A general stochastic maximum principle for mixed relaxed-singular control problems

Seid Bahlali
Laboratory of Applied Mathematics
University Med Khider
Po. Box 145, Biskra 07000, Algeria
Email : sbahlali@yahoo.fr

August 28, 2008

Abstract
We consider in this paper, mixed relaxed-singular stochastic control problems, where the control variable has two components, the first being measure-valued and the second singular. The control domain is not necessarily convex and the system is governed by a nonlinear stochastic differential equation, in which the measure-valued part of the control enters both the drift and the diffusion coefficients. We establish necessary optimality conditions, of the Pontryagin maximum principle type, satisfied by an optimal relaxed-singular control, which exist under general conditions on the coefficients. The proof is based on the strict singular stochastic maximum principle established by Bahlali-Mezerdi, Ekeland’s variational principle and some stability properties of the trajectories and adjoint processes with respect to the control variable.

AMS Subject Classification. 93Exx

Keywords. Stochastic differential equation, relaxed-singular control, optimal control, maximum principle, adjoint equation, variational inequality, variational principle.

1 Introduction
In this paper we study a stochastic control problems of nonlinear systems, where the control variable has two components, the first being measure-valued and the second singular. The system is governed by a stochastic differential equation
(SDE for short) of the type

\[
\begin{aligned}
dx_t^q &= \int_{A_1} b(t, x_t^q, a) q_t(da) dt + \int_{A_1} \sigma(t, x_t^q, a) q_t(da) dW_t + G_t d\eta_t, \\
x_0^q &= x_0,
\end{aligned}
\]

where \(b, \sigma\) and \(G\) are given deterministic functions, \(x_0\) is the initial data and \((W_t)_{t \geq 0}\) is a \(d\)-dimensional standard Brownian motion, defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), satisfying the usual conditions. The control variable is a suitable process \((q, \eta)\) where \(q : [0, T] \times \Omega \rightarrow \mathbb{P}(A_1)\), \(\eta : [0, T] \times \Omega \rightarrow A_2 = ((0, \infty))^m\) are \(B[0, T] \otimes \mathcal{F}\)-measurable, \((\mathcal{F}_t)\)-adapted, and \(\eta\) is an increasing process (componentwise), continuous on the left with limits on the right with \(\eta_0 = 0\).

The pair \((q, \eta)\) is called mixed relaxed-singular control (relaxed control for short) and we denote by \(\mathcal{R}\) the class of relaxed controls.

The functional cost, to be minimized over \(\mathcal{R}\), has the form

\[
J(q, \eta) = \mathbb{E} \left[ g(x_T^q) + \int_0^T \int_{A_1} h(t, x_t^q, a) q_t(da) dt + \int_0^T l_t d\eta_t \right].
\]

A relaxed control \((\mu, \xi)\) is called optimal if it satisfies

\[
J(\mu, \xi) = \inf_{(q, \eta) \in \mathcal{R}} J(q, \eta).
\]

Singular control problems have been studied by many authors including Benes-Shepp-Witsenhausen [5], Chow-Menaldi-Robin [8], Karatzas-Shreve [18], Davis-Norman [9], Haussmann-Suo [14, 15, 16]. See [15] for a complete list of references on the subject. The approaches used in these papers, to solve the problem are mainly based on dynamic programming. It was shown in particular that the value function is solution of a variational inequality, and the optimal state is a reflected diffusion at the free boundary. Note that in [14], the authors apply the compactification method to show existence of an optimal relaxed-singular control.

The other major approach to solve control problems is to derive necessary conditions satisfied by some optimal control, known as the stochastic maximum principle. The first version of the stochastic maximum principle that covers singular control problems was obtained by Cadenillas-Haussmann [7], in which they consider linear dynamics, convex cost criterion and convex state constraints. necessary optimality conditions for non linear SDEs were obtained by Bahvalal-Chala [1] and Bahvalali-Mezerdi [2].

The common fact in this works is that an optimal strict singular control does not necessarily exist, the set \(U\) of strict singular controls \((v, \eta)\), where \(v : [0, T] \times \Omega \rightarrow A_1 \subset \mathbb{R}^k\), is too narrow and not being equipped with a good topological structure. The idea is then to introduce the class \(\mathcal{R}\) of relaxed controls in which the controller chooses at time \(t\), a probability measure \(q_t(da)\) on the set \(A_1\), rather than an element \(v_t\) of \(A_1\). The relaxed control problem find its interest in two essential points. The first is that it is a generalization.
of the strict control problem, indeed if $q_t(da) = \delta_{v_t}(da)$ is a Dirac measure concentrated at a single point $v_t$, then we get a strict control problem as a particular case of the relaxed one. The second is that an optimal relaxed control exists.

Stochastic maximum principle for relaxed controls (without the singular part) was obtained by Mezerdi-Bahlali [21] in the case of uncontrolled diffusion and Bahlali-Mezerdi-Djehiche [3] where the drift and the diffusion coefficients depends explicitly on the relaxed control variable. Necessary optimality conditions for relaxed-singular controls and uncontrolled diffusion are derived by Bahlali-Djehiche-Mezerdi [4].

Our main goal in this paper, is to establish a maximum principle for relaxed-singular controls, where the first part of the control is a measure-valued process and enters both the drift and the diffusion coefficients. This leads to necessary optimality conditions satisfied by an optimal relaxed control, which exists under general conditions on the coefficients (see [14]). To achieve this goal, we use the maximum principle for strict singular controls established by Bahlali-Mezerdi [2] and Ekeland’s variational principle. We are able to prove necessary conditions for near optimality satisfied by a sequence of strict controls, converging in some sense to the relaxed optimal control, by the so called chattering lemma. The relaxed maximum principle is then derived by using some stability properties of the trajectories and the adjoint processes with respect to the control variable.

This result generalizes at the same time the results of Bahlali-Mezerdi [2], Bahlali-Mezerdi-Djehiche [3] and Bahlali-Djehiche-Mezerdi [4]. We note that the result of [2] and [3] are the extensions of the Peng’s stochastic maximum principle [22] respectively to the singular and relaxed controls.

The paper is organized as follows. In Section 2, we formulate the strict and relaxed control problems and give the various assumptions used throughout the paper. Section 3 is devoted to the proof of the main approximation. In Section 4, we establish the near stochastic maximum principle. In the last Section, we state and prove the main result of this paper, which is the stochastic maximum principle for relaxed-singular controls.

Along this paper, we denote by $C$ some positive constant and for simplicity, we need the following matrix notations. We denote by $\mathcal{M}_{n \times d} (\mathbb{R})$ the space of $n \times d$ real matrix and $\mathcal{M}^d_{n \times n} (\mathbb{R})$ the linear space of vectors $M = (M_1, ..., M_d)$ where $M_i \in \mathcal{M}_{n \times n} (\mathbb{R})$. For any $M, N \in \mathcal{M}^d_{n \times n} (\mathbb{R})$, $L, S \in \mathcal{M}_{n \times d} (\mathbb{R})$, $Q \in \mathcal{M}_{n \times n} (\mathbb{R})$, $\alpha, \beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^d$, we use the following notations

- $\alpha \beta = \sum_{i=1}^{n} \alpha_i \beta_i \in \mathbb{R}$ is the product scalar in $\mathbb{R}^n$;
- $LS = \sum_{i=1}^{d} L_i S_i \in \mathbb{R}$, where $L_i$ and $S_i$ are the $i^{th}$ columns of $L$ and $S$;
- $ML = \sum_{i=1}^{d} M_i L_i \in \mathbb{R}^n$;
\[ M \alpha \gamma = \sum_{i=1}^{d} (M_i \alpha) \gamma_i \in \mathbb{R}^n; \]
\[ MN = \sum_{i=1}^{d} M_i N_i \in \mathcal{M}_{n \times n}(\mathbb{R}); \]
\[ MQN = \sum_{i=1}^{d} M_i QN_i \in \mathcal{M}_{n \times n}(\mathbb{R}); \]
\[ MQ \gamma = \sum_{i=1}^{d} M_i Q \gamma_i \in \mathcal{M}_{n \times n}(\mathbb{R}). \]

We denote by \( L^* \) the transpose of the matrix \( L \) and \( M^* = (M^*_1, \ldots, M^*_d) \).

2 Formulation of the problem

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P}) \) be a filtered probability space satisfying the usual conditions, on which a \( d \)-dimensional Brownian motion \( W = (W_t)_{t \geq 0} \) is defined. We assume that \( (\mathcal{F}_t) \) is the \( \mathcal{P} \)-augmentation of the natural filtration of \( W \).

Let \( T \) be a strictly positive real number and consider the following sets

- \( A_1 \) is a nonempty subset of \( \mathbb{R}^k \) and \( A_2 = ([0, \infty))^m \).
- \( U_1 \) is the class of measurable, adapted processes \( v : [0, T] \times \Omega \rightarrow A_1 \).
- \( U_2 \) is the class of measurable, adapted processes \( \eta : [0, T] \times \Omega \rightarrow A_2 \) such that \( \eta \) is nondecreasing (componentwise), left-continuous with right limits and \( \eta_0 = 0 \).

2.1 The strict control problem

**Definition 1** An admissible strict control is an \( \mathcal{F}_t \)-adapted process \((v, \eta) \in U_1 \times U_2\) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |v_t|^2 + |\eta_T|^2 \right] < \infty.
\]

We denote by \( \mathcal{U} \) the set of all admissible controls.

For any \((v, \eta) \in \mathcal{U}\), we consider the following SDE

\[
\begin{cases}
    d x_t^v = b(t, x_t^v, v_t) \, dt + \sigma(t, x_t^v, v_t) \, dW_t + G_t \, d\eta_t, \\
    x_0^v = x_0,
\end{cases}
\]

(1)

where

- \( b : [0, T] \times \mathbb{R}^n \times A_1 \rightarrow \mathbb{R}^n \),
- \( \sigma : [0, T] \times \mathbb{R}^n \times A_1 \rightarrow \mathcal{M}_{n \times d}(\mathbb{R}) \),
- \( G : [0, T] \rightarrow \mathcal{M}_{n \times m}(\mathbb{R}) \).

The expected cost, to be minimized over the class \( \mathcal{U} \), has the form

\[
J(v, \eta) = \mathbb{E} \left[ g(x_T^v) + \int_0^T h(t, x_t^v, v_t) \, dt + \int_0^T l_t \, d\eta_t \right],
\]

(2)
where

\[ g : \mathbb{R}^n \longrightarrow \mathbb{R}, \]
\[ h : [0, T] \times \mathbb{R}^n \times A_1 \longrightarrow \mathbb{R}, \]
\[ l : [0, T] \longrightarrow ([0, \infty))^m. \]

A control \((u, \xi) \in \mathcal{U}\) is called optimal, if that solves

\[
J(u, \xi) = \inf_{(v, \eta) \in \mathcal{U}} J(v, \eta),
\]

The following assumptions will be in force throughout this paper

1. \(b, \sigma, g, h\) are twice continuously differentiable with respect to \(x\).
2. The derivatives \(b_x, b_{xx}, \sigma_x, \sigma_{xx}, g_x, g_{xx}, h_x, h_{xx}\) are continuous in \((x, v)\) and uniformly bounded.
3. \(b, \sigma\) are bounded by \(C(1 + |x| + |v|)\).
4. \(G\) and \(l\) are continuous and \(G\) is bounded.

Under the above hypothesis, for every \((v, \eta) \in \mathcal{U}\), equation (1) has a unique strong solution and the cost functional \(J\) is well defined from \(\mathcal{U}\) into \(\mathbb{R}\).

### 2.2 The relaxed model

The strict control problem \{\(1, 2, 3\)\} formulated in the last subsection may fail to have an optimal solution. Let us begin by two deterministic examples who show that even in simple cases, existence of a strict optimal control is not ensured.

**Example 1.** The problem is to minimize, over the set of measurable functions \(v : [0, T] \rightarrow \{-1, 1\}\), the following functional cost

\[
J(v) = \int_0^T (x^v_t)^2 dt,
\]

where \(x^v_t\) denotes the solution of

\[
\begin{cases}
    dx^v_t = v_t dt, \\
    x^v_0 = 0.
\end{cases}
\]

We have

\[
\inf_{v \in \mathcal{U}} J(v) = 0.
\]

Indeed, consider the following sequence of controls

\[
v^n_t = (-1)^k \quad \text{if} \quad \frac{k}{n} T \leq t \leq \frac{k+1}{n} T, \quad 0 \leq k \leq n - 1.
\]
Then clearly
\[|x^n_t| \leq \frac{T}{n},\]
\[|J(v^n)| \leq \frac{T^3}{n^2}.
\]
Which implies that
\[\inf_{v \in U} J(v) = 0.\]

There is however no control \(v\) such that \(J(v) = 0\). If this would have been the case, then for every \(t, x^n_t = 0\). This in turn would imply that \(v_t = 0\), which is impossible. The problem is that the sequence \((v^n)\) has no limit in the space of strict controls. This limit if it exists, will be the natural candidate for optimality.

If we identify \(v^n_t\) with the Dirac measure \(\delta_{v^n_t}(da)\) and set \(q_n(dt, dv) = \delta_{v^n_t}(dv) dt\), we get a measure on \([0, 1] \times U\). Then, the sequence \((q_n(dt, dv))_n\) converges weakly to \(\frac{1}{2} dt \cdot [\delta_{-1} + \delta_1](da)\).

**Example 2.** Consider the control problem where the system is governed by the SDE
\[
\begin{cases}
    dx_t = v_t dt + dW_t, \\
    x_0 = 0.
\end{cases}
\]

The functional cost to be minimized is given by
\[J(v) = \mathbb{E} \int_0^T \left[ x^2_t + (1 - v_t^2)^2 \right] dt.
\]

\(U = [-1, 1]\) and \(x, v, W\) are one dimensional. The control \(v\) (open loop) is a measurable function from \([0, T]\) into \(U\). We assume that \(\mathbb{E} [x^2_t] = (\mathbb{E} [x_t])^2\).

The separation principle applies to this example, the optimal control minimizes
\[\int_0^T \left[ \tilde{x}_t^2 + (1 - v_t^2)^2 \right] dt,
\]
where \(\tilde{x}_t = \mathbb{E} [x_t]\) satisfies
\[
\begin{cases}
    d\tilde{x}_t = v_t dt, \\
    \tilde{x}_0 = 0.
\end{cases}
\]

This problem has no optimal strict control. A relaxed solution is to let
\[\mu_t = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1},\]
where \(\delta_a\) is an Dirac measure concentrated at a single point \(a\).

This suggests that the set \(U\) of strict controls is too narrow and should be embedded into a wider class with a richer topological structure for which the control problem becomes solvable. The idea of relaxed control is to replace the absolutely continuous part \(v_t\) of the strict control by a \(\mathbb{P}(A_1)\)-valued process \((q_t)\), where \(\mathbb{P}(A_1)\) is the space of probability measures on \(A_1\) equipped with the topology of weak convergence.
Definition 2 A relaxed control is a pair \((q, \eta)\) of processes such that \(\eta \in U_2\) and \(q\) is a \(\mathbb{P}(A_1)\)-valued process, progressively measurable with respect to \((\mathcal{F}_t)\) and such that for each \(t\), \(1_{[0,t]}q\) is \(\mathcal{F}_t\)-measurable.

We denote by \(R = R_1 \times U_2\) the set of relaxed controls.

For any \((q, \eta) \in R\), we consider the following relaxed SDE

\[
\begin{aligned}
dx^q_t &= \int_{A_1} b(t, x^q_t, a) q_t(da) dt + \int_{A_1} \sigma(t, x^q_t, a) q_t(da) dW_t + G_t d\eta_t, \\
x^q_0 &= x_0.
\end{aligned}
\]

The expected cost, to be minimized over the class \(R\) of relaxed controls, is defined as follows

\[
J(q, \eta) = E \left[ g(x^q_T) + \int_0^T \int_{A_1} h(t, x^q_t, a) q_t(da) dt + \int_0^T l_t d\eta_t \right].
\]

A relaxed control \((\mu, \xi)\) is called optimal, if it satisfies

\[
J(\mu, \xi) = \inf_{(q, \eta) \in R} J(q, \eta).
\]

The set \(U_1\) of absolutely component of strict controls is embedded into the set \(R_1\) of measure-valued processes by the mapping

\[
\Psi : v \in U_1 \rightarrow \Psi(v)_t(da) = \delta_{v_t}(da) \in R_1,
\]

where, \(\delta_v\) is the Dirac measure at a single point \(v\).

Throughout this paper we suppose moreover that

\[
b, \sigma \text{ and } h \text{ are bounded},
\]

\(A_1\) is compact.

Haussmann and Suo [14] have proved that the relaxed control problem admits an optimal solution under general conditions on the coefficients. Indeed, by using a compactification method and under some mild continuity hypotheses on the data, it is shown by purely probabilistic arguments that an optimal control for the problem exists. Moreover, the value function is shown to be Borel measurable. See Haussmann and Suo [14], Section 3, page 925 to page 934 and essentially Theorem 3.8, page 933. See also [11, 13] for a complete study of relaxed controls.

Remark 3 If we put

\[
\begin{aligned}
\overline{b}(t, x^q_t, q_t) &= \int_{A_1} b(t, x^q_t, a) q_t(da), \\
\overline{\sigma}(t, x^q_t, q_t) &= \int_{A_1} \sigma(t, x^q_t, a) q_t(da), \\
\overline{h}(t, x^q_t, q_t) &= \int_{A_1} h(t, x^q_t, a) q_t(da),
\end{aligned}
\]

7
then equation (5) becomes
\[
\begin{cases}
    dx_t^q = \mathbf{b}(t, x_t^q, q_t) \, dt + \mathbf{\sigma}(t, x_t^q, q_t) \, dW_t + G_t \, d\eta_t, \\
    x_0^q = x_0,
\end{cases}
\]
with a functional cost given by
\[
J(q, \eta) = \mathbb{E} \left[ g(x_T^q) + \int_0^T \mathbf{r}(t, x_t^q, q_t) \, dt + \int_0^T l_t \, d\eta_t \right].
\]

Hence by introducing relaxed controls, we have replaced \( A_1 \) by a larger space \( \mathbb{P}(A_1) \). We have gained the advantage that \( \mathbb{P}(A_1) \) is both compact and convex. Moreover, the drift, the diffusion and the running cost coefficients are linear with respect to the measure-valued process \( q \).

Remark 4 The coefficients \( \mathbf{b} \) and \( \mathbf{\sigma} \) (defined in the above remark) check respectively the same assumptions as \( b \) and \( \sigma \). Then, under assumptions (4), \( \mathbf{b} \) and \( \mathbf{\sigma} \) are uniformly Lipschitz and with linear growth. Then, by classical results on SDEs, for every \( q \in \mathcal{R} \), equation (5) admits a unique strong solution.

On the other hand, it is easy to see that \( \mathbf{r} \) checks the same assumptions as \( h \). Then, the functional cost \( J \) is well defined from \( \mathbb{R} \) into \( \mathbb{R} \).

Remark 5 If \( q_t = \delta_{v_t} \) is an atomic measure concentrated at a single point \( v_t \in A_1 \), then for each \( t \in [0, T] \) we have
\[
\int_{A_1} b(t, x_t^q, a) \, q_t(da) = \int_{A_1} b(t, x_t^q, v_t) \, \delta_{v_t}(da) = b(t, x_t^q, v_t),
\]
\[
\int_{A_1} \sigma(t, x_t^q, a) \, q_t(da) = \int_{A_1} \sigma(t, x_t^q, v_t) \, \delta_{v_t}(da) = \sigma(t, x_t^q, v_t),
\]
\[
\int_{A_1} h(t, x_t^q, a) \, q_t(da) = \int_{A_1} h(t, x_t^q, v_t) \, \delta_{v_t}(da) = h(t, x_t^q, v_t).
\]

In this case \( x^q = x^v \), \( J(q, \eta) = J(v, \eta) \) and we get a strict control problem. So the problem of strict controls \( \{(1), (2), (3)\} \) is a particular case of relaxed control problems \( \{(5), (6), (7)\} \).

Remark 6 The relaxed control problems studied e.g. in El Karoui et al [11] and Bahlali-Mezerdi-Djehiche [3] is different to ours, in that they relax the corresponding infinitesimal generator of the state process, which leads to a martingale problem for which the state process driven by an orthogonal martingale measure. In our setting the driving martingale measure \( q_t(da) \, dW_t \) is however not orthogonal. See Ma-Yong [20] for more details.

3 Approximation of trajectories

The next lemma, known as the Chattering Lemma, tells us that any measure-valued process is a weak limit of a sequence of absolutely continuous processes. This lemma was proved for deterministic measures and then extended to random measures in [13].
Lemma 7 (Chattering Lemma). Let \((q_t)\) be a predictable process with values in the space of probability measures on \(A_1\). Then there exists a sequence of predictable processes \((u^n_t)\) with values in \(A_1\) such that
\[
\delta_{u^n_t}(da) \ dt \text{ converges weakly to } q_t(da) \ dt, \ \mathcal{P} - a.s. \quad (9)
\]

Proof. See Fleming [13]. ■

The next lemma gives the stability of the controlled SDE with respect to the control variable.

Lemma 8 Let \((q, \eta)\) ∈ \(\mathcal{R}\) be a relaxed control and \(x^q\) the corresponding trajectory. Then there exists a sequence \((v^n, \eta)_n \subset \mathcal{U}\) such that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |x^n_t - x^q_t|^2 \right] = 0, \quad (10)
\]
\[
\lim_{n \to \infty} J(v^n, \eta) = J(q, \eta), \quad (11)
\]
where \(x^n\) denotes the solution of equation (1) associated with \((v^n, \eta)\).

Proof. i) Let \(q\) be a relaxed control, then from the chattering lemma (lemma7), there exists a sequence of strict controls \((v^n)_n\) such that
\[
\delta_{v^n_t}(da) \ dt \text{ converges weakly to } q_t(da) \ dt, \ \mathcal{P} - a.s.
\]
where \(\delta_{v^n_t}\) is a Dirac measure concentrated at a single point \(v^n_t\)

Let \(x^n\) and \(x^q\) be the trajectories of the system associated, respectively, with \((q, \eta)\) and \((v^n, \eta)\), and \(t \in [0, T]\), then
\[
x^n_t - x^q_t = \int_0^t \left[ \int_{A_1} b(s, x^n_s, a) \delta_{v^n_s}(da) - \int_{A_1} b(s, x^q_s, a) q_s(da) \right] ds
+ \int_0^t \left[ \int_{A_1} \sigma(s, x^n_s, a) \delta_{v^n_s}(da) - \int_{A_1} \sigma(s, x^q_s, a) q_s(da) \right] dW_s,
\]
do not depend on the singular part. Then, (10) is proved by using the results and the same proof that in Bahlali-Mezerdi-Djehiche [3, Lemma 4.1, page 12].

ii) On the other hand, (11) is proved in [3, Lemma 4.1, page 12]. ■

Remark 9 As a consequence, it is easy to see that the strict and relaxed optimal control problems have the same value function.

4 Maximum principle for near optimal controls

In this section we establish necessary condition of near optimality satisfied by a sequence of nearly optimal strict controls. This result is based on Ekeland’s variational principle which is given by the following.
Lemma 10 Let $(E, d)$ be a complete metric space and $f : E \to \mathbb{R}$ be lower-semicontinuous and bounded from below. Given $\varepsilon > 0$, suppose $u^\varepsilon \in E$ satisfies $f(u^\varepsilon) \leq \inf f + \varepsilon$. Then for any $\lambda > 0$, there exists $v \in E$ such that

1. $f(v) \leq f(u^\varepsilon)$.
2. $d(u^\varepsilon, v) \leq \lambda$.
3. $f(v) < f(w) + \frac{\varepsilon}{\lambda}d(v, w)$, $\forall w \neq v$.

Proof. See Ekeland [10].

To apply Ekeland’s variational principle, we have to endow the set $\mathcal{U}$ of strict controls with an appropriate metric. For any $(u, \xi), (v, \eta) \in \mathcal{U}$, we set

\[
    \begin{align*}
        d_1(u, v) &= \mathcal{P} \otimes dt \{ (\omega, t) \in \Omega \times [0, T], \ u(t, \omega) \neq v(t, \omega) \}, \\
        d_2(\xi, \eta) &= \left( \mathbb{E} \int_0^T \sup_{t \in [0, T]} |\xi_t - \eta_t|^2 dt \right)^{1/2}, \\
        d[(u, \xi), (v, \eta)] &= d_1(u, v) + d_2(\xi, \eta),
    \end{align*}
\]

where $\mathcal{P} \otimes dt$ is the product measure of $\mathcal{P}$ with the Lebesgue measure $dt$.

Let us summarize some of the properties satisfied by $d$.

Lemma 11

1. $(\mathcal{U}, d)$ is a complete metric space.
2. The cost functional $J$ is continuous from $\mathcal{U}$ into $\mathbb{R}$.

Proof. 1. It is clear that $(\mathcal{U}_2, d_2)$ is a complete metric space. Moreover, it was shown in [12] that $(\mathcal{U}_1, d_1)$ is a complete metric space. Hence $(\mathcal{U}, d)$ is a complete metric space as product of two complete metric spaces.

2. is proved in [12].

Now let $(\mu, \xi) \in \mathcal{R}$ be an optimal relaxed control and denote by $x^\mu_t$ the trajectory of the system controlled by $(\mu, \xi)$. From (9) and (10), there exists a sequence $(u^n)_n$ in $U_1$ such that

\[
dt \mu^n_t(da) = dt \delta_{x^n_t} (da) \n\to \infty \quad dt \mu_t (da) \text{ weakly, } \quad \mathcal{P} - \text{a.s},
\]

\[
    \mathbb{E} \left[ \sup_{t \in [0, T]} |x^n_t - x^\mu_t|^2 \right] \n\to \infty \quad 0,
\]

where $x^n_t$ is the solution of equation (5) corresponding to the control $(\mu^n, \xi)$.

According to the optimality of $(\mu, \xi)$ and (9), there exists a sequence $(\varepsilon_n)$ of positive real numbers with $\lim_{n \to \infty} \varepsilon_n = 0$ such that

\[
    J(u^n, \xi) = J(\mu^n, \xi) \leq J(\mu, \xi) + \varepsilon_n.
\]

A suitable version of lemma 10 (see [12] theorem 4.1) implies that a given any $\varepsilon_n > 0$, there exists $(u^n, \xi) \in \mathcal{U}$ such that

\[
    J(u^n, \xi) \leq \inf_{(v, \eta) \in \mathcal{U}} J(v, \eta) + \varepsilon_n, \quad (12)
\]

\[
    J(u^n, \xi) \leq J(v, \eta) + \varepsilon_n d[(u^n, \xi); (v, \eta)] \quad \forall (v, \eta) \in \mathcal{U}.
\]
Define the following two perturbations

\[
(u^n_\tau, \xi_t) = \begin{cases} 
(v, \xi_t) & \text{if } t \in [\tau, \tau + \theta], \\
(u^n_\tau, \xi_t) & \text{Otherwise}, 
\end{cases}
\]

(13)

\[
(u^n_\tau, \xi_t^\theta) = (u^n_\theta, \xi_t + \theta (\eta_t - \xi_t)),
\]

(14)

where \( v \) is a \( A_1 \)-valued, \( \mathcal{F}_t \)-measurable random variable and \( \eta \) is an increasing process with \( \eta_0 = 0 \) such that \( \mathbb{E} \left[ |v| + |\eta_T| \right] < \infty \).

From (12) we have

\[
0 \leq J (u^n_\tau, \xi_t) - J (u^n, \xi) + \varepsilon_n d \left[ (u^n, \xi); (u^n, \xi) \right],
\]

\[
0 \leq J (u^n, \xi_t^\theta) - J (u^n, \xi) + \varepsilon_n d \left[ (u^n, \xi); (u^n, \xi^\theta) \right].
\]

From the definition of the metric \( d \), we obtain

\[
0 \leq J (u^n_\tau, \xi_t) - J (u^n, \xi) + \varepsilon_n d_1 (u^n, u^n_\tau),
\]

\[
0 \leq J (u^n_\tau, \xi_t^\theta) - J (u^n, \xi) + \varepsilon_n d_2 (\xi, \xi^\theta).
\]

Using the definitions of \( d_1 \) and \( d_2 \), it holds that

\[
0 \leq J (u^n_\tau, \xi_t) - J (u^n, \xi) + \varepsilon_n C_\theta,
\]

(15)

\[
0 \leq J (u^n_\tau, \xi_t^\theta) - J (u^n, \xi) + \varepsilon_n C_\theta.
\]

(16)

From these above inequalities, we shall establish the near maximum principle in integral form.

**Theorem 12** (The near maximum principle in integral form). For each \( \varepsilon_n > 0 \), there exists \((u^n, \xi) \in \mathcal{U}\) such that there exists two unique couples of adapted processes

\[
(p^n, P^n) \in L^2 ([0, T]; \mathbb{R}^d) \times L^2 ([0, T]; \mathbb{R}^{n \times d}),
\]

\[
(k^n, K^n) \in L^2 ([0, T]; \mathbb{R}^{n \times n}) \times (L^2 ([0, T]; \mathbb{R}^{n \times n}))^d,
\]

solution of the following backward stochastic differential equations

\[
\begin{aligned}
&-dp^n_t = H_x (x^n_t, u^n_t, p^n_t, P^n_t) dt - P^n_t dW_t \\
p^n_T = g_x (x^n_T),
\end{aligned}
\]

(17)

\[
\begin{aligned}
&-dk^n_t = \left[ \alpha^n_x (t, x^n_t, u^n_t) k^n_t + k^n_t b_x (t, x^n_t, u^n_t) \right] dt \\
&\quad + \sigma^n_x (t, x^n_t, u^n_t) k^n_t \sigma_x (t, x^n_t, u^n_t) dt \\
&\quad + \left[ \sigma^n_x (t, x^n_t, u^n_t) K^n_t + K^n_t \sigma_x (t, x^n_t, u^n_t) \right] dt \\
&\quad + H_{xx} (x^n_t, u^n_t, p^n_t, P^n_t) dt - K^n_t dW_t, \\
k^n_T = g_{xx} (x^n_T).
\end{aligned}
\]

(18)
such that for all \((v, \eta) \in \mathcal{U}\),

\[
H(t, x^n_t, u^n_t, p^n_t, P^n_t) - k^n_t \sigma(t, x^n_t, u^n_t) + \frac{1}{2} \text{Tr} [\sigma \sigma^* (t, x^n_t, u^n_t)] k^n_t \tag{19}
\]

\[
\leq H(t, x^n_t, v, p^n_t, P^n_t) - k^n_t \sigma(t, x^n_t, u^n_t) + \frac{1}{2} \text{Tr} [\sigma \sigma^* (t, x^n_t, v)] k^n_t + C \varepsilon_n.
\]

where the Hamiltonian \(H\) is defined from \([0, T] \times \mathbb{R}^n \times A_1 \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R})\) into \(\mathbb{R}\) by

\[
H(t, x, v, P) = h(t, x, v) + pb(t, x, v) + \sigma(t, x, v) P.
\]

**Proof.** From inequalities (15) and (16), we use respectively the same method as in [2] with index \(n\). \(\blacksquare\)

### 5 The relaxed stochastic maximum principle

For simplicity, we note by \(f(t, x^n_t, \mu_t) = \int_{A_1} f(t, x^n_t, a) \mu_t(da)\), where \(f\) stands for \(b_x, \sigma_x, h_x, H_x, H_{xx}\).

Let \((\mu, \xi)\) be an optimal relaxed control and \(x^n\) be the corresponding optimal trajectory. Let \((p^n, P^n)\) and \((k^n, K^n)\) be the solutions of the following backward stochastic differential equations

\[
\begin{cases}
-dp^n_t = H_x (x^n_t, \mu_t, p^n_t, P^n_t) \, dt - P^n_t \, dW_t, \\
p^n_0 = g_x(x^n_0),
\end{cases}
\tag{21}
\]

\[
\begin{cases}
-dk^n_t = [b^n_x(t, x^n_t, \mu_t) k^n_t + k^n_t b_x(t, x^n_t, \mu_t)] \, dt \\
+ \text{Tr} (b^n_x(t, x^n_t, \mu_t) k^n_t) \, dt \\
+ \text{Tr} (b^n_x(t, x^n_t, \mu_t) P^n_t) \, dt - K^n_t \, dW_t,
\end{cases}
\tag{22}
\]

**Lemma 13** We have

\[
\lim_{n \to \infty} \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |p^n_t - p^\mu_t|^2 \right] + \mathbb{E} \int_0^T |P^n_t - P^\mu_t|^2 \, ds \right) = 0. \tag{23}
\]

\[
\lim_{n \to \infty} \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |k^n_t - k^\mu_t|^2 \right] + \mathbb{E} \int_0^T |K^n_t - K^\mu_t|^2 \, ds \right) = 0. \tag{24}
\]

where \((p^n, P^n)\) and \((k^n, K^n)\) are respectively the solutions of (17) and (18).

**Proof.** We have the same proof that in Bahlali-Mezerdi-Djehiche [3, Lemma 4.11, page 19]. \(\blacksquare\)

We can now state the relaxed maximum principle in integral form.
Theorem 14 (The relaxed maximum principle in integral form). Let \((\mu, \xi)\) be an optimal relaxed control minimizing the cost \(J\) over \(\mathcal{R}\) and \(x_{\mu}^t\) the corresponding optimal trajectory. There exists two unique couples of adapted processes
\[
(p^\mu, P^\mu) \in L^2 \left([0, T]; \mathbb{R}^n\right) \times L^2 \left([0, T]; \mathbb{R}^{n \times d}\right),
\]
\[
(k^\mu, K^\mu) \in L^2 \left([0, T]; \mathbb{R}^{n \times n}\right) \times \left(L^2 \left([0, T]; \mathbb{R}^{n \times n}\right)\right)^d,
\]
which are respectively solutions of backward stochastic differential equations (21) and (22) such that
\[
H(t, x^\mu_t, \mu_t, p^\mu_t, P^\mu_t, k^\mu_t, K^\mu_t) = \inf_{q \in \mathcal{P}(A_1)} H(t, x^\mu_t, q, p^\mu_t, P^\mu_t, k^\mu_t, K^\mu_t),
\]
\[
0 \leq \mathbb{E} \int_0^T \left[l_i + G^*_i(x^\mu_t, p^\mu_t)\right] d(\eta - \xi). \tag{26}
\]
Where
\[
H(t, x^\mu_t, \mu_t, p^\mu_t, P^\mu_t, k^\mu_t, K^\mu_t) = H(t, x^\mu_t, \mu_t, p^\mu_t, P^\mu_t - k^\mu_t \sigma(t, x^\mu_t, \mu_t))
+ \frac{1}{2} \text{Tr} \left[\sigma \sigma^* \right] k^\mu_t.
\]

Proof. Let \((\mu, \xi)\) be an optimal relaxed control. From theorem 12, there exists a sequence \((u^n, \xi)\) in \(\mathcal{U}\) such that for all \((v, \eta)\) in \(\mathcal{U}\), the variational equations (19) and (20) holds. Then by using (10), (23), (24) and by letting \(n\) go to infinity, the results follows immediately. \(\blacksquare\)

We are ready now state the main result of this paper, which is the relaxed stochastic maximum principle for singular control problems in its global form.

Theorem 15 (The relaxed maximum principle in global form). Let \((\mu, \xi)\) be an optimal control minimizing the functional cost \(J\) over \(\mathcal{R}\) and \(x_{\mu}^t\) the trajectory of the system controlled by \((\mu, \xi)\). Then, there exist two unique couples of adapted processes
\[
(p^\mu, P^\mu) \in L^2 \left([0, T]; \mathbb{R}^n\right) \times L^2 \left([0, T]; \mathbb{R}^{n \times d}\right),
\]
\[
(k^\mu, K^\mu) \in L^2 \left([0, T]; \mathbb{R}^{n \times n}\right) \times \left(L^2 \left([0, T]; \mathbb{R}^{n \times n}\right)\right)^d,
\]
which are respectively solutions of backward stochastic differential equations (21) and (22) such that
\[
H(t, x^\mu_t, \mu_t, p^\mu_t, P^\mu_t, k^\mu_t, K^\mu_t) = \inf_{q \in \mathcal{P}(A_1)} H(t, x^\mu_t, q, p^\mu_t, P^\mu_t, k^\mu_t, K^\mu_t),
\]
\[
\mathcal{P}\left\{\forall t \in [0, T], \forall i ; l_i(t) + G^*_i(x^\mu_t, p^\mu_t) \geq 0\right\} = 1, \tag{28}
\]
\[
\mathcal{P}\left\{\sum_{i=1}^m 1\{l_i(t) + G^*_i(x^\mu_t, p^\mu_t) \geq 0\} d\xi^i = 0\right\} = 1. \tag{29}
\]
Proof. From (25), we deduce immediately (27) and from (26), assertions (28) and (29) are proved exactly as in [7].

Remark 16 1) If \( G = l = 0 \), we recover the relaxed stochastic maximum principle for classical controls, see Bahlali-Mezerdi-Djehiche [3].

2) If \( \mu_t(da) = \delta_{u(t)}(da) \), we recover the strict singular stochastic maximum principle established by Bahlali-Mezerdi [2].

3) If \( \mu_t(da) = \delta_{u(t)}(da) \) and \( G = l = 0 \), we obtain Peng’s stochastic maximum principle [22].

4) If the diffusion coefficient \( \sigma \) does not contain the measure-valued part, we recover the result of Bahlali-Djehiche-Mezerdi [4].

Acknowledgement. The authors would like to thank the referees for valuable remarks and suggestions that improved the first version of the paper.

References

[1] S. Bahlali and A. Chala, The stochastic maximum principle in optimal control of singular diffusions with non linear coefficients, Rand. Operat. and Stoch. Equ, Vol. 18, 2005, no 1, pp 1-10.

[2] S. Bahlali and B. Mezerdi, A general stochastic maximum principle for singular control problems, Elect. J. of Probability, Vol. 10, 2005, Paper no 30, pp 988-1004.

[3] S. Bahlali, B. Mezerdi and B. Djehiche, Approximation and optimality necessary conditions in relaxed stochastic control problems, Journal of Applied Mathematics and Stochastic Analysis, Volume 2006, pp 1-23.

[4] S. Bahlali, B. Djehiche and B. Mezerdi, The relaxed maximum principle in singular control of diffusions, SIAM J. Control and Optim, 2007, Vol 46, Issue 2, pp 427-444.

[5] V.E Benêş, L.A Shepp and H.S Witsenhausen, Some solvable stochastic control problems, Stochastics, 4 (1980), pp. 39-83.

[6] A. Bensoussan, Lecture on stochastic control, in non linear filtering and stochastic control, Lecture notes in mathematics 972, 1981, Proc. Cortona, Springer Verlag.

[7] A. Cadenillas and U.G. Haussmann, The stochastic maximum principle for a singular control problem. Stochastics and Stoch. Reports., Vol. 49, 1994, pp. 211-237.

[8] P.L. Chow, J.L. Menaldi and M. Robin, Additive control of stochastic linear system with finite time horizon, SIAM J. Control and Optim., 23, 1985, pp. 858-899.
[9] M.H.A. Davis and A. Norman, *Portfolio selection with transaction costs*, Math. Oper. Research, 15, 1990, pp. 676 - 713.

[10] I. Ekeland, *On the variational principle*. J. Math. Anal. Appl., Vol. 47, 1974, pp 324-353.

[11] N. El Karoui, N. Huu Nguyen and M. Jeanblanc Piqué, *Compactification methods in the control of degenerate diffusions*. Stochastics, Vol. 20, 1987, pp 169-219.

[12] R.J. Elliott and M. Kohlmann, *The variational principle and stochastic optimal control*. Stochastics, 1980, 3, pp 229-241.

[13] W.H. Fleming, *Generalized solutions in optimal stochastic control*, Differential games and control theory 2, (Kingston conference 1976), Lect. Notes in Pure and Appl. Math.30, 1978.

[14] U.G. Haussmann and W. Suo, *Singular optimal stochastic controls I: Existence*, SIAM J. Control and Optim., Vol. 33, 1995, pp. 916-936.

[15] U.G. Haussmann and W. Suo, *Singular optimal stochastic controls II: Dynamic programming*, SIAM J. Control and Optim., Vol. 33, 1995, pp 937-959.

[16] U.G. Haussmann and W. Suo, *Existence of singular optimal control laws for stochastic differential equations*, Stochastics and Stoch. Reports, 48, 1994, pp 249 - 272.

[17] J. Jacod and J. Mémin, *Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité*. Sem. Proba.XV, Lect. Notes in Math. 851, Springer Verlag, 1980.

[18] I. Karatzas and S. Shreve, *Connections between optimal stopping and stochastic control I: Monotone follower problem*, SIAM J. Control Optim., 22, 1984, pp 856 - 877.

[19] H.J. Kushner, *Necessary conditions for continuous parameter stochastic optimization problems*, SIAM J. Control Optim., Vol. 10, 1973, pp 550-565.

[20] J. Ma and J. Yong, *Solvability of forward-backward SDEs and the nodal set of Hamilton-Jacobi-Bellman equations*. A Chinese summary appears in Chinese Ann. Math. Ser. A 16 (1995), no. 4, 532. Chinese Ann. Math. Ser. B 16, 1995, no. 3, pp 279–298.

[21] B. Mezerdi and S. Bahlali, *Necessary conditions for optimality in relaxed stochastic control problems*, Stochastics And Stoch. Reports, 2002, Vol 73 (3-4), pp 201-218.

[22] S. Peng, *A general stochastic maximum principle for optimal control problems*, SIAM Jour. Cont. Optim, 1990, 28, No 4, pp 966-979.
[23] J. Yong and X.Y. Zhou, Stochastic controls, Hamilton systems and HJB equations, vol 43, Springer, New York, 1999.