The Double Complex of a Blow-up

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Abstract

We compute the double complex of smooth complex-valued differential forms on projective bundles over and blow-ups of compact complex manifolds up to a suitable notion of quasi-isomorphism. This simultaneously yields formulas for 'all' cohomologies naturally associated with this complex (in particular, de-Rham, Dolbeault, Bott-Chern and Aeppli).

Given a compact complex manifold $X$ and a submanifold $Z \subseteq X$, the blow-up $\tilde{X}$ of $X$ along $Z$ is a new complex manifold, roughly obtained from $X$ by replacing $Z$ with the space of all directions into $Z$, i.e. the projectivized normal bundle. A natural task is to express cohomological invariants of $\tilde{X}$ in terms of those of $X$ and $Z$.

For the de-Rham cohomology, this task has been solved in [5, p. 605 f.], using the Thom-isomorphism. In [12, 7.3.3] the Dolbeault-cohomology (the Hodge Structure on the de-Rham cohomology) of $\tilde{X}$ is computed explicitly in the case when $X$ is Kähler.

Recently, there has been a lot of activity on extending these results to cohomologies other than de-Rham in the non-Kähler case: For the Dolbeault-cohomology, this has been done in [1] under the assumption that $Z$ admits an open neighborhood with a holomorphic retract to $Z$, using a Thom-isomorphism for Dolbeault-cohomology by T. Suwa. Independently, in [10] and [14], the dimensions of Dolbeault- and Bott-Chern cohomology were computed by sheaf-theoretic methods without additional assumptions on $Z$ or $X$ and without using a Thom-isomorphism. Some questions were left open: For example, the isomorphism was not made explicit and a formula for Bott-Chern cohomology of projective bundles was conjectured. Building on the results in [14], the isomorphism for Dolbeault cohomology was made explicit in [8]. Furthermore, the Morse-Novikov cohomology of $\tilde{X}$ was computed in [9].

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\footnote{In addition, I was not able to understand some details in the proofs. It seems to me that in [10 lemma 3.10] higher direct image sheaves are not treated and I did not understand an identification in the proof of [14 property 3.4].}
The goal of this short note is to give another proof of an explicit formula for the Dolbeault-cohomology of a blow-up. The main new ingredient, when compared to [10] and [14], is a result from [6] and the consideration of Leray-spectral sequences. In addition, we avoid passing to cohomology and stay on the level of double complexes as long as possible and indicate how this also yields the exactly analogous formulas for Bott-Chern and Aeppli cohomology. For this last step only we use a result which will be proved elsewhere.

We refer to the articles [1], [10] [14] for several beautiful applications of such formulas and some follow-up questions.

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1 Preliminaries

It will be convenient to abstract the algebraic properties of the double complex of \(\mathbb{C}\)-valued smooth differential forms on a complex manifold.

We will be considering bounded double complexes over the complex numbers with real structure, i.e. quadruples

\[(A^{\bullet, \bullet}, \partial_1, \partial_2, \sigma)\]

consisting of

- A (not necessarily finite dimensional) \(\mathbb{Z}^2\)-graded \(\mathbb{C}\)-vector space \(A^{\bullet, \bullet}\), s.t. \(A^{p,q} = 0\) for almost all \((p, q) \in \mathbb{Z}^2\).
- Two \(\mathbb{C}\)-linear maps \(\partial_1\) and \(\partial_2\) of degrees \((1, 0)\) and \((0, 1)\) which satisfy
  \[
  \partial_1^2 = \partial_2^2 = 0 \quad \text{and} \quad \partial_1 \partial_2 + \partial_2 \partial_1 = 0.
  \]
- A conjugation-antilinear involution \(\sigma\) on \(A^{\bullet, \bullet}\), satisfying \(\sigma A^{p,q} = A^{q,p}\) and \(\sigma \partial_1 \sigma = \partial_2\).

To ease notation and language, in the following, we will say double complex instead of bounded double complex over the complex numbers with real structure and write \(A\) instead of \((A^{\bullet, \bullet}, \partial_1, \partial_2, \sigma)\). By a map of double complexes, we mean a \(\mathbb{C}\)-linear map of the underlying vector spaces, compatible with the bigrading, differentials and real structure.

**Example 1.** Let \(X\) be a connected compact complex manifold of dimension \(n\).

- The double complex \(A_X = (A^n_X, \partial, \bar{\partial}, \sigma)\) of \(\mathbb{C}\)-valued smooth differential forms on \(X\).
• The double complex $D^{top}A_X$, consisting in degree $(p,q)$ of the (topological) dual of $A_X^{n-p,n-q}$ (i.e. ‘currents’), with differentials

$$\partial_{D^{top}A_X}^p := (\varphi \mapsto (-1)^{p+q+1} \varphi \circ \partial^{n-p-1,n-q})$$

(and similarly $\overline{\partial}_{D^{top}A_X}$).

Associated with any double complex $A$ are several cohomologies:

- de-Rham (or ‘total’) cohomology $H^{dR}_k(A) := H^k \left( \bigoplus_{p+q=k} A^{p,q}, \partial_1 + \partial_2 \right)$
- Dolbeault (or ‘column’) cohomology: $H^{p,q}_\partial(A) := H^q(A^p, \partial_2)$
- Bott-Chern cohomology: $H^{p,q}_{BC}(A) := \left( \ker \partial_1 \circ \partial_2 \cap \ker \partial_1 + \ker \partial_2 \right)^p \cap \ker \partial_1$ (im $\partial_2$ + im $\partial_2$)
- Aeppli: $H^{p,q}_A(A) := \left( \ker \partial_1 \cap \ker \partial_2 \right)^p \cap \ker \partial_1$ (im $\partial_2$ + im $\partial_2$)

There is the ‘Frölicher spectral sequence’, converging from Dolbeault to de-Rham cohomology:

$$FS : E_1^{p,q} = H^{p,q}_\partial(A) \Rightarrow H^{p+q}_{dR}(A)$$

We call a morphism of double complexes $E_1$-quasi-isomorphism, if it induces an isomorphism in Dolbeault cohomology. It is well-known that it then automatically induces an isomorphism on all later pages of the Frölicher spectral sequence and on the de-Rham cohomology. A new observation seems to be the following result:

**Lemma 2.** Any $E_1$-quasi-isomorphism induces an isomorphism in Bott-Chern and Aeppli cohomology.

This follows from structure theory of double complexes over a field. A proof will be given in the forthcoming article [11].

**Example 3.** For any connected compact complex manifold, the map

$$\Phi : A_X \rightarrow D^{top}A_X$$

$$\omega \mapsto \int_X \omega \wedge -$$

is an $E_1$-quasi-isomorphism by Serre duality.

## 2 Theorems and Proofs

The following is essentially proved (in slightly different language) in [10]. We repeat it here for convenience of the reader and to indicate the necessary changes to work on the double complex level.

**Proposition 4.** Let $\tilde{\pi} : E \rightarrow X$ be a complex vector bundle of rank $n$ over a compact complex manifold, $\pi : \mathbb{P}(E) \rightarrow X$ the associated projective bundle. There is a double complex $K$ and a commutative diagram

$$\begin{align*}
\xymatrix{ A_X \ar[r]^{\pi^*} & \ar[d] \mathbb{P}(E) \ar[l]_{\otimes A^{p-1}} \\
A_X & K,}
\end{align*}$$

(⋆)
such that the horizontal maps are $E_1$-quasi-isomorphisms and the others are injective in Dolbeaut-cohomology.

Proof. Let
\[ T := \{(e, p) \in E \times \mathbb{P}(E) \mid e \in p \} \subseteq \pi^*E \]
denote the tautological bundle on $\mathbb{P}(E)$. For any fibre $F_x := \pi^{-1}(x) \cong \mathbb{P}^{n-1}$, there is an identification $T|_{F_x} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

Choose some hermitian metric $g$ on $T$ and let $\theta \in \mathcal{A}_{\mathbb{P}(E)}^2$ be the curvature of the Chern connection defined by $g$, s.t.
\[ c_1(T) = \left[ \frac{1}{2\pi i} \theta \right] \in H^2(\mathbb{P}(E)). \]

It is known that $\theta$ is a closed $(1,1)$-form and because $0 \neq c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = c_1(T)|_{F_x}$, it is not trivial and not exact. Denote $\theta_x := \theta|_{F_x}$ and by $\mathcal{A}(\theta)$, resp. $\mathcal{A}(\theta_x)$ the finite dimensional (as $\mathbb{C}$-vector space) subcomplexes of $\mathcal{A}_{\mathbb{P}(E)}$, resp. $\mathcal{A}_{F_x}$ with basis $\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$ resp. $\{1, \theta_x, \theta_x^2, \ldots, \theta_x^{n-1}\}$. With this, we can define $K := \mathcal{A}_X \otimes \mathcal{A}(\theta)$. The bigraded Dolbeaut cohomology algebra of $\mathbb{P}^{n-1}$ is given by $H(\mathbb{P}^{n-1}) = \mathbb{C}[t]/(t^{n-1})$ with $t = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$. In particular, restriction and projection to cohomology (all forms in $\mathcal{A}(\theta_x)$ are closed) yield isomorphisms of differential bigraded bidifferential algebras $\mathcal{A}(\theta) \cong \mathcal{A}(\theta_x) \cong H(\mathbb{P}^{n-1})$ and the inclusion
\[ \mathcal{A}(\theta_x) \hookrightarrow \mathcal{A}_{F_x} \]
is an isomorphism on the first page of the Frölicher spectral sequence. Thus, we can define the left hand map in $(\ast)$ as the composite
\[ \mathcal{A}_X \otimes \mathcal{A}(\theta) \longrightarrow \mathcal{A}_X \otimes \mathcal{A}(\theta_x) \longrightarrow \mathcal{A}_X \otimes \mathcal{A}_{\mathbb{P}^{n-1}}, \]
where the maps are the identity on the first factor and restriction and inclusion on the second factor.

The right hand map in $(\ast)$ is given by $\pi^*$ on the first factor and the inclusion on the second. It is an isomorphism on the first page of the Frölicher spectral sequence by the Hirsch Lemma for Dolbeaut cohomology. The left diagonal and the vertical maps are inclusions to the first factor of the tensor product and commutativity is clear by definition.

Remark 5. Since the maps are defined on the level of complexes, this result allows in particular the computation not only of the Dolbeaut, but also of the Bott-Chern and Aeppli cohomologies of a projective bundle, thereby confirming a formula conjectured in [7].

A natural next goal is to compute the double complex of blow-ups up to $E_1$-quasi-isomorphism. Here is a general computation yielding a partial answer:

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2Proven in [8] lem. 18], as a consequence of a spectral sequence introduced by Borel in the appendix to [2].

Beware of the following potentially confusing notation in [3]. In their terminology, they assume $H^{\ast, \ast}(F)$ to be a free algebra with transgressive basis. It suffices however, to have a graded vector space basis for $H^{\ast, \ast}(F)$ which consists of transgressive elements as their proof does not make use of the internal multiplicative structure of $H^{\ast, \ast}(F)$. It seems this is also what is meant, as otherwise in their corollary 19 the hypotheses of the Hirsch Lemma do not seem to be satisfied.
Definition 6. For a map $p : X \to Y$ between connected compact complex manifolds the pushforward $p_*$ is, with the notation of section \ref{sec:top}, the composite

$$A_X \xrightarrow{\Phi} D^{\text{top}} A_X \xrightarrow{D^p_*} D^{\text{top}} A_Y.$$ 

Lemma 7. For a surjective holomorphic map $f : Y \to X$ of connected compact complex manifolds of the same dimension, the map

$$A_Y (f_*, \text{pr}) \xrightarrow{D^{\text{top}} A_X \oplus A_Y / f^* A_X}$$

is an $E_1$-quasi-isomorphism.

Proof. Since $f$ is a finite covering with $\deg(f) > 0$ sheets when restricted to appropriate dense open subsets of $X$ and $Y$\footnote{c.f. \cite[p. 179]{ref}} one obtains an exact sequence

$$0 \to A_X \xrightarrow{j^*} A_Y \to A_Y / f^* A_X \to 0. \quad (*)$$

As noted in \cite{ref}, one has $\int_Y f^* \omega = \deg(f) \int_X \omega$ for any form $\omega$ on $X$, so that the diagram

$$
\begin{array}{ccc}
A_X & \xrightarrow{j^*} & A_Y \\
\downarrow{\deg(f) \Phi} & & \downarrow{\Phi} \\
D^{\text{top}} A_X & \xleftarrow{D^f} & D^{\text{top}} A_Y
\end{array}
$$

commutes. Since $\Phi$ induces an isomorphism on the first page of the Fr"{o}licher spectral sequence, for every $p \in \mathbb{Z}$, in the long exact sequence of terms on the first page of the Fr"{o}licher spectral sequence induced by $(*)$

$$\ldots \to H_{\text{Fr}}^{p,q} (A_Y) \xrightarrow{j^*} H_{\text{Fr}}^{p,q} (A_Y) \xrightarrow{\text{pr}} H_{\text{Fr}}^{p,q} (A_Y / f^* A_X) \xrightarrow{\delta} \ldots$$

the map $f^*$ is a split injection (with left inverse $\frac{1}{\deg(f)} f_*$) and hence $\text{pr}$ is surjective. This implies that the morphism $(f_*, \text{pr})$ in the statement induces an isomorphism on the first page of the Fr"{o}licher spectral sequence. \hfill \square

Since $\Phi : A_X \to D^{\text{top}} A_X$ is an $E_1$-quasi-isomorphism, this implies in particular that the (Dolbeault, Bott-Chern, Aeppli or de-Rham) cohomology of $X$ is a direct summand in that of $Y$. Thus, in order to compute the double complex of a blow-up, one just has to take care of the quotient-type summand.

Theorem 8. Let $X$ be a connected compact complex manifold, $Z \subseteq X$ a closed submanifold and $\tilde{X}$ the blow-up of $X$ at $Z$ and $E \subseteq \tilde{X}$ the exceptional divisor, so that the following diagram is cartesian:

$$\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\downarrow{\pi_E} & & \downarrow{\pi} \\
Z & \xrightarrow{i} & X
\end{array}$$

The map

$$A_{\tilde{X}} (\pi_*, \pi^*) \xrightarrow{D A_X \oplus A_E / \pi_*^* A_Z}$$

is an $E_1$-quasi-isomorphism, where $\pi^* = \text{pr} \circ j^*$.\footnote{c.f. \cite[p. 179]{ref}}
Proof. By lemma\[7\] it remains to show that the induced map

\[ \tilde{j}^* : A_X/\pi^*A_X \rightarrow A_E/\pi_E^*A_Z \]

is an $E_1$-quasi-isomorphism. This map sits inside a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & A_X & \xrightarrow{\pi^*} & A_{\tilde{X}} & \rightarrow & A_{\tilde{X}}/\pi^*A_X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A_Z & \xrightarrow{\pi_E^*} & A_E & \rightarrow & A_E/\pi_E^*A_Z & \rightarrow & 0,
\end{array}
\]

which, when applying Dolbeault cohomology, yields for every $p$ a map of two long exact sequences

\[
\cdots \rightarrow H^{p,q}_b(A_X) \xrightarrow{\pi^*} H^{p,q}_b(A_{\tilde{X}}) \rightarrow H^{p,q}_b(A_{\tilde{X}}/\pi^*A_X) \rightarrow \cdots \\
\cdots \rightarrow H^{p,q}_b(A_Z) \xrightarrow{\pi_E^*} H^{p,q}_b(A_E) \rightarrow H^{p,q}_b(A_E/\pi_E^*A_Z) \rightarrow \cdots.
\]

But $\pi^*$ and $\pi_E^*$ in these sequences are injective: In fact, for $\pi^*$ this was shown in the proof of lemma\[4\] and since $E$ is a projective bundle over $Z$, for $\pi_E$ it follows from proposition\[4\]. In particular, the long exact sequences decompose into short exact ones and we obtain one diagram for every pair $(p, q) \in \mathbb{Z}^2$:

\[
\begin{array}{cccccc}
0 & \rightarrow & H^{p,q}_b(A_X) & \xrightarrow{\pi^*} & H^{p,q}_b(A_{\tilde{X}}) & \rightarrow & H^{p,q}_b(A_{\tilde{X}}/\pi^*A_X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^{p,q}_b(A_Z) & \xrightarrow{\pi_E^*} & H^{p,q}_b(A_E) & \rightarrow & H^{p,q}_b(A_E/\pi_E^*A_Z) & \rightarrow & 0
\end{array}
\]

\[ (** \) \]

Now, note that $H^{p,q}_b(A_{\tilde{X}}) = H^q(X, \Omega^p_{\tilde{X}})$ is the cohomology of the sheaf of holomorphic $p$ forms (and similarly for $E$). Let us consider the Leray spectral sequences associated with $\pi$ and $\pi_E$ and the sheaves $\Omega^p_{\tilde{X}}$ and $\Omega^p_E$:

\[ L_{\pi, \Omega^p_{\tilde{X}}} : E_2^{r,s} = H^r(X, R^s \pi_* \Omega^p_{\tilde{X}}) \Rightarrow (H^{r+s}(\tilde{X}, \Omega^p_{\tilde{X}}), F^r_L) \]

and

\[ L_{\pi_E, \Omega^p_E} : E_2^{r,s} = H^r(Z, R^s \pi_E* \Omega^p_E) \Rightarrow (H^{r+s}(E, \Omega^p_E), F^r_L) \]

where $F^r_L$ denotes in both cases the Leray-filtration on the target. Pullback by $j$ induces a morphism of spectral sequences

\[ j^*_L : L_{\pi, \Omega^p_{\tilde{X}}} \rightarrow L_{\pi_E, \Omega^p_E} \]

which approximates $j^*$ on the target, i.e., on the $E_\infty$-page the $(r, s)$-component coincides with the $s$-th graded part of $j^* : H^{r+s}(\tilde{X}, \Omega^p_{\tilde{X}}) \rightarrow H^{r+s}(E, \Omega^p_E)$.

The sheaves $R^s \pi_* \Omega^p_{\tilde{X}}$ and $R^s \pi_E* \Omega^p_E$ have been investigated in \[6\] prop. 3.3, using a vanishing result by Bott and Serre’s theorem (B)\[4\] The result is that

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\[ The \text{first isomorphism is a consequence of the Hartogs principle and was also noted in } 10.\]

\[ The \text{computation is done there for smooth schemes. However, all tools used are also available in the holomorphic category, c.f. } 2, \text{ and the proof can be copied verbatim.} \]
the following maps are isomorphisms for all \( p \in \mathbb{Z} \):

\[
\begin{align*}
\pi^* : \Omega^p_X & \cong \pi_* \Omega^p_{\tilde{X}} \quad \text{(i)} \\
\pi^*_E : \Omega^p_Z & \cong \pi_{E*} \Omega^p_E \quad \text{(ii)} \\
j^* : R^s \pi_* \Omega^p_{\tilde{X}} & \cong i_* R^s \pi_{E*} \Omega^p_E \quad \text{for each } s \geq 1 \quad \text{(iii)}
\end{align*}
\]

The first two isomorphisms imply that the \( E^r_{\pi,0} \)-terms can be identified with \( H^p_{\partial}(A_X) \), resp. \( H^p_{\partial}(A_E) \) and the edge maps with \( \pi^* \), resp. \( \pi^*_E \). Since these are injective, the diagram (**) is canonically isomorphic to

\[
\begin{array}{ccc}
0 & \rightarrow & F^0_L H^p_{\partial}(A_{\tilde{X}}) \xleftarrow{\subseteq} H^p_{\partial}(A_{\tilde{X}}) \rightarrow H^p_{\partial}(A_{\tilde{X}})/F^0_L H^p_{\partial}(A_{\tilde{X}}) \rightarrow 0 \\
\downarrow F^0_L j^* & & \downarrow j^* \\
0 & \rightarrow & F^0_L H^p_{\partial}(A_E) \xleftarrow{\subseteq} H^p_{\partial}(A_E) \rightarrow H^p_{\partial}(A_E)/F^0_L H^p_{\partial}(A_E) \rightarrow 0
\end{array}
\]

Finally, all differentials with target in degree \((r,0)\) for some \( r \in \mathbb{Z} \) vanish since the edge maps are injective and the identification (iii) implies that \( j^*_L \) is an isomorphism on the \( E_2 \)-page in bidegrees \((r,s)\) for all \( r \in \mathbb{Z}, s \geq 1 \). Therefore, \( j^* \) induces isomorphisms

\[
j^* : \text{gr}^*_{F_L} H^p_{\partial}(A_{\tilde{X}}) \cong \text{gr}^*_L H^p_{\partial}(A_E)
\]

for all \( p, q \in \mathbb{Z} \) and \( s \geq 1 \). In particular, since a filtered morphism of vector spaces with finite filtrations is an isomorphism if its associated graded is, \( \tilde{j}^* \) is an isomorphism in Dolbeault cohomology. \( \square \)

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