OPTIMAL EMBEDDING THEOREM FOR FLEXIBLE VARIETIES

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Abstract. Let \( Z \) be an affine algebraic variety and \( X \) be a smooth flexible variety. We show that if \( \dim X \geq \max(2 \dim Z + 1, \dim TZ) \), then \( Z \) admits a closed embedding into \( X \).

1. Introduction

All algebraic varieties which appear in this paper are considered over an algebraically closed field \( k \) of characteristic zero. If \( Z \) is an affine algebraic variety and \( TZ \) is its Zariski tangent bundle then we call \( ED(Z) = \max(2 \dim Z + 1, \dim TZ) \) the embedding dimension of \( Z \).

Holme’s theorem [Hol, Theorem 7.4] (later rediscovered in [Ka91] and [Sr]) states that \( Z \) admits a closed embedding into any affine space \( \mathbb{A}^n \) with \( n \geq ED(X) \). In the smooth case (when \( ED(Z) = 2 \dim Z + 1 \)) this fact was proven earlier by Swan [Swan, Theorem 2.1].

The latter result is sharp - examples of smooth irreducible \( d \)-dimensional affine algebraic varieties with \( d \geq \frac{n}{2} \) such that they do not admit closed embeddings in \( \mathbb{A}^n \) were constructed in [BMS]. Recently Feller and van Santen [FvS21] proved that if \( X \) is an affine variety isomorphic to a simple linear algebraic group and \( Z \) is smooth, then \( Z \) admits a closed embedding into \( X \), provided that \( \dim X > ED(Z) \). They also proved that for every \( n \)-dimensional algebraic group \( G \) (with \( n > 0 \)) there exist smooth irreducible \( d \)-dimensional affine algebraic varieties with \( d \geq \frac{n}{2} \) such that they do not admit closed embeddings in \( G \) [FvS21, Corollary 4.4]. In particular, their embedding result is optimal for even dimensions of \( X \). However, they did not know whether their result is sharp in the case of an odd dimension of \( X \) and a specific question posed in [FvS21] asks whether a smooth affine algebraic variety of dimension 7 can be embedded properly into \( SL_4(k) \). We consider a more general situation. Namely, starting from dimension 2 affine spaces and linear algebraic groups without nontrivial characters are examples of so-called flexible varieties (a normal quasi-affine variety \( X \) of dimension at least 2 is flexible if \( SAut(X) \) acts transitively on the smooth part \( X_{\text{reg}} \) of \( X \) where \( SAut(X) \) is the subgroup of the
group $\text{Aut}(X)$ of algebraic automorphisms of $X$ generated by all one-parameter unipotent subgroups). The main result of this paper is the following.

**Theorem 1.1.** Let $Z$ be an affine algebraic variety and $X$ be a smooth flexible variety such that $\dim X \geq \text{ED}(Z)$. Then $Z$ admits a closed embedding into $X$.

In particular, the question of Feller and van Santen has a positive answer. As we mentioned by [BMS] and [FvS21] this result is optimal for some classes of flexible varieties. It admits also the following generalization.

**Corollary 1.2.** Let $Z$ be an affine algebraic variety, $Y$ be a smooth flexible variety. Suppose that $\varphi : Y \to X$ is a finite morphism into a normal variety $X$ and $\dim X \geq \text{ED}(Z)$. Let $S$ be a closed subvariety of $X$ such that it contains $X_{\text{sing}}$ and $\dim Z < \text{codim}_X S$. Then $Z$ admits a closed embedding into $X$ with the image contained in $X \setminus S$.

The proof of Theorem 1.1 is heavily based on the theory of flexible varieties and the technique developed in [AFKKZ], [Ka20], [KaUd] and [Ka21] whose survey can be found in Section 2. As a part of this survey we describe injective immersions of affine algebraic varieties into smooth flexible varieties. In section 3 we develop a criterion of properness for such injective immersion which yields Theorem 1.1.

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2. Flexible varieties

Let us start with the main definitions for the theory of flexible varieties.

**Definition 2.1.** (1) Given an irreducible algebraic variety $A$ and a map $\varphi : A \to \text{Aut}(X)$ we say that $(A, \varphi)$ is an **algebraic family of automorphisms of** $X$ if the induced map $A \times X \to X, (\alpha, x) \mapsto \varphi(\alpha).x$ is a morphism (see [Ra]).

(2) If we want to emphasize additionally that $\varphi(A)$ is contained in a subgroup $G$ of $\text{Aut}(X)$, then we say that $A$ is an **algebraic $G$-family of automorphisms of** $X$.

(3) In the case when $A$ is a connected algebraic group and the induced map $A \times X \to X$ is not only a morphism but also an action of $A$ on $X$ we call this family a **connected algebraic subgroup of** $\text{Aut}(X)$.

(4) Following [AFKKZ, Definition 1.1] we call a subgroup $G$ of $\text{Aut}(X)$ **algebraically generated** if it is generated as an abstract group by a family $G$ of connected algebraic subgroups of $\text{Aut}(X)$.
Definition 2.2. (1) A nonzero derivation $\delta$ on the ring $A$ of regular functions on an affine algebraic variety $X$ is called \textit{locally nilpotent} if for every $a \in A$ there exists a natural $n$ for which $\delta^n(a) = 0$. This derivation can be viewed as a vector field on $X$ which we also call \textit{locally nilpotent}. The set of all locally nilpotent vector fields on $X$ will be denoted by $\text{LND}(X)$. The flow of $\delta \in \text{LND}(X)$ is an algebraic $\mathbb{G}_a$-action on $X$, i.e., the action of the group $(\mathbb{k}, +)$ which can be viewed as a one-parameter unipotent group $U$ in the group $\text{Aut}(X)$ of all algebraic automorphisms of $X$. In fact, every $\mathbb{G}_a$-action is a flow of a locally nilpotent vector field (e.g., see [Fr, Proposition 1.28]).

(2) If $X$ is a quasi-affine variety, then an algebraic vector field $\delta$ on $X$ is called \textit{locally nilpotent} if $\delta$ extends to a locally nilpotent vector field $\tilde{\delta}$ on some affine algebraic variety $Y$ containing $X$ such that $\tilde{\delta}$ vanishes on $Y \setminus X$ where $\text{codim}_Y(Y \setminus X) \geq 2$. Note that under this assumption $\delta$ generates a $\mathbb{G}_a$-action on $X$ and we use again the notation $\text{LND}(X)$ for the set of all locally nilpotent vector fields on $X$.

Definition 2.3. (1) For every locally nilpotent vector fields $\delta$ and each function $f \in \text{Ker} \delta$ from its kernel the field $f\delta$ is called a \textit{replica} of $\delta$. Recall that such a replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on $X$ and $G_{\mathcal{N}} \subset \text{Aut}(X)$ denotes the group generated by all flows of elements of $\mathcal{N}$. We say that $G_{\mathcal{N}}$ is \textit{generated by $\mathcal{N}$}.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called \textit{saturated} if $\mathcal{N}$ is closed under conjugation by elements in $G_{\mathcal{N}}$ and for every $\delta \in \mathcal{N}$ each replica of $\delta$ is also contained in $\mathcal{N}$.

Definition 2.4. Let $X$ be a normal quasi-affine algebraic variety of dimension at least 2, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on $X$ and $G = G_{\mathcal{N}}$ be the group generated by $\mathcal{N}$. Then $X$ is called $G$-flexible if for every point $x$ in the smooth part $X_{\text{reg}}$ of $X$ the vector space $T_xX$ is generated by the values of locally nilpotent vector fields from $\mathcal{N}$ at $x$ (which is equivalent to the fact that $G$ acts transitively on $X_{\text{reg}}$ [FKZ, Theorem 2.12]). In the case of $G = \text{SAut}(X)$ we call $X$ flexible without referring to $\text{SAut}(X)$ (recall that $\text{SAut}(X)$ is the subgroup of $\text{Aut}(X)$ generated by all one-parameter unipotent subgroups).

Notation 2.5. Further in this paper $X$ is always a smooth quasi-affine variety and $G$ is group acting transitively on $X$ such that $G$ is algebraically generated by a collection $\mathcal{G}$ of connected algebraic subgroups of $G$. Given a sequence $\mathcal{H} = (H_1, \ldots, H_s)$ of elements of $\mathcal{G}$ we consider the map

\[(1) \quad \Phi_{\mathcal{H}} : H \times X \rightarrow X \times X, (h_s, \ldots, h_1, x) \mapsto ((h_s \cdot \ldots \cdot h_1).x, x)\]

where $H = H_s \times \ldots \times H_1$. By $\varphi_{\mathcal{H}} : H \rightarrow X$ we denote the restriction of $\Phi_{\mathcal{H}}$ to $H \times x_0$ where $x_0$ is a fixed point of $X$. 
Proposition 2.6. Suppose that \( G \) is closed under conjugation by \( G \). Then \( H \) can be chosen so that for a dense open subset \( U \) of \( H \) the morphism \( \Phi_H \) is smooth on \( U \times X \) (in particular, \( \varphi_H \) is smooth on \( U \)). Furthermore, one can suppose that the codimension of \( H \setminus U \) in \( H \) is arbitrarily large.

Proof. The first statement follows from [AFKKZ, Proposition 1.16] and the second statement from [AFKKZ, p. 778, footnote]. □

We shall use the notion of a perfect (algebraic) \( G \)-family of automorphisms of \( X \) (see [Ka21, Definition 2.7]). Without stating the formal definition of such families we need to emphasize some of their properties.

Proposition 2.7. ([Ka21, Proposition 2.8]) Let \( A \) be a perfect \( G \)-family of automorphisms of a smooth \( G \)-flexible variety \( X \) and \( H_0 \in G \). Then \( H_0 \times A \) and \( A \times H_0 \) are also perfect \( G \)-families of automorphisms of \( X \). Furthermore, \( A \) satisfies the transversality theorem ([AFKKZ, Theorem 1.11], see also [Ka21, Theorem 2.2]), e.g., if \( Z \) and \( W \) are subvarieties of \( X \) with \( \text{dim} \ Z + \text{dim} \ W < \text{dim} \ X \), then one has \( \alpha(Z) \cap W = \emptyset \) for a general \( \alpha \in A \).

Theorem 2.8. Let \( X \) be a smooth quasi-affine \( G \)-flexible variety, \( A \) be a perfect \( G \)-family of automorphisms of \( X \), \( Q \) be a normal algebraic variety and \( \varrho : X \to Q \) be a dominant morphism. Suppose that \( Q_0 \) is a smooth open dense subset of \( Q \), \( X_0 \) is an open subset of \( X \) contained in \( \varrho^{-1}(Q_0) \) and

\[
(2) \quad X_0 \times_{Q_0} X_0 = 2 \text{dim} \ X - \text{dim} \ Q.
\]

Let \( Y \) be the closure of \( \bigcup_{x \in X_0} \ker \{ \varrho_* : T_x X_0 \to T_{\varrho(x)} Q_0 \} \) in \( TX \) and

\[
(3) \quad \text{dim} \ Y = 2 \text{dim} \ X - \text{dim} \ Q.
\]

Let \( Z \) be a locally closed reduced subvariety of \( X \) with \( \text{ED}(Z) \leq \text{dim} Q \) and \( \text{dim} Z < \text{codim}_{\varrho^{-1}(Q_0)}(\varrho^{-1}(Q_0) \setminus X_0) \). Then for a general element \( \alpha \in A \) the morphism \( \varrho|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \to Q_0 \) is an injective immersion.

Proof. In the case of \( X_0 = \varrho^{-1}(Q_0) \) the statement is the combination of [Ka21, Theorem 2.6] and [Ka21, Proposition 2.8(5)]. In the general case the proof goes without change if one observes that \( \alpha(Z) \) does not meet \( \varrho^{-1}(Q_0) \setminus X_0 \) for a general \( \alpha \in A \) by the transversality theorem. □

Proposition 2.9. Let the assumptions and conclusions of Proposition 2.6 hold. Suppose that \( H \) itself is an \( F \)-flexible variety. Let \( Z \) be a locally closed reduced subvariety of \( H \) with \( \text{ED}(Z) \leq \text{dim} X \) (and by the conclusions of Proposition 2.6 with \( \text{dim} Z < \text{codim}_H(H \setminus U) \)). Then for a general element \( \beta \in B \) in any perfect \( F \)-family \( B \) of automorphisms of \( H \) the morphism \( \varphi_H|_{\beta(Z)} : \beta(Z) \to X \) is an injective immersion.
Proof. Since $\varphi_H|_U : U \to X$ is a smooth morphism Formulas (2) and (3) hold with $\varphi : X \to Q, Q_0$ and $X_0$ replaced by $\varphi_H : H \to X, X$ and $U$, respectively. Hence, the desired conclusion follows form Theorem 2.8.

Corollary 2.10. Let the assumptions and conclusions of Proposition 2.6 hold and $Z$ be an affine algebraic variety with $\text{ED}(Z) \leq \dim X$ (and by the conclusions of Proposition 2.6 with $\dim Z < \text{codim}_H(H \setminus U)$). Suppose that each element of $\mathcal{G}$ is a unipotent group, i.e. $H \simeq \mathbb{A}^t$ where $t \geq \dim X$. Then $Z$ can be treated as a closed subvariety of $H$ and for a general element $\beta \in \mathcal{B}$ in any perfect $F$-family $\mathcal{B}$ of automorphisms of $H$ the morphism $\varphi_H|_{\beta(Z)} : \beta(Z) \to X$ is an injective immersion.

Proof. The first statement follows from Holme’s theorem and the second from Proposition 2.9. □

Since every smooth flexible variety $X$ admits a morphism $\varphi_H : H \to X$ as in Corollary 2.10 we have the following.

Theorem 2.11. ([Ka21, Theorem 3.7]) Let $Z$ be an affine algebraic variety and $X$ be a smooth quasi-affine flexible variety of dimension at least $\text{ED}(Z)$. Then $Z$ admits an injective immersion into $X$.

Remark 2.12. It is worth mentioning that if $\varphi : Z \to X$ is an injective immersion, then it may happen that $Z$ is not isomorphic to $\varphi(Z)$. As an example one can consider the morphism $\mathbb{A}^1 \setminus \{1\} \to \mathbb{A}^2, t \mapsto (t^2 - 1, t(t^2 - 1))$. It maps $\mathbb{A}^1 \setminus \{1\}$ onto the polynomial curve given in $\mathbb{A}^2$ by the equation $y^2 = x^2(x + 1)$.

We have also in our disposal the following slightly improved version of ([Ka21, Theorem 3.2].

Theorem 2.13. Let $\psi : X \to Y$ be a finite morphism where $X$ is a smooth flexible variety and $Y$ is normal. Let $Z$ be a quasi-affine algebraic variety which admits a closed embedding in $X$. Suppose also that $S$ is a closed subvariety of $Y$ such that it contains $Y_{\text{sing}}$ and $\dim Z < \text{codim}_Y S$. Then $Z$ admits a closed embedding in $Y$ with the image contained in $Y \setminus S$.

Proof. One can treat $Z$ as a closed subvariety of $X$. By [AFKKZ, Theorem 1.11] there exists an algebraic family $\mathcal{A}$ of automorphisms of $X$ such that for a general $\alpha \in \mathcal{A}$ the variety $\alpha(Z)$ does not meet $\psi^{-1}(S)$. By Proposition 2.7 enlarging $\mathcal{A}$ we can suppose that it is a perfect family. Theorem 2.8 implies now that $\psi|_{\alpha(Z)} : \alpha(Z) \to Y_{\text{reg}} \subset Y$ is an injective immersion. Since $\psi$ is finite $\psi|_{\alpha(Z)}$ is also proper. Hence, we are done. □

3. Main Theorem

Recall the definition of weighted degree functions on polynomial rings.
Definition 3.1. Let $k[x_1, \ldots, x_n]$ be a polynomial ring and $r_1, \ldots, r_n$ be real numbers. For every monomial $\mu = c \prod_{i=1}^m x_i^{m_i}$ (where $c \in k$ is nonzero) we let $d(\mu) = \sum m_i r_i$. We also let $d(0) = -\infty$. If $p \in k[x_1, \ldots, x_m]$ is the sum $\sum_{\mu \in M(p)} \mu$ of a collection $M(p)$ of monomials, then we put $d(p) = \max\{d(\mu) | \mu \in M(p)\}$. Such function $d$ is called a weighted degree function on $k[x_1, \ldots, x_n]$ (with weights $r_1, \ldots, r_n$).

Remark 3.2. Suppose that $q_1, \ldots, q_k \in k[x_1, \ldots, x_n]$ are homogeneous polynomials of degrees $m_1, \ldots, m_k$, $P \in k[y_1, \ldots, y_k]$, $R(x_1, \ldots, x_n) = P(q_1(x_1, \ldots, x_n), \ldots, q_k(x_1, \ldots, x_n))$ and $d$ is the weighted degree function on $k[y_1, \ldots, y_k]$ with weights $m_1, \ldots, m_k$. We want to emphasize that if $P$ is a $d$-homogeneous polynomial, then $R$ is a homogeneous polynomial such that $\deg R = d(P)$. In particular, if $P$ is a not necessarily homogeneous and $Q$ is the leading $d$-homogeneous part of $P$, then $R(x_1, \ldots, x_n) = Q(q_1(x_1, \ldots, x_n), \ldots, q_k(x_1, \ldots, x_n))$.

Proposition 3.3. Let the assumptions and conclusions of Corollary 2.10 be satisfied (in particular, $Z$ is affine, $\varphi_H : H \simeq A^t \rightarrow X$ is dominant and $\text{ED}(Z) \leq \dim X$). Suppose that $X$ is a subvariety of $\mathbb{A}^m$ where $\mathbb{A}^m$ is equipped with a coordinate system such that the origin $o \in \mathbb{A}^m$ is contained in $X$ and $\varphi_H^{-1}(o)$ is a general fiber of $\varphi_H$ (one can always make this assumption since $X$ is quasi-affine). Let $I \subset \mathbb{A}^t$ be the defining ideal of $\varphi_H^{-1}(o)$ and and $V$ be the zero locus of the ideal $\tilde{I} = \{\tilde{a} | a \in I\}$. Let the codimension of $V$ in $H$ be at least $\dim Z$. Then $Z$ admits a closed embedding into $X$.

Proof. Let $\mathcal{A}$ be a perfect $\text{SAut}(H)$-family of automorphisms of $H \simeq A^t$. Consider the natural embedding $A^t \to \mathbb{P}^t$, $D = \mathbb{P}^t \setminus A^t \simeq \mathbb{P}^{t-1}$ and $K = \text{SL}_t(k)$. Then we have the natural $K$-action on $\mathbb{P}^t$ such that $D$ is invariant under it and the restriction of the action to $D$ is transitive. By Proposition 2.7 $K \times A$ is still a perfect $\text{SAut}(H)$-family of automorphisms of $H$. That is, for a general $\beta$ in $K$ and a general $\alpha$ in $A$ the morphism $\varphi_H|_{\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X$ is still an injective immersion by Corollary 2.10.

Note that every $a \in I$ can be extended to a rational function on $\mathbb{P}^t$. The intersection $R$ of the indeterminacy sets of these extensions is given by the common zeros of the homogeneous polynomials $\hat{a}$, $a \in I$ in $D$. In particular, $R$ has codimension at least $\dim Z$ in $D$. Furthermore,
since \( \hat{I} \) is finitely generated the embedding \( X \hookrightarrow \mathbb{A}^m \) can be chosen so that \( \hat{I} \) is generated by \( \hat{f}_1, \ldots, \hat{f}_m \) where \( f_1, \ldots, f_m \) are the coordinate functions of \( \varphi_{H} : \mathbb{A}^l \to X \subset \mathbb{A}^m \). In particular, \( R \) is the set of common zeros of the homogeneous polynomials \( \hat{f}_1, \ldots, \hat{f}_m \) in \( D \). As before we treat \( Z \) as a closed subvariety of \( \mathbb{A}^l \). Let \( P \) be the intersection of \( D \) with the closure of \( \beta \circ \alpha(Z) \) in \( \mathbb{P}^d \), i.e., \( \dim P \leq \dim Z - 1 \). Since the restriction of the \( K \)-action to \( D \) is transitive, \( P \) does not meet \( R \) for general \( \beta \in K \) and \( \alpha \in A \) by [AFKKZ, Theorem 1.15]. Hence, \( \varphi_{H} \mid_{\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X \) is a proper morphism by [Ka20, Corollary 5.4]. Consequently, it is a closed embedding which concludes the proof. \( \square \)

**Proposition 3.4.** Under the assumptions of Proposition 3.3 the codimension of \( V \) is least \( \dim X \).

**Proof.** Let \( \pi : \mathbb{A}^{[l]} \to A = \mathbb{A}^{[l]}/I \) and \( \hat{\pi} : \mathbb{A}^{[l]} \to \hat{A} = \mathbb{A}^{[l]}/\hat{I} \) be the natural homomorphisms. The standard degree function \( \deg \) on \( \mathbb{A}^{[l]} \) generates the grading \( \hat{I} = \bigoplus_{j=0}^{\infty} \hat{I}_j \). Hence, \( \hat{A} \) is the graded ring \( \hat{A} = \bigoplus_{j=0}^{\infty} \hat{A}_j \) where \( \hat{A}_j \) is the image of the subspace of homogeneous polynomials of degree \( j \) under \( \hat{\pi} \). Suppose that for some \( k > 0 \) there exist \( k \) algebraically independent elements \( a_1, \ldots, a_k \) in \( \hat{A} \). Since \( \hat{A} \) is a graded ring one can assume that each \( a_i \) is a homogeneous element, i.e., \( a_i \in \hat{A}_{r_i} \) and \( a_i = \hat{\pi}(p_i) \) where \( p_i \in \mathbb{A}^{[l]} \) is a homogeneous polynomial such that \( \deg p_i = r_i \). Let \( Q(x_1, \ldots, x_k) \in \mathbb{A}^{[k]} \) be a nonzero \( d \)-homogeneous polynomial where \( d \) is the weighted degree function with weights \( r_1, \ldots, r_k \). The fact that \( a_1, \ldots, a_k \) is algebraically independent is equivalent to the fact that for every \( Q \) as above \( Q(p_1, \ldots, p_k) \) does not belong to \( \hat{I} \). Since by Remark 3.2 \( Q(p_1, \ldots, p_k) \) is a homogeneous polynomial this implies that it does not belong to \( I \). Let \( P(x_1, \ldots, x_k) \in \mathbb{A}^{[k]} \) be nonzero, \( Q \) be the leading \( d \)-homogeneous part of \( P \) and \( f := P(p_1, \ldots, p_k) \). By Remark 3.2 \( \hat{f} = Q(p_1, \ldots, p_k) \). Hence, \( f \notin I \) and, in particular, \( P(\pi(p_1), \ldots, \pi(p_k)) \neq 0 \). Thus, we have \( k \) algebraically independent elements of \( A \) which implies that the transcendence degree of \( A \) is no less than the transcendence degree of \( \hat{A} \).

Note that \( \hat{A} \) is the ring of regular functions on \( V \) and \( A \) is the ring of regular functions of the fiber \( \varphi_{H}^{-1}(o) \). Hence, \( \dim V \leq \dim \varphi_{H}^{-1}(o) \). Since \( \varphi_{H}^{-1}(o) \) is a general fiber of \( \varphi_{H} \) one has \( \dim \varphi_{H}^{-1}(o) = \dim H - \dim X \) which yields the desired conclusion. \( \square \)

Now we can prove our main result.

**Theorem 3.5.** Let \( Z \) be an affine algebraic variety and \( X \) be a smooth flexible variety such that \( \dim X \geq ED(Z) \). Then \( Z \) admits a closed embedding into \( X \).

\(^1\)One can easily show that these transcendence degrees coincide as a consequence of the Chevalley semi-continuity theorem (e.g., see [KaML, Lemma 6.2]).
Proof. Since every smooth flexible variety $X$ admits a morphism $\varphi_H : H \to X$ as in Corollary 2.10 the conclusions of Proposition 3.3 are valid. By Proposition 3.4 $\text{codim}_HV \geq \dim X > \dim Z$. Thus, Proposition 3.3 yields the desired conclusion. □

Theorem 2.13 implies the following.

**Corollary 3.6.** Let $Z$ be an affine algebraic variety, $Y$ be a smooth flexible variety. Suppose that $\varphi : Y \to X$ is a finite morphism into a normal variety $X$ and $\dim X \geq \text{ED}(Z)$. Let $S$ be a closed subvariety of $X$ such that it contains $X_{\text{sing}}$ and $\dim Z < \text{codim}_XS$. Then $Z$ admits a closed embedding into $X$ with the image contained in $X \setminus S$.

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