NO PERIODIC NORMAL GEODESICS IN $J^k(\mathbb{R}, \mathbb{R}^n)$

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Abstract. The space of $k$-jets of $n$ real function of one real variable $x$ admits the structure of a Carnot group, which then has an associated Hamiltonian geodesic flow. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does the space of $k$-jets have periodic geodesics? This study will demonstrate the integrability of sub-Riemannian geodesic flow, characterize and classify the sub-Riemannian geodesics in the space of $k$-jets, and show that they are never periodic.

1. Introduction

This paper is the generalization of [10, 8, 9]: In [10], the space of $k$-jets of real function of a single variable was presented as a sub-Riemannian manifold, the sub-Riemannian geodesic flow was defined and its integrability was verified. In [8], the sub-Riemannian geodesics were classified and some of their minimizing properties were studied. In [9], the non-existence of periodic geodesics on the space of $k$-jets of a real function of a single variable was proved.

The $k$-jets space of $n$ real functions of a single real variable, denoted here by $J^k(\mathbb{R}, \mathbb{R}^n)$ or $J^k$ for short, is a $(n(k + 1) + 1)$-dimensional manifold endowed with a canonical rank $n + 1$ distribution, i.e., a linear sub-bundle of its tangent bundle. This distribution is globally framed by $n$ vector fields, denoted by $X_1, \ldots , X_{n+1}$ in Section 2 whose iterated Lie brackets give $J^k(\mathbb{R}, \mathbb{R}^n)$ the structure of a stratified group. Declaring $X_1, \ldots , X_{n+1}$ to be orthonormal endows $J^k(\mathbb{R}, \mathbb{R}^n)$ with the structure of a sub-Riemannian manifold, which is left-invariant under the group multiplication. Like any sub-Riemannian structure, the geodesics are projection of the solution to a Hamiltonian system defined on $T^*J^k$, called the geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$.

This paper has three main goals, the following theorem is the first.

Theorem A. The sub-Riemannian geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$ is integrable.

The bijection between geodesics on $J^k(\mathbb{R}, \mathbb{R})$ and the pairs $(F, I)$ will be generalized, module translation $F(x) \rightarrow F(x - x_0)$, where $F(x)$ is a polynomial of degree $k$ or less and $I$ is a closed interval associated to $F(x)$, made by Monroy-Perez and Anzaldo-Meneses [2, 3, 4], also described in [8] (see pg. 4). In the present paper it will be a bijection between the geodesic

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in $J^k(\mathbb{R},\mathbb{R}^n)$ and the pairs $(F,I)$, module translation $F(x) \to F(x-x_0)$, where $F(x) = (F^1(x), \cdots, F^n(x))$ is a polynomial vector of degree $k$ or less and $I$ is a closed interval associated to $F(x)$, see Definition 3.1 for more detail of $I$.

In Section 3 it will be described how to build a geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ given a pair $(F,I)$ and prove the following main result.

**Theorem B.** The prescription described in Section 3 yields a geodesic in $J(\mathbb{R},\mathbb{R}^n)$ parameterized by arclength. Conversely, any arc-length parameterized geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ can be achieved by this prescription applied to some polynomial vector $F(x)$ of degree $k$ or less.

$J^k(\mathbb{R},\mathbb{R}^n)$ comes with a projection $\Pi : J^k(\mathbb{R},\mathbb{R}^n) \to \mathbb{R}^{n+1}$ onto the Euclidean plane, which projects the frame $X_1, \cdots, X_{n+1}$ onto the standard coordinate frame $\{\partial/\partial x, \partial/\partial \theta_1, \cdots, \partial/\partial \theta_n\}$ of $\mathbb{R}^{n+1}$, see Section 2 for the meaning of the coordinates.

Using Theorem B the geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ will be classified into two main families: line-geodesics and non-line-geodesics: We say that a geodesic $\gamma(t)$ is a line-geodesics if $\gamma(t)$ corresponds to a constant polynomial vector and its projection to $\mathbb{R}^{n+1}$ is a line. We say that a geodesic $\gamma(t)$ is a non-line-geodesic if $\gamma(t)$ corresponds to a non-constant polynomial and its Hill interval is compact. Moreover, if $I = [x_0, x_1]$, we say that a non-line-geodesic $\gamma(t)$ is $x$-periodic (or regular), if $x_0$ and $x_1$ are regular points of $||F(x)||^2$, that is, exist $L(F,I)$ such that $x(t+L(F,I)) = x(t)$. While, $\gamma(t)$ is critical if one point or both are critical points of $||F(x)||^2$; in this case the $x$-coordinate has an asymptotic behavior to the critical point and then the $x$-coordinate has an infinite period.

The third main result is the answer to a question by Enrico Le Donne: Does $J^k(\mathbb{R},\mathbb{R}^n)$ have periodic geodesics?

**Theorem C.** $J^k(\mathbb{R},\mathbb{R}^n)$ does not have periodic normal geodesics.

Following this classification, the only candidates to be periodic are $x$-periodic geodesics; so the focus is on non-constant vectors correspondig to $x$-periodic geodesics.

Remark 1: Viewing $J^k(\mathbb{R},\mathbb{R}^n)$ as a Carnot group, Theorem C is a particular case of the conjecture made by Enrico Le Donne.

**Conjecture 1.** Carnot groups do not have periodic geodesics.

Remark 2: In control theory a “chained normal form” is a control system that is locally diffeomorphic to the canonical distribution for $J^k(\mathbb{R},\mathbb{R}^n)$, see [15].

1.1. **Outline of paper.** The outline of the paper is as follows. In Section 2 the $k$-th jet space $J^k(\mathbb{R},\mathbb{R}^n)$ is presented as a subRiemannian manifold, as well as, the notation that will be followed throughout the work. The subRiemannian geodesic flow is defined and the proof of Theorem A is given.
Finally, the Carnot structure of $J^k(\mathbb{R}, \mathbb{R}^n)$ is presented. In Section 3, the prescription for constructing geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$ given the pair $(F, I)$ is described, the Hamilton equation are computed and Theorem B is proved. In Section 4, the proof of Theorem C is given.

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2. $J^k(\mathbb{R}, \mathbb{R}^n)$ as a subRiemannian manifold

The $k$-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}^n$ at a point $x_0 \in \mathbb{R}$ is its $k$-th order Taylor expansion at $x_0$. We will encode this $k$-jet as a $(k+2)$-tuple of real numbers as follows:

$$(j^k f) = (x_0, f^k(x_0), \cdots, f^1(x_0), f(x_0)) \in \mathbb{R}^{n(k+1)+1}.$$  

As $f$ varies over smooth functions and $x_0$ over $\mathbb{R}$, these $k$-jets sweep out the $k$-jet space. $J^k(\mathbb{R}, \mathbb{R}^n)$ is diffeomorphic to $\mathbb{R}^{n(k+1)+1}$ and we will use the global coordinates

$$(x, u_k, \cdots, u_1, u_0) \in \mathbb{R}^{n(k+1)+1}.$$  

Where, $u_i = (u_i^1, \cdots, u_i^n)$ and, if $f = u_0$, then $u_1 = du_0/dx$, and more general, $u_{i+1} = du_i/dx$, $j \geq 1$. These equations are rewritten into $du_0 = u_1 dx$, and in general, $du_i = u_{i+1} dx$, we see that $J^k(\mathbb{R}, \mathbb{R}^n)$ is endowed with a natural rank $(n+1)$ distribution $D \subset T J^k$ characterized by the $nk$ Pfaffian equations

$$0 = du_0 - u_1 dx$$
$$0 = du_1 - u_2 dx$$
$$\vdots = \vdots$$
$$0 = du_k - u_{k-1} dx.$$  

$J^k(\mathbb{R}, \mathbb{R}^n)$ has a natural definition using the coordinates $u_i$, but they do not reflect the symmetries of the dynamics, see the proof of Theorem A in Section 3. We will introduce the alternate coordinates $\theta_i$ for $J^k(\mathbb{R}, \mathbb{R}^n)$ describes in [2, 3] and also introduced in [8, 9], they are exponential coordinates of the second type, see [6] Section 6.2.;

$$\theta_0 = u_k$$
$$\theta_1 = xu_k - u_{k-1}$$
$$\vdots = \vdots$$
$$\theta_k = \frac{x^k}{k!} u_k - \frac{x^k}{k!} u_{k-1} dx + \cdots + (-1)^k u_0.$$
\( D \) is globally framed by \((n + 1)\) vector fields:

\[
X_0 = \frac{\partial}{\partial x}, \quad X_j^0 = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i^j} \quad \text{for } 1 \leq j \leq n.
\]

A subRiemannian structure on \( J^k(\mathbb{R}, \mathbb{R}^n) \) is defined by declaring these \((n + 1)\) vector fields to be orthonormal. In these coordinates the subRiemannian metric is defined by restricting \( ds^2 = dx^2 + (d\theta_1^0)^2 + \cdots + (d\theta_k^0)^2 \) to \( D \).

During this work we will use the convention \( \theta_i^j \), where \( i = 0, \cdots, k \) and \( j = 1, \cdots, n \), that is, \( i \) is used to denotes the vector \( \theta_i \) and \( j \) denote the \( j \)-th entry of the vector \( \theta_i \).

2.1. Hamiltonian. Let \((p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)\) be the traditional co-ordinates for the cotangent bundle \( T^*J^k \), or abbreviated as \((p, q)\). Also, let \( P_{X_0}, P_{X_1^0}, \cdots, P_{X_k^0} : T^*J^k \rightarrow \mathbb{R} \) be the momentum functions of the vector fields \( X_0, X_1^0, \cdots, X_k^0 \), in the coordinates \((p, q)\); the momentum functions are given by

\[
P_{X_0} = p_x, \quad P_{X_j^0} = \sum_{i=0}^{k} \frac{x^i}{i!} p_{\theta_i^j} \quad \text{for } 0 \leq j \leq k.
\]

Then the Hamiltonian governing the subRiemannian geodesic flow on \( J^k(\mathbb{R}, \mathbb{R}^n) \) is

\[
H = \frac{1}{2}(P_{X_0}^2 + P_{X_1^0}^2 + \cdots + P_{X_k^0}^2)
\]

(see [13], pg 8). We will see in Section 3 that the condition \( H = 1/2 \) implies that the geodesics are parameterized by arc-length.

2.2. Proof of Theorem [A]

**Proof.** The Hamiltonian \( H \) does not depend on the coordinate \( \theta_i^j \) because the Hamilton equations \( p_{\theta_i^j} \) is a constant of motion. Then \( \{H, p_{\theta_i^j}\} \) is a set of \( n(k + 1) + 1 \) constants of motion that Poisson commute and they are linearly independent. \( \square \)

2.3. Carnot Group structure. The frame \( \{X_0, X_1^0, \cdots, X_k^0\} \) generates \((n(k + 1) + 1)\)-dimensional nilpotent Lie algebra, under the iterated bracket. That is,

\[
X_1^i = [X_0, X_0^1], \quad \cdots, \quad X_j^i = [X_0, X_k^{i-1}], \quad \cdots = 0 = [X_0, X_j^k],
\]

all the other Lie brackets \( [X_i, X_j^l] \) are zero. Then the frame \( \{X_0, X_j^i\} \) with \( 0 \leq i \leq k \) and \( 1 \leq j \leq n \) forms a \( n(k + 1) + 1 \)-dimensional graded nilpotent Lie algebra:

\[
\mathfrak{g}_k = V_1 \oplus \cdots \oplus V_{k+1}, \quad V_1 = \{X_0, X_0^1\}, \quad V_i = \{X_i^{i-1}\}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n.
\]

Like any graded nilpotent Lie algebra, this algebra has an associated Lie group which is a Carnot group \( G \) w.r.t the subRiemannian structure. We
can identify $G$ with $J^k(\mathbb{R}, \mathbb{R}^n)$, using the flows of \{\(X_0, X_j^i\}\}. For more detail on the jets space as a Carnot group see [7].

3. Geodesic in \(J^k(\mathbb{R}, \mathbb{R}^n)\)

This Section describes how to build a geodesic on \(J^k(\mathbb{R}, \mathbb{R}^n)\): Let us formalize the definition of the interval \(I\).

**Definition 3.1.** We say that a closed interval \(I\) is a Hill interval, associated to \(F(x)\), if \(F^2(x) < 1\) for all \(x\) in the interior of \(I\) and \(G^2(x) = 1\) for \(x\) in the boundary of \(I\). Then, \(i\) is compact if and only if \(F(x)\) is not a constant polynomial, if \(I\) is in the form \([x_0, x_1]\), \(x_0\) and \(x_1\) are called endpoints of the Hill interval.

Consider the Hamiltonian system of one degree of freedom defined on the plane phase space \((p_x, x)\) and with potential \(1/2||F(x)||^2\), in other words, a Hamiltonian function given by

\[
H_F(p_x, x) = \frac{1}{2} p_x^2 + \frac{1}{2} ||F(x)||^2;
\]

then, the Hamilton equations are give by

\[
\dot{x} = p_x, \quad \dot{p}_x = (\frac{dF}{dx}, F(x)),
\]

where \(dF/dx\) is the derivative of the polynomial vector and \((, )\) is the Euclidean dot product on \(\mathbb{R}^n\). Since the Hamiltonian is autonomous, we choose \(H_F = 1/2\); then the dynamic takes place in the point where \(||F(x)||^2 \leq 1\). If \(F(x)\) is not the constant polynomial vector, and \(I = [x_0, x_1]\) is the Hill interval, then \(\dot{x} = 0\) if and only if \(x = x_0, x_1\). Moreover, \(x_0\) and \(x_1\) are equilibrium points, if and only if, \(x_0\) and \(x_1\) are critical points of \(||F(x)||^2\), in other words, \(0 = (dF/dx, F(x))\).

Having found the solution \(x(t)\), next we solve

\[
\dot{\theta}^j_0(t) = F^j(x(t)),
\]

for \(\theta^i_0\). Then, \(c(t) = (x(t), \theta_0(t))\) is a curve on \(\mathbb{R}^{n+1}\) parameterized by arc-length. Finally, we solve the horizontal lift equation associated to the curve \(c(t)\)

\[
\dot{\theta}^j_1 = x(t)F^j(x(t)),
\]

\[
\dot{\theta}^j_2 = \frac{x^2(t)}{2!}F^j(x(t)),
\]

\[
\vdots = \dot{\theta}^j_k = \frac{x^k(t)}{k!}F^j(x(t)).
\]
3.1. **Hamilton equations.** To prove Theorem B, we need to write down the Hamilton equations for the geodesic flow. Since the Hamiltonian function is a left invariant function on the cotangent bundle of the Lie group \( G \), the 'Lie-Poisson bracket' structure can be used for such Hamiltonian flows to find the equations, see Appendix [5] or chapter 4 [12]. That is, if \( X \) and \( Y \) are left invariant vector fields then

\[
\{P_X, P_Y\} = -P_{[X,Y]}.
\]

In this context, the Hamilton equations are read as \( \dot{f} = \{f, H\} \). With the Hamiltonian of this system, they expand to

\[
\dot{f} = \{f, P_0\} P_0 + \{f, P_{X_0^1}\} P_{X_0^1} + \cdots \{f, P_{X_0^n}\} P_{X_0^n}.
\]

Using \( \{P_0, P_{X_0^i}\} = -P_{X_0^i} \), we see that \( P_0 \) and \( P_{X_0^i} \) evolves according to the equations

\[
\dot{P}_0 = -P_{X_0^1} P_{X_0^1} - \cdots - P_{X_0^n} P_{X_0^n} \quad \text{and} \quad \dot{P}_{X_0^i} = P_0 P_{X_0^i}
\]

for \( 1 \leq i \leq n \). We also compute the Hamilton equations for the coordinates \( (x, \theta_0, \ldots, \theta_n) \),

\[
\dot{x} = P_0 \quad \dot{\theta}_i^j = \frac{\partial}{\partial \theta_i^j} P_{X_0^j} \text{ for } 0 \leq i \leq k \text{ and } 1 \leq j \leq n.
\]

3.2. **Proof of Theorem B.**

**Proof.** Let \( \gamma(t) \) be a curve corresponding to the pair \((F, I)\), that is, the coordinates \( x, \theta_0, \theta_j^i \) are solutions to the equations (3.1), (3.3) and (3.4), we will associate to \( \gamma(t) \) some momentum functions and show that they hold equations (3.6) and (3.7), respectively.

Let \( (p_x(t), x(t)) \) be the solution to the equation (3.2) with \( x(t) \) laying in the \( I \), comparing with the geodesic equation from (3.8), we define \( P_0 := p_x \). In the same way, comparing the equations (3.2) and (3.4) with the Hamilton equations (3.8) and (3.7) for \( \theta_0^j \) and \( \theta_i^j \), we define \( P_{X_0^j}(t) := F^j(x(t)) \) and \( P_{X_1^i}(t) := \frac{d}{dx} F^j(x(t)) \). Then using the change rule we have

\[
\dot{P}_{X_0^i}(t) = \frac{d}{dt} F^j(x(t)) = \frac{dF^j}{dx} x = P_{X_1^i} P_0,
\]
which is the equation (3.6). In the same way

\[ \dot{P}_{X_i}(t) = \frac{d}{dt} \frac{d^i F_j}{dx^j}(x(t)) = \frac{d^{i+1} F_j}{dx^{i+1}} \cdot \dot{x} = P_{X_i}P_0. \]

Since \( F^i(x) \) is a polynomial of degree \( k \) or less, we obtain \( \dot{P}_{X_i}(t) = 0 \) for all \( j = 1, \ldots, n \), and the equation (3.9) is the same as equation (3.7).

Conversely, let \( \gamma(t) \) be a geodesic parameterized by arc-length with the initial condition \( \gamma(0) \), that is, \( \gamma(t) \) is the projection to the solution \((p(t), \gamma(t))\) of the Hamiltonian function (2.3), we will show that the coordinates \( x, \theta_0^i, \theta_i^j \) of the geodesic \( \gamma(t) \) hold the equations (3.11), (3.28) and (3.31), respectively.

Using the bijection between the Hamilton equations \( p_{\theta_i^j} \) is constant, if \( a_i^j := i!p_{\theta_i^j} \) and \( F^j(x) := a_i^0 + a_i^1 \cdot x + \cdots + a_i^n \cdot x^n \) for all \( 1 \leq j \leq n \), then, using these expressions and \( x_p = P_{X_0} \), the Hamiltonian function (2.3) became

\[ H = \frac{1}{2}(P_{X_0}^2 + P_{X_1}^2 + \cdots + P_{X_n}^2) = \frac{1}{2}(p_x^2 + ||F(x)||^2) = H_F. \]

Thus the \( x \)-coordinate of the geodesic \( \gamma(t) \) is a solution to the Hamiltonian system of one degree of freedom with potential \( \frac{1}{2} ||F(x)|| \), defined by equation (3.1), where the initial condition \( x(0) \) lays in a Hill interval \( I \) and, so does \( x(t) \). In the same way, using the solution \( x(t) \) and the Hamilton equation for \( \theta_0^i \), that is, \( \dot{\theta}_0^i = \partial H / \partial (p_{\theta_0^i}) = F^j(x(t)) \), thus the \( \theta_0^i \)-coordinate of the geodesic \( \gamma(t) \) is a solution to equation (2.3). Finally, the Hamilton equation for \( \theta_i^j \), that is, \( \dot{\theta}_i^j = \partial H / \partial (p_{\theta_i^j}) = \frac{1}{2} F^j(x(t)) \) is equivalent to the horizontal equation (3.1). Thus, \( \gamma(t) \) is a geodesic corresponding to the pair \((F, I)\).

### 3.3 Geodesics Classification in \( J^k(\mathbb{R}, \mathbb{R}^n) \)

Using the bijection between geodesics in \( J^k(\mathbb{R}, \mathbb{R}^n) \) and the pair \((F, I)\), the geodesics are classified. Let \( \gamma(t) \) be a geodesic corresponding to \((F, I)\), as said before the first dichotomy is if the projected curve \( \pi(\gamma(t)) = c(t) \) is a line or not.

- We say that \( \gamma(t) \) is a line-geodesic if \( F(x) \) is the constant polynomial vector, since equation (3.3) implies that the curve \( c(t) = (x(t), \theta_0(t)) \) in \( \mathbb{R}^{n+1} \) is a line.
- We say that \( \gamma(t) \) is a non-line-geodesic if \( F(x) \) is not the constant polynomial vector with Hill interval \( I = [x_0, x_1] \), since equation (3.2) implies that the \( x \)-dynamics takes place in \( I \) and curve \( c(t) = (x(t), \theta_0(t)) \) in \( \mathbb{R}^{n+1} \) is not a line.

Let \( \gamma(t) \) be a non-line-geodesic corresponding to \((F, I)\), where \( I = [x_0, x_1] \), the second dichotomy refers to the qualitative behavior of the \( x(t) \) dynamic.

- We say that \( \gamma(t) \) is \( x \)-periodic or regular, that is, exist \( L(F, I) \) such that \( x(t + L(F, I)) = x(t) \), if \( x_0 \) and \( x_1 \) are regular points of the potential \( \frac{1}{2} ||F(x)||^2 \), if and only if, \( x_0 \) and \( x_1 \) are simple roots of \( 1 - ||F(x)||^2 \), if and only if, \( 1 - ||F(x)||^2 = (x - x_0)(x_1 - x)q(x) \), where \( q(x) \) is not zero if \( x \) is in \( I \).
We say that $\gamma(t)$ is critical, if one or both endpoints $x_0$ and $x_1$ are critical points of the potential $1/2||F(x)||^2$, if and only if, one or both endpoints $x_0$ and $x_1$ are not simple roots of $1-||F(x)||^2$. Then, by equation (3.11), the critical points are equilibrium points of a one degree of freedom system, and the solution $x(t)$ has an asymptotic behavior to the critical points.

3.3.1. Periods. $x$-periodic geodesics have the property that the change undergone by the coordinates $\theta_{ij}$ after one $x$-period $L(F,I)$ is finite and does not depend on the initial point. This is summarized in the following proposition.

**Proposition 3.1.** Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ in $J^k(\mathbb{R}, \mathbb{R}^n)$ be an $x$-periodic geodesic corresponding to the pair $(F,I)$. Then the $x$-period is

$$L(F,I) = 2\int_I \frac{dx}{\sqrt{1-||F(x)||^2}},$$

and is twice the time it takes for the $x$-curve to cross its Hill interval exactly once. After one period, the changes $\Delta \theta^i_j := \theta^i_j(t_0 + L) - \theta^i_j(t_0)$ for $i = 0,1,\ldots,k$ and $j = 1,\cdots,n$ undergone by $\theta^i_j$ are given by

$$\Delta \theta^i_j(F,I) = \frac{2}{i!} \int_I \frac{x^iF^j(x)dx}{\sqrt{1-||F(x)||^2}}.$$ 

The proof of this Proposition is equivalent to the proofs of Proposition 4.1 from [8] (pg. 13) or Proposition 2.1 from [9] (pg. 2). In [8] an argument of classical mechanics was used, see [11] pg. 25 equation (11.5); while, in [9], a generating function to find action-angle coordinates for Hamiltonian systems was constructed, see [5] Section 50.

Then a $x$-periodic geodesic $\gamma(t)$ corresponding to the pair $(F,I)$ is periodic if and only if $\Delta \theta^i_j(F,I) = 0$ for all for $i = 0,1,\ldots,k$ and $j = 1,\cdots,n$.

4. Proof of Theorem C

Because that period $L(F,I)$ in equation (3.10) is finite, we can define an inner product in the space of polynomials of degree $k$ or less as follows

$$(4.1) \quad <P_1(x), P_2(x)>_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1-F^2(x)}}.$$ 

This inner product is not degenerated and will be the key to the proof of Theorem C.

4.1. Proof of Theorem C

**Proof.** It will be proceeded by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on $J^k(\mathbb{R}, \mathbb{R}^n)$ corresponding to the pair $(F,I)$, where $F(x)$ is not a constant polynomial vector; then $\Delta \theta^i_j(F,I) = 0$ for all $i = 0,\cdots,k$ and $j = 1,\cdots,n$. 


In the context of the space of polynomials of degree \( k \) or less with inner product \( < , >_F \), the condition \( \Delta \theta^j_i(F,I) = 0 \) for all \( i \) and \( j \) is equivalent to each \( F^j(x) \) being perpendicular to \( x^i \) for all \( i \in 0, 1, \cdots, k \) (0 = \( \Delta \theta^j_i(F,I) = < x^i, F^j(x) >_F \)). But \( \{x^i\} \) is a basis for the space of polynomials of degree \( k \) or less, then each \( F^j(x) \) is perpendicular to any vector, so each \( F^j(x) \) is zero since the inner product is not degenerated. This is a contradiction to the assumption that \( F(x) \) is not a constant polynomial. □

References

[1] A. Agrachev and D. Barilari and U. Boscain, A Comprehensive Introduction to Sub-Riemannian Geometry , Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press , Vol. 181, 2019.

[2] Alfonso Anzaldo-Meneses and Felipe Monroy-Perez Goursat distribution and sub-Riemannian structures, December 2003, Journal of Mathematical Physics

[3] Alfonso Anzaldo-Meneses and Felipe Monroy-Perez, Integrability of nilpotent sub-Riemannian structures, preprint; INRIA; inria-00071749, (2003).

[4] Alfonso Anzaldo-Meneses and Felipe Monroy-Perez, Optimal Control on Nilpotent Lie Groups Journal of Dynamical and Control Systems, October 2002.

[5] Arnold, Vladimir Igorevich, Mathematical methods of classical mechanics, Springer Science, (1988).

[6] Z.M. Balogh and J.T. Tyson and B. Warhurst Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups, Advances in Mathematics, vol. 220, pg 560-619, 2009.

[7] Ben Warhurst, Jet spaces as nonrigid Carnot groups Journal of Lie Theory, Volume 15, 341–356, 2005.

[8] A. Bravo-Doddoli and R. Montgomery, Geodesics in Jet Space Regular and Chaotic Dynamics, Volume 27, 151–182, 2002.

[9] A. Bravo-Doddoli, Non periodic geodesic on the Jet Space https://arxiv.org/abs/2203.16178

[10] A. Bravo-Doddoli, The Higher Euler: Geodesics in Jet Space https://arxiv.org/abs/2003.08022

[11] L. Landau and E. Lifshitz, Mechanics, vol. 1 of a Course of Theoretical Physics, Pergamon Press, [1976].

[12] Marsden, Jerrold E and Ratiu, Tudor S, Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems, Springer Science & Business Media, [2013]

[13] R. Montgomery, A Tour of SubRiemannian Geometry, Mathematical Surveys and Monographs, vol. 91, American Math. Society, Providence, Rhode Island, 2002.

[14] R. Montgomery and M. Zhitomirskii Points and Curves in the Monster Tower, American Mathematical Soc. 2010.

[15] D. Tilbury and O.J. Sordalen and L. Bushnell and S.S. Sastry A multisteering trailer system: conversion into chained form using dynamic feedback, IEEE Transactions on Robotics and Automation, vol. 11, 1995.

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