QUADRATIC CHABAUTY AND $p$-ADIC GROSS–ZAGIER

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Abstract. Let $X$ be a quotient of the modular curve $X_0(N)$ whose Jacobian $J_X$ is a simple factor of $J_0(N)_{\text{new}}$ over $\mathbb{Q}$. Let $f$ be the newform of level $N$ and weight 2 associated with $J_X$; assume $f$ has analytic rank 1. We give analytic methods for determining the rational points of $X$ using quadratic Chabauty by computing two $p$-adic Gross–Zagier formulas for $f$. Quadratic Chabauty requires a supply of rational points on the curve or its Jacobian; this new method eliminates this requirement. To achieve this, we give an algorithm to compute the special value of the anticyclotomic $p$-adic $L$-function of $f$ constructed by Bertolini, Darmon, and Prasanna, which lies outside of the range of interpolation.

1. Introduction

Let $X$ be a smooth projective geometrically integral curve of genus $g > 1$ over $\mathbb{Q}$. Let $p$ be a prime of good reduction for $X$. Chabauty’s method is a family of $p$-adic methods developed to try to determine $X(\mathbb{Q})$; the goal is to show that $X(\mathbb{Q})$ lies in the zero set of nontrivial locally analytic functions from $X(\mathbb{Q}_p)$ to $\mathbb{Q}_p$. Since a locally analytic function has only finitely many zeros on each residue disk of $X(\mathbb{Q}_p)$, this exhibits $X(\mathbb{Q})$ inside a finite set.

Let $\rho(J_X)$ denote the Néron–Severi rank of the Jacobian $J_X$ of $X$ and $r$ the Mordell–Weil rank of $J_X(\mathbb{Q})$ over $\mathbb{Q}$. When $r < g + \rho(J_X) - 1$, the quadratic Chabauty method [BBM16, BD18, BD21] makes effective M. Kim’s program [Kim09] for an explicit Faltings’s theorem by studying quotients of the unipotent fundamental group at depth 2. It uses $p$-adic height functions to determine a finite set containing $X(\mathbb{Q})$. Crucially, the construction of the locally analytic function using $p$-adic heights requires knowing sufficiently many rational points on $X$ or $J_X$ [BDM+21, Section 3.3].

We develop a quadratic Chabauty method for certain quotients of modular curves that replaces this requirement for knowing rational points on $X$ or $J_X$ with computations of special values of $p$-adic $L$-functions. Our methods apply to quotients $X$ of $X_0(N)$ whose Jacobians are simple quotients of $J_0(N)_{\text{new}}$ over $\mathbb{Q}$.

In particular, we consider the case when $J_X$ is a simple factor of $J_0(N)_{\text{new}}$. Then $J_X$ has the property that $g = \rho(J_X)$ and $r$ is a multiple of $g$. By requiring $r = g \geq 2$, the inequality $r < g + \rho(J_X) - 1$, is satisfied and we may apply quadratic Chabauty. Studying the isogeny decomposition of $J_0(N)_{\text{new}}$ shows that the $L$-function of $J_X$ can be written in terms of the $L$-function of a newform $f_X$ of weight 2 and level $N$ and its Galois conjugates. In this way, we can rephrase our study of $J_X$ as a study of the newform $f_X$ and its $L$-function.

Our main theorem is Theorem 5.10 where we give a new analytic method for computing the quadratic Chabauty function on these quotients in terms of two special values of $p$-adic $L$-functions. The quadratic Chabauty function can be written as the difference between a local and global $p$-adic height: this theorem gives a novel way of expressing the global height as a locally analytic function. We do not discuss the computation of the local height in this

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paper. To rewrite the global height, we use $p$-adic Gross–Zagier formulas, taking advantage of the fact that $X$ is a modular curve, to construct locally analytic functions for quadratic Chabauty.

Our method proceeds by studying Rankin–Selberg $L$-functions associated with $f_X$ and an imaginary quadratic field $K$. This analytic study of the $L$-functions of modular forms contains deep arithmetic information through the computation of special values of $L$-functions. When $K$ satisfies the Heegner hypothesis, we can construct a Heegner divisor $y_K \in J_0(N)(K)$. The Gross–Zagier theorem [GZ86] relates the derivative of the Rankin–Selberg $L$-function $L'(f_X, \chi, 1)$ of a weight 2 newform $f_X$ to the canonical height of the $f_X$-isotypical component of $y_K$ twisted by ring class characters $\chi$.

Assume the prime $p$ is ordinary for the newform $f_X$. Perrin-Riou [PR87] developed a $p$-adic version of Gross and Zagier’s formula, relating the derivative of a $p$-adic $L$-function to the cyclotomic $p$-adic height of the $f_X$-isotypical component of the Heegner divisor. This height was also studied in papers of Mazur–Tate [MT83], Schneider [Sch82], and Coleman–Gross [CG89]. We give an algorithm to compute the $p$-adic height of the Heegner divisor using overconvergent modular symbols [PS11].

Bertolini, Darmon, and Prasanna [BDP13] constructed an anticyclotomic $p$-adic Rankin $L$-series $L_p$ that also interpolates the central values of the Rankin–Selberg $L$-function $L(f_X, \chi, 1)$ for a different set of Hecke characters of $K$. The special value of the anticyclotomic $p$-adic $L$-function is proportional to $(\log_{f_X dq/q} y_K)^2$. We give an algorithm to compute this special value, which lies outside of the range of interpolation.

Background material and notational conventions are found in Section 2. We compute the special value of Perrin-Riou’s $p$-adic $L$-function Section 4. Section 3 discusses the special value of the anticyclotomic $p$-adic $L$-function. Finally, Section 5 gives the quadratic Chabauty algorithms and several examples of the method.

\section{2. Background and notation}

2.1. Modular forms and modular curves. For background on modular curves, we loosely follow [BGJGP05] but take the perspective of Katz [BDP13, Section 1.1] on viewing modular forms as functions on marked elliptic curves with level structure. Let $\mathcal{H}$ denote the complex upper half plane. Throughout we use the convention that $z := x + iy$.

\textbf{Definition 2.1.} Let $R$ be a ring. An elliptic curve $E$ with $\Gamma_0(N)$-level structure over $R$ is a pair $(E, C_N)$ where $E \to \text{Spec} R$ is an elliptic curve over $\text{Spec} R$ and $C_N$ a cyclic a finite locally free sub-group scheme of $E$ of order $N$, meaning that locally f.p.p.f on the base we can find a point $P$ such that $C_N = \sum_{a \mod N}[aP]$ as Cartier divisors.

A marked elliptic curve $E$ with $\Gamma_0(N)$-level structure over $R$ is a triple $(E, C_N, \omega)$ such that $\omega$ is a nonzero global section of $\Omega^1_E$ over $\text{Spec} R$ and $(E, C_N)$ is an elliptic curve with $\Gamma_0(N)$-level structure.

\textbf{Definition 2.2.} Let $F$ be a field and $R$ an $F$-algebra. A weakly holomorphic algebraic modular form $f$ of weight $k$ for $\Gamma_0(N)$ defined over $F$ is a rule that assigns to every isomorphism class of marked elliptic curves with $\Gamma_0(N)$-level structure $(E, C_N, \omega)$ over $R$ an element $f(E, C_N, \omega) \in R$ such that

1. for every homomorphism of $F$-algebras $j : R \to R'$, $f((E, C_N, \omega) \otimes j R') = j(f(E, C_N, \omega))$;
2. for all $\lambda \in R^\times$, $f(E, C_N, \lambda \omega) = \lambda^{-k}(E, C_N, \omega)$.
Denote by \((\text{Tate}(q), T_N, du/u)\) the Tate curve \(\mathbb{G}_m / q^\mathbb{Z}\) with \(\Gamma_0(N)\)-structure \(T_N\) and canonical differential \(du/u\) where \(u\) is the usual parameter in \(\mathbb{G}_m\). The Tate curve is defined over \(F((q^{1/d}))\) for some \(d|N\).

**Definition 2.3.** An algebraic modular form \(f\) of weight \(k\) for \(\Gamma_0(N)\) defined over \(F\) is a weakly holomorphic one such that \(f(\text{Tate}(q), T_N, du/u) \in F[[q^{1/d}]]\) for all \(T_N\).

Algebraic modular forms are sections of line bundles in the following way. If \(N \geq 3\), the modular curve \(Y_0(N)\) is a moduli space: for any \(\mathbb{Z}[1/N]\)-algebra \(R\), the points \(Y_0(N)(R)\) can be identified with the set of isomorphism classes of elliptic curves with \(\Gamma_0(N)\)-level structure over \(R\). Let \(\mathcal{E}\) be the universal elliptic curve with \(\Gamma_0(N)\)-level structure and \(\tau : \mathcal{E} \to Y_0(N)\). Let \(\omega := \pi_* \Omega^1_{\mathcal{E}/Y_0(N)}\) be the sheaf of relative differentials. If \(g\) is a weakly holomorphic modular form of weight \(k\) for \(\Gamma_0(N)\), we view \(g\) as a global section of \(\omega^k\) by \(g(E, C_N) = g(E, C_N, \omega)\omega^k\) where \(\omega\) is a generator of \(\Omega^1_{E/R}\). Furthermore, \(\omega\) extends to a line bundle on \(X_0(N)\), characterized by the property that the global sections \(H^0(X_0(N), \omega^k)\) are exactly the space of weight \(k\) modular forms for \(\Gamma_0(N)\).

Specializing to the complex numbers, \(X_0(N)(\mathbb{C})\) is a compact Riemann surface and the map \(\mathcal{H} \to \Gamma_0(N) \backslash \mathcal{H}\) sending

\[
\tau \mapsto (\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), 1/N)
\]

identifies \(Y_0(N)(\mathbb{C})\) with \(\Gamma_0(N) \backslash \mathcal{H}\), where we identify \(1/N\) with the cyclic subgroup it generates by abuse of notation.

**Definition 2.4.** If \(g\) is a weakly holomorphic modular form of weight \(k\) it gives a holomorphic section of the sheaf \(\omega^k\) (viewed as an analytic sheaf on the Riemann surface \(X_0(N)(\mathbb{C})\)) and gives rise to a holomorphic function on \(\mathcal{H}\) by the definition

\[
g(\tau) := g(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), 1/N, 2\pi i\tau)
\]

where \(\tau\) is the usual coordinate on \(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\).

Let \(S_2(N)\) denote the weight 2 cuspforms for \(\Gamma_0(N)\). We can decompose

\[S_2(N) = S_2(N)_{\text{old}} \oplus S_2(N)_{\text{new}}.\]

The space \(S_2(N)\) is equipped with an action of Hecke operators \(T_n\) for \(n \geq 2\), and we write \(T = \mathbb{Z}[T_2, T_3, \ldots]\) for the Hecke algebra. Furthermore, \(S_2(N)\) has a basis of cuspforms \(f\) that are eigenforms for all the Hecke operators, that is \(T_n f = a_n(f) f\) for all \(n \geq 2\). We say \(f \in S_2(N)_{\text{new}}\) is a **newform** if it is an eigenform and \(a_1(f) = 1\). Write \(\text{New}_N\) for the set of newforms of level \(N\). If \(f \in \text{New}_N\), then \(f\) is defined over a number field which we denote by \(E_f := \mathbb{Q}(a_2, a_3, \ldots)\). There is action of \(G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) on the set of newforms.

Shimura attached to each newform \(f\) a \(\mathbb{Q}\)-simple isogeny factor \(A_f\) of \(J_0(N)\) of dimension \([E_f : \mathbb{Q}]\). We have an isogeny decomposition over \(\mathbb{Q}\)

\[J_0(N) \sim \bigoplus_{M | N} \bigoplus_{f \in G_\mathbb{Q} \backslash \text{New}_M} A_f^{\sigma_0(N/M)}\]

where \(\sigma_0(n)\) is the divisor function.

**Definition 2.5.** Let \(X/\mathbb{Q}\) be a smooth projective geometrically integral curve. We say \(X\) is \(\Gamma_0(N)\)-modular if there is a non-constant morphism \(\phi : X_0(N) \to X\) over \(\mathbb{Q}\).

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Lemma 2.6. Suppose $X$ is $\Gamma_0(N)$-modular of genus $g$, and its Jacobian $J_X$ is simple over $\mathbb{Q}$. Then $g = \dim J_X = \dim \text{End}_\mathbb{Q}^0(J_X)$ and there exists an integer $M|N$ and $f_X \in \text{New}_M$ such that $J_X$ is $\mathbb{Q}$-isogenous to $A_{f_X}$. Furthermore, $f_X$ is unique up to Galois conjugacy.

Proof. If $X$ is $\Gamma_0(N)$-modular, then $J_X$ is a quotient of $J_0(N)$ of dimension $g$. Then $J_X$ is simple. So by the isogeny decomposition of $J_0(N)$ and Poincaré reducibility, $J_X$ is isogenous over $\mathbb{Q}$ to some $A_{f_X}$, with $f_X \in \text{New}_M$ for some $M|N$, where $f_X$ is unique up to Galois conjugacy. Then $\dim J_X = \dim A_f = [E_f : \mathbb{Q}] = \dim \text{End}_\mathbb{Q}^0(A_f) = \dim \text{End}_\mathbb{Q}^0(J_X)$. □

Definition 2.7. Let $\phi : X_0(N) \to X$ be $\Gamma_0(N)$-modular and genus $g$, and suppose $J_X$ is simple. By Lemma 2.6, we have an associated $f_X \in \text{New}_M$, unique up to Galois conjugacy. We say $X$ is a simple new $\Gamma_0(N)$-modular curve if $X$ satisfies all these assumptions and furthermore $f_X \in \text{New}_N$ (that is, $M = N$).

Assumption 2.8. We assume throughout the whole paper that $\phi : X_0(N) \to X$ is a simple new $\Gamma_0(N)$-modular curve with Jacobian $J_X$ associated to a Galois orbit of newforms $\{f^n_\sigma : \sigma \in \text{Gal}(E_{f_X}/\mathbb{Q})\}$. We write $\pi : J_0(N) \to J_X$ for the map induced by $\phi$.

2.2. Heegner points. We now introduce definitions and notation for Heegner points and discuss relevant background. More details on Heegner points and references for this section can be found in [Gro84, GZ86, GKZ87].

Let $K$ be an imaginary quadratic field of class number one. Let $N$ be a positive integer.

Definition 2.9. The Heegner hypothesis for $K$ and $N$ is the assumption that every prime $q|N$ splits in $K$.

If $N$ satisfies the Heegner hypothesis then we can write $(N) = n\overline{n}$ in $\mathcal{O}_K$. Write $n = bN + Z^b\sqrt{D}$ for some $b \in \mathbb{Z}$. Then under the map (2.1.1), the point

$$\tau_n := \frac{b + \sqrt{D}}{2N}$$

corresponds to the elliptic curve with $\Gamma_0(N)$-level structure $(C/\overline{n}^{-1}, 1/N)$. Then $C/\overline{n}^{-1}$ is an elliptic curve with CM by $\mathcal{O}_K$ and hence has a model $A$ defined over $\mathcal{O}_K$.

Definition 2.10. We define the Heegner point to be the point $P_K := (A, A[n]) \in X_0(N)(K)$. The point $P_K$ gives rise to a divisor class $y_K := [P_K - \infty] \in J_0(N)(K)$.

We now discuss the choice of a differential form for $A$ in order to evaluate modular forms at the Heegner point. Choose $\omega_A$ a Néron differential on $A$. Then the period lattice of $\omega_A$ is $\Omega_K \cdot \mathcal{O}_K \subset C$. Then for any modular form $g$ of weight $k$ for $\Gamma_0(N)$ we have

$$g(A, A[n], \omega_A) = g(C/(Z + Z\tau_n), 1/N, \Omega_K\overline{\sigma}dz) = \frac{g(\tau_n)}{(\Omega_K)^k}$$

where $\sigma$ is a generator of $\overline{n}$. Let $\Omega_A$ be $1/(2\pi i)$ times the real period of $A$. By [BDP17, p.25] these are related by $\Omega_K = \Omega_A/\sqrt{D}$.

Remark 2.11. Note that $\Omega_A$ does not depend on the choice of $N$, only on $K$.

Consider the vector space $V = J_0(N)(K) \otimes \overline{Q}$. The height pairing gives an inner product on $V$ for which the Hecke operators are self-adjoint: the adjoint of an endomorphism of an abelian variety with respect to the height pairing is the Rosati involution by [MT83, (3.4.3)]
c.f. Proposition 4.3. Since the Hecke algebra on $X_0(N)$ is a product of totally real fields, the Rosati involution (which is totally positive) is forced to be the identity.

By the spectral theorem, the fact that the Hecke operators are self-adjoint for the height pairing yields a decomposition into eigenspaces

$$V = \bigoplus_{f \in \text{New}_N} V_f.$$

**Definition 2.12.** Let $f \in \text{New}_N$ and $\sigma \in \text{Gal}(E_f/\mathbb{Q})$. We denote by $y_{K,f^\sigma}$ the $f^\sigma$-isotypical component of $y_K$, with $y_{K,f^\sigma} \in V_{f^\sigma} \subset J_0(N)(K) \otimes E_f$.

Because distinct eigenvectors are orthogonal, for any $\sigma, \tau \in \text{Gal}(E_f/\mathbb{Q})$

$$\langle y_{K,f^\sigma}, y_{K,f^\tau} \rangle = 0$$

whenever $\sigma \neq \tau$.

Let $f \in \text{New}_N$. The Gross–Zagier formula [GZ86, Theorem I.6.3] says

$$L'(f, 1) L(f^\epsilon, 1) = h_{NT}(y_{K,f})$$

where $h_{NT}$ denotes the real valued canonical height on $J_0(N)$.

A theorem of Waldspurger then guarantees infinitely many $K$ such that $L(f^\epsilon, 1) \neq 0$ by relating the value $L(f^\epsilon, 1)$ to the $|D|$th Fourier coefficient of another modular form of weight $3/2$ obtained via the Shimura correspondence. We do not know an explicit way to construct a suitable $K$ a priori; instead we can verify this condition computationally.

The Fricke involution $w_N$ on $X_0(N)$ induces an involution on $S_2(N)$. Since $f$ has analytic rank 1, we have that $L'(f, 1) \neq 0$ only if the Fricke sign is 1, so we require this.

The morphism $\phi : X_0(N) \to X$ induces a morphism $\pi : J_0(N) \to J_X$ via pushforward. The action of complex conjugation on $\pi(y_K)$ is given by

$$\pi(w_N(P_K) - \infty)$$

therefore, $\pi(y_K) \in J_X(\mathbb{Q})$ [Gro84, (5.2)]. The Hecke action on $J_X$ can be identified with an order $\mathcal{O}_{J_X}$ in $K$. Finally, $\mathcal{O}_{J_X} \pi(y_K)$ generates a finite index subgroup of $J_X(\mathbb{Q})$ [DLF21, 7 Appendix].

**Remark 2.13.** We will often denote (Galois orbits of) newforms by labels. These labels are LMFDB labels [LMF22]. Where appropriate, we will also reference curves with LMFDB labels. All of the labels in this paper are LMFDB labels.

### 2.3. Hecke characters

We give a short background section on Hecke characters and their varied incarnations in this section. For this section we fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Recall that $K$ is still an imaginary quadratic field.

Let $\mathfrak{c}$ be an integral ideal of $K$. Let $I_\mathfrak{c}$ denote the group of fractional ideals of $K$ prime to $\mathfrak{c}$. Let $J_\mathfrak{c}$ be the set of ideals in $K$ satisfying

$$J_\mathfrak{c} = \{(\alpha) : \text{for all prime ideals } q|\mathfrak{c}, v_q(\alpha - 1) \geq \text{ord}_q(\mathfrak{c})\} \subset I_\mathfrak{c}.$$

**Definition 2.14.** Let $(n_1, n_2) \in \mathbb{Z}^2$. An algebraic Hecke character of infinity type $(n_1, n_2)$ and conductor dividing $\mathfrak{c}$ is a homomorphism

$$\chi : I_\mathfrak{c} \to \mathbb{C}^\times$$

such that

$$\chi((\alpha)) = \alpha^{n_1} \overline{\alpha}^{n_2}, \text{ for all } \alpha \in J_\mathfrak{c}.$$
It is possible that $\chi$ can be extended to some Hecke character of conductor dividing $\mathfrak{c}'$; the smallest such integral ideal $\mathfrak{c}'$ is the conductor of $\chi$.

**Example 2.15.** Let $\mathfrak{q}$ be an $\mathcal{O}_K$-ideal. The norm character $N_K$ sending $\mathfrak{q} \mapsto \#(\mathcal{O}_K/\mathfrak{q})$ has infinity type $(1, 1)$ and conductor $\mathcal{O}_K$.

We can associate to a Hecke character an idèle class character $\chi : \mathbb{A}_K^\times/K^\times \rightarrow \mathbb{C}^\times$ such that $\chi_\infty(z) = z^{-n_1}\overline{z}^{-n_2}$, where $\chi_\infty$ denotes the component of $\chi$ at $(K \otimes \mathbb{R})^\times$. The map $(K \otimes \mathbb{R})^\times \rightarrow \mathbb{C}^\times$ is constructed using the embedding $i_\infty$.

**Remark 2.16.** The sign convention for the infinity type is picked to agree with [BDP13], and is negative in some other papers.

Algebraic Hecke characters are in bijection with algebraic $p$-adic Hecke characters, as we now describe. Let $p = \mathfrak{p}\mathfrak{p}$ be a prime that splits in $K$. A $p$-adic Hecke character is a continuous homomorphism $\chi_{\mathfrak{p}} : \mathbb{A}_K^\times/K^\times \rightarrow \overline{\mathbb{Q}}_p^\times$. It is algebraic if there are integers $n_1$ and $n_2$ such that the local factors $\chi_\mathfrak{p}$ on $K_p^\times \simeq \mathbb{Q}_p^\times$ and $\chi_q$ on $K_q^\times$ are of the form $\chi_q(z) = z^{-n_1}$ and $\chi_q(z) = -n_2$. Then $\chi_C$ is an algebraic Hecke character of infinity type $(n_1, n_2)$ corresponds to the $p$-adic Hecke character $\chi_{\mathfrak{p}}$ via the formula

$$\chi_{\mathfrak{p}}(z) = t_\mathfrak{p} \circ \iota_\infty^{-1}(\chi_C(z)z_\infty^{n_1}\overline{z}_\infty^{n_2})z_\mathfrak{p}^{-n_1}\overline{z}_\mathfrak{p}^{-n_2}$$

for an idèle $z = (z_v)$.

If $\chi_{\mathfrak{p}}$ is algebraic, then $\chi_{\mathfrak{p}}$ factors through $\text{Gal}(K^{ab}/K)$. We call this the associated $p$-adic Galois representation $\chi_C : \text{Gal}(K^{ab}/K) \rightarrow \overline{\mathbb{Q}}_p^\times$.

Conversely, given any $p$-adic Galois representation $\text{Gal}(K^{ab}/K) \rightarrow \overline{\mathbb{Q}}_p^\times$, we can obtain an idèle class character $\mathbb{A}_K^\times/K^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ by precomposing with the Artin map.

### 3. The special value of the anticyclotomic $p$-adic $L$-function

In this section, we explain how to compute the special value of the $p$-adic Rankin $L$-series $L_p(f, 1)$ introduced by Bertolini, Darmon, and Prasanna [BDP13] attached to a newform $f \in S_2(N)$ and the imaginary quadratic field $K$ satisfying the hypotheses below. This value occurs at the norm character $\mathbf{N}$ with infinity type $(1, 1)$. Since the norm character $\mathbf{N}$ lies outside of the range of interpolation for $L_p$, this value is not readily accessible by computing a classical $L$-value. We follow a method of Rubin [Rub81] for evaluating the Katz 2-variable $p$-adic $L$-function outside the range of interpolation. Our method requires us to first evaluate the $p$-adic $L$-function at certain characters in the range of interpolation. In the case of the $p$-adic Rankin $L$-series of Bertolini, Darmon, and Prasanna, if $\chi$ is a character in the range of interpolation, then the value of $L_p(f)$ at $\chi$ is shown to be an explicit multiple of the central value of the Rankin $L$-series $L(f, \chi, 1)$ as in (3.0.2). An explicit Waldspurger’s formula (Theorem 3.8) relates the central $L$-values $L(f, \chi, 1)$ to the square of the Shimura–Maass derivative of $f$ at the Heegner point. By considering $p$-adic characters in the “anticyclotomic” direction, Bertolini, Darmon, and Prasanna obtain the special value formula, relating the square of the logarithm of $y_K$ to $L_p(f, 1)$. We will refer to $L_p(f)$ as the anticyclotomic $p$-adic $L$-function.

The section is divided into two subsections. Section 3.1 is devoted to explaining how to compute the values of $L_p(f)$ in the range of interpolation. The strategy here is to compute
the Shimura–Maass derivatives of $f$ evaluated at the Heegner point. Section 3.2 develops the computation of the special value $L_p(f, 1)$ following Rubin’s method. The key proposition is Proposition 3.13, which relates the value $L_p(f, 1)$ to values inside the range of interpolation.

Let $f$ be a newform in $S_q(N)$ with coefficient field $E_f$. We start by collecting some running assumptions that will be used for the remainder of the section.

**Assumption 3.1.**

1. Let $p > 2$ be a prime number not dividing $N$;
2. assume $p$ splits in $E_f$ and fix an embedding $e : E_f \to \mathbb{Q}_p$;
3. assume $f$ has analytic rank 1;
4. let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of class number 1;
5. assume $(p) = p\mathfrak{p}$ splits in $K$ and $K$ has odd discriminant $D < -3$;
6. assume every prime $q$ dividing $N$ splits in $K$ (the Heegner hypothesis).

Assumptions (4) and (2) are simplifying assumptions to avoid working in field extensions of $\mathbb{Q}_p$ for ease of computation and clarity of exposition, (5) is required for the construction of the $p$-adic $L$-function, and (6) ensures the existence of $y_K := [P_K - \infty] \in J_0(N)(K)$.

Now, we are ready to state the main theorem of [BDP13, Main Theorem]: they show that

$$
(3.0.1) \quad L_p(f, 1) = \left(1 - \frac{a_p(f) + p}{p}\right)^2 \log_{fdq/q}(y_K)^2,
$$

where $\log_{fdq/q}$ denotes the logarithm on $J_0(N)$ and $L_p(f, 1)$ denotes the value of $L_p(f)$ at the character $\mathfrak{N}$ (the notation is to emphasize the similarity to the special value of Perrin-Riou’s $p$-adic $L$-function, see Section 4). Our goal in this section is to provide a method for computing $L_p(f, 1)$.

**Remark 3.2.** Given a surjective morphism $\pi : J_0(N) \to J$, the logarithm on $J_0(N)$ induces an inclusion $\pi^* : H^0(J, \Omega^1) \hookrightarrow H^0(J_0(N), \Omega^1)$. Since $fdq/q$ is in the image of $\pi^*$, by applying change of variables [Col85, Proposition 2.4 (iii)] the logarithm on $J_0(N)$ with respect to $fdq/q$ can be considered as the logarithm on $J$.

We start by recalling the interpolation property of the $p$-adic $L$-function $L_p$ in [BDP13] associated to $f$. The interpolation property will provide enough information to work with the $p$-adic $L$-function for our purposes; for more details on the construction of the anticyclotomic $p$-adic $L$-function see the main reference [BDP13] or [BCD+14] for an expository article. Let $K^{ac}/K$ be the anticyclotomic $\mathbb{Z}_p$-extension of $K$, and $\Gamma^- := \text{Gal}(K^{ac}/K)$ denote its Galois group. Write $\hat{\mathcal{O}}$ for the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$ and define $\Lambda^{ac} := \mathbb{Z}_p[[\Gamma^-]] \hat{\otimes}_{\mathbb{Z}_p} \hat{\mathcal{O}}$.

Let $S \subset \text{Hom}_{cts}(\Gamma^-, \hat{\mathbb{Q}}_p^\times)$ be the subset of Galois characters associated to Hecke characters of $K$ with infinity type $(1 + r, 1 - r)$ for some integer $r \geq 1$. Bertolini, Darmon, and Prasanna prove that there exists $L_p(f) \in \Lambda^{ac}$ interpolating the algebraic $L$-values $L_{\text{alg}}(f, \chi^{-1}, 0)$ for all $\chi \in S$ in the following sense: each $\chi \in S$ determines a map $\Lambda^{ac} \to \hat{\mathbb{Q}}_p$ by $\hat{\mathcal{O}}$-linear extension. The interpolation property [BDP13, (5.2.3)] for $L_p(f)$ says that for all $\chi \in S$ and $\Omega_\chi$ a $p$-adic period associated to $y_K$ defined in [BDP13, (5.2.2)], we have

$$
(3.0.2) \quad \chi(L_p(f))/\Omega_p^{1r} = (1 - \chi^{-1}(\mathfrak{p}))a_p + \chi^{-2}(\mathfrak{p})p^2L_{\text{alg}}(f, \chi^{-1}, 0).
$$
The left hand side of (3.0.2) is our notation for evaluating the $p$-adic $L$-function $L_p(f)$ at $\chi$ for $\chi$ in the range of interpolation. The norm character $N$, with infinity type $(1,1)$, is not in $S$, so we cannot use (3.0.2) to evaluate $\text{alg}(L_p(f))$. However, for $\chi \in S$, we can evaluate $\chi(L_p(f))$, which we will now explain.

**Remark 3.3.** The value $L_{\text{alg}}(f, \chi^{-1}, 0)$ is defined as an explicit constant multiple of the special value of the Rankin $L$-series $L(f, \chi^{-1}, 0)$. This $L$-series $L(f, \chi^{-1}, s)$ can be written explicitly (in some right half plane of $C$) as an Euler product over prime ideals $q$ of $O_K$,

$$L(f, \chi^{-1}, s) = \prod_q \left( 1 - \alpha_q(f) \chi^{-1}(q) Nq^{-s} \right)^{-1} \left( 1 - \beta_q(f) \chi^{-1}(q) Nq^{-s} \right)$$

where $\alpha_q(f)$ and $\beta_q(f)$ are the roots of the Hecke polynomial $x^2 - a_q(f)x + q$ and if $Nq = q^l$ then we set $\alpha_{Nq} := \alpha_q(f)^l$ and $\beta_{Nq} := \beta_q(f)^l$. Using Rankin’s method, one can show that $L(f, \chi^{-1}, s)$ has an analytic continuation to the entire complex plane.

**3.1. Evaluating inside the range of interpolation.** In order to $p$-adically interpolate the values $L(f, \chi^{-1}, 0)$ and evaluate the anticyclotomic $p$-adic $L$-function, we interpret $L_{\text{alg}}(f, \chi^{-1}, 0)$ as values of a $p$-adic modular form. This is done using an explicit form of Waldspurger’s formula [BDP13, Theorem 5.4]. We precede the statement of this formula with a discussion of the Shimura–Maass operator.

Recall that we view modular forms as global sections $H^0(X_0(N), \omega^k)$ as in Section 2.1 and Definition 2.4. We also rely on the background and notation on Heegner points in Section 2.2. In particular, recall that the marked elliptic curve with $\Gamma_0(N)$-torsion corresponding to the Heegner point in $P_K \subseteq X_0(N)(K)$ is denoted by

$$(A, A[n], \omega_A) = (C/(\mathbb{Z} + \mathbb{Z} \tau_n), 1/N, \Omega_K \delta dz).$$

Let $M_k(\Gamma_0(N))$ denote the space of modular forms of weight $k$ and $g \in M_k(\Gamma_0(N))$.

**Definition 3.4.** The Shimura–Maass derivative $\delta_k$ is defined as

$$\delta_k(g) = \frac{1}{2\pi i} \left( \frac{\partial}{\partial z} + \frac{k}{2\pi i} \right) g(z).$$

**Definition 3.5.** We say $g$ is a nearly holomorphic modular form of weight $k$ and order less than or equal to $P$ if $g$ is $C^\infty$ on $H$ and we can express $g$ as a sum

$$g(z) = \sum_{j=0}^P g_j(z) y^{-j}$$

where $g_j(z)$ are holomorphic functions on $H$, the function $g$ transforms like a modular form of weight $k$ for $\Gamma_0(N)$, and $g$ has finite limit at the cusps. We denote the space of such forms $N_k^P(\Gamma_0(N))$.

The following lemmas can be proved with simple calculations.

**Lemma 3.6 ([Urb14, Lemma 2.1.3]).** Let $g \in N_k^P(\Gamma_0(N))$. Assume $P > 2k$. There exist $g_0, \ldots, g_P$ with $g_i \in M_{k-2i}(\Gamma_0(N))$ such that

$$g = g_0 + \delta_{k-2g_1} + \cdots + \delta_{k-2P} g_P.$$
Lemma 3.7 ([Zag08, (52)]). If \( g \in M_k(\Gamma_0(N)) \) and \( \gamma \in \Gamma_0(N) \) such that \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( \delta_k(\gamma g) = (c + d)^{k+2} \delta_k(g) \).

Thus, Lemma 3.6 and Proposition 3.7 imply that the Shimura–Maass derivative is an operator \( \delta_k : N_f^2(\Gamma_0(N)) \to N_f^{2r+2}(\Gamma_0(N)) \). We denote by \( \delta^r \) the \( r \)-fold composition \( \delta_{k+2r-2} \circ \cdots \circ \delta_k \). Let \( \delta \) be the Atkin–Serre derivative that acts on \( q \)-expansions by \( qd/dq \). Shimura [Shi75] showed that the values of \( \delta^r f \) and \( d^r f \) agree and are algebraic on CM points. In particular, we have the following:

\[
(3.1.1) \quad d^r f(A, A[n], \Omega_A) = \delta^r f(A, A[n], \omega_A) \in E_f K.
\]

We are ready to state Waldspurger’s formula [BDP13, Theorem 5.4].

Theorem 3.8 (Waldspurger’s formula).

\[
(\delta^{r-1} f(A, A[n], \omega_A))^2 = 1/2(2\pi/\sqrt{D})^{2r-1}(r-1)! \cdot L(f, \chi_r, 0)/(2\pi i\alpha\Omega_K)^{4r}.
\]

To interpolate \( L(f, \chi_r, 0) \) and therefore \( (\delta^{r-1} f(A, A[n], \omega_A))^2 = (d^{r-1} f(A, A[n], \omega_A))^2 \) \( p \)-adically, we take the \( p \)-depletion

\[
d^{r-1} f^{[p]}(q) := \sum_{(n,p)=1} n^{r-1} a_n(f) q^n.
\]

The set \( \{d^{r-1} f^{[p]}\} \) is a \( p \)-adic family of modular forms, and there exists a \( p \)-adic period \( \Omega_p \in \mathbb{C}_p^* \) such that

\[
L_p(f, \chi_r) := \Omega_p^{4r} (d^{r-1} f^{[p]}(A, A[n], \omega_A))^2
\]

extends to a \( p \)-adic analytic function of \( r \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p \) [BDP13, Theorem 5.9].

Remark 3.9. The period pair \((\Omega_K, \Omega_p)\) is only well-defined as a pair: both periods depend linearly on the choice of \( \omega_A \), but their ratio is independent of this choice of scalar multiple.

To compute the anticyclotomic \( p \)-adic \( L \)-function in the range of interpolation, we need to compute the Shimura–Maass derivative \( \delta^{r-1} f(\tau_n) \). Let \( \chi_r \in S \) be the Galois character associated to the Hecke character of \( K \) of infinity type \((1 + r, 1 - r)\) with \( r \geq 1 \). Then

\[
(3.1.2) \quad L_{\text{alg}}(f, \chi_r^{-1}, 0) = \delta^{r-1} f(A, A[n], \omega_A)^2 = (\delta^{r-1} f(\tau_n))^2 / (\alpha\Omega_K)^{4r}.
\]

We would like to evaluate the right hand side of (3.1.2). As \( r \) gets large in the usual absolute value, this value also gets large, and a naive strategy like applying the equality (3.1.1) and evaluating a truncated \( q \)-expansion of \( d^{r-1} f(\tau_n) \) does not approximate the true algebraic value well.

However, [Zag08, Section 6.3] and [VZ93] show that the values of the Shimura–Maass derivative of a modular form evaluated at a CM point satisfy a recurrence relation due to a large amount of algebraic structure on the ring of modular forms \( M_*(\Gamma_0(N)) \). We recall briefly some of the essential ideas involved in the proof.
We introduce the \( \vartheta \) differential operator that acts on a weight \( k \) modular form \( g \) by

\[
\vartheta g = dg - \frac{k}{12} E_2 g
\]

where \( E_2 \) is the weight 2 Eisenstein series. We modify the \( \vartheta \) operator by the following recursive definition. Let \( \vartheta^{[0]} g = g \) and

\[
\vartheta^{[r+1]} g = \vartheta(\vartheta^{[r]} g) - r(k + r - 1)(E_4/144)\vartheta^{[r-1]} g \quad \text{for} \quad r \geq 1.
\]

The Cohen–Kuznetsov series are formal generating series attached to the differential operators that have nice transformation properties under \( \Gamma_0(N) \). We provide the details necessary for our calculations here; the interested reader can see [Zag08, Section 5.2] or [Zag94] for more details, as well as [VZ93, Section 7] for some example calculations. The relationships between the Cohen–Kuznetsov series show that the set of values \( \{ \delta^r(\tau_n) \}_{r \geq 0} \) inherits a recursive relation coming from the recursive definition of \( \vartheta^{[i]} \). We can define a Cohen–Kuznetsov series associated to \( \vartheta \) and \( \delta \) by

\[
\tilde{g}_\vartheta(z, X) = \sum_{n=0}^{\infty} \frac{\vartheta^{[i]} g(z)}{n!(k)_n} X^n \quad \text{and} \quad \tilde{g}_\delta(z, X) = \sum_{n=0}^{\infty} \frac{\delta^n g(z)}{n!(k)_n} X^n.
\]

These power series satisfy

\[
\tilde{g}_\vartheta(z, X) = e^{-XE_2'(z)/12} \tilde{g}_\delta(z, X)
\]

where \( E_2'(z) := E_2(z) - 3/(\pi y) \).

The key idea is that if \( E_2'(z_0) = 0 \), then (3.1.4) implies that \( \vartheta^{[i]} g(z_0) = \delta^i g(z_0) \) for all \( i \geq 0 \). Then, because \( \vartheta^{[i]} g(z_0) \) is defined recursively, we can compute \( \delta^i g(z_0) \) recursively. This method requires two pieces of input.

1. We modify the operator \( \vartheta^{[i]} \) for the CM point \( \tau_n \) with an appropriate holomorphic function \( \phi(z) \) such that \( \phi^*(z) := \phi(z) - 1/(4\pi y) \) transforms like a modular form of weight 2 on \( \Gamma_0(N) \). Define \( \vartheta_\phi := dg - k\phi g \). The resulting relationship (3.1.4) becomes \( \tilde{g}_\vartheta_\phi(z, X) = e^{-X\phi^*(z)} \tilde{g}_\delta(z, X) \) on \( \mathcal{H} \), and we require \( \phi^*(\tau_n) = 0 \).

2. We need generators \( g_1, \ldots, g_n \) for \( M_*(\Gamma_0(N)) \) so that we can compute \( \vartheta_\phi \) of each generator as well as the values \( g_i(\tau_n) \) and \( (\vartheta_\phi g_i)(\tau_n) \) for each generator to determine the recurrence relation.

We can always take \( \phi = (1/12)E_2 + A \) for some weight 2 holomorphic or meromorphic function \( A \) on \( \Gamma_0(N) \) [Zag08, Section 5.2]. Therefore to compute \( \phi \) we evaluate both \( E_2 \) and a basis of weight 2 level \( N \) forms on \( \tau_n \). Then we solve for \( \phi \) by finding a linear combination of the weight 2 level \( N \) forms that when evaluated at \( \tau_n \) equal to \( E_2 \) evaluated at \( \tau_n \).

Define the modular form \( \Phi := d\phi - \phi^2 \). We also obtain operators \( \vartheta_\phi^{[i]} \) satisfying the recurrence relation

\[
\vartheta_\phi^{[0]} g = g, \quad \vartheta_\phi^{[r+1]} g = \vartheta_\phi(\vartheta_\phi^{[r]} g) + r(k + r - 1)\Phi(\vartheta_\phi^{[r-1]} g).
\]

To rigorously evaluate \( g_i(\tau_n) \) and \( (\vartheta_\phi g_i)(\tau_n) \) as algebraic numbers we need to bound the denominators of these values that, a priori, lie in \( K \) by (3.1.1). With work, one can explicitly bound these denominators. Let \( t := (p - 1)^{-1} \). By [BDP17, Theorem B.3.2.1], when \( \text{ord}_p(N) = 1, 2, \text{or} 3 \), the denominators are at most \( p^{[k(t+1)]} \), \( p^{[3kt(2t+1)]} \), or \( p^{[2kt(t+1)(t+2)]} \) respectively.
We give some intuition following [BDP17, Appendix B]. Consider the \( \mathcal{O}_K \)-module \( M_k(\Gamma_0(N), \mathcal{O}_K) \) consisting of weight \( k \) modular forms \( g \) for \( \Gamma_0(N) \) whose \( q \)-expansions have coefficients in \( \mathcal{O}_K \). For \( g \in M_k(\Gamma_0(N), \mathcal{O}_K) \), the evaluation of \( g(A, A[n], \omega_A) \) belongs to \( \mathcal{O}_K[1/N] \). One can construct a finite index \( \mathcal{O}_K \)-module \( M_{k,\mathcal{O}_K} \) of the module \( M_k(\Gamma_0(N), \mathcal{O}_K) \) such that for every \( g \in M_{k,\mathcal{O}_K} \), the evaluation of \( g(A, A[n], \omega_A) \) is \( \mathcal{O}_K \)-integral. This is tensor-compatible with the corresponding \( \mathbb{Z} \)-modules:

\[
\mathcal{O}_K \otimes_{\mathbb{Z}} M_k,\mathbb{Z} = M_{k,\mathcal{O}_K} \otimes_{\mathbb{Z}} M_k(\Gamma_0(N), \mathbb{Z}) = M_k(\Gamma_0(N), \mathcal{O}_K)
\]

so the exponent of the finite abelian group \( M_k(\Gamma, \mathbb{Z})/M_k,\mathbb{Z} \) multiples \( M_k(\Gamma_0(N), \mathcal{O}_K) \) into \( M_{k,\mathcal{O}_K} \) and is an explicit bound on the denominator. (In fact, it gives a much stronger result: this gives a bound on all denominators for all number fields.)

**Example 3.10.** Let \( f \) be the modular form \( 37.2.a.a \) with weight 2 and level 37. We have a basis of weight 2 forms on \( \Gamma_0(37) \) given by

\[
\begin{align*}
f_1 &= 1 - 2q^3 + 10q^4 + 2q^5 + 14q^6 + 6q^7 + 10q^8 + 18q^9 + O(q^{10}) \\
f_2 &= q + q^3 - 2q^4 - q^7 - 2q^9 + O(q^{10}) \\
f_3 &= q^2 + 2q^3 - 2q^4 + q^5 - 3q^6 - 4q^9 + O(q^{10}).
\end{align*}
\]

Let \( \tau_n = \frac{27 + \sqrt{-11}}{2\cdot 3^7} \). Let \( p = 5 \). By the discussion above, we can bound the denominators of \( f_i(\tau_n) \) and \( E_2^*(\tau_n) \) by \( 37^{[2p/(p-1)]} = 37^3 \). We can compute \( E_2^*(\tau_n)/(\Omega_A)^2 = 4400 - 3696\sqrt{-11} \).

We also compute \( f_i(\tau_n) \):

\[
\begin{align*}
\tau_1(\tau_n)/\Omega_A^2 &= 2420 + 572\sqrt{-11} \\
\tau_2(\tau_n)/\Omega_A^2 &= -726 + 154\sqrt{-11} \\
\tau_3(\tau_n)/\Omega_A^2 &= -1210 - 286\sqrt{-11}.
\end{align*}
\]

Therefore \( \phi = 1/12E_2 - 1/12(-28/11f_1 - 160/11f_2) \).

Given an expression \( M_k(\Gamma_0(N)) \approx \mathbb{Q}[g_1, \ldots, g_n]/I \), we can use the iterative relation

\[
\varphi^{[r+1]} f = \varphi^{[r]}(\varphi^{[r]} f) + r(r+1)\Phi(\varphi^{[r-1]} f) \quad \text{for } r \geq 1
\]

and apply the Leibniz rule to the monomials in \( \varphi^{[r]} f \) to apply \( \varphi \) iteratively. By [VZB19, Corollary 1.5.1] to obtain generators and relations for \( M_k(\Gamma_0(N)) \) we need generators up to degree 6 and relations up to degree 12.

**Example 3.11.** Continuing Example 3.10, we can use these methods to compute

\[
f(\tau_n)/\Omega_A^2 = 1694 + 726\sqrt{-11} \quad \text{and} \quad \delta f(\tau_n)/\Omega_A^2 = 532400 - 447216\sqrt{-11}.
\]

Note we have fixed an embedding \( K \to \mathbb{Q}_p \) given by the splitting \( p = p\bar{p} \). We define \( p \) to be the prime with valuation 1. Then \( K_p \simeq \mathbb{Q}_p \).

**Remark 3.12.** When \( [E_f: \mathbb{Q}] > 1 \), then \( \delta^{-1} f(\tau_n)/\Omega_A^{2r} \) belong to the compositum of \( K \) and \( E_f \) and we embed \( e : E_f \to \mathbb{Q}_p \). Changing the embedding \( e \) is equivalent to picking a Galois conjugate \( f^\sigma \) for \( \sigma \in \text{Gal}(E_f/\mathbb{Q}) \). Since the operator \( d \) is Galois-equivariant, and we have the equality (3.1.1), it follows

\[
\delta^{-1} f^\sigma(\tau_n) = \sigma(\delta^{-1} f(\tau_n)).
\]
In this way, we can obtain all \([E_f : \mathbb{Q}]\) \(p\)-adic values of \(\delta^{r-1} f^\sigma(\tau_n)\) for \(\sigma \in \text{Gal}(E_f/\mathbb{Q})\) from knowing a single value \(\delta^{r-1} f(\tau_n)\).

3.2. Evaluating outside of the range of interpolation. We now discuss an adaptation of Rubin’s method to compute the value \(L_p(f, 1)\) outside of the range of interpolation.

Let \(r \in \mathbb{N}\). Let \(\chi_r \in S\) be the Galois character associated to the Hecke character with infinity type \((1 + r, 1 - r)\). Define

\[
\ell(0) := L_p(f; \chi_r)\Omega_{p}^{-4r}.
\]

We want to compute \(\ell(0)\). Since this is not in the range of interpolation, we compute auxiliary values \(\ell((p-1))/2, \ell(2(p-1))/2, \ldots, \ell(B(p-1))/2\) in the range of interpolation and recover \(\ell(0)\) modulo \(p^B\) from a version of [Rub94, Theorem 9, Proposition 7] for the anticyclotomic \(p\)-adic \(L\)-function of Bertolini, Darmon, and Prasanna.

The main result of this section is the following proposition.

Proposition 3.13. Let \(\ell(r)\) be defined as above. Then for any \(B \in \mathbb{N}\), we have

\[
\ell(0)^{(p-1)/2} \equiv \prod_{j=1}^{B} \left( \sum_{i=0}^{B} (-1)^{j-1} \binom{i}{j-1} \ell(j(p-1))^{(p-1)/2} \right) \bmod p^B.
\]

Furthermore, \(\ell(0) \equiv \ell((p-1)^2)/2 \bmod p\).

Assuming \(\ell(0) \not\equiv 0 \bmod p\), Proposition 3.13 allows us to uniquely recover \(\ell(0)\) from the auxiliary values \(\ell(j(p-1))\). We now prove Proposition 3.13.

Following Rubin, we introduce a ring \(\mathcal{I}\) of generalized Iwasawa functions. A function \(g\) on \(\mathbb{Z}_p^\infty\) is in \(\mathcal{I}\) if there exist units \(u_1, \ldots, u_m \in 1 + p\hat{\mathcal{O}}\) and a power series \(H \in \hat{\mathcal{O}}[[X_1, \ldots, X_m]]\) such that \(g(s) = H(u_1^{s}-1, \ldots, u_m^{s}-1)\) for all \(s \in \mathbb{Z}_p\).

Recall \(\chi_{i(p-1)} \in S\) is a Galois character associated to a Hecke character of \(K\) with infinity type \((1 + i(p-1), 1 - i(p-1))\). By composing with a projection arising from \(\mathbb{Z}_p^\infty \simeq (\mathbb{Z}/p\mathbb{Z})^\infty \times (1 + p\mathbb{Z}_p)\), we have

\[
\langle \chi_{i(p-1)} \rangle : \Gamma^- \to \mathbb{Z}_p^\infty \to 1 + p\mathbb{Z}_p
\]

since \(\chi_{i(p-1)}\) already takes values in \(1 + p\mathbb{Z}_p\) [dS87, II.4.17]. For \(F \in \Lambda^{ac}\), we have \(\langle \chi_{i(p-1)} \rangle(F) \in \hat{\mathcal{Q}}_p\), and furthermore for \(s \in \mathbb{Z}_p\) we can define \(\chi_{s(p-1)} := \langle \chi(p-1) \rangle^s\) and evaluate \(\chi_{s(p-1)}(F) \in \hat{\mathcal{Q}}_p\) by continuity.

Define

\[
H(s) := (\Omega^{(p-1)/2})^{-2s}L_p(f; \chi_{s(p-1)})^{(p-1)/2}.
\]

By [dS87, II.4.3(10)], we have

\[
\Omega^{(p-1)/2} \in 1 + p\hat{\mathcal{O}}
\]

so \(H\) is well-defined. For \(i \in \mathbb{Z}\), note that

\[
H(i) = \ell((p-1)i)^{(p-1)/2}.
\]

Analogously to [Rub94, Proposition 7], we have the following proposition.

Proposition 3.14. Let \(\ell(r)\) and \(H\) be defined as in (3.2.1) and (3.2.4). Then

(1) \(H \in \mathcal{I}\).
\[(2) \ell(0) \equiv \ell((p - 1)^2/2) \mod p.\]

**Proof.** We define functions \(f_1\) and \(f_2\) by
\[
\begin{align*}
  f_1(s) &:= L_p(f, \chi_{s(p-1)}) \\
  f_2(s) &:= (\Omega_p^{(p-1)^2})^{-2s}.
\end{align*}
\]
Then \(H = f_1^{(p-1)/2} f_2\). We know \(f_1 \in \mathcal{I}\) since \(\chi_{i(p-1)}\) is a character into \(1 + p\mathbb{Z}_p\) for all \(i \in \mathbb{N}\) (see (3.2.3)) and by (3.2.5) we know \(f_2 \in \mathcal{I}\), so \(H \in \mathcal{I}\).

Finally, since \(f_1(s) \equiv f_1(s') \mod p\) for all \(s, s' \in \mathbb{Z}_p\) we have
\[
\ell(0) = f_1(0) \equiv \Omega_p^{-2(p-1)^2} f_1((p - 1)/2) = \ell((p - 1)^2/2) \mod p. \tag{3.2.8}
\]

\[\square\]

**Remark 3.15.** Proposition 3.14 (2) is only helpful if \(\ell((p - 1)^2/2) \not\equiv 0 \mod p\). More generally, one can see that by (3.2.5), for \(n \geq 1\) we have
\[
\ell(0) = f_1(0) \equiv \Omega_p^{-2(p-1)^2} f_1((p - 1)p^{n-1}/2) = \ell((p - 1)^2p^{n-1}/2) \mod p^n. \tag{3.2.9}
\]
The main difficulty in applying this congruence is computing \(\ell((p - 1)^2p^{n-1}/2)\).

From here it is straightforward to follow Rubin’s proof to obtain a proof of Proposition 3.13: we give a brief summary. He defines a difference operator \(\Delta\) on \(\mathcal{I}\) by \(\Delta(g)(s) := g(s+1) - g(s)\). If \(g \in \mathcal{I}\) then \(\Delta(g) \in p\mathcal{I}\) [Rub94, Lemma 8].

By inverting \((1 + \Delta)^{-1} = \sum_{i=0}^{\infty} (-1)^i \Delta^i\) and applying the congruence, we obtain the desired formula [Rub94, Theorem 9]
\[
g(0) = \sum_{j=1}^{B} \left( \sum_{i=j}^{B} (-1)^{i-1} \binom{i-1}{j-1} \right) g(j) \mod p^B. \tag{3.2.10}
\]
By applying this to \(H \in \mathcal{I}\) we can compute the special values.

| \(f\) | \(p\) | \(D\) | time | Sturm Bound |
|-------|------|------|------|------------|
| 37.2.a.a | 5   | −11  | 13.970 | 7          |
| 43.2.a.a | 5   | −19  | 18.480 | 8          |
| 58.2.a.a | 11  | −7   | 1583.380 | 15       |
| 61.2.a.a | 5   | −19  | 34.240 | 11         |
| 83.2.a.a | 5   | −19  | 73.400 | 14         |
| 89.2.a.a | 3   | −11  | 15.730 | 15         |
| 77.2.a.a | 5   | −19  | 65.350 | 16         |
| 101.2.a.a| 5   | −19  | 95.850 | 17         |
| 131.2.a.a| 5   | −19  | 326.160 | 22        |

**Table 1.** Timings for fixed \(B = 5\) and varying \(N\) for modular forms with rational Fourier coefficients.
Remark 3.16. We have written code to compute \( \ell(0) \) and \( L_p(f, 1) \) in Magma V2.26-11. It can be found at [Has]. See Table 1 for some timings. These timings were done on a 2017 Macbook Pro with a 2.3 GHz Dual-Core Intel Core i5 processor and 8 GB of RAM. All times are given in seconds.

\[
\begin{array}{|c|c|c|c|}
\hline
r & \ell(r) \mod p^{10} & r & \ell(r) \mod p^{10} \\
\hline
4 & -2341944 & 6 & 19467645 \\
8 & 830906 & 12 & 27057027 \\
12 & -3933069 & 18 & -443168 \\
16 & -35494 & 24 & 72608418 \\
20 & 1760756 & 30 & -32171562 \\
24 & 1706556 & 36 & -95344303 \\
28 & -3662194 & 42 & -68492493 \\
32 & 3734381 & 48 & -129070518 \\
36 & 4015256 & 54 & -81233717 \\
40 & & 60 & 24574313 \\
\hline
\end{array}
\]

(A) \( \ell(r) \) for 37.2.a.a  
(B) \( \ell(r) \) for \( f \) in 85.2.a.b

Table 2. Example computations of \( \ell(r) \)

Example 3.17. Let \( f \) be the newform with LMFDB label 37.2.a.a, \( D = -11 \), and \( p = 5 \). We compute the values of \( \ell(r) \) in Table 2A. So Proposition 3.13 implies that

\[
H(0) = L_p(f, 1)^2 \equiv 2502536 \mod p^{10}
\]

and

\[
L_p(f, 1) \equiv \ell(8) \equiv 830906 \mod p
\]

so

\[
L_p(f, 1) \equiv 4635631 \mod p^{10}.
\]

Example 3.18. For 77.2.a.a, \( D = -19 \), and \( p = 5 \), we can similarly compute \( \ell(0) \) mod \( B = 7 \):

\[
L_p(f, 1) \equiv 4 + 2 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 5^4 + 3 \cdot 5^6 \mod p^7.
\]

This agrees with the computed value of \( \left( \frac{1-a_p(f)+p}{p} \right)^2 \log(\pi(y_K))^2 \) on the associated elliptic curve \( E \). In this case \( \pi_E(y_K) \) has index 2 in \( E(\Q) \).

Example 3.19. Let \( f \) be a newform in the orbit 85.2.a.b. Then \( f \) has coefficient field \( \Q(\sqrt{2}) \), and we denote

\[
f(q) := q + (\sqrt{2} - 1)q^2 + (-\sqrt{2} - 2)q^3 + (-2\sqrt{2} + 1)q^4 - q^5 - \sqrt{2}q^6 + O(q^7)
\]

Let \( D = -19 \), and \( p = 7 \) and fix the embedding \( \sqrt{2} \mapsto 4 + 5 \cdot 7 + 4 \cdot 7^2 + O(7^4) \). We compute the values of \( \ell(r) \) in Table 2B.
So Proposition 3.13 implies that

\[ L_p(f, 1) \equiv -25026440 \cdot 7^2 \mod p^{10}. \]

Per Remark 3.12, we can use the same algebraic values used to compute Table 2b under the other \( p \)-adic embedding to compute

\[ L_p(f^\sigma, 1) \equiv -107584760 \cdot 7^{-2} \mod p^{10}, \]

where \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \) is the nontrivial automorphism.

In other words, let \( X_0(85)^* := X_0(85)/\langle w_5, w_17 \rangle \) be the full Atkin–Lehner quotient of \( X_0(85) \) with Jacobian \( J_0(85)^* \) and \( \pi : J_0(85) \to J_0(85)^* \). We have determined \( (\log_{f dq/q} \pi(y_K))^2 \) and \( (\log_{f^\sigma dq/q} \pi(y_K))^2 \), the values of the square of the logarithm of the \( \pi(y_K) \) on the Jacobian of \( X_0(85)^* \) with respect to the basis \( f dq/q, f^\sigma dq/q \) of \( H^0(X_0(85)^*, \Omega^1) \).

### Table 3. \( \ell(r) \) for 169.2.a.b

| \( r \) | \( \ell(r) \mod p^9 \) |
|---|---|
| 10 | \(-297386117 \cdot \nu^2 + 592900508 \cdot \nu + 979493595\) |
| 20 | \(476184186 \cdot \nu^2 - 623749256 \cdot \nu - 274455717\) |
| 30 | \(-716852945 \cdot \nu^2 + 898090404 \cdot \nu - 230653046\) |
| 40 | \(-813805284 \cdot \nu^2 + 94452192 \cdot \nu - 205004852\) |
| 50 | \(652023685 \cdot \nu^2 + 765243602 \cdot \nu + 477480226\) |
| 60 | \(-730757902 \cdot \nu^2 + 146316051 \cdot \nu - 343684998\) |
| 70 | \(-529440885 \cdot \nu^2 - 683691281 \cdot \nu + 99641402\) |
| 80 | \(-834057296 \cdot \nu^2 + 568271068 \cdot \nu - 580138952\) |
| 90 | \(84670657 \cdot \nu^2 + 476068012 \cdot \nu + 511028466\) |

**Example 3.20.** Let \( f \) be a newform in the orbit 169.2.a.b. Then \( f \) has coefficient field \( E_f = \mathbb{Q}(\zeta_{14})^+ \), and we denote

\[ f(q) := q + (-\nu^2 + 1)q^2 + (\nu^2 - \nu - 2)q^3 + (\nu^2 + \nu - 2)q^4 + (-\nu^2 + \nu)q^5 + O(q^6) \]

where \( \nu \) satisfies the minimal polynomial \( x^3 - x^2 - 2x + 1 \). Let \( D = -43 \), and \( p = 11 \). Then \( p \) is inert in \( E_f \), so does not satisfy one of our assumptions. However, it is still possible to compute \( L_p(f, 1) \) by working in an extension of \( \mathbb{Q}_p \). The \( p \)-adic completion of the compositum \( KE_f \) is a degree 3 unramified extension of \( \mathbb{Q}_p \). Let \( \nu \mapsto 16978 \cdot \nu^2 + 53324 \cdot \nu - 31046 + O(11^5) \) be the embedding of \( E_f \) into this extension.

We compute the values of \( \ell(r) \) in Table 3. Then Proposition 3.13 implies that

\[ L_p(f, 1) \equiv -1049872412 \cdot \nu^2 - 976527363 \cdot \nu + 889741537 \mod p^9. \]

By considering the other two embeddings from \( E_f \) into the \( p \)-adic completion of \( KE_f \), we obtain the values of \( L_p(f^\sigma, 1) \) for \( \sigma \in \text{Gal}(E_f/\mathbb{Q}) \). This allows us to compute \( (\log_{f^\sigma dq/q} y_K)^2 \) for the genus 3 modular curve \( X_0(169)^+ \) which is isomorphic over \( \mathbb{Q} \) to the split Cartan modular curve \( X_8(13) \).
4. Perrin-Riou’s $p$-adic Gross–Zagier formula

Recall that $f \in S_2(N)$ is a weight 2 newform for $\Gamma_0(N)$. In this section we discuss the computation of the cyclotomic $p$-adic height of the $f$-isotypical component of the Heegner point using the $p$-adic Gross–Zagier formula of Perrin-Riou [PR87]. Like the anticyclotomic $L$-function of the previous section, Perrin-Riou’s $p$-adic $L$-function $L_p(f)$ also interpolates the central value of the Rankin $L$-series $L(f, \chi, 1)$ for certain Hecke characters $\chi$, and she also provides a $p$-adic Gross–Zagier formula with a special value at 1. The nature of Perrin-Riou’s construction of the $p$-adic Rankin $L$-series associated to $f$ and $K$ leads $L_p(f)$ to vanish, and instead she considers the derivative in the cyclotomic direction (4.0.3), which she shows is proportional to the $p$-adic height of the Heegner point.

We outline how to compute the derivative $L_p(f)$ at 1 in the cyclotomic direction, as well as the other constants appearing in the $p$-adic Gross–Zagier formula, and therefore the height of the Heegner point. In order to do this, we will use the relationship between Perrin-Riou’s $p$-adic $L$-function and the $p$-adic $L$-function of Amice–Vélu and Vishik. We then use the theory of overconvergent modular symbols [PS11] to compute values of the latter $p$-adic $L$-function.

For this section we have the following assumptions.

**Assumption 4.1.**

1. Assume $p > 2$ is a prime number not dividing $N$;
2. assume $p$ splits in $E_f$, the coefficient field of $f$, and fix an embedding $e : E_f \to \mathbb{Q}_p$;
3. assume $f^{\sigma}$ is ordinary in $e$, that is, $e(a_p(f^{\sigma}))$ is a unit for all $\sigma \in \text{Gal}(E_f/\mathbb{Q})$;
4. assume $f$ has analytic rank 1;
5. let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of class number 1 with odd discriminant $D < -3$;
6. assume $p$ splits in $K$;
7. assume every prime $q$ dividing $N$ splits in $K$.

Like in the previous section, Assumptions (2), (6), and (5) are simplifying assumptions to avoid working in field extensions of $\mathbb{Q}_p$. We require Assumption (3) for the construction of the $p$-adic $L$-function: [Kob13] constructs an analogous $p$-adic $L$-function in the case of supersingular reduction.

Let $K_{\infty}/K$ be the (unique) $\mathbb{Z}_p^2$-extension of $K$ with Galois group $\Gamma := \text{Gal}(K_{\infty}/K)$. Write $\mathcal{O}$ for the ring of integers in $\overline{\mathbb{Q}}_p$. Let $\psi : \Gamma \to \mathcal{O}^\times$ be a finite order character. The theta series

\begin{equation}
\Theta_\psi := \sum_{a \in \mathcal{O}_K} \psi(a)q^{Nm(a)}
\end{equation}

is a weight 1 modular form. We denote the Rankin–Selberg convolution by $L(f, \psi, s) := L(f, \Theta_\psi, s)$. Perrin-Riou constructs a $p$-adic $L$-function $L_p(f) \in \mathbb{Z}_p[[\Gamma]]$ characterized by the interpolation property that for any finite order character $\psi : \Gamma \to \mathcal{O}^\times$, the values

\begin{equation}
\psi(L_p(f)) \doteq L(f, \psi, 1)
\end{equation}

are proportional. (As in (3.0.2), the left hand side is our notation for evaluation $L_p(f)$ at $\psi$ in the range of interpolation.) Using the functional equation for $L_p(f)$ we can show that that $L_p(f)$ vanishes at the trivial character 1.
Perrin-Riou relates the derivative of this $p$-adic $L$-function at $1$ to a $p$-adic height pairing defined by Schneider and Mazur–Tate [Sch82, MT83] on abelian varieties. This coincides also with the height pairing of Coleman–Gross [CG89] in the case of Jacobians, when the required splitting of the Hodge filtration is given by the unit root subspace of $H^1(X_{Q_p}, \Omega^1)$ [Col91].

This height pairing $\langle \cdot, \cdot \rangle_{\ell_K}$ depends on a choice of idèle class character: $$\ell : \mathbf{A}^\times_K/K^\times \to \mathbf{Q}_p$$ equivalently, by class field theory, a homomorphism $\ell : \text{Gal}(K^\infty/K) \to \mathbf{Q}_p$, which we will take to be the cyclotomic character. Let $\ell_K$ be the restriction of $\ell$ to $\text{Gal}(\overline{\mathbf{Q}}/K)$. We can decompose $\ell$ as the composition $\ell = \log_p \circ \lambda$ where $\lambda : \Gamma \to \mathbf{Z}_p$ is the cyclotomic character.

The derivative of $L_p(f)$ in the direction of $\ell$ at the trivial character $1$ is defined as

$$L'_p,\ell(f, 1) := \left( \frac{d}{ds} \lambda^s(L_p(f)) \right) \bigg|_{s=0}. \tag{4.0.3}$$

Recall that $y_K \in J_0(N)(K)$ denotes the Heegner point associated to $K$, and for $\sigma \in \text{Gal}(E_f/K)$, we write $y_{K,f,\sigma}$ for the $\sigma$-isotypical part of $y_K$, which belongs to $J_0(N)(K) \otimes E_f$.

By Assumption 4.1 (3), $f$ is ordinary in $e$; let $\alpha_p(f)$ denote the unit root of the Frobenius polynomial $x^2 - e(a_p)x + p$ in $\mathbf{Q}_p$.

**Theorem 4.2** ([PR87, Theorem 1.3]). The function $L_p(f)$ vanishes at $1$ and the derivative at $1$ in the direction $\ell$ is

$$L'_p,\ell(f, 1) = \left(1 - \frac{1}{\alpha_p(f)} \right)^4 \langle y_{K,f}, y_{K,f} \rangle_{\ell_K}. \tag{4.0.4}$$

We now discuss how to modify Theorem 4.2 to obtain the heights of the images of Heegner points on quotients $J_X$ of $J_0(N)$. These formulas lay the groundwork for doing quadratic Chabauty on quotients of $X_0(N)$.

**Proposition 4.3** ([MT83, (3.4.3)]). Let $g : A \to B$ be a homomorphism of principally polarized abelian varieties over $\mathbf{Q}$, $a \in A(K)$, and $b \in B(K)$. Let $g^\vee$ denote the dual map $g^\vee : B^\vee \to A^\vee$, while $\lambda_A : A \to A^\vee$ and $\lambda_B : B \to B^\vee$ denote the principal polarizations. Then

$$\langle a, (\lambda_A^{-1} \circ g^\vee \circ \lambda_B)(b) \rangle_{\ell_K} = \langle g(a), b \rangle_{\ell_K}. \tag{4.0.5}$$

We deduce the following proposition from Mazur and Tate’s formula.

**Proposition 4.4.** Let $\phi : X_0(N) \to X$ be a simple new $\Gamma_0(N)$-modular curve, with Jacobian $J_X$ and an associated newform $f_X$. Let $\pi : J_0(N) \to J_X$ be induced by $\phi_*$. We continue to suppose $p$ is ordinary for $J_X$ (see Assumption 4.1 (3)). Then

$$L'_p,\ell(f_X, 1) = \left(1 - \frac{1}{\alpha_p(f_X)} \right)^4 \left( \frac{\pi(y_{K,f_X}), \pi(y_{K,f_X})}{\deg \phi} \right)_{\ell_K}. \tag{4.0.5}$$

**Proof.** Let $\pi^\vee : J_X^\vee \to J_0(N)^\vee$ be the dual map on the dual abelian varieties and $\theta_J : J_0(N)^\vee \to J_0(N)$ be the principal polarization on $J_0(N)$. Consider the polarization $\theta_X = \pi \circ \theta_J \circ \pi^\vee : J_X^\vee \to J_X$. Let $\lambda : J_X^\vee \to J_X$ be the principal polarization of $J_X$. By identifying $\phi_* = \pi$ and $\phi^* = \theta_J \circ \pi^\vee \circ \lambda^{-1}$ we see that $(\deg \phi)\lambda = \theta_X$. In other words $\phi_* \phi^* = \text{id}_{J_X} \deg \phi$, so letting

$$e_X := \left( \frac{1}{\deg \phi} \right) \theta_J \circ \pi^\vee \circ \lambda^{-1} \circ \pi$$
we see that $e_X$ is an idempotent in $\text{End}^0(J_0(N))$ which gives projection onto the component $\text{End}^0(J_X)$.

Proposition 4.3 gives

$$\langle y_{K,f_X}, (\theta_J \circ \pi \circ \lambda^{-1})(\pi(y_{K,f_X})) \rangle_{\ell_K} = \langle \pi(y_{K,f_X}), \pi(y_{K,f_X}) \rangle_{\ell_K}.$$ 

But the left hand side is also equal to $(\deg \phi)\langle y_{K,f_X}, e_X(y_{K,f_X}) \rangle_{\ell_K} = (\deg \phi)\langle y_{K,f_X}, y_{K,f_X} \rangle_{\ell_K}$, since $y_{K,f_X} = e_X(y_{K,f_X})$ and $e_X$ is idempotent.

So $\langle y_{K,f_X}, y_{K,f_X} \rangle_{\ell_K} = \langle \pi(y_{K,f_X}), \pi(y_{K,f_X}) \rangle_{\ell_K} / (\deg \phi)$, as claimed.

The following corollary follows from the formula for taking the trace [MT83, (1.10.5)]. Let $\ell_Q$ be the cyclotomic character of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

**Corollary 4.5.** Let $\phi : X_0(N) \to X$ be a simple new $\Gamma_0(N)$-modular curve, with Jacobian $J_X$, and an associated newform $f_X$. Let $\pi : J_0(N) \to J_X$ be induced by $\phi_*$. Make an identification $\text{End}^0(J_0(N)) \simeq E_{J_X}$. We have

$$\langle \pi(y_K), \pi(y_K) \rangle_{\ell_Q} = \frac{1}{2} \sum_{\sigma \in \text{Gal}(E_{J_X}/Q)} \langle \pi(y_{K,f_X}), \pi(y_{K,f_X}) \rangle_{\ell_K},$$

where on the left hand side we are considering $\pi(y_K)$ as a point in $J_X(Q)$.

**Proof.** We have that $y_K = \sum_{g \in \text{New}_N} \sum_{\sigma \in \text{Gal}(E_g/Q)} y_{K,f_g}$. For $g \neq f_X$ the $g$-isotypical subspace of $J_0(N)$ is in the kernel of $\pi$, so

$$\langle \pi(y_K), \pi(y_K) \rangle_{\ell_K} = \sum_{\sigma \in \text{Gal}(E_{J_X}/Q)} \langle \pi(y_{K,f_X}), \pi(y_{K,f_X}) \rangle_{\ell_K}.$$ 

Then by [MT83, (1.10.5)]

$$[K : Q]\langle \pi(y_K), \pi(y_K) \rangle_{\ell_K} = \langle 2\pi(y_K), 2\pi(y_K) \rangle_{\ell_Q}$$

where $2\pi(y_K)$ is the trace of $\pi(y_K)$ from $K/Q$.

Perrin-Riou provides a comparison of the $p$-adic $L$-function described here to the $p$-adic $L$-function $L_{p,\text{MTT}}(f)$ of Amice–Vélu and Vishik [PR87, (1.1)] discussed in the paper of Mazur, Tate, and Teitelbaum [MTT86]. This comparison will be important computationally, since the latter $L$-function can be computed in Sage. We first give a brief description of the interpolation property of the $p$-adic $L$-function $L_{p,\text{MTT}}(f)$; more details can be found in [MTT86]. Let $Q_\infty$ be the cyclotomic $Z_p$-extension of $Q$ and $\Gamma_Q := \text{Gal}(Q_\infty/Q)$. Let $\gamma$ be a topological generator for $\Gamma_Q$. Let $\zeta$ be a primitive $p'$th root of unity. We write $\psi_\zeta$ for the associated character of $\Gamma_Q$ sending $\gamma \mapsto \zeta$. We can also think of this as a Dirichlet character by considering it as a character of $\text{Gal}(Q(\zeta_{p'})/Q) \simeq (\mathbb{Z}/p^r\mathbb{Z})^\times$. Write $\tau(\psi_\zeta)$ for the Gauss sum. Then there exists an element

$$(4.0.6) \quad \psi_\zeta(L_{p,\text{MTT}}(f)) = e_p(\zeta) \frac{L(f, \psi_\zeta^{-1}, 1)}{\Omega_f^{\text{sgn}(\psi_\zeta)}}$$

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where $\Omega_f^+$ are certain periods associated to $f$, which we will elaborate on later (see (4.0.6) and Algorithm 4.8), and

\[
\epsilon_p(\zeta) = \begin{cases} 
\alpha_p^{-r-1} \frac{p^s+1}{\tau(\zeta^*)} & \text{if } \zeta \neq 1 \\
\alpha_p(f)^{-1} \left(1 - \frac{1}{\alpha_p(f)}\right)^2 & \text{if } \zeta = 1.
\end{cases}
\]

Let $\Omega_f := 8\pi^2\|f\|$ be the period of the modular form $f$ [PR87, p.458]. Let $\varepsilon$ denote the quadratic character associated to $K$ with conductor $|D|$. Then

\[
\ell_K(L_p(f, 1)) = \ell_Q(L_{p,MTT}(f))\ell_Q(L_{p,MTT}(f^\varepsilon)) \left(\frac{\sqrt{|D|}}{\Omega_f}\right).
\]

Sage has an implementation of the $p$-adic $L$-function of Amice–Vélu and Vishik, so in practice, we use (4.0.8) and the fact that $L(f, 1) = 0$ to translate Theorem 4.2 from a statement about the derivative of $L_p(f, 1)$ in the direction of $\ell$ into a statement about this $p$-adic $L$-function to compute the cyclotomic $p$-adic height of $y_{K,f}$:

\[
\mathcal{L}_{p,\ell}'(f, 1) = L'_{p,MTT}(f, 1)L_{p,MTT}(f^\varepsilon, 1) \left(\frac{\Omega_f^+\Omega_f^+\sqrt{|D|}}{\Omega_f}\right).
\]

To compute $L_{p,MTT}(f^\varepsilon, 1)$ we use the interpolation property of the $p$-adic $L$-function (4.0.6):

\[
L_{p,MTT}(f^\varepsilon, 1) = (1 - 1/\alpha_p(f^\varepsilon))^2 L(f^\varepsilon, 1) / \Omega_f^+.
\]

Since we chose $K$ to be a field where $p$ splits, $a_p(f^\varepsilon) = \varepsilon(p)a_p(f) = a_p(f)$ and therefore $\alpha_p(f^\varepsilon) = \alpha_p(f)$. We use the equality $(1 - 1/\alpha_p(f^\varepsilon))^2 = (1 - 1/\alpha_p(f))^2$ to cancel some factors. When we combine (4.0.10) with (4.0.9), we have

\[
\langle y_{K,f}, y_{K,f}\rangle_{\ell_K} = \left(\frac{\Omega_f^+\Omega_f^+\sqrt{|D|}}{\Omega_f}\right) \left(1 - \frac{1}{\alpha_p(f)}\right)^{-2} L(f^\varepsilon, 1) \frac{d}{dT} L_{p,MTT}(f, T) \bigg|_{T=0} \log_p(1 + p).
\]

The conversion from $L_{p,MTT}(f, s)$ to the series expansion $L_{p,MTT}(f, T)$ requires a choice of topological generator $1 + p$ for the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\operatorname{Gal}(K_{\text{cyc}}/K)$.

Let $\sigma \in \operatorname{Gal}(E_f/\mathbb{Q})$. By substituting $f^\sigma$ into the right hand side of (4.0.11), we obtain $\langle y_{K,f^\sigma}, y_{K,f^\sigma}\rangle_{\ell_K}$.

**Algorithm 4.6** (The cyclotomic $p$-adic height over $K$ of the $f$-isotypical component of the Heegner point $y_{K,f}$).

**Input:**
- $f \in S_2(N)$ newform with coefficients in $E_f$;
- $K$ imaginary quadratic field of class number 1 and discriminant $D < -3$ satisfying the Heegner hypothesis for $N$;
- $p$ a prime split in $K$; and
- an embedding $e : E_f \to \mathbb{Q}_p$ such that $f$ is ordinary in $e$.

**Output:** The cyclotomic $p$-adic height $\langle y_{K,f}, y_{K,f}\rangle_{\ell_K}$ over $K$ of $y_{K,f} \in J_0(N)(K) \otimes E_f$.

1. Compute $\frac{d}{dT} L_{p,MTT}(f, T) \bigg|_{T=0}$ using overconvergent modular symbols [PS11].
2. Compute $L(f^\varepsilon, 1)$ using Dokchitser’s algorithms [Dok04].
(3) Compute $\Omega_f^+$ using Algorithm 4.8 (normalized to agree with the normalization on the overconvergent modular symbols).

(4) Compute $\|f\|$, for example, using [Col18].

(5) Return $\frac{\Omega_f^+ \sqrt{|D|}}{8\pi^2\|f\|} \cdot \left(1 - \frac{1}{\alpha_p(f)}\right)^{-2} \cdot L(f^e, 1) \cdot \frac{d}{dT} L_{p,MTT}(f, T)|_{T=0} \log(1 + p)$.

**Remark 4.7.** The convention of the sign of the height in Perrin-Riou differs from the convention chosen in Mazur–Tate–Teitelbaum and Pollack–Stevens. To achieve the correct normalization for $p$-adic BSD we negate the sign of the height returned by Algorithm 4.6.

To compute the quantity $\Omega_f^+$ we exploit the relationship [MTT86, I §8 (8.6)] between $f$ and quadratic twists of $f$ by fundamental discriminants $D' > 0$. Let $\tau(\chi)$ denote the Gauss sum

\[(4.0.12) \quad \tau(\chi) := \sum_{a \mod D'} \chi(a) e^{2\pi i a / D'}.
\]

Since $D'$ is a fundamental discriminant, $\tau(\chi) = \sqrt{D'}$.

Before we state the formula we need, we establish some background on modular symbols, following [PS11, Pol14]. Write $\Delta_0 := \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. We can act on $\Delta_0$ by elements of $\Gamma_0(N)$ via fractional linear transformation. Via this action, the set of additive homomorphisms $\text{Hom}(\Delta_0, \mathbf{C})$ has an action

$$\varphi|\gamma := \varphi(\gamma E)$$

where $\varphi \in \text{Hom}(\Delta_0, \mathbf{C})$, $E \in \Delta_0$, and $\gamma \in \Gamma_0(N)$. The $\mathbf{C}$-valued modular symbols are those symbols that are invariant under the action of all $\gamma \in \Gamma_0(N)$, i.e. $\varphi|\gamma = \varphi$. We denote the space of these symbols as $\text{Symb}_{\Gamma}(\mathbf{C})$.

For any weight 2 newform $g$ there exists a $\mathbf{C}$-valued modular symbol $\psi_g \in \text{Symb}_{\Gamma}(\mathbf{C})$ given by

$$\{s\} - \{r\} \mapsto 2\pi i \int_{r}^{s} g(z) dz.$$

In this context, $\{s\} - \{r\}$ denotes the divisor with support +1 on $s \in \mathbf{Q}$ and −1 on $r \in \mathbf{Q}$. This symbol encodes information about the twisted $L$-values of $g$.

There is a 2-dimensional subspace of $\text{Symb}_{\Gamma}(\mathbf{C})$ where the action of Hecke is equal to the eigenvalues of $g$. The set $\text{Hom}(\Delta_0, \mathbf{C})$ has an involution $\iota = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ and so $\psi_g$ can be decomposed as a sum of modular symbols $\psi_g = \psi_g^+ + \psi_g^-$. There exist complex numbers $\Omega_g^+$ and $\Omega_g^-$ such that $\varphi_g^+ := \psi_g^+/\Omega_g^+$ and $\varphi_g^- := \psi_g^-/\Omega_g^-$ take values in $E_g$ (see [BMS16, Theorem 2.2]).

The period $\Omega_g^+$ is only well-defined up to an element of $\overline{\mathbf{Q}}$ that is a unit in $\mathbf{Q}_p$. By [MTT86, I §8 (8.6)] we can write the following relationship between modular forms and modular symbols

\[(4.0.13) \quad \frac{L(f^\chi, 1)}{\Omega_f^+} = \frac{\tau(\chi)}{D'} \sum_{\substack{a_1 = 1 \\ \gcd(a, D') = 1}}^{[D'/2]} \chi(a)(\varphi_f^+(\{a\} - \{D'\}) + \varphi_f^+(\{-a\} - \{D'\})).
\]

If $f^\chi$ is rank 0, the sum will be non-zero.

In practice, when evaluating the symbol $\psi_g^+$ and therefore the $p$-adic $L$-values of $g$ in a computer algebra program, a choice of $\Omega_f^+$ must be fixed. For example, Sage makes a
random choice of generator in the Hecke-eigenspace of \( \text{Hom}(\Delta_0, \mathbb{C})^\pm \) corresponding to \( g \) [Ste00, Section 3.5.3]. To extract this choice, we can evaluate the modular symbols \( \varphi_f^+ \) in (4.0.13). This leads to the following algorithm for determining the period \( \Omega_f^+ \) that is compatible with the normalization on \( \varphi_f^+ \).

**Algorithm 4.8** (The plus period, normalized to agree with the overconvergent modular symbols).

**Input:**
- \( p \) a prime
- \( f \in S_2(N) \) newform with coefficients in \( E_f \) and an embedding \( e : E_f \to \mathbb{Q}_p \) such that \( f \) is ordinary in \( e \)

**Output:** The period \( \Omega_f^+ \) (normalized to agree with the overconvergent modular symbols)

1. Set \( D' := 5 \).
2. Compute the right hand side of (4.0.13) by evaluating the modular symbols and set \( R \) equal to this value.
3. If \( R \) is equal to 0, set \( D' \) to the next largest fundamental discriminant, and go back to Step (2).
4. Compute \( L(f^\infty, 1) \) using Dokchitser’s algorithms [Dok04].
5. Return \( R/L(f^\infty, 1) \).

**Remark 4.9.** When \( f \) is the modular form associated to an elliptic curve \( E \), we can take \( \Omega_f^+ \) to be the real period \( \Omega_E^+ \) of the elliptic curve.

**Example 4.10.** Let \( f_E \) be the modular form associated to the elliptic curve with LMFDB label 61.a1 and \( p = 5 \) a prime of good ordinary reduction. Let \( \pi : X_0(61) \to E \) denote the modular parametrization. Choose \( D = -19 \) a Heegner discriminant for \( E \). Then the \( p \)-adic \( L \)-series expansion for \( E/\mathbb{Q} \) can be computed in Sage using [PS11]

\[
L_{p,MTT}(f_E, T) = O(5^{10}) + (1 + 2 \cdot 5^2 + 5^3 + 5^4 + 3 \cdot 5^5 + 2 \cdot 5^7 + O(5^8)) \cdot T + (1 + 4 \cdot 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 2 \cdot 5^4 + 5^5 + O(5^6)) \cdot T^2 + O(T^3).
\]

Using the interpolation property, we have \( \ell_{\mathbb{Q}}(L_{p,MTT}(f^\infty)) = \left(1 - \frac{1}{\alpha_p}\right)^2 L(f_E^\infty, 1)/\Omega_{f_E}^+ \) and we can evaluate

\[
L(f_E^\infty, 1)/\Omega_{f_E}^+ = 2.
\]

Finally, by [Cre95, Proposition 1], we have \( \Omega_{f_E} = 2\deg \pi \cdot \text{Vol} E \), so

\[
\left( \frac{\Omega_{f_E}^+ \Omega_{f_E}^+ \sqrt{|D|} \deg \pi}{\Omega_f^+} \right) = \left( \frac{\Omega_{f_E}^+ \Omega_{f_E}^+ \sqrt{|D|}}{2 \text{Vol} E^+} \right) = 1.
\]

Altogether, we evaluate the following formula for the \( p \)-adic height of \( \pi(y_{K,f_E}) \):

\[
\frac{1}{2} \langle \pi_E(y_{K,f_E}), \pi_E(y_{K,f_E}) \rangle_{\ell_K} = \langle \pi_E(y_{K,f_E}), \pi_E(y_{K,f_E}) \rangle_{\ell_{\mathbb{Q}}}
= \left( \frac{\Omega_{f_E}^+ \Omega_{f_E}^+ \sqrt{|D|}}{2 \text{Vol} E^+} \right) \left( 1 - \frac{1}{\alpha_p} \right)^{-2} \left. \frac{L(f_E^\infty, 1)}{\Omega_{f_E}^+} \frac{d}{dT} L_{p,MTT}(f_E, T) \right|_{T=0} \log_p(1 + p)
= 4 \cdot 5 + 4 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + 4 \cdot 5^5 + 5^6 + 2 \cdot 5^7 + 4 \cdot 5^8 + O(5^9).
\]
Since \( f_E \) has analytic rank 1, by Gross–Zagier–Kolyvagin, the rank of \( E(\mathbb{Q}) \) is one and the trace of the \( f_E \)-isotypical component of the Heegner point found here generates \( E(\mathbb{Q}) \) up to finite index.

**Example 4.11.** Let \( p = 11 \) and \( D = -19 \). Let \( f \) and \( f^\sigma \) be the modular forms in the newform orbit 73.2.a.b given by

\[
\begin{align*}
    f &= q + (-\nu - 1)q^2 + (\nu - 2)q^3 + 3\nu q^4 + (-\nu - 1)q^5 + q^6 - 3q^7 + O(q^8) \\
    f^\sigma &= q + (\nu - 2)q^2 + (-\nu - 1)q^3 + (-3\nu + 3)q^4 + (\nu - 2)q^5 + q^6 - 3q^7 + O(q^8).
\end{align*}
\]

This has coefficient field \( E_f = \mathbb{Q}(\nu) \) where \( \nu \) has minimal polynomial \( \nu^2 - \nu - 1 \). Then \( f \) and \( f^\sigma \) are associated newforms for the simple new \( \Gamma_0(11) \)-modular curve \( X = X_0(73)^+ \). Fix the embedding \( e : E_f \to \mathbb{Q}_p \)

\[
(4.0.14) \quad \nu \mapsto 8 + 7 \cdot 11 + 10 \cdot 11^2 + 7 \cdot 11^3 + O(11^4).
\]

By computing the derivatives of the \( p \)-adic \( L \)-functions in Sage, we get the values

\[
(4.0.15) \quad \frac{d}{dT} L_{p,MTT}(f, T) \bigg|_{T=0} = 7 + 7 \cdot 11 + 2 \cdot 11^2 + 9 \cdot 11^3 + 4 \cdot 11^4 + O(11^5)
\]

\[
(4.0.16) \quad \frac{d}{dT} L_{p,MTT}(f^\sigma, T) \bigg|_{T=0} = 2 + 5 \cdot 11 + 6 \cdot 11^2 + 3 \cdot 11^3 + 8 \cdot 11^4 + O(11^5).
\]

We have that \( a_p(f) = \nu - 2 \) and \( a_p(f^\sigma) = -\nu - 1 \), and we embed via \( e \). It remains to compute the twisted \( L \)-value and the periods. We fix a complex embedding \( \epsilon_c : E_f \to \mathbb{C} \) given by

\[
(4.0.17) \quad \nu \mapsto 1.618.
\]

We can compute the Petersson norm of \( f \) and \( f^\sigma \) under (4.0.17). We get

\[
\|f\| = 0.986763 \text{ and } \|f^\sigma\| = 0.368434.
\]

Changing the complex embedding would swap the values of the norms.

Using Dokchitser’s package for computing values of \( L \)-functions we now compute \( L(f^\epsilon, 1) \) where \( \epsilon \) is the quadratic character twisting by \( D = -19 \). This yields \( L(f^\epsilon, 1) = 4.771908 \) under the embedding \( \epsilon_c \).

Finally, we compute \( \Omega_f^+ \). For \( D' = 5 \) we find that the right hand side of (4.0.13) is \(-4/\sqrt{5}\). We compute \( L(f^\chi, 1) = 6.34683 \). Therefore \( \Omega_f^+ = 3.5479 \). Combining the complex terms, we find

\[
\frac{\Omega_f^+ L(f^\epsilon, 1) \sqrt{|D|}}{\Omega_f} = 0.94721.
\]

Numerically, by computing the quantities to higher precision, we recognize this as being close to an algebraic number having minimal polynomial \( 20x^2 - 20x + 1 \), and so belongs to \( E_f \). It appears to be \( \epsilon_c(r_1) \) where \( r_1 := 2/5\nu + 3/10 \). Repeating the calculations for \( f^\sigma \), we obtain \( \Omega_f^+ L(f^{\sigma\epsilon}, 1) \sqrt{|D|}/\Omega_f = r_2 := -2/5\nu + 7/10 \), the other root of this minimal polynomial.

The values \( \epsilon(r_1), (4.0.15) \), and \( \epsilon(\alpha_p(f)) \) can be combined using (4.0.11) to obtain the height of \( y_{K,f} \in J_0(N)(K) \). To project \( \pi : J_0(73) \to J_0(73)^+ \), we multiply by the degree of the quotient map \( X_0(73) \to X_0(73)^+ \) which is 2. We have

\[
(4.0.18) \quad \langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{\epsilon_K} = 6 \cdot 11 + 5 \cdot 11^2 + 3 \cdot 11^3 + 8 \cdot 11^5 + 7 \cdot 11^6 + O(11^7).
\]
Similarly, we can compute $\langle \pi(y_{K,f^*}), \pi(y_{K,f^*}) \rangle_{t_K}$ by using the quantities $e(r_2), (4.0.16)$, and $\alpha_p(f^*)$ in the formula (4.0.11). We get
\begin{equation}
(4.0.19) \quad \langle \pi(y_{K,f^*}), \pi(y_{K,f^*}) \rangle_{t_K} = 7 \cdot 11 + 8 \cdot 11^2 + 10 \cdot 11^3 + 11^4 + 2 \cdot 11^5 + 3 \cdot 11^6 + O(11^7).
\end{equation}
Using Corollary 4.5 we also obtain $\langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{t_Q}$.

5. Quadratic Chabauty

Let $X$ be a smooth projective geometrically integral curve defined over $\mathbb{Q}$ of genus $g > 1$. Quadratic Chabauty [BD18, BD21] is a technique for studying rational points on $X$ that computes a finite set of $p$-adic points containing $X(\mathbb{Q})$ in some cases when the rank of $J$ is greater than or equal to the genus of $X$.

Quadratic Chabauty uses local and global $p$-adic height functions to construct a quadratic Chabauty function $\rho(z)$ that is used to cut out the finite set of $p$-adic points of $X$ containing $X(\mathbb{Q})$. We will denote the global $p$-adic cyclotomic height on $y \in J(\mathbb{Q})$ by $h(y) := \langle y, y \rangle_{t_Q}$ when the field of definition and choice of idèle class character is clear. Otherwise we will use the notation in Section 4. The global height $h$ is a sum of local heights $h = \sum_{\ell} h_{\ell}$ where $\ell$ ranges over finite primes. For $\ell \neq p$, the local height $h_{\ell}$ is a biadditive, continuous, and symmetric function on pairs of disjoint $\mathbb{Q}$-rational divisors of degree zero on $X$. For $\ell = p$, Coleman and Gross describe $h_p$ as a Coleman integral of a third kind differential form. For more background on $p$-adic heights, which we do not describe in detail here, see [CG89, BB15, BBM16].

In Section 5.1 we discuss the case of rank 1 elliptic curve $E$ and construct a locally analytic quadratic Chabauty function $\rho(z)$ whose solutions contain the integer points of $E$. In Section 5.2 we discuss the case of rational points on higher genus curves. We give our main theorem, Theorem 5.10, explicitly constructing the quadratic Chabauty function $\rho(z)$ as a locally analytic function without knowing any infinite order points in the case of simple new $\Gamma_0(N)$-modular curves. We also provide several examples of how to apply Theorem 5.10 in practice.

5.1. Integral points on rank one elliptic curves. In this section, we study the case of determining integral points on a rank 1 genus 1 (elliptic) curve $E/\mathbb{Q}$. A consequence of Faltings's theorem is that the affine curve obtained by removing a point $X := E - \mathcal{O}$ has finitely many integral points $X(\mathbb{Z})$. Quadratic Chabauty for integral points on rank 1 elliptic curves requires an infinite order point in $E(\mathbb{Q})$. We replace this requirement with the computation of special values of two $p$-adic $L$-functions constructed by Perrin-Riou (see Section 4) and Bertolini, Darmon, and Prasanna (see Section 3) that determine the height and logarithm of a Heegner point for $E$, respectively. This allows us to determine $X(\mathbb{Z})$ without knowing a rational point of infinite order. Since we rely on the results from the previous sections, we make the assumptions 3.1 and 4.1.

Let $E$ be a rank one elliptic curve over $\mathbb{Q}$ with conductor $N$ given by a Weierstrass equation
\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]
Let $p > 2$ be a prime of good ordinary reduction. Let $\mathcal{E}/\mathbb{Z}$ denote the minimal regular model of $E$ and $\mathcal{X} = \mathcal{E} - \mathcal{O}$ the complement of the zero section in $\mathcal{E}$. Fix also differentials $\omega_0 = \frac{dx}{2y + a_1 x + a_3}$ and $\omega_1 = x \omega_0$. 

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Let $b$ be a tangential basepoint at the point at infinity or an integral 2-torsion point (see [Bes12, Section 1.5.4] for more on tangential basepoints). Consider the two functions, the double Coleman integral
\[(5.1.1)\quad D_2(z) := \int_b^z \omega_0 \omega_1\]
as well as the logarithm, which can be expressed as the Coleman integral
\[(5.1.2)\quad \log(z) := \int_b^z \omega_0.\]

In [BB15, BBM16, BBM17] Balakrishnan, Besser, and Müller give algorithms to compute a finite set of $p$-adic points containing the integral points. We recall a related theorem.

Suppose that $p > 3$ and let $E_2$ be the Katz $p$-adic weight 2 Eisenstein series [MST06]. Define the constant
\[(5.1.3)\quad c := \frac{a_1^2 + 4a_2 - E_2(E, \omega_0)}{12}.\]
For any non-torsion point $P \in E(\mathbb{Q})$ define $\gamma$ by
\[(5.1.4)\quad \gamma := \frac{h(P)}{\log(P)^2}.\]

**Remark 5.1.** The quantity $\gamma$ does not depend on $P$. The $\mathbb{Z}_p$-module $\mathbb{Z}_p \otimes E(\mathbb{Q})$ is 1-dimensional, and has only a 1-dimensional space of quadratic forms, and $\log(P) \neq 0$ when $P$ is non-torsion.

**Theorem 5.2 ([Bia20, Theorem 1.7]).** Let $E$ be a rank 1 elliptic curve over $\mathbb{Q}$ with good ordinary reduction at $p$ and bad reduction at the primes in a finite set $S$. There is a computable finite set $W \subset \mathbb{Q}_p$, $W = \prod_{q \in S} W_q$ such that $W_q$ is the possible local height contributions for an integral point at bad places, and $W_q$ is determined by the Kodaira type of the reduction of $E$ at $q$. For $w \in W$ define $\|w\|$ to be the sum of its elements.

If $E$ has good reduction at $q = 2$ or $q = 3$, and $E(\mathbb{F}_q) = \{O\}$, or if $E$ has split multiplicative reduction of Kodaira type $I_1$ at 2, then $\mathcal{X}(\mathbb{Z}) = \emptyset$.

Otherwise,
\[\mathcal{X}(\mathbb{Z}) \subset \bigcup_{w \in W} \psi(w),\]
where
\[(5.1.5)\quad \psi(w) := \{ z \in \mathcal{X}(\mathbb{Z}_p) : 2D_2(z) + c\log(z)^2 + \|w\| = \gamma \log(z)^2 \}.

We describe $\gamma$ in terms of two different special values of $p$-adic $L$-functions associated to $f \in S_2(N)$ the cusp form related to $E$ by modularity. This allows us to obtain new input into quadratic Chabauty as described in Theorem 5.2, by replacing the $\gamma$ in (5.1.5) with one determined by special values of $L$-functions.

**Theorem 5.3.** Let $f$ be the modular form associated to $E$ and $\pi : X_0(N) \to E$ the modular parametrization. Assume 4.1 and 3.1. We have the equality
\[(5.1.6)\quad \gamma = \frac{\frac{1}{2}(\deg \pi) \left(1 - \frac{1}{\alpha_p}\right)^{-4} \mathcal{L}_{p,1}^0(f, 1)}{\left(1 - a_p(f) + \frac{p}{\alpha_p}\right)^{-2} L_p(f, 1)}.\]
whenever $L(f^ε, 1) \neq 0$. Furthermore, $γ$ is computable.

In other words, $ρ(z) = h_p(z) - γ \log(z)^2$ is a computable locally analytic function from $X(Z_p)$ to $Q_p$ that takes values on a finite computable set when evaluated on $X(Z)$.

Theorem 5.3 allows us to determine a finite set of $p$-adic points of $X$ containing $X(Z)$ without knowing an infinite order point of $E(Q)$.

**Proof.** Corollary 4.4 shows that

$$\langle π(y_K, f), π(y_K, f) \rangle_{ℓ_K} = \deg π L_{p, ℓ}′(f, 1) \left(1 - \frac{1}{α_p(f)}\right)^{-4}$$

while (3.0.1) shows $L_p(f, 1) \left(\frac{1-a_p(f)+p}{p}\right)^{-2}$ is equal to $(\log_{fdq/q} π(y_K))^2$. Corollary 4.5 implies that $\frac{1}{2} \langle π(y_K, f), π(y_K, f) \rangle_{ℓ_K} = h(π(y_K))$. □

**Remark 5.4.** Suppose $X/Q$ is a smooth projective geometrically integral curve of genus 2 with Jacobian isogenous over $Q$ to $E_1 × E_2$ having Mordell–Weil rank 2. These methods allow us to obtain a finite set of $p$-adic points containing $X(Q)$ without knowing an infinite order point of the Jacobian. One simply follows the formula [BD18, Theorem 1.4] using the $γ_1$ and $γ_2$ in Theorem 5.3 associated with each $E_i$ as input.

**Example 5.5.** Let $E$ be the elliptic curve with LMFDB label 43.a1 and consider $p = 11$ a prime of good ordinary reduction. This is a model for the modular curve $X_0(43)^+$. We choose $D = −7$ a Heegner discriminant for $E$ in which $p$ and $N = 43$ split. Fix a model for $E$

$$X : y^2 + y = x^3 + x^2.$$ 

As in Examples 4.10 and 3.17, we compute the constant $γ$:

$$γ = \frac{h(π(y_K, f))}{\log(π(y_K, f))^2} = \frac{L_{p, ℓ}′(f, 1) \left(\frac{1}{2}\right) \left(1 - \frac{1}{α_p}\right)^{-4} \deg π}{L_p(f, 1) \left(\frac{1-a_p(f)+p}{p}\right)^{-2}}$$

$$= \frac{9 \cdot 11 + 5 \cdot 11^2 + 5 \cdot 11^3 + 3 \cdot 11^4 + 7 \cdot 11^6 + 4 \cdot 11^7 + 4 \cdot 11^8 + O(11^9)}{11^2 + 8 \cdot 11^3 + 9 \cdot 11^4 + 6 \cdot 11^5 + 8 \cdot 11^6 + 6 \cdot 11^7 + 4 \cdot 11^8 + 4 \cdot 11^9 + O(11^{10})}$$

$$= 9 \cdot 11^{-1} + 10 + 2 \cdot 11 + 4 \cdot 11^2 + 5 \cdot 11^4 + 8 \cdot 11^5 + 10 \cdot 11^6 + O(11^7).$$

We proceed to solve the equations described by (5.1.5). The only prime of bad reduction for $E$ is 43, and the Kodaira type of $E$ over 43 is $I_1$ so $W = \{0\}$, and so $X(Z) \subset \{h_p(z) = γ \log(z)^2\}$. Using a modified version of the code associated to [Bia20], we obtain the finite set:

$$\{(−1, −1), (−1, 0), (0, −1), (0, 0), (1, −2), (1, 1), (2, −4), (2, 3), (21, −99), (21, 98), (10 \cdot 11 + 7 \cdot 11^2 + O(11^3), 10 + 10 \cdot 11 + 9 \cdot 11^2 + 5 \cdot 11^3 + O(11^4)), (10 \cdot 11 + 7 \cdot 11^2 + O(11^3), 11^2 + 5 \cdot 11^3 + O(11^4)), (1 + 6 \cdot 11 + 2 \cdot 11^2 + O(11^3), 9 + 3 \cdot 11^2 + O(11^3)), (1 + 6 \cdot 11 + 2 \cdot 11^2 + O(11^3), 1 + 10 \cdot 11 + 7 \cdot 11^2 + O(11^3)), (2 + 9 \cdot 11 + 7 \cdot 11^2 + O(11^3), 3 + 8 \cdot 11 + 11^2 + O(11^3)), (2 + 9 \cdot 11 + 7 \cdot 11^2 + O(11^3), 7 + 2 \cdot 11 + 9 \cdot 11^2 + O(11^3))\}.$$
This contains the 10 integral points on \( X \) as well as 3 pairs of \( \mathbb{Z}_{11} \)-points conjugate under the hyperelliptic involution.

5.2. Rational points on higher genus curves. We now discuss how to extend the results of the previous section to the case of rational points on higher genus curves. For rank 1 genus 1 curves, we constructed the locally analytic function \( \rho(z) \) used in quadratic Chabauty by writing the global height \( h \) in terms of \( \log_{\rho_0}(z)^2 \), a locally analytic basis for \((H^0(X_{\mathbb{Q}_p}, \Omega^1)^{\vee} \otimes H^0(X_{\mathbb{Q}_p}, \Omega^1)^{\vee})\). This strategy generalizes for finding rational points on higher genus curves.

However, the Coleman–Gross local height functions \( h_v \) at each prime \( v \) do not immediately extend to functions on \( X(\mathbb{Q}_v) \); more sophisticated heights machinery is needed to deal with the heights of rational points. For this, we turn to Nekovář’s theory of \( p \)-adic heights \([\text{Nek93}, \text{BD18}]\) to define heights \( h_{\text{Nek}} \) and \( h_{\text{Nek}}^v \) of mixed extensions of Galois representations associated to points \( x \in X(\mathbb{Q}_v) \).

For this section we assume the following.

**Assumption 5.6.**

1. Assume \( \phi : X_0(N) \to X \) is a simple new \( \Gamma_0(N) \)-modular curve;
2. assume \( X/\mathbb{Q} \) has genus \( g > 1 \);
3. assume \( X(\mathbb{Q}) \neq \emptyset \) and fix a basepoint \( b \in X(\mathbb{Q}) \);
4. assume its Jacobian \( J_X(\mathbb{Q}) \) has rank \( r = g \);
5. assume \( p \) is a prime of good reduction for \( X \) such that \( \log : J(\mathbb{Q}) \otimes \mathbb{Q}_p \to H^0(X_{\mathbb{Q}_p}, \Omega^1)^{\vee} \) is an isomorphism.

Since we rely on the results from the previous sections, we also make the assumptions in 3.1 and 4.1. Note that by Assumption 5.6 (1), and Lemma 2.6, we have \( \rho(J_X) = r = g \).

Since \( \rho(J_X) \geq 2 \), we can find some nontrivial correspondence \( Z \in \ker(\text{NS}(J_X) \to \text{NS}(X)) \).

Let \( K = \mathbb{Q} \) or \( \mathbb{Q}_p \). As explained in \([\text{BD18}, \text{Section 5}]\), the choice of \( Z \) can be used to construct a certain quotient of the two step \( \mathbb{Q}_p \)-pro-unipotent fundamental group, and, by a twisting construction, for every \( x \in X(K) \) we obtain an equivalence class of Galois representations

\[
A_{\mathbb{Z}}(b, x) \in \{ G_K \to \text{GL}_{2g+2}(\mathbb{Q}_p) \} / \sim .
\]

These Galois representations are mixed extensions: they admit a \( G_K \)-stable weight filtration with graded pieces \( \mathbb{Q}_p(1), V := H^1_{et}(X_K, \mathbb{Q}_p)^{\vee}, \mathbb{Q}_p \). Nekovář’s theory of \( p \)-adic heights \([\text{Nek93}]\) yields a height function \( X(\mathbb{Q}) \to \mathbb{Q}_p \) by sending \( x \in X(\mathbb{Q}) \) to \( h_{\text{Nek}}(A_{\mathbb{Z}}(b, x)) \).

Similar to the story for Coleman–Gross heights, the global height decomposes as a sum of local heights \( h_{\text{Nek}} = \sum_v h_{\text{Nek}}^v \). Furthermore, \( h_{\text{Nek}} \) is bilinear in the following sense: for each \( z \in X(\mathbb{Q}) \) we have projection maps

\[
\pi_1(A_{\mathbb{Z}}(b, z)) = [W_0A_{\mathbb{Z}}(b, z)/W_{-2}A_{\mathbb{Z}}(b, z)] \in H^1_f(G_{\mathbb{Q}}, V)
\]

\[
\pi_2(A_{\mathbb{Z}}(b, z)) = [W_{-1}A_{\mathbb{Z}}(b, z)] \in H^1_f(G_{\mathbb{Q}}, V^* (1))
\]

where \( W \) denotes the weight filtration on the mixed extension. The height \( h_{\text{Nek}} \) is bilinear on \( H^1_f(G_{\mathbb{Q}}, V) \times H^1_f(G_{\mathbb{Q}}, V^* (1)) \). Under our assumptions, these cohomology groups are both isomorphic to \( H^0(X_{\mathbb{Q}_p}, \Omega^1)^{\vee} \) and so \( \pi_i \) can also be seen as a map into \( H^0(X_{\mathbb{Q}_p}, \Omega^1)^{\vee} \).

The local height \( h_{\text{Nek}}^p \) at \( p \) can be described in terms of linear algebraic data given by the filtered \( \phi \)-module associated to \( A_{\mathbb{Z}}(b, x) \). For more details see \([\text{BDM}^{+21}, \text{Section 3.3.2}]\) or \([\text{BD18}, \text{Section 4.3.2}]\). We will simply write \( h_{\text{Nek}}^p(z) \) for the local height of \( z \in X(\mathbb{Q}_p) \), omitting the dependence on \( Z \) and \( b \).
The quadratic Chabauty function $\rho(z)$ is equal to the difference between the local height at $p$ and the global height $h_{\text{Nek}}(A_Z(b, z)) = h_{\text{Nek}}^p(z)$. The following theorem about $\rho(z)$ is the analog of Theorem 5.2. For this theorem it is not necessary to assume $X$ is a simple new modular $\Gamma_0(N)$-curve.

**Theorem 5.7** ([BD18, Proposition 5.5]). Let $\psi_1, \ldots, \psi_M$ be a basis for $(H^0(X_{Q_p}, \Omega^1)^\vee \otimes H^0(X_{Q_p}, \Omega^1)^\vee)^\vee$. There are finite computable constants $\alpha_1, \ldots, \alpha_M \in \mathbb{Q}_p$ such that the function $\rho(z)$ is a locally analytic function. Furthermore, there exists a finite set $S \subset \mathbb{Q}_p$ such that $\{\rho(x) = s : x \in X(\mathbb{Q}_p), s \in S\}$ contains $X(\mathbb{Q})$.

The set $S$ is given by computing local heights away from $p$ at primes $v$ of bad reduction. When $X$ has a semistable regular model with geometrically irreducible special fibers, then $S = \{0\}$ [BDM+21, Theorem 3.2].

To solve for the $\alpha_i$ in Theorem 5.7, our goal is to find constants such that $\sum_{i=1}^M \alpha_i \psi_i \circ (\pi_1, \pi_2)(A_Z(b, z)) = h_{\text{Nek}}(A_Z(b, z))$, thus rewriting the global height as a locally analytic function.

Computing the $\alpha_i$ requires knowing sufficiently many rational points on $X$ [BDM+21, Section 3.3]. We need enough $z \in X(\mathbb{Q})$ to find a basis of $H^0(X_{Q_p}, \Omega^1) \otimes H^0(X_{Q_p}, \Omega^1)$ of the form $(\pi_1(A_Z(b, z)), \pi_2(A_Z(b, z)))$ where $Z \in \ker(\text{NS}(J_X) \to \text{NS}(X))$ is nontrivial. The number of required rational points can also be decreased by working with symmetric $\text{End}(J_X)$-equivariant heights.

If we do not have sufficiently many rational points, we can also use $r = g$ independent points on $J_X(\mathbb{Q})$. Note that $(\pi_1(A_Z(b, z)), \pi_2(A_Z(b, z)))$ can be expressed in terms of a dual basis of $H^0(X_{Q_p}, \Omega^1)$, and $H^0(X_{Q_p}, \Omega^1)^\vee \simeq J_X(\mathbb{Q}) \otimes \mathbb{Q}_p$. The formula to represent $\pi_i(A_Z(b, z))$ in terms of the dual basis is given in [BBB+21, (41)]. Furthermore, Besser [Bes04] gives an equivalence between the construction of the height due to Coleman and Gross and that of Nekovář. In particular, they can be related through the study of a certain divisor, also studied in [DRS12].

**Definition 5.8.** Define $D_Z(b, z)$ to be the degree zero divisor on $X$ given by $D_Z(b, z) := Z|_\Delta - Z|_{X \times b} - Z|_{z \times X}$.

**Theorem 5.9** ([BD18, Theorem 6.3]). Let $z \neq b$ be an element of $X(\mathbb{Q})$. Then $h_{\text{Nek}}(A_Z(b, z)) = h(z - b, D_Z(b, z))$.

Therefore it is sufficient to express the height pairing in terms a basis for symmetric bilinear pairings on $J_X(\mathbb{Q}) \otimes \mathbb{Q}_p$ for a basis of $J_X(\mathbb{Q})$. Then, given a choice of $Z$ and $b$ this determines a locally analytic function $h_{\text{Nek}} : X(\mathbb{Q}_p) \to \mathbb{Q}_p$.

This strategy for determining the height from Jacobian points is basis of the strategy we use in the proof of our main theorem, Theorem 5.10. However, because of the modular nature of our arguments, we do not need to explicitly describe a basis for $J_X(\mathbb{Q})$. We also have a simplified calculation when computing the height pairing in terms of a basis of symmetric bilinear pairings on $J_X(\mathbb{Q})$ because of the choice of dual basis.
We now present a construction of $\rho(z)$ as a locally analytic function for simple new $\Gamma_0(N)$-modular curves that does not require knowing rational points on $X$ or $J_X$, other than the basepoint $b$. The theorem relies on Assumptions 4.1, 3.1, and 5.6. This is the main result of this section.

**Theorem 5.10.** Let $\phi : X_0(N) \to X$ be a simple new $\Gamma_0(N)$-modular curve with associated $f \in \text{New}_N$ of analytic rank 1 and Jacobian $J_X$. Let $Z \in \ker(\NS(J_X) \to \NS(X))$ be a nontrivial correspondence and $b \in X(\Q)$ a choice of basepoint, and recall the divisor $D_Z(b, z)$ from Definition 5.8.

Let $p$ be a good prime that is ordinary for all $f^\sigma$, for $\sigma \in \Gal(E_f/\Q)$. Assume $p$ splits in $E_f$ and let $e$ be a choice of embedding $e : E_f \to \Q_p$. Recall that $\varepsilon$ is the quadratic character associated with the imaginary quadratic field $K$.

Define constants

$$\alpha_\sigma := \frac{\frac{1}{2}(\deg(\phi)L'_{p,\ell}(f^\sigma, 1)\left(1 - \frac{1}{e(\alpha_\rho(f^\sigma))}\right)^{-4}}{L_p(f^\sigma, 1)e\left(\frac{1 - \alpha_\rho(f^\sigma) + p}{p}\right)^{-2}}$$

for $\sigma \in \Gal(E_f/\Q)$. Assume $L(f^\varepsilon, 1) \neq 0$.

The $\alpha_\sigma$ are computable and for $z \in X(\Q)$ we have

$$\sum_{\sigma \in \Gal(E_f/\Q)} \alpha_\sigma (\log f^\sigma dq/q(z))^2 = h(z).$$

Hence

$$\rho(z) = \sum_{\sigma \in \Gal(E_f/\Q)} \alpha_\sigma \log_{f^\sigma dq/q}(z - b) \log_{f^\sigma dq/q}(D_Z(b, z)) - h_{\text{Nek}}^p(z)$$

is a locally analytic function on $X(\Q_p)$ away from $b$. Furthermore, there exists a finite set $S \subset \Q_p$ such that $\{\rho(z) = s : z \in X(\Q_p), s \in S\}$ contains $X(\Q)$.

We give algorithms to compute the numerator and denominator of the constants $\alpha_\sigma$ appearing in Theorem 5.10 in Section 4 and 3 respectively.

**Proof.** Since

$$\{f^\sigma dq/q \text{ for } \sigma \in \Gal(E_f/\Q)\}$$

is a basis for $H^0(X_{\Q_p}, \Omega^1)$ the functions

$$\frac{1}{2}(\log_{f^\sigma dq/q}(D) \log_{f^\tau dq/q}(E) + \log_{f^\sigma dq/q}(E) \log_{f^\tau dq/q}(D)) \text{ for } \sigma, \tau \in \Gal(E_f/\Q)$$

form a basis for the symmetric bilinear pairings on $J_X(\Q) \otimes \Q_p$ by Assumption 5.6 (5).

We will show that for $z \in X(\Q)$, we have the equality $\rho(z) = h_{\text{Nek}}(A_Z(b, z)) - h_{\text{Nek}}^p(z)$, and in particular

$$(5.2.1) \sum_{\sigma \in \Gal(E_f/\Q)} \alpha_\sigma (\log_{f^\sigma dq/q}(z))^2 = h(z).$$

Let $\pi : J_0(N) \to J_X$ be induced by the pushforward of $\phi$. Corollary 4.4 shows that

$$\langle \pi(y_K, f^\sigma), \pi(y_K, f^\tau) \rangle_{\ell_K} = \deg(\phi)L'_{p,\ell}(f^\sigma, 1)\left(1 - \frac{1}{e(\alpha_\rho(f^\sigma))}\right)^{-4}$$
while (3.0.1) shows $L_p(f^\sigma, 1)e\left(\frac{1-a_p(f^\sigma)+p}{p}\right)^{-2}$ is equal to $(\log f^{\sigma}_{d/q} \pi(y_K))^2$.

Corollary 4.5 implies that
\[
\sum_{\sigma \in \text{Gal}(E_f/\mathbb{Q})} \alpha_{\sigma} (\log f^{\sigma}_{d/q} \pi(y_K))^2 = h(\pi(y_K))
\]
where $h : J(\mathbb{Q}) \to \mathbb{Q}_p$ is the global $p$-adic cyclotomic height of Coleman and Gross.

Recall from Section 2.2 that Hecke acts via an order $\mathcal{O}_f$ in $K$ and $\mathcal{O}_f y_K$ generates a finite index subgroup of $J_X(\mathbb{Q})$. Consider the action of $\mathcal{O}_f$ as through the embedding $e : \mathcal{O}_f \to \mathbb{Q}_p$ so that $\mathcal{O}_f \pi(y_K) \subseteq J_X(\mathbb{Q}) \otimes \mathbb{Q}_p$. The bilinearity of the height implies that for all $C_1, C_2 \in \mathbb{Q}_p$,
\[
\langle C_1 \pi(y_K), C_2 \pi(y_K) \rangle_{\mathbb{Q}} = C_1 C_2 \langle \pi(y_K), \pi(y_K) \rangle_{\mathbb{Q}}.
\]

Every $D \in J_X(\mathbb{Q})$ can be written as $C \pi(y_K)$ for some $C \in \mathbb{Q}_p$. Therefore, since the logarithm is linear
\[
\langle \pi(y_K), \pi(y_K) \rangle_{\mathbb{Q}} = \sum_{\sigma \in \text{Gal}(E_f/\mathbb{Q})} \alpha_{\sigma} \log f^{\sigma}_{d/q} (\pi(y_K)) \log f^{\sigma}_{d/q} (\pi(y_K))
\]
implies that
\[
(5.2.2) \quad \langle D, E \rangle_{\mathbb{Q}} = \sum_{\sigma \in \text{Gal}(E_f/\mathbb{Q})} \alpha_{\sigma} \log f^{\sigma}_{d/q} (D) \log f^{\sigma}_{d/q} (E)
\]
for all $D, E \in J_X(\mathbb{Q})$.

Then $\log f^{\sigma}_{d/q}(z)$ has a power series expansion in each residue disk, and is locally analytic on $X(\mathbb{Q})$. We can then extend $h^{\text{Nek}}$ to a locally analytic function on $x \in X(\mathbb{Q}_p)$ away from $b$. By Theorem 5.9, we have the equalities
\[
h^{\text{Nek}}(AZ(b, z)) = h(\pi_1(AZ(b, z)), \pi_2(AZ(b, z))) = h(z-b, DZ(b, z)).
\]
Then $z-b$ and $DZ(b, z)$ can be viewed as elements of $J_X(\mathbb{Q}) \otimes \mathbb{Q}_p$ and therefore $h(z-b, DZ(b, z))$ can be evaluated using (5.2.2), so
\[
h^{\text{Nek}}(AZ(b, z)) = \sum_{\sigma \in \text{Gal}(E_f/\mathbb{Q})} \alpha_{\sigma} \log f^{\sigma}_{d/q} (z-b) \log f^{\sigma}_{d/q} (DZ(b, z)).
\]

For each $v \neq p$, the local height $h^{\text{Nek}}_v(X(\mathbb{Q}_v)) \subseteq S_v \subset \mathbb{Q}_p$ has finite image [KT08] and we define $S := \{s_v : s_v \in S_v\}$. Then
\[
\rho(z) = h^{\text{Nek}}(AZ(b, z)) - h^{\text{Nek}}_p(z) = \sum_{v \neq p} h^{\text{Nek}}_v(z)
\]
and therefore $\{\rho(z) : s : z \in X(\mathbb{Q}_p), s \in S\}$ contains $X(\mathbb{Q})$.

Finally, $h_p(z)$ is the solution to a $p$-adic differential equation and therefore also locally analytic [BDM+19, Lemma 3.7]. Thus $\rho(z)$ is a locally analytic function on $X(\mathbb{Q}_p)$ away from $b$. \hfill $\Box$

**Example 5.11.** We consider the case of $X_0(67)^+$, a genus 2 rank 2 hyperelliptic curve. The rational points for $X_0(67)^+$ were previously determined in [BBB+21], but we give a new approach here. Let $f$ and $f^\sigma$ be the newforms in the orbit 67.2.a.b. Then $E_f = \mathbb{Q}(\nu)$ where $\nu$ has minimal polynomial $z^2 - z - 1$. Let $f$ be the newform with the $q$-expansion
\[
f(q) = q + (-\nu - 1)q^2 + (\nu - 2)q^3 + 3\nu q^4 - 3q^5 + q^6 + O(q^7).
\]
Let $p = 11$ and $D = -7$. We fix the embedding $e : E_f \to \mathbb{Q}_p$ sending $\nu \mapsto 4 + 3 \cdot 11 + 3 \cdot 11^3 + O(11^4)$. Using the methods of Section 3 we find that
\[
\log_{f_{dq/q}}(\pi(y_K))^2 = 3 \cdot 11^2 + 9 \cdot 11^3 + 10 \cdot 11^5 + 4 \cdot 11^6 + 8 \cdot 11^6 + O(11^7),
\]
\[
\log_{f_{\nu, dq/q}}(\pi(y_K))^2 = 11^2 + 11^4 + 11^5 + 9 \cdot 11^6 + 6 \cdot 11^7 + O(11^8).
\]

To specify $\rho(z)$ in terms of a basis of symmetric bilinear forms on $J(\mathbb{Q}) \otimes \mathbb{Q}_p$, we need to relate the basis back to the coordinates of $X$. Let $g_1$ and $g_2$ be a basis of modular forms for $S_2(67)_{\text{new}}$ given by
\[
g_1(q) := q - 3q^3 - 3q^4 - 3q^5 + q^6 + 4q^7 + 3q^8 + 5q^9 - O(q^{10})
g_2(q) := q^2 - q^3 - 3q^4 + 3q^7 + 4q^8 + 3q^9 - O(q^{10}).
\]

We can construct a model $X$ of $X_0(67)^+$ over $\mathbb{Q}$ where under the identification $H^0(X, \Omega^1) \simeq S_2(67)_{\text{new}}$, the differential $dx/y$ is $g_1$ and $xdx/y$ is $g_2$ by letting $x = g_2/g_1$ and $y = qdx/g_1$ and solving for the linear dependence in the monomials $1, x, x^2, \ldots, x^6, y^2$. The resulting model is
\[
X : y^2 = h(x) = 9x^6 - 14x^5 + 9x^4 - 6x^3 + 6x^2 - 4x + 1.
\]

Then since $f = g_1 - (\nu + 1)g_2$, we have
\[
f_{dq/q} = dx/y - (\nu + 1)xdx/y.
\]

Let $Z$ be the trace zero correspondence associated to $T_p$. We fix an arbitrary choice of basepoint $b = [1 : 1 : 2]$. Write $A := 1$ and $B := -\nu - 1$.

Using the techniques in Section 4 we get that
\[
\langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{\ell_K} = 4 \cdot 11^2 + 3 \cdot 11^3 + 9 \cdot 11^4 + 11^5 + 3 \cdot 11^6 + 2 \cdot 11^7 + O(11^8)
\]
\[
\langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{\ell_K} = 10 \cdot 11^2 + 3 \cdot 11^3 + 8 \cdot 11^4 + 10 \cdot 11^5 + 7 \cdot 11^6 + 9 \cdot 11^7 + O(11^8)
\]

Let
\[
\alpha_1 := \frac{\langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{\ell_K}}{(\log_{f_{dq/q}}(\pi(y_K))^2)} \quad \text{and} \quad \alpha_2 := \frac{\langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{\nu}}{(\log_{f_{\nu, dq/q}}(\pi(y_K))^2)}.
\]

Then using linearity of the logarithm and (5.2.4), by setting $\alpha_{00} = \alpha_1 A^2 + \alpha_2 A^{\sigma_2}$, $\alpha_{01} = 2(\alpha_1 AB + \alpha_2 A^\sigma B^\sigma)$ and $\alpha_{11} = \alpha_1 B^2 + \alpha_2 B^{2\sigma}$ we obtain the relation
\[
\langle D, E \rangle_{\ell_Q} = \alpha_{00} \log_{dx/y}(D) \log_{dx/y}(E) + \alpha_{01} \frac{1}{2} (\log_{dx/y}(D) \log_{dx/\!y}(E) + \log_{\omega_1}(D) \log_{dx/y}(E)) + \alpha_{11} \log_{dx/\!y}(D) \log_{dx/\!y}(E).
\]

We use the basis $dx/y, xdx/y$ for convenience: this is the default basis in the code [BDM+].

Let $\rho = h_{\text{Nek}} - h_{\text{p,Nek}}$. We can construct $\rho$ as a locally analytic function and solve for $\rho = 0$ using the code [BDM+]: the input to this construction is the coefficients $\alpha_{ij}$. From this, the code writes $h_{\text{Nek}}$ as a locally analytic function by expressing $\pi_i(A_Z(b, x))$ in terms of a dual basis. It also writes $h_{p,\text{Nek}}$ as a locally analytic function. This process recovers the points found in [BBB+21, Table 1].

Galbraith [Gal96] computed the Heegner point for some Atkin–Lehner quotients of modular curves; his computations show $\pi(y_K) = [1 : -1 : 1] - [0 : 1 : 1]$ on the model (5.2.3). Using this, and forthcoming work of Gajović for computing local Coleman–Gross heights on even degree hyperelliptic curves we verified the above logarithm and height calculations.
Example 5.12. Let $f$ and $f^\sigma$ be the newforms in the orbit $85.2.a.b$ defined over $E_f = \mathbb{Q}(\sqrt{2})$. Let
\[ f = q + (\sqrt{2} - 1)q^2 + (-\sqrt{2} - 2)q^3 + (-2\sqrt{2} + 1)q^4 - q^5 - \sqrt{2}q^6 + O(q^7). \]
We finish Example 3.19 by studying the height the Heegner point $X^*_0(85)$. The rational points of $X^*_0(85)$ were determined by [BGX21].

Let $p = 7$ and $D = -19$. In this example, we let $Z$ be the trace zero correspondence associated with $T_p$, and $b = [2 : 38 : 5]$ on the model below. Recall we have fixed a $p$-adic embedding $e : E_f \to \mathbb{Q}_p$ by $\sqrt{2} \mapsto 3 + 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + O(7^4)$. Using the methods described in Section 4 and already exhibited in the previous examples, we find
\[ \langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{\epsilon_K} = 3 \cdot 7^{-3} + 7^{-2} + 2 + 3 \cdot 7 + 3 \cdot 7^2 + 7^3 + 7^4 + 5 \cdot 7^5 + 7^6 + O(7^7) \]
\[ \langle \pi(y_{K,f^\sigma}), \pi(y_{K,f^\sigma}) \rangle_{\epsilon_K} = 3 \cdot 7 + 6 \cdot 7^2 + 4 \cdot 5 \cdot 7 + 4 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + 6 \cdot 7^6 + 5 \cdot 7^7 + O(7^8). \]

Note that in this case the degree of the quotient $X_0(85) \to X^*_0(85)$ is 4.

Picking a basis of newforms $g_1 = q^2 - q^3 + \cdots$, $g_2 = q - 3q^3 + \cdots$ with rational coefficients for the weight 2 and level 85 space of newforms with Atkin–Lehner signs equal to $+1$, we can construct a rational model for $X_0(85)^*$ with $g_1 dq/q = dx/y$ and $g_2 = xdx/y$. The model is
\[ y^2 = x^6 - 4x^5 + 12x^4 - 22x^3 + 32x^2 - 40x + 25. \]
Furthermore we have the relationship $(\sqrt{2} - 1)dx/y + xdx/y = f dq/q$. On this model, $(\log_{f dq/q}([1 : -1 : 0] - [1 : 1 : 0])^2$ agrees with $(\log_{f dq/q} \pi(y_K))^2$. Furthermore, since $[2 : 5 : 1] - [2 : -5 : 1]$ is linearly equivalent to twice $[1 : -1 : 0] - [1 : 1 : 0]$, we can compute the global height of this divisor using forthcoming work of Gajović for computing local Coleman–Gross heights on even degree hyperelliptic curves:
\[ h([1 : -1 : 0] - [1 : 1 : 0]) = \frac{1}{2} h_p([1 : -1 : 0] - [1 : 1 : 0], [2 : 5 : 1] - [2 : -5 : 1]) \]
\[ = 5 \cdot 7^{-3} + 1 + 4 \cdot 7 + 6 \cdot 7^3 + 7^4 + 6 \cdot 7^5 + 3 \cdot 7^6 + 3 \cdot 7^7 + 3 \cdot 7^8 + 5 \cdot 7^9 + O(7^{10}). \]

In this case, there are local height contributions away from $p$, which we did not compute.

Example 5.13. Let $f$ and $f^\sigma$ be the newforms in the newform orbit $107.2.a.a$ defined over $E_f = \mathbb{Q}(\nu)$ where $\nu$ satisfies the polynomial $z^2 - z - 1$. Let
\[ (5.2.5) \quad f = q - \nu q^2 + (\nu - 2)q^3 + (\nu - 1)q^4 + (\nu - 2)q^5 + (\nu - 1)q^6 + O(q^7). \]
We consider the curve $X_0(107)^+$. The rational points of this curve were determined [BDM+21, Example 5.3]. In this case, there were not sufficiently many points on the curve to determine the height pairing, but one can use a pair of independent infinite order points on the Jacobian to set the Coleman–Gross height. We offer an alternative strategy.

Let $p = 11$ and let $K$ be the imaginary quadratic field with discriminant $D = -7$. Let $e : E_f \to \mathbb{Q}_p$ sending $\nu \mapsto 4 + 3 \cdot 11 + 3 \cdot 11^3 + O(11^4)$. Again, by picking a basis of newforms $j_1 = q - 2q^3 - q^4 + \cdots$, $j_2 = q^2 - q^3 + \cdots$ with rational coefficients for the weight 2 and level 107 space of newforms with Atkin–Lehner sign $+1$, we can construct a rational model for $X_0(107)^+$ with $j_1 dq/q = dx/y$ and $j_2 = xdx/y$. Our model is
\[ y^2 = x^6 - 10x^5 + 17x^4 - 18x^3 + 10x^2 - 4x + 1. \]
On this model \( dx/y - \nu xdx/y = fdq/q \). Let \( Z \) be the trace zero correspondence associated to \( T_p \), and \( b = [1 : 1 : 2] \). We can find a finite index subgroup of the Mordell–Weil group generated by the classes of \( Q_1 := [0 : 1 : 1] - [0 : -1 : 1] \) and \( Q_2 = [1/2 : -1/8 : 1] - [0 : -1 : 1] \). The logarithms \( L_{Q_i} := \log_{fdq/q} Q_i \) (under the embedding \( e \)) are

\[
L_{Q_1} = 3 \cdot 11 + 4 \cdot 11^2 + 2 \cdot 11^3 + 9 \cdot 11^4 + 10 \cdot 11^5 + 7 \cdot 11^8 + 4 \cdot 11^9 + O(11^{10})
\]

\[
L_{Q_2} = 2 \cdot 11 + 8 \cdot 11^2 + 7 \cdot 11^4 + 4 \cdot 11^5 + 6 \cdot 11^6 + 3 \cdot 11^7 + 3 \cdot 11^8 + O(11^{10}).
\]

We also can compute the logarithm of the Heegner point using the techniques described in Section 3. The values of \( \ell(r) \) are given in Table 4A.

| \( r \) | \( \ell(r) \mod p^5 \) | \( r \) | \( \ell(r) \mod p^5 \)
|-----|------------------|-----|------------------|
| 10  | -22250          | 10  | 39142          |
| 20  | -17899          | 20  | 70280          |
| 30  | -70252          | 30  | 39031          |
| 40  | 28890           | 40  | -40900        |
| 50  | 56376           | 50  | 49703          |

(A) \( \ell(r) \) for \( f \)    (B) \( \ell(r) \) for \( f^\sigma \)

Table 4. Computations for \( f \) in 107.2.a.a

Then \( L_p(f, 1) = -50731 + O(11^5) \) and the logarithm of the Heegner point is

\[
(\log_{fdq/q} \pi(y_K))^2 = 4 \cdot 11^2 + 8 \cdot 11^4 + 2 \cdot 11^6 + O(11^7).
\]

Then we can check that when \( A = \pm 2 \) and \( B = \pm 1 \) we have the relation

\[
A^2(L_{Q_1}/2)^2 + 2AB(L_{Q_1}/2)(L_{Q_2}/2) + B^2(L_{Q_2}/2)^2 = (\log_{fdq/q} \pi(y_K))^2.
\]

The division by two occurs because the \( Q_i \) are not 2-saturated in the Mordell–Weil group. For the conjugate modular form \( f^\sigma dq/q \) we have the values given in Table 4B and so

\[
L_p(f^\sigma, 1) = 37471 + O(11^5)
\]

and therefore

\[
(\log_{f^\sigma dq/q} \pi(y_K))^2 = 3 \cdot 11^2 + 4 \cdot 11^3 + 2 \cdot 11^5 + 10 \cdot 11^6 + O(11^7).
\]

We finish the example by computing the heights of \( \pi(y_{K,f}) \). To do this we fix the complex embedding \( e_c : E_f \to \mathbb{C} \) by \( \nu \mapsto 1.61803 \) First, using [Col18] we can compute

\[
\Omega_f = 42.114698 \quad \Omega_{f^\sigma} = 51.071742.
\]

In Sage, we can also compute

\[
\frac{d}{dT} L_{p,MTT}(f, T) \bigg|_{T=0} = 4 + 7 \cdot 11 + 7 \cdot 11^2 + 7 \cdot 11^3 + 8 \cdot 11^4 + 11^5 + 3 \cdot 11^6 + O(11^7)
\]

\[
\frac{d}{dT} L_{p,MTT}(f^\sigma, T) \bigg|_{T=0} = 6 + 11 + 6 \cdot 11^2 + 9 \cdot 11^3 + 7 \cdot 11^5 + 7 \cdot 11^6 + O(11^7).
\]
Let $\chi$ be the quadratic character with $D = 5$ and recall that $\varepsilon$ denotes the quadratic character associated with $K$. Following Algorithm 4.8 we find that the periods are $\Omega_1 = L(f^x, 1)/(-4/\sqrt{5})$ and $\Omega_1^+ = L(f^\sigma x, 1)/((-4)/\sqrt{5})$. We compute
\[
L(f^x, 1) = 3.948128 \quad L(f^\sigma x, 1) = 2.407825 \\
L(f^e, 1) = 0.996703 \quad L(f^\sigma e, 1) = 5.188648.
\]
Combining the complex values, we find
\[
(5.2.6) \quad \frac{\Omega_1^+ L(f^e, 1)\sqrt{|D|}}{\Omega_1} = -0.138196 \quad \frac{\Omega_1^+ L(f^\sigma e, 1)\sqrt{|D|}}{\Omega_1} = -0.361803.
\]
Let $r_1 := 1/10\nu - 3/10$ and $r_2 := -1/10\nu - 1/5$ be the roots of the polynomial $20x^2 + 10x + 1$. The complex values of (5.2.6) computed to 30 digits are within $10^{-28}$ of the algebraic numbers $e_c(r_1)$ and $e_c(r_2)$. Under the assumption that the values of (5.2.6) are $e_c(r_1)$ and $e_c(r_2)$, we have
\[
\langle \pi(y_{K,f}), \pi(y_{K,f}) \rangle_{K} = 8 \cdot 11 + 4 \cdot 11^2 + 10 \cdot 11^3 + 7 \cdot 11^4 + 4 \cdot 11^5 + 8 \cdot 11^6 + O(11^7).
\]
\[
\langle \pi(y_{K,f^\sigma}), \pi(y_{K,f^\sigma}) \rangle_{K} = 6 \cdot 11 + 11^2 + 2 \cdot 11^3 + 2 \cdot 11^5 + 2 \cdot 11^6 + O(11^7).
\]
We ran the quadratic Chabauty code [BDM+] using the resulting $\rho(z)$ from Theorem 5.10 and recovered a finite superset of the rational points on $X_0(107)^+$. 

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