\textit{L}^p \textit{ estimates for angular maximal functions associated with Stieltjes and Laplace transforms}

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Abstract. Maximal angular operator sends a function defined in a sector of the complex plane to a function of modulus obtained by maximizing over all admissible values of the argument for the given modulus. The compositions of the so obtained maximal angular operator with the Poisson, Stieltjes and Laplace transform (in the sectors of their respective ranges) are shown to be bounded (nonlinear) operators from \textit{L}^p to \textit{L}^q for the naturally expected values of \(p\) and \(q\).

1. Suppose \(g(z)\) is a complex-valued function defined in the sector
\[ \mathbb{C}(\theta_1, \theta_2) = \{ z \in \mathbb{C} \mid \theta_1 < \arg z < \theta_2 \}. \]

The \textit{angular maximal function} for \(g\) with respect to \(\mathbb{C}(\theta_1, \theta_2)\) is a function \(\mathbb{R}_+ \to [0, \infty] \) defined as follows:
\[ \mathcal{M}_g^{\theta_1, \theta_2}(\rho) = \text{ess sup}_{\theta \in (\theta_1, \theta_2)} |g(\rho e^{i\theta})|. \]

In this work, we will only deal with continuous (in fact, harmonic) functions \(g(z)\), so the question whether \(g(\rho e^{i\theta})\) is measurable and in what sense will never arise. Also, in this situation \text{ess sup} in (1) can be replaced by \text{sup}. For convenience we introduce shorter notation for the angular maximal functions corresponding to the three sectors that will be predominantly used: the plane cut along the real axis \(\mathbb{C}^* = \mathbb{C}_{(0,2\pi)}\), the upper half-plane \(\mathbb{H} = \mathbb{C}_{(0,\pi)}\), and the right half-plane \(\mathbb{C}_+ = \mathbb{C}_{(-\pi/2,\pi/2)}\):
\[ \mathcal{M}_g^*(\rho) = \sup_{0<\theta<2\pi} |g(\rho e^{i\theta})|, \]
\[ \mathcal{M}_g^a(\rho) = \sup_{0<\theta<\pi} |g(\rho e^{i\theta})|, \]
\[ \mathcal{M}_g^+(\rho) = \sup_{|\theta|<\pi/2} |g(\rho e^{i\theta})|. \]

2. The objects of our study are the angular maximal functions associated with the Poisson transform, the Stieltjes transform, and the Laplace transform.
If \( f(t) \) is defined on \( \mathbb{R}_+ \) and \( \int_0^\infty f(t)(t+1)^{-1} dt < \infty \), then the Poisson transform of \( f \) is a harmonic function in \( \mathbb{H} \) (we identify \( z = x + iy \) with the coordinate pair \((x,y)\)),

\[
\mathcal{P}f(x + iy) = \frac{1}{\pi} \int_0^\infty \frac{y}{(t-x)^2 + y^2} f(t) \, dt. \tag{2}
\]

Under the same assumptions, the Stieltjes transform of \( f \) is an analytic function in \( \mathbb{C}_\ast \),

\[
Sf(z) = \int_0^\infty \frac{f(t)}{t-z} \, dt. \tag{3}
\]

(The usage of the name “Stieltjes transform” is common in the case \( -z \in \mathbb{R}_+ \).)

The Laplace transform of function \( f(t) \) defined on \( \mathbb{R}_+ \) (say, \( f \in L^p(\mathbb{R}_+) \) for some \( p \in [1, \infty] \)) is an analytic function in \( \mathbb{C}_+ \),

\[
\mathcal{L}f(z) = \int_0^\infty f(t)e^{-zt} \, dt. \tag{4}
\]

The angular maximal Poisson transform \( \hat{\mathcal{P}} \) sends a function \( f(t) \) defined on \( \mathbb{R}_+ \) into a function of \( \rho \in \mathbb{R}_+ \),

\[
\hat{\mathcal{P}}f(\rho) = \mathcal{M}_{PF}^u(\rho) = \frac{1}{\pi} \sup_{0 < \theta < \pi} \left| \int_0^\infty \frac{\rho \sin \theta}{(t - \rho \cos \theta)^2 + (\rho \sin \theta)^2} f(t) \, dt \right|. \tag{5}
\]

The angular maximal Stieltjes transform \( \hat{S} \) sends a function \( f(t) \) defined on \( \mathbb{R}_+ \) into a function of \( \rho \in \mathbb{R}_+ \),

\[
\hat{S}f(\rho) = \mathcal{M}_{SF}^u(\rho) = \sup_{0 < \theta < 2\pi} \left| \int_0^\infty \frac{f(t)}{t - \rho e^{i\theta}} \, dt \right|. 
\]

The angular maximal Laplace transform \( \hat{\mathcal{L}} \) sends a function \( f(t) \) defined on \( \mathbb{R}_+ \) into a function of \( \rho \in \mathbb{R}_+ \),

\[
\hat{\mathcal{L}}f(\rho) = \mathcal{M}_{LF}^+(\rho) = \sup_{|\theta| < \pi/2} \left| \int_0^\infty e^{-t \rho e^{i\theta}} f(t) \, dt \right|. 
\]

3. Our goal is to obtain \( L^p \) estimates for the maximal transformations introduced above. Here the results are formulated; the proofs follow in the subsequent sections.

The key technical result is the weak \((1,1)\) estimate for the maximal radial Poisson transform.

We use the notation

\[
E_f(\lambda) = \{ x : |f(x)| > \lambda \}, \quad \mu_f(\lambda) = |E_f(\lambda)|.
\]

(Here \(| \cdot |\) denotes the Lebesgue measure of a subset of \( \mathbb{R} \).)
**Theorem 1** The map \( f \mapsto \hat{P}f \) is of weak \((1,1)\) type. Specifically, if \( f \in L^1(\mathbb{R}) \), then for any \( \lambda > 0 \)

\[
\mu_{Pf}(\lambda) \leq K_1 \frac{\|f\|}{\lambda}
\]

(6)

with constant \( K_1 \) independent of \( f \).

An easy corollary, by means of the Marcinkiewicz interpolation theorem, is

**Theorem 2** Let \( f \in L^p(\mathbb{R}) \), \( 1 < p \leq \infty \). Then

\[
\|\hat{P}f\|_p \leq K_2 \|f\|_p,
\]

(7)

where \( K_2 \) depends only on \( p \) but not on \( f \).

Throwing in the \( L^p \) boundedness of the Hilbert transform, the maximal theorem for the Stieltjes transform (aka Cauchy integral) is obtained.

**Theorem 3** Let \( f \in L^p(\mathbb{R}_+) \), \( 1 < p < \infty \). Then

\[
\|\hat{S}f\|_p \leq K_3 \|f\|_p,
\]

(8)

where \( K_3 \) depends only on \( p \) but not on \( f \).

Finally, the maximal theorem for the Laplace transform will be derived by means of Cauchy formula and using an estimate for the Laplace transform observed along rays in the right half-plane. We always assume the familiar relation \( p^{-1} + p'^{-1} = 1 \) between the exponents of the spaces \( L^p \) and \( L^{p'} \).

**Theorem 4** Let \( f \in L^p(\mathbb{R}_+) \), \( 1 \leq p \leq 2 \). Then

\[
\|\hat{L}f\|_{p'} \leq K_4 \|f\|_p,
\]

(9)

where \( K_4 \) depends only on \( p \) but not on \( f \).

4. We start the proofs from an easy end, by demonstrating the relations between Theorem 1 and Theorems 2 and 3 stated above.

If \( f \in L^\infty(\mathbb{R}) \), then for any \( x \in \mathbb{R} \) and \( y > 0 \) the estimate

\[
|Pf(x + iy)| \leq \|f\|\infty
\]

holds due to the “approximate identity” properties of the Poisson kernel. Hence \( \|\hat{P}f\|\infty \leq \|f\|\infty \). This estimate together with (6) immediately leads to (7) due to Marcinkiewicz’s theorem.
Now, if $z \in \mathbb{H}$ and $f$ is a function defined on $\mathbb{R}$, then the Cauchy integral
\[
g(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} \, dt
\]
can be written as $(\mathcal{P}(I + i\mathcal{H})f)(z)$, where $\mathcal{H}$ is the Hilbert transform (convolution with $(\pi t)^{-1}$ understood in the principal value sense). Since the operator $\mathcal{H}$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$, we deduce from (7):
\[
\|M_{\mathcal{H}}(0, \pi \mathcal{H}) f\|_p \leq K_2(p) (1 + \|\mathcal{H}\|_{L^p}) \|f\|_p, \quad 1 < p < \infty.
\]
For $f$ supported on $[0, \infty)$, the function $g$ is analytic in $\mathbb{C}^*$ and the estimate identical to the one just obtained holds true for $\|M_{\mathcal{H}}(\pi, 2\pi) f\|_p$.

Hence (8) follows with
\[
K_3(p) = 2\pi K_2(p) (1 + \|\mathcal{H}\|_{L^p}).
\]

5. To prove Theorem 4, we refer to the following yet unpublished result by the author and A.E. Merzon [2].

Consider the restriction of the Laplace transform to a ray $\arg z = \theta$, where $|\theta| < \pi/2$:
\[
\mathcal{L}_\theta f(\rho) = \int_0^{\infty} f(t) e^{-t\rho e^{i\theta}} \, dt.
\]
Thus, $\mathcal{L}_\theta$ is an operator that sends a function defined on $\mathbb{R}_+$ to a function defined on $\mathbb{R}_+$. The result mentioned extends the Hausdorff-Young theorem (which formally corresponds to the limiting cases $\theta = \pm \pi/2$):
\[
\|\mathcal{L}_\theta f\|_{p'} \leq K_5 \|f\|_p, \quad 1 \leq p \leq 2,
\]
where $K_5$ depends on $p$, but does not depend on $f$ and $\theta$. We emphasize that the estimate is uniform with respect to $\theta$.

If $-\pi/2 < \theta_1 < \theta_2 < \pi/2$ and $f(t)$ is a simple (finitely-supported, with finitely many values) function defined on $\mathbb{R}_+$, then from an estimate $\mathcal{L} f(z) = O(z^{-1})$ uniformly in $C(\theta_1, \theta_2)$ the Cauchy representation easily follows:
\[
\mathcal{L} f(z) = \frac{1}{2\pi i} \left( \int_{\arg \zeta = \theta_2} - \int_{\arg \zeta = \theta_1} \right) \frac{\mathcal{L} f(\zeta)}{\zeta - z} \, d\zeta,
\]
provided that $\theta_1 < \arg z < \theta_2$.

Suppose $1 < p \leq 2$ (the case $p = 1$ is trivial). Using the estimate (11) on both rays of integration and combining it with Theorem 3 we conclude that the inequality
\[
\|M_{\mathcal{L} f}(\theta_1, \theta_2) (\rho)\|_{p'} \leq C \|f\|_p
\]
holds for all simple functions $f$ with $C$ independent of $f$ and of $\theta_1, \theta_2$. Taking the supremum over $(\theta_1, \theta_2) \subset (-\pi/2, \pi/2)$, we obtain (9) for simple functions $f$; the general case follows by density. (The operator $\hat{L}$ is nonlinear; however, we can fix some measurable function $\theta(\rho) : \mathbb{R}_+ \rightarrow (-\pi/2, \pi/2)$, then consider the operator $f(t) \mapsto Lf(\rho e^{i\theta(\rho)})$; this operator is linear and the standard density argument can be applied; finally we take supremum over all functions $\theta(\rho)$, noticing that the constant in the inequality is independent of $\theta(\rho)$.)

6. Finally, we prove Theorem 1. Without loss of generality we may assume $f \geq 0$.

Suppose for simplicity that the supremum in (5) is attained, specifically — at $\theta = \theta_*(R)$. (The set of all simple functions for which this is true is dense in $L^1(\mathbb{R})$.) We will estimate the measure of the set

$$E_{\hat{P}_f}(\lambda) \cap \{R \mid \theta_*(R) \in (0, \pi/2]\}.$$

An identical estimate can be obtained for $|E_{\hat{P}_f}(\lambda) \cap \{R \mid \theta_*(R) \in [\pi/2, \pi]\}|$.

Denote $x_*(R) = R \cos \theta_*(R)$, $y_*(R) = R \sin \theta_*(R)$, and $\delta = \delta(R) = R - x_*(R)$. By our assumption, $x_*(R) \geq 0$. Let us split the Poisson kernel as follows:

$$P(t, y) = \pi^{-1} \frac{y}{y^2 + t^2} = P_1(t, y) + P_2(t, y),$$

where

$$P_1(t, y) = \min(P(t, y), P(\delta, y)),$$

$$P_2(t, y) = \begin{cases} 0, & |t| \geq \delta; \\ P(t, y) - P(\delta, y), & |t| < \delta. \end{cases}$$

Denote

$$g_k(x, y) = \int P_k(x - t, y) f(t) dt, \quad k = 1, 2.$$ 

Weak $L^1$ estimate for $g_1$.

The Poisson kernel is a convex combination of the normalized characteristic functions of centered segments, and $P_1$ is a sub-convex combination of those corresponding to the intervals containing the point $\delta$. The precise statement is given by the following lemma.

**Lemma 1** Let the function $\varphi_y(a), a > \delta$ be defined as

$$\varphi_y(a) = -2a \frac{d}{da} P_1(a, y).$$

Then $\varphi_y(a) > 0$ on $(\delta, \infty)$, $\int_{\delta}^{\infty} \varphi_y(a) da < 1$, and

$$P_1(t, y) = \int_{\delta}^{\infty} \frac{1}{2a} \chi_{[-a,a]}(t) \varphi_y(a) da. \quad (12)$$
Proof. The inequality \( \varphi_y(a) > 0 \) is obvious, since \( P_1(t, y) \) is monotonely decreasing in \( t \) when \( t > \delta \). Then,

\[
\int_{\delta}^{\infty} \varphi_y(a) \, da = -2 \int_{\delta}^{\infty} a \, d_a P_1(a, y)
\]

\[
= -2a P_1(a, y) \bigg|_{a=\delta}^{\infty} + 2 \int_{\delta}^{\infty} P_1(a, y) \, da
\]

\[
= 2\delta P_1(\delta, y) + 2 \int_{\delta}^{\infty} P_1(a, y) \, da
\]

\[
= \int_{-\infty}^{\infty} P_1(a, y) \, da < \int_{-\infty}^{\infty} P(a, y) \, da = 1.
\]

Let us check formula (12). Since the functions \( P_1(t, y) \) and \( \chi_{[-a,a]}(t) \) are even (as functions of \( t \)), we may assume that \( t > 0 \). If \( t < \delta \), then \( \chi_{[-a,a]}(t) = 1 \) for all \( a \geq \delta \) and the right-hand side of (12) becomes

\[
\int_{\delta}^{\infty} \frac{1}{2a} \varphi_y(a) \, da = -\int_{\delta}^{\infty} \frac{d}{da} P_1(a, y) \, da = P_1(\delta, y).
\]

If \( t \geq \delta \), then the right-hand side of (12) becomes

\[
\int_{a>\delta} \frac{1}{2a} \varphi_y(a) \, da = -\int_{t}^{\infty} \frac{d}{da} P_1(a, y) \, da = P_1(t, y),
\]

as required. \( \square \)

We will obtain separate estimates of the measure of “large value sets” for the functions \( g_1 \) and \( g_2 \). Below, \( \lambda_1 \) and \( \lambda_2 \) are some arbitrarily chosen positive numbers. After the separate estimates are obtained, we will specify \( \lambda_1 \) and \( \lambda_2 \) in terms of the threshold value \( \lambda \) from (6).

If \( g_1(x, y) > \lambda_1 \), then, due to Lemma 1, there exists \( a > \delta \) such that

\[
\frac{1}{2a} \int_{x-a}^{x+a} f(t) \, dt > \lambda_1.
\]

With \( x = x_*(R) \), \( y = y_*(R) \), \( \delta = \delta(R) \) we get

\[
\frac{1}{2a} \int_{R-(\delta+a)}^{R+(\delta-a)} f(t) \, dt > \lambda_1.
\]

According to the non-centered Hardy-Littlewood maximal theorem, the set of all \( R \) for which such an inequality holds, has measure not exceeding

\[
\mu_1 = \frac{C_1 \|f\|_1}{\lambda_1}, \quad (13)
\]
with some universal constant $C$. It is important that the segments $[R - (\delta + a), R + (a - \delta)]$ contain $R$—because $a > \delta$.

**Weak $L^1$ estimate for $g_2$.**

We have

$$P_2(t, y) \leq P_2(0, y) = \frac{y}{\pi} \left( \frac{1}{y^2} - \frac{1}{y^2 + \delta^2} \right) = \frac{\delta^2}{\pi y(\delta^2 + y^2)}.$$ 

Therefore, if

$$\int P_2(x_* - t, y_*) f(t) \, dt > \lambda_2,$$  \hspace{1cm} (14) 

then

$$\frac{\delta^2}{\pi y_*(\delta^2 + y_*^2)} > \frac{\lambda_2}{\|f\|_1}.$$ 

Now,

$$R^2 = x_*^2 + y_*^2 = (R - \delta)^2 + y_*^2 = R^2 - 2\delta R + \delta^2 + y_*^2,$$

so

$$y_*^2 + \delta^2 = 2R\delta.$$ 

Thus, the inequality (14) implies

$$\frac{\delta}{2\pi y_* R} > \frac{\lambda_2}{\|f\|_1},$$

hence

$$R < \frac{\delta\|f\|_1}{2\pi y_* \lambda_2}.$$ 

Note that $x_* > 0$, so $0 < R - \delta$, hence $\delta < 2R - \delta$. It follows that

$$\left( \frac{\delta}{y_*} \right)^2 = \frac{\delta^2}{\delta(2R - \delta)} < 1,$$

and

$$R < \frac{\|f\|_1}{2\pi \lambda_2}.$$ 

Thus the measure of the set of all such $R$ for which $g_2 > \lambda_2$ does not exceed

$$\mu_2 = \frac{\|f\|_1}{2\pi \lambda_2}.$$ 

Put $\lambda_1 = \lambda_2 = \lambda/2$. If $g(R) = g_1(R) + g_2(R) > \lambda$, then at least one of the two inequalities holds: $g_1(R) > \lambda_1$ or $g_2(R) > \lambda_2$. Thus

$$E_g(\lambda) \subset E_{g_1}(\lambda_1) \cup E_{g_2}(\lambda_2),$$
so

\[ \mu_g(\lambda) \leq \mu_1 + \mu_2 \leq \frac{C_1 \| f \|_1}{\lambda/2} + \frac{1}{2\pi} \frac{\| f \|_1}{\lambda/2} = C \| f \|_1, \]

with \( C = 2C_1 + \pi^{-1}. \)

The classical theorems of harmonic analysis used in this paper — Marcinkiewicz’s theorem, \( L^p \)-boundedness of the Hilbert transform, properties of the Poisson integral, Hardy-Littlewood maximal theorem — can be found, for example, in [1].

References

[1] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education Inc., 2004.

[2] A. Merzon and S. Sadov, Hausdorff-Young type theorems for the Laplace transform restricted to a ray or to a curve in the complex plane, 2009, (in preparation).