LOWER BOUNDS FOR THE INDEX OF COMPACT CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{R}^3$ AND $S^3$

MARCOS P. CAVALCANTE AND DARLAN F. DE OLIVEIRA

Abstract. Let $M$ be a compact constant mean curvature surface either in $S^3$ or $\mathbb{R}^3$. In this paper we prove that the stability index of $M$ is bounded from below by a linear function of the genus. As a by-product we obtain a comparison theorem between the spectrum of the Jacobi operator of $M$ and those of Hodge Laplacian of 1-forms on $M$.

1. Introduction

Let $\overline{M}^3$ be a complete Riemannian 3-manifold and let $M \subset \overline{M}^3$ be a compact surface immersed in $\overline{M}^3$. It is well known that $M$ is a minimal surface if it is a critical point of the area functional, and $M$ is a constant non zero mean curvature (CMC) surface if it is a critical point of the area functional for those variations that preserve the enclosed volume. When the ambient space is the Euclidean three-space, it means that minimal surfaces are the mathematical models of soap films while constant mean curvature surfaces are those of soap bubbles.

The stability properties of minimal and CMC surfaces are given by the study of the second variation of the area functional. In order to give a precise definition let us assume that $M$ is closed and two-sided, and denote by $N$ a unit normal vector field along $M$. Given $u \in C^\infty(M)$, a smooth function on $M$ and considering variations given by $V := uN$, the second variation formula is given by the quadratic form

$$Q(u, u) := \int_M \|\nabla u\|^2 - (\overline{\text{Ric}}(N) + \|A\|^2)u^2 \, dM,$$

where $\overline{\text{Ric}}(N)$ is the Ricci curvature of $\overline{M}$ in the direction of $N$ and $\|A\|^2$ stands for the square norm of the second fundamental form of $M$ (see Section 3 for details). For CMC surfaces the requirement that variations preserve the enclosed volume is equivalent to consider functions
satisfying $\int_M u \, dM = 0$. For future reference we will denote by $\mathcal{F}$ the space of smooth functions satisfying this property.

The Morse index of a minimal surface $M$ is defined as the maximal dimension of a vector subspace of $C^\infty(M)$ where $Q$ is negative defined and will be denoted by $\text{Ind}(M)$. If $M$ is a CMC surface, we define the weak index $\text{Ind}_w(M)$ as the maximal dimension of a vector subspace of $\mathcal{F}$ where $Q$ is negative defined. The index is always finite when $M$ is compact and if the index is zero, we say that the surface is stable. In fact, the index indicates the number of directions whose variations decrease area. If $M$ is non compact, we can extend the notion of index by taking a limit of the indices of an exhaustion on $M$. Finally, we recall that all these concepts can be given in higher dimensions as well.

A classical theorem proved independently by do Carmo and Peng [14], Fischer-Colbrie and Schoen [18] and Pogorelov [40] asserts that a complete stable minimal surface in $\mathbb{R}^3$ is a flat plane.

If $M$ is a complete minimal surface with finite index immersed in a manifold $M$ with nonnegative scalar curvature, Fischer-Colbrie [17] and Gulliver [20], also independently, proved that $M$ is conformally diffeomorphic to a compact Riemann surface with genus $g$ and punctured at finitely many points $p_1, \ldots, p_r$, corresponding to the ends of $M$. In [34], using harmonic 1-forms to construct test functions, Ros proved that the index of a minimal surface immersed in $\mathbb{R}^3$ or in quotients of $\mathbb{R}^3$ is bounded from below by a linear function of its genus, namely he proved that $\text{Ind}(M) \geq 2g/3$ if $M$ is orientable and $\text{Ind}(M) \geq g/3$ if $M$ is nonorientable. In the case of oriented minimal surfaces $M \subset \mathbb{R}^3$, Chodosh and Máximo [16], improved Ros’ ideas and proved that $\text{Ind}(M) \geq 2/3(g + r) - 1$.

For closed minimal hypersurfaces in the unit sphere $M \subset S^{n+1}$, Savo [37] proved that the Morse index is bounded from below by a linear function of its first Betti number (the genus in dimension 2) of $M$. This result was generalized recently by Ambrozio, Carlotto and Sharp in [5] and by Mendes and Radesh in [29] for a wide class of ambient manifolds with positive curvature. These results provide a partial answer for a conjecture of Marques and Neves which asserts that the index of a compact minimal hypersurface immersed in ambient spaces with positive Ricci curvature is bounded from below by a linear function of the first Betti number (see [28] and [30]).

Also recently, Chao Li proved in [25] that the index and the nullity (the dimension of the subspace of solutions to $J = 0$) of a complete minimal hypersurface with finite total curvature in the Euclidean space is bounded from below by a linear function of the number of ends and its first Betti number.

In the case of CMC surfaces, Barbosa, Do Carmo and Eschenburg [7] proved that geodesic spheres are the only compact constant mean curvature hypersurfaces in space forms that are stables, that is, whose
weak index is zero. On the other hand, there are few results in the literature about index estimates of other examples of CMC hypersurfaces. To the best of the authors’ knowledge, estimates were given by Lima-Sousa Neto-Rossman \cite{26} and Rossman \cite{35} for CMC tori in $\mathbb{R}^3$, by Perdomo-Brasil \cite{32} for CMC hypersurfaces in the sphere, but in terms of the dimension, and by Rossman-Sultana \cite{36} and Cañete \cite{13} for CMC tori in $S^3$.

The purpose of the present paper is to generalize Ros and Savos’s estimates to obtain lower bounds for the week stability index of compact CMC surfaces immersed in either $\mathbb{R}^3$ or in $S^3$ in terms of the genus.

To do that we apply Ros’ ideas \cite{34} making use of harmonic 1-forms to construct test functions related to the topology of the surface.

We point out that the similar ideas were used by Palmer \cite{31} to obtain a lower bound for the index of the energy functional and also by Torralbo and Urbano in \cite{39} to classify stable CMC spheres in homogeneous spaces.

This paper is organized as follows. Section 2 is devoted to state the main results of the paper. In Section 3 we present the precise definition of index we discuss some known examples of CMC surfaces and they indices. In Section 4 we present some auxiliares results that will be used in Section 5, a section dedicated to the proofs of the theorems.

2. Results

In this section we present the precise statements of our results. For simplicity let us denote by $\bar{M}^3_c$ the space form of constant curvature $c \in \{0, 1\}$, that is, the 3-dimensional unit sphere $S^3$ for $c = 1$ and the 3-dimensional Euclidean space $\mathbb{R}^3$ for $c = 0$.

Our main theorem read as follows.

**Theorem 2.1.** Let $M^2$ be a compact constant mean curvature surface with genus $g$ immersed in $\bar{M}^3_c$. Then,

$$\text{Ind}_w(M) \geq \frac{g}{3 + c}.$$

Note that if $M$ is stable CMC surface in $\bar{M}^3_c$, then Theorem 2.1 implies that $M$ is a topological sphere and by Chern-Hopf Theorem \cite{15,31} it is a round sphere. So we recover Barbosa-do Carmo-Eschenburg Theorem \cite{17} in dimension $n = 2$.

**Remark 2.2.** It is a natural question to wonder whether there is a lower bound for the weak index of compact CMC hypersurfaces in $S^{n+1}$ or in $\mathbb{R}^{n+1}$, $n \geq 3$, in terms of the first Betti number.

As a by product of the technique, we obtain a comparison between the eigenvalues $\lambda^J_0$ of the Jacobi operator of a CMC surface immersed in $\bar{M}^3_c$ and the eigenvalues $\lambda^\Delta_0$ of the Hodge Laplacian acting on 1-forms, in the same spirit as Savo did for minimal hypersurfaces in $S^{n+1}$. 

Theorem 2.3. Let $M^2$ be a compact immersed surface in $\bar{M}_3^3$ with constant mean curvature $H$. Then for all positive integers $\alpha$ we have

$$\lambda_{\alpha}^J \leq -2(c + H^2) + \lambda_{m(\alpha)},$$

where $m(\alpha) > 2(3 + c)(\alpha - 1)$.

Remark 2.4. Bingqing Ma and Guangyue Huang obtained in [27] a similar inequality as in Theorem 2.3 for CMC hypersurfaces in $\mathbb{S}^{n+1}$, but involving the norm of the second fundamental form of $M$. They comparison theorem does not imply index estimates for CMC hypersurfaces.

Remark 2.5. If $M$ is a compact minimal hypersurface in $\mathbb{S}^{n+1}$ which is not totally geodesic, then $-n$ is an eigenvalue of $J$ with multiplicity at least $n + 2$ (see [42], [19], [37]), in particular its index is at least $n + 3$, since the first eigenvalue is simple. Savo [37] used this fact to improve his estimates for the index of minimal hypersurfaces in the sphere. We would like to point out here that Perdomo and Brasil proved in [32] that if $M$ is a compact CMC hypersurface in $\mathbb{S}^{n+1}$ which is not totally umbilical, then $\text{Ind}_w(M) \geq n + 1$, however the negative eigenvalues are not explicit and thus we cannot use it to improve our estimates.

3. Preliminaries

In this section we will considerer hypersurfaces in any dimension.

3.1. The index of constant mean curvature hypersurfaces. Let $M^{n+1}$ be a Riemannian manifold and let $\psi : M^n \rightarrow M^{n+1}$ be an immersed two sided compact hypersurface without boundary. We consider in $M$ the Riemannian metric $g$ induced by $\psi$. Let $\nabla$ and $\bar{\nabla}$ be the Levi-Civita connections on $M$ and $\bar{M}$, respectively. Fixed a unit normal vector field $N$ along $M$, we will denote by $A$ its associated shape operator, that is,

$$AX = -\bar{\nabla}_X N \quad \text{for all } X \in TM.$$

The mean curvature function of $M$ is then defined as $H = (1/n) trA$.

It is well known that every smooth function $u \in C^\infty(M)$ induces a normal variation $\psi_t : M^n \rightarrow M^{n+1}$ given by

$$\psi_t(x) = exp_{\psi(x)}(tu(x)N_x),$$

where $exp$ denotes the exponential map in $M^{n+1}$. Since $M$ is closed and $\psi_0 = \psi$, there exists $\epsilon > 0$ such that

$$M_{u,\epsilon} = \{ exp_{\psi(x)}(tu(x)N) ; x \in M \}$$

are immersed hypersurfaces for all $t \in (-\epsilon, \epsilon)$. We can consider the area functional $A_u : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ which is given by

$$A_u(t) = \int_M dM_{u,t},$$
where $dM_{u,t}$ is the $n$-dimensional area element of the metric induced on $M$ by $ψ_t$. The first variation formula for the area is given by

$$A_u'(0) = -n \int_M uHdM.$$ 

As a direct consequence, minimal hypersurfaces are characterized as critical points of the area functional, while constant mean curvature (CMC) hypersurfaces are the critical points of the area functional restricted to variations that preserves volume, that is, $\int_M udM = 0$. For such critical points, the second variation of the area functional is given by the following quadratic form

$$A_u''(0) = \int_M \|\nabla u\|^2 - (\text{Ric}(N) + \|A\|^2)u^2 dM.$$ 

Here $\|A\|^2 = tr(A^2)$ is the Hilbert-Schmidt norm of $A$ and $\text{Ric}(N)$ denotes the Ricci curvature of $\tilde{M}$ in the direction of $N$. Integrating by parts we can write

$$A_u''(0) = \int_M uJu dM.$$ 

where $J = \Delta - \text{Ric}(N) - \|A\|^2$ is the so called Jacobi operator or stability operator of $M$. We also point out that we are using the geometric definition of the Laplace-Beltrami operator, that is, $\Delta u = \delta du$ where $\delta w = -tr\nabla w$ for $w \in \Omega^1(M)$.

The index of a CMC hypersurface $M$ is denoted by $\text{Ind}_w(M)$ and defined as the maximum dimension of any subspace $V$ of $F = \{u \in C^\infty(M); \int_M u = 0\}$ on which $A_u''(0)$ is negative definite. In other words, $\text{Ind}_w(M)$ is the number of negative eigenvalues of $J$, which is necessarily finite for closed hypersurfaces. In the special case that $\tilde{M}^3$ has constant sectional curvature $c$, the Jacobi operator reads as

$$J = \Delta - \|A\|^2 - 2c,$$

and this form will be used in the rest of the paper.

### 3.2. Examples of compact CMC hypersurfaces and their indices.

The most simple examples of closed CMC surfaces in the Euclidean space are the geodesic spheres, which are the only stables ones [6]. In 1986, Wente [43] constructed the first examples of CMC tori in $\mathbb{R}^3$ solving a question posed by Hopf in [21]. After that, all CMC tori in space forms were classified in a series of works (see [1], [8], [33] and [41]) and their indices were estimated by Lima, Sousa Neto and Rosmann in [26] and by Rosmann in [35]. One can summarizing they results as follows.
The index of a CMC torus in $\mathbb{R}^3$ is at least 8 and there are CMC tori with arbitrarily large index.

Many examples of compact CMC surfaces with genus $g \geq 2$ were constructed by Kapouleas in [22, 23, 24] using the gluing method.

When the ambient space is the round sphere $S^{n+1}$ many examples of CMC hypersurfaces are known. Again, geodesic spheres are the only stable CMC hypersurfaces [4]. The next important family is given by the CMC Clifford tori, including the case $H = 0$. In fact, it was proved by S. Brendle [9] that that Clifford tori are the only minimal embedded tori in $S^3$, confirming longstanding conjecture of H. Lawson. It is proved by B. Andrews and H. Li [4] that all CMC embedded tori in $S^3$ are rotational surfaces, confirming longstanding conjectures of Pinkall-Sterling. We also note that Andrews and Li [4] gave a complete classifications of all CMC embedded tori in $S^3$. It follows from the work of Simons [38] that any compact minimal hypersurface not totally geodesic in $S^{n+1}$ has index $\text{Ind}(M) \geq n + 3$ and it is also well known that the minimal Clifford torus has index $n + 3$. It is a natural problem to classify the minimal hypersurfaces $M \subset S^{n+1}$ with $\text{Ind}(M) = n + 3$. This problem was solved when $n = 2$ by Urbano in [42], showing that the minimal Clifford tori are the only minimal surface in $S^3$ whose index is 5. This problem is still open in higher dimensions.

In the case $H \neq 0$, Perdomo and Brasil [32] proved that if $M \subset S^{n+1}$ is a compact CMC hypersurface not totally umbilical, then $\text{Ind}_w(M) \geq n + 1$ (see also [2] for a nice survey). Recently, Alias and Piccione [3] showed the existence of infinite sequences of isometric embeddings of tori with constant mean curvature in Euclidean spheres that are not isometrically congruent to the CMC Clifford tori, and accumulating at some CMC Clifford torus. In the same spirit as above, the index of CMC tori of revolution in $S^3$ were estimated by Rossman and Sultana in [36] and by Cañete in [13]. On the other hand, higher genus CMC surfaces in $S^3$ were constructed by Butscher-Pacard [12] (see also [10, 11] for examples higher dimensions), but no index estimates is known.

4. Test functions based on coordinates of vector fields

In this section we will consider CMC immersed surfaces $\psi : M \to \mathbb{R}^3$. In the spherical case, $c = 1$, we will also consider the unit normal vector field $\nu = -\psi$ along $S^3$, such that the second fundamental form of the inclusion map, $\tilde{\psi} : S^3 \to \mathbb{R}^4$, is the identity. That is, $D_X \nu = -X$ where $D$ denotes the Levi-Civita connection in the Euclidean space. It follows immediately that

$$D_Y \nu = -\nabla_Y \nu = \langle X, Y \rangle \nu + \langle AX, Y \rangle N,$$

for all $X, Y \in TM$.

for $M$ immersed in $S^3$ and

$$D_Y X - \nabla_Y X = \langle AX, Y \rangle N,$$

for all $X, Y \in TM$. 


for $M$ immersed in $\mathbb{R}^3$.

Fixed an orthonormal basis $\mathcal{E} = \{\vec{E}_1, \ldots, \vec{E}_{3+c}\}$ of parallel vector fields on $\mathbb{R}^{3+c}$ we will denote by

$$E_i := \vec{E}_i - \langle \vec{E}_i, N \rangle N - c \langle \vec{E}_i, \nu \rangle \nu$$

the vector fields given by the orthogonal projection of $\vec{E}_i$ on $TM$. For the fixed basis $\mathcal{E}$ we will consider the smooth support functions $f_i, g_i : M \to \mathbb{R}$ given by

$$f_i = \langle \vec{E}_i, \nu \rangle \quad \text{and} \quad g_i = \langle \vec{E}_i, N \rangle,$$

for $1 \leq i \leq 3 + c$.

Let $\xi \in TM$ be a smooth vector field on $M$ and let $\omega$ denote its dual 1-form, that is $\xi = \omega^\#$. Inspired in the works of Ros and Savo we will use the coordinates of $\xi \in TM$ as test functions. They are given by

$$w_i := \langle E_i, \xi \rangle, \quad 1 \leq i \leq 3 + c.$$

Let us denote by $\nabla^* \nabla$ the rough Laplacian acting on vector fields and by $\Delta$ the Hodge Laplacian acting on 1-forms. They are defined, respectively, by

$$\nabla^* \nabla \xi = - \text{tr} \nabla^2 \xi \quad \text{and} \quad \Delta \omega = d\delta \omega + \delta d\omega,$$

where $d$ is the exterior differential and $\delta = - * d *$ is the formal adjoint of $d$ with respect to the canonical $L^2$-inner product on 1-forms induced by the Riemannian metric of $M$. We define the Laplacian of the vector fields $\xi$ as being $\Delta \xi = (\Delta \omega)^\#$ and so these Laplacians are related by the well known Bochner formula

$$\Delta \xi = \nabla^* \nabla \xi + K \xi,$$

where $K$ is the Gauss curvature of $M$.

In order to compute the Jacobi operator of $w_i$ we need the following lemma.

**Lemma 4.1.** Let $M^2$ be an orientable CMC surface immersed in $\bar{M}_c^3$. Then, using the above notation we have

$$\Delta w_i = (\|A\|^2 - 4H^2)w_i + 2H \langle AE_i, \xi \rangle - 2g_i \langle A, \nabla \xi \rangle + 2c f_i \text{div} \xi + \langle E_i, \Delta \xi \rangle,$$

for $1 \leq i \leq 3 + c$.

**Proof.** Fixed a point $p \in M$, we consider a local orthonormal frame $\{e_1, e_2\}$ on $M$ which is geodesic at $p$. A direct computation shows that
\[ \nabla_{e_\ell} E_i = g_i A e_\ell + c f_i e_\ell, \quad \ell = 1, 2. \] Thus, using Einstein summation notation, we get

\[
\Delta w_i = -e_\ell e_\ell \langle E_i, \xi \rangle = -e_\ell (\langle \nabla_{e_\ell} E_i, \xi \rangle + \langle E_i, \nabla_{e_\ell} \xi \rangle) = -\langle \nabla_{e_\ell} (g_i A e_\ell + c f_i e_\ell), \xi \rangle - 2 \langle g_i A e_\ell + c f_i e_\ell, \nabla_{e_\ell} \xi \rangle + \langle E_i, \nabla^* \nabla \xi \rangle = \langle A^2 E_i, \xi \rangle - 2 g_i \langle A, \nabla \xi \rangle + 2 c f_i \nabla \xi + \langle E_i, \Delta \xi \rangle - 2 H^2 \langle E_i, \xi \rangle + \|A\|^2 \langle E_i, \xi \rangle,
\]

where in the last equality we used the Bochner equation (4.1) and the Gauss equation in the form below:

\[ K = c + 2 H^2 - \frac{1}{2} \|A\|^2. \]

To conclude the proof we note that shape operator \( A \) satisfies the following equation

\[ A^2 = \frac{1}{2} (\|A\|^2 - 4 H^2) I_2 + 2 H A. \]

We end this section noting that the coordinates of harmonic vector fields are admissible functions to compute the index of CMC surfaces. Moreover, we have:

**Lemma 4.2.** If \( \xi \in TM \) is a harmonic vector field, then \( w_i = \langle E_i, \xi \rangle \) and \( \bar{w}_i = \langle E_i, \Star \xi \rangle \) satisfy

\[ \int_M w_i = \int_M \bar{w}_i = 0 \]

for \( 1 \leq i \leq 3 + c \).

**Proof.** If \( \xi \) is harmonic we have \( \text{div} \xi = 0 \) and therefore

\[ \int_M w_i = -\int_M \langle \nabla f_i, \xi \rangle = -\int_M f_i \text{div} \xi = 0. \]

Now, recall that, in an orthonormal basis \( \{e_1, e_2\} \) of \( TM \), the Hodge star operator is defined by

\[ \Star e_1 = e_2, \quad \Star e_2 = -e_1. \]

Since \( \Delta \) commutes with \( \Star \) it follows that \( \Star \xi \) is also a harmonic vector field on \( M \) and this concludes the proof. \( \square \)
5. Proofs of the theorems

For simplicity we will present the proofs in the case of CMC surfaces in the unit sphere $S^3$. The case of CMC surfaces in $\mathbb{R}^3$ follows the same steps.

Let us denote by $L_\Delta^m$ the vector space given by the direct sum of the eigenspaces generated by $\xi_1, \xi_2, \ldots, \xi_m$, the first $m$ eigenfunctions of the Hodge Laplacian $\Delta$ and let us denote by $\mathcal{H}^1(M)$ the vector space of the harmonic vector fields on $M$. Notice that $\dim \mathcal{H}^1(M) = 2g$, where $g$ is the genus of surface $M$, and $\mathcal{H}^1(M) \subset L_\Delta^m$.

5.1. Proof of Theorem 2.3. Since $J$ is an elliptic self-adjoint operator, it admits a sequence of eigenvalues diverging to infinity, 

$$
\lambda_1^J \leq \lambda_2^J \leq \cdots \leq \lambda_k^J \leq \cdots
$$

Fix an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of $C^\infty(M)$ given by eigenfunctions of the Jacobi operator, that is, $J\phi_i = \lambda_i^J \phi_i$. We denote by $J^p := \langle \phi_1, \cdots, \phi_p \rangle$ the linear space orthogonal to the first $p$ eigenfunctions of the Jacobi operator.

Initially, we look for vector fields $\xi \in L_\Delta^m$ such that the functions $w_i, \bar{w}_i \in J^{\alpha-1}$, for some $\alpha \in \mathbb{N}$ and $i \in \{1, \ldots, 4\}$. In other words, we have a system with $8(\alpha - 1)$ homogenous linear equations in the variable $\xi$

\[
\int_M w_i \phi_k = \int_M \bar{w}_i \phi_k = 0, \quad 1 \leq i \leq 4 \quad \text{and} \quad 1 \leq k \leq \alpha - 1.
\]

Therefore, if $m(\alpha) = \dim L_\Delta^m > 8(\alpha - 1)$, then the system (5.1) has at least a non trivial solution $\xi \in L_\Delta^m$ such that $w_i, \bar{w}_i \in J^{\alpha-1}$ for all $1 \leq i \leq 4$. By the min-max principle we have

$$
\lambda_1^J \leq \frac{\int_M w_i J w_i}{\int_M w_i^2} \quad \text{and} \quad \lambda_1^J \leq \frac{\int_M \bar{w}_i J \bar{w}_i}{\int_M \bar{w}_i^2}.
$$

Now, using Lemma 4.1 we get

$$
\lambda_1^J \int_M w_i^2 \leq -(2 + 4H^2) \int_M \langle E_i, A\xi \rangle w_i + \int_M \langle E_i, \Delta \xi \rangle w_i - 2 \int_M g_i \langle A, \nabla \xi \rangle w_i + 2 \int_M f_i \delta w_i.
$$

Summing upon $i = 1, \ldots, 4$ we obtain

$$
\lambda_1^J \int_M \|\xi\|^2 \leq -(2 + 4H^2) \int_M \|\xi\|^2 + 2 \int_M \langle A\xi, \xi \rangle + \int_M \langle \Delta \xi, \xi \rangle.
$$

Analogously, we do the same to the test functions $\bar{w}_i$:

$$
\lambda_1^J \int_M \|\xi\|^2 \leq -(2 + 4H^2) \int_M \|\xi\|^2 + 2 \int_M \langle A \ast \xi, \ast \xi \rangle + \int_M \langle \Delta \ast \xi, \ast \xi \rangle.
$$
Summing these last two inequalities we have

\[(5.2)\lambda_{\alpha}^J \int_M \|\xi\|^2 \leq - (2 + 4H^2) \int_M \|\xi\|^2 + H \int_M \langle A\xi, \xi \rangle + \langle A^* \xi, *\xi \rangle + \frac{1}{2} \int_M (\langle \Delta \xi, \xi \rangle + \langle \Delta^* \xi, *\xi \rangle) \]

Now, we observe that

\[(5.3) \quad \langle A\xi, \xi \rangle + \langle A^* \xi, *\xi \rangle = 2H\|\xi\|^2 \]

for any \(\xi \in TM\). If \(\xi \in \mathcal{L}_m^\alpha\) we can \(\xi = \alpha_i \xi_i\) and then

\[(5.4) \quad \int_M \langle \Delta^* \xi, *\xi \rangle = \lambda_i^\Delta \int_M \alpha_i \alpha_k \langle \xi_i, \xi_k \rangle \leq \lambda_{m(\alpha)} \int_M \|\xi\|^2. \]

Plugging (5.3) and (5.4) into (5.2) we obtain

\[\lambda_{\alpha}^J \leq -2(1 + H^2) + \lambda_{m(\alpha)} \]

where \(m(\alpha) > 8(\alpha - 1)\).

5.2. **Proof of Theorem 2.1.** We start as in proof of Theorem 2.3 but now choosing an orthonormal basis \(\{\phi_1, \phi_2, \ldots\}\) of the space \(\mathcal{F} = \{u \in C^\infty(M); \int_M u = 0\}\) given by eigenfunctions of the Jacobi operator. We already know from Lemma 4.2 that for any \(\xi \in \mathcal{H}^1(M)\) the test functions \(w_i, \bar{w}_i \in J\alpha - 1\), for some \(\alpha \in \mathbb{N}\) and \(i \in \{1, \ldots, 4\}\). As before, if \(\dim \mathcal{H}^1(M) = 2g > 8(\alpha - 1)\), then the system (5.1) has at least a non trivial solution \(\xi \in \mathcal{H}^1(M)\). Following the same steps as above, we use Lemma 4.1 but now for the harmonic vector fields \(\xi\) and its dual \(*\xi\). We obtain

\[\lambda_{\alpha}^J \int_M \|\xi\|^2 \leq -2(1 + H^2) \int_M \|\xi\|^2. \]

Hence, we conclude that \(\lambda_{\alpha}^J < 0\) and then \(\text{Ind}_w(M) \geq \alpha\). Since \(\alpha\) can be chosen as the largest integer such that \(2g > 8(\alpha - 1)\) we get

\[\text{Ind}_w(M) \geq \frac{g}{4}. \]

**References**

1. Uwe Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. 374 (1987), 169–192.
2. Luis Alias, *On the stability index of minimal and constant mean curvature hypersurfaces in spheres*, Revista Unión Math. Argent. 47 (2006), 39–61.
3. Luis Álías and Paolo Piccione, *Bifurcation of constant mean curvature tori in Euclidean spheres*, J. Geom. Anal. 23 (2013), no. 2, 677–708.

4. Ben Andrews and Haizhong Li, *Embedded constant mean curvature tori in the three-sphere*, J. Differential Geom. 99 (2015), no. 2, 169–189.

5. Lucas Ambrozio, Alessandro Carlotto and Ben Sharp, *Comparing the Morse index and the first Betti number of minimal hypersurfaces*, J. Differential Geom. 108 (2018), no. 3, 379–410.

6. João Lucas Barbosa and Manfredo do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. 185 (1984), 339–353.

7. João Lucas Barbosa, Manfredo do Carmo and J. Eschenburg, *Stability of hypersurfaces with constant mean curvature in Riemannian manifolds*, Math. Z. 197 (1988), 123–138.

8. Alexander Ivanovich Bobenko, *All constant mean curvature tori in $\mathbb{R}^3$, $S^3$ and $H^3$ in terms of theta-functions*, Math. Ann. 290 (1991), 209–245.

9. Simon Brendle, *Embedded minimal tori in $S^3$ and the Lawson conjecture*, Acta Math. 211 (2013), no. 2, 177–190.

10. Adrian Butscher, *Constant mean curvature hypersurfaces in $S^{n+1}$ by gluing spherical building blocks*, Math. Z. 263 (2009), no. 1, 1–25.

11. Adrian Butscher, *Gluing constructions amongst constant mean curvature hypersurfaces of $S^{n+1}$*, Ann. Glob. Anal. Geom. 36 (2009), 221–274.

12. Adrian Butscher and Frank Pacard, *Generalized doubling constructions for constant mean curvature hypersurfaces in $S^{n+1}$*, Ann. Global Anal. Geom. 32 (2007), no. 2, 103–123.

13. Antonio Cañete, *A new bound on the Morse index of constant mean curvature tori of revolution in $S^n$*, Calc. Var. Partial Differential Equations 45 (2012), no. 3-4, 467–479.

14. Manfredo do Carmo, Chiakuei Peng, *Stable complete minimal surfaces in $\mathbb{R}^3$ are planes*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 6, 903–906.

15. Chern, Shiing Shen Chern, *On surfaces of constant mean curvature in a three-dimensional space of constant curvature*, Geometric dynamics (Rio de Janeiro, 1981), 104–108, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

16. Otis Chodosh, Davi Maximo, *On the topology and index of minimal surfaces*, J. Differential Geom. 104 (2016), no. 3, 399–418.

17. Doris Fischer-Colbrie, *On complete minimal surfaces with finite Morse index*, Inventiones Math., 82 (1985), 121–132.

18. Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. 33(2), 199–211 (1980)

19. Ahmad El Soufi, *Applications harmoniques, immersions minimales et transformations conformes de la sphère*, Compositio Math. 85 (1993), 281–298.

20. Robert Gulliver, *Index and total curvature of complete minimal surfaces*, Geometric measure theory and the calculus of variations, (Arcata, Calif., 1984), Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 207–211.

21. Heinz Hopf, *Differential geometry in the large*, Notes taken by Peter Lax and John W. Gray. With a preface by S. S. Chern. Second edition. With a preface by K. Voss. Lecture Notes in Mathematics, 1000. Springer-Verlag, Berlin, 1989. viii+184 pp.

22. Nikolaos Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. (2) 131 (1990), no. 2, 239–330.

23. Nikolaos Kapouleas, *Compact constant mean curvature surfaces in Euclidean three-space*, J. Differential Geom. 33 (1991), no. 3, 683–715
24. Nikolaos Kapouleas, *Constant mean curvature surfaces constructed by fusing Wente tori*, Invent. Math. 119(3) (1995), 443–518.

25. Chao Li, *Index and topology of minimal hypersurfaces in \( \mathbb{R}^n \)*, arXiv:1605.09693v1. To appear in Calculus of Variation and PDE.

26. Levi Lopes de Lima, Vicente Francisco de Sousa Neto and Wayne Rossman, *Lower bounds for index of Wente tori*, Hiroshima Math. J. 31 (2001), no. 2, 183–199.

27. Bingqing Ma and Guangyue Huang, *Eigenvalue relationships between Laplacians of constant mean curvature hypersurfaces in \( S^{n+1} \)*, Commun. Math. 21 (2013), no. 1, 31–38.

28. Fernando Codá Marques, *Minimal surfaces - variational theory and applications*, Proceedings of the International Congress of Mathematicians, Seoul 2014.

29. Ricardo Mendes and Marco Radeshi, *Generalized immersions and minimal hypersurfaces in compact symmetric spaces*, arXiv:1708.05881 [math.DG].

30. André Neves, *New applications of Min-max Theory*, Proceedings of the International Congress of Mathematicians, Seoul 2014

31. Bennett Palmer, *Index and stability of harmonic Gauss maps*, Math. Z. 206 (1991), no. 4, 563–566.

32. Oscar Perdomo and Aldir Brasil, *Stability index jump for CMC hypersurfaces of spheres*, Arch. Math. 99 (2012), 493–500.

33. Ulrich Pinkall and Ivan Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. (2) 130 (1989), 407–451.

34. Antonio Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom. 74 (2006), no. 1, 69–92.

35. Wayne Rossman, *Lower bounds for Morse index of constant mean curvature tori*, Bull. London Math. Soc. 34 (2002), no. 5, 599–609.

36. Wayne Rossman and Nahid Sultana, *Morse index of constant mean curvature tori of revolution in the 3-sphere*, Illinois J. Math. 51 (2007), no. 4, 1329–1340.

37. Alessandro Savo, *Index bounds for minimal hypersurfaces of spheres*, Indiana Univ. Math. J. 59 (2010), 823–837.

38. James Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) 88 1968 62–105.

39. Francisco Torralbo and Francisco Urbano, *Compact Stable Constant Mean Curvature Surfaces in Homogeneous 3-manifolds*, Indiana Univ. Math. J. 61 (2012), 1129–1156.

40. Aleksei Vasil'evich Pogorelov, *On the stability of minimal surfaces*, Dokl. Akad. Nauk SSSR 260(2), 293–295 (1981)

41. Masaki Umehara and Kotaro Yamada, *A deformation of tori with constant mean curvature in \( \mathbb{R}^3 \) to those in other space forms*, Trans. Am. Math. Soc. 330 (1992), 845–857.

42. Francisco Urbano, *Minimal surfaces with low index in the three-dimensional sphere*, Proc. Amer. Math. Soc. 108 (1990), 989–992.

43. Henry Wente, *Counterexample to a Hopf conjecture*, Pacific J. Math. 121 (1986), no. 1, 193–243.
Instituto de Matemática
Universidade Federal de Alagoas (UFAL)
Campus A. C. Simões, BR 104 - Norte, Km 97, 57072-970.
Maceió - AL - Brazil
E-mail address: marcos@pos.mat.ufal.br

Departamento de Ciências Exatas
Universidade Estadual de Feira de Santana (UEFS)
Avenida Transnordestina, S/N, Novo Horizonte, 44036-900
Feira de Santana - BA - Brazil
E-mail address: darlanfdeoliveira@gmail.com