Robustness of vortex-bound Majorana zero modes against correlated disorder*

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We investigate the effect of correlated disorder on Majorana zero modes (MZMs) bound to magnetic vortices in two-dimensional topological superconductors. By starting from a lattice model of interacting fermions with a $p_x \pm i p_y$ superconducting ground state in the disorder-free limit, we use perturbation theory to describe the enhancement of the Majorana localization length at weak disorder and a self-consistent numerical solution to understand the breakdown of the MZMs at strong disorder. We find that correlated disorder has a much stronger effect on the MZMs than uncorrelated disorder and that it is most detrimental if the disorder correlation length $\ell$ is on the same order as the superconducting coherence length $\xi$. In contrast, MZMs can survive stronger disorder for $\ell \ll \xi$ as random variations cancel each other within the length scale of $\xi$, while an MZM may survive up to very strong disorder for $\ell \gg \xi$ if it is located in a favorable domain of the given disorder realization.

Topological phases of matter harbor exotic nonlocal quasi-particles and have been proposed as a promising platform for fault-tolerant quantum computation [11,12]. In particular, topological superconducting systems, including one-dimensional (1D) and two-dimensional (2D) heterostructures [3-5] as well as intrinsic 2D superconductors with $p$-wave pairing symmetry [9], are predicted to host Majorana zero modes (MZMs) [10,12] which can implement the Clifford gate set via braiding. While most proposals for MZM braiding have focused on 1D systems, such as nanowire T-junctions [13], MZMs bound to superconducting vortices in 2D systems have distinct advantages as the 2D geometry allows a greater degree of freedom in the motion of the MZMs.

Due to their inherently nonlocal nature, MZMs are known to be protected against infinitesimal local perturbations, including random disorder. However, given that real-world materials contain disorder in varying forms and strength, it is also important to understand the robustness of MZMs against disorder beyond the infinitesimal limit. For example, weak disorder may make the MZMs less localized, leading to a smaller qubit density and/or more gate errors, whereas strong disorder may lead to a complete breakdown of the MZMs. While there have been numerous studies along these lines, most of them focus on 1D nanowires [14-23], while those studying 2D superconductors do not consider vortex-bound MZMs [24] or only concentrate on uncorrelated disorder [25,26].

In this Letter, we consider a simple microscopic model of interacting fermions with a $p_x \pm i p_y$ superconducting ground state [27,28] and study the effect of correlated disorder by combining analytical and numerical approaches. Specifically, we investigate vortex-bound MZMs in this model and understand how their robustness depends on the correlation length of the disorder. Our main result is that correlated disorder is significantly more detrimental to the MZMs than uncorrelated disorder. In particular, disorder has the most adverse effect if its correlation length $\ell$ is similar to the superconducting coherence length $\xi$, while disorders with $\ell \ll \xi$ and $\ell \gg \xi$ are both more benign, even though for completely different reasons. Since our results naturally extend to the continuum limit of the model and are expressed in terms of measurable length and energy scales, they should apply universally for $p_x \pm i p_y$ superconductors and provide useful guidelines for the realization of MZM braiding in realistic experimental systems.

Model.—We consider a tight-binding Hamiltonian of interacting spinless fermions on the square lattice,

$$\hat{H} = -\sum_{\langle r,r' \rangle} t_{r,r'} c^{\dagger}_{r} c_{r'} - \sum_{\langle r,r' \rangle} \left( t_{r,r'} \delta^{\dagger}_{r} \delta_{r'} + h.c. \right) - g \sum_{\langle r,r' \rangle} c^{\dagger}_{r} \delta_{r} c_{r'},$$

(1)

where the three terms describe a site-dependent chemical potential, a nearest-neighbor hopping amplitude, and a nearest-neighbor attractive interaction, respectively. In the presence of a magnetic field, the hopping amplitude is spatially modulated by the vector potential $A(\mathbf{r})$ through the Peierls substitution, $t_{r,r'} = t e^{i A_{r'} \cdot \mathbf{r}}$, where $A_{r'} = \int_{r'} A(\mathbf{r}) \cdot d\mathbf{r}$. We expand the chemical potential as $\mu_{r} = \bar{\mu} + \delta \mu_{r}$, where $\bar{\mu}$ is a constant background, while $\delta \mu_{r}$ describes random disorder of strength $\bar{\mu}$ that is correlated within a length scale $\ell$. Mathematically, $\delta \mu_{r}$ are real Gaussian random variables characterized by

$$\bar{\delta} \mu_{r} = 0, \quad \delta \mu_{r} \delta \mu_{r'} = \delta \mu^2 e^{-|r-r'|^2/\ell^2},$$

(2)

where the overline denotes averaging over many disorder realizations. In practice, these real-space random variables are generated through $\delta \mu_{r} = \sum_{k} \Re \{ e^{i k \cdot r} \delta \mu_{k} \}$ from the independent momentum-space complex variables $\delta \mu_{k}$ satisfying

$$\bar{\delta} \mu_{k} = \delta \mu_{k} = 0, \quad \delta \mu_{k} \delta \mu_{k'} = \delta \mu^2 \delta_{kk'}, \quad \delta \mu_{k} \delta \mu_{k'} = 2N \delta \mu^2 \delta_{kk'} e^{-|k|^2/\ell^2/4},$$

(3)

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where the normalization constant is $N = \left[ \sum_r e^{-\epsilon_r^2 |r|^2/4} \right]^{-1}$ for a large enough system size $L \gg \ell$.

In the absence of interactions ($g = 0$), disorder ($\mu_r = \bar{\mu}$), and magnetic field ($t_{r,r'} = t$), the tight-binding Hamiltonian in Eq. (1) is quadratic and translation invariant. By means of a Fourier transform, one then obtains a single fermion band with energy-momentum dispersion $\varepsilon_k = -\bar{\mu} - 2t(\cos k_x + \cos k_y)$ for a normalized lattice constant $a = 1$. For $|\bar{\mu}| < 4t$, the low-energy physics is governed by a Fermi surface characterized by $\varepsilon_F = 0$. In the following, we consider $\bar{\mu} = -4t + \varepsilon_F$ with $0 < \varepsilon_F < t$ to get an approximately circular Fermi surface around $k = 0$. From an expansion to the lowest order in $k$, the energy-momentum dispersion is then $\varepsilon_k = -\varepsilon_F + |k|^2/2m$, where $m = 1/(2t)$ is an effective mass. Thus, in this approximation, the Fermi surface is indeed circular with Fermi energy $\varepsilon_F$ and Fermi wave vector $k_F = \sqrt{2m\varepsilon_F} = \sqrt{\varepsilon_F/t}$.

**Bulk superconductivity.**—We first consider the Hamiltonian in Eq. (1) with attractive interactions ($g > 0$) but without disorder ($\mu_r = \bar{\mu}$) or magnetic field ($t_{r,r'} = t$). It has been shown numerically [27] and analytically [28] that the ground state is then a gapped $p_x \pm ip_y$ superconductor which spontaneously breaks time-reversal symmetry. To describe this ground state on the mean-field (i.e., saddle-point) level, we employ a standard Hubbard-Stratonovich decoupling in Eq. (1) to obtain a quadratic Bogoliubov-de Gennes (BdG) Hamiltonian,

$$H = -\sum_r \mu_r c_r^\dagger c_r - \sum_{(r,r')} \left( t_{r,r'} c_{r'}^\dagger c_r + t^*_{r,r'} c_r c_{r'} \right) - \sum_{(r,r')} \left( \Delta^+_{r,r'} c_r c_{r'} + \Delta^{-}_r c_r^\dagger c_{r'}^\dagger \right),$$

(4)

which must be solved self-consistently in terms of the superconducting pairing potentials,

$$\Delta_{r,r'} = -\Delta_{r',r} = g \langle \epsilon_r c_r c_{r'} \rangle,$$

(5)

where $\langle \cdot \rangle$ means the expectation value of the operator $\cdot$ with respect to the ground state of $H$. These pairing potentials can generally be parameterized as

$$\Delta^x = \Delta_{r,r+\hat{r}_x} = \sum_q \left( \Delta^+_q + \Delta^-_q \right) e^{iq \cdot r_x},$$

$$\Delta^y = \Delta_{r,r+\hat{r}_y} = i \sum_q \left( \Delta^+_q - \Delta^-_q \right) e^{iq \cdot r_y},$$

(6)

where $\hat{r}_x = (1,0)$ and $\hat{r}_y = (0,1)$ are the lattice vectors, and the component $\Delta^+_q$ corresponds to $p_x \pm ip_y$ superconductivity with a spatial modulation of wave vector $q$. In the absence of disorder ($\mu_r = \bar{\mu}$) and magnetic field ($t_{r,r'} = t$), the superconductivity is translation symmetric [27,28]. Assuming $p_x + ip_y$ pairing symmetry without loss of generality, the components in Eq. (5) then become

$$\Delta^+_q = \bar{\Delta} \delta_{q,0}, \quad \Delta^-_q = 0,$$

(7)

corresponding to $\Delta_r = \Delta^x = -i \Delta^y = \bar{\Delta}$. The constant $\bar{\Delta}$ can be determined from a self-consistent solution of Eqs. (4) and (5). In the universal continuum limit ($k_F \ll 1$), we show in the Supplemental Material (SM) [29] that $\bar{\Delta}$ satisfies

$$1 = \frac{g}{N} \sum_k \frac{|k|^2}{\varepsilon_k^2 + 4|k|^4/\Delta^2} \approx g \nu \int \frac{d\varepsilon k_F^2}{\sqrt{\varepsilon^2 + 4k_F^2/\Delta^2}},$$

(8)

where $N$ is the number of lattice sites, and $\nu$ is the density of states at the Fermi level. If we then choose $\bar{\Delta}$ to be real and positive without loss of generality, it is approximately given by the standard superconducting gap formula,

$$\Delta \sim \frac{E}{2k_F} \exp \left( -\frac{1}{2g k_F^2 \nu} \right),$$

(9)

where $E$ is an energy scale governing the high-energy cutoff (whose precise value is irrelevant), while $2g k_F^2 \nu$ is an effective interaction strength reflecting the $p$-wave symmetry of the superconductivity. Importantly, because of the factor $k_F^2 \nu \propto \varepsilon_F$ within the exponential, the pairing potential $\bar{\Delta}$ strongly depends on the Fermi energy $\varepsilon_F$ [28].

Next, we include a weak disorder in the chemical potential ($\delta \bar{\mu} \ll \bar{\mu}$) and study its effect on the pairing potentials $\Delta_{r,r'}$ via perturbation theory. Formally, we restore $\mu_r = \bar{\mu} + \delta \mu_r$ in Eq. (1) and modify Eq. (7) by writing $\Delta^+_q = \Delta \delta_{q,0} + \delta \Delta^+_q$ and $\Delta^-_q = \delta \Delta^-_q$. We can then employ $\delta \mu_r \equiv \sum_q \delta \mu_r e^{iq \cdot r}$, where $\delta \mu_q = q \Delta \delta \mu_q$, and obtain the self-consistent solution of Eqs. (4) and (5) up to linear order in $\delta \mu_q$ and $\delta \Delta_q$. In the continuum limit ($|q| \ll k_F \ll 1$) of weak superconductivity ($\xi^{-1} \ll k_F$), this approach gives (see the SM [29])

$$\delta \Delta^+_q = f \left( \frac{|q|}{2} \right) \frac{\partial \bar{\Delta}}{\partial \varepsilon_F} \delta \mu_q,$$

$$\delta \Delta^-_q = -h \left( \frac{|q|}{2} \right) e^{\Omega q} \frac{\partial \bar{\Delta}}{\partial \varepsilon_F} \delta \mu_q,$$

(10)

where $\Omega = v_F/(2k_F \Delta) = 1/(2m \Delta)$ is the superconducting coherence length, $v_F = k_F/m$ is the Fermi velocity, $\delta \mu_q$ is the angle between $q$ and $\hat{r}_x$, while $f(x)$ and $h(x)$ are dimensionless functions with asymptotic forms

$$f(x) \approx \begin{cases} 1 - \frac{x^2}{8} & (x \ll 1), \\ \frac{1}{\ln x} & (x \gg 1), \end{cases}$$

$$h(x) \approx \begin{cases} \frac{x^2}{8} & (x \ll 1), \\ \frac{1}{2 \ln x} & (x \gg 1). \end{cases}$$

(11)

For $q = 0$, the disorder component $\delta \mu_q$ simply corresponds to a shift in the Fermi energy $\varepsilon_F$, and the pairing potential $\bar{\Delta}$ with $p_x + ip_y$ symmetry is renormalized accordingly. For finite $q$, however, the disorder gives rise to reduced variations in the $p_x + ip_y$ pairing due to $f(|q|/2) < 1$ and also generates a finite $p_x - ip_y$ pairing due to $h(|q|/2) > 0$. Both of these effects are more pronounced if the disorder wave vector $q$ exceeds the inverse coherence length $\xi^{-1}$. We note that, while the mean-field results in Eqs. (10) and (11) may not be quantitatively right for $|q| \gg \xi^{-1}$, any corrections beyond the
mean-field level are expected to strengthen our main conclusions by suppressing \( f(\xi|q|/2) \) and \( h(\xi|q|/2) \).

Finally, we describe the real-space correlations in the pairing potentials \( \Delta r_r \) as a result of disorder. Since \( h(x) \ll f(x) \) for all \( x \), we neglect the components \( \delta \Delta^\pm \) and use Eq. (9) to introduce \( \delta \Delta_r = \delta \Delta_r^x = -i \delta \Delta_r^y = \sum_q \delta \Delta_q^r e^{i q \cdot r} \). From Eqs. (3) and (10), the disorder correlations in \( \delta \Delta_r \) are then

\[
\delta \Delta_r \delta \Delta_r' = \alpha_\epsilon^2 \bar{\mu}_r^2 R e \sum_q e^{-\frac{4}{\ell^2} q^2 + a(r-r')^2 f^2 \left( \frac{\xi|q|}{2} \right)},
\]

where \( \alpha = \partial \Delta / \partial \epsilon_F \) and \( f^2(x) = |f(x)|^2 \). Since \( f(x) \) depends only logarithmically on its argument, it is a reasonable approximation to substitute \( f(\xi|q|/2) \) with \( f(\xi|\ell|) \) in Eq. (12) and work with the resulting simplified correlations,

\[
\delta \Delta_r \delta \Delta_r' = \alpha_\epsilon^2 f^2(\xi|\ell|) \mu_\epsilon e^{-(r-r')^2/2\ell^2}.
\]

From a direct comparison with Eq. (7), this result has a simple physical interpretation. For \( \ell \gg \xi \), the local pairing potential is determined by the local chemical potential via Eq. (9). For \( \ell \ll \xi \), the variations in the pairing potential still follow those in the chemical potential, but the constant of proportionality is reduced by a factor \( f^2(\xi|\ell|) \ll 1 \).

**Majorana localization length.**—We now consider a superconducting vortex hosting a MZM and understand the effect of weak disorder on the localization length of the MZM. Taking the continuum limit, \( r \rightarrow \psi(r) \), assuming a pure \( p_x + ip_y \) pairing symmetry, \( \Delta_r \approx \Delta_r^x = -i \Delta_r^y = \Delta(r) \), and including a magnetic field, the BdG Hamiltonian in Eq. (4) takes the form

\[
\mathcal{H} = \int d^2 r \psi^\dagger \mathcal{H} \psi, \quad \psi = (\psi_x, \psi_y)^T
\]

Choosing a single vortex at the origin and using polar coordinates, \( r = (r, \theta) \), the \( \pi \) magnetic flux of the vortex can be represented by a vector potential with components

\[
A_r(r) = 0, \quad A_\theta(r) = \frac{2\pi \delta(\theta) - a(r)}{2r},
\]

where the term \( \propto \delta(\theta) \) corresponds to a \( \mathbb{Z}_2 \) flux string \( \gamma + 1 \), while \( a(r) \approx 1 \) for \( r \ll \lambda \) and \( a(r) \approx e^{-r/\lambda} \) for \( r \gg \lambda \) in terms of the London penetration depth \( \lambda \). In this gauge, the pairing potential \( \Delta(r) \) does not have any angular winding and simply takes its bulk value for \( r \gg \xi \). The Hamiltonian matrix in Eq. (14) can then be written in polar coordinates as

\[
\mathcal{H} = \frac{1}{2} \left[ \begin{array}{cc}
-\frac{1}{2m}(D_r^2 + D_\theta^2) - \epsilon_F & 2\Delta e^{i\theta} (\partial_r + i \partial_\theta) \\
-2\Delta^* e^{-i\theta} (\partial_r - i \partial_\theta) & \frac{1}{2m}(D_r^2 + D_\theta^2) + \epsilon_F
\end{array} \right],
\]

where \( D_r^2 = \partial_r^2 + (1/r) \partial_r \) and \( D_\theta \pm \equiv (\partial_r \mp i a/2r) \), while the \( \mathbb{Z}_2 \) flux string induces antiperiodic boundary conditions, \( \psi(r, 2\pi) = -\psi(r, 0) \), in the polar angle \( \theta \). If we take \( \Delta \in \mathbb{R} \) without loss of generality, searching for the MZM in the form

\[
\gamma = \int d^2 r \phi(r) \left[ e^{-i\theta/2} \psi(r) - e^{i\theta/2} \psi^\dagger(r) \right],
\]

which naturally satisfies the antiperiodic boundary conditions, and demand \( \gamma = \gamma^\dagger \) as well as \( [H, \gamma] = 0 \), the radial MZM wave function \( \phi(r) \) must be a real solution of

\[
\frac{1}{2m} \left[ \frac{d^2 \phi}{dr^2} + \frac{d \phi}{r \, dr} - \frac{(1-a)^2 \phi}{4r^2} \right] + e_F \phi + 2\Delta \left[ \frac{d \phi}{dr} + \frac{\phi}{2r} \right] = 0.
\]

For large distances, \( r \gg \lambda \), in the disorder-free limit, we can set \( \Delta = \Delta \) and neglect \( a(r) \sim e^{-r/\lambda} \ll 1 \). The exact general solution of Eq. (18) then takes the form

\[
\phi(r) = \frac{C}{\sqrt{r}} \exp \left( -2m\Delta r \cos \left( \sqrt{2m\epsilon_F - (2m\Delta)^2} r + \varphi \right) \right)
\]

where \( C \) and \( \varphi \) are arbitrary constants, while \( \xi \) is the coherence length and \( q_F = \sqrt{k_F^2 - \xi^{-2}} \approx k_F \) is the Fermi wave vector for weak superconductivity. Importantly, the solution in Eq. (19) is approximately valid even for \( \xi \ll r \ll \lambda \) as the correction to Eq. (18) from a finite \( a(r) \) is subdominant due to \( |\phi(r)|^2 \ll |d^2 \phi/dr^2|^2 \) for any \( r \gg \xi \). As expected, the Majorana localization length is thus simply the coherence length \( \xi \) in the disorder-free limit.

If we include a weak disorder in the chemical potential \( \mu \) (i.e., the Fermi energy \( \epsilon_F \)), it affects the decay of the MZM wave function \( \phi(r) \) and, hence, the localization length via the pairing potential \( \Delta \). Ignoring the power-law prefactor, the approximate disorder average of \( |\phi(r)| \) from Eq. (19) is

\[
|\phi(r)| \sim \exp \left[ -2m \int_0^r dr' \frac{\Delta + \delta \Delta(\hat{r})}{\Delta + \delta \Delta(\hat{r})} \right].
\]

Utilizing the Gaussian nature of the random component \( \delta \Delta(\hat{r}) \) and taking its correlations from Eq. (13), the disorder average for \( r \gg \ell \gg k_F^{-1} \) then becomes

\[
|\phi(r)| \sim \exp \left[ -2m \Delta + 2m^2 \int_0^r dr' \frac{\delta \Delta(\hat{r}) \delta \Delta(\hat{r'})}{\Delta + \delta \Delta(\hat{r})} \right]
\]

\[
\approx \exp \left[ -r \frac{\sqrt{\pi}}{\xi} + \kappa \frac{\sqrt{\pi}}{\xi} \frac{\ell}{2} f^2 \left( \frac{\xi}{\ell} \right) \right],
\]

and corresponds to an enhanced Majorana localization length

\[
\xi' = \frac{1}{\kappa} \left[ 1 - \frac{\sqrt{\pi}}{\xi} \frac{\ell}{2} f^2 \left( \frac{\xi}{\ell} \right) \right]^{-1},
\]

where \( \kappa = \delta \mu(\partial \Delta / \partial \epsilon_F) / \Delta \) is the relative change in the pairing potential as a result of a shift \( \delta \mu \) in the Fermi energy. According to Eq. (22), the localization length is more sensitive to disorder with larger correlation length \( \ell \). Indeed, for \( \xi \ll \ell \), the correction to the localization length is suppressed due to both \( \ell / \xi \ll 1 \) and \( f(\xi|\ell|) \ll 1 \). However, for \( \ell \gg \xi \), it should be emphasized that the disorder average leading to Eq. (22) is only appropriate for \( r \gg \ell \). Instead, for \( \xi \ll r \ll \ell \), the behavior is determined by the specific disorder realization, and the localization length may even decrease if the MZM is located in a region with \( \mu > \bar{\mu} \).
ln \epsilon \\

4 ln \epsilon \\

8 ln \epsilon \\

-4 ln \epsilon \\

-10 ln \epsilon \\

-12 ln \epsilon \\

FIG. 1: (a) Two vortices centered at the yellow plaquettes with separation \( \mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2 = (5, 2) \). The \( \mathbb{Z}_2 \) flux string (dashed line) intersects several links denoted by thick lines. (b) MZM hybridization energy \( \epsilon \) as a function of the separation \( \mathbf{R} = (R, 0) \) for a 50 x 30 system. The dotted line is a fit of Eq. (19) with \( \xi = 2.8 \) and \( q_F = 0.65 \).

\begin{align*}
\text{ln } \epsilon & \\
-4 & \\
-8 & \\
-12 & \\
-16 & \\
-20 & \\
R & \\
-10 & \\
-8 & \\
-6 & \\
-4 & \\
-2 & \\
0 & \\
2 & \\
4 & \\
6 & \\
8 & \\
10 & \\
12 & \\
14 & \\
16 & \\
18 & \\
20 & \\
\xi & \\
2.8 & \\
q_F & \\
0.65 & \\
\end{align*}

Numerical solution.—To qualitatively check the validity of our results, we numerically obtain self-consistent solutions of Eqs. (4) and (5) through an iterative procedure. Since MZMs must appear in pairs for any closed system, we consider two superconducting vortices centered at two square plaquettes with positions \( \mathbf{R}_{1,2} \) [see Fig. 1(a)]. In this case, the \( \mathbb{Z}_2 \) flux string connects the two vortices, and the hopping amplitudes in Eq. (4) become \( t_{r,r'} = tu_{r,r'}e^{iA_{r,r'}} \), where \( u_{r,r'} \) is \(-1 \) (+1) if the \( \mathbb{Z}_2 \) flux string intersects (does not intersect) the link \( \langle r, r' \rangle \), while \( A_{r,r'} \) is only nonzero within a radius \( \lambda \) of each vortex. The precise form of \( A_{r,r'} \) and the details of the iterative procedure are described in the SM [29, 31].

We choose the parameters of Eq. (1) to be \( t = 1, \mu = -3.5 \), and \( g = 5.0 \), which correspond to \( m = 0.5, \varepsilon_F = 0.5 \), and \( k_F = 0.7 \). In the absence of disorder, the self-consistent solution for a vortex-free system gives a bulk pairing potential \( \Delta \approx 0.33 \) and a bulk fermion gap \( E_0 \approx 0.41 \). If we then include two vortices with separation \( \mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2 \), we find a low-energy fermion in the bulk gap whose energy decays exponentially with \( R \equiv |\mathbf{R}| \) [see Fig. 1(b)]. Since this fermion consists of the two MZMs bound to the vortices, and its finite energy results from a hybridization between the MZM wave functions, we fit its energy \( \epsilon \) with the functional form of Eq. (19) to extract \( \xi \approx 2.8 \) and \( q_F \approx 0.65 \). We note that these values agree with \( 1/(2m\Delta) \approx 3.0 \) and \( k_F \approx 0.7 \) even though the system is not in the continuum limit.

Finally, we include two vortices with \( R \gg 1 \) and investigate how the energy \( \epsilon \) of the lowest-energy fermion behaves as the disorder strength \( \delta \mu \) is gradually increased. The disorder-averaged results are shown in Fig. 2 for two different disorder correlation lengths, corresponding to \( \ell < \xi \) and \( \ell > \xi \), respectively. In both cases, we find that the energy \( \epsilon \) increases from the MZM result, \( \epsilon \sim e^{-R/\xi} \), to the generic disordered result, \( \epsilon \sim 1/N \), which indicates the breakdown of the MZMs. This breakdown occurs at \( \delta \mu \sim \varepsilon_F \) due to a hybridization between the MZMs and the gapless edge modes that surround disorder-induced non-superconducting regions with local \( \varepsilon_F < 0 \). Remarkably, this breakdown is in qualitative agreement with our weak-disorder results in at least three different ways. First, the breakdown at \( \delta \mu \sim \varepsilon_F \) roughly corresponds to \( \kappa \sim 1 \) at which Eq. (22) predicts a divergent localization length in the case of \( \ell \sim \xi \). Second, the MZMs can generally survive stronger disorder for \( \ell < \xi \). Third, the energy \( \epsilon \) has larger variations for \( \ell > \xi \) as the MZMs can survive even very strong disorder for certain disorder realizations.

Discussion.—We have studied the effect of correlated disorder on vortex-bound MZMs in \( p_x \pm ip_y \) superconductors and demonstrated that it is much more detrimental than uncorrelated disorder. The general picture is that disorder gradually increases the MZM localization length until the MZMs eventually break down due to a divergent localization length. However, according to Eq. (22), the correction to the localization length strongly depends on the disorder correlation length \( \ell \) and is suppressed for short-range-correlated disorder (\( \ell \ll \xi \)) because random variations cancel each other within the superconducting coherence length \( \xi \). We note that, while Eq. (22) is only valid for \( \ell \gg k_F \lambda \), our numerical results confirm this suppression even in the uncorrelated limit (\( \ell \to 0 \)).

For long-range-correlated disorder (\( \ell \gg \xi \)), the MZM localization length, which characterizes the decay of the wave function \( \phi(r) \) at large distances, \( r \gg \ell \), is strongly renormalized and thus rapidly diverges. Nevertheless, if the MZM is located within a large (size \( \ell \)) “disorder domain” with \( \mu > \mu_c \), it survives even in the presence of strong disorder because its wave function is already exponentially small, \( \phi(\ell) \sim e^{-\ell/\xi} \), at the boundary, \( r \sim \ell \), of the disorder domain. While any actual braiding of the MZMs is then restricted to such favorable disorder domains, effective braiding may still be achievable through a measurement-only protocol [32, 33].

Therefore, we conclude that disorder has the most adverse effect on the MZMs if its correlation length is similar to the superconducting coherence length. In this regime, the MZMs break down if disorder is strong enough to induce topologically distinct regions surrounded by gapless edge modes. We emphasize that, while we focus on a specific lattice model and only include disorder in the chemical potential, our results naturally extend to the continuum limit and should be universally applicable to disordered \( p_x \pm ip_y \) superconductors.
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Supplemental Material

MEAN-FIELD THEORY OF BULK SUPERCONDUCTIVITY

General formulation

Here we derive the mean-field theory for the $p_x \pm ip_y$ superconducting ground state of our model. Employing the path-integral formulation, the partition function corresponding to the Hamiltonian in Eq. (1) of the main text reads

$$Z = \int D(\psi, \psi^*) \exp (-S[\psi]),$$

$$S[\psi] = \int_0^\beta d\tau \left\{ \sum_r \psi_r^* (\tau) [\partial_\tau - \mu_r] \psi_r (\tau) - \sum_{(r,r')} \left[ t_{r,r'} \psi_r^* (\tau) \psi_{r'} (\tau) + c.c. \right] - g \sum_{(r,r')} \psi_r^* (\tau) \psi_r (\tau) \right\},$$

where $\beta$ is the inverse temperature, while $\psi_r (\tau)$ and $\psi_r^* (\tau)$ are Grassmann fields representing the fermionic operators $c_r$ and $c_r^\dagger$, respectively. Introducing the bosonic Hubbard-Stratonovich fields $\Delta_{r,r'} (\tau)$ and $\Delta_{r,r'}^* (\tau)$, the partition function then becomes

$$Z = \int D(\Delta, \Delta^*) \int D(\psi, \psi^*) \exp (-S[\Delta, \psi]),$$

$$S[\Delta, \psi] = \int_0^\beta d\tau \left\{ \sum_r \psi_r^* (\tau) [\partial_\tau - \mu_r] \psi_r (\tau) - \sum_{(r,r')} \left[ t_{r,r'} \psi_r^* (\tau) \psi_{r'} (\tau) + t_{r,r'} \Delta_{r,r'} (\tau) \psi_{r'} (\tau) \right] + \frac{1}{4g} \sum_{(r,r')} \left| \Delta_{r,r'} (\tau) \right|^2 \right\}.$$

Since the action $S[\Delta, \psi]$ is quadratic in the Grassmann fields $\psi_r (\tau)$ and $\psi_r^* (\tau)$, these Grassmann fields can be integrated out to obtain an effective action $S[\Delta]$ exclusively in terms of the Hubbard-Stratonovich fields $\Delta_{r,r'} (\tau)$ and $\Delta_{r,r'}^* (\tau)$. To this end, it is useful to introduce Fourier transforms in both space and (imaginary) time for both the Grassmann fields,

$$\psi_r (\tau) = \frac{1}{\sqrt{\beta N}} \sum_{k,\omega_n} \psi_{k,\omega_n} e^{i(k \cdot r - \omega_n \tau)},$$

as well as the Hubbard-Stratonovich fields,

$$\Delta^x_r (\tau) \equiv \Delta_{r,r+x} (\tau) = \sum_{q,\Omega_n} \left( \Delta^+_{q,\Omega_n} + \Delta^-_{q,\Omega_n} \right) e^{i(q \cdot r - \delta \Omega_n \tau)},$$

$$\Delta^y_r (\tau) \equiv \Delta_{r,r+y} (\tau) = i \sum_{q,\Omega_n} \left( \Delta^+_{q,\Omega_n} - \Delta^-_{q,\Omega_n} \right) e^{i(q \cdot r - \delta \Omega_n \tau)},$$

where $N$ is the number of lattice sites, $\bar{r}_x = (1, 0)$ and $\bar{r}_y = (0, 1)$ are the lattice vectors, $\omega_n = (2n + 1)\pi/\beta$ are the fermionic Matsubara frequencies, $\Omega_n = 2n\pi/\beta$ are the bosonic Matsubara frequencies, while $\Delta^\pm_{q,\Omega_n}$ correspond to pairing potentials with $p_x \pm ip_y$ pairing symmetry and a spatial modulation of wave vector $q$ [see also Eq. (6) in the main text]. Setting $t_{r,r'} = t$ and $\mu_r = \mu + \sum_q \hat{\mu}_q e^{iq \cdot r}$, the action $S[\Delta, \psi]$ in Eq. (24) can then be written as

$$S[\Delta, \psi] = \frac{2N\beta}{g} \sum_{q,\Omega_n} \left( |\Delta^+_{q,\Omega_n}|^2 + |\Delta^-_{q,\Omega_n}|^2 \right) + \frac{1}{2} \sum_{k,\omega_n} \sum_{k',\omega'_n} \left( \psi_{k,\omega_n}^* \psi_{k',\omega'_n} - \psi_{-k,-\omega_n}^* \psi_{-k',-\omega'_n} \right) \cdot G_{(k,\omega_n),(k',\omega'_n)}^{-1} [\Delta] \cdot \left( \psi_{k',\omega'_n}^* \psi_{-k,-\omega_n}^* \right),$$

$$G_{(k,\omega_n),(k',\omega'_n)}^{-1} [\Delta] = \left( \begin{array}{cc} \delta_{\omega_n,\omega'_n} \left( -i \omega_n + \varepsilon_k \right) & -i p_{k,k'} \Delta^+_{k'-k,\omega'_n-\omega_n} - i p_{k,k'}^* \Delta^-_{k-k',\omega_n-\omega'_n} \\ -i p_{k,k'}^* \left( \Delta^+_{k-k',\omega_n-\omega'_n} \right)^* & \delta_{\omega_n,\omega'_n} \left( -i \omega_n - \varepsilon_{k'} \right) \delta_{k,k'} + i \hat{\mu}_{k,k'} \end{array} \right).$$

In the continuum limit, corresponding to $|k| \sim |k'| \sim k_F \ll 1$, the functions $\varepsilon_k$ and $p_{k,k'}$ can be expanded as

$$\varepsilon_k = -\mu - 2t \left( \cos k_x + \cos k_y \right) \approx -\varepsilon_F + \frac{|k|^2}{2m},$$

$$p_{k,k'} = -i \left[ e^{ik' \cdot \bar{r}_x + i\varepsilon_{k'} \cdot \bar{r}_y} - e^{-i k' \cdot \bar{r}_x - i\varepsilon_{k'} \cdot \bar{r}_y} \right] \approx (k_x + ik_y) + (k'_x + ik'_y),$$

(28)
where $m = 1/(2t)$ is the effective mass, $\varepsilon_F = \mu + 4t$ is the Fermi energy, and $k_F = \sqrt{2m\varepsilon_F} = \sqrt{\varepsilon_F/\mu}$ is the Fermi wave vector. Integrating out the Grassmann fields, the partition function in Eq. (24) then takes the form

$$Z = \int D(\Delta, \Delta^*) \exp (-S[\Delta]),$$

$$S[\Delta] = \frac{2N\beta}{g} \sum_{q,\Omega_n} \left( |\Delta_{q,\Omega_n}^+|^2 + |\Delta_{q,\Omega_n}^-|^2 \right) - \text{Tr} \ln G^{-1}[\Delta].$$

The infinitely large matrix $G^{-1}[\Delta]$ simultaneously acts in particle-hole (Nambu) space, momentum space, and frequency space, while its $2 \times 2$ blocks corresponding to Nambu space are given by $G^{-1}_{(k,\omega_n),(k',\omega_{n'})}$ in Eq. (27).

### Mean-field theory in the disorder-free limit

On the level of mean-field theory, we restrict our attention to the saddle points of the effective action in Eq. (29). Differentiating $S[\Delta]$ with respect to $(\Delta_{q,\Omega_n}^\pm)^\ast$, the general saddle-point equation becomes

$$\frac{\partial S[\Delta]}{\partial (\Delta_{q,\Omega_n}^\pm)^\ast} = \frac{2N\beta}{g} \Delta_{q,\Omega_n}^\pm - \text{Tr} \left\{ G[\Delta] \cdot \frac{\partial G^{-1}[\Delta]}{\partial (\Delta_{q,\Omega_n}^\pm)^\ast} \right\} = 0,$$

where $G[\Delta]$ is the inverse matrix of $G^{-1}[\Delta]$. In the disorder-free limit ($\delta\mu_q = 0$), the saddle point with spatially homogeneous $p_x + ip_y$ superconductivity, corresponding to the known ground state of the model, is characterized by

$$\Delta_{q,\Omega_n}^+ = \Delta_0 \delta_\Omega, \quad \Delta_{q,\Omega_n}^- = 0.$$

The matrices $G^{-1}[\Delta]$ and $G[\Delta]$ are then block diagonal in both $k$ and $\omega_n$, and their respective $2 \times 2$ blocks are given by

$$G^{-1}_{k,\omega_n}[\Delta] = G^{-1}_{(k,\omega_n),(k,\omega_n)}[\Delta] = \left( \begin{array}{cc} -i\omega_n + \varepsilon_k & ip_k \Delta \\ -ip_k \Delta^* & -i\omega_n - \varepsilon_k \end{array} \right),$$

$$G_{k,\omega_n}[\Delta] = G_{(k,\omega_n),(k,\omega_n)}[\Delta] = \frac{1}{\omega_n^2 + \varepsilon_k^2 + |p_k|^2} \left( \begin{array}{cc} i\omega_n + \varepsilon_k & ip_k \Delta \\ -ip_k \Delta^* & i\omega_n - \varepsilon_k \end{array} \right),$$

where $p_k \equiv p_{k,k} \approx 2(k_x + ik_y)$, while the saddle-point equation in Eq. (30) takes the form

$$\bar{\Delta} = \frac{g}{2N\beta} \sum_{k,\omega_n} \text{Tr} \left\{ G_{k,\omega_n}[\Delta] \cdot \frac{\partial G^{-1}_{k,\omega_n}[\Delta]}{\partial \Delta^*} \right\} = \frac{g}{2N\beta} \sum_{k,\omega_n} |p_k|^2 \Delta^* = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{|p_k|^2 \Delta^*}{\omega_n^2 + \varepsilon_k^2 + |p_k|^2 |\Delta|^2}.$$

At zero temperature ($\beta \to \infty$), the summation in the Matsubara frequency $\omega_n$ can be turned into an integral. Dividing both sides of Eq. (33) by $\Delta$, and using $|p_k|^2 = 4|k|^2$, the saddle-point equation then becomes

$$1 = \frac{g}{4\pi N} \int_{\infty}^{\infty} df \frac{|p_k|^2}{\omega^2 + \varepsilon_k^2 + |p_k|^2 |\Delta|^2} = \frac{g}{4N} \sum_k \frac{|p_k|^2}{\sqrt{\varepsilon_k^2 + 4|k|^2 |\Delta|^2}} = \frac{g}{N} \sum_k \frac{|k|^2}{\sqrt{\varepsilon_k^2 + 4|k|^2 |\Delta|^2}}.$$

This final form of the saddle-point equation is equivalent to Eq. (8) in the main text.

### Disorder corrections through perturbation theory

In the presence of disorder ($\delta\mu_q \neq 0$), we consider perturbative corrections to the disorder-free saddle point. To this end, we modify Eq. (35) by including spatially inhomogeneous corrections as

$$\Delta_{q,\Omega_n}^+ = (\Delta_0 + \delta\mu_q) \delta_\Omega, \quad \Delta_{q,\Omega_n}^- = \delta\Delta_{q,\Omega_n}.$$

The spatial inhomogeneities $\delta\mu_q$ and $\delta\Delta_q$ give corrections to the matrices $G^{-1}[\Delta]$ and $G[\Delta]$ that are still block diagonal in $\omega_n$ but no longer in $k$. The $2 \times 2$ blocks of the correction matrices $\delta G^{-1}[\Delta]$ and $\delta G[\Delta]$ are

$$\delta G^{-1}_{k,k',\omega_n}[\Delta] = \delta G^{-1}_{(k,\omega_n),(k',\omega_n)}[\Delta] = \left( \begin{array}{cc} -i\bar{p}_{k,k'} \delta\Delta_{k-k'} + ip_{k,k'}^* \delta\Delta_{k-k'} & ip_{k,k'} \delta\mu_{k-k'} \\ -i\bar{p}_{k,k'} \delta\mu_{k-k'} & -i\bar{p}_{k,k'} \delta\mu_{k-k'} \end{array} \right),$$

$$\delta G_{k,k',\omega_n}[\Delta] = \delta G_{(k,\omega_n),(k',\omega_n)}[\Delta] = -G_{k,\omega_n}[\Delta] \cdot \delta G^{-1}_{k,k',\omega_n}[\Delta] \cdot G_{k,\omega_n}[\Delta].$$
Up to linear order in both \( \delta \mu_q \) and \( \delta \Delta_q^\pm \), the resulting correction to the saddle-point equation in Eq. (33) is then

\[
\delta \Delta_q^\pm = \frac{g}{2N\beta} \sum_{k',\omega_n} \text{Tr} \left\{ G_{k',k',\omega_n} \frac{\partial (G_{k',k',\omega_n}^\pm (\Delta))}{\partial (\delta \Delta_q^\pm)^*} \right\}
\]

(37)

Substituting Eqs. (32) and (36) into Eq. (37), the saddle-point equations for \( \delta \Delta_q^+ \) and \( \delta \Delta_q^- \) can be written as

\[
\delta \Delta_q^+ = A_q^+ \delta \mu_q + (1 - B_q^{++}) \delta \Delta_q^+ - B_q^{+-} \delta \Delta_q^- - C_q^{++} (\delta \Delta_q^+)^* - C_q^{+-} (\delta \Delta_q^-)^*,
\]

\[
\delta \Delta_q^- = A_q^- \delta \mu_q - B_q^{+-} \delta \Delta_q^+ + (1 - B_q^{--}) \delta \Delta_q^- - C_q^{+-} (\delta \Delta_q^+)^* - C_q^{--} (\delta \Delta_q^-)^*.
\]

(38)

For \( |q| \ll |k| \), where \(|p_{k+q'} - p_k| \ll |p_k| \) and thus \( p_{k+q'} \approx p_k \) in terms of \( q' \equiv q/2 \), the coefficients in Eq. (38) are

\[
A_q^+ = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{|p_k|^2 (\varepsilon_{k+q'} + \varepsilon_{k-q'})}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

\[
A_q^- = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{\bar{\Delta} p_k^2 (\varepsilon_{k+q'} + \varepsilon_{k-q'})}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

\[
B_q^{++} = B_q^{--} = 1 - \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{|p_k|^2}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

\[
B_q^{+-} = (B_q^{--})^* = -\frac{g}{2N\beta} \sum_{k,\omega_n} \frac{p_k^2 (\varepsilon_{k+q'} - \varepsilon_{k-q'})^2 + 2\bar{\Delta}^2 |p_k|^2}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

\[
C_q^{++} = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{\bar{\Delta}^2 |p_k|^4}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

\[
C_q^{+-} = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{\bar{\Delta}^2 p_k^2 |p_k|^2}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

\[
C_q^{--} = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{\bar{\Delta}^2 |p_k|^4}{(\omega_n^2 + \varepsilon_{k+q'}^2 + \Delta|p_k|^2)(\omega_n^2 + \varepsilon_{k-q'}^2 + \Delta|p_k|^2)},
\]

where we assume without loss of generality that \( \Delta \) is real and positive. Using \( p_{k+q'} \approx p_k \), the terms in the curly brackets vanish for \( B_q^{++} \) because of Eq. (33) and for \( B_q^{+-} \) because \( p_k^2 \) changes sign under fourfold rotation symmetry. Since \( \varepsilon_{k+q'} + \varepsilon_{k-q'} \approx 2\varepsilon_k \) for \(|q| \ll |k| \), the summands of \( A_q^\pm \) change sign at the Fermi surface, \( \varepsilon_k = 0 \), and the dominant contributions to the resulting sums are from regions far away from the Fermi surface. Given that \( |\varepsilon_{k+q'} - \varepsilon_k| \ll |\varepsilon_k| \) in those regions, it is then reasonable to approximate \( A_q^+ \) with \( A_0^+ \) for \(|q| \ll k_F \). In this approximation, we obtain

\[
A_q^+ \approx A_0^+ = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{2|p_k|^2 \varepsilon_k}{(\omega_n^2 + \varepsilon_k^2 + \Delta|p_k|^2)^2} = \Delta \frac{\partial}{\partial \varepsilon_{F'}} \left[ \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{|p_k|^2}{(\omega_n^2 + \varepsilon_k + \Delta|p_k|^2)^2} \right],
\]

(40)

where \( A_0^- \) vanishes because \( p_k^2 \) changes sign under fourfold rotation symmetry. In contrast, the dominant contributions to the sums of \( B_q^{\pm} \) and \( C_q^{\pm} \) are from the vicinity of the Fermi surface. Therefore, we can write \( p_k \approx 2k_F \varepsilon_k \equiv p_{F'} \varepsilon_{k_F} \) and

\[
\varepsilon_{k+q'} \approx \varepsilon_k \pm \cos(\theta_k - \theta_q') v_F |q'| = \varepsilon_k \pm \frac{1}{2} \cos(\theta_k - \theta_q) v_F |q| = \varepsilon_k \pm \cos(\theta_k - \theta_q) \Delta p_{F'} \frac{\xi|q|}{2},
\]

(41)
where \( v_F = k_F/m \) is the Fermi velocity, \( \xi = v_F/(2k_F\Delta) = 1/(2m\Delta) \) is the superconducting coherence length, while \( \vartheta_k \) is the angle between \( k \) and \( \hat{r}_x \). If we then take the zero-temperature limit (\( \beta \to \infty \)), and turn the summation in \( k \) into an integral,

\[
\frac{1}{N} \sum_k \to \frac{\nu}{2\pi} \int_{-\infty}^{+\infty} d\varepsilon \int_0^{2\pi} d\vartheta, \tag{42}
\]

where \( \varepsilon = \varepsilon_k \) and \( \vartheta = \vartheta_k - \vartheta_q \), while \( \nu \) is the density of states at the Fermi level, the coefficients \( B_q^{\pm \pm} \) and \( C_q^{\pm \pm} \) become

\[
B_q^{++} = B_q^{--} = B_0^{++} \left[ I_0 \left( \frac{\xi|q|}{2} \right) + I_0' \left( \frac{\xi|q|}{2} \right) \right],
\]

\[
B_q^{-+} = (B_q^{++})^* = B_0^{++} e^{2i\vartheta_q} \left[ I_2 \left( \frac{\xi|q|}{2} \right) + I_2' \left( \frac{\xi|q|}{2} \right) \right],
\]

\[
C_q^{++} = B_0^{++} \left( \frac{\xi|q|}{2} \right), \tag{43}
\]

\[
C_q^{-+} = C_q^{++} = B_0^{++} e^{2i\vartheta_q} I_2 \left( \frac{\xi|q|}{2} \right),
\]

\[
C_q^{--} = B_0^{++} e^{4i\vartheta_q} I_4 \left( \frac{\xi|q|}{2} \right),
\]

where the common constant of proportionality is given by

\[
B_0^{++} = \frac{g}{2N\beta} \sum_{k,\omega_n} \left( \frac{\Delta^2 |p_k|^4}{(e^\omega_n + e^{\Delta} + \Delta^2 |p_k|^2)^2} \right) = -\frac{1}{2} \frac{\partial}{\partial \Delta} \left[ \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{|p_k|^2}{e^\omega_n + e^{\Delta + \Delta^2 |p_k|^2}} \right], \tag{44}
\]

while the dimensionless functions are appropriate integrals,

\[
I_n(x) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\varepsilon \int_0^{2\pi} d\vartheta \left[ 1 + \omega^2 + (\varepsilon + x \cos \vartheta)^2 \right] \frac{e^{in\vartheta}}{\left[ 1 + \omega^2 + (\varepsilon + x \cos \vartheta)^2 \right] \left[ 1 + \omega^2 + (\varepsilon + x \cos \vartheta)^2 \right]}, \tag{45}
\]

with \( \tilde{\omega} = \omega/(\Delta p_F) \) and \( \tilde{\varepsilon} = \varepsilon/(\Delta p_F) \), taking the exact analytical forms

\[
I_0(x) = \frac{\arctan x}{x}, \quad I_2(x) = \frac{\log(1 + x^2) - x \arctan x}{x^2}, \quad I_4(x) = \frac{x^2(2x \arctan x) - 2(1 + x^2) \log(1 + x^2)}{x^4},
\]

\[
I_0'(x) = \log(1 + x^2), \quad I_2'(x) = 1 - \frac{\log(1 + x^2)}{x^2}. \tag{46}
\]

Using Eqs. (40) and (43), the solution of Eq. (38) for \( \delta \Delta_q^{\pm} \) then becomes

\[
\delta \Delta_q^{+} = f \left( \frac{\xi|q|}{2} \right) \frac{A_0^+}{2B_0^+} \delta \hat{\mu}_q, \quad \delta \Delta_q^{-} = -h \left( \frac{\xi|q|}{2} \right) e^{2i\vartheta_q} \frac{A_0^+}{2B_0^+} \delta \hat{\mu}_q, \tag{47}
\]

where the dimensionless functions,

\[
f(x) = \frac{2I_0(x) + 2I_4(x) + 2I_0'(x)}{[2I_0(x) + I_0'(x)][I_0(x) + I_4(x) + I_0'(x)] - [2I_2(x) + I_2'(x)]^2},
\]

\[
h(x) = \frac{4I_2(x) + 2I_2'(x)}{[2I_0(x) + I_0'(x)][I_0(x) + I_4(x) + I_0'(x)] - [2I_2(x) + I_2'(x)]^2}, \tag{48}
\]

are plotted in Fig. 3 and have asymptotic forms

\[
f(x) \approx \begin{cases} \frac{1}{1 + x^2} & (x \ll 1), \\ \frac{x^2}{6} & (x \gg 1), \end{cases} \quad h(x) \approx \begin{cases} \frac{x^2}{6} & (x \ll 1), \\ \frac{1}{2|x|^2} & (x \gg 1). \end{cases} \tag{49}
\]
Finally, if we rewrite the disorder-free saddle-point equation in Eq. (33) as

\[ P[\varepsilon_F, \Delta(\varepsilon_F)] = \frac{g}{2N\beta} \sum_{k,\omega_n} \frac{|p_k|^2}{\omega_n^2 + \varepsilon_k^2 + \Delta^2 |p_k|^2} = 1, \]

(50)

take its total derivative with respect to \(\varepsilon_F\), and substitute \(\partial P/\partial \varepsilon_F\) and \(\partial P/\partial \Delta\) from Eqs. (40) and (44), we obtain

\[ \frac{dP}{d\varepsilon_F} = \frac{\partial P}{\partial \varepsilon_F} + \frac{\partial P}{\partial \Delta} \frac{\partial \Delta}{\partial \varepsilon_F} = \frac{1}{\Delta} \left( A^+_0 - 2B_0^{++} \frac{\partial \Delta}{\partial \varepsilon_F} \right) = 0. \]

(51)

Therefore, \(A^+_0/(2B_0^{++}) = \Delta/\varepsilon_F\), and Eq. (47) is equivalent to Eq. (10) in the main text.

**SELF-CONSISTENT NUMERICAL SOLUTION**

Here we describe the details of the numerical procedure that we use to obtain the self-consistent solution of the Bogoliubov-de Gennes (BdG) Hamiltonian [see Eq. (4) in the main text],

\[ H = -\sum_r \mu_r c^+_r c_r - \sum_{(r,r')} \left( t_{r,r'} c^+_r c_{r'} + t^*_{r',r} c^+_r c_{r'} \right) - \sum_{(r,r')} \left( \Delta_{r,r'} c_r c_{r'} + \Delta^*_{r,r'} c^+_r c^+_{r'} \right), \]

(52)

in terms of the superconducting pairing potentials [see Eq. (5) in the main text],

\[ \Delta_{r,r'} = -\Delta_{r',r} = g \langle c_r c_{r'} \rangle. \]

(53)

The site-dependent chemical potentials are \(\mu_r = \tilde{\mu} + \delta \mu_r\), while the hopping amplitudes are \(t_{r,r'} = t\) for a vortex-free system and \(t_{r,r'} = tu_{r,r'} e^{iA'_{r,r'}}\) if there are two vortices at positions \(R_{1,2}\) connected by a \(\mathbb{Z}_2\) flux string [see Fig. 1(a) in the main text], where \(u_{r,r'}\) is \(-1 (+1)\) if the \(\mathbb{Z}_2\) flux string intersects (does not intersect) the link \(\langle r, r' \rangle\), and

\[ A'_{r,r'} = \int_{r'} [A'(\hat{r} - R_1) + A'(\hat{r} - R_2)] \cdot d\hat{r}, \]

(54)

corresponds to an effective vector potential which is only nonzero within a London penetration depth \(\lambda\) of each vortex. Indeed, the components of \(A'(r)\) in polar coordinates, \(r = (r, \theta)\), can be written as

\[ A'_r(r) = 0, \quad A'_\theta(r) = -\frac{a(r)}{2r}, \]

(55)

where \(a(r)\) must asymptotically satisfy \(a(r) \approx 1\) for \(r \ll \lambda\) and \(a(r) \sim e^{-r/\lambda}\) for \(r \gg \lambda\). We choose \(a(r) = (1 + r/\lambda) e^{-r/\lambda}\) but note that the precise form of \(a(r)\) does not matter as long as the asymptotic conditions are satisfied.

To find a self-consistent solution of Eqs. (52) and (53), we first make an initial guess for the pairing potentials \(\Delta_{r,r'}\). Then, we solve the BdG Hamiltonian in Eq. (52) by substituting these pairing potentials, and compute the ground-state expectation values in Eq. (53) to obtain an updated set of pairing potentials. Finally, we repeat this procedure iteratively until the pairing potentials

![FIG. 3: Exact forms (solid lines) and asymptotic forms (dotted lines) of the dimensionless functions \(f(x)\) and \(h(x)\).](image-url)
converge up to the desired accuracy. In practice, we always start from a vortex-free system in the disorder-free limit. To describe a $p_x + ip_y$ superconductor with the correct pairing symmetry, the appropriate initial guess is

$$\Delta_{r,r+\hat{r}_x} = \Delta_0, \quad \Delta_{r,r+\hat{r}_y} = i\Delta_0. \quad (56)$$

Due to the symmetries of the system, the iterative procedure does not change the form of Eq. (56) but only makes $\Delta_0$ converge to the right value $\bar{\Delta}$. The next step is to introduce two vortices at positions $\mathbf{R}_{1,2}$ with an appropriate initial guess,

$$\Delta_{r,r+\hat{r}_x} = u_{r,r+\hat{r}_x} \bar{\Delta} \tanh \left[ \frac{1}{\xi} \left( \mathbf{r} + \frac{1}{2} \hat{r}_x - \mathbf{R}_1 \right) \right] \tanh \left[ \frac{1}{\xi} \left( \mathbf{r} + \frac{1}{2} \hat{r}_x - \mathbf{R}_2 \right) \right],$$

$$\Delta_{r,r+\hat{r}_y} = i u_{r,r+\hat{r}_y} \bar{\Delta} \tanh \left[ \frac{1}{\xi} \left( \mathbf{r} + \frac{1}{2} \hat{r}_y - \mathbf{R}_1 \right) \right] \tanh \left[ \frac{1}{\xi} \left( \mathbf{r} + \frac{1}{2} \hat{r}_y - \mathbf{R}_2 \right) \right], \quad (57)$$

where $\xi = 1/(2m\bar{\Delta})$ is the superconducting coherence length. Once the iterative procedure is converged, we use the converged set of pairing potentials as the initial guess when we finally introduce disorder. For each disorder realization, we turn on disorder smoothly by fixing the given disorder realization and only rescaling it (i.e., increasing its overall strength $\delta\bar{\mu}$) in small steps. At each step, the initial guess for the pairing potentials is the converged set of pairing potentials from the previous step.