 1, k \leq 1} \int_{U} \left( \frac{\partial}{\partial x^i} \right)^i \left( \frac{\partial}{\partial x^j} \right)^j \left( \frac{\partial}{\partial x^k} \right)^k f_n \right)^2 \, dx \lesssim C_0 \tag{B.1} \]

and

\[ \sup_{n} \sum_{i \leq 1, j \leq 1, k \leq 1} \int_{U} \left( \frac{\partial}{\partial x^i} \right)^i \left( \frac{\partial}{\partial x^j} \right)^j \left( \frac{\partial}{\partial x^k} \right)^k h_n \right)^2 \, dx \lesssim C_0. \tag{B.2} \]

By (B.1), (B.2) and one-dimensional Sobolev embedding, we have

\[ \|f_n h_n\|_{L^2(U)} \lesssim \|f_n\|_{L^\infty_{x^1} L^2_{x^2} L^2_{x^3} L^4_{x^4}} \|h_n\|_{L^2_{x^1} L^\infty_{x^2} L^\infty_{x^3} L^\infty_{x^4}} \lesssim C_0. \]

Therefore, by a simple density argument, it suffices to show that after passing to a subsequence,

\[ \int_{U} (f_n h_n - f_{\infty} h_{\infty}) \psi \, dx \to 0 \]

as $k \to \infty$ for all $\psi \in L^\infty \cap L^2$.

For the rest of the proof fix a function $\psi \in L^\infty \cap L^2$ and fix $\epsilon > 0$.

Step 2: Frequency decomposition. Denote by either $` or $\mathcal{F}$ the Fourier transform on $\mathbb{R}^4$. First, notice that since $\psi$ is in $L^2(\mathbb{R}^4)$, there exists some $C_1 > 1$ such that $\psi$ satisfies

\[ \int_{|\xi| \geq C_1} |\hat{\psi}|^2 \, d\xi \leq \epsilon. \tag{B.3} \]

Let $\chi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a non-negative, smooth function which is compactly support in $[-2, 2]$ and is identically $1$ in $[-1, 1]$. For each $n$, we then define

\[ f_n = f_{n,H,1} + f_{n,H,2} + f_{n,L}, \quad h_n = h_{n,H,1} + h_{n,H,2} + h_{n,L}. \]

where

\[ \mathcal{F}(f_{n,L}) := \hat{f}_n(\xi) \chi(\frac{\xi}{2C_1}), \quad \mathcal{F}(f_{n,H,1}) := \hat{f}_n(\xi) \left( 1 - \chi(\frac{\xi}{2C_1}) \right) \frac{100C_1 |\xi_1|}{|\xi|^2}, \]

and

\[ \mathcal{F}(h_{n,L}) := \hat{h}_n(\xi) \chi(\frac{\xi}{2C_1}), \quad \mathcal{F}(h_{n,H,2}) := \hat{h}_n(\xi) \left( 1 - \chi(\frac{\xi}{2C_1}) \right) \frac{100C_1 |\xi_2|}{|\xi|^2}. \]

We note explicitly that the definitions for $f_{n,H,1}$ and $h_{n,H,1}$ are different. Moreover, in the support of $\mathcal{F}(f_{n,H,1})$, $50C_1 |\xi_2| \leq |\xi_1|$ and $|\xi| \geq 4C_1$; while in the support of $\mathcal{F}(h_{n,H,2})$, $50C_1 |\xi_1| \leq |\xi_2|$ and $|\xi| \geq 4C_1$.

Similarly, we define

\[ f_{\infty} = f_{\infty,H,1} + f_{\infty,H,2} + f_{\infty,L}, \quad h_{\infty} = h_{\infty,H,1} + h_{\infty,H,2} + h_{\infty,L} \]

where

\[ \mathcal{F}(f_{\infty,L}) := \hat{f}_{\infty}(\xi) \chi(\frac{\xi}{2C_1}), \quad \mathcal{F}(f_{\infty,H,1}) := \hat{f}_{\infty}(\xi) \left( 1 - \chi(\frac{\xi}{2C_1}) \right) \frac{100C_1 |\xi_1|}{|\xi|^2}, \]

and

\[ \mathcal{F}(h_{\infty,L}) := \hat{h}_{\infty}(\xi) \chi(\frac{\xi}{2C_1}), \quad \mathcal{F}(h_{\infty,H,2}) := \hat{h}_{\infty}(\xi) \left( 1 - \chi(\frac{\xi}{2C_1}) \right) \frac{100C_1 |\xi_2|}{|\xi|^2}. \]

Let us briefly explain this decomposition. For say $f_n$, we decompose into a piece $f_{n,H,1}$ where the $\xi_1$ frequency dominates and a piece $f_{n,H,2}$ where the $\xi_2$ frequency dominates. We add in an extra low

\footnote{We note that in the appendix, $\chi, \eta$ will be used as real-valued functions and are not to be confused with the notation for the Ricci coefficient in the main text!}

\footnote{We note that of course $f_{n,H}, f_{n,L}, g_{n,H}, g_{n,L}$ are no longer supported in $U$.}
frequency piece $f_{n,L}$ in the decomposition to ensure in particular that all the Fourier multipliers are smooth. We also make a similar decomposition for $g_n$.

**Step 3: Completion of the argument.** It is clear that we have the weak convergences $f_{n,L} \rightharpoonup f_{\infty,L}$, $f_{n,H_1} \rightharpoonup f_{\infty,H_1}$, $f_{n,H_1} \rightharpoonup f_{\infty,H_1}$, $h_{n,H_1} \rightharpoonup h_{\infty,H_1}$, $h_{n,H_1} \rightharpoonup h_{\infty,H_1}$ in $L^2$. More importantly, by Propositions B.1, B.2 and B.3 below, after passing to subsequences $f_{n,k}, f_{n,H_2}; h_{n,k}, h_{n,H_2}$, $h_{n_k,H_1}$ in fact converges in the $L^2$ norm. Therefore,

$$
\int_{\mathbb{R}^4} (f_{n,k}(h_{n_k,H_1} + h_{n,k,L}) + (f_{n,k,L} + f_{n,k,L})h_{n_k}) \psi \, dx
\to
\int_{\mathbb{R}^4} f_{\infty}(h_{\infty,H_1} + h_{\infty,L}) + (f_{\infty,L} + f_{\infty,L})h_{\infty}) \psi \, dx.
$$

In order to conclude the proof, it therefore suffices to show that

$$
\int_{\mathbb{R}^4} f_{n,H_1} f_{n,H_2} \psi \, dx \lesssim C_0, \quad \int_{\mathbb{R}^4} f_{\infty,H_1} f_{\infty,H_2} \psi \, dx \lesssim C_0. \quad (B.4)
$$

The point is that that the cutoffs have been chosen such that $\text{supp}(F(f_{n,H_1} f_{n,H_2})) \subset \{ |\xi| \geq C_1 \}$ and $\text{supp}(F(f_{\infty,H_1} f_{\infty,H_2})) \subset \{ |\xi| \geq C_1 \}$. Therefore, using (B.3),

$$
\int_{\mathbb{R}^4} f_{n,H_1} f_{n,H_2} \psi \, dx = \int_{\mathbb{R}^4} F(f_{n,H_1} f_{n,H_2}) (\hat{\psi}(\xi) d\xi = \int_{|\xi| \geq C_1} F(f_{n,H_1} f_{n,H_2}) (\hat{\psi}(\xi) d\xi
\leq \| F(f_{n,H_1} f_{n,H_2}) \|_{L^2(\mathbb{R}^4)} \
\leq \| f_{n,H_1} f_{n,H_2} \|_{L^2(\mathbb{R}^4)} \
\leq C \| f_{n,H_1} \|_{L^2(\mathbb{R}^4)} \| f_{n,H_2} \|_{L^2(\mathbb{R}^4)} \
\leq C \| n \|_{H^1_x L^2_x H^1_x H^1_x} \| n \|_{H^1_x L^2_x H^1_x H^1_x}.
$$

where in the last step, we have used one-dimensional Sobolev embedding theorem in three of the directions and in the last step, we have used the boundedness of the Fourier multipliers. By (B.1) and (B.2), we have

$$
\| f_{n} \|_{L^2_x H^1_x H^1_x L^2_x} + \| h_{n} \|_{H^1_x L^2_x H^1_x H^1_x} \lesssim C_0 1
$$

Therefore,

$$
\int_{\mathbb{R}^4} f_{n,H_1} f_{n,H_2} \psi \, dx \lesssim C_0.
$$

Similarly,

$$
\int_{\mathbb{R}^4} f_{\infty,H_1} f_{\infty,H_2} \psi \, dx \lesssim C_0.
$$

so that we have proven (B.4). This concludes the proof.

It now remains to prove the compactness results and the support properties that we have used in the proof of Lemma 6.17. First, the low frequency part of $f_n$ converges in $L^2$ norm:

**Proposition B.1.** Let $f_{n,L}, f_{\infty,L}$ be as in the proof of Lemma 6.17. Then, after passing to a subsequence,

$$
\| f_{n,L} - f_{\infty,L} \|_{L^2(\mathbb{R}^4)} \to 0
$$

as $k \to +\infty$.

**Proof.** By a standard extension of Rellich’s theorem to weighted $H^1$ spaces in non-compact domain, it suffices to show that

$$
\sup_{n} \sum_{|\alpha| \leq 1, |\beta| \leq 1} \| x^\alpha \partial_\xi^\beta f_{n,L} \|_{L^2(\mathbb{R}^4)} < \infty.
$$

(Here, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ are multi-indices.) This obviously holds since $f_n$ have uniform $L^2$ norm and $\mathcal{A}(\frac{q}{2C_1})$ is smooth and compactly supported.

We also have norm convergence for $f_{n,H_2}$. The key point is to notice that the cutoff in frequency guarantees that the $|\xi_1|$ frequency is controlled by the $|\xi_2|$ frequency. This then allows us to use the assumption (B.1).

**Proposition B.2.** Let $f_{n,H_2}, f_{\infty,H_2}$ be as in the proof of Lemma 6.17. Then, after passing to a subsequence,

$$
\| f_{n,H_2} - f_{\infty,H_2} \|_{L^2(\mathbb{R}^4)} \to 0
$$

as $k \to +\infty$.
Proof. As in the proof of Proposition B.1, it suffices to show that
\[
\sup_n \sum_{|\alpha| \leq 1, |\beta| \leq 1} \|x^{\alpha} \partial_x^\beta f_n,h,2\|_{L^2(\mathbb{R}^4)}
\]
is bounded.
By definition,
\[
\mathcal{F}(f_n,h,2) := \hat{f}(\xi) \left(1 - \chi\left(\frac{|\xi|}{2C_1}\right)\right) \left(1 - \chi\left(\frac{100C_1|\xi_2|}{|\xi_1|}\right)\right).
\]
To simplify notation, we denote
\[
\tilde{\chi}(\xi) := \left(1 - \chi\left(\frac{|\xi|}{2C_1}\right)\right) \left(1 - \chi\left(\frac{100C_1|\xi_2|}{|\xi_1|}\right)\right).
\]
By Plancherel's theorem,
\[
\|x^{\alpha} \partial_x^\beta f_n,h,2(x)\|_{L^2_x} \lesssim \|\partial_\xi^\beta (\xi^\alpha \tilde{\chi}(\xi) \hat{f}(\xi))\|_{L^2_x}.
\]
Using the product rule and the fact that 100C_1|\xi_2| \geq |\xi_1| in the support of \(\tilde{\chi}\), we have
\[
\|\partial_\xi^\beta (\xi^\alpha \tilde{\chi}(\xi) \hat{f}(\xi))\|_{L^2_x} \lesssim c_1, \|(1 + |\xi_2| + |\xi_3| + |\xi_4|)\hat{f}(\xi)\|_{L^2_x} + \|(1 + |\xi_2| + |\xi_3| + |\xi_4|)(\partial_\xi^\beta \hat{f}(\xi))\|_{L^2_x}.
\]
Therefore, combining the above estimates and using Plancherel's theorem again, we obtain
\[
\|x^{\alpha} \partial_x^\beta f_n,h,2(x)\|_{L^2_x} \lesssim c_1, \|f_n\|_{L^2_x} + \|\partial_x^\beta f_n\|_{L^2_x} + \|\partial_x f_n\|_{L^2_x} + \|\partial_x^\beta f_n\|_{L^2_x} + \|\partial_x^2 (\partial_x f_n)\|_{L^2_x} + \|\partial_x^2 (f_n)\|_{L^2_x}
\]
where in the last line we use \(\text{supp}(f_n) \subset U\) and (B.1).

Similar convergence statements can also be proved for \(h_{n,L}\) and \(h_{n,H,1}\). The proof is similar to that of Propositions B.1 and B.2 and is omitted.

**Proposition B.3.** Let \(h_{n,L}, h_{\infty,L}, h_{n,H,1}, h_{\infty,H,1}\) be as in the proof of Lemma 6.17. Then, after passing to a subsequence,
\[
\|h_{n,L} - h_{\infty,L}\|_{L^2(\mathbb{R}^4)} \to 0, \quad \|h_{n,H,1} - h_{\infty,H,1}\|_{L^2(\mathbb{R}^4)} \to 0
\]
as \(k \to +\infty\).

Finally, we conclude with the frequency support properties of the products \(f_{n,H,1}h_{n,H,2}\) and \(f_{\infty,H,1}h_{\infty,H,2}\):

**Proposition B.4.**
\[
\text{supp}(\mathcal{F}(f_{n,H,1}h_{n,H,2})) \subset \{|\xi| \geq C_1\} \quad \text{for all } n
\]
and
\[
\text{supp}(\mathcal{F}(f_{\infty,H,1}h_{\infty,H,2})) \subset \{|\xi| \geq C_1\}.
\]

**Proof.** We claim that if \(\xi^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}, \xi_4^{(1)})\) and \(\xi^{(2)} = (\xi_1^{(2)}, \xi_2^{(2)}, \xi_3^{(2)}, \xi_4^{(2)})\) satisfy
\[
|\xi_1^{(1)}| \geq 50C_1|\xi_2^{(1)}| \tag{B.5}
\]
and
\[
|\xi_2^{(2)}| \geq 50C_1|\xi_1^{(2)}|, \quad |\xi_2^{(2)}| \geq 4C_1, \tag{B.6}
\]
then \(|\xi^{(1)} - \xi^{(2)}| \geq C_1\). This then implies the conclusion of the proposition.

To prove the claim, assume for the sake of contradiction that it is not true, i.e. assume \(|\xi^{(1)} - \xi^{(2)}| < C_1\).

This immediately implies that
\[
|\xi_1^{(2)}| \geq |\xi_1^{(1)}| - C_1, \quad |\xi_2^{(2)}| \leq |\xi_2^{(1)}| + C_1.
\]
Combining this with (B.5) and (B.6), we get
\[
|\xi_2^{(2)}| \leq \frac{1}{50C_1} |\xi_1^{(1)}| + C_1 \leq \frac{1}{2500C_1^2} |\xi_2^{(2)}| + 2C_1.
\]
Rearranging, this gives
\[
|\xi_2^{(2)}| \leq \frac{2C_1}{1 - \frac{1}{2500C_1^2}}. \tag{B.7}
\]
Combining (B.7) and (B.8), and using the fact that $C_1 > 1$, we obtain

$$\left| \xi^{(2)}_1 \right| \leq \frac{1}{25(1 - \frac{1}{25000})},$$

(B.8)

which contradicts (B.6).

□

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