OPTIMAL ADAPTIVITY FOR A STANDARD FINITE ELEMENT METHOD FOR THE STOKES PROBLEM

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Abstract. We prove that the a standard adaptive algorithm for the Taylor-Hood discretization of the stationary Stokes problem converges with optimal rate. This is done by developing an abstract framework for indefinite problems which allows us to prove general quasi-orthogonality proposed in [10]. This property is the main obstacle towards the optimality proof and therefore is the main focus of this work. The key ingredient is a new connection between the mentioned quasi-orthogonality and LU-factorizations of infinite matrices.

1. Introduction. We consider an adaptive mixed finite element method for the stationary Stokes problem

\[-\Delta u + \nabla p = f,\]
\[\text{div} u = 0\]

with standard Dirichlet boundary conditions in two space dimensions. We discretize the problem with standard Taylor-Hood elements and employ a standard adaptive algorithm with Dörfler marking.

The theory of rate optimal adaptive algorithms for finite element methods originated in the seminal paper [39] by Stevenson and was further improved in [14] by Cascon, Kreuzer, Nochetto, and Siebert. These papers prove essentially, that a standard adaptive algorithm of the form

\[
\begin{array}{c}
\text{Solve} \\
\rightarrow \\
\text{Estimate} \\
\rightarrow \\
\text{Mark} \\
\rightarrow \\
\text{Refine}
\end{array}
\]

generates asymptotically optimal meshes for the approximation of the solution of a Poisson problem. The new ideas sparked a multitude of papers applying and extending the techniques to different problems, see e.g., [33, 15] for conforming methods, [36, 5, 7, 11, 34] for nonconforming methods, [16, 13, 31] for mixed formulations, and [26, 40, 2, 23, 24] for boundary element methods (the list is not exhausted, see also [10] and the references therein). All the mentioned results, however, focus on symmetric problems in the sense that the underlying equation induces a symmetric operator. The first proof of rate optimality for a non-symmetric problem which does not rely on additional assumptions is given in [25] for a general second order elliptic operator with non-vanishing diffusion coefficient of the form

\[-\text{div}(A\nabla u) + b \cdot \nabla u + cu = f.\]

This approach, however, relies heavily on the fact that the non-symmetric part of the operator \((b \cdot \nabla u + cu)\) is only a compact perturbation (one differentiation instead of two for the diffusion part). The first optimality proof of a strongly non-symmetric problem was given in the recent work [22] for a finite-element/boundary-element discretization of a transmission problem. The present work aims to generalize the approach from [22]...
to include indefinite problems. While the work is concerned with the Stokes problems, the applied methods are quite general and may be useful for other indefinite problems.

Currently available convergence and optimality theory for the Stokes problem is building on the seminal works \cite{17, 4}. For certain non-standard (Uzawa type) algorithms for the Stokes problem, the work \cite{32} proves optimal convergence. For nonconforming finite element methods, rate optimality and convergence has been investigated and achieved in \cite{6, 29, 12}.

For the standard Taylor-Hood element, the first proofs of adaptive convergence were presented in \cite{35, 38}, while the first a posteriori error estimator was presented in \cite{41}. The work \cite{27} gives an optimality proof under the assumption that general quasi-orthogonality is satisfied. This assumption is verified in the present work.

Since the level of technicality is already considerably high in the present two dimensional case, we refrain from presenting the general case. However, there seems to be no inherent barrier and the proof techniques are expected to transfer.

The remainder of the work is organized as follows: After presenting basic assumption in Section 2, we develop the framework for LU-factorizations of infinite matrices and connect it with general quasi-orthogonality in Section 3. We construct a Riesz basis for the Stokes problem in Section 4\&5 and apply the abstract approach to the Stokes problem in Section 6.

2. General assumptions.

2.1. Preliminaries. In the following, $\Omega \subseteq \mathbb{R}^2$ is a polygonal domain with boundary $\Gamma := \partial \Omega$. Given a Lipschitz domain $\omega \subseteq \mathbb{R}^2$, we denote by $H^s(\omega)$ the usual Sobolev spaces for $s \geq 0$. For non-integer values of $s$, we use real interpolation to define $H^s(\omega)$. Their dual spaces $\tilde{H}^{-s}(\omega)$ are defined by extending the $L^2$-scalar product. We denote by $H^s_0(\Omega)$ the $H^1$-functions with vanishing trace and the subscript $\star$ denotes vanishing mean of the functions in the given space, i.e., $L^2_\star(\Omega) := \{v \in L^2(\Omega) : \langle v, 1 \rangle_\Omega = 0\}$ (the definition generalizes to $\tilde{H}^{-s}(\Omega)$ in a straightforward fashion). Finally, $\mathcal{P}^p(\omega)$ denotes the polynomials of total degree less or equal to $p$.

2.2. Mesh refinement. Let $\mathcal{T}_0$ be a triangulation of $\Omega$. Given two triangulations $\mathcal{T}, \mathcal{T}'$, we write $\mathcal{T}' = \text{refine}(\mathcal{T}, \mathcal{M})$ for some $\mathcal{M} \subseteq \mathcal{T}$ if $\mathcal{T}'$ is generated from $\mathcal{T}$ by refinement of all $\mathcal{T} \in \mathcal{M}$ via newest vertex bisection. We write $\mathcal{T}' \in \text{refine}(\mathcal{T})$ if $\mathcal{T}'$ is generated from $\mathcal{T}$ by a finite number of iterated newest-vertex-bisection refinements and we denote the set of all possible refinements by $\mathcal{T} := \text{refine}(\mathcal{T}_0)$. Given $\omega \subseteq \Omega$, we call $\mathcal{T}'|_{\omega}$ a local refinement of $\mathcal{T}$, if there exists $\mathcal{T}'' \in \text{refine}(\mathcal{T})$ such that $\mathcal{T}'|_{\omega} = \mathcal{T}''|_{\omega}$. Given $T \in \mathcal{T}$ for some $\mathcal{T} \in \mathcal{T}$, level($\mathcal{T}$) denotes the number of bisections necessary to generate $T$ from a parent element in $\mathcal{T}_0$.

We define $\mathcal{N}(\mathcal{T})$ as the set of nodes of $\mathcal{T}$ and $\mathcal{E}(\mathcal{T})$ as the set of edges of $\mathcal{T}$. For any triangulation, we define

\begin{align*}
\mathcal{P}^p(\mathcal{T}) &:= \{v \in L^2(\Omega) : v|_T \in \mathcal{P}^p(T), T \in \mathcal{T}\}, \\
\mathcal{S}^p(\mathcal{T}) &:= \mathcal{P}^p(\mathcal{T}) \cap H^1(\Omega), \\
\mathcal{S}^p_0(\mathcal{T}) &:= \mathcal{P}^p(\mathcal{T}) \cap H^1_0(\Omega), \\
\mathcal{S}^p_\star(\mathcal{T}) &:= \mathcal{P}^p(\mathcal{T}) \cap \tilde{H}^1(\Omega).
\end{align*}

We define $h_T \in \mathcal{P}^0(\mathcal{T})$ as the mesh-size function by $h_T|_T := \text{diam}(T)$ for all $T \in \mathcal{T}$.
Given a subset $\Omega' \subseteq \Omega$, we define the patch

$$\omega(\Omega', \mathcal{T}) := \{ T_1 \in \mathcal{T} : \exists T_2 \in \mathcal{T}, T_1 \cap T_2 \neq \emptyset, \int_{T_2 \cap \Omega'} 1 \, dx > 0 \}.$$ 

The extended patches $\omega^k(\Omega', \mathcal{T})$ are defined iteratively by

$$\omega_1(\Omega', \mathcal{T}) := \omega(\Omega', \mathcal{T}), \quad \text{and} \quad \omega^k(\Omega', \mathcal{T}) := \omega(\bigcup_{i=1}^{k-1} \omega^i(\Omega', \mathcal{T}), \mathcal{T}).$$

### 2.3. Variational form

While we focus on the Stokes problem in Section 6, we start by considering a general variational problem. For a given Hilbert space $\mathcal{X}$, let $a( \cdot, \cdot): \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ denote a continuous bilinear form which satisfies the standard LBB-stability conditions, i.e.,

$$\inf_{u \in \mathcal{X}} \sup_{v \in \mathcal{X}} \frac{a(u,v)}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}} > 0. \tag{2.1}$$

Suppose that each triangulation $\mathcal{T} \in \mathcal{T}$ induces a corresponding closed subspace $\mathcal{X}_\mathcal{T} \subset \mathcal{X}$. We assume

$$\inf_{T \in \mathcal{T}} \inf_{u \in \mathcal{X}_T} \sup_{v \in \mathcal{X}_T} \frac{a(u,v)}{\|u\|_{\mathcal{X}_T} \|v\|_{\mathcal{X}_T}} > 0 \tag{2.2}$$

and define the unique continuous solution $u \in \mathcal{X}$ as well as the unique discrete solution $u_\mathcal{T} \in \mathcal{X}_\mathcal{T}$ by

$$a(u,v) = \langle f, v \rangle \quad \text{for all } v \in \mathcal{X} \quad \text{and} \quad a(u_\mathcal{T},v) = \langle f, v \rangle \quad \text{for all } v \in \mathcal{X}_\mathcal{T}. \tag{2.3}$$

Existence and uniqueness of the solutions is guaranteed by (2.1)–(2.2).

### 2.4. Adaptive algorithm

Given a triangulation $\mathcal{T} \in \mathcal{T}$, we assume that we can compute an error estimator $\eta(\mathcal{T}) = \sqrt{\sum_{T \in \mathcal{T}} \eta_T(\mathcal{T})^2}$. In the application to the stationary Stokes problem below, we have to restrict to adaptive triangulations with mild grading in the sense that there exists $D_{\text{grad}} \in \mathbb{N}$ such that

$$|\text{level}(T) - \text{level}(T')| \leq 1 \quad \text{for all } T' \in \omega^{D_{\text{grad}}} (T, \mathcal{T}) \quad \text{and all } T \in \mathcal{T}. \tag{2.4}$$

This condition is necessary for the present proof and also appears in [19] to prove optimal convergence in the $L^2$-norm. By $\mathcal{T}_{\text{grad}} \subseteq \mathcal{T}$, we denote all triangulations which satisfy (2.4) for a given $D_{\text{grad}} \in \mathbb{N}$. The result [22, Lemma 2.3] shows that the restriction does not alter the optimal convergence rate. Numerical experiments suggest that the restriction is not even necessary for optimal convergence rate, and thus might just be an artifact of the proof. In the following, we assume that $D_{\text{grad}}$ is sufficiently large to satisfy all the conditions in the proofs below.

We assume that the sequence $\mathcal{T}_\ell$ is generated by an adaptive algorithm of the form

### 2.5. Rate optimality

We aim to analyze the best possible algebraic convergence rate which can be obtained by the adaptive algorithm. This is mathematically characterized as follows: For the exact solution $u \in \mathcal{X}$, we define an approximation class $A_s$ by

$$u \in A_s \iff \|u\|_{A_s} := \sup_{N \in \mathbb{N}} \min_{\mathcal{T} \in \mathcal{T}_N} \|u\|_{\mathcal{X}} N_s \eta(\mathcal{T}) < \infty. \tag{2.7}$$
Input: \( \ell = 0, T_0, D_{\text{grad}} \in \mathbb{N}, 0 \leq \theta \leq 1, f \in X^* \).
For \( \ell = 0, 1, \ldots \) do:
1. Compute \( u_\ell \in X_\ell \) as the unique solution of
   \[
   a(u_\ell, v) = \langle f, v \rangle \quad \text{for all } v \in X_\ell. 
   \]
2. Compute error estimator \( \eta(T_\ell) \) for all \( T \in T_\ell \).
3. Mark set of minimal cardinality \( M_\ell \subseteq T_\ell \) such that
   \[
   \sum_{T \in M_\ell} \eta(T_\ell) \geq \theta \sum_{T \in T_\ell} \eta(T_\ell).
   \]
4. Refine at least the elements \( M_\ell \) of \( T_\ell \) to obtain \( T_{\ell+1} \).
5. Refine additional elements to ensure that \( T_{\ell+1} \) satisfies (2.4) (see, e.g., [19, Section A.3] for a valid mesh-refinement algorithm).
Output: sequence of meshes \( T_\ell \) and corresponding solutions \( u_\ell \).

By definition, a convergence rate \( \eta(T) = O(N^{-s}) \) is theoretically possible if the optimal meshes are chosen. In view of mildly graded triangulations, we define
\[
\|u\|_{A^s} \overset{\text{def}}{=} \|u\|_{A^s} := \sup_{N \in \mathbb{N}} \min_{\ell \in T_{\text{grad}}} N^s \eta(T) < \infty.
\]
In [22, Lemma 2.3], we show that in many situations (including the present setting) \( A^s \) exists. In the spirit of [10], rate optimality of the adaptive algorithm means that there exists a constant \( C_{\text{opt}} > 0 \) such that
\[
C_{\text{opt}}^{-1} \|u\|_{A^s} \leq \sup_{\ell \in \mathbb{N}} \frac{\eta(T_\ell)}{(#T_\ell - #T_0 + 1)^{-s}} \leq C_{\text{opt}} \|u\|_{A^s},
\]
for all \( s > 0 \) with \( \|u\|_{A^s} < \infty \).

**2.6. The Axioms.** As proved in [10], we need to check the axioms (A1)–(A4) to ensure rate optimality for a given adaptive algorithm: There exist constant \( C_{\text{red}}, C_{\text{stab}}, C_{\text{qo}}, C_{\text{dir}}, C_{\text{ref}} > 0 \), and \( 0 \leq q_{\text{red}} < 1 \) such that

A1. **Stability on non-refined elements:** For all refinements \( \tilde{T} \in T \) of a triangulation \( T \in \mathcal{T} \), for all subsets \( S \subseteq T \cap \tilde{T} \) of non-refined elements, it holds that
\[
\left( \sum_{T \in S} \eta(T) \right)^{1/2} \leq C_{\text{stab}} \|u_T - u_{\tilde{T}}\|_X.
\]

A2. **Reduction property on refined elements:** Any refinement \( \tilde{T} \in T \) of a triangulation \( T \in \mathcal{T} \) satisfies
\[
\sum_{T \in \mathcal{T} \setminus \tilde{T}} \eta(T) \leq q_{\text{red}} \sum_{T \in \mathcal{T} \setminus \tilde{T}} \eta(T)^2 + C_{\text{red}} \|u_T - u_{\tilde{T}}\|_X^2.
\]

A3. **General quasi-orthogonality:** For one sufficiently small \( \varepsilon \leq 0 \) the output of Algorithm 2.4 satisfies for all \( \ell, N \in \mathcal{N} \)
\[
\sum_{k=\ell}^{\ell+N} \left( \|u_{k+1} - u_k\|_X^2 - C_{\text{qo}} \varepsilon \|u - u_k\|_X^2 \right) \leq C_{\text{qo}} \|u - u_\ell\|_X^2.
\]
A4. **Discrete reliability**: For all refinements \( \mathcal{T} \subset \mathbb{T} \) of a triangulation \( \mathcal{T} \subset \mathbb{T} \), there exists a subset \( \mathcal{R}(\mathcal{T}, \mathcal{T}) \subset \mathcal{T} \) with \( \mathcal{T} \setminus \mathcal{T} \subset \mathcal{R}(\mathcal{T}, \mathcal{T}) \) and \( |\mathcal{R}(\mathcal{T}, \mathcal{T})| \leq C_{\text{ref}}|\mathcal{T} \setminus \mathcal{T}| \) such that

\[
\|u_{\mathcal{T}} - u_{\mathcal{F}}\|_{X}^2 \leq C_{\text{dir}}^2 \sum_{T \in \mathcal{R}(\mathcal{T}, \mathcal{T})} \eta(T)^2.
\]

While the axioms (A1), (A2), and (A4) are already known in various forms in the literature, the general quasi-orthogonality (A3) seems to be the main obstacle for the optimality proof.

### 3. General quasi-orthogonality and LU-factorization

In this section, we establish the link between general quasi-orthogonality (A3) and LU-factorization of infinite matrices. To that end, we first introduce exponentially decaying matrices.

#### 3.1. Jaffard class matrices

Jaffard class matrices generalize the notion of matrices which decay exponentially away from the diagonal. The generalization allows to replace the distance \(|i - j|\) between indices by a general metric \(d(i, j)\). This class was introduced and analyzed in [30].

**Definition 3.1 (Jaffard class).** We say that an infinite matrix \( M \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}} \) is of Jaffard class, \( M \in J(d, \gamma, C) \) for some metric \(d(i, j)\) and some \(\gamma > 0\) if for all \(0 < \gamma' < \gamma\) there exists \(C(\gamma') > 0\) such that

\[
|M_{ij}| \leq C(\gamma') \exp(-\gamma'd(i, j)) \quad \text{for all } i, j \in \mathbb{N}.
\]

Moreover, the metric \(d(\cdot, \cdot)\) must satisfy for all \(\varepsilon > 0\)

\[
\sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \exp(-\varepsilon d(i, j)) < \infty.
\]

We also write \( M \in J \) to state the existence of parameters \(d, \gamma, C\) such that \( M \in J(d, \gamma, C) \).

**Definition 3.2 (banded matrix).** We say that an infinite matrix \( M \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}} \) is banded with respect to some metric \(d(\cdot, \cdot)\) if there exists a bandwidth \(b \geq 1\) such that

\[
d(i, j) > b \implies M_{ij} = 0 \quad \text{for all } i, j \in \mathbb{N}.
\]

In this case, we write \( M \in \mathcal{B}(d, b) \). Note that we do not require \(d(\cdot, \cdot)\) to satisfy (3.2). We also write \( M \in \mathcal{B} \) or \( M \in \mathcal{B}(d) \) to state that the missing parameters exist.

The following technical lemmas state some straightforward facts about infinite matrices and can be found in [22].

**Lemma 3.3.** Let \( M^{i,j} \in \mathcal{B}(d, b_j) \), \( i = 1, \ldots, n \), \( j = 1, \ldots, m \) for some \(m, n \in \mathbb{N} \) with respect to some metric \(d(\cdot, \cdot)\) and respective bandwidths \(b_j \in \mathbb{N} \). Then, there holds

\[
\sum_{i=1}^{n} \prod_{j=1}^{m} M^{i,j} \in \mathcal{B}(d, \sum_{j=1}^{m} b_j).
\]

**Lemma 3.4.** Let \( M \in \mathcal{J}(d, \gamma, C) \), then \( |M|: \ell_2 \to \ell_2 \) is a bounded operator (the modulus \(|M|\) is understood entry wise).
Given a block structure in the sense that there exist numbers \( n_1, n_2, \ldots \in \mathbb{N} \) with \( n_1 = 1 \) and \( n_i < n_j \) for all \( i \leq j \), we denote matrix blocks by

\[
M(i, j) := M|_{(n_i, \ldots , n_i+1-1) \times (n_j, \ldots , n_j+1-1)} \in \mathbb{R}^{(n_i+1-n_i) \times (n_j+1-n_j)}.
\]

By \( M[k] \in \mathbb{R}^{(n_k+1-1) \times (n_k+1-1)} \), we denote the restriction of \( M \) to the first \( k \times k \) blocks.

**Lemma 3.5.** Let \( M \in \mathcal{J}(d, \gamma, C) \) and assume a block structure \( n_1, n_2, \ldots \in \mathbb{N} \) such that \( M \) satisfies the inf-sup condition

\[
sup \|(M[k])^{-1}\|_2 = C_{\inf-sup} < \infty.
\]

Then,

\[
\overline{M} \in \mathbb{R}^{N \times N} \quad \text{with} \quad \overline{M}_{ij} := \sup_{n_k \geq \max(i, j)} |(M[k])^{-1}|_{ij}
\]

is of exponential class \( (d, \tilde{\gamma}, \tilde{C}) \) and thus a bounded operator \( \overline{M}: \ell_2 \rightarrow \ell_2 \). The constant \( \tilde{\gamma} \) depends only on \( C_{\inf-sup} > 0 \), \( d \), and \( \gamma \), whereas for all \( 0 < \gamma' < \tilde{\gamma} \), \( \tilde{C}(\gamma') \) depends only on an upper bound for \( C(\gamma') \) and on \( C_{\inf-sup} > 0 \).

**Proof.** The result [30, Proposition 2] shows that \( (M[k])^{-1} \in \mathcal{J}(d, \tilde{\gamma}, \tilde{C}) \). Inspection of the proof reveals that \( \tilde{\gamma} \) depends only on \( \gamma \), \( d \), and \( \tilde{C}(\gamma') \) depends only on an upper bound for \( C(\gamma') \) from Definition 3.1 and on \( C_{\inf-sup} > 0 \). Therefore, we have for all \( 0 < \gamma' < \tilde{\gamma} \)

\[|\overline{M}_{ij}| \leq \tilde{C}(\gamma') \exp(-\gamma'd(i, j)) \quad \text{for all} \quad i, j \in \mathbb{N}\]

and hence \( \overline{M} \in \mathcal{J}(d, \tilde{\gamma}, \tilde{C}) \). Lemma 3.4 concludes the proof.

**3.2. LU-factorization.** A matrix \( M \in \mathbb{R}^{N \times N} \) has an LU-factorization if \( M = LU \) for matrices \( L, U \in \mathbb{R}^{N \times N} \) such that

\[
L_{ij} = 0 \quad \text{and} \quad U_{ji} = 0 \quad \text{for all} \quad i, j \in \mathbb{N}, \ i < j.
\]

**Lemma 3.6.** Let \( M \in \mathbb{R}^{N \times N} \) such that \( M : \ell_2 \rightarrow \ell_2 \) is bounded and satisfies (3.4). Moreover, let \( M \in \mathcal{B}(d, b_0) \). Assume a block structure \( n_1, n_2, \ldots \in \mathbb{N} \). Then, given \( \epsilon > 0 \), there exists a bandwidth \( b \in \mathbb{N} \) such that for all \( k \in \mathbb{N} \), there exist \( R, R_k \in \mathcal{B}(d, b) \) such that

\[
\|M^{-1} - R\|_2 + \sup_{k \in \mathbb{N}} \|M[k]^{-1} - R_k\|_2 \leq \epsilon.
\]

If \( M \) is additionally block-banded in the sense \( M(i, j) = 0 \) for all \( |i - j| > b_0 \), then, \( R_k \) and \( R \) will additionally be block-banded with bandwidth \( b \). If \( M \) is block-diagonal, also \( R \) and \( R_k \) will be block-diagonal. The bandwidth \( b \) depends only on \( b_0, C_{\inf-sup}, \|M\|_2, \) and \( \epsilon \).

**Proof.** Let \( A := M[k]^T M[k] \) or \( A := MM^T \). Due to (3.4), \( A \) is elliptic with some constant \( C_{\inf-sup}^{-2} \). We obtain for \( \alpha := C_{\inf-sup}^{-2} \) and \( x \in \ell_2 \)

\[
\|x - \alpha Ax\|_{\ell_2}^2 = \|x\|_{\ell_2}^2 - 2\alpha \langle x, Ax \rangle_{\ell_2} + \alpha^2 \|Ax\|_{\ell_2}^2
\leq (1 - 2\alpha C_{\inf-sup}^{-2} + \alpha^2 \|A\|_{\ell_2}^2)\|x\|_{\ell_2}^2 \leq (1 - C_{\inf-sup}^{-2} \|M\|_{\ell_2}^2)\|x\|_{\ell_2}^2.
\]
This shows \( \|I - \alpha A\|_2 \leq (1 - C_{\inf-sup}^{-2}/\|M\|_2^2) := q < 1 \). We obtain
\[
A^{-1} = (\alpha A)^{-1} = \alpha (I - (I - \alpha A))^{-1} = \alpha \sum_{k=0}^{\infty} (I - \alpha A)^k.
\]

We define \( R := (\sum_{k=0}^{N}(I - \alpha A)^k)M^T \) and \( R_k := (\sum_{k=0}^{N}(I - \alpha A)^k)M[k]^T \) for some \( N \in \mathbb{N} \) such that \( \|\sum_{k=N+1}^{\infty}(I - \alpha A)^k\|_2 \leq \sum_{k=N+1}^{\infty} \|g^k\| \leq \varepsilon/\|M\|_2 \). Since \( (I - \alpha A), M \in B(d, b_0) \), Lemma 3.3 shows that \( R \) and \( R_k \) are banded as well. The bandwidth depends only on \( b_0, q, \) and \( N \). If \( M \) is additionally block-banded, also \( A \) and \( (I - \alpha A) \) will be block-banded with bandwidth \( 2b_0 \). Hence \( (I - \alpha A)^k \) will be block-banded with bandwidth \( 2k_b0 \). The same argumentation proves the statement for block-diagonal \( M \). This concludes the proof.

The following results prove that block-banded matrices \( M \) hand down some structure to their \( LU \)-factors. This is used in Section 6 to construct suitable hierarchical bases for the Stokes problem. This section is similar to \([22, \text{Section 3.2}]\), however, with the difference that we include indefinite problems instead of elliptic ones.

**Lemma 3.7.** Let \( M \in \mathbb{R}^{n \times n} \) such that \( M : \ell_2 \to \ell_2 \) is bounded and satisfies (3.4). Assume a block structure in the sense that there exist numbers \( n_1, n_2, \ldots \in \mathbb{N} \) with \( n_1 = 1 \) and \( n_i < n_j \) for all \( i < j \). Then, the block-\( LU \)-factorization \( M = LU \) for block-upper/block-lower triangular matrices \( L, U \in \mathbb{R}^{n \times n} \) such that \( L(i, i) = I \) for all \( i \in \mathbb{N} \) and satisfies
\[
\sup_{k \in \mathbb{N}} \|U(k, k)\|_2 \leq (1 + \|M\|_2^2)C_{\inf-sup}.
\]

**Proof.** It is well-known that the block-\( LU \)-factorization exists. We further note that \( M[k] = L[k]U[k] \) for all \( k \in \mathbb{N} \), i.e., restriction to principal submatrices commutes with the block-\( LU \)-factorization. Therefore, to see that \( U(k, k) \) is uniformly bounded, we may restrict to \( M[k] \) for \( k \in \mathbb{N} \). For matrices \( R_1, R_2 \in \mathbb{R}^{(n_i-1) \times (n_i+1)} \) and \( R_3 \in \mathbb{R}^{(n_i-1) \times (n_i+1)} \) the \((2 \times 2)\)-block-\( LU \)-factorization reads
\[
M[k] = \begin{pmatrix}
M[k-1] & R_1 \\
R_2 & R_3
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
R_2 & M[k-1]^{-1}
\end{pmatrix} \begin{pmatrix}
M[k-1] & R_1 \\
0 & R_3 - R_2 M[k-1]^{-1}R_1
\end{pmatrix}.
\]

Uniqueness of normalized block-\( LU \)-factorizations (further factorization of \( M[k-1] \)) will not alter the lower-right block of the \( U \)-factor implies \( U(k, k) = R_3 - R_2 M[k-1]^{-1}R_1 \) and hence \( \|U(k, k)\|_2 \leq \|M[k]\|_2 + \|M[k]\|^2C_{\inf-sup} \leq (1 + \|M\|_2^2)C_{\inf-sup} \).

**Lemma 3.8.** Let \( M \in \mathbb{R}^{n \times n} \) such that \( M : \ell_2 \to \ell_2 \) is bounded and satisfies (3.4). Assume a block structure in the sense that there exist numbers \( n_1, n_2, \ldots \in \mathbb{N} \) with \( n_1 = 1 \) and \( n_i < n_j \) for all \( i \leq j \). Moreover, let \( M \) be block-banded in the sense \( M(i, j) = 0 \) for \( |i - j| > b_0 \) for some \( b_0 \in \mathbb{N} \). Then, the block-\( LU \)-factorization \( M = LU \) for block-upper/block-lower triangular matrices \( L, U \in \mathbb{R}^{n \times n} \) such that \( L(i, i) = I \) for all \( i \in \mathbb{N} \) exists, is block-banded with bandwidth \( b_0 \), and satisfies
\[
\|L\|_2 + \|U\|_2 + \|L^{-1}\|_2 + \|U^{-1}\|_2 < \infty.
\]

Moreover, the block-diagonal matrix \( D \in \mathbb{R}^{n \times n}, D(i, i) := U(i, i) \) as well as its inverse are bounded and satisfy (3.4).
Proof. Since \( M[k] \) is invertible for all \( k \in \mathbb{N} \), it is a well-known fact that the block-LU-factorization exists. Since \( M[k] \) and \( L[k] \) are invertible by definition, \( U[k] \) is invertible for all \( k \in \mathbb{N} \). The block-triangular structure guarantees that \( L[k]^{-1} = (L^{-1})[k] \), \( U[k]^{-1} = (U^{-1})[k] \), and hence existence of \( L^{-1}, U^{-1} \) as matrices in \( \mathbb{R}^{N \times N} \). Moreover, it is well-known that \( L \) and \( U \) are block-banded with bandwidth \( b_0 \). By definition, we have \( M[k]^{-1} = U[k]^{-1}L[k]^{-1} \). Since \( L \) is lower-triangular with normalized block-diagonal (only identities in the diagonal blocks), the same is true for \( L[k]^{-1} \). Therefore, we obtain

\[
M[k]^{-1}(i, k) = \sum_{r=1}^{k} U[k]^{-1}(i, r)L[k]^{-1}(r, k) = U[k]^{-1}(i, k) = U^{-1}(i, k),
\]

where the last identity follows from the fact that \( U^{-1} \) is upper-block triangular.

We see that \( \sup_{i,j} \|U^{-1}(i, j)\|_2 \leq \sup_{k \in \mathbb{N}} \|M[k]^{-1}\|_2 \leq C_{\inf-sup} < \infty \). Hence, there holds that \( \|D(k,k)^{-1}\|_2 = \|U^{-1}(k,k)\|_2 \leq C_{\inf-sup} \). Moreover, we obtain

\[
\sup_{i,j \in \mathbb{N}} \|L(i,j)\|_2 \leq \sup_{i,j \in \mathbb{N}} \sum_{|k-i| \leq b_0} \|M(i,k)U^{-1}(k,j)\|_2 \leq 2b_0 \|M\|_2 \sup_{i,j} \|U^{-1}(i,j)\|_2 < \infty.
\]

Since \( L \) is block-banded with bandwidth \( b_0 \), the result [22, Lemma 8.4] shows \( \|L\|_2 < \infty \). This implies \( \|U^{-1}\|_2 \leq \|M^{-1}\|_2 \|L\|_2 < \infty \). Lemma 3.7 shows that \( D(k,k) = U(k,k) \) is uniformly bounded. Thus, we proved \( \|D\|_2 + \|D^{-1}\|_2 < \infty \) depending only on \( C_{\inf-sup} \) and \( \|M\|_2 \).

Let \( M^T = \tilde{L}\tilde{U} \) be the analogous block-LU-factorization for the transposed matrix (note that \( M^T \) still satisfies (3.4) and is bounded and banded). Since normalized \( LU \)-factorizations are uniquely defined, we see that

\[
\tilde{L} = U^TD^{-T} \quad \text{and} \quad \tilde{U} = D^TL^T.
\]

Repeating the above arguments shows \( \|\tilde{L}\|_2 + \|\tilde{U}^{-1}\|_2 < \infty \). With boundedness of \( D \) and \( D^{-1} \), (3.6) shows \( \|\tilde{U}\|_2 + \|\tilde{U}^{-1}\|_2 < \infty \) and hence concludes the proof.

Lemma 3.9. Under the assumptions of Lemma 3.8, assume that additionally \( M \in \mathcal{B}(d, b_0) \). Given \( \varepsilon > 0 \), there exists \( b \in \mathbb{N} \) and block-upper triangular \( U_{\varepsilon}^{-1} \in \mathcal{B}(d, b) \) which is additionally block-banded in the sense \( U_{\varepsilon}^{-1}(i, j) = 0 \) for \( |i - j| > b \) such that

\[
\|U^{-1} - U_{\varepsilon}^{-1}\|_2 < \varepsilon.
\]

The approximation \( U_{\varepsilon}^{-1} \) is invertible with bounded inverse such that \( \sup_{\varepsilon>0} \|U_{\varepsilon}\|_2 + \|U_{\varepsilon}^{-1}\|_2 < \infty \). Moreover, there exists block-diagonal \( D_{\varepsilon} \in \mathcal{B}(d, b) \) which is bounded and satisfies (3.4) such that \( \|D - D_{\varepsilon}\|_2 \leq \varepsilon \).

Proof. Lemma 3.6 shows that there exist \( R, R_k \in \mathcal{B}(d, b) \) which are block-banded with bandwidth \( b = b(\varepsilon) \) such that \( \|M^{-1} - R\|_2 + \|M[k]^{-1} - R_k\|_2 \leq \varepsilon \) for all \( k \in \mathbb{N} \). Choosing \( \varepsilon > 0 \) sufficiently small, we ensure that also \( R \) and \( R_k \) are bounded and satisfy (3.4) with uniform constant.

Inspired by (3.5), we define a first approximation to \( U^{-1} \) by

\[
T(i,j) := \begin{cases} 
0 & \text{for } j < i \text{ or } i < j - b, \\
R_j(i,j) & \text{for } j - b \leq i \leq j.
\end{cases}
\]
This ensures that \( T \in \mathcal{B}(d, b) \) and that \( T \) is block-banded with bandwidth \( b \). Additionally, we obtain
\[
\sup_{i,j \in \mathbb{N}} \|T(i, j) - U^{-1}(i, j)\|_2 \leq \varepsilon. \tag{3.8}
\]
We define an approximation to \( L \) (which is block-banded with bandwidth \( b_0 \)) by
\[
S(i, j) := \begin{cases} 
0 & \text{for } j < i - b_0 \text{ or } j > i, \\
I & \text{for } j = i, \\
(MT)(i, j) & \text{for } i - b_0 \leq j < i.
\end{cases}
\]
The definition and (3.8) imply
\[
\|L(i, j) - S(i, j)\|_2 \leq \| \sum_{k=i-b_0}^{i+b_0} M(i, k)(U^{-1}(k, j) - T(k, j)) \|_2 \leq \|M\|_2 \|U^{-1}(k, j) - T(k, j)\|_2 \lesssim \|M\|_2 b_0 \varepsilon. \tag{3.9}
\]
Since both \( L \) and \( S \) are block-banded with bandwidth \( b_0 \), the result [22, Lemma 8.4] shows even
\[
\|L - S\|_2 \lesssim \varepsilon, \tag{3.11}
\]
where the hidden constant is independent of \( \varepsilon \). Moreover, Lemma 3.3 shows that \( L \in \mathcal{B}(d, \tilde{b}) \), for some \( \tilde{b} \in \mathbb{N} \) which depends only on \( b_0 \) and \( b \).
Recall \( R \) from above with \( \|M^{-1} - R\|_2 \leq \varepsilon \), \( R \in \mathcal{B}(d, b) \) and \( R \) is block-banded with bandwidth \( b \). This allows to define \( U_{\varepsilon}^{-1} \) by
\[
U_{\varepsilon}^{-1} := RS.
\]
We obtain from the definition and with (3.11)
\[
\|U_{\varepsilon}^{-1} - U^{-1}\|_2 \leq \|RS - L\|_2 + \|(R - M^{-1})L\|_2 \lesssim \|R\|_2 \varepsilon + \|L\|_2 \varepsilon \leq (C_{\inf-sup} + \varepsilon + 1) \varepsilon, \tag{3.12}
\]
where the hidden constant does not depend on \( \varepsilon \). Moreover, Lemma 3.3 shows (since \( S \) and \( R \) are block-banded), that \( U_{\varepsilon}^{-1} \in \mathcal{B}(d) \) with bandwidth depending on \( \varepsilon \). Analogously, we see \( U_{\varepsilon}^{-1} \) is block-banded with bandwidth \( b_0 + b \). Since \( U^{-1} \) is invertible with bounded inverse, choosing \( \varepsilon > 0 \) sufficiently small ensures that \( U_{\varepsilon}^{-1} \) is invertible, with bounded inverse uniformly in \( \varepsilon \).
Let \( \tilde{D} \) denote the block-diagonal of \( U_{\varepsilon}^{-1} \). Obviously, \( \tilde{D} \in \mathcal{B}(d) \) and (3.12) implies
\[
\|\tilde{D} - D^{-1}\|_2 \lesssim \varepsilon. \tag{3.13}
\]
Lemma 3.8 shows that \( D \) and \( D^{-1} \) are bounded, thus sufficiently small \( \varepsilon > 0 \) guarantees the same for \( \tilde{D} \) and \( D^{-1} \). Since \( \tilde{D} \) is block-diagonal, this implies (3.4) for \( \tilde{D} \). Hence, Lemma 3.6 ensures that there exists block-diagonal \( D_{\varepsilon} \in \mathcal{B}(d) \) (with bandwidth depending only on \( \varepsilon > 0 \)), such that \( \|\tilde{D}^{-1} - D_{\varepsilon}\|_2 \leq \varepsilon. \) From this, we obtain
\[
\|D - D_{\varepsilon}\|_2 \leq \|\tilde{D}^{-1} - D_{\varepsilon}\|_2 + \|\tilde{D}^{-1} - D\|_2 \leq \varepsilon + \|\tilde{D}^{-1}\|_2 \|D\|_2 \|\tilde{D} - D^{-1}\|_2 \lesssim (1 + \|\tilde{D}^{-1}\|_2 \|D\|_2) \varepsilon.
\]
For sufficiently small \( \varepsilon > 0 \), \( \|\tilde{D}^{-1}\|_2 \) is bounded in terms of \( \|D\|_2 \). This ensures that the constant above does not depend on \( \varepsilon \) and thus concludes the proof. \( \square \)
Lemma 3.10. Under the assumptions of Lemma 3.8–3.9, there exists \( b \in \mathbb{N} \) and block-lower triangular \( L_{\varepsilon}^{-1} \in \mathcal{B}(d, b) \) with \( L_{\varepsilon}^{-1}(i, i) = I \), which is additionally block-banded in the sense \( L_{\varepsilon}^{-1}(i, j) = 0 \) for \( |i - j| > b \) such that
\[
\|L^{-1} - L_{\varepsilon}^{-1}\|_2 \leq \varepsilon.
\]
The approximation \( L_{\varepsilon}^{-1} \) is invertible such that \( \sup_{\varepsilon > 0}(\|L_{\varepsilon}\|_2 + \|L_{\varepsilon}^{-1}\|_2) < \infty \).

Proof. Recall that \( M^T \) satisfies all the assumptions of Lemma 3.8–3.9. Let \( M^T = \tilde{L}\tilde{U} \). We apply Lemma 3.9 to \( M^T \) to obtain an approximation \( \tilde{U}_{\varepsilon}^{-1} \in \mathcal{B}(d, b) \), block-banded with bandwidth \( b \), bounded with bounded inverse (uniformly in \( \varepsilon \)) such that
\[
\|\tilde{U}^{-1} - \tilde{U}_{\varepsilon}^{-1}\|_2 \leq \varepsilon.
\]
The identity (3.6) shows \( L^{-1} = D\tilde{U}^{-T} \) and thus motivates the definition
\[
L_{\varepsilon}^{-1}(i, j) := \begin{cases} (D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T})(i, j) & i \neq j, \\ I & i = j. \end{cases}
\]
with \( D_{\varepsilon} \in \mathcal{B}(d, b) \) from Lemma 3.9 applied to \( M \). Lemma 3.3 shows \( L_{\varepsilon}^{-1} \in \mathcal{B}(d) \) and \( L_{\varepsilon}^{-1} \) is also block-banded with bandwidth \( b \). We obtain with the approximation estimates from Lemma 3.8
\[
\|L_{\varepsilon}^{-1} - L^{-1}\|_2 \leq \|D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T} - D\tilde{U}^{-T}\|_2 + \sup_{i \in \mathbb{N}} \|(D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T})(i, i) - I\|_2
\]
The first term on the right-hand side is bounded by
\[
\|D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T} - D\tilde{U}^{-T}\|_2 \leq \|(D_{\varepsilon} - D)\tilde{U}_{\varepsilon}^{-T}\|_2 + \|D(\tilde{U}_{\varepsilon}^{-T} - \tilde{U}^{-T})\|_2
\]
\[
\leq \varepsilon(\|\tilde{U}^{-1}\|_2 + \varepsilon) + \|D\|_2 \varepsilon \lesssim \varepsilon.
\]
The second term satisfies
\[
\sup_{i \in \mathbb{N}} \|(D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T})(i, i) - I\|_2 \leq \|D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T} - L\|_2 = \|D_{\varepsilon}\tilde{U}_{\varepsilon}^{-T} - D\tilde{U}^{-T}\|_2 \lesssim \varepsilon.
\]
Choosing \( \varepsilon > 0 \) sufficiently small ensures that \( L_{\varepsilon}^{-1} \) is invertible with bounded inverse uniformly in \( \varepsilon > 0 \). This concludes the proof.

Theorem 3.11. Under the assumptions of Lemma 3.8–3.10, there exists an approximate block-LDU-decomposition \( \|M - LDU\|_2 \leq \varepsilon \) for block-upper/block-lower triangular factors \( L, U \) such that \( L(i, i) = U(i, i) = I \) for all \( i \in \mathbb{N} \), and a block diagonal factor \( D \). The factors \( L, D, U: \ell_2 \to \ell_2 \) are bounded with bounded inverses uniformly in \( \varepsilon \) and satisfy \( L^{-1}, D, U^{-1} \in \mathcal{B}(d, b) \) for some bandwidth \( b \). Moreover, \( L^{-1}, U^{-1} \) are block-banded with bandwidth \( b \), i.e., \( L^{-1}(i, j) = U^{-1}(i, j) = 0 \) for all \( |i - j| > b \). Additionally, \( D \) satisfies (3.4). The constant \( b \) depends only on \( M \), \( b_0 \), and \( \varepsilon \).

Proof. To avoid confusion, we denote the LU-factorization of \( M \) from Lemma 3.8 by \( \tilde{L} \) and \( \tilde{U} \), with diagonal matrix \( \tilde{D} \). With Lemma 3.9–3.10, we set \( D := D_{\varepsilon} \) and \( L^{-1} := L_{\varepsilon}^{-1} \). This ensures \( L^{-1}, D \in \mathcal{B}(d) \) and that \( L^{-1} \) is block-banded. Moreover, \( D \) is bounded and satisfies (3.4). This motivates the definition
\[
U^{-1}(i, j) := \begin{cases} (U_{\varepsilon}^{-1}D_{\varepsilon})(i, j) & i \neq j, \\ I & i = j. \end{cases}
\]
Lemma 3.3 shows that $U^{-1} \in \mathcal{B}(d)$ with bandwidth depending on $\varepsilon$ and moreover $U^{-1}$ is block-banded. We obtain
\[
\|M^{-1} - U^{-1}D^{-1}L^{-1}\|_2 \leq \|M^{-1} - U^{-1}L^{-1}\|_2 + \|(U^{-1} - U^{-1}D^{-1})L^{-1}\|_2
\]

\[
\leq \|M^{-1} - U^{-1}L^{-1}\|_2 + \sup_{i \in \mathbb{N}} \|(U^{-1}D\varepsilon(i, i) - I\|_2 \|L^{-1}\|_2
\]

\[
\leq \|M^{-1} - U^{-1}L^{-1}\|_2 + \|U^{-1}D\varepsilon - \bar{U}^{-1}\bar{D}\|_2 \|L^{-1}\|_2.
\]

The first term on the right-hand side can be bounded by use of Lemma 3.9–3.10 by
\[
\|M^{-1} - U^{-1}L^{-1}\|_2 \leq \|\bar{U}^{-1}\|_2 \|\bar{L}^{-1} - L^{-1}\|_2 + \|U^{-1} - U^{-1}\|_2 \|L^{-1}\|_2 \lesssim \varepsilon,
\]

where the hidden constant does not depend on $\varepsilon > 0$. The second term can be bounded in a similar fashion by
\[
\|U^{-1}D\varepsilon - \bar{U}^{-1}\bar{D}\|_2 \|L^{-1}\|_2 \leq \|U^{-1}\|_2 \|D\varepsilon - \bar{D}\|_2 \|L^{-1}\|_2 + \|U^{-1} - \bar{U}^{-1}\|_2 \|\bar{D}\|_2 \|L^{-1}\|_2 \lesssim \varepsilon
\]

with $\varepsilon$-independent hidden constant. Altogether we proved
\[
\|M^{-1} - U^{-1}D^{-1}L^{-1}\|_2 \lesssim \varepsilon.
\]

Moreover, there holds
\[
\|M - LDU\|_2 \leq \|M\|_2 \|LDU\|_2 \|M^{-1} - U^{-1}D^{-1}L^{-1}\|_2
\]

\[
\leq (\|M\|_2^2 + \|M\|_2 \|LDU - M\|_2) \|M^{-1} - U^{-1}D^{-1}L^{-1}\|_2
\]

\[
\lesssim \|M\|_2^2 \varepsilon + \|M\|_2 \|M - LDU\|_2 \varepsilon.
\]

Sufficiently small $\varepsilon > 0$ shows
\[
\|M - LDU\|_2 \lesssim \varepsilon,
\]

where the hidden constant does not depend on $\varepsilon$. This concludes the proof. \qed

The following results establishes existence of a bounded $LU$-factorization for particular Jaffard class matrices.

**Theorem 3.12.** Let $M \in \mathbb{R}^{N \times N} \in \mathcal{F}(d, \gamma, C)$ and additionally satisfy (3.4) for some given block-structure $n_1, n_2, \ldots \in \mathbb{N}$ with $n_1 = 1$ and $n_i < n_j$ for all $i \leq j$. Then, $M$ has a block-$LU$-factorization such that $L, U, L^{-1}, U^{-1}; \ell_2 \to \ell_2$ are bounded operators with operator norms depending only on $\mathcal{F}(d, \gamma, C)$, $C_{inf-sup}$, and $\|M\|_2$.

**Proof.** We repeat the proof of [1, Theorem 2] to show that it also works for block-$LU$-factorizations. The existence of block-upper/block-lower triangular matrices $L, U \in \mathbb{R}^{N \times N}$ such that $M = LU$ is well-known. The identity (3.5) shows for $x \in \ell_2$ and $\bar{M}$ from Lemma 3.5
\[
\|U^{-1}x\|_{\ell_2}^2 \leq \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} |M[k]^{-1}(i, k)||x|_{\{n_k, \ldots, n_{k+1}-1\}}^2
\]

\[
\leq \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \bar{M}(i, k)||x|_{\{n_k, \ldots, n_{k+1}-1\}}^2 \leq \|\bar{M}\|_2 \|x\|_{\ell_2} \lesssim \|x\|_{\ell_2}.
This shows that \( \|U^{-1}\|_2 < \infty \) and we deduce immediately \( \|L\|_2 \leq \|M\|_2 \|U^{-1}\|_2 < \infty \). Lemma 3.7 shows additionally that the block-diagonal matrix \( D(i, i) := U(i, i), \; i \in \mathbb{N} \) is bounded \( \|D\|_2 < \infty \) in terms of \( C_{\inf-sup} \) and \( \|M\|_2 \). The same argumentation proves for \( M^T = \tilde{L}U \) that \( \|\tilde{L}\|_2 + \|U^{-1}\|_2 < \infty \). From (3.6), we see
\[
\|U\|_2 = \|\tilde{L}\|_2 \|D^T\|_2 < \infty \quad \text{and hence} \quad \|L^{-1}\|_2 \leq \|U\|_2 \|M^{-1}\|_2 < \infty .
\]

This concludes the proof. \qed

The following three theorems connect existence of bounded \( LU \)-factors with general quasi-orthogonality.

**Theorem 3.13.** Let there exist a Riesz basis \( (w_n)_{n \in \mathbb{N}} \) of \( X \) and a constant \( C > 0 \) such that all \( x = \sum_{n \in \mathbb{N}} \lambda_n w_n \in X \) satisfy
\[
C^{-1} \|x\|_2^2 \leq \sum_{n \in \mathbb{N}} \lambda_n^2 \|x\|_2^2
\]
and there holds \( X_\ell = \text{span}\{w_n : n \in \{1, \ldots, N_\ell\}\} \) for some constants \( N_\ell \in \mathbb{N} \) with \( N_\ell < N_{\ell+1} \). If \( M_{ij} := a(w_i, w_j) \) and \( M \in \mathcal{J} \), then there holds general quasi-orthogonality (2.8) even with \( \varepsilon = 0 \). The constant \( C_{\text{qo}} \) depends only on the basis \( (w_n) \), \( C_{\inf-sup} \), the norm of \( a(\cdot, \cdot) \), \( C, c_0 \), the Jaffard class \( \mathcal{J} \), and \( X \).

**Proof.** The \( N_\ell \) induce a block structure. By (3.13) and with (2.1)–(2.2), the matrix \( M \) is bounded and satisfies (3.4). Since \( M \in \mathcal{J} \), Theorem 3.12 shows that there exists a bounded block-\( LU \)-factorization \( M = LU \). Let \( u = \sum_{n=1}^{\infty} \lambda_n w_n \) and \( u_\ell = \sum_{n=1}^{\infty} \lambda(\ell)_n w_n \). With \( \lambda := (\lambda_1, \lambda_2, \ldots) \) and \( \lambda(\ell) := (\lambda(\ell)_1, \lambda(\ell)_2, \ldots, \lambda(\ell)_N, 0, \ldots) \), \( F := (\langle f, w_1 \rangle, \langle f, w_2 \rangle, \ldots) \in \mathbb{R}^{N} \), there holds
\[
M\lambda = F \quad \text{and} \quad M[\ell] \lambda(\ell) = F[\ell] \quad \text{for all } \ell \in \mathbb{N},
\]
where \( F[\ell] := (F_1, \ldots, F_{N_\ell}, 0, \ldots) \). Moreover, there holds \( M[\ell] = L[\ell] U[\ell] \) for any \( LU \)-factorization. Due to the lower-triangular structure of \( L \) (note that \( L(i, i) = I \)), there holds for all \( 1 \leq i \leq N_\ell \) that
\[
(L[\ell] U[\ell] \lambda(\ell))_{i} = (M[\ell] \lambda(\ell))_{i} = F[\ell]_{i} = F_{i} = (M\lambda)_{i} = (LU\lambda)_{i} = (L[\ell] U\lambda)_{i}.
\]

Since \( L \) and hence also \( L[\ell] \) is regular, this shows that \( (U\lambda)_{i} = (U[\ell] \lambda(\ell))_{i} \) for all \( 1 \leq i \leq N_\ell \). Moreover, there holds \( U[\ell] \lambda(\ell) = U\lambda(\ell) \) due to the block-upper triangular structure of \( U \). Altogether, this proves
\[
(U\lambda)_{i} = (U\lambda(\ell))_{i} \quad \text{for all } 1 \leq i \leq N_\ell \quad \text{and} \quad (U\lambda(\ell))_{i} = 0 \quad \text{for all } i > N_\ell.
\]
Hence, we have, by use of the boundedness of \( U \) and \( U^{-1} \) and (3.13), that
\[
\|u_{k+1} - u_{k}\|_X \simeq \|\lambda(k+1) - \lambda(k)\|_{\ell_2} \\
\simeq \|U\lambda(k+1) - U\lambda(k)\|_{\ell_2} = \|(U\lambda)|_{\{N_k, \ldots, N_{k+1} - 1\}}\|_{\ell_2}.
\]
This shows
\[
\sum_{k=\ell}^{\infty} \|u_{k+1} - u_{k}\|_X^2 \simeq \sum_{k=\ell}^{\infty} \|(U\lambda)|_{\{N_k, \ldots, N_{k+1} - 1\}}\|_{\ell_2}^2 = \|(U\lambda)|_{\{N_\ell, \ldots\}}\|_{\ell_2}^2 \\
= \|U\lambda - U\lambda(\ell)\|_{\ell_2}^2 \simeq \|\lambda - \lambda(\ell)\|_{\ell_2}^2 \simeq \|u - u_\ell\|_{\lambda}^2.
\]
Hence, we conclude the proof. \qed
THEOREM 3.14. With the spaces and basis functions from Theorem 3.13 assume that for some \( \varepsilon > 0 \), there exists \( M^\varepsilon \in \mathbb{R}^{N \times N} \) such that \( M_{ij} := a(v_j, w_i), M \in \mathbb{R}^{N \times N} \) satisfies \( \| M - M^\varepsilon \|_2 \leq \varepsilon \). If \( M^\varepsilon \) satisfies (3.4) and \( M^\varepsilon \in \mathcal{J} \), then there holds general quasi-orthogonality (2.8). The constant \( C_{qo} > 0 \) depends only on the basis \( (w_n), a, C, \) the Jaffard class \( \mathcal{J} \), and \( X \).

Proof. Note that \( M^\varepsilon \in \mathcal{J} \) implies \( \| M^\varepsilon \|_2 < \infty \) (Lemma 3.4). With the notation from the proof of Theorem 3.13, we apply Theorem 3.13 to the bilinear form \( a^\varepsilon : \ell_2 \times \ell_2 \rightarrow \mathbb{R} \), \( a^\varepsilon (x, y) := \langle M^\varepsilon x, y \rangle_{\ell_2} \) and \( f^\varepsilon := M^\varepsilon \lambda \in \ell_2 \) and the spaces \( \mathcal{X}_\ell := \{ x \in \ell_2 : x_i = 0 \text{ for } i < N \ell \} \). Boundedness of \( M^\varepsilon \) together with (3.4) imply boundedness and the inf-sup condition (2.1) for \( a^\varepsilon (\cdot, \cdot) \). We use the \( \ell_2 \) unit vectors as the Riesz basis to obtain with Theorem 3.13

\[
\sum_{k=\ell}^{\infty} \| \lambda^\varepsilon (k + 1) - \lambda^\varepsilon (k) \|^2_{\ell_2} \lesssim \| \lambda - \lambda^\varepsilon (\ell) \|^2_{\ell_2}.
\]

Here, we used that \( \lambda^\varepsilon = \lambda \) by definition of \( f^\varepsilon \).

We identify vectors in \( \mathbb{R}^n \) with vectors in \( \mathbb{R}^N \) by adding zeros. Then, there holds with (2.1)

\[
\| \lambda^\varepsilon (k) - \lambda (k) \|^2_{\ell_2} \lesssim \sup_{\mu \in \mathcal{S}_k \atop \| \mu \|_{\ell_2} = 1} a^\varepsilon (\lambda^\varepsilon (k) - \lambda (k), \mu) = \sup_{\mu \in \mathcal{S}_k \atop \| \mu \|_{\ell_2} = 1} a^\varepsilon (\lambda - \lambda (k), \mu) = \sup_{\mu \in \mathcal{S}_k \atop \| \mu \|_{\ell_2} = 1} \langle (M^\varepsilon - M)(\lambda - \lambda (k)), \mu \rangle_{\ell_2} \leq \| M^\varepsilon - M \|_2 \| \lambda - \lambda (k) \|_{\ell_2}.
\]

Hence, we have

\[
\| \lambda^\varepsilon (k + 1) - \lambda^\varepsilon (k) \|_{\ell_2} - \| \lambda (k + 1) - \lambda (k) \|_{\ell_2} \lesssim \varepsilon (\| \lambda - \lambda (k) \|_{\ell_2} + \| \lambda - \lambda (k + 1) \|_{\ell_2}),
\]

where the hidden constant is independent of \( \varepsilon > 0 \). With (3.15), this concludes the proof.

THEOREM 3.15. With the spaces and basis function from Theorem 3.13 assume that there exists another Riesz basis \( (v_n)_{n \in \mathbb{N}} \) which satisfies the same conditions as \( (w_n) \) in Theorem 3.13. Assume that for some \( \varepsilon > 0 \), there exists \( M^\varepsilon \in \mathbb{R}^{N \times N} \) such that \( M_{ij} := a(v_j, w_i), M \in \mathbb{R}^{N \times N} \) satisfies \( \| M - M^\varepsilon \|_2 \leq \varepsilon \). If \( M \) and \( M^\varepsilon \) satisfy (3.4) and \( M^\varepsilon \in \mathcal{J} \), then there holds general quasi-orthogonality (2.8). The constant \( C_{qo} > 0 \) depends only on the basis \( (w_n), a, C_{ell}, C, \) the Jaffard class \( \mathcal{J} \), and \( X \).

Proof. With the notation from the proof of Theorem 3.13, we apply Theorem 3.14 to the bilinear form \( \tilde{a} : \ell_2 \times \ell_2 \rightarrow \mathbb{R} \) defined by \( \tilde{a}(x, y) := \langle M x, y \rangle_{\ell_2} \). Let \( M^\varepsilon \) as in the statement and choose the \( \ell_2 \)-unit vectors as the Riesz bases and \( \mathcal{X}_\ell \) as in the proof of Theorem 3.14. Note that the Riesz bases condition (3.13) ensures that \( M \) and \( M^\varepsilon \) are bounded operators in \( \ell_2 \) and thus Theorem 3.14 is applicable. Thus, we obtain for all \( \ell, N \in \mathbb{N} \)

\[
\sum_{k=\ell}^{\ell+N} \| \lambda (k + 1) - \lambda (k) \|^2_{\ell_2} - C'_{qo} \varepsilon \| \lambda - \lambda (k) \|^2_{\ell_2} \leq C'_{qo} \| \lambda - \lambda (\ell) \|^2_{\ell_2}.
\]
By definition, the vectors $\lambda$ and $\lambda(k)$ satisfy the equations (3.14). Definition of $M$ implies

$$a(\sum_{j=1}^{\infty} \lambda_j v_j, w_i) = \langle f, w_i \rangle \quad \text{for all } i \in \mathbb{N}.$$  

Hence, we know $\sum_{j=1}^{\infty} \lambda_j v_j = u$. We get analogously

$$a(\sum_{j=1}^{N_\ell} \lambda(\ell)_j v_j, w_i) = \langle f, w_i \rangle \quad \text{for all } 1 \leq i \leq N_\ell.$$  

Since $(v_n)$ and $(w_n)$ span the same subspaces $X_\ell$, this shows $\sum_{j=1}^{N_\ell} \lambda(\ell)_j v_j = u_\ell$. Thus, by use of (3.13) for $(v_n)$, we rewrite (3.16) and conclude

$$\sum_{k=\ell}^{\ell+N} \|u_{k+1} - u_k\|_{L_2}^2 - C^2 C_{q_0} \|u - u_\ell\|_{L_2}^2 \leq \sum_{k=\ell}^{\ell+N} C \|\lambda(k + 1) - \lambda(k)\|_{L_2}^2 - C C_{q_0} \|\lambda - \lambda(\ell)\|_{L_2}^2 \leq C C_{q_0} \|\lambda - \lambda(\ell)\|_{L_2}^2 \leq C^2 C_{q_0} \|u - u_\ell\|_{L_2}^2.$$

We conclude the proof with $C_{q_0} := C^2 C_{q_0}'$ which is independent of $\varepsilon > 0$. 

4. A modified Scott-Zhang projection.

**Definition 4.1.** Given a triangulation $T$, define the hat functions $v_z \in S^1(T)$ associated with a certain node $z$ of $T$. For an edge $E$ of $T$ with endpoints $z_1$ and $z_2$, define the edge bubble $v_E := \alpha_E v_{z_1} v_{z_2}$ with $\alpha_E > 0$ such that $\|v_E\|_{L^\infty(\Omega)} = 1$. Let $S_p^p(T)$ for $p \in \{1, 2\}$ denote the set of all hat-, resp. hat and edge bubble-functions defined on $T$ and let $S_p(T)$ define the linear span of $S_p^p(T)$. For a refinement $\bar{T}$ of $T$, we denote by $S_p^p(\bar{T} \setminus T)$ all hat functions $v_z$ associated with new nodes $z \in \mathcal{N}(\bar{T}) \setminus \mathcal{N}(T)$ and (for $p = 2$) also all edge bubble functions $v_E$ associated with new edges $E \in \mathcal{E}(\bar{T}) \setminus \mathcal{E}(T)$. By $S_{p,0}^p(T)$ and $S_{p,0}^p(\bar{T} \setminus T)$, we denote the bases which vanish on $T$.

In the following, we define a particular basis of $S^1_+(T)$ with a certain moment condition.

**Definition 4.2.** Given a triangulation $T$, consider the following basis of $S^1_+(T)$: Assume a numbering $\{z_1, \ldots, z_{n_T}\} = \mathcal{N}(T)$ such that $z_{i+1} \in \text{supp}(v_{z_i})$ for all $i = 1, \ldots, n_T - 1$ and the hat-functions $v_{z_i}$. Then, define

$$v_{z_i, *} := v_{z_i} + \alpha_i v_{z_{i+1}} \quad \text{for all } i = 1, \ldots, n_T - 1,$$

where $\alpha_i \in \mathbb{R}$ is chosen such that $\int_\Omega v_{z_i,*} \, dx = 0$. Obviously, $S_{p,0}^p(\{v_{z_i,*} : i = 1, \ldots, n - 1\})$ is a linear independent subset of $S^1_+(T)$. To see that it is also a basis, note that $\dim(S^1_+(T)) = \#\mathcal{N}(T) - 1 = \#S_{p,0}^p(T)$.

We also define the dual basis functions as follows. For all $i = 1, \ldots, n_T - 1$, define $T_i \in T$ such that $z_i, z_{i+1} \in T_i$. Define the local function space $S_i := \text{span}\{v_{z_j,*} | T_j : \text{supp}(v_{z_j,*})|T_j| \neq 0\}$ and define $v_{z_j,*}^{i} \in S_i$ to be the dual basis function to $v_{z_j,*}|T_i$. This allows us to define

$$v_{z_i,*}^{i} := v_{z_i, *i} \quad \text{for all } i = 1, \ldots, n_T - 1.$$
We define the Scott-Zhang projection as

\[ J_T^1 v := \sum_{i=1}^{n_T-1} v_{z_i,*,}(v, v'_{z_i,*,})_{T_i}. \]

**Definition 4.3.** Define the Sobolev-Slobodeckij semi norm

\[ |v|^s_{H^s(\omega)} := \int_\omega \int_\omega |v(x) - v(y)|/(|x - y|^{2s+2}) \, dx \, dy \quad \text{for} \quad 0 < s < 1. \]

For \( s = \nu + r \in \mathbb{R} \) with \( \nu \in \mathbb{N} \) and \( s \in (0, 1) \), define \( |\cdot|_{H^s(\omega)} := |\nabla^\nu(\cdot)|_{H^r(\omega)} \), where \( \nabla^\nu \) denotes the tensor of all partial derivatives of order \( \nu \). As shown in [28], \( |\cdot|_{H^s(\omega)} + |\cdot|_{H^r(\omega)} \) is equivalent to the \( H^s \)-norm obtained via (real) interpolation. The norm equivalence constants depend only on the shape of \( \omega \).

**Lemma 4.4.** The Scott-Zhang operator \( J_T^1 : L^2_s(\Omega) \to \mathcal{S}_1^1(T) \) from Definition 4.2 is a projection which satisfies for all \( 0 \leq s \leq 1 \) and all \( v \in H^s_s(\Omega) \)

\[
\begin{align*}
(4.1) \quad \|J_T^1 v\|_{H^s(T)} &\leq C_{sz}\|v\|_{H^s(\omega(\omega(T,T)))}, \\
(4.2) \quad \|J_T^1 v\|_{H^s(\Omega)} &\leq C_{sz}\|v\|_{H^s(\Omega)},
\end{align*}
\]

as well as for all \( 0 \leq r \leq s \)

\[
\begin{align*}
(4.3) \quad \|(1 - J_T^1)v\|_{H^r(T)} &\leq C_{sz}\text{diam}(T)^s-r\|v\|_{H^r(\omega(\omega(T,T)))}, \\
(4.4) \quad \|(1 - J_T^1)v\|_{H^r(\Omega)} &\leq C_{sz}\|h_T^{-s}\nabla v\|_{L^2(\Omega)}.
\end{align*}
\]

For \( 0 < s < 1/2 \) and \( v \in L^2_s(\Omega) \), \( w \in \tilde{H}^{-s}(\Omega) \), there holds

\[
\begin{align*}
(4.5) \quad \|J_T^1 w\|_{\tilde{H}^{-s}(\Omega)} &\leq C_{sz}\|w\|_{\tilde{H}^{-s}(\Omega)}, \\
(4.6) \quad \|(1 - J_T^1)v\|_{\tilde{H}^{-s}(\Omega)} &\leq C_{sz}\|h_T^{-s}\nabla v\|_{L^2(\Omega)}.
\end{align*}
\]

The constant \( C_{sz} > 0 \) depends only on the shape regularity of \( T \), the fact that \( T \) is generated from \( T_0 \) by newest vertex bisection and on a lower bound on \( s \). The function \( (J_T^1v)|_T \) depends only on \( v|_{\omega(\omega(T,T))} \).

**Proof.** For the projection property, let \( v \in \mathcal{S}_1^1(T) \) with \( v = \sum_{j=1}^{n_T-1} \alpha_j v_{z_j,*,} \) for some coefficients \( \alpha_j \in \mathbb{R} \). Then, there holds

\[ J_T^1 v := \sum_{i,j=1}^{n_T-1} \alpha_j v_{z_i,*,}(v_{z_j,*,}, v'_{z_i,*,})_{T_i} = v \]

by use of \( (v_{z_i,*,}, v'_{z_i,*,})_{T_i} = 1 \) only if \( i = j \). Obviously, \( (J_T v)|_T \) depends only on \( v|_{T_i} \) for all \( i = 1, \ldots, n_T \) with \( v_{z_i,*,}|_T \neq 0 \). By construction, we have \( T_i \in \omega(T, T) \). Thus, \( (J_T v)|_T \) depends only on \( v|_{\omega(T,T)} \).

The estimates (4.1)–(4.4) follow analogously to the proof of [3, Section 3.2]. It remains to prove the estimates for \(-s\). To that end, note that the \( L^2 \)-adjoint operator of \( J_T^1 \) has the form

\[ (J_T^1)' v := \sum_{i=1}^{n_T-1} v'_{z_i,*,}(v, v_{z_i,*,})_{\Omega}. \]
We aim to prove that \((J^1_T)^0\colon H^s(\Omega) \to H^s(\Omega)\) is bounded for all \(0 \leq s < 1/2\). To that end, standard scaling arguments from [28] show

\[
\|v_{z_i,*}\|_{L^2(T_i)} \simeq |T_i|^{-1/2} \quad \text{and} \quad |v_{z_i,*}|_{H^s(T_i)} \simeq h_{T_i}^{-s}|T_i|^{-1/2}.
\]

Moreover, we obtain with the fact \(\int_{T_1} v_{z_i,*} \, dx = 0\) and a standard Poincaré inequality that

\[
\|v_{z_i,*}\|_{H^{-s}(\text{supp}(v_{z_i,*}))} \simeq h_{T_i}^{s}|T_i|^{1/2}.
\]

This and the results [20, 21] allow us to estimate

\[
\|(J^1_T)^0 v\|_{H^s(\Omega)}^2 \lesssim \sum_{i=1}^{n_T-1} \left( |v_{z_i,*}|_{H^s(T_i)}^2 + h_{T_i}^{-2s}|T_i|^{-1} \|v_{z_i,*}\|_{H^{-s}(\text{supp}(v_{z_i,*}))}^2 \right) \langle v, v_{z_i,*} \rangle_{\Omega}^2
\]

\[
\lesssim \sum_{i=1}^{n_T-1} h_{T_i}^{-2s}|T_i|^{-1} \|v_{z_i,*}\|_{H^{-s}(\text{supp}(v_{z_i,*}))}^2 \|v\|_{H^s(\text{supp}(v_{z_i,*}))}^2
\]

\[
\lesssim \sum_{i=1}^{n_T-1} \|v\|_{H^s(\text{supp}(v_{z_i,*}))}^2 \lesssim \|v\|_{H^s(\Omega)}^2,
\]

where we used that the \(\text{supp}(v_{z_i,*})\) have finite overlap. This shows boundedness of \((J^1_T)^0\colon H^s(\Omega) \to H^s(\Omega)\) and therefore implies (4.5) by duality. From this and the projection property, we immediately infer (4.6) for \(0 < s < 1/2\) via

\[
\|(1 - J^1_T)v\|_{H^{-s}(\Omega)} = \|(1 - J^1_T)(1 - \Pi^1_T)v\|_{H^{-s}(\Omega)}
\]

\[
\lesssim \|(1 - \Pi^1_T)v\|_{H^{-s}(\Omega)} \lesssim \|h_{T_i}^s v\|_{L^2(\Omega)},
\]

where \(\Pi^1_T\colon L^2(\Omega) \to S^1(T)\) is the \(L^2\)-orthogonal projection. Note that the corresponding estimate for \(\Pi^1_T\) is well-known and follows from duality in combination with quasi-interpolation operators of Clemént type. This concludes the proof.

**Definition 4.5.** We consider an auxiliary sequence \((\widehat{T}_\ell)_{\ell \in \mathbb{N}}\) of uniform refinements such that \(\widehat{T}_0 = T_0\) and

\[
\widehat{T}_{\ell+1} = \text{refine}^k(\widehat{T}_\ell, \widehat{T}_0),
\]

which means that each element of \(\widehat{T}_\ell\) is bisected \(k\)-times to obtain \(\widehat{T}_{\ell+1}\). There exist constants \(C_{\text{base}}, C_{\text{mesh}} \geq 1\) which depend on \(k\) and on \(T_0\) such that

\[
C_{\text{base}}^{-\ell} C_{\text{mesh}}^{-\ell} \leq \text{diam}(T) \leq C_{\text{base}}^{-\ell} C_{\text{mesh}}^{-\ell}
\]

for all \(T \in \widehat{T}_\ell\) and all \(\ell \in \mathbb{N}\). We choose \(k = k_{\text{mesh}}\) sufficiently large such that \(C_{\text{mesh}} \geq (C_{\text{sz}} + 1)^4\), where \(C_{\text{sz}}\) is defined in Lemmas 4.48 and 4.7.

**Theorem 4.6.** Recall \(\widehat{T}_\ell\) and \(C_{\text{mesh}}\) from Definition 4.5. With \(J^1_{\widehat{T}_\ell} := J^1_{\widehat{T}_0}\), define

\[
S^1_{\ell} v := \lim_{N \to \infty} (J^1_{\widehat{T}_\ell} J^1_{\widehat{T}_{\ell+1}} \ldots J^1_{\widehat{T}_{\ell+N}}) v \in S^1_{\ast}(\widehat{T}_\ell)
\]
for all $v \in L^2_s(\Omega)$. Then, the operator $S^1_s : L^2_s(\Omega) \to S^1_s(\mathcal{T}_0)$ is well-defined and satisfies

\begin{align}
(4.7) \quad & \| (1 - S^1_s) v \|_{\tilde{H}^{-s}(\Omega)} \leq C_s C^{-s \ell \ell}_{\text{mesh}} \| v \|_{L^2(\Omega)} \quad \text{for all } v \in L^2_s(\Omega) \text{ and } 0 < s < 1/2, \\
(4.8) \quad & \| (1 - S^1_s) v \|_{\tilde{H}^{-s}(\Omega)} \leq C_s C^{-s-r \ell \ell}_{\text{mesh}} \| v \|_{\tilde{H}^{-r}(\Omega)} \quad \text{for all } v \in \tilde{H}^{-r}(\Omega), 0 < r \leq s - 1/4, \\
(4.9) \quad & \| (1 - S^1_s) v \|_{L^2(\Omega)} \leq C_s C^{-s \ell/4}_{\text{mesh}} \| v \|_{H^{1/4}(\Omega)} \quad \text{for all } v \in H^{1/4}_s(\Omega), \\
(4.10) \quad & \| S^1_s v \|_{\tilde{H}^{-1/4}(\Omega)} \leq C_s \| v \|_{\tilde{H}^{-1/4}(\Omega)} \quad \text{for all } v \in \tilde{H}^{-1/4}(\Omega), \\
(4.11) \quad & \| S^1_s v \|_{H^s(\Omega)} \leq C_s \| v \|_{H^s(\Omega)} \quad \text{for all } v \in H^s_s(\Omega) \text{ and } s \in \{0, 1/4\}.
\end{align}

Moreover, there holds $S^1_s S^1_k = S^1_{\min(s, k)}$ for all $\ell, k \in \mathbb{N}_0$ as well as

\begin{equation}
S^1_s v = J^1_s J^1_{s+1} \cdots J^1_{s+k} v \quad \text{for all } v \in S^1_s(\mathcal{T}_0)
\end{equation}

for all $\ell, k \in \mathbb{N}$. The constant $C_s > 0$ depends only on $C_{sz}$ and $C_{\text{mesh}}$, whereas the constant $r$ depends only on $T_0$.

**Proof.** For brevity of presentation, we also write $H^{-s}(\Omega)$ for $\tilde{H}^{-s}(\Omega)$ in this proof. From Lemma 4.4, we obtain for $-1/2 < r \leq 1$ and $s = 1$

\begin{equation}
\| (1 - J^1_s) v \|_{H^r(\Omega)} \lesssim C^{-1-r}_{\text{mesh}} \| v \|_{H^r(\Omega)} \quad \text{for all } v \in H^r_s(\Omega).
\end{equation}

Moreover, from (4.2), we even get $\| (1 - J^1_s) v \|_{H^r(\Omega)} \lesssim \| v \|_{H^r(\Omega)}$ for all $0 \leq r \leq 1$. Interpolation arguments prove that (4.13) holds for all $s$ with $r \leq s \leq 1$. Let $u \in H^\mu_s(\Omega)$. Since $J^1_s : H^s_s(\Omega) \to H^s_s(\Omega)$ is continuous for $-1/2 < \nu \leq 1$ (see Lemma 4.4), there holds for $-1/2 < \nu \leq 1$ and $\nu + 1/4 \leq \mu \leq 1$ with $\mu \geq 0$ and by use of (4.13) that

\begin{equation}
\| (1 - J^1_s J^1_{s+1} \cdots J^1_{s+N}) u \|_{H^r(\Omega)} \\
\leq \| (1 - J^1_s) u \|_{H^r(\Omega)} + \sum_{k=0}^{N-1} \| ((J^1_s \cdots J^1_{s+k}) - (J^1_s \cdots J^1_{s+k+1})) u \|_{H^r(\Omega)} \\
\leq \| (1 - J^1_s) u \|_{H^r(\Omega)} + \sum_{k=0}^{N-1} \| (J^1_s \cdots J^1_{s+k}) \|_{H^r(\Omega) \to H^r(\Omega)} \| (1 - J^1_{s+k+1}) u \|_{H^r(\Omega)} \\
\lesssim \sum_{k=0}^{N} C_{\text{mesh}}^{-s} C^{-s \mu \ell}_{\text{mesh}} \| u \|_{H^r(\Omega)} \lesssim C^{-s \mu \ell}_{\text{mesh}} \| u \|_{H^r(\Omega)},
\end{equation}

where we used $C_{\text{mesh}}^{-s} C^{-s \mu \ell}_{\text{mesh}} \geq C_{\text{mesh}}^{-s} C^{-s \mu \ell}_{\text{mesh}}$ from Definition 4.5. Hence, (4.14) for $(\nu_1, \mu_1) =$
and $(\nu_2, \mu_2) = (1/4, 1/2)$ implies for $M \leq N$

$$
\|(J_{\ell}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1) - (J_{\ell}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1)\|_{L^2(\Omega)} \\
= \|(J_{\ell}^1 \ldots J_{\ell+N}^1)((1 - (J_{\ell+M+1}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1))u\|_{L^2(\Omega)} \\
\leq \|((1 - (J_{\ell+M+1}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1))u\|_{L^2(\Omega)} \\
+ \|((1 - (J_{\ell+M+1}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1))((1 - (J_{\ell+M+1}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1))u\|_{L^2(\Omega)} \\
\leq \|(1 - (J_{\ell+M+1}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1))u\|_{H^{1/4}(\Omega)} \\
\leq C_{\text{mesh}}^{-\ell(M)/4}\|u\|_{H^{1/2}(\Omega)}.
$$

This shows that $(J_{\ell}^1 J_{\ell+1}^1 \ldots J_{\ell+N}^1)u$ is a Cauchy-sequence in $S^1(\hat{T}_\ell)$ with respect to the $L^2(\Omega)$-norm as $N \to \infty$. Thus, for $v \in H^{1/2}(\Omega)$, the limit $S^1_k u \in S^1(\hat{T}_\ell)$ exists. The estimates (4.7)–(4.9) follow from (4.14), the convergence $\lim_{N \to \infty}(J_{\ell}^1 \ldots J_{\ell+N}^1)v = S^1_k v$ in $L^2(\Omega)$, and density arguments.

To see (4.10)–(4.11), we apply inverse estimates for $s \in \{-1/4, 0, 1/4\}$ as well as Lemma 4.4 and (4.7)–(4.8) to see

$$
\|S_k S^1_k u\|_{H^s(\Omega)} \lesssim \|S^1_k u\|_{H^s(\Omega)} + \|u\|_{H^s(\Omega)} \\
\lesssim C^{(2/5+s)}_{\text{mesh}} \|(S_k^1 - J_{\ell}^1)u\|_{H^{3/5}(\Omega)} + \|u\|_{H^s(\Omega)} \\
\lesssim C^{(2/5+s)}_{\text{mesh}} (\|(1 - S_k^1)u\|_{H^{3/5}(\Omega)} + \|(1 - J_{\ell}^1)u\|_{H^{3/5}(\Omega)}) + \|u\|_{H^s(\Omega)} \\
\lesssim \|u\|_{H^s(\Omega)},
$$

The proof of $S_k S^1_k u = S_k u$, (4.12), and the projection property follows analogously to [22, Theorem 5.6].

**Lemma 4.7.** Define the Scott-Zhang operator $J^2_T : H^1(\Omega) \to S^2(T)$ from [37]. $J^2_T$ is a projection which satisfies for all $1/2 < s < 3/2$ and all $v \in H^s(\Omega)$

$$
(4.15) \quad \|J^2_T v\|_{H^s(T)} \leq C_{sz} \|v\|_{H^s(\omega(T,T))},
$$

$$
(4.16) \quad \|J^2_T v\|_{H^s(\Omega)} \leq C_{sz} \|v\|_{H^s(\Omega)},
$$

as well as for all $0 \leq r \leq s$

$$
(4.17) \quad \|(1 - J^2_T)v\|_{H^r(\Omega)} \leq C_{sz} \text{diam}(T)^{s-r} |v|_{H^s(\omega(T,T))},
$$

$$
(4.18) \quad \|(1 - J^2_T)v\|_{H^r(\Omega)} \leq C_{sz} h_T^{1-s} \nabla v\|_{L^2(\Omega)}.
$$

The constant $C_{sz} > 0$ depends only on the shape regularity of $T$, the fact that $T$ is generated from $T_0$ by newest vertex bisection and on a lower bound on $s > 1/2$. The function $(J^2_T v)|_T$ depends only on $v|_{\omega(T,T)}$.

**Theorem 4.8.** With $J^2_T := J^2_T : H^1(\Omega) \to S^2(\hat{T}_\ell)$ denoting the standard Scott-Zhang projection from Lemma 4.7, define

$$
S^2_T v := \lim_{N \to \infty} (J^2_T J_{\ell+1}^2 \ldots J_{\ell+N}^2)v \in S^2(\hat{T}_\ell)
$$

for all $v \in H^1(\Omega)$. Then, the operator $S^2_T : H^1(\Omega) \to S^1(\hat{T}_\ell)$ is well-defined and
satisfies
\begin{equation}
\|(1 - S^2)v\|_{H^{1-s}(\Omega)} \leq C_SC_{\text{mesh}}^{-sf}\|v\|_{H^1(\Omega)} \text{ for all } v \in H^1(\Omega) \text{ and } 0 < s < 1/2,
\end{equation}
\begin{equation}
\|(1 - S^2)v\|_{H^{3/4}(\Omega)} \leq C_SC_{\text{mesh}}^{-\ell/4}\|v\|_{H^1(\Omega)} \text{ for all } v \in H^1(\Omega),
\end{equation}
\begin{equation}
\|S^2v\|_{H^r(\Omega)} \leq C_S\|v\|_{H^r(\Omega)} \text{ for all } v \in H^*_r(\Omega) \text{ and } s \in \{3/4, 1, 5/4\}.
\end{equation}
Moreover, there holds $S^2S^2 = S^2_{\min(\ell,k)}$ for all $\ell, k \in \mathbb{N}_0$ as well as
\begin{equation}
S^2v = J^2_{\ell}J^2_{\ell+1}\cdots J^2_{\ell+k}v \text{ for all } v \in S^2_1(\mathcal{T}_{\ell+k})
\end{equation}
for all $\ell, k \in \mathbb{N}$. The constant $C_S > 0$ depends only on $C_{sz}$ and $C_{\text{mesh}}$, whereas the constant $r$ depends only on $T_0$.

Proof. The proof follows analogously to [22, Theorem 5.6] and is therefore omitted.

5. Riesz bases. This section constructs suitable Riesz bases of $H^1_0(\Omega)$ and $L^2_0(\Omega)$ for Section 6. To that end, we use the that $(H^{3/4}(\Omega), H^1(\Omega), H^{5/4}(\Omega))$ form a Gelfand triple with $H^1(\Omega)$ as its pivot space.

Lemma 5.1 (from [22]). For $0 < s < 1/2$, the interpolation spaces $H^{1-s}(\Omega)$ and $H^{1+s}(\Omega)$ form a Gelfand triple in the sense $H^{1+s}(\Omega) \subset H^1(\Omega) \subset H^{1-s}(\Omega)$.

The following theorem establishes the Riesz basis for $H^1_0(\Omega)$.

Theorem 5.2. With the spaces from Definition 4.1, define
\begin{align*}
B^1_0 := \left\{ \frac{v_0}{\|v_0\|_{H^1(\Omega)}} : v_0 \in S^2_{B^1_0}(T_0) \right\}
\end{align*}
and for $\ell \geq 1$
\begin{align*}
B^1_\ell := \left\{ \frac{(1 - J^2_{\ell-1})v_0}{\|1 - J^2_{\ell-1}\|_{H^1(\Omega)}} : v_0 \in \left( \bigcup_{k \in \mathbb{N}} S^2_{B^1_0}(T_k \setminus T_{k-1}) \right) \cap (S^2(\mathcal{T}_\ell) \setminus S^2(\mathcal{T}_{\ell-1})) \right\}.
\end{align*}
Define $B^1 := \bigcup_{\ell \in \mathbb{N}} B^1_\ell$. Then, $B^1$ is Riesz bases of $\bigcup_{\ell \in \mathbb{N}} S^2(\mathcal{T}_\ell) \subseteq H^1_0(\Omega)$, i.e.,
\begin{equation}
\|\sum_{v \in B^1} \alpha_v v\|_{H^1(\Omega)} \simeq \left( \sum_{v \in B^1} \alpha_v^2 \right)^{1/2}.
\end{equation}
Moreover, $\text{diam}(\supp(v)) \simeq C_{\text{mesh}}^{-\ell}$ for all $v \in B^1_\ell$ and there holds
\begin{equation}
\|v\|_{H^s(\supp(v))} \simeq C_{\text{mesh}}^{-sf}\|v\|_{L^2(\supp(v))} \text{ for all } v \in B^1_\ell \text{ and all } 0 \leq s < 3/2.
\end{equation}

Proof. We aim to employ [18] with the operators $(S^2_{\ell})_{\ell \in \mathbb{N}_0}$ from Theorem 4.8. The $S^2_{\ell}$ are uniformly $H^1(\Omega)$ bounded and satisfy $S^2_{\ell}S^2_{k} = S^2_{\ell}$ for all $\ell \leq k$. Moreover, their ranges $S^2(\mathcal{T}_\ell)$ form a dense and nested sequence of subspaces of $H^1(\Omega)$. Lemma 5.1 confirms that $H^{3/4}(\Omega)$ is the dual space of $H^{5/4}(\Omega)$ with respect to the $H^1(\Omega)$-scalar product. Theorem 4.8 confirms the approximation estimates
\begin{align*}
\|\left(1 - S^2_{\ell}\right)u\|_{H^{3/4}(\Omega)} &\lesssim C_{\text{mesh}}\|u\|_{H^1(\Omega)},
\|\left(1 - S^2_{\ell}\right)u\|_{H^1(\Omega)} &\lesssim C_{\text{mesh}}\|u\|_{H^{5/4}(\Omega)}.
\end{align*}
as well as uniform boundedness $S^2_k : H^{3/4}(\Omega) \rightarrow H^{3/4}(\Omega)$. Standard inverse estimates prove
\[
\|S^2_k u\|_{H^{3/4}(\Omega)} \lesssim \|S^2_k u\|_{H^1(\Omega)}.
\]
Therefore, we may apply [18, Theorems 3.1&3.2] to prove
\[
\|u\|_{H^1(\Omega)}^2 \approx \sum_{\ell=0}^\infty \| (S^2_k - S^2_{k-1}) u \|_{H^1(\Omega)}^2,
\]
where we define $S^2_{-1} := 0$. The identity (4.12) implies that for $v_0 \in S^2(\mathcal{T}_k) \setminus S^2(\mathcal{T}_{k-1})$, there holds
\[
(1 - J_{k-1}) v_0 = (1 - S^2_{k-1}) v_0
\]
and Theorem 4.8 shows
\[
(S^2_k - S^2_{k-1})(1 - S^2_{k-1}) v_0 = S^2_k v_0 - S^2_k S^2_{k-1} v_0 - S^2_{k-1} v_0 + S^2_{k-1} S^2_{k-1} v_0
\]
\[
= \begin{cases} 
  S^2_k v_0 - S^2_{k-1} v_0 & k < \ell, \\
  (1 - S^2_{k-1}) v_0 & k = \ell, \\
  0 & k > \ell.
\end{cases}
\]
Thus, writing $w = \sum_{\ell \in \mathbb{N}_0} \sum_{v \in B^1_\ell} \alpha_v v$, we get with (5.3)
\[
\|u\|_{H^1(\Omega)}^2 \approx \sum_{\ell \in \mathbb{N}_0} \| (S^2_k - S^2_{k-1}) u \|_{H^1(\Omega)}^2 \approx \sum_{\ell \in \mathbb{N}_0} \| \sum_{v \in B^1_\ell} \alpha_v v \|_{H^1(\Omega)}^2.
\]
We define
\[
\overline{B}_\ell^1 := \left( \bigcup_{k \in \mathbb{N}} S^2_{B,0}(\mathcal{T}_k \setminus \mathcal{T}_{k-1}) \right) \cap (S^2(\mathcal{T}_{k}) \setminus S^2(\mathcal{T}_{k-1})).
\]
Each $v \in B^1_\ell$ is of the form $v = (1 - J^2_{k-1}) v_0$ for some $v_0 \in \overline{B}_\ell^1$.

As in the proof of [22, Theorem 6.3], we obtain
\[
\|u\|_{H^1(\Omega)}^2 \approx \sum_{\ell \in \mathbb{N}_0} \sum_{v \in B^1_\ell} \alpha_v^2 \| v \|_{H^1(\text{supp}(v))}^2.
\]
Therefore, the operator $\iota : \ell_2(B^1) \rightarrow H^1(\Omega)$, $\iota(\alpha) := \sum_{v \in B^1} \alpha_v v$ is bounded and has a bounded inverse on its closed range. Obviously, the range is dense in $\bigcup_{k \in \mathbb{N}} S^2_{B,0}(T_k) \subseteq H^1(\Omega)$ and hence $\iota$ is bijective. This shows that $B^1$ is a Riesz basis of $H^1(\Omega)$.

The scaling estimate (5.2) can be proved as follows: Let $v = (1 - J^2_{k-1}) v_0 \in B^1_\ell$ for some $v_0 \in \overline{B}_\ell^1$ and let $\omega := \text{supp}(v)$. The approximation property and the projection property of $J^2_{k-1}$ show
\[
\|w\|_{L^2(\omega)} \lesssim \|w\|_{H^s(\omega)}
\]
for all $0 \leq s < 3/2$. The converse estimates $\|w\|_{L^2(\omega)} \gtrsim C_{\text{mesh}}^{-s} \|w\|_{H^s(\omega)}$ for $0 \leq s < 3/2$ follow from standard inverse estimates. This concludes (5.2) and thus concludes the proof. \qed
Definition 5.3. We define the basis \( \tilde{B}^0 \) as  \( S_{B,\ast}(T_0) \). For all \( \ell \in \mathbb{N} \), define

\[
\tilde{B}^\ell_0 := \left\{ \frac{v + \alpha_v w_v}{\|v + \alpha_v w_v\|_{L^2(\Omega)}} : v \in \left( \bigcup_{k \in \mathbb{N}} S_k^1(T_k \setminus T_{k-1}) \right) \cap (S_k^1(\widehat{T}_\ell) \setminus S_k^1(\widehat{T}_{\ell-1})) \right\},
\]

where \( \alpha_v \in \mathbb{R} \) and \( w_v \in S_k^1(\widehat{T}_{\ell-1}) \) are such that \( \text{supp}(v) \subset \text{supp}(w_v) \) and

\[
\int_\Omega v + \alpha_v w_v \, dx = 0.
\]

Note that \( \text{supp}(v + \alpha_v w_v) \subseteq \bigcup \omega(\text{supp}(v), \widehat{T}_{\ell-1}) \) for all \( v \in \tilde{B}^0 \).

Theorem 5.4. With the spaces from Definition 4.1, define

\[
B^0 := \left\{ \frac{v_0}{\|v_0\|_{L^2(\Omega)}} : v_0 \in S_{B,\ast}(T_0) \right\}
\]

and for \( \ell \geq 1 \)

\[
B^\ell_0 := \left\{ \frac{(1 - J_{\ell-1}^1)v_0}{\|(1 - J_{\ell-1}^1)v_0\|_{L^2(\Omega)}} : v_0 \in \tilde{B}^\ell_0 \right\}.
\]

Define \( B^0 := \bigcup_{\ell \in \mathbb{N}} B^\ell_0 \). Then, \( B^0 \) is Riesz bases of \( \bigcup_{\ell \in \mathbb{N}} S^1_{\ell}(T_k) \subseteq L^2(\Omega) \), i.e.,

\[
\| \sum_{v \in B^0} \alpha_v v \|_{L^2(\Omega)} \lesssim \left( \sum_{v \in B^0} \| \alpha_v \|^2 \right)^{1/2}.
\]

Moreover, \( \text{diam}(\text{supp}(v)) \lesssim C_{\text{mesh}}^{-\ell} \) for all \( v \in B^\ell_0 \) and there holds for all \( v \in B^\ell_0 \)

\[
\|v\|_{H^{-s}(\text{supp}(v))} \lesssim C_{\text{mesh}}^{-s\ell} \|v\|_{L^2(\text{supp}(v))} \quad \text{for all } 0 \leq s < 3/2,
\]

\[
\|v\|_{\tilde{H}^{-s}(\text{supp}(v))} \lesssim C_{\text{mesh}}^{-s\ell} \|v\|_{L^2(\text{supp}(v))} \quad \text{for all } 0 \leq s \leq 1.
\]

Proof. Again, we use [18] with the operators \( (S^1_\ell)_{\ell \in \mathbb{N}_0} \) from Theorem 4.6. Additionally, we use \( \Pi_0 : L^2(\Omega) \rightarrow \mathbb{R} \) defined by \( \Pi_0(w) := |\Omega|^{-1} \int_\Omega w \, dx \). Obviously, \( \Pi_0 : \mathcal{Y} \rightarrow \mathcal{Y} \) is bounded for \( \mathcal{Y} \in \{H^{-1/4}(\Omega), L^2(\Omega), H^{1/4}(\Omega)\} \). The operators \( S^1_\ell \) are uniformly \( L^2(\Omega) \) bounded and satisfy for all \( \ell \leq k \)

\[
S^1_\ell S^1_k = S^1_\ell (1 - \Pi_0)(S^1_k (1 - \Pi_0) + \Pi_0) + \Pi_0(S^1_k (1 - \Pi_0) + \Pi_0)
\]

\[
= S^1_\ell S^1_k (1 - \Pi_0) + \Pi_0 = S^1_\ell.
\]

Moreover, their ranges \( S^1(\widehat{T}_\ell) \) form a dense and nested sequence of subspaces of \( L^2(\Omega) \). Theorem 4.6 confirms the approximation estimates

\[
\|(1 - S^1_\ell)u\|_{\tilde{H}^{-1/4}(\Omega)} \lesssim C_{\text{mesh}}^{-\ell/4} \|(1 - \Pi_0)u\|_{L^2(\Omega)} \leq C_{\text{mesh}}^{-\ell/4} \|u\|_{L^2(\Omega)} ,
\]

\[
\|(1 - S^1_\ell)u\|_{L^2(\Omega)} \lesssim C_{\text{mesh}}^{-\ell/4} \|(1 - \Pi_0)u\|_{H^{1/4}(\Omega)} \leq C_{\text{mesh}}^{-\ell/4} \|u\|_{H^{1/4}(\Omega)}
\]

as well as uniform boundedness \( \overline{S}^1_\ell : \tilde{H}^{-1/4}(\Omega) \rightarrow \tilde{H}^{-1/4}(\Omega) \) by

\[
\|\overline{S}^1_\ell u\|_{\tilde{H}^{-1/4}(\Omega)} \lesssim \|(1 - \Pi_0)u\|_{\tilde{H}^{-1/4}(\Omega)} + \|\Pi_0 u\|_{\tilde{H}^{-1/4}(\Omega)} \lesssim \|u\|_{\tilde{H}^{-1/4}(\Omega)}.
\]
Standard inverse estimates prove
\[
\|S_\ell^1 u\|_{H^{1/4}(\Omega)} \lesssim C_{\text{mesh}} \|S_\ell^1 u\|_{L^2(\Omega)},
\]
\[
\|S_\ell^1 u\|_{L^2(\Omega)} \lesssim C_{\text{mesh}} \|S_\ell^1 u\|_{H^{-1/4}(\Omega)}.
\]

Therefore, we may apply [18, Theorems 3.1&3.2] to prove
\[
(5.8) \quad \|u\|_{L^2(\Omega)}^2 \simeq \sum_{\ell=0}^\infty \|S_\ell^1 - S_{\ell-1}^1 u\|^2_{H^1(\Omega)},
\]
where we define \(S_{-1}^1 := S_1^1 := 0\). The identity (4.12) implies that for \(v_0 \in \bar{B}_k^0\) for \(\ell \geq 1\), there holds
\[
J_k^1 v_0 = S_k^1 v_0 = S_k^1 v_0 \quad \text{for all } k \geq 1.
\]
The identity (5.7) shows for \(v_0 \in \bar{B}_k^0\)
\[
(S_\ell^1 - S_{\ell-1}^1)(1 - S_{\ell-1}^1)v_0 = S_\ell^1 v_0 - S_\ell^1 S_{\ell-1}^1 v_0 - S_{\ell-1}^1 v_0 + S_{\ell-1}^1 S_{\ell-1}^1 v_0
\]
\[
= \begin{cases} 
S_\ell^1 v_0 - S_{\ell-1}^1 v_0 = 0 & k < \ell, \\
(1 - S_{\ell-1}^1)v_0 = (1 - J_{\ell-1}^1 v_0) & k = \ell, \\
0 & k > \ell.
\end{cases}
\]
Thus, writing \(w = \sum_{\ell \in \mathbb{N}_0} \sum_{v \in B_\ell^0} \alpha_v v\), we get with (5.8)
\[
(5.9) \quad \|w\|_{L^2(\Omega)}^2 \simeq \sum_{\ell \in \mathbb{N}_0} \|S_\ell^1 - S_{\ell-1}^1 w\|_{L^2(\Omega)}^2 \simeq \sum_{\ell \in \mathbb{N}_0} \| \sum_{v \in B_\ell^0} \alpha_v v \|^2_{L^2(\Omega)}.
\]

We prove that for \(T \in \hat{T}_{\ell-1}\) and the orthogonal projection \(\Pi_T : L^2(T) \to \mathcal{P}^1(T)\), there holds
\[
\| \sum_{v \in B_\ell^0} \alpha_v v \|_{L^2(T)}^2 \geq \| \sum_{v \in B_\ell^0} \alpha_v (1 - \Pi_T)v \|_{L^2(T)}^2
\]
\[
\simeq \sum_{v \in B_\ell^0} \alpha_v^2 \| (1 - \Pi_T)v \|_{L^2(T)}^2 \simeq \sum_{(1 - \Pi_T)v \neq 0} \alpha_v^2.
\]

To see this, we use that \(|v|_T \in v_00|_T + \mathcal{P}^1(T)|\), where \(v_00 \in \left( \bigcup_{k \in \mathbb{N}} S_k^1(\mathcal{T}_k \setminus \mathcal{T}_{k-1}) \right) \cap (S_1^1(\hat{T}_\ell) \setminus S_1^1(\hat{T}_{\ell-1}))\) is a hat-function associated with a node in \(\omega(T, \hat{T}_{\ell-1})\). Hence, all non-zero \(v_00|_T\) form a linear independent set. Therefore, also the non-zero \((1 - \Pi_T)(v|_T) = (1 - \Pi_T)(v_00|_T)\) are linearly independent. Since additionally, \((1 - \Pi_T)(v|_T) \in \mathcal{P}^1(T \cap \hat{T}_\ell)\), a scaling argument and norm equivalence on finite dimensional spaces proves (5.10). Summing up over all \(T \in \hat{T}_{\ell-1}\) shows \(\|w\|_{L^2(\Omega)}^2 \gtrsim \sum_{\ell \in \mathbb{N}_0} \sum_{v \in B_\ell^0} \alpha_v^2 \|v\|_{L^2(\text{supp}(v))}^2\). Using the finite overlap of the \(v \in B_\ell^0\), we finally prove
\[
\|w\|_{L^2(\Omega)}^2 \simeq \sum_{\ell \in \mathbb{N}_0} \sum_{v \in B_\ell^0} \alpha_v^2 \|v\|_{L^2(\text{supp}(v))}^2.
\]
Therefore, the operator \( \iota : \ell_2(B^0) \to L^2_\omega(\Omega) \), \( \iota(\alpha) := \sum_{v \in B^0} \alpha_v v \) is bounded and has a bounded inverse on its closed range. Obviously, the range is dense in \( \bigcup_{k \in \mathbb{N}} S^1_\ast(T_k) \subseteq L^2(\Omega) \) and hence \( \iota : \ell_2(B^0) \to \bigcup_{k \in \mathbb{N}} S^1_\ast(T_k) \) is bijective. This concludes that \( B^0 \) is a Riesz basis of \( \bigcup_{k \in \mathbb{N}} S^1_\ast(T_k) \subseteq L^2(\Omega) \).

The scaling estimates (5.6) can be proved as follows: Let \( v = (1 - J_{\ell-1}) v_0 \in B^0 \) for some \( v_0 \in B^0 \) and let \( \omega := \text{supp}(v) \). By construction of \( J_{\ell-1} \), we see that \( J_{\ell-1} v_0 \) has integral mean zero on its support. Thus also \( v \) has zero mean on its support. Therefore, a Poincaré estimate proves

\[
\|v\|_{\dot{H}^{-s}(\omega)} \lesssim C^{-st}_{\text{mesh}} \|v\|_{L^2(\omega)}
\]

for all \( 0 \leq s \leq 1 \). The approximation property (4.3) and the projection property of \( J_{\ell-1} \) show

\[
\|v\|_{L^2(\omega)} \lesssim C^{-st}_{\text{mesh}} \|v\|_{H^s(\omega)}
\]

for all \( 0 \leq s < 3/2 \). The converse estimates \( \|w\|_{L^2(\omega)} \gtrsim C^{st}_{\text{mesh}} \|w\|_{H^s(\omega)} \) for \( 0 \leq s < 3/2 \) as well as \( \|v\|_{\dot{H}^{-s}(\omega)} \gtrsim C^{st}_{\text{mesh}} \|v\|_{L^2(\omega)} \) for \( 0 \leq s \leq 1 \) follow from standard inverse estimates. This concludes the proof.

**Lemma 5.5.** Let \( T \in T_{\text{grad}} \) and \( \ell \in \mathbb{N} \). Let \( T \in T \setminus \hat{T}_\ell \) such that \( T|_T \) is a strict local refinement of \( \hat{T}_\ell \). Then, \( T|_{\omega^{D_{\text{grad}}}(T, \hat{T}_\ell)} \) is a local refinement of \( \hat{T}_\ell \).

**Proof.** Let \( L \in \mathbb{N} \) denote the level of the elements in \( \hat{T}_\ell \). By assumption, there holds \( \text{level}(T) > L \). Assume there exists \( T' \in T|_{\omega^{D_{\text{grad}}}(T, \hat{T}_\ell)} \) with \( \text{level}(T') < L \). Assumption (2.4) implies \( \text{level}(T'') \leq L \) for all \( T'' \in \omega^{D_{\text{grad}}}(T', T) \). This shows that there holds

\[
T \in \bigcup \omega^{D_{\text{grad}}}(T', \hat{T}_\ell) \subseteq \bigcup \omega^{D_{\text{grad}}}(T', T).
\]

With (2.4), this implies \( \text{level}(T) \leq L \), which contradicts the assumption that \( T|_T \) is a strict local refinement of \( \hat{T}_\ell \).

**Lemma 5.6.** Given a mesh \( T \) with satisfies (2.4) for \( D_{\text{grad}} \geq 3 \), there holds

\[
S^2_0(T) = \text{span}(B^1 \cap S^2(T)) \quad \text{and} \quad S^1_0(T) = \text{span}(B^0 \cap S^1(T)).
\]

**Proof.** We note that \( S^2_0(T) = \text{span}\{v_0 \in \bigcup_{i=0}^\infty \tilde{B}^1_i : v_0 \in S^2_0(T)\} \). For each \( v_0 \in S^2_0(T) \cap \tilde{B}^1_0 \), we note that \( J_{\ell-1} v_0 \) is supported on \( \omega(\text{supp}(v_0), \hat{T}_{\ell-1}) \). Since \( v_0 \in S^2_0(T) \cap \tilde{B}^1_0 \) implies that \( T|_{\text{supp}(v_0)} \) is a true local refinement of \( \tilde{T}_{\ell-1} \), there exists at least one \( T \in T \setminus \hat{T}_{\ell-1} \) with \( T \subseteq \text{supp}(v_0) \). By definition of \( v = (1 - J^2_{\ell-1}) v_0 \), we know that \( \text{supp}(v) \subseteq \bigcup \omega^2(T, \hat{T}_{\ell-1}) \). Lemma 5.5 proves that \( T|_{\omega^2(T, \hat{T}_{\ell-1})} \) is a local refinement of \( \tilde{T}_{\ell-1} \) and hence \( v \in S^2_0(T) \). This concludes \( S^2_0(T) = \text{span}\{v \in B^1 : v \in S^2_0(T)\} \).

For \( S^1_0(T) \), Definition 5.3 states that each \( v_0 \in S^1_0(T) \cap \tilde{B}^1_0 \) is of the form \( v_0 \in v_{00} + w_{00} \), where \( w_{00} \in S^2_0(\tilde{T}_{\ell-1}) \) and \( \text{supp}(w_{00}) \subseteq \text{supp}(w_{00}) \). Therefore, we have for \( v = (1 - J^1_{\ell}) v_0 \) that \( \text{supp}(v) \subseteq \omega^1(T, \hat{T}_{\ell-1}) \) for all \( T \in T \) with \( T \subseteq \text{supp}(v_{00}) \). There exists at least one such \( T \) which satisfies \( T \in T \setminus \hat{T}_{\ell-1} \). Hence, Lemma 5.5 concludes the proof as in the previous case.
6. Application. The abstract theory developed in the previous sections allows us to prove optimality of the adaptive algorithm for the stationary Stokes problem.

6.1. Model problem. Although the framework developed above seems to be fairly general, our main goal here is to prove general quasi-orthogonality for the stationary Stokes problem, which reads

\[-\Delta u + \nabla p = f \quad \text{in } \Omega,
\]
\[\text{div} u = 0 \quad \text{in } \Omega,
\]
\[u = 0 \quad \text{on } \Gamma := \partial \Omega,
\]
\[\int_{\Omega} p \, dx = 0
\]

for given given functions \( f \in L^2(\Omega) \) with weak solutions \( u \in H^1_0(\Omega)^2 \) and \( p \in L^2(\Omega) \). We define the space \( \mathcal{X} := H^1_0(\Omega)^2 \times L^2(\Gamma) \).

The weak formulation of (6.1) reads: Find \((u, p) \in \mathcal{X}\) such that all \((v, q) \in \mathcal{X}\) satisfy

\[
a((u, p), (v, q)) := \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Omega} p \, \text{div} v \, dx - \int_{\Omega} q \, \text{div} u \, dx = \int_{\Omega} fv \, dx.
\]

Existence of unique solutions of the Stokes problem via (2.1) is guaranteed by [9]. For the purpose of discretization, we choose standard Taylor-Hood elements defined by \(X_T := S^2(T)^2 \times S^1(T)\).

Thus, the Galerkin formulation reads: Find \((u_T, p_T) \in X_T\) such that all \((v, q) \in X_T\) satisfy

\[
a((u_T, p_T), (v, q)) := \int_{\Omega} \nabla u_T \cdot \nabla v \, dx - \int_{\Omega} p_T \, \text{div} v \, dx - \int_{\Omega} q \, \text{div} u_T \, dx = \int_{\Omega} fv \, dx.
\]

The Galerkin formulation satisfies (2.2) if \( T_0 \) contains at least three triangles, as proved in [8].

We use a locally equivalent variation proposed in [27] of the classical error estimator proposed by Verfürth [41], i.e., for all \( T \in \mathcal{T} \) define

\[
\eta_T(T)^2 := \text{diam}(T)^2 \| f + \Delta u_T - \nabla p_T \|_{L^2(T)}^2 + \text{diam}(T) \| \partial_n u_T \|_{L^2(\partial T \cap \Omega)}^2
\]
\[+ \text{diam}(T) \| \text{div}(u_T) \|_{L^2(T)}^2 \| \partial T \cap \Omega \|_{L^2(\partial T)},
\]

where \([\cdot]\) denotes the jump across an edge of \( T \). (Note that there are also other error estimators which could be used here, e.g., [35].) The overall estimator reads

\[
\eta(T) := \left( \sum_{T \in \mathcal{T}} \eta_T(T)^2 \right)^{1/2} \quad \text{for all } T \in \mathcal{T}.
\]

We define \( \mathcal{X}_T := S^2(T) \times S^1(T) \) to denote the adaptively generated spaces from Algorithm 2.4.

6.2. Main result. The following result shows rate optimality of the adaptive algorithm and is the main result of the paper.
THEOREM 6.1 (Optimality of the adaptive algorithm). Given sufficiently small \( \theta > 0 \) and sufficiently large \( D_{\text{grad}} \geq 1 \), Algorithm 2.4 applied to the stationary Stokes problem as described above guarantees rate-optimal convergence, i.e., there exists a constant \( C_{\text{opt}} > 0 \) such that

\[
C_{\text{opt}}^{-1}\|u\|_{A_s} \leq \sup_{\ell \in \mathbb{N}_0} \frac{\eta(\ell)}{(#\mathcal{T}_\ell - #\mathcal{T}_0 + 1)^{-s}} \leq C_{\text{opt}}\|u\|_{A_s},
\]

for all \( s > 0 \) with \( \|u\|_{A_s} < \infty \).

Proof. We have to specify a mesh-refinement strategy which ensures (2.4) and fits into the framework of [10, Section 2.4]. To that end, we use the strategy specified in [19, Section A.3] (note that the condition in [19, Section A.3] and (2.4) are equivalent up to shape regularity). Then, the result follows immediately from [10, Theorem 4.1] and [22, Lemma 2.3], after we prove the axioms (A1)–(A4) in the sections below. (Note that instead of this proof, we could also use Theorem 6.3 together with [27], in which general quasi-orthogonality is assumed in order to prove optimality.) □

6.3. Proof of (A1), (A2), and (A4). The proofs of the axioms (A1), (A2), and (A4) will not be surprising to experts. They can be found in [27].

Proof of (A1)/\( \ell \)(A2). The proof follows from standard arguments as for the Poisson problem (see, e.g., [14]) and can be found in condensed form in [27, Lemma 3.2].

Proof of (A4). The property (A4) can be found in [27, Lemma 3.1]. □

6.4. Proof of (A3). The main innovation of this paper is the proof of general quasi-orthogonality (A3). To that end, we first need to show that we can restrict the problem to the space \( \tilde{\mathcal{X}} = \bigcup_{\ell \in \mathbb{N}_0} \mathcal{X}_\ell \subset \mathcal{X} \). From (2.2), we know that for all \( \varepsilon > 0 \) and all \((\tilde{u}, \tilde{p}) \in \tilde{\mathcal{X}}\), there exists \( \ell \in \mathbb{N} \) and a Galerkin approximation \((u_\ell, p_\ell) \in \mathcal{X}_\ell\) with \( a((u_\ell, p_\ell), (v, q)) = a((\tilde{u}, \tilde{p}), (v, q)) \) for all \((v, q) \in \mathcal{X}_\ell\) such that \( \| (\tilde{u}, \tilde{p}) - (u_\ell, p_\ell) \|_X \leq \varepsilon \).

This implies

\[
\sup_{(v, q) \in \tilde{\mathcal{X}}} a((\tilde{u}, \tilde{p}), (v, q)) \geq \sup_{(v, q) \in \mathcal{X}_\ell} a((\tilde{u}, \tilde{p}), (v, q)) = \sup_{(v, q) \in \mathcal{X}_\ell} a((u_\ell, p_\ell), (v, q)) \geq \| (u_\ell, p_\ell) \|_X \geq \| (\tilde{u}, \tilde{p}) \|_X - \varepsilon
\]

with uniform hidden constant from (2.2). Choosing \( \varepsilon \) sufficiently small proves the inf-sup condition for \( a(\cdot, \cdot) \) and \( \tilde{\mathcal{X}} \). Let \((\tilde{u}, \tilde{p}) \in \tilde{\mathcal{X}}\) denote the unique solution of (6.2) with respect to \( \tilde{\mathcal{X}} \). Since \((u_\ell, p_\ell)\) from (2.5) are also Galerkin approximations of \((\tilde{u}, \tilde{p})\), there holds \( \lim_{\ell \to \infty} \| (\tilde{u}, \tilde{p}) - (u_\ell, p_\ell) \|_X = 0 \). Hence, [10, Corollary 4.8] shows \((u, p) = (\tilde{u}, \tilde{p}) \in \tilde{\mathcal{X}}\).

To fit the problem into our abstract framework, we choose the following Riesz basis of \( \tilde{\mathcal{X}} \) from Theorems 5.2&5.4:

\[
B := \{ (v, 0, 0) : v \in B^1 \} \cup \{ (0, v, 0) : v \in B^1 \} \cup \{ (0, 0, w) : w \in B^0 \}.
\]

We recall that \( \mathcal{X}_\ell \subset \mathcal{X}_{\ell+1} \subset \mathcal{X} \) are nested finite dimensional spaces generated by the adaptive algorithm described in Section 2.2. We introduce the level function \( L(w) = \ell \) for all \( w \in (B^1_\ell)^2 \times B^0_\ell \) and also \( L(v) = \ell \) for all \( v \in B^1_\ell \cup B^0_\ell \). We order the functions in \( B \) such that \( \mathcal{X}_\ell = \text{span}\{w_1, w_2, \ldots, w_{N_\ell}\} \) for particular \( N_\ell \in \mathbb{N} \) and all \( \ell \in \mathbb{N} \) (note that this is possible due to Lemma 5.6).

The proofs below will use several metrics defined in [22], which we recall in the following:
Definition 6.2. For $B := B^0 \cup B^1$, define the following functions:
- $\delta: B \times B \to \{0, 1\}$ is defined by $\delta(v, w) = 1$ if $v \neq w$ and $\delta(v, w) = 0$ if $v = w$.
- $\delta_k: B \times B \to \mathbb{N}$ is defined by
  \[
  \delta_k(v, w) := \min \{n \in \mathbb{N} : \exists T_1, \ldots, T_n \in \hat{T}_k, \text{mid}(v) \cap T_1 \neq \emptyset, \text{mid}(w) \cap T_n \neq \emptyset, T_i \cap T_{i+1} \neq \emptyset, i = 1, \ldots, n-1\},
  \]
where mid($\cdot$) denotes the barycenter of the support of the function.
- $d_1: B \times B \to \mathbb{N}$ is defined by
  \[
  d_1(v, w) := \delta_{\min\{L(v), L(w)\}}(v, w).
  \]
- Given $\beta > 0$, $d_2: B \times B \to [0, \infty]$ is defined by
  \[
  d_2(v, w) := \delta(v, w) + \beta|L(v) - L(w)| + \log(\delta(v, w) + d_1(v, w)).
  \]
- Given $\gamma > 0$, $d_3: B \times B \to [0, \infty]$ is defined by
  \[
  d_3(v, w) := \begin{cases} 
  \gamma^{\max\{L(v), L(w)\}} & L(v) \neq L(w), \\
  \delta(v, w) + d_1(v, w) - 1 & L(v) = L(w).
  \end{cases}
  \]

It is shown in [22, Section 4] that for sufficiently large $\beta, \gamma > 0$, $d_2$ and $d_3$ are metrics on $B$. If $B$ is identified with $\mathbb{N}, d_3$ even satisfies (3.2).

Theorem 6.3. Given $\varepsilon > 0$ and under all previous assumptions, there exists $D_{\text{grad}} > 0$ sufficiently large such that the solutions (2.3) of the stationary Stokes problem (6.1) satisfy general quasi-orthogonality (A3).

Proof. With the basis $B$, define the matrix $A \in \mathbb{R}^{N \times N}$ by
\[
A_{ij} := a(w_j, w_i).
\]
The spaces $A_\ell, \ell \in \mathbb{N}$ induce the natural block structure $N_1, N_2, \ldots$ on $A$. Since $B$ is a Riesz basis, (2.1)–(2.2) of $a(\cdot, \cdot)$ imply that $A$ satisfies (3.4). By reordering $B$ such that $L(w_{k_i}) \leq L(w_{k_j})$ for all $i \leq j$, we obtain a permuted matrix
\[
\bar{A}_{ij} := A_{k_i, k_j} \text{ or } \bar{A} = P^T A
\]
for some permutation matrix $P \in \{0, 1\}^{N \times N}$ defined by $P_{ij} = 1$ if and only if $i = k_j$. We introduce a block-structure on $B$ with $n_1, n_2, \ldots$ such that $\{w_{k_i} : i = n_r, \ldots, n_{r+1} - 1\} = \{w \in B : L(w) = r\}$. Then, Lemmas 6.6–6.7 below show that there exists $\bar{A}^c$ which is block-banded for some bandwidth $b$ such that
\[
\|\bar{A} - \bar{A}^c\|_2 \leq \varepsilon.
\]
Moreover, if we identify $i \mapsto w_{k_i}$, there holds $\bar{A}^c \in B(d_2)$ with the metric $d_2(\cdot, \cdot)$ from Definition 6.2. By definition of $B$, we observe that $\{w_{k_i} : i = 1, \ldots, n_{r+1} - 1\}$ spans the space
\[
\left(S_0^0(\hat{T}_\ell)^2 \times S_1^1(\hat{T}_\ell)\right) \cap \bigcup_{\ell \in \mathbb{N}} \left(S_0^0(T_\ell)^2 \times S_1^1(T_\ell)\right) = S_0^0(\mathcal{T})^2 \times S_1^1(\mathcal{T}),
\]
where $\mathcal{T} \in \mathbb{T}$ is the finest mesh, which is both coarser than $\hat{T}_\ell$ and there exists $\ell \in \mathbb{N}$ such that $T_\ell$ is a refinement of $\mathcal{T}$. Therefore, [8] shows that (2.2) is satisfied for
\( \mathcal{X}_r := \mathcal{S}_r^2(\mathcal{T})^2 \times \mathcal{S}_r^1(\mathcal{T}) \) (for all \( r \in \mathbb{N} \)) and hence (3.4) holds also for the block-structure on \( \tilde{A} \). Choosing \( \varepsilon > 0 \) sufficiently small, we ensure that \( \tilde{A}^\varepsilon \) satisfies (3.4) as well. Thus, Theorem 3.11 shows that there exists an approximate block-\( LDU \)-factorization with \( \| \tilde{A} - LDU \|_2 \leq 2\varepsilon \). Moreover, the factors \( L^{-1}, D, U^{-1} \in \mathcal{B}(d_2) \) are block-banded with bandwidth \( b \). Since \( D \) is block-diagonal, we also have \( D \in \mathcal{B}(d_3) \) for the metric \( d_3(\cdot, \cdot) \) from Definition 6.2. This shows \( D \in \mathcal{F}(d_3) \). We define a new basis \( B^U \) by

\[
B^L := \bigcup_{\ell \in \mathbb{N}} B^U_\ell, \quad \text{and} \quad B^U_\ell := \{ v_{kj} := \sum_{i=1}^{\infty} (U^{-1})_{ij} w_{ki} : j \in \mathbb{N}, L(w_{kj}) = \ell \}.
\]

and we define \( B^L \) by

\[
B^L := \bigcup_{\ell \in \mathbb{N}} B^L_\ell, \quad \text{and} \quad B^L_\ell := \{ w_{kj} := \sum_{i=1}^{\infty} (L^{-1})_{ji} w_{ki} : j \in \mathbb{N}, L(w_{kj}) = \ell \}.
\]

Since \( L^{-1}, U^{-1} \) are inversely bounded uniformly in \( \varepsilon \), we see that also \( B^U \) and \( B^L \) are a Riesz basis of \( \mathcal{X} \). Moreover, since \( L^{-1}, U^{-1} \in \mathcal{B}(d_2, b') \) and block-banded with a certain bandwidth \( b' \), and since the diagonal blocks of \( L^{-1} \) and \( U^{-1} \) are just identities, we have

\[
v_{kj} = w_{kj} + \sum_{\ell = \overline{L(w_{kj}) - b'}}^L (U^{-1})_{ij} w_{ki},
\]

\[
w_{kj} = w_{kj} + \sum_{\ell = \overline{L(w_{kj}) - b'}}^L (L^{-1})_{ji} w_{ki},
\]

for some \( n \in \mathbb{N} \) which depends only on \( b' \). The result [22, Lemma 7.2] shows that since all \( T_\ell \) satisfy (2.4) for sufficiently large \( D_{\text{grad}} \geq n + C_{\text{grad}} \), we have that \( w_j \in \mathcal{X}_\ell \) implies \( v_j, w_j \in \mathcal{X}_\ell \). Therefore, we have

\[
\mathcal{X}_\ell = \text{span}\{ v_1, v_2, \ldots, v_{N_\ell} \} = \text{span}\{ w_1, w_2, \ldots, w_{N_\ell} \}.
\]

Moreover, with \( C_{ij} := a(v_{kj}, w_{kj}) \), we see \( C = L^{-1}(\tilde{A}U^{-1}) \) and hence

\[
\| C - D \|_2 = \| L^{-1}(\tilde{A} - LDU)U^{-1} \|_2 \lesssim \varepsilon,
\]

where we used that \( L^{-1} \) and \( U^{-1} \) are bounded uniformly in \( \varepsilon \) (see Theorem 3.11). Considering \( M := PCP^T \) and \( M^\varepsilon := PDP^T \), we obtain with the above \( \| M - M^\varepsilon \|_2 \lesssim \varepsilon \) as well as \( M^\varepsilon \in \mathcal{F}(d_3) \). Since \( M \) is the Galerkin matrix of \( a(\cdot, \cdot) \) with respect to \( \mathcal{X}_\ell \), \( \ell \in \mathbb{N} \), (2.2) implies that \( M \) satisfies (3.4). Choosing \( \varepsilon > 0 \) sufficiently small, we ensure that also \( M \) is satisfies (3.4). Thus, Theorem 3.15 (with \( \mathcal{X} \) instead of \( \mathcal{X} \)) applies and concludes the proof.

6.5. Almost bandedness of differential operator matrices.

**Lemma 6.4.** Let \( v \in H^s(\omega) \) for some \( 0 \leq s < 1/2 \) and some Lipschitz domain \( \omega \subseteq \Omega \). Then, there holds for the extension of \( v \) to \( \Omega \) by zero

\[
\| v \|_{H^s(\Omega)} \leq C_{\text{res}} \| v \|_{H^s(\omega)}.
\]

The constant \( C_{\text{res}} > 0 \) does only depend on the shape of \( \omega \).
Proof. There holds
\[
|v|_{H^s(\Omega)}^2 - |v|_{H^s(\omega)}^2 = 2 \int_{\Omega \setminus \omega} \int_{\omega} |v(x) - v(y)|^2 \frac{1}{|x-y|^{2s+2}} \, dy \, dx \\
= 2 \int_\omega |v(y)|^2 \int_{\Omega \setminus \omega} \frac{1}{|x-y|^{2s+2}} \, dx \, dy \\
\lesssim \int_\omega |v(y)|^2 \, dy = \|v\|_{L^2(\text{supp}(v))}^2,
\]
where we used that in polar coordinates centered at \(y\), the inner integral reads
\[
\int_{\Omega \setminus \omega} \frac{1}{|x-y|^{2s+2}} \, dx = \int_{\Omega \setminus \omega} \frac{1}{r^{2s}} \, dr \lesssim 1.
\]
This, together with norm equivalence discussed in Definition 4.3 concludes the proof.\[\Box\]

**Lemma 6.5.** Let \(k \leq \ell \in \mathbb{N}\), let \(v \in P^2(\tilde{T}_k)\) with \(\text{supp}(v) \subseteq \bigcup \omega^n(T, \tilde{T}_k)\) for some \(T \in \tilde{T}_k\). Additionally, let \(w \in B_1^k \cup B_0^k\). Then, there holds for \(0 \leq s < 3/2\) if \(v \in H^s(\text{supp}(v) \cup \text{supp}(w))\) that
\[
\|v\|_{H^s(\text{supp}(v))} \leq C_{w} C_{\text{mesh}} \|v\|_{H^s(\text{supp}(v))}.
\]
The constant \(C_{w} > 0\) depends only on \(T_0\) and \(n \in \mathbb{N}\).

**Proof.** First, we prove the statement for \(s = 0\). To that end, consider the affine transformations \(\phi_v, \phi_w : \mathbb{R}^2 \to \mathbb{R}^2\) with \(\phi_v(\omega_v) = \text{supp}(v)\) and \(\phi_w(\omega_w) = \text{supp}(w)\), where \(\omega_v\) and \(\omega_w\) have unit area and belong to a finite family of shapes depending only on \(T_0\) and \(n \in \mathbb{N}\). We obtain
\[
\|v\|_{L^2(\text{supp}(w))} = \|\text{supp}(w)\| \|v \circ \phi_w\|_{L^2(\omega_w)} \leq \|\text{supp}(w)\| \|v \circ \phi_w\|_{L^\infty(\omega_w)} \\
\leq \|\text{supp}(w)\| \|v \circ \phi_v\|_{L^\infty(\omega_v)},
\]
where we used the fact that \(\phi_v(\omega_v) = \text{supp}(v)\) in the last estimate. Since \(v \circ \phi_v\) belongs to a finite dimensional space which depends only on \(\omega_v\), we further conclude
\[
|\text{supp}(w)| \|v \circ \phi_v\|_{L^2(\omega_v)} \lesssim \|\text{supp}(w)\| \|v \circ \phi_v\|_{L^2(\omega_v)} \approx \frac{|\text{supp}(w)|}{|\text{supp}(v)|} \|v\|_{L^2(\text{supp}(v))}.
\]
Both estimates and the support size estimates in Theorems 5.2 & 5.4 conclude the statement for \(s = 0\). By applying the case \(s = 0\) to \(\nabla v\) and \(w\), we immediately prove the case \(s = 1\). To get the intermediate cases, we employ the interpolation lemma for the operators \(T^k_w : P^2(\omega^n(T, \tilde{T}_k)) \cap H^s(\Omega) \to P^2(\omega^n(T, \tilde{T}_k) \cap \text{supp}(w)) \cap H^s(\Omega)\). The above shows that \(T^1_w\) and \(T^0_w\) are uniformly bounded for all \(w \in B_1^k \cup B_0^k\), with operator norms \(C_{w} C_{\text{mesh}}^{k-1}\). The interpolation lemma implies boundedness for all \(T^s_w, 0 \leq s \leq 1\), where the operators norms are independent of \(T\) or \(w\). The cases \(1 < s < 3/2\) follow from the same argument applied to \(\nabla v\).

**Lemma 6.6.** Let \(M_{ij} := \langle \nabla v_i, \nabla v_j \rangle_{\Omega}\) for all \(i, j \in \mathbb{N}\) with \(v_i, v_j \in B^1\). Given \(\varepsilon > 0\), there exists \(M^\varepsilon \in \mathbb{R}^{N \times N}\) and a constant \(C_M > 0\) such that
\[
\|M^\varepsilon - M\|_2 \leq \varepsilon.
\]
as well as
\[
(6.4) \quad \left( |L(v_i) - L(v_j)| > C_M \text{ or } d_1(v_i, v_j) > C_M \right) \implies M_{ij} = 0.
\]
Proof. Assume $k := L(v_i) \leq \ell := L(v_j)$. There holds

$$|\langle \nabla v_i, \nabla v_j \rangle_\Omega| \leq \|\nabla v_i\|_{H^{1/4}(\text{supp}(v_i))} \|\nabla v_j\|_{H^{-1/4}(\text{supp}(v_j))}.$$ 

Lemma 6.5 proves $\|\nabla v_i\|_{H^{1/4}(\text{supp}(v_i))} \lesssim \|v_i\|_{H^{5/4}(\text{supp}(v_i))} \lesssim C^{-1}_{\text{mesh}} \|v_i\|_{H^{5/4}(\text{supp}(v_i))}$ as well as

$$\|\nabla v_j\|_{H^{-1/4}(\text{supp}(v_j))} = \sup_{w \in H^1(\text{supp}(v_j))^2} \frac{\langle \nabla v_j, w \rangle_{\text{supp}(v_j)}}{\|w\|_{H^1(\text{supp}(v_j))}}$$

$$= \sup_{w \in H^1(\text{supp}(v_j))^2} \frac{\langle v_j, \text{div} w \rangle_{\text{supp}(v_j)}}{\|w\|_{H^1(\text{supp}(v_j))}} \leq \|v_j\|_{L^2(\text{supp}(v_j))}.$$

Together with the obvious estimate $\|\nabla v_j\|_{L^2(\text{supp}(v_j))} \leq \|v_j\|_{H^1(\text{supp}(v_j))}$, interpolation concludes $\|\nabla v_j\|_{H^{-1/4}(\text{supp}(v_j))} \leq \|v_j\|_{H^{3/4}(\text{supp}(v_j))}$. The above, together with (5.2) shows

$$(6.5) \quad |\langle \nabla v_i, \nabla v_j \rangle_\Omega| \lesssim C^{-5/4}_{\text{mesh}}^{k-\ell}.$$ 

Symmetry of the problem shows the above also for $\ell \leq k$ and hence for all $i, j \in \mathbb{N}$. We restrict the index set by

$$I := \{(i, j) \in \mathbb{N}^2 : |L(v_i) - L(v_j)| \leq r \}$$

and define $M_{ij}^\varepsilon := M_{ij}$ for all $(i, j) \in I$ and zero elsewhere. Note that $\#\left\{v_j \in B_k^1 : M_{ij} \neq 0 \right\} \lesssim C_{\text{mesh}}^{2(k-\min(\ell,k))}$ and $\#\left\{v_i \in B_k^1 : M_{ij} \neq 0 \right\} \lesssim C_{\text{mesh}}^{2(\ell-\min(k,\ell))}.\) Estimate (6.5) and [22, Lemma 8.3] with $q = C_{\text{mesh}}^{-2}$ show $\|M - M^\varepsilon\|_2 \lesssim C_{\text{mesh}}^{-r/4}$. The implication (6.4) follows by definition of $I$. Thus, we conclude the proof by choosing $r \in \mathbb{N}$ sufficiently large. 

\[\square\]

**Lemma 6.7.** Let $M_{ij} := \langle \text{div}(v_i), w_j \rangle_\Omega$ or $M_{ij} := \langle \text{div}(0, v_i), w_j \rangle_\Omega$ for all $i, j \in \mathbb{N}$ with $v_i \in B^1$ and $w_j \in B^0$. Given $\varepsilon > 0$, there exists $M^\varepsilon \in \mathbb{R}^{N \times N}$ and a constant $C_M > 0$ such that

$$\|M^\varepsilon - M\|_2 \leq \varepsilon.$$ 

as well as

$$(6.6) \quad \left|\left|L(v_i) - L(v_j)\right| \geq C_M \text{ or } d_1(v_i, w_j) > C_M\right) \implies M_{ij}^\varepsilon = 0.$$ 

**Proof.** Assume $k := L(v_i) \leq \ell := L(v_j)$. Then, there holds with (5.2)

$$|M_{ij}| = |\langle (v_i, 0), \nabla w_j \rangle| \leq \|v_i\|_{L^2(\text{supp}(v_i))} \|w_j\|_{H^1(\text{supp}(v_i))}$$

$$\lesssim C_{\text{mesh}}^{-k} \|w_j\|_{H^1(\text{supp}(v_i))}.$$ 

From this, Lemma 6.5 together with (5.6) conclude

$$\|w_j\|_{H^1(\text{supp}(v_i))} \lesssim C_{\text{mesh}}^{-k} \|w_j\|_{H^1(\text{supp}(w_j))} \lesssim C_{\text{mesh}}^{-k+\ell}.$$
Altogether, we obtain $|M_{ij}| \lesssim C_{\text{mesh}}^{-2[k-\ell]}$. For $k < \ell$, we have with Lemma 6.5 as well as (5.2)\&(5.6) for some $0 < s < 1/2$

$$|M_{ij}| \leq \|v_i\|_{H^{1+s}(\text{supp}(w_j))} \|w_j\|_{H^{-s}(\text{supp}(w_j))} \lesssim C_{\text{mesh}}^{-sf} \|v_i\|_{H^{1+s}(\text{supp}(w_j))} \lesssim C_{\text{mesh}}^{-s|\ell-k|} \|v_i\|_{H^{1+s}(\text{supp}(v_i))} \lesssim C_{\text{mesh}}^{-|1+s|\ell-k}.$$ 

We restrict the index set by

$$I := \{(i,j) \in \mathbb{N}^2 : |L(v_i) - L(v_j)| \leq r \}$$

and define $M_{ij}^r := M_{ij}$ for all $(i,j) \in I$ and zero elsewhere. Note that $\#\{v_j \in B_k^1 : M_{ij} \neq 0\} \lesssim C_{\text{mesh}}^{2(k-\min\{k,\ell\})}$ and $\#\{v_i \in B_k^1 : M_{ij} \neq 0\} \lesssim C_{\text{mesh}}^{2(\ell-\min\{k,\ell\})}$. The above estimates and [22, Lemma 8.3] with $q = C_{\text{mesh}}^{-2}$ show $\|M - M^r\|_2 \lesssim C_{\text{mesh}}^{-sr}$. The implication (6.4) follows by definition of $I$. Thus, we conclude the proof by choosing $r \in \mathbb{N}$ sufficiently large.

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