NEWTON POLYGONS AND FAMILIES OF POLYNOMIALS

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Abstract. We consider a continuous family \((f_s)_s \in [0, 1]\) of complex polynomials in two variables with isolated singularities, that are Newton non-degenerate. We suppose that the Euler characteristic of a generic fiber is constant. We firstly prove that the set of critical values at infinity depends continuously on \(s\), and secondly that the degree of the \(f_s\) is constant (up to an algebraic automorphism of \(\mathbb{C}^2\)).

1. Introduction

We consider a family \((f_s)_s \in [0, 1]\) of complex polynomials in two variables with isolated singularities. We suppose that coefficients are continuous functions of \(s\). For all \(s\), there exists a finite bifurcation set \(\mathcal{B}(s)\) such that the restriction of \(f_s\) above \(\mathbb{C} \setminus \mathcal{B}(s)\) is a locally trivial fibration. It is known that \(\mathcal{B}(s) = \mathcal{B}_{\text{aff}}(s) \cup \mathcal{B}_\infty(s)\), where \(\mathcal{B}_{\text{aff}}(s)\) is the set of affine critical values, that is to say the image by \(f_s\) of the critical points; \(\mathcal{B}_\infty(s)\) is the set of critical values at infinity. For \(c \notin \mathcal{B}(s)\), the Euler characteristic verifies

\[
\chi(f_s^{-1}(c)) = \mu(s) + \lambda(s),
\]

where \(\mu(s)\) is the affine Milnor number and \(\lambda(s)\) is the Milnor number at infinity.

We will be interested in families such that the sum \(\mu(s) + \lambda(s)\) is constant. These families are interesting in the view of \(\mu\)-constant type theorem, see [HZ, HP, Ti, Bo, BT]. We say that a multi-valued function \(s \mapsto F(s)\) is continuous if at each point \(\sigma \in [0, 1]\) and at each value \(c(\sigma) \in F(\sigma)\) there is a neighborhood \(I\) of \(\sigma\) such that for all \(s \in I\), there exists \(c(s) \in F(s)\) near \(c(\sigma)\). \(F\) is closed, if, for all points \(\sigma \in [0, 1]\), for all sequences \(c(s) \in F(s), s \neq \sigma\), such that \(c(s) \to c(\sigma) \in \mathbb{C}\) as \(s \to \sigma\), then \(c(\sigma) \in F(\sigma)\). It is well-known that \(s \mapsto \mathcal{B}_{\text{aff}}(s)\) is a continuous multi-valued function. But it is not necessarily closed: for example \(f_s(x, y) = (x - s)(xy - 1)\), then for \(s \neq 0\), \(\mathcal{B}_{\text{aff}}(s) = \{0, s\}\) but \(\mathcal{B}_{\text{aff}}(0) = \emptyset\).

We will prove that \(s \mapsto \mathcal{B}_\infty(s)\) and \(s \mapsto \mathcal{B}(s)\) are closed continuous functions under some assumptions.

**Theorem 1.** Let \((f_s)_{s \in [0, 1]}\) be a family of complex polynomials such that \(\mu(s) + \lambda(s)\) is constant and such that \(f_s\) is (Newton) non-degenerate for all \(s \in [0, 1]\), then the multi-valued function \(s \mapsto \mathcal{B}_\infty(s)\) is continuous and closed.

**Remark.** As a corollary we get the answer to a question of D. Siersma: is it possible to find a family with \(\mu(s) + \lambda(s)\) constant such that \(\lambda(0) > 0\) (equivalently \(\mathcal{B}_\infty(0) \neq \emptyset\)) and \(\lambda(s) = 0\) (equivalently \(\mathcal{B}_\infty = \emptyset\)) for \(s \in [0, 1]\).
Corollary 2. expressed in the following corollary (of Theorems 1 and 3): $\mu$ closed even if $\mu$ s

Remark. Theorem 1 does not imply that $\mu(s)$ and $\lambda(s)$ are constant. For example let the family $f_s(x, y) = x^2y^2 + sxy + x$. Then for $s = 0$, $\mu(0) = 0$, $\lambda(0) = 2$ with $B_\infty(0) = \{0\}$, and for $s \neq 0$, $\mu(s) = 1$, $\lambda(s) = 1$ with $B_{\text{aff}}(s) = \{0\}$ and $B_\infty(s) = \{-s^2\}$.

The multi-valued function $s \mapsto B_{\text{aff}}(s)$ is continuous but not necessarily closed even if $\mu(s) + \lambda(s)$ is constant, for example (see [Ti]): $f_s(x, y) = x^4 - x^2y^2 + 2xy + s x^2$, then $\mu(s) + \lambda(s) = 5$. We have $B_{\text{aff}}(0) = \{0\}$, $B_\infty(0) = \{1\}$ and for $s \neq 0$, $B_{\text{aff}} = \{0, 1 - \frac{s^2}{4}\}$, $B_\infty(s) = \{1\}$. We notice that even if $s \mapsto B_{\text{aff}}(s)$ is not closed, the map $s \mapsto B(s)$ is closed. This is expressed in the following corollary (of Theorems 1 and 3):

Corollary 2. Let $(f_s)_{s \in [0, 1]}$ be a family of complex polynomials such that $\mu(s) + \lambda(s)$ is constant and such that $f_s$ is non-degenerate for all $s \in [0, 1]$. Then the multi-valued function $s \mapsto B(s)$ is continuous and closed.

We are now interested in the constancy of the degree; in all hypotheses of global $\mu$-constant theorems the degree of the $f_s$ is supposed not to change (see [HZ, HP, Bo, BT]) and it is the only non-topological hypothesis. We prove that for non-degenerate polynomials in two variables the degree is constant except for a few cases, where the family is of quasi-constant degree. We will define in a combinatoric way in paragraph 3 what a family of quasi-constant degree is, but the main point is to know that such a family is of constant degree up to some algebraic automorphism of $\mathbb{C}^2$. More precisely, for each value $\sigma \in [0, 1]$ there exists $\Phi \in \text{Aut} \mathbb{C}^2$ such $f_s \circ \Phi$ is of constant degree, for $s$ in a neighborhood of $\sigma$. For example the family $f_s(x, y) = x + sy^2$ is of quasi-constant degree while the family $f_s(x, y) = sxy + x$ is not.

Theorem 3. Let $(f_s)_{s \in [0, 1]}$ be a family of complex polynomials such that $\mu(s) + \lambda(s)$ is constant and such that $f_s$ is non-degenerate for all $s \in [0, 1]$, then either $(f_s)_{s \in [0, 1]}$ is of constant degree or $(f_s)_{s \in [0, 1]}$ is of quasi-constant degree.

Remark. In theorem 3, $f_0$ may be degenerate.

As a corollary we get a $\mu$-constant theorem without hypothesis on the degree:

Theorem 4. Let $(f_s)_{s \in [0, 1]}$ be a family of polynomials in two variables with isolated singularities such that the coefficients are continuous function of $s$. We suppose that $f_s$ is non-degenerate for $s \in [0, 1]$, and that the integers $\mu(s) + \lambda(s)$, $\#B(s)$ are constant $(s \in [0, 1])$ then the polynomials $f_0$ and $f_1$ are topologically equivalent.
It is just the application of the $\mu$-constant theorem of [Bo], [BT] to the family $(f_s)$ or $(f_s \circ \Phi)$. Two kinds of questions can be asked: are Theorems 1 and 3 true for degenerate polynomials? are they true for polynomials in more than 3 variables? I would like to thank Prof. Günter Ewald for discussions concerning Theorem 3 in $n$ variables (that unfortunately only yield that the given proof cannot be easily generalized).

2. Tools

2.1. Definitions. We will recall some basic facts about Newton polygons, see [Ko], [CN], [NZ]. Let $f \in \mathbb{C}[x, y]$, $f(x, y) = \sum_{(p, q) \in \mathbb{N}^2} a_{p,q} x^p y^q$. We denote $\text{supp}(f) = \{(p, q) \mid a_{p,q} \neq 0\}$, by abuse $\text{supp}(f)$ will also denote the set of monomials $\{x^p y^q \mid (p, q) \in \text{supp}(f)\}$. $\Gamma_-(f)$ is the convex closure of $\{(0, 0)\} \cup \text{supp}(f)$, $\Gamma(f)$ is the union of closed faces which do not contain $(0, 0)$. For a face $\gamma$, $f_\gamma = \sum_{(p,q)\in\gamma} a_{p,q} x^p y^q$. The polynomial $f$ is (Newton) non-degenerate if for all faces $\gamma$ of $\Gamma(f)$ the system

$$\frac{\partial f_\gamma}{\partial x}(x, y) = 0; \quad \frac{\partial f_\gamma}{\partial y}(x, y) = 0$$

has no solution in $\mathbb{C}^* \times \mathbb{C}^*$.

We denote by $S$ the area of $\Gamma_-(f)$, by $a$ the length of the intersection of $\Gamma_-(f)$ with the $x$-axis, and by $b$ the length of the intersection of $\Gamma_-(f)$ with the $y$-axis (see Figure 1). We define

$$\nu(f) = 2S - a - b + 1.$$  

![Figure 1. Newton polygon of $f$ and $\nu(f) = 2S - a - b + 1$.](image)

2.2. Milnor numbers. The following result is due to Pi. Cassou-Noguès [CN], it is an improvement of Kouchnirenko’s result.

**Theorem 5.** Let $f \in \mathbb{C}[x, y]$ with isolated singularities. Then

1. $\mu(f) + \lambda(f) \leq \nu(f)$.
2. If $f$ is non-degenerate then $\mu(f) + \lambda(f) = \nu(f)$. 


2.3. **Critical values at infinity.** We recall the result of A. Néméthi and A. Zaharia on how to estimate critical values at infinity. A polynomial \( f \in \mathbb{C}[x, y] \) is *convenient for the x-axis* if there exists a monomial \( x^a \) in \( \text{supp}(f) \) \((a > 0)\); \( f \) is *convenient for the y-axis* if there exists a monomial \( y^b \) in \( \text{supp}(f) \) \((b > 0)\); \( f \) is *convenient* if it is convenient for the x-axis and the y-axis. It is well-known (see [Br]) that:

**Lemma 6.** A non-degenerate and convenient polynomial with isolated singularities has no critical value at infinity: \( B_\infty = \emptyset \).

Let \( f \in \mathbb{C}[x, y] \) be a polynomial with \( f(0,0) = 0 \) not depending only on one variable. Let \( \gamma_x \) and \( \gamma_y \) the two faces of \( \Gamma_{-}(f) \) that contain the origin. If \( f \) is convenient for the x-axis then we set \( C_x = \emptyset \) otherwise \( \gamma_x \) is not included in the x-axis and we set

\[
C_x = \left\{ f_{\gamma_x}(x, y) \mid (x, y) \in \mathbb{C}^* \times \mathbb{C}^* \text{ and } \frac{\partial f_{\gamma_x}}{\partial x}(x, y) = \frac{\partial f_{\gamma_x}}{\partial y}(x, y) = 0 \right\}.
\]

In a similar way we define \( C_y \).

A result of [NZ, Proposition 6] is:

**Theorem 7.** Let \( f \in \mathbb{C}[x, y] \) be a non-degenerate and non-convenient polynomial with \( f(0,0) = 0 \), not depending only on one variable. The set of critical values at infinity of \( f \) is

\[
B_\infty = C_x \cup C_y \quad \text{or} \quad B_\infty = \{0\} \cup C_x \cup C_y.
\]

Unfortunately this theorem does not determine whether \( 0 \in B_\infty \) (and notice that the value 0 may be already included in \( C_x \) or \( C_y \)). This value 0 is treated in the following lemma.

**Lemma 8.** Let \( f \in \mathbb{C}[x, y] \) be a non-degenerate and non-convenient polynomial, with isolated singularities and with \( f(0,0) = 0 \). Then

\[
B_\infty = B_{\infty,x} \cup B_{\infty,y}
\]

where we define:

1. if \( f \) is convenient for the x-axis then \( B_{\infty,x} := \emptyset \);
2. otherwise there exists \( x^py \) in \( \text{supp}(f) \) where \( p \geq 0 \) is supposed to be maximal;
   a. If \( x^py \) is in a face of \( \Gamma_{-}(f) \) then \( B_{\infty,x} := C_x \) and \( 0 \notin B_{\infty,x} \);
   b. If \( x^py \) is not in a face of \( \Gamma_{-}(f) \) then \( B_{\infty,x} := \{0\} \cup C_x \);
3. we set a similar definition for \( B_{\infty,y} \).

Theorem 7 and its refinement Lemma 8 enable to calculate \( B_\infty \) from \( \text{supp}(f) \). The different cases of Lemma 8 are pictured in Figures 2 and 3.

**Proof.** As \( f \) is non-convenient with \( f(0,0) = 0 \) we may suppose that \( f \) is non-convenient for the x-axis so that \( f(x, y) = yk(x, y) \). But \( f \) has isolated singularities, so \( y \) does not divide \( k \). Then there is a monomial \( x^py \in \text{supp}(f) \), we can suppose that \( p \geq 0 \) is maximal among monomials \( x^ky \in \text{supp}(f) \).
Let $d = \deg f$. Let $\bar{f}(x, y, z) - cz^d$ be the homogenization of $f(x, y) - c$; at the point at infinity $P = (1 : 0 : 0)$, we define $g_c(y, z) = \bar{f}(1, y, z) - cz^d$. Notice that only $(1 : 0 : 0)$ and $(0 : 1 : 0)$ can be singularities at infinity for $f$. The value 0 is a critical value at infinity for the point at infinity $P$ (that is to say $0 \in \mathcal{B}_{\infty,x}$) if and only if $\mu_P(g_0) > \mu_P(g_c)$ where $c$ is a generic value.

The Newton polygon of the germ of singularity $g_c$ can be computed from the Newton polygon $\Gamma(f)$, for $c \neq 0$, see [NZ, Lemma 7]. If $A, B, O$ are the points on the Newton diagram of coordinates $(d, 0), (0, d), (0, 0)$, then the Newton diagram of $g_c$ has origin $A$ with $y$-axis $AB$, $z$-axis $AO$, and the convex closure of $\text{supp}(g_c)$ corresponds to $\Gamma_-(f)$.

![Newton polygon of $g_c$. First case: $0 \notin \mathcal{B}_{\infty,x}$.

![Newton polygon of $g_c$. Second case: $0 \in \mathcal{B}_{\infty,x}$.

We denote by $\Delta_c$ the Newton polygon of the germ $g_c$, for a generic value $c$, $\Delta_c$ is non-degenerate and $\mu_P(g_c) = \nu(\Delta_c)$. The Newton polygon $\Delta_0$ has no common point with the $z$-axis $AO$ but $\nu$ may be defined for non-convenient series, see [Ko, Definition 1.9].

If $x^py$ is in the face $\gamma_x$ of $\Gamma_-(f)$ then $\Delta_0$ is non-degenerate and $\nu(\Delta_0) = \nu(\Delta_c)$, then by [Ko, Theorem 1.10] $\mu_P(g_0) = \nu(\Delta_0)$ and $\mu_P(g_c) = \nu(\Delta_c)$. So $\mu_P(g_0) = \mu_P(g_c)$ and 0 is not a critical value at infinity for the point $P : 0 \notin \mathcal{B}_{\infty,x}$.

If $x^py$ is not in a face of $\Gamma_-(f)$ then there is a triangle $\Delta_c$ that disappears in $\Delta_0$, by the positivity of $\nu$ (see below) we have $\nu(\Delta_0) > \nu(\Delta_c)$, then by [Ko, Theorem 1.10]: $\mu_P(g_0) \geq \nu(\Delta_0) > \nu(\Delta_c) = \mu_P(g_c)$. So we have $0 \in \mathcal{B}_{\infty,x}$. □
2.4. Additivity and positivity. We need a variation of Kouchnirenko’s number $\nu$. Let $T$ be a polytope whose vertices are in $\mathbb{N} \times \mathbb{N}$, $S > 0$ the area of $T$, $a$ the length of the intersection of $T$ with the $x$-axis, and $b$ the length of the intersection of $T$ with the $y$-axis. We define

$$\tau(T) = 2S - a - b,$$

so that, $\nu(T) = \tau(T) + 1$.

It is clear that $\tau$ is additive: $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2) - \tau(T_1 \cap T_2)$, and in particular if $T_1 \cap T_2$ has null area then $\tau(T_1 \cup T_2) = \tau(T_1) + \tau(T_2)$. This formula enables us to argue on triangles only (after a triangulation of $T$).

Let $T_0$ be the triangle defined by the vertices $(0, 0), (1, 0), (0, 1)$, we have $\nu(T_0) = -1$. We have the following facts, for every triangle $T \neq T_0$:

1. $\nu(T) \geq 0$;
2. $\nu(T) = 0$ if and only if $T$ has an edge contained in the $x$-axis or the $y$-axis and the height of $T$ (with respect to this edge) is 1.

Remark. The formula of additivity can be generalized in the $n$-dimensional case, but the positivity can not. Here is a counter-example found by Günter Ewald: Let $n = 4$, $a$ a positive integer and let $T$ be the polytope whose vertices are: $(1, 0, 0, 0), (1 + a, 0, 0, 0), (1, 1, 1, 0), (1, 2, 1, 0), (1, 1, 1, 1)$ then $\tau(T) = \nu(T) + 1 = -a < 0$.

2.5. Families of polytopes. We consider a family $(f_s)_{s \in [0,1]}$ of complex polynomials in two variables with isolated singularities. We suppose that $\mu(s) + \lambda(s)$ remains constant. We denote by $\Gamma(s)$ the Newton polygon of $f_s$.

We will always assume that $f_s$ is non-degenerate for $s \in (0,1]$. We will always assume that the only critical parameter is $s = 0$. We will say that a monomial $x^py^q$ disappears if $(p, q) \in \text{supp}(f_s) \setminus \text{supp}(f_0)$ for $s \neq 0$. By extension a triangle of $\mathbb{N} \times \mathbb{N}$ disappears if one of its vertices (which is a vertex of $\Gamma(s)$, $s \neq 0$) disappears. Now after a triangulation of $\Gamma(s)$ we have a finite number of triangles $T$ that disappear (see Figure 4, on pictures of the Newton diagram, a plain circle is drawn for a monomial that does not disappear and an empty circle for monomials that disappear).

![Figure 4. Triangles that disappear.](image-url)
Lemma 9. Let \( T \neq T_0 \) be a triangle that disappears then \( \tau(T) = 0 \).

Proof. We suppose that \( \tau(T) > 0 \). By the additivity and positivity of \( \tau \) we have for \( s \in [0, 1] \):

\[
\nu(s) = \nu(G(s)) \geq \nu(G(0)) + \tau(T) > \nu(0).
\]

Then by Theorem 5,

\[
\mu(s) + \lambda(s) = \nu(s) > \nu(0) \geq \mu(0) + \lambda(0).
\]

This gives a contradiction with \( \mu(s) + \lambda(s) = \mu(0) + \lambda(0) \).

We remark that we do not need \( f_0 \) to be non-degenerate because in all cases we have \( \nu(0) \geq \mu(0) + \lambda(0) \). \( \square \)

3. Constancy of the degree

3.1. Families of quasi-constant degree. Let \( \sigma \in [0, 1] \), we choose a small enough neighborhood \( I \) of \( \sigma \). Let \( M_\sigma \) be the set of monomials that disappear at \( \sigma \): \( M_\sigma = \text{supp}(f_s) \setminus \text{supp}(f_{s'}) \) for \( s \in I \setminus \{\sigma\} \). The family \( (f_s)_{s \in [0,1]} \) is of quasi-constant degree at \( \sigma \) if there exists \( x^p y^q \in \text{supp}(f_\sigma) \) such that

\[
(\forall x^{p'} y^{q'} \in M_\sigma \ (p > p') \text{ or } (p = p' \text{ and } q > q'))
\]

or

\[
(\forall x^{p'} y^{q'} \in M_\sigma \ (q > q') \text{ or } (q = q' \text{ and } p > p')).
\]

The family \( (f_s)_{s \in [0,1]} \) is of quasi-constant degree if it is of quasi-constant degree at each point \( \sigma \) of \([0, 1]\). The terminology is justified by the following remark:

Lemma 10. If \( (f_s) \) is of quasi-constant degree at \( \sigma \in [0, 1] \), then there exists \( \Phi \in \text{Aut } \mathbb{C}^2 \) such that \( \text{deg } f_s \circ \Phi \) is constant in a neighborhood of \( \sigma \).

The proof is simple: suppose that \( x^p y^q \) is a monomial of \( \text{supp}(f_\sigma) \) such that for all \( x^{p'} y^{q'} \in M_\sigma , p > p' \text{ or } (p = p' \text{ and } q > q') \). We set \( \Phi(x,y) = (x + y^\ell, y) \) with \( \ell \gg 1 \). Then the monomial of highest degree in \( f_s \circ \Phi \) is \( y^{q+p\ell} \) and does not disappear at \( \sigma \). For example let \( f_s(x,y) = xy + sy^3 \), we set \( \Phi(x,y) = (x + y^\ell, y) \) then \( f_s \circ \Phi(x,y) = y^4 + xy + sy^3 \) is of constant degree.

We prove Theorem 3. We suppose that the degree changes, more precisely we suppose that \( \text{deg } f_s \) is constant for \( s \in ]0,1[ \) and that \( \text{deg } f_0 < \text{deg } f_s , s \in ]0,1[ \). As the degree changes the Newton polygon \( \Gamma(s) \) cannot be constant, that means that at least one vertex of \( \Gamma(s) \) disappears.

3.2. Exceptional case. We suppose that \( f_0 \) is a one-variable polynomial, for example \( f_0 \in \mathbb{C}[y] \). As \( f_0 \) has isolated singularities then \( f_0(x,y) = ay + b_0 \), so \( \mu(0) = \lambda(0) = 0 \), then for all \( s \), \( \mu(s) = \lambda(s) = 0 \). So \( \nu(s) = \nu(\Gamma(s)) = 0 \), then \( \text{deg } y f_s = 1 \), and \( f_s(x,y) = asy + b_s(x), \) so \( (f_s)_{s \in [0,1]} \) is a family of quasi-constant degree (see Figure 5). We exclude this case for the end of the proof.
3.3. **Case to exclude.** We suppose that a vertex $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ disappears. Then there exists a triangle $T$ that disappears whose faces are not contained in the axis. Then $\tau(T) > 0$ that contradicts Lemma 9 (see Figure 6).

![Figure 5](image1.png)

**Figure 5.** Case $f_0 \in \mathbb{C}[y]$.

3.4. **Case where a monomial $x^a$ or $y^b$ disappears (but not both).** If, for example the monomial $y^b$ of $\Gamma(s)$ disappears and $x^a$ does not, then we choose a monomial $x^p y^q$, with maximal $p$, among monomials in $\text{supp}(f_s)$. Certainly $p \geq a > 0$. We also suppose that $q$ is maximal among monomials $x^p y^k \in \text{supp}(f_s)$. If $q = 0$ then $p = a$, and the monomial $x^p y^a = x^a$ does not disappear (by assumption). If $q > 0$ then $x^p y^q$ cannot disappear (see above). In both cases the monomial $x^p y^q$ proves that $(f_s)$ is of quasi-constant degree.

3.5. **Case where both $x^a$ and $y^b$ disappear.**
Sub-case: No monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$. Then there is an area $T$ with $\tau(T) > 0$ that disappears (see Figure 7). Contradiction.

![Figure 6](image2.png)

**Figure 6.** Case where a monomial $x^p y^q$, $p > 0, q > 0$ of $\Gamma(s)$ disappears.

![Figure 7](image3.png)

**Figure 7.** Sub-case: no monomial $x^p y^q$ in $\Gamma(s)$, $p > 0, q > 0$. 
Sub-case: there exists a monomial $x^py^q$ in $\Gamma(s)$, $p > 0, q > 0$. We know that $x^py^q$ is in $\Gamma(0)$ because it cannot disappear. As $\deg f_0 < \deg f_s$, a monomial $x^py^q$ that does not disappear verifies $\deg x^py^q = p + q < \deg f_s$, \((s \in ]0, 1[)\). So the monomial of highest degree is $x^a$ or $y^b$. We will suppose that it is $y^b$, so $d = b$, and the monomial $y^b$ disappears. Let $x^py^q$ be a monomial of $\Gamma(s)$, $p, q > 0$ with minimal $q$. By assumption such a monomial exists. Then certainly we have $q = 1$, otherwise there exists a region $T$ that disappears with $\tau(T) > 0$ (on Figure 8 the regions $T_1$ and $T_2$ verify $\nu(T_1) = 0$ and $\nu(T_2) = 0$). For the same reason the monomial $x^py^q'$ with minimal $p'$ verifies $p' = 1$.

We look at the segments of $\Gamma(s)$, starting from $y^b = y^d$ and ending at $x^a$. The first segment is from $y^d$ to $xy^q'$, ($p' = 1$) and we know that $p' + q' < d$ so the slope of this segment is strictly less than $-1$. By the convexity of $\Gamma(s)$ all the following slopes are strictly less than $-1$. The last segment is from $x^py$ to $x^a$, with a slope strictly less than $-1$, so $a \leq p$. Then the monomial $x^py$ gives that $(f_s)_{s \in [0, 1]}$ is of quasi-constant degree.

4. Continuity of the critical values

We now prove Theorem 1. We will suppose that $s = 0$ is the only problematic parameter. In particular $\Gamma(s)$ is constant for all $s \in ]0, 1[$.

4.1. The Newton polygon changes. That is to say $\Gamma(0) \neq \Gamma(s)$, $s \neq 0$. As in the proof of Theorem 3 (see paragraph 3) we remark:

- If $f_0$ is a one-variable polynomial then $\mathcal{B}_\infty(s) = \emptyset$ for all $s \in [0, 1]$.
- A vertex $x^py^q$, $p > 0, q > 0$ of $\Gamma(s)$ cannot disappear.

So we suppose that a monomial $x^a$ of $\Gamma(s)$ disappears (a similar proof holds for $y^b$). Then for $s \in ]0, 1[$ the monomial $x^a$ is in $\Gamma(s)$, so there are no critical values at infinity for $f_s$ at the point $P = (1 : 0 : 0)$. If $\Gamma(0)$ contains a monomial $x^a'$, $a' > 0$ then there are no critical values at infinity for $f_0$ at the point $P$. So we suppose that all monomials $x^k$ disappear.
Then a monomial $x^{p}y^{q}$ of $\text{supp}(f_{0})$ with minimal $q > 0$, verifies $q = 1$, otherwise there would exist a region $T$ with $\tau(T) > 0$ (in contradiction with the constancy of $\mu(s) + \lambda(s)$, see Lemma 9). And for the same reason if we choose $x^{p}y$ in $\text{supp}(f_{0})$ with maximal $p$ then $p > 0$ and $x^{p}y \in \Gamma(0)$. Now the edge of $\Gamma_{-}(f_{0})$ that contains the origin and the monomial $x^{p}y$ (with maximal $p$) begins at the origins and ends at $x^{p}y$ (so in particular there is no monomial $x^{2p}y^{2}$, $x^{3p}y^{3}$ in $\text{supp}(f_{0})$). Now from Theorem 7 and Lemma 8 we get that there are no critical values at infinity for $f_{0}$ at $P$.

So in case where $\Gamma(s)$ changes, we have for all $s \in [0, 1]$, $B_{\infty}(s) = \emptyset$.

4.2. The Newton polygon is constant : case of non-zero critical values. We now prove the following lemma that ends the proof of Theorem 1.

**Lemma 11.** Let a family $(f_{s})_{s \in [0, 1]}$ such that $f_{s}$ is non-degenerate for all $s \in [0, 1]$ and $\Gamma(s)$ is constant, then the multi-valued function $s \mapsto B_{\infty}(s)$ is continuous and closed.

In this paragraph and the next one we suppose that $f_{s}(0, 0) = 0$, that is to say the constant term of $f_{s}$ is zero. We suppose that $c(0) \in B_{\infty}(0)$ and that $c(0) \neq 0$. Then $c(0)$ has been obtained by the result of Némethi-Zaharia (see Theorem 7). There is a face $\gamma$ of $\Gamma_{-}(f_{0})$ that contains the origin such that $c(0)$ is in the set:

$$C_{\gamma}(0) = \left\{ (f_{0})_{\gamma}(x, y) \mid (x, y) \in (\mathbb{C}^{*})^{2} \text{ and } \frac{\partial (f_{0})_{\gamma}}{\partial x}(x, y) = \frac{\partial (f_{0})_{\gamma}}{\partial y}(x, y) = 0 \right\}.$$  

Now, as $\Gamma(s)$ is constant, $\gamma$ is a face of $\Gamma_{-}(s)$ for all $s$. There exists a family of polynomials $h_{s} \in \mathbb{C}[t]$ and a monomial $x^{p}y^{q}$ ($p, q > 0, \gcd(p, q) = 1$) such that $(f_{s})_{\gamma}(x, y) = h_{s}(x^{p}y^{q})$. The family $(h_{s})$ is continuous (in $s$) and is of constant degree (because $\Gamma(s)$ is constant). The set $C_{\gamma}(0)$ and more generally the set $C_{\gamma}(s)$ can be computed by

$$C_{\gamma}(s) = \left\{ h_{s}(t) \mid t \in \mathbb{C}^{*} \text{ and } h'_{s}(t) = 0 \right\}.$$  

As $c(0) \in C_{\gamma}(0)$ there exists a $t_{0} \in \mathbb{C}^{*}$ with $h'_{s}(t_{0}) = 0$, and for $s$ near $0$ there is a $t_{s} \in \mathbb{C}^{*}$ near $t_{0}$ with $h'_{s}(t_{s}) = 0$ (because $h'_{s}(t)$ is a continuous function of $s$ of constant degree in $t$). Then $c(s) = h_{s}(t_{s})$ is a critical value at infinity near $c(0)$ and we get the continuity.

4.3. The Newton polygon is constant : case of the value 0. We suppose that $c(0) = 0 \in B_{\infty}(0)$ and that $f_{s}(x, y) = yk_{s}(x, y)$. We will deal with the point at infinity $P = (1 : 0 : 0)$, the point $(0 : 1 : 0)$ is treated in a similar way. Let $x^{p}y$ be a monomial of $\text{supp}(f_{s})$ with maximal $p \geq 0$, $s \neq 0$. If $x^{p}y$ is not in a face of $\Gamma(s)$ then $0 \in B_{\infty}(s)$ for all $s \in [0, 1]$, and we get the continuity. Now we suppose that $x^{p}y$ is in a face of $\Gamma(s)$; then $x^{p}y$ disappears otherwise 0 is not a critical value at infinity (at the point $P$) for all $s \in [0, 1]$. As $\Gamma(s)$ is constant then the face $\gamma$ that contains the
origin and $x^py$ for $s \neq 0$ is also a face of $\Gamma(0)$, then there exists a monomial $(x^py)^k$, $k > 1$ in $\text{supp}(f_0)$. Then $(f_\gamma)_\gamma = h_s(x^py)$, $h_s \in \mathbb{C}[t]$. We have $\deg h_s > 1$, with $h_s(0) = 0$ (because $f(0,0) = 0$) and $h'_s(0) = 0$ (because $x^py$ disappears). Then $0 \in C_\gamma(0) \subset B_\infty(0)$ but by continuity of $h_s$ we have a critical value $c(s) \in C_\gamma(s) \subset B_\infty(s)$ such that $c(s)$ tends towards 0 (as $s \to 0$). It should be noticed that for $s \neq 0$, $c(s) \neq 0$.

In all cases we get the continuity of $B_\infty(s)$.

4.4. **Proof of the closeness of $s \mapsto B_\infty(s)$**. We suppose that $c(s) \in B_\infty(s)$, is a continuous function of $s \neq 0$, with a limit $c(0) \in \mathbb{C}$ at $s = 0$. We have to prove that $c(0) \in B_\infty(0)$. As there are critical values at infinity we suppose that $\Gamma(0)$ is constant.

Case $c(0) \neq 0$. Then for $s$ near 0, $c(s) \neq 0$ by continuity, then $c(s)$ is obtained as a critical value of $h_s(t)$. By continuity $c(0)$ is a critical value of $h_0(t)$: $h_0(t_0) = c(0)$, $h'_0(t_0) = 0$; as $c(0) \neq 0$, $t_0 \neq 0$ (because $h_0(0) = 0$). Then $c(0) \in B_\infty(0)$.

Case $c(0) = 0$. Then let $x^py$ be the monomial of $\text{supp}(f_s)$, $s \neq 0$, with maximal $p$. By Lemma 8 if $x^py \notin \Gamma(s)$ for $s \in [0,1]$ then $0 \in B_\infty(s)$ for all $s \in [0,1]$ and we get closeness. If $x^py \in \Gamma(s)$, $s \neq 0$, then as $c(s) \to 0$ we have that $x^py$ disappears, so $x^py \notin \Gamma(0)$, then by Lemma 8, $c(0) = 0 \in B_\infty(0)$.

4.5. **Proof of the closeness of $s \mapsto B(s)$**. We now prove Corollary 2. The multi-valued function $s \mapsto B(s)$ is continuous because $B(s) = B_{\text{aff}}(s) \cup B_\infty(s)$ and $s \mapsto B_{\text{aff}}(s)$, $s \mapsto B_\infty(s)$ are continuous. For closeness, it remains to prove that if $c(s) \in B_{\text{aff}}(s)$ is a continuous function with a limit $c(0) \in \mathbb{C}$ at $s = 0$ then $c(0) \in B(0)$.

We suppose that $c(0) \notin B_{\text{aff}}(0)$. There exist critical points $Q_s = (x_s, y_s) \in \mathbb{C}^2$ of $f_s$ with $f_s(x_s, y_s) = c(s)$, $s \neq 0$. We can extract a countable set $S$ of $[0,1]$ such that the sequence $(Q_s)_{s \in S}$ converges towards $P$ in $\mathbb{C}P^2$. As $c(0) \notin B_{\text{aff}}(0)$ we have that $P$ relies on the line at infinity and we may suppose that $P = (0 : 1 : 0)$.

By Theorem 3 we may suppose, after an algebraic automorphism of $\mathbb{C}^2$, that $d = \deg f_s$ is constant. Now we look at $g_{s,c}(x, z) = f_s(x, 1, z) - cz^d$. The critical point $Q_s$ of $f_s$ with critical value $c(s)$ gives a critical point $Q'_s = (\frac{x_s}{y_s}, \frac{1}{y_s})$ of $g_{s,c}(s)$ with critical value 0 (see [Bo, Lemma 21]). Then by semi-continuity of the local Milnor number on the fiber $g_{s,c}(s)(0)$ we have $\mu_P(g_{0,c}(0)) \geq \mu_P(g_{s,c}(s)) + \mu_{Q'_s}(g_{s,c}(s)) > \mu_P(g_{s,c}(s))$. As $\mu(s) + \lambda(s)$ is constant we have $\mu_P(g_{s,c})$ constant for a generic $c$ (see [ST, Corollary 5.2] or [BT]). Then we have $\mu_P(g_{0,c}(0)) - \mu_P(g_{0,c}) > \mu_P(g_{s,c}(s)) - \mu_P(g_{s,c}) \geq 0$. Then $c(0) \in B_\infty(0)$ and we get closeness for $s \mapsto B(s)$.

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