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On the integrability of evolving membranes

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Abstract. We analyze an integrable sector of the space of solutions of the four dimensional (super) membrane theory, where the target space is locally Minkowskian and hence a solution of the Supergravity equations. By performing a duality transformation we relate it to the 1+1 Born-Infeld integrable equation.

1. Introduction
The membrane field equations are a nonlinear coupled system which describes the evolution of the membrane on a target space. The equations are very interesting, they have the same dynamical structure as the Einstein equations of General Relativity. The initial data is restricted by first class contraints which are preserved by the membrane field equations. The constraints are the generators of diffeomorphisms on the base manifold, a three dimensional pseudo riemannian manifold. The evolution equations are not directly hyperbolic but they can be rewritten as an equivalent hyperbolic system which has a well-posed initial value problem and a blow up criterion [1]. From a physical point of view the membrane theory or more generally the supermembrane theory has the remarkable property of interacting with supergravity theories. In particular, the 11-dimensional supermembrane [6] is the unique geometrical object which interacts with the unique 11-dimensional supergravity. It is believed that the non-perturbative quantization of supermembranes is a fundamental step in the understanding of string theory. In this work we study the integrable properties of a class of solutions of the membrane field equations.

2. The duality transformation
In this section we analyze an integrable sector of Born-Infeld theory and of the membrane theory in a 4-dimensional target space. The duality among the theories allows to relate the 1+1 integrable Born-Infeld (BI) equation:

\[ \left[ 1 + \phi'^2 \right] \ddot{\phi} + \left[ -1 + \dot{\phi}^2 \right] \phi'' - 2 \dot{\phi} \phi' \dot{\phi}' = 0 \]  

where \( \phi' = \frac{\partial}{\partial \sigma} \phi \) and \( \dot{\phi} = \frac{\partial}{\partial \tau} \phi \), \( \sigma \) and \( \tau \) are local coordinates on the 1+1 formulation of Born-Infeld theory, to an integrable sector on the space of solutions of the membrane field equations. The transformation between the BI action and the membrane action in the case we will consider here, that is, in which the target space is a three dimensional Minkowski space times \( S^1 \) is a \( S \)-duality transformation in the string terminology.
In [2] the Cauchy problem of the 1+1 BI equation was solved and the quantization of the system was analyzed. In [3, 5, 4], for the same equation, an integrable criterion was shown (it has an infinite number of conserved quantities). In [5] the existence of a multi-Hamiltonian structure associated to it was proven.

The membrane theory in 4-dimensions is defined in terms of an embedding

\[ X = (X^\mu) : M_3 \to T \]

from the base manifold \( M_3 \) to the target space \( T \) which we will take as a product of a Minkowski space times \( S^1 \). \( M_3 \) is assumed to be a product of a compact closed Riemann surface \( \Sigma \) times \( \mathbb{R} \), the one dimensional vector space on which we define a time coordinate \( \xi^0 \).

The induced metric

\[ g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \]  

(2)

defines the induced volume element \( d^3\xi \sqrt{-g} \) where \( g \) is the determinant of the induced metric \( g_{ij} \).

The action of the theory is defined by the volume of \( M_3 \) with the induced metric (2):

\[ S(X) = \int_{M_3} \sqrt{-g}. \]  

(3)

An equivalent formulation of the membrane theory expressed in terms of a polynomial lagrangian density is given by

\[ S(\gamma, X) = -\frac{1}{2} \int_{M_3} d^3\xi \sqrt{-\gamma} \left( \gamma^{ij} \partial_i X^m \partial_j X^n \eta_{mn} - 1 \right) \]  

(4)

where the metric \( \gamma_{ij} \) on the 3-dimensional manifold \( M_3 \) is now taken as an independent field and not to be the induced metric. The \(-1\) follows from the equivalence of this Polyakov type of lagrangian with the Nambu-Goto type of lagrangian (3).

If some of the embedding maps \( X^\mu \), say \( X \), takes value on \( S^1 \) then we impose

\[ \int_C dX = 2\pi q_c \]  

(5)

where \( C \) is a basis of homology of dimension 1 elements on the base manifold \( M_3 \), while \( q_c \) is an integral number associated to each element of the basis. The set of \( q_c \) defines the winding of the map over \( S^1 \).

The \( S \)-dual formulation to (4) may be obtained by considering the master action

\[ S(\gamma, X, L, A) = -\frac{1}{2} \int_{M_3} d^3\xi \sqrt{-\gamma} \left( \gamma^{ij} L_i L_j + \gamma^{ij} \partial_i X^m \partial_j X^n \eta_{mn} - 1 \right) - \int_{M_3} d^3\xi \partial_j A_k L_i \]  

(6)

where the closed two form with components \( F_{ij} = \partial_i A_j - \partial_j A_i \) is restricted by

\[ \oint_{\Sigma} F(A) = 2\pi p. \]  

(7)

Here \( L_i \) is a vector field defined on the \( M_3 \) base manifold or in geometrical terms a section on a real vector bundle over \( M_3 \), \( A_k d\xi^k \) is a \( U(1) \) connection one-form over \( M_3 \) which arises from a connection on a \( U(1) \) principal bundle with base manifold \( M_3 \) acting on an associated vector bundle and \( F_{ij} \) are the components of the curvature two-form associated to the connection one form \( A_k d\xi^k \). The connection one form on \( M_3 \), valued on \( U(1) \), is obtained by choosing a (local)
section on the principal bundle over $M_3$ and taking a pullback of the connection on the principal bundle.

If we take variations of $S(\gamma, X, L, A)$ with respect to $A_k$ we obtain locally

$$L_i = \partial_i X$$

for some scalar field $X$. If we replace the solution for $L_i$ into the action (6) we obtain (4) or equivalently (3).

If instead we take variations of $S(\gamma, X, L, A)$ with respect to $L_i$, we obtain

$$L_i = \gamma^{ij} \partial_i X^m \partial_j X^n \eta_{mn} + \frac{1}{2} \gamma^{ij} \gamma^{kl} F_{ik} F_{jl} - 1$$

which can be proven to be equivalent to the following Born-Infeld action in terms of the induced metric $g_{ij} = \partial_i X^m \cdot \partial_j X^n \eta_{mn}$

$$S(X, A) = \int_{M_3} \left[ -\sqrt{\det (g_{ij} + F_{ij})} \right].$$

For a global argument showing the equivalence of the action of the membrane theory compactified on a circle and the Born-Infeld action, in terms of the induced metric $g_{ij}$, see [10, 11, 12]. The condition (5) ensures that the closed two-form $F$ is the curvature of a one-form connection on a principle bundle over the base manifold and it is a necessary condition to prove the equivalence.

We now consider a class of configurations which leads to an integrable system: the 1+1 Born-Infeld equation (1). The manipulation that we are going to do should be performed directly on the field equations. However the resulting action from them is the same as if we perform the manipulation directly on the action.

We consider

$$X^0 = \xi^0, X^1 = \xi^1, X^2 = \xi^2$$

$$A_0 = A_1 = 0, \partial \xi^2 A_2 = 0,$$

then

$$\det (g_{ij} + F_{ij}) = -1 + (\dot{\phi})^2 - (\phi')^2$$

where we have denoted $A_2 \equiv \phi$.

We now perform a dimensional reduction of the Born-Infeld action on $M_3$, it reduces to a two-dimensional formulation

$$\int_{S^1 \times \mathbb{R}} d^2 \xi \left( -\sqrt{1 - (\dot{\phi})^2 + (\phi')^2} \right)$$

which under variations with respect to $\phi$ yields the 1+1 Born-Infeld integrable equation (1) [13].

3. Solutions to the membrane field equations

We may now analyze the dual configurations in terms of the solutions of the membrane field equations.

They arise as the critical points of the action (3). The field equations obtained by taking variations of (3) with respect to $X^\mu$ give rise to submanifolds with vanishing mean curvature [1]:

$$\partial_i \left[ \sqrt{-g} g^{ij} \partial_j X^\mu \right] = 0$$
where $g_{ij}$ is the induced metric constructed from the maps $X^\mu$, see (2). When $\sqrt{-g} \neq 0$ these equations can be expressed as

$$\left( \delta_{\mu\nu} - g^{ij} \partial_i X^\mu \partial_j X^\nu \right) g^{kl} \partial_k \partial_l X^\nu = 0. \quad (9)$$

If we consider the class of solutions such that

$$X^0 = \xi^0$$
$$X^1 = \xi^1$$
$$X^2 = \xi^2$$
$$X^3 = \phi$$

it follows that any solution of

$$g^{kl} \partial_k \partial_l \phi = 0$$

is a solution of the field equations (9). If we evaluate $g^{kl}$ on the above configurations we obtain exactly the 1+1 Born-Infeld equation (1), as expected since the theories are related by a duality transformation. Although this equation resembles a laplacian operator acting of $\phi$ it isn’t. In fact, $g^{kl}$ itself depends on $\phi$.

4. Conclusions

We have then characterized a class of solutions of the membrane field equations which describe an integrable equation. By performing a suitable change of variables this second order (in time) partial differential equation (1) reduces to a coupled system [5]

$$u_t + uu_x - ww_x = 0$$
$$w_t + uw_x - uu_x = 0$$

describing a Chaplygin gas. The polynomial expression of this integrable system and its generalization

$$u_t + u^p u_x - ww_x = 0$$
$$w_t + uw_x - uu_x = 0 \quad \text{(10)}$$

($p$ an integer) can be compared to the known parametric coupled Korteweg-de Vries system [14, 15, 16].

The analysis of system (10) and generalizations of it in the context of membrane theory will be considered elsewhere.

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