Constraints on heterotic M-theory from s-cobordism

Hisham Sati

Department of Mathematics
University of Maryland
College Park, MD 20742

Abstract

We interpret heterotic M-theory in terms of $h$-cobordism, that is the eleven-manifold is a product of the ten-manifold times an interval is translated into a statement that the former is a cobordism of the latter which is a homotopy equivalence. In the non-simply connected case, which is important for model building, the interpretation is then in terms of $s$-cobordism, so that the cobordism is a simple-homotopy equivalence. This gives constraints on the possible cobordisms depending on the fundamental groups and hence provides a characterization of possible compactification manifolds using the Whitehead group–a quotient of algebraic K-theory of the integral group ring of the fundamental group–and a distinguished element, the Whitehead torsion. We also consider the effect on the dynamics via diffeomorphisms and general dimensional reduction, and comment on the effect on F-theory compactifications.

Contents

1 Introduction 1
2 Heterotic M-theory as an $h$-cobordism and $s$-cobordism 3
3 The Whitehead group and Whitehead torsion 4
   3.1 The Whitehead group ...................................................... 4
   3.2 Whitehead torsion ....................................................... 6
4 Further examples in heterotic M-theory 8
5 Dynamical aspects 11
   5.1 Automorphisms ............................................................. 11
   5.2 Compactification ......................................................... 13

1 Introduction

A major goal of string theory is to provide a unification of fundamental interactions. This includes constructing the standard model via string compactifications [31], most notably via heterotic M-theory [35] [36]. Eleven-dimensional spacetime is taken to be an interval $I$ times a ten-manifold $M^{10}$, and with the two boundaries each supporting an $E_8$ gauge theory. One of the boundaries is called the hidden sector and the other is the visible sector, in which the structure group is broken down to a realistic symmetry group. The ten-manifold $M^{10}$ is typically taken to be Minkowski space $\mathbb{R}^{1,3}$ times a Calabi-Yau threefold $X^6$. In the visible sector one usually works with SU(5) or SO(10) $\subset E_8$ and breaks this group further (at least in principle) to the standard model group (ideally) SU(3)$\times$SU(2)$\times$U(1). Physical and mathematical constraints on the (Calabi-Yau) manifold $X^6$ and bundles on $X^6$ are recently reviewed in [33].

*e-mail: hsati@math.umd.edu

Current address: Department of Mathematics, University of Pittsburgh, 139 University Place, Pittsburgh, PA 15260.
Wilson lines are needed to break the gauge group from the grand unified (GUT) group to the standard model group $[15, 60]$. In order to introduce Wilson lines, the manifolds $M^{10}$ must have a nontrivial fundamental group. Starting with a simply connected Calabi-Yau manifold, one gets a smooth non-simply connected Calabi-Yau manifold by dividing by a freely acting discrete symmetry $X^6 \to X^6 / \Gamma$, where $\Gamma$ is a discrete group of finite order $|\Gamma|$, and the resulting fundamental group is $\pi_1(X^6 / \Gamma) = \Gamma$. Important choices for the finite group include $\Gamma = \mathbb{Z}_2$, which breaks SU(5) down to SU(3)×SU(2)×U(1), and $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_6$, which break SO(10) down to SU(3)×SU(2)×U(1)½ $[31]$.  

A major area of research involves choosing $\Gamma$ so that one gets the standard model, not just as far as the symmetry groups are concerned but also accounting for example for correct generations and spectra of particles. A sampler of fundamental groups of Calabi-Yau threefolds $X^6$ applied in the heterotic setting include: $\mathbb{Z}_2$ $[8]$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ $[24]$, $\mathbb{Z}_4 \times \mathbb{Z}_3$ $[14, 12]$, $\mathbb{Z}_6 \times \mathbb{Z}_8$ constructed in $[32]$ on which rank 5 bundles are constructed in $[3]$, abelian surface fibrations over CP$^1$ with (abelianization of) fundamental group $\mathbb{Z}_n \times \mathbb{Z}_n$ are considered in $[50, 23]$, complete intersection Calabi-Yau manifolds with fundamental groups which include $[16]$, $\mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_5 \times \mathbb{Z}_2, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_5$, and the quaternion group $\mathbb{Q}_8$, the latter being closely related to construction in $[7]$ of Calabi-Yau threefolds with nonabelian fundamental groups, roughly speaking a semidirect product of $\mathbb{Z}_8$ with a quaternion group. Torsion curves, important for instanton corrections to the heterotic minimally supersymmetric standard model (MSSM), are studied in $[13]$ for the quintic as well as for threefolds with fundamental groups $\mathbb{Z}_3 \times \mathbb{Z}_3$.  

Almost all known Calabi-Yau threefolds are simply connected. For example, only 16 out of about 500 million hypersurfaces in complex 4-dimensional toric varieties have nontrivial fundamental groups, and the only groups which occur are $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_5$ $[5]$. All elliptically fibered Calabi-Yau threefolds are simply connected, with the exception of fibrations over an Enriques base. In $[25]$ elliptic fibrations without section, i.e. torus bundles, with nontrivial fundamental group are constructed. Another class of examples with no section is the Schoen family $[55]$ which are fiber products of two rational elliptic surfaces. Free finite group actions on these are classified (under certain conditions) $[9]$ giving fundamental groups $\pi_1(X) \in \{ \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_3 \}$. $[\dagger]$ In another class of threefolds, the complete intersections in products of projective spaces, an exhaustive search $[10]$ of the 7890 such threefolds leads to many interesting fundamental groups including $\mathbb{Z}_i$ (for $i = 2, 3, 4, 5, 6, 8, 10, 12$, $\mathbb{Z}_2 \times \mathbb{Z}_j$ (for $j = 2, 4, 8, 10$, $\mathbb{Z}_4 \times \mathbb{Z}_k$ (for $k = 4, 8$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{Z}_8 \times \mathbb{Z}_8$, as well as semidirect products $\mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_5 \times \mathbb{Z}_{10}$ and groups involving the quaternion group $\mathbb{Q}_8$, namely $\mathbb{Z}_2 \times \mathbb{Q}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Q}_8, \mathbb{Z}_4 \times \mathbb{Q}_8, \mathbb{Z}_8 \times \mathbb{Q}_8$ (for a complete list see $[10]$).  

In this paper we seek constraints on the possible fundamental groups coming from global considerations, namely from looking at the relation between the heterotic boundary and bounding M-theory. We first interpret this relation as a cobordism which connects one boundary component to the other through the eleven-dimensional bulk. We take one of the two boundary components and the bulk to be of the same homotopy type. It is natural to ask when such cobordisms are trivial, that is when are they of product (or “cylinder”) form, as is usually the case in heterotic M-theory. When the fundamental groups of both the eleven-manifold $Y^{11}$ and the ten-manifold $M^{10}$ are trivial then we consider the cobordism as an $h$-cobordism ($h$ is for homotopy). When the fundamental groups are equal but nontrivial then we view heterotic M-theory as an $s$-cobordism ($s$ is for simple homotopy). Since the dimension of the nontrivial part of $M^{10}$, namely the Calabi-Yau threefold, is six then the $h$-cobordism $[40]$ and the $s$-cobordism $[38, 45, 61]$ theorems can be applied. In both cases we are assuming that inclusions of the boundaries in $Y^{11}$ are homotopy equivalences. The case when $\pi_1(Y^{11})$ is nontrivial is discussed extensively in $[52]$ in relation the partition functions and to type IIA string theory.  

The obstruction to finding a cobordism that is of the cylinder type is the Whitehead torsion of the inclusion $\tau(Y^{11}, M^{10})$, which is an element of the Whitehead group of the fundamental group $\text{Wh}(\pi_1(M^{10}))$. The Whitehead group is extensively studied and is well-known for finite groups (see $[48]$), which is the case we mainly study as such groups seem to be the most interesting for model building. Given our identification

$^{\dagger}$A free quotient of the manifold corresponding to $\mathbb{Z}_3 \times \mathbb{Z}_3$ by the quaternion group is given in $[16]$.  

2
of heterotic M-theory as an $s$-cobordism, we are able to identify fundamental groups that allow trivial cobordisms from the ones which do not. We view this as providing global consistency constraints on heterotic compactifications in view for model building.

We summarize the main point of this article with

**Theorem 1** Consider heterotic M-theory with the $E_8$ heterotic string theory on each of the two the boundary components. Then
(i) $M$-theory is an $s$-cobordism for one of the two connected components of the boundary.
(ii) Consistency requires the Whitehead torsion, in the Whitehead group of the integral group ring of the fundamental group of the boundary component, to vanish.

Since the use of $h$- and $s$-cobordism and the Whitehead torsion is novel in the context of heterotic M-theory and is perhaps not widely known in theoretical physics in general, we choose to take an expository route to arrive at our conclusions. We provide the description of heterotic M-theory in terms of $h$ and $s$-cobordism in section 2. Then in section 3 we look at constraints on the fundamental group, coming from the Whitehead group in section 3.1 and from the Whitehead torsion in section 3.2. We provide many examples along the way and then in section 4 we consider representative examples explicitly appearing in model building. We then consider the dynamical aspects in section 5, emphasizing the main points of this article. We first consider automorphisms, including diffeomorphisms and issues of orientation, in section 5.1 and then we consider dynamical aspects of compactifications in section 5.2.

### 2 Heterotic M-theory as an $h$-cobordism and $s$-cobordism

In this section we set up heterotic M-theory as a cobordism, first as an $h$-cobordism and then as $s$-cobordism. Viewed from M-theory the data involves an eleven-dimensional manifold $Y^{11}$ which is a product $[0,1] \times M^{10}$ together with an $E_8$ bundle on each of $M^{10} \times \{0\}$ and $M^{10} \times \{1\}$. We will consider this from a ten-dimensional point of view, where we will have a cobordism taking one boundary component to the other.

**$H$-cobordism.** A compact connected eleven-manifold $Y^{11}$ whose boundary $\partial Y^{11}$ is the disjoint union of two closed manifolds $M^{10}$ and $M^{10}$, $\partial Y^{11} = M^{10} \cup M^{10}$, is called an $h$-cobordism, provided the inclusions of $M^{10}$ into $Y^{11}$ and of $M^{10}$ into $Y^{11}$ are both homotopy equivalences. The pair $(Y^{11}, M^{10})$ is called a $h$-cobordism with base $M^{10}$ and top $M^{10}$. A smooth $h$-cobordism is one where $Y^{11}$ is a smooth manifold. A trivial or product $h$-cobordism is of the form $M^{10} \times [0,1]$. If $Y^{11}$ is simply-connected, then the $h$-cobordism theorem can be applied (see [16]) to give that $Y^{11}$ is diffeomorphic to the product $M^{10} \times [0,1]$. This is the configuration that is usually considered in heterotic M-theory [35, 36].

We can consider a more detailed description, which will be useful in section 3.2 and section 5. An eleven-dimensional cobordism $(Y^{11}; M^{10}, f_0, M^{10}, f_1)$ consists of a compact oriented eleven-manifold $Y^{11}$, two closed ten-manifolds $M^{10}$ and $M^{10}$, a disjoint decomposition $\partial Y^{11} = \partial_0 Y^{11} \sqcup \partial_1 Y^{11}$ of the boundary $\partial Y^{11}$ of $Y^{11}$ and orientation-preserving diffeomorphisms $f_0 : M^{10} \to \partial_0 Y^{11}$ and $f_1 : (M^{10})^{-} \to \partial_1 Y^{11}$. By $X^{-}$ we mean the manifold $X$ taken with the opposite orientation. On the boundary $\partial Y^{11}$ we use the orientation with respect to the decomposition $TY^{11} = T\partial Y^{11} \oplus \mathbb{R}$ coming from an inward normal field to the boundary. If $\partial_0 Y^{11} = M^{10}$, $\partial_1 Y^{11} = (M^{10})^{-}$, and $f_0$ and $f_1$ are the identity maps, then the $h$-cobordism can be referred to as $(Y^{11}; (M^{10})^{-}, (M^{10})^{-})$. An $h$-cobordism over $M^{10}$ is trivial if it is diffeomorphic relative $M^{10}$ to the trivial $h$-cobordism $(M^{10} \times [0,1]; M^{10} \times \{0\}, (M^{10} \times \{1\})^{-})$.

**The fundamental group.** The $h$-cobordism theorem can be applied only when the fundamental group is trivial. Next we consider the more interesting case when the fundamental group is not necessarily trivial. We will assume that $\pi_1(Y^{11}) \cong \pi_1(M^{10})$. The fundamental group functor takes products to products, that is, the fundamental group is multiplicative. For $M^{10} = \mathbb{R}^{1,3} \times X$, we have $\pi_1(M^{10}) \cong \pi_1(\mathbb{R}^{1,3}) \times \pi_1(X) \cong \pi_1(X)$, so
that the fundamental group of \( M^{10} \) is determined by that of the Calabi-Yau threefold \( X \). The generalization from Minkowski to other four-dimensional spacetimes gives an obvious modification, which depends on whether or not the latter is simply connected. Next we consider the appropriate description of heterotic M-theory when \( \pi_1(M^{10}) \neq 0 \). By our assumption, this is equivalent to taking \( \pi_1(Y^{11}) \neq 0 \), considered in \cite{52}.

**S-cobordism.** Let \( M^{10} \) be a connected compact 10-manifold with fundamental group \( \Gamma \), and consider the family \( \mathcal{F} \) of all \( h \)-cobordisms built on \( M^{10} \). These are connected compact 11-manifolds \( Y^{11} \) with exactly two boundary components, one of which is \( M^{10} \) and the other of which is some other manifold \( M'^{10} \) such that \( Y^{11} \) is homotopy equivalent to both \( M^{10} \) and \( M'^{10} \). There is a map \( \tau : \mathcal{F} \to \text{Wh}(\Gamma) \) called the Whitehead torsion which induces a natural one-to-one correspondence from \( \mathcal{F}/\sim \to \text{Wh}(\Gamma) \), where \( \sim \) is the equivalence relation induced by diffeomorphisms \( Y^{11} \to Y'^{11} \) which are the identity on \( M^{10} \). If \( Y^{11} \) is the “trivial” \( h \)-cobordism \( Y^{11} = M^{10} \times [0,1] \), then \( \tau(Y^{11}) = 1 \). This is an application of the Barden-Mazur-Stallings theorem \cite{38,55,61} (see \cite{51} for a review).

If the fundamental group \( \Gamma \) is such its Whitehead group \( \text{Wh}(\Gamma) \) is trivial, then certainly the Whitehead torsion will vanish and we are back to the case of an \( h \)-cobordism. Consequently, \( Y^{11} \) is diffeomorphic (relative \( M^{10} \)) to a product \( M^{10} \times [0,1] \). More generally, the other boundary component \( M'^{10} \) is diffeomorphic to \( M^{10} \). This is induced by a homotopy equivalence. It is an open conjecture of Borel from 1955 that this can be extended to a homeomorphism. The strengthening to smooth manifolds fails \cite{20}. To show this, we would choose the category of topological spaces. This implies, for example, that orbifolds are also included in our discussion, for which we would choose the category of topological spaces.

**3 The Whitehead group and Whitehead torsion**

We now consider the Whitehead group and Whitehead torsion in our setting of heterotic M-theory via algebraic K-theory of the group ring of the fundamental group and give the main properties which are useful for us.

**3.1 The Whitehead group**

Algebraic K-theory roughly characterizes how, in passing from a field to an arbitrary ring, notions of linear algebra related to the general linear group and vector spaces might extend. One measure of failure of such an extension is \( K_1(R) \), the algebraic K-theory of an associative ring \( R \). Let \( \bar{K}_1(R) \) be the cokernel of the map \( K_1(\mathbb{Z}) \to K_1(R) \) induced by the canonical ring homomorphism \( \mathbb{Z} \to R \). Since \( \mathbb{Z} \) is a ring with Euclidean algorithm then the homomorphism \( \text{det} : K_1(\mathbb{Z}) \to \{\pm 1\} \), given by \( [A] \mapsto \text{det}(A) \), is a bijection. Hence \( \bar{K}_1(R) \) is the quotient of \( K_1(R) \) by a cyclic group of order two generated by the class of the \( 1 \times 1 \)-matrix \( (-1) \). We are interested in the case when \( R \) is a group algebra \( \mathbb{Z}[\Gamma] \) of the fundamental group \( \Gamma = \pi_1(M^{10}) \), that is in integer linear combinations of elements of \( \Gamma \). Define the **Whitehead group** \( \text{Wh}(\Gamma) \) of a group \( \Gamma \) to be the
torsion in the algebraic K-group is \( \text{Tor}(K_1(\mathbb{Z}[\Gamma])) \) which sends \((\gamma, \pm 1)\) to the class of the invertible \(1 \times 1\)-matrix \((\pm \gamma)\). In other words, \(\text{Wh}(\Gamma)\) is the quotient of \(K_1(\mathbb{Z}[\Gamma])\) by the image of \(\{\pm \gamma : \gamma \in \Gamma\}\), that is

\[
\text{Wh}(\Gamma) = K_1(\mathbb{Z}[\Gamma])/\{\pm \gamma : \gamma \in \Gamma\}.
\] (3.1)

The zero element \(0 \in \text{Wh}(\Gamma)\) is represented by the identity matrix \(I_n\) for any positive integer \(n\).

Note that one can define the Whitehead group of the fundamental group by choosing a base point, as is usual in the fundamental group. However, the end result will be independent of the choice of the base point. Therefore, one should think of \(\pi_1(M)\) in \(\text{Wh}(\pi_1(M))\) as the fundamental groupoid of \(M\). Note also that the Whitehead group can be viewed either additively or multiplicatively. In the first point of view, this corresponds to adding two cobordisms by connecting one ‘cylinder’ to another over a ten-dimensional section, while an instance of the second point of view is a ‘flip’.

**Example 1. Trivial case.** Consider the case when the fundamental group is trivial. Then the group algebra \(\mathbb{Z}[1] = \mathbb{Z}\) is a ring with a Gaussian algorithm, so that the determinant induces an isomorphism \(K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm\}\) and the Whitehead group \(\text{Wh}\{1\}\) of the trivial group vanishes. Hence any \(h\)-cobordism over a simply-connected closed \(M^{10}\) is trivial. Thus, as expected in this case, \(s\)-cobordism reduces to \(h\)-cobordism.

**Example 2. Finite cyclic groups.** \(\text{Wh}(\Gamma)\) is torsion-free for a finite cyclic group. For example, \(\text{Wh}(\mathbb{Z}_p)\), \(p\) odd prime, is the free abelian group of rank \((p - 3)/2\) and \(\text{Wh}(\mathbb{Z}_2) = 0\).

We will consider many more examples in section 4.

**Properties of the Whitehead group.** We are interested in the case when the fundamental group \(\Gamma\) is a finite group. For such a group the following useful properties hold [47, 2, 48]

1. **Functoriality:** \(\text{Wh}(\Gamma)\) is a covariant functor of \(\Gamma\), that is, any homomorphism \(f : \Gamma_1 \to \Gamma_2\) induces a homomorphism \(f_* : \text{Wh}(\Gamma_1) \to \text{Wh}(\Gamma_2)\).

2. **Trivial group:** Let \(\Gamma = \pi_1(M)\) be trivial. Then from \(K_1(\mathbb{Z}) = \mathbb{Z}_2\) one gets \(\text{Wh}(\pi_1(M)) = \text{Wh}(1) = 1\). This is example 1 above.

3. **Low rank:** Whitehead showed that \(\text{Wh}(\Gamma) = 1\) if \(|\Gamma| \leq 4\). This implies, for instance, that \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4\) and \(\mathbb{Z}_2 \times \mathbb{Z}_2\) have trivial Whitehead group and hence lead to (desirable) trivial \(h\)-cobordisms.

4. **Rank:** By a result of Bass, \(\text{Wh}(\Gamma)\) is a finitely generated abelian group of rank \(r(\Gamma) - q(\Gamma)\), where \(r(\Gamma)\) is the number of irreducible real representations of \(\Gamma\) and \(q(\Gamma)\) is the number of irreducible rational representations of \(\Gamma\). Explicitly, \(q(\Gamma)\) is the number of conjugate classes of cyclic subgroups of \(\Gamma\) and \(r(\Gamma)\) is the number of conjugate classes of unordered pairs \(\{\gamma, \gamma^{-1}\}\).

5. **Free product:** The Whitehead group of a free product is multiplicative, \(\text{Wh}(G \ast H) = \text{Wh}(G) \oplus \text{Wh}(H)\). Unfortunately, there is no corresponding formula for Cartesian products. For example, \(\text{Wh}(\mathbb{Z}_3) = 0\) and \(\text{Wh}(\mathbb{Z}_4) = 0\) but \(\text{Wh}(\mathbb{Z}_3 \times \mathbb{Z}_4) \cong \mathbb{Z}\). More on this will be discussed in section 2.

**The torsion subgroup.** We have seen above that the Whitehead group of a cyclic group \(\mathbb{Z}_p\) of prime order \(p\) is torsion-free. While these groups form an important class of fundamental groups we are considering, we should consider other cases as well. In particular, there could be groups \(\Gamma\) for which \(\text{Wh}(\Gamma)\) is torsion. The torsion in the algebraic K-group is \(\text{Tor}(K_1(\mathbb{Z}[\Gamma]))\) = \(\{\pm\} \times \Gamma^{ab} \times SK_1(\mathbb{Z}[\Gamma])\), where \(\Gamma^{ab}\) is the abelianization of \(\Gamma\) (that is the first homology group \(H_1(M^{10})\)) and \(SK_1(\mathbb{Z}[\Gamma]) = \ker(K_1(\mathbb{Z}[\Gamma]) \to K_1(\mathbb{Q}[\Gamma]))\). This kernel of the change of coefficients homomorphism is the full torsion subgroup of \(\text{Wh}(\mathbb{Z}[\Gamma])\).
Properties of the torsion subgroup. The torsion subgroup $SK_1(\mathbb{Z}[\Gamma])$ of $Wh(\Gamma)$ is highly nontrivial [2] [64] [48]. Some of the useful properties are

1. The torsion subgroup of $Wh(\Gamma)$ is isomorphic to $SK_1(\mathbb{Z}[\Gamma])$.
2. The torsion in $Wh(\Gamma)$ comes from $SL(2,\mathbb{Z}[\Gamma])$.
3. $SK_1(\mathbb{Z}[\Gamma])$ is non-vanishing for all groups of the form $\Gamma \cong (\mathbb{Z}_p)^n$, $n \geq 3$ and $p$ an odd prime.
4. $SK_1(\mathbb{Z}[\Gamma]) = 1$ if $\Gamma \cong \mathbb{Z}_{p^n}$ or $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$ (for any prime $p$, and any $n$), if $\Gamma \cong (\mathbb{Z}_2)^n$ (any $n$), or if $\Gamma$ is any dihedral, quaternion, or semidihedral 2-group.
5. The classes of finite groups $\Gamma$ for which $Wh(\Gamma) = 1$, or $SK_1(\mathbb{Z}[\Gamma]) = 1$, are not closed under products. This provides many nontrivial examples using products.

For a finitely generated fundamental group $\Gamma$ the vanishing of the Whitehead group $Wh(\Gamma)$ is equivalent to the statement that each $h$-cobordism over a closed connected $M^{10}$ is trivial. Knowing that all $h$-cobordisms over a given manifold are trivial is useful, but strong. Alternatively, we could have $Wh(\Gamma)$ nontrivial yet the distinguished element, the Whitehead torsion $\tau$ is zero.

3.2 Whitehead torsion

The Whitehead torsion, which is essentially a linking matrix for handles in the handle decomposition of the manifold, serves as an obstruction to the reduction of an $h$-cobordism to a product. We have encountered above many situations where the Whitehead group is not trivial. In certain cases these elements, including the distinguished element given by the Whitehead torsion, can be characterized. This characterization can be geometric due to the realization theorem which says that every Whitehead torsion comes from a manifold (see [38]).

First, note that a map $f : Y^{11} \to M^{10}_0$ induces a homomorphism $f_* : Wh(\pi_1(Y^{11})) \to Wh(\pi_1(M^{10}_0))$ on the corresponding Whitehead groups such that $id_* = id$, $(g \circ f)_* = g_* \circ f_*$, and $f \simeq g$ implies that $f_* = g_*$. Next, the Whitehead torsion of our eleven-dimensional $h$-cobordism $(Y^{11}; M^{10}_0, f_0, M^{10}_1, f_1)$ over $M^{10}_0$,

$$\tau(Y^{11}, M^{10}_0) \subset Wh(\pi_1(M^{10}_0)),$$

is defined to be the preimage of the Whitehead torsion $\tau(M^{10}_0 \xrightarrow{f_0} \partial Y^{11} \xrightarrow{t_0} Y^{11}) \subset Wh(\pi_1(Y^{11}))$ under the isomorphism $(\iota_0 \circ f_0)_* : Wh(\pi_1(M^{10}_0)) \cong Wh(\pi_1(Y^{11}))$, where $t_0 : \partial Y^{11} \hookrightarrow Y^{11}$ is the inclusion (see [39]). Next we will consider the simple situation when the diffeomorphisms are the identity.

Geometric definition of Whitehead torsion. There is a description of Whitehead torsion at the level of chain complexes [47] [19]. Let $W(M^{10})$ be the collection of all pairs of finite complexes $(Y^{11}, M^{10})$ such that $M^{10}$ is a strong deformation retract of $Y^{11}$. For any two objects $(Y^{11}_1, M^{10}_1), (Y^{11}_2, M^{10}_2) \in W$ define an equivalence $(Y^{11}_1, M^{10}_1) \sim (Y^{11}_2, M^{10}_2)$ if and only if $Y^{11}_1$ and $Y^{11}_2$ are simple homotopically equivalent relative to the subcomplex $M^{10}$. Define $Wh(M^{10}) = W/ \sim$ and let $[Y^{11}_1, M^{10}_1]$ and $[Y^{11}_2, M^{10}_2]$ be two classes in $Wh(M^{10})$. For $Y^{11}_1 \amalg M^{10}_1 Y^{11}_2$, the disjoint union of $Y^{11}_1$ and $Y^{11}_2$ identified along the common subcomplex $M^{10}$, an abelian group structure can be defined on the Whitehead group $Wh(M^{10})$ by $[Y^{11}_1, M^{10}] \oplus [Y^{11}_2, M^{10}] = [Y^{11}_1 \amalg M^{10}_1 Y^{11}_2, M^{10}]$.

The universal cover $(\widetilde{Y}^{11}, \widetilde{M}^{10})$ of an element $(Y^{11}, M^{10})$ in $W$ can be equipped with the CW-complex structure lifted from the CW-structure of $(Y^{11}, M^{10})$. The inclusion $\widetilde{M}^{10} \subset \widetilde{Y}^{11}$ is a homotopy equivalence. Let $C_*(\widetilde{Y}^{11}, \widetilde{M}^{10})$ be the cellular chain complex of $(\widetilde{Y}^{11}, \widetilde{M}^{10})$. The covering action of $\pi_1(Y^{11})$ on $(\widetilde{Y}^{11}, \widetilde{M}^{10})$ induces an action on $C_*(\widetilde{Y}^{11}, \widetilde{M}^{10})$ and makes it a finitely generated free acyclic chain complex of $\mathbb{Z}[\pi_1(Y^{11})]$-modules. In addition to the boundary map $\partial$, there is a contraction map $\delta$ of
degree +1 on $C_*(\tilde{Y}^1, \tilde{M}^{10})$ such that $\partial \delta + \delta \partial = \text{id}$ and $\delta^2 = 0$. The module homomorphism $\partial + \delta : \bigoplus_{i=0}^{\infty} C_{2i+1}(Y^{11}, M^{10}) \to \bigoplus_{i=0}^{\infty} C_{2i}(Y^{11}, M^{10})$ is an isomorphism of $\mathbb{Z}[\pi_1(Y^{11})]$-modules. The image and the range of this homomorphism are finitely generated free modules with a basis we choose coming from the CW-structure on $(Y^{11}, \tilde{M}^{10})$. Consider the matrix of this homomorphism $\partial + \delta$ which is an invertible matrix with entries in $\mathbb{Z}[\pi_1(Y^{11})]$ and hence lies in $GL(n, \mathbb{Z}[\pi_1(Y^{11})])$ for some $n$. Now take the image of this matrix in $\text{Wh}(\pi_1(Y^{11}))$ via an isomorphism $\tau$, sending $(Y^{11}, \tilde{M}^{10})$ to $\text{Wh}(\pi_1(Y^{11}))$.

More explicitly, let $\cdots \xrightarrow{\partial} C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$ be the complex which calculates the homology $H_*(Y^{11}, M^{10}; \mathbb{Z}[\Gamma])$ of the inclusion $M^{10} \subset Y^{11}$. Each $C_i$ is a finitely generated free $\mathbb{Z}[\Gamma]$-module. Up to orientation and translation by an element in $\Gamma$, each $C_i$ has a preferred basis over $\mathbb{Z}[\Gamma]$ coming from the $i$-simplices added to get from $M^{10}$ to $Y^{11}$ in some triangulation of the universal covering spaces. The group $Z_i$ of $i$-cycles is the kernel of $\partial : C_i \to C_{i-1}$ and the group $B_i$ of $i$-boundaries is the image of $\partial : C_{i+1} \to C_i$. Since $M^{10} \subset Y^{11}$ is a deformation retract, homotopy invariance of homology gives that $B_* = 0$, so that $B_* = Z_*$. Let $\mathcal{M}_i \in GL(\mathbb{Z}[\pi_1(M^{10})])$ be the matrices representing the isomorphism $B_i \oplus B_{i-1} \cong C_i$ coming from a choice of section $0 \to B_i \to C_i \to B_{i-1} \to 0$. Let $[\mathcal{M}_i] \in \text{Wh}(\pi_1(M^{10}))$ be the corresponding equivalence classes. The Whitehead torsion is then

$$
\tau(Y^{11}, M^{10}) = \sum (-1)^i [\mathcal{M}_i] \in \text{Wh}(\pi_1(M^{10})).
$$

Note that the Whitehead group is identified as a quotient of $K_1(\mathbb{Z}[\Gamma])$ by the subgroup generated by the units of the form $\pm \gamma$ for $\gamma \in \Gamma = \pi_1(M^{10})$. In the present context, this ensures the independence of the choice of $\mathbb{Z}[\Gamma]$-basis within the cellular equivalence class of $\mathbb{Z}[\Gamma]$-bases.

**Properties of Whitehead torsion.** The Whitehead torsion has existence and uniqueness properties.

1. **Existence.** Given $\alpha \in \text{Wh}(\pi_1(M^{10}))$, there exists an $h$-cobordism $Y^{11}$ with $\tau(Y^{11}) = \alpha$. This implies that if the Whitehead group is nontrivial then we can find a cobordism for every element in that group. In order to get a trivial $h$-cobordism, that is one of cylinder type, we have to make sure that the element $\text{Wh}(\Gamma)$ we identify for our spaces will be the zero element. This is of course not guaranteed to occur.

2. **Uniqueness.** $\tau(Y^{11}) = \tau(Y^{11})$ if and only if there exists a diffeomorphism $f : Y^{11} \to Y^{11}$ such that $f|_M = \text{id}_M$. This tells us that we are allowed to “deform” $Y^{11}$ in a nice way and still be able to get the same type of cobordism. In particular, for $Y^{11}$ with $\tau(Y^{11}) = 0$ we can always find a diffeomorphic $Y^{11}$ for which the property that the Whitehead torsion is zero is preserved.

**Elements of finite order in the Whitehead group.** We have seen that the Whitehead group of products of finite cyclic groups may contain torsion. Elements of finite order can be characterized as follows [47]. Consider an orthogonal representation $\Gamma \to O(n)$ of the finite group $\Gamma$. This representation gives rise to a ring homomorphism $\rho : \mathbb{Z}[\Gamma] \to M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is the algebra of $n \times n$ matrices over the real numbers. This induces a group homomorphism $\rho_* : K_1(\mathbb{Z}[\pi]) \to K_1(M_n(\mathbb{R})) \cong K_1(\mathbb{R}) \cong \mathbb{R}^+$. Since $\mathbb{R}^+$ has no elements of finite order then there is the corresponding homomorphism $\text{Wh}(\Gamma) \to \mathbb{R}^+$. Therefore, an element $\omega \in \text{Wh}(\Gamma)$ has finite order if and only if $\rho_*(\omega) = 1$ for every orthogonal representation $\rho$ of $\Gamma$.

**Elements of $\text{Wh}(\Gamma)$ as matrices and the representation dimension.** Nontrivial elements of the Whitehead group can be represented by matrices, usually of small size. The representation dimension of a group $\Gamma$ is said to be less than or equal to $m$, with notation $r \text{-dim} \Gamma \leq m$, if every element of $\text{Wh}(\Gamma)$ can be realized as a matrix in $GL(m, \mathbb{Z}[\Gamma])$. If $\Gamma$ is finite then $r \text{-dim} \Gamma \leq 2$. Furthermore, the representation dimension of the finite group $\Gamma$ satisfies $r \text{-dim} \Gamma \leq 1$ if and only if $\Gamma$ admits no epimorphic mapping onto the following (see [57])

1. the generalized quaternion group,
2. the binary tetrahedral, octahedral, or icosahedral groups,
3. and the groups $\mathbb{Z}_p^2 \times \mathbb{Z}_p^2$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_4 \times \mathbb{Z}_4$, for $p$ a prime.
Thus $r\text{-dim } \Gamma \leq 1$ for all finite simple groups. However, if we take products then the size of the matrix can grow (see expression (4.1) for an explicit matrix).

4 Further examples in heterotic M-theory

We have already seen many classes of examples both for the Whitehead group in section 3.1 and for the Whitehead torsion in section 3.2 we now provide more examples and in particular ones which appear explicitly in model building (cf. the introduction).

Tori and free abelian groups. The fundamental group of the circle is the free abelian group $\mathbb{Z}$, so that the corresponding Whitehead torsion is zero, $\text{Wh}(\mathbb{Z}) = 0$. For the $n$-torus $T^n$, the fundamental group $\pi_1(T^n) = \mathbb{Z}^n$. This free abelian group of rank $n$ has a trivial Whitehead torsion $\text{Wh}(\pi_1(T^n)) = 0$, since $\text{Wh}(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) = 0$ by the multiplicative property of Whitehead torsion under free product (section 3.1). It follows from the theorem of Bass about the rank of the Whitehead group that $\text{Wh}(\Gamma)$ of a free abelian group $\Gamma$ is zero if and only if $\Gamma$ has exponent 1, 2, 3, 4, or 6 [47].

Cyclic groups. Suppose $\Gamma$ is a finite group. Then $\text{Wh}(\Gamma)$ is finitely generated, and $\text{rank}(\text{Wh}(\Gamma))$ is the difference between the number of irreducible representations of $\Gamma$ over $\mathbb{R}$ and the number of irreducible representations of $\Gamma$ over $\mathbb{Q}$. For $\Gamma$ a cyclic group $\mathbb{Z}_p$ of order $p$, an odd prime, the numbers of representations are $q(\mathbb{Z}_p) = 2$ and $r(\mathbb{Z}_p) = \frac{1}{2}(p + 1)$, respectively. This implies that $\text{Wh}(\mathbb{Z}_p)$ is the free abelian group of rank $(p - 3)/2$ and that $\text{Wh}(\mathbb{Z}_2) = 0$. Alternatively, note that $\mathbb{Z}_p$ has $(p - 1)/2$ inequivalent two-dimensional irreducible representations over $\mathbb{R}$, but one $(p - 1)$-dimensional irreducible representation over $\mathbb{Q}$ (since $Q[\mathbb{Z}_p] \cong Q \times Q(\zeta)$, $\zeta$ a primitive $p$-th root of unity, and $[Q(\zeta) : Q] = p - 1$), so $\text{rank}(\text{Wh}(\mathbb{Z}_p)) = \frac{p - 1}{2} + 1 - 2 = (p - 3)/2$. Note that we have already seen that $\text{Wh}(\mathbb{Z}_k) = 0$ for $k = 2, 3, 4, 6$.

Units in the group ring. Consider the integral group ring $\mathbb{Z}[\mathbb{Z}_p]$ of the finite cyclic group $\mathbb{Z}_p$, and let $\zeta$ be a primitive $p^{th}$ root of unity with corresponding group ring $\mathbb{Z}[\zeta]$. The pullback of rings

$$
\begin{diagram}
\mathbb{Z}[\mathbb{Z}_p] \arrow{e} \mathbb{Z}[\zeta] \arrow{s} \\
\mathbb{Z} \arrow{n} \mathbb{F}_p
\end{diagram}
$$

where $\mathbb{F}_p$ is the field with $p$ elements, implies that the $(p - 1)st$ power of any unit in $\mathbb{Z}[\zeta]$ comes from a unit in $\mathbb{Z}[\mathbb{Z}_p]$. An example of a unit in $\mathbb{Z}[\zeta]$ is $(\zeta + \zeta^{-1})^t$. This is invariant under complex conjugation in $\mathbb{Z}[\zeta]$ (this corresponds to invariance under the orientation duality discussed in section 3.1).

The quintic and the cyclic group of order 5. The quintic threefold plays an important role as a prototype example of compactification on Calabi-Yau manifolds. Consider the one-parameter family of quintic threefolds $Q := \{z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + \psi^5z_1z_2z_3z_4z_5 = 0\} \subset \mathbb{CP}^4$. The defining equation is invariant under the $\mathbb{Z}_5 \times \mathbb{Z}_5 \subset \text{PGL}(5, \mathbb{C})$ group action

$$
[z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [z_2 : z_3 : z_4 : z_5 : z_1], \quad [z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [\zeta z_1 : \zeta^2 z_2 : \zeta^3 z_3 : \zeta^4 z_4 : z_5],
$$

where $\zeta = e^{2\pi i/5}$. The fixed points lie on $\mathbb{CP}^4 - Q$, so that $Q/\mathbb{Z}_5$ and $Q/\mathbb{Z}_5 \times \mathbb{Z}_5$ are smooth Calabi-Yau threefolds. The six different $\mathbb{Z}_5$ subgroups in $\mathbb{Z}_5 \times \mathbb{Z}_5$ can be used. The Whitehead group of $\mathbb{Z}_5 = \{t | t^5 = 1\}$ is $\text{Wh}(\mathbb{Z}_5) = \mathbb{Z}$ with generator the torsion $\tau(u)$ of the unit $u = 1 - t + t^2 \in \mathbb{Z}[\mathbb{Z}_5]$ [47]. The identity $(t + t^{-1} - 1)(t^2 + t^{-2} - 1) = 1$ indeed shows that $u$ is a unit. The homomorphism $\alpha : \mathbb{Z}[\mathbb{Z}_5] \to \mathbb{C}$, sending $t$ to $\zeta$, also sends $\{\pm \gamma : \gamma \in \Gamma\}$ to the roots of unity in $\mathbb{C}$, and hence $x \mapsto |\alpha(x)|$ defines a homomorphism from $\text{Wh}(\mathbb{Z}_5)$ into $\mathbb{R}^+$, the nonzero positive real numbers. Then the map $u \mapsto 1 - \zeta - \zeta^{-1} = 1 - 2 \cos(2\pi/5)$ can be used to show that no power of $u$ is equal to 1. Indeed, $|\alpha(u)| = |1 - 2 \cos(2\pi/5)| \approx 0.4$, so that $\alpha$ defines
an element of infinite order in \( \text{Wh}(\mathbb{Z}_5) \). Note that the unit \( u \) is self-conjugate, and that the automorphism \( t \mapsto t^2 \) of \( \mathbb{Z}_5 \) carries \( u \) to \( u^{-1} \). In fact, for \( \Gamma \) finite abelian, every element of \( \text{Wh}(\Gamma) \) is self-conjugate \(^{[17]} \) (see the last paragraph in section \(^{[22]} \)).

We see from the example of the quintic that, a priori, there are countably infinitely many elements in the Whitehead group of the fundamental group of the quintic. Unless the Whitehead torsion is the zero element, there will be an obstruction to having a trivial \( h \)-cobordism and hence to a consistent relation to heterotic M-theory. Therefore, it is an interesting problem to compute the Whitehead torsion of the quintic.

Recall from the end of section \(^{[3.1]} \) that the full torsion subgroup of the Whitehead group is given by \( SK_1(\mathbb{Z}[\Gamma]) \). Therefore, one way to tell that \( \text{Wh}(\Gamma) \) is nontrivial is to detect torsion via \( SK_1(\mathbb{Z}[\Gamma]) \).

**Products of abelian groups.** We now consider products of abelian groups, in particular of cyclic groups.

1. **Products of groups of even order.** For even order, we have already seen that the Whitehead group of the lowest rank non-simple group, \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), is zero. Next we consider products of \( \mathbb{Z}_2 \) with \( \mathbb{Z}_4 \) and so on. We use the following two general formulae \(^{[48]} \) for the torsion part of the Whitehead group \( SK_1(\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_{2^n}]) \equiv \left[ \oplus_{r=1}^{k} \left( \mathbb{Z}/r \right) \cdot (\mathbb{Z}_{2^r-1}) \right] \oplus \left[ \oplus_{s=2}^{n} (\mathbb{Z}_{2^s}) \right] \) and \( SK_1(\mathbb{Z}[\mathbb{Z}_2^2 \times \mathbb{Z}_{2^n}]) \equiv \mathbb{Z}_2^{n-1} \). For instance, the following cases can then be deduced:

   1. \( SK_1(\mathbb{Z}[\mathbb{Z}_4 \times \mathbb{Z}_2]) \equiv \mathbb{Z}_2 \).
   2. \( SK_1(\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4]) \equiv \mathbb{Z}_2 \).
   3. \( SK_1(\mathbb{Z}[\mathbb{Z}_2^2 \times \mathbb{Z}_4]) \equiv (\mathbb{Z}_2)^3 \times \mathbb{Z}_4 \). This last case is curious in that \( \text{Wh}(\Gamma) = \Gamma \).

   We can also use the general formula \( SK_1(\mathbb{Z}[\mathbb{Z}_4 \times \mathbb{Z}_{2^n}]) \equiv (\mathbb{Z}_2)^{(a-1)} \) to deduce other relevant groups. For example, \( SK_1(\mathbb{Z}[\mathbb{Z}_4 \times \mathbb{Z}_8]) \equiv (\mathbb{Z}_2)^2 \), \( SK_1(\mathbb{Z}[\mathbb{Z}_4 \times \mathbb{Z}_{16}]) \equiv (\mathbb{Z}_2)^3 \), \( SK_1(\mathbb{Z}[\mathbb{Z}_4 \times \mathbb{Z}_{32}]) \equiv (\mathbb{Z}_2)^4 \), etc.

2. **Products of groups of odd order.** Next we consider the case when the orders of the groups in the products are odd. We will look at groups of the form \( (\mathbb{Z}_p)^k \), \( \mathbb{Z}_p^2 \times \mathbb{Z}_p^m \), and \( \mathbb{Z}_p^2 \times \mathbb{Z}_p^n \), as well as combinations involving three factors, using general results from reference \(^{[48]} \).

   (i) The torsion subgroup \( SK_1(\mathbb{Z}[\Gamma]) \) is trivial if \( \Gamma \) is cyclic or an elementary 2-group, or of type \( \mathbb{Z}_p \oplus \mathbb{Z}_p^n \). However, \( SK_1(\mathbb{Z}[\Gamma]) \) is nontrivial form most abelian groups \(^{[2]} \). If \( \Gamma = (\mathbb{Z}_p)^k \), \( p \) odd, then \( SK_1(\mathbb{Z}[\Gamma]) \) is a \( \mathbb{Z}_p^r \)-vector space of dimension \( (p^k - 1)/(p - 1) - (p^k+1)/p \). For example, for \( \Gamma = (\mathbb{Z}_3)^3 \), the torsion subgroup is \( SK_1(\mathbb{Z}[\mathbb{Z}_3^3]) \equiv (\mathbb{Z}_3)^3 \).

   (ii) For \( p \) an odd prime, \( SK_1(\mathbb{Z}[\mathbb{Z}_p^2 \times \mathbb{Z}_p^m]) \equiv (\mathbb{Z}/p)^{(p-1)(n-1)} \).

   (iii) For \( p \) an odd prime, \( SK_1(\mathbb{Z}[\mathbb{Z}_p^2 \times \mathbb{Z}_p^n]) \equiv (\mathbb{Z}/p)^{np(n-1)/2} \).

Let \( p \) be an odd prime and \( \Gamma \) an elementary abelian \( p \)-group of rank \( k \). Then \(^{[63]} \) \( SK_1(\mathbb{Z}[\Gamma]) \) is an elementary abelian \( p \)-group of rank \( (p^k - 1)/(p - 1) - (p^{k+1})/p \). In particular \( SK_1(\mathbb{Z}[\Gamma]) \neq 0 \) for \( k \geq 3 \). For example, the following table can be formed (see also \(^{[63]} \))

\[
\begin{array}{|c|c|}
\hline
\Gamma & SK_1(\mathbb{Z}[\Gamma]) \\
\hline
\mathbb{Z}_p^2 \times \mathbb{Z}_p^2 & (\mathbb{Z}/p)^{p-1} \\
\mathbb{Z}_p^2 \times \mathbb{Z}_p \times \mathbb{Z}_p & (\mathbb{Z}/p)^{p(p-1)} \\
\mathbb{Z}_p \times \mathbb{Z}_3 & (\mathbb{Z}_3)^4 \\
\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 & (\mathbb{Z}_9)^9 \\
\mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_9 & (\mathbb{Z}_9)^{10} \times (\mathbb{Z}_9)^2 \\
\hline
\end{array}
\]

**Nonabelian groups.** We have already seen examples of nonabelian groups in section \(^{[3.1]} \). In addition,

1. **Crystallographic groups.** \( SK_1(\mathbb{Z}[\Gamma]) = 0 \) for \( \Gamma \) a dihedral, the binary tetrahedral or icosahedral group \(^{[43]} \) \(^{[63]} \).
2. The quaternion group. The Whitehead group \( \text{Wh}(\mathbb{Z}[Q_8]) \) of the quaternion group \( Q_8 \) of order 8 is isomorphic to \( \pm V \), where \( V = \mathbb{Z}_2 \times \mathbb{Z}_2 \) is Klein’s 4-group \([37]\). Note that \( V \) is the factor group \( Q_8/\{\pm\} \), where \( \{\pm\} \) is the commutator subgroup of \( Q_8 \).

3. Products with abelian groups. If \( \Gamma \) is any (nonabelian) quaternion or semidihedral 2-group, then for all \( k \geq 0 \), the torsion subgroup is \( SK_1(\mathbb{Z}[\Gamma \times (\mathbb{Z}_2)^k]) \cong (\mathbb{Z}_2)^{2^k-k-1} \).

4. Nonabelian groups with specified abelianization. For instance, if for order \( |\Gamma| = 16 \) the torsion subgroup is given by

\[
SK_1(\mathbb{Z}[\Gamma]) \cong \begin{cases} 
1 & \text{if } \Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\mathbb{Z}_2 & \text{if } \Gamma^{ab} \cong \mathbb{Z}_4 \times \mathbb{Z}_2.
\end{cases}
\] (4.4)

Finding Whitehead groups via transfer. Looking at inclusions tells us about the corresponding Whitehead groups. We will consider several situations.

1. Consider the cyclic group \( \mathbb{Z}_{2k+1} \) of order \( 2k + 1 \) as a subgroup of the cyclic group \( \mathbb{Z}_{4k+2} \) of order \( 4k + 2 \). Then the transfer \( i^* : \text{Wh}(\mathbb{Z}_{4k+2}) \to \text{Wh}(\mathbb{Z}_{2k+1}) \), corresponding to \( i : \mathbb{Z}_{2k+1} \to \mathbb{Z}_{4k+2} \), is onto for all \( k \) \([11]\).

2. Now consider the inclusion \( i : \mathbb{Z}_{2k} \to \mathbb{Z}_{2k} \oplus \mathbb{Z}_2 \). Then the transfer \( i^* : \text{Wh}(\mathbb{Z}_{2k} \oplus \mathbb{Z}_2) \to \text{Wh}(\mathbb{Z}_{2k}) \) is onto if and only if \( k = 1, 2 \) or 3 \([12]\). Since \( \text{Wh}(\mathbb{Z}_{2k}) = 0 \) for \( k = 1, 2 \) and 3, this means that \( \text{Wh}(\mathbb{Z}_{2k} \oplus \mathbb{Z}_2) \) is trivial for these values of \( k \).

3. Now let \( \Gamma \) be a finite abelian group of odd order. Then \( i^* : \text{Wh}(\Gamma \oplus \mathbb{Z}_2) \to \text{Wh}(\Gamma) \) is onto \([12]\). This then can tell us whether \( \Gamma \oplus \mathbb{Z}_2 \) is trivial from whether or not the Whitehead group of \( \Gamma \) itself is trivial.

In general, if \( \Gamma \to \Gamma' \) is a surjection of finite abelian groups induces a surjection \( SK_1(\mathbb{Z}[\Gamma]) \to SK_1(\mathbb{Z}[\Gamma']) \) \([2]\).

Semidirect products. For finite \( \Gamma \), the torsion subgroup of the Whitehead group is trivial \( SK_1(R[\Gamma]) = 1 \) for all rings of integers in number fields if and only if \( \Gamma \) is a semidirect product of two cyclic groups of relatively prime orders \([1]\). In general, we can determine the ranks of the (torsion-free part) of these groups using Bass’ theorem.

Given the above rules and results, it is a straightforward exercise to find the Whitehead groups of the fundamental groups appearing in the literature of model building (reviewed partially in the introduction). This includes, for instance, the groups appearing in \([10]\).

Whitehead torsion. The approach in this paper can also guide us to anticipate conditions on cobordisms when constructing Calabi-Yau threefolds with fundamental groups of certain types. Recall that just because the Whitehead group is nontrivial does not mean that the particular element, the Whitehead torsion, is a nontrivial element. That is, one still has to compute the Whitehead torsion (geometrically), which we do not do here. We consider examples where elements in the torsion subgroup of the Whitehead group can be explicitly characterized (see \([48]\)).

(i) For \( \Gamma = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g \rangle \langle h_1 \rangle \langle h_2 \rangle \), the torsion subgroup is \( SK_1(\mathbb{Z}[\Gamma]) \cong \mathbb{Z}/2 \), and the nontrivial element is represented by the matrix

\[
\begin{bmatrix}
1 + 8(1 - g^2)(1 + h_1)(1 + h_2)(1 - g) & -(1 - g^2)(1 + h_1)(1 + h_2)(3 + g) \\
-13(1 - g^2)(1 + h_1)(1 + h_2)(3 - g) & 1 + 8(1 - g^2)(1 + h_1)(1 + h_2)(1 + g)
\end{bmatrix} \in \text{GL}(2, \mathbb{Z}[\Gamma]).
\] (4.5)

In this case, one would have to check for a given \( h \)-cobordism built out of \( Y^{11} \) and \( M^{10} \) whether the corresponding Whitehead torsion is the zero element or the nontrivial element represented by matrix \([13]\).

(ii) For \( \Gamma = \mathbb{Z}_4 \times Q_8 = \langle g \rangle \times \langle a, b \rangle \), where \( Q_8 \) is a quaternion group of order 8, the torsion subgroup is \( SK_1(\mathbb{Z}[\Gamma]) \cong \mathbb{Z}/2 \), and the nontrivial element is represented by the unit

\[
1 + (2 - g - g^2)(1 - a^2)(3g + a + 4g^2a + 4(g^2 - g)b + 8ab) \in (\mathbb{Z}[\Gamma])^*.
\] (4.6)
Again, one would check the geometry to see which of the two elements one gets.

It would be very interesting to calculate the Whitehead torsion explicitly for interesting classes of non-simply connected Calabi-Yau manifolds. As far as we know, no such calculations exist. One approach could be to find an explicit Morse function (which seems not easy).

5 Dynamical aspects

In this section we consider some dynamical aspects of heterotic M-theory as they arise in connection to the Whitehead group and Whitehead torsion. We consider the effect of diffeomorphisms as well as orientation characters in section 5.1 and then consider dynamical constraints on general compactifications in heterotic M-theory in section 5.2.

5.1 Automorphisms

Diffeomorphism. We study the effect of diffeomorphisms on our cobordisms, starting with a visible sector $M^{10}_0$. Two eleven-dimensional cobordisms $(Y^{11}; M^{10}_0, f_0, M^{10}_1, f_1)$ and $(Y'^{11}; M'^{10}_0, f'_0, M'^{10}_1, f'_1)$ over $M^{10}_0$ are diffeomorphic relative $M^{10}_0$ if there is an orientation preserving diffeomorphism $F : Y^{11} \to Y'^{11}$ such that $F \circ f_0 = f'_0$. Indeed in [29] the quantum integrand in the M-theory effective action is shown to be invariant under the group of Spin diffeomorphisms of $Y^{11}$ which act freely on the space of metrics. On the other hand, the effective action of the heterotic string is invariant under diffeomorphisms $\varphi : M^{10} \to M^{10}$ which lift to the Spin bundle and to the $E_8$ vector bundles [67]. The global anomaly is absent for arbitrary choices of the Spin $M^{10}$ and the two $E_8$ vector bundles.

In addition to the many examples that we have considered so far, one might be able to generate others using diffeomorphism. In a sense, constructing manifolds with cobordisms for which the Whitehead torsion is nontrivial would be easier than calculating the Whitehead torsion for a fixed cobordism. The idea is to take a cobordism and and glue it to another after a ‘twist’ via an automorphism, i.e. a diffeomorphism in our case. This may give rise to a nonzero Whitehead torsion. This requires the study of the mapping torus as is done with the global anomalies in the heterotic effective action, e.g. in [67].

Scale and intervals. In the discussion so far we have used unit intervals $[0, 1]$ to characterize the cobordism. In the physical set-up of Horava-Witten [35] [36] we have a length scale imposed by the dynamics in the theory. In the above formulation, we can introduce this length scale by simply replacing the unit interval by the interval $[0, L]$ or $[-L, L]$, with $L$ the (dynamical) length in the eleventh direction.

Manifolds with non-positive sectional curvature. It is interesting to note that $\text{Wh}(\Gamma)$ is trivial for $\Gamma$ the fundamental group of closed manifolds with all the sectional curvatures $\leq 0$ [28]. Therefore, although not Calabi-Yau (see [24] Theorem 2.3), such spaces are admissible for $s$-cobordism (see [39]).

The Whitehead torsion relative to left vs. right boundary. We ask whether it makes a difference to take the Whitehead torsion relative to the left boundary vs. taking it relative to the right boundary. There is a duality theorem which relates the Whitehead torsion relative to one boundary to that of the second boundary [47]. For any orientable $h$-cobordism $(Y^{11}, M^{10}, M'^{10})$ we have the relation between $\tau(Y^{11}, M^{10})$ and $\tau(Y^{11}, M'^{10})$ as

$$\tau(Y^{11}, M'^{10}) = \tau(Y^{11}, M^{10})$$

where $\tau$ is the conjugate of $\tau$, defined as follows. If $a = \sum n_i \gamma_i$ is an element of $\mathbb{Z}[\Gamma]$, with $n_i \in \mathbb{Z}, \gamma_i \in \Gamma$, then the conjugate of $a$ is the element $\sum n_i \gamma_i^{-1}$. This conjugation operation is an anti-automorphism of the group ring with corresponding automorphism on $GL(\mathbb{Z}[\Gamma])$ given by sending each matrix to its conjugate transpose. Passing to the abelianized group $K_1(\mathbb{Z}[\Gamma])$ gives an automorphism and hence an automorphism also of the quotient $\text{Wh}(\Gamma)$. We see that ‘reversing’ the direction of the cobordism, that is taking $M'^{10}$ to
In our current context. Nevertheless, next we provide an explanation of this problem it uses boundary conditions which lead to a Bianchi identity for the C-field which is different from the one in [36]. We should keep these subtleties in mind when dealing with bundles, which are always there.

Remark on the $E_8$ gauge bundles. General boundary conditions for M-theory on a manifold with boundary are considered in [22, 53]. The left and right boundaries in heterotic M-theory each carries an $E_8$ bundle which, in the process of model building is desired to be broken down to a realistic group. Each of the two bundles is characterized with a degree four characteristic class, $a_L$ for left and $a_R$ for right. As explained in [22], when $a_L = a_R$ then the eleven-dimensional spacetime provides a homotopy of the left and right connections so that the $E_8$ bundles on the boundaries necessarily have $a_L = a_R$, which is the case in the non-supersymmetric model in [27]. However, in (the supersymmetric) Horava-Witten theory, $a_L + a_R = \frac{1}{2} p_1(\gamma^{11})$. In order to overcome this difficulty, the authors of [22] give a parity-invariant formulation of the C-field in M-theory by passing from $\gamma^{11}$ to $\gamma_d^{11}$, the orientation double cover of $\gamma^{11}$, and defining the C-field to be a parity invariant $E_8$ cocycle on $\gamma_d^{11}$. This is done via a nontrivial deck transformation $\sigma$ on $\gamma_d^{11}$, so that a parity-invariant $E_8$ cocycle is one for which the differential character corresponding to the C-field satisfies $\sigma^*(\overline{C}) = [\overline{C}]^P$, where the action of the parity $P$ is $[\overline{C}]^P = [\overline{C}]^*$. While this solves the parity problem it uses boundary conditions which lead to a Bianchi identity for the C-field which is different from the one in [30]. We should keep these subtleties in mind when dealing with bundles, which are always there (but we do not directly deal with them in this paper). Nevertheless, next we provide an explanation of this in our current context.

Orientation characters and twisted group algebras of the fundamental group. The orientation character $\omega(M_{10}^0): \pi_1(M_{10}^0) \to \mathbb{Z}_2 = \{\pm1\}$ sends a loop $\gamma: S^1 \to M_{10}^0$ to $\omega(\gamma) = +1$ (respectively, -1) if $\gamma$ is orientation-preserving (respectively, orientation-reversing). Thus, in the oriented case $\omega(e) = +1$ for all $e \in \Gamma$, that is $\omega$ is trivial if and only if $M_{10}^0$ is orientable. This has the following effect on the integral group ring of the fundamental group. The orientation character defines a twisted involution (an anti-automorphism) on the group ring $\mathbb{Z}[\Gamma]$ given by $a \mapsto \omega(a)a^{-1}$, i.e. $\pm a$ according to whether $a$ is orientation preserving or reversing. The resulting group ring is denoted $\mathbb{Z}[\Gamma]^{\omega}$.

Let us consider this in more detail. An involution on $\mathbb{Z}[\Gamma]$ is a function $\mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$, taking an element $a$ to an element $\overline{a}$ satisfying: $[a + b] = \overline{a} + \overline{b}$, $[ab] = \overline{b} \cdot \overline{a}$, $[a] = a$, and $1 = 1 \in \mathbb{Z}[\Gamma]$. This gives rise to the $\omega$-twisted involution on $\mathbb{Z}[\Gamma]$, defined as the map from $\mathbb{Z}[\Gamma]$ to $\mathbb{Z}[\Gamma]$ given by
\[
a = \sum_{\gamma \in \Gamma} n_\gamma \gamma \mapsto \overline{a} = \sum_{\gamma \in \Gamma} \omega(\gamma) n_\gamma \gamma^{-1}, \quad n_\gamma \in \mathbb{Z}. \tag{5.2}
\]

In this case we have to use $\omega$-twisted cohomology and fundamental class in evaluating expressions in the theory. Starting from the cellular $\mathbb{Z}[\pi_1(M_{10}^0)]$-module chain complex $C(M_{10}^0)$, the $\omega(M_{10}^0)$-twisted involution on $\mathbb{Z}[\pi_1(M_{10}^0)]$ can be used to define the left $\mathbb{Z}[\pi_1(M_{10}^0)]$-module structure on the dual cochain complex $C(M_{10}^0) = \text{Hom}_{\mathbb{Z}[\pi_1(M_{10}^0)]}(\mathbb{Z}[\pi_1(M_{10}^0)], [\overline{C}(\tilde{M}_{10}^0), \mathbb{Z}[\pi_1(M_{10}^0)])$. When $M_{10}^0$ is compact, the fundamental class is given by $[\tilde{M}_{10}^0] \in H_{10}(M_{10}^0, \mathbb{Z}_{\omega}(M))$ such that the cap product defines $\mathbb{Z}[\pi_1]$-module isomorphisms
\[
[M_{10}^0] \cap - : H^*_{\omega(M)}(\tilde{M}_{10}^0) \xrightarrow{\approx} H_{10-\ast}(\tilde{M}_{10}^0) \tag{5.3}
\]
with $\tilde{M}_{10}^0$ the universal cover of $M_{10}^0$. Quantities, e.g. ones appearing the effective action and the corresponding partition function, should be formulated using this fundamental class.

\[\dagger M_{10}^0\] does not necessarily have to be a manifold, but just a Poincaré duality complex.
There is a duality formula for the Whitehead torsion which takes into account the orientation character. Let $Y^{11}$ be an eleven-dimensional $h$-cobordism and let $\omega: \Gamma \to \{\pm 1\}$ be the orientation character. This gives rise to an anti-involution on the integral group ring $\mathbb{Z}^*$. Then Milnor’s duality formula is cast as

$$\tau(Y^{11}, M^{10}) = \tau(Y^{11}, M^{10})^*. \quad (5.4)$$

**Effect on F-theory.** Recently there has been a lot of research activity in model building using F-theory (see [21] and references therein). F-theory can be considered as a limit of M-theory on a 2-torus when the volume of the two-torus becomes very small. This means that constraints on the possible fundamental groups in [21]. Therefore, we expect that our discussion in the heterotic/M-theory setting will have, via duality, consequences for fundamental groups in F-theory. This is strengthened by the fact that in a class of models which admit perturbative heterotic duals, the F-theory and heterotic computations match [20]. It would be interesting to perform explicit checks of this in relevant examples.

### 5.2 Compactification

We have considered in general the relation between M-theory on a general eleven-manifold and heterotic string theory on a general ten-manifold $M_0^{10}$. There are two aspects to this. First, for consistency the theory should make sense on any admissible manifold and so studying this might give insight into understanding the theory further. Second, there are certain favorable types of spaces for model building. We have in mind that $M_0^{10}$ is a product (or a bundle) of a Calabi-Yau threefold $X^6$ with a four-dimensional spacetime. In general, the latter can be taken to be a general four-manifold that solves the equations of motion and does not break supersymmetry according to the goal one has in mind. It can be taken to be flat Minkowski or something close. We study such situations in this section and consider whether the choice of four-dimensional spacetime changes the discussion we have had so far.

We take M-theory on an eleven-manifold $Y^{11} = Z^7 \times N^4$, where $N^4$ is spacetime and $Z^7$ is a seven-dimensional cobordism of the Calabi-Yau threefold $X^6$. This always exists because the Stiefel-Whitney numbers of a Calabi-Yau threefold are zero: $w_1 = 0$ because of orientation, $w_2 = 0$ because of Spin, and $w_3 = 0$ because both $w_1$ and $w_2$ are zero; then the Stiefel-Whitney numbers $w_1 w_5[X^6], w_2 w_4[X^6]$, and $w_3 w_5[X^6]$ are all zero. The heterotic ten-manifold is of the form $M_0^{10} = X^6 \times N^4$.

**The $h$-cobordism of a product.** Let $(Z^7; X^6_0, X^6_i)$ be a seven-dimensional $h$-cobordism for the Calabi-Yau threefold $X^6_0$, and let $N^4$ be a closed four-manifold. Then we can form an eleven-dimensional $h$-cobordism $(Z^7 \times N^4; X^6_0 \times N^4, X^6_i \times N^4)$. From the cut and paste properties of the Whitehead torsion (see [47] [63] [89]), we get that the torsion are related as follows

$$\tau(Z^7 \times N^4, X^6_0 \times N^4) = \tau(Z^7, X^6_0) \chi(N^4), \quad (5.5)$$

where $\chi(N^4)$ is the Euler characteristic of $N^4$. Thus the value of this invariant will determine whether there we can relate the discussion of torsion in eleven/two dimensions to that in seven/six dimensions. The former is the global picture we have built so far, and the latter correspond to the actual situation studied in model building, that is the fundamental groups appearing as examples are those of $X^6$ and not (necessarily) of $M_0^{10}$.

If spacetime were compact and odd-dimensional then the Euler characteristic would vanish identically. In that case, the torsion would vanish. For example, if we take spacetime to be the circle $S^1$ then $Z^7 \times S^1 \approx X^6_0 \times S^8 \times [0,1] \approx X^6_0 \times S^1 \times [0,1]$, i.e. the torsion vanishes. In particular, this gives $X^6_0 \times S^1 \approx X^6_1 \times S^1$. 

13
Product with a torus and Wall’s finiteness obstruction. The circle $S^1$ has fundamental group $\pi_1(S^1) \cong \mathbb{Z}$. If we consider the product $S^1 \times Y$, then what is the corresponding Whitehead group in terms of that of the factors? There is in fact a direct sum decomposition $\text{Wh}(Z \times Y) \cong \text{Wh}(Z) \oplus K_0(\mathbb{Z}[\pi_1(X)]) \oplus N$, for some Nil-group $N$. For the 2-torus with fundamental group $\mathbb{Z}^2$, the process can be repeated. It might seem that for this product we can have nonzero Whitehead group for the product manifold even though that group for the factors might not be zero. However, elements in $K_0(\mathbb{Z}[\pi_1(X)])$, called Wall’s finiteness obstruction, detects whether or not $X^k$ has the homotopy type of a CW-complex. If we are within the category of such spaces then this element within the class group vanishes.

Spacetime with flat structure. A manifold admits a flat structure if the tangent bundle is isomorphic to a flat vector bundle, i.e., admits a flat connection. Even for such manifolds, one can have nonzero Euler-characteristic. For example, if we take the connected sum $N^4 = (\Sigma_3 \times \Sigma_3) \#_i^6 (S^1 \times S^3)$, where $\Sigma_3$ is a surface of genus 3. The product $\Sigma_3 \times \Sigma_3$ is almost parallelizable and the product of spheres $S^1 \times S^3$ is parallelizable. Then the Euler characteristic is $\chi(N^4) = 4$ (see [60]). In this example, the fundamental group is the free product $\pi_1(N^4) = \Gamma_1 \ast \Gamma_2$, where $\Gamma_1$ is the direct product of two copies of a non-abelian surface group and $\Gamma_2$ is of rank 6. In fact, $S^1 \times S^3$ can be replaced by any parallelizable four-manifold.

Compact vs. noncompact spacetime. So far we have taken $N^4$ to be compact. For compact manifolds, the existence of a smooth Lorentzian metric is equivalent to the manifold having a vanishing Euler characteristic (see [62]). However, the situation gets modified in the presence of singularities (see [44]). What if it is not compact? Noncompact spacetimes are more desirable for the purpose of equipping spacetime with a Lorentzian structure; all noncompact manifolds admit a Lorentzian metric. On the other hand, every noncompact manifold admits vector fields with any specified set of isolated zeros. This suggests that noncompact manifolds with nonzero (appropriate notion of) $Euler$ characteristic are abundant. Note that for noncompact Riemann surfaces, the Cohn-Vossen theorem gives the inequality $\int_{\Sigma} K dA \leq 2\pi\chi(\Sigma)$ (see e.g. [40]). In general one works with $L^2$-Euler characteristics. For example, the Euler characteristic of an Asymptotically Locally Euclidean (ALE) space corresponding to the Lie algebra of type $A_n$ is $n + 1$. It is important to note that it should be checked whether equation (6.5) extends to the noncompact case. Furthermore, strictly speaking, in the noncompact case we have to use the noncompact version of the s-cobordism theorem, for which the Whitehead torsion lives a new group, which fits into an exact sequence involving the Whitehead group and algebraic $K_0$, as well as information about the ends [59]. Some aspects of behavior of ends in M-theory are discussed in [54].

In the following few paragraphs we describe a way for studying the Whitehead torsion via other invariants, namely the Reidemeister torsion [47] and the Ray-Singer torsion [50]. This then provides a setting for making some direct connections to phenomenology.

Relation of the Whitehead torsion to Reidemeister torsion. The Whitehead torsion $\tau$ is closely related to Reidemeister torsion (or R-torsion) $\Delta$; the former generalizes the latter but is a more delicate invariant. Algebraically, the Whitehead torsion is more general than R-torsion in that it is also defined for noncommutative rings (such as the group ring of the fundamental group $\mathbb{Z}[\pi_1(X)]$) for which the determinant, needed for the R-torsion, is not defined. The R-torsion is a topological invariant which distinguishes spaces which are homotopy equivalent but not homeomorphic, and is defined for spaces whose fundamental group $\pi$ is finite and for which the homology with coefficients in a certain $\pi$-representation vanishes. The R-torsion is defined in more general situations than Whitehead torsion, since any homotopy equivalence is a homology equivalence. Furthermore, R-torsion has two advantages over the Whitehead torsion:

(i) It is more likely to be defined.

(ii) Its value is an honest real number, instead of being an element of a somewhat esoteric group. On the other hand, when defined, the Whitehead torsion is a sharper invariant. When they are both defined, the R-torsion is a function of the Whitehead torsion. That is, for each unitary (orthogonal) representation $\phi$ of $\pi_1(X)$, the Whitehead torsion vanishes identically if and only if the R-torsion vanishes identically. In general, one has a relation of the form $\tau(\phi) = \phi(\Delta)$ (see e.g. [40]).
\(\rho\) of the fundamental group \(\pi\), the R-torsion is the real part of the determinant of the complex (real) matrix induced by \(\rho\) from any matrix representation of the Whitehead torsion. One can find a useful criterion for when the Whitehead torsion is zero by studying the R-torsion. For concreteness, let \(h : \pi_1(M^{10}) \to O(n)\) be an orthogonal representation of the fundamental group \(\pi = \pi_1(M^{10})\). Then \(h\) extends to a unique homomorphism from the group ring \(\mathbb{Z}[\pi]\) to the ring \(\mathcal{M}_n(\mathbb{R})\) of all real \(n \times n\) matrices and determines a homomorphism \(h_* : \text{Wh}(\pi) \to K_1(\mathbb{R}) \cong \mathbb{R}^+\). Suppose that the Whitehead torsion \(\tau(Y^{11}; M^{10}) \in \text{Wh}(\pi)\) is defined and suppose that \(\pi\) is a finite group. Then it follows from the identity relating the two torsions \([47]\)

\[
\Delta_h(Y^{11}; M^{10}) = h_* \tau(Y^{11}; M^{10})
\]

that \(\tau(Y^{11}; M^{10})\) is an element of finite order in \(\text{Wh}(\pi)\) if and only if the R-torsion is \(\Delta_h(Y^{11}; M^{10}) = 1\) for all possible orthogonal representations \(h\) of \(\pi\). If \(\pi\) is finite abelian, then \(\tau(Y^{11}; M^{10}) = 0\) if and only if \(\Delta_h(Y^{11}; M^{10}) = 1\) for all possible such representations \(h\). Since the R-torsion is easier to calculate, this gives a concrete way of checking whether the Whitehead torsion vanishes without having to go through the difficult task of calculating it explicitly.

**Examples of when R-torsion is defined and the Whitehead torsion is not.** There are examples in which the Whitehead torsion cannot be defined but the R-torsion can (see [47]). For instance, the Whitehead torsion \(\tau(S^1)\) of the circle \(S^1\) cannot be defined since the module \(H_0(S^1)\) for the universal cover \(\hat{S}^1\) is not zero, and is not a free \(\mathbb{Z}[\pi]\)-module. On the other hand, the R-torsion is defined; if the homomorphism \(h\) from the fundamental group \(\pi_1(S^1)\) to the units \(\mathbb{F}^\times\) in a field \(\mathbb{F}\) maps a generator into the field element \(x \neq 1\), then the associated R-torsion \(\Delta_h(S^1) \in \mathbb{F}^\times/\pm h(\pi_1)\) is well-defined and equal to \(1 - h\), up to multiplication by \(h^m\) for some \(m \in \mathbb{Z}^\times\). Another example is a knot complement \(X\) in the 3-sphere with \(h : \pi_1(X) \to \mathbb{F}^\times\) mapping each loop with linking number \(+1\) into the field element \(x \neq 1\). Then the R-torsion is well-defined, and is equal to \((1 - h)/A(h)\), where \(A(h)\) is the Alexander polynomial of the knot.

**Effect on phenomenology.** The Ray-Singer torsion, which is an analytic analog of R-torsion and which coincides with it for Riemannian manifolds, has direct physical applications. The Ray-Singer torsion can be defined using determinants of Laplacians. In this form it has natural connection to one-loop amplitudes. For example, this torsion governs the threshold corrections for the heterotic string [6]. In M-theory compactifications on manifolds with \(G_2\) holonomy, the GUT scale \(\mathcal{M}_{\text{GUT}}\) is essentially given by the Ray-Singer torsion \(\Delta_{RS}(\Sigma)\) via \(\mathcal{M}^3_{\text{GUT}} = \Delta_{RS}(\Sigma)/V_\Sigma\), where \(V_\Sigma\) is the volume of the corresponding 3-cycle \(\Sigma\) [80]. For example, when \(\Sigma = S^3/\mathbb{Z}_q\) is a lens space, on which there is a Wilson line of eigenvalues \((e^{2\pi i(2m/q)}, e^{2\pi i(2m/q)}, e^{2\pi i(m/q)}, e^{-2\pi i(3m/q)}, e^{-2\pi i(3m/q)})\) with \(m\) and \(q\) coprime integers, then the Ray-Singer torsion for the lens space is \(\Delta_{RS}(\Sigma) = 4q\sin^2(5\pi m/q)\). Now, the more delicate Whitehead torsion can be partially studied by considering the R-torsion (or Ray-Singer torsion) as above. It should be an obstruction to supersymmetry in heterotic M-theory. The breaking scale would be the intermediate 5-dimensional scale, and only gravitationally mediate to the visible sector. It would be interesting to see how this works explicitly.

**Higher-dimensional compactifications.** If we take our eleven-manifold \(Y^{11}\) to be a product of two manifolds, where the internal manifold is of dimension lower than 6 then we can no longer apply the s-cobordism arguments we have been using. In particular, the s-cobordism theorem fails in dimensions five and it is an open problem in dimension four (see [17] [18]). For example, there exists an \(h\)-cobordism \((W^5, T^4, T^4)\), where \(T^4\) is the four-dimensional torus, for which there is no diffeomorphism from \(W^5\) to \(T^4 \times [0, 1]\). Since \(\text{Wh}(\pi_1(T^4)) = 0\), the s-cobordism indeed fails in five dimensions. For topological spaces, the theorem fails in both four and five dimensions [85]; one might say that we could apply the s-cobordism in this case to the spacetime part rather than the internal part, now that spacetime has grown to admissible dimensions. This certainly can be done and will give consistency conditions depending on fundamental groups of spacetime (the arguments we have outlined will go through with the obvious changes). However, we would then not be studying fundamental groups for purposes of particle physics but rather for purposes of cosmology.
Acknowledgement

The author would like to thank Jonathan Rosenberg for useful discussions on the Whitehead torsion and Kenji Fukaya and the referee for useful comments. He also acknowledges the hospitality of the Department of Physics and the Department of Mathematics at the National University of Singapore where part of this work was done.

References

[1] R. C. Alperin, R. K. Dennis, R. Oliver, and M. R. Stein, \( SK_1 \) of finite abelian groups II, Invent. Math. 87 (1987), no. 2, 253–302.

[2] R. C. Alperin, R. K. Dennis and M. R. Stein, The nontriviality of \( SK_1(\mathbb{Z}_\pi) \), in Orders, Group Rings and Related Topics, Lecture Notes in Math., vol. 353, Springer-Verlag, New York, 1973, 1–7.

[3] A. Bak, V. Bouchard, and R. Donagi, Exploring a new peak in the heterotic landscape, J. High Energy Phys. 06 (2010) 108, 1–31, [arXiv:0811.1242 [hep-th]].

[4] H. Bass, A. Heller, and R. Swan, The Whitehead group of a polynomial extension, Publ. de l’Inst. des Hautes Etudes Sci. 22 (1964) 61–79.

[5] V. Batyrev and M. Kreuzer, Integral cohomology and mirror symmetry for Calabi-Yau 3-folds, Mirror symmetry V, 255–270, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc., Providence, RI, 2006, math.AG/0505432.

[6] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (1994) 311–428, [arXiv:hep-th/9309140].

[7] L. Borisov and Z. Hua, On Calabi-Yau threefolds with large nonabelian fundamental groups, Proc. Amer. Math. Soc. 136 (2008), no. 5, 1549–1551, [arXiv:math/0609728 [math.AG]].

[8] V. Bouchard and R. Donagi, An SU(5) heterotic standard model, Phys. Lett. B633 (2006) 783–791, arXiv:hep-th/0512149.

[9] V. Bouchard and R. Donagi, On a class of non-simply connected Calabi-Yau threefolds, Comm. Numb. Theor. Phys. 2 (2008) 1–61, arXiv:0704.3096 [math.AG].

[10] V. Braun, On free quotients of complete intersection Calabi-Yau manifolds, arXiv:1003.3235 [hep-th].

[11] V. Braun, Discrete Wilson lines in F-theory, arXiv:1010.2520 [hep-th].

[12] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, The exact MSSM spectrum from string theory, J. High Energy Phys. 0605 (2006) 043, arXiv:hep-th/0512177.

[13] V. Braun, M. Kreuzer, B. A. Ovrut, and E. Scheidegger, Worldsheet instantons and torsion curves, part A: Direct computation, J. High Energy Phys. 0710 (2007) 022, arXiv:hep-th/0703182.

[14] V. Braun, B. A.Ovrut, T. Pantev, and R. Reinbacher, Elliptic Calabi-Yau threefolds with \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) Wilson lines, J. High Energy Phys. 0412 (2004) 062, arXiv:hep-th/0410055.

[15] J. D. Breit, B. A. Ovrut, and G. C. Segre, \( E_6 \) symmetry breaking in the superstring theory, Phys. Lett. B158 (1985) 33–39.

[16] P. Candelas and R. Davies, New Calabi-Yau manifolds with small Hodge numbers, arXiv:0809.4681 [hep-th].

[17] S. Cappell and J. Shaneson, On 4-dimensional s-cobordisms, J. Differential Geom. 22 (1985), no. 1, 97–115.
[18] W. Chen, *Smooth s-cobordisms of elliptic 3-manifolds*, J. Differential Geom. **73** (2006), no. 3, 413–490.

[19] M. M. Cohen, *A Course in Simple-Homotopy Theory*, GTM 10, Springer-Verlag, New York-Berlin, 1973.

[20] M. Davis and J.-C. Hausmann, *Aspherical manifolds without smooth or PL structure*, Lect. Notes in Math., Vol. 1370, Springer-Verlag, New York, 1989, pp. 135-142.

[21] F. Denef, *Les Houches lectures on constructing string vacua*, [arXiv:0803.1194 [hep-th]].

[22] E. Diaconescu, D. S. Freed and G. Moore, *The M-theory 3-form and E8 gauge theory*, Elliptic cohomology, 44–88, Cambridge Univ. Press, Cambridge, 2007, [arXiv:hep-th/0312069].

[23] R. Donagi, P. Gao, and M. B. Schulz, *Abelian fibrations, string junctions, and flux/geometry duality*, [arXiv:0810.5195 [hep-th]].

[24] R. Donagi, B. A. Ovrut, T. Pantev, and R. Reinbacher, *SU(4) instantons on Calabi-Yau threefolds with Z2 × Z2 fundamental group*, J. High Energy Phys. **0401** (2004) 022, [arXiv:hep-th/0307273].

[25] R. Donagi, B. A. Ovrut, T. Pantev, and D. Waldram, *Standard models from heterotic M-theory*, Adv. Theor. Math. Phys. **5** (2002) 93–137, [arXiv:hep-th/9912208].

[26] R. Donagi and M. Wijnholt, *Model building with F-theory*, [arXiv:0802.2969 [hep-th]].

[27] M. Fabinger and P. Horava, *Casimir effect between world-branes in heterotic M-theory*, Nucl. Phys. **B580** (2000) 243–263, [arXiv:hep-th/0002073].

[28] F.T. Farrell and L.E. Jones, *Topological rigidity for compact nonpositively curved manifolds*, Proc. Sympos. Pure Math. **54**, Part 3, American Mathematical Society, Providence, R.I. (1993), 229–274.

[29] D. S. Freed and G. W. Moore, *Setting the quantum integrand of M-theory*, Commun. Math. Phys. **263** (2006) 89–132, [arXiv:hep-th/0409135].

[30] T. Friedmann and E. Witten, *Unification scale, proton decay, and manifolds of G2 holonomy*, Adv. Theor. Math. Phys. **7** (2003) 577–617, [arXiv:hep-th/0211269].

[31] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, vol 2, Cambridge Univ. Press, Cambridge, 1988.

[32] M. Gross and S. Popescu, *Calabi-Yau threefolds and moduli of abelian surfaces I*, Compositio Math. **127** (2001), 169–228.

[33] Y.-H. He, *An algorithmic approach to heterotic string phenomenology*, Mod. Phys. Lett. A **25** (2010) 79–90, [arXiv:1001.2419 [hep-th]].

[34] G. Heier, S. S. Y. Lu, and B. Wong, *On the canonical line bundle and negative holomorphic sectional curvature*, Math. Res. Lett. **17** (2010) 1101–1110.

[35] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** (1996) 506–524, [arXiv:hep-th/9510209].

[36] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl. Phys. **B475** (1996) 94–114, [arXiv:hep-th/9603142].

[37] M. E. Keating, *On the K-theory of the quaternion group*, Mathematika **20** (1973) 59–62.

[38] M. Kervaire, *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. **40** (1965) 31–42.

[39] M. Kreck and W. Lück, *The Novikov Conjecture: Geometry and Algebra*, Birkhäuser, Basel, 2005.
[40] W. Kühnel, Differential Geometry: Curves– Surfaces– Manifolds, Amer. Math. Soc., Providence, RI, 2006.

[41] K. W. Kwun, Transfer homomorphisms of Whitehead groups of some cyclic groups, Amer. J. Math. 93 (1971) 310–316.

[42] K. W. Kwun, Transfer homomorphisms of Whitehead groups of some cyclic groups II, Lecture Notes in Math. 298, 437–440, Springer, Berlin, 1972.

[43] B. Magurn, SK$_1$ of dihedral groups, J. Algebra 51 (1978), no. 2, 399–415.

[44] L. Markus, Line element fields and Lorentz structures on differentiable manifolds, Ann. of Math. (2) 62 (1955), 411–417.

[45] B. Mazur, Relative neighborhoods and the theorems of Smale, Ann. of Math. (2) 77 (1963) 232–249.

[46] J. Milnor, Lectures on the h-cobordism theorem, Princeton Univ. Press, Princeton, NJ, 1965.

[47] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc. 72 (1966) 358–426.

[48] R. Oliver, Whitehead groups of finite groups, Cambridge Univ. Press, Cambridge, 1988.

[49] R. Rares, On the topology and differential geometry of Kähler threefolds, PhD Dissertation, Stony Brook University, 2005.

[50] D. B. Ray and I. M. Singer, Analytic torsion for complex manifolds, Ann. Math. (2) 98 (1973) 154–177.

[51] J. Rosenberg, Algebraic K-theory and its Applications, Springer-Verlag, New York, 1994.

[52] H. Sati, Geometry of Spin and Spin$^c$ structures in the M-theory partition function [arXiv:1005.1700 [hep-th]].

[53] H. Sati, Duality and cohomology in M-theory with boundary, [arXiv:1012.4495 [hep-th]].

[54] H. Sati, Corners in M-theory, J. Phys. A44 (2011) 255402, [arXiv:1101.2793 [hep-th]].

[55] C. Schoen, On fiber products of rational elliptic surfaces with section, Math. Zeitschrift 197(2) (1988) 177–199.

[56] M. B. Schulz, Calabi-Yau duals of torus orientifolds, J. High Energy Phys. 0605 (2006) 023, [arXiv:hep-th/0412270].

[57] V. V. Sharko, Functions on Manifolds: Algebraic and Geometric Aspects, American Mathematical Society, Providence, RI, 1993.

[58] L. C. Siebenmann, Disruption of low-dimensional handlebody theory by Rohlin’s theorem, Topology of Manifolds, 57–76, Markham, Chicago, Ill, 1970.

[59] L. C. Siebenmann, Infinite simple homotopy types, Indag. Math. 32 (1970) 479–495.

[60] J. Smillie, Flat manifolds with non-zero Euler characteristic, Comment. Math. Helv. 52 (1977), no. 3, 453–455.

[61] J. Stallings, On infinite processes leading to differentiability in the complement of a point, in Differential and Combinatorial Topology, 245–254, Princeton Univ. Press, Princeton, NJ, 1965.

[62] N. Steenrod, The Topology of Fiber Bundles, Princeton Univ. Press, Princeton, NJ, 1951.

[63] M. R. Stein, Whitehead groups of finite groups, Bull. Amer. Math. Soc. 84 (1978) 201–212.
[64] C. T. C. Wall, *Norms of units in group rings*, Proc. London Math. Soc. 29 (1974), 593–632.

[65] S. Weinberger, The Topological Classification of Stratified Spaces, Univ. of Chicago Press, Chicago, Ill 1994.

[66] E. Witten, *Symmetry breaking patterns in superstring models*, Nucl. Phys. B258 (1985) 75–100.

[67] E. Witten, *Topological tools in 10-dimensional physics*, Int. J. Mod. Phys. A 1 (1986) 39–64.