Rephasing invariance and neutrino mixing

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Abstract

A rephasing invariant parametrization is introduced for three flavor neutrino mixing. For neutrino propagation in matter, these parameters are shown to obey evolution equations as functions of the induced neutrino mass. These equations are found to preserve (approximately) some characteristic features of the mixing matrix, resulting in solutions which exhibit striking patterns as the induced mass varies. The approximate solutions are compared to numerical integrations and found to be quite accurate.

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I. INTRODUCTION

Flavor mixing plays a central role in the physics of flavors. For quarks, the CKM ($V_{\text{CKM}}$) matrix has stood the test of time and is found to be sufficient in describing all of the relevant physics. Similarly, the PMNS ($V_\nu$) matrix has been used to analyze neutrino oscillation with no known discrepancies. Mathematically, both matrices belong to elements of $U(3)$, the $3 \times 3$ unitary matrices. Physically, for quarks, since the phases of individual quark states are unobservable, the rephasing transformation, $V_{\text{CKM}} \rightarrow PV_{\text{CKM}}P'$, where $P$ and $P'$ are arbitrary phase matrices, leaves the physics unchanged. Thus, only the rephasing invariant part of $V_{\text{CKM}}$ is physical. For $V_\nu$, while the charged lepton phases are unobservable, for Majorana neutrinos actually there are two observable, CP-violating, phases \[1\]. However, as long as one restricts oneself to lepton number conserving processes, such as in neutrino oscillations, these phases also become unphysical so that the physical $V_\nu$ is again rephasing invariant.

Related to the rephasing invariance of mixing matrices is their parametrization. While it may appear that the choice of parametrization is not important, since, at the end of the day, the physical quantities must be grouped into rephasing invariant combinations. However, when one deals with a situation where relations between parameters are considered, a particular choice may be advantageous over others. For instance, when the mixing depends on the energy scale, as in the RGE for mass matrices, we have a set of evolution equations relating parameters at neighboring scales. Another example deals with neutrino mixing in matter. Here, the mixing depends on the density of matter and neutrino energy, contributing to an induced neutrino mass. One can establish the relation between parameters for neighboring densities, resulting in a set of evolution equations as a function of the neutrino effective mass. They are very similar to those of the energy scale as described by a set of RGE. It turns out that, in both cases, the use of explicitly rephasing invariant parameters simplifies the evolution equations.

In the following we will derive a set of evolution equations, as a function of the effective mass of neutrinos, for neutrino parameters in matter. These equations are based on the use of rephasing invariant parameters developed earlier. We find that they have simple, analytic, albeit approximate, solutions. It is interesting that the parameters in matter preserve a number of salient features of those in vacuum, resulting in a matter-dependent PMNS matrix that can be grasped at a glance.

The paper is organized as follows. Section II is a brief summary of the rephasing invariant parametrization that is adopted in this work. In stead of directly solving the eigenvalue problem, we derive in Section III the evolution equations for the neutrino mixing parameters and masses from the effective Hamiltonian in matter. Certain well-known invariants are also derived using the symmetric properties of the equations. Section IV is devoted first to solving the two-flavor problem using this rephasing invariant formulation, and then the three-flavor case. Making use of the known properties of measured neutrino parameters, analytic, approximate, solutions are obtained. In Section V, the accuracy of the solutions are confirmed by comparison with numerical integration of the equations. Section VI is the summary. In appendix A, we also derive the neutrino transition probabilities in matter using the adopted rephasing invariant parametrization.
II. REPHASING INVARIANT PARAMETRIZATION

In this section, we briefly summarize the rephasing invariant parametrization introduced earlier for quark mixing [2], and will now be adopted for neutrino mixing, valid for lepton number conserving processes.

For the PMNS matrix \( V \), without loss of generality, we can impose the condition \( \det V = +1 \). There are then a set of rephasing invariants

\[
\Gamma_{ijk} = V_{i1}V_{j2}V_{k3} = R_{ijk} - iJ,
\]

where their common imaginary part can be identified with the Jarlskog invariant \( J \) [3]. Their real parts are defined as

\[
(R_{123}, R_{312}; R_{132}, R_{213}, R_{321}) = (x_1, x_2, x_3; y_1, y_2, y_3).
\]

These variables are bounded by \( \pm 1 \): \(-1 \leq (x_i, y_j) \leq +1\), with \( y_j \leq x_i \) for any \((i, j)\). They satisfy two constraints

\[
\det V = (x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1,
\]

\[
(x_1x_2 + x_2x_3 + x_3x_1) - (y_1y_2 + y_2y_3 + y_3y_1) = 0.
\]

In addition, it is found that

\[
J^2 = x_1 x_2 x_3 - y_1 y_2 y_3.
\]

The \((x, y)\) parameters are related to \( |V_{ij}|^2 \) by

\[
W = [|V_{ij}|^2] = \begin{pmatrix}
  x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\
  x_3 - y_3 & x_1 - y_1 & x_2 - y_2 \\
  x_2 - y_2 & x_3 - y_3 & x_1 - y_1
\end{pmatrix}.
\]

One can readily obtain the parameters \((x, y)\) from \( W \) by computing its cofactors, which form the matrix \( w \) with \( w^TW = (\det W)I \), and is given by

\[
w = \begin{pmatrix}
  x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\
  x_3 + y_3 & x_1 + y_1 & x_2 + y_2 \\
  x_2 + y_2 & x_3 + y_3 & x_1 + y_2
\end{pmatrix}.
\]

Eqs. (6) and (7) establish the close relationship between the two rephasing invariant parametrizations \((x, y)\) and \(|V_{ij}|^2\). Besides the obvious difference in the number of constraints (two for \((x, y)\) and five for \(|V_{ij}|^2\)), the set \((x, y)\) has built-in symmetry amongst the three states considered, which, as we will see, helps to make the evolution equations simpler.

For the PMNS matrix in vacuum, its elements squared are well-approximated by

\[
W_0 = \begin{pmatrix}
  \frac{2(1-\epsilon^2)}{1+2\epsilon^2} - 2\eta & \frac{1-\epsilon^2}{3} + 2\eta & \frac{\epsilon^2}{3} - \beta - \eta \\
  \frac{1+2\epsilon^2-6\epsilon}{6} - \beta + \eta & \frac{2+2\epsilon^2-2\xi}{6} - \beta - \eta & \frac{\epsilon^2}{3} - \beta - \eta \\
  \frac{2+2\epsilon^2+2\xi}{6} - \beta + \eta & \frac{1+2\epsilon^2+2\xi}{6} - \beta - \eta & \frac{\epsilon^2}{3} - \beta - \eta
\end{pmatrix},
\]

with \((\epsilon, \eta, \beta, \xi) \ll 1\). \( W_0 \) reduces to the tri-bimaximal [4] matrix when \( \epsilon = \eta = \beta = \xi = 0 \). If we allow the parameters \((\epsilon, \eta, \beta, \xi)\) to take on arbitrary values, the matrix above can be used as a general parametrization of the mixing matrix. Also, it is related to the
familiar “standard parametrization” \[3\] by \(S_{13}^2 = \epsilon^2, S_{12}^2 = \frac{1}{3} + \frac{2\eta}{1 - \eta}, S_{23}^2 = \frac{1}{2} + \frac{1}{2(1 - \eta)}\), and

\(2\beta = (S_{23}^2 - C_{23}^2)[-\frac{2}{3}C_{13}^2 + C_{12}^2 - S_{12}^2 S_{13}^2] + 4S_{12} C_{12} S_{13} C_{23} S_{23} \cos \phi\), so that, if \((\epsilon, \eta, \xi) \ll 1\), \(\beta \simeq \sqrt{2} C_{\phi} S_{13}\).

The matrix \(W_0\) in Eq. \[8\] exhibit several interesting features. When \(\epsilon = \eta = \beta = \xi = 0\), we find \(x_{10} = 1/3, x_{20} = 1/6, x_{30} = 0\), and \(x_{i0} + y_{i0} = 0\) \((i = 1, 2, 3)\). The conditions \(x_{30} = y_{30} = 0\) come from \(W_{13} = 0\) \((so\ also\ V_{13} = 0)\). The conditions \(x_{i0} + y_{i0} = 0\) are equivalent to \(W_{2i} = W_{3i}\ \[3\]. From known experimental bounds, for non-vanishing \((\epsilon, \eta, \beta, \xi)\), these conditions are valid to \(\mathcal{O}(10^{-2})\).

III. EVOLUTION OF NEUTRINO MIXING PARAMETERS

It is well-established that neutrino mixing is modified by the presence of matter \[7\]. Their effect has been used in the analyses of solar neutrinos, and is expected to impact those of the supernova neutrinos, when and if they become available. Closer to home, there is a plethora of long baseline experiments either in operation or in the planning stage. For these studies, it is essential to include the matter effects in order to understand neutrino mixing at the fundamental level.

In the literature, effort has been devoted to solving problems along this line (see, e.g., \[5\]). However, the process involves the complication of the cubic eigenvalue problems, and the results are usually far from transparent for a clear extraction of the physical implications.

In this work we study this problem from another angle. It is well-known that, when neutrinos propagate through matter, the latter contributes an induced mass to the neutrinos. Similar to the case of RGE, we may write down, as a function of the induced mass, a set of evolution equations for the neutrino parameters. It turns out that, with the initial conditions given by \(W_0\) in Eq. \[8\], we can find simple, approximate, solutions to these equations, as we will detail in Sec. IV. These results were summarized in a previous publication \[9\].

To derive these equations, we start from the effective Hamiltonian for neutrino propagation in matter

\[
H_{\text{eff}} = H/2E,
\]

where \(H\) is given, in the flavor basis, by

\[
H = \begin{bmatrix}
V_0 \left( \begin{array}{cc}
m_1^2 & m_2^2 \\
m_2^2 & m_3^2
\end{array} \right) V_0^\dagger + 
\begin{pmatrix}
A \\
0 \\
0
\end{pmatrix}
\end{bmatrix},
\]

where \(m_1, m_2, \text{ and } m_3\) are the neutrino masses in vacuum, \(V_0\) is the mixing matrix in vacuum, \(E\) is the neutrino energy, and the induced mass \(A = \sqrt{2}G_F n_e E\).

The matrix \(H\) can be diagonalized,

\[
H = VDV^\dagger = V\begin{pmatrix}
D_1 \\
D_2 \\
D_3
\end{pmatrix} V^\dagger,
\]

where \(D_i = M_i^2\) is the squared mass in matter. To study how the elements of \(V\) evolve in matter, one may start with \(dH/dA\), which leads to

\[
V^\dagger \frac{d}{dA} [VVD^\dagger] V = \begin{pmatrix}
|V_{11}|^2 & V_{12} V_{11}^* & V_{13} V_{11}^* \\
V_{11} V_{12}^* & |V_{12}|^2 & V_{13} V_{12}^* \\
V_{11} V_{13}^* & V_{12} V_{13}^* & |V_{13}|^2
\end{pmatrix}.
\]
Taking the diagonal terms of Eq. (12), we find
\[
\frac{dD_i}{dA} = |V_{ii}|^2 = x_i - y_i, \quad (i = 1, 2, 3).
\]

The off-diagonal terms yield
\[
[(\frac{dV^\dagger}{dA})V]_{ik} = \frac{V_{i}^{*}V_{ik}}{D_i - D_k}, \quad (i \neq k).
\]

The diagonal elements \([dV^\dagger/dA)V]_{ii}\) are not constrained by Eq. (12). Fortunately, it is rephasing dependent \[^{10}\), and we can set it to vanish by a proper choice of the phase. This means that, when we multiply Eq. (15) by \((V^\dagger)_{kj}\), and sum over \(k \neq i\) on the right hand side, we may sum over all \(k\)-values on the left. The result is
\[
\frac{dV_{ij}}{dA} = \sum_{k \neq j} \frac{V_{ik}V_{kj}}{D_j - D_k} V_{ik}^*.
\]

Note that the dependence is only on the mass differences, \((D_j - D_k)\), in accordance with the invariance of \(V_{ij}\) if \(H \rightarrow H + \text{constant}\).

While Eq. (15) is valid only with a particular choice of phase, this rephasing ambiguity is removed if one uses it to compute rephasing invariant quantities, \(e.g.,\)
\[
\frac{d\Gamma_{123}}{dA} = \frac{d}{dA}(V_{11}V_{22}V_{33}) = \frac{dx_1}{dA} - \frac{dJ}{dA}.
\]

After some algebra, separating the real and imaginary parts, in addition to using different \(\Gamma_{ijk}\)'s, we obtain the evolution equations for all \((x_i, y_i)\) and \(d\ln J/dA\), which are collected in Table I.

Note that, since Eq. (15) can be obtained from Eq.(3.6) (Ref. \[^{10}\)) in appropriate limits, the entries in Table I can be identified with those in Table II of Ref. \[^{10}\). Indeed, it can be verified that \(dx_i/dA = \sum (\bar{B}_i)_{2r} - (\bar{B}_s)_{3s}]/(D_s - D_i)\), \(dy_j/dA = \sum (\bar{B}_j')_{2r} - (\bar{B}_t')_{3r}]/(D_s - D_j)\), where the sum is over cyclicly permuted \((r, s, t) = (1, 2, 3)\), and \(\bar{B}_i(\bar{B}_j')\) are obtained from \(B_i(\bar{B}_j')\) in Table II of Ref. \[^{10}\) by exchanging \(x_2 \leftrightarrow x_3\), since \(V \leftrightarrow V^\dagger\) under the usual conventions in going from quarks to neutrinos.
The symmetric form of these equations allows us to find readily the result:

\[
\frac{d}{dA} \ln[J(D_1 - D_2)(D_2 - D_3)(D_3 - D_1)] = 0,
\]

(17)
i.e., the product \( J(D_1 - D_2)(D_2 - D_3)(D_3 - D_1) \) is a constant as \( A \) changes, a well known result derived with different methods \([11]\).

From Table I, we find

\[
\frac{d}{dA} \ln(x_1 - y_1) = \frac{x_2 - y_2}{D_1 - D_2} - \frac{x_3 - y_3}{D_3 - D_1},
\]

(18)

\[
\frac{d}{dA} \ln(x_2 - y_2) = -\frac{x_1 - y_1}{D_1 - D_2} + \frac{x_3 - y_3}{D_2 - D_3},
\]

(19)

\[
\frac{d}{dA} \ln(x_3 - y_3) = -\frac{x_2 - y_2}{D_2 - D_3} + \frac{x_1 - y_1}{D_3 - D_1}.
\]

(20)

We see that there is another “matter invariant”:

\[
\frac{d}{dA} \left[ J^2(x_1 - y_1)(x_2 - y_2)(x_3 - y_3) \right] = 0.
\]

(21)

Or,

\[
J^2/(|V_{11}|^2|V_{12}|^2|V_{13}|^2) = \text{constant}.
\]

(22)

When we use the “standard parametrization”, it is seen that \( J^2/(|V_{11}|^2|V_{12}|^2|V_{13}|^2) = S_\phi^2S_{23}^2C_{23}^2 \), i.e., \( S_\phi \sin 2\theta_{23} \) is independent of \( A \), a result obtained earlier \([12]\).

The evolution equations for \((x, y)\) also have a structure akin to that of the fixed point of single variable equations. It can be verified that, if \( x_i + y_i = 0 \) \((i = 1, 2, 3)\), then

\[
\frac{d}{dA} (x_j + y_j) = 0, \quad j = (1, 2, 3).
\]

(23)

This result is understandable since the conditions \( x_i + y_i = 0 \) are equivalent to \( W_{2i} = W_{3i} \), which, in turn, imply that the effective Hamiltonian \( H \) has a \( \mu - \tau \) exchange symmetry \([13]\). This symmetry is clearly independent of \( A \) in Eq. (10), resulting in Eq. (23). Note also that there are actually only two independent constraints in \( x_i + y_i = 0 \). Given any two of them, say for \( i = 1, 2 \), we can use Eq.(10) to derive \( x_3 + y_3 = 0 \). Thus, the set of evolution equations has a “fixed surface” (in the four-dimensional parameter space), points on the surface defined by \( x_i + y_i = 0 \) stay on it as \( A \) varies.

IV. APPROXIMATE SOLUTIONS

While analytical solutions to the equations in Table I are not available, as we will see, given the known physical parameters, one can exploit certain characteristic properties thereof to arrive at simple, but fairly accurate, solutions to these equations. Before we do that, it is instructive to first study the two flavor problem, which can be compared to the traditional approach, since exact solutions can be obtained in both cases.
A. Two-flavor problem

For two flavors, we have

\[
\frac{dH}{dA} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

(24)

with the familiar diagonalization matrix

\[
V = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\]

(25)

so that

\[
x = V_{11}V_{22} = \cos^2 \theta, \quad y = V_{12}V_{21} = -\sin^2 \theta, \quad x - y = 1.
\]

The evolution equations are

\[
\frac{dD}{dA} = -(x + y), \quad \frac{dx}{dA} = 2xy = \frac{dy}{dA},
\]

(26)

where \(D \equiv D_2 - D_1\). It follows that

\[
\frac{d}{dA}(xyD^2) = 0,
\]

(27)

\[
\frac{d}{dA}[D(x + y)] = -(x + y)^2 + 4xy = -1.
\]

(28)

Eq. (27) is the familiar result

\[
D^2 \sin^2 2\theta = D_0^2 \sin^2 2\theta_0.
\]

(29)

Eq. (28) gives

\[
D(x + y) = -A + D_0(x + y)_0,
\]

(30)

and thus

\[
D^2 = [(A - D_0 \cos 2\theta_0)^2 + D_0^2 \sin^2 2\theta_0],
\]

(31)

which is the well-known resonance formula with \(D = \text{min. at } A = D_0 \cos 2\theta_0\).

These results show that the use of evolution equations is equivalent to the traditional method, that of finding directly the eigenvalues of the effective Hamiltonian. We now turn to the case of three flavors.

B. Three-flavor problem

Experimentally, it is known that \(\delta_0 = m_3^2 - m_2^2 \approx 7.6 \times 10^{-5} \text{eV}^2\), \(\Delta_0 = m_3^2 - m_1^2 \approx 2.4 \times 10^{-3} \text{eV}^2\), so that \(\delta_0/\Delta_0 \ll 1\) (We assume the “normal” ordering of neutrino masses. The “inverted” case can be similarly treated). Note that these values are relevant to long baseline experiments since \(A = \sqrt{2}G_F n_e E \sim (7.6 \times 10^{-5} \text{eV}^2)(E/\text{GeV})(\rho/\text{gcm}^{-3})\).

Since \(\delta_0 \ll \Delta_0\), we expect that the three-flavor problem can be approximated by a pair of well separated two-flavor problems \[14\]. Indeed, the structure of the differential equations in Table I shows that the variables \((x_i, y_i)\) evolve slowly as a function of \(A\) except for two regions, where \(D_1 \approx D_2\) and \(D_2 \approx D_3\), corresponding to the two resonance regions. More precisely,
let us denote by \((A_0, A_l, A_i, A_h, A_d)\) the values of \(A\) in vacuum \((A_0 = 0)\), at the lower resonance \((A_l, [d(D_1 - D_2)/dA]_{A_l} = 0)\), the intermediate range \((A_i)\), the higher resonance \((A_h, [d(D_2 - D_3)/dA]_{A_h} = 0)\), and for dense medium \((A_d)\). Rapid evolution for \((x_i, y_i)\) only occurs for \(A \approx A_l\) and \(A \approx A_h\). Near the lower resonance region, \((D_2 - D_1) \ll (D_3 - D_1)\) or \((D_3 - D_2)\). For the higher resonance region, \((D_3 - D_2) \ll (D_3 - D_1)\) or \((D_2 - D_1)\). Thus, for these two regions, we need only to keep terms \(\propto 1/(D_2 - D_1)\) and \(1/(D_3 - D_2)\), respectively. This approximation is generally valid to \(O(10^{-2})\). We now turn to a detailed analysis.

For \(0 < A < A_i\), in the neighborhood of \(A_l\), we keep terms \(\propto 1/(D_1 - D_2)\) in Table I. Let’s concentrate on the variables \(x_i - y_i = |V_{1i}|^2\) and use Eq. (13) and Eqs. (18)-(20). We define

\[
\begin{align*}
X &= x_1 - y_1, \\
Y &= x_2 - y_2, \\
Z &= x_3 - y_3, \\
\delta &= D_2 - D_1.
\end{align*}
\]

Then,

\[
\begin{align*}
\frac{dX}{dA} &= -\frac{2XY}{\delta} = -\frac{dY}{dA}, \\
\frac{dZ}{dA} &= 0, \\
\frac{d\delta}{dA} &= -(X - Y).
\end{align*}
\]

These equations are identical to those for the two flavor problem, Eq. (26), with \(X \rightarrow x\), \(Y \rightarrow -y\), and \(\delta \rightarrow D\). Also, in place of \(x - y = 1\), we have

\[
X + Y = p_l,
\]

where \(p_l\) is a constant (since \(d(X + Y)/dA = 0\)), and \(p_l = 1 - Z\), \(Z = |V_{13}|^2 = \text{constant}\). The solutions are

\[
\begin{align*}
XY\delta^2 &= X_0Y_0\delta_0^2, \\
(X - Y)\delta &= -p_l^2A + q_l\delta_0, \\
q_l &= X_0 - Y_0.
\end{align*}
\]

Explicitly, we have

\[
\begin{align*}
\delta^2 &= \frac{p_l^2A^2 - 2q_l\delta_0A + \delta_0^2}, \\
X &= \frac{1}{2}[p_l - (p_l^2A - q_l\delta_0)/\delta], \\
Y &= \frac{1}{2}[p_l + (p_l^2A - q_l\delta_0)/\delta].
\end{align*}
\]

Thus, \((\delta, X, Y)\) exhibit the classic resonance behavior, with the resonance location at the minimum of \(\delta\):

\[
A_l = \left(\frac{q_l}{p_l^2}\right)\delta_0.
\]
Substituting in the vacuum input values, $X_0 \simeq 2/3$, $Y_0 \simeq 1/3$, $A_l \simeq \delta_0/3$. The width of the resonance is

$$(\delta A)_l = (1 - q_l^2/p_l^2)^{1/2}(\delta_0/p_l).$$

(38)

For the physical PMNS matrix, $(\delta A)_l \simeq \delta_0$. This means that the intermediate $A$ value, $A_i$, already starts at $A \gtrsim (2 - 3)\delta_0$. For $A \sim A_i$, $\delta \rightarrow A$, $X \rightarrow 0$, $Y \rightarrow 1$, with $p_l \simeq 1$.

Turning to the higher resonance, we define

$$\Delta = D_3 - D_2.$$  

(39)

The evolution equations are

$$\frac{dX}{dA} = 0,$$
$$\frac{dY}{dA} = -2YZ = -dZ,$$
$$\frac{d\Delta}{dA} = -(Y - Z),$$

(40)

with the solutions

$$YZ\Delta^2 = Y_0Z_0\Delta_0^2,$$
$$Y - Z)\Delta = -p_h^2A + q_h\Delta_0,$$

(41)

or,

$$\Delta^2 = p_h^2A^2 - 2q_h\Delta_0A + \Delta_0^2,$$
$$Y = \frac{1}{2}[p_h - (p_h^2A - q_h\Delta_0)/\Delta],$$
$$Z = \frac{1}{2}[p_h + (p_h^2A - q_h\Delta_0)/\Delta],$$

(42)

where $p_h = Y_0 + Z_0$, $q_h = Y_0 - Z_0$. Here, the values $Y_0$ and $Z_0$ are taken at $A = A_i \gg \delta_0$, so that $Y_0 \simeq 1$, $Z_0 \simeq |V_{13}|^2 \simeq 0$, from the solutions for the lower resonance. The position of the higher resonance is at

$$A_h = (\frac{q_h}{p_h})\Delta_0 \simeq \Delta_0.$$  

(43)

Its width is

$$(\delta A)_h = (1 - q_h^2/p_h^2)^{1/2}(\Delta_0/p_h) \simeq 2Z_0\Delta_0 \ll \Delta_0.$$  

(44)

The above analyses show that the two-flavor approximation yields simple solutions to the mixing parameters $|V_{1i}|^2$, for all $A$ values. However, the vacuum mixing, given by $W_0$ in Eq. (35), has another important feature, namely, $(W_0)_{2i} \simeq (W_0)_{3i}$, or $x_{i0} + y_{i0} \simeq 0$. This feature, according to Eq. (23), is preserved by the evolution equations and so $W_{2i} \simeq W_{3i}$, or $x_i + y_i \simeq 0$, for all $A$. Thus, with the known solutions for $W_{1i}$ from above, all elements of $W$ are determined by unitarity. Explicit solutions for $W$ or $(x_i, y_i)$ were presented in Ref. [9], obtained by using both approximations.

We may divide the full range of $A$ values into a low-$A$ and a high-$A$ regions. The former covers the range from $A = 0$ to a value below the higher resonance region, while the later starts from beyond the lower resonance region and ends at $A = \infty$. In these regions, the
resonance. For the high- 

to appreciably only in two regions: 1) lower resonance, the solution can be characterized by the higher resonance. The mixing parameters full range of are dominated by pole terms E.g. These "partial matter invariants" are useful in understanding some detail properties of the evolutions are dominated by contributions from pole terms, \(1/(D_1 - D_2)\) for low-A and \(1/(D_2 - D_3)\) for high-A. The exact demarcation between low-A and high-A is not important, since in the intermediate region contributions from either pole are small, and there can be considerable overlap between low-A and high-A, corresponding to the large range of \(A_i\).

It should be noted that "pole dominance", which was used to go from Eqs. \((18-20)\) to Eqs. \((33)\) and \((40)\), is an excellent approximation in this situation. This is because the terms dropped are doubly suppressed, first by the large denominators, and then by the small numerators \((|x_i - y_j| \ll 1 \text{ throughout the low-A region and } |x_i - y_i| \ll 1 \text{ for high-A})\). We should also emphasize that Eqs. \((33)\) and \((40)\) are derived independently of the approximations, such as \(J_2\), valid only for either the low-A or the high-A regions. Thus, from Eq. \((35)\) and using Table I,

\[
(D_1 - D_2)^2|V_{11}|^2|V_{12}|^2 \approx \text{constant, \quad (low-A)}
\]

\[
J^2(D_1 - D_2)^2 \approx \text{constant. \quad (45)}
\]

Similarly,

\[
(D_2 - D_3)^2|V_{12}|^2|V_{13}|^2 \approx \text{constant, \quad (high-A)}
\]

\[
J^2(D_2 - D_3)^2 \approx \text{constant. \quad (46)}
\]

These "partial matter invariants" are useful in understanding some detail properties of the parameters. E.g., for \(A \sim A_i\), \(|V_{11}|^2 \approx (2/9)\left(\delta_0/A\right)^2\). The behavior of \(J^2\) is also clarified, as we will see in the discussion on Fig. 4.

In summary, the solution to the three flavor problem can be made simple by dividing the full range of \(A\) into a low-A and a high-A regions. In the low-A region, the evolution equations are dominated by pole terms \(\propto 1/(D_1 - D_2)\), and the solution centers around the lower resonance. For the high-A region, correspondingly, pole terms \(\propto 1/(D_2 - D_3)\) dominate, and the solution can be characterized by the higher resonance. The mixing parameters change appreciably only in two regions: 1) lower resonance, \([A_l - (\delta A)_l] \approx \Delta_0 \lesssim [A_l + (\delta A)_l]\), \(A_l \approx \delta_0/3\), \((\delta A)_l \approx \delta_0\); 2) higher resonance, \([A_h - (\delta A)_h] \approx \Delta_0 \lesssim [A_h + (\delta A)_h]\), \(A_h \approx \Delta_0\).
\((\delta A)_h \simeq 2|V_{13}|^2 \Delta_0 \ll \Delta_0\). The solutions for \(X(|V_{11}|^2), Y(|V_{12}|^2), \text{and } Z(|V_{13}|^2)\) are: 1) \(A = 0, X_0 \simeq 2/3, Y_0 \simeq 1/3, Z_0 \simeq \epsilon^2 \ll 1\), which are the given vacuum values; 2) \(A = A_l\), \(X \simeq Y \simeq 1/2, Z \simeq \epsilon^2 \ll 0\); 3) \(A = A_i, A_i\) covers roughly the range, \(2\delta_0 \lesssim A_i \lesssim \Delta_0(1 - 2\epsilon^2)\), \(X \simeq 0, Y \simeq 1, Z \simeq \epsilon^2\); 4) \(A = A_h, X \simeq 0, Y \simeq Z \simeq 1/2\); 5) \(A = A_d, \text{with } A_d \gtrsim \Delta_0(1 + 2\epsilon^2), X \simeq Y \simeq 0, Z \simeq 1\).

When we incorporate the other feature of the vacuum PMNS matrix, that \((W_0)_{3i} \simeq (W_0)_3\), which is preserved by the evolution equations, the result is that \(W_{2i} \simeq W_{3i}, \text{for all } A\). Given \(W_{1i}\) from above, the matrix \(W\) is then completely determined by unitarity.

Our results can be put together by giving the matrices \(W\) at \(A = (A_0, A_l, A_i, A_h, A_d)\):

\[
W_0 \simeq \begin{pmatrix}
2/3 & 1/3 & 0 \\
1/6 & 1/3 & 1/2 \\
1/6 & 1/3 & 1/2
\end{pmatrix}, \quad W_i \simeq \begin{pmatrix}
1/2 & 1/2 & 0 \\
1/4 & 1/4 & 1/2 \\
1/4 & 1/4 & 1/2
\end{pmatrix},
\]

\[
W_l \simeq \begin{pmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
1/2 & 0 & 1/2
\end{pmatrix}, \quad W_h \simeq \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/2 & 1/4 & 1/4 \\
1/2 & 1/4 & 1/4
\end{pmatrix},
\]

\[
W_d \simeq \begin{pmatrix}
0 & 0 & 1 \\
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0
\end{pmatrix}.
\]

As a group, these matrices exhibit the remarkable simplicity of the PMNS matrix as \(A\) varies from 0 to \(\infty\). Note that all of the matrices have at least one zero, \(W_{1i} = 0\), implying \(x_i = y_i = 0\).

The other feature, as mentioned before, is that they have equal elements in their second and third rows, \(W_{2i} = W_{3i}\). This means that the \(W\) matrix is completely fixed by its first row, \(W_{1i}\). These elements, in turn, control \(dD_i/dA, \text{Eq. (13)}\). Thus, the progression of \(W\) as a function of \(A\) can be read off from the plot of \(D_i(A)\), which is given in Fig. 1. Generally, the accuracy of the entries in Eq. (47) is of the order \(10^{-2}\). Note also that \(A_i\) covers a rather large range, \(2\delta_0 \lesssim A_i \lesssim \Delta_0(1 - 2\epsilon^2)\). The validity of Eq. (47) will be confirmed by numerical integrations, to be given in the next section. Where applicable, they also agree with numerical results in the literature \([5]\), after the proper change of variables is carried out.

V. NUMERICAL SOLUTIONS

It is straightforward to numerically integrate the evolution equations for \((x, y)\). To do this we first obtain the vacuum expressions for the \((x, y)\) parameters from Eqs. (6) and (8):

\[
x_{10} = \frac{1}{6}(2 - 3\beta - 2\epsilon^2), \quad y_{10} = \frac{1}{6}(-2 - 3\beta + 2\epsilon^2),
\]

\[
x_{20} = \frac{1}{6}(1 - 3\beta - \epsilon^2), \quad y_{20} = \frac{1}{6}(-1 - 3\beta + \epsilon^2),
\]

\[
x_{30} = \frac{1}{2}(\beta + \epsilon^2), \quad y_{30} = \frac{1}{2}(\beta - \epsilon^2),
\]

where \(\xi = \eta = 0\) is chosen and the terms in \(O(\beta^2)\) are ignored. In addition, we choose the initial values \(\epsilon = 0.17, \beta = 0.02\), corresponding to the experimental bounds \(|V_{e3}|^2 \leq 0.03\) \([5]\).
FIG. 2: The numerical (solid) and approximate (dot-dashed) solutions for $x_1(A)$, $x_2(A)$, and $x_3(A)$. Note that $y_i(A) \simeq -x_i(A)$.

and an assumed CP violation phase $\cos \phi = 1/4$. With the input hierarchy $\delta_0/\Delta_0 = 1/32$, the numerical results for $D_i$ and that for $(x_i, y_i)$ are then compared with the approximate solutions obtained earlier (Eqs. (36), (42)) in Fig. 1 and Fig. 2, respectively. The agreements are quite good.

Given the possible normal or inverted mass hierarchies, Fig. 3 summaries the evolution of the mixing parameters for both the $\nu$ and the $\bar{\nu}$ sectors. Note that the parameters $(\bar{x}_i, \bar{y}_i)$ for the $\bar{\nu}$ sector can be obtained by replacing $A$ with $-A$ and $V$ with $V^*$ in that for the $\nu$ sector. With the establishment of these “basic solutions” for the mixing parameters, we may easily study other physical quantities that are relevant to practical calculations for the neutrino propagation in matter. For illustration purpose, we plot some of the quantities numerically under the normal hierarchy in the following. The corresponding solutions under the inverted hierarchy can be manipulated likewise.

The evolution of $J^2 = x_1x_2x_3 - y_1y_2y_3$ ($\bar{J}^2 = \bar{x}_1\bar{x}_2\bar{x}_3 - \bar{y}_1\bar{y}_2\bar{y}_3$) in matter is shown in Fig. 4. Compared to its vacuum value, it is seen that, except for some enhancement for $J^2$ near $A = A_l$ and $A = A_h$, the general trend is for it to decrease with $A$. We note that near $A_l$, in the $1/(D_1 - D_2)$ dominance approximation, $J^2(D_1 - D_2)^2 \approx$ constant, so while $(D_2 - D_1)$ goes through a dip, $J^2$ has a bump near $A_l$, after which $J^2/J_0^2 \approx (\delta_0/A)^2$, for $\delta_0 \ll A \lesssim \Delta_0$. A similar behaviour occurs near $A_h$, with $J^2(D_2 - D_3)^2 \approx$ constant, although the effects are hardly noticeable. Similar numerical results were also reached by solving directly the eigenvalue problem [8]. In addition, Fig. 5 shows the evolution of the first two rows of $W_{ij}$.
FIG. 3: The mixing parameters in matter under the normal hierarchy (left column) and the inverted hierarchy (right column), for both the $\nu$ sector (solid, $x_i$) and the $\bar{\nu}$ sector (dashed, $\bar{x}_i$). Note that $x_i + y_i \simeq 0$ ($\bar{x}_i + \bar{y}_i \simeq 0$) for all $A$, and $x_i = \bar{x}_i$ in vacuum.

FIG. 4: The evolution of $J^2 = x_1x_2x_3 - y_1y_2y_3$ and $\bar{J}^2 = \bar{x}_1\bar{x}_2\bar{x}_3 - \bar{y}_1\bar{y}_2\bar{y}_3$, obtained from the numerical solutions for $\nu$ (solid) and $\bar{\nu}$ (dashed) sectors.

in matter. We do not present the plots of $W_{3i}$ since they are almost indistinguishable from those of $W_{2i}$. The patterns of $W_{ij}$ in Eq. (47) are clearly seen from the plots. Furthermore, the mixing angles of the standard parametrization: $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$, and $\sin^2 \theta_{13}$, are related to the $(x, y)$ parameters:

$$\sin^2 \theta_{12} = 1/(1 + \frac{x_1 - y_1}{x_2 - y_2}),$$

(49)
FIG. 5: $W_{ij}$ for both the $\nu$ (solid lines) and $\bar{\nu}$ (dashed lines) sectors are plotted as functions of $A/\delta_0$. Note that $W_{2i} = W_{3i}$, and only the first two rows of $W_{ij}$ are shown in the figure. The patterns of $W_{ij}$ can be readily deduced from Fig. 3.

$$\sin^2 \theta_{12} = \frac{x_1 - y_2}{x_2 - y_1}, \quad (50)$$

$$\sin^2 \theta_{13} = x_3 - y_3. \quad (51)$$

The numerical results for both the $\nu$ and $\bar{\nu}$ sectors are shown in Fig. 6. These plots are in agreement with those in the literature [8]. For $\theta_{12}$ and $\theta_{13}$, note the characteristic step-

FIG. 6: The mixing angles for the $\nu$ (left) and the $\bar{\nu}$ (right) sectors in the standard parametrization: $\sin^2 \theta_{12}$ (solid), $\sin^2 \theta_{23}$ (dot-dashed), and $\sin^2 \theta_{13}$ (dashed), are plotted using the $(x, y)$ values in Fig. 3. Note that $\theta_{23}$ for the $\nu$ sector remains constant ($\theta_{23} = \pi/4$) for $A < A_h$ (higher resonance) and deviates slightly from $\pi/4$ for $A > A_h$. 

The numerical results for both the $\nu$ and $\bar{\nu}$ sectors are shown in Fig. 6. These plots are in agreement with those in the literature [8]. For $\theta_{12}$ and $\theta_{13}$, note the characteristic step-
function resonance behaviors near \(A_L\) and \(A_H\). Also, \(\theta_{23} \approx \pi/4\) is a reflection of \(W_{23} \approx W_{33}\), for all \(A\). The phase angle \((\phi)\) is not plotted since it also remains constant due to the invariance of \(S_{\phi} \sin 2\theta_{23}\).

VI. CONCLUSION

Understanding the propagation of neutrinos through matter is one of the core problems in neutrino physics. In a medium of constant density, it is well-known that the electron neutrino acquires an induced mass which alters both the eigenvalues and the mixing matrix of the neutrinos. Traditionally, one studies directly the eigenvalue problem of the effective Hamiltonian. The neutrino parameters are expressed as complicated formulas in terms of the induced mass and their values in vacuum. One then resorts to numerical plots by assuming specific values for these partially known parameters. The drawback of this method is the lack of insights into the nature of the solutions, and it is not easy to gain an overview of the mixing as a function of the induced mass.

In this paper we try a different approach, by finding the evolution equations of the neutrino parameters as a function of the induced mass. The resulting equations, when written in terms of a rephasing invariant parametrization, turn out to be manageable and we are able to find simple, approximate, solutions with the help of two important features of the vacuum neutrino parameters. 1) The two measured mass differences are widely separated so that the two-flavor resonance approximation becomes applicable. 2) The vacuum PMNS matrix has an approximate \(\mu - \tau\) symmetry, which is preserved by the set of evolution equations. The result is summarized in Eq. (47), showing the striking simplicity of the neutrino mixing matrix as a function of \(A\). Approximate solutions for the parameters are explicitly given in Eqs. (36) and (42). The evolution equations also facilitate the derivation of “matter invariants”, given in Eq. (17) and (22). In addition, there are also “partial matter invariants”, Eqs. (45) and (46). These are useful in obtaining properties of the various parameters without performing detailed calculations.

Based on the incomplete measurements that exist for the vacuum parameters, our analyses show that those in matter, to a good approximation, can already be determined. We hope that these results will be helpful in the exploration of the physics of neutrino propagation in matter.

Acknowledgments

S.H.C. is supported by the National Science Council of Taiwan, grant No. NSC 98-2112-M-182-001-MY2.

Appendix A: Neutrino transition probabilities in \((x, y)\) parameters

As the neutrinos travel through a baseline \(L\) in matter of constant density, the flavor transition probability is given by

\[
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{j>i} Re(V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta j}) \sin^2(D_{ij})
\]
\[ + 2 \sum_{j>i} Im(V_{\alpha_i} V_{\beta_i}^* V_{\alpha_j}^* V_{\beta_j}) \sin(2D_{ij}), \]  
(A1)

where \( D_{ij} \equiv (D_i - D_j)L/4E \). For \( \alpha \neq \beta \), we obtain the explicit expression,

\[
P(\nu_\alpha \rightarrow \nu_\beta) = - 4[Re(V_{\alpha_1} V_{\beta_1}^* V_{\alpha_2} V_{\beta_2}) \sin^2(D_{12}) + Re(V_{\alpha_1} V_{\beta_1}^* V_{\alpha_3} V_{\beta_3}) \sin^2(D_{13}) + Re(V_{\alpha_2} V_{\beta_2}^* V_{\alpha_3} V_{\beta_3}) \sin^2(D_{23})] \\
+ 2[Im(V_{\alpha_1} V_{\beta_1}^* V_{\alpha_2} V_{\beta_2}) \sin(2D_{12}) + Im(V_{\alpha_1} V_{\beta_1}^* V_{\alpha_3} V_{\beta_3}) \sin(2D_{13}) + Im(V_{\alpha_2} V_{\beta_2}^* V_{\alpha_3} V_{\beta_3}) \sin(2D_{23})]. \]  
(A2)

For a specific process, e.g., \( P(\nu_\mu \rightarrow \nu_e) \), we have

\[
Re(V_{21} V_{12}^* V_{22}^* V_{11}) = x_2 x_3 + x_1 y_2 - y_1 y_2 - y_2 y_3 = F^{\mu e}_{21}, \\
Re(V_{21} V_{13}^* V_{23}^* V_{11}) = -x_1 x_3 - x_2 x_3 + x_3 y_1 + y_2 y_3 = F^{\mu e}_{31}, \\
Re(V_{22} V_{13}^* V_{23}^* V_{12}) = x_1 x_3 + x_2 y_3 - y_1 y_3 - y_2 y_3 = F^{\mu e}_{32}, \]  
(A3)

and the probability,

\[
P(\nu_\mu \rightarrow \nu_e) = - 4[F^{\mu e}_{21} \sin^2(D_{21}) + F^{\mu e}_{31} \sin^2(D_{31}) + F^{\mu e}_{32} \sin^2(D_{32})] + 8 J \sin(D_{21}) \sin(D_{31}) \sin(D_{32}). \]  
(A4)

where \( Im[V_{\alpha_1} V_{\beta_j}^* V_{\alpha_3}^* V_{\beta_3}] = J \sum_{\gamma,k} \epsilon_{\alpha_\beta_\gamma} \epsilon_{ijk} \) has been used in reducing the sum of the imaginary parts in Eq. (A2). In addition, the probability for the T-conjugate process takes the form

\[
P(\nu_e \rightarrow \nu_\mu) = - 4[F^{\mu e}_{21} \sin^2(D_{21}) + F^{\mu e}_{31} \sin^2(D_{31}) + F^{\mu e}_{32} \sin^2(D_{32})] - 8 J \sin(D_{21}) \sin(D_{31}) \sin(D_{32}), \]  
(A5)

where

\[
F^{\mu e}_{21} = -x_2 x_3 - x_1 x_3 + x_1 y_2 + y_1 y_3, \\
F^{\mu e}_{31} = x_2 x_3 + y_1 y_3 - y_2 y_3, \\
F^{\mu e}_{32} = -x_1 x_2 - x_2 x_3 + x_2 y_3 + y_1 y_2. \]  
(A6)

We may verify the relation \( F^{\mu e}_{ij} = F^{\mu e}_{ji} \) using Eq. (4).

The explicit probabilities for other processes can be derived following the same procedure:

\[
P(\nu_e \rightarrow \nu_\tau) = - 4[(x_1 x_3 + x_2 y_1 - y_1 y_2 - y_1 y_3) \sin^2(D_{21})] + (-x_1 x_2 - x_1 x_3 + x_1 y_2 + y_1 y_3) \sin^2(D_{31}) + (x_1 x_2 + x_3 y_2 - y_1 y_2 - y_2 y_3) \sin^2(D_{32})] + 8 J \sin(D_{21}) \sin(D_{31}) \sin(D_{32}) \]  
(A7)

\[
P(\nu_\tau \rightarrow \nu_e) = - 4[(-x_1 x_2 - x_2 x_3 + x_2 y_1 + y_2 y_3) \sin^2(D_{21})] + (x_2 x_3 + x_1 y_3 - y_1 y_3 - y_2 y_3) \sin^2(D_{31}) + (-x_1 x_3 - x_2 x_3 + x_3 y_2 + y_1 y_3) \sin^2(D_{32})] - 8 J \sin(D_{21}) \sin(D_{31}) \sin(D_{32}) \]  
(A8)
The evolution equations and the analytic, approximate, solutions for different ways of reducing $Re F^\alpha\beta$ following the same method outlined in this work. The numerical solutions for $\bar{\nu}_e \rightarrow \nu_e$ may also be written as $F^{\alpha\beta}_{ij}$ by replacing the parameters for the $\nu$ sector with that for the $\bar{\nu}$ sector: $x \rightarrow \bar{x}$, $y \rightarrow \bar{y}$, $D_{ij} \rightarrow \bar{D}_{ij}$, and thus $F^{\alpha\beta}_{ij} \rightarrow F^{\alpha\beta}_{ij}$, $J \rightarrow \bar{J}$. As an example, the probability $P(\bar{\nu}_e \rightarrow \nu_e)$ is given by

$$P(\nu_\mu \rightarrow \nu_\tau) = -4 \left[ (-x_1 x_3 - x_2 x_3 + x_3 y_3 + y_1 y_2) \sin^2(D_{21}) + (x_1 x_3 + x_2 y_2 - y_1 y_2 - y_2 y_3) \sin^2(D_{31}) \right. \\
+ \left. (-x_1 x_2 - x_1 x_3 + x_1 y_1 + y_2 y_3) \sin^2(D_{32}) \right] - 8J \sin(D_{21}) \sin(D_{31}) \sin(D_{32})$$  \hspace{1cm} (A9)

$$P(\nu_\tau \rightarrow \nu_\mu) = -4 \left[ (x_1 x_2 + x_3 y_3 - y_1 y_3 - y_2 y_3) \sin^2(D_{21}) + (-x_1 x_2 - x_2 x_3 + x_2 y_2 + y_1 y_3) \sin^2(D_{31}) \right. \\
+ \left. (x_2 x_3 + x_1 y_1 - y_1 y_2 - y_1 y_3) \sin^2(D_{32}) \right] + 8J \sin(D_{21}) \sin(D_{31}) \sin(D_{32})$$  \hspace{1cm} (A10)

We may also write down the expressions for the $\bar{\nu}$ sector, $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$, by replacing the parameters for the $\nu$ sector with that for the $\bar{\nu}$ sector: $x \rightarrow \bar{x}$, $y \rightarrow \bar{y}$, $D_{ij} \rightarrow \bar{D}_{ij}$, and thus $F^{\alpha\beta}_{ij} \rightarrow F^{\alpha\beta}_{ij}$, $J \rightarrow \bar{J}$. As an example, the probability $P(\bar{\nu}_\mu \rightarrow \nu_\mu)$ is given by

$$P(\bar{\nu}_\mu \rightarrow \nu_\mu) = -4 \left[ \bar{F}_{21}^{\mu\nu} \sin^2(\bar{D}_{21}) + \bar{F}_{31}^{\mu\nu} \sin^2(\bar{D}_{31}) + \bar{F}_{32}^{\mu\nu} \sin^2(\bar{D}_{32}) \right] - 8\bar{J} \sin(\bar{D}_{21}) \sin(\bar{D}_{31}) \sin(\bar{D}_{32}),$$  \hspace{1cm} (A11)

where

$$\bar{F}_{21}^{\mu\nu} = \bar{x}_2 \bar{x}_3 + \bar{x}_1 \bar{y}_2 - \bar{y}_1 \bar{y}_2 - \bar{y}_2 \bar{y}_3,$$

$$\bar{F}_{31}^{\mu\nu} = -\bar{x}_1 \bar{x}_3 - \bar{x}_2 \bar{x}_3 + \bar{x}_3 \bar{y}_1 + \bar{y}_2 \bar{y}_3,$$

$$\bar{F}_{32}^{\mu\nu} = \bar{x}_1 \bar{x}_3 + \bar{x}_2 \bar{y}_3 - \bar{y}_1 \bar{y}_3 - \bar{y}_2 \bar{y}_3.$$  \hspace{1cm} (A12)

The evolution equations and the analytic, approximate, solutions for $(\bar{x}_i, \bar{y}_i)$ can be obtained following the same method outlined in this work. The numerical solutions for $(\bar{x}_i, \bar{y}_i)$ are shown in Fig. 3. Note that 1) $F^{\alpha\beta}_{ij} = F^{\alpha\beta}_{ij} = F^{\beta\alpha}_{ij} = F^{\beta\alpha}_{ij}$ in vacuum; and 2) the functions $F^{\alpha\beta}_{ij}$, $F^{\beta\alpha}_{ij}$, $F^{\alpha\beta}_{ij}$, and $F^{\beta\alpha}_{ij}$ can take varied forms in terms of $(x, y)$ and $(\bar{x}, \bar{y})$ since there are different ways of reducing $Re(V_{\alpha\beta}V_{\beta\alpha}^*V_{\alpha\beta}^*V_{\beta\alpha})$.

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