The momentum-dependent effective sound speed and multi-field inflation

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For any physical system satisfying the Einstein’s equations the comoving curvature perturbations satisfy an equation involving the momentum-dependent effective sound speed (MESS), valid for any system with a well defined energy-stress tensor (EST), including multi-fields models of inflation. While the power spectrum of adiabatic perturbations may generically receive contributions from many independent quantum degrees of freedom present in these models, there is often a single mode that dominates adiabatic perturbations of a given wavelength and evolves independently of other modes, with evolution entirely described by MESS. We study a number of two-field models with a kinetic coupling between the fields, identifying this single effective mode and showing that MESS fully accounts for the predictions for the power spectrum of adiabatic perturbations. Our results show that MESS is a convenient scheme for describing all inflationary models that admit a single-field effective theory, including the effects of entropy perturbations present in multi-fields systems, which are not included in the effective theory of inflation.

I. INTRODUCTION

The study of cosmological perturbations is one of the foundations of modern cosmology, since it allows to make quantitative predictions of different observables such as the cosmic microwave background (CMB) radiation or large scale structure formation. In the simplest models of inflation, consisting in a scalar field minimally coupled to gravity, the scalar field is driving the accelerated expansion of the Universe, and its perturbations induce metric perturbations which, in the comoving gauge, obey an evolution equation containing a Laplacian whose coefficient is called sound speed. In these models the sound speed is only a function of time, but it has been shown \cite{1} that a similar equation, but with a space or momentum dependent sound speed, is satisfied by an arbitrary physical system satisfying Einstein’s equations, including multifields, modified gravity, or a combination of the two. The momentum-dependent effective sound speed (MESS) is given by the ratio between the Fourier transform of the pressure and energy density perturbations in the comoving gauge, obtained from the effective stress-energy-momentum tensor (EST) appearing on the right hand side of the Einstein’s equations.

From the point of view of the action formalism the for curvature perturbations is obtained by integrating out the entropy field, but in the effective energy momentum tensor formulation on which the MESS definition is based, this is just a special case of the general equation for comoving curvature perturbation which can be derived in a model independent way manipulating the Einstein’s equations, without specifying the matter content of the theory.

The MESS can be defined for any system, including systems with multiple scalar fields, but it does not guarantee the existence of an effective single field quantum theory. This follows from the fact that in general a given mode of adiabatic perturbations can include contributions from different degrees of freedom. However, there exist a broad class of models, including models with a strong kinetic coupling between the adiabatic and entropy perturbations, in which the mode of adiabatic perturbations responsible for generation of observable CMB anisotropies evolves independently of other modes.

In this paper, we show that the evolution of the effective adiabatic mode is correctly described within the MESS formalism, clarifying the notion of effective single-field theory for inflationary perturbations and providing a set of numerical calculations corresponding to specific two-field inflationary models that have attracted considerable attention.

The paper is organized as follows. In Section [II] we briefly introduce the MESS formalism. In Section [III] we analyze decoupling of heavy degrees of freedom and calculate the sound speed in models with a constant turning rate of the inflationary trajectory from the geodesic line. In Section [IV] we discuss the normalization of perturbations and appropriate initial conditions. Section [V] is devoted to numerical examples corroborating our analytical calculations. After a short discussion of the results in Section [VI] we conclude in Section [VII]. A technical derivation of the sound speed in two-field models with arbitrary field-space metric is relegated to the Appendix.
II. THE MOMENTUM EFFECTIVE SOUND SPEED

A. Derivation of the effective equation of motion

We use the following notation for the metric and energy momentum tensor perturbations

\[ ds^2 = -(1 + 2 \gamma) dt^2 + 2a \partial_i \mu dx^i dt + a^2 \{ \delta_{ij}(1 + 2 \zeta) + 2 \partial_i \partial_j \nu \} dx^i dx^j, \]

\[ T^0_0 = -(\rho + \delta \rho), \quad T^0_i = (\rho + P) \partial_i (v + B), \]

\[ T^i_j = (P + \delta P) \delta^i_j + \delta^{ik} \partial_k \partial_j \Pi - \frac{1}{3} \delta^i_j \Delta \Pi, \]

where \( v \) is the velocity potential and \( \Delta^{(3)} = \delta^{ik} \partial_k \partial_i \).

Note that the above equations are completely general since according to scalar-vector-tensor (SVT) decomposition they give the most general way to perturb the metric and the energy momentum tensor of a homogeneous and isotropic universe. For any physical system an appropriate effective stress-energy tensor (EST) can be defined and can be written in the form given above. All the results derived from this set up are consequently general and can be applied to any physical system for which an EST can be defined. In particular for any theory admitting a lagrangian formulation the EST can be obtained by taking the variation with respect to the metric.

The comoving slices gauge, from now on for brevity called comoving gauge, is defined by the condition \( (T^0_0)_c = 0 \), where from now on we will be denoting with a subscript \( c \) quantities evaluated on comoving slices. We will denote the metric and the perturbed EST in the comoving gauge as

\[ ds^2 = -(1 + 2 \gamma) dt^2 + 2a \partial_i \mu dx^i dt + a^2 \{ \delta_{ij}(1 + 2 \zeta) + 2 \partial_i \partial_j \nu \} dx^i dx^j, \]

\[ (T^0_0)_c = -(\rho + \beta), \]

\[ (T^i_j)_c = (P + \alpha) \delta^i_j + \delta^{ik} \partial_k \partial_j \Pi - \frac{1}{3} \delta^i_j \Delta \Pi, \]

where we have defined the gauge invariant quantities \( \alpha = \delta P_c, \beta = \delta \rho_c, \gamma = A_c, \mu = B_c, \zeta = C_c, \nu = E_c \).

In the case of a single scalar field, the comoving gauge coincides with the uniform field gauge, also known as unitary gauge, but for multi-field systems they are different.

The standard approach to the study of the evolution of cosmological perturbation in multi-fields systems involves the solution of a system of coupled differential equations for fields perturbations in the flat gauge \( Q_i \), which are gauge invariant by construction. The comoving curvature perturbation \( \zeta \) is then obtained using the gauge invariant relation between the \( Q_i \) and \( \zeta \).

Recently it was shown [1] that an alternative approach could be adopted, based on the solution of a single differential equation

\[ \ddot{\zeta} + \frac{\partial_i (Z^2 \dot{\zeta})}{Z^2} \zeta - \frac{v_s^2}{a^2} \Delta \zeta + \frac{v_s^2}{\epsilon} \Delta \Pi + \frac{1}{3Z^2} \dot{\partial}_i \left( \frac{Z^2}{H \epsilon} \Delta \Pi \right) = 0, \]

where \( Z^2 \equiv \epsilon a^3/v_s^2 \) and an effective space dependent sound speed (SESS) has been defined as

\[ v_s^2(t, x^i) = \frac{\alpha(t, x^i)}{\beta(t, x^i)}. \]

In this picture the entropy perturbations are not appearing explicitly in the equation for curvature perturbations, and are "hidden" in the SESS.

In fact in the standard approach [2] entropy perturbations \( \Gamma \) are defined by

\[ \alpha(t, x^i) = c_s(t)^2 \beta(t, x^i) + \Gamma(t, x^i), \]

where \( c_s \) is interpreted as sound speed and is a function of time only. Combining eq. (8) and (7) we get the relation between SESS and entropy

\[ v_s^2 = c_s^2 \left[ 1 + \frac{\Gamma}{2H \epsilon \left( \zeta + \frac{1}{3H \epsilon} \Delta \Pi \right)} \right]^{-1}. \]
After defining the momentum dependent effective sound speed (MESS) $\hat{v}_k(t)^2$ as

$$\hat{v}_k(t)^2 = \frac{\alpha_k(t)}{\beta_k(t)},$$  \hspace{1cm} (10)

and following a procedure mathematically similar to the one used to derive eq. (9) we can obtain this equation in momentum space [3]

$$\ddot{\zeta}_k + \left(3H + \frac{\partial_t(\hat{Z}_k^2)}{\hat{Z}_k^2}\right) \dot{\zeta}_k + \frac{\hat{v}_k^2}{a^2} k^2 \zeta_k - \frac{\hat{v}_k^2}{\epsilon} \hat{Z}_k^2 k^2 \Pi_k - \frac{1}{3} \hat{Z}_k^2 \partial_t \left(\frac{\hat{Z}_k^2}{H} k^2 \Pi_k\right) = 0,$$  \hspace{1cm} (11)

where $\hat{Z}_k^2 \equiv \epsilon/\hat{v}_k^2$. In this paper we will consider scalar fields with isotropic EST, for which eq. (11) simplifies to

$$\ddot{\zeta}_k + \left(3H + \frac{\partial_t(\hat{Z}_k^2)}{\hat{Z}_k^2}\right) \dot{\zeta}_k + \frac{\hat{v}_k^2}{a^2} k^2 \zeta_k = 0.$$  \hspace{1cm} (12)

It can be shown that eq. (12) reduces to the Sasaki-Mukhanov equation when $\hat{v}_k$ is a function of time only. It is important to note that the MESS $\hat{v}_k(t)$ defined in eq. (10) is not simply the Fourier transform of the SESS $v_s(x^\mu)$ defined in eq. (7), because the product of the Fourier transforms of two functions is the transform of the convolution of the two functions.

Quite remarkably the equations above can be applied to any system described by an appropriate EST, including multi-fields, supergravity, and modified gravity theories.

III. EFFECTIVE EQUATION OF MOTION VS FULL THEORY IN MULTI-FIELD MODELS

As a particular example, we will consider models involving $N = 2$ scalar fields minimally coupled to Einstein gravity, and whose action reads:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} G_{IJ} (\phi^K) \partial_\mu \phi^I \partial^\mu \phi^J - V (\phi^K) \right].$$  \hspace{1cm} (13)

In eq. (13), uppercase Latin letters refer to the field space directions and summation over repeated indices is assumed. It is then convenient to project the evolution of homogeneous fields and the perturbations in the field space onto the adiabatic/entropic basis $(e^I_\sigma, e^J_\sigma)$ [4 5], where $e^I_\sigma \equiv \hat{\phi}^I/\hat{\sigma}$ is the unit vector pointing along the background trajectory in field space, and where $e^I_\sigma$ is such that the basis $(e^I_\sigma, e^J_\sigma)$ is orthonormal and right-handed for definiteness; the velocity of the system in the field space reads $\dot{\sigma} = (G_{IJ} \hat{\phi}^I \hat{\phi}^J)^{1/2}$. The adiabatic perturbation $Q_\sigma \equiv e^I_\sigma Q^I$ is directly proportional to the comoving curvature perturbation $\zeta = H/\hat{\sigma} Q_\sigma$, while the genuine multifield effects are embodied by the entropic fluctuation $Q_s$, perpendicular to the background trajectory.

In this basis, the equations of motion take the form

$$\ddot{Q}_\sigma + 3H \dot{Q}_\sigma + \left( \frac{k^2}{a^2} + \frac{m^2}{a^2} \right) Q_\sigma = 2H \eta_\perp \dot{Q}_s - \left( \frac{H}{\hat{H}} + \frac{V_s}{\hat{\sigma}} \right) 2H \eta_\perp Q_s,$$  \hspace{1cm} (14)

$$\ddot{Q}_s + 3H \dot{Q}_s + \left( \frac{k^2}{a^2} + \frac{m^2}{a^2} \right) Q_s = -2\hat{\sigma} \eta_\perp \dot{\zeta},$$  \hspace{1cm} (15)

where

$$\eta_\perp = \frac{-V_s}{H \hat{\sigma}}$$  \hspace{1cm} (16)

is the dimensionless parameter, describing the rate (in Hubble times) at which the trajectory in the field space deviates from a geodesic line [6]. Here $V_s \equiv e^I_\sigma V_I$, the adiabatic mass (squared) is given by $m^2/Q^2 = -\frac{2}{\hat{\sigma}} e_2 + \ldots$ with the slow-roll parameters given by $\epsilon_1 \equiv -\frac{H}{H^2}$, $\epsilon_2 \equiv \frac{H}{H^3}$, and the dots representing terms of higher order in the slow-roll parameters, and the entropic mass squared reads $m^2_s = V_{ss} - 2(H \eta_\perp)^2$. 
In order to connect the system of equations of motion (14) and (15) to the effective equation (11), we note that
\[
\alpha_k(t) = -\frac{\zeta}{H} \dot{\sigma} = -\frac{H^2 \dot{\sigma}^2}{H} \frac{k^2}{a^2 H^2} \Psi - 2\eta_{\perp} HQ_s,
\]
(17)
\[
\beta_k(t) = \alpha + 2\eta_{\perp} HQ_s = -\frac{H^2 \dot{\sigma}^2}{H} \frac{k^2}{a^2 H^2} \Psi,
\]
(18)
where \(\Psi\) is the Bardeen potential, which can be expressed as:
\[
\Psi = -\frac{H}{a} \int \frac{\dot{H} a}{H^2} dt'.
\]
(19)
Inserting (17) and (18) into (10), we find that
\[
v_k^{-2} = 1 - \frac{2\eta_{\perp} H^2 Q_s}{\zeta \dot{\sigma}}.
\]
(20)
Plugging (20) into (6), we find that the latter equation, upon setting \(\Pi = 0\), which is appropriate for the system of scalar fields, is equivalent to (14), i.e. it describes the evolution of the adiabatic perturbations if it is supplemented by (15) that dictates the evolution of the entropy perturbations.

And important comment is now in order. Eq. (20) is well defined in classical theory with \(\zeta\) and \(Q_s\) being number-valued functions of time. However, the interpretation of (20) becomes murky when the perturbations are quantized, because the formula would contain a ratio of two quantum operators. We would therefore like to dedicate more attention to the proper effective description of perturbations in that case. Therefore, we would like to discuss a number of cases that have already been studied and for which the predictions for the power spectrum of curvature perturbations have been calculated.

A. Geodesic trajectory

If the trajectory in the field space follows a geodesic line, the entropy perturbations do not affect the adiabatic perturbations, which evolve as if the entropy perturbations were entirely absent. We can, therefore, set \(Q_s = 0\) in eq. (20) and conclude that the speed of adiabatic perturbations is that of light, \(v_k = 1\).

B. Sourcing on super-Hubble scales

If the amplitude of the entropy modes are not significantly smaller than those of after the adiabatic ones after Hubble-radius crossing and the trajectory in the field space does not follow a geodesic line, adiabatic perturbations are sourced by the entropy ones. The rate of this sourcing can be read from eq. (17); as the first term on the r.h.s. is negligible on super-Hubble scales, we arrive at \(\zeta \dot{\sigma} \approx 2\eta_{\perp} H^2 Q_s^2\) and the two terms in eq. (20) cancel. This can be interpreted as infinite sound speed. This should not come as a surprise, because on super-Hubble scales, the amplitude of the adiabatic perturbations grows coherently over distances exceeding the size of the horizon.

C. Strongly coupled perturbations and sub-Hubble freeze-in

If the turn rate is large, \(\eta_{\perp} \gg 1\) and slowly varying, the adiabatic and entropy perturbations exhibit interesting dynamics, leading to the adiabatic perturbations freezing in before the Hubble radius crossing and to enhancement of the power spectrum compared to the predictions of a single-field scenario with the same Hubble and slow-roll parameters [6–8]. This happens after the amplitude of the more massive of the solutions of the system of eqs. (14) and (15) becomes negligible and the lighter and more slowly changing mode becomes dominant. The relation between the adiabatic and entropy component of that mode can be read from (15):
\[
\left(\frac{k^2}{a^2} + m_s^2\right) Q_s = -2\dot{\sigma} \eta_{\perp} \zeta.
\]
(21)
Substituting eq. (21) to (20), we obtain:

\[ \tilde{v}_k^{-2} = 1 + \frac{4\eta_1^2}{k^2 + \frac{m^2}{\eta^2}}, \]  

(22)

If the sound speed of perturbations deviates significantly from one, the second term in eq. (22) must dominate; depending on the relative size of the two terms in the denominator, we arrive at:

\[ \tilde{v}_k^2 \approx \frac{m_s^2}{4\eta_1^2 H^2} \quad \text{for} \quad k/a \ll m_s \]  

(23)

or

\[ \tilde{v}_k^2 \approx \frac{k^2}{4\eta_1^2 a^2 H^2} \approx \frac{k^2\eta_s^2}{4\eta_1^2} \quad \text{for} \quad k/a \gg m_s. \]  

(24)

The first limit shown in eq. (23) corresponds to constant reduced sound speed and has been extensively studied in the literature. The positive and negative frequency solutions of eq. (11) reads:

\[ \zeta = A_{\pm} e^{\mp i\tilde{v}_k k\eta} \left( 1 \mp \frac{i}{\tilde{v}_k k\eta} \right), \]  

(25)

where \( A \) is a normalization constant and the symbol \( \pm \) refers to positive- and negative-frequency solutions.

The second limit shown in eq. (24) was first studied in [7] and later in [8]; because of the explicit dependence of \( \tilde{v}_k \) on \( k \), we shall refer to these models as models with modified dispersion relations. Upon substitution \( \zeta = aw/\eta \) eq. (12) becomes

\[ w'' + \left( \frac{k^4\eta_s^2}{4\eta_1^2} - \frac{6}{\eta^2} \right) = 0. \]  

(26)

The solutions of eq. (26) reads [8]:

\[ w = B_{\pm} \sqrt{-k\eta} H_{1,2}^{(1,2)} \left( \frac{k^2\eta_s^2}{4\eta_1^2} \right), \]  

(27)

where \( H_{1,2}^{(1,2)} \) are the Hankel functions of the first and second kind, respectively.

The examples discussed in this subsection offer a route to a consistent interpretation of eq. (20) in a class of multi-field models that allow an effective field theory with just one field. If the amplitudes of all the perturbations except for the freezing-in adiabatic perturbations decay quickly, either because they are massive or, according to eq. (21), the entropy perturbations are suppressed after freeze-in of curvature perturbations, we can describe the evolution of the adiabatic perturbations in the single-field model with an effective sound speed \( v_k \), which depends both on time and the wavenumber of the mode.

In Section V we shall present a set of numerical examples, corroborating the assertion above and show that the predictions of the effective theory are consistent with those of the full theory for all times.

IV. LIOUVILLE FORMULA

A. Single-field limits

The Liouville formula states that for a function \( y(\eta) \), which solves the equation:

\[ \frac{d^2 u}{d\eta^2} + b_1(\eta) \frac{du}{d\eta} + b_0(\eta) u = 0, \]  

(28)

where \( b_1 \) and \( b_0 \) are real-values functions, the Wronskian defined as:

\[ W(\eta) = u \frac{du}{d\eta} - \left( \frac{du}{d\eta} \right)^* u \]  

(29)
frequency solution \((27)\) into eq. \((37)\) and using the fact that \(W\) this formula agrees very well with numerical results presented in [7].

Perturbations deep inside the Hubble radius have \(\tilde{v}_k = 1\). If this value was constant throughout the entire inflationary evolution, the solution to eq. \((12)\) would have a familiar form corresponding to standard single-field inflation:

\[
\zeta = Ce^{-ik\eta} \left(1 - \frac{i}{k}\right). \tag{34}
\]

With \(\tilde{v}_k\) assuming an asymptotic value according to eq. \((23)\), we can write eq. \((33)\) as:

\[
\frac{P}{P_{sf}} = \frac{|A_+|^2}{|C|^2 \tilde{v}_k^2} = \frac{1}{\tilde{v}_k}. \tag{35}
\]

Comparing late-time asymptotics of the positive-frequency solution \((25)\) and eq. \((34)\), which in the limit \(\eta \to 0^-\) read \(-iA_+/k\tilde{v}_k\eta\) and \(-iC/k\eta\), respectively, we conclude that for adiabatic perturbations freezing in when described by an effective theory with \(\tilde{v}_k\) given by eq. \((23)\), the resulting power spectrum of adiabatic perturbations is enhanced by a factor of \(\frac{1}{\tilde{v}_k}\) with respect to the single-field case with \(\tilde{v}_k = 1\):

\[
\frac{P}{P_{sf}} = \frac{|A_+|^2}{|C|^2 \tilde{v}_k^2} = \frac{1}{\tilde{v}_k}. \tag{36}
\]

The calculation for \(\tilde{v}_k\) assuming an asymptotic value according to eq. \((24)\) is analogous, and in de Sitter limit we find that \(w = \zeta \eta/a \approx -H\eta^2 \zeta\) satifies:

\[
w \ast \frac{dw}{d\eta} - \left(\frac{dw}{d\eta}\right)^* w = W(\eta_0) \frac{k^2}{4\eta_\perp^2}, \tag{37}
\]

where \(W(\eta_0)\) is calculated adhering to the definition in \((29)\) in the sub-Hubble regime. Substituting the positive frequency solution \((27)\) into eq. \((37)\) and using the fact that \(W(\eta_0) = -i|C|^2\) for the sub-Hubble solution \((25)\), we find

\[
\frac{8k}{\pi} |B_+|^2 = \frac{k^2}{4\eta_\perp^2} |C|^2. \tag{38}
\]

The asymptotic form of eq. \((27)\) at late times, i.e. for \(\eta \to 0^-\) is

\[
w \sim \frac{i2^{15/4} \eta_\perp^{5/4} \Gamma \left(\frac{5}{4}\right)}{\pi k^2 \eta_\perp^2}. \tag{39}
\]

Using eq. \((38)\) we can compare moduli squared of the functions \(u\) given in \((23)\) and \(\eta w\) with \(w\) given by \((27)\) at late times to conclude that the power spectrum of the adiabatic perturbations \(P\) is enhanced in comparison to the single field value \(P_{sf}\) by a factor:

\[
\frac{P}{P_{sf}} = \frac{8\sqrt{2} \left(\frac{\Gamma \left(\frac{5}{4}\right)}{\pi}\right)^2 \eta_\perp^{1/2}}{\eta_\perp^{1/2}} \sim 2.96 \eta_\perp^{1/2}. \tag{40}
\]

This formula agrees very well with numerical results presented in [7].
B. Multi-field case

The calculation given in Section IV A can be easily generalized to a system of \( N \) coupled linear and homogeneous equations, which can be written as:

\[
\frac{d^2 \vec{U}}{d\eta^2} + L(\eta) \frac{d\vec{U}}{d\eta} + M(\eta)\vec{U} = 0, \quad (41)
\]

where \( \vec{U} = (U_1(\eta), \ldots, U_N(\eta)) \) and \( L(\eta), M(\eta) \) are real-valued \( N \times N \) matrices, which are functions of the independent variable \( \eta \). It is easy to show that for \( L = 0 \) and \( M^T = M \) the Wronskian defined as:

\[
W(\eta) \equiv \vec{U}^\dagger \frac{d\vec{U}}{d\eta} - \left( \frac{d\vec{U}}{d\eta} \right)^\dagger \vec{U} \quad (42)
\]
does not depend on \( \eta \).

The equations of motion for the two-field system of adiabatic and entropy perturbations [14]–[15] can be transformed so that we can make use of this fact. We first redefine perturbations as \( \vec{u} = (aQ_\sigma, aQ_\sigma) \) and identify \( \eta \) with conformal time. We obtain a system of equations of the form (41) with:

\[
L = \begin{pmatrix}
0 & \frac{3\eta_\perp}{\eta} \\
-\frac{3\eta_\perp}{\eta} & 0
\end{pmatrix} \quad (43)
\]
and

\[
M = \left(k^2 - \frac{2}{\eta^2}\right) \mathbf{1} + \begin{pmatrix}
0 & -\frac{2\eta_\perp}{\eta^2} \\
-\frac{2\eta_\perp}{\eta^2} & \frac{3\eta_\perp}{\eta^2}
\end{pmatrix}, \quad (44)
\]

where \( \eta = \frac{m^2}{2\eta^2} - 2\eta_\perp^2 \) and we used de Sitter approximation again [17]. We then define

\[
\vec{U} = \mathbb{R}\vec{u} \quad (45)
\]
with

\[
\mathbb{R}(\eta) = \begin{pmatrix}
\cos \left( \eta_\perp \log \left( \frac{\eta_0}{\eta} \right) \right) & \sin \left( \eta_\perp \log \left( \frac{\eta_0}{\eta} \right) \right) \\
-\sin \left( \eta_\perp \log \left( \frac{\eta_0}{\eta} \right) \right) & \cos \left( \eta_\perp \log \left( \frac{\eta_0}{\eta} \right) \right)
\end{pmatrix}, \quad (46)
\]
where \( \eta_0 \) is an arbitrary constant. In terms of the new variable \( \vec{U} \), the equation of motion (41) reads:

\[
\frac{d^2 \vec{U}}{d\eta^2} + \left( \left(k^2 + \frac{\eta_\perp^2}{\eta^2}\right) \mathbf{1} + \frac{1}{\eta^2} \mathbb{R} \mathbb{M} \mathbb{R}^T \right) \vec{U} = 0, \quad (47)
\]

where

\[
\mathbb{M} = \begin{pmatrix}
0 & -3\eta_\perp \\
-3\eta_\perp & \nu
\end{pmatrix}, \quad (48)
\]
The conserved Wronskian (42) reads:

\[
W(\eta) = \vec{u}^\dagger \frac{d\vec{u}}{d\eta} - \left( \frac{d\vec{u}}{d\eta} \right)^\dagger \vec{u} + \frac{2\eta_\perp}{\eta} \vec{u}^\dagger \mathbb{E} \vec{u}, \quad (49)
\]
where we denoted:

\[
\mathbb{E} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad (50)
\]
and made use of the fact that \( \frac{d\mathbb{R}}{d\eta} = \frac{\eta_0}{\eta} \mathbb{RE} \).
We are now in the position to comment on the choice of the Bunch-Davies vacuum as an initial state for the adiabatic and entropy perturbations. Deep inside the Hubble radius, i.e. for \( \eta \to -\infty \), eq. (47) becomes an equation of motion for a harmonic oscillator and it has two independent positive-frequency solutions:

\[
\vec{U}^{(1)}(\eta) \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \vec{U}^{(1)}_0 \quad \text{and} \quad \vec{U}^{(2)}(\eta) \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \vec{U}^{(2)}_0 ,
\]

where \( \vec{U}^{(1)}_0 \) and \( \vec{U}^{(2)}_0 \) are constant vectors satisfying

\[
\vec{U}^{(1)}_0 \vec{U}^{(2)}_0 = \delta_{IJ} .
\]

These vectors can be parametrized as:

\[
\vec{U}^{(1)}_0 = \left( \frac{\cos \theta_0}{\sin \theta_0}, e^{i\phi_0} \right) \quad \text{and} \quad \vec{U}^{(2)}_0 = \left( -\frac{\sin \theta_0 e^{-i\phi_0}}{\cos \theta_0} \right) .
\]

In terms of perturbations \( \vec{u} \), the solution (51) reads:

\[
\vec{u}^{(1)} \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \left( \frac{\cos \theta_0 \cos \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right) - e^{i\phi_0} \sin \theta_0 \sin \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right)}{\cos \theta_0 \sin \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right) + e^{i\phi_0} \sin \theta_0} \right)
\]

and

\[
\vec{u}^{(2)} \sim \frac{e^{-ik\eta}}{\sqrt{2k}} \left( \frac{-e^{-i\phi_0} \sin \theta_0 \cos \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right) - \cos \theta_0 \sin \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right)}{-e^{-i\phi_0} \sin \theta_0 \sin \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right) + \cos \theta_0 \cos \left( \eta \log \left( \frac{n_0}{\eta_0} \right) \right)} \right).
\]

The modulus squared of the upper (adiabatic) component in (54) reads:

\[
|\vec{u}_s^{(1)}|^2 = \frac{1}{4k} \left( 1 + \cos 2\theta_0 \cos \left( 2 \eta \log \left( \frac{n_0}{\eta_0} \right) \right) - \cos \phi_0 \sin 2\theta_0 \sin \left( 2 \eta \log \left( \frac{n_0}{\eta_0} \right) \right) \right).
\]

This expression is constant for \( \theta_0 = \pm \pi/4 \) and \( \phi_0 = \pi/2 \), which also corresponds to constant \(|\vec{u}_s^{(1)}|^2\), \(|\vec{u}_s^{(2)}|^2\) and \(|\vec{u}_s^{(2)}|^2\). Our final results is, therefore:

\[
\vec{u}^{(1)} \sim \frac{e^{-ik\eta+i\eta \log \left( \frac{n_0}{\eta_0} \right)}}{2\sqrt{k}} \left( \frac{1}{-i} \right) \quad \text{and} \quad \vec{u}^{(2)} \sim \frac{e^{-ik\eta-i\eta \log \left( \frac{n_0}{\eta_0} \right)}}{2\sqrt{k}} \left( \frac{-i}{1} \right).
\]

Note that eq. (57) exhibits some redundancy, which was not visible in the intermediate steps leading to that result. A change in arbitrary constant \( \eta_0 \) can be extracted as an unphysical phase factor multiplying the solution.

The approximate solution (57) is reliable as long as the last term in eq. (47) is negligible. This is satisfied for \( (kn)^2 > \max \{ \nu, 3\eta_\perp \} \).

It is also interesting to study the late-time behavior of the system of equations (47) with (43) and (44), following the treatment in [7]. In the limit \( \eta \to 0^+ \), we can neglect the \( k \)-dependent term and assume solutions of the form:

\[
\vec{u} = \left( \frac{\eta}{\eta_0} \right)^p \left( \frac{A_\sigma}{A_\sigma} \right)^p ,
\]

where \( \eta_0 \) represents the value of the conformal time at which the solution should be matched with the early-time solution. We obtain an algebraic equation:

\[
\left( \begin{array}{cccc}
p(p-1) - 2 & 2 \eta _\perp (p-2) & 2\eta _\perp (p-1) - 2 + \nu & 0 \\
-2\eta _\perp (p+1) & p(p-1) - 2 + \nu & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \left( \begin{array}{cccc}
A_\sigma \\
A_\sigma \\
A_\sigma \\
A_\sigma
\end{array} \right) = 0.
\]

Eq. (59) has four nontrivial solutions for \( p \):

\[
p_1 = -1, \quad \text{with} \quad \frac{A_\sigma^{(1)}}{A_\sigma^{(1)}} = 0
\]

\[
p_2 = 2, \quad \text{with} \quad \frac{A_\sigma^{(2)}}{A_\sigma^{(2)}} = \frac{6\eta _\perp}{\nu}
\]

\[
p_{3,4} = \frac{1}{2} + i \sqrt{\nu + 4\eta _\perp^2} - \frac{9}{4}, \quad \text{with} \quad \frac{A_\sigma^{(3,4)}}{A_\sigma^{(3,4)}} = -\frac{\nu + 4\eta _\perp^2}{\eta _\perp \left( 3 + 2i \sqrt{\nu + 4\eta _\perp^2} - \frac{9}{4} \right)}.
\]
The last two solutions \( \nu \) correspond to the positive and negative frequency solutions for a massive mode, of mass squared \( (\nu + 4\eta_{\perp}^2)H^2 \). The first two solutions, eqs. (60)-(62), correspond to the growing and decaying part of a massless mode. It is also clear that the growing mode \( \sim 1/\kappa t \) carries only the adiabatic component, i.e. in the considered model adiabatic perturbations can freeze in at some scale, while all entropy perturbations decay at late times.

The mode corresponding to the exponent \( p_4 \) corresponds to negative frequency. If the relative change of the sound speed is not much larger than one, this mode is not excited during the evolution of the perturbations. It is instructive to analyze the relations between the sub-Hubble solutions (57) and the solutions (60)-(62). This is particularly simple in the limit \( \nu \to 0 \), which will correspond to numerical examples to be discussed later. In this limit, we have:

\[
A_{s}^{(1)} = 0, \quad A_{s}^{(2)} \approx 0, \quad A_{s}^{(3)} \approx iA_{s}^{(3)}.
\]  

Matching (57) with (60)-(62), we find that \( \tilde{w}^{(2)} \) corresponds to a massive mode with \( p_3 \), which decays on super-Hubble scales, while \( \tilde{w}^{(4)} \) is a combination of a growing mode corresponding to \( p_1 \) and the decaying massive mode corresponding to \( p_2 \), with \( A_{s}^{(1)} \approx -iA_{s}^{(2)} \).

A general late-times solution of (64) can therefore be written as:

\[
\tilde{w} = \sum_{I=1}^{4} \left( \frac{\eta}{\eta_0} \right)^{p_I} \left( \frac{A_{s}^{(I)}(\nu)}{A_{s}^{(I)}(0)} \right),
\]  

where for a given \( I \) the coefficients \( A_{s}^{(I)}(\nu) \) and \( A_{s}^{(I)}(0) \) satisfy the relations in respective eqs. (60)-(62). Plugging (64) into the expression for the conserved Wronskian, we find:

\[
W = -\frac{i(\nu + 4\eta_{\perp}^2)}{\eta_{\perp}\eta_0} \Im \left( A_{s}^{(1)}(\nu)A_{s}^{(2)*}(\nu) - i(\nu + 4\eta_{\perp}^2)\sqrt{\nu + 4\eta_{\perp}^2 - \frac{9}{4}} \left( |A_{s}^{(3)}|^2 - |A_{s}^{(4)}|^2 \right) \right). 
\]  

In the limit \( \nu \to 0 \) considered above, this reduces to:

\[
W = -\frac{4\eta_{\perp}}{\eta_0}|A_{s}^{(1)}|^2.
\]  

As the Wronskian (66) is conserved and equal \( -i \), we find that \( |A_{s,1}|^2 = \eta_0/4\eta_{\perp} \), which leads to the following prediction for the power spectrum of the adiabatic perturbations:

\[
\frac{P}{P_{ef}} = \frac{|k\eta_{0}|^3}{2\eta_{\perp}}.
\]  

Since \( \eta_0 \) corresponds to matching between the early- and late-time solutions, and we argued that for \( \nu \to 0 \) we have \( \eta_0 = -\sqrt{3\eta_{\perp}}/k \), we obtain:

\[
\frac{P}{P_{ef}} = \frac{3\sqrt{3}}{2} - \frac{1}{\eta_{\perp}}.
\]  

We note that this equation has the same parametric form as eq. (40) and the numerical prefactor \( \sim 2.6 \) in eq. (68) is very close to that eq. (40). This is a remarkable consistency, given our crude approach to solving the equations of motion for the two-field system, relying on matching between the early- and late-time asymptotic solutions.

V. NUMERICAL EXAMPLES

In Section II, we have put forth a number a hypotheses. We argued that slow-roll fast-turn two-field inflationary models can be effectively described by a single-field theory with a time and \( k \)-dependent sound speed. We also proposed which combination of modes serves as an effective degree of freedom in the single-field theory. In this Section, we would like to corroborate those findings by presenting results of numerical calculations.

We study the evolution of the perturbations in the model described by the Lagrangian:

\[
\mathcal{L} = \frac{e^{-2\phi_2/M}}{2} (\partial\phi_1)^2 - V_{int}(\phi_1) + \frac{1}{2} (\partial\phi_2)^2 - \frac{1}{2} m_2^2 \phi_1^2. 
\]
In this model, the interactions stemming from the non-canonical kinetic term can compensate the potential force acting on the field $\phi_2$. As a consequence, there may exist an inflationary trajectory, for which $\phi_1$ rolls slowly and $\phi_2$ stays constant. This model has been analyzed by many authors and it was found that for certain values of the parameters one can describe the curvature perturbations with a single-field effective theory, either one with an effective sound speed smaller than one or one with modified dispersion relations.

Here we consider the approximation of quasi-de Sitter space, i.e. we assume that the Hubble parameter is practically constant and that the field $\phi_1$ moves negligibly during inflation, so all the quantities defined in terms of the homogeneous background are also practically constant. In this approximation, equations of motion resulting from (69) assume the form (41) with (43) and (44), where $\eta_\perp = \frac{\dot{\phi}_1}{H}$ can be much larger than 1.

For numerical calculations, we use initial conditions (51) and (53) with $\theta_0 = 0$, integrating the equations of motion (41) with (43) and (44) twice: to cover both initial conditions. In order to isolate the adiabatic mode that dominates after Hubble radius crossing, we perform the following unitary transformations of the two results corresponding to initial conditions. If the first initial condition leads to $\tilde{u}^{(1)}_{\sigma} = z_1$ and the second initial condition leads to $\tilde{u}^{(2)}_{\sigma} = z_2$, we consider combinations of the two solutions, corresponding to rotated vectors in (51):

\[
\begin{pmatrix}
\tilde{u}^{(1)}_{\sigma} \\
\tilde{u}^{(2)}_{\sigma}
\end{pmatrix} = \frac{1}{\sqrt{|z_1|^2 + |z_2|^2}} \begin{pmatrix}
z_1^* & z_2^* \\
z_2 & z_1
\end{pmatrix} \begin{pmatrix}
\tilde{u}^{(1)}_{\sigma} \\
\tilde{u}^{(2)}_{\sigma}
\end{pmatrix}.
\]

At the end of numerical evolution, we have $\tilde{u}^{(2)}_{\sigma} \rightarrow 0$, and therefore we identify the freezing mode with $\tilde{u}^{(1)}_{\sigma}$ and the decaying mode with $\tilde{u}^{(2)}_{\sigma}$. According to our discussion in Section IV B, with freeze-in at sub-Hubble scales the freezing mode should correspond to $z_2 = -iz_1$ and we confirm this in our numerical examples.

We represent perturbations as instantaneous power spectra and normalize them to the corresponding instantaneous power spectra of curvature perturbations in single-field models, as described in detail in [9]. We use color coding for different components and different initial conditions described in Table I.

### A. Single-field effective theories with reduced sound speed

For the first numerical example, we assume $\eta_\perp = 30$ and $\nu = 10^2$, which leads to the effective sound speed $\tilde{v}_k^2 = 0.0.0265 \approx 1/37.7$. Evolution of the effective sound speed calculated from (10) and evolution of adiabatic perturbations is shown in Figure 1. We find exquisite consistency at all scales between the predictions of the full two-field model and the effective single-field theory with a MESS sound speed.

### B. Single-field effective theories with modified dispersion relations

For the second numerical example, we assume $\eta_\perp = 300$ and $\nu = 10$. This model is not described by an effective single-field theory with a constant, reduced sound speed, but rather by an effective single-field theory with modified dispersion relations. Evolution of the effective sound speed calculated from (10) and
evolution of adiabatic perturbations is shown in Figure 2. We find exquisite consistency at all scales between the predictions of the full two-field model and the effective single-field theory with a MESS sound speed.
If the Lagrangian mass term for the entropy perturbations is small compared to other scales, the mass of these perturbations is dominated by the ‘geometrical’ $-2H^2\eta_\perp^2$ term, which in our example is related to the negative curvature of the field space. Such a negative mass term leads to instability and to a very strong enhancement of the amplitude of the perturbations. This phenomenon was first described in [7], which dubbed it transient tachyonic instability around the Hubble radius, and after a decade it was rediscovered in [10], which called it hyperinflation, and further analyzed in [11, 12].

It is interesting to note that hyperinflation can also be described in our effective single-field approach, albeit with a sound speed $\tilde{v}_k^2$ which changes sign during evolution. We demonstrate this numerically by an example with $\eta_\perp = 300$ and $\nu = -10^4$. Evolution of the effective sound speed calculated from (10) and evolution of adiabatic perturbations is shown in Figure 3. We find exquisite consistency at all scales between the predictions of the full two-field model and the effective single-field theory with a MESS sound speed.

In [7], hyperinflation was described as an intrinsically two-field phenomenon. However, [10] hinted at a curious property, determined numerically, that the freezing adiabatic mode is obtained from a single, well-defined initial mode. Here we confirm this observation and show that the evolution of that mode can be understood in effective theory with a time-dependent sound speed that starts at a canonical value of 1 and then goes imaginary.

D. Light entropy perturbations

Our last example is intended to show that the existence of an effective single-field theory is not always guaranteed. To this end, we consider a model with light entropy perturbations, $\nu = 0$ and moderate kinetic coupling between perturbations, $\eta_\perp = 0.3$. Such models were proposed in [13] to explain in an alternative way the red tilt of the power spectrum of adiabatic perturbations; later they were rediscovered and analyzed anew in an improved way, invoking symmetries of the theory [14]. Our particular model has entropy perturbations slowly decaying, so the sourcing of the adiabatic perturbations eventually becomes ineffective; had we chosen $\nu = -2\eta_\perp^2$, the amplitude of entropy perturbations would remain constant and the sourcing could last indefinitely.

In these models, adiabatic perturbations are sourced by entropy perturbations on super-Hubble scales, which corresponds to the situation described in Section III B, with the sound speed diverging to infinity. A closer inspection shows [15] that the amplitude of the adiabatic perturbations grows as $\sim \eta_\perp N$ on super-Hubble scales, hence the sound speed increases as $\sim a^{2-\eta_\perp}$, according to eq. (20).
In Figure 4 we show that, similarly to the case of hyperinflation, the sound speed \( \tilde{v}_k^2 \) changes sign during evolution. We also show there evolution of adiabatic and entropy perturbations.

The evolution of the freezing and decaying modes of the adiabatic perturbations is compared to the evolution of a single-field effective theory with an effective sound speed given by (10) with an appropriate set of initial conditions. We find a good agreement between the predictions of the full theory and two single-field effective theories with different effective sound speeds. Depending on the phase of the evolution, either the freezing or the decaying mode dominates the instantaneous power spectrum and the late-time domination of the freezing mode starts only after Hubble radius crossing. This shows that the model cannot be approximated by an effective single-field theory – we need to combine two single-field theories with two effective, independent sound speeds to obtain correct predictions for the curvature perturbations around and after Hubble radius crossing.

VI. DISCUSSION

In the context of cosmological perturbations, the existence of a single-field effective theory requires that the degree of freedom corresponding to the freezing mode, accounting for the entire amplitude of adiabatic perturbations at the end of inflation, evolves independently of all other perturbations. Those perturbations can be dynamical, but as their masses are larger than the Hubble parameter, their amplitudes decrease as power law functions of the scale factor. Hence the notion of the effective theory in cosmology is different from the one used in particle physics, where decoupling normally means that other degrees of freedom are too heavy to be excited.

At face value, our effective theory of single-field inflation resembles the quadratic part of the action for adiabatic perturbations derived in [15]. However, we would like to point out that the sound speed in that reference is a function of time only. Using a very simple model with a large and constant turning rate, analyzed previously in [7, 8], we have shown that the evolution of the adiabatic perturbations is correctly accounted for by a sound speed that is both time- and momentum-dependent. Hence our approach generalizes the effective theory of inflation of [15] in a non-trivial way, including the effects of entropy, which are implicitly ignored in [15].

The MESS approach is more general because it only relies on the validity of the Einstein's equations and the scalar-vector-tensor (SVT) decomposition of cosmological perturbations of the metric and of the EST, making it valid for any physical system with a well defined EST, including multi-fields systems, while the effective action studied in [15] assumes the existence of a single physical scalar degree of freedom, and as such can be applied only to a restricted class of EST, while the MESS approach can be applied to any arbitrary EST. The
MESS approach is more general because the SVT decomposition ensures that all possible theories are effectively described by the perturbed EST, making the EST the fundamental object of the effective theory, instead of the action.

A truly effective single-field theory has only one relevant degree of freedom that fully accounts for both the power spectrum of the adiabatic perturbations and for higher-order correlation functions of adiabatic perturbations. Although such a mode has both the adiabatic and the entropic components, a known effective sound speed (10) provides an algebraic relation between these two components, so the entropic component is no longer an independent quantity. Such an effective theory is described just by one effective sound speed, because other degrees of freedom are assumed to have decayed before the Hubble radius crossing and thus do not contribute to correlation functions of adiabatic perturbations. In this sense, the models analyzed in Sections V A, V C have a single-field effective theory, while the model described in Section V D does not have it, because another independent degree of freedom significantly contributes to the amplitude of adiabatic perturbations around the Hubble crossing. We can therefore conclude that a momentum-dependent effective sound speed parametrizes single-field effective theories of inflation.

We have identified the effective degree of freedom and shown how its evolution can be treated independently of other degrees of freedom, even at scales at which the amplitudes the latter are not suppressed yet. Hence we have demonstrated that the MESS approach, which generalizes the notion of single-field effective theory of inflation, is a powerful and useful scheme for studying a wide range of inflationary models.

VII. CONCLUSIONS

In this work, we presented a formulation of a single-field effective theory of inflation, making use of a recently advocated approach based on the momentum-dependent effective sound speed (MESS) [1]. We have shown that this formulation includes a number of multi-field models that were considered in the literature in the last decade. We have identified the effective degree of freedom and shown how its evolution can be treated independently of other degrees of freedom, even at scales at which the amplitudes the latter are not suppressed yet. Hence we have demonstrated that the MESS approach, which generalizes the notion of single-field effective theory of inflation, is a powerful and useful scheme for studying a wide range of inflationary models.
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Appendix A: MESS of multiple scalar fields

The EST for the system described by the action given in eq. (13) is

\[ T^\mu_\nu = G_{IJ} \left( \Phi^K \right) \partial^\mu \Phi^I \partial_\nu \Phi^J + \delta^\mu_\nu \left[ -\frac{1}{2} G_{IJ} \left( \Phi^K \right) \partial_\lambda \Phi^I \partial^\lambda \Phi^J - V \left( \Phi^K \right) \right]. \]  

(1)

The scalar fields at linear order can be expanded as \( \Phi^K(x^\mu) = \phi^K(t) + \delta \phi^K(x^\mu) \), where the background parts of the scalar fields satisfy the following equations of motion

\[ \ddot{\phi}^I + 3H \dot{\phi}^I + \Gamma^I_{JK} \dot{\phi}^J \dot{\phi}^K + G^{IJ} (\phi^K) V_{,J} (\phi^J) = 0, \]

(2)

where \( \Gamma^I_{JK} \) are the Christoffel symbols corresponding to the fields space metric \( G^{IJ} \), and we denote the partial derivative respect to the field \( \phi^J \) according to \( V_{,J} (\phi^J) = \frac{\partial V}{\partial \phi^J} \). The background energy density and pressure are

\[ \rho = \frac{1}{2} \dot{\phi}^2 + V (\phi^K), \]

(3)

\[ P = \frac{1}{2} \dot{\phi}^2 - V (\phi^K), \]

(4)

where \( \dot{\phi}^2 = G_{IJ} (\phi^K) \dot{\phi}^I \dot{\phi}^J \). The components of the perturbed EST of the two scalar fields system, without gauge fixing, are

\[ \delta T^0_0 = -\frac{1}{2} G_{IJ} (\phi^K) \left( \dot{\phi}^I \delta \phi^J + \dot{\phi}^J \delta \phi^I \right) + \dot{\phi}^2 - \delta \dot{\phi} \left( \frac{1}{2} \dot{\phi}^I \dot{\phi}^J G_{IJ,K} (\phi^K) + V_{,K} (\phi^K) \right), \]

\[ \delta T^i_j = \delta^i_j \left[ \frac{1}{2} G_{IJ} (\phi^K) \left( \dot{\phi}^I \delta \phi^J + \dot{\phi}^J \delta \phi^I \right) - \dot{\phi}^2 A + \delta \dot{\phi} \left( \frac{1}{2} \dot{\phi}^I \dot{\phi}^J G_{IJ,K} (\phi^K) - V_{,K} (\phi^K) \right) \right], \]

\[ \delta T^0_i = -\partial_i \left[ G_{IJ} (\phi^K) \dot{\phi}^I \delta \phi^J \right]. \]

(5)

Under an infinitesimal time translation \( t \to t + \delta t \) the fields perturbations transform according to the gauge transformation

\[ \delta \phi^K = \delta \phi^K - \dot{\phi}^J \delta \phi^J. \]

(6)

From these equations we can find the time translation \( \delta t_c \), necessary to go to the comoving gauge, by imposing the comoving gauge condition \( \delta T^0_i = 0 \to G_{IJ} (\phi^K) \dot{\phi}^J \delta \phi^J = 0 \), obtaining

\[ \delta t_c = \frac{G_{IJ} (\phi^K) \dot{\phi}^I \delta \phi^J}{\dot{\phi}^2}. \]

(7)

We can now compute the gauge invariant comoving field perturbations according to

\[ U^K = \delta \phi^K - \dot{\phi}^J \delta \phi^J = \delta \phi^K - \dot{\phi}^J \frac{G_{IJ} (\phi^K) \dot{\phi}^I \delta \phi^J}{\dot{\phi}^2}, \]

(8)

and the comoving pressure and energy density perturbations

\[ \alpha = \delta P_c = \frac{1}{2} G_{IJ} (\phi^K) \left( \dot{\phi}^I \dot{\phi}^J + \dot{\phi}^J \dot{\phi}^I \right) - \dot{\phi}^2 \gamma + U^K \left( \frac{1}{2} \dot{\phi}^I \dot{\phi}^J G_{IJ,K} (\phi^K) - V_{,K} (\phi^K) \right), \]

(9)

\[ \beta = \delta \rho_c = \frac{1}{2} G_{IJ} (\phi^K) \left( \dot{\phi}^I \dot{\phi}^J + \dot{\phi}^J \dot{\phi}^I \right) - \dot{\phi}^2 \gamma + U^K \left( \frac{1}{2} \dot{\phi}^I \dot{\phi}^J G_{IJ,K} (\phi^K) + V_{,K} (\phi^K) \right). \]

(10)
After replacing eq. (8) and eq. (2) into these expressions we find

\[ U^k V_K (\phi^K) = \frac{1}{2} G_{IJ} (\phi^K) \left( \dot{\phi}^I \dot{\phi}^J + \phi^I \phi^J \right) + U^k \frac{1}{2} \dot{\phi}^I \dot{\phi}^J G_{IJ,K} (\phi^K) = -\dot{\sigma}^2 \frac{\Theta}{4}, \]  

(11)

\[ \alpha = -\dot{\sigma}^2 \gamma = -\dot{\sigma}^2 \frac{\dot{\zeta}}{H}, \]

(12)

\[ \beta = -\dot{\sigma}^2 \left( \gamma + \frac{\Theta}{2} \right) = -\dot{\sigma}^2 \left( \frac{\dot{\zeta}}{H} + \frac{\Theta}{2} \right), \]

(13)

where we have used the perturbed Einstein’s equation \( \gamma = \dot{\zeta}/H \), and we have defined the function \( \Theta \) according to

\[ \Theta = -4 \frac{\dot{\phi}_1 \dot{\phi}_2}{\dot{\sigma}^3} \sqrt{G} \left( \frac{\delta \phi_1}{\phi_1} - \frac{\delta \phi_2}{\phi_2} \right) V_s = \frac{4}{\dot{\sigma}^2} Q_s V_s, \]

(14)

where \( G \) is the determinant of the fields space metric \( G_{IJ} (\phi^K) \), i.e. \( G \equiv \text{det} \left( G_{IJ} \right) \), \( Q_s \equiv Q_s \), \( V_s \equiv V_s \), and

\[ e^K_s = (e^K_1, e^K_2) = \left( \frac{G_{21} \dot{\phi}_1 + G_{22} \dot{\phi}_2}{\dot{\sigma} \sqrt{G}}, -\frac{G_{11} \dot{\phi}_1 + G_{12} \dot{\phi}_2}{\dot{\sigma} \sqrt{G}} \right). \]

(15)

Finally the MESS of this system is given by

\[ \tilde{v}_k^2 (t) = \left( 1 + \frac{H \Theta}{2 \dot{\zeta}} \right)^{-1} = \left( 1 + \frac{2 H V_s Q_s}{\dot{\zeta} \dot{\sigma}^2} \right)^{-1} = \left( 1 - \frac{2 H^2 \eta_\perp Q_s}{\dot{\zeta} \dot{\sigma}} \right)^{-1}, \]

(16)

where

\[ \eta_\perp = -\frac{V_s}{H \dot{\sigma}}. \]

(17)