On the strong coupling problem in cosmologies with “strong gravity in the past”

Y. Ageeva\textsuperscript{a,b,c,1}, P. Petrov\textsuperscript{a,2}

\textsuperscript{a} Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect, 7a, 117312 Moscow, Russia

\textsuperscript{b} Department of Particle Physics and Cosmology, Physics Faculty, M.V. Lomonosov Moscow State University, Leninskie Gory 1-2, 119991 Moscow, Russia

\textsuperscript{c} Institute for Theoretical and Mathematical Physics, M.V. Lomonosov Moscow State University, Leninskie Gory 1, 119991 Moscow, Russia

Abstract

We examine the potential strong coupling problem at early times in a bouncing cosmological model with “strong gravity in the past” (Jordan frame), which is conformally related to inflation (Einstein frame). From naive dimensional analysis in the Jordan frame one would conclude that the quantum strong coupling energy scale can be lower than the classical energy scale. However, from the Einstein frame prospective this should not be the case. We illustrate this point by calculation in the Jordan frame which shows cancellations of the dangerous contributions in the tree level amplitude.

1 Introduction

In scalar-tensor gravities, there is a possibility of bouncing or genesis cosmology in the Jordan frame, with the effective Planck mass depending on time and tending to zero in the asymptotic past (“strong gravity in the past”). Such a scenario has been discussed [1, 2, 3, 4], in particular, in the context of Horndeski theories [5, 6, 7, 8, 9], where it has been proposed to avoid instabilities otherwise guaranteed by a no-go theorem [1, 10]. Similar situations have been considered in other contexts, see, e.g., Ref. [11] and references therein.

\textsuperscript{1}email: ageeva@inr.ac.ru

\textsuperscript{2}email: petrov@inr.ac.ru
Once the effective Planck mass tends to zero in the asymptotic past, one may worry that the theory is in the strong coupling regime at early times, so the classical treatment of the background is not legitimate. Whether or not this is the case depends on the relationship between the quantum strong coupling energy scale and the classical scale determined by the Hubble parameter and its time derivatives. One way to approach this issue is to make use of naive dimensional analysis of the interacting theory [3, 12, 13]. The purpose of this note is to point out that naive dimensional analysis may sometimes badly fail in estimating the strong coupling scale.

Our example is the bouncing Universe in the Jordan frame which is conformally related to the inflationary Universe in the Einstein frame [14]. For an appropriate inflationary scalar potential, the Einstein frame picture guarantees that there is no strong coupling problem, i.e., the classical treatment of the background is fully legitimate. We will see that, on the other hand, the naive dimensional analysis in the Jordan frame would show the opposite. This is the problem of the naive dimensional analysis, however: our direct calculation of tree level amplitude in the Jordan frame shows strong cancellations yielding consistency with the Einstein frame inflationary considerations.

We introduce the model in Sec. 2, derive the action for scalar perturbations in the Jordan frame at quadratic and cubic orders in Sec. 3.1, consider the strong coupling issue at the level of naive dimensional analysis in Sec. 3.2 and finally calculate the tree level amplitude in Sec. 3.3. We conclude in Sec. 4.

\section{Bounce conformally related to inflation}

\subsection{Actions}

Following Ref. [14], we consider a class of bouncing models (Jordan frame) that are conformally related to cosmological inflation. The action in the Jordan (bounce) frame is given by

\[ S_b = \int d^4x \sqrt{-g} \left[ P(\phi, X) + \frac{M_P^2 f^2(\phi)}{2} R \right], \]  

with

\[ P(\phi, X) = \omega(\phi)X - V(\phi), \]

where \( M_P = (8\pi G)^{-1/2} \) is reduced Planck mass, \( R \) is Ricci scalar and

\[ X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \]

\[ \omega(\phi) = f^2 - 6M_P^2 \left( \frac{df}{d\phi} \right)^2, \quad V(\phi) = f^4(\phi)V_I(\phi). \]
Here \( f(\phi) \) is a yet undetermined function, and \( V_I(\phi) \) is the scalar potential in the Einstein frame. We do not use special notation for quantities in the Jordan frame; notations here agree with Ref. [15], modulo definition \( F(\phi) = f^2(\phi) \).

By conformal transformation
\[
g_{\mu\nu} = f^{-2}(\phi)g_{I\mu\nu}
\]
the theory (1) is related to the following inflationary model in the Einstein (inflation) frame:
\[
S_I = \frac{1}{2} \int d^4 x \sqrt{-g_I} \left[ M_P^2 R_I - g_I^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V_I(\phi) \right],
\]
where subscript “I” refers to quantities in the Einstein frame.

### 2.2 Einstein frame: inflation

We consider inflation potential that flattens out at large fields,
\[
V_I(\phi) \rightarrow V_\infty, \quad \text{as} \quad \phi \rightarrow \infty; \quad V_\infty \ll M_P^4,
\]
so that the energy density is always sub-Planckian. Viewed from the Einstein frame, the classical description of inflating background and semiclassical treatment of cosmological perturbations are perfectly legitimate. Inflation occurs in the slow roll regime at early times, \( \epsilon \ll 1, \eta \ll 1 \), where we use the standard notations
\[
\epsilon = \frac{(V'_I)^2 M_P^2}{2V^2}, \quad \eta = \frac{V''_I M_P^2}{V}.
\]
The slow roll equations are
\[
\frac{d\phi(\tau)}{d\tau} = -\frac{M_P V'_I}{\sqrt{3V_I}}, \quad H_I = \frac{\sqrt{V_I}}{3 M_P},
\]
where \( \tau \) is cosmic time in the Einstein frame.

### 2.3 Jordan frame: bounce

We follow Ref. [14] and choose the function defining the conformal transformation as follows:
\[
f(\phi) = f_0 \exp \left[ -\frac{(\alpha + 1)}{M_P^2} \int d\phi \frac{V_I}{V'_I} \right], \quad \alpha > 0,
\]
where the value of \( f_0 \) is irrelevant for our purposes. Then the Jordan frame metric is
\[
ds^2 = f^{-2}(\phi(\tau))d\tau^2 - f^{-2}(\phi(\tau))a_I^2(\tau)d\mathbf{x}^2
\]
and the Hubble parameter in the Jordan frame is given by
\[ H = f \frac{d}{d\tau} \ln(a_I f^{-1}) = -f \cdot \frac{\alpha}{M_P} \sqrt{\frac{V_I}{3}}, \tag{5} \]
where we make use of the slow roll equations (4). The Jordan frame universe contracts (and, at the end of the Einstein frame inflation, experiences the bounce).

It is worth emphasizing that \( f \to 0 \) as \( t \to -\infty \). So, the Jordan frame Hubble parameter vanishes in the asymptotic past. The Jordan frame effective Planck mass \( M_{P(\text{eff})} = f M_P \) also tends to zero as \( t \to -\infty \); this situation is dubbed “strong gravity in the past”.

3 Strong coupling and absence thereof

From now on we work in the Jordan frame.

We concentrate on the scalar sector of perturbations about the contracting solution (5). We use the unitary gauge
\[ \delta \phi = 0, \]
then the scalar perturbation is parameterized with the field \( \zeta \), entering the spatial metric, so that the full metric in the Jordan frame cosmic time is [15]
\[ ds^2 = -\left[(1 + \alpha)^2 - a^{-2} e^{-2\zeta}(\partial\psi)^2\right]dt^2 + 2\partial_i\psi dt dx^i + a^2 e^{2\zeta} dx^2, \]
where \( \alpha \) and \( \psi \) are perturbations of the lapse and shift. Upon solving the constraints, one arrives at the unconstrained action written in terms of \( \zeta \). We consider its quadratic and cubic parts. To this end, we adapt the results of Ref. [15].

3.1 Quadratic and cubic actions

The quadratic action for scalar perturbation is
\[ S^{(2)}_{\zeta \zeta} = \int dtd^3xa^3 G_S \left[ \dot{\zeta}^2 - \frac{1}{a^2} \zeta_{,i} \zeta_{,i} \right], \]
where, using formulas given in Ref. [15, 16], we obtain
\[ G_S = \frac{\alpha}{2H_I^2} = \frac{f^2}{2H_I^2} \left( \frac{d\phi}{dr} \right)^2. \]
This is an exact expression, which is actually a straightforward Jordan frame reformulation of the standard Einstein frame result. In the slow roll case (4) one has
\[ G_S = f^2 \cdot \frac{M_P^4(V_I)^2}{2V_I^2}. \]
Note that the perturbations propagate luminally, which is again a Jordan frame counterpart of the standard Einstein frame property.

The terms in the cubic action for scalars, which do not vanish in the model (1) either identically or due to background equations and field redefinition [15, 16, 17], are (we use notations of Ref. [15])

\[ S^{(3)} = \int dt d^3x \, a^3 \left\{ C_1 \dot{\zeta}^2 + \frac{1}{a^2} C_2 \zeta (\partial \zeta)^2 + C_4 \dot{\zeta} (\partial_i \zeta) (\partial_i \mathcal{X}) + C_5 \partial^2 \zeta (\partial \mathcal{X})^2 \right\} , \quad (6) \]

where \( \partial^2 = \partial_i \partial_i \) and \( \partial^2 \mathcal{X} = \dot{\zeta} \). The coefficients are straightforwardly calculated. To the leading order in the slow roll parameters we have

\[ C_1 = f^2 \cdot \frac{M_p^6 (V'_I)^2}{4V'_I} (4V'_I V''_I - 3(V'_I)^2) , \quad (7a) \]
\[ C_2 = f^2 \cdot \frac{M_p^6 (V''_I)^2}{4V'_I} (5(V'_I)^2 - 4V'_I V''_I) , \quad (7b) \]
\[ C_4 = f^2 \frac{M_p^6 (V'_I)^4}{16V'_I^2} (M_p^2 (V'_I)^2 - 8V'_I^2) , \quad (7c) \]
\[ C_5 = f^2 \frac{M_p^8 (V'_I)^6}{32V'_I^6} . \quad (7d) \]

### 3.2 Naive dimensional analysis

We now proceed with the naive dimensional analysis of the strong coupling problem. The classical energy scale is of order of the Hubble parameter (5),

\[ |E^{(class)}| = |H| \sim \frac{f \sqrt{V_I}}{M_P} . \quad (8) \]

To obtain an estimate of the strong coupling scale through naive dimensional analysis, we set, at a given moment of time, \( a = 1 \) and introduce canonically normalized field

\[ \zeta_c = \sqrt{2} g_s \zeta . \]

In terms of the canonically normalized field, the cubic action still has the form (6) with the replacement

\[ \tilde{C}_i = (2g_s)^{-3/2} C_i , \]

so that

\[ \tilde{C}_1 = \frac{1}{f} \cdot \frac{(-3(V'_I)^2 + 4V'_I V''_I)}{4V'_I V'_I} , \]
\[ \tilde{C}_2 = \frac{1}{f} \cdot \frac{(5(V'_I)^2 - 4V'_I V''_I)}{4V'_I V'_I} . \]
while

\[ \tilde{C}_4 \sim \frac{1}{f} \cdot \frac{V_I'}{V_I}, \quad \tilde{C}_5 \sim \frac{1}{f} \cdot M_P^2 \left( \frac{V_I'}{V_I} \right)^3. \]  

(9)

All operators in the resulting cubic Lagrangian are dimension-5, so one immediately finds naive estimates for the associated strong coupling scales,

\[ E_i^{(naive)} \sim |\tilde{C}_i|^{-1}. \]

Naively, the most relevant of these scales are the lowest ones, which are associated with the largest \( C_i \).

For asymptotically flat inflaton potential (2), one typically has \( \eta \gg \epsilon \), so the largest couplings in (7) are \( C_1 \) and \( C_2 \). The two naive strong coupling scales are of the same order:

\[ E^{(naive)} \sim f \frac{V_I'}{V_I}. \]  

(10)

Depending on the shape of the inflaton potential, classical energy scale (8) may exceed strong coupling energy scale (10). As an example, for the inflaton potential

\[ V_I = V_\infty \left( 1 - e^{\phi^2/\mu^2} \right) \]

one has

\[ \frac{E^{(naive)}}{E^{(class)}} \sim \frac{\mu^2}{\phi H_I} \]

which is less than 1 at large \( \phi \).

We conclude that naive dimensional analysis in the Jordan frame suggests that there is a quantum strong coupling energy scale which, for appropriate inflaton potential, is below the classical scale. If not for the Einstein frame considerations, one would be tempted to dismiss such a model.

To end up this Section, we notice that the cubic couplings \( \tilde{C}_4 \) and \( \tilde{C}_5 \) are not enhanced, see (9). So, at large \( \phi \), their associated strong coupling scales are much higher than the classical energy scale (8). In other words, the third and fourth terms in the integrand in (6) per se do not imply strong coupling, even naively. Thus, we do not have to consider the terms with couplings \( C_4 \) and \( C_5 \) in our analysis of the amplitudes.

### 3.3 Scattering amplitude

Making use of the first and second terms in the cubic action (6), with \( C_{1,2} \) replaced by \( \tilde{C}_{1,2} \) and \( \zeta \) by canonically normalized \( \zeta_c \), it is straightforward to calculate \( 2 \to 2 \) scattering amplitude.
Before giving the result, we note that if we set, for the sake of argument, \( \tilde{C}_2 = 0 \), then the matrix element would be given by

\[
M_{\tilde{C}_1; \tilde{C}_2=0} = -\frac{E^2}{f^2} \cdot \frac{(9x^2 - 5)(3(V'_I)^2 - 4V_I V''_I)^2}{64(x^2 - 1)V_I^2(V'_I)^2},
\]

where \( x = \cos \theta \) and \( \theta \) is scattering angle. Were this the correct matrix element, our naive expectation would be confirmed: the partial wave amplitudes

\[
a^{(l)} = \frac{1}{32\pi} \int dx \ P_l(x) M_{\tilde{C}_1; \tilde{C}_2=0}
\]

where \( P_l \) is the Legendre polynomials, would hit the unitarity bound \( |a^{(l)}| = 1/2 \) at \( E \sim E^{(naive)} \). The same situation would occur if we set \( \tilde{C}_1 = 0 \).

However, there are strong cancellations. Indeed, the matrix elements in \( s \)-, \( t \)- and \( u \)-channels are, respectively

\[
M_s = -\frac{E^2}{4} (3\tilde{C}_1 + \tilde{C}_2)^2,
M_t = \frac{E^2}{2(1-x)} [\tilde{C}_1 + \tilde{C}_2(2-x)]^2,
M_u = \frac{E^2}{2(1+x)} [\tilde{C}_1 + \tilde{C}_2(2+x)]^2.
\]

The resulting matrix element is

\[
M = M_s + M_t + M_u = \frac{E^2}{f^2} \cdot \frac{(41x^2 - 45)(V'_I)^2 - 40(x^2 - 1)V_I V''_I}{16(x^2 - 1)V_I^2}.
\]

We see that the strong coupling scale is actually given by\(^1\)

\[
E^{(strong)} \sim f \cdot \left( \frac{V_I}{V''_I} \right)^{1/2} \sim f \cdot \frac{M_P}{\eta^{1/2}},
\]

where \( \eta \) is the slow roll parameter (3). As anticipated, this scale is much higher than the classical energy scale (8) for \( V_I \ll M_P^4 \). Our calculation of the amplitude confirms the absence of the strong coupling problem.

4 Conclusion

Of course, the model we have considered in this paper is nearly trivial. Still, it illustrates the main point: naive dimensional analysis may grossly underestimate the quantum strong

\(^1\)We still consider the case \( V_I V''_I \gg (V'_I)^2 \); we cannot trust the term with \( (V'_I)^2 \) in the numerator anyway, since we neglected terms with \( C_{4,5} \) in the cubic action (6).
coupling energy scale. There may be less trivial situations where this property holds, e.g., due to kinematical or dynamical symmetries. It would be interesting to have more examples and see whether the mismatch between the dimensional analysis and actual strong coupling scale can always be understood via field redefinitions.

Acknowledgments

The authors are grateful to Valery Rubakov for useful comments and fruitful discussions as well as for careful reading of the early versions of this manuscript. This work has been supported by Russian Science Foundation Grant No. 19-12-00393.

References

[1] T. Kobayashi, Phys. Rev. D 94, no.4, 043511 (2016) doi:10.1103/PhysRevD.94.043511 [arXiv:1606.05831 [hep-th]].

[2] A. Ijjas and P. J. Steinhardt, Phys. Lett. B 764, 289-294 (2017) doi:10.1016/j.physletb.2016.11.047 [arXiv:1609.01253 [gr-qc]].

[3] Y. Ageeva, P. Petrov and V. Rubakov, JHEP 12, 107 (2020) doi:10.1007/JHEP12(2020)107 [arXiv:2009.05071 [hep-th]].

[4] Y. Ageeva, P. Petrov and V. Rubakov, Phys. Rev. D 104, no.6, 063530 (2021) doi:10.1103/PhysRevD.104.063530 [arXiv:2104.13412 [hep-th]].

[5] G. W. Horndeski, Int. J. Theor. Phys. 10, 363-384 (1974) doi:10.1007/BF01807638

[6] D. B. Fairlie, J. Govaerts and A. Morozov, Nucl. Phys. B 373, 214-232 (1992) doi:10.1016/0550-3213(92)90455-K [arXiv:hep-th/9110022 [hep-th]].

[7] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79, 064036 (2009) doi:10.1103/PhysRevD.79.064036 [arXiv:0811.2197 [hep-th]].

[8] C. Deffayet, O. Pujolas, I. Sawicki and A. Vikman, JCAP 10, 026 (2010) doi:10.1088/1475-7516/2010/10/026 [arXiv:1008.0048 [hep-th]].

[9] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. Lett. 105, 231302 (2010) doi:10.1103/PhysRevLett.105.231302 [arXiv:1008.0603 [hep-th]].

[10] M. Libanov, S. Mironov and V. Rubakov, JCAP 08, 037 (2016) doi:10.1088/1475-7516/2016/08/037 [arXiv:1605.05992 [hep-th]].
[11] C. Wetterich, Phys. Rev. D **104**, no.10, 104040 (2021) doi:10.1103/PhysRevD.104.104040 [arXiv:2104.14013 [gr-qc]].

[12] Y. A. Ageeva, O. A. Evseev, O. I. Melichev and V. A. Rubakov, EPJ Web Conf. **191**, 07010 (2018) doi:10.1051/epjconf/201819107010 [arXiv:1810.00465 [hep-th]].

[13] Y. Ageeva, O. Evseev, O. Melichev and V. Rubakov, Phys. Rev. D **102**, no.2, 023519 (2020) doi:10.1103/PhysRevD.102.023519 [arXiv:2003.01202 [hep-th]].

[14] D. Nandi, Phys. Lett. B **809**, 135695 (2020) doi:10.1016/j.physletb.2020.135695 [arXiv:2003.02066 [astro-ph.CO]].

[15] A. De Felice and S. Tsujikawa, JCAP **04**, 029 (2011) doi:10.1088/1475-7516/2011/04/029 [arXiv:1103.1172 [astro-ph.CO]].

[16] X. Gao, T. Kobayashi, M. Shiraishi, M. Yamaguchi, J. Yokoyama and S. Yokoyama, PTEP **2013**, 053E03 (2013) doi:10.1093/ptep/ptt031 [arXiv:1207.0588 [astro-ph.CO]].

[17] J. M. Maldacena, JHEP **05**, 013 (2003) doi:10.1088/1126-6708/2003/05/013 [arXiv:astro-ph/0210603 [astro-ph]].