HALF SPACE THEOREM FOR THE ALLEN-CAHN EQUATION

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Abstract. We prove the following half-space theorem for Allen-Cahn equation: Let \( n = 2 \) or \( 3 \) and \( u \) be a bounded solution of Allen-Cahn equation \( \Delta u + u - u^3 = 0 \) in \( \mathbb{R}^n \) such that the level set \( \{ u = 0 \} \) is contained in a half space \( \{ x_n > 0 \} \). Then \( u \) must be one-dimensional.

1. Introduction and main results

We consider rigidity results for bounded entire solutions of the Allen-Cahn equation
\begin{equation}
-\Delta u = u - u^3, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.
\end{equation}

For \( n = 1 \), (1) has a heteroclinic solution \( H(x) = \tanh \left( \frac{x}{\sqrt{2}} \right) \). Up to a translation and rotation, this is the unique monotone increasing solution in \( \mathbb{R} \).

De Giorgi ([5]) conjectured that for \( n \leq 8 \), if a solution to (1) is monotone in one direction, then up to translation and rotation it must be one dimensional. This conjecture has been proved to be true for \( n = 2 \) (Ghoussoub-Gui [10]), \( n = 3 \) (Ambrosio-Cabre [3]), and for \( 4 \leq n \leq 8 \) (Savin [12]), under an additional limiting condition
\[ \lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1. \]

On the other hand, for \( n \geq 9 \), Del Pino, Kowalczyk and the third author ([6]) constructed counterexamples, i.e. monotone solutions which are not one-dimensional.

The second rigidity result concerns the global minimizers: Savin [12] proved that global minimizers are one-dimensional up to dimensions \( n \leq 7 \). In dimension 8 we ([11]) constructed counterexamples, i.e. global minimizers which are not one-dimensional. For related results on rigidity of solutions to Allen-Cahn, we refer to [2, 4, 7, 9] and the references therein.

In this short note we prove the following half-space rigidity result:

Theorem 1. (Half space theorem) Let \( n \) be 2 or 3. Suppose \( \{ u > 0 \} \) is contained in \( \{ x_n > 0 \} \). Then \( u \) is one-dimensional.

Note that if \( \{ u = 0 \} \) is empty, then using comparison, we can show that \( u = \pm 1 \). Hence we can assume without loss of generality that \( \{ u = 0 \} \) is nonempty. The assumption \( \{ u > 0 \} \) is contained in \( \{ x_n > 0 \} \) is equivariant to \( \{ u = 0 \} \) is contained in \( \{ x_n > 0 \} \). Note that here we don’t assume that the nodal set \( \{ u = 0 \} \) is an epigraph. For rigidity results in the epi-graph case we refer to [8] and the references therein.

We remark that the half space theorem is not true for minimal hypersurfaces in \( \mathbb{R}^n \) with \( n \geq 4 \), for example the higher dimensional catenoid. However, for Allen-Cahn equation, this is still open in higher dimensions. (In view of the construction
of solutions concentrated on higher dimensional catenoid [1], we turn to believe that half space theorem should be true for all \( n \geq 4. \)

2. THE CASE OF \( n = 2 \)

We first give a simple proof when \( n = 2. \)

**Proof.** We can assume that there exists a sequence \((t_n^+, s_n^+)\) with \( t_n^+ \to +\infty \) such that \( u(t_n^+, s_n^+) = 0 \) and \( s_n^+ \to 0. \)

Then using sweeping principle, we see that locally around \((t_n^+, s_n^+)\), \( u \) converges to the one dimensional solution \( H(x_2) \).

For each \( x_1 \), we define \( g(x_1) = \min \{ t : u(x_1, t) = 0 \} \).

Note that \( g(x_1) > 0. \) Let us set \( \alpha = \lim \inf_{x_1 \to -\infty} g(x_1). \)

We first assume \( \alpha < +\infty. \) (The case of \( \alpha = +\infty \) is similar and easier). There exists a sequence \((t_n^-, s_n^-)\) with \( t_n^- \to -\infty \) such that \( u(t_n^-, s_n^-) = 0 \) and \( s_n^- \to \alpha. \) Still using sweeping principle, we know that locally around \((t_n^-, s_n^-)\), \( u \) converges to the one dimensional solution \( H(x_2) \).

Consider the region

\[
\Omega_n := \{(x_1, x_2) : s_n^- < x_1 < s_n^+, -\infty < x_2 < \alpha \}.
\]

Let \( X = (0, 1). \) Pohozaev type identity tells us that

\[
\int_{\partial \Omega_n} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X \cdot \nu - (\nabla u \cdot X)(\nabla u \cdot \nu) \, ds = 0.
\]

Let \( \Omega_{n,1} \) be the upper boundary of \( \Omega_n. \) Then

\[
\int_{\Omega_{n,1}} \left[ \frac{1}{2} |\nabla u|^2 + F(u) - |\partial_{x_2} u|^2 \right] \, ds \to 0, \quad \text{as } n \to +\infty.
\]

Hence on \( \Omega_{n,1}, \) we infer from Modica estimate that \( |\nabla u|^2 = 2F(u). \) This implies that \( u \) is the one dimensional solution \( H(x_2). \) \( \square \)

3. THE CASE OF \( n = 3 \)

Next, we shall consider the case of dimension 3. The arguments in this section could also be applied in the 2d case. Let \( \Omega = \Omega_{r_0} \) be the cylinder

\[
\left\{(x, y, z) : z < 1, \quad r = \sqrt{x^2 + y^2} < r_0 \right\}.
\]

Let \( X = (0, 0, 1). \) Then we have the following Pohozaev’s identity on \( \Omega: \)

\[
\int_{\partial \Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X \cdot \nu - (\nabla u \cdot X)(\nabla u \cdot \nu) \, ds = 0.
\]

Let \( B_1 \) be the vertical part of the boundary \( \partial \Omega, \) and \( B_2 \) be horizontal part. We denote the integrand by \( I. \)
Note that
\[
\int_{B_1} I = \int_{B_2} I.
\]
We define
\[
g(r_0) = -\int_{B_1} I.
\]
Since \(I \geq 0\) on \(B_2\), \(g\) is monotone increasing. Define
\[
\alpha = \lim_{r_0 \to +\infty} g(r_0).
\]
Then \(0 \leq \alpha \leq +\infty\).

We have, at \(z = 1\), using \((2)\),
\[
\int_0^{2\pi} \int_0^{+\infty} \frac{1}{2} ru_r^2 drd\theta \leq \alpha.
\]
Let us define the function \(K(s, z) = \int_0^{2\pi} \int_0^s ru_r^2 drd\theta\). The half space assumption yields suitable exponential decay of \(u_r\) along the \(z\)-direction, from which we get the following key property of \(K\).

**Lemma 2.** There exist universal positive constants \(C\), such that for \(z \leq 0\),
\[
K(l, z) \leq CK(l + 10 \ln l, 0) + C, \text{ for } l \text{ large.}
\]

**Proof.** The function \(u_x\) satisfies
\[
-\Delta u_x + (3u^2 - 1) u_x = 0.
\]
Note that \(u^2\) converges to 1, uniformly and exponentially in \(z\).

Let us consider the model problem
\[
\begin{cases}
-\Delta \phi + \phi = 0, \\
\phi(x, y, 0) = |u_x(x, y, 0)|.
\end{cases}
\]
To estimate \(\phi\), we take the Fourier transform in \((x, y)\) variable to obtain
\[
\phi(x, y, z) = \phi(x, y, 0) \ast \left( e^{-\sqrt{[\xi]^2 + 1}z} \right).
\]
Let us denote
\[
G(x, y, z) = \left( e^{-\sqrt{[\xi]^2 + 1}z} \right).
\]
Then
\[
\phi(x, y, z) = \int_{R^2} \phi(s, t, 0) G(x - s, y - t, z) dsdt.
\]
Note that we are interested in the estimate of \(\phi\) in the region
\[
D = D_l := \{(x, y) : x^2 + y^2 \leq l\}.
\]
Let us write
\[
\phi = \left( \int_{|s, t| \geq l + 10 \ln l} + \int_{|s, t| \leq l + 10 \ln l} \right) \phi(s, t, 0) G(x - s, y - t, z) dsdt
\]
\[
\quad := \phi_1 + \phi_2.
\]
For \((x, y)\) in this region \(\Omega\), due to the exponential decay of \(G\) in the first two variables, we have
\[
|\phi_1| \leq Ct^{-5}.
\]
We have in mind that \( \phi \) resembles \( u \). Observe that 
\[
\int_0^{2\pi} \int_0^l ru^2_r dr d\theta = \int_{\Omega} (u_x^2 + u_y^2).
\]
Now with the previous estimate at hand, we deduce using Cauchy-Schwarz that 
\[
\int_D \phi^2 dx dy \leq \int_{\Omega} \left( \int_{|s,t| \leq l + a \ln l} \phi(s,t,0) G(x-s, y-t, z) ds dt \right)^2 dxdy
\]
\[
\leq \int_{\Omega} \left\{ \int_{|s,t| \leq l + a \ln l} \phi^2(s,t,0) Gds dt \int Gdtds \right\} dxdy
\]
\[
\leq C \int_{|s,t| \leq l + a \ln l} \phi^2(s,t,0) ds dt.
\]
The estimates for \( \phi_1 \) and \( \phi_2 \) together with maximum principles concludes the proof.

□

Let us now define 
\[
f(s) = \int_0^s \int_{-\infty}^0 \int_0^{2\pi} u_r u_z d\theta dz dr.
\]
Furthermore, we introduce \( t = s + 10 \ln s \).

**Lemma 3.** As \( t \to +\infty \),
\[
\frac{f(t) - f(s)}{f(t)^3} \to 0.
\]

**Proof.** We have, using the fact that \( g(s) \to +\infty \),
\[
f(t) = \int_1^t \frac{g(s)}{s} ds \geq C \ln t.
\]
On the other hand, since \( t = s + 10 \ln s \),
\[
f(t) - f(s) = \int_s^{s+10\ln s} \frac{g(s)}{s} ds.
\]
We observe that \( g(s) = s \int_{-\infty}^1 \int_0^{2\pi} u_r u_z d\theta dz \leq Cs \). Hence
\[
f(t) - f(s) \leq C \ln s.
\]
We then conclude that
\[
\frac{f(t) - f(s)}{f(t)^3} \leq C \frac{\ln s}{(\ln t)^3} \to 0, \text{ as } t \to +\infty.
\]

□

**Proposition 4.** The constant \( \alpha = 0 \).

**Proof.** We assume to the contrary that \( \alpha > 0 \).
Using the fact that \( |u_z| \leq C e^{-\beta |z|} \), we get

□
\[ f(s) \leq C \int_{-\infty}^{0} \sqrt{\left( \int ru^2 \int \frac{u^2}{r} \right) dz} \]
\[ = C \int_{-\infty}^{0} \sqrt{K(s, z) e^{-\beta|z|} \ln sdz} \]
\[ \leq C \int_{-\infty}^{0} \sqrt{(K(s + a \ln s, 0) + 1) e^{-\beta|z|} \ln sdz} \]
\[ \leq C \sqrt{f'(s + a \ln s) s \ln s}. \]

That is,
\[ f'(s + 10 \ln s) \geq \frac{C f^2(s)}{s \ln s}. \]

Hence we find that
\[ f'(t) \geq \frac{C f(t)^2}{t \ln t} = \frac{C f(t)^2 f(s)^2}{f(t)^2 t \ln t} \]
\[ = \frac{C f(t)^2}{t \ln t} + \frac{C}{t \ln t} \left( \frac{f(s)}{f(t)} \right)^2 - 1 \]
\[ = \frac{C f(t)^2}{t \ln t} + \frac{C}{t \ln t} \left( 1 - \frac{f(t) - f(s)}{f(t)} \right)^2 - 1 \]
\[ \geq \frac{C f(t)^2}{t \ln t} - \frac{C}{t \ln t} \frac{f(t) - f(s)}{f(t)}. \]

We write it in the form
\[ \frac{f'(t)}{f(t)^2} \geq \frac{C}{t \ln t} - \frac{C}{t \ln t} \left( \frac{f(t) - f(s)}{f(t)^2} \right). \]

Now we obtain, for some \( t_0 \),
\[ \frac{f'(t)}{f(t)^2} \geq \frac{C}{t \ln t}, \text{ for } t \geq t_0. \]

This inequality can be written as
\[ (f^{-1})' + C \ln \ln t)' \leq 0, \text{ for } t \geq t_0. \]

It follows that the function
\[ \frac{1}{f(t)} + C \ln \ln t \]

is decreasing. But we know that \( f \) is positive and \( \frac{1}{f(t)} + C \ln \ln t \) tends to infinity as \( t \) goes to \( +\infty \). This is a contradiction.

The fact that \( \alpha = 0 \) together with the Pohozaev identity implies that \(|\nabla u|^2 = 2F(u)|, \text{ for } z = 1, \text{ which tells us that } u \text{ is one-dimensional. The proof of the half space theorem is thus completed.} \]

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