"HOMOGENEOUS" GRAVITATIONAL FIELD
IN GENERAL RELATIVITY?

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The gravitation field of the flat plate was investigated. It have been shown that there exist the internal solution of Einstein equations sewed together with external one, which described a "homogeneous" gravitational field.

Introduction. The interest to the problems with planar symmetry has again increased in the last years. First of all it is connected to the fact that this type of solution plays an important role in the theories of superstrings and supergravity [1].

There are series of papers devoted to the study of gravitational field of a flat plate (see Ref. in [2]). The external solution of this problem have been found for the first time by Taub [3]. The numerical integration of internal solution and its sewing together with external one was performed in [4]. Here, we stress that Einstein equations accept two types of internal solutions: with negative and positive derivative of metric tensor at the centre of plate.

We will show in this paper, that only solution with negative derivative can be sewing together Taub solution. The second one corresponds to gravitational field for which the Riemannian tensor is zero outside of the plate.

Flat plate: general. Let’s consider a static gravitational field of infinite flat plate. Following the paper [4], space - time metric can be written as (axis $Ox$ is perpendicular to the plate, and the plane $yz$ is located at the middle of the plate)

$$ds^2 = e^{\nu(x)} dt^2 - dx^2 - e^{\lambda(x)} (dy^2 + dz^2).$$

(1)

Obviously, $\nu(x)$ and $\lambda(x)$ are even functions of $x$. Below for definiteness we shall consider the solutions in $x > 0$.

Einstein’s equations and equations of hydrodynamics (with energy-momentum tensor for ideal liquid) appropriate to the metric (1) are as follows (using system of units $c = G = 1$):

$$\lambda'' + \frac{3}{4} \lambda'^2 = -8\pi\rho,$$

(2)

$$\frac{\lambda'^2}{4} + \frac{\lambda'\nu'}{2} = 8\pi p,$$

(3)

$$\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\lambda''}{4} + \frac{\lambda'^2}{4} + \frac{\lambda'\nu'}{4} = 8\pi p,$$

(4)

$$p' = -\frac{\nu'}{2} (p + \rho),$$

(5)
where $\rho(x)$ is the energy density, $p(x)$ is the pressure, and prime denotes the differentiation with respect to $x$. We should add to this system the state equation $p = p(\rho)$. We would like to remind that one of the equations of system (2) - (5) is a consequence of the others.

As boundary conditions for system (2) - (5) we have to specify values for $\nu(0), \nu'(0), \lambda(0), \lambda'(0)$ and $p(0)$ at the centre. The functions $\nu(x)$ and $\lambda(x)$ are determined to up additive constants (as they are not contained in the equations evidently), which can be removed by simple scale transformation: $t \rightarrow \alpha t, \ y \rightarrow \beta y, \ z \rightarrow \beta z$. Therefore, we can put $\nu(0) = \lambda(0) = 0$.

It is obvious from symmetry, that derivative of pressure (force) at the centre of the configuration is equal to zero, i.e.

$$p'(0) = 0,$$

therefore

$$\nu'(0) = 0.$$

Inside the configuration pressure decreases monotonically ($p' < 0$), and $\nu(x)$ increases monotonically ($\nu' > 0$). The bound of configuration $x_s$ is determined by condition

$$p(x_s) = 0.$$

The derivative of $\lambda$ at the centre of the plate is determined from the equation (3)

$$\lambda'(0) = \pm \sqrt{\frac{32}{3}\pi p_0},$$

where $p_0 \equiv p(0)$. At $x = x_s$ the pressure is equal to zero, therefore, as it follows from the equation (3), $\lambda'(x_s) = 0$ or $\lambda''(x_s) = 0$ (from the equation (2)), if $\lambda'(0) = -\sqrt{\frac{32}{3}\pi p_0}$ then $\lambda'(x_s) = 0$ and $\lambda''(x_s) = 0$.

According to this, depending on sign of $\lambda'(0)$ the internal solution will be sewed either with the external solution of Taub [3] (for $\lambda'(0) = -\sqrt{\frac{32}{3}\pi p_0}$)

$$ds^2 = \frac{A}{(1 - Bx)^{\frac{3}{2}}} dt^2 - dx^2 - C (1 - Bx)^{\frac{3}{4}} \left( dy^2 + dz^2 \right),$$

where $A > 0, B$ and $C > 0$ are constants of integration, or with external solution distinct essentially from it (for $\lambda'(0) = \sqrt{\frac{32}{3}\pi p_0}$)

$$ds^2 = (a + bx)^2 dt^2 - dx^2 - c \left( dy^2 + dz^2 \right),$$

where $a > 0, b$ and $c > 0$ are also constants of integration.

Model of ideal liquid. We study in detail the case of homogeneous liquid $\rho = \rho_0 = \text{const}$. The equation (2) can be integrated easily and taking into account the boundary conditions (3) and (10) we have:

$$e^{\lambda(x)} = (1 + 3q_0)^{\frac{2}{3}} \left[ \cos \left( \frac{1}{\sqrt{6\pi \rho_0 x}} \pm \arccos \frac{1}{\sqrt{1 + 3q_0}} \right) \right]^\frac{4}{3},$$

(13)
where \( q_0 = \frac{p_0}{\rho_0} \) the ratio of the central pressure to the density. In the formula (13) upper sign corresponds to the solution with boundary condition \( \lambda'(0) = -\sqrt{\frac{32}{\pi p_0}} \), and lower sign corresponds to \( \lambda'(0) = \sqrt{\frac{32}{\pi p_0}} \). As in the second case \( \lambda'(x_s) = 0 \), from (13) we can find also the bound of the configuration \( x_s \). Differentiating the right hand side of (13) and equating to zero we find:

\[
x_s = \arccos \frac{1}{\sqrt{1+3q_0}} \frac{1}{\sqrt{6\pi\rho_0}}.
\] (14)

In the second case the value of \( \lambda \) on bound is equal to

\[
\lambda(x_s) = \frac{2}{3} \ln (1 + 3q_0).
\] (15)

The equation of hydrodynamics can be also integrated

\[
p = (p + \rho) e^{-\nu} - \rho_0.
\] (16)

Therefore, in both cases the value of \( \nu \) on bound is equal to

\[
\nu(x_s) = 2 \ln (1 + q_0).
\] (17)

Substituting (13) and (16) into (3), and taking into account the boundary conditions we find:

\[
e^{\nu(x)} = (1 + q_0)^2 \left[ 1 + \frac{\sin \left( \sqrt{6\pi\rho_0} x \pm \arccos \frac{1}{\sqrt{1+3q_0}} \right)}{\cos^3 \left( \sqrt{6\pi\rho_0} x \pm \arccos \frac{1}{\sqrt{1+3q_0}} \right)} \pm \sqrt{\frac{q_0}{3}} \left( \frac{1}{1+q_0} \right)^{\frac{3}{2}} \right]
\]

\[
+ \frac{\sqrt{6\pi\rho_0}}{3} \int_0^x \frac{dx}{\cos^3 \left( \sqrt{6\pi\rho_0} x \pm \arccos \frac{1}{\sqrt{1+3q_0}} \right)}
\]

(18)

To find the constants of integration \( A, B \) and \( C \) or \( a, b \) and \( c \) we should require continuity of the components of metric tensor and their first derivative on the bound of the configuration. However, as (14) (with the account (18)) is rather complicated, the bound of the configuration could not be found explicitly, and sewing together can not be made analytically. Therefore it’s convenient to find the internal solution by numerical calculation. For this purpose we should write down system of the equations (2) - (5) in the convenient form. It’s reasonable, for that, besides \( \nu(x), \lambda(x) \) and \( p(x) \) to calculate the following functions

\[
\sigma(x) = 2 \int_0^x \left( T_0^0 - T_1^1 - T_2^2 - T_3^3 \right) \sqrt{-g} dx = 2 \int_0^x (p + 3\rho) e^{\frac{x}{2} + \lambda} dx,
\] (19)

\[
\chi(x) = 2 \int_0^x \left( T_0^0 + T_1^1 \right) \sqrt{-g} dx = 2 \int_0^x (\rho - p) e^{\frac{x}{2} + \lambda} dx.
\] (20)

The function \( \sigma(x) \) is ”accumulated” surface density (it can be easily verified passing to the Newtonian limit [1]). Taking into account (19), (20) and replacing the independent variable
we can write equations (3) - (5) in the following form:

\[ \ddot{\sigma} = 2 (1 + 3q) e^{\frac{\nu}{2} + \lambda}, \]  
\[ \dot{\nu} = \frac{1}{2} \sim \dot{\sigma} e^{-\frac{\nu}{2} - \lambda}, \]  
\[ \ddot{\chi} = 2 (1 - q) e^{\frac{\nu}{2} + \lambda}, \]  
\[ \dot{\lambda} = -\frac{1}{2} \sim \dot{\chi} e^{-\frac{\nu}{2} - \lambda}, \]  
\[ \dot{q} = -\frac{1}{4} \sim \dot{\sigma} e^{-\frac{\nu}{2} - \lambda} (1 + q), \]  
where \( q \equiv p/\rho_0, \sim \equiv \sqrt{8\pi/\rho_0} \sigma, \) \( \dot{\sigma} \equiv \sqrt{8\pi/\rho_0} \sigma, \) and dot denotes differentiation with respect to \( \sim x. \) For numerical calculation, starting from the centre of the configuration \( \sim x = 0 \) we should specify values of \( \ddot{\sigma} (0), \ddot{\chi} (0) \) and \( q (0) \) besides \( \nu (0) \) and \( \lambda (0) \) (see eq. (6)). Obviously, \( \ddot{\sigma} (0) = 0. \) From the equation (3), using (19), (20) and (21) we find:

\[ \ddot{\chi} = \pm 4 \sqrt{2\pi\rho_0} \sim \nu \]  

Constants of integration in terms of boundary values of \( \nu_s, \lambda_s, \sim_s \) and \( x_s, \) have the following form:

\[ A = e^{\frac{\nu}{2} + \lambda_s} C^{-\frac{1}{2}}; \quad B = \frac{6\sqrt{2\pi\rho_0} \sim \dot{\sigma}_s}{4e^{\frac{\nu}{2} + \lambda_s} + 3 \dot{\sigma}_s x_s}; \quad C = \left( e^{\frac{\nu}{2} + \lambda_s} + \frac{3}{4} \sim \dot{\sigma}_s x_s e^{-\frac{\nu}{2} - \lambda_s} \right)^{\frac{1}{2}}, \]  
\[ a = e^{\frac{\nu}{2}} - \frac{1}{4} \sim \dot{\sigma}_s \sim x_s e^{-\lambda_s}; \quad b = \sqrt{\frac{\pi\rho_0}{2}} \sim \dot{\sigma}_s e^{-\lambda_s}; \quad c = e^{\lambda_s}. \]  

The numerical calculation of the equations (22) - (26) was carried out for configurations with the parameter values \( 10^{-2} \leq q_0 \leq 1. \) The results of numerical calculation are given on figures.

External solutions. We study now external solutions (11) and (12). In Taub solution (11) constant \( B > 0 \) (as \( \lambda' (x_s) < 0 \)), therefore, on final distance from the centre of the plate in the point \( x = 1/B \) the metric has real singularity: \( |g_{ik}| = 0. \) And, it can be easily convinced that \( x_s < 1/B, \) i.e. the plate is inside of the singular planes. Indeed, from the equations it can be seen that the function \( \lambda(x) \) monotonically decreases in all space (\( \lambda' < 0 \) and \( \lambda'' < 0 \) ) and becomes \( "-\infty" \) only outside the plate, as function is a finite quantity inside the plate.

In the solution (12) constant of integration \( b \) is also positive, as \( \nu' (x_s) > 0. \) The solution is 4-flat, what can be convinced by evaluating Riemannian tensor for this metric. However, it can be checked easier using transformations
Figure 1: The dependence of "surface density" (on the left) and of "proper thickness" of the plate (on the right) on the parameter $q_0$ (dotted curve corresponds to the case of $\tilde{\chi}(0) < 0$, and continuous curve $-\tilde{\chi}(0) > 0$).

Figure 2: Dependence of functions $e^{\nu(\tilde{x})}$ and $e^{\lambda(\tilde{x})}$ on the dimensionless coordinate $\tilde{x}$. The dotted vertical line is the bound of the configuration.
Thus, the metric (12) takes the Galilean form

\[ ds^2 = dt_1^2 - dx_1^2 - dy^2 - dz^2, \]  

(31)

Thus, this solution represents realization of a homogeneous gravitational field in sense of GR, which can be excluded by one global transformation (30) in all space outside the plate.

Conclusions. It is conventionally assumed that in the Einstein’s theory the presence of gravitational field in some area is indicated by nonzero Riemannian tensor in this area. However, the analysis of this solution leads to the following conclusion: the metric (12) describes a homogeneous gravitational field of the plate in reference system, fixed with respect to the plate, but for it the Riemannian tensor \( R^i_{klm} \) is zero. Really, in this field the trial particle moves under the law

\[ a + bx = \frac{a + bx_0}{\cosh bt}, \]  

(32)

and falls on the plate in finite time \( t_0 = \text{arccosh} \left( \frac{a + bx_0/a + bx_s}{a + bx_s} \right) \), where \( x_0 > x_s \) is the coordinate of the point, where the particle was at rest at the moment \( t = 0 \).

We think that namely this solution (but not Taub solution) is a relativistic analogue of the appropriate solution in the Newton’s theory.

References

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