Mirror Symmetry, D-brane Superpotential and Ooguri-Vafa Invariants of Compact Calabi-Yau Manifolds

Shan-Shan Zhang, Fu-Zhong Yang

College of Physical Sciences, Graduate University of Chinese Academy of Sciences  
YuQuan Road 19A, Beijing 100049, China

Abstract

The D-brane superpotential is very important in the low energy effective theory. As the generating function of all disk instantons from the worldsheet point of view, it plays a crucial role in deriving some important properties of the compact Calabi-Yau manifolds. By using the GKZ-generalized hypergeometric system, we will calculate the B-brane superpotentials of two non-fermat type compact Calabi-Yau hypersurfaces in toric varieties, respectively. Then according to the mirror symmetry, we obtain the A-model superpotentials and the Ooguri-Vafa invariants for the mirror Calabi-Yau manifolds.

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*Corresponding author  E-mail: fzyang@gucas.ac.cn
1 Introduction

The theory of topological string which is derived from the two dimensional $(N, \hat{N}) = (2, 2)$ superconformal field theory has gained a great development over the past years and it has influenced deeply on the mathematics. The D-brane superpotential, the generating functional of correlation function, is particularly vital physical quantity which is a section of a special holomorphic line bundles of the moduli space from the mathematical perspective. Through the superpotential we can derive a series of important properties for the CY manifolds such as Yukuwa couplings, Ooguri-Vafa invariants and so on. Therefore the calculation of the superpotential is very meaningful.

Some important properts of the moduli spaces for various Calabi-Yau manifolds [1–3] has been well studied via the mirror symmetry which was first mentioned in the local operator algebra of the $N = 2$ string theory [4]. It is well known that mirror symmetry connects two different moduli spaces which are respectively parameterized by kahler geometry deformation and complex geometry deformation in A- and B-model. In A-model there exist contributions from the instantons while there is none in B-model. So calculating superpotential directly in A-model is considerably difficult. In fact only in several special cases we know the corresponding brane configuration on mirror A-model side for a given brane configuration in the compact CY manifold [6] derived from the GKZ system in B-model. In GKZ system the superpotential is related to period integral. And the Hodge theoretic approach [5] provides a useful insight on studying the period integrals for CY manifolds period integral satisfies Picard-Fuchs differential equation which is closely related to the GKZ system.

Recently, for compact CY manifolds, there are some great development in calculating the quantum corrected domain wall tensions on the CY threefolds via open-closed mirror symmetry [7, 8]. The properties of some compact Calabi-Yau manifolds has been studied in refs. [25, 26, 28]. In this note, we compute the D-brane superpotentials for two non-fermat CY threefolds in detail via mirror maps and GKZ hypergeometric system.

The structure of this paper is as follows. In section 2 we describe the generalized GKZ hypergeometric system. The solution of GKZ hypergeometric system is just the integral period. We also outline the approach to constructing the corresponding polyhedron $\Delta$ and its mirror polyhedron $\Delta^*$ for the Calabi-Yau manifold. Then we review how to calculate superpotential. In section 3 we analyze two non-fermat type compact
CY manifolds in toric varieties, respectively. and compute their superpotentials as well as some disk invariants with the method referred previously. The last section is the conclusion.

2 Toric Geometry, Relative Period Integrals and GKZ System

We divide this section into two parts to review some related background.

2.1 Superpotential on D-brane

In the presence of some background fluxes and space-filling D-branes, the type II string theory compactification on Calabi-Yau manifold gives rise to the $N = 1$ low effective energy theories [25], whose effective superpotential is captured by the relative period integral of the holomorphic three form $\Omega(z)$ around the integral relative cycle with boundaries in the D-branes [14]. As is listed in the refs. [15–18] that the above integral is derived from the action of a holomorphic Chern-Simons theory on the brane which wraps the holomorphic curves.

For a $D$-brane wrapping internal cycles of Calabi-Yau manifold $X$, the corresponding effective superpotential is [16]

$$W_{brane} = \int_X \Omega \wedge Tr[A \wedge \bar{\partial}A + \frac{2}{3} A \wedge A \wedge A]$$

(2.1)

where if there are $N$ branes, $A$ is a holomorphic $U(N)$ gauge connection on $X$ and $\Omega$ is the holomorphic three form on $X$. For type IIB string, the effective superpotential is a linear combination of the relative period integrals [18,19,21].

$$W_{brane}(\varphi, \xi) = \hat{N}_a \hat{\Pi}^a(\varphi, \xi)$$

(2.2)

where $\hat{N}_a$ stands for the homology class, which is wrapped with the D-brane. $\hat{\Pi}^a(\varphi, \xi)$ represents the period integral

$$\hat{\Pi}^a(\varphi, \xi) = \int_{\gamma^a(\xi)} \Omega(\varphi)$$

(2.3)

Here, $\xi$ and $\varphi$ stands for the open- and closed-string moduli respectively. The internal background fluxes $H = H^{RR} + H^{NS}$ lead to an effective superpotential [22–28] which is defined by

$$W_{Flux} = \int \Omega \wedge H = \int \Omega \wedge (H^{RR} + \tau H^{NS})$$

(2.4)
where $\Omega$ denote the holomorphic three-form on the Calabi-Yau manifold, and $\tau$ denote the complex couplings for the type II string of B-model. In this note, we only consider the RR flux, the induced superpotential becomes

$$W_{\text{flux}} = \int \Omega \wedge H_{RR} = \sum_{\alpha} N_\alpha \Pi_\alpha(\varphi)$$  \hspace{0.5cm} (2.5)$$

Therefore the combined superpotential generated by D-brane and flux is \cite{29,30}

$$W(\varphi, \xi) = W_{\text{brane}}(\varphi, \xi) + W_{\text{flux}}(\varphi) = \sum N_\Sigma \Pi_\Sigma(\varphi, \xi)$$  \hspace{0.5cm} (2.6)$$

here the coefficient $N_\Sigma$ denotes both the D-brane topological charge and the RR flux quantum data and $\Pi_\Sigma(\varphi, \xi)$ denotes the integral of the three-form $\Omega(\varphi)$ over the three-chains in the relative integer homology group, which is defined by

$$\Pi_\Sigma(\varphi, \xi) = \int_{\Gamma(\xi)} \Omega(\varphi), \quad \Gamma^\alpha(\xi) \in H_3(Y, S, Z)$$  \hspace{0.5cm} (2.7)$$

The relative period integral referred previously $\hat{\Pi}^\alpha(\varphi, \xi)$ is equal to the domain wall tension $T(\varphi, \xi)$ \cite{6,17,18,29,30}. $T(\varphi, \xi)$ is defined as

$$T(\varphi, \xi) = W(C^+_{(\varphi, \xi)}) - W(C^-_{(\varphi, \xi)})$$  \hspace{0.5cm} (2.8)$$

At its critical point $\xi = z$, $T(\varphi, \xi)$ is identical to the on-shell domain wall tension $T = W(C^+_{\varphi}) - W(C^-_{\varphi})$. As is depicted in the refs. \cite{3,8,31,32}, at the critical points, the domain wall tensions are considered as normal function from which the Abel-Jacobi invariants can be derived.

For the D-brane in A-model, the superpotential which is expressed in terms of the flat closed/open coordinates can be calculated as the generating functional of the correlation functions \cite{18,29,33–35}. It is defined by

$$W(t, \hat{t}) = \sum_{\vec{k}, \vec{m}} G_{\vec{k}, \vec{m}} q^{\vec{k}} \hat{q}^{\vec{m}} = \sum_{\vec{k}, \vec{m}} \sum_{d} n_{\vec{k}, \vec{m}} q^{d_1} \hat{q}^{d_2} / k^2$$  \hspace{0.5cm} (2.9)$$

Here, $q = e^{2\pi it}$, $\hat{q} = e^{2\pi i\hat{t}}$ and $n_{\vec{k}, \vec{m}}$ is the Ooguri-Vafa invariant. Mirror symmetry, which indicates that the two superpotentials for D-branes in A- and B-model, respectively, are related to each other by the mirror map, gives us a method to computing the Ooguri-Vafa invariant which is closely related to the open Gromov-Witten invariant $G_{\vec{k}, \vec{m}}$ \cite{6}. The superpotentials and the Ooguri-Vafa invariants are as follows \cite{7,36}:

$$\frac{W(\pm)(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k \in \text{odd}} \sum_{d_1, d_2 \in \text{odd}} n_{d_1, d_2}^{(\pm)} q^{kd_1} \hat{q}^{kd_2 / 2} / k^2$$  \hspace{0.5cm} (2.10)$$
where \( q_a = e^{t_a} \ (a = 1, 2) \) and

\[
\omega_0(z) = \sum_n c(n)z^n = \sum_n \frac{\prod_j \Gamma(\sum a_{ij} n_a + 1)}{\prod_i \Gamma(\sum a_i n_i + 1)} z^n \quad (2.11)
\]

Here the mirror map \( t_a \) is defined as \( t_a = \frac{\partial q_a}{\partial \omega_0} \).

2.2 Toric Geometry and GKZ System

The generalized hypergeometric system was first introduced in refs. [37], and soon after gained a fast development in mirror symmetry [5, 38–41]. Define a mirror pair of hypersurfaces \((X, X^*)\) in two toric ambient spaces \((V, V^*)\), respectively. The toric varieties \((V, V^*)\) are related to the fans \((\Sigma(\Delta), \Sigma(\Delta^*))\) induced by two dual polyhedron \((\Delta, \Delta^*)\). The defining polynomial for the hypersurface is defined as:

\[
\mathcal{P} = \sum_{i=0}^{p-1} a_i \prod_{k=1}^4 X_k^{v_{i,k}} \quad (2.12)
\]

Or we can write the above equation in another way

\[
\mathcal{P} = \sum_{i=0}^{p-1} a_i \prod_{v_j \in \Delta} x_j^{<v_{j,v^*_{i,j}>}+1} \quad (2.13)
\]

Here \( a_i \) is complex parameter and \( X_k \) are inhomogeneous coordinates on the open torus, \( x_i \) is the homogeneous coordinates.

The general integral period is expressed as

\[
\Pi(a_i) = \frac{1}{(2\pi i)^4} \int_{|X_k|=1} \frac{1}{\mathcal{P}} \prod_{k=1}^4 dX_k
\]

(2.14)

It is referred in [38, 39] that the period can be annihilated by a GKZ hypergeometric differential system

\[
\mathcal{D}_l \Pi(a) = 0 \ (l \in L) \quad \mathcal{Z}_i \Pi(a) = 0 \ (i = 0, 1, \cdots, p) \quad (2.15)
\]

the operators \( \mathcal{D}_l \) and \( \mathcal{Z}_j \) are expressed as

\[
\mathcal{D}_l = \prod_{l_i > 0} \frac{\partial}{\partial a_i}^{l_i} - \prod_{l_j < 0} \frac{\partial}{\partial a_j}^{-l_j} \quad (l \in L) \quad (2.16)
\]
\[ Z_j = \sum_{i=0}^{p} \bar{v}_{i,j}^* \theta_{a_i} - \beta_j \quad (j = 0, 1, \cdots, n) \]  

(2.17)

The torus invariant algebraic coordinates \( z_a \) in the large complex structure limit [3]

\[ z_a = (-1)^{l_0} \prod_j a_j^{l_a} \]  

(2.18)

where \( l^a \) is the set of basic vectors which denote the generators of the Mori cone. Then, by \( \theta_a = z_a \partial z_a \), (2.16) changes into

\[ D_l = \prod_{k=1}^{l_0} (\theta_0 - k) \prod_{l_i > 0} \prod_{k=0}^{l_i-1} (\theta_i - k) - (-)^{l_0} \prod_{k=1}^{l_0} (\theta_0 - k) \prod_{l_i < 0} \prod_{k=0}^{l_i-1} \]  

(2.19)

where \( l \) is the linear combination of \( l^a \). One can refer [31, 42, 43] for more details. The result to the GKZ system is described as

\[ B_{\{l^a\}}(z_a; \rho_a) = \sum_{n_1, \cdots, n_N \in \mathbb{Z}^+} \frac{\Gamma(1 - \sum_a l_0^a (n_a + \rho_a))}{\prod_{i>0} \Gamma(1 + \sum_a l_i^a (n_a + \rho_a))} \prod_a z_a^{(n_a + \rho_a)} \]  

(2.20)

3 Study of two compact non-fermat type Calabi-Yau manifolds

In this section we will calculate the superpotentials and disk invariants for two compact CY in the weighted projective space, respectively, with the method mentioned in the section 2.

3.1 Calabi-Yau hypersurface \( X_7(1, 1, 1, 1, 3) \)

\( X_7(1, 1, 1, 1, 3) \) is a hypersurface in the weighted projective space \( \mathbb{P}^4(1, 1, 1, 1, 3) \). Let \( X \) denote this hypersurface, then, its mirror manifold \( X^* \) is denoted by \( X^* = \hat{X}/H \), where \( \hat{X} \) represents the CY 3-fold \( X_{14}(1, 2, 2, 2, 7) \) and \( H \) is defined by \( (h_i^j) = \frac{1}{7} (1, 0, 6, 0, 0) \). So, \( X_7(1, 1, 1, 1, 3) \) is isomorphic to \( X_{14}(1, 2, 2, 2, 7) \), which had been checked in refs. [44]. The hypersurface \( X_7(1, 1, 1, 1, 3) \) is defined as the zero locus of the polynomial \( \mathcal{P} \).

\[ \mathcal{P} = x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^2 \]  

(3.1)
The weighted projected space $\mathbb{P}^4(1,1,1,1,3)$ as a toric variety, the vertices of its corresponding polyhedron $\Delta$ are as follows:

\[
v_1 = (-1, -1, -1, 1) , \quad v_2 = (-1, -1, -1, -1) , \quad v_3 = (-1, -1, 1, 0) , \\
v_4 = (6, -1, -1, -1) , \quad v_5 = (-1, 1, -1, 0) , \quad v_6 = (-1, 3, -1, -1) , \\
v_7 = (-1, -1, 3, -1) , \quad v_8 = (0, -1, 3, -1) , \quad v_9 = (0, 3, -1, -1) .
\]

The vertices of the corresponding dual polyhedron $\Delta^*$ are

\[
v_1^* = (-1, -1, -1, -3) , \quad v_2^* = (1, 0, 0, 0) , \quad v_3^* = (0, 1, 0, 0) , \\
v_4^* = (0, 0, 1, 0) , \quad v_5^* = (0, 0, 0, 1) , \quad v_6^* = (0, 0, 0, -1) ,
\]

There exists only one integral point denoted as $v_0^* = (0, 0, 0, 0)$ in $\Delta^*$. For $\Delta^*$, the charge vector of the Mori cone is

\[
l^{(1)} = (-2, 0, 0, 0, 0, 1, 1) , \quad l^{(2)} = (-1, 1, 1, 1, 1, 0, -3)
\]

Consider the divisor

\[
Q(D) = x_3^7 + z_3 x_4^7
\]

at the critical point $z_3 = 1$. Let

\[
u_1 = -\frac{z_1}{z_3}(1 - z_3)^2 , \quad u_2 = z_2
\]

then according to (2.19) the GKZ system of the two-parameters family become

\[
\mathcal{D}_1 = \tilde{\theta}_1(\tilde{\theta}_1 - 3\tilde{\theta}_2) - (2\tilde{\theta}_1 + \tilde{\theta}_2)(2\tilde{\theta}_1 + \tilde{\theta}_2 - 1)z_1
\]

\[
\mathcal{D}_2 = \tilde{\theta}_2^2(7\tilde{\theta}_2 - 2\tilde{\theta}_1) + 4\tilde{\theta}_2^2(2\tilde{\theta}_1 + \tilde{\theta}_2 - 1)z_1 - 7 \prod_{i=1}^{3}(2\tilde{\theta}_2 - \tilde{\theta}_1 - i)
\]

The solution to this GKZ system is written as

\[
\Pi_1(u_1, u_2) = \frac{c}{2} B_{\{\tilde{\theta}\}}(u_1, u_2, 0, \frac{1}{2})
\]

\[
\Pi_2(u_1, u_2) = \frac{c}{2} B_{\{\tilde{\theta}\}}(u_1, u_2, \frac{1}{2}, \frac{1}{2})
\]
At the critical point $z_3 = 1$, the on-shell superpotential satisfies $W^+_C = W^-_C$ according to the $Z_2$ symmetry. So the on-shell superpotential is described as

$$W^\pm(z_1, z_2) = \frac{1}{2\pi i} \int_{\xi_0}^{\pm \sqrt{\pi i}} \Pi(z_1, z_2, \xi^2) \frac{d\xi}{\xi} \bigg|_{z_3 = 1} \tag{3.8}$$

For this model the on-shell superpotentials are expressed as

$$W^+_1 = \pm \frac{c}{8} \sum_{n_1, n_2 \geq 0} \frac{\Gamma(2n_1 + n_2 + \frac{3}{2})z_1^{n_1}z_2^{n_2 + \frac{1}{2}}}{\Gamma(n_1 + 1)\Gamma(n_1 - 3n_2 - \frac{1}{2})\Gamma(n_2 + \frac{3}{2})^4} \tag{3.9}$$

$$W^-_2 = \pm \frac{c}{8} \sum_{n_1, n_2 \geq 0} \frac{\Gamma(2n_1 + n_2 + \frac{3}{2})z_1^{n_1 + \frac{1}{2}}z_2^{n_2 + \frac{1}{2}}}{\Gamma(n_1 + \frac{3}{2})\Gamma(n_1 - 3n_2)\Gamma(n_2 + \frac{3}{2})^4} \tag{3.10}$$

The flat coordinates in A- and B-model are connected via mirror map $t_0 = \frac{a_{\infty}}{a_0}$. The mirror map is as follows:

$$t_1 = \log(z_1) + 2z_1 + 3z_1^2 + \frac{20}{3}z_1^3 + 68z_1^2z_2 - 10z_1z_2^2 + 2z_2 - 15z_2^2 + 66z_1z_2^3 + \frac{560}{3}z_2^3 + o(z^3)$$

$$t_2 = \log(z_2) - 6z_2 + 45z_2^2 - 560z_2^3 - 198z_2z_1 + 30z_2z_1 + 9z_1 - 204z_2z_1^2 + \frac{43}{2}z_1^2 + 62z_2^3 + o(z^3)$$

and the corresponding inverse mirror map is

$$z_1 = q_1 - 2q_1^2 + 3q_1^3 - 2q_1q_2 + 5q_1^2q_2 + 36q_1^2q_2 + o(q^4)$$

$$z_2 = q_2 + 6q_2^2 + 9q_2^3 - 9q_1q_2 - 120q_1q_2^2 + 37q_1^2q_2 + o(q^4)$$

According to (2.10) we can derive the Ooguri-Vafa invariants from the on-shell superpotentials. The results are listed in the following tables:

### 3.2 Calabi-Yau hypersurface $X_9(1, 1, 2, 2, 3)$

$X_9(1, 1, 2, 2, 3)$ is a hypersurface defined in the weighted projective space $\mathbf{P}^4(1, 1, 2, 2, 3)$. Let $X$ denotes the corresponding Calabi-Yau threefold, then, its mirror manifold $X^*$ is denoted by $X^* = \hat{X}/H$, where $\hat{X}$ represents the CY 3-fold $X_{12}(1, 1, 3, 3, 4)$ and $H$ is defined by $(h_i) = \frac{1}{8}(1, 8, 0, 0, 0) + \frac{1}{4}(1, 0, 3, 0, 0)$. So, $X_9(1, 1, 2, 2, 3)$ is isomorphic to $X_{12}(1, 1, 3, 3, 4)$. The polyhedron $\Delta$ for this model has the vertices

$$v_1 = (-1, -1, -1, 2), \quad v_2 = (-1, -1, -1, -1), \quad v_3 = (-1, -1, 2, 0), \quad$$
\[
\begin{array}{c|cccccccc}
  d_1 \setminus d_2 & 1 & 3 & 5 & 7 & 9 & 11 \\
  \hline
  0 & -2 & 2 & -10 & 84 & -858 & 13820 \\
  1 & 28 & -28 & 252 & -2828 & 36400 & -729130 \\
  2 & 70 & 112 & -2702 & 42910 & -714410 & 17644120 \\
  3 & 0 & 252 & 16716 & -391580 & 8645280 & -262033434 \\
  4 & 0 & 6832 & -83286 & 2543170 & -74112654 & 2741674588 \\
  5 & 0 & 84364 & 315980 & -13445796 & 496350736 & -22527070394 \\
\end{array}
\]

Table 1: \( n_{d_1,d_2}^{(1,+)} \)

\[
\begin{array}{c|cccccccc}
  d_1 \setminus d_2 & 1 & 3 & 5 & 7 & 9 & 11 \\
  \hline
  1 & 0 & 0 & 0 & 0 & 0 & 137970 \\
  3 & -70 & 0 & 0 & 0 & 8008 & 140 \\
  5 & -28 & 56 & -140 & 896 & -8008 & -5747938 \\
  7 & 2 & -792 & 3164 & -27608 & 315050 & 108056554 \\
  9 & 0 & -38962 & -32424 & 391104 & -5784408 & 1230725514 \\
  11 & 0 & -84364 & 358288 & -3814860 & 68819744 & 10291361734 \\
\end{array}
\]

Table 2: \( n_{d_1,d_2}^{(2,+)} \)

\[
v_4 = (8, -1, -1, -1) , \ v_5 = (-1, 2, -1, 0) , \ v_6 = (-1, 3, -1, -1) , \\
v_7 = (-1, -1, 3, -1) , \ v_8 = (0, -1, 3, -1) , \ v_9 = (0, 3, -1, -1) ,
\]

then the dual polyhedron \( \Delta^* \) are

\[
v_1^* = (-1, -2, -2, -3) , \ v_2^* = (1, 0, 0, 0) , \ v_3^* = (0, 1, 0, 0) , \\
v_4^* = (0, 0, 1, 0) , \ v_5^* = (0, 0, 0, 1) , \ v_6^* = (0, -1, -1, -1) ,
\]

There exist no points inside the polyhedron \( \Delta^* \) but the original point \( v_0^* = (0, 0, 0, 0) \). We define the hypersurface \( X_9(1,1,2,3) \) as the zero locus of the above polynomial \( \mathcal{P} \).

\[
\mathcal{P} = x_1^9 + x_2^9 + x_1 x_3^4 + x_2 x_4^4 + x_5^3
\]

\[
l^{(1)} = (-3, -1, -1, 1, 1, 0, 3) , \ l^{(2)} = (-1, 1, 1, 0, 0, 1, -2)
\]


are the generators of the Mori cone related to this model.

Consider the divisor
\[ Q(D) = x_1^9 + z_3 x_2^9 \]  
(3.12)

at the critical point \( z_3 = 1 \). Let
\[
\begin{align*}
u_1 &= -\frac{z_1 z_3}{(1 - z_3)^2} , \\
u_2 &= -\frac{z_2}{z_3}(1 - z_3)^2
\end{align*}
\]  
(3.13)

according to (2.16) the GKZ system for the two-parameters family become
\[
\begin{align*}
D_1 &= \tilde{\theta}_2(\tilde{\theta}_2 - \tilde{\theta}_1)^2 - (3\tilde{\theta}_1 + \tilde{\theta}_2)(3\tilde{\theta}_1 - 2\tilde{\theta}_2 + 2)(3\tilde{\theta}_1 - 2\tilde{\theta}_2 + 1)z_2 \\
D_2 &= (\tilde{\theta}_2 - \tilde{\theta}_1)^2 - (\tilde{\theta}_2 - \tilde{\theta}_1)(3\tilde{\theta}_1 - 2\tilde{\theta}_2) + 4\tilde{\theta}_1(3\tilde{\theta}_1 - 2\tilde{\theta}_2) + 3z_1(3\tilde{\theta}_1 - 2\tilde{\theta}_2)(3\tilde{\theta}_1 - 2\tilde{\theta}_2 - 1) \\
&- 48z_1 z_2(3\tilde{\theta}_1 + \tilde{\theta}_2 + 1)(3\tilde{\theta}_1 + \tilde{\theta}_2 + 2) - 48z_1 z_2(3\tilde{\theta}_1 + \tilde{\theta}_2 + 1)(3\tilde{\theta}_1 - 2\tilde{\theta}_2) - 16z_1(\tilde{\theta}_2 - \tilde{\theta}_1)^2
\end{align*}
\]  
(3.14)

The solution to this GKZ system are written as
\[
\begin{align*}
\Pi_1(u_1, u_2) &= \frac{c}{2} B_{i\bar{j}}(u_1, u_2, 1, 0) \quad (3.15) \\
\Pi_2(u_1, u_2) &= \frac{c}{2} B_{i\bar{j}}(u_1, u_2, \frac{1}{2}, \frac{1}{2}) \quad (3.16)
\end{align*}
\]

Similarly, in this model the on-shell superpotentials satisfy \( W_C^+ = W_C^- \) according to the \( Z_2 \) symmetry. So the superpotentials is described as
\[
W^\pm(z_1, z_2) = \frac{1}{2\pi i} \int_{\xi_0}^{\pm\sqrt{\pi i}} \Pi(z_1, z_2, \xi^2) \frac{d\xi}{\xi^2}|_{z_3=1}
\]  
(3.17)

At the critical point \( z_3 = 1 \), on-shell superpotentials are expressed as
\[
\begin{align*}
W_1^+ &= \frac{c}{8} \sum_{n_1, n_2 \geq 0} \frac{\Gamma(3n_1 + n_2 + \frac{5}{2}) z_1^{-n_1 + \frac{1}{2}} z_2^{-n_2}}{\Gamma(-n_1 + n_2 + \frac{1}{2})^2 \Gamma(3n_1 - 2n_2 + \frac{3}{2}) \Gamma(n_2 + 1) \Gamma(n_1 + \frac{3}{2})^2} \\
W_2^+ &= \frac{c}{8} \sum_{n_1, n_2 \geq 0} \frac{\Gamma(3n_1 + n_2 + 3) z_1^{-n_1 + \frac{1}{2}} z_2^{-n_2 + \frac{1}{2}}}{\Gamma(-n_1 + n_2 + 1)^2 \Gamma(3n_1 - 2n_2 + \frac{3}{2}) \Gamma(n_2 + \frac{3}{2}) \Gamma(n_1 + \frac{3}{2})^2}
\end{align*}
\]  
(3.18)

(3.19)

the mirror maps are
\[
t_1 = \log(z_1) + 30 z_1 z_2 - 3 z_2 + 252 z_1 z_2^2 - 10 z_2^3 + 927 z_1^2 z_2^2 + 288 z_1 z_2^3 - \frac{105}{4} z_2^4 + o(z^4)
\]
\[ t_2 = \log(z_2) + 2z_2 + 3z_2^2 + \frac{20}{3}z_2^3 + 74z_1z_2 - 168z_1z_2^2 - 192z_1z_2^3 + 8853z_1^2z_2^3 + \frac{35}{2}z_2^4 + o(z^4) \]

and the corresponding inverse mirror maps are

\[ z_1 = q_1 + 3q_1q_2 + 3q_1q_2^2 + q_1q_2^3 - 30q_1^2q_2^2 - 594q_1^2q_2^2 + o(q^4) \]
\[ z_2 = q_2 - 2q_2^2 + 3q_2^3 - 4q_2^4 - 74q_1q_2^2 + 390q_1q_3^2 + o(q^4) \]

Analogous to computing the disk invariants of the \(X_7(1, 1, 1, 1, 3)\), we have the results listed in the following tables:

| \(d_1 \backslash d_2\) | 0 | 1 | 2 | 3 | 4 | 5 |
|----------------------|---|---|---|---|---|---|
| 1                    | 2 | 18| 2 | 2 | 2 | 2 |
| 3                    | 0 | 0 | 1584| -710| -626| -616|
| 5                    | 0 | 0 | 0 | 38018| 208244| 49382|
| 7                    | 0 | 0 | 0 | 0 | 1745190| 111219514|
| 9                    | 0 | 0 | 0 | 0 | 0 | 90081018|
| 11                   | 0 | 0 | 0 | 0 | 0 | -94519326|

Table 3: \(n_{d_1,d_2}^{(1,+)}\)

| \(d_1 \backslash d_2\) | 1 | 3 | 5 | 7 | 9 | 11 |
|----------------------|---|---|---|---|---|-----|
| 1                    | 16| 0 | -2| -2| -2| 46  |
| 3                    | 0 | -80| 2208| 608| 584| 26332|
| 5                    | 0 | 0 | 720| 158784| -4120| 3039542|
| 7                    | 0 | 0 | 0 | -8848| 15904928| 94908448|
| 9                    | 0 | 0 | 0 | 0 | 126608| 3473058320|
| 11                   | 0 | 0 | 0 | 0 | 0 | -434510208|

Table 4: \(n_{d_1,d_2}^{(2,+)}\)

4 Conclusion

The D-brane superpotential plays a crucial role in both physics and mathematics. From the physical point of view, they determine the vacuum of the low energy
effective theory. From the A-model worldsheet viewpoint, it is the generating functional of the Ooguri-Vafa Invariants of Calabi-Yau manifold and the submanifold on which is wrapped by the D-branes in the A-model. These Ooguri-Vafa Invariants are closely related to the number of the BPS states. From the mirror geometric viewpoint, it is the integral period which is the solution to the generalized GKZ system. It is very hard to calculate directly the D-brane superpotential for compact Calabi-Yau manifold in A-model because these superpotentials are non-perturbative in essential, and are impossibly obtained from the perturbative or localization way which is important methods to compute the D-brane superpotential in non-compact Calabi-Yau manifold in A-model. An effective approach to obtain the D-brane superpotential is by using the blown-up geometry of target space along the submanifold wrapped by the D-branes [10, 47]. The alternative approach to compute the superpotential of the D-brane in compact Calabi-Yau manifold in A-model is via the algebraic geometric method and mirror symmetry.

In this paper, we extend the generalized GKZ system in a Fermat Calabi-Yau threefolds to the compact non-Fermat Calabi-Yau threefolds which are less study so far in contrast to the Fermat type Calabi-Yau threefolds. We first construct the generalized GKZ system for the compact non-fermat type Calabi-Yau manifolds, then work out the corresponding D-brane superpotential in the mirror B-model by the algebraic geometric method. The superpotential in the A-model is obtained according to mirror symmetry. Finally the Ooguri-Vafa invariants are extracted from the A-model superpotential.

These superpotential have potential phenomenological applications. Furthermore, according to the type II string/M-theory/F-theory duality, in the weak decoupling limit \( g_s \to 0 \), these superpotentials of Type II string give the Gukov-Vafa-Witten superpotentials \( W_{GVW} \) of F-theory compactified on the dual fourfold. On the other hand, since there is not a systematic mathematical method to compute them by now, after all, it is difficult to get, from other approach, those Ooguri-Vafa Invariants predicted in this paper. Those Ooguri-Vafa Invariants provide some concrete data which could potentially be checked by an independent mathematical calculation.

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