$U_q[\widehat{sl(2|1)}]$ Vertex Operators, Screen Currents and Correlation Functions at Arbitrary Level

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Abstract

Bosonized $q$-vertex operators related to the 4-dimensional evaluation modules of the quantum affine superalgebra $U_q[\widehat{sl(2|1)}]$ are constructed for arbitrary level $k = \alpha$, where $\alpha \neq 0, -1$ is a complex parameter appearing in the 4-dimensional evaluation representations. They are intertwiners among the level-$\alpha$ highest weight Fock-Wakimoto modules. Screen currents which commute with the action of $U_q[\widehat{sl(2|1)}]$ up to total differences are presented. Integral formulae for $N$-point functions of type I and type II $q$-vertex operators are proposed.

Mathematics Subject Classifications (1991): 17B37, 81R10, 81R50, 16W30
I Introduction

The notion of $q$-vertex operators as certain intertwiners of highest weight modules of quantum affine algebras was introduced by Frenkel and Reshetikhin in their work on the $q$-deformation of the Wess-Zumino-Novikov-Witten model. These $q$-vertex operators give rise to $q$-analogues of the primary fields in conformal field theory.

Similar to the classical case, $q$-vertex operators are characterized by the intertwining property defined from the relevant quantum affine algebras. However, it is non-trivial to obtain explicit expressions of them. A powerful tool for constructing such explicit formulae is the bosonization technique, initiated by Wakimoto in the theory of affine Lie algebras. This method enables one in principle to determine $q$-vertex operators in terms of certain free bosonic fields. So far, level-one bosonized $q$-vertex operators have been constructed for most quantum affine algebras and the type I quantum affine superalgebras $U_q[\widehat{sl}(M|N)]$, $M \neq N$ and $U_q[\widehat{gl}(N|N)]$. In the case of arbitrary level, bosonized formulae have been known only for the type I $q$-vertex operators of $U_q[\widehat{sl}(2)]$ and $U_q[\widehat{sl}(N)]$.

One of the central issues in conformal field theory and massive integrable models is the computation of correlation functions, which are matrix elements of certain products of vertex operators. The explicit bosonized expressions of vertex operators play an essential role. They enable one to compute correlators exactly in the form of integral representations. This was demonstrated by the Kyoto group and collaborators in their ground-breaking work on the diagonalization of the $XXZ$ spin chain. In certain correlation functions of other quantum affine (super)algebras at level one were computed via the bosonization procedure, generalizing the work of the Kyoto group and collaborators.

The case of arbitrary level is more complicated. Due to the existence of nontrivial background charges, the naive solutions to the intertwining relations in terms of free bosonic fields do not give rise to proper bosonizations of the $q$-vertex operators, which ensure the nonvanishing of correlation functions. As in conformal field theory, $q$-screen currents which balance the background charges are generally needed. $q$-screen currents are dimension 1 operators which (anti-)commute with the relevant quantum algebra generators up to total differences. Bosonized $q$-screen currents have been obtained for $U_q[\widehat{sl}(N)]$ and $U_q[\widehat{sl}(2)]$ vertex operators.

In this paper, by using the free field realization of $U_q[\widehat{sl}(2)]$ at arbitrary level $k \neq$
0, −1 [20] we investigate the bosonization of $q$-vertex operators related to the 4-dimensional evaluation modules of $U_{q}[\hat{sl}(2|1)]$. It is worth mentioning that our 4-dimensional representation contains an extra complex parameter $\alpha \neq 0, -1$. For arbitrary level $k = \alpha$, the $q$-vertex operators are mappings of certain highest weight Fock-Wakimoto modules in a bosonic Fock space. Screen currents which (anti-)commute with the action of $U_{q}[\hat{sl}(2|1)]$ are obtained and bosonized $q$-vertex operators dressed with the screen charges are proposed. This provides a natural way to write down an integral representation for correlation functions of the bosonized $q$-vertex operators.

The results obtained in this paper will be useful in analysing the supersymmetric integrable model introduced in [21]. This is a quantum spin chain model arising from the R-matrix for the 4-dimensional $U_{q}[\hat{sl}(2|1)]$ evaluation representation and can be interpreted as a model describing strongly correlated electrons.

II Prelimilaries

II.1 Quantum affine superalgebra $U_{q}[\hat{sl}(2|1)]$

The simple roots of the affine superalgebra $sl(2|1)$ [22] are

$$\alpha_0 = \delta - \varepsilon_1 + \delta_1, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \delta_1,$$

where $\delta$ is the null root and $\{\varepsilon_1, \varepsilon_2, \delta_1\}$ are orthonormal basis satisfying

$$\langle \delta, \delta \rangle = \langle \delta, \varepsilon_i \rangle = \langle \delta_1, \varepsilon_i \rangle = 0, \quad i = 1, 2,$$

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}, \quad \langle \delta_1, \delta_1 \rangle = -1.$$

The fundamental weights are

$$\Lambda_0, \quad \Lambda_1 = \Lambda_0 - \varepsilon_2 + \delta_1, \quad \Lambda_2 = \Lambda_0 - \varepsilon_1 - \varepsilon_2 + 2\delta_1,$$

where $\Lambda_0$ is the affine weight obeying $\langle \Lambda_0, \Lambda_0 \rangle = \langle \Lambda_0, \varepsilon_i \rangle = 0, \quad i = 1, 2$ and $\langle \Lambda_0, \delta \rangle = 1$.

The symmetric Cartan matrix $(a_{ij})$ of the affine Lie superalgebra $sl(2|1)$ has elements $a_{ij} = (\alpha_i, \alpha_j), \ i, j = 0, 1, 2$. Explicitly,

$$(a_{ij}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Quantum affine superalgebra $U_{q}[\hat{sl}(2|1)]$ is a $q$-analogue of the universal enveloping algebra of $sl(2|1)$ generated by the Chevalley generators $\{\varepsilon_i, f_i, q^{h_i}, d| i = 0, 1, 2\}$, where $d$
is the usual derivation operator. The \( Z_2 \)-grading of the generators are \([e_0] = [f_0] = [e_2] = [f_2] = 1 \) and zero otherwise. The defining relations are

\[
\begin{align*}
    h_i h_j &= h_j h_i, \quad h_i d = d h_i, \quad [d, e_i] = \delta_{i,0} e_i, \quad [d, f_i] = -\delta_{i,0} f_i, \\
    q^{b_i} e_j q^{-b_i} &= q^{a_{ij}} e_j, \quad q^{b_i} f_j q^{-b_i} = q^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{q^{b_i} - q^{-b_i}}{q - q^{-1}}, \\
    [e_i, e_j] &= [f_i, f_j] = 0, \quad \text{for } a_{ij} = 0, \\
    [e_1, [e_1, e_1] q^{-1}] &= 0, \quad [f_1, [f_1, f_1] q^{-1}] = 0, \quad l = 0,2. \quad (\text{II.1})
\end{align*}
\]

Here and throughout, \([X, Y]_\epsilon = X Y - (-1)^{|X||Y|} X Y\) and \([X, Y] = [X, Y]_1\).

\( U_q[sl(2|1)] \) is a quasi-triangular Hopf superalgebra endowed with the \( Z_2 \)-graded Hopf algebra structure:

\[
\begin{align*}
    \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i, \quad \Delta(d) = d \otimes 1 + 1 \otimes d, \\
    \Delta(e_i) &= e_i \otimes 1 + q^{b_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-b_i} + 1 \otimes f_i, \\
    \epsilon(h_i) &= \epsilon(d) = \epsilon(e_i) = \epsilon(f_i) = 0, \\
    S(e_i) &= -q^{-b_i} e_i, \quad S(f_i) = -f_i q^{b_i}, \quad S(h_i) = -h_i, \quad S(d) = -d. \quad (\text{II.2})
\end{align*}
\]

Note the antipode \( S \) is a \( Z_2 \)-graded algebra anti-automorphism. Namely for homogeneous elements \( a, b \in U_q[sl(2|1)] \), \( S(a b) = (-1)^{|a||b|} S(b) S(a) \). The multiplication rule for the tensor product is \( Z_2 \)-graded and is defined for homogeneous elements \( a, b, a', b' \in U_q[sl(2|1)] \) by \((a \otimes b)(a' \otimes b') = (-1)^{|b'||a'|}(a a' \otimes b b')\), which extends to inhomogeneous elements through linearity.

\( U_q[sl(2|1)] \) can also be realized by the Drinfeld generators \( [23] \) \( \{X^{\pm,i}, h^i_n, q^{b_i}, c, d| i = 1,2, m \in Z, n \in Z_{\neq 0}\} \). The \( Z_2 \)-grading of the Drinfeld generators are \([X^{\pm,2}] = 1 \) (\( m \in Z \)) and zero otherwise. The relations read \( [24], [25] \)

\[
\begin{align*}
    c &: \text{ central element,} \\
    [h^j_0, h^j_m] &= 0, \quad [d, h^j_0] = 0, \quad [d, h^j_m] = m h^j_m, \\
    [h^i_m, h^j_n] &= \delta_{m+n,0} \frac{[a_{ij}] q^{\frac{nc}{2}}}{m}, \\
    q^{b_i} X^\pm_m q^{-b_i} &= q^{\pm a_{ij}} X^\pm_m, \quad [d, X^\pm_m] = m X^\pm_m, \\
    [h^i_m, X^\pm_n] &= \pm \frac{[a_{ij}] q^{\frac{k|m|c}{2}}} {m} X^\pm_{m+n}, \\
    [X^+_m, X^-_n] &= \frac{\delta_{ij}}{q - q^{-1}} (q^{-(m-n)c/2} \psi^+_m \psi^-_n - q^{-(m-n)c/2} \psi^-_m \psi^+_n), \\
    [X^\pm_1, X^\pm_2] &= 0, \\
    [X^\pm_i, X^\pm_j] &= \frac{\delta_{ij}}{q - q^{-1}} (q^{-(m-n)c/2} \psi^+_m \psi^-_n - q^{-(m-n)c/2} \psi^-_m \psi^+_n), \\
    [X^\pm_1, [X^\pm, X^\pm] q^{-1}] &= (n_1 \leftrightarrow n_2) = 0, \quad (\text{II.3})
\end{align*}
\]
where \([m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}\) and \(\psi^\pm_n, i\) are defined by

\[
\sum_{n \in \mathbb{Z}} \psi^\pm_n z^{-n} = q^{\pm h_b^0} \exp \left( \pm (q - q^{-1}) \sum_{n > 0} h^i_{\pm n} z^n \right).
\]

The Chevalley generators are related to the Drinfeld generators by the formulae:

\[
\begin{aligned}
h_i &= h^0_i, & e_i &= X^+_0, & h_0 &= c - h^0_1 - h^0_2, & f_i &= X^-_0, & i &= 1, 2, \\
e_0 &= -[X^+_0, X^-_1]_{q^{-1}} q^{-h^1_0 - h^2_0}, & f_0 &= q^{h^1_0 + h^2_0} [X^+_{-1}, X^+_0]_q.
\end{aligned}
\]  \hspace{1cm} \text{(II.4)}

\section*{II.2 Bosonization of \(U_q[sl(2|1)]\) at arbitrary level \(k\)}

In this subsection we briefly recall the free boson realization of \(U_q[sl(2|1)]\) at arbitrary level \(k\) \cite{Zhang et al: Vertex Operators, Screen Currents and Correlation Functions}. Let us introduce the bosonic \(q\)-oscillators \(\{a^1_n, a^2_n, b^ij_n, c_n, Q_a^1, Q_a^2, Q_b^j, Q_c^k\mid n \in \mathbb{Z}, 1 \leq i < j \leq 3\}\) which satisfy the commutation relations

\[
\begin{aligned}
[a^i_m, a^j_n] &= \delta_{m+n, 0} \frac{[a^i_m][a^j_n]}{m}, & \quad [a^i_0, Q_a^j] &= (k + 1) a^i, \\
[b^i_m, b^j_n] &= (-1)^{ij} \delta_{ij} \delta_{m+n, 0} \frac{[m]_q^2}{m}, & \quad [b^i_0, Q_w^j] &= (-1)^{ij} \delta_{ij} \delta_{m+n, 0}, \\
[c_m, c_n] &= \delta_{m+n, 0} \frac{[m]_q^2}{m}, & \quad [c_0, Q_c] &= 1.
\end{aligned}
\]  \hspace{1cm} \text{(II.5)}

The remaining commutators vanish. Here and throughout \(k \neq 0, -1\) is a complex parameter. For any pair \((a_n, Q_a)\), we define

\[
\begin{aligned}
a(z; \kappa) &= -\sum_{n \neq 0} \frac{a_n}{[n]_q} q^{-|n||z|^{-n}} + Q_a + a_0 \ln z, \\
a^{\pm}(z) &= \pm (q - q^{-1}) \sum_{n > 0} a_{\pm n} z^{-n} \pm a_0 \ln q.
\end{aligned}
\]  \hspace{1cm} \text{(II.6)}

We have

\textbf{Theorem 1} \cite{Zhang et al: Vertex Operators, Screen Currents and Correlation Functions}: Define the fields \(X^{\pm, i}(z)\) by

\[
X^{\pm, i}(z) = \sum_{n \in \mathbb{Z}} X^{\pm, i}_n z^{-n-1}.
\]

Then at arbitrary level \(k \neq 0, -1\), \(U_q[sl(2|1)]\) is realized by the free boson fields as follows

\[
\begin{aligned}
c &= k, & h^1 &= a^1 + 2b^1 + b^3, & h^2 &= a^2 - b^1 - b^3, \\
h^1_m &= a^1_m q^{-|m|/2} + b^1_m q^{-(k+1)|m|} (q^{2|m|} - q^{-2|m|}) + b^3_m q^{-(k+1)|m|}, \\
h^2_m &= a^2_m q^{-|m|/2} - b^1_m q^{-(k+1)|m|} - b^3_m q^{-(k+1)|m|}, \\
X^{+, 1}(z) &= \frac{1}{(q - q^{-1})z} e^{-b^1_2 z^{-1}} (e^{c(qz; 0)} - e^{-c(qz; 0)}),
\end{aligned}
\]
\[ X^{+,2}(z) = -e^{b_{12}^2(qz) - b_{13}^2(qz) + b_{23}^2(qz;0)} : e^{\sqrt{-1}\pi(c_0 + b_{10}^2 + b_{10}^3 + b_{20}^3)} : e^{b_{12}^2(z;0) + b_{13}^2(z;0) + c(z;0)} ; \]
\[ X^{-,1}(z) = \frac{1}{(q - q^{-1})z} \left( c_{a_1}^2(q^a_{k+1}z) + b_{12}^2(q^a_{k+2}z;1) + b_{13}^2(q^a_{k+2}z) - b_{23}^2(q^{a+1}z) + c(q^{a+1}z;0) \right) ; e^{-\sqrt{-1}\pi(c_0 + b_{10}^2)} \]
\[ X^{-,2}(z) = \frac{1}{(q - q^{-1})z} \left( c_{a_2}^2(q^a_{k+1}z) - b_{23}^2(q^{a+1}z;0) + b_{23}^2(q^{a+1}z;1) ; e^{\sqrt{-1}\pi(b_{10}^3 + b_{20}^3)} \right) . \]

(II.7)

### III Level-zero representations

We discuss level-zero representations of \( U_q[\widehat{sl}(2\mid 1)] \), which are needed in next section for the investigation of \( q \)-vertex operators.

Let \( V_\alpha \) is the one parameter family of the 4-dimensional typical irreducible representation of \( U_q[sl(2\mid 1)] \). Here and throughout, \( \alpha \neq 0, -1 \) is a complex parameter. We choose the basis vectors \( \{v_1, v_2, v_3, v_4\} \) of \( V_\alpha \) and assign them the \( \mathbb{Z}_2 \) gradings \([v_1] = [v_4] = 0, [v_2] = [v_3] = 1\). Let \( e_{ij} \) be the \( 4 \times 4 \) matrices satisfying \((e_{ij})_{kl} = \delta_{ik}\delta_{jl}\). In the homogeneous gradation, the evaluation representation \( V_{\alpha; z} \) of \( U_q[\widehat{sl}(2\mid 1)] \) is given by

\[
\begin{align*}
 e_1 &= e_{23}, \quad f_1 = e_{32}, \quad h_1 = e_{22} - e_{33}, \\
 e_2 &= \sqrt{[\alpha]_q}e_{12} + \sqrt{[\alpha + 1]_q}e_{34}, \\
 f_2 &= \sqrt{[\alpha]_q}e_{21} + \sqrt{[\alpha + 1]_q}e_{43}, \\
 h_2 &= \alpha(e_{11} + e_{22}) + (\alpha + 1)(e_{33} + e_{44}), \\
 e_0 &= -z(-\sqrt{[\alpha]_q}e_{31} + \sqrt{[\alpha + 1]_q}e_{42}), \\
 f_0 &= z^{-1}(-\sqrt{[\alpha]_q}e_{13} + \sqrt{[\alpha + 1]_q}e_{24}), \\
 h_0 &= -\alpha(e_{11} + e_{33}) - (\alpha + 1)(e_{22} + e_{44}).
\end{align*}
\]

(III.1)

We define the dual module \( V_{\alpha; z}^* \) of \( V_{\alpha; z} \) by \( \pi_{V_{\alpha; z}^*}(a) = \pi_{V_{\alpha; z}}((S(a))^*), \forall a \in U_q[sl(2\mid 1)] \), where \( st \) is the supertransposition operation. On \( V_{\alpha; z}^* \), the Chevalley generators are represented by

\[
\begin{align*}
 e_1 &= -q^{-1}e_{32}, \quad f_1 = -qe_{23}, \quad h_1 = -e_{22} + e_{33}, \\
 e_2 &= q^{-\alpha}\sqrt{[\alpha]_q}e_{21} - q^{-\alpha - 1}\sqrt{[\alpha + 1]_q}e_{43}.
\end{align*}
\]
Proposition 1: The Drinfeld generators are represented on $V_{\alpha,z}$ by

\[ f_2 = -q^\alpha \sqrt{[\alpha]_q e_{12}} + q^{\alpha+1} \sqrt{[\alpha + 1]_q e_{34}}, \]
\[ h_2 = -\alpha(e_{11} + e_{22}) - (\alpha + 1)(e_{33} + e_{44}), \]
\[ e_0 = -z(q^\alpha \sqrt{[\alpha]_q e_{13}} + q^{\alpha+1} \sqrt{[\alpha + 1]_q e_{24}}), \]
\[ f_0 = -z^{-1}(q^{-\alpha} \sqrt{[\alpha]_q e_{31}} + q^{-\alpha-1} \sqrt{[\alpha + 1]_q e_{42}}), \]
\[ h_0 = \alpha(e_{11} + e_{33}) + (\alpha + 1)(e_{22} + e_{44}). \] (III.2)

We state

\begin{align*}
  h_0^1 &= e_{22} - e_{33}, & h_0^2 &= \alpha(e_{11} + e_{22}) + (\alpha + 1)(e_{33} + e_{44}), \\
  X_m^{+1} &= (zq^{\alpha+1})^m e_{23}, & X_m^{-1} &= (zq^{\alpha+1})^m e_{32}, \\
  X_m^{+2} &= (zq^{\alpha+1})^m (q^{-m} \sqrt{[\alpha]_q e_{12}} + q^m \sqrt{[\alpha + 1]_q e_{34}}), \\
  X_m^{-2} &= (zq^{\alpha+1})^m (q^{-m} \sqrt{[\alpha]_q e_{21}} + q^m \sqrt{[\alpha + 1]_q e_{43}}), \\
  h_m^1 &= (zq^{\alpha+1})^m [m]_q (q^{-m} e_{22} - q^m e_{33}), \\
  h_m^2 &= \frac{z^m}{m} ([\alpha m]_q (e_{11} + e_{22}) + q^m [(\alpha + 1) m]_q (e_{33} + e_{44})), \tag{III.3}
\end{align*}

and on $V_{\alpha,z}^{*S}$ by

\begin{align*}
  h_0^1 &= -e_{22} + e_{33}, & h_0^2 &= -\alpha(e_{11} + e_{22}) - (\alpha + 1)(e_{33} + e_{44}), \\
  X_m^{+1} &= -z^m q^{-m\alpha - m - 1} e_{32}, & X_m^{-1} &= -z^m q^{-m\alpha - m + 1} e_{23}, \\
  X_m^{+2} &= z^m q^{-(1+m)\alpha} (\sqrt{[\alpha]_q e_{21}} - q^{-2m-1} \sqrt{[\alpha + 1]_q e_{43}}), \\
  X_m^{-2} &= z^m q^{(1-m)\alpha} (-\sqrt{[\alpha]_q e_{12}} + q^{-2m+1} \sqrt{[\alpha + 1]_q e_{34}}), \\
  h_m^1 &= -(zq^{-\alpha-1})^m [m]_q (q^m e_{22} - q^{-m} e_{33}), \\
  h_m^2 &= \frac{z^m}{m} \big( [\alpha m]_q (e_{11} + e_{22}) + q^{-m} [(\alpha + 1) m]_q (e_{33} + e_{44}) \big). \tag{III.4}
\end{align*}

IV  Vertex operators at arbitrary level $k = \alpha$

Let $V(\lambda)$ be a level-$k$ highest weight $U_q[sl(2|1)]$-module with highest weight $\lambda$ and highest weight vector $|\lambda>$. Consider the following intertwiners of $U_q[sl(2|1)]$-modules:

\[ \Phi^\mu_V(z) : V(\lambda) \to V(\mu) \otimes V_{\alpha,z}, \quad \Phi^\mu_{V^*} (z) : V(\lambda) \to V(\mu) \otimes V_{\alpha,z}^{*S}, \]
\[ \Psi^\mu_V(z) : V(\lambda) \to V_{\alpha,z} \otimes V(\mu), \quad \Psi^\mu_{V^*} (z) : V(\lambda) \to V_{\alpha,z}^{*S} \otimes V(\mu). \] (IV.1)

They are intertwiners in the sense that for any $x \in U_q[sl(2|1)]$,

\[ \Theta(z), x = \Delta(x) \cdot \Theta(z), \quad \Theta(z) = \Phi(z), \Phi^*(z), \Psi(z), \Psi^*(z). \] (IV.2)
The intertwiners are even operators, that is their grading is $[\Theta(z)] = 0$. $\Phi(z)$ ($\Phi^*(z)$) is called type I (dual) vertex operator and $\Psi(z)$ ($\Psi^*(z)$) type II (dual) vertex operator.

Expand these vertex operators in terms of their components

$$
\Phi(z) = \sum_{r=1}^{4} \Phi_r(z) \otimes v_r, \quad \Phi^*(z) = \sum_{r=1}^{4} \Phi^*_r(z) \otimes v^*_r, \quad (IV.3)
$$

$$
\Psi(z) = \sum_{r=1}^{4} v_r \otimes \Psi_r(z), \quad \Psi^*(z) = \sum_{r=1}^{4} v^*_r \otimes \Psi^*_r(z), \quad (IV.4)
$$

where $v_r \in V_\alpha$ and $v^*_r \in V^*_\alpha$. Then we have

**Proposition 2**: The operators $\Phi(z)$ and $\Psi(z)$ with respect to $V_{\alpha,z}$ are determined by the components $\Phi_4(z)$ and $\Psi_1(z)$, respectively. More explicitly,

$$
\Phi_3(z) = -\frac{1}{\sqrt{\alpha + 1}}[\Phi_4(z), f_2]_{q^{-\alpha + 1}},
\Phi_2(z) = [\Phi_3(z), f_1]_q, \quad \Phi_1(z) = -\frac{1}{\sqrt{\alpha}}[\Phi_2(z), f_2]_{q^{-\alpha}},
\Psi_2(z) = \frac{1}{\sqrt{\alpha}}[\Psi_1(z), e_2]_{q^{\alpha}}, \quad \Psi_3(z) = [\Psi_2(z), e_1]_q,
\Psi_4(z) = \frac{1}{\sqrt{\alpha + 1}}[\Psi_3(z), e_2]_{q^{\alpha + 1}}. \quad (IV.5)
$$

With respect to $V^*_\alpha$, the operators $\Phi^*(z)$ and $\Psi^*(z)$ are determined by $\Phi^*_1(z)$ and $\Psi^*_1(z)$, respectively:

$$
\Phi^*_3(z) = \frac{q^{-\alpha}}{\sqrt{\alpha}}[\Phi^*_4(z), f_2]_{q^{\alpha}}, \quad \Phi^*_2(z) = -q^{-1}[\Phi^*_2(z), f_1]_q,
\Phi^*_4(z) = -\frac{q^{-\alpha - 1}}{\sqrt{\alpha + 1}}[\Phi^*_3(z), f_2]_{q^{\alpha + 1}},
\Psi^*_3(z) = -q^{\alpha + 1}[\Psi^*_4(z), e_2]_{q^{-\alpha - 1}},
\Psi^*_2(z) = -q[\Psi^*_3(z), e_1]_q, \quad \Psi^*_4(z) = \frac{q^{\alpha}}{\sqrt{\alpha}}[\Psi^*_2(z), e_2]_{q^{-\alpha}}. \quad (IV.6)
$$

Next we determine the relations between the components $\Phi_4(z), \Phi^*_4(z), \Psi_1(z), \Psi^*_1(z)$ and the Drinfeld generators. We have

**Proposition 3**: For $\Phi(z)$ associated with $V_{\alpha,z}$,

$$
[\Phi_4(z), X^+(i)] = 0, \quad i = 1, 2,
q^{h_0} \Phi_4(z) q^{-h_0} = q^{-(\alpha + 1)h_0} \Phi_4(z),
[h^+_{n}, \Phi_4(z)] = -\delta_{2q^{(1+\frac{1}{2})}} n \frac{[\alpha + 1]_q}{n} z^n \Phi_4(z),
[h^-_{n}, \Phi_4(z)] = -\delta_{2q^{-(1+\frac{1}{2})}} n \frac{[\alpha + 1]_q}{n} z^{-n} \Phi_4(z); \quad (IV.7)
$$
for $\Phi^*(z)$ associated with $V^S_{a,z}$,

$$[\Phi^*_1(z), X^{\pm i}(w)] = 0, \quad i = 1, 2,$$

$$q^{b_0^i}\Phi^*_1(z)q^{-b_0^i} = q^{\alpha \delta_{i2}}\Phi^*_1(z),$$

$$[h_n^i, \Phi^*_1(z)] = \delta_{i2}q^{\frac{1}{2}kn}\frac{[\alpha n]_q}{n}z^n\Phi^*_1(z),$$

$$[h_{-n}^i, \Phi^*_1(z)] = \delta_{i2}q^{-\frac{1}{2}kn}\frac{[\alpha n]_q}{n}z^{-n}\Phi^*_1(z); \quad (IV.8)$$

for $\Psi(z)$ associated with $V_{a,z}$,

$$[\Psi_1(z), X^{-i}(w)] = 0, \quad i = 1, 2,$$

$$q^{b_0^i}\Psi_1(z)q^{-b_0^i} = q^{-\alpha \delta_{i2}}\Psi_1(z),$$

$$[h_n^i, \Psi_1(z)] = -\delta_{i2}q^{\frac{1}{2}kn}\frac{[\alpha n]_q}{n}z^n\Psi_1(z),$$

$$[h_{-n}^i, \Psi_1(z)] = -\delta_{i2}q^{-\frac{1}{2}kn}\frac{[\alpha n]_q}{n}z^{-n}\Psi_1(z); \quad (IV.9)$$

and for $\Psi^*(z)$ associated with $V^S_{a,z}$,

$$[\Psi^*_4(z), X^{-i}(w)] = 0, \quad i = 1, 2,$$

$$q^{b_0^i}\Psi^*_4(z)q^{-b_0^i} = q^{(\alpha + 1)\delta_{i2}}\Psi^*_4(z),$$

$$[h_n^i, \Psi^*_4(z)] = \delta_{i2}q^{\frac{1}{2}kn}\frac{[\alpha n]_q}{n}z^n\Psi^*_4(z),$$

$$[h_{-n}^i, \Psi^*_4(z)] = \delta_{i2}q^{-\frac{1}{2}kn}\frac{[\alpha n]_q}{n}z^{-n}\Psi^*_4(z). \quad (IV.10)$$

To obtain bosonized expressions of the intertwining operators, we introduce the combinations of bosonic oscillators for $m \in \mathbb{Z}$,

$$A_m^* = -(a_m^1 + \frac{[2m]_q a_m^2}{[m]_q})q^{\frac{|m|}{2}},$$

$$B_m^* = -(a_m^1 + \frac{[2m]_q a_m^2}{[m]_q})q^{-\frac{|m|}{2}},$$

$$\tilde{B}_m^* = -(a_m^1 + \frac{[2m]_q a_m^2}{[m]_q})q^{-\frac{|m|}{2}} + (b_m^{13} + q^{-|m|}b_m^{23})q^{-\frac{3|m|}{2}},$$

$$\tilde{A}_m^* = -(a_m^1 + \frac{[2m]_q a_m^2}{[m]_q})q^{-\frac{|m|}{2}} - (q^{|m|}b_{m}^{13} + b_{m}^{23})q^{-\frac{3|m|}{2}},$$

$$Q_{A^*} = -Q_{a^1} - 2Q_{a^2}, \quad Q_{B^*} = -\frac{\alpha}{\alpha + 1}(Q_{a^1} + 2Q_{a^2}),$$

$$Q_{\tilde{B}^*} = -Q_{a^1} - 2Q_{a^2} + Q_{b^{13}} + Q_{b^{23}},$$

$$Q_{\tilde{A}^*} = -\frac{\alpha}{\alpha + 1}(Q_{a^1} + 2Q_{a^2}) - Q_{b^{13}} - Q_{b^{23}}. \quad (IV.11)$$

For $k = \alpha$, these operators obey the commutation relations, among others,

$$[A_m^*, h_{n}^i] = \delta_{i2}\delta_{m+n,0}\frac{[m]_q([\alpha + 1]m)_q}{m} = [\tilde{A}_m^*, h_{n}^i],$$

$$[B_m^*, h_{n}^i] = \delta_{i2}\delta_{m+n,0}\frac{[m]_q[\alpha m]_q}{m} = [\tilde{B}_m^*, h_{n}^i]. \quad (IV.12)$$
Then

**Theorem 2**: For \( k = \alpha \), the bosonized forms \( \phi_4(z), \phi_4^*(z), \psi_1(z) \) and \( \psi_1^*(z) \) of the vertex operator components \( \Phi_4(z), \Phi_4^*(z), \Psi_1(z) \) and \( \Psi_4^*(z) \) are given by

\[
\begin{align*}
\phi_4(z) & = : e^{-A^*(q^{a+1}; z; -\frac{\phi}{2})} :, \\
\phi_4^*(z) & = : e^{B^*(q^a z; -\frac{\phi}{2})} :, \\
\psi_1(z) & = : e^{-\hat{A}^*(q^{a+1}; z; -\frac{\phi}{2})} :, \\
\psi_1^*(z) & = : e^{-\hat{A}^*(q^{a-1}; z; \frac{\phi}{2})} : e^{\sqrt{-\pi}(b_{13}^2 + b_{23}^2)}. 
\end{align*}
\]

The other components \( \phi_r(z), \phi_r^*(z), \psi_r(z) \) and \( \psi_r^*(z) \) are represented by multiple contour integrals of the Drinfeld currents (c.f. proposition 2).

Vertex operators (IV.13) are referred to as “elementary \( q \)-vertex operators” and are determined solely from their commutation relations with the bosonized \( U_q[\widehat{sl(2)}] \) generators. The construction is completely independent of which infinite dimensional modules the vertex operators intertwine. In next section, we shall clarify on which space these bosonized vertex operators act.

**V Fock space and Fock-Wakimoto modules**

In this section we study bosonic Fock space on which the \( U_q[\widehat{sl(2)}] \) generators and the bosonized vertex operators act. As we will see, all highest weight modules of \( U_q[\widehat{sl(2)}] \) can be embedded in the bosonic Fock space. Note that \( k = \alpha \neq 0, -1 \).

Let \( |0> \) be the vacuum vector, which is defined by \( a_n^i|0> = b_n^{12}|0> = b_n^{13}|0> = b_n^{23}|0> = c_n|0> = 0 \) for \( n \geq 0 \). Introduce the vector

\[
|\lambda_1^1, \lambda_2^1, \lambda_{b_{12}}, \lambda_{b_{13}}, \lambda_{b_{23}}, \lambda_c >
\]

which carries the weight \( (\frac{\lambda_1^1}{1+\Gamma}, \frac{2\lambda_2^1}{1+\Gamma}, \lambda_{b_{12}}, \lambda_{b_{13}}, \lambda_{b_{23}}, \lambda_c) \in \mathbb{C}^6 \). Denote by

\[
F_{\frac{1}{\alpha+1}\lambda_1^1, \frac{2\lambda_2^1}{\alpha+1}, \lambda_{b_{12}}, \lambda_{b_{13}}, \lambda_{b_{23}}, \lambda_c}
\]

the module generated by the creation operators \( a_n^1, a_n^2, b_n^{12}, b_n^{13}, b_n^{23} \) and \( c_n \) \((n < 0)\) over the vector \( |\lambda_1^1, \lambda_2^1, \lambda_{b_{12}}, \lambda_{b_{13}}, \lambda_{b_{23}}, \lambda_c >\). Introduce the bosonic Fock space

\[
F_{(\lambda_1^1, \lambda_2^1, \lambda_{b_{12}}, \lambda_{b_{13}}, \lambda_{b_{23}}, \lambda_c)} = \bigoplus_{i,j \in \mathbb{Z}} F_{\frac{1}{\alpha+1}\lambda_1^1, \frac{2\lambda_2^1}{\alpha+1}, \lambda_{b_{12}} + i + j, \lambda_{b_{13}} + j, \lambda_{b_{23}} + l, \lambda_c + i + j}.
\]

(V.2)
It can be shown that the action of $U_q[sl(2|1)]$ on this space is closed, i.e. $U_q[sl(2|1)] F_\ast = F_\ast$ for $\ast = (\lambda_{\alpha 1}, \lambda_{\alpha 2}, \lambda_{\beta 12}, \lambda_{\beta 13}, \lambda_{\beta 23}, \lambda_c)$. Hence the Fock space $F_\ast$ constitutes a $U_q[sl(2|1)]$-module. The elementary $q$-vertex operators are mapps of the following Fock spaces:

$$
\phi_r(z), \psi_r(z) : \quad F(\lambda_{a1}, \lambda_{a2}, \lambda_{b12}, \lambda_{b13}, \lambda_{b23}, \lambda_c) \longrightarrow F(\lambda_{a1} + \alpha + 1, \lambda_{a2} + \alpha + 1, \lambda_{b12}, \lambda_{b13}, \lambda_{b23}, \lambda_c),
$$

$$
\phi^*_r(z), \psi^*_r(z) : \quad F(\lambda_{a1}, \lambda_{a2}, \lambda_{b12}, \lambda_{b13}, \lambda_{b23}, \lambda_c) \longrightarrow F(\lambda_{a1} - \alpha, \lambda_{a2} - \alpha, \lambda_{b12}, \lambda_{b13}, \lambda_{b23}, \lambda_c), \quad (V.3)
$$

for all $r = 1, 2, 3, 4$.

Let us now discuss the embedding of the highest weight module $V(\lambda)$ in the bosonic Fock space $F_\ast$. We impose the highest weight conditions on the vector $|\lambda_{a1}, \lambda_{a2}, \lambda_{b12}, \lambda_{b13}, \lambda_{b23}, \lambda_c>$, where $\beta$ and $\gamma$ are arbitrary complex parameters. The corresponding highest weight is

$$
\lambda_{\beta, \gamma} = (\alpha - \beta + 2\gamma) \Lambda_0 + 2(\beta - \gamma) \Lambda_1 - \beta \Lambda_2.
$$

Thus we have the identification

$$
|\lambda_{\beta, \gamma} > = |\beta, \gamma, 0, 0, 0, 0 > . \quad (V.5)
$$

Denote by

$$
F(\beta, \gamma) = \bigoplus_{i, j, l \in \mathbb{Z}} F_{\frac{1}{\alpha + 1} \beta, \frac{2}{\alpha + 1} \gamma, i + j, j, i + j} \quad (V.6)
$$

the Fock space associated to this highest weight vector. It is easy to see that the $U_q[sl(2|1)]$ action on the subspace $F(\beta, \gamma)$ is still closed and therefore $F(\beta, \gamma)$ is a $U_q[sl(2|1)]$-module. Using the highest weight vector $|\lambda_{\beta, \gamma} >$, we construct the level-$\alpha$ highest weight module of $U_q[sl(2|1)]$,

$$
V(\lambda_{\beta, \gamma}) = U_q[sl(2|1)] |\lambda_{\beta, \gamma} > . \quad (V.7)
$$

This module is not irreducible in general, but contains a maximal proper submodule $M(\lambda_{\beta, \gamma})$ such that $V(\lambda_{\beta, \gamma})/M(\lambda_{\beta, \gamma})$ yields an irreducible $U_q[sl(2|1)]$ module. It is clear that the module $V(\lambda_{\beta, \gamma})$ can be embedded in the bosonic Fock space $F(\beta, \gamma)$. Moreover, from $(V.3)$ the elementary $q$-vertex operators are mappings of the Fock spaces:

$$
\phi_r(z), \psi_r(z) : \quad F(\beta, \gamma) \longrightarrow F(\beta + \alpha + 1, \gamma + \alpha + 1),
$$

$$
\phi^*_r(z), \psi^*_r(z) : \quad F(\beta, \gamma) \longrightarrow F(\beta - \alpha, \gamma - \alpha). \quad (V.8)
$$

However, the Fock space $F(\beta, \gamma)$ contains some redundancies arising from the free bosonic field $c(z; 0)$. To see this, we define the fermionic ghost system $(\eta, \xi)$ of dimension

$$
|\lambda_{\beta, \gamma} > = |\beta, \gamma, 0, 0, 0, 0 > . \quad (V.5)
$$

Denote by

$$
F(\beta, \gamma) = \bigoplus_{i, j, l \in \mathbb{Z}} F_{\frac{1}{\alpha + 1} \beta, \frac{2}{\alpha + 1} \gamma, i + j, j, i + j} \quad (V.6)
$$

the Fock space associated to this highest weight vector. It is easy to see that the $U_q[sl(2|1)]$ action on the subspace $F(\beta, \gamma)$ is still closed and therefore $F(\beta, \gamma)$ is a $U_q[sl(2|1)]$-module. Using the highest weight vector $|\lambda_{\beta, \gamma} >$, we construct the level-$\alpha$ highest weight module of $U_q[sl(2|1)]$,

$$
V(\lambda_{\beta, \gamma}) = U_q[sl(2|1)] |\lambda_{\beta, \gamma} > . \quad (V.7)
$$

This module is not irreducible in general, but contains a maximal proper submodule $M(\lambda_{\beta, \gamma})$ such that $V(\lambda_{\beta, \gamma})/M(\lambda_{\beta, \gamma})$ yields an irreducible $U_q[sl(2|1)]$ module. It is clear that the module $V(\lambda_{\beta, \gamma})$ can be embedded in the bosonic Fock space $F(\beta, \gamma)$. Moreover, from $(V.3)$ the elementary $q$-vertex operators are mappings of the Fock spaces:

$$
\phi_r(z), \psi_r(z) : \quad F(\beta, \gamma) \longrightarrow F(\beta + \alpha + 1, \gamma + \alpha + 1),
$$

$$
\phi^*_r(z), \psi^*_r(z) : \quad F(\beta, \gamma) \longrightarrow F(\beta - \alpha, \gamma - \alpha). \quad (V.8)
$$

However, the Fock space $F(\beta, \gamma)$ contains some redundancies arising from the free bosonic field $c(z; 0)$. To see this, we define the fermionic ghost system $(\eta, \xi)$ of dimension
We are now in a position to consider a restriction of the Fock space relations obviously, \( \eta \) and \( \xi \) into a direct sum of subspaces

\[
Ker F = \eta \xi_n^0 \oplus \xi \eta_0 F
\]

Proposition 5: The Fock-Wakimoto module \( U \) constitutes a Fock-Wakimoto module with highest weight vector belonging to the smaller space \( Ker \eta \).

Proposition 4: \( \eta \) commutes (or anticommutes) with the action of \( U_q[sl(\hat{2}|1)] \). Thus \( Ker \eta \) and \( Coker \eta \) are both \( U_q[sl(\hat{2}|1)] \)-modules.

We are now in a position to consider a restriction of the Fock space \( F \) to a smaller space \( F \), referred to as the Fock-Wakimoto space.

Proposition 5: The restricted Fock space

\[
F = \eta \xi_0 F \oplus \xi \eta_0 F
\]

commutes (or anticommutes) with the action of \( U_q[sl(\hat{2}|1)] \).

One can check that \( \eta \lambda \beta, \gamma \geq 0 \) for any \( \beta, \gamma \in C \). Thus \( |\lambda \beta, \gamma \rangle \) is a \( U_q[sl(\hat{2}|1)] \) highest weight vector belonging to the smaller space \( Ker \eta \). It follows that

Proposition 6: The Fock-Wakimoto module \( F \) is a highest weight \( U_q[sl(\hat{2}|1)] \)-module with highest weight vector \( |\lambda \beta, \gamma \rangle \) and highest weight \( \lambda \beta, \gamma \).

Using the projection operator \( \eta \xi_0 \), we define the “projected q-vertex operators” \( \tilde{\phi}_r(z), \tilde{\phi}_r^*(z), \tilde{\psi}_r(z) \) and \( \tilde{\psi}_r^*(z) \) as follows

\[
\tilde{\Theta}(z) = \eta \xi_0 \Theta(z) \eta \xi_0, \quad \Theta(z) = \phi_r(z), \phi_r^*(z), \psi_r(z) \text{ or } \psi_r^*(z).
\]

Since \( \eta \) commutes with the elementary q-vertex operators, we can deduce from that the projected q-vertex operators are mappings of the highest weight Fock-Wakimoto modules:

\[
\tilde{\phi}_r(z) : F \rightarrow F, \quad \tilde{\psi}_r(z) : F \rightarrow F,
\]

\[
\tilde{\phi}_r^*(z) : F \rightarrow F, \quad \tilde{\psi}_r^*(z) : F \rightarrow F.
\]
VI Screen currents and correlation functions

Due to the existence of background charges, the projected \( q \)-vertex operators are not yet the proper bosonizations of the \( q \)-vertex operators (IV.1). In this section we construct \( q \)-screen currents which balance the background charges and thus ensure the nonvanishing of correlation functions of the bosonized \( q \)-vertex operators.

Let us introduce the oscillators
\[
a_{m}^{*,i} = \frac{[m]_q}{[(k+1)m]_q} a_{m}, \quad Q_{a^{*,i}} = \frac{1}{k+1} Q_{a^{i}}, \quad i = 1, 2
\]
and define the corresponding currents \( S^{i}(z) \) by
\[
S^{i}(z) = : e^{-a^{*,i}(z;\frac{k+1}{2})} : S^{i}(z), \quad (VI.2)
\]
\[
\tilde{S}^{1}(z) = : e^{-b_{12}(z;0)-b_{12}(q^{-1}z)-b_{12}(q^{-1}z)+b_{13}(z)} \partial_z e^{-c(q^{-1}z;0)} : e^{\sqrt{-1}\pi(c_{0}+b_{12}^{0})} \]
\[
+q : e^{b_{13}(z;0)-b_{23}(z;0)+b_{23}(z)} : e^{-\sqrt{-1}\pi(b_{13}^{0}+b_{23}^{0})}, \quad (VI.3)
\]
\[
\tilde{S}^{2}(z) = -q^{-1} : e^{b_{23}(z;0)} : e^{-\sqrt{-1}\pi(c_{0}+b_{12}^{0}+b_{13}^{0}+b_{23}^{0})}. \quad (VI.4)
\]

Here we have used the notation
\[
k \partial_z f(z) = \frac{f(q^k z) - f(q^{-k} z)}{(q - q^{-1}) z}. \quad (VI.5)
\]

Then we can verify

**Theorem 3** : The currents \( S^{i}(z) \) satisfy the following commutation relations with the \( U_q[sl(2|1)] \) generators
\[
[h_{n}^{i}, S^{j}(w)] = 0, \quad n \in \mathbb{Z},
\]
\[
[X^{+,i}(z), S^{j}(w)] = 0,
\]
\[
[X^{-,i}(z), S^{j}(w)] = \delta^{ij} k_{+1} \partial_w \left( -z^{-1} \cdot \delta \left( \frac{w}{z} \right) : e^{-a^{*,i}(w;\frac{k}{2})} : \right). \quad (VI.6)
\]

That is, the currents \( S^{i}(z) \) (anti-)commute with the action of \( U_q[sl(2|1)] \) up to total differences. The currents \( S^{i}(z) \) are referred to as the \( q \)-screen currents of \( U_q[sl(2|1)] \).

For \( p \in \mathbb{C}, \ |p| < 1 \) and \( s \in \mathbb{C} - \{0\} \), one defines the Jackson integral
\[
\int_{0}^{s \infty} f(z) d_p z = s(1-p) \sum_{m \in \mathbb{Z}} f(s p^m) p^m, \quad (VI.7)
\]

The Jackson integral enjoys the following property, among others,
\[
\int_{0}^{s \infty} f(z) d_p z = \int_{0}^{s \infty} p f(p z) d_p z, \quad (VI.8)
\]
which implies that for \( p = q^{2k} \),
\[
\int_{0}^{s\infty} k \partial_z f(z) d_p z = 0. \tag{VI.9}
\]
Note that the right hand side of (VI.6) is a total \( p = q^{2(k+1)} \) difference. We have

**Corollary 1** : The screen charges

\[
Q^i = \int_{0}^{s\infty} S^i(z) d_p z, \quad p = q^{2(k+1)}, \tag{VI.10}
\]
assuming that the Jackson integrals are convergent, (anti-)commute with all the generators of \( U_q[sl(2|1)] \).

Since \( \eta_0 \) commutes with \( S^i(z) \), \( i = 1, 2 \), the screen charges with \( k = \alpha \) give rise to the following mappings of the Fock-Wakimoto modules:

\[
Q^1 : \quad \mathcal{F}_{(\beta, \gamma)} \rightarrow \mathcal{F}_{(\beta-1, \gamma)}, \tag{VI.11}
\]
\[
Q^2 : \quad \mathcal{F}_{(\beta, \gamma)} \rightarrow \mathcal{F}_{(\beta, \gamma-\frac{1}{2})}. \tag{VI.12}
\]

Introduce the screened \( q \)-vertex operators

\[
\tilde{\phi}_r^{(x_1, \bar{x}_1)}(z) = (Q^1)^{x_1}(Q^2)^{\bar{x}_1} \tilde{\phi}_r(z),
\]
\[
\tilde{\phi}_r^*(y_1, \bar{y}_1)(z) = (Q^1)^{y_1}(Q^2)^{\bar{y}_1} \tilde{\phi}_r^*(z),
\]
\[
\tilde{\psi}_r^{(x'_1, \bar{x}'_1)}(z) = (Q^1)^{x'_1}(Q^2)^{\bar{x}'_1} \tilde{\psi}_r(z),
\]
\[
\tilde{\psi}_r^*(y'_1, \bar{y}'_1)(z) = (Q^1)^{y'_1}(Q^2)^{\bar{y}'_1} \tilde{\psi}_r^*(z). \tag{VI.13}
\]

We are now in a position to state

**Theorem 4** : The \( q \)-vertex operators (IV.1) are bosonized as

\[
\Phi^{\lambda^1_{\beta^1_+}(x), \gamma^1_+}(z) = \sum_{r=1}^{4} \phi_r^{(x_1, \bar{x}_1)}(z) \otimes v_r,
\]
\[
\Phi^{\lambda^1_{\beta^1_-}(y), \gamma^1_-}(z) = \sum_{r=1}^{4} \phi_r^*(y_1, \bar{y}_1)(z) \otimes v_r^*,
\]
\[
\tilde{\psi}_r^{\lambda^1_{\beta^1_+}(x'), \gamma^1_+}(z) = \sum_{r=1}^{4} v_r \otimes \tilde{\psi}_r^{(x'_1, \bar{x}'_1)}(z),
\]
\[
\tilde{\psi}_r^*^{\lambda^1_{\beta^1_-}(y'), \gamma^1_-}(z) = \sum_{r=1}^{4} v_r^* \otimes \tilde{\psi}_r^*(y'_1, \bar{y}'_1)(z), \tag{VI.14}
\]

where

\[
\beta^1_+(x) = \beta + \alpha + 1 - x_1, \quad \gamma^1_+(\bar{x}) = \gamma + \alpha + 1 - \frac{1}{2} \bar{x}_1,
\]
\[
\beta^1_-(y) = \beta - \alpha - y_1, \quad \gamma^1_-(\bar{y}) = \gamma - \alpha - \frac{1}{2} \bar{y}_1. \tag{VI.15}
\]
for certain choices of nonnegative integers \(x_1, \tilde{x}_1, y_1, \text{ and } \tilde{y}_1\). These operators are intertwiners of the highest weight \(U_q[\widehat{sl(2|1)}]\)-modules:

\[
\Phi_{\beta, \gamma}^{(s)}(z) : \mathcal{F}(\beta, \gamma) \rightarrow \mathcal{F}(\beta, \gamma) \otimes V_{\alpha, z}, \\
\Psi_{\beta, \gamma}^{(s)}(z) : \mathcal{F}(\beta, \gamma) \rightarrow \mathcal{F}(\beta, \gamma) \otimes V_{\alpha, z}^*.
\]

In the following we compute \(N\)-point correlation function which is defined to be the trace of the bosonized \(q\)-vertex operators over the \(U_q[\widehat{sl(2|1)}]\)-module \(\mathcal{F}(\beta, \gamma)\), that is

\[
\text{Tr}_{\mathcal{F}(\beta, \gamma)} \left( q^L \Theta_{r_1}(z_1) \cdots \Theta_{r_N}(z_N) \right). \tag{VI.17}
\]

Here \(\Theta_{r_i}(z_i)\) stands for the type I \(q\)-vertex operators \(\varphi_{r_{1i}}^{(s)}(z_i)\), \(\tilde{\varphi}_{r_{1i}}^{(s)}(z_i)\) or the type II \(q\)-vertex operators \(\tilde{\psi}_{r_{1i}}^{(s)}(z_i)\), \(\tilde{\psi}_{r_{1i}}^{(s)}(z_i)\); \(L_0 = -d\) is the \(q\)-Virasoro operator which is bosonized as (for \(k = \alpha \neq 0, -1\))

\[
-L_0 = \sum_{n>0} \left( \frac{n^2}{n_q^{\alpha+1} n} (a_{-n}^1 a_n^2 + a_{-n}^2 a_n^1 + (q^n + q^{-n}) a_{-n}^2 a_n^2) \\
+ \frac{n^2}{n_q^2} (b_{-n}^1 b_n^2 - b_{-n}^2 b_n^1 - b_{-n}^3 b_n^3 - c_{-n} c_n) \\
+ \frac{1}{\alpha + 1} (a_{-n}^1 a_n^2 + (a_{-n}^2)^2 + a_{-n}^1 + 3a_{-n}^3) \\
+ \frac{1}{2} (b_{-n}^1)^2 - b_{-n}^2 (b_{-n}^1 + 1) - b_{-n}^3 (b_{-n}^1 + 1) - (c_n)^2 \right). \tag{VI.18}
\]

The zero mode part of the \(a_n^1, a_n^2\) oscillators is added to the \(L_0\) operator so that its eigenvalue on \(|\lambda_{\beta, \gamma}\rangle\) is \(\frac{1}{2(\alpha+1)}(\lambda_{\beta, \gamma}^2 + 2\rho)\), where \(\rho = \Lambda_0 + \Lambda_1 + \Lambda_2\).

Let us define the Fock spaces for \(s \in \mathbb{Z}\),

\[
F_{(\beta, \gamma)}^{(s)} = \bigoplus_{i,j \in \mathbb{Z}} F_{i+j}^{(s)}, \tag{VI.19}
\]

We have \(F_{(\beta, \gamma)}^{(0)} = F_{(\beta, \gamma)}\). It can be shown that \(\eta_0, \xi_0\) intertwine various Fock spaces

\[
\eta_0 : F_{(\beta, \gamma)}^{(s)} \rightarrow F_{(\beta, \gamma)}^{(s+1)}, \quad \xi_0 : F_{(\beta, \gamma)}^{(s)} \rightarrow F_{(\beta, \gamma)}^{(s-1)}.
\]

Since \(\eta_0^2 = 0\), we obtain the following BRST complex:

\[
\cdots \overset{Q_{s+1}}{\rightarrow} F_{(\beta, \gamma)}^{(s)} \overset{Q_s}{\rightarrow} F_{(\beta, \gamma)}^{(s+1)} \overset{Q_{s+1}}{\rightarrow} F_{(\beta, \gamma)}^{(s)} \overset{Q_s}{\rightarrow} \cdots \tag{VI.20}
\]

It follows from \(\eta_0 \xi_0 + \xi_0 \eta_0 = 1\), that \(\text{Ker}Q_s = \text{Im}Q_{s-1}\) for any \(s \in \mathbb{Z}\). We have
Proposition 7: The $N$-point correlation function of the type I vertex operators

$$\text{Tr}_{F_{(\beta, \gamma)}} \left( q^{L_0} \tilde{\phi}^{(x_N, \bar{x}_N)}(z_N) \cdots \tilde{\phi}^{(x_1, \bar{x}_1)}(z_1) \right) \neq 0$$

iff $\alpha \in \mathbb{N}$ and $\sum_{i=1}^{N} x_i = \frac{1}{2} \sum_{i=1}^{N} \bar{x}_i = N(\alpha + 1)$. For such $\alpha$ and $x_i$, $\bar{x}_i$, the above trace is given by

$$\sum_{s=1}^{\infty} (-1)^{s+1} \text{Tr}_{F_{(\beta, \gamma)}} \left( q^{L_0} (Q^{1})^{x_N} (Q^{2})^{\bar{x}_N} \phi^{(z_N)}(z_N) \cdots (Q^{1})^{x_1} (Q^{2})^{\bar{x}_1} \phi^{(z_1)}(z_1) \right). \quad (VI.21)$$

Similarly, the $N$-point correlator of the type II vertex operators

$$\text{Tr}_{F_{(\beta, \gamma)}} \left( q^{L_0} \tilde{\phi}^{(x'_N, \bar{x}'_N)}(z_N) \cdots \tilde{\phi}^{(x'_1, \bar{x}'_1)}(z_1) \right)$$

$$= \sum_{s=1}^{\infty} (-1)^{s+1} \text{Tr}_{F_{(\beta, \gamma)}} \left( q^{L_0} (Q^{1})^{x'_N} (Q^{2})^{\bar{x}'_N} \phi^{(z_N)}(z_N) \cdots (Q^{1})^{x'_1} (Q^{2})^{\bar{x}'_1} \phi^{(z_1)}(z_1) \right) \quad (VI.22)$$

is nonvanishing iff $\alpha \in \mathbb{N}$ and $\sum_{i=1}^{N} x'_i = \frac{1}{2} \sum_{i=1}^{N} \bar{x}'_i = N(\alpha + 1)$.

We now consider the $N$-point correlation function involving also dual vertex operators,

$$\text{Tr}_{F_{(\beta, \gamma)}} \left( q^{L_0} \tilde{\phi}^{(y_N, \bar{y}_N)}(z_N) \cdots \tilde{\phi}^{(y_{l+1}, \bar{y}_{l+1})}(z_{l+1}) \tilde{\phi}^{(x_l, \bar{x}_l)}(z_l) \cdots \tilde{\phi}^{(x_1, \bar{x}_1)}(z_1) \right). \quad (VI.23)$$

Then we have

Proposition 8: For $\alpha \in \mathbb{N}$, $(VI.23)$ is non-zero iff $\sum_{i=1}^{l} x_i + \sum_{i=l+1}^{N} y_i = \frac{1}{2} (\sum_{i=1}^{l} \bar{x}_i + \sum_{i=l+1}^{N} \bar{y}_i) = (2l - N) \alpha + l$. And for $\alpha \not\in \mathbb{N}$ it is nonvanishing iff $N$ is even, i.e. $N = 2L$, and $l = L = \sum_{i=1}^{L} x_i + \sum_{i=L+1}^{N} y_i = \frac{1}{2} (\sum_{i=1}^{L} \bar{x}_i + \sum_{i=L+1}^{N} \bar{y}_i)$. In both case, the trace $(VI.23)$ can be written as the following unified formula

$$(VI.23) = \sum_{s=1}^{\infty} (-1)^{s+1} \text{Tr}_{F_{(\beta, \gamma)}} \left( q^{L_0} (Q^{1})^{y_N} (Q^{2})^{\bar{y}_N} \phi^{(z_N)}(z_N) \cdots (Q^{1})^{y_{l+1}} (Q^{2})^{\bar{y}_{l+1}} \phi^{(z_{l+1})}(z_{l+1}) \right.$$  

$$(Q^{1})^{x_l} (Q^{2})^{\bar{x}_l} \phi^{(z_l)}(z_l) \cdots (Q^{1})^{x_1} (Q^{2})^{\bar{x}_1} \phi^{(z_1)}(z_1) \right). \quad (VI.24)$$

An integral formula for the $N$-point functions of type II (dual) vertex operators can be written down in a similar way, which we omit.

Acknowledgements

This work has been financially supported by the Australian Research Council large, small and QEII fellowship grants. Discussions with Wen-Li Yang are gratefully acknowledged.
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