SKYRMIONS AND THE PION MASS

Richard A. Battye\textsuperscript{1} and Paul M. Sutcliffe\textsuperscript{2}

\textsuperscript{1} Jodrell Bank Observatory, Macclesfield, Cheshire SK11 9DL U.K.
& School of Physics and Astronomy, Schuster Laboratory,
University of Manchester, Brunswick St, Manchester M13 9PL, U.K.
Email : rbattye@jb.man.ac.uk

\textsuperscript{2} Institute of Mathematics, University of Kent, Canterbury, CT2 7NF, U.K.
Email : P.M.Sutcliffe@kent.ac.uk

October 2004

Abstract

We present numerical evidence that suggests the introduction of a non-zero pion mass might dramatically affect the structure of minimal energy Skyrmions. It appears that the shell-like Skyrmions which are the minima when the pions are massless can fail to be minimal energy bound states for particular baryon numbers, with a strong dependence upon the value of the pion mass. The effects of a pion mass may include the replacement of shell-like configurations with crystal chunks and the loss of shell-like bound states with baryon numbers five and eight; which is in agreement with expectations based on real nuclei.
1 Introduction

The Skyrme model [15] is a nonlinear theory of pions which is an approximate, low energy effective theory of quantum chromodynamics, obtained in the limit of a large number of quark colours [16]. Skyrmions are topological soliton solutions of the model and are candidates for an effective description of nuclei, with an identification between soliton and baryon numbers.

The Lagrangian of the Skyrme model contains only three free parameters; two of these set the energy and length units and the third corresponds to the (tree-level) pion mass. In Ref.[2] the energy and length units were calculated by fitting to the masses of the proton and delta resonance assuming massless pions, and in Ref.[1] this calculation was repeated using the measured value for the pion mass. As the pion mass is relatively small its inclusion modifies the computed energy and length units by only a few percent in this approach. The main effect of the pion mass is that the Skyrmion becomes exponentially localized, rather than the algebraic asymptotic behaviour of the field in the massless pion model.

It is often said that the pion mass has little qualitative effect and so it is frequently neglected in the study of Skyrmions. For more than two Skyrmions there are very few studies that include massive pions [8, 3, 11] and furthermore these results merely reinforce the view that there are few qualitative differences, since the Skyrmions show no substantial variations from those in the massless pion model [7]. In addition, as far as we are aware, there are no studies of the properties of multi-Skyrmions as a function of the pion mass. In this paper we show that the inclusion of a pion mass may have dramatic effects, if either the baryon number or the pion mass is not small.

In section 3 we discuss the properties of shell-like Skyrmions with baryon numbers upto 40, using the standard value of the pion mass, as calculated in Ref.[1]. We show that this pion mass leads to an interesting modification to the asymptotic behaviour of the energy per baryon at large baryon number. Above baryon number 12 this quantity starts to increase, in constrast to the decreasing trend in the massless case. This means that for large enough baryon numbers there can be no shell-like bound states; for example, the charge 24 Skyrmion shell is unstable to the decay into two charge 12 Skyrmions. This is because the pion mass produces an energy contribution which depends upon the volume enclosed within the shell of the Skyrmion.

Candidate replacements for shell-like Skyrmions involve chunks of the Skyrme crystal. The most natural crystal chunk occurs at charge 32, so in section 4 we compare the energy of this crystal chunk with the minimal energy charge 32 shell, as a function of the pion mass. For massless pions the shell has lowest energy, but at a value well below the standard pion mass we find that there is a cross-over so that the crystal chunk has lowest energy.

When the pion mass is included it is invariably set to the value calculated in Ref.[1], which we are referring to as the standard value. As we have already pointed out, this calculation involves first determining the energy and length units by fitting to the masses of the proton and delta resonance, and then converting the measured pion mass into these units. At the time of this calculation there were no results available for multi-Skyrmions,
so this approach was essentially the only one available. However, there are now substantial results available (at least classically) for a large number of multi-Skyrmions, and it is our opinion that it is worth reconsidering whether fitting to an unstable resonance is the best way to determine the parameters of the Skyrme model, when the main application is to try and model nuclei of all baryon numbers.

With this point of view in mind there is clearly more flexibility for the physical value of the pion mass, since we are prepared to sacrifice the approximation of the delta resonance in favour of a better fit to nuclei with baryon number greater than one. This motivates our study in section 5, where we consider shell-like Skyrmions with baryon numbers one to eight, and pion masses from zero up to a value several times larger than the standard one. We find evidence for some interesting behaviour at relatively large pion masses. Notably, there appears to be a range of pion masses at which the shell-like Skyrmions with baryon numbers five and eight are no longer bound states. This result is in agreement with real nuclei, since there are no stable states with five or eight nucleons.

Finally, in section 6 we summarize our findings and describe the future work that needs to be done to investigate the validity of some of our conjectures, and to test our ideas regarding the comparisons with real nuclei.

2 Skyrmions

2.1 The Skyrme model

The field $U$ of the Skyrme model [15] is an $SU(2)$-valued scalar with the Lagrangian

$$L = \int \left\{ \frac{F_\pi^2}{16} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{Tr}([\partial_\mu UU^\dagger, \partial_\nu UU^\dagger][\partial^\mu UU^\dagger, \partial^\nu UU^\dagger]) + \frac{m_\pi^2 F_\pi^2}{8} \text{Tr}(U - 1) \right\} d^3 x. \quad (2.1)$$

Here $F_\pi$, $e$ and $m_\pi$ are parameters, whose values are fixed by comparison with experimental data. $F_\pi$ may be interpreted as the pion decay constant, and so could be fixed by its experimental value, but this approach is not usually taken (see for example Refs.[2, 1]). $e$ is a dimensionless constant and $m_\pi$ is the pion mass, which if set to be non-zero is generally taken to be the measured value once $F_\pi$ and $e$ are fixed in some way.

The parameters $F_\pi$ and $e$ can be scaled away by using energy and length units of $F_\pi/4e$ and $2/eF_\pi$ respectively, which we adopt from now on. In terms of these units the Skyrme Lagrangian becomes

$$L = \int \left\{ -\frac{1}{2} \text{Tr}(R_\mu R^\mu) + \frac{1}{16} \text{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + m^2 \text{Tr}(U - 1) \right\} d^3 x, \quad (2.2)$$

where we have introduced the $su(2)$-valued current $R_\mu = (\partial_\mu U)U^\dagger$, and the normalized pion mass $m = 2m_\pi/(F_\pi e)$.

From Ref.[1] the standard values used are $F_\pi = 108\text{MeV}$, $e = 4.84$ and $m_\pi = 138\text{MeV}$, giving $m = m_\pi = 0.526$. So, if the energy and length units are fixed by fitting the masses
of the proton and delta resonance then the pion mass in our units is \( m^* \). However, if a different approach is taken to determining \( F_\pi \) and \( e \), for example, by requiring a fit to the properties of nuclei with baryon numbers greater than one, then a pion mass other than \( m^* \) would result. This motivates our later study in which we will vary \( m \).

The boundary condition is that \( U \to 1 \) at spatial infinity, and such maps are characterized by an integer-valued winding number. This topological charge is interpreted as the baryon number and has the expression

\[
B = -\frac{1}{24\pi^2} \int \epsilon_{ijk} \text{Tr} \left( R_i R_j R_k \right) \, d^3x. \tag{2.3}
\]

In this paper we are only concerned with static fields, in which case the Skyrme energy derived from the Lagrangian (2.2) is

\[
E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}([R_i, R_j][R_i, R_j]) + m^2 \text{Tr}(1 - U) \right\} \, d^3x, \tag{2.4}
\]

where we have introduced the additional factor of \( 1/12\pi^2 \) for later convenience. With this normalization of the energy the Faddeev-Bogomolny bound reads \( E \geq |B| \).

To make contact with the nonlinear pion theory \( U \) is written as

\[
U = \sigma + i \pi \cdot \tau \tag{2.5}
\]

where \( \tau \) denotes the triplet of Pauli matrices, \( \pi = (\pi_1, \pi_2, \pi_3) \) is the triplet of pion fields and \( \sigma \) is determined by the constraint \( \sigma^2 + \pi \cdot \pi = 1 \).

### 2.2 The rational map ansatz and energy prediction

For massless pions the minimal energy Skyrmions have been obtained for all \( B \leq 22 \) [7]. For low charges the Skyrmions are typically very symmetric (often having Platonic symmetries) and agree qualitatively with the Skyrmions in the case of the standard pion mass [8], though only charges up to five have been computed accurately in this latter case. For higher charges the Skyrmions become even more shell-like, with the baryon and energy densities being localized on the edges of polyhedra which are generically of the fullerene type. All these known minimal energy Skyrmions (and some others) can be very well approximated using the rational map ansatz [10], which we now briefly review.

In this approach a Skyrme field with baryon number \( B \) is constructed from a degree \( B \) rational map between Riemann spheres. Although this ansatz does not give exact solutions of the static Skyrme equations, it produces approximations which have energies only a few percent above the numerically computed solutions. Briefly, use spherical coordinates in \( \mathbb{R}^3 \), so that a point \( \mathbf{x} \in \mathbb{R}^3 \) is given by a pair \((r, z)\), where \( r = |\mathbf{x}| \) is the distance from the origin, and \( z \) is a Riemann sphere coordinate giving the point on the unit two-sphere which intersects the half-line through the origin and the point \( \mathbf{x} \). Now, let \( R(z) \) be a degree \( B \) rational map between Riemann spheres, that is, \( R = p/q \) where \( p \) and \( q \) are polynomials...
in $z$ such that $\max[\deg(p), \deg(q)] = B$, and $p$ and $q$ have no common factors. Given such a rational map the ansatz for the Skyrme field is

$$U(r, z) = \exp \left[ \frac{if(r)}{1 + |R|^2} \left( 1 - \frac{|R|^2}{2R^2} \frac{2\tilde{R}}{|R|^2 - 1} \right) \right],$$

where $f(r)$ is a real profile function satisfying the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$, which is determined by minimization of the Skyrme energy of the field (2.6) given a particular rational map $R$.

Substitution of the rational map ansatz (2.6) into the Skyrme energy (2.4) results in the following expression

$$E = \frac{1}{3\pi} \int \left( r^2 f'^2 + 2B(f'^2 + 1) \sin^2 f + \mathcal{I} \frac{\sin^4 f}{r^2} + 2m^2 r^2 (1 - \cos f) \right) \, dr ,$$

where $\mathcal{I}$ denotes the integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left( \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i \, dzd\bar{z}}{(1 + |z|^2)^2} .$$

To minimize the energy (2.7) one first determines the rational map which minimizes $\mathcal{I}$, then given the minimum value of $\mathcal{I}$ it is a simple exercise to find the minimizing profile function. Thus, within the rational map ansatz, the problem of finding the minimal energy Skyrmion reduces to the simpler problem of calculating the rational map which minimizes the function $\mathcal{I}$. Note that since $\mathcal{I}$ is independent of $m$ then the same minimizing maps for massless pions will also be the minimizing maps for all values of the pion mass. These maps have been determined numerically [5] for all $B \leq 40$. The effect of the pion mass is merely to change the profile function $f(r)$ and hence the energy.

Typically the energy of the rational map ansatz overestimates the true energy of the Skyrmion by a few percent (except for $B = 1$ where it is exact). A simple, but as it turns out quite accurate, method to predict the energy of a given Skyrmion as a function of the pion mass is to assume that the percentage error is independent of the pion mass. Let $E(m)$ denote the energy of the rational map ansatz for a given Skyrmion with pion mass $m$. Then if the true energy at zero pion mass, denoted by $E_0$, is known we define the predicted energy to be

$$\tilde{E}(m) = E(m) \frac{E_0}{E(0)} .$$

As we shall show in later sections, by comparison with the results of full field simulations, this simple prediction method gives a surprisingly accurate approximation to the energy of the given Skyrmion with pion mass $m$.

### 2.3 The Skyrme crystal

Studies of an hexagonal Skyrme lattice [6] suggest that for massless pions the limiting energy per baryon for large shell-like fullerene Skyrmions is 1.061, and this is consistent
Table 1: The numerically computed values of $E/B$ for $B = 1$ and $B = 4n$ with $1 \leq n \leq 10$. The first column is the baryon number. The second column is that computed in a full field theory simulation with $m = 0$, while the third column is the equivalent for $m = m_\ast$. The final column is that computed for $m = m_\ast$ using the rational map prediction (2.9) and the data in the second column. Notice that the third and fourth columns are very close to each other, the differences typically being in the third decimal place.

with results obtained for icosahedral Skyrmions with large baryon numbers [5]. However, this is larger than the value 1.036 obtained for the infinite triply periodic Skyrme crystal [14, 9, 6], again for massless pions, which has the lowest energy per baryon of any known Skyrme field.

The Skyrme crystal has cubic symmetry and is composed of half-Skyrmions. A good approximation, which has the correct symmetry structure, is given by the formulae [9]

$$\sigma = s_1 s_2 s_3, \quad \pi_1 = c_1 \sqrt{1 - \frac{c_2^2}{2} - \frac{c_3^2}{2} + \frac{c_2 c_3^2}{3}} \quad \text{and cyclic,} \quad (2.10)$$

where $s_i = \sin(\pi x_i/L)$ and $c_i = \cos(\pi x_i/L)$. Here $2L$ is the side length of the cube forming the unit cell of the crystal, and the energy is minimal when $L = 4.7$.

The Skyrme crystal, and the approximation (2.10), will play a role in section 4.

3 Shells with a pion mass

From the point of view of our discussion here, the most important feature of the shell-like structure of Skyrmions is that the value of the field is close to $U = -1$ in the centre. Since $U = 1$ at spatial infinity the Skyrmion can be thought of as a novel spherical domain wall solution, with only a finite point group symmetry, interpolating between 1 and $-1$. This is important since the term in the energy density associated with the pion mass is non-zero inside the shell and hence the overall energy will contain an extra term which will scale with the volume enclosed by the shell. If the windings associated with the topological
charge are associated only with the shell, then the baryon number $B$ will scale like the surface area of the shell, and hence it is easy to see that there will be a term in $E/B$ which is proportional to $\sqrt{B}$, and therefore it grows with $B$.

This is very much in contrast to the case of $m = 0$ for which all evidence appears to suggest that the value of $E/B$ tends to some asymptotic value which is only a few percent above the Faddeev-Bogomolny bound of $E/B \geq 1$. If the above discussion is correct then there is one important consequence: for sufficiently large values of $B$ the shell-like solutions will be unbound to a decay into smaller shells for any non-zero value of $m$. Hence, we must conclude that there are either another class of solutions for which $E/B$ tends to a constant for large $B$, or more radically that there are no bound static solutions for very large $B$.

In order to test this radical proposal we have used the same methods as developed in Ref.[7] to compute accurate numerical energies of shell-like solutions with $m \neq 0$, concentrating initially on the case of $m = m_*$ for $B = 4n$ with $1 \leq n \leq 10$. We use an initial configuration created using the rational map ansatz with the symmetry thought to be that of minimal energy [7]. This was then relaxed toward a minimum using numerical field theory methods on a parallel supercomputer in 3D to find the numerical solution. We note that imposition of the exact symmetry from the beginning will ensure that, even if the solution is unstable, it will last sufficiently long within our simulations for an accurate energy to be computed.

The results of these simulations are presented in Table 1 and Fig. 1. We see that for

Figure 1: The values of $E/B$ presented in Table 1 plotted against $B$. The open circles are the second column, the solid squares are the third column and the open squares are the final column. Included also are the two fits discussed in section 3.
For the range of baryon numbers we are considering, the energies per baryon presented in Fig. 1 can be well approximated by a fit of the form

\[ \frac{E}{B} = \frac{\alpha}{\sqrt{B}} + \beta + \gamma \sqrt{B}. \]  

(3.1)

For \( m = 0 \) the best fitting parameters are \( \alpha = 0.215, \beta = 1.007, \gamma = 0.005 \) and for \( m = m_\ast \) they are \( \alpha = 0.276, \beta = 1.016, \gamma = 0.020 \). These fits are shown as the curves in Fig. 1. The most substantial difference between these two sets of parameters is the fourfold increase in \( \gamma \) with the inclusion of the pion mass. As discussed above this term has the interpretation of a volume contribution to the energy, and this increase appears to uphold the associated explanation. It is important to note that the fit (3.1) is only appropriate for the range of values currently under consideration, that is, \( B \leq 40 \). It is not necessarily appropriate for larger values of \( B \), where, for example, in the case of massless pions we expect an asymptotic approach to the constant value \( 1.061 \) associated with large shells.

For values of \( B \) beyond the minimum in \( E/B \) for \( m = m_\ast \) the possibility of the corresponding shell-like solution being unbound arises. On inspection of the computed energies we see that the energy of the \( B = 24 \) shell solution is greater than that of two with \( B = 12 \). Hence we conclude that (at least) for \( B = 24 \), and probably all \( B \) beyond this, the shell solutions are unbound for this value of \( m \). It is our expectation that the critical value of \( B \) at which the solutions become unbound is smaller for larger values of \( m \).

The pion mass term considered in this paper is linear in the Skyrme field and this is the one that is generally used in the literature. However, this term is only really determined up to quadratic order in the pion fields so alternative possibilities exist, such as a pion mass term which is quadratic in the Skyrme field. Our results suggest that there will be significant qualitative differences between these two types of pion mass term, since a term quadratic in the Skyrme field does not penalize the region where \( U = -1 \) so shells should again be favoured as in the massless pion theory.

### 4 Thirytwo Skyrmions: Shell vs Crystal

In the previous section we have presented some numerical evidence that shell-like solutions do not have minimal energy if \( m = m_\ast \) for \( B \geq 24 \). One possibility is that the solutions become like chunks of the Skyrme crystal discussed earlier. Natural chunks of the crystal, with cubic symmetry, can be constructed with \( B = 4n^3 \) for \( n = 1, 2, \ldots \) using the method discussed below. In this section we compute \( E/B \) as a function of \( m \) for a solution which is constructed from the minimal energy rational map with degree 32 and for a chunk of the Skyrme crystal with \( B = 32 \). It is clear that the value of \( E/B \) for a chunk of the Skyrme...
crystal will not have the same properties as shell-like solutions since the baryon number will scale like volume in contrast to the surface area dependence in the case of shells.

For $B = 32$ the minimal energy rational map has $\mathcal{I} = 1297.3 = 1.2668 \times 32^2$. It has the dihedral symmetry $D_5$ and is given by

$$R = \frac{a_1 z^2 + a_2 z^7 + a_3 z^{12} + a_4 z^{17} + a_5 z^{22} + a_6 z^{27} + z^{32}}{1 + a_6 z^5 + a_5 z^{10} + a_4 z^{15} + a_3 z^{20} + a_2 z^{25} + a_1 z^{30}},$$

where $a_1 = -0.1 - 33.3$, $a_2 = 1992.1 + i906.8$, $a_3 = 312.0 + i20441.1$, $a_4 = 24183.5 + i8826.7$, $a_5 = -8633.2 - i4867.6$, $a_6 = 383.3 - i559.6$.

To create a Skyrme field with baryon number 32 from a chunk of the Skyrme crystal we follow the approach of Ref. [4], in which an interior portion of the crystal is cut out and the fields at the boundary of this chunk are interpolated to the vacuum configuration.

The approximate formulae (2.10) are used to describe the fields of the Skyrme crystal, though it is first convenient to perform a chiral $SO(4)$ rotation and work with the rotated fields

$$\bar{\sigma} = \frac{1}{\sqrt{3}}(\bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3), \quad \bar{\pi}_i = -\frac{\sigma}{\sqrt{3}} - \frac{1}{3}(\bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3) + \pi_i.$$

A cutoff $\lambda$ is chosen, which is slightly less than the crystal cell period $2L$, and within the cube $-\lambda \leq x_i \leq \lambda$ the above rotated fields are used. At a point $\mathbf{x}$ outside this cube, the direction of the pion fields is fixed to the value taken at the point where the half-line from the origin through $\mathbf{x}$ meets the surface of the cube. The $\sigma$ field outside the cube is determined by a simple linear profile function of the distance from the cube’s surface, which interpolates between the value on the surface and the vacuum value $\sigma = 1$ at the boundary of the grid.

The results of our numerical relaxations are presented in Fig. 3 for $m = 0$ and $m = 2^{n-5}m_*$ for $1 \leq n \leq 6$. We see that for low values of $m$ the shell-like solution created from the rational map is the lowest energy solution, but for larger values of $m$ the crystal chunk
is clearly much more tightly bound. The cross-over from a shell to the chunk of crystal being lower energy appears to take place around $m \approx 0.16$. We should note that this does not by itself prove that the minimum energy Skyrmion is a chunk of the crystal, but we have shown that there is a solution with lower energy than the best shell.

It is interesting to note that for most of the range of $m$ displayed in Fig. 3 the behaviour of $E/B$ indicates that $E/B \propto m$, and only for very small $m$ is the quadratic $E/B \propto m^2$ behaviour seen, which is expected from the naive examination of the contributions to the energy density. It is clear that there is an important energy dependence on $m$ which comes from a modification to the profile of the Skyrmion. In fact, within the rational map description, it is possible to show that for very large $m$ then the profile function is deformed so that $E/B \propto \sqrt{m}$, and this is evident in the figures displayed in the following section, where we study large pion masses for low baryon numbers.

As we discussed in the previous section, the term in the energy density associated with the pion mass will produce a total contribution to the energy of a shell-like Skyrmion which scales like the volume enclosed by the shell. In Fig. 4 we display this effect by plotting the energy density averaged over the sphere of radius $r$ for the charge 32 shell. These results were obtained using the rational map ansatz. The solid curve is the massless pion case $m = 0$, where the energy density is localized around the shell. The dashed curve is for $m = m_*$ and it can be seen that although the energy density is still peaked around a shell it attains a constant value inside the shell, producing a volume contribution to the energy. As the pion mass is increased (the dotted curve corresponds to $m = 1$) the density inside the shell increases and the radius of the shell decreases to compensate.
Figure 4: The energy density averaged over the sphere of radius $r$ for the charge 32 shell, using the rational map ansatz; $m = 0$ (solid curve), $m = m_*$ (dashed curve), $m = 1$ (dotted curve).

5 Large pion masses

As discussed in the introduction, the value of the pion mass parameter in the Skyrme model depends not only on the experimentally measured pion mass, but also on how the other parameters of the Skyrme model are fixed by comparison with experimental data. In recent years there has been a substantial increase in the results available for multi-Skyrmions and so it is perhaps time to reconsider alternative approaches to fitting the Skyrme parameters to experimental data, with a view to modelling nuclei of all baryon numbers, rather than restricting attention only to the single baryon to fit the Skyrme parameters, as is done in Refs.[2, 1] from which the standard value originates. This motivates our investigations in this section, where we consider shell-like Skyrmions with baryon numbers one to eight, and pion masses from zero up to a value several times larger than the standard one.

In Fig. 5 we plot (solid circles) the results of full field simulations to compute the energy per baryon with $1 \leq B \leq 8$ for a range of pion masses from zero up to twice the standard value. The curves are the rational map predicted energies using the formula (2.9). As can be seen from this figure, there is a good agreement between these two different methods, which provides evidence for the validity of both.

Recall that for $B = 1$ the rational map ansatz produces the exact solution, so in this case the curve gives the true value. The profile function must be computed numerically, but since this requires the solution of only an ordinary differential equation this can be done with a high accuracy. It is curious to note that the largest discrepancy (though it is still quite small) between the rational map prediction and the full field simulations...
occurs for $B = 1$, with the pion mass being the largest value considered. As the rational map approximation is exact in this case then we know that the error here is in the full field simulation. The reason for this error is that it becomes more difficult to accurately calculate the energy in a full field computation as the pion mass increases because the fields have a more rapid spatial variation. This effect is greatest for $B = 1$ because this is the smallest Skyrmion, so it is the most localized. The increased localization with larger pion mass means that we are unable to determine accurate energy values much beyond twice the standard pion mass using full field simulations. However, since the rational map predicted energies are in such good agreement with the full field computations this gives us confidence to trust the rational map predicted energies for larger pion masses.

As a further test of the rational map energies, we can compare with exact calculations of the energy of the $B = 2$ Skyrmion. The $B = 2$ Skyrmion is axially symmetric, so the
Figure 6: The energy of the $B = 2$ Skyrmion (solid curve and solid circles) and twice the energy of the $B = 1$ Skyrmion (dashed curve and open circles). The circles are the full field simulations assuming axial symmetry and curves are the rational map predictions.

computation is effectively 2D and this allows larger grids to be used than in 3D, producing accurate results for larger pion masses. The axial 2D computation was done using the method described in Ref. [13] and the results are shown as the solid circles in Fig. 6. For comparison, the open circles show twice the energy of the $B = 1$ Skyrmion, computed using the same axial code. The curves are the rational map predicted energies for both these quantities. This figure supports the use of the rational map energies even for very large pion masses, with the errors still being small for pion masses upto ten times the standard value, though beyond this even the accuracy of the 2D simulations is not good enough to make quantitative statements.

Fig. 6 suggests that the binding energy of the $B = 2$ Skyrmion decreases with increasing pion mass. In fact the rational map prediction is that for a large enough pion mass the $B = 2$ Skyrmion is not bound into the break-up to two single Skyrmions. However, the pion mass at which this occurs is so large that the rational map energies can no longer be trusted, so it remains an open problem as to whether the axial $B = 2$ Skyrmion is bound for all pion masses.

For massless pions the $B = 5$ and $B = 8$ Skyrmions have relatively low binding energies. This is an encouraging result of the Skyrme model since there are no real nuclei which are stable with five or eight nucleons. There is therefore a possibility that the quantization of these classical solutions may reproduce the experimental result of no bound states. However, let us investigate the classical results further and consider how the binding energy of these shell Skyrmions varies with the pion mass.

In Fig. 7 we display, as a function of the pion mass, the energy of the $B = 5$ Skyrmion
and for comparison the energy of the $B = 4$ Skyrmion plus the $B = 1$ Skyrmion. As in Fig. 6, and all the subsequent figures in this section, the solid and open circles are the numerical field theory simulations (computed upto pion masses at which the accuracy is reliable) and the lines are the rational map predictions. In Fig. 7 the solid curve ($B = 5$) is below the dashed curve ($B = 4 + 1$) for low pion masses, indicating that the $B = 5$ Skyrmion is bound against losing a single Skyrmion, but at $m \approx 3.3$ the curves cross, so that the $B = 5$ Skyrmion is no longer a bound state above this pion mass. Assuming the rational map energy is accurate upto these values (for which we have presented reasonable evidence) this still does not prove that there is no stable $B = 5$ Skyrmion, but it does show that if a stable Skyrmion exists then it can not be of the simple shell type.

It is interesting that the experimental result of no stable baryon number five nucleus can be reproduced in the classical Skyrme model if the pion mass is set to be much larger than the standard value. This prompts us to examine the similar situation for $B = 8$. In Fig. 8 we display the energy of the $B = 8$ Skyrmion (solid curve) and for comparison twice the energy of the $B = 4$ Skyrmion (dashed curve). The results are similar, with the $B = 8$ Skyrmion becoming unbound at an even smaller pion mass $m \approx 1.8$

We have checked all other possible decay combinations for the $B = 5$ and $B = 8$ Skyrmions and found that the two described above are the relevant ones. Furthermore, we have examined all Skyrmions with $B \leq 8$ for all pion masses $m \leq 4$ and found that only the $B = 5$ and $B = 8$ Skyrmions are unbound in this range. Thus it appears a possibility that there are values of the pion mass at which $B = 5$ and $B = 8$ are the only low charge Skyrmions for which there is no bound state. For larger baryon numbers we
Figure 8: The energy of the $B = 8$ Skyrmion (solid curve and solid circles) and twice the energy of the $B = 4$ Skyrmion (dashed curve and open circles). The circles are the full field simulations and curves are the rational map predictions.

expect shell Skyrmions to be replaced by other configurations, as we have seen in detail for $B = 32$, but for small values of $B$ it is not as clear that there are alternative bound state configurations that appear at large pion mass. Whichever of these possibilities arises it is clearly an interesting phenomenon which requires further investigation.

6 Conclusion

We have studied some aspects of Skyrmions with a pion mass using both full field simulations and the rational map approximation, with the results from these two different approaches being in good agreement. These results suggest that the introduction of a non-zero pion mass might dramatically change both the quantitative and qualitative features of minimal energy Skyrmions. In particular, shell-like Skyrmions which are minimal energy bound states for massless pions fail to be bound states for massive pions, and we have presented some evidence that crystal chunks may play an important role in the structure of minimal energy Skyrmions with a pion mass.

The replacement of shell Skyrmions with crystal chunks, or some similar more three-dimensional structure, has an important consequence which improves a qualitative feature of the Skyrme model in comparison with experimental data. Namely, for large baryon number $B$ the size of nuclei scales like $B^{1/3}$, which is observed experimentally, whereas for shell-like Skyrmions the size grows like $\sqrt{B}$. Thus the inclusion of the pion mass is clearly important in comparing the results of the Skyrme model with the properties of real nuclei.

As the pion mass is increased its effects become important even for very low baryon
numbers and it appears that shell-like bound states may fail to exist for baryon numbers five and eight. This is an interesting feature since it reproduces the experimental result that there are no stable nuclei with five or eight nucleons.

In this paper we have demonstrated that interesting phenomena arise when the pion mass is included in the Skyrme model. It is clearly desirable to know more about the details of Skyrmions with massive pions and further numerical investigations are currently in progress. In a zero mode quantization of Skyrmions the symmetry of the classical solution plays an important role in providing constraints on the spin, isospin and parity quantum numbers. This has been studied in detail [12] for \( B \leq 22 \) with massless pions, with some limited success in comparing with experimental data. However, the results we have presented suggest that the symmetries of Skyrmions will be quite different for massive pions, so it is of interest to determine these symmetries and calculate the new constraints on quantum numbers that arise.

Finally, we have shown that a number of Skyrmion properties appear to be quite sensitive to the value of the pion mass, so a detailed study should provide the data necessary to reconsider the way in which the parameters of the Skyrme model are set by experimental data. In particular, with a view to try and model nuclei of all baryon numbers it would seem sensible to fit the parameters of the Skyrme model by comparison with multi-Skyrmions over a range of baryon numbers, rather than restricting the fit to a single Skyrmion.

Acknowledgements

RAB acknowledges PPARC for an Advanced Fellowship.

References

[1] G. S. Adkins and C. R. Nappi, The Skyrme model with pion masses, *Nucl. Phys. B* 233, 109 (1984).

[2] G. S. Adkins, C. R. Nappi and E. Witten, Static properties of nucleons in the Skyrme model, *Nucl. Phys. B* 228, 552 (1983).

[3] C. Barnes, W. Baskerville and N. Turok, Normal modes of the \( B = 4 \) Skyrme soliton, *Phys. Rev. Lett.* 79, 367 (1997); Normal mode spectrum of the deuteron in the Skyrme model, *Phys. Lett. B* 411, 180 (1997).

[4] W. K. Baskerville, Making nuclei out of the Skyrme crystal, *Nucl. Phys. A* 596, 611 (1996).

[5] R. A. Battye, C. J. Houghton and P. M. Sutcliffe, Icosahedral Skyrmions, *J. Math. Phys.* 44, 3543 (2003).

[6] R. A. Battye and P. M. Sutcliffe, A Skyrme lattice with hexagonal symmetry, *Phys. Lett. B* 416, 385 (1998).
[7] R. A. Battye and P. M. Sutcliffe, Symmetric Skyrmions, *Phys. Rev. Lett.* **79**, 363 (1997); Solitonic fullerene structures in light atomic nuclei, *Phys. Rev. Lett.* **86**, 3989 (2001); Skyrmions, fullerenes and rational maps, *Rev. Math. Phys.* **14**, 29 (2002).

[8] E. Braaten, S. Townsend and L. Carson, Novel structure of static multisoliton solutions in the Skyrme model, *Phys. Lett.* **B235**, 147 (1990).

[9] L. Castillejo, P. S. J. Jones, A. D. Jackson, J. J. M. Verbaarschot and A. Jackson, Dense Skyrmion systems, *Nucl. Phys.* **A501**, 801 (1989).

[10] C. J. Houghton, N. S. Manton and P. M. Sutcliffe, Rational maps, monopoles and Skyrmions, *Nucl. Phys.* **B510**, 507 (1998).

[11] V. Kopeliovich, Characteristic predictions of topological soliton models, *J. Exp. Theor. Phys.* **93**, 435 (2001).

[12] S. Krusch, Homotopy of rational maps and the quantization of Skyrmions, *Ann. Phys.* **304**, 103 (2003).

[13] S. Krusch and P.M.Sutcliffe, Sphalerons in the Skyrme model, *J. Phys.* **A37**, 9037 (2004).

[14] M. Kugler and S. Shtrikman, A new Skyrmion crystal, *Phys. Lett.* **B208**, 491 (1988); Skyrmion crystals and their symmetries, *Phys. Rev.* **D40**, 3421 (1989).

[15] T. H. R. Skyrme, A nonlinear field theory, *Proc. R. Soc. Lond.* **A260**, 127 (1961).

[16] E. Witten, Global aspects of current algebra, *Nucl. Phys.* **B223**, 422 (1983); Current algebra, baryons, and quark confinement, *ibid* **B223**, 433 (1983).