THE LOWER BOUND ON THE HK MULTIPLICITIES OF QUADRIC HYPERSURFACES

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Abstract. Here we prove that the Hilbert-Kunz multiplicity of a quadric hypersurface of dimension $d$ and odd characteristic $p \geq 2d - 4$ is bounded below by $1 + m_d$, where $m_d$ is the $d^{th}$ coefficient in the expansion of $\sec + \tan$. This proves a part of the long standing conjecture of Watanabe-Yoshida. We also give an upper bound on the HK multiplicity of such a hypersurface.

We approach the question using the HK density function and the classification of ACM bundles on the smooth quadrics via matrix factorizations.

1. Introduction

Let $R$ be a Noetherian ring containing a field of characteristic $p > 0$ and $I$ be an ideal of finite colength in $R$. For such a pair Monsky ([M]) had introduced a characteristic $p$ invariant known as the Hilbert-Kunz (HK) multiplicity $e_{HK}(R, I)$. This is a positive real number ($\geq 1$) given by

$$e_{HK}(R, I) = \lim_{n \to \infty} \frac{\ell(R/I[q])}{q^d}. $$

If $(R, m, k)$ is a formally unmixed Noetherian local ring then it was proved by Watanabe-Yoshida (Theorem 1.5 in [WY1]) that $e_{HK}(R, m) = 1$ if and only if $R$ is regular. For the next best class of rings, namely quadric hypersurfaces they made the following (Conjecture 4.2 in [WY2])

**Conjecture [WY]** Let $p > 2$ be prime and $K = \overline{\mathbb{F}_p}$ and let $(R, m_R, K)$ be a formally unmixed nonregular local ring of dimension $n + 1$. Then

$$e_{HK}(R, m_R) \geq e_{HK}(R_{p,n+1}, m) \geq 1 + m_{n+1}. $$

Here $R_{p,n+1} = K[x_0, \ldots, x_{n+1}]/(x_0^2 + \cdots + x_{n+1}^2)$ and $m_{n+1}$ are the constants occuring as the coefficients of the following expression

$$\sec(x) + \tan(x) = 1 + \sum_{n=0}^{\infty} m_{n+1} x^{n+1}, \text{ where } |x| < \pi/2. $$

In the same paper ([WY2]) they showed that the conjecture holds for $n \leq 3$. The second inequality of the conjecture for $n \leq 5$ was proved by Yoshida in [Y]. Later the conjecture up to $n \leq 5$ was proved by Aberbach-Enescu in [AE2].

In the context of this conjecture, we recall the following result (around 2010) of Gessel-Monsky:

$$\lim_{p \to \infty} e_{HK}(R_{p,n+1}, m) = 1 + m_{n+1}. $$

In higher dimensional cases for the class of local formally unmixed nonregular rings of fixed dimension $d$, various people ([AE1], [AE2], Celikbas-Dao-Huneke-Zhang in [CDHZ]) have given a lower bound $C(d) (> 1)$ on the HK multiplicity $e_{HK}(R, m)$. 
However such lower bounds $C(d)$ are weaker than the bound given in the above conjecture as implied by the above result of Gessel-Monsky.

Enescu and Shimomoto in [ES] have proved the first inequality $e_{HK}(R) \geq e_{HK}(R_{p,n+1})$, where $R$ belongs to the class of complete intersection local rings.

The conjecture [WY] and related problems have been revisited in the recent paper [JNSWY].

Here we focus on the second inequality of the above mentioned conjecture and prove the following in Section 4.

**Theorem 1.1.** Let $p \neq 2$ and let $p \geq n - 2$ for $n$ even and let $p \geq 2n - 4$ for $n$ odd. Then
\[
1 + m_{n+1} + \left( \frac{2n-4}{p} \right) \geq e_{HK}(R_{p,n+1}, m) \geq 1 + m_{n+1}.
\]

We approach the invariant by considering the Hilbert-Kunz (HK) density functions for the pair $(R_{p,n+1}, m)$, where $k$ is a perfect field of characteristic $p > 0$. The notion of HK density function for $(R, I)$, where $R$ is a $\mathbb{N}$-graded ring and $I$ is a homogeneous ideal in $R$ of finite colength, was introduced by the author ([T]) for standard graded rings and later generalized by the author and Watanabe ([TW2]) for $\mathbb{N}$-graded rings.

We recall that the HK density function is a compactly supported continuous function $f_{R,I} : [0, \infty] \rightarrow [0, \infty)$ defined as
\[
f_{R,I}(x) = \lim_{s \to \infty} \ell((R/I^q)|_{xq}), \quad \text{where} \quad q = p^s
\]

and
\[
e_{HK}(R, I) = \int_0^\infty f_{R,I}(x) dx.
\]

To prove Theorem 1.1 we prove a stronger result about char $p$ vis-a-vis char 0 (in Section 4):

**Theorem 1.2.** The function $f_{R_{n+1}^\infty} : [0, \infty] \rightarrow [0, \infty)$ given by
\[
f_{R_{n+1}^\infty}(x) := \lim_{p \to \infty} f_{R_{p,n+1}, m}(x)
\]
is a well defined continuous function such that $\int_0^\infty f_{R_{n+1}^\infty} = 1 + m_{n+1}$.

Moreover, if $p \geq 2n - 4$ and $n$ is odd, or $p \geq n - 2$ and $n$ even then
\[
f_{R_{p,n+1}, m}(x) = f_{R_{n+1}^\infty}(x) \quad x \in [0, \frac{n+2}{2} - \frac{n-2}{2p}]
\]
\[
\geq f_{R_{n+1}^\infty}(x) \quad x \in [\frac{n+2}{2} - \frac{n-2}{2p}, \frac{n+2}{2} + \frac{n-2}{2p}]
\]
\[
= f_{R_{n+1}^\infty}(x) \quad x \in [\frac{n+2}{2} + \frac{n-2}{2p}, \infty).
\]

Note that for $n = 1$ and $n = 2$ the ring $R_{p,n+1}$ is the homogeneous coordinate ring of $\mathbb{P}_k^1$ and $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ respectively. In both the cases the invariants $e_{HK}(R_{p,n+1})$ and $f_{R_{p,n+1}, m}$ are independent of the characteristic (see Eto-Yoshida [EY] and [T]). Hence we can assume $n \geq 3$.

Here given $n$ we explicitly write the function $f_{R_{n+1}^\infty}$ in Theorem 1.1 by first writing the function $f_{R_{p,n+1}, m}(x)$ for $x \in [0, \infty) \setminus [\frac{n+2}{2} - \frac{n-2}{2p}, \frac{n+2}{2} + \frac{n-2}{2p}]$. Hence we have a computation of $F$-thresholds as a (see Corollary 1.5)

**Corollary** The $F$-thresholds $e^m(m)$ are $n$ for $R_{p,n+1}$ defined over a perfect field of characteristic $p \neq 2$, where $p \geq 2n - 4$ and $n$ is odd, or $p \geq n - 2$ and $n$ even.
Theorem 1.1 and the result of [ES] prove the Conjecture [WY] for the class of complete local rings (for large $p$):

Let $p \neq 2$ and let $p \geq n - 2$ for $n$ even and let $p \geq 2n - 4$ for $n$ odd. Let $(R, m_R, K)$ be a formally unmixed nonregular local ring of dimension $n + 1$. Then $R$ is a complete intersection ring implies

$$e_{HK}(R, m_R) \geq e_{HK}(R_{p,n+1}, m) \geq 1 + m_{n+1}.$$  

We go about computing the HK density function as follows. Recall that there exists the complete classification of indecomposable Arithmetically Cohen-Macaulay (ACM) bundles (due to Buchweitz-Eisenbud-Herzog [BEH]) on smooth quadrics $Q_n = \text{Proj } R_{p,n+1}$ in terms of line bundles $\mathcal{O}(t) = \mathcal{O}_{Q_n}(t)$ and twisted spinor bundles $S(t)$ (see Section 2). Since $F_{s}^{*}(\mathcal{O}(a))$ and $F_{s}^{*}(\mathcal{S}(a))$ are ACM bundles on $Q_n$, for every $s^{th}$ iterated Frobenius map $F_{s}^{*} : Q_n \to Q_n$ we have

$$F_{s}^{*}(\mathcal{O}(a)) = \bigoplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\nu^{s}(t,a)} \bigoplus \bigoplus_{t \in \mathbb{Z}} \mathcal{S}(t)^{\nu^{s}(t,a)}$$

and

$$F_{s}^{*}(\mathcal{S}(a)) = \bigoplus_{t \in \mathbb{Z}} \mathcal{O}(t)^{\varphi^{s}(t,a)} \bigoplus \bigoplus_{t \in \mathbb{Z}} \mathcal{S}(t)^{\varphi^{s}(t,a)},$$

Achinger in [A] showed that the ranks of the bundles $\mathcal{O}(t)$ and $\mathcal{S}(t)$ are related to the graded components of the ring $R_{p,n+1}/m^{[q]}$ by the formula

$$(1.1) \quad \ell(R_{p,n+1}/m^{[q]})_{a} = \nu^{s}(0, a) + 2\lambda_{0} \mu^{s}(1, a),$$

where $m = (x_{0}, \ldots, x_{n+1})$. This at once implies that to compute $f_{R_{p,n+1}}$ it is enough to compute all the pairs

$$\nu^{s}(t, a) + 2\lambda_{0} \mu^{s}(t + 1, a), \quad \text{for } q = p^{s} \gg 0, \quad \text{where } t \in \mathbb{Z}, \quad \text{and } \quad 0 \leq a < q.$$  

Now we use another result (Theorem 2 in [A]) which determines, in terms of $q = p^{s}$, $a$ and $n$, the occurrence of the bundle $\mathcal{O}(t)$ or $\mathcal{S}(t)$ in the decomposition of $F_{s}^{*}(\mathcal{O}(a))$ and $F_{s}^{*}(\mathcal{S}(a))$.

The layout of the paper is as follows:

In Section 2 we recall the known results.

In Section 3 we prove that the pairs are computable if the decomposition of $F_{s}^{*}(\mathcal{O}(a))$ has only one type of spinor bundles. However this is not always the case, as the existence of only one type of spinor bundle would imply that the HK density function $f_{R_{p,n+1}}$ and therefore the HK multiplicity $e_{HK}(R_{p,n+1})$ are independent of the characteristic $p$. However, for large enough $p$ one can ensure that there are at the most two types of spinor bundles, as observed in Lemma 3.2.

We analyse the difficult range in the interval $[0, 1)$, with the property that if $a/q$ is outside this range, then the bundle $F_{s}^{*}(\mathcal{O}(a))$ has atmost one type of spinor bundle. In particular every pair $\{\nu^{s}(t, a) + 2\lambda_{0} \mu^{s}(t + 1, a)\}_{t}$ is computable provided $a/q$ avoids this range.

Notably this range keeps shrinking as $p \to \infty$. We use this observation in Section 4 to explicitly write down the HK density function everywhere except on the range (as in Theorem 1.2) and also get a closed formula for the function $f_{R_{p,n+1}}$.

On this range too the HK density function $R_{p,n+1}$ can be computed as suggested by the Lemma 3.2 and the computation done in Section 5 for $n = 3$ case. However the expression will get more complicated as the case $n = 3$ shows; here the function $f_{R_{p,4}}$ is a piecewise polynomial and, on the range $[2 + (p - 1)/2p, 2 + (p + 1)/p)$, it is given by infinitely many polynomial functions, defined using a nested sequence of intervals.
Looking further, this suggests possible computations for the HK density and related invariants in other situations, where we have information on ACM bundles using matrix factorizations.

2. PRELIMINARIES

In this section we recall the relevant results which are known in the literature.

Definition 2.1. A vector bundle $E$ on a smooth $n$-dimensional hypersurface $X = \text{Proj } S/(f)$, where $S = k[x_0, \ldots, x_{n+1}]$ is called arithmetically Cohen-Macaulay (ACM) if $H^i(X, E(m)) = 0$, for $0 < i < n$ and for all $m$.

It is easy to check that a vector bundle $E$ on $X$ is ACM if and only if the corresponding graded $S/(f)$ module is maximal Cohen-Macaulay (MCM).

Let $Q_n = \text{Proj } S/(f)$ be the quadric given by the hypersurface $x_0^2 + \cdots + x_{n+1}^2 = 0$ in $\mathbb{P}^{n+1}_k = \text{Proj } S$, where $n \geq 3$. Let $k$ be an algebraically closed field. Henceforth we assume $n > 2$.

By B-E-H classification ([BEH]) of indecomposable graded MCM modules over quadrics we have: Other than free modules on $\mathbb{P}^n$, Section 2.2) gives an exact sequence of locally free sheaves on $\mathbb{P}^n$ (such an equivalence is given by Eisenbud in [E], for more general hypersurfaces $f$), which is a pair $(\phi, \psi)$ of square matrices of polynomials, of the same size, such that $\phi \cdot \psi = f \cdot \text{id} = \psi \cdot \phi$ and the MCM module is the cokernel of $\phi$.

Now the matrix factorization $(\phi_n, \psi_n)$ for indecomposable bundles on $Q_n$ (see Langer [L], Section 2.2) gives an exact sequence of locally free sheaves on $\mathbb{P}^{n+1}$

\[
(2.1) \quad 0 \to \mathcal{O}_{\mathbb{P}^n_k}(-2)^{2(n/2)+1} \xrightarrow{\Phi_n} \mathcal{O}_{\mathbb{P}^n_k}(-1)^{2(n/2)+1} \to i_* S \to 0,
\]

$S = \Sigma$ and $\Phi = \phi_n = \psi_n$ for $n$ odd and $S = \Sigma_+ \oplus \Sigma_-$ and $\Phi_n = \phi_n \oplus \psi_n$ for $n$ even. Since $S$ is supported on $Q_n$ it is sheaf on $Q_n$. Moreover the above description gives the short exact sequences of vector bundles on $Q_n$: If $n$ odd then

\[
0 \to S \to \mathcal{O}_{Q_n}^{2(n/2)+1} \to S(1) \to 0.
\]

If $n$ is even then

\[
0 \to \Sigma_- \to \mathcal{O}_{Q_n}^{2(n/2)} \to \Sigma_+(1) \to 0
\]

and

\[
0 \to \Sigma_+ \to \mathcal{O}_{Q_n}^{2(n/2)} \to \Sigma_-(1) \to 0.
\]

We also have the natural exact sequence

\[
(2.2) \quad 0 \to \mathcal{O}_{\mathbb{P}^n_k}(-2) \to \mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{Q_n} \to 0.
\]

We denote

\[
R_{p,n+1} = \frac{k[x_0, \ldots, x_{n+1}]}{(x_0^2 + \cdots + x_{n+1}^2)} = \oplus_{m \geq 0} H^0(Q_n, \mathcal{O}_{Q_n}(m)) \quad \text{and} \quad n \geq 3,
\]

where $k$ is a field of characteristic $p > 2$. In particular the $m^{th}$ graded component of $R_{p,n+1}$ is $H^0(Q_n, \mathcal{O}_{Q_n}(m))$. We will be using the following set of equalities in our forthcoming computations.
\( \ell(R_{p,n+1})_m = h^0(Q_n, \mathcal{O}_{Q_n}(m)) = h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{\mathbb{P}^{n+1}}(m)) - h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{\mathbb{P}^{n+1}}(m-2)), \)

\( h^0(Q_n, S(m)) = 2\lambda_0 \left[ h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{\mathbb{P}^{n+1}}(m-1)) - h^0(\mathbb{P}^{n+1}_k, \mathcal{O}_{\mathbb{P}^{n+1}}(m-2)) \right], \)

where \( 2\lambda_0 = 2^{\lfloor n/2 \rfloor} + 1. \)

By Serre duality, \( (\omega Q_n = \mathcal{O}_{Q_n}(-n)) \) and \( S^\vee = S(1) \)

\( h^n(Q_n, O(m)) = h^0(Q_n, O(-m-n)) \) and \( h^n(Q_n, S(m)) = h^0(Q_n, S(1-m-n)). \)

The rank of \( Q_n \)-bundle \( S = \lambda_0. \)

Now we recall other relevant facts from \([A]\). We also use the following result of Achinger (Theorem 2 in \([A]\)) which determines, in particular

\( \nu_s(0, a) = 2\lambda_0\mu_s(1, a), \)

where \( \mathbf{m} = (x_0, \ldots, x_{n+1}). \)

We use this observation of \([A]\), for the computation of the HK density function \( f_{R_{p,n+1}, \mathbf{m}}. \) Note that for any integer \( m \geq 0, \) there is an integer \( i \geq 0 \) such that \( iq \leq m < (i+1)q. \) Hence by the projection formula

\( F^*_s(O(m)) = F^*_s(O(m - iq) \otimes F^{ss}(O(iq))) = F^*_s(O(m - iq)) \otimes O(i). \)

In particular

\( \ell(R_{p,n+1}/\mathbf{m}^{[q]})_m = \nu^s(-i, m - iq) + 2\lambda_0\mu^s(-i + 1, m - iq). \)

Therefore to know the HK density function \( f_{R_{p,n+1}, \mathbf{m}} \), it is enough to compute the pair \( \nu^s(-i, a) + 2\lambda_0\mu^s(-i + 1, a), \) for all \( i \) and for \( 0 \leq a < q. \)

We also use the following result of Alekseev (Theorem 2 in \([A]\)) which determines, in terms of \( s, a \) and \( n, \) when the numbers \( \nu^s(i, a) \) and \( \mu^s(i, a) \) are nonzero in the decomposition of \( F^*_s(O(a)). \) Langer in \([L]\) has given such formula for the occurrence of line bundles in the Frobenius direct image.
Theorem [A]. Let $p \neq 2$, $s \geq 1$ and $n \geq 3$ and
\[ F_s^*(O(a)) = \oplus_{t \in \mathbb{Z}} O(t)^{\nu^*(t,a)} \oplus \oplus_{t \in \mathbb{Z}} S(t)^{\mu^*(t,a)}. \]

Then
(1) $F_s^*(O(a))$ contains $O(t)$ if and only if $0 \leq a - tq \leq n(q-1)$.
(2) $F_s^*(O(a))$ contains $S(t)$ if and only if
\[ \left( \frac{(n-2)(p-1)}{2} \right) \frac{q}{p} \leq a - tq \leq \left( \left( \frac{(n-2)(p-1)}{2} + n - 2 + p \right) \frac{q}{p} - n. \]
(3) $F_s^*(S(a))$ contains $O(t)$ if and only if $1 \leq a - tq \leq n(q-1)$.
(4) $F_s^*(S(a))$ contains $S(t)$ if and only if
\[ \left( \frac{(n-2)(p-1)}{2} \right) \frac{q}{p} + 1 - \delta_{s,1} \leq a - tq \leq \left( \left( \frac{(n-2)(p-1)}{2} + n - 2 + p \right) \frac{q}{p} - n + \delta_{s,1}. \]

3. Formula for the pairs $\nu^*(i,a) + 2\lambda_0 \mu^*(i+1,a)$

In the rest of the paper
\[ R_{p,n+1} = \frac{k[x_0, \ldots, x_{n+1}]}{(x_0^2 + \cdots + x_{n+1}^2)} \quad \text{and} \quad Q_n = \text{Proj } R_{p,n+1}, \]
where $n \geq 3$ and $k$ is a perfect field of characteristic $p > 2$.

Notations 3.1.
(1) $\nu^*(-t,a) = \nu^*_t(a)$ and $\mu^*(-t,a) = \mu^*_t(a)$, where
\[ F_s^*(S(a)) = \oplus_{t \in \mathbb{Z}} O(t)^{\tilde{\nu}^*(t,a)} \oplus \oplus_{t \in \mathbb{Z}} S(t)^{\tilde{\mu}^*(t,a)}. \]
(2) We also denote $O_{Q_n}(m)$ by $O(m)$.
(3) $h^0(Q_n, O(m)) = Y_m$ and $h^0(\mathbb{P}^{n+1}, O_{\mathbb{P}^{n+1}}(m)) = X_m$.
(4) Let
\[ n_0 = \left\lfloor \frac{(n-2)(p-1)}{2p} \right\rfloor \quad \text{and} \quad \Delta = n_0 - \frac{(n-2)(p-1)}{2p}. \]

Now
\[ n \text{ even } \implies \Delta = \frac{n-2}{2p} \quad \text{and} \quad n \text{ odd } \implies \Delta = \frac{n-2}{2p} + \frac{1}{2}. \]

(5) A spinor bundle is of type $t$ if it is isomorphic to $S(t)$. We say two spinor bundles $S(t)$ and $S(t')$ are of the same type if $t = t'$.

(6) An invariant such as $\nu_t^*(a)$ is computable if there exists a polynomial $F_t(X,Y) \in \mathbb{Q}[X,Y]$ of degree $\leq n$ such that $\nu_t^*(a) = F_t(p^*, a)$ (similarly for $\mu_t^*(a)$).

In the first lemma we prove that for sufficiently large $p$ (compare to $n$) there are almost three types of spinor bundles in the decomposition of $F_s^*(a)$, for any $0 \leq a < q$. Moreover for a fixed such an integer $a$, there are almost two types of spinor bundles.

However one can not do better than this, because if the decomposition of $F_s^*(O(a))$ contains only one type of the spinor bundle then all the pairs $\nu_t^*(a) + 2\lambda_0 \mu_t^{*+1}(a)$ are computable as will be shown in Lemma 3.6. But then the HK density function $f_{R_{p,n+1}}$ and therefore $e_{HK}(R_{p,n+1})$ will be independent of characteristic $p$, which is a contradiction due to the examples of [WY2].

Lemma 3.2. If $0 \leq a < q = p^*$ and $p > 2$ then
(1) $F_s^*(O(a))$ contains $O(t)$ if and only if $t \in \{0, -1, \ldots, -n+1\}$. Moreover
(2) \(F^s_*(\mathcal{O}(a))\) contains \(S(t)\) implies

\[
    t \in \left\{ -(n_0 - 1), -n_0, -(n_0 + 1), \ldots, \left( n_0 + \left\lfloor \frac{n - 2}{p} \right\rfloor \right) \right\}.
\]

In particular, if \(n - 2 \leq p\) then \(n_0 = \lfloor n/2 \rfloor - 1\) and \(t \in \{ -n_0 - 1, -n_0, -n_0 + 1 \} \).

Moreover,

(a) if \(n\) is even then

(i) \(\mu^s_{-n_0-1}(a) \neq 0 \implies 0 \leq a/q < \frac{n-2}{2p}\)

(ii) \(\mu^s_{-n_0}(a) \neq 0 \implies 0 \leq a/q < 1\)

(iii) \(\mu^s_{-n_0+1}(a) \neq 0 \implies 1 - \frac{n-2}{2p} \leq a/q\).

(b) If \(n\) is odd then

(i) \(\mu^s_{-n_0-1}(a) = 0\)

(ii) \(\mu^s_{-n_0}(a) \neq 0 \implies 0 \leq a/q < \frac{1}{2} + \frac{n-2}{2p}\)

(iii) \(\mu^s_{-n_0+1}(a) \neq 0 \implies \frac{1}{2} - \frac{n-2}{2p} \leq a/q\).

Proof. The assertion (1) is just restating the assertion (1) of \([A]\).

By the assertion (2) of \([A]\), if \(S(t)\) occurs in \(F^s_*(\mathcal{O}(a))\) then

\[
    (n_0 - \Delta)q \leq a - tq \leq (n_0 - \Delta)q + (n - 2)q/p + q - n.
\]

\[
    (n_0 - \Delta) \leq a/q - t \leq (n_0 - \Delta) + (n - 2)/p + 1 - n/q.
\]

Hence

\[
    0 \leq \frac{a}{q} + \Delta - t - n_0 \leq \frac{n-2}{p} + 1 - \frac{n}{q},
\]

Now \(n_0 - 1 \leq -t\) as \(a/q + \Delta < 2\). On the other hand

\[
    -t - n_0 - 1 \leq \frac{n-2}{p} - \frac{n}{q} \implies -t - n_0 - 1 \leq \frac{n-2}{p} \leq \left\lfloor \frac{n-2}{p} \right\rfloor
\]

This implies \(-t \leq n_0 + \left\lfloor \frac{n-2}{p} \right\rfloor\).

This proves assertion (2): \(n_0 - 1 \leq -t \leq \lfloor (n - 2)/p \rfloor + n_0\).

In particular \(n - 2 \leq p\) implies \(-t \in \{ n_0 - 1, n_0, n_0 + 1 \} \).

Now the rest of the assertion follows from the following three possibilities:

1. If \(-t = n_0 - 1\) then we have \(0 \leq a/q + \Delta - 1 \leq 1 + \frac{n-2}{p} - \frac{a}{q} < 1 + \frac{n-2}{p}\).
2. If \(-t = n_0\) then we have \(0 \leq a/q + \Delta \leq 1 + \frac{n-2}{p} - \frac{n}{q} < 1 + \frac{n-2}{p}\).
3. If \(-t = n_0 + 1\) then we have \(0 \leq a/q + \Delta \leq \frac{n-2}{p} - \frac{n}{q} < \frac{n-2}{p}\).

\(\square\)

**Remark 3.3.** If \(n\) is even and \(p \geq n - 2\) then \(F^s_*(\mathcal{O}(a))\) as atmost three types of spinor bundles. If \(0 \leq a < q\), then they all belong to the set \(\{S(-n_0 + 1), S(-n_0), S(-n_0 - 1)\}\) (we will see that the only first two are relevant for our computation). Moreover for a given choice of such an integer \(a\), there are atmost two types of spinor bundles, namely \(S(-n_0 + 1), S(-n_0)\) or \(S(-n_0), S(-n_0 - 1)\).

If \(n\) is odd and \(p \geq n - 2\) and \(0 \leq a < q\) then the only possible spinor bundles are \(S(-n_0 + 1), S(-n_0)\).

**Notations 3.4.** Let \(0 \leq a < q = p^r\) be an integer. We define iteratively, \(Z^s_{-i}(a)\) for \(0 \leq i \leq n_0 + 2\) as follows.

Let \(Z^s_0(a) = Y_a\), \(Z^s_{-1}(a) = Y_{a+q} - Y_1Y_a\)

and let

\[
    Z^s_{-i}(a) = Y_{a+i} - [Y_1Z^s_{-i+1}(a) + Y_2Z^s_{-i+2}(a) + \cdots + Y_iZ^s_0(a)].
\]
Similarly we define iteratively, $L^s_{i-1}(a)$ for $n_0 + 1 \leq i \leq n - 1$ as follows. Let $L^s_{n+1}(a) = Y_{q-a-n}$ and for $n_0 + 1 \leq i < n - 1$ we denote 
\[ L^s_{i-1}(a) = Y_{(n-i)q-a-n} - \left[ Y_{n-i-1}L^s_{n-i+1}(a) + \cdots + Y_1L^s_{i-1}(a) \right]. \]

**Remark 3.5.** By construction 
\[ Z^s_{i-1}(a) = r_{i0}Y_a + r_{i1}Y_{a+q} + \cdots + r_{i(i-1)}Y_{a+(i-1)q} + Y_{a+i} \]
and 
\[ L^s_{i-1}(a) = s_{i0}Y_{q-a-n} + s_{i1}Y_{2q-a-n} + \cdots + s_{i(n-i-2)}Y_{(n-i-1)q-a-n} + Y_{(n-i)q-a-n}, \]

where $\{r_{ij}, s_{ik}\}_{j,k}$ are some rational numbers independent of $s$ and $a$. On the other hand for an integer $m \geq 0$ 
\[ Y_m = \left( \frac{m + n + 1}{n + 1} \right) - \left( \frac{m + n - 1}{n + 1} \right) = \frac{2mn}{n!} + O(m^{n-1}). \]

Hence both $Z^s_{i-1}(a)$ and $L^s_{i-1}(a)$ are computable in the sense of Notations 3.1.

The following lemma implies that except for $i = n_0 + 1$ and $i = n_0 + 2$ all the pairs $\nu^s_{i-1}(a) + 2\lambda_0\mu^s_{i-1}(a)$ are computable.

**Lemma 3.6.** If $p \geq n - 2$ is an odd prime and $n \geq 3$. Then for given $1 \leq s$ and $0 \leq a < q = p^s$

1. $\nu^s_{i-1}(a) + 2\lambda_0\mu^s_{i-1}(a) = Z^s_{i-1}(a)$, if $0 \leq i \leq n_0$.
2. $\nu^s_{n_0-1}(a) + 2\lambda_0\mu^s_{n_0-1}(a) = Z^s_{n_0-1}(a) + 2\nu^s_{n_0-1}(a)$.
3. $\nu^s_{n_0-2}(a) + 2\lambda_0\mu^s_{n_0-2}(a) = Z^s_{n_0-2}(a) + 2\nu^s_{n_0-2}(a)$.
4. $\nu^s_{i}(a) + 2\lambda_0\mu^s_{i+1}(a) = L^s_{i-1}(a)$, if $n_0 + 3 \leq i \leq n - 1$.
5. $\nu^s_{n_0-2}(a) = L^s_{n_0-2}(a)$.
6. $\nu^s_{n_0-1}(a) = L^s_{n_0-1}(a) - 2\lambda_0\mu^s_{n_0-1}(a)$.
7. $\nu^s_{i}(a) = 0$, for $i \geq n$ and $\mu^s_{j}(a) = 0$ if $j \not\in \{n_0 + 1, n_0, n_0 - 1\}$.

**Proof.** We fix $0 \leq a < q = p^s$. Then, by Lemma 3.2 
\[
F^s_* (\mathcal{O}(a)) = \mathcal{O}(-n + 1)\nu^s_{n+1}(a) \oplus \cdots \oplus \mathcal{O}(-1)\nu^s_1(a) \oplus \mathcal{O}^0(a) \\
\oplus \mathcal{S}(-n_0 + 1)\mu^s_{n_0+1}(a) \oplus \mathcal{S}(-n_0)\mu^s_{n_0}(a) \oplus \mathcal{S}(-n_0 - 1)\mu^s_{n_0-1}(a).
\]

Tensoring the above equation by $\mathcal{O}(i)$ and by projection formula, we get 
\[
F^s_* (\mathcal{O}(a + iq)) = \mathcal{O}(i - n + 1)\nu^s_{n+1}(a) \oplus \cdots \oplus \mathcal{O}(i - 1)\nu^s_1(a) \oplus \mathcal{O}(i)\nu^0_0(a) \\
\oplus \mathcal{S}(i - n_0 + 1)\mu^s_{n_0+1}(a) \oplus \mathcal{S}(i - n_0)\mu^s_{n_0}(a) \oplus \mathcal{S}(i - n_0 - 1)\mu^s_{n_0-1}(a).
\]

Applying the the functor $H^0(Q_{n-1})$ we get 
\[
\nu^0_0(a) = Y_a = Z^s_0(a) \\
\nu^s_{-1}(a) = Y_{a+q} - Y_1Y_a = Z^s_{-1}(a)
\]

(3.1)

In general, for $i < n_0$, 
\[
Y_{a+i} = Y_0\nu^s_{i-1}(a) + Y_1\nu^s_{i-1}(a) + \cdots + Y_i\nu^s_0(a)
\]
which implies

\[ \nu^s_{-1}(a) = \nu^s_{-1}(a) + 2\lambda_0 \mu^s_{-i+1}(a) = Z^s_{-1}(a). \]

For \( i = n_0 \)

\[ Y_{a+n_0q} = Y_0 \nu^s_{n_0}(a) + Y_1 \nu^s_{n_0+1}(a) + \cdots + Y_{n_0} \nu^s_0(a) + 2\lambda_0 \mu^s_{n_0+1}(a) \]

\[ = \nu^s_{n_0}(a) + Y_1 Z^s_{n_0+1}(a) + \cdots + Y_{n_0} Z^s_0(a) + 2\lambda_0 \mu^s_{n_0+1}(a) \]

\[ \implies \nu^s_{-n_0}(a) + 2\lambda_0 \mu^s_{n_0+1}(a) = Z^s_{-n_0}(a) \]

This proves assertion (1).

For \( i = n_0 + 1 \)

\[ Y_{a+(n_0+1)q} = Y_0 \nu^s_{n_0-1}(a) + Y_1 \nu^s_{n_0-2}(a) + \cdots + Y_{n_0+1} \nu^s_0(a) + 2\lambda_0 [(Y_1 - Y_0) \mu^s_{n_0+1}(a) + \mu^s_{n_0}(a)]. \]

\[ \implies \nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{n_0}(a) = Z^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{n_0+1}(a) \]

This proves assertion (2).

For \( i = n_0 + 2 \)

\[ Y_{a+(n_0+2)q} = Y_0 \nu^s_{n_0-2}(a) + Y_1 \nu^s_{n_0-1}(a) + \cdots + Y_{n_0+2} \nu^s_0(a) \]

\[ + 2\lambda_0 (X_2 - X_1) \mu^s_{n_0+1}(a) + 2\lambda_0 (X_1 - X_0) \mu^s_{n_0}(a) + 2\lambda_0 \mu^s_{n_0-1}(a). \]

This implies

\[ \nu^s_{n_0-2}(a) + 2\lambda_0 \mu^s_{n_0-1}(a) = Z^s_{n_0-2}(a) + 2\lambda_0 (Y_1 - Y_0) \mu^s_{n_0+1}(a) + 2\lambda_0 \mu^s_{n_0}(a). \]

This proves assertion (3).

Now we tensor the decomposition of \( F^s_+(O(a)) \) by \( O(-j) \) and apply the functor \( H^0(Q_n, -) \). By duality \( h^0(Q_n, O_{Q_n}(m)) = h^0(Q_n, O_{Q_n}(-m - n)) = Y_{m-n} \), hence

\[ Y_{jg-y-n} = Y_{j-1} \nu^s_{n+1}(a) + Y_{j-2} \nu^s_{n+2}(a) + \cdots + Y_0 \nu^s_{n+j}(a) + \mu^s_{-n+1} h_0(Q_n, S(n_0 + j)) \]

\[ + \mu^s_{-n} h_0(Q_n, S(n_0 + j + 1 - n)) + \mu_{-n-1} h_0(Q_n, S(n_0 + j + 2 - n)). \]

Hence \( 1 \leq j \leq n_0 - 2 \), we get

\[ Y_{jg-y-n} = Y_{j-1} \nu^s_{n+1}(a) + Y_{j-2} \nu^s_{n+2}(a) + \cdots + Y_0 \nu^s_{n+j}(a). \]

Hence

\[ \nu_{-n+j} = Y_{jg-y-n} - [Y_{j-1} \nu^s_{n+1}(a) + \cdots + Y_2 \nu^s_{n+j-2}(a) + Y_1 \nu^s_{n+j-1}(a)] = L_{n+j}. \]

(3.2)

For \( j = 1 \), \( \nu^s_{n+1}(a) = Y_{q-a-n} \)

For \( j = 2 \), \( \nu^s_{n+2}(a) = Y_{2q-a-n} - Y_1 \nu^s_{n+1}(a) \)

In general

\[ \nu^s_{i}(a) = \nu^s_{i}(a) + 2\lambda_0 \mu^s_{i+1}(a) = L^s_{i+1}(a), \quad \text{for} \quad n_0 + 3 \leq i \leq n - 1 \]

\[ \nu^s_{n+2}(a) = L^s_{n+2}(a). \]

This proves assertions (4) and (5).

For \( j = n - n_0 - 1 \) we get

\[ Y_{(n-n_0-1)q-a-n} = Y_{n-n_0-2} \nu^s_{n+1}(a) + \cdots + Y_0 \nu^s_{n-1}(a) + 2\lambda_0 \mu^s_{n-1}(a). \]
\[ \nu^s_{-n_0-1}(a) = L^s_{-n_0-1}(a) - 2\lambda_0 \mu^s_{-n_0-1}(a). \]

This proves assertion (6) and hence the lemma. \[\square\]

**Remark 3.7.** By the above set of equalities it follows that, for a given \( a \), if there is at the most one type of spinor bundle in the decomposition of \( F^s_* (\mathcal{O}(a)) \) then all the pairs \( \nu^s_{-i}(a) + 2\lambda_0 \mu^s_{-i-1}(a) \) are computable.

In the next lemma we use this observation to classify the \((a, q)\) for which all the pairs are computable. It is enough to check this for the pairs \( \nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{-n_0}(a) \) and \( \nu^s_{-n_0-2}(a) + 2\lambda_0 \mu^s_{-n_0-1}(a) \), as, by Lemma 3.6, rest of the other pairs are computable.

**Lemma 3.8.** Let \( 0 \leq a < q = p^s \), where \( p > 2 \) and \( n \geq 3 \).

1. \( \nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{-n_0}(a) = Z^s_{-n_0-1}(a) \) if \( 0 \leq \frac{a}{q} < 1 - \frac{(n - 2)}{2p} \).
2. \( \nu^s_{-n_0-2}(a) + 2\lambda_0 \mu^s_{-n_0-1}(a) \) if \( 0 \leq \frac{a}{q} < \frac{(n - 2)}{2p} \).
   
   In particular all the pairs are computable for \( a/q \in [0, 1) \setminus [0, \frac{n - 2}{p}) \cup [1 - \frac{n - 2}{p}, 1) \).

Proof. Let \( \delta = 1 \) if \( n \) is even and \( \delta = 1/2 \) if \( n \) is odd.

1. If \( 0 \leq a/q < \delta - (n - 2)/2p \) then \( \mu^s_{-n_0-1}(a) = 0 \). Hence the assertion

\[ \nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{-n_0}(a) = Z^s_{-n_0-1}(a) \]

follows from Lemma 3.6 (2).

2. If \( \delta + (n - 2)/2p \leq a/q < 1 \) (which holds only if \( n \) is odd) then \( \mu^s_{-n_0-1}(a) = 0 \) and \( \mu^s_{-n_0}(a) = 0 \) hence by Lemma 3.6 (6)

\[ \nu^s_{-n_0-1}(a) + 2\lambda_0 \mu^s_{-n_0}(a) = \nu^s_{-n_0-1}(a) = L^s_{-n_0-1}(a). \]

3. If \( 0 \leq a/q < (n - 2)/2p \) then, by Lemma 3.2 (2), \( \mu^s_{-n_0+1}(a) = 0 \) and hence the equality

\[ \nu^s_{-n_0-2}(a) + 2\lambda_0 \mu^s_{-n_0-1}(a) = Z^s_{-n_0-2}(a) + 2\lambda_0 \mu^s_{-n_0}(a) \]

follows from Lemma 3.6 (3).

4. If \( (n - 2)/2p \leq a/q < 1 \) or if \( n \) is odd then \( \mu^s_{-n_0-1}(a) = 0 \). Hence by Lemma 3.6 (5), we have the equality \( \nu^s_{-n_0-2}(a) + 2\lambda_0 \mu^s_{-n_0-1}(a) = L^s_{-n_0-2}(a) \).

\[\square\]
Proposition 5.2 of [L] states that if $s = 1$ then $F_*(\mathcal{O}(a))$ and $F_*(\mathcal{S}(a))$ both have at most one type of spinor bundle. Here, using explicit formulation of [A] we write them down explicitly. In particular all the pairs and $\nu_1^1(a)$, $\tilde{\nu}_1^1(a)$, and $\tilde{\mu}_1^1(a)$ are computable, where $\tilde{\nu}_1^1(a) = \tilde{\nu}_1^1(i, a)$ and $\tilde{\mu}_1^1(a) = \tilde{\mu}_1^1(i, a)$ are the integers as in [274].

**Lemma 3.9.** If $p \neq 2$ and $n = 3$ then for $0 \leq a < p$, we have

$$F_*(\mathcal{O}(a)) = \begin{cases} O(-2)^{\nu_2^1(a)} \oplus O(-1)^{\nu_1^1(a)} \oplus O_{0}^{\nu_0^1(a)} \oplus S(-1)^{\mu_1^1(a)}, & \text{if } a \leq \frac{p-1}{2} - 2 \\ O(-2)^{\nu_2^1(a)} \oplus O(-1)^{\nu_1^1(a)} \oplus O_{0}^{\mu_0^1(a)}, & \text{if } a = \frac{p-1}{2} - 1 \\ O(-2)^{\nu_2^1(a)} \oplus O(-1)^{\nu_1^1(a)} \oplus O_{0}^{\mu_0^1(a)} \oplus S(0)^{\bar{\mu}_0^1(a)}, & \text{if } a \geq \frac{p-1}{2}. \end{cases}$$

Moreover $4 \mu_1^1(a) = Y_{a+2p} - Y_1 Y_{a+p} + (Y_1^2 - Y_2) Y_a - Y_{p-a-3}$, if $a \leq \frac{p-1}{2} - 2$.

Also

$$F_*(\mathcal{S}(a)) = \begin{cases} O(-2)^{\nu_2^1(a)} \oplus O(-1)^{\nu_1^1(a)} \oplus O_{0}^{\nu_0^1(a)} \oplus S(-1)^{\tilde{\mu}_1^1(a)}, & \text{if } a \leq \frac{p-1}{2} - 1 \\ O(-2)^{\nu_2^1(a)} \oplus O(-1)^{\nu_1^1(a)} \oplus O_{0}^{\nu_0^1(a)} \oplus S(0)^{\bar{\mu}_0^1(a)}, & \text{if } a \geq \frac{p-1}{2}. \end{cases}$$

Moreover, if $a \leq \frac{p-1}{2} - 1$ then

$$4 \mu_1^1(a) = h^0(Q_3, \mathcal{S}(a+2p)) - Y_1 h^0(Q_3, \mathcal{S}(a+p)) + (Y_1^2 - Y_2) h^0(Q_3, \mathcal{S}(a)) - h^0(Q_3, \mathcal{S}(p-a-2)).$$

In general, for any given $n \geq 3$ and $0 \leq a < p$ the bundle $F_*(\mathcal{O}(a))$ (similarly $F_*(\mathcal{S}(a))$) cannot contain both $S(t)$ and $S(t')$, where $t \neq t'$.

**Proof.** We first prove the last assertion for $n \geq 3$. By Theorem 2 of [A], $F_*(\mathcal{O}(a))$ contains $S(t)$ if and only if

$$\frac{(n-2)(p-1)}{2} \leq a - tp \leq \frac{(n-2)(p-1)}{2} + (p-2).$$

Since the difference between the maximum and minimum is $\leq p - 1$, there can not be two different $t$ and $t'$ satisfying such equation. Similar assertion holds for $F_*(\mathcal{S}(a))$ $F_*(\mathcal{S}(a))$ contains $S(t)$ if and only if

$$\frac{(n-2)(p-1)}{2} \leq a - tp \leq \frac{(n-2)(p-1)}{2} + (p-1).$$

It is easy to work out $n = 3$ case. The formula for $\tilde{\mu}_1^1(a)$ can be worked out as follows:

Let $a < (p - 1)/2$. Tensoring the equation

$$F_*(\mathcal{S}(a)) = O(-2)^{\nu_2^1(a)} \oplus O(-1)^{\nu_1^1(a)} \oplus O_{0}^{\nu_0^1(a)} \oplus S(-1)^{\tilde{\mu}_1^1(a)}$$

by $O(i)$ we get

$$F_*(\mathcal{S}(a + ip)) = O(i - 2)^{\nu_2^1(a)} \oplus O(i - 1)^{\nu_1^1(a)} \oplus O(i)^{\nu_0^1(a)} \oplus S(i - 1)^{\tilde{\mu}_1^1(a)}.$$
4. The HK Density Function $f_{R_{p,n+1},m}$ and $f_{R_{n+1,m}}$

Remark 4.1. Let $Z_{-i}^s(a)$ and $L_{-i}^s(a)$ be the numbers as Notations 3.3 Then we can write

$$Z_{-i}^s(a) = \sum_{j=0}^{i} r_{ij} Y_{a+j}$$

and

$$L_{-i}^s(a) = \sum_{j=0}^{n-i-1} s_{ij} Y_{(j+1)q-a-n}$$

where $\{r_{ij}, s_{ik}\}_{j,k}$ are rational numbers independent of $a$ and $s$. Now if $x \geq 0$ such that $xq_0 \in \mathbb{Z}_{\geq 0}$ for some $q_0 = p^{s_0}$ and if $i$ is the integer such that $0 \leq xq_0 - iq_0 < q_0$ then $\lim_{q \to \infty} (xq - iq)/q = x - i$. This observation implies that, if we define the functions $Z_{-i}$ and $L_{-i}$ on the interval $[i, i+1)$ by

$$Z_{-i}(x) := \lim_{q \to \infty} \frac{Z_{-i}^s([xq] - iq)}{q^n}$$

and

$$L_{-i}(x) := \lim_{q \to \infty} \frac{L_{-i}^s([xq] - iq)}{q^n}.$$ 

then we have

$$Z_{-i}(x) = \frac{2}{n!} [r_{i0}(x - i)^n + r_{i1}(x - i + 1)^n + \cdots + r_{ii}(x)^n]$$

and

$$L_{-i}(x) = \frac{2}{n!} [s_{i0}(i + 1 - x)^n + s_{i1}(i + 2 - x)^n + \cdots + s_{i(n-i-1)}(n - x)^n].$$

Lemma 4.2. (1) If $n \geq 4$ is an even number and $p \geq n - 2$ and $p \neq 2$ Then

$$f_{R_{p,n+1},m}(x) =$$

$$\begin{cases}
Z_{-i}(x), & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{-n_0-1}(x), & \text{if } (n_0 + 1) \leq x < (n_0 + 2) - \frac{n-2}{2p} \\
Z_{-n_0-1}(x) + 2\lambda_0 \lim_{q \to \infty} \frac{\mu_{-n_0+1}([xq] - (n_0 + 1)q)}{q^n}, & \text{if } 1 - \frac{n-2}{2p} \leq x - (n_0 + 1) < 1 \\
Z_{-n_0-2}(x) + 2\lambda_0 \lim_{q \to \infty} \frac{\mu_{-n_0}([xq] - (n_0 + 2)q)}{q^n}, & \text{if } 0 \leq x - (n_0 + 2) < \frac{n-2}{2p} \\
L_{-n_0-2}(x), & \text{if } (n_0 + 2) + \frac{n-2}{2p} \leq x < (n_0 + 3) \\
L_{-i}(x), & \text{if } i \leq x < i + 1 \text{ and } n_0 + 3 \leq i < n \\
\end{cases}$$

and $f_{R_{p,n+1},m}(x) = 0$ otherwise.

(2) If $n \geq 3$ is an odd number and $2n - 4 \leq p$ and $p \neq 2$ then

$$f_{R_{p,n+1},m}(x) =$$
By Lemma 3.8 and Lemma 3.6 (2), we have,

\[
\begin{align*}
Z_{-t}(x) & \text{ if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 \\
Z_{-n_0-1}(x) & \text{ if } (n_0 + 1) \leq x < (n_0 + \frac{3}{2}) - \frac{n-2}{2p} \\
Z_{-n_0-1}(x) + 2\lambda_0 \lim_{q \to \infty} \mu_n^{s-1}\left([xq] - (n_0 + 1)q\right) & \text{ if } \frac{1}{2} - \frac{n-2}{2p} \leq x - (n_0 + 1) < \frac{1}{2} + \frac{n-2}{2p} \\
L_{-n_0-1}(x) & \text{ if } (n_0 + 1) + \frac{1}{2} + \frac{n-2}{2p} \leq x < (n_0 + 2) \\
L_{-i}(x) & \text{ if } i \leq x < i + 1 \text{ and } n_0 + 2 \leq i < n \\
\end{align*}
\]

and \(f_{R_{p,n+1}}(x) = 0\) otherwise.

**Proof.** Let \(q = p^s\) and \(m \in \mathbb{Z}\) and let \(\nu_i^s(m)\) and \(\mu_i^s(m)\) be the numbers occuring in the decomposition

\[F^s(O(m)) = \oplus_{t \in \mathbb{Z}} O(t)\nu_t^s(m) \oplus \oplus_{t \in \mathbb{Z}} S(t)\mu_t^s(m).\]

If \(m \geq 0\) is integer then there is \(i \geq 0\) an integer such that \(0 \leq m - iq < q\). Now, by (2.6),

\[\ell(R_{p,n+1}/m[q])_m = \nu_0^s(a + iq) + 2\lambda_0 \mu_1^s(a + iq) = \nu_i^s(a) + 2\lambda_0 \mu_{i+1}^s(a).\]

Here we write the details when \(n\) is even, the case when \(n\) is odd follows along the same lines.

Lemma 3.6 (1) gives

\[\ell(R_{p,n+1}/m[q])_{a+iq} = Z_i^s(a) \text{ for every } 0 \leq i \leq n_0 \text{ and for } 0 \leq a < q\]

By Lemma 3.8 and Lemma 3.6 (2), we have,

\[
\ell(R_{p,n+1}/m[q])_{a+(n_0+1)q} = \begin{cases} 
Z_{-n_0-1}(a) & \text{if } 0 \leq a < q(1 - \frac{n-2}{2p}) \\
Z_{-n_0-1}(a) + 2\lambda_0 \mu_{-n_0+1}^s(a) & \text{if } q - \frac{n-2}{2p} q \leq a < q \\
\end{cases}
\]

\[
\ell(R_{p,n+1}/m[q])_{a+(n_0+2)q} = \begin{cases} 
Z_{-n_0-2}(a) + 2\lambda_0 \mu_{-n_0}^s(a) & \text{if } 0 \leq a < \frac{n-2}{2p} q \\
L_{-n_0-2}(a) & \text{if } \frac{n-2}{2p} q \leq a < q \\
\end{cases}
\]

By Lemma 3.6 (4)

\[\ell(R_{p,n+1}/m[q])_{a+jq} = L_{-j}(a) \text{ for every } n_0 + 3 \leq j \leq n - 1 \text{ and for } 0 \leq a < q,\]

and \(\ell(R_{p,n+1}/m[q])_m = 0\) otherwise.

By definition

\[f_{R_{p,n+1}}(x) = \lim_{q \to \infty} \frac{1}{q^n} \ell(R_{p,n+1}/m[q])_{[xq]}\]

and is a continuous function and the set \(\{x \in \mathbb{R} \mid xq \in \mathbb{Z}, \text{ for some } q = p^s\}\) is a dense of \(\mathbb{R}\). Hence the theorem follows from Remark 4.1.
Theorem 4.3. The function $f_{R_{n+1}}^\infty : [0, \infty) \to [0, \infty)$ given by
\[
f_{R_{n+1}}^\infty(x) := \lim_{p \to \infty} f_{R_p, n+1}(x)
\]
is partially symmetric continuous function, that is
\[
f_{R_{n+1}}^\infty(x) = f_{R_{n+1}}^\infty(n - x), \quad \text{for} \quad 0 \leq x \leq (n - 2)/2
\]
and is described as follows:

(1) If $n \geq 4$ is even then
\[
f_{R_{n+1}}^\infty(x) = \begin{cases} 
Z_{-1}(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n_0 + 1 \\
L_{-n_0-2}(x) & \text{if } (n_0 + 2) \leq x < (n_0 + 3) \\
L_{-i}(x) & \text{if } i \leq x < i + 1 \text{ and } n_0 + 3 \leq i < n
\end{cases}
\]
and $f_{R_{n+1}}^\infty(x) = 0$ otherwise.

(2) If $n \geq 3$ is an odd number then
\[
f_{R_p, n+1}(x) = \begin{cases} 
Z_{-1}(x) & \text{if } i \leq x < i + 1 \text{ and } 0 \leq i \leq n \\
Z_{-n_0-1}(x) & \text{if } (n_0 + 1) \leq x < (n_0 + 1 + \frac{3}{4}) \\
L_{-n_0-1}(x) & \text{if } (n_0 + 1 + \frac{3}{4}) \leq x < (n_0 + 2) \\
L_{-i}(x) & \text{if } i \leq x < i + 1 \text{ and } n_0 + 2 \leq i < n
\end{cases}
\]
and $f_{R_{n+1}}^\infty(x) = 0$ otherwise.

Proof. The description of the function $f_{R_{n+1}}^\infty : [0, \infty) \to [0, \infty)$ follows from Lemma 4.2.
To prove the symmetry, we consider the $Z_{-j} : [0, \infty) \to [0, \infty)$ and $L_{-j} : [0, \infty) \to [0, \infty)$ and

Claim. $Z_{-j}(x) = L_{n-1-j}(n - x)$, if $j \leq x < j + 1$.

Proof of the claim: By induction on $j \geq 0$, first we prove the assertion that
\[
\lim_{q \to \infty} Z_{-i}^s(a)/q^n = \lim_{q \to \infty} L_{-i}^s(a)/q^n
\]
for $0 \leq a < q$.

If $j = 0$ then
\[
\lim_{q \to \infty} Z_0^s(a)/q^n = \lim_{q \to \infty} Y_a/q^n = \lim_{q \to \infty} Y_{a+n}/q^n = \lim_{q \to \infty} L_{-(n-1)}^s(a)/q^n.
\]
Assume that the assertion holds for $0 \leq j < i$. Now
\[
\lim_{q \to \infty} Z_{-i}(a)/q^n = \lim_{q \to \infty} Y_a/q^n - [Y_i Z_{-i+1}(a) + \cdots + Y_i Z_{i}(a)]/q^n
\]
\[
= \lim_{q \to \infty} Y_{a+n}/q^n - [Y_i L_{-(n-1)}^s(a) + \cdots + Y_i L_{-(n-1)}^s(a - q)]/q^n
\]
\[
\lim_{q \to \infty} L_{-(n-1-i)}(q - a)/q^n.
\]

Now to prove the claim, it is enough to prove for $x = m/q$, where $m \in \mathbb{Z}_{\geq 0}$. If $j \leq x < j + 1$ then $m = a + jq$, where $0 \leq a < q$. Now
\[
Z_{-j}(x) = \lim_{q \to \infty} Z_{-j}^s(m - jq)/q^n = \lim_{q \to \infty} L_{-(n-1-j)}^s((j + 1)q - m))/q^n
\]
\[
= \lim_{q \to \infty} L_{-(n-1-j)}^s((n - m)q - (nq - jq))/q^n = L_{-(n-1-j)}^s(n - x).
\]
This proves the claim.
If \( n \) is even then \( n_0 = n/2 - 1 \). Let \( 0 \leq x < (n - 2)/2 = n_0 \) then \( i \leq x < (i + 1) \) for some \( 0 \leq i \leq (n_0 - 1) \). Now

\[
f_{R_{n+1}}^{\infty}(x) = Z_{-i}(x) = L_{-(n-1-i)}(n-x) = f_{R_{n+1}}^{\infty}(n-x).
\]

where the second equality follows as \( n - (i + 1) < n - x \leq n - i \).

If \( n \) is odd then \( n_0 = (n - 1)/2 \). Let \( 0 \leq x < (n - 2)/2 = n_0 - (1/2) \).

If \( i < x < (i + 1) \), where \( i \leq n_0 - 1 \) then

\[
f_{R_{n+1}}^{\infty}(x) = Z_{-i}(x) = L_{-(n-1-i)}(n-x) = f_{R_{n+1}}^{\infty}(n-x).
\]

If \( (n_0 - 1) \leq x < n_0 - 1/2 \) then again

\[
f_{R_{n+1}}^{\infty}(x) = Z_{-(n_0-1)}(x) = L_{-(n_0+1)}(n-x) = f_{R_{n+1}}^{\infty}(n-x).
\]

\[\square\]

**Remark 4.4.** The same argument as above proves that \( f_{R_{p,n+1}} \) is partially symmetric and the symmetry is given by

\[
f_{R_{p,n+1}}(x) = f_{R_{p,n+1}}(n-x) \quad \text{for} \quad 0 \leq x \leq \frac{n-2}{2} \left( 1 - \frac{1}{p} \right).
\]

**Proof of Theorem 1.2** If \( n \) is even then \( n_0 = \frac{n-2}{2} \) and the interval

\[
\left[ n_0 + 2 - \frac{n-2}{2p}, n_0 + 2 + \frac{n-2}{2p} \right) = \left[ \frac{n+2}{2} - \frac{n-2}{2p}, \frac{n+2}{2} + \frac{n-2}{2p} \right)
\]

If \( n \) is odd then \( n_0 = \frac{n-1}{2} \). and the interval

\[
\left[ n_0 + \frac{3}{2} - \frac{n-2}{2p}, n_0 + \frac{3}{2} + \frac{n-2}{2p} \right) = \left[ \frac{n+2}{2} - \frac{n-2}{2p}, \frac{n+2}{2} + \frac{n-2}{2p} \right)
\]

Note that, by Lemma 4.2 and Theorem 4.3

\[
f_{R_{p,n+1}}(x) = f_{R_{n+1}}^{\infty}(x) \quad \text{if} \quad x \not\in \left[ \frac{n+2}{2} - \frac{n-2}{2p}, \frac{n+2}{2} + \frac{n-2}{2p} \right).
\]

Since both \( f_{R_{p,n+1}} \) and \( f_{R_{n+1}}^{\infty} \) are continuous functions on \( \mathbb{R} \), it is enough to prove the rest of the assertion for \( x \in \mathbb{Z}[1/p] \). Now let \( xq_0 \in \mathbb{Z} \) for some \( q_0 = p^{s_0} \).

1. Let \( n \geq 4 \) be an even number with \( n - 2 \leq p \).
   1. (a) Let \( n_0 + 2 - \frac{n-2}{2p} \leq x < n_0 + 2 \). For a fix \( q \geq q_0 \) let \( a_q = xq - (n_0 + 1)q \).
      Then \( 0 \leq a_q < q \) for all \( q \geq q_0 \) and by Lemma 3.6 (2)

\[
\ell \left( \frac{R_{p,n+1}}{m[q]} \right)_{xq} = \nu^{s}_{-n_0-1}(a_q) + 2 \lambda_0 \mu^{s}_{-n_0}(a_q) = Z^{s}_{-n_0-1} + 2 \lambda_0 \mu^{s}_{-n_0+1}(a_q).
\]

Hence

\[
f_{R_{p,n+1}}(x) = Z_{-n_0-1}(x) + \lim_{q \to \infty} 2 \lambda_0 \frac{\mu^{s}_{-n_0+1}(a_q)}{q^n}
\]

whereas \( f_{R_{n+1}}^{\infty}(x) = Z_{-n_0-1}(x) \).

(b) Let \( n_0 + 2 \leq x < n_0 + 2 + \frac{n-2}{2p} \). For a fix \( q \geq q_0 \) let \( a_q = xq - (n_0 + 2)q \)
      then \( 0 \leq a_q < q \) and, by Lemma 3.6 (5)

\[
\ell \left( \frac{R_{p,n+1}}{m[q]} \right)_{xq} = \nu^{s}_{-n_0-2}(a_q) + 2 \lambda_0 \mu^{s}_{-n_0-1}(a_q) = L^{s}_{-n_0-2} + 2 \lambda_0 \mu^{s}_{-n_0-1}(a_q).
\]
Hence
\[ f_{R, n+1}(x) = L_{-n-2}(x) + \lim_{q \to \infty} 2\lambda_0 \frac{\mu_{-n-1}(a_q)}{q^n} \]
whereas \( f_{R, n+1}(x) = L_{-n-2}(x) \).

(2) Let \( n \geq 3 \) be an odd number with \( 2n - 4 \leq p \).

(a) Let \( n_0 + \frac{3}{2} - \frac{n-2}{2p} \leq x < n_0 + \frac{3}{2} + \frac{n-2}{2p} \).

For a fix \( q = p^\ell \geq q_0 \) let \( a_q = xq - (n_0 + 1)q \). Then \( 0 \leq a_q < q \) and
\[
\ell \left( \frac{R_{n+1}}{m^{(n)}} \right)_{xq} = \nu_{-n-1}(a_q) + 2\lambda_0 \mu_{-n-1}(a_q) = \lambda_{-n-1} - 2\lambda_0 \mu_{-n-1}(a_q),
\]
where the last equality follows as, by Lemma 3.6(6) and Lemma 3.2(2)(b)(i) \( \nu_{-n-1}(a_q) = L_{-n-1}(a_q) \).

Hence we can write
\[
f_{R, n+1}(x) = L_{-n-1}(x) + \lim_{q \to \infty} 2\lambda_0 \frac{\mu_{-n-1}(a_q)}{q^n}.
\]

Whereas
\[
f_{R, n+1}(x) = \begin{cases} 
L_{-n-1}(x) & \text{if } n_0 + \frac{3}{2} - \frac{n-2}{2p} \leq x < n_0 + \frac{3}{2} \\
L_{-n-1}(x) & \text{if } n_0 + \frac{3}{2} - \frac{n-2}{2p} \leq x < n_0 + \frac{3}{2} + \frac{n-2}{2p} 
\end{cases}
\]
This proves the theorem. □

**Proof of Theorem 1.1**

We note that, for any integer \( 0 \leq a < q \) and \( q = p^\ell \), we have the decomposition
\[
F_\ell^{(n)}(x) = \sum_{n_0}^n \mathcal{O}(a) \mathcal{O}(-i)^\nu_{-i}(a) \oplus \cdots \oplus \sum_{n_0}^n \mathcal{S}(-i)^\mu_{-i}(a).
\]

By computing the ranks we get \( q^n = \sum_{n_0}^n \nu_{-i}(a) + \sum_{n_0}^n \lambda_0 \mu_{-i}(a) \). In particular \( 0 \leq \lambda_0 \mu_{-i}(a) / q^n \leq 1 \).

Therefore, by Lemma 3.2 and by the proof of Theorem 1.2 we have
\[
0 \leq \int_0^q f_{R, n+1}(x) dx - \int_0^q f_{R, n+1}(x) dx
\]
\[
= \int_0^{n+2} f_{R, n+1}(x) - f_{R, n+1}(x) dx \leq \frac{2n-1}{p}.
\]

On the other hand by Theorem 1.1 of [T] we have
\[
e_{HK}(R_{p,n+1}, m) = \int_0^q f_{R, n+1}(x) dx.
\]

This gives
\[
1 + m_{n+1} = \lim_{p \to \infty} e_{HK}(R_{p,n+1}, m) = \lim_{p \to \infty} \int_0^q f_{R, n+1}(x) dx = \int_0^q f_{R, n+1}(x) dx,
\]
where the first equality follows by the result of Gessel-Monsky [GM], this can also be derived using Theorem 4.3 in principle. □

**Corollary 4.5.** Let \( p > 2 \).

(1) If \( n \) even and \( p \geq n - 2 \), or
(2) if \( n \) and \( p \geq 2n - 4 \) then the F-threshold of the ring \( R_{p,n+1} \) is \( c^m(m) = n \).

**Proof.** By Theorem E of [TW1], the F-threshold \( c^m(m) = \max\{x \mid f_{R_{p,n+1}}(x) \neq 0\} \).

Now, by Lemma 1.2 \( f_{R_{p,n+1}}(x) = 0 \), for \( x \geq n \) and for \( n - 1 \leq x \leq n \),

\[
f_{R_{p,n+1}}(x) = L_{n+1}(x) = \lim_{q \to \infty} \frac{L_{n+1}(\lfloor qx \rfloor) - (n - 1)q}{q^n} = \frac{2(n - x)^n}{n!},
\]

where the last equality follows as \( L_{n+1}(a) = Y_{q-a-n} \). \(\Box\)

5. The HK density function for \( R_{p,4} \)

**Notations 5.1.** Let \( p \geq 5 \) be a prime and

\[
P_0 = \frac{p-1}{2} \quad \text{and} \quad P_i = \frac{p-1}{2p} \left[ \frac{1}{p^i} + \cdots + \frac{1}{p} + 1 \right] \quad \text{for} \quad i \geq 1.
\]

then

\[
P_1 < \cdots < P_j < P_{j+1} < \cdots < \frac{1}{2} < \cdots < \left( P_{j+1} + \frac{1}{p^{j+1}} \right) < \left( P_j + \frac{1}{p^j} \right) < \cdots < \left( P_1 + \frac{1}{p} \right).
\]

We divide the interval

\[
[2, 3] = [2, 2 + \frac{p-1}{2p}] \cup [2 + \frac{p-1}{2p}, 2 + \frac{p+1}{2p}] \cup [2 + \frac{p+1}{2p}, 3),
\]

then \( [2 + \frac{p-1}{2p}, 2 + \frac{p+1}{2p}] = [2 + P_1, 2 + P_1 + \frac{1}{p}] \) can be further divided as

\[
[2 + P_1, 2 + P_1 + \frac{1}{p}] = \bigcup_{j=1}^{\infty} [2 + P_j, 2 + P_{j+1}) \cup \left( 2 + \frac{1}{2} \right) \cup \bigcup_{j=1}^{\infty} \bigcup_{j=1}^{\infty} [2 + P_{j+1} + \frac{1}{p^{j+1}}, 2 + P_j + \frac{1}{p^j}].
\]

Let

\[
\mu_{-1} = \mu_{-1}^1(P_0 - 2) \quad \text{and} \quad \overline{\mu_{-1}} = \overline{\mu_{-1}}^1(P_0 - 1),
\]

where the formula for \( \mu_{-1}^1(a) \) and \( \overline{\mu}_{-1}^1(a) \) is given in Lemma 3.9

**Theorem 5.2.** Let \( k \) be a perfect field of characteristic \( p \geq 5 \) and let

\[
R_{p,4} = \frac{k[x_0, x_1, x_2, x_3, x_4]}{(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2)}.
\]

Then

\[
f_{R_{p,4},m}(x) = \begin{cases} 
\frac{x^3}{3} & \text{for } 0 \leq x < 1 \\
\frac{x^3}{3} - \frac{5}{3}(x - 1)^3 & \text{for } 1 \leq x < 2 \\
\frac{1}{3}x^3 - \frac{5}{3}(x - 1)^3 + \frac{11}{3}(x - 2)^3 & \text{for } 2 \leq x < 2 + P_1 \\
\frac{3}{3}(x - 2 - P_j)^3 - \frac{4}{3p} + P_i + 2 - x \right]^{3} (\mu_{-1}^1)^{i-1}) + \left[ \frac{8}{3} [x - 2 - P_j]^3 - \frac{4}{3p} [x - 2 - P_j]^2 + \frac{2}{3p^2} \right] (\overline{\mu}_{-1})^j, & \text{for } 2 + P_j \leq x < 2 + P_{j+1} \quad \text{and for} \quad j \geq 1
\end{cases}
\]

for \( 2 + P_j \leq x < 2 + P_{j+1} \) and for \( j \geq 1 \).
Therefore, by Lemma 3.8 (2) (ii),

\[
\mu_{-1}(b) = Z_{-2}(b) - \nu_{-2}^s(b) = Y_{b+2p^m} - Y_{b+p^m} + (Y_1^2 - Y_1)Y_b - Y_{p^m-b-3}
\]

and \(b_{m-1} < P_0 \implies \mu_{-1}(b) = 0\). Moreover for \(m = 1\), by Lemma 3.9 \(b = b_0 \geq P_0\) implies \(\mu_{-1}(b) = 0\).

Consider the \(p\)-adic expansion \(a_0 + a_1p + \cdots + a_{s-1}p^{s-1}\) of \(a\). Then by the hypothesis on \(a\) we have \(a_{s-1} = P_0\).

In general if \(1 \leq j \leq s - 1\) is an integer such that \(a_{s-j} = \cdots = a_{s-1} = P_0\), then \(P_j \leq a/q < P_j + 1/p\). Moreover

1. \(a_{s-j-1} < P_0 \iff P_j \leq a/q < P_j + 1/p\).
2. \(a_{s-j-1} = P_0 \iff P_{j+1} \leq a/q < P_{j+1} + 1/p^{j+1}\).
3. \(a_{s-j-1} > P_0 \iff P_{j+1} + 1/p^{j+1} \leq a/q < P_j + 1/p^j\).

We choose
(1) \( j = s - 1 \) if \( a_0 = a_1 = \cdots = a_{s-1} = P_0 \). Otherwise
(2) \( 1 \leq j \leq s - 1 \) is the integer such that \( a_{s-j} = \cdots = a_{s-1} = P_0 \) and \( a_{s-j-1} \neq P_0 \).

We denote \( A_{s-i} = a_0 + a_1p + \cdots + a_{s-i-1}p^{s-i-1} \). Therefore
\[
A_{s-j} = a_0 + a_1p + \cdots + a_{s-j-1}p^{s-j-1}m \quad \text{where} \quad a_{s-j-1} \neq P_0.
\]

Hence \( \mu_{s-j}^{-1}(A_{s-j}) \) and, for all \( i \), \( \nu_{s-i}^{-1}(A_{s-i}) \) are computable. By Lemma 3.9 the numbers \( \mu_{s-j}^{-1}(b) \) and \( \mu_{s-j}^{-1}(b) \) are computable.

**Claim.** Let \( \mu = \mu_{s-j}(P_0 - 2) \) and \( \mu = \mu_{s-j}(P_0 - 1) \). Then
\[
\mu = \nu_{s-j}^{s-j}(A_{s-j})(\mu) + \nu_{s-j}^{s-j}(A_{s-j})(\mu)
\]
\[
+ \cdots \nu_{s-j}^{s-j}(A_{s-j})(\mu) + \nu_{s-j}^{s-j}(A_{s-j})(\mu) \]

**Proof of the claim:** For an integer \( m \) and \( q = p^s \), we have the decomposition (by [A])
\[
F^s_*(\mathcal{O}(m)) = \mathcal{O}(-1)\nu_{s-j}(m) \oplus \mathcal{O}(m)\nu_{s-j}(m) \oplus M(0)m\nu_{s-j}(m) \oplus M(-1)m\nu_{s-j}(m)
\]
\[
F^s_*(\mathcal{S}(m)) = \mathcal{O}(-1)\nu_{s-j}(m) \oplus \mathcal{O}(m)\nu_{s-j}(m) \oplus M(0)m\nu_{s-j}(m) \oplus M(-1)m\nu_{s-j}(m).
\]

By the projection formula
\[
F^s_*(\mathcal{O}(a)) = F^j_*(F^s_*(\mathcal{O}(A_{s-j}))) \otimes \mathcal{O}(P_0 + \cdots + P_0p^{j-1}).
\]

Therefore

(5.1)

\[
[\nu_{s-j}^{s-j}(A_{s-j}), \nu_{s-j}^{s-j}(A_{s-j}), \nu_{s-j}^{s-j}(A_{s-j}), \nu_{s-j}^{s-j}(A_{s-j})] = [\nu_{s-j}^{s-j}(A_{s-j}), \cdots, \nu_{s-j}^{s-j} (A_{s-j})], [b_{kl}] \times j \times [b_{kl}],
\]

where \( [b_{kl}] \) is the matrix
\[
[b_{kl}] = \begin{bmatrix}
\nu_{1-1}(P_0 - 2) & \nu_{1-1}(P_0 - 2) & \nu_{1-1}(P_0 - 2) & 0 & \mu_{s-j}^{-1} \\

\nu_{1-1}(P_0 - 1) & \nu_{1-1}(P_0 - 1) & \nu_{1-1}(P_0 - 1) & 0 & 0 \\

\nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \mu_{s-j}^{-1} \\

\nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \mu_{s-j}^{-1} \\

\nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \mu_{s-j}^{-1} \\

\nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \nu_{1-1}(P_0) & \mu_{s-j}^{-1} \\

\end{bmatrix}
\]

Now the claim follows by induction on \( j \).

If \( a_0 = \cdots = a_{s-1} = P_0 \) then \( A_{s-j} = a_0 \) and \( \mu_{s-j}^{-1}(a_0) = 0 \).

We recall that
\[
Y_a = \frac{1}{6} (2a^3 + 9a^2 + 13a + 6) = a^3 / 3 + O(a^2).
\]

Hence
\[
\lim_{s \to \infty} \frac{\nu_{s-j}^{s-j}(A_{s-j})}{p^{s-j}} = \lim_{s \to \infty} \frac{Y_{p^{s-j}(a_0p^{s-j-1}+\cdots+p^{s-1})}}{3p^{s-j}} = \frac{1}{3} \left[ \frac{1}{p^j} + P_1 - x \right]^3.
\]

Now
(1) If there is $1 \leq j \leq s - 1$ such that $P_j \leq a/q < P_{j+1}$ then $a_{s-j-1} < P_0$ and
\[
\lim_{q \to \infty} \frac{(4)\mu_{s-1}^{-1}(A_{s-j})}{q^3} = \lim_{q \to \infty} Z_{s-2}^{-j}(A_{s-j}) - \nu_{s-2}^{-j}(A_{s-j}) = \frac{8}{3} [x - P_j]^{3} - \frac{4}{p^j} [x - P_j]^2 + \frac{2}{3p^j}.
\]

Hence
\[
\lim_{q \to \infty} \frac{\nu_{s-2}^j(a) + 4\mu_{s-1}^j(a)}{q^3} = \frac{(1-x)^3}{3} + \frac{4}{3} \sum_{i=1}^{j} \left[ \frac{1}{p^i} + P_i - x \right]^{3} (\mu_{s-1}^{-j-1}) + \left( \frac{4}{3} [x - P_j]^3 - \frac{4}{p^j} [x - P_j]^2 + \frac{2}{3p^j} \right) (\mu_{s-1}^{-j}).
\]

(2) If there is $1 \leq j \leq s - 1$ such that $P_j + \frac{1}{p^{j+1}} < a/q < P_j + \frac{1}{p^j}$. Then $a_{s-j-1} > P_0$ and hence $\mu_{s-1}^{-j}(A_{s-j}) = 0$. This gives
\[
\lim_{q \to \infty} \frac{\nu_{s-2}^j(a) + 4\mu_{s-1}^j(a)}{q^3} = \frac{(1-x)^3}{3} + \frac{4}{3} \sum_{i=1}^{j} \left[ \frac{1}{p^i} + P_i - x \right]^{3} (\mu_{s-1}^{-j-1}).
\]

(3) If there is no $j$ satisfying the any of the above two cases then $a/q = P_s$ and $j = s-1$ and $A_{s-j} = A_1 = P_0$. But $\mu_{s-1}(P_0) = 0$. Hence
\[
\lim_{q \to \infty} \frac{\nu_{s-2}^j(a) + 4\mu_{s-1}^j(a)}{q^3} = \frac{(1-x)^3}{3} + \frac{4}{3} \sum_{i=1}^{s-1} \left[ \frac{1}{p^i} + P_i - x \right]^{3} (\mu_{s-1}^{-j-1}).
\]

This proves the theorem. \hfill \Box

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