Geodesic completeness and the lack of strong singularities in effective loop quantum Kantowski–Sachs spacetime

Sahil Saini and Parampreet Singh

Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA

E-mail: ssaini3@lsu.edu and psingh@phys.lsu.edu

Received 8 July 2016, revised 19 October 2016
Accepted for publication 27 October 2016
Published 1 December 2016

Abstract

Resolution of singularities in the Kantowski–Sachs model due to non-perturbative quantum gravity effects is investigated. Using the effective spacetime description for the improved dynamics version of loop quantum Kantowski–Sachs spacetimes, we show that even though expansion and shear scalars are universally bounded, there can exist events where curvature invariants can diverge. However, such events can occur only for very exotic equations of state when pressure or derivatives of energy density with respect to triads become infinite at a finite energy density. In all other cases curvature invariants are proved to remain finite for any evolution in finite proper time. We find the novel result that all strong singularities are resolved for arbitrary matter. Weak singularities pertaining to above potential curvature divergence events can exist. The effective spacetime is found to be geodesically complete for particle and null geodesics in finite time evolution. Our results add to a growing evidence for generic resolution of strong singularities using effective dynamics in loop quantum cosmology by generalizing earlier results on isotropic and Bianchi-I spacetimes.

Keywords: loop quantum cosmology, Kantowski–Sachs spacetime, strength of singularities, singularity resolution

1. Introduction

In general relativity (GR), occurrence of singularities brings forth the underlying limitations of the classical continuum spacetime. A singular event in classical theory has many
characteristics, generally captured via curvature divergences and break down of geodesic evolution in a finite proper time. However, not all singularities are necessarily the boundaries of classical spacetime. Their strength matters. It is believed that strong singularities [1–3], which cause inevitable complete destruction of arbitrarily strong in-falling detectors, are the true boundaries of the classical continuum spacetime. Strong singular events, such as the big bang or a central singularity inside a black hole, are conjectured to be associated with geodesic incompleteness [2, 3]. In contrast, weak singularities can be considered harmless. Even though some curvature components may diverge at such events, a sufficiently strong detector survives such a singularity. Geodesics can be extended beyond such singularities in the classical spacetime. Thus, not all space-like singularities are necessarily harmful. A fundamental challenge for any theory going beyond Einsteinian gravity is whether it can resolve all of the strong singularities.

In the absence of quantum gravitational effects which profoundly modify the structure of the underlying spacetime, singularity resolution has been elusive. In the last decade, applications of loop quantum gravity (LQG) to cosmological spacetimes in loop quantum cosmology (LQC) indicate that non-perturbative quantum gravitational modifications play an important role in singularity resolution [5]. Various examples of cosmological spacetimes, including isotropic [6, 7] and anisotropic models [8, 9], and also hybrid Gowdy models [10–12], have been thoroughly studied at a rigorous quantum level. Extensive numerical simulations of quantum cosmological models have been performed [6, 13], which show that the singularities such as the big bang and big crunch are resolved and replaced by a non-singular bounce. Singularity resolution has also been understood in terms of quantum probabilities using consistent histories approach [14].

At the fundamental level, geometry in LQC is discrete resulting in a classical continuum quickly below the Planck curvature scale. In fact, the quantum Hamiltonian constraint in LQC on quantum geometry can be very well approximated by the Wheeler-DeWitt Hamiltonian constraint on classical continuum as soon as the spacetime curvature becomes less than a percent of the Planck curvature. Interestingly, numerical simulations show that an effective quantum continuum description exists which captures the quantum dynamics in LQC at all scales. This effective spacetime description, or effective dynamics, has been used in a plenty of investigations. In relation to the objectives of this manuscript, notable results include the following. Using effective dynamics, expansion and shear scalars have been found to be generically bounded for isotropic and anisotropic spacetimes [9, 15–20]. Strong singularities are shown to be absent and effective spacetime is found to be geodesically complete for loop quantized isotropic cosmological and Bianchi-I spacetimes for matter with a vanishing anisotropic stress [16–18]. However, weak and non-curvature singularities can exist [16–18], of which various examples have been studied in isotropic and Bianchi-I spacetimes in LQC [21].

The goal of this manuscript is to investigate the resolution of strong singularities and geodesic completeness in the Kantowski–Sachs model in LQC using the effective spacetime description. We wish to understand under what conditions, for arbitrary non-viscous matter, quantum geometry effects as understood in LQC lead to a generic resolution of strong singularities, and whether there exist any weak singularities. Essentially, our aim is to generalize the results of geodesic completeness and generic resolution of strong singularities in isotropic and Bianchi-I spacetime in LQC to the Kantowski–Sachs model. The Kantowski–Sachs spacetime is an interesting avenue to study for various reasons. In the absence of matter, it captures the interior \((r < 2m)\) of the Schwarzschild black hole. In the presence of matter, it is an anisotropic cosmological model with a spatial curvature. This spacetime has
been loop quantized with different regularizations of the Hamiltonian constraint, given by Ashtekar–Bojowald [22] (see also [23, 24]), Corichi–Singh [25], and Boehmer–Vandersloot [26] (see also [27]). The first quantization prescription is a reminiscent of ‘fixed area of the loop’ procedure in LQC [28, 29] which is known to have various phenomenological issues due to fiducial cell dependence [30]. The second and third quantization prescriptions overcome this limitation in their unique ways, and give qualitatively different physics of singularity resolution. In this manuscript, we study the Kantowski–Sachs spacetime with Boehmer–Vandersloot prescription which is an avatar of the improved dynamics prescription in LQC [6]. It should be noted that resolution of strong singularities and geodesic completeness for isotropic and Bianchi-I spacetime in LQC have been achieved for this particular prescription. Interestingly, a loop quantization of this spacetime, in absence of matter and also in presence of cosmological constant, results in a singularity resolution with a pre-bounce spacetime which is a product of two constant curvature spaces and with an almost Planckian curvature [31]. An important result pertinent to our investigations is that the expansion and shear scalars in this quantization turn out to be universally bounded [32]. In the following, our reference to loop quantized Kantowski–Sachs model will imply Boehmer–Vandersloot prescription.

Our analysis assumes the validity of the effective dynamics in LQC at all scales. In LQC, effective Hamiltonian is obtained using a geometrical formulation of quantum theory, and, as noted earlier, turns out to be an excellent approximation for isotropic and anisotropic models. This is true at least for states which correspond to macroscopic spacetimes at late times. A specific example is the case of homogeneous and isotropic spatially flat spacetimes where effective Hamiltonian has been derived explicitly using coherent states for the case of the massless scalar [33]. The resulting effective dynamics has been tested rigorously using numerical simulations [6, 13], which validate the analytical derivation of the effective Hamiltonian for the above family of physical states. In terms of the gravitational phase space variables, Ashtekar–Barbero connection components and conjugate triads, the effective Hamiltonian contains trigonometric terms of the connections, apart from the triads and the matter variables. These trigonometric or the polymerized terms arise from expressing field strength of the connection in terms of holonomies over a minimum physical area in the improved dynamics. The resulting Hamilton’s equations from the effective Hamiltonian encode the quantum gravitational repulsiveness and result in singularity resolution. In the effective Hamiltonian, there can also be modifications coming from expressing inverse powers of triads in terms of Poisson brackets between holonomies and triads. The role of these modifications in singularity resolution is generally found to be negligible when compared to the effects originating from the polymerized terms. In our analysis, we ignore the latter modifications. Interestingly, the polymerized Hamiltonian of LQC can also be obtained using an inverse procedure without any prior hints of the canonical structure or assuming a Lagrangian, just by demanding a form of repulsive nature of gravity at high curvature scales and general covariance [36].

The main results from our investigation are the following. Considering non-viscous minimally coupled matter with a general equation of state and anisotropic stress, we show that for any finite proper time, energy density is always finite in the loop quantized Kantowski–Sachs model. This is the first novel result of our analysis. In previous works, such as in

---

3 These modifications in absence of polymerized terms can also lead to singularity resolution and interesting phenomenology in spatially curved models, see e.g. [34, 35].

4 It should be noted that in models where these terms have been argued to become significant, their overall effect is to strengthen the singularity resolution effects [9, 19, 38]. We will find that the effective Hamiltonian even in the absence of these terms suffices to obtain generic resolution of strong singularities.
[26, 31, 32], energy density was found not to diverge dynamically using numerical simulations. Since such simulations do not cover the entire set of solutions, an analytical understanding of the behavior of energy density was very much needed. Our second result is to show that the physical volume remains non-zero and finite throughout the finite time evolution. Along with our result on energy density finiteness, this rules out big bang/crunch, big rip and big freeze singularities. Our third result is to show that even though expansion and shear scalars are universally bounded in the effective spacetime, curvature invariants can still diverge. Albeit, this only happens when pressure or derivatives of energy density with respect to the triads diverge while energy density remains finite. Such exotic equations of state are known to cause sudden singularities in cosmological models (see e.g. [16]). It turns out that the divergence in curvature invariants in finite proper time correspond to weak curvature singularities. Our fourth result is to show that all strong singularities are absent for any finite time evolution. Finally, analysis of the time-like and null geodesics shows that the effective spacetime is geodesically complete. There is no breakdown of geodesics for any finite proper time in the effective dynamics evolution in loop quantized Kantowski–Sachs spacetime. The effective quantum spacetime in loop quantized Kantowski–Sachs model is geodesically complete.

Our manuscript is organized in the following way. In section 2, we provide a brief review of the classical Hamiltonian formulation of the Kantowski–Sachs spacetime in Ashtekar variables and obtain the dynamical equations. We calculate the expressions for the expansion and shear scalar, and for curvature invariants in terms of connection and triad variables. The effective Hamiltonian from LQC based on Boehmer–Vandersloot quantization is studied in section 3 where we obtain the quantum gravitational modified dynamical equations and obtain the bounded behavior of expansion and shear scalars. We obtain the analytical bounds on the triad variables from the dynamical equations for finite time evolution, which then imply that the energy density is finite for any finite proper time. We show that for any finite time evolution, the physical volume is non-zero and that the curvature invariants are non-divergent except for singular events where pressure and energy density derivatives with respect to triads diverge at finite energy density. In section 4, we consider the special case of matter with vanishing anisotropic stress, and show that the above results turn out to be true using a simpler argument. Behavior of the geodesics is investigated in section 5.1, where they are shown to not break down in any finite proper time evolution. In section 5.2, we show that the Kantowski–Sachs spacetime in effective dynamics does not satisfy the necessary conditions for the existence of strong curvature singularities. Hence we conclude that the above mentioned pressure singularities are weak singularities. We summarize with conclusions in section 6.

2. Classical dynamics of Kantowski–Sachs spacetime

In this section, we summarize the basic features of classical Kantowski–Sachs spacetime in Ashtekar variables. The homogeneity of the Kantowski–Sachs spacetime leads to a simple diagonal form for the Ashtekar–Barbero connection components and conjugate triads [22]:

$$A^t_i dx^a = \tilde{\epsilon} \tau_3 dx + \tilde{b} \tau_2 d\theta - \tilde{b} \tau_1 \sin \theta d\phi + \tau_3 \cos \theta d\phi,$$

(2.1)

Unlike big bang/crunch, big rip singularity occurs at infinite volume with an infinite energy density (see e.g. [37]). A big freeze singularity occurs at finite volume but has infinite energy density (see e.g. [16]).
where \( \tau_i = -i\sigma_i/2 \), and \( \sigma \) are the Pauli spin matrices. The symmetry reduced triads \( \tilde{p}_b \) and \( \tilde{p}_c \) are related to the metric components of the spacetime line element:

\[
dx^2 = -N^2 dt^2 + g_{xx} \ dx^2 + g_{\Omega\Omega}(d\theta^2 + \sin^2\theta d\phi^2)
\]

as

\[
g_{xx} = \frac{\tilde{p}_b^2}{\tilde{p}_c}, \quad \text{and} \quad g_{\Omega\Omega} = |\tilde{p}_b|.
\]

The modulus sign arises due to two orientations of the triad. Since the matter considered in this analysis is non-fermionic, we can fix one orientation. In the following, the orientation of the triads is chosen to be positive without any loss of generality. The Kantowski–Sachs spacetime is naturally foliated with spatial slices of topology \( \mathbb{R} \times S^2 \). The spatial slices are non-compact in \( x \)-direction. In order to define a symplectic structure on the spatial slices, we need to restrict the integration along the \( x \)-direction to a fiducial length, say \( L_o \). The resulting symplectic structure is:

\[
\Omega = \frac{L_o}{2G\gamma}(2d\hat{b} \wedge dp_b + dc \wedge dp_c),
\]

where \( \gamma \) is the Barbero–Immirzi parameter, its value is set to 0.2375 from black hole entropy calculations in LQG. The fiducial length is a non-physical parameter in our theory, and can be arbitrarily re-scaled. In order to make the symplectic structure independent of \( L_o \), we introduce the new triad and connection variables \( p_b, p_c, \) and \( b, c \) obtained by re-scaling the symmetry reduced triad and connection variables:

\[
p_b = L_o\tilde{p}_b, \quad p_c = \tilde{p}_c, \quad b = \hat{b}, \quad c = L_o\hat{c}.
\]

The non-vanishing Poisson brackets between these new variables are given by,

\[
\{ b, p_b \} = G\gamma, \quad \{ c, p_c \} = 2G\gamma.
\]

In terms of these phase space variables, the classical Hamiltonian constraint is the following for lapse \( N = 1 \):

\[
H_{cl} = -\frac{1}{2G\gamma^2} \left[ 2bc\sqrt{R} + (b^2 + \gamma^2) \frac{p_b}{\sqrt{R}} \right] + 4\pi p_b \sqrt{R} \rho.
\]

Here \( \rho \) is the energy density, related to matter Hamiltonian as \( \rho = H_m/V \), and \( V \) is the physical volume of the fiducial cell: \( V = 4\pi p_b \sqrt{R} \). The energy density is taken to depend only on triad variables, and not on connection variables. The Hamilton’s equations for the triad and connection variables are:

\[
p_b = -G\gamma \frac{\partial H_{cl}}{\partial b} = \frac{1}{\gamma} \left( c\sqrt{R} + \frac{bp_b}{\sqrt{R}} \right),
\]

\[
p_c = -2G\gamma \frac{\partial H_{cl}}{\partial c} = \frac{1}{\gamma} 2b\sqrt{R},
\]

\[
b = G\gamma \frac{\partial H_{cl}}{\partial p_b} = -\frac{1}{2\gamma} (b^2 + \gamma^2) + 4\pi G\gamma \sqrt{R} \left( \rho + p_b \frac{\partial \rho}{\partial p_b} \right).
\]
\[ \dot{c} = 2G\gamma \frac{\partial \mathcal{H}_{cl}}{\partial \rho} = \frac{-1}{\sqrt{\rho}} \left( b c - (b^2 + \gamma) \frac{p_b}{2\rho} \right) + 4\pi G\gamma \frac{p_b}{\sqrt{\rho}} \left( \rho + 2p_c \frac{\partial \rho}{\partial \rho} \right). \]

(2.12)

Here ‘dot’ refers to derivative with respect to proper time. Using the above equations, a useful result follows:

\[ \frac{d}{dt} (\dot{c} - b p_b) = \frac{\gamma p_b}{\sqrt{\rho}} + G\gamma V \left( 2p_c \frac{\partial \rho}{\partial p_c} - p_b \frac{\partial \rho}{\partial p_b} \right). \]

(2.13)

As will be proved in section 3, it turns out that the same expression also holds in the presence of quantum gravitational modifications in LQC.

Let us now find some useful expressions to understand singularities in Kantowski–Sachs spacetime. The simplest to obtain is the expression for energy density in terms of the gravitational phase space variables, by imposing the vanishing of the Hamiltonian constraint, \( \mathcal{H}_{cl} \approx 0 \):

\[ \rho = \frac{1}{8\pi G} \left( \frac{2bc}{c^2p_b} + \frac{b^2}{\gamma c^2p_c} + \frac{1}{\rho} \right). \]

(2.14)

Two useful quantities of interest to understand the behavior of geodesics as singularities are approached are the expansion and shear scalars. The expansion scalar \( \theta \) is given by

\[ \theta = \frac{\dot{V}}{V} = \frac{\dot{p}_b}{p_b} + \frac{\dot{p}_c}{2p_c}. \]

(2.15)

The shear scalar \( \sigma^2 \) expressed in terms of the directional Hubble rates \( H_i = \sqrt{\mathcal{H}_i} / \sqrt{\mathcal{H}_0} \) is given by

\[ \sigma^2 = \frac{1}{2} \sum_{i=1}^{3} \left( H_i - \frac{1}{3} \theta \right)^2 = \frac{1}{3} \left( \frac{\dot{p}_b}{p_b} - \frac{\dot{p}_c}{p_c} \right)^2. \]

(2.16)

Next we find the expressions for curvature invariants, which when diverge signal singularities (though not necessarily strong ones). In terms of the reduced triad and connection variables, the expressions for the Ricci scalar, the square of the Weyl scalar and the Kretschmann scalar are respectively as follows:

\[ R = 2 \frac{\dot{p}_b}{p_b} + \frac{\dot{p}_c}{p_c} + \frac{2}{p_c}, \]

(2.17)

\[ C_{abcd} C^{abcd} = \frac{1}{3} \left[ 3 \left( \frac{\dot{p}_b}{p_b} - \frac{\dot{p}_c}{p_c} \right) - 2 \left( \frac{\dot{p}_b}{p_b} - \frac{\dot{p}_c}{p_c} \right) - \frac{2}{p_c} \right]^2. \]

(2.18)

and

\[ K = 6 \left( \frac{\dot{p}_b \dot{p}_b}{p_b p_c} \right)^2 - 8 \left( \frac{\dot{p}_b \dot{p}_b \dot{p}_c}{p_b p_c p_b} \right)^2 + 4 \left( \frac{\dot{p}_b}{p_b} \right)^2 + 6 \left( \frac{\dot{p}_b}{p_b} \right)^2 - 4 \frac{\dot{p}_b \dot{p}_b}{p_b p_c}, \]

\[ - 8 \left( \frac{\dot{p}_b}{p_b} \right)^3 + 4 \left( \frac{\dot{p}_b \dot{p}_b}{p_b p_c} \right)^2 + \frac{7}{2} \left( \frac{\dot{p}_b}{p_b} \right)^4 + \frac{2}{p_c} \left( \frac{\dot{p}_b}{p_b} \right)^2 \frac{1}{p_c}, \]

\[ - 5 \left( \frac{\dot{p}_b}{p_b} \right)^2 \frac{\dot{p}_b}{p_b} + \frac{4}{p_c} + 3 \left( \frac{\dot{p}_b}{p_b} \right)^2. \]

(2.19)
We notice that the expansion and shear scalar, and all the curvature invariants depend on the following five quantities: $\dot{p}_b, \ddot{p}_b, \dot{p}_c, \ddot{p}_c$ and $\frac{1}{\rho}$. The behavior of $\dot{p}_b/\rho_b$ and $\dot{p}_c/\rho_c$ is obtained from the Hamilton’s equations. Taking their time derivatives, we obtain

$$\frac{\ddot{p}_b}{p_b} = \frac{b c}{\gamma^2 p_b} + 8\pi G \rho + 4\pi G \left( p_b \frac{\partial \rho}{\partial p_b} + 2p_b \frac{\partial \rho}{\partial \rho_b} \right),$$

(2.20)

$$\frac{\ddot{p}_c}{p_c} = -\frac{1}{\rho_c} + \frac{b^2}{\gamma^2 \rho_c} + 8\pi G \rho + 8\pi G p_b \frac{\partial \rho}{\partial p_b}.$$

(2.21)

It is clear from the classical Hamilton’s equations (2.9)–(2.12) and the above equations that the expansion scalar, shear scalar, curvature invariants and energy density all diverge as the triad components vanish, and/or the connection components diverge and/or the terms $\frac{\partial \rho}{\partial p_b}$ and $\frac{2p_b}{\rho_b}$ diverge. Generic physical solutions obtained from the classical Hamiltonian constraint (2.8) turn out to be of this form and are singular.

3. Effective loop quantum cosmological dynamics

In the previous section, we obtained the classical singular dynamical equations from the classical Hamiltonian constraint of the Kantowski–Sachs spacetime. Let us now see the way quantum gravitational modifications result in non-singular dynamics. Our starting point is the effective Hamiltonian constraint [26]:

$$\mathcal{H} = -\frac{p_b \sqrt{\rho_b}}{2\gamma^2 \Delta} \left[ 2 \sin(b \delta_b) \sin(c \delta_c) + \sin^2(b \delta_b) + \frac{\gamma^2 \Delta}{p_c} \right] + 4\pi p_b \sqrt{\rho_b} \rho,$$

(3.1)

where $\Delta$ denotes the minimum non-zero eigenvalue of the area operator in LQG: $\Delta = 4\sqrt{3}\pi \gamma^2 \rho_0$, and

$$\delta_b = \sqrt{\frac{\Delta}{p_b}}, \quad \delta_c = \sqrt{\frac{\Delta \rho_c}{p_b}}.$$  

(3.2)

It should be noted that the above effective Hamiltonian corresponds to the improved dynamics prescription in LQC [6]. For $\delta_b$ and $\delta_c$ which are arbitrary functions of phase space variables, this prescription turns out to be unique in the sense that it yields physics independent of the fiducial length $L_0$, and as discussed below universal bounds on expansion and shear scalars [32].

Using the Hamilton’s equations, we obtain the modified dynamical equations for the gravitational phase space variables (assuming that the energy density depends only on triad variables, and not on connection variables):

$$\dot{p}_b = \frac{p_b \cos(b \delta_b)}{\gamma \sqrt{\Delta}} \left( \sin(c \delta_c) + \sin(b \delta_b) \right),$$

(3.3)

$$\dot{p}_c = \frac{2p_b}{\gamma \sqrt{\Delta}} \sin(b \delta_b) \cos(c \delta_c),$$

(3.4)
\[ \dot{b} = - \frac{\sqrt{p_c}}{2\gamma\Delta} \left[ 2\sin(b\delta_b)\sin(c\delta_c) + \sin^2(b\delta_b) + \frac{\gamma^2\Delta}{P_c} \right] \\
+ \frac{c p_c}{\gamma\sqrt{\Delta} P_b} \sin(b\delta_b)\cos(c\delta_c) + 4\pi G\frac{\gamma\sqrt{R}}{P_c} \left( \rho + p_b\frac{\partial \rho}{\partial p_b} \right) \] (3.5)

and

\[ \dot{c} = - \frac{p_b}{2\gamma\Delta\sqrt{R}} \left[ 2\sin(b\delta_b)\sin(c\delta_c) + \sin^2(b\delta_b) + \frac{\gamma^2\Delta}{P_c} \right] \\
- \frac{c}{\gamma\sqrt{\Delta}} \sin(b\delta_b)\cos(c\delta_c) + \frac{b p_b}{\gamma\sqrt{\Delta} P_b} \cos(b\delta_b)(\sin(c\delta_c) + \sin(b\delta_b)) \\
+ \frac{\gamma p_b}{\sqrt{R}} + 4\pi G\frac{p_b}{\sqrt{R}} \left( \rho + 2\rho\frac{\partial \rho}{\partial \rho} \right). \] (3.6)

As in the classical theory (see equation (2.13)), it turns out that time derivative of \((c p_b - b p_c)\) is given by

\[ \frac{d}{dt}(c p_b - b p_c) = \frac{\gamma p_b}{\sqrt{R}} + G\gamma V \left( 2\rho\frac{\partial \rho}{\partial \rho} - p_b\frac{\partial \rho}{\partial \rho} \right). \]

The change in the Hamiltonian evolution from classical theory to LQC, results in \(\dot{p}_c/\dot{p}_c\) and \(\dot{p}_b/\dot{p}_b\) as bounded functions. This in turn yields a non-divergent behavior of expansion and shear scalars. The expansion scalar is given by [32]

\[ \theta = \frac{1}{\gamma\sqrt{\Delta}} \left( \sin(b\delta_b)\cos(c\delta_c) + \cos(b\delta_b)\sin(c\delta_c) + \sin(b\delta_b)\cos(b\delta_b) \right), \] (3.7)

which is bounded above due to discrete quantum geometric effects inherited via area gap \(\Delta\): \(|\theta| \leq 2.78/\ell_P^2\). The shear scalar becomes,

\[ \sigma^2 = \frac{1}{3\gamma^2\Delta} \left( 2\sin(b\delta_b)\cos(c\delta_c) - \cos(b\delta_b)(\sin(c\delta_c) + \sin(b\delta_b)) \right)^2, \] (3.8)

which is also universally bounded [32]: \(\sigma^2 \leq 5.76/\ell_P^2\).

From (3.3) and (3.4) an important result follows on the permitted values of \(p_b\) and \(p_c\). Let \(t_0\) be some time in the evolution at which \(p_c\) and \(p_b\) have some given non-zero finite values \(p_c^0\) and \(p_b^0\). Then from (3.4) we have

\[ \int_{p_c^0}^{p_c(t)} \frac{dp_c}{\dot{p}_c} = \int_{0}^{t} \frac{2}{\gamma\sqrt{\Delta}} \sin(b\delta_b)\cos(c\delta_c)dt \] (3.9)

which implies

\[ p_c(t) = p_c^0 \exp \left\{ \frac{1}{\gamma\sqrt{\Delta}} \int_{0}^{t} (\sin(b\delta_b + c\delta_c) + \sin(b\delta_b - c\delta_c))dt \right\}. \] (3.10)

Since \(\sin(b\delta_b + c\delta_c) + \sin(b\delta_b - c\delta_c) \leq 2\), the integration (inside the exponential) over a finite time is finite. Hence, at any finite proper time, in the past or in the future:

\[ 0 < p_c(t) < \infty. \] (3.11)

Similarly using (3.3), we get
\[ p_b(t) = p_b^0 \exp \left\{ \frac{1}{\gamma \sqrt{\Delta}} \int_0^t \cos(b \delta_b)(\sin(c \delta_c) + \sin(b \delta_b)) \, dt \right\} \] (3.12)

and since the integration inside the exponential is again over a bounded function, we obtain a finite integral over finite range of time. Hence, we obtain

\[ 0 < p_b(t) < \infty \] (3.13)

for any given finite time in past or future. Therefore, we reach an important result that \( p_b, p_c \) and consequently the volume \( V = 4\pi p_b \sqrt{\rho} \) are finite, positive and non-zero at any finite time. Note that a similar argument was used in [12] to show the finiteness of the triad variables for any finite time in the effective dynamics of the Gowdy model.

From the vanishing of the Hamiltonian constraint, we can get the energy density in terms of dynamical variables:

\[ \rho = \frac{1}{8\pi G \gamma^2 \Delta} \left[ 2 \sin(b \delta_b) \sin(c \delta_c) + \sin^2(b \delta_b) + \frac{\gamma^2 \Delta}{\rho} \right]. \] (3.14)

Hence the energy density remains finite by virtue of (3.11) and (3.13) for any finite proper time.

So far we have seen that the expansion and shear scalars are generically bounded for all time. Energy density \( \rho \) remains finite under evolution over a finite proper time. And \( p_b, p_c \) and \( V \) remain non-zero, positive and finite under evolution over a finite proper time. The terms \( \dot{p}_c/p_c \) and \( \dot{p}_b/p_b \) are bounded due to (3.4) and (3.3). Hence the divergence in curvature invariants given by (2.17), (2.18) and (2.19) may only come from divergence in \( \dot{p}_b/p_b \) and \( \dot{p}_c/p_c \).

After a straightforward calculation, the expressions for \( \dot{p}_b \) and \( \dot{p}_c \) turn out to be the following:

\[
\dot{p}_b = p_b \left[ \frac{\cos(b \delta_b) \cos(c \delta_c)}{p_c} + \frac{\cos^2(b \delta_b)}{\gamma^2 \Delta} (\sin(b \delta_b) + \sin(c \delta_c))^2 \right.
\]

\[
- \frac{4\pi}{\gamma^2 \sqrt{\Delta}} \frac{(c \rho - b p_b)}{V} \cos(c \delta_c) (\sin(c \delta_c) + \sin^2(b \delta_b))
\]

\[
+ 4\pi G \left( 2p_c \frac{\partial \rho}{\partial p_c} \cos(b \delta_b) \cos(c \delta_c) - p_b \frac{\partial \rho}{\partial p_b} \sin(b \delta_b) \sin(c \delta_c) \right)
\]

\[ + \frac{p_b}{p_b} \frac{\partial \rho}{\partial p_b} \cos(2b \delta_b) \right]. \] (3.15)

and

\[
\dot{p}_c = p_c \left[ -2 \sin(b \delta_b) \sin(c \delta_c) + \frac{4 \sin^2(b \delta_b) \cos^2(c \delta_c)}{\gamma^2 \Delta} \right.
\]

\[
+ \frac{4\pi}{\gamma^2 \sqrt{\Delta}} \frac{(c \rho - b p_b)}{V} \sin(2b \delta_b) (1 + \sin(b \delta_b) \sin(c \delta_c))
\]

\[
+ 8\pi G \left( p_b \frac{\partial \rho}{\partial p_b} \cos(b \delta_b) \cos(c \delta_c) - 2p_c \frac{\partial \rho}{\partial p_c} \sin(b \delta_b) \sin(c \delta_c) \right). \] (3.16)

The unboundedness in above terms can arise from terms containing \( (c \rho - b p_b) \) and/or from terms with \( \frac{\partial \rho}{\partial p_c} \) and \( \frac{\partial \rho}{\partial p_b} \). It turns out that any potential divergences from the first type are tied to the second type in the following way. We have earlier found, below equation (3.6), that the time derivative of this difference is given by the same expression in the classical theory and LQC (equation (2.13)). In equation (2.13), first term on the R.H.S is finite due to (3.11).
and (3.13). Integrating the right hand side, we see that the quantity \((c_R - b_{p_0})\) may diverge if the derivatives of the energy density with respect to triads diverge. Otherwise \((c_R - b_{p_0})\) will be finite at any finite past or future time.

In conclusion, any divergences in \(\frac{\rho}{\rho_0}\) and \(\frac{p}{p_0}\), and consequently in the curvature invariants come from the terms \(\frac{\partial \rho}{\partial p_0}\) and \(\frac{\partial \rho}{\partial p_e}\). Since energy density is always finite for any finite time, we need matter with an equation of state which has divergent triad derivatives of energy density with energy density being finite. Only then \(\dot{p}_0\) and \(\dot{p}_e\) can diverge in finite time in this loop quantized Kantowski–Sachs model. And only then the curvature invariants can diverge. These divergences in curvature invariants in case of special matter types lead to weak curvature singularities as will be shown in section 5. For all other types of matter the curvature invariants are non-divergent for all finite times indicating the absence of any singularities.

4. Effective dynamics: matter with vanishing anisotropic stress and pressure singularities

In the previous section we found that for effective spacetime description in LQC, the only way curvature invariants can diverge is when derivatives of energy density with respect to triads diverge at finite energy density. We will now show that for the special case of matter having a vanishing anisotropic stress, these divergences are related to the pressure divergences. For such a matter, the energy density is a function of volume only, i.e. \(\rho(p_b, p_e) = \rho(p_b, \sqrt{V})\). Then the pressure \(P\) can be written as,

\[
P = -\frac{\partial \mathcal{H}_{\text{mat}}}{\partial V} = -\rho - V \frac{\partial \rho}{\partial V}.
\]

(4.1)

The derivatives of the energy density with respect to triads can be written in terms of the energy density and pressure as:

\[
p_b \frac{\partial \rho}{\partial p_b} = 2p_b \frac{\partial \rho}{\partial p_e} = -\rho - P.
\]

(4.2)

Using the above equations, the expression for \(\dot{p}_b\) in the case of vanishing anisotropic stress can be obtained from equation (3.15). It turns out to be:

\[
\dot{p}_b = p_b \left[ \frac{\cos(b_{\dot{b}_b}) \cos(c_{\dot{b}_c})}{p_c} + \frac{\cos^2(b_{\dot{b}_b})}{\gamma^2 \Delta} (\sin(b_{\dot{b}_b}) + \sin(c_{\dot{b}_c}))^2 \right. \\
- \frac{4\pi}{\gamma^2 \sqrt{\Delta}} \frac{(c_{\dot{b}_c} - b_{\dot{p}_b})}{V} \cos(c_{\dot{b}_c}) (\sin(c_{\dot{b}_c}) + \sin^2(b_{\dot{b}_b})) \\
- 4\pi G \left( \cos(b_{\dot{b}_b} + c_{\dot{b}_c}) + \cos(2b_{\dot{b}_b})(\rho + P) \right].
\]

(4.3)

Similarly, equation (3.16) yields,

\[
\dot{p}_c = p_c \left[ -2 \sin(b_{\dot{b}_b}) \sin(c_{\dot{b}_c}) \frac{\rho}{p_c} + \frac{4 \sin^2(b_{\dot{b}_b}) \cos^2(c_{\dot{b}_c})}{\gamma^2 \Delta} \right. \\
+ \frac{4\pi}{\gamma^2 \sqrt{\Delta}} \frac{(c_{\dot{b}_c} - b_{\dot{p}_b})}{V} \sin(2b_{\dot{b}_b})(1 + \sin(b_{\dot{b}_b}) \sin(c_{\dot{b}_c})) \\
- 8\pi G \left( \cos(b_{\dot{b}_b} + c_{\dot{b}_c}) (\rho + P) \right].
\]

(4.4)
Before we analyze the nature of potential singularities, it is interesting to note that the time derivative of \((c_p - b_p)\) in case of matter with vanishing anisotropic stress is given by
\[
\frac{d}{dt}(c_p - b_p) = \gamma \frac{p_b}{\sqrt{\rho_c}}.
\] (4.5)
This is easily checked by using equation (4.2) in equation (2.13). The right hand side of (4.5) is bounded by virtue of (3.11) and (3.13), which implies that the quantity \((c_p - b_p)\) is also bounded at any finite past or future time.

We have shown earlier in section 3 that \(p_b, p_c\) and \(V\) remain non-zero, positive and finite under evolution over a finite proper time. Equation (4.5) implies that \((c_p - b_p)\) is finite upon evolution over a finite time. Hence in effective dynamics in LQC for matter with a vanishing anisotropic stress, both \(p_p\) and \(p_b\), and consequently the curvature invariants given by (2.17), (2.18) and (2.19) are non-divergent except when the pressure diverges at a finite value of energy density, shear scalar and expansion scalar and non-zero volume. In the next section, we would show that such pressure singularities are weak singularities and geodesics evolution does not break down at such events.

5. Analysis of geodesics and strength of possible singularities

In this section, we analyze whether the effective spacetime description of the Kantowski–Sachs model in LQC results in geodesic evolution which breaks down in finite time, and the strength of potential singularities. We start with an analysis of geodesics, followed by the strength of singularities using Króliczak’s condition [3].

5.1. Geodesics

We noted in the previous section that the curvature invariants are generically bounded in effective dynamics except for very specific type of pressure singularities. This means that there may be potential singularities in the effective spacetime description of Kantowski–Sachs spacetime. A commonly used criterion to characterize singularities is that all the geodesics that go into the singularity must end at the singularity, i.e. the geodesics must not be extendible beyond the singularity. However, if geodesics can be extended beyond the point where the curvature invariants diverge, then it may not be a strong enough singularity to be physically significant. The spacetime may be extendable in such a case. Geodesic (in)completeness analysis is therefore important to understand the exact nature of singularities or lack thereof.

For the metric of the Kantowski–Sachs spacetime (2.3), the geodesic equations yield the following second order equations in the affine parameter \(\tau\):
\[
(g_{xx}x')' = 0, \quad (g_{\theta\theta})\sin^2(\theta)\phi' = 0,
\] (5.1)
\[
(g_{\theta\theta})' = g_{\theta\theta} \sin \theta \cos \theta \phi'^2
\] (5.2)
and
\[
-2t' t'' = g_{xx} x'^2 + g_{\theta\theta} (\theta'^2 + \sin^2 \theta \phi'^2).
\] (5.3)
Here prime denotes derivative with respect to the affine parameter. And, we recall that the metric components, \(g_{xx}\) and \(g_{\theta\theta}\) are related to the triads \(p_b\) and \(p_c\) via equation (2.4).

---

6 For examples in GR and LQC, see [4] and [16] respectively.
To find the solutions, we notice that one can rotate angular coordinates in such a way that initially when affine parameter $\tau = 0$, $\theta(0) = \pi/2$ and $\theta'(0) = 0$. Then $\theta(\tau) = \pi/2$ for all $\tau$ is a solution of the above $\theta$ geodesic equation with these initial conditions. Due to the uniqueness of solutions of second order differential equations with given initial conditions, this is the unique solution. Therefore, we will assume that $\theta = \pi/2$ hereafter. Using this result, the solutions to the remaining geodesic equations in $x$, $\phi$ and $t$ are:

$$x' = \frac{C_x}{g_{xx}}, \quad \phi' = \frac{C_\phi}{g_{\Theta\Theta}}$$

and

$$t'^2 = \epsilon + \frac{C_x^2}{g_{xx}} + \frac{C_{\phi}^2}{g_{\Theta\Theta}}.$$  

Here $C_x$ and $C_\phi$ are constants of integration, and $\epsilon = 1$ for timelike geodesics and $\epsilon = 0$ for null geodesics.

In classical GR, the geodesic equations break down if either $g_{xx}$ or $g_{\Theta\Theta}$ vanishes at a finite value of the affine parameter. This is certainly the case for the classical singularity in the Kantowski–Sachs spacetime which results in geodesic incompleteness. The situation changes dramatically, when quantum gravitational effects in LQC are in play. Due to the bounds on the values of $p_b$ and $p_c$ given in (3.13) and (3.11), both $g_{xx}$ and $g_{\Theta\Theta}$, as defined in equation (2.4), are finite, non-zero, positive functions for any finite time. This implies that the geodesic evolution never breaks down in effective dynamics in loop quantized Kantowski–Sachs model. For any finite time evolution, effective spacetime is geodesically complete.

### 5.2. Strength of singularities

Apart from the analysis of geodesics, important information about the nature of singularities can be found by analyzing their strength. This is determined by considering what happens to an object as it falls into the singularity. A strong curvature singularity is defined as one that crushes any in-falling objects to zero volume irrespective of the properties or composition of the objects [1, 2]. Basically the curvature squeezes any in-falling objects to infinite density. Infinite tidal forces completely destroy any arbitrary in-falling object. In contrast to the strong singularities, weak singularities do not imply a complete destruction of the in-falling objects. Even though some curvature components or curvature invariants may diverge, it is possible to construct a sufficiently strong detector which survives large tidal forces and escapes the singular event. These qualitative notions has been put in precise mathematical terms by Tipler [2] and Królak [3]. It has been conjectured that the physical singularities in the sense of geodesic incompleteness are those which are also strong curvature type [2, 3]. It has been argued that if the conjecture is satisfied then a weak form of Penrose’s cosmic censorship hypothesis can be proved [3].

The necessary conditions for a strong curvature singularity derived by Królak are broader than Tipler’s conditions. Any singularity which is weak by Królak’s conditions will be weak by Tipler criteria, but the converse is not true. So we use the Królak conditions in order to search for the signs of strong singularities in the broadest sense. According to necessary conditions due to Królak [3], if a singularity is a strong curvature singularity, then for some non-spacelike geodesic running into the singularity, the following integral diverges as the singularity is approached:

$$\int \frac{d\tau}{\sqrt{-g}} = \infty.$$

Infinite tidal forces completely destroy any arbitrary in-falling object. In contrast to the strong singularities, weak singularities do not imply a complete destruction of the in-falling objects. Even though some curvature components or curvature invariants may diverge, it is possible to construct a sufficiently strong detector which survives large tidal forces and escapes the singular event. These qualitative notions has been put in precise mathematical terms by Tipler [2] and Królak [3]. It has been conjectured that the physical singularities in the sense of geodesic incompleteness are those which are also strong curvature type [2, 3]. It has been argued that if the conjecture is satisfied then a weak form of Penrose’s cosmic censorship hypothesis can be proved [3].

The necessary conditions for a strong curvature singularity derived by Królak are broader than Tipler’s conditions. Any singularity which is weak by Królak’s conditions will be weak by Tipler criteria, but the converse is not true. So we use the Królak conditions in order to search for the signs of strong singularities in the broadest sense. According to necessary conditions due to Królak [3], if a singularity is a strong curvature singularity, then for some non-spacelike geodesic running into the singularity, the following integral diverges as the singularity is approached:
That is, if there is a strong curvature singularity in the region, then for a non-spacelike geodesic running into the singularity the following necessary condition is satisfied:

\[
\lim_{\tau \to \tau_0} K^j_i \to \infty,
\]

where \( \tau_0 \) is the value of the affine parameter at which the singularity is located.

Considering the behavior of the integrand, i.e. components of Riemann tensor, can lead us to understand which terms may potentially diverge and result in strong singularities. The non-zero components of the Riemann tensor for the Kantowski–Sachs metric in terms of the triads are:

\[
R^1_{212} = g_{\alpha \beta} R^2_{211} = \left( \frac{p_b^2}{p_c^2 p_c} \right)^2 - \frac{3}{4} \left( \frac{p_b}{p_c} \right)^2 - \left( \frac{p_b}{p_b} \right) \left( \frac{p_b}{p_c} \right) + \frac{1}{2} \left( \frac{p_b}{p_c} \right)^2,
\]

\[
R^1_{441} = \sin^2 \theta R^3_{331} = p_c \sin^2 \theta R^4_{441} = p_c \sin^2 \theta R^4_{441}
\]

\[
= p_c \sin^2 \theta \left[ \frac{1}{4} \left( \frac{p_b}{p_c} \right)^2 - \frac{1}{2} \left( \frac{p_b}{p_c} \right)^2 \right],
\]

\[
R^2_{442} = \sin^2 \theta R^3_{332} = -\frac{p_c}{g_{xx}} R^3_{332} = -\frac{p_c}{g_{xx}} R^4_{442}
\]

\[
= -p_c \sin^2 \theta \left[ \frac{1}{2} \left( \frac{p_b}{p_b} \right) \left( \frac{p_b}{p_c} \right) - \frac{1}{4} \left( \frac{p_b}{p_c} \right)^2 \right] \quad (5.10)
\]

and

\[
R^3_{443} = -\sin^2 \theta R^4_{443} = -\sin^2 \theta \left[ 1 + \frac{p_c}{4} \left( \frac{p_b}{p_c} \right)^2 \right].
\]

Note that the factors of \( \sin^2 \theta \) can be ignored in this analysis, as we can always choose \( \theta = \pi/2 \) along our geodesics as discussed in section 5.1.

Most of the terms in all the Riemann tensor components are of the type \( \left( \frac{p_b}{p_c} \right)^m \left( \frac{p_c}{p_b} \right)^n \left( \frac{p_c}{p_c} \right)^q f(p_1, p_2, p_3) \), which are made out of products of powers of \( p_b, p_c, \frac{p_b}{p_c} \) and \( \frac{p_c}{p_c} \), which are functions of the affine parameter. The other type of terms are \( g(p_b, p_c) \frac{p_b}{p_c} \) or \( g(p_b, p_c) \), where \( g(p_b, p_c) \) is a function only of \( p_b \) and \( p_c \), without involving any of their derivatives.

First note that the integral in (5.6) involves an integral of the absolute value of Riemann tensor components, and in turn each Riemann tensor component is a sum of several terms. Since the integral of the absolute value of a sum is always less than or equal to the integral of the sum of the absolute value of each term, for our purposes it would suffice to look individually at the integrals of the absolute value of each term separately. So we consider the different types of terms present in the Riemann tensor components mentioned above one by one.
We first look at terms of type \((\frac{p_1}{p_2})^m (\frac{p_2}{p_3})^n \frac{p_3}{p_1} f(p_1, p_2, p_3)\). We can split the integral from 0 to \(\tau\) into pieces where the integrand takes a definite sign (positive or negative), e.g.

\[
\int_0^{\tau} \, d\tau \left| \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3) \right|
\]

\[
= \int_0^{\tau_1} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3)
\]

\[
- \int_{\tau_1}^{\tau_2} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3)
\]

\[
+ \int_{\tau_2}^{\tau_3} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3)
\]

\[
= \int_{\tau_k}^{\tau_{k+1}} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3).
\]

Now focus on any one of the terms from the above expression, say \(\tau_k\) to \(\tau_{k+1}\).

\[
\int_{\tau_k}^{\tau_{k+1}} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3)
\]

\[
= \int_{\tau_k}^{\tau_{k+1}} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3)
\]

\[
= \int_{\tau_k}^{\tau_{k+1}} \, d\tau \left( \frac{p_1}{p_2} \right)^m \left( \frac{p_2}{p_3} \right)^n \frac{p_3}{p_1} f(p_1, p_2, p_3)
\]

\[
= \frac{1}{\sqrt{\epsilon + c^2 L_{B, B} p_2^2 + \frac{c^4}{p_1^2}}}
\]

Here we have used equation (5.5), and note that \(\epsilon\) is 1 for timelike geodesics and 0 for null geodesics. We have shown earlier in section 3, particularly equations (3.4), (3.3), (3.11) and (3.13) that \(\frac{p_0}{p_0}\) and \(\frac{p_i}{p_i}\) are bounded functions, and \(p_0, p_i\) are non-zero and finite as well for all finite values of the coordinate time \(t\). The quantity under the square root in the denominator of equation (5.12) is therefore positive definite because of the bounds on \(p_0\) and \(p_i\). In the following, let us consider the special case when both of the integration constants of the geodesic equations, \(C_1\) and \(C_2\), happen to be simultaneously zero.

In the timelike case, since \(\epsilon\) is equal to unity, we see from (5.4) and (5.5) that it represents the worldline of a massive particle sitting at rest at a location in space, and the denominator in (5.12) becomes unity. However, in the case of null geodesics (photons), since \(\epsilon\) is zero it seems that the denominator in the R.H.S. of (5.12) could be zero if both \(C_1\) and \(C_2\) vanish simultaneously. But we find from (5.4) and (5.5) that if both \(C_1\) and \(C_2\) are simultaneously zero, then we have the peculiar situation with coordinates \(x, \phi\) and \(t\) being constant as a function of the affine parameter. This means that the whole geodesic will be just one event in the spacetime manifold, i.e. the photon appears for one moment at some location and disappears immediately. Such a case is hence not relevant for our discussion of the strength of singularities as it does not correspond to a physically suitable null geodesic.
Thus, we show that the integrand in R.H.S. of (5.12) is well-defined, real and finite for all finite values of the time $t$. That means that the integral will be finite if the upper limit $t_0$ is finite.

If initially both $\tau$ and $t$ start from zero, then

$$\tau_0 = \int_0^{\tau_0} d\tau = \int_0^{t_0} \frac{dt}{c + \frac{c_1^2 \dot{r}^2}{P_0^2} + \frac{c_2^2}{P_0}}. \quad (5.13)$$

Note that for observers comoving with respect to the matter world lines (the fundamental observers), the proper time is given by $t$, hence $\tau_0 = t_0$. Hence for a finite $\tau_0$, $t_0$ is always finite for such observers. The integral in (5.12) is then finite for finite $\tau_0$. In general, the integrand on the R.H.S. of (5.13) is positive definite and finite for finite $t$. It is possible that for certain geodesics there can be potential cases where the upper limit $t_0$ is infinite even when $\tau_0$ is finite. If such a case exists and if the integral (5.12) diverges in such a case, then this divergence occurs at an infinite proper time for fundamental observers. Further, for such a potential divergence the energy density is still finite in the finite time evolution for the matter world lines (using the results from section 3). Hence, such a potential divergence would not correspond to any known strong singularity such as big bang/crunch, big rip and big freeze singularities which are characterized by divergence in energy density in finite proper time for fundamental observers. Thus, we can conclude that terms in the Riemann curvature components of the type $\left(\frac{\xi}{\dot{p}}\right)^n \left(\frac{\xi}{\dot{p}}\right)^m \left(\frac{\xi}{\dot{p}}\right)^q f(p_1, p_2, p_3)$ in (5.7) will not contribute to any potential divergences in finite proper time measured by comoving observers.

Let us now consider the other type of terms in the Riemann tensor components. These are of the type $g(p_0, P_1)\frac{\dot{P}_1}{P_1}$ or $g(p_0, P_1)\frac{\dot{P}_0}{P_0}$ with $g(p_0, P_1)$ independent of any derivatives. Again as before, we split the integral of the absolute value into pieces where the integrand has a definite sign, and then look at one of those pieces. These terms will be integrated at least once in (5.7), and it can be seen that on integrating by parts, there are no terms left with double derivatives of $p_0$ or $P_0$. For example,

$$\int g(p_0, P_1)\frac{\dot{P}_1}{P_1} d\tau = \int \frac{g(p_0, P_1)}{c + \frac{c_1^2 \dot{r}^2}{P_0^2} + \frac{c_2^2}{P_0}} \frac{\dot{P}_1}{P_1} dt = \int g_1(p_0, P_1)\dot{P}_1 dt$$

$$= g_1(p_0, P_1)\dot{P}_1 - \int \dot{P}_1 \left(\frac{d g_1(p_0, P_1)}{dt}\right) dt$$

$$= g_2(p_0, P_1)\frac{\dot{P}_1}{P_1} - \int f_2 \left(p_0, P_1, \frac{\dot{P}_1}{P_1}, \frac{\dot{P}_0}{P_0}\right) dt. \quad (5.14)$$

Here in the last line, we have defined $g_2(p_0, P_1) := g_1(p_0, P_1)$, and $f_2 \left(p_0, P_1, \frac{\dot{P}_1}{P_1}, \frac{\dot{P}_0}{P_0}\right)$. Note that $f_2 \left(p_0, P_1, \frac{\dot{P}_1}{P_1}, \frac{\dot{P}_0}{P_0}\right)$ is a term of type $\left(\frac{\xi}{\dot{p}}\right)^m \left(\frac{\xi}{\dot{p}}\right)^n \left(\frac{\xi}{\dot{p}}\right)^q f(p_1, p_2, p_3).$ We have already shown in (5.12) that integrals of terms like $f_2 \left(p_0, P_1, \frac{\dot{P}_1}{P_1}, \frac{\dot{P}_0}{P_0}\right)$ over a finite range of proper time for fundamental observers are non-divergent. And $g_2(p_0, P_1)\frac{\dot{P}_1}{P_1}$ is finite for finite values of time $t$. As for the case of (5.12), integral (5.14) does not result in a divergence occurring in a finite proper time for comoving observers. The terms containing quantities like $\frac{\dot{P}_1}{P_1}$ or $\frac{\dot{P}_0}{P_0}$, which could lead to potential
divergences arising from pressure or derivative of energy density with respect to the triads, as mentioned in sections 3 and 4, are removed upon integrating once.

Hence we have established that the necessary condition (5.7) for the presence of strong curvature singularity, i.e equation (5.7), is not satisfied in effective dynamics of Kantowski–Sachs spacetime for any finite proper time measured by fundamental observers. Therefore, all the potential curvature divergent events associated with pressure and derivatives of energy density with triads discussed in sections 3 and 4 turn out to be weak singularities.

6. Conclusions

A key question for any quantum theory of gravity is whether it can successfully resolve various spacelike singularities. As classical singularities are generic features of the classical continuum spacetime, the analogous question is whether non-existence of singularities is a generic result of quantum spacetime. As there are singularity theorems in classical GR, is there an analogous non-singularity theorem in quantum gravity? Since we do not have a full theory of quantum gravity, these questions cannot be fully answered at the present stage. Yet, valuable insights can be gained by understanding whether and how quantum gravitational effects lead to singularity resolution in spacetimes which can be quantized. By systematically studying such spacetimes with increasing complexity, one expects that key features of singularity resolution in general in quantum gravity can be uncovered.

LQC provides a very useful avenue for these studies. In recent years, a rigorous quantization of various spacetimes has been performed, and resolution of singularities in different models has been found [5]. The effective spacetime description of LQC enables us to understand singularity resolution in considerable detail. In previous works, using this description above questions on generic resolution of singularities have been addressed in isotropic and Bianchi-I spacetimes [16–18]. In these works, it was found that in the effective spacetime description of LQC all strong singularities are resolved and spacetime is geodesically complete. Our goal in this manuscript was to probe these issues in Kantowski–Sachs spacetime in LQC using Boehmer–Vandersloot prescription [26].

In contrast to the previous investigations on this topic, Kantowski–Sachs spacetime is additionally non-trivial. Unlike the isotropic and Bianchi-I spacetime in LQC, energy density is not universally bounded because of the presence of inverse power of a triad component \( p_\alpha \). Note that universal bound on energy density played an important role in proving geodesic completeness and resolution of strong singularities in isotropic and Bianchi-I spacetimes [16–18]. It was recently found using numerical simulations that dynamical bounds exist on energy density in loop quantized Kantowski–Sachs model [32, 39]. However, an analytical proof was needed to reach general conclusions about singularity resolution. A novel result in our analysis is that in any finite time range, energy density remains finite. Coupled with another result from our present analysis, that volume never becomes zero or infinite in finite time evolution, we find that singularities such as big bang/crunch which occur at zero volume with infinite energy density, big rip singularities occurring at infinite volume with infinite energy density, and big freeze singularities occurring at finite volume but infinite energy density are avoided.

The finiteness of energy density, expansion and shear scalars does not imply that curvature invariants are also finite. Investigating their behavior, we find that the curvature invariants remain bounded for all finite times except under certain circumstances. If the matter present is such that the derivatives of the energy density with respect to the triad variables can diverge even though the energy density is finite, then the curvature invariants diverge at these events. By considering the case of matter with vanishing anisotropic stress we show that these
triad-derivatives of energy density are related to the pressure. In other words these divergences occur due to divergences in pressure at finite value of energy density. Do these events where curvature invariants diverge imply strong singularities and geodesic incompleteness? The answer turns out to be negative.

Analyzing geodesics to understand the nature of the potential singularities indicated by divergences in curvature invariants, we find that the Kantowskâ–Sachs spacetime is geodesically complete in the effective dynamics of LQC. That means geodesics can be extended beyond the potential singularities where pressure or triad derivatives of energy density diverge at finite energy density, expansion and shear scalars. Using Królak’s condition of the strength of the singularities, these potential singularities turn out to be weak. We find that all known strong curvature singularities are non-existent in finite time evolution in effective spacetime. Thus, the only possible singularities in effective spacetime of Kantowski–Sachs model in LQC are weak singularities beyond which geodesic can be extended.

Our analysis, thus generalizes previous results on geodesic completeness and strong singularity resolution in LQC to Kantowski–Sachs spacetime providing useful insights on singularity resolution in black hole interior and in presence of anisotropies and spatial curvature. Note that our analytical results though show strong singularity avoidance in any finite time evolution, the question of how exactly the singularity is resolved for a specific matter can be answered only using numerical simulations. Such numerical investigations carried out for scalar fields, massless and in presence of potentials, show that classical singularity is replaced by bounces of triads [26, 27, 32, 39]. All these results obtained in different spacetimes imply robust signs of quantum geometric effects as understood in LQG yielding a generic resolution of strong singularities. Future investigations with more complex and richer spacetimes are important in this direction.

Acknowledgments

We thank an anonymous referee for useful comments and suggestions on the manuscript which led to its improvement. This work is supported by NSF grants PHYS1404240 and PHYS1454832.

References

[1] Ellis G F R and Schmidt B G 1977 Gen. Rel. Grav. 8 915
[2] Tipler F J 1977 Phys. Lett. 64A 8
   Tipler F J, Clarke C J S and Ellis G F R 1980 General Relativity and Gravitation ed A Held (New York: Plenum)
[3] Clarke C J S and Krolak A 1985 J. Geom. Phys. 2 127
   Krolak A 1986 Class. Quantum Grav. 3 267
[4] Fernandez-Jambrina L and Lazkoz R 2004 Phys. Rev. D 70 121503
[5] Ashtekar A and Singh P 2011 Class. Quantum Grav. 28 213001
[6] Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. D 74 084003
[7] Ashtekar A, Corichi A and Singh P 2008 Phys. Rev. D 77 024046
   Bentivegna E and Pawlowski T 2008 Phys. Rev. D 77 124025
   Szulc L, Kaminski W and Lewandowski J 2007 Class. Quantum Grav. 24 2621
   Ashtekar A, Pawlowski T, Singh P and Vandersloot K 2006 Phys. Rev. D 75 0240035
   Vandersloot K 2007 Phys. Rev. D 75 023523
   Szulc L 2007 Class. Quantum Grav. 24 6191
   Kaminski W and Pawlowski T 2010 Phys. Rev. D 81 024014
   Mena Marugan G A, Olmedo J and Pawlowski T 2011 Phys. Rev. D 84 064012
   Corichi A and Karam A 2011 Phys. Rev. D 84 044003

17
Pawlowski T and Ashtekar A 2012 Phys. Rev. D 85 064001
Pawlowski T, Pierini R and Wilson-Ewing E 2014 Phys. Rev. D 90 123538
[8] Ashtekar A and Wilson-Ewing E 2009 Phys. Rev. D 79 083535
Ashtekar A and Wilson-Ewing E 2009 Phys. Rev. D 80 123532
Wilson-Ewing E 2010 Phys. Rev. D 82 043508

[9] Singh P and Wilson-Ewing E 2014 Class. Quantum Grav. 31 035010
[10] Martin-Benito M, Mena Marugan G A and Wilson-Ewing E 2010 Phys. Rev. D 82 084012
[11] Garay L J, Martin-Benito M and Mena Marugan G A 2010 Phys. Rev. D 82 044048
[12] Tarrio P, Mendez M F and Mena Marugan G A 2013 Phys. Rev. D 88 084050
[13] Singh P 2012 Class. Quantum Grav. 29 244002
Brizuela D, Cartin D and Khanna G 2012 SIGMA 8 001
Diener P, Gupta B and Singh P 2014 Class. Quantum Grav. 31 025013
Diener P, Gupta B and Singh P 2014 Class. Quantum Grav. 31 105015
Diener P, Gupta B, Megevand M and Singh P 2014 Class. Quantum Grav. 31 165006

[14] Craig D and Singh P 2011 Found. Phys. 41 371
Craig D A and Singh P 2013 Class. Quantum Grav. 30 205008
Craig D and Singh P 2016 The vertex expansion in the consistent histories formulation of spin foam loop quantum cosmology arXiv:1603.09671 [gr-qc]

[15] Corichi A and Singh P 2009 Phys. Rev. D 80 044024
[16] Singh P 2009 Class. Quantum Grav. 26 125005
[17] Singh P 2012 Phys. Rev. D 85 104011
[18] Singh P 2014 Bull. Astron. Soc. India 42 121
[19] Gupta B and Singh P 2012 Phys. Rev. D 85 044011
[20] Corichi A and Karami A 2016 Int. J. Mod. Phys. D 25 1642011
[21] Cailleteau T, Cardoso A, Vandersloot K and Wands D 2008 Phys. Rev. Lett. 101 251302
Samir M, Singh P and Tsujikawa S 2006 Phys. Rev. D 74 043514
Naskar T and Ward J 2007 Phys. Rev. D 76 063514
Samart D and Gumjudpai B 2007 Phys. Rev. D 76 043514
Singh P and Vidotto F 2011 Phys. Rev. D 83 064027
Singh P 2016 Int. J. Mod. Phys. D 25 1642001

[22] Ashtekar A and Bojowald M 2006 Class. Quantum Grav. 23 391
[23] Modesto L 2004 Phys. Rev. D 70 124009
Modesto L 2006 Int. J. Theor. Phys. 45 2235
[24] Campiglia M, Gambini R and Pullin J 2008 AIP Conf. Proc. 977 52
Gambini R and Pullin J 2013 Phys. Rev. Lett. 110 211301
[25] Corichi A and Singh P 2016 Class. Quantum Grav. 33 055006
[26] Boehmer C G and Vandersloot K 2007 Phys. Rev. D 76 104030
[27] Chiou D W 2008 Phys. Rev. D 78 044019
[28] Ashtekar A, Bojowald M and Lewandowski J 2003 Adv. Theor. Math. Phys. 7 233
[29] Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. Lett. 96 141301
Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. D 73 124038
[30] Corichi A and Singh P 2008 Phys. Rev. D 78 024034
[31] Dadhich N, Joe A and Singh P 2015 Class. Quantum Grav. 32 185006
[32] Joe A and Singh P 2015 Class. Quantum Grav. 32 015009
[33] Tavera V 2008 Phys. Rev. D 78 064072
[34] Singh P and Toporensky A 2004 Phys. Rev. D 69 104008
[35] Magueijo J and Singh P 2007 Phys. Rev. D 76 023510
[36] Singh P and Soni S K 2016 Class. Quantum Grav. 33 125001
[37] Singh P, Sami M and Dadhich N 2003 Phys. Rev. D 68 023522
[38] Corichi A and Karami A 2014 Class. Quantum Grav. 31 035008
Corichi A, Karami A and Montoya E 2014 Loop quantum cosmology: anisotropy and singularity resolution Relativity and Gravity (Springer Proceedings in Physics vol 157) (Berlin: Springer) 469–77
[39] Joe A and Singh P Numerical investigations in loop quantum Kantowski–Sachs model (to appear)