Production of $P_c(4312)$ state in electron-proton collisions

In Woo Park, Sungtae Cho, Yongsun Kim, and Su Houng Lee

1 Department of Physics and Institute of Physics and Applied Physics, Yonsei University, Seoul 03722, Korea
2 Division of Science Education, Kangwon National University, Chuncheon 24341, Korea
3 Center for Extreme Nuclear Matters (CENuM), Korea University, Seoul, Korea
4 Department of Physics, Sejong University, Seoul, Korea

We study the cross sections for the electro-production of $P_c(4312)$ particle, a recently discovered pentaquark state, in electron-proton collisions assuming possible quantum numbers to be $J^P = \frac{1}{2}^\pm$, $\frac{3}{2}^\pm$. $\sqrt{s}$ is set to the energy of the future Electron Ion Collider at Brookhaven National Laboratory, in order to assess the possibility of the measurement in this facility. One can discriminate the spin of $P_c(4312)$ by comparing the pseudorapidity distribution in two different polarization configurations for proton and electron beams. Furthermore, the parity of $P_c(4312)$ can be discerned by analyzing the decay angle in the $P_c \rightarrow p + J/\psi$ channel. As the multiplicity of $P_c$ production in our calculation is large, the EIC can be considered as a future facility for precision measurement of heavy pentaquarks.

I. INTRODUCTION

Recent years have witnessed the observation of a series of pentaquark state candidates from the measurements at the Large Hadron Collider (LHC); the first observation of probable pentaquark states $P_c(4380)$ and $P_c(4450)$ was reported by LHCb collaboration in 2015 [1], and later the observation of $P_c(4312)$, $P_c(4440)$, and $P_c(4457)$ was made in 2019 [2]. The $P_c(4450)$ measured earlier in 2015 was confirmed, but revealed to consist of two narrow overlapping peaks $P_c(4440)$, and $P_c(4457)$ by the investigation of $J/\psi + p$ decays in $pp$ collisions at $\sqrt{s} = 7$, 8 and 13 TeV. More recently, the LHCb collaboration discovered a strange pentaquark state $P_{cs}(4558)$ in the $J/\psi \Lambda$ invariant mass distribution from an analysis of the $\Xi_{c0}^+ \rightarrow J/\psi + \Lambda + K^-$ decay channel [3].

These heavy pentaquark states confirmed the existence of exotic hadrons and inspired a diverse discussion about their internal structures and the quantum numbers: Are they in molecular configurations or compact multiquark states? Do just kinematical effects generate these resonances? [4] [5] What are the spins and parities of them? There have been several theoretical approaches to answer these questions, including quark models, meson-based models, diquark-based models, and QCD sum rules, yet without making no consensus.

Given the observation of the $P_c$ decay into $J/\psi$ and proton, we can expect to create the $P_c$ by colliding a proton with a photon which couples to $J/\psi$. Thereby, we propose electron-proton ($e+p$) collision experiment to create statistically meaningful $P_c$ states, thus providing critical evidence for their quantum numbers. One of the standard methods to determine the spins and parity of an unknown particle is to examine their angular distribution. Moreover, it would be beneficial if that experiment could adjust the spin polarity of colliding particles. In that sense, the $e+p$ collision with polarized beams will provide desirable circumstances.

The Electron Ion Collider (EIC) is a future collider to be built at Brookhaven National Laboratory (BNL) [6] which is designed to collide an electron beam with proton, deuteron and various heavy ion beams at high luminosity. The EIC can be a great factory for the $P_c$ production. A large coverage of detector system will be useful to measure $P_c \rightarrow p + J/\psi \rightarrow p + e^+ + e^-$ Two prospective experiments proposed at the EIC, ECCE [7] and ATHENA [8], meet this requirement well.

In this paper, we study the angular distribution of $P_c(4312)$ production at the EIC’s design energy $\sqrt{s} = 126 \text{ GeV} (E_e = 16 \text{ GeV} \text{ and } E_p = 250 \text{ GeV})$. The differential cross sections are formulated for possible combinations of spin and parity. For the technical evaluation, we use the vector meson dominance (VMD) approach. The interaction strength is derived from the decay width of $P_c(4312)$ measured by the LHCb collaboration.

This paper is organized as follows. In Section II, we introduce the VMD model to determine the coupling strength of a proton, a $\gamma$, and a $P_c(4312)$. In Section III, we calculate the cross section of $P_c$ production under four situation of spin($\frac{1}{2}$ or $\frac{3}{2}$) and parity($\pm$). In Section IV, the analysis of differential cross section is presented. The last section is given for the summary.

II. COUPLING STRENGTH : $g_{e+pP_c}$

We consider the pentaquark which is electro-produced from a proton target; $P_c$ is produced by the interaction between the proton and a photon ($\gamma$) emitted from the electron. Fig. 1(a) describes the process to the leading order with an effective coupling strength $g_{e+pP_c}$ between a proton, a $\gamma$, and a pentaquark. Although our calculation is carried out only for the $F_c(4312)$ in this paper, it can be generalized to other pentaquark states.

To compute the coupling strength $g_{e+pP_c}$, we use the VMD hypothesis and assume that the experimental esti-
mate of the $P_c(4312)$ width (9.8 MeV) is dominated by its $P_c \rightarrow p + J/\psi$ decay. This approximation provides an upper bound for $g_{\gamma p P_c}$ because all the measured pentaquark states could in principle also decay into a charmed baryon and meson such as $P_c \rightarrow \Lambda_c + D$.

A. Coupling between $J/\psi$, $p$, and $P_c$: $g_{\gamma p P_c}$

The VMD model states that photon interacts with hadrons through vector mesons as shown in Fig. 1(b). In the $P_c$-creating channels, $J/\psi$ acts as the main player because it contains a $c\bar{c}$ pair. Therefore, the first step is to determine the coupling between $P_c$, $J/\psi$, and $p$, called $g_{\gamma p P_c}$. The form of interaction depends on the quantum numbers of $P_c$, and we choose the following derivative effective Lagrangians depending on the spin-parity ($J^P$) state.

\[
\mathcal{L}_{\text{int}} = \begin{cases} 
\frac{g_{\gamma p P_c}}{m_{J/\psi}} \bar{\psi}_p \gamma^\mu F_{\mu\nu}^J F_{\mu\nu}^J \psi_{P_c} & J^P = \frac{1}{2}^+, \\
\frac{g_{\gamma J p}}{m_{J/\psi}} \bar{\psi}_p \gamma_5 \gamma^\mu F_{\mu\nu}^J F_{\mu\nu}^J \psi_{P_c} & J^P = \frac{1}{2}^-, \\
\frac{g_{\gamma p P_c}}{m_{J/\psi}} \bar{\psi}_p \gamma^\mu F_{\mu\nu}^J F_{\mu\nu}^J \psi_{P_c} & J^P = \frac{3}{2}^+, \\
\frac{g_{\gamma p P_c}}{m_{J/\psi}} \bar{\psi}_p \gamma_5 \gamma^\mu F_{\mu\nu}^J F_{\mu\nu}^J \psi_{P_c} & J^P = \frac{3}{2}^{-}.
\end{cases}
\]

Based on Eq. (1), the decay width can be calculated as

\[
\Gamma_{P_c \rightarrow p + J/\psi} = \frac{1}{16 \pi m_{P_c}^2} |\mathcal{M}|^2
\]

with $\mathcal{M}$ being the invariant matrix amplitude, and $\mathcal{M}$ being the momentum of the decayed particle in the center of mass (CM) frame: we summarize relevant formulas in Appendix A A-1. The masses of $P_c(4312)$ and $J/\psi$ are taken from the Particle Data Group: $m_{P_c} = 4311.9$ MeV, $m_{J/\psi} = 3096.9$ MeV. By equating Eq. (2) with the LHCb result, we can derive $g_{\gamma p P_c}$ as summarized in Table. I.

B. Coupling between $J/\psi$ and $\gamma$: $g_{J}$

Regarding $J/\psi \rightarrow e^- + e^+$, we adopt the following interaction Lagrangians for $J/\psi$-$\gamma$ and $\gamma$-dilepton interactions, respectively,

\[
\mathcal{L}_{J/\psi \gamma} = -\frac{e}{2g_J} F_{\mu\nu}^J F_{\mu\nu}^J, \\
\mathcal{L}_{\gamma e^- e^+} = -e\bar{\psi}\gamma^\mu A_\mu \psi.
\]

where $g_J$ is the coupling constant between the $J/\psi$ and the $\gamma$. Using the invariant matrix element given in Appendix A A-2, we can relate $g_J$ to the decay width of $J/\psi \rightarrow e^- + e^+$:

\[
\Gamma = \frac{4\pi}{3} \frac{\alpha^2}{g_J^2} \sqrt{m_{J/\psi}^2 - 4m_e^2} (1 + \frac{2m_e^2}{m_{J/\psi}^2}) \\
= 92.9 \text{ keV} \times 0.05971,
\]

, from which we obtain $g_J = 11.2$.

C. Relationship between $g_{\gamma p P_c}$, $g_{\gamma J p}$, and $g_J$

Finally, we can derive $g_{\gamma p P_c}$ from $g_{\gamma J p}$ and $g_J$ using the Lagrangians given in Eq. (3).

\[
g_{\gamma p P_c} = \frac{-e g_{\gamma J p} g_J^2}{q^2 - m_{J/\psi}^2}. 
\]

where $q$ is the momentum of the $J/\psi$. 

FIG. 1. (a) The electro-production of a pentaquark on the proton target. The effective proton-$\gamma$-pentaquark coupling is described in the VMD framework. (b) The coupling between a proton, a $\gamma$, and a pentaquark is mediated by the $J/\psi$ meson in the VMD model.
III. CROSS SECTION CALCULATION

In this section, we calculate the invariant amplitudes for the production of $P_c$ state in four possible spin-parity situation. The relevant diagram is given in Fig. [3](a).

A. Cross sections with unpolarized beams

Invariant matrix amplitudes for each Lagrangian shown in Eq. (1) are given by,

$$M = \begin{cases} \frac{g_p p_c}{m_{J/\psi}} \bar{u}(k') \gamma^\nu u(k) \frac{2\gamma^\mu}{q^2} \bar{u} P'(p') \sigma_{\mu\nu} u^N(p) & J^P = \frac{1}{2}^+, \\ \frac{2g_p p_c}{m_{J/\psi}} \bar{u}(k') \gamma^\nu u(k) \frac{2\gamma^\mu}{q^2} \bar{u} P'(p') \gamma_5 \sigma_{\mu\nu} u^N(p) & J^P = \frac{1}{2}^-, \end{cases} \quad \begin{cases} \frac{g_p p_c}{m_{J/\psi}} \bar{u}(k') \gamma^\alpha u(k) \frac{2g_0 g_{2m} - g_{2m} g_0}{q^2} \bar{u} P' \gamma_5 \gamma^\nu u^N(p) & J^P = \frac{3}{2}^+, \\ \frac{2g_p p_c}{m_{J/\psi}} \bar{u}(k') \gamma^\alpha u(k) \frac{2g_0 g_{2m} - g_{2m} g_0}{q^2} \bar{u} P' \gamma_5 \gamma^\nu u^N(p) & J^P = \frac{3}{2}^-. \end{cases} \quad (5)$$

We sum the square of the results for final spins and take the average of the initial spin polarizations of the incoming electron and proton. The detailed computation is shown in Appendix [B]. The results show that the differences in the spin-averaged square of the invariant amplitudes between opposite parities, Eqs. (B2) and (B4), appear in the differences in the sign for the $m_p$ term.

B. Cross sections with polarized beams

Considering the operation of spin-polarized beams of electron and proton, we also study the polarization dependencies of the electro-production cross section. In order to describe polarized electrons and protons, we use the projection operator, $P_{R/L} = \frac{1 \pm \gamma_5 \not{p}}{2}$, which satisfies

$$\frac{1 + \gamma_5 \not{p}}{2} u(p, s) = u(p, s), \quad \frac{1 - \gamma_5 \not{p}}{2} u(p, -s) = u(p, -s), \quad (6)$$

with the spin 4-vector, $s^\mu = (0, \vec{s}) = (0, \vec{p}/p)$. $\vec{s}$ is the spin polarization vector in the rest frame and $\vec{p}$ is the momentum of polarized particle. The spin 4-vector becomes in the Lorentz transformation,

$$s^\mu = \left( \frac{\vec{p} \cdot \vec{s}}{m}, \vec{s} + \frac{\vec{p} \cdot \vec{s}}{m(E + m)} \not{p} \right) = (\frac{p}{m}, \frac{E \vec{p}}{m p}). \quad (7)$$

It results in

$$\sum_{i=1}^{2} u_i(p) \bar{u}_i(p) \frac{1 + \gamma_5 \not{p}}{2} = (\not{p} + m) \frac{1 + \gamma_5 \not{p}}{2} = \not{p} + m \pm \not{p} \gamma_5 \not{p} \pm m \gamma_5 \not{p}. \quad (8)$$

In high energy limit, $(m \to 0)$, $s^\mu \approx p^\mu/m$, Eq. (8) becomes $\not{p} \frac{1 + \gamma_5}{2}$, and therefore we can approximate the projection operator for massless particles, or electrons, as $P_{R/L} = \frac{1 + \gamma_5}{2}$.

With the above spin projection operator, we consider the invariant amplitudes for the cross sections with polarized electrons and protons. Here, electrons and protons are chosen to be $RR, RL, LR, LL$, where $R$ and $L$ represent the right-handed and the left-handed polarity, respectively.
\[ \mathcal{M} = \begin{cases} \frac{2g_{\gamma PPc}}{m_{\gamma PPc}} \bar{u}^i(k)\gamma^\beta \frac{1}{2} \frac{1}{2} u^j(k)\bar{u}^P(p)q^\nu \sigma_{\nu\beta} \frac{1}{2} u^N(p) & J^P = \frac{1}{2}^+ \text{ RR, LL,} \\
\frac{2g_{\gamma PPc}}{m_{\gamma PPc}} \bar{u}^i(k)\gamma^\beta \frac{1}{2} \frac{1}{2} u^j(k)\bar{u}^P(p)q^\nu \sigma_{\nu\beta} \frac{1}{2} u^N(p) & J^P = \frac{1}{2}^+ \text{ RL, LR,} \\
\frac{2g_{\gamma PPc}}{m_{\gamma PPc}} \bar{u}^i(k)\gamma^\beta \frac{1}{2} \frac{1}{2} u^j(k)\bar{u}^P(p)q^\nu \sigma_{\nu\beta} \frac{1}{2} u^N(p) & J^P = \frac{1}{2}^+ \text{ RR, LL,} \\
\frac{2g_{\gamma PPc}}{m_{\gamma PPc}} \bar{u}^i(k)\gamma^\beta \frac{1}{2} \frac{1}{2} u^j(k)\bar{u}^P(p)q^\nu \sigma_{\nu\beta} \frac{1}{2} u^N(p) & J^P = \frac{1}{2}^+ \text{ RL, LR,} \\
\frac{2g_{\gamma PPc}}{m_{\gamma PPc}} \bar{u}^i(k)\gamma^\beta \frac{1}{2} \frac{1}{2} u^j(k)\bar{u}^P(p)q^\nu \sigma_{\nu\beta} \frac{1}{2} u^N(p) & J^P = \frac{1}{2}^+ \text{ LL, RL,} \end{cases} \quad (9) \]

When calculating the polarized invariant amplitude, we use the same coupling constants which were derived previously as only a given initial polarization state is taken. Note that among the four possible combinations of electron and proton polarizations, only two cases are independent as RR and LL, as well as RL and LR, result in the same invariant amplitudes. In the result, the different handedness under the same parity changes the sign of \( m_{p+} \) term only. More details are shown in Appendix C.

C. Cross section as functions of pseudorapidity and transverse momentum

We evaluate the differential cross section as functions of pseudorapidity (\( \eta \)) and transverse momentum (\( p_T \)). \( \eta \) is chosen as the main observable instead of rapidity because \( \eta \) is directly connected to detector geometries in experiment. Using the squared matrix amplitudes for \( e + p \) scattering which is detailed in Appendix B and Appendix C, the differential cross section in the CM frame is:

\[ \frac{d\sigma}{d\eta}_{CM} = \frac{2\pi \sin \theta}{64\pi^2 E_{CM}^2} \left| \frac{\vec{p}_f}{\vec{p}_i} \right| |\mathcal{M}|^2 \]

where \( |\vec{p}_i| = \frac{s-m^2}{2\sqrt{s}} \) and \( |\vec{p}_f| = \frac{s-m^2}{2\sqrt{s}} \) are the initial and final momentum in the CM frame, respectively. \( \theta \) is the polar angle of electrons after scattering in the CM frame. After all, the 4-momentum of \( p_c \) is boosted back to the lab frame to obtain the \( \eta \)-differential cross section.

IV. RESULT

In this section, we present the differential cross sections of \( p_c(4312) \) production in the \( e + p \) collision at \( \sqrt{s} = 126 \) GeV which is the EIC energy. In accordance with the previous section, the results are studied as functions of \( \eta \) and \( p_T \) under four cases of \( J^P = \frac{1}{2}^\pm \) and \( \frac{3}{2}^\pm \). \( \eta \) of \( p_c(4312) \) is computed in the lab frame, thus we can judge whether it arrives in the typical detector coverage proposed for the EIC (\( |\eta| < 4 \)). Fig. 2 shows the differential cross sections for unpolarized \( e + p \) collision.

The numbers of \( p_c(4312) \) expected to be produced at the EIC with an integrated luminosity of 10 \( fb^{-1} \) is tabulated in Tab. II. This luminosity value, 10 \( fb^{-1} \), can be reached by running the EIC for about a month at the peak intensity (10\(^{34} \) cm\(^{-2} \)s\(^{-1} \)), 8 hours a day. We found that the expected yields for the positive parity is larger than those for the negative parity by a factor 5, independent of \( p_T \) and \( \eta \). The largest yield is expected for \( J^P = \frac{3}{2}^+ \). Supposing a detector system measures the \( J/\psi \) via \( e^+ + e^- \) decay (branching ratio = 5.94\%) with the 100\% efficiency for electron and proton, \( \mathcal{O}(10^8) \) \( p_c \)'s are expected to be observed in the data accumulated for one month.
FIG. 2. Differential cross section of $P_c$ production in the unpolarized $e + p$ collision for each case of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ with positive and negative parity states. The results are calculated as a function of (a) $\eta$ and (b) $p_T(|\eta| < 4)$.

TABLE II. Expected number of $P_c(4312)$ produced at the EIC with 10 fb$^{-1}$.

| $J^P$ of $P_c$ | Yield  |
|---------------|--------|
| $\frac{1}{2}^+$ | $5.09 \times 10^9$ |
| $\frac{1}{2}^-$ | $1.01 \times 10^9$ |
| $\frac{3}{2}^+$ | $4.51 \times 10^8$ |
| $\frac{3}{2}^-$ | $7.46 \times 10^7$ |

A. Polarized cross section

The differential cross sections for the polarized electron and proton beams are shown in Fig. 3 (spin-$\frac{1}{2}$), and Fig. 4 (spin-$\frac{3}{2}$). In the case of spin-$\frac{1}{2}$, the cross sections of RR (same handedness) and RL (opposite handedness) configuration are almost identical for the backward rapidity region (proton-going direction), and they split in the forward region, $\eta > 2$ (electron-going direction). In the case of spin-$\frac{3}{2}$, a more dramatic behavior is observed: the cross section curves for RR and RL begin to separate early from $\eta \approx -2$, making RL cross section larger than RR one by two orders of magnitude at $\eta = 4$. For clear observation of this effect in experiment, we propose to measure the forward-to-backward ratio (RFB) and the beam spin asymmetry (BSA), which are defined as follows.

$$RFB(\eta) = \frac{d\sigma/d\eta(+\eta)}{d\sigma/d\eta(-\eta)} \text{, where } \eta > 0$$

$$BSA(\eta) = \frac{d\sigma/d\eta[RL] - d\sigma/d\eta(RR)}{d\sigma/d\eta[RL] + d\sigma/d\sigma(RR]}$$

These observables have experimental benefit because some of uncertainties, such as luminosity, tracking correction, and geometric acceptance, are cancelled out. As shown in Fig. 3, the spin of $P_c$ can be clearly determined by measuring the BSA in the mid-rapidity region. Yet, we found that both BSA and RFB are not much useful to judge the parity. In particular, if $P_c$ was in the spin-$\frac{3}{2}$ state, the BSA and RFB are completely insensitive to the parity.

B. Determination of $P_c$’s parity using $J/\psi$ polarization

As shown above, it is hard to identify the parity of $P_c$ with only the cross section result. To cope with this problem, we further investigate the polarization of $J/\psi$. $J/\psi$ is a spin-1 massive vector boson with two transverse and one longitudinal polarization, thus having an anisotropic angular distribution for $J/\psi \rightarrow e^+ + e^-$. The decay angle ($\theta$) is defined, in the rest frame of $J/\psi$, as the angle between the electron momentum and boost direction of the $J/\psi$ in the lab frame. By measuring $\theta$, one can experimentally tune the transverse-to-longitudinal ratio as shown in Fig. 6 (a). After tagging the polarity of $J/\psi$, we study the dependence of matrix amplitude on $\phi$ which is defined as the decay angle of $J/\psi$ from $P_c$ in the rest frame of $P_c$.

As shown in Fig. 6, the $\phi$ distribution is significantly sensitive to the polarity of $J/\psi$ for both spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ states. In either cases, the difference between the transverse $J/\psi$ events (T) and the longitudinal ones (L) is more dramatic in the positive parity state than in the negative parity state.
FIG. 3. The differential cross sections for spin $-\frac{1}{2}$ cases with polarized collision. R and L mean right-handed and left-handed, respectively.

FIG. 4. The differential cross sections for spin $-\frac{3}{2}$ cases. R and L mean right-handed and left-handed, respectively.

FIG. 5. The forward-to-backward ratio (RFB) for spin $-\frac{3}{2} P_c$ state. (b) Beam spin asymmetry (BSA) results for $J^P = \frac{1}{2}^\pm$ and $\frac{3}{2}^\pm$ states.
V. SUMMARY

The cross section for the $P_c(4312)$ production in $e+p$ collision is studied under various assumptions for its potential quantum states; $J^P = \frac{1}{2}^\pm$ and $J^P = \frac{3}{2}^\pm$.

The interaction strength of the electro-production of $P_c(4312)$, created by scattering $\gamma$ onto a proton, is calculated using the vector meson dominance hypothesis to the leading order. We also assume that the $P_c(4312) \rightarrow J/\psi + p$ channel is dominant in the decay width of $P_c(4312)$ that was measured by the LHCb collaboration.

The cross section is larger for the spin-$\frac{3}{2}$ state than for the spin-$\frac{1}{2}$ state, and larger for the positive parity case than for the negative parity. With one month of operation at the EIC in its nominal condition, millions of $P_c(4312)$'s are expected to be measured via $p + e^+ + e^-$ channel. Furthermore, more kinds of pentaquarks can be produced by electro-production onto a neutron using $e+d$ collision at the EIC. Hence, the EIC can be considered as a factory of heavy pentaquarks and will provide an excellent opportunity for a comprehensive understanding of exotic particles.

Given the availability of polarized beams at the EIC, we suggest that the analysis of pseudorapidity distribution of $P_c$ can confirm its spin number. The forward-to-backward ratio and the beam-spin asymmetry results are unambiguously distinct for the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ states. These observables are also useful to reduce the experimental uncertainties as well.

In addition, we prove that the decay kinematics of $P_c \rightarrow p + J/\psi$ is sensitive to the parity of $P_c$. The distribution of the decay angle of $P_c$ depends on the polarization of the $J/\psi$, which can be statistically determined by measuring its decay angle of $e^- + e^+$. Therefore, the parity of $P_c$ can be determined by the analysis of angular distribution. For this purpose, a hermetic detector with efficient calorimeters and tracking systems, such as ATHENA and ECCE, is necessary.

ACKNOWLEDGMENTS

This work was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1901-04, the POSCO Science Fellowship of POSCO TJ Park Foundation, and the National Research Foundation of Korea (NRF) of the Korea government (MSIT) (No. 2018R1A5A1025563 and No. 2019R1A2C1087107).
Appendix A: Invariant matrix amplitude for three-particles involving a $P_c$

A-1. $P_c(p') \rightarrow J/\psi(q) + N(p)$

Here, we present invariant matrix elements for the decay of the $P_c$ in all possible spin-parity states, i.e., four possible states: $J^{P} = \frac{1}{2}^{+}, \frac{3}{2}^{+}$, using the interaction Lagrangians given in Eq. [1].

\[ |\mathcal{M}|^2 = \begin{cases} 
\frac{g_s^2 p' \cdot (-8(p \cdot p')m_{J/\psi}^2 + 32(q \cdot p)(q' \cdot p') - 24m_p m_{J/\psi} m_{J/\psi}^2)}{m_{J/\psi}^2} & J^P = \frac{1}{2}^+, \\
\frac{g_s^2 p' \cdot (-8(p \cdot p')m_{J/\psi}^2 + 32(q \cdot p)(q' \cdot p') + 24m_p m_{J/\psi} m_{J/\psi}^2)}{m_{J/\psi}^2} & J^P = \frac{1}{2}^-, \\
2g_s^2 p' \cdot \frac{2[(q'p')^2/(p'p)] + 2(q \cdot p)(q' \cdot p') - m_{J/\psi}^2(p \cdot p' + 3m_p m_{J/\psi})}{m_{J/\psi}^2} & J^P = \frac{3}{2}^+, \\
2g_s^2 p' \cdot \frac{2[(q'p')^2/(p'p)] - 2(q \cdot p)(q' \cdot p') - m_{J/\psi}^2(p \cdot p' - 3m_p m_{J/\psi})}{m_{J/\psi}^2} & J^P = \frac{3}{2}^-.
\end{cases} \tag{A1}
\]

where, $p \cdot p' = \frac{1}{2}(m_p^2 + m_{J/\psi}^2 - m_{J/\psi}^2)$, $q \cdot p = \frac{1}{2}(m_{p'}^2 - m_p^2 - m_{J/\psi}^2)$, $q \cdot p' = \frac{1}{2}(m_{p'}^2 + m_{J/\psi}^2 - m_p^2)$, $p^2 = m_p^2$, $q^2 = m_{J/\psi}^2$, $p'^2 = m_{J/\psi}^2$. The decay rate is then given by,

\[ \Gamma = \frac{1}{8\pi E_{CM}^2} |\mathcal{M}|^2, \tag{A2} \]

where $p'_J$ is the momentum of $J/\psi$ and proton in the CM frame after decay of $P_c$,

\[ |p'_J| = \frac{1}{2m_p} \sqrt{(m_p - m_{J/\psi})^2}, \tag{A3} \]

To find the 4-momentum and polarization vector of the $J/\psi$, we take an inverse Lorentz transformation of them from the rest frame of the $J/\psi$ to the Lab frame. In the $P_c$ decay, the $P_c$ is boosted along the z-axis and the $J/\psi$ is boosted along an arbitrary direction.

\[
\begin{align*}
q^\mu &= (m_{J/\psi}, 0, 0, 0) \rightarrow (q_0, q_1, q_2, q_3), \\
\varepsilon^{\mu}_{(1)} &= (0, 1, 0, 0) \rightarrow \left\{ \frac{q_{\mu}}{m_{J/\psi}}, 1 + \frac{q_0^2}{m_{J/\psi}(q_0 + m_{J/\psi})}, \frac{q_1 q_2}{m_{J/\psi}(q_0 + m_{J/\psi})}, \frac{q_1 q_3}{m_{J/\psi}(q_0 + m_{J/\psi})} \right\}, \\
\varepsilon^{\mu}_{(2)} &= (0, 0, 1, 0) \rightarrow \left\{ \frac{q_{\mu}}{m_{J/\psi}}, \frac{q_1 q_2}{m_{J/\psi}(q_0 + m_{J/\psi})}, 1 + \frac{q_0^2}{m_{J/\psi}(q_0 + m_{J/\psi})}, \frac{q_2 q_3}{m_{J/\psi}(q_0 + m_{J/\psi})} \right\}, \\
\varepsilon^{\mu}_{(3)} &= (0, 0, 0, 1) \rightarrow \left\{ \frac{q_{\mu}}{m_{J/\psi}}, \frac{q_1 q_3}{m_{J/\psi}(q_0 + m_{J/\psi})}, \frac{q_1 q_2}{m_{J/\psi}(q_0 + m_{J/\psi})}, 1 + \frac{q_0^2}{m_{J/\psi}(q_0 + m_{J/\psi})} \right\},
\end{align*}
\]

where $q_0$ is the energy of the $J/\psi$, and $q_i (i = 1, 2, 3)$ is the 3-momentum of the $J/\psi$. We adopt the Metric tensor, $g_{00} = 1, g_{0i} = g_{i0} = 0, g_{ij} = -\delta_{ij}$.

In order to distinguish the difference between positive and negative spin-parity states, we exhibit transverse and longitudinal parts of the matrix amplitude by using

\[ P_{\mu\nu} = \sum_{a=1}^{3} \varepsilon^{(a)}_{\mu} \varepsilon^{(a)}_{\nu} = \varepsilon^{T}_{\mu} \varepsilon^{T}_{\nu} + \varepsilon^{L}_{\mu} \varepsilon^{L}_{\nu} = P_{\mu\nu}^T + P_{\mu\nu}^L = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_{J/\psi}^2}, \tag{A4} \]

where superscripts, $T$ and $L$ stand for transverse and longitudinal directions, respectively. Using the 4-momentum and polarization vectors given above, we can get transverse and longitudinal part of the polarization tensor,

\[ P_{\mu\nu}^T = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix}, \quad P_{\mu\nu}^L = \begin{pmatrix} \frac{q_0^2}{m_{J/\psi}^2} & -\frac{q_0 q_i}{m_{J/\psi}^2} \\ -\frac{q_0 q_i}{m_{J/\psi}^2} & \frac{q_i q_j}{m_{J/\psi}^2} \end{pmatrix}. \tag{A5} \]
and matrix amplitude.

\[
J^P = \frac{1}{2} \left\{ \begin{array}{l}
|\mathcal{M}|^2_T = \frac{32g_p^2}{m_{J/\psi}^2}
\left( 2(q \cdot p)(q \cdot p') + \frac{m_{J/\psi}^2}{q^2}(\vec{q} \cdot \vec{p})(\vec{q} \cdot \vec{p}') - m_{J/\psi}^2(E_{p}E_{p'} + m_pm_p') \right), \\
|\mathcal{M}|^2_L = \frac{16g_p^2}{m_{J/\psi}^2}
\left( -m_{J/\psi}^2(p \cdot p' + m_pm_p') + 2m_{J/\psi}E_{p}E_{p'} - \frac{2m_{J/\psi}^2}{q^2}(\vec{q} \cdot \vec{p})(\vec{q} \cdot \vec{p}') \right),
\end{array} \right.
\]  

(A6)

\[
J^P = -\frac{1}{2} \left\{ \begin{array}{l}
|\mathcal{M}|^2_T = \frac{32g_p^2}{m_{J/\psi}^2}
\left( 2(q \cdot p)(q \cdot p') + \frac{m_{J/\psi}^2}{q^2}(\vec{q} \cdot \vec{p})(\vec{q} \cdot \vec{p}') - m_{J/\psi}^2(E_{p}E_{p'} - m_pm_p') \right), \\
|\mathcal{M}|^2_L = \frac{16g_p^2}{m_{J/\psi}^2}
\left( -m_{J/\psi}^2(p \cdot p' - m_pm_p') + 2m_{J/\psi}E_{p}E_{p'} - \frac{2m_{J/\psi}^2}{q^2}(\vec{q} \cdot \vec{p})(\vec{q} \cdot \vec{p}') \right),
\end{array} \right.
\]  

(A7)

\[
J^P = \frac{3}{2} \left\{ \begin{array}{l}
|\mathcal{M}|^2_T = \frac{8g_p^2}{3m_{J/\psi}^2}
\left( 2m_{p}^2(q \cdot p)(q \cdot p') + 2(p \cdot p')(q \cdot p')^2 - 2m_pm_{J/\psi}^2 - m_{p}^2(m_{p}^2 - p \cdot p')m_{J/\psi}^2 \\
+ \frac{m_{J/\psi}^2}{q^2}(\vec{q} \cdot \vec{p})^2 + \frac{m_{J/\psi}^2}{q^2}(\vec{p} \cdot \vec{q})^2 - m_{J/\psi}^2p^{2}(p \cdot p') - m_{p}^2E_{p}E_{p'}m_{J/\psi}^2 \right), \\
|\mathcal{M}|^2_L = \frac{8g_p^2}{3m_{J/\psi}^2}
\left( -m_pm_{J/\psi}^2 - m_{p}^2(\vec{q} \cdot \vec{p})(\vec{q} \cdot \vec{p}') \frac{m_{J/\psi}^2}{q^2} - (p \cdot p')(\vec{q} \cdot \vec{p})^2 \frac{m_{J/\psi}^2}{q^2} \right), \\
+ m_{J/\psi}^2\dot{p}^2(p \cdot p') + m_{p}^2E_{p}E_{p'}m_{J/\psi}^2 \right) \\
\end{array} \right.
\]  

(A8)

\[
J^P = \frac{3}{2} \left\{ \begin{array}{l}
|\mathcal{M}|^2_T = \frac{8g_p^2}{3m_{J/\psi}^2}
\left( 2m_{p}^2(q \cdot p)(q \cdot p') + 2(p \cdot p')(q \cdot p')^2 - 2m_pm_{J/\psi}^2 - m_{p}^2(m_{p}^2 - p \cdot p')m_{J/\psi}^2 \\
+ \frac{m_{J/\psi}^2}{q^2}(\vec{q} \cdot \vec{p})^2 + \frac{m_{J/\psi}^2}{q^2}(\vec{p} \cdot \vec{q})^2 - m_{J/\psi}^2p^{2}(p \cdot p') - m_{p}^2E_{p}E_{p'}m_{J/\psi}^2 \right), \\
|\mathcal{M}|^2_L = \frac{8g_p^2}{3m_{J/\psi}^2}
\left( -m_pm_{J/\psi}^2 - m_{p}^2(\vec{q} \cdot \vec{p})(\vec{q} \cdot \vec{p}') \frac{m_{J/\psi}^2}{q^2} - (p \cdot p')(\vec{q} \cdot \vec{p})^2 \frac{m_{J/\psi}^2}{q^2} \right), \\
+ m_{J/\psi}^2\dot{p}^2(p \cdot p') + m_{p}^2E_{p}E_{p'}m_{J/\psi}^2 \right) \\
\end{array} \right.
\]  

(A9)

We obtain the 4-momentum of particles by performing an inverse Lorentz transformation from the CM frame (rest frame of the $P_c$) to the Lab frame. The result is given by,

\[
p^\mu = (\vec{E}_{p'}, \vec{p'}) = (\gamma m_{p}, 0, 0, \gamma m_{p}, \beta),
\]

\[
p^\mu = (E_{p'}, \vec{p'}) = \left( \gamma \sqrt{p_{1}^2 + m_{p}^2 + \beta p_{1} | \cos \phi |}, |p_{1}| \sin \phi, 0, \gamma (|p_{1}| \cos \phi + \beta \sqrt{p_{1}^2 + m_{p}^2}) \right),
\]

\[
q^\mu = (E_{\vec{q}}, \vec{q}) = \left( \gamma \sqrt{q_{1}^2 + m_{J/\psi}^2 - \beta q_{1} | \cos \phi |}, -|q_{1}| \sin \phi, 0, \gamma (-|q_{1}| \cos \phi + \beta \sqrt{q_{1}^2 + m_{J/\psi}^2}) \right). \quad (A10)
\]

with $|\vec{p}_{f}|$ being the momentum defined in (A3), and $\phi$ being the polar angle of the proton in the CM frame with respect to boost axis of the $P_c$.

**A-2.** $J/\psi \rightarrow \gamma \rightarrow e^- + e^+$

The invariant matrix amplitude of the $J/\psi$ decaying into a positron and an electron is given as below.

\[
\mathcal{M} = (-i) \frac{e^2 q^2}{g_J} \frac{-i(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{m_{J/\psi}^2})}{q^2} \vec{u}_{e^-} \gamma^\nu \vec{v}_{e^+} = -\frac{e^2}{g_J} \vec{v}_{e^+} \vec{u}_{e^-} \gamma_\mu \vec{v}_{e^+}. \quad (A11)
\]

Averaging over initial polarization of the $J/\psi$, we get,

\[
|\mathcal{M}|^2 = \frac{64\pi^2 \alpha^2}{3g_J^2} (m_{J/\psi}^2 + 2m_{J/\psi}^2). \quad (A12)
\]

with $m_{e}$ being the electron mass and $\alpha = e^2/4\pi$ is a fine structure constant. Using the method in Appendix A A-1 we can also obtain the transverse and longitudinal matrix amplitude of $J/\psi(q) \rightarrow e^-(q) + e^+(q')$. 

\[ |\mathcal{M}|_j^2 = \frac{128\pi^2\alpha^2}{g_J^2} \left( \frac{E_k E_{k'} + m_l^2}{Q_j^2} - \frac{Q_j^2}{Q_j^2} \right), \]
\[ |\mathcal{M}|_L^2 = \frac{64\pi^2\alpha^2}{g_J^2} \left( k \cdot k' + m_l^2 + \frac{2E_k E_{k'} Q_j^2}{m_{J/\psi}^2} + \frac{2E_k^2 (q_j \cdot k')(q_j \cdot k')}{Q_j^2 m_{J/\psi}^2} - \frac{2E_k E_{k'} (q_j \cdot k')(q_j \cdot k')}{m_{J/\psi}^2} - 2E_k E_{k'} (q_j \cdot k') \right). \] (A13)

Similarly as shown in Eq. (A10), we obtain the momentum in the Lab frame by taking an inverse Lorentz transformation from the CM frame (rest frame of the J/ψ) to the Lab frame,
\[ q" = (E_{q"}, \vec{q}"
\[ k" = (E_{k"}, \vec{k}"
\[ k"" = (E_{k""}, \vec{k}""
with the magnitude of momentum in the CM frame, \(|\vec{p}_J| = \frac{1}{2}\sqrt{m_{J/\psi}^2 - 4m_l^2} \) and \( E_{\vec{p}_J} = \sqrt{\vec{p}_J^2 + m_l^2} \). \( \theta \) is the polar angle of an electron in the CM frame with respect to the boost axis of the J/ψ. Then, the decay rate becomes,
\[ \Gamma = \frac{4\pi}{3} \frac{\alpha^2}{g_J^2} \sqrt{m_{J/\psi}^2 - 4m_l^2} (1 + \frac{2m_l^2}{m_{J/\psi}^2}), \] (A15)
and from the decay rate, we obtain \( g_J = 11.2 \).

Appendix B: Invariant matrix amplitudes for the scattering between unpolarized electrons and protons

Here, we present the square of the invariant matrix elements shown in Eq. (5) for the electro-production of the \( P_c \) in all possible four spin-parity states, \( J^{P} = \frac{1}{2}^{\pm}, \frac{3}{2}^{\pm} \) obtained from the interaction Lagrangians given in Eq. (1). The invariant matrix amplitude square for the scattering between unpolarized electrons and protons are as follows.
\[ |\mathcal{M}|_{J^{P} = \frac{1}{2}}^2 = \frac{64\pi\alpha\gamma_{\gamma\gamma} P_{P}}{m_{J/\psi}^2 q^4} \left( -2(k \cdot p)(k' \cdot p')q^2 - 2(k' \cdot p)(k \cdot p')q^2 + (k \cdot k')(p \cdot p')q^2 + 2(k \cdot p)(q \cdot k')(q \cdot p') \right. \]
\[ + 2(k \cdot p')(q \cdot k')(q \cdot p) + 2(k' \cdot p')(q \cdot k)(q \cdot p) + 2(k \cdot p')(q \cdot k')(q \cdot p') - 2(p \cdot p')(q \cdot k)(q \cdot k') \]
\[ - 2m_p m_P (q \cdot k)(q \cdot k') - m_p m_P (k \cdot k')q^2 \right), \] (B1)
\[ |\mathcal{M}|_{J^{P} = \frac{3}{2}}^2 = \frac{64\pi\alpha\gamma_{\gamma\gamma} P_{P}}{m_{J/\psi}^2 q^4} \left( -2(k \cdot p)(k' \cdot p')q^2 - 2(k' \cdot p)(k \cdot p')q^2 + 2(k \cdot p')(q \cdot k)(q \cdot p) + 2(k' \cdot p)(q \cdot k')(q \cdot p') \right. \]
\[ + 2(k \cdot p')(q \cdot k')(q \cdot p) + 2(k' \cdot p')(q \cdot k)(q \cdot p) + 2(k \cdot p')(q \cdot k')(q \cdot p') - 2(p \cdot p')(q \cdot k)(q \cdot k') \]
\[ + 2m_p m_P (q \cdot k)(q \cdot k') + m_p m_P (k \cdot k')q^2 \right), \] (B2)

Similarly for the \( P_c \) with its spin \( \frac{3}{2} \),
\[ |M|_{J^p=\frac{1}{2}^+}^{2} = \frac{32\pi\alpha g_{pP_1}^2}{m_{P_1}^2 m_{\gamma/\psi}^4} \left( -2m_p m_{P_1}^2 (q \cdot k)(q \cdot k') - m_{P_1}^2 q^2 (k \cdot k') (m_p m_{P_1} - p \cdot p') - 2m_{P_1}^2 (q \cdot k)(p \cdot p')(q \cdot k') \\
+ m_{P_1}^2 (q \cdot k)(k' \cdot p)(q \cdot p') + m_{P_1}^2 (k \cdot p)(q \cdot k')(q \cdot p') + m_{P_1}^2 (q \cdot k)(q \cdot p)(k' \cdot p') - m_{P_1}^2 q^2 (k \cdot p)(k' \cdot p') \\
+ (k' \cdot p') \left( m_{P_1}^2 q^2 (k' \cdot p) - m_{P_1}^2 (q \cdot p)(q \cdot k') + 2(p \cdot p')(q^2 (k' \cdot p') - (q \cdot k')(q \cdot p')) \right) \right) + 2(q \cdot k)(p \cdot p')(k' \cdot p')(q \cdot p'), \quad (B3) \]

\[ |M|_{J^p=\frac{1}{2}^-}^{2} = \frac{32\pi\alpha g_{pP_1}^2}{m_{P_1}^2 m_{\gamma/\psi}^4} \left( 2m_p m_{P_1}^2 (q \cdot k)(q \cdot k') + m_{P_1}^2 q^2 (k \cdot k')(m_p m_{P_1} + p \cdot p') - 2m_{P_1}^2 (q \cdot k)(p \cdot p')(q \cdot k') \\
+ m_{P_1}^2 (q \cdot k)(k' \cdot p)(q \cdot p') + m_{P_1}^2 (k \cdot p)(q \cdot k')(q \cdot p') + m_{P_1}^2 (q \cdot k)(q \cdot p)(k' \cdot p') - m_{P_1}^2 q^2 (k \cdot p)(k' \cdot p') \\
+ (k' \cdot p') \left( -m_{P_1}^2 q^2 (k' \cdot p) + m_{P_1}^2 (q \cdot p)(q \cdot k') + 2(p \cdot p')(q^2 (k' \cdot p') - q^2 (k' \cdot p')) \right) \right) + 2(q \cdot k)(p \cdot p')(k' \cdot p')(q \cdot p'), \quad (B4) \]

with \( k \) being the momentum of incoming electrons, \( p \) being the momentum of incoming protons, \( k' \) being the momentum of outgoing electrons, and \( p' \) being the momentum of outgoing \( P_1 \). \( q \equiv k - k' = p' - p \). In the CM frame, these momenta can be written in terms of \( |\vec{p}_1| \), \( |\vec{p}_f| \) and \( \theta \) defined in Eq. \( 10 \).

\[ k^\mu = (|\vec{p}_1|, 0, 0, |\vec{p}_1|), \]
\[ p^\mu = (\sqrt{p_1^2 + m_{P_1}^2}, 0, 0, -|\vec{p}_1|), \]
\[ k'^\mu = (|\vec{p}_f|, |\vec{p}_f| \sin \theta, 0, |\vec{p}_f| \cos \theta), \]
\[ p'^\mu = (\sqrt{p_f^2 + m_{P_1}^2}, -|\vec{p}_f| \sin \theta, 0, -|\vec{p}_f| \cos \theta). \quad (B5) \]

Appendix C: Invariant matrix amplitudes for the scattering between polarized electrons and protons

Here, we present the invariant matrix elements for the scattering between polarized electrons and protons shown in Eq. \( \ref{eq:invariant} \). We use the notation \( s_p \) to refer to the 4-spin vector of the proton. The absolute value square of invariant matrix elements for polarized electrons and protons are given by,

\[ |M|_{J^p=\frac{1}{2}^+}^{2} = \frac{16\pi\alpha g_{pP_1}^2}{m_{\gamma/\psi}^4} \left( q^2 (k \cdot k')(p \cdot p' - m_p m_{P_1}) - 2((q \cdot k)((q \cdot k')(p \cdot p' + m_p m_{P_1}) + m_p(s_p \cdot k')(q \cdot p') \\
+ m_p(q \cdot s_p)(k' \cdot p') + (k' \cdot p)(m_p(q \cdot s_p) - q \cdot p') - m_p(q \cdot p)(s_p \cdot k') - (q \cdot p)(k' \cdot p')) \\
- m_p(s_p \cdot k)(q \cdot k'(q \cdot p') - (k \cdot p'((q \cdot k')(m_p(q \cdot s_p) + q \cdot p) - q^2 (k' \cdot p)) + m_p(s_p \cdot k)(q \cdot p)(q \cdot k') \\
- m_p(k \cdot p)(q \cdot s_p)(q \cdot k') - (k \cdot p)(q \cdot k')(q \cdot p') + q^2 (k \cdot p)(k' \cdot p')) \right), \quad (C1) \]

\[ |M|_{J^p=\frac{1}{2}^-}^{2} = \frac{16\pi\alpha g_{pP_1}^2 m_{P_1}^2}{m_{\gamma/\psi}^4} \left( q^2 (k \cdot k')(p \cdot p' - m_p m_{P_1}) - 2((q \cdot k)((q \cdot k')(p \cdot p' + m_p m_{P_1}) - m_p(s_p \cdot k')(q \cdot p') \\
- m_p(q \cdot s_p)(k' \cdot p') - (k' \cdot p)(m_p(q \cdot s_p) + q \cdot p') + m_p(q \cdot p)(s_p \cdot k') - (q \cdot p)(k' \cdot p')) \\
+ m_p(s_p \cdot k)(q \cdot k'(q \cdot p') + (k \cdot p'((q \cdot k')(m_p(q \cdot s_p) - q \cdot p) + q^2 (k' \cdot p)) - m_p(s_p \cdot k)(q \cdot p)(q \cdot k') \\
+ m_p(k \cdot p)(q \cdot s_p)(q \cdot k') - (k \cdot p)(q \cdot k')(q \cdot p') + q^2 (k \cdot p)(k' \cdot p')) \right), \quad (C2) \]

and for the tensor coupling in the negative parity \( J^p = \frac{1}{2}^- \),

\[ |M|_{J^p=\frac{1}{2}^-}^{2} = \frac{16\pi\alpha g_{pP_1}^2}{m_{\gamma/\psi}^4} \left( q^2 (k \cdot k')(p \cdot p' + m_p m_{P_1}) + 2((q \cdot k)((q \cdot k')(m_p m_{P_1} - p \cdot p') - m_p(s_p \cdot k')(q \cdot p') \\
- m_p(q \cdot s_p)(k' \cdot p') + (k' \cdot p)(m_p(q \cdot s_p) + q \cdot p') - m_p(q \cdot p)(s_p \cdot k') + (q \cdot p)(k' \cdot p') \right) \]
Similarly for the coupling in the positive parity $J^p = \frac{3}{2}^+$,
\[
|M|^2_{J^p=\frac{3}{2}^+_{\text{RR,LL}}} = \frac{16\pi\alpha^2_{\gamma\gamma}}{m_{J^p}^2 q_\gamma^4} \left( q^2(k')(p'p) + m_p m_P \right) + 2 \left( (q+k)(m_p m_P - p'p) + m_p (k'p)(q'p) 
- m_p (k,p) (q'p) - q^2(k'p)(k'p) \right).
\] (C3)

Similarly for the coupling in the positive parity $J^p = \frac{3}{2}^+$,
\[
|M|^2_{J^p=\frac{3}{2}^+_{\text{R,LL}}} = -\frac{8\pi\alpha^2_{\gamma\gamma}}{m_{J^p}^2 m_P^2 q_\gamma^4} \left( 2m_p m_P^2 (q'k')(q'k) - m_p m_P (q'k')(q'k) \right) + m_p m_P^2 (q'k')(q'k) + m_p m_P^2 (q'k')(q'k) - 2m_p m_P^2 (q'k')(q'k) + 2m_p m_P^2 (q'k')(q'k) + 2m_p m_P^2 (q'k')(q'k) - 2m_p m_P^2 (q'k')(q'k)
\] (C4)

and, for the coupling in the negative parity $J^p = \frac{3}{2}^-$,
\[
|M|^2_{J^p=\frac{3}{2}^-_{\text{R,LL}}} = \frac{8\pi\alpha^2_{\gamma\gamma}}{m_{J^p}^2 m_P^2 q_\gamma^4} \left( 2m_p m_P^2 (q'k')(q'k) - m_p m_P^2 (q'k')(q'k) \right) + m_p m_P^2 (q'k')(q'k) + m_p m_P^2 (q'k')(q'k) - 2m_p m_P^2 (q'k')(q'k) + 2m_p m_P^2 (q'k')(q'k) + 2m_p m_P^2 (q'k')(q'k) - 2m_p m_P^2 (q'k')(q'k)
\] (C6)
\[ |\mathcal{M}|^2_{\nu=\frac{1}{2} RLLR} = \frac{8\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) - 2m_{\nu}m_{\bar{\nu}}^2 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) + \frac{4\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) - \frac{2\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) + \frac{4\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) - \frac{2\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) + \frac{4\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) - \frac{2\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) + \frac{4\pi\alpha g^2_{\nu\bar{\nu}}}{3m_{\nu}^2 m_{\bar{\nu}}^2 m_{\gamma}^2} \left( 2m_{\nu}m_{\bar{\nu}}^3 (q \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k)(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') + m_{\nu}m_{\bar{\nu}}^2 (s_{p'} \cdot k')(q \cdot k') \right) \right). \]