Tridiagonal pairs of $q$-Racah type and the $\mu$-conjecture

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Abstract

Let $\mathbb{K}$ denote a field and let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of $A, A^*$ is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V^*_i\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that $AV^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1}$ for $0 \leq i \leq \delta$, where $V^*_{-1} = 0$ and $V^*_{\delta+1} = 0$; (iv) there is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a tridiagonal pair on $V$. It is known that $d = \delta$ and for $0 \leq i \leq d$ the dimensions of $V_i, V_{d-i}, V^*_i, V^*_{d-i}$ coincide. We say the pair $A, A^*$ is sharp whenever $\dim V_0 = 1$. It is known that if $\mathbb{K}$ is algebraically closed then $A, A^*$ is sharp.

A conjectured classification of the sharp tridiagonal pairs was recently introduced by T. Ito and the second author. Shortly afterwards we introduced a conjecture, called the $\mu$-conjecture, which implies the classification conjecture. In this paper we show that the $\mu$-conjecture holds in a special case called $q$-Racah.

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1 Tridiagonal pairs

Throughout the paper $\mathbb{K}$ denotes a field.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we call $W$ an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case $\theta$ is the eigenvalue of $A$ associated with $W$. We say that $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$. 

Definition 1.1  [1, Definition 1.1] Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a tridiagonal pair on $V$ we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of $A, A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W$ of $V$ such that $AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V$.

We say that the pair $A, A^*$ is over $\mathbb{K}$.

Note 1.2 According to a common notational convention $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

We refer the reader to [1–12, 14–16] for background on tridiagonal pairs.

In order to motivate our results we recall some facts about tridiagonal pairs. Let $A, A^*$ denote a tridiagonal pair on $V$, as in Definition 1.1. By [1, Lemma 4.5] the integers $d$ and $\delta$ from (ii), (iii) are equal; we call this common value the diameter of the pair. An ordering of the eigenspaces of $A$ (resp. $A^*$) is said to be standard whenever it satisfies (i) (resp. (ii)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of $A$. By [1, Lemma 2.4], the ordering $\{V_d-i\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of $A^*$. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of $A$ (resp. $A^*$). By [1, Corollary 5.7], for $0 \leq i \leq d$ the spaces $V_i, V_i^*$ have the same dimension; we denote this common dimension by $\rho_i$. By [1, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the shape of $A, A^*$. We say $A, A^*$ is sharp whenever $\rho_0 = 1$. If $\mathbb{K}$ is algebraically closed then $A, A^*$ is sharp [11, Theorem 1.3].

We now summarize the present paper. A conjectured classification of the sharp tridiagonal pairs was introduced in [7, Conjecture 14.6] and studied carefully in [9–11]; see Conjecture 3.1 below. Shortly afterwards we introduced a conjecture, called the $\mu$-conjecture, which implies the classification conjecture. The $\mu$-conjecture is roughly described as follows. We start with a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ of scalars taken from $\mathbb{K}$ that satisfy the known constraints on the eigenvalues of a tridiagonal pair over $\mathbb{K}$ of diameter $d$; these are conditions (i), (ii) of Conjecture 3.1. Following [11, Definition 2.4] we associate with $p$ an
associative \( \mathbb{K} \)-algebra \( T \) defined by generators and relations; see Definition \( \text{4.1} \) below. We are interested in the \( \mathbb{K} \)-algebra \( e_{0}^{*}Te_{0}^{*} \) where \( e_{0}^{*} \) is a certain idempotent element of \( T \). Let \( \{x_{i}\}_{i=1}^{d} \) denote mutually commuting indeterminates. Let \( \mathbb{K}[x_{1}, \ldots, x_{d}] \) denote the \( \mathbb{K} \)-algebra consisting of the polynomials in \( \{x_{i}\}_{i=1}^{d} \) that have all coefficients in \( \mathbb{K} \). In [12, Corollary 6.3] we displayed a surjective \( \mathbb{K} \)-algebra homomorphism \( \mu : \mathbb{K}[x_{1}, \ldots, x_{d}] \to e_{0}^{*}Te_{0}^{*} \). The \( \mu \)-conjecture [12, Conjecture 6.4] asserts that \( \mu \) is an isomorphism. We have shown that the \( \mu \)-conjecture implies the classification conjecture [12, Theorem 10.1] and that the \( \mu \)-conjecture holds for \( d \leq 5 \) [12, Theorem 12.1]. In the present paper we obtain the following additional evidence that the \( \mu \)-conjecture is true. There is a general class of parameters \( p \) said to have \( q \)-Racah type [8, Definition 3.1]. In [8, Theorem 3.3] T. Ito and the second author verified the classification conjecture for the case in which \( p \) has \( q \)-Racah type and \( \mathbb{K} \) is algebraically closed; see Proposition \( \text{3.2} \) below. Making heavy use of this result, we verify the \( \mu \)-conjecture for the case in which \( p \) has \( q \)-Racah type, with no restriction on \( \mathbb{K} \). Our main result is Theorem \( \text{5.3} \). On our way to Theorem \( \text{5.3} \) we obtain two related results Theorem \( \text{5.1} \) and \( \text{5.2} \), which might be of independent interest.

2 Tridiagonal systems

When working with a tridiagonal pair, it is often convenient to consider a closely related object called a tridiagonal system. To define a tridiagonal system, we recall a few concepts from linear algebra. Let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. Let \( \text{End}_{\mathbb{K}}(V) \) denote the \( \mathbb{K} \)-algebra of all linear transformations from \( V \) to \( V \). Let \( A \) denote a diagonalizable element of \( \text{End}_{\mathbb{K}}(V) \). Let \( \{V_{i}\}_{i=0}^{d} \) denote an ordering of the eigenspaces of \( A \) and let \( \{\theta_{i}\}_{i=0}^{d} \) denote the corresponding ordering of the eigenvalues of \( A \). For \( 0 \leq i \leq d \) define \( E_{i} \in \text{End}_{\mathbb{K}}(V) \) such that \((E_{i} - I)V_{i} = 0 \) and \( E_{i}V_{j} = 0 \) for \( j \neq i \) \((0 \leq j \leq d) \). Here \( I \) denotes the identity of \( \text{End}_{\mathbb{K}}(V) \). We call \( E_{i} \) the primitive idempotent of \( A \) corresponding to \( V_{i} \) (or \( \theta_{i} \)). Observe that (i) \( I = \sum_{i=0}^{d} E_{i} \); (ii) \( E_{i}E_{j} = \delta_{i,j}E_{i} \) \((0 \leq i, j \leq d) \); (iii) \( V_{i} = E_{i}V \) \((0 \leq i \leq d) \); (iv) \( A = \sum_{i=0}^{d} \theta_{i}E_{i} \). Moreover

\[
E_{i} = \prod_{0 \leq j \leq d}^{i \neq j} \frac{A - \theta_{j}I}{\theta_{i} - \theta_{j}}. \tag{3}
\]

Now let \( A, A^{*} \) denote a tridiagonal pair on \( V \). An ordering of the primitive idempotents or eigenvalues of \( A \) (resp. \( A^{*} \)) is said to be standard whenever the corresponding ordering of the eigenspaces of \( A \) (resp. \( A^{*} \)) is standard.
**Definition 2.1** [1, Definition 2.1] Let $V$ denote a vector space over $K$ with finite positive dimension. By a *tridiagonal system* on $V$ we mean a sequence

$$
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)
$$

that satisfies (i)–(iii) below.

(i) $A, A^*$ is a tridiagonal pair on $V$.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A$.

(iii) $\{E^*_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A^*$.

We say $\Phi$ is *over* $K$. We call $V$ the *vector space underlying* $\Phi$.

The notion of isomorphism for tridiagonal systems is defined in [9, Definition 3.1]. The following result is immediate from lines (1), (2) and Definition 2.1.

**Lemma 2.2** [10, Lemma 2.5] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a tridiagonal system. Then for $0 \leq i, j, k \leq d$ the following (i), (ii) hold.

(i) $E_i A^k E_j = 0$ if $k < |i-j|$.

(ii) $E^*_i A^k E^*_j = 0$ if $k < |i-j|$.

**Definition 2.3** Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a tridiagonal system on $V$. For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta^*_i$) denote the eigenvalue of $A$ (resp. $A^*$) associated with the eigenspace $E_i V$ (resp. $E^*_i V$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of $\Phi$. We observe that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) are mutually distinct and contained in $K$. We say $\Phi$ is *sharp* whenever the tridiagonal pair $A, A^*$ is sharp.

Let $\Phi$ denote a tridiagonal system over $K$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$. By [1, Theorem 11.1] the expressions

$$
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}
$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$. For this constraint the “most general” solution is

$$
\theta_i = a + bq^{2i-d} + cq^{d-2i} \quad (0 \leq i \leq d), \quad (5)
$$

$$
\theta^*_i = a^* + b^* q^{2i-d} + c^* q^{d-2i} \quad (0 \leq i \leq d), \quad (6)
$$

$$
q, a, b, c, a^*, b^*, c^* \in \overline{K}, \quad (7)
$$

$$
q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1, \quad bb^* cc^* \neq 0, \quad (8)
$$

where $\overline{K}$ denotes the algebraic closure of $K$. For this solution $q^2 + q^{-2} + 1$ is the common value of (4). The tridiagonal system $\Phi$ is said to have *q-Racah type* whenever (5)–(8) hold.

The following definition is more general.
Definition 2.4 Let \(d\) denote a nonnegative integer and let \((\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)\) denote a sequence of scalars taken from \(K\). We call this sequence \(q\)-Racah whenever the following (i), (ii) hold.

(i) \(\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*\) if \(i \neq j\) \((0 \leq i, j \leq d)\).

(ii) There exist \(q, a, b, c, a^*, b^*, c^*\) that satisfy (5)–(8).

We will return to the subject of \(q\)-Racah a bit later. We now recall the split sequence of a sharp tridiagonal system. We will use the following notation.

Definition 2.5 Let \(\lambda\) denote an indeterminate and let \(K[\lambda]\) denote the \(K\)-algebra consisting of the polynomials in \(\lambda\) that have all coefficients in \(K\). Let \(d\) denote a nonnegative integer and let \((\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)\) denote a sequence of scalars taken from \(K\). Then for \(0 \leq i \leq d\) we define the following polynomials in \(K[\lambda]\):

\[
\begin{align*}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*), \\
\eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*). \\
\end{align*}
\]

Note that each of \(\tau_i, \eta_i, \tau_i^*, \eta_i^*\) is monic with degree \(i\).

Definition 2.6 [12, Definition 2.5] Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a sharp tridiagonal system over \(K\), with eigenvalue sequence \(\{\theta_i\}_{i=0}^d\) and dual eigenvalue sequence \(\{\theta_i^*\}_{i=0}^d\). By [11, Lemma 5.4], for \(0 \leq i \leq d\) there exists a unique \(\zeta_i \in K\) such that

\[
E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}.
\]

Note that \(\zeta_0 = 1\). We call \(\{\zeta_i\}_{i=0}^d\) the split sequence of the tridiagonal system.

Definition 2.7 [9, Definition 6.2] Let \(\Phi\) denote a sharp tridiagonal system. By the parameter array of \(\Phi\) we mean the sequence \((\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)\) where \(\theta_i\) \((resp. \theta_i^*)_{i=0}^d\) is the eigenvalue sequence (resp. dual eigenvalue sequence) of \(\Phi\) and \(\{\zeta_i\}_{i=0}^d\) is the split sequence of \(\Phi\).

3 The classification conjecture

To motivate our results we recall a conjectured classification of the tridiagonal systems due to T. Ito and the second author.
**Conjecture 3.1** [7, Conjecture 14.6] Let $d$ denote a nonnegative integer and let
\begin{equation}
(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d) \tag{9}
\end{equation}
denote a sequence of scalars taken from $\mathbb{K}$. Then there exists a sharp tridiagonal system $\Phi$ over $\mathbb{K}$ with parameter array (9) if and only if (i)–(iii) hold below.

(i) $\theta_i \neq \theta_j, \theta^*_i \neq \theta^*_j$ if $i \neq j$ ($0 \leq i, j \leq d$).

(ii) The expressions
\begin{align*}
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, & \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{10}
\end{align*}
are equal and independent of $i$ for $2 \leq i \leq d - 1$.

(iii) $\zeta_0 = 1, \zeta_d \neq 0$ and
\begin{equation}
0 \neq \sum_{i=0}^d \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i. \tag{11}
\end{equation}

Suppose (i)–(iii) hold. Then $\Phi$ is unique up to isomorphism of tridiagonal systems.

In [9, Section 8] we proved the “only if” direction of Conjecture 3.1. In [11, Theorem 1.6] we proved the last assertion of Conjecture 3.1. Concerning the “if” direction of Conjecture 3.1, we proved this for $d \leq 5$ in [12, Corollary 12.2]. We also have the following result due to T. Ito and the second author.

**Proposition 3.2** [8, Theorem 3.3] Assume $\mathbb{K}$ is algebraically closed. Let $d$ denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{K}$ that is $q$-Racah in the sense of Definition 2.4. Let $(\zeta_i)_{i=0}^d$ denote a sequence of scalars taken from $\mathbb{K}$ that satisfies condition (iii) of Conjecture 3.1. Then there exists a sharp tridiagonal system over $\mathbb{K}$ that has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.

### 4 The $\mu$-conjecture

In this section we recall the $\mu$-conjecture. It has to do with the following algebra.
Definition 4.1 [11, Definition 2.4] Let $d$ denote a nonnegative integer, and let $p = (\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d})$ denote a sequence of scalars taken from $\mathbb{K}$ that satisfies conditions (i), (ii) of Conjecture 3.1. Let $T = T(p, \mathbb{K})$ denote the associative $\mathbb{K}$-algebra with 1, defined by generators $a_i$, $\{e_i\}_{i=0}^{d}$, $a^*$, $\{e_i^*\}_{i=0}^{d}$ and relations

$$e_i e_j = \delta_{i,j} e_i, \quad e_i^* e_j^* = \delta_{i,j} e_i^* \quad (0 \leq i, j \leq d),$$

$$1 = \sum_{i=0}^{d} e_i, \quad 1 = \sum_{i=0}^{d} e_i^*,$$  

$$a = \sum_{i=0}^{d} \theta_i e_i, \quad a^* = \sum_{i=0}^{d} \theta_i e_i^*,$$

$$e_i^* a^k e_j = 0 \quad \text{if} \quad k < |i-j| \quad (0 \leq i,j,k \leq d),$$

$$e_i a^k e_j = 0 \quad \text{if} \quad k < |i-j| \quad (0 \leq i,j,k \leq d).$$

The algebra $T$ is related to tridiagonal systems as follows.

Lemma 4.2 [11, Lemma 2.5] Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $(A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$ denote a tridiagonal system on $V$ with eigenvalue sequence $\{\theta_i\}_{i=0}^{d}$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^{d}$. For the sequence $p = (\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d})$ let $T = T(p, \mathbb{K})$ denote the algebra from Definition 4.1. Then there exists a unique $T$-module structure on $V$ such that $a, a^*, e_i, e_i^*$ acts on $V$ as $A, A^*, E_i, E_i^*$, respectively. Moreover this $T$-module is irreducible.

For the rest of this section, let $d$ denote a nonnegative integer and let $p = (\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d})$ denote a sequence of scalars taken from $\mathbb{K}$ that satisfies conditions (i), (ii) of Conjecture 3.1. Let $T = T(p, \mathbb{K})$ denote the corresponding algebra from Definition 4.1. Observe that $e_0^* T e_0^*$ is a $\mathbb{K}$-algebra with multiplicative identity $e_0^*$.

Lemma 4.3 [11, Theorem 2.6] The algebra $e_0^* T e_0^*$ is commutative and generated by

$$e_0^* \tau_i(a) e_0^* \quad (1 \leq i \leq d).$$

Definition 4.4 Let $\{x_i\}_{i=1}^{d}$ denote mutually commuting indeterminates. We denote by $\mathbb{K}[x_1, \ldots, x_d]$ the $\mathbb{K}$-algebra consisting of the polynomials in $\{x_i\}_{i=1}^{d}$ that have all coefficients in $\mathbb{K}$.

From Lemma 4.3 we immediately obtain the following.

Lemma 4.5 [12, Corollary 6.3] There exists a surjective $\mathbb{K}$-algebra homomorphism

$$\mu : \mathbb{K}[x_1, \ldots, x_d] \to e_0^* T e_0^*$$

that sends $x_i \mapsto e_0^* \tau_i(a) e_0^*$ for $1 \leq i \leq d$.

The following conjecture was introduced in [12, Conjecture 6.4].
Conjecture 4.6 (μ-conjecture) The map μ from Lemma 4.5 is an isomorphism.

We have shown that the μ-conjecture implies Conjecture 3.1 [12, Theorem 10.1], and that the μ-conjecture holds for \( d \leq 5 \) [12, Theorem 12.1]. In this paper we prove Conjecture 4.6 for the case in which \( p \) is \( q \)-Racah. In our proof we make heavy use of Proposition 3.2.

Our main result is Theorem 5.3.

5 The main result

In this section we obtain our main result, which is Theorem 5.3. On our way to this result we obtain two other results Theorem 5.1 and 5.2, which may be of independent interest. Throughout this section we denote a nonnegative integer and let \( p = (\{ \theta_i \}_{i=0}^d; \{ \theta_i^* \}_{i=0}^d) \) denote a sequence of scalars taken from \( \mathbb{K} \) that satisfies conditions (i), (ii) of Conjecture 3.1.

Theorem 5.1 Assume the field \( \mathbb{K} \) is infinite and let \( T = T(p, \mathbb{K}) \) denote the \( \mathbb{K} \)-algebra from Definition 4.1. Assume that, for every sequence \( \{ \zeta_i \}_{i=0}^d \) of scalars taken from \( \mathbb{K} \) that satisfies condition (iii) of Conjecture 3.1, there exists a sharp tridiagonal system over \( \mathbb{K} \) that has parameter array \((\{ \theta_i \}_{i=0}^d; \{ \theta_i^* \}_{i=0}^d; \{ \zeta_i \}_{i=0}^d)\). Then the map \( \mu : \mathbb{K}[x_1, \ldots, x_d] \to e_0^p T e_0^p \) from Lemma 4.3 is an isomorphism.

Proof. We assume \( d \geq 1 \); otherwise the result is obvious. The map \( \mu \) is surjective by Lemma 4.5 so it suffices to show that \( \mu \) is injective. We pick any \( f \in \mathbb{K}[x_1, \ldots, x_d] \) such that \( \mu(f) = 0 \), and show \( f = 0 \). Instead of working directly with \( f \), it will be convenient to work with the product \( \psi = fgh \), where \( g = \eta_d^*(\theta_0^*)x_d \) and \( h = \eta_d^*(\theta_0^*) \) times

\[
\eta_d(\theta_0) + \sum_{i=1}^d \eta_{d-i}(\theta_0)x_i.
\]

By construction \( \eta_d^*(\theta_0^*) \neq 0 \) so each of \( g, h \) is nonzero. To show that \( f = 0 \), we show \( \psi = 0 \) and invoke the fact that \( \mathbb{K}[x_1, \ldots, x_d] \) is a domain [13, page 129]. We now show that \( \psi = 0 \). Since the field \( \mathbb{K} \) is infinite it suffices to show that \( \psi(\xi_1, \ldots, \xi_d) = 0 \) for all \( d \)-tuples \((\xi_1, \ldots, \xi_d)\) of scalars taken from \( \mathbb{K} \) [13, Proposition 6.89]. Let \((\xi_1, \ldots, \xi_d)\) be given. Define

\[
\zeta_i = \xi_i(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*) \quad (1 \leq i \leq d)
\]

and \( \zeta_0 = 1 \). Observe

\[
g(\xi_1, \ldots, \xi_d) = \zeta_d.
\]

\[
h(\xi_1, \ldots, \xi_d) = \sum_{i=0}^d \eta_{d-i}(\theta_0)\eta_{d-i}(\theta_0^*)\zeta_i.
\]

First assume \( \{ \zeta_i \}_{i=0}^d \) does not satisfy condition (iii) of Conjecture 3.1. Then either \( \zeta_d = 0 \), in which case \( g(\xi_1, \ldots, \xi_d) = 0 \), or \( 0 = \sum_{i=0}^d \eta_{d-i}(\theta_0)\eta_{d-i}(\theta_0^*)\zeta_i \), in which case \( h(\xi_1, \ldots, \xi_d) = 0 \).
0. Either way \( \psi(\xi_1, \ldots, \xi_d) = 0 \). Next assume \( \{\zeta_i\}_{i=0}^d \) does satisfy condition (iii) of Conjecture 3.1. By the assumption of the present theorem, there exists a sharp tridiagonal system \( \Phi = \{A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\} \) over \( K \) that has parameter array \( \{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d\) . Let \( V \) denote the vector space underlying \( \Phi \). By Definition 2.6 the following holds on \( V \):
\[
E_0^*\tau_i(A)E_0^* = \frac{\zeta_iE_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_1^*) \cdots (\theta_0^* - \theta_i^*)} \quad (1 \leq i \leq d). \tag{18}
\]
Consider the \( T \)-module structure on \( V \) from Lemma 4.2. Using (17), (18) we find that the following holds on \( V \):
\[
e_0^*\tau_i(a)e_0^* = \xi_ie_0^* \quad (1 \leq i \leq d). \tag{19}
\]
Pick an integer \( i \) \((1 \leq i \leq d)\). By (19) the element \( e_0^*\tau_i(a)e_0^* \) acts on \( e_0^*V \) as \( \xi_i \) times the identity map. Recall that \( \mu \) sends \( x_i \) to \( e_0^*\tau_i(a)e_0^* \), so \( \mu(x_i) \) acts on \( e_0^*V \) as \( \xi_i \) times the identity map. By these comments \( \mu(f) \) acts on \( e_0^*V \) as \( f(\xi_1, \ldots, \xi_d) \) times the identity map. But \( \mu(f) = 0 \) and \( e_0^*V \neq 0 \) so \( f(\xi_1, \ldots, \xi_d) = 0 \), and therefore \( \psi(\xi_1, \ldots, \xi_d) = 0 \). We have shown \( \psi(\xi_1, \ldots, \xi_d) \), \( 0 \) for all \( d \)-tuples \( (\xi_1, \ldots, \xi_d) \) of scalars taken from \( K \), and therefore \( \psi = 0 \). The result follows.

**Theorem 5.2** Let \( F \) denote a field extension of \( K \). Let the algebras \( T_K = T(p, K) \) and \( T_F = T(p, F) \) be as in Definition 1.1. Let
\[
\begin{align*}
\mu_K : K[x_1, \ldots, x_d] &\to e_0^*T_Ke_0^*, \\
\mu_F : F[x_1, \ldots, x_d] &\to e_0^*T_Fe_0^*
\end{align*}
\]
denote the maps from Lemma 4.5 and assume that \( \mu_F \) is an isomorphism. Then \( \mu_K \) is an isomorphism.

**Proof.** The map \( \mu_K \) is surjective by Lemma 4.5 so it suffices to show that \( \mu_K \) is injective. Since \( K \) is a subfield of \( F \) we may view any \( F \)-algebra as a \( K \)-algebra. The inclusion map \( K \to F \) induces an injective \( K \)-algebra homomorphism \( \iota : K[x_1, \ldots, x_d] \to F[x_1, \ldots, x_d] \). In the \( K \)-algebra \( T_K \) the defining generators \( a, \{a_i\}_{i=0}^d; a^*; \{a_i^*\}_{i=0}^d \) satisfy the defining relations for \( T_K \). Therefore there exists a \( K \)-algebra homomorphism \( N : T_K \to T_F \) that sends each of the \( T_K \) generators \( a, \{a_i\}_{i=0}^d; a^*; \{a_i^*\}_{i=0}^d \) to the corresponding generator in \( T_F \). The restriction of \( N \) to \( e_0^*T_Ke_0^* \) is a \( K \)-algebra homomorphism \( e_0^*T_Ke_0^* \to e_0^*T_Fe_0^* \); we denote this homomorphism by \( \nu \). By the following diagram commutes:
\[
\begin{array}{ccc}
K[x_1, \ldots, x_d] & \xrightarrow{\iota} & F[x_1, \ldots, x_d] \\
\mu_K \downarrow & & \downarrow \mu_F \\
e_0^*T_Ke_0^* & \xrightarrow{\nu} & e_0^*T_Fe_0^*
\end{array}
\]
The maps \( \iota \) and \( \mu_F \) are injective so their composition \( \mu_F \circ \iota \) is injective. But \( \mu_F \circ \iota = \nu \circ \mu_K \) so \( \nu \circ \mu_K \) is injective. Therefore \( \mu_K \) is injective and hence an isomorphism.

The following is our main result.
Theorem 5.3 Let $K$ denote a field and let $d$ denote a nonnegative integer. Let $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $K$ that is $q$-Racah in the sense of Definition 2.4. Let the $K$-algebra $T = T(p, K)$ be as in Definition 4.1. Then the corresponding map $\mu : K[x_1, \ldots, x_d] \to e_0^* T e_0^*$ from Lemma 4.5 is an isomorphism.

Proof. Abbreviate $F = \overline{K}$ for the algebraic closure of $K$, and note that $F$ is infinite. Let $T_F = T(p, F)$ denote the $F$-algebra from Definition 4.1 and let $\mu_F : F[x_1, \ldots, x_d] \to e_0^* T_F e_0^*$ be the corresponding map from Lemma 4.5. By Proposition 3.2, for every sequence $\{\zeta_i\}_{i=0}^d$ of scalars taken from $F$ that satisfies condition (iii) of Conjecture 3.1, there exists a sharp tridiagonal system over $F$ that has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$. By this and Theorem 5.1 the map $\mu_F$ is an isomorphism. Now $\mu$ is an isomorphism by Theorem 5.2. □

We finish with a comment.

Lemma 5.4 Proposition 3.2 remains true if we drop the assumption that $K$ is algebraically closed.

Proof. Immediate from Theorem 5.3 and [12, Theorem 10.1]. □

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