Spin Hall effect in infinitely large and finite-size diffusive Rashba two-dimensional electron systems: A helicity-basis nonequilibrium Green’s function approach

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A nonequilibrium Green’s function approach is employed to investigate the spin-Hall effect in diffusive two-dimensional electron systems with Rashba spin-orbit interaction. Considering a long-range electron-impurity scattering potential in the self-consistent Born approximation, we find that the spin-Hall effect arises from two distinct interband polarizations in helicity basis: a disorder-unrelated polarization directly induced by the electric field and a polarization mediated by electron-impurity scattering. The disorder-unrelated polarization is associated with all electron states below the Fermi surface and produces the original intrinsic spin-Hall current, while the disorder-mediated polarization emerges with contribution from the electron states near the Fermi surface and gives rise to an additional contribution to the spin-Hall current. Within the diffusive regime, the total spin-Hall conductivity vanishes in infinitely large samples, independently of temperature, of the spin-orbit coupling constant, of the impurity density, and of the specific form of the electron-impurity scattering potential. However, in a finite-size Rashba two-dimensional semiconductor, the spin-Hall conductivity no longer always vanishes. Depending on the sample size in the micrometer range, it can be positive, zero or negative with a maximum absolute value reaching as large as $e/8\pi$ order of magnitude at low temperatures. As the sample size increases, the total spin-Hall conductivity oscillates with a decreasing amplitude. We also discuss the temperature dependence of the spin-Hall conductivity for different sample sizes.

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I. INTRODUCTION

The proposed spin-Hall effect, namely, the appearance of a spin current along the direction perpendicular to the driving electric field, has attracted much recent theoretical and experimental attention. In early studies, it was shown that an extrinsic spin-Hall effect may arise from a spin-orbit (SO) interaction of electrons induced by electron-impurity scattering potential. Recently, a scattering-independent intrinsic spin-Hall effect, which entirely originates from a spin-orbit coupling in the free-carrier system itself, has been predicted respectively in $p$-type bulk semiconductors and $n$-type two-dimensional (2D) systems with Rashba and Dresselhaus SO interaction. The experimental observations of the spin-Hall effect have been also reported in a $n$-type bulk semiconductor and in a two-dimensional heavy-hole system.

In two-dimensional electron systems with Rashba SO interaction, ignoring the effect of disorders, Sinova et al. showed that the spin-Hall conductivity, $\sigma_{SH}$, has a universal intrinsic value $e/8\pi$ at zero temperature. Subsequently, a great deal of research work was focused on the influence of disorders on this intrinsic spin-Hall effect. When Rashba two-dimensional electron systems are sufficiently dirty and the Anderson localization is dominant, Sheng et al. found that the spin-Hall conductivity can be much greater or smaller than the universal value $e/8\pi$. In the diffusive regime, it was demonstrated that the collisional broadening in the density of states of electrons leads to a reduction of the spin-Hall current and the $e/8\pi$ value of $\sigma_{SH}$ can be reached only for relatively weak electron-impurity scattering. However, further investigation revealed that the electron-impurity scattering can also produce an additional contribution to spin-Hall current, which is independent of the impurity density and has a sign opposite to the original one. As a result, the total spin-Hall current is completely suppressed for a short-range electron-impurity scattering potential. This conclusion has been confirmed by different methods, such as Kubo formula, Keldysh formalism and spin-density method. Also, it made clear that this cancellation of spin-Hall current is not due to any symmetry.

However, in most previous studies, the vanishing of the spin-Hall current was found only for a short-range electron-impurity scattering. It is well known that, in realistic 2D semiconductor systems, the dominant electron-impurity collisions are long-ranged. It is interesting to see whether the spin-Hall current survives in the case of long-range electron-impurity scattering. In Ref., the authors argued that the total spin-Hall current should be nonvanishing in the case of long-range electron-impurity collisions. However, Raimondi and Schwab again got a vanishing spin-Hall conductivity considering a weakly momentum-dependent potential: this potential depends on the cosine of the angle between the initial and the final momenta, but is independent of their magnitudes.

In this paper, we carefully investigate the spin-Hall current in 2D electron systems with Rashba SO coupling by means of a nonequilibrium Green’s function approach. We consider a quite general form of the electron-impurity
scattering potential: it depends not only on the directions but also on the magnitudes of the electron momenta. Such a potential can be used to describe the realistic Coulomb interaction between electrons and impurities in 2D semiconductors. Besides, in contrast to all of the previous discussions concerning electron behaviors in the spin basis, our formalism is presented in the helicity basis. Such a treatment allows us to interpret the origin of the spin-Hall effect in terms of interband polarization processes. We clarify that the spin-Hall current arises from two mechanisms: disorder-unrelated and disorder-mediated mechanisms, which correspond to two distinct helicity-basis interband polarizations. The disorder-unrelated mechanism is associated with a polarization directly induced by dc electric field, which results in the original intrinsic spin-Hall current with contribution from all electron states in the Fermi sea. The disorder-mediated mechanism relates to a polarization mediated by electron-impurity scattering and is associated mainly with the electron states in the vicinity of the Fermi surface. We find that in infinitely large 2D Rashba semiconductors, the total spin-Hall current vanishes, independently of the specific form of the electron-impurity scattering potential, of the impurity density, of the SO coupling constant, and of temperature.

However, we also make clear that the "always vanishing" of the spin-Hall current occurs only for infinitely large samples. Care must be taken in regard with this conclusion when the sample size reduces. The discretization of the energy levels in finite-size system may lead to a nonvanishing total spin-Hall current in finite Rashba 2D semiconductors even in the quasiclassical regime. Numerical calculation for square shape Rashba 2D electron systems of size in micrometer regime, indicates that depending on the system size, the total spin-Hall conductivity can be positive, zero or negative, with a maximum absolute value reaching up to the order of magnitude of $e/8\pi$ at low temperatures. As a function of the sample size, $\sigma_{xH}$ oscillates around zero with a decreasing amplitude when increasing sample size. Such a size effect can be observable only at low temperatures. When temperature increases that $T$ becomes comparable with the finite-size induced energy separation of the electron states at the Fermi surface, $\sigma_{xH}$ oscillation disappears and the spin-Hall conductivity returns to a small nonvanishing value before it slowly approaches zero with further increasing sample size.

This paper is organized as follows. In Sec. II, we present a general formalism for the nonequilibrium Green’s functions. Based on this, the mechanisms of the spin-Hall effect are clarified. In Sec. III, we give analytical and numerical calculations of the spin-Hall conductivity in both the infinitely large and finite-size Rashba 2D semiconductors. Finally, we review our results in Sec. IV.

**II. GENERAL FORMALISM**

**A. Kinetic equation for lesser Green’s function**

We consider a quasi-2D electron semiconductor in $x-y$ plane, subjected to a Rashba SO interaction. The single-particle noninteracting Hamiltonian of the system can be written as

$$\hat{h} = \frac{\mathbf{p}^2}{2m} + \alpha \mathbf{p} \cdot (\mathbf{n} \times \hat{\sigma}),$$

where $\alpha$ is the Rashba SO coupling constant, $\hat{\sigma} \equiv (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the Pauli matrices, $m$ is the electron effective mass, $\mathbf{p} \equiv (p_x, p_y) \equiv (p \cos \phi_p, p \sin \phi_p)$ is the 2D electron momentum, and $\mathbf{n}$ is the unit vector perpendicular to the 2D electron plane. This Hamiltonian can be diagonalized, resulting in two eigenvalues $\varepsilon_\mu(p) = \frac{p^2}{2m} + (-1)^\mu \alpha p$ and eigen wave functions $\varphi_\mu(p) = u_\mu(p)e^{i\mathbf{n} \cdot \mathbf{r}}$ with

$$u_\mu(p) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ (-1)^{\mu+1}i e^{i\phi_p} \end{array} \right),$$

and $\mu = 1, 2$. It is useful to introduce a unitary transformation $U_p = [u_1(p), u_2(p)]$, by which the basis of the system is transformed from a spin basis to a helicity basis.

In 2D systems, the electrons experience scattering by impurities. The previous studies were concerned only with a short-range interaction between electrons and impurities, corresponding to a potential independent of electron momentum. However, in realistic 2D semiconductors, such as heterojunctions and quantum wells etc, the potential of the Coulomb interaction between the electrons and impurities essentially is long-ranged. In this paper, we assume that the electron-impurity scattering can be described by an isotropic potential $V(\mathbf{p} - \mathbf{k})$, which corresponds to scattering an electron from momentum state $\mathbf{p}$ to state $\mathbf{k}$. This potential depends not only on the angle $\phi_p - \phi_k$ but also on the magnitudes of the momenta $\mathbf{p}$ and $\mathbf{k}$. The latter dependence of the potential has been ignored in Refs. 11, 12, 13, 14, 15, 17. In helicity basis, the scattering potential takes a transformed form, $\hat{T}(\mathbf{p}, \mathbf{k}) = \hat{U}^+(\mathbf{p})\hat{V}(\|\mathbf{p} - \mathbf{k}\|)\hat{U}(\mathbf{k})$.

We are interested in the spin-Hall current driven by a dc electric field $\mathbf{E}$ along the $x$ axis. In Coulomb gauge, this electric field can be described by a scalar potential $V = -e\mathbf{E} \cdot \mathbf{r}$, with $\mathbf{r}$ as the electron coordinate. In Rashba 2D
electron systems driven by the electric field $E$, the only nonvanishing component of the spin-Hall current is just the spin-Hall current polarized along the $z$-direction and flow along the $y$-axis, $j_y^z$. Its single-particle operator, defined in the spin basis as $j_y^z = (\hat{j}_y^z \hat{\sigma}_z + \hat{\sigma}_x \hat{j}_y^z)/4e$ with the electric current operator $\hat{j}_y^z$ reduces to an off-diagonal matrix in the helicity basis. Taking a statistical ensemble average, the net spin-Hall current can be determined in the helicity basis via

$$J_y^z = -i \sum_{\mathbf{p}} \frac{p_y}{2m} \int \frac{d\omega}{2\pi} \left[ \hat{G}_{12}^z(\mathbf{p},\omega) + \hat{G}_{21}^z(\mathbf{p},\omega) \right] = \sum_{\mathbf{p}} \frac{p_y}{m} \int \frac{d\omega}{2\pi} \text{Im} \hat{G}_{12}^z(\mathbf{p},\omega),$$

(3)

with $\hat{G}_{12}^z(\mathbf{p},\omega)$ as the helicity-basis nonequilibrium lesser Green’s function. We see that the spin-Hall effect in Rashba 2D electron systems arises only from interband polarization processes. The contribution to $j_y^z$ from the diagonal elements of $\hat{G}_{12}^z(\mathbf{p},\omega)$ vanishes because $U^+(\mathbf{p})\sigma_z U(\mathbf{p})$ is an off-diagonal matrix.

In order to investigate the spin-Hall effect, it is necessary to study the nonequilibrium lesser Green’s function $\hat{G}_{12}^z$. For brevity, hereafter, we employ a subscript $\mathbf{p}$ to denote the arguments of the Green’s functions and self-energies, $(\mathbf{p},\omega)$. In Rashba 2D electron systems with short-range disorders, the kinetic equation for Keldysh function, which simply relates to the lesser Green’s function, has already been constructed in the spin basis.$^{25}$ However, from Eq. (3), we see that it is most convenient to study $\hat{G}_{12}^z$ in the helicity basis. In this basis, the noninteracting retarded and advanced Green’s functions $\hat{G}_{12}^{r,a}$ are diagonal, $\hat{G}_{12}^{r,a} = \text{diag} (\omega - \varepsilon_1(\mathbf{p}) \pm i\delta)^{-1}$, $(\omega - \varepsilon_2(\mathbf{p}) \pm i\delta)^{-1}$, as well as the interacting unperturbed ones $\hat{G}_{12}^{r,a} = [1 + \hat{\Sigma}^{r,a}_{12}]^{-1} \hat{G}_{12}^{r,a}$ with diagonal unperturbed self-energies $\hat{\Sigma}^{r,a}_{12}$.

Under steady and homogeneous conditions, the kinetic equation for the helicity-basis lesser Green’s function $\hat{G}_{12}^z$ reads

$$ie \cdot \mathbf{E} \cdot \nabla_p \hat{G}_{12}^z + \frac{i\nabla_p \phi_p}{2} \hat{G}_{12}^z + \alpha \hat{\Sigma}^{r,a}_{12} \hat{G}_{12}^z = \hat{\Sigma}^{r,a}_{12} \hat{G}_{12}^z - \hat{G}_{12}^z \hat{\Sigma}^{r,a}_{12} - \hat{\Sigma}^{r,a}_{12} \hat{G}_{12}^z + \hat{\Sigma}^{r,a}_{12} \hat{G}_{12}^z,$$

(4)

where, $\hat{G}_{12}^{r,a}$ and $\hat{\Sigma}^{r,a}$, respectively, are the nonequilibrium Green’s functions and self-energies. Eq. (3) is derived from the kinetic equation in the spin basis by application of the local unitary transformation $U(\mathbf{p})$. At the same time, in this, only the lowest order of gradient expansion is taken into account.$^{19}$

In present paper, we consider the electron-impurity scattering in the self-consistent Born approximation. It is widely accepted that this is sufficiently accurate to analyze the transport properties in diffusive regime. Accordingly, the self-energies take the forms

$$\hat{\Sigma}^{r,a}_{12} = n_i \sum_{\mathbf{k}} T(\mathbf{p}, \mathbf{k}) \hat{G}_{12}^{r,a}_{\mathbf{k}} \hat{T}^+(\mathbf{p}, \mathbf{k}),$$

(5)

with impurity density $n_i$. Substituting explicit form of the matrix $U(\mathbf{p})$ into Eq. (5), we get

$$\hat{\Sigma}^{r,a}_{12} = \frac{1}{2} n_i \sum_{\mathbf{k}} |V(\mathbf{p} - \mathbf{k})|^2 \left\{ a_1 \hat{G}_{12}^{r,a}_{\mathbf{k}} + a_2 \hat{\sigma}_x \hat{G}_{12}^{r,a}_{\mathbf{k}} \hat{\sigma}_x + a_3 \hat{\sigma}_y \hat{G}_{12}^{r,a}_{\mathbf{k}} \hat{\sigma}_y \right\},$$

(6)

Here $a_i (i = 1, 2, 3)$ are the factors associated with the directions of momenta, $a_1 = 1 + \cos(\phi_p - \phi_k)$, $a_2 = 1 - \cos(\phi_p - \phi_k)$, $a_3 = \sin(\phi_p - \phi_k)$.

Further, our considerations are restricted to the linear response regime. In connection with this, all the functions, such as the nonequilibrium Green’s functions and self-energies, can be expressed as sums of two terms: $A = A_0 + A_1$, with $A$ as the Green’s functions or self-energies. $A_0$ and $A_1$, respectively, are the unperturbed part and the linear electric field part of $A$. In this way, the kinetic equation for $\hat{G}_{12}^z_{1p}$ can be written as

$$- \alpha \hat{C}_{1p} + ie \mathbf{E} \cdot \nabla_p \hat{G}_{12}^z_{1p} - \frac{1}{2} e \mathbf{E} \cdot \nabla_p \phi_p \hat{D}_{0p} = \hat{\Sigma}^{r,a}_{1p} \hat{G}_{12}^z_{1p} - \hat{G}_{12}^z_{1p} \hat{\Sigma}^{r,a}_{1p} - \hat{\Sigma}^{r,a}_{1p} \hat{G}_{12}^z_{1p} + \hat{\Sigma}^{r,a}_{1p} \hat{G}_{12}^z_{1p},$$

(7)

where the matrices $\hat{C}_{1p}$ and $\hat{D}_{0p}$, respectively, are

$$\hat{C}_{1p} = \begin{pmatrix} 0 & -2(\hat{G}_{12}^z)_{12} \\ 2(\hat{G}_{12}^z)_{21} & 0 \end{pmatrix},$$

(8)

and

$$\hat{D}_{0p} = \begin{pmatrix} 0 & (\hat{G}_{12}^z)_{11} - (\hat{G}_{12}^z)_{22} \\ (\hat{G}_{12}^z)_{21} - (\hat{G}_{12}^z)_{12} & 0 \end{pmatrix},$$

(9)
and \( \hat{G}_{0p}^{r} \) is the unperturbed lesser Green’s function, \( \hat{G}_{0p}^{r} = -2i n_{F}(\omega) \text{Im} \hat{G}_{0p}^{r} \), with \( n_{F}(\omega) \) as the Fermi function. Here, to derive Eq. (4), we have employed the vanishing of the contributions to spin-Hall current from \( \hat{G}_{r1p}^{r,a} \) and \( \hat{G}_{00p}^{r,a} \) involved in the terms on the right-hand side of Eq. (4), which can be easily demonstrated by considering Eqs. (21) and (3), as well as Eq. (21) presented below.

It is obvious that the driving force in Eq. (7) comprises two components: \( ie \mathbf{E} \cdot \nabla_{p} \hat{G}_{0p}^{r} \) and \( -e \mathbf{E} \cdot \nabla_{p} \hat{G}_{0p}^{r} / 2 \). Due to the linearity of Eq. (7) that its solution can be assumed to be a sum of two terms \( (\hat{G}_{1p}^{r})^{I} \) and \( (\hat{G}_{1p}^{r})^{II} \), which, respectively, obey the following equations,

\[
- \alpha p \hat{G}_{1p}^{I} + i e \mathbf{E} \cdot \nabla_{p} \hat{G}_{0p}^{r} = \hat{\Sigma}_{op}^{r}(\hat{G}_{1p}^{r})^{I} - (\hat{G}_{1p}^{r})^{I} \hat{\Sigma}_{op}^{a} - \hat{G}_{0p}^{r}(\hat{\Sigma}_{1p}^{a})^{I} + (\hat{\Sigma}_{1p}^{a})^{I} \hat{G}_{0p}^{r},
\]

\[
- \alpha p \hat{G}_{1p}^{II} - \frac{1}{2} e \mathbf{E} \cdot \nabla_{p} \hat{G}_{0p}^{r} / 2 = \hat{\Sigma}_{op}^{r}(\hat{G}_{1p}^{r})^{II} - (\hat{G}_{1p}^{r})^{II} \hat{\Sigma}_{op}^{a} - \hat{G}_{0p}^{r}(\hat{\Sigma}_{1p}^{a})^{II} + (\hat{\Sigma}_{1p}^{a})^{II} \hat{G}_{0p}^{r}.
\]

Here, \( (\hat{\Sigma}_{1p}^{r})^{I} \) and \( (\hat{\Sigma}_{1p}^{r})^{II} \) are the corresponding self-energies, corresponding to the Green’s functions \( (\hat{G}_{1p}^{r})^{I} \) and \( (\hat{G}_{1p}^{r})^{II} \), respectively.

### B. disorder-unrelated mechanism of the spin-Hall effect

The solution of Eq. (11) is off-diagonal and can be derived analytically,

\[
(\hat{G}_{1p}^{r})_{12}^{II} = (\hat{G}_{1p}^{r})_{21}^{II} = \frac{ieE}{2\alpha p} \sin \phi_{p} n_{F}(\omega) \text{Im}[(\hat{G}_{0p}^{r})_{11} - (\hat{G}_{0p}^{r})_{22}].
\]

Substituting Eq. (12) into Eq. (8), we obtain the contribution from \( (\hat{G}_{1p}^{r})^{II} \) to the spin-Hall conductivity:

\[
\sigma_{sH}^{\mu} = \frac{-e}{4m_{\alpha}} \sum_{p} \frac{p_{\mu}^{2}}{p^{2}} [f_{1}(p) - f_{2}(p)],
\]

with \( (\mu = 1, 2) \)

\[
f_{\mu}(p) = -2 \int \frac{d\omega}{2\pi} n_{F}(\omega) \text{Im}[(\hat{G}_{0p}^{r})_{\mu\mu}]
\]

as the unperturbed distribution function. \( \sigma_{sH}^{\mu} \) arises from the off-diagonal driving force, \( -e \mathbf{E} \cdot \nabla_{p} \hat{G}_{0p}^{r} / 2 \), which is associated with the electric dipole matrix

\[
- \int d\mathbf{r} \varphi_{\mu}^{*}(\mathbf{p}) e \mathbf{E} \cdot \mathbf{r} \varphi_{\nu}(\mathbf{p}') = -ieE \frac{\partial}{\partial \mathbf{p}} \int d\mathbf{r} \varphi_{\mu}^{*}(\mathbf{p}) \varphi_{\nu}(\mathbf{p}') + ieE \frac{\partial}{\partial \mathbf{p}} u_{\nu}^{*}(\mathbf{p}) u_{\mu}(\mathbf{p}') \int d\mathbf{r} e^{i(p'-p) \cdot \mathbf{r}}.
\]

To the first order of the electric field, the off-diagonal elements of the electric dipole moment \( (i.e. \text{ the second term on the right hand side of Eq. (15)}) \) describes a polarization process between different bands, directly induced by the dc electric field. Hence, in essence, \( \sigma_{sH}^{\mu} \) originates from this polarization process and becomes disorder-unrelated, although it may depend on the scattering through the collisional broadening in \( \hat{G}_{0p}^{r} \).

We note that the polarization process directly induced by the dc electric field is not restricted to the electron states near the Fermi surface: it comes from all electron states in the Fermi sea. As a result, \( \sigma_{sH}^{\mu} \) depends on the distribution function \( f_{\mu}(p) \) itself, rather than its derivative.

### C. Disorder-mediated mechanism of the spin-Hall effect

The off-diagonal element of solution of Eq. (10), \( (\hat{G}_{1p}^{r})_{12}^{II} \), can be formally expressed as

\[
(\hat{G}_{1p}^{r})_{12}^{II} = \frac{1}{2\alpha p} \hat{I}_{12},
\]

where \( \hat{I} \) is the term on the right hand side of Eq. (10):

\[
\hat{I}_{\mu\nu} = 2i \text{Im}[(\hat{\Sigma}_{0p}^{r})_{\mu\nu} (\hat{G}_{1p}^{r})_{\mu\nu}^{I} - \text{Im}(\hat{G}_{0p}^{r})_{\mu\nu} (\hat{\Sigma}_{1p}^{a})_{\mu\nu}^{I}],
\]
even when considering a collisional broadening in \( \hat{G} \), that function, \( f \), the contribution to spin-Hall current from \( \hat{G}_{1p}^< \), \( J_y^{<f} \), is given by

\[
J_y^{<f} = \frac{1}{2m\alpha} \sum_{p,k} \int \frac{d\omega}{2\pi} \frac{p_y}{2m\alpha p} \left\{ \text{Re}(\hat{G}_{1p}^<) \left[ \text{Im}(\Sigma_{0p}^<)_{12} + \text{Im}(\Sigma_{0p}^>)_{22} \right] + \text{Im}(\hat{G}_{0p}^<) \left[ \text{Re}(\Sigma_{0p}^<)_{12} + \text{Re}(\Sigma_{0p}^>)_{22} \right] - (\Sigma \leftrightarrow \hat{G}) \right\}.
\]

By inserting the explicit forms of the self-energies, it can be further simplified as

\[
J_y^{<f} = \frac{1}{2m\alpha} \sum_{p,k} \int \frac{d\omega}{2\pi} |V(p - k)|^2 \cos \phi_k \sin(\phi_p - \phi_k) \text{Re}(\hat{G}_{0k}^<) \left[ (\hat{G}_{0p}^<)_{12} + (\hat{G}_{0p}^>)_{22} \right] \left[ \text{Im}(\Sigma_{0p}^<)_{12} + \text{Im}(\Sigma_{0p}^>)_{22} \right].
\]

Here, we have considered the vanishing of the real parts of the diagonal elements of \( \hat{G}_{1p}^< \) according to its definition. Also we have used the relation

\[
\sum_k |V(p - k)|^2 \sin(\phi_p - \phi_k) [G_{0k}^<]_{22} = 0,
\]

which is derived from angular independence of \( \hat{G}_{0p}^< \). Combining the terms proportional to \( \cos \phi_p \cos(\phi_p - \phi_k) \) in the second line of Eq. (20) with a similar term in the first line, we obtain

\[
J_y^{<f} = \frac{1}{4m\alpha} \sum_p \cos \phi_p \left\{ \text{Im}(\hat{G}_{0p}^<)_{12} + \text{Im}(\hat{G}_{0p}^>)_{22} \right\} \left[ \text{Im}(\Sigma_{0p}^<)_{12} + \text{Im}(\Sigma_{0p}^>)_{22} \right].
\]

Further, we note that in the self-consistent Born approximation, there is a vanishing quantity

\[
\mathcal{K} = \frac{1}{4m\alpha} \sum_p \cos \phi_p \left\{ \text{Im}(\hat{G}_{0p}^<)_{12} + \text{Im}(\hat{G}_{0p}^>)_{22} \right\} \left[ \text{Im}(\Sigma_{0p}^<)_{12} + \text{Im}(\Sigma_{0p}^>)_{22} \right] - (\Sigma \leftrightarrow \hat{G}).
\]

The fact of \( \mathcal{K} = 0 \) can be shown by inserting the explicit forms of the self-energies, Eq. (16), into the right hand side of Eq. (20) and using Eq. (21). Adding \( \mathcal{K} \) to the right hand side of Eq. (22), we find

\[
J_y^{<f} = \frac{1}{4m\alpha} \sum_p \cos \phi_p (-1)^{\mu+1} \left\{ \text{Im}(\hat{G}_{0p}^<)_{12} \text{Im}(\hat{G}_{0p}^>)_{22} - (\Sigma \leftrightarrow \hat{G}) \right\}.
\]

Considering the diagonal parts of Eq. (18), we finally obtain

\[
\sigma_{sH}^< \equiv \frac{J_y^{<f}}{E} = \sum_p \frac{e p_x}{4m\alpha \partial p_x} \left[ f_1(p) - f_2(p) \right].
\]

Although \( \sigma_{sH}^< \) looks independent of the impurity density, this spin-Hall conductivity arises essentially from a disorder-mediated interband polarization. The longitudinal transport of electrons driven by a dc electric field leads to diagonal elements of the nonequilibrium distribution function \( \hat{G}_{0p}^< \), proportional to \( n_\gamma^{-1} \). Also, these electrons participating in transport are scattered by impurities to give rise to an interband polarization, which becomes independent of the impurity density in the diffusive regime. It is evident that the disorder plays an intermediate role during such a polarization process.

When ignoring the collisional broadening in \( \hat{G}_{0p}^< \), distribution function \( f_\mu(p) \) becomes the conventional Fermi function, \( f_\mu(p) \to n_F[\varepsilon_\mu(p)] \). The appearance of its derivative \( \partial f_\mu(p)/\partial p_x \), rather than \( f_\mu(p) \) itself, in Eq. (26) implies that \( \sigma_{sH}^< \) mainly relates to the electron states near the Fermi surface. Note that this point still remains reasonable even when considering a collisional broadening in \( \hat{G}_{0p}^< \) which is much smaller than the Fermi energy.
III. RESULTS AND DISCUSSIONS

A. Vanishing spin-Hall current in infinitely large Rashba 2D semiconductors

We first consider the spin-Hall effect in an infinitely large Rashba 2D semiconductor. In this case, the electron momentum is continuous and the summation over the electron momentum in Eq. (25) can be replaced by a momentum integral. Performing this momentum integral by parts, we find \( \sigma_{sH}^I = -\sigma_{sH}^{II} \), i.e., the total spin-Hall conductivity vanishes. Thus, we have analytically proven the vanishing of the total spin-Hall current in infinitely large Rashba 2D electron systems within the diffusive regime. Obviously, this elimination of the spin-Hall current occurs quite generally: it is independent of the specific form of scattering potential \( V(p) \), of the impurity density, of the SO coupling constant \( \alpha \), and of temperature \( T \).

Ignoring the collisional broadening, \( \sigma_{sH}^{II} \) becomes independent of any scattering and takes the form

\[
\sigma_{sH}^{II} = \frac{-e}{16\pi m\alpha} \int_0^\infty dp \left( n_F [\varepsilon_1(p) - \mu_c] - n_F [\varepsilon_2(p) - \mu_c] \right),
\]

with the chemical potential \( \mu_c \). At zero temperature, \( \sigma_{sH}^{II} \) is equal to \( e/8\pi \), in agreement with the previous studies.\(^9,11,13\) It is noted that in the case of short-range scattering, there exists a simple relationship between our result and the conclusion in the studies by means of Kubo formula.\(^11,12,13\) the \( \sigma_{sH}^I \) and \( \sigma_{sH}^{II} \) in our treatment, respectively, correspond to the bubble diagram and its vertex correction in the Kubo formalism.

B. Spin-Hall effect in finite-size Rashba 2D semiconductors

In the proof presented above, the summation over momentum was replaced by a momentum integral. This is accurate only for samples with large sizes. When the sample size is reduced to be comparable with \( 2\pi/k_F \) \( [k_F \equiv (k_{1F} + k_{2F})/2 \) and \( k_{\mu_F} \) is the Fermi momentum of electrons in spin-orbit coupled (helicity) band \( \mu \), the discretization of the electron momentum can not be ignored. For definiteness, in present paper, we consider a square Rashba 2D electron system with length \( L \) in both \( x \) and \( y \) directions. The possible values of the electron momentum are

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FIG. 1: Sample-size dependence of the total spin-Hall conductivity \( \sigma_{sH} \) in Rashba two-dimensional GaAs-based semiconductors at different temperatures: (a) \( T = 0.5 \) K, (b) \( T = 0.8 \) K, and (c) \( T = 1 \) K. The electron density and broadening parameter are \( n_e = 5 \times 10^{10} \) cm\(^{-2} \) and \( \gamma_0 = 0.1 \) meV. The SO coupling constant is \( \alpha = 1 \) meV·nm.
FIG. 2: Temperature dependence of the total spin-Hall conductivity in 2D semiconductors of different sample sizes, \( L = 1.47, 1.5, 1.52, \) and 1.54 \( \mu \text{m} \). Other parameters are the same as in Fig. 1.

FIG. 3: Sample-size dependence of the total spin-Hall conductivity in Rashba two-dimensional GaAs-based semiconductors at temperature \( T = 1 \text{ K} \). Other parameters are the same as in Fig. 1.

\( p_x = 2\pi n_x/L \) and \( p_y = 2\pi n_y/L \) with integers \( n_x \) and \( n_y \). Since the disorder-mediated spin-Hall current is associated mainly with the states near the Fermi surface while the disorder-unrelated one is related to all electron states in the Fermi sea, the discretization of the electron momentum has an effect on \( \sigma_{sH}^I \) stronger than that on \( \sigma_{sH}^{II} \). As a result, the total spin-Hall conductivity may not be always vanishing in finite size samples.

In considering the effect of an energy (or momentum) discretization, the collisional broadening of the retarded Green’s function should be taken into account. For this, we assume that the imaginary part of the electron self-energy \( \Sigma_{0p} \) can be described by a constant parameter \( \gamma_0 \): \( \text{Im}(\Sigma_{0p})_{\mu\mu} = \gamma_0 \). In this way, the distribution function \( f(p) \) takes a form, \( f(p) = \text{Im}[\Psi(1/2 + C_p)]/\pi + 1/2 \), with the Digamma function \( \Psi(x) \) and \( C_p = [\gamma_0 - i(\epsilon_\mu - \mu_c)]/2\pi T \). [From Eqs. (13) and (25), we can see that the total \( \sigma_{sH} \) also vanishes in infinitely large samples considering such a collisional broadening.] In present paper, we restrict our discussion on the SHE in finite size samples within the quasiclassical regime. In this regime, the sample size \( L \) is still much larger than \( 2\pi/k_F \) that a large number of electron states are contained inside the Fermi surface (in the case \( L < 2\pi/k_F \), the quantum size effect is important and the motion of electrons will no longer be quasiclassical). Also, we restrict our discussion to the diffusive regime: the impurities should be enough dense that the electron mean free path \( l = v_F \tau \) \( (v_F = k_F/m) \) is the average Fermi velocity and \( \tau \) is the scattering time) is much less than \( L \). Otherwise, the electron motion will become ballistic. Under these considerations, all the derivations in Sec. II, as well as Eqs. (13) and (25), remain valid for 2D systems of finite size.

Within the quasiclassical and diffusive regime, we have performed a numerical study on the spin-Hall conductivity in a finite GaAs-based heterojunction with a Rashba spin-orbit interaction. In calculation, the SO coupling constant is chosen to be \( \alpha = 1 \text{ meV} \cdot \text{nm} \). The electron density is \( n_e = 5 \times 10^{10} \text{ cm}^{-2} \), which indicates a Fermi wavevector \( k_F \approx 5.6 \times 10^7 \text{ m}^{-1} \), a Fermi velocity \( v_F \approx 0.95 \times 10^5 \text{ m/s} \), and a Fermi energy \( \epsilon_F \approx 1.8 \text{ meV} \) (the electron effective
mass of GaAs is \( m = 0.068 \, m_e \) with the free electron mass \( m_e \)). The impurity density is assumed to give rise to a broadening parameter \( \gamma_0 = 0.1 \, \text{meV} \), indicating a scattering \( \tau \approx 3.3 \times 10^{-12} \, \text{s} \), a mean free path \( l \approx 0.3 \, \mu\text{m} \), and an electron mobility of order of \( 10 \, \text{m}^2/\text{Vs} \). We consider only the samples with sizes \( L \geq 1 \, \mu\text{m} \), which are much larger than \( 2\pi/k_F \) and the mean free path \( l \). The numerical results obtained from Eqs. (18) and (20) are plotted in Figs. 1, 2 and 3.

In Fig. 1, the total spin-Hall conductivity \( \sigma_{sH} = \sigma_{sH}^I + \sigma_{sH}^{II} \), is shown as a function of the sample size at three different temperatures \( T = 0.5, 0.8 \) and 1.0 K. We see that sensitively depending on the size of micrometer samples, the spin-Hall conductivity can be positive, zero and negative. At a given temperature, \( \sigma_{sH} \) actually oscillates with a decreasing amplitude when increasing the sample size from \( L = 1 \, \mu\text{m} \). The period of the oscillation is approximately equal to \( 2\pi/k_F \). Besides, there actually exists another period, \( 2\pi/k_{Fm} \) with \( k_{Fm} = (k_F - k_2)/2 \). This period is quite large for the chosen parameter \( \alpha \) and its effect on \( \sigma_{sH} \) becomes almost unobservable. When temperature rises, the oscillation amplitude decreases. Note that at low temperature, the maximum value of the amplitude can be as large as \( e/8\pi \).

In Fig. 2, we plot the temperature dependence of the spin-Hall conductivity at several different sample sizes, \( L = 1.47, 1.5, 1.52, \) and 1.54 \( \mu\text{m} \) (the spin-Hall conductivities for samples of \( L = 1.47 \, \mu\text{m} \) and \( L = 1.54 \, \mu\text{m} \), respectively, correspond to a peak and a trough in Fig. 1). At high temperature, \( \sigma_{sH} \) approaches a small (nonzero) constant value. When temperature goes down from 1 K, \( \sigma_{sH} \) of different \( L \) spreads and approaches different values between \(-e/8\pi \) and \( 1.4e/8\pi \). As a matter of fact, the finite-size effect of \( \sigma_{sH} \) originates from the rapid change of the electron distribution around the Fermi surface, which is relevant to three energy scales: (i) the finite-size induced energy separation of the electron states around the Fermi surface, \( \Delta_F = 2\pi v_F/L \), which is about 0.27 meV for \( L = 1.5 \, \mu\text{m} \), (ii) the collisional broadening of the energy level described by the parameter \( \gamma_0 \approx 0.1 \, \text{meV} \), and (iii) the temperature \( T \), which leads to a smearing of the distribution function. Note that the finite size effect on \( \sigma_{sH} \) remains nonvanishing when \( \gamma_0 \) and \( T \) are smaller than \( \Delta_F \). When temperature \( T \) increases from zero to \( \gamma_0 \), the collisional broadening dominates the smearing of the electron distribution and the total spin-Hall conductivity exhibits a plateau due to the temperature independence of \( \gamma_0 \). As \( T \) further increases to the range of \( T > \gamma_0 \), the temperature smearing dominates and \( |\sigma_{sH}| \) shrinks with increasing \( T \). When temperature becomes larger than \( \Delta_F, T > \Delta_F \), the effect of the energy level separation is washed out and \( \sigma_{sH} \) approaches a small nonzero constant.

Note that the strong sample-size and temperature dependencies of \( \sigma_{sH} \) discussed above, come almost entirely from the change of the disorder-mediated spin-Hall conductivity, \( \sigma_{sH}^I \), with variation of \( L \) and \( T \). Besides, there exists another finite size effect arising mainly from the change of the disorder-unrelated spin-Hall conductivity, \( \sigma_{sH}^{II} \), with sample size. It becomes important in the larger \( L \) scale, because in this case the finite size effect on \( \sigma_{sH}^{II} \) is washed out. In Fig. 3, we plot the total spin-Hall conductivity of finite Rashba two-dimensional GaAs-based semiconductors having size from \( L = 2 \, \mu\text{m} \) to \( L = 20 \, \mu\text{m} \) at temperature \( T = 1 \, \text{K} \). We see that, though at this temperature the \( \sigma_{sH} \) oscillation disappears (Fig. 1c) when \( L > 2 \, \mu\text{m} \), the spin-Hall conductivity remains to have a small finite value. Only when the sample size increases to \( L \geq 20 \, \mu\text{m} \), can \( \sigma_{sH} \) close to zero, the result of an infinitely large sample.

### IV. CONCLUSIONS

Employing a helicity-basis nonequilibrium Green’s function approach, we have investigated the spin-Hall effect in both the infinitely large and finite-size Rashba two-dimensional electron systems. A long-range electron-impurity scattering has been considered in the self-consistent Born approximation. We found that the spin-Hall effect originates from two different mechanisms in helicity basis: disorder-unrelated and disorder-mediated mechanisms. The disorder-unrelated mechanism corresponds to a polarization process directly induced by dc electric field and is associated with all electron states in the Fermi sea, while the disorder-mediated one is the result of a polarization relating to the nonequilibrium electrons participating in longitudinal transport. In infinitely large diffusive Rashba 2D semiconductors, the total spin-Hall current vanishes, independently of the temperature, of the impurity density, of the specific form of the isotropic scattering potential, and of the spin-orbit coupling constant. However, when the sample size reduces, the spin-Hall conductivity no longer always vanishes. Depending on the sample size in the micrometer regime, the total \( \sigma_{sH} \) can be positive, zero or negative, with a maximum absolute value reaching up to the order of magnitude of \( e/8\pi \) at low temperatures. Such a size effect shows up only at low temperatures. When temperature increases that \( T \) becomes comparable with the finite-size induced energy separation of the electron states at the Fermi surface, the \( \sigma_{sH} \) oscillations disappear and spin-Hall conductivity takes a small finite value before slowly approaching zero with further increasing sample size to \( L \geq 20 \, \mu\text{m} \).

The present study indicates that a nonvanishing spin-Hall conductivity may be obtained in a 2D Rashba electron systems of micrometer size, notwithstanding its disappearance in infinitely large samples. For it to appear, one has to accurately control the shape and size of the sample. In addition, the mobility of the sample should be high and the temperature should be low that both the collisional broadening of the electron energy level and the temperature
smearing are smaller than the finite-size induced energy separation of the electron states around the Fermi surface.

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Note added.—After our work was completed and submitted, Adagideli and Bauer also reported the vanishing of spin-Hall current in the presence of long-range scattering.\footnote{I. Adagideli and G. E. W. Bauer, Phys. Rev. Lett. 95, 256602 (2005).}

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