Characterizing the entanglement of bipartite quantum systems

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We derive a separability criterion for bipartite quantum systems which generalizes the already known criteria. It is based on observables having generic commutation relations. We then discuss in detail the relation among these criteria.

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I. INTRODUCTION

Entangled states have been known almost from the very beginning of quantum mechanics [1, 2], and their somewhat unusual features have been investigated for many years. However, recent developments in the theory of quantum information [3] have required a deeper knowledge of their properties.

The simplest system where one can study entanglement is represented by a bipartite system. In such a system, either with discrete or continuous variable, the inseparability of pure states, is now well understood and the von Neumann entropy of either subsystem quantifies the amount of entanglement [4]. Instead, the question of inseparability of mixed states is much more complicated and involves subtle effects. For discrete variable systems the Peres-Horodecki criterion [5] constitutes a theoretical tool to investigate the separability. Recently, different criteria have been also proposed for continuous variable systems [6, 7, 8, 9, 10, 11]. Nevertheless a unifying criterion, of practical use, does not yet exist. Needless to say that also the entanglement quantification for mixed states is not well assessed [12]. On the other hand, the lack of knowledge for bipartite entanglement is not only a serious drawback in the study of mixed-state entanglement, but also a limitation for understanding multipartite entanglement.

The aim of this paper is to throw some light on the plethora of entanglement criteria for bipartite systems. In particular, we shall derive a general separability criterion valid for any state of any bipartite system. We shall then discuss its relation with the already known criteria.

II. A GENERAL SEPARABILITY CRITERION

Let us consider a bipartite system whose subsystems, not necessarily identical, are labeled as 1 and 2, and a separable state $\hat{\rho}_{\text{sep}}$ on the Hilbert space $\mathcal{H}_{\text{tot}} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Such state can be written as

$$\hat{\rho}_{\text{sep}} = \sum_k w_k \hat{\rho}_{k1} \otimes \hat{\rho}_{k2},$$

where $\hat{\rho}_{kj}$ ($j = 1, 2$) are normalized density matrices on $\mathcal{H}_j$ while $w_k \geq 0$ with $\sum_k w_k = 1$.

Let us now choose a generic couple of observables for each subsystem, say $\hat{r}_j, \hat{s}_j$ on $\mathcal{H}_j$ ($j = 1, 2$), and define the operators

$$\hat{C}_j = i [\hat{r}_j, \hat{s}_j], \quad j = 1, 2. \quad (2)$$

Notice that the two couples $\hat{r}_j, \hat{s}_j$ may represent completely different observables, e.g. one couple may refer to a continuous variable subsystem while the other to a discrete variable subsystem. Furthermore, $\hat{C}_j$ is typically nontrivial Hermitian operator on the Hilbert subspaces.

We now introduce the following observables on $\mathcal{H}_{\text{tot}}$:

$$\hat{u} = a_1 \hat{r}_1 + a_2 \hat{r}_2, \quad \hat{v} = b_1 \hat{s}_1 + b_2 \hat{s}_2, \quad (3)$$

where $a_j, b_j$ are real parameters. From the standard form of the uncertainty principle [13], it follows that every state $\hat{\rho}$ on $\mathcal{H}_{\text{tot}}$ must satisfy

$$\langle (\Delta \hat{u})^2 \rangle / \langle (\Delta \hat{v})^2 \rangle \geq \frac{|a_1 b_1 \langle \hat{C}_1 \rangle + a_2 b_2 \langle \hat{C}_2 \rangle|^2}{4}, \quad (4)$$

where $\langle \hat{\Theta} \rangle \equiv \text{Tr}[\hat{\Theta} \hat{\rho}]$ is the expectation value over $\hat{\rho}$ of the operator $\hat{\Theta}$, and $\Delta \hat{\Theta} \equiv \hat{\Theta} - \langle \hat{\Theta} \rangle$. However, for separable states, a stronger bound exists. As a matter of fact, the following theorem holds

**Theorem:**

$$\hat{\rho}_{\text{sep}} \implies \langle (\Delta \hat{u})^2 \rangle / \langle (\Delta \hat{v})^2 \rangle \geq \check{\Theta}^2, \quad (5)$$

with

$$\check{\Theta} = \frac{1}{2} ( |a_1 b_1| \check{\Theta}_1 + |a_2 b_2| \check{\Theta}_2 ), \quad (6)$$

where

$$\check{\Theta}_j \equiv \sum_k w_k \langle \hat{\Theta}_j \rangle_k, \quad j = 1, 2, \quad (7)$$

being $\langle \hat{\Theta}_j \rangle_k \equiv \text{Tr}_j[\hat{\Theta}_j \hat{\rho}_{kj}]$ the expectation value of the operator $\hat{\Theta}_j$ onto $\hat{\rho}_{kj}$. 
**Proof:** From the definitions of $\langle (\Delta \hat{u})^2 \rangle$, and $\hat{\rho}_{\text{sep}}$, it is easy to see that

$$
\langle (\Delta \hat{u})^2 \rangle = \sum_k w_k \left[ a_1^2 \langle (\Delta \hat{r}_1^{(k)})^2 \rangle_k + a_2^2 \langle (\Delta \hat{r}_2^{(k)})^2 \rangle_k \right]
+ \sum_k w_k \langle \hat{u} \rangle_k^2 - \left( \sum_k w_k \langle \hat{u} \rangle_k \right)^2,
$$

where the quantity $\Delta \hat{r}_j^{(k)} \equiv \hat{r}_j - \langle \hat{r}_j \rangle_k$ is the variance of the operator $\hat{r}_j$ on the state $\hat{\rho}_{kj}$. An analogous expression holds for $\langle (\Delta \hat{v})^2 \rangle$. By applying the uncertainty principle to the inequality on the last two terms of the r.h.s. of (8) we obtain

$$
\langle (\Delta \hat{u})^2 \rangle \geq \sum_k w_k \left[ b_1^2 \langle (\Delta \hat{s}_1^{(k)})^2 \rangle_k + b_2^2 \langle (\Delta \hat{s}_2^{(k)})^2 \rangle_k \right],
$$

and analogously

$$
\langle (\Delta \hat{v})^2 \rangle \geq \sum_k w_k \left[ b_1^2 \langle (\Delta \hat{s}_1^{(k)})^2 \rangle_k + b_2^2 \langle (\Delta \hat{s}_2^{(k)})^2 \rangle_k \right].
$$

Now, taking any two real nonnegative numbers $\alpha$ and $\beta$, and using the relations (9) and (10), we get the following inequality

$$
\alpha \langle (\Delta \hat{u})^2 \rangle + \beta \langle (\Delta \hat{v})^2 \rangle \geq \sum_k w_k \left[ \alpha a_1^2 \langle (\Delta \hat{r}_1^{(k)})^2 \rangle_k + \beta b_1^2 \langle (\Delta \hat{s}_1^{(k)})^2 \rangle_k \right]
+ \sum_k w_k \left[ \alpha a_2^2 \langle (\Delta \hat{r}_2^{(k)})^2 \rangle_k + \beta b_2^2 \langle (\Delta \hat{s}_2^{(k)})^2 \rangle_k \right].
$$

Furthermore, by applying the uncertainty principle to the operators $\hat{r}_j$ and $\hat{s}_j$ on the state $\hat{\rho}_{kj}$, it follows

$$
\alpha a_j^2 \langle (\Delta \hat{r}_j^{(k)})^2 \rangle_k + \beta b_j^2 \langle (\Delta \hat{s}_j^{(k)})^2 \rangle_k \geq \alpha a_j^2 \langle (\Delta \hat{r}_j^{(k)})^2 \rangle_k + \beta b_j^2 \frac{|\langle \hat{C}_j \rangle_k|^2}{4 \langle (\Delta \hat{r}_j^{(k)})^2 \rangle_k} \geq \sqrt{\alpha \beta |\langle \hat{a}_j \rangle_k^2 |} |\langle \hat{C}_j \rangle_k|.
$$

where $\alpha$ and $\beta$ have been defined in Eq. (9).

The last inequality of Eq. (12) comes from the behavior, for $x \geq 0$, of the function

$$
f(x) = \gamma_1 x + \gamma_2 / x,
$$

with $\gamma_1, \gamma_2 \geq 0$. Such function takes the minimum value $f_{\text{min}} = 2 \sqrt{\gamma_1 \gamma_2}$. Then, inserting Eq. (12) into (11) we obtain

$$
\alpha \langle (\Delta \hat{u})^2 \rangle + \beta \langle (\Delta \hat{v})^2 \rangle \geq 2 \sqrt{\alpha \beta} \hat{O}.
$$

where $\hat{O}$ has been defined in Eq. (9).

Notice that, for a given system state, the quantities $\langle (\Delta \hat{u})^2 \rangle$, $\langle (\Delta \hat{v})^2 \rangle$ and $\hat{O}$ are fixed, and the inequality (14) must be satisfied for every positive value of $\alpha$ and $\beta$. Thus, we can write

$$
\langle (\Delta \hat{u})^2 \rangle \geq \max_{\alpha>0; \beta>0} \left\{ \alpha \beta \hat{O} - \alpha \beta \langle (\Delta \hat{u})^2 \rangle \right\} = \frac{\hat{O}^2}{\langle (\Delta \hat{u})^2 \rangle},
$$

where the equality has been obtained by maximizing the function $g(x) = 2 x \hat{O} - x^2 \langle (\Delta \hat{u})^2 \rangle$ over $x \geq 0$. This concludes the proof of Eq. (15).

Practically, proving the theorem, we have created a family of linear inequalities (14), which must be always satisfied by separable states. The “convolution” of such relations gives the condition (15), representable by a region in the $\langle (\Delta \hat{u})^2 \rangle$, $\langle (\Delta \hat{v})^2 \rangle$ plane delimited by a hyperbola (see Fig. 1). Notice also that, since $\hat{O}_j = \sum_k w_k |\langle \hat{C}_j \rangle_k| \geq |\sum_k w_k \langle \hat{C}_j \rangle_k| = |\langle \hat{C}_j \rangle|$, the following inequalities hold

$$
\hat{O} \geq \frac{1}{2} \left( |a_{1b1}| + |a_{2b2}| \langle \hat{C}_1 \rangle \right),
$$

$$
\geq \frac{1}{2} \left( |a_{1b1}| \langle \hat{C}_1 \rangle + a_{2b2} \langle \hat{C}_2 \rangle \right).
$$

In particular, Eq. (16) tells us that the bound (5) for separable states is much stronger than Eq. (15) for generic states. Moreover, Eq. (16) gives us a simple separability criterion. In fact, while $\hat{O}$ is not easy to evaluate directly, as it depends on the type of convex decomposition (1) one is considering, the right hand side of Eq. (16) is easily measurable, as it depends on the expectation value of the observables $\hat{C}_j$. In this sense we can claim that Eq. (5) is a necessary criterion for separability, i.e.

$$
\langle (\Delta \hat{u})^2 \rangle > \langle (\Delta \hat{v})^2 \rangle \quad \Rightarrow \quad \hat{\rho} \quad \text{entangled. (18)}
$$

An important simplification applies when the observable $\hat{C}_j$ is proportional to the identity operator (e.g., $\hat{r}_j$ is the position and $\hat{s}_j$ is the momentum operator of a particle), or more generally when it is positive (or negative)
definite. In this case the inequality \[10\] reduces to an identity and the quantity \(\tilde{O}\) does not depend on the convex decomposition \[11\].

The criterion \[5\] can be further generalized if one adopts the strong version of the uncertainty principle \[13\] in deriving the inequality \[14\]. In this situation the quantity \(\tilde{O}_j\) of Eq. \[6\] becomes,

\[
\tilde{O}_j = 2 \sum_k w_k |(\Delta\delta_j^{(k)})|, \quad j = 1, 2,
\]

where \(\Delta\delta_j^{(k)}\) and \(\Delta\delta_j^{(k)}\) are the same objects we have introduced in Eqs. \[9\] and \[10\]. Also in this case \(\tilde{O}\) depends in general on the convex decomposition \[11\] of the state \(\hat{\rho}\).

III. RELATION WITH OTHER CRITERIA

In this Section we analyze the relation between the criterion \[5\] and other necessary criteria for separability that have been proposed in the past.

First of all it is possible to show that the “sum” criterion of Ref. \[2\] represents a particular case of Eq. \[5\]. As a matter of fact the “sum” criterion is given by Eq. \[14\] with \(\alpha = \beta = 1\),

\[
\langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle \geq 2 \tilde{O} \geq |a_1 b_1| ||(\hat{C}_1)| + |a_2 b_2| ||(\hat{C}_2)||,
\]

where we have exploited Eq. \[10\] to get a rhs independent from the convex decomposition of \(\hat{\rho}_{sep}\). The fact that the “sum” criterion comes from condition \[5\] is a consequence of the fact that the latter has been derived by maximizing over the family of inequalities \[14\] [see Eq. \[15\] and Fig. \[1\]]. However, a straightforward derivation is easy to obtain as well. In fact, from Eq. \[5\] we have

\[
\langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle \geq \langle (\Delta u)^2 \rangle + \frac{\tilde{O}^2}{\langle (\Delta u)^2 \rangle} \geq 2\tilde{O},
\]

where, for the second inequality we have used the property of \(f(x)\) in Eq. \[13\].

Let us now compare the criterion developed in Section \[11\] with the “product” criterion developed in Ref. \[11\]. The latter, with the generic operators \(\hat{u}\) and \(\hat{v}\) of Eq. \[3\], can be written as

\[
\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle \geq |a_1 a_2 b_1 b_2| \frac{||\hat{C}_1 \otimes \hat{C}_2||^2}{||\hat{C}_1|| ||\hat{C}_2||},
\]

where

\[
||\hat{C}_j|| = \sup_{|\psi\rangle \in \mathcal{H}_j} \left\{ |\langle \psi | \hat{C}_j | \psi \rangle| \right\}.
\]

is the norm of the operator \(\hat{C}_j\) \((j = 1, 2)\). In order to prove that Eq. \[22\] comes from Eq. \[5\], we first note that the properties of the function \(f(x)\) of Eq. \[15\] allows us to write

\[
\tilde{O} = \sqrt{|a_1 a_2 b_1 b_2|} \left[ \frac{|a_1 b_1|}{|a_2 b_2|} \hat{O}_1 + \sqrt{\frac{|a_2 b_2|}{|a_1 b_1|} \hat{O}_2} \right] \geq \sqrt{|a_1 a_2 b_1 b_2|} \sqrt{\hat{O}_1 \hat{O}_2}.
\]

Then, by applying the Cauchy-Schwarz inequality and using the definitions \[17\] and \[23\] we can build up the following chain of relations

\[
||\langle \hat{C}_1 \otimes \hat{C}_2 \rangle ||^2 \equiv \left| \sum_k w_k \langle \hat{C}_1 \rangle_k \langle \hat{C}_2 \rangle_k \right|^2 \\
\leq \left[ \sum_k w_k |\langle \hat{C}_1 \rangle_k|^2 \right] \left[ \sum_k w_k |\langle \hat{C}_2 \rangle_k|^2 \right] \\
\leq ||\hat{C}_1|| ||\hat{C}_2|| \left[ \sum_k w_k |\langle \hat{C}_1 \rangle_k| \right] \left[ \sum_k w_k |\langle \hat{C}_2 \rangle_k| \right] \\
eq ||\hat{C}_1|| ||\hat{C}_2|| \hat{O}_1 \hat{O}_2,
\]

or

\[
\hat{O}_1 \hat{O}_2 \geq \frac{||\hat{C}_1 \otimes \hat{C}_2||^2}{||\hat{C}_1|| ||\hat{C}_2||}. \tag{26}
\]

Substituting Eqs. \[24\] and \[26\] into Eq. \[5\] we finally get Eq. \[22\]. Equations \[5\] and \[22\] give the same separability criterion when \(\hat{C}_j\) is a real number \(c_j\), and the parameters \(a_j, b_j\) satisfy the condition \(a_1 b_1 c_1 = \pm a_2 b_2 c_2\).

An example of this situation has been presented in \[11\].

Summarizing, we have proved that condition \[5\] is stronger than the criteria of Refs. \[7\] and \[11\]. This is depicted in Fig. \[2\] where the inequality \[15\] determines a zone under the solid hyperbola where we can only find entangled states: separable states must lie above this curve. Notice however that entangled states could also lie above the solid hyperbola since the condition \[5\] is only sufficient for entanglement. On the other hand, also the condition \[24\] determines a portion of the plane where only entangled states can live: that below the dashed hyperbola. However, this part is entirely included in the portion subtended by the solid hyperbola. Finally, criterion \[20\] determines a straight line inclined at \(-45^\circ\) which in general is not tangent to the solid hyperbola representing condition \[5\]. Also in this case the portion of the plane reserved to an entangled states is included in the portion delimited by the hyperbola of Eq. \[5\].

This shows the generality of the criterion presented in Section \[11\].

A couple of interesting connections can be also established when comparing the criterion of Eq. \[15\] with the weaker EPR criterion discussed in \[7\] and with Simon criterion \[10\]. In fact Eq. \[15\] and the weaker EPR criterion are essentially equivalent when applied to observables \(\hat{r}_j\),
$\hat{s}_j$ with trivial commutation rules. In order to show this, it is sufficient to observe that the uncertainties $\langle (\Delta \hat{u})^2 \rangle$ and $\langle (\Delta \hat{v})^2 \rangle$ give an upper bound for the errors in the inferred measurements of the observables $\hat{r}_1$ and $\hat{s}_1$ obtained through a direct measurement of the operators $-a_2 \hat{q}_2$ and $-b_2 \hat{s}_2$ on $\hat{\rho}$ (see Fig. 2 for more details about the definition of the inferred measurements). The comparison with Simon’s criterion is obtained considering the case in which $\hat{r}_j, \hat{s}_j$ are linear combinations of the position $\hat{q}_j$ and momentum $\hat{p}_j$ operators of the $j$-th system, i.e.

\[
\hat{r}_1 \equiv \hat{q}_1 + \frac{a_3}{a_1} \hat{p}_1, \quad \hat{s}_1 \equiv \hat{p}_1 + \frac{b_3}{b_1} \hat{q}_1,
\]

\[
\hat{r}_2 \equiv \hat{q}_2 + \frac{a_4}{a_2} \hat{p}_2, \quad \hat{s}_2 \equiv \hat{p}_2 + \frac{b_4}{b_2} \hat{q}_2 , \tag{27}
\]

where $a_3, a_4, b_3$ and $b_4$ are generic real parameters. Since in this case $[\hat{q}_j, \hat{p}_j] = i$, Eq. (10) becomes

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle \geq \frac{1}{4} \left( |a_1 b_1 - a_3 b_3| + |a_2 b_2 - a_4 b_4| \right)^2 , \tag{28}
\]

that has to be compared with the corresponding equation of Ref. [10], i.e.

\[
\langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle \geq |a_1 b_1 - a_3 b_3| + |a_2 b_2 - a_4 b_4| . \tag{29}
\]

It is easy to verify that given $a_j, b_j$ ($j = 1, \cdots, 4$), the “product” condition of Eq. (28) implies the “sum” condition of Eq. (29). However, the necessary criterion for separability of Simon requires that Eq. (29) should be verified for all possible values of the coefficients $a_j, b_j$ (see Eq. (11) of [10]). In this case, Eqs. (28) and (29) are equivalent since one can reobtain the first from the second using the same convolution trick already used in deriving (15) from Eq. (14). In particular this means that Eq. (28), when considered for all possible values of $a_j, b_j$, provides a criterion for separability which is necessary and sufficient if applied to Gaussian states.

Finally, it is also possible to establish a connection with the two subsystems, the above equation reduces to Eq.(4) of Ref. [15] with $\hat{r}_j, \hat{s}_j$ the fluctuations of the Stokes parameters.

### IV. CONCLUSION

In this paper we have studied the connections between the separability condition of the initial state of a bipartite system and the uncertainty relation of a couple of non-local observables $\hat{u}, \hat{v}$ of the two subsystems. In the case where $\hat{u}, \hat{v}$ are linear combinations of generic operators $\hat{r}_j, \hat{s}_j$ of the two subsystems, we have derived a mathematical constraint, Eq. (6), that has to be satisfied by a separable system. In general this relation depends on terms which are not measurable, meaning that Eq. (6) can not be directly used to test experimentally the separability of the system. However, in many cases of experimental relevance Eq. (6) can be expressed in terms of measurable quantities (see the discussion at the end of Section 10), providing a very general necessary criterion for separability, i.e. a sufficient criterion for entanglement. Most importantly, Eq. (6) represents a powerful theoretical tool which can be used to derive new measurable criteria (see for instance Eqs. (10) and (22)) and to compare them with other already known (e.g. those given in Refs. [6, 7, 8, 10, 11, 14, 15]).

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