PASSIVE LINEAR DISCRETE-TIME SYSTEMS - CHARACTERIZATION THROUGH STRUCTURE

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Abstract. We here show that discrete-time passive linear systems are intimately linked to the structure of maximal, matrix-convex sets, closed under multiplication among their elements. Moreover, this observation unifies three setups: (i) difference inclusions, (ii) matrix-valued rational functions, (iii) realization arrays associated with rational functions.

It turns out that in the continuous-time case, the associated structure is of maximal matrix-convex, cones, closed under inversion.

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1. Introduction

The focus of this work is on showing that

Discrete-time passive linear systems are closely associated with maximal open multiplicative, matrix-convex sets.

Furthermore, this formulation suits well, three different frameworks:

(i) Difference inclusions,
(ii) Bounded Real rational functions,
(iii) Families of realization arrays of Bounded Real rational functions.
This work is organized as follows. In Section 2 we resort as an initial motivation to the problem of stability of difference inclusions. Then, in Section 3 we lay the foundation for the sequel and present maximal matrix-convex sets of matrices which are closed under multiplication among the elements. Subsequently, this structure is used to characterize discrete-time passive linear systems, see Section 4. We conclude by presenting in Section 5 a complete analogy to continuous-time passive linear systems.

2. Stability of difference inclusion

We start with notations. Let $(\mathbb{C}_R)$ $\mathbb{C}_R$ denote the (closed) open right half of the complex plane. Similarly, we shall denote by $\mathbb{H}_n (\mathbb{H}_n)$ the set of $n \times n$ Hermitian (non-singular) matrices and by $(\mathbb{P}_n)$ $\mathbb{P}_n$ the subsets of $n \times n$ positive (semi)-definite matrices.

Now, for a prescribed $H \in \mathbb{H}_n$, consider the set of all $n \times n$ matrices $A$ sharing the same Stein factor $H \in \mathbb{H}_n$,

$$\text{Stein}_H = \{ A : H - A^* HA \in \mathbb{P}_n \}$$

The set $\text{Stein}_H$ may be viewed as the closure of the open set $\text{Stein}_H$ in the sense that $\mathbb{P}_n$ is the closure in $\mathbb{H}_n$ of the open set $\mathbb{P}_n$.

One can refine the above definition by adding a parameter $\alpha > 0$, to obtain,

$$\frac{1}{\alpha} \text{Stein}_H = \{ A : H - \frac{1}{\alpha} A^* HA \in \mathbb{P}_n \}$$

We now examine the structure of this set.

**Theorem 2.1.** For $H \in \mathbb{H}_n$ the set $\text{Stein}_H$ is open, convex, closed under multiplication by $c \in \mathbb{C}$, $1 \geq |c|$ and under product among its elements, i.e. whenever $A \in \frac{1}{\alpha} \text{Stein}_H$ and $B \in \frac{1}{\beta} \text{Stein}_H$, for some $\alpha, \beta > 0$, then their product satisfies $AB \in \frac{1}{\alpha \beta} \text{Stein}_H$.

**Proof**: Although classical, for completeness we show this part. Convexity, closure under multiplication by $c \in \mathbb{C}$, $1 \geq |c|$ and the fact that this set is open, are all trivial.

Assume that for some $H \in \mathbb{H}_n$ one has that $A \in \frac{1}{\alpha} \text{Stein}_H$ and $B \in \frac{1}{\beta} \text{Stein}_H$, for some $\alpha, \beta > 0$ namely,

$$H - \frac{1}{\alpha} A^* HA = Q_a \quad \text{for some} \quad Q_a \in \mathbb{P}_n$$

$$H - \frac{1}{\beta} B^* HB = Q_b \quad \text{for some} \quad Q_b \in \mathbb{P}_n .$$

Multiplying the first equation by $\frac{1}{\beta} B^*$ and the second equation by $\frac{1}{\alpha} B$ from the left and from the right respectively, and adding the result to the second equation yields,

$$H - \frac{1}{\alpha \beta} (AB)^* HAB = \frac{1}{\alpha \beta} B^* Q_a B + Q_b ,$$

and as the right hand side is positive definite, so the claim is established.

In the sequel, we focus our attention on the case where in Eq. (2.2) one has that $H \in \mathbb{P}_n$.

---

1The subscript stands for Right.
Corollary 2.2. Consider the description in Theorem 2.1 of the set \( \alpha^{\text{Stein}_H} \) in Eq. (2.2). Whenever \( H \in \mathcal{P}_n \) this is in addition a family of matrices whose spectral radius is bounded by \( \alpha \).

Indeed, when \( H \in \mathcal{P}_n \), one can multiply the Stein matrix inclusion in Eq. (2.2) by \( H^{-\frac{1}{2}} \) from both sides to obtain,

\[
\begin{align*}
\frac{1}{\alpha}^{\text{Stein}_H} &= \{ A : \alpha > \| H^{\frac{1}{2}} A H^{-\frac{1}{2}} \|_2 \} \quad H \in \mathcal{P}_n \\
\frac{1}{\alpha}^{\text{Stein}_H} &= \{ A : \alpha \geq \| H^{\frac{1}{2}} A H^{-\frac{1}{2}} \|_2 \} \quad \alpha > 0.
\end{align*}
\]

Thus, in particular, the spectral norm of \( A \) is bounded by \( \alpha \).

We conclude this section by pointing out that a complete characterization of the set \( \text{Stein}_H \) in Eq. (2.1), for an arbitrary \( H \in \mathcal{H}_n \), appeared in [5, Theorem 3.5]. This remarkable result is quite involved. Now, on the expense of restricting the case to \( H = I_n \), in Proposition 3.4 below, we obtain, through matrix-convexity, a much simpler characterization. Subsequently, this advantage is exploited in two ways: (i) To describe Bounded Real functions and (ii) To set-up an analogy with Positive Real functions.

3. Maximal multiplicative matrix-convex sets of matrices

We next resort to the notion of a matrix-convex set, see e.g. [15] and more recently, [16], [17], [21], [31].

Definition 3.1. Let \( v_j \in \mathbb{C}^{n \times n} \), where \( j = 1, \ldots, k \), be so that, for all natural \( k \),

\[
\sum_{j=1}^{k} v_j^* v_j = I_n.
\]

A set \( M \) of \( n \times n \) matrices\(^2\) is said to be matrix-convex if having \( A_1, \ldots, A_k \) in \( M \), implies that for all natural \( k \), and for all \( v_j \in \mathbb{C}^{n \times n} \),

\[
\sum_{j=1}^{k} v_j^* A_j v_j \in M.
\]

The definition suggests that a notation like \( v_{j,k} \) is more appropriate, however for simplicity we drop the subscript \( k \).

In the sequel, Skew-Hermitian matrices are denoted by, \( \mathcal{H}_n \). It is common to take \( \overline{\mathcal{H}} \) and \( i\mathcal{H} \) as the matricial extension of \( \mathbb{R} \) and \( i\mathbb{R} \), respectively.

Remark 3.2. In [27] it was shown that there are not-too-many, non-trivial \( n \times n \) matrix-convex sets, among them:

\[
\mathcal{H}_n, \quad i\mathcal{H}_n, \quad \overline{\mathcal{P}}_n, \quad \mathcal{P}_n.
\]

\(^2\)We do not assume that \( M \subset \overline{\mathcal{H}} \).
Note that matrix-convexity is rather stringent. Specifically, by definition, matrix-convexity implies both classical convexity and being unitarily-invariant. The following Example 3.3 illustrates the fact that the converse is not true.

**Example 3.3.** The matrix Frobenius (a.k.a. Euclidean or Hilbert-Schmidt) norm, see e.g. [23, p. 291], is both convex and unitarily invariant, but it is not matrix-convex.

Consider the set of matrices \( \{ A : 5 \geq \| A \| \text{Frobenius} \} \). Now from \( A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \) which belong to this set (\( \| A \| \text{Frobenius} = 5 \)), construct the matrix

\[
\hat{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 4I_2 .
\]

Now, since \( \| \hat{A} \| \text{Frobenius} = 4\sqrt{2} \approx 5.7 \), this set is not matrix-convex. □

We next present the key player in this work.

**Proposition 3.4.** An open (closed), matrix-convex family of matrices whose spectral radius is less or equal to some \( \alpha > 0 \), is the set \( \frac{\alpha}{\alpha} \text{Stein}_n \) (\( \frac{\alpha}{\alpha} \text{Stein}_n \)).

Furthermore, the converse is true as well.

If in addition this is a maximal family of matrices which is closed under multiplication among its elements, this is equivalent to

\[ \alpha = 1. \]

**Proof:** First recall from Eq. (2.3) that

\[
\frac{\alpha}{\alpha} \text{Stein}_n = \{ A : \alpha > \| A \|_2 \} \\
\frac{\alpha}{\alpha} \overline{\text{Stein}}_n = \{ A : \alpha \geq \| A \|_2 \}.
\]

Next, recall that every induced norm, a set of the form \( \{ A : \alpha > \| A \| \} \), is convex and the spectral radius of all matrices in it, is bounded by \( \alpha \), see e.g. [23, Section 5.6].

To guarantee matrix-convexity, we must take the spectral norm (in fact already unitarily-invariant induced norm implies \( \| \|_2 \)).

Next, we show that the closed set \( \frac{\alpha}{\alpha} \text{Stein}_n \) (the case of the open set \( \frac{\alpha}{\alpha} \text{Stein}_n \) is similar and thus omitted) is matrix-convex. For a natural parameter \( k \) let \( \Upsilon \in \mathbb{C}^{kn \times n} \) be an isometry, i.e. \( \Upsilon^* \Upsilon = I_n \), then

\[
\| \Upsilon^* \left( \begin{array}{c} A_1 \\ \vdots \\ A_k \end{array} \right) \Upsilon \|_2 \leq \| \Upsilon^* \|_2 \| \left( \begin{array}{c} A_1 \\ \vdots \\ A_k \end{array} \right) \|_2 \| \Upsilon \|_2 \quad \text{sub-multiplicative norm}
\]

\[
\leq \left\| \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix} \right\|_2 \quad \Upsilon \text{ is an isometry}
\]

\[
= \max (\| A_1 \|_2, \ldots, \| A_k \|_2) \quad \text{induced norm}
\]

\[
\leq \alpha \quad \text{assumption},
\]
so this part of the claim is established.

For maximality of the spectral norm under product of elements, let \( B \not\in \mathbf{Stein}_I \) one can always find within \( \mathbf{Stein}_I \) a matrix \( A \) so that the spectral radius of the product \( AB \), is larger than one (and thus the spectral radius of \( (AB)^k \) is diverging with \( k \)). Indeed, let the Singular Value Decomposition, see e.g. [23, Theorem 7.35], of a matrix \( B \) be

\[
B = \sum_{j=1}^{n} \sigma_j u_j v_j^* \quad \text{with} \quad u_j \in \mathbb{C}^n, \quad v_j \in \mathbb{C}^n, \quad \sigma_j \geq 0, \quad \epsilon > 0
\]

where \( \delta_{j,k} \) is the Kronecker delta. To avoid triviality, assume that the spectral radius of \( B \) is less than one (Schur stable). This implies that

\[
\frac{1}{1+\epsilon} > |u_1 v_1^*| \quad \text{when} \quad \sigma_2 = 0, \quad \text{is in fact sufficient}.
\]

Take now \( A = \frac{1}{1+2\epsilon} B^* \). By construction \( \|A\|_2 = \frac{1+\epsilon}{1+2\epsilon} \), so indeed \( A \in A \). Next,

\[
AB = \frac{1}{1+2\epsilon} B^* B = \frac{1}{1+2\epsilon} \sum_{j=1}^{n} \sigma_j^2 v_j v_j^* = \frac{1}{1+2\epsilon} \left( (1+\epsilon)^2 v_1 v_1^* + \sum_{j=2}^{n} \sigma_j^2 v_j v_j^* \right).
\]

Thus, in fact \( AB \in \mathbf{P}_n \) and \( \|AB\|_2 = \frac{(1+\epsilon)^2}{1+2\epsilon} = 1 + \frac{\epsilon^2}{1+2\epsilon} \), which is also the spectral radius of \( AB \), so this part of the construction is complete.

The converse direction is to show that the set \( \frac{1}{\alpha} \mathbf{Stein}_I \) is of this structure. This is easy and thus omitted.

Finally, to obtain a set which is closed under multiplication among its elements, one needs to take \( 1 \geq \alpha \), and maximality requires \( 1 = \alpha \). Thus the proof is complete. \( \square \)

As an application consider the following

**Stability of difference inclusion**

Recall that the solution \( x(j) \) of an autonomous difference equation \( x(j+1) = Ax(j) \) converges to zero for all \( x(0) \), if and only if the spectral radius of \( A \) is less than one. Recall also that the set of matrices whose spectral radius is less than one (colloquially, “Schur stable”) is not closed under multiplication, e.g. both matrices \( A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \) and \( B = A^* \), have a zero spectral radius. However, the spectral radius of the product \( AB \), is four.

A difference inclusion

\[
(3.2) \quad x(j + 1) \in Mx(j) \quad x(j) \in \mathbb{R}^n \quad j = 0, 1, 2, ...
\]

can be interpreted as having

\[
x(j + 1) = A(j) x(j) \quad j = 0, 1, 2, \ldots \quad A(j) \text{ is arbitrary, within } M.
\]

From Proposition 3.4 it follows that:

**Corollary 3.5.** There exists \( \alpha \in (0, 1 - \epsilon) \), with \( 1 >> \epsilon > 0 \), so that the difference inclusion in Eq. (3.2) satisfies Eq. (3.3),

\[
(3.3) \quad \|x(0)\|_2 \alpha^j \geq \|x(j)\|_2 \quad \forall j = 0, 1, 2, ...
\]
if and only if for the same \( \alpha \),

\[
M \subset \{ x \in \mathbb{R}^{n} \mid x(0) \geq \alpha x(j) \forall j \geq 0 \}.
\]

For completeness we recall that if for some \( \alpha \in (0, 1 - \epsilon] \), with \( 1 >> \epsilon > 0 \) the condition is relaxed to \( M \subset \{ A : \alpha > \| A \| \} \), for some induced matrix norm, see e.g. [23, Section 5.6], then Eq. (3.3) holds when \( \| \cdot \|_2 \) is substituted by the above induced norm.\(^3\)

In the next section we use the set \( \text{Stein}_{I_n} \) to describe a family of rational functions.

4. Multiplicative matrix-convex sets of Passive Systems

4.1. Bounded real rational functions. The renowned family of \( m \times m \)-valued Bounded Real rational functions, denoted by \( \mathcal{BR} \), is classically associated with the scattering matrix of a \( R - L - C \) circuits, see e.g. [4, Chapter 11], [6, Chapter 6]. The set \( \mathcal{BR} \) can be described as the family of rational functions \( F(s) \) where \( 1 \geq \| F(s) \|_2 \) for all \( s \in \mathbb{C}_R \) (and real on the real axis) i.e.,

\[
(I_m - (F(s))^* F(s)) \in \mathbb{P}_m \quad \forall s \in \mathbb{C}_R.
\]

These functions are pivotal to our discussion.

**Observation 4.1.** A family of \( m \times m \)-valued rational functions which is matrix-convex and a maximal set closed under multiplication among its elements, is the family \( \mathcal{BR} \), of Bounded Real functions.

The converse is true as well.

Using previous notation the set \( \mathcal{BR} \) can be formally written as \( m \times m \)-valued rational functions \( F(s) \) so that,

\[
F(s) \in \begin{cases} \mathbb{R}^{m \times m} & s \in \mathbb{R} \\ \text{Stein}_{I_m} & \text{and analytic} \quad s \in \mathbb{C}_R. \end{cases}
\]

Thus in fact the claim follows from Proposition 3.4.

In particular, whenever \( F_a \) and \( F_b \) are two \( m \times m \)-valued \( \mathcal{BR} \) rational functions, then also the product \( F_a F_b \) belongs to \( \mathcal{BR} \).

4.2. Matrix-convex sets of realization arrays. Let \( F(s) \) be an \( m \times m \)-valued rational function \( F(s) \) with no pole at infinity and let \( R_F \) be a corresponding \( (n + m) \times (n + m) \) state-space realization array, i.e.

\[
F(s) = C(sI_n - A)^{-1} B + D \quad R_F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

The realization \( R_F \) in Eq. (4.3) is called minimal, if \( n \) is the McMillan degree of \( F(s) \). Recall that the classical Bounded Real Lemma, see e.g. [4, Section 7.2], [7, Subsection 2.7.3], [14, Lemma 7.4], [19, Section 3.7] and [23, Section 20.1], says the following.

\(^3\)In principle this can further relaxed in two ways: (i) to having possibly another norm and (ii) \( \beta \geq 1 \) so that \( \beta \| x(0) \|_\alpha \geq \| x(j) \| \forall j = 0, 1, 2, \ldots \)
Lemma 4.2. Let $F(s)$ be an $m \times m$-valued rational function and let $R_F$ in Eq. (4.3) be a corresponding realization.

(I) If there exists a matrix $H \in P_n$ so that

\[
\begin{pmatrix}
-HA - A^*H & -HB \\
-B^*H & I_m
\end{pmatrix}
- \begin{pmatrix}
C^* \\
D^*
\end{pmatrix}
\begin{pmatrix}
C \\
D
\end{pmatrix} \in P_{n+m},
\]

then $F(s)$ is Bounded Real.

If the realization in Eq. (4.4) is minimal, i.e. $n$ is the McMillan degree of $F(s)$, then the converse is true as well.

(II) Up to change of coordinates, one can substitute in Eq. (4.4) $H = I_n$ so that.

\[
\begin{pmatrix}
-A - A^* & -B \\
-B^* & I_m
\end{pmatrix}
- \begin{pmatrix}
C^* \\
D^*
\end{pmatrix}
\begin{pmatrix}
C \\
D
\end{pmatrix} \in P_{n+m}.
\]

In particular, this is the case when the realization is balanced.

For a quantitative refinement of B\textsuperscript{R} Lemma, see [2, Theorem 6.3].

We now introduce families of realization arrays associated with rational functions. To this end, we adopt the elegant idea from [13] and [32] to treat the above $(n + m) \times (n + m)$ $R_F$ as having two faces:

(i) of an array and (ii) of a matrix.

Before that, a word of caution: For example, $R_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $R_2 = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$ are two realization of the same rational function. Furthermore, $R_1$ is minimal (balanced) if and only if $R_2$ is minimal (balanced). However, $R_3 = \frac{1}{2}(R_1 + R_2) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is only a non-minimal realization of a zero degree rational function $F(s) \equiv D$.

To further study families of realization simultaneously satisfying Eq. (4.5) we need to introduce a relaxed version of matrix-convexity.

Definition 4.3. For all $k$, let $v_j \in \mathbb{C}^{(n+m)\times(n+m)}$, $j = 1, \ldots, k$ be block-diagonal so that

\[
\sum_{j=1}^{k} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix}^* \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}.
\]

A set $R$ of $(n+m) \times (n+m)$ matrices is said to be $n, m$-matrix-convex if having $R_1, \ldots, R_k$ in $R$, implies that

\[
\sum_{j=1}^{k} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix}^* \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \in R,
\]

for all natural $k$ and all $v_j \in \mathbb{C}^{(n+m)\times(n+m)}$.

In [27] it was pointed out that the notion of $n, m$-matrix-convexity is intermediate between the (more strict) matrix-convexity, and (weaker) classical convexity.

\footnote{Like Janus in the Roman mythology}
For a natural parameter $k$, let $F_1(s), \ldots, F_k(s)$ be a family of $m \times m$-valued rational functions, admitting $(n + m) \times (n + m)$ realizations, i.e.

\begin{equation}
R_{F_j} = \begin{pmatrix} \hat{A}_j & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{pmatrix}
\end{equation}

\(j = 1, \ldots, k\).

Using block-diagonal structured isometry from Eq. (4.6) along with the realizations $R_{F_j}$ in Eq. (4.7), let $R_F$ be of the form,

\begin{equation}
R_F = \sum_{j=1}^{k} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \begin{pmatrix} \hat{A}_j & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{pmatrix} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix}.
\end{equation}

Let now $F(s)$ be an $m \times m$-valued rational function whose realization $R_F$ is given by Eq. (4.8). We now address the following problem: Under what conditions having $F_1(s), \ldots, F_k(s)$ in Eq. (4.7) Bounded Real, implies that the resulting $F(s)$ in Eq. (4.8) is Bounded Real as well?

**Proposition 4.4.** For a natural parameter $k$, let $F_1(s), \ldots, F_k(s)$ be a family of $m \times m$-valued rational functions, admitting $(n + m) \times (n + m)$ realizations as in Eq. (4.7), and without loss of generality, assume that these realizations are in the form of Eq. (4.5), i.e.

\begin{equation}
\begin{pmatrix} -\hat{A}_j & -\hat{B}_j \\ -\hat{C}_j & I_m \end{pmatrix} - \begin{pmatrix} \hat{C}_j & \hat{D}_j \end{pmatrix} = Q_j, \quad j = 1, \ldots, k
\end{equation}

\(Q_j \in \mathbb{P}_{n+m} \).

Then, an arbitrary realization $R_F$ defined by Eq. (4.8), satisfies Eq. (4.5) and thus the associated rational function, $F(s)$, is Bounded Real.

**Proof:** It suffices to show that for all $v_{1,n}, \ldots, v_{k,n}, v_{1,m}, \ldots, v_{k,m}$ and for all $k$,

\[
\left( \sum_{j=1}^{k} \begin{pmatrix} -v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \begin{pmatrix} -\hat{A}_j & -\hat{B}_j \\ -\hat{C}_j & I_m \end{pmatrix} - \begin{pmatrix} \hat{C}_j & \hat{D}_j \end{pmatrix} v_{j,m} v_{j,m}^* \right) \in \mathbb{P}_{n+m}.
\]

Using the fact that $v_{j}$, $j = 1, \ldots, k$ are block-diagonal isometries, the above condition can be more compactly written as having

\begin{equation}
\sum_{j=1}^{k} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \left( \begin{pmatrix} -\hat{A}_j & -\hat{B}_j \\ -\hat{C}_j & I_m \end{pmatrix} - \begin{pmatrix} \hat{C}_j & \hat{D}_j \end{pmatrix} v_{j,m} v_{j,m}^* \right) \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \end{equation}

positive semi-definite. We establish in steps Eq. (4.10). Indeed, from Eq. (4.9) we have that

\[
\begin{pmatrix} -\hat{A}_j & -\hat{B}_j \\ -\hat{C}_j & I_m \end{pmatrix} = \begin{pmatrix} \hat{C}_j & \hat{D}_j \end{pmatrix} (\hat{C}_j \hat{D}_j) + Q_j
\]

and thus Eq. (4.10) can be written as,

\[
\sum_{j=1}^{k} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \left( \begin{pmatrix} \hat{C}_j & \hat{D}_j \end{pmatrix} (\hat{C}_j \hat{D}_j) + Q_j - \begin{pmatrix} \hat{C}_j & \hat{D}_j \end{pmatrix} v_{j,m} v_{j,m}^* (\hat{C}_j \hat{D}_j) \right) \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix}
\]
and in turn,

\[
\sum_{j=1}^{k} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix} \begin{pmatrix} \hat{C}^*_{j} & 0 \\ 0 & \hat{D}^*_{j} \end{pmatrix} \begin{pmatrix} I_{m} - v_{j,m}v_{j,m}^* + Q_j \end{pmatrix} \begin{pmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{pmatrix}.
\]

Note now that each \(\Pi_j\) is an \(m \times m\) orthogonal projection i.e,

\[
\Pi_j^2 = \Pi_j = \Pi_j^* \quad \text{rank}(\Pi_j) \in [0, m].
\]

In addition, by Eq. (4.9) each \(Q_j\) is positive semi-definite, thus Eq. (4.10) is satisfied. □

This result suggests that out of a small number of “extreme points” of balanced realizations of \(\mathcal{BR}\) functions, one can construct a whole “matrix-convex-hull” realizations of \(\mathcal{BR}\) functions. This may enable one to perform a simultaneous balanced truncation model reduction of a whole family of bounded real functions, in the spirit of [9, Section 5].

As already indicated, even when the “extreme points” realizations are balanced, the resulting realization may be not minimal.

5. Analogy with continuous-time passive systems

Recall that the Cayley transform of a matrix \(A \in \mathbb{C}^{n \times n}\), \(\mathcal{C}(A)\), is given by

\[
\mathcal{C}(A) := (I_n - A)(I_n + A)^{-1} = -I_n + 2(I_n + A)^{-1}, \quad -1 \notin \text{spect}(A).
\]

Recall also that this transform is involutive, i.e. whenever defined,

\[
\mathcal{C}(\mathcal{C}(A)) = A.
\]

It is known that (see e.g. [4, Example 2.7.1]), from physical point of view the Cayley transform relates between:

- Discrete-time passive systems
- and
- Continuous-time passive systems.

In this section we show that the Cayley transform maps between:

- Maximal matrix-convex set closed under product of its elements
- and
- Maximal matrix-convex cones closed under inversion.

Here are the details.

5.1. (Constant) Matrices. To start building this analogy, we next consider the open (closed) set of all \(n \times n\) matrices \(A\) sharing the same Lyapunov factor \(H \in \mathbb{H}_n\),

\[
\mathcal{L}_H = \{A : HA + A^*H \in \mathbb{P}_n\}
\]

\[
\overline{\mathcal{L}_H} = \{A : HA + A^*H \in \overline{\mathbb{P}_n}\}.
\]

\footnote{In the sense that within \(\overline{\mathbb{P}_n}, \overline{\mathbb{P}_n}\) is the closure of the open set \(\mathbb{P}_n\).}
Proposition 5.1. Let $\text{Stein}_H$, $\mathcal{C}$ and $\mathbf{L}_H$ be as in Eqs. (2.1), (5.1), (5.2), respectively, then,
\[ \mathcal{C}(\text{Stein}_H) = \mathbf{L}_H \]
\[ \mathcal{C}(\overline{\text{Stein}_H}) = \mathbf{L}_H. \]

Proof: For completeness we repeat this classical derivation. Assuming $-1 \notin \text{spect}(A)$, having $A \in \mathcal{C}(\text{Stein}_H)$ means that,
\[ H - (\mathcal{C}(A))^* H \mathcal{C}(A) = Q \quad \text{for some} \quad Q \in \mathbb{P}_n. \]
Thus,
\[ (I + A^*)H(I + A) - (I - A^*)H(I - A) = (I + A^*)Q(I + A) \]
multiplying by $(I + A^*)$ and $(I + A)$ from the left and the right respectively
\[ HA + A^*H = \frac{1}{2}(I + A^*)Q(I + A) \]
and as by assumption, $\frac{1}{2}(I + A^*)Q(I + A) \in \mathbb{P}_n$, one can conclude that $A \in \mathbf{L}_H$. □

It is only natural now to look at properties of the sets $\mathbf{L}_H$ and $\overline{\mathbf{L}_H}$ in Eq. (5.2). To this end we need some preliminaries.

Definition 5.2. A set of $n \times n$ matrices is said to be a convex cone if it is closed under positive scaling and summation.
A set of $n \times n$ matrices is said to be invertible (= “closed under inversion”) if whenever a matrix $M$ in it is non-singular, its inverse $M^{-1}$ belongs to the same set.
A set of $n \times n$ matrices combining both properties is called a convex invertible cone. □

Example 5.3. Note that the set of $2 \times 2$ matrices with det = 1 is not convex, while under the above definition, its subset of matrices of the form $\begin{pmatrix} 0 & -1 \\ 1 & c \end{pmatrix}$, $c \in \mathbb{C}$, is convex but not invertible, as $\begin{pmatrix} 0 & -1 \\ 1 & c \end{pmatrix}^{-1} = \begin{pmatrix} -c & 1 \\ 1 & 0 \end{pmatrix}$.
In contrast, the set $\mathbb{P}_n$ is a convex invertible cone, although it contains singular matrices. □

Here are structural properties of the set $\mathbf{L}_H$. For details, see [27].

Proposition 5.4. For $H \in \mathbb{H}_n$ the set $\mathbf{L}_H$ is a cone closed under inversion, and a maximal open convex set of non-singular matrices.

In the sequel we shall use a couple of useful properties of the Cayley transform.

Observation 5.5. Consider the Cayley transform Eq. (5.1).
(I) Let $A \in \mathbb{C}^{n \times n}$ be so that $-1, \ 0 \notin \text{spect}(A)$ then
\[ -\mathcal{C}(A) = \mathcal{C}(A^{-1}). \]
(II) Let $A \in \mathbb{C}^{n \times n}$ be so that $-1, \ 0, \pm i \notin \text{spect}(A)$ then
\[ (5.3) \quad \mathcal{C}(\overline{\mathcal{C}(A)}) = \frac{1}{2}(A + A^{-1}). \]

$^6$ Convex Invertible Cones were originally defined over any real unital algebra, see [8], [11], [29]. For simplicity of exposition, we here start with matrices.
Proof (I)

\[ C(A^{-1}) = (I_n - A^{-1})(I_n + A^{-1})^{-1} \]
\[ = (A - I_n)A^{-1}((A + I_n)A^{-1})^{-1} \]
\[ = (A - I_n)(A + I_n)^{-1} = -C(A). \]

(II) Under the above assumption, the following chain of equalities is well defined,

\[ - (C(A))^2 = -((I_n - A)(I_n + A)^{-1})^2 \]
\[ = -(I_n - 2A + A^2)(I_n + 2A + A^2)^{-1} \]
\[ = -((\frac{1}{2}A^{-1} - I_n + \frac{1}{2}A)2A) \left( (\frac{1}{2}A^{-1} + I_n + \frac{1}{2}A)2A \right)^{-1} \]
\[ = (I_n - \frac{1}{2}(A + A^{-1})) \left( I_n + \frac{1}{2}(A + A^{-1}) \right)^{-1} \]
\[ = C(\frac{1}{2}(A + A^{-1})). \]

Applying \( C \) to both sides, completes the construction. \( \square \)

To illustrate Observation 5.5, let us take \( A \in \mathbf{L}_H \), for some \( H \in \mathbf{H}_n \), and denote \( \hat{A} := C(A) \). By Proposition 5.1 \( \hat{A} \in \text{Stein}_H \) and by Theorem 2.1 also \( -\hat{A}^2 \in \text{Stein}_H \).

Resorting again to Proposition 5.1 \( C(-\hat{A}^2) \in \mathbf{L}_H \), which by Observation 5.3 says that also \( \frac{1}{2}(A + A^{-1}) \) belongs to \( \mathbf{L}_H \). This suggests the \( \mathbf{L}_H \) is closed under positive scaling, summation and inversion, see Proposition 5.4.

Specifically, iterative operations of the form \( A_{j+1} = \frac{1}{2}(A_j + A_j^{-1}) \) \( j = 0, 1, 2, \ldots \) are known as the Matrix Sign Function Algorithm and are studied in details in [22, Chapter 5], [25, Chapter 22], [28] and [29].

Noting that in Eq. (5.2), for all \( H \in \mathbf{H}_n \),

\[ HL_H = \mathbf{L}_{I_n}, \]

suggests that it is of particular interest to focus on the set \( \mathbf{L}_{I_n} \), i.e. on all matrices \( A \) with positive (semi)-definite Hermitian part,

\[ \mathbf{L}_{I_n} := \{ A : A + A^* \in \mathbf{P}_n \} \]

(5.4)

\[ \mathbf{C}_{I_n} := \{ A : A + A^* \in \mathbf{P}_n \}. \]

Note that these sets may be viewed as matricial extensions of \( \mathbb{C}_R, \mathbb{C}_R \), respectively.

We can now characterize the set \( \mathbf{L}_{I_n} \) through its structure. For details, see [27].

**Theorem 5.6.** The following statements are true.

(i) The set \( \mathbf{L}_{I_n} \) in (5.4), is a matrix-convex cone of matrices, closed under inversion. Moreover, it is a maximal open convex set of matrices whose spectrum is in \( \mathbb{C}_R \).

(ii) Conversely, a maximal open, matrix-convex, cone of non-singular matrices, closed under inversion, containing the matrix \( I_n \), is the set \( \mathbf{L}_{I_n} \) in Eq. (5.4).

Recall that Eq. (3.1) says that \( \mathbb{C}_{\text{Stein}_{I_n}} \) is the set of all matrices whose spectral norm is bounded from above by \( \alpha \). Thus for all \( \alpha \in (0, 1) \), these are strict contractions (weak
contractions for $\alpha = 1$). Thus, Proposition 5.1 suggests that the set $L_{H_n}$ should be “contractive in some sense”. In fact, using Eq. (2.3) one can make a quantitative refinement of Proposition 5.1.

Observation 5.7. For all $H \in H_n$ and all $\alpha \in (0, 1]$,

$$\mathcal{C}(\frac{\alpha}{\alpha} \text{Stein}_H) = \{ A : (HA + A^*H - \frac{\alpha^2}{1+\alpha^2} (H + A^*HA) ) \in P_n \}.$$ 

In scalar terminology this means that for all $\alpha \in (0, 1)$ the Cayley transform maps disks of a radius $\alpha$, centered at the origin, see the left hand side of Figure 1, to disks in $\mathbb{C}_R$, centered at $\frac{1}{1+\alpha} + 0i$ and closed under inversion, see the right hand side of Figure 1.

The corresponding rational functions are called Quantitatively Hyper-Positive Real, and are associated with absolute stability (a.k.a. the Lurie problem). The parameter $\alpha$ quantifies stability, roughly speaking, the smaller $\alpha$, the “more stable” this set is. For further details see [2].

5.2. Rational Functions. Following Eq. (4.2) along with Proposition 5.1 one can expect a family of $m \times m$-valued functions $F(s)$ so that

$$F(s) \in \begin{cases} \mathbb{R}^{m \times m} & s \in \mathbb{R} \\ L_{I_m} \text{ and analytic} & s \in \mathbb{C}_R \end{cases}$$

Indeed, this is exactly the classical family of Positive Real functions, denoted by $\mathcal{PR}$. It also well known, see e.g. [4, Example 2.7.1], [19, Problem 3.17], [30, Section 2.11], that

$$\mathcal{C}(\mathcal{BR}) = \mathcal{PR}.$$ 

Indeed we have a continuous-time version of Observation 4.1.

Theorem 5.8. The family $\mathcal{PR}$, of $m \times m$-valued positive real rational functions, is a maximal, matrix-convex, cone, closed under inversion, of functions analytic in $\mathbb{C}_R$.

Conversely, a maximal matrix-convex cone of $m \times m$-valued rational functions, containing the zero degree function $F_0(s) \equiv I_m$, is the set $\mathcal{PR}$.

For more details, see [27].
5.3. Realization Arrays. The celebrated Positive Real Lemma, see e.g. [1], [3], [4], Chapters 5, 6], [13], [19, Problem 3.25] and [32], characterizes \( PR \) functions with no pole at infinity through the realization. We here cite a version which suits our framework.

**Theorem 5.9.** Let \( F(s) \) be an \( m \times m \)-valued rational function \( F(s) \) admitting a state-space realization \( R_F \) as in Eq. (4.3).

\( F(s) \) is Positive Real if and only if it admits a balanced realization satisfying,

\[
\begin{pmatrix}
-1_n & 0 \\
0 & 1_m
\end{pmatrix} R_F + R_F^* \begin{pmatrix}
-1_n & 0 \\
0 & 1_m
\end{pmatrix} \in \mathbb{P}_{n+m}.
\]

We next show that in a way similar to what we had before, out of a small number of “extreme points” of balanced realizations of \( PR \) functions, one can construct a whole “matrix-convex-hull” realizations of \( PR \) functions.

**Theorem 5.10.** Given a family of \((n+m) \times (n+m)\) realizations \( R_F \) satisfying Eq. (5.5).

Then (as matrices), this family is an \( n, m \)-matrix-convex cone, closed under inversion. Thus every element in this \( n, m \)-matrix-convex cone, closed under inversion, is a realization of an \( m \times m \)-valued \( PR \) function of a McMillan degree of at most \( n \).

In brief, we have shown above that,

| Passive linear time-invariant systems and maximal matrix-convexity |
|---------------------------------------------------------------|
| discrete-time | continuous-time |
| a set closed under product among its elements | a cone closed under inversion |

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