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Causality in the relativistic bound-state problem

In memory of John A. Tjon

Abstract Although the exact Bethe-Salpeter equation is certainly the appropriate field-theoretic framework to describe the non-perturbative problem of scattering and bound states, the inevitable truncations introduce inconsistencies such as loss of symmetries or incorrect one-body limit. I conjecture that these problem can be overcome if the truncation preserves the field-redefinition invariance of the exact equation. A sum rule for light-by-light scattering can provide a testing ground of this conjecture.

Keywords Bethe-Salpeter equation · Gross equation · Charge conjugation · Analyticity · Sum rules

1 Introduction

John Tjon has had about 30 PhD students over the course of his remarkable life in physics. I am privileged to be one of them, one of the very last in fact. In these proceedings I shall go through a piece of physics I was learning from John. It concerns the bound-state problem in relativistic theory, or, more specifically, the quest for a relativistic analog of the Lippmann-Schwinger equation which would reduce to the appropriate one-body equation (Klein-Gordon or Dirac, depending on the spin) in the limit when one of the particles is infinitely heavy, viz., one-body limit. This is quite an old and rich subject and I would never be able to present it fairly, but the bottom line is that the only fully consistent equation is the exact Bethe-Salpeter equation (BSE) [1], derived from quantum field theory [2]. The problem with the BSE is that in practice it can not be solved exactly because the kernel consists of infinitely many two-particle irreducible (2PI) graphs. One needs to truncate, and that is very difficult to do in a systematics fashion. For example, keeping only the trees (one-particle exchanges) in the kernel leads to the ladder BSE which does not have the correct one-body limit.

An improvement of the kernel, e.g., the inclusion of the 2PI one-loop graphs, presumably improves the situation. But in the earlier days of nuclear physics it was not clear how to include such loops, which in the case of the nucleon-nucleon (NN) interaction, for example, would be entering with higher orders of the large $\pi NN$ coupling. It is only with the advent of the effective-field theories (EFTs) [3], that we have learned how systematically expand the $NN$ kernel in powers of small energy scales [4, 5, 6, 7]. To date, however, just a handful of the EFT calculations in few-nucleon systems include relativistic effects, let alone attempt to solve the full BSE. It is usually argued that relativistic effects are negligible because the nucleon mass is much greater than the relative energy of nucleons in nuclei. I am not very happy with this argument, because by a similar argument the heavy-baryon expansion (HBChPT) in the single-nucleon sector would always converge well, and we know it is not, e.g. [8, 9]. In fact, calculations
in the single-nucleon sector show that by the time the $\Delta(1232)$-resonance excitation becomes important relativity kicks in [10]. A dramatic example is provided by the magnetic polarizability of the nucleon, where the $\Delta$-excitation and relativistic effects largely cancel each other, leaving us with the small value for this quantity [11]. As the $\Delta(1232)$-isobar has recently been included in the effective $NN$ potential and shown to play an important role [12], further relativistic EFT studies will hopefully be done in the near future.

2 What Gross equation and infrared regularization of BChPT have in common

Going back to the pre-EFT era, when the loops were out of question, it became customary to improve the kernel by an additional approximation, however paradoxically that might sound. The additional approximation is called quasi-potential reduction of the Bethe-Salpeter equation (BSE). If we write BSE, for the scattering amplitude in momentum space, as (see Fig. 1):

$$T(p', p) = V(p', p) + i \int \frac{d^4 k}{(2\pi)^4} V(p', k) G(k) T(k, p) ,$$

(1)

where $V,G$ is the kernel, then the 3-dimensional reduction is done by manipulating the dependence of $V,G$, and subsequently $T$, on $k_0$, the energy component of the integration four-momentum. It is assumed that $k_0$ is not an independent variable, but is fixed some way in terms of the external momenta and possibly $k$. Any choice of fixing $k_0$ is okay as long as the unitarity and Lorentz invariance are not violated. This procedure lifts the integration over $k_0$, leaving us with a much simpler 3-dimensional equation.

Sometimes a quasi-potential equation is just introduced or postulated, and has no evident connection to BSE, but still, they all are 3-dimensional and we refer to them as ‘relativistic 3D equations’.

![Fig. 1](image) Graphical representation of a two-body scattering equation.

In 1969 Franz Gross described a relativistic 3D equation [13], bearing now his name, which has the correct one-body limit for one-boson-exchange potential. It thus provides a consistent formulation of the relativistic Yukawa problem, but with some caveats. The Gross equation is obtained in a 3D reduction of the ladder BSE by putting the heavier particle on the mass shell. This means that in doing the $k_0$ integration in Eq. (1) by counting the poles in the complex $k_0$ plane, one takes the contribution of one and only one pole: the positive-energy pole of the heavy-particle propagator, contained in the two-particle propagator $G$. The contribution of all the other poles in either $V, G$, or $T$ is discarded. This seems as a harsh approximation, but it does the trick with the one-body limit, cf. Chapter 12 in [14] for more details.

The first caveat is that the one-body limit is recovered for one-neutral-boson-exchange potentials only [13], which makes it of limited help in practice but is not a conceptual problem. The caveat that concerned John Tjon, during my work with him, was the apparent lack of charge conjugation symmetry [16]. It is interesting, although perhaps obvious to some, that while the equation is Lorentz-invariant, the loss of charge-conjugation symmetry leads to violation of (micro-) causality. The relativistic covariance is thus compromised at the quantum level — virtual states may propagate outside the light cone. It is clear that the resulting unphysical effects must diminish in the strict one-body limit, but it is hard to assess their size away from the limit, where the equation is actually applied.

Years later I encountered a similar situation in the infrared regularization (IR) of baryon chiral perturbation theory (BChPT), introduced by Becher and Leutwyler in 1999 [8]. The IR is a scheme to calculate chiral loops with nucleons such that no positive powers of the nucleon mass $M_N$ occur — the positive powers were thought to violate the power counting of BChPT [9]. The original formulation of IR

\[1\] When being a postdoc at Flinders University in Adelaide between 1999 and 2001, I learned from a graduate student at our department (Jambul Gegelia) that the positive powers of $M_N$ do not break power counting because their effect is always absorbed by low-energy constants [17]. But note that [17] first appeared in 1999 and was only published in 2003. It took some time before this work was considered as a legitimate solution to the power-counting problem.
is done in terms of Feynman parametrization of loop integrals, but when checking the Ward-Takahashi
identities in this scheme, I found that all the IR does is to remove the negative-energy pole of nucleon
propagators. Namely, if we consider a one-loop graph with one pion propagator, \( S_\pi(k) = (k^2 - m_\pi^2)^{-1} \),
and any number of nucleon propagators, \( S_N(p) = (\gamma \cdot p - M_N)^{-1} \), then as the result of the IR procedure
every nucleon propagator is replaced as follows:

\[
S_N(p) \xrightarrow{\text{IR}} S_N(p)
\left(1 + \frac{1}{S_\pi(k)(p^2 - M_N^2)}\right) = \frac{\gamma \cdot p + M_N}{p^2 - k^2 - M_N^2 + m_\pi^2}
\]

Since both \( p \) and \( k \) depend linearly on the loop momentum, the denominator of the modified nucleon
propagator is linear too, the quadratic term cancels. Hence this nucleon propagator has only one pole,
the positive-energy one. This procedure has been generalized by Tim Ledwig to any number of pion
propagators \[18\]. Graphs with no pion propagators vanish in IR, since all the poles lie in the same
half-plane of the complex energy plane.

So, once again, deleting the negative-energy pole violates charge-conjugation symmetry, hence
causality. It is manifested in the unphysical cuts whose appearance was already noticed in the original
work \[8\]. It is argued nonetheless that the cuts lie far away from the domain of interest of BChPT,
and hence their effect should be negligible. Unfortunately this argument is not always corroborated in
actual calculations, e.g. \[9, 19, 20, 21\].

To summarize thus far, the Gross equation as well as IR-BChPT neglect the negative-energy nucleon
pole in the loop calculations. This seems as a very safe approximation at low energies, since the nucleon
is rather heavy and it costs a lot of energy to produce an anti-nucleon. However, in loops we integrate
over all energies, and the absence of virtual anti-nucleons induces the unphysical (acausal) high-energy
contributions. The net effect of these contributions on low-energy physics is expected to be small, but
cannot be assessed a priory. In EFT the high-energy physics probed in the loops is compensated by
low-energy constants, but in order to use this argument in the IR case one needs to use an effective
Lagrangian without the charge-conjugation symmetry.

3 Sum rules for light-by-light scattering as a test of causality

In a given calculation it is usually not difficult to check gauge invariance or unitarity. A test of causality
is less obvious, but a test at the final stage of a calculation can be provided by sum rules such as GDH.
The sum rules involve cross sections, the observables which should be computable in any physical
theory, even the string theory. The GDH sum rule itself involves other quantities, such as the anomalous
magnetic moment, which might obscure the test. However, there is at least one exact sum rule which
involves cross sections only:

\[
\int_0^\infty \frac{ds}{s} \left[ \sigma_2(s) - \sigma_0(s) \right] = 0,
\]

where \( \sigma_3 \) are polarized total cross sections for \( \gamma \gamma \) fusion into anything; \( \sigma_0 \) is for circularly polarized
photons with the same helicity, and \( \sigma_2 \) with opposite. The integration is over the total invariant energy,
the Mandelstam variable \( s \). This sum rule is derived from general properties of unitarity, analyticity,
crossing, and gauge invariance of the light-by-light scattering amplitude \[22\], and as such it provides
a test ground of these principles.

To perform this test for a relativistic scattering equation one should first be able to construct
the particle-antiparticle scattering amplitude, e.g. \( N \bar{N} \) if we talk about the nucleons. Based on that,
one should obtain the gauge-invariant \( \gamma \gamma \rightarrow N \bar{N} \) amplitude (see Fig. 2), calculate the corresponding
polarized cross sections, and integrate them as in Eq. (3) to find zero, or not. In the latter case, one
can compare the result with the size of the cross sections. If the integral is much smaller than cross
sections themselves, the violation is small.

![Fig. 2 Graphical representation of the \( \gamma \gamma \rightarrow N \bar{N} \) amplitude. Crossed graphs are omitted.](image-url)
4 Field-redefinition invariance

The two-body scattering equations have issues bigger than causality. Fundamental local symmetries, such as the e.m. gauge invariance, are not easy to maintain either, see e.g., \[23, 24\]. Even the non-relativistic EFT calculations face this problem when the photon comes in, e.g., in the description of electron-deuteron scattering. And even without the photons, in plain Lippmann-Schwinger description of $NN$ (with non-perturbative pions), because the chiral symmetry is obscured by iterations and it is not clear how to renormalize \[7\].

It seems the source of all these issues lies in field-redefinition invariance (FRI). Or more precisely, in the fact that the equations are not invariant under field redefinitions.

In quantum field theory the FRI is a simple consequence of independence of the partition function on the choice of integration variables. In a two-body equations, this is much less trivial, since already the separation of graphs into reducible and irreducible is not invariant. Of course when all graphs are present the answer is invariant, but, if truncations are made as they must, it is generally not.

To see an example, consider the leading order $NN$ interaction in the chiral effective theory of the nuclear force (in Weinberg’s counting \[4\]). The leading-order potential consists of a four-nucleon contact interaction and a pion exchange with the $\pi NN$ vertices bearing the pseudo-vector coupling. After making a redefinition of the nucleon field as \[11\]:

\[
N(x) \rightarrow \exp \left( \frac{gA}{2f} \tau^a \pi^a(x) \gamma_5 \right) N(x),
\]

(4)

where $\pi(x)$ is the pion field, we can see that the only change in the potential is that the pseudo-vector $\pi NN$ coupling is replaced by the pseudo-scalar one. The on-shell potential has not changed, but the off-shell one has, and hence the solution changes. The change is quite dramatic, e.g., the pion exchange contribution is finite and one does not need a regulator (i.e., the finite cutoff) which otherwise is always present in such calculation \[7\]. The bottom line is that there is a strong dependence of the result on the choice of the nucleon field, a feature which is not desirable given the fact that the FRI is assumed in constructing the chiral Lagrangian.

Finding a truncation of BSE which maintains the field-redefinition invariance, will solve many, if not all, of the aforementioned problems. Whether it can be done in principle is another question.

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