ELECTRODYNAMIC TWO-BODY PROBLEM FOR PRESCRIBED INITIAL DATA ON THE STRAIGHT LINE

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Abstract. Due to the finite speed of light, direct electrodynamic interaction between point charges can naturally be described by a system of ordinary differential equations involving delays. As electrodynamics is time-symmetric, these delays appear as time-like retarded as well as advanced arguments in the fundamental equations of motion – the so-called Fokker-Schwarzschild-Tetrode (FST) equations. However, for special initial conditions breaking the time-symmetry, effective equations can be derived which are purely retarded. Dropping radiation terms, which in many situations are very small, the latter equations are called Synge equations. In both cases, few mathematical results are available on existence of solutions, and even fewer on uniqueness. We investigate the situation of two like point-charges in 3 + 1 space-time dimensions restricted to motion on a straight line. We give a priori estimates on the asymptotic motion and, using a Leray-Schauder argument, prove: 1) Existence of solutions to the FST equations on the future or past half-line given finite trajectory strips; 2) Global existence of the Synge equations for Newtonian Cauchy data; 3) Global existence of a FST toy model that involves advanced and retarded terms. Furthermore, we give a sufficient criterion that uniquely distinguishes solutions by means of finite trajectory strips.

1. Introduction and Main Results

The direct electrodynamic interaction between point-charges is a prime example for systems of ordinary differential equations involving delays. Its fundamental equations of motion can be inferred by means of an informal variational principle of the action $S$ which is given as a functional of the world-lines of the point-charges, $z_i : \mathbb{R} \to \mathbb{R}^4$, $\tau \mapsto z_i(\tau)$:

$$S[z_{i=1,\ldots,n}] = -\sum_{i=1}^N \int m_i \sqrt{dz_i(\tau)d\bar{z}_i(\tau)} - \frac{e_i e_j}{2} \int dz_{ij} \int d\bar{z}_{ij} \delta((z_i(\tau) - z_j(\sigma))^2).$$

In this relativistic notation, $\tau$ is the parametrization of the world-line, and $z_i = (z_i^{\mu})_{\mu=0,1,2,3}$ denotes the time and space coordinates, $z_i^0$ and $z = (z_i^{j})_{j=1,2,3}$, respectively. The integral $\int dz_i^\mu$ is to be interpreted as the line integral $\int d\tau dz_i(\mu)$, where dots denote derivatives w.r.t. $\tau$. Furthermore, the summation convention $a_{\mu} b^{\mu} = a^{0} b^{0} - a \cdot b$ is used so that $a^2 = a_{\mu} a^\mu$ equals the square of the relevant, indefinite Minkowski metric in relativistic space-time. The symbol $\delta$ denotes the one-dimensional Dirac delta distribution, $m_i$ denotes the mass of the particles, $e_i$ the respective charge, and we chose units such that the speed of light and the electric constant equal one. The integral in the first summand in (1) measures the arc length of the $i$-th world line using the Minkowski metric, and the double integral in the second summand gives rise to an interaction between pairs of world lines whenever the Minkowski distance between $z_i$ and $z_j$ is zero. The extrema of the action $S$, i.e., $z_i$ such that $\frac{d}{dz_i} S[z_i + \epsilon \delta z_i] = 0$, fulfill the FST equations (also known...
as Wheeler-Feynman equations):

$$m_i \ddot{z}_i^\mu(\tau) = e_i \sum_{j=1, j \neq i}^N \frac{1}{2} \sum_{\pm} F_{j \pm}^{\mu\nu}(z_i(\tau)) \dot{z}_j^\nu(\tau), \quad i = 1, 2, \ldots, N$$

(2)

with the electromagnetic field tensors $F_{j \pm}^{\mu\nu}(x) = \partial / \partial x_\mu A_j^\nu(x) - \partial / \partial x_\nu A_j^\mu(x)$ given by means of the four-vector potentials

$$A_j^\mu(x) = e_j \frac{\dot{z}_j^\mu}{(x - z_j^{\mu \pm})_{\mu} \dot{z}_j^{\mu \pm}}, \quad \dot{z}_j^\mu = z_j^\mu(\tau_j^\pm), \quad x^0 - z_j^0(\tau_j^\pm) = \pm |x - z_j(\tau_j^\pm)|$$

(3)

Equation (2) is the special relativistic form of Newton’s force law, in electrodynamics referred to as Lorentz equation. The field tensors $F_{j \pm}^{\mu\nu}$ are the so-called advanced (+) and retarded (−) electrodynamic Liénard-Wiechert fields [11] which are generated by the $j$-th charge, respectively. They are given in terms of the corresponding potentials $A_{j \pm}, A_{j -}$, which are functionals of the world line $\tau \mapsto z_j(\tau)$ since the parameters $\tau_j^\pm$ are defined implicitly as solutions to the last equation in line (3). This implicit equation is due to the delta function in (1) and has a nice geometrical interpretation. When evaluating $F_{j \pm}^{\mu\nu}(x)$ at $x = z_i$ as in the Lorentz force (2), the respective $\tau_j^\pm$ identify the space-time points $z_j^\pm$ which can be reached with speed of light. The existence of both $\tau_j^+$ and $\tau_j^-$ is ensured as long as the world lines have velocities smaller than speed of light. Their values are however not bounded a priori. Since computing $F_{j \pm}^{\mu\nu}(x)$ involves taking another derivative of $A_j^\mu(x)$ w.r.t. to $x$, and $\tau_j^\pm$ depends on $x$, the right-hand side of the Lorentz equations (2) involves advanced and retarded four-vectors $\dot{z}_j^\mu$, four-velocities $z_j^\mu$, and four-accelerations $\ddot{z}_j^\mu$ − hence, (2) is a neutral equation of mixed-type, state-dependent, and has unbounded delay.

The formulation of electrodynamics by means of direct interaction as in (2) is due to ideas and works of Gauss [9], Fokker, [8] Tetrode [13], Schwarzschild [12]. Wheeler and Feynman [14, 15] showed that this formulation is capable of explaining the irreversible nature of radiation. Beyond that, it is the only candidate for a singularity free formulation of classical electrodynamics. For a more detailed discussion, see also the overview article [4]. Mathematically, however, even global existence of solutions to (2) is an open problem. The few rigorous results available apply to special situations only. In the special case of two like charges restricted on the straight line, global existence for prescribed asymptotic data was proven in [2]. In the case of $N$ arbitrary extended charges in $3 + 1$ space-time dimensions, the existence of solutions for prescribed Newtonian Cauchy data on finite, but arbitrary large time intervals was shown in [3]. If, furthermore, the two charges are initially sufficiently far apart and have zero velocities the corresponding solutions were shown to be unique [7].

In this paper we also study the case of two like charges and restrict us to solutions that describe motion on a straight line as in [2]. However, in contrast to [2], we aim at existence of solutions for prescribed data at finite times instead of asymptotic data. As long as the motion takes place on a straight line, the equations of motion (2) can be simplified as follows. To keep the notation short, we express (2) in coordinates and introduce the trajectories $a, b$ as maps $\mathbb{R} \to \mathbb{R}$, $t \mapsto a(t)$ and $t \mapsto b(t)$, such that $z_1 = (t, a(t), 0, 0)$ and...
Figure 1. Initial data for the FST equations required for (i) the existence result in Theorem 1.1; (ii) the alternative existence result mentioned in remark 1.1; (iii) the unique reconstruction of solutions as in theorem 1.4.

\[ z_2 = (t, b(t), 0, 0). \]  

Equations (2)-(3) then turn into

\[
\frac{d}{dt} \left( \frac{\dot{a}(t)}{\sqrt{1 - \dot{a}(t)^2}} \right) = \kappa_a \left[ \epsilon_- - \frac{\rho(b(t^-_2))}{(a(t) - b(t^-_2))^2} + \epsilon_+ \frac{\sigma(b(t^-_2))}{(a(t) - b(t^-_2))^2} \right],
\]

\[
\frac{d}{dt} \left( \frac{\dot{b}(t)}{\sqrt{1 - \dot{b}(t)^2}} \right) = -\kappa_b \left[ \epsilon_- - \frac{\sigma(a(t^-_1))}{(b(t) - a(t^-_1))^2} + \epsilon_+ \frac{\rho(a(t^-_1))}{(b(t) - a(t^-_1))^2} \right]
\]  

(4)

for

\[
\rho(v) = \frac{1 + v}{1 - v}, \quad \sigma(v) = \frac{1 - v}{1 + v}.
\]  

(5)

Here, we introduced the following parameters for convenience of our discussion: \( \kappa_{a/b} := \frac{e_a e_b}{m_{a/b}} \) which is the coupling constant, and \( \epsilon_+ \) and \( \epsilon_- \) which allow to individually switch the advanced or retarded terms on and off. The time-symmetric FST equations (2) are recovered by setting \( \epsilon_+ = \frac{1}{2} = \epsilon_- \). The analogs of the parameters \( \tau_{i\pm} \) given in (3), expressed in coordinates, are the advanced and retarded times \( t_{i\pm}^-(a, b, t) \), \( t_{i\pm}^+(a, b, t) \). These times are functions of the trajectories \( a, b \) and time \( t \) defined by

\[
t_{i\pm}^-(a, b, t) = t \pm |a(t^-_{i\pm}(a, b, t)) - b(t)|, \quad t_{i\pm}^+(a, b, t) = t \pm |a(t) - b(t_{i\pm}^+(a, b, t))|.
\]  

(6)

Their dependence on \( a, b, t \) will often be suppressed in the notation. Beside \( \epsilon_{\pm} = \frac{1}{2} \), another interesting case of (4) is given by \( \epsilon_+ = 0 \) and \( \epsilon_- = 1 \), which results in the so-called Synge equations. It was shown by Wheeler and Feynman that solutions to the time-symmetric fundamental equations of motion (2), in the case of special initial configurations that break the time symmetry, are effectively also solutions to equations of motion involving a radiation reaction term and only retarded delays. Upon neglecting radiation reaction terms, which for small charges (like the electron charge) and small accelerations give only small corrections, these approximate equations take the form of the Synge equations. For them, the following mathematical results are available: In 3 + 1 space-time dimensions existence of solutions for times \( t \geq 0 \) and given admissible trajectory histories was discussed in [1]. In the case of \( N \) extended charges, existence and uniqueness of solutions to the Synge equations for prescribed histories was shown in [3]. On the straight line, uniqueness w.r.t. Newtonian Cauchy data for like charges that are initially sufficiently far apart and have zero velocities was shown in [6].

Our present work extends these results as follows. For the case of the FST equations, i.e., \( \epsilon_- = \frac{1}{2} = \epsilon_+ \), we prove existence of solutions on the future or past half-line for initial data that consist of position and velocity of charge \( a \) and a trajectory strip of charge \( b \) whose ends are intersection points of the light-cone through \( (0, a(0)) \), see Figure 1(i):
Figure 2. Smoothness of solutions to (4) with initial data (7), (8) such that \( b_0 \in C^{1+n} \).

**Theorem 1.1 (Fokker-Schwarzschild-Tetrode Equations).** Let \( \epsilon_+ = \frac{1}{2} = \epsilon_- \). Given initial position and velocity

\[ a_0 \in \mathbb{R}, \dot{a}_0 \in ]-1,1[ \]  

(7)

of charge \( a \) and an initial trajectory strip

\[ b_0 \in C^1 (\left[ T^{-}, T^{+}\right], -\infty, a_0] \]  

(8)

of charge \( b \) with \( T^\pm = \pm(a_0 - b_0(T^\pm)) \) and \( \| \dot{b}_0 \|_\infty < 1 \), the following holds:

a) There is at least one pair of trajectories

\[ a \in C^2(\mathbb{R}^+_0) \text{ and } b \in C^1([T^-, \infty]) \cap C^2([T^+, \infty)) \]  

(9)

such that the first equation of (4) together with (5)-(6) is satisfied for all \( t \geq 0 \) and the second one of (4) holds for all \( t \geq T^+ \), and furthermore

\[ a(0) = a_0, \dot{a}(0) = \dot{a}_0, \quad b|_{T^-, T^+} = b_0. \]  

(10)

b) If \( b_0 \in C^{1+n}, n \in \mathbb{N}_0 \), then the regularity of any solution is characterized as follows: Let \( \sigma_0 := 0, \tau_0 := T^+, \sigma_1 := t^+_1(T^+), \tau_1 := t^+_2(\sigma_1), \text{ and } \sigma_{k+1} := t^+_k(\tau_k), \tau_{k+1} := t^+_2(\sigma_{k+1}) \), then \( a \) is \( 2k \) times differentiable at \( \sigma_k \) for \( k \in \mathbb{N} \), \( b \) is \( 2k + 1 \) times differentiable at \( \tau_k \) for \( k \in \mathbb{N}_0 \), and \( a|_{\sigma_k, \sigma_{k+1}} \in C^{2+n+2k} \) and \( b|_{\tau_k, \tau_{k+1}} \in C^{3+n+2k} \) for any \( k \in \mathbb{N}_0 \); see Figure 2.

Here, the set \( C^n(D, E) \) refers to \( n \) times continuously differential functions on \( D \) with values in \( E \) where derivatives at boundary points are to be understood as one-sided ones. By a similar argument, one can also show existence of solutions \((a, b)\) in the past of \( a(0) \) and \( b(T^-) \). The proof is based on an a priori estimate on the asymptotic behavior of FST solutions provided by Proposition 2.1 below. The trajectory data needed to establish the a priori estimate is exactly the required data (7)-(8). This estimate allows an application of Leray-Schauder’s fixed-point theorem.

For the Synge equations, i.e., (4)-(6) for \( \epsilon_+ = 0 \) and \( \epsilon_- = 1 \), we are able to control the asymptotic behavior of solutions knowing Newtonian Cauchy data only, i.e., positions and velocities at one time instant. In turn, this enables us to show global existence of solutions to the Synge equations for any given Newtonian Cauchy data.

**Theorem 1.2.** Let \( \epsilon_- = 1, \epsilon_+ = 0 \). For any Newtonian Cauchy data \( a_0 > b_0 \) and \( \dot{a}_0, \dot{b}_0 \in ]-1,1[ \), there is at least one pair of trajectories \( a, b \in C^\infty(\mathbb{R}) \) that solves (4)-(6)
with

\[ a(0) = a_0, \quad \dot{a}(0) = \dot{a}_0, \quad b(0) = b_0, \quad \dot{b}(0) = \dot{b}_0. \]  

(11)

With the same technique one can also treat the case \( \epsilon_- = 0, \epsilon_+ = 1 \). We note that, although we only regard motion along the straight line, this global existence results goes somewhat beyond the existence results given in [6, 1, 3], which treat existence of solutions only for the half-line \( t \geq 0 \) given prescribed histories.

**Remark 1.1.** Due to the Lorentz invariance of the Synge equations it is also possible to specify the “initial” positions and velocities at different times \( t_1 \) and \( t_2 \), provided the space-time points \( (t_1, a(t_1)) \) and \( (t_2, b(t_2)) \) are space-like separated. Alternative initial data for the time-symmetric equations are obtained by choosing \( (t_1, a(t_1)) \) and \( (t_2, a(t_2)) \) space-like separated and prescribing both trajectories up to the corresponding retarded times, i.e., specifying \( a\big|_{[T_1^-, t_1]} \) and \( b\big|_{[T_2^-, t_2]} \) such that \( T_1^- = t_2 - [a(T_1^-) - b(t_2)] \) and \( T_2^- = t_1 - [a(t_1) - b(T_2^-)] \); see Figure 1(ii).

One may wonder if the reason why global existence of solution to the Synge equations can be shown already for any Newtonian Cauchy data instead of giving initial strips of the trajectories stems from the fact that only retarded delays are involved and advanced delays are absent (or vice versa). This is however not the case. To see this, we also regard a FST toy model inferred from (4)-(6) by setting \( \epsilon_+ = \frac{1}{2} = \epsilon_- \) and \( \sigma = 1 = \rho \) which results in:

\[
\frac{d}{dr} \left( \frac{\dot{a}(t)}{\sqrt{1-\dot{a}(t)^2}} \right) = \frac{1}{(a(t) - b(t_2^+))} + \frac{1}{(a(t) - b(t_2^-))},
\]

\[
\frac{d}{dr} \left( \frac{\dot{b}(t)}{\sqrt{1-\dot{b}(t)^2}} \right) = -\frac{1}{(b(t) - a(t_1^+))} - \frac{1}{(b(t) - a(t_1^-))};
\]

(12)

see [5] for a more detailed discussion of this model in 3+1 space-time dimensions. Although the resulting equation of motion (12) involves advanced as well as retarded delays, we were able to prove global existence of solutions for Newtonian Cauchy data with the same technique. In conclusion, the technical obstacle to infer a similar result for (4) is due to the lack of control of the denominators of the velocity factors \( \sigma \) and \( \tau \) in (5) by means of Newtonian Cauchy data only, and not necessarily due to the time-symmetry.

**Theorem 1.3.** For any Newtonian Cauchy data \( a_0 > b_0 \) and \( \dot{a}_0, \dot{b}_0 \in ]-1,1[ \), there is at least one pair of trajectories \( a, b \in C^\infty(\mathbb{R}) \) that solves (12) and (6) with

\[ a(0) = a_0, \quad \dot{a}(0) = \dot{a}_0, \quad b(0) = b_0, \quad \dot{b}(0) = \dot{b}_0. \]

(13)

These existence results do not touch upon the question of uniqueness. In the case of the toy model (12) it was shown in [5] that at least for finite times solutions to (12) and (6) can be constructed by a what is commonly called a “method of steps” in the field of delay differential equations. Here, we give necessary conditions to identify solutions to (4)-(6) uniquely:

**Theorem 1.4.** Let \( \epsilon_+ = \frac{1}{2} = \epsilon_- \). Any solution \((a, b)\) to equations (4)-(6) can be uniquely reconstructed knowing only the trajectory strips

\[ a\big|_{[T_1^-, T_1^+]}, \quad b\big|_{[T_2^-, T_2^+]}, \]

(14)

with times \( T_1^+ \in \mathbb{R}, T_2^+ = t_2^+(T_1^+), T_2^- = t_2^-(T_1^+) \) and \( T_1^- = t_1^-(T_2^-) \).

Similar results are possible for other combinations of \( \epsilon_\pm \). In the following sections we provide the proofs of the presented theorems.
2. Proof of Theorem 1.1

Similarly to [2], our proof is based on the following version of Leray-Schauder’s fixed point theorem:

**Theorem 2.1 (Leray-Schauder Theorem [10]).** Let \((\mathcal{B}, \|\cdot\|)\) be a Banach space, \(\mathcal{O} \subset \mathcal{B}\) a bounded open subset containing the origin and \(\mathcal{H} : [0, 1] \times \overline{\Omega} \to \mathcal{B}\) a compact homotopy such that \(\mathcal{H}(0, \cdot)\) is the zero mapping and none of the mappings \(\mathcal{H}(\lambda, \cdot)\) for \(\lambda \in [0, 1]\) has a fixed point on \(\partial \mathcal{O}\). Then \(\mathcal{H}(1, \cdot)\) has a fixed point.

In order to apply the theorem, we need a Banach space consisting of pairs of trajectories. For \((x, y) \in C^2(\mathbb{R}^+_* \times [0, T]) \cap C^2([T^-, \infty) \times [T^+, \infty])\), we define

\[
\|(x, y)\| := \max \left(\|\dot{x}\|_\infty, \|\dot{y}\|_\infty, \sup_{t \geq 0} |(1 + |t|)\dot{x}(t)|, \sup_{t > T^+} |(1 + |t|)\dot{y}(t)|\right),
\]

choose

\[
B := \left\{ (x, y) \in C^2(\mathbb{R}^+_* \times [0, T]) \cap C^2([T^-, \infty) \times [T^+, \infty]) \right\},
\]

\[
x(0) = \dot{x}(0) = 0, \quad y|_{[T^-, T^+]} \equiv 0, \quad \|(x, y)\| < \infty
\]

for the Banach space, and equip it with the norm \(\|\cdot\|\). Its particular choice will become clear below. From now on, we consider the values of the initial data \(a_0\) and \(b_0\) as fixed. Since nonzero initial data are not compatible with linearity, they cannot directly be included in the definition of the Banach space \(B\). Instead, we fix a pair of reference trajectories \((x_0, y_0) \in C^2(\mathbb{R}^+_* \times [0, T]) \cap C^2([T^-, \infty) \times [T^+, \infty])\) that satisfy the initial data and scatter apart without reaching too large velocities or accelerations or too small distances. Precisely, we require

\[
\begin{align*}
\|\dot{x}_0\|, \|\dot{y}_0\| &< 1, \\
\inf_{t \geq 0}(x_0(t) - y_0(t)) &> 0, \\
(\dot{x}_0 - \dot{y}_0)(T^+ + 1) &> 0, \\
x_0(t) &\geq 0 \text{ for } t \geq 0, \quad y_0(t) \leq 0 \text{ for } t > T^+ \\
x_0(t) = y_0(t) &\equiv 0 \text{ for } t \geq T^+ + 1.
\end{align*}
\]

The actual trajectories shall then given by

\[
X := x + x_0, \quad Y := y + y_0.
\]

Moreover, on a suitable subset of \(B\), we define the homotopy \(H = (H_1, H_2)\) with range in \(B\) and components

\[
\begin{align*}
H_1(\lambda, x, y)(t) := &-x_0(t) + a_0 + \int_0^t \frac{P_1(\lambda, X, Y)(s)}{\sqrt{1 + P_1(\lambda, X, Y)(s)^2}} ds, \\
H_2(\lambda, x, y)(t) := &\begin{cases} 
0 & \text{for } T^- \leq t \leq T^+ \\
-y_0(t) + b_0(T^+) + \int_{T^+}^t \frac{P_2(\lambda, X, Y)(s)}{\sqrt{1 + P_2(\lambda, X, Y)(s)^2}} ds & \text{for } t \geq T^+.
\end{cases}
\end{align*}
\]
Here, we used the abbreviations

\[
P_1(\lambda, X, Y)(t) := (1 - \lambda) \frac{\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} + \lambda \frac{\dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} + \lambda \int_0^t \frac{\kappa_a}{2} F_1(X, Y)(s) \, ds ,
\]

\[
P_2(\lambda, X, Y)(t) := (1 - \lambda) \frac{\dot{y}_0(t)}{\sqrt{1 - \dot{y}_0(t)^2}} + \lambda \frac{\dot{b}_0(T^+)}{\sqrt{1 - \dot{b}_0(T^+)^2}} + \lambda \int_{T^+}^t \frac{\kappa_b}{2} F_2(X, Y)(s) \, ds ,
\]

and

\[
F_1(X, Y)(t) := \frac{1}{1 - \dot{Y} (t_2^+ (X, Y, t))} \left( \frac{1}{1 - \dot{Y} (t_2^+ (X, Y, t))} \left( X(t) - Y(t_2^+ (X, Y, t)) \right) \right) ^2 + \frac{1}{1 - \dot{Y} (t_2^+ (X, Y, t))} \left( X(t) - Y(t_2^+ (X, Y, t)) \right) ^2 ,
\]

\[
F_2(X, Y)(t) := - \frac{1}{1 + \dot{X} (t_1^+ (X, Y, t))} \left( Y(t) - X(t_1^+ (X, Y, t)) \right) ^2 + \frac{1}{1 + \dot{X} (t_1^+ (X, Y, t))} \left( Y(t) - X(t_1^+ (X, Y, t)) \right) ^2 ,
\]

with \( t_{1/2} \) given by (6), now depending on \( X \) and \( Y \) instead of \( a \) and \( b \). By definition, \( H(0, \cdot) \) is the zero mapping and, if \((x, y)\) is a fixed point of \( H(\lambda, \cdot) \), then

\[
X(t) = a_0 + \int_0^t \frac{P_1(\lambda, X, Y)(s)}{\sqrt{1 + P_1(\lambda, X, Y)(s)^2}} \, ds ,
\]

\[
Y(t) = b_0(T^+) + \int_0^{T^+} \frac{P_2(\lambda, X, Y)(s)}{\sqrt{1 + P_2(\lambda, X, Y)(s)^2}} \, ds ,
\]

which fulfill the equations

\[
\frac{\dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} = P_1(\lambda, X, Y)(t) , \quad \frac{\dot{Y}(t)}{\sqrt{1 - \dot{Y}(t)^2}} = P_2(\lambda, X, Y)(t) ,
\]

i.e.,

\[
\frac{d}{dt} \left( \frac{\dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \right) = \frac{\dot{X}(t)}{(1 - \dot{X}(t)^2)^{3/2}} = (1 - \lambda) \frac{\ddot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{3/2}} + \frac{\lambda \kappa_a}{2} F_1(X, Y)(t) ,
\]

\[
\frac{d}{dt} \left( \frac{\dot{Y}(t)}{\sqrt{1 - \dot{Y}(t)^2}} \right) = \frac{\dot{Y}(t)}{(1 - \dot{Y}(t)^2)^{3/2}} = (1 - \lambda) \frac{\ddot{y}_0(t)}{(1 - \dot{y}_0(t)^2)^{3/2}} + \frac{\lambda \kappa_b}{2} F_2(X, Y)(t) .
\]

Hence, for \( \lambda = 1 \), \((X, Y)\) solve the equations of motion (4) respecting the initial data (10). In Proposition 2.1 below we provide a priori estimates ensuring that \( H \) has no fixed points on the boundary of

\[
\Omega := \left\{ (x, y) \in B \left| \| X \|_{\infty}, \| Y \|_{\infty} < V, \inf_{t \in [0, T]} (X(t) - Y(t)) > D, \inf_{t \in [T, \infty]} (\dot{X}(t) - \dot{Y}(t)) > \nu, \sup_{t \geq 0} (1 + |t|) |\dot{X}(t)|, \sup_{t \geq T^+} (1 + |t|) |\dot{Y}(t)| < A \right\}
\]

(23)
for suitably chosen constants $v,V \in [0,1]$ and $A,D,T > 0$; recall definition (18) of $X,Y$ in terms of $x,y$. The estimates for $\lambda \approx 1$ will come from energy considerations, the ones for $\lambda \approx 0$ from the properties (17) of $(x_0,y_0)$.

On $\Omega$, it is rather straightforward to prove continuity and compactness of $t$ for all $\lambda$. For any Lemma 2.1. Existence is clear; the compactness proof, which is of order one on all of $t$. A priori estimate given in (24d) below suffices for the continuity proof - had we needed in terms of $v,V$ for suitably chosen constants $\lambda < A,D,T > 0$.

Proof. We apply the Arzela-Ascoli theorem to $\dot{x}$ and $(1 + |t|)\dot{x}$. Hence, it is fortunate that the $\frac{1}{1+|t|}$ a priori estimate given in (24d) below suffices for the continuity proof - had we needed $\frac{1}{1+|t|}$ there, we would have had to take $1 + t^2$ into the norm and to consider $(1 + t^2)\ddot{x}$ in the compactness proof, which is of order one on all of $\mathbb{R}_0^+$.

Now we perform the detailed calculations, starting with a technicality:

**Lemma 2.1.** For any $C^1$-trajectories $(x,y)$ with $\|\dot{x}\|_\infty, \|\dot{y}\|_\infty \leq C < 1$ and $x(t) > y(t)$ for all $t \in \mathbb{R}$, the advanced and retarded times $t^\pm_1$ introduced in (6) are well-defined and

\[
\frac{x(t) - y(t)}{2} \leq x(t) - y(t^\pm_1(t)) \leq \frac{x(t) - y(t)}{1 - \|\dot{y}\|_\infty},
\]

\[
\frac{x(t) - y(t)}{2} \leq x(t^\pm_1(t)) - y(t) \leq \frac{x(t) - y(t)}{1 - \|\dot{x}\|_\infty}.
\]

**Proof.** Existence is clear;

\[
x(t) - y(t^\pm_1) = x(t) - y(t) + y(t) - y(t^\pm_1) = x(t) - y(t) + \dot{y}(\tau)(t - t^\pm_1)
\]

\[
= x(t) - y(t) \mp \dot{y}(\tau)[x(t) - y(t^\pm_1)]
\]

with $\tau$ between $t$ and $t^\pm_1$, so

\[
x(t) - y(t^\pm_1) = \frac{x(t) - y(t)}{1 \pm \dot{y}(\tau)}.
\]

$\square$

In particular, the lemma shows that $H$ is well-defined on

\[M := \{(x,y) \in C^1(\mathbb{R}_0^+) \times C^1([T^- , \infty[) \mid \|\dot{X}\|_\infty, \|\dot{Y}\|_\infty < 1, \forall t \in \mathbb{R} X(t) > Y(t)\}.
\]

After these preparations, we provide the required global a priori estimates on fixed points of $H$ in terms of the initial data. We partly adapt and generalize ideas from [2].

**Proposition 2.1.** There are constants $\bar{v},\bar{V} \in [0,1]$ and $\hat{A},\hat{D},\hat{T} > 0$ such that, for any fixed point $(x,y)$ of $H : [0,1] \times M \to C^1(\mathbb{R}_0^+) \times C^1([T^- , \infty[)$,

\[
\|\dot{X}\|_\infty, \|\dot{Y}\|_\infty < \bar{V},
\]

\[
\inf_{t \geq 0} (X - Y)(t) > \hat{D},
\]

\[
\inf_{t \geq T} (\dot{X} - \dot{Y})(t) > \bar{v},
\]

\[
|\dot{X}(t)|, |\dot{Y}(t)| < \frac{\hat{A}}{1 + |t|}.
\]
We remark that this result implies in particular that any solution \((X, Y)\) of (4) with \(\epsilon_{\pm} = \frac{1}{T}\) satisfying (10) obeys all bounds (24a)-(24d). Physically, inequalities (24a) make sure that the velocities of the charges are bounded away from one, and inequality (24b) guarantees that the charges obey a minimal distance. More precisely, inequality (24c) ensures that, at least starting from a certain time, the charges move away from each other in the future, and inequality (24d) describes how the accelerations of the charges decay and the trajectories relax to their corresponding incoming and outgoing asymptotes. All possible fixed-points therefore describe scattering solutions.

The key in the following proof of the desired a priori bounds is a comparison of the time derivative of the special relativistic kinetic energy with the one of the potential energy carried in the delayed Coulomb potentials:

**Proof.** According to (22) and (21), the time derivative of the “kinetic energy” of the first component of a fixed point of \(H\) satisfies

\[
\frac{d}{dt} \left( \frac{1}{\sqrt{1 - \dot{X}(t)^2}} \right) = \frac{\ddot{X}(t)\dot{X}(t)}{(1 - \dot{X}(t)^2)^{3/2}}
\]

\[
= (1 - \lambda) \frac{\dot{X}(t)\dot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{3/2}} + \frac{\lambda \kappa_a}{2} \left[ \frac{\dot{X}(t) + \dot{X}(t)\dot{Y}(t_2^+)}{1 - Y(t_2^+)^2} \frac{1}{(X(t) - Y(t_2^+))^2} \right. \\
+ \left. \frac{\dot{X}(t) - \dot{X}(t)\dot{Y}(t_2^-)}{1 + Y(t_2^-)^2} \frac{1}{(X(t) - Y(t_2^-))^2} \right].
\]  

(25)

By definition (6) we have

\[
i_{t_2}^\pm = 1 \pm \dot{X}(t) \mp \dot{Y}(t_2^\pm) = \frac{1 \pm \dot{X}(t)}{1 \pm Y(t_2^\pm)},
\]  

(26)

so that the derivative of the “potential energy” is given by

\[
\frac{d}{dt} \left( -\frac{1}{X(t) - Y(t_2^\pm)} \right) = \frac{\dot{X}(t) - \dot{Y}(t_2^\pm) i_{t_2}^\pm}{[X(t) - Y(t_2^\pm)]^2} = \frac{\dot{X}(t) - \dot{Y}(t_2^\pm)}{1 \pm Y(t_2^\pm)^2} \frac{1}{[X(t) - Y(t_2^\pm)]^2}.
\]

(27)

If \(\dot{Y}(s)\) happens to be nonpositive for all \(s \in [T^-, T^+]\), then, since the acceleration is always negative according to (22), \(\dot{Y}(t_2^-) \leq 0\) for all \(t \geq T^-\). Consequently, \(X(t) \mp \dot{X}(t)\dot{Y}(t_2^\pm) \leq X(t) - Y(t_2^\pm)\), and the last two terms on the right-hand side of (25) can be estimated by a total differential (27):

\[
\frac{d}{dt} \left( \frac{1}{\sqrt{1 - \dot{X}(t)^2}} \right) \leq (1 - \lambda) \frac{\dot{X}(t)\dot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{3/2}} - \frac{\lambda \kappa_a}{2} \frac{d}{dt} \left[ \frac{1}{X(t) - Y(t_2^+)} + \frac{1}{X(t) - Y(t_2^-)} \right].
\]

Furthermore, \(\dot{X}(t)\) in the first summand can be estimated by one, and integration from 0 to \(t\) gives

\[
\frac{1}{\sqrt{1 - \dot{X}(t)^2}} - \frac{1}{\sqrt{1 - \dot{a}_0^2}} \leq (1 - \lambda) \left[ \frac{\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} - \frac{\dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} \right] \\
+ \frac{\lambda \kappa_a}{2} \left[ \frac{1}{a_0 - b_0(T^-)} - \frac{1}{X(t) - Y(t_2^+)} + \frac{1}{a_0 - b_0(T^+)} - \frac{1}{X(t) - Y(t_2^-)} \right].
\]
which implies
\[
\frac{1}{\sqrt{1 - \dot{X}(t)^2}} \leq \frac{2}{\sqrt{1 - (\dot{a}_0 \sqrt{\|\dot{x}_0\|})^2}} + \frac{\kappa_a}{3} \left[ \frac{1}{a_0 - b(T^-)} + \frac{1}{a_0 - b(T^+)} \right] - \frac{\lambda \kappa_a}{2} \left[ \frac{1}{X(t) - Y(t_2^-)} + \frac{1}{X(t) - Y(t_2^+)} \right].
\]

Here, \( a \lor b \) denotes the maximum of \( a \) and \( b \). Should we instead have \( \dot{Y}(s) > 0 \) for some \( s \in [T^-, T^+] \), the Lorentz invariance of the equations of motion suggests to repeat the calculation for \( \frac{1 - \|\dot{b}_0\|_\infty}{\sqrt{1 - \|\dot{b}_0\|_\infty^2}} \), which is the “kinetic energy” term in an inertial frame with relative velocity \( \|\dot{b}_0\|_\infty \):

\[
\frac{d}{dt} \left( \frac{1 - \|\dot{b}_0\|_\infty \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \right) = \frac{(\dot{X}(t) - \|\dot{b}_0\|_\infty \dot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{\frac{3}{2}}} + \frac{\lambda \kappa_a}{2} \left[ \frac{(\dot{X}(t) - \|\dot{b}_0\|_\infty)(1 + \dot{Y}(t_2^-))}{1 - \dot{Y}(t_2^-)} - \frac{1}{(X(t) - Y(t_2^+))^2} \right] + \frac{(\dot{X}(t) - \|\dot{b}_0\|_\infty)(1 - \dot{Y}(t_2^+))}{1 + \dot{Y}(t_2^+)} - \frac{1}{(X(t) - Y(t_2^+))^2}.
\]

We observe the inequality
\[
(\dot{X}(t) - \|\dot{b}_0\|_\infty)(1 \mp \dot{Y}(t_2^\pm)) = (1 \mp \|\dot{b}_0\|_\infty)(\dot{X}(t) - \dot{Y}(t_2^\pm)) + (\dot{Y}(t_2^\pm) - \|\dot{b}_0\|_\infty)(1 \mp \dot{X}(t))
\]

\[
\leq (1 \mp \|\dot{b}_0\|_\infty)(\dot{X}(t) - \dot{Y}(t_2^\pm)),
\]

where \( \dot{Y}(t_2^\pm) \leq \|\dot{b}_0\|_\infty \) holds true. The latter can be seen from the fact that it is satisfied for \( t = 0 \) and that \( \dot{Y} < 0 \) implies this also for all later times. Using equations (26) and (27) again and upon integration from 0 to \( t \), we now arrive at

\[
\frac{1 - \|\dot{b}_0\|_\infty \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} - \frac{1 - \|\dot{b}_0\|_\infty \dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} \leq (1 - \lambda) \left[ \frac{\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} - \frac{\dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} \right] + \frac{\lambda \kappa_a}{2} \left[ \frac{1 + \|\dot{b}_0\|_\infty}{a_0 - b(T^-)} - \frac{1}{X(t) - Y(t_2^-)} + \frac{1 - \|\dot{b}_0\|_\infty}{a_0 - b(T^+)} - \frac{1}{X(t) - Y(t_2^+)} \right].
\]

In consequence, we get

\[
\frac{1 - \|\dot{b}_0\|_\infty}{\sqrt{1 - \dot{X}(t)^2}} \leq \frac{4}{\sqrt{1 - (\dot{a}_0 \sqrt{\|\dot{x}_0\|})^2}} + \frac{\lambda \kappa_a}{2} \left[ \frac{2}{a_0 - b(T^-)} + \frac{1}{a_0 - b(T^+)} \right] - \frac{\lambda \kappa_a}{2} \left[ \frac{1 - \|\dot{b}_0\|_\infty}{X(t) - Y(t_2^-)} + \frac{1 - \|\dot{b}_0\|_\infty}{X(t) - Y(t_2^+)} \right].
\]

Using Lemma 2.1, we estimate \( a_0 - b(T^-) \geq \frac{a_0 - \kappa_0}{2} \) to find

\[
\sup_{t \geq 0} |\dot{X}(t)| \leq \frac{1}{\sqrt{\frac{4}{\sqrt{1 - (\dot{a}_0 \sqrt{\|\dot{x}_0\|})^2}} + \frac{3 \kappa_a}{a_0 - b_0(0)}}} =: \tilde{V}_X.
\]
The estimate \( \sup_{t \geq T^+} |\dot{Y}(t)| \leq \tilde{V}_Y \) is obtained in an analogous way. This concludes the proof of (24a) for \( \tilde{V} := \tilde{V}_X \lor \tilde{V}_Y \lor \|\dot{b}_0\|_{\infty} \).

In order to prove (24b) we go back to inequality (29). First, we note that \( X(t) - Y(t) \leq \frac{X(t) - Y(t)}{1 - \tilde{V}} \infty \) according to Lemma 2.1. The resulting inequality ensures

\[
\inf_{t \geq 0} (X(t) - Y(t)) \geq \frac{\lambda \kappa_a (1 - \tilde{V})^2}{1 - \tilde{a}_0} + \frac{3 \kappa_a}{\alpha_0 - \beta_0(0)}. \tag{30}
\]

This estimate has to be improved for small values of \( \lambda \), but before doing that, it is more convenient to proceed with the estimates (24c) and (24d) involving \( \tilde{T} \) and \( \tilde{v} \). Integrating the difference of the equations (22) results in

\[
\frac{\dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} - \frac{\dot{Y}(t)}{\sqrt{1 - \dot{Y}(t)^2}} = \lambda \left( \frac{\dot{a}_0}{\sqrt{1 - a_0^2}} - \frac{\dot{b}_0(0)}{\sqrt{1 - \dot{b}_0(0)^2}} \right) + (1 - \lambda) \left( \frac{\dot{x}_0(t)}{\sqrt{1 - x_0(t)^2}} - \frac{\dot{y}_0(t)}{\sqrt{1 - y_0(t)^2}} \right) + \frac{\lambda}{2} \int_0^t (\kappa_a F_1(X,Y,s) - \kappa_b F_2(X,Y,s)) ds \tag{31}
\]

\[
\geq - \frac{4}{\sqrt{1 - \tilde{V}^2}} + \frac{\lambda}{2} \int_0^t (\kappa_a F_1(X,Y,s) - \kappa_b F_2(X,Y,s)) ds. \tag{32}
\]

The function \( \frac{u}{\sqrt{1 - u^2}} \) is strictly monotone in \( u \), so \( a > b \) is equivalent to \( \frac{a}{\sqrt{1 - a^2}} > \frac{b}{\sqrt{1 - b^2}} \). Furthermore, because of \( \dot{x}_0 \geq 0 \) and \( \dot{y}_0 \leq 0 \), \( F_1 \) is positive and \( F_2 \) negative according to (21). Therefore, equation (31) shows that, if \( \dot{a}_0 \geq \dot{b}_0(0) \), then \( \dot{X}(t) \geq \dot{Y}(t) \) holds for all larger \( t \). On the contrary, if \( \dot{a}_0 < \dot{b}_0(0) \) and \( \dot{X}(t) < \dot{Y}(t) \), then

\[
X(s) - Y(s) = a_0 - b_0(0) + \int_0^s (\dot{X}(r) - \dot{Y}(r)) dr < a_0 - b_0(0)
\]

for all \( s \in [0,t] \). Thus, we find

\[
|F_i(X,Y,s)| > \frac{(1 - \tilde{V})^2}{a_0 - b_0(0)}. \]

That means, for

\[
t \geq S_1(\lambda) := \frac{8(a_0 - b_0(0))}{\lambda(\kappa_a + \kappa_b)\sqrt{1 - \tilde{V}^2}(1 - \tilde{V})^2},
\]

(32) implies \( \dot{X}(t) - \dot{Y}(t) \geq 0 \). Recall further that, from time \( T^+ + 1 \) on, the accelerations of \( x_0 \) and \( y_0 \) are zero. Hence, again by (22), for \( t \geq (S_1(\lambda) \lor T^+) + 2 \) we have

\[
\frac{\dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} - \frac{\dot{Y}(t)}{\sqrt{1 - \dot{Y}(t)^2}} \geq \frac{\lambda}{2} \int_{(S_1(\lambda) \lor T^+) + 1}^{(S_1(\lambda) \lor T^+) + 2} (\kappa_a F_1(X,Y,s) - \kappa_b F_2(X,Y,s)) ds.
\]

Since \( \frac{du}{\sqrt{1 - u^2}} = \frac{1}{(1 - u^2)^{1/2}} \leq \frac{1}{(1 - V^2)^{1/2}} \), it holds

\[
\dot{X}(t) - \dot{Y}(t) \geq (1 - \tilde{V}^2)^{1/2} \left( \frac{\dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} - \frac{\dot{Y}(t)}{\sqrt{1 - \dot{Y}(t)^2}} \right);
\]
moreover, we have $X(s) - Y(s) \leq a_0 - b_0(0) + 2\hat{V}s$, so that we arrive at
\[ |F_t(X, Y, s)| \geq \frac{(1 - \hat{V})^2}{a_0 - b_0(0) + 2\hat{V}s}. \]
We therefore infer
\[ \inf_{t \geq (S_1(\lambda) \lor T^+)^+} \left( \dot{X}(t) - \dot{Y}(t) \right) \geq v_1(\lambda) := \frac{\lambda(1 - \hat{V}^2)^{\frac{3}{2}}}{2} \int_{(S_1(\lambda) \lor T^+)^+}^{(S_1(\lambda) \lor T^+)^++1} \frac{(\kappa_a + \kappa_b)(1 - \hat{V})^2}{2(a_0 - b_0 + 2\hat{V}s)} ds. \]
To get our hands on the missing estimates for small values of $\lambda$ we rewrite and estimate the integrated version of equation (22) for the velocity according to
\[
\frac{\dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} - \frac{\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} = \lambda \left( \frac{\dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} - \frac{\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} \right) + \frac{\lambda \kappa_a}{2} \int_0^t F_1(X, Y, s) ds
\geq - \frac{2\lambda}{\sqrt{1 - V^2}},
\]
and conclude
\[
\dot{X}(t) - \dot{x}_0(t) \geq - \frac{2\lambda}{\sqrt{1 - V^2}}.
\]
The latter clearly holds true if $\dot{X}(t) - \dot{x}_0(t)$ is non-negative, and in the opposite case, it follows from the fact that $\frac{\dot{a}}{\lambda \sqrt{1 - a^2}} = \frac{1}{(1 - a^2)^{\frac{3}{2}}} \geq 1$. Consequently, we have
\[
X(t) \geq x_0(t) - \frac{2\lambda}{\sqrt{1 - V^2}};
\]
similarly we find the corresponding estimates for the other charge:
\[
\dot{Y}(t) - \dot{y}_0(t) \leq \frac{2\lambda}{\sqrt{1 - V^2}} \quad \text{and} \quad Y(t) \leq y_0(t) + \frac{2\lambda}{\sqrt{1 - V^2}}.
\]
Hence, the last four estimates imply
\[
X(t) - Y(t) \geq \inf_{t \geq 0} (x_0(t) - y_0(t)) - \frac{4\lambda|t|}{\sqrt{1 - V^2}} \quad \text{(33)}
\]
and
\[
\inf_{t \geq T^+} \left( \dot{X}(t) - \dot{Y}(t) \right) \geq \inf_{t \geq T^+} (\dot{x}_0(t) - \dot{y}_0(t)) - \frac{4\lambda}{\sqrt{1 - V^2}} \\
\geq (\dot{x}_0 - \dot{y}_0)(T^+ + 1) - \frac{4\lambda}{\sqrt{1 - V^2}} =: v_2(\lambda),
\]
where in the last inequality we used the properties of $x_0$ and $y_0$ given in (17).
Combining the estimate
\[
\inf_{t \geq T^+} \left( \dot{X}(t) - \dot{Y}(t) \right) \geq v_2(\lambda') > 0
\]
for all fixed points corresponding to values
\[
\lambda \leq \lambda' := \frac{\sqrt{1 - V^2}}{8}(\dot{x}_0 - \dot{y}_0)(T^+ + 1),
\]
with the estimate
\[
\inf_{t \geq (S_1(\lambda') \lor T^+)^+} \left( \dot{X}(t) - \dot{Y}(t) \right) \geq v_1(\lambda') > 0
\]
that holds for all fixed points corresponding to some \( \lambda \geq \lambda' \), we obtain \( \tilde{T} \) and \( \tilde{v} \). Furthermore, equation (33) gives a positive lower bound for the distance \( X(t) - Y(t) \) up to time \( t = \tilde{T} \) if \((X,Y)\) is a fixed point corresponding to a \( \lambda \) fulfilling

\[
\lambda \leq \frac{\sqrt{1 - V^2} \inf_{t \geq 0} (x_0(t) - y_0(t))}{16T}.
\]

This estimate extends to all times and, together with the above estimate (30) corresponding to larger values of \( \lambda \), yields the constant \( \tilde{D} \), which concludes the proof of inequalities (24b) and (24c).

The existence of \( \tilde{D} \) and \( \tilde{v} \) imply \( X(t) - Y(t) \geq D_1 + v_3|t| \) for suitable \( D_1 \leq \tilde{D} \) and \( v_3 \leq \tilde{v} \), so that an estimate for \( \tilde{X} \) and \( \tilde{Y} \), which proves the remaining bound in (24d), can now be directly read off equation (22):

\[
|\tilde{X}(t)|, |\tilde{Y}(t)| \leq \left[ \frac{\|\tilde{x}_0\|_\infty \vee \|\tilde{y}_0\|_\infty}{(1 - V^2)^{\frac{3}{4}}} + \frac{2(\kappa_a + \kappa_b)}{(1 - V)(D_1 + v_3|t|)^2} \right] \vee \sup_{T - \tilde{T} \leq t \leq T + \tilde{T}} |\tilde{y}_0| \leq \frac{\tilde{A}}{1 + |t|}.
\]

For the proof of Theorem 1.1, which is given at the end of this section, we need the following technical lemmata. In their proofs, the symbols \( C, C_1, C_2, \ldots \) will be used for positive constants whose values may vary from line to line.

**Lemma 2.2.** \( H \) is Lipschitz continuous on \([0,1] \times \overline{\Omega}\).

**Proof.** We will repeatedly use the following decay property of the Coulomb-like force term in \( F \) given in (21): Since, according to definition (23) of \( \Omega \), \(|X(t) - Y(t)| \geq D\) holds for all \( t \) and \(|X(t) - Y(t)| \geq D + v|t - T| \) does for \(|t| \geq T \), we get the estimate \(|X(t) - Y(t)| \geq C(1 + |t|)\) for all \( t \). Using Lemma 2.1,

\[
|X(t) - Y(t)|, |X(t_1(t)) - Y(t)| \geq C(1 + |t|)
\]

is satisfied, and, together with the velocity bound

\[
|\dot{X}(t)|, |\dot{Y}(t)| \leq V
\]

from the definition of \( \Omega \), the definition (21) of \( F \) leads to

\[
|F_i(X,Y)(t)| \leq \frac{C}{1 + t^2},
\]

which ensures that \( H \) maps into \( B \).

To show Lipschitz continuity, we choose \((\lambda, x, y), (\tilde{\lambda}, \tilde{x}, \tilde{y}) \in [0,1] \times \overline{\Omega} \) and \( t \in \mathbb{R} \), and, recalling definition (19) of \( H \), consider

\[
|\dot{H}_1(\lambda, x, y)(t) - \dot{H}_1(\tilde{\lambda}, \tilde{x}, \tilde{y})(t)|
\]

\[
= \left| \frac{P_1(\lambda, X, Y)(t)}{\sqrt{1 + P_1(\lambda, X, Y)(t)^2}} - \frac{P_1(\tilde{\lambda}, \tilde{X}, \tilde{Y})(t)}{\sqrt{1 + P_1(\tilde{\lambda}, \tilde{X}, \tilde{Y})(t)^2}} \right|.
\]

This inequality holds because

\[
\left| \frac{d}{dt} \frac{u}{\sqrt{1 + u^2}} \right| = \left| \frac{1}{(1 + u^2)^{\frac{3}{2}}} \right| \leq 1.
\]

Substituting definition (20) of \( P_1 \), splitting the summands via triangle inequality, and using the velocity bound (35) as
well as estimate (36) on $F$, we find
\[
|P_1(\lambda, X, Y)(t) - P_1(\tilde{\lambda}, \tilde{X}, \tilde{Y})(t)|
\leq \frac{\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}}|\lambda - \tilde{\lambda}| + \frac{\dot{a}_0}{\sqrt{1 - a_0^2}}|\lambda - \tilde{\lambda}| + \int_0^t \frac{\kappa_a}{2} |F_1(X, Y)(s)| ds |\lambda - \tilde{\lambda}|
\]
\[+ |\lambda| \int_0^t \frac{\kappa_a}{2} |F_1(X, Y)(s) - F_1(\tilde{X}, \tilde{Y})(s)| ds \leq \frac{V}{\sqrt{1 - V^2}}|\lambda - \tilde{\lambda}| + \frac{V}{\sqrt{1 - V^2}}|\lambda - \tilde{\lambda}| + \int_0^\infty \frac{\kappa_a C}{1 + t^2} dt |\lambda - \tilde{\lambda}|
\]
\[+ \int_0^\infty \frac{\kappa_a}{2} |F_1(X, Y)(t) - F_1(\tilde{X}, \tilde{Y})(t)| dt \leq C|\lambda - \tilde{\lambda}| + \int_0^\infty \kappa_a |F_1(X, Y)(t) - F_1(\tilde{X}, \tilde{Y})(t)| dt.
\]
To keep the notation short, we write
\[
\dot{x} + x_0 = \tilde{X}, \; \dot{y} + y_0 = \tilde{Y},
\]
\[
t^{\pm}_{1/2}(X, Y, t) = t^{\pm}_{1/2}, \; t^{\pm}_{1/2}(\tilde{X}, \tilde{Y}, t) = \tilde{t}^{\pm}_{1/2},
\]
\[
P_1(\lambda, X, Y)(t) = P(t), \; P_1(\tilde{\lambda}, \tilde{X}, \tilde{Y})(t) = \tilde{P}(t),
\]
\[
F_1(X, Y)(t) = F(t), \; F_1(\tilde{X}, \tilde{Y})(t) = \tilde{F}(t),
\]
from now on. We continue with
\[
|F(t) - \tilde{F}(t)| \leq \left| \frac{1 - \tilde{Y}(t^+_2)}{1 + \tilde{Y}(t^+_2)} - \frac{1 - \hat{Y}(t^+_2)}{1 + \hat{Y}(t^+_2)} \right| \frac{1}{(X(t) - Y(t^+_2))^2}
\]
\[+ \left| \frac{1 + \tilde{Y}(t^-_2)}{1 - \tilde{Y}(t^-_2)} - \frac{1 + \hat{Y}(t^-_2)}{1 - \hat{Y}(t^-_2)} \right| \frac{1}{(X(t) - Y(t^-_2))^2}
\]
\[+ \left| \frac{1 - \hat{Y}(t^-_2)}{1 + \hat{Y}(t^-_2)} \right| \frac{1}{(X(t) - Y(t^-_2))^2} - \frac{1}{(\tilde{X}(t) - \tilde{Y}(t^-_2))^2}
\]
\[+ \left| \frac{1 + \hat{Y}(t^-_2)}{1 - \hat{Y}(t^-_2)} \right| \frac{1}{(X(t) - Y(t^-_2))^2} - \frac{1}{(\tilde{X}(t) - \tilde{Y}(t^-_2))^2}.
\]
Exploiting $\frac{d}{du} \left( \frac{1 + u}{1 + u^2} \right) = \frac{-2}{(1 + u^2)^2}$, $\frac{d}{du} \left( \frac{1}{u^2} \right) = -\frac{2}{u^3}$, (34), and $||\dot{y}||_\infty, ||\tilde{y}||_\infty \leq V$ leads to
\[
|F(t) - \tilde{F}(t)| \leq \frac{2}{(1 - V)^2} \left[ |\dot{Y}(t^+_2) - \dot{\hat{Y}}(t^+_2)| + |\dot{Y}(t^-_2) - \dot{\hat{Y}}(t^-_2)| \right] \frac{C}{1 + t^2}
\]
\[+ \frac{1 + V}{1 - V + ||t||^2} \left[ |x(t) - \tilde{x}(t)| + |Y(t^+_2) - \tilde{Y}(t^+_2)| + |Y(t^-_2) - \tilde{Y}(t^-_2)| \right].
\]
According to definition (6) we have $|t^\pm_2| \leq |t| + |a_0| + |b_0(0)| + V|t| + V|t^\pm_2|$, so that
\[
|t^\pm_2| \leq C(1 + |t|)
\]
holds. Using this bound and, again, the definition of \( t^+ \) and \( \hat{t}_2^+ \), we arrive at

\[
|Y(t^+_2) - \hat{Y}(\hat{t}_2^+)| \leq |y(t^+_2) - \hat{y}(\hat{t}_2^+)| + |\hat{Y}(t^+_2) - \hat{Y}(\hat{t}_2^+)|
\]

\[
\leq \|y - \hat{y}\|_\infty |t^+_2| + \|\hat{Y}(t^+_2) - \hat{Y}(\hat{t}_2^+)|
\]

\[
\leq \|y - \hat{y}\|_\infty |t^+_2| + V|t^+_2 - \hat{t}_2^+|
\]

\[
\leq C(1 + |t|)\|\dot{y} - \dot{\hat{y}}\|_\infty + V \left(|x(t) - \dot{x}(t)| + |Y(t^+_2) - \hat{Y}(\hat{t}_2^+)|\right).
\]

Rearranging terms, we find

\[
|Y(t^+_2) - \hat{Y}(\hat{t}_2^+)| \leq \frac{C}{1 - V} (1 + |t|)\|\dot{y} - \dot{\hat{y}}\|_\infty + \frac{V}{1 - V} |x(t) - \dot{x}(t)|
\]

\[
\leq C(1 + |t|) \left(\|\dot{y} - \dot{\hat{y}}\|_\infty + \|\dot{x} - \dot{\hat{x}}\|_\infty\right).
\]

This bound and the definition of \( t^+_2 \) and \( \hat{t}_2^+ \) imply

\[
|t^+_2 - \hat{t}_2^+| \leq C(1 + |t|) \left(\|\dot{y} - \dot{\hat{y}}\|_\infty + \|\dot{x} - \dot{\hat{x}}\|_\infty\right)
\]

Therefore, using the decay of the accelerations according to the definition of \( \Omega \), it holds

\[
|\dot{Y}(t^+_2) - \hat{\dot{Y}}(\hat{t}_2^+)| \leq |\dot{Y}(t^+_2) - \dot{Y}(\hat{t}_2^+) + |\hat{\dot{Y}}(t^+_2) - \hat{\dot{Y}}(\hat{t}_2^+)\|
\]

\[
\leq \|\dot{y} - \hat{\dot{y}}\|_\infty + \|\hat{\dot{Y}}(t^+_2) - \hat{\dot{Y}}(\hat{t}_2^+)\|
\]

\[
\leq C \left(\|\dot{y} - \hat{\dot{y}}\|_\infty + \|\dot{x} - \dot{\hat{x}}\|_\infty\right).
\]

Coming back to (39), we exploit the last bound together with (41) and \( |x(t) - \dot{x}(t)| \leq |t|\|\dot{x} - \dot{\hat{x}}\|_\infty \) to find

\[
|F(t) - \hat{F}(t)| \leq \frac{C}{1 + t^2} \|(x, y) - (\dot{x}, \dot{y})\|
\]

and, by (37) and (38),

\[
|\dot{H}_1(\lambda, x, y)(t) - \hat{\dot{H}}_1(\hat{\lambda}, \dot{x}, \dot{y})(t)| \leq |P(t) - \hat{P}(t)| \leq C_1 |\lambda - \hat{\lambda}| + C_2 \|(x, y) - (\dot{x}, \dot{y})\|.
\]

To complete the proof, we have to take care of the second derivatives of \( H \):

\[
|\dddot{H}_1(\lambda, x, y)(t) - \hat{\dddot{H}}_1(\hat{\lambda}, \dot{x}, \dot{y})(t)| = \left| \frac{d}{dt} \left( \frac{P(t)}{\sqrt{1 - P(t)^2}} - \frac{\hat{P}(t)}{\sqrt{1 - \hat{P}(t)^2}} \right) \right|
\]

\[
= \left| \frac{\dot{P}(t)}{(1 + P(t)^2)^{3/2}} - \frac{\dot{\hat{P}}(t)}{(1 + \hat{P}(t)^2)^{3/2}} \right|
\]

\[
\leq |\dot{P}(t)| \cdot \left| \frac{1}{(1 + P(t)^2)^{3/2}} - \frac{1}{(1 + \hat{P}(t)^2)^{3/2}} \right| + \frac{1}{(1 + P(t)^2)^{3/2}} \left| \dot{P}(t) - \dot{\hat{P}}(t) \right|
\]

Since \( \left| \frac{d}{du} \frac{1}{(1 + u^2)^{3/2}} \right| = \left| - \frac{3u}{(1 + u^2)^{5/2}} \right| \leq 1 \) for \( |u| \leq 1 \) and, by the definition of \( P \) and the decay of \( F \) given in (36), we get

\[
|P(t)| \leq C_1 + C_2 \int_0^t |F(s)| ds \leq C_1 + C_2 \int_0^\infty \frac{ds}{1 + s^2} \leq C.
\]

Furthermore, since \( \ddot{x}_0(t) = 0 \) for sufficiently large \( |t| \) as can be inferred from (17),

\[
|\dot{P}(t)| \leq (1 - \lambda) \left| \frac{\ddot{x}_0(t)}{(1 - \dot{x}_0(t))^2} \right| + \lambda |F(t)| \leq \frac{C}{1 + t^2}
\]
holds. Thus, inequality (44) results in
\[ |\tilde{H}_1(\lambda, x, y)(t) - \hat{H}_1(\tilde{\lambda}, \tilde{x}, \tilde{y})(t)| \leq \frac{C_1}{1 + t^2} |P(t) - \tilde{P}(t)| + C_2 |\tilde{P}(t) - \hat{P}(t)|. \]

Here, estimate (43) can be used for the first modulus and
\[
|\tilde{P}(t) - \hat{P}(t)| = \kappa_a \left| \lambda F(t) ds - \tilde{\lambda} \tilde{F}(t) \right|
\leq |\lambda - \tilde{\lambda}| C_1 |F(t)| + |\tilde{\lambda}| F(t) - \tilde{\lambda} F(t)|
\leq \frac{C_1}{1 + t^2} |\lambda - \tilde{\lambda}| + \frac{C_2}{1 + t^2} \|(x, y) - (\tilde{x}, \tilde{y})\|
\]
according to bound (42). Therefore, it holds that
\[ (1 + |t|)|\tilde{H}_1(\lambda, x, y)(t) - \hat{H}_1(\tilde{\lambda}, \tilde{x}, \tilde{y})(t)| \leq \frac{C_1}{1 + |t|} |\lambda - \tilde{\lambda}| + \frac{C_2}{1 + |t|} \|(x, y) - (\tilde{x}, \tilde{y})\|. \]

Together with (43) and the corresponding estimates for \( H_2 \), we finally find
\[
\|H(\lambda, x, y)(t) - H(\tilde{\lambda}, \tilde{x}, \tilde{y})(t)\| \leq C_1 |\lambda - \tilde{\lambda}| + C_2 \|(x, y) - (\tilde{x}, \tilde{y})\|,
\]
i.e., that \( H \) is Lipschitz continuous, which was to show. \( \square \)

**Lemma 2.3.** If a sequence \( f_1, f_2, \ldots \) of bounded continuous functions on \( \mathbb{R} \) is uniformly bounded and equicontinuous and
\[
\lim_{S \to \infty} \sup_{S, n \in \mathbb{N}} \left( |f_n(t) - f_n(S)| \lor |f_n(-t) - f_n(-S)| \right) = 0, \tag{47}
\]
then it has a uniformly convergent subsequence.

**Proof.** Let \((S_k)\) be a sequence such that
\[
\sup_{t > S_k, n \in \mathbb{N}} \left( |f_n(t) - f_n(S_k)| \lor |f_n(-t) - f_n(-S_k)| \right) \leq \frac{1}{k}
\]
for all \( k \in \mathbb{N} \). The Arzela-Ascoli Theorem guarantees the existence of a subsequence \((f_{n_k}^1)\) such that, for all \( l \in \mathbb{N} \), \( \sup_{|t| \leq S_1} |f_{n_k}^1(t) - f_{n_m}^1(t)| < 1 \) for all \( l, m \in \mathbb{N} \). For \( |t| > S_1 \), we find
\[
|f_{n_k}^1(t) - f_{n_m}^1(t)| \leq |f_{n_k}^1(t) - f_{n_k}^1(-S_1)| + |f_{n_k}^1(-S_1) - f_{n_m}^1(-S_1)| + |f_{n_m}^1(S_1) - f_{n_m}^1(t)| \leq 3
\]
so that \( \|f_{n_k}^1 - f_{n_m}^1\|_{\infty} < 3 \) holds. This construction can be iterated for every \( k \in \mathbb{N} \) to yield a subsequence \( \left( f_{n_k}^{i+1} \right)_{i \in \mathbb{N}} \) extracted from \( \left( f_{n_k}^i \right)_{i \in \mathbb{N}} \) such that \( \sup_{i, m \in \mathbb{N}} \|f_{n_k}^i - f_{n_m}^i\|_{\infty} < \frac{3}{k} \).

The diagonal sequence \( \left( f_{n_k}^k \right)_{k \in \mathbb{N}} \) is then uniformly Cauchy, and thus, uniformly convergent. \( \square \)

**Lemma 2.4.** \( H([0, 1] \times \overline{\Omega}) \) is precompact in \( B \).

**Proof.** Let \((\lambda_n, x_n, y_n)\) be a sequence in \([0, 1] \times \overline{\Omega}\). It is convenient to introduce the following abbreviations: \( H^{(n)} := H(\lambda_n, x_n, y_n) \), \( P^{(n)} := P(\lambda_n, x_n, y_n) \), and \( F^{(n)} := F(\lambda_n, x_n, y_n) \). According to the definition of the norm on \( B \) in (15), we have to show the existence of uniformly convergent subsequences of \( (H^{(n)}) \) and \((1 + |\cdot|)\tilde{H}^{(n)}\). We will use the preceding Lemma 2.3 and carry out the proof for \((H_1^{(n)}) \) and \((1 + |\cdot|)\tilde{H}_1^{(n)}\) only as \((H_2^{(n)}) \)
and \((1 + |x|) \bar{H}_2^{(n)}\) can be treated analogously. Using the bound on \(|P_1^{(n)}(t)|\) from (45), we infer from definition (19) of \(H\) that
\[
|\dot{H}_1^{(n)}(t)| \leq \|\dot{x}_0\|_\infty + \left\| \frac{P_1^{(n)}}{\sqrt{1 + \left(\frac{P_1^{(n)}}{P_1^{(n)}}\right)^2}} \right\|_\infty \leq 2
\]
and
\[
\ddot{H}_1^{(n)}(t) = -\ddot{x}_0(t) + \frac{\ddot{P}_1^{(n)}(t)}{(1 + P_1^{(n)}(t)^2)^2}.
\]
Together with the bound on \(\dot{P}_1^{(n)}(t)\) from (46) and the fact that \(\dot{x}_0(t) = 0\) for sufficiently large \(t\) from definition (17) we infer
\[
(1 + |t|)|\dot{H}_1^{(n)}(t)| \leq (1 + |t|) \left( |\dot{x}_0(t)| + |\dot{P}_1^{(n)}(t)| \right)
\leq \frac{C}{1 + |t|}.
\]
This means that both \((\dot{H}_1^{(n)}(t))\) and \(((1 + |x|) \bar{H}_1^{(n)}(t))\) are uniformly bounded; using in addition that \(|\frac{d}{du} \sqrt{1 + u^2}| \leq 1\) and \(|\frac{d}{du} \sqrt{1 - u^2}| = \left| \frac{1}{(1 - u^2)^{3/2}} \right| \leq \frac{1}{(1 - u^2)^{3/2}}\) for \(|u| \leq V\), we deduce for \(t \geq s\)
\[
|\dot{H}_1^{(n)}(t) - \dot{H}_1^{(n)}(s)| \leq |\dot{x}_0(t) - \dot{x}_0(s)| + \left| P_1^{(n)}(t) - P_1^{(n)}(s) \right|
\leq C|\dot{x}_0(t) - \dot{x}_0(s)| + \frac{\kappa a}{2} \int_s^t |F_1^{(n)}(r)| dr
\leq C_1 \|\dot{x}_0\|_\infty |t - s| + C_2 \int_s^t \frac{1}{1 + r^2} dr.
\]
Recalling (48), using the bounds on \(|P_1^{(n)}(t)|\) and \(|\dot{P}_1^{(n)}(t)|\) from (45) and (46), and employing the mean value theorem (similarly as for inequality (44) in the proof of Lemma 2.2), we obtain
\[
|\ddot{H}_1^{(n)}(t) - \ddot{H}_1^{(n)}(s)|
\leq |\ddot{P}_1(t)| \cdot \left| \frac{1}{(1 + P_1(t)^2)^{3/2}} - \frac{1}{(1 + P_1(s)^2)^{3/2}} \right| + \left| \frac{1}{(1 + P_1(s)^2)^{3/2}} |\dot{P}_1(t) - \dot{P}_1(s)| \right|
\leq \frac{C_1}{1 + s^2} |P_1^{(n)}(t) - P_1^{(n)}(s)| + C_2 |\dot{P}_1^{(n)}(t) - P_1^{(n)}(s)|
\leq \frac{C_1}{1 + s^2} \int_s^t \frac{1}{1 + r^2} dr + C_2 \left[ \frac{\ddot{x}_0(t)}{(1 - \ddot{x}_0(t)^2)^{3/2}} - \frac{\ddot{x}_0(s)}{(1 - \ddot{x}_0(s)^2)^{3/2}} \right] + \frac{\kappa a}{2} |F_1^{(n)}(t) - F_1^{(n)}(s)|
\]
\[
(51)
\]
The summand in front of the bracket and the first one within the bracket can both be estimated by \(\frac{C_{1|s|}}{1 + s^2}\) because \(\dot{x}_0\) was chosen uniformly continuous and zero for large \(t\). Moreover, using definition (21) of \(F_1^{(n)}\), splitting the summands via triangle inequality, applying the mean value theorem, and taking into account the bounds (34) and (35) on
\[ |X(t) - Y(t^\pm)\| \] and \[ |Y(t)| \], respectively, we get
\[
|F_1^{(n)}(t) - F_1^{(n)}(s)| \\
\leq \frac{C_1}{1 + s^2} \left| \dot{Y}^{(n)}(t^+)(t) \right| - \left| \dot{Y}^{(n)}(t^-)(s) \right| + \frac{C_2}{1 + s^2} \left| \dot{Y}^{(n)}(t^-)(t) \right| - \left| \dot{Y}^{(n)}(t^-)(s) \right| \\
+ \frac{C_3}{1 + |s|^3} \left| Y^{(n)}(t^+)(t) \right| - \left| Y^{(n)}(t^+)(s) \right| + \frac{C_4}{1 + |s|^3} \left| Y^{(n)}(t^-)(t) \right| - \left| Y^{(n)}(t^-)(s) \right| \\
+ \frac{C_5}{1 + |s|^3} \left| X^{(n)}(t) - X^{(n)}(s) \right| \\
\leq \frac{C}{1 + |s|^3} \left( |t^+(t) - t^-_1(s)| + |t^-_1(t) - t^-_2(s)| + |t - s| \right) .
\]

Going back to definition (6) of \( t^\pm \), we observe
\[
|t^+(t) - t^-_1(s)| \leq |t - s| + |X^{(n)}(t) - X^{(n)}(s)| + |Y^{(n)}(t^+_2)(t) - Y^{(n)}(t^+_2)(s)| \\
\leq |t - s| + C|t^+_2(t) - t^-_2(s)| \leq C|t - s| ,
\]
so that
\[
|F_1^{(n)}(t) - F_1^{(n)}(s)| \leq \frac{C|t - s|}{1 + |s|^3}
\]
is satisfied. Substituting this into (51) gives
\[
|\tilde{H}_1^{(n)}(t) - \tilde{H}_1^{(n)}(s)| \leq \frac{C|t - s|}{1 + |s|^3} .
\]

Consequently, recalling estimate (49) for \(|\tilde{H}_1^{(n)}(t)|\), we find
\[
|\tilde{H}_1^{(n)}(t) - \tilde{H}_1^{(n)}(s)| \leq C|t - s| (1 + |s|) \bar{\tilde{H}}^{(n)}(t) - \bar{\tilde{H}}^{(n)}(s)| \\
\leq \frac{C|t - s|}{1 + s^2} .
\]

This, together with (50), implies equicontinuity and condition (47) for both \((\tilde{H}_1^{(n)})\) and \(((1 + |\cdot|)\bar{\tilde{H}}^{(n)})\) as required by Lemma 2.3.

**Lemma 2.5.** \( a \) is \((r \wedge s) + 1 \) times differentiable at \( t \) if \( b \) is \( r \) times differentiable at \( t^-_2(t) \) and \( s \) times differentiable at \( t^+_2(t) \) for \( r, s \in \mathbb{N} \). Correspondingly, \( b \) is \((r \wedge s) + 1 \) times differentiable at \( t \) if \( a \) is \( r \) times differentiable at \( t \) and \( b \) is \( r \) times differentiable at \( t^+_2(t) \) and \( s \) times differentiable at \( t^-_1(t) \).

**Proof.** According to formula (26) for their derivatives, \( t^\pm_2 \) are \( r \) times differentiable at \( t \) if \( a \) is \( r \) times differentiable at \( t \) and \( b \) is \( r \) times differentiable at \( t^+_2(t) \), and correspondingly for \( t^+_1 \). Therefore, equations (4) for \( \tilde{a} \) and \( \tilde{b} \) contain one more derivative on the left-hand side than on the right, and the claim follows.

**Proof of Theorem 1.1:** Choosing \( V \in \tilde{V}, 1, D \in \tilde{D}, v \in \tilde{v}, T \geq \tilde{T} \) and \( A \geq \tilde{A} \) in the definition (23) of \( \Omega \) ensures, according to Proposition 2.1, that \( H \) defined in (19) cannot have fixed points on \( \partial \Omega \). Obviously, \( \Omega \) is a bounded open subset of Banach space \( B \) defined in (16) containing the origin. Lemma 2.2 shows that \( H \) is a homotopy, Lemma 2.4 proves its compactness. Furthermore, by definition, \( H(0, \cdot) \) is the zero mapping and fixed points of \( H \) satisfy (22). In particular, fixed points of \( H(1, \cdot) \) satisfy (4) with initial data (7) and (8). Thus, the existence of a solution to (4), (7)-(8) follows from the Leray-Schauder Theorem 2.1.

Concerning the regularity statement b), we observe that \( b|_{[T^-_0, T^+_0]} \in C^{n+1} \) by assumption, \( b \) is once differentiable at \( T^+ \) by the piecewise definition of \( H \) and \( b|_{[T^+_0, \infty]}, a|_{[0, \infty]} \in C^{n+1} \) by assumption.
C^2 by lemma 2.5. Consequently, also by Lemma 2.5, \( b|_{\mathbb{T}^+\infty} \in C^3 \). If \( n \geq 1 \), we can now apply the Lemma another \( n \) times, alternately to \( a \) at \( t > 0 \) and \( b \) at \( t > T^+ \), to find \( a|_{\mathbb{T}^+\infty\setminus\{\sigma_1,\sigma_2,\ldots\}} \in C^{2+n} \) and \( b|_{\mathbb{T}^+\infty\setminus\{\tau_2,\tau_3,\ldots\}} \in C^{3+n} \). Now, another application of the Lemma e.g. at \( t \in [0,\sigma_1[ \) yields no more regularity than \( C^{2+n} \) because \( b(t_2) \) is only \( C^{1+n} \), but for \( t \in ]\sigma_1,\infty[ \setminus\{\sigma_2,\sigma_3,\ldots\} \), \( b(t_2) \in C^{3+n} \), so one finds \( a|_{\sigma_1,\infty}\setminus\{\sigma_2,\sigma_3,\ldots\} \in C^{2+n} \) and, by three more applications, \( b|_{\tau_2,\infty}\setminus\{\tau_3,\tau_4,\ldots\} \in C^{4+n} \). Iteratively, the regularity of the segments as claimed in Theorem 1.1b) is proven. The proof for \( (\sigma_k) \) and \( (\tau_k) \) works in the same way, the only difference being the lower initial regularity at \( \sigma_1 \) and \( \tau_1 \). \( \square \)

3. Proof of Theorem 1.2

In this section we turn to the Synge equations, i.e., the FST equations without the advanced terms. In order to prove Theorem 1.2 we have to adopt our strategy slightly. We define again
\[
\|(x,y)\| := \max\left(\|\dot{x}\|, \|\dot{y}\|, \sup_{t \in \mathbb{R}}(1 + |t|)|\ddot{x}(t)|, \sup_{t \in \mathbb{R}}(1 + |t|)|\ddot{y}(t)|\right)
\]
and make
\[
B' := \{(x,y) \in C^2(\mathbb{R},\mathbb{R}^2) \mid x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0, \|(x,y)\| < \infty\}
\]
a Banach space w.r.t. that norm. Furthermore, we define a map \( H' \) in almost the same way as we defined \( H \) in (19) in the preceding section except that this time we omit the advanced term:
\[
\begin{align*}
\quad H'_1(\lambda,x,y)(t) & := -x_0(t) + a_0 + \int_0^t \frac{P'_1(\lambda,X,Y)(s)}{\sqrt{1 + P'_1(\lambda,X,Y)(s)^2}} \, ds \\
\quad H'_2(\lambda,x,y)(t) & := -y_0(t) + b_0 + \int_0^t \frac{P'_2(\lambda,X,Y)(s)}{\sqrt{1 + P'_2(\lambda,X,Y)(s)^2}} \, ds,
\end{align*}
\]
where
\[
X := x + x_0, Y := y + y_0
\]
with \( C^2 \) trajectories \((x_0,y_0)\) satisfying the Cauchy data and
\[
\begin{align*}
\|\dot{x}\|, \|\dot{y}\| & < 1, \\
\inf_{t \in \mathbb{R}}(x_0(t) - y_0(t)) & > 0, \\
(\dot{x}_0 - \dot{y}_0)(-1) & < 0, \ (\dot{x}_0 - \dot{y}_0)(1) > 0, \\
\ddot{x}_0(t) & \geq 0, \ \ddot{y}_0(t) \leq 0 \text{ for } t \in \mathbb{R}, \\
\dot{x}_0(t) = \dot{y}_0(t) & = 0 \text{ for } |t| \geq 1.
\end{align*}
\]
The velocity and force terms are given by
\[
\begin{align*}
P'_1(\lambda,X,Y)(t) & := (1 - \lambda) \frac{\dot{x}_0(t)}{\sqrt{1 - x_0(t)^2}} + \lambda \frac{\dot{a}_0}{\sqrt{1 - a_0^2}} + \lambda \int_0^t \kappa_2 F'_1(X,Y)(s) \, ds \\
P'_2(\lambda,X,Y)(t) & := (1 - \lambda) \frac{\dot{y}_0(t)}{\sqrt{1 - y_0(t)^2}} + \lambda \frac{\dot{b}_0}{\sqrt{1 - b_0^2}} + \lambda \int_0^t \kappa_2 F'_2(X,Y)(s) \, ds
\end{align*}
\]
and
\[
F_1'(X, Y)(t) := \frac{1 + \dot{Y}(t_2^-(X, Y, t))}{1 - \dot{Y}(t_2^-(X, Y, t))} \frac{1}{(X(t) - Y(t_2^-(X, Y, t)))^2},
\]
\[
F_2'(X, Y)(t) := -\frac{1 - \dot{X}(t_1^-(X, Y, t))}{1 + \dot{X}(t_1^-(X, Y, t))} \frac{1}{(Y(t) - X(t_1^-(X, Y, t)))^2}.
\]

Again, \(H'(0, \cdot)\) is the zero mapping whereas, if \((x, y)\) is a fixed point of \(H'(\lambda, \cdot)\), then
\[
\frac{d}{dt} \left( \frac{\dot{X}(t)}{\sqrt{1 - X(t)^2}} \right) = \frac{\dot{X}(t)}{(1 - X(t)^2)^{3/2}} = (1 - \lambda) \frac{\ddot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{3/2}} + \lambda \kappa_a F_1'(X, Y)(t),
\]
\[
\frac{d}{dt} \left( \frac{\dot{Y}(t)}{\sqrt{1 - Y(t)^2}} \right) = \frac{\dot{Y}(t)}{(1 - Y(t)^2)^{3/2}} = (1 - \lambda) \frac{\ddot{y}_0(t)}{(1 - \dot{y}_0(t)^2)^{3/2}} + \lambda \kappa_b F_2'(X, Y)(t)
\]
holds – in particular, if \(\lambda = 1\), then \((X, Y)\) solve the equations of motion \((4)\) for \(\epsilon_+ = 0, \epsilon_- = 1\) and Cauchy data \((10)\).

This time we want to obtain global a priori bounds on possible fixed points of \(H' : [0, 1] \times M' \to C^1(\mathbb{R}, \mathbb{R}^2)\) with
\[
M' := \{(x, y) \in C^1(\mathbb{R}, \mathbb{R}^2) \mid x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0, \|\dot{X}\|_\infty, \|\dot{Y}\|_\infty < 1, \forall_{t \in \mathbb{R}} X(t) > Y(t)\}
\]
in terms of Newtonian Cauchy data only:

**Proposition 3.1.** For any given Newtonian Cauchy data \((10)\), there are constants \(\tilde{v}', \tilde{V}' \in [0, 1[\) and \(\tilde{A}', \tilde{D}', \tilde{T}' > 0\) such that, for any fixed point \((x, y)\) of \(H'\),
\[
\|\dot{X}\|_\infty, \|\dot{Y}\|_\infty < \tilde{v}', \quad \inf_{t \in \mathbb{R}} (X - Y)(t) > \tilde{D}',
\]
\[
\sup_{t \leq -\tilde{T}'} (\dot{X} - \dot{Y})(t) < -\tilde{v}', \quad \inf_{t \geq \tilde{T}'} (\dot{X} - \dot{Y})(t) > \tilde{v}',
\]
\[
|\ddot{X}(t)|, |\ddot{Y}(t)| < \frac{\tilde{A}'}{1 + |t|}.
\]

The strategy of proof stems from the following observation: Imitating the proof of Proposition 2.1 in order to get an estimate for \(\dot{Y}\) at times \(t > 0\) and omitting the advanced terms appearing there, we would need an upper bound for \(\dot{Y}(t_2^-)\) which in Proposition 2.1 above was provided by the prescribed trajectory strip. In contrast, we observe that an upper bound for the now missing term \(\dot{Y}(t_2^-)\) would already be given by \(\dot{b}_0\) because \(\dot{Y}\) is always negative. This leads to the idea of estimating the velocity for negative times first. In this case, by reversing the corresponding signs in the corresponding calculation in the proof of Proposition 2.1 above, the retarded term takes the role the advanced term played in the estimate for \(t > 0\) above, and a lower bound for \(\dot{Y}(t_2^-)\) at times \(t < 0\) is needed, which is again given by \(\dot{b}_0\). An estimate for \(\dot{X}(t), t < 0,\) is obtained in the same way, and in a second step, the obtained estimates provide the velocity bounds required in order to get hands on \(X, Y\) at future times.
Proof. We start the calculation by deducing from equation (53)
\[
\frac{d}{dt} \left( \frac{1 - \dot{b}_0 \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \right) = \frac{(\dot{X}(t) - b_0)\dot{X}(t)}{(1 - X(t)^2)^{\frac{3}{2}}} \]

\[
= (1 - \lambda) \frac{(\dot{X}(t) - \dot{b}_0)\ddot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{\frac{3}{2}}} + \lambda \kappa_a \left[ \frac{(\dot{X}(t) - \dot{b}_0)(1 + \dot{Y}(t_2^-))}{1 - Y(t_2^-)} - \frac{1}{(X(t) - Y(t_2^-))^2} \right]. \tag{54}
\]

**Step 1** \((t \leq 0):\) As in the proof of Proposition 2.1, we want to use (26) and (27) to estimate the right-hand side by a total differential. Instead of equation (28), we now know, since \(\dot{Y}(t) > \dot{b}_0\) for \(t < 0\), that
\[
(\dot{X}(t) - \dot{b}_0)(1 + \dot{Y}(t_2^-)) = (1 + \dot{b}_0)(\dot{X}(t) - \dot{Y}(t_2^-)) + (\dot{Y}(t_2^-) - \dot{b}_0)(1 + \dot{X}(t)) \\
\geq (1 + \dot{b}_0)(\dot{X}(t) - \dot{Y}(t_2^-)).
\]

Integration of (54) from \(t\) to 0 gives
\[
\frac{1 - \dot{b}_0 \dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} - \frac{1 - \dot{b}_0 \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \geq (1 - \lambda) \left[ \frac{2\ddot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} - \frac{2\dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} \right] + \lambda \kappa_a \left[ \frac{1 + \dot{b}_0}{\dot{X}(t) - Y(t_2^-)} - \frac{1 + \dot{b}_0}{a_0 - Y(t_2(0))} \right],
\]
and similar estimates as in the time-symmetric case lead to
\[
\frac{1 - |\dot{b}_0|}{\sqrt{1 - \dot{X}(t)^2}} \leq \frac{4}{\sqrt{1 - \dot{a}_0^2}} + \frac{4\kappa_a}{a_0 - \dot{b}_0} - \lambda \kappa_a \frac{1 + \dot{b}_0}{X(t) - Y(t_2^-)}
\]
and
\[
\sup_{t \leq 0} |X(t)| \leq \sqrt{1 - \left( \frac{4 + \frac{4\kappa_a}{a_0 - \dot{b}_0}}{\sqrt{1 - \dot{a}_0^2}} \right)^2} =: V_a^-.
\]

The estimate \(V_a^-\) for \(\sup_{t \leq 0} |Y(t)|\) is obtained in an analogous way.

**Step 2** \((t > 0):\) Now that we know bounds on the retarded velocities, we can bound \(\dot{X}(t)\) for \(t > 0\) by considering \(\frac{1 - \dot{V}_a^- \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}}\) and proceeding as in the proof of Proposition 2.1 above. Again, the estimate for \(\dot{Y}\) for positive times is obtained in the same way, so \(\dot{V}'\) exists as required.

\(\dot{v}', \dot{A}', \dot{D}', \dot{T}'\) can then be derived as in the time-symmetric case in the proof of Proposition 2.1 above.

**Proof of Theorem 1.2:** The continuity and compactness proof in Lemma 2.2 and 2.4 for \(H\) carries over to \(H'\) by crossing out the advanced terms, hence the existence of a fixed point of \(H'\) can be inferred as in the proof of Theorem 1.1.
4. Proof of Theorem 1.3

In this section we turn to the FST toy model. Theorem 1.3 exemplifies that the technique used so far is in principle also applicable of proving the existence of global solutions to mixed-type equations, i.e., equations involving advanced as well as retarded terms, satisfying Cauchy data. In order to treat equations (12), it is sufficient to employ the Banach space

\[ B := \{ (x, y) \in C^1(\mathbb{R}, \mathbb{R}^2) \mid x(0) = x_0, y(0) = y_0, \| (x, y) \| < \infty \} \]

with

\[ \| (x, y) \| := \max(\| x \|_\infty, \| y \|_\infty) . \]

The second derivatives in the previous proofs came only into play for the sake of estimating differences of the velocities appearing in the force law. \( H'' \) is defined as \( H' \) in (52) ff., where now

\[
F_1''(X, Y)(t) := \frac{1}{(X(t) - Y(t_2^+ (X, Y, t)))^2} + \frac{1}{(X(t) - Y(t_1^+ (X, Y, t)))^2},
\]

\[
F_2''(X, Y)(t) := -\frac{1}{(Y(t) - X(t_1^+ (X, Y, t)))^2} - \frac{1}{(Y(t) - X(t_2^+ (X, Y, t)))^2} .
\]

Fixed points again satisfy (53), which for \( \lambda = 1 \) means that they solve the equations of motion of the FST toy model (12) for the given Cauchy data (13). Now the following a priori estimates are needed:

**Proposition 4.1.** For any given Newtonian Cauchy data, there are constants \( \tilde{v}'', \tilde{V}'' \in \]0, 1\[ and \( \tilde{D}'', \tilde{T}'' > 0 \) such that, for any fixed point \( (x, y) \) of \( H'' \),

\[
\| \dot{X} \|_\infty, \| \dot{Y} \|_\infty < \tilde{V}'',
\]

\[
\inf_{t \in \mathbb{R}} (X - Y)(t) > \tilde{D}'',
\]

\[
\sup_{t \leq -\tilde{T}''} (X - Y)(t) < -\tilde{v}'', \inf_{t \geq \tilde{T}''} (X - Y)(t) > \tilde{v}'' .
\]

**Proof.** We first assume that \( \dot{a}_0 - \dot{b}_0 \leq 0 \) (and, consequently, \( \dot{X}(t) - \dot{b}_0 \leq 0 \) for \( t < 0 \)) and consider \( t \leq 0 \). Even though (12) is not Lorentz invariant, we again compute the “kinetic energy” in the Lorentz frame boosted by \( b_0 \):

\[
\frac{d}{dt} \left( \frac{1 - \dot{b}_0 \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \right) = \frac{(\dot{X}(t) - \dot{b}_0) \dot{X}(t)}{(1 - \dot{X}(t)^2)^{\frac{3}{2}}}
\]

\[
= (1 - \lambda) \frac{(\dot{X}(t) - \dot{b}_0) \ddot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{\frac{3}{2}}} + \lambda \frac{\dot{X}(t) - \dot{b}_0}{(X(t) - Y(t_2^+))^2} + \lambda \frac{\dot{X}(t) - \dot{b}_0}{(X(t) - Y(t_1^+))^2} .
\]

Since the acceleration of \( Y \) is never positive, the closest possible route to \( X \) that \( Y \) can take is \( \dot{Y}(t) := b_0 + \dot{b}_0 t \), and, using Lemma 2.1,

\[
X(t) - Y(t_2^+) \geq X(t) - \dot{Y}(t_2^+) \geq \frac{1}{2} (X(t) - \dot{Y}(t))
\]

holds. Equation (55) then reduces to

\[
\frac{d}{dt} \left( \frac{1 - \dot{b}_0 \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \right) \geq \frac{-2 \ddot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{\frac{3}{2}}} + 4\lambda \frac{\dot{X}(t) - \dot{b}_0}{(X(t) - \dot{b}_0)^2} ;
\]
by integration from $t < 0$ to $0$ one arrives at

$$\frac{1 - \dot{b}_0 \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \leq \frac{1 - \dot{b}_0 \dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} + \frac{2\dot{a}_0}{\sqrt{1 - \dot{a}_0^2}} - \frac{2\dot{x}_0(t)}{\sqrt{1 - \dot{x}_0(t)^2}} + \frac{4\lambda}{a_0 - b_0} - \frac{4\lambda}{X(t) - b_0 - \dot{b}_0 t}$$

$$\leq \frac{4}{\sqrt{1 - \dot{a}_0^2}} + \frac{4}{a_0 - b_0}.$$

From this, one finds the bound

$$\sup_{t \leq 0} |\dot{X}(t)| \leq \sqrt{1 - \frac{(1 - |\dot{b}_0|)^2}{\left(\frac{4}{\sqrt{1 - \dot{a}_0^2}} + \frac{4}{a_0 - b_0}\right)^2}} =: V_a^-.$$

The estimate $V_b^-$ for $|\dot{Y}(t)|$ at negative times is inferred in the same way. For positive times, we look at

$$\frac{d}{dt} \left( \frac{1 - V_b^- \dot{X}(t)}{\sqrt{1 - \dot{X}(t)^2}} \right) = \frac{(\dot{X}(t) - V_b^-) \dot{X}(t)}{1 - \dot{X}(t)^2}$$

$$\leq (1 - \lambda) \frac{(\dot{X}(t) - V_b^-) \ddot{x}_0(t)}{(1 - \dot{x}_0(t)^2)^{\frac{3}{2}}} + \lambda \frac{\dot{X}(t) - \ddot{Y}(t_2^-)}{(X(t) - Y(t_2^-))^2} + \lambda \frac{\dot{X}(t) - \ddot{Y}(t_2^+)}{(X(t) - Y(t_2^+))^2}$$

and use

$$\dot{X}(t) - \ddot{Y}(t_2^+) \leq 2 \frac{\ddot{X}(t) - \ddot{Y}(t_2^+)}{1 + \ddot{Y}(t_2^+)}.$$

We can proceed with the help of equation (27) as in the proof of Proposition 2.1. The same procedure can be applied to get the corresponding bound for $\dot{Y}$. If, contrary to our assumption, $\dot{a}_0 > \dot{b}_0$, we have to reverse the order of our proof, i.e., do the first part for positive and the second part for negative times.

The remaining inequalities are then proved analogous to the arguments in Proposition 2.1. \hfill \Box

**Proof of Theorem 1.3**: The proof that $H''$ is continuous and compact is similar to the one of Lemmata 2.2 and 2.4, with considerable simplifications due to the absence of the velocity terms $\sigma$ and $\rho$ introduced in (5). Again, the existence of a fixed point of $H''$ can be inferred as in the proof of Theorem 1.1. \hfill \Box

### 5. Proof of Theorem 1.4

In this last section we prove that only finites strips of the solutions are needed to identify them uniquely.

**Proof of Theorem 1.4**: The idea is to solve equation (4) for $b$ for the advanced term in order to compute $a$ up to time $t_1^+(T_2^+)$ and to iterate this procedure as indicated in Figure 3. To reconstruct $a(t)$ for $t \in [T_1^+, t_1^+(T_2^+)]$, we rewrite equation (4) for $\dot{b}$ as

$$\dot{a}(t_1^+) = f \left( -2 \left( b(t) - a(t_1^+) \right)^2 \left[ \frac{\dot{b}(t)}{\kappa b(1 - b(t)^2)^{\frac{3}{2}}} + \frac{1}{2} \frac{1}{1 + \dot{a}(t_1^+)} \right] + \frac{1}{2} \dot{a}(t_1^+) \right).$$
well-defined for all \((t, a(t))\) from \(b(t_2^-)\) and \(a(t_2^-)\) and their derivatives; (ii) Step 2: Reconstruction of the dotted part \(b[t_2^+, t_2^-((t_2^+)\).}

\[
\dot{a}(t) = f \left( -2 (b(t_2^-) - a(t))^2 \left[ \frac{\dot{b}(t_2^-)}{\kappa_b(1 - \dot{b}(t_2^-))} + \frac{1}{2 \left( 1 + \dot{a}(t_2^-) (b(t_2^-) - a(t_2^-))^2 \right)} \right] \right)
\]

with \(f(u) = \frac{u-1}{u+1}. \) Trajectory \(b\) at the retarded times as well as \(a\) at the double retarded times being the given input for the reconstruction, this equation can be interpreted as an ordinary differential equation

\[ 
\dot{a}(t) = f(g(t, a(t))) \tag{56} 
\]

for \(a\) with given initial value \(a(T_1^+)\), where

\[ 
g(t, x) := -2 (b(t_2^-) - x)^2 \left[ \frac{\dot{b}(t_2^-)}{\kappa_b(1 - \dot{b}(t_2^-))} + \frac{1}{2 \left( 1 + \dot{a}(t_2^-) (b(t_2^-) - a(t_2^-))^2 \right)} \right] \]

and each term \(t_2^-\) also depends on \(t\) and \(x\) according to definition (6). The map \(g\) is well-defined for all \((t, x)\) such that \(t_2^-\) exists, i.e., on

\[ 
D_g := \{ (t, x) \mid \exists s \in [T_2^-, T_2^+]: t - s = |x - b(s)| \}, \]

the set of all space-time points which can be reached from the initial trajectory segment with speed of light.

The velocity field \(f \circ g\) is thus well-defined on

\[ 
D_{fog} := \{ (t, x) \in D_g \mid g(t, x) \neq -1 \}. \]

By definition of the initial segments, we have \(T_1^+ - T_2^- = a(T_1^+) - b(T_2^-)\), which implies that \((T_1^+, a(T_1^+))\) is contained in \(D_g\). That this point also lies in \(D_{fog}\) follows from the fact that the equation of motion (4) for \(b\) is satisfied at time \(T_2^-\). The strips \(a[T_1^-, T_1^+]\) and \(b[T_2^-, T_2^+]\) are \(C^\infty\) because the equations of motion have one more derivative on the left-hand side than on the right. Therefore, \(f \circ g\) is differentiable w.r.t. \(x\), in particular locally Lipschitz continuous, and (56) has a unique solution with a certain maximal lifetime. Since (56) is just the reordered equation of motion, \(a\) itself solves it as long as \((t, a(t))\) remains in \(D_{fog}\). As shown in Proposition 2.1, we have

\[ 
|\dot{a}| \leq C < 1 \quad \text{and} \quad \inf_{t \in \mathbb{R}} |a(t) - b(t)| =: D > 0. \tag{57} 
\]

Therefore, such a maximal lifetime \(T_{max}\) exists and \(T_{max} - T_2^- = |a(T_{max}) - b(T_2^+)| \geq D\), i.e., \(T_{max} = t_1^+(a, b, T_2^-)\). Consequently, \(a[T_1^+, T_1^+(a, b, T_2^-)]\) is the unique maximal solution to
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In an analogous way, \( b_{[(T^+_2, t^+_2(t^+_1(a, b; T^+_2)))]} \) can be uniquely reconstructed and, iteratively, the whole solution in the future because the bounds \((57)\) are uniform in time. The reconstruction of the solution in the past is performed in the same way, solving \((4)\) for the retarded instead of advanced terms. □

Remark 5.1. Note that, if this construction is performed with arbitrarily prescribed (smooth) trajectory strips (not a priori gained from a global solution), it can end after a finite lifetime. More precisely, this is always the case if the prescribed trajectories or the ones obtained after finite iterations exhibit accelerations that require an attractive or zero force between the particles, reach the speed of light in finite time or approach a light line as an asymptote. Moreover, the obtained trajectories need not be differentiable at times \( T^+_1, T^+_2, t^+_1(T^+_2) \), etc. Apart from very particular cases, it is unknown which reasonable additional conditions on the trajectory strips ensure global existence of this construction. The corresponding uniqueness assertion for the toy model \((12)\) in three spacial dimensions can be found in [5]. There, it is also discussed how the initial strips can be restricted in order to increase the regularity at times \( T^+_1, T^+_2, t^+_1(T^+_2) \), etc.

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