Perfect teleportation and superdense coding through an asymmetric five qubit state

Sreraman Muralidharan
Loyola College, Nungambakkam, Chennai - 600 034, India
Prasanta K. Panigrahi
Indian Institute of Science Education and Research (IISER) Kolkata, Salt Lake, Kolkata - 700106, India and Physical Research Laboratory, Navrangpura, Ahmedabad - 380 009, India

We explicate an example of the so called Task oriented Maximaly Entangled states (TMES) in the context of teleportation and superdense coding. In physical situation, this state can emerge from decoherence of more entangled state or may be prepared for the purpose at hand, with less entanglement resource. We find that a variant of the Brown state can be utilized for perfect teleportation of single and two qubit states. In case of superdense coding, it is observed that five qubits can be transmitted by sending only three qubits.

PACS numbers: 03.67.Hk, 03.65.Ud
Keywords: Entanglement, Teleportation, Superdense coding

I. INTRODUCTION

Entanglement is a completely quantum characteristic which has given rise to a number of counter-intuitive phenomena. The most well known of this is the Einstein-Podolsky-Rosen (EPR) paradox, which arises because of quantum correlations. With the advent of quantum computation, this correlation has come in handy for a number of purposes, not achievable in the classical world. In a remarkable result [1], it has been shown that an unknown single qubit state can be teleported to a desired destination through an EPR state. Entanglement is well understood in the case of two particles. In the multi-particle scenario, much remains to be understood and explored. In the case of three particles, the well known states are the GHZ and the W states. Interestingly, the former can be used for teleportation but the latter is unusable for the same purpose. In the case of four particles, entangled Bell pair and the cluster states have found much applications in Quantum computation. These states have been experimentally realized in laboratory conditions [2]. Higher particle generalizations are of recent origin. Prominent among them is the five particle Brown state [3], which has been numerically shown to be highly entangled. Recently, the utility of this state for teleportation, Quantum state sharing and superdense coding have been illustrated by us [4]. These kind of states for the purpose of specific tasks are called Task oriented Maximally entangled states (TMES) [5].

Superdense coding is intimately related to teleportation, whereas in the case of state-splitting , multiple parties need to cooperate for a given member to recover the state. In reality, entangled states are prone to decoherence. In many scenarios, highly entangled states may not be required for the purpose at hand. Hence, it is worth investigating a less entangled state for physical applications. Below, we consider a variant of the Brown state for usefulness towards teleportation and dense coding. Also, the state is highly robust from the point of view of entanglement. This state is asymmetric in the sense that it does not have the same form for all \((3+2)\) splits. This state has the form :

\[
|\psi_5\rangle = \frac{1}{2} (|000\rangle|\phi_-\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle),
\]

(1)

where, \(|\psi_\pm\rangle = |00\rangle \pm |11\rangle\) and \(|\phi_\pm\rangle = |01\rangle \pm |10\rangle\) are Bell states. \(|\psi_5\rangle\) exhibits genuine multi-partite entanglement according to both negative partial transpose measure, as well as von Neumann entropy measure. The von Neumann entropy between \((1234|5)\) is equal to one and between \((123|45)\) is two. These are the maximum possible entanglement values between the respective subsystem and thus, is highly ‘robust’. Also, the state is maximally mixed, after we trace out 1, or 2, or 3 or 4 qubits which is an indication of genuine multi-particle entanglement for the five-qubit state \(|\psi_5\rangle\). Four-qubit states do not show such characteristic behaviour and fail to attain maximal entropy [6]. Thus, the five-qubit state can provide an edge over the four-qubit state for state transfer and coding. We recall the standard protocol for

*Electronic address: sreraman@loyolacollege.edu
†Electronic address: prasanta@prl.res.in
teleportation for the sake of completeness. Alice and Bob share a maximally entangled state \(|\varphi\rangle_{AB}\), where \(A\) and \(B\) respectively refer to the subsystems of Alice and Bob. Alice wants to teleport \(|\psi\rangle_a\), in her possession, to Bob. Thus, she prepares the combined state,

\[
|\psi\rangle_a|\varphi\rangle_{AB} = \frac{1}{\sqrt{D}} \sum_{x=1}^{D} |\phi_x\rangle_a U_x |\psi\rangle_B. \tag{2}
\]

Here, \(U_x\) are unitary operators on subsystem \(B\) and \(|\phi_x\rangle_a\) are mutually orthogonal states of the joint system. Now, Alice makes a projective measurement on the joint system and classically communicates the result to Bob, who then recovers the state, after applying appropriate unitary operations.

II. A SINGLE QUBIT STATE

In the seminal paper, Bennett et al. [1] considered the teleportation of a single qubit state given by \(a|0\rangle + \beta|1\rangle\) using the Bell state as an entangled channel. In their scheme, Alice first combines the unknown qubit with her state and performs a Bell measurement and communicates the result of her measurement to Bob via two bits of information. Bob, then performs an unitary operation and obtains the unknown qubit. We now demonstrate, the utility of \(|\psi_5\rangle\) for teleportation. Let us first consider the situation in which Alice possesses qubits 1, 2, 3, 4 and particle 5 belongs to Bob. Alice wants to teleport \(a|0\rangle + \beta|1\rangle\) to Bob. So, Alice prepares the combined state,

\[
(a|0\rangle + \beta|1\rangle)|\psi_5\rangle = |\phi_1\rangle_{a_1+} (a|0\rangle + \beta|1\rangle) + |\phi_2\rangle_{a_1-} a|0\rangle - \beta|1\rangle) + |\phi_3\rangle_{a_2+} (\beta|0\rangle + a|1\rangle) + |\phi_4\rangle_{a_2-} (\beta|0\rangle - a|1\rangle), \tag{3}
\]

where, the \(|\phi_x\rangle_{a_1,\pm}\) are mutually orthogonal states of the measurement basis. The states \(|\phi_x\rangle_{a_1,\pm}\) are given as,

\[
|\phi_x\rangle_{a_1\pm} = (-|00001\rangle_A + |00100\rangle_A + |01010\rangle_A + |01110\rangle_A \pm (|10000\rangle_A - |10101\rangle_A + |11000\rangle_A + |11111\rangle_A),
\]

\[
|\phi_x\rangle_{a_2\pm} = (-|00001\rangle_A + |01010\rangle_A + |10110\rangle_A + |11110\rangle_A \pm (|00000\rangle_A - |01011\rangle_A + |10001\rangle_A + |01111\rangle_A).
\]

Alice can now make a five-particle measurement using \(|\phi_x\rangle_{a_1,\pm}\). Bob can apply suitable unitary operations given by \((1, \sigma_1, i\sigma_2, \sigma_3)\) to recover the original state \((a|0\rangle + \beta|1\rangle)\). This completes the teleportation protocol for the teleportation of a single qubit state using the state \(|\psi_5\rangle\).

III. AN ARBITRARY TWO QUBIT STATE

Teleportation of an arbitrary two qubit state was first studied, by Rigolin [2], using two entangled Bell pairs as a quantum channel. Later, a genuinely entangled four qubit state was introduced by Yeo and Chua [8] for the same purpose. In this section, we shall investigate the efficacy of \(|\psi_5\rangle\) for the same purpose. Alice has an arbitrary two qubit state,

\[
|\psi\rangle = a|00\rangle + \mu|10\rangle + \gamma|01\rangle + \beta|11\rangle, \tag{4}
\]

which she has to teleport to Bob. Qubits 1, 2, 3 and 4, 5 respectively, belong to Alice and Bob. Alice prepares the combined state,

\[
|\psi\rangle|\psi_5\rangle = \frac{1}{4} [ |\psi_5\rangle_1(a|01\rangle + \gamma|00\rangle + \mu|11\rangle + \beta|10\rangle) + |\psi_5\rangle_2(a|01\rangle + \gamma|00\rangle + \mu|11\rangle - \beta|10\rangle) + |\psi_5\rangle_3(a|01\rangle - \gamma|00\rangle + \mu|11\rangle - \beta|10\rangle) + |\psi_5\rangle_4(a|01\rangle - \gamma|00\rangle - \mu|11\rangle + \beta|10\rangle) + |\psi_5\rangle_5(a|11\rangle + \gamma|10\rangle + \mu|01\rangle + \beta|00\rangle) + |\psi_5\rangle_6(a|11\rangle - \gamma|10\rangle + \mu|01\rangle - \beta|00\rangle) + |\psi_5\rangle_7(a|11\rangle + \gamma|10\rangle - \mu|01\rangle + \beta|00\rangle) + |\psi_5\rangle_8(a|11\rangle - \gamma|10\rangle - \mu|01\rangle + \beta|00\rangle) + |\psi_5\rangle_9(a|00\rangle + \gamma|01\rangle + \mu|10\rangle + \beta|11\rangle) + |\psi_5\rangle_10(a|00\rangle - \gamma|01\rangle + \mu|10\rangle - \beta|11\rangle) + |\psi_5\rangle_11(a|00\rangle + \gamma|01\rangle - \mu|10\rangle - \beta|11\rangle) + |\psi_5\rangle_12(a|00\rangle - \gamma|01\rangle + \mu|10\rangle - \beta|11\rangle) + |\psi_5\rangle_13(a|10\rangle + \gamma|11\rangle + \mu|00\rangle + \beta|01\rangle) + |\psi_5\rangle_14(a|10\rangle + \gamma|11\rangle + \mu|00\rangle - \beta|01\rangle) + |\psi_5\rangle_15(a|10\rangle + \gamma|11\rangle + \mu|00\rangle - \beta|01\rangle) + |\psi_5\rangle_16(a|10\rangle - \gamma|11\rangle - \mu|00\rangle + \beta|01\rangle)]. \tag{5}
\]
Here, |ψ5⟩i’s forming the mutual orthogonal basis of measurement are given by:

|ψ5⟩1 = 1/2(|φ−⟩010 + |φ+⟩111 + |ψ−⟩000 + |ψ+⟩100); |ψ5⟩2 = 1/2(|φ+⟩010 + |φ−⟩111 + |ψ+⟩000 + |ψ−⟩100); |ψ5⟩3 = 1/2(|ψ−⟩000 + |ψ+⟩100 - |φ+⟩010 - |φ−⟩111); |ψ5⟩4 = 1/2(|ψ−⟩000 + |ψ+⟩100 - |φ+⟩010 - |φ−⟩111); |ψ5⟩5 = 1/2(|ψ+⟩111 - |ψ−⟩010 - |φ+⟩000 + |φ−⟩100); |ψ5⟩6 = 1/2(|ψ−⟩111 - |ψ+⟩010 + |φ+⟩000 - |φ−⟩100); |ψ5⟩7 = 1/2(|ψ−⟩111 - |ψ+⟩010 - |φ+⟩000 + |φ−⟩100); |ψ5⟩8 = 1/2(|ψ+⟩111 - |ψ−⟩010 + |φ+⟩000 - |φ−⟩100); |ψ5⟩9 = 1/2(|ψ+⟩111 + |ψ−⟩010 + |φ−⟩000 + |φ+⟩100); |ψ5⟩10 = 1/2(|ψ+⟩111 + |ψ−⟩010 - |φ−⟩000 - |φ+⟩100); |ψ5⟩11 = 1/2(|ψ−⟩111 + |ψ+⟩010 + |φ−⟩000 + |φ+⟩100); |ψ5⟩12 = 1/2(|ψ−⟩111 + |ψ+⟩010 - |φ−⟩000 - |φ+⟩100); |ψ5⟩13 = 1/2(|ψ−⟩100 - |ψ−⟩000 - |φ−⟩010 + |φ+⟩111); |ψ5⟩14 = 1/2(|ψ−⟩100 - |ψ−⟩000 + |φ−⟩010 - |φ+⟩111); |ψ5⟩15 = 1/2(|ψ−⟩100 - |ψ+⟩000 - |φ+⟩010 + |φ−⟩111); |ψ5⟩16 = 1/2(|ψ+⟩100 - |ψ−⟩000 + |φ−⟩010 - |φ+⟩111).

Alice can make a five-particle measurement and then convey her results to Bob. Bob then retrieves the original state |ψ⟩b by applying any one of the unitary transforms shown in Table I to the respective states. As is evident, each of the above states are obtained with equal probability. This successfully completes the teleportation protocol of a two qubit state using |ψ5⟩.

**TABLE I: Set of Unitary operators needed to obtain |ψ⟩b**

| State | Unitary Operation |
|-------|------------------|
| (α|01⟩ + γ|00⟩ + μ|11⟩ + β|10⟩) | I ⊗ σ1 |
| (α|01⟩ + γ|00⟩ - μ|11⟩ - β|10⟩) | I ⊗ σ2 |
| (α|01⟩ - γ|00⟩ + μ|11⟩ - β|10⟩) | I ⊗ σ3 |
| (α|01⟩ - γ|00⟩ - μ|11⟩ + β|10⟩) | I ⊗ σ4 |
| (α|11⟩ + γ|10⟩ + μ|01⟩ + β|00⟩) | I ⊗ σ5 |
| (α|11⟩ - γ|10⟩ + μ|01⟩ - β|00⟩) | I ⊗ σ6 |
| (α|11⟩ + γ|10⟩ - μ|01⟩ + β|00⟩) | I ⊗ σ7 |
| (α|11⟩ - γ|10⟩ - μ|01⟩ - β|00⟩) | I ⊗ σ8 |
| (α|00⟩ + γ|01⟩ + μ|10⟩ + β|11⟩) | I ⊗ I |
| (α|00⟩ - γ|01⟩ + μ|10⟩ - β|11⟩) | I ⊗ σ9 |
| (α|00⟩ + γ|01⟩ - μ|10⟩ - β|11⟩) | I ⊗ σ10 |
| (α|00⟩ - γ|01⟩ - μ|10⟩ + β|11⟩) | I ⊗ σ11 |
| (α|10⟩ + γ|11⟩ + μ|00⟩ + β|01⟩) | I ⊗ σ12 |
| (α|10⟩ - γ|11⟩ + μ|00⟩ - β|01⟩) | I ⊗ σ13 |
| (α|10⟩ - γ|11⟩ - μ|00⟩ + β|01⟩) | I ⊗ σ14 |
| (α|10⟩ + γ|11⟩ - μ|00⟩ - β|01⟩) | I ⊗ σ15 |

**IV. SUPERDENSE CODING**

We now proceed to show the utility of |ψ5⟩ for superdense coding. Entanglement is quite handy in communicating information efficiently, in a quantum channel. Suppose Alice and Bob share an entangled state, namely |ψ⟩AB then Alice can convert her state into different orthogonal states by applying suitable unitary transforms on her particle. Bob then does appropriate Bell measurements on his qubits to retrieve the encoded information. It is known that two classical bits per qubit can be exchanged by sending information through a Bell state. In this section, we shall discuss the suitability of |ψ5⟩ as a resource for superdense coding. Let us assume that Alice has three qubits, and Bob has last two qubits. Alice can apply the set of unitary transforms on her particle and generate 64 states out of which 32 are mutually orthogonal as shown below:

\[ U_x^3 \otimes I \otimes I \rightarrow |\psi_5⟩_{x_i}. \]
Bob can then perform a five-partite measurement in the basis of $|\psi_5\rangle_x$, and distinguish these states. The appropriate unitary transforms applied and the respective states obtained by Alice are shown in the Table II.

| Unitary Operation | State |
|-------------------|-------|
| $I \otimes I \otimes I$ | $|000\rangle|\phi_+\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle$ |
| $I \otimes \sigma_x \otimes I$ | $|000\rangle|\phi_+\rangle - |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle - |111\rangle|\psi_+\rangle$ |
| $\sigma_x \otimes I \otimes I$ | $|000\rangle|\phi_+\rangle - |010\rangle|\psi_-\rangle - |100\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle$ |
| $\sigma_x \otimes \sigma_x \otimes I$ | $|110\rangle|\phi_+\rangle + |100\rangle|\psi_-\rangle + |010\rangle|\phi_+\rangle + |001\rangle|\psi_+\rangle$ |
| $\sigma_x \otimes \sigma_z \otimes I$ | $|100\rangle|\phi_+\rangle - |110\rangle|\psi_-\rangle + |000\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle$ |
| $\sigma_x \otimes \sigma_z \otimes I$ | $|110\rangle|\phi_+\rangle - |100\rangle|\psi_-\rangle - |010\rangle|\phi_+\rangle - |001\rangle|\psi_+\rangle$ |
| $|\alpha_2 \otimes \alpha_2 \otimes I$ | $|110\rangle|\phi_+\rangle - |100\rangle|\psi_-\rangle - |010\rangle|\phi_+\rangle - |001\rangle|\psi_+\rangle$ |
| $I \otimes \sigma_z \otimes I$ | $|000\rangle|\phi_+\rangle + |000\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |100\rangle|\psi_+\rangle$ |
| $I \otimes i\sigma_z \otimes I$ | $|010\rangle|\phi_+\rangle - |000\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle - |100\rangle|\psi_+\rangle$ |
| $\sigma_z \otimes \sigma_z \otimes I$ | $|100\rangle|\phi_+\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |010\rangle|\psi_+\rangle$ |
| $\sigma_z \otimes i\sigma_z \otimes I$ | $|000\rangle|\phi_+\rangle - |010\rangle|\psi_-\rangle + |000\rangle|\phi_+\rangle + |010\rangle|\psi_+\rangle$ |
| $\sigma_z \otimes \sigma_z \otimes I$ | $|000\rangle|\phi_+\rangle + |010\rangle|\psi_-\rangle + |000\rangle|\phi_+\rangle - |010\rangle|\psi_+\rangle$ |
| $i\sigma_2 \otimes i\sigma_2 \otimes I$ | $|000\rangle|\phi_+\rangle - |010\rangle|\psi_-\rangle - |000\rangle|\phi_+\rangle - |010\rangle|\psi_+\rangle$ |
| $I \otimes \sigma_z \otimes I$ | $|000\rangle|\phi_+\rangle + |000\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |100\rangle|\psi_+\rangle$ |
| $I \otimes i\sigma_z \otimes I$ | $|010\rangle|\phi_+\rangle - |000\rangle|\psi_-\rangle - |100\rangle|\phi_+\rangle - |100\rangle|\psi_+\rangle$ |
| $\sigma_z \otimes \sigma_z \otimes I$ | $|100\rangle|\phi_+\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |010\rangle|\psi_+\rangle$ |
| $\sigma_z \otimes i\sigma_z \otimes I$ | $|000\rangle|\phi_+\rangle - |010\rangle|\psi_-\rangle + |000\rangle|\phi_+\rangle + |010\rangle|\psi_+\rangle$ |
| $\sigma_z \otimes \sigma_z \otimes I$ | $|000\rangle|\phi_+\rangle + |010\rangle|\psi_-\rangle + |000\rangle|\phi_+\rangle - |010\rangle|\psi_+\rangle$ |
| $i\sigma_2 \otimes i\sigma_2 \otimes I$ | $|000\rangle|\phi_+\rangle - |010\rangle|\psi_-\rangle - |000\rangle|\phi_+\rangle - |010\rangle|\psi_+\rangle$ |

The capacity of superdense coding is defined as,

$$X(\rho^{AB}) = \log_2 d_A + S(\rho^B) - S(\rho^{AB}),$$

where $d_A$ is the dimension of Alice’s system, $S(\rho)$ is von-Neumann entropy. For the state $|\psi_5\rangle$, $X(\rho^{AB}) = 3 + 2 - 0 = 5$. The Holevo bound of a multipartite quantum state gives the maximum amount of classical information that can be encoded. It is equal to five, for the five-qubit state $(log_2 N)$. Thus, the super dense coding reaches the "Holevo bound" allowing five classical bits to be transmitted through three quantum bits.

V. CONCLUSION

We have illustrated an example of a TMES for teleportation of an unknown single and two qubit states. This state is also a very useful resource for superdense coding. The superdense coding capacity for the state reaches Holevo bound of five classical bits. This also gives a picture, about what kind of lesser entangled states could be useful for teleportation and superdense coding. The study of the decoherence properties of this state and the Brown state also needs careful investigation in case of any practical application. The usefulness of $|\psi_5\rangle$, and the Brown state for many other applications like the Quantum error correction and one way quantum computing needs extensive investigation.
The physical realization of these states in laboratory conditions is yet another challenge.

[1] C. H. Bennett et al., Phys. Rev. Lett. 70, 1895 (1993).
[2] C. Y. Lu et al., Nature 3, 91 (2007).
[3] Iain D. K. Brown et al., J. Phys. A: Math. Gen. 38, 1119 (2005).
[4] S. Muralidharan and P. K. Panigrahi, To appear in Phys. Rev. A, eprint quant-ph/0708.3785v3.
[5] P. Agrawal and B. Pradhan, eprint quant-ph/0707.4295v2.
[6] A. Higuchi and A. Sudbery, Phys. Lett. A 273, 213 (2000).
[7] G. Rigolin, Phys. Rev. A 71, 032303 (2005).
[8] Y. Yeo and W. K. Chua, Phys. Rev. Lett. 96, 060502 (2006).
[9] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[10] D. Bruß et al., Phys. Rev. Lett. 93, 210501 (2004).