Introduction. The properties of the simplest magnetic monopoles in Yang-Mills-Higgs theory in flat spacetime are by now quite well understood. Moreover the fact that almost any grand unified theory will admit monopole solutions has had a profound impact on modern cosmology since it provides a very strong argument in favour of inflation as a possible solution of the monopole problem. It is therefore rather surprising that comparatively little is known about the gravitational properties of magnetic monopoles and the relation between monopoles and black holes. In the past couple of years or so this situation has begun to change and it is my purpose in these lectures to review the current situation and make some comments on it. I have included a fairly complete list of references, not all of which are referred to explicitly in the text, for the convenience of those wishing to follow up these topics.

Contents

1. Monopoles in Flat Spacetime
2. Static Solutions of Einstein’s Equations without Horizons
3. Bogomolnyi Bounds for Einstein-Yang-Mills-Higgs initial data
4. Static solutions of the Einstein-Yang-Mills Equations with Horizons
5. Solutions of the Einstein-Yang-Mills-Higgs Equations with Horizons
6. Global Monopoles and Black Holes
7. Black Hole Monopole Pair Production
1. **Monopoles in Flat Spacetime.** The virial theorem tells us that any finite energy time-independent solution of the equations of motion must satisfy:

\[ \int_{\mathbb{R}^3} T_{ik} d^3x = 0 \]  

where \( T_{ik} \) are the spatial components of the energy-momentum tensor. Equation (1.1) follows from the conservation equation \( T_{ij,j} = 0 \) and the obvious consequence:

\[ (T_{ij}x_k), j = T_{ik} \]

using the divergence theory and discarding the boundary term. The physical meaning of (1.1) is that the total stresses in an extended object must balance. In particular the components of \( T_{ik} \) cannot have a fixed sign - for example the spatial trace \( \sum_i T_{ii} \) equals the sum of the "principle pressures" so there must be regions where the matter is in tension and regions where it is in compression. For a pure Higgs field (assumed throughout these lectures to lie in the adjoint representation of \( SU(2) \))

\[ T_{ij} = +D_i \Phi . D_j \Phi - \frac{1}{2} \delta_{ij} (D_k \Phi . D_k \Phi) - \frac{1}{2} \delta_{ij} W(\Phi) \]  

and (1.1) cannot possibly be satisfied as long as \( W(\Phi) \geq 0 \). The potential term gives an isotropic negative pressure. The gradient term gives a positive pressure along the gradient direction and equal tensions in the orthogonal directions. The sum of principal pressures is thus negative. In fact this result holds for an arbitrary harmonic map with non-negative potential.

By contrast for a Yang-Mills field

\[ T_{ij} = -B_i . B_j + \frac{1}{2} B_k . B_k \delta_{ij} \]  

where

\[ B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \]

is the magnetic field strength. As Faraday taught Maxwell there is now a **tension** along the direction of \( B_i \) and an equal pressure orthogonal to the field lines. The sum of the principal pressures is now positive. Thus for pure Yang-Mills there can be no static solution either.

However as ’t Hooft and Polyakov showed there is a solution of the combined Yang-Mills Higgs system which does satisfy (1.1) and which is stable. Infact it minimizes the “total energy”:

\[ \int T_{00} d^3x = \int d^3x \left[ \frac{1}{2} B^2 + \frac{1}{2} (D\Phi)^2 + W(\Phi) \right] \]

where \( D \) is of course now a gauge covariant derivative. Moreover if \( \Phi \) transforms by the adjoint representation of the gauge group \( SU(2) \) the total energy is bounded below by

\[ \frac{4\pi \eta}{e} = g\eta \]
where $e$ is the gauge coupling constant, $g = 4\pi/e$ the magnetic charge of the monopole and $|\Phi| \to \eta$ at infinity. This Bogomolnyi bound can only be attained if $W(\Phi)$ vanishes identically (the Prasad Sommerfeld limit) and the Bogomolnyi equations:

$$\pm B_i = D_i \Phi$$  \hspace{1cm} (1.7)$$

hold, moreover from (1.7), it follows that the stresses vanish pointwise i.e. $T_{ij} = 0$.

The Prasad Sommerfeld limit and the associated Bogomolnyi equations are of great mathematical interest and have many geometrical consequences. They are also related to supersymmetry: the system with $W(\Phi) = 0$ admits an $N = 4$ supersymmetric extension. It has, as a consequence, received a great deal of attention. Of greater physical relevance is the case when

$$W(\Phi) = \lambda(\Phi^2 - \eta^2)^2$$  \hspace{1cm} (1.8)$$

and the total energy is given by

$$\frac{4\pi \eta}{e} f(\lambda/e^2)$$  \hspace{1cm} (1.9)$$

where $f(\lambda/e^2)$ is a dimensionless function of the dimensionless ratio $\lambda/e^2$ with $f(0) = 1$.

In addition to the finite energy solutions there is a static solution of the pure Higgs field equations (with $\Phi$ again a triplet of $SU(2)$) $\Phi$ satisfies a Hedgehog Ansatz:

$$\Phi^a = (x^a/r) F(r)$$  \hspace{1cm} (1.10)$$

with $F(0) = 0$ and $F(\infty) = \eta$. Since

$$T_{00} \to \frac{\eta^2}{r^2}$$  \hspace{1cm} (1.11)$$

this solution has infinite energy. It is called a global monopole.

It is a non-singular solution of the Higgs equations of motion. It is however well approximated by a singular solution of the non-linear $\sigma$-model obtained by enforcing the constraint that

$$|\Phi|^2 = \eta^2$$  \hspace{1cm} (1.12)$$

everywhere. Such singular solutions arise in the theory of nematic liquid crystals. They are known to be unstable in that a lower energy configuration is available with the energy concentrated along lines (“strings”) with energy per unit length equal to $4\pi \eta^2$. Global monopoles have recently been considered in connection with “cosmic textures”. I will discuss them further in a later section. Let us first see to what extent these basic facts are modified when we consider self-gravitating monopoles.
2. Static Solutions of Einstein’s Equations Without Horizons

Globally static metrics (i.e. time independent, time reversal invariant and without event horizons) may be cast in the form:

\[ ds^2 = -V^2(x)dt^2 + g_{ij}(x)dx^i dx^j \]  

(2.1)

The field equations are then:

\[ \nabla^2 g V = 4\pi GV(T_{00} + \sum T_{jj}) \]  

(2.2)

\[ R_{ij}[g] = V^{-1} \nabla_i \nabla_j V + 4\pi G g_{ij}(T_{00} - \sum T_{jj}) + 8\pi G T_{ij} \]  

(2.3)

where \( \nabla^2_g \) is the Laplacian of \( g_{ij} \) and \( \nabla_i \) its covariant derivative. \( T_{00} \) and \( T_{jk} \) are the components of the energy momentum tensor in an orthonormal frame with \( e_0 = V^{-1} \frac{\partial}{\partial t} \).

If the metric is asymptotically flat then

\[ V \sim 1 - 2Gm/r + O\left(\frac{1}{r^2}\right) \]  

and \( g_{ij} \sim (1 + 2Gm/r)\delta_{ij} + 0\left(\frac{1}{r^2}\right) \)  

(2.4)

where \( m \) is the A.D.M. mass of the spacetime. From (2.2) we have

\[ m = \int_{\Sigma} V(T_{00} + \sum T_{ii}) \sqrt{g} d^3x \]  

(2.5)

where \( \Sigma \) is a surface of constant time (assumed complete).

For some purposes it is convenient to rescale the 3-metric \( g_{ij} \) and re-write (2.1) as

\[ ds^2 = -e^{2U}dt^2 + e^{-2U} \gamma_{ij}dx^i dx^j \]  

(2.6)

The quantity \( U \) may be called the Newtonian potential. The field equations now become:

\[ \nabla^2_\gamma U = 4\pi Ge^{-2U} (T_{00} + \sum T_{ii}) \]  

(2.7)

and

\[ \tilde{R}_{ij}[\gamma] = 2\partial_i U \partial_j U + +8\pi G(T_{ij} - \gamma_{ij} e^{-2U} \sum T_{ii}) \]  

(2.8)

where \( \tilde{R}_{ij} \) is the Ricci tensor of \( \gamma_{ij} \). Now note:

(1) from (2.4) it follows that \( \gamma_{ij} \) is a complete asymptotically flat 3-metric with zero ADM mass

(2) from (2.8) the Ricci scalar \( \tilde{R} \) of \( \gamma_{ij} \) is given by

\[ \tilde{R} = 2\gamma^{ij} \partial_i U \partial_j U - 16\pi Ge^{-2U} (\sum T_{ii}) \]  

(2.9)

Using the positive mass theorem we now deduce the following

**Theorem 1** There are no globally static asymptotically flat solutions of Einstein’s with \( \sum_i T_{ii} \leq 0 \). In other words since gravity is attractive we need some pressure to resist collapse inwards. Note that to prove theorem 1 we do not need to assume that, the matter has positive energy.
If $T_{\alpha\beta} = 0$ theorem 1 is just Lichnerowicz’s theorem. If the matter is a scalar field however we obtain a new result:

**Cor 1.** There are no globally static asymptotically solutions of Einstein’s equations with a minimally coupled scalar field source with a non-negative potential.

It is interesting to note that the solutions recently derived by Vilenkin and Bariola giving the gravitational fields of global monopoles escape cor. 1 by virtue of not being asymptotically flat, as I shall describe later. It is also important to point out that there do exist solutions in which a complex scalar field varies harmonically with time in such a way that $T_{\mu\nu}$ and the metric $g_{\mu\nu}$ are static. Such scalar fields are said to consist of $Q$-matter.

On the other hand for pure Einstein-Yang-Mills we cannot deduce from Theorem 1 that there are no static solutions without horizons, since $\sum T_{ii} \geq 0$. In fact for pure Einstein-Maxwell theory there are in fact no static solutions without horizons (for a proof see Breitenlohner, Gibbons and Maison (1988)). It came as a surprise therefore when Bartnik and McKinnon (1988) announced the existence of an integer’s worth of static spherically symmetric solutions. Their metric ansatz was

$$ds^2 = -e^{2U(r)}dt^2 + \frac{dr^2}{1 - \frac{2Gm(r)}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (2.10)$$

with

$$eA = a(r)\tau_3 dt + w(r)(\tau_1 d\theta + \tau_2 \sin \theta d\phi) + \tau_3 \cos \theta d\phi \quad (2.11)$$

where $\tau_1, \tau_2, \tau_3$ is the usual basis for the Lie algebra of $SU(2)$. Equation (2.11) gives spherically symmetric $SU(2)$ connection over $S^2$ if we set $r =$ constant, $t =$ constant. Purely magnetic solutions have $a(r) = 0$ and $eF = w' dr \wedge d\theta \tau_1 + w' \sin \theta dr \wedge d\phi \tau_2 - (1-w^2) \sin \theta d\theta \wedge d\phi \tau_3$.

According to Bartnik and McKinnon the assumption of suitable asymptotics and of finite energy implies that the electric potential $a(r)$ must vanish. There results a system of radial ordinary differential equations for $U(r), w(r),$ and $m(r)$. If $'$ denotes differentiation with respect to $r$ we have:

$$\frac{m'}{4\pi G} = (1 - \frac{2Gm}{r})(w')^2 + \frac{1}{2}\frac{1}{r^2} (1-w^2)$$

$$r^2(1-\frac{2Gm}{r})w'' + (2Gm - \frac{(1-w^2)^2}{r^2} )w' + (1-w^2)w = 0$$

and

$$U = \frac{1}{2} \ln(1 - \frac{2Gm}{r}) - 4\pi G \int_r^\infty \frac{1}{r} (w')^2 dr$$

These equations may be combined into a single 3rd order differential equation as shown by Ray (1978) who may be said to have anticipated some aspects of the results of Bartnik and Mckinnon. In any event Bartnik and Mckinnon presented numerical evidence that there exist solutions with

$$m(r) = O(r^3); \quad w(r) = 1 + O(r^2) \quad \text{at} \quad r = 0$$
and
\[ m(r) \sim m(\infty) - c^2/r^3; \quad w(r) \sim \pm(1 - c/r) \text{ at } r = \infty \]
for some constant \( c \).

The solutions are indexed by the number \( k \) of zeros of \( w(r) \); \( k = 1, 1, 2 \ldots \). The solutions have 3 regions. Region I is an inner core. Region III is the asymptotic region where \( F = 0(1/r^3) \) and the metric becomes Schwarzschildian. The middle region has \( w \approx 0 \) and the solution behaves rather like an abelian \( U(1) \) Dirac monopole and the geometry resembles the throat region of an extreme Reissner-Nordstrom solution.

The results of Bartnik and McKinnon have been confirmed by Kunzle and Masood-ul-Alam (1990) and by Maison (private communication).

The question immediately arises: are these solutions stable? Bartnik and McKinnon themselves felt that the cases \( k \geq 3 \) were unstable. A stability analysis was carried out by Straumann and Zhou (1990) and also (private communications) by Maison. These analyses show that these solutions are unstable for all values of \( k \). Presumably a small perturbation would cause them either to collapse to form a black hole or (rather less likely) to explode and dissipate. Since they have no magnetic moment the expected hole will be Schwarzschildian. A noteworthy feature of the analysis of Straumann and Zhou was that the configurations they considered were spherically symmetric and yet time-dependent. In other words Birkhoff’s theorem is not valid for Einstein-Yang-Mills unlike Einstein-Maxwell.

Thus Einstein-Yang-Mills admits unstable finite energy static non-singular solutions. We have seen above that the Einstein-Higgs equations do not. What about Einstein-Yang-Mills-Higgs? It is physically clear that for small values of \( Gm/R \) where \( R \) is a typical radius and \( m \) a typical total energy the effects of gravity on a ’t Hooft-Polyakov monopole will be negligible. Since typically
\[ m \sim \frac{4\pi\eta}{e}, \quad R \sim \frac{1}{e\eta} \]
gravity will be negligible so long as:
\[ 4\pi G\eta^2 \ll 1. \]

If, on the other hand we consider a one parameter family of static solutions labelled by the dimensionless number \( 4\pi G\eta^2 \) (keeping \( \lambda/e^2 \) fixed) we might expect to encounter a critical value beyond which no static solutions are possible because they will undergo gravitational collapse. It is also likely that the static family will already have become unstable at some smaller value of \( 4\pi G\eta^2 \). The intuition one is drawing on here is of course an analogy with the theory of white dwarf stars, the critical value of \( 4\pi G\eta^2 \) corresponding to the famous Chandrasekhar limit.

As far as I know a detailed analysis of this situation has not been carried out until recently. Miguel Ortiz in his Ph.D thesis has begun a numerical study and his results confirm that for fixed \( \lambda/e^2 \) there is a maximum value of \( 4\pi G\eta^2 \) beyond which regular solutions without event horizons cease to exist. This maximum value is about 2.5 in the Prasad-Sommerfeld limit and decreases as the quartic coupling constant \( \lambda \) increases.
The metric at large distances appears to approach the Reissner-Nordstrom form with an approximately minimal mass for a magnetic charge $g$, that is the monopole appears to collapse as soon as the Cosmic Censorship allows. The exterior gauge field and Higgs field appear to approach a Wu-Yang like configuration with the Higgs field being covariantly constant. This will be described in sections 4 and five in more detail.

The basic equations for self-gravitating ’t Hooft Polyakov monopoles in the spherically symmetric case were in fact written down some time ago by Perry, Van Nieuwenhuizen and Wilkinson (1976). A variational principle was established but the equations were not analysed in detail. A qualitative physical discussion along the lines indicated in the previous paragraph has been given by Frieman and Hill (1987). Some more exact information can possibly be obtained by considering the generalizations of the Bogomolnyi bound in the gravitational setting so we now turn to that topic.
3. Bogomolnyi Bounds for Einstein-Yang-Mills-Higgs Initial Data

We shall consider time-symmetries initial data for simplicity, that is the second fundamental form $K_{ij}$ of the initial surface is assumed to vanish. In addition we assume that the non abelian electric field vanishes, as well as the time component of the Higgs field’s covariant derivative. Thus the Ricci scalar $R$ of the 3-metric $g_{ij}$ satisfies

$$R = 16\pi GT_{\hat{0}\hat{0}}$$

where

$$T_{\hat{0}\hat{0}} = \frac{1}{2} B^2 + \frac{1}{2} (D\Phi)^2 + W(\Phi).$$

Let us define the “total amount of matter” on the initial surface $\Sigma$ (assumed to be complete) by:

$$M = \int_{\Sigma} T_{\hat{0}\hat{0}} \sqrt{\det g} \, d^3x$$

Note that $M$ does not, in general, equal the ADM mass $m$ of the 3-metric $g_{ij}$. Even if it were the case that the data were such as to evolve to a static solution a comparison of (3.3) and (2.5) shows that $M$ and $m$ cannot be expected to coincide. Another measure of the total energy of the matter in a static spacetime would be the “Killing Energy” $E$ defined by

$$E = \int_{\Sigma} \sqrt{-g_{00}} \, T_{\hat{0}\hat{0}} \sqrt{\det g} \, d^3x$$

In general we have (when they are defined)

$$M \neq E \neq M \neq m$$

Now Bogomolnyi’s original argument may trivially be “covariantised” with respect to spatial diffeomorphisms using the covariantly constant alternating tensor $\epsilon_{ijk}$ of the 3-metric $g_{ij}$ (I prefer not to use tensor densities and I am of course assuming that the initial surface is oriented). Thus we have

**Theorem 2.** The total amount of matter $M$ of a time-symmetric initial dates set for the $SU(2)$ Einstein-Yang-Mills-Higgs equations with non negative potential $W(\Phi)$ is bounded below by

$$M \geq g\eta$$

where $g$ is the asymptotic magnetic monopole moment. For a single monopole $g = 4\pi/e$. Moreover equality in (3.5) implies that the covariant Bogomolnyi equations hold.

$$D_i \Phi = \pm \frac{1}{2} \epsilon_{i}^{jk} F_{jk}$$

hold.

The existence of solution of (3.6) on a curved metric has been studied by Floer (1987). Although of some mathematical interest the following result shows that these solutions are never of relevance if the monopole self-gravitates.
Theorem 3. Static solutions of the Einstein-Yang-Mills-Higgs equations satisfying the Bogomolnyi equations (3.6) or equivalently saturating the Bogomolnyi bound (3.5) do not exist.

Proof The Bogomolnyi equation (3.6) imply that the spatial components of the stress tensor $T_{ij} = 0$. We can thus invoke our previous theorem 1.

It is known that to form an abelian black hole of ADM mass $m$ and magnetic charge $g$ we must have

$$m \geq \frac{g}{\sqrt{4\pi G}}$$

(recall that we are using rationalized units for electromagnetic or Yang-Mills fields). Equality in corresponds to the Papapetrou-Majumdar metrics describing the equipoise of an arbitrary number of extreme Reissner-Nordström magnetic black holes. As mentioned in section 2 equation (3.7) shows that for fixed magnetic charge $g$ a 't Hooft Polyakov monopole cannot collapse until its ADM mass $m$ satisfies $\sqrt{4\pi G}m \geq g$. Since $m \sim M \sim \eta g$ we need $4\pi G\eta^2 \geq 1$ which agrees approximately with what has been found by Ortiz. It thus seems very reasonable to expect that the configuration to which it gives rise is similar, if not identical to, an extreme Reissner-Nordström solution. We shall consider static solutions with horizons with horizons. That we in the next section. Before doing so we remark that some information about initial data for Einstein-Yang-Mills-Higgs has been obtained by Malec and and Koc (1990) and Chmaj and Malec (1989).

If one is merely interested in the Yang-Mills equations in a gravitational background it is possible to find a modified set of Bogomolnyi equations:

$$D_i \Phi = B_i - \Phi \nabla_i U$$

(3.8)

where $U$ is the Newtonian potential as defined by (2.6). If the background metric satisfies

$$\nabla^2 gU = 0,$$

(3.9)

then (3.8) implies the second order Yang-Mills equations in the background (see Comtet (1980) and Comtet Forgacs and Horvathy (1984). In general (3.9) will be incompatible with the Einstein-Yang-Mills equations. An interesting case for which (3.9) is compatible with the Einstein-Yang-Mills equations is when $\gamma_{ij}$ is flat. This gives the Papapetrou-Majumdar metrics for which the left and right hand sides of (3.8) are separately zero and $B$ and $\Phi$ point in a constant direction in internal space. For more detail about these equations see Horvathy ((1987)
4. Static solutions of the Einstein-Yang-Mills equations with Horizons

It has been known for many years that one has the abelian black hole solutions with

\[ A = \tau_3 \left( \frac{q}{r} dt + g \cos \theta d\phi \right) \]  (4.1)

\[ ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  (4.2)

\[ \Delta = 1 - \frac{2Gm}{r} + G\frac{(q^2 + g^2)}{4\pi r^2} \]  (4.3)

where \( m \) is an arbitrary constant satisfying

\[ m \geq \frac{1}{\sqrt{4\pi G}} (q^2 + g^2)^{\frac{1}{2}} \]  (4.4)

If \( \tau_3 \) has a normalization such that

\[ \exp(4\pi i \tau_3) = 1 \]  (4.5)

we must demand that

\[ \frac{eg}{2\pi} = 0, 1, 3, 5, \ldots \]  (4.6)

if \( A \) is an \( SO(3) \) connection and

\[ \frac{eg}{4\pi} = 0, 1, 2, 3 \]  (4.7)

if \( A \) is an \( SU(2) \) connection. Note that the \( SO(3) \) case is only possible because the presence of the horizon means that the singularity which would otherwise result at \( r = 0 \) is hidden inside the horizon. It cannot occur if there are no horizons.

Note that (4.1) will always yield a spherically symmetric energy-momentum tensor although it is not spherically symmetric as an \( SU(2) \) connection unless

\[ \frac{eg}{4\pi} = 1 \]  (4.8)

This latter case corresponds to \( w = 1 \) in (2.11). Some authors prefer to use a different gauge. Let

\[ x = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]

then the connection

\[ e\tilde{A}^1 = \frac{dx^2 \wedge dx^3}{r^2} (w - 1) \text{ etc} \]  (4.9)

is gauge equivalent to (2.11) with \( a(r) = 0 \). Thus the Reissner-Nordstrom metric with \( q = 0 \) and \( \tilde{A} \) given by (4.9) with \( w = 0 \), whence (4.8) represents the simplest purely magnetic \( SU(2) \) solution. This solution will extend trivially to a solution of the Einstein-Yang-Mills-Higgs equations if one appends the covariantly constant Higgs field

\[ \Phi^i = \eta \frac{x^i}{r} \]  (4.10)
The resulting solution is said to satisfy the Wu-Yang ansatz. Of course in the Abelian gauge it reduces to

\[ eA = 4\pi \tau_3 \cos \theta d\phi \]  
\[ \Phi = \eta \tau_3 \]  
\[ ds^2 = -(1 - \frac{2Gm}{r} + \frac{4\pi G}{e^2r^2}) dt^2 + (1 - \frac{2Gm}{r} + \frac{4\pi G}{e^2r^2})^{-1} dr^2 \]  
\[ + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

Until the work of Bartnik and McKinnon it had long been felt that these abelian solutions would be the only solutions - rather by analogy with the No-Hair Theorems for the Einstein-Maxwell equations. However recent work has clearly indicated this conjecture to be false. Kunzle and Masood-ul-Alam (1990), Bizon (1990), and Volkov and Gal’tsov (1989) have shown that there exist analogues of the Bartnik and McKinnon solutions with horizons. As one might expect these are unstable, as shown by Straumann and Zhou (1990). The no hair conjecture in its naive form thus fails. In the spherically symmetric case however Gal’tsov and Ershov (1990) have argued that as long as there is a net Yang-Mills charge measurable at infinity, i.e. that \( w_2 \neq 1 \) at infinity then the abelian solutions are unique.

What about the stability of the abelian solutions? This was studied some time ago by Lohiya (1982). Following the analogous analysis of singular monopoles in flat spacetime. The stability is determined by the large distance behaviour of the fields in a manner described by Coleman (1983) and Brandt and Neri (1979). The analysis shows that the purely magnetic solutions are unstable stable if the connection is topologically trivial restricted to a 2-sphere at infinity. This means that all \( SU(2) \) connections i.e.

\[ \frac{eg}{4\pi} \in \mathbb{Z} \]

are unstable. Of the remaining \( SO(3) \) connections only the lowest one with

\[ \frac{eg}{2\pi} = 1 \]

is stable.

A sufficient condition for instability of the electric solutions is that the electric charge \( q \) exceeds \( 3e/2 \). Again this is consistent with the flat space results.

The conclusion would seem to be that in the absence of Higgs fields regardless of uniqueness the non abelian-Einstein-Yang-Mills solutions are not of very much physical interest. It is therefore appropriate to turn to the case when Higgs fields are included.
5. Solutions of the Einstein-Yang-Mills Higgs Equations with Horizons

The obvious first remark is to recall that solutions always exist if the Higgs field is covariantly constant. Choosing a gauge in which the direction of $\Phi$ is everywhere the same in internal space we see that Yang-Mills potentials associated with rotations about that direction satisfy the abelian equations and thus must belong to the Reissner-Nordstrom family described earlier. Of course the stability results of Lohiya do not necessarily apply now because the Higgs mechanism might well stabilize these Dirac-type monopoles (for sufficiently large Higgs mass) against instabilities in the non-abelian directions. To my knowledge this has not been looked at by anybody in detail.

In the case that the electric charge vanishes the results of Ortiz suggest that gravitational collapse of a ’t Hooft-Polyakov monopole will result in an exterior field in which the Higgs field is covariantly constant and given by (4.10), and the gauge field by (4.9) and $w = 0$. This solution was originally written down by Cho and Freund (1975) and Bais and Russell (1975). As mentioned earlier these correspond to the Reissner-Nordstrom solutions with $q = 0$ and $g = \frac{4\pi}{e}$. My conjecture is that these are indeed classically stable. Moreover quantum mechanically they should evolve by Hawking evaporation to the extreme, zero temperature state. Such objects should behave like stable solitons and have been studied extensively by Hajicek and his collaborators from that point of view. Thus if $4\pi G \eta^2$ is large enough the monopole problem of cosmology is in fact a primordial black-hole monopole problem. In fact it it seems rather likely that ’tHooft-Polyakov monopoles will be unstable for values of $4\pi G \eta^2$ which are somewhat smaller than the maximum allowed value.

An important question now arises: are there any other solutions? For example are there any electrically charged solutions in which the electric charges are associated with the broken $SU(2)$ generators for example? Experience and intuition based on both the physical ideas behind the Higgs mechanism (charge should be screened) and the non-hair properties of black holes would have suggested until very recently that the answer is no. At present however the answer is less clear because of two developments. The first is the Bartnik-McKinnon-Bizon-Kunzle-Masood-ul-Alam - Vokkov-Gal’tsov solutions. The second is the issue of fractional charges raised by Krauss and Wilczek (1990), see Preskill and Krauss (1990) and Preskill (1990). Even in the simpler Abelian-Higgs model the situation is not entirely clear. For that reason I will discuss what is known in that case.

The simplest question to ask is are there any static solutions of the Einstein-Higgs equations with horizons? By static I mean that not only is the metric static but that the complex Higgs field which I shall now call $\phi$ is strictly independent of time. If one doesn’t make that assumption one might expect to find shells of “Q-matter” surrounding a black hole. It is generally expected that as long as $W(\phi)$ is positive with an absolute minimum at $|\phi| = \eta$ then the only solution must have $\phi = \text{constant}$, with the constant real with no loss of generality. If $W(\phi)$ vanishes it is easy to establish this result. If $W(\phi)$ is convex it is also possible to establish this result using a “Bochner Identity”. Suppose, more generally, that a field $\phi^A(x)$ takes its values in some riemannian target manifold $N$ with metric $G_{AB}(\phi)$ and potential function $W(\phi)$. The Bochner identity tells us that:

$$
\left( \frac{1}{2} G_{AB} \frac{\partial \phi^A}{\partial x^\alpha} \frac{\partial \phi^B}{\partial x^\beta} g^{\alpha\beta} \right)_{;\mu} = \phi^{A;\alpha;\beta} \phi^{B;\alpha;\beta} G_{AB}
$$
+ \phi^{A;\alpha}(G_{AB}R_{\alpha\beta} - g_{\alpha\beta}R_{ACBD}\phi^C_{\mu\nu}\phi^D_{\mu\nu}g^{\mu\nu})\phi^{B;\beta}
+ \phi^{A;\alpha}(\phi^{B;\beta} ;\beta) ;\alpha G_{AB}
\tag{5.1}

where all covariant derivatives are covariant with respect to the spacetime metric \( g_{\alpha\beta} \) and the target-space metric \( G_{AB} \) in the manner described by Misner (1978). The field equations are:

\[ \phi^{A;\beta} ;\beta = G^{AB}\nabla_B W \tag{5.2} \]

and

\[ R_{\alpha\beta} = 8\pi G[G_{AB}\phi^{A}_{;\alpha}\phi^{B}_{;\beta} + g_{\alpha\beta}W(\phi)] \tag{5.2} \]

If one integrates (5.1) over the region exterior to the black hole and confined within 2 spacelike surfaces \( \Sigma_1 \) and \( \Sigma_2 \) such that \( \Sigma_1 \) is the time translation of \( \Sigma_2 \) and uses the field equations and the boundary conditions:

\[ \phi^{A}_{;\alpha} l^\alpha = 0 \]

on the horizon and \( \phi^A \rightarrow \text{constant at } \infty \) and where \( l^\alpha \) is the null generator of the horizon one obtains the following

**Theorem 4 (Scalar No Hair theorem)** There are no non-trivial static scalar fields on a static black hole solution of the Einstein-Higgs equations for which \( W(\phi) \geq 0 \), and the sectional curvature of the target manifold is non positive and \( W(\phi) ;A;B \) is non negative

Note that unlike Cor. 1 of Theorem 1 we need a stronger assumption on \( W(\phi) \) and \( G_{AB} \). If \( G_{AB} \) is flat and \( \phi^A \) takes its value in a vector space we could have used the simpler identity:

\[ (\phi^{A}G_{AB}\phi^{B}_{;\alpha})^{\alpha} = \phi^{A}_{;\alpha}G_{AB}\phi^{B}_{;\beta} g^{\alpha\beta} \tag{5.4} \]

and the field equation (5.2) one obtains

**Theorem 5**: There are no non-trivial static scalar fields on a static black hole solution of the Einstein-Higgs equations for which \( \phi^A W(\phi);B \geq 0 \)

Remark: The proof of theorems 4 and 5 also applies to the case where infinity is replaced by a cosmological event horizon.

Neither theorem 4 nor theorem 5 applies to even the simplest case of a single real scalar field \( \phi \) with potential

\[ W(\phi) = \lambda(\phi^2 - \eta^2)^2 \tag{5.5} \]

\( \lambda > 0 \). Thus for non-linear field equations of this type the no-hair conjecture remains - to use a standard term in Scots law - "Not Proven" although there is some suggestive work by Sawyer (1977) and Brumbaugh (1978).

Let us turn to the work of Adler and Pearson (1978). They assume spherical symmetry and the Einstein-Maxwell-Higgs equations, with a complex scalar \( \phi \). They assume that that there is only an electric field present and a gauge exists in which:

1. \( \phi \) is independent of time and real
2. \( A = A_0 dt \) with \( A_0 \) everywhere bounded

Actually their assumptions are unnecessarily restrictive and their arguments both incomplete and in part wrong. We shall assume to begin with that

\[ A_\mu = A_0 dt \tag{5.6} \]
with

(1) $\phi, A_0$ independent of time
(2) $A_0 \to 0$ at $\infty$
(3) the one form $A_0 dt = A_\mu dx^\mu$ has bounded “length’. That is $g^{\mu\nu} A_\mu A_\nu < \infty$ on the horizon.

The equation for $A_0$ is

$$\nabla_j (V^{-1} \nabla^j A_0) + e^2 |\phi|^2 A_0 = 0 \quad (5.7)$$

where $\nabla$ is taken with respect to the 3-metric $g_{ij}$. We have dropped the assumption that the metric is spherically symmetric and that $\phi$ is real.

If one multiplies (5.7) by $A_0$ and integrates over a surface of constant time one obtains:

$$\int_\Sigma \sqrt{g} (V^{-1} e^2 |\phi|^2 (A_0)^2 + V^{-1} (\nabla^i A_0 \nabla^j A_0) g_{ij})$$

$$= \int_{\partial \Sigma} (V^{-1} A_0 \nabla^j A_0) d\sigma_j \quad (5.8)$$

Now if $|\phi| \to \eta$ and $V \to 1$ at infinity solutions of (5.7) at infinity to like $\frac{1}{r} \exp \pm e^{\eta r}$. Thus if $A_0$ is to be bounded it must fall to zero exponentially and the boundary term at infinity (5.9) will vanish. On the other hand if the field strength $F_{i0} = \partial_i A_0$ is to have bounded scalar invariant on the horizon we require that $V^{-2} (\nabla_j A_0) (\nabla^j A_0)$ should be bounded near the horizon. Now if in addition $A_\mu A_\nu g^{\mu\nu}$ is to be bounded we require that $A_0$ vanishes at least as fast as $V$ at the horizon and so the boundary term at the horizon in (5.9) must vanish.

We have thus established the following:

**Lemma 1** There are no regular time independent electrostatic fields with time independent vector potentials and Higgs field which are bounded with bounded length $A_\mu g^{\mu\nu} A_\nu$ around a static black hole.

Unfortunately lemma 1 is not sufficient to establish that there can be no time independent electrostatic fields around a black hole because it is not clear that there should exist a global gauge in which the vector potentials and Higgs fields are both time independent and bounded. In the usual electromagnetic case without symmetry breaking the potential $A_\mu$ cannot in fact be cast in a such a gauge. Thus it is necessary to investigate the case when either the gauge variant fields $A_\mu$ and $\phi$ vary with time or do not fall off at infinity. To my knowledge this has not been done.

Even if one assumes that the electromagnetic field vanishes and that the Higgs field is time independent and bounded and even if one assumes further that it is real I know of no rigorous proof that it must be constant in the case that the potential $W$ has the (non-convex) form (5.5). The argument given by Adler and Pearson for example appears to be incorrect. Although the no-hair property seems very plausible physically it is clear that much remains to be done to establish it rigorously even in the abelian case with symmetry breaking let alone in the non-abelian case.
6. Global Monopoles and Black Holes

Barriola and Vilenkin (1989) have pointed out that the gravitational field of a global monopole has some interesting properties. Far from the core one has

\[ \Phi^i \simeq \eta X^i / r \] (6.1)
\[ T^o_\omega \simeq \eta^2 1 / r^2 \] (6.2)
\[ T^r_\omega \simeq \eta^2 1 / r^2 \] (6.3)
\[ T^\theta_\phi = T^\phi_\theta \simeq -\eta^2 / r^2 \] (6.4)

with asymptotic metric

\[ ds^2 \simeq -dt^2 + \frac{dr^2}{1 - 8\pi G\eta^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (6.5)

This metric is not asymptotically flat but rather asymptotically the product of a flat time direction (i.e. the Newtonian potential tends to zero) with a 3-dimensional cone over \(S^2\) with solid angular deficit \(32\pi^2 G\eta^2\). For a non singular (but infinite total energy) \(\Phi^i\) must vanish at \(r = 0\), and the metric acquires some corrections. Nevertheless in the "\(\sigma\)-model approximation" in which \(\Phi^i\) always remains in the global minimum of \(W(\Phi)\) one may replace the \(\simeq\) in (6.1) - (6.5) by = signs.

They are exact solutions of the Einstein equations with \(\sigma\)-model source. Moreover one may consider in addition a black hole. Then (6.1) - (6.4) continue to hold as equalities and (6.5) is replaced by

\[ ds^2 = -(1 - \frac{2Gm}{r})dt^2 + \frac{dr^2}{(1 - 8\pi G\eta^2)(1 - \frac{2Gm}{r})} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (6.6)

The metric (6.6) (which was worked out by myself and Fernando Ruiz-Ruiz) thus represents a global monopole inside a black hole. Unfortunately however, as pointed out by Goldhaber (1989), the global monopole is likely to be unstable against a sort of angular collapse in which all the \(\Phi\) field energy becomes concentrated along a line defect or string leaving the point defect at \(r = 0\). This string has an energy per unit length of \(4\pi\eta^2\). The analysis of Goldhaber is consistent with the work of a number of people on defects in liquid crystals which are modelled using the a free-energy functional [the "Frank Oseen free energy in the one constant approximation" which particle physicists would refer to as a \(\sigma\)-model action and mathematicians as an harmonic map energy functional. Point defects have strings emerging from them which tend to the zero thickness limit of the \(\sigma\)-model cosmic strings introduced by Comtet and myself (1989). Despite this instability there continue to appear preprints analyzing there properties and those of similar objects. An interesting feature is that under some circumstances there can be repulsive gravitational effects. In particular Harari and Lousto (1990) have drawn attention to a repulsive region near the core. A similar feature was found by Ortiz near the core of a \('t Hooft-Polyakov monopole. An interesting question to ask is whether for large enough \(4\pi\eta^2\) gravitational collapse is inevitable and what is the critical value? In effect this is a limiting case of the Ortiz problem when \(\lambda/e^2\) is large.
The gravitational field of an infinite straight $\sigma$-model strings was given by myself and Comtet (1989). What about that due to a string emerging from a black hole? Such a string would cause the black hole to accelerate and so the appropriate solution (in the thin string approximation) is the C-metric:

$$ds^2 = \frac{1}{A^2(x+y)^2} [\frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x) d\alpha^2 - F(y) dt^2]$$

where

$$G(x) = -F(-x) = 1 - x^2 - 2GmA x^3; \quad 0 < GmA < 1/\sqrt{27}$$

$$= 2GmA(x - x_2)(x - x_3)(x - x_4)$$

I have labelled the 3 real roots of $G(x)$ $x_2$, $x_3$, $x_4$ in ascending magnitude ($x_2$ and $x_3$ are both negative and $x_4$ is positive).

The range of the “radial” variable $y$ is

$$-x_3 \leq y \leq -x_2$$

with $y = |x_3|$ being an acceleration horizon and $y = |x_2|$ a black hole horizon. The range of the “angular” variable $x$ is $x_3 \leq x \leq x_4$. The 2-surfaces $x = x_3$ and $x = x_4$ are axes of symmetry for the angular Killing vector $\frac{\partial}{\partial \alpha}$.

In order to understand what the coordinates used it is helpful to consider the case when the mass parameter $m$ vanishes. Then the metric is flat and one may transform to flat inertial coordinates using the formulae:

$$X^1 \pm iX^2 = \frac{(1 - x^2)^{\frac{1}{2}}}{A(x+y)} \exp(\pm i\alpha)$$

$$X^3 \pm X^0 = \frac{(y^2 - 1)^{\frac{1}{2}}}{A(x+y)} \exp(\pm t)$$

Evidently the coordinate singularity at $x = \pm 1$ is a rotation axis while the coordinate singularity at $y = \pm 1$ corresponds to a pair of intersecting null hyperplanes forming the past and future event horizons for a family of uniformly accelerating worldlines. A similar interpretation may be given in the case that $m \neq 0$ but there is in addition a Black Hole horizon. A detailed description was given by Kinnersley and Walker (1970)

If $0 \leq \alpha \leq \Delta \alpha$ there will be angular deficits:

$$\frac{\delta_4}{2\pi} = \frac{\Delta \alpha - \Delta \alpha_4}{\Delta \alpha_4}; \quad \frac{\delta_3}{2\pi} = \frac{\Delta \alpha - \Delta \alpha_3}{\Delta \alpha_3}$$

where

$$\Delta \alpha_4 = \frac{4\pi}{|G'(x_4)|}; \quad \Delta \alpha_3 = \frac{4\pi}{|G'(x_3)|}$$

Since (unless $mA = 0$) $\Delta \alpha_4 \neq \Delta \alpha_3$ it is not possible to eliminate both of these by choosing $\Delta \alpha$. One can eliminate $\delta_3$ in which case the black hole is pulled along by a
string, or $\delta_4$ in which case it is pushed along by a rod. In general the net “force” on the hole is

$$F = \frac{\delta_4}{8\pi G} - \frac{\delta_3}{8\pi G} = \frac{\Delta \alpha}{4G} \left( \frac{1}{\Delta \alpha_4} - \frac{1}{\Delta \alpha_3} \right) = \frac{\Delta \alpha m A}{8\pi} (x_4 - x_3)^2$$

The black hole event horizon area $A$ is given by

$$A = \frac{\Delta \alpha}{A^2} \frac{x_4 - x_3}{(x_4 - x_2)(x_3 - x_2)}$$

The black hole horizon surface gravity $\kappa_{BH}$ and the acceleration horizon surface gravity $\kappa_R$ are given by:

$$\kappa_{BH} = Gm A^2 (x_3 - x_2)(x_4 - x_2)$$
$$\kappa_R = Gm A^2 (x_3 - x_2)(x_4 - x_3)$$

If $GmA << 1$ one obtains:

$$\kappa_{BH} \sim \frac{1}{4Gm}$$
$$\kappa_R \sim A$$
$$A \sim 8\Delta \alpha G^2 m^2$$
$$F \sim \Delta \alpha \frac{mA}{2\pi}$$

whence

$$F \sim \frac{A \kappa_{BH}}{8\pi G} \cdot \kappa_R$$

which is equivalent to Newton’s second law of motion. However for finite $mA$ one does not obtain such a simple expression. This is perhaps not surprising since if $mA$ is not small the Schwarzschild radius of the black hole is comparable with the radius of curvature of its world line. Nevertheless it would be nice to understand the relation between mass, acceleration and force in this non-linear situation. Some attempts in this direction, which also relate to black hole thermodynamics were made by Aryal, Ford and Vilenkin (1986), (see also Martinez and York (1990). Note that Aryal et al. use the representation of accelerating black hole metrics in terms of Weyl-metrics using the “rod representation” of Schwarzschild (Israel and Khan (1964). The relationship between this picture and the C-metric including the co-ordinate transformation between the finite rod plus semi-infinite rod (each of mass per unit length $\frac{1}{2}$) and the C-metric form quoted above may be found in (Godfrey (1972) see also (Bonnor (1983,1990). More about strings and black holes may be found in Chandraskhar and Xanthopouls (1989).
7. Black Hole Monopole Pair-Production

In the quantum theory we know that charged particle anti-particle pairs may be created by a sufficiently strong electric field - a process sometimes called the Schwinger Process. It is plausible that magnetic monopoles should similarly be created by strong magnetic fields. This process was investigated in Yang-Mills-Higgs theory by Affleck and Manton (1982) using instanton methods. The use of instanton methods to calculate the rate of production by the Schwinger process is discussed in (Affleck, Alvarez and Manton).

Some time ago I suggested that the same process should occur in quantized Einstein-Maxwell theory (Gibbons 1986). The idea has recently been taken up again by Strominger and Garfinkle (1990). Since pure Einstein-Maxwell theory has invariance under the duality transformation

$$F_{\mu\nu} \rightarrow (\exp \alpha \ast) F_{\mu\nu}$$

where $\ast$ is the Hodge star operation on 2-forms. There is no invariant distinction between electric and magnetic, so let us concentrate on the purely magnetic case. In any event it is this case which is physically most interesting in more realistic models.

To begin we need to model a strong magnetic field coupled to gravity. The natural choice is the Melvin solution which represents an infinitely long straight self-gravitating Faraday flux tube in equilibrium, the gravitational attraction being in equipoise with the transverse magnetic pressure (Melvin (1964). The metric is:

$$ds^2 = (1 + \pi GB^2 \rho^2)^2 (-dt^2 + dz^2 + d\rho^2) + \rho^2 d\phi^2 (1 + \pi GB^2 \rho^2)^{-2}$$

The magnetic field is given by:

$$F = \frac{B \rho d\rho \wedge d\phi}{(1 + \pi GB^2 \rho^2)^2}$$

The Melvin solution possesses a degree of uniqueness. For example Hiscock (1981) has shown

Theorem: The only axisymmetric, static solution of the Einstein-Maxwell field equations without an horizon which is asymptotically Melvin is in fact the Melvin Solution.

In fact Hiscock also allows for a neutral or electrically charged black hole as well.

I myself can show:

Theorem: The only translationally invariant, static solution of the Einstein-Maxwell field equations without horizon which is asymptotically Melvin is in fact the Melvin solution.

Proof: assume the metric is static and has reflection invariance with respect to the $z-$direction. These two assumptions may easily be justified. The metric takes the form

$$ds^2 = -V^2 dt^2 + Y^2 dz^2 + g_{AB} dx^A dx^B$$

with $A = 1, 2$. The field equations are:

$$\nabla^A (VY \nabla_A \ln(V/Y)) = VY 8\pi G (T_{zz} + T_{t0})$$

$$\nabla^A (VY \nabla_A \ln(V/Y)) = VY 8\pi G T^A_A$$
\[ V^{-1}\nabla_A \nabla_B V + Y^{-1}\nabla_A \nabla_B Y = Kg_{AB} - 8\pi(T_{AB} - \frac{1}{2}g_{AB}(T^A_A + T^{zz} + T_{0\bar{0}})) \]

where \( K \) is the Gauss curvature of the 2-metric \( g_{AB} \). The electromagnetic field is assumed to be of the form:

\[ F = \frac{1}{2}F_{AB} dx^A dx^B. \]

It follows that \( T_{\bar{0}\bar{0}} + T_{\bar{z}\bar{z}} = 0 \) and hence:

\[ \nabla^A (VY\nabla_A (V/Y)) \equiv 0. \]

Now \( V/Y \) tends to one at infinity (asymptotic boost invariance) and so we may invoke the Maximum Principle to show that \( V = Y \) everywhere. Thus the metric must be boost invariant. It now follows that

\[ \nabla_A \nabla_B V = fg_{AB} \]

for some scalar \( f \). Thus

\[ K^A = \epsilon^{AB}\nabla_B V \]

is a Killing vector field of the 2-metric \( g_{AB} \) and since \( K^A \partial_A V = 0 \) it is also a Killing vector field of the entire 4-metric. It is not difficult to see that this Killing vector field corresponds to rotational symmetry of the solution.

The argument just given will generalise in various ways to cover some other stress tensors and as mentioned above the staticity assumption and the assumption that \( g_{\alpha z} = g_{zz} \delta_{\alpha z} \) is not difficult to justify using standard methods on the global theory of black holes. Interestingly however it does not seem to be possible to show using this method that the metric of a local cosmic string must be axisymmetric. Even in flat spacetime this seems to be a very difficult problem, i.e. to show that all time independent Nielsen-Olesen vortex solutions of the abelian Higgs model (in the non-supersymmetric case) must have axial symmetry. Having established the credentials of the Melvin solution as uniquely suitable model of a static magnetic field in general relativity we turn to looking for instanton solutions representing the creation of a black hole monopole anti-monopole pair. If there were no external magnetic field the obvious candidate instantons would be the magnetically charged C-metric for which

\[ G(x) = 1 - x^2 - 2GmA x^3 - G(g^2/4\pi)A^2 x^4. \]

However this has nodal singularities. In fact since the metric is boost invariant it has zero ADM mass and thus it cannot be regular by the positive mass theorem generalised to include apparent horizons. However it was pointed out by Ernst ((1976) that the nodal singularity may be eliminated if one appends a suitable magnetic field. The resulting metric is of the same form as (6.7) but the first three terms are multiplied by and the last term divided by the factor:

\[ (1 + GBgx/2)^4. \]
If \( m = 0 = g = A \) we get the Melvin solution but the limit must be taken carefully. The nodal singularity may be eliminated if \( B \) is chosen so that

\[
G'(x_3)/(1 + GBg x_3/2)^4 + G'(x_4)/(1 + Gg x_4/2)^4 = 0.
\]

Where \( x_3 \) and \( x_4 \) are two larger roots of \( G(x) \) and we assume now that there are 4 roots. The smallest root \( x_1 \) is thus inside the acceleration horizon. This equation may be regarded as an equation for \( B \) the magnetic field necessary to provide the force to accelerate the magnetically charged black hole. It is difficult to find an explicit solution in terms of \( g, m \) and \( A \) except when \( GmA \) is small in which case one finds the physically sensible result:

\[
gB \approx mA.
\]

In order to obtain an instanton which is regular on the Riemannian section obtained by allowing the time coordinate \( t \) to be pure imaginary it is necessary that the \( \tau = it \) is periodic with period given by the surface gravity. This leads to the condition that

\[
G'(x_2) + G'(x_3) = 0.
\]

It appears that the the only way to satisfy this condition is to set:

\[
m = |g|/\sqrt{4\pi G}
\]

Note that this equation does not mean that the horizons have vanishing surface gravity as I mistakenly asserted in (1986). It is not difficult to see that the topology of the Riemann section is \( S^2 \times S^2 \) with a point removed. In fact topologically one can obtain this manifold from \( R^4 \), which is the topology of the Melvin solution, by surgery along an \( S^1 \). That is by cutting out a neighbourhood of a circle which has topology \( D^3 \times S^1 \) with boundary \( S^2 \times S^1 \) and replacing by \( S^2 \times D^2 \) which has the same boundary. This surgery is also what is needed to convert \( R^3 \times S^1 \) to \( R^2 \times S^2 \) i.e. to convert a manifold with the topology of "Hot Flat Space" to that with the topology of the Riemannian section of the Schwarzschild solution.

The existence of this instanton would seem to be rather important. It seems to imply for example that it would be inconsistent not to consider the effects of black hole monopoles since given strong enough magnetic fields they will be spontaneously created. Once they are created they should evolve by thermal evaporation to the extreme zero temperature soliton state. Another reason why I believe that this process is so important is that it seems to show that while one may have one’s doubts about the effects of wormholes because of the absence of suitable solutions of the classical equations of motion with positive definite signature, the solutions described here do indicate that some sort of topological fluctuations in the structure of spacetime must be taken into account in a satisfactory theory of gravity coupled to Maxwell or Yang-Mills theory.
References

S L Adler and R P Pearson, No Hair theorem for the Abelian Higgs and Goldstone models. Phys Rev D18, 2798-2803 (1978)
I K Affleck, O Alvarez and N S Manton, Pair production at strong coupling in weak external fields. Nucl. Phys B197 509-519 (1982)
I K Affleck and N S Manton Monopole pair production in a magnetic field. Nucl Phys B194 38-64 (1982)
F J Almgren and E H Lieb, Counting Singularities in liquid crystals. Proc. IXth International Congress on Mathematical Physics, eds B Simon, A Truman and I M Davies, Adam Hilger 1989
F Almgren and E Lieb, Singularities of energy minimizing maps from the ball to the sphere: examples counter examples and bounds, Ann of Math. 128 483-430 (1988)
M Aryal, L H Ford and A Vilenkin, Cosmic Strings and Black Holes, Phys Rev D34 2263-2266 (1986)
A Ashtekar and T Dray, On the Existence of solutions to Einstein’s Equation with Non-Zero Bondi News. Commun. Math. Phys 79 581-589 (1981)
F A Bais and R J Russell, Magnetic-monopole solution of the non-Abelian gauge theory in curved spacetime. Phys Rev D11 2692-2695 (1975)
R Bartnik and J McKinnon, Particle like solutions of the Einstein-Yang-Mills equations. Phys Rev Lett 61 141-144 (1988)
M Barriola and A Vilenkin, Gravitational field of a global monopole. Phys. Rev. Lett 63 341-343 (1989)
J Bicak, The motion of a charged black hole in an electromagnetic field. Proc. Roy. Soc. A371 429-438 (1980)
W B Bonnor, The sources of the vacuum C-metric. Gen. Rel. Grav. 15 535-551 (1983)
W B Bonnor, The C-metric in Bondi’s coordinates. Class. Quant. Grav. 7 L229-L230 (1990)
P Bizon, Colored Black Holes. Phys. Rev. Letts 64 2844-2847 (1990)
P J Braam, A Kaluza-Klein approach to hyperbolic three-manifolds. Enseign. Math 34 275-311 (1985)
R A Brandt and F Neri, Stability Analysis for Singular Non-Abelian Magnetic monopoles. Nucl. Phys. B161 253-282 (1979)
P B Breitenlohner, G W Gibbons and D Maison, 4-dimensional Black Holes from Kaluza-Klein Theory. Commun. Math. Phys. 120 295-334 (1988)
H Brezis, J M Coron and E Lieb, Harmonic Maps with defects. Commun. Math. Phys 107 649-705 (1986)
B E Brumbaugh, Nonlinear scalar field dynamics in Schwarzschild geometry. Phys. Rev. D18 1335-1338 (1978)
S Chandrasehkar and B C Xanthopoulos two Black Holes attached to strings. Proc. Roy. Soc. A423 387-400 (1989)
T Chmaj and E Malec, Magnetic monopoles and gravitational collapse. Class and Quantum Grav. 6 1687-1696 (1989)
Y M Cho and P G O Freund, Gravitating ’t Hooft monopoles. Phys. Rev. D12 1588-1589, (1975)
S Coleman in “The Unity of Fundamental interactions” ed. A Zichichi (Plenum, New York) (1983)
A Comtet, Magnetic Monopoles in curved spacetimes. Ann. Inst. H Poincare 23 283-293 (1980)
A Comtet, P Forgacs and P A Horvathy, Bogomolnyi-type equations in curved spacetime. Phys. Rev D30 468-471 (1984)
A Comtet and G W Gibbons, Bogomol’nyi Bounds for Cosmic Strings. Nucl. Phys. B299 719-733 (1989)
A D Dolgov, Gravitational Dipole. JETP Lett. 51 393-396 (1990)
T Dray, On the Asymptotic Flatness of the C Metrics at Spatial Infinity. Gen.Rel.Grav. 14 109-112 (1982)
T Dray and M Walker, On the regularity of Ernst’s generalized C-metric. L.I.M.P. 4 15-18 (1980)
F J Ernst, Black holes in a magnetic universe. J.M.P. 17 54-56 (1976)
F J Ernst, Removal of the nodal singularity of the C-metric. J.M.P. 17 515-516 (1976)
F J Ernst, Generalized C-metric. J.M.P. 19 1986-1987 (1978)
F J Ernst and W J Wild, Kerr black holes in a magnetic universe. J.M.P. 17 182-184 (1976)
A A Ershov and D V Gal’tsoy, Non Existence of regular monopoles and dyons in the SU(2) Einstein-Yang-Mills theory. Phys. Letts. 150A 159-162 (1990)
J A Frieman and C T Hill, Imploding Monopoles. SLAC-PUB-4283 Oct. 1987 T/AS
A Floer, Monopoles on asymptotically Euclidean manifolds. Bull. AMS 16 125-127 (1987)
D V Gal’tsoy and A A Ershov, Non-abelian baldness of coloured black holes. Phys. Letts A138 160-164 (1989)
D Garfinkle and A Strominger, Semi-classical Wheeler Wormhole Production. UCSB-TH-90-17
G W Gibbons Non-existence of Equilibrium Configurations of Charged Black holes. Proc. Roy. Soc. A372 535-538 (1980)
G W Gibbons, Quantised Flux-Tubes in Einstein-Maxwell theory and non-compact internal spaces, in Fields and Geometry Proc. of XII Karpac Winter School of Theoretical Physics 1986, ed A Jadezyk, World Scientific
B B Godfrey, Horizons in Weyl metrics exhibiting extra symmetries. G.R.G. 3 3-15 (1972)
A Goldhaber, Collapse of a Global Monopole. Phys. Rev. Letts 63 2158(c) (1989)
Gu Chao-hao, On Classical Yang-Mills Fields. Phys. Rep. 80 251-337 (1981)
P Hajicek, Wormhole solutions in the Einstein-Yang-Mills-Higgs system. I General theory of zero-order structure. Proc. Roy. Soc A 386 223-240 (1983)
P Hajicek, Wormhole solutions in Einstein-Yang-Mills-Higgs system II Zeroth-order structure for $G = SU(2)$. J. Phys. A16 1191-1205 (1983)
P Hajicek, Classical Action Functional for the system of fields and wormholes. Phys. Rev. D26 3384-2295 (1982)
P Hajicek, Generating functional $Z_0$ for the one-wormhole sector. Phys. Rev. **D26** 3396-3411 (1982)

P Thomi, B Isaak and P Hajicek, Spherically Symmetric Systems of Fields and Black Hole. I Definition and properties of Apparent Horizon. Phys. Rev. **D30** 1168-1171 (1984)

P Hajicek, Spherically symmetric systems of fields and black holes. II Apparent horizon in canonical formalism. Phys. Rev. **D30** 1178-1184 (1984)

P Hajicek, Spherically symmetric systems of fields and black holes. III Positively of enemy and a new type of Euclidean action. Phys. Rev. **D30** 1185-1193 (1984)

P Hajicek, Spherically symmetric systems of fields and black holes. IV No room for black hole evaporation in the reduced configuration space? Phys. Rev. **D31** 785-795 (1985)

P Hajicek, Spherically symmetric systems of fields and black holes. V Predynamical properties of causal structure. Phys. Rev. **D31** 2452-2458 (1985)

P Hajicek, Quantum theory of wormholes. Phys. Letts **106B** 77-80 (1981)

P Hajicek, Quantum wormholes (I). Choice of the classical solutions. Nuc. Phys. **B185** 254-268 (1981)

P Hajicek, Duality in Klein-Kaluza Theories. BUTP-9/82

P Hajicek, Exact Models of Charged Black Holes. I: Geometry of totally geodesic null hypersurface. Commun. M. Phys. **34** 37-52 (1973)

P Hajicek, Exact Models of Charged Black Holes II: Axisymmetries Sationary Horizons. Commun. M. Phys **34** 53-76 (1973)

P Hajicek, Can outside fields destroy black holes. J. Math. Phys. **15** 1554 (1974)

D Harari and C Lousto, Repulsive gravitational effects of global monopoles. Buenos Aires preprint GTCRG-90-4

W A Hiscock, Magnetic Monopoles and evaporating black holes. Phys. Rev. Letts **50** 1734-1737 (1982)

W A Hiscock, On black holes in magnetic universes. J. Math. Phys. **22** 1828-1833 (1981)

W A Hiscock, Magnetic monopoles and evaporating black holes. Phys. Rev. Lett **50** 1734-1737 (1983)

W A Hiscock, Astrophysical bounds on global monopoles.

W A Hiscock, Gravitational particle production in the formation of global monopoles.

P A Horvathy, Bogomolny-type equations in curved space. Proc. 2nd Hungarian Relativity Workshop (Budapest 1987) ed. Z Peres, World Scientific

H S Hu, Non existence theorems for Yang-Mills fields and harmonic maps in the Schwarzschild spacetime (I). Lett. Math. Phys. **14** 253-262 (1987)

H S Hu and S Y Wu, Non existence theorems for Yang-Mills fields and harmonic maps in the Schwarzschild spacetime (II). Lett Math. Phys. **14** 343-351 (1987)

W Israel and K A Khan Collinear paricles and Bondi dipoles in general relativity. Nuovo Cimento **33** 331 (1964)

W Kinnersley and M Walker, Uniformly accelerating charged mass in general relativity. Phys. Rev. **D2** 1359-1370 (1970)

L M Krauss and F Wilczek, Discrete gauge symmetry in continuum theories. Phys Rev Letts **62** 1221-1223 (1989)
H P Kunzle and A K M Masood-ul-Alam, Spherically symmetric static $SU(2)$ Einstein-Yang-Mills fields. J. Math. Phys. 31 928-935 (1990)
A S Lapedes and M J Perry, Type D Gravitational instantons. Phys. Rev. D24 1478-1483 (1983)
D Lohiya, Stability of Einstein-Yang-Mills Monopoles and Dyons. Ann. Phys. 14 104-115 (1982)
M Magg, Simple proof for Yang-Mills instability. Phys. Letts 74B 246-248 (1978)
E Malec and P Koc, Trapped surfaces in monopole-like Cauchy data of Einstein-Yang-Mills-Higgs equations. J. Math. Phys. 31 1791-1795 (1990)
J E Mandula, Classical Yang-Mills potentials. Phys. Rev. D14 3497-3507 (1976)
J E Mandula, Color screening by a Yang-Mills instability. Phys. Letts 67B 175-178 (1977)
J E Mandula, Total Charge Screening. Phys. Lett. 69B 495-498 (1977)
E A Martinez and J W York, Thermodynamics of black holes and cosmic strings. IFP-342 UNC: May 1990
A K M Masood-ul-Alam and Pan Yanglian (Y L Pan) Non Existence theorems for Yang-Mills fields outside a black hole of the Schwarschild spacetime. Lett in Math. Phys. 17 129-139 (1989)
M A Melvin, Pure magnetic and electric geons. Phys. Lett 8 65-67 (1964)
C W Misner, Harmonic maps as models for physical theories. Phys. Rev D18 4510-4524 (1978)
M J Perry, Black holes are coloured. Phys. Letts 71B 234 (1977)
J Preskill and L M Krauss, Local discrete symmetry and quantum-mechanical hair. Nucl. Phys. B34150-100 (1990)
J Preskill Quantum Hair. Caltech preprint CALT-68-1671 (1990)
D Ray, Solutions of coupled Einstein-SO(3) gauge field equations. Phys. Rev. D18 1329-1331 (1978)
P Ruback A New Uniqueness Theorem for Charged Black Holes. Class. and Quant. Grav. 5 L155-L159 (1988)
R F Sawyer, The possibility of a static scalar field in the Schwarzschild geometry. Phys. Rev. D15, 1427-1434 (1977). Erratum Phys. Rev. D16 (1977) 1979
P Sikivie and N Weiss, Screening Solutions to Classical Yang-Mills theory. Phys. Rev. Letts 40 1411-1413 (1978)
N Straumann and Z-H Zhou, Instability of colored black hole solution. Phys. Letts 141B 33-35 (1990)
N Straumann and S-H Shou, Instability of the Bartnik-McKinnon solutions of the Einstein-Yang-Mills equations. Phys. Letts 237 353-356 (1990)
P van Nieuwenhuizen, D Wilkinson and M J Perry, Regular solution of ’t Hooft’s magnetic monopole in curved space. Phys. Rev. D13 778-784 (1976)
M Y Wang, A solution of coupled Einstein-SO(3) gauge field equations. Phys. Rev. D12 3069-3071 (1975)
M S Volkov and D V Gal’tsov, Non-Abelian Einstein-Yang-Mills black holes. JETP. Letts 50 346-350 (1989)
D V Galt’sov and A A Ershov. Yad. Fiz. 47 560 (1988)