ON A CLASS OF SEMIPOSITONE PROBLEMS WITH SINGULAR TRUDINGER-MOSER NONLINEARITIES

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Abstract. We prove the existence of positive solutions for a class of semi-positone problems with singular Trudinger-Moser nonlinearities. The proof is based on compactness and regularity arguments.

1. Introduction. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and let $f$ be a Carathéodory function on $\Omega \times [0, \infty)$. The semilinear elliptic boundary value problem
\[
\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
is said to be of semipositone type if $f(\cdot, 0) < 0$ on a set of positive measure. It is notoriously difficult to find positive solutions of this class of problems due to the fact that $u = 0$ is not a subsolution (see, e.g., Castro and Shivaji [5], Ali et al. [2], Ambrosetti et al. [3], Chhetri et al. [6], Castro et al. [4], Costa et al. [7], and their references).

The purpose of the present paper is to study a class of semipositone problems with singular exponential nonlinearities in dimension $N = 2$. We consider the problem
\[
\begin{cases}
-\Delta u = \lambda u \frac{e^{\alpha u^2}}{|x|^\gamma} + \mu g(u) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  

(1.1)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$ containing the origin, $\alpha > 0$, $0 \leq \gamma < 2$, $\lambda, \mu > 0$ are parameters, and $g$ is a continuous function on $[0, \infty)$ satisfying
\[
\lim_{t \to \infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \forall \beta > 0
\]  

(1.2)

and
\[
\sup_{t \in [0, \infty)} (2G(t) - tg(t)) < \infty,
\]  

(1.3)

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where \( G(t) = \int_0^t g(s) \, ds \). We make no assumptions about the sign of \( g(0) \) and hence allow the semipositone case \( g(0) < 0 \). For example, the functions \( g(t) = -1 \), \( g(t) = t^p - 1 \), where \( p \geq 1 \), and \( g(t) = e^t - 2 \) all satisfy (1.2), (1.3), and \( g(0) < 0 \).

The motivation for problem (1.1) comes from the following singular Trudinger-Moser embedding of Adimurthi and Sandeep [1]:

\[
\int_{\Omega} e^{\alpha u^2} \, dx < \infty \quad \forall u \in H^1_0(\Omega)
\]

for all \( \alpha > 0 \) and \( 0 \leq \gamma < 2 \), and

\[
\sup_{\|u\|_{H^1_0(\Omega)} \leq 1} \int_{\Omega} e^{\alpha u^2} \, dx < \infty
\]

if and only if \( \alpha/4\pi + \gamma/2 \leq 1 \). Our problem is critical with respect to this embedding and hence the variational functional associated with this problem lacks compactness, which is an additional difficulty in finding solutions.

Let \( \lambda_1(\gamma) > 0 \) be the first eigenvalue of the singular eigenvalue problem

\[
\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^\gamma} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

given by

\[
\lambda_1(\gamma) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} \frac{u^2}{|x|^\gamma} \, dx}.
\]

We will show that problem (1.1) has a positive solution for all \( 0 < \lambda < \lambda_1(\gamma) \) and \( \mu > 0 \) sufficiently small. We have the following theorem.

**Theorem 1.1.** Assume that \( \alpha > 0 \) and \( 0 \leq \gamma < 1 \) satisfy

\[
\frac{\alpha}{4\pi} + \frac{\gamma}{2} \leq 1,
\]

\( 0 < \lambda < \lambda_1(\gamma) \), and \( g \) satisfies (1.2) and (1.3). Then there exists a \( \mu^* > 0 \) such that for all \( 0 < \mu < \mu^* \), problem (1.1) has a solution \( u_\mu \).

We note that this result does not follow from standard arguments based on the maximum principle since \( g(0) \) is not assumed to be nonnegative. Our proof is based on regularity arguments and will be given in Section 3, after establishing a suitable compactness property of an associated variational functional in the next section.

2. A compactness result. In this section we consider the modified problem

\[
\begin{cases}
-\Delta u = \lambda u^+ \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} + \mu \bar{g}(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( u^+(x) = \max \{ u(x), 0 \} \) and

\[
\bar{g}(t) = \begin{cases}
0, & t \leq -1 \\
(1 + t) g(0), & -1 < t < 0 \\
g(t), & t \geq 0.
\end{cases}
\]
Weak solutions of this problem coincide with critical points of the $C^1$-functional

$$E_\mu(u) = \int_\Omega \left[ \frac{1}{2} \nabla u^2 - \frac{\lambda}{2\alpha} \frac{e^{\alpha(u^+)^2} - 1}{|x|\gamma} - \mu \tilde{G}(u) \right] \, dx, \quad u \in H^1_0(\Omega),$$

where $\tilde{G}(t) = \int_0^t \tilde{g}(s) \, ds$. The main result of this section is the following compactness result.

**Theorem 2.1.** Assume that $\alpha > 0$ and $0 \leq \gamma < 2$ satisfy $\alpha/4\pi + \gamma/2 \leq 1$ and $g$ satisfies (1.2) and (1.3). If $\mu_j > 0$, $\mu_j \to \mu \geq 0$, $(u_j) \subset H^1_0(\Omega)$, and

$$E_{\mu_j}(u_j) \to c, \quad E'_{\mu_j}(u_j) \to 0$$

for some $c \neq 0$ satisfying

$$c < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) - \frac{\mu \theta}{2} |\Omega|,$$  \hspace{1cm} (2.2)

where

$$\theta = \sup_{t \in \mathbb{R}} \left(2\tilde{G}(t) - t\tilde{g}(t)\right)$$

and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$, then a subsequence of $(u_j)$ converges to a critical point of $E_\mu$ at the level $c$. In particular, $E_\mu$ satisfies the (PS)$_c$ condition for all $c \neq 0$ satisfying (2.2).

First we prove the following lemma.

**Lemma 2.2.** If $(u_j)$ is a sequence in $H^1_0(\Omega)$ converging a.e. to $u \in H^1_0(\Omega)$ and

$$\sup_j \int_\Omega (u_j^+)^2 \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx < \infty,$$  \hspace{1cm} (2.3)

then

$$\int_\Omega \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx \to \int_\Omega \frac{e^{\alpha(u^+)^2}}{|x|\gamma} \, dx.$$

**Proof.** For $M > 0$, write

$$\int_\Omega \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx = \int_{\{u_j^+ < M\}} \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx + \int_{\{u_j^+ \geq M\}} \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx.$$

By (2.3),

$$\int_{\{u_j^+ \geq M\}} \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx \leq \frac{1}{M^2} \int_\Omega (u_j^+)^2 \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx = O\left(\frac{1}{M^2}\right) \quad \text{as} \quad M \to \infty.$$

Hence

$$\int_\Omega \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx = \int_{\{u_j^+ < M\}} \frac{e^{\alpha(u_j^+)^2}}{|x|\gamma} \, dx + O\left(\frac{1}{M^2}\right),$$

and the conclusion follows by first letting $j \to \infty$ and then letting $M \to \infty$. 

We will also need the following result from Adimurthi and Sandeep [1, Theorem 2.3].
Lemma 2.3. Let $0 \leq \gamma < 2$. If $(u_j)$ is a sequence in $H^1_0(\Omega)$ with $\|u_j\| = 1$ for all $j$ and converging weakly to a nonzero function $u$, then

$$\sup_j \int_{\Omega} e^\beta u_j^2 \, dx < \infty$$

for all $\beta < 4\pi(1 - \gamma/2)/(1 - \|u\|^2)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We have

$$E_{\mu_j}(u_j) = \frac{1}{2} \|u_j\|^2 - \frac{\lambda}{2\alpha} \int_{\Omega} \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx - \mu_j \int_{\Omega} \tilde{G}(u_j) \, dx = c + o(1) \quad (2.4)$$

and

$$E'_{\mu_j}(u_j) u_j = \|u_j\|^2 - \lambda \int_{\Omega} (u_j^+)^2 \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx - \mu_j \int_{\Omega} u_j \tilde{g}(u_j) \, dx = o(\|u_j\|) \quad (2.5)$$

Multiplying (2.4) by 4 and subtracting (2.5) gives

$$\|u_j\|^2 + \lambda \int_{\Omega} \left( (u_j^+)^2 - \frac{2}{\alpha} \right) e^{\alpha(u_j^+)^2} \frac{dx}{|x|^\gamma} + \mu_j \int_{\Omega} (u_j \tilde{g}(u_j) - 4\tilde{G}(u_j)) \, dx$$

$$= 4c + o(\|u_j\|^2 + 1),$$

and this together with (1.2) implies that $(u_j)$ is bounded in $H^1_0(\Omega)$. Hence a renamed subsequence converges to some $u$ weakly in $H^1_0(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e. in $\Omega$. Moreover,

$$\sup_j \int_{\Omega} e^\beta u_j^2 \, dx < \infty$$

for all $\beta < 4\pi/(\sup_j \|u_j\|)$ by (1.4), and hence $\int_{\Omega} u_j \tilde{g}(u_j) \, dx$ is bounded by (1.2). Then

$$\sup_j \int_{\Omega} (u_j^+)^2 \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx < \infty \quad (2.6)$$

by (2.5), and hence

$$\int_{\Omega} \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx \to \int_{\Omega} \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} \, dx \quad (2.7)$$

by Lemma 2.2. Denoting by $C$ a generic positive constant,

$$|u_j \tilde{g}(u_j)| \leq |u_j| \left( e^{\alpha(u_j^+)^2/2} + C \right) \leq \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} + C (u_j^+ + 1)$$

by (1.2), so it follows from (2.7) and the dominated convergence theorem that

$$\int_{\Omega} u_j \tilde{g}(u_j) \, dx \to \int_{\Omega} u \tilde{g}(u) \, dx. \quad (2.8)$$

Similarly,

$$\int_{\Omega} \tilde{G}(u_j) \, dx \to \int_{\Omega} \tilde{G}(u) \, dx. \quad (2.9)$$

We claim that the weak limit $u$ is nonzero. Suppose $u = 0$. Then

$$\int_{\Omega} \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} \, dx \to \int_{\Omega} \frac{dx}{|x|^\gamma}; \quad \int_{\Omega} u_j \tilde{g}(u_j) \, dx \to 0, \quad \int_{\Omega} \tilde{G}(u_j) \, dx \to 0 \quad (2.10)$$
by (2.7)–(2.9). So (2.4) implies that \( c > 0 \) and
\[
\| u_j \| \to (2c)^{1/2}. \tag{2.11}
\]
Noting that \( c < 2\pi (1 - \gamma/2)/\alpha \) by (2.2), let \( 2c < \nu < 4\pi (1 - \gamma/2)/\alpha \). Then (2.11) implies that \( \| u_j \| \leq \nu^{1/2} \) for all \( j \geq j_0 \) for some \( j_0 \). Let \( q = 4\pi (1 - \gamma/2)/\alpha \nu > 1 \) and let \( 1/(1 - 1/q) < r < 2/(1 - 1/q) \). By the Hölder inequality,
\[
\int_{\Omega} (u_j^+)^{2} \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx \leq \left( \int_{\Omega} |u_j|^{2p} \, dx \right)^{1/p} \left( \int_{\Omega} e^{\nu u_j^2} \, dx \right)^{1/q} \left( \int_{\Omega} \frac{dx}{|x|^\gamma (1-1/q)} \right)^{1/r},
\]
where \( 1/p + 1/q + 1/r = 1 \). The first integral on the right-hand side converges to zero since \( u = 0 \), the second integral is bounded for \( j \geq j_0 \) by (1.4) since \( q_0 u_j^2 = 4\pi (1 - \gamma/2) \tilde{u}_j^2 \), where \( \tilde{u}_j = u_j/\nu^{1/2} \) satisfies \( \| u_j \| \leq 1 \), and the last integral is finite since \( \gamma r (1 - 1/q) < 2 \), so
\[
\int_{\Omega} (u_j^+)^{2} \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx \to 0.
\]
Then \( u_j \to 0 \) by (2.5) and (2.10), and hence \( c = 0 \) by (2.11), a contradiction. So \( u \) is nonzero.

Since \( E'_{\mu_j}(u_j) \to 0 \),
\[
\int_{\Omega} \nabla u_j \cdot \nabla v \, dx - \lambda \int_{\Omega} u_j^+ \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, v \, dx - \mu_j \int_{\Omega} g(u_j) \, v \, dx \to 0 \tag{2.12}
\]
for all \( v \in H^1_0(\Omega) \). For \( v \in C^\infty_0(\Omega) \), an argument similar to that in the proof of Lemma 2.2 using the estimate
\[
\left| \int_{\{u_j^+ \geq M\}} u_j^+ \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, v \, dx \right| \leq \frac{\sup |v|}{M} \int_{\Omega} (u_j^+)2 \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx
\]
and (2.6) shows that \( \int_{\Omega} u_j^+ \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, v \, dx \to \int_{\Omega} u^+ \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} \, v \, dx \). Moreover, denoting by \( C \) a generic positive constant,
\[
|g(u_j) v| \leq \sup |v| \left( e^{\alpha(u_j^+)^2} + C \right) \leq C \sup |v| \left( \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} + 1 \right)
\]
by (1.2), so it follows from (2.7) and the dominated convergence theorem that
\[
\int_{\Omega} g(u_j) \, v \, dx \to \int_{\Omega} g(u) \, v \, dx.
\]
So it follows from (2.12) that
\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} u^+ \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} \, v \, dx + \mu \int_{\Omega} g(u) \, v \, dx.
\]
Then this holds for all \( v \in H^1_0(\Omega) \) by density, and taking \( v = u \) gives
\[
\| u \|^2 = \lambda \int_{\Omega} (u^+)2 \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} \, dx + \mu \int_{\Omega} u \tilde{g}(u) \, dx. \tag{2.13}
\]
Next we claim that
\[
\int_{\Omega} (u_j^+)^2 \frac{e^{\alpha(u_j^+)^2}}{|x|^\gamma} \, dx \to \int_{\Omega} (u^+)^2 \frac{e^{\alpha(u^+)^2}}{|x|^\gamma} \, dx. \tag{2.14}
\]
We have
\[(u_j^+)^2 e^{\alpha (u_j^+)^2} \leq u_j^2 e^{\alpha u_j^2} \leq u_j^2 e^{\alpha \|u_j\|^2 \bar{v}^2_j}, \tag{2.15}\]
where \(\bar{v}_j = u_j / \|u_j\|\).

Setting
\[\kappa = \frac{\lambda}{2\alpha} \int_{\Omega} \frac{e^{\alpha (u^+)^2}}{|x|^{\gamma}} dx - \mu \int_{\Omega} \bar{G}(u) dx,\]
we have
\[\|u_j\|^2 \to 2 (c + \kappa)\]
by (2.4), (2.7), and (2.9), so \(\bar{v}_j\) converges weakly and a.e. to \(\bar{v} = u/[2 (c + \kappa)]^{1/2}\).

Then
\[\|u_j\|^2 \left(1 - \|\bar{u}\|^2\right) \to 2 (c + \kappa) - \|u\|^2. \tag{2.16}\]

Since \(te^t \geq e^t - 1\) for all \(t \geq 0,\)
\[\int_{\Omega} (u^+)^2 \frac{e^{\alpha (u^+)^2}}{|x|^{\gamma}} dx \geq \frac{1}{\alpha} \int_{\Omega} \frac{e^{\alpha (u^+)^2}}{|x|^{\gamma}} dx - \mu \int_{\Omega} \bar{G}(u) dx,\]
and
\[\int_{\Omega} u \bar{g}(u) dx \geq 2 \int_{\Omega} \bar{G}(u) dx - \theta |\Omega|\]
since \(\theta \geq 2 \bar{G}(t) - t \bar{g}(t)\) for all \(t \in \mathbb{R},\) so it follows from (2.13) that \(\|u\|^2 \geq 2 \kappa - \mu \theta |\Omega|\).

Hence
\[2 (c + \kappa) - \|u\|^2 \leq 2c + \mu \theta |\Omega| < \frac{4\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) \tag{2.17}\]
by (2.2).

We are done if \(\|\bar{u}\| = 1\), so suppose \(\|\bar{u}\| < 1\) and let
\[\frac{2c + \mu \theta |\Omega|}{1 - \|\bar{u}\|^2} < \bar{v} - 2\varepsilon < \bar{v} < \frac{4\pi (1 - \gamma/2)/\alpha}{1 - \|\bar{u}\|^2}.\]

Then \(\|u_j\|^2 \leq \bar{v} - 2\varepsilon\) for all \(j \geq j_0\) for some \(j_0\) by (2.16) and (2.17), and
\[\sup_j \int_{\Omega} \frac{e^{\alpha \bar{v} \bar{u}_j^2}}{|x|^\gamma} dx < \infty \tag{2.18}\]
by Lemma 2.3. For \(M > 0\) and \(j \geq j_0,\) (2.15) then gives
\[
\int_{\{u_j^+ \geq M\}} \left(\frac{u_j^+}{\|u_j\|}\right)^2 e^{\alpha (\bar{u}_j^+)^2} \frac{\bar{u}_j}{|x|^\gamma} dx \\
\leq \int_{\{u_j^+ \geq M\}} u_j^2 e^{\alpha (\bar{v} - 2\varepsilon)^2 \bar{u}_j^2} \frac{\bar{u}_j}{|x|^\gamma} dx \\
= \|u_j\|^2 \int_{\{u_j^+ \geq M\}} \bar{u}_j^2 e^{-\varepsilon \alpha \|u_j\|^2} e^{-\varepsilon \alpha (u_j/\|u_j\|)^2} \frac{\bar{u}_j}{|x|^\gamma} dx \\
\leq \left(\max_{t \geq 0} te^{-\varepsilon \alpha t}\right) \|u_j\|^2 e^{-\varepsilon \alpha (M/\|u_j\|)^2} \int_{\Omega} e^{\alpha \bar{v} \bar{u}_j^2} dx.
\]
The last expression goes to zero as \(M \to \infty\) uniformly in \(j\) since \(\|u_j\|\) is bounded and (2.18) holds, so (2.14) now follows as in the proof of Lemma 2.2.
Now it follows from (2.5), (2.14), (2.8), and (2.13) that
\[
\|u_j\|^2 \to \lambda \int_\Omega (u^+)^2 \frac{e^{\alpha u^+}}{|x|^\gamma} \, dx + \mu \int_\Omega u \bar{g}(u) \, dx = \|u\|^2
\]
and hence \(\|u_j\| \to \|u\|\), so \(u_j \to u\). Clearly, \(E_{\mu}(u) = c\) and \(E'_{\mu}(u) = 0\). \(\square\)

3. Proof of Theorem 1.1. In this section we prove our main result. By Theorem 2.1, \(E_\mu\) satisfies the (PS)\(_c\) condition for all \(c \neq 0\) satisfying
\[
c < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) - \frac{\mu \theta}{2} |\Omega|.
\]
First we show that \(E_\mu\) has a uniformly positive mountain pass level below this threshold for compactness for all sufficiently small \(\mu > 0\). Take \(r > 0\) so small that \(\overline{B_r(0)} \subset \Omega\) and let
\[
v_j(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
\sqrt{\log j}, & |x| \leq r/j \\
\frac{\log(r/|x|)}{\sqrt{\log j}}, & r/j < |x| < r \\
0, & |x| \geq r.
\end{cases}
\]
It is easily seen that \(v_j \in H^1_0(\Omega)\) with \(\|v_j\| = 1\) and
\[
\int_\Omega v_j^2 \, dx = O(1/\log j) \quad \text{as} \quad j \to \infty.
\]

Lemma 3.1. There exist \(\mu_0, \rho, c_0 > 0\), \(j_0 \geq 2\), \(R > \rho\), and \(\tilde{\vartheta} < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right)\) such that the following hold for all \(\mu \in (0, \mu_0)\):
(i) \(\|u\| = \rho \implies E_\mu(u) \geq c_0\),
(ii) \(E_\mu(\varrho v_{j_0}) \leq 0\),
(iii) denoting by \(\Gamma = \{ \gamma \in C([0, 1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = \varrho v_{j_0}\}\) the class of paths joining the origin to \(\varrho v_{j_0}\),
\[
c_0 \leq c_\mu := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0, 1])} E_\mu(u) \leq \tilde{\vartheta} + C\mu^2
\]
for some constant \(C > 0\),
(iv) \(E_\mu\) has a critical point \(u_\mu\) at the level \(c_\mu\).

Proof. Set \(\rho = \|u\|\) and \(\tilde{u} = u/\rho\). Since \(e^t - 1 \leq t + t^2 e^t\) for all \(t \geq 0\),
\[
\frac{1}{\alpha} \int_\Omega \frac{e^{\alpha (u^+)^2} - 1}{|x|^\gamma} \, dx \leq \int_\Omega \frac{u^2}{|x|^\gamma} \, dx + \alpha \int_\Omega u^4 \frac{e^{\alpha u^2}}{|x|^\gamma} \, dx.
\]
By (1.5),
\[
\int_\Omega \frac{u^2}{|x|^\gamma} \, dx \leq \frac{\rho^2}{\lambda_1(\gamma)}.
\]
Let \(2 < r < 4/\gamma\). By the Hölder inequality,
\[
\int_\Omega u^4 \frac{e^{\alpha u^2}}{|x|^\gamma} \, dx \leq \left( \int_\Omega u^{4p} \, dx \right)^{1/p} \left( \int_\Omega e^{2\alpha u^2} \, dx \right)^{1/2} \left( \int_\Omega \frac{dx}{|x|^\gamma} \right)^{1/r},
\]
where \(1/p + 1/r = 1/2\). The first integral on the right-hand side is bounded by \(C\rho \tilde{u}^2\) for some constant \(C > 0\) by the Sobolev embedding. Since \(2\alpha u_\mu^2 = 2\alpha \rho^2 \tilde{u}^2\) and
$\|\tilde{u}\| = 1$, the second integral is bounded when $\rho^2 \leq 2\pi (1 - \gamma/2)/\alpha$ by (1.4). The last integral is finite since $\gamma r < 4$. So combining (3.3)–(3.5) gives

$$
\frac{1}{\alpha} \int_\Omega \frac{e^{\alpha (u^+)^2} - 1}{|x|^\gamma} dx \leq \frac{\rho^2}{\lambda_1(\gamma)} + O(\rho^4) \quad \text{as} \ \rho \to 0.
$$

On the other hand, it follows from (1.2) that $\int_\Omega \tilde{G}(u) dx$ is bounded on bounded subsets of $H^1_0(\Omega)$. So

$$
E_\mu(u) \geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1(\gamma)}\right) \rho^2 + O(\rho^4) - C\mu \quad \text{as} \ \rho \to 0
$$

for some constant $C > 0$. Since $\lambda(\gamma) < \lambda_1$, (i) follows from this for sufficiently small $\rho, \mu, c_0 > 0$.

Since $\|v_j\| = 1$ and $v_j \geq 0$,

$$
E_\mu(tv_j) = \frac{t^2}{2} - \int_\Omega \left[ \frac{\lambda}{2\alpha} \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} + \mu G(tv_j) \right] dx
$$

for $t \geq 0$. For $\mu \leq \lambda/2$, this gives

$$
E_\mu(tv_j) \leq \frac{t^2}{2} - \int_\Omega \left[ \frac{\lambda}{2\alpha} \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} + \mu F(x, tv_j) \right] dx,
$$

where

$$
F(x, t) = \frac{1}{2\alpha} \frac{e^{\alpha t^2} - 1}{|x|^\gamma} + G(t) = \int_0^t \left( s \frac{e^{\alpha s^2}}{|x|^\gamma} + g(s) \right) ds \geq -Ct
$$

for some generic positive constant $C$ by (1.2), so

$$
E_\mu(tv_j) \leq \frac{t^2}{2} - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} dx + C\mu \int_\Omega v_j dx.
$$

Since

$$
C\mu \int_\Omega v_j dx \leq C\mu t \left( \int_\Omega v_j^2 dx \right)^{1/2} \leq C\mu^2 + \frac{t^2}{2} \int_\Omega v_j^2 dx,
$$

then

$$
E_\mu(tv_j) \leq H_j(t) + C\mu^2,
$$

where

$$
H_j(t) = \frac{t^2}{2} \left(1 + \int_\Omega v_j^2 dx\right) - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} dx \to -\infty \quad \text{as} \ t \to \infty.
$$

So to prove (ii) and (iii), it suffices to show that $\exists j_0 \geq 2$ such that

$$
\vartheta := \sup_{t \geq 0} H_{j_0}(t) < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right).
$$

Suppose $\sup_{t \geq 0} H_j(t) \geq 2\pi (1 - \gamma/2)/\alpha$ for all $j$. Since $H_j(t) \to -\infty$ as $t \to \infty$, there exists $t_j > 0$ such that

$$
H_j(t_j) = \frac{t_j^2}{2} (1 + \varepsilon_j) - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t_j^2 v_j^2} - 1}{|x|^\gamma} dx = \sup_{t \geq 0} H_j(t) \geq \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) \quad (3.6)
$$
and

\[ H_j'(t_j) = t_j \left( 1 + \varepsilon_j - \frac{\lambda}{2} \int_{\Omega} v_j^2 e^{\alpha t_j v_j^2} \frac{dx}{|x|^\gamma} \right) = 0, \quad (3.7) \]

where \( \varepsilon_j = \int_{\Omega} v_j^2 dx \to 0 \) by (3.1). The inequality in (3.6) gives

\[ \alpha t_j^2 \geq \frac{4\pi}{1 + \varepsilon_j} \left( 1 - \frac{\gamma}{2} \right), \]

and then (3.7) gives

\[ \frac{2}{\lambda} (1 + \varepsilon_j) = \int_{\Omega} v_j^2 e^{\alpha t_j v_j^2} \frac{dx}{|x|^\gamma} \geq \int_{B_{r/j}(0)} v_j^2 e^{4\pi (1-\gamma/2) v_j^2/(1+\varepsilon_j)} \frac{dx}{|x|^{1-\gamma/2}} \]

\[ = \frac{j^2 (1-\gamma/2)}{2 (1-\gamma/2)} \frac{\log j}{j^2 (1-\gamma/2) \varepsilon_j/(1+\varepsilon_j)}.\]

This is impossible for large \( j \) since

\[ j^2 (1-\gamma/2) \varepsilon_j/(1+\varepsilon_j) \leq j^2 (1-\gamma/2) \varepsilon_j = e^{2(1-\gamma/2) \varepsilon_j \log j} = O(1) \]

by (3.1).

By (i)–(iii), \( E_\mu \) has the mountain pass geometry and the mountain pass level \( c_\mu \) satisfies

\[ 0 < c_\mu \leq \vartheta + C \mu^2 < \frac{2\pi}{\alpha} \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu \vartheta}{2} |\Omega| \]

for all sufficiently small \( \mu > 0 \), so \( E_\mu \) satisfies the \( (PS)_{c_\mu} \) condition. So \( E_\mu \) has a critical point \( u_\mu \) at this level by the mountain pass theorem. \( \square \)

Next we prove the following lemma.

**Lemma 3.2.** If \((u_j)\) is a convergent sequence in \( H_0^1(\Omega) \), then

\[ \sup_j \int_{\Omega} e^{\beta u_j^2} \frac{dx}{|x|^\gamma} < \infty \]

for all \( \beta > 0 \) and \( 0 \leq \gamma < 2 \).

**Proof.** Let \( u \in H_0^1(\Omega) \) be the limit of \((u_j)\). Since \( u_j^2 \leq (|u| + |u_j - u|)^2 \leq 2u^2 + 2(u_j - u)^2 \),

\[ \int_{\Omega} e^{\beta u_j^2} \frac{dx}{|x|^\gamma} \leq \left( \int_{\Omega} e^{4\beta u_j^2} \frac{dx}{|x|^\gamma} \right)^{1/2} \left( \int_{\Omega} e^{4\beta (u_j - u)^2} \frac{dx}{|x|^\gamma} \right)^{1/2}. \]

The first integral on the right-hand side is finite, and the second integral equals

\[ \int_{\Omega} e^{4\beta \|u_j - u\|^2} \frac{dx}{|x|^\gamma}, \]

where \( w_j = (u_j - u)/\|u_j - u\| \). Since \( \|w_j\| = 1 \) and \( \|u_j - u\| \to 0 \), this integral is bounded by (1.4). \( \square \)

Now we show that \( u_\mu \) is positive in \( \Omega \), and hence a solution of problem (1.1), for all sufficiently small \( \mu \in (0, \mu_0) \). It suffices to show that for every sequence.
\(\mu_j > 0, \mu_j \to 0\), a subsequence of \(u_j = u_{\mu_j}\) is positive in \(\Omega\). By (3.2), a renamed subsequence of \(c_{\mu_j}\) converges to some \(c\) satisfying
\[
0 < c < \frac{2\pi}{\alpha} \left(1 - \frac{1}{2}\right).
\]
Then a renamed subsequence of \((u_j)\) converges in \(H^1_0(\Omega)\) to a critical point \(u\) of \(E_0\) at the level \(c\) by Theorem 2.1. Since \(c > 0\), \(u\) is nontrivial.

Since \(u_j\) is a critical point of \(E_{\mu_j}\),
\[
-\Delta u_j = \lambda u_j^+ e^{\alpha(u_j^+)^2/|x|^\gamma} + \mu_j \tilde{g}(u_j)
\]
in \(\Omega\). Let \(2 < p < 2/\gamma\) and \(1 < r < 2/\gamma p\). By the Hölder inequality,
\[
\int_\Omega \left| u_j^+ e^{\alpha(u_j^+)^2/|x|^\gamma} \right|^p dx \leq \left( \int_\Omega |u_j|^pq dx \right)^{1/q} \left( \int_\Omega e^{p\alpha u_j^2/\gamma p} dx \right)^{1/r},
\]
where \(1/q + 1/r = 1\). The first integral on the right-hand side is bounded by the Sobolev embedding, and so is the second integral by Lemma 3.2 since \(\gamma p r < 2\), so \(u_j^+ e^{\alpha(u_j^+)^2/|x|^\gamma}\) is bounded in \(L^p(\Omega)\). By (1.2) and Lemma 3.2 again, \(\tilde{g}(u_j)\) is also bounded in \(L^p(\Omega)\). By the Calderon-Zygmund inequality, then \((u_j)\) is bounded in \(W^{2,p}(\Omega)\). Since \(W^{2,p}(\Omega)\) is compactly embedded in \(C^1(\overline{\Omega})\) for \(p > 2\), it follows that a renamed subsequence of \(u_j\) converges to \(u\) in \(C^1(\overline{\Omega})\).

Since \(u\) is a nontrivial solution of the problem
\[
\begin{cases}
-\Delta u = \lambda u^+ e^{\alpha(u^+)^2/|x|^\gamma} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

\(u > 0\) in \(\Omega\) by the strong maximum principle and its interior normal derivative \(\partial u/\partial \nu > 0\) on \(\partial \Omega\) by the Hopf lemma. Since \(u_j \to u\) in \(C^1(\overline{\Omega})\), then \(u_j > 0\) in \(\Omega\) for all sufficiently large \(j\). This concludes the proof of Theorem 1.1.

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