The Nielsen Identities for the generalized $R_\xi$-gauge

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Abstract

We show that it is possible, in opposition to a previous conjecture, to derive a Nielsen identity for the effective action in the case of the generalized $R_\xi$-gauge, where the gauge function explicitly depends on the gauge parameter $\xi$. Also the Nielsen identity for the effective potential is verified to one-loop in the Abelian Higgs model and the corresponding identity for the physical Higgs mass is derived.

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I. INTRODUCTION

The gauge dependence of the effective action $\Gamma(\varphi)$ and in particular, that of the effective potential $V_{\text{eff}}(\varphi)$ is known since a long time ago [1]. This fact raised the question about the gauge invariance of the results obtained from the effective potential. For example, it was not known whether spontaneous symmetry breaking was a gauge invariant effect.

The investigation carried out in [2] and [3] showed that gauge independent physical quantities can be obtained from a gauge-dependent effective potential and also that spontaneous symmetry breaking is a gauge-invariant phenomenon.

In [2] a set of identities was derived that implement the physical requirement that the value of the effective potential at its extrema should be left invariant under simultaneous variations of the gauge parameter $\xi$ and the solution of the extremal equation. The works [2] and [3], especially this last one, also allowed to understand the origin of the problem studied in [4] and further analysed in [5], on the definition of the effective potential in the $R_\xi$-gauge.

In [6] the formal work done in [2] was supplemented by an explicit and complete calculation at the one-loop level in the Abelian Higgs model. There the gauge-fixing function that defines the modified $R_\xi$-gauge was chosen as

$$F = -\partial_\mu A_\mu + ev \Phi_2,$$

where $v$ is an additional arbitrary gauge parameter that in principle is not related to any property of the scalar field and $\Phi_2$ is the imaginary component of the scalar field.

Although the work done in [6] is very complete, they were unable to prove that even in the simplest case the $\xi$-dependence of the effective action $\Gamma$ could be expressed by means of a Nielsen identity in the case of the generalized $R_\xi$-gauge where $v = f(\xi)u$. In fact, they conjectured that no dependence of $v$ on the gauge parameter is possible if the Nielsen identities are to hold.

However, in a recent paper, [7] a generalization of the Nielsen identities has been derived,
which are also valid for the case of background field gauges, in which the gauge fixing \[1\] is included.

In this letter we show explicitly, following the simple arguments shown in \[7\], that it is possible to derive a Nielsen identity for the effective action in the case of the generalized \(R_\xi\)-gauge where \(v\) depends on the gauge parameter \(\xi\), thus showing that the conjecture done in \[3\] was not correct. The derivation is such that is readily generalizable to non-abelian gauge theories.

For completeness we will perform an explicit calculation to one loop order in the Abelian Higgs model in order to verify the Nielsen identity for the effective potential. In addition we will show that the corresponding identity for the physical Higgs mass is straightforwardly derived from the identity for the effective action.

II. DERIVATION OF THE NIELSEN IDENTITY

Let us consider a gauge theory in euclidean space defined by some Lagrangian \(\mathcal{L}(\phi_i)\), which includes a gauge-fixing Lagrangian given by

\[
\mathcal{L}_{g.f.} = \frac{1}{2\xi} [F(\phi_i; \xi)]^2,
\]

where \(\phi_i\) denote the set of gauge and scalar fields. We assume that with such a gauge-fixing function the Fadeev-Popov ghost fields do not decouple. Therefore the generating functional of connected Green functions is

\[
e^{-W(J,\xi)} = \int \mathcal{D}\phi_i \mathcal{D}\eta \mathcal{D}\overline{\eta} e^{-S - \frac{1}{2\xi} \int d^dxF^2 + \int d^dxF(x) \frac{d^dF}{d^d\eta(x)} \Delta_i \eta(x) + \int d^dxF_i(x) \phi_i(x)},
\]

where \(\Delta_i\) is defined via the BRS transformation of the fields \(\phi_i\) and \(S\) is the classical action of the theory.

Using the fact that the integral over a single Grassman variable vanishes and that the full action is invariant under the BRS transformations

\[
\delta\phi_i = \Delta_i \eta \omega, \quad \delta\overline{\eta} = -\frac{1}{\xi} F \omega, \quad \delta\eta = 0,
\]

\[3\]
it is easy to see that the following identity is satisfied
\[
\left\langle -\frac{1}{\xi} F(\phi_i(x); \xi) K(\phi_i(x)) + \eta(x) \frac{\delta K(\phi_i(x))}{\delta \phi_i(x)} \Delta_i \eta(x) - \int d^d y J_i(y) \Delta_i \eta(y) \eta(x) K(\phi_i(x)) \right\rangle = 0,
\]
(5)

where use of eqs. (4) has been done as well as the properties of Grassman variables. Above, 
\(K\) is an arbitrary functional of the fields \(\phi_i\) and the expectation value of the operator 
\(O(\phi_i, \eta, \eta)\) is given by
\[
\langle O \rangle = e^{W(J, \xi)} \int D\phi_i D\eta D\eta \ e^{-S_F + \int d^d x J_i(x) \phi_i(x)}
\]
(6)

and \(S_F\) is the full action, including ghost and gauge fixing terms.

Now we determine how the functional \(W(J, \xi)\) varies under infinitesimal changes in the 
gauge parameter \(\xi\), i.e. when \(\xi \to \xi + \Delta \xi\). From eq. (3) we have that to leading order in \(\Delta \xi\)
\[
\Delta W = -\Delta \xi \int d^d x \left[ \left\langle \frac{1}{2 \xi^2} F^2 \right\rangle + \left\langle \frac{F}{\xi} \frac{\delta F}{\delta \phi_i(x)} \left( \frac{\partial F}{\partial \xi} \right) \Delta_i \eta(x) \right\rangle \right].
\]
(7)

Using the identity (5) with \(K = \frac{1}{2 \xi} F\) we find that the first term on the r.h.s. of eq. (7) 
equals
\[
\frac{1}{2 \xi^2} \left\langle F^2 \right\rangle = -\frac{1}{2 \xi} \int d^d y J_i(y) \langle \Delta_i \eta(y) \eta(x) F \rangle + \frac{1}{2 \xi} \langle \eta(x) \frac{\delta F}{\delta \phi_i(x)} \Delta_i \eta(x) \rangle
\]
(8)

and if we choose \(K\) to be equal to \(\frac{\partial F}{\partial \xi}\) the second term on the r.h.s. of eq. (6) can be written as
\[
\left\langle \frac{F}{\xi} \frac{\delta F}{\delta \phi_i(x)} \left( \frac{\partial F}{\partial \xi} \right) \Delta_i \eta(x) \right\rangle = \int d^d y J_i(y) \left\langle \Delta_i \eta(y) \eta(x) \frac{\partial F}{\partial \xi} \right\rangle.
\]
(9)

Therefore, eq. (7) reads
\[
\Delta W = \Delta \xi \int d^d x \int d^d y J_i(y) \left\langle \Delta_i \eta(y) \eta(x) \left[ \frac{F}{2 \xi} - \frac{\partial F}{\partial \xi} \right] \right\rangle,
\]
(10)

where we have dropped a constant term \(\frac{\Delta \xi}{2 \xi}\) arising from the second term on the r.h.s. of eq. (5).

The Nielsen identity for the effective action \(\Gamma(\varphi_i)\) is obtained by performing the Legendre 
transformation of eq. (10), with the condition that the classical fields \(\varphi_i\) are kept fixed under
variations of the gauge parameter $\xi$. Therefore it is easy to see that in the limit when $\Delta \xi$ tends to zero such identity reads

$$
\xi \frac{\partial \Gamma}{\partial \xi} = \int d^d x \int d^d y \frac{\delta \Gamma}{\delta \varphi_i(y)} \left\langle \Delta_i \eta(y) \eta(x) \left[ \frac{1}{2} F^2 - \xi \frac{\partial F}{\partial \xi} \right] \right\rangle, \quad (11)
$$

where in this case

$$
\langle O \rangle_\Gamma = e^\Gamma \int D\phi_i D\eta D\eta O e^{-S_F + \int d^d x \frac{\delta F}{\delta \phi_i} (\phi_i - \varphi_i)} \quad (12)
$$

and thus only one-particle irreducible Green function are taken into account.

III. VERIFICATION OF THE NIelsen IDENTITY TO ONE LOOP ORDER

In this section we verify as an example that the Nielsen identity for the one-loop effective potential is satisfied. To this end we consider the Abelian Higgs model in four-diemsional euclidean space with Lagrangian

$$
\mathcal{L} = \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + |D_\mu \Phi|^2 - m^2 \Phi^* \Phi + \frac{\lambda}{3!} (\Phi^* \Phi)^2, \quad (13)
$$

with $\Phi = [\Phi_1 + i \Phi_2]/\sqrt{2} = [(H + \varphi) + i G]/\sqrt{2}$ and choose the gauge-fixing Lagrangian to be

$$
\mathcal{L}_{g.f.} = \frac{1}{2\xi} \left[ - \partial_\mu A_\mu + \xi u G \right]^2, \quad (14)
$$

where for simplicity the simplest case of the generalized $R_\xi$ -gauge, $v = \xi u$, has been considered. For this kind of gauge fixing there is a mixing between the gauge and Goldstone degrees of freedom and the ghosts fields do not decouple.

Assume a constant classical field $\varphi$ for the Higgs field component and all other classical fields equal to zero. Therefore, from eq. (11) the Nielsen identity for the effective potential reads

$$
\xi \frac{\partial V}{\partial \xi} = C \frac{\partial V}{\partial \varphi}, \quad (15)
$$

where $C$ is defined by
\[ C = -\frac{1}{2} \int d^4 x \int d^4 y \langle (\partial_\mu A_\mu(x) + \xi e u G(x)) e G(y) \eta(y) \widetilde{\eta}(x) \rangle. \] (16)

The integrals have to be evaluated by dimensional regularization and the infinite parts are discarded (the scale of dimensional regularization will be set equal to one throughout this letter). Thus, expanding \( V \) and \( C \) in powers of \( \bar{\hbar} \)

\[ V = V^{(0)} + \bar{\hbar} V^{(1)} + \ldots \]

\[ C = C^{(0)} + \bar{\hbar} C^{(1)} + \ldots \] (17)

one finds that to one-loop order

\[ \xi \frac{\partial V^{(1)}}{\partial \xi} = C^{(1)} \frac{\partial V^{(0)}}{\partial \varphi}, \] (18)

where \( C^{(1)} \) has a Feynman diagram representation shown in Fig. 1. The evaluation of \( C^{(1)} \) requires the knowledge of the Goldstone, ghost and mixed propagators, which are given by

\[ \Delta_G = \frac{k^2 + \xi m_A^2}{D_N}; \quad \Delta_{gh} = -\frac{1}{k^2 + m_{gh}^2}; \quad \Delta_\mu = i \xi e (\varphi - u) \frac{k_\mu}{D_N} \] (19)

respectively. Here,

\[ D_N = k^4 + k^2 (m_2^2 + 2 \xi e^2 \varphi u) + m_A^2 [(\xi e u)^2 + \xi m_2^2], \]
\[ m_A^2 = e^2 \varphi^2; \quad m_2^2 = -m^2 + \frac{\lambda}{6} \varphi^2; \quad m_{gh}^2 = \xi e^2 u \varphi, \] (20)

and in \( \Delta_\mu \) the momentum flow is from \( G \) to \( A_\mu \).

Therefore, in momentum space, the sum of both terms in Fig. 1 gives

\[ C^{(1)} = \frac{e^2 \xi}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_N (k^2 + m_{gh}^2)} \left\{ (\varphi - 2u) k^2 - \xi u e^2 \varphi^2 \right\}. \] (21)

Also, the \( \xi \)-dependent part of the one-loop contribution to the effective potential is

\[ V^{(1)} = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln D_N - \int \frac{d^d k}{(2\pi)^d} \ln (k^2 + m_{gh}^2) \] (22)

and thus its derivative with respect to the gauge parameter reads

\[ \xi \frac{\partial V^{(1)}}{\partial \xi} = \frac{1}{2} (m_2^2 \varphi) \xi e^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_N (k^2 + m_{gh}^2)} \left\{ (\varphi - 2u) k^2 - \xi u e^2 \varphi^2 \right\}. \] (23)
Now, combining
\[ \frac{\partial V^{(0)}}{\partial \varphi} = (-m^2 + \frac{\lambda}{6} \varphi^2) \varphi = m_2^2 \varphi \] (24)
and eqs. (21) and (23) we observe that the Nielsen identity for the effective potential, eq. (18) is satisfied.

**IV. NIELSEN IDENTITY FOR THE PHYSICAL HIGGS MASS**

The physical Higgs mass \( M^2 \) is defined as the pole in the physical part of the Higgs propagator (in Minkowski space). In the case of the abelian Higgs model the inverse propagator is given by
\[
G^{-1}(x - y) = \left. \frac{\delta^2 \Gamma}{\delta \Phi_1(x) \delta \Phi_1(y)} \right|_{\Phi_1 = \varphi} \tag{25}
\]
and vanishes on the euclidean mass shell: \( G^{-1}(p^2 = -M^2) = 0. \)

Therefore, by differentiating the Nielsen identity for the effective action twice with respect to \( \Phi_1 \), evaluating at \( \Phi_1(x) = \varphi \) and the using eq. (23) we obtain
\[
\xi \frac{\partial}{\partial \xi} G^{-1}(w - z) = \int d^4x \int d^4y \frac{\delta}{\delta \varphi} G^{-1}(w - z) \left< \Delta i \eta(y) \eta(x) \left[ \frac{1}{2} F - \xi \frac{\partial F}{\partial \xi} \right] \right> \Gamma, \tag{26}
\]
where as before all classical fields except that of the Higgs field have been set equal to zero. This equation can be rewritten as
\[
\xi \frac{\partial}{\partial \xi} G^{-1}(w - z) = C \frac{\delta}{\delta \varphi} G^{-1}(w - z), \tag{27}
\]
with \( C \) defined in eq. (16). Or, in Fourier space
\[
\xi \frac{\partial}{\partial \xi} G^{-1}(p^2) = C \frac{\delta}{\delta \varphi} G^{-1}(p^2). \tag{28}
\]
This equation means that if the inverse propagator vanishes for particular values of the momentum squared, the gauge parameter and \( \varphi \), it will also vanish for the same value of \( p^2 \) if \( \xi \to \xi + \delta \xi \) and \( \varphi \to \varphi + C \delta \xi / \xi \). Therefore, the Nielsen identity for the Higgs mass squared reads
\[ \xi \frac{\partial}{\partial \xi} M^2 = C \frac{\delta}{\delta \varphi} M^2. \] (29)

This equation can be directly verified to the (first nontrivial) one-loop order and this will be done elsewhere.

V. CONCLUSIONS

Therefore we have seen that it is possible to obtain in a rather simple way a Nielsen identity for the effective action in the generalized $R_{\xi}$-gauge and verified that the corresponding identity for the effective potential is satisfied to one-loop order, in scalar electrodynamics. This result shows that the a previous conjecture on the impossibility of deriving such an identity is not correct. It is clear that although the verification of the identity for the effective potential was done for the special case $v = f(\xi)u$, $f(\xi) = \xi$, our results are valid for an arbitrary, but well behaved, function $f(\xi)$.

In addition, we have shown that the corresponding identity for the physical Higgs mass can be straightforwardly obtained from eq. (11).

Finally, it is clear that these ideas are easily generalizable to the case of non-abelian gauge theories because in essence we have just made use of BRS symmetries and the properties of Grassman variables.

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VII. FIGURE CAPTIONS

Fig. 1. Feynman representation of $C^{(1)}$. The dashed line represents the ghost propagator and the full line, the Goldstone propagator. The mixed propagator is shown as a wiggly and full line. The crosses represent $e$ times numerical factors.
