Some Characterizations of a Normal Subgroup of a Group

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Abstract

Let $G$ be a group and $H$ be a subgroup of $G$ which is either finite or of finite index in $G$. In this note, we give some characterizations for normality of $H$ in $G$. As a consequence we get a very short and elementary proof of the Main Theorem of [5], which avoids the use of the classification of finite simple groups.

Key words: Right loop, Normalized Right Transversal, Right Inverse Property.

1 Introduction

Let $G$ be a group and $H$ be a subgroup of $G$. A normalized right transversal (NRT) $S$ of $H$ in $G$ is a subset of $G$ obtained by choosing one and only one element from each right coset of $H$ in $G$ and $1 \in S$. Then $S$ has a induced binary operation $\circ$ given by $\{x \circ y\} = Hxy \cap S$, with respect to which $S$ is a right loop with identity 1, that is, a right quasigroup with both sided identity (see [8 Proposition 4.3.3, p.102],[4]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [4 Theorem 3.4, p.76]).

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Let \( T(G, H) \) denote the set of all NRTs (normalized right transversals) of \( H \) in \( G \). We say that \( S, T \in T(G, H) \) are isomorphic, if their induced right loop structures are isomorphic. If \( H \) is normal subgroup in \( G \), then each \( S \in T(G, H) \) is isomorphic to the quotient group \( G/H \). The converse of this statement was proved in [5, Main Theorem, p.643] for finite groups:

**Theorem 1.1** (Main Therem [5]). Let \( G \) be a finite group and \( H \) a subgroup of \( G \). If all NRTs of \( H \) in \( G \) are isomorphic, then \( H \) is normal in \( G \).

The proof of the Main Theorem in [5] used the classification of finite simple groups (the knowledge of order of automorphism groups of finite non-abelian simple groups). In this note, we obtain an elementary short proof of Theorem 1.1, which avoids the use of the classification of finite simple groups.

Let \( S \in T(G, H) \). For \( x \in S \), we denote the map \( y \mapsto y \circ x \) \((y \in S)\) by \( R_x \), where \( \circ \) is the binary operation on \( S \) defined in the first paragraph of Section 1. We say that (1) \( S \) has right inverse property (RIP), if there is a map \( r : S \to S \) such that \( R_{x}^{-1} = R_{r(x)} \), for all \( x \in S \), (2) \( S \) is right conjugacy closed (RCC), if for each pair \((x, y) \in S \times S \) there exists \( z \in S \) such that \( R_x R_y R_z = R_z \), (3) \( S \) is \( A_r \)-transversal if \( H \subseteq N_G(S) \), where \( N_G(S) \) denotes the normalizer of \( S \) in \( G \). Now, we state the main result of this note:

**Theorem 1.2.** Let \( H \) be subgroup of \( G \) such that either the order \(|H|\) of \( H \) or the index \([G:H]\) is finite. Then following are equivalent:

1. \( H \) is a normal subgroup of \( G \).
2. All \( S \in T(G, H) \) are both sided transversals.
3. All \( S \in T(G, H) \) are isomorphic.
4. All \( S \in T(G, H) \) have RIP.
5. All \( S \in T(G, H) \) are RCC.

### 2 Proof of the Theorem 1.2

Let \( G \) be a group and \( H \) a subgroup of \( G \). It is shown in [1] and [6] that if \( H \) is finite subgroup of \( G \) then there exists a common set of representatives for the left and right cosets of \( H \) in \( G \). Let us call such a transversal as both
sided transversal. In [9, Theorem 3, p. 12], it is observed that if the index $[G : H]$ of $H$ in $G$ is finite, then both sided transversal exists. O. Ore has generalized these results in [7].

Let $S \in T(G, H)$ and $\circ$ be the binary operation on $S$ defined in the first paragraph of Section 1. Let $x, y \in S$ and $h \in H$. Then $x.y = f(x, y)(x \circ y)$ for some $f(x, y) \in H$ and $x \circ y \in S$. Also $x.h = \sigma_x(h)x\theta h$ for some $\sigma_x(h) \in H$ and $x\theta h \in S$. This gives us a map $f : S \times S \to H$ and a map $\sigma : S \to H^H$ defined by $f((x, y)) = f(x, y)$ and $\sigma(x)(h) = \sigma_x(h)$. Also $\theta$ is a right action of $H$ on $S$. The quadruple $(S, H, \sigma, f)$ is a c-groupoid (see [4, Definition 2.1, p.71]). Infact, every c-groupoid comes in this way (see [4, Theorem 2.2, p.72]). The same is observed in [2] but with different notations (see [2, Section 3, p. 289]). We need following result of [2] to prove Theorem 1.2:

Proposition 2.1 ([2], Proposition 3.5, p. 292). Let $S \in T(G, H)$ and $(S, H, \sigma, f)$ be the associated c-groupoid. Then following are equivalent:

1. $\sigma_x : S \to S$ is surjective, for all $x \in S$.
2. The equation $x \circ X = 1$, where $X$ is unknown, has a solution, for all $x \in S$.
3. $S$ is a both sided transversal.

The equivalence of (2) and (3) has also been proved in [3, Lemma 7*, p.30]

Lemma 2.2. Let $S, T \in T(G, H)$ be isomorphic and $S$ be a both sided transversal. Then $T$ is also both sided transversal.

Proof. Let $\circ$ and $\circ'$ be the induced binary operations on $S$ and $T$ respectively. Fix an isomorphism $p : S \to T$. Let $y \in T$ and $x = p^{-1}(y)$. Since $S$ is a both sided transversal, by Proposition 2.1 there exists $a \in S$ such that $x \circ a = 1$. Hence $y \circ' p(a) = p(x) \circ' p(a) = p(1) = 1$. Thus by Proposition 2.1 $T$ is a both sided transversal.

Proof. 

Lemma 2.3. Let $G$ be a group and $H$ be a non-normal subgroup of $G$. Then there exists $S \in T(G, H)$, which is not a left transversal of $H$ in $G$. 

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Proof. Since \( H \not\trianglelefteq G \), there exists \( x \in G \) such that \( xH \neq Hx \). We may assume that \( xH \setminus Hx \neq \emptyset \), for \( xH \subsetneq Hx \) if and only if \( Hx^{-1} \subsetneq x^{-1}H \) (and so we may replace \( x \) by \( x^{-1} \), if necessary). Choose \( y \in xH \setminus Hx \). Then \( xH = yH \) but \( Hx \neq Hy \). Let \( S \in \mathcal{T}(G, H) \) containing \( 1, x, y \). Clearly \( S \) is a right transversal but not left transversal of \( H \) in \( G \).

**Proposition 2.4.** Let \( G \) be a group and \( H \) be a subgroup of \( G \). If all \( S \in \mathcal{T}(G, H) \) are \( A_r \)-transversals, then \( H \trianglelefteq G \).

Proof. Assume that all \( S \in \mathcal{T}(G, H) \) are \( A_r \)-transversals. Let \( S \in \mathcal{T}(G, H) \). Let \( x \in S \) and \( h \in H \). Since \( S \) is an \( A_r \)-transversal, \( h^{-1}xh \in S \). Hence \( xh = h(h^{-1}xh) \) implies that \( \sigma_x = I_H \), for all \( x \in S \), where \( I_H \) is the identity map on \( H \). By Proposition 2.1 all \( S \) are both sided transversals. Thus by Lemma 2.3 \( H \trianglelefteq G \).

**Remark 2.5.** The converse of Proposition 2.4 is not true. For example, let \( G = \text{Sym}(3) \), the symmetric group of degree 3. Let \( H \) and \( S \) be subgroups of \( G \) of order 3 and 2 respectively. Then \( H \trianglelefteq G \), \( S \in \mathcal{T}(G, H) \) and \( N_G(S) = S \). Thus \( S \) is not an \( A_r \)-transversal.

**Proof of Theorem 1.2** The statement (1) implies each of the statements (2)-(6) (for all \( S \in \mathcal{T}(G, H) \) are isomorphic to the group \( G/H \)).

2 \( \Rightarrow \) 1: Follows from Lemma 2.3.

3 \( \Rightarrow \) 2: Assume that (3) holds. Let \( S \) be a both sided transversal of \( H \) in \( G \) (See first paragraph of Section 2). By Lemma 2.2 all \( S \in \mathcal{T}(G, H) \) are both sided transversals.

4 \( \Rightarrow \) 2: Assume that (4) holds. Let \( S \in \mathcal{T}(G, H) \). Since \( S \) has RIP, there exists a map \( r : S \to S \) such that \( R_x^{-1} = R_{r(x)} \) for all \( x \in S \). Fix \( x \in S \). Then \( R_x R_{r(x)}(x) = x \), that is, \( (x \circ r(x)) \circ x = x = 1 \circ x \), where \( \circ \) is the binary operation on \( S \) defined in the first paragraph of Section 1. By right cancellation in \( S \), \( x \circ r(x) = 1 \). Hence by Proposition 2.1 \( S \) is a both sided transversal. Thus (2) holds.

5 \( \Rightarrow \) 2: Assume that (5) holds. Let \( S \in \mathcal{T}(G, H) \) and \( x' \) be the left inverse of \( x \) in \( S \). Since \( S \) is RCC, there exists \( z \in S \) such that \( R_x R_{x'} R_x^{-1} = R_z \). Hence \( R_x R_{x'} R_x^{-1}(x) = R_z(x) \), i.e. \( x \circ z = x' \circ x = 1 \), where \( \circ \) is the binary operation on \( S \) defined in the first paragraph of Section 1. Hence by Proposition 2.1
$S$ is a both sided transversal. Thus (2) holds.

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