NORMAL BASES USING 1-DIMENSIONAL ALGEBRAIC GROUPS

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ABSTRACT. This paper surveys and illustrates geometric methods for constructing normal bases allowing efficient finite field arithmetic. These bases are constructed using the additive group, the multiplicative group and the Lucas torus. We describe algorithms with quasi-linear complexity to multiply two elements given in each one of the bases.

1. INTRODUCTION

Consider two fields $K$ and $L$, such that $L$ is a degree $n$ cyclic extension of $K$. Denote by $\sigma$ a generator of the Galois group $\text{Gal}(L/K)$. A normal basis of $L$ over $K$ is a basis $(\theta, \sigma(\theta), \ldots, \sigma^{n-1}(\theta))$ generated by some $\theta$ in $L^{*}$. Such a $\theta$ is called a normal element of $L$ over $K$. The normal basis theorem ensures that $L$ possesses at least one normal element over $K$.

Let $\Theta = (\theta_i)_{0 \leq i \leq n-1}$ be an arbitrary basis of $L/K$. Given

\[ a = \sum_{i=0}^{n-1} a_i \theta_i \quad \text{and} \quad b = \sum_{j=0}^{n-1} b_j \theta_j \]

in $L$, the sum $a + b$ is component-wise and easy to implement. The product $a \times b$ may be more difficult. Let $\Gamma$ be a straight-line program computing the coordinates of $a \times b$ in $\Theta$, from the coordinates of $a$ and $b$. We assume that $\Gamma$ consists of additions, subtractions, multiplications of a register by a constant, and additions, subtractions, multiplications between two registers. The complexity of $\Gamma$ is the total number of such operations. We define the complexity of $\Theta$ to be the minimal possible complexity of a straight-line program computing the coordinates of $a \times b$ from the ones of $a$ and $b$. Let $t_{i,j}^k$ be coefficients in $K$ such that

\[ \theta_i \theta_j = \sum_{k=0}^{n-1} t_{i,j}^k \theta_k. \]

Then

\[ a \times b = \sum_{k=0}^{n-1} c_k(a,b) \theta_k, \]

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where \( c_k \) is a bilinear form on \( L \times L \) defined by
\[
c_k(a, b) = \sum_{i,j} t_{i,j}^k a_i b_j.
\]

Assume that \( \Theta \) is a normal basis. Then every \( x \) with vector coordinate \((x_0, x_1, \ldots, x_{n-1})\) in \( \Theta \) is such that \( \sigma^k(x) \) has coordinate vector \((x_{-k}, x_{-k+1}, \ldots, x_{-k-1})\). Since the coordinate vector of the product \( \sigma^{n-k}(a) \times \sigma^{n-k}(b) \) is equal to
\[
(c_0(\sigma^{n-k}(a), \sigma^{n-k}(b)), c_1(\sigma^{n-k}(a), \sigma^{n-k}(b)), \ldots, c_{n-1}(\sigma^{n-k}(a), \sigma^{n-k}(b))),
\]
we have
\[
c_k(a, b) = c_0(\sigma^{n-k}(a), \sigma^{n-k}(b)).
\]

This means that \( c_k \) is obtained from \( c_0 \) by a \( k \)-fold cyclic shift of coordinates of the variables involved. Hence we define the \textit{weight}, denoted \( w \), of a normal basis to be the number of non-zero terms in the form \( c_0 \). There is a straightforward algorithm with complexity \( 2nw + n(w - 1) \) for computing the coordinates of \( a \times b \) from the ones of \( a \) and \( b \) in a normal basis with weight \( w \). The weight is sometimes called the complexity of the normal basis, but we prefer to use a different terminology. Using action of \( \sigma \) on equation (1), it is easily checked that the weight of a normal basis \( \Theta = (\theta_i)_{0 \leq i \leq n-1} \) is also equal to the number of non-zero coefficients in the linear combinations
\[
(2) \quad \theta_0 \theta_i = \sum_{j=0}^{n-1} t_{0,i}^j \theta_j, \quad \text{for} \ 1 \leq i \leq n - 1.
\]

Mullin, Onyszchuk, Vanstone and Wilson [11] showed that the weight of any normal basis of \( F_q^n \) over \( F_q \) is greater than \( 2n - 1 \). This lower bound is reached by the so-called optimal normal bases. It is appropriate here to define \textit{Gauss periods}.

\begin{definition}
Let \( q \) be a prime power. Let \( n \) and \( k \) be two integers such that \( r = nk + 1 \) is a prime number not dividing \( q \). Denote by \( \mathcal{K} \) the unique subgroup of \( (\mathbb{Z}/r\mathbb{Z})^* \) of order \( k \). A Gauss period of type \((n, k)\) over \( F_q \) is a sum of the form
\[
\vartheta = \sum_{a \in \mathcal{K}} \theta^a
\]
where \( \theta \) is an arbitrary primitive \( r \)-th root of unity in \( F_{q^n} \).
\end{definition}

It is easy to see that a Gauss period of type \((n, k)\) over \( F_q \) lies in \( F_{q^n} \). Moreover, a Gauss period of type \((n, k)\) over \( F_q \) generates a normal basis of \( F_{q^n}/F_q \) if and only if \( \gcd(e, n) = 1 \), where \( e \) denotes the index of \( q \) modulo \( r = nk + 1 \) [see [7], or [11]]. Optimal normal bases occur when \( k = 1 \) or \( k = 2 \). In fact normal bases with low weight and low complexity are usually constructed using \textit{Gauss periods} [see [11], [3], [6], [7], [9], [10]]. But Gao described in [15], chapter 5] another way to construct normal bases with low weight. The Lucas torus and its isogenies play an important, though implicit, role in Gao’s construction. Further, Couveignes and Lercier constructed normal bases using elliptic curves [4]. The resulting elliptic normal bases allow quasi-linear finite field arithmetic. Our work is concerned by efficient normal bases constructed with 1-dimensional algebraic groups. It is known that there are only three 1-dimensional connected affine algebraic groups over a perfect field (up to isomorphism): the additive group \( G_a \),
the multiplicative $G_m$ and the Lucas torus $\mathbb{T}_\alpha$ [see [10], chapter II, section 3, page 54]. Moreover, any connected projective algebraic group of dimension 1 is an elliptic curve. So a 1-dimensional connected algebraic group $G$ over a perfect field is either $G_a$, $G_m$, $\mathbb{T}_\alpha$ or an elliptic curve. In this paper, we consider normal bases constructed from $G_a$, $G_m$, and $\mathbb{T}_\alpha$. We adapt the construction of elliptic normal bases proposed by Couveignes and Lercier to these contexts. That results in natural and efficient algorithms. We prove the following theorems:

**Theorem 1.1.** Let $K$ be a field with characteristic $p > 0$. Let $a \in K$ be an element which does not lie in $\{x^p - x | x \in K\}$. Then $L := K[\alpha]/(X^p - X - a)$ is a degree $p$ cyclic extension of $K$, and there is a normal basis of $L$ over $K$ with weight $\leq 3p - 2$ and complexity $O(p \log p)(\log(\log p))$.

**Theorem 1.2.** Let $K$ be a field with characteristic $p > 0$. Assume that $m \geq 2$ and $n \geq 2$ are two integers such that $mn$ is prime to $p$. Assume that $K$ possesses a primitive $mn$-th root of unity. Let $a \in K$ be a non-zero element such that the order of the class of $a$ in $K^*/K^{*n}$ is equal to $n$. Then $L := K[\alpha]/(X^n - a)$ is a degree $n$ cyclic extension of $K$, and there is a normal basis of $L$ over $K$ with weight $\leq 3n - 2$ and complexity $O(n(\log n)(\log(\log n)))$.

If $K = F_q$ is a finite field, then the first two requirements in theorem 1.2 are equivalent to saying that $mn$ divides $q - 1$. A sufficient condition for the last requirement in theorem 1.2 in case $K = F_q$, is to take for $a$ a generator of $F_q^*$. Gao constructed low weight normal bases of $F_{q^n}/F_q$ in [5], chapter 5, section 5.3 by using irreducible polynomials of degree $n$ which divide $cX^{q + 1} + dX^q - aX - b \in F_q[X]$, where $c \neq 0$, $ad - bc \neq 0$, and $n$ divides $q - 1$. The weights of the resulting normal bases have the same upper bound as the one in theorem 1.2 and the one in theorem 1.1 when $n$ is equal to the characteristic of $F_q$.

**Theorem 1.3.** Let $q$ be a prime power and $n$ a non-trivial divisor of $q + 1$. Then there exists a normal basis $\Theta$ of $F_{q^n}$ over $F_q$ with complexity $O(n(\log n)(\log(\log n)))$.

Note that elliptic normal bases of $F_{q^n}/F_q$ constructed by Couveignes and Lercier [4] have complexity $O(n(\log n)^2(\log(\log n)))$ when they exist. Gao, von zur Gathen, Panario and Shoup showed [7] that fast multiplication methods (like FFT) can be adapted to normal bases of $F_{q^n}/F_q$ constructed with Gauss periods. They proved that the complexity of a normal basis of $F_{q^n}/F_q$ generated by a Gauss period of type $(n, k)$ is equal to $O(nk(\log nk)(\log(\log nk)))$.

**Plan.** In section 2 we prove theorem 1 using the additive group. In section 3 we use the multiplicative group to prove theorem 2. In sections 4 we explain how the Lucas Torus can be used to prove theorem 3. At the end of each section, we give detailed examples.

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2. Constructing normal bases with the additive group

Consider a field $K$ with characteristic $p > 0$. We denote by $\bar{K}$ an algebraic closure of $K$. We identify the additive group $G_a$ over $K$ with the affine line $\mathbb{A}^1$ over $\bar{K}$ endowed with the
$x$-coordinate. Any point $P$ in $G_a$ is given by its $x$-coordinate. The unit element $O_{G_a}$ has $x$-coordinate equal to 0. The group law $G_a \oplus G_a$ is defined by

$$x(P_1 \oplus G_a P_2) = x(P_1) + x(P_2).$$

2.1. Specializing isogenies of the additive group. The $F_p$-rational points of $G_a$ form a cyclic subgroup of $G_a(K)$. Let $I : G_a \to G_a$ be the quotient isogeny of $G_a$ by $G_a(F_p)$. In terms of $x$-coordinates, $I$ is given by

$$x(I(P)) = x(P)^p - x(P).$$

Let $a$ be a $K$-rational point in $G_a$ outside of the image $I(G_a(K))$. Then the subfield $L = K(I^{-1}(a))$ of $K$ is a cyclic extension of $K$ with degree $p$. Indeed, fix $b$ in $I^{-1}(a)$ and denote by $\theta$ the $x$-coordinate of $b$. Since $L$ is the splitting field of the separable polynomial

$$X^p - X - x(a) = (X - \theta)(X - (\theta - 1) \cdots (X - (\theta - (p - 1))) \in K[X],$$

it is a normal and separable extension of $K$. If $P(X) \in K[X]$ is an irreducible factor of $X^p - X - x(a)$ with degree $1 \leq r < p$, then $r\theta \in K$. There exist $u, v \in \mathbb{Z}$ such that $ur + vp = 1$. So $\theta = ur\theta$ lies in $K$, contradicting the assumption that $a = I(b)$ does not lie in $I(G_a(K))$. Hence $X^p - X - x(a)$ is irreducible over $K$, and $L$ is a degree $p$ Galois extension of $K$. The Galois group $\text{Gal}(L/K)$ is made of $K$-automorphism $a_k$ such that $a_k(\theta) = \theta + k$, for $0 \leq k \leq p - 1$. So $\text{Gal}(L/K)$ is generated by $a_1$.

We set $t = a_1(b) \ominus G_a b$. Then the fiber of $I$ above $a$ is given by

$$I^{-1}(a) = [b] + [b \oplus G_a t] + \cdots + [b \oplus G_a (p-1)t]].$$

The additive group is an open subset of the projective line $\mathbb{P}^1$. Consider the divisor

$$D = [O_{G_a}] + [t] + [2t] + \cdots + [(p-1)t] - [\infty] \in \text{Div}(\mathbb{P}^1).$$

The linear space

$$L = \mathbb{H}^0(\mathbb{P}^1, O_{\mathbb{P}^1}(D))$$

has dimension $p$ over $K$. The translation $\tau : P \mapsto P \ominus G_a t$ is an automorphism of $G_a$ which extends to the whole $\mathbb{P}^1$ by setting $\tau(\infty) = \infty$. The divisor $D$ is invariant by $\tau$, then so is the space $L$. For $0 \leq k \leq p - 1$, the functions

$$\frac{1}{x} \circ \tau^k \in K(G_a)$$

lie in $L$. Examination of poles shows that they are linearly independent. So the system

$$\left(\frac{1}{x}, \frac{1}{x-1}, \ldots, \frac{1}{x-(p-1)}\right)$$

is a basis of $L$ invariant by $\tau$. Evaluation at $b$ results in a normal basis

$$\Theta = \left(\frac{1}{\theta}, \frac{1}{\theta-1}, \ldots, \frac{1}{\theta-(p-1)}\right)$$

of $L$ over $K$. Indeed, let $\lambda_0, \ldots, \lambda_{p-1}$ be scalars in $K$ such that

$$\sum_{k \in F_p} \lambda_k \frac{1}{\theta - k} = 0.$$
Then the function $\sum_{k \in F_p} \lambda_k \frac{1}{x-k} = \frac{F(x)}{x(x-1)\cdots(x-(p-1))}$ cancels at $b$, where $F(X)$ is a polynomial in $K[X]$. If $F(X)$ is non-zero, then its degree is at most $p-1$. But $F$ cancels at $\theta$ and also at all its $p$ conjugates, this is impossible. So $F$ is the zero polynomial. All $\lambda_k$ are 0 because the functions $\frac{1}{x-k}$ are linearly independent. Recall that $\text{Gal}(L/K) = \{a_k|0 \leq k \leq p-1\}$ is generated by $a_1$ which satisfies $a_1(\theta) = \theta + 1$. So

$$a_1\left(\frac{1}{\theta - k}\right) = \frac{1}{\theta - (k - 1)}$$

for $k$ in $F_p$.

We conclude that $\Theta$ is a normal basis. If $i \neq 0$, then

$$\frac{1}{\theta} \cdot \frac{1}{\theta - i} = \frac{1}{i} \left(\frac{1}{\theta - i} - \frac{1}{\theta}\right).$$

Since the weight can be defined using linear combinations in equation (2), we conclude that the weight of $\Theta$ is at most $3p - 2$.

2.2. Complexity. In this section we describe an FFT-like algorithm which computes the product of two elements of $L$ in the normal basis $\Theta$ defined in (5). We adapt the construction proposed by Couveignes and Lercier in [4], section 4.3] to our context.

**Notation:** Let $\overrightarrow{\alpha} = (\alpha_k)_{0 \leq k \leq p-1}$ and $\overrightarrow{\beta} = (\beta_k)_{0 \leq k \leq p-1}$ be two vectors in $K^p$. We denote by $\overrightarrow{\alpha} \star_k \overrightarrow{\beta} = \sum_{0 \leq i \leq p-1} \alpha_i \beta_{k-i}$ the $k$-th component of the convolution product. We denote by $\sigma(\overrightarrow{\alpha}) = (\alpha_{k-1})_k$ the cyclic shift of $\overrightarrow{\alpha}$. We denote by $\overrightarrow{\alpha} \circ \overrightarrow{\beta} = (\alpha_k \beta_k)_k$ the component-wise product and by $\overrightarrow{\alpha} \ast \overrightarrow{\beta} = (\overrightarrow{\alpha} \star_k \overrightarrow{\beta})_k$ the convolution product.

**Reduction and evaluation.** We fix $u = \frac{1}{x}$, and $\theta_0 = \frac{1}{\theta}$. For $0 \leq k \leq p - 1$, we set

$$u_k = u \circ r^k, \theta_k = a_{-k}(\theta_0), \xi_k = \theta_k^2.$$

We also set $\xi_0 = \sum_{k \in F_p} u_k \theta_k$ and $\overrightarrow{\xi} = (\xi_k)_{0 \leq k \leq p-1}$. We want to reduce a linear combination of the $\xi$’s into a linear combination of the $\theta$’s. We have

$$\xi_i = a_i^{-1}(\xi_0) = \sum_{k \in F_p} u_k \theta_{k+i} = \sum_{k \in F_p} u_{k-i} \theta_k$$

for $i \leq p - 1$.

Let $\overrightarrow{\alpha} = (\alpha_i)_{0 \leq i \leq p-1}$ and $\overrightarrow{\beta} = (\beta_j)_{0 \leq j \leq p}$ be two vectors in $K^p$ such that

$$\sum_{i \in F_p} \alpha_i \xi_i = \sum_{j \in F_p} \beta_j \theta_j.$$

Since

$$\sum_i \alpha_i \xi_i = \sum_i \alpha_i \sum_k u_{k-i} \theta_k = \sum_k \theta_k \sum_i \alpha_i u_{k-i} = \sum_k (\overrightarrow{\xi} \star_k \overrightarrow{\alpha}) \theta_k.$$

We have

(6) $$\beta_j = \overrightarrow{\xi} \star_k \overrightarrow{\alpha},$$

that is $\overrightarrow{\beta} = \overrightarrow{\xi} \ast \overrightarrow{\alpha}$. 
We now focus on evaluation of some functions in $K(G_u)$. Let $R$ be a point in $G_u(K)$ outside of the subgroup generated by $t$. We want to evaluate $f = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \alpha_i u_i$ at $R + jt$ for $0 \leq j \leq p - 1$. We have

$$f((R + jt) = \sum_{i \in \mathbb{F}_p} \alpha_i u_i(R + jt) = \sum_{i \in \mathbb{F}_p} \alpha_i u_0(R + (j - i)t) = \vec{\alpha} \star_j \vec{u}_R$$

where $\vec{u}_R = (u_0(R + kt))_{k \in \mathbb{F}_p}$. So the evaluation of $f$ is given by the convolution product (7)

$$\vec{u}_R \star \vec{\alpha}.$$  

Similarly, for $f = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \alpha_i u_i^2$ we have

$$f(R + jt) = \sum_{i \in \mathbb{F}_p} \alpha_i u_i^2(R + jt) = \sum_{i \in \mathbb{F}_p} \alpha_i u_0^2(R + (j - i)t) = \vec{\alpha} \star_j \vec{u}_R$$

where $\vec{u}_R = (u_0^2(R + kt))_{k \in \mathbb{F}_p}$. So the evaluation of $f$ is given by the convolution product (8)

$$\vec{u}_R \star \vec{\alpha}.$$  

**Interpolation.** The evaluation map $f \mapsto (f(R + jt))_{j \in \mathbb{F}_p}$ is a bijection from the linear space $L$ onto $K^p$. Indeed two functions $f_1, f_2$ in $L$ have the same evaluation if and only if the function $f_1 - f_2$ cancels at $R, R + t, \ldots, R + (p - 1)t$ and $\infty$. But $f_1 - f_2$ has at most $p$ poles. So $f_1 = f_2$.

Given a vector $\vec{\beta} = (\beta_0, \ldots, \beta_{p-1})$ in $K^p$, we can compute the function $f$ in $L$ such that $f(R + jt) = \beta_j$ by inverting the evaluation map. That corresponds to the inverse $\vec{u}_R^{-1}$ of $\vec{u}_R$ for the convolution product.

**An efficient multiplication algorithm.** We want to compute the coordinates in $\Theta$ of the product

$$\left( \sum_{i \in \mathbb{F}_p} \alpha_i \theta_i \right) \times \left( \sum_{i \in \mathbb{F}_p} \beta_i \theta_j \right).$$

Define the functions

$$A = \sum_{i} \alpha_i u_i, \quad B = \sum_{i} \beta_i u_i,$$

$$C = \sum_{i} \alpha_i \beta_i u_i^2,$$

$$D = AB - C.$$  

The product we want to compute is $A(b) B(b) = C(b) + D(b)$.

If $i, j \in \mathbb{F}_p$ are such that $i \neq j$, then $u_i u_j$ lies in $L$. So

$$(\sum_{i} \alpha_i u_i)(\sum_{i} \alpha_i u_j) = \sum_{i} \alpha_i \beta_i u_i^2 \mod L,$$

that is $D$ is in $L$. From equation (8), we deduce that the coordinates in $\Theta$ of $C(b)$ are given by the vector

$$\vec{t} \star (\vec{\alpha} \diamond \vec{\beta}).$$

From equation (7), the evaluation of $A$ at the points $(R + jt)_j$ is given by $\vec{u}_R \star \vec{\alpha}$. The evaluation of $D$ at these points is

$$(\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) - \vec{w}_R \star (\vec{\alpha} \diamond \vec{\beta}).$$
We get the coordinates of $D$ in the basis $(u_0, \ldots, u_{p-1})$ by applying $\vec{u}_R^{-1}$ on the left to this vector. These are also the coordinates of $D(b)$ in the basis $\Theta$.

Altogether, the coordinates in $\Theta$ of the product $(\sum_{i \in \mathbb{F}_p} \alpha_i \theta_i) \times (\sum_{j \in \mathbb{F}_p} \beta_j \theta_j)$ are given by

$$\vec{\alpha} \star (\vec{\alpha} \diamond \vec{\beta}) + \vec{u}_R^{-1} \star \left( (\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) - \vec{w}_R \star (\vec{\alpha} \diamond \vec{\beta}) \right).$$

That consists in 5 convolution products, 2 component-wise products, 1 addition and 1 subtraction between vectors in $\mathbb{K}^p$.

Each convolution product can be computed at the expense of $O(p \log p | \log \log p |)$ operations in $\mathbb{K}$ using algorithms due to Schönhage and Strassen [12], Schönhage [13], Cantor and Kaltofen [2] (see [[15], section 8.3] for a survey). So the above multiplication algorithm has complexity $O(p \log p | \log \log p |)$.

2.3. **Example.** Take $p = 5$ and $\mathbb{K} = \mathbb{F}_5[X]/(X^3 + 3X + 2)$. We set $\epsilon = X \mod X^3 + 3X + 2$. We denote by $R$ the point in $G_a$ with coordinate $x(R) = \epsilon$. The point $a \in G_a(\mathbb{K})$ with coordinate $x(a) = 1$ does not lie in the image $I(G_a(\mathbb{K}))$ because $Y^5 - Y - 1$ is relatively prime to $Y^{125} - Y$. So $Y^5 - Y - 1$ is irreducible over $\mathbb{K}$. We set $L = \mathbb{K}[Y]/(Y^5 - Y - 1)$ and $\theta = Y \mod Y^5 - Y - 1$. So $\Theta = (\theta_k)_{0 \leq k \leq 4}$ is a normal basis of $L/\mathbb{K}$. The weight of $\Theta$ is equal to 13, according to following equations:

$$\theta_0^2 = 4\theta_0 + 4\theta_1 + 2\theta_2 + 3\theta_3 + \theta_4, \quad \theta_0 \times \theta_1 = -\theta_0 + \theta_1$$

$$\theta_0 \times \theta_2 = \frac{1}{2}(-\theta_0 + \theta_2), \quad \theta_0 \times \theta_3 = \frac{1}{3}(-\theta_0 + \theta_3), \quad \theta_0 \times \theta_4 = \frac{1}{4}(-\theta_0 + \theta_4).$$

Now we compute the coordinates in $\Theta$ of the product

$$\left( \sum_{i \in \mathbb{F}_5} \alpha_i \theta_i \right) \times \left( \sum_{i \in \mathbb{F}_5} \beta_j \theta_j \right)$$

where $\vec{\alpha} = (1, 3, 1, 1, 2)$ and $\vec{\beta} = (2, 1, 1, 4, 2)$.

We know that $\vec{\gamma} = (4, 4, 2, 3, 1)$. We compute

$$\vec{u}_R = (2\epsilon^2 + 1, 4\epsilon^2 + 4\epsilon + 1, 4\epsilon^2 + 3\epsilon + 3, 3\epsilon^2 + 4\epsilon + 1, 3\epsilon^2 + 2\epsilon + 2),$$

$$\vec{u}_R^{-1} = (3\epsilon + 4, 2\epsilon^2 + \epsilon + 4, 2\epsilon + 2 + 4\epsilon + 2, 3\epsilon, 3\epsilon^2 + \epsilon + 4),$$

$$\vec{u}_R = (2\epsilon^2 + 2\epsilon + 1, \epsilon^2 + 2 + 4\epsilon + 1, 3\epsilon + 3, 4\epsilon^2 + 4\epsilon).$$

So

$$\vec{\gamma} \star (\vec{\alpha} \diamond \vec{\beta}) = (3, 1, 1, 1, 0)$$

$$\vec{u}_R \star \vec{\alpha} = (\epsilon^2 + \epsilon + 3, 4\epsilon^2 + \epsilon + 3, 2\epsilon^2 + 1 + 2\epsilon^2 + \epsilon + 1, 4\epsilon^2 + \epsilon + 1)$$

$$\vec{u}_R \star \vec{\beta} = (4\epsilon^2 + \epsilon + 4, 3\epsilon^2 + 2\epsilon + 2\epsilon + \epsilon + 3, 3\epsilon^2 + 4\epsilon + 4, 3\epsilon^2 + 2\epsilon + 4)$$

$$\vec{w}_R \star (\vec{\alpha} \diamond \vec{\beta}) = (2, 2\epsilon^2 + 3\epsilon + 3, \epsilon^2 + 3\epsilon + 1, 2\epsilon, 4\epsilon + 2)$$

$$\vec{u}_R \star (\vec{\alpha} \diamond \vec{\beta}) = (4\epsilon + 2, 4\epsilon + 3, \epsilon^2 + 2\epsilon + 4 + 2, 2\epsilon^2 + 3\epsilon + 2, 4\epsilon + 2).$$

Therefore

$$\vec{u}_R^{-1} \star \left( (\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) - \vec{w}_R \star (\vec{\alpha} \diamond \vec{\beta}) \right) = (2, 2, 4, 4, 3).$$
Finally we get

$$\left( \sum_{i \in \mathbb{F}_5} \alpha_i \theta_i \right) \times \left( \sum_{j \in \mathbb{F}_5} \beta_j \theta_j \right) = 3\theta_1 + 3\theta_1.$$

### 3. The multiplicative group case

Let $K$ be a field with characteristic $p > 0$. Consider the affine line $\mathbb{A}^1$ over $K$ endowed with the $x$-coordinate. We identify the multiplicative group $G_m$ with the open subset $\{ x \neq 0 \}$ of $\mathbb{A}^1$. Any point $P$ in $G_m$ is given by its $x$-coordinate. The unit element $O_{G_m}$ has $x$-coordinate equal to 1. The group law $\oplus_{G_m}$ is defined by

$$x(P_1 \oplus_{G_m} P_1) = x(P_1) \times x(P_2).$$

### 3.1. Specializing isogenies of the multiplicative group

Let $m \geq 2$ and $m \leq 2$ be two integers such that $mn$ is prime to $p$. We assume that $K$ contains a primitive $mn$-th root of unity which we denote by $\zeta_{mn}$. If $K = F_q$ is a finite field, this is equivalent to saying that $mn$ divides $q - 1$. We set $\zeta = (\zeta_{mn})^m$. The $n$-torsion $G_m[n]$ is a cyclic subgroup of order $n$ of $G_m(K)$. Let $I : G_m \to G_m$ be quotient isogeny of $G_m$ by $G_m[n]$. This is the multiplication by $n$ isogeny. Let $a$ be a $K$-rational point in $G_m$ such that $a \mod I(G_m(K))$ has order $n$ in $G_m(K)/I(G_m(K))$. In case $K = F_q$ is a finite field, take for $a$ a generator of $\mathbb{F}_q^*$ a sufficient condition for this last requirement. In any case, the subfield $L = K(I^{-1}(a))$ of $K$ is a cyclic extension of $K$ with degree $n$. Indeed, fix $b$ in $I^{-1}(a)$ and denote by $\theta$ the $x$-coordinate of $b$. Since $L$ is the splitting field of the separable polynomial

$$X^n - x(a) = (X - \theta)(X - \zeta_n \theta)(X - \zeta_n^2 \theta) \cdots (X - \zeta_n^{n-1} \theta) \in K[X],$$

it is a normal and separable extension of $K$. Let $P(X) \in K[X]$ be a monic irreducible factor of $X^n - x(a)$ with degree $r \leq n$ such that $r$ is the smallest element among the degrees of irreducible polynomials dividing $X^n - x(a)$. Then $r$ is the smallest integer $> 0$ such that $\theta^r \in K$. This means that $r$ is the smallest integer $> 0$ such that $[r](a) \in I(G_m(K))$. Since $a \mod I(G_m(K))$ has order $n$ in $G_m(K)/I(G_m(K))$, we have $r = n$ and $P(X) = X^n - x(a)$. Hence $L$ is a degree $n$ Galois extension of $K$. The Galois group $\text{Gal}(L/K)$ is made of $K$-automorphisms $a_k$ such that $a_k(\theta) = \zeta_n^k \theta$, for $0 \leq k \leq n - 1$. So $\text{Gal}(L/K)$ is generated by $a_1$. We set $t = a_1(b) \oplus_{G_m} b$. Then the fiber of $I$ above $a$ is given by

$$(9) \quad I^{-1}(a) = [b] + [b \oplus_{G_m} t] + \cdots + [b \oplus_{G_m} (n - 1)t].$$

The multiplicative group is an open subset of the projective line $\mathbb{P}^1$. Consider the divisor

$$D = [O_{G_m}] + [t] + [2t] + \cdots + [(n - 1)t] - [\infty] \in \text{Div}(\mathbb{P}^1).$$

The linear space

$$(10) \quad \mathcal{L} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D))$$

has dimension $n$. The translation $\tau : P \mapsto P \oplus_{G_m} t$ is an automorphism of $G_m$ which extends to the whole $\mathbb{P}^1$ ($\tau(\infty) = \infty$). The divisor $D$ is invariant by $\tau$, then so is the space $\mathcal{L}$. The functions

$$\frac{1}{x - 1}, \frac{1}{\zeta_n^{-1}x - 1}, \cdots, \frac{1}{\zeta_n^{-(n-1)}x - 1} \in K(G_m)$$
lie in \( \mathcal{L} \). Examination of poles shows that they are linearly independent. So they form a basis of \( \mathcal{L} \) invariant by \( \tau \). Evaluation of these functions at \( b \), results in a normal basis

\[
\Theta = \left( \frac{1}{\theta - 1}, \frac{1}{\zeta^{-1}\theta - 1}, \ldots, \frac{1}{\zeta^{-(n-1)}\theta - 1} \right)
\]

of \( \mathcal{L} \) over \( K \). Indeed, let \( \lambda_0, \ldots, \lambda_{n-1} \) be scalars in \( K \) such that

\[
\sum_{0 \leq k \leq n-1} \lambda_k \frac{1}{\zeta^{-k}\theta - 1} = 0.
\]

Then the function \( \sum_{0 \leq k \leq n-1} \lambda_k \frac{1}{\zeta^{k}x - 1} = \frac{F(x)}{(\zeta^{-1}x - 1)(\zeta^{-2}x - 1)\cdots(\zeta^{-(n-1)}x - 1)} \) cancels at \( b \), where \( F(X) \) is a polynomial in \( K[X] \). If \( F(X) \) is non-zero, then its degree is at most \( \leq n - 1 \). But \( F \) cancels at \( \theta \) and also at all its \( n \) conjugates, this is impossible. So \( F \) is the zero polynomial. All \( \lambda_k \) are 0 because the functions \( \frac{1}{\zeta^{-k}\theta - 1} \) are linearly independant. Recall that \( \text{Gal}(L/K) = \{ \alpha_k | 0 \leq k \leq n - 1 \} \) is generated by \( \alpha_1 \) which satisfies \( \alpha_1(\theta) = \zeta_n \theta \). So

\[
\alpha_1\left( \frac{1}{\zeta_n^{-k}\theta - 1} \right) = \frac{1}{\zeta_n^{-(k-1)}\theta - 1}.
\]

We conclude that \( \Theta \) is a normal basis. If \( i \neq 0 \), then

\[
\frac{1}{\theta - 1} \times \frac{1}{\zeta_n^{-i}\theta - 1} = \frac{1}{\zeta_n^{-i} - 1} \left( \frac{1}{\theta - 1} - \frac{\zeta_n^{-i}}{\zeta_n^{-i}\theta - 1} \right).
\]

Since the weight can be defined using linear combinations in equation \( [2] \), we conclude that the weight of \( \Theta \) is at most \( 3n - 2 \).

3.2. Complexity. We use the same procedure as in section [2,2] The notation is also the same, except that

\[
u = \frac{1}{x - 1}, \quad \theta_0 = \frac{1}{\theta - 1},
\]

and for \( 0 \leq k \leq p - 1 \)

\[
u_k = \nu \circ \tau^k, \quad \theta_k = \alpha_k(\theta_0), \quad \xi_k = \theta_k^2.
\]

We set \( \xi_0 = \sum_{0 \leq k \leq n-1} \nu_k \theta_k \) and \( \theta^R = (\nu_k)_{0 \leq k \leq n-1} \). We denote by \( R \) the point in \( G_m(K) \) with coordinate \( x(R) = \zeta_{mn} \). The evaluation map

\[
f \mapsto (f(R + jR))_{0 \leq k \leq n-1}
\]

is a bijection from the linear space \( \mathcal{L} \) onto \( K^n \). Its inverse map is

\[
(a_i)_{0 \leq i \leq n-1} \mapsto \mathbf{u}_R^{-1} \ast (a_i)_{0 \leq i \leq n-1},
\]

where \( \mathbf{u}_R = (\nu_0(R + kR))_{0 \leq k \leq n-1} \) and \( \mathbf{u}_R^{-1} \) is the inverse of \( \mathbf{u}_R \) for the convolution product.

The coordinates in \( \Theta \) of the product

\[
\left( \sum_{0 \leq k \leq n-1} \alpha_k \theta_k \right) \times \left( \sum_{0 \leq k \leq n-1} \beta_k \theta_k \right)
\]
are given by
\[
\vec{t} \star (\vec{\alpha} \diamond \vec{\beta}) + \vec{u}_R^{-1} \star \left( (\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) \right),
\]
where \(\vec{u}_R = (u_{0}^2(R + kt))_{0 \leq k < n-1}\). This multiplication algorithm consists in 5 convolution products, 2 component-wise products, 1 addition and 1 subtraction between vectors in \(K^n\). Using fast algorithms for the convolution products, this multiplication algorithm has complexity \(O(n(\log n)|\log |\log n|)|\).

3.3. Example. Take \(p = 61\) and \(n = 6\), we set \(K = F_{61}\). The point \(a \in G_m(K)\) with coordinate \(x(a) = 2\) mod 61 generates the group \(G_m(K)\), and 48 mod 61 is a primitive 6-th root of unity in \(K\). So \(a \mod \mathcal{I}(G_m(K))\) has order 6 in \(G_m(F_{61})/\mathcal{I}(G_m(K))\). Hence \(X^6 - 2 \in K[X]\) is irreducible over \(K\). We set \(L = K[X]/(X^6 - 2)\) and \(\theta = X \mod X^6 - 2\). Thus \(\Theta = (\theta_k)_{0 \leq k \leq 5}\) is a normal basis of \(L/K\). The weight of \(\Theta\) is equal to 15, according to following equations:
\[
\theta_0^2 = 53\theta_1 + 40\theta_2 + 23\theta_3 + 50\theta_4 + 18\theta_5, \quad \theta_0 \times \theta_1 = 47(\theta_0 - 14\theta_1), \quad \theta_0 \times \theta_2 = 56(\theta_0 - 13\theta_2),
\]
\[
\theta_0 \times \theta_3 = 30(\theta_0 + \theta_3), \quad \theta_0 \times \theta_4 = 4(\theta_0 - 47\theta_4), \quad \theta_0 \times \theta_5 = 13(\theta_0 - 48\theta_5).
\]
We are going to compute the coordinates in \(\Theta\) of the product
\[
\left( \sum_{0 \leq i \leq 5} \alpha_i \theta_i \right) \times \left( \sum_{0 \leq j \leq 5} \beta_j \theta_j \right)
\]
where \(\vec{\alpha} = (1, 3, 1, 1, 2, 1)\) and \(\vec{\beta} = (2, 1, 1, 4, 2, 1)\).
We know that \(\vec{t} = (0, 53, 40, 23, 50, 18)\). We compute
\[
\begin{align*}
\vec{u}_a &= (1, 9, 21, 20, 22, 52) \\
\vec{u}_a^{-1} &= (43, 11, 37, 55, 46, 32), \\
\vec{u}_a^2 &= (1, 20, 14, 34, 57, 20).
\end{align*}
\]
So
\[
\begin{align*}
\vec{t} \star (\vec{\alpha} \diamond \vec{\beta}) &= (43, 29, 46, 36, 12, 32) \\
\vec{u}_a \star \vec{\alpha} &= (6, 25, 43, 36, 44, 56) \\
\vec{u}_a \star \vec{\beta} &= (26, 14, 4, 27, 29, 16) \\
\vec{w}_a \star (\vec{\alpha} \diamond \vec{\beta}) &= (26, 14, 4, 27, 29, 16) \\
(\vec{u}_a \star \vec{\alpha}) \diamond (\vec{u}_a \star \vec{\beta}) &= (52, 45, 50, 57, 56, 42).
\end{align*}
\]
Therefore
\[
\vec{u}_a^{-1} \star \left( (\vec{u}_a \star \vec{\alpha}) \diamond (\vec{u}_a \star \vec{\beta}) - \vec{w}_a \star (\vec{\alpha} \diamond \vec{\beta}) \right) = (24, 25, 20, 28, 33, 54).
\]
Finally we get
\[
\left( \sum_{0 \leq i \leq 5} \alpha_i \theta_i \right) \times \left( \sum_{0 \leq j \leq 5} \beta_j \theta_j \right) = 6\theta_0 + 54\theta_1 + 5\theta_2 + 3\theta_3 + 45\theta_4 + 25\theta_5.
\]

4. The Lucas Torus Case
This section is devoted to the use of Lucas torus for constructing normal bases.
### 4.1. Basic facts concerning the Lucas Torus.

Let $K$ be a field with characteristic different from 2. Let $\alpha \in K$ be a nonsquare element. The Lucas torus $T_\alpha$ over $K$ is the affine plane curve defined by

$$T_\alpha : x^2 - \alpha y^2 = 1.$$  \hfill (12)

This is a commutative algebraic group with group law $\oplus_{T_\alpha}$ defined by

$$(x, y) \oplus_{T_\alpha} (x', y') = (xx' + \alpha yy', xy' + x'y).$$  \hfill (13)

Its unit element $O_{T_\alpha}$ has coordinates equal to $(1, 0)$.

It is easily checked that the map

$$\varphi : T_\alpha(K) \longrightarrow \{\xi = (a + b\sqrt{\alpha}) \in K(\sqrt{\alpha}) \mid \text{Norm}(\xi) = 1\}$$

$$(x, y) \longmapsto x + y\sqrt{\alpha}$$

is a group isomorphism.

Assume that $K = F_q$ is a finite field with characteristic $p > 2$, and $\alpha \in F_q$ is a nonsquare element. We have an exact sequence

$$0 \longrightarrow T_\alpha(F_q) \overset{1}{\longrightarrow} F_q^* \overset{\text{Norm}}{\longrightarrow} F_q^* \longrightarrow 0$$

Indeed the $\iota$ map is injective because the above map $\varphi$ is injective. Further the $\text{Norm}$ map is surjective [see [14], proposition 4, page 34]. So $T_\alpha(F_q)$ is a cyclic group with order $(q^2 - 1)/(q - 1) = q + 1$.

Actually the map $\varphi : (x, y) \mapsto z := x + y\sqrt{\alpha}$ is an isomorphism of $K(\sqrt{\alpha})$-varieties, from the Lucas torus onto the multiplicative group. The inverse map is given by $z \mapsto (\sqrt{\frac{z^2+1}{2z^2}}, \sqrt{\frac{z^2-1}{2z}})$. So $T_\alpha$ is a twist of the multiplicative group.

### 4.2. Specializing isogenies of the Lucas torus.

Assume that $n$ is a non-trivial divisor of $q + 1$. We denote by $m$ the cofactor of $n$, that is $q + 1 = nm$. The $n$-torsion $T_\alpha(F_q)[n]$ is a cyclic subgroup of $T_\alpha(F_q)$. Let $\Phi : F_q^n \rightarrow F_q^n$ be the $q$-Frobenius automorphism. Let $I : T_\alpha \rightarrow T_\alpha$ be the quotient isogeny of $T_\alpha$ by $T_\alpha[n]$. This is the multiplication by $n$ isogeny. We have

$$\varphi(I(x, y)) = \varphi(x, y)^n = (x + \sqrt{\alpha}y)^n$$

$$= \sum_{0 \leq 2k \leq 2n} \binom{n}{2k} x^{n-2k} y^{2k} \alpha^k + \sqrt{\alpha} \sum_{0 \leq 2k+1 \leq 2n} \binom{n}{2k + 1} x^{n-2k-1} y^{2k+1} \alpha^k.$$

The quotient group $T_\alpha(F_q)/I(T_\alpha(F_q))$ is cyclic of order $n$. Let $a$ be a generator of $T_\alpha(F_q)$. Then $a \text{ mod } I(T_\alpha(F_q))$ generates $T_\alpha(F_q)/I(T_\alpha(F_q))$. The subfield $L = F_q(I^{-1}(a))$ of $F_q^n$ is a cyclic extension of $F_q$ with degree $n$. Indeed $L$ is separable because it is an algebraic extension of the perfect field $F_q^n$. The fiber $I^{-1}(a)$ is defined over $F_q$ because our Lucas torus is defined over $F_q$ and $a$ is a $F_q$-rational point. Thus any $F_q$-automorphism of $F_q^n$ maps $I^{-1}(a)$ into itself. So the $F_q$-automorphisms of $F_q^n$ map $L$ into itself, that is $L$ is normal over $F_q$. Let $b$ be a point in $I^{-1}(a)$. For any $\sigma \in \text{Gal}(L/F_q)$ we have

$$\sigma b \ominus_{T_\alpha} b \in \text{Ker}(I),$$  \hfill (14)
that is there exists \( t_\sigma \in \mathbb{T}_\alpha[n] \) such that
\[
(15) \quad \sigma b = b \oplus_{\mathbb{T}_\alpha} t_\sigma.
\]
If \( \sigma, \sigma' \) lie in \( \text{Gal}(L/F_\alpha) \) then
\[
\sigma \circ \sigma' b = (\sigma(\sigma' b)) = \sigma(b \oplus_{\mathbb{T}_\alpha} t_{\sigma'}) = b \oplus_{\mathbb{T}_\alpha} t_{\sigma'} \oplus_{\mathbb{T}_\alpha} t_\sigma.
\]
So \( \sigma \circ \sigma' b = \sigma' \circ \sigma b \). Hence \( \text{Gal}(L/F_\alpha) \) is an abelian group with exponent dividing \( n \). Let \( \langle a \mod I(\mathbb{T}_\alpha(F_\alpha)) \rangle \) be the subgroup of \( \mathbb{T}_\alpha(F_\alpha)/I(\mathbb{T}_\alpha(F_\alpha)) \) generated by the class of \( a \). The map
\[
\kappa: \langle a \mod I(\mathbb{T}_\alpha(F_\alpha)) \rangle / I(\mathbb{T}_\alpha(F_\alpha)) \times \text{Gal}(L/F_\alpha) \rightarrow \mathbb{T}_\alpha[n]
\]
\[
(k\overline{\alpha}, \sigma) \mapsto (kb)^\sigma - (kb),
\]
is a non-degenerate pairing. It induces a group isomorphism
\[
\kappa_L: \text{Gal}(L/F_\alpha) \rightarrow \text{Hom}(\langle a \mod I(\mathbb{T}_\alpha(F_\alpha)) \rangle / I(\mathbb{T}_\alpha(F_\alpha)), \mathbb{T}_\alpha[n]).
\]
The group \( \text{Hom}(\langle a \mod I(\mathbb{T}_\alpha(F_\alpha)) \rangle / I(\mathbb{T}_\alpha(F_\alpha)), \mathbb{T}_\alpha[n]) \) has order \( n \). So \( \# \text{Gal}(L/F_\alpha) = n \), that is \( L \) is a degree \( n \) cyclic extension of \( F_\alpha \) with Galois group generated by the \( q \)-Frobenius automorphism \( \Phi \). We set \( t = b^\Phi \odot_{\mathbb{T}_\alpha} b \). This a generator of the \( n \)-torsion \( \mathbb{T}_\alpha[n] \). So the fiber of \( I \) above \( a \) is given by
\[
(16) \quad I^{-1}(a) = \{ b + [b \oplus_{\mathbb{T}_\alpha} t] + \cdots + [b \oplus_{\mathbb{T}_\alpha} (n-1)t] \}.
\]

The projective closure of \( \mathbb{T}_\alpha \) is the locus
\[
(17) \quad \overline{\mathbb{T}_\alpha}: X^2 - \alpha Y^2 - Z^2 = 0
\]
in the projective plane \( \mathbb{P}^2 \). This is a genus 0 projective curve with two points \( \infty_1 = (\sqrt{\alpha} : 1 : 0) \) and \( \infty_2 = (-\sqrt{\alpha} : 1 : 0) \) on the line at infinity. The translation \( \tau: P \mapsto P \odot_{\mathbb{T}_\alpha} t \) is an automorphism of the Lucas torus which extends to its projective closure \( \overline{\mathbb{T}_\alpha} \) by setting \( \tau(\infty_1) = \infty_1 \), and \( \tau(\infty_2) = \infty_2 \).

Denote by \( \text{Div}^0(\mathbb{T}_\alpha) \) the subgroup of \( \text{Div}^0(\overline{\mathbb{T}_\alpha}) \) made of divisors \( D \) with support contained in \( \mathbb{T}_\alpha(F_\alpha) \). A divisor \( D \) in \( \text{Div}^0(\mathbb{T}_\alpha) \) is said to be principal if there exists a non-zero function \( f \) in \( F_\alpha(\mathbb{T}_\alpha) \) such that \( f(\infty_1) = f(\infty_2) \) and \( D = \text{div}(f) \).

**Claim.** If \( D = \text{div}(f) \) is a principal divisor in \( \text{Div}^0(\overline{\mathbb{T}_\alpha}) \), then \( \sum_{P \in \overline{\mathbb{T}_\alpha}} |\text{ord}_P(f)|P = O_{\mathbb{T}_\alpha} \) and \( f \) has at least two poles.

Indeed, using arguments similar to that used in [\cite{S}, section 4], one shows that a point \( P \) in \( \mathbb{T}_\alpha \) is a zero (resp. a pole) of \( f \) if and only if \( -P \) is a zero (resp. a pole) of \( f \). More precisely, for any \( P \) in \( \mathbb{T}_\alpha \), we have \( \text{ord}_P(f) = \text{ord}_{-P}(f) \). So
\[
\sum_{P \in \mathbb{T}_\alpha} |\text{ord}_P(f)|P = O_{\mathbb{T}_\alpha}.
\]

Since \( \text{div}(f) \) has degree zero, we are done.

Set
\[
x = X/Z, \quad y = Y/Z \quad \text{and} \quad v = \frac{x - 1}{y}.
\]
Using equation (17), we obtain
\[ y = \frac{(x + 1)v}{\alpha}, \quad x = \frac{(x + 1)v^2}{\alpha} + 1. \]

Besides,
\[ \text{div}(v) = [O_{\alpha}] - [\tilde{O}_\alpha], \]
where \( \tilde{O}_\alpha \) is the point in \( \mathbb{T}_\alpha \) with coordinate \((-1, 0)\). Let \( u_{O,t} \) be the function in \( F_q(\mathbb{T}_\alpha) \) defined by
\[ u_{O,t} = 1 + \frac{1}{y - \frac{y(t)}{x(t) - 1}(x - 1)}. \]
Its polar divisor is equal to \(-[O_{\alpha}] - [t]\), and we have \( u_{O,t}(\infty_1) = u_{O,t}(\infty_2) = 1 \).
For \( 0 \leq k \leq n - 1 \), we set
\[ (18) \quad y_k = y \circ \tau^k, \quad x_k = x \circ \tau^k, \quad v_k = v \circ \tau^k \text{ and } u_{kt,(k+1)t} := u_{O,t} \circ \tau^k. \]
The function \( v_k \) is a uniformizer at \( kt \). Besides, the function \( u_{kt,(k+1)t} \) has two simple poles \((kt)\) and \((k+1)t\), and \( \text{div}(u_{kt,(k+1)t}) \) is a principal divisor on \( \mathbb{T}_\alpha \). Since
\[ u_{kt,(k+1)t} = 1 + \frac{1}{y_k} \times \frac{1}{1 - \frac{y(t)}{x(t) - 1} \times v_k}, \]
the Taylor expansion of \( u_{kt,(k+1)t} \) at \( kt \) is
\[ (19) \quad u_{kt,(k+1)t} = 1 + \frac{\alpha}{2} \left( \frac{1}{y_k} + \frac{y(t)}{(x(t) - 1)} + \frac{y^2(t)}{(x(t) - 1)^2} v_k + O(v_k^2) \right). \]
Since
\[ u_{kt,(k+1)t} = 1 + \frac{1}{y(P - (k + 1)t + t) - \frac{y(t)}{x(t) - 1}(x(P - (k + 1)t + t) - 1 + x_{k+1} - x_{k+1})} \]
\[ = 1 - \frac{1}{y_{k+1}} \times \frac{1}{1 + \frac{y(t)}{x(t) - 1} \times v_{k+1}}, \]
the Taylor expansion of \( u_k \) at \((k+1)t\) is
\[ u_{kt,(k+1)t} = 1 - \frac{\alpha}{2} \left( \frac{1}{y_{k+1}} - \frac{y(t)}{(x(t) - 1)} + \frac{y^2(t)}{(x(t) - 1)^2} v_{k+1} - O(v_{k+1}^2) \right). \]
The divisor
\[ D = [0_{\alpha}] + [t] + [2t] + \cdots + [(n - 1)t]. \]
induces a linear space
\[ \mathcal{L} = H^0(\mathbb{T}_\alpha, O_{\mathbb{T}_\alpha}(D)) \]
with dimension \( n + 1 \). Denote by \( L_1 \) the subspace of \( \mathcal{L} \) made of functions \( f \in \mathcal{L} \) such that \( f(\infty_1) = f(\infty_2) \). This is a \( n \)-dimensional \( F_q \)-vector space, as shown by the following lemma.

**Lemma 4.1.** With the above notation:

1. The sum \( \sum_{0 \leq k \leq n - 1} u_{kt,(k+1)t} \) is a constant \( c \in F_q \), and there exists two scalars \( a \neq 0 \) and \( b \) in \( F_q \) such that \( ac + nb = 1 \).
(2) We set \( u_k = a u_{kt,(k+1)t} + b \). Then \((u_0, u_1, \ldots, u_{n-1})\) is a basis of \( \mathcal{L}_1 \).

(3) The system \( \Theta = (\theta_0, \theta_1, \ldots, \theta_{n-1}) \) defined by \( \theta_k = u_k(b) \) is a normal basis of \( \mathbb{L} \) over \( \mathbb{F}_q \).

Proof. (1) Since \( \text{div}(\sum_{0 \leq k \leq n-1} u_{kt,(k+1)t}) \) is a principal divisor on \( \mathbb{T}_\alpha \), the proof uses the arguments of [4, proof of lemma 4]. First we denote by \( T \) the subgroup generated by \( t \). The sum \( \sum_{0 \leq k \leq n-1} u_{kt,(k+1)t} \) is invariant by translations in \( T \). So it can be seen as a function on the quotient of \( \mathbb{T}_\alpha \) by \( T \), with no more than one pole and taking the same value at two distinct points. Hence it is a constant which we denote by \( c \). If the characteristic \( p \) of \( \mathbb{F}_q \) divides \( n \), it is easily checked that \( c \neq 0 \) using the function \( \sum_{0 \leq k \leq n-1} ku_{kt,(k+1)t} \).

So either \( n \) is prime to \( p \) or \( c \neq 0 \). In any case, there exist two scalars \( a \neq 0 \) and \( b \) in \( \mathbb{F}_q \) such that \( ac + nb = 1 \).

(2) and 3. Let \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) be scalars in \( \mathbb{F}_q \) such that

\[
\sum_{k \in \mathbb{Z}/n\mathbb{Z}} \lambda_k \theta_k = 0.
\]

Then the function \( F = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \lambda_k u_k \in \mathbb{F}_q[X] \) cancels at \( b \) and also at all its \( n \) conjugates over \( \mathbb{F}_q \). Since \( \text{div}(\sum_{k \in \mathbb{Z}/n\mathbb{Z}} \lambda_k u_k) \) is a principal divisor on \( \mathbb{T}_\alpha \) and \( b \) does not lie in the \( n \)-torsion, we deduce that \( \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \lambda_k u_k = 0 \). Examination of poles shows that all \( \lambda \)'s are equal. So

\[
\lambda_0(\sum_{k \in \mathbb{Z}/n\mathbb{Z}} u_k) = 0.
\]

But \( \sum_{k \in \mathbb{Z}/n\mathbb{Z}} u_k = 1 \). So all \( \lambda \)'s are equal to zero.

Recall that \( \text{Gal}(\mathbb{L}/\mathbb{F}_q) \) is generated by \( \Phi \) which satisfies \( b^\Phi = b + t \). So \( \Phi(\theta_k) = \theta_{k-1} \).

\( \square \)

If \( k, l \in \mathbb{Z}/n\mathbb{Z} \) and \( k \neq l, l + 1, l - 1 \mod n \), then

\[
u_k u_l \in \mathcal{L}_1.\]

Further

\[
u_k u_l + \frac{\alpha^2}{4v_k^2} \in \mathcal{L}_1,
\]

and

\[
u_k^2 - \frac{\alpha^2}{4v_k^2} - \frac{\alpha^2}{4v_{k+1}^2} \in \mathcal{L}_1.
\]

If \( \sum_k a_k u_k \) and \( \sum_k b_k u_k \) are two functions in \( \mathcal{L}_1 \), then

\[
(\sum_k a_k u_k)(\sum_k b_k u_k) = \frac{\alpha^2}{4} \sum_k a_k b_k \left( \frac{1}{v_k^2} + \frac{1}{v_{k+1}^2} \right) - \frac{\alpha^2}{4} \sum_k a_{k-1} b_k - \frac{\alpha^2}{4} \sum_k b_{k-1} a_k \mod \mathcal{L}_1,
\]

that is

\[
(\sum_k a_k u_k)(\sum_k b_k u_k) = \frac{\alpha^2}{4} \sum_k \frac{(a_k - a_{k-1})(b_k - b_{k-1})}{v_k^2} \mod \mathcal{L}_1.
\]
4.3. Complexity. We use the same procedure as in section 2.2. The notation is also the same, except that

\[
u = a(1 + \frac{1}{y - \frac{y(t)}{x(t) - 1}(x - 1)}) + b, \quad v = \frac{x - 1}{y},\]

and for \(0 \leq k \leq n - 1\) we set

\[
u_k = u \circ \tau^k, \quad \theta_k = u_k(b), \quad v_k = v \circ \tau^k, \quad \xi_k = \frac{1}{v_k^2(b)}.
\]

We also set \(\xi_0 = \sum_{0 \leq k \leq n-1} v_k \theta_k\) and \(\vec{r} = (v_k)_{0 \leq k \leq n-1}\). A linear combination of the \(\xi\)'s can be reduced into a linear combination of the \(\theta\)'s

\[
\sum_{0 \leq i \leq n-1} x_i \xi_i = \sum_{0 \leq k \leq n-1} (\vec{r} \ast_k \vec{r}) \theta_k,
\]

where \(\vec{r} = (x_0, x_1, \ldots, x_{n-1})\). The evaluation map \(f \mapsto (f(a + j t))_{0 \leq j \leq n-1}\) is a bijection from the linear space \(L_1\) onto \(F_q^n\) because \([n](a) \neq O_{T, a}\). Its inverse map is

\[
\vec{r} \mapsto \vec{u}_{a-1} \ast \vec{r},
\]

where \(\vec{u}_{a-1}\) is the inverse of \(\vec{u}_a = (u_0(a + j t))_{0 \leq j \leq n-1}\) for the convolution product.

So the coordinates in \(\Theta\) of a product

\[
\left( \sum_{0 \leq i \leq n-1} x_i \theta_i \right) \times \left( \sum_{0 \leq j \leq n-1} y_j \theta_j \right)
\]

are given by

\[
(\alpha \vec{r} \ast \vec{r}) \ast (\sigma(\vec{r}) - \sigma(\vec{r})) +
\]

\[
\vec{u}_{a-1} \ast \left( [\vec{u}_a \ast \vec{r}) \ast (\vec{u}_a \ast \vec{y}) - (\alpha^2 \vec{u}_a \ast \vec{r}) \ast (\vec{r} - \sigma(\vec{r})) \right) +
\]

where \(\vec{y} = (y_0, y_1, \ldots, y_{n-1})\) and \(\vec{u}_a = \left(\frac{1}{\nu(a + j t)}\right)_{0 \leq j \leq n-1}\). That consists in 5 convolution products, 2 component-wise products, 1 additions and 3 subtraction between vectors in \(F_q^n\). So this multiplication algorithm has complexity \(O(n \log n) \log |\log n|\).

4.4. Example. Take \(p = q = 7\), \(n = 4\), and \(\alpha = 3\). We set \(K = F_7\). The point \(a = (5, 1)\) generates the group \(T_3(F_7)\). The point \(t = (0, 3)\) generates the 4-torsion subgroup. The multiplication by 4 isogeny \(I : T_3 \to T_3\) is given in terms of coordinates by

\[
I(x, y) = (x^4 + 4y^2x^2 + 2y^4, 4yx^3 + 5y^3x).
\]

We focus on the intersection points of the two planes curves

\[
\{x^4 + 4y^2x^2 + 2y^4 - 5 = 0\} \quad \text{and} \quad \{4yx^3 + 5y^3x - 1 = 0\}
\]

Since their defining polynomials are relatively prime, they have finitely many intersection points. The polynomial \(P(x) = x^4 + x^2 + 3\) is an irreducible factor of the resultant of the defining polynomials. We set \(L = K[x]/(x^4 + x^2 + 3)\), \(\theta = x \mod x^4 + x^2 + 3\), and \(b = (\theta, \theta^2 + 5\theta)\).
We find
\[(u_{kt,(k+1)t})_{k \in \mathbb{Z}/4\mathbb{Z}} = \left( \frac{1}{y + 3(x - 1)}, -\frac{1}{3(x + 9y + 1)}, \frac{1}{3(9y + 27x - 1)}, -\frac{1}{3(27x + 243y + 1)} \right),\]
so that \(c = 0, a = 1, b = 2\) and
\[
\begin{align*}
\theta_0 &= 2\theta^3 + 6\theta^2 + \theta + 1, & \theta_1 &= \theta^3 + \theta^2 + \theta + 2, \\
\theta_2 &= 5\theta^3 + 6\theta^2 + 6\theta + 1, & \theta_3 &= 6\theta^3 + \theta^2 + 6\theta + 2.
\end{align*}
\]
We are going to compute the coordinates in \(\Theta = (\theta_k)_{0 \leq k \leq 3}\) of the product
\[
\left( \sum_{0 \leq i \leq 3} x_i \theta_i \right) \times \left( \sum_{0 \leq j \leq 3} y_j \theta_j \right)
\]
with \(\overrightarrow{x} = (1, 3, 1, 1)\) and \(\overrightarrow{y} = (2, 1, 1, 4)\).

We know that \(\overrightarrow{t} = (3, 0, 0, 3)\).

We compute
\[
\begin{align*}
\overrightarrow{u}_{a} &= (1, 4, 4, 0) \\
\overrightarrow{u}_{a}^{-1} &= (0, 2, 6, 3), \\
\overrightarrow{w}_{a} &= (1, 2, 2, 0).
\end{align*}
\]
So
\[
\begin{align*}
\left(\frac{\alpha^2}{4} \overrightarrow{t} \right) \ast (\overrightarrow{x} - \sigma(\overrightarrow{x})) \ast (\overrightarrow{y} - \sigma(\overrightarrow{y})) &= (4, 4, 0, 0) \\
\overrightarrow{u}_{a} \ast \overrightarrow{\alpha} &= (2, 4, 3, 3) \\
\overrightarrow{u}_{a} \ast \overrightarrow{\beta} &= (1, 4, 6, 5) \\
\left(\frac{\alpha^2}{4} \overrightarrow{u}_{a} \right) \ast (\overrightarrow{x} - \sigma(\overrightarrow{x})) \ast (\overrightarrow{y} - \sigma(\overrightarrow{y})) &= (0, 6, 5, 5) \\
\overrightarrow{u}_{a} \ast \overrightarrow{\alpha} \ast (\overrightarrow{u}_{a} \ast \overrightarrow{\beta}) &= (2, 2, 4, 1).
\end{align*}
\]
Therefore
\[
\overrightarrow{u}_{a}^{-1} \ast (\overrightarrow{u}_{a} \ast \overrightarrow{\alpha}) \ast (\overrightarrow{u}_{a} \ast \overrightarrow{\beta}) - \left(\frac{\alpha^2}{4} \overrightarrow{u}_{a} \right) \ast (\overrightarrow{x} - \sigma(\overrightarrow{x})) \ast (\overrightarrow{y} - \sigma(\overrightarrow{y})) = (2, 3, 6, 3).
\]
Finally we get
\[
\begin{align*}
\left( \sum_{0 \leq i \leq 3} \alpha_i \theta_i \right) \times \left( \sum_{0 \leq j \leq 3} \beta_j \theta_j \right) &= 6\theta_0 + 6\theta_2 + 3\theta_3.
\end{align*}
\]
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