A Note on the Asymptotic Expansion of Matrix Coefficients over $p$-adic Fields

Zahi Hazan

Abstract

In this note, presented as a “community service”, followed by the PhD research of the author, we draw the relation between Casselman’s theorem [Cas93] regarding the asymptotic behavior of matrix coefficients of reductive algebraic groups over $p$-adic fields and its expression as a finite sum of finite functions. In addition, we write the expansion explicitly for general linear groups.

1 Introduction

Let $k$ be a non-Archimedean locally compact field with $\mathcal{O}$, its ring of integers, and $\mathcal{P}$, the maximal ideal in $\mathcal{O}$. Denote a uniformizer of $\mathcal{P}$ by $\varpi$, and the cardinality of the residue field by $q$. Let $G$ be the group of $k$-rational points of a $k$-split reductive algebraic group. Let $(\pi, V)$ be a (complex) smooth, admissible, irreducible representation of $G$. For a parabolic subgroup $P$ of $G$ with Levi decomposition $P = MN$, we denote by $V(N)$ the subspace of $V$ generated by $\{ \pi(n) v - v | n \in N, v \in V \}$. We also denote $V_N = V/V(N)$. This is the space of a smooth representation $(\pi_N, V_N)$ of $M$, called the Jacquet module of $(\pi, V)$ along $N$.

Let $P_{\phi} = M_{\phi} N_{\phi}$ be a minimal parabolic subgroup. Let $T_{\phi} = Z(M_{\phi})$ be the center of $M_{\phi}$. This is the maximal split torus of $G$. Let $\Sigma$ be the set of all roots corresponding to the pair $(G, T_{\phi})$, i.e., the non-trivial eigencharacters of the adjoint action of $T_{\phi}$ on the lie algebra $\mathfrak{g}$ of $G$. Let $\Sigma^+$ be the subset of positive roots determined by $P_{\phi}$, so that $n = \bigoplus_{\alpha \in \Sigma^+} g_{\alpha}$, where $g_{\alpha}$ is the eigenspace of $\alpha$, and $n$ is the lie algebra of $N_{\phi}$. Let $\Delta$ be the basis of $\Sigma^+$, so that every root in $\Sigma^+$ is a sum of roots in $\Delta$.

For each $\Theta \subseteq \Delta$ and $0 < \varepsilon \leq 1$, we define

$$\Theta T_{\phi}^{-} (\varepsilon) = \left\{ a \in T_{\phi} : |\alpha(a)| \leq \varepsilon, \forall \alpha \in \Delta \setminus \Theta \right\}$$
and $T_{\Theta}^-(\varepsilon) = \Theta T_{\phi}^- (\varepsilon) \cap T_{\Theta}$. These are subsets of

$$T_{\phi}^- = \{ a \in T_{\phi} \mid |\alpha(a)| \leq 1, \forall \alpha \in \Delta \}. $$

For any $0 < \varepsilon \leq 1$ one sees that $T_{\phi}^-$ is the disjoint union of the $\Theta T_{\phi}^- (\varepsilon)$ as $\Theta$ ranges over all subsets of $\Delta$. Moreover, by [BP12, Proof of Lemma 2.3.2] we have the following decomposition of $T_{\phi}^-$. 

**Lemma A.** Let $v \in V$. Let $K_v$ be an open subgroup of

$$T_1 = \{ a \in T_{\phi} \| \alpha(a) \| = 1, \forall \alpha \in \Delta \},$$

that stabilizes $v$. For all $0 < \varepsilon < 1$,

$$\Theta T_{\phi}^- (\varepsilon) = \bigcup_{\gamma \in \Gamma_{\Theta}^-} T_{\Theta}^- (\varepsilon) \gamma K_v, \tag{1}$$

where $\Gamma_{\Theta}^- \subseteq T_{\phi}^-$ is a finite set. Hence,

$$T_{\phi}^- = \bigcup_{\Theta \subseteq \Delta} \bigcup_{\gamma \in \Gamma_{\Theta}^-} T_{\Theta}^- (\varepsilon) \gamma K_v.$$

Let $N_{\Theta}^-$ be the unipotent radical opposite to $N_{\Theta}$. We define a canonical non-degenerate pairing of $V_{N_{\Theta}}$ with $\tilde{V}_{N_{\Theta}^-}$ according to the formula

$$\langle u_{\Theta}, \tilde{u}_{\Theta} \rangle_{N_{\Theta}} = \langle v, \tilde{v} \rangle,$$

where $v \in V, \tilde{v} \in \tilde{V}$ are any two canonical lifts of $u_{\Theta}, \tilde{u}_{\Theta}$. We have the following theorem from [Cas93, Corollary 4.3.4].

**Theorem A (Casselman).** Let $v \in V$ and $\tilde{v} \in \tilde{V}$ be given. For $\Theta \subseteq \Delta$, let $u_{\Theta}, \tilde{u}_{\Theta}$ be their images in $V_{N_{\Theta}}$, $\tilde{V}_{N_{\Theta}^-}$. There exists $\varepsilon > 0$ such that for any $\Theta \subseteq \Delta$ and $a \in \Theta T_{\phi}^- (\varepsilon)$ one has

$$\langle \pi(a)v, \tilde{v} \rangle = \langle \pi_{N_{\Theta}}(a)u_{\Theta}, \tilde{u}_{\Theta} \rangle_{N_{\Theta}}.$$ 

For a subgroup $H$ of $G$, we say that a function is an $H$-finite function (or simply finite) if the space spanned by its right $H$-translations is finite dimensional. By Jacquet and Langlands [JL06, Lemma 8.1] we have an explicit basis for all $H$-finite function when $H$ is a locally compact abelian group.

**Theorem B (Jacquet and Langlands).** Let $H$ be a locally compact abelian group of the form

$$H = K \times \mathbb{Z}^r \times \mathbb{R}^n$$

where $K$ is a compact group. For $1 \leq i \leq r + n$ let $\xi_i : (h_0, x_1, \ldots, x_{r+n}) \to x_i$ be the projection map. Then, for any sequence of non-negative integers $p_1, \ldots, p_{r+n}$ and any quasi-character $\chi$ of $H$, the function $\chi \prod_{i=1}^{r+n} \xi_i^{p_i}$ is continuous and finite. These functions form a basis of the space of continuous finite functions on $H$.

Our main goal is to show how Theorem A and Theorem B give an asymptotic expansion for matrix coefficients in terms of a finite linear combination of $T_{\Theta}$-finite functions. In detail,
Theorem 1. Let \( v \in V \) and \( \tilde{v} \in \tilde{V} \). There exists \( \varepsilon > 0 \) such that for any \( \Theta \subseteq \Delta \) and \( a \in gT_\phi^- (\varepsilon) \) there exist finite sets of vectors, that depend on \( \{ \pi, v, \tilde{v} \}, p' = (p'_1, \ldots, p'_r) \in \mathbb{R}^r, \ p = (p_1, \ldots, p_r) \in \mathbb{Z}_{\geq 0}^r, \) and \( \chi = (\chi_1, \ldots, \chi_r) \) where for all \( 1 \leq i \leq r, \chi_i : k^\times \to \mathbb{C}^\times \) is a unitary character, so that

\[
\langle \pi(a) v, \tilde{v} \rangle = \sum_{p', p, \chi} \alpha_{p', p, \chi} \prod_{i=1}^{r} \chi_i(b_i) |b_i|^{p'_i} \log_{|q_i|} |b_i|,
\]

where \( r \) is such that \( T_\Theta^- (\varepsilon) \cong (k^\times)^r \) by the map \( b \mapsto (b_1, \ldots, b_r) \). There exist finite sets of vectors, that depend on \( \{ \pi, u_\Theta, \tilde{u}_\Theta \}, p' = (p'_1, \ldots, p'_r) \in \mathbb{R}^r, \ p = (p_1, \ldots, p_r) \in \mathbb{Z}_{\geq 0}^r, \) and \( \chi = (\chi_1, \ldots, \chi_r) \), where for all \( 1 \leq i \leq r, \chi_i : k^\times \to \mathbb{C}^\times \) is a unitary character, such that for all \( b \in T_\Theta \), one has

\[
\langle \pi_{N_\Theta} (b) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta} = \sum_{p', p, \chi} \alpha_{p', p, \chi} \prod_{i=1}^{r} \chi_i(b_i) |b_i|^{p'_i} \log_{|q_i|} |b_i|,
\]

where \( \alpha_{p', p, \chi} \in \mathbb{C} \) such that \( \alpha_{p', p, \chi} = 0 \) for all but finitely many \( p_i, p, \chi \).

Section 2 is devoted to the proof of the following proposition.

Proposition 1. Let \( \Theta \subseteq \Delta, u_\Theta \in V_{N_\Theta} \) and \( \tilde{u}_\Theta \in \tilde{V}_{N_\Theta} \). Let \( r \) be such that \( T_\Theta \cong (k^\times)^r \) by the map \( b \mapsto (b_1, \ldots, b_r) \). There exist finite sets of vectors, that depend on \( \{ \pi, u_\Theta, \tilde{u}_\Theta \}, p' = (p'_1, \ldots, p'_r) \in \mathbb{R}^r, \ p = (p_1, \ldots, p_r) \in \mathbb{Z}_{\geq 0}^r, \) and \( \chi = (\chi_1, \ldots, \chi_r) \), where for all \( 1 \leq i \leq r, \chi_i : k^\times \to \mathbb{C}^\times \) is a unitary character, so that for all \( b \in T_\Theta \), one has

\[
\langle \pi_{N_\Theta} (b) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta} = \sum_{p', p, \chi} \alpha_{p', p, \chi} \prod_{i=1}^{r} \chi_i(b_i) |b_i|^{p'_i} \log_{|q_i|} |b_i|,
\]

where \( \alpha_{p', p, \chi} \in \mathbb{C} \) such that \( \alpha_{p', p, \chi} = 0 \) for all but finitely many \( p_i, p, \chi \).

The proof of Theorem 1 is followed immediately from Proposition 1 and Lemma A. Indeed,

**Proof of Theorem 1.** Let \( \Theta \subseteq \Delta \) and \( a \in gT_\phi^- (\varepsilon) \). Using the decomposition (1), for any \( 0 < \varepsilon \leq 1 \), there exist \( b \in T_\Theta^- (\varepsilon) \) (corresponding to \( a \) as in Lemma A), \( \gamma \in T_\phi^- \), and \( k_1 \in K_\varepsilon \), such that we can write \( a = b \gamma k_1 \). Hence,

\[
\langle \pi(a) v, \tilde{v} \rangle = \langle \pi(b \gamma k_1) v, \tilde{v} \rangle = \langle \pi(\gamma) v, \tilde{v} \rangle.
\]

Since \( b \in T_\Theta^- (\varepsilon) \) and \( \gamma \in T_\phi^- \), we have \( b \gamma \in T_\Theta^- (\varepsilon) \). Thus, we can apply Theorem A, i.e. there exists \( \varepsilon > 0 \) such that

\[
\langle \pi(\gamma) v, \tilde{v} \rangle = \langle \pi_{N_\Theta} (\gamma) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta}.
\]

We write

\[
\langle \pi_{N_\Theta} (b \gamma) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta} = \langle \pi_{N_\Theta} (b) (\pi_{N_\Theta} (\gamma) u_\Theta), \tilde{u}_\Theta \rangle_{N_\Theta}.
\]

We have \( b \in T_\Theta^- (\varepsilon) \subseteq T_\Theta \), so the result is obtained by applying Proposition 1, where \( u_\Theta \) is replaced with \( \pi_{N_\Theta} (\gamma) u_\Theta \). \( \square \)

We note that Lemma A does not give an explicit expression of \( b \in T_\Theta^- (\varepsilon) \) in terms of its correspondent \( a \in gT_\phi^- (\varepsilon) \). In Section 3 we give a constructed proof of Lemma A for the case \( G = \text{GL}_n \) for any positive integer \( n \). This allows us to write an explicit asymptotic expansion of the matrix coefficient at \( a \) in terms of its coordinates.
2 Proof of Proposition 1

First, we use Theorem B to write an explicit basis for all finite function in our case,

**Corollary 1.** Let \( \Theta \subseteq \Delta \) and \( f : T_\Theta \to \mathbb{R} \) a continuous finite function. Let \( r \) be such that \( T_\Theta \cong (k^\times)^r \) by the map \( b \mapsto (b_1, \ldots, b_r) \). Then, for all \( b \in T_\Theta \), \( f(b) \) is spanned by

\[
\prod_{i=1}^r \chi_i(b_i) |b_i|^{p_i' \log q |b_i|},
\]

where \( (p_1', \ldots, p_r') \in \mathbb{R}^r, (p_1, \ldots, p_r) \in \mathbb{Z}_{\geq 0}^r \), and for all \( 1 \leq i \leq r \), \( \chi_i : k^\times \to \mathbb{C}^\times \) is a unitary character.

**Proof.** We have \( k^\times \cong \mathcal{O}^\times \times \mathbb{Z} \) by the map \( x \mapsto \left( \frac{x}{|x|}, \log q |x| \right) \). A character of \( k^\times \) is of the form \( \chi'(|\cdot|) : |\cdot| \), where \( \chi' \) is a unitary character and \( 0 \neq s \in \mathbb{C} \). We can assume \( s \in \mathbb{R} \) by attaching the imaginary part to \( \chi' \). Let \( (b_1, \ldots, b_r) \in (k^\times)^r \) be the image of \( b \in T_\Theta \). Let \( \chi \) be a quasi character of \( (\mathcal{O}^\times)^r \). Then \( \chi(b) = \prod_{i=1}^r \chi_i(b_i)^{p_i'} |b_i|^{p_i} \), where \( \chi_i \) is unitary and \( p_i' \in \mathbb{R} \) for all \( 1 \leq i \leq r \). Applying Theorem B for \( H = (k^\times)^r \) implies that the space of continuous finite function on \( T_\Theta \) is spanned by \( \chi \prod_{i=1}^r \xi_i^{p_i} \), where \( \chi \) is a quasi character of \( (\mathcal{O}^\times)^r \) and for all \( 1 \leq i \leq r \), \( \xi_i \) is the projection map to each coordinate of \( \mathbb{Z}^r \), i.e. \( \xi_i(b) = \log q |b_i| \). Therefore, this space is spanned by \( \prod_{i=1}^r \chi_i(b_i) |b_i|^{p_i' \log q |b_i|^{p_i}} \). □

In order to deduce Proposition 1 from Theorem A and Corollary 1, it is left to show that for each \( \Theta \subseteq \Delta \), the function \( x \mapsto \langle \pi_{N_\Theta}(x) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta} \) is a \( T_\Theta \)-finite function. Namely,

**Proposition 2.** Let \( u_\Theta \in V_{N_\Theta} \), \( \tilde{u}_\Theta \in \tilde{V}_{N_\Theta} \). There exist finite sets \( \{b_i\}_{1 \leq i \leq \ell} \subseteq T_\Theta \) and \( \{c_i(b)\}_{1 \leq i \leq \ell} \subseteq \mathbb{C} \), such that for all \( b \in T_\Theta \) and \( m \in M_\Theta \) we have

\[
\langle \pi_{N_\Theta}(mb) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta} = \sum_{i=1}^\ell c_i(b) \langle \pi_{N_\Theta}(mb_i) u_\Theta, \tilde{u}_\Theta \rangle_{N_\Theta}.
\]

Before proving Proposition 2 we need the following lemma.

**Lemma 1.** Let \( R \) be a group with center \( Z(R) \cong K \times \mathbb{Z}^r \), where \( K \) is a compact group. Denote the standard basis of \( \mathbb{Z}^r \) by \( \{e_1, \ldots, e_r\} \). Let \( (L, \sigma) \) be a (complex) smooth \( R \)-module of finite length.

(i.) For all finite dimensional spaces \( W \subseteq L \) and all \( 1 \leq j \leq r \) there exists a finite dimensional \( Z(R) \)-invariant space \( W_j \subseteq L \), such that

\[
\sigma(e_j) W \subseteq W + W_j.
\]

(ii.) For all finite dimensional spaces \( W \subseteq L \), there exists a finite dimensional \( Z(R) \)-invariant space \( W' \subseteq L \), such that

\[
\sigma(Z^r) W \subseteq W + W'.
\]

(iii.) Let \( v \in L \). The \( Z(R) \)-module generated by \( v \) is finite dimensional.
Proof. We begin by proving part (i.). It is sufficient to prove this part for a one dimensional space \( W \) as the general case follows directly. Hence, we assume that \( W \) is spanned as a \( R \)-module by \( v \in L \). We prove this part by induction on the length of \( L \). First, assume the length is 1. i.e. \( L \) is irreducible. Then, by Schur’s lemma, \( Z(R) \) acts on \( L \) as a scalar. Thus, the \( Z(R) \)-module generated by \( v \) is of dimension 1 and all the parts of the lemma follow. Now, assume that the assertion is true for modules of length \( d - 1 \). Let \( L \) be a \( R \)-module of length \( d \). That is, there exists a sequence of \( R \)-modules

\[
0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq L_d = L
\]

such that \( L_{i+1}/L_i \) is irreducible for all \( 0 \leq i < d \).

By the fact that \( L_d/L_{d-1} \) is irreducible, and by Schur’s lemma, for all \( 1 \leq j \leq r \) there exists \( \alpha_j \in \mathbb{C} \), such that

\[
\sigma(e_j)(v + L_{d-1}) = \alpha_jv + L_{d-1}.
\]

In particular,

\[
\sigma(e_j)v = \alpha_jv + h_j,
\]

where \( h_j \in L_{d-1} \).

Let \( w = \sum_{i=1}^{\ell} c_i \sigma(g_i)v \in W \), where \( c_i \in \mathbb{C} \) and \( g_i \in R \) for all \( 1 \leq i \leq \ell \). Then, by eq. (4)

\[
\sigma(e_j)w = \sigma(e_j)\left( \sum_{i=1}^{\ell} c_i \sigma(g_i)v \right) = \sum_{i=1}^{\ell} c_i \sigma(g_i)\sigma(e_j)v = \sum_{i=1}^{\ell} c_i \sigma(g_i)(\alpha_jv + h_j).
\]

Denote by \( U_j \) the \( R \)-module spanned by \( h_j \). In this notation, eq. (5) gives

\[
\sigma(e_j)w \in \mathbb{C}w + U_j.
\]

We have \( U_j \subseteq L_{d-1} \). Thus, by the induction hypothesis, there exists a finite dimensional \( Z(R) \)-invariant space \( W'_j \subseteq L_{d-1} \), such that

\[
\sigma(e_j)U_j \subseteq U_j + W'_j.
\]

We take \( W_j = U_j + W'_j \). Thus, \( U_j \subseteq W_j \), so eq. (6) gives \( \sigma(e_j)w \in \mathbb{C}w + W_j \) and by eq. (7) gives

\[
\sigma(e_j)W_j = \sigma(e_j)\left( U_j + W'_j \right) \subseteq W_j + W'_j = W_j.
\]

Next, we prove part (ii.). Let \( 1 \leq j \leq r \). First,

\[
\sigma(0e_j)W = W.
\]

Let \( 0 \neq n \in \mathbb{Z} \). By eq. (3) we have

\[
\sigma(ne_j)W \subseteq \sigma((n - \text{sgn}(n))e_j)(W + W_j)
\]

\[
= \sigma((n - \text{sgn}(n))e_j)W + W_j
\]

\[
= \ldots = \sigma(e_j)(W + W_j) \subseteq \sigma(e_j)W + W_j,
\]

where eq. (9) is due to the \( Z(R) \)-invariant property of \( W_j \), and eq. (10) is followed by repeating eqs. (8) and (9) \( n \) times. Thus, by taking \( W' = \sum_{j=1}^{r} W_j \) the statement readily follows.
In order to deduce part (iii.) we first note that \( \sigma \) is smooth, so for all \( v \in L \), the space \( \text{sp}\{\sigma(x)v \mid x \in K\} \) is of finite dimension. Hence, for a finite dimensional space \( W \subseteq L \), the space

\[
\sigma(K)W = \text{sp}\{\sigma(x)w \mid x \in K, w \in W\}
\]
is also of finite dimension. It follows that \( \sigma(K)Cv \) is finite dimensional. Therefore, by part 2 there exists \( W' \subseteq L \) a finite dimensional \( Z(R) \)-invariant space such that

\[
\text{sp}_C\{\sigma(z)v \mid z \in Z(R)\} = \sigma(Z') (\sigma(K)Cv) \subseteq \sigma(K)Cv + W'.
\]

\( \square \)

We are now ready to prove Proposition 2.

Proof of Proposition 2. The Jacquet module is a smooth \( G \)-module of finite length [Cas93, Theorems 3.3.1 and 6.3.10]. Applying part (iii.) of Lemma 1 for \( R = M_\Theta \) (with \( Z(R) = T_\Theta \)), \( L = V_{N_\Theta}, \sigma = \pi_{N_\Theta}, \) and \( v = u_\Theta \in V_{N_\Theta} \), gives that \( \{\pi_{N_\Theta}(b) u_\Theta \mid b \in T_\Theta\} \) is of finite dimension. Let \( \{\pi_{N_\Theta}(b_1) u_\Theta, \ldots, \pi_{N_\Theta}(b_l) u_\Theta\} \) be its basis. Let \( b \in T_\Theta \) and \( m \in M_\Theta \). Then,

\[
\pi_{N_\Theta}(b) u_\Theta = \sum_{i=1}^{l} c_i(b) \pi_{N_\Theta}(b_i) u_\Theta.
\]

Therefore,

\[
\langle \pi_{N_\Theta}(mb) u_\Theta, \bar{u}_\Theta \rangle_{N_\Theta} = \sum_{i=1}^{l} c_i(b) \langle \pi_{N_\Theta}(mb_i) u_\Theta, \bar{u}_\Theta \rangle_{N_\Theta}.
\]

\( \square \)

3 General linear groups

Let \( n \) be a positive integer. We provide a proof of Lemma A for the \( \text{GL}_n \) case. This will allow us to express \( b \in T_{\Theta}^{-}(\varepsilon) \), corresponding to \( a \in T_{\Phi}^{-}(\varepsilon) \) in the decomposition (1), in terms of the coordinates of \( a \).

Proof of Lemma A (for \( \text{GL}_n \)). Let \( 0 < \varepsilon < 1, \Theta \subseteq \Delta, \) and \( a \in \Theta T_{\Phi}^{-}(\varepsilon) \). There exists a positive integer \( \ell \) such that \( T_{\Phi}^{-} \cong (k^\times)^\ell \). Under this isomorphism we identify \( a \) with \( (a_1, \ldots, a_\ell) \). We denote \( \Delta = \{\alpha_1, \ldots, \alpha_{\ell-1}\} \), where for all \( 1 \leq i \leq \ell - 1, \alpha_i(a) = \frac{a_i}{a_{i+1}}. \)

Denote \( I_\Theta = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, \ell - 1\}, \) such that \( \Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\} \). We write \( \{1, \ldots, \ell - 1\} \backslash I_\Theta = \{i_{m+1}, \ldots, i_{\ell - 1}\} \). Thus,

\[
\begin{cases}
\varepsilon < \frac{a_{i_j}}{a_{i_{j+1}}} \leq 1, & \forall 1 \leq j \leq m, \\
\left|\frac{a_{i_j}}{a_{i_{j+1}}}\right| \leq \varepsilon, & \forall m + 1 \leq j \leq \ell - 1.
\end{cases}
\] (11)

We take \( a' \) identified with \( (a'_1, \ldots, a'_\ell) \) such that for all \( 1 \leq i \leq \ell, \) the following change of variables is satisfied \( a_i = \prod_{j=1}^{i} a'_j \). We have \( \alpha_i(a) = a'_i. \) Therefore, (11) gives

\[
\begin{cases}
\varepsilon < |a'_{i_j}| \leq 1, & \forall 1 \leq j \leq m, \\
|a'_{i_j}| \leq \varepsilon, & \forall m + 1 \leq j \leq \ell - 1.
\end{cases}
\] (12)
We take \( b \) identified with \((b_1, \ldots, b_\ell)\) as follows. For \( 1 \leq i \leq \ell \),

\[
b_i = a'_\ell \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij}.
\]

Note that \( b_\ell = a'_\ell = a_\ell \). Let \( 1 \leq j' \leq m \). Then,

\[
\alpha_{i,j'}(b) = \frac{b_{i,j'}}{b_{i,j'+1}} = \frac{a'_\ell \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij}}{a'_\ell \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij+1}} = 1,
\]

i.e. \( \alpha(b) = 1 \) for all \( \alpha \in \Theta \). Let \( m < j' < \ell \). Then,

\[
\alpha_{i,j'}(b) = \frac{b_{i,j'}}{b_{i,j'+1}} = \frac{a'_\ell \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij}}{a'_\ell \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij+1}} = a'_{i,j'},
\]

i.e. \( |\alpha(b)| \leq \varepsilon \) for all \( \alpha \in \Delta \setminus \Theta \). Thus, \( b \in T^-_{\Theta}(\varepsilon) \).

We take \( c = b^{-1}a \), so \( c_\ell = 1 \) and for all \( 1 \leq i \leq \ell - 1 \),

\[
c_i = \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij} = \prod_{\substack{i \leq j < \ell \\
m < j < \ell}} a'_{ij}.
\]

By Equation (12), for all \( 1 \leq j \leq m \) we have \( \varepsilon < |a'_{ij}| \leq 1 \). Thus, \( c_i \) is bounded for all \( 1 \leq i \leq \ell - 1 \) as a product of bounded elements. For all \( 1 \leq i < \ell \), we write \( c_i = u_i \varpi^r_i \) where \( u_i \in \mathcal{O}^\times \), and \( r_i \) is an integer in a bounded set. Next, we split \( c = c_\varpi c_u \), such that \( c_u \) is identified with \((u_1, \ldots, u_{\ell-1}, 1)\) and \( c_\varpi \) is identified with \((\varpi^{r_1}, \ldots, \varpi^{r_{\ell-1}}, 1)\).

The stabilizer of \( v \) is a congruence subgroup of \( K = \text{GL}_n(\mathcal{O}) \), the maximal compact subgroup of \( G \). Hence, \( K_v \leq K \cap T_1 \cong (\mathcal{O}^\times)^n \), and as an open subgroup, it has a finite index in this compact group. i.e., there exists \( d \) and \( \{x_i\}_{i=1}^d \) such that

\[
K \cap T_1 = \bigcup_{i=1}^d x_i K_v.
\]

Now \( c_u \in K \cap T_1 \). Therefore,

\[
c = c_\varpi c_u \in \bigcup_{i=1}^d c_\varpi x_i K_v \subseteq \bigcup_{\gamma \in \Gamma_{\Theta}^\varepsilon} \gamma K_v,
\]

where \( \Gamma_{\Theta}^\varepsilon \) is a finite set consisting of all the (finitely many) possibilities for \( c_\varpi \) multiplied by all the finitely many representatives \( x_i \). Hence,

\[
a = bc \in T^-_{\Theta}(\varepsilon) \bigcup_{\gamma \in \Gamma_{\Theta}^\varepsilon} \gamma K_v.
\]

\( \square \)
From this proof we conclude that for a given $a \in \Theta_\phi^- (\varepsilon)$ identified with $(a_1, \ldots, a_\ell)$, $\Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$, and $\Delta \setminus \Theta = \{\alpha_{i_{m+1}}, \ldots, \alpha_{i_{\ell-1}}\}$, the corresponding $b \in T_\phi^- (\varepsilon)$ from Lemma A is a block diagonal matrix with scalars blocks where the different scalars are

\[
\left\{ b_{j'} \right\}_{m < j' < \ell} \cup \left\{ a'_\ell \right\} = \left\{ a'_\ell \prod_{i_j \geq i_{j'}} \frac{a'_{i_j}}{m < j < \ell} \right\}_{m < j' < \ell},
\]

where for all $1 \leq i \leq \ell$, $a_i = \prod_{j=i}^\ell a'_j$, or equivalently $a'_\ell = a_{\ell}$ and for all $1 \leq i \leq \ell - 1$, $a'_i = a_{i+1}^{-1} a_i$. Therefore, the different scalars in $b$ are

\[
\left\{ a_\ell \prod_{i_j \geq i_{j'}} \frac{a^{-1}_{i_{j+1}} a_{i_j}}{m < j < \ell} \right\}_{m < j' < \ell} \cup \left\{ a_\ell \right\}.
\]

By the multiplicity of characters and absolute values, and by the additivity of logarithm, we can write (2) explicitly.

**Corollary 2.** Let $G = \text{GL}_n$. In the setting of Theorem 1, and by writing $\Delta \setminus \Theta = \{\alpha_{i_{r+1}}, \ldots, \alpha_{i_{n-1}}\}$, formula (2) takes the following form.

\[
\langle \pi(a)v, \bar{v} \rangle = \sum_{\rho', \rho, \lambda} \alpha_{\rho'} \rho \lambda X_{n-r}(a_n) \left| a_n \right|^{p'_{n-r} \log p_{n-r}} |a_n| \prod_{j=r+1}^{n-1} X_j(a_{ij} |a_{ij}|^{p'_{j-r} \log p_{j-r}} |a_{ij}|).
\]

(13)

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**References**

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