Typed Generic Traversal
With Term Rewriting Strategies

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Abstract

A typed model of strategic term rewriting is developed. The key innovation is that generic traversal is covered. To this end, we define a typed rewriting calculus $S'_\gamma$. The calculus employs a many-sorted type system extended by designated generic strategy types $\gamma$. We consider two generic strategy types, namely the types of type-preserving and type-unifying strategies. $S'_\gamma$ offers traversal combinators to construct traversals or schemes thereof from many-sorted and generic strategies. The traversal combinators model different forms of one-step traversal, that is, they process the immediate subterms of a given term without anticipating any scheme of recursion into terms. To inhabit generic types, we need to add a fundamental combinator to lift a many-sorted strategy $s$ to a generic type $\gamma$. This step is called strategy extension. The semantics of the corresponding combinator states that $s$ is only applied if the type of the term at hand fits, otherwise the extended strategy fails. This approach dictates that the semantics of strategy application must be type-dependent to a certain extent. Typed strategic term rewriting with coverage of generic term traversal is a simple but expressive model of generic programming. It has applications in program transformation and program analysis.

Key words: Term rewriting, Strategies, Generic programming, Traversal, Type systems, Program transformation

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1 Preface

Strategic programming  Term rewriting strategies are of prime importance for the implementation of term rewriting systems. In the present paper, we focus on another application of strategies, namely on their utility for programming. Strategies can be used to describe evaluation and normalisation strategies, e.g., to explicitly control rewriting for a system that is not confluent or terminating. Moreover, strategies can be used to perform traversal, and to describe reusable traversal schemes. In fact, the typeful treatment of generic traversal is the primary subject of the present paper. To perform traversal in standard rewriting without extra support for traversal, one has to resort to auxiliary function symbols, and rewrite rules have to be used to encode the actual traversal for the signature at hand. This usually implies one rewrite rule per term constructor, per traversal. This problem has been identified in \([BSV97, VBT98, LVK00, BSV00, BKV01, Vis01]\) from different points of view. In a framework, where traversal strategies are supported, the programmer can focus on the term patterns which require problem-specific treatment. All the other patterns can be covered once and for all by the generic part of a suitable strategy.

Application potential  Language concepts for generic term traversal support an important dimension of generic programming which is useful, for example, for the implementation of program transformations and program analyses. Such functionality is usually very uniform for most patterns in the traversed syntax. In \([Vis00]\), untyped, suitably parameterised traversal strategies are used to capture algorithms for free variable collection, substitution, unification in a generic, that is, language-independent manner. In \([LV01]\), typed traversal strategies are employed for the specification of refactorings for object-oriented programs in a concise manner. There are further ongoing efforts to apply term rewriting strategies to the modular development of interpreters, to language-independent refactoring, to grammar engineering, and others.

\(S'_\gamma\) and relatives  In the present paper, the rewriting calculus \(S'_\gamma\) is developed. The calculus corresponds to a simple but expressive language for generic programming. The design of \(S'_\gamma\) was influenced by existing rewriting frameworks with support for strategies as opposed to frameworks which assume a fixed built-in strategy for normalisation / evaluation. Strategies are supported, for example, by the specification formalisms Maude \([CELM96, CDE+99]\) and ELAN \([BKK+98, BKKR01]\). The \(\rho\)-calculus \([CK99]\) provides an abstract model for rewriting including the definition of strategies. The programming language Stratego \([VBT98]\) based on system \(S\) \([VB98]\) is entirely devoted to strategic programming. In fact, the “\(S\)” in \(S'_\gamma\) refers to system \(S\) which was most influential in the design of \(S'_\gamma\). The “\(\gamma\)” in \(S'_\gamma\) indicates that even the untyped part of \(S'_\gamma\) does not coincide with system \(S\). The “\(\gamma\)” in \(S'_\gamma\) stands for the syntactical domain \(\gamma\) of generic strategy types. The idea of rewriting
strategies goes back to Paulson’s work on higher-order implementation of rewriting strategies [Pau83] in the context of the implementation of tactics and tacticals for theorem proving. The original contribution of $S'_\gamma$ is the typeful approach to generic traversal strategies in a many-sorted setting of term rewriting.

**Examples of generic traversal** In Figure 1, five examples (I)–(V) of intentionally generic traversal are illustrated. In (I), all naturals in the given term (say, tree) are incremented as modelled by the rewrite rule $N \rightarrow \text{succ}(N)$. We need to turn this rule into a traversal strategy because the rule on its own is not terminating when considered as a rewrite system. The strategy should be generic, that is, it should be applicable to terms of any sort. In (II), a particular pattern is rewritten according to the rewrite rule $g(P) \rightarrow g'(P)$. Assume that we want to control this replacement so that it is performed in bottom-up manner, and the first (i.e., bottom-most) matching term is rewritten only. Indeed, in Figure 1, only one $g$ is turned into a $g'$, namely the deeper one. The strategy to locate the desired node in the term is completely generic. While (I)–(II) require type-preserving traversal, (III)–(V) require type-
changing traversal. We say that type-unifying traversal [LVK00] is needed because the results of (III)–(V) are of a fixed, say a unified type. In (III), we test some property of the term, namely if naturals occur at all. The result is of type Boolean. In (IV), we collect all the naturals in the term using a left-to-right traversal. That is, the result is a list of integers. Finally, in (V), we count all the occurrences of the function symbol $g$.

**The tension between genericity and specificity** In addition to a purely many-sorted type system, the rewriting calculus $S'_\gamma$ offers two designated generic strategy types, namely the type TP denoting generic type-preserving strategies, and the type TU(τ) denoting generic type-unifying strategies with the unified result type $\tau$. Generic traversal strategies typically employ many-sorted rewrite rules. Hence, we need to cope with both many-sorted and generic types, and we somehow need to mediate between the two levels. Since a traversal strategy must be applicable to terms of any sort, many-sorted ingredients must be lifted in some way to a generic type before they can be used in a generic context. As a matter of fact, a traversal strategy might attempt to apply lifted many-sorted ingredients to subterms of different sorts. For the sake of type safety, we have to ensure that many-sorted ingredients are only applied to terms of the appropriate sort. $S'_\gamma$ offers a corresponding type-safe combinator for so-called strategy extension. The many-sorted strategy $s$ is lifted to the generic strategy type $\gamma$ using the form $s \lhd \gamma$. The extended strategy will immediately fail when applied to a term of a sort that is different from the domain of $s$. Generic strategies are composed in a manner that they recover from failure of extended many-sorted ingredients by applying appropriate generic defaults or by recursing into the given term.

**Value of typing** The common arguments in favour of compile-time as opposed to run-time type checking remain valid for strategic term rewriting. Let us reiterate some of these arguments in our specific setting to justify our contribution of a typed rewriting calculus. To start with, the type system of $S'_\gamma$ and the corresponding reduction semantics should obviously prevent us from constructing ill-typed terms. Consider, for example, the rewrite rule $\text{INC} = N \rightarrow \text{succ}(N)$ of type $\text{Nat} \rightarrow \text{Nat}$ for incrementing naturals in the context of example (I) above. The left-hand side of rewrite rule $\text{INC}$ would actually match with all terms of all sorts, but it only produces well-typed terms when applied to naturals. A typed calculus prevents us from applying a rewrite rule to a term of an inappropriate sort. Admittedly, most rewrite rules use pattern matching to destruct the input term. In this case, ill-typed terms cannot be produced. Still, a typed calculus prevents us from even attempting the application of a rewrite rule to a term of an inappropriate sort. This is very valuable because such attempts are likely to represent a design flaw in a strategy. Furthermore, a typed calculus should also prevent the programmer from combining specific and generic strategies in certain undesirable ways. Consider, for example, an asymmetric choice $\ell \leftrightarrow \epsilon$ where a rewrite rule $\ell$ is strictly preferred over the
identity strategy $\epsilon$, and only if $\ell$ fails, the identity strategy $\epsilon$ triggers. This choice is controlled by success and failure of $\ell$. One could argue that this strategy is generic because the identity strategy $\epsilon$ is applicable to any term. Actually, we favour two other possible interpretations. One option is to refuse this composition altogether because we would insist on the types of the branches in a choice to be the same. Another option is to favour the many-sorted argument type for the type of the compound strategy. In fact, strategies should not get generic too easily since we otherwise lose the valuable precision of a many-sorted type system. Untyped strategic programming suffers from symptoms such that strategies fail in unexpected manner, or generic defaults apply to easily. This is basically the same problem as for untyped programming in Prolog. $S'_\gamma$ addresses all the aforementioned issues, and it provides static typing for many-sorted and generic strategies.

**Beyond parametric polymorphism** Some strategy combinators are easier to type than others. Combinators for sequential composition, signature-specific congruence operators and others are easy to type in a many-sorted setting. By contrast, generic traversal primitives, e.g., a combinator to apply a strategy $s$ to all immediate subterms of a given term, are more challenging since standard many-sorted types are not applicable, and the well-established concept of parametric polymorphism is insufficient to model the required kind of genericity. Let us consider the type schemes underlying the two different forms of generic traversal:

- **TP** $\equiv \forall \alpha. \alpha \rightarrow \alpha$ (i.e., type-preserving traversal)
- **TU**($\tau$) $\equiv \forall \alpha. \alpha \rightarrow \tau$ (i.e., type-unifying traversal)

In the schemes, we point out that $\alpha$ is a universally quantified type variable. It is easy to see that these schemes are appropriate. A type-preserving traversal processes terms of any sort (i.e., $\alpha$), and returns terms of the same sort (i.e., $\alpha$); similarly for the type-unifying case. In fact, $S'_\gamma$ does not enable us to inhabit somewhat arbitrary type schemes. The above two schemes are the only schemes which can be inhabited with the traversal combinators of $S'_\gamma$. This is also the reason that we do not favour type schemes to represent types of generic strategies in the first place, but we rather employ the designated constants TP and TU($\tau$). If we read the above type schemes in the sense of parametric polymorphism [Rey83, Wad89], we can only inhabit them in a trivial way. The first scheme can only be inhabited by the identity function. The second scheme can only be inhabited by a constant function returning some fixed value of type $\tau$. Generic traversal goes beyond parametric polymorphism for two reasons. Firstly, traversal strategies can observe the structure of terms, that is, they can descend into terms of arbitrary sorts, test for leafs and compound terms, count the number of immediate subterms, and others. Secondly, traversal strategies usually exhibit non-uniform behaviour, that is, there are different branches for certain distinguished sorts in a traversal. Although strategies are statically typed in $S'_\gamma$, the latter property implies that the reduction semantics of strategies is type-dependent.
Structure of the paper  In Section 2, we provide a gentle introduction to the subject of strategic programming, and to the rewriting calculus $S'_\gamma$. Examples of traversal strategies are given. The design of the type system is motivated. As an aside, we use the term “type” for types of variables, constant symbols, function symbols, terms, strategies, and combinators. We also use the term “sort” in the many-sorted sense if it is more suggestive. In Section 3, we start the formal definition of $S'_\gamma$ with its many-sorted core. In this phase, we cannot yet cover the traversal primitives. A minor contribution is here that we show in detail how to cope with type-changing rewrite rules. In Section 4, we provide a type system for generic strategies. The two aforementioned schemes of type preservation and type unification are covered. A few supplementary issues to complement $S'_\gamma$ are addressed in Section 5. Implementation issues and related work are discussed in Section 6 and Section 7. The paper is concluded in Section 8.

Objective  An important meta-goal of the present paper is to develop a simple and self-contained model of typeful generic programming in the sense of generic traversal of many-sorted terms. To this end, we basically resort to a first-order setting of term rewriting. We want to clearly identify the necessary machinery to accomplish generic traversal in such a simple setting. We also want to enable a simple implementation of the intriguing concept of typed generic traversal. The $S'_\gamma$ expressiveness is developed in a stepwise manner. In the course of the paper, we show that our type system is sensible from a strategic programmer’s point of view. We contend that the type system of $S'_\gamma$ disciplines strategic programs in a useful and not too restrictive manner.

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2 Rationale

We set up a rewriting calculus $S'_\gamma$ inspired by ELAN \cite{BKK98, BKK01} and system $S$ \cite{VB98}. Some basic knowledge of strategic rewriting is a helpful background for the present paper (cf. \cite{BKK96, VB98, CK99}). First, we give an overview on the primitive strategy combinators of $S'_\gamma$. Then, we illustrate how to define new combinators by means of strategy definitions. Afterwards, we pay special attention to generic traversal, that is, we explain the meaning of the traversal primitives, and we illustrate their expressiveness. In the last part of the section, we sketch the type system of $S'_\gamma$. The subsequent sections 3–5 provide a formal definition of $S'_\gamma$.

2.1 Primitive combinators

In an abstract sense, a term rewriting strategy is a partial mapping from a term to a term, or to a set of terms. In an extreme case, a strategy performs normalisation, that is, it maps a term to a normal form. We use $s$ and $t$, possibly subscripted or primed, to range over strategy expressions, or terms, respectively. The application of a strategy $s$ to a term $t$ is denoted by $s \circ t$. The result $r$ of strategy application is called a reduct. It is either a term or “↑” to denote failure. The primitive combinators of the rewriting calculus $S'_\gamma$ are shown in Figure 2. Note that we use the term “combinator” for all kinds of operators on strategies, even for constant strategies like $\varepsilon$ and $\delta$ in Figure 2.

**Rewrite rules as strategies** There is a form of strategy $t_l \rightarrow t_r$ for first-order, one-step rules to be applied at the top of the term. The idea is that if the given term matches the left-hand side $t_l$, then the input is rewritten to the right-hand side $t_r$ with the variables in $t_r$ bound according to the match. Otherwise, the rewrite rule considered as a strategy fails. We adopt some common restrictions for rewrite rules. The left-hand side $t_l$ determines the bound variables. (Free) variables on the right-hand $t_r$ side also occur in $t_l$.

**Basic combinators** Besides rule formation, there are standard primitives for the identity strategy ($\varepsilon$), the failure strategy ($\delta$), sequential composition ($\cdot ; \cdot$), non-deterministic choice ($\cdot + \cdot$), and negation by failure ($\neg \cdot$). Non-deterministic choice means that there is no prescribed order in which the two argument strategies are considered. Negation by failure means that $\neg s$ fails if and only if $s$ succeeds. In case of success of $\neg s$, the input term is simply preserved. In addition to non-deterministic choice, we should also allow for asymmetric choice, namely left- vs. right-biased choice. We assume the following syntactic sugar:
\[ s ::= t \rightarrow t \quad \text{(Rewrite rule)} \]
\[ \mid \epsilon \quad \text{(Identity)} \]
\[ \mid \delta \quad \text{(Failure)} \]
\[ \mid s; s \quad \text{(Sequential composition)} \]
\[ \mid s + s \quad \text{(Non-deterministic choice)} \]
\[ \mid \neg s \quad \text{(Negation by failure)} \]
\[ \mid c \quad \text{(Congruence for constant symbol)} \]
\[ \mid f(s, \ldots, s) \quad \text{(Congruence for function symbol)} \]
\[ \mid \Box(s) \quad \text{(Apply strategy to all children)} \]
\[ \mid \Diamond(s) \quad \text{(Apply strategy to one child)} \]
\[ \mid \bigcirc^s(s) \quad \text{(Reduce all children)} \]
\[ \mid \#(s) \quad \text{(Select one child)} \]
\[ \mid \bot \quad \text{(Build empty tuple, i.e., \langle \rangle)} \]
\[ \mid s \parallel s \quad \text{(Apply two strategies to input)} \]
\[ \mid s < \gamma \quad \text{(Extend many-sorted strategy)} \]

Fig. 2. Primitives of \( S'_\gamma \)

\[
\begin{align*}
s_1 \leftarrow s_2 & \equiv s_1 + (\neg s_1; s_2) \\
s_1 \rightarrow s_2 & \equiv s_2 \leftarrow s_1
\end{align*}
\]

That is, in \( s_1 \leftarrow s_2 \), the left argument has higher priority than the right one. \( s_2 \) will only be applied if \( s_1 \) fails. From an operational perspective, it would very well make sense to consider asymmetric choice as a primitive since the above reconstruction suggests the repeated attempt to perform the preferred strategy. We do not include asymmetric choice as a primitive because we want to keep the calculus \( S'_\gamma \) as simple as possible.

**Congruences** Recall that rewrite rules when considered as strategies are applied at the top of a term. From here on, we use the term “child” to denote an immediate subterm of a term, i.e., one of the \( t_i \) in a term of the form \( f(t_1, \ldots, t_n) \). The congruence strategy \( f(s_1, \ldots, s_n) \) provides a convenient way to apply strategies to the children of a term with \( f \) as outermost symbol. More precisely, the argument strategies \( s_1, \ldots, s_n \) are applied to the parameters \( t_1, \ldots, t_n \) of a term of the form \( f(t_1, \ldots, t_n) \). If all these strategy applications deliver proper term reducts \( t_1', \ldots, t_n' \), then the term \( f(t_1', \ldots, t_n') \) is constructed, i.e., the outermost function symbol is preserved. If any child cannot be processed successfully, or if the outermost
function symbol of the input term is different from $f$, then the strategy fails. The congruence $c$ for a constant $c$ can be regarded as a test for the constant $c$. One might consider congruences as syntactic sugar for rewrite rules which apply strategies to subterms based on where-clauses as introduced later. We treat congruences as primitive combinators because this is helpful for our presentation: the generalisation of congruences ultimately leads to the notion of a generic traversal combinator.

**Notational conventions**  *Slanted* type style is used for constant symbols, function symbols, and sorts. The former start in lower case, the latter in upper case. *SMALL CAPS* type style is used for names of strategies. Variables in term patterns are potentially subscripted letters in upper case. We use some common notation to declare constant and function symbols such as $\text{fork} : \text{Tree} \times \text{Tree} \rightarrow \text{Tree}$. Here, “$\times$” denotes the Cartesian product construction for the parameters of a function symbol.

**Example 1** We can already illustrate a bit of strategic rewriting with the combinators that we have explained so far. Let us consider the following problem. We want to flip the top-level subtrees in a binary tree with naturals at the leafs. We assume the following symbols to construct such trees:

- $\text{zero} : \text{Nat}$
- $\text{succ} : \text{Nat} \rightarrow \text{Nat}$
- $\text{leaf} : \text{Nat} \rightarrow \text{Tree}$
- $\text{fork} : \text{Tree} \times \text{Tree} \rightarrow \text{Tree}$

$N$ and $T$ optionally subscripted or primed are used as variables of sort Nat and Tree, respectively. We can specify the problem of flipping top-level subtrees with a standard rewrite system. We need to employ an auxiliary function symbol $\text{fliptop}$ in order to operate at the top-level.

$$\text{fliptop}(\text{fork}(T_1, T_2)) \rightarrow \text{fork}(T_2, T_1)$$

Note that there is no rewrite rule which eliminates $\text{fliptop}$ when applied to a leaf. We could favour the invention of an error tree for that purpose. Now, let us consider a strategy $\text{FLIPTOP}$ to flip top-level subtrees:

$$\text{FLIPTOP} = \text{fork}(T_1, T_2) \rightarrow \text{fork}(T_2, T_1)$$

That is, we define a strategy, in fact, a rewrite rule $\text{FLIPTOP}$ which rewrites a fork tree by flipping the subtrees. Note that this rule is non-terminating when considered as a standard rewrite system. However, when considered as strategy, the rewrite rule is only applied at the top of the input term, and application is not iterated in
any way. Note also that an application of the strategy \texttt{FLIPTOP} to a leaf will simply fail. There is no need to invent an error element. If we want \texttt{FLIPTOP} to succeed on a leaf, we can define the following variant of \texttt{FLIPTOP}. We show two equivalent definitions:

\[
\texttt{FLIPTOP}' = \texttt{FLIPTOP} \oplus \epsilon = \texttt{FLIPTOP} + \texttt{leaf}(\epsilon)
\]

In the first formulation, we employ left-biased choice and the identity \(\epsilon\) to recover from failure if \texttt{FLIPTOP} is not applicable. In the second formulation, we use a case discrimination such that \texttt{FLIPTOP} handles the constructor \texttt{fork}, and the constructor \texttt{leaf} is covered by a separate congruence for \texttt{leaf}.

**Generic traversal combinators**

Congruences can be used for type-specific traversal. Generic traversal is supported by designated \(S'_t\) combinators \(\square(\cdot), \Diamond(\cdot), \bigcirc(\cdot), \text{ and } \sharp(\cdot)\). These traversal combinators have with congruences in common that they operate on the children of a term. Since traversal combinators have to cope with any number of children, one might view them as list-processing functions. The strategy \(\square(s)\) applies the argument strategy \(s\) to all children of the given term. The strategy \(\Diamond(s)\) applies the argument strategy \(s\) to exactly one child of the given term. The selection of the child is non-deterministic but constrained by the success-and-failure behaviour of \(s\). The strategies \(\square(s)\) and \(\Diamond(s)\) are meant to be type-preserving since they preserve the outermost function symbol. The remaining traversal combinators deal with type-unifying traversal. The strategy \(\bigcirc^{s_0}(s)\) reduces all children. Here \(s\) is used to process the children, and \(s_0\) is used for the pairwise composition of the intermediate results. We will later discuss the utility of different orders for processing children. The strategy \(\sharp(s)\) processes one child via \(s\). The selection of the child is non-deterministic but constrained by the success-and-failure behaviour of \(s\) as in the case of the type-preserving \(\Diamond(s)\).

There are two trivial combinators which are needed for a typeful treatment of type-unifying strategies. They do not perform traversal but they are helpers. The strategy \(\bot\) builds the empty tuple \(\langle\rangle\). The strategy \(\downarrow\) allows us to discard in a sense the current term of whatever sort, and replace it by the trivial term \(\langle\rangle\). This is useful if we want to migrate to the fixed and content-free empty tuple type. Such a migration is sometimes needed if we are not interested in the precise type of the term at hand, e.g., if want to encode constant strategies, that is, strategies which return a fixed term. The strategy \(s_1 \parallel s_2\) applies the two strategies \(s_1\) and \(s_2\) to the input term, and forms a pair from the results. This is a fundamental form of decomposition relevant for type-unifying traversal. Obviously, one can nest the application of the combinator \(\cdot \parallel \cdot\) if more than two strategies should be applied to the input term.

The last combinator \(\cdot \triangleleft \cdot\) in Figure 2 serves for lifting a many-sorted strategy such
\[ \text{TRY}(\nu) = \nu + \epsilon \quad \text{(Apply } \nu \text{ if possible, succeed otherwise)} \]
\[ \text{REPEAT}(\nu) = \text{TRY}(\nu; \text{REPEAT}(\nu)) \quad \text{(Apply } \nu \text{ as often as possible)} \]
\[ \text{CHI}(\nu, \nu_t, \nu_f) = (\nu; \nu_t) + (\bot; \nu_f) \quad \text{("Characteristic function")} \]

Fig. 3. Reusable strategy definitions

as a rewrite rule to the generic level. We postpone discussing this combinator until Section 2.4 when typed strategic programming is discussed.

2.2 Strategy definitions

New strategy combinators can be defined by means of the abstraction mechanism for strategy definition. We use \( \nu \), possibly subscripted, for formal strategy parameters in strategy definitions. A strategy definition \( \phi(\nu_1, \ldots, \nu_n) = s \) introduces an \( n \)-ary combinator \( \phi \). Strategy definitions can be recursive. When we encounter an application \( \phi(s_1, \ldots, s_n) \) of \( \phi \), then we replace it by the instantiation \( s\{\nu_1 \mapsto s_1, \ldots, \nu_n \mapsto s_n\} \) of the body \( s \) of the definition of \( \phi \). This leads to a sufficiently lazy style of unfolding strategy definitions.

In Figure 3, three simple strategy definitions are shown. These definitions embody idioms which are useful in strategic programming. Firstly, \( \text{TRY}(s) \) denotes the idiom to try \( s \) but to succeed via \( \epsilon \) if \( s \) fails. Secondly, \( \text{REPEAT}(s) \) denotes exhaustive iteration in the sense that \( s \) is performed as many times as possible. Thirdly, \( \text{CHI}(s, s_t, s_f) \) is intended to map success and failure of \( s \) to “constants” \( s_t \) and \( s_f \), respectively. To this end, \( s \) is supposed to compute \( \langle \rangle \) (if it succeeds), while \( s_t \) and \( s_f \) map the “content-free” \( \langle \rangle \) to some term. The helper \( \bot \) is used in the right branch to prepare for the application of the constant strategy \( s_f \).

Example 2  Recall Example 1 where we defined a strategy \( \text{FLIPTOP} \) for flipping top-level subtrees. Let us define a strategy \( \text{FLIPALL} \) which flips subtrees at all levels:

\[ \text{FLIPALL} = \text{TRY}(\text{FLIPTOP}; \text{fork}(\text{FLIPALL}, \text{FLIPALL})) \]

Note how the congruence for fork trees is used to apply \( \text{FLIPALL} \) to the subtrees of a fork tree.

Polyadic strategies  Many strategies need to operate on several terms. Consider, for example, a strategy for addition. It is supposed to take two naturals. There are several ways to accomplish strategies with multiple term arguments. Firstly, the programmer could be required to define function symbols for grouping. Although
this is a very simple approach to deal with polyadic strategies, it is rather inconvenient for the programmer because (s)he has to invent designated function symbols. Secondly, we could introduce a special notation to allow a kind of polyadic strategy application with multiple term positions. This will not lead to an attractive simple calculus. Thirdly, we could consider curried strategy application. This would immediately lead to a higher-order calculus. Recall that we want stay in a basically first-order setting. Fourthly, polyadic strategies could be based on polymorphic tuple types. This is the option we choose. There are distinguished constructors for tuples. The constant symbol $\langle \rangle$ represents the empty tuple, and a pair is represented by $\langle t_1, t_2 \rangle$. The notions of rewrite rules and congruence strategies are immediately applicable to tuples. For simplicity, we do not consider arbitrary polymorphic types in $S'_\gamma$, but we restrict ourselves to polymorphic tuples in $S'_\gamma$.

Example 3  To map a pair of naturals to the first component, the rewrite rule $\langle N_1, N_2 \rangle \to N_1$ is appropriate. To flip the top-level subtrees of a pair of fork trees, the congruence $\langle \text{FLIPTOP}, \text{FLIPTOP} \rangle$ is appropriate.

Example 4  The following confluent and terminating rewrite system defines addition of naturals in the common manner:

\[
\begin{align*}
\text{add} : & \text{Nat} \times \text{Nat} \to \text{Nat} \\
\text{add}(N, \text{zero}) & \to N \\
\text{add}(N_1, \text{succ}(N_2)) & \to \text{succ}(\text{add}(N_1, N_2))
\end{align*}
\]

That is, add is a function symbol to group two naturals to be added. We rely on a normalisation strategy such as innermost to actually perform addition. By contrast, we can also define a polyadic strategy $\text{ADD}$ which takes a pair of naturals:

\[
\begin{align*}
\text{DEC} & = \text{succ}(N) \to N \\
\text{INC} & = N \to \text{succ}(N) \\
\text{ADD}_\text{base} & = \langle N, \text{zero} \rangle \to N \\
\text{ADD}_\text{step} & = \langle \epsilon, \text{DEC} \rangle; \text{ADD}; \text{INC} \\
\text{ADD} & = \text{ADD}_\text{base} + \text{ADD}_\text{step}
\end{align*}
\]

For clarity of exposition, we defined a number of auxiliary strategies. $\text{DEC}$ attempts to decrement a natural. $\text{INC}$ increments a natural. Actual addition is performed according to the scheme of primitive recursion with the helpers $\text{ADD}_\text{base}$ and $\text{ADD}_\text{step}$ for the base and the step case. Both cases are mutually exclusive. The base case is applicable if the second natural is a zero. The step case is applicable if the second natural is a non-zero value since $\text{DEC}$ will otherwise fail. Notice how a congruence for pairs is employed in the step case.
Where-clauses For convenience, we generalise the concept of rewrite rules as follows. A rewrite rule is of the form \( t \rightarrow b \) where \( t \) is the term of the left-hand side as before, and \( b \) is the right-hand side body of the rule. In the simplest case, a body \( b \) is a term \( t' \) as before. However, a body can also involve where-clauses. Then \( b \) is of the following form:

\[
b' \text{ where } x = s @ t'
\]

The meaning of such a body with a where-clause is that the term reduct which results from the strategy application \( s @ t' \) is bound to \( x \) for the evaluation of the remaining body \( b' \). For simplicity, we assume a linear binding discipline, that is, \( x \) is not bound elsewhere in the rule.

Example 5 We illustrate the utility of where-clauses by a concise reconstruction of the strategy \( \text{ADD} \) from Example 4:

\[
\text{ADD} = \langle N, \text{zero} \rangle \rightarrow N \\
+ \langle N_1, \text{succ}(N_2) \rangle \rightarrow \text{succ}(N_3) \text{ where } N_3 = \text{ADD} @ \langle N_1, N_2 \rangle
\]

The strategy takes roughly the form of an eager functional program with pattern-match case à la SML.

2.3 Generic traversal

Let us discuss the core asset of \( S'_\gamma \), namely its combinators for generic traversal in some detail. To prepare the explanation of the corresponding primitives, we start with a discussion of how to encode traversal in standard rewriting. By “standard rewriting”, we mean many-sorted, first-order rewriting based on a fixed normalisation strategy. We derive the strategic style from this encoding. Afterwards, we will define a number of reusable schemes for generic traversal in terms of the \( S'_\gamma \) primitives. Ultimately, we will provide the encodings for the traversal problems posed in the introduction.

Traversal functions Suppose we want to traverse a term of a certain sort. In the course of traversing into the term, we need to process the subterms of it at maybe all levels. In general, these subterms are of different sorts. If we want to encode traversal in standard rewriting, we basically need an auxiliary function symbol for each traversed sort to map it to the corresponding result type. Usually, one has to define one rewrite rule per constructor in the signature at hand.
Example 6  Let us define a traversal to count leafs in a tree. Note that a function from trees to naturals is obviously type-changing. Consider the following rewrite rule:

\[
\text{COUNT}_{\text{leaf}} = \text{leaf}(N) \rightarrow \text{succ}(\text{zero})
\]

This rule directly models the essence of counting leafs, namely it says that a leaf is mapped to 1, i.e., \(\text{succ}(\text{zero})\). In standard rewriting, we cannot employ the above rewrite rule since it is type-changing. Instead, we have to organise a traversal with rewrite rules for an auxiliary function symbol \text{count}:

\[
\text{count} : \text{Tree} \rightarrow \text{Nat}\\
\text{count(leaf}(N)) \rightarrow \text{succ}(\text{zero})\\
\text{count(fork}(T_1, T_2)) \rightarrow \text{add} (\text{count}(T_1), \text{count}(T_2))
\]

The first rewrite rule restates \(\text{COUNT}_{\text{leaf}}\) in a type-preserving manner. The second rewrite rule is only there to traverse into fork trees. In this manner, we can cope with arbitrarily nested fork trees, and we will ultimately reach the leafs that have to be counted. Note that if we needed to traverse terms which involve other constructors, then designated rewrite rules had to be provided along the schema used in the second rewrite rule for \text{fork} above. That is, although we are only interested in leafs, we still have to skip all other constructors to reach leafs. To be precise, we have to perform addition all over the place to compute the total number of leafs from the number of leafs in subterms.

Traversal strategies  Traversal based on such auxiliary function symbols and rewrite rules gets very cumbersome when larger signatures, that is, more constructors, are considered. This problem has been clearly articulated in [BSV97, VBT98, LKV00, BSV00, BVK01, Vis01]. An application domain which deals with large signatures is program transformation. Signatures correspond here to language syntaxes. The aforementioned papers clearly illustrate the inappropriateness of the manual encoding of traversal functions for non-trivial program transformation systems. The generic traversal facet of strategic programming solves this problem in the most general way. In strategic rewriting, we do not employ auxiliary function symbols and rewrite rules to encode traversal, but we rely on expressiveness to process the children of a term in a uniform manner. In fact, traversal combinators allow us to process children regardless of the outermost constructor and the type of the term at hand.

Example 7  Let us attempt to rephrase Example 6 in strategic style. We do not want to employ auxiliary function symbols, but we want to employ the type-changing
rewrite rule \( \text{COUNT}_{\text{leaf}} \) for handling the terms of interest. In our first attempt, we do not yet employ traversal combinators. We define a strategy \( \text{COUNT} \) as follows:

\[
\begin{align*}
\text{UNWRAP}_{\text{fork}} &= \text{fork}(T_1, T_2) \rightarrow (T_1, T_2) \\
\text{COUNT}_{\text{leaf}} &= \text{leaf}(N) \rightarrow \text{succ}(\text{zero}) \\
\text{COUNT}_{\text{fork}} &= \text{UNWRAP}_{\text{fork}}; \langle \text{COUNT}, \text{COUNT} \rangle; \text{ADD} \\
\text{COUNT} &= \text{COUNT}_{\text{leaf}} + \text{COUNT}_{\text{fork}}
\end{align*}
\]

The helper \( \text{COUNT}_{\text{fork}} \) specifies how to count the leafs of a proper fork tree. That is, we first turn the fork tree into a pair of its subtrees via \( \text{UNWRAP}_{\text{fork}} \), then we perform counting for the subtrees by means of a type-changing congruence on pairs, and finally the resulting pair is fed to the strategy for addition. Note that the recursive formulation of \( \text{COUNT} \) allows us to traverse into arbitrarily nested fork trees. In order to obtain a more generic version of \( \text{COUNT} \), we can use a traversal combinator to abstract from the concrete constructor in \( \text{COUNT}_{\text{fork}} \). Here is a variant of \( \text{COUNT} \) which can cope with any constructor with one or more children:

\[
\begin{align*}
\text{COUNT}_{\text{leaf}} &= \text{leaf}(N) \rightarrow \text{succ}(\text{zero}) \\
\text{COUNT}_{\text{any}} &= \bigcirc \text{ADD}(\text{COUNT}) \\
\text{COUNT} &= \text{COUNT}_{\text{leaf}} + \text{COUNT}_{\text{any}}
\end{align*}
\]

That is, we use the combinator \( \bigcirc (\cdot) \) to reduce children accordingly. Note that left-biased choice is needed in the new definition of \( \text{COUNT} \) to make sure that \( \text{COUNT}_{\text{leaf}} \) is applied whenever possible, and we only descend into the term for non-leaf trees. We should finally mention that the strategy \( \text{COUNT} \) is not yet fully faithful regarding typing because we pass the many-sorted strategy \( \text{COUNT} \) to \( \bigcirc (\cdot) \) whereas the argument for processing the children is intentionally generic.

**Traversal schemes** In Figure 4 and Figure 5, we derive some combinators for generic traversal. Most of the combinators should actually be regarded as reusable definitions of traversal schemes. The definitions immediately illustrate the potential of the generic traversal combinators. Several of the definitions from the type-preserving group are adopted from [VB98]. We postpone discussing typing issues for a minute. Let us read a few of the given definitions. The strategy \( \text{TD}(s) \) applies \( s \) to each node in top-down manner. This is expressed by sequential composition such that \( s \) is first applied to the current node, and then we recurse into the children. It is easy to see that if \( s \) fails for any node, the traversal fails entirely. A similar derived combinator is \( \text{STOPTD} \). However, left-biased choice instead of sequential composition is used to transfer control to the recursive part. Thus, if the strategy succeeds for the node at hand, the children will not be processed anymore. Another insightful, intentionally type-preserving example is \( \text{INNERMOST} \) which directly models the innermost normalisation strategy known from standard rewriting. The first three
CON \(= \square(\delta)\) (Test for a constant)

FUN \(= \Diamond(e)\) (Test for a compound term)

\(\ominus^*(\nu)\) \(= \Square(\text{TRY}(\nu))\) (Process several children)

\(\ominus^+(\nu)\) \(= -\Square(-\nu); \ominus^*(\nu)\) (Process at least one child)

TD(\(\nu\)) \(= \nu; \square(\text{TD}(\nu))\) (Top-down traversal)

BU(\(\nu\)) \(= \square(\text{BU}(\nu)); \nu\) (Bottom-up traversal)

ONCETD(\(\nu\)) \(= \nu + \Diamond(\text{ONCETD}(\nu))\) (Process one node in top-down manner)

ONCEBU(\(\nu\)) \(= \nu + \Diamond(\text{ONCEBU}(\nu))\) (Process one node in bottom-up manner)

INNERMOST(\(\nu\)) \(= \text{REPEAT}(\text{ONCEBU}(\nu))\) (Innermost evaluation strategy)

STOPTD(\(\nu\)) \(= \nu + \Diamond(\text{STOPTD}(\nu))\) (Top-down traversal with “cut”)

Fig. 4. Definitions of type-preserving combinators

ANY(\(\nu\)) \(= \nu + \sharp(\text{ANY}(\nu))\)

TM(\(\nu\)) \(= \nu + \sharp(\text{TM}(\nu))\)

BM(\(\nu\)) \(= \nu + \sharp(\text{BM}(\nu))\)

CF(\(\nu, \nu_\text{u}, \nu_\circ\)) = (CON; \(\downarrow; \nu_\text{u}\)) + (FUN; \(\circ^\nu(\nu)\))

CRUSH(\(\nu, \nu_\text{u}, \nu_\circ\)) = (\(\nu \parallel \text{CF(CRUSH}(\nu, \nu_\text{u}, \nu_\circ))\); \(\nu_\circ\))

STOPCRUSH(\(\nu, \nu_\text{u}, \nu_\circ\)) = \(\nu + \text{CF(S}O\text{P CRUSH}(\nu, \nu_\text{u}, \nu_\circ), \nu_\text{u}, \nu_\circ)\)

Fig. 5. Definitions of type-unifying combinators

type-unifying combinators ANY, TM, and BM deal with the selection of a subterm. They all have in common that they resort to the selection combinator \(\sharp(\cdot)\) to determine a suitable child. They differ in the sense that they perform search either non-deterministically, or in top-down manner, or in bottom-up manner. One might wonder whether it is sensible to vary the horizontal order as well. We will discuss this issue later. The combinator CF complements \(\circ(\cdot)\) to also cope with a constant. To this end, there is an additional parameter \(\nu_\text{u}\) for the “neutral element” to be applied when a constant is present. The combinators CRUSH and STOPCRUSH model deep reduction based on the same kind of monoid-like argument strategies as CF. As an aside, the term crushing has been coined in the related context of polytypic programming [Mec96]. The combinator CRUSH evaluates each node in the tree, and hence, it needs to succeed for each node. The reduction of the current node and the recursion into the children is done in parallel based on \(\cdot \parallel \cdot\). The corresponding pair of intermediate results is reduced with the binary monoid operation. STOPCRUSH is similar to STOPTD in the sense that the current node is first evaluated, and only if evaluation fails, then we recurse into the children.
Combinators on Booleans

\[
\begin{align*}
\text{FALSE} & : \langle \rangle \to \text{false} \\
\text{TRUE} & : \langle \rangle \to \text{true}
\end{align*}
\]

(Build “false”)

(Build “true”)

Combinators on naturals

\[
\begin{align*}
\text{NAT} & : \text{zero} + \text{succ}(\epsilon) \\
\text{ZERO} & : \langle \rangle \to \text{zero} \\
\text{ONE} & : \langle \rangle \to \text{succ} (\text{zero}) \\
\text{INC} & : N \to \text{succ}(N) \\
\text{ADD} & : \ldots
\end{align*}
\]

(Test by congruences)

(Build “0”)

(Build “1”)

(Increment)

(Addition; see Example 4)

Combinators on lists of naturals

\[
\begin{align*}
\text{NIL} & : \langle \rangle \to \text{nil} \\
\text{SINGLETON} & : N \to \text{cons}(N, \text{nil}) \\
\text{APPEND} & : \ldots
\end{align*}
\]

(Build “nil”)

(Construct a singleton list)

(Append two lists; definition postponed)

Actual encodings of (I)–(IV)

\[
\begin{align*}
\text{(I)} & = \text{STOPTD}(\text{NAT}; \text{INC}) \\
\text{(II)} & = \text{ONCEBU}(g(P) \to g'(P)) \\
\text{(III)} & = \text{CHI}(\text{ANY}(\text{NAT}; \perp), \text{TRUE}, \text{FALSE}) \\
\text{(IV)} & = \text{STOPCRUSH}(\text{NAT}; \text{SINGLETON}, \text{NIL}, \text{APPEND}) \\
\text{(V)} & = \text{CRUSH} (\text{CHI}(g(\epsilon); \perp, \text{ONE}, \text{ZERO}), \text{ZERO}, \text{ADD})
\end{align*}
\]

Fig. 6. Untyped encodings for traversal problems from Figure 1

Example 8 Let us solve the problems (I)–(V) illustrated in Figure 1 in the introduction of the paper. In Figure 6, we first define some auxiliary strategies on naturals, Booleans, and lists of naturals, and then, the ultimate traversals (I)–(V) are defined in terms of the combinators from Figure 3–Figure 5. Note that the encodings are not yet fully faithful regarding typing. We will later revise these encodings accordingly.

Let us explain the strategies in detail.

(I) We are supposed to increment all naturals. The combinator \text{STOPTD} is employed to descend into the given term as long as we do not find a natural recognised via \text{NAT}. When we encounter a natural in top-down manner, we apply the rule \text{INC} for incrementing naturals. Note that we must not further descend
into the term. In fact, if we used TD instead of STOPTD, we describe a non-terminating strategy. Also note that a bottom-up traversal is not an option either.

If we used BU instead of STOPTD, we model the replacement of a natural \( N \) by \( 2N + 1 \).

**II** We want to replace terms of the form \( g(P) \) by \( g'(P) \). As we explained in the introduction, the replacement must not be done exhaustively. We only want to perform one replacement where the corresponding redex should be identified in bottom-up manner. These requirements are met by the combinator \( \text{ONCE} \text{BU} \).

**III** We want to find out if naturals occur in the term. The result should be encoded as a Boolean; hence, the two branches \( \text{TRUE} \) and \( \text{FALSE} \) in \( \text{CHI} \). We look for naturals again via the auxiliary strategy \( \text{NAT} \). The kind of deep matching we need is provided by the combinator \( \text{ANY} \) which non-deterministically looks for a child where \( \text{NAT} \) succeeds. \( \text{NAT} \) is followed by \( \perp \) to express that we are not looking for actual naturals but only for the property if there are naturals at all. The application of \( \text{CHI} \) turns success and failure into a Boolean.

**IV** To collect all naturals in a term, we need to perform a kind of deep reduction. Here, it is important that reduction with cut (say, \( \text{STOPCRUSH} \)) is used because a term representing a non-zero natural \( N \) “hosts” the naturals \( N - 1, \ldots, 0 \) due to the representation of naturals via the constructors \( \text{succ} \) and \( \text{zero} \). These hosted naturals should not be collected. Recall that crushing uses monoid-like arguments. In this example, \( \text{APPEND} \) is the associative operation of the monoid, and the strategy \( \text{NIL} \) to build the empty list represents the unit of \( \text{APPEND} \).

**V** Finally, we want to count all occurrences of \( g \). In order to locate these occurrences, we use the congruence \( g(e) \). In this example, it is important that we perform crushing exhaustively, i.e., without cut, since terms rooted by \( g \) might indeed host further occurrences of \( g \). We assume that all occurrences of \( g \) have to be counted.

Note the genericity of the defined strategies (I)–(V). They can be applied to any term. Of course, the strategies are somewhat specific because they refer to some concrete constant or function symbols, namely \( \text{true} \), \( \text{false} \), \( \text{zero} \), \( \text{succ} \), \( g \), and \( g' \).

### 2.4 Typed strategies

Let us now motivate the typeful model of strategic programming underlying \( S'_\gamma \). The ultimate challenge is to assign types to generic traversal strategies like TD, STOPTD, or CRUSH. Recall our objective for \( S'_\gamma \) to stay in a basically first-order many-sorted term rewriting setting. The type system we envisage should be easy to define and implement.

**Many-sorted types** Let us start with a basic, many-sorted fragment of \( S'_\gamma \) without support for generic traversal. We use \( \tau \) and \( \pi \), possibly subscripted or primed, to
range over term types or strategy types, respectively. Term types are sorts and tuple types. We use \( \langle \tau_1, \tau_2 \rangle \) to denote the product type for pairs \( \langle t_1, t_2 \rangle \). The type of the empty tuple \( \langle \rangle \) is simply denoted as \( \langle \rangle \). A strategy type \( \pi \) is a first-order function type, that is, \( \pi \) is of the form \( \tau \rightarrow \tau' \). Here, \( \tau \) is the type of the input term, and \( \tau' \) is the type of the term reduct. We also use the terms domain and co-domain for \( \tau \) or \( \tau' \), respectively. The type declaration for a strategy combinator \( \varphi \) which does not take any strategy arguments is of the form \( \varphi : \pi \). The type declaration for a strategy combinator \( \varphi \) with \( n \geq 1 \) arguments is represented in the following format:

\[
\varphi : \pi_1 \times \cdots \times \pi_n \rightarrow \pi_0
\]

Here, \( \pi_1, \ldots, \pi_n \) denote the strategy types for the argument strategies, and \( \pi_0 \) denotes the strategy type of an application of \( \varphi \). All the \( \pi_i \) are again of the form \( \tau_i \rightarrow \tau'_i \). Consequently, strategy combinators correspond to second-order functions on terms. This can be checked by counting the level of nesting of arrows “\( \rightarrow \)” in a combinator type.

**Example 9** We show the type of \( \text{FLIPALL} \) from Example 2, the type of the congruence combinator \( \text{fork}(\cdot, \cdot) \) for the function symbol \( \text{fork} \) used in Example 2, and the type of \( \text{ADD} \) from Example 4.

\[
\begin{align*}
\text{FLIPALL} & : \text{Tree} \rightarrow \text{Tree} \\
\text{fork} & : (\text{Tree} \rightarrow \text{Tree}) \times (\text{Tree} \rightarrow \text{Tree}) \rightarrow (\text{Tree} \rightarrow \text{Tree}) \\
\text{ADD} & : (\text{Nat}, \text{Nat}) \rightarrow \text{Nat}
\end{align*}
\]

**Type inference vs. type checking** For simplicity, we assume that the types of all function and constant symbols, variables, and strategy combinators are explicitly declared. This is well in line with standard practice in term rewriting and algebraic specification. Declarations for variables, rewriting functions and strategies are common in several frameworks for rewriting, e.g., in CASL, ASF+SDF, and ELAN. Note however that this assumption is not essential. Inference of types for all symbols is feasible. In fact, type inference is simple because the special generic types of \( S' \gamma \) are basically like constant types, and their inhabitation is explicitly marked by the combinator \( \cdot \cdot \cdot \cdot \). We will eventually add a bit of parametric polymorphism to \( S' \gamma \) but since we restrict ourselves to top-level quantification, type inference will still be feasible.

**Example 10** To illustrate type declarations, we define a strategy \( \text{APPEND} \) to append two lists. For simplicity, we do not consider a polymorphic \( \text{APPEND} \), but one that appends lists of naturals. We declare all the constant and function symbols (namely \( \text{nil} \) and \( \text{cons} \)), and variables for lists (namely \( L_1, L_2, L_3 \)) and naturals as
Generic types  In order to provide types for generic strategies, we need to extend our basically many-sorted type system. To this end, we identify distinguished generic types for strategies which are applicable to all sorts. We use $\gamma$ to range over generic strategy types. There are two generic strategy types. The type $TP$ models generic type-preserving strategies. The type $TU(\tau)$ models type-unifying strategies where all types are mapped to $\tau$. These two forms correspond to the main characteristics of $S^\gamma$. The types $TP$ and $TU(\cdot)$ can be integrated into an initially many-sorted system in a simple manner.

**Example 11** The following types are the intended ones for the illustrative strategies defined in Figure 6.

(I), (II) : $TP$

(III) : $TU(\text{Boolean})$

(IV) : $TU(\text{NatList})$

(V) : $TU(\text{Nat})$

**Parametric polymorphism**  Generic strategy types capture the kind of genericity needed for generic traversal while being able to mix uniform and sort-specific behaviour. In order to turn $S^\gamma$ in a somewhat complete programming language, we also need to enable parametric polymorphism. Consider, for example, the combinator CRUSH for deep reduction in Figure 5. The result type of reduction should be a parameter. The overall scheme of crushing is in fact not dependent on the actual unified type. The arguments passed to CRUSH are the only strategies to operate on the parametric type for unification. We employ a very simple form of parametric polymorphism. Types of strategy combinators may contain type variables which are explicitly quantified at the top level [Mil78, CW85]. We use $\alpha$, possibly subscripted, for term-type variables. Thus, in general, a type of a strategy combinator

\[
\begin{align*}
\text{nil} & : \text{NatList} \\
\text{cons} & : \text{Nat} \times \text{NatList} \rightarrow \text{NatList} \\
L_1, L_2, L_3 & : \text{NatList} \\
N & : \text{Nat} \\
\text{APPEND} & : \langle \text{NatList, NatList} \rangle \rightarrow \text{NatList} \\
\text{APPEND} = & \langle \text{nil}, L \rangle \rightarrow L \\
& + \langle \text{cons}(N, L_1), L_2 \rangle \rightarrow \text{cons}(N, L_3) \text{ where } L_3 = \text{APPEND} \odot \langle L_1, L_2 \rangle
\end{align*}
\]
is of the following form:

$$\varphi : \forall \alpha_1, \ldots \forall \alpha_m. \pi_1 \times \cdots \times \pi_n \to \pi_0$$

We assume that any type variable in $$\pi_0, \ldots, \pi_n$$ is contained in the set $$\{\alpha_1, \ldots, \alpha_m\}$$. Furthermore, we assume explicit type application, that is, the application of a type-parameterised strategy combinator $$\varphi$$ involves type application using the following form:

$$\varphi[\tau_1, \ldots, \tau_m](s_1, \ldots, s_n)$$

For convenience, an actual implementation of $$S'_\gamma$$ is likely to support implicit type application. Also, a more complete language design would include support for parameterised datatypes such as parameterised lists as opposed to lists of naturals in Example 10. For brevity, we omit parameterised datatypes in the present paper since they are not strictly needed to develop a typeful model of generic traversal, and a corresponding extension is routine. Parameterised algebraic datatypes are well-understood in the context of algebraic specification and rewriting. The instantiation of parameterised specifications or modules is typically based on signature morphisms as supported, e.g., in CASL [ABK+01] or ELAN [BKK+98]. A more appealing approach to support parameterised datatypes would be based on a language design with full support for polymorphic functions and parameterised data types as in the functional languages SML and Haskell.

**Example 12**  Here are the types for the strategy combinators from Figure 3–Figure 5. All traversal schemes which involve a type-unifying facet, need to be parameterised by the unified type.

|                  | Type                                  |
|------------------|---------------------------------------|
| **TRY, REPEAT**  | : TP → TP                             |
| **CHI**          | : $$\forall \alpha. \text{TU}(\langle \rangle) \to (\langle \rangle \to \alpha) \to (\langle \rangle \to \alpha) \to \text{TU}(\alpha)$$ |
| **CON, FUN**     | : TP                                  |
| $$\otimes^*, \ldots, \text{STOPTD}$$ | : TP → TP |
| **ANY, TM, BM**  | : $$\forall \alpha. \text{TU}(\alpha) \to \text{TU}(\alpha)$$ |
| **CF, CRUSH, STOPCRUSH** | : $$\forall \alpha. \text{TU}(\alpha) \times (\langle \rangle \to \alpha) \times (\langle \alpha, \alpha \rangle \to \alpha) \to \text{TU}(\alpha)$$ |
We update all definitions which involve type parameters:

\[
\begin{align*}
\text{CHI}[\alpha](\nu, \nu_t, \nu_f) &= (\nu; \nu_t) \leftrightarrow (\perp; \nu_f) \\
\text{ANY}[\alpha](\nu) &= \nu + \sharp(\text{ANY}[\alpha](\nu)) \\
\text{TM}[\alpha](\nu) &= \nu + \sharp(\text{TM}[\alpha](\nu)) \\
\text{BM}[\alpha](\nu) &= \nu + \sharp(\text{BM}[\alpha](\nu)) \\
\text{CF}[\alpha](\nu, \nu_u, \nu_o) &= (\text{CON}; \perp; \nu_u) + (\text{FUN}; \bigcirc^{\nu_o}(\nu)) \\
\text{CRUSH}[\alpha](\nu, \nu_u, \nu_o) &= (\nu \parallel \text{CF}[\alpha](\text{CRUSH}[\alpha](\nu, \nu_u, \nu_o), \nu_u, \nu_o)); \nu_o \\
\text{STOPCRUSH}[\alpha](\nu, \nu_u, \nu_o) &= \nu \leftrightarrow \text{CF}[\alpha](\text{STOPCRUSH}[\alpha](\nu, \nu_u, \nu_o), \nu_u, \nu_o)
\end{align*}
\]

**Example 13** Let us also give an example of a polymorphic strategy definition which does not rely on the generic strategy types \(TP\) and \(TU(\cdot)\) at the same time. Consider the declaration \(\text{TRY} : TP \rightarrow TP\) from Example 12. This type is motivated by the use of \(\text{TRY}\) in the definition of traversal strategies, e.g., in the definition of \(\mathcal{D}^*(\cdot)\) in Figure 4. However, the generic type of \(\text{TRY}\) invalidates the application of \(\text{TRY}\) in Example 2 where it was used to recover from failure of a many-sorted strategy. We resolve this conflict of interests by the introduction of a polymorphic combinator \(\text{TRY}'\) for many-sorted strategies, and we illustrate it by a corresponding revision of Example 2:

\[
\begin{align*}
\text{TRY}' & : \forall \alpha. (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha) \\
\text{TRY}'[\alpha](\nu) &= \nu \leftrightarrow \epsilon \\
\text{FLIPALL} &= \text{TRY}'[\text{Tree}](\text{FLIP}; \text{fork}(\text{FLIPALL}, \text{FLIPALL}))
\end{align*}
\]

Hence, we strictly separate many-sorted vs. generic recovery from failure.

**Strategy extension** The remaining problem with generic strategies is the mediation between many-sorted and generic strategy types. If we look back to the simple-minded definition of (I) in Figure 6, we see that \(\text{NAT}; \text{INC}\) is used as an argument for \(\text{STOPTD}\). The argument is of the many-sorted type \(\text{Nat} \rightarrow \text{Nat}\). However, the combinator \(\text{STOPTD}\) should presumably insist on a generic argument because the argument strategy is potentially applied to nodes of all possible sorts. Obviously, \(\text{NAT}; \text{INC}\) will fail for all terms other than naturals because \(\text{NAT}\) performs a type check via congruences for the constructors of sort \(\text{Nat}\). It turns out that failure of \(\text{NAT}; \text{INC}\) controls the traversal scheme \(\text{STOPTD}\) in an appropriate manner. However, if the programmer would have forgotten the type guard \(\text{NAT}\), the traversal is not type-safe anymore. In general, we argue as follows:
A programmer has to explicitly turn many-sorted strategies into generic ones. The reduction semantics is responsible for the type-safe application of many-sorted ingredients in generic contexts.

To this end, $S'_\gamma$ offers the combinator $\cdot \triangleleft \cdot$ to turn a many-sorted strategy into a generic one. A strategy of the form $s \triangleleft \gamma$ models the extension of the strategy $s$ to be applicable to terms of all sorts. In $s \triangleleft \gamma$, the $\gamma$ is a generic type, and the strategy $s$ must be of a many-sorted type $\tau \to \tau'$. The type $\tau \to \tau'$ of $s$ and the generic type $\gamma$ must be related in a certain way, namely the type scheme underlying $\gamma$ has to cover the many-sorted type $\tau \to \tau'$. Strategy extension is performed in the most basic way, namely $s \triangleleft \gamma$ fails for all terms of sorts which are different from the domain $\tau$ of $s$. The reduction semantics of $s \triangleleft \gamma @ t$ is truly type-dependent, that is, reduction involves a check to see whether the type of $t$ coincides with the domain of $s$ to enable the application of $s$. One should not confuse this kind of explicit type test and the potential of failure with an implicit dynamic type check that might lead to program abort. In typed strategic rewriting, strategy extension is a programming idiom to create generic strategies. In a sense, failure is the initial generic default for an extended strategy. Subsequent application of $\cdot + \cdot$ and friends can be used to establish behaviour other than failure. That is, one can recover from failure caused by $\cdot \triangleleft \cdot$, and one can resort to a more useful generic default, e.g., $\epsilon$, or the recursive branch of a generic traversal. Strategy extension is essential for the type-safe application of many-sorted ingredients in the course of a generic traversal.

**Example 14** We revise Example 8 to finally supply the typeful solutions for the traversal problems (I)–(V) from the introduction. The following definitions are in full compliance with the $S'_\gamma$ type system:

(I) = STOPTD(INC $\triangleleft$ TP)

(II) = ONCEBU($g(P) \to g'(P) \triangleleft$ TP)

(III) = CHI[Boolean]([ANY[\{\}]](NAT $\triangleleft$ TP; $\perp$), TRUE, FALSE)

(IV) = STOPCRUSH[NatList]((NAT $\triangleleft$ TU(Nat); SINGLETON, NIL, APPEND)

(V) = CRUSH[Nat]([CHI[Nat]]((g(\epsilon) < TP; $\perp$, ONE, ZERO), ZERO, ADD)

The changes concern the inserted applications of $\cdot \triangleleft \cdot$, and the actual type parameters for type-unifying combinators. In the definition of (I), the strategy INC clearly needs to be lifted to TP; similarly for the rewrite rule in (II). Note that the original test for naturals is gone in the revision of (I). The mere type of INC sufficiently restricts its applicability. In the definition of (III), the strategy NAT is used to check for naturals, and it is lifted to TP. The type-unifying facet of (III) is enforced by the subsequent application of $\perp$, and it is also pointed out by the application of CHI. In the definition of (IV), the strategy NAT is used to select naturals, and it is lifted to TU(Nat). The subsequent application of SINGLETON converts naturals to singleton lists of naturals. The extension performed in (V) can be justified by
similar arguments as for (III). In both cases, CHI is applied to map the success and failure behaviour of a strategy to distinguished constants.

**Static type safety** The resulting typed calculus $S'_\gamma$ obeys a number of convenient properties. Firstly, $S'_\gamma$ supports statically type-safe strategic programming. Secondly, each strategy expression is strictly either many-sorted or generic. Thirdly, many-sorted strategies cannot become generic just by accident, say due to the context in which they are used. Strategies rather become generic via explicit use of $\cdot \bowtie \cdot$. Fourthly, the type-dependent facet of the reduction semantics is completely restricted to $\cdot \bowtie \cdot$. The semantics of all other strategy combinators does not involve type dependency. There are no implicit dynamic type checks.

Admittedly, any kind of type-dependent reduction is somewhat non-standard because type systems in the tradition of the $\lambda$-cube are supposed to meet the type-erasure property \[Bar92, BLRU97\]. That is, reduction is supposed to lead to the same result even if type annotations are removed. An application of the combinator $\cdot \bowtie \cdot$ implies a type inspection at “run time”, but this inspection is concerned with the treatment of different behaviours depending on the actual term type. Also, the inspection is requested by the programmer as opposed to an implicit dynamic type check that is performed providently by a run-time system. Similar expressiveness has also been integrated into other statically typed languages (cf. \[ACPP91, ACPR92, HM95, CGL95, DRW95, CWM99, Gle99\]).

### 3 Many-sorted strategies

We start the formal definition of $S'_\gamma$. As a warm-up, we discuss many-sorted strategies. For simplicity, we postpone formalising strategy definitions until Section 5.1. First, we will define the reduction semantics of a basic calculus $S'_0$ corresponding to an initial untyped fragment of $S'_\gamma$. The corresponding piece of syntax is shown in Figure 7. Then, we develop a simple type system starting with many-sorted type-preserving strategies. We will discuss some standard properties of the type system. Afterwards, we elaborate the type system to cover type-changing strategies and tuples for polyadic strategies. We use inference rules, say deduction rules, in the style of Natural semantics \[Kah87, Des88, Pet92\] for both the reduction semantics and the type system.

#### 3.1 The basic calculus $S'_0$

**Reduction of strategy applications** As for the dynamic semantics of strategies, say the reduction semantics, we employ the judgement $s @ t \rightsquigarrow r$ for the reduction
Syntax

c (Constant symbols)

f, g (Function symbols)

x (Term variables)

t ::= c | f(t, . . . , t) | x (Terms)

r ::= t | ↑ (Reducts of rewriting)

s ::= t → b | ε | δ | s; s | s + s | ¬ s | c | f(s, . . . , s) (Strategies)

b ::= t (Rule bodies)

Fig. 7. Syntax of the basic calculus \( S_0' \)

of strategy applications. Here, \( r \) is the reduct that results from the application of the strategy \( s \) to the term \( t \). Recall that a reduct is either a term \( t \) or “↑” denoting failure (cf. Figure 7). We assume that strategies are only applied to ground terms, and then also yield ground terms. The latter assumption is not essential but it is well in line with standard rewriting. In Figure 8, we define the reduction semantics of strategy application for the initial calculus \( S_0' \). The inference rules formalise our informal explanations from Section 2.1. The reduction semantics of \( S_1' \) is a big-step semantics, that is, \( r \) in \( s \mathbin{@} t \leadsto r \) models the final result of the execution of the strategy \( s \).

Notational conventions We use the common mix-fix notation for judgements in Natural semantics, that is, a judgement basically amounts to a mathematical relation over the ingredients such as \( s \), \( t \), and \( r \) in the example \( s \mathbin{@} t \leadsto r \). The remaining symbols “@” and “\( \leadsto \)” only hint at the intended meaning of the judgement. As for \( s \mathbin{@} t \leadsto r \), we say that the reduct \( r \) is “computed” from the application of \( s \) to \( t \). The direction of computation is indicated by “\( \leadsto \)”. Deduction rules are tagged so that we can refer to them. Deduction rules define, as usual, how to derive valid judgements from given valid judgements. Hence, semantic evaluation or type inference amounts to a proof starting from the axioms in a Natural semantics specification. As for the reduction semantics, we use rule tags that contain “+” whenever the reduct is known to be a proper term whereas “−” is used for remaining cases with failure as the reduct. We also use the terms “positive” vs. “negative” rules. To avoid confusion, we should point out that the term “reduction” has two meanings in the present paper, namely reduction in the sense of the reduction semantics for strategies, and reduction in the sense of traversal where the children of a term are reduced by monoid-like combinators (recall \( \bigcirc (\cdot) \)).
Reduction of strategy applications

Positive rules

\[ \exists \theta. (\theta(t_l) = t \land \theta(t_r) = t') \]
\[ t_l \rightarrow t_r @ t \leadsto t' \]

[rule⁺]

Negative rules

\[ \exists \theta. \theta(t_l) = t \]
\[ t_l \rightarrow t_r @ t \leadsto \uparrow \]

[rule⁻]

\[ e @ t \leadsto t \]

[id⁺]

\[ s @ t \leadsto \uparrow \]

[neg⁺]

\[ s @ t \leadsto t' \]

[neg⁻]

\[ s_1 @ t \leadsto t^* \land s_2 @ t^* \leadsto t' \]
\[ s_1; s_2 @ t \leadsto t' \]

[seq⁺]

\[ s_1 @ t \leadsto \uparrow \]
\[ s_1; s_2 @ t \leadsto \uparrow \]

[seq⁻.1]

\[ s_1 @ t \leadsto t' \]
\[ s_1 + s_2 @ t \leadsto t' \]

[choice⁺.1]

\[ s_1 @ t \leadsto \uparrow \]
\[ \land s_2 @ t \leadsto \uparrow \]
\[ s_1 + s_2 @ t \leadsto \uparrow \]

[choice⁻]

\[ c @ c \leadsto c \]

[cong⁺.1]

\[ s_1 @ t_1 \leadsto t'_1 \]
\[ \land \ldots \land s_n @ t_n \leadsto t'_n \]
\[ f(s_1, \ldots, s_n) @ \]
\[ f(t_1, \ldots, t_n) \leadsto f(t'_1, \ldots, t'_n) \]

[cong⁺.2]

\[ f @ \]

[f \neq g]

[cong⁻.2]

\[ f(s_1, \ldots, s_n) @ \]
\[ g(t_1, \ldots, t_m) \leadsto \uparrow \]

\[ \exists i \in \{1, \ldots, n\}, s_i @ t_i \leadsto \uparrow \]
\[ f(s_1, \ldots, s_n) @ \]
\[ f(t_1, \ldots, t_n) \leadsto \uparrow \]

[cong⁻.3]

Fig. 8. Reduction semantics for the basic calculus \( S'_0 \)

Deduction rules  The axioms for \( \epsilon \) and \( \delta \) are trivial. Let us read, for example, the rules for negation. The application \( \neg s @ t \) returns \( t \) if the application \( s @ t \) returns \( \uparrow \) (cf. \([\text{neg}⁺]\)). If \( s @ t \) results in a proper term reduct, then \( \neg s @ t \) evaluates to \( \uparrow \) (cf. \([\text{neg}⁻]\)). These rules also illustrate why we need to include failure as reduct.
Syntax

\[ b ::= \cdots \mid b \text{ where } x = s \oplus t \]

Evaluation of rule bodies

\[ b \rightsquigarrow r \]

Positive rules

\[ t \rightsquigarrow t \]

\[ s \oplus t \rightsquigarrow t' \land b[x \mapsto t'] \rightsquigarrow t'' \]

where \( x = s \oplus t \rightsquigarrow t'' \)

Negative rules

\[ s \oplus t \rightsquigarrow \uparrow \]

where \( x = s \oplus t \rightsquigarrow \uparrow \)

\[ s \oplus t \rightsquigarrow t' \land b[x \mapsto t'] \rightsquigarrow \uparrow \]

where \( x = s \oplus t \rightsquigarrow \uparrow \)

Fig. 9. Extension for where-clauses

Otherwise, a judgement could not query whether a certain strategy application did not succeed. Recall that asymmetric choice also depends on this ability. Let us also look at the rules for the other combinators. The rule \([\text{seq}^+]\) directly encodes the idea of sequential composition where the intermediate term \( t^* \) that is obtained via \( s_1 \) is then further reduced via \( s_2 \). Sequential composition fails if one of the two ingredients \( s_1 \) or \( s_2 \) fails (cf. \([\text{seq}^-1]\), \([\text{seq}^-2]\)). As for choice, there is one positive rule for each operand of the choice (cf. \([\text{choice}^+1]\), \([\text{choice}^+2]\)). Choice allows recovery from failure because if one branch of the choice evaluates to \( \uparrow \), the other branch can still succeed. Choice fails if both options do not admit success (cf. \([\text{choice}^-]\)). The congruences for constants are trivially defined (cf. \([\text{cong}^+1]\), \([\text{cong}^-1]\)). The congruences for function symbols are defined in a schematic manner to cover arbitrary arities (cf. \([\text{cong}^+2]\), \([\text{cong}^-2]\), \([\text{cong}^-3]\)).

Where-clauses In Figure 8, rule bodies were assumed to be terms. In Figure 9, an extension is supplied to cope with where-clauses as motivated earlier. The semantics of rewrite rules as covered in Figure 8 is surpassed by the new rules in Figure 9. Essentially, we resort to a new judgement for the evaluation of rule bodies. A rule body which consists of a term, evaluates trivially to this term (cf. \([\text{body}^+1]\)). A rule
body of the form \( b \) where \( x = s \odot t \) is evaluated by first performing the strategy application \( s \odot t \), and then binding the intermediate term reduct \( t' \) (if any) to \( x \) in the remaining body \( b \) (cf. \([\text{body}^+.2]\)). Obviously, a rewrite rule can now fail for two reasons, either because of an infeasible match (cf. \([\text{rule}^-.1]\)), or due to a failing sub-computation in a where-clause (cf. \([\text{rule}^-.2], [\text{body}^-1], \text{and [body}^-2]\)). For brevity, we will abstract from where-clauses in the formalisation of the type system for \( S''_\gamma \). As the reduction semantics indicates, where-clause do not pose any challenge for formalisation.

### 3.2 Type-preserving strategies

We want to provide a type system for the basic calculus \( S'_0 \). We first focus on type-preserving strategies. We use \( S'_{tp} \) to denote the resulting calculus. In fact, type-changing strategies are not standard in rewriting. So we will consider them in a separate step in Section 3.3. In general, the typed calculus \( S''_\gamma \) is developed in a stepwise and modular fashion.

**Type expressions**  We already sketched the type syntax in Section 2.4. As for purely many-sorted strategies, the forms of term and strategy types are trivially defined by the following grammar:

\[
\begin{align*}
\sigma & : \text{(Sorts)} \\
\tau & ::= \sigma \quad \text{(Term types)} \\
\pi & ::= \tau \rightarrow \tau \quad \text{(Strategy types)}
\end{align*}
\]

**Contexts**  In the upcoming type judgements, we use a context parameter \( \Gamma \) to keep track of sorts \( \sigma \), and to map constant symbols \( c \), function symbols \( f \) and term variables \( x \) to types. Initially, we use the following grammar for contexts:

\[
\begin{align*}
\Gamma & ::= \emptyset \mid \Gamma, \Gamma \quad \text{(Contexts as sets)} \\
& \mid \sigma \mid c : \sigma \mid f : \sigma \times \cdots \times \sigma \rightarrow \sigma \quad \text{(Signature part)} \\
& \mid x : \tau \quad \text{(Term variables)}
\end{align*}
\]

We will have to consider richer contexts when we formalise strategy definitions in Section 5.1. Let us state the requirements for a well-formed context \( \Gamma \). We assume that there are different name spaces for the various kinds of symbols and variables. Also, we assume that constant symbols, function symbols and variables are not associated with different types in \( \Gamma \). That is, we do not consider overloading. All
sorts used in some type declaration in \( \Gamma \) also have to be declared themselves in \( \Gamma \). Finally, when contexts are composed via \( \Gamma_1, \Gamma_2 \) we require that the sets of symbols and variables in \( \Gamma_1 \) and \( \Gamma_2 \) are disjoint. Note that disjoint union of contexts will not be used before Section 5.1. In fact, our contexts are completely static until then.

**Typing judgements** The principal judgement of the type system is the type judgement for strategies. It is of the form \( \Gamma \vdash s : \pi \), and it holds if the strategy \( s \) is of strategy type \( \pi \) in the context \( \Gamma \). Here is a complete list of all well-formedness and well-typedness judgements:

- \( \Gamma \vdash \tau \) (Well-formedness of term types)
- \( \Gamma \vdash \pi \) (Well-formedness of strategy types)
- \( \Gamma \vdash t : \tau \) (Well-typedness of terms)
- \( \Gamma \vdash \neg \pi \rightsquigarrow \pi' \) (Negatable types)
- \( \Gamma \vdash \pi_1 ; \pi_2 \rightsquigarrow \pi \) (Composable types)
- \( \Gamma \vdash s \circledast t : \tau \) (Well-typedness of strategy applications)
- \( \Gamma \vdash s : \pi \) (Well-typedness of strategies)

**Typing rules** The corresponding deduction rules are shown in Figure 10. The present formulation is meant to be very strict regarding type preservation. For some of the rules, one might feel tempted to immediately cover type-changing strategies, e.g., for the rules \([\text{apply}]\) for strategy application or \([\text{comp.1}]\) for composable types in sequential composition. However, we want to enable type changes in a subsequent step. Let us read some inference rules for convenience. Type preservation is postulated by the well-formedness judgement for strategy types (cf. \([\text{pi.1}]\)). Rule \([\text{apply}]\) says that a strategy application \( s \circledast t \) is well-typed if the strategy \( s \) is of type \( \tau \rightarrow \tau \), and the term \( t \) is of type \( \tau \). Obviously, the strategies \( \epsilon \) and \( \delta \) have many types, namely any type \( \tau \rightarrow \tau \) where \( \Gamma \vdash \tau \) holds (cf. \([\text{id}]\) and \([\text{fail}]\)). In turn, compound strategies can also have many types. The strategy types for compound strategies are regulated by the rules \([\text{neg}]\), \([\text{seq}]\), \([\text{choice}]\), and \([\text{cong.2}]\). The typing rules for negation and sequential composition (cf. \([\text{neg}]\) and \([\text{seq}]\)) refer to auxiliary judgements for negatable and composable types. Their definition is straightforward for the initial case of many-sorted type-preserving strategies (cf. \([\text{negt.1}]\) and \([\text{comp.1}]\)). The compound strategy \( s_1 + s_2 \) for choice is well-typed if both strategies \( s_1 \) and \( s_2 \) are of a common type \( \pi \). This common type constitutes the type of the choice.

**Properties** We use \( S'_{tp} \) to denote the composition of \( S'_{0} \) defined in Figure 8, and the type system from Figure 10. The following theorem is concerned with properties of \( S'_{tp} \). It says that actual strategy types adhere to the scheme of type preservation, strategy applications are uniquely typed, and the reduction semantics is properly abstracted in the type system.
| Well-formedness of term types | Well-typedness of term types |
|-------------------------------|-----------------------------|
| $\sigma \in \Gamma$          | $\Gamma \vdash \tau$        |
| $\Gamma \vdash \sigma$       | $\tau : \tau \land \Gamma \vdash t : \tau$ $\quad \Gamma \vdash s \@ t : \tau$ [apply] |

| Well-formedness of strategy types | Well-typedness of strategy types |
|----------------------------------|----------------------------------|
| $\Gamma \vdash \pi$             | $\Gamma \vdash \pi$             |
| $\Gamma \vdash \tau$            | $\Gamma \vdash \tau$            |
| $\Gamma \vdash \pi$             | $\Gamma \vdash \pi \land \Gamma \vdash \tau \Rightarrow \pi' \Rightarrow \pi' \Rightarrow \pi'$ [neg] |
| $\Gamma \vdash \tau \Rightarrow \tau$ [pi] | $\Gamma \vdash \tau \Rightarrow \tau \Rightarrow \tau \Rightarrow \tau$ [comp.1] |

| Well-typedness of terms          | Negatable types                 |
|----------------------------------|----------------------------------|
| $c : \sigma \in \Gamma$         | $\Gamma \vdash \neg \pi \Rightarrow \pi'$ |
| $\Gamma \vdash c : \sigma$      | $\Gamma \vdash \neg \pi \Rightarrow \pi'$ [neg.1] |
| $\Gamma \vdash c : \tau$        | $\Gamma \vdash c : \sigma \Rightarrow \sigma$ [cong.1] |
| $\Gamma \vdash f : \sigma_1 \times \cdots \times \sigma_n \Rightarrow \sigma_0 \in \Gamma$ | $\Gamma \vdash f : \sigma_1 \times \cdots \times \sigma_n \Rightarrow \sigma_0 \in \Gamma$ [fun] |

| Composable types                 |                                    |
|----------------------------------|-----------------------------------|
| $\Gamma \vdash \pi_1; \pi_2 \Rightarrow \pi$ | $\Gamma \vdash \pi_1; \pi_2 \Rightarrow \pi$ [comp.1] |

Fig. 10. Many-sorted type-preserving strategies
Theorem 1  The calculus $S'_{tp}$ for many-sorted type-preserving strategies obeys the following properties:

(1) Actual strategy types adhere to the scheme of type preservation, i.e., for all well-formed contexts $\Gamma$, strategies $s$ and term types $\tau$, $\tau'$:
$$\Gamma \vdash s : \tau \rightarrow \tau' \implies \tau = \tau'.$$

(2) Strategy applications satisfy unicity of typing (UOT, for short), i.e., for all well-formed contexts $\Gamma$, strategies $s$, term types $\tau$, $\tau'$ and terms $t$:
$$\Gamma \vdash s @ t : \tau \land \Gamma \vdash s @ t : \tau' \implies \tau = \tau'.$$

(3) Reduction of strategy applications satisfies subject reduction, i.e., for all well-formed contexts $\Gamma$, strategies $s$, term types $\tau$ and terms $t$, $t'$:
$$\Gamma \vdash s : \tau \rightarrow \tau \land \Gamma \vdash t : \tau \land s @ t \leadsto t' \implies \Gamma \vdash t' : \tau.$$

In the further development of $S'_{tp}$, we will use refinements of these properties to prove the formal status of the evolving type system. UOT and subject reduction are basic desirable properties of type systems (cf. [Bar92, Gnu93, Sch94]). We claim UOT for strategy applications but not for strategies themselves because of the typing rules for the constant combinators $\epsilon$ and $\delta$. UOT for strategy application means that the result type of a strategy application is determined by the type of the input term. Subject reduction means that if we initiate a reduction of a well-typed strategy application, then we can be sure that the resulting term reduct (if any) is of the prescribed type. The following proof is very verbose to prepare for the elaboration of the proof in the context of generic types.

Proof 1  (IH abbreviates induction hypothesis in all the upcoming proofs.)

(1) We show adherence to the scheme of type preservation by induction on $s$ in $\Gamma \vdash s : \pi$. Base cases: Type preservation is directly enforced for rewrite rules, $\epsilon$, $\delta$, and congruences for constants by the corresponding typing rules (cf. [rule], [id], [fail], and [comp.1]), that is, the type position in the conclusion is instantiated according to the type-preserving form of strategy types. Induction step: Type preservation for $\neg \cdot$ (cf. [neg]) is implied by the rule $\neg t$.1 for negatable types. Strictly speaking, we do not need to employ the IH since the type-preserving shape of the result type is enforced by $\neg t$.1 regardless of the argument type. As for $s_1; s_2$, the auxiliary judgement for composable types enforces type preservation (cf. $\Gamma \vdash \cdots ; \cdots \leadsto \pi \rightarrow \pi$ in $\text{comp.1}$). Again, the IH does not need to be employed. As for $s_1 + s_2$, the result type coincides with the argument types, and hence, type preservation is implied by the IH. Finally, type preservation for congruences $f(s_1, \ldots, s_n)$ is directly enforced by the corresponding typing rule (cf. the type position in the conclusion of $\text{comp.2}$).

(2) Let us first point out that UOT obviously holds for terms because the inductive definition of $\Gamma \vdash t : \tau$ enforces a unique type $\tau$ for $t$. Here it is essential that we ruled out overloading of function and constant symbols, and variables. According to the rule $\text{apply}$, the result type of a strategy application is equal
to the type of the input term. Hence, \( s \otimes t \) is uniquely typed.

(3) In the type-preserving setting, subject reduction actually means that the reduction semantics for strategy applications is type-preserving as prescribed by the type system. That is, if the reduction of a strategy application \( s \otimes t \) with \( s : \tau \rightarrow \tau, t : \tau \) yields a proper term reduct \( t' \), then \( t' \) is also of type \( \tau \). We show this property by induction on \( s \) in \( s \otimes t \sim t' \) while we assume \( \Gamma \vdash s : \tau \rightarrow \tau \) and \( \Gamma \vdash t : \tau \). To this end, it is crucial to maintain that the IH can only be employed for a premise \( s_i \otimes t_i \sim t'_i \) and a corresponding type \( \tau_i \), if we can prove the following side condition:

\[
\Gamma \vdash s : \tau \rightarrow \tau \land \Gamma \vdash t : \tau \land \ldots
\]

implies \( \Gamma \vdash s_i : \tau_i \rightarrow \tau_i \land \Gamma \vdash t_i : \tau_i \)

With the “…” we indicate that actual side conditions might involve additional requirements. The judgements \( \Gamma \vdash s_i : \tau_i \rightarrow \tau_i \) and \( \Gamma \vdash t_i : \tau_i \) have to be approved by consulting the corresponding typing rules that relate \( t \) to \( t_i \), and \( s \) to \( s_i \), and by other means. Note there are no proof obligations for deduction rules which do not yield a proper term reduct, namely for negative rules. In particular, there is no case for \( \delta \) in the sequel, that is, \( \delta \) is type-preserving in a degenerated sense. Base cases: As for rewrite rules, we know that both the left-hand side \( t_1 \) and the right-hand side \( t_r \) are of type \( \tau \) as prescribed by \([\text{rule}]\). The substitution \( \theta \) in \([\text{rule}^+]\) preserves the type of the right-hand side as implied by basic properties of many-sorted unification and substitution. Hence, rule application is type-preserving. \( \epsilon \) preserves the very input term, and hence, it is type-preserving. The same holds for congruences for constants. Induction step: Negation is type-preserving because the very input term is preserved as for \( \epsilon \). Thus, we do not need to employ the IH for \( s \) in \( \neg s \). In fact, the IH tells us here that we do not even attempt to apply \( s \) in an ill-typed manner. Let us consider sequential composition \( s_1; s_2 @ t \). By the rules \([\text{seq}]\) and \([\text{comp},1]\), we know that the types of \( s_1, s_2 \) and \( s_1; s_2 \) coincide, that is, the common type is \( \tau \rightarrow \tau \). We want to show that \( t' \) in \( s_1; s_2 @ t \sim t' \) is of the same type as \( t \). As \( s_1 \) must be of the same type as \( s_1; s_2 \), the IH is enabled for \( s_1 @ t \sim t' \). Thereby, we know that \( t^* \) is of type \( \tau \). Since we also know that \( s_2 \) must be of the same type as \( s_1; s_2 \), the IH is enabled for the second premise \( s_2 @ t^* \sim t' \). Hence, \( t' \) of the same type as \( t \), and sequential composition is type-preserving. As for choice, reduction of \( s_1 + s_2 @ t \) directly resorts to either \( s_1 @ t \) or \( s_2 @ t \) (cf. \([\text{choice}^+,1]\) and \([\text{choice}^+,2]\)). We also know that \( s_1, s_2 \) and \( s_1 + s_2 \) have to be of the same type (cf. \([\text{choice}]\)). Hence, the IH is enabled for the reduction of the chosen strategy, be it \( s_1 \) or \( s_2 \). Finally, let us consider congruence strategies where \( f(s_1, \ldots, s_n) @ f(t_1, \ldots, t_n) \) is reduced to \( f(t'_1, \ldots, t'_{n}) \) while the \( t'_i \) are obtained by the reduction of the \( s_i @ t_i \) (cf. \([\text{cong},2]\)). Let \( f : \sigma_1 \times \cdots \times \sigma_n \rightarrow \sigma_0 \) be in \( \Gamma \). Then, we know that for a well-typed term \( f(t_1, \ldots, t_n) \), the \( t_i \) must be of type \( \sigma_i \) (cf. \([\text{fun}]\)). We also know that for a well-typed strategy \( f(s_1, \ldots, s_n) \), the \( s_i \) must be of type \( \sigma_i \rightarrow \sigma_i \) (cf. \([\text{cong},2]\)). Hence, the IH is enabled for the var-
3.3 Type-changing strategies

In standard rewriting, as a consequence of a fixed normalisation strategy, rewrite rules are necessarily type-preserving. It does not make sense to repeatedly look for a redex in a compound term, and then to apply some type-changing rewrite rule to the redex since this would potentially lead to an ill-typed compound term. In strategic rewriting, it is no longer necessary to insist on type-preserving rewrite rules. One can use strategies to apply type-changing rewrite rules or other strategies in a disciplined manner making sure that intermediate results are properly combined as opposed to the type-changing replacement of a redex in a compound term.
Type system update  In Figure 11, the type system for type-preserving strategies is updated to enable type-changing strategies. We use $S'_{tc}$ to denote the refinement of $S'_{tp}$ according to the figure. The refinements amounts to the following adaptations. We replace rule $[\pi.1]$ to characterise potentially type-changing strategies as well-formed. We also replace the rule $[apply]$ for strategy application, and the rule $[rule]$ to promote type-changing strategies. Furthermore, the auxiliary judgements for negatable and composable strategy types have to be generalised accordingly (cf. $[negt.1]$ and $[comp.1]$). The relaxation for composable types is entirely obvious but we should comment on the typing rule for negatable types. Negation is said to be type-preserving regardless of the argument’s type. This is appropriate because the only possible term reduct admitted by negation is the input term itself. The argument strategy is only tested for failure. Hence, negation by itself is type-preserving even if the argument strategy would be type-changing. All the other typing rules carry over from $S'_{tp}$. As an aside, we do not generalise the type of $\delta$ to go beyond type preservation. In fact, one could say that the result type of $\delta$ is arbitrary since no term reduct will be returned anyway. However, such a definition would complicate our claim of UOT.

Theorem 2  The calculus $S'_{tc}$ for potentially type-changing strategies obeys the following properties:

1. Co-domains of strategies are determined by domains. i.e., for all well-formed contexts $\Gamma$, strategies $s$ and term types $\tau_1$, $\tau'_1$, $\tau_2$, $\tau'_2$:
   $$\Gamma \vdash s : \tau_1 \rightarrow \tau'_1 \land \Gamma \vdash s : \tau_2 \rightarrow \tau'_2 \land \tau'_1 \neq \tau'_2 \text{ implies } \tau_1 \neq \tau_2.$$ 

2. Strategy applications satisfy UOT, i.e., ... (cf. Theorem 1).

3. Reduction of strategy applications satisfies subject reduction, i.e., for all well-formed contexts $\Gamma$, strategies $s$, term types $\tau$, $\tau'$ and terms $t$, $t'$:
   $$\Gamma \vdash s : \tau \rightarrow \tau' \land \Gamma \vdash t : \tau \land s @ t \leadsto t' \text{ implies } \Gamma \vdash t' \vdash \tau'.$$

The first property is the necessary generalisation of adherence to the scheme of type preservation in Theorem 1. We require that the co-domain of a strategy type is uniquely determined by its domain. That is, there might be different types for a strategy, but once the type of the input term is fixed, the type of the result is determined. The second property carries over from Theorem 1. The third property needs to be generalised compared to Theorem 1 in order to cover type-changing strategies.

Proof 2

1. Note that the property trivially holds for type-preserving strategies. We show the property by induction on $s$ in $\Gamma \vdash s : \pi$. Base cases: The co-domain of a rewrite rule is even uniquely defined regardless of the domain as an implication of UOT for terms. The remaining base cases are type-preserving, and hence, they are trivial. Induction step: Negation is trivially covered because it is type-preserving. As for sequential composition, the domain of $s_1; s_2$ coin-
cides with the domain of $s_1$, the co-domain of $s_1$ coincides with the domain of $s_2$, and the co-domain of $s_2$ coincides with the co-domain of $s_1; s_2$ (cf. [comp.1]). By applying the IH to $s_1$ and $s_2$, we obtain that the co-domain of $s_1; s_2$ is transitively determined by its domain. As for choice, the property follows from the strict coincidence of the types of $s_1$, $s_2$, and $s_1 + s_2$ (cf. [choice]) which immediately enables the IH. Congruences $f(s_1, \ldots, s_n)$ are trivially covered because they are type-preserving.

(2) The simple argument from Proof 1 regarding the rule [apply] can be generalised as follows. The domain of the strategy in $s \otimes t$ needs to coincide with the type of $t$. Since the co-domain of $s$ is determined by the type of $t$ (cf. (1) above), we know that the type of the reduct is uniquely defined.

(3) We need to elaborate our induction proof for Proof 1 where we argued that subject reduction for type-preserving strategies can be proved by showing that the reduction semantics is type-preserving, too. As for potentially type-changing strategies, we need to show that reduction obeys the strategy types. Hence, the side condition for the employment of the IH has to be revised, too. That is, the IH can be employed for a premise $s_i @ t_i \sim t'_i$ and corresponding types $\tau_i$ and $\tau'_i$, if we can prove the following side condition:

\[
\Gamma \vdash s : \tau \rightarrow \tau' \land \Gamma \vdash t : \tau \land \ldots
\]

implies \[\Gamma \vdash s_i : \tau_i \rightarrow \tau'_i \land \Gamma \vdash t_i : \tau_i\]

Base cases: Subject reduction for rewrite rules is implied by basic properties of many-sorted unification and substitution. The remaining base cases are type-preserving, and hence, they are covered by Proof 1. Induction step: The strategy $s$ in $\neg s$ is not necessarily type-preserving anymore but negation by itself adheres to type preservation as prescribed by the type system (cf. [negt.1]). As for sequential composition, we start from the assumptions $\Gamma \vdash s_{1}; s_{2} : \tau \rightarrow \tau'$ and $\Gamma \vdash t : \tau$, we want to show that $t'$ in $s_{1}; s_{2} @ t \sim t'$ is of type $\tau'$. There must exist a $\tau^*$ such that $\Gamma \vdash s_{1} : \tau \rightarrow \tau^*$ and $\Gamma \vdash s_{2} : \tau^* \rightarrow \tau'$ (by [comp.1] and [seq]). In fact, $\tau^*$ is uniquely defined because it is the co-domain of $s_1$ determined by the domain of $s_1$ which coincides with the domain of $s_1; s_2$. We apply the IH for $s_{1} @ t$, and hence, we obtain that the reduction of $s_{1} @ t$ delivers a term $t^*$ of type $\tau^*$. This enables the IH for the second operand of sequential composition. Hence, we obtain that the reduction of $s_{2} @ t^*$ delivers a term $t'$ of type $\tau'$. As for choice, the arguments from Proof 1 are still valid since we did not rely on type preservation. That is, we know that the reduction of the choice directly resorts to one of the argument strategies, and the type of the choice has the same type as the two argument strategies. Hence, subject reduction for choice follows from the IH. Congruences $f(s_1, \ldots, s_n)$ and the involved arguments are type-preserving, and hence, subject reduction carries over from Proof 1.
Syntax

\[ \tau ::= \cdots \mid \langle \rangle \mid \langle \tau_1, \tau_2 \rangle \]

\[ t ::= \cdots \mid \langle \rangle \mid \langle t_1, t_2 \rangle \]

\[ s ::= \cdots \mid \langle \rangle \mid \langle s_1, s_2 \rangle \]

Well-formedness

of term types

\[ \Gamma \vdash \tau \]

Well-typedness

of strategies

\[ \Gamma \vdash s : \pi \]

Well-formedness

of term types

\[ \Gamma \vdash \langle \rangle \quad \text{[tau.2]} \]

\[ \Gamma \vdash \langle \rangle : \langle \rangle \quad \text{[empty-tuple]} \]

\[ \Gamma \vdash \tau_1 \land \Gamma \vdash \tau_2 \]

\[ \Gamma \vdash \langle \tau_1, \tau_2 \rangle \quad \text{[tau.3]} \]

\[ \Gamma \vdash \langle \tau_1, \tau_2 \rangle : \langle \tau_1', \tau_2' \rangle \quad \text{[cong.4]} \]

Well-typedness

of terms

\[ \Gamma \vdash \langle \rangle : \langle \rangle \]

\[ \Gamma \vdash t_1 : \tau_1 \land \Gamma \vdash t_2 : \tau_2 \]

\[ \Gamma \vdash \langle t_1, t_2 \rangle : \langle \tau_1, \tau_2 \rangle \quad \text{[pair]} \]

The reduction semantics of tuple congruences is defined precisely as for ordinary many-sorted constant and function symbols.

Fig. 12. Tuple types, tuples, and tuple congruences

3.4 Polyadic strategies

As we motivated in Section 2, we want to employ tuples to describe polyadic strategies, that is, strategies which process several terms. In principle, the following extension for tuples can be composed with both \( S'_tp \) and \( S'_tc \). However, tuples are only potent in \( S'_tc \) with type-changing strategies enabled.

In Figure 12, we extend the basic calculus \( S'_0 \) with concepts for polyadic strategies in a straightforward manner. There are distinguished symbols \( \langle \rangle \) for the empty tuple, and \( \langle \cdot, \cdot \rangle \) for pairing terms. We use the same symbols for tuple types, tuples, and congruences on tuples. The judgements for well-formedness of term types and well-typedness of terms are extended accordingly (cf. [tau.2], [tau.3], [empty-tuple], and [pair]). We also introduce special typing rules for congruences on tuples (cf. [cong.3] and [cong.4]). Note that the typing rules for congruences for ordinary symbols relied on a context lookup (cf. [cong.1] and [cong.2] in Figure 10) while this is not the case for polymorphic congruences on tuples. Moreover, congruences on pairs can be type-
changing (cf. $\text{cong.4}$) whereas this is not an option for many-sorted congruences. We should point out that tuples are solely intended for argument and result lists while constructor terms should be purely many-sorted. This intention is enforced because the argument types of ordinary function symbols are still restricted to sorts as opposed to tuple types (cf. $\text{run}$ in Figure 10).

**Example 15** The strategy $\text{ADD}$ from Example 4 and the strategy $\text{COUNT}$ from Example 7 are well-typed as type-changing strategies.

\[
\text{ADD} : \langle \text{Nat}, \text{Nat} \rangle \rightarrow \text{Nat} \\
\text{COUNT} : \text{Tree} \rightarrow \text{Nat}
\]

As for $\text{ADD}$, we rely on tuple types since addition is encoded as a strategy which takes a pair of naturals. As for $\text{COUNT}$, the definition from Example 7 involves a type-changing congruence on pairs, namely $\langle \text{COUNT}, \text{COUNT} \rangle$. This congruence applies $\text{COUNT}$ to the two subtrees of a fork tree independently.

### 4 Generic strategies

In the present section, we extend our basic calculus for many-sorted strategies by types and combinators for generic strategies. First, we spell out the reduction semantics of type-preserving combinators, and we formalise the corresponding generic type $\text{TP}$. Then, the problem of mediation between many-sorted and generic strategies is addressed. There are two directions for mediation. When we qualify a many-sorted strategy to become generic, then we perform extension. When we instantiate the type of a generic strategy for a given sort, then we perform restriction. Afterwards, we define the type-unifying traversal combinators and the corresponding generic type (constructor) $\text{TU}(\cdot)$.

#### 4.1 Strategies of type $\text{TP}$

**Combinators** In Figure 13, we define the reduction semantics of the generic traversal primitives $\square(\cdot)$ and $\Diamond(\cdot)$ adopted from system $S$. The rule $[\text{all}^{+}.1]$ says that $\square(s)$ applied to a constant immediately succeeds because there are no children which have to be processed. The rule $[\text{all}^{+}.2]$ directly encodes what it means to apply $s$ to all children of a term $f(t_1, \ldots, t_n)$. Note that the function symbol $f$ is preserved in the result. The reduction scheme for $\Diamond(s)$ is similar. The rule $[\text{one}^{+}]$ says that $s$ is applied to some subterm $t_i$ of $f(t_1, \ldots, t_n)$ such that it succeeds for this child. The semantics is non-deterministic as for choice of the child. One could also think of a different semantics where the children are tried from left to right or vice versa until one child is processed successfully. The negative rule $[\text{one}^{-}.1]$ says
Syntax

\[ s ::= \cdots | \Box(s) | \Diamond(s) \]

Reduction of strategy applications

Positive rules

\[
\begin{align*}
\Box(s) @ c & \leadsto c & [\text{all}^+.1] \\
\Delta s @ t_1 & \leadsto t'_1 \\
& \wedge \cdots \\
& \wedge s @ t_n \leadsto t'_n \end{align*}
\]

\[
\begin{align*}
\Box(s) @ f(t_1, \ldots, t_n) & \leadsto f(t'_1, \ldots, t'_n) & [\text{all}^+.2] \\
\exists i \in \{1, \ldots, n\}. s @ t_i & \leadsto t'_i & [\text{one}^+.1] \\
\Diamond(s) @ f(t_1, \ldots, t_n) & \leadsto f(t_1, \ldots, t'_i, \ldots, t_n) & [\text{one}^+.2]
\end{align*}
\]

Negative rules

\[
\begin{align*}
\exists i \in \{1, \ldots, n\}. s @ t_i & \leadsto \uparrow & [\text{all}^-] \\
\Box(s) @ f(t_1, \ldots, t_n) & \leadsto \uparrow \\
\Diamond(s) @ c & \leadsto \uparrow & [\text{one}^-.1] \\
\Delta s @ t_1 & \leadsto \uparrow \\
& \wedge \cdots \\
& \wedge s @ t_n \leadsto \uparrow & [\text{one}^-.2]
\end{align*}
\]

| Fig. 13. Type-preserving traversal combinators |

that \(\Diamond(s)\) applied to a constant fails because there is no child that could be processed by \(s\). The negative rule \([\text{one}^-.2]\) says that \(\Diamond(s)\) fails if \(s\) fails for all children of \(f(t_1, \ldots, t_n)\). Dually, \(\Box(s)\) fails if \(s\) fails for some child (cf. \([\text{all}^-]\)).

The generic type \(\text{TP}\) In Figure 14, we extend the typing judgements to formalise \(\text{TP}\), and to employ \(\text{TP}\) for the relevant combinators. We establish a syntactical domain \(\gamma\) of generic types. We integrate \(\gamma\) into the grammar for types by stating that \(\gamma\) corresponds to another form of strategy types \(\pi\) complementing many-sorted strategy types. We start the definition of \(\gamma\) with the generic type \(\text{TP}\). We use a relation \(\prec_{\Gamma}\) on types to characterise generic types, say the genericity of types. The relation \(\preceq_{\Gamma}\) denotes the reflexive closure of \(\prec_{\Gamma}\). By \(\pi \prec_{\Gamma} \pi'\), we mean that \(\pi'\) is more generic than \(\pi\). If we view generic types as type schemes, we can also say that the type \(\pi\) is an instance of the type scheme \(\pi'\). Rule \([\text{less}.1]\) axiomatises \(\text{TP}\). The rule says that \(\tau \rightarrow \tau\) is an instance of \(\text{TP}\) for all well-formed \(\tau\). This formulation indeed suggests to consider \(\text{TP}\) as the type scheme \(\forall \alpha. \alpha \rightarrow \alpha\). We urge the reader not to confuse \(\preceq_{\Gamma}\) with subtyping. The remaining rules in Figure 14 deal with well-typedness of generic strategies. The constant combinators \(\epsilon\) and \(\delta\) are defined to be
Syntax

\[ \pi ::= \cdots \mid \gamma \]
\[ \gamma ::= \text{TP} \]

Well-formedness of strategy types

\[ \Gamma \vdash \pi \]

Composable types

\[ \Gamma \vdash \pi_1; \pi_2 \leadsto \pi \]

Genericity relation

\[ \pi \prec_\Gamma \pi' \]

Well-typedness of strategies

\[ \Gamma \vdash s : \pi \]

Negatable types

\[ \Gamma \vdash \neg \pi \leadsto \pi' \]

\[ \Gamma \vdash \neg \gamma \leadsto \text{TP} \]

\[ \Gamma \vdash \neg \pi \leadsto \pi' \]

\[ \Gamma \vdash s : \text{TP} \]

\[ \Gamma \vdash \Box(s) : \text{TP} \]

\[ \Gamma \vdash \Diamond(s) : \text{TP} \]

Fig. 14. TP—The type of generic type-preserving strategies

generic type-preserving strategies (cf. [id] and [fail]). As for negation, we add another rule to the auxiliary judgement \( \Gamma \vdash \neg \pi \leadsto \pi' \) for negatable types. The enabled form of negation is concerned with generic strategies. The rule [negt.2] states that any strategy of a generic type \( \gamma \) can be negated. As for sequential composition, we also add a rule to the auxiliary judgement \( \Gamma \vdash \pi_1; \pi_2 \leadsto \pi \) to cover the case that a generic type-preserving strategy and another generic strategy are composed (cf. [comp.2]). The typing rules for the traversal combinators state simply that \( \Box(\cdot) \) and \( \Diamond(\cdot) \) can be used to derive a strategy of type TP from an argument strategy of type TP (cf. [all] and [one]).

Well-defined generic strategy types In general, we assume that a well-defined generic type should admit an instance for every possible term type since a generic strategy should be applicable to terms of all sorts. To be precise, there should be exactly one instance per term type. It is an essential property that there is only one instance per term type. Otherwise, the type of a generic strategy application would
be ambiguous. The type TP is obviously well-defined in this sense.

Separation of many-sorted and generic strategies  Note that there are now two levels in our type system, that is, there are many-sorted types and generic types. The type system strictly separates many-sorted strategies (such as rewrite rules) and generic strategies (such as applications of $\Box(\cdot)$). Since there are no further intermediate levels of genericity, there are only chains of length 1 in the partial order $\preceq_T$. Longer chains will be needed in Section 5.3 when we consider a possible sophistication of $S'_\gamma$ to accomplish overloaded strategies. As the type system stands, we cannot turn many-sorted strategies into generic ones, nor the other way around. Also, strategy application, as it was defined for $S'_tp$ and $S'_tc$, only copes with many-sorted strategies. The type system should allow us to apply a generic strategy to any term. We will now develop the corresponding techniques for strategy extension and restriction.

4.2 Strategy extension

Now that we have typed generic traversal combinators at our disposal, we also want to inhabit the generic type TP. So far, we only have two trivial constants of type TP, namely $\epsilon$ and $\delta$. We would like to construct generic strategies from rewrite rules. We will formalise the corresponding combinator $\cdot < \cdot$ for strategy extension. To this end, we also examine other approaches in order to justify the design of $\cdot < \cdot$.

Infeasible approaches  In the untyped language Stratego, no distinction is made between rewrite rules and generic strategies. We might attempt to lift this design to a typed level. We study two such approaches. One infeasible approach to the inhabitation of generic strategy types like TP is that the typing requirements for generic strategy arguments would be relaxed: whenever we require a generic strategy argument, e.g., for $s$ in $\Box(s)$, we also would accept a many-sorted $s$. This implicit approach to the inhabitation of generic types would only be type-safe if extra dynamic type checks are added. Consider, for example, the strategy $\Box(s)$ where $s$ is of some many-sorted type $\tau \rightarrow \tau$. For the sake of subject reduction (say, type-safe strategy application), the semantics of $\Box(s)$ had to ensure that every child at hand is of type $\tau$ before it even attempts to apply $s$ to it. If some child is not of type $\tau$, $\Box(s)$ must fail. From the programmer’s point of view, the approach makes it indeed too easy for many-sorted strategies to get accepted in a generic context. The resulting applicability failures of many-sorted strategies in generic contexts are not approved by the strategic programmer. By contrast:

We envision a statically type-safe style of strategic programming, where the employment of many-sorted strategies in a generic context is approved by the pro-
grammer. Moreover, the corresponding calculus should correspond to a conservative extension of $S'_{tp}$ (or $S'_{tc}$).

Another infeasible approach to the inhabitation of generic strategy types is to resort to a choice combinator ($\cdot + \cdot$ or $\cdot \lhd \cdot$) to compose a many-sorted strategy and a generic default. We basically end up having the same problem as above. Let us attempt to turn a rewrite rule $\ell$ into a generic strategy using the form $\ell \lhd s$ where $s$ is some generic strategy, e.g., $\varepsilon$ or $\delta$. We could assume that the result type of a choice corresponds to the least upper bound of the argument types w.r.t. $\preceq_{\Gamma}$. One possible argument to refuse such a style arises from the following simple derivation:

$$s \lhd \delta \leadsto s + s; \delta \leadsto s + \delta \leadsto s$$

That is, we show that $\delta$ is the unit of $\cdot \lhd \cdot$ while assuming that it is the zero of sequential composition. Since the derivation resembles desirable algebraic identities of choice, sequential composition and failure that are also met by the formalisation of $S'_{\gamma}$, we should assume that all strategies in the derivation are of the same type. This is in conflict with the idea to use $\cdot \lhd \cdot$ or $\cdot + \cdot$ to inhabit generic types. Furthermore, the approach would also affect the reduction semantics in a way that goes beyond a conservative extension. We had to redefine the semantics of $\cdot + \cdot$ to make sure that the argument strategies are applied only if their type and the type of the term at hand fits. Finally, a too liberal typing rule for choice makes it easy for a strategic programmer to confuse two different idioms:

- recovery from failure, and
- the inhabitation of generic types.

This confusion can lead to unintentionally generic strategies which then succeed (cf. $\cdots \lhd \varepsilon$) or fail (cf. $\cdots \lhd \delta$) in a surprising manner. The avoidance of this confusion is among the major advantages that a typed system offers when compared to the currently untyped Stratego.

**Inhabitation by extension** The combinator $\cdot \lhd \cdot$ serves for the explicit extension of a many-sorted strategy to become applicable to terms of all sorts. Suppose the type of $s$ is $\tau \rightarrow \tau$. Then, of course, $s$ can only be applied to terms of sort $\tau$ in a type-safe manner. It is the very meaning of $s \lhd \text{TP} \oslash t$ to apply $s$ if and only if $t$ is of sort $\tau$. Otherwise, $s \lhd \text{TP} \oslash t$ fails. Hence, $s$ is extended in a trivial sense, that is, to behave like $\delta$ for all sorts different from $\tau$. Well-typedness and the reduction semantics of the combinator $\cdot \lhd \cdot$ are defined in Figure 15. We should point out a paradigm shift, namely that the typing context $\Gamma$ is now also part of the reduction judgement. That is, the new judgement for the reduction of strategy applications takes the form $\Gamma \vdash s \oslash t \leadsto r$. In the typing rule $[\text{extend}]$, we check if the actual type $\pi'$ of $s$ in $s \lhd \pi$ is an instance of the type $\pi$ for the planned extension. In the reduction semantics, in rule $[\text{extend}^+]$, we check if the type $\tau$ of the term $t$ is covered
Syntax

\[ s ::= \cdots \mid s \triangleleft \pi \]

Well-typedness of strategies

\[
\Gamma \vdash s : \pi
\]

\[
\Gamma \vdash s : \pi' \land \pi' \prec \pi
\]

\[
\Gamma \vdash s \triangleleft \pi : \pi
\] [extend]

Reduction of strategy applications

Positive rule

\[
\Gamma \vdash s : \pi'
\]

\[
\land \Gamma \vdash t : \tau
\]

\[
\land \exists \pi', \tau \rightarrow \tau' \preceq \Gamma \pi'
\]

\[
\land \Gamma \vdash s \circ t \rightsquigarrow t'
\]

\[
\Gamma \vdash s \triangleleft \pi \circ t \rightsquigarrow t'
\] [extend⁺]

Negative rules

\[
\Gamma \vdash s : \pi'
\]

\[
\land \Gamma \vdash t : \tau
\]

\[
\land \exists \pi', \tau \rightarrow \tau' \preceq \Gamma \pi'
\]

\[
\land \Gamma \vdash s \circ t \leftarrow t'
\]

\[
\Gamma \vdash s \triangleleft \pi \circ t \leftarrow t'
\] [extend⁻.1]

\[
\Gamma \vdash s : \pi'
\]

\[
\land \Gamma \vdash t : \tau
\]

\[
\land \exists \pi', \tau \rightarrow \tau' \preceq \Gamma \pi'
\]

\[
\Gamma \vdash s \triangleleft \pi \circ t \leftarrow \uparrow
\] [extend⁻.2]

Fig. 15. Turning many-sorted strategies into generic ones

by the type \( \pi' \) of \( s \) in \( s \triangleleft \pi \). Clearly, this check made the addition of the typing context \( \Gamma \) necessary. Note that the generic type \( \pi \) from \( s \triangleleft \pi \) does not play any role during reduction. The type \( \pi \) is only relevant in the typing rule for strategy extension to point out the result type of strategy extension.

**Type dependency** The combinator \( \cdot \triangleleft \cdot \) makes it explicit where we want to become generic. There is no hidden way how many-sorted ingredients may become generic—accidentally or otherwise. As the reduction semantics of \( \cdot \triangleleft \cdot \) clearly points out, reduction is truly type-dependent. That is, the reduction of an extended strategy depends on the run-time comparison of the types of \( s \) and \( t \) in \( s \triangleleft \pi \circ t \). We assume that all previous rules of the reduction semantics are lifted to the new form of judgement by propagating \( \Gamma \). Otherwise, all rules stay intact, and hence we may claim that the incorporation of \( \cdot \triangleleft \cdot \) corresponds to a conservative extension. One should not confuse type-dependent reduction with dynamic type checks. Type dependency merely means that a generic strategy admits different behaviours for different sorts.
Syntax
\[ s ::= \cdots | s \triangleright \pi \]

Well-typedness
of strategies

\[ \Gamma \vdash s : \pi \]

Reduction
of strategy applications

\[ \Gamma \vdash s @ t \leadsto r \]

\[ \frac{\Gamma \vdash s : \pi' \land \pi \prec_{\Gamma} \pi'}{\Gamma \vdash s \triangleright \pi : \pi} \] [restrict]

\[ \frac{\Gamma \vdash s @ t \leadsto r}{\Gamma \vdash s \triangleright \pi @ t \leadsto r} \] [restrict\(\pm\)]

Fig. 16. Explicit strategy restriction

As an aside, in Section 5.2, we will discuss a convenient approach to eliminate the typing judgements in the reduction semantics for strategy extension. The basic idea is to resort to tagged strategy applications such that types do not need to be determined at run-time, but a simple tag comparison is sufficient to perform strategy extension.

4.3 Restriction

So far, we only considered one direction of mediation between many-sorted and generic strategy types. We should also refine our type system so that generic strategies can be easily applied in specific contexts. Actually, there is not just one way to accommodate restriction. Compared to extension, restriction is conceptually much simpler since restriction is immediately type-safe without further precautions. In general, we are used to the idea that a generic entity is used in a specific context, e.g., in the sense of parametric polymorphism.

Explicit restriction  We consider a strategy combinator \( s \triangleright \pi \) which has no semantic effect, but at the level of typing it allows us to consider a generic strategy \( s \) to be of type \( \pi \) provided it holds \( \pi \prec_{\Gamma} \pi' \) where \( \pi' \) is the actual type of \( s \). Explicit restriction is defined in Figure 16. The combinator for restriction is immediately sufficient if we want to apply a generic strategy \( s \) to a term \( t \) of a certain sort \( \tau \). If we assume, for example, that \( s \) is of type \( TP \), then the well-typed strategy application \( s \triangleright \tau \rightarrow \tau @ t \) can be employed. Thus, the rule [apply] for strategy applications from Figure 10 or the updated rule from Figure 11 can be retained without modifications.

Extension and restriction in concert  For completeness, let us assume that we also can annotate strategies by their types, say by the form \( s : \pi \). This is well-typed
if \( s \) is indeed of type \( \pi \). The reduction of \( s : \pi \) simply resorts to \( s \). Then, in a sense, the three forms \( s \triangleright \pi \) (i.e., explicit restriction), \( s : \pi \) (i.e., type annotation), and \( s \triangleleft \pi \) (i.e., strategy extension) complement each other as they deal with the different ways how a strategy \( s \) and a type \( \pi \) can be related to each other via the partial order \( \preceq_\Gamma \). The three forms interact with the type system and the reduction semantics in the following manner. Type annotations can be removed without any effect on well-typedness and semantics. By contrast, a replacement of a restriction \( s \triangleright \pi \) by \( s \) will result in an ill-typed program, although \( s \) is semantically equivalent to \( s \triangleright \pi \). Finally, a replacement of an extension \( s \triangleleft \pi \) by \( s \) will not just harm well-typedness, but an ultimate application of \( s \) is not even necessarily type-safe.

**Inter-mezzo** The concepts that we have explained so far are sufficient to assemble a calculus \( S'_{\text{TP}} \) which covers generic type-preserving traversal. Generic type-preserving strategies could be combined with both \( S'_{\text{tp}} \) and \( S'_{\text{tc}} \). For simplicity, we have chosen the former as the starting point for \( S'_{\text{TP}} \). For convenience, we summarise all ingredients of \( S'_{\text{TP}} \):

- The basic calculus \( S'_0 \) (cf. Figure 8)
- Many-sorted type-preserving strategies (cf. Figure 10)
- Generic traversal primitives \( \Box (\cdot) \) and \( \Diamond (\cdot) \) (cf. Figure 13)
- The generic type \( \text{TP} \) (cf. Figure 14)
- Strategy extension (cf. Figure 15)
- Explicit restriction (cf. Figure 16)

In Theorem 1, and Theorem 2, we started to address properties of the many-sorted fragments \( S'_{\text{tp}} \) and \( S'_{\text{tc}} \) of \( S'_{\gamma} \). Let us update the theorem for \( S'_{\text{TP}} \) accordingly.

**Theorem 3** The calculus \( S'_{\text{TP}} \) obeys the following properties:

1. **Strategies satisfy UOT, and their types adhere to the scheme of type preservation**, i.e., for all well-formed contexts \( \Gamma \), strategy types \( \pi, \pi' \) and strategies \( s \):
   
   (a) \( \Gamma \vdash s : \pi \ \land \ \Gamma \vdash s : \pi' \implies \pi = \pi' \), and
   (b) \( \Gamma \vdash s : \pi \implies \pi \preceq_\Gamma \text{TP} \).

2. **Strategy applications satisfy UOT**, i.e., ... (cf. Theorem 1).
3. **Reduction of strategy applications satisfies subject reduction**, i.e., for all well-formed contexts \( \Gamma \), strategies \( s \), terms \( t, t' \), term types \( \tau \) and strategy types \( \pi \):
   
   \( \Gamma \vdash s : \pi \ \land \ \Gamma \vdash t : \tau \ \land \ \tau \rightarrow \tau \preceq_\Gamma \pi \ \land \ \Gamma \vdash s @ t \sim t' \implies \Gamma \vdash t' : \tau. \)

This theorem is an elaboration of Theorem 1 for many-sorted type-preserving strategies. The first property is strengthened since we now claim UOT for strategies. This becomes possible because the previously “overloaded” combinators \( \epsilon \) and \( \delta \) are now of type \( \text{TP} \). Further, we need to rephrase what it means that a strategy type adheres to the scheme of type preservation. Also, the formulation of subject reduction needs
to be upgraded to take TP into account.

Proof 3

(1) (a) We prove this property by induction on \( s \) in \( \Gamma \vdash s : \pi \). Base cases: UOT for rewrite rules is implied by UOT for terms. Congruences meet UOT because of the requirement for well-formed contexts. The two base cases for \( \epsilon \) and \( \delta \) trivially satisfy UOT by simple inspection of the type position in the rules \([\text{a}]\) and \([\text{tail}]\). Induction step: UOT for \( \neg s \) is implied by the IH and by the fact that the auxiliary judgement for negatable types encodes a function from the argument type to the type of the negated strategy. UOT for \( s_1; s_2 \) and \( s_1 + s_2 \) is implied by the IH. In both cases, the type of the compound strategy coincides with the type of the arguments (cf. \([\text{comp.1}]\), \([\text{comp.2}]\), \([\text{seq}]\), \([\text{choice}]\)). UOT for \( f(s_1, \ldots, s_n) \) follows from well-formedness of contexts. As for, \( \sqcap(s) \), \( \Diamond(s) \), the property can be inferred by inspection of the type position in the corresponding typing rules. As for \( s \triangleleft \pi \) and \( s \triangleright \pi \), UOT follows from the fact that the specified \( \pi \) directly constitutes the type of the extended or restricted strategy.

(b) We prove this property by induction on \( s \) in \( \Gamma \vdash s : \pi \). We only need to cover the cases which were updated or newly introduced in the migration from \( S'_tp \) to \( S'_TP \). Base cases: The types of \( \epsilon \) and \( \delta \) are uniquely defined as TP, and hence, they trivially adhere to the required scheme. Induction step: As an aside, we hardly have to employ the IH for the compound strategies. The type of \( \Box(s) \) and \( \Diamond(s) \) is defined as TP. The forms \( \neg s \) and \( s_1; s_2 \) are type-preserving in \( S'_tp \). There are still type-preserving in \( S'_TP \) since the added rules for negatable and composable types only admit TP as an additional possible result type (cf. \([\text{negt.2}]\) and \([\text{comp.2}]\)). The property holds for choice because the arguments are type preserving by the IH, and the result type of choice coincides with the type of the arguments (cf. \([\text{choice}]\)). The type \( \pi \) in \( s \triangleleft \pi \), and hence the type of \( s \triangleleft \pi \) itself, must coincide with TP since this is the only possible strategy type admitted as the right argument of \( \prec_\pi \) in \( S'_TP \) (cf. \([\text{extend}]\) and \([\text{less.1}]\)). Dually, the type \( \pi \) in \( s \triangleright \pi \), and hence the type of \( s \triangleright \pi \) itself, must coincide with a type of the form \( \tau \to \tau \) because this is the only possible form admitted as the left argument of \( \prec_\pi \) in \( S'_TP \) (cf. \([\text{restrict}]\) and \([\text{less.1}]\)).

(2) The property follows immediately from
- UOT for terms,
- UOT for strategies, and
- the fact that TP is a “well-defined generic type”, that is, fixing the term type \( \tau \) processed by a generic strategy, the result type of strategy application is determined, in fact, it is \( \tau \) in the case of TP.

(3) Note that we only deal with type-preserving strategies which allows us to adopt Proof 1 to a large extent. Of course, the side condition for the employment of the IH has to be revised. That is, the IH can be employed for a premise \( s_i \oplus t_i \leadsto t'_i \) and corresponding types \( \tau_i \) and \( \pi_i \) if we can prove the following
**side condition:**

\[ \Gamma \vdash s : \pi \land \Gamma \vdash t : \tau \land \tau \to \tau \preceq_{\Gamma} \pi \land \ldots \]

implies \[ \Gamma \vdash s_i : \pi_i \land \Gamma \vdash t_i : \tau_i \land \tau_i \to \tau_i \preceq_{\Gamma} \pi_i \]

**Base cases:** Proof 1 is still intact as for rewrite rules and congruences for constants since these forms of strategies were completely preserved in \( S_{TP}' \). The strategy \( \epsilon \) is said to be generically type-preserving according to \([\text{id}]\) while it was “overloaded” before. Reduction of \( \epsilon \circ t \) yields \( t \). Hence, reduction of \( \epsilon \) is type-preserving. Induction step: The property holds for negation because \([\text{neg}^+]\) only admits the input term as proper term reduct. The proof for sequential composition can be precisely repeated as in Proof 1 with the only exception that we now have to consider two cases according to \([\text{comp.1}]\) and \([\text{comp.2}]\).

In both cases, the type of \( s_1, s_2 \) and \( s_1; s_2 \) coincide, and the types adhere to the scheme of type preservation. This is all what the original proof relied on.

Proof 1 also remains valid as for choice and congruences. As for the traversal combinators, the property follows from the IH, and the fact that the shape of the processed term is preserved. The IH is enabled for any \( s \circ t_i \sim t'_i \) because the strategy \( s \) in \( \Box(s) \) and \( \Diamond(s) \) is required to be of type TP, and hence it can cope with any term. The interesting case is \( s \ll \pi \circ t \) where we assume that \( t \) is of type \( \tau \). We want to employ the IH for the premise \( \Gamma \vdash s \circ t \sim t' \) in \([\text{extend}^+]\).

Hence we are obliged to show that the type \( \pi' \) of \( s \) covers \( \tau \). This obligation is precisely captured by the premise \( \exists \tau'. \tau \to \tau' \preceq_{\Gamma} \pi' \) in \([\text{extend}^+]\). Thus the IH is enabled, and subject reduction holds. As for \( s \gg \pi \), we directly resort to \( s \). The IH for the reduction of \( s \) is trivially implied since \( \tau \to \tau \preceq_{\Gamma} \pi \) implies \( \tau \to \tau \preceq_{\Gamma} \pi' \) for \( \pi \ll_{\Gamma} \pi' \) (cf. \([\text{restrict}]\)) by transitivity of \( \preceq_{\Gamma} \).

**Implicit restriction** While extension required a dedicated combinator for the reasons we explained earlier, we do not need to insist on explicit restriction. Implicit restriction is desirable because otherwise a programmer needs to point out a specific type whenever a generic strategy is applied in a many-sorted context. Implicit restriction is feasible because restriction has no impact on the reduction semantics of a strategy. Let us stress that implicit restriction does not harm type safety in any way. In the worst case, implicit restriction might lead to accidentally many-sorted strategies. However, such accidents will not go unnoticed. If we attempt to apply the intentionally generic strategy in a generic context or to assign a generic type to it, then the type system will refuse such attempts.

**Example 16** Consider the strategy \( \text{NAT} \) which was defined in Figure 6. It involves a congruence \( \text{succ}(\epsilon) \), where \( \epsilon \) is supposed to be applied to a natural. In \( S_{TP}' \), we have to rephrase the congruence as \( \text{succ}(\epsilon \gg \text{Nat}) \). In a calculus with implicit restriction, \( \text{succ}(\epsilon) \) can be retained.
Composable types: $\Gamma \vdash \pi_1 ; \pi_2 \rightsquigarrow \pi$

Well-typedness of strategies:

- $\Gamma \vdash \pi \Rightarrow \pi' \quad \text{[comp.3]}$
  $\Gamma \vdash s_1 : \pi_1$
  $\land \Gamma \vdash s_2 : \pi_2$

- $\Gamma \vdash \pi \Rightarrow \pi' \Rightarrow \pi' \quad \text{[comp.4]}$
  $\land \Gamma \vdash \pi_1 \sqcap \pi_2 \rightsquigarrow \pi$
  $\Gamma \vdash s_1 + s_2 : \pi$

Well-typedness of strategy applications:

- $\Gamma \vdash s : \pi \land \Gamma \vdash t : \tau$
  $\land \exists \tau'. \pi \Rightarrow \tau' \preceq_\Gamma \pi$
  $\Gamma \vdash s @ t : \tau'$

Implicit restriction made explicit

Let us first consider one form of implicit restriction where we update all typing rules which have to do with potentially many-sorted contexts. Basically, we want to state that the type of a compound strategy $s_1; s_2$ or $s_1 + s_2$ is dictated by a many-sorted argument (if any). As for congruences, we want to state that generic strategies can be used as argument strategies. Finally, we also need to relax strategy application so that we can apply a generic strategy without further precautions to any term. This approach to implicit restriction is formalised in Figure 17. The updated rule [apply] for well-typedness of strategy applications states that a strategy $s$ of type $\pi$ can be applied to a term $t$ of type $\tau$, if the domain of $\pi$ covers $\tau$. As an aside, note that the definition is sufficiently general to cope with type-changing strategies. As for $\cdot; \cdot$, we relax the definition of composable types to cover composition of a many-sorted type and the generic type $\text{TP}$ in both possible orders (cf. [comp.3] and [comp.4]). As for $\cdot +$, we do not insist on equal argument types anymore, but we employ an auxiliary judgement $\Gamma \vdash \pi_1 \sqcap \pi_2 \rightsquigarrow \pi$ for the greatest lower bound w.r.t. $\preceq_\Gamma$ (cf. [choice]). Finally, we relax the argument types for congruences via the $\preceq_\Gamma$ relation.

Unicity of typing vs. principal types

The value of the refinement in Figure 17 is that we are very precise about where restriction might be needed. Moreover, we can maintain UOT for this system. A problem with the above approach is that several typing rules need to be refined to become aware of $\preceq_\Gamma$. There is a simpler approach to implicit restriction. We can include a typing rule which models that a generic strategy can also be regarded as a many-sorted strategy. The rule is shown in Fig-
Well-typedness of strategies

\[ \Gamma \vdash s : \pi \]

\[ \Gamma \vdash s : \pi \land \pi' \preceq_{\Gamma} \pi \]

\[ \Gamma \vdash s : \pi' \]

Fig. 18. Implicit restriction relying on principal types

This new approach implies that UOT does not hold anymore for strategies. However, one can easily see that the multiple types arising from implicit restriction are closed under \( \preceq_{\Gamma} \). Thus, one can safely replace UOT by the existence of a principal type. We simply regard the most generic type of a strategy as its principal type. Note that UOT still holds for strategy application.

4.4 Strategies of type \( \text{TU}(\cdot) \)

Combination In Figure 19, the reduction semantics of the combinators for type-unifying traversal is defined. Thereby, we complete the combinator suite of \( S'_{\gamma} \). For brevity, we omit the typing context which might be needed for the type-dependent reduction of \( \cdot \sqcup \cdot \). The combinator \( \sqcup' (\cdot) \) is defined in [\( \text{red}^+ \)]. Every child \( t_i \) is processed, and pairwise composition is used to compute a final term \( t'_{2n-1} \) from all the intermediate results. Note that pairwise composition is performed from left to right. This is a kind of arbitrary choice at this point, and we will come back to this issue in Section 5.5. Note also that the reduction semantics for \( \sqcup (\cdot) \) does not specify a total temporal order on how pairwise composition is intertwined with processing the children. There are at least two sensible operational readings of [\( \text{red}^+ \)]. Either we first process all children, and then we perform pairwise composition, or we immediately perform pairwise composition whenever a new child has been processed. The negative rules for \( \sqcup (\cdot) \) are similar to those of \( \square (\cdot) \) and \( \diamond (\cdot) \). A constant cannot be reduced as in the case of \( \diamond (\cdot) \) (cf. [\( \text{red}^- 1 \)]. Reduction fails if any of the children cannot be processed as in the case of \( \square (\cdot) \) (cf. [\( \text{red}^- 2 \)]. There is also the possibility that pairwise composition fails (cf. [\( \text{red}^- 3 \) and [\( \text{red}^- 4 \)].

Selection of a child is more easily explained. The overall scheme regarding both the positive rule and the two negative rules for \( \sharp (\cdot) \) is very similar to the combinator \( \Diamond (\cdot) \). The combinator \( \sharp (\cdot) \) differs from \( \Diamond (\cdot) \) only in that the shape of the input term is not preserved. Recall that in the reduct of an application of \( \Diamond (\cdot) \), the outermost function symbol and all non-processed children carry over from the input term. Instead, selection simply yields the processed child. As in the case of \( \Diamond (\cdot) \), we cannot process constants ([\( \text{sel}^- 1 \)], and we also need to fail if none of the children admits selection (cf. [\( \text{sel}^- 2 \)].

Let us finally consider the auxiliary combinators \( \bot \) and \( s_1 \parallel s_2 \). The combinator \( \bot \)
Syntax

\[ s ::= \cdots | \bigcirc s | \#(s) | \perp | s \parallel s \]

Reduction of strategy applications

Positive rules

\[ s @ t_1 \leadsto t'_1 \]
\[ \cdots \]
\[ s @ t_n \leadsto t'_n \]
\[ s_0 @ \langle t'_1, t'_2 \rangle \leadsto t'_{n+1} \]
\[ s_0 @ \langle t'_n+1, t'_3 \rangle \leadsto t'_{n+2} \]
\[ \cdots \]
\[ s_0 @ \langle t'_{2n-2}, t'_{2n} \rangle \leadsto t'_{2n-1} \] [red^+]

\[ \exists i \in \{1, \ldots, n\}. s @ t_i \leadsto t'_i \] [sel^+]

\[ \#(s) @ f(t_1, \ldots, t_n) \leadsto t'_i \] [void^+]

\[ \perp @ t \leadsto \langle \rangle \] [spawn^+]

\[ s_1 @ t \leadsto t_1 \land s_2 @ t \leadsto t_2 \]
\[ s_1 \parallel s_2 @ t \leadsto \langle t_1, t_2 \rangle \] [red^-3]

\[ s @ t_n \leadsto t'_n \]
\[ s_0 @ \langle t_1, t_2 \rangle \leadsto \uparrow \]

\[ \bigcirc s @ f(t_1, t_2, \ldots, t_n) \leadsto \uparrow \]

\[ \exists i \in \{3, \ldots, n\}. ( \]
\[ s @ t_1 \leadsto t'_1 \]
\[ \cdots \]
\[ s @ t_n \leadsto t'_n \]
\[ s_0 @ \langle t_1, t_2 \rangle \leadsto t'_{n+1} \]
\[ \cdots \]

\[ \exists i \in \{n+3, \ldots, n\}. ( \]
\[ s @ t_{n+i-2} \leadsto t'_{i-2} \]
\[ s_0 @ \langle t_1, t_2 \rangle \leadsto \uparrow \] [red^-4]

\[ \bigcirc s @ f(t_1, t_2, \ldots, t_n) \leadsto \uparrow \]

\[ s @ t_1 \leadsto \uparrow \]
\[ \cdots \]
\[ s @ t_n \leadsto \uparrow \] [sel^-2]

\[ \#(s) @ c \leadsto \uparrow \] [sel^-1]

\[ \bigcirc s @ f(t_1, \ldots, t_n) \leadsto \uparrow \] [spawn^-]

\[ s_1 @ t \leadsto \uparrow \lor s_2 @ t \leadsto \uparrow \]

\[ s_1 \parallel s_2 @ t \leadsto \uparrow \]

Fig. 19. Type-unifying combinators
Syntax

\[ \gamma ::= \cdots | \text{TU}(\tau) \]

Well-formedness of strategy types

\[
\begin{align*}
\Gamma \vdash \tau & \quad \text{[pi.3]} \\
\Gamma \vdash \text{TU}(\tau)
\end{align*}
\]

Well-typedness of strategies

\[
\begin{align*}
\Gamma \vdash s : \pi & \quad \text{[red]} \\
\Gamma \vdash s : \text{TU}(\tau)
\end{align*}
\]

Genericity relation

\[
\begin{align*}
\pi \prec_\Gamma \pi' & \quad \text{[less.2]} \\
\Gamma \vdash \tau \land \Gamma \vdash \tau' & \quad \text{[spawn]}
\end{align*}
\]

Composable types

\[
\begin{align*}
\Gamma \vdash \pi_1 ; \pi_2 \leadsto \pi & \quad \text{[comp.5]} \\
\Gamma \vdash \tau \land \Gamma \vdash \tau' & \quad \text{[spawn]}
\end{align*}
\]

\[ \text{Fig. 20. } \text{TU}(\cdot) — \text{The type of generic type-unifying strategies} \]

simply accepts any term, and reduction yields the empty tuple \( \langle \rangle \). The combinator is in a sense similar to \( \epsilon \) as it succeeds for every term. However, the term reduct is of a trivial type, namely \( \langle \rangle \) regardless of the type of the input term. The strategy \( s_1 \parallel s_2 \) applies both strategies to the input term, and the intermediate results are paired (cf. [\text{spawn}^+]). If either of the strategy applications fails, \( s_1 \parallel s_2 \) fails, too (cf. [\text{spawn}^-]). Note that one could attempt to describe the behaviour underlying \( \bot \) and \( \cdot \parallel \cdot \) by the following strategies that employ rewrite rules:

\[
\begin{align*}
\text{VOID} & = X \rightarrow \langle \rangle \\
\text{SPAWN}(\nu_1, \nu_2) & = X \rightarrow \langle Y_1, Y_2 \rangle \text{ where } Y_1 = \nu_1 \odot X \text{ where } Y_2 = \nu_2 \odot X
\end{align*}
\]

However, the above rewrite rules and the pattern variables would have to be generically typed. This is in conflict with the design decisions that were postulated by us for \( S'_\gamma \). The types of rewrite rules in \( S'_\gamma \) are required to be many-sorted. All genericity should arise from distinguished primitive combinators. Recall that these requirements are meant to support a clean separation of genericity and specificity, a simple formalisation of \( S'_\gamma \), and a simple implementation of the calculus.
The generic type $\text{TU}(\cdot)$ The formalisation of the generic type (constructor) $\text{TU}(\cdot)$ is presented in Figure 20. We basically have to perform the same steps as we discussed for $\text{TP}$. Firstly, well-formedness of $\text{TU}(\cdot)$ is defined (cf. [pi.3]). Secondly, the type scheme underlying $\text{TU}(\tau)$ is defined (cf. [less.2]). Thirdly, the auxiliary judgement for sequential composition is updated (cf. [comp.5]) to describe how $\text{TU}(\cdot)$ is promoted. Consider a sequential composition $s_1; s_2$ where $s_1$ is of type $\text{TU}(\tau)$, and $s_2$ is of type $\tau \rightarrow \tau'$. The result is of type $\text{TU}(\tau')$. Note that a many-sorted strategy followed by a type-unifying strategy or the sequential composition of two type-unifying strategies do not amount to a generic strategy. Still we could include these constellations in order to facilitate implicit restriction. The typing rules for the type-unifying strategy combinators are easily explained. Reduction of all children to a type $\tau$ via $\boxtimes^\circ(s)$ requires $s_0$ to be able to map a pair of type $\langle \tau, \tau \rangle$ to a value of type $\tau$ in the sense of pairwise composition, and the strategy $s$ for processing the children has to be type-unifying w.r.t. the same $\tau$ (cf. [red]). The typing rule for the combinator $\sharp(\cdot)$ directly states that the combinator is a transformer on type-unifying strategies (cf. [sel]). The typing rule for $\downarrow(\cdot)$ states that every type is mapped to the most trivial type (cf. [void]). Finally, $\cdot || \cdot$ takes two type-unifying strategies, and produces another type-unifying strategy. If $\text{TU}(\tau_1)$ and $\text{TU}(\tau_2)$ are the types of the argument strategies in $s_1 || s_2$, then $\text{TU}(\langle \tau_1, \tau_2 \rangle)$ is the type of the resulting strategy (cf. [spawn]).

Assembly of $S'_{\gamma}$ Let us compose the ultimate calculus $S'_{\gamma}$. It accomplishes both type-preserving and type-changing strategies. Furthermore, tuples are supported. Ultimately, the generic types $\text{TP}$ and $\text{TU}(\cdot)$ are enabled. We favour implicit restriction for $S'_{\gamma}$. For convenience, we summarise all ingredients for $S'_{\gamma}$:

- The basic calculus $S'_0$ (cf. Figure 8)
- Many-sorted type-preserving strategies (cf. Figure 10)
- Many-sorted type-changing strategies (cf. Figure 11)
- Polyadic strategies (cf. Figure 12)
- The combinators $\boxtimes(\cdot)$ and $\diamond(\cdot)$ (cf. Figure 13)
- The generic type $\text{TP}$ (cf. Figure 14)
- The combinators $\bigcirc(\cdot)$, $\#(\cdot)$, $\downarrow$ and $\cdot || \cdot$ (cf. Figure 19)
- The generic type $\text{TU}(\cdot)$ (cf. Figure 20)
- Strategy extension (cf. Figure 15)
- Implicit restriction (cf. Figure 17)

Theorem 4 The calculus $S'_{\gamma}$ obeys the following properties:

1. Strategies satisfy UOT, i.e., ... (cf. Theorem 3).
2. Strategy applications satisfy UOT, i.e., ... (cf. Theorem 1).
3. Reduction of strategy applications satisfies subject reduction, i.e., for all well-formed contexts $\Gamma$, strategies $s$, terms $t$, $t'$, term types $\tau$, $\tau'$, and strategy types $\pi$:  

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We omit the proof because it is a simple combination of the ideas from Proof 2 and Proof 3. In the former proof, we generalised the scheme for many-sorted type-preserving strategies from Proof 1 to cope with type-changing strategies as type-unifying strategies are, too. In the latter, we generalised Proof 1 in a different dimension, namely to cope with generic strategies as type-unifying strategies are, too. It is easy to cope with implicit restriction instead of explicit restriction in \( S'_{\text{TP}} \), neither does the introduction of tuples pose any challenge.

5 Sophistication

In the previous two sections we studied the reduction semantics and the type system for all the \( S'_{\gamma} \) primitives. In this section, we want to complement this development with a few supplementary concepts. Firstly, we will consider a straightforward abstraction mechanism for strategy combinators, that is, strategy definitions. Secondly, we will refine the model underlying the formalisation of \( S'_{\gamma} \) to obtain a reduction semantics which does not employ typing judgements in the reduction semantics anymore. Thirdly, we describe a form of overloaded strategies, that is, strategies which are applicable to terms of several types. Fourthly, we introduce some syntactic sugar to complement strategy extension by a sometimes more convenient approach to the inhabitation of generic types, namely asymmetric type-dependent choice. Finally, we will discuss the potential for more general or additional traversal combinators.

5.1 Strategic programs

The syntax and semantics of strategic programs is shown in Figure 21. A strategic program is of the form \( \Gamma \Delta s \). Here \( \Gamma \) corresponds to type declarations for the program, \( \Delta \) is a list of strategy definitions, and \( s \) is the main expression of the program. A strategy definition is of the form \( \varphi(\nu_1, \ldots, \nu_n) = s \) where \( \nu_1, \ldots, \nu_n \) are the formal parameters. The parentheses are omitted if \( \varphi \) has no parameters. We assume that the RHS \( s \) does not contain other strategy variables than \( \nu_1, \ldots, \nu_n \). Furthermore, we assume \( \alpha \)-conversion for the substitution of strategy variables. In the judgement for the reduction of strategy applications, we propagate the strategy definitions as context parameter \( \Delta \) (cf. \([\text{prog}^{+/-}]\)) so that occurrences of strategy combinators can be expanded accordingly (cf. \([\text{comb}^{+/-}]\)). Note that the reduction judgement for strategy applications carries \( \Gamma \) in the context in order to enable strategy extension.

To consider well-formedness and well-typedness of strategic programs we need to
**Syntax**

\[
p ::= \Gamma \Delta s \quad \text{(Programs)}
\]

\[
\nu
\]

\[
\nu ::= \cdots | \varphi : \pi \times \cdots \times \pi \rightarrow \pi | \nu : \pi \quad \text{(Strategy variables)}
\]

\[
\Gamma ::= \cdots | \nu
\]

\[
\Gamma ::= \cdots | \varphi : \pi \times \cdots \times \pi \rightarrow \pi | \nu : \pi \quad \text{(Contexts)}
\]

\[
\Delta ::= \emptyset | \Delta, \Delta | \varphi(\nu, \cdots, \nu) = s \quad \text{(Definitions)}
\]

\[
s ::= \cdots | \nu | \varphi(s, \ldots, s) \quad \text{(Strategies)}
\]

**Reduction of programs**

\[
\frac{\Gamma, \Delta \vdash s \circ t \leadsto r}{\Gamma \Delta s \circ t \leadsto r} \quad \text{[prog+/−]}
\]

\[
\frac{\varphi(\nu_1, \cdots, \nu_n) = s \in \Delta}{\varphi(\nu_1, \cdots, \nu_n) = s \in \Delta}
\]

\[
\wedge \Delta' = s_1 \mapsto s_1, \ldots, s_n \mapsto s_n \quad \text{[comb+/−]}
\]

**Reduction of strategy applications**

\[
\frac{\Gamma, \Delta \vdash s \circ t \leadsto r}{\Gamma, \Delta \vdash s \circ t \leadsto r}
\]

\[
\frac{\varphi(\nu_1, \cdots, \nu_n) = s \in \Delta}{\varphi(\nu_1, \cdots, \nu_n) = s \in \Delta}
\]

\[
\wedge \Gamma, \Delta \vdash s' \circ t \leadsto r \quad \text{[comb+/−]}
\]

**Fig. 21. Strategic programs: syntax and reduction semantics**

extend the grammar for contexts \(\Gamma\) as it was already indicated in Figure 21. Contexts may contain type declarations for strategy combinators and types of strategy variables. A strategic program is well-formed if the strategy definitions and the main expression of a program are well-typed (cf. \([\text{prog}]\)). A strategy definition is well-typed if the body can be shown to have the declared result type of the combinator while assuming the appropriate types of the formal parameters in the context (cf. \([\text{def.3}]\)). When a strategy variable is encountered by the well-typedness judgement, its type is determined via the context (cf. \([\text{arg}]\)). An application of a combinator is well-typed if the types of the actual parameters are equal to the types of the formal parameters (cf. \([\text{comb}]\)). We could also elaborate the latter typing rule to facilitate implicit restriction. This would allow us to place generic strategies as actual parameters on many-sorted parameter positions of strategy combinators.

**Type-parameterised strategy definitions** Let us also enable type-parameterised strategy definitions. In Figure 23, we give typing rules to cope with type parameters in strategy definitions and combinator applications. The formalisation is pretty standard. We assume \(\alpha\)-conversion for the substitution of type variables. Term-type variables are regarded as another form of a term type. The extension of the grammar rule for \(\Gamma\) details that types of strategy combinators might contain type variables that are quantified at the top level. Type variables are scoped by the corresponding strategy definition (cf. \([\text{def.4}]\)). If the well-formedness judgements for types encounter a term type variable, it has to be in the context (cf. \([\text{tau.4}]\)). The
Well-typedness of programs

\[ \Gamma \vdash p : \pi \]

Well-typedness of strategies

\[ \Gamma \vdash s : \pi \]

Well-typedness of strategy definitions

\[ \Gamma \vdash \Delta \land \Gamma \vdash s : \pi \]

\[ \Gamma \vdash \Delta \land \Gamma \vdash s : \pi \]

Well-typedness of strategies

\[ \varphi : \pi_1 \times \cdots \times \pi_n \rightarrow \pi_0 \in \Gamma \land \Gamma \vdash s_1 : \pi_1 \land \cdots \land \Gamma \vdash s_n : \pi_n \]

\[ \Gamma \vdash \varphi(s_1, \ldots, s_n) : \pi_0 \]

Application of a combinator \( \varphi \) involves type application, namely substitution of the type variables by the actual types (cf. [comb–forall]). For brevity, we do not refine the reduction semantics from Figure 21.

5.2 \( \Gamma \)-free strategy extension

When we introduced strategy extension, we encountered a complication regarding the reduction semantics. In order to define the type-safe application of a many-sorted strategy \( s \) in a generic context, we have to perform a run-time comparison of the type of the given term and the type of \( s \). To this end, we added the typing context \( \Gamma \) to the judgement for the reduction of strategy applications, and typing judgements were placed as premises in the rule for \( \cdot \triangleleft \cdot \) (cf. Figure 15). We would like to obtain a form of semantics where typing and reduction judgements are strictly separated. We will employ an intermediary static elaboration judgement to annotate strategies accordingly. Furthermore, we assume that terms are tagged by their types. The resulting reduction semantics is better geared towards implementation.
Syntax

\[\begin{align*}
\alpha & \quad (\text{Term-type variables}) \\
\tau & ::= \cdots | \alpha \quad (\text{Term types}) \\
\Gamma & ::= \cdots | \varphi \forall \alpha, \ldots, \alpha. \pi \times \cdots \times \pi \to \pi \quad (\text{Contexts}) \\
\Delta & ::= \cdots | \varphi[\alpha, \ldots, \alpha](\nu_1, \ldots, \nu) = s \quad (\text{Definitions}) \\
s & ::= \cdots | \varphi[\tau, \ldots, \tau](s, \ldots, s) \quad (\text{Strategies})
\end{align*}\]

Well-formedness of term types

\[
\frac{\alpha \in \Gamma}{\Gamma \vdash \alpha} \quad [\text{tau.4}]
\]

Well-typedness of strategy definitions

\[
\frac{\varphi : \forall \alpha_1, \ldots, \alpha_m. \pi_1 \times \cdots \times \pi_n \to \pi_0 \in \Gamma \\
\land \Gamma, \nu_1 : \pi_1, \ldots, \nu_n : \pi_n, \alpha_1, \ldots, \alpha_m \vdash s : \pi_0}{\Gamma \vdash \varphi[\alpha_1, \ldots, \alpha_m](\nu_1, \ldots, \nu_n) = s} \quad [\text{def.4}]
\]

Well-typedness of strategies

\[
\frac{\varphi : \forall \alpha_1, \ldots, \alpha_m. \pi_1 \times \cdots \times \pi_n \to \pi_0 \in \Gamma \\
\land \Gamma \vdash \tau_1 \land \cdots \land \Gamma \vdash \tau_m \\
\land \Gamma \vdash s_1 : \pi_1\{\alpha_1 \mapsto \tau_1, \ldots, \alpha_m \mapsto \tau_m\} \\
\land \cdots \\
\land \Gamma \vdash s_n : \pi_n\{\alpha_1 \mapsto \tau_1, \ldots, \alpha_m \mapsto \tau_m\}}{\Gamma \vdash \varphi[\tau_1, \ldots, \tau_m](s_1, \ldots, s_n) : \pi_0\{\alpha_1 \mapsto \tau_1, \ldots, \alpha_m \mapsto \tau_m\}} \quad [\text{comb–forall}]
\]

Fig. 23. Type-parametrised strategy definitions

Static elaboration  So far, we only considered well-typedness and reduction judgements. We want to refine the model for the formalisation of \(S'_\gamma\) to include a static elaboration judgement of the following form:

\[
\Gamma \vdash s \leadsto s'
\]
Static elaboration of strategies

\[ \Gamma \vdash \text{rule'~} t_l \rightarrow t_r \leadsto t_l \rightarrow t_r \]
\[ \Gamma \vdash \text{id'~} \epsilon \leadsto \epsilon \]
\[ \Gamma \vdash \text{fail'~} \delta \leadsto \delta \]
\[ \Gamma \vdash \text{neg'~} \neg s \leadsto \neg s' \]
\[ \Gamma \vdash \text{seq'~} s_1; s_2 \leadsto s_1'; s_2' \]
\[ \Gamma \vdash \text{choice'~} s_1 + s_2 \leadsto s_1' + s_2' \]
\[ \Gamma \vdash \text{cong'~} 1 \]
\[ \Gamma \vdash \text{cong'~} 2 \]
Syntax

\[ s ::= \cdots \mid s : \pi \text{ (Strategies)} \]

\[ t ::= \cdots \mid t : \tau \text{ (Terms)} \]

Static elaboration

of strategies

\[ \Gamma \vdash s \leadsto s' \]

\[ \Gamma \vdash s \triangleleft \pi \leadsto s' : \pi' \triangleleft \pi \]

Reduction

of strategy applications

\[ s \circ t \leadsto r \]

Positive rule

\[ \frac{s \circ t : \tau \leadsto t'} {s : \tau \rightarrow \tau' \triangleleft \circ t : \tau \leadsto t'} \]

Negative rules

\[ \frac{s \circ t : \tau \leadsto \uparrow} {s : \tau \rightarrow \tau' \triangleleft \circ t : \tau \leadsto \uparrow} \]

\[ \tau \neq \tau'' \]

\[ s : \tau \rightarrow \tau' \triangleleft \circ t : \tau'' \leadsto \uparrow \]

Fig. 25. Strategy extension relying on type tags

(1) The given strategy \( s \) is checked to be well-typed.
(2) \( s \) is elaborated resulting in a strategy \( s' \).
(3) Given a suitable term \( t \), the strategy \( s' \) is applied to \( t \) to derive a reduct.

These phases obviously map nicely to an implementational model where type checking and elaboration is done once and for all statically, that is, without insisting on an input term. In general, static elaboration might be type-dependent, that is, the typing context \( \Gamma \) is part of the elaboration judgement as in the case of the well-formedness and well-typedness judgements. In Figure 24, we initiate the general scheme of static elaboration. We give trivial rules for all combinators of the basic calculus \( S^0 \) such that we descend into compound strategy expressions. So far, the judgement encodes the identity function on strategy expressions. Below, we will provide a special rule for the elaboration of applications of \( \triangleleft \cdot \).

Type tags In order to eliminate the typing premise for the extended strategy in the reduction semantics of \( \triangleleft \cdot \) we replace strategy expressions of the form \( s \triangleleft \pi \) by \( s : \pi' \triangleleft \pi \) where \( \pi' \) denotes the actual type of \( s \). Here, we reanimate the notation of type-annotated strategies that was already proposed earlier. Since the type is captured in the elaborated strategy expression, the type of \( s \) does not need to be determined during reduction anymore. Furthermore, we assume that terms are
Reduction of strategy applications

\[
\begin{align*}
\text{c} @ c : \tau & \sim c : \tau \\
s_1 @ t_1 & \sim t_1' \\
\cdots \\
\land \quad \land \\
& \quad \land \quad \land \\
s_n @ t_n & \sim t_n'
\end{align*}
\]

\[f(s_1, \ldots, s_n) @ f(t_1, \ldots, t_n) : \tau \sim f(t_1', \ldots, t_n') : \tau\]

\[\Box(s) @ c : \tau \sim c : \tau\]

\[\text{all}^+\]

\[\Box(s) @ f(t_1, \ldots, t_n) : \tau \sim f(t_1', \ldots, t_n') : \tau\]

\[\text{all}^+\]

Fig. 26. Refined reduction semantics to cope with tagged terms

tagged by their sorts. Obviously, this assumption is useful to also get rid of the type judgement for the term \(t\) in the reduction semantics for \(\cdots \in \pi \circledast t\). Thus, the original type dependency reduces to a simple comparison of type tags of the extended strategy and the term at hand. The rules for static elaboration and the new reduction semantics of strategy extension is shown in Figure 25. The elaboration rule \([\text{extend}^-]\) deviates from the trivial default scheme of static elaboration by actually adding the inferred type as a tag. The deduction rule \([\text{extend}^+]\) defines the new reduction semantics of strategy extension.

**Tagged terms** The assumption that terms are tagged by a type has actually two implications which need to be treated carefully. Firstly, we should better assume that terms are consistently tagged at all levels. This means that the terms constituting a rewrite rule have to be tagged, too. Secondly, we need to make sure that all reduction rules appropriately deal with tagged terms. In fact, we need to update the reduction semantics of congruences and generic traversal because they are not prepared to deal with tags. In Figure 26, we illustrate the new style of traversal. For brevity, we only show the positive rules for congruences and for the combinator \(\square(\cdot)\).
5.3 Overloaded strategies

We want to consider an intermediate form of genericity, namely overloaded strategies. Overloading means that we can cope with strategies which are applicable to terms of a number of sorts. We introduce a designated combinator $\cdot \& \cdot$ to gather strategies of different types in an overloaded strategy. The combinator $\cdot \& \cdot$ is type-dependent in the same way as strategy extension via $\cdot < \cdot$. In fact, we say that $\cdot \& \cdot$ performs symmetric type-dependent choice. The type of the ultimate term decides which side of the choice is attempted. Hence, this choice is not left- or right-biased, nor is it controlled by success and failure. We use the notation $\cdot \& \cdot$ for the construction of both overloaded strategies and the corresponding strategy types.

**Example 17** Consider the following constructors for naturals and integers:

\[
\begin{align*}
\text{one} &: \text{NatOne} \\
\text{succ} &: \text{NatOne} \to \text{NatOne} \\
\text{zero} &: \text{NatZero} \\
\text{notzero} &: \text{NatOne} \to \text{NatZero} \\
\text{positive} &: \text{NatZero} \to \text{Int} \\
\text{negative} &: \text{NatOne} \to \text{Int}
\end{align*}
\]

NatZero includes 0, whereas NatOne starts with 1. Integers are constructed via two branches, one for positive integers including zero, and another for negative integers. We use NO as stem of variables of sort NatOne. Let us define two overloaded strategies INC and DEC which are capable of incrementing and decrementing terms of the three above sorts:

\[
\begin{align*}
\text{INC} &: \text{NatOne} \to \text{NatOne} \& \text{NatZero} \to \text{NatZero} \& \text{Int} \to \text{Int} \\
\text{INC} &= \text{NO} \to \text{succ(NO)} \\
&\quad \& \text{zero} \to \text{notzero(one)} + \text{notzero(INC)} \\
&\quad \& \text{positive(INC)} + \text{negative(DEC)} + \text{negative(one)} \to \text{positive(zero)} \\
\text{DEC} &: \text{NatOne} \to \text{NatOne} \& \text{NatZero} \to \text{NatZero} \& \text{Int} \to \text{Int} \\
\text{DEC} &= \text{succ(NO)} \to \text{NO} \\
&\quad \& \text{notzero(one)} \to \text{zero} + \text{notzero(DEC)} \\
&\quad \& \text{positive(DEC)} + \text{negative(INC)} + \text{positive(zero)} \to \text{negative(one)}
\end{align*}
\]

The strategies are defined via symmetric type-dependent choice with three cases, one for each sort. Otherwise, the functionality to increment and decrement is defined by rewrite rules or in terms of congruences on the appropriate constructors. As an aside, it is necessary to assume implicit restriction for overloaded strategies in order to claim well-typedness for the above definitions. This is because the using
Syntax

\[ \pi = \cdots | \pi \& \pi \]
\[ s = \cdots | s \& s \]

Well-formedness of strategy types

\[ \Gamma \vdash \pi \]
\[ \Gamma \vdash \pi_1 \land \text{DOM}(\pi_1) \to \tau_{S_1} \]
\[ \land \Gamma \vdash \pi_2 \land \text{DOM}(\pi_2) \to \tau_{S_2} \]
\[ \land \tau_{S_1} \cap \tau_{S_2} = \emptyset \]
\[ \Gamma \vdash \pi_1 \land \pi_2 \quad \text{[pi.4]} \]

Domains of strategies

\[ \text{DOM}(\pi) \to \tau_S \quad \text{[dom.1]} \]
\[ \text{DOM}(\pi_1) \to \tau_{S_1} \]
\[ \land \text{DOM}(\pi_2) \to \tau_{S_2} \quad \text{[dom.2]} \]
\[ \text{DOM}(\pi_1 \land \pi_2) \to \tau_{S_1} \cup \tau_{S_2} \quad \text{[dom.2]} \]

Genericity relation

\[ \pi \prec_{\Gamma} \pi' \quad \text{[less.3]} \]
\[ \pi \prec_{\Gamma} \pi \land \pi_1 \prec_{\Gamma} \pi \land \pi_2 \prec_{\Gamma} \pi \quad \text{[less.4]} \]

Composable types

\[ \Gamma \vdash \pi ; \pi' \leadsto \pi'' \quad \text{[comp.6]} \]
\[ \pi_1 \equiv \pi'_1 \land \pi''_1 \]
\[ \land \pi_2 \equiv \pi'_2 \land \pi''_2 \]
\[ \land \Gamma \vdash \pi'_1 ; \pi'_2 \leadsto \pi'_3 \]
\[ \land \Gamma \vdash \pi''_1 ; \pi''_2 \leadsto \pi''_3 \]
\[ \Gamma \vdash \pi_1 ; \pi_2 \leadsto \pi'_3 \land \pi''_3 \quad \text{[comp.6]} \]

Well-typedness of strategies

\[ \Gamma \vdash s : \pi \quad \text{[amp]} \]
\[ \Gamma \vdash s_1 : \pi_1 \]
\[ \land \Gamma \vdash s_2 : \pi_2 \]
\[ \land \Gamma \vdash \pi_1 \land \pi_2 \]
\[ \Gamma \vdash s_1 \land s_2 : \pi_1 \land \pi_2 \quad \text{[amp]} \]

Reduction of strategy applications

\[ \Gamma \vdash s @ t \leadsto r \quad \text{[amp+/−]} \]
\[ \exists i \in \{1, 2\} . ( \]
\[ \Gamma \vdash t : \tau \]
\[ \land \Gamma \vdash s : \pi_i \]
\[ \land \text{DOM}(\pi_i) \to \tau_{S_i} \]
\[ \land \tau \in \tau_{S_i} \]
\[ \land \Gamma \vdash s_i @ t \leadsto r \)
\[ \Gamma \vdash s_1 \land s_2 @ t \leadsto r \quad \text{[amp+/−]} \]

Fig. 27. Overloaded strategies

Occurrences of INC and DEC are used for specific sorts covered by the overloaded types of INC and DEC.
Typing rules  In Figure 27, the reduction semantics for overloaded strategies and the corresponding typing rules are defined. The type of an overloaded strategy is of the form \(\tau_1 \rightarrow \tau'_1 \& \cdots \& \tau_n \rightarrow \tau'_n\). The type models strategies which are applicable to terms of types \(\tau_1, \ldots, \tau_n\). If such a strategy is actually applied to a term of type \(\tau_i\), the result will be of type \(\tau'_i\). We use an auxiliary judgement \(\text{DOM}(\pi) \sim \tau_s\) to obtain the finite set \(\tau_s\) of term types admitted as domains by a strategy type \(\pi\). We do not attempt to cover generic types in this judgement because symmetric type-dependent choice cannot involve a generic strategy. This is because if one branch would be generic, there are no sorts left to be covered by the other branch. Indeed, we require that the domains of the types composed by \(\cdot \& \cdot\) must be disjoint (cf. \([\pi.4]\)). This requirement enforces immediately UOT of strategy applications. Furthermore, the requirement also ensures that type-dependent choice is deterministic, and hence does not overlap with \(\cdot \& \cdot\) and \(\cdot + \cdot\), i.e., choice controlled by success and failure. In \([\text{less.3}]-[\text{less.4}]\), we update the relation \(\prec_\Gamma\) on strategy types. To this end, we employ an equivalence \(\equiv\) on strategy types modulo associativity and commutativity of \(\cdot \& \cdot\). Rule \([\text{less.3}]\) models that \(\pi\) is less generic than any type \(\pi'\) which is equivalent to \(\pi \& \pi''\). Clearly, this rule is needed to relate simple many-sorted and overloaded strategy types to each other. The rule also relates overloaded strategy types among each other. Rule \([\text{less.4}]\) models that the type of an overloaded strategy is less generic than another type \(\pi\), if both components \(\pi_1\) and \(\pi_2\) of the overloaded type are also less generic than \(\pi\). This rule relates overloaded strategy types and generic types to each other. In this elaboration of \(\prec_\Gamma\), the simple many-sorted strategies are the least elements, and the generic types are the greatest elements.

Reduction semantics  An overloaded strategy is constructed by symmetric type-dependent choice \(s_1 \& s_2\) where the types of the arguments \(s_1\) and \(s_2\) have to admit the construction of an overloaded strategy type (cf. \([\text{amp}]\)). As for the reduction semantics of \(s_1 \& s_2\), the appropriate \(s_i\) is chosen depending on the type of the input term (cf. \([\text{amp}^+]\)). The kind of typing premises in the reduction semantics are similar to the original definition of strategy extension, and static elaboration could be used again to eliminate them. We should note that the refined reduction semantics from Section 5.2 is not prepared to cope with overloaded strategies. A corresponding generalisation does not pose any challenge.

Expressiveness  Although overloading is convenient in strategic programming, it can usually be circumvented with some additional coding effort. To reconstruct Example 17 without overloading, we had to define separate strategies for the different sorts \(\text{NatZero}, \text{NatOne}, \text{Int}\). Overloading is convenient to describe many-sorted ingredients of a traversal in the case that the traversal deals with several term types \(\tau_1, \ldots, \tau_n\) in a specific manner. If we use overloading we can compose the many-sorted ingredients for \(\tau_1, \ldots, \tau_n\) in one overloaded strategy. It is then still possible to extend the overloaded strategy in different ways before we pass it to the ultimate...
traversal scheme. Without overloading, we need to immediately represent the several many-sorted ingredients as a generic strategy by iterating strategy extension for each $\tau_i$. Also, while the type system enforces that the $\tau_1, \ldots, \tau_n$ are distinct in the case of overloading, there is no such guarantee without overloading. In addition to the convenience added by overloading, it is also worth mentioning that overloading can be used to reconstruct generic strategies in some restricted manner. If we consider a fixed signature, then we can represent the signature-specific instantiations of generic strategy types as overloaded strategy types. Consider, for example, the type $TP$. We can reconstruct $TP$ by overloading all $\tau \rightarrow \tau$ for all well-formed $\tau$ according the given signature. Note that this construction becomes infinite if we enable tuple types, but it is finite if we restrict ourselves to traversal of many-sorted terms. Based on these signature-exhausting overloaded types, we could represent the generic traversal combinators as signature-specific overloaded combinators defined in terms of the many-sorted congruences for all the available function symbols.

**Bibliographical notes** When we compare symmetric type-dependent choice to other notions of overloading or ad-hoc polymorphism [CW85, WB89, Jon95], we should note that these other notions are usually based on a form of declaration as opposed to a combinator. Also, other models of overloading usually perform overloading resolution at compile time whereas the dispatch for overloaded strategies happens at run-time. In [CGL95], an extended $\lambda$-calculus $\lambda\&$ is defined that employs type-dependent reduction in a way very similar to our approach. Type-dependent reduction is used to model late binding in the object-oriented sense. More precisely, type-dependent reduction is used in $\lambda\&$ to resort to the most appropriate “branch” of a function based on the run-time type of the argument. This work also discusses the relation of overloading and intersection types [CDCV81, BCDG95]. This is interesting because, at a first glance, one could envision that intersection types might be useful in modelling overloading. For short, intersection types are not appropriate to model overloading if type-dependent reduction is involved. Using intersection types, we say that a function $f$ is of type $a \cap b$ if $f$ can play the role of both an element of type $a$ and of type $b$. Overloading in the sense of $S'_{\gamma}$ and $\lambda\&$ relies on type-dependent reduction, and thereby the selection of the role is crucial for the computation. This facet goes beyond the common interpretation of intersection types.

### 5.4 Asymmetric type-dependent choice

So far, the only way to turn a many-sorted strategy $s$ into a generic one is based on the form $s \ll \pi$. This kind of casting implies that the lifted strategy will fail at least for all term types different from the domain of $s$. This is often not desirable, and hence, an extension usually entails a complementary choice. In the present
section, we want to argue that the separation of lifting (by $\cdot \triangleleft \cdot$) and completion by $\cdot \triangleleft \cdot$ and friends is problematic. It is however possible to support a different style of inhabitation of generic types. We will define a corresponding combinator for *asymmetric type-dependent choice*. While the combinator $\cdot \& \cdot$ for *symmetric* type-dependent choice from Section 5.3 was linked to the notion of overloading, the upcoming asymmetric form does not rely on overloading. In fact, the corresponding left-biased and right-biased forms $\cdot \triangleleft \& \cdot$ and $\cdot \triangleright \& \cdot$ can be regarded as syntactic sugar defined in terms of strategy extension. Asymmetric type-dependent choice means to apply the less generic strategy if this is type-safe, and to resort to a more generic strategy otherwise. If we do not consider overloading, then this form of choice favours the many-sorted operand if this is type-safe, and it resorts to the generic default otherwise.

**Example 18** To motivate the idea of asymmetric type-dependent choice, let us reconsider the traversal scheme $\text{STOPTD}$ that was defined earlier. We repeat its definition for convenience:

$$\text{STOPTD} : \text{TP} \rightarrow \text{TP}$$

$$\text{STOPTD}(\nu) = \nu \triangleleft (\Box(\text{STOPTD}(\nu)))$$

*Left-biased choice controlled by success and failure* is used here to first try the generic argument $s$ of $\text{STOPTD}(s)$ but to descend into the children if $s$ fails. Let us assume that $s$ was obtained from a many-sorted strategy $s'$ by strategy extension as in $s' \triangleleft \text{TP}$. It is important to note that $s$ could fail for two reasons. Firstly, $s$ is faced with a term of a sort different from the domain of $s'$. Secondly, $s'$ is applicable as for the typing, but $s'$ is defined in a way to refuse the given term, e.g., because of unsatisfied preconditions. These two sources of failure are not separated in the definition of $\text{STOPTD}$. In the present formulation, $\text{STOPTD}$ will always recover from failure of $s$ and descend into the children. In fact, $\text{STOPTD}$ will always succeed because $\Box(\cdot)$ at least succeeds for leafs.

**Syntactic sugar** We conclude from the above example that type-mismatch and other sources of failure are hard to separate in a programming style based on $\cdot \triangleleft \cdot$. We improve the situation as follows. We introduce asymmetric type-dependent choice. In the left-biased notation $s_1 \triangleleft \& s_2$, the left operand $s_1$ is regarded as an update for the default $s_2$. Hence, we call this form left-biased type-dependent choice. The many-sorted strategy $s_1$ should be applied if the type of the term at hand fits, and we resort to the generic default $s_2$ otherwise. For brevity, we do not take overloaded strategies into account. One essential ingredient of the definition of asymmetric type-dependent choice is a type guard, that is, a generic strategy which is supposed to accept terms of a certain sort and to refuse all other terms. A type guard is constructed from a many-sorted restriction of $\epsilon$ which is then lifted to the generic type of choice. Here is the syntactic sugar for type guards and asymmetric
type-dependent choice:

\[ \tau \triangleleft \gamma \equiv (\epsilon \triangleright \tau \rightarrow \tau) \triangleleft \gamma \]
\[ s_1 \triangleleft \& s_2 \equiv s_1 \triangleleft \pi + (\neg (\tau \triangleleft \text{TP}); s_2) \] where \( s_1 : \tau \rightarrow \tau' \), \( s_2 : \pi \)
\[ s_1 \triangleleft \& s_2 \equiv s_2 \triangleleft \& s_1 \]

A fully formal definition of this syntactic sugar could be given via the elaboration judgement discussed earlier but we omit this definition for brevity. The definition of \( s_1 \triangleleft \& s_2 \) employs a negated type guard \( \neg (\tau \triangleleft \text{TP}) \) to block the application of the generic default \( s_2 \) in case \( s_1 \) is applicable as for typing.

**Example 19** Let us define a variant of \textit{STOPTD} which interprets failure of the argument strategy as global failure. This can be used for some form of “design by contract”. If the argument strategy ever detects that some precondition is not met, the corresponding failure will be properly propagated as opposed to accidental descent.

\[
\text{STOPTD}' : \forall \alpha. (\alpha \rightarrow \alpha) \rightarrow \text{TP}
\]
\[
\text{STOPTD}'[\alpha](\nu) = \nu \triangleleft \& \square(\text{STOPTD}'[\alpha](\nu))
\]

\text{STOPTD}' is different from \text{STOPTD} in that the argument of \text{STOPTD} is a generic strategy whereas it is many-sorted in the case of \text{STOPTD}'. To this end, the type of \text{STOPTD}' involves a type parameter for the sort of the argument. The asymmetric type-dependent choice to derive a generic strategy from the argument is part of the definition of \text{STOPTD}'.

**Example 20** We should mention that type guards are useful on their own. Recall the illustrative traversal problem (IV) to collect all natural numbers in a tree. The encoding from Example 14 relies on a user-defined strategy \text{NAT} to test for naturals based on the congruences for the constructors of sort \text{Nat}. The syntactic sugar for type guards allows us to test for arbitrary sorts without the cumbersome style of enumerating all constructors. This is illustrated in the following definition of (IV) where we use the notation for a type guard for naturals instead of relying on the user-defined strategy \text{NAT}:

\[
(IV) = \text{STOPCRUSH}[\text{NatList}](\text{Nat} \triangleleft \text{TU}(\text{Nat}); \text{SINGLETON}, \text{NIL}, \text{APPEND})
\]

**Complementary forms of choice** It is instructive to compare the different forms of asymmetric choice encountered in the present paper. In the case of \( s_1 \leftrightarrow s_2 \), the success of \( s_1 \) rules out the application of \( s_2 \). In the case of \( s_1 \triangleleft \& s_2 \), the mere type of \( s_1 \) decides if the application of \( s_2 \) will be ever considered. To understand this
twist, consider the following strategy approximating $s_1 \llcorner s_2$:

$$
\begin{align*}
    s_1 &\llcorner \pi_2 \Leftarrow s_2 \quad \text{where } s_2 \text{ is of type } \pi_2 \\
\end{align*}
$$

This formulation attempts to compensate for the type guard in the definition of asymmetric type-dependent choice by resorting to left-biased choice controlled by success and failure. That is, we attempt to simulate left-biased type-dependent choice by left-biased choice controlled by success and failure. This attempt is not faithful since $s_2$ might be applied to a term $t$ even if the types of $s_1$ and $t$ fit, namely if $s_1$ fails on $t$.

To summarise, choice between strategies of the same type is solely modelled by the combinators $\cdot \leftarrow + \cdot$ and friends that are controlled by success and failure. Non-deterministic and asymmetric choice differ in the sense if there is a preferred order on the arguments of the choice. For convenience, we might accept different types for the argument strategies of $\cdot \leftarrow + \cdot$ and friends. But then we restrict the type of the choice to the greatest lower bound of the types of the arguments. By contrast, type-dependent choice composes strategies of different types, and the type of the choice extends to the least upper bound of the types of the arguments. The corresponding combinators are not at all controlled by success and failure. Instead, the type of the term at hand determines the branch to be taken. The arguments in an asymmetric type-dependent choice are related via $\prec_{\Gamma}$, whereas the domains of the arguments in a symmetric type-dependent choice are required to be disjoint. In conclusion, choice by success and failure and type-dependent choice complement each other. The division of labour between the two kinds of choice was also nicely illustrated in Example 17.

### 5.5 Variations on traversal

The selection of the traversal primitives of $S'_{\gamma}$ has been driven by the requirement not to employ any universal representation type. For that reason the children are never directly exposed to the strategic program. Instead, one has to select the appropriate combinator to process the children. We want to indicate briefly that there is a potential for generalised or additional traversal primitives while keeping in mind the aforementioned requirement.

**Order of processing children** The reduction semantics of the traversal primitives left the order of processing children largely unspecified. As for $\sqcap(\cdot)$, the order does not seem to be an issue since all the children are processed anyway and independently of each other. Note however that the order becomes an issue if we anticipate the possibility that processing fails for one child or several children. Then, different orders will not just lead to different execution times, but even program termination...
might depend on the order. As for $\text{Diamond}(\cdot)$, a flexible order is desirable for yet another reason. That is, one might favour the left-most vs. the right-most child that can be processed. The actual choice might be a correctness issue as opposed to a mere efficiency issue. To cope with such variations, one can consider refined traversal combinators such as $\text{Diamond}_o(s)$ where we assume that the order of processing children is constrained by $o$. There are the following options for such an order constraint $o$:

- “$\rightarrow$” — processing from left to right
- “$\leftarrow$” — processing from right to left
- unspecified

As for the type-unifying traversal combinators, order constraints make sense as well. In $\text{O}_{s_0}^o(s)$, the constraint $o$ could be used to control how the pairwise composition $s_0$ is applied to the processed children. A simple investigation of the original formalisation of $\text{O}_{s_0}^o(s)$ in Figure 19 makes clear that a left-to-right reduction was specified (although it was not constrained if pairwise composition is intertwined with processing the children). A certain order $o$ for reduction might be relevant to cope with combinators $s_0$ which do not admit associativity and/or commutativity. As for selection via $\text{Pick}_o(s)$, basically the same arguments apply as to $\text{Diamond}_o(s)$.

**Pairwise composition vs. folding** It turns out that reduction as modelled by the combinator $\text{O}^\cdot(\cdot)$ can be generalised. Instead of separating the aspects of processing the children and composing intermediate results, we can also define reduction so that the way how a child is processed depends on previously processed children. In fact, one can define a combinator $\langle\cdot, \cdot\rangle$ which folds directly over the children of a term very much in the sense of the folklore pattern for folding a list. Note however that we have to cope with an intentionally heterogeneous list corresponding to the children of a given term. That is, in folding over the children of a term, we need a generic ingredient to operate on a given child and the intermediate result obtained from previous folding steps. The reduction semantics of the strategy $\langle s_0, s_c \rangle$ is defined in Figure 28. The first argument $s_0$ encodes the initial value for folding. In the case of a constant symbol, $s_0$ defines the result of folding (cf. $\text{fold}^+ 1$). For nontrivial terms, we fold over their children by repeated application of the second argument $s_c$ (cf. $\text{fold}^+ 2$). Without loss of generality, $\langle\cdot, \cdot\rangle$ is a right-associative fold.

**Example 21** Let us attempt a reconstruction of the strategy $\text{CF}$ from Figure 5. For convenience, we first show the original definition in terms of $\text{O}^\cdot(\cdot)$. Then, we show a reconstruction which employs the combinator $\langle\cdot, \cdot\rangle$.

$$\text{CF}(\nu, \nu_u, \nu_o) = (\text{CON}; \bot; \nu_u) + (\text{FUN}; \text{O}_{\nu_o}^o(\nu))$$

$$= (\langle\nu_u, \langle\nu, \epsilon\rangle; \nu_o\rangle)$$

This reconstruction immediately illustrates why the combinator $\langle\cdot, \cdot\rangle$ is more powerful than the combinator $\text{O}^\cdot(\cdot)$. As the second argument in the above application
Syntax

\[ s ::= \cdots \mid (s, s) \]

Reduction of strategy applications

Positive rules

\[
\begin{align*}
  s_0 @ () & \leadsto t_0' \\
  (s_0, s_c) @ c & \leadsto t_0' \\
  s_0 @ () & \leadsto t_n' \\
  s_c @ (t_n, t_n') & \leadsto t_{n-1}' \\
  \cdots \\
  s_c @ (t_1, t_1') & \leadsto t_0' \\
  (s_0, s_c) @ f(t_1, \ldots, t_n) & \leadsto t_0'
\end{align*}
\]

\[ \text{[fold}^+1] \]

Negative rules

\[
\begin{align*}
  s_0 @ () & \leadsto \uparrow \\
  (s_0, s_c) @ c & \leadsto \uparrow \\
  s_0 @ () & \leadsto t_n' \\
  s_c @ (t_n, t_n') & \leadsto t_{n-1}' \\
  \cdots \\
  \exists i \in \{1, \ldots, n\}. \\
  s_c @ (t_i, t_i') & \leadsto \uparrow \\
  (s_0, s_c) @ f(t_1, \ldots, t_n) & \leadsto \uparrow
\end{align*}
\]

\[ \text{[fold}^-2] \]

Fig. 28. An intentionally type-unifying traversal combinator for folding the children of \((\cdot, \cdot)\) points out, a child is processed independent of the intermediate value of reduc- tion (cf. the congruence \(\langle \nu, \epsilon \rangle\)), and both values are composed in a subsequent step by \(\nu_o\). This is precisely the scheme underlying \(\bigcirc (\cdot)\).

We cannot type the combinator \((\cdot, \cdot)\) in a simple way in our present type system. Consider the intended type of the second argument. The strategy should process a pair consisting of a term of any type (corresponding to some child), and a term of the distinguished type for type unification. This amounts to the type scheme \(\forall \alpha. \langle \alpha, \tau \rangle \rightarrow \tau\) where \(\tau\) is the unified type for reduction. One could introduce a designated generic type for that purpose. Unfortunately, more extensions would be needed to effectively use the additional generality. It is not obvious how to stay in a many-sorted setting in this case. Due to these complications we do not attempt to work out typing rules for \((\cdot, \cdot)\).

Environments and states There are other useful type schemes than just TP and TU(\(\cdot\)). In the following table, we repeat the definition of TP and TU(\(\cdot\)), and we list
Reduction of strategy applications

\[ s @ t \sim t' \]

TP

\[ \exists i \in \{1, \ldots, n\}. s \oplus t_i \sim t'_i \]
\[ \diamond (s \oplus f(t_1, \ldots, t_n)) \sim f(t_1, \ldots, t'_1, \ldots, t_n) \]

TE(\cdot)

\[ \exists i \in \{1, \ldots, n\}. s \oplus \langle t_i, e \rangle \sim t'_i \]
\[ \diamond (s \oplus (f(t_1, \ldots, t_n), e)) \sim f(t_1, \ldots, t'_1, \ldots, t_n) \]

TS(\cdot)

\[ \exists i \in \{1, \ldots, n\}. s \oplus \langle t_i, a \rangle \sim \langle t'_i, a' \rangle \]
\[ \diamond (s \oplus (f(t_1, \ldots, t_n), a)) \sim \langle f(t_1, \ldots, t'_1, \ldots, t_n), a' \rangle \]

Fig. 29. Variants of \( \diamond (\cdot) \)

three further schemes:

TP \( \equiv \forall \alpha. \alpha \rightarrow \alpha \) \hspace{1cm} (Type preservation)
TU(\tau) \( \equiv \forall \alpha. \alpha \rightarrow \tau \) \hspace{1cm} (Type unification)
TA(\tau) \( \equiv \forall \alpha. \langle \alpha, \tau \rangle \rightarrow \tau \) \hspace{1cm} (Accumulation)
TE(\tau) \( \equiv \forall \alpha. \langle \alpha, \tau \rangle \rightarrow \alpha \) \hspace{1cm} (TP with environment passing)
TS(\tau) \( \equiv \forall \alpha. \langle \alpha, \tau \rangle \rightarrow \langle \alpha, \tau \rangle \) \hspace{1cm} (TP with state passing)

A strategy of type TA(\tau) takes a pair \( \langle x, a \rangle \) where \( x \) can be of any term type and \( a \) is of type \( \tau \), and it returns the resulting value \( a' \) of type \( \tau \). When thinking of traversal, TA(\tau) suggest accumulation of a value whereas the earlier TU(\tau) rather suggests synthesis of a value. Both schemes of traversal are interchangeable, in principle. Then, the type scheme TE(\tau) denotes all strategies that take a pair \( \langle x, e \rangle \) where \( x \) can be of any term type and \( e \) is of type \( \tau \), and it returns a resulting term \( x' \). When thinking of traversal, TE(\tau) amounts to a combination of type-preserving traversal and environment passing. Finally, TS(\tau) can be regarded as a combination of TP and TA(\tau). In this combination, it is suggestive to speak of state passing.

Designated combinators vs. monads The ultimate question is how to inhabit the above type schemes. \( S'_r \) is not sufficiently expressive to derive traversal com-
binators for the additional generic types from the existing combinators that cover TP and TU(·). However, it is not difficult to define corresponding variations on the existing traversal primitives. Let us illustrate this idea for the generic type TE(·). Dedicated traversal combinators should not simply apply a given strategy to the children, but an environment has to be pushed through the term, too. Let us characterise a corresponding variation on ∘(·). If ∘(s) is applied to ⟨f(t₁, . . . , tₙ), e⟩, then the strategy application to rewrite a child tᵢ is of the form s @ ⟨tᵢ, e⟩. In Figure 29, the positive rules for variants of ∘(·) for TP, TE(·) and TS(·) are shown. All other traversal primitives admit similar variations. In a higher-order functional programming context, monads [Spi90, Wad92] can be employed to merge effects like environment or state passing with the basic scheme of type-preserving or type-unifying traversal. Monads would also immediately allow us to deal with reducts other than optional terms, namely lists or sets of terms. In the reduction semantics of S′, we hardwired the choice of an optional term as reduct. This choice corresponds to the maybe monad.

6 Implementation

In the sequel, we discuss a Prolog-based implementation of S′, and we report on an investigation regarding the integration of the S′ expressiveness into the rewriting framework ELAN. The Prolog implementation is convenient to verify our ideas and the formalisation, but also to prove the simplicity of the approach. We have chosen Prolog due to its suitability for prototyping language syntax, typing rules, and dynamic semantics (cf. [LR01]). The ELAN-centered investigation backs up our claim that the proposed form of generic programming can be easily integrated into an existing, basically first-order, many-sorted rewriting framework.
It is well-known that deduction rules in the style of Natural semantics map nicely to Prolog clauses (cf. [Des88]). Prolog’s unification and backtracking enable the straight execution of a large class of deduction systems. In fact, the Natural semantics definitions from the present paper are immediately implementable in this manner. The judgements were mapped to Prolog in the following manner. Well-formedness, well-typedness, static elaboration and reduction judgements constitute corresponding predicate definitions. Terms are represented as ground and basically untyped Prolog terms. Strategic programs are represented as files of period-terminated Prolog terms encoding type declarations and strategy definitions. In this manner, Prolog I/O can be used instead of parsing. Prolog variables are used to encode term variables in rewrite rules, strategy variables in strategy definitions, and term-type variables in type declarations.

**Strategies in Prolog** The encoding of strategies is illustrated in Figure 30. On the left side, the Prolog encoding for some reusable strategies from Figure 3 and Figure 4 are shown. On the right side, the strategies for the introductory traversal problems (I) and (II) from the introduction are shown. The rewrite rule to increment a natural is for example represented as \( N \rightarrow \text{succ}(N) \). One can see that the encoding basically deals with notational conventions of Prolog such as the period “.” to terminate a term to be read from a file. The term \( tp \) denotes the type \( TP \). The \texttt{data} directive is used to declare algebraic datatypes contributing to the context \( \Gamma \) of a strategic program. We do not declare types of term variables since it is very easy to infer their types using the non-ground representation for rewrite rules.

**Prolog encodings of the judgements** The implementation of \( S' \) is illustrated with a few excerpts in Figure 31. We show some clauses for the predicates encoding the reduction of strategy applications and static elaboration of strategies. The left-most excerpt shows the very simple implementation of the combinator \( \Box(\cdot) \) in Prolog. Here we resort to the Prolog operator “=..” to access the children as a list, and we employ a higher-order predicate \( \text{map/3} \) to map the argument strategy over the children. In the middle, we show the encoding of the static elaboration rule from Figure 25. The right-most excerpt implements the reduction semantics of an annotated application of \( \cdot \langle \cdot \cdot \cdot \rangle \). It deviates from the formalisation in Figure 25 in that we do not assume tagged terms but we rather look up the type of the given term by retrieving the outermost symbol’s result type from a simple context parameter.

**Prological strategies** The proposed implementational model is geared towards a direct implementation of the calculus’ formalisation in Prolog, that is, judgements
Fig. 31. Implementation of $S'_\gamma$ in Prolog

become predicates. Strategic programming can also be integrated into Prolog in a more seamless way from the logic programmer’s point of view. Essentially, strategy combinators can be represented as higher-order predicates. Prolog programmers are used to this idea which is for example used for list processing. Furthermore, we abandon rewrite rules altogether, and we assume that many-sorted functionality is defined in terms of ordinary Prolog predicates. This approach is not just convenient for logic programmers, but it also leads to a very compact implementation of strategic programming expressiveness. In such a Prolog incarnation of strategic programming, the most complicated issue is typing. In general, all attempts to impose type systems on Prolog restrict Prolog’s expressiveness to a considerable extent. We cannot expect that all the implementations of the strategy combinators themselves can be typed-checked. In particular, the use of the univ operator “=..” for generic term destruction and construction is hardly typeable. Recall that “=..” would be needed for the implementation of traversal combinators. Hence, we need an approach where type-checking is optional, that is, it can be switched off maybe per Prolog module. We refer to [LR01] for a discussion of the Prological incarnation of strategic programming.

6.2 Integration into ELAN

The rewriting framework ELAN supports many-sorted rewriting strategies. However, generic traversal combinators are not offered. ELAN’s type system is indeed a many-sorted one. ELAN’s module system offers parameterisation of modules by sorts. One can import the same parameterised module for different sorts. This leads to a style of programming where function symbols and strategy combinators are potentially overloaded. In the sequel, we explain how combinators for generic traversal and strategy extension can be made available in ELAN based on the $S'_\gamma$ model of typed strategies. The simplicity of the integration model indeed further backs up our claim that $S'_\gamma$ is straightforward to implement. We should point out that there are ongoing efforts to revise the specification formalism and the system architecture underlying ELAN. We base our explanations on ELAN as of [BKK+98, BKKR01].

The module strat[X] Let us recall some characteristics of many-sorted strategies as supported by ELAN. There is a designated library module strat[X] for
strategy combinators parameterised by a sort \( X \). In fact, certain ELAN strategy combinators are built-in, but for the sake of a homogeneous situation we assume that all combinators are provided by the module \( \text{strat}[X] \). ELAN offers a notation for strategy application which can be used in the where-clauses of a rewrite rule and in the user interface. If strategies should be composed and applied to terms of a certain sort, one needs to import the module \( \text{strat}[X] \) where the formal parameter \( X \) is instantiated by the given sort. By importing this module for several sorts, the strategy combinators are overloaded for all the sorts accordingly. This approach implies that parsing immediately serves for type checking. ELAN also allows one to define new many-sorted strategy combinators. One can also define combinators for the sort parameter of a module so that the definitions are reusable for different sorts. As an aside, ELAN’s parameterised modules can be used as a substitute for type-parameterised strategies in the sense of \( S'_\gamma \).

**The module \( \text{any}[X] \)** In addition to parameterised modules, ELAN offers further means to define generic functionality, that is, functionality dealing with terms of arbitrary sorts. We review these techniques to see whether they are suitable for the implementation of the \( S'_\gamma \) combinators for generic traversal and strategy extension. There is a designated library module \( \text{any}[X] \) which supports a form of dynamic typing and generic term destruction / construction per sort \( X \). The module uses a universal datatype \( \text{any} \) in the sense of dynamic typing. The datatype \( X \) and \( \text{any} \) are mediated via an injection function defined by the module. Further, the module hosts \( \text{explode} \) and \( \text{implode} \) functions to destruct and construct terms of sort \( \text{any} \). The children of a term are made accessible as a list of terms of sort \( \text{any} \). Internally, ELAN uses a pre-processor to generate the rewrite rules for explosion and implosion.

**Naive encoding of \( S'_\gamma \)** For brevity, we restrict ourselves to type-preserving strategies in the sequel. \( S'_\gamma \) strategies of type TP can be encoded as ELAN strategies of type \( \text{any} \rightarrow \text{any} \). One can define traversal combinators in terms of implosion and explosion based on the functionality of the module \( \text{any}[X] \). The combinator \( \Box(\cdot) \), for example, would be defined in roughly the same manner as in the above Prolog encoding. First, the given term of sort \( \text{any} \) is exploded to access the functor and the children. Then, the argument strategy is mapped over the children via a dedicated strategy for list processing. Finally, the original functor and the processed children are imploled. The combinator for strategy extension can be encoded in ELAN as follows. Given a strategy \( s \) of type \( X \rightarrow X \), strategy extension derives a strategy of type \( \text{any} \rightarrow \text{any} \). The application of the extended \( s \) entails the attempt to take away the injection of type \( X \rightarrow \text{any} \) from the term at hand. If the given term is not of sort \( X \), the application of the extended strategy fails in accordance with type safety. The combinator for strategy extension is overloaded for all possible \( X \), that is, it needs to be placed in a module parameterised by \( X \). If the strategic program-
mer wants to apply a “generic” strategy, (s)he has to inject the given term into \texttt{any} prior to application, and to unwrap the injection from the result.

**Fully typed encoding of** $S'_\gamma$ The above encoding suffers from the following problem. The sort \texttt{any} is exposed to the strategic programmer in the sense that generic strategies are known to operate on terms of sort \texttt{any}. Hence, there is no guarantee that generic strategies are well-typed in a many-sorted sense. To give an example, an intentionally type-preserving strategy can map a term of sort $X$ to a term of sort $Y$ while this type change would go unnoticed as long as terms are represented inside the union type \texttt{any}. Furthermore, the exposition of \texttt{any} allows a strategic programmer to manipulate compound terms in an inconsistent manner.

Note that explosion and implosion involves lists of terms. That is, the ELAN type system does not ensure that the manipulated exploded terms form valid terms in the many-sorted sense. This implies a potential for implosion failure at run-time. A fully typed encoding requires the following elaboration of the naive approach. In abstract terms, we need to hide the employment of \texttt{any} for strategic programmers who want to apply generic strategies, inhabit generic strategy types via strategy extension, or define new combinators in terms of the basic combinators. Then, a strategic programmer cannot define ill-typed generic strategies, neither can (s)he cause implosion failures provided all the basic combinators are implemented in accordance with the $S'_\gamma$ reduction judgement that is known to be type-safe. In order to hide the employment of \texttt{any}, we assume the introduction of designated sorts for generic strategy types where these sorts are known to the strategic programmer but not their definition. To give an example, we assume a sort \texttt{tp} for the $S'_\gamma$ type \texttt{TP} with the hidden definition \texttt{any} $\rightarrow$ \texttt{any} in ELAN. All the combinators for a generic strategy type are defined in a module together with the designated sort. Since strategy application and strategy extension work per sort, we need a parameterised module, e.g., \texttt{tp [X]} for generic type-preserving strategies which can be applied to terms of sort $X$, and which can be derived by extending many-sorted strategies of type \texttt{X} $\rightarrow$ \texttt{X}. Clearly, \texttt{tp [X]} can be regarded as an abstract datatype (ADT) for generic type-preserving strategies.

To summarise, the described integration model relies on the following concepts:

- parameterised modules to overload strategy combinators per sort,
- type-checking by parsing overloaded many-sorted strategies,
- dynamic typing to achieve the needed degree of polymorphism, and
- support for generic term destruction / construction.

Because these features are present in ELAN, the support of $S'_\gamma$-like strategies does not require any internal modification of ELAN. Instead of relying on features like a pre-processor for term implosion and explosion, we could favour an extension of the rewrite engine to directly support traversal combinators, and strategy extension as well. This approach would be, in general, appropriate to implement $S'_\gamma$-
like strategies in other frameworks for rewriting or algebraic specification, e.g., in ASF+SDF [BHK89, Kli93, BHJ+01].

7 Related work

Specific pointers to related work were placed in the technical sections. It remains to comment on related work from a more general point of view. First, we relate $S'_\gamma$ to existing strategic rewriting calculi. Then, we discuss other efforts in the rewriting community to enable some form of generic programming. Finally, we discuss genericity in functional programming because this paradigm is very much related to rewriting.

7.1 Strategic rewriting calculi

Let us relate the calculus $S'_\gamma$ to those frameworks for strategic programming which were most influential for its design, namely system $S$ underlying Stratego [VB98, VBT98], and ELAN [BKK+98, BKKR01].

$S'_\gamma$ vs. system $S$ and Stratego Our typed rewriting calculus $S'_\gamma$ adopts the untyped system $S$ to a large extent. We stick to the same semantic model. We also adopt its traversal combinators \(\Box(\cdot)\) and \(\Diamond(\cdot)\). System $S$ suggests a hybrid traversal combinator \(\Box(\cdot)\) where the application of the argument strategy has to succeed for at least one child but the application is attempted for all children. We leave out \(\Box\cdot\) in $S'_\gamma$ in order to minimise the operator suite which needs to be covered by the formalisation. The main limitation of $S'_\gamma$ compared to system $S$ is that we favour standard first-order rewrite rules with where-clauses as primitive form of strategy. By contrast, system $S$ provides less standard primitives which are however sufficient to model rewrite rules as syntactic sugar. These primitives are matching to bind variables, building terms relying on previous bindings, and scoping of variables. The additional flexibility which one gains by this separation is that arbitrary strategies can be performed between matching and building. One can simulate this style by using where-clauses in $S'_\gamma$. The key innovation of $S'_\gamma$ when compared to system $S$ is the combinator \(\cdot\triangleleft\cdot\) for strategy extension. Since the combinator \(\cdot\triangleleft\cdot\) relies on a type-dependent reduction semantics, one cannot even expect any combinator like this in untyped systems such as Stratego or system $S$. Furthermore, $S'_\gamma$ also introduces combinators which are not expressible in system $S$, namely the combinators \(\Box(\cdot)\) and \(\sharp(\cdot)\) for intentionally type-unifying traversal. Stratego provides a combinator which can be used to encode type-unifying traversal, namely \(\cdot\sharp\cdot\). This combinator is meant for generic destruction and construction of terms very much.
in the style of the standard univ operator “=..” in Prolog. Interestingly, the combiners \( \Box(\cdot) \) and \( \Diamond(\cdot) \) could be encoded in terms of \( \cdot\#\cdot \). A crucial problem with \( \cdot\#\cdot \) is that it leads to a hopelessly untyped model of traversal since the programmer accesses the children of a term as a list. While the system \( S \) combiners \( \Box(\cdot) \) and \( \Diamond(\cdot) \) suggest a typeful treatment, typeful type-unifying strategies cannot be based on \( \cdot\#\cdot \) but other combinators are needed. This is the reason that we designed the traversal combinators \( \bigcirc(\cdot) \) and \( \bigtriangledown(\cdot) \) for type-unifying traversal in \( S'_\gamma \).

**\( S'_\gamma \) vs. ELAN**

The influence of ELAN is also traceable in \( S'_\gamma \). We adopt the model of rewrite rules with where-clauses from ELAN. We also adopt recursive strategy definitions from ELAN while system \( S \) favours a special recursion operator \( \mu \cdot\cdot\cdot \). In the initial design of a basically many-sorted type system we also received inspiration from the ELAN specification language. ELAN and \( S'_\gamma \) differ in the semantic model assumed for reduction. ELAN offers a faithful model of non-determinism via sets or lists of possible results where the empty set represents failure. The type system of \( S'_\gamma \) does not rely on the simple model of system \( S \). In fact, our typeful approach to generic traversal could be integrated with the ELAN-like semantic model without changing any detail in the type system.

### 7.2 Genericity in rewriting

In [CDMO01], polytypic entities are defined in terms of the reflection and metaprogramming capabilities of Maude. This approach is hard to compare with \( S'_\gamma \) which is based on the idea of static typing and a designated type system. Furthermore, the mixture of many-sorted and generic functionality is not considered. Also, the Maude approach—as any other polytypic approach—does not propose traversal combinators but traversal is based on polytypic induction.

In [BKVO1], a fixed set of traversal strategies is supported by so-called traversal functions extending the algebraic specification formalism ASF [BHK89]. The central idea is to declare designated function symbols for traversal according to predefined strategies for top-down, bottom-up and accumulating traversal. The programmer refines a traversal function by rewrite rules for specific sorts. This approach is less general than the \( S'_\gamma \) approach because the programmer cannot define new traversal schemes. Also, it is more difficult to separate many-sorted and generic functionality. However, this approach is sufficient for many common scenarios in program transformation and analysis [BKVO1, LW01]. In fact, traversal functions are very convenient to use because of their seamless integration into ASF+SDF [BHK89, Kli93, BHJ+01].

In [BKKR01], dynamic typing [ACPP91, ACPR92] and generic implosion / explosion à la Prolog’s “=..” are used to traverse terms. Dynamics tend to spread all over
a program which clearly goes against a many-sorted typing discipline. Also, the
use of explosion and implosion in a program implies a basically untyped manipula-
tion of exploded terms. Furthermore, basic traversal combinators are not identified.
Their benefits were first identified in the early work on Stratego [LV97]. We al-
ready explained in Section 6.2 how ELAN’s features can be used to implement the
$S'_\gamma$ combinators in a typeful manner. The formalisation of $S'_\gamma$ avoids all kinds of
typing problems in the first place because terms are not converted to a universal
type.

We envision that the design of $S'_\gamma$ could be useful in the further elaboration of other
formal models for rewriting so that typed generic traversal will be covered. One
prime candidate is the $\rho$-calculus [CK99] which provides an abstract and very gen-
eral formal model for rewriting including strategies. Generic traversal combinators
have been defined in the $\rho$-calculus (cf. $\Psi(s)$ and $\Phi(s)$ in [CK99], corresponding to
$\square(s)$ and $\Diamond(s)$) but these definitions cannot be typed in the available typed frag-
ments of the $\rho$-calculus [CKL01]. There is ongoing research to organise typed cal-
culi in a so-called $\rho$-cube, very much in the sense of the $\lambda$-cube [Bar92]. It is not
obvious how certain typing notions interact with each other if we attempt to cover
generic traversal in this cube, e.g., type-dependent reduction à la $S'_\gamma$ vs. dependent
types.

7.3 Genericity in functional programming

The most established notion of genericity or polymorphism is certainly parametric
polymorphism [Mil78, Gir72, Rey74, Rey83, CW85, Wad89]. It is clear that para-
metric polymorphism is not sufficient to model typed generic traversal strategies.
Firstly, it does not allow us to descend into terms. Secondly, parametric polymor-
phism is also in conflict with non-uniform behaviour as it can be assembled in terms
of strategy extension. Several forms of polymorphism were proposed that go be-
yond parametric polymorphism, namely dynamic typing [ACPP91, ACPR92], ex-
tensional polymorphism [DRW95], intensional polymorphism [HM95], and poly-
typism [JJ97]. A general observation is that none of the available systems subsumes
$S'_\gamma$. Dynamic typing was already discussed in the previous section on rewriting. The
remaining forms are reviewed in the sequel. We also refer to [LV00, LV01] where
we report on actual efforts to encode generic traversal strategies as generic func-
tions.

Let us check the requirement for generic traversal, that is, the ability to descend into
terms. Clearly, algebraic datatypes model sets of typed terms in functional pro-
gramming. Extensional polymorphism, intensional polymorphism and polytypism have
in common that they offer some form of generic function definition based on struc-
tural pattern matching on types. These forms can be used to encode traversal. In the
cases of extensional and intensional polymorphism, type-based induction involves
cases for basic datatypes, products and functions. The mere structure of algebraic datatypes implies that a case for sums is also needed. In fact, polytypic programming considers algebraic datatypes as sums of products, and adds a corresponding pattern for type induction. It would be straightforward to extend extensional and intensional polymorphism accordingly.

The idea of strategy extension implies that generic strategies are aware of many sorts, say systems of named algebraic datatypes in the sense of functional programming. However, all the aforementioned forms of polymorphism are geared towards structural induction on types, that is, they do not involve a notion of checking the coincidence of two (names of) types as it is required for strategy extension. This crucial difference is discussed in [Gle99]. This shortcoming has been addressed in recent work on polytypic programming to some extent, namely different proposals for Generic Haskell include support for some form of type-specific cases (cf. ad-hoc definitions in [Hin99]) in an otherwise structural induction on types.

In fact, Generic Haskell appears to offer the most complete feature list for an encoding of rewriting strategies because generic term traversal and specific type cases are offered by the language design of Generic Haskell. However, we cannot reconstruct $S'_\gamma$ in this language setup for the following reasons. Firstly, polytypic functions are not first-class citizens. In particular, one cannot pass a polytypic function as an argument to another polytypic function. First-class functions are needed to model traversal combinators, traversal schemes or other parameterised strategies. Secondly, type case is based on polytypic function definition as a top-level form of declaration. This restricts the separation and composition of type-specific vs. generic functionality. More generally, polytypic programming does not support combinator style of generic programming whereas strategic programming relies on combinator style.

8 Conclusion

Typed generic traversal strategies In the present paper, we developed a typed calculus $S'_\gamma$ for term rewriting strategies. The main contribution of the paper is that generic traversal is covered. The idea of generic traversal combinators is already present in previous work on strategic rewriting, however, only in untyped settings. It turned out that existing combinators for intentionally type-preserving traversal could be easily typed. However, the typical approach to type-unifying traversal is hopelessly untyped (cf. generic term destruction and construction à la · #· in Stratego, “=..” in Prolog). To resolve this problem, we proposed designated traversal combinators for type-unifying traversal. The key idea underlying our type system is the $S'_\gamma$-like style of type-safe extension of many-sorted strategies. This approach allows us to combine many-sorted and generic functionality in a very flexible manner without confusing different kinds of strategy composition (cf. · ⊲· vs. · ⊾·).
The type system separates many-sorted and generic strategies in a way that the precision of the underlying many-sorted type system is preserved.

**Simple generic programming**  At a design level, our declared goal was to obtain a simple, self-contained model of typed generic traversal on the grounds of basically many-sorted, first-order term rewriting. In fact, the type system of $S'_\gamma$ is simple, and the complete calculus is straightforward to implement. To explain what we mean by “simple type system” and “straightforward implementation”, we mention that the development of the Prolog prototype, which we discussed earlier, took two days. Contrast that with other approaches to generic programming such as PolyP and Generic Haskell which usually require(d) several man years of design and (prototype) implementation. Generic term traversal based on the designed suite of traversal combinators is very potent but it certainly does not cover the full range of generic programming (cf. kind-indexed polytypic definitions, generic anamorphisms, and others). Also, the overall setting of $S'_\gamma$, especially the restriction to a basically first-order, many-sorted setting, rules out several powerful programming idioms, e.g., higher-order functions. Nevertheless, the prime justification for the restricted approach is the well-defined application domain covered by the chosen expressiveness, namely program transformation and analysis for large language syntaxes (cf. [VBT98, LV00, Vis01, BKV01, LV01]).

**Functional strategies**  In our ongoing work, we transpose strategic term rewriting to the functional programming paradigm (cf. the Haskell-based generic programming bundle Strafunski; see http://www.cs.vu.nl/Strafunski/). In [LV01], we motivated and characterised a corresponding notion of functional strategies, and we provided a corresponding combinator library for generic functions. This approach complements existing approaches to generic functional programming in that it supports *first-class* generic functions which can traverse into terms of systems of algebraic datatypes while mixing uniform and type-specific behaviour. In fact, we investigate different models to support strategies in functional programming. One model which we also discuss in [LV01] is based on the formula “strategies as functions on a universal representation type” as discussed for ELAN in Section 6.2.

**Future work**  Besides the notion of functional strategies, we are also interested in the further development of the strategic rewriting paradigm in general. We indicate an open-ended list of challenges for future work:

- Typed-based optimisation of traversals.
- Typeful treatment of impure extensions of Stratego.
- Fusion-like principles for traversal strategies [LV01].
- Systematic derivation of one-step traversal combinators.
- Interaction of constraint mechanisms and traversal strategies.
• Application of strategic programming to document processing.
• Coverage of generic datatype-changing transformations \cite{LL01}.
• More precise types as for success and failure behaviour \cite{Mor99}.
• More precise types as for kinds of involved polymorphic behaviour.
• Coverage of generic term construction in the sense of anamorphisms.
• Comparison of attribute grammar approaches and strategic programming.
• Comparison of strategic programming and adaptive programming \cite{LPS97}.
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