MEAN ERGODIC THEOREMS IN HILBERT-KAPLANSKY SPACES

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Abstract. We prove the mean ergodic theorem of von Neumann in a Hilbert—Kaplansky space. We also prove a multiparameter, modulated, subsequential and a weighted mean ergodic theorems in a Hilbert—Kaplansky space

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1. Introduction

It is well known fact that for a contracting map $T$ on a Hilbert space $H$, the ergodic average $\frac{1}{n} \sum_{i=0}^{n-1} T^i x$ is norm convergent for all $x \in H$ [13], [17]. This theorem is referred as mean ergodic theorem or von Neumann ergodic theorem. Since then, various generalizations of mean ergodic theorem were given. For example, in [1], the convergence of a modulated and a subsequential ergodic averages have been studied. Also, in [16], the convergence of weighted ergodic averages in a Hilbert space have been given. An analogue of mean ergodic theorem in $L_2$ for multiple contractions is due to T.Tao [19] under assumption that transformations commute.

Banach—Kantorovich spaces were firstly introduced by A. V. Kantorovich in 1938 (see, for example [10], [11]), which have a rich applications in analysis. Later, the theory of Banach—Kantorovich spaces was developed in [8], [9], [14], [15]. One of the interesting problems in a Banach—Kantorovich space is to study the convergence of ergodic averages. However, just few results are known in this direction. For example, an analogue of individual ergodic theorem for positive contractions on a Banach—Kantorovich lattice $L_p(\nabla, \mu)$ has been given in [2]. Later in [20], this result was extended to an Orlicz—Kantorovich space. In addition, “zero-two” law for positive contractions on Banach—Kantorovich lattice $L_p(\nabla, \mu)$ has been proven in [6]. An analogue of Doob’s martingale convergence theorem in a Banach—Kantorovich lattice $L_p(\nabla, \mu)$ was given in [3]. So, the study of the convergence of ergodic averages in Banach—Kantorovich spaces is no doubt of interest.

Since the work of Kaplansky [12], a special case of Banach—Kantorovich space, Hilbert—Kaplansky spaces (AW*-modules) have been introduced. Noncommutative algebras consummated as subalgebras of the algebra of bounded operators on a Hilbert—Kaplansky space were considered by A.G. Kusraev [14],[15]. In [5], a Hilbert—Kaplansky space over $L^0$ is represented as a measurable bundle of Hilbert spaces $L^0$ bounded operators are given as a measurable bundle of bounded operators in layers.

There are several approaches that may be used to get the ergodic type theorems. One of them is a direct method. We can repeat all the steps provided in the proof of classical

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Banach or Hilbert spaces, taking into consideration the distinctions of $L^0$—valued norms. Another way is to use boolean analysis developed in [14], which gives a possibility to reduce $AW^*$-modules to Hilbert spaces. Finally, a Hilbert—Kaplansky space can be represented as a measurable bundle of classical Hilbert spaces, which is based on the existence of respective liftings.

The first method is ineffective, because we need to repeat all the known steps of proofs for classical Hilbert spaces, modifying these steps into $L^0$—valued inner product. The second method is connected with the use of sufficiently complicated apparatus of Boolean analysis, realization of which requires a huge preliminary work, connected with the establishment of interrelation of Hilbert—Kaplansky spaces in an ordinary and boolean models of the set theory. More effective way, in our opinion, is the use of the third method. This is because, the theory of measurable bundles of Banach as well as Hilbert spaces has been developed sufficiently well (see, for example [15]).

In the present paper we give a description of contractions in a Hilbert—Kaplansky space. We prove an analogue of von Neumann ergodic theorem and its multiparameter analogue in a Hilbert—Kaplansky space. Besides, we study the convergence modulated, subsequential and weighted ergodic averages. To do so, we use a theory of measurable bundles.

The paper is organized as follows. We give some necessary notations and give the definition for Hilbert—Kaplansky space in the next section. In section 3 we give the description of contractions and unitary operators in a Hilbert—Kaplansky space. We also prove the mean ergodic theorem and multiparameter mean ergodic theorem in a Hilbert—Kaplansky space. Finally, in section 4 we study the convergence of modulated, subsequential and weighted ergodic averages in a Hilbert—Kaplansky space.

2. Preliminaries

Let $(\Omega, \Sigma, \mu)$ be a space with a complete finite measure, and $L^0 = L^0(\Omega)$ be an algebra of classes of complex measurable functions in $(\Omega, \Sigma, \mu)$.

Consider a vector space $H$ over the complex numbers $\mathbb{C}$. A transformation $||:|| : H \to L^0$ is said to be a vector or an $L^0$—valued norm on $H$ if it the following conditions are fulfilled:

1) $||x|| \geq 0$ for all $x \in H$; $||x|| = 0 \iff x = 0$; 2) $||\lambda x|| = ||\lambda|| ||x||$ for all $\lambda \in C$ and $x \in H$;
3) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in H$.

A pair $(H, ||:||)$ is said to be lattice-normed space over $L^0$. Lattice-normed space $H$ is said to be disjunctively decomposable, or $d$—decomposable if the following condition holds: for any $x \in H$ and disjunctive elements $e_1, e_2 \in L^0$ satisfying $||x|| = e_1 + e_2$, there exist $x_1, x_2 \in H$ such that $x = x_1 + x_2$ with $||x_1|| = e_1$ and $||x_2|| = e_2$. A net $x_\alpha$ in $H$ is said to be a (bo)–convergent to an element $x \in H$ if there exists a decreasing net $(e_\gamma)_{\gamma \in \Gamma}$ with $\inf_{\gamma \in \Gamma} e_\gamma = 0$ such that for any $\gamma \in \Gamma$ there is $\alpha = \alpha(\gamma)$ such that $||x - x_\alpha|| \leq e_\gamma$ for all $\alpha \geq \alpha(\gamma)$. A lattice-normed space is said to be (bo)–complete if any fundamental net $x_\alpha$ in $H$ is (bo)–convergent to an element of $H$. Any $d$-decomposable (bo)–complete lattice-normed space is said to be a Banach—Kantorovich space.

Definition 1 [15](14). A transformation $\langle \cdot, \cdot \rangle : H \times H \to L^0$ is called an $L^0$-valued inner product, if for any $x, y, z \in H$ and $\alpha \in C$, the following conditions hold:

1) $\langle x, y \rangle \geq 0$, $\langle x, x \rangle = 0 \iff x = 0$; 2) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$; 3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$; 4) $\langle x, y \rangle = \langle y, x \rangle$.

It is known [15], that $||x|| = \sqrt{\langle x, x \rangle}$ defines an $L^0$-valued norm in $H$. If $(H, ||:||)$ is a Banach—Kantorovich space, then $(H, \langle \cdot, \cdot \rangle)$ is said to be a Hilbert—Kaplansky space. Examples of such spaces can be found in [14], [15].

Let $H$ be a map that assigns some Hilbert space $H(\omega)$ to any point $\omega \in \Omega$. Function $u$, defined a.e. in $\Omega$, with the values $u(\omega) \in H(\omega)$, for all $\omega$ in the domain $dom(u)$ of $u$, is
said to be a section on $\mathcal{H}$. For the set of sections $L$, following [9], we call a pair $(\mathcal{H}, L)$ a measurable bundle of Hilbert spaces over $\Omega$, if

1) $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in C$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in \text{dom } (c_1) \bigcap \text{dom } (c_2) \rightarrow \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;

2) a function $||c|| : \omega \in \text{dom } (c) \rightarrow ||c(\omega)||_{\mathcal{H}(\omega)}$ is measurable for all $c \in L$;

3) for any $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in \text{dom } (c)\}$ is dense in $\mathcal{H}(\omega)$.

A section $s$ is called a step-section, if $s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega)c_i(\omega)$, where $c_i \in L, A_i \in \Sigma, i = \overline{1,n}$. A section $u$ is called measurable, if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of step-sections such that $||s_n(\omega) - u(\omega)||_{\mathcal{H}(\omega)} \rightarrow 0$ for almost all $\omega \in \Omega$.

Let $M(\Omega, \mathcal{H})$ be the set of all measurable sections. By $L^0(\Omega, \mathcal{H})$ we denote the factor space of $M(\Omega, \mathcal{H})$ with respect to a.e. equality. By $\hat{u}$ we denote the class from $L^0(\Omega, \mathcal{H})$, containing the section $u$. Note that the function $\omega \rightarrow ||u(\omega)||_{\mathcal{H}(\omega)}$ is measurable for any $u \in M(\Omega, \mathcal{H})$, and therefore, the function $(u(\omega), v(\omega))_{\mathcal{H}(\omega)} = \frac{1}{2}([||u(\omega) + v(\omega)||^2_{\mathcal{H}(\omega)} - ||u(\omega) - v(\omega)||^2_{\mathcal{H}(\omega)}])$ is measurable for all $u, v \in M(\Omega, \mathcal{H})$.

We denote by $(\hat{u}, \hat{v})$ the element of $L^0$ containing $(u(\omega), v(\omega))_{\mathcal{H}(\omega)}$. Clearly, $\langle \cdot, \cdot \rangle$ is an $L^0$-valued inner product. We also denote by $||\hat{u}||_{\mathcal{H}(\omega)}$ the element of $L^0$ containing the function $||u(\omega)||$, for any $u \in M(\Omega, \mathcal{H})$. Then $||\hat{u}||^2 = ||\overline{u(\omega)}||^2_{\mathcal{H}(\omega)} = \langle \overline{u(\omega)}, u(\omega) \rangle_{\mathcal{H}(\omega)} = \langle \hat{u}, \hat{u} \rangle$, that is $||u|| = \sqrt{\langle \hat{u}, \hat{u} \rangle}$.

From theorem 4.1.14 ([9] page 144) $\langle L^0(\Omega, \mathcal{H}), || \cdot || \rangle$ is a Banach —Kantorovich space. That is why $(L^0(\Omega, \mathcal{H}), \langle \cdot, \cdot \rangle)$ is a Hilbert —Kaplansky space over $L^0$.

**Definition 2.** The collection $\{T_\omega : \mathcal{H}(\omega) \rightarrow \mathcal{H}(\omega), \omega \in \Omega\}$ of linear operators is called a measurable bundle of linear operators, if $T_\omega(x) \in M(\Omega, \mathcal{H})$ for any $x \in M(\Omega, \mathcal{H})$.

$H$ be a Hilbert —Kaplansky space. An operator $T : H \rightarrow H$ on a Hilbert —Kaplansky space $H$ is said to be an $L^0$-linear, if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $\alpha, \beta \in L^0(\Omega)$ and $x, y \in H$. $L^0$-linear operator is called an $L^0$-bounded, if for any bounded set $B$ in $L^0$, the set $T(B)$ is bounded as well. For the $L^0$-bounded operator $T$ we define its norm as $||T|| = \sup\{||Tx|| : ||x|| \leq 1\}$, where $1$ is an identity element of $L^0$.

If every $T_\omega$ is contraction (unitary), then $\{T_\omega : \omega \in \Omega\}$ is called a measurable bundle of contractions (unitary operators).

**Theorem 2.1.** [7] Let $\{T_\omega : \mathcal{H}(\omega) \rightarrow \mathcal{H}(\omega), \omega \in \Omega\}$ be a measurable bundle of linear operators. Then the operator $\hat{T} : L^0(\Omega, \mathcal{H}) \rightarrow L^0(\Omega, \mathcal{H})$, defined by $\hat{T}x = T_\omega x(\omega)$ is $L^0$-linear and $L^0$-bounded operator.

Let $\mathcal{L}^\infty(\Omega)$ be the set of all bounded measurable functions on $\Omega$ and $L^\infty(\Omega)$ be the factor space of $\mathcal{L}^\infty(\Omega)$ with respect to a.e. equality. By $\mathcal{L}^\infty(\Omega, \mathcal{H})$ we denote the set of those points $||u|| \in M(\Omega, \mathcal{H})$ for which $||u(\omega)||_{\mathcal{H}(\omega)} \in \mathcal{L}^\infty(\Omega)$ and by $L^\infty(\Omega, \mathcal{H})$ we denote the factor space of $\mathcal{L}^\infty(\Omega, \mathcal{H})$ with respect to equality a.e.

Let $p : L^\infty(\Omega) \rightarrow L^\infty(\Omega, \mathcal{H})$ be a lifting (see [9],[15]).

**Definition 3.** A map $l : L^\infty(\Omega, \mathcal{H}) \rightarrow L^\infty(\Omega, \mathcal{H})$ is said to be a vector valued lifting associated with lifting $p$ if for all $\hat{u}, \hat{v} \in L^\infty(\Omega, \mathcal{H})$ the following conditions hold:

1) $l(\hat{u}) \in \hat{u}, \text{ dom}(u) = \Omega$;
2) $||l(\hat{u})(\omega)||_{\mathcal{H}(\omega)} = p(||\hat{u}||)(\omega)$;
3) $l(\hat{u} + \hat{v}) = l(\hat{u}) + l(\hat{v})$;
4) $l(\lambda \hat{u}) = p(\lambda)l(\hat{u})$ for $\lambda \in L^\infty(\Omega, \mathcal{H})$;
5) $l(l(\hat{u})) = l(l(\hat{u}))$;
6) $\{l(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, \mathcal{H})\}$ is dense in $\mathcal{H}(\omega)$, for all $\omega \in \Omega$. 


Theorem 2.2. [5] For any Hilbert—Kaplansky space $H$ over $L^0$ there exists a measurable bundle of Hilbert spaces $(\mathcal{H}, L)$, with vector valued lifting, such that $H$ isometrically isomorphic to $L^0(\Omega, \mathcal{H})$.

3. MEAN EROGIC THEOREMS

In this section we give a description of contractions and prove an analogue of von Neumann ergodic theorem in a Hilbert—Kaplansky space. Besides, we prove the convergence multiparameter ergodic averages.

An $L^0$— linear and $L^0$— bounded operator $T : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H})$ is called a contraction (unitary) in $L^0(\Omega, \mathcal{H})$ if $||T|| \leq 1$ ($||T|| = 1$) The following theorem describes contractions in Hilbert—Kaplansky space.

Theorem 3.1. For any contraction $T : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H})$ and for all $\omega \in \Omega$ there exists a measurable bundle of contractions $T_\omega : \mathcal{H}(\omega) \to \mathcal{H}(\omega)$ such that

$$ (T_\omega)(\omega) = T_\omega(u(\omega)) $$

for almost all $\omega \in \Omega$ and for all $\hat{u} \in L^0(\Omega, \mathcal{H})$.

Proof. Since $T$ is a contraction in $L^0(\Omega, \mathcal{H})$, then $||T\hat{u}|| \leq ||\hat{u}||$. Let $\hat{u} \in L^\infty(\Omega, \mathcal{H})$. Then we have $||\hat{u}|| \in L^\infty(\Omega)$, hence $||T\hat{u}|| \in L^\infty(\Omega)$, which means that $T\hat{u} \in L^\infty(\Omega, \mathcal{H})$.

Let $T_\omega(l(\hat{u})(\omega)) = l(T\hat{u}(\omega))$, where $l$ is a vector valued lifting on $L^\infty(\Omega, \mathcal{H})$, associated with a lifting $p$. From

$$ ||T_\omega(l(\hat{u})(\omega)))||_{\mathcal{H}(\omega)} = ||l(T\hat{u})(\omega)||_{\mathcal{H}(\omega)} = p(||T\hat{u}||)(\omega) \leq p(||\hat{u}||)(w) = ||l(\hat{u})(\omega)||_{\mathcal{H}(\omega)} $$

we imply that the linear operator

$$ T_\omega : \{l(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, \mathcal{H})\} \to \mathcal{H}(\omega) $$

is well defined and bounded. Moreover $||T_\omega|| \leq 1$.

Since $\{l(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, \mathcal{H})\}$ is dense in $\mathcal{H}(\omega)$, then for all $\omega \in \Omega$, $T_\omega$ can be extended to operator $T_\omega : \mathcal{H}(\omega) \to \mathcal{H}(\omega)$ preserving the norm. That is why we get contractions $T_\omega : \mathcal{H}(\omega) \to \mathcal{H}(\omega)$ for all $\omega \in \Omega$.

According to definition of $T_\omega$ and property 1) of a vector valued lifting we have

$$ T_\omega(l(\hat{u})(\omega)) = l(T\hat{u})(\omega) = (T\hat{u})(\omega) $$

for almost all $\omega \in \Omega$.

Now, let $\hat{u} \in L^0(\Omega, \mathcal{H})$. Since $L^\infty(\Omega, \mathcal{H})$ is (bo)-dense in $L^0(\Omega, \mathcal{H})$ (see [4]), then there is a sequence $\hat{u}_n \in L^\infty(\Omega, \mathcal{H})$ such that $||\hat{u}_n - \hat{u}|| \to 0$. Then $||\hat{u}_n(\omega) - \hat{u}(\omega)||_{\mathcal{H}(\omega)} \to 0$ for almost all $\omega \in \Omega$. From

$$ T(\hat{u}) = \lim_{n \to \infty} T(\hat{u}_n) $$

we get

$$ ||T_\omega(\hat{u}_n(\omega)) - T(\hat{u})(\omega)||_{\mathcal{H}(\omega)} = ||T_\omega(\hat{u}_n(\omega)) - T(\hat{u})(\omega)||_{\mathcal{H}(\omega)} \to 0 $$

for almost all $\omega \in \Omega$. Therefore, $T(\hat{u})(\omega) = \lim_{n \to \infty} T_\omega(u_n(\omega))$ for almost all $\omega \in \Omega$. On the other hand, the continuity of $T_\omega$ yields $\lim_{n \to \infty} T_\omega(u_n(\omega)) = T_\omega(u(\omega))$. Hence for every $\hat{u} \in L^0(\Omega, \mathcal{H})$ we have $T(\hat{u})(\omega) = T_\omega(u(\omega))$ for almost all $\omega \in \Omega$. 

□
Corollary 3.2. For any unitary operator \( U : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H}) \) and for all \( \omega \in \Omega \) there exists a measurable bundle of unitary operators \( U_\omega : \mathcal{H}(\omega) \to \mathcal{H}(\omega) \) such that

\[
(U\hat{u})(\omega) = U_\omega(u(\omega))
\]

for almost all \( \omega \in \Omega \) and for all \( \hat{u} \in L^0(\Omega, \mathcal{H}) \).

For \( \hat{u} \in L^0(\Omega, \mathcal{H}) \) and \( T : L^0(\Omega, \mathcal{H}) \to L_0(\Omega, \mathcal{H}) \) we define \( A_n(T, \hat{u}) = \frac{1}{n} \sum_{k=0}^{n-1} T^k\hat{u} \).

The following theorem is an analogue of von Neumann ergodic theorem for Hilbert—Kaplansky space.

Theorem 3.3. If \( T : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H}) \) is a contraction, then \( A_n\hat{u} \to \hat{P}(\hat{u}) \) in \( L^0(\Omega, \mathcal{H}) \) for all \( \hat{u} \in L^0(\Omega, \mathcal{H}) \), where \( \hat{P} \) is a projection in \( L^0(\Omega, \mathcal{H}) \).

Proof. Let \( T \) be a contraction in \( L^0(\Omega, \mathcal{H}) \). According to Theorem 3.1 there exist the corresponding contractions \( T_\omega \) in \( \mathcal{H}(\omega) \). Then we have

\[
A_n(T, \hat{u})(\omega) = \left(\frac{1}{n} \sum_{k=0}^{n-1} T^k\hat{u}\right)(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(u(\omega)) = A_n(u(\omega))
\]

for any \( \hat{u} \in L^0(\Omega, \mathcal{H}) \) and almost all \( \omega \in \Omega \).

From mean ergodic theorem (see [13]) there exists a projection \( P(\omega) \) such that \( A_n(u(\omega)) \to P(\omega)(u(\omega)) \) in \( \mathcal{H}(\omega) \). Since \( A_n(u(\omega)) \) is a measurable section, then from Gutman’s theorem [9], it follows that the section \( P(\omega)(u(\omega)) \) is measurable. Further, since projections \( P(\omega) \) map a measurable bundle to a measurable bundle, then Theorem 2.1 implies the existence of \( L^0 \)-linear and \( L^0 \)-bounded operator \( \hat{P} : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H}) \) given by

\[
\hat{P}(\hat{u}) = P(\omega)(u(\omega)).
\]

Finally, from

\[
||A_n(T, \hat{u}) - \hat{P}(\hat{u})|| = ||A_n(u(\omega)) - P(\omega)(u(\omega))||_{\mathcal{H}(\omega)} \to 0
\]

we get \( A_n(T, \hat{u}) \to \hat{P}(\hat{u}) \) in \( L^0(\Omega, \mathcal{H}) \). We show that \( \hat{P} \) is a projection. Indeed, since \( P(\omega) \) is a projection, then we have \( P(\omega)^2 = P(\omega) \) and \( P(\omega)^* = P(\omega) \), where \( P(\omega)^* \) is the adjoint of \( P(\omega) \). Since \( P(\omega) \) is measurable bundle of operators, then \( \{P(\omega)^* : \omega \in \Omega\} \) and \( \{P(\omega)^2 : \omega \in \Omega\} \) are measurable bundles of operators. We have

\[
\hat{P}^2(\hat{u}) = P(\omega)^2u(\omega) = P(\omega)u(\omega) = \hat{P}(\hat{u}).
\]

Similarly,

\[
\hat{P}^*(\hat{u}) = P(\omega)^*u(\omega) = P(\omega)u(\omega) = \hat{P}(\hat{u}).
\]

So, the ergodic average tends to some projection.

\[\square\]

In the following examples we illustrate the above proved theorem.

Example 1. Let \( \Omega = \mathbb{N} \), then \( L^0(\Omega) = s \) – the space of sequences. We define

\[
H = s[\ell_2] = \{ f = (f_1, f_2, \ldots, f_n, \ldots) : f_i \in \ell_2, \forall i \in \mathbb{N} \}.
\]

The norm is defined as \( ||f|| = (||f_1||_2, ||f_2||_2, \ldots, ||f_n||_2, \ldots) \in s \), and the inner product as \( (f, g) = ((f_1, g_1), (f_2, g_2), \ldots, (f_n, g_n), \ldots) \in s \), where \((,\cdot)\) is an inner product in \( \ell_2 \). Then \( H \) is a Hilbert—Kaplansky space.

Now, let \( f_i = (f_{i1}^{(1)}, f_{i1}^{(2)}, \ldots, f_{in}^{(n)}, \ldots) \in \ell_2 \) and \( Tf_i = (0, f_{i1}^{(1)}, f_{i1}^{(2)}, \ldots) \). Define \( s \)-linear operator \( T : s[\ell_2] \to s[\ell_2] \) by

\[
Tf = (Tf_1, Tf_2, \ldots, Tf_n, \ldots).
\]
One can see that
\[ |Tf| = (|Tf_1|, |Tf_2|, \cdots) = (|f_1|, |f_2|, \cdots) = |f|, \]
hence \( T \) is a contraction. Then according to Theorem 3.3 \( \lim n^{-1} \sum_{k=0}^{n-1} T^k f \) is norm convergent. Let us find its limit. Since \( \frac{1}{n} \sum_{k=0}^{n-1} T^k f_i \to 0 \) as \( n \to \infty \) in \( \ell_2 \) for all \( i \in \mathbb{N} \), then
\[ \frac{1}{n} \sum_{k=0}^{n-1} T^k f = \left( \frac{1}{n} \sum_{k=0}^{n-1} T^k f_1, \frac{1}{n} \sum_{k=0}^{n-1} T^k f_2, \cdots \right) \to (0, 0, \cdots) \]
as \( n \to \infty \). Therefore, \( \lim n^{-1} \sum_{k=0}^{n-1} T^k f = 0 \) in \( s[\ell_2] \).

**Example 2.** Let \( ([0, \infty), B, m) \) be measurable space and \( L_2 = L_2([0, \infty), B, m) \) be a Banach space of square integrable functions. Let \( L^0[L_2] \) be the space of all measurable function \( K \) on \( \Omega \times [0, \infty) \) containing the class of equivalency of functions \( y \to K(\omega, y) \) belonging to \( L_2 \) such that \( \omega \to ||K(\omega, \cdot)||_{L_2} \) is measurable.

We define the norm of \( K = K(\omega, y) \) as \( ||K|| = ||K(\omega, \cdot)||_{L_2} \in L^0 \). Then \( L^0[L_2] \) is a Banach —Kantorovich space [15]. We define the inner product as \( \langle K_1, K_2 \rangle = \langle K_1(\omega, y), K_2(\omega, y) \rangle_{L_2} \), where \( \langle \cdot, \cdot \rangle_{L_2} \) denotes the inner product in \( L_2 \). Then \( ||K|| = \sqrt{\langle K, K \rangle} \) implies that \( L^0[L_2] \) is a Hilbert —Kaplansky space.

Now let \( P(t, x, B) \) be a Markov process with invariant measure \( m \). Then the operator
\[ T(K) = \int_0^\infty K(\omega, y)P(1, x, dy) \]
is a contraction [18]. Then according to Theorem 3.3 \( \lim n^{-1} \sum_{k=0}^{n-1} T^k(K) \) is norm convergent in \( L^0[L_2] \).

In the following example we assume \( \Omega \) to be a countable.

**Example 3.** Let \( \Omega \) be a countable set. Then \( L^0[L_2] = s[L_2] \). For \( K \in s[L_2] \) we define \( ||K|| = (||K_1||_{L_2}, |K_2||_{L_2}, \cdots ||K_n||_{L_2}, \cdots) \in s \) and \( (K, M) = ((K_1, M_1)_{L_2}, (K_2, M_2)_{L_2}, \cdots) \).

Then \( s[L_2] \) is a Hilbert —Kaplansky space.

We define \( T: s[L_2] \to s[L_2] \) as
\[ T(K) = (T(K_1), T(K_2), \cdots) \]
where \( T(K_i) = K_i(y + 1) \). Since \( T \) is a contraction in \( L_2 \), then \( T \) is a contraction in \( s[L_2] \). Then
\[ A_n(T, K) = (A_n(T, K_1), A_n(T, K_2), \cdots). \]

Since \( A_n(T, K_i) \) converges as \( n \to \infty \) in \( L_2 \) for all \( i \) then \( A_n(T, K) \) converges in \( s[L_2] \).

Let \( T_1, T_2, \cdots, T_d \) be \( d \) linear operators on \( L^0(\Omega, H) \). We define the **multiparameter ergodic average** by
\[ A_{n_1, n_2, \cdots, n_d}(T_1, T_2, \cdots, T_d, \hat{u}) = \frac{1}{n_1 n_2 \cdots n_d} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \cdots \sum_{i_d=0}^{n_d-1} T_{i_1}^{1} T_{i_2}^{2} \cdots T_{i_d}^{d}(\hat{u}). \]

**Theorem 3.4.** Let \( T_1, T_2, \cdots, T_d \) be contractions in \( L^0(\Omega, H) \). Then for any \( \hat{u} \in L^0(\Omega, H) \), the average \( A_{n_1, n_2, \cdots, n_k}(T_1, T_2, \cdots, T_d, \hat{u}) \) is norm convergent as \( n_1, n_2, \cdots, n_d \to \infty \) independently.
Proof. We proceed by induction, noting that the theorem is true for \( d = 1 \) by the previous theorem. Assume that multiparameter ergodic average is norm convergent for any \( d - 1 \) contractions in \( L^0(\Omega, \mathcal{H}) \). Let \( P_1, P_2, \cdots, P_d \) be projections such that

\[
\lim_{n_i \to \infty} A_{n_i}(T_i, \hat{u}) = P_i(\hat{u}) \quad i = 1, d.
\]

Then

\[
\lim_{n_1, n_2, \cdots, n_d \to \infty} A_n(T_1, T_2, \cdots, T_d, \hat{u}) = P_d P_{d-1} \cdots P_1(\hat{u}).
\]

Indeed,

\[
\|A_{n_1, n_2, \cdots, n_d}(T_1, T_2, \cdots, T_d, \hat{u}) - P_d P_{d-1} \cdots P_1(\hat{u})\| \leq \|A_{n_2, \cdots, n_d}(T_2, \cdots, T_d, \hat{u})(A_{n_1}(T_1) - P_1)\hat{u}\| + \|(A_{n_2, \cdots, n_d}(T_2, \cdots, T_d, \hat{u}) - P_d \cdots P_2)P_1(\hat{u})\|
\]

\[
\leq \|A_{n_1}(T_1)(\hat{u}) - P_1(\hat{u})\| + \|(A_{n_2, \cdots, n_d}(T_2, \cdots, T_d, \hat{u}) - P_d \cdots P_2)P_1(\hat{u})\|
\]

Due to the assumption of induction both expressions \( \|A_{n_1}(T_1)(\hat{u}) - P_1(\hat{u})\| \) and \( \|(A_{n_2, \cdots, n_d}(T_2, \cdots, T_d, \hat{u}) - P_d \cdots P_2)P_1(\hat{u})\| \) converge to 0 as \( n_1, n_2, \cdots, n_d \to \infty \). Therefore, multiparameter ergodic average is norm convergent.

\[\square\]

4. Modulated, subsequential and weighted ergodic theorems

In this section we study a modulated, subsequential and a weighted ergodic theorems in a Hilbert — Kaplansky space.

Let \( \{a_k\}_{k \geq 0} \) be a sequence of complex numbers. For \( \hat{u} \in L^0(\Omega, \mathcal{H}) \) and \( T : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H}) \) we define \( A_n(a_k, T, \hat{u}) = \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k \hat{u} \). Following [1], we call this average a modulated.

**Theorem 4.1.** Let \( \{a_k\}_{k \geq 0} \) be a sequence of complex numbers satisfying the following conditions:

a) For every complex \( \lambda \) with \( |\lambda| = 1 \) there exists \( c(\lambda) \) such that

\[
\frac{1}{n} \sum_{k=0}^{n-1} a_k \lambda^k \to c(\lambda)
\]

as \( n \to \infty \)

b)

\[
\sup_{n \geq 1} \sup_{|\lambda| = 1} \left| \frac{1}{n} \sum_{k=0}^{n-1} a_k \lambda^k \right| < \infty.
\]

Then for any contraction \( T \) in \( L^0(\Omega, \mathcal{H}) \), the average \( A_n(a_k, T, \hat{u}) \) converges in norm for any \( \hat{u} \in L^0(\Omega, \mathcal{H}) \);

**Proof.** Let \( T \) be a contraction in \( L^0(\Omega, \mathcal{H}) \) and \( T_\omega \) be the corresponding contractions in \( \mathcal{H}(\omega) \). Then

\[
A_n(a_k, T, \hat{u})(\omega) = \left( \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k \hat{u} \right)(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k_\omega(u(\omega)) = A_n(a_k, T_\omega, u(\omega))
\]

for any \( \hat{u} \in L^0(\Omega, \mathcal{H}) \) and almost all \( \omega \in \Omega \).

Since the sequence \( \{a_k\}_{k \geq 0} \) satisfies a) and b), then according to Corollary 2.3. from [1], there exists \( u^*(\omega) \) such that

\[
A_n(a_k, T_\omega, u(\omega)) = \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k_\omega(u(\omega)) \to u^*(\omega) \text{ in } \mathcal{H}(\omega)
\]

for almost all \( \omega \in \Omega \) as \( n \to \infty \). Note that \( A_n(a_k, T_\omega, u(\omega)) \) is a measurable section,
and therefore according to [9], \( u^*(\omega) \) is a measurable section and \( \hat{u}^* = \hat{u}^* \in L^0(\Omega, \mathcal{H}) \).

Therefore, from

\[
||A_n(a_k, T, \hat{u}) - \hat{u}^*|| = ||A_n(a_k, T, \omega, \hat{u}(\omega)) - u^*(\omega)||_{\mathcal{H}(\omega)} \to 0
\]

we get \( A_n(a_k, T, \hat{u}) \to \hat{u}^* \) in \( L^0(\Omega, \mathcal{H}) \).

\( \square \)

**Corollary 4.2.** If the conditions of above Theorem 4.1 hold, then for any unitary operator \( U \) in \( L^0(\Omega, \mathcal{H}) \), the average \( A_n(a_k, U, \hat{u}) \) converges in norm for any \( \hat{u} \in L^0(\Omega, \mathcal{H}) \).

**Proof.** Applying Corollary 2.3 from [1] and Corollary 3.2 and the arguments given in the proof of we get the desired result. \( \square \)

Now, we turn our attention to the study of subsequential ergodic averages. For a sequence \( \{k_j\} \), \( \hat{u} \in L^0(\Omega, \mathcal{H}) \) and \( T : L^0(\Omega, \mathcal{H}) \to L^0(\Omega, \mathcal{H}) \) we define a subsequential ergodic average \( A_n(k_j, T, \hat{u}) = \frac{1}{n} \sum_{k=0}^{n-1} T^{k_j} \hat{u} \).

**Theorem 4.3.** Let \( \{k_j\} \) be a strictly increasing sequence of positive integers satisfying

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \lambda^{k_j} = 0 \text{ for every } |\lambda| = 1, \lambda \neq 1.
\]

Then for a given contraction \( T \) in \( L^0(\Omega, \mathcal{H}) \), the average \( A_n(k_j, T, \hat{u}) \) converges in norm for all \( \hat{u} \in L^0(\Omega, \mathcal{H}) \).

**Proof.** Let \( T \) be an \( L^0 \)- contraction and \( T_\omega \) be the corresponding measurable bundle of contractions in \( \mathcal{H}(\omega) \). Then

\[
A_n(k_j, T, \hat{u})(\omega) = \left( \frac{1}{n} \sum_{j=1}^{n} T^{k_j} \hat{u} \right)(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} T^{k_j}(u(\omega)) = A_n(k_j, T_\omega, u(\omega))
\]

for any \( \hat{u} \in L^0(\Omega, \mathcal{H}) \) and almost all \( \omega \in \Omega \).

The sequence \( \{k_j\} \) satisfies the condition of the Theorem, then Proposition 3.2 from [1] implies that there exists \( u^*(\omega) \), such that \( A_n(k_j, T_\omega, u(\omega)) = \frac{1}{n} \sum_{j=1}^{n} T^{k_j}_\omega(u(\omega)) \to u^*(\omega) \) in \( \mathcal{H}(\omega) \) as \( n \to \infty \). Note that \( A_n(k_j, T_\omega, u(\omega)) \) is a measurable section, hence \( u^*(\omega) \) is measurable and \( \hat{u}^* = \hat{u}^* \in L^0(\Omega, \mathcal{H}) \). Therefore, from

\[
||A_n(k_j, T, \hat{u}) - \hat{u}^*|| = ||A_n(k_j, T_\omega, \hat{u}(\omega)) - u^*(\omega)||_{\mathcal{H}(\omega)} \to 0
\]

we get \( A_n(a_k, T, \hat{u}) \to \hat{u}^* \) in \( L^0(\Omega, \mathcal{H}) \).

\( \square \)

Let \( \{w_k\}_{k \geq 0} \) be a non-null sequence of nonnegative numbers and denote its partial sums by \( W_n \). We also define \( A_n(w_k, T, \hat{u}) = \frac{1}{W_n} \sum_{k=0}^{n-1} w_k T^k \hat{u} \). The following theorem is a weighted ergodic theorem in a Hilbert—Kaplansky space.

**Theorem 4.4.** If for any complex \( \lambda \) with \( |\lambda| = 1 \) we have

\[
\frac{1}{W_n} \sum_{k=0}^{n-1} w_k \lambda^k \to c(\lambda)
\]

as \( n \to \infty \). Then

a). For any contraction \( T \) in \( L^0(\Omega, \mathcal{H}) \), the average \( A_n(w_k, T, \hat{u}) \) converges in norm for any \( \hat{u} \in L^0(\Omega, \mathcal{H}) \);

b). For any contraction \( T \) in \( L^0(\Omega, \mathcal{H}) \)

\[
||A_n(w_k, T, \hat{u}) - A_n(T, \hat{u})|| \to 0.
\]
Proof. a). This part can be proven by providing all arguments given in the proof of the previous theorem and applying Theorem 2.1 from [16].

b). Let $T$ be a contraction in $L^0(\Omega, \mathcal{H})$ and $T_\omega$ be the corresponding contractions in $\mathcal{H}(\omega)$. Then

$$A_n(w_k, T, \hat{u})(\omega) = \left(\frac{1}{W_n} \sum_{k=0}^{n-1} w_k T^k \hat{u}\right)(\omega) = \frac{1}{W_n} \sum_{k=0}^{n-1} w_k T_\omega^k (u(\omega)) = A_n(w_k, T_\omega, u(\omega))$$

for any $\hat{u} \in L^0(\Omega, \mathcal{H})$ and almost all $\omega \in \Omega$.

From Corollary 2.2 of [16] we get

$$||A_n(w_k, T_\omega, u(\omega)) - A_n(T_\omega, u(\omega))|| \rightarrow 0$$

in $\mathcal{H}(\omega)$ for almost all $\omega \in \Omega$. Using the technique of the previous theorems we obtain

$$||A_n(w_k, T, \hat{u}) - A_n(T, \hat{u})|| = ||A_n(w_k, T_\omega, u(\omega)) - A_n(T_\omega, u(\omega))||_{\mathcal{H}(\omega)} \rightarrow 0.$$ 

□

Let $k_j$ be a strictly increasing sequence of positive integers; put $w_{k_j} = 1$, and $w_k = 0$ if $k \notin \{k_j\}$. Then the weighted averages become the averages along the subsequence $k_j$.

More examples of weights $w_k$, satisfying the conditions of Theorem 4.4 can be found in [16].

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