Detecting topological orders through continuous quantum phase transition

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(Dated: June 2005)

We study a continuous quantum phase transition that breaks a Z_2 symmetry. We show that the transition is described by a new critical point which does not belong to the Ising universality class, despite the presence of well defined symmetry breaking order parameter. The new critical point arises since the transition not only break the Z_2 symmetry, it also changes the topological/quantum order in the two phases across the transition. We show that the new critical point can be identified in experiments by measuring critical exponents. So measuring critical exponents and identifying new critical points is a way to detect new topological phases and a way to measure topological/quantum orders in those phases.

PACS numbers: 73.43.Nq, 71.10.Hf, 71.27.+a

For a long time, Landau symmetry-breaking theory [1, 2] is believed to be the theory that describes all possible phases and phase transitions. The Ginzburg-Landau theory [3] based on order parameters and long range order became the standard theory for all kinds of continuous phase transitions.

However, after the discovery of fractional quantum Hall (FQH) effect,[4] people realized that different FQH states all have the same symmetry. So the order in FQH states cannot be described by the Landau’s symmetry breaking theory. The new order is called topological order. [5, 6] Topological order is new since it has nothing to do with symmetry breaking, long range correlation, or local order parameters. None of the usual tools that we used to describe a symmetry breaking phase applies to topological order. Despite this, topological order is not an empty concept since it can be described by a new set of tools, such as the number of degenerate ground states [7, 8], quasiparticle statistics [9], and edge states [6, 10, 11].

The existence of topological orders has consequences on our understanding of continuous phase transitions. If there exists phases that are not described by symmetry breaking, then it is reasonable to guess that there exist continuous phase transitions that are not described by changes of symmetries and the associated order parameters. Indeed continuous phase transitions exist between two phases with the same symmetry [12–16] and between two phases with the incompatible[26] symmetries.[17]. In this paper, we will show that even some symmetry breaking continuous phase transitions are beyond Landau’s symmetry breaking paradigm in the sense that critical properties of the transition are not described by fluctuating symmetry breaking order parameters and not described by Ginzburg-Landau effective theories. As a results, the critical exponents of those symmetry breaking transitions are different for those obtained from Ginzburg-Landau theory.

Why some symmetry breaking transitions give rise to new class of critical points? One reason is that those transitions not only change the symmetry of the states, they also changes the topological/quantum order in the states. So the appearance of the new critical points, in many cases, imply the appearance of new state of matter with non-trivial topological/quantum orders. It is known that frustrated spin systems on Kagome or pyrochlore lattices contain many different quantum phases. Those different quantum phases in general contain different spin orders as shown by magnetic susceptibility measurements. So one naturally assume those spin ordered phases are described by symmetry breaking, and the continuous transition between those phases are symmetry breaking transitions described by Ginzburg-Landau theory. The main message of this paper is that those T = 0 spin ordered phases may contain additional topological orders and represent new states of matter. The additional topological orders can be detected by measuring critical exponents at continuous quantum transition points between those T = 0 quantum phases (even when the continuous transitions are symmetry breaking transitions). If the measured critical exponents do not agree with the values obtained from Ginzburg-Landau theory, then the transition point will be a new quantum critical point and the two phases separated by the phase transition will contain non-trivial topological orders.

But why zero temperature is important? Can we find new states of matter and new continuous phase transitions at finite temperatures? The answer is yes, but it is more difficult to find new states of matter and new continuous transitions at finite temperatures. This is because most of the known new states of matter are due to string-net condensations. [18, 19] String-net condensations and the continuous phase transition between different string-net condensed states only exist at zero temperature. So it is much easier to find new states of matter and new continuous transitions at zero temperature.

To illustrate our point that a continuous T = 0 symmetry breaking phase transition can have a new set of critical exponents beyond Ginzburg-Landau symmetry breaking theory, we consider a frustrated spin-1/2 model

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The effective field theory that describes the transition from the $SU(2)$-linear phase to the chiral spin phase.

on square lattice:

$$H = \sum_{ij} J_{ij} S_i \cdot S_j$$

where the coupling between the nearest neighbors is $J_1$ and between the second nearest neighbors is $J_2$. We have used $SU(2)$ slave-boson theory [20, 21] to study the possible spin liquid phases of the above frustrated spin model. Under mean-field approximation, we find a continuous phase transition between the two spin liquids at $J_2/J_1 \approx 0.52$. [22, 23]

The spin liquid phase for $J_2 < 0.52J_1$ is called $SU(2)$-linear spin liquid (or $\pi$-flux phase) [20, 21, 24] whose low energy effective field theory is a $SU(2)$ gauge theory
couple to massless Dirac fermions [23]

$$L = \sum_{i=1}^{N} \bar{\psi}_a (\partial_{\mu} - ia_{\mu}^{l} \tau_{ab}) \gamma_{\mu} \psi_{b} + \frac{1}{4g^2} f_{\mu\nu}^{l} f_{\mu\nu}^{l}$$

where $\psi_{ai}$ is four-component Dirac fermion field, $a = 1, 2, i = 1, 2, ..., N$, and $N = 1$. $\gamma_{\mu}$, $\mu = 0, 1, 2, 3, 5$, are Dirac matrices and the summation of $\mu$ run through $0, 1, 2, \tau^l$, $l = 1, 2, 3$, are Pauli matrices and $a_{\mu}^l$ is $SU(2)$ gauge field.

The spin liquid phase for $J_2 > 0.52J_1$ is called chiral spin liquid [22] whose low energy effective field theory is a $SU(2)$ gauge theory couple to fermions with a chiral mass [22, 23]

$$L = \sum_{i=1}^{N} \bar{\psi}_a (\partial_{\mu} - ia_{\mu}^{l} \tau_{ab}) \gamma_{\mu} \psi_{b} + m\bar{\psi}_a (i\gamma^{3}\gamma^{5}) \psi_{ai} + \frac{1}{4g^2} f_{\mu\nu}^{l} f_{\mu\nu}^{l}$$

The effective field theory that describes the transition connects (2) and (3) and is given by [23]

$$L = \sum_{i=1}^{N} \bar{\psi}_i (\partial_{\mu} - ia_{\mu}^{l} \tau^{l}) \gamma_{\mu} \psi_{i} + \frac{1}{4g^2} f_{\mu\nu}^{l} f_{\mu\nu}^{l} + \sigma \bar{\psi} [i\gamma_{3}\gamma_{5}] \psi + \frac{1}{2g^2} (\partial_{\mu} \sigma)^2 + V(\sigma)$$

where $V(\sigma) = V(-\sigma)$.

We note that the $SU(2)$-linear state does not break any symmetry and the chiral spin state breaks the time reversal and parity symmetry. The real $\sigma$ field is the order parameter of the symmetry breaking. The potential $V(\sigma)$ controls the phase transition (see Fig. 1). $\sigma = 0$ gives rise to the $SU(2)$-linear state and $\sigma \neq 0$ gives rise to the symmetry breaking chiral spin state. The order parameter $\sigma$ is related to the following combination of physical spin operators: $\sigma \propto S_i \cdot (S_{i+x} \times S_{i+y})$ [22].

The transition between the $SU(2)$-linear and the chiral spin state is a $Z_2$ symmetry breaking transition. So we may expect the critical point to belong to the universality class of 3D Ising model. For 3D Ising model, the order parameter has scaling dimension $[\sigma]_{\text{Ising}} \approx 0.51$ (or a correlation $\langle \sigma(x)\sigma(0) \rangle = x^{-2-2[\sigma]_{\text{Ising}}}$) at the critical point. One may conclude that, at the transition point between the $SU(2)$-linear and the chiral spin state, the order parameter $S_i \cdot (S_{i+x} \times S_{i+y})$ also has the correlation $\langle S_i \cdot (S_{i+x} \times S_{i+y}) \rangle \propto |i-j|^{-2[\sigma]_{\text{Ising}}}$.

In fact the above guess is incorrect. For our case, even though the transition breaks a $Z_2$ symmetry and has a well defined $Z_2$ order parameter, the critical point does not belong to the 3D Ising class.

Why the $SU(2)$-linear state to the chiral spin state transition belongs to a new universality class? The reason is that, at the critical point, not only the fluctuations of the order parameter $\sigma$ give rise to gapless excitations, the fermion field $\psi$ and the $SU(2)$ gauge field $a_{\mu}^{l}$ also give rise to gapless excitations. Had all gapless excitations come from the order parameter $\sigma$, then the transition would be belong to the 3D Ising class. So the key to understand the existence of the new critical point is to understand why $\psi$ and $a_{\mu}^{l}$ can give rise to gapless excitations.

At first sight, one may expect both $\psi$ and $a_{\mu}^{l}$ are gapped due to their interaction. In fact, the self energy term (see Fig. 2) from the $SU(2)$ gauge interaction, in general, can generate a fermion mass term $\delta m \bar{\psi} \psi$. Once the fermions are gapped, the $SU(2)$ gauge field is always confined phase in 1+2 dimensions and the gauge bosons are also gapped. So in order for the new critical point described by the gapless $\sigma$, $\psi$ and $a_{\mu}^{l}$ field to exist, we must find the reason that protect the gaplessness of $\psi$ and $a_{\mu}^{l}$.

To find such a reason, we like to point out that both the $SU(2)$-linear state and the chiral spin state contain non-trivial quantum or topological orders. Such quantum/topological orders are characterized by projective symmetry groups (PSG) which describe symmetry of the effective theory. [19, 24] The PSGs of the $SU(2)$-linear state and the chiral spin state are studied in detail and will be described in a forthcoming paper. [23] We find that it is PSG that protects the gaplessness of $\psi$ and $a_{\mu}^{l}$.  

![FIG. 2: The fermion self energy due to the gauge interaction. The solid lines represent the fermion propagator and wiggled line gauge propagator.](image-url)
In other words, if we regulate the effective field theory (4) in a way that does not break the symmetry described by the PSG, then the self energy term in Fig. 2 cannot generate fermion mass. [23] With the gapless excitations from $\sigma$, $\psi$ and $\sigma'$, the effective field theory (4) describes a new critical point at the transition between the $SU(2)$-linear state and the chiral spin state.

The scaling dimensions of operators are in general difficult to calculate. However, if we assume the number of the fermion fields $N$ to be large (instead of $N = 1$), then they can be calculated systematically in $1/N$ expansion. [23] We find that the order parameter $\sigma$ has a scaling dimension $[\sigma] = 1 + O(\frac{1}{N})$ at the new critical point, which is different from the scaling dimension for the 3D Ising universality class given by $[\sigma]_{\text{Ising}} = 0.51$. Near the transition, there is a diverging length scale $\xi \propto |t - t_c|^{-\nu}$ where $t$ is a parameter (such as $J_2/J_1$) that controls the transition. For the new critical point, the coherent length exponent $\nu$ is found to be $\nu = 1 + O(\frac{1}{N})$, while for the 3D Ising universality class $\nu_{\text{Ising}} \approx 0.63$.

We can also calculate the staggered spin-spin correlations which is easier to measure. In $SU(2)$-linear phase and at the critical point, spins have algebraic correlations with different exponents. In the chiral spin phase, the spins have short ranged correlation (see Fig. 3).

The scaling dimension of staggered-spin-correlation function $\langle (-)^x S(x)S(0) \rangle$ is calculated by the large-$N$ expansion of quantum field theory. In our formalism, one can show that the staggered-spin-correlation function is just the correlation function of the fermion mass operator $\langle \psi \bar{\psi}(x) \bar{\psi}(0) \rangle$ in the effective theory Eq.(4). By power counting, the scaling behavior should be $\langle \psi \bar{\psi}(x) \bar{\psi}(0) \rangle \propto x^{-4}$, but quantum fluctuation change the it into $\langle \psi \bar{\psi}(x) \bar{\psi}(0) \rangle \propto x^{-4-2\gamma_{\psi}}$, where $\gamma_{\psi}$ is called the anomalous dimension of fermion mass operator.

In the following, we will use the spin correlation as an example to demonstrate how various correlations are calculated in the large $N$ limit. It turned out that the easiest way of calculating $\gamma_{\psi}$ is not to calculate $\langle \psi \bar{\psi}(x) \bar{\psi}(0) \rangle$ directly, but to calculate the correlation function of fermion field $\bar{\psi}(x) \bar{\psi}(0)$, and the three-point correlation function $\langle \psi \bar{\psi}(x) \bar{\psi}(y) \rangle$. Let us firstly calculate the staggered spin-spin correlation function in $SU(2)$-linear phase, where the low energy effective theory is Eq.(2). We will do our calculations in Landau gauge.

In the large-$N$ limit, the gauge field is strongly screened by fermions. To the leading order of $\frac{1}{N}$, the dressed gauge propagator is shown in Fig. 4. The dressed gauge propagator in Landau gauge is found to be:

$$C_{\mu \nu, \text{dressed}}(k) = \frac{N \delta^{ab}}{16k} \left( \delta_{\mu \nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right)$$

(5)

The fermion correlation to the first order in $\frac{1}{N}$, is given by (see Fig. 4):

$$S_{\text{dressed}}(k) = -\frac{i k^2}{\lambda} \left( 1 + \Sigma \right)$$

(6)

$$p \Sigma = i \int \frac{dq}{(2\pi)^3} \gamma_{\psi}(-i)(k + \bar{g}) \frac{3}{4} \cdot \frac{16}{Nq} \left( \delta_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right)$$

(7)

Therefore the anomalous dimension of $\psi$ is $\gamma_{\psi} = -\frac{1}{2} \frac{4}{\pi^2 N}$.

Then we look at the dressed three-point correlation function $\langle \bar{\psi}(x) \bar{\psi}(y) \psi(0) \rangle$ at order of $\frac{1}{N}$, as shown in Fig. 5. Suppose we fix the momentum of $\bar{\psi}$ to be $2k$, while $\bar{\psi}$ and $\psi$ each carry momentum $k$, then the tree level three point correlation function will be $\frac{G_3(2k,k,k) = \frac{1}{k^2} \left( 1 + (A + B + C) \log \left( \frac{k}{\Lambda} \right) \right)}{A, B, C}$.

From the contributions of diagrams in Fig. 5, the dressed three point correlation function is:

$$G_{3, \text{dressed}}(2k,k,k) = \frac{1}{k^2} \left( 1 + (A + B + C) \log \left( \frac{k}{\Lambda} \right) \right)$$

where $A, B, C$ are the contribution from each corresponding diagram. Actually we know that $A + B + C = \gamma_{\psi} + 2 \gamma_{\psi}$.

It is easy to see that $A, B$ come from the dressed fermion propagator: $A = B = 2 \gamma_{\psi}$. New calculation
Thus the difference is that the transition at the critical point in a similar fashion. The only need to be done for vertex correction in C.

\[
C \log\left(\frac{k}{\Lambda}\right) = \frac{4}{N} \cdot 16 \int \frac{d \mathbf{q}}{(2\pi)^3} \sum_{\mu \nu} \gamma_{\mu} \gamma_{\nu} \left(\frac{k_{\mu} k_{\nu}}{k^2}\right) \delta_{\mu \nu} \frac{q^2}{q^2} + \frac{16}{\pi^2 N} \log\left(\frac{k}{\Lambda}\right).
\]

(8)

Thus \( \gamma_{\psi \psi} = A + B + C - 2\gamma_{\psi} = -\frac{16}{\pi^2 N} \).

We can also calculate the spin-spin correlation function at the critical point in a similar fashion. The only difference is that the \( \sigma \)-boson becomes massless at critical point and contributes to the anomalous dimension of correlation functions. As shown in Fig.6 and Fig.7, after similar calculations, we found that at the critical point, \( \gamma_{\psi \psi} = -\frac{16}{\pi^2 N} + \frac{4}{3\pi^2 N} \), where the second term comes from contribution of massless \( \sigma \)-boson.

In this paper, we study a continuous quantum phase transition that breaks a \( Z_2 \) symmetry. Despite being a symmetry breaking transition with well defined order parameter, the transition is described by a new critical point which does not belong to the Ising universality class. The new critical point is due to the fact that the transition not only break the \( Z_2 \) symmetry, it also changes the topological/quantum order in the two phases across the transition. The additional gapless excitations protected by the PSG change the scaling behavior and the universality class of the critical point. So even a symmetry breaking continuous transition may be described by a new critical point. Thus it is very important to measure critical exponents even for seemingly ordinary symmetry breaking transition. A new critical point can be identified by measuring critical exponents and confirming the critical exponents to be different from those predicted by Ginzburg-Landau theory. Since a new critical point is usually due to a change in topological/quantum order at the transition, a discovery of new critical point usually implies a discovery of new states of matter with non-trivial topological/quantum orders on the two sides of the transition.

This research was supported by NSF grant No. DMR-0433632.

\[\begin{align*}
\text{FIG. 6: The contribution of } \sigma \text{-boson to fermion propagator at order of } \frac{1}{k}, \text{ where the double dashed line is the dressed } \\
\sigma \text{-boson propagator at leading order.}
\end{align*}\]

\[\begin{align*}
\text{FIG. 7: Contributions of } \sigma \text{-boson to three point correlation function at order of } \frac{1}{k}.
\end{align*}\]