Variations of primeness and factorization of ideals in Leavitt path algebras

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**ABSTRACT**

In this paper we describe three different variations of prime ideals: strongly irreducible ideals, strongly prime ideals and insulated prime ideals in the context of Leavitt path algebras. We give necessary and sufficient conditions under which a proper ideal of a Leavitt path algebra \(L\) is a product as well as an intersection of finitely many of these different types of prime ideals. Such factorizations, when they are irredundant, are shown to be unique except for the order of the factors. We also characterize the Leavitt path algebras \(L\) in which every ideal admits such factorizations and also in which every ideal is one of these special type of ideals.

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1. Introduction

The multiplicative ideal theory in commutative algebra has been an active area of research with contributions from many researchers including Robert Gilmer and William Heinzer. Recently, the development of the multiplicative ideal theory of Leavitt path algebras has become an active area of research. Leavitt path algebras are algebraic analogues of graph \(C^*\)-algebras and are also natural generalizations of Leavitt algebras of type \((1, n)\) constructed by William Leavitt. What stands out quite surprising is that, even though Leavitt path algebras are highly non-commutative, the multiplicative ideal theory of Leavitt path algebras is quite similar to that of commutative algebras. Specifically, Leavitt path algebras satisfy a number of characterizing properties of special types of commutative integral domains such as the Bézout domains, the Dedekind domains, the Prüfer domains etc., in terms of their ideal properties (see [6, 7, 20]). These integral domains are well-known for admitting satisfactory ideal factorizations. Because of this, investigating the factorizations of ideals in a Leavitt path algebra as products or intersections of special types of ideals such as the prime, the semiprime, the irreducible and the primary ideals are quite promising. Prime ideals and their various generalizations play essential role in developing the multiplicative ideal theory of commutative algebras. So it is natural to study these notions in the case of Leavitt path algebras to develop its multiplicative ideal theory and the various types of ideal factorizations. The theory of prime ideals and that of semi-prime ideals for Leavitt path algebras were developed in [3, 17]. In this paper we study three different variations in the notion of prime ideals and irreducible ideals in the context of Leavitt path algebras.
Recall that if $P$ is a prime ideal of a ring $R$, then for any two ideals $A, B$ of $R$, $A \cap B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$. The converse of this statement is not true. For example, in the ring $\mathbb{Z}$ of integers, it can be verified that the ideal $8\mathbb{Z}$ has this property by using the prime power factorization of integers in $\mathbb{Z}$, but $8\mathbb{Z}$ is not a prime ideal. Ideals of a ring having this property were first studied by Fuchs [8] who called them primitive ideals. Blair [4] called them strongly irreducible ideals. The idea was clearly inspired by strengthening the conditions required for an ideal to be an irreducible ideal. Recall that an ideal $I$ of a ring $R$ is said to be irreducible if, for any two ideals $A, B$ of $R$, $A \cap B = I$ implies that $A = I$ or $B = I$. Interestingly, strongly irreducible ideals are mentioned in Bourbaki’s treatise on commutative algebras [5] where they are referred to as quasi-prime ideals. In [12], Heinzler, Ratliff Jr. and Rush investigated non-prime strongly irreducible ideals of commutative noetherian rings. Recently, Schwartz [21] studied the truncated valuations induced by strongly irreducible ideals in commutative rings. In general, an irreducible ideal need not be strongly irreducible [21], however, in the case of Leavitt path algebras, our description of the strongly irreducible ideals in Section 3 shows that these two notions coincide. The main result in this section gives necessary and sufficient conditions under which a proper ideal $I$ of a Leavitt path algebra $L$ can be represented as a product as well as an intersection of finitely many strongly irreducible ideals. Interestingly, the graded part $\text{gr}(I)$ of such an ideal $I$ plays an important role and, in this case, $I/\text{gr}(I)$ must be finitely generated. We also prove uniqueness theorems by showing that factorizations of an ideal $I$ of $L$ as an irredundant product or an irredundant intersection of finitely many strongly irreducible ideals are unique except for the order of the factors (Theorems 3.12 and 3.13). We give several characterizations (both algebraic and graphical) of Leavitt path algebras in which every proper ideal is uniquely an irredundant intersection/product of finitely many strongly irreducible ideals. This answers, in the context of Leavitt path algebras, an open question by W. Heinzler and B. Olberding ([11]) raised for the case of commutative rings. As a by-product, we obtain a characterization of the Leavitt path algebras which are Laskerian.

We also provide characterizations of Leavitt path algebras in which each ideal is strongly irreducible.

It is well-known that if a prime ideal $P$ contains an intersection of finitely many ideals $A_i, i = 1, 2, \ldots, n$, then $P$ contains at least one of the ideals $A_i$. However, this statement fails to hold for a prime ideal if we consider an infinite intersection of ideals. An ideal $P$ is called strongly prime ([13]) if the above statement holds even for an infinite intersection of ideals. In Section 4, we characterize the strongly prime ideals of a Leavitt path algebra $L$ and describe when a given ideal of $L$ can be factored as a product of strongly prime ideals. We also describe when every ideal of $L$ admits such a factorization. In [13], the authors call a commutative ring $R$ strongly zero-dimensional if every prime ideal of $R$ is strongly prime and they prove a number of interesting properties of strongly zero-dimensional commutative rings. We end the Section 4 by characterizing all the strongly zero-dimensional Leavitt path algebras.

Recall that an arbitrary ring $R$ is a prime ring if for all $a, b \in R$, whenever $a \neq 0, b \neq 0$, then there is an element $c \in R$ such that $acb \neq 0$. In their attempts to consider the non-commutative version of Kaplansky’s conjecture on prime von Neumann regular rings, Handelman and Lawrence [9] strengthen this concept of prime rings and consider rings with a stronger property. They do this by restricting, for each $a \neq 0$, the choice of the $c$ to a finite set (independent of $b$, but depending on $a$). To make this definition precise, they define a (right) insulator for $a \in R$ to be a finite subset $S(a)$ of $R$, such that the right annihilator $\text{ann}_R\{ac : c \in S(a)\} = 0$. A ring $R$ is said to be a right insulated prime ring if every non-zero element of $R$ has a right insulator; and an ideal $I$ of ring $R$ to be a right-insulated prime ideal if $R/I$ is a right insulated prime ring. It is known that, in general, the notion of insulated prime ring is not left-right symmetric. In fact, Handelman and Lawrence constructed a ring that is right insulated prime but not left insulated prime. In Section 5, we first describe when a Leavitt path algebra is a left/right insulated prime ring. Interestingly, the distinction between left and right insulated primeness vanishes for Leavitt path algebras. We show (Theorem 5.6) that a Leavitt path algebra is a left/right insulated prime ring exactly when it is a simple ring or it is isomorphic to the matrix ring $M_n(K[x, x^{-1}])$ some
integer \( n \geq 1 \). We characterize the insulated prime ideals of Leavitt path algebras and also describe conditions under which each ideal of a Leavitt path algebra can be factored as a product of insulated prime ideals.

### 2. Basics of Leavitt path algebras

A (directed) graph \( E = (E^0, E^1, r, s) \) consists of two sets \( E^0 \) and \( E^1 \) together with maps \( r, s : E^1 \to E^0 \). The elements of \( E^0 \) are called vertices and the elements of \( E^1 \) edges. A vertex \( v \) is called a sink if it emits no edges and a vertex \( v \) is called a regular vertex if it emits a non-empty finite set of edges. An infinite emitter is a vertex which emits infinitely many edges. For each edge \( e \in E^1 \), we consider an edge in the opposite direction, called ghost edge and denote it as \( e^* \). So we have \( r(e^*) = s(e) \), and \( s(e^*) = r(e) \).

A path \( \mu \) of length \( n > 0 \) is a finite sequence of edges \( \mu = e_1 e_2 \cdots e_n \) with \( r(e_i) = s(e_{i+1}) \) for all \( i = 1, \ldots, n - 1 \). We denote the length of the path \( \mu \) as \( |\mu| \). Thus for \( \mu = e_1 e_2 \cdots e_n \), we have \( |\mu| = n \). In this case \( \mu^* = e_n^* \cdots e_2^* e_1^* \) is the corresponding ghost path. A vertex is considered a path of length 0. The set of all vertices on a path \( \mu \) is denoted as \( \mu^0 \). A path \( \mu = e_1 \cdots e_n \) in \( E \) is closed if \( r(e_n) = s(e_1) \), in which case \( \mu \) is said to be a base at the vertex \( s(e_1) \). A closed path \( \mu \) as above is called simple provided it does not pass through its base more than once, i.e., \( s(e_i) \neq s(e_j) \) for all \( i = 2, \ldots, n \). The closed path \( \mu \) is called a cycle if it does not pass through any of its vertices twice, that is, if \( s(e_i) \neq s(e_j) \) for every \( i \neq j \).

A graph \( E \) is said to satisfy Condition (\( K \)), if any vertex \( v \) on a simple closed path \( c \) is also the base of a another simple closed path \( c' \) different from \( c \). An exit for a path \( \mu = e_1 \cdots e_n \) is an edge \( e \) such that \( s(e) = s(e_i) \) for some \( i \) and \( e \neq e_i \). A graph \( E \) is said to satisfy Condition (\( L \)), if every cycle in \( E \) has an exit.

If there is a path from vertex \( u \) to a vertex \( v \), we write \( u \geq v \). A subset \( D \) of vertices is said to be downward directed if for any \( u, v \in D \), there exists a \( w \in D \) such that \( u \geq w \) and \( v \geq w \). When we say that a graph \( E \) is downward directed, then it means \( E^0 \) is downward directed. A subset \( H \) of \( E^0 \) is called hereditary if, whenever \( v \in H \) and \( w \in E^0 \) satisfy \( v \geq w \), then \( w \in H \). A hereditary set is saturated if, for any regular vertex \( v \), \( r(s^{-1}(v)) \subseteq H \) implies \( v \in H \).

Given an arbitrary graph \( E \) and a field \( K \), the Leavitt path algebra \( L_K(E) \) is defined to be the \( K \)-algebra generated by a set \( \{v : v \in E^0\} \) of pair-wise orthogonal idempotents together with a set of variables \( \{e, e^* : e \in E^1\} \) which satisfy the following conditions:

1. \( s(e)e = e = er(e) \) for all \( e \in E^1 \).
2. \( r(e)e^* = e^* = e^*s(e) \) for all \( e \in E^1 \).
3. (The “CK-1 relations”) For all \( e, f \in E^1 \), \( e^*e = r(e) \) and \( e^*f = 0 \) if \( e \neq f \).
4. (The “CK-2 relations”) For every regular vertex \( v \in E^0 \),

\[
\gamma = \sum_{e \in E^1, s(e) = v} ee^*. 
\]

Every Leavitt path algebra \( L_K(E) \) is a \( \mathbb{Z} \)-graded algebra, namely, \( L_K(E) = \bigoplus L_n \) induced by defining, for all \( v \in E^0 \) and \( e \in E^1 \), \( \deg(v) = 0, \deg(e) = 1, \deg(e^*) = -1 \). Here, the \( L_n \) are abelian subgroups satisfying \( L_ml_n \subseteq L_{m+n} \) for all \( m, n \in \mathbb{Z} \). Further, for each \( n \in \mathbb{Z} \), the homogeneous component \( L_n \) is given by

\[
L_n = \left\{ \sum k_i z_i^* \beta_i^* \in L : |z_i| - |\beta_i| = n \right\}.
\]

Elements of \( L_n \) are called homogeneous elements. An ideal \( I \) of \( L_K(E) \) is said to be a graded ideal if \( I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n) \). If \( A, B \) are graded modules over a graded ring \( R \), we write \( A \cong_{gr} B \) if \( A \) and \( B \) are graded isomorphic and we write \( A \oplus_{gr} B \) to denote a graded direct sum. We will also be
using the usual grading of a matrix of finite order. For this and for the various properties of
graded rings and graded modules, we refer to [10] and [16].

For any ideal \( I \) of a Leavitt path algebra \( L_K(E) \), \( I \cap E^0 \) is a hereditary saturated subset
[1, Lemma 2.4.3]. A breaking vertex of a hereditary saturated subset \( H \) is an infinite emitter \( w \in E^0 \backslash H \) with the property that \( 0 < |s^{-1}(w) \cap r^{-1}(E^0 \backslash H)| < \infty \). The set of all breaking vertices of \( H \)
is denoted by \( B_H \). For any \( v \in B_H \), \( v^H \) denotes the element \( v - \sum_{s(e)}=v, r(e) \in H} ee^* \). Given a hereditary
saturated subset \( H \) and a subset \( S \subseteq B_H \), \( (H, S) \) is called an admissible pair. Given an admissible pair \( (H, S) \), the ideal generated by \( H \cup \{v^H : v \in S\} \) is denoted by \( I(H, S) \). It was shown in [22] that the graded ideals of \( L_K(E) \) are precisely the ideals of the form \( I(H, S) \) for some admissible pair \( (H, S) \). We have a partial ordering on the set of all admissible pairs of \( L_K(E) \) defined as \( (H_1, S_1) \leq (H_2, S_2) \) if and only if \( H_1 \subseteq H_2 \) and \( S_1 \subseteq H_2 \cup S_2 \). Moreover, \( L_K(E)/I(H, S) \cong L_K(E \backslash \{H, S\}) \) and \( (E \backslash \{H, S\})^1 = \{e \in E^1 : r(e) \not\in H\} \cup \{e' : e \in E^1 \text{ with } r(e) \in B_H \backslash S\} \) and \( r, s \) are extended to \( (E \backslash \{H, S\})^1 \) by setting \( s(e') = s(e) \) and \( r(e') = r(e) \).

A maximal tail is a subset \( M \) of \( E^0 \) satisfying the following three properties:

(1) \( M \) is downward directed;
(2) If \( u \in M \) and \( v \in E^0 \) satisfies \( v \geq u \), then \( v \in M \);
(3) If \( u \in M \) emits edges, there is at least one edge \( e \) with \( s(e) = u \) and \( r(e) \in M \).

A graph \( E \) is called row-finite if \( s^{-1}(v) \) is finite for each \( v \in E^0 \). A graph \( E \) is called a comet if it is row-finite, there is a cycle \( c \) without exits in \( E \) and every path in \( E \) ends at a vertex on \( c \).

We will be using the fact that the Jacobson radical (and in particular, the prime/Baer radical)
of \( L_K(E) \) is always zero (see [1]).

We will make the convention that if \( c \) is a cycle in the graph \( E \) based at a vertex \( v \), and if \( f(x) = 1 + k_1 x + \cdots + k_n x^n \in K[x] \), then \( f(c) = v + k_1 c + \cdots + k_n c^n \in L_K(E) \).

In the following, “ideal” means “two-sided ideal” and, given a subset \( S \) of \( L_K(E) \), we shall denote by \( < S > \), the ideal generated by \( S \) in \( L_K(E) \).

We begin with listing the various results and basic observations from the literature about
ideals in Leavitt path algebras that we will be using throughout this paper. The next theorem
describes a generating set for ideals in a Leavitt path algebra.

**Theorem 2.1.** (Theorem 4, [19]) Let \( L_K(E) \) be a Leavitt path algebra and let \( I \) be an ideal of
\( L_K(E) \) with \( I \cap E^0 = H \) and \( S = \{v \in B_H : v^H \in I\} \). Then \( I = I(H, S) + \sum_{t \in T} < f_t(c_t) > \) where \( T \) is
an index set (may be empty) such that for each \( t \in T \), \( c_t \) is a cycle without exits in \( E \backslash \{H, S\} \) and
\( f_t(x) \in K[x] \) with a non-zero constant term.

For convenience, some times we will denote \( I(H, S) \) by \( gr(I) \) and call it the graded part of the
ideal \( I \) described above.

The next result describes the prime ideals of Leavitt path algebras.

**Theorem 2.2.** (Theorem 3.12, [17]) An ideal \( P \) of \( L_K(E) \) with \( P \cap E^0 = H \) is a prime ideal if and
only if \( P \) satisfies one of the following properties:

(1) \( P = I(H, B_H) \) and \( E^0 \backslash H \) is downward directed;
(2) \( P = I(H, B_H \backslash \{u\}), v \geq u \) for all \( v \in E^0 \backslash H \) and the vertex \( u \) that corresponds to \( u \) in
\( E \backslash (H, B_H \backslash \{u\}) \) is a sink;
(3) \( P \) is a non-graded ideal of the form \( P = I(H, B_H) + < p(c) > \), where \( c \) is a cycle without exits
based at a vertex \( u \) in \( E \backslash (H, B_H) \), \( v \geq u \) for all \( v \in E^0 \backslash H \) and \( p(x) \) is an irreducible polynomial
in \( K[x, x^{-1}] \) such that \( p(c) \in P \).

We shall also be using the following two results.
Lemma 2.3. (Theorem 5.7, [20]) If an ideal $I$ of $L_K(E)$ is irreducible, then $I = P^n$, a power of a prime ideal $P$ for some $n \geq 1$. Also $\text{gr}(I) = \text{gr}(P)$ is a prime ideal.

Lemma 2.4. (Lemma 3.1, [20]) Let $A$ be a graded ideal of $L = L_K(E)$.

(a) For any ideal $B$ of $L$, $AB = A \cap B$; In particular, $A^2 = A$;
(b) $A = I_1 \ldots I_n$ is a product of ideals if and only if $A = I_1 \cap \ldots \cap I_n$ is their intersection.

For a ring $R$, and an infinite set $\Lambda$, we will denote by $M_\Lambda(R)$, the ring of $\Lambda \times \Lambda$ matrices in which all except at most finitely many entries are non-zero.

For more details on results in Leavitt path algebras, we refer the reader to [1] and [18].

3. Strongly irreducible ideals of leavitt path algebras

In this section we describe the strongly irreducible ideals of Leavitt path algebras. We give necessary and sufficient conditions under which a proper ideal $I$ of a Leavitt path algebra admits a factorization as product as well as an intersection of finitely many strongly irreducible ideals. Interestingly the graded part $\text{gr}(I)$ of this ideal $I$ also admits such a factorization and in this case $I/\text{gr}(I)$ is finitely generated. We characterize the Leavitt path algebras in which every proper ideal can be factored as an irredundant intersection/product of finitely many strongly irreducible ideals.

Two uniqueness theorems are established showing that such factorizations are unique except for the order of the factors. This answers an open question of Heinzer and Olberding ([11]) in the context of Leavitt path algebras. We also describe when every ideal of $L$ is strongly irreducible.

As a biproduct, Leavitt path algebras which are Laskerian are described.

Definition 3.1. An ideal $I$ of a ring $R$ is said to be irreducible if, for any two ideals $A, B$ of $R$, $A \cap B = I$ implies that $A = I$ or $B = I$.

Definition 3.2. An ideal $I$ of a ring $R$ is said to be a strongly irreducible ideal if, for any two ideals $A, B$ of $R$, $A \cap B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.

Clearly a prime ideal of a ring is strongly irreducible and a strongly irreducible ideal is always irreducible. In general, an irreducible ideal need not be strongly irreducible (see for e.g. [21], where it is shown that in the polynomial ring $\mathbb{Q}[x, y]$, the ideal $< x, y^2 >$ is irreducible, but not strongly irreducible) and as we have noted earlier the ideal $8\mathbb{Z}$ is a strongly irreducible ideal but not a prime ideal in the ring $\mathbb{Z}$ of integers. Irreducible ideals of Leavitt path algebras are described in [20].

We first list some elementary (perhaps known) properties of strongly irreducible ideals of any ring.

(i) An ideal $I$ of a ring $R$ is strongly irreducible if, for all $a, b \in R$, $(aR \cap bR) \subseteq I$ (similarly, $(Ra \cap Rb) \subseteq I$) implies that $a \in I$ or $b \in I$.
(ii) If $I$ is a strongly irreducible ideal in $R$, then for any ideal $K \subseteq I$, $I/K$ is strongly irreducible in $R/K$.

Proof of (i): Suppose $A \cap B \subseteq I$ for some ideals of $R$ and $A \nsubseteq I$. Choose $a \in A$ with $a \notin I$. Then for any $b \in B$, $(aR \cap bR) \subseteq A \cap B \subseteq I$ and so $b \in I$. Hence $B \subseteq I$.

Proof of (ii) is straightforward.

Next, we list some useful results on ideals of Leavitt path algebras over graphs containing a cycle without exits.
Proposition 3.3. Suppose $c$ is a cycle without exits in a graph $E$.

(i) ([1], Theorem 2.7.1) If $M$ is the ideal of $L$ generated by $c^0$, then $M \cong M_\Lambda(K[x,x^{-1}])$ for a suitable index set $\Lambda$.

(ii) ([20], Lemma 3.3) $<f(c)> < g(c) >= <f(c)g(c)>$ for any two $f(x), g(x) \in K[x]$. In particular, $<f(c)>^n = <f^n(c)>$ for any positive integer $n$.

(iii) ([6], Proposition 1) The map $A \mapsto M_\Lambda(A)$ defines a lattice isomorphism between the lattices of ideals of $K[x,x^{-1}]$ and $M_\Lambda(K[x,x^{-1}])$. Moreover, $M_\Lambda(AB) = M_\Lambda(A)M_\Lambda(B)$ for any two ideals $A, B$ of $K[x,x^{-1}]$.

Lemma 3.4. Suppose $E$ is a downward directed graph containing a cycle $c$ without exits based at a vertex $v$. If $M$ is the ideal generated by $c^0$, then for every non-zero ideal $A$ of $L_K(E)$ either $M \subseteq A$ or $A = <f(c)> \subseteq M$ where $f(x) \in K[x]$ with a non-zero constant term according as $A$ contains a vertex or not.

Proof. First observe that $u \geq v$ for every vertex $u \in E$. This is because, by downward directness of $E$, corresponding to $u, v$, there is a vertex $w$ such that $u \geq w$ and $v \geq w$. Since $v$ sits on the cycle $c$ without exits, $w$ must be a vertex on $c$ and so $w \geq v$. Thus, $u \geq v$. So if an ideal $A$ of $L_K(E)$ contains a vertex $u$, then since $A \cap E^0$ is hereditary, $A$ contains $v$ and hence $c^0$. Consequently, $M = <c^0> \subseteq A$. Suppose the non-zero ideal $A$ does not contain any vertices. Since $E$ is downward directed, we appeal to Lemma 3.5 of [17] to conclude that $A = <f(c)>$ where $f(x) \in K[x]$. In this case, clearly $A \subseteq M$.

The next theorem describes the strongly irreducible ideals of a Leavitt path algebra $L_K(E)$ and shows that in the case of Leavitt path algebras, the notions of irreducible ideals and strongly irreducible ideals coincide.

Theorem 3.5. The following properties are equivalent for an ideal $I$ of a Leavitt path algebra $L := L_K(E)$;

(1) $I$ is a strongly irreducible ideal of $L$;
(2) $I$ is an irreducible ideal of $L$;
(3) $I = P^n$, a power of a prime ideal $P$.

Proof. Clearly (1) $\Rightarrow$ (2) and the implication (2) $\Rightarrow$ (3) is proved in ([20], Theorem 5.7).

Assume (3), so that $I = P^n$ for some prime ideal $P$ and integer $n \geq 1$. If $P$ is graded, then by Lemma 2.4, $I = P^n = P$ is a prime ideal and so is strongly irreducible. Suppose now that $P$ is a non-graded ideal. Then, by Theorem 2.2, we have $P = I(H, B_{H}) + <p(c)>$ where $H = P \cap E^0, (E \setminus (H, B_{H}))^0$ is downward directed, $c$ is a cycle without exits in $E \setminus (H, B_{H})$ and $p(x)$ is an irreducible polynomial in $K[x,x^{-1}]$. Then, using Proposition 3.3(ii) and Lemma 2.4, one can show that $I = P^n = I(H, B_{H}) + <p^n(c)>$. Suppose $A \cap B \subseteq I$ for some ideals $A, B$ in $L$. Let $\bar{L} = L/I(H, B_{H}) \cong L_K(E \setminus (H, B_{H})), A = (A + I(H, B_{H}))/I(H, B_{H}), B = (B + I(H, B_{H}))/I(H, B_{H})$ and $\bar{I} = I/I(H, B_{H})$. Since the ideals of $L$ satisfy the distributive law ([20], Theorem 4.3),

\[(A + I(H, B_{H})) \cap (B + I(H, B_{H}))\]

simplifies to $(A \cap B) + I(H, B_{H})$ and so

$\bar{A} \cap \bar{B} = [(A \cap B) + I(H, B_{H})]/I(H, B_{H}) \subseteq \bar{I} = <p^n(c)>$.

Let $M = <c^0>$, the ideal generated by $c^0$ in $\bar{L}$. By Lemma 3.4, each of $\bar{A}, \bar{B}$ either contains $M$ or is contained in $M$. Now both $\bar{A}$ and $\bar{B}$ cannot contain $M$, since otherwise, $\bar{A} \cap \bar{B} \supseteq M \not\subseteq <p^n(c)> \subseteq \bar{I}$, a contradiction. If only one of them is contained in $M$, say $\bar{A} \subseteq M$ and $\bar{B} \supseteq M$,
then \( \bar{A} = \bar{A} \cap \bar{B} \subseteq \bar{I} \) and this implies that \( A \subseteq I \). Suppose both \( \bar{A} \subseteq M \) and \( \bar{B} \subseteq M \). By Lemma 3.4, \( \bar{A} = (f(c)) \) and \( \bar{B} = (g(c)) \) where \( f(x), g(x) \in K[x] \) with non-zero constant terms. Now, by Proposition 3.3(ii), \( M \cong M_{\Lambda}(K[x,x^{-1}]) \) for a suitable index set \( \Lambda \) and by Proposition 3.3(iii), the ideal lattices of \( M_{\Lambda}(K[x,x^{-1}]) \) and the principal ideal domain \( K[x,x^{-1}] \) are isomorphic. So, in \( K[x,x^{-1}] \),

\[
< f(x) > \cap < g(x) > \subseteq p^n(x) > .
\]

Now \( < f(x) > \cap < g(x) > = h(x) > \), where \( h(x) = \text{lcm}(f(x), g(x)) \).

Thus \( p^n(x)|\text{lcm}(f(x), g(x)) \) and since \( p^n(x) \) is a prime power, by the uniqueness of prime power factorization in \( K[x,x^{-1}] \), \( p^n(x)f(x) \) or \( p^n(x)g(x) \). This means either \( < f(x) > \subseteq p^n(x) > \) or \( < g(x) > \subseteq p^n(x) > \). We then conclude that either \( A \subseteq I \) or \( B \subseteq I \). This proves (1).

Next, we give conditions under which a proper ideal \( I \) of a Leavitt path algebra \( L_K(E) \) is a product as well as an intersection of finitely many strongly irreducible ideals of \( L_K(E) \).

We begin by proving a series of preparatory lemmas the first of which is well-known.

**Lemma 3.6.** Let \( R \) be a Principal ideal domain. Then every non-zero proper ideal \( I \) of \( R \) is an intersection of finitely many powers of distinct prime ideals.

**Proof.** Let \( I = (a) \) with \( a \neq 0 \) being a non-unit. Let \( a = p_1^{n_1} \cdots p_k^{n_k} \) be the factorization of \( a \) as a product of powers of distinct prime (equivalently, irreducible) elements \( p_1, \ldots, p_k \) of \( R \). Since \( \gcd(p_1^{n_1}, \ldots, p_k^{n_k}) = 1 \), \( \text{lcm}(p_1^{n_1}, \ldots, p_k^{n_k}) = p_1^{n_1} \cdots p_k^{n_k} \). Consequently,

\[
<p_1^{n_1} > \cap \cdots \cap < p_k^{n_k} > \subseteq \text{lcm}(p_1^{n_1}, \ldots, p_k^{n_k}) = p_1^{n_1} \cdots p_k^{n_k} > = (a) > .
\]

\( \square \)

**Lemma 3.7.** Suppose \( I \) is a non-graded ideal of a Leavitt path algebra \( L_K(E) \) such that \( \text{gr}(I) \) is a prime ideal. Then \( I \) is an intersection of finitely many (strongly) irreducible (= prime power) ideals.

**Proof.** By Theorem 2.1, \( I = (H(S) + \sum_{t \in T} f_t(c_t)) \) where \( T \) is a non-empty index set, for each \( t \in T \), \( c_t \) is a cycle without exits in \( E \setminus (H(S) \setminus f_t(x)) \) in \( K[x] \) with a non-zero constant term. Since \( \text{gr}(I) = (H(S)) \) is a prime ideal, by Theorem 2.2, \( (E \setminus (H(S)))^0 \) is downward directed and so there can be only one cycle, say \( c \) without exits in \( E \setminus (H(S)) \). Hence we can write \( I = (H(S) + < f(s) > \) where \( c \) is then a unique cycle without exits in \( E \setminus (H(S)) \). From the description of the prime ideals in Theorem 2.2 and the fact that \( I(H(S)) \) is a prime ideal such that \( E \setminus (H(S)) \) has a cycle without exits, we conclude that \( S = B_H \), so we can write \( I = (H(S), B_H) + < f(s) > \). Since \( L_K(E) = L_K(E)/I(H(S), B_H) \), \( < f(s) > \subseteq M = \{ c^0 \} \). Then, by Proposition 3.3 and Lemma 3.6, we conclude that \( < f(s) > = p_1^{n_1} \cap \cdots \cap p_k^{n_k} \) where \( p_j(x) \) are distinct irreducible polynomials in \( K[x,x^{-1}] \) and \( f(x) = p_1^{n_1}(x) \cdots p_k^{n_k}(x) \) is a prime power factorization of \( f(x) \) in \( K[x,x^{-1}] \). Then \( I = p_1^{n_1} \cap \cdots \cap p_k^{n_k} \) where \( p_j = (H(S), B_H) + < f_j(s) > \) is a prime ideal for all \( j = 1, \ldots, k \). By Theorem 3.5, each \( p_j^{n_j} \) is (strongly) irreducible ideal.

\( \square \)

The next technical lemma is obtained by modifying parts of the proof of Theorem 6.2 in [20] and is used in Theorem 3.9. Recall that given a collection of sets \( \{ A_i : i \in I \} \), the intersection \( \cap_{i \in I} A_i \) is called irredundant, if \( \cap_{i \in I \setminus \{ j \}} A_i \not\subseteq A_j \) for any \( j \in I \). In particular, \( A_i \not\subseteq A_j \) for any two \( i / j \in I \). Similarly the union \( \cup_{i \in I} A_i \) is called irredundant if \( A_i \not\supset A_j \) for any \( j \in I \).

**Lemma 3.8.** Suppose \( I = (H(S) + \sum_{t \in T} f_t(c_t)) \) is a non-graded ideal of \( L \), where \( T \) is a non-empty index set, for each \( t \in T \), \( c_t \) is a cycle without exits in \( E \setminus (H(S)) \) based at a vertex \( v_t \) with \( v_s^0 \cap v_t^0 = \emptyset \) for all \( s, t \in T \) with \( s \neq t \) and \( f_t(x) \in K[x] \) with a non-zero constant term. Suppose further that \( I(H(S)) = \cap_{i=1}^m P_i \) is an irredundant intersection of \( m \) graded prime ideals \( P_i = (H(S), S_i) \).

Then we can take \( T = \{ 1, \ldots, k \} \), \( k \leq m \) and \( I = (H(S) + \sum_{i=1}^k f_t(c_t)) \). After any needed rearrangement of indices, we have, for each \( t \in T \), \( v_t \not\in P_i \) but \( v_t \in P_j \) for all \( j = 1, \ldots, m \) with \( j \neq t \). Thus \( c_t \in E \setminus (H(S), S_t) \) for all \( t \in T \).
Proof. Clearly for each \( t \in T \), there is a \( j_t \) such that \( v_t \not\in P_{j_t} = I(H_{j_t}, S_{j_t}) \), since, otherwise, \( v_t \in \bigcap_{j=1}^m P_j = I(H, S) \), a contradiction. We claim that, \( v_t \in P_{j_t} \) for all \( j = 1, \ldots, m \). Suppose, on the contrary, \( v_t \not\in P_{j_t} = I(H_{j_t}, S_{j_t}) \) for some \( i \neq j_t \). Since both \( E(H_{j_t}, S_{j_t}) \) and \( E(H, S) \) contain the cycle \( c_t \) without exits, it is clear from the description of the graded prime ideals in Theorem 2.2 that \( P_{j_t} = I(H_{j_t}, B_{H_i}) \) and \( P_i = I(H_i, B_{H_i}) \). We wish to show that \( P' = P_{j_t} \cap P_i = (P_{j_t} \cap P_i) \) is a graded prime ideal. Let \( P' = I(H', B_{H_i}) \) so that \( H' = P' \cap E^0 = H_{j_t} \cap H_i \). Now \( c_t \) is a cycle without exits in \( (E^0 \setminus H_{j_t}) \cup (E^0 \setminus H_i) = E^0 \setminus (H_{j_t} \cup H_i) = E^0 \setminus H_i \). Since both \( E^0 \setminus H_{j_t} = (E(H_{j_t}, B_{H_i}))^0 \) and \( E^0 \setminus H_i = (E(H_i, B_{H_i}))^0 \) are downward directed, \( u \geq v_t \) for every vertex \( u \in E^0 \setminus H_i \cup E^0 \setminus H_{j_t} \). Hence \( E^0 \setminus H_i \cup E^0 \setminus H_{j_t} \) is an irredundant intersection of graded prime ideals. We claim that \( P' = I(H', B_{H_i}) \). To see this, let \( u \in B_{H_i} \). We need to show that \( u^{H'} = u - \sum_{c \in E^{-1}(u), c \not\in H'} e_c \in P' \). Noting that \( e_c \not\in P_{j_t} \) if \( c \not\in H_{j_t} \), we have

\[
\begin{align*}
\mu^{H'} &= u - \sum_{c \in E^{-1}(u), c \not\in H'} e_c - \sum_{c \in E^{-1}(u), c \not\in H', c \in H'_{j_t}} e_c \\
&= u^H - \sum_{c \in E^{-1}(u), c \not\in H', c \in H_{j_t}} e_c \in P_{j_t}.
\end{align*}
\]

By the similar argument, \( u^{H'} \not\in P_{j_t} \). Hence \( u^{H'} \in P_{j_t} \cap P_i = P' \). It is then clear that \( P' = I(H', B_{H_i}) \). Since \( (E(H', B_{H_i}))^0 = E^0 \setminus H' \) is downward directed, \( P' \) is a prime ideal (Theorem 2.2). But then \( P_{j_t} \cdot P_i = P_{j_t} \cap P_i \subseteq P' \) implies that \( P_{j_t} \subseteq P' \) or \( P_i \subseteq P' \). This implies that \( P_{j_t} \subseteq P_i \) or \( P_i \subseteq P_{j_t} \) contradicting that \( I(H, S) = \bigcap_{j=1}^m P_j \) is an irredundant intersection. Hence, we conclude that for each \( t \in T \) there is a \( P_{j_t} \) such that \( v_t \not\in P_{j_t} \) but \( v_t \in P_j \) for all \( j \neq j_t \). It is also clear that if \( s \in T \) with \( s \neq t \) (so that \( c_s \cap c_t = \emptyset \)), then the corresponding prime ideal \( P_{j_s} \neq P_{j_t} \). Thus the map \( t \mapsto P_{j_t} \) is an injective map from \( T \) to \( \{P_1, \ldots, P_m\} \). Consequently, \( |T| \leq m \), say \( |T| = k \). After rearranging the indices, we may assume that \( T = \{1, \ldots, k\} \) and, for each \( t = 1, \ldots, k, v_t \not\in P_{j_t} \), but \( v_t \in P_j \) for all \( j = 1, \ldots, m \) with \( j \neq t \). Clearly, \( I = I(H, S) + \sum_{t=1}^{k} <f_t(c_t)> \).

Theorem 3.9. The following properties are equivalent for an ideal \( I \) of a Leavitt path algebra \( L = L_K(E) \):

1. \( I \) is an intersection of finitely many (strongly) irreducible ideals;
2. \( \gamma(I) = P_1 \cap \cdots \cap P_m \) is an irredundant intersection of graded prime ideals;
3. \( \gamma(I) = I(H, S) = P_1 \cap \cdots \cap P_m \) is an irredundant intersection of graded prime ideals, \( I = I(H, S) + \sum_{k=1}^{m} <f_t(c_t)> \), where \( k \leq m \), for each \( t = 1, \ldots, k, c_t \) is a cycle with no exits in \( E \setminus (H, S) \) based at a vertex \( v_t \not\in P_t \) and \( f_t(x) \in K[x] \) with a non-zero constant term;
4. \( I \) is a product of (finitely many) strongly irreducible ideals.

Proof. If \( I \) is a graded ideal, then conditions (1) and (4) are equivalent by Lemma 2.4. Also, since \( I = \gamma(I) \), conditions (1) and (2) are easily seen to be equivalent by using Theorem 3.5 and Lemma 2.3. Finally, for a graded ideal, condition (3) simplifies to condition (2).

So we may take \( I \) to be a non-graded ideal.

Assume (1) so \( I = \bigcap_{j=1}^m Q_j \) is an intersection of (strongly) irreducible ideals of \( L \). Then \( \gamma(I) = \bigcap_{j=1}^m \gamma(Q_j) \). If needed remove appropriate ideals \( \gamma(Q_j) \) and, after re-indexing, assume \( \gamma(I) = \bigcap_{j=1}^m \gamma(Q_j) \) is an irredundant intersection. By Theorem 3.5, each \( Q_j \) is a power of a graded prime ideal and so, by Lemma 2.3, \( \gamma(Q_j) = P_j \) is a graded prime ideal. Thus we get a representation of \( \gamma(I) \) as an irredundant intersection of graded prime ideals, \( \gamma(I) = \bigcap_{j=1}^m P_j \). This proves (2).

Assume (2) so \( \gamma(I) = I(H, S) = \bigcap_{j=1}^m P_j \) is an irredundant intersection of graded prime ideals \( P_j \). By Lemma 3.8, we then have

\[
I = I(H, S) + \sum_{t=1}^{k} <f_t(c_t)> = (P_1 \cap \cdots \cap P_m) + \sum_{t=1}^{k} <f_t(c_t)>
\]
where \( k \leq m \), for each \( t = 1, \ldots, k \), \( c_t \) is a cycle without exits in \( E \setminus (H, S) \) based at a vertex \( v_t \) and \( f_t(x) \in K[x] \) with a non-zero constant term. Moreover, for each \( t = 1, \ldots, k, v_t \not\in P_t \), but \( v_t \in P_j \) for all \( j = 1, \ldots, m \) with \( j \neq t \). This proves (3).

Assume (3). For each \( t = 1, \ldots, k \), define \( A_t = P_t + < f_t(c_t) > \). By Lemma 3.7, each ideal \( A_t \) is an intersection of finitely many (strongly) irreducible ideals of \( L \). So we are done if we show that

\[
(P_1 \cap \cdots \cap P_m) + \sum_{t=1}^{k} < f_t(c_t) > = A_1 \cap \cdots \cap A_k \cap P_{k+1} \cap \cdots \cap P_m.
\]

We prove this by induction on \( k \). Assume \( k = 1 \). Consider \( A_1 \cap P_2 \cap \cdots \cap P_m \). Since the ideal lattice of \( L \) is distributive ([20], Theorem 4.3), we have

\[
A_1 \cap P_2 \cap \cdots \cap P_m = (P_1 + < f_1(c_1) >) \cap (P_2 \cap \cdots \cap P_m)
\]

\[
= (P_1 \cap \cdots \cap P_m) + < f_1(c_1) > \cap (P_2 \cap \cdots \cap P_m)
\]

\[
= (P_1 \cap \cdots \cap P_m) + < f_1(c_1) >
\]

as \( c_1 \in P_j \) for all \( j > 1 \) (Lemma 3.8).

Suppose \( k > 1 \) and assume that

\[
A_1 \cap \cdots \cap A_k \cap P_k \cap \cdots \cap P_m = (P_1 \cap \cdots \cap P_m) + \sum_{t=1}^{k-1} < f_t(c_t) >.
\]

Then

\[
A_1 \cap \cdots \cap A_k \cap P_{k+1} \cap \cdots \cap P_m
\]

\[
= A_1 \cap \cdots \cap A_{k-1} \cap (P_k + < f_k(c_k) >) \cap P_{k+1} \cap \cdots \cap P_m
\]

\[
= (A_1 \cap \cdots \cap A_{k-1} \cap P_k \cap \cdots \cap P_m) + A_1 \cap \cdots \cap A_{k-1} \cap < f_k(c_k) > \cap P_{k+1} \cap \cdots \cap P_m
\]

\[
= (P_1 \cap \cdots \cap P_m) + \sum_{t=1}^{k-1} < f_t(c_t) > + < f_k(c_k) >
\]

as \( < f_k(c_k) > \subseteq P_j \) for all \( j \neq k \) (Lemma 3.8).

By induction, we conclude that \( I \) is an intersection of finitely many (strongly) irreducible ideals. This proves (1).

Finally, the equivalence of conditions (4) and (2) follows from the fact that a product of strongly irreducible ideals is also a product of prime ideals (as a strongly irreducible ideal is a power of a prime ideal by Theorem 3.5) and that the equivalence of condition (2) with the existence of the prime factorization of \( I \) is established in Theorem 6.2 of [20]. \( \square \)

**Remark 3.10.** In general, a product of finitely many distinct prime ideals in a general ring \( R \) need not be equal to their intersection. We use the example from [15]. Let \( R = K[x, y] \) be the commutative ring of polynomials in two variables \( x, y \) over a field \( K \). Now \( P = < x, y > \) and \( Q = < x > \) are prime ideals of \( R \). Then \( PQ = < x^2, xy > \) and \( P \cap Q = Q \). Clearly, \( PQ \neq P \cap Q \). In contrast, the preceding theorem states that an ideal \( I \) of a Leavitt path algebra is a product of (finitely many) strongly irreducible ideals if and only if \( I \) is an intersection of finitely many strongly irreducible ideals.

Our next goal is to prove the uniqueness of factorization of an ideal of \( L \) as a product as well as an intersection of finitely many strongly irreducible ideals. Since strongly irreducible ideals are powers of prime ideals, we consider the uniqueness of representing an ideal of \( L \) as a product/intersection of powers of distinct prime ideals. This is done in the next two theorems. Here, we say a product \( I = A_1 \cdots A_k \) of distinct ideals \( A_j \) is an irredundant product if \( I \) is not the product of any proper subset of ideals of the set \( \{A_1, \ldots, A_k\} \). Likewise, an intersection of distinct ideals \( I = A_1 \cap \cdots \cap A_m \) is said to be irredundant if \( I \) is not the intersection of any proper subset of \( \{A_1, \ldots, A_m\} \). In the proof of the theorem, we shall be using the following properties of prime ideals of \( L \).
Lemma 3.11. (a) Let $P$ be a prime ideal of $L$ and let $A$ be any ideal with $P \subseteq A$. Then

(i) (Lemma 5.3, [20]) Either $P \subseteq \text{gr}(A)$ or $P = A$;

(ii) (Corollary 4.5, [20]) If $P \neq A$, then $P = AP$.

(iii) (Lemma 5.2, [20]) Suppose $P$ be a prime ideal of $L$ and $A$ is an ideal such that $P^n \subseteq A \subseteq P$ for some $n > 1$. Then $A = P^r$ for some $1 \leq r \leq n$.

Theorem 3.12. Suppose

$$I = P_1^m \cdot P_m^n = Q_1^m \cdot Q_m^n$$

are two representations of an ideal $I$ of a Leavitt path algebra $L$ as an irredundant product of powers of distinct prime ideals. Then $m = n$ and $\{P_1^m, \ldots, P_m^n\} = \{Q_1^m, \ldots, Q_n^n\}$.

Proof. Now the prime ideal $P_1$ contains the product $Q_1^m \cdot Q_2^n$ and so $P_1 \supseteq Q_i_j$ for some index $j_1$. In the same way, the prime ideal $Q_i_j$ contains $P_1^m \cdot P_2^n$ and so $Q_i_j \supseteq P_i_j$ for some $i_j$. So $P_1 \supseteq P_i_j$. We claim that $P_1 = P_i_j$. Because, $P_1 \supseteq P_i_j$ implies, by Lemma 3.11 (a)(i), that gr$(P_1) \supseteq P_i_j$ and so gr$(P_1) = (\text{gr}(P_1))^{i_j} \supseteq P_i_j^{i_j}$ which implies that gr$(P_1)P_i_j^{i_j} = P_i_j^{i_j}$. By Lemma 2.4, we have $P_1^m P_i_j^{i_j} = P_1^m (\text{gr}(P_1)P_i_j^{i_j}) = (P_1^m \cap \text{gr}(P_1))P_i_j^{i_j} = P_i_j^{i_j}$. Thus, in the product $P_1^m \cdot P_m^n$, using the commutativity of the ideal multiplication ([20], Theorem 3.4; [1], Corollary 2.8.17), $P_1^m P_i_j^{i_j}$ can be replaced by $P_i_j^{i_j}$. This contradicts the irredundancy of the product. Thus $P_1 = P_i_j$ and consequently, $P_1 = Q_i_j$. Re-arranging the factors, we write, without loss of generality,

$$I = P_1^i P_2^j \cdots P_m^n = P_1^i Q_1^j \cdots Q_n^n. \quad (*)$$

Repeating this process, using the irredundancy and successively replacing $Q_2, \ldots, Q_m$ by $P_2, \ldots, P_m$ respectively, we get $m \leq n$. Likewise, starting with the prime ideals $Q_1, \ldots, Q_n$ and replacing them by the ideals $P_1, \ldots, P_n$, we conclude that $n \leq m$. Consequently $m = n$, the map $j \mapsto i_j$ is a permutation $\sigma$ on the set $\{1, \ldots, m\}$ such that $P_j = Q_{\sigma(j)}$. In particular, $\{P_1, \ldots, P_m\} = \{Q_1, \ldots, Q_m\}$. Thus

$$I = P_1^i P_2^j \cdots P_m^n = P_1^i P_2^j \cdots P_m^n \quad (\star)$$

is an irredundant product of powers of distinct prime ideals where $t_j = s_{\sigma(j)}$ for all $j = 1, \ldots, m$ and where we assume that $r_j = 1 = t_j$ if $P_j$ is a graded ideal (Lemma 2.4). Note that necessarily $P_i \not\subseteq P_j$ for all $i \neq j$; because, $P_i \subseteq P_j$ and $P_i \not\subseteq P_j$ implies that $P_i \subseteq \text{gr}(P_j)$ and $P_i = P_j P_j$ (Lemma 3.11 (a)(ii)). As before, we can then derive that $P_1^i P_j^j = P_1^i$ and this will lead to contradicting the irredundancy of the product $(\star)$.

Next, we wish to show that the exponents $r_j = t_j$ for all $j = 1, \ldots, m$. If all the ideals $P_j$ are graded, then $P_1^i = P_j^j$ and we can conclude that $r_j = 1 = t_j$ for all $j = 1, \ldots, m$. So assume that at least one of the ideals is non-graded. Let $j$ be an arbitrary index for which $P_j$ is a non-graded (prime) ideal. So we are done if we show that $r_j = t_j$. Using the commutativity of the ideal multiplication in $L$ and re-indexing, we may assume, for convenience in writing, that $j = 1$ and so $P_1$ is a non-graded prime ideal, say $P_1 = I(H, B_H) + \langle p(c) \rangle$, where $H = P_1 \cap E^0, E(H, B_H)$ is downward directed, $c$ is a cycle with exits in $E(H, B_H)$ based at a vertex $v$ and $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$. Now $\overline{L} = L/I(H, B_H) \cong L_K(E(H, B_H))$ and under this isomorphism, we identify $\overline{L}$ with $L_K(E(H, B_H))$. Let $\overline{P}_1 = (P_1 + I(H, B_H))/I(H, B_H)$ for all $j = 1, \ldots, m$ and let $M$ be the ideal of $\overline{L}$ generated by $c^p$. Clearly $P_1 = \langle p(c) \rangle \not\subseteq M$. By Lemma 3.4, we have, for each $j \geq 2$, either $M \subseteq \overline{P}_j$ or $\overline{P}_j \subseteq M$ according as $\overline{P}_j$ contains a vertex or not.

If $M \subseteq \overline{P}_j$ for all $j = 2, \ldots, m$ then, since $M$ is a graded ideal, we have, by Lemma 2.4, $\overline{P}_j M = M \cap \overline{P}_j = M$ for all $j = 2, \ldots, m$ and also $\overline{P}_1 M = \overline{P}_1 \cap M = \overline{P}_1$. Using these equations repeatedly,
we then have,
\[ P_1^n P_2^n \cdots P_m^n = P_1^n M P_2^n \cdots P_m^n = P_1^n M = P_1^n = < p^n(c) >. \]

Likewise,
\[ P_1^n P_2^n \cdots P_m^n = P_1^n M P_2^n \cdots P_m^n = P_1^n M = P_1^n = < p^n(c) >. \]

From the equation (*) we have \( < p^n(c) > = < p^n(c) > \subseteq v_{L,K}(E(H, B_H))v \), noting that \( v_{p(c)}v = p(c) \). Now \( v_{L,K}(E(H, B_H))v \cong K[x, x^{-1}] \) where the isomorphism \( \theta \) maps \( v \) to 1, \( c \) to \( x \), \( c^\ast \) to \( x^{-1} \) and thus maps \( p(c) \) to \( p(x) \). Hence \( < p^n(c) > = < p^n(c) > \) implies that \( < p^n(c) > = < p^n(c) > \in K[x, x^{-1}] \). Since \( K[x, x^{-1}] \) is a unique factorization domain and \( p(x) \) is irreducible, we then conclude that \( r_1 = t_1 \).

Suppose not all the \( P_j \) contain \( M \). Without loss of generality, assume \( P_j \subseteq M \) for \( j = 2, ..., k \) and \( P_j \supseteq M \) for \( j = k + 1, ..., m \). By Lemma 3.4, \( P_j^n = < f_j(c) > \) for all \( j = 2, ..., k \) where \( f_j(c) \in K[x] \) with a non-zero constant term and moreover, \( P_j^n \) does not contain any vertex in \( \tilde{L} \). This means that \( gr(P_j) = gr(P_j^n) \subseteq I(H, B_H) \). Notice that this implies that the ideal \( P_j \) is not graded (since otherwise, \( P_j = gr(P_j) \subseteq I(H, B_H) \subseteq P_1 \) which implies, by Lemma 3.11, that \( P_j = P_1 P_1 \) and this leads to contradicting the irredundancy of the product (*)). Let \( P_j = I(H_j, B_{H_j}) + < q_j(c_j) > \) where \( c_j \) is a cycle without exits in \( E(H_j, B_{H_j}) \) which is downward directed and \( q_j(x) \) is an irreducible polynomial in \( K[x, x^{-1}] \). We claim that \( H_j = H \). Otherwise, there will be a vertex \( u \in H \setminus H_j \) and, as \( u \geq w \) for some vertex \( w \in c_j^0 \) (due to downward directedness), \( H \) will contain \( w \) and hence \( c_j^0 \). This will imply that \( P_j \supseteq P_j \), a contradiction as \( P_j \not\subseteq P_j \) for all \( i \neq j \). Thus \( E(H_j, B_{H_j}) = E(H_j, B_{H_j}) \), \( c_j = c \) and \( P_j = I(H, B_H) + < q_j(c) > \) which holds for all \( j = 2, ..., k \). Clearly, \( p(x) \neq q_j(x) \) for any \( j = 2, ..., k \). Now, as noted in the preceding paragraph, \( M P_{k+1}^n \cdots P_m^n = M \) and that \( P_1^n \cdots P_k^n M = P_1^n \cdots P_k^n \) and so we have
\[ P_1^n P_2^n \cdots P_m^n = P_1^n \cdots P_m^n M P_{k+1}^n \cdots P_m^n = P_1^n \cdots P_m^n = < p^n(c) > < q_2^n(c) > \cdots < q_k^n(c) >. \]

Similarly,
\[ P_1^n P_2^n \cdots P_m^n = P_1^n \cdots P_m^n M P_{k+1}^n \cdots P_m^n = P_1^n \cdots P_m^n = < p^n(c) > < q_2^n(c) > \cdots < q_k^n(c) >. \]

From the equation (*), we have
\[ < p^n(c) > < q_2^n(c) > \cdots < q_k^n(c) > = < p^n(c) > < q_2^n(c) > \cdots < q_k^n(c) >. \]

Again, as noted in the previous paragraph, we use the isomorphism
\[ v_{L,K}(E(H, B_H))v \cong K[x, x^{-1}] \]

to conclude that, in \( K[x, x^{-1}] \)
\[ < p^n(x) > < q_2^n(x) > \cdots < q_k^n(x) > = < p^n(x) > < q_2^n(x) > \cdots < q_k^n(x) >. \]

Now \( p(x), q_2(x), ..., q_k(x) \) are all distinct irreducible polynomials in \( K[x, x^{-1}] \) and so, by the uniqueness of prime power factorization in \( K[x, x^{-1}] \), we conclude that \( r_1 = t_1 \). By repeating this argument for every \( j \) for which \( P_j \) is a non-graded ideal, we conclude that \( r_j = t_j \) for all \( j = 1, ..., m \). This proves Theorem 3.12.

The next theorem considers the uniqueness of representing an ideal \( I \) of \( L \) as an irredundant intersection of strongly irreducible ideals. Again, we state the theorem in terms of powers of distinct prime ideals. It states, in particular, that if \( I \) is an irredundant intersection of finitely many
powers of distinct prime ideals, then \( I \) cannot be an irredundant intersection of infinitely powers of distinct prime ideals.

**Theorem 3.13.** Suppose

\[
I = P_1^i \cap \cdots \cap P_m^i = \bigcap_{j \in J} Q_j^i
\]

are two irredundant intersections of an ideal \( I \) of \( L \), where \( I \) is an arbitrary index set and the ideals \( P_j^i, Q_j^i \) are powers of distinct prime ideals. Then \( |J| = m \) and \( \{P_1^i, \ldots, P_m^i\} = \{Q_j^i : j \in J\} \).

**Proof.** The idea of the proof is essentially the one used in the proof of Theorem 3.12. After noting that \( P_1 P_i = P_i \) implies that \( P_1 \cap P_i = P_i \) and likewise, \( P_i^k P_j^l = P_i^k \) implies that \( P_i^k \cap P_j^l = P_i^k \), proceed as in the proof of Theorem 3.12 to conclude that for each \( i = 1, \ldots, m, P_i = Q_i \). We claim that \( |I| = m \). Suppose not. Let \( J' = J \setminus \{j_1, \ldots, j_m\} \). Consider the irredundant intersections

\[
I = P_1^i \cap \cdots \cap P_m^i = (P_1^i \cap \cdots \cap P_{m'}^i) \cap \bigcap_{j \in J'} Q_j^i
\]

Now, for a \( j \in J' \), the corresponding prime ideal \( Q_j \) contains \( P_1^i \cap \cdots \cap P_m^i \) and so \( Q_j \supseteq P_i \) for some \( i \in \{1, \ldots, m\} \). Note that \( Q_j \neq P_i \), because otherwise \( Q_j^i = P_i^i \) and \( P_i^k \cap Q_j^i = P_i^k \cap P_i^i = P_i^k \) or \( Q_j^i \) leading to contradicting the irredundancy of the intersection \( (P_1^i \cap \cdots \cap P_{m'}^i) \cap \bigcap_{j \in J'} Q_j^i \). Appealing to Lemma 3.11 (a)(ii), we then get \( P_i \subseteq \text{gr}(Q_j) \) which implies that \( P_i^k \subseteq (\text{gr}(Q_j))^* = \text{gr}(Q_j) \subseteq Q_j^i \). Then \( P_i^k \cap Q_j^i = P_i^k \) which again contradicts the irredundancy of the intersection \( (P_1^i \cap \cdots \cap P_{m'}^i) \cap \bigcap_{j \in J'} Q_j^i \). Thus \( |I| = m \) and we have the irredundant intersections

\[
I = P_1^i \cap \cdots \cap P_m^i = Q_1^i \cap \cdots \cap Q_m^i = P_1^i \cap \cdots \cap P_m^i.
\]

Then proceed as in the proof of Theorem 3.12 to reach the conclusion that \( r_j = s_j \) for all \( j = 1, \ldots, m \) after noting that, in the unique factorization domain \( K[x, x^{-1}] \), \( <p^k(x)> \cap <q^n(x)> = <p^k(x)>\cap <q^n(x)> \) for any two irreducible polynomials \( p(x) \) and \( q(x) \).

\[ \Box \]

In [11], Heinzer and Olberding raised the following question which (according to Bruce Olberding) is still open.

**Question 3.14.** (Heinzer - Olberding, [11]) Under what conditions every ideal in a commutative ring \( R \) can be uniquely represented as an irredundant intersection of irreducible ideals?

We completely answer this question in the context of Leavitt path algebras. Interestingly, it turns out (see Theorem 3.12) that, in a Leavitt path algebra, if an ideal \( I \) is represented as an irredundant intersection/product of finitely many irreducible ideals, then such a representation is automatically unique.

**Theorem 3.15.** The following properties are equivalent for a Leavitt path algebra \( L = L_K(E) \):

1. Every ideal of \( L \) is an irredundant intersection of finitely many (strongly) irreducible ideals;
2. Every ideal of \( L \) is a product of (finitely many) strongly irreducible ideals;
3. \( L \) is a generalized ZPI ring, that is, every ideal of \( L \) is a product of prime ideals;
4. Every non-zero homomorphic image of \( L \) is either a prime ring or contains only finite number of minimal prime ideals;
5. For every admissible pair \((H, S)\), \((E(H,S))^0\) is the union of a finite number of maximal tails.

**Proof.** Now (1) \( \iff \) (2) follows from the equivalence of conditions (1) and (4) in Theorem 3.9. Assume (2). Let \( I \) be an arbitrary ideal of \( L \). We are given that \( I = Q_1 \cdots Q_n \) is a product of strongly irreducible ideals. By Theorem 3.5, each \( Q_i = P_j^k_i \), where the \( P_j \) are prime ideals with \( k_j \geq 1 \). Expanding each \( P_j^k_i \), \( I \) then becomes a product of prime ideals. Thus \( L \) is a generalized ZPI ring. This proves (3).
Since every prime ideal is strongly irreducible, condition (3) implies condition (2).

The equivalence of conditions (3) and (4) has been established in (Theorem 6.5, [20]).

Assume (1). Given an admissible pair \((H, S)\), consider the graded ideal \(A = I(H, S)\). By hypothesis, \(A = \bigcap_{j=1}^{m} A_j\) where each \(A_j\) is strongly irreducible and \(m\) is a positive integer. Since \(A\) is graded, \(A = \bigcap_{j=1}^{m} \text{gr}(A_j) = \bigcap_{j=1}^{m} P_j\) where \(P_j = \text{gr}(A_j)\) is a (graded) prime ideal as \(A_j\) is a power of a prime ideal by Theorem 3.5. Then, in \(L_K((E\setminus (H, S)) \cong L/I(H, S), \{0\} = \bigcap_{j=1}^{m} Q_j\) where each \(Q_j = P_j/I(H, S)\) is a graded prime ideal, say \(Q_j = I(H_j, S_j)\) where \(H_j = Q_j \cap (E\setminus (H, S))^0\) and \((E\setminus (H, S))^0\setminus H_j\) is downward directed. Now \(\bigcap_{j=1}^{m} H_j = \emptyset\) and so \((E\setminus (H, S))^0 = \bigcup_{j=1}^{m} M_j\) where \(M_j = (E\setminus (H, S))^0 \setminus H_j\) is a maximal tail. This proves (5).

Assume (5). We shall prove (1). In view of Theorem 3.9(2), it is enough if we show that every graded ideal \(I\) of \(L\) is an intersection of finitely many graded prime ideals. This means that \(I\) is an intersection of finitely many graded prime ideals in \(L/I\).

Example 3.17. Let \(E\) be the following graph

The graph is row-finite and satisfies the Condition (K). The proper non-empty hereditary saturated subsets of this graph are precisely the sets \(A_1 = \{v_n : n \geq 1\}; A_2 = \{v_n : n \geq 2\}; B_1 = \{w_n : n \geq 1\}; B_2 = \{w_n : n \geq 2\}\). For each \(i = 1, 2\), \(E^0 \setminus A_i\) and also \(E^0 \setminus B_i\) is the union of at most two maximal tails. Thus, every non-zero ideal of \(L_k(E)\) is the intersection as well as a product of at most two strongly irreducible ideals. If \(P_1 = <A_1>\) and \(P_2 = <B_1>\), then \(P_1\) and \(P_2\) are prime and hence strongly irreducible ideals and \(P_1 \cap P_2 = \{0\}\). Thus every ideal of \(L_k(E)\) is a product/intersection of at most two strongly irreducible ideals.
Remark 3.18. Recall, an ideal I of a ring R is said to be a primary ideal if, for all ideals A, B of R, \( AB \subseteq I \) and \( A \not\subseteq I \) implies that \( B \subseteq \text{rad}(I) \). A ring R is said to be Laskerian (or simply, Lasker) if every ideal of R is an intersection of finitely many primary ideals. It was shown in Theorem 5.7 of [20], that an ideal I is a primary ideal of L if and only if I is a power of a prime ideal which, by Theorem 3.5, is equivalent to being strongly irreducible. Thus Theorem 3.15 gives a complete description of Leavitt path algebras which are Laskerian.

Next we consider when every ideal of a Leavitt path algebra \( L_K(E) \) is strongly irreducible.

Theorem 3.19. The following are equivalent for any Leavitt path algebra \( L = L_K(E) \):

1. Every ideal of \( L \) is strongly irreducible;
2. All the ideals of \( L \) are graded and form a chain under set inclusion;
3. The graph \( E \) satisfies Condition (K) and the admissible pairs \( (H, S) \) form a chain under the partial ordering of the admissible pairs;
4. Every ideal of \( L \) is a prime ideal.

Proof. Assume (1). Let \( A, B \) be any two ideals of \( L \). Now \( I = A \cap B \) is strongly irreducible and hence irreducible. So \( I = A \) or \( I = B \). This means \( A = A \cap B \subseteq B \) or \( B = A \cap B \subseteq A \). Thus ideals of \( L \) form a chain under set inclusion. Suppose, by way of contradiction, that \( L \) contains a non-graded ideal \( J \). By Theorem 2.1, \( J = I(H, S) + \Sigma_{i \in X} < f_i(c_i) > \), where \( X \) is a non-empty index set, for each \( i \in X \), \( c_i \) is a cycle without exits in \( E \rangle \langle H, S \rangle \) based at a vertex \( c_i \) and \( f_i(x) \in K[x] \) with a non-zero constant term which, without loss of generality, we may assume to be 1. Then \( v_i \) will be the non-zero constant term of \( f_i(c_i) \). It is clear that, for a fixed \( i \in X \), \( H_i = \{ u \in E^0 : u \not\sim v_i \} \) is a hereditary saturated set and \( (E \rangle \langle H, B_{H_i} \rangle)^\circ = E^0 \setminus H_i \) is downward directed. Then, for two distinct non-conjugate irreducible polynomials \( p(x), q(x) \in K[x, x^{-1}] \), we have \( P = I(H_i, B_{H_i}) + < p(c_i) > \) and \( Q = I(H_i, B_{H_i}) + < q(c_i) > \) are prime ideals of \( L \) such that neither contains the other. To see this, note that in \( \bar{L} = L/I(H_i, B_{H_i}) \), \( M = < c_i^q > \) contains both 

\[ \bar{P} = P/(H_i, B_{H_i}) = < p(c_i) > \quad \text{and} \quad \bar{Q} = Q/(H_i, B_{H_i}) = < q(c_i) > . \]

By Proposition 3.3, \( M \cong M_A(K[x, x^{-1}]) \) and that the ideal lattices of \( M_A(K[x, x^{-1}]) \) and \( K[x, x^{-1}] \) are isomorphic. Consequently, \( P \) and \( Q \) are maximal ideals of \( M \) and hence neither contains the other. This contradiction shows that all the ideals of \( L \) must be graded. This proves (2).

Now (2) implies (4), because, if \( A, B, I \) are ideals of \( L \) such that \( AB \subseteq I \). Since ideals of \( L \) are all graded, \( AB = A \cap B \) (Lemma 2.4). Thus \( A \cap B \subseteq I \). Since \( A \cap B = A \) or \( B \), we have \( A \subseteq I \) or \( B \subseteq I \).

Also (4) implies (1), since a prime ideal is always strongly irreducible.

Finally, (2) \( \iff \) (3), since, by (Proposition 2.9.9, [1]), the graph \( E \) satisfies Condition (K) if and only if every ideal of \( L \) is graded and, by (Theorem 2.5.8, [1]), the map \( I(H, S) \mapsto (H, S) \) is an order preserving bijection between graded ideals of \( L \) and admissible pairs \( (H, S) \).

\[ \square \]

Example 3.20. Suppose \( E \) is the following graph

\[ \ldots \rightarrow v_3 \rightarrow v_{12} \rightarrow v_1 \]

Then clearly \( E \) satisfies Condition (K) and is row-finite. In particular, \( B_{H} \) is empty for every hereditary saturated subset of vertices \( H \subseteq E^0 \). Moreover, the non-empty proper hereditary saturated subsets of vertices in \( E \) are precisely the sets of the form \( \{v_i : i \geq n\} \), where \( n \) is a positive integer and these form an ascending chain under set inclusion. It follows that the admissible pairs
(H, S) form a chain under the partial order of the admissible pairs as defined in Section 2. Thus, the graph E satisfies Condition (3) of Theorem 3.19.

4 Strongly prime ideals of Leavitt path algebras

As noted earlier, a prime ideal P containing the intersection \( \cap_{i=1}^{n} A_i \) of finitely many ideals \( A_i \) will contain one of the ideals \( A_i \). But, for the intersection of infinitely many ideals, the corresponding statement does not hold. For example, in the ring \( \mathbb{Z} \) of integers, the zero ideal \( \{0\} = \cap_{n=1}^{\infty} 2^n \mathbb{Z} \), but \( 2^n \mathbb{Z} \neq \{0\} \). In [13], Jayaram, Oral and Tekir study the ideals of a commutative ring having the desired property for infinite intersections and call them strongly prime ideals.

Definition 4.1. ([13]) An ideal \( P \) of a ring \( R \) is called a strongly prime ideal if \( P \supseteq \cap_{i \in X} A_i \), where \( X \) is an arbitrary index set and the \( A_i \) are ideals of \( R \) implies that \( P \supseteq A_i \) for some \( i \in X \). A ring \( R \) is called strongly zero-dimensional, if every prime ideal of \( R \) is strongly prime.

In this section we characterize the strongly prime ideals of a Leavitt path algebra \( L \). We give necessary and sufficient conditions under which a given ideal \( I \) of \( L \) can be factored as a product of strongly prime ideals. We describe both algebraically and graphically when every ideal of \( L \) admits a factorization as a product of strongly prime ideals. Finally, Leavitt path algebras which are strongly zero-dimensional are fully characterized.

Remark 4.2. In their definition of a strongly prime ideal \( P \) in [13], the authors assume to start with that the ideal \( P \) is a prime ideal satisfying the stated property. As is clear from our definition above, we do not assume a priori that \( P \) is a prime ideal. We will show in Theorem 4.11 that, for Leavitt path algebras, such an ideal \( P \) is always a prime ideal.

Clearly a prime (and hence a strongly prime) ideal of a ring is strongly irreducible. But a strongly irreducible ideal need not be strongly prime. For instance, in the ring \( \mathbb{Z} \) of integers, \( 2\mathbb{Z} \) is strongly irreducible (and also a prime ideal), but \( 2\mathbb{Z} \) is not strongly prime since \( \cap_{n=1}^{\infty} 3^n \mathbb{Z} = 0 \in 2\mathbb{Z} \), but \( 3^n \mathbb{Z} \not\subseteq 2\mathbb{Z} \) for any \( n \geq 1 \).

We begin with some preliminary results.

Lemma 4.3. No ideal in a principal ideal domain \( R \) is strongly prime.

Proof. Let \( I = \langle a \rangle \) be a non-zero ideal of \( R \). Choose a prime (equivalently; irreducible) element \( p \) in \( R \) such that \( p \not\mid a \). Now \( \cap_{n=1}^{\infty} \langle p^n \rangle = \{0\} \not\subseteq \langle a \rangle \), but \( \langle p^n \rangle \not\subseteq \langle a \rangle \) for any \( n \), since otherwise, \( p^n = ab \) for some \( b \in R \) and by the uniqueness of prime factorization, \( a = p^k \) for some \( k \), a contradiction. Also the zero ideal \( \langle 0 \rangle \) of \( R \) cannot be strongly prime, since for any prime element \( p \) in \( R \), \( \cap_{n=1}^{\infty} \langle p^n \rangle = \langle 0 \rangle \).

Corollary 4.4. No ideal in the ring \( M_{n}(K[x,x^{-1}]) \) is strongly prime.

Proof. By Proposition 3.3, the ideal lattices of \( M_{n}(K[x,x^{-1}]) \) and the principal ideal domain \( K[x,x^{-1}] \) are isomorphic. Then the result follows from Lemma 4.3.

The next Proposition states that a strongly prime ideal of \( L = L_{K}(E) \) must be a graded ideal.

Proposition 4.5. If \( P \) is a non-graded prime ideal of a Leavitt path algebra \( L_{K}(E) \), then, for any \( n \geq 1 \), \( P^n \) is not a strongly prime ideal.

Proof. By Theorem 2.2, \( P = I(H,B_H) = \langle p(c) \rangle \) where \( H = P \cap E^{0}, E \setminus (H,B_H) \) is downward directed, \( c \) is a cycle without exits in \( E \setminus (H,B_H) \) and \( p(x) \) is an irreducible polynomial in \( K[x,x^{-1}] \). Then \( P^n = I(H,B_H) = \langle p^n(c) \rangle \). In \( L_{K}(E) = L_{K}(E)/I(H,B_H) \cong L_{K}(E \setminus (H,B_H)) \),
\[
P^n = P^n/I(H, B_H) = < p^n(c) > \subseteq < c^0 > = M.
\]

Now, by Proposition 3.3, \( M \cong M_\Lambda(K[x, x^{-1}]) \) for a suitable index set \( \Lambda \). Then, by Corollary 4.4, \( (P^n \text{ and hence} ) P^n \) is not a strongly prime ideal in \( L_K(E) \). \( \square \)

We now proceed to give a complete description of the strongly prime ideals in a Leavitt path algebra \( L_K(E) \). In particular, it shows that a strongly prime ideal must be a prime ideal. In its proof, we shall be using the following definition.

**Definition 4.6.** ([2]) Given a graph \( E \), we say that \( E^0 \) satisfies the countable separation property (for short, CSP), if there is a non-empty countable subset \( S \) of \( E^0 \) such that for every \( u \in E^0 \) there is a \( v \in S \) such that \( u \geq v \).

The CSP condition turns out to be essential in the description of primitive ideals of Leavitt path algebras as noted in the next theorem. Recall that a ring \( R \) is called a right primitive ring if there exists a faithful simple right \( R \)-module. An ideal \( I \) of \( R \) is called a right primitive ideal if \( R = I \) is a right primitive ring.

**Theorem 4.7.** (Theorem 4.3, [17]) Let \( E \) be an arbitrary graph and let \( P \) be an ideal of \( L_K(E) \) with \( P \cap E^0 = H \). Then \( P \) is a primitive ideal if and only if \( P \) satisfies one of the following:

1. \( P \) is a non-graded prime ideal;
2. \( P \) is a graded prime ideal of the form \( I(H, B_H \backslash \{u\}) \);
3. \( P \) is a graded ideal of the form \( I(H, B_H) \) (where \( B_H \) may be empty) and \( (E \backslash (H, B_H))^0 = E^0 \backslash H \) is downward directed and satisfies Condition (L) and the CSP.

**Definition 4.8.**

(i) We say \( E^0 \) satisfies the strong CSP, if \( E^0 \) satisfies the CSP such that the corresponding non-empty countable set \( S \) of vertices is contained in every non-empty hereditary saturated subset of \( E^0 \).

(ii) A primitive ideal \( P \) of \( L_K(E) \) with \( gr(P) = I(H, S) \) is called strongly primitive if \( (E \backslash (H, S))^0 \) satisfies the strong CSP.

(iii) We say \( L_K(E) \) is strongly primitive, if \( L_K(E) \) is a primitive ring such that \( E^0 \) satisfies the strong CSP.

**Example 4.9.** Any non-graded prime ideal \( P \) of a Leavitt path algebra \( L \) is strongly primitive. Because, by Theorem 2.2, \( P = I(H, B_H) + < p(c) > \) where \( c \) is a cycle without exits in the downward directed set \( (E \backslash (H, B_H))^0 \) based at a vertex \( v \). Then \( (E \backslash (H, B_H))^0 \) satisfies the strong CSP with respect to \( \{v\} \), thus making \( I \) strongly primitive. Also any graded prime ideal of the form \( I(H, B_H \backslash \{u\}) \) is strongly primitive since it is primitive ([17]Theorem 4.3) and \( (E \backslash (H, B_H \backslash \{u\}))^0 \) satisfies the strong CSP with respect to \( \{u'\} \).

**Lemma 4.10.** Every strongly prime ideal of a Leavitt path algebra is graded prime.

**Proof.** Let \( I \) be a strongly prime ideal of a Leavitt path algebra \( L \). In particular, \( I \) is irreducible and so, by Lemma 2.3, \( I = P^n \), a power of a prime ideal \( P \). By Proposition 4.5, \( P \) must be a graded ideal and so \( I = P^n = P \) is a graded prime ideal. \( \square \)

**Theorem 4.11.** The following properties are equivalent for an ideal \( I \) of \( L = L_K(E) \) with \( I \cap E^0 = H \):

1. \( I \) is a strongly prime ideal;
2. \( I \) is a graded and strongly primitive ideal;
(3) \( I = I(H, S) \) is graded and \( E \setminus (H, S) \) is downward directed satisfying Condition (L) and the strong CSP.

Proof. Assume (1). By above lemma \( I \) is a graded prime ideal, say, \( I = I(H, S) \) where \( H = I \cap E^0 \) and \( (E \setminus (H, S))^0 \) is downward directed. We claim that \( I \) must be a primitive ideal of \( L \). Because, otherwise, \( \{0\} \) is not a primitive ideal of \( L/I \), so the primitive ideals of \( L/I \) are non-zero and their intersection is \( \{0\} \) since the Jacobson radical of \( L/I \) is \( \{0\} \) due the fact that \( L/I \cong L_K(E \setminus (H, S)) \). This means that \( I \) is the intersection of all the primitive ideals properly containing \( I \), contradicting the fact that \( I \) is strongly prime. Thus \( I \) must be a (graded) primitive ideal and, by Theorem 4.13, \( (E \setminus (H, S))^0 \) is downward directed, satisfies Condition (L) and has the CSP with respect to a non-empty countable subset \( C \) of vertices. Clearly, \( \{0\} \) is strongly prime in \( L/I \cong L_K(E \setminus (H, S)) \). Let \( X = \bigcap_{j \in J} H_j \) be the intersection of all non-empty hereditary saturated subsets \( H_j \) of vertices in \( E \setminus (H, S) \). Now \( X \) is not empty since, otherwise, \( \{0\} = \bigcap_{j \in J} < H_j > \) in \( L/I \) and this would contradict the fact that \( \{0\} \) is strongly prime. We claim that, for each vertex \( v \in E \setminus (H, S) \), there is a vertex \( w \in X \cap C \) such that \( v \geq w \). To see this, let \( u \) be a fixed vertex in \( X \). By downward directness, there is a vertex \( w' \in E \setminus (H, S) \) such that \( u \geq w' \) and \( v \geq w' \). By CSP, there is a \( w \in C \) such that \( w' \geq w \). Since \( X \) is hereditary, \( u \geq w \) implies that \( w \in X \). Thus \( v \geq w \in X \cap C \) and we conclude that \( E \setminus (H, S) \) satisfies the strong CSP with respect to \( X \cap C \). Consequently, \( I \) is a graded and strongly primitive ideal. This proves (2).

Assume (2). Let \( I \) be a graded and strongly primitive ideal of the form \( I(H, S) \) with \( E \setminus (H, S) \) satisfying Condition (L) and the strong CSP with respect to a non-empty countable set \( C \) of vertices. To show that \( I \) is strongly prime, suppose \( \{A_j : j \in J\} \) is an arbitrary family of ideals of \( L \) such that \( A_j \nsubseteq I \) for all \( j \in J \). Consider \( L = L/I(H, S) \cong L_K(E \setminus (H, S)) \). Let, for each \( j \), \( A_j = (A_j + I(H, S))/I(H, S) \). Identifying \( L/I(H, S) \) with \( L_K(E \setminus (H, S)) \) under this isomorphism, observe that the non-zero ideal \( \tilde{A}_j \) must contain a vertex. Otherwise, as \( (E \setminus (H, S))^0 \) is downward directed, Lemma 3.5 in [17] will imply that \( A_j = \langle f(c) \rangle \) where \( f(x) \in K[x] \) is a polynomial with a non-zero constant term and \( c \) is a cycle without exits in \( E \setminus (H, S) \). This is a contradiction, since \( E \setminus (H, S) \) satisfies Condition (L). Thus \( H_j = A_j \cap (E \setminus (H, S))^0 \not= \emptyset \) for all \( j \in J \). By the strong CSP of \( (E \setminus (H, S))^0 \), the countable set \( C \subseteq H_j \) for all \( j \in J \). Consequently, \( \bigcap_{j \in J} A_j \supseteq C \not= \emptyset \). This means that \( \bigcap_{j \in J} A_j \nsubseteq I(H, S) \). Hence \( I \) is strongly prime, thus proving (1).

The equivalence of (2) and (3) follows from the definition of strongly primitive ideal.

Remark 4.12. We shall call a ring \( R \) a strongly prime ring if \( \{0\} \) is a strongly prime ideal. It is then clear from Theorem 4.11 that a Leavitt path algebra \( L_K(E) \) is strongly prime if and only if \( E^0 \) is downward directed satisfying the strong CSP and Condition (L) if and only if \( L_K(E) \) is a strongly primitive ring.

The next result describes conditions under which an ideal \( I \) of a Leavitt path algebra \( L_K(E) \) can be factored as a product of strongly prime ideals.

Theorem 4.13. The following properties are equivalent for an ideal \( I \) of a Leavitt path algebra \( L = L_K(E) \):

(1) \( I = P_1 \cdots P_n \) is a product of strongly prime ideals \( P_i \);
(2) \( I = P_1 \cap \cdots \cap P_n \) is an intersection of finitely many strongly prime ideals \( P_i \);
(3) \( I \) is a graded ideal, say \( I = I(H, S) \) where \( H = I \cap E^0 \) and \( S \subseteq B_H \) such that \( E \setminus (H, S)^0 \) is an irredundant union of finitely many maximal tails \( M_1, \ldots, M_t \) with each subset \( M_j \) satisfying Condition (L) and the strong CSP with respect to a countable subset \( C_j \subseteq M_j \).

Proof. To prove the equivalence of (1) and (2), note that each \( P_i \) is a graded ideal, by Theorem 4.11. Then, by Lemma 2.4, \( P_1 \cdots P_n = P_1 \cap \cdots \cap P_n \).
Assume (2). If necessary remove some of the ideal $P_j$ and assume that $I = P_1 \cap \cdots \cap P_t$ is an irredundant intersection of strongly prime ideals. By Theorem 4.11, each ideal $P_j$ is graded, say, $P_j = I(H_j, S_j)$ such that $E\backslash (H_j, S_j)$ is downward directed and satisfies both Condition (L) and the strong CSP with respect to a countable subset of vertices $C_j$. So $I = I(H, S)$ where $H = I \cap E^0$ and $S \subseteq B_H$. In $L/I \cong L_K(E \backslash (H, S), \{0\}) = \bar{P}_1 \cap \cdots \cap \bar{P}_t$ is an irredundant intersection, where $\bar{P}_j = P_j/I(H, S)$ is strongly prime and hence a prime ideal for all $j = 1, \ldots, t$. Let $H_j = P_j \cap E\backslash (H, S)$.

Then $M_j = \left[\left(E\backslash (H, S)^0\right)\backslash H_j\right]$ is a maximal tail. As $\bigcap_{j=1}^t H_j = \emptyset$, we have $E\backslash (H, S)^0 = \bigcup_{j=1}^t M_j$ is an irredundant union of the maximal tails $M_j$. Since each $P_j/I(H, S)$ is strongly prime, each subset $M_j$ satisfies Condition (L) and the strong CSP with respect to a countable subset $C_j \subseteq M_j$. This proves (3).

Assume (3). In $L_K(E\backslash (H, S), H_j = \left(E\backslash (H, S)^0\right)\backslash M_j)$ is a hereditary saturated set for each $j = 1, \ldots, t$ and let $\bar{P}_j = I(H_j, B_{H_j})$ Since $\bigcap_{j=1}^t H_j = \emptyset$, $\bigcap_{j=1}^t P_j = \{0\}$. Identifying $L/I$ with $L_K(E\backslash (H, S))$, each $\bar{P}_j = P_j/I$ for some ideal $P_j$ of $L$ containing $I$. Using Theorem 4.11 and the hypothesis on $M_j$, we conclude that each $P_j$ is strongly prime and that $I = P_1 \cap \cdots \cap P_t$. This proves (2).

To illustrate the above, we consider the following simple example.

Example 4.14. Let $E$ be the following graph

\begin{center}
\begin{tikzpicture}
\node (u) at (0,0) {$u$};
\node (v) at (2,0) {$v$};
\node (w) at (4,0) {$w$};
\draw (u) -- (v);
\draw (v) -- (w);
\draw (w) -- (u);
\end{tikzpicture}
\end{center}

Clearly, $P = \langle u \rangle = \langle \{u, v\} \rangle$ and $Q = \langle w \rangle = \langle \{v, w\} \rangle$ are strongly prime ideals and $I = \langle v \rangle = PQ = P \cap Q$.

Remark 4.15. Just as we did in Section 3, Theorem 3.12, a natural question is to inquire about the uniqueness of factorization of an ideal $I$ of $L$ as an irredundant product or an irredundant intersection of finitely strongly prime ideals. Since a strongly prime ideal is, in particular, a prime ideal, the uniqueness of such a factorization follows from the uniqueness of factorizing an ideal $I$ as an irredundant product/intersection of prime ideals (see e.g. Propositions 2.6 and 3.4 in [7]).

In the next theorem, we characterize when every ideal of a Leavitt path algebra is a product of strongly prime ideals.

Theorem 4.16. The following properties are equivalent for a Leavitt path algebra $L_K(E)$:

(1) Every ideal of $L_K(E)$ is a product of strongly prime ideals;
(2) Every ideal of $L_K(E)$ is graded, $L$ is a generalized ZPI ring, every homomorphic image of $L$ is a Leavitt path algebra and is either strongly prime or contains only a finite number of minimal prime ideals each of which is strongly primitive;
(3) The graph $E$ satisfies Condition (K) and for every quotient graph $E\backslash (H, S), E\backslash (H, S)^0$ is either downward directed satisfying the strong CSP or is the union of finitely many maximal tails $S_j$, each of which satisfies the strong CSP with respect to a countable subset $C_j \subseteq S_j$.

Proof. Assume (1). First note that every ideal of $L$ is, in particular, a product of prime ideals and so $L$ is a generalized ZPI ring. If $I$ is an arbitrary ideal, then $I = P_1 \cdots P_n$ is a product of strongly prime ideals $P_j$ implies $I = P_1 \cap \cdots \cap P_n$ as the $P_j$ are all graded. Thus $I$ is a graded ideal. Removing appropriate factors $P_j$ and re-indexing, we may assume that $I = P_1 \cap \cdots \cap P_m$ is an irredundant intersection of graded strongly primitive ideals. If $I$ is a prime ideal, then $I = P_j$ for some $j$ and so $I$ is a strongly prime ideal of $L$. Suppose $I$ is not a prime ideal. In $L/I \cong L_K(E\backslash (H, S), \{0\})$.

Theorem 6.5 of [20] and its proof implies that $L/I$ contains finitely many minimal prime ideals $Q_1, \ldots, Q_k$ which are all graded and that we have an irredundant intersection $Q_1 \cap \cdots \cap Q_k = \{0\}$.
By the minimality of the prime ideals $Q$, and by irredundancy of the two intersections, we obtain that $k = m$ and $\{Q_1, \ldots, Q_m\} = \{P_1, \ldots, P_m\}$. Thus $L/I$ contains finitely many minimal prime ideals each of which is strongly primitive. This proves (2).

Assume (2). Since every ideal is graded, the graph $E$ satisfies Condition (K) (Proposition 2.9.9, [1]). For a given graded ideal $I(H, S)$, we are given that $L_K(E(H, S)) \cong L/I(H, S)$ is strongly prime or contains only a finite number of minimal prime ideals. In the former case, since Condition (K) implies Condition (L), we obtain from Theorem 4.11 that $E(H, S)$ is downward directed and satisfies the strong CSP. On the other hand, suppose $L_K(E(H, S)) \cong L/I(H, S)$ contains only a finite number of minimal prime ideals $P_1, \ldots, P_k$ all of which are graded and strongly prime. For each $j = 1, \ldots, k$, write as $P_j = I(H_j, S_j)$. Since every non-zero prime ideal of $L_K(E(H, S))$ contains one of these $P_j$ and since the intersection of all the prime ideals of the Leavitt path algebra $L_K(E(H, S))$ is zero, we conclude that $\bigcap_{j=1}^m I(H_j, S_j) = \{0\}$. As $I(H_j, S_j)$ are graded strongly primitive, $M_j = [E(H, S) \setminus (H_j, S_j)]^0$ is a maximal tail satisfying the strong CSP with respect to a countable subset $C_j \subseteq S_j$ and $E(H, S)^0 = \bigcup_{j=1}^m M_j$. This proves (3).

Assume (3). Let $I$ be an arbitrary ideal of $L$. Condition (K) implies that $I$ is graded, say, $I = I(H, S)$ and that $E(H, S)^0 = \bigcup_{j=1}^m S_j$ where each $S_j$ is a maximal tail satisfying the strong CSP (and Condition (L)). Then, for each $j = 1, \ldots, m$, $H_j = E(H, S)^0 \setminus S_j$ will be a hereditary saturated subset of $E(H, S)^0$ and, by Theorem 4.11, $Q_j = I(H_j, B_{H_j})$ will be a graded strongly prime ideal of $L_K(E(H, S))$. Clearly $\bigcap_{j=1}^m I(H_j, B_{H_j}) = \{0\}$. Now $L/I(H, S) \cong L_K(E(H, S))$ and identify these two rings under this isomorphism. For each $j = 1, \ldots, m$, let $P_j$ be the ideal of $L$ such that $P_j/I(H, S) = Q_j$. Then $P_j$ is a strongly prime ideal and $I = \bigcap_{j=1}^m P_j$. As $I$ is graded, we appeal to Lemma 2.4 to conclude that $I = P_1 \cdots P_m$, a product of strongly prime ideals. This proves (1).

For an illustration for the above theorem, let us recall the Example 3.17 from previous section.

Example 4.17. Let $E$ be the following graph:

![Graph](https://example.com/graph_example)

The graph is row-finite and satisfies the Condition (K). The proper hereditary saturated subsets of this graph are precisely the empty set $\emptyset$ and the sets $A_1 = \{v_n : n \geq 1\}$; $A_2 = \{v_n : n \geq 2\}$; $B_1 = \{w_n : n \geq 1\}$; $B_2 = \{w_n : n \geq 2\}$. For each $i = 1, 2$, $E^0 \setminus A_i$ and also $E^0 \setminus B_i$ is the union of at most two maximal tails each of which satisfy strong CSP with respect to itself. Since $A_1 \cap A_2 = \emptyset$, $E^0 \setminus \emptyset = E^0 \setminus (A_1 \cap A_2) = (E^0 \setminus A_1) \cup (E^0 \setminus A_2)$ is also the union of two maximal tails satisfying the strong CSP with themselves. Thus the graph satisfies Condition (3) of Theorem 4.16.

We proceed to characterize Leavitt path algebras in which each ideal is strongly prime.

Theorem 4.18. The following are equivalent for any Leavitt path algebra $L = L_K(E)$:

1. Every ideal of $L$ is strongly prime;
2. All the ideals of $L$ are graded strongly primitive and form a chain under set inclusion;
The graph $E$ satisfies Condition (K), the admissible pairs $(H, S)$ form a chain under the partial ordering of the admissible pairs and for each admissible pair $(H, S)$, $E \setminus (H, S)$ is downward directed and satisfies the strong CSP.

**Proof.** Assume (1). Let $I$ be any ideal of $L$. Since $I$ is strongly prime, it is graded and strongly primitive, by Theorem 4.11. To show that the ideals form a chain, observe that every ideal of $L$ is prime, as a strongly prime ideal is always prime. Then, for any two ideals $A$, $B$, $A \cap B$ is a prime ideal and so $AB \subseteq A \cap B$ implies that $A \subseteq A \cap B$ or $B \subseteq A \cap B$ and this implies that either $A \subseteq B$ or $B \subseteq A$. Thus the ideals of $L$ form a chain under inclusion. This proves (2).

Assume (2). Since every ideal of $L$ is graded, the graph $E$ satisfies Condition (K), by Proposition 2.9.9 of [1]. Now every ideal of $L$ form a chain, it is clear from the order preserving bijection between the graded ideals of $L$ and the admissible pairs that the admissible pairs form a chain under their partial ordering. Now, for a given admissible pair $(H, S)$, the corresponding ideal $I(H, S)$ is strongly prime and so, by Theorem 4.11, $E \setminus (H, S)$ is downward directed and satisfies the strong CSP. This proves (3).

Assume (3). Let $I$ be an arbitrary ideal of $L$. Since $E$ satisfies Condition (K), $I$ is a graded ideal (Proposition 2.9.9, [1]), say $I = I(H, S)$ where $H = I \cap E^0$. Now $E \setminus (H, S)$ satisfies Condition (K) and hence Condition (L). Since, by supposition, it is downward directed and satisfies the strong CSP, we appeal to Theorem 4.11 to conclude that $I$ is a strongly prime ideal, thus proving (1). \[\square\]

**Example 4.19.** Consider the graph $E$ below.

The proper hereditary saturated subset of vertices are $H_0 = \emptyset$, $H_1 = \{v_1\}$, $H_2 = \{v_1, v_2\}$ and $H_3 = \{v_1, v_2, v_3\}$. It is easy to see that, $E^0 \setminus H_0 = E^0, E^0 \setminus H_1, E^0 \setminus H_2, E^0 \setminus H_3$ are all downward directed and satisfy the strong CSP with respect to $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$, respectively. Hence every proper ideal of $L_K(E)$ is strongly prime. The graph $E$ is finite and satisfies Condition (K), thus illustrating Theorem 4.18.

The next theorem describes when a Leavitt path algebra is strongly zero-dimensional, that is, when every prime ideal of $L$ is strongly prime. In its proof, we shall be using the following concept of an extreme cycle.

**Definition 4.20.** [1] A cycle $c$ in a graph $E$ is said to an extreme cycle, if it has exits and for every path $a$ with $s(a) \in c^0$, there is a path $b$ with $s(b) = r(a)$ such that $r(b) \in c^0$.

**Theorem 4.21.**

1. Suppose $E$ is a finite graph (or more generally suppose $E^0$ is finite). Then every prime ideal of $L_K(E)$ is strongly prime if and only if the graph $E$ satisfies Condition (K);
(2) Let $E$ be an arbitrary graph. Then every prime ideal of $L_K(E)$ is strongly prime if and only if the graph $E$ satisfies Condition (K) and every quotient graph $E\setminus(H,S)$ which is downward directed satisfies the strong CSP.

**Proof.** Suppose every prime ideal of $L_K(E)$ is strongly prime, then every prime ideal of $L_K(E)$ is graded, since a strongly prime ideal always graded (Theorem 4.11). This implies, by Corollary 3.13 of [17], that the graph $E$ satisfies Condition (K) which, by Proposition 2.9.9 of [1], is equivalent to every ideal of $L_K(E)$ being a graded ideal.

(1) Assume now that $E^0$ is finite and that $E$ satisfies Condition (K). Let $P$ be any prime ideal of $L_K(E)$ which, being graded, will be of the form $P = I(H,S)$ where $H = P \cap E^0$. Since $(E\setminus(H,S))^0$ is finite, Lemma 3.7.10 of [1] implies that every path in $E\setminus(H,S)$ ends at a sink, a cycle without exits or an extreme cycle. If there is a sink in $(E\setminus(H,S))^0$, then by downward directedness, there can be only one sink, say $w$ in $(E\setminus(H,S))^0$ and, moreover, every non-empty hereditary subset of the downward directed set $(E\setminus(H,S))^0$ will contain $w$ and hence $(E\setminus(H,S))^0$ will satisfy the strong CSP with respect to $\{w\}$. If there are no sinks in $(E\setminus(H,S))^0$, then, since $E$ satisfies Condition (K), every cycle will have exits and so every path in $E\setminus(H,S)$ will end at an extreme cycle. Since $(E\setminus(H,S))^0$ is finite, there are only finitely many extreme cycles, say $c_1, \ldots, c_n$. Fix a vertex $v \in E^0$. We claim that every vertex $u \in (E\setminus(H,S))^0$ satisfies $u \geq v$. To see this, note that, since every path ends at one of the cycles $c_1, \ldots, c_n$, $u \geq w$ in $c_j^0$ for some $j$. Since $(E\setminus(H,S))^0$ is downward directed, there is a vertex $v_0$ such that $v \geq v_0$ and $w \geq v_0$. Suppose $\alpha$ denotes the path connecting $v$ to $v_0$ and $\beta$ denotes the path connecting $w$ to $v_0$. Since $c_1$ is an extreme cycle, there is a path $\gamma$ with $s(\gamma) = v_0$ and $r(\gamma) \in c_1^0$. Then the path $\beta \gamma$ can be elongated to a path connecting $w$ to $v$. Hence $w \geq v$ which implies $u \geq v$. It is then clear that $(E\setminus(H,S))^0$ satisfies the strong CSP with respect to $\{v\}$. Since Condition (K) always implies Condition (L), in both cases $E\setminus(H,S)$ is downward directed satisfying the strong CSP and Condition (L). Hence, by Theorem 4.11, $P$ is strongly prime.

(2) Suppose $E$ is an arbitrary graph. If every prime ideal of $L_K(E)$ is strongly prime, then as noted earlier, $E$ satisfies Condition (K) and so every ideal is graded. So any prime ideal $P$ will be of the form $P = I(H,S)$ where $E\setminus(H,S)$ is downward directed and satisfies Condition (K) and therefore Condition (L). Thus, in the context of Condition (K), Theorem 4.11 implies that the condition that the downward directed graph $E\setminus(H,S)$ satisfies strong CSP is equivalent to the prime ideal $I(H, S)$ being strongly prime. $\square$

**Example 4.22.** Let $E$ be the following graph.

```
\begin{center}
\begin{tikzpicture}
  \node (v) at (0,0) [circle, draw] {$v$};
  \node (w) at (1,1) [circle, draw] {$w$};
  \node (u) at (-1,1) [circle, draw] {$u$};
  \draw (v) -- (w);
  \draw (v) -- (u);
  \draw (u) -- (v);
\end{tikzpicture}
\end{center}
```

The graph is finite and satisfies Condition (K). Hence every ideal of $L_K(E)$ is graded. Now the proper hereditary saturated subsets of $E^0$, are $H_0 = \emptyset, H_1 = \{v\}, H_2 = \{u, v\}$ and $H_3 = \{v, w\}$. Hence $I_i = <H_i>$, $i = 0, 1, 2, 3$ are all the proper ideals of $L_K(E)$. It is easy to check that the prime ideals of $L_K(E)$ are $\{0\} = I_0, I_2$ and $I_3$ as $E\setminus(H_i)$ is downward directed for $i = 0, 2, 3$. It is also easy to see that each of these $E\setminus(H_i)$ satisfies strong CSP. Hence the prime ideals $I_0, I_1, i = 0, 2, 3$ are actually strongly prime. So every prime ideal of $L_K(E)$ is strongly prime, thus illustrating Theorem 4.21.

5. **Insulated Prime Leavitt path algebras and insulated prime ideals**

For non-commutative rings, the concept of a left/right strongly prime ring was introduced in [9] while dealing with Kaplansky’s conjecture on prime von Neumann regular rings. Following this,
the definition of a left/right strongly prime ideal was given in [14] which is different from the one introduced in [13]. To avoid confusion with the concept of strongly prime rings and ideals that we investigated in Section 4, we rename this concept in Definition 5.2 below.

**Definition 5.1.** ([9]) Let $R$ be a ring. A right insulator of an element $a \in R$ is defined to be a finite subset $S(a)$ of $R$, such that the right annihilator $ann_R\{ac : c \in S(a)\} = 0$.

Similarly, left insulator of an element can be defined.

**Definition 5.2.** ([9]) A ring $R$ is called a right insulated prime ring if every non-zero element of $R$ has a right insulator. Equivalently, every non-zero two-sided ideal of such a ring $R$ contains a finite non-empty subset $S$ whose right annihilator is zero. This finite set $S$ is called an insulator of $I$.

A left insulated prime ring is defined similarly.

Following [9], Kaucikas and Wisbauer [14] define the concept of a right/left strongly prime ideals. Again to avoid confusion with the concept strongly prime ideals of Section 4, we rename these ideals as indicated below.

**Definition 5.3.** An ideal $I$ of a ring $R$ is called a right/left insulated prime ideal, if $R/I$ is a right/left insulated prime ring.

In this section, we first describe when a Leavitt path algebra $L_K(E)$ is a left/right insulated prime ring. Interestingly, the distinction between the left and right insulated primeness vanishes for Leavitt path algebras. So we may just state $L_K(E)$ as being an insulated prime ring. We show (Theorem 5.6) that a Leavitt path algebra $L_K(E)$ is an insulated prime ring exactly when $L_K(E)$ is a simple ring or $L_K(E)$ is isomorphic to the matrix ring $M_n(K[x,x^{-1}])$ some integer $n \geq 1$.

Equivalent graphical conditions on $E$ are also given. Next, we characterize the insulated prime ideals $P$ of $L_K(E)$. Non-graded insulated prime ideals of $L_K(E)$ are precisely the (non-graded) maximal ideals of $L_K(E)$. A graded ideal $P$ with $P \cap E^0 = H$ will be an insulated prime ideal of $L_K(E)$ if and only if $P = I(H,B_H)$ and $P$ is either a maximal graded ideal of $L_K(E)$ or is a maximal ideal of $L_K(E)$ (which is graded). It is then clear that an insulated prime ideal of $L_K(E)$ is always a prime ideal. Graphically, if $gr(P) = I(H,B_H)$, then $E_n(H,B_H)$ contains only finitely many vertices, is downward directed and has no non-empty proper hereditary saturated subset of vertices. Examples are constructed, showing that the concepts of strongly prime ideals and insulated prime ideals are independent in the case of Leavitt path algebras.

We first prove the following useful proposition which states that the matrix rings $M_\Lambda(K[x,x^{-1}])$ are precisely the Leavitt path algebras which are graded-simple but not simple.

**Proposition 5.4.** The following properties are equivalent for a Leavitt path algebra $L = L_K(E)$:

1. $L_K(E)$ is graded-simple but not simple, that is, $L_K(E)$ contains non-zero proper graded ideals but does not contain any non-zero proper graded ideals;
2. $E$ is row-finite, downward directed and contains a cycle $c$ without exits based at a vertex $v$;
3. $L_K(E) \cong \oplus \Lambda M_\Lambda(K[x,x^{-1}])$ under the matrix grading of $M_\Lambda(K[x,x^{-1}])$, where $\Lambda$ is some index set.

**Proof.** Assume (1). Let $I$ be a proper non-zero ideal of $L = L_K(E)$. Then there is a vertex $u \notin I$. If $P$ is an ideal maximal with respect to $u \notin P$, then $P$ is a prime ideal (because, if $a \notin P, b \notin P$, then $u \in LaL + P$ and $u \in LbL + P$ and then $u = u^2 \in (LaL + P)(LbL + P) = LaLbL + P$. Since $u \notin P$, this implies that $aLb \not\subseteq P$). Since $L_K(E)$ is graded-simple, $P$ must be non-graded and is of the form $P = I(H,B_H) + < p(c) >$ where $c$ is a cycle without exits in $E_n(H,B_H)$ based at a vertex $v$, $E^n_H = E_n(H,B_H)^0$ is downward directed and $p(x)$ is an irreducible polynomial in $K[x,x^{-1}]$ (Theorem 2.2). By hypothesis, the graded ideal $I(H,B_H) = \{0\}$ and so $H = \emptyset$. Consequently, $E^0$
is downward directed and \( \epsilon \) is the only cycle without exits in \( E \). We claim that \( E \) is row-finite. Suppose, on the contrary, there is a vertex \( w \) which is an infinite emitter. Since \( w \) is not in the hereditary set \( \mathcal{O} \), it follows from the definition of its saturated closure, that \( w \) is not in the saturated closure of \( \mathcal{O} \). Hence \( w \not\in < \mathcal{O} > \). This is a contradiction, since the non-zero graded ideal \( < \mathcal{O} > \) is a left/right insulated prime ring; by hypothesis. We thus conclude that \( E \) must be row-finite. This proves (2).

Assume (2). Now, by downward directness, every path in \( E \) ends at the vertex \( v \). Then, by Theorem 4.2.12 in [1], \( L_E = < \mathcal{O} > \cong \Lambda_M(K[x, x^{-1}]) \) where \( \Lambda \) denotes the set of all paths in \( E \) that end at \( v \), but do not go through the entire cycle \( c \). It is shown in (the paragraph “grading of matrix rings” in Section 2, [10]) that this isomorphism is a graded isomorphism under the matrix-grading of \( \Lambda_M(K[x, x^{-1}]) \). Hence (3) follows.

Now (3) \( \Rightarrow \) (1) follows from the fact that \( \Lambda_M(K[x, x^{-1}]) \) is a graded direct sum of copies of \( K[x, x^{-1}] \) and that \( K[x, x^{-1}] \) has no non-zero proper graded ideals under its natural \( \mathbb{Z} \)-grading. \( \square \)

The next corollary points out a property of non-graded maximal ideals in a Leavitt path algebra that will be used later.

**Corollary 5.5.** If \( I \) is a non-graded maximal ideal of \( L = L_K(E) \), then \( \text{gr}(I) \) is a maximal graded ideal of \( L \) and \( L/\text{gr}(I) \cong \text{gr}_{\Lambda} M_{\Lambda}(K[x, x^{-1}]) \) for some index set \( \Lambda \).

**Proof.** Now \( I \) is, in particular, a (non-graded) prime ideal and so \( I = I(H, B_H) + < p(\epsilon) > \), where \((E\setminus (H, B_H))^0 = E^0 \setminus H \) is downward directed, \( c \) is a cycle without exits in \( E\setminus (H, B_H) \) based at a vertex \( v \) and \( p(x) \) is an irreducible polynomial in \( K[x, x^{-1}] \). If \( A = I(H', B_H') \) is a graded ideal such that \( A \supseteq I(H, B_H) \) then \( H' \supseteq H \). If \( u \in H' \setminus H \), then by downward directness, \( u \geq v \) and this implies that \( (v \text{ and hence}) \in A \) and so \( I \subseteq A \). By the maximality of \( I \), \( A = L \). Thus \( \text{gr}(I) = I(H, B_H) \) is a maximal graded ideal of \( L \). Then, by Proposition 5.4, \( L/\text{gr}(I) \cong \text{gr}_{\Lambda} M_{\Lambda}(K[x, x^{-1}]) \) for some index set \( \Lambda \). \( \square \)

**Theorem 5.6.** The following properties are equivalent for a Leavitt path algebra \( L = L_K(E) \) of an arbitrary graph \( E \):

(1) \( L \) is a left/right insulated prime ring;
(2) Either (a) \( L \) is a simple ring or (b) \( L \cong \text{gr}_{\Lambda} M_{\Lambda}(K[x, x^{-1}]) \) where \( \Lambda \) is some positive integer;
(3) Either (a) \( E \) satisfies Condition (L) and has no non-empty proper hereditary saturated subsets of vertices or (b) \( E \) is a finite “comet”, that is, a downward directed finite graph containing a cycle without exits.

**Proof.** Assume (1) and that \( L \) is a right insulator ring. If \( L \) is a simple ring, we have nothing to prove. Assume \( L \) is not a simple ring. We claim that \( L \) is graded-simple. Suppose, on the contrary, there is a non-zero graded ideal \( I \) in \( L \). By hypothesis, \( I \) has a right insulator \( S \). Now, by Theorem 2.5.22 in [1], the graded ideal \( I \) is isomorphic to a Leavitt path algebra and hence has local units. This means that corresponding to the finite subset \( S \), there is an idempotent \( \epsilon \) (depending on \( S \)) in \( I \) such that \( S \subseteq \epsilon I \). Let \( \epsilon = \sum_{j=1}^n x_j \beta_j^0 \in I \) and let \( X = \{ s(x_j), r(x_j) = r(\beta_j^0), s(\beta_j^0) : j = 1, ..., n \} \). Now \( E^0 \) cannot be an infinite set, because, otherwise, we can find a vertex \( v \in E^0 \setminus X \) and as \( \epsilon v = 0, \epsilon v L = 0 \) which implies that \( S \cdot vL = 0 \), a contradiction to the hypothesis. Thus \( E^0 \) is finite. This means that \( L \) has a multiplicative identity \( 1 = \sum_{v \in E^0} v \). Moreover, \( S \subseteq \epsilon \epsilon I \epsilon \) implies that \( (1 - \epsilon) S = 0 = S(1 - \epsilon) \). Since \( S \) is the insulator for \( I \), \( 1 - \epsilon \neq 0 \) or \( 1 - \epsilon \in I \). Hence \( I = L \), thus proving that \( L \) is graded-simple. We then appeal to Proposition 5.4 to conclude that \( L \cong \text{gr}_{\Lambda} M_{\Lambda}(K[x, x^{-1}]) \). Since \( L \) has a multiplicative identity \( 1 \), the index set \( \Lambda \) must be finite and we conclude that \( L \cong \text{gr}_{\Lambda} M_{\Lambda}(K[x, x^{-1}]) \) for some positive integer \( \Lambda \). This proves (2).
Assume (2). If $L$ is a simple ring, it is trivially insulated prime. Suppose $L \cong M_n(K[x, x^{-1}])$ for some positive integer $n$. Now $K[x, x^{-1}]$, being an integral domain, is clearly an insulated prime ring. We show (following the ideas in the proof of Proposition II.1. [9]), that $M_n(K[x, x^{-1}])$ is also insulated prime. Suppose $0 \neq A \in M_n(K[x, x^{-1}])$ with a non-zero entry, say $a_{ij} \neq 0$. For each $i = 1, \ldots, n$ and $j = 1, \ldots, n$, let $E_{ij} \in M_n(K[x, x^{-1}])$ be the matrix unit having 1 at the $(i, j)$-entry and 0 everywhere else. Let $0 \neq b \in K[x, x^{-1}]$. Then $\{E_{ij} : i = 1, \ldots, n; j = 1, \ldots, n\}$ is an insulator for $A$. Because, if $N = (n_{ij})$ is any non-zero matrix with a non-zero entry $n_{ki}$, then $AE_{ik}bN \neq 0$ since its $(i, l)$-entry is $a_{il}b_{kl} \neq 0$. This proves (1).

Now (2) $\iff$ (3) by Theorem 2.9.1 and Theorem 4.2.12 in [1].

In the next theorem we describe the insulated prime ideals of a Leavitt path algebra.

**Theorem 5.7.** The following properties are equivalent for an ideal $P$ of a Leavitt path algebra $L = L_K(E)$ with $P \cap E^0 = H$:

1. $P$ is an insulated prime ideal of $L_K(E)$;
2. Either $P$ is a maximal ideal or $P$ is not a maximal ideal but a maximal graded ideal such that $L/P \cong M_n(K[x, x^{-1}])$ where $n$ is a positive integer and, in particular, $E^0 \setminus P$ is a finite set.

**Proof.** Assume (1). Suppose $P$ is an insulated prime ideal so that $L_K(E)/P$ is an insulated prime ring. So, by Theorem 5.6, $L_K(E)/P$ is a simple ring or $L_K(E)/P \cong M_n(K[x, x^{-1}])$.

If $L_K(E)/P$ is a simple ring, then clearly, $P$ is a maximal ideal of $L_K(E)$.

Suppose $L_K(E)/P \cong M_n(K[x, x^{-1}])$ for some $n \geq 1$. As $M_n(K[x, x^{-1}])$ is a prime ring, $P$ is clearly a prime ideal. We claim that $P$ must be a graded ideal. Assume to the contrary that $P$ is a non-graded ideal. By Theorem 2.2, we can then write $P = I(H, B_H) + < p(c) >$, where $H = P \cap E^0$, $E^0 \setminus H = (E \setminus (H, B_H))^0$ is downward directed, $c$ is a cycle without exits in $E \setminus (H, B_H)$ based at a vertex $v$ and $p(x) \in K[x, x^{-1}]$ is an irreducible polynomial. Then, in $\overline{L_K(E)} = L_K(E)/I(H, B_H)$, $\overline{P} = P/I(H, B_H) = < p(c) > \not\subseteq M = < \{c^0\} >$. Now, being a graded ideal, $(M)^2 = M$ and this implies that $(\overline{M/\overline{P}})^2 = M/\overline{P}$ in $\overline{L_K(E)}/\overline{P} \cong M_n(K[x, x^{-1}])$. Since $I^2 \neq I$ for any non-zero proper ideal $I$ in $K[x, x^{-1}], M_n(K[x, x^{-1}])$ satisfies the same property as the ideal lattices of $M_n(K[x, x^{-1}])$ and $K[x, x^{-1}]$ are isomorphic (Proposition 3.3(ii)). Hence $M = \overline{L_K(E)}$. By Proposition 3.3(ii), $\overline{P} = < p(c) >$ is a maximal ideal of $\overline{M = \overline{L_K(E)}}$ and so $\overline{L_K(E)}/\overline{P}$ must be a simple ring. This is a contradiction, since the ring $M_n(K[x, x^{-1}])$ is not simple. Thus $P$ must be a graded ideal of $L_K(E)$, say $P = I(H, S)$ where $H = P \cap E^0$. Now, by Proposition 5.4, $L_K(E)/P \cong M_n(K[x, x^{-1}])$ has no non-zero proper graded ideals and hence $P$ is a maximal graded ideal of $L_K(E)$ and hence $S = B_H$ and $P = I(H, B_H)$. Since $M_n(K[x, x^{-1}])$ contains a multiplicative identity, $E^0 \setminus P = (E \setminus (H, B_H))^0$ is a finite set. This proves (2).

Assume (2). If $P$ is a maximal ideal, then clearly $P$ is an insulated prime ideal as the simple ring $L_K(E)/P$ is an isolated prime. Suppose $P$ is a maximal graded ideal with $E^0 \setminus P$, a finite set. It is easy to check that $P$ is a graded prime ideal and, as $L_K(E)$ is $\mathbb{Z}$-graded, $P$ is a prime ideal of $L_K(E)$ (see Proposition II.1.4, Chapter II in [16]). Hence $P = I(H, S)$ where $H = P \cap E^0$ and $(E \setminus (H, S))^0$ is downward directed. By the maximality of $P$, $P = I(H, B_H)$ and $L_K(E)/P \cong L_K(E \setminus (H, B_H))$ has no non-zero proper graded ideals. Hence, by Proposition 5.4, $L_K(E)/P \cong M_\Lambda(K[x, x^{-1}])$, where $\Lambda$ is some index set. Now, by hypothesis, $(E \setminus (H, B_H))^0 = E^0 \setminus P$ is a finite set and so $L_K(E \setminus (H, B_H))$ has a multiplicative identity. Hence $\Lambda$ must be a finite set and we conclude that $L_K(E)/P \cong M_n(K[x, x^{-1}])$ for some positive integer $n$. This proves (1).
Remark 5.8. The property of being insulated prime is independent of the property of being strongly prime. Note that any non-graded maximal ideal of $L$ is insulated prime by Theorem 5.7, but it is not strongly prime since, by Theorem 4.11, a strongly prime ideal of $L$ must be graded. Likewise, a graded prime ideal of the form $I(H,B_H \setminus u)$ is strongly prime since it is graded strongly primitive (Theorem 4.11), but is not a maximal graded ideal as it is properly contained in the ideal $I(H,B_H)$ and consequently it cannot be an insulated prime ideal.

Next, we give two examples. The first one is example of an insulated prime that is not strongly prime whereas the second one is example of a strongly prime ideal that is not insulated prime.

Example 5.9. Let $E$ be the following graph.

Let us denote the loops on vertex $v_1$ by $c_1$, $c_2$ and the loop on vertex $v_3$ by $c_3$. Now $H = \{v_1\}$ is a hereditary saturated set. Let $P = I(H) + \langle p(c_3) \rangle$ where $p(x)$ is an irreducible polynomial in $K[x,x^{-1}]$. Clearly $P$ is a non-graded ideal. Now the graph $E \setminus H$ consists of a single edge ending at the cycle $c_3$ without exits and so $L_K(E \setminus H) \cong M_2(K[x,x^{-1}])$. Since $p(x)$ is irreducible in $K[x,x^{-1}]$, $\langle p(c_3) \rangle$ will be a maximal ideal in $L_K(E \setminus H) \cong M_2(K[x,x^{-1}])$, by Proposition 3.3(ii). Thus $P/I(H) = \langle p(c_3) \rangle$ will be a maximal ideal of $L_K(E)/I(H) \cong L_K(E \setminus H)$ and hence $P$ is a maximal (non-graded) ideal in $L_K(E)$. By Theorem 5.6, $P$ is an insulated prime ideal. But $P$ is not strongly prime since $P$ is not a graded ideal.

Example 5.10. Let $E^0 = \{v_1, v_2, v_3, v_4\}$, and the graph $E$ be as below,

Let $H = \{v_1, v_2\}$. Then $H$ is a hereditary saturated subset of $E^0$ and $B_H = \{v_3, v_4\}$. Consider the graded ideal $P = I(H,B_H \setminus \{v_3\})$. Now $(E \setminus (H,B_H \setminus \{v_3\}))^0 = \{v_3, v_4, v'_3\}$ is downward directed and satisfies (Condition (K) and hence) Condition (L) and the strong CSP with respect to $\{v'_3\}$. Hence $P$ is strongly prime by Theorem 4.10. But $P$ is not a maximal graded ideal, as $P \subsetneq I(H,B_H)$ and hence $P$ not an insulated prime, by Theorem 5.6.

Next, we describe conditions under which an ideal of a Leavitt path algebra can be factored as a product of (finitely many) insulated prime ideals. In the proof, we shall be using the following basic result.

Lemma 5.11. (Lemma 2.7.1 [1]) A graph $E$ is a comet (with a no exit cycle $c$ based at a vertex $v$) if and only if $L_K(E) \cong M_\Lambda(K[x,x^{-1}])$ where $\Lambda$ is the set of all paths that end at $v$ but not include the entire cycle $c$. 
Theorem 5.12. The following properties are equivalent for an ideal $I$ of a Leavitt path algebra $L = L_K(E)$ with $gr(I) = I(H,S)$:

1. $I$ is a product of (finitely many) insulated prime ideals of $L$;
2. $I(H,S) = gr(I) = Q_1 \cap \cdots \cap Q_m$ is an irredundant intersection of $m$ graded ideals each of which is either an insulated prime (hence maximal graded) ideal or a maximal graded ideal which is not insulated prime;
3. $I(H,S) = gr(I) = Q_1 \cap \cdots \cap Q_m$ is an irredundant intersection of $m$ graded ideals each of which is either an insulated prime (which is maximal graded) ideal or a maximal graded ideal which is not insulated prime and $I = I(H,S) + \sum_{t=1}^{k} f_t(c_t)$, where $k \leq m$, for each $t = 1, \ldots, k$, $c_t$ is a cycle without exits in $E \setminus (H,S)$ based at a vertex $v_t$ and $f_t(x) \in K[x]$ with a non-zero constant term.
4. $L/I(H,S) = L_1 \oplus \cdots \oplus L_m$, where $\oplus$ is a graded ring direct sum and, for each $j = 1, \ldots, m$, $L_j \cong M_{n_j}(K[x,x^{-1}])$ where $n_j$ is a finite or infinite index set;
5. The quotient graph $E \setminus (H,S)$ is an irredundant union of finitely many finite comets.

Proof. Assume (1) so $I = P_1 \cdots P_n$ is a product of insulated prime ideals $P_i$. By Theorem 5.7, each ideal $P_i$ is either a maximal ideal or a maximal graded ideal such that $L/P_i \cong M_{n_i}(K[x,x^{-1}])$ where $n_i$ is a positive integer. Let $Q_1 = gr(P_1)$ for $j = 1, \ldots, n$. Then $g(I) = Q_1 \cap \cdots \cap Q_n$, by Lemma 2.4. If necessary, after removing appropriate ideals and after re-indexing, we get $g(I) = Q_1 \cap \cdots \cap Q_m$, an irredundant intersection of graded ideals with $m \leq n$. Here, for each $j = 1, \ldots, m$, either $Q_j$ is a graded insulated prime ideal or $Q_j = gr(P_j)$ is a maximal graded ideal which is not insulated prime with $P_j$ a non-graded maximal ideal of $L$ (Corollary 5.5. This proves (2)).

Now (2) $\iff$ (3). Because, a maximal graded ideal of $L$ is a prime ideal and so the equivalence of (2) and (3) follows from the equivalence of conditions (2) and (3) of Theorem 3.9.

Assume (2). Then, in $\bar{L} = L/I(H,S), 0 = \bar{Q}_1 \cap \cdots \cap \bar{Q}_m$, where, for each $j = 1, \ldots, m$, $\bar{Q}_j = Q_j/I(H,S)$. Consider the map $\theta : L \to L/\bar{Q}_1 \oplus \cdots \oplus L/\bar{Q}_m$ given by $a \mapsto (a + Q_1, \ldots, a + Q_m)$, where $\oplus$ is a graded direct sum. Now $\theta$ is clearly a monomorphism. It is also a graded morphism, since the coset map $L \to L/\bar{Q}_j$ is a graded morphism for all $j$. To show that $\theta$ is an epimorphism, first note that the Chinese Remainder Theorem holds in the Leavitt path algebra $\bar{L} \cong L_K(E \setminus (H,S))$ (see Remark after Theorem 4.3 n [20]). Also, by maximality, $Q_i + Q_j = \bar{L}$ for all $i, j$ with $i \neq j$. Consequently, given any element $x = (x_1 + Q_1, \ldots, x_m + Q_m) \in L/\bar{Q}_1 \oplus \cdots \oplus L/\bar{Q}_m$, there is an element $y \in L$ such that $y \equiv x_j (mod Q_i)$ for all $j = 1, \ldots, m$. It is then clear that $\theta(y) = x$. Thus $\theta$ is a graded isomorphism and $\bar{L} \cong gr(L/\bar{Q}_1 \oplus \cdots \oplus L/\bar{Q}_m)$. If the graded ideal $Q_j$ is an insulated prime ideal, then, by Theorem 5.7 $\bar{L}/\bar{Q}_j \cong L/\bar{Q}_j \cong M_{n_j}(K[x,x^{-1}])$ where $n_j$ is some positive integer. On the other hand, if $Q_j = gr(P_j)$ is a maximal graded ideal which is not insulated prime (with $P_j$ a non-graded maximal ideal of $L$), then, by Corollary 5.5, $\bar{L}/\bar{Q}_j \cong L/\bar{Q}_j \cong M_{n_j}(K[x,x^{-1}])$ where $n_j$ is an infinite index set. This proves (4).

Assume (4) so $\bar{L} = L/I(H,S) = L_1 \oplus \cdots \oplus L_m$ where $\oplus$ is a graded ring direct sum and, for each $j = 1, \ldots, m$, $L_j \cong M_{n_j}(K[x,x^{-1}])$ where $n_j$ is a finite or infinite index set. As $L_j \cong M_{n_j}(K[x,x^{-1}])$ is graded-simple, we have for each $j = 1, \ldots, m$, $A_j = \oplus_{i \neq j, i=1}^{m} L_i$ is a maximal graded ideal of $\bar{L} \cong L_K(E \setminus (H,S))$ and so will be of the form $A_j = I(H_j, B_{H_j})$ where $H_j = A_j \cap (E \setminus (H,S))^0$. Now $L_K(E \setminus (H,S))/I(H_j, B_{H_j}) \cong \bar{L}/\bar{A}_j \cong L_j \cong M_{n_j}(K[x,x^{-1}])$ and so, by Lemma 5.11, $M_j = (E \setminus (H,S)) \setminus (H_j, B_{H_j})$ is a comet. Since $\cap_{j=1}^{m} A_j = \{0\}, E \setminus (H,S)^0 = \cup_{j=1}^{m} M_j$, a union of finite number of comets. This proves (5).

Assume (5). So $E \setminus (H,S) = \cup_{j=1}^{m} M_j$ is a union of comets. If $H_j = (E \setminus (H,S))/M_j$, then $\bar{Q}_j = I(H_j, B_{H_j})$ is a graded ideal of $\bar{L} \cong L_K(E \setminus (H,S)) \cong L/I(H,S)$ and $\cap_{j=1}^{m} \bar{Q}_j = \{0\}$. Then $\bar{L}/\bar{Q}_j \cong L_K[(E \setminus (H,S)) \setminus (H_j, B_{H_j})] \cong L_K(M_j) \cong M_{n_j}(K[x,x^{-1}])$
where $\Lambda_j$ is an finite or infinite index set according as $M_j$ is a finite or infinite index set. Now, for each $j$, $Q_j = Q_j/I(I(H,S)$ for some ideal $Q_j \supseteq I(H,S)$. Clearly, $gr(I) = I(H,S) = \cap_{j=1}^{m} Q_j$ where the $Q_j$ are graded ideals which are either insulated prime or non-maximal ideals which are maximal graded ideals according as $\Lambda_j$ is a finite or an infinite index set. This proves (2).

**Remark 5.13.** As noted in Remark 4.15, factorization of an ideal $I$ of $L$ as an irredundant product or intersection of finitely many insulated prime ideals is unique except for the order of the factors due to the fact that an insulated prime ideal is always a prime ideal.

We next consider the case when every ideal of $L$ is a product of finitely many insulated prime ideals.

**Theorem 5.14.** The following properties are equivalent for any Leavitt path algebra $L = L_K(E)$:

1. Every ideal of $L$ is a product of (finitely many) insulated prime ideals;
2. $L \cong \bigoplus_{j=1}^{m} L_j$ is a graded ring direct sum of matrix rings $L_j \cong gr M_{\Lambda_j}(K[x,x^{-1}])$ where $\Lambda_j$ is a suitable index set.

**Proof.** Assume (1). Since $\{0\}$ is a product of insulated prime ideals, applying Theorem 5.12 (iv) with $I = \{0\}$, we obtain (2).

Assume (2), so that $L = L_1 \oplus \cdots \oplus L_m$, where $\oplus$ is a graded ring direct sum and, for each $j = 1, \cdots, m$, $L_j \cong M_{\Lambda_j}(K[x,x^{-1}])$ where $\Lambda_j$ is a finite or infinite index set. Note that $\{0\}$ is a maximal graded ideal of $K[x,x^{-1}]$ and so is an insulated prime ideal. Also every non-zero ideal of $K[x,x^{-1}]$ is a product of maximal (hence insulated prime) ideals of $K[x,x^{-1}]$. By Proposition 3.3(ii), the same properties hold for ideals of $L_j \cong M_{\Lambda_j}(K[x,x^{-1}])$ for all $j = 1, \ldots, m$. Since $\oplus$ is a graded ring direct sum, a simple induction on $m$ shows that every ideal of $L$ is a product of insulated prime ideals. This proves (1).

As an illustration, we have the following example.

**Example 5.15.** Let $E = F \cup F$ be the disjoint union of two copies of the graph $F$ as shown below:

Now $L_K(F) \cong gr M_2(K[x,x^{-1}])$ and so $L_K(E) \cong gr (M_2(K[x,x^{-1}]), M_2(K[x,x^{-1}]))$.

The next theorem describes when every ideal of a Leavitt path algebra is insulated prime.

**Theorem 5.16.** Let $L = L_K(E)$ be any Leavitt path algebra. Then we have the following:

1. Every ideal of $L$ is insulated prime if and only if $L$ is a simple ring;
2. Every non-zero ideal of $L$ is insulated prime if and only if either $L$ is a simple ring or $L$ contains exactly one non-zero ideal $I$ which is graded and $L/I \cong M_n(K[x,x^{-1}])$ for some positive integer $n$.

**Proof.**

(1) Assume every ideal of $L$ is insulated prime. We claim $L$ is a simple ring. Suppose, by way of contradiction, $L$ contains non-zero ideals. Since $\{0\}$ is an insulated prime and not a maximal ideal of $L$, it follows from Theorem 5.7, that $L \cong M_n(K[x,x^{-1}])$ for some positive integer $n$. But $M_n(K[x,x^{-1}])$ contains non-zero ideals which are not maximal and if they were to be insulated prime, they must be (non-zero) maximal graded ideals, by Theorem 5.7. But this is not possible, since $M_n(K[x,x^{-1}])$ is graded-simple, that is, it contains no non-zero graded ideals. Hence $L$ must be a simple ring. The converse is obvious.
(2) Suppose every non-zero ideal is insulated prime. If $L$ is a simple ring, we are done. Suppose $L$ contains non-zero ideals. We first claim that every ideal of $L$ must be graded. Suppose, on the contrary, there is a non-graded ideal $I$. By Theorem 2.1, $I$ will be of the form $I = I(H, S) + \sum_{t \in T} f_t(c_t) > 0$, where $H = I \cap E^0, S = \{v \in B_H : v^H \in I\}$, $T$ is a non-empty index set, for each $t \in T$, $c_t$ is a cycle without exits in $E \setminus (H, S)$ and $f_t(x) \in K[x]$ with a non-zero constant term. But then, for a fixed $t \in T$, we can construct the ideal $A = I(H, S) + < 1 - c_t^2 >$ and, as $1 - x^2$ is not an irreducible polynomial in $K[x, x^{-1}]$, $A$ is not a prime ideal of $L$ and hence not insulated prime. This contradiction shows that every non-zero ideal $I$ of $L$ is a graded ideal. By Theorem 5.7, every non-zero ideal of $L$ must then be a maximal graded ideal such that $L/I = M_n(K[x, x^{-1}])$ for some $n > 0$. We claim that $L$ has exactly one non-zero ideal. Suppose, on the contrary, there are two distinct non-zero ideals $A, B$ in $L$. By the maximality of $A$ and $B$, $A \not\subset B$ and $B \not\subset A$ and $A + B = L$. If $A \cap B$ is non-zero, then, by hypothesis, it is insulated prime and hence prime and so $AB \not\subseteq A \cap B$ will imply $A \not\subseteq A \cap B$ or $B \not\subseteq A \cap B$, a contradiction. Hence $A \cap B = 0$. But then $L = A \oplus B$ and this again leads to a contradiction since $A \cong L/B \cong M_n(K[x, x^{-1}])$ will then contain non-graded ideals contradicting the fact that every ideal of $L$ is graded. Thus $L$ contains exactly one non-zero ideal $I$ which is graded and $L/I \cong M_n(K[x, x^{-1}])$ for some positive integer $n$. Since such an ideal $I$ is always insulated prime, the converse holds.

\[ \square \]

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