Comments on the Chern Number Argument of the Integer Quantum Hall Effect

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The Chern number argument of the integer quantum Hall effect is invalid. Because a process of calculation does not change its result, the result of our numerical calculation means that the argument fails. We briefly explain why the misuse of the theory of fiber bundles happens.

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I. INTRODUCTION

Since its discovery by von Klitzing et al., the quantum Hall effect (QHE) is a fascinating phenomenon for physicists, especially for those with a close connection to fundamental physics. Among logics to explain quantization of Hall conductance, one theory utilizes the theory of fiber bundles. Although it is natural to connect Hall conductance which is represented by integers in units of e²/h with the integration of the Chern class in a cycle with integer coefficient being an integer, it cannot be grounds for being integers. In fact, our numerical calculation of Hall conductance does not become an integer in units of e²/h, which means the failure of the argument. A brief explanation of the weakness of the argument follows.

II. MODEL

Our first consideration is the solution of bloch electrons in a uniform magnetic field \( B = \frac{eB}{\hbar} \), where \( \phi_0 \) is the flux quantum, \( S \) is the size of a unit cell, and \( p \) and \( q \) are positive integers which are mutually prime. The Hamiltonian is

\[
H = \frac{1}{2m}(\hat{p} + e\hat{A})^2 + V(\hat{x})
\]

where \( m, e, \hat{p}, \hat{A}, \) and \( V(\hat{x}) \) are the electron mass, the electron charge, the momentum operator, a vector potential, and a periodic potential, respectively. We take the Landau gauge \( \hat{A} = (0, B\hat{x}_1) \). We present here briefly a calculation method which enables band calculation in a magnetic field.

The followings are magnetic translation operators.

\[
\tau(n_1a_1) = e^{-i\frac{n_1^2q^2\sin \theta \cos \theta}{2\hbar^2}} e^{-i\frac{\theta (\hat{p}_1 + \frac{e}{\hbar}\hat{A}_1)}{n_1a_1} \sin \theta} e^{-i\frac{\theta \hat{p}_2 n_1 a_1 \cos \theta}{n_2q^2}}
\]

\[
= e^{-i\frac{\theta n_1q_1 a_1 \cos \theta}{n_2q^2}} e^{-i\frac{\theta \hat{p}_1 n_1 a_1 \sin \theta}{n_2q^2}} e^{-i\frac{\theta \hat{p}_2 n_1 a_1 \cos \theta}{n_2q^2}},
\]

\[
\tau(n_2a_2) = e^{-i\frac{\theta n_2 a_2}{n_2q^2}},
\]

where \( a_1 \) and \( a_2 \) are primitive translation vectors of a lattice, and they form an angle \( \theta \). These operators commute with a Hamiltonian \( H_0 = \frac{1}{2m}(\hat{p} + e\hat{A})^2 \). They become commutative when the vectors are enlarged from \( (a_1, a_2) \) to \( (qa_1, a_2) \). Then, we shall produce from the Landau level wave functions which satisfy periodic boundary condition in terms of the magnetic translation operators, i.e., \( \tau(N_1a_1)\psi(x) = \psi(x) \) and \( \tau(N_2a_2)\psi(x) = \psi(x) \) where \( N_1 \) and \( N_2 \) are periodicity of the system, and \( N_1 \) is assumed to be a multiple of \( q \). On these periodic fuctions, the magnetic translation operators form a representation of abelian magnetic translation group (MTG) which is isomorphic to \( (\mathbb{Z}/(N_1/q)\mathbb{Z}) \times (\mathbb{Z}/N_2\mathbb{Z}) \). Finally, multiplying a projection operator of an irreducible representation of MTG onto the obtained wave functions, we obtain:

\[
\psi_{N_1, \frac{n_1}{N_1}, \frac{n_2}{N_2}a_1, a_2, m}(x)
\]

\[
= \sqrt{\frac{q}{N_1}} \sum_{N_1} \sum_{\tilde{n}_1=1}^{N_1} \sum_{\tilde{n}_2=-\infty}^{\infty} e^{i\frac{\pi}{4}n_1\tilde{n}_1 a_1 \cos \theta} e^{-i\frac{\pi}{4}n_1\tilde{n}_1 a_1 \sin \theta} e^{i\frac{\pi}{4}n_2\tilde{n}_2 a_2 \cos \theta} e^{-i\frac{\pi}{4}n_2\tilde{n}_2 a_2 \sin \theta} \psi_{N_1, \frac{n_1}{N_1}, \frac{n_2}{N_2}a_1, a_2, m}(x_1)
\]

\[
\times \phi_{N_1, \frac{n_1}{N_1}, \frac{n_2}{N_2}a_1, a_2, m, \tilde{n}_1, \tilde{n}_2}(x_2) \frac{1}{\sqrt{L_2}} e^{i\frac{\pi}{4} n_2(\tilde{n}_2 + N_2m - N_2(p\tilde{n}_1 + \frac{p+q}{q}N_1)) x_2},
\]
where \( L_1 = N_1a_1 \) and \( L_2 = N_2a_2 \). \( \phi_{N,X}(x) = \frac{1}{\sqrt{2\pi N\sqrt{\pi l}}} (\frac{x}{2\pi} - l \frac{\partial}{\partial x})^N \exp(-\frac{1}{2\pi}(x - X)^2) \) where \( l \) is the magnetic length, \( I^2 = \frac{h^2}{2\pi} = \frac{q^2}{2\pi^2} \).

The wave function of equation (4) satisfies \( \tau(n_1q_1a_1) \psi_{N, \frac{x}{2\pi}a_1, \frac{y}{2\pi}a_2, m} = e^{-i\frac{q_1}{2\pi}a_1q_2} \psi_{N, \frac{x}{2\pi}a_1, \frac{y}{2\pi}a_2, m} \).

\( \tau(n_2a_2) \psi_{N, \frac{x}{2\pi}a_1, \frac{y}{2\pi}a_2, m} = e^{-i\frac{a_1q_1}{2\pi}a_2} \psi_{N, \frac{x}{2\pi}a_1, \frac{y}{2\pi}a_2, m} \). \( (k_1, k_2) = (\frac{q_1}{2\pi}a_1, \frac{q_2}{2\pi}a_2) \) are the wave vectors in the magnetic Brillouin zone (MBZ), i.e., \(-\frac{q_1}{2\pi} \leq k_1 \leq \frac{q_1}{2\pi}, -\frac{q_2}{2\pi} \leq k_2 \leq \frac{q_2}{2\pi}\). It also satisfies

\[ H_0 \psi_{N, \frac{x}{2\pi}a_1, \frac{y}{2\pi}a_2, m} = (N + \frac{1}{2}) \hbar \omega \psi_{N, \frac{x}{2\pi}a_1, \frac{y}{2\pi}a_2, m}. \]

Because a periodic potential is invariant under the magnetic translation operations, the representation matrix in terms of the basis (4) is block diagonalized. The matrix elements of a periodic potential is given in Appendix A.

Note that inter Landau level index \( m \) varies as 0, \ldots, \( p - 1 \); that is, each Landau level is \( p \)-fold degenerated and splits into \( p \) bands in general when a periodic potential is applied. This is often misunderstood as \( q \)-fold as Douglas Hofstadter himself did.

Although a similar method was given by Springsguth et al., the reason why the method is justified is not presented and is different from ours. Although subtlety of our method is the periodic boundary condition in the presence of a magnetic field, this subtlety comes from validity of using periodic boundary condition to analyze a solid which is not periodic actually. Moreover, people who use tight binding model with Peierls-Onsager substitution can not criticize our method because phase factors are the same as theirs. Note that, of course, hopping integral changes in general as magnetic field strength varies. Our method shows Hofstadter butterfly structure as well (FIG.1). In FIG.1, as mentioned above, the \( p \)-fold degenerated lowest flat Landau level splits into \( p \) distinct bands.

III. HALL CONDUCTANCE

For the Hamiltonian (1) regardless of its potential, the Kubo formula for Hall conductance is

\[ \sigma_{12} = \frac{-\hbar e^2}{iVM^2} \sum_{n,n',k}\sum_{E_{nk}\leq E_F} \frac{<nk|\pi_1|n'k> <n'k|\pi_2|nk> - <nk|\pi_2|n'k> <n'k|\pi_1|nk>}{(E_{nk} - E_{n'k})^2}, \]

where \( \pi_1 = \hat{p}_1, \pi_2 = \hat{p}_2 + eB\hat{x}_1 \). When the Fermi level lies in a gap between subbands of the split lowest Landau level, the calculation of Hall conductance corresponds to the calculation,

\[ \frac{1}{2\pi} \sum_{N,m} \int_{MBZ} d^2k \int d^2x (\frac{\partial u^*_N}{\partial k_2} N_{k,m} \frac{\partial u_{N,k,m}}{\partial k_1} - \frac{\partial u^*_N}{\partial k_1} N_{k,m} \frac{\partial u_{N,k,m}}{\partial k_2}) \]

where \( u^*_N, N_{k,m} (x) = e^{-ik_1x_1 - ik_2x_2} \psi_{N,k,m}^*(x) \). Kohmoto claims that this quantity is a Chern number and is therefore an integer. To clarify the invalidity of the argument, let us calculate the Hall conductance directly and concretely. Equation (6) is transformed into

\[ -\frac{ne}{B} - \frac{2me}{VB^3} \sum_{N,k,m,E_{nk}\leq E_F} \frac{|<N,k,m|\frac{\partial V}{\partial x_1}|N',k,m'||^2}{E_{N,k,m} - E_{N',k,m'}} \]

by using the commutation relation. The first term is equal to \(-\frac{e^2}{\hbar}\) when \( j \) bands are below the Fermi energy, by utilizing the number of states of MBZ of \( N_1N_2/q \). Therefore, if the second term is negligible, the Chern number argument is invalid. In fact, this occurs. The result of our numerical calculation is shown in FIG.2. Plateaus appear as non-integers. In fact, the second term is at most of the order \( 10^{-56} \) in units of \( \frac{2\pi}{q} \), thus negligible.

What the author wants to say is that although Hall conductance has a form of integration of Chern class, nevertheless, it does not become an integer. Why the integration did not become an integer is rather simple: It forms no connection. When we take one of the largest open coverings, i.e., whole base space minus measure zero subspace, a quantity on the covering can not be always regarded as a connection. In our case, this occurred. In the argument, the basis for being integers is only that Hall conductance is written in the same form as the Chern number, which lacks confirmation whether the quantity made by wave functions can form a connection. And in fact, it cannot; hence, the reason this failure occurred.
FIG. 1: The periodic potential is taken as \( V(x) = \sum_{l_1= \pm 1, l_2= \pm 1} V_{l_1,l_2} e^{2\pi i \left( \frac{l_1 x_1}{L_1} + \frac{l_2 x_2}{L_2} \right)} \) where \( V_{l_1,l_2} = -\frac{e^2}{4 \pi^2 \varepsilon_0} \frac{1}{l_1^2 + l_2^2} \). The lattice is square with the lattice constant 1 Å. Calculation was done up to \( p = 50 \). The horizontal axis is \( \Phi_0/\Phi(= q/p) \). The vertical axis is \( (E - \hbar \omega/2) / 4 \pi^2 \varepsilon_0 / e^2 \). It can be seen that the broadening of the lowest Landau level due to the periodic potential gets larger as magnetic field strength tends to infinity.

**IV. DISCUSSION**

Believers of the Chern number argument say that the Chern number within a Landau level has yet to be observed since the energy gap is too small. But can such an anomalous behavior of Hall conductance actually be observed? Experiments in a magnetic field so far show us monotonic behavior when gate voltage or magnetic field varies \( 5-7 \). Our calculation here is consistent with those experiments in being monotonic. Development of an experimental method that sheds light on small energy gaps will clarify which is the correct perspective.

**Appendix A: Matrix Elements of a Periodic Potential**

Let us consider the following quantity:

\[
\langle N, \frac{2\pi}{L_1} n_1, \frac{2\pi}{L_2} n_2, m | e^{i \frac{2\pi}{L_1} \left( \frac{l_1 x_1}{L_1} - \frac{l_2 x_2}{L_2} \right)} | N', \frac{2\pi}{L_1} n_1', \frac{2\pi}{L_2} n_2', m' \rangle.
\]

(A1)

Let us write \( l_2 = p l_2 + l''_2 \), \( l''_2 \) is always taken to be \( 0 \leq l''_2 \leq p - 1 \). We shall assume that \( |l_2| < N'_1 = \frac{N_1}{q} \). Then, (i) If \( l''_2 \neq 0 \), \( m' \leq m \), and \( l'_2 \geq 0 \)
\[ \langle N, \frac{2\pi}{L_1} n_1, \frac{2\pi}{L_2} n_2, m | e^{i \frac{2\pi x_1}{L_1}} (\frac{L_2}{L_2}) x_2 | N', \frac{2\pi}{L_1} n_1, \frac{2\pi}{L_2} n_2, m' \rangle = \delta_{m', m''} e^{i \frac{2\pi}{L_1} n_1 l'_2 q a_1} e^{-i \frac{2\pi}{L_2} n_2 l'_2 q a_1 \cos \theta} e^{i \frac{\pi q a_1}{N_2} \sin \theta (l'_2 - 2m)} \times \int_{-\infty}^{\infty} dx_1 \phi_{N, (-\frac{a(n_2 + N_2 m)}{2N_2}) a_1} \sin \theta (x_1) \phi_{N', (-\frac{a(n_2 + N_2 m')}{2N_2}) a_1} \sin \theta (x_1) \nonumber \times e^{i \frac{2\pi x_1}{L_1} (\frac{L_2}{L_1}) x_2}, \] 

(A2)

(ii) If \( l'_2 \neq 0, m' \leq m, \) and \( l'_2 < 0, \)

\[ = \delta_{m', m''} m e^{i \frac{2\pi}{L_1} n_1 (l'_2 + 1) q a_1} e^{-i \frac{2\pi}{L_2} n_2 (l'_2 + 1) q a_1 \cos \theta} e^{i \frac{\pi q a_1}{N_2} \sin \theta (l'_2 - 2m)} \times \int_{-\infty}^{\infty} dx_1 \phi_{N, (-\frac{a(n_2 + N_2 m)}{2N_2}) a_1} \sin \theta (x_1) \phi_{N', (-\frac{a(n_2 + N_2 m')}{2N_2}) a_1} \sin \theta (x_1) \nonumber \times e^{i \frac{2\pi x_1}{L_1} (\frac{L_2}{L_1}) x_2}, \] 

(A3)

(iii) If \( l'_2 \neq 0, m' \geq m, \) and \( l'_2 + 1 \geq 0, \)

\[ = \delta_{m', m''} m e^{i \frac{2\pi}{L_1} n_1 (l'_2 + 1) q a_1} e^{-i \frac{2\pi}{L_2} n_2 (l'_2 + 1) q a_1 \cos \theta} e^{i \frac{\pi q a_1}{N_2} \sin \theta (l'_2 + 2m)} \times \int_{-\infty}^{\infty} dx_1 \phi_{N, (-\frac{a(n_2 + N_2 m)}{2N_2}) a_1} \sin \theta (x_1) \phi_{N', (-\frac{a(n_2 + N_2 m')}{2N_2}) a_1} \sin \theta (x_1) \nonumber \times e^{i \frac{2\pi x_1}{L_1} (\frac{L_2}{L_1}) x_2}, \] 

(A4)

(iv) If \( l'_2 \neq 0, m' \geq m, \) and \( l'_2 + 1 < 0, \)

\[ = \delta_{m', m''} m e^{i \frac{2\pi}{L_1} n_1 (l'_2 + 1) q a_1} e^{-i \frac{2\pi}{L_2} n_2 (l'_2 + 1) q a_1 \cos \theta} e^{i \frac{\pi q a_1}{N_2} \sin \theta (l'_2 - 2m)} \times \int_{-\infty}^{\infty} dx_1 \phi_{N, (-\frac{a(n_2 + N_2 m)}{2N_2}) a_1} \sin \theta (x_1) \phi_{N', (-\frac{a(n_2 + N_2 m')}{2N_2}) a_1} \sin \theta (x_1) \nonumber \times e^{i \frac{2\pi x_1}{L_1} (\frac{L_2}{L_1}) x_2}, \] 

(A5)

(v) If \( l'_2 = 0 \) and \( l'_2 \geq 0, \)

\[ = \delta_{m', m''} m e^{i \frac{2\pi}{L_1} n_1 l'_2 q a_1} e^{-i \frac{2\pi}{L_2} n_2 l'_2 q a_1 \cos \theta} e^{i \frac{\pi q a_1}{N_2} \sin \theta (l'_2 + 2m)} \times \int_{-\infty}^{\infty} dx_1 \phi_{N, (-\frac{a(n_2 + N_2 m)}{2N_2}) a_1} \sin \theta (x_1) \phi_{N', (-\frac{a(n_2 + N_2 m')}{2N_2}) a_1} \sin \theta (x_1) \nonumber \times e^{i \frac{2\pi x_1}{L_1} (\frac{L_2}{L_1}) x_2}, \] 

(A6)

(vi) If \( l'_2 = 0 \) and \( l'_2 < 0, \)

\[ = \delta_{m', m''} m e^{i \frac{2\pi}{L_1} n_1 l'_2 q a_1} e^{-i \frac{2\pi}{L_2} n_2 l'_2 q a_1 \cos \theta} e^{i \frac{\pi q a_1}{N_2} \sin \theta (l'_2 - 2m)} \times \int_{-\infty}^{\infty} dx_1 \phi_{N, (-\frac{a(n_2 + N_2 m)}{2N_2}) a_1} \sin \theta (x_1) \phi_{N', (-\frac{a(n_2 + N_2 m')}{2N_2}) a_1} \sin \theta (x_1) \nonumber \times e^{i \frac{2\pi x_1}{L_1} (\frac{L_2}{L_1}) x_2}, \] 

(A7)
FIG. 2: The horizontal axis is \((E_F - \hbar\omega/2)\frac{e^2}{4\pi^2\epsilon_0}\), where \(E_F\) is the Fermi energy. The vertical axis is the Hall conductance in units of \(\frac{e^2}{\hbar}\). The condition is the same as FIG.1. The calculation is done at \(\frac{q_p}{r_p} = \frac{2}{3}\). At \(\frac{q_p}{r_p} = \frac{2}{3}\), the lowest Landau level splits into three bands as can be seen in FIG. 1. When the Fermi energy is in the gaps, the Hall conductance exhibits plateaus of approximately \(-\frac{1}{3}, -\frac{2}{3}, -1\).

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1. K. von Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. 45, 449 (1980).
2. M. Kohmoto, Ann. of Phys. (New York) 160,343 (1985).
3. D. Springguth, R. Ketzerick, and T. Geisel, Phys. Rev. B 56, 2036 (1997).
4. D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
5. S. Kawaji and J. Wakabayashi in *Physics in High Magnetic Fields*, edited by S. Chikazumi and N. Miura (Springer, Berlin, 1981), p. 284.
6. R. Willet, J. P. Eisenstein, H. L. Storner, D. C. Tsui, A. C. Gossard and J. H. English, Phys. Rev. Lett. 59, 1776 (1987).
7. We do not take into account the Rashba term.