Generalized Minimum Error Entropy for Adaptive Filtering

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Abstract—Error entropy is a important nonlinear similarity measure, and it has received increasing attention in many practical applications. The default kernel function of error entropy criterion is Gaussian kernel function, however, which is not always the best choice. In our study, two novel concepts, called generalized error entropy and quantized generalized error entropy, utilizing the generalized Gaussian density (GGD) function as the kernel function are proposed. We further derive the generalized minimum error entropy (GMEE) and quantized generalized minimum error entropy (QGMEE) criterion, and two novel adaptive filtering algorithms called GMEE and QGMEE are derived by utilizing GMEE and QGMEE criterion, respectively. The stability, steady-state performance, and computational complexity of the proposed algorithms are investigated. Some simulations indicate that the proposed two adaptive filtering algorithms outperform some existing adaptive filtering algorithms in Gaussian, sub-Gaussian, and super-Gaussian noise environments, respectively. Finally, the proposed algorithms are applied to acoustic echo cancelation and performs well.

Index Terms—Generalized Gaussian density, generalized minimum error entropy, quantized generalized minimum error entropy, GMEE algorithm, QGMEE algorithm.

I. INTRODUCTION

The adaptive filtering algorithms have been extensively utilized in a variety of practical applications, such as active noise control (ANC) [1]–[4], acoustic echo cancelation (AEC) [5], [6], and noise cancelation [7]. How to choose or construct an appropriate cost function is a key issue for adaptive filtering algorithm.

The distribution of the noise is an essential factor in determining the choice of the adaptive filtering cost function. Generally speaking, the common noise distributions are mainly divided into Gaussian, sub-Gaussian, and super-Gaussian distributions [8]–[10]. Scholars, motivated by different noise distributions, proposed various optimization criteria (cost functions). The minimum mean square error (MMSE), as an important optimization criterion, plays a critical role when dealing with Gaussian noises. Some widely known algorithms based on MMSE criterion [11] are derived, such as least mean square (LMS) [12], normalized least mean square (NLMS) [13], variable step-size LMS (VSSLMS) [14], and fractional order modified least square (FOMLMS) algorithms and some other algorithms [16], [17]. Sub-Gaussian noises, such as uniform and binary noises, are also common noise distributions in the real environment. The least mean fourth (LMF) algorithm [18] and the least mean p-power [19] algorithm outperform the LMS algorithm in sub-Gaussian noises environment. To address super-Gaussian noises (e.g., heavy-tailed impulse noises, Laplace, α-stable, etc.), typical cost functions such as mixed norm [20], [21], M-estimate cost [22], [23], and correntropy [24], [27] are utilized. Error entropy, as a widely known theory [28], [29], takes higher order moments into account. Therefore, those algorithms founded on the minimum error entropy (MEE) criterion perform very well with the presence of impulsive (heavy-tailed) noises [30]–[33]. Especially, the MEE criterion has been successfully applied to adaptive filtering [32], [34]–[37].

The Gaussian kernel function is favored because of its smoothness and strict positive-definiteness, and it is always treated as the kernel function of error entropy. However, the default kernel function is not necessarily the best option [8]. In our study, a new error entropy, called generalized error entropy, utilizing the generalized Gaussian density (GGD) [38] function as kernel function, is proposed. Moreover, we also propose a new learning criterion (or optimization criterion) called generalized minimum error entropy (GMEE) and a novel adaptive filtering based on GMEE criterion. However, the estimate of information potential (IP) of GMEE can be performed by way of a double summation, which increases the computational burden of the GMEE algorithm. To overcome this shortcoming, a novel criterion, call quantized generalized minimum error entropy, be proposed by quantizing the error set with a quantization means [30]. Some important theoretical analysis of GME and QGMEE algorithms, such as stability, steady-state performance, and computational complexity, are investigated. Moreover, we compare the performance of GMEE and QGMEE algorithms with some existing algorithms in respect of convergence speed and MSD in Gaussian, sub-Gaussian and super-Gaussian noise environments, respectively. The effect of these key parameters on the algorithm is also investigated in our paper, and some guidances for parameter selection are provided. Finally, the two new adaptive filtering algorithms are applied to AEC to verify the practicality of the two proposed algorithms.

The organization of the rest of our work is presented below. The generalized error entropy and quantized generalized error entropy are defined in Section II. In Section III, two new adaptive filtering algorithms are derived based on the GME and QGMEE criterion. In Section IV the stability, steady-state performance, and computational complexity of GMEE
and QGMEE algorithms are investigated. In Section VI some simulation examples and AEC experiment are presented to validate the theoretical results and the capabilities of the proposed algorithms. The conclusion and acknowledgements are given in Section VII and VIII respectively.

II. DEFINITIONS OF GENERALIZED ERROR ENTROPY AND QUANTIZED GENERALIZED ERROR ENTROPY

A. Generalized Error Entropy

Rényi’s seminal work on information theory is called Renyi’s $\mu$ entropy:

$$H_\mu(e) = \frac{1}{1-\mu} \log V_\mu(e).$$

(1)

Here, $\mu (\mu \neq 1, \mu > 0)$ represent the order of Renyi’s entropy. The $\mu$ information potential ($\mu$ IP) $V_\mu(e)$ of continuous variables is expressed as

$$V_\mu(e) = \int p^\mu(e) dx = E[p^{\mu-1}(e)].$$

(2)

Here $p(\cdot)$ denotes the probability density function (PDF) with respect to $e$, and $E[\cdot]$ stands for the expectation operator. In fact, PDF $p(x)$ is always estimated utilizing Parzen’s window strategy:

$$\hat{p}(x) = \frac{1}{L} \sum_{i=1}^{L} G_\sigma(x-e_i),$$

(3)

where $G_\sigma(x) = (1/\sqrt{2\pi} \sigma) \exp(-x^2/2\sigma^2)$ stands for the Gaussian kernel function, and $\sigma$ stands for kernel bandwidth, and $\{e_i\}_{i=1}^{L}$ are $L$ error samples. Combining (2) and (3), the estimation of the quadratic IP $V_2(e)$ can be obtained

$$\hat{V}_2(e) = \frac{1}{L} \sum_{i=1}^{L} \hat{p}(e) = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} G_\sigma(e_i-e_j).$$

(4)

The kernel function of conventional error entropy is always a Gaussian kernel function, however, it is not necessarily the best option. The GGD function is a widely known extension of Gaussian density function [38]:

$$G_{\alpha,\beta}(e) = \frac{\alpha}{2\beta T (1/\alpha)} \exp\left(-\left|\frac{e}{\beta}\right|^\alpha\right).$$

(5)

In Eq. (5), parameter $\alpha$ is referred to as a shape parameter, denotes the exponential rate of decay. Parameter $\beta$, often called scale parameter, denotes the dispersion of the distribution. When shape parameter is set to 1 or 2, the GGD distribution turns to Laplacian or Gaussian distribution. Fig. II shows the GGD distribution with different shape parameters. According to Fig. II it is obvious that the smaller the value of $\alpha$, the heavier the tail of the GGD distribution shape. When it satisfies $\alpha \rightarrow \infty$, the GGD distribution turns to the uniform distribution. When it satisfies $\alpha \rightarrow 0 +$, GGD distribution turns to $\delta$-distribution.

In our study, the GGD function is treated as the new kernel function of error entropy, and we define IP

$$V_{\alpha,\beta}(e) = \int \hat{p}_{\alpha,\beta}^\mu(e) de = E[\hat{p}_{\alpha,\beta}^{\mu-1}(e)],$$

(6)

where $\hat{p}_{\alpha,\beta}(x) = \frac{1}{L} \sum_{i=1}^{L} G_{\alpha,\beta}(x-e_i)$. ($\alpha = 1, \beta = 1$)

In practical application, only a finite number of error set $\{e_i\}_{i=1}^{L}$ can be obtained. Substituting (7) into (6) yields ($\mu = 2$)

$$\hat{V}_{\alpha,\beta}(e) = \frac{1}{L} \sum_{i=1}^{L} \hat{p}_{\alpha,\beta}(e_i) = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} G_{\alpha,\beta}(e_i-e_j).$$

(8)

B. Quantized Generalized Error Entropy

According to (8), the IP can be calculated by using double summation method. This method causes a huge computational burden to get the IP, especially for large data sets. Therefore, on the basis of the generalized error entropy, we refer to previous studies [39] to propose quantized generalized error entropy

$$\hat{V}_{\alpha,\beta}(e) = \frac{1}{L} \sum_{i=1}^{L} \hat{p}_{\alpha,\beta}(e_i)$$

$$\approx \hat{V}_{\alpha,\beta}^Q(e_i) = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} G_{\alpha,\beta}^Q[e_i-e_j]$$

(9)

$$= \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} H^Q_{\alpha,\beta}^Q [e_i-c_h]$$

$$= \frac{1}{L^2} \hat{p}_{\alpha,\beta}^Q(e_i).$$

Here, where quantization operator [39] $Q[e_j, \gamma] \in C$ ($\gamma$ denotes the quantization threshold) is used to obtain a codebook $C = \{c_1, c_2, \cdots, c_H \in \mathbb{R}^1\}$ (in general $H \leq L$). $H$ denotes the number of error set that are quantized to the code, $\hat{p}_{\alpha,\beta}^Q(e) = (1/L) \sum_{i=1}^{H} H^Q_{\alpha,\beta}(e-c_h)$ is the PDF estimator based on the quantized error samples. Obviously, one has $L = \sum_{h=1}^{H} H$ and $\int \hat{p}_{\alpha,\beta}^Q(e) de = 1$.

Remark 1: When $H = L$, one can obtain $\hat{p}_{\alpha,\beta}^Q(e) = \hat{p}_{\alpha,\beta}(e)$, which means that the quantized
For linear adaptive filtering, under the generalized minimum error entropy criterion, when \( \alpha \) is random gradient-based novel adaptive filtering, and call it GMEE algorithm. Under new optimization criterion (12), we derive a kind of algorithms in kernel space and virtually nonlinear algorithms in original space [43]. The kernel adaptive filtering algorithm consists of two main types of algorithms, kernel least mean square (KLMS) and kernel recursive least squares (KRLS). The KRLS algorithms outperform KLMS algorithms at the cost of higher computational complexity, and kernel recursive minimum error entropy (KRMEE) [44] and kernel recursive maximum correlation entropy (KMC) [45] are good examples. The derivation of the new kernel adaptive filtering algorithm is shown below.

Remark 3: Various variants of the GMEE algorithm can be derived, such as novel recursive least squares (RLS) based on GMEE and variable kernel width GMEE algorithm.

Remark 4: Similar to some cost functions (e.g., MCC [40], MEE [41], and GMCC [42]), GMEE can also be applied to Kalman filter.

B. Least Mean Square Based on QGMEE

Similar to the derivation of (12) and (13), we can obtain the cost function based on quantized generalized error entropy [9]

\[
J_{QGMEE} (w_n) = \arg\min_{w_n} \frac{1}{L^2} \sum_{i=n}^{n+L-1} \sum_{h=1}^{H} H_h G_{\alpha,\beta} [e_i - c_h]
\]

and an updated form of the weight vector

\[
w_{n+1} = w_n + \eta \nabla J_{QGMEE} (w_n)
\]

where

\[
a_i = \sum_{h=1}^{H} \left[ H_h G_{\alpha,\beta} (e_i - c_h) \times |e_i - c_h|^{\alpha-1} \text{sign} (e_i - c_h) \right].
\]

The computational burden of the GMEE algorithm can be effectively reduced by quantifying the set of errors \( \{e_i\}_{i=1}^{L} \), and the computational complexity and performance of the proposed QGMEE are shown in Sections IV and V, respectively.

C. Kernel adaptive filtering Based on QGMEE

In this subsection, a new kernel adaptive filtering algorithm based on GMEE criterion. The kernel adaptive filtering algorithms are efficient online learning methods for nonlinear and nonstationary system modeling, which are linear adaptive algorithms in kernel space and virtually nonlinear algorithms in original space [43]. The kernel adaptive filtering algorithm consists of two main types of algorithms, kernel least mean square (KLMS) and kernel recursive least squares (KRLS). The KRLS algorithms outperform KLMS algorithms at the cost of higher computational complexity, and kernel recursive minimum error entropy (KRMEE) [44] and kernel recursive maximum correlation entropy (KMC) [45] are good examples. The derivation of the new kernel adaptive filtering algorithm is shown below.

The cost function, based on GMEE criterion, of kernel adaptive filtering is given as following:

\[
J (w) = \frac{1}{L^2} \sum_{i=n}^{n+L-1} \sum_{j=n}^{n+L-1} G_{\alpha,\beta} (e_i - e_j) - \frac{1}{2} \gamma_1 ||w||^2
\]
and the gradient of \(J(w)\) with respect to \(w\) can be obtained

\[
\frac{\partial J(w)}{\partial w} = 2\alpha \frac{L^2}{L^2 + \beta^2} \Phi_L B_L \varepsilon_L - \gamma_1 \|w\|
\]

with
\[
\Phi_L = [\varphi_1, \ldots, \varphi_{n+L-1}], \quad \varepsilon_L = [e_n, \ldots, e_{n+L-1}],
\]

\[
[M]_{ij} = \begin{cases} [G_{\alpha,\beta} (e_i - e_j)]_i, & i, j = n, \ldots, n + L - 1, \\ 0, & i \neq j, \end{cases}
\]

\[
[N]_{ij} = \sum_{k=n}^{n+L-1} [G_{\alpha,\beta} (e_i - e_k)]_i, \quad \psi_L = \sum_{k=n}^{n+L-2} G_{\alpha,\beta} (e_k - e_i) |e_i - e_k|^\alpha.
\]

Here, \(\gamma_1\) represents regularization factor, \(e_i\) is error.

Setting \(\frac{\partial J(w)}{\partial w} = 0\), and one can obtain

\[
w = \Phi_L a_L
\]

with
\[
a_L = \begin{bmatrix} a_{L-1} - z_L r_L^{-1} e_L \\ r_L^{-1} e_L \end{bmatrix}, \\
z_L = C_L^{-1} h_L, \\
h_L = \Phi_L^{-1} \varphi_L, \\
r_L = \varphi_L^T \varphi_L + \beta^2 \varphi_L^{-1} - \varphi_L^T \Phi_L^{-1} \varphi_L, \\
C_L = r_L^{-1} \begin{bmatrix} C_{L-1} r_L + z_L \psi_L - z_L \psi_L^{-1} 1 \\ 1 \end{bmatrix}.
\]

A new KRLS type algorithm based on GMEEM (KRGMEEM) can be obtained from the derivation above, and its pseudo-code is displayed in Algorithm 1.

**Remark 6:** The KRGMEEM is an extension of KRMEEM, and it reduces to KRMEEM, when \(\alpha = 2\).

**IV. STABILITY AND STEADY-STATE PERFORMANCE**

In this part, some theoretical analysis of GMEEM and QGMEEM algorithms are investigated including stability, steady-state mean square behavior, and computational complexity. Before proceeding, two necessary assumptions are given as follows:

A1: The element of priori errors \(\varepsilon_{a,n}\) and posteriori errors \(\varepsilon_{p,n}\) are independent of the noise.

A2: The input signal and noise are uncorrelated at different instant.

\[
E[u_m u_n] = \begin{cases} \sigma_m^2 M, & m = n, \\ 0, & m \neq n, \end{cases}
\]

\[
E[u_n^T u_n] = \begin{cases} M \sigma_n^2, & m = n, \\ 0, & m \neq n. \end{cases}
\]

**Algorithm 1: KRGMEEM**

**Input:** sample sequences \(\{d_n, u_n\}, n = 1, 2, \cdots\)

**Output:** function \(f_n(\cdot)\)

1. **Parameters setting:** Select the proper parameters including \(\gamma_1, \alpha,\) and \(\beta;\)

2. **Initialization:**

   \(C_1 = [\beta^{\alpha} \gamma_2 + \kappa (u_1, u_1)]^{-1},\)

3. **while** \(\{d_n, u_n\} \neq \emptyset\)**

   4. **Compute** \(h_L, \varphi_L, \) and \(e_L\) by

   \[
   \begin{aligned}
   h_L &= \kappa (u_{n+L-1}, u_n, \cdots, \kappa (u_{n+L-1}, u_{n+L-2}), \\
   \varphi_L &= d_L - y_L, \\
   e_L &= d_L - y_L; \\
   \end{aligned}
   \]

5. **Compute** \(z_L, \psi_L, \) and \(r_L\) by Eqs. (23b), (21f), and (23d);

6. **Update** \(C_L\) and \(a_L\) by Eqs. (23a) and (23e).

7. **end**

In this part the derivation of the theoretical analysis of the GMEEM algorithm are displayed. Since the derivation process of the theoretical analysis of the QGMEEM algorithm is similar to that of GMEEM, we directly give the results of the theoretical analysis of the QGMEEM algorithm.

**A. Stability Analysis**

The equation (13) it is further formulated as below

\[
\hat{w}_{n+1} = \hat{w}_n - \eta \frac{\alpha}{L^2 \beta^2} U_n (P_n - Q_n^T), \tag{27}
\]

where \(\hat{w}_n = \hat{w} - w_n\), and priori and posteriori errors of the GMEEM algorithm are defined by

\[
\begin{aligned}
\varepsilon_{a:n} &= U_n^T \hat{w}_n, \\
\varepsilon_{p:n} &= U_n^T \hat{w}_{n+1}. \\
\end{aligned}
\]

\[
\varepsilon_{a:n} \equiv \begin{bmatrix} \varepsilon_{a, n} \varepsilon_{a, n+1} \cdots \varepsilon_{a, n+L-1} \end{bmatrix},
\]

Some expressions can obtain for simplicity

\[
D_n = \begin{bmatrix} d_n & d_{n+1} & \cdots & d_{n+L-1} \end{bmatrix}, \\
V_n = \begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+L-1} \end{bmatrix}, \\
\varepsilon_n = \begin{bmatrix} e_n & e_{n+1} & \cdots & e_{n+L-1} \end{bmatrix} \\
U_n = D_n - U_n^T U_n.
\]

Moreover, by combining the defining (28) and (29c), one can obtain

\[
\begin{aligned}
\varepsilon_n &= \varepsilon_{a:n} + V_n, \\
\varepsilon_{p:n} &= \varepsilon_{a:n} - \eta \frac{\alpha}{L^2 \beta^2} U_n^T U_n (P_n - Q_n^T). \\
\end{aligned}
\]

Left multiply both sides of (27) by \(U_n^T\), and one can obtain

\[
\varepsilon_{p:n} = \varepsilon_{a:n} - \eta \frac{\alpha}{L^2 \beta^2} (P_n - Q_n^T)^T (\varepsilon_{a:n} - \varepsilon_{p:n}). \tag{31}
\]

we can further get the following expression with assumption that matrix \(U_n^T U_n\) is invertible

\[
\begin{aligned}
\varepsilon_{p:n} &= \varepsilon_{a:n} - \eta \frac{\alpha}{L^2 \beta^2} (P_n - Q_n^T)^T (U_n^T U_n)^{-1} (\varepsilon_{a:n} - \varepsilon_{p:n}). \tag{32}
\end{aligned}
\]
Substituting (32) into (27), and one can obtain
\[ \tilde{w}_{n+1} = \tilde{w}_n - U_n (U_n^T U_n)^{-1} (\varepsilon_{a,n} - \varepsilon_{p,n}) . \] (33)

We then square both sides to get
\[ \| \tilde{w}_{n+1} \|^2 = \| \tilde{w}_n - U_n (U_n^T U_n)^{-1} (\varepsilon_{a,n} - \varepsilon_{p,n}) \|^2 \times \] (34)
This yields, after some straightforward manipulations, the relation
\[ \begin{aligned}
E \left[ \| \tilde{w}_{n+1} \|^2 \right] &= E \left[ \| \tilde{w}_n \|^2 \right] - 2\eta \frac{\alpha}{L^2 \beta^2} E \left[ \varepsilon_{a,n}^T (P_n^T - Q_n^T) \right] \\
&+ \left( \eta \frac{\alpha}{L^2 \beta^2} \right) E \left[ (P_n - Q_n) U_n^T U_n (P_n^T - Q_n^T) \right].
\end{aligned} \] (35)

From (35), when it meets the conditions
\[ E \left[ \| \tilde{w}_{n+1} \|^2 \right] = E \left[ \| \tilde{w}_n \|^2 \right], \]
one can obtain
\[ \eta \lesssim \frac{2L^2 \beta^2 \sigma^2 \varepsilon_{a,n}^T (P_n^T - Q_n^T)}{\alpha E \left[ (P_n - Q_n) U_n^T U_n (P_n^T - Q_n^T) \right]} \] (36)

According to assumption A2, the equation (36) can be rewritten as
\[ \eta \lesssim \frac{2L^2 \beta^2 \sigma^2 \varepsilon_{a,n}^T (P_n^T - Q_n^T)}{\alpha E \left[ (P_n - Q_n) U_n^T U_n (P_n^T - Q_n^T) \right]}. \] (37)

According to assumption A2, it is easy to obtain
\[ E \left[ U_n^T U_n \right] = M \sigma_a^2 I_L, \] (38)
then one can obtain
\[ \eta \lesssim \frac{2L^2 \beta^2 \sigma^2 \varepsilon_{a,n}^T (P_n^T - Q_n^T)}{\alpha M \sigma_a^2 E \left[ (P_n - Q_n) U_n^T U_n (P_n^T - Q_n^T) \right]}, \] (39)

When time point \( n \to \infty \), one can get \( \varepsilon_n \to v_n \), then matrices \( P_n \) and \( Q_n \) can be written as
\[ \begin{aligned}
\hat{P}_n &= \left[ \begin{array}{c}
\hat{p}_{n;0} \\
\hat{p}_{n;1} \\
\vdots \\
\hat{p}_{n;n+L-1}
\end{array} \right], \\
\hat{Q}_n &= \left[ \begin{array}{c}
\hat{q}_{n;0} \\
\hat{q}_{n;1} \\
\vdots \\
\hat{q}_{n;n+L-1}
\end{array} \right],
\end{aligned} \] (40)
with
\[ \begin{aligned}
\hat{p}_{n;j} &= E \left[ \sum_{j=0}^{n+L-1} G_{a,\beta} (v_{t+j} - v_j) \times \\
& \left( v_{t+j} - v_j \right)^{\alpha-1} \text{sign} (v_{t+j} - v_j) \right], \\
\hat{q}_{n;j} &= E \left[ \sum_{j=0}^{n+L-1} G_{a,\beta} (v_{t+j} - v_j) \times \\
& \left( v_{t+j} - v_j \right)^{\alpha-1} \text{sign} (v_{t+j} - v_j) \right].
\end{aligned} \] (41)

When the system is ergodic in a general sense, (41) can be further written as
\[ \begin{aligned}
\hat{p}_{n;j} &\approx \frac{1}{L} \sum_{t=1}^{i} \sum_{j=0}^{n+L-1} G_{a,\beta} (v_{t+j} - v_j) \times \\
& \left( v_{t+j} - v_j \right)^{\alpha-1} \text{sign} (v_{t+j} - v_j), \\
\hat{q}_{n;j} &\approx \frac{1}{L} \sum_{t=1}^{i} \sum_{j=0}^{n+L-1} G_{a,\beta} (v_{t+j} - v_j) \times \\
& \left( v_{t+j} - v_j \right)^{\alpha-1} \text{sign} (v_{t+j} - v_j).
\end{aligned} \] (42)

It is obvious that if step-size \( \eta \) satisfies (39), then the sequence of \( E \left[ \| \tilde{w}_n \|^2 \right] \) is convergent.

Similar to the derivation above (the stability analysis of the QGMEE algorithm is very similar to that of the GMEE algorithm, so we present the convergence conditions of the QGMEE algorithm directly), one can obtain the convergence condition of the QGMEE algorithm as the following form
\[ \eta \lesssim \frac{2L^2 \beta^2 \sigma^2 \varepsilon_{a,n}^T E \left[ A_n^T \right]}{\alpha M \sigma_a^2 E \left[ A_n^T \right]}, \] (43)
with
\[ E \left[ a_n \right] \approx \frac{1}{i} \sum_{t=1}^{i} \sum_{h=1}^{H} H h G_{a,\beta} (v_t - v_h) \times \\
\left( v_t - v_h \right)^{\alpha-1} \text{sign} (v_t - v_h). \] (44)

One can further obtain from (35)
\[ \begin{aligned}
E \left[ \varepsilon_{a,n}^T \right] &\approx \frac{1}{2} \left( \frac{\alpha}{L^2 \beta^2} \right) E \left[ (P_n^T - Q_n^T)^T E \left[ U_n^T U_n \right] (P_n^T - Q_n^T) \right].
\end{aligned} \] (45)

Left multiplying both sides of (46) by left inverse of \( E \left[ (P_n - Q_n) \right] \):
\[ \begin{aligned}
E \left[ (P_n^T - Q_n^T) (P_n - Q_n)^{-1} (P_n^T - Q_n^T) \right],
\end{aligned} \] (47)

and we obtain
\[ \begin{aligned}
E \left[ \varepsilon_{a,n} \right] &\approx \frac{\eta}{2} \left( \frac{\alpha}{L^2 \beta^2} \right) \times E \left[ (P_n^T - Q_n^T) (P_n - Q_n)^{-1} \right] \\
&\times E \left[ (P_n^T - Q_n^T)^T E \left[ U_n^T U_n \right] (P_n^T - Q_n^T) \right].
\end{aligned} \] (48)

Substituting (38) into (48), one can obtain
\[ \begin{aligned}
E \left[ \varepsilon_{a,n} \right] &\approx \frac{\eta}{2} \left( \frac{\alpha}{L^2 \beta^2} \right) M \sigma_a^2 E \left[ (P_n^T - Q_n^T) \right].
\end{aligned} \] (49)

Assuming that the proposed adaptive filter is stable, we can get the steady-state excess mean square error (EMSE) of GMEE algorithm.

\[ \begin{aligned}
\lim_{n \to \infty} E \left[ \varepsilon_{a,n}^2 \right] &\approx \frac{1}{L} \text{tr} \left( E \left[ \varepsilon_{a,n} \varepsilon_{a,n}^T \right] \right) \\
&= \frac{\eta^2 \alpha^2 M^2 \sigma_a^4}{4L^5 \beta^2 \alpha^2} \text{tr} \left( E \left[ (P_n^T - Q_n^T) (P_n - Q_n)^{-1} \right] \right).
\end{aligned} \] (50)

The steady-state excess mean square error (EMSE) of QGMEE algorithm can be obtained by a similar derivation process as above
\[ \begin{aligned}
\lim_{n \to \infty} E \left[ \varepsilon_{a,n}^2 \right] &\approx \frac{\eta^2 \alpha^2 M^2 \sigma_a^4}{4L^5 \beta^2 \alpha^2} \text{tr} \left( E \left[ \Lambda_n^T \right] E \left[ \Lambda_n \right] \right).
\end{aligned} \] (51)
When time point \( n \to \infty \), one can get \( e_n \to v_n \), then matrices \( P_n \) and \( Q_n \) can be computed. Knowing the noise distribution, the theoretical value of the EMSE can be calculated by (50).

### C. Computational Complexity

As shown in Table I we compare the computational complexities of the proposed algorithms (GMEE and QGMEE) with several adaptive filtering algorithms including LMS, LMF, and GMCC algorithms. The method [40] is utilized to approximate the computational complexity of the algorithm, and the results are displayed as follows:

\[
\begin{align*}
S_{\text{LMS}} &= 4M + 1, \\
S_{\text{LMF}} &= 4M + 2, \\
S_{\text{GMCC}} &= 4M + 8, \\
S_{\text{GMEE}} &= 4M + 2ML + 20L^2 + 5, \\
S_{\text{QGMEE}} &= 2M + 2ML + 11HL + 5.
\end{align*}
\]  

From Eq. (52), we can obtain the GMEE algorithm has a slightly higher computational complexity compared with LMS, LMF, and GMCC algorithms. In general, the number of real-valued code words \( H \) is much less than the length of the sliding window \( L \), as it shown in Table 2. We can infer that the adoption of the quantization mechanism can effectively reduce the computational complexity of the GMEE algorithm, and which has only a minimal negative impact on the performance of the GMEE algorithm (detailed simulations are shown in Section V).

### V. Simulations

We show some simulations to demonstrate the theoretical results, in this part, and verify the outstanding performance of GMEE, QGMEE, and KRGMEE algorithms. Four noise distributions (Gaussian, sub-Gaussian, and super-Gaussian noise) are considered:

1. Gaussian noise with zero-mean and unit variance.
2. The normalized kurtosis of uniform sub-Gaussian noise is -1.25.
3. Super-Gaussian noise is composed of a kind of mixed Gaussian distribution with the form \( v_n \sim 0.95N(0,0.01) + 0.05N(0,100) \).
4. Super-Gaussian noise with zero mean Rayleigh distribution, which can be generated as \( v_n = b_nR_n \) (\( R_n \) is zero mean Rayleigh noise), where \( b_n \) is a Bernoulli process with \( P\{b_i = 1\} = 0.3 \).

#### A. Steady-State Performance

The theoretical and simulated steady-state value of the GMEE algorithm are investigated with the four kinds of noise mentioned above, respectively. In those simulation, we set \( \alpha = 2, \beta = 1, \) and \( L = 10 \). To evaluate EMSEs, 100 independent simulations are performed, and 50000 iterations are performed to guarantee that the GMEE algorithm reaches a steady state in each simulation. Fig. 2 presents the steady state of EMSEs at various step sizes, and we can obtain: 1) the steady state of EMSE increases with increasing step-size; 2) the steady state values of EMSEs calculated by simulation are very consistent with the theory values computed by (50), when the step-size is equal to a small value.

#### B. Linear System Identification

A linear system identification problem is utilized to verify the effectiveness of the proposed algorithms, and the length of the adaptive filtering is equal to that of the unknown linear system impulse response. The symbol \( w_s \) is utilized to denote the weight vector of the unknown system, and the performance of the proposed algorithms is measured by the mean-square deviation (MSD):

\[
\text{MSD} = \mathbb{E}[\|w_s - w_{n}\|^2].
\]  

The weight vector \( w_n \) is \( 10 \times 1 \) vector. The signal \( u \) is white Gaussian random sequences with covariance \( \mathbb{E}\{ uu^T\} = I_{10} \) and \( \mathbb{E}\{ uu^T\} = 10 \). In this subsection, without mentioning otherwise, 100 independent simulations were performed, 4000 iterations were run to evaluate MSD in every test.

First, the performance of the GMEE and QGMEE with proper parameters algorithms is compared with several adaptive filtering algorithms, such as the LMS, LMF, and GMCC [8] algorithms with the presence of four different noises, respectively. In this subsection, the step-sizes that enables all of the algorithms to have the similar initial convergence speed are chosen, the sliding data length of the proposed algorithms is set to 30. The convergence curves of all algorithms in respect to MSD and final values of these parameters are displayed in Fig. 3. It is clear to observe that the GMEE and QGMEE significantly outperform LMS, LMF, and GMCC algorithms under Gaussian, sub-Gaussian, and super-Gaussian noises. Second, we study the influence of the \( L \) on the performance of the GMEE and QGMEE algorithms, and the values of these parameters are the same as those of the previous experiments, except for \( L \). In this simulation, the parameter \( L \) are set as \( L = 10, 20, 30, 50, 100, 150, 200 \), respectively. The distribution of the additional noise is same as that of previous simulation, and simulation results are displayed in Fig. [4] and

### Table I

Computational complexities of different algorithms

| Algorithms | ×/÷ | +/− | Exponentiation |
|------------|-----|-----|----------------|
| LMS        | 2M+1| 2M  | 0              |
| LMF        | 2M+1| 2M  | 1              |
| GMCC       | 2M+4| 2M+1| 3              |
| GMEE       | 2M+ML+6L^2+3 | 2M+ML+8L^2 | 6L^2+2 |
| QGMEE      | M+ML+4HL+3 | M+ML+4HL | 3HL+2 |

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| Algorithms | ×/÷ | +/− | Exponentiation |
|------------|-----|-----|----------------|
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| GMEE       | 2M+ML+6L^2+3 | 2M+ML+8L^2 | 6L^2+2 |
| QGMEE      | M+ML+4HL+3 | M+ML+4HL | 3HL+2 |
Table II: One can obtain that the steady-state error of the GMEE and QGMEE algorithms decreases significantly as the number of error samples L increases with the presence of Gaussian, sub-Gaussian, and super-Gaussian noises.

**Remark 7:** It is clear that increasing L is effective in improving the performance of the GMEE and QGMEE algorithms, but it also increases the computational burden of the proposed algorithms. It is worth noting that the quantization parameters γ of the QGMEE algorithm can significantly reduce the computational burden of GMEE algorithm while stabilising its performance.

Third, the influence of parameters α and β on the performance of the GMEE and QGMEE algorithms are studied, and we show the MSD (the initial convergence speed of GMEE and QGMEE algorithms) are studied, and the performance of the GMEE and QGMEE algorithms increases with the increase of parameter γ. Results, one can obtain the following: a) The larger α is the better the proposed algorithms are at handling sub-Gaussian noise; the smaller α is, the better the proposed algorithms at handling super-Gaussian noise; b) the performance of GMEE and QGMEE algorithms increases with the increase of parameter β under sub-Gaussian noises; c) the GMEE and QGMEE algorithms with proper α and β can outperform GMCC, LMF, and LMS algorithms.

Fourth, the effect of parameter γ on the performance of the QGMEE algorithms is investigated with the four kinds of noise mentioned above. The quantization threshold is set as γ = 0.00, 0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.8, 1.00, 5.00. Other parameters of QGMEE algorithm are the same as those of the previous simulations. Simulation results of the QGMEE algorithm with different γ are presented in Fig. 7 and Table IV. As one observes that the performance of the QGMEE algorithm and the parameter H decrease as the parameter γ increases, and the computational burden of the QGMEE algorithm decreases considerably as a result. A reasonable choice of the value of γ can effectively reduce the computational burden of the QGMEE algorithm without significantly degrading the performance of the algorithm.

According to above simulations, one can infer that the GMEE and QGMEE algorithms perform well with Gaussian, super-Gaussian, and sub-Gaussian noise, especially in super- and sub-Gaussian noise. QGMEE algorithm have similar performance as GMEE with fewer calculations.

The effect of parameters L, α, β, and γ on the performance of the proposed adaptive filtering algorithms is investigated in the above simulations, which provides guidance for the selection of parameters for the proposed algorithms.

1. From Fig. 4 and Table I, the choice of the parameter L requires a trade-off between the computational complexity and performance of the proposed algorithms, and we set L = 30.

2. The selection of parameters α and β can be referred to the simulation results in Tables III and IV.

3. According to Fig. 7 and Table V, the value of H decreases as the parameter γ increases, and the computational complexity and performance of the algorithm decreases. When γ satisfies 0.05 ≤ γ ≤ 0.2, compared with GMEE algorithm, QGMEE algorithm with lower computational complexity has similar performance to GMEE algorithm.

C. Mackey–Glass time series prediction

In this subpart, a benchmark data set called Mackey–Glass (MG) chaotic time series is utilized to test the nonlinear learning capability of the KRGMEE algorithm. MG equation is a nonlinear delay differential equation with the form of

\[
\frac{ds(t)}{dt} = \frac{0.2s(t - \tau)}{1 + s^{10}(t - \tau)} - 0.1s(t). \tag{54}
\]

The 1000 training data with mixed Gaussian noise and 100 test data are generated by solving the MG equation.

The performance of KRGME is compared with KRME and KRMC [45]. Convergence curves in terms of MSD and the parameters of those algorithms are presented in Fig. 8. One can obtain that the proposed KRLS algorithm perform best with proper parameters. Furthermore, the effects of parameters α and β on the performance of the KRGME algorithm were
Fig. 3. Convergence curves under different noises

| Table III | Simulation steady-state error (dBs) of GMEE and QGMEE with different $\alpha$. |
|-----------|--------------------------------------------------------------------------------------------------|
| $\alpha$  | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE |
|-----------|------|------|------|------|------|------|------|------|------|------|------|------|
| Gaussian  | -5.4 | -5.2 | -9.6 | -9.3 | -12.4 | -12.1 | -12.9 | -12.7 | -13.2 | -13.0 |
| Sub-Gaussian | -4.8 | -4.5 | -7.2 | -9.9 | -11.5 | -11.2 | -23.7 | -23.5 | -29.5 | -29.2 |
| Mixed Gaussian | -20.6 | -20.4 | -3.4 | -3.1 | -0.8 | -0.6 | -0.3 | -0.1 | 0.1 | 0.1 |
| Rayleigh | -27.0 | -26.7 | -7.3 | -6.9 | -6.1 | -5.9 | -5.0 | -4.8 | -4.9 | -4.7 |

| Table IV | Simulation steady-state error (dBs) of GMEE and QGMEE with different $\beta$. |
|-----------|--------------------------------------------------------------------------------------------------|
| $\beta$  | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE | GMEE | QGMEE |
|-----------|------|------|------|------|------|------|------|------|------|------|
| Gaussian  | 2.3  | 2.5  | -9.2 | -9.0 | -14.4 | -14.1 | -14.8 | -14.6 | -14.9 | -14.7 |
| Sub-Gaussian | -0.1 | 0.2  | -26.0 | -25.8 | -2.54 | -2.3 | 2.5 | 2.7 | 5.8 | 5.9 |
| Mixed Gaussian | -10.1 | -9.8 | -18.3 | -18.1 | -20.6 | -20.4 | -22.1 | -21.8 | -22.6 | -22.4 |
| Rayleigh | -11.7 | -11.5 | -21.6 | -21.4 | -29.9 | -29.7 | -32.3 | -32.1 | -33 | -32.8 |
TABLE V
SIMULATION STEADY-STATE ERROR (dBs) OF THE QGMEE WITH DIFFERENT γ.

|                | γ = 0.00 | γ = 0.01 | γ = 0.05 | γ = 0.1  | γ = 0.2  | γ = 0.3  | γ = 0.4  | γ = 0.8  | γ = 1.00 | γ = 5.00 |
|----------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| H              |          |          |          |          |          |          |          |          |          |          |
| Gaussian       | -12.9    | -12.8    | -12.5    | -12.2    | -12.1    | -12.0    | -11.9    | -11.6    | -11.5    | -11.0    |
| sub-Gaussian   | -30.0    | -29.8    | -29.6    | -29.5    | -29.4    | -29.2    | -29.1    | -22.8    | -22.7    | -22.4    |
| Mixed Gaussian | -27.9    | -27.7    | -27.6    | -27.4    | -26.2    | -24.7    | -23.6    | -23.0    | -22.8    | -22.6    |
| Rayleigh       | -34.8    | -34.1    | -33.2    | -32.7    | -32.5    | -32.3    | -31.4    | -31.3    | -30.9    |          |

TABLE VI
MSDS (dBs) OF THE KRGME WITH DIFFERENT β.

|                | β = 0.1  | β = 0.2  | β = 0.5  | β = 1.0  | β = 1.5  | β = 2.0  | β = 4.0  | β = 8.0  | β = 20.0 |
|----------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| KRGME          | -21.03   | -21.89   | -22.87   | -24.44   | -23.38   | -23.75   | -22.80   | -21.16   | -20.30   |

Fig. 4. MSDs (dB) of GME and QGME algorithms under different parameters (L) under sub-Gaussian noises.

Fig. 5. MSDs (dB) of GME and QGME algorithms under different parameters (α) under sub-Gaussian noises.

Fig. 6. MSDs (dB) of the GME and QGME algorithms under different parameters (β) under Rayleigh noises.

Fig. 7. MSDs (dB) of the QGME algorithm under different quantization threshold (γ) under mixed Gaussian noises.

investigated, and these results are presented in Tables VI and VII respectively. We can obtain that the KRGME algorithm obtains the best performance when α = 0.5 and β = 1.0.
TABLE VII

| MSDs (dBs) of the KRGME with different $\alpha$. |
|-----------------------------------------------|
| $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 1.0$ | $\alpha = 2.0$ | $\alpha = 4.0$ | $\alpha = 8.0$ | $\alpha = 10.0$ |
| KRGME | -16.08 | -23.40 | -24.13 | -24.96 | -23.77 | -21.51 | -19.52 | -16.79 | -14.68 |

VI. CONCLUSION

The last few years, error entropy is a similarity measurement method, and the MEE criterion has also been successfully applied to many practical fields. In the available literature, the default kernel function of error entropy is Gaussian kernel function, which is not necessarily the best option. In this paper, the GGD function is utilized as kernel function of error entropy, one can further obtain generalized error entropy and quantized generalized error entropy. In addition, we also propose two novel adaptive filtering algorithms called GMEE and QGMEE algorithm using GMEE and QGMEE criterion, respectively. Further, the stability, steady-state performance, and computation complexity of the proposed algorithms are analyzed. Some simulation results indicate that GMEE and QGMEE adaptive filter algorithms outperform some existing adaptive filtering algorithms in Gaussian, sub-Gaussian, and super-Gaussian noises respectively. This experiment of applying GME and QGME to AEC further demonstrates the practicality of the proposed algorithms.

The GMEE optimization criterion with its superior learning performance has been successfully applied to the LMS algorithm. On top of this, there are still some issues that need to be studied. As we know, the RLS algorithm outperforms LMS algorithm, it is natural to develop a new RLS algorithm based on GMEE to improve the performance of the RLS algorithm. This point is the focus of our research in the future.

VII. ACKNOWLEDGEMENTS

This study was founded by the National Natural Science Foundation of China under Grant 61371182 and 51975107, and by Sichuan Science and Technology Major Project No.2019ZDZX0020.

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