Coherent Flag Codes

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Abstract

In this paper we study flag codes on $\mathbb{F}_q^n$, being $\mathbb{F}_q$ the finite field with $q$ elements, paying special attention to the connection between the parameters and properties of a flag code and the ones of a family of constant dimension codes naturally associated to it (the projected codes). More precisely, we focus on coherent flag codes, that is, flag codes whose distance and size are completely determined by their projected codes. We explore some aspects of this family of codes and present interesting examples of them by generalizing the concepts of equidistant and sunflower subspace code to the flag codes setting. Finally, we present a decoding algorithm for coherent flag codes that fully exploits the coherence condition.

Keywords: Network coding, flag codes, constant dimension codes, equidistant codes, sunflower.

1 Introduction

The concept of network coding was introduced in [1] as a method to increase the information flow within a network modelled as an acyclic directed graph with possibly more than one source and sink. This network operates with vectors of a given vector space $\mathbb{F}_q^n$ over the finite field of $q$ elements $\mathbb{F}_q$, being $q$ a prime power. The intermediate nodes transmit random linear combinations of these vectors, instead of simply routing them. In [7], Koetter and Kschischang presented an algebraic aproach to network coding. Since vector spaces are invariant by linear combinations, the authors suggested using vector subspaces, in lieu of vectors, as codewords. In this context, a subspace code of length $n$ is just a nonempty collection of subspaces of $\mathbb{F}_q^n$. In case we restrict ourselves to subspaces with the same dimension, we speak about constant dimension codes. The study of

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constant dimension codes has lead to many papers in recent years. We refer the reader to [10] and references therein for the basics on these codes.

Subspace codes defined as above require a single use of the channel in order to send each codeword. In contrast, if we use the channel more than once, say \( r \geq 2 \) times, we talk about multishot subspace codes of length \( r \) or \( r \)-shot codes, for short. These codes were introduced in [9] and their codewords are sequences consisting of \( r \) vector subspaces of \( \mathbb{F}_q^n \). As it was showed in that paper, fixed the values \( n \) and \( q \), multishot codes could achieve better cardinality and distance that one-shot codes just by introducing a new parameter: the number of the channel uses.

In this paper we focus on flag codes, a specific family of multishot codes whose codewords are given by sequences of nested subspaces with prescribed dimensions (flags). In the network coding framework, flag codes were introduced in [8]. In that work, the authors studied flag codes as orbits of groups of the general linear group and provided some constructions of them as well as a new channel model for flags.

The goal of the present work is the study of the connection between the parameters and properties of a flag code and the ones of its projected codes, that is, the constant dimension codes used at each shot when sending flags of a flag code. In this direction, we introduce the concept of coherent flag codes, a family of flag codes whose cardinality and distance are perfectly described in terms of its projected codes. This notion of coherence (cardinality-coherence together with distance-coherence) will allow us to easily translate distance and cardinality properties of a flag code to the subspace code level and vice versa. Moreover, in a coherent flag code, some structural properties satisfied at flag codes level are transferred as the equivalent properties at the subspace codes level, that is, they are properly inherited by the projected codes (and conversely). We will exhibit this fact providing two specific families of coherent flag codes coming from the natural generalization of equidistant and sunflower constant dimension codes (see [4, 5]). The coherence condition will be exploited to give an efficient decoding algorithm, which translates the problem of decoding a flag code to the one of decoding a constant dimension code.

The paper is is organized as follows. In Section 2, we provide the basic background on subspace codes, focusing on two well-known families of constant dimension codes: equidistant and sunflower codes. Besides, some definitions and known facts about flag codes are presented, together with the channel model to be used later on. Section 3 is devoted to properly define the concept of coherence of a flag code with respect to its projected codes. In Section 4, we present some families of coherent flag codes by generalizing the concepts of equidistant and sunflower code to the flag codes scenario. Furthermore, we will see that the only coherent equidistant (resp. sunflower) flag codes are the ones that have equidistant (resp. sunflower) projected codes. Finally, in Section 5 we study the problem of decoding coherent flag codes on the erasure channel by exhibiting a
suitable decoding algorithm.

2 Preliminaries

This section is devoted to recall some background needed along this paper. The first part concerns subspace codes, focusing on two important families of constant dimension codes. In the second part we remind some known facts and definitions related to flag codes.

2.1 Subspace codes

Let $q$ be a prime power and $\mathbb{F}_q$ the finite field of $q$ elements. For every $n \geq 1$, we denote by $\mathcal{P}_q(n)$ the projective geometry of the vector space $\mathbb{F}_q^n$, which consists of the set of all the $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_q^n$. This set can be endowed with a metric, the subspace distance, given by

$$d_S(U, V) = \dim(U + V) - \dim(U \cap V), \ \forall U, V \in \mathcal{P}_q(n). \quad (1)$$

The Grassmannian $\mathcal{G}_q(k, n)$ (or Grassmann variety) of dimension $k \leq n$ of $\mathbb{F}_q^n$ is just the set of $k$-dimensional subspaces in $\mathcal{P}_q(n)$. The subspace distance induces in turn a metric in $\mathcal{G}_q(k, n)$ and, in this case, its expression becomes

$$d_S(U, V) = 2(k - \dim(U \cap V)), \ \forall U, V \in \mathcal{G}_q(k, n).$$

A subspace code of $\mathbb{F}_q^n$ is a nonempty subset $C$ of $\mathcal{P}_q(n)$. If every subspace in $C$ has the same dimension, say $k$, then it is said to be a constant dimension code in the Grassmannian $\mathcal{G}_q(k, n)$. The (minimum) distance of a subspace code $C$ is

$$d_S(C) = \min\{d_S(U, V) \mid U, V \in C, U \neq V\}.$$ 

For codes consisting of just one element, we put $d_S(C) = 0$. In any other case, the minimum distance of a subspace code is a positive integer. For a constant dimension code $C$ in the Grassmannian $\mathcal{G}_q(k, n)$, the minimum distance $d_S(C)$ is an even integer with

$$d_S(C) \leq \min\{2k; 2(n - k)\}. \quad (2)$$

For the basic background on constant dimension codes we refer the reader to [7], the seminal paper in this area, and to [10].

A constant dimension code $C \subseteq \mathcal{G}_q(k, n)$ is said to be equidistant if its distance is attained by every pair of different subspaces in $C$. In this situation, there exists an integer $c$ such that $\max\{0, 2k - n\} \leq c \leq k$ and $d_S(C) = 2(k - c)$. This value $c$ represents the dimension of the intersection between every pair of different subspaces in $C$. Due to this reason, these codes are also known as equidistant $c$-intersecting constant dimension codes. Trivial codes consisting just of one
element are trivially equidistant of distance zero. Equidistant subspace codes have been widely studied in [4, 5]. In Section 3 we will generalize this concept to the flag codes setting.

Observe that constant dimension codes in the Grassmannian $G_q(k, n)$ attaining the maximum distance are, in particular, equidistant $c$-intersecting constant dimension codes with $c = \max\{0, 2k - n\}$. For dimensions up to $[n/2]$, they are better known as partial spread codes. For further information on this family of codes, consult [6].

Regarding the intersections between couples of codewords, there is another interesting class of equidistant constant dimension codes to take into account. A constant dimension code $C \subseteq G_q(k, n)$ is said to be a sunflower if there exists a subspace $C$ such that, for every pair of different subspaces $U, V \in C$, it holds $U \cap V = C$. In this case, the subspace $C$ is called the center of the sunflower. Observe that a sunflower is an equidistant $c$-intersecting code with $c = \dim(C)$. These codes have been also studied and constructed in [4, 5]. Observe that every subspace code with just one codeword $C = \{U\}$ can be seen as a trivial sunflower of center $U$.

Concerning the way we use the channel to transmit some information encoded in a subspace code, note that, to send a codeword (a subspace) we just need to use the channel once, that is, we perform one shot. Under this viewpoint, subspace codes can be called one-shot codes. In contrast, when the number of required uses of the channel is bigger, say $r$, we speak about multishot codes of length $r$ or, simply, $r$-shot codes. More precisely, $r$-shot codes are nonempty subsets of $P_q(n)^r$, i.e., their codewords are sequences of length $r$ of subspaces of $\mathbb{F}_q^n$. The subspace distance defined in (1) can be naturally generalized to this setting. Given two sequences of subspaces $U = (U_1, \ldots, U_r)$ and $V = (V_1, \ldots, V_r)$, their extended subspace distance is given by

$$d_S(U, V) = \sum_{i=1}^r d_S(U_i, V_i).$$

For further information on multishot codes, see [9].

2.2 Flag codes

Flag codes are a special family of multishot subspace codes, in which codewords are sequences of nested subspaces of a vector space over a finite field. In the network coding setting, they were first introduced in [8]. Given integers $1 \leq t_1 < t_2 < \cdots < t_r < n$, a flag of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a sequence of nested vector subspaces $F = (F_1, \ldots, F_r) \in G_q(t_1, n) \times \cdots \times G_q(t_r, n)$. The subspace $F_i$ is called the $i$-th subspace of the flag $F$. When we consider flags of full type vector, that is, $(1, \ldots, n-1)$, we speak about full flags.

The flag variety of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is denoted by $F_q((t_1, \ldots, t_r), n)$ and it is the set of flags of the corresponding type. As a subset of $P_q(n)^r$, the
flag variety can be seen as a metric space, equipped with the extended subspace distance given in (3). We call it the flag distance and denote it by $d_f$ in this setting. More precisely, if $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$ are flags of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$, the flag distance between them is given by

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i).$$

A flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a nonempty subset $C$ of the flag variety $\mathcal{F}_q((t_1, \ldots, t_r), n)$ and its (minimum) distance is given by

$$d_f(C) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in C, \mathcal{F} \neq \mathcal{F}'\}.$$

Observe that, if $C$ contains at least two flags, its distance is a positive even integer. On the other hand, if $|C| = 1$, we put $d_f(C) = 0$. The bounds for the distance between $t_i$-dimensional subspaces given in (2) yield to the following upper bound for the flag distance:

$$d_f(C) \leq 2 \left( \sum_{t_i \leq \lfloor \frac{n}{2} \rfloor} t_i + \sum_{t_i > \lfloor \frac{n}{2} \rfloor} (n - t_i) \right).$$

Flag codes attaining this bound are called optimum distance flag codes (see [2, 3]).

To finish this section, we present a channel for flags following the general idea of the channel model introduced in [8]. If we see flag codes as a particular case of multishot codes, sending a flag of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ (as a codeword of a flag code) requires using the subspace channel $r$ times to send $r$ nested subspaces of $\mathbb{F}_q^n$. The nested structure allows us reduce the amount of sent information in every shot. Let us precise this.

The network can be modelled as a finite directed acyclic multigraph with a single source and several receivers. Assume that we want to send a flag $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$. By virtue of the nested structure of flags, one can find vectors $v_1, \ldots, v_{t_r} \in \mathbb{F}_q^n$ such that for every $1 \leq i \leq r$, the subspace $\mathcal{F}_i$ is spanned by the set of vectors

$$\{v_1, \ldots, v_{t_i}\}.$$ 

In order to send the flag $\mathcal{F}$, we proceed as follows. We fix $t_0 = 0$ and for every value of $1 \leq i \leq r$, at the $i$-th shot:

- the source injects the set of vectors $\{v_{t_{i-1}+1}, \ldots, v_{t_i}\}$. One of each through a different outgoing edge.
- Then, intermediate nodes construct random linear combinations of the received vectors up to this moment (included the ones received in previous shots) and send each of them through an outgoing edge.
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- The receivers get random linear combinations of the vectors \( \{v_1, \ldots, v_t\} \) and put them as the rows of a matrix \( Z_i \) to construct the subspace

\[
X_i = \text{rowsp} \begin{pmatrix} Z_1 \\ \vdots \\ Z_t \end{pmatrix},
\]

that is, the vector space spanned by the rows of \( Z_1, \ldots, Z_r \).

After \( r \) shots, every receiver is able to form a stuttering flag \( X = (X_1, \ldots, X_r) \), i.e., a sequence of nested subspaces where equalities are allowed. Note that, in absence of errors, this sequence coincides with the sent flag \( F \). Nevertheless, erasures and insertions can occur at every shot. Let us precise these two concepts. If we write \( X_i = F_i \oplus E_i \), with \( F_i \subseteq F \) and \( F_i \cap E_i = \{0\} \), then the number of erasures at the \( i \)-th shot is

\[
d_S(F_i, \bar{F}_i) = \dim(F_i) - \dim(\bar{F}_i)
\]

and it represents the number of dimension losses from \( F_i \) in \( X_i \). The number of insertions at the \( i \)-th shot is \( \dim(E_i) \) and it measures the dimension of the vector space generated by the set of vectors in \( X_i \) but not in \( F_i \). Hence, the number of errors at the \( i \)-th shot is given by \( e_i = d_S(F_i, X_i) \) and it counts both the number of erasures and insertions occurred in this single shot. Finally, the total number of errors of the communication is denoted by \( e \) and it is computed by

\[
e = \sum_{i=1}^{r} e_i = \sum_{i=1}^{r} d_S(F_i, X_i) = d_f(F, X).
\]

In a general channel, this number \( e \) tells us how many erasures and insertions have occurred in the transmission of the flag \( F \). Through this paper, we will focus on a more particular channel, the erasure channel for flags, in which insertions are not allowed. Hence, only erasures can occur and it holds \( X_i \subseteq F \) for every \( 1 \leq i \leq r \). In this context, we will call total number of erasures and number of erasures at the \( i \)-th shot to the numbers \( e \) and \( e_i \), respectively.

3 Coherent flag codes

Following the ideas in [2], given a flag code \( C \), we can naturally associate to it a set of constant dimension codes that we obtain when we gather the subspaces of the same dimension used at a fixed shot in the process of sending flags. Let us precise the definition of these codes.
**Definition 3.1.** Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. For every $1 \leq i \leq r$, we call the $i$-projected code of $\mathcal{C}$ to the constant dimension code $\mathcal{C}_i \subseteq \mathcal{G}_q(t_i, n)$ given by the set of $i$-th subspaces of flags in $\mathcal{C}$. More precisely,

$$\mathcal{C}_i = \{ \mathcal{F}_i \mid (\mathcal{F}_1, \ldots, \mathcal{F}_i, \ldots, \mathcal{F}_r) \in \mathcal{C} \}.$$ 

Due to the close relationship between a flag code $\mathcal{C}$ and its projected codes, it is a natural question to explore how far properties and structure of these codes determine the structure of $\mathcal{C}$ and conversely. In this direction, this section is devoted to the study of flag codes that are coherent with respect to their projected codes or just coherent flag codes. This is a family of flag codes in which the cardinality and distance are completely determined by the ones of their projected codes. We will show how, in some specific cases, this property of coherence goes beyond size and distance and gives rise to a stronger structural coherence. Furthermore, as we will see in Section 5, the property of being coherent makes it possible to give a decoding algorithm in the erasure channel for flags.

Let us first point out the connection between the cardinality of a flag code $\mathcal{C}$ of type $(t_1, \ldots, t_r)$ and the ones of its projected codes. It is clear that the size of any projected code is upper-bounded by the cardinality of $\mathcal{C}$, that is,

$$|\mathcal{C}_i| \leq |\mathcal{C}|, \ i = 1, \ldots, r.$$ 

The first condition of coherence we will impose to our family of flags is the property of disjointness.

**Definition 3.2.** A flag code $\mathcal{C}$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is said to be disjoint if its cardinality coincides with the ones of its projected codes, that is, if

$$|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|.$$ 

The notion of disjoint flag codes was introduced in [2] in order to characterize optimum distance flag codes in terms of their projected codes. Observe that the cardinality of a disjoint flag code is determined by the one of any of its projected codes. In this sense, we can say that disjoint flag codes are coherent with respect to the cardinality of their projected codes, or just cardinality-coherent, for short.

Just as the cardinality of a disjoint flag code is determined by the ones of its projected codes, we introduce the concept of coherence of flag codes with respect to the distance of their projected codes taking into account the pairs of flags attaining the minimum distance of the flag code.
**Definition 3.3.** Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. We say that $\mathcal{C}$ is *distance-coherent* if for every pair of different flags $\mathcal{F}, \mathcal{F}'$ in $\mathcal{C}$, the following statements are equivalent:

1. $d_f(\mathcal{F}, \mathcal{F}') = d_f(\mathcal{C})$.
2. $d_S(\mathcal{F}_i, \mathcal{F}'_i) = d_S(\mathcal{C}_i)$, for all $i = 1, \ldots, r$.

Notice that, in a distance-coherent flag code, a pair of flags provides the minimum distance if, and only if, the (subspace) distance between their subspaces is the minimum (subspace) distance of the corresponding projected code. Hence, closest flags in a distance-coherent flag code are given by nested sequences of the closest subspaces in the projected codes. That does not occur in a general flag code, as we can see in the following example:

**Example 3.4.** Let $\mathcal{C}$ be the flag code of type $(1, 2, 3)$ on $\mathbb{F}_q^5$ given by the set of flags

- $\mathcal{F}^1 = \langle \langle u_1 \rangle, \langle u_1, u_3 \rangle, \langle u_1, u_3, u_4 \rangle \rangle$,
- $\mathcal{F}^2 = \langle \langle u_1 \rangle, \langle u_1, u_5 \rangle, \langle u_1, u_2, u_5 \rangle \rangle$,
- $\mathcal{F}^3 = \langle \langle u_2 \rangle, \langle u_1, u_2 \rangle, \langle u_1, u_2, u_4 \rangle \rangle$,

where $\{u_i\}_{i=1}^5$ denotes the standard basis of $\mathbb{F}_q^5$ over $\mathbb{F}_q$.

In this case, we have $d_f(\mathcal{C}) = d_f(\mathcal{F}^1, \mathcal{F}^2) = 6$. However, the flag code $\mathcal{C}$ is not distance-coherent since the distance of every projected code is $d_S(\mathcal{C}_i) = 2$ but $d_S(\mathcal{F}_1^1, \mathcal{F}_1^2) = 0$ and $d_f(\mathcal{F}_2^1, \mathcal{F}_2^2) = 4$.

Under the distance-coherence condition, there is a consistent link between what is close both at flag level and at subspace level. Moreover, it follows a clear connection between the distance of a flag code and the ones of its projected codes.

**Proposition 3.5.** The distance of a distance-coherent flag code coincides with the sum of the ones of its projected codes.

The previous example shows that the converse of this result is not true in general. Just observe that the distance of $\mathcal{C}$ is 6, which is the sum of the distances of its projected codes, whereas $\mathcal{C}$ is not distance-coherent. However, we can notice that, in this example, the minimum distance is attained by two different flags $\mathcal{F}^1, \mathcal{F}^2 \in \mathcal{C}$ with a common subspace. If we exclude this situation and focus on flag codes where different flags have all their subspaces different, i.e., disjoint flag codes, we have the following characterization.

**Proposition 3.6.** Let $\mathcal{C}$ be a disjoint flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. The following statements are equivalent:

1. $\mathcal{C}$ is distance-coherent and
2. $d_f(\mathcal{C}) = \sum_{i=1}^r d_S(\mathcal{C}_i)$. 

Proof. By means of Proposition 3.5, we just need to prove that a disjoint flag code with distance \( d_f(C) = \sum_{i=1}^{r} d_s(C_i) \) needs to be distance-coherent. To do so, consider a pair of different flags \( F, F' \) in \( C \) giving the distance of the code. Since \( C \) is disjoint, we know that \( F_i \neq F'_i \) for every \( 1 \leq i \leq r \). As a result, for every value of \( i \), the distance \( d_s(F_i, F'_i) \) cannot be zero and hence it is, at least, \( d_s(C_i) \).

On the other hand, we have that \( d_f(F, F') = d_f(C) = \sum_{i=1}^{r} d_s(C_i) \), which happens if, and only if, \( d_s(F_i, F'_i) = d_s(C_i) \) for every \( 1 \leq i \leq r \), that is, if \( C \) is distance-coherent.

These two concepts of coherence, with respect either to the cardinality or to the distance, give rise to a more general idea of coherence which gathers both of them.

Definition 3.7. A flag code is said to be coherent (w.r.t. its projected codes), if it is both cardinality-coherent (disjoint) and distance-coherent.

This definition along with Proposition 3.6 provides the following characterization of coherent flag codes.

Theorem 3.8. Let \( C \) be a flag code of type \( (t_1, \ldots, t_r) \) on \( \mathbb{F}_q^n \). The following statements are equivalent:

1. The code \( C \) is coherent.
2. The code \( C \) is disjoint and \( d_f(C) = \sum_{i=1}^{r} d_s(C_i) \).

Observe that, by means of this result, in order to determine if a flag code is coherent, we just need to compute the distance and cardinalities of the given flag code and the ones of its projected codes. This is notably easier than checking the distance-coherence condition, i.e., that every pair of flags in the code gives the minimum distance if, and only if, the distance between their subspaces coincides with the minimum distance of every projected code. In particular, every flag code consisting of a single flag is automatically coherent with \( d_f(C) = 0 \).

To finish this section, we deepen the structure of coherent flag codes in order to give some crucial definitions and properties for the design of the decoding algorithm described in Section 5. Let us fix \( C \) a coherent flag code of type \( (t_1, \ldots, t_r) \) on \( \mathbb{F}_q^n \). Observe that, if \( F, F' \in C \) attain the minimum distance of \( C \), by virtue of the distance-coherence property, it holds

\[
d_s(C_i) = d_s(F_i, F'_i) = 2(t_i - \dim(F_i \cap F'_i))
\]
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for every \(1 \leq i \leq r\). Equivalently, the distance of \(\mathcal{C}\) is attained by a pair of flags \(\mathcal{F}, \mathcal{F}' \in \mathcal{C}\) if, and only if, we have \(\dim(\mathcal{F}_i \cap \mathcal{F}'_i) = m_i\), where

\[
m_i = t_i - \frac{d_S(\mathcal{C}_i)}{2}, \quad i = 1, \ldots, r.
\]

for every \(1 \leq i \leq r\).

Recall also that, as the code \(\mathcal{C}\) is coherent, in particular, it is disjoint. Hence, if \(|\mathcal{C}| \geq 2\), every projected code contains at least two different subspaces and its distance is a positive even integer. As a result, for every value \(1 \leq i \leq r\), we have that \(m_i < t_i\). Moreover, since subspaces in a flag are nested, associated to \(\mathcal{C}\) we obtain a non-decreasing sequence of integers \(0 \leq m_1 \leq m_2 \leq \cdots \leq m_r < t_r\) such that, for each pair \(\mathcal{F}, \mathcal{F}'\) with \(d_f(\mathcal{F}, \mathcal{F}') = d_f(\mathcal{C})\), we can construct a (stuttering) flag

\[
\mathcal{F} \cap \mathcal{F}' := (\mathcal{F}_1 \cap \mathcal{F}_1', \ldots, \mathcal{F}_r \cap \mathcal{F}_r')
\]

of type \((m_1, \ldots, m_r)\). With this notation, we give the following definition.

**Definition 3.9.** Let \(\mathcal{C}\) be a coherent flag code of type \((t_1, \ldots, t_r)\) on \(\mathbb{F}_q^n\). We define the minimum distance intersection code of \(\mathcal{C}\) as the stuttering flag code of type \((m_1, \ldots, m_r)\) given by the family

\[
\{\mathcal{F} \cap \mathcal{F}' \mid d_f(\mathcal{F}, \mathcal{F}') = d_f(\mathcal{C})\}.
\]

Notice that, for not coherent flag codes, the set given in (5) has not necessarily a fixed type. For instance, if we consider the the code \(\mathcal{C}\) in the Example 3.4, the set

\[
\{\mathcal{F} \cap \mathcal{F}' \mid d_f(\mathcal{F}, \mathcal{F}') = 6\} = \{\mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{F}_1 \cap \mathcal{F}_3, \mathcal{F}_1 \cap \mathcal{F}_3\}
\]

contains stuttering flags of types \((1, 1, 1)\) and \((0, 1, 2)\).

The sequence of numbers \((m_1, \ldots, m_r)\) defined as in (4) provides upper bounds for the dimension of the intersection of subspaces in every projected code of a coherent flag code.

**Proposition 3.10.** Let \(\mathcal{C}\) be a coherent flag code \(\mathcal{C}\) of type \((t_1, \ldots, t_r)\) and consider a pair of different flags \(\mathcal{F}, \mathcal{F}' \in \mathcal{C}\). Then \(\dim(\mathcal{F}_i \cap \mathcal{F}'_i) \leq m_i\) for every \(1 \leq i \leq r\).

**Proof.** Let \(\mathcal{F}\) and \(c\mathcal{F}'\) be a pair of flags in a coherent flag code \(\mathcal{C}\). Since this code is disjoint, from the condition \(\mathcal{F} \neq \mathcal{F}'\), we obtain that \(\mathcal{F}_i \neq \mathcal{F}'_i\) for every value of \(1 \leq i \leq r\). Hence, it holds

\[
d_S(\mathcal{C}_i) \leq d_S(\mathcal{F}_i, \mathcal{F}'_i) = 2(t_i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i))
\]
or, equivalently,

\[
\dim(\mathcal{F}_i \cap \mathcal{F}'_i) \leq t_i - \frac{d_S(\mathcal{C}_i)}{2} = m_i.
\]

\[\blacksquare\]
Notice that, in the previous result, the condition of coherence cannot be relaxed in none of its two sides. On the one hand, being distance-coherent is necessary to properly define the numbers $m_1, \ldots, m_r$. On the other hand, the next example shows that the condition of cardinality-coherence can neither be removed.

**Example 3.11.** Consider the flag code $C$ of type $(1, 2, 3, 4)$ on $\mathbb{F}_q^6$ consisting of the set of three flags

\[
\mathcal{F}^1 = \langle \langle u_1 \rangle, \langle u_1, u_2 \rangle, \langle u_1, u_2, u_3 \rangle \rangle, \\
\mathcal{F}^2 = \langle \langle u_2 \rangle, \langle u_2, u_3 \rangle, \langle u_2, u_3, u_4 \rangle \rangle, \\
\mathcal{F}^3 = \langle \langle u_1 \rangle, \langle u_1, u_5 \rangle, \langle u_1, u_4, u_5 \rangle \rangle, \\
\]

where $\{u_1, \ldots, u_6\}$ is the standard basis of $\mathbb{F}_q^6$ over $\mathbb{F}_q$.

Observe that $C$ is not disjoint since $C_1$ contains only two subspaces. The distance of every projected code is 2 and the distance of the flag code is $d_f(C) = 8$. This distance is only attained by the pair of flags $\mathcal{F}^1$ and $\mathcal{F}^2$ and, for this couple of flags, it holds $d_S(\mathcal{F}^1, \mathcal{F}^2) = 2 = d_S(C_i)$ for every $i = 1, 2, 3, 4$. Hence, $C$ is a distance-coherent flag code and it makes sense to consider the values

\[m_i = t_i - \frac{d_S(C_i)}{2} = i - 1, \text{ for } 1 \leq i \leq 4,
\]

defined as in (4). However, notice that $\dim(\mathcal{F}^1 \cap \mathcal{F}^3) = 1 > 0 = m_1$, in contrast to what happens in the context of Proposition 3.10, where the extra condition of cardinality-coherence (disjointness) is required.

### 4 Some families of coherent flag codes

Given that a flag code $C$ of type $(t_1, \ldots, t_r)$ can be seen as a subset of the product of the Grassmannians $\mathcal{G}_q(t_1, n) \times \cdots \times \mathcal{G}_q(t_r, n)$, it seems quite natural, beyond cardinality and distance questions, to study the relationship between flag codes satisfying certain property and flag codes with projected codes fulfilling the equivalent property at the subspace codes level.

With this objective in mind, in this section we introduce two couples of special families of flag codes. We start studying and comparing equidistant flag codes and flag codes with equidistant projected codes. We show that, in general, these concepts are not equivalent whereas under the assumption of being coherent they coincide. Next, we focus on the study of sunflower flag codes and flag codes with sunflower (subspace) codes as their projected codes. Again, we see that, in general, these two families do not coincide but they turn to be the same when we impose the coherence condition. This study allows us to conclude that the coherence condition defined in Section 3 leads as well to a structural coherence that strongly relates the nature of a flag code with the ones of its projected codes. Let us precise these ideas.
4.1 Coherent equidistant flag codes

Recall from Section 2 that a subspace code $C$ in $G_q(k, n)$ is called equidistant if for each pair of codewords $U, V$ in $C$, we have that $d_S(U, V) = d_S(C)$. Let us generalize this concept to the flag codes scenario in two different ways:

**Definition 4.1.** A flag code $C \subseteq F_q((t_1, \ldots, t_r), n)$ is said to be *equidistant* if, for every pair of different flags $F, F' \in C$, it holds $d_f(C) = d_f(F, F')$.

**Definition 4.2.** A flag code $C$ is said to be *projected-equidistant* if all its projected codes $C_i$ are equidistant constant dimension codes.

These two concepts do not represent the same notion in the general framework of flag codes as the reader can realize from the following two examples. First, we see that equidistant flag codes do not need to be projected-equidistant.

**Example 4.3.** Let $\{u_1, \ldots, u_5\}$ denote the standard basis of $F_q^5$ over $F_q$ and consider the flag code $C$ of type $(2, 3)$ on $F_q^5$, consisting of the set of flags

$$F^1 = (\langle u_1, u_2 \rangle, \langle u_1, u_2, u_3 \rangle),$$
$$F^2 = (\langle u_1, u_4 \rangle, \langle u_1, u_4, u_5 \rangle),$$
$$F^3 = (\langle u_3, u_4 \rangle, \langle u_2, u_3, u_4 \rangle).$$

This code is equidistant with $d_f(C) = 6$ but none of its projected code is equidistant. For instance, $d_s(F^1_1, F^2_1) = 2 \neq 4 = d_s(F^1_1, F^3_1)$.

The other way round, being projected-equidistant does not imply equidistance.

**Example 4.4.** Let $C$ be a full flag code on $F_q^3$ given by the set of flags

$$F^1 = (\langle u_1 \rangle, \langle u_1, u_3 \rangle),$$
$$F^2 = (\langle u_1 \rangle, \langle u_1, u_2 + u_3 \rangle),$$
$$F^3 = (\langle u_2 \rangle, \langle u_1, u_2 \rangle),$$

where $\{u_1, u_2, u_3\}$ is the standard basis of $F_q^3$ over the field $F_q$. Observe that $C_1$ is and $C_2$ are equidistant constant dimension codes. In other words, the flag code $C$ is projected-equidistant. Nevertheless, it is clear that $C$ is not equidistant since

$$d_f(F^1, F^2) = 2 \neq 4 = d_f(F^1, F^3).$$

In light of Example 4.4, we can see that projected-equidistant flag codes are not in general coherent since they might not even be cardinality-coherent (disjoint). When we require them to satisfy this extra condition, we obtain the following results.

**Proposition 4.5.** Let $C \subseteq F_q((t_1, \ldots, t_r), n)$ be a projected-equidistant flag code. If $C$ is cardinality-coherent, then it is equidistant with distance $d_f(C) = \sum_{i=1}^r d_s(C_i)$. 


Proof. Assume that $\mathcal{C}$ is a cardinality-coherent projected-equidistant flag code, i.e., a disjoint flag code with equidistant projected codes. We distinguish two cases:

- If $|\mathcal{C}| = 1$, we have $|\mathcal{C}_i| = 1$ for each $1 \leq i \leq r$. Hence, $d_f(\mathcal{C}) = d_S(\mathcal{C}_i) = 0$ and the result trivially holds.

- If $|\mathcal{C}| \geq 2$, consider an arbitrary pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$. The cardinality-coherence condition implies that $\mathcal{F}_i \neq \mathcal{F}'_i$ for every $1 \leq i \leq r$. Now, given that every projected code $\mathcal{C}_i$ is equidistant, we have that $d_S(\mathcal{F}_i, \mathcal{F}'_i) = d_S(\mathcal{C}_i)$ for every $1 \leq i \leq r$. Consequently, $d_f(\mathcal{C}) = d_S(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{C}_i)$ and the flag code $\mathcal{C}$ is equidistant.

As a consequence, and by means of Theorem 3.8, we obtain the next corollary.

**Corollary 4.6.** Disjoint projected-equidistant flag codes are coherent.

**Remark 4.7.** Observe that if $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ is a projected-equidistant flag code, then there exist integers $c_1, \ldots, c_r$ such that every projected code $\mathcal{C}_i$ is an equidistant $c_i$-intersecting constant dimension code in $\mathcal{G}_q(t_i, n)$. Moreover, if $\mathcal{C}$ is disjoint, by means of the previous corollary, it is coherent. In this case, the sequence $(c_1, \ldots, c_r)$ is precisely the type vector of the minimum distance intersection code defined as in (5) associated to the coherent flag code $\mathcal{C}$.

According to these two results, it is clear that disjoint projected-equidistant flag codes are equidistant and coherent. In the next result, we prove that they are exactly the only codes that can be equidistant and coherent simultaneously.

**Theorem 4.8.** A coherent flag code is equidistant if, and only if, it is projected-equidistant.

**Proof.** The “if” part is consequence of Proposition 4.5. For the “only if” part, assume that $\mathcal{C}$ is a coherent and equidistant flag code. Let us see that, under these two conditions, every projected code is equidistant. In case $\mathcal{C}$ contains a single flag, it is clear that every projected code is equidistant (with distance equal to zero).

Let us study the case $|\mathcal{C}| \geq 2$. Observe that, since $\mathcal{C}$ is coherent, in particular the cardinality of every projected code coincides with the one of $\mathcal{C}$ and, given an index $1 \leq i \leq r$, we can always find two different subspaces $U, V \in \mathcal{C}_i$. By the definition of projected code, there must exist different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $\mathcal{F}_i = U$ and $\mathcal{F}'_i = V$. Moreover, by means of Theorem 3.8, the code $\mathcal{C}$ is disjoint and equidistant with $d_f(\mathcal{C}) = \sum_{i=1}^{r} d_S(\mathcal{C}_i)$. Hence, it holds $d_f(\mathcal{C}) = d_f(\mathcal{F}, \mathcal{F}')$. This fact together with the distance-coherence property implies that $d_S(\mathcal{F}_j, \mathcal{F}'_j) = d_S(\mathcal{C}_j)$ for every $1 \leq j \leq r$. In particular, we conclude that $d_S(\mathcal{C}_j) = d_S(\mathcal{F}_j, \mathcal{F}'_j) = d_S(U, V)$, which proves that the projected code $\mathcal{C}_i$ is equidistant. 


Observe that coherence completely translates the property of being equidistant from a flag code to its projected codes and vice versa. As a particular case of equidistant flag codes, we mention the family of optimum distance flag codes.

### 4.1.1 Optimum distance flag codes

Optimum distance flag codes were introduced in [2] as a generalization of constant dimension codes attaining the maximum distance. Constructions of them can be found as well in [3]. These codes were characterized as disjoint flag codes with projected codes attaining the maximum possible distance for the corresponding dimension. In particular, the projected codes of every optimum distance flag code are equidistant subspace codes. Hence, optimum distance flag codes are an example of disjoint projected-equidistant flag codes. As a result, by means of Corollary 4.6, the next result holds.

**Corollary 4.9.** Optimum distance flag codes are coherent.

Once again, coherence allows to perfectly translate the property of attaining maximum distance at the subspace code level to flag codes and conversely.

### 4.2 Coherent sunflower flag codes

Following the ideas in [5], here below we introduce the concept of sunflower flag code.

**Definition 4.10.** A flag code \( C \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n) \) is said to be a sunflower if there exists a stuttering flag \( C = (C_1, \ldots, C_r) \) such that, for every pair of different flags \( \mathcal{F}, \mathcal{F}' \in C \) it holds

\[
\mathcal{F} \cap \mathcal{F}' = (\mathcal{F}_1 \cap \mathcal{F}'_1, \ldots, \mathcal{F}_r \cap \mathcal{F}'_r) = (C_1, \ldots, C_r) = C.
\]

In this case, the stuttering flag \( C \) is called the center of the sunflower flag code \( \mathcal{C} \).

By analogy with the concept of projected-equidistant flag code, we define now the family of flag codes having sunflowers as projected codes.

**Definition 4.11.** A flag code \( C \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n) \) is said to be projected-sunflower if all its projected codes are sunflowers. In this situation, there exist subspaces \( C_1, \ldots, C_r \) such that every \( i \)-projected code \( \mathcal{C}_i \) of \( C \) is a sunflower of center \( C_i \). We say that \( C_1, \ldots, C_r \) are the centers of the projected-sunflower \( \mathcal{C} \).

One has the following relationship between these two concepts.

**Proposition 4.12.** Let \( C \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n) \) be a sunflower flag code of center \( C = (C_1, \ldots, C_r) \). Then the code \( \mathcal{C} \) is a projected-sunflower flag code of centers \( C_1, \ldots, C_r \).
Proof. Assume that $\mathcal{C}$ is a sunflower flag code of type $(t_1, \ldots, t_r)$. If $|\mathcal{C}| = 1$, the result trivially holds. Suppose now that $|\mathcal{C}| \geq 2$. For every index $1 \leq i \leq r$, we must prove that the projected code $\mathcal{C}_i$ is a sunflower of center $C_i$. We distinguish to possibilities.

- If $|\mathcal{C}_i| = |\mathcal{C}| \geq 2$, we can find two different subspaces $U, V \in \mathcal{C}_i$. Hence, there exist flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $\mathcal{F}_i = U$ and $\mathcal{F}'_i = V$. Since $\mathcal{C}$ is a sunflower of center $C$, we have
  \[ U \cap V = \mathcal{F}_i \cap \mathcal{F}'_i = \mathcal{C}_i, \]
  which proves that $\mathcal{C}_i$ is a sunflower of center $C_i$.

- In case of $|\mathcal{C}_i| < |\mathcal{C}|$, there must exist two different flags $\mathcal{F}, \mathcal{F}'$ in $\mathcal{C}$ such that $\mathcal{F}_i = \mathcal{F}'_i$. Since $\mathcal{C}$ is a sunflower of center $C$, we have that $C_i = \mathcal{F}_i \cap \mathcal{F}'_i = \mathcal{F}_i$. On the other hand, notice that every subspace in the projected code $\mathcal{C}_i$ contains the subspace $C_i$. Using that $\dim(C_i) = t_i$, we conclude that $\mathcal{C}_i = \{C_i\}$, which is the trivial sunflower of center $C_i$.

The converse of Proposition 4.12 is not true in general. The next example shows that projected-sunflower flag codes are not necessarily sunflowers.

Example 4.13. Consider the flag code $\mathcal{C} \subseteq \mathcal{F}_q((2, 3), 4)$ given by the set of flags

$\mathcal{F}^1 = \{(u_1, u_2), (u_1, u_2, u_3)\}$,
$\mathcal{F}^2 = \{(u_1, u_3), (u_1, u_2, u_3)\}$,
$\mathcal{F}^3 = \{(u_1, u_4), (u_1, u_2, u_4)\}$,

where $\{u_1, u_2, u_3, u_4\}$ denotes the standard basis of $\mathbb{F}_q^4$ over $\mathbb{F}_q$. Observe that the projected codes are sunflowers of center $C_1 = \langle u_1 \rangle$ and $C_2 = \langle u_1, u_2 \rangle$, respectively. However, the code $\mathcal{C}$ is not a projected sunflower of center $(C_1, C_2)$ since $\mathcal{F}_2^1 \cap \mathcal{F}_2^2 \neq C_2$.

Notice that the code in the previous example is not disjoint (cardinality-coherent) since $\mathcal{F}^1$ and $\mathcal{F}^2$ have the same second subspace. If we require the cardinality-coherence condition, we have the following characterization.

Theorem 4.14. Let $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ be a cardinality-coherent flag code. They are equivalent:

1. $\mathcal{C}$ is a sunflower of center $(C_1, \ldots, C_r)$ and
2. $\mathcal{C}$ is a projected-sunflower of centers $C_1, \ldots, C_r$. 

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Proof. By means of Proposition 4.12, it suffices to see that (2) implies (1). Assume that $C$ is a projected-sunflower and, for every $1 \leq i \leq r$, denote by $C_i$ the center of $C_i$. Consider two different flags $F, F' \in C$. Since $C$ is disjoint, we get that $F_i \neq F'_i$ for every $1 \leq i \leq r$. Hence, we have $F_i \cap F'_i = C_i$ for every choice of $i$ and we conclude that $C$ is a sunflower flag code of center $(C_1, \ldots, C_r)$.

Observe that projected-sunflower flag codes are, in particular, projected-equidistant. Hence, by the previous theorem together with Corollary 4.6, the next result holds straightforwardly.

Corollary 4.15. Assume that $C$ is a cardinality-coherent sunflower flag code of center $C = (C_1, \ldots, C_r)$. Then it is coherent and its minimum distance intersection code is given by its center $\{C\}$.

Remark 4.16. Notice that Theorem 4.8 still holds true if we replace “equidistant” by “sunflower”. However, in the latter case, the condition of being coherent can be relaxed to just the one of cardinality-coherence (disjointness), as shown in Theorem 4.14. This is due to the fact that distance-coherence is an inherent property of sunflowers. More precisely, if $C \subseteq F_q((t_1, \ldots, t_r), n)$ is a sunflower of center $(C_1, \ldots, C_r)$, then this center not only determines the distance of the flag code $C$ that is given by

$$d_f(C) = 2 \sum_{i=1}^{r} (t_i - \dim(C_i)),$$

but also the one of every projected code $C_i$, which is $d_S(C_i) = 2(t_i - \dim(C_i))$.

Once again we see that, under the coherence condition, one can naturally translate properties between the flag codes and the subspace codes frameworks. In this case, we can identify the flag codes property of being a sunflower with the one of having sunflowers as projected codes.

5 A decoding algorithm for coherent flag codes

In this section we provide a decoding algorithm on the erasure channel for coherent flag codes. In particular, it can be applied to the mentioned families in the previous section. This algorithm generalizes the ideas given in [2], where a decoding process for optimum distance flag codes was given. As we will see through this section, both distance-coherence and cardinality-coherence play a key role in this process and they allow us to reduce the problem of decoding a flag code to the one of just decoding one of its projected codes.

Let $C$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. Assume we have sent a flag $F = (F_1, \ldots, F_r) \in C$ through the erasure channel for flags defined in Section 2.2 (see [8]) and, after $r$ shots, a receiver gets a stuttering flag

$\mathcal{X} = (X_1, \ldots, X_r)$. 

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As we are using an erasure channel, only erasures (no insertions) can occur. Hence, every subspace $\mathcal{X}_i$ is contained in the subspace $\mathcal{F}_i$ sent at the corresponding shot. Recall from Subsection 2.2 that, at every shot, some information can be lost. This amount of information is called number of erasures at the $i$-th shot and computed as $e_i = d_S(\mathcal{F}_i, \mathcal{X}_i) = \dim(\mathcal{F}_i) - \dim(\mathcal{X}_i)$. The total number of erasures of the communication is given by

$$e = \sum_{i=1}^{r} e_i = d_f(\mathcal{F}, \mathcal{X})$$

and it is said to be correctable by the flag code $\mathcal{C}$ if

$$e \leq \left\lfloor \frac{d_f(\mathcal{C}) - 1}{2} \right\rfloor.$$ 

In this case, the received sequence $\mathcal{X}$ can be decoded into $\mathcal{F}$ by minimum distance in $\mathcal{C}$. This means that $\mathcal{F}$ is the closest flag in $\mathcal{C}$ to the received stuttering flag $\mathcal{X}$. Analogously, for every $1 \leq i \leq r$, the number of erasures at the $i$-th shot $e_i$ is said to be correctable by the projected code $\mathcal{C}_i$ of $\mathcal{C}$ if

$$e_i \leq \left\lfloor \frac{d_S(\mathcal{C}_i) - 1}{2} \right\rfloor.$$

The next proposition relates the correctability of the total number of erasures by a flag code $\mathcal{C}$ and with the number of erasures occurred at each shot, under certain conditions on the distance of the code.

**Proposition 5.1.** Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ and suppose that $d_f(\mathcal{C}) \leq \sum_{i=1}^{r} d_S(\mathcal{C}_i)$. If the total number of erasures $e$ is correctable by $\mathcal{C}$, then there is some $e_i$ that is correctable by the corresponding $\mathcal{C}_i$.

**Proof.** Arguing by contradiction, assume that $e \leq \left\lfloor \frac{d_f(\mathcal{C}) - 1}{2} \right\rfloor$ but no $e_i$ is correctable, that is, $e_i > \frac{d_S(\mathcal{C}_i)}{2} - 1$, for all $1 \leq i \leq r$. Thus, $e_i \geq \frac{d_S(\mathcal{C}_i)}{2}$ and the total number of erasures satisfies

$$e = \sum_{i=1}^{2k-1} e_i \geq \sum_{i=1}^{r} \frac{d_S(\mathcal{C}_i)}{2} \geq \frac{d_f(\mathcal{C})}{2} > \frac{d_f(\mathcal{C}) - 1}{2},$$

which is a contradiction, since $e$ is correctable.

The next result provides a characterization of a correctable number of erasures at some shot in terms of the dimension of the received subspace. The proof follows straightforwardly from the definition of the number of erasures at some shot, together with the condition of correctability.
Proposition 5.2. The number of erasures at the $i$-th shot $e_i$ is correctable if, and only if, the dimension of the subspace $X_i$ is greater than $t_i - \frac{d_S(C_i)}{2}$.

Observe that distance-coherent flag codes introduced in Section 3, by means of Proposition 3.5, satisfy the required condition on the distance of Proposition 5.1. Moreover, the quantity $t_i - \frac{d_S(C_i)}{2}$ that appears in Proposition 5.2 is precisely the value $m_i$ defined in (4). Hence, assuming that a correctable number of erasures have occurred, we can always decode by minimum distance at least one subspace $F_i$ of the sent flag $F$ in a distance-coherent flag code. If we add the condition of cardinality-coherence, we obtain the next result.

Theorem 5.3. Let $C$ be a coherent flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ and assume that $e$, the total number of erasures of the communication, is correctable. Then there exists some $1 \leq j \leq r$ such that $e_j$ is correctable and we can recover the sent flag $F$ as the unique flag in $C$ such that $X_j$ is contained in its $j$-th subspace.

Proof. Observe that, by the property of distance-coherence, the distance of the code is $d_f(C) = \sum_{i=1}^{r} d_S(C_i)$. Hence, since $e$ is correctable, by applying Proposition 5.1, the number of erasures at some shot, say the $j$-th one, must be correctable as well. Thus, from Proposition 5.2, it holds

$$\dim(X_j) > t_j - \frac{d_S(C_j)}{2} = m_j.$$ 

Recall that, since we are sending flags through an erasure channel, the subspace $X_j$ is contained in $F_j$, i.e., the $j$-th subspace of the sent flag. Moreover, by means of Proposition 3.10, $F_j$ is the unique subspace in $C_j$ containing $X_j$. Finally, given that $C$ is a disjoint flag code, we can recover $F$ as the unique flag in $C$ having $F_j$ as its $j$-th subspace.

Observe that this result guarantees the success of this decoding algorithm, starting from an index $j$ such that the number of erasures $e_j$ is correctable. Such an index can be easily identified just by checking the dimensions of the received subspaces $X_1, \ldots, X_r$ and applying Proposition 5.2. Observe that we do not need to wait to receive the whole stuttering flag $X$ to start to decode. Our idea is doing it sequentially during the transmission process. At every shot, we check the dimension of the received subspace until we obtain a subspace $X_j$ of dimension greater than $m_j = t_j - \frac{d_S(C_j)}{2}$. At that moment, we can easily recover $F_j$ and determine the flag $F$ in $j \leq r$ shots. We sum up this ideas in the following algorithm.
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Decoding Algorithm

Assumptions: We send a flag $F$ in a coherent flag code $C$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. At each shot, the $i$-th subspace $F_i$ is sent and a subspace $X_i$ is received. The total number of erasures $e$ is correctable.

Input: The received stuttering flag $X = (X_1, \ldots, X_r)$.
Output: The sent flag $F = (F_1, \ldots, F_r) \in C$.

Define $i = 1$

\[
\text{if } \dim(X_i) > m_i = t_i - \frac{d_2(C_i)}{2}, \\
\text{then decode } X_i \text{ into the only } F_i \in C_i \text{ that contains } X_i.
\]

\[
\text{return: the unique flag } F \in C \text{ that has } F_i \text{ as } i\text{-th subspace.}
\]

else $i := i + 1$

This decoding process takes advantage of the coherence condition in two ways. First, under the assumption of a correctable number of erasures, the distance-coherence property makes it possible to reduce the problem of decoding $X$ into $F$ to the one of decoding some $X_i$ into the corresponding sent subspace $F_i$. After that, the cardinality-coherence condition allows us to come back to the flags setting and recover $F$ from any of its subspaces. Again, the use of coherence in flag codes transfers a problem at the flag codes level -the one of decoding on the erasure channel- to the equivalent problem in the subspace codes scenario corresponding to its projected codes.

6 Conclusions and future work

In this paper we have introduced the concept of coherence for flag codes. This new notion allows us to measure in some way how far the parameters and properties of a flag code are determined by the ones of its projected codes. Moreover, in our search for families of coherent flag codes, we have generalized the concepts of equidistant and sunflower code to the flag codes framework in several ways and proved that, under the assumption of coherence, they coincide. In this way, coherence plays an important role in the study of properties of flag codes that can be induced by their projected codes and vice versa. A decoding algorithm for coherent flag codes in the erasure channel has been provided.

In future works, we would like to generalize the decoding algorithm presented in this paper in two possible ways. On the one hand, exploring how to decode coherent flag codes in a general channel where insertions were allowed. On the
other hand, we want to study the decoding process for more general families of flag codes on the erasure channel, by relaxing the property of coherence in one of its two sides: either the distance-coherence or the cardinality-coherence.

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