1 Introduction and Main results

This paper, the second of a series, deals with the function space of all smooth Kähler metrics in any given closed complex manifold $M$ in a fixed cohomology class. This function space is equipped with a pre-Hilbert manifold structure introduced by T. Mabuchi [10], where he also showed formally it has non-positive curvature. The previous result of the second author [4] showed that the space is a path length space and it is geodesically convex in the sense that any two points are joined by a unique path, which is always length minimizing and of class $C^{1,1}$. This already confirms one of Donaldson’s conjecture completely and verifies another one partially (cf. [8]). In the present paper, we show first of all, that the space is, as expected, a path length space of non-positive curvature in the sense of A. D. Alexanderov. A second result is related to the theory of extremal Kähler metrics, namely that the gradient flow of the K energy is strictly length decreasing on all paths except those induced by a path of holomorphic automorphisms of $M$. This result, in particular, implies that extremal Kähler metric is unique up to holomorphic transformations, provided that Donaldson’s conjecture on the regularity of geodesic is true.

1.1 Riemannian metrics and Non Positive Curved space.

Let $(V, \omega_0)$ be a polarized Kähler manifold without boundary. Consider the space of Kähler distortion potentials

$$\mathcal{H} = \{ \varphi \in C^\infty(V) : \omega_\varphi = \omega_0 + \partial \bar{\partial} \varphi > 0 \text{ on } V \}.$$  

Clearly, the tangent space $T\mathcal{H}$ is $C^\infty(V)$. Each Kähler potential $\phi \in \mathcal{H}$ defines a measure $d\mu_\phi = \frac{1}{n!} \omega_\phi^n$. A Weil-Peterson type metric was defined on this infinite dimensional manifold $\mathcal{H}$, using the $L^2$ norm provided by these measures (cf. Section 2.1 for historical remark of this metric). In 1997, following a program of Donaldson [8], the second author proved that this space is convex by $C^{1,1}$ geodesic; and used this fact to prove that $\mathcal{H}$ is indeed a path length space.

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*The second author is supported partially by NSF postdoctoral fellowship.

*The K energy is defined by T. Mabuchi in 1987 [10] while the flow was introduced by the first author in 1982 [2]. And it is commonly known as the “Calabi flow” in the literature.
If one could prove that the resulting geodesic were $C^4$ instead of $C^{1,1}$, then the formal calculations in [10] would yield that the curvature is non-positive. Since we only have $C^{1,1}$ geodesics, an important question is whether this is a non-positive curved space in the sense of A.D. Aleksandrov. We give an affirmative answer to this question here:

**Theorem 1.1.** The space of Kähler potentials in a fixed Kähler class is Non-Positive Curved space: Suppose $A, B, C$ are three points in the space of Kähler potentials and $P_\lambda$ is a geodesic interpolation point of $B$ and $C$ for $0 \leq \lambda \leq 1$: the geodesic distance from $P_\lambda$ to $B$, and the distance from $P_\lambda$ to $C$ are $\lambda d(B,C)$ and $(1 - \lambda)d(B,C)$ respectively (here we use $d(P,Q)$ to denote the distance between $P$ and $Q$ in $\mathcal{H}$).

Then the following inequality holds:

$$d(A,P_\lambda)^2 \leq (1 - \lambda)d(A,B)^2 + \lambda d(A,C)^2 - \lambda \cdot (1 - \lambda)d(B,C)^2.$$  

In [4], the second author proved that the geodesic minimizes all possible length. However, it is not clear whether a sequence of curves which minimize the length between any two points in $\mathcal{H}$ converges to a geodesic or not. Now we can give an affirmative answer to this question:

**Theorem 1.2.** For any two metrics $\varphi_0, \varphi_1$ in $\mathcal{H}$, let $\{C_i\}$ be any sequence of curves in $\mathcal{H}$ which connect between $\varphi_0$ and $\varphi_1$. Suppose the length of this sequence of curves approaches the infimum of length over all possible curves between $\varphi_0$ and $\varphi_1$, then $C_i$ converges to the unique $C^{1,1}$ geodesic which connects $\varphi_0$ and $\varphi_1$ in the sense of distance.

### 1.2 The gradient flow

In [2], the first author introduced the notion of extremal Kähler metrics and proposed to use a fourth order heat equation to attack the existence problem of extremal Kähler metrics:

$$\frac{\partial \varphi}{\partial s} = R(\varphi) - \mathbf{R},$$

where $R(\varphi)$ is the scalar curvature of the Kähler metric $\omega_\varphi$ and $\mathbf{R}$ is the average scalar curvature — a constant depending only the Kähler class. Somewhat surprisingly, we observed that this flow actually decreases the length of any smooth curve in $\mathcal{H}$:

**Theorem 1.3.** Given any two Kähler potentials $\varphi_1$ and $\varphi_2$ in $\mathcal{H}$ and a smooth curve $C(t)$ in $\mathcal{H}$ connecting them, the length of this curve strictly decreases under gradient flow (1.1) unless this curve in $\mathcal{H}$ represents a path of holomorphic transformations. More specifically, if $\varphi(t), 0 \leq t \leq 1$ is a curve in $\mathcal{H}$, and $L$ is the length of this curve; and suppose $\varphi(s,t)$ is the family of curves under the gradient flow (1.1), then

$$\frac{dL}{ds} = - \int_0^1 \left( \int_V |D \frac{\partial \varphi}{\partial s}(\varphi(s,t))|^2 d\varphi(s,t) \cdot \sqrt{\int_V |D \frac{\partial \varphi}{\partial t}(\varphi(s,t))|^2 d\varphi(s,t)} \right)^{-\frac{1}{2}} dt.$$
Here $g(s,t)$ is the Kähler metric corresponding to the Kähler potential $\varphi(s,t)$, while $D$ is the Lichnerowicz operator (cf. Definition 1.4 below).

**Definition 1.4.** The Lichnerowicz operator $D$: For any smooth function $f$ in $V$, $D(f) = \sum_{\alpha,\beta=1}^{n} f_{,\alpha\beta} dz^\alpha \otimes dz^\beta$ where $f_{,\alpha\beta}$ is the second covariant derivatives of $f$.

This flow is known to have short time existence. In Riemann surface, Chrusciel [7] proved that the global existence and convergence of this flow if there is a constant scalar curvature metric apriori. The second author gave a new geometric proof to the Chrusciel’s theorem [6]. Following approach taken in [6], M. Struwe [13] gave a unified treatment of both Ricci flow and Calabi flow in Riemann surfaces. In higher dimensional Kähler manifold cases, very little is known about the global existence of the flow.

**Theorem 1.5.** The following two statements hold

1. If the gradient flow (1.1) exists for all time for any smooth initial data, then the distance between any two metrics decreases under the gradient flow (1.1).

2. If the $K$ energy is weakly convex along geodesic, then the gradient flow decreases distance. In particular, if the first Chern class $C_1(V) \leq 0$, then the gradient flow (1.1) decreases distance.

The following question is very interesting:

**Question 1.6.** If $C_1(V) < 0$, is the distance function in $\mathcal{H}$ strictly decrease under the gradient flow (1.1)?

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## 2 $\mathcal{H}$ is a non positive curved space

In this section, we want to show that $\mathcal{H}$ is a non positive curved space in the sense of Aleksandrov.

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[A function $f(t)(0 \leq t \leq 1)$ is weakly convex if for any $t$, we have $f(t) \leq (1-t)f(0)+tf(1)$. For this theorem, we additionally assume that $f$ is differentiable at both end points, i.e., $f'(1) \geq f'(0)$.]
2.1 A Riemannian metric in the infinite dimensional space.

Mabuchi ([10]) in 1987 defined a Riemannian metric on the space of Kähler metrics, under which it becomes (formally) a non-positive curved infinite dimensional symmetric space. Apparently unaware of Mabuchi’s work, Semmes [12] and Donaldson [8] re-introduced this same metric again from different angles. Let us now introduce this metric here. A tangent vector in $\mathcal{H}$ is just a function in $V$.

For any vector $\psi \in T_{\phi} \mathcal{H}$, we define the length of this vector as

$$\|\psi\|^2 = \int_V \psi^2 \, d\mu_{\phi}.$$  

For two “vectors” $f_1, f_2$ in $T_{\phi} \mathcal{H}$, we use the standard notation in Riemannian geometry to denote their inner product:

$$\langle f_1, f_2 \rangle_{\phi} = \int_V f_1 \cdot f_2 \, d\mu_{\phi}.$$  

When no confusion is arisen, we just write

$$\langle f_1, f_2 \rangle = \int_V f_1 \cdot f_2 \, d\mu_{\phi}.$$  

For a path $\phi(t) \in \mathcal{H}(0 \leq t \leq 1)$, the length is given by

$$\int_0^1 \langle \frac{\partial \phi}{\partial t}(t), \frac{\partial \phi}{\partial t}(t) \rangle_{\phi(t)} \, dt = \int_0^1 \sqrt{\int_V \left( \frac{\partial \phi}{\partial t}(t) \right)^2 \, d\mu_{\phi(t)}} \, dt$$

and the geodesic equation is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} \left( \nabla \frac{\partial \phi}{\partial t}(t) \right)_{\phi(t)}^2 = 0,$$  

(2.1)

where the derivatives and norm in the second term of the left hand side are taken with respect to the metric $\omega_{\phi(t)}$.

This geodesic equation shows us how to define a connection on the tangent bundle of $\mathcal{H}$. The notation is simple if one thinks of such a connection as a way of differentiating vector fields along paths. Thus, if $\phi(t)$ is any path in $\mathcal{H}$ and $\psi(t)$ is a field of tangent vectors along the path (that is, a function on $V \times [0, 1]$), we define the covariant derivative along the path to be

$$D_t \psi = \frac{\partial \psi}{\partial t} - \frac{1}{2} (\nabla \psi, \nabla')_\phi.$$  

This connection is torsion-free because in the canonical “co-ordinate chart”, which represents $\mathcal{H}$ as an open subset of $C^\infty(V)$, the “Christoffel symbol”

$$\Gamma : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$$
at $\phi$ is just
\[
\Gamma(\psi_1, \psi_2) = -\frac{1}{2}(\nabla \psi_1, \nabla \psi_2)_\phi
\]
which is symmetric in $\psi_1, \psi_2$. It is easy to verify that the connection is metric-compatible. By a direct calculation, it was proved formally in [10] (and later re-proved in [12] and [8]) that $\mathcal{H}$ is a non-positive curved space.

Donaldson [8] in 1996 introduced a connection between this formal Riemannian metric in the infinite dimensional space $\mathcal{H}$ and the traditional Kähler geometry through a series of important conjectures and theorems. In 1997, following his program, the second author proves some of his conjectures:

**Theorem B** [4] The following statements are true:

1. The space of Kähler potentials $\mathcal{H}$ is convex by $C^{1,1}$ geodesics. More specifically, if $\varphi_0, \varphi_1 \in \mathcal{H}$ and $\varphi(t)$ ($0 \leq t \leq 1$) is a geodesic connecting these two points in $\mathcal{H}$, then the second order mixed covariant derivatives of $\varphi(t)$ are uniformly bounded from above.

2. $\mathcal{H}$ is a metric space. In other words, the infimum of the lengths of all possible curves between any two different points in $\mathcal{H}$ is strictly positive.

3. If $C_1(V) < 0$, then the extremal Kähler metric is unique in each Kähler class.

### 2.2 An approximate geodesic Lemma

In a local coordinate of $V$, let $\omega_0 = \sum_{\alpha,\beta=1}^{n} g_{\alpha \beta} \, dz_\alpha \, d\overline{z_\beta}$ and
\[
\omega_\varphi = \sum_{\alpha,\beta=1}^{n} g_{\alpha \beta} \, dz_\alpha \, d\overline{z_\beta}, \quad \text{where} \quad g_{\alpha \beta} = g_{0 \alpha \beta} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z_\beta}}.
\]

In this subsection, $z_1, z_2, \ldots, z_n$ are local coordinates in $V$; and we always use the following notations:

\[
i, j, k = 1, 2, \ldots, n, n + 1 \quad \text{and} \quad \alpha, \beta, \gamma = 1, 2, \ldots, n.
\]

For any path $\varphi(\cdot, t) : [0, 1] \to \mathcal{H}$, we can view it as a function defined in the product manifold $V \times [0, 1]$. Following an idea of S. Semmes, we introduce a dummy variable $\theta$ such that $V \times ([0, 1] \times S^1)$ is a $(n + 1)$ dimensional Kähler manifold and $t = re(z_{n+1})$. Here $S^1$ is the unit circle. Consider the projection
\[
\pi : V \times ([0, 1] \times S^1) \to V \\
(z, t, \theta) \to z.
\]

Consider the pull back metric $\pi^* g_0$. Note that $\pi^* \omega_0$ is a degenerate Kähler form of co-rank 1 in $V \times ([0, 1] \times S^1)$.

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\[\text{This is a conjecture of Donaldson}^{[8]}].\]
Definition 2.1. A path $\varphi(t)(0 < t < 1)$ in $\mathcal{H}$ is a convex path if
\[
\det \left( \pi^* g_{0\overline{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right)_{(n+1)(n+1)} > 0, \quad \text{in } V \times (I \times S^1).
\]

Definition 2.2. A convex path $\varphi(t)$ in the space of Kähler metrics is called an $\epsilon$-approximate geodesic if the following holds:
\[
\det \left( \pi^* g_{0\overline{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right)_{(n+1)(n+1)} = \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} |\nabla \frac{\partial \varphi}{\partial t}|^2 g(t) \right) \det g(t) = \epsilon \cdot \det g_0
\]
where $g(t)_{\alpha \overline{\beta}} = g_{0\alpha \overline{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta} (1 \leq \alpha, \beta \leq n)$.

Also from [4], we have the following:

Lemma 2.3. [4] (Geodesic approximation lemma): Suppose $\varphi_1(\cdot, s), \varphi_2(\cdot, s) : [0, 1] \to \mathcal{H}$ are two smooth curves in $\mathcal{H}$. For $\epsilon_0$ small enough, there exist two parameters smooth families of curves $\varphi(\cdot, t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, \epsilon_0)(0 \leq t, s \leq 1, \text{ and } 0 < \epsilon \leq \epsilon_0) \to \mathcal{H}$ (from $\varphi_1(\cdot, s)$ to $\varphi_2(\cdot, s)$) such that the following properties hold:

1. For any fixed $s$ and $\epsilon$, $\varphi(\cdot, t, s, \epsilon)$ is an $\epsilon$-approximate geodesic connecting $\varphi_1(\cdot, s)$ and $\varphi_2(\cdot, s)$. More precisely, $\varphi(\cdot, t, s, \epsilon)$ solves the corresponding Monge-Ampere equation
\[
\det \left( \pi^* g_{0\overline{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right) = \epsilon \cdot \det g_0, \quad \text{in } V \times \mathbb{R}; \quad (2.3)
\]

and
\[
\varphi(\cdot, 0, s, \epsilon) = \varphi_1(\cdot, s), \quad \varphi(\cdot, 1, s, \epsilon) = \varphi_2(\cdot, s).
\]

Here $\varphi$ is independent of $\text{Im}(z_{n+1})$.

2. There exists a uniform constant $C$ which depends only on $\varphi_1(\cdot, s), \varphi_2(\cdot, s)$ such that
\[
|\varphi| + \left| \frac{\partial \varphi}{\partial s} \right| + \left| \frac{\partial \varphi}{\partial t} \right| < C; \quad 0 < \left| \frac{\partial^2 \varphi}{\partial t^2} \right| < C, \quad \frac{\partial^2 \varphi}{\partial s^2} < C.
\]

3. For fixed $s$, let $\epsilon \to 0$, the $\epsilon$-approximating geodesic $\varphi(\cdot, t, s, \epsilon)$ converges to the unique geodesic between $\varphi_1(\cdot, s)$ and $\varphi_2(\cdot, s)$ in weak $C^{1,1}$ topology.

4. Define energy element along $\varphi(\cdot, t, s, \epsilon)$ by
\[
E(t, s, \epsilon) = \int_V \left| \frac{\partial \varphi}{\partial t} \right|^2 g(t, s, \epsilon),
\]
where \( g(t, s, \epsilon) \) is the corresponding Kähler metric defined by the Kähler potentials \( \varphi(t, s, \epsilon) \). Then there exists a uniform constant \( C \) such that
\[
\max_{t, s} \left| \frac{\partial E}{\partial t} \right| \leq \epsilon \cdot C.
\]
In other words, both the energy and length element converge to a constant along each convex curve if \( \epsilon \to 0 \).

This is a crucial lemma needed in the proof in the next subsection.

2.3 Length of Jacobi vector field grows super-linearly

In this subsection, we use the same notation as in Lemma 2.3. We want to prove that the Jacobi vector field along any geodesic grows sup-linearly.

**Lemma 2.4.** Let \( \varphi(\cdot, t, s, \epsilon) \) be the two parameter families of approximating geodesics defined as in Lemma 2.3. Let \( Y(\cdot, t, s, \epsilon) = \frac{\partial \varphi}{\partial s} \) be the deformation vector fields and \( X(\cdot, t, s, \epsilon) = \frac{\partial \varphi}{\partial t} \) the tangential vector fields along the approximating geodesic. Then the second derivatives of \( Y \) along the approximating geodesic are positive:
\[
\nabla_X \nabla_X Y \geq 0.
\]

Note that \( Y \) converges to a Jacobi vector field as \( \epsilon \to 0 \). Moreover, we have
\[
\langle Y, \nabla_X Y \rangle \geq \langle Y, Y \rangle.
\]

**Proof.** The equation for a family of \( \epsilon \)-approximate geodesics is:
\[
\left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \nabla \left( \frac{\partial \varphi}{\partial t} \right)^2_{g(t)} \right) \det g(t) = \epsilon \cdot \det g_0, \quad 0 \leq s, t \leq 1.
\]

Denote
\[
X = \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial s}, \quad Y' = \nabla_X Y;
\]
and
\[
H = \frac{\det g_0}{\det g}, \quad f = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \nabla \left( \frac{\partial \varphi}{\partial t} \right)^2_{g(t)} = \nabla_X X.
\]

Then the approximating geodesic equation becomes
\[
f = \nabla_X X = \epsilon \cdot H.
\]

Note that any two-parameter family of smooth functions \( F(s, t) \) can be viewed as a two parameter family of tangent vectors at \( T_{\varphi(\cdot, t, s, \epsilon)} \mathcal{H} \). Then, the Riemannian metric in \( \mathcal{H} \) gives the following covariant derivatives:
\[
\nabla_X F = \nabla_{\frac{\partial}{\partial t}} F(s, t) = \frac{\partial F}{\partial t} - \frac{1}{2} \nabla_{\frac{\partial}{\partial t}} \frac{\partial \varphi}{\partial t} \nabla_{\frac{\partial}{\partial t}} \frac{\partial F(s, t)}{\partial t}.
\]
and
\[ \nabla_Y F = \nabla_{\frac{\partial F}{\partial s}} F(s, t) = \frac{\partial F}{\partial s} - \frac{1}{2} \nabla_s \frac{\partial \varphi}{\partial s} \cdot \nabla_g \frac{\partial F(s, t)}{\partial s}. \]

Clearly, as \( \epsilon \to 0 \), \( Y \) is the Jacobi vector field along the geodesic. By definition, the length of \( Y \) at \( t \) is:
\[ |Y|^2(t, s) = \int_V \left| \frac{\partial \varphi}{\partial s} \right|^2 \det g. \]

Then
\[ \frac{1}{2} \frac{\partial}{\partial t} |Y|^2 = \langle \nabla_X Y, Y \rangle = \langle \nabla_Y X, Y \rangle. \]

Let \( K(X, Y) \) denote the sectional curvature of the space of Kähler metrics at point \( \varphi(\cdot, t, s, \epsilon) \). By a formal calculation (cf. [10], [12] and [8]), we have
\[ K(X, Y) = -|\{X, Y\}_\varphi|^2 \leq 0. \]

Therefore, we have
\[ \frac{1}{2} \frac{\partial^2}{\partial t^2} |Y|^2 = \langle \nabla_Y X, \nabla_X Y \rangle + \langle \nabla_X \nabla_Y X, Y \rangle \]
\[ \geq |Y'|^2 - K(X, Y) + \langle \nabla_Y \nabla_X X, Y \rangle \]
\[ \geq |Y'|^2 + \int_V \epsilon \frac{\partial \varphi}{\partial s} \frac{\partial H}{\partial s} \det g \]
\[ = |Y'|^2 + \int_V \epsilon \frac{\partial \varphi}{\partial s} \left( \frac{\partial H}{\partial s} - \frac{1}{2} \nabla \frac{\partial \varphi}{\partial s} \cdot \nabla H \right) \det g \]
\[ = |Y'|^2 + \frac{1}{2} \int_V \left| \nabla \frac{\partial \varphi}{\partial s} \right|^2 H \cdot \det g \geq |Y'|^2. \]

The last equality holds since
\[ \frac{\partial H}{\partial s} = \frac{\partial}{\partial s} \left( \frac{\det g_0}{\det g} \right) = -\Delta_g \frac{\partial \varphi}{\partial s} \cdot H. \]

and
\[ - \int_Y \frac{\partial \varphi}{\partial s} \Delta_g \frac{\partial \varphi}{\partial s} \cdot H \det g \]
\[ = \frac{1}{2} \int_V \left| \nabla \frac{\partial \varphi}{\partial s} \right|^2 H \cdot \det g + \frac{1}{2} \int_V \frac{\partial \varphi}{\partial s} \nabla \frac{\partial \varphi}{\partial s} \nabla H \det g. \]

It follows that
\[ \frac{\partial^2}{\partial t^2} |Y| \geq 0. \]

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\(^5\) For any two functions \( f_1, f_2 \) and a Kähler form \( \omega_\varphi \), the term \( \{f_1, f_2\}_\varphi \) is defined to be the Possion brake of \( f_1 \) and \( f_2 \) with respect to the sympletic form \( \omega_\varphi \).
In other words, \(|Y(t)|(0 \leq t \leq 1)\) is a convex function of \(t\). Since \(Y(0) = 0\), we have
\[
\frac{\partial}{\partial t}|Y(t)|_{t=1} \geq \frac{|Y(1)|}{1}.
\]
or at time \(t = 1\)
\[
\langle Y, Y' \rangle \geq \langle Y, Y \rangle. \tag{2.4}
\]

2.4 Proof of Theorem 1.1

In this subsection, we want to show that \(\mathcal{H}\) is a non-positive curved space. We follow again the notations in Lemma 2.3 and the preceding subsection.

**Proof.** Consider a special case of Lemma 2.3 when \(\phi_1(\cdot, s) = \phi_1\) is one point on \(\mathcal{H}\) (instead of a curve). We denote this point as \(P\). Let \(Q = \phi_2(\cdot, 0) \in \mathcal{H}\) and \(R = \phi_2(\cdot, 1) \in \mathcal{H}\). Furthermore, we assume that \(\phi_2(\cdot, s)\) (denoted as by \(QR\)) is an \(\epsilon\)-approximate geodesic connecting \(Q\) and \(R\). In other words, it satisfies the following equation:
\[
\nabla_Y Y \cdot g = \left(\frac{\partial^2 \varphi}{\partial t^2}_{ss} - \frac{1}{2} \nabla \frac{\partial \varphi}{\partial t} \frac{\partial g}{\partial s} \right) \det g = \epsilon \cdot \det g_0.
\]

Denote \(Q(s)\) the point \(\phi_2(\cdot, s)\); and denote \(E(s)\) the energy of \(\epsilon\)-approximate geodesic from \(P\) to \(Q(s)\). As \(\epsilon \to 0\), \(E(s) \to\) a constant which, by our normalization, is the square of the geodesic distance from \(P\) to \(Q(s)\). Thus it is enough to work with \(E(s)\). Next
\[
E(s) = \int_0^1 \langle X, X \rangle dt = \int_0^1 \int_V \left(\frac{\partial \varphi}{\partial t}\right)^2 g dt d t
\]
and
\[
E(QR) = \int_0^1 \langle Y, Y \rangle ds = \int_0^1 \int_V \left(\frac{\partial \varphi}{\partial s}\right)^2 g ds d s.
\]
Thus
\[
\frac{1}{2} \frac{dE(s)}{ds} = \int_0^1 \langle \nabla_Y X, X \rangle dt = \int_0^1 \langle X \langle X, Y \rangle - \langle \nabla_X X, Y \rangle \rangle dt
\]
\[
= \langle X, Y \rangle_{t=1} - \int_0^1 \int_V \frac{\partial \varphi}{\partial s} \cdot \epsilon H \det g dt
\]
\[
= \langle X, Y \rangle_{t=1} - \epsilon \cdot \int_0^1 \int_V \frac{\partial \varphi}{\partial s} \det g_0 dt.
\]
Now the second derivatives:
\[
\frac{1}{2} \frac{d^2 E(s)}{ds^2} = \frac{d}{ds} \langle X, Y \rangle_{t=1} - \epsilon \int_0^1 \int_V \frac{\partial^2 \varphi}{\partial s^2} \det g_0 \, dt
\]
\[
\geq \langle Y', Y \rangle_{t=1} + \langle X, \nabla_Y Y \rangle_{t=1} - C \epsilon \int_V \det g_0
\]
\[
\geq \langle Y, Y \rangle_{t=1} + \int_V \frac{\partial \varphi}{\partial t} \epsilon \cdot \det g_0 - C \epsilon \int_V \det g_0
\]
\[
\geq E(QR) - C \cdot \epsilon \cdot vol(V).
\]

Here we have used the inequality (2.6) in the second inequality from the top. And \(E(QR)\) denotes the energy of the path \(\phi_2(\cdot, s)\). For the energy elements of curves, the following inequality holds
\[
E(s) \leq (1 - s)E(0) + sE(1) - s(1 - s)E(QR) - C \cdot \epsilon \cdot vol(V)
\]

Now fix \(s \neq 1\) and let \(\epsilon \to 0\), each energy element of a path approaches the square of the length of that path. Thus the above inequality reduces to:
\[
|PQ(s)|^2 \leq (1 - s)|PQ|^2 + s|PR|^2 - s(1 - s)|QR|^2.
\]

Thus the space of Kähler metrics satisfies the defining inequality for non-positive curved space and hence it is a non-positive space. Here \(|PQ(s)|\) represents the distance from \(P\) to \(Q(s)\); \(|PQ|\) represents the distance from \(P\) to \(Q\); \(|PR|\) represents the distance from \(P\) to \(R\); and \(|QR|\) represents the distance from \(Q\) to \(R\).

Next we prove Theorem 1.3.

Proof. Let \(\varphi_0\) and \(\varphi_1\) be two points in \(H\) with distance \(l > 0\). Suppose that \(\varphi(\cdot, t)(0 \leq t \leq 1)\) is a \(C^{1,1}\) geodesic which connects these two points in \(H\). Let \(\varphi_i(t)(0 \leq t \leq 1)\) be an arbitrary family \((i = 1, 2, \cdots, n \cdots)\) of curves between \(\varphi_0\) and \(\varphi_1\) with length \(l_i \geq l > 0\). Next we assume that this is a distance minimizing sequence of curves. In other words,
\[
\lim_{i \to \infty} l_i = l.
\]

Then, we need to show that \(\varphi_i(\cdot, t)(0 \leq t \leq 1)\) converges to \(\varphi(\cdot, t)(0 \leq t \leq 1)\) in some reasonable topology. For convenience, we assume that every curve involved has been parameterized proportional to the arc-length. Then we only need to show that for each fixed \(s > 0\), we have
\[
\lim_{i \to \infty} d(\varphi_i(\cdot, s), \varphi(\cdot, s)) \to 0.
\]

Actually, using successive subdivision one sees that knowing the inequality (2.7) holds for \(s = \frac{1}{2}\) is enough to prove it for all \(0 < \lambda < 1\), cf. [9].
Since $\mathcal{H}$ is a non-positive curved space, we have (comparing with the Euclidean space):

$$d(\varphi(\cdot, s), \varphi(\cdot, s)) \leq \sqrt{\frac{l_i^2 - l_j^2}{4}} \to 0.$$ 

Theorem 1.3 is then proved. □

3 The gradient flow of the K energy

In this section, we prove Theorem 1.4.

Proof. Let $\varphi(\cdot, t) : [0, 1] \to \mathcal{H}$ be any smooth curve in $\mathcal{H}$. Suppose that $\varphi(\cdot, t, s)$ is the image of this curve under the gradient flow after time $s$. Recall

$$\frac{\partial \varphi}{\partial s} = R(\varphi) - L.$$ 

Denote $g(s, t)$ as the Kähler metric associated with the Kähler potentials $\varphi(s, t)$. Use $\triangle$ to denote the complex Laplacian operator of metric $g(s, t)$.

Following a calculation in [2], we have

$$\frac{\partial R}{\partial t} = -D^* D \frac{\partial \varphi}{\partial t} + \sum_{\alpha=1}^n \left( \frac{\partial \varphi}{\partial t} \right) \varphi^\alpha R_{\alpha}$$

and

$$\frac{\partial}{\partial t} \det g = \Delta \frac{\partial \varphi}{\partial t} \det g.$$

Recall that the energy of the path $\varphi(\cdot, t, s)$ (at time $s$ fixed) is:

$$E(s) = \int_0^1 \int_V \left( \frac{\partial \varphi}{\partial t} \right)^2 \det g \, dt.$$

Under the gradient flow (1.4), we have

$$\frac{dE}{ds} = \int_0^1 \int_V \frac{\partial^2 \varphi}{\partial t^2} \, \det g \, dt + \int_0^1 \int_V \left( \frac{\partial \varphi}{\partial t} \right)^2 \Delta \frac{\partial \varphi}{\partial t} \, \det g \, dt$$

$$= \int_0^1 \int_V \frac{\partial^2 \varphi}{\partial t^2} \, \det g \, dt - \int_0^1 \int_V 2 \frac{\partial \varphi}{\partial t} \left( \varphi^\alpha R_{\alpha} \right) \, \det g \, dt$$

$$= - \int_0^1 \int_V |D \frac{\partial \varphi}{\partial t}|^2 \, \det g \, dt.$$ 

It follows that

$$\frac{dL}{ds} = - \int_0^1 \int_V |D \frac{\partial \varphi}{\partial t}|^2 \, g(s, t) \cdot \sqrt{\int_V \frac{\partial \varphi}{\partial t}^2 \, d g(s, t)} \, dt.$$
where $L(s)$ is the length of the evolved curve at time $s > 0$. From this formula, if the length of a smooth curve is not decreasing, then

$$\int_0^1 \left( \int_V |D \frac{\partial \varphi}{\partial t}|_{\varphi(s,t)}^2 \det g(s,t) \sqrt{\int_V \left| \frac{\partial \varphi}{\partial t} \right|^2 dg(s,t)} \right) dt = 0.$$  

It follows that

$$\int_0^1 \int_V |D \frac{\partial \varphi}{\partial t}|_{\varphi(s,t)}^2 = 0$$

or

$$\left( \frac{\partial \varphi}{\partial t} \right)_{,\alpha\beta} \equiv 0, \quad \forall \alpha, \beta = 1, 2, \cdots n; \quad \forall t \in [0, 1].$$

In other words, the curve $\varphi(t)(0 \leq t \leq 1)$ is either trivial (depending only on $t$) or it represents a family of holomorphic transformation. Theorem 1.4 is then proved.

Next we give a proof of the first part of Theorem 1.5.

Proof. For any $\varphi_0, \varphi_1 \in \mathcal{H}$, consider the space of all smooth curves which connect $\varphi_0$ with $\varphi_1$. We denote it by $\mathcal{L}(\varphi_0, \varphi_1)$. For any curve $c \in \mathcal{L}(\varphi_0, \varphi_1)$, we denote its length by $L(c)$. Then the distance between the two points $\varphi_0$ and $\varphi_1$ can be defined as

$$d(\varphi_0, \varphi_1) = \inf_{c \in \mathcal{L}(\varphi_0, \varphi_1)} L(c).$$

We also define a map in $\mathcal{H}$ via the gradient flow (1.1): for a fixed time $s$, and for any $\varphi \in \mathcal{H}$, we define that the image of $\varphi$ under the map $\pi_s$ is the image of $\varphi$ along the gradient flow after time $s > 0$, provided the gradient flow initiated at $\varphi$ does exist for time $s > 0$. It is clear that for any $\varphi$, the map is defined for small $s > 0$. However, for a fixed $s > 0$, $\pi_s$ is not necessarily defined for all $\varphi \in \mathcal{H}$ since we don’t know the global existence of the gradient flow.

On the other hand, if the gradient flow exists for all the time for any smooth initial metric, then this induces a well defined map from $\mathcal{L}(\varphi_0, \varphi_1)$ to $\mathcal{L}(\pi_s(\varphi_0), \pi_s(\varphi_1))$ for any $s > 0$. Since the length of any smooth curve in $\mathcal{H}$ is decreased under the gradient flow, we have

$$\inf_{c \in \mathcal{L}(\pi_s(\varphi_0), \pi_s(\varphi_1))} L(c) \leq \inf_{c \in \mathcal{L}(\varphi_0, \varphi_1)} L(c), \quad \forall s > 0.$$  

Thus,

$$d(\pi_s(\varphi_0), \pi_s(\varphi_1)) \leq d(\varphi_0, \varphi_1), \quad \forall s > 0.$$
Before proving the second part of Theorem 1.5, we need to use a theorem in [4] where an explicit formula for the first derivatives of the distance function in $\mathcal{H}$ is given. For the convenience of the readers, we will include this theorem here. The second part of Theorem 1.5 is essentially a corollary of this theorem.

**Theorem 3.1.** [4] For any two Kähler potentials $\varphi_0, \varphi_1$, the distance function $d(\varphi_0, \varphi_1)$ is at least $C_1$. More specifically, if $\varphi_0, \varphi_1$ move along two curves $\varphi_0(s)$ and $\varphi_1(s)$ respectively, and if we denote the distance between $\varphi_1(s)$ and $\varphi_2(s)$ is $L(s)$, then

$$\frac{dL(s)}{ds} \bigg|_{s=0} = \langle X, Y \rangle \frac{|X|^2}{2} \bigg|_{t=1} - \langle X, Y \rangle \frac{|X|^2}{2} \bigg|_{t=0},$$

where $X = \frac{\partial \varphi_1}{\partial t} \in T_{\varphi_1(1)} \mathcal{H}$ and $Y_i = \frac{\partial \varphi_i}{\partial s} \in T_{\varphi_i} \mathcal{H}$ ($i = 0, 1$).

Here $\varphi(t)(0 \leq t \leq 1)$ denotes the $C^{1,1}$ geodesic connecting the two metrics $\varphi_0$ and $\varphi_1$; and $X = \frac{\partial \varphi}{\partial t} \in T_{\varphi(t)} \mathcal{H}$.

Now we complete the proof of Theorem 1.5.

**Proof.** If the gradient flow (1.1) exists for all the time, then it is straightforward to show that flow (1.1) decreases the distance between any two points in $\mathcal{H}$ unless they are connected by a holomorphic transformation. Thus, we only deal with the case when the K energy is weakly convex. By definition, for any curve $\varphi(t) \in \mathcal{H}$, the K energy is defined as

$$\frac{dM(\varphi(t))}{dt} = -\int_V \frac{\partial \varphi}{\partial t} (R - \bar{R}) \det g.$$

Along a $C^{1,1}$ geodesic, the second derivative of the K energy is convex in the weak sense that

$$\frac{d^2 M(\varphi(t))}{dt^2} \geq 0.$$

In particular, we have

$$\frac{dM(\varphi(t))}{dt} \bigg|_{t=1} \geq \frac{dM(\varphi(t))}{dt} \bigg|_{t=0}. \quad (3.1)$$

Suppose that $\varphi(t)(0 \leq t \leq 1)$ is the unique $C^{1,1}$ geodesic which connects $\varphi_1$ and $\varphi_2$, and suppose it is parameterized proportional to arc length. If we flow $\varphi_1$ and $\varphi_2$ by the gradient flow (1.1), we have

$$\frac{\partial \varphi_1}{\partial s} = R(\varphi_1(s)) - R \quad \text{and} \quad \frac{\partial \varphi_2}{\partial s} = R(\varphi_2(s)) - R.$$
Plugging this into the corresponding formula in Theorem 3.1, we have

\[
\frac{d L(s)}{ds} = \left\{ \int_V \left( R(\varphi) - R \frac{\partial \varphi}{\partial t} \right) dg(s) \right\}^{\frac{1}{2}} \cdot \left( \int_V \left( R(\varphi) - R \frac{\partial \varphi}{\partial t} \right) dg(s) \right|_{t=1} - \int_V \left( R(\varphi) - R \frac{\partial \varphi}{\partial t} \right) dg(s) \right|_{t=0}
\]

\[
= \left\{ \int_V \left( \frac{\partial^2 \varphi}{\partial t^2} \right) dg(s) \right\}^{-\frac{1}{2}} \cdot \left( \frac{d M}{dt} \right|_{t=1} - \frac{d M}{dt} \right|_{t=0}
\]

\[
= \left\{ \int_V \left( \frac{\partial^2 \varphi}{\partial t^2} \right) dg(s) \right\}^{-\frac{1}{2}} \cdot \int_0^1 \frac{d^2 M}{dt^2} dt \leq 0.
\]

4 Some further corollaries, remarks and the relationship with stability

**Corollary 4.1.** If all geodesics are smooth, then the extremal Kähler metric is unique up to some holomorphic automorphisms.

**Proof.** Suppose that there exist two extremal Kähler metrics in a fixed Kähler class. It was proved in [3] that any extremal Kähler metric must be symmetric with respect to a maximal compact sub-group. Without loss of generality, one may assume that both metrics are symmetric under the same maximal compact sub-group. Then every metric in the geodesic which connects this two extremal metrics must also have the same symmetry group (via Maximum principle). If the scalar curvature is constant, then an argument of Donaldson [8] on the convexity of the K energy implies that the extremal metric must be unique. If the scalar curvature is not a constant, then the gradient vector field of the scalar curvature is a holomorphic vector field and it is unique in each Kähler class once the maximal compact sub-group is fixed. In particular, the gradient flow (1.1) restricted to the two extremal metrics induces the exact same holomorphic transformation. Thus the distance of these two extremal Kähler metrics under the gradient flow is unchanged. Suppose \( \varphi_1, \varphi_2 \) are the two extremal metrics and \( \varphi(\cdot, s) \) is the unique geodesic connecting them. Since the distance of \( \varphi_1 \) and \( \varphi_2 \) is not decreased under the gradient flow, by Theorem 1.3, the path \( \varphi(\cdot, s) \) must either be totally trivial or represent a holomorphic transformations. \( \square \)

**Remark 4.2.** Donaldson in 1997 gave a proof to this corollary in the case of constant scalar curvature metric; and suggested to the second author that a modified proof of his works for general extremal Kähler metrics. In this paper, we presented a new proof.

**Remark 4.3.** For the uniqueness of the extremal Kähler metric, the known results are as follows: 1) in 1950s, the first author showed the uniqueness of Kähler-Einstein metric if \( C_1 \leq 0 \). 2) in 1987, T. Mabuchi and S. Bando [1].
showed the uniqueness of Kähler-Einstein metrics up to holomorphic transformation if the first Chern class is positive. In [4], the 2nd author proved that the constant scalar curvature metric is unique in each Kähler class if \( C_1(V) < 0 \). The problem for the general case is still open. However, the second author [5] had examples of non-uniqueness of some degenerated extremal Kähler metrics in \( S^2 \).

S. T. Yau predicted in [14] that the existence of Kähler-Einstein metrics is related to the stability in the sense of Hilbert Schemes and Geometric invariant theory. His conjecture should be extended to include the case of extremal Kähler metrics. From Theorem 1.3, we observe some kind of link, perhaps still a bit mysterious, between the the existence of extremal metrics and “stability” of the infinite dimensional space \( \mathcal{H} \) in some sense. At least formally, it fits nicely to the general picture Yau’s conjecture describes. The following paragraph is essentially speculative in the effort to explaining this point. If we are willing to put aside the regularity issue, then Theorem 1.3 implies that the gradient flow of the K energy is a distance contracting flow in \( \mathcal{H} \). In this infinite dimensional path length space \( \mathcal{H} \), we choose a large enough ball, which hopefully contains any possible candidates for extremal Kähler metrics. Now flow the entire ball by this gradient flow, if global solution of the gradient flow always exist for all smooth initial metric, then the contracting nature of the flow will shrink the size of the ball. At the end, the ball shall be contracted to a point, and the limit point must be an extremal Kähler metric we are looking for. However, this formal picture is not quite complete. A dichotomy can possibly taken place: As the size of the ball shrinks, the ball may also be drifted away to infinity. In the first possibility when the ball stays in a finite domain, the infinite dimensional manifold is considered ”stable” in some sense and we arrive at the unique extremal Kähler metric in the limit of the flow. In the second case when the ball drifts to the infinity, then the infinite dimensional space is considered as ”unstable” in some sense, and the gradient flow converges to an extremal Kähler metric in a different Kähler manifold.

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