Non-Abelian Black Holes and Catastrophe Theory II: Charged Type

T. Tachizawa\textsuperscript{(a)}, K. Maeda\textsuperscript{(b)} and T. Torii\textsuperscript{(c)}

Department of Physics, Waseda University, Tokyo 169, Japan

Abstract

We reanalyze the gravitating monopole and its black hole solutions in the Einstein-Yang-Mills-Higgs system and we discuss their stabilities from the point of view of catastrophe theory. Although these non-trivial solutions exhibit fine and complicated structures, we find that stability is systematically understood via a swallow tail catastrophe. The Reissner-Nordström trivial solution becomes unstable from the point where the non-trivial monopole black hole appears. We also find that, within a very small parameter range, the specific heat of a monopole black hole changes its sign.

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\textsuperscript{(a)} electronic mail:63L507@cfi.waseda.ac.jp JSPS Research Fellow
\textsuperscript{(b)} electronic mail: maeda@cfi.waseda.ac.jp
\textsuperscript{(c)} electronic mail: 64L514@cfi.waseda.ac.jp
1 Introduction

The magnetic monopole was first discussed by Dirac in the context of the $U(1)$ gauge field\[1\]. 't Hooft and Polyakov found a new type of monopole solution in the $SO(3)$ Yang-Mills-Higgs system\[2\]. Later on we realized that such a type of monopole usually appears during a cosmological phase transition of gauge symmetries based on the grand unified theories (GUTs). Since it is a non-trivial structure in the Yang-Mills-Higgs system, there may be many interesting points to be investigated. Among them, from the viewpoint of cosmology, we may address the following questions: Does it really exist and is it stable? How was it created in the history of the universe? What are the cosmological implications? Can we observe it now?

One important outcome of such research is the so-called inflationary scenario in the early universe. Because the thermal production of such monopoles in the early universe results in destruction in the conventional Big Bang scenario, the idea of inflation was proposed in the early 80’s to resolve this monopole problem. As a result, we may not find so many GUTs-monopoles in the present epoch. Although the present amount of such a massive monopole is very much constrained within the present observation as well as by the inflationary scenario, there may have existed some unknown important effect in the early stage of the universe. For example, Vilenkin and Linde have recently proposed a new type of inflationary scenario, which occurs inside of topological defects such as monopoles at the Planck energy scale\[3\].

Since a monopole in the GUTs or at the Planck energy scale is very massive, a gravitational effect may become important. Breitenlohner et al.\[4\], Lee et al.\[5\], and Ortiz\[6\] recently found a gravitating monopole and its black hole solutions (monopole black hole). Although the role of such objects in the history of the universe is not yet clear, we have found several important results from a fundamental viewpoint. For example, because of the existence of this non-trivial structure, the trivial charged black hole (Reissner-Nordström solution) becomes unstable. However, as pointed out by Aichelburg and Bizon\[7\], mass energy may not be a good indicator of stability when more than two non-trivial solutions exist. In order to find a universal picture of such non-trivial structures and investigate the role of those objects in the early universe, we need more detailed investigations. This is the purpose of the present paper.

As for non-trivial gravitating structures including black hole solutions, we know, so far, several numerical solutions\[8\] (Paper I. see also references therein). Because neither the vacuum
Einstein nor the pure Yang-Mills system has a non-singular finite energy solution, the first discovery of a particle-like solution in the Einstein-Yang-Mills system by Bartnik and Mckinnon [9] was surprising. After this discovery, the so-called colored black hole was also discovered in the same system [10]. This colored black hole has a non-Abelian hair, which does not contribute to a global charge. Thus this casts doubt on the no-hair conjecture about black holes. These examples suggest that a non-Abelian(NA) field coupled to gravity throws a new light upon general relativity. The monopole and its black holes also provide us with a candidate for a counterexample to the no-hair conjecture. The trivial Reissner-Nordström black hole becomes unstable when those non-trivial structures exist and then goes to the stable monopole black hole. In this sense, the monopole black hole could be one of the most plausible counterexamples. To discuss the evolution of those structures, we have to know their properties and stabilities. That is another point to be discussed here. Apart from the monopole and its black holes, we presented a universal picture of NA black holes in [11, 8] from the point of view of catastrophe theory. In those papers, we analyzed mainly globally neutral black holes. In particular, we found that the stability of the black holes was naturally understood by catastrophe theory.

In this report, we will focus on the charged case, that is, the Einstein-Yang-Mills system with the real triplet Higgs field. A particle-like solution in this system is a gravitating ’t Hooft-Polyakov magnetic monopole. Hence, the black hole in this system is globally charged. The properties of the globally charged hole are very different from those of the neutral cases, as we shall see later. Nonetheless, we will show that we can again apply catastrophe theory. Thus, catastrophe theory not only is a powerful tool to treat the stability problem but also gives us a universal picture of NA black holes.

The plan of this paper is as follows. In Sec. II, we derive the basic equations. In Sec. III, we briefly review studies of the gravitating monopole and its black holes and summarize the known results. In Sec. IV, we discuss the stability of the non-trivial structures via catastrophe theory. In Sec. V, we discuss a new feature of the monopole black hole obtained from the present analysis and also show its spacetime structure. In Sec. VI, we give our conclusion and some final remarks.
2 BASIC EQUATIONS

The \( SO(3) \) EYMH system is described by the action:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R + \mathcal{L}_{\text{matter}} \right],
\]

with

\[
\mathcal{L}_{\text{matter}} = \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D_{\mu} \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} \left( \Phi^a \Phi^a - v^2 \right)^2,
\]

\[
F_{\nu}^a = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu - e\epsilon^{abc} A^b_\mu A^c_\nu,
\]

\[
D_{\mu} \Phi^a = \partial_{\mu} \Phi^a + e \epsilon^{abc} A^b_{\mu} \Phi^c,
\]

where \( A^a_\mu \) and \( F_{\nu}^a \) are the \( SO(3) \) Yang-Mills field potential and its field strength, respectively, and \( \Phi^a \) is the real triplet Higgs field. \( D_{\mu} \) is the covariant derivative. \( G, e, v \) and \( \lambda \) are the Newton's gravitational constant, the gauge coupling constant, the vacuum expectation value of the Higgs field, and the Higgs self coupling constant, respectively. We use the unit \( c = \hbar = 1 \). \( M_{Pl} \equiv 1/\sqrt{G} \) is the Planck mass.

We consider a static and spherically symmetric solution. The metric is written as

\[
ds^2 = -\left(1 - \frac{2Gm(r)}{r}\right) e^{-2\delta(r)} dt^2 + \left(1 - \frac{2Gm(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2.
\]

For the matter fields, we adopt the so-called hedgehog ansatz given by

\[
\Phi^a = vr^a h(r),
\]

\[
A^a_i = \omega^c_i \epsilon^{cab} \frac{1 - w(r)}{er}, \quad A^a_0 = 0,
\]

where \( r^a \) and \( \omega^c_i \) are a unit radial vector in the internal space and a triad, respectively. Introducing dimensionless variables,

\[
\tilde{r} = evr, \quad \text{and} \quad \tilde{m} = evm/M_{Pl}^2,
\]

and dimensionless parameters,

\[
\tilde{v} = v/M_{Pl}, \quad \text{and} \quad \tilde{\lambda} = \lambda/e^2,
\]
the Einstein equations and the field equations derived from the action \( \Pi \) are reduced to

\[
\frac{d\delta}{d\tilde r} = -8\pi \tilde v^2 \tilde r \left[ \frac{1}{\tilde r^2} \left( \frac{dw}{d\tilde r} \right)^2 + \frac{1}{2} \left( \frac{dh}{d\tilde r} \right)^2 \right],
\]

\[\frac{d\tilde m}{d\tilde r} = 4\pi \tilde v^2 \left( 1 - 2\tilde m/\tilde r \right) \left\{ \left( \frac{dw}{d\tilde r} \right)^2 + \frac{\tilde r^2}{2} \left( \frac{dh}{d\tilde r} \right)^2 \right\} + (1 - w^2)^2 + w^2 h^2 \]

\[+ \pi \tilde \lambda \tilde v^2 \tilde r^2 (h^2 - 1)^2 \left( = \frac{4\pi G}{\epsilon^2 v^2} r_0 \tilde v^2 \right),\]

\[
\frac{d^2 w}{d\tilde r^2} = \frac{1}{\tilde r^2 (1 - 2\tilde m/\tilde r)} \left[ -w(1 - w^2 - \tilde r^2 h^2) - 2\tilde m \frac{dw}{d\tilde r} \right.
\]

\[+ 8\pi \tilde v^2 \tilde r \frac{dw}{d\tilde r} \left\{ \left( 1 - w^2 \right)^2 + w^2 h^2 + \tilde r^2 (h^2 - 1)^2 \right\} \bigg],
\]

\[
\frac{d^2 h}{d\tilde r^2} = -2 \frac{dh}{d\tilde r} + \frac{1}{\tilde r^2 (1 - 2\tilde m/\tilde r)} \left[ 2hw^2 + \tilde \lambda \tilde r^2 h (h^2 - 1) - 2\tilde m \frac{dh}{d\tilde r} \right.
\]

\[+ 8\pi \tilde v^2 \tilde r \frac{dh}{d\tilde r} \left\{ \left( 1 - w^2 \right)^2 + w^2 h^2 + \tilde r^2 (h^2 - 1)^2 \right\} \bigg],
\]

We choose the normalization of \( t \) by setting \( \delta(\infty) = 0 \).

The boundary conditions of a globally regular monopole solution are

\[ w(\infty) = 0, \quad h(\infty) = 1, \quad \tilde m(\infty) = \text{finite}, \]

at spatial infinity and

\[ w(\tilde r) = 1 - c_w \tilde r^2 + \cdots, \]

\[ h(\tilde r) = c_h \tilde r + \cdots, \]

\[ \tilde m(\tilde r) = \frac{4\pi}{3} \tilde v^2 \left( 6c_w^2 + \frac{3}{2} c_h^2 + \frac{1}{4} \tilde \lambda \right) \tilde r^3 + \cdots, \]

near the origin, where \( c_w \) and \( c_h \) are constants determined iteratively so that \( w \) and \( h \) satisfy the boundary condition (14).

As for the black hole solution, the boundary condition at spatial infinity is the same as that of the regular monopole solution while that at the horizon is given by

\[ \tilde m(\tilde r_H) = \frac{\tilde r_H}{2}, \]

where \( \tilde r_H \) is the value of \( \tilde r \) at the horizon. Moreover the square brackets in (12) and (13) must vanish at \( \tilde r_H \) for the horizon to be regular. Hence,

\[
\frac{dw}{d\tilde r} \bigg|_{\tilde r = \tilde r_H} = \frac{w_H (1 - w_H^2 - h_H^2 \tilde r_H^2)}{2\pi \tilde v^2 \tilde r_H \left[ 2\tilde r_H^{-2} (1 - w_H^2)^2 + 4w_H^2 h_H^2 + \tilde \lambda \tilde r_H^2 (h_H^2 - 1)^2 \right] - \tilde r_H},
\]
\[
\frac{dh}{d\tilde{r}}\bigg|_{\tilde{r}=\tilde{r}_H} = \frac{-h_H[2w_H^2 + \tilde{\lambda}\tilde{r}_H^2(h_H^2 - 1)]}{2\pi\tilde{v}^2\tilde{r}_H^2\left[2\tilde{r}_H^2(1 - w_H^2)^2 + 4w_H^2h_H^2 + \tilde{\lambda}\tilde{r}_H^2(h_H^2 - 1)^2\right] - \tilde{r}_H},
\]
where \(w_H \equiv w(\tilde{r}_H)\) and \(h_H \equiv h(\tilde{r}_H)\). \(w_H\) and \(h_H\) are shooting parameters and should be determined iteratively so that the boundary condition (14) is satisfied. Therefore, this is a kind of eigenvalue problem though the equations do not contain a physical constant parameter which becomes an eigenvalue.

3 OVERVIEW OF A MONOPOLE BLACK HOLE

So far, several research groups have investigated the gravitating monopole and its black hole solutions and found very fine and complicated structures in a solution space. Here we shall briefly review those non-trivial structures and summarize their properties.

(1) Monopole:
Breitenlohner et al.\[4\], Lee et al.\[5\], and Ortiz\[6\] solved Eqs. (10), (11) numerically under the boundary conditions (14)-(17), and found a gravitating monopole. There exists a maximal value of \(\tilde{\lambda}\), \(\tilde{\lambda}_{\text{max}}\) (∼ 0.3958)\[12\] for \(\tilde{\lambda} = 0\), beyond which no solution exists\[4\]. Only a trivial RN black hole solution can exist. We understand the existence of this critical value of \(\tilde{\lambda}\) intuitively as follows: The mass of monopole and its core radius are \(\sim 4\pi\tilde{v}/e\) and \(\sim 1/ev\), respectively. Then, as \(v\) gets large, the monopole radius decreases while its gravitational radius \((8\pi G v/e)\) increases. The monopole radius eventually becomes smaller than its gravitational radius and it collapses into a black hole. This happens when \(v \sim M_{\text{Pl}}/\sqrt{8\pi}\). For \(e \ll 1\), since the energy density \(\sim e^2 v^4 \ll M_{\text{Pl}}^4\), then we can still ignore the effect of quantum gravity.

Near the critical point \(\tilde{\lambda}_{\text{max}}\), there exist two monopole solutions with different mass \(M\) at least for small values of \(\tilde{\lambda}\) (see Fig. 1). One is more massive than the other and it disappears at some value of \(\tilde{v}\), \(\tilde{v}_{\text{extreme}}(< \tilde{v}_{\text{max}})\), where the solution turns out to be the extreme black hole solution. The more massive branch is unstable while the less massive branch is stable\[23\]. Since a cusp appears at \(\tilde{v}_{\text{max}}\) (see Fig. 1), we can understand this stability via catastrophe theory (see later).

There are two interesting limiting cases: (i) \(G = 0\) and (ii) \(v = 0\). \(\tilde{v} = 0\) denotes \(G = 0\), i.e., gravity is switched off. (Notice that the limit of \(\tilde{v} = 0\) does not recover \(v = 0\) because \(\tilde{r}\)
contains $v$.) We find the solutions in a flat spacetime. In the limit of $\tilde{\lambda} = 0$ with $\tilde{v} = 0$ (i.e., flat spacetime), namely, in the Bogomol’nyi-Prasad-Sommerfield limit\[19\], we have the following analytic solution:

$$w(\tilde{r}) = \frac{\tilde{r}}{\sinh \tilde{r}}, \quad h(\tilde{r}) = \coth \tilde{r} - \frac{1}{\tilde{r}}.$$

(1)

On the other hand, in the limit of $v \to 0$ keeping $G$ fixed, there is no solution which satisfies the above boundary conditions. Instead, if we set $h \equiv 0$, the solution is no longer a monopole solution but the Bartnik-McKinnon (BM) solution in the Einstein-Yang-Mills system\[9\]. The typical mass scale of the BM particle is $\sqrt{4\pi M_{Pl}/e}$, which is the same as that of the extreme Reissner-Nordström black hole with a magnetic charge $1/e$.

(2) Black holes:
The Reissner-Nordström (RN) solution always exists for arbitrary values of $\tilde{v}$ and $\tilde{\lambda}$. It is a trivial solution in this system, i.e.,

$$w(r) \equiv 0, \quad h(r) \equiv 1, \quad m(r) = M - \frac{2\pi}{e^2 r}, \quad \text{and} \quad \delta(r) \equiv 0,$$

(2)

where $M$ is the gravitational mass. On the other hand, a non-trivial solution exists only for a finite range of $\tilde{v}$ just as in the case of a monopole. We may regard the NA black hole as a black hole lying inside a ‘t Hooft-Polyakov monopole. Therefore, we shall call the NA black hole a monopole black hole.

Breitenlohner et al.\[4\] solved the above equations (10)-(13) numerically and showed explicitly the monopole black hole solution. Lee et al.\[5\] suggested the existence of the monopole black hole solution from consideration of a simplified model. They assumed that the distribution of the energy density is

$$\rho = \begin{cases} \rho_0 \ (\text{const.}) & (r < R) \\ 1/2e^2 r^4 & (r > R) \end{cases}$$

(3)

where $R$ denotes the core radius of a monopole. The mass function is obtained by integration

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr + m(0).$$

(4)

(Later we will also discuss the validity of this model and use it to show the Penrose diagram.)

Roughly speaking, the NA black hole can exist when the horizon radius ($\sim GM$) is less than the monopole radius ($\sim 1/e\nu$), and it is stable. This stability may be naively understood from the following argument (but see \S.4 for the details): As the horizon radius $r_H$ shrinks, the energy
density of the magnetic field near the horizon gets as large as \( \sim 1/r_H^4 \) (see eq.(4)-(9)). Hence, the energy density inside the core is lowered and is energetically favorable to restore symmetry and to shield the magnetic field in the monopole core. Such is the case of the monopole black hole.

Depending on the parameters \( \tilde{\lambda} \) and \( \tilde{v} \), a variety of solutions are found. If \( \tilde{\lambda} \) is large, e.g. \( \tilde{\lambda} = 1 \), we have a rather simple structure, that is, there is only one NA black hole branch in addition to the RN branch. The two branches merge at the bifurcation point \( B \). However, if \( \tilde{\lambda} \) is small enough, e.g. \( \tilde{\lambda} = 0.1 \), the NA solutions exist in two branches and show a cusp \( C \). The bifurcation point \( B \) where the NA black hole and the RN branch merge also exist. In a small range of parameters, there are three black hole solutions. Which one is stable? Aichelburg and Bizon\cite{7} claimed that the mass is not a good indicator of stability. Because of such a complexity in the EYMH system, they also studied a simpler case of \( \tilde{\lambda} = \infty \), although it turns out that the case of \( \tilde{\lambda} = \infty \) is not the same as the case of \( \tilde{\lambda} = \text{finite} \). Later we will show a unified picture via catastrophe theory.(3) Higher excited modes:

Besides a fundamental solution, in which \( h \) and \( w \) are monotonic functions of \( r \), there exists a discrete family of radial excitations with an increasing number of zeros of \( w \)\cite{4}. The inside structure of the solution in this family approaches the BM solution in the limit of \( v \to 0 \), although the behaviors at large distances are not the same because of different boundary conditions.

(4) Stabilities:

One of the most important properties of those non-trivial structures is stability.

The trivial RN black hole seems to be stable, but it turns out to be unstable against linear radial perturbation for some range of parameters\cite{5}. Then, the stable non-trivial monopole black hole appears.

When we have more than two solutions, the less massive one seems to be more stable because it is energetically more favorable. The result in the present case, however, is not so simple. As discussed in (2), there are three solutions (one RN and two monopole BHs) in some parameter range. The smallest mass black hole is of course stable, but one of the heavier holes is also stable at least against linear radial perturbations. To understand this situation, we shall adopt catastrophe theory and show in the next section how stability is naturally understood.
(5) Thermodynamical properties:
A RN black hole changes the sign of its specific heat when the charge becomes larger than the critical value \( Q_{\text{cr}} = \sqrt{3} M/2 \). Davies claimed that it is a second order phase transition\(^2\). This may indicate the stability change of the charged black hole in the heat bath system at the critical value. How about in the EYMH theory? Lee et al. claimed that the specific heat of the NA bole is always negative\(^5\), but as we discuss in §5, there is a very narrow range of parameters where the specific heat becomes positive.

(6) Fate of charged black hole:
What happens on such a small black hole when we take into account the Hawking evaporation process? In the Einstein-Maxwell theory, as the RN black hole loses its mass via the Hawking thermal radiation, if the charge is conserved, the black hole approaches the extreme one and the temperature goes to zero. The evaporation will cease. Hence, the extreme RN black hole is stable and it may become a cosmological remnant.

On the other hand, in the Einstein-Yang-Mills case, the RN black hole becomes classically unstable when its horizon radius is less than the monopole radius \( \sim 1/ev \)\(^5\). Thus, as the RN black hole evaporates, it turns out to be unstable and probably becomes a monopole black hole. Since the specific heat of the stable monopole black hole is negative\(^5\) (but see later for the unstable one), the monopole black hole does not stop evaporating. Then, we will find a regular monopole at the end.

4 STABILITY OF A MONOPOLE BLACK HOLE AND CATASTROPHE THEORY

When we dealt with neutral black holes\(^1\),\(^8\), the mass-horizon radius diagram played an important role. The cusp structure appeared in the diagram, and it gave us a clue to apply catastrophe theory. Here, we again begin with similar diagrams. For a review of catastrophe theory, see, for example, reference\(^17\), and for its application to physics, reference\(^18\).

Fig. 2 shows the relation between the mass of black holes and their horizon radius when \( \tilde{\lambda} = 1 \) and \( \tilde{v} = 0.05 \). (\( \tilde{M} = \tilde{m}(\infty) \)) The mass of the RN black hole has a lower bound (the
extreme case) because its magnetic charge is fixed as $1/e$. From Fig. 2, unlike in the neutral case, we cannot find any cusp structure.

On the other hand, when $\tilde{\lambda} = 0.1$ and $\tilde{v} = 0.05$, the relation becomes as shown in Fig. 3(a). This looks similar to Fig. 2 at first glance. However, as shown in Fig. 3(b), which is an enlarged diagram near the bifurcation point $B$, where the branch of the NA solutions and that of the RN solutions merge, a cusp structure appears.

So, we have two behaviors of the solutions depending on the value of $\tilde{\lambda}$. From our analysis of the neutral case and the discussion by Lee et al. [5] of the stability of the RN black hole, we can guess the stability of the solution as follows: In the first case, as in Fig. 2, the NA solution is always stable. While, the RN solution is stable when the mass is larger than $M_{N-R}$, which is the mass at the bifurcation point, $B$, but it becomes unstable if it is less than $M_{N-R}$. In the second case, as in Fig. 3, the NA solution is stable in the branch of the larger horizon radius, $AC$, and the stability changes at the cuspidal edge $C$. It becomes unstable in the branch of the smaller horizon radius, $CB$. The RN solution is stable when the mass is larger than $M_{N-R}$, but otherwise it is unstable. We will confirm this below via catastrophe theory.

Although we divide the solutions into two classes above, since our discussion is based on the numerical results, we cannot exclude the possibility that a cusp structure may also exist in the case of Fig. 2 but is so small that we cannot see it. Catastrophe theory, however, explains very naturally the existence of two classes depending on the parameters, and the change of the stability described above. Which elementary catastrophe explains these behaviors? We find that it is a swallow tail type.

In catastrophe theory, we discuss changes in the shape of a potential function as we change control parameters. Thom’s theory guarantees that if a system which is described by any potential function and has less than or equal to four control parameters experiences catastrophic change, the potential function can be made to coincide with one of the seven elementary catastrophe’s potential functions by diffeomorphism in the neighborhood of the point where the catastrophe occurs. Of course, we assume in the above theorem that the system is structural stable (see, for example, reference [17]). One or two state variables characterize the system, and the minimal and the maximal points of the potential function represent a stable and an unstable configuration, respectively [20].
Any potential function of elementary catastrophe has one minimal and one maximal point at least. Ordinary black holes belonging to the Kerr-Newman family have only three hairs. If global charges (mass, angular momentum, and electric or magnetic charge) are given, the configuration is uniquely determined. On the other hand, NA black holes can violate the no-hair conjecture, and given global charges do not necessarily determine the configuration completely. That is, even if control parameters are given, several stationary points in the potential function are possible. This is why catastrophe theory is applicable in analyzing the stability of NA black holes. As for the stability of black holes belonging to the Kerr-Newman family, see reference [21].

A swallow tail catastrophe has three control parameters, \( a, b, c \), and one state variable, \( x \), and its potential function is described as

\[
V = \frac{1}{5} x^5 + \frac{1}{3} a x^3 + \frac{1}{2} b x^2 + c x. \tag{1}
\]

A set of the points where catastrophe occurs, that is, a catastrophe set, is schematically shown in Fig. 4, which looks like a swallow’s tail. In our present model, we set \( c = 0 \) and reduce the three-dimensional control parameter space to a two-dimensional one. We need this reduction in order that the RN solution (\( x = 0 \)) can always exist as a trivial solution. A control parameter \( a \) depends on \( \tilde{\lambda} \) and \( \tilde{\nu} \), and \( b \) depends on \( \tilde{M}(\equiv evM/M_{Pl}^2) \) and \( \tilde{\nu} \) as

\[
a = a(\tilde{\lambda}, \tilde{\nu}), \quad b = b(\tilde{M}, \tilde{\nu}). \tag{2}
\]

As we shall soon see below, the condition of \( a = 0 \) divides the solutions into the two cases of whether the cusp exists or not. That is to say, a cusp structure appears in the tail of the swallow, while it does not appear in its body.

We denote the value of \( \tilde{\lambda} \) corresponding to the case of \( a = 0 \) as \( \tilde{\lambda}_{\text{crit}} \). \( \tilde{\lambda}_{\text{crit}} \) is a function of \( \tilde{\nu} \). Fig. 5 shows the schematic potential in the case of \( a > 0 \), namely, \( \tilde{\lambda} > \tilde{\lambda}_{\text{crit}} \). We regard the entropy of the black hole (\( S = \text{area}/4 \)) as the potential function. When we consider the entropy, the maximal point represents the stable configuration. Hence, we put \( S = -V \). The ordinate represents the entropy \( S \). The abscissa represents a state variable. We adopt \( \delta_H \), which is the value of \( \delta(r) \) at the horizon, as the state variable. We may adopt any variable as a state variable as long as it can be mapped to \( \delta_H \) by diffeomorphism. From our numerical
calculation, it seems that the “effective magnetic charge” of the black hole defined by

\[(1/4\pi) \int_{r=r_H} F = (1 - w_H^2)/e,\]  

(3)
or the minimum value of the function \((1 - 2\tilde{m}/\tilde{r})\) is related to \(\delta_H\) by diffeomorphism once we fix \(\tilde{v}\) and \(\tilde{\lambda}\). Here, we take \(\delta_H\) because we can also use it when we discuss the non-singular monopole.

Furthermore, we have to cut down the region of \(\delta_H < 0\) because such a region is unphysical. We can easily show that \(\delta_H\) is always positive from (10). (Remember that \(\delta(\infty) = 0\).) The maximal points in Fig. 5 represent stable solutions and the minimal points unstable ones. The origin \((\delta_H = 0)\) is always a stationary point by virtue of setting \(c \equiv 0\), and it corresponds to the RN solution.

To discuss dynamical stability, we shall consider a physical process increasing the mass of the black hole. Physically, it can be realized by matter accretion. We fix parameters in the theory, i.e., \(\tilde{v}\) and \(\tilde{\lambda}\). Only \(b\) varies by a change of the mass. We assume that \(b(\tilde{M}, \tilde{v})\) is a monotonically increasing function of \(\tilde{M}\). When \(b\) is negative, the shape of the potential function is shown in Fig.5(a). At this time, the RN solution is unstable and the NA solution is stable. Since our potential function is entropy, the maximal point represents a stable solution and the minimal one an unstable one. Only the NA black hole can really exist. As we increase \(b\), the minimal value increases and the minimal point approaches the origin(Fig. 5(b)). When \(b = 0\), the minimal and the maximal points eventually merge (Fig.5(c)). This is the bifurcation point \(B\) in Fig. 2. At this point, the NA black hole turns into the RN black hole continuously. Then the NA solution disappears, and the RN solution begins to be stable(Fig.5(d)).

We can discuss the inverse process decreasing the mass through the Hawking radiation. Suppose there exists a RN black hole, which is heavier than the mass at the bifurcation point \(B\). When its mass decreases, the RN black hole eventually reaches the bifurcation point \(B\) and it changes to the RN black hole continuously. It continues to evaporate, then probably turns into a non-singular monopole\[5\].

Next we consider the case of \(a < 0\) \((\tilde{\lambda} < \tilde{\lambda}_{crit})\). In this case, a cusp structure exists. We again discuss the case increasing \(b\) from a negative value. The change of the potential form is shown in Fig. 6. Initially, one unstable RN solution \((R)\) and one stable NA solution \((N_1)\) exist. Then, only the NA black hole can really exist(Fig.6(a)). As \(b\) increases, another pair
of a maximal and minimal point \((N_2, S)\) comes to appear in the unphysical region (Fig.6(b)). Then the minimal point \(N_2\) merges with the point \(R\) at \(b = 0\) (Fig.6(c)), and the RN solution becomes stable. However, since the NA solution \(N_1\) is still stable, nothing happens at this point. And another branch of unstable NA solutions \((N_2)\) bifurcates from the branch of the RN solution (Fig.6(d)). Then, two stable solutions \((R, N_1)\) and one unstable solution \((N_2)\) exist at the same time. That is, even if the global charge (=mass) is fixed, two stable solutions become possible. This means a violation of the weak no-hair conjecture\([7, 22]\). As \(b\) increases further, the stable and unstable NA solutions approach each other, and eventually they merge to become an inflection point (Fig.6(e)). This causes catastrophe. The NA black hole jumps to a RN black hole discontinuously as a solid arrow. After that only the stable RN solution remains (Fig.6(f)).

The inverse process, i.e., the case decreasing \(b(\tilde{M}, \tilde{v})\), is described as follows: Suppose there exists only a stable RN solution \((R)\) initially. As \(b\) decreases, a pair of stable and unstable NA solutions \((N_1, N_2)\) appears (Fig.6(e)). At this point, since the RN solution remains stable, nothing happens. When \(b\) decreases further and reaches zero, the stable RN \((R)\) and the unstable NA branches \((N_2)\) merge. Then, the RN solution begins to be unstable (Fig.6(c)). The RN black hole \((R)\) jumps to the NA black hole \((N_1)\) discontinuously as the dotted arrow (Fig.6(b)). Only the NA solution remains as a stable solution. Then, the NA black hole will evaporate to be a stable monopole in the same way as the case of \(\tilde{\lambda} > \tilde{\lambda}_{\text{crit}}\).

As we have seen above, a swallow tail catastrophe can explain the behavior of solutions. Moreover, note that the RN branch and the NA branch always merge at \(b = 0\) (i.e., \(\tilde{M} = \tilde{M}_{N=\text{R}}\)) independently of the value of \(\tilde{\lambda}\). We can confirm the fact that \(\tilde{M}_{N=\text{R}}\) is independent of the value of \(\tilde{\lambda}\) from Fig. 7, which shows the critical values of mass at the bifurcation points. If \(\tilde{\lambda} < \tilde{\lambda}_{\text{crit}}\), there are two critical values, \(\tilde{M}_{N=\text{R}}\) and \(\tilde{M}_{N=\text{N}}\), which correspond to the bifurcations of \(R\) and \(N_2\) (Fig.6(c)) and of \(N_1\) and \(N_2\) (Fig.6(e)) , respectively. On the other hand, if \(\tilde{\lambda} > \tilde{\lambda}_{\text{crit}}\), there is only one bifurcation of \(R\) and \(N\) (Fig.5(c)) appearing at \(\tilde{M}_{N=\text{R}}\). From Fig. 7, we find that \(\tilde{M}_{N=\text{R}}\) is in fact independent of \(\tilde{\lambda}\).

The dependence of \(\tilde{M}_{N=\text{R}}\) on \(\tilde{v}\) is shown in Fig. 8. This relation is approximated as

\[
\tilde{M}_{N=\text{R}} = 7.256\tilde{v}^{1.745} + 0.273. \tag{4}
\]

We can compare this with the result of Aichelburg and Bizon\([7]\) who dealt with the case of
\( \lambda = \infty \). According to their result, at \( \tilde{v} = 0.288/\sqrt{4\pi} \approx 0.0812, \tilde{M}_{N-R} = 1.25 \times 0.288 \approx 0.360. \) From our analysis (4), we can extrapolate and guess the value of \( M_{N-R} \) at \( \lambda \) as

\[
\tilde{M}_{N-R} = 7.256 \times \left( \frac{0.288}{\sqrt{4\pi}} \right)^{1.745} + 0.273 \approx 0.363. \tag{5}
\]

Those two values of \( \tilde{M}_{N-R} \) agree well. Thus, we find that \( \tilde{M}_{N-R} \) is independent of the \( \tilde{\lambda} \) until \( \tilde{\lambda} = \infty \). Therefore, we can deal with the range of the parameter \( 0 \leq \tilde{\lambda} < \infty \) by one potential function of the swallow tail catastrophe.

When we fix \( \tilde{M} \) and \( \tilde{v} \), and change \( \tilde{\lambda} \) from 0 to \( \infty \), the stable branch of the NA black hole neither bifurcates nor merges with another branch. This means that the stability of a NA black hole in the stable branch does not change. Since Aichelburg and Bizon showed that the NA black hole with \( \tilde{\lambda} = \infty \) is stable[7], we can conclude that a NA black hole in the “stable” branch is always stable.

As for a non-singular monopole, as shown in some references[4, 7, 23], a cusp structure appears in a \( \tilde{v}-M \) diagram when \( \tilde{\lambda} \) is small (Fig. 1). Therefore, taking the mass as a potential function and \( \delta_H \) as a state variable, we can understand the stability of the non-singular monopole by fold catastrophe in the same way as we treated neutral NA black holes[11, 8]; the lower branch is stable and the upper branch is unstable[23]. We find that the branch is really stable for the following reason. The branch changing \( \tilde{\lambda} \) with fixed \( \tilde{v} \) and \( \tilde{M} \) neither bifurcates nor merges with another branch. Therefore, the stability of the branch does not change. Since the monopole in the stable branch in the \( \tilde{v}-M \) diagram is stable when \( \tilde{\lambda} \) is small[23], then the branch is stable even when \( \tilde{\lambda} \) is large.

5 NEW PROPERTIES OF A MONOPOLE BLACK HOLE

In this section, we present new properties of monopole black holes which have not been discussed in §. 2, i.e., their thermodynamical properties and global spacetime structure.

(1) Thermodynamical properties:

First, we show the inverse temperature of the black holes in Fig. 9. For small values of \( \tilde{\lambda} \), we find the turning point \( C \) in Fig. 9, which corresponds to the cuspidal edge in a \( \tilde{M}-\tilde{r}_H \) diagram in Fig. 3(b). We can understand this coincidence as follows: Because the black hole is non-

\( \tilde{\lambda} = \infty \). According to their result, at \( \tilde{v} = 0.288/\sqrt{4\pi} \approx 0.0812, \tilde{M}_{N-R} = 1.25 \times 0.288 \approx 0.360. \) From our analysis (4), we can extrapolate and guess the value of \( M_{N-R} \) at \( \lambda \) as

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rotating and its charge is fixed, the first law of the black hole in the present system can be written as

$$\delta M = T \delta S, \quad (1)$$

where $S = \pi r_H^2$ is the entropy of the black hole. From this, we can write as

$$\frac{dS}{dM} = \frac{1}{T}, \quad (2)$$

or

$$\frac{dr_H}{dM} = \frac{1}{2\pi r_H T}. \quad (3)$$

So, if $1/T$ depends on $M$ as in Fig. 9, that is, if a turning point $C$ exists, then a cusp must appear in a $M-r_H$ diagram. Hence, the stability changes at the turning point $C$ on a vertical tangent in the $M-1/T$ diagram, where $d(1/T)/dM = \infty$. Thus, we can also discuss the stability of a black hole by using the $M-1/T$ diagram. As for this method, see reference [21].

The specific heat of monopole black holes in the stable branch is always negative, while that in the unstable one changes its sign a few times depending on the parameters $\tilde{v}$ and $\tilde{\lambda}$. This feature is similar to that of neutral NA black holes. When $\tilde{\lambda} < \tilde{\lambda}_{crit}(\tilde{v})$ and $\tilde{v}$ is small enough, the sign of the specific heat in the unstable branch changes at two points (one is the turning point $C$ on the vertical tangent and the other is another turning point $A$ on the horizontal tangent). The specific heat between two turning points is positive. For example, for $\tilde{v} = 0.05$ and $\tilde{\lambda} = 0.1$, we find the two turning points $C$ and $A$ at $M = 0.32495$ and $\tilde{M} = 0.32401$, respectively. Note that the specific heat at $C$ vanishes while that at $A$ diverges just like that of the RN black hole with the critical charge. As $\tilde{v}$ increases, the unstable NA branch merges with the RN branch before the specific heat turns to negative. The sign of the specific heat changes only at the point $C'$ as in Fig.9(a). As $\tilde{v}$ increases further, $\tilde{\lambda}_{crit}(\tilde{v})$ becomes smaller than $\tilde{\lambda}$. Then, the stability change does not occur, and the specific heat is always negative.

(2) Spacetime structure

Next, we consider the distribution of energy density around a monopole to discuss the spacetime structure. The energy density $\rho$ can be written as

$$\rho = -T^0_0 = \left(1 - \frac{2\tilde{m}}{r}\right) \rho_{\text{grad}} + \rho_{\text{pot}}, \quad (4)$$
where the gradient term $\rho_{\text{grad}}$ and the potential term $\rho_{\text{pot}}$ have forms

$$\rho_{\text{grad}} = e^2v^4\left\{\frac{1}{\tilde{r}^2}\left(\frac{dw}{d\tilde{r}}\right)^2 + \frac{1}{2}\left(\frac{dh}{d\tilde{r}}\right)^2\right\},$$  \hspace{1cm} (5)$$

$$\rho_{\text{pot}} = \rho_{\text{pot}(F^2)} + \rho_{\text{pot}(D\Phi^2)} + \rho_{\text{pot}(V)},$$  \hspace{1cm} (6)

with

$$\rho_{\text{pot}(F^2)} = e^2v^4\left(1 - w^2\right)^2,$$  \hspace{1cm} (7)

$$\rho_{\text{pot}(D\Phi^2)} = e^2v^4\frac{h^2w}{\tilde{r}^2},$$  \hspace{1cm} (8)

$$\rho_{\text{pot}(V)} = \frac{1}{4}\lambda e^2v^4\left(h^2 - 1\right)^2.$$  \hspace{1cm} (9)

$\rho_{\text{pot}(F^2)}$, $\rho_{\text{pot}(D\Phi^2)}$ and $\rho_{\text{pot}(V)}$ come from the potential parts of $-(1/4)F^2$, $-(1/2)(D\Phi)^2$ and $-(\lambda/4)(\Phi^2 - v^2)^2$ in the matter Lagrangian, respectively.

We show the energy density distribution around the non-singular monopole in Fig.10. The energy density is almost constant inside the monopole core ($\tilde{r} < \sim 1$), but it decays as $1/\tilde{r}^4$ outside ($\tilde{r} > \sim 1$). This indicates that Eq.(3) which Lee et al. used is a good approximation.

This simplified model may tell us the global spacetime structures of the gravitating monopole and its black holes as follows: To determine the mass function $m(r)$ and the core radius $R$, we adopt the approximation of (3) and minimize $M$. Then, we obtain:

$$\frac{m(r)}{r} = \begin{cases} 
\frac{(M - M_{\text{mon}})/r + M_{\text{mon}}}{4R^2}, & (r < R) \\
\frac{M}{r} - 3M_{\text{mon}}R/4r^2, & (r > R) 
\end{cases}$$  \hspace{1cm} (10)

where $M_{\text{mon}} = 8\pi/3e^2R$ and $R = (2e^2\rho_0)^{-1/4}$. Using the known relation $M_{\text{mon}} = 4\pi v/e$, we can fix the relation between $\rho_0$ and $v$, i.e., $\rho_0 = 81e^2v^4/32$.

From Fig.12, we can approximate a form of $\delta(r)$ to be a step function:

$$\delta = \begin{cases} 
\delta_0 = \text{constant}, & (r < R) \\
\delta_\infty = \text{constant}, & (r > R) 
\end{cases}$$  \hspace{1cm} (11)

From our approximation (10) and (11), we regard the inside of the monopole core as the Schwarzschild-de Sitter spacetime. (de Sitter spacetime for “monopole” solution, $m(0) = 0$) and the outside as the Reissner-Nordström spacetime. Let us first discuss the case of the “monopole” ($m(0) = 0$). The maximal value of $\tilde{m}(\tilde{r})/\tilde{r}$ is $\tilde{M}_{\text{mon}}^2/8\pi$ and $\tilde{m}(\tilde{R})/\tilde{R} = 3\tilde{M}_{\text{mon}}^2/32\pi$, where $\tilde{M} \equiv eM/M_{Pl}$ and $\tilde{R} \equiv evR$(see Fig. 13). Since $\tilde{m}/\tilde{r}_H = 1/2$, if

$$\frac{\tilde{M}_{\text{mon}}^2}{8\pi} < \frac{1}{2} \iff \tilde{v} < \frac{1}{\sqrt{4\pi}},$$  \hspace{1cm} (12)
a horizon does not exist and the surface \( R = \text{constant} \) is timelike. The Penrose diagram is shown in Fig. 14(a). When

\[
\frac{3\dot{M}_{\text{mon}}^2}{32\pi} < \frac{1}{2} \leq \frac{\dot{M}_{\text{mon}}^2}{8\pi} \Leftrightarrow \frac{1}{\sqrt{4\pi}} \leq \tilde{\nu} < \frac{1}{\sqrt{3\pi}},
\]

(13)

horizons exist and \( R = \text{constant} \) is timelike (Fig. 14(b)), but when

\[
\frac{1}{2} \leq \frac{3\dot{M}_{\text{mon}}^2}{32\pi} \Leftrightarrow \tilde{\nu} \geq \frac{1}{\sqrt{3\pi}},
\]

(14)

horizons exist and \( R = \text{constant} \) becomes spacelike. See Fig. 14(c) for the Penrose diagram.

That is, the “monopole” is no longer static and probably the core itself expands exponentially. This behavior may suggest a topological inflation\(^3\). However, following the discussion on stability of Cauchy horizon\(^{29}\), which exists in the present Penrose diagram, the de Sitter phase in the black hole will probably be unstable. The existence of a maximum value of \( \tilde{\nu} \) (\( \tilde{\nu}_{\text{max}} \)) in a \( \tilde{\nu}-M \) diagram (Fig. 1) may also give some indication for the topological inflation. Notice that the critical vacuum expectation value of the Higgs field is almost the same as that in our simplified model.

Next we classify the spacetime of the black hole solutions \( m(0) \neq 0 \) in the same way and describe the Penrose diagrams (Fig. 15). Using the simplified model, we find the classification as follows:

1. \( \dot{M} < (3/2)\dot{M}_{\text{mon}} \) and \( \dot{M} < \sqrt{4\pi} \),
2. \( \dot{M} < (3/2)\dot{M}_{\text{mon}} \) and \( \sqrt{4\pi} < \dot{M} < 4\pi/(3\dot{M}_{\text{mon}}) + (3/4)\dot{M}_{\text{mon}} \),
3. \( \dot{M} < (3/2)\dot{M}_{\text{mon}} \) and \( 4\pi/(3\dot{M}_{\text{mon}}) + (3/4)\dot{M}_{\text{mon}} < \dot{M} < (32\pi^{3/2})/(27\dot{M}_{\text{mon}}^2) + \dot{M}_{\text{mon}} \),
4. \( \dot{M} < (3/2)\dot{M}_{\text{mon}} \) and \( (32\pi^{3/2})/(27\dot{M}_{\text{mon}}^2) + \dot{M}_{\text{mon}} < \dot{M} \),
5. \( \dot{M} > (3/2)\dot{M}_{\text{mon}} \) and \( \dot{M} < 4\pi/(3\dot{M}_{\text{mon}}) + (3/4)\dot{M}_{\text{mon}} \),
6. \( \dot{M} > (3/2)\dot{M}_{\text{mon}} \) and \( 4\pi/(3\dot{M}_{\text{mon}}) + (3/4)\dot{M}_{\text{mon}} < \dot{M} \).

We have required here the condition of no naked singularity, \( M > M_{\text{mon}} \). The corresponding Penrose diagrams for the cases (1) \text{~} (6) are shown in Fig. 15. If:

- (7) \( \dot{M} < \dot{M}_{\text{mon}} \) and \( (3/2)\dot{M}_{\text{mon}} < \dot{M} \),
we have a naked singularity.

When we see the spacetime structures of (13) and (14) for the “monopole” case, and (2), (3), (4), and (6) for the black hole case from the outside of the event horizon, we cannot distinguish them from the structure of the RN spacetime. However, their global structures
may be completely different from the RN spacetime as shown in Fig. 14 and Fig. 15. In some spacetimes, a Cauchy horizon does not appear. The existence of the NA field changes the global structure of the spacetime.

(3) Physical understanding of the stabilities:
As discussed in Paper I, we may understand the physics of the stabilities of NA black holes from a comparison of energy distributions between stable and unstable branches. For the stable and the unstable monopoles with the same value of \( \tilde{v} \) (Fig. 11), we find that \( \rho_{\text{pot}(D\Phi^2)} \) is almost the same in the two monopoles, while \( (1 - 2\tilde{m}/\tilde{r})\rho_{\text{grad}} \) and \( \rho_{\text{pot}(F^2)} \) are larger in the unstable one than in the stable one. We can regard \( \rho_{\text{pot}(D\Phi^2)} \) as a “stabilizer” (see Paper I). From the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium in the EYMH system[25], the term \( (1 - 2\tilde{m}/\tilde{r})\rho_{\text{grad}} \) works as an “attractive force” and the term \( \rho_{\text{pot}(F^2)} \) works as a “repulsive force”. Thus, we may say that the stronger “attractive force” by the gradient term balances the stronger “repulsive force” by the magnetic field in an unstable monopole. However, since the “stabilizer”\( (\rho_{\text{pot}(D\Phi^2)}) \) of the unstable monopole is almost the same as that of the stable monopole, it cannot keep the balance of the heavier burden(namely, the balance between the stronger “attractive” and “repulsive” force) stable.

6 SUMMARY AND CONCLUDING REMARKS

We have re-investigated regular monopole and monopole black hole solutions in the Einstein-Yang-Mills-Higgs system. When we discussed neutral NA holes in our previous papers[11, 8], we found that catastrophe theory is applicable in analyzing the stability of NA holes. The properties of the monopole black hole are very complicated and different from those of neutral holes in many respects. However, we have found that catastrophe theory is still applicable to the monopole black hole, i.e., a swallow tail catastrophe can explain many features of the monopole black hole very naturally. It explains the transition from the monopole black hole to the Reissner-Nordström black hole and its opposite process. This means that we not only can understand its stability, but also can get an insight into the structure of the solution space. Thus, catastrophe theory is not only a powerful tool to study the stability of NA black holes
but also gives us a universal picture of NA black holes.

The monopole black hole violates the weak no-hair conjecture\[7, 22\], that is, two distinct stable solutions can exist for a given mass. But the monopole black hole is not the first black hole violating the weak no-hair conjecture. The Skyrme black hole is also stable against linear perturbation and the non-singular Skyrmion may be stable even against nonlinear perturbation\[28\]. However, the entropy of the Skyrme black hole is always smaller than that of the Schwarzschild black hole with the same mass. Therefore, the Skyrme hair would be lost in the formation process of a black hole and it would become a Schwarzschild black hole at last. On the other hand, the monopole and the monopole black holes are really classically stable because they have the maximum entropy among black holes with the same mass. Such objects can be the final remnant in the universe. Even if we start with the RN black hole, it will lose its mass energy via the Hawking evaporation process and a transition from the RN black hole to the monopole black hole will occur at some critical point. The gravitating monopole may be found at last. Thus the monopole black hole may be the first real example which violates the no-hair conjecture and which can be formed in the universe.

We have also discussed the thermal properties of monopole black holes. The sign of the specific heat of unstable monopole black holes changes one or two times depending on the parameters. On the other hand, the specific heat of stable monopole black holes is always negative. This feature is the same as that found in neutral NA black holes.

We have shown the energy density distributions of fields for a gravitating monopole and its black holes and discussed the global spacetime structure by using a simplified model. If the vacuum expectation value of the Higgs field $\psi$ is larger than a critical value ($\sim 0.370821 M_{Pl}$ for $\lambda = 0.1$), the core of the monopole cannot be static. It may expand exponentially. This may suggest a topological inflation, recently proposed by Linde and Vilenkin\[3\].

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Figure Captions

Figure 1: The $\tilde{v} - M$ diagram of a gravitating monopole for $\tilde{\lambda} = 0.1$ (the solid line) and the extreme Reissner-Nordström black hole (the dotted line).

Figure 2: The mass-horizon diagram of the Reissner-Nordström black hole (the dotted line) and of the monopole black hole (the solid line) for $\tilde{\lambda} = 1$ and $\tilde{v} = 0.05$.

Figure 3: (a) The same diagram as Fig.2 for $\tilde{\lambda} = 0.1$ and $\tilde{v} = 0.05$. (b) The difference between the horizon radius of a monopole black hole and that of a Reissner-Nordström black hole is plotted near the bifurcation point $B$. There exists a small cusp structure.

Figure 4: The catastrophe set of a swallow tail type.

Figure 5: The behavior of a potential function (entropy of the black hole) for $\tilde{\lambda} > \tilde{\lambda}_{\text{crit}}$. The maximum and minimum points correspond to the stable and the unstable solutions, respectively. There are two solutions $R$ (the RN black hole) and $N$ (the NA black hole).

Figure 6: The same as Fig. 5 for $\tilde{\lambda} < \tilde{\lambda}_{\text{crit}}$. There are three solutions $R$ (the RN black hole) and $N_1, N_2$ (the stable and unstable NA black holes).

Figure 7: The critical values of mass $\tilde{M}_{N-R}$, where the RN black hole and the NA black hole merge, and $\tilde{M}_{N-N}$, where two NA solutions merge. $\tilde{M}_{N-R}$ is independent of $\tilde{\lambda}$.

Figure 8: The dependence of $\tilde{M}_{N-R}$ on $\tilde{v}$. The line represents $\tilde{M}_{N-R} = 7.256\tilde{v}^{1.745} + 0.273$.

Figure 9: The mass-inverse temperature diagrams of the Reissner-Nordström black hole (the dotted line) and of the monopole black hole for $\tilde{\lambda} = 0.1$ and $\tilde{v} = 0.03$ (the solid line), $\tilde{v} = 0.05$ (the dashed line), $\tilde{v} = 0.15$ (the double-dot dashed line). (b) The enlarged figure near the bifurcation point of Fig.9(a) for $\tilde{\lambda} = 0.1$ and $\tilde{v} = 0.05$.

Figure 10: The energy density of the monopole black hole for $\tilde{\lambda} = 0.1$ and $\tilde{v} = 0.05$. It is almost constant inside the monopole core $\tilde{r} \lesssim 1$ but decays as $1/\tilde{r}^4$ outside the core.
Figure 11: (a) The potential parts of the energy density ($\rho_{pot}(F^2)$ and $\rho_{pot}(D\phi^2)$) of the stable monopole (the solid line) and the unstable monopole (the dotted line) for $\tilde{\lambda} = 0.1$ and $\tilde{v} = 0.37083$. We do not show the term $\rho_{pot}(V)$ because it is small. (b) The gradient part of the energy density ($\left((1 - 2 \tilde{m}/\tilde{r})\rho_{grad}\right)$ of the stable monopole (the solid line) and the unstable monopole (the dotted line) for the same values of $\tilde{\lambda}$ and $\tilde{v}$ as Fig. 11(a).

Figure 12: The function $\delta(\tilde{r})$ for a gravitating monopole for $\tilde{\lambda} = 1$ and $\tilde{v} = 0.2$ (the solid line), $\tilde{v} = 0.3$ (the dashed line), $\tilde{v} = 0.35$ (the dot dashed line), and $\tilde{v} = 0.37$ (the dotted line). As $\tilde{v}$ increases, the approximation by a step function may be justified.

Figure 13: The function form of $\tilde{m}(\tilde{r})/\tilde{r}$. The horizontal dot dashed lines represent $\tilde{m}/\tilde{r} = 1/2$ for (i) $\tilde{v} < 1/\sqrt{4\pi}$, (ii) $1/\sqrt{4\pi} \leq \tilde{v} < 1/\sqrt{3\pi}$, and (iii) $\tilde{v} \leq \sqrt{3\pi}$.

Figure 14: The Penrose diagrams of the gravitating “monopole” when (a) $\tilde{v} < 1/\sqrt{4\pi}$, (b) $1/\sqrt{4\pi} \leq \tilde{v} < 1/\sqrt{3\pi}$, and (c) $\tilde{v} \leq \sqrt{3\pi}$. The dashed line represents the surface of the “monopole”. The shaded and the unshaded regions are the de Sitter and the Reissner-Nordstrom spacetimes, respectively.

Figure 15: The Penrose diagrams of the monopole black hole for (a) Case (1) and (5), (b) Case (2), (c) Case (3), and (d) Case (4) and (6). The shaded and the unshaded regions are the Schwarzschild-de Sitter and the Reissner-Nordstrom spacetimes, respectively. The double lines represent singularities.
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