Morita Duality and Large-\(N\) Limits

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ABSTRACT

We study some dynamical aspects of gauge theories on noncommutative tori. We show that Morita duality, combined with the hypothesis of analyticity as a function of the noncommutativity parameter \(\Theta\), gives information about singular large-\(N\) limits of ordinary \(U(N)\) gauge theories, where the large-rank limit is correlated with the shrinking of a two-torus to zero size. We study some non-perturbative tests of the smoothness hypothesis with respect to \(\Theta\) in theories with and without supersymmetry. In the supersymmetric case this is done by adapting Witten’s index to the present situation, and in the nonsupersymmetric case by studying the dependence of energy levels on the instanton angle. We find that regularizations which restore supersymmetry at high energies seem to preserve \(\Theta\)-smoothness whereas nonsupersymmetric asymptotically free theories seem to violate it. As a final application we use Morita duality to study a recent proposal of Susskind to use a noncommutative Chern-Simons gauge theory as an effective description of the Fractional Hall Effect. In particular we obtain an elegant derivation of Wen’s topological order.

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1. Introduction

Noncommutative Field Theories (NCFTs) provide an interesting generalization of the framework of local quantum field theory, allowing for some degree of non-locality while still keeping an interesting mathematical structure [1,2,3]. From a different point of view, the perturbative dynamics of NCFT mimics in many respects that of string theory. To be more specific, NCFT arises as a peculiar low-energy limit of open-string theory, [4,5] so that the string becomes a rigid, extended object, i.e. a rigid ‘dipole’ [6]. This is achieved by placing the strings in a large magnetic field. The resulting low-energy theory lives effectively on a noncommutative version of $\mathbb{R}^d$, where the non-locality is parametrized by a quantum phase-space structure of space-time: $[x^\mu, x^\nu] = i \theta^{\mu\nu}$, with $\theta^{\mu\nu}$ related to the string magnetic field $B_{\mu\nu}$ via $\theta = B^{-1}$. The stringy nature of NCFT is not really hidden in the short-distance structure, but apparent at low energies, since the size of the ‘dipoles’ is approximately linear in the momentum: $\ell_{\text{eff}} = |\theta^{\mu\nu} p^\nu|$, so that high energy particles are macroscopic in size. This is the basis for the so-called UV/IR connection, [7] that represents the main novelty of NCFT dynamics. In particular, if time enters the noncommutativity relations, the particles are non-local in time, which poses a problem for Hamiltonian methods. In fact, these theories appear to be inconsistent when defined as field theories with a finite number of particle degrees of freedom [8,9]. We henceforth restrict the noncommutativity of space–time to the purely spatial sections.

Another ‘stringy’ property of NCFT is the so-called Morita duality of gauge theories on noncommutative tori [1,4,10] a low-energy remnant of the T-duality symmetry of the underlying string model [11] (see also [12].) It acts on the periods of the string magnetic field $\int B/2\pi$ by fractional linear transformations. In particular, if the matrix of periods has rational entries, there is a Morita transformation that maps the given noncommutative theory to an ordinary $U(N)$ gauge theory on a smaller torus, with a larger rank and some non-vanishing magnetic fluxes. This opens up the interesting possibility of using the information encoded in the $\theta$-dependence of physical quantities to learn about ordinary gauge theories at finite volume, i.e. the complicated structure of confining, oblique-confining, Higgs and Coulomb phases of gauge theories could be encoded in the noncommutative language in terms of some interesting modular properties as a function of $\theta$. Conversely, the exotic features of perturbation theory in NCFT, notably the UV/IR phenomenon, put into question most of our standard expectations for non-perturbative dynamics in NCFT. In this context, Morita duality can be used to translate the standard body of knowledge about ordinary confining gauge theories into the context of their noncommutative cousins. In particular, it was suggested in [13] that smooth behaviour of physical quantities in $\theta$ would imply very nontrivial constraints on the large-$N$ limit of ordinary gauge theories on tori.
The interplay between noncommutativity and the large-$N$ limit is at the root of the subject in its (hidden) beginnings [14], since NCFT is a formal continuum limit of the twisted reduced Eguchi–Kawai models [15] (see also [16,17]). In this article we sharpen this relationship with a number of qualitative and quantitative tests. We propose an analyticity criterion on $\theta$-dependence based on the limiting behaviour of rational approximations to generic NCFTs. We show that this criterion of $\theta$-smoothness gives information about ordinary gauge theories in a singular large-$N$ limit. Although this singular limit is difficult to study in the language of ordinary gauge theory, we argue that its dynamics can be very rich.

We also add some comments on the inverse problem, i.e. we study degenerate limits of NCFTs which are tuned to reproduce the standard confinement regime of ordinary gauge theories in the large-$N$ limit. Morita duality implies that all these limits necessarily involve infinite noncommutativity $\theta \to \infty$, a property already hinted at by the behaviour of perturbation theory [15,18,7].

Some exact information about the singular large-rank limits is obtained by considering rational approximations of $\mathcal{N} = 1$ theories in three and four dimensions. We define an appropriate Witten index that probes aspects of confinement dynamics in the non-abelian sector of the models, and show that this index varies smoothly with $\theta$. In some cases this behaviour depends on subtle properties such as oblique confinement in ordinary gauge theories.

In section five we go one step further by considering the dependence of energy levels on the instanton angle in non-supersymmetric theories. For a specific choice of electric and magnetic fluxes, a reliable estimate is possible within an instanton-gas approximation, since the corresponding energy splittings receive no contribution in perturbation theory. This is a rather sensitive test of $\theta$-smoothness, because of the occurrence of level-crossing phenomena that depend explicitly on the rank of the gauge theory. We find that a regularization with restored $\mathcal{N} = 4$ supersymmetry at high energies preserves $\theta$-smoothness of the low-energy physics, whereas asymptotically free non-supersymmetric theories seem to violate the smoothness constraints. Hence, some form of supersymmetry, albeit ‘softly broken’ seems to be necessary to maintain continuity of the physics as a function of $\Theta$.

We end with an application of Morita duality to the Quantum Hall Effect by calculating Wen’s topological order [19] for the proposed noncommutative Chern–Simons effective description of the Fractional Hall Effect, [20].
2. Morita Equivalence of Gauge Theories

We consider $U(N)$ Noncommutative Yang–Mills Theories (NCYM) on $S^1_\beta \times T^3_\theta$, with $S^1_\beta$ representing a compact euclidean time direction of length $\beta$, and $T^3_\theta$ a noncommutative three-torus with flat metric. The microscopic parameters of the $U(N)$ gauge theory are the Yang–Mills coupling at some cutoff scale, $g$, the instanton angle, $\vartheta$, and the spatial noncommutativity parameters $\theta^{ij}$. There is also a background field $\phi_{ij}$ that enters the physics as a constant $U(1)$ shift of the field strength, so that the action reads

$$S = \frac{1}{2g^2} \int \text{tr} \left( (F + \phi)_{\mu\nu}(F + \phi)^{\mu\nu} + \frac{i\vartheta}{8\pi^2} \int \text{tr} \left( (F + \phi) \wedge (F + \phi) \right) + \ldots \right), \quad (2.1)$$

where the dots stand for other terms depending on extra fields, such as fermions or scalars in supersymmetric theories, all of them in the adjoint representation of the gauge group, and $F$ is the noncommutative gauge field strength.

The Feynman rules of this theory are obtained from those of the ordinary $U(N)$ theory by the following replacement of the $U(N)$ structure constants:

$$f^{\alpha\beta\gamma} \rightarrow f^{\alpha\beta\gamma} \cos \left( \frac{1}{2} \theta^{ij} k_i k_j' \right) + d^{\alpha\beta\gamma} \sin \left( \frac{1}{2} \theta^{ij} k_i k_j' \right). \quad (2.2)$$

$k, k'$ are any two momenta entering the trilinear vertex. In particular, we see that the global $U(1)$ is self-coupled and coupled to the $SU(N)$ subgroup. Indeed, the rank-one $U(1)$ theory shows asymptotic freedom [21].

It is useful to define a dimensionless noncommutativity parameter and background field by

$$\Theta^{ij} = \frac{2\pi}{L_i L_j} \theta^{ij}, \quad \Phi_{ij} = \frac{L_i L_j}{2\pi} \phi_{ij}. \quad (2.3)$$

When considering the Hamiltonian quantization on $T^3_\theta$, the Hilbert space splits into sectors labelled by the integer magnetic fluxes

$$m_{ij} = \int_{(ij)} \text{tr} \frac{F}{2\pi}, \quad (2.4)$$
determined by the first Chern class of the $U(N)$ gauge bundle, together with the usual integer momenta

$$p_j = -i L_j \int_{T^3_\theta} \text{tr} \frac{F_{jk} \delta}{2\pi} \delta A_k. \quad (2.5)$$

There are also integrally quantized electric fluxes $w^j$ of the form (c.f. [22]):

$$w^j = -i \frac{L_j}{L_i} \int_{T^3_\theta} \text{tr} \frac{\delta}{\delta A_j} - \Theta^{jk} p_k. \quad (2.6)$$
We shall frequently use the vector notation for the magnetic fluxes and the background field:

\[ m^k = \frac{1}{2} \epsilon^{kij} m_{ij}, \quad \Phi^k = \frac{1}{2} \epsilon^{kij} \Phi_{ij}. \]

For simplicity, we consider orthogonal three-tori of the form \( T^3 = T^2_\theta \times S^1_e \) with \( \Theta \) and \( \Phi \) of rank two and aligned with the noncommutative two-torus \( T^2_\theta \), which will be assumed squared with side length \( L \). The length of \( S^1_e \) is \( L_e \), so that the volume of the three-torus is \( V = L_e L^2 \). In this simple case Morita duality is represented by the \( SL(2, \mathbb{Z}) \) action:

\[
\begin{align*}
\Theta' &= \frac{b\Theta - a}{s + r\Theta}, \\
\Phi' &= (s + r\Theta)^2 \Phi - r(s + r\Theta), \\
\tau' &= \frac{\tau}{|s + r\Theta|}, \\
L' &= |s + r\Theta| L \tag{2.7}
\end{align*}
\]

on the parameters, where \( a, b, s, r \in \mathbb{Z} \) and \( sb + ar = 1 \). We have collected the gauge coupling and instanton angle into the complex coupling \( 2\pi \tau = \vartheta + 8\pi^2 i / g^2 \). The action on the other quantum numbers is

\[
\begin{pmatrix}
m' \\
N'
\end{pmatrix} = \begin{pmatrix} s & r \\ -a & b \end{pmatrix} \begin{pmatrix} m \\
N
\end{pmatrix}, \quad
\begin{pmatrix}
p' \\
* w'
\end{pmatrix} = \begin{pmatrix} s & r \\ -a & b \end{pmatrix} \begin{pmatrix} p \\
* w
\end{pmatrix}, \tag{2.8}
\]

where \( m \) is the magnetic flux through \( T^2_\theta \) and \( w, p \) are the two-dimensional vectors of electric fluxes and momenta along the same torus. The notation \( *w \) refers to the Hodge duality operation on the \( T^2_\theta \), i.e. \((*w)_i = \epsilon_{ij} w^j\).

In perturbation theory, Morita equivalence follows as an algebraic identity of the Feynman diagram expansion [15], or as T-duality of the regularized version in terms of an underlying string model [11]. More generally, there is a very explicit proof as a formal change of variables in a path integral [16].

Rational theories are characterized by a rational dimensionless noncommutativity parameter. If \( \Theta = a/b \), so that \( s + r\Theta = 1/b \), the Morita transformation (2.7) sends the theory to an ordinary one with \( \Theta' = 0 \) and different parameters, involving in particular a rescaling of the rank of the gauge group. In this case, the details of the transformation at the level of fields are rather elementary (c.f. [13, 23]). Consider, for example, a \( U(N) \)-valued noncommutative connection on \( T^2_\theta \times Y \), which is also periodic on the two-torus \( T^2_\theta \) of length \( L \), and has arbitrary boundary conditions on the commutative space \( Y \):

\[
A(x, y) = \sum_{\ell \in \mathbb{Z}^2} a_\ell(y) e^{-2\pi i \ell \cdot x / L}. \tag{2.9}
\]

The coefficient matrices \( a_\ell \) satisfy \( a_\ell^\dagger = a_{-\ell} \) and have arbitrary dependence on the commutative coordinates \( y \in Y \).
The Morita-dual is a twisted $U(N')$ theory with $N' = Nb$, with a connection given by a particular superposition of matrices in the subgroup $U(N) \otimes U(b)$:

$$A'(x, y) + A_{\phi'}(x, y) = \sum_{\ell \in \mathbb{Z}^2} a_\ell(y) \otimes V^{-a_\ell_1 U\ell_2} \omega^{-a_\ell_1 \ell_2/2} e^{-2\pi i \ell \cdot x/bL'},$$

(2.10)

where $\omega \equiv e^{2\pi i/b}$ and the pair $U$ and $V$ are the standard clock and shift matrices of $SU(b)$ satisfying:

$$UV = \omega VU.$$  

(2.11)

The connection $A_{\phi'}$ is a constant-curvature abelian gauge field in the diagonal $U(1)$ subgroup of $U(N')$ that satisfies $\phi' = dA_{\phi'}$. For our particular choice of periodic noncommutative connection $A$, the effect of $A_{\phi'}$ is to cancel the first Chern class induced by $A'$:

$$\int_{T^2} \text{tr} (F' + dA_{\phi'}) = \int_{T^2} \text{tr} (F' + \phi') = 0.$$  

(2.12)

The traceless part of the ordinary connection $A'$ furnishes a twisted bundle of $SU(N')/\mathbb{Z}_{N'}$ with 't Hooft magnetic flux $[m'] = rN \pmod{N'}$ and periodicity conditions

$$A'(x_j + L') = \Gamma_j A'(x_j) \Gamma_j^\dagger.$$  

(2.13)

The twist matrices may be chosen as

$$\Gamma_1 = 1_N \otimes U', \quad \Gamma_2 = 1_N \otimes V.$$  

(2.14)

Notice that the same matrix Fourier components $a_\ell$ appear in (2.9) and (2.10). Therefore, when expressed in terms of the $a_\ell$, both the action and the integration measure in the path integral remain formally invariant under the Morita transformation, which acquires the simple interpretation of a change of variables in the position-space representation. For our example of a periodic noncommutative two-torus we have the identity

$$\frac{1}{2g'^2} \int_{\mathcal{M}_\theta} \text{tr} |F|^2 + \frac{i\theta'}{8\pi^2} \int_{\mathcal{M}_\theta} \text{tr} F \wedge F$$

$$= \frac{1}{2g^2} \int_{\mathcal{M}'} \text{tr} |F' + \phi'|^2 + \frac{i\theta'}{8\pi^2} \int_{\mathcal{M}'} \text{tr} (F' + \phi') \wedge (F' + \phi'),$$

(2.15)

with

$$\mathcal{M}_\theta = T^2(L) \times Y, \quad \mathcal{M}' = T^2(L') \times Y,$$

and the couplings mapping according to the rules:

$$g'^2 N' = g^2 N, \quad \theta' = \theta b, \quad \Phi' = -\frac{r}{b}.$$  

(2.16)
Similar manipulations can be performed for other fields in the adjoint representation of the gauge group.

We assume that this Morita equivalence is a true isomorphism of the physical Hilbert space of the theory that, in particular, preserves the exact finite-volume spectrum, including non-perturbative dynamical scales related, for example, to confinement. In this case, the proper interpretation of (2.7) is as a mapping of \textit{bare} parameters at some cutoff scale \( \Lambda_{UV} \) that remains fixed under the duality. A convenient regularization that respects the duality at the short-distance scales is to consider a \( \mathcal{N} = 4 \) SYM theory with appropriate mass terms at the scale \( M_s = \Lambda_{UV} \). In this case, the map (2.7) applies to the \( \mathcal{N} = 4 \) microscopic parameters.

Thus, barring possible ‘Morita anomalies’, (2.7) should hold non-perturbatively for the continuum theory. In \( \mathcal{N} = 4 \)NCYM theory, the explicit computation of the spectrum of \( \frac{1}{4} \) and \( \frac{1}{8} \) BPS states by a number of groups [24,11] gives a Morita-invariant result. In particular, the energies of abelian electric and magnetic fluxes are Morita-invariant. Based on the evidence provided by these examples, in the rest of this paper we assume that Morita equivalence is free from anomalies and holds as a true quantum symmetry of the continuum theory.

3. Rational Approximations and Singular Large-\( n \) Limits

Given an infinite convergent sequence of rational numbers determining some noncommutativity parameter: \( \Theta_n \to \Theta \), we may define the noncommutative theory arising as a limit of the Morita-dual ordinary theories, provided that this limit actually exists. The existence of this limit is a necessary condition for the smoothness of the physics as a function of \( \Theta \). We symbolically denote the resulting limiting theory by

\[
\lim_{n \to \infty} U(N, \Theta_n) \equiv \overline{U(N)}_{\Theta}.
\]

Writing \( \Theta_n = a_n/b_n \), with \( (a_n, b_n) = 1 \) (i.e. relatively prime,\(^2\)) and \( b_n > 0 \), we determine appropriate Morita transformations to ordinary theories by defining numbers \( r_n, s_n \) with the property \( s_nb_n + r_na_n = 1 \) and performing the transformation (2.7) for each value of \( n \):

\[
T_n = \begin{pmatrix} s_n & r_n \\ -a_n & b_n \end{pmatrix}.
\]

For given values of \( a_n \) and \( b_n \), we can fix the freedom in the choice of the pair \( r_n, s_n \) by picking \( 0 \leq r_n < b_n \).

\(^2\) The symbol \((a, b)\) for any two integers \(a, b\) represents their greatest common divisor.
After Morita duality, we obtain a series of $U(N'_n)$ models with rank and magnetic flux:

$$N'_n = b_n N, \quad m'_n = r_n N,$$

(3.3)

where we have assumed that the initial magnetic flux of all $U(N, \Theta_n)$ models vanishes, a condition that may be enforced by an appropriate Morita transformation.

Since $b_n \to \infty$ in the limit, any such rational approximation is a large-rank limit. However, since the length of the ordinary tori also shrinks by

$$L'_n = \frac{L}{b_n},$$

(3.4)

we actually have a particular singular large-$n$ limit, in which the large-rank limit is correlated with the scaling of a two-torus to zero size. One can say that the dynamics of the model $\overline{U(N)_{\Theta}}$ resolves such singular limits of ordinary gauge theories. In principle, it is not guaranteed that the model $\overline{U(N)_{\Theta}}$ exists in all situations, in the sense of all gauge-invariant operators having smooth matrix elements in the limit. The evidence from BPS sectors in $\mathcal{N} = 4$ NCYM suggests that such limit makes sense at least for sufficiently supersymmetric theories.

There is evidence that $\Theta$-smoothness, if true, must involve rather non-trivial dynamical phenomena. In Refs [25,26], Morita duality was used as a tool to provide an optimum set of quasilocal descriptions of the physics of $\mathcal{N} = 4$ NCYM theories on a torus. A ‘quasilocal description’ is defined by requiring that the elementary degrees of freedom are well-contained in the box, taking into account both the quantum size given by the Compton wave-length and the ‘classical size’ given by the ‘dipolar extent’ of noncommutative particles: the effective size scales with energy as $\ell(E)_{\text{eff}} = \max(1/E, E \theta)$. Demanding $\ell(E)_{\text{eff}} < L$ gives the definition of a given quasilocal patch $1/L < E < 2\pi/\Theta L$.

It is found in [20] that the structure of quasilocal patches depends sensitively on $\Theta$. Not only it depends on the rational or irrational character of $\Theta$, but even on the ‘degree of irrationality’ according to some well-defined criteria. Although some non-BPS quantities such as the entropy in the planar approximation behave smoothly as a function of $\Theta$, it is much less clear that the generic non-BPS physical quantity (particularly at the level of non-planar corrections) will behave smoothly given the ‘multifractal’ nature of the renormalization-group flows.

Ignoring for the moment these caveats, we can entertain a strong form of the smoothness hypothesis (see [13]) and conjecture that, in appropriate situations, the $\overline{U(N)_{\Theta}}$ model is actually equivalent to some other ab initio definition of the $U(N)_{\Theta}$ theory. This strong form of the conjecture severely constrains the large-$n$ limit of the ordinary gauge theories. Unfortunately, it concers a singular large-$n$ limit, and it is unclear what practical information could be extracted for the regular large-$n$ limit of ’t Hooft [27].
For irrational $\Theta$, we can use the $\overline{U(N)_\Theta}$ theory as a tentative non-perturbative definition of the model. In this case, the best rational approximations are given by continued fractions. For rational $\Theta$, the $U(N)_\Theta$ is itself Morita-dual to some finite-rank gauge theory. Therefore, in this case, the putative limiting theory can be rigorously defined, and the equivalence of $\overline{U(N)_\Theta}$ and $U(N)_\Theta$ imposes particularly strong constraints.

Perhaps the most radical statement of $\Theta$-analiticity along these lines would be the one associated to the sequence $\Theta_n = 1/n \to 0$ in a rank-one model. In this case, the series of Morita-dual ordinary theories consists of $U(n)$ models with one unit of magnetic flux on a torus of volume $L_e L^2 / n^2$. The limit would be equivalent to an ordinary $U(1)$ theory on a torus of volume $L_e L^2$, i.e. a free theory. If true, such an equivalence is rather surprising in view of the UV/IR effects discovered in [7], which tend to render the theory non-analytic around $\theta = 0$.

In the remainder of this paper we evaluate the case for the equivalence $U(N)_\Theta \equiv \overline{U(N)_\Theta}$ from different points of view. Notably, we use an appropriately defined Witten index as a useful criterion for $\mathcal{N} = 1$ supersymmetric theories, and the dependence on the instanton angle as a (less robust) criterion for non-supersymmetric theories.

In the remainder of this section, we collect a number of observations on the qualitative physics of the $\overline{U(N)_\Theta}$ limiting models. As a final remark, if we try to define the noncommutative gauge theories using a lattice formulation along the lines proposed in [16], it seems that one would end up with a prescription similar to the limiting procedure described in these sections. According to [10] in their lattice formulation one is forced to have $\Theta$ rational, thus if we are interested in studying a theory with an irrational value of $\Theta$, as we take the continuum limit we should also describe a rational sequence depending on the lattice spacing converging to the value of interest. Whether this is the only way to define noncommutative theories on the lattice is an open question.

### 3.1. Cutoffs and Dynamical Scales

We can define two variants of the singular large-$n$ limits, depending on whether the ultraviolet scale $\Lambda_{UV}$ remains fixed in the large-$n$ limit, or it is scaled appropriately. If $\Lambda_{UV}$ remains fixed, eventually $L_n \Lambda_{UV} < 1$ and the small torus shrinks below the cutoff scale. In this case we must interpret $\Lambda_{UV}$ as a $\mathcal{N} = 4$ supersymmetry breaking scale, $M_s$, so that the short-distance definition of the $\overline{U(N)_\Theta}$ model refers to the $\mathcal{N} = 4$ theory.

Alternatively, we can take the continuum limit $\Lambda_{UV} \to \infty$ for each $n$ and define the large-$n$ limit of the series of continuum field theories. In this case, with $\mathcal{N} = 0, 1$ supersymmetry, we have asymptotically free theories for which the microscopic coupling parameter transmutes into a dynamical scale $\Lambda_n$. Since the value of each $\Lambda_n$ is adjustable,
we can specify its large-$n$ limit $\Lambda_\infty$ as part of our specification of the limiting procedure. One possible uniform definition is given by the localization of the one-loop infrared Landau pole in the planar approximation:

$$\Lambda_n = \Lambda_{\text{UV}} \exp \left( \frac{-8\pi^2}{\beta_0 \lambda_n(\Lambda_{\text{UV}})} + \cdots \right),$$

(3.5)

where the dots stand for higher (planar) loop contributions. In this formula, $\beta_0$ is the (positive) one-loop beta function coefficient with normalization

$$\beta_0 = \frac{11 - 2n_f - n_s}{3},$$

(3.6)

for a theory with $n_f$ Majorana fermions and $n_s$ complex scalars, all in the adjoint representation of the gauge group, and $\lambda_n(\Lambda_{\text{UV}}) = g_n^2(\Lambda_{\text{UV}})N'_n$ is the bare 't Hooft coupling of the $U(N'_n)$ theory. Under Morita duality, the 't Hooft coupling is invariant $g_n^2N'_n = g^2N$, with $g^2(\Lambda_{\text{UV}})$ the bare coupling of the $\overline{U(N)}_\Theta$ theory. In the limit, $\Lambda_n$ converges to $\Lambda_\infty$, the standard dynamical scale of the $U(\infty)$ gauge theory arising in 't Hooft’s planar limit.

It is important to realize that, despite the two-torus of vanishing volume $L'_n \to 0$, there is no dimensional reduction on the series of ordinary theories, i.e. the low-energy physics is not ‘trivialized’ into a $(1 + 1)$-dimensional renormalization-group flow. The reason is that all of the ordinary gauge theories in the series have non-zero magnetic flux through the vanishing torus, so that they support ‘light’ electric-flux excitations down to energies of order $1/L'_n b_n = 1/L$, which remains fixed in the limit, and is the true infrared threshold for the transition to $(1 + 1)$-dimensional physics. The interactions of these light delocalized modes between the high scale $b_n/L$ and the low scale $1/L$ are governed by a renormalization-group flow with beta function coefficient $\beta_0$ ([21], see also [28]). Therefore, asymptotically free theories with $L\Lambda_\infty \gg 1$ can develop strong coupling before the reduction to $(1 + 1)$-dimensional dynamics takes place.

In summary, the physics of the $\overline{U(N)}_\Theta$ models can be very rich, including possible non-perturbative phenomena at intermediate scales $\Lambda_\infty$. It must be emphasized that our heuristic reasoning is based on the planar one-loop approximation to the renormalization-group flow of the Yang–Mills coupling. In principle, non-planar diagrams can yield important contributions at energy scales of order $1/\sqrt{\Theta}$. For example, non-planar one-loop diagrams in the non-compact theory turn the screening behaviour of the global $U(1)$ modes into antiscreening, c.f. [29,30]. In this case, one should further impose the hierarchy $L\Lambda_\infty \gg L/\sqrt{\Theta} \gg 1$.

### 3.2. Decoupled Photons

Each of the rational $U(N,\Theta_n)$ theories in the approximating series $\Theta_n \to \Theta$ has a free ordinary Maxwell field effectively living on a torus of volume $V'_n = V/b_n^2$. This is a
general property of any rational noncommutative theory on a finite torus, as follows from the structure of the vertices in (2.2). Since momenta are quantized as \(k_i = 2\pi n_i / L_i\) and \(\theta^{ij} = \Theta^{ij} L_i L_j / 2\pi\), we obtain for the symmetric structure constants

\[
d^{\alpha\beta\gamma} \sin \left( \pi n_i \Theta^{ij} n_j' \right),
\]

so that modes in the diagonal \(U(1)\) subgroup of \(U(N)\) and with momenta given by integral multiples of \(b_n\) decouple. In the language of the ordinary \(U(N')\) Morita dual, the free \(U(1)\) is simply the diagonal subgroup of \(U(N')\). Moreover, since \(b_n \to \infty\), the excitation gap for these free photons diverges in the limit \(\Theta_n \to \Theta\), so that they are irrelevant for the \(\overline{U(N)}\) theory at finite volume. The decoupled photon modes at exceptional momenta \(k_i = 2\pi \ell_i b_n / L_i\) are responsible for ultraviolet divergences in non-planar diagrams, unlike the case of irrational theories in perturbation theory (c.f. [31]). Notice, however, that the infinite gap that is generated in the irrational limit for these free photons should render the perturbation theory of \(\overline{U(N)}\) equivalent to the standard irrational perturbation theory.

In the Hamiltonian formalism, the decoupled \(U(1)\) contributes an energy

\[
E_{U(1)} = E_\gamma + E_{\text{flux}},
\]

where \(E_\gamma\) is the energy in photons and \(E_{\text{flux}}\) denotes the contribution of electric and magnetic fluxes, which reads, in Morita-covariant form [11, 24]:

\[
E_{\text{flux}} = E_e + E_m = \frac{g^2}{4N_\Theta} \sum_i \frac{L_i^2}{V} (\Theta^{ij} p_j + w^i)^2 + \frac{1}{g^2 N_\Theta} \sum_i \frac{L_i^2}{V} (2\pi m^i + 2\pi N_\Theta \Phi^i)^2,
\]

where \(N_\Theta = N + \frac{1}{2} \Theta^{ij} m_{ij}\) is the dimension of the noncommutative module and \(g^2\) stands for the bare coupling at the cutoff scale \(\Lambda_{\text{UV}}\) (the \(\mathcal{N} = 4\) high-energy coupling for softly broken models.) This expression is Morita-invariant and depends smoothly on \(\Theta\). However, for asymptotically free models we take \(g^2(\Lambda_{\text{UV}}) \to 0\) in the continuum limit, yielding a divergent magnetic energy for \(m^i + N_\Theta \Phi^i \neq 0\). Thus, in many of the applications, we shall assume that the \(\overline{U(N)}\) model has \(m^i + N_\Theta \Phi^i = 0\). Since we can set \(m_{ij} = 0\) by an appropriate Morita transformation, we can assume with no loss of generality that \(\Phi^i = m^i = 0\). For the series of ordinary \(U(N')\) theories in (3.3), we have

\[
m'_n + N'_n \Phi'_n = 0.
\]

Notice that, under the same conditions: \(g^2(\Lambda_{\text{UV}}) \to 0\), the contribution to the energy from abelian electric fluxes becomes degenerate for all values of the electric flux.
3.3. Rational Constructions of Theories on $\mathbb{R}^3_\theta$

A variant of the limit discussed here can be introduced to provide a ‘purely rational’ definition of a noncommutative theory on $\mathbb{R}^3_\theta$ by a blow-up of $T^3_\theta$. Consider a sequence of rational theories $U(N, \Theta_n)$ with $\Theta_n = 1/n \to 0$ on $\mathbb{R} \times T^2_\theta$, where the noncommutative tori have size $L_n = \sqrt{2\pi n\theta} \to \infty$. The commutative description involves a series of $U(nN)$ ordinary theories defined on shrinking two-tori of size $L'_n = \sqrt{2\pi \theta/n}$ and supporting exactly $N$-units of magnetic flux. In terms of the series of ordinary theories, this limit is even more singular than the previous one, since the perturbative gap of electric fluxes vanishes as $O(1/\sqrt{n})$.

This ‘rational’ definition of the non-compact noncommutative models may be useful in discussing the non-perturbative physics associated to various UV/IR phenomena [7,32,33,34,29,30,35].

3.4. Large $N$ versus Large $\theta$

It is somewhat disappointing that the simplest criterion of $\Theta$-analyticity imposes constraints on singular large-$n$ limits, rather than the ordinary large-$n$ limit of 't Hooft [27]. Thus, an interesting variation of the limits constructed here would involve stabilizing the size of the shrinking commutative torus by hand. We can simply consider a combination of the $\Theta_n \to \Theta$ limit in the $U(N, \Theta_n)$ models, together with a large-volume scaling of the noncommutative torus, $L_n \to \infty$. Since $L'_n = L_n/b_n$, we must define the sequence of rational noncommutative theories on tori whose size, $L_n$, grows at least linearly with $b_n$.

More specifically, consider a $\overline{U(1)}_\Theta$ limiting model on growing tori of size $L_n = L'|b_n|^{\alpha}$ with $\alpha > 1$, and $L'$ a fixed length scale. The Morita-dual tori have sizes $L'_n = L'|b_n|^{\alpha-1}$, which grow in the limit. Hence, in this case we end up with exactly the dynamics of an ordinary $U(\infty)$ theory on $S^1_e \times \mathbb{R}^2$, with a free photon and a tower of free glueballs at the confining scale $\Lambda_\infty$. This rather standard large-$n$ spectrum is encoded in the variables of the noncommutative theory in a very nonlocal way, since the noncommutativity scale diverges as $\theta_n \sim L_n^2 \Theta_n \sim b_n^2 L'^2 \Theta \to \infty$. This is a simple way of using Morita duality to argue that the ordinary large-$n$ limit is captured by a $\theta \to \infty$ limit [15,14,7,36].

Hence, we find that the $\Theta$-analyticity criterion, when forced to reproduce a standard large-$n$ limit, relates it to an extremely non-local description of physics. As an illustration of the arbitrariness of these limits, let us consider the marginal case of the scaling above with $\alpha = 1$. The resulting noncommutative geometry is superficially identical to that of $\alpha > 1$, i.e. $S^1_e \times \mathbb{R}^2_\infty$. However, the Morita-dual tori have fixed volume $L'_n = L'$, so that we still have confining phenomenology on a large box, provided $L' \Lambda_\infty \gg 1$, except that now
we also have a photon with a discrete spectrum of gap $1/L'$. This is rather exotic when viewed from the point of view of the limiting noncommutative theory, since the length scale $L'$ is not directly associated to the limiting geometry. Thus, we learn that what could be called ‘the $U(1)_{\Theta}$ theory on $\mathbb{R}^2_\infty$ ’ is not unique, having at least a one-parameter family of theories labelled by the hidden scale ratio $L'\Lambda_\infty$.

In fact, it is likely that this marginal limit model with $\alpha = 1$ is not well defined in the absence of supersymmetry. Although the energy scale of the finite box $1/L'$ is largely irrelevant to the dynamics of the $U(\infty)$ glueballs provided $\Lambda_\infty L' \gg 1$, the (exponentially small) finite-size effects in this regime can be quantified by ’t Hooft’s dual confinement criterion in terms of screening of magnetic-flux energy. Under very general assumptions, the non-abelian contribution to the magnetic-flux energy on a large box is given by (c.f. [37]):

$$E(m'_n)_{SU(N'_n)} = C_n \frac{L}{\Delta^2} \exp (-\sigma L'^2),$$

(3.11)

where $\Delta \sim \Lambda_\infty^{-1}$ is the effective thickness of the large-$n$ confining string and $\sigma \sim \Lambda_\infty^2$ is the corresponding string tension. The numerical constant is

$$C_n = 1 - \cos (2\pi m'_n / N'_n) = 1 - \cos (2\pi r_n / b_n).$$

(3.12)

This quantity is well-defined in the $n \to \infty$ limit provided the trigonometric constant $C_n$ has a limit. In general, this is not the case for arbitrary sequences $a_n/b_n$. One can easily find examples of approximations to irrational numbers, for which the ratio $r_n/b_n$ behaves erratically as $n \to \infty$. In these cases, there are exponentially small quantities that are not well-defined in this limit. This conclusion can be extended to any physical quantity whose commutative evaluation depends explicitly on the ratio $m'_n / N'_n$ (see also [38].)

4. The Witten Index

Having established that rational approximations of irrational theories, denoted $\overline{U(N)}_{\Theta}$, serve as a ‘blow-up’ of a specific singular large-$n$ limit of ordinary theories, we would like to study this limit in more detail. In supersymmetric theories, a significant amount of non-perturbative information is obtained by looking at the subset of BPS-saturated states. The most primitive BPS quantity is perhaps the supersymmetric (Witten) index, $\Tr (-1)^F$, that counts the number of Bose minus Fermi vacuum states [34]. This index gives valuable information, not only about dynamical supersymmetry breaking [35], but also about the dynamics of confinement and the breaking of various global symmetries [10-11]. For example, for $\mathcal{N} = 1$ pure Yang–Mills theories in four dimensions with gauge group $SU(N)$ one has $I = \Tr (-1)^F = N$, reflecting the spontaneous breaking
of the $\mathbb{Z}_{2N}$ R-symmetry acting on the gluinos, $\lambda \rightarrow e^{i\pi/N} \lambda$, to the $\mathbb{Z}_2$ subgroup of $2\pi$-rotations: $\lambda \rightarrow -\lambda$. One can also define refinements of the index that depend on electric and magnetic fluxes through a torus, and probe the standard hypothesis about confinement dynamics, i.e. the confinement of electric flux, and the screening of magnetic flux \[37\].

In the large-$N$ limit, the number of confining vacua is infinite. Naively, this reflects the restoration of the classical $U(1)_R$ symmetry. Since the classical $U(1)_R$ symmetry was broken to $\mathbb{Z}_{2N}$ by instantons, the restoration of the continuous symmetry is compatible with the general—and sometimes naive—idea that instanton effects turn-off in the large-$N$ limit. Here too the situation is subtle, since domain walls separating adjacent vacua still have divergent $O(N)$ tension (c.f. \[12\]).

We would like to test the hypothesis of $\Theta$-analyticity in minimal supersymmetric theories by computing the Witten index of the limiting model $U(N)_\Theta$, defined as a limit over the rational series of $\mathcal{N} = 1$ theories

$$I_{U(N)_\Theta} = \lim_{n \rightarrow \infty} I_{U(N,\Theta_n)}, \quad (4.1)$$

where $I_{U(N,\Theta_n)}$ is defined in terms of the index of the ordinary $U(N'_n)$ Morita dual. Incidentally, although this quantity is an integer, if we do not have continuity in $\Theta$ there is no reason why the index should not jump in an uncontrollable way as we move along the sequence $\Theta_n$ approximating $\Theta$. This makes the computation of the index meaningful and less trivial than it might naively seem at first sight. Before applying these results to our problem we must sort out some technical issues related to the proper treatment of the diagonal $U(1)$ subgroup of each ordinary $U(N'_n)$ theory.

### 4.1. A Supersymmetric Index for $U(N)$ Theories

In $U(N)$ gauge theories, the naive definition of the index as $\text{Tr} (-1)^F$, in terms of a trace over the full Hilbert space of the $U(N)$ theory, gives a trivial vanishing result from the contribution of the ground states in the diagonal $U(1)$ sector, obtained by the action the photino zero-mode operators, $\int \text{tr} \lambda$, on the vacuum. In fact, the vanishing of the index for a theory with a $U(1)$ factor is associated to the possibility of adding a Fayet–Iliopoulos term that breaks supersymmetry \[39\].

The usual remedy in the study of $\mathcal{N} = 1$ $U(1)$ theories is to consider instead the refined index $\text{Tr} C (-1)^F$, where $C$ is the charge-conjugation operator. Such an index can be defined precisely when the Fayet–Iliopoulos term is zero, and is nonvanishing for the free $U(1)$ theory. This solution is not directly applicable in our case because, although the charge conjugation is a symmetry of the $U(N'_n)$ theories, it acts non-trivially on the
magnetic fluxes $m'_n \rightarrow -m'_n$. Thus, the corresponding index is again trivially vanishing for a generic value of the magnetic flux $m'_n$.

On the other hand, the physical excitations of the decoupled diagonal $U(1)$ multiplet are rather uninteresting and even become infinitely massive in the $n \rightarrow \infty$ limit. Roughly speaking, the Hilbert space of a $U(N)$ theory factorizes between $U(1)$ excitations and $SU(N)$ excitations, the latter being the interesting ones in our singular large-$n$ limit. Therefore, it should be possible to find a refinement of the standard index so that it does not vanish and gives information about the degeneracy of ground-states in the interacting (non-abelian) sector.

The details of the factorization of the diagonal $U(1)$ subgroup are complicated by the global structure of the gauge group: $U(N) = (U(1) \times SU(N))/Z_N$. This has two main consequences for the structure of the canonical quantization on a torus. First, a given $U(N)$ bundle over $T^3$, with fixed integer-valued first Chern classes $m_{ij} \in \mathbb{Z}$, can be seen as a particular $U(1) \times (SU(N)/Z_N)$ bundle with quantized magnetic fluxes $m$ on the $U(1)$ and ’t Hooft flux $[m] \equiv m \pmod{N}$ on the adjoint factor $SU(N)/Z_N$.

A second, related subtlety is that the factorization of gauge transformations into a diagonal $U(1)$ part and an $SU(N)$ part is ambiguous by elements of the center $Z_N$ of $SU(N)$:

$$U = e^{i \alpha} \cdot U' = e^{i \alpha} z \cdot z^{-1} U', \quad (4.2)$$

where $z^N = 1$ and $U' \in SU(N)$. Since $Z_N$ is a discrete set, it follows by continuity that this ambiguity only affects the periodicity conditions. In particular, a strictly periodic $U(N)$ transformation can be factorized into non-periodic factors satisfying:

$$e^{i \alpha(x + L_j)} = z_j e^{i \alpha(x)}, \quad U'(x + L_j) = z_j^{-1} \Gamma_j U'(x) \Gamma_j^\dagger, \quad (4.3)$$

where we have considered arbitrary twisted boundary conditions on $T^3$. The entangled action of $Z_N$ means that we cannot simply factorize the Hilbert space into $U(1)$ and $SU(N)$ parts, even at a fixed value of the magnetic flux. Rather, we should quantize the theory without explicitly dividing by the gauge transformations (4.3), and then impose the $Z_N$-invariance at the end by an averaging procedure. This is facilitated by the fact that (4.3) act as global symmetries on the Hilbert space obtained by satisfying the Gauss law within the space of $z_j = 1$ gauge transformations.

The characters of the $Z_N$ action on the Hilbert space of the $SU(N)/Z_N$ gauge theory define the ’t Hooft electric fluxes: $[w]$, a three-vector of integers modulo $N$. If we construct the $SU(N)/Z_N$ Hilbert space in terms of gauge-invariant Wilson-loop operators, each component of ’t Hooft electric flux corresponds to the action of a Wilson line, $W'([w^j])$, wrapping the $j$-th torus direction and carrying an irreducible representation with ‘$N$-ality’
equal to \([w^j]\). We may now add the diagonal \(U(1)\) gauge field \((\text{tr} A)/N\) and build the general operator:

\[
W'([w^j]) \times \exp \left( i w^j \oint \frac{1}{N} \text{tr} A \right).
\] (4.4)

The integer \(w^j\) is interpreted as the abelian electric flux in the \(j\)-th direction. Invariance under (4.3) imposes the expected constraint between abelian and non-abelian electric fluxes:

\[
[w^j] = w^j \mod N.
\] (4.5)

Hence, we learn that the complete Hilbert space can be constructed by a decomposition with respect to value of the total electric flux:

\[
\mathcal{H}(w, m)_{U(N)} = \mathcal{H}(w, m)_{U(1)} \otimes \mathcal{H}([w], [m])_{SU(N)/\mathbb{Z}_N},
\] (4.6)

which induces a general decomposition of the \(U(N)\) index:

\[
I(m)_{U(N)} = \sum_{w \in \mathbb{Z}} I(w, m)_{U(1)} I([w], [m])_{SU(N)/\mathbb{Z}}.
\] (4.7)

Thus, the vanishing of the index is a trivial consequence of the vanishing of all the \(U(1)\) factors, even for vanishing electric flux. In order to include the effects of the instanton angle in more detail, it will be instructive to rederive (4.7) using a path integral argument.

Actually, we can be slightly more general with no extra complications. Since the covering group of \(U(N)\) is non-compact, we can define a \textit{continuous} electric flux valued on a torus, labelling representations of the covering group, \(\mathbb{R} \times SU(N)\), that are not representations of \(U(N)\). Intuitively, it corresponds to inserting unquantized \(U(1)\) probe charges into the system, and can be incorporated by adding a topological ‘theta-term’ to the action of the form

\[
S_e = \frac{i}{N} \sum_{j} e^j \int_{(0j)} \text{tr} F_{0j} = \frac{2\pi i}{N} \sum_{j} e^j m_{0j}.
\] (4.8)

With this normalization, \(e^j\) are \textit{real numbers} defined modulo \(N\). The vector of integers \(m_{0j} \equiv k_j\) combines with the magnetic fluxes \(m^k = \frac{1}{2} \epsilon^{kij} m_{ij}\) to specify the first Chern classes of the \(U(N)\) bundle over the euclidean four-torus. In addition, there is an integer-valued second Chern class that is canonically dual to the standard instanton angle, entering the action as

\[
S_\theta = \frac{i \theta}{8\pi^2} \int \text{tr} (F + \phi) \wedge (F + \phi).
\] (4.9)
In this formula, the constant $U(1)$ background field $\phi_{ij}$ is added for notational convenience in the applications involving Morita duality. The extra terms amount to a constant term and a redefinition of the electric flux:

$$e \rightarrow e_\theta = e + \frac{\theta}{2\pi} (m + N\Phi). \quad (4.10)$$

The projected index at a given value of the instanton angle can be given a path-integral interpretation as follows.

$$I(e, m, \theta)_{U(N)} = \text{Tr} \, P(e, m, \theta) (-1)^F \, e^{-\beta H} = \mathcal{N} \sum_{k \in \mathbb{Z}} \sum_{\nu \in \mathbb{Z}} e^{-S_e - S_\theta} \, Z(k, m, \nu)_{U(N)}, \quad (4.11)$$

where $\mathcal{N}$ is a normalization constant and $Z(k, m, \nu)_{U(N)}$ is the path integral of the $U(N)$ theory (with the Yang–Mills action) on a given topological sector of $\mathbb{T}^4$ bundles specified by magnetic fluxes $m_{ij} = \epsilon_{ijk} m^k$ and $m_{0i} = k_i$, in addition to the integral Pontriagin number $\nu \in \mathbb{Z}$.

On a fixed bundle topology, the partition function factorizes between $U(1)$ and $SU(N)$ parts:

$$Z(k, m, \nu)_{U(N)} = Z(k, m)_{U(1)} \cdot Z([k], [m], \nu)_{SU(N)/\mathbb{Z}_N}, \quad (4.12)$$

where the non-abelian partition function depends on $k, m$ only through their mod $N$ reductions $[k], [m]$. The abelian partition function can be further factorized, according to (3.8), into the contribution of the free photons (the ‘Maxwell term’) and that of topological fluxes:

$$Z(k, m)_{U(1)} = Z_{\text{Maxwell}} \cdot e^{-\beta E_m} \cdot e^{-\pi k \cdot \Omega_e \cdot k}, \quad \Omega_e \equiv \text{diag} \left( \frac{4\pi}{g^2 N} \left( \frac{V_\perp}{\beta L_i} \right) \right), \quad (4.13)$$

with $(V_\perp)_i$ denoting the spatial volume orthogonal to the $i$-th direction, and $E_m$ the magnetic energy in (3.9).

The $\theta$-dependence of the action $S_\theta$ is conveniently factorized into abelian and non-abelian contributions by the decomposition of the $U(N)$ curvature:

$$F = \frac{1}{N} \text{tr} \, F + \left( F - \frac{1}{N} \text{tr} \, F \right). \quad (4.14)$$

The first term yields a contribution

$$\exp \left[ -\frac{i\theta}{N} k \cdot (m + N\Phi) \right], \quad (4.15)$$

to the partition function, whereas the non-abelian contribution is

$$\exp \left[ i\theta \left( \nu + \frac{[k] \cdot [m]}{N} \right) \right], \quad (4.16)$$
with $\nu \in \mathbb{Z}$. The fractional contribution to the instanton charge in (4.16) cancels the
analogous term in (4.15), so that the total second Chern class $\int \text{tr} (F \wedge F)/8\pi^2$ of the
$U(N)$ bundle is an integer.

Let us separate explicitly the $\vartheta$-dependence induced by integral instanton numbers by
defining the function:

$$Z([k], [m], \vartheta)_{SU(N)/\mathbb{Z}_N} = \sum_{\nu \in \mathbb{Z}} e^{i\nu \vartheta} Z([k], [m], \nu)_{SU(N)/\mathbb{Z}_N}.$$  

(4.17)

Then, putting all factors together we have the following expression for the index:

$$I(e, m, \vartheta)_{U(N)} = \mathcal{N} Z_{\text{Maxwell}} \sum_{[k]=1}^{N} e^{-\pi[k] \cdot \Omega_e [k] - \frac{2\pi i [k] \cdot (e_\vartheta + [m] \frac{\vartheta}{2\pi})}{N}} Z([k], [m], \vartheta)_{SU(N)/\mathbb{Z}_N}$$

$$\times \sum_{k' \in \mathbb{Z}^3} e^{-2\pi i k' \cdot e_\vartheta - 2\pi N \Omega_e [k] k'} \times e^{-\pi N^2 k' \cdot \Omega_e k'},$$

(4.18)

where we have split the sum over $k$ as $k = [k] + Nk'$. We find, upon Poisson resummation
in the integers $k'$:

$$I(e, m, \vartheta)_{U(N)} = Z_{\text{Maxwell}} \sum_{w \in \mathbb{Z}^3} e^{-\pi (e_\vartheta + w) \cdot \frac{1}{N^2} \Omega_e (e_\vartheta + w)} I([w], [m], \vartheta)_{SU(N)/\mathbb{Z}_N},$$

(4.19)

where

$$[w]_\vartheta \equiv [w] + [m] \frac{\vartheta}{2\pi}.$$  

(4.20)

In writing down (4.19), we have adjusted the normalization constant so that $\mathcal{N}^2 = \det(\Omega_e)$,
and we have introduced the standard index of the $SU(N)/\mathbb{Z}_N$ theory as a function of the
‘t Hooft electric and magnetic fluxes:

$$I([w], [m], \vartheta)_{SU(N)/\mathbb{Z}_N} = \frac{1}{N^3} \sum_{[k]=1}^{N} e^{2\pi i [k] \cdot [w]/N} Z([k], [m], \vartheta)_{SU(N)/\mathbb{Z}_N}.$$  

(4.21)

Notice the appearance of the integrally quantized electric fluxes $w$, whose reduction modulo
$N$ defines the ‘t Hooft electric flux of the $SU(N)$ sector. The effective electric fluxes (4.10)
and (4.20) incorporate the fact, discovered in [43], that magnetic fluxes induce an electric
charge in the presence of an instanton angle.

The exponential term in (4.19) can be recognized as the contribution of the abelian
electric fluxes to the energy, i.e. for an abelian electric flux of quantum $\varepsilon^i$ we have:

$$\beta E(\varepsilon)_{\text{electric}} = \beta \frac{g^2}{4N} \sum_i \frac{L^2_i}{V} (\varepsilon^i)^2 = \sum_i \frac{\pi}{N^2 (\Omega_e)_{ii}} (\varepsilon^i)^2.$$  

(4.22)
Notice that these energies become all degenerate in the continuum limit, which involves
\( g^2 N \to 0 \). However, the electric fluxes contribute a strictly positive energy in the regular-
ized theory. We define the index as a limit from the regularized theory, so that only states
with effective abelian flux \( \varepsilon = 0 \) correspond to vacua.

A final compact expression for the \( U(N) \) index is given as a convolution of abelian
and non-abelian ones that generalizes the formula (4.7):

\[
I(e, m, \vartheta)_{U(N)} = \sum_{w \in \mathbb{Z}^3} I(e_\vartheta + w, m)_{U(1)} \ I([w], [m], \vartheta)_{SU(N)/\mathbb{Z}_N}.
\]  

(4.23)

Thus, the vanishing of the index is due to the trivial vanishing of the pure \( U(1) \) terms, even
for zero effective electric flux \( e_\vartheta + w = 0 \). This suggests a natural definition of a purely ‘non-
abelian’ index that measures the degeneracy of ground states in the non-abelian sector.
We just formally divide by the purely abelian index at zero effective electric flux:

\[
I'(e, m, \vartheta)_{U(N)} \equiv \frac{1}{I(\varepsilon = 0, m)_{U(1)}} \times \text{Tr} \ P(e, m, \vartheta) \ P(\varepsilon = 0) \ (-1)^F e^{-\beta H},
\]

(4.24)

where \( \varepsilon = e_\vartheta + w \). This formal operation cancels the trivial zero due to the contribution
of photino zero-modes.

In the case that we set \( m + N \Phi = 0 \) in the initial series of \( U(N, \Theta_n) \) models, i.e. we
cancel the magnetic vacuum energy, we can give a more elegant definition of the refined
index by twisting the purely abelian degrees of freedom by the charge-conjugation operator
\( C_{U(1)} \). We can specify the action of \( C_{U(1)} \) exactly due to the factorization (4.6):

\[
I'(e, m, \vartheta)_{U(N)} = \frac{1}{4} \text{Tr} \ C_{U(1)} P(e, m, \vartheta) (-1)^F e^{-\beta H}.
\]

(4.25)

Since the abelian charge-conjugation operator inverts the sign of the effective abelian electric
flux \( \varepsilon \to -\varepsilon \), only the \( U(1) \) ground states with \( \varepsilon = 0 \) contribute, and we have a sensible
index measuring the degeneracy of non-abelian degrees of freedom.

We are now ready to use the results of [40,41]. First, notice that for pure \( \mathcal{N} = 1 \)
\( SU(N) \) SYM theories in four dimensions, the classical \( U(1)_R \) symmetry is anomalous, so
that we can absorb the \( \vartheta \) angle into a phase redefinition of the gluino field. This means that
the physics will be actually \( \vartheta \)-independent. On the other hand, the symmetry \( \vartheta \to \vartheta + 2\pi \)
is realized by a permutation of vacua. The consequence is that, in a sector with both electric
and magnetic fluxes, electric fluxes differing by a redefinition \( [w] \to [w]+[m] \) give equivalent
physics, since this transformation is generated by the \( 2\pi \)-shift in the instanton angle. This
effect, associated to ‘t Hooft’s mechanism of ‘oblique confinement’ [44], partitions the \( N \)
supersymmetric vacua of the \( SU(N) \) theory into \( N/c_m \) sets, each one with degeneracy \( c_m \),
where

\[
c_m \equiv \max \ ([m^i], N), \quad i = 1, 2, 3.
\]

(4.26)
Thus, the result for the $SU(N)$ index is

$$I ([w], [m], \vartheta)_{SU(N)/\mathbb{Z}_N} = c_m \quad \text{for } [w] = 0 \text{ modulo } [m] \mathbb{Z}, \quad (4.27)$$

and it vanishes otherwise.

The abelian index $I' (e, w + \vartheta, m)_{U(1)}$ vanishes unless the effective electric-flux energy is zero, which requires $e_\vartheta = -w$ or, equivalently, $e_\vartheta$ must be an integer. Combining the two selection rules with the convolution (4.23) we find the result for our refined $U(N)$ index:

$$I' (e, m, \vartheta)_{U(N)} = c_m \quad \text{for } e_\vartheta = 0 \text{ modulo } [m] \mathbb{Z}, \quad (4.28)$$

and it vanishes otherwise.

In particular, (4.28) implies that the non-abelian index for a $U(N)$ theory with periodic boundary conditions $m = 0$ is concentrated at zero electric flux $e = 0$ and has value $N$. Thus, it diverges in the large-$N$ limit.

### 4.2. The Supersymmetric Index of the $\overline{U(N)}_\Theta$ Model

We are ready to apply these results to our problem of determining the number of non-trivial supersymmetric vacua of the $\overline{U(N)}_\Theta$ limiting theory. Since the $\overline{U(N)}_\Theta$ model was defined with periodic boundary conditions, a similar behaviour to that of the commutative counterpart would suggest an infinite index at zero electric flux, given the fact that we define the model via a large-$n$ limit. However, this naive expectation is not borne out by the explicit calculation. In our limit definition:

$$I' (e)_{\overline{U(N)}_\Theta} = \lim_{n \to \infty} I' (e, m'_n)_{U(N'_n)} \quad (4.29)$$

we use the electric flux $e$ of the ordinary theories $U(N'_n)$ to label the states. Note that, for the components parallel to the noncommutative directions, this is Morita dual to fixing a projector over some combination of momenta and electric fluxes in the series of noncommutative theories. In particular, a fixed value of the integer electric flux in the ordinary $U(N'_n)$ theories maps under the duality to the combination:

$$w'_n = b_n (w + \Theta_n * p), \quad (4.30)$$

where $p$ and $w$ are momenta and electric flux on the noncommutative $T^2_{\theta_n}$. Since we find more transparent the language of the ordinary unitary theories, we choose to parametrize the index in terms of the $U(N'_n)$ electric flux $e$. Notice also that, since $N'_n \to \infty$, this flux is asymptotically defined as an arbitrary vector in $\mathbb{R}^3$ in the large-$n$ limit.
The important point is that \((m'_n, N'_n) = N\) is bounded and constant in the limit. Hence, the result \((4.28)\) translates into

\[
I'(e)_{U(N)_{\Theta}} = N \quad \text{for} \quad e = 0 \mod (0, 0, N) \mathbb{Z}.
\]

Notice that the definition of the index with the extra phase depending on the rank of the gauge group, as in \([41]\), would yield an extra factor of \((-1)^{N'_n-1}\), rendering the index erratic in the large-\(n\) limit.

Thus, the refinement of the index by the electric flux is smooth. The index has been ‘fractionalized’ due to the physics of oblique confinement in the series of ordinary theories.

The refinement by the electric flux is essential in obtaining a smooth answer. Since the index is computed in a Born–Oppenheimer approximation, the result is determined by the structure of the space of classical ground states, which in turn depends sensitively on the \textit{rational} value of \(\Theta\). For a periodic \(U(N)_{\Theta}\) model with \(\Theta = a/b\), the corresponding moduli space has \(b\) connected components, each one of dimension \(N\). This is clear in the Morita-dual picture of the gauge bundle, where we have (for our particular choice of background field), a commutative \(U(1) \times SU(N')/\mathbb{Z}_{N'}\) bundle with \(N' = Nb\) and \(Nr \mod N'\) units of ‘t Hooft flux through \(T^2\). The flat connections are characterized by holonomies in the non-abelian factor \(U_1, U_2, U_e\) satisfying

\[
U_1 U_2 = U_2 U_1 e^{2\pi ir/b},
\]

and

\[
U_e U_j = U_j U_e, \quad \text{for} \quad j = 1, 2.
\]

The solution has \(b\) components

\[
U_j = (H_j \otimes 1_b) \cdot \Gamma_j, \quad U_e = H_e \otimes e^{2\pi iq/b} 1_b,
\]

where \(H_j, H_e\) are in the Cartan torus of \(SU(N)\) and \(q = 0, 1, \ldots, b - 1\). The different components correspond to the different sectors of ‘t Hooft’s electric flux that can be related by tunneling via fractional instantons. This explains why the index is only smooth in the large-\(b\) limit once it is refined at a fixed value of the electric flux. The relation between the fractional instantons and the structure of the space of flat connections suggests that the contribution of fractional instantons, if non-vanishing, tends to work \textit{against} continuity in \(\Theta\). In the next section we present a quantitative test of this idea.

5. Instanton-induced \(\vartheta\)-dependence

Our treatment of the index in \(U(N)\) theories can be formally generalized to other physical quantities. For example, if we consider the antiperiodic spin structure for the
gauginos, we are computing the finite-temperature partition function. This partition function admits a similar factorization on electric-flux labels:

\[ e^{-\beta F(e,m,\vartheta)_{U(N)}} = \sum_{w \in \mathbb{Z}^3} e^{-\beta F(e+w)_{U(1)}} e^{-\beta F([w]_{\vartheta},[m],\vartheta)_{SU(N)}}, \]  

\[ (5.1) \]

where now we have a non-trivial \( \vartheta \) dependence due to the breakdown of supersymmetry.

It is often the case that, particularly when the instanton-gas approximation suffers from infrared problems, the \( \vartheta \)-dependence of physical quantities is apparently invariant under \( \vartheta \rightarrow \vartheta + 2\pi N \) only. This is especially obvious when considering a soft breaking of \( \mathcal{N} = 1 \) supersymmetry by the addition of a small gluino mass. This has the effect of splitting the vacuum energies of the \( N \) vacua, that are not equivalent any more. Since \( \vartheta \rightarrow \vartheta + 2\pi \) shifts the vacua one by one, we need \( N \) steps to return to the same vacuum.

For \( SU(N)/\mathbb{Z}_N \) theories in finite volume, the fractional nature of the instanton number means that, in principle, physics is only periodic in \( \vartheta \) modulo \( 2\pi N \). However, for the case of electric-flux energies, the \( 2\pi \) periodicity of the instanton angle is restored by a non-trivial level-crossing \[ [5.2] \]. The fractional \( \vartheta \)-dependence implicit in the redefinition of \( SU(N)/\mathbb{Z}_N \) electric fluxes:

\[ [w]_{\vartheta} = [w] + [m] \frac{\vartheta}{2\pi}, \]

\[ (5.2) \]

gives the appropriate spectral flow \( [w] \rightarrow [w] + [m] \).

Similar behaviour should be found in the \( U(N') \) theories that appear as Morita duals of rational noncommutative theories, provided the abelian contribution to \( \vartheta' \)-dependence cancels out. The condition for this is the same as the vanishing of the magnetic ground-state energy: \( m' + N'\Phi' = 0 \).

Under Morita duality, the instanton angle is fractionalized as

\[ \vartheta' = b \vartheta. \]

\[ (5.3) \]

Thus, \( 2\pi \)-periodicity in the \( U(N,\Theta_n) \) model translates into naive \( 2\pi b_n \) periodicity in the \( U(N'_n) \) model. If the physics is \( \vartheta \)-dependent, such as in the absence of supersymmetry, we may have some level-crossing phenomena that recover the \( 2\pi \)-periodicity in \( \vartheta'_n \). As a result, the \( U(N,\Theta_n) \) model would actually show a hidden \( 2\pi/b_n \) periodicity in the spectrum as a function of \( \vartheta \). Thus, level-crossing phenomena of order \( b_n \) could be present in rational theories with \( \Theta_n = a_n/b_n \). In the large-\( n \) limit we have \( b_n \rightarrow \infty \), so that the \( \vartheta \)-dependence of the \( U(N)_{\Theta} \) model would be either trivial or infinitely discontinuous. In both cases, we would violate the strongest form of the \( \Theta \)-smoothness conjecture, since the \( \vartheta \)-dependence for rational \( \Theta \) must be non-trivial and smooth.
We can test this scenario with an explicit example. Let us consider the non-perturbative splittings of the electric-flux energies in the non-supersymmetric theory, calculated using a dilute gas of fractional instantons (see \cite{15,16,17}).

We consider the simplest case of a rank-one rational noncommutative theory on a periodic torus $S^1(L_e) \times T^2(L)$, with $\Theta = a/b$. The ordinary dual $U(N')$ theory lives in $S^1(L_e) \times T^2(L')$, with $N' = b, L' = L/b$. We have $m' + N'\Phi' = 0$ so that the diagonal $U(1)$ has effectively zero magnetic flux. The $SU(N')/\mathbb{Z}_{N'}$ bundle has 't Hooft flux

$$[m'] = (0, 0, [m'_e]) = (0, 0, r),$$

where $sb + ra = 1$ and $0 \leq r < b$. Then, the non-abelian contribution to the energies of electric fluxes of the form $[w] = (0, 0, [w_e])$ is exactly degenerate to all orders in perturbation theory. The leading effect lifting this degeneracy is the tunneling contribution by fractional instantons of charge proportional to $1/N'$. We can estimate this splitting by a standard computation in the dilute-gas approximation.

Let us consider the sum over all chains of $k_+$ fractional instantons and $k_-$ anti-instantons of minimal charge along the euclidean time direction, in the limit $\beta/L' \to \infty$. Using $ar + bs = 1$, we can write the total instanton charge in each term in the sum as

$$Q = \frac{k_+ - k_-}{N'} = \frac{(k_+ - k_-) a [m'_e]}{N'} + (k_+ - k_-) s.$$  \hspace{1cm} (5.4)

Using the standard parametrization of the topological charge

$$Q = \frac{[k'] \cdot [m']}{N'} + \nu,$$  \hspace{1cm} (5.5)

with $\nu \in \mathbb{Z}$, we can identify $[(k_+ - k_-) a] = [k'_e]$ as the conjugate variable to the electric flux $[w'_e]$. In particular, in computing the discrete Fourier transform

$$e^{-\beta E([w'], [m'], \vartheta')} = \sum_{[k']=1}^{N'} e^{2\pi i[k'] \cdot [m'] / N'} \sum_{\nu \in \mathbb{Z}} e^{i\vartheta' Q} Z ([k'], [m'], \nu)$$  \hspace{1cm} (5.6)

within the instanton-gas approximation, we can replace the sum over $[k'_e]$ by a free sum over integers $(k_+ - k_-)$. Assuming the usual factorization of the instanton measure in the dilute instanton-gas approximation, the sum over $k_+$ and $k_-$ exponentiates and we find for the instanton-induced splitting:

$$\Delta E ([w'_e], [m'_e], \vartheta') = -2K \cos \left[ \frac{1}{N'} (2\pi a [w'_e] + \vartheta') \right].$$  \hspace{1cm} (5.7)
Going back to the variables of the original noncommutative $U(1)$ theory, we notice that $[w_e']$ is Morita invariant in our particular case, and we have

$$\Delta E ([w_e], \vartheta) = -2K \cos (2\pi [w_e] \Theta + \vartheta). \quad (5.8)$$

The numerical prefactor $K$ is interesting. It takes the form:

$$\beta K \propto \beta L_e L' L'^2 \cdot N' \cdot (\Lambda_{\text{UV}})^4 \cdot (\sqrt{S_{\text{cl}}})^4 \cdot e^{-S_{\text{cl}}} \cdot \left( \left| \frac{\text{Det}_F'}{\text{Det}_B'} \right| + \ldots \right). \quad (5.9)$$

The first term in this expression gives the contribution of the four translational zero modes of the fractional instanton. There is a degeneracy factor $N' \cdot 2$ coming from the various independent tunnelings for a given topological charge. The terms containing the classical action

$$S_{\text{cl}} = \frac{8\pi^2}{g'^2(\Lambda_{\text{UV}})N'}$$

include the usual Jacobian $\sqrt{S_{\text{cl}}}$ for each collective coordinate. A factor $(\Lambda_{\text{UV}})^4$ comes from the zero-modes in the regularization of the one-loop determinants, whose non-zero-mode contribution for bosons and fermions is also indicated. The dots in (5.9) stand for higher-loop contributions.

Let us consider $\Lambda_{\text{UV}} = M_s$ as the scale of soft breaking of $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 0$. In terms of this fixed scale, we can rewrite the numerical prefactor in $U(1)_{\Theta}$ variables as

$$K \propto L_e L' L'^2 \cdot (M_s)^4 \cdot \left( \frac{8\pi^2}{g^2(M_s)} \right)^2 \cdot e^{-8\pi^2/g^2(M_s)} \cdot \left( \left| \frac{\text{Det}_F'}{\text{Det}_B'} \right| + \ldots \right). \quad (5.10)$$

Notice that the factor of $(L'N')^2$ in (5.9) combines into a single factor of $L^2$, as corresponds to a single position collective coordinate for the noncommutative $U(1)$ instanton (c.f. [49]) that appears as a Morita dual of the ordinary fractional instanton.

The important property of (5.8) and (5.10) is that the size of the instanton-induced $\vartheta$-dependence is non-perturbative in the $\mathcal{N} = 4$ coupling $g^2(M_s)$. This is to be compared to the splittings induced by the energies of the abelian fluxes (3.9):

$$\Delta E(w_e)_{\text{abelian}} = \frac{g^2(M_s)}{4} \frac{L_e}{L^2} (w_e)^2. \quad (5.11)$$

---

3 There are $N'^2$ inequivalent instanton solutions on $\mathbb{R} \times T^3$ that tunnel between fixed pairs of vacua on $T^3$. They are obtained by the action of discrete translations in the plane of twisted boundary conditions (c.f. [48,49,47]).
Clearly, within the conditions of applicability of the instanton expansion, the non-abelian contribution is completely negligible in comparison to the abelian one. As a result, there are no possible level-crossing phenomena induced by (5.8), and energy levels are clearly continuous under rational approximations of any given $\Theta$.

The qualitative picture changes considerably if we decouple completely the $\mathcal{N} = 4$ regularization. Namely, let us remove completely the $\Lambda_{\text{UV}}$ cutoff scale by proper renormalization in the $g^2(\Lambda_{\text{UV}}) \to 0$ limit. In this limit, the abelian splittings (5.11) vanish, and we are left with the non-abelian contributions (5.8). Therefore, level-crossing becomes possible.

In order to correctly renormalize (5.9) we must identify the effective infrared cutoff of the one-loop determinants. This is given by the size of the noncommutative $U(1)$ instanton in the noncommutative box of volume $L_eL^2$. In the limit that $\sqrt{\theta} \ll L_e, L$, this size is of $\mathcal{O}(\sqrt{\theta})$, since the instantons must be a smooth deformation of the instantons at $\theta = 0$. In this case, the instanton at $\theta = 0$ is point-like, and the only available scale for the resolution is given by $\theta$.

Therefore, provided the theory is still perturbative at the energy scale $M_\theta = 1/\sqrt{\theta}$, i.e. $g^2(M_\theta) \ll 1$, we can eliminate $\Lambda_{\text{UV}}$ in favour of $\theta$ and the dynamical scale

$$\Lambda = \Lambda_{\text{UV}} e^{-8\pi^2/\beta_0 g^2(\Lambda_{\text{UV}})},$$

with $\beta_0$ given in (3.6). We obtain

$$K = L_e L^2 \cdot (M_\theta)^4 \cdot \left( \frac{\Lambda}{M_\theta} \right)^{\beta_0} \cdot f \left( g^2(M_\theta) \right), \quad (5.12)$$

where we have summarized in the function $f(g^2)$ the renormalized perturbative expansion in the instanton background.

Consider now a rational approximation of some generic noncommutativity parameter: $a_n/b_n = \Theta_n \to \Theta$. The induced splittings of electric fluxes are given by

$$\Delta E([w_e], \vartheta)_n = -2 K_n \cos \left( 2\pi [w_e] \Theta_n + \vartheta \right). \quad (5.13)$$

In the large-$n$ limit, $K_n \to K_\infty$ with

$$K_\infty = L_e L^2 (M_\theta)^4 \left( \frac{\Lambda_\infty}{M_\theta} \right)^{\beta_0} f \left( g^2(M_\theta) \right),$$

and each individual curve for fixed $[w_e]$ converges to the limiting curve

$$-2 K_\infty \cos \left( 2\pi [w_e] \Theta + \vartheta \right).$$
However, the different curves obtained by varying \([w_e]\) cross one another as a function of \(\vartheta\). Therefore, the ground-state energy as a function of \(\vartheta\) is defined by the minimum:

\[
E(\vartheta)_{\text{gs}} = \lim_{n \to \infty} \min_{0 \leq [w_e] < b_n} \left[ -2 K_n \cos \left( 2\pi [w_e] \Theta_n + \vartheta \right) \right].
\]

(5.14)

If \(\Theta\) is irrational, the minima of the cosine function are uniformly distributed in the limit. Therefore, in this case, the ground-state energy is a constant, independent of \(\vartheta\). In general, let us write \(\Theta_n = \Theta + \Delta \Theta_n\) and \([w_e] = x b_n\). In the limit, \(x\) becomes a continuous variable in the unit interval and we have,

\[
E(\vartheta)_{\text{gs}} \to -2 K_\infty \min_{0 \leq x \leq 1} \cos \left( 2\pi [w_e] \Theta + \vartheta + 2\pi x b_n \Delta \Theta_n \right).
\]

(5.15)

For rational \(\Theta = a/b\), the direct computation of the resulting ground-state energy yields

\[
E(\vartheta)_{\text{gs}} \propto \min_{0 \leq [w_e] < b} \cos \left( 2\pi [w_e] \frac{a}{b} + \vartheta \right).
\]

(5.16)

Hence, the condition for (5.15) to approach (5.16) is that the combination

\[
b_n \Delta \Theta_n = b_n \left( \frac{a_n}{b_n} - \Theta \right) \to 0
\]

in the large-\(n\) limit. However, this is true if and only if the limiting value \(\Theta\) is an irrational number. We conclude that, for rational \(\Theta\), the term \(2\pi x b_n \Delta \Theta_n\) shifts the argument of the cosine function in (5.15) by an arbitrary amount in the large-\(n\) limit, so that the minimum is given by \(E(\vartheta)_{\text{gs}} = -2 K_\infty\) for any \(\vartheta\).

Approximations by rationals give a constant ground-state energy, in disagreement with the direct evaluation (5.16). This implies that the strongest possible conjecture of \(\Theta\)-analyticity over the rationals fails this test. It is very interesting that the \(\Theta\)-discontinuity that we have discussed involves a complete decoupling of the supersymmetric energy scale, i.e. \(M_s \to \infty\) faster than the large-\(n\) limit.

6. Models in 2 + 1 Dimensions

Many of the previous results can be extended to the case of rational approximations of minimal supersymmetric models in 2 + 1 dimensions. In this case we borrow the results of Ref. [40] for the case of \(SU(N)\) gauge group. The index is a function of the topological Chern–Simons coupling \(k\), governing the supersymmetric Chern–Simons mass term

\[
CS(A)_{SU(N)} = \frac{k}{4\pi} \int \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A + \bar{\lambda} \lambda \right).
\]

(6.1)
This quantity satisfies a topological quantization condition $\bar{k} \in \mathbb{Z}$ where $\bar{k} \equiv k - N/2$, the shift by $N/2$ being a consequence of the contribution of the fermionic measure. The result of [10] for $SU(N)_k$ with $k > 0$ and periodic boundary conditions is

$$I(k, N)_{SU(N)} = \frac{(N + \bar{k} - 1)!}{k!(N - 1)!}.$$  (6.2)

We also have $I(-k, N) = (-1)^{N-1}I(k, N)$ and the index vanishes for $|k| < N/2$.

For the group $SU(N)/\mathbb{Z}_N$, i.e. allowing 't Hooft’s magnetic fluxes $[m] \in \mathbb{Z}_N$ the quantization condition is instead

$$k = \frac{N}{2} \mod \frac{N}{(N, [m])}\mathbb{Z},$$  (6.3)

due to the fractionalization of the instanton number. In particular, for such non-trivial gauge bundles the Chern–Simons action must be defined in terms of a four-dimensional topological action. Extending the bundle to an appropriate four-manifold $X$ whose boundary is the desired three-manifold $M_3 = \partial X$, we define the Chern–Simons term as:

$$CS(A)_{M_3} = \frac{k}{4\pi} \int_X \tr \left( F \wedge F + d(\bar{\lambda} \lambda) \right).$$  (6.4)

For example, if $M_3 = S^1 \times T^2$, we can take $X = D \times T^2$, where $D$ is a two-dimensional disk over which the bundle extends trivially. The fermionic term in (6.4) does not need to be extended to the interior of $X$ because the action is already well-defined on its boundary.

### 6.1. Morita Duality

The definition (6.4) is suitable to the study of Morita duality in the noncommutative case, since the Morita transformation of the right hand side is the same as that of the Yang–Mills action (after properly redefining the field strength $F \rightarrow F + \phi$). Working on $\mathcal{M}_\theta = S^1_{\beta} \times T^2_{\theta}(L)$, with periodic boundary conditions on the noncommutative torus, under the Morita duality (2.7) we have:

$$\frac{4\pi}{k} NCCS(A)_{\mathcal{M}_\theta} = \int_{\mathcal{M}_\theta} \tr \left( A \wedge dA + \frac{2}{3} A \wedge_A A \wedge_A A + \bar{\lambda} \lambda \right) = (s + r \Theta) \int_{X'} \tr \left[ (F' + \phi') \wedge (F' + \phi') + d(\bar{\lambda}' \lambda') \right],$$  (6.5)

where $X' = D \times T^2(L')$ and the notation $A \wedge_A A$ means $A_\mu \star A_\nu \, dx^\mu \wedge dx^n$. In the particular case of interest to us, we have $s + r \Theta = 1/b$ and $L' = L/b$. Thus, we learn that the T-duality mapping of the topological Chern–Simons coupling is

$$k \rightarrow k' = k (s + r \Theta)^{-1} = k b.$$  (6.6)
Notice that the rescaling by a factor of $b$ agrees with the quantization condition in the ordinary $U(N')$ theory, in the presence of 't Hooft fluxes, since the Chern–Simons coupling of the (periodic) noncommutative theory, $k$, is quantized modulo integers $[50]$. Our normalization conventions for the Chern–Simons action, together with the general action of Morita duality for rational $\Theta$, gives the general Morita-covariant quantization condition in sectors with arbitrary magnetic flux $m$,

$$k = \frac{N_\Theta}{2} \pmod{\frac{N_\Theta}{(N, m)} \mathbb{Z}},$$  
(6.7)

where $N_\Theta = N - \Theta m$ is the dimension of the noncommutative module. This quantization condition should extend to general irrational $\Theta$ by analytic continuation. This is a reasonable expectation since all we use to obtain (6.7) are classical properties of the classical action which is analytic in $\Theta$.

In particular, for our rational sequence of T-dualities between $U(N, \Theta_n)$ and $U(N'_n)$, we have

$$k'_n = b_n k,$$  
(6.8)

where $k$ is the fixed topological coupling in the noncommutative $U(N, \Theta_n)$ theories, which acquires the quantization condition:

$$k = \frac{N}{2} \pmod{\mathbb{Z}}.$$  
(6.9)

It is interesting to notice that the shift by $1/2$ in the rank-one case can be understood directly by integrating-out the noncommutative Majorana fermions $[52]$.

### 6.2. Level Normalization and the Born–Oppenheimer Approximation

In the microscopic evaluation of the index one considers the effective dynamics of the gauge and fermion zero-modes $[39,40,41]$. Unlike the four-dimensional case, where the index was independent of the value of couplings in the Lagrangian, in $2 + 1$ dimensions the index depends on the Chern–Simons level. Therefore, its absolute normalization is essential in the computation.

A naive zero-mode reduction of the noncommutative Chern–Simons Lagrangian on the noncommutative torus with periodic boundary conditions yields the effective Born–Oppenheimer Lagrangian (we only consider the bosons in this discussion)

$$\frac{k}{4\pi} \int_{\mathbb{R} \times T^2_\phi} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \rightarrow \frac{k}{4\pi} \int dt \epsilon^{ij} C_i^\alpha \partial_t C_j^\alpha,$$  
(6.10)

---

4 The quantization condition (6.7) should agree with that in Ref. [51], given the proper notational conventions.
where
\[
C^\alpha_j \equiv \text{tr} \left( H^\alpha (a_0)_j L \right),
\]
(6.11)
is defined in terms of the constant component \(a_0\) of the noncommutative gauge field in the Fourier expansion (2.3). The \(N\) matrices \(H^\alpha\) are a convenient basis of the Cartan subalgebra of \(U(N)\), normalized as \(\text{tr}(H^\alpha H^\beta) = \delta^{\alpha\beta}\).

After the Morita transformation, in terms of the twisted ordinary \(U(N')\) gauge theory, the flat connections are parametrized simply by those in the non-abelian sector. The moduli space of flat connections of the twisted \(SU(N')/\mathbb{Z}_N'\) bundle on a two-torus is isomorphic to that of \(SU(N)\) flat connections on the periodic torus. It can be constructed explicitly in terms of the embedding \(SU(N) \otimes SU(b) \subset SU(bN) = SU(N')\) that is realized in formula (2.10).

The Born–Oppenheimer reduction to zero modes reads
\[
\frac{ik'}{4\pi} \int_{D \times T^2(L')} \text{tr} \left( F' + \phi' \right) \wedge (F' + \phi') \rightarrow \frac{ik'}{4\pi} \int dt (L')^2 \epsilon^{ij} (a_0)_i^\alpha \partial_t (a_0)_j^\beta \text{tr} \left( H^\alpha \otimes 1_b \right)(H^\beta \otimes 1_b).
\]
(6.12)
Using now
\[
\text{tr} \left( H^\alpha \otimes 1_b \right)(H^\beta \otimes 1_b) = b \delta^{\alpha\beta},
\]
together with \(L' = L/b\) and \(k' = kb\) we find exactly the same effective action
\[
\frac{k}{4\pi} \int dt \epsilon^{ij} C_i^\alpha \partial_t C_j^\alpha,
\]
(6.13)
when expressed in terms of the angular variables \(C^\alpha_j = (a_0)_j^\alpha L\), just as before. Thus, we learn that, despite the rescaling of the Chern–Simons level under Morita duality \(k \rightarrow kb\), which is required by our trace normalizations, the space of flat connections is sensitive to the same effective Chern–Simons level before and after the duality. The reason for this is of course that the Fourier components \(a_\ell\) remain the same in both representations and, in particular, their periodicity should not change. This is also clear from the fact that the zero modes know little about noncommutativity.

The periodicity of the zero-mode variables \(C^\alpha_j\) is enforced by gauge transformations that are periodic on the noncommutative torus and satisfy the condition of Moyal-unitarity: \(U \star U^\dagger = U^\dagger \star U = 1_N\). Expanding in Fourier series:
\[
U(x) = \sum_{\ell \in \mathbb{Z}^2} u_\ell e^{-2\pi i \ell \cdot x/L},
\]
(6.14)
\footnote{This is a consequence of our choice of background field and magnetic fluxes in the original model, making the diagonal \(U(1)\) effectively flat (2.12).}
where the coefficient matrices $u_\ell$ are not unitary in general. It is then easy to check that the twisted gauge transformations

$$U'(x) = \sum_{\ell \in \mathbb{Z}^2} u_\ell \otimes V^{-a_1} U^{\ell_2} \omega^{-a_1 \ell_2/2} e^{-2\pi i \ell \cdot x/L},$$

(6.15)

satisfying

$$U'(x + L_j) = \Gamma_j U'(x) \Gamma_j^\dagger,$$

(6.16)

enforce exactly the same periodicity conditions on $C^0_j$ when acting on $A' + A_{\phi'}$, as defined by (2.10).

6.3. The $2 + 1$ Dimensional Index

The upshot of the discussion in the preceding subsection is that, when calculating the index of $(2 + 1)$-dimensional gauge theories by quantization of the moduli space of flat connections, the effective Chern–Simons level is exactly given by that of the noncommutative theory, which remains fixed. Thus, the index is $\Theta$-independent and can be calculated as if the $U(N)_\Theta$ model was an ordinary gauge theory with periodic boundary conditions on the torus.

This means that, whatever its value, the index will be only a function of $k$ and $N$, and it is obviously smooth under rational approximations of $\Theta$. Although this settles our main concern, it is nevertheless interesting to pursue this matter and compute the index of the $U(N)_\Theta$ model.

Following our general discussion (4.7), the result factorizes into abelian and non-abelian components for each value of the electric flux. The linking between the $U(1)$ and $SU(N)$ sectors is done by the $\mathbb{Z}_N$ quotient acting on Wilson loops in each direction, enforcing the constraint $[w^i] = w^i \pmod N$, $i = 1, 2$.

Since the infrared behaviour of the $2 + 1$ models under consideration is dominated by the Chern–Simons term, the degeneracy of the ground states of the full theory is naturally related to the dimension of the Hilbert space of the topological Chern–Simons model describing the infrared limit.

One subtlety of the description in terms of the effective topological theory is that Wilson lines wrapping homologically inequivalent cycles of the torus are canonical conjugates of one another. Thus, the states are labelled by the eigenvalues of only one set of Wilson lines. In the abelian case, this can be understood in elementary terms. For an abelian Maxwell–Chern–Simons model with action

$$S_{U(1)} = \frac{1}{4e^2} \int |dA|^2 + \frac{i\kappa}{4\pi} \int A \wedge dA + \text{Fermi terms},$$

(6.17)
we can explicitly find the zero energy states on the torus with periodic boundary conditions (see, for example [53] and references therein). Only the holonomies of the gauge field \( C_i = \oint_i A \) are important, with effective Lagrangian

\[
L_{\text{eff}} = \frac{1}{2e^2} (\partial_t C_i)^2 + \frac{\kappa}{4\pi} \epsilon^{ij} C_i \partial_t C_j,
\]

since the fermions are massive and free, and do not contribute to the ground-state degeneracy. This Lagrangian is equivalent to that of a non-relativistic particle of mass \( 1/e^2 \) in a torus of length \( 2\pi \), threaded by \( \kappa \) units of magnetic flux. Thus, for integer \( \kappa \), the ground state is a degenerate Landau level of \( \kappa \) states. This generalizes for rational values of \( \kappa = q/p \), with \( (q, p) = 1 \), since we may allow multivaluated wave-functionals. It is enough to focus on the purely topological term that is obtained by neglecting the Maxwell action (the kinetic energy of the particle). Then, canonical quantization of the holonomies gives

\[
e^{iC_1} e^{iC_2} = e^{2\pi ip/q} e^{iC_2} e^{iC_1},
\]

which can be represented by the \( q \)-dimensional pair of clock and shift matrices. Hence, states are labelled by the \( q \) eigenvalues of, say \( \exp(iC_1) \). Notice that the zero-modes \( C_i \) are normalized as angular variables with period \( 2\pi \).

The topological nature of the Chern–Simons theory allows us to obtain this spectrum by the general construction of [54]. A basis for the Hilbert space of the Chern–Simons theory on \( T^2 \) can be obtained by computing path integrals on the ‘filled torus’ \( D \times S^1 \), with \( D \) a disk carrying an insertion of a Wilson line \( \exp(\i w_1 \oint A_1) \). By the usual Chern–Simons ‘holography’, this basis is in one-to-one correspondence with the set of integrable representations of the corresponding level-\( \kappa \) WZW model on the boundary of the disk. This restricts the possible values of \( w_1 \) to the set

\[
w_1 = 0, 1, \ldots, \kappa - 1.
\]

The fact that only one component of the electric flux is relevant is seen here by the topological impossibility of ‘filling in’ both homologically nontrivial circles of the torus at the same time.

A similar construction can be carried out for the non-abelian component of the index, in terms of the set of integrable representations of the corresponding \( SU(N) \) level-\( \bar{k} \) WZW model. Thus, the index can be computed by imposing the \( \mathbb{Z}_N \) modding on the Hilbert space of the product theory \( U(1)_\kappa \times SU(N)_{\bar{k}} \).

Notice that there is no shift of the abelian level by effects of the fermionic measure, since there are no other fields in the theory that are charged with respect to the diagonal
$U(1)$ subgroup. Thus, we can freely adjust it depending on the physical definition we adopt for the abelian level.

If we wish the $U(N)$ symmetry to act classically, we define the abelian field $A = A \cdot \mathbf{1}_N$. The periodic $U(1)$-gauge transformations impose an angular identification, modulo $2\pi$, to the holonomy variables $\oint A$. Hence, the effective abelian level is $\kappa = kN$. However, since the space of ground states is equivalent to that of the effective Chern–Simons model that arises after fermion integration, it is more natural to define the $U(N)$ symmetry referred to this effective bosonic Lagrangian. In this case, we would have $\kappa = \bar{k}N$ and the classical theory would be defined with an extra abelian counterterm with coupling $\delta \kappa = -N^2/2$. Yet another possibility would be to define the abelian level so that the $U(N)$ symmetry has a simple action on expectation values of Wilson lines. Since these are naturally functions of $\bar{k} + N$, one would have a natural definition $\kappa = N(\bar{k} + N)$.

In the particular case $\kappa = \bar{k}N$, the purely bosonic low-energy Chern–Simons model is a standard $U(N)_{\bar{k}}$ model that can be analyzed directly in terms of the $U(1)^N$ Cartan subalgebra. The problem is exactly given by the multi-particle generalization of (6.18) to a system of $N$ identical bosons. The degeneracy is given by all the wave functions on $N$ variables that can be constructed out of $k$ elementary single-particle ‘orbitals’. This is

$$I_{U(N)_{\bar{k}}} = \frac{(N + \bar{k} - 1)!}{N!(\bar{k} - 1)!}. \quad (6.20)$$

This is the total dimension of the Hilbert space of the product theory $U(1)_{\bar{k}N} \times SU(N)_{\bar{k}}$, divided by $N^2$, and one can think of the modding by $\mathbb{Z}_N$ as dividing by a factor of $N$ for each independent cycle of the torus. The general case with asymmetric levels is more involved. In general, one can argue that the consistency of the $\mathbb{Z}_N$ modding procedure requires $\kappa = \bar{k}N \pmod{N^2}$. Then, a natural conjecture would be that the index is given by that of the product $U(1)_\kappa \times SU(N)_{\bar{k}}$ theory, divided by $N^2$. Notice that the selection rule for $\kappa$ ensures that the result is always an integer.

7. A Formal Digression

On general grounds, the large-$n$ limit implied in our construction of $\overline{U(N)}_{\Theta}$ is intimately connected with the definition of the gauge group for noncommutative theories. In [55,56] (see also [3]) a preliminary analysis is made of this issue. In the traditional Weyl–Moyal correspondence we consider the space of functions on some space, in our case...
$\mathbb{T}^D$, with some particular smoothness conditions, and the standard product of functions is deformed into a $\star$-product. Under appropriate mathematical conditions, the functions on the noncommutative torus can be mapped into operators on an infinite-dimensional, separable Hilbert space $\mathcal{H}$. The noncommutative gauge symmetry acts as a subgroup of the group of unitary transformations on $\mathcal{H}$, $U(\mathcal{H})$. Which subgroup we choose will determine the gauge-group topology and it is likely also to determine important non-perturbative properties of the theory.

In our construction of $\mathcal{U}(N)_{\Theta}$ we approximate $\Theta$ by rationals $\Theta_n$ and then, through a Morita transformation, we relate the noncommutative theories with an ordinary $\mathcal{U}(N'_n)$ theory with a certain amount of magnetic flux, and $N'_n \to \infty$ as $\Theta_n \to \Theta$, irrational. The details of the construction imply that in this procedure we are keeping the topology of the gauge group for each intermediate $N'_n$. From this point of view the gauge group looks like the inductive limit of $\mathcal{U}(N)$, i.e. $U(\infty)$, whose homotopy groups are given by Bott periodicity. A theorem of Palais (c.f. [57]) implies that

$$U(\infty) \subset U_1(\mathcal{H}) \subset \cdots \subset U_p(\mathcal{H}) \subset \cdots \subset U_{\text{cpt}}(\mathcal{H}),$$

(where $U \in U_p(\mathcal{H})$ iff $U = 1 + \mathcal{O}$, with $\mathcal{O}^p$ being trace class, and $U_{\text{cpt}}(\mathcal{H})$ are compact operators) all have the same homotopy type. Hence the proposal in ref. [55] to consider $U_{\text{cpt}}(\mathcal{H})$ as the appropriate gauge group arises as a natural one.

If we were to carry out the naive quantization of the theory with such a gauge group, it seems clear that the electric and magnetic sectors of the corresponding gauge theory would ‘remember’ the electric and magnetic sectors of $U(N)$ for each $N$. One could in principle imagine bigger gauge groups (all the way to $U(\mathcal{H})$, which is contractible). In this case once Gauss’ law is imposed we will lose many of the electric and magnetic sectors of the standard analysis, or perhaps the new theory would correspond to specific averages of the standard sectors. In the case of $U(1)$ on $\mathbb{T}^2$, the classical algebra of infinitesimal $U(1)$-gauge transformations is equivalent to the Fairlie–Fletcher–Zachos algebra, a trigonometric deformation of the algebra of area-preserving diffeomorphisms on $\mathbb{T}^2$, $w_\infty(\mathbb{T}^2)$ (cf. [58]). Once again, there are many choices that could be made to define the quantum theory with this gauge group. One of them in particular uses the $U(\infty)$ construction above. Other constructions might involve the different definition of $W_\infty$ starting with $w_\infty$ (see [59] for details and references.) It is clear that the physics depends on the choice of the gauge group, but at this stage it is not known to what extent. Addressing this question is likely to be one of the critical elements in the direct construction of noncommutative gauge theories for arbitrary values of $\Theta$. 

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8. Morita Duality and Topological Order in the Quantum Hall Effect

The Morita transformation of the Chern–Simons action (6.5) and, in particular, the topological coupling (6.8), has an interesting application to the model of the Fractional Quantum Hall Effect (FQHE) proposed in [20] (see also [60, 61]). Noncommutative Chern–Simons with rank one was proposed in [20] as a more refined low-energy description of the FQHE than the usual $U(1)$ Chern–Simons model [62, 63]. Specifically, one considers the rank-one model

$$S = \frac{1}{4\pi \nu} \int_{\mathbb{R} \times \mathbb{R}^2} \left( A \wedge dA + \frac{2}{3} A \wedge_\theta A \wedge_\theta A \right),$$

(8.1)

with a noncommutativity parameter determined in terms of the electron’s fluid density: $\theta = 1/2\pi \rho_e$. The topological coupling is $1/\nu$, with $\nu$ the FQHE filling fraction. We restrict for the time being to the Laughlin series $\nu = 1/q$ with odd $q$. Since noncommutative Chern–Simons theories have a very smooth ultraviolet behaviour [64], we expect the $\theta$-dependence of physical quantities to be analytic near $\theta = 0$, so that the effects of the electron’s ‘granularity’ reflected in $\theta$ can be expanded in powers of $\theta$ via the Seiberg–Witten map [5].

The real impact of the proposal (8.1) on the physics of the FQHE should be evaluated on the basis of the induced theory for edge states. Extracting the boundary dynamics induced by the bulk action (8.1) is technically non-trivial. In fact, it is natural to expect that, because of the non-locality of the Moyal products, some boundary action with non-locality on length scales of $O(\sqrt{\theta})$ must be added to the bulk Lagrangian in (8.1) (see [65, 66] for results in this direction.) It is also possible that the applicability of (8.1) to FQHE phenomena is reduced to the finite matrix aproximations, such as those studied in [60, 61], which show encouraging similarities with the microscopic structure of the relevant wave-function hierarchies.

One way of testing the smooth dependence on $\theta$ using discrete criteria is to look at Wen’s topological order [19], i.e. the universal degeneracy of Laughlin’s ground state on a torus. This degeneracy is equal to the ordinary Chern–Simons level $1/\nu$. If the noncommutative model is to give a good description of the FQHE fluid, we expect that the degeneracy on the torus is correctly accounted for.

Working on a spatial two-torus, the dimensionless noncommutativity parameter is rational

$$\Theta = \frac{1}{n_e},$$

(8.2)

where $n_e$ is the number of electrons in the torus. Thus, we can use Morita duality to define the noncommutative model on the torus via the ordinary $U(n_e)$ Chern–Simons theory with one unit of magnetic flux and level $k' = n_e/\nu$. As in the previous sections, if we choose
the background field and magnetic flux to vanish for the noncommutative $U(1)_{\Theta}$ Chern–Simons model, the resulting ordinary model has an effectively periodic $U(1)$ sector. On the other hand, the $SU(n_e)$ sector is twisted with one unit of 't Hooft’s magnetic flux. This means that the non-abelian sector does not contribute to the degeneracy of ground states, since there are no $SU(n_e)/\mathbb{Z}_{n_e}$ flat connections on the completely twisted torus.

Therefore, the topological order is determined by the abelian factor with level $k' = n_e/\nu$. As in the previous section, we must present the abelian level in the effective normalization relevant to the physics of the zero modes:

$$\kappa'_{\text{eff}} = \frac{1}{\nu}. \quad (8.3)$$

Thus, the effective abelian level is invariant under Morita transformations and we recover the expected result for the topological order.

In fact, since the ordinary non-abelian Chern–Simons model is a topological theory with a trivial Hilbert space on the twisted torus (only the vacuum remains), we can say that the physics of (8.1) on the torus is rigorously equivalent to that of the ordinary abelian Chern–Simons model with the same level. Possible non-trivial $\theta$-dependence could only arise when including the Maxwell term in the effective Lagrangian, i.e. the mixing with higher Landau levels. On the other hand, if the mixing with higher Landau levels is significant, it is unlikely that the present field-theoretical degrees of freedom (the ‘statistical’ Chern–Simons gauge field) will furnish a good description of the dynamics.

Our results in the previous sections indicate that this situation should also generalize to non-abelian models that have been proposed as effective theories of the incompressible Hall fluid with more general filling fractions [20]. Namely, according to (2.10), a noncommutative $U(N)$ model with $\Theta = 1/n_e$ can be traded by a twisted ordinary theory with gauge-field momentum modes in $U(N) \otimes U(n_e)$ and flat connections living only on the first factor. Since the effective level, as seen by the flat connections, does not change under Morita duality, we conclude that the ground-state degeneracy is independent of $\theta$ also in this generalized situation.

For example, if a $\nu = p/q$ multi-layer Hall fluid is represented by a $U(p)$ noncommutative model at level $q$, the results in the previous section yield a value of the topological order:

$$\dim \mathcal{H}(T_2) = \frac{(p+q-1)!}{p!(q-1)!}. \quad (8.4)$$

What is specifically ‘noncommutative’ in this prediction is the particular ‘linking’ of the global $U(1)$ group and the $SU(p)$ non-abelian part. If this model is regarded as a single-layer Hall fluid with non-abelian statistics, the predicted topological order differs in general from other schemes. For example, an abelian model of type [37] for the main Jain sequences,
\( \nu = p/q \) with \( q = 2ps + 1 \) and \( s \) integer, has enhanced \( U(1) \times SU(p)_1 \) affine symmetry, but the topological order is still given by

\[
\dim \mathcal{H}(T_2)_{\text{Jain}} = q = 2ps + 1. \tag{8.5}
\]

The success of the standard schemes suggests that the noncommutative non-abelian models are unlikely to describe single-layer fluids along the main sequences. Therefore, their possible applications would be restricted to multi-layer fluids.

8.1. A Comment on Level Normalizations

In the Conformal Field Theory approach to the QHE (see \[68,69\] for a summary,) the topological order is given by the level of the abelian current algebra of edge excitations. This is in turn equal to the classical coupling \( k \) of the Chern–Simons Lagrangian. In the noncommutative case, many aspects of the perturbation theory of the rank-one model (8.1) are similar to the behaviour of non-abelian Chern–Simons theory. One such instance is the quantum shift \( k \rightarrow k + 1 \) of the level, analogous to the shift \( k \rightarrow k + N \) in ordinary \( SU(N) \) Chern–Simons theory \[70\]. To a large extent, the quantum shift in ordinary Chern–Simons theory is a matter of renormalization prescription, although many observables, such as expectation values of Wilson lines, are conveniently written in terms of the shifted level.

The important point for us is that such a shift does not affect the evaluation of the dimension of the Hilbert space on the torus, which is still given in terms of the classical Chern–Simons coupling. This is rather clear in our computation: the topological order is only sensitive to the diagonal \( U(1) \), for which there is no level-shift, since no fields are charged with respect to this subgroup.

In the context of the finite-matrix models of \[60\], the link between the filling fraction and the Chern–Simons level does reflect the shift: \( k + 1 = 1/\nu \). However, these models contain additional ‘matter’ fields in the fundamental representation of the \( U(N) \) group. Hence, direct comparison of the finite-matrix level and the one appearing in (8.1) may require due attention to renormalization effects induced when ‘integrating-out’ the matter fields.

9. Conclusions

We have studied some properties of models in the dense completion of rational noncommutative gauge theories, defined in finite volume. These models, denoted \( U(N,\Theta,\Theta_n) \), are defined by \( \Theta_n \rightarrow \Theta \) limits of rational theories \( U(N,\Theta_n) \).
Assuming that these limiting models exist, they are interesting for two reasons. First, one could try to use them in giving a constructive definition of the generic noncommutative gauge theory. Even if such a program should fail and the $U(N)_{\Theta}$ theory turned out to be generically inequivalent to some other independent definition of the irrational $U(N)_{\Theta}$ theory, the physics of the $U(N)_{\Theta}$ is certainly interesting in itself.

The second interesting aspect of the $U(N)_{\Theta}$ models is that, using Morita duality, we may as well define them as certain large-$n$ limits of ordinary gauge theories. Thus, to the extent that smooth behaviour in $\Theta$ could have consequences for the physics of ordinary theories, one would be interested in precisely the $U(N)_{\Theta}$ models. There may be an interesting space of large-$n$ limits in ordinary gauge theories that remain to be explored, and of which the noncommutative theories on tori are just examples.

The evidence for continuity of the physics as a function of $\Theta$ is strong for theories with an underlying $\mathcal{N} = 4$ supersymmetry. In models with less supersymmetry, smoothness in $\Theta$ is expected in perturbation theory, except perhaps at $\Theta = 0$. There are exceptional values of the momenta in perturbation theory for which the rational and irrational theories behave very differently, but these modes seem to decouple in the irrational limit. At any rate, the most interesting tests of $\Theta$-continuity would involve non-perturbative physics such as that of confining $\mathcal{N} = 1, 0$ models.

Using the definition of $U(N)_{\Theta}$ in terms of series of ordinary theories, we have used various expectations about the dynamics of ordinary confining theories to put constraints on the $\Theta$-dependence of the $U(N)_{\Theta}$ models. In particular, we find that the Witten index (when properly defined, so that it gives non-trivial information) of minimal $\mathcal{N} = 1$ models in three and four dimensions is smooth under rational approximations. In the four-dimensional case, this result depends on the subtle interplay between the instanton angle and the magnetic flux, exactly as in the dynamics of oblique confinement.

A related interesting question is the dependence of the energy on the instanton angle, a truly non-BPS quantity. An estimate of the vacuum energy as a function of $\vartheta$ can be given by a dilute-gas fractional-instanton approximation in the series of ordinary theories. This corresponds to the dilute-gas of ordinary noncommutative instantons after Morita duality. We find that the non-trivial level-crossing phenomena that ensure $2\pi$-periodicity of physics in ordinary theories translate into a $2\pi/b_n$-periodicity in the noncommutative theories. In the $b_n \to \infty$ limit, the non-abelian contribution to the $\vartheta$-dependence becomes either trivial or discontinuous.

We have checked that the instanton-induced functional dependence is smooth in $\Theta$ provided no level-crossing phenomena takes place. Such is the case when $\mathcal{N} = 4$ supersymmetry is restored at some high energy scale. On the other hand, if supersymmetry is not restored, level-crossing may occur, resulting in strictly $\vartheta$-independent energies in the
Such a result violates $\Theta$-smoothness when the rational series approximate a rational number. Therefore, supersymmetry still acts as a ‘custodian’ of continuity in $\Theta$. It would be interesting to know whether there are other scaling or continuum limits that would guarantee continuity without necessarily requiring an underlying $\mathcal{N}=4$ supersymmetric theory in the ultraviolet. In the non-supersymmetric cases, our treatment of the ultraviolet cutoff is not rigorous, in the sense that we assume implicitly a commutativity of Morita equivalence and continuum limit. It is thus possible that, when the correct physical questions are asked, continuity of the physics in $\theta$ is borne out.

There are still quite a number of open questions; in particular a viable construction of the quantum theory directly in the noncommutative setting. In this case one of the problems to be solved is the question of the appropriate definition of the gauge group. It may turn out that some definitions of the noncommutative theory are completely disconnected from the naive approximations presented in this paper. A less ambitious problem that should be tractable is to look again at the UV/IR problem at finite volume. In theories with rational values of $\Theta$ (whether they are gauge theories or not) the problem can be expressed in terms of ordinary field theories whose fields are matrix valued. In this context the standard renormalization of the theory can be carried out without problems, and in particular one can use standard techniques to define the Wilsonian effective action (with due attention paid to the presence of electric and magnetic fluxes.) The peculiar UV/IR mixing pointed out in [7,32] would appear as we take combinations of limits where $\Theta$ becomes irrational, or the volume of the torus goes to infinity or both. In any of these cases (see subsection 3.4) we end up with limits involving large noncommutativity or non-standard large-$n$ limits. By looking at rational approximations to the UV/IR mixing we may gain some understanding of this phenomenon and of its physical significance.

Finally we have shown that there are a number of interesting things that can be learned from Morita duality when applied to the formulation of the Fractional Hall Effect proposed in [20]; in particular we get an elegant derivation of Wen’s topological order [19]. Since Chern–Simons theory is rather soft in the ultraviolet, we do not expect any problems concerning continuity in the noncommutative parameter. For these applications however, this problem is not relevant since the dimensionless deformation parameter is always rational, and given by the inverse of the number of electrons in the torus. A very relevant question is how to extract the boundary dynamics of edge states induced by the bulk action (8.1). It would be interesting to know how to define noncommutative spaces with ‘boundaries’. Perhaps in this context the use of the Morita equivalence to an ordinary gauge theory with gauge group $U(n_e)$ should provide useful guidance.

*Note added.* When this paper was being finished, a paper by Z. Guralnik [71] appeared studying noncommutative QED using also Morita equivalence and large-$\mathcal{N}$ limits.
of ordinary gauge theories.

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