Multiplicity of the Spiral Roll State in Heat Convection between Non-Rotating Concentric Double Spherical Boundaries

Takahiro Ninomiya, Keito Konno, Masako Sugihara-Seki, and Tomoaki Itano

Department of Pure and Applied Physics, Faculty of Engineering Science, Kansai University, Osaka, 564-8680, Japan

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The single-arm spiral roll state in the system of Boussinesq fluid confined between non-rotating double concentric spherical boundaries with an opposing temperature gradient previously reported by Zhang et al. (2002) [1] has been numerically investigated in detail. It is presented that the state can exist even in a relatively thicker gap rather than that used in our previous report [2].

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Introduction.— Recent global seismological observation has established the existence of fluid core at the depth of our planet, the earth which is exhibiting a lot of unsolved phenomena [3]. This evidence now practically provides us the convincing grounds that thermal convection in the core is the prime mover inside the apparently solid but pulsating planet, whereas the direct measurement of convection is still beyond our reach. While some more realistic configurations could be examined to elucidate the mechanism of the secular variations of terrestrial geomagnetic field [4], the variety and beauty of its possible flow patterns exhibited even by the simpler model has been only perceived by a few pioneers from the viewpoint of mathematical science. One of prototypes which allow us to model the convection is the system of Boussinesq fluid confined between non-rotating double concentric spherical boundaries with an opposing temperature gradient, which has been extensively investigated for recent several decades since Chandrasekhar’s introduction [5].

In this system, at the transition from the thermal conductive to convective states, the unstable modes emerge simultaneously due to the spherical symmetry inherited in the system, so that either axisymmetric or relatively highly-symmetric convective states invariant under a set of point groups may bifurcate from the static state via some nonlinear interactions [6, 7]. Although it has been so far intuitively believed that only such highly-symmetric states are to be predominant rather than any other less-symmetric equilibrium or asymmetric chaotic state around the transitional stage, a series of recent numerical studies [1, 8] elucidated that a less-symmetric equilibrium state, single-arm spiral roll state, coexists with other equilibrium states and is moreover stable. In what follows, we will disclose a few interesting but unforeseen aspects on the spiral roll state, which has remained obscure from lack of its detail investigation.

Formulation.— Firstly following Chandrasekhar [3], under the assumption that the distribution of internal heat generation inside the outer spherical boundary is uniform, we obtain the nondimensionalised form of the governing equations of the system,

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + Ra e_r \Theta + \nabla^2 u,$$

$$Pr(\partial_t \Theta + u \cdot \nabla \Theta) = ru_r + \nabla^2 \Theta,$$

where $Ra$ and $Pr$ are Rayleigh and Prandtl numbers, and $u$, $p$, $\Theta$ are the flow, pressure and temperature deviation fields, respectively. Non-slip and isothermal boundary conditions are imposed at the concentric inner and outer spherical boundaries, and the half width of the gap is taken to be unity via our non-dimensionalization. Thus, the system is determined only by the three control parameters, $Ra$, $Pr$ and the radius ratio of the inner to outer boundaries, $\eta$ ($0 < \eta < 1$). The flow field, $u$, should be determined by toroidal $\Psi$ and poloidal $\Phi$ components with regard to the radial direction due to incompressibility constraint. However we will hereafter neglect the former, which is to vanish in non-rotating case, because...
the absence of the energy source term to sustain it.

Linear stability of static state.— The neutral curves of the static conductive state on the map of \((\eta, Ra)\) is shown in Fig. 1, which is obtained by solving the eigenvalue problem deduced via the liberalisation from the aforementioned governing equations. The conductive state loses its stability regardless of the value of \(Pr\) over a curve specified by the degree, \(l\), against an infinitesimal superimposed perturbation composed by a set of \(2l + 1\) degenerated spherical harmonic functions of a degree \(l\) but different azimuthal order, that is, \(Y_{l}^{-i}, Y_{l}^{i+1}, \ldots, Y_{l}^{l}\). It should be noted that, at the limit of the narrow gap, our system is identical to the so-called Rayleigh-Bénard convection in a planar layer between a couple of horizontal plates with infinite extent. As a spherical harmonic function of degree \(l\) may represents \(l\) eddies in the meridian section from the north to the south poles, the length of a meridian line is approximated to the product of the gap width by the number of the eddies. This estimation gives us an approximation on the most dangerous degree \(l_{cr}\) as the function of \(\eta\) at the limit \(\eta \to 1\), \(l_{cr}(\eta) \approx \sqrt[4]{\frac{1+\eta}{1-\eta}}\). As a result, a higher degree spherical harmonic function is associated with the characteristic convective state at higher \(\eta\).

Existence of spiral roll state in wide gap.— The triangle symbol plotted in the figure indicates the position at which a single-arm spiral roll state was confirmed to be stable for a great variety of values of \(10^{-1} < Pr < 10^{2}\) in numerical simulations conducted by Zhang’s group. At fixed \(\eta\), the static state firstly loses its stability at the critical Rayleigh number \(Ra_{cr} = 0.729\) only against the perturbation with the degree \(l = 18\), and moreover, with a slight increase of \(Ra\) up to 0.933, the number of the degree of its unstable eigenmodes drastically diverges, as suggested by dense curves in the inset of the figure. This would imply that the number of effective degree of freedoms in the system is huge at relatively large \(\eta\) even if the increase of \(Ra\) from \(Ra_{cr}\) is relatively small, and that, thus, the spiral roll state would not have been exactly identified as a steady state if it would have taken infinite time interval to ensure its steadiness and stability just by a direct numerical simulation. This could be also a reason why many subspecifics of the spiral state with some dislocations were reported in their investigation.

On the other hand, according to Ref. 8 reporting the ubiquitous existence of spiral patterns in reaction-diffusion systems on the spherical geometry, a spiral convective state may exist even at lower \(\eta\) in our thermal convection system, in general, as not steady but travelling wave solution with a finite angular velocity. Thus, we have developed a numerical scheme to pursue exact spiral roll states at a wider gap than studied previously, restricting our investigation to the case \(Pr\) as unity. With the aid of a numerical library provided by Ref. 11 for the expansion of the spherical harmonic functions, the velocity and temperature deviation fields are expanded into a series of modified Chebyshev polynomials in the radial direction and spherical harmonic functions in the polar and azimuthal directions, respectively truncated by \((N_{radial}, N_{spherical})\). Suppose a set of unknown coefficients to determine the fields is represented by \(X\), then the governing equation is written as \(\Omega^{\nu}X + \mathcal{L}(X) + \mathcal{F}(X, X) = 0\), where \(\Omega^{\nu}\) is a tensor corresponding to a rotation of the fields determined by an angular velocity, and \(\mathcal{L}, \mathcal{F}\) are linear and nonlinear operators respectively. By fixing the azimuthal phase of the fields, the above equation can be numerically solved by the Newton-Raphson method, under the assumption of the existence of travelling wave solution.

In Fig. 2, the equilibrium states solved at \(\eta = 0.3\) and 0.75, which correspond to both ends of the scope of the present survey, are visualised by an isosurface of temperature deviation. If it would be expanded into a plane spanned by spherical ordinates \((\theta, \phi)\), the state would be consisted by a steady couple of rolls with positive and negative vorticity as observed in the conventional planar Rayleigh-Bénard convection, which is equivalent to the narrow gap limit of the spherical one. It should be again emphasized that the spiral roll state however exists not only at narrow gaps but also at wider gaps, as a not steady but travelling wave solution. As a result, it is expected that the the spiral roll state in a wider gap has a larger angular velocity. Moreover, it is also plausible that the tips of a roll cause some geometrical instability inducing dislocations or meandering the roll itself, so that one might imagine the spiral roll state could be sustained as an equilibrium state only in the case of narrow gap, where the ratio of tips to the full length of a roll surrounding the sphere is relatively small.

Numerical accuracy and angular velocity.— In order to evaluate the sufficient truncation numbers to represent the fields with good accuracy, the non-dimensional ratio
of convective to conductive heat transfer across the inner spherical boundary, $Nu$, is here introduced as

$$Nu = 1 - \frac{Pr}{4\pi r_n^3} \int \frac{1}{r_n} \partial_\phi \theta \bigg|_{r_n} (r_n^2 \sin \theta d\theta d\phi).$$

For reference, we calculated $Nu$ of the spiral state obtained at $(Ra, \eta) = (80, 0.3)$ (see Tab.1), which tells us that $(N_{\text{radial}}, N_{\text{spherical}}) = (33, 24)$ is required to ensure a convergence with six-digit accuracy in $Nu$ at least. In addition, it is also shown in the table that $Nu$ of the spiral roll state is slightly larger than that of the axisymmetric state, which suggests the spiral roll state is rather preferred at the control parameters.

The single-arm spiral roll state satisfies the 2-fold rotational symmetry, $C_2$, which means invariance under the a rotation by an angle of $\pi$ with respect to a fixed axis, referred as the $z$ axis hereafter. As described before, the state, in general, evolves with a rotation with a finite constant angular velocity $\Omega_z$ around the $z$ axis. Fig.3 illustrates the $z$ component of the angular velocity $\Omega$ of the spiral roll state obtained at $\eta = 0.5$. The magnitude of $\Omega_z$ is on the order of $10^{-3}$, which is an enough large quantity to make the rotation measurable by a numerical simulation, whereas it becomes fairly smaller with increase of $\eta$. This could be a reason why a single-arm spiral roll state was detected as a steady state in the early studies for relatively large $\eta$.

It should be also noted that, in the system subject to the spherical homogeneous geometry, a solution reflected by the mirroring transformation (parity), is another exact solution of the system. However, the spiral roll state exhibits the chirality, which means the presence of an identical exact solution in non-superposable mirroring transformation, where the sign of the angular velocity is inverted. In the figure, the black solid curves correspond to the spiral roll pattern screwed in a manner of anti-clockwise rotation along the line connecting both ends of its roll (e.g. states presented in Fig.2), while the gray solid curves is that of the corresponding state reflected under the mirroring transformation, the spiral screwed in a manner of clockwise rotation.

**Conclusion and ongoing works** The single-arm spiral roll state is numerically obtained in spherical Rayleigh-Bénard system with a relatively wider gap than that used in our previous reports[2]. In general, the state is a travelling wave solution around the axis, with respect to which the solution is invariant under the a rotation by an angle of $\pi$. This work has been also supported in part by KAKENHI (23760164) and by the Kansai University Special Research Fund 2012. T.I. would like to thank Prof. Takuji Kawahara for his kindly providing private teachings on the subject of nonlinear interaction.

| $N_{\text{spherical}}$ | 12 | 24 | 48 |
|------------------------|----|----|----|
| $N_{\text{radial}}$    | 5.32947 | 5.741696 | 5.741741 | 5.741739 |
|                        | 5.520933 | 5.739515 | 5.739561 | 5.739561 |
|                        | 5.520932 | 5.739515 | 5.739561 | 5.739561 |

**TABLE I.** Nusselt number of (upper) the spiral state and (lower) the axisymmetrical state at $(Ra, \eta) = (80, 0.3)$ for a different set of the truncation levels $(N_{\text{spherical}}, N_{\text{radial}})$. 

**Fig. 3.** The angular velocity of the spiral roll state obtained at $\eta = 0.5$ against $Ra$. The black solid curves correspond to the spiral roll pattern screwed in a manner of anti-clockwise rotation along the line connecting both ends of its roll (e.g. states presented in Fig.2), while the gray solid curves is that of the corresponding state reflected under the mirroring transformation, the spiral screwed in a manner of clockwise rotation.

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