Maximal symmetry and metric-affine $f(R)$ gravity

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Abstract
The affine connection in a spacetime with a homogenous and isotropic subspace is derived using the properties of maximally symmetric tensors. The number of degrees of freedom in metric-affine gravity is thereby considerably reduced while the theory allows spatio-temporal torsion and remains non-metric. The Ricci tensor and scalar are calculated in terms of the connection and the field equations derived for the Einstein–Hilbert as well as for $f(R)$ Lagrangians.

By considering specific forms of $f(R)$, we demonstrate that the resulting Friedmann equations in the so-called Palatini formalism without torsion and metric-affine formalism with maximal symmetry are in general different in the presence of matter.

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1. Introduction

Based on the cosmological principle derived from the Copernican principle of mediocrity and large-scale observations, standard cosmology assumes a homogeneous and isotropic universe. One finds that there are several studies backing this assumption (e.g. [1–4]). Although the cosmological principle still holds its position as the bedrock of most cosmological models, recently the claim for homogeneity has nonetheless been seriously contested (e.g. [5–7]): one finds this credible as at least on small scales the universe is indeed very inhomogeneous.

The idea of homogeneity and isotropy of the universe has been around for a long time. Its cosmological implications have been studied thoroughly in the context of the metric formalism of the general relativity (GR). Metric-affine formulation of gravity is also an early idea (for its history, see e.g. [8]) based on general concepts of pseudo-Riemannian theory of manifolds where no $a$ priori relation between the metric and the connection is assumed. However, there have been few studies into the effects of homogeneity and isotropy on the independent connection in metric-affine gravity, probably because the Einstein–Hilbert action does not make a distinction between the two formalisms.
After the initial interest, metric-affine gravity received only marginal attention until it flared again in the 1970s [9, 10]. There were high hopes that metric-affine gravity might lead us closer to quantum gravity. Failure to do so leads us to put metric-affine gravity aside once again. It functioned merely as curiosity until lately the interest in metric-affine gravity has grown rapidly since Vollick [11] argued that it is possible to explain the accelerating expansion of the universe without the cosmological constant by modifying the Einstein–Hilbert action.

In metric-affine gravity the connection is independent of the metric and has 64 components which are functions of temporal and spatial coordinates. It is clear that by assuming symmetries of the universe, say homogeneity and isotropy, the degrees of freedom should decrease. This is indeed well known to be true also for the affine connection and the consistent use of symmetry principles forms the basis of the present paper. Our aim is to study the structure of metric-affine formalism, in the context of $f(R)$ theories of gravity exploiting the symmetries of homogeneous and isotropic universe, i.e. we seek solutions in the cosmological case. More formal studies of $f(R)$ gravity with torsion have also been conducted recently, see e.g. [12, 13] and references therein.

The difference between metric and metric-affine formalisms is manifested by two important fundamental features. Torsion is allowed in metric-affine gravity unlike in GR (for a review, see [14]). The connection can also deviate from GR in non-metricity. According to Sotiriou [15] both can be induced by matter. However, there is not much experimental evidence to rule out torsion (nor non-metricity) or to prove its existence [16–19]. The debate on the possibility to measure torsion with the data from Gravity Probe B [20, 21] is also interesting. In the latter, it is found that the coupling between the physical objects with the geometrical objects is such that the non-Riemannian geometric quantities couple to the internal degrees of freedom. Therefore, torsion cannot be measured when the experiment does not contain microstructure (spin, dilaton charge and intrinsic shear). One possibility is to use nuclear magnetic resonance gyroscopes instead of mechanical gyroscopes in future experiments. One problem with measurements is the different role it plays in different theories—e.g. in teleparallelism torsion represents the field strength of gravitation while in GR torsion vanishes by definition and curvature geometrizes gravity.

By using symmetry to reduce the degrees of freedom in metric the field equations become much more simple. Comparing the results in standard cosmology and in metric-affine formalism it is possible to better see the role which the independent connection plays. The present study is organized as follows: in section 2 we devise the general tools needed for the following sections. Many parts of this section can be found derived in a slightly different manner in [23]. In section 3, we consider a homogeneous and isotropic space and derive the independent components of the connection and calculate the Ricci tensor and scalar as a function of the found components. The results of section 3 are put into use in section 4. In the case of the Einstein–Hilbert Lagrangian we allow hypermomentum and calculate the Friedmann equation and see how the results relate to standard cosmology. Then we generalize to $f(R)$ actions. In section 5, we discuss our results.

2. Symmetry in spacetime

The symmetry of space can be formalized in terms of isometry and form invariance. A space is form invariant [24] under an isometric coordinate transformation $x \rightarrow \bar{x}$ if corresponding metric tensors are related by $\bar{g}_{\alpha\beta}(y) = g_{\alpha\beta}(y)$ for all $y$. In the case of infinitesimal transformations defined by Killing vectors $\bar{x}^\mu = x^\mu + X^\mu(x)$ this is easily seen to be equivalent.
with the requirement of vanishing Lie derivative $\mathcal{L}_X g_{\mu\nu} = 0$ [25, 26]. The Lie derivative can be expressed in terms of Lévi-Civita connection, i.e. Christoffel symbol $\left\{ ^\alpha_{\mu\nu} \right\}$ as
\[
\mathcal{L}_X g_{\mu\nu} = 2\partial_{(\mu} X_{\nu)} - 2X_\alpha \left\{ ^\alpha_{\mu\nu} \right\}.
\]
(1)

The affine connection can be most generally written as a sum of a Christoffel symbol, a torsion part and a non-metricity part [9]. However, if the connection is metric, i.e. the non-metricity tensor ($Q_{\alpha\mu\nu} = -\nabla_\alpha g_{\mu\nu}$) vanishes, form invariance can be characterized by the Killing equation
\[
\nabla_\nu (X_\mu) = 0.
\]
(2)
The Killing equation still allows for a non-zero torsion tensor [27] as connections of the form
\[
\Gamma^\alpha_{\mu\nu} = \left\{ ^\alpha_{\mu\nu} \right\} + \frac{1}{2} C_{\mu\nu}^\alpha,
\]
(3)
where $C_{\mu\nu}^\alpha$ is antisymmetric in the first two indices, fulfil (2) when (1) vanishes. From now on we assume the connection to be of the form (3) in order to use the Killing equation. This is in accordance with the argument of [15] that the connection must be constrained in some way to produce a viable theory.

For a general tensor we require invariance in an infinitesimal isometric transformation as for all $y$
\[
T_{\mu\nu\ldots\alpha\beta\ldots}^{(y)} = T_{\mu\nu\ldots\alpha\beta\ldots}^{(y)}
\]
leading to the conditions
\[
0 = \frac{\partial X^\alpha}{\partial x^\mu} T_{\alpha\nu\ldots}(x) + \frac{\partial X^\beta}{\partial x^\nu} T_{\mu\beta\ldots}(x) + \cdots + X^\lambda(x) \frac{\partial}{\partial x^\lambda} T_{\mu\nu\ldots}(x).
\]
(5)
In a maximally symmetric space, the requirement that the number of independent Killing vectors is maximal, i.e. equations (5) are satisfied, strongly restricts invariant tensors [24].

A form invariant scalar in a maximally symmetric space must always be a constant. For higher rank tensors the invariance equation can be written as
\[
\delta_\mu^\alpha T_{\nu\ldots}^\beta + \delta_\nu^\beta T_{\mu\ldots}^\alpha + \cdots = \delta_\mu^\alpha T_{\nu\ldots}^\beta + \delta_\nu^\beta T_{\mu\ldots}^\alpha + \cdots.
\]
(6)

For our purposes the invariance conditions for tensors of ranks 1, 2 and 3 in a four-dimensional spacetime with a maximally symmetric three-dimensional subspace are needed. The first two can be easily found in the literature, e.g. [24]. For rank 3 tensor the result is seldom calculated explicitly. From here on we use Latin indices for the maximally symmetric subspace while the Greek indices refer to four-dimensional spacetime.

The cases of form invariant covariant tensors of ranks 1 and 2 easily yield that
\[
A_i = 0,
\]
(7a)
\[
B_{ij} = fg_{ij},
\]
(7b)
where the function $f$ does not depend on the coordinates of the maximally symmetric subspace. Applying (6) to a form invariant rank 3 tensor and contracting indices we get three equations
\[
(N - 1)C_{njk} + C_{jnk} + C_{kjn} = 0,
\]
(8a)
\[
C_{jnk} + (N - 1)C_{njk} + C_{nkj} = 0,
\]
(8b)
\[
C_{njk} + C_{knj} + (N - 1)C_{jnk} = 0,
\]
(8c)
where we have adopted a more general notation with $N$ indicating the dimensions of the maximally symmetric subspace. From these we obtain two useful conditions for form invariant tensors: they are invariant under cyclic index permutations,

$$C_{k_{ij}} = C_{n_{kj}},$$

and they are antisymmetric in the first two indices, except for $N = 3$, since

$$(N - 3)C_{(n)kj} = 0.$$  

From the set of conditions above ((8a), (9), (10)) it follows that all form invariant tensors of rank 3 vanish unless the maximally symmetric subspace is three dimensional, i.e. $N = 3$. As the torsion and non-metricity tensors are of rank 3, they may hence exist only in three-dimensional maximally symmetric (sub)spaces (see also [27]). One might consider that torsion and non-metricity need not be maximally symmetric tensors. However, physically it is not sensible. With $N \neq 3$ the connection is then necessarily the Lévi-Civita connection.

3. Homogeneous and isotropic space

3.1. Affine connection

A metric with a homogeneous and isotropic subspace can be written in spherical coordinates as [24]

$$ds^2 = b^2(t) dt^2 - a^2(t) \tilde{g}_{ij} dx^i dx^j,$$

where

$$\tilde{g}_{ij} dx^i dx^j = \frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

is the metric of the spatial part with $k \in \{-1, 0, 1\}$. Usually a rescaling of the time coordinate is performed [28] to remove the function $b(t)$ but at this point we postpone doing this. This ensures that we can calculate the equations of motion by varying the action with respect to $a(t)$ and $b(t)$ instead of varying with respect to the full metric tensor.

Taking advantage of the symmetries of spacetime, we require that covariant derivative of a maximally symmetric tensor preserves invariance, i.e. maximal symmetry. Thereupon, we can reduce the number of degrees of freedom in the connection by utilizing results of the previous section. First we consider a maximally symmetric covariant vector $V_\nu$. According to (7a) only $V_0 \neq 0$ and $V_0 = \dot{V}_0(t)$. Hence

$$V_0 V_0 = \partial_0 V_0 - \Gamma^0_{00} V_0 = d(t) \Rightarrow \Gamma^0_{00} \equiv c_0(t),$$

where $d(t)$ is some function of time. We see that $\Gamma^0_{00}$ depends on time only. Moreover

$$0 = \nabla_0 V_i = \partial_0 V_i - \Gamma^0_{0i} V_a \Rightarrow \Gamma^0_{0i} = 0,$$

$$0 = \nabla_i V_0 = \partial_i V_0 - \Gamma^0_{i0} V_a \Rightarrow \Gamma^0_{i0} = 0,$$

$$f(t) \tilde{g}_{ij} = \nabla_i V_j = \partial_i V_j - \Gamma^a_{ij} V_a \Rightarrow \Gamma^0_{ij} = -\frac{f(t)}{V_0} \tilde{g}_{ij} \equiv c_n(t) \tilde{g}_{ij}.$$  

Keeping in mind that maximally symmetric contravariant vectors only have one nonvanishing component, one finds that

$$\Gamma^{i}_{0j} = c_i(t) \tilde{g}^i_j.$$  

Similar constraints can be derived for rank 2 tensors, for example

$$0 = \nabla_0 B_{0i} = \partial_0 B_{0i} - \Gamma^0_{0i} B_{0\ell} - \Gamma^0_{i0} B_{0\ell} = -\Gamma^0_{00} B_{ii}.$$  

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(no sum in the last form), implying that \( \Gamma^i_{00} = 0 \). Correspondingly the 0\(ij\)-component gives

\[
\Gamma^i_{j0} = c_i(t)\delta^i_j.
\]

(17)

The discussion above covers 37 components of the connection reducing them to four independent components \( c_0, c_t, c_n \) and \( c_s \). The last 27 components are found using the results of section 2 in three dimensions. Assuming that the non-metricity tensor vanishes in the maximally symmetric subspace the connection can be written as

\[
\Gamma^k_{ij} = \begin{pmatrix} k \end{pmatrix} + K(t)\epsilon^k_{ij},
\]

(18)

where \( \epsilon_{ijk} \) is the three-dimensional Lévi-Civita symbol. Note that here the second term, i.e. the contortion tensor, is invariant under cyclic permutations leaving only one degree of freedom.

Thus the connection preserving maximal symmetry in a three-dimensional homogeneous and isotropic subspace can be reduced to four spatio-temporal components \( c_i(t) \), one component, \( K(t) \), characterizing spatial torsion and the usual metric Christoffel symbols of a maximally symmetric subspace. Their usual metric counterparts are

\[
\begin{align*}
    c_0 &= 0 \\
    c_t &= c_t = \frac{\dot{a}}{a} \\
    c_n &= a \dot{a} \\
    K &= 0
\end{align*}
\]

(19a)

(19b)

(19c)

(19d)

with \( b(t) = 1 \).

3.2. The Ricci tensor and scalar

The Ricci tensor and curvature scalar are now straightforwardly calculable. The Ricci tensor is given by [29]

\[
R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\gamma_{\nu\beta} \Gamma^\alpha_{\gamma\mu} - \Gamma^\gamma_{\mu\beta} \Gamma^\alpha_{\gamma\nu}.
\]

(20)

The components 0\(i\) and \(i0\) vanish as they are maximally symmetric vectors of rank 1 in the subspace. The temporal 00 component reads as

\[
R_{00} = 3(\dot{c}_s + c_0 c_s - c_t c_s)
\]

(21)

and the spatial components can be expressed as

\[
R_{ij} = \tilde{R}_{ij} + (\dot{c}_n + c_n c_0 + 2c_n c_s - c_t c_n)\tilde{g}_{ij} + S_{ij},
\]

(22)

where \( \tilde{R}_{ij} \) is the standard Ricci tensor of the spatial part. Here the last term carries information on spatial torsion,

\[
S_{ij} \equiv \partial_k (K \epsilon_{ik}^j) + K \epsilon_{ij}^k \begin{pmatrix} l \end{pmatrix} \tilde{g}_{lk} + 2K \epsilon_{ij}^k \begin{pmatrix} l \end{pmatrix} \tilde{g}_{lk} - K^2 \epsilon_{li}^j \epsilon_{jk}.
\]

(23)

As \( S_{ij} \) is antisymmetric and \( \tilde{g}_{ij} \) symmetric, contraction of the Ricci tensor yields

\[
R = -\frac{3}{a^2} \left( 2k + \dot{c}_s + \frac{\dot{c}_s a^2}{b^2} + 2c_n c_s + C \left( c_n - c_t \frac{a^2}{b^2} \right) - 2K^2 \right),
\]

(24)

where we have used the fact that \( \tilde{R} = -6k/a^2 \) and denoted \( C \equiv c_0 - c_t \). Note that one can also derive the curvature scalar by using only the torsion tensor instead of the connection, as was done in [27].
4. Field equations

4.1. The Einstein–Hilbert action with hypermomentum

Although our goal is to study the results of the previous section in a general $f(R)$ model, let us first consider the Einstein–Hilbert action in a universe containing matter with non-zero hypermomentum and perfect fluid style energy–momentum. This is by no means the first time these equations of motion are derived (cf [23]) but this provides us a way to fix our notation. We use the energy–momentum and hypermomentum tensors defined as

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}, \quad \Delta_{\mu}{}^{\alpha} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta \Gamma_{\mu}{}^{\alpha}{}_{\nu\rho}}. \]  

(25)

In the case of the Einstein–Hilbert action we set

\[ f(R) = R. \]  

(26)

We also set $b(t) = 1$ which we know must be a solution [30]. Now the equations of motion are

\[
2\kappa T_{i}{}^{i} = -\frac{1}{a^2} (2k + c_n (C + 2c_s) - 2K^2 + 3\dot{c}_n - 2\dot{c}_s) + 3(Cc_s - \dot{c}_s) \tag{27a}
\]

\[
2\kappa T_{0}{}^{0} = -\frac{3}{a^2} (2k + c_n (C + 2c_s) - 2K^2 + \dot{c}_n) + 3(Cc_s - \dot{c}_s) \tag{27b}
\]

\[
\frac{\Delta_{0}{}^{0}}{6} = c_s - \frac{c_n}{a^2} \tag{27c}
\]

\[
\frac{a^2}{6} \Delta_{i}{}^{i} \tilde{g}_{ij} = \frac{\dot{a}}{a} - C - 2c_s \tag{27d}
\]

\[
\frac{\Delta_{i}{}^{i}}{6} = C + \frac{3a\dot{a} - 2c_n}{a^2} \tag{27e}
\]

\[
\frac{a^2}{12} \Delta_{k}{}^{i} \epsilon_{ij}{}^{k} = K. \tag{27f}
\]

From these equations the functions $c_i$ can be solved

\[
c_n = -\frac{a}{24} (2a\Delta_{0}{}^{0} + a\Delta_{i}{}^{i} + a^2 \Delta_{0}{}^{i} \tilde{g}_{ij} - 24\dot{a}) \tag{28a}
\]

\[
C = \frac{1}{12} \left( \Delta_{i}{}^{i} - a^2 \Delta_{0}{}^{i} \tilde{g}_{ij} - 12 \frac{\dot{a}}{a} \right) \tag{28b}
\]

\[
c_s = \frac{1}{24} (2\Delta_{0}{}^{0} - \Delta_{i}{}^{i} - a^2 \Delta_{0}{}^{i} \tilde{g}_{ij}) + \frac{\dot{a}}{a} \tag{28c}
\]

Inserting these into the equations of motion for $a$ and $b$ and choosing perfect fluid energy–momentum tensor we find the Friedmann equation

\[
H^2 = \frac{k\rho_0}{3a^2} = \frac{k}{a^2} + HA_1 + A_2 + a^2 A_3 - a^4 A_4. \tag{29}
\]

with

\[
A_1 = \frac{\Delta_{0}{}^{0}}{4} + \frac{a^2}{6} \Delta_{0}{}^{i} \tilde{g}_{ij} \tag{30a}
\]
\[ A_2 = \frac{1}{144} \left( (\Delta_0^0)^2 - 4 \Delta_0^0 \Delta_i^0 + \frac{(\Delta_i^0)^2}{4} \right) \]  
\[ A_3 = \frac{1}{144} \left( (\Delta_i^j \epsilon_i^j)^2 + \frac{4 \Delta_0^0 \Delta_k^0 \tilde{g}_i^0}{\Delta_k^0} - \Delta_0^0 \Delta_0^0 \tilde{g}_i^0 \right) \]  
\[ A_4 = \frac{1}{192} (\Delta_0^0 \tilde{g}_i^0)^2. \]  

From this form we clearly see that setting hypermomentum to zero yields general relativity as was to be expected.

### 4.2. A General f(R) Lagrangian without hypermomentum

Modified gravity theories in which the Lagrangian is a function of the curvature scalar have received much attention (e.g. [31–34]). Adding terms of the type \( \mathcal{L} = \mathcal{R} \) is a natural and simple modification to the general relativity. This type of terms can produce early time inflation [35] and late time accelerating expansion [31].

Further motivation for the \( f(R) \) gravity can be found in the fact that is equivalent to a certain class of scalar tensor theories [36]. One might ask why restrict to functions of only the curvature scalar. Simplicity is one reason but there are also underlying problems with Lagrangians depending upon more than one time derivative [37, 38]. Functions of curvature scalar only avoid the linear instability troubling other possibilities.

However, using the metric formalism there are problems with the \( f(R) \) gravity [39]. Most of the work on \( f(R) \) gravity is done in metric formalism. Here we look into the possibilities of using metric-affine formalism and maximal symmetry together with \( f(R) \) gravity.

The analysis in a general \( f(R) \) theory with matter follows along similar lines as above. We assume that the matter Lagrangian \( \mathcal{L}_m \) does not depend explicitly on the connection, i.e. the hypermomentum is zero. This does not necessarily hold for the cosmic fluid but it still possesses some interesting characteristics. In this case the gravitational Lagrangian is given by \( \mathcal{L} = ba^3 f(R(a, b, C, \dot{c}_n, \dot{c}_a, \dot{c}_s, \ddot{c}_s, K)) \) and the field equations are now

\[ 2 \kappa T_0^0 = f'(R) + 6 \frac{\dot{b}}{b^2} (c_s - C c_n) f'(R) \]
\[ 0 = f'(R) \left( \frac{c_s}{b^2} - \frac{c_n}{a^2} \right) \]
\[ f''(R) \frac{\dot{R}}{R} = f'(R) \left( C + 2 c_s - \frac{b}{a} - \frac{\dot{a}}{a} \right) \]
\[ f''(R) \frac{\dot{K}}{K} = -f'(R) \left( C - 2 c_n \frac{b^2}{a^2} - \frac{b}{a} + 3 \frac{\dot{a}}{a} \right) \]
\[ 0 = f'(R) K. \]

If \( f'(R) \neq 0 \), the third and the last equations are readily solvable,

\[ c_s = \frac{b^2}{a^2} c_n, \quad K = 0. \]
Summing equations (31d) and (31e) and using (32) we find
\[ C = \dot{b}b - \dot{a}a. \]  
(33)

Combining (31a), (31b), (32) and (33) gives
\[ \frac{b^2}{a^2}c_n^2 + k + c_n \left( \frac{\dot{a}}{a} - \frac{\dot{b}}{b} \right) - \dot{c}_n = \frac{\kappa a^2 (T^i_j - 3T^0_0)}{6f'(R)}. \]  
(34)

Because the curvature scalar \( R \) can be expressed in terms of \( c_n, a \) and \( b \), equation (34) is a nonlinear first-order equation for \( c_n \). It can be solved, at least in principle, for a given \( f(R) \).

In the absence of matter summing the first two equations gives the trace equation,
\[ f'(R)R - 2f(R) = 0. \]  
(35)

This differential equation is readily solved for
\[ f(R) = cR^2 \]  
(36)

with some constant \( c \). Because by plugging in a given \( f(R) \) we can solve the equation for a constant \( R \), empty space is necessarily a space with constant scalar curvature. Equations (31d) and (31e) then yield
\[ c_n = \frac{\dot{a}}{a}. \]  
(37)

Thus we end up with same components for the connection as for the case of the Einstein–Hilbert action without matter. We can hence conclude that in a homogeneous and isotropic space without matter, the metric-affine formalism results in the same equations as metric formalism. As an easy check shows, adding the cosmological constant leaves the situation unaltered. Therefore, the possible new effects of metric-affine formalism are due to matter.

With matter that is not coupled to the independent connection, we still get equations (32)–(34). The trace equation, however, changes. If the matter energy–momentum tensor is of perfect fluid form with \( T^0_0 = -\rho \) and \( T^i_i = 3p \) we have
\[ f'(R)R - 2f(R) = \kappa (3p - \rho). \]  
(38)

Here we note that in the special case of radiation filled universe the right-hand side vanishes and once again we reproduce the results of metric formalism. Moreover, if the hypermomentum were present all the aforementioned equations would change. Even the simple (31f) would become non-trivial and giving \( K \propto (a^3f'(R))^{-1} \). As the nature of the gravitation–matter coupling is not completely clear even this approach has some potential interest.

Although a radiation-dominated universe reproduces the metric cosmology, this is not a general property. For example, if we choose \( f(R) = R + \lambda R^2 \), with \( \lambda \) some small constant, and examine a non-relativistic matter filled universe, the trace equation (38) yields
\[ R = \kappa \rho = \frac{\kappa \rho_0}{a^3}, \]  
(39)

where \( \rho_0 \) is a constant and we have rescaled time so that \( b = 1 \). From equations (32), (33) and (31d) we then get
\[ c_n = \frac{a\dot{a}(a^3 - \kappa \rho_0 \lambda)}{2\kappa \rho_0 \lambda + a^5}. \]  
(40)

Clearly we need \( \rho_0 = 0 \) in order to reproduce \( c_n = \dot{a}a \) (i.e. the metric solution), leaving empty space as the only possibility. If, however, we allow for non-Lévi-Civitá connections there are
other possibilities. Inserting equation (40) into (31b) and (39) we can eliminate $\dot{a}$ to obtain an effective Friedmann equation

$$H^2 = \frac{(2\kappa + a^3)(\frac{1}{2}k^2\lambda\rho_0^2 + 6\kappa\lambda ka - \kappa\rho_0 a^3 + 3ka^4)}{3a^3(a^3 - \kappa\lambda\rho_0)^2}. \quad (41)$$

If we expand this equation in $\lambda$, the result is

$$H^2 = \frac{\kappa\rho_0}{3a^3} \cdot \left( \frac{k}{a^2} + \left( \frac{7k^2\rho_0^2}{6a^6} - \frac{6\kappa\rho_0 k}{a^2} \right) \lambda + \mathcal{O}(\lambda^2). \quad (42)$$

The limit $\lambda \rightarrow 0$ coincides with standard cosmology as expected. Note, that the correction $\propto a^{-6}$ can be created also by adding non-metric matter coupling, i.e. hypermomentum, as in [27], but here it is created solely by the form of the gravitational Lagrangian. Comparing equation (41) to the results in the so-called Palatini formalism\(^2\) [40–42] we find that they agree.

This raises the question, whether our maximally symmetric approach generally coincides with the results in the more commonly considered Palatini formalism. Another reason to suspect similar results is that in [12] metric-affine formalism with fully vanishing non-metricity is found to produce the same dynamics as the Palatini formalism.

In order to answer this question, we consider a toy model where the Lagrangian is of the form $f(R)$. Here we can check that equation (34) accepts this kind of Lagrangian and easily see that it is acceptable. Following the procedure above results in an effective Friedmann equation

$$H^2 = -\frac{4n^2k}{(n-3)^2a^2} - \frac{2n(n+1)}{3(n-3)^2} A^2, \quad (43)$$

where $A = \frac{\kappa\rho_0}{n-2na}$. The corresponding equation in the Palatini formalism reads as

$$H^2 = \frac{2n\left((1-n)A^2a^3 + 2\kappa\rho_0 A^2 - 6nka\right)}{3a^3(7n+6)n - 9}. \quad (44)$$

Hence, we see that the coincidence in the $\lambda R^2$ model was an exception: the maximally symmetric formalism and the Palatini formalism in general lead to different dynamical equations. The difference is pronounced in the case of $n = 3$, where the Palatini formalism is well behaved but here we find that our approach is singular in the sense that no Friedmann equation can be derived. Note, that there is also singularity at $n = 2$ in both cases as the trace equation for $f(R) = R^2$ holds only in empty space. Our result should be compared with the result of [12] where it was found that metric-affine formalism with torsion only does coincide with Palatini formalism.

A much studied special case of the $f(R)$ models is the $f(R) = R - \frac{\mu^4}{R}$ (cf [11, 31]). Inserting this $f(R)$ and a dust-dominated universe yields the Friedmann equation. The corrections to general relativity in this formalism are most easily seen when we expand in $\mu$

$$H^2 = \frac{\kappa\rho_0}{3a^3} \cdot \left( \frac{k}{a^2} + \frac{2a^3(9ka - 4\kappa\rho_0)}{3k^2\rho_0^2} \mu + \mathcal{O}(\mu^2). \quad (45)$$

Clearly this is different from that in metric formalism. This is important as in [39] it is shown that in the metric formalism this action leads to instability effectively ruling it out as a viable

\(^2\) The term Palatini formalism is widely used but misleading as it was in fact Einstein who first varied independently metric and connection [8]. We have adopted the convention of Sotiriou, i.e. we will call $f(R)$ theories in which the matter action is chosen to be independent of the connection, $f(R)$ theories of gravity in the Palatini formalism, to make the distinction from metric-affine $f(R)$ gravity [15].
model. As the different Friedmann equations mean different dynamics it is possible to avert this instability.

5. Conclusions

In this paper, we have studied a homogeneous and isotropic spacetime with a maximally symmetric formalism in $f(R)$ theories of gravity. Even though this is not the most general case homogenous and isotropic spacetime is an important special case in cosmology. The effects of homogeneity and isotropy in the standard Einstein–Hilbert case has been discussed before [27] but here we have shown that even in more general $f(R)$ theories, only one spurious extra degree of freedom appears in empty space.

Interesting possibilities begin to emerge, when one includes matter in the system. In the case of the Einstein–Hilbert action, the addition of matter without hypermomentum does not change the solutions of the field equations from those of metric formalism. New types of solutions appear only if the matter Lagrangian has an explicit dependence on the connection [27], in which case the connection is even less determined for general $f(R)$ Lagrangians. These results are in accordance with those of [15] where it was argued that torsion is caused by the antisymmetric part of the hypermomentum.

However, even for ordinary matter (i.e. no hypermomentum), the construction of the Friedmann equations reveals that the maximally symmetric formalism is dynamically different from the corresponding Palatini formalism although they may coincide in some special cases. This appears to be a consequence of inclusion of spatio-temporal non-metricity. Indeed the difference between the two formalisms is due to the fact that in the Palatini formalism torsion is assumed to vanish a priori whereas here only spatial non-metricity is assumed to vanish. Therefore the degrees of freedom in these two approaches are dissimilar resulting in a differently constrained system. Physically it is unclear which approach one should adopt. As there is almost no evidence for torsion, the usual pick would be Palatini formalism. Metric-affine formalism, however, is more general and is based on the explicit use of the cosmological principle.

In [15] it is argued that all constraints on non-metricity also place a constraint on the form of the Lagrangian and should therefore be avoided. We agree with the first argument. Our equation (34) is an example of these constraints. However, in this case the constraint allows non-trivial forms of $f(R)$. Thus, we do not see the necessity for the latter argument. We find that our formalism reduces to Palatini formalism only in special cases. This is not in contradiction with [15] since our assumption of vanishing spatial components of non-metricity differs from the assumption of [15] (i.e. the Weyl vector vanishes).

In all cases a spurious degree of freedom which has little or no physical meaning remains. It emerges because two components of the connection appear only as a certain combination in the Lagrangian. As they affect the physics of the universe only via this combination, their geometrical interpretation can be found if there are non-metric matter couplings present.

The cosmological consequences of the maximally symmetric formalism are an interesting possible direction of studies as well as generalization to spherically symmetric systems. Both are likely to give at least some constraints for a given $f(R)$ theory. Cosmological data (e.g. CMB and supernova data) could be fitted to metric-affine gravity models with maximal symmetry in order to find the constraints in this formalism.

Furthermore, although isotropy is commonly accepted there have been numerous articles investigating the possibility of an inhomogeneous universe [5–7, 43, 44], motivating further study of the connection in an inhomogeneous and isotropic space. These results could be used to ease the usage of metric-affine formalism in spherically symmetric universes.
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