Quantum random walks and thermalisation II

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Abstract

A convergence theorem is obtained for quantum random walks with particles in an arbitrary normal state. This result unifies and extends previous work on repeated-interactions models, including that of the author (2010, J. London Math. Soc. (2) 81, 412–434; 2010, Comm. Math. Phys. 300, 317–329). When the random-walk generator acts by ampliation and multiplication or conjugation by a unitary operator, necessary and sufficient conditions are given for the quantum stochastic cocycle which arises in the limit to be driven by an isometric, co-isometric or unitary process.

Key words: thermalization; heat bath; repeated interactions; toy Fock space; non-commutative Markov chain; quantum stochastic cocycle.

MSC 2010: 81S25 (primary); 46L53, 46N50, 60F17, 82C10, 82C41 (secondary).

1 Introduction

The repeated-interactions framework, also called the theory of quantum random walks or non-commutative Markov chains, has attracted much attention. Physically, it describes a small quantum-mechanical system interacting with a heat bath which is modelled by a chain of identical particles. There have been many applications of this model; for example, to quantum optics [8, 9, 11], to quantum control [7, 12] and to the dilation of quantum dynamical semigroups [16]; for the latter, see also [5, Section 6]. There are interesting connexions between non-commutative Markov chains and multivariate operator theory [10].

Many results in this area (for example, those contained in [2], [8] and [9]) focus only on the reduced dynamics, i.e., the expectation semigroup which arises in the limit. In contrast, the results obtained below provide a full quantum-stochastic description of the limit dynamics. They may be considered to be quantum analogues of Donsker’s theorem, which gives the convergence of suitably scaled classical random walks to Brownian motion.

In previous work, the particles of the model were required to be either in a vector state [3, 5] or in a faithful normal state [6]. Here a generalisation is obtained, Theorem 4.5, which applies to quantum random walks with particles in an arbitrary normal state; the previous results appear as special cases.
Let \( \rho \) be a normal state on the particle algebra \( B(K) \) and suppose that the linear map \( \Phi : B(h) \rightarrow B(h \otimes K) \), which describes the interaction between the system and a particle, depends on the step-size parameter \( \tau \). (For simplicity, the domain of the generator is taken to be \( B(h) \) throughout this introduction; below it may be a general concrete operator space, or a von Neumann algebra for the applications in Section [5].) In order that the random walk with generator \( \Phi \) and particle state \( \rho \) converges to a limit cocycle, the mapping \( \Phi \) must behave correctly as the step size \( \tau \rightarrow 0 \).

As shown in [5], when \( \rho \) is a vector state given by \( \omega \in K \) then it is required that

\[
\begin{bmatrix}
\frac{\Phi(a)^0}{\tau} - a & \frac{\Phi(a)\times}{\sqrt{\tau}} \\
\frac{\Phi(a)^0}{\sqrt{\tau}} & \Phi(a)\times - a \otimes I_k
\end{bmatrix} \rightarrow
\begin{bmatrix}
\Psi(a)^0 & \Psi(a)\times \\
\Psi(a)^0 & \Psi(a)\times
\end{bmatrix}
\quad \text{as } \tau \rightarrow 0 \quad \text{for all } a \in B(h),
\]

where the convergence holds in a suitable topology and the matrix decomposition

\[
B(h \otimes K) \ni T = \begin{bmatrix}
T_0^0 & T_0^\times \\
T_0^\times & T_\times
\end{bmatrix} \in \begin{bmatrix}
B(h) & B(h \otimes k; h) \\
B(h; h \otimes k) & B(h \otimes k)
\end{bmatrix}
\]

corresponds to the Hilbert-space decomposition \( K = \mathbb{C} \omega \oplus k \).

For the other extreme, where the normal state \( \rho \) is faithful, a conditional expectation \( d \) on \( B(K) \) which preserves \( \rho \) is required, and then

\[(\tau^{-1} \delta + \tau^{-1/2} \delta^\perp)(\Phi(a) - a \otimes I_K) = \frac{\delta(\Phi(a) - a \otimes I_K)}{\tau} + \frac{\delta^\perp(\Phi(a))}{\sqrt{\tau}} \]

must converge to \( \Psi(a) \) as \( \tau \rightarrow 0 \), where \( \delta := I_{B(h) \otimes d} \) and \( \delta^\perp = I_{B(h \otimes K)} - \delta \); see [6].

The general case is resolved below. Let \( \varnothing \) be the density matrix that corresponds to the normal state \( \rho \), decompose \( K \) by letting \( K_0 := (\ker \varnothing)^\perp \), and let \( d_0 \) be a conditional expectation on \( B(K_0) \) which preserves the faithful state

\[
\rho_0 : B(K_0) \rightarrow \mathbb{C}; \quad Z \mapsto \rho\left(\begin{bmatrix}
Z & 0 \\
0 & 0
\end{bmatrix}\right).
\]

The direct sum \( K = K_0 \oplus K_0^\perp \) provides a matrix decomposition of operators in \( B(h \otimes K) \) and the appropriate modification of \( \Phi(a) \) has the form

\[
\begin{bmatrix}
\delta_0(\Phi(a)^0_0 - a \otimes I_{K_0})_0 + \delta_0^\perp(\Phi(a)^0_0) & \frac{\Phi(a)^0_0}{\sqrt{\tau}} \\
\frac{\Phi(a)^0_0}{\sqrt{\tau}} & \Phi(a)^\times_0 - a \otimes I_{K_0^\perp}
\end{bmatrix}, \quad (1.1)
\]
where \( \delta_0 := I_{\mathcal{B}(h)} \otimes d_0 \) and similarly for \( d^\perp \). The top-left corner, where \( \rho \) is faithful, is scaled by \( \tau^{-1} \) on the range of \( \delta_0 \) and by \( \tau^{-1/2} \) off it; elsewhere, the scaling is as for a vector state, with \( \mathbb{C}_{\omega} \) and \( k \) replaced by \( K_0 \) and \( K_0^\perp \), respectively.

A concrete realisation \((\hat{k}, \pi, \omega)\) of the GNS representation for \( \rho \) is employed to obtain the main result, Theorem 4.5; this circumvents problems which arise from taking a quotient, when the state is not faithful, in the standard approach. Let \( \delta \) be the conditional expectation on \( \mathcal{B}(h \otimes K) \) obtained by extending \( d_0 \) and ampliating, so that

\[
\delta(a \otimes X) = a \otimes \begin{bmatrix} d_0(X_0^0) & 0 \\ 0 & 0 \end{bmatrix}
\]

for all \( a \in \mathcal{B}(h) \) and \( X \in \mathcal{B}(K) \);

further, let \( \tilde{\pi} := I_{\mathcal{B}(h)} \otimes \pi \) and \( \tilde{\rho} := I_{\mathcal{B}(h)} \otimes \rho \). If the modification (1.1) converges to a limit \( \Psi \) in a suitable manner then the embedded random walk with generator \( \tilde{\pi} \circ \Phi \) converges to a limit cocycle \( j^\psi \) with generator

\[
\psi : a \mapsto \begin{bmatrix} \tilde{\rho} \circ \Psi(a) & (\tilde{\pi} \circ \delta^\perp \circ \Psi)(a)^{0_x} \\ (\tilde{\pi} \circ \delta^\perp \circ \Psi)(a)^{0_x} & \tilde{\pi}(\tilde{P}_x \Psi(a)\tilde{P}_x)_x \end{bmatrix},
\]

where the matrix decomposition here is that induced by writing \( \hat{k} \) as \( \mathbb{C}_{\omega} \oplus k \) and \( \tilde{P}_x \) is the orthogonal projection from \( h \otimes K \) onto \( h \otimes K_0^\perp \).

The presence of the conditional expectation \( \delta \) and the orthogonal projection \( \tilde{P}_x \) in the formula (1.2) implies that, in general, the number of independent noises in the quantum stochastic differential equation satisfied by the cocycle \( j^\psi \) is fewer than might be expected. This thermalisation phenomenon, which was first described in [1], is quantified for particles with finite degrees of freedom in Proposition 4.7.

As is well known, if the cocycle generator \( \psi \) acts by right multiplication, \( i.e. \), has the form

\[
\psi : a \mapsto (a \otimes I_k) G
\]

for some \( G \in \mathcal{B}(h \otimes \hat{k}) \), then \( j^\psi_t(a) = (a \otimes I_k)X_t \) for all \( t \geq 0 \) and \( a \in \mathcal{B}(h) \), and the driving process \( (X_t := j^\psi_t(I_h))_{t \geq 0} \) is isometric or co-isometric if and only if

\[
G + G^* + G^* \Delta G = 0 \quad \text{or} \quad G + G^* + G \Delta G^* = 0,
\]

respectively, where \( \Delta \) is the orthogonal projection from \( h \otimes \hat{k} \) onto \( h \otimes k \). If \( \Psi \) acts by right multiplication then so does the map \( \psi \) given by (1.2), and Theorem 5.2 provides necessary and sufficient conditions on \( \Psi \) for the process which drives \( j^\psi \) to be isometric or co-isometric. This is used in Theorems 5.5 and 5.8 to show that random-walk generators of the form

\[
a \mapsto (a \otimes I_k) \exp(-i\tau H(\tau)) \quad \text{and} \quad a \mapsto \exp(i\tau H(\tau))(a \otimes I_k) \exp(-i\tau H(\tau)),
\]

respectively.
where the Hamiltonian $H(\tau)$ behaves correctly as $\tau \to 0$, give rise to limit cocycles which are driven by unitary processes, \textit{i.e.}, they are of the form
\[ a \mapsto (a \otimes I_\hat{k})U_t \quad \text{and} \quad a \mapsto U_t^*(a \otimes I_\hat{k})U_t \quad \text{for all } t \geq 0, \]
where the process $(U_t)_{t \geq 0}$ is composed of unitary operators.

This article is organised as follows. The basics of quantum random walks on operator spaces are reviewed in Section 2. Section 3 contains the concrete GNS representation and some subsidiary results. The main theorem is established in Section 4, and the final section, Section 5, gives some applications of the general theory.

1.1 Conventions and notation

For the most part, the conventions and notation of [5, 6] are followed; some innovations have been introduced in an attempt to increase clarity. Vector spaces have complex scalar field; inner products are linear in the second variable. An empty sum or product equals the appropriate additive or multiplicative unit.

The indicator function of a set $S$ is denoted by $1_S$; the sets of non-negative integers and non-negative real numbers are denoted by $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ := [0, \infty)$. The identity transformation on a vector space $V$ is denoted by $I_V$ and the linear span of $A \subseteq V$ is denoted by $\text{lin} A$ and the image and kernel of a linear transformation $T$ on $V$ are denoted by $\text{im} T$ and $\ker T$; the sets of $m \times n$ and $n \times n$ matrices with entries in $V$ are denoted by $M_{m,n}(V)$ and $M_n(V)$. If the vectors $u$ and $v$ lie in an inner-product space $V$ then $|uangle\langle v|$ is the linear operator on $V$ such that $w \mapsto \langle v, w \rangle u$; the orthogonal complement of $A \subseteq V$ is denoted by $A^\perp$. Algebraic, Hilbert-space and ultraweak tensor products are denoted by $\odot$, $\otimes$ and $\bar{\odot}$, respectively. The von Neumann algebra of bounded operators on a Hilbert space $H$ is denoted by $\mathcal{B}(H)$, and $\mathcal{B}(H_1; H_2)$ denotes the Banach space of bounded operators from Hilbert space $H_1$ to Hilbert space $H_2$.

2 Walks with particles in the vacuum state

2.1 Toy and Boson Fock space

\textbf{Definition 2.1.} Let $\hat{k}$ be a Hilbert space containing the distinguished unit vector $\omega$ and let $k := \hat{k} \ominus \mathbb{C}\omega$ be the orthogonal complement of $\mathbb{C}\omega$ in $\hat{k}$. Given $x \in k$, let $\hat{x} := \omega + x \in \hat{k}$.

The \textit{toy Fock space over} $k$ is $\Gamma := \bigotimes_{n=0}^\infty \hat{k}(n)$, where $\hat{k}(n) := \hat{k}$ for all $n \in \mathbb{Z}_+$, with respect to the stabilising sequence $(\omega(n) := \omega)_{n=0}^\infty$; the suffix $(n)$ is used to indicate the relevant copy of $\hat{k}$. Note that $\Gamma = \Gamma_n \otimes \Gamma_{[n]}$, where $\Gamma_n := \bigotimes_{m=0}^{n-1} \hat{k}(m)$ and $\Gamma_{[n]} := \bigotimes_{m=n}^\infty \hat{k}(m)$, for all $n \in \mathbb{Z}_+$. 

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Notation 2.2. Let $\mathcal{F} = \mathcal{F}_+(L^2(\mathbb{R}_+;k))$ be the Boson Fock space over $L^2(\mathbb{R}_+;k)$, the Hilbert space of square-integrable $k$-valued functions on the half line. Recall that $\mathcal{F}$ may be considered as the completion of $\mathcal{E}$, the linear span of exponential vectors $\epsilon(f)$ labelled by $f \in L^2(\mathbb{R}_+;k)$, with respect to the inner product

$$\langle \epsilon(f), \epsilon(g) \rangle := \exp\left(\int_0^\infty \langle f(t), g(t) \rangle \, dt\right) \quad \text{for all } f, g \in L^2(\mathbb{R}_+;k).$$

Proposition 2.3. For all $\tau > 0$ there is a unique co-isometry $D_\tau : \mathcal{F} \to \Gamma$ such that

$$D_\tau \epsilon(f) = \bigotimes_{n=0}^\infty f(n;\tau), \quad \text{where } f(n;\tau) := \tau^{-1/2} \int_{n\tau}^{(n+1)\tau} f(t) \, dt,$$

for all $f \in L^2(\mathbb{R}_+;k)$. Furthermore, $D_\tau^* D_\tau \to I_\mathcal{F}$ strongly as $\tau \to 0$.

Proof. See [4, Section 2].

2.2 Matrix spaces

For more detail on the topics of this subsection and the next, see [13].

Henceforth $V$ is a fixed concrete operator space, i.e., a norm-closed subspace of $B(h)$, where $h$ is a Hilbert space.

Definition 2.4. For a Hilbert space $H$, the matrix space

$$V \otimes M B(H) := \{ T \in B(h \otimes H) : E^x T E_y \in V \text{ for all } x, y \in H \}$$

is an operator space, where $E^x \in B(h \otimes H; h)$ is the adjoint of

$$E_x : h \to h \otimes H; \quad u \mapsto u \otimes x.$$

Note that $V \otimes B(H) \subseteq V \otimes M B(H) \subseteq V \otimes B(H)$, with the latter an equality if $V$ is ultraweakly closed, and $(V \otimes M B(H_1)) \otimes M B(H_2) = V \otimes M B(H_1 \otimes H_2)$.

Definition 2.5. If $W$ is an operator space and $H$ is a non-zero Hilbert space then a linear map $\phi : V \to W$ is $H$ bounded if $||\phi||_{Hb} < \infty$, where

$$||\phi||_{Hb} := \begin{cases} \dim H ||\phi|| & \text{if } \dim H < \infty, \\ ||\phi||_{cb} & \text{if } \dim H = \infty, \end{cases}$$

with $|| \cdot ||$ and $|| \cdot ||_{cb}$ the operator and completely bounded norms, respectively. The Banach space of all such $H$-bounded maps, with norm $|| \cdot ||_{Hb}$, is denoted by $HB(V;W)$. 

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Proposition 2.6. Let $\Phi \in \mathcal{HB}(V; W)$. The unique map $\Phi \otimes_M I_{B(H)} : V \otimes_M B(H) \to W \otimes_M B(H)$ such that
\[
E^x(\Phi \otimes_M I_{B(H)}(T)) E_y = \Phi(E^x T E_y)
\]
for all $x, y \in H$ and $T \in V \otimes_M B(H)$ is the $H$ lifting of $\Phi$. This lifting is linear, $H$ bounded and such that $\|\Phi \otimes_M I_{B(H)}\| \leq \|\Phi\|_{\text{Hb}}$; if $\Phi$ is completely bounded then so is $\Phi \otimes_M I_{B(H)}$, with $\|\Phi \otimes_M I_{B(H)}\|_{\text{cb}} \leq \|\Phi\|_{\text{cb}}$.

Proof. See [5] Theorem 2.5. \qed

Proposition 2.7. Let $\Phi \in \mathcal{HB}(V; \otimes_M B(H))$. There exists a unique family of maps $\Phi^{(n)} : V \to V \otimes_M B(H^{\otimes n})$ indexed by $n \in \mathbb{Z}_+$, the quantum random walk with generator $\Phi$, such that $\Phi^{(0)} = I_V$ and
\[
E^x \Phi^{(n+1)}(a) E_y = \Phi^{(n)}(E^x \Phi(a) E_y)
\]
for all $x, y \in H$, $a \in V$ and $n \in \mathbb{Z}_+$.

Each map is linear, $H$ bounded and such that $\|\Phi^{(n)}\|_{\text{Hb}} \leq \|\Phi\|^n_{\text{Hb}}$ for all $n \geq 1$; if $\Phi$ is completely bounded then so is $\Phi^{(n)}$, with $\|\Phi^{(n)}\|_{\text{cb}} \leq \|\Phi\|^n_{\text{cb}}$ for all $n \in \mathbb{Z}_+$.

Proof. See [5] Theorem 2.7. \qed

2.3 Quantum stochastic cocycles

Definition 2.8. An $h$ process $X$ is a family $(X_t)_{t \in \mathbb{R}_+}$ of linear operators in $h \otimes \mathcal{F}$, such that the domain of each operator contains $h \otimes E$ and the map $t \mapsto X_t u \varepsilon(f)$ is weakly measurable for all $u \in h$ and $f \in L^2(\mathbb{R}_+; k)$; this process is adapted if
\[
\langle u \varepsilon(f), X_t v \varepsilon(g) \rangle = \langle u \varepsilon([0,t], f), X_t v \varepsilon([0,t], g) \rangle \langle \varepsilon([1,t,\infty), f), \varepsilon([1,t,\infty), g) \rangle
\]
for all $u, v \in h$, $f, g \in L^2(\mathbb{R}_+; k)$ and $t \in \mathbb{R}_+$. (As is conventional, the tensor-product sign is omitted between elements of $h$ and exponential vectors.)

A mapping process $j$ is a family $(j_t(a))_{a \in V}$ of $h$ processes such that the map $a \mapsto j_t(a)$ is linear for all $t \in \mathbb{R}_+$; this process is adapted if each $h$ process $j_t(a)$ is, and is strongly regular if
\[
j_t(\cdot) E_{\varepsilon(f)} \in \mathcal{B}(V; B(h; h \otimes \mathcal{F})) \quad \text{for all } f \in L^2(\mathbb{R}_+; k) \text{ and } t \in \mathbb{R}_+,
\]
with norm locally uniformly bounded as a function of $t$.

Theorem 2.9. Let $\psi \in \hat{kB}(V; \otimes_M B(\hat{k}))$. There exists a unique strongly regular adapted mapping process $j^\psi$, the quantum stochastic cocycle generated by $\psi$, such that
\[
\langle u \varepsilon(f), (j^\psi_t(a) - a \otimes I_{\mathcal{F}}) v \varepsilon(g) \rangle = \int_0^t \langle u \varepsilon(f), j^\psi_s( E^s f \psi(a) E_{g(s)}^{-1} ) v \varepsilon(g) \rangle \, ds
\]
for all $u, v \in h$, $f, g \in L^2(\mathbb{R}_+; k)$, $a \in V$ and $t \in \mathbb{R}_+$. The process $j^\psi$ has the Feller property, in the sense that $E^x(f) j^\psi_t(a) E_{g(t)}(g) \in V$ for all $f, g \in L^2(\mathbb{R}_+; k)$, $a \in V$ and $t \in \mathbb{R}_+$. If $\psi$ is completely bounded then so is $j^\psi_t(\cdot) E_{\varepsilon(f)}$, for all $f \in L^2(\mathbb{R}_+; k)$ and $t \in \mathbb{R}_+$. 


Remark 2.10. The fact that \((2.2)\) holds is equivalent to saying that the strongly regular adapted mapping process \(j^\psi\) satisfies the quantum stochastic differential equation
\[
dj^\psi_t(a) = j^\psi_t d\Lambda_{\psi(a)}(t) \quad \text{for all } t \in \mathbb{R}_+,
\]
with the initial condition \(j^\psi_0(a) = a \otimes I_T\), for all \(a \in A\).

Definition 2.11. Let \(\tau > 0\) and \(\Phi \in \hat{k}B(V; V \otimes_M B(\hat{k}))\). The embedded random walk with generator \(\Phi\) and step size \(\tau\) is the mapping process \(J^\Phi_{\cdot \tau}\) such that
\[
J^\Phi_{t \tau}(a) := (I_h \otimes D_\tau)^*(\Phi^{(n)}(a) \otimes I_{[n]})(I_h \otimes D_\tau) \quad \text{if } t \in [n\tau, (n + 1)\tau)
\]
for all \(a \in V\) and \(n \in \mathbb{Z}_+\).

Notation 2.12. Let \(\tau > 0\) and \(\Phi \in \hat{k}B(V; V \otimes_M B(\hat{k}))\), and let \(\Delta\) denote the orthogonal projection from \(h \otimes \hat{k}\) onto \(h \otimes k\), with \(\Delta^\perp := I_{h \otimes k} - \Delta\). The modification
\[
m(\Phi, \tau) : V \to V \otimes_M B(\hat{k}); \quad a \mapsto (\tau^{-1/2}\Delta^\perp + \Delta)(\Phi(a) - a \otimes I_{\hat{k}})(\tau^{-1/2}\Delta^\perp + \Delta)
\]
is \(\hat{k}\) bounded, and is completely bounded whenever \(\Phi\) is.

Theorem 2.13. Let \(\tau_n > 0\) and \(\Phi_n, \psi \in \hat{k}B(V; V \otimes B(\hat{k}))\) be such that
\[
\tau_n \to 0 \quad \text{and} \quad m(\Phi_n, \tau_n) \otimes_M I_{B(\hat{k})} \to \psi \otimes_M I_{B(\hat{k})} \text{ strongly,}
\]
i.e., pointwise in norm, as \(n \to \infty\). Then
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| J^{\Phi_n, \tau_n}(a) E_{\varepsilon(f)} - J^\psi_t(a) E_{\varepsilon(f)} \| = 0 \quad \text{for all } a \in V, f \in L^2(\mathbb{R}_+; k) \text{ and } T \in \mathbb{R}_+.
\]
If, further, \(\| m(\Phi_n, \tau_n) - \psi \|_{k\hat{b}} \to 0\) as \(n \to \infty\) then
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| J^{\Phi_n, \tau_n}(\cdot) E_{\varepsilon(f)} - J^\psi_t(\cdot) E_{\varepsilon(f)} \|_{k\hat{b}} = 0 \quad \text{for all } f \in L^2(\mathbb{R}_+; k) \text{ and } T \in \mathbb{R}_+;
\]
when \(\Phi_n\) and \(\psi\) are completely bounded, the same holds with \(\| \cdot \|_{k\hat{b}}\) replaced by \(\| \cdot \|_{cb}\).

Proof. See [5, Theorem 7.6].

Notation 2.14. For brevity, the conclusion \((2.4)\) will be denoted by \(J^{\Phi, \tau} \to j^\psi\); the stronger conclusion \((2.5)\) will be denoted by \(J^{\Phi, \tau} \to_{k\hat{b}} j^\psi\), or by \(J^{\Phi, \tau} \to_{cb} j^\psi\) if the completely bounded version holds.

Remark 2.15. If \(H\) is infinite dimensional and \(\phi_n \in HB(V; W)\) then \(\phi_n \otimes_M I_{B(H)} \to 0\) strongly if and only if \(\| \phi_n \|_{cb} \to 0\) [5, Lemma 2.13].
3 A concrete GNS representation

**Definition 3.1.** If $H$ is a Hilbert space then $H^\dagger$ denotes the Hilbert space *conjugate* to $H$; thus $H^\dagger := \{ u^\dagger : u \in H \}$, with

$$ u^\dagger + v^\dagger := (u + v)^\dagger, \quad \lambda u^\dagger := (\overline{\lambda} u)^\dagger \quad \text{and} \quad \langle u^\dagger, v^\dagger \rangle := \langle v, u \rangle $$

for all $u, v \in H$ and $\lambda \in \mathbb{C}$. Note that the map

$$ \dagger : B(H) \to B(H^\dagger); \quad T^\dagger(u^\dagger) := (Tu)^\dagger \quad \text{for all } T \in B(H) \text{ and } u \in H $$

is anti-linear and isometric, and it commutes with the adjoint.

**Notation 3.2.** If $H$ is a Hilbert space then $B_2(H)$ is the Hilbert space of Hilbert–Schmidt operators on $H$, with inner product $\langle S, T \rangle := \text{tr}(S^*T)$ where $\text{tr}$ is the standard trace on $B(H)$. Recall that $B_2(H)$ is a two-sided $*$-ideal in the $*$-algebra $B(H)$.

**Proposition 3.3.** The isometric isomorphism $U_H : B_2(H) \to H \otimes H^\dagger$ determined by the requirement that $U_H(|u\rangle\langle v|) = u \otimes v^\dagger$ for all $u, v \in H$ is such that

$$ U_H(XTY^*) = (X \otimes Y^\dagger)U_H(T) \quad \text{for all } T \in B_2(H) \text{ and } X, Y \in B(H). \quad (3.1) $$

**Proof.** This is elementary. \hfill \Box

**Notation 3.4.** Let $\rho$ be a normal state on $B(K)$ with density matrix $\varrho \in B(K)$, so that

$$ \varrho \geq 0, \quad \varrho^{1/2} \in B_2(K), \quad \| \varrho^{1/2} \|_2 = 1 \quad \text{and} \quad \rho(X) = \text{tr}(\varrho X) \quad \text{for all } X \in B(K). $$

Let $P_0$ denote the orthogonal projection from $K$ onto $K_0 := \overline{\text{im } \varrho^{1/2}} = (\ker \varrho^{1/2})^\perp$, where $\overline{\cdot}$ denotes norm closure.

**Proposition 3.5.** Let $\hat{k} := K \otimes K_0^\dagger$. The injective normal unital $*$-homomorphism

$$ \pi : B(K) \to B(\hat{k}); \quad X \mapsto X \otimes I_{K_0^\dagger}, $$

the concrete GNS representation, has cyclic vector $\omega := U_K(\varrho^{1/2}) \in \hat{k}$ such that

$$ \langle \omega, \pi(X)\omega \rangle_{\hat{k}} = \rho(X) \quad \text{for all } X \in B(K) $$

and

$$ \rho(XP_0) = \rho(X) = \rho(P_0X) \quad \text{for all } X \in B(K). \quad (3.2) $$
Proof. Note that \( q^{1/2} = q^{1/2}P_0 \), since \( K_0^\perp = \ker q^{1/2} \), and so, by Proposition \[3.3\]

\[
\omega = U_K(q^{1/2}P_0) = (I_K \otimes P_0^\dagger)U_K(q^{1/2}) \in K \otimes K_0^\dagger = \hat{k}
\]

and

\[
\langle \omega, \pi(X)\omega \rangle_{\hat{k}} = \langle U_K(q^{1/2}), U_K(Xq^{1/2}) \rangle_{\hat{k}} = \text{tr}(qX) = \rho(X) \quad \text{for all } X \in B(K).
\]

By Proposition \[3.3\]

\[
\pi(\langle u \rangle \langle v \rangle \omega = U_K(\langle u \rangle \langle v \rangle q^{1/2}) = u \otimes (q^{1/2}v)^\dagger \quad \text{for all } u, v \in K,
\]

thus

\[
\{ \pi(X)\omega : X \in B(K) \} \supseteq \text{lin}\{ u \otimes (q^{1/2}v)^\dagger : u, v \in K \} = K \otimes (\text{im } q^{1/2})^\dagger
\]

and \( \omega \) is cyclic for \( \pi \). Finally,

\[
\rho(P_0X) = \text{tr}(qP_0X) = \text{tr}(qX) = \rho(X) \quad \text{for all } X \in B(K)
\]

and, similarly, \( \rho(XP_0) = \rho(X) \).

\[ \square \]

**Notation 3.6.** For brevity, let

\[
[X] := \pi(X)\omega = U_K(Xq^{1/2}) \quad \text{for all } X \in B(K),
\]

where \( U_K \) is as in Proposition \[3.3\]. Note that \( [X] \in \hat{k} := (\mathbb{C} \omega)^\perp \) if and only if \( X \in \ker \rho \).

**Proposition 3.7.** The amplified representation

\[
\tilde{\pi} := I_{B(h)} \otimes \pi : B(h \otimes K) \to B(h \otimes \hat{k}); \; T \mapsto T \otimes I_0^\dagger
\]

is an injective normal unital \( * \)-homomorphism such that \( \tilde{\pi}(V \otimes M B(K)) \subseteq V \otimes M B(\hat{k}) \). The slice map

\[
\tilde{\rho} := I_{B(h)} \otimes \rho : B(h \otimes K) \to B(h)
\]

is completely positive, normal and such that

\[
E^{[X]}\tilde{\pi}(T)E^{[Y]} = \tilde{\rho}((I_h \otimes X)^*T(I_h \otimes Y)) \quad \text{for all } T \in B(h \otimes K) \text{ and } X, Y \in B(K); \quad (3.3)
\]

in particular, \( \tilde{\rho}(V \otimes M B(K)) \subseteq V \). Furthermore,

\[
\tilde{\rho}(\tilde{P}_0T) = \tilde{\rho}(T) = \tilde{\rho}(T \tilde{P}_0) \quad \text{for all } T \in B(h \otimes K), \quad (3.4)
\]

where \( \tilde{P}_0 := I_h \otimes P_0 \) is the orthogonal projection from \( h \otimes K \) onto \( h \otimes K_0 \).
Proof. The existence of $\pi$ and $\tilde{\rho}$ is standard; see [17] Theorem IV.5.2 & Proposition IV.5.13. Furthermore,

$$\pi(V \otimes_M B(k)) \subseteq (V \otimes_M B(k)) \otimes B(K_0^1) \subseteq V \otimes_M B(K \otimes K_0^1) = V \otimes_M B(\hat{k}).$$

Next, observe that if $D$ is a faithful normal state.

$$K := \pi(\rho_0 X F_0^*) = \rho(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}) = \text{tr}(F_0^* g F_0 X)$$

is a normal positive linear functional. As $\varrho$ is compact and positive, there exists an orthonormal set $\{e_j : j \in J\} \subseteq K$ such that

$$\varrho = \sum_{j \in J} \lambda_j |e_j\rangle \langle e_j|,$$

where $\lambda_j > 0$ for all $j \in J$ and $\sum_{j \in J} \lambda_j = 1$. Since $e_j \in K_0$ for all $j \in J$, it follows that

$$\rho_0(X) = \text{tr}(gF_0XF_0^*) = \sum_{j \in J} \lambda_j \langle e_j, F_0 XF_0^* e_j \rangle = \sum_{j \in J} \lambda_j \langle e_j, X e_j \rangle \quad \text{for all } X \in B(K_0);$$

in particular, $\rho_0(I_{K_0}) = 1$. Furthermore, $\{e_j : j \in J\}$ is total in $K_0$, so if $X \in B(K_0)$ then

$$\rho_0(X^* X) = 0 \iff \sum_{j \in J} \lambda_j \|X e_j\|^2 = 0 \iff X = 0. \quad \square$$

Notation 3.10. Fix a conditional expectation $d_0$ from $B(K_0)$ onto a *-subalgebra $D_0$, and suppose $d_0$ preserves the state $\rho_0$.

By definition, $d_0$ is a completely positive linear idempotent which is $D_0$ linear, i.e., a module map for the natural $D_0 - D_0$-bimodule structure on $B(K_0)$. As $\rho_0 \circ d_0 = \rho_0$ and $\rho_0$ is faithful, it follows that $d_0$ is ultraweakly continuous, so $D_0$ is a von Neumann algebra, and $d_0$ is unital, i.e., $d_0(I_{K_0}) = I_{K_0}$.
Proposition 3.11. The ultraweakly continuous map

\[ d : B(K) \to B(K); \quad X = \begin{bmatrix} x_0^* & x_0^* \\ x_0^* & x_0^* \end{bmatrix} \mapsto F_0 d_0 (F_0^* X F_0) F_0^* = \begin{bmatrix} d_0(x_0^*) & 0 \\ 0 & 0 \end{bmatrix} \]

is a conditional expectation onto \( F_0 D_0 F_0^* \) such that \( \rho \circ d = \rho \), so the ultraweakly continuous ampliation

\[ \delta := I_{B(h)} \otimes d : B(h \otimes K) \to B(h \otimes K) \]

is a conditional expectation onto \( B(h) \otimes F_0 D_0 F_0^* \) such that \( \tilde{\rho} \circ \delta = \tilde{\rho} \) and

\[ \delta(V \otimes_M B(K)) \subseteq V \otimes_M B(K). \quad (3.5) \]

Proof. The maps \( d \) and \( \delta \) inherit linearity, idempotency, complete positivity and ultraweak continuity from \( d_0 \); furthermore,

\[ d(d(X)Y) = F_0 d_0 (F_0^* X F_0) d_0 (F_0^* Y F_0) F_0^* = F_0 d_0 (F_0^* X F_0) = d(X) d(Y) \quad \text{for all } X, Y \in B(K) \]

and, using the adjoint, \( d(X d(Y)) = d(X) d(Y) \). Thus \( d \) and \( \delta \) are conditional expectations.

To see that states are preserved, let \( X \in B(K) \) and recall that \( d_0 \) preserves \( \rho_0 \), so

\[ \rho(d(X)) = \rho_0(d_0(F_0^* X F_0)) = \rho_0(F_0^* X F_0) = \rho(P_0 X P_0) = \rho(X), \]

where the final equality follows from (3.2).

Finally, let \( T \in B(h \otimes K) \) and note that \( T \in V \otimes_M B(K) \) if and only if \( (I_{B(h)} \otimes \phi)(T) \in V \) for every normal linear functional \( \phi \) on \( B(K) \). As \( d \) is ultraweakly continuous, the inclusion (3.5) follows.

4 Walks with an arbitrary normal particle state

Throughout this section, \( \rho \) is a normal state on \( B(K) \) corresponding to the density matrix \( \varrho \), the subspace \( K_0 = \text{im } \varrho^{1/2} \) and \( (\hat{k}, \pi, \omega) \) is the concrete GNS representation of Proposition 3.5.

Notation 4.1. Given \( \Phi \in KB(V; V \otimes_M B(K)) \), let \( \Phi'(a) := \Phi(a) - a \otimes I_K \) for all \( a \in V \). Recall that \( \tilde{P}_0 \) is the orthogonal projection from \( h \otimes K \) onto \( h \otimes K_0 \).

The following definition gives the correct modification of a generator for a quantum random walk with particle state \( \rho \) and conditional expectation \( d_0 \).
Definition 4.2. Let $\tau > 0$ and $\Phi \in KB(V; V \otimes_M B(K))$. The modification

$$m_\delta(\Phi, \tau) : V \to V \otimes_M B(K); \ a \mapsto \tilde{P}_0(\tau^{-1}\delta + \tau^{-1/2}\delta \perp^\perp)(\Phi'(a)) \tilde{P}_0 + \tau^{-1/2} \tilde{P}_0 \Phi(a) \tilde{P}_0^\perp + \tau^{-1/2} \tilde{P}_0^\perp \Phi(a) \tilde{P}_0 + \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp \quad (4.1)$$

is $K$ bounded, and is completely bounded whenever $\Phi$ is.

Remark 4.3. The modification $(4.1)$ acts as follows: on the block corresponding to $K_0 \times K_0$, the scaling régime appropriate for a faithful normal state is adopted $[6]$; on the blocks corresponding to $K_0 \times K_0^\perp$, $K_0^\perp \times K_0$ and $K_0^\perp \times K_0^\perp$, the scaling is that used for the vector-state situation, Theorem 2.13 with $K_0$ playing the rôle of $\CC \omega$ and $K_0^\perp$ that of $k$.

In particular, if $\rho$ is faithful then $(4.1)$ is the same modification as in [6] Definition 11], whereas if $\rho$ is a vector state then $d_0$ must be the identity map and the modification is the same as that given in [2,3].

Lemma 4.4. Let $\tau > 0$ and $\Phi \in KB(V; V \otimes_M B(K))$. Then

$$(\tau \delta + \tau^{1/2}\delta \perp^\perp)(m_\delta(\Phi, \tau)(a)) = \Phi'(a) + (\tau^{1/2} - 1) \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp \quad \text{for all } a \in V.$$

**Proof.** Note first that, as $d_0(I_{K_0}) = I_{K_0}$, it follows that $d(I_K) = P_0$ and so, by the bimodule property for a conditional expectation,

$$d(P_0 X) = d(X) = d(X P_0) \quad \text{for all } X \in B(K).$$

Hence, using the bimodule property again,

$$\tilde{P}_0 \delta(T) = \delta(\tilde{P}_0 T) = \delta(T) = \delta(T \tilde{P}_0) = \delta(T) \tilde{P}_0$$

and

$$\tilde{P}_0^\perp \delta(T) = \delta(\tilde{P}_0^\perp T) = 0 = \delta(T \tilde{P}_0^\perp) = \delta(T) \tilde{P}_0^\perp \quad \text{for all } T \in B(h \otimes K). \quad (4.2)$$

Consequently,

\begin{align*}
(\tau \delta + \tau^{1/2} \delta \perp^\perp)(m_\delta(\Phi, \tau)(a)) &= \delta(\Phi'(a)) + \tau^{1/2} m_\delta(\Phi, \tau)(a) - \tau^{-1/2} \delta(\Phi'(a)) \\
&= (1 - \tau^{-1/2}) \delta(\Phi'(a)) + \tilde{P}_0(\tau^{-1/2} \delta + \delta \perp^\perp)(\Phi'(a)) \tilde{P}_0 \\
&\quad + \tilde{P}_0 \Phi(a) \tilde{P}_0^\perp + \tilde{P}_0^\perp \Phi(a) \tilde{P}_0 + \tau^{1/2} \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp \\
&= (1 - \tau^{-1/2}) \delta(\Phi'(a)) + (\tau^{-1/2} - 1) \tilde{P}_0 \delta(\Phi'(a)) \tilde{P}_0 \\
&\quad + \tilde{P}_0 \Phi(a) \tilde{P}_0 + \tilde{P}_0^\perp \Phi(a) \tilde{P}_0 + \tilde{P}_0^\perp \Phi(a) \tilde{P}_0 + \tau^{1/2} \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp \\
&= \Phi'(a) + (\tau^{1/2} - 1) \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp. \quad \square
\end{align*}
The following theorem gives a convergence result for quantum random walks with particles in the arbitrary normal state \( \rho \). Recall that \( \Delta \) denotes the orthogonal projection from \( \mathfrak{h} \otimes \mathfrak{k} \) onto \( \mathfrak{h} \otimes \mathfrak{k} \).

**Theorem 4.5.** Let \( \tau_n > 0 \) and \( \Phi_n, \Psi \in \mathcal{KB}(\mathcal{V}; \mathcal{V} \otimes \mathcal{M} \mathcal{B}(\mathcal{K})) \) be such that

\[
\tau_n \to 0 \quad \text{and} \quad m_\delta(\Phi_n, \tau_n) \otimes \mathcal{M} I_{\mathcal{B}(\mathcal{K})} \to \Psi \otimes \mathcal{M} I_{\mathcal{B}(\mathcal{K})} \quad \text{strongly as } n \to \infty.
\]

Define \( \psi \in \hat{\mathcal{K}} \mathcal{B}(\mathcal{V}; \mathcal{V} \otimes \mathcal{M} \mathcal{B}(\mathcal{K})) \) by setting

\[
\psi (a) := \Delta (\hat{\pi} \circ \Psi)(a) \Delta + \Delta (\hat{\pi} \circ \delta \circ \Psi)(a) \Delta + \Delta \hat{\pi} \delta (\tilde{P}_0 \Psi(a) \tilde{P}_0 \Delta) \quad \text{for all } a \in \mathcal{V},
\]

and note that \( \psi \) is completely bounded if \( \Psi \) is. Then \( J_{\tilde{\pi} \circ \Phi, \tau} \to j_\psi \); furthermore,

\[
\text{if } \|m_\delta(\Phi_n, \tau_n) - \Psi\|_K \to 0 \quad \text{then} \quad J_{\tilde{\pi} \circ \Phi, \tau} \to_k, \quad \text{and, when } \Phi_n \text{ and } \Psi \text{ are completely bounded,}
\]

\[
\text{if } \|m_\delta(\Phi_n, \tau_n) - \Psi\|_{cb} \to 0 \quad \text{then} \quad J_{\tilde{\pi} \circ \Phi, \tau} \to_{cb} j_\psi.
\]

**Proof.** Let \( a \in \mathcal{V} \) and, for brevity, let \( \tau = \tau_n \) and \( \Phi = \Phi_n \). Note first that

\[
E_\omega m(\hat{\pi} \circ \Phi, \tau)(a) E_\omega = \tau^{-1} E_\omega \hat{\pi} (\Phi'(a)) E_\omega = \tau^{-1} \rho_\delta(\Phi'(a)),
\]

by (2.3) and (3.3), whereas

\[
E_\omega \hat{\pi} (m_\delta(\Phi, \tau)(a)) E_\omega = \rho(m_\delta(\Phi, \tau)(a)) = \tau^{-1} \rho(\Phi'(a)),
\]

with the second equality a consequence of (3.3) and the fact that \( \delta \) preserves \( \tilde{\rho} \). Hence

\[
E_\omega m(\hat{\pi} \circ \Phi, \tau)(a) E_\omega = E_\omega \hat{\pi} (m_\delta(\Phi, \tau)(a)) E_\omega.
\]

Next, let \( X \in \ker \rho \) and use (2.3) and (3.3) again to see that

\[
E_\omega m(\hat{\pi} \circ \Phi, \tau)(a) E_{[X]} = \tau^{-1/2} \rho_\delta(\Phi(a)(I_h \otimes X)) = \tilde{\rho}((\tau^{-1/2} \Phi'(a)(I_h \otimes X)),
\]

where the second equality holds because \( \tilde{\rho}(a \otimes X) = \rho(X)a = 0 \). As \( \delta \) preserves \( \tilde{\rho} \), so

\[
\tilde{\rho}(\tilde{P}_0 \Phi'(a) \tilde{P}_0 (I_h \otimes X)) = (\tilde{\rho} \circ \delta)(\tilde{P}_0 \Phi'(a) \tilde{P}_0 (I_h \otimes X)) = 0,
\]

by (4.2), and then Lemma 4.4 gives that

\[
E_\omega m(\hat{\pi} \circ \Phi, \tau)(a) E_{[X]} = \tilde{\rho}((\tau^{1/2} \delta + \delta)(m_\delta(\Phi, \tau)(a)) (I_h \otimes X))
\]

\[
= E_\omega \hat{\pi} ((\tau^{1/2} \delta + \delta)(m_\delta(\Phi, \tau)(a))) E_{[X]};
\]

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similar working produces the identity
\[ E^{[X]} m(\tilde{\pi} \circ \Phi, \tau)(a) E_{\omega} = E^{[X]} \tilde{\pi} \left( (\tau^{1/2} \delta + \delta^\perp)(m_\delta(\Phi, \tau)(a)) \right) E_{\omega}. \]

Finally, if \( X, Y \in \ker \rho \) then \([2,3]\) and Lemma \([1,4]\) give that
\[ E^{[X]} m(\tilde{\pi} \circ \Phi, \tau)(a) E_{[Y]} = E^{[X]} \tilde{\pi} \left( \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp + \tau^{1/2} R_1(a, \tau) \right) E_{[Y]}, \]
where
\[ R_1(a, \tau) := (\tau^{1/2} \delta + \delta^\perp)(m_\delta(\Phi, \tau)(a)) - \tilde{P}_0^\perp \Phi'(a) \tilde{P}_0^\perp. \]

Hence
\[ (m(\tilde{\pi} \circ \Phi, \tau) - \psi)(a) = \Delta^\perp \tilde{\pi}((m_\delta(\Phi, \tau) - \Psi)(a)) \Delta^\perp + \Delta^\perp (\tilde{\pi} \circ \delta^\perp)((m_\delta(\Phi, \tau) - \Psi)(a)) \Delta^\perp \]
\[ + \Delta \tilde{\pi}(\tilde{P}_0^\perp(m_\delta(\Phi, \tau) - \Psi)(a) \tilde{P}_0^\perp) \Delta + \tau^{1/2} R_2(a, \tau), \]
where
\[ R_2(a, \tau) := \Delta^\perp (\tilde{\pi} \circ \delta \circ m_\delta(\Phi, \tau))(a) \Delta + \Delta^\perp (\tilde{\pi} \circ \delta \circ m_\delta(\Phi, \tau))(a) \Delta + \Delta \tilde{\pi}(R_1(a, \tau)) \Delta. \]

The result now follows from Theorem \([2,13]\). \(\square\)

**Remark 4.6.** Theorem \([4,5]\) is an extension of previous results. If \( \rho \) is faithful or a vector state then it reduces to \([6, \text{Theorem 3}] \) or \([5, \text{Theorem 7.6}] \), respectively; the former theorem has \([11, \text{Theorem 7}] \) as a special case, whereas the latter is a generalisation of Attal and Pautrat's convergence theorem \([3, \text{Theorem 13}] \).

**Proposition 4.7.** Let \( X, Y \in \ker \rho \). If \( \psi(a) \) is given by \((4.3)\) then
\[ E^{[\omega]} \psi(a) E_{\omega} = E^{[\omega]} \tilde{\pi}(\Psi(a)) E_{\omega} = \tilde{\rho}(\Psi(a)), \]
\[ E^{[\omega]} \psi(a) E_{[Y]} = E^{[\omega]} \tilde{\pi}(\Psi(a)) E_{[d^+(Y)]}, \]
\[ E^{[X]} \psi(a) E_{\omega} = E^{[d^+(X)]} \tilde{\pi}(\Psi(a)) E_{\omega} \]
and
\[ E^{[X]} \psi(a) E_{[Y]} = E^{[d^+(X) \tilde{P}_0^\perp Y]} \tilde{\pi}(\Psi(a)) E_{[d^+(Y)]}. \]

Thus if \( N := \dim K < \infty \) then there can be no more than
\[ 2(Nk - l) + (N - k)^2 k^2 \]
(independent noises in the quantum stochastic differential equation \((2.2)\) satisfied by the limit cocycle \( j^\psi \), where
\[ k := \dim K_0 \in \{1, \ldots, N\} \quad \text{and} \quad l := \rank d_0 \in \{1, \ldots, k^2\}. \]
Proof. If \( Y \in \ker \rho \) then (3.3) implies that
\[
E^\omega \psi(a)E_{[Y]} = \tilde{\rho}((\delta^\perp \circ \Psi)(a)(I_h \otimes Y)) = \tilde{\rho}(\Psi(a)(I_h \otimes Y)) - \tilde{\rho}((\delta \circ \Psi)(a)(I_h \otimes Y))
\]
However, as \( \delta \) preserves \( \tilde{\rho} \), it follows from the bimodule property that
\[
\tilde{\rho}((\delta \circ \Psi)(a)(I_h \otimes Y)) = (\tilde{\rho} \circ \delta)((\delta \circ \Psi)(a)(I_h \otimes Y)) = (\tilde{\rho} \circ \delta)(\Psi(a)\delta(I_h \otimes Y))
\]
and therefore
\[
E^\omega \psi(a)E_{[Y]} = \tilde{\rho}(\Psi(a)\delta^\perp(I_h \otimes Y)) = E^\omega \pi(\Psi(a))E_{[d^\perp(Y)]},
\]
as required. The other identities are may be established similarly.

Henceforth, suppose that \( K \) is finite dimensional. From the previous working, there can be no more than \( 2n_1 + n_2^2 \) independent noises in the quantum stochastic differential equation (2.2), where
\[
n_1 := \dim\{[d^\perp(X)] : X \in \ker \rho\} \quad \text{and} \quad n_2 := \dim\{[P^\perp_0 X] : X \in \ker \rho\}.
\]
To find \( n_2 \), note that
\[
[P^\perp_0 X] = (P^\perp_0 \otimes I_{K^\perp_0})\pi(X)\omega = U_K(P^\perp_0 X\varrho^{1/2}) \quad \text{for all } X \in B(K),
\]
so that, in particular, \([P^\perp_0] = 0\); as \( \omega \) is a cyclic vector for the representation \( \pi \), it follows that
\[
n_2 = \operatorname{rank}(P^\perp_0 \otimes I_{K^\perp_0}) = \dim(K^\perp_0 \otimes K^\perp_0) = (N - k)k.
\]
For \( n_1 \), note first that \( d^\perp(I_K) = P^\perp_0 \) and \([P^\perp_0] = 0\), hence
\[
n_1 = \dim\{[d^\perp(X)] : X \in B(K)\} = \dim\{d^\perp(X)\varrho^{1/2} : X \in B(K)\}.
\]
Writing \( X = \begin{bmatrix} x^0_0 & x^x_0 \\ x^0_\times & x^\times_\times \end{bmatrix} \) and \( \varrho = \begin{bmatrix} \varrho_0 & 0 \\ 0 & 0 \end{bmatrix} \), it follows that
\[
d^\perp(X)\varrho^{1/2} = \begin{bmatrix} d^\perp_0(x^0_0) & x^0_\times \\ x^0_\times & x^\times_\times \end{bmatrix} \begin{bmatrix} \varrho_0^{1/2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} d^\perp_0(x^0_0)\varrho_0^{1/2} & 0 \\ x^0_\times \varrho_0^{1/2} & 0 \end{bmatrix}.
\]
As \( \varrho_0 \) is faithful, the operator \( \varrho_0^{1/2} \) is invertible, therefore
\[
\dim\{X^\times_0 \varrho_0^{1/2} : X^\times_0 \in B(K_0; K^\perp_0)\} = \dim B(K_0; K^\perp_0) = k(N - k).
\]
Similarly,
\[
\{d^\perp_0(Z)\varrho_0^{1/2} : Z \in B(K_0)\} \cong \{d^\perp_0(Z) : Z \in B(K_0)\} = \{d_0(Z) : Z \in B(K_0)\}^\perp,
\]
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where the orthogonal complement is taken with respect to the inner product
\[ \langle Z, W \rangle := \rho_0(Z^*W) \quad \text{for all } Z, W \in \mathcal{B}(K_0); \]
the last equality holds because the bimodule property and the fact that \( d_0 \) preserves \( \rho_0 \) imply that \( d_0 \) is a self-adjoint linear idempotent, i.e., an orthogonal projection, on this inner-product space.

As
\[ \dim \{ d_0(Z) : Z \in \mathcal{B}(K_0) \}^\perp = \dim \mathcal{B}(K_0) - \text{rank } d_0 = k^2 - l, \]
the result now follows.

Remark 4.8. Suppose \( N := \dim K < \infty \) and let \( k := \dim K_0 \) and \( l := \text{rank } d_0 \), as in Proposition 4.7. Since \( \hat{k} = K \otimes K_0^* \), in principle \( \dim \hat{k} = N^2k^2 - 1 \) quantum noises can appear in the quantum stochastic differential equation (2.2).

If \( \rho \) is a vector state then \( k = 1 \) and \( l = 1 \), so (4.4) equals
\[ 2(N - 1) + (N - 1)^2 = N^2 - 1, \]
as expected. At the other extreme, if \( \rho \) is a faithful state then (4.4) equals \( 2(N^2 - l) \).

In general,
\[ N^2k^2 - 1 - (2(Nk - l) + (N - k)^2k^2) = 2Nk^3 - k^4 - 2Nk + 2l - 1 \]
\[ = (k^2 - 1)((2N - k)k - 1) + 2l - 2 \]
and this equals zero if and only if \( k = 1 \). Hence the thermalisation phenomenon, the loss of noises in the quantum stochastic differential equation which governs the limit cocycle, occurs exactly when \( \rho \) is not a vector state.

5 Applications

Notation 5.1. Let \( A \subseteq \mathcal{B}(h) \) be a von Neumann algebra; recall that \( A \otimes_M \mathcal{B}(H) = A \otimes \mathcal{B}(H) \) for any Hilbert space \( H \), and \( \Phi \otimes_M \mathcal{B}(H) = \Phi \otimes \mathcal{B}(H) \) for any ultraweakly continuous, \( H \)-bounded map \( \Phi \).

Let \( \rho \) be a normal state on \( \mathcal{B}(K) \), with density matrix \( \varrho \), and let \( K_0 := \overline{\text{im } \varrho^{1/2}} \) as in Section 3. Suppose \( d_0 \) is a conditional expectation on \( \mathcal{B}(K_0) \) which preserves the faithful state \( \rho_0 \) defined in Lemma 3.9; let \( \delta_0 := \mathcal{I}_{\mathcal{B}(h)} \otimes d_0 \) and \( \tilde{\rho}_0 := \mathcal{I}_{\mathcal{B}(h)} \otimes \rho_0 \).
5.1 Hudson–Parthasarathy evolutions

The following theorem is a generalisation of both [6, Remark 7] and the well-known Hudson–Parthasarathy conditions for processes to be isometric, co-isometric or unitary.

**Theorem 5.2.** Let $F \in A \otimes B(K)$ and define

$$\Psi : A \rightarrow A \otimes B(K); \ a \mapsto (a \otimes I_K)F.$$  

If $\psi : A \rightarrow A \otimes B(\hat{k})$ is given by (1.3) then $\psi(a) = (a \otimes I_\hat{k})G$ for all $a \in A$, where

$$G := \Delta^\perp \pi(F)\Delta^\perp + \Delta^\perp (\pi \circ \delta^\perp)(F)\Delta + \Delta(\pi \circ \delta^\perp)(F)\Delta^\perp + \Delta \tilde{\pi}(\tilde{P}_0^\perp F \tilde{P}_0^\perp)\Delta \in A \otimes B(\hat{k}). \quad (5.1)$$

The cocycle $j^\psi$ is such that $j^\psi_t(a) = (a \otimes I_\hat{k})X_t$ for all $a \in A$ and $t \in \mathbb{R}_+$, where the adapted process $X = (X_t)_{t \geq 0}$ satisfies the right Hudson–Parthasarathy equation

$$X_0 = I_{h \otimes F}, \quad dX_t = d\Delta_G(t)X_t \quad \text{for all } t \in \mathbb{R}_+. \quad (5.2)$$

The process $X$ is composed of isometric, co-isometric or unitary operators if and only if

$$F = \begin{bmatrix}
-\mathrm{i}(H_d + H_o) - \frac{1}{2}K & -D^*V \\
D & V - I_{h \otimes K_0^\perp}\end{bmatrix}, \quad (5.3)$$

where

(i) $H_d, \ H_o \in A \otimes B(K_0)$ are self adjoint, with $H_d = \delta_0(H_d)$ and $H_o = \delta_0^*(H_o)$,

(ii) $K \in A \otimes B(K_0)$ is self adjoint, with $K = \delta_0(K)$ and $\tilde{\rho}_0(K) = \tilde{\rho}_0(H_o^2 + D^*D)$,

(iii) $D \in A \otimes B(K_0; K_0^\perp)$

and (iv) $V \in A \otimes B(K_0^\perp)$ is isometric, co-isometric or unitary, respectively.

**Proof.** The first claim is immediate, and the second follows from [14, Proof of Theorem 7.1]. For the final part, recall that $\tilde{\rho} \circ \delta = \tilde{\rho}$, by Proposition 3.11 it follows from this, (3.3) and (3.4) that

$$G + G^* + G^*\Delta G = \Delta^\perp \pi(F_1)\Delta^\perp + \Delta^\perp \pi(F_2)\Delta + \Delta \tilde{\pi}(F_3^*)\Delta + \Delta \tilde{\pi}(F_3)\Delta,$$

where

$$F_1 := \delta(F + F^* + \delta^\perp(F^*)\delta^\perp(F)), \quad F_2 := \tilde{P}_0(\delta^\perp(F + F^*) + \delta^\perp(F^*)\tilde{P}_0^\perp F\tilde{P}_0^\perp)$$

and $F_3 := \tilde{P}_0^\perp(F + F^* + F^*\tilde{P}_0^\perp F)\tilde{P}_0^\perp,$

so

$$G + G^* + G^*\Delta G = 0 \iff \Delta^\perp \pi(F_1)\Delta^\perp + \Delta^\perp \pi(F_2)\Delta + \Delta \tilde{\pi}(F_3)\Delta = 0, \quad (5.2)$$

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Let \( F = \begin{bmatrix} A+D & C \\ D & E \end{bmatrix} \), where \( \delta_0(A) = A \) and \( \delta_0^\perp(B) = B \); after some working, it may be shown that

\[
F_1 = \begin{bmatrix} \tilde{\rho}_0(A + A^* + B^*B + D^*D) & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} B + B^* & C + D^* + D^*E \\ 0 & 0 \end{bmatrix}
\]

and \( F_3 = \begin{bmatrix} 0 & 0 \\ 0 & E + E^* + E^*E \end{bmatrix} \).

If \( X \in \mathcal{B}(K) \) then

\[
E^\omega \tilde{\pi}(F_3)E_{[X]} = \tilde{\rho}(\tilde{P}_0F_3(I_h \otimes X)) = 0 = \tilde{\rho}((I_h \otimes X)^*F_3\tilde{P}_0) = E^{[X]}\tilde{\pi}(F_3)E_\omega,
\]

so \( \Delta^\perp\tilde{\pi}(F_3) = \widetilde{\pi}(F_3)\Delta^\perp = 0 \) and therefore \( \Delta\tilde{\pi}(F_3)\Delta = \widetilde{\pi}(F_3) \). Hence

\[
\Delta\tilde{\pi}(F_3)\Delta = 0 \iff E + E + E^*E = 0 \iff V^*V = I_{h \otimes K_0},
\]

where \( V = E + I_{h \otimes K_0} \). Next, note that

\[
E^\omega \tilde{\pi}(F_2)E_\omega = (\rho \circ \delta)(\delta^\perp(F + F^*)) + \rho(\delta^\perp(F^*)\delta(F)\tilde{P}_0^\perp F\tilde{P}_0^\perp \tilde{P}_0) = 0,
\]

so \( \Delta^\perp\tilde{\pi}(F_2)\Delta = \Delta^\perp\tilde{\pi}(F_2) \). If \( Y = \begin{bmatrix} Y_0^0 & Y_0^x \\ Y_0^x & Y_0^x \end{bmatrix} \in \mathcal{B}(K) \) then

\[
E^\omega \tilde{\pi}(F_2)E_{[Y]} = \begin{bmatrix} \tilde{\rho}_0((B + B^*)(I_h \otimes Y_0^0) + (C + D^*V)(I_h \otimes Y_0^x)) & 0 \\ 0 & 0 \end{bmatrix},
\]

therefore \( \Delta^\perp\tilde{\pi}(F_2)\Delta = 0 \) if and only if

\[
\tilde{\rho}_0((B + B^*)(I_h \otimes Y_0^0)) = \tilde{\rho}_0((C + D^*V)(I_h \otimes Y_0^x)) = 0
\]

for all \( Y_0^0 \in \mathcal{B}(K_0) \) and \( Y_0^x \in \mathcal{B}(K_0; K_0^\perp) \). Suppose \( T \in \mathcal{B}(h \otimes K_0) \) is such that

\[
\tilde{\rho}_0(T(I_h \otimes Y_0^0)) = 0 \quad \text{for all } Y_0^0 \in \mathcal{B}(K_0);
\]

with the notation as in Lemma \( 3.9 \)

\[
0 = \langle u, \tilde{\rho}_0(T(I_h \otimes Y_0^0))v \rangle = \sum_{j \in J} \lambda_j \langle u \otimes e_j, T(v \otimes Y_0^0 e_j) \rangle \quad \text{for all } u, v \in h,
\]

where \( \lambda_j > 0 \) for all \( j \in J \) and \( \{ e_j : j \in J \} \) is an orthonormal basis for \( K_0 \). With \( Y_0^0 = |y\rangle\langle e_j| \) for arbitrary \( y \in K_0 \) and \( j \in J \), this gives that

\[
T(h \otimes K_0) \perp h \otimes \text{lin}\{ e_j : j \in J \}
\]
and therefore $T = 0$. Taking $T = B + B^*$ and $T = (C + D^*V) |x\rangle\langle y|$, where $x \in K_0^\perp$ and $y \in K_0$ are arbitrary, it follows that
\[ \Delta^\perp \tilde{\pi}(F_2) \Delta = 0 \iff B + B^* = 0 \quad \text{and} \quad C + D^*V = 0. \]

Finally,
\[ \Delta^\perp \tilde{\pi}(F_1) \Delta^\perp = 0 \iff \tilde{\rho}_0(A + A^*) = -\tilde{\rho}_0(B^*B + D^*D) \]
and the result now follows from [14, Theorem 7.5]. \hfill \square

**Remark 5.3.** By definition, the adapted $h$ process $X$ satisfies equation (5.2) if and only if
\[ \langle u \varepsilon(f), (X_t - I_{h \otimes F}) v \varepsilon(g) \rangle = \int_0^t \langle u \varepsilon(f), (E_{\tilde{\pi}(s)} G E_{\tilde{\pi}(g)} \otimes I_F) X_s v \varepsilon(g) \rangle \, ds \]
for all $u, v \in h, f, g \in L^2(\mathbb{R}_+; k)$ and $t \in \mathbb{R}_+.

**Notation 5.4.** Define the decapitated exponential functions
\[ \exp_1(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \quad \text{and} \quad \exp_2(z) = \sum_{n=2}^{\infty} \frac{z^{n-2}}{n!} \quad \text{for all} \quad z \in \mathbb{C}. \]

Note that
\[ \exp_1(z) \exp(-z) = \exp_1(-z) \quad \text{and} \quad \exp_1(z) \exp_1(-z) = \exp_2(z) + \exp_2(-z) \quad (5.4) \]
for all $z \in \mathbb{C}.$

**Theorem 5.5.** Let the total Hamiltonian
\[ H_\tau(\tau) = \begin{bmatrix} H_d + \tau^{-1/2}H_o & \tau^{-1/2}L^* \\ \tau^{-1/2}L & \tau^{-1}H_\times \end{bmatrix} + \begin{bmatrix} R^0_0(\tau) & \tau^{-1/2}R^0_0(\tau) \\ \tau^{-1/2}R^0_0(\tau) & \tau^{-1}R^0_\times(\tau) \end{bmatrix} \in A \otimes B(K) \]
for all $\tau > 0$, where
\begin{enumerate}
    \item the self-adjoint operators $H_d, H_o \in A \otimes B(K_0)$ are such that $\delta_0(H_d) = H_d$ and $\delta_0^\perp(H_o) = H_o$, 
    \item $L \in A \otimes B(K_0; K_0^\perp)$,
    \item $H_\times \in A \otimes B(K_0^\perp)$ is self-adjoint
\end{enumerate}
and (iv) the functions $R^0_0, R^\times_0, R^\times_\times$ and $R^\times_\times$ are such that
\begin{enumerate}
    \item $R^0_0(\tau) = R^0_0(\tau)^*$, $R^\times_0(\tau) = R^\times_0(\tau)^*$ and $R^\times_\times(\tau) = R^\times_\times(\tau)^*$ for all $\tau > 0,$
    \item the function $\tau \mapsto \|R^0_0(\tau)\|$ is bounded on a neighbourhood of 0 \quad (5.5)
\end{enumerate}
and
\begin{enumerate}
    \item $\lim_{\tau \to 0} \delta_0(R^0_0(\tau)) = \lim_{\tau \to 0} R^0_0(\tau) = \lim_{\tau \to 0} R^\times_\times(\tau) = 0,$ \quad (5.6)
\end{enumerate}
where the convergence holds in the norm topology.
Then the completely isometric map

$$
\Phi(\tau) : A \to A \otimes B(K); \quad a \mapsto (a \otimes I_K) \exp(-i\tau H_t(\tau))
$$

is such that $$\|m_\delta(\Phi(\tau), \tau) - \Psi\|_{cb} \to 0$$ as $$\tau \to 0$$, where

$$
\Psi : A \to A \otimes B(K); \quad a \mapsto (a \otimes I_K)F
$$

and

$$
F = \begin{bmatrix}
-i(H_d + H_o) - \delta_0(\frac{1}{2}H_o^2 + L^* \exp_2(-iH_\times)L) & -iL^* \exp_1(-iH_\times) \\
-i \exp_1(-iH_\times)L & \exp(-iH_\times) - I_{h \otimes \mathcal{K}_d} \end{bmatrix}.
$$

(5.7)

Consequently, $$J_{\hat{\pi}_0} \Phi(\tau)t \to_{cb} j^{\psi}$$, where the completely bounded map $$\psi : A \to A \otimes B(\mathcal{K})$$ is as defined in (4.3). The adapted $$\mathfrak{h}$$ process $$(U_t := j^{\psi}_t(I_{h}))_{t \in \mathbb{R}_+}$$ is unitary for all $$t \in \mathbb{R}_+$$ and such that $$j^{\psi}_t(a) = (a \otimes I_k)U_t$$ for all $$t \in \mathbb{R}_+$$ and $$a \in A$$.

**Proof.** Let $$G := \tau H_t(\tau) = A + \tau^{1/2}B + \tau C$$, where

$$
A = \begin{bmatrix} 0 & 0 \\ 0 & H_\times + R_\times^x(\tau) \end{bmatrix}, \quad B = \begin{bmatrix} H_o & L^* + R_\times^0(\tau) \\ L + R_\times^0(\tau) & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} H_d + R_0^0(\tau) & 0 \\ 0 & 0 \end{bmatrix};
$$

by (5.5) and (5.6), there exists $$\tau_0 \in (0, 1)$$ such that

$$c := \sup\{\|A\|, \|B\|, \|C\| : 0 < \tau < \tau_0\} < \infty.$$

Then

$$m_\delta(\Phi(\tau), \tau)(a) = (a \otimes I_K)\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} m(G^n)$$

for all $$\tau > 0$$, where the linear map

$$m : T \mapsto \tilde{P}_0(\tau^{-1}\delta + \tau^{-1/2}\delta^\perp)(T)\tilde{P}_0 + \tau^{-1/2}\tilde{P}_0T\tilde{P}_0^\perp + \tau^{-1/2}\tilde{P}_0^\perp T\tilde{P}_0 + \tilde{P}_0^\perp T\tilde{P}_0^\perp.$$

Note that

$$G^n = A^n + \tau^{1/2} \sum_{j=0}^{n-1} A^jBA^{n-1-j} + \tau \sum_{j=0}^{n-1} A^jCA^{n-1-j}$$

$$+ \tau \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} A^jBA^kB A^{n-2-j-k} + \tau^{3/2} D_n$$

for all $$n \geq 1$$, where $$\|D_n\| \leq 3^n c^n$$ for all $$\tau \in (0, \tau_0)$$. As $$AC = CA = 0$$ and $$ABA = 0$$, this simplifies to give that

$$G^n = A^n + \tau^{1/2}(BA^{n-1} + A^{n-1}B) + \mathbb{1}_{n \geq 3} \tau BA A^{n-2} B + \tau \sum_{j=0}^{n-2} A^jB^2A^{n-2-j} + \tau^{3/2} D_n$$
for all $n \geq 2$. (Here and below, the expression $1_P$ has the value 1 if $P$ is true and 0 if $P$ is false.) Furthermore, if $p \geq 1$, $0 \leq j \leq p$ and

$$r_\tau(T) := \tilde{P}_0 \delta(T) \tilde{P}_0 + \tilde{P}_0 T \tilde{P}_0 + \tilde{P}_0 T \tilde{P}_0 + \tau^{1/2} \tilde{P}_0 T \tilde{P}_0$$

then

$$m(A^p) = A^p,$$

$$m(\tau^{1/2}(B A^p + A^p B)) = B A^p + A^p B,$$

$$m(\tau B^2) = \delta(B^2) + \tau^{1/2} r_\tau(B^2),$$

$$m(\tau B A^p B) = \delta(B A^p B) + \tau^{1/2} \delta(B A^p B)$$

and

$$m(\tau A^j B^2 A^{p-j}) = \tau^{1/2} r_\tau(A^j B^2 A^{p-j}).$$

Hence, omitting the argument $\tau$ from $R_0^0$, $R_0^x$, $R_0^x$ and $R_x^x$ for brevity,

$$m(G) = \begin{bmatrix}
H_d + H_o & L^* + R_0^0 \\
L + R_0^x & H_x + R_x^x
\end{bmatrix} + D'_1,$$

where

$$D'_1 = \begin{bmatrix}
\delta_0(R_0^0) + \tau^{1/2} \delta_0^+(R_0^0) & 0 \\
0 & 0
\end{bmatrix},$$

and

$$m(G^n) = A^n + B A^{n-1} + A^{n-1} B + \delta(B A^{n-2} B) + \tau^{1/2} D'_n$$

$$= \left[ 1_{n=2} H_o^2 + (L^* + R_0^0)(H_x + R_x^x)^{-2}(L + R_0^0) \begin{bmatrix}
(L^* + R_x^x)(H_x + R_x^x)^{-1} \\
(H_x + R_x^x)^{-2}(L + R_0^0)
\end{bmatrix} + \tau^{1/2} D'_n\right]$$

for all $n \geq 2$, where

$$D'_n = 1_{n \geq 2} \delta^{1/2}(B A^{n-2} B) + \sum_{j=0}^{n-2} r_\tau(A^j B^2 A^{n-2-j}) + m(\tau D_n);$$

in particular, if $n \geq 2$ and $\tau \in (0, \tau_0)$ then

$$\|D_n'\| \leq 2e^n + 5(n - 1)c^n + 6(3c)^n = (5n - 3 + 2 \cdot 3^{n+1})c^n.$$

An $M$-test argument now gives that $\|m_\delta(\Phi(\tau), \tau) - \Psi\|_{cb} \to 0$ as $\tau \to 0$ and therefore $J^{\Phi(\tau), \tau} \to cb$, by Theorem [4,5]. Using the identities [5,1], it is readily verified that $F$ satisfies the unitarity conditions of Theorem [5,2] in the notation of that theorem, but with $H_d$ and $H_o$ there replaced by $H'_d$ and $H'_o$, $H'_d = H_d - \frac{i}{2} \delta_0(L^*(\exp_2(-iH_x) - \exp_2(iH_x))L)$, $H'_o = H_o,$ $K = \delta_0(H_o^2 + L^*(\exp_2(-iH_x) + \exp_2(iH_x))L)$, $D = -i \exp_1(-iH_x)L$ and $V = \exp(-iH_x).$
Remark 5.6. When the state $\rho$ is faithful or a vector state, Theorem 5.5 is a generalisation of [6, Theorem 4] or [3, Theorem 19], respectively; for the latter case, see also [11, Theorem 4.1].

The following example is the simplest which illustrates the various features of Theorem 5.5.

Example 5.7. Suppose $K = \mathbb{C}^3$ and take the density matrix

$$\varrho = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \in M_3(\mathbb{C}), \quad \text{where } \lambda_1, \lambda_2 \in (0, 1) \text{ are such that } \lambda_1 + \lambda_2 = 1.$$

Then $K_0 = \mathbb{C}^2$; let the $\rho_0$-preserving conditional expectation

$$d_0 : M_2(\mathbb{C}) \to M_2(\mathbb{C}); \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto \begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix}.$$

Let $e_{ij} \in M_3(\mathbb{C})$ be the elementary matrix with 1 in the $(i, j)$ entry and 0 elsewhere, let

$$f_{ij} = \lambda_j^{-1/2}e_{ij} \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2,$$

and let $\{ e_i : i = 1, 2, 3 \}$ be the canonical basis of $\mathbb{C}^3$, so that $\hat{k}$ has the basis

$$\{ [f_{ij}] = e_i \otimes e_j^\dagger : i = 1, 2, 3, \ j = 1, 2 \}.$$

Note also that $d(f_{ij}) = 0$ unless $i = j = 1$ or $i = j = 2$, and $\{ [d^\dagger(X)] : X \in \ker \rho \}$ has basis

$$\{ [d^\dagger(f_{ij})] = e_i \otimes e_j^\dagger : (i, j) \in \{(1, 2), (2, 1), (3, 1), (3, 2)\} \};$$

similarly, $P_0f_{3k} = 0$ for $k = 1$ and $k = 2$, and $\{ [P_0^\dagger X] : X \in \ker \rho \}$ has the basis

$$\{ [P_0^\dagger f_{3k}] = e_3 \otimes e_k^\dagger : k = 1, 2 \}.$$

If $H_d, H_o, L$ and $H_\times$ are as in Theorem 5.5 then

$$H_d = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}, \quad H_o = \begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix} \in M_2(\text{A}), \quad L = \begin{bmatrix} l & m \end{bmatrix} \in M_{1,2}(\text{A}), \quad \text{and } H_\times = h \in \text{A},$$

where $b, c, h \in \text{A}$ are self adjoint. With the notation of Theorem 5.5

$$H_t(\tau) = \begin{bmatrix}
b & \tau^{-1/2}g^* & \tau^{-1/2}l^* \\
\tau^{-1/2}g & c & \tau^{-1/2}m^* \\
\tau^{-1/2}l & \tau^{-1/2}m & \tau^{-1/2}h \\
\end{bmatrix} \quad \text{for all } \tau > 0$$

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In particular, there are 10 independent quantum noises in the quantum stochastic differential equations satisfied by the limit cocycle \( j \).

As the upper bound (4.4) is not achieved: in this case, the upper bound equals

\[
\begin{pmatrix}
-ib - \frac{1}{2}g^*g - l^* \exp(-ih)l & -ig^* & -il^* \exp_1(-ih) \\
-ig & -ic - \frac{1}{2}g^*g - m^* \exp(-ih)m & -im^* \exp_1(-ih) \\
-i \exp_1(-ih)l & -i \exp_1(-ih)m & \exp(-ih) - I_h
\end{pmatrix}.
\]

As \( \omega = \lambda_1^{1/2}e_1 \otimes e_1^\dagger + \lambda_2^{1/2}e_2 \otimes e_2^\dagger \), it follows that

\[
E^\omega \psi(a)E_\omega = a(\lambda_1 F_{11} + \lambda_2 F_{22}),
\]

\[
E^{[f_{ij}]} \psi(a)E_\omega = \lambda_j^{1/2} a F_{ij}
\]

\[
E^\omega \psi(a)E_{[f_{ij}]} = \lambda_j^{1/2} a F_{ji}
\]

and \( E^{[f_{ki}]} \psi(a)E_{[f_{kl}]} = 1_{k = l} a F_{33} \)

for all \((i, j) \in \{(1, 2), (2, 1), (3, 1), (3, 2)\}\) and \( k, l \in \{1, 2\} \), where \( F_{pq} \) denotes the \((p, q)\) entry of the matrix \( F \) and \( 1_{k = l} \) equals 1 if \( k = l \) and 0 otherwise.

In particular, there are 10 independent quantum noises in the quantum stochastic differential equations satisfied by the limit cocycle \( j^\psi \) and the unitary process \( U \) given by Theorem 5.5 so the upper bound (4.4) is not achieved: in this case, the upper bound equals

\[
2(3 \times 2 - 2) + (3 - 2)^2 2^2 = 12.
\]

### 5.2 Evans–Hudson evolutions

The following result is a generalisation of [6, Remark 8].

**Theorem 5.8.** For any \( F \in A \otimes B(K) \), let

\[
\Psi : A \to A \otimes B(K); \quad a \mapsto (a \otimes I_K)F + F^*(a \otimes I_K) + \delta(\delta^\perp(F)^*(a \otimes I_K)\delta^\perp(F))
\]

\[
+ F^*\tilde{P}_0^\perp(a \otimes I_K)\tilde{P}_0^\perp F - \tilde{P}_0 F^*\tilde{P}_0^\perp(a \otimes I_K)\tilde{P}_0^\perp F\tilde{P}_0
\]

(5.8)

and let \( G \in A \otimes B(\hat{k}) \) be given by (5.1). Then \( \psi : A \to A \otimes B(\hat{k}) \) as defined in (4.3) is such that

\[
\psi(a) = (a \otimes I_\hat{k})G + G^*(a \otimes I_\hat{k}) + G^*\Delta(a \otimes I_\hat{k})\Delta G \quad \text{for all } a \in A.
\]

(5.9)

The cocycle \( j^\psi \) is such that \( j^\psi_t(a) = X^*_t(a \otimes I_K)X_t \) for all \( a \in A \) and \( t \in \mathbb{R}_+ \), where the adapted process \( X = (X_t)_{t \geq 0} \) satisfies the right Hudson–Parthasarathy equation (5.2).

**Proof.** Using Theorem 5.2 linearity and the adjoint, it suffices to show that if

\[
\Upsilon(a) = \delta(\delta^\perp(F)^*(a \otimes I_K)\delta^\perp(F)) + F^*\tilde{P}_0^\perp(a \otimes I_K)\tilde{P}_0^\perp F - \tilde{P}_0 F^*\tilde{P}_0^\perp(a \otimes I_K)\tilde{P}_0^\perp F\tilde{P}_0
\]

(5.9)
then

\[ G^* \Delta(a \otimes \mathcal{I}_k) \Delta G = \Delta^\perp(\tilde{\pi} \circ \Upsilon)(a) \Delta^\perp + \Delta^\perp(\tilde{\pi} \circ \delta^\perp \circ \Upsilon)(a) \Delta \]

\[ + \Delta(\tilde{\pi} \circ \delta^\perp \circ \Upsilon)(a) \Delta^\perp + \Delta \tilde{\pi}(\tilde{P}_0^\perp \Upsilon(a) \tilde{P}_0^\perp) \Delta \]

\[ = \Delta^\perp(\tilde{\pi} \circ \Upsilon)(a) \Delta^\perp + \Delta^\perp \tilde{\pi}(\tilde{P}_0^\perp(\delta^\perp \circ \Upsilon)(a)) \Delta \]

\[ + \Delta \tilde{\pi}((\delta^\perp \circ \Upsilon)(a) \tilde{P}_0) \Delta^\perp + \Delta \tilde{\pi}(\tilde{P}_0^\perp \Upsilon(a) \tilde{P}_0^\perp) \Delta, \]

where the latter equality follows by using (3.3) together with the identities \( \tilde{\rho} \circ \delta^\perp = 0 \) and

\[ \tilde{\rho}(\tilde{P}_0 T) = \tilde{\rho}(T \tilde{P}_0) = \tilde{\rho}(T) \quad \text{for all } T \in \mathcal{B}(h \otimes \mathcal{K}). \]

Letting \( F = [X \ Y \ W] \), a little algebra shows that

\[
 \delta(\delta^\perp(F)^*(a \otimes \mathcal{I}_K)\delta^\perp(F)) = \begin{bmatrix}
 \delta_0(\delta_0^\perp(X)^*(a \otimes I_{K_0})\delta_0^\perp(X) + Z^*(a \otimes I_{K_0})Z) & 0 \\
 0 & 0
 \end{bmatrix},
\]

\[
 F^* \tilde{P}_0^\perp(a \otimes \mathcal{I}_K) \tilde{P}_0 \partial F = \begin{bmatrix}
 Z^*(a \otimes I_{K_0})Z & Z^*(a \otimes I_{K_0})W \\
 W^*(a \otimes I_{K_0})Z & W^*(a \otimes I_{K_0})W
 \end{bmatrix}
\]

and

\[
 \Upsilon(a) = \begin{bmatrix}
 \delta_0(\delta_0^\perp(X)^*(a \otimes I_{K_0})\delta_0^\perp(X) + Z^*(a \otimes I_{K_0})Z) & Z^*(a \otimes I_{K_0})W \\
 W^*(a \otimes I_{K_0})Z & W^*(a \otimes I_{K_0})W
 \end{bmatrix}.
\]

Furthermore, with \( G \) given by (5.11), a short calculation shows that

\[
 G^* \Delta(a \otimes \mathcal{I}_k) \Delta G
 \]

\[ = \Delta^\perp \tilde{\pi}(\delta^\perp(F)^*(a \otimes \mathcal{I}_K)\delta^\perp(F)) \Delta^\perp + \Delta^\perp \tilde{\pi}(\tilde{P}_0^\perp(\delta^\perp(F)^*(a \otimes \mathcal{I}_K)\tilde{P}_0^\perp F \tilde{P}_0^\perp) \Delta \]

\[ + \Delta \tilde{\pi}(\tilde{P}_0 \partial^* F^* \tilde{P}_0^\perp(a \otimes \mathcal{I}_K) \tilde{P}_0^\perp F \tilde{P}_0^\perp F \tilde{P}_0^\perp) \Delta. \]

Now,

\[ E^{\omega} \tilde{\pi}(\Upsilon(a)) E_{\omega} = (\tilde{\rho} \circ \delta(\delta^\perp(F)^*(a \otimes \mathcal{I}_K)\delta^\perp(F)) = E^{\omega} \tilde{\pi}(\delta^\perp(F)^*(a \otimes \mathcal{I}_K)\delta^\perp(F)) E_{\omega}, \]

and, since

\[ \delta^\perp(F)^*(a \otimes \mathcal{I}_K) \tilde{P}_0^\perp F = F^* \tilde{P}_0^\perp(a \otimes \mathcal{I}_K) \tilde{P}_0^\perp F = F^* \tilde{P}_0^\perp(a \otimes \mathcal{I}_K) \delta^\perp(F), \]

so

\[
 \tilde{P}_0^\perp \delta^\perp(\Upsilon(a)) = \begin{bmatrix}
 0 & Z^*(a \otimes I_{K_0})W \\
 0 & 0
 \end{bmatrix} = \tilde{P}_0 \delta^\perp(F)^*(a \otimes I_{K_0}) \tilde{P}_0^\perp F \tilde{P}_0^\perp \tilde{P}_0^\perp.
\]
and
\[ \delta^+(Y(a))\tilde{P}_0 = \begin{bmatrix} 0 & 0 \\ W^*(a \otimes I_{K^+})Z & 0 \end{bmatrix} = \tilde{P}_0^\perp F^*\tilde{P}_0^\perp (a \otimes I_K)\delta^+(F)\tilde{P}_0. \]

Finally, as \( \tilde{P}_0^\perp Y(a)\tilde{P}_0^\perp = \tilde{P}_0^\perp F^*\tilde{P}_0^\perp (a \otimes I_K)\tilde{P}_0^\perp F \tilde{P}_0^\perp \), the first result holds as claimed. The second is an immediate consequence of [14, Theorem 7.4].

**Theorem 5.9.** Let \( H_t(\tau) \) be defined as in Theorem 5.5 for all \( \tau > 0 \). Then the normal \(*\)-homomorphism
\[ \Phi(\tau) : A \to A \otimes B(K); a \mapsto \exp(i\tau H_t(\tau))(a \otimes I_K)\exp(-i\tau H_t(\tau)) \]
is such that \( \|m_\delta(\Phi(\tau), \tau) - \Psi\|_{cb} \to 0 \) as \( \tau \to 0 \), where \( \Psi : A \to A \otimes B(K) \) is as defined in (5.8) and \( F \) is given by (5.1).

Hence \( J^{\bar{\tau} \circ \Phi(\tau)} \to_{cb} j^\psi \), where the completely bounded map \( \psi : A \to A \otimes B(\tilde{K}) \) is given by (5.9) and \( G \) is given by (5.1). The limit cocycle \( j^\psi \) is such that
\[ j^\psi_t(a) = U^*_t(a \otimes I_K)U_t \quad \text{for all } t \in \mathbb{R}_+ \text{ and } a \in A, \]
where the adapted \( \mathfrak{h} \) process \( (U_t)_{t \in \mathbb{R}_+} \) is unitary for all \( t \in \mathbb{R}_+ \) and satisfies the quantum stochastic differential equation (5.2); in particular, the map \( j^\psi_t \) is a normal \(*\)-homomorphism for all \( t \geq 0 \).

**Proof.** Fix \( a \in A \) and let \( m, G, A, B, C, c, \tau_0 \) and \( r_\tau \) be as in the proof of Theorem 5.5 so that, in particular,
\[ m_\delta(\Phi(\tau), \tau)(a) = \sum_{j=1}^{\infty} \frac{1}{j!} m((a \otimes I_K)(-iG)^j + (iG)^j(a \otimes I_K)) + \sum_{j,k=1}^{\infty} \frac{j^j - k^k}{j! k!} m(G^j(a \otimes I_K)G^k). \]

From the working in that proof, the first series converges to \( (a \otimes I_K)F + F^*(a \otimes I_K) \) as \( \tau \to 0 \) and, considered as a function of \( a \), the convergence holds in the completely bounded sense.

For the double series, note that
\[ A(a \otimes I_K)BA = AB(a \otimes I_K)A = 0 \quad \text{and} \quad A(a \otimes I_K)C = C(a \otimes I_K)A = 0; \]
therefore, after some working,
\[ G(a \otimes I_K)G = A(a \otimes I_K)A + \tau^{1/2}(B(a \otimes I_K)A + A(a \otimes I_K)B) + \tau B(a \otimes I_K)B + \tau^3/2 D_{1,1} \]
and, if \( j \) and \( k \) are not both 1,
\[ G^j(a \otimes I_K)G^k = A^j(a \otimes I_K)A^k + \tau^{1/2}(BA^{j-1}(a \otimes I_K)A^k + A^j(a \otimes I_K)A^{k-1}B) \]
\[ + \tau(A^jB(a \otimes I_K)BA^{k-1} + BA^{j-1}(a \otimes I_K)A^{k-1}B) \]
\[ + \tau \sum_{l=0}^{j-2} A^l B^2 A^{j-2-l}(a \otimes I_K)A^k + \tau \sum_{l=0}^{k-2} A^j(a \otimes I_K)A^l B^2 A^{k-2-l} + \tau^{3/2} D_{j,k}, \]

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where $\|a \mapsto D_{j,k}\|_{cb} \leq (3c)^{j+k}$ for all $\tau \in (0, \tau_0)$.

Furthermore, if $j, k \geq 1$ then

\[
\begin{align*}
\text{m}(A^i(a \otimes I_k)A^k) &= A^i(a \otimes I_k)A^k, \\
\text{m}(\tau^{1/2}BA^j(a \otimes I_k)A^k) &= BA^{j-1}(a \otimes I_k)A^k \\
\text{and} \quad \text{m}(\tau^{1/2}A^j(a \otimes I_k)A^{k-1}B) &= A^j(a \otimes I_k)A^{k-1}B.
\end{align*}
\]

Also,

\[
\text{m}(\tau B(a \otimes I_k)B) = \delta(B(a \otimes I_k)B) + \tau^{1/2}r_\tau(B(a \otimes I_k)B),
\]

whereas, if $j$ and $k$ are not both 1,

\[
\begin{align*}
\text{m}(\tau A^{j-1}B(a \otimes I_k)BA^{k-1}) &= \tau^{1/2}r_\tau(A^{j-1}B(a \otimes I_k)BA^{k-1}) \\
\text{and} \quad \text{m}(\tau BA^{j-1}(a \otimes I_k)A^{k-1}B) &= \delta(BA^{j-1}(a \otimes I_k)A^{k-1}B).
\end{align*}
\]

Finally, if

\[
S_{j,k} := \sum_{l=0}^{j-2} A^lB^2A^{j-2-l}(a \otimes I_k)A^k + \sum_{l=0}^{k-2} A^l(a \otimes I_k)A^lB^2A^{k-2-l}
\]

then $\text{m}(\tau S_{j,k}) = \tau^{1/2}r_\tau(S_{j,k})$. Hence

\[
\text{m}(G^j(a \otimes I_k)G^k) = A^j(a \otimes I_k)A^k + BA^{j-1}(a \otimes I_k)A^k + A^j(a \otimes I_k)A^{k-1}B \\
+ \delta(BA^{j-1}(a \otimes I_k)A^{k-1}B) + \tau^{1/2}D'_{j,k},
\]

where

\[
D'_{j,k} = r_\tau(A^{j-1}B(a \otimes I_k)BA^{k-1}) + r_\tau(S_{j,k}) + m(\tau D_{j,k}),
\]

for all $j, k \geq 1$. Since

\[
\|a \mapsto D'_{j,k}\|_{cb} \leq (5 + 5(j - 1 + k - 1) + 6 \cdot 3^{j+k})c^{j+k}
\]

for all $\tau \in \tau_0$ and $j, k \geq 1$, the result now follows by an M-test argument, the identity \([5,10]\) and Theorems \([5.2,5.8]\) as $\tau \to 0$, the double series $\sum_{j,k=1}^{\infty} i^{j+k}m(G^j(a \otimes I_K)G^k)/(j!k!)$ tends to

\[
\left[
\begin{array}{cc}
\delta_0(H_0(a \otimes I_{K_0})H_0 + L^*e_1^{i\mathbf{H}_x}(a \otimes I_{K_0})e_1^{-i\mathbf{H}_x} L) & iL^*e_1^{i\mathbf{H}_x}(a \otimes I_{K_0}) (e^{-i\mathbf{H}_x} - I_{h \otimes K_0}) \\
-i(e^{i\mathbf{H}_x} - I_{h \otimes K_0}) (a \otimes I_{K_0}) e_1^{-i\mathbf{H}_x} L) & (e^{i\mathbf{H}_x} - I_{h \otimes K_0}) (a \otimes I_{K_0}) (e^{-i\mathbf{H}_x} - I_{h \otimes K_0})
\end{array}
\right],
\]

where $e_1^{-i\mathbf{H}_x}$ is an abbreviation for $\exp_1(-i\mathbf{H}_x)$ \textit{et cetera}.

\[\Box\]
Remark 5.10. Theorem [5.8] is a generalisation of [6, Theorem 5]; see also [11, Theorem 4.1 and Remark 3] for the vector-state case. It provides an explicit description of the Lindblad generator $L$ for expectation semigroup of the cocycle $j^\psi$ which arises in the limit: if $a \in A$ then

$$L(a) := E^\omega \psi(a) E_\omega$$

$$= \tilde{\rho}(\Psi(a))$$

$$= -i[a, \tilde{\rho}_0(H_d)] - \frac{1}{2} \{a, \tilde{\rho}_0(H^2_d)\} - a \tilde{\rho}_0(L^* \exp_2(-iH_\times) L) - \tilde{\rho}_0(L^* \exp_2(iH_\times) L) a$$

$$+ \tilde{\rho}_0(H_o(a \otimes I_{K_d}) H_o) + \tilde{\rho}_0(L^* \exp_1(iH_\times)(a \otimes I_{K_d}) \exp_1(-iH_\times)L)$$

$$= -i[a, \tilde{\rho}_0(H'_d)] - \frac{1}{2} \{a, \tilde{\rho}_0(D^* D)\} + \tilde{\rho}_0(D^*(a \otimes I_{K_d}) D)$$

$$- \frac{1}{2} \{a, \tilde{\rho}_0(H^2_o)\} + \tilde{\rho}_0(H_o(a \otimes I_{K_o}) H_o),$$

where

$$H'_d := H_d - \frac{1}{2} \delta_0(L^*(\exp_2(-iH_\times) - \exp_2(iH_\times)) L)$$

and

$$D := -i \exp_1(-iH_\times)L.$$

This goes beyond the results for Gibbs states contained in [1].

Acknowledgements

This work was begun while the author was an Embark Postdoctoral Fellow in the School of Mathematical Sciences, University College Cork, funded by the Irish Research Council for Science, Engineering and Technology. A significant part of it was completed while a guest of Professor Rajarama Bhat at the Indian Statistical Institute, Bangalore, whose hospitality is gratefully acknowledged; this visit was made as part of the UKIERI research network Quantum Probability, Noncommutative Geometry and Quantum Information. Thanks are extended to Professor Martin Lindsay for helpful comments on a previous draft.

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