AN IMPROVED EXPLICIT ESTIMATE FOR ζ(1/2 + it)

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Abstract. An explicit subconvex bound for the Riemann zeta function ζ(s) on the critical line s = 1/2 + it is proved. Previous subconvex bounds relied on an incorrect version of the Kusmin–Landau lemma. After accounting for the needed correction in that lemma, we recover and improve the record explicit bound for |ζ(1/2 + it)|.

1. Introduction

An important type of inequality in analytic number theory is upper bounds on the exponential sum $S = |\sum_{a \leq n < b} e^{2\pi i f(n)}|$ for some smooth phase function $f$. Suppose that $f'$ is monotonic and $\theta \leq f' \leq 1 - \theta$ for some $\theta \in [0, 1/2]$ and throughout $[a, b)$. One would like to derive inequalities of the form $S \leq A/\theta$, where $A$ is a constant. Among other applications, such results are used to derive explicit upper bounds for $|\zeta(1/2 + it)|$.

Kusmin [Kuz27, p. 239] writes that inequalities for $S$ were first introduced by Vinogradov in 1916. According to Landau [Lan28, p. 21], van der Corput was the first to prove a bound on $S$ depending only on $\theta$, independent of the length of the interval $[a, b)$. Using results in [Cor21, p. 58] and [Lan26, p. 221], which rely on the Poisson summation formula, it follows easily that $A = 4$ is admissible. Kusmin [Kuz27, p. 237] gave a geometric argument to improve this to $A = 1$. He then immediately noted that it follows from his proof that $A = 2/\pi$ is also admissible. Landau [Lan28, p. 21] refined the Kusmin bound to $S \leq \cot(\pi \theta/2)$ and constructed examples for which the equality $S = \cot(\pi \theta/2)$ holds.

In [CG04, Lemma 2], the bound $S \leq 1/(\pi \theta) + 1$ was derived. Unfortunately, this bound misses an extra factor of 2, i.e. the leading constant in the bound should be $2/\pi$ instead of $1/\pi$. This was recently pointed out by K. Ford and also discussed in a preprint by J. Arias de Reyna. Indeed, Kusmin [Kuz27, p. 237] and Landau [Lan28, p. 21] had shown that $A = 2/\pi$ is sharp. Therefore, there is no hope of recovering the constant $1/\pi$ in [CG04, Lemma 2] in the general case, but only in special (though important) cases such as when $f$ is linear.

The incorrect constant $1/\pi$ has impacted all published explicit subconvex bounds on $|\zeta(1/2 + it)|$. In particular, the bound $|\zeta(1/2 + it)| \leq 0.63t^{1/6}\log t$ in [Hia16, Theorem 1.1] is affected. This bound relied on an explicit $B$ process (from the method of exponent pairs) that is derived in [CG04, Lemma 3]. In turn, this explicit $B$ process relied on the version of the Kusmin–Landau lemma in [CG04, Lemma 2] from which the incorrect constant $1/\pi$ arises. After accounting for the correct constant $2/\pi$ in the Kusmin–Landau lemma, the constant 0.63 in [Hia16, Theorem 1.1] increases substantially to 0.77. In other words, the revised bound becomes $|\zeta(1/2 + it)| \leq 0.77t^{1/6}\log t$ for $t \geq 3$.

Date: July 2022.
Since the missing factor of 2 in [CG04, Lemma 3] is sizeable and the method of proof in [Hia16] is already optimized, our savings had to come in small quantities from multiple places.

We specialize the phase function to our specific application of bounding zeta, and derive better explicit $B$ and $AB$ processes in lemmas 2.5 and 2.6, as well as a generalized Kusmin-Landau lemma in Lemma 2.3.

Moreover, we are exceedingly careful in treating boundary terms in the explicit $A$ and $B$ processes we derive. Boundary term may be all that arises in the intermediate region where $t$ is too large for the Riemann–Siegel–Lehman bound to be useful, but too small for the asymptotic savings from the $A$ and $B$ processes to be realized. This intermediate region (bottleneck region) is the subject of subsection 3.3.

Also, our treatment for large $t$ improves on [Hia16] in one important aspect that enables reducing the coefficient of $t^{1/6} \log t$ considerably, as detailed at the end of subsection 3.4.

Put together, our main result is to recover and improve the constant $0.63$.

**Theorem 1.1.** For $t \geq 3$, we have

$$|\zeta(1/2 + it)| \leq 0.618 t^{1/6} \log t.$$ 

Note that if one employs the Riemann–Siegel–Lehman bound for any range of $t$ at all, then the leading constant cannot break 0.541. The constant we obtain is not too far from this barrier.

1.1. **Notation.** Throughout this work let $\|x\|$ denote the distance to the integer nearest to $x$, i.e. $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. We write $e(x) := \exp(2\pi ix)$.

2. **Required lemmas**

**Lemma 2.1.** If $t \geq 200$ and $n_1 := \lfloor \sqrt{t/(2\pi)} \rfloor$, then

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{n=1}^{n_1} n^{-1/2 + it} \right| + R(t)$$

where $R(t) := 1.48t^{-1/4} + 0.127t^{-3/4}$.

**Proof.** One starts with the Riemann–Siegel formula, and applies the triangle inequality to the main sum and to the Gabcke remainder term, bounding the latter by $R(t)$. See Lemma 2.1 in [Hia16].

**Lemma 2.2** (Riemann-Siegel-Lehman bound). If $t \geq 200$, then

$$|\zeta(1/2 + it)| \leq \frac{4t^{1/4}}{(2\pi)^{1/4}} - 2.08.$$  

**Proof.** Follows from Lemma 2.1 on bounding the main sum for $n > 5$ by an integral and using monotonicity to bound $R(t)$ for $t \geq 200$. See Lemma 2.3 in [Hia16].

**Lemma 2.3** (Generalised Cheng-Graham lemma). Let $f(x)$ be a real-valued function with a monotonic and continuous derivative on $[a, b]$, satisfying

$$\ell + U^{-1} \leq f'(x) \leq \ell + 1 - V^{-1}, \quad x \in [a, b],$$
for some $U, V > 1$ and some integer $\ell$. Then,

$$\left| \sum_{a \leq n < b} e(f(n)) \right| \leq \frac{U + V}{\pi}.$$ 

Proof. The proof follows from a similar argument to Lemma 2.1 in Patel [Pat p] and Kuzmin [Kuz27] and Landau [Lan28]. See Section 4 for details. This generalized lemma is essential to the cases considered at the end of the proof of Lemma 2.5 \end{proof}

Lemma 2.4 (Weyl differencing). Let $f(n)$ be a real-valued function and $L$ and $M$ positive integers. Then

$$\left| \sum_{n=N+1}^{N+L} e(f(n)) \right|^2 \leq \left( \frac{L + M - 1}{M} \right) \left( L + 2 \sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) |s'_m(L)| \right)$$

where if $m < L$, then

$$s'_m(L) := \sum_{n=N+1}^{N+L-1} e(f(r + m) - f(r)),$$

and if $m \geq L$, then $s'_m(L) = 0$.

Proof. See Cheng and Graham [CG04, Lemma 5] and Platt and Trudgian [PT15]. Lemma 5 in [CG04] appears with $\max_{L \leq L} |s'_m(L)|$ instead of $|s'_m(L)|$. It was pointed out in [Hia16], though, that one can remove the max by using the more precise form presented at the bottom of page 1273 in [CG04]. \end{proof}

Lemma 2.5 (Improved second derivative test). Let $m, r, K$ be positive integers, and let $t$ and $K_0$ be a positive numbers. Suppose $K \geq K_0 > 1$ and $m < K$. Let

$$g(x) := \frac{t}{2\pi} \log \left( 1 + \frac{m}{rKx} \right), \quad W := \frac{\pi(r+1)^3 K^3}{t}, \quad \lambda := \frac{(1+r)^3}{r^3}.$$

So, by construction,

$$\frac{m}{W} \leq |g''(x)| \leq \frac{m\lambda}{W}, \quad (0 \leq x \leq K - m).$$

For each positive integer $L \leq K$, and each positive integer $m < L$,

$$\left| \sum_{n=0}^{L-1-m} e(g(n)) \right| \leq \frac{4\mu K}{\sqrt{\pi W}} m^{1/2} + \frac{\mu K}{W} m + 4 \sqrt{\frac{W}{\pi}} m^{-1/2} + 2 - \frac{4}{\pi},$$

where

$$\mu := \frac{1}{2} \lambda^{2/3} \left( 1 + \frac{1}{(1-K_0^{-1})\lambda^{1/3}} \right).$$

If $m \geq L$ or $L = 1$, then the sum on the left-side is zero and the bound still holds.

Proof. See Section 5. \end{proof}

Remark. We have $g(x) = f(x + m) - f(x)$, where $f(x)$ is defined in Lemma 2.6. Thus, $g(x)$ is the phase function that arises after we apply the Weyl differencing from Lemma 2.4 to a contiguous portion of the main sum in Lemma 2.2.
Lemma 2.6 (Improved third derivative test). Let $r$ and $K$ be positive integers, and let $t$ and $K_0$ be positive numbers. Suppose $K \geq K_0 > 1$. Let

$$f(x) := \frac{t}{2\pi} \log (rK + x).$$

Furthermore let $W$ and $\lambda$ be as defined in Lemma 2.5, so that

$$\frac{1}{W} \leq |f'''(x)| \leq \frac{\lambda}{W}.$$

For each positive integer $L \leq K$, each integer $N$, and any $\eta > 0$,

$$\left| \sum_{n=N+1}^{N+L} e(f(n)) \right|^2 \leq \left( \frac{K}{W^{1/3}} + \eta \right) (\alpha K + \beta W^{2/3})$$

where

$$\alpha := \frac{1}{\eta} + \frac{\eta \mu}{3W^{1/3}} + \frac{\mu}{3W^{2/3}} + \frac{32\mu}{15\sqrt{\pi}} \sqrt{\eta + W^{-1/3}},$$

$$\beta := \frac{32}{3\sqrt{\pi} \eta} + \left( 2 - \frac{4}{3} \right) \frac{1}{W^{1/3}}.$$

Here, $\mu$ is defined as in Lemma 2.5.

Proof. See Section 6.

Remark. Versions of these derivative tests for a general phase function, as well as several other derivative tests, can be found in [Pat p].

3. Proof of Theorem 1

We divide the proof into four regions.

3.1. Proof for $3 \leq t < 200$. In this range we rely on the interval-arithmetic computations carried out in Hiary [Hia16], which established

$$|\zeta(1/2 + it)| \leq 0.595t^{1/6} \log t$$

for $3 \leq t < 200$.

3.2. Proof for $200 \leq t < 5.5 \cdot 10^7$. For this region we use the Riemann-Siegel-Lehman formula combined with the triangle inequality. Firstly, in preparation for using Lemma 2.2 we note

$$\frac{4t^{1/4}}{(2\pi)^{1/4}} - 2.08 < 0.592t^{1/6} \log t, \quad 200 \leq t \leq 10^7.$$

This can be seen by verifying that the difference of the two sides is unimodal (monotonically increasing then monotonically decreasing), and so it suffices to check that the difference is positive at the endpoints 200 and $10^7$. Hence, our main theorem follows from Lemma 2.2 for that range of $t$.

Assume now that $10^7 \leq t < 5.5 \cdot 10^7$. We follow a similar argument to Lemma 2.3 in [Hia16]. By Lemma 2.2

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{n=1}^{n_1} n^{-1/2+it} \right| + 1.48t_0^{-1/4} + 0.127t_0^{-3/4}, \quad t \geq t_0$$
Also, let \( t_0 = 10^7 \). Next, we observe that if \( h \) is a real-valued function such that \( h''(x) > 0 \) for \( a - 1/2 \leq x \leq b + 1/2 \), then by Jensen’s inequality

\[
\sum_{n=a}^{b} h(n) = \frac{b}{n-\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(x) \, dx \leq \frac{b}{n-\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(x) \, dx = \int_{a-\frac{1}{2}}^{b+\frac{1}{2}} h(x) \, dx. \tag{3.1}
\]

Therefore, using \( h(x) = x^{-1/2} \) and the fact that \( n_1 \geq \lceil \sqrt{t_0/(2\pi)} \rceil = 1261 \),

\[
\left| \sum_{n=1}^{n_1} n^{-1/2 + it} \right| \leq \sum_{n=1}^{\lfloor t_0/\pi \rfloor} \left| \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \int_{1260.5}^{n_1+\frac{1}{2}} dx \right|
\leq 69.575 + 2\sqrt{n} - 2\sqrt{1260.5} + \frac{1}{\sqrt{n}} \int_{n_1}^{n_1+\frac{1}{2}} dx
\leq 2\sqrt{n} + \frac{1}{2\sqrt{1261}} - 1.432
= \frac{2t^{1/4}}{(2\pi)^{1/4}} - 1.417.
\]

However, \( 1.48t_0^{-1/4} + 0.127t_0^{-3/4} \leq 0.027 \), so

\[
|\zeta(1/2 + it)| \leq \frac{4t^{1/4}}{(2\pi)^{1/4}} - 2.807 \leq 0.618t^{1/6} \log t
\]

for \( 10^7 \leq t < 5.5 \cdot 10^7 \). One need verify the last inequality at the endpoints \( 10^7 \) and \( 5.5 \cdot 10^7 \) since the difference of the two sides is monotonic in between.

3.3. **Proof for** \( 5.5 \cdot 10^7 \leq t < 10^{12} \). In this range of \( t \) we use the following modified third-derivative test in Lemma 2.6.

Throughout this subsection, let \( r_0 > 1 \) be an integer and \( 0 < \phi < \frac{1}{2} \) be a constant, both to be chosen later. Furthermore, let

\[
K := \lceil t^{\phi} \rceil, \quad n_1 := \lceil t^{1/2} \pi \rceil, \quad R := \lfloor n_1/K \rfloor.
\]

Also, let \( t_0 \geq 5.5 \cdot 10^7 \), to be chosen later, and suppose \( t \geq t_0 \). By Lemma 2.6 we have

\[
|\zeta(1/2 + it)| \leq T_1 + T_2,
\]

where

\[
T_1 := 2 \sum_{1 \leq n < r_0 K} \frac{1}{\sqrt{n}} + 1.48t_0^{-1/4} + 0.127t_0^{-3/4},
\]

\[
T_2 := 2 \sum_{r=r_0}^{R-1} \left| \sum_{rK \leq n < (r+1)K} e^{it \log n} \sqrt{n} \right| + 2 \left| \sum_{rK \leq n \leq n_1} e^{it \log n} \sqrt{n} \right|.
\]

Define

\[
I_0(r_0, t_0) := 2 \sum_{n=1}^{\lceil r_0 t_0^{\phi} \rceil - 1} \frac{1}{\sqrt{n}} - 4 \sqrt{\lceil r_0 t_0^{\phi} \rceil} - 1.
\]
Then, noting that $K \leq t^{\phi} + 1$, and following the arguments in Hiary [Hia16],

$$T_1 \leq 4 \sqrt{r_0K} + I_\phi(r_0, t_0) + 1.48t_0^{-1/4} + 0.127t_0^{-3/4}$$

$$\leq 4 \sqrt{r_0} \left( 1 + t_0^{-\phi} \right) t^{\phi/2} + I_\phi(r_0, t_0) + 1.48t_0^{-1/4} + 0.127t_0^{-3/4}, \quad (3.2)$$

for all $t \geq t_0$. Meanwhile, by partial summation,

$$\sum_{r=r_0}^{R-1} \sum_{rK \leq n < (r+1)K} \frac{e^{it \log n}}{\sqrt{n}} \leq \sum_{r=r_0}^{R-1} \frac{1}{\sqrt{rK}} \max_{\Delta \leq K} \left| \sum_{k=0}^{\Delta-1} e^{it \log (rK+k)} \right|. \quad (3.3)$$

So, since $n_1 \leq (R+1)K$,

$$\left| \sum_{rK \leq n \leq n_1} \frac{e^{it \log n}}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{rK}} \max_{\Delta \leq K} \left| \sum_{k=0}^{\Delta-1} e^{it \log (rK+k)} \right|, \quad (3.4)$$

it follows

$$T_2 \leq 2 \sum_{r=r_0}^{R} \frac{1}{\sqrt{rK}} \max_{\Delta \leq K} \left| \sum_{k=0}^{\Delta-1} e^{it \log (rK+k)} \right|. \quad (3.5)$$

We apply Lemma 2.6 with $K_0 := t_0^{\phi} > 1$ to obtain for any $\eta > 0$,

$$T_2 \leq 2 \sum_{r=r_0}^{R} \frac{1}{\sqrt{rK}} \max_{\Delta \leq K} \left| \sum_{k=0}^{\Delta-1} e^{it \log (rK+k)} \right| \leq 2 \frac{1}{\pi^{1/6}} \sum_{r=r_0}^{R} \frac{1}{\sqrt{r(r+1)}} \left[ \frac{K}{W^{1/3}} + \eta \right] \left( \alpha K + \beta W^{2/3} \right)$$

$$= \frac{2t^{1/6}}{\pi^{1/6}} \sum_{r=r_0}^{R} \frac{1}{\sqrt{r(r+1)}} \left[ \frac{\eta \alpha W^{1/3}}{K} + \frac{\beta W^{2/3}}{K} + \frac{\eta \beta W}{K^2} \right]. \quad (3.6)$$

where $W, \lambda, \alpha, \beta, \alpha, \beta$, and $\mu$ are defined in Lemma 2.6.

Using the same trick with the Jensen inequality as in the previous section, we observe that

$$\sum_{r=r_0}^{R} \frac{1}{\sqrt{r(r+1)}} \leq \sum_{r=r_0}^{R} \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{dx}{\sqrt{x(x+1)}} = \int_{r_0-\frac{1}{2}}^{R+\frac{1}{2}} \frac{dx}{\sqrt{x(x+1)}} = \frac{2}{\pi} \left[ \arcsinh \sqrt{r_0} - \arcsinh \sqrt{R+1} \right]$$

$$= 2 \sinh^{-1} \sqrt{R + \frac{1}{2}} - 2 \sinh^{-1} \sqrt{r_0 - \frac{1}{2}} \quad (3.7)$$

where the inequality follows from using $h(x) = 1/\sqrt{x(x+1)}$ in (3.1). In addition,

$$2 \sinh^{-1} \sqrt{x} = \log x + 2 \log \left( 1 + \sqrt{1 + \frac{1}{x}} \right) \quad (3.8)$$

for $x \geq 0$, and

$$R \geq R_0 := \left[ \sqrt{\frac{t_0/2\pi - 1}{t_0^{\phi} + 1}} - 1 \right],$$

hence

$$2 \sinh^{-1} \sqrt{R + \frac{1}{2}} = \log R + \log \left( 1 + \frac{1}{2R} \right) + 2 \log \left( 1 + \sqrt{1 + \frac{1}{R + \frac{1}{2}}} \right) \leq \log R + J(R_0),$$
where

\[ J(R_0) := \log \left( 1 + \frac{1}{2R_0} \right) + 2 \log \left( 1 + \sqrt{1 + \frac{1}{R_0 + \frac{1}{2}}} \right). \]

Next, let

\[ \rho := \frac{1}{\sqrt{2\pi}^{1/6}} \left( 1 + \frac{1}{R} \right). \]

Since \( R \leq t^{1/2 - \phi}/\sqrt{2\pi} \) and \( K \leq t^\phi + 1 \), we have

\[
W^{1/3} = \frac{\pi^{1/3}(r + 1)K}{t^{1/3}} \leq \frac{\pi^{1/3}(R + 1)(t^\phi + 1)}{t^{1/3}} \leq \rho t^{1/6} \left( 1 + \frac{1}{t^\phi} \right) \tag{3.9}
\]

and

\[
\frac{W^{1/3}}{K} = \frac{\pi^{1/3}(r + 1)K}{t^{1/3}K^{1/3}} \leq \frac{\pi^{1/3}R}{t^{1/3}} \left( 1 + \frac{1}{R} \right) \leq \rho t^{1/6 - \phi}. \tag{3.10}
\]

Therefore,

\[
\frac{\eta\alpha W^{1/3}}{K} \leq \frac{\eta\alpha \rho}{t^{\phi - 1/6}},
\]

\[
\frac{\beta W^{2/3}}{K} = \frac{\beta W^{1/3}}{K} \left( \frac{W^{1/3}}{K} \right)^2 \leq \frac{\beta \rho^2 (1 + t^{-\phi})}{t^{\phi - 1/3}}, \tag{3.11}
\]

\[
\frac{\eta\beta W}{K^2} = \frac{\eta\beta W^{1/3}}{K} \left( \frac{W^{1/3}}{K} \right)^2 \leq \frac{\eta\beta^3 (1 + t^{-\phi})^2}{t^{2\phi - 1/2}}.
\]

We apply the above inequalities to (3.6), together with the inequality

\[ \rho_0 := \frac{1}{\sqrt{2\pi}^{1/6}} \left( 1 + \frac{1}{R_0} \right) \geq \rho, \]

to obtain

\[
T_2 \leq \frac{2t^{1/6}}{\pi^{1/6} \rho} \sum_{r=r_0}^{R} \frac{1}{\sqrt{r(r + 1)}} \sqrt{\alpha + \frac{\eta\alpha \rho_0}{t^{\phi - 1/6}} + \frac{\beta \rho^2_0 (1 + t^{-\phi})}{t^{\phi - 1/3}} + \frac{\eta\beta^3_0 (1 + t^{-\phi})^2}{t^{2\phi - 1/2}}} \tag{3.12}
\]

We observe since \( \lambda \) is monotonically decreasing with \( r \), \( \mu \) is decreasing with \( r \). Since, in addition, \( W \) is monotonically increasing with \( r \), \( \alpha \) and \( \beta \) are both decreasing with \( r \). Denoting the values of \( W \), \( \alpha \) and \( \beta \) at \( r = r_0 \) by \( W_0 \), \( \alpha_0 \) and \( \beta_0 \), we see that \( W \geq W_0 \), \( \alpha \leq \alpha_0 \) and \( \beta \leq \beta_0 \). Therefore, using (3.7) and the subsequent estimates, we obtain

\[
T_2 \leq \frac{2t^{1/6}}{\pi^{1/6}} \left[ \left( \frac{1}{2} - \phi \right) \log t - \log \sqrt{2\pi} + J(R_0) - 2 \sinh^{-1} \sqrt{r_0 - \frac{1}{2}} \right] \kappa_\phi \tag{3.13}
\]

where, as we also have \( \phi > 1/3 \) and \( t \geq t_0 \), \( \kappa_\phi \) may be taken to be

\[
\kappa_\phi := \sqrt{\alpha_0 + \frac{\eta\alpha_0 \rho_0}{t_0^{\phi - 1/6}} + \frac{\beta_0 \rho^2_0 (1 + t_0^{-\phi})}{t_0^{\phi - 1/3}} + \frac{\eta\beta_0^3_0 (1 + t_0^{-\phi})^2}{t_0^{2\phi - 1/2}}} \tag{3.14}
\]

Explicitly, combining with (3.2), yields

\[ |\zeta(1/2 + it)| \leq a_1 t^{1/6} \log t + a_2 t^{1/6} + a_3 t^{\phi/2} + a_4, \quad t \geq t_0, \]
where,

\[
\begin{align*}
a_1 & := \frac{1 - 2\phi}{\pi^{1/6}} \kappa_\phi, \\
a_2 & := -\frac{2}{\pi^{1/6}} \left[ \log \sqrt{2\pi} - J(R_0) + 2 \sin^{-1} \left( \sqrt{r_0 - \frac{1}{2}} \right) \right] \kappa_\phi, \\
a_3 & := 4 \sqrt{r_0 \left( 1 + t_0^{-\phi} \right)}, \\
a_4 & := I_\phi(r_0, t_0) + 1.48t_0^{-1/4} + 0.127t_0^{-3/4}.
\end{align*}
\]

(3.15)

We choose \( \phi = 0.3414, \eta = 1.8, r_0 = 4 \) and \( t_0 = 5.5 \cdot 10^7 \). This choice of \( r_0 \) is valid since it satisfies \( r_0 \leq R_0 \). Also, in view of the chosen values for \( t_0 \) and \( \phi \), the inequality \( W_0 \geq \pi(r_0 + 1)^3[K_0]^3/t_0 \), for \( t \geq t_0 \), is valid. We use this inequality to simplify the bounds for \( a_0 \) and \( \beta_0 \), considering they are both monotonically decreasing with \( W_0 \). Together, we obtain

\[
|\zeta(1/2 + it)| \leq 0.59289t^{1/6} \log t - 8.0314t^{1/6} + 8.0092t^{17/6} - 2.8796, \quad t \geq 5.5 \cdot 10^7 \leq 0.618t^{1/6} \log t, \quad \text{for } 5.5 \cdot 10^7 \leq t < 10^8.
\]

Similarly, choosing \( \phi = 0.3414, \eta = 1.8, r_0 = 4 \) and \( t_0 = 10^8 \), we obtain

\[
|\zeta(1/2 + it)| \leq 0.58589t^{1/6} \log t - 8.0115t^{1/6} + 8.0074t^{17/6} - 2.8843, \quad t \geq 10^8 \leq 0.618t^{1/6} \log t, \quad \text{for } 10^8 \leq t < 8.5 \cdot 10^{10}.
\]

Finally, choosing \( \phi = 0.3414, \eta = 1.8, r_0 = 4 \) and \( t_0 = 8.5 \cdot 10^{10} \), we obtain

\[
|\zeta(1/2 + it)| \leq 0.55305t^{1/6} \log t - 7.8629t^{1/6} + 8.0008t^{17/6} - 2.9111, \quad t \geq 8.5 \cdot 10^{10} \leq 0.618t^{1/6} \log t, \quad \text{for } 8.5 \cdot 10^{10} \leq t < 10^{12},
\]

hence the desired result holds for \( 5.5 \cdot 10^7 \leq t < 10^{12} \).

### 3.4. Proof for \( t \geq 10^{12} \)

For the region \( t \geq 10^{12} \) we use a similar method as the previous subsection, but with \( \phi = 1/3 \). Analogously to before, let \( K = [t^{1/3}] \), \( n_1 = \lfloor \sqrt{t/2\pi} \rfloor \), \( R = \lfloor n_1/K \rfloor \) and, this time, let \( t_0 = 10^{12} \). Suppose \( t \geq t_0 \).

We bound \( T_2 \) differently, by splitting the square root in (3.10) as follows. Let

\[
\mathcal{B}_r := \alpha + \frac{\eta_0 W^{1/3}}{K} + \frac{\beta W^{2/3}}{K} + \frac{\eta_3 W}{K^2},
\]

so that, recalling (3.10), we have

\[
T_2 \leq \frac{2t^{1/6}}{\pi^{1/6}} \sum_{r=r_0}^{R} \frac{\mathcal{B}_r}{\sqrt{r(r+1)}}.
\]

(3.17)

Substituting \( \phi = 1/3 \) into (3.11), and noting that \( t \geq t_0 \) and \( \rho \leq \rho_0 \), we deduce

\[
\frac{\eta_0 W^{1/3}}{K} \leq \eta_0 t_0^{-1/6} \rho_0, \quad \frac{\eta_3 W}{K^2} \leq \eta_3 t_0^{-1/6} \rho_0^3 (1 + t_0^{-1/3})^2.
\]

(3.18)

Next, using the definition of \( W \), and since \( K \leq t^{1/3} + 1 \), \( r \geq r_0 \) and \( t \geq t_0 \),

\[
\frac{1}{r(r+1)} \frac{\beta W^{2/3}}{K} = \frac{\beta}{r(r+1)} \frac{\pi^{2/3}(r+1)^2 K}{t^{2/3}} \leq \frac{\beta \pi^{2/3}(1 + r_0^{-1})(1 + t_0^{-1/3})}{t^{2/3}}.
\]

(3.19)
Furthermore, using $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, valid for any nonnegative numbers $x$ and $y$, and combining (3.16), (3.19), (3.18) and (3.21), as well as the observation $\alpha \leq \alpha_0$ and $\beta \leq \beta_0$ which follows by the same reasoning as in subsection 3.3, we thus see

$$\sqrt{\frac{\beta_0}{r(r+1)}} \leq \sqrt{\frac{1}{r(r+1)}} \left( \frac{\alpha + \eta_0 W^{1/3}}{K} + \frac{\eta_0 W^2}{K} \right) + \sqrt{\frac{1}{r(r+1)}} \frac{\beta_0 W^{2/3}}{K},$$

for all $t \geq t_0$, which follows by the same reasoning as in subsection 3.3, we thus see

$$\frac{\alpha + \eta_0 W^{1/3}}{K} + \frac{\eta_0 W^2}{K} \leq \frac{\alpha_0 + \eta_0 W^{1/3} \rho_0 + \eta_0 W^2 \rho_0}{K},$$

valid for $t \geq t_0$, and implying that

$$2 \sinh^{-1} \sqrt{R + \frac{1}{2}} = \log \left( \frac{R + 1}{2} \right) + 2 \log \left( 1 + \log \left( 1 + \frac{1}{39.5} \right) \right) \leq \log R + 1.412, \quad (3.21)$$

in that range of $t$. Note, in addition, that $R \leq t^{1/6}/\sqrt{2\pi}$. Therefore, combined with (3.21) we obtain

$$\sum_{r=r_0}^{R} \frac{1}{\sqrt{r(r+1)}} \leq \frac{1}{6} \log t + \omega_0$$

(3.22)

where

$$\omega_0 := -\log \sqrt{2\pi} + 1.412 - 2 \sinh^{-1} \sqrt{r_0 - 1/2}. \quad (3.23)$$

Using this together with (3.20), it follows on recalling (3.17) that

$$T_2 \leq \frac{2t^{1/6}}{\pi^{1/6}} \left( \frac{\log t}{6} + \omega_0 \right) \kappa'_0 + \frac{2t^{1/6}}{\pi^{1/3}} \sqrt{\beta_0 (1 + r_0^{-1})(1 + t_0^{-1/3})},$$

(3.24)

for $t \geq t_0$.

As for $T_1$, we bound it the same way as in (3.2) but with $\phi = 1/3$, which gives

$$T_1 \leq 4 \sqrt{r_0 \left( 1 + t_0^{-1/3} \right)} t^{1/6} + I_4 (r_0, t_0) + 1.48 t_0^{-1/4} + 0.127 t_0^{-3/4}. \quad (3.25)$$

for all $t \geq t_0$.

Finally, combining (3.23) and (3.25), we arrive at

$$|\zeta(1/2 + it)| \leq b_1 t^{1/6} \log t + b_2 t^{1/6} + b_3,$$

for all $t \geq t_0$, where

$$b_1 := \frac{\kappa'_0}{3\pi^{1/6}},$$

$$b_2 := 4 \sqrt{r_0 \left( 1 + t_0^{-1/3} \right)} + \frac{2}{\pi^{1/6}} \omega_0 \kappa'_0 + \frac{\sqrt{2}}{\pi^{1/3}} \sqrt{\beta_0 (1 + r_0^{-1})(1 + t_0^{-1/3})},$$

$$b_3 := I_4 (r_0, t_0) + 1.48 t_0^{-1/4} + 0.127 t_0^{-3/4}.\]
Choosing \( r_0 = 4 \) and \( \eta = 1.6 \), and using the inequality \( W_0 \geq \pi (r_0 + 1)^3 \) to remove remaining dependence of \( \alpha_0 \) and \( \beta_0 \) on \( t \), yields
\[
|\zeta(1/2 + it)| \leq 0.478013 t^{1/6} \log t + 3.853165 t^{1/6} - 2.914229 \\
\leq 0.618 t^{1/6} \log t, \quad (t \geq t_0),
\]
as required.

We point out one of main reasons for the improvement over [Hia16] obtained in this subsection. After we invoke the explicit \( AB \) process from Lemma 2.6 to arrive at (3.3), we pay greater attention to the cross term \((K/W^{1/3})(\beta W^{2/3}) = \betaKW^{1/3}\). Specifically, we arrange for this cross term to contribute to the coefficient of \( t^{1/6} \) in the overall bound in (3.24), rather than to the coefficient of \( t^{1/6} \log t \), as done in [Hia16]. This saves a factor of \( \log t \) from the contribution of this term, which is a considerable saving.

4. Proof of Lemma 2.3

The proof proceeds similarly to Lemma 2 in [CG04] and Lemma 2.1 in Patel [Pat n], with only a few differences. We include the complete proof here for convenience.

Let \( f \) be a function satisfying the conditions of Lemma 2.6, that is, \( f \) has a continuous and monotonically increasing derivative on \([a, b]\) and
\[
\ell + U^{-1} \leq f'(x) \leq \ell + 1 - V^{-1}, \quad x \in [a, b).
\]
for some \( U, V > 1 \) and some integer \( \ell \). We may assume that \( \ell = 0 \), i.e. that \( U^{-1} \leq f'(x) \leq 1 - V^{-1} \), since
\[
\left| e(-\ell n) \sum_{a \leq n < b} e(f(n)) \right| = \left| \sum_{a \leq n < b} e(f(n) - \ell n) \right|.
\]
We may also assume that \( f'(x) \) is increasing, since we may replace \( f(x) \) with \(-f(x)\) without changing the magnitude of the sum.

Now, define \( g(x) := f(x + 1) - f(x) \). Since by assumption \( f'(x) \) is increasing in \( x \) over \( x \in [a, b) \), \( g'(x) = f'(x + 1) - f'(x) \geq 0 \) over \( x \in [a, b - 1] \). Therefore, \( g(x) \) is increasing in \( x \) over that interval. Furthermore, by the mean-value theorem, \( g(x) = (x + 1 - x)f'(\xi) = f'(\xi) \) for some \( \xi \in (x, x + 1) \). Therefore,
\[
U^{-1} \leq f'(x) \leq f'(\xi) = g(x) \leq f'(x + 1) \leq 1 - V^{-1},
\]
for every \( x \in [a, b - 1] \). Thus, for instance, \( 0 < g(x) < 1 \) over \( x \in [a, b - 1] \).

Next, let
\[
G(n) := \frac{1}{1 - e(g(n))} = \frac{1 + i \cot(\pi g(n))}{2},
\]
so that
\[
G(n) [e(f(n)) - e(f(n + 1))] = \frac{e(f(n)) - e(f(n + 1))}{1 - e(g(n))} = \frac{e(f(n))[1 - e(g(n))]}{1 - e(g(n))} = e(f(n)),
\]
and also
\[
G(n) - G(n - 1) = \frac{\cot(\pi g(n - 1)) - \cot(\pi g(n))}{2i}.
\] (4.5)

Let \(L = \lceil a \rceil\). If \(b\) is not an integer, let \(M = \lfloor b \rfloor\). If \(b\) is an integer, let \(M = b - 1\). In either cases, the summation over \(n \in [a, b)\) is the same as the summation over \(n \in [L, M]\), and \(g'(x)\) is increasing in \(x \in [L, M - 1]\).

Suppose that \(M = L\), so there is only one term in the sum. Then, by the trivial bound, we have
\[
\left| \sum_{a \leq n < b} e(f(n)) \right| \leq 1 \leq \frac{U + V}{\pi}.
\] (4.6)

Here we use the fact that the condition (4.1) implies that
\[
U - 1 \leq 1 - V - 1,
\]
so by the inequality of harmonic and arithmetic means,
\[
U + V \geq \frac{4}{U - 1 + V - 1} \geq 4.
\] (4.7)

Therefore, the result of the lemma follows when \(M = L\).

Next, suppose \(M - L \geq 1\). By (4.4), and after a few rearrangements,
\[
\left| \sum_{n=L}^{M} e(f(n)) \right| = \left| \sum_{n=L}^{M-1} G(n)[e(f(n)) - e(f(n + 1))] + e(f(M)) \right|
\]
\[
= \left| \sum_{n=L+1}^{M-1} e(f(n))(G(n) - G(n - 1)) + e(f(L))G(L) + e(f(M))(1 - G(M - 1)) \right|
\]
\[
\leq |e(f(L))G(L)| + |e(f(M))(1 - G(M - 1))| + \sum_{n=L+1}^{M-1} |e(f(n))(G(n) - G(n - 1))|
\]
\[
= |G(L)| + |1 - G(M - 1)| + \sum_{n=L+1}^{M-1} |G(n) - G(n - 1)|.
\]

Note that if \(M - L = 1\), then the last sum is empty and should be interpreted as equal to 0. Now, since \(g(x)\) is increasing in \(x \in [L, M - 1]\), \(\cot(\pi g(x))\) is decreasing in \(x\) over the same interval. Hence, by (4.5),
\[
|G(n) - G(n - 1)| = \frac{\cot(\pi g(n - 1)) - \cot(\pi g(n))}{2}.
\]
for \( L + 1 \leq n \leq M - 1 \). Therefore,

\[
\left| \sum_{n=L}^{M} e(f(n)) \right| = \sum_{n=L+1}^{M-1} \left| \left( \cot(\pi g(n-1)) - \cot(\pi g(n)) \right) + \frac{1}{2} + \frac{i}{2} \cot(\pi g(L)) \right| + \left| \frac{1}{2} - \frac{i}{2} \cot(\pi g(M-1)) \right| \tag{4.8}
\]

\[
= \frac{1}{2} \cot(\pi g(L)) - \frac{1}{2} \cot(\pi g(M-1)) + \frac{1}{2} \sin(\pi g(L)) + \frac{1}{2} \sin(\pi g(M-1)) \tag{4.9}
\]

\[
= \frac{1}{2} \cot\left( \frac{\pi}{2} g(L) \right) + \frac{1}{2} \tan\left( \frac{\pi}{2} g(M-1) \right) \tag{4.10}
\]

\[
\leq \frac{1}{2} \cot\left( \frac{\pi}{2U} \right) + \frac{1}{2} \tan\left( \frac{\pi}{2} \left( 1 - \frac{1}{V} \right) \right) \tag{4.11}
\]

\[
\leq \frac{U}{\pi} + \frac{V}{\pi},
\]

as required. Going from (4.9) to (4.10), we combined the various terms using the formulas \((1 + \cos(\pi x))/\sin(\pi x) = \cot(\pi x/2)\) and \((1 - \cos(\pi x))/\sin(\pi x) = \tan(\pi x/2)\), valid for \(0 < x < 1\). In (4.11) we used the monotonicity of \(\cot(\pi x)\) and \(\tan(\pi x)\) over \(0 < x < 1/2\). In the last line we used the inequalities \(\cot(\pi x) < 1/(\pi x)\) and \(\tan(\pi x/2) < 2/(\pi(1 - x))\), valid for \(0 < x < 1\). Lastly, we note that passing from (4.8) to (4.9) presents no difficulty if \(M - L = 1\) since \(\cot(\pi g(L)) - \cot(\pi g(M-1)) = 0\) in this case.

5. Proof of Lemma 2.5

We divide the summation interval into about \(2k\) suitable subintervals, chosen so that we may apply the generalized Cheng–Graham Lemma 2.3 on about half of the subintervals and the trivial bound on the remaining half. The special form of \(g\) allows a sharper bound on \(k\), so that in the definition of \(\mu\) in Lemma 2.6 the contribution of \(\lambda\) can be reduced to \(\lambda^{2/3}\). Moreover, we adjust the definition of the boundary subintervals (determined by \(C_0\) and \(C_k\) below) to further reduce the number of subintervals in the sum. Recall that \(r\) is a positive integer, \(\lambda = (r + 1)^3/r^3\), and \(K \geq K_0 > 1\) where \(K\) is an integer. We will use the following elementary inequality.

\[
\frac{rK}{rK + K - 1} = \frac{r}{r + (1 - 1/K)} \leq (1 - K^{-1})^{-1} \frac{r}{r + 1} \leq \frac{1}{(1 - K_0^{-1})\lambda^{1/3}}. \tag{5.1}
\]

Let us recall that \(W = \pi(r+1)^3K^3/t\) and the phase function \(g(x) = f(x+m) - f(x)\) where \(f(x) = \frac{1}{2\pi} \log(rK + x)\). We compute, for \(m < L \leq K\),

\[
g'(L - 1 - m) - g'(0) \leq g'(K - 1 - m) - g'(0)
\]

\[
= \frac{t}{2\pi r^3K^3} \cdot \frac{m}{rK + K - 1 - m} \left( \frac{rK(K - 1 - m)}{rK + K - 1 - m} \right) \left( 1 + \frac{rK}{rK + K - 1} \right) \leq \frac{\lambda}{2W} \cdot m \cdot K \lambda^{-1/3} \cdot 1 \left( \frac{1}{K_0^{-1}} \right) \lambda^{1/3},
\]
where in the first line we used that \( g' \) is monotonically increasing, and in the last line we used the inequality (5.1), as well as the observation
\[
\frac{rK(K - 1 - m)}{rK + K - 1 - m} \leq \frac{rK \cdot K}{rK + K} = K^{1/3}.
\]
This observation follows since for any real number \( \alpha \) the expression \( \alpha y/(\alpha + y) \) is positive for positive \( \alpha \) and \( y \), as we have with \( \alpha = rK \) and \( y \in [K - 1 - m, K] \), and is increasing in \( y \) away from the possible discontinuity at \( y = -\alpha \). Therefore, by definition of \( \mu \), we obtain
\[
g'(L - 1 - m) - g'(0) \leq \frac{mK}{W} \mu. \tag{5.2}
\]
We next define
\[
C_0 := [g'(0)], \quad C_k := [g'(L - 1 - m)],
\]
and let \( \{g'(0)\} = \epsilon_1 \) and \( \{g'(L - 1 - m)\} = \epsilon_2 \), where \( \{x\} \) denotes the fractional part of \( x \). Let \( \Delta \) be a number such that \( 0 < \Delta < 1/2 \), to be chosen later, and let
\[
C_j := C_{j-1} + 1, \quad 1 \leq j \leq k - 1,
\]
\[
x_j := \max\{(g')^{-1}(C_j - \Delta), 0\}, \quad 1 \leq j \leq k,
\]
\[
y_j := \min\{(g')^{-1}(C_j + \Delta), L - 1 - m\}, \quad 0 \leq j \leq k.
\]
So that
\[
k = C_k - C_0 = g'(L - 1 - m) - \epsilon_2 - (g'(0) - \epsilon_1) \leq \frac{mK}{W} \mu + \epsilon_1 - \epsilon_2, \tag{5.3}
\]
where we used (5.2) in the last inequality. Furthermore, since both \( g' \) and \( (g')^{-1} \) are increasing, we have, for \( 1 \leq j \leq k \),
\[
y_j - x_j = \min\{(g')^{-1}(C_j + \Delta), L - 1 - m\} - \max\{(g')^{-1}(C_j - \Delta), 0\}
\]
\[
= (g')^{-1}(\min\{C_j + \Delta, g'(L - 1 - m)\}) - (g')^{-1}(\max\{C_j - \Delta, g'(0)\})
\]
\[
\leq 2\Delta((g')^{-1}(\nu_j)),
\]
for some \( \nu_j \) satisfying
\[
\max\{C_j - \Delta, g'(0)\} \leq \nu_j \leq \min\{C_j + \Delta, g'(L - 1 - m)\}.
\]
However this implies that \( \xi_j := (g')^{-1}(\nu_j) \in [0, L - 1 - m] \) and hence
\[
y_j - x_j \leq 2\Delta((g')^{-1}(\nu_j)) = \frac{2\Delta}{|g''(\xi_j)|} \leq \frac{2W \Delta}{m}. \tag{5.4}
\]
Next, we observe that \( 0 \leq x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k \leq L - m - 1 \), so the \( x_j \) and \( y_j \) interlace. We treat the intervals \([x_1, y_1], [x_2, y_2], \ldots, [x_k, y_k]\) using the trivial bound, and the intervals \([y_1, x_2], [y_2, x_3], \ldots, [y_k - 1, x_k]\) using the Kusmin–Landau lemma. As for the boundary intervals \([0, x_1]\) and \([y_k, L - m - 1]\), they will require more careful analysis and a separate treatment.

By the triangle inequality, we have for any real numbers \( a \) and \( b \) such that \( a \leq b \),
\[
\left| \sum_{a \leq n \leq b} e(g(n)) \right| \leq b - a + 1.
\]
Hence, applying (5.4),

\[
\left| \sum_{x_j \leq n < y_j} e(g(n)) \right| \leq y_j - x_j + 1 \leq \frac{2W\Delta}{m} + 1.
\]  

(5.5)

As for the complementary intervals \([y_j, x_{j+1})\), we have, by construction, \(\|g'(x)\| \geq \Delta\) for all \(x \in [y_j, x_{j+1})\). So by Lemma 2.3 for \(j = 1, \ldots, k-1\),

\[
\left| \sum_{y_j \leq n < x_{j+1}} e(g(n)) \right| \leq \frac{2}{\pi\Delta}.
\]  

(5.6)

It remains to consider the boundary intervals, starting with \([0, x_1)\). Let

\[
S_0 := \left| \sum_{0 \leq n < x_1} e(g(n)) \right|.
\]

We consider the following three cases.

**Case 1:** \(\epsilon_1 \in (1 - \Delta, 1]\). Since \(\epsilon_1 > 1 - \Delta\) and \((g')^{-1}\) is increasing,

\[
(g')^{-1}(C_1 - \Delta) < (g')^{-1}(|g'(0)| + \epsilon_1) = (g')^{-1}(g'(0)) = 0,
\]

hence \(x_1 = 0\) and thus \(S_0 = 0\).

**Case 2:** \(\epsilon_1 \in [0, \Delta)\). Since \(\epsilon_1 < \Delta\),

\[
y_0 = (g')^{-1}(|g'(0)|) + \Delta > (g')^{-1}(g'(0)) = 0.
\]

Therefore, \(\|g'(n)\| \geq \Delta\) for \(n \in [y_0, x_1)\), so by Lemma 2.3 and on using the trivial bound to estimate the subsum over \([0, y_0)\) we obtain

\[
S_0 \leq \left| \sum_{0 \leq n < y_0} e(g(n)) \right| + \left| \sum_{y_0 \leq n < x_1} e(g(n)) \right| \leq \frac{W(\Delta - \epsilon_1)}{m} + 1 + \frac{2}{\pi\Delta}.
\]

Here, we also used the analog of (5.4) to bound the length of the first subsum.

**Case 3:** \(\epsilon_1 \in [\Delta, 1 - \Delta]\). In this case, \(y_0 \leq 0 \leq x_1\) and \(\ell + \epsilon_1 \leq g'(n) \leq \ell + 1 - \Delta\) for all \(n \in [0, x_1)\), where \(\ell = C_0\). So by Lemma 2.3

\[
S_0 \leq \frac{1}{\pi} \left( \frac{1}{\epsilon_1} + \frac{1}{\Delta} \right).
\]

Combining the three cases, we conclude that

\[
S_0 \leq \frac{1}{\pi\Delta} + h(\epsilon_1),
\]

where

\[
h(\epsilon) := \begin{cases} 
\frac{W(\Delta - \epsilon)}{m} + 1 + \frac{1}{\pi\Delta} & \text{if } \epsilon \in [0, \Delta) \\
\frac{1}{\pi\epsilon} + \frac{1}{\Delta} & \text{if } \epsilon \in [\Delta, 1 - \Delta] \\
-\frac{1}{\pi\Delta} & \text{if } \epsilon \in (1 - \Delta, 1]
\end{cases}
\]

(5.7)
AN IMPROVED EXPLICIT ESTIMATE FOR $\zeta(1/2 + it)$

Similarly, the sum corresponding to the other boundary interval $[y_k, L - 1 - m]$ satisfies

$$S_k := \left| \sum_{y_k \leq n \leq L - 1 - m} e(g(n)) \right| \leq \frac{1}{\pi \Delta} + h(1 - \epsilon_2). \quad (5.8)$$

The only difference in the treatment of $S_k$ is that if $\epsilon_2 \in [\Delta, 1 - \Delta]$, then we use a slightly modified version of Lemma 2.3 that holds for sums over the closed interval $[a, b]$ instead of $[a, b)$. This is easy to produce since the bound from that lemma is independent of the length of summation.

Together with (5.5) and (5.6), and using (5.3) to bound the sums over $k$, we obtain

$$\left| \sum_{0 \leq n \leq L - 1 - m} e(g(n)) \right| \leq S_0 + k \sum_{j=1}^{k-1} \left| \sum_{y_j \leq n < y_{j+1}} e(g(n)) \right| + S_k$$

$$\leq 2kW \Delta + k + \frac{2(k-1)}{\pi \Delta} + \frac{1}{\pi \Delta} + h(\epsilon_1) + \frac{1}{\pi \Delta} + h(1 - \epsilon_2)$$

$$\leq 2 \left( \frac{mK\mu}{W} + \epsilon_1 - \epsilon_2 \right) \left( \frac{1}{\pi \Delta} + \frac{W \Delta}{m} + \frac{1}{2} \right) + h(\epsilon_1) + h(1 - \epsilon_2).$$

To balance the first two terms in the second factor, we choose

$$\Delta = \sqrt{\frac{m}{\pi W}}. \quad (5.9)$$

Note that if this choice of $\Delta$ is $\geq 1/2$, then the bound we are trying to prove in Lemma 2.5 follows anyway since the contribution of the first term of the bound will already be $4\mu K \Delta \geq 2\mu K \geq K$, which is no better than the trivial bound. Therefore, we may assume that our choice of $\Delta$ satisfies $\Delta < 1/2$. Also, with our choice of $\Delta$ we have

$$\frac{W \Delta}{m} = \frac{1}{\pi \Delta}. \quad (5.10)$$

Put together, on defining

$$H(\epsilon) := \left( \frac{4}{\pi \Delta} + 1 \right) \epsilon + h(\epsilon), \quad (5.11)$$

and using (5.10) to simplify in the second line next, we arrive at

$$\left| \sum_{0 \leq n \leq L - 1 - m} e(g(n)) \right| \leq 2 \left( \frac{mK\mu}{W} + \epsilon_1 - \epsilon_2 \right) \left( \frac{1}{\pi \Delta} + \frac{1}{2} \right) + h(\epsilon_1) + h(1 - \epsilon_2)$$

$$= \frac{4K\mu}{\sqrt{\pi W}} m^{1/2} + \frac{\mu K}{W} m - \frac{4}{\pi \Delta} - 1 + H(\epsilon_1) + H(1 - \epsilon_2). \quad (5.12)$$

We will show that for $\epsilon \in [0, 1]$,

$$H(\epsilon) \leq \frac{4}{\pi \Delta} - \frac{2}{\pi} + \frac{3}{2}. \quad (5.13)$$

Substituting this bound back into (5.12), the lemma follows. We prove the inequality (5.13) by considering three cases.
Case 1: \( \epsilon \in [0, \Delta) \). Then recalling (5.7) and using (5.10),

\[
H(\epsilon) = \left( \frac{4}{\pi \Delta} + 1 \right) \epsilon + \frac{W(\Delta - \epsilon)}{m} + 1 + \frac{1}{\pi \Delta} = \left( 1 + \frac{4}{\sqrt{\pi}} \frac{\sqrt{W}}{m} - \frac{W}{m} \right) \epsilon + 1 + \frac{2}{\pi \Delta}.
\]

Noting that \( 1 + \frac{4}{\sqrt{\pi}} x - x^2 \leq 1 + \frac{4}{\pi} \) for all \( x \), we have, since \( 0 \leq \epsilon < \Delta < 1/2 \),

\[
H(\epsilon) < \left( 1 + \frac{4}{\pi} \right) \Delta + 1 + \frac{2}{\pi \Delta} \leq \frac{3}{2} + \frac{2}{\pi} + \frac{2}{\pi \Delta} < \frac{4}{\pi \Delta} - \frac{2}{\pi} + \frac{3}{2}.
\]

(5.14)

So the claimed bound on \( H(\epsilon) \) in (5.13) follows in this case.

Case 2: \( \epsilon \in [\Delta, 1 - \Delta] \). Then,\[
H(\epsilon) = \left( \frac{4}{\pi \Delta} + 1 \right) \epsilon + \frac{1}{\pi}.
\]

We observe that \( H \) is convex over \( x > 0 \) since \( H''(x) > 0 \). So,

\[
H(\epsilon) \leq \max\{H(\Delta), H(1 - \Delta)\}.
\]

(5.15)

On the other hand, since \( \Delta < 1/2 \),

\[
H(\Delta) = \frac{4}{\pi} + \Delta + \frac{1}{\pi \Delta} < \frac{4}{\pi} + \frac{1}{2} + \frac{4}{\pi \Delta} - \frac{6}{\pi} < \frac{4}{\pi \Delta} - \frac{2}{\pi} + \frac{1}{2}.
\]

(5.16)

Also,

\[
H(1 - \Delta) = \frac{4(1 - \Delta)}{\pi \Delta} + 1 - \Delta + \frac{1}{\pi (1 - \Delta)}.
\]

Since \( x + \frac{1}{\pi x} \) is convex for \( x > 0 \) and \( 1 - \Delta \in (1/2, 1) \), we have

\[
1 - \Delta + \frac{1}{\pi (1 - \Delta)} \leq \max \left( \frac{1}{2} + \frac{2}{\pi}, 1 + \frac{1}{\pi} \right) = 1 + \frac{1}{\pi}.
\]

Hence,

\[
H(1 - \Delta) \leq \frac{4}{\pi \Delta} - \frac{3}{\pi} + 1.
\]

(5.17)

Combining (5.10) and (5.17), and using that \(-2/\pi + 1/2\) and \(-3/\pi + 1\) are both \(-2/\pi + 3/2\), the claim follows in this case as well.

Case 3: \( \epsilon \in (1 - \Delta, 1) \). Then, using \( \Delta < 1/2 \),

\[
H(\epsilon) = \left( \frac{4}{\pi \Delta} + 1 \right) \epsilon - \frac{1}{\pi \Delta} < \frac{4}{\pi \Delta} + 1 - \frac{2}{\pi} < \frac{4}{\pi \Delta} - \frac{2}{\pi} + \frac{3}{2}.
\]

(5.18)

as claimed.

6. Proof of Lemma 2.6

In this lemma we propagate the improvements from Lemma 2.5 through to the third derivative test. The approach remains the same as Hiary [Hia16]. We first apply Weyl differencing from Lemma 2.4 to the given exponential, then we estimate the differenced sums \( s_m'(K) \). By Lemma 2.5 we have

\[
|s_m'(K)| \leq \frac{4K\mu}{\sqrt{\pi W}} m^{1/2} + \frac{\mu K}{W} m + 4\sqrt{\frac{W}{\pi}} m^{-1/2} - \frac{4}{\pi} + 2
\]
where \( \mu \) is as defined in Lemma 2.5. We apply the inequalities
\[
\sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) \sqrt{m} \leq \frac{4}{15} M^{3/2}, \quad \sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) \frac{1}{\sqrt{m}} \leq \frac{4}{3} \sqrt{M},
\]
\[
\sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) \leq \frac{1}{2} M, \quad \sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) m \leq \frac{1}{6} M^2,
\]
which appear after [CG04, Lemma 7], which gives
\[
\sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) \left| s_m^\prime(K) \right| \leq \frac{16\mu K}{15\sqrt{\pi W}} M^{3/2} + \frac{\mu K}{6W} M^2 + \frac{16}{3} \sqrt{\frac{W}{\pi}} M^{1/2} + \frac{1}{2} \left( 2 - \frac{4}{\pi} \right) M.
\]
Upon choosing \( M = \lceil \eta W^{1/3} \rceil \), for some \( \eta > 0 \), we have \( \eta W^{1/3} \leq M \leq \eta W^{1/3} + 1 \) and
\[
\frac{2}{M} \sum_{m=1}^{M} \left( 1 - \frac{m}{M} \right) \left| s_m^\prime(K) \right|
\leq \frac{32\mu K}{15\sqrt{\pi W}} M^{1/2} + \frac{\mu K}{3W} M + \frac{32}{3} \sqrt{\frac{W}{\pi}} M^{-1/2} + \left( 2 - \frac{4}{\pi} \right)
\leq \frac{32\mu K}{15\sqrt{\pi W}} \sqrt{\eta W^{1/3}} + 1 + \frac{\mu K}{3W} (\eta W^{1/3} + 1) + \frac{32}{3} \sqrt{\frac{W}{\pi}} \frac{1}{\sqrt{\eta W^{1/3}}} + \left( 2 - \frac{4}{\pi} \right)
\leq \frac{K}{W^{1/3}} \left[ \frac{\eta\mu}{3W^{1/3}} + \frac{\mu}{3W^{2/3}} + \frac{32\mu}{15\sqrt{\pi}} \sqrt{\eta + W^{-1/3}} \right]
+ W^{1/3} \left[ \frac{32}{3\sqrt{\pi\eta}} + \left( 2 - \frac{4}{\pi} \right) W^{-1/3} \right]
\]
Substituting into Lemma 2.4, we obtain the desired result.

7. Concluding remarks

It appears difficult to substantially improve the constant 0.618 in Theorem 1.1 without resorting to a large-scale numerical computation. Such a numerical computation would probably need to be extensive enough to allow us to, both, avoid using the Riemann–Siegel-Lehman bound and increase the threshold value of \( t_0 \) where we start applying explicit van der Corput lemmas.

We briefly describe two theoretical approaches that may achieve modest improvements. Firstly, in Lemma 2.5 we used the inequality \( \Delta < 1/2 \), but this can be replaced with a sharper inequality. This change will ultimately lower the constant term appearing in the main bound of Lemma 2.5.

Secondly, our application of the third derivative test in (3.4) may be inefficient on the last piece in the main sum subdivision. This last piece contains \( n_1 - RK + 1 \) terms, yet we always bound it as though it contains \( K \) terms. This could be wasteful if \( n_1 - RK + 1 \) is much smaller than \( K \). More careful treatment of this boundary piece may produce savings for certain ranges of \( t \) in the “intermediate” region, i.e. in the region \( 5.5 \cdot 10^7 \leq t < 10^{12} \).

Lastly, the context of this work highlights the sensitivity of explicit estimates in number theory to errors (even minor ones) that could compound quickly, and points to the need for a more automated approach in the future to verify explicit theoretical results. Overall, the incorrect explicit Kusmin–Landau lemma affected
all the published subconvex explicit estimates on zeta, as well as other important explicit estimates in number theory. For example, the explicit $AB$ process derived in [Hia16, Lemma 1.2] was impacted, and in turn, as mentioned earlier, the constant $0.63$ in the main theorem there was also impacted. Our work restores the $0.63$ constant and improves it. Therefore, we leave intact several works that have subsequently used the bound in [Hia16, Theorem 1.1].
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