Finite-time stabilization control of quantum systems

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Abstract

This paper proposes a non-smooth control scheme to achieve finite-time convergence to an eigenstate of the internal Hamiltonian of a given closed quantum system. First, we define finite-time stability for quantum systems, and then present a Lyapunov stability criterion to identify the finite-time stability of quantum systems. Second, we propose a new non-smooth control law for two-level quantum systems by selecting a distance between quantum states as a Lyapunov function and using the Lyapunov stability theory, and prove the existence and uniqueness of solutions of the quantum system under the action of the non-smooth controller in the framework of Bloch vectors. Further, an equivalent transformation is made for the system model and the control target by expressing quantum states in terms of complex exponentials, and the finite-time stability of the system is proved by combining the finite-time Lyapunov stability criterion with the homogeneity theorem. Finally, we perform numerical simulations on a spin-1/2 particle system and demonstrate the effectiveness of the finite-time stabilization control scheme.

Key words: quantum systems, finite-time stability, non-smooth control, quantum control, finite-time convergence

1 Introduction

The frontier technologies such as nanotechnology, nuclear magnetic resonance (NMR), and ultrafast laser make it possible to manipulate matter at the level of a single atom or a single electron, which has been motivating the rise of quantum control. As a new multidisciplinary research field, quantum control promotes the development of quantum computing, quantum communication, quantum optics, quantum chemistry and other fields [1]. One of important approaches to studying the control of quantum systems is to extend the concepts and methods in control theory to microscopic quantum systems. For instance, quantum optimal control [2,3], quantum Lyapunov control [4,5,6,7], sliding mode control [8,9], quantum $H_{\infty}$ control [10,11], structure decomposition of linear quantum systems [12], fault-tolerant quantum control and filtering [13,14], and quantum control based on machine learning [15,16,17] have been extensively investigated. The designed control laws based on these methods are usually smooth or discontinuous. Generally, smooth control laws achieve infinite-time convergence of the system state to the target state, while discontinuous control laws may lead to the non-existence of solutions of the system dynamics so that the system stops evolving toward the target state although they can speed up the control process.

In order to incorporate the rapidness and convergence of the control process, the non-smooth control between smooth control and discontinuous control was proposed in 1961 [18] and can be used to achieve finite-time convergence of the system. At present, the design of non-smooth controllers mainly focuses on some systems with special structures, e.g., upper triangular systems [19], lower triangular systems [20], and integral systems [21]. For upper triangular nonlinear systems, Ref. [22] proposed a non-smooth stabilizing controller by combining the nested saturation method [23] and the Lyapunov method [24] and achieved the global stabilization of the system in an infinite time interval. On the basis of [22], Ref. [19] designed a global finite-time stabilizing controller, which is suitable for a broader class of upper triangular systems, by further considering the adding-a-power integrator technique [25]. For lower triangular nonlinear systems, several global finite-time stability results were obtained in [20] by using the same adding-a-power integrator technique when the system parameters...
satisfy certain conditions. Ref. [26] considered more universal $P$-normal-form lower triangular systems and designed a more general finite-time stabilizing controller by constructing a Lyapunov function different from that in [20]. For integral systems, Ref. [27] designed a globally finite-time stabilizing controller for third-order systems via the adding-a-power integrator technique and gave a method for the selection of parameters contained in the controller. Ref. [28] designed a global finite-time stabilizing controller for higher-order systems by considering the Hurwitz stability of the system polynomial. However, it is hard to determine the parameters in the controller. In the above mentioned literature, to prove the finite-time stability of the closed-loop systems, the methods based on the Lyapunov stability theory [19,20,22,26,27] and the homogeneity theory [28] have been used. It should be noted that there is no uniform non-smooth control framework for general nonlinear systems without special structures. In this case, dependent on different systems, a special dealing is usually necessary.

Another property that non-smooth control is different from continuously differentiable control or smooth control is that the Lipschitz continuity condition in the usual sense does not hold any longer in the non-smooth control systems. Therefore, the existence and uniqueness of solutions cannot be guaranteed. However, for the aforementioned three types of systems, the existing literatures do not discuss the uniqueness of solutions. This is because that the corresponding closed-loop systems do not satisfy the Lipschitz condition only at the origin. For an other system, e.g., the double integral system in [29], the closed-loop system under the action of the designed non-smooth controller may be non-Lipschitz continuous at points other than the origin. In this case, it is essential to discuss the uniqueness of solutions. For example, the notion of transversality has been used to demonstrate the uniqueness of solutions of the closed-loop system at points dissatisfying the Lipschitz continuity [30]. Readers can refer to [31] and [32] for more other methods to show the existence and uniqueness of solutions of non-smooth systems.

This paper considers the non-smooth control problem of quantum systems and aims at achieving the finite-time convergence of a given system to the target state. Since the models of quantum control systems are not of the above special structures, the design methods of finite-time stabilizing controllers and the stability proof methods existing in classical systems cannot be directly applied to quantum systems. The main contributions of this paper are summarized as follows. First, we consider the finite-time convergence control problem of quantum systems. Although the optimal control problem with a fixed terminal time also can be regarded as a finite-time control problem, the exact convergence of the system to the target state within a finite time may not be able to guarantee. Furthermore, the optimal control often suffers from complicated numerical computation problems. Second, the definition of finite-time stability of quantum systems is presented and a finite-time Lyapunov stability criterion is proposed to identify finite-time stability of quantum systems. Third, via the Lyapunov stability theory, we further propose a non-smooth control law (i.e., continuous and non-differentiable control law) with a fractional power factor, where the fractional power factor is a function of the system state. This new control law enables the finite-time convergence of the system. Fourth, for two-level quantum systems, we prove the existence and uniqueness of solutions of the system with the non-smooth control. Due to the non-smoothness of the control law, the vector field of the controlled quantum system no longer satisfies the Lipschitz continuity condition. Therefore, it is important to ensure that the quantum system under the control field has a unique solution, which does not need to be considered in the usual smooth control. Finally, by expressing quantum states in terms of complex exponentials, the homogeneity theory and the finite-time Lyapunov stability criterion are simultaneously used to prove the finite-time stability of the controlled quantum system, i.e., the target state is reached within a finite time.

This paper is organized as follows. Section 2 introduces the considered quantum system models in the Schrödinger picture and their counterparts in the corresponding Bloch space, and then presents the definition of finite-time stability of quantum systems and a Lyapunov criterion for finite-time stability. In Section 3, based on the Lyapunov stability theory, we design a non-smooth controller for a two-level quantum system and prove the existence and uniqueness of solutions of the “closed-loop” system under the action of the controller. The finite-time convergence of the “closed-loop” system to the target state, i.e., finite-time stability of the “closed-loop” system, is analyzed and proved in detail in Section 4. Section 5 presents several numerical examples to demonstrate the effectiveness of the proposed finite-time control scheme. Conclusions are presented in Section 6.

Notation. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ be the set of non-negative real numbers, $\nabla$ be a vector differential operator, $\langle \cdot, \cdot \rangle$ denote an inner product operation, and $[A, B]$ denote the commutator between $A$ and $B$. The two unit state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ satisfying $|\psi_1\rangle = e^{i\theta}|\psi_2\rangle$ are called equivalent state vectors, and the set of all equivalence state vectors of $|\psi\rangle$ forms an equivalence class of $|\psi\rangle$. In physics, equivalent state vectors have the same observation meaning, and therefore we can ignore the global phase between equivalent states and regard all states in an equivalence class as the same state. A continuous function is called $C^0$-smooth; a function with up to $n$—order continuous derivatives is called $C^n$-smooth; a function with any order continuous derivative is called a smooth function or a $C^\infty$ function.
2 Finite-time stability of quantum systems

In this section, we present several concepts of finite-time stability of quantum systems and the finite-time stability discrimination criteria based on Lyapunov stability theory.

2.1 Basic concepts of finite-time stability

For a general $n$-dimensional closed controllable quantum system, its state is fully described by a unit column vector $|\psi\rangle$ in the Hilbert space defined on $\mathbb{C}^n$ and its dynamics evolution obeys the following Schrödinger equation

$$|\dot{\psi}(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle = -\frac{i}{\hbar}\left(\mathcal{H}_0 + \sum_{k=1}^{r}H_ku_k\right)|\psi(t)\rangle,$$  \hspace{0.5cm} (1)

where $H_0$ is the internal Hamiltonian of the system, $H_k$ is the control Hamiltonian that describes the interaction between the external control fields and the system, $H_0$ and $H_k$ are time-independent; $\hbar$ is the reduced Planck constant and we set $\hbar = 1$ in this paper; $u_k$ is an external control field to be designed.

This paper considers the control problem of finite-time convergence to an eigenstate $|\psi_f\rangle$ of the internal Hamiltonian $H_0$ of the system (1), that is, the aim is to achieve finite-time convergence to $|\psi_f\rangle$ by designing $u_k$ as a certain non-smooth control law. It should be pointed out that the solutions of the system (1) under the non-smooth control is continuously differentiable. Since the quantum system with the non-smooth control no longer satisfies the Lipschitz condition, we can prove the uniqueness of solutions via the method based on the transversality between the system vector field and a non-Lipschitz set [30]. In the Bloch vector framework, the quantum state $|\psi\rangle$ in the Hilbert space can be expressed in terms of the vector $s$ in the Bloch space as

$$|\psi\rangle\langle\psi| = \rho = s_0\sigma_0 + \frac{1}{2}\sum_{\kappa=1}^{n^2-1}s_\kappa\sigma_\kappa = \frac{I_n}{n} + \frac{1}{2}\sum_{\kappa=1}^{n^2-1}s_\kappa\sigma_\kappa,$$  \hspace{0.5cm} (2)

where $\rho$ denotes the density operator, $\{\sigma_\kappa\}_{\kappa=0}^{n^2-1}$ is a set of orthogonal bases of the $n \times n$ complex Hermitian matrix space, $\sigma_0 = \sqrt{n}$, and $s_0 = \sqrt{n}$. The $(n^2 - 1)$-dimensional real vector $(s_1, \ldots, s_{n^2-1}) \triangleq s = (\text{tr}(\rho\sigma_1), \ldots, \text{tr}(\rho\sigma_{n^2-1})) \in \mathbb{R}^{n^2-1}$ ($n > 1$) is called the Bloch vector in a selected basis, and the set of all Bloch vectors forms the Bloch space $\mathcal{B}(\mathbb{R}^{n^2-1})$ [33].

For simplicity, we only consider the case of one control field and the generalization to multiple control fields is straightforward. By expressing the quantum state in terms of the Bloch vector $s$, the quantum system (1) can be written as [34]

$$\dot{s}_i(t) = (A_0 + u_1A_1)s_i(t),$$  \hspace{0.5cm} (3)

$$\dot{s}_f(t) = A_0s_f(t),$$  \hspace{0.5cm} (4)

where $s_f(t)$ is the Bloch vector associated with the target state $|\psi_f\rangle$, $A_0$ and $A_1$ are the following antisymmetric matrices

$$A_0 (m, n) = \text{tr}(i\mathcal{H}_0[\sigma_m, \sigma_n]),$$  \hspace{0.5cm} (5)

$$A_1 (m, n) = \text{tr}(i\mathcal{H}_1[\sigma_m, \sigma_n]).$$  \hspace{0.5cm} (6)

Denote the system (3) as

$$\dot{s}(t) = f(s(t)), \ s(t) \in \mathcal{B}(\mathbb{R}^{n^2-1})$$  \hspace{0.5cm} (7)

where $f : \mathcal{B}(\mathbb{R}^{n^2-1}) \to \mathcal{B}(\mathbb{R}^{n^2-1})$ is a continuous function defined on $\mathcal{B}(\mathbb{R}^{n^2-1})$.

Let the initial moment be $t_0 = 0$. For $t \in \mathbb{R}^+$, if there exists a continuously differentiable function $s(t)$ such that (7) holds, then the function $s(t) : \mathbb{R}^+ \to \mathcal{B}(\mathbb{R}^{n^2-1})$ is called a solution of the system (7). To illustrate the concept of finite-time stability, we assume that the quantum system (7) has a unique solution in the space $\mathcal{B}(\mathbb{R}^{n^2-1})$.

For any initial vector $s_0 \in \mathcal{B}(\mathbb{R}^{n^2-1})$, we denote the unique solution of the system (7) as $s(t, s_0)$ ($t \geq 0$). Then, $s(t, s_0)$ defines a flow from $\mathbb{R}^+ \times \mathcal{B}(\mathbb{R}^{n^2-1})$ to $\mathcal{B}(\mathbb{R}^{n^2-1})$ when $s_0$ varies.

Now, we give the definition of finite-time stability of the quantum system (7).

**Definition 1** For the quantum system (7), the target vector $s_f$ is said to be finite-time stable if for an arbitrarily given initial vector $s_0 \in \mathcal{B}(\mathbb{R}^{n^2-1})$, there exists a continuous function $T(s_0) : \mathcal{B}(\mathbb{R}^{n^2-1}) \to [0, \infty)$ such that the unique solution $s(t, s_0)$ of the system (7) satisfies $\lim_{t \to T(s_0)} s(t, s_0)$ and $s(t, s_0) = s_f$ for $t \geq T(s_0)$. Particularly, we define $T(s_f) = 0$ when the initial vector is $s_0 = s_f$. The continuous function $T(s_0)$ is referred to as the finite convergence time corresponding to $s_0$.

Some properties of the solution $s(t, s_0)$ and the finite convergence time $T(s_0)$ can be summarized in the following proposition.

**Proposition 2** Assume that $s_f$ is the finite-time stable target vector of the system (7). Let $s_0 \in \mathcal{B}(\mathbb{R}^{n^2-1})$, $t, t_1 \in \mathbb{R}^+$, and the finite convergence time function be $T(s_0) : \mathcal{B}(\mathbb{R}^{n^2-1}) \to [0, \infty)$ as in Definition 1, then the following conclusions hold:

(i) $s(0, s_0) = s_0$;

(ii) $s(T(s_0) + t, s_0) = s_f$;
(iii) \( s(t, s(t_1, s_0)) = s(t + t_1, s_0) \);

(iv) The function \( T(s_0) \) can be written as

\[
T(s_0) = \inf \{ t \in \mathbb{R}_+ : s(t, s_0) = s_f \};
\]

(v) Further, it follows from (ii)-(iv) that

\[
T(s(t, s_0)) = \max \{ T(s_0) - t, 0 \}.
\]

**Example 3** Consider the scalar differential equation

\[
\dot{y}(t) = -k \text{sign}(y(t)) |y(t)|^\alpha
\]

where \( \text{sign}(0) = 0, k > 0, \) and \( \alpha \in (0, 1) \).

Since the right-hand side of (8) is continuous everywhere and the local Lipschitz condition is always satisfied outside the origin, the system (8) has a unique solution for any initial condition \( y_0 \in \mathbb{R} \). By direct integration, the solution of the system (8) can be obtained as

\[
\mu(t, y_0) = \begin{cases} 
\text{sign}(y_0) \frac{|y_0|^{1-\alpha} - k (1-\alpha) t^{1-\alpha}}{k(1-\alpha)}, & (t < \frac{|y_0|^{1-\alpha}}{k(1-\alpha)}, y_0 \neq 0) \\
0, & (t \geq \frac{|y_0|^{1-\alpha}}{k(1-\alpha)}, y_0 \neq 0) \\
0, & (t \geq 0, y_0 = 0).
\end{cases} \tag{9}
\]

It is known from (9) that the finite-time convergence function is \( T(y_0) = \frac{1}{k(1-\alpha)}|y_0|^{1-\alpha} \). The Lyapunov function \( V(y) = y^2 \) can be used to prove that the origin of the system (8) is globally finite-time stable. Here, we omit the proof for brevity.

### 2.2 Lyapunov theorem of finite-time stability

We first give the following comparison lemma:

**Lemma 4** [35] Let \( V \) be a Lyapunov function defined on \( \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^{n-1}) \) and assume that

\[
\dot{V}_E(t, m) \leq \gamma(t, V(t, m)), \tag{10}
\]

where \( (t, m) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^{n-1}) \); \( E \) denotes the differential equation \( \dot{x} = F(t, x); \dot{V}_E(t, m) \) represents the time derivative of the Lyapunov function \( V \) along any trajectory of the differential equation \( E \); \( \gamma : \mathbb{R}_+ \times \mathcal{B} \rightarrow \mathbb{R} \) is a continuous function. Further, we assume that the initial value problem \( \dot{m} = \gamma(t, m), m(t_0) = m_0 \) has a unique solution \( m(t, m_0) \) in the interval \( [t_0, T] \), where \( 0 \leq t_0 < T \leq +\infty \). Let \( x(t) = x(t_0, T) \) be any solution of \( E \) with \( V(t_0, x(t_0)) \leq m_0 \). Then, \( V(t, x(t)) \leq m(t, m_0) \) holds for every \( t \in [t_0, T] \).

Based on Lemma 4, we can give the following finite-time stability theorem for the quantum system (7).

**Theorem 5** For the quantum system (7), suppose that \( s_f \) is a target vector and there exists a continuously differentiable function \( V : \mathcal{B}(\mathbb{R}^{n-1}) \rightarrow \mathbb{R} \) such that the following conditions are satisfied:

(i) \( V \) is positive definite;

(ii) For \( s_0 \in \mathcal{B}(\mathbb{R}^{n-1}) \), there exist two positive real numbers \( c > 0 \) and \( \alpha \in (0, 1) \) such that

\[
\dot{V}(s(t, s_0)) + c(V(s(t, s_0)))^\alpha \leq 0. \tag{11}
\]

Then, the system (7) is finite-time stable, that is, it converges to the target vector \( s_f \) within a finite time. And the finite convergence time function \( T(s_0) \) satisfies

\[
T(s_0) \leq \frac{1}{c(1-\alpha)}V(s_0)^{1-\alpha}. \tag{12}
\]

**proof.** Let us consider (8) in Example 3. When \( y(t) = V(s(t, s_0)) \) and \( k = c \), it can be simplified as

\[
\dot{V}(s(t, s_0)) = -c(V(s(t, s_0)))^\alpha. \tag{13}
\]

For \( t \in \mathbb{R}_+ \) and \( s_0 \in \mathcal{B}(\mathbb{R}^{n-1}) \), applying Lemma 4 to the differential inequality (11) and the scalar differential equation (13) yields

\[
V(s(t, s_0)) \leq \mu(t, V(s_0)), \tag{14}
\]

where \( \mu \) can be written from (9) as

\[
\mu(t, V(s_0)) = \begin{cases} 
\frac{|V(s_0)^{1-\alpha} - c (1-\alpha) t^{1-\alpha}}{c(1-\alpha)}, & (t < \frac{V(s_0)^{1-\alpha}}{c(1-\alpha)}, s_0 \neq s_f) \\
0, & (t \geq \frac{V(s_0)^{1-\alpha}}{c(1-\alpha)}, s_0 \neq s_f) \\
0, & (t \geq 0, s_0 = s_f).
\end{cases} \tag{15}
\]

According to (15), when \( t \geq \frac{1}{c(1-\alpha)}(V(s_0))^{1-\alpha} \) with \( s_0 \in \mathcal{B}(\mathbb{R}^{n-1}) \), the right-hand side of (14) is equal to zero and therefore \( V(s(t, s_0)) = 0 \), that is,

\[
s(t, s_0) = s_f. \tag{16}
\]

Since \( s(t, s_0) \) is continuous, it follows that \( \inf \{ t \in \mathbb{R}_+ : s(t, s_0) = s_f \} > 0 \) for \( s_0 \in \mathcal{B}(\mathbb{R}^{n-1})\setminus s_f \); and \( \inf \{ t \in \mathbb{R}_+ : s(t, s_0) = s_f \} < \infty \) for \( s_0 \in \mathcal{B}(\mathbb{R}^{n-1}) \). Define \( T(s_0) : \mathcal{B}(\mathbb{R}^{n-1}) \rightarrow \mathbb{R}_+ \) as \( \inf \{ t \in \mathbb{R}_+ : s(t, s_0) = s_f \} \), then Definition 1 and Proposition 2 guarantee that the system (7) is finite-time stable and \( s_f \) is the finite-time stable target vector of the system (7). Furthermore, it is clear from (14)-(16) that (12) holds.

Theorem 5 can be regarded as a Lyapunov criterion for the finite-time stability of the quantum system (7). In
addition, the theory based on homogeneity also can be used to determine the finite-time stability of the system. We list some concepts and results associated with homogeneity in the Appendix, and prove the finite-time stability of the quantum system (1) with two energy levels by combining Theorem 5 and the homogeneity theory in Section 4.

3 Design of finite-time convergent controller for two-level quantum systems

A two-level quantum system may form a two-state qubit as an information unit in quantum communication and quantum computation, and information processing can be achieved by controlling the quantum state. In this section, for two-level quantum systems, we design the control law $u_k$ in (1) via the Lyapunov stability theory to realize the finite-time convergence of the system to the eigenstate $|\psi_f\rangle$ of $H_0$.

We assume that the internal and control Hamiltonians of the system (1) when $n = 2$ are given as

$$H_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \quad (17)$$

Note that the two eigenstates of $H_0$ are $|0\rangle = [1 \ 0]^T$ and $|1\rangle = [0 \ 1]^T$, respectively. Further, we assume that the target state is $|\psi_f\rangle = |1\rangle$ and therefore have

$$H_0|\psi_f\rangle = -|\psi_f\rangle. \quad (19)$$

We choose the following function based on the Hilbert-Schmidt distance as a Lyapunov function [36], that is,

$$V = 1 - |\langle \psi_f | \psi \rangle|^2. \quad (20)$$

Its first-order time derivative can be calculated as

$$\dot{V} = -2u_1|\langle \psi | \psi_f \rangle|\text{Imag}[e^{i\zeta \langle \psi | \psi_f \rangle} \langle \psi_f | H_1 | \psi \rangle]. \quad (21)$$

To guarantee $\dot{V} \leq 0$, we design a control law with a fractional power as

$$u_1 = K\text{sign}(\phi_\alpha(|\psi\rangle))|\phi_\alpha(|\psi\rangle)|^\alpha, \quad (22)$$

where $K > 0$, $\phi_\alpha(|\psi\rangle) = \text{Imag}[e^{i\zeta \langle \psi | \psi_f \rangle} \langle \psi_f | H_1 | \psi \rangle]$, and $\alpha \in (0, 1)$. It is easily verified that the control law (22) is non-smooth, namely, continuous and non-differentiable.

We apply the homogeneity criterion for finite-time stability of Lemma 6 in the Appendix to prove that the controller (22) will achieve the finite-time stability of the system (1). To this end, we need to calculate the homogeneous degree of the system. By expressing complex numbers in terms of their complex exponentials, the controlled quantum state can be written as

$$|\psi\rangle = [x_1, x_2]^T = r_1 e^{i\phi_\alpha}|0\rangle + r_2 e^{i\phi_\alpha}|1\rangle, \quad (23)$$

where $r_1$ and $r_2$ are non-negative real numbers satisfying $r_1^2 + r_2^2 = 1$; $e^{i\phi_\alpha}$ and $e^{i\phi_\alpha}$ are the global phase factors of $|\psi\rangle$ with $\phi_\alpha, \phi_\beta \in \mathbb{R}$, $\phi_\beta - \phi_\alpha \neq \phi$ is called the relative phase of $|\psi\rangle$. Particularly, we define the phase of $x_j$ as 0 when $x_j = 0$ ($j = 1, 2$).

Considering (23), we have $\langle \psi | \psi_f \rangle = r_2 e^{-i\phi_\beta}$, $e^{i\zeta \langle \psi | \psi_f \rangle}$ $\langle \psi_f | H_1 | \psi \rangle = e^{-i\phi_\beta}ir_1 e^{i\phi_\alpha} = ir_1 e^{i(\phi_\alpha - \phi_\beta)}$, and $\phi_\alpha(|\psi\rangle) = \text{Imag}[e^{i\zeta \langle \psi | \psi_f \rangle} \langle \psi_f | H_1 | \psi \rangle] = r_1 \cos(\phi_\alpha - \phi_\beta) = r_1 \cos \phi$.

Thus, (20)-(22) can be written as

$$V = 1 - |\langle \psi_f | \psi \rangle|^2 = r_1^2, \quad (24)$$

$$\dot{V} = -2u_1|\langle \psi | \psi_f \rangle|\text{Imag}[e^{i\zeta \langle \psi | \psi_f \rangle} \langle \psi_f | H_1 | \psi \rangle] = -2Kr_1|\cos \phi|^{\alpha+1},$$

$$u_1 = K\text{sign}(\phi_\alpha(|\psi\rangle))|\phi_\alpha(|\psi\rangle)|^\alpha = K\text{sign}(r_1 \cos \phi)|r_1 \cos \phi|^\alpha. \quad (26)$$

Note that $\dot{V} = 0$ will occur in the process of the transition of the system state toward the target state $|\psi_f\rangle$, which means that $r_2 = 0$ or $r_1 = 0$ or $\cos \phi = 0$. When $r_2 = 0$, the system is in $|\psi\rangle = [1 \ 0]^T$. In this case, it follows from (26) that $u_1 \neq 0$, which shows that the system state is transferring toward the target state $|\psi_f\rangle$. When $r_1 = 0$, the system is in $|\psi\rangle = |\psi_f\rangle$. In this case, it follows from (24)-(26) that $V = 0, \dot{V} = 0$, and $u_1 = 0$. Further, considering the positive definiteness of $V$ and the negative definiteness of $\dot{V}$, we know that the system will be stabilized in the equivalence class of the target state after the target state $|\psi_f\rangle$ is reached. For the case of $\cos \phi = 0$, we denote the quantum state before the target state is achieved by $\psi_q$. Since $u_1 = 0$ is satisfied at $t_q$ and the two eigenvalues of the internal Hamiltonian are mutually different, the relative phase $\phi$ will continue evolving, that is, there exists $t_1 > t_q$ such that $\cos \phi(t) \neq 0$ ($t_q < t_1 \leq t_1$). It can be known from (26) that $u_1(t) \neq 0$ ($t_q < t_1 \leq t_1$). Thus, the system state will keep evolving toward the target state $|\psi_f\rangle$ and will not remain at $|\psi_q\rangle$ forever, that is, the transition moment $t_q$ associated with the transition state $|\psi_q\rangle$ forms a zero-measure set. This means that the transition states and the corresponding transition moments do not change the finite-time stability of the control system.

Since the system (1) under the action of the controller (26) does not satisfy the Lipschitz continuity condition,
The Bloch spherical coordinate.

The correspondence between the system state (27) and the angle between the vector on the Bloch sphere, the evolution trajectory of the system (1) in the time interval [0, 1] within a finite time for any initial state.

Theorem 6 Under the action of the controller (26), the two-level quantum system (1) with the Hamiltonians as shown in (17) and (18) has a unique continuously differentiable solution for every initial state.

Proof. In the Bloch spherical coordinate frame (as shown in Fig. 1), any Bloch vector of the two-level quantum system is represented by the vector $s = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ in the unit sphere of the three-dimensional Euclidean space. In this case, the wave function of the system can be written as

$$|\psi(\theta, \phi)\rangle = \begin{bmatrix} \cos \frac{\theta}{2}, \ e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix}^T,$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ [37]. The relative phase $\phi$ of $|\psi\rangle$ is the angle between the projection of the vector $s$ on the $-y$ plane and the positive $x$-axis, and $\theta$ is the angle between the vector $s$ and the positive $z$-axis. The correspondence between the state (27) and the Bloch spherical coordinate $s$ is one-to-one.

For any initial state outside the set $\mathcal{O} = \{|\psi\rangle : \cos \phi = 0\}$, the vector field of the system (1) with the control law (26) is Lipschitz everywhere. The system (1) in this case has a unique solution. In what follows, we discuss the solution of the system (1) for the initial state in the set $\mathcal{O}$.

When the initial state is given in the set $\{|\psi\rangle : \cos \phi = 0\}$, all quantum states satisfying $\cos \phi = 0$ form a longitude circle such that $\phi = \frac{\pi}{2} + q\pi (q = \ldots, -1, 0, +1, \ldots)$ holds on the Bloch sphere. Since each quantum state of $\{|\psi\rangle : \cos \phi = 0\}$ is a transition state $|\psi_q\rangle$, when $|\psi\rangle = |\psi_q\rangle$ at $t_q$, there exists $t_1$ such that the relative phase $\phi$ changes from $\phi(t_0) = \frac{\pi}{2} + q\pi$ to $\phi(t) \neq \frac{\pi}{2} + q\pi (q = \ldots, -1, 0, +1, \ldots)$ in the time interval $[t_0, t_1]$. Reflected on the Bloch sphere, the evolution trajectory of the system (1) in the time interval $[t_0, t_1]$ intersects the longitude circle of $\phi = \frac{\pi}{2} + q\pi (q = \ldots, -1, 0, +1, \ldots)$ in a non-overlap manner. That is to say, the vector field $f$ of the "closed-loop" system (1) is transversal to the non-Lipschitz set $\{|\psi\rangle : \cos \phi = 0\}$ on the Bloch sphere. It is known from [30] that the system (1) in this case has a unique solution for every initial condition in $\{|\psi\rangle : \cos \phi = 0\}$. When the initial state satisfies $r_1 = 0$, we have $|\psi_0\rangle = |\psi_f\rangle$. In this case, the Lyapunov stability theorem guarantees that the state of the system always stays in $|\psi_0\rangle = |\psi_f\rangle$, that is, the system (1) has a unique solution $|\psi_f\rangle$.

To sum up, the system (1) has a unique solution for any initial state.

Remark 7 The notion of transversality is involved in the proof of Theorem 6, which is a description of how two objects intersect. For two intersecting curves, if they are not tangent, then these two curves are said to be transversal each other. By the phrase that the two curves are tangent, we mean that the angle between the tangent lines of the two curves at the intersection point is zero. Readers can refer to [38] for more general concepts and criteria of transversality.

4 Analysis of finite-time stability of two-level quantum control systems

For two-level quantum systems, we have the following finite-time stability theorem.

Theorem 8 Under the action of the control law (26), the system (1) with the Hamiltonians in (17) and (18) is globally finite-time stable, that is, the system will be stabilized in the equivalence class of the target state $|\psi_f\rangle = \{0, 1\}^T$ within a finite time for any initial state.

Proof. By considering (23) and the relative phase $\phi = \phi_b - \phi_a$ of $|\psi\rangle$, (1) can be written as

$$\begin{bmatrix} r_1 \ e^{i\phi_a} + ir_1 \ e^{i\phi_a} \phi_a \\ r_2 \ e^{i\phi_a} + ir_2 \ e^{i\phi_a} \phi_b \end{bmatrix} = -i \begin{bmatrix} r_1 \ e^{i\phi_a} \\ -r_2 \ e^{i\phi_a} \end{bmatrix} - iu_1 \begin{bmatrix} -ir_2 \ e^{i\phi_b} \\ ir_1 \ e^{i\phi_a} \end{bmatrix},$$

that is,

$$\begin{align*}
\dot{r}_1 &= -u_1 r_2 \cos \phi = -Kr_1^\alpha r_2 |\cos \phi|^{\alpha+1} \\
\dot{r}_1 \phi_a &= -r_1 - u_1 r_2 \sin \phi = -r_1 - Kr_1^\alpha r_2 |\cos \phi|^{\alpha} \sin \phi \\
\dot{r}_2 &= -u_1 r_1 \cos \phi = K[r_1 \cos \phi]^{\alpha+1} \\
\dot{r}_2 \phi_b &= r_2 - u_1 r_1 \sin \phi = r_2 - Kr_1^\alpha |\cos \phi|^{\alpha} \sin \phi.
\end{align*}$$

Since the system (29) is obtained by an equivalent transformation of the system (1), it is known from Theorem 6 that the system (29) also has a unique solution. Thus, we can regard $|\cos \phi|^{\alpha+1}$ in (29) as a function of time $t$. 

---

Fig. 1. The Bloch vector space of two-level quantum systems.

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Let $|\cos \phi|^{\alpha+1} = g(t)$, and then we have
\[
\begin{aligned}
\dot{r}_1 &= -Kr_1^\alpha r_2 g(t) \\
\dot{r}_2 &= Kr_1^{\alpha+1} g(t).
\end{aligned}
\tag{30}
\]

The analysis of the transition of the controlled quantum state $|\psi(t)\rangle = [x_1, x_2]^T$ defined on $\mathbb{C}^2$ from the initial state $|\psi_0\rangle$ to the target state $|\psi_f\rangle = [0, 1]^T$ is equivalent to the analysis on whether the controlled variable $(r_1, r_2)^T$ defined on $\mathbb{R}^2$ can be stabilized to the target point $[0, 1]^T$ from the initial point $(r_1(0), r_2(0))^T$. Since $r_1^2 + r_2^2 = 1$, we only need to consider whether the controlled variable $r_1$ defined on $\mathbb{R}_+$ can be stabilized to the origin 0 from the initial point $r_1(0)$. Expressing $r_2$ with $r_1$, we have
\[
r_2 = (1 - r_1^2)^{\frac{1}{2}} = 1 - \sum_{j=1}^{\infty} \frac{C_{2j}^j}{2^{2j} \times (2j - 1)} r_1^{2j}.
\tag{31}
\]

Substituting (31) into the first equation of (30) gives
\[
\dot{r}_1 = -Kr_1^\alpha g(t) + \sum_{j=1}^{\infty} \frac{C_{2j}^j K r_1^{\alpha+2j} g(t)}{2^{2j} \times (2j - 1)}.
\tag{32}
\]

For convenience in analysis, we write (32) as
\[
\dot{r}_1 = f(r_1) = p_0(r_1) + \sum_{j=1}^{\infty} p_j (r_1) = \sum_{j=0}^{\infty} p_j (r_1),
\tag{33}
\]
where $p_0(r_1) = -Kr_1^\alpha g(t)$ and $p_j(r_1) = \frac{C_{2j}^j K r_1^{\alpha+2j} g(t)}{2^{2j} \times (2j - 1)} (j \geq 1)$.

Thus, we only need to prove the system (33) is finite-time stable. The proof can be divided into two steps.

**Step 1** The following system defined by
\[
\dot{r}_1 = p_0 (r_1)
\tag{34}
\]
is finite-time stable.

**Step 2** The system (33) is globally finite-time stable.

**Proof of Step 1.**

According to Lemma 15 in the Appendix, in order to prove the finite-time stability of the system (34), we only need to verify that the system (34) is asymptotically stable and has a negative degree of homogeneity.

**Asymptotic stability.** For the Lyapunov function $V(r_1) = r_1^2$, we calculate its Lie derivative along any trajectory of the system (34) and have
\[
L_{p_0} V(r_1) = (\nabla V(r_1), p_0(r_1)) = 2r_1 p_0(r_1) = -2Kr_1^{\alpha+1} g(t).
\tag{35}
\]

It can be known from (35) that the Lyapunov function $V(r_1)$ is non-increasing and $L_{p_0} V(r_1)$ is bounded. The latter implies that $L_{p_0} V(r_1)$ is uniformly continuous, and therefore the Barbalat’s lemma [35] guarantees that $L_{p_0} V(r_1) \to 0$ as $t \to \infty$. Considering $g(t) > 0$, we have $r_1 \to 0$, that is, the system (34) is asymptotically stable.

**Degree of homogeneity.** According to Definition 12 in the Appendix, when $0 < \alpha < 1$ and the dilation is taken as $\delta_1^L$, the vector field of $p_0(r_1)$ satisfies
\[
p_0(\varepsilon r_1) = \varepsilon^\alpha p_0(r_1) = \varepsilon^{1+\alpha} p_0(r_1).
\tag{36}
\]

Therefore, the degree of homogeneity of the vector field $p_0(r_1)$ with respect to the dilation $\delta_1^L$ is $\kappa_0 = \alpha - 1 < 0$. It follows from Lemma 15 in the Appendix that the origin of the system (34) is finite-time stable.

**Proof of Step 2.**

For $j = 1, 2, 3, \ldots$, we calculate the degree of homogeneity of the vector field $p_j(r_1)$ in (33) with respect to the dilation $\delta_1^L$, $k_j$, and have
\[
p_j(\varepsilon r_1) = \frac{C_{2j}^j}{2^{2j} \times (2j - 1)} K \varepsilon^{\alpha+2j} r_1^{\alpha+2j} g(t)
= \varepsilon^{1+\alpha+2j} p_j(r_1)
= \varepsilon^{1+k_j} p_j(r_1),
\tag{37}
\]
that is, $k_j = \alpha + 2j - 1 (j = 1, 2, 3, \ldots)$.

Noticing the facts that the degree of homogeneity of $V(r_1)$ with respect to the dilation $\delta_1^L$ is $l_1 = 2$ and $\langle \nabla V(r_1), p_j(r_1) \rangle$ is continuous and its degree of homogeneity with respect to $\delta_1^L$ is $l_1 + k_j$, we take $V_1 = V(r_1)$ and $V_2 = \langle \nabla V(r_1), p_j(r_1) \rangle$ for Lemma 14 of the Appendix. Since $l_1 = 2 > 0$ and $l_2 = l_1 + k_j = \alpha + 2j + 1 > 0$, Lemma 14 in the Appendix implies
\[
\langle \nabla V(r_1), p_j(r_1) \rangle \leq -c_j V(r_1)^{\frac{\alpha+2j+1}{2}},
\tag{38}
\]
where $c_j = -\max_{(r_1 : V(r_1) = 1)} \langle \nabla V(r_1), p_j(r_1) \rangle \in \mathbb{R} (j = 0, 1, 2, \ldots)$. Thus,
\[
\langle \nabla V(r_1), f(r_1) \rangle 
\leq -c_0 V(r_1)^{\frac{\alpha+1}{2}} - \cdots - c_j V(r_1)^{\frac{\alpha+2j+1}{2}} - \cdots
\tag{39}
\]
\[
= V(r_1)^{\frac{\alpha+1}{2}} (-c_0 + \mathcal{U}(r_1)),
\]
where \( \mathcal{U}(r_1) \triangleq -c_1 V(r_1)^{\frac{3}{2}} - \cdots - c_j V(r_1)^{\frac{2j}{2}} - \cdots \). Since \( \frac{2j}{2} > 0 \) for \( j \geq 1 \), \( \mathcal{U}(r_1) \) is a continuous function with \( \mathcal{U}(0) = 0 \).

Now, we show that (39) satisfies the condition in (11). Assume that there exists an open neighborhood \( \mathcal{V} \) of the origin such that \( \mathcal{U}(r_1) < \frac{c_0}{2} \) holds for any \( r_1 \in \mathcal{V} \). Then, (39) can be written as

\[
\langle \nabla V(r_1), f(r_1) \rangle < -\frac{c_0}{2} V(r_1)^{\frac{2j+1}{2}},
\]

where \( c_0 > 0 \) and \( \frac{c_0}{2} \in (0, 1) \). Thus, the condition (11) in Theorem 5 is satisfied. In view of the positive definiteness of \( V(r_1) \), Theorem 5 guarantees that the origin is a finite-time stable equilibrium point of the system (33).

Next, we verify the existence of the open neighborhood \( \mathcal{V} \), that is, there exist \( r_1 \) such that \( \mathcal{U}(r_1) < \frac{c_0}{2} \) holds. Considering that \( c_j = -\max_{V(r_1)=1} \langle \nabla V(r_1), p_j(r_1) \rangle (j = 0, 1, 2, \ldots) \) and \( r_1 = 1 \) holds when \( V(r_1) = 1 \), we can calculate \( c_0 \) and \( c_j \) as

\[
c_0 = -\langle \nabla V(r_1), p_0(r_1) \rangle = 2Kg(t),
\]

\[
c_j = -\langle \nabla V(r_1), p_j(r_1) \rangle = -\frac{2KC_j g(t)}{2^{j} \times (2j-1)}.
\]

It follows from (42) that

\[
\mathcal{U}(r_1) = -c_1 V(r_1)^{\frac{3}{2}} - \cdots - c_j V(r_1)^{\frac{2j}{2}} - \cdots
\]

\[
= 2Kg(t) [\frac{1}{2} V(r_1) + \cdots + \frac{C_j}{2^{j}} V(r_1)^{\frac{2j}{2}} + \cdots ]
\]

\[
= 2Kg(t) [1 - (1 - V(r_1)^{\frac{3}{2}}) ].
\]

Substituting (43) and (41) into \( \mathcal{U}(r_1) < \frac{c_0}{2} \), we have \( r_1 < \frac{\sqrt{2}}{2} \), that is, \( \mathcal{V} = \{ r_1 : r_1 < \frac{\sqrt{2}}{2} \} \). This shows the existence of \( \mathcal{V} \) in (40). Further, considering that all the moments \( t_q \) corresponding to the transition state \( |\psi_q \rangle \) constitute a zero measure set and any non-transition state satisfies \( V < 0 \), it is easy to draw the conclusion that \( r_1 \) always can converge into \( \mathcal{V} \) within a finite time for every initial state \( r_1(0) \notin \mathcal{V} \).

Thus, the origin of the system (33) is a global finite-time stable equilibrium point, that is, \( r_1 \) can be stabilized to the origin within a finite time, equivalently, the quantum state is stabilized to the equivalence class of the target state \( |\psi_f \rangle \) = \( [0, 1]^T \) within a finite time.

\begin{remark}
According to the proof of Theorem 8, (40) always holds for the system (33). Therefore, from Theorem 5, we know that the finite convergence time function of the system (33) satisfies \( T(r_1(0)) < \frac{1}{c_1(1-\alpha)} V(r_1(0))^{\frac{3}{2}} \), where the finite convergence time \( T(r_1(0)) \) represents the evolution time from the initial state \( r_1(0) \) to the origin. It should be noted that the above inequality relation of the finite convergence time function can be obtained only when \( r_1(0) \notin \mathcal{V} \). When \( r_1(0) \notin \mathcal{V} \), the calculation of the finite convergence time relies on the system equation (33), and therefore it is hard to give a range of \( T(r_1(0)) \) restricted by a certain analytical expression. In addition, since \( c_1 \), \( V(r_1(0)) \), and \( \alpha \in (0, 1) \) are bounded, \( T(r_1(0)) \) is also bounded. However, its value will vary with \( \alpha \).
\end{remark}

5 Numerical examples

In order to demonstrate the validity of the non-smooth control law designed in this paper, we choose a spin \( \frac{1}{2} \) particle system for numerical simulation experiments. In simulations, we set \( K = 0.5 \) and \( \alpha = \frac{2}{3} \) for the control law in (26). In addition, we also consider simulation results under the standard Lyapunov control law \( u^*_f \) [36] and the standard bang-bang Lyapunov control law \( u^* \) [6] to compare the results under the control law in this paper, where \( u^*_f \) and \( u^* \) are

\[
u^*_f = K\text{Imag}[e^{i\zeta(\psi_f)} (\psi_f | H_1 | \psi_f)],
\]

\[
u^* = K\text{Imag}[e^{i\zeta(\psi_f)} (\psi_f | H_1 | \psi_f)].
\]

Assume that the initial state is given as \( |\psi(0)\rangle = [1, 0]^T \notin \mathcal{V} \). We choose \( K = 0.5 \) for the standard Lyapunov control law \( u^*_f \) and the standard bang-bang Lyapunov control law \( u^* \). The simulation results are shown in Fig. 2 and Fig. 3.

The simulations show that the finite convergence time of the system with the initial state \( |\psi(0)\rangle = [1, 0]^T \) is \( t_f \approx 9.96 \) a.u.. According to Fig. 2 and simulation data, the population of the target state under the standard Lyapunov control \( u^*_f \) is only 97.7% at \( t = 9.96 \) a.u., and the population of the target state under the standard bang-bang Lyapunov control law \( u^*_b \) is 96.82% at \( t = 9.96 \) a.u. and no longer changes. It can be seen from Fig. 3 that the non-smooth control in this paper is continuous and non-differentiable, that is, continuous in the whole time interval but non-differentiable at some points; while the standard Lyapunov control is continuously differentiable in \([0, \infty)\), and the standard bang-bang Lyapunov control is discontinuous.

To verify the estimation relation of the finite convergence time, we also perform simulation experiments for the initial state \( |\psi(0)\rangle = [\frac{1}{2}, \frac{\sqrt{3}}{2}]^T \in \mathcal{V} \). The simulation results are shown in Fig. 4, which indicates that the convergence time of the system in this case is \( t_f \approx 5.33 \) a.u.. According to Remark 9, the finite convergence time corresponding to the initial state \( |\psi(0)\rangle = [\frac{1}{2}, \frac{\sqrt{3}}{2}]^T \) satisfies
To achieve the finite-time stabilization of an arbitrary eigenstate of the internal Hamiltonian of a quantum system, this paper proposed a new non-smooth control scheme. We provided the concept of finite-time stabilization control law. In order to achieve an optimal convergence performance, the optimization of the parameter \( \alpha \) needs further research. In addition, this paper only considers the finite-time stabilization control of two-dimensional quantum systems. Extending the scheme to high-dimensional and mixed-state systems is also a topic worth exploring.

\section{Conclusion}

To achieve the finite-time stabilization of an arbitrary eigenstate of the internal Hamiltonian of a quantum system, this paper proposed a new non-smooth control scheme. We provided the concept of finite-time stability of quantum systems and corresponding identification criteria. Via the exponential representation of quantum states, an equivalent transformation was performed for the system model and the control target. The Lyapunov stability theory and the homogeneity theorem of finite-time stability were used to prove the finite-time stabilization of two-level quantum systems to the target state.

At the same time, the estimation problem of the finite convergence time was discussed. Based on the transversality condition in the Bloch space, we also proved the existence and uniqueness of solutions of the quantum system under the non-smooth control law. The effectiveness of the non-smooth control law was illustrated by numerical examples.

Since the control law designed in this paper contains two adjustable parameters, it is more flexible in the use of the control law. In order to achieve an optimal convergence performance, the optimization of the parameter \( \alpha \) needs further research. In addition, this paper only considers the finite-time stabilization control of two-dimensional quantum systems. Extending the scheme to high-dimensional and mixed-state systems is also a topic worth exploring.

\section*{Appendix: Homogeneity theory of finite-time stability}

Here, we list some notions related to homogeneity and the finite-time stability criterion based on the homogeneity theory, which can be found in [28].

\begin{definition}
Let \( d = (d_1, d_2, \ldots, d_{n-1}) \) be a set of positive real numbers. For a set of coordinates \( r = (r_1, r_2, \ldots, r_{n-1}) \) in \( \mathbb{R}^{n-1} \), define the dilation \( \delta_\varepsilon \) of \( r \) as the following coordinate vector
\[
\delta_\varepsilon (r) = (\varepsilon^{d_1} r_1, \ldots, \varepsilon^{d_{n-1}} r_{n-1}), \forall \varepsilon > 0
\]
where \( d_i \) is the weight of the coordinate \( r_i \). The dilation with \( d_1 = \cdots = d_{n-1} = 1 \) is called a standard dilation.
\end{definition}

\begin{definition}
A function \( V : \mathbb{R}^{n-1} \to \mathbb{R} \) is said to be homogeneous of degree \( m \) \( (m \in \mathbb{R}) \) with respect to \( \delta_\varepsilon \) if
\[
V(\delta_\varepsilon (r)) = \varepsilon^m V(r), \forall r \in \mathbb{R}^{n-1}, \forall \varepsilon > 0.
\]
\end{definition}

\begin{definition}
A vector field \( f(r) : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) with \( f(r) = (f_1(r), \ldots, f_{n-1}(r))^T \) is said to be homogeneous of degree \( m \) \( (m \in \mathbb{R}) \) if
\[
f(\delta_\varepsilon (r)) = \varepsilon^m f(r), \forall r \in \mathbb{R}^{n-1}, \forall \varepsilon > 0.
\]
\end{definition}
of degree $k(k \in \mathbb{R})$ with respect to $\delta_i^d$ if for each $i = 1, \ldots, n - 1$, $f_i$ is homogeneous of degree $k + d_i$, that is,

$$f_i(\delta_i^d(r)) = \varepsilon^{k+d_i} f_i(r), \quad \forall r \in \mathbb{R}^n, \quad \forall \varepsilon > 0.$$  \hspace{1cm} (48)

**Lemma 13** Assume that the function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is homogeneous of degree $k(k \in \mathbb{R})$ with respect to $\delta_i^d$ and the origin is a locally asymptotically stable equilibrium point if and only if it is an asymptotically stable equilibrium point. Then, when $m > \max\{-k,0\}$, there exists a Lyapunov function $V$ such that $V$ and its time derivative $\dot{V}$ are homogeneous of degrees $m$ and $m + k$ with respect to $\delta_i^d$, respectively.

**Lemma 14** Let $V_1$ and $V_2$ be continuous real-valued functions defined on $\mathbb{R}^{n - 1}$ and $V_1$ be positive definite. Suppose that $V_1$ and $V_2$ are homogeneous of degrees $l_1 > 0$ and $l_2 > 0$ with respect to $\delta_i^d$ respectively. Then, for every $r \in \mathbb{R}^n$, the following holds:

$$\left(\min_{\{z:V_1(z)=1\}} V_2(z)\right) \left(V_1(r)\right)^{\frac{l_1}{l_2}} \leq V_2(r),$$

$$\leq \left(\max_{\{z:V_1(z)=1\}} V_2(z)\right) \left(V_1(r)\right)^{\frac{l_1}{l_2}}.$$  \hspace{1cm} (49)

The following lemma shows the application of the homogeneity theory to the finite-time stability.

**Lemma 15** Let $f(r) = (f_1(r), \ldots, f_n-1(r))^T: \mathbb{R}^{n - 1} \rightarrow \mathbb{R}^{n - 1}$ be a continuous vector function and be homogeneous of degree $k(k \in \mathbb{R})$ with respect to $\delta_i^d$, where $d = (d_1, d_2, \ldots, d_{n-1})$ is a set of positive real numbers and $\varepsilon > 0$. Then, the origin is a finite-time stable equilibrium point if and only if it is an asymptotically stable equilibrium point and $k < 0$.

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