Negative Imaginary State Feedback Equivalence for Systems of Relative Degree One and Relative Degree Two

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Abstract—This paper presents necessary and sufficient conditions under which a linear system of relative degree either one or two is state feedback equivalent to a negative imaginary (NI) system. More precisely, we show for a class of linear time-invariant strictly proper systems, that such a system can be rendered minimal and NI using full state feedback if and only if it is controllable and weakly minimum phase. A strongly strict negative imaginary state feedback equivalence result is also provided. The NI state feedback equivalence result is then applied in a robust stabilization problem for an uncertain system with a strongly negative imaginary uncertainty.

Index Terms—Negative imaginary systems, feedback equivalence, stabilization, robust control.

I. INTRODUCTION

The negative imaginary (NI) systems theory was established in [1]–[3] to address robust control problems for systems with colocated force actuators and position sensors. Motivated by the control of flexible structures [4]–[6], negative imaginary systems theory has been applied in many fields. For example, it has achieved success in the control of lightly damped structures [7]–[9] and nano-positioning [10]–[13]. NI systems theory provides an important alternative to positive real (PR) systems theory [14] in order to achieve robust stability when the output of a mechanical system is position rather than velocity. Unlike PR systems theory which uses negative velocity feedback control, NI systems can be controlled using positive position feedback. One advantage of NI systems theory is that it can deal with systems of relative degrees zero, one or two, while PR systems theory can only deal with systems of relative degrees zero or one [14].

Roughly speaking, a square transfer matrix is NI if it is stable and its Hermitian imaginary part is negative semidefinite for all frequencies \( \omega \geq 0 \). In particular, the frequency response of a single-input single-output (SISO) NI system has a phase lag between 0 to \( 2\pi \) radians for all frequencies \( \omega > 0 \). The negative imaginary lemma states that a system is NI if it is dissipative, with its supply rate being the inner product of its input and the derivative of its output [3], [15], [16]. Also note that for the positive feedback interconnection of an NI system \( R(s) \) and a strictly negative imaginary (SNI) system \( R_n(s) \) with \( R(\infty)R_n(\infty) = 0 \) and \( R_n(\infty) \geq 0 \), internal stability is achieved if and only if the DC loop gain of the interconnection is strictly less than unity; i.e., \( \lambda_{\text{max}}(R(0)R_n(0)) < 1 \) (see [1]).

Feedback stabilization problems have been addressed in many papers using the PR systems theory (see [17], [18], etc.). In these papers, a system with a specified nonlinearity is stabilized by a state feedback control law that renders the linear part of the system PR. The essence of these papers is deriving conditions for such PR state feedback equivalence, based on which stabilization results can then be achieved. For example, [18] renders a linear system PR and this result is then generalized to nonlinear systems in [19] using passivity theory. Further nonlinear generalizations of these ideas are presented in the papers [20]–[23]. However, because of the limitations of passivity and PR systems theory, the systems investigated in these papers are only allowed to have relative degree one. This rules out a wide variety of control systems with relative degree two, such as mechanical systems with force actuators and position sensors. Therefore, we seek to solve the problem of state feedback equivalence to an NI system, for systems of relative degree one or two, to complement the existing results that are based on passivity and PR systems theory.

In this paper, we investigate the NI state feedback equivalence problem for systems of relative degree one and relative degree two. A system with no zero at the origin and of relative degree one or two can be made minimal and NI via the use of state feedback if and only if it is controllable and weakly minimum phase (see for example [24] and [25] for details about feedback linearization). In particular, a controllable system of relative degree one is state feedback equivalent to a strongly strict negative imaginary (SSNI) system if and only if it is minimum phase. The proposed NI state feedback equivalence results are then applied to a robust stabilization problem for an uncertain system with SNI uncertainty.

In addition to complementing the existing feedback equivalence results to allow for relative degree two, the present research provides alternative feedback equivalence results for systems of relative degree one based on NI systems theory, as NI systems arise naturally in a wide variety of applications [26]. This work enables NI systems theory to be applied to a broader class of systems of relative degree one or two when full state information is available.

This paper is organised as follows: Section III provides the essential background on NI systems theory. Section IV defines the class of systems under consideration and states the objectives of this paper. The problem is separated into the relative degree one and relative degree two cases. We present
in Lemmas 4 and 6 necessary and sufficient conditions under which there exist state feedback matrices that render the system NI. Also, for the special cases when the internal dynamics have zero dimension, we show that there always exist state feedback matrices that make the system NI. The main result of this paper is presented in Theorem 1, which combines the NI feedback equivalence results of both the relative degree one and relative degree two cases. In Section V, an SSNI state feedback equivalence result is also provided for systems of relative degree one while it is explained that systems with relative degree two can never be rendered SSNI via state feedback. Section VI applies the NI state feedback equivalence results presented in Section IV in stabilizing an uncertain system with SNI uncertainty. Section VII illustrates the presented results with a numerical example. Section VIII concludes the paper and discusses possible future work. A full arXiv version of the paper including proofs of the results and some remarks can be found in [27].

II. NOTATION

The notation in this paper is standard. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers, respectively. \( \mathbb{R}^{m \times n} \) and \( \mathbb{C}^{m \times n} \) denote the spaces of real and complex matrices of dimension \( m \times n \), respectively. \( \mathcal{R}[\cdot] \) is the real part of a complex number. \( A^T \) denotes the transpose of a matrix \( A \). \( A^{-T} \) denotes the transpose of the inverse of \( A \); i.e., \( A^{-T} = (A^{-1})^T = (A^T)^{-1} \). \( \det(A) \) denotes the determinant of \( A \). \( \ker(A) \) denotes the kernel of \( A \). \( \text{spec}(A) \) denotes the spectrum of \( A \). \( \lambda_{\text{max}}(A) \) denotes the largest eigenvalue of a matrix \( A \) with real spectrum. For a symmetric matrix \( P \), \( P > 0 \) (\( P \geq 0 \)) denotes the fact that the matrix \( P \) is positive definite (positive semidefinite) and \( P < 0 \) (\( P \leq 0 \)) denotes the fact that the matrix \( P \) is negative definite (negative semidefinite). For a positive definite matrix \( P \), we denote by \( P^{\frac{1}{2}} \) the unique positive definite square root of \( P \). \( \text{OLHP} \) and \( \text{CLHP} \) are the open and closed left half-planes of the complex plane, respectively.

III. PRELIMINARIES

**Definition 1:** (Negative Imaginary Systems) [3] A square real-rational proper transfer function matrix \( R(s) \) is said to be negative imaginary (NI) if:
1. \( R(s) \) has no poles at the origin and in \( \mathcal{R}[s] > 0 \);
2. \( j[R(j\omega) - R^*(j\omega)] \geq 0 \) for all \( \omega \in (0, \infty) \) except for values of \( \omega \) where \( j\omega \) is a pole of \( R(s) \);
3. if \( j\omega_0 \) with \( \omega_0 \in (0, \infty) \) is a pole of \( R(s) \), then it is a simple pole and the residue matrix \( K_0 = \lim_{\omega \to j\omega_0} (s - j\omega_0) R(s) \) is Hermitian and positive semidefinite.

**Definition 2:** (Strictly Negative Imaginary Systems) [3] A square real-rational proper transfer function \( R(s) \) is said to be strictly negative imaginary (SNI) if the following conditions are satisfied:
1. \( R(s) \) has no poles in \( \mathcal{R}[s] \geq 0 \);
2. \( j[R(j\omega) - R^*(j\omega)] > 0 \) for all \( \omega \in (0, \infty) \).

**Definition 3:** (Strongly Strictly Negative Imaginary Systems) [28] A square real-rational proper transfer function matrix \( R(s) \) is said to be strongly strictly negative imaginary (SSNI) if the following conditions are satisfied:
1. \( R(s) \) is SNI.
2. \( \lim_{\omega \to \infty} j\omega [R(j\omega) - R^*(j\omega)] > 0 \) and \( \lim_{\omega \to 0} j\frac{1}{\omega} [R(j\omega) - R^*(j\omega)] > 0 \).

**Lemma 1:** (NI Lemma) [3] Let \((A, B, C, D)\) be a minimal state-space realisation of an \( n \times n \) real-rational proper transfer function matrix \( R(s) \) where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times p} \), \( C \in \mathbb{R}^{m \times n} \), \( D \in \mathbb{R}^{m \times p} \). Then \( R(s) \) is NI if and only if:
1. \( \det(A) \neq 0 \), \( D = D^T \);
2. There exists a matrix \( Y = Y^T > 0 \), \( Y \in \mathbb{R}^{n \times n} \)

\[ AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0. \]

**Definition 4:** (Lyapunov Stability) [29] A square matrix \( A \) is said to be Lyapunov stable if \( \text{spec}(A) \subset \text{CLHP} \) and every purely imaginary eigenvalue of \( A \) is semisimple.

IV. STATE FEEDBACK EQUIVALENCE TO A NEGATIVE IMAGINARY SYSTEM

Consider a linear multiple-input multiple-output (MIMO) system with the following state-space model

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx.
\end{align*}
\]

where \( x \in \mathbb{R}^n \) and \( u, y \in \mathbb{R}^p \). We consider the necessary and sufficient conditions under which the system (2) is state feedback equivalent to an NI system. State feedback equivalence to an NI system is defined as follows:

**Definition 5:** The system (2) is said to be state feedback equivalent to an NI system if there exists a state feedback control law

\[
u = K_ux + Kv
\]
such that the closed-loop system with the new input \( v \in \mathbb{R}^p \) is minimal and NI.

Let us recall the definition of relative degree.

**Definition 6:** (Relative Degree) [18] The system (2) is said to have relative degree \( r \) if its first \( r - 1 \) Markov parameters are zero, i.e., \( CA^iB = 0 \) for \( i = 0, 1, \ldots, r-2 \); and \( CA^{r-1}B \) is nonsingular.

We discuss the relative degree one and two cases in the following two subsections.

A. Relative Degree One Case

Suppose the system (2) has relative degree one, that is, \( \det(\mathcal{C}B) \neq 0 \). Then \( m := n - p \geq 0 \) and hence, without loss
of generality, the system (2) can be considered to be in the special coordinate basis (SCB) (see [30], [31])
\[
\dot{z} = A_{11}z + A_{12}y, \quad (3a)
\]
\[
\dot{y} = A_{21}z + A_{22}y + CBu, \quad (3b)
\]
\[
y = [0 \quad I] \begin{bmatrix} z \\ y \end{bmatrix}, \quad (3c)
\]
where \( z \in \mathbb{R}^m \). This can be realized using a state transformation, according to Lemma 3.

**Lemma 3:** Suppose the system (2) has relative degree one, that is, \( \det(CB) \neq 0 \). Then there exists a state transformation such that the resulting transformed system is of the form (3).

For the system (3), since \( \det(CB) \neq 0 \), then the input \( u \) can be represented as
\[
u = (CB)^{-1}(v + (K_1 - A_{21})z + (K_2 - A_{22})y),
\]
and the system (3) takes the form
\[
\dot{z} = A_{11}z + A_{12}y, \quad (4a)
\]
\[
\dot{y} = K_1z + K_2y + v, \quad (4b)
\]
\[
y = [0 \quad I] \begin{bmatrix} z \\ y \end{bmatrix}. \quad (4c)
\]
We need to find the state feedback matrices \( K_1 \in \mathbb{R}^{p \times m} \) and \( K_2 \in \mathbb{R}^{p \times p} \) such that the system (4) is NI. We show in the following lemma the necessary and sufficient conditions under which such state feedback matrices exist.

**Lemma 4:** Suppose the system (4) satisfies \( \det(A_{11}) \neq 0 \). Then there exist \( K_1 \in \mathbb{R}^{p \times m} \) and \( K_2 \in \mathbb{R}^{p \times p} \) such that the system (4) is an NI system with minimal realisation if and only if the pair \( (A_{11}, A_{12}) \) is controllable and \( A_{11} \) is Lyapunov stable.

**B. Relative Degree Two Case**

Suppose the system (2) has relative degree two, that is \( CB = 0 \) and \( \det(CAB) \neq 0 \). Then \( m := n - 2p \geq 0 \) and hence without loss of generality, the system (2) can be considered to be in the SCB (see [30], [31])
\[
\dot{z} = A_{11}z + A_{12}x_1 + A_{13}x_2, \quad (5a)
\]
\[
\dot{x}_1 = x_2, \quad (5b)
\]
\[
\dot{x}_2 = A_{31}z + A_{32}x_1 + A_{33}x_2 + CABu, \quad (5c)
\]
\[
y = [0 \quad I \quad 0] \begin{bmatrix} z \\ x_1 \\ x_2 \end{bmatrix}, \quad (5d)
\]
where \( z \in \mathbb{R}^m \) and \( x_1, x_2 \in \mathbb{R}^p \). This can be realized using a state transformation, according to Lemma 5.

**Lemma 5:** Suppose the system (2) has relative degree two, that is, \( CB = 0 \) and \( \det(CAB) \neq 0 \). Then there exists a state transformation such that the resulting transformed system is of the form (5).

Let the input of system (5) be
\[
u = (CAB)^{-1}(v + (K_1 - A_{31})z + (K_2 - A_{32})x_1 + (K_3 - A_{33})x_2), \quad (6)
\]
then the system (5) takes the form
\[
\dot{z} = A_{11}z + A_{12}x_1 + A_{13}x_2, \quad (7a)
\]
\[
\dot{x}_1 = x_2, \quad (7b)
\]
\[
\dot{x}_2 = K_1z + K_2x_1 + K_3x_2 + v, \quad (7c)
\]
\[
y = [0 \quad I \quad 0] \begin{bmatrix} z \\ x_1 \\ x_2 \end{bmatrix}. \quad (7d)
\]

We need to find the state feedback matrices \( K_1 \in \mathbb{R}^{p \times m} \), \( K_2 \in \mathbb{R}^{p \times p} \) and \( K_3 \in \mathbb{R}^{p \times p} \) such that the system (7) is NI. We show in the following lemma the necessary and sufficient conditions under which such state feedback matrices exist.

**Lemma 6:** Suppose the system (7) satisfies \( \det(A_{11}) \neq 0 \). Then there exist \( K_1, K_2 \) and \( K_3 \) such that the system (7) is an NI system with minimal realisation if and only if \( A_{11} \) is Lyapunov stable and the pair \( (A_{11}, A_{11}A_{13} + A_{12}) \) is controllable.

**C. Main Theorem**

To summarize the NI state feedback equivalence results for the relative degree one and two cases, we recall the following terminologies (see [24], [25]).

The systems (3) and (5) are said to be the normal forms of the system (2) in the relative degree one and the relative degree two cases, respectively. For these two cases, the dynamics described in (3a) and (5a) are not controlled by the input \( u \) directly or through chains of integrators, and are called the internal dynamics. Setting the other states to be zero in the internal dynamics, we obtain the zero dynamics. That is \( \dot{z} = A_{11}z \) for both relative degree one and two cases, with a minor abuse of notation. We now give the definitions of the weakly minimum phase property.

**Definition 7:** (Weakly Minimum Phase) [18], [19] The system (2) is said to be weakly minimum phase if its zero dynamics are Lyapunov stable.

We now combine the NI state feedback equivalence results shown in Lemmas 4 and 6 in the following theorem.

**Theorem 1:** Suppose a system with the state-space model (2) is of relative degree one or relative degree two and has no zero at the origin. Then it is state feedback equivalent to an NI system if and only if it is controllable and weakly minimum phase.

**V. STATE FEEDBACK EQUIVALENCE TO A STRONGLY STRICT NEGATIVE IMAGINARY SYSTEM**

We now consider necessary and sufficient conditions under which the system (2) is state feedback equivalent to an SSNI system. State feedback equivalence to an SSNI system is defined as follows:

**Definition 8:** The system (2) is said to be state feedback equivalent to an SSNI system if there exists a state feedback control law
\[
u = K_xx + K_vv
\]
such that the closed-loop system with the new input \( v \in \mathbb{R}^p \) is an SSNI system.
Similarly, we first consider the existence of state feedback matrices for systems of relative degree one and two in the normal forms.

**Lemma 7:** Suppose the system (4) has \((A_{11}, A_{12})\) controllable. Then the following statements are equivalent:
1. \(A_{11}\) is Hurwitz;
2. There exist \(K_1\) and \(K_2\) such that the system (4) is an SSNI system with realisation \((A, B, C)\), where \(A\) is Hurwitz, and the transfer function \(R(s) := C(sI - A)^{-1}B\) is such that \(R(s) + R(-s)^T\) has full normal rank.

**Remark 1:** Unlike the relative degree one case, the system (7) can not be an SSNI system. Indeed, considering the particular form of the system (7), the condition \(B + AYC = 0\) in (1) requires the middle diagonal block of \(AY\) be 0. Therefore, the matrix \(AY + YA^T\) can never be sign definite. Hence, the system (7) can never be SSNI.

Therefore, we conclude the SSNI state feedback equivalence result in the following theorem. First we give the definition of the minimum phase property.

**Definition 9:** (Minimum Phase) [19], [24] The system (2) is said to be minimum phase if its zero dynamics is asymptotically stable.

**Theorem 2:** Consider a system with the state-space model (2), suppose it is controllable and has relative degree one. Then following statements are equivalent:
1. The system is minimum phase;
2. The system is state feedback equivalent to an SSNI system with realisation \((A, B, C)\), where \(A\) is Hurwitz, and the transfer function \(R(s) := C(sI - A)^{-1}B\) is such that \(R(s) + R(-s)^T\) has full normal rank.

**VI. CONTROL OF SYSTEMS WITH SNI UNCERTAINTY**

Let us consider the system
\[
\begin{align*}
\dot{x} &= Ax + Bu + w, \\
y &= Cx, \\
w &= \Delta(s)y,
\end{align*}
\]
where the uncertainty transfer function \(\Delta(s)\) is assumed to be SNI with \(\Delta(\infty) \geq 0\) and \(\lambda_{\max}(\Delta(0)) \leq \gamma\) for some constant \(\gamma < \infty\). We consider the relative degree one and two cases separately.

A. Relative Degree One Case

Suppose the system (8) has relative degree one, that is \(CB \neq 0\). Without loss of generality, the system (8) can be considered to be in the SCB (see [30], [31])
\[
\begin{align*}
\dot{z} &= A_{11}z + A_{12}y, \\
\dot{y} &= A_{21}z + A_{22}y + CB(u + w), \\
y &= [0 \ I \ z]^T, \\
w &= \Delta(s)y.
\end{align*}
\]
With the result in Section IV-A, the following stabilization theorem is obtained.

**Lemma 8:** Suppose the uncertain system (9) satisfies \(\det(CB) \neq 0\). Then without loss of generality, the system (8) can be stabilized by the state-feedback control law
\[
u = (CB)^{-1}((K_1 - A_{21})z + (K_2 - A_{22})y),
\]
where \(K_1\) and \(K_2\) are as defined in the proof of Lemma 4 (see [27]) with \(\gamma_2\) in \(K_2\) also satisfying \(\lambda_{\max}(\gamma_2) < \frac{1}{\gamma}\).

**Remark 2:** For the uncertain system (9), if the uncertainty \(\Delta(s)\) is NI, then it can be robustly stabilized by applying Lemma 7 in a similar way to make the nominal closed-loop system SSNI.

B. Relative Degree Two Case

Suppose the system (8) has relative degree two, that is \(CB = 0\) and \(\det(CAB) \neq 0\). Without loss of generality, the system (8) can be considered to be in the SCB (see [30], [31])
\[
\begin{align*}
\dot{z} &= A_{11}z + A_{12}x_1 + A_{13}x_2, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= A_{31}z + A_{32}x_1 + A_{33}x_2 + CAB(u + w), \\
y &= [0 \ I \ z]^T, \\
w &= \Delta(s)y.
\end{align*}
\]
With the result in Section IV-B, the following stabilization theorem is obtained.

**Lemma 9:** Suppose the uncertain system (10) satisfies \(\det(A_{11}) \neq 0\), \(A_{11}\) Lyapunov stable and \((A_{11}, A_{11}A_{13} + \ldots + A_{1n}A_{1n+1})\) is Hurwitz.
A12) controllable. Then the system (10) can be stabilized by
the state-feedback control law
\[ u = (CA)B^{-1}(K_1 - A_{31})z + (K_2 - A_{32})x_1 + (K_3 - A_{33})x_2, \]
where \( K_1, K_2 \) and \( K_3 \) are as defined in the proof of Lemma 6 (see [27]) with \( K_2 \) in \( K_2 \) also satisfying \( \lambda_{\max}(K_2) < \frac{1}{7} \).

C. Existence of a Stabilizing State Feedback Control Law

The results in Lemmas 8 and 9 are concluded in the following theorem.

Theorem 3: Consider the uncertain system (8), suppose it has relative degree one or relative degree two, \((A, B)\) is controllable and the realization \((A, B, C)\) has no zero at the origin. Also, suppose the realization \((A, B, C)\) is weakly minimum phase. Then there always exists state feedback in the form \( u = Kx \) that asymptotically stabilizes this system.

VII. ILLUSTRATIVE EXAMPLE

Consider an uncertain system with the state-space model
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & 1 \\ 0 \\ 1 \end{bmatrix} x, \\
w &= \Delta(s)y,
\end{align*}
\]
where the transfer function \( \Delta(s) \) is SNI with \( \lambda_{\max}(0) < 1 \) and \( \Delta(\infty) \geq 0 \). Let
\[
A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.
\]
The nominal plant with the state-space realisation \((A, B, C)\) is not an NI system because \( A \) is unstable. Therefore, we need to apply the proposed state feedback equivalence result to make it NI. We have \( CB = 0 \) and \( CAB = 1 \). Hence, the system (12) has relative degree two. With a state transformation
\[
\begin{bmatrix} z \\ x_1 \\ x_2 \end{bmatrix} = T x,
\]
where \( T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \)
the system (12) becomes
\[
\begin{align*}
\dot{z} &= -z + x_1, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_1 - 2x_2 + w + u, \\
y &= x_1, \\
w &= \Delta(s)y.
\end{align*}
\]
In comparison to the system (10), here we have \( A_{11} = -1, A_{12} = 1, A_{13} = 0, A_{31} = 0, A_{32} = 1, A_{33} = -2 \). The assumptions in Lemma 9 that \( A_{11} \) is Lyapunov stable and \((A_{11}, A_{11}A_{13} + A_{12})\) is controllable are satisfied. According to (11), choose the state-feedback matrices to be
\[
K_1 = 1, \quad K_2 = -1 - Y_2^{-1}, \quad \text{and} \quad K_3 = -1.
\]
Choose \( Y_2 = 0.5 < \frac{1}{\lambda_{\max}(\Delta(0))} \), then \( K_2 = -3 \). According to (11), let
\[
u = z_1 - 4x_1 + x_2.
\]
Then the system (13) becomes
\[
\begin{align*}
\dot{z} &= -z + x_1, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= z_1 - 3x_1 - x_2 + y, \\
y &= x_1, \\
w &= \Delta(s)y.
\end{align*}
\]
The transfer function of the nominal closed-loop system described by (15a)-(15d) is
\[
R(s) = \frac{s + 1}{s^3 + 2s^2 + 4s + 2},
\]
which has a Bode plot shown in Fig. 2. Since \( \angle R(s) \in [-\pi, 0] \) for positive frequencies, \( R(s) \) is NI. Also, the magnitude of the DC gain of \( R(s) \) is less than unity. In fact, \( R(0) = \frac{1}{2} \). Therefore, \( \lambda_{\max}(R(0)\Delta(0)) < 1 \). Because \( R(\infty)\Delta(\infty) = 0 \) and \( \Delta(\infty) \geq 0 \), the system (15) is asymptotically stable. Thus, the system (13) is robustly stabilized by the control law (14).

VIII. CONCLUSION AND FUTURE WORK

In this paper, we have provided necessary and sufficient conditions under which a system of relative degree one or two is state feedback equivalent to an NI system. As is stated in Theorem 1, the system (2), which is of relative degree one or two and has no zeros at the origin, is state feedback equivalent NI if and only if it is controllable and weakly minimum phase. A similar SSNI feedback equivalence result is presented in Theorem 2. The state feedback NI results are then applied to solve the robust stabilization problem for an uncertain system with a specific uncertainty. An example is also provided to illustrate stabilizing process for an uncertain system.

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The results of this paper have been recently extended by the authors in [32]. The paper [32] completes the present paper by considering the case when a system has mixed relative degree one and two, while the present paper considers the relative degree one and relative degree two cases separately. It is provided in [32] the necessary and sufficient conditions for a system in the form of (2) to be state feedback equivalent to an NI system.

Considering the emergence of the nonlinear negative imaginary systems theory (see [33]–[35]), it is also worth investigating the feedback equivalence problem for nonlinear systems using the nonlinear NI systems theory. This future state feedback equivalent nonlinear NI research is planned to be a complement of the work done by Byrnes, Isidori and Williams in [19], which investigates the feedback passivity problem for a nonlinear system of relative degree one. It can be also regarded as an extension of the present paper to nonlinear systems.

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