A study of generalized roughness in \((\varepsilon, \in \vee qk)\)-fuzzy filters of ordered semigroups

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ABSTRACT

Concept of generalized rough approximations for fuzzy filters in ordered semigroups is introduced. Then this idea is extended to rough approximations of fuzzy bi-filters and fuzzy quasi-filters in ordered semigroups. \((\varepsilon, \in \vee q)\)-fuzzy filters are a generalization of fuzzy filters and these can be further generalized by \((\varepsilon, \in \vee qk)\)-fuzzy filters. Generalized roughness has been studied for both \((\varepsilon, \in \vee q)\) and \((\varepsilon, \in \vee qk)\)-fuzzy filters of ordered semigroups.

1. Introduction

In daily life, there are many situations where certain order exist among the elements of a set. For example, prices of commodities can be described by terms like very cheap, cheap, affordable, costly, very costly. It is clear that there is an order among these terms and commodities can be arranged with the help of order among their prices. Algebraically ordered semigroups are sets with associative binary operation having a certain partial order. Ordered semigroups have wide ranging applications in computer science, automata theory and coding theory.

Rough set theory and fuzzy set theory are two distinct concepts, but both of them are very handy to deal with uncertainty. These two can be hybridized in a very fruitful manner. Therefore, concepts of fuzzy rough sets and rough fuzzy sets are introduced in [1].

Rosenfeld introduced the idea of fuzzy algebraic structures by introducing fuzzy subgroups [2]. Study of fuzzy semigroup is initiated by Kuroki [3,4]. However fuzzy ordered groupoids and ordered semigroup were investigated by Kehayopulu and Tsingelis [5,6] for the first time. The idea of \((\alpha, \beta)\)-fuzzy subgroups is presented by Bhakat and Das [7–9]. Among different \((\alpha, \beta)\)-fuzzy subgroups \((\in, \in \vee q)\)-fuzzy subgroup are the most interesting. In study of \((\in, \in \vee q)\)-fuzzy subgroups quasi coincident fuzzy points play a basic role. This notion is introduced in [10]. Fuzzy filters for various algebraic structures have been studied by many authors. Generalized fuzzy filters of \(R_0\) algebra have been studied by Ma et al. [11]. Kehayopulu and Tsingelis presented the notion of fuzzy filter in ordered semigroups [6].

Pawlak is the founder of rough set theory [12]. Many applications of this theory have been reported. Actually it is a nice tool to discuss uncertainty among the elements of a set. Equivalence relations play a fundamental role in it. Due to limited knowledge about the elements of a set, it is too complicated to determine the equivalence relation among the objects of a set. So authors studied different models with less restrictions. Generalized rough sets has been studied in [13]. Instead of equivalence relation, set valued maps are used to define approximations of a set in generalized rough set theory. In algebraic structures roughness has be discussed by many authors. Kuroki studied roughness in semigroups and fuzzy semigroups in [14]. Then this concept is studied for prime ideals in semigroups [15]. Study of roughness in \((\varepsilon, \in \vee q)\)-fuzzy ideals of hemirings initiated in [16]. In ordered semigroups rough approximations as proposed in [14] can not be a good idea. As in ordered semigroup there is a partial order associated with the semigroup, therefore non-trivial equivalence relations for such semigroups are difficult to find. Perhaps, this is the major reason that no study of roughness in case of ordered semigroups has been reported till now according to our knowledge. Therefore in this paper some weaker tool to study roughness for fuzzy filters in ordered semigroups have been introduced. Set valued maps give rise to binary relations in general. These maps with monotone or isotope order help us to study roughness in fuzzy filters of ordered semigroups.

Organization of this paper is as the following. In Section 2, some basic concepts about ordered semigroups, fuzzy sets and rough sets are given. These
notions will be useful in later sections. Section 3, is devoted for discussion about approximations of fuzzy filters, fuzzy bi-filters and quasi-filters in ordered semigroup. It is seen that set valued monotone and isotone homomorphism play basic role for approximation of fuzzy filters. In Section 4, concept of approximations is extended to \((\in, \in \vee \theta)\)-fuzzy filters, fuzzy bi-filters and fuzzy quasi-filters. Approximations of \((\in, \in \vee \theta)\)-fuzzy filters, fuzzy bi-filters and fuzzy quasi-filters in ordered semigroup have been studied in Section 5.

2. Preliminaries

In this section some fundamental concepts about ordered semigroups, fuzzy sets and rough sets are presented. These notions will be helpful in later sections.

An algebraic system \((S, \cdot, \leq)\) is called a partially ordered semigroup (po-semigroup) if it satisfies

\[(c_1)\] S is a semigroup with respect to ",",

\[(c_2)\] It is a po-set with respect to \(\leq \),

\[(c_3)\] If \(y_1 \leq y_2 \rightarrow ay_1 \leq ay_2 \) and \(y_1 a \leq y_2 a\), \(\forall a, y_1, y_2 \in S\).

If \(K\) is any non-empty subset of a po-semigroup \(S\), then \(K\) is an ordered subsemigroup of \(S\), if \(K^2 \subseteq K\).

**Definition 2.1:** [17] A subset \(F \neq \emptyset\) of \(S\) is known as a left (right) filter of \(S\) if the following hold

\[(F_1)\] \(y_2 \leq y_1\) implies \(y_1 \in F, \forall y_1 \in S\) and \(\forall y_2 \in F, \forall y_1 \in S, \forall y_2 \in S, \forall y_1 \in F, \forall y_2 \in F\).

\[(F_2)\] \(y_1, y_2 \in F\) implies \(y_1 y_2 \in F, \forall y_1, y_2 \in S, \forall y_1, y_2 \in S, \forall y_1 \in F, \forall y_2 \in F, \forall y_2 \in S\).

\(F\) is known as a filter if it is both a left filter and a right filter of \(S\).

Next definitions of bi-filters and quasi-filters are given. These are actually generalizations of filters in po-semigroups.

**Definition 2.2** ([17]): A subset \(F \neq \emptyset\) of \(S\) is known as bi-filter of \(S\) if \((F_1), (F_2)\) hold and

\[(F_3)\] \(y_1, y_2, y_1 \in F\) implies \(y_1 \in F, \forall y_1, y_2 \in S\).

**Definition 2.3** ([18]): A subset \(F \neq \emptyset\) of \(S\) is known as a quasi-filter if it satisfies \((F_1), (F_2)\) and

\[(F_3)\] \(y_1 y_2 = y_3 y_1 \in F\) implies \(y_1 \in F, \forall y_1, y_2, y_3 \in S\).

In the following some basic concepts about fuzzy filters and their generalizations are given.

**Definition 2.4** ([19]): A function \(\Psi\) from \(S\) to a unit closed interval \([0, 1]\) is known as fuzzy subset.

**Definition 2.5** ([10]): A fuzzy subset

\[
\Psi(x) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}
\]

represent fuzzy point with value \(t\) and support by \(y\). It is written as \(y_t\).

A fuzzy point \(y_t\) of \(S\) is said to “belong to” fuzzy subset \(\Psi\) denoted as \(y_t \in \Psi\), if \(\Psi(y_t) \geq t\), and is said to “quasi-coincident” to \(\Psi\) denoted by \(y_t q \Psi\), if \(\Psi(y_t) + t > 1\).

From here onward in discussion below \(S\) stands for ordered semigroup and \(\Psi\) for fuzzy subset on ordered semigroup \(S\) unless stated otherwise.

**Definition 2.6:** \(\Psi\) is known as a fuzzy ordered subsemigroup of \(S\) if it satisfies

\[\Psi(y_1 y_2) \geq \min\{\Psi(y_1), \Psi(y_2)\}, \forall y_1, y_2 \in S.\]

In the following this inequality will be denoted by \((FF_2)\).

**Definition 2.7** ([17]): \(\Psi\) is known as a fuzzy left (right) filter of \(S\) if the following assertions hold

\[(FF_1)\] \(y_1 \leq y_2\) implies \(\Psi(y_1) \leq \Psi(y_2)\), \(\forall y_1, y_2 \in S, \forall y_1, y_2 \in S, \forall y_1 \in F, \forall y_2 \in F, \forall y_2 \in S, \forall y_1, y_2 \in S, \forall y_1 \in F, \forall y_2 \in F, \forall y_2 \in S\).

A fuzzy subset \(\Psi\) is known as a fuzzy filter of \(S\), if \(\Psi\) is a fuzzy left and a fuzzy right filter of \(S\).

**Definition 2.8** ([17]): \(\Psi\) is called a fuzzy bi-filter of \(S\) if it holds \((FF_1), (FF_2)\) and

\[(FF_3)\] \(\Psi(y_1 y_2, y_1) \geq \Psi(y_1), \forall y_1, y_2 \in S.\]

**Definition 2.9** ([18]): \(\Psi\) is said to be a fuzzy quasi filter of \(S\) if it holds \((FF_1), (FF_2)\) and

\[(FF_4)\] \(y_1, y_2 = y_2, y_1 \implies \Psi(y_1, y_2) \leq \Psi(y_1), \forall y_1, y_2, y_3 \in S.\]

In the following we recall some basic concepts of rough set theory introduced by Pawlak [12].

Consider an equivalence relation \(K\) on a universal set \(U\). The pair \((U, K)\) is known as an approximation space. Let \(\emptyset \neq A \subseteq U\), then \(A\) is definable if we can express it in the form of some equivalence classes of \(U\), else it is not definable. If \(A\) is not definable, then it may be approximated in the form of definable subsets called lower approximation and upper approximation of \(A\), defined as

\[
\mathcal{app}(A) = \{y_1 \in U : [y_1]_K \subseteq A\}
\]

\[
\mathcal{ap}(A) = \{y_1 \in U : [y_1]_K \cap A \neq \emptyset\}
\]

The pair \((\mathcal{app}(A), \mathcal{ap}(A))\) is known as a rough set. If \(\mathcal{app}(A) = \mathcal{ap}(A)\), then \(A\) is a definable set.

This notion of lower and upper approximations can be generalized for fuzzy sets as well.
**Definition 2.10 ([20]):** Let us consider the approximation space \((U, \mathcal{K})\) and \(\Psi\) as a fuzzy subset of \(U\). Define the lower and upper approximation of a fuzzy subset \(\Psi\) as the following.

\[
\text{app}\Psi(y) = \frac{\wedge}{y \in [y]_k} \Psi(y') \quad \text{and} \quad \text{cpp}\Psi(y) = \frac{\vee}{y \in [y]_k} \Psi(y').
\]

For any \(y \in U\),

A rough fuzzy set is the pair \((\text{app}\Psi, \text{cpp}\Psi)\), if \(\text{app}\Psi \neq \text{cpp}\Psi\).

**Definition 2.11:** Consider the ordered semigroups \(S_1\) and \(S_2\). A mapping \(H : S_1 \rightarrow P^\ast(S_2)\) is said to be a set-valued homomorphism in short \((SVH)\) if it holds

(i) \(H(y_1)H(y_2) = H(y_1y_2)\).

Where \(P^\ast(S_2)\) denotes the collection of all non-empty subsets of \(S_2\).

**Definition 2.12:** Suppose that \(S_1\) and \(S_2\) are ordered semigroups. A mapping \(H : S_1 \rightarrow P^\ast(S_2)\) is said to be a set-valued monotone homomorphism in short \((SV MH)\) if it satisfies condition (i) of Definition 2.11, and

(ii) if \(y_1 \leq y_2\) then \(H(y_1) \subseteq H(y_2)\) for each \(y_1, y_2 \in S_1\).

**Definition 2.13:** Let us consider the ordered semigroups \(S_1\) and \(S_2\). A mapping \(H : S_1 \rightarrow P^\ast(S_2)\) is said to be a set-valued isotope homomorphism in short \((SV IH)\) if it hold condition (i) of Definition 2.11, and

(ii) if \(y_1 \leq y_2\) then \(H(y_1) \subseteq H(y_2)\) for each \(y_1, y_2 \in S_1\).

In the following concept of roughness in fuzzy sets is being generalized by SVMH and SVIH.

**Definition 2.14:** Let \(H : S \rightarrow P^\ast(S)\) be an SVMH or SVIH. Then for every \(y \in S\), we define the generalized lower and upper approximation of \(\Psi\) with respect to mapping \(H\) as,

\[
\tilde{H}(\Psi)(y) = \frac{\wedge}{y' \in H(y)} \Psi(y') \quad \text{and} \quad \check{H}(\Psi)(y) = \frac{\vee}{y' \in H(y)} \Psi(y').
\]

A rough fuzzy set is the pair \((\tilde{H}(\Psi), \check{H}(\Psi))\), with respect to \(H\), if \(\tilde{H}(\Psi) \neq \check{H}(\Psi)\).

3. Approximations of fuzzy filters in ordered semigroups

In this section study of roughness for fuzzy filters of ordered semigroups is being initiated. The following result is for the upper approximation of a fuzzy ordered subsemigroup of a fuzzy ordered semigroup.

**Theorem 3.1:** Let \(\Psi\) be a fuzzy ordered subsemigroup of \(S\) and \(H : S \rightarrow P^\ast(S)\) be a SVH. Then \(\check{H}(\Psi)\) is a fuzzy ordered subsemigroup of \(S\).

**Proof:** For any \(y_1, y_2 \in S\), consider

\[
\check{H}(\Psi)(y_1y_2) = \frac{\vee}{y_1' \in H(y_1), y_2' \in H(y_2)} \Psi(y_1'y_2')
\]

\[
= \frac{\vee}{a \in H(y_1), b \in H(y_2)} \Psi(ab)
\]

\[
= \frac{\vee}{a \in H(y_1), b \in H(y_2)} \Psi(ab)
\]

\[
(\text{as } y_1' = ab \text{ such that } a \in H(y_1) \quad \text{and} \quad b \in H(y_2))
\]

implies \(\tilde{H}(\Psi)(y_1y_2) \geq \check{H}(\Psi)(y_1)\).

**Theorem 3.2:** Let \(\Psi\) be a fuzzy ordered subsemigroup of \(S\) and \(H : S \rightarrow P^\ast(S)\) be a SVH. Then \(\tilde{H}(\Psi)\) is a fuzzy ordered subsemigroup of \(S\).

**Proof:** Similar to the proof of Theorem 3.1.

In the following, study of roughness in fuzzy filters of ordered semigroups is being initiated. Certain restrictions are required on SVH for this study.

**Theorem 3.3:** Let \(H : S \rightarrow P^\ast(S)\) be an SVMH and \(\Psi\) be a fuzzy left (right) filter of \(S\). Then \(\tilde{H}(\Psi)\) is a fuzzy left (resp. right) filter of \(S\).

**Proof:** For any \(y_1, y_2 \in S\), let \(y_1 \leq y_2\) then \(H(y_1) \subseteq H(y_2)\). Now consider the following

\[
\tilde{H}(\Psi)(y_1) = \frac{\vee}{y_1' \in H(y_1)} \Psi(y_1')
\]

\[
\leq \frac{\vee}{y_1' \in H(y_1)} \Psi(y_1')
\]

That is \(\tilde{H}(\Psi)(y_1) \leq \tilde{H}(\Psi)(y_2)\).

It is easy to see from Theorem 3.1, that \(\tilde{H}(\Psi)(y_1y_2) \geq \min\{\tilde{H}(\Psi)(y_1), \tilde{H}(\Psi)(y_2)\}\). Next consider

\[
\tilde{H}(\Psi)(y_1y_2) = \frac{\vee}{y_1' \in H(y_1), y_2' \in H(y_2)} \Psi(y_1'y_2')
\]

\[
= \frac{\vee}{a \in H(y_1), b \in H(y_2)} \Psi(ab)
\]

\[
(\text{as } y_1' = ab \text{ such that } a \in H(y_1) \quad \text{and} \quad b \in H(y_2))
\]

implies \(\tilde{H}(\Psi)(y_1y_2) \geq \tilde{H}(\Psi)(y_1)\).
The following example shows that if \( H \) is an SVMH, then for a fuzzy filter \( \Psi \), its lower approximation \( H(\Psi) \) may not be a fuzzy filter.

**Example 3.1:** Consider a set \( S = \{a, b, c, d, e, f\} \) with Table 1 and order relation \( \leq \) defined as follows: \( (a, a), (b, b), (c, c), (d, d), (e, e), (f, f) \), \( (a, d), (a, e), (d, e), (b, f), (b, e), (c, f), (c, e), (f, e) \). Then \( (\cdot, \leq) \) is an ordered semigroup [17]. Define a fuzzy subset \( \Psi \) of \( S \) by \( \Psi(e) = \Psi(f) = 0.8 \), \( \Psi(d) = 0.7 \), \( \Psi(a) = 0.3 \), \( \Psi(c) = 0.4 \). Then \( \Psi \) is a fuzzy filter of \( S \). Now consider \( H: S \rightarrow P(S) \) defined as \( H(a) = \{a, d\} \), \( H(b) = \{d, e, f\} \), \( H(c) = \{d, e, f\} \), \( H(d) = \{a, d, f\} \), \( H(e) = \{a, d, e, f\} \), \( H(f) = \{d, e, f\} \). Then \( H \) is an SVMMH. Now as \( b \leq e \Rightarrow H(b) \subseteq H(e) \) but \( H(\Psi)(b) \subseteq H(\Psi)(e) \), also \( f \leq e \Rightarrow H(f) \subseteq H(e) \) but \( H(\Psi)(f) \subseteq H(\Psi)(e) \). Hence in SVMMH \( H(\Psi) \) is not a fuzzy filter of \( S \).

In above example, it has been seen that in case of SVM MH lower approximation of a fuzzy filter may not be a fuzzy filter. Now we turn over attention to SVIH and have the following:

**Theorem 3.4:** Let \( H: S \rightarrow P(S) \) be an SVIH and \( \Psi \) be a fuzzy left (right) filter of \( S \). Then \( H(\Psi) \) is a fuzzy left (right) filter of \( S \).

**Proof:** For any \( y_1, y_2 \in S \), let \( y_1 \leq y_2 \), then \( H(y_2) \subseteq H(y_1) \). Now consider the following

\[
H(\Psi)(y_1) = \bigvee_{y_i \in H(y_1)} \Psi(y_i) \\
\leq \bigvee_{y_i \in H(y_2)} \Psi(y_i) \\
\text{That is } H(\Psi)(y_1) \leq H(\Psi)(y_2)
\]

From Theorem 3.1, it is easy to see that \( H(\Psi)(y_1, y_2) \geq \min[H(\Psi)(y_1), H(\Psi)(y_2)] \).

Next consider

\[
H(\Psi)(y_1, y_2) = \bigwedge_{y_i \in H(y_1, y_2)} \Psi(y_i) \\
= \bigwedge_{y_i \in H(y_1) \cap H(y_2)} \Psi(y_i) \\
= \bigwedge_{a \in H(y_1) \cap H(y_2)} \Psi(ab)
\]

as \( y_i = ab \) such that \( a \in H(y_1) \) and \( b \in H(y_2) \)

\[
= \bigwedge_{a \in H(y_1) \cap H(y_2)} \Psi(ab)
\]

implies \( H(\Psi)(y_1, y_2) \geq H(\Psi)(y_1) \).

Hence \( H(\Psi) \) on \( S \) is a fuzzy bi-filter of \( S \).

**Theorem 3.6:** Let \( H: S \rightarrow P(S) \) be an SVIH and \( \Psi \) be a fuzzy bi-filter of \( S \). Then \( H(\Psi) \) is a fuzzy bi-filter of \( S \).

**Proof:** From Theorem 3.3, if \( y_1 \leq y_2 \) then \( H(y_1) \subseteq H(y_2) \). Then \( H(\Psi)(y_1) \leq H(\Psi)(y_2) \), for all \( y_1, y_2 \in S \). Also \( H(\Psi)(y_1, y_2) \geq \min[H(\Psi)(y_1), H(\Psi)(y_2)] \). Therefore consider the following for any \( y_1, y_2 \in S \).

\[
H(\Psi)(y_1, y_2) = \bigwedge_{y_i \in H(y_1, y_2)} \Psi(y_i) \\
= \bigwedge_{y_i \in H(y_1) \cap H(y_2)} \Psi(y_i) \\
= \bigwedge_{a \in H(y_1) \cap H(y_2)} \Psi(ab)
\]

as \( y_i = ab \) such that \( a \in H(y_1) \) and \( b \in H(y_2) \)

\[
= \bigwedge_{a \in H(y_1) \cap H(y_2)} \Psi(ab)
\]

implies \( H(\Psi)(y_1, y_2) \geq H(\Psi)(y_1) \).

Hence \( H(\Psi) \) on \( S \) is a fuzzy bi-filter of \( S \).

**Theorem 3.6:** Let \( H: S \rightarrow P(S) \) be an SVIH and \( \Psi \) be a fuzzy bi-filter of \( S \). Then \( H(\Psi) \) is a fuzzy bi-filter of \( S \).
Theorem 3.7: Let $H : S \rightarrow P^*(S)$ be an SVMH and $\Psi$ be a fuzzy quasi filter of $S$. Then $H(\Psi)$ is a fuzzy quasi filter of $S$.

Proof: From Theorem 3.3, $(FF_1)$ and $(FF_2)$ hold for $H(\Psi)$. For $(FF_3)$ if for each $y_1, y_2, y_3 \in S$, if $y_1 y_2 = y_3 y_1$, then consider the following
\[ H(\Psi)(y_1 y_2) = \bigvee_{y'_1 \in H(y_1)} \Psi(y'_1) \]
\[ = \bigvee_{y'_1 \in H(y_1)} \Psi(y'_1) \]
\[ = \bigvee_{a \in H(y_1), b \in H(y_2)} \Psi(ab) \]
\[ = \bigvee_{a \in H(y_1), b \in H(y_2)} \Psi(ab) \]
\[ \leq \bigvee_{a \in H(y_1), b \in H(y_2)} \Psi(ab) \]
implies $H(\Psi)(y_1 y_2) \leq H(\Psi)(y_1)$

Hence $H(\Psi)$ on $S$ is a fuzzy quasi filter of $S$.

4. Approximations of $(\in, \notin)\text{-fuzzy filters in ordered semigroups}$

In the following, study of roughness in $(\in, \notin)\text{-fuzzy filters of ordered semigroups}$ is being initiated.

Definition 4.1 ([17]): A fuzzy subset $\Psi$ on $S$ is said to be $(\in, \notin)\text{-fuzzy left (right) filter of } S$ if

$(FF_6)$
\[ (y_1 \leq y_2, y_1 \in \Psi \implies y_2 \in \nu \Psi) \]
\[ (\forall y_1, y_2 \in S \text{ and } \forall t_1 \in (0, 1)) \]
$(FF_7)$
\[ (y_1 \in \Psi, y_2 \in \Psi \implies (y_1 y_2)_{\min(t_1, t_2)} \in \nu \Psi) \]
\[ (\forall y_1, y_2 \in S \text{ and } \forall t_1, t_2 \in (0, 1)) \]
$(FF_8)$
\[ (y_1 \in \Psi \implies (y_1 y_2)_{t_1} \in \nu \Psi) \]
\[ q\Psi(\text{resp. } (y_2 y_1)_{t_1} \in \nu \Psi) \]
\[ (\forall y_1, y_2 \in S \text{ and } \forall t_1 \in (0, 1)) \]

If $\Psi$ is both $(\in, \notin)\text{-fuzzy left filter and } (\in, \notin)\text{-fuzzy right filter of } S$, then $\Psi$ is called $(\in, \notin)\text{-fuzzy quasi filter of } S$, or equivalently, $\Psi$ is known as $(\in, \notin)\text{-fuzzy filter of } S$ if it hold conditions $(FF_6)$, $(FF_7)$, and
\[(y_1 y_2)_{t_1} \in \Psi \implies (y_1)_{t_1} \in \nu \Psi, (y_2)_{t_1} \in \nu \Psi) \]
\[ \times (\forall y_1, y_2 \in S \text{ and } \forall t_1 \in (0, 1)) \]

Definition 4.2 ([17]): A fuzzy subset $\Psi$ is called $(\in, \notin)\text{-fuzzy bi-filter of } S$ if it hold conditions $(FF_6)$, $(FF_7)$ and
\[(FF_8) (y_1, y_2 \in \Psi \implies (y_1 y_2)_{t_1} \in \nu \Psi(\forall y_1, y_2 \in S \text{ and } \forall t_1 \in (0, 1)) \]

Definition 4.3: $\Psi$ is known as $(\in, \notin)\text{-fuzzy quasi filter of } S$ if it satisfies $(FF_6)$, $(FF_7)$ and
\[(FF_10) (y_1 y_2 = y_3 y_1 \text{ then } (y_1 y_2)_{t_1} \in \nu \Psi(\forall y_1, y_2 \in S \text{ and } \forall t_1 \in (0, 1))) \]

Lemma 4.1 ([17]): $\Psi$ is an $(\in, \notin)\text{-fuzzy left (resp. right) filter of } S$ if and only if it satisfies:

$(FF_6)$ (for $y_1 \leq y_2, \Psi(y_2) \geq \min(\Psi(y_1), 0.5))$
\[ (\forall y_1, y_2 \in S) \]
$(FF_7)$ (for $\Psi(y_1 y_2) \geq \min(\Psi(y_1), \Psi(y_2), 0.5)) (\forall y_1, y_2 \in S)$
\[ (FF_8) \Psi(y_1 y_2) \geq \min(\Psi(y_1), 0.5) (\text{resp. } \Psi(y_1 y_2) \geq \min(\Psi(y_2), 0.5)) \]
\[ (\forall y_1, y_2 \in S) \]

Lemma 4.2 ([17]): A fuzzy subset $\Psi$ is an $(\in, \notin)\text{-fuzzy bi-filter of } S$ if and only if it holds $(FF_6)$, $(FF_7)$ of Lemma 4.1, and
\[(FF_9) \Psi(y_1 y_2) \in \nu \Psi(\forall y_1, y_2 \in S) \]

Lemma 4.3: A fuzzy subset $\Psi$ is an $(\in, \notin)\text{-fuzzy quasi filter of } S$ if and only if it satisfies $(FF_6)$, $(FF_7)$ of Lemma 4.1, and
\[(FF_{10}) (y_1 y_2 = y_3 y_1 \text{ then } \Psi(y_1) \geq \min(\Psi(y_1), 0.5)) (\forall y_1, y_2 \in S) \]

Theorem 4.1: Let $H : S \rightarrow P^*(S)$ be an SVMH and $\Psi$ be an $(\in, \notin)\text{-fuzzy left (resp. right) filter of } S$. Then $H(\Psi)$ is an $(\in, \notin)\text{-fuzzy left (resp. right) filter of } S$.

Proof: To prove this theorem, it must be seen $H(\Psi)$ satisfies $(FF_6)$, $(FF_7)$ and $(FF_8)$. For each $y_1, y_2 \in S$, if
y_1 \leq y_2$, then $H(y_1) \subseteq H(y_2)$. Now consider

$$
\tilde{H}(\Psi)(y_2) = \bigwedge_{y_1 \in H(y_2)} \Psi(y_1) 
\geq \bigwedge_{y_1 \in H(y_2)} \min(\Psi(y_1), 0.5) 
= \min\left(\bigwedge_{y_1 \in H(y_2)} \Psi(y_1), 0.5\right)
$$

That is $\tilde{H}(\Psi)(y_2) \geq \min(\tilde{H}(\Psi)(y_1), 0.5)$.

Next consider

$$
\tilde{H}(\Psi)(y_1y_2) = \bigwedge_{y_1 \in H(y_1), y_2 \in H(y_2)} \Psi(y_1) 
= \bigwedge_{a \in H(y_1), b \in H(y_2)} \Psi(ab) 
\geq \bigwedge_{a \in H(y_1), b \in H(y_2)} \min(\Psi(a), \Psi(b), 0.5) 
= \min\left(\bigwedge_{a \in H(y_1)} \Psi(a), \bigwedge_{b \in H(y_2)} \Psi(b), 0.5\right)
$$

implies $\tilde{H}(\Psi)(y_1y_2) \geq \min(\tilde{H}(\Psi)(y_1), \tilde{H}(\Psi)(y_2), 0.5)$.

Further

$$
\tilde{H}(\Psi)(y_1y_2) = \bigwedge_{y_1 \in H(y_1), y_2 \in H(y_2)} \Psi(y_1) 
= \bigwedge_{a \in H(y_1), b \in H(y_2)} \Psi(ab) 
\geq \bigwedge_{a \in H(y_1), b \in H(y_2)} \min(\Psi(a), 0.5) 
= \min\left(\bigwedge_{a \in H(y_1)} \Psi(a), 0.5\right).
$$

Hence $\tilde{H}(\Psi)(y_1y_2) \geq \min(\tilde{H}(\Psi)(y_1), 0.5)$.

Therefore, $\tilde{H}(\Psi)$ is an $(\varepsilon, \bar{\varepsilon})$-fuzzy left filter of $S$. Similarly, it can be shown that $\tilde{H}(\Psi)$ is an $(\varepsilon, \varepsilon)$-fuzzy right filter of $S$.

**Theorem 4.2:** Let $H : S \rightarrow P^*(S)$ be an SVIH and $\Psi$ be $(\varepsilon, \bar{\varepsilon})$-fuzzy left (resp. right) filter of $S$. Then $\tilde{H}(\Psi)$ is $(\varepsilon, \varepsilon)$-fuzzy left (resp. right) filter of $S$. □

**Proof:** It is required that $\tilde{H}(\Psi)$ satisfies $(FF_{\theta})$, $(FF_{\bar{\theta}})$, and $(FF_{\bar{\theta}})$. If for each $y_1, y_2 \in S$, we have $y_1 \leq y_2$, then $H(y_1) \subseteq H(y_2)$, therefore $\tilde{H}(\Psi)(y_2) \geq \min(\tilde{H}(\Psi)(y_1), 0.5)$. Further, it is obligatory that $\tilde{H}(\Psi)$ satisfies $(FF_{\theta})$. Therefore, consider the
following for each \(y_1, y_2 \in S\).

\[
\hat{H}(\Psi)(y_1, y_2 y_1) = \bigvee_{y_i \in H(y_1, y_2)} \Psi(y_i) \\
= \bigvee_{y_i \in H(y_1, H(y_2))} \Psi(y_i) \\
= \min_{\omega \in H(y_1, H(y_2))} \Psi(\omega) \\
\]

as \(y_1 = \omega\) such that \(a \in H(y_1)\) and \(b \in H(y_2)\).

\[
= \min_{\omega \in H(y_1, H(y_2))} \Psi(\omega) \\
\geq \min_{\omega \in H(y_1, H(y_2))} \min(\Psi(a), 0.5) \\
\text{(by Lemma 4.2)} \\
= \min(\hat{H}(\Psi)(y_1), 0.5).
\]

implies \(\hat{H}(\Psi)(y_1, y_2 y_1) \geq \min(\hat{H}(\Psi)(y_1), 0.5)\).

Hence, it is clear that \(\hat{H}(\Psi)\) on \(S\) is a \((\in, \vee qk)\)-fuzzy bi-filter.

\[\square\]

**Theorem 4.4:** Let \(H : S \rightarrow P(S)\) be an SVIH and a fuzzy subset \(\Psi\) be an \((\in, \vee q)-\)fuzzy bi-filter of \(S\). Then \(\hat{H}(\Psi)\) is \((\in, \vee qk)\)-fuzzy bi-filter of \(S\).

**Proof:** Proof directly follows from Theorems 4.2 and 4.3.

\[\square\]

**Theorem 4.5:** Let \(H : S \rightarrow P(S)\) be an SVMH and a fuzzy subset \(\Psi\) be an \((\in, \vee qk)\)-fuzzy quasi filter of \(S\). Then \(\hat{H}(\Psi)\) is \((\in, \vee qk)\)-fuzzy quasi filter of \(S\).

**Proof:** From Theorem 4.1, if \(y_1 \leq y_2\), then \(H(y_1) \subseteq H(y_2)\), therefore \(\hat{H}(\Psi)(y_2) \geq \min(\hat{H}(\Psi)(y_1), 0.5)\), for all \(y_1, y_2 \in S\). Also \(\hat{H}(\Psi)(y_1, y_2) \geq \min(\hat{H}(\Psi)(y_1), \hat{H}(\Psi)(y_2), 0.5)\). Next, it must be seen that \(\hat{H}(\Psi)\) satisfies (FF10). Therefore, if \(y_1 y_2 = y_3 y_1\), for each \(y_1, y_2, y_3 \in S\), consider the following

\[
\min(\hat{H}(\Psi)(y_1, y_2), 0.5) = \min_{\omega \in H(y_3)} \min(\psi_{\omega}(y_1), 0.5) \\
= \min_{\omega \in H(y_3)} \min(\psi_{\omega}(y_1), 0.5) \\
= \min_{\omega \in H(y_3)} \min(\psi_{\omega}(y_1), 0.5) \\
= \min_{\omega \in H(y_3)} \min(\psi_{\omega}(y_1), 0.5) \\
\text{(as \(y_1 = ab\) where \(a \in H(y_1)\) and \(b \in H(y_2)\))} \\
= \min_{\omega \in H(y_3)} \min(\psi_{\omega}(y_1), 0.5) \\
\leq \min_{\omega \in H(y_3)} \min(\psi_{\omega}(y_1), 0.5) \\
\text{implies } \min(\hat{H}(\Psi)(y_1, y_2), 0.5) \leq \hat{H}(\Psi)(y_1).
\]

Hence, it is clear that \(\hat{H}(\Psi)\) on \(S\) is a \((\in, \vee qk)\)-fuzzy quasi filter.

\[\square\]

**Theorem 4.6:** Let \(H : S \rightarrow P(S)\) be an SVIH and a fuzzy subset \(\Psi\) be an \((\in, \vee qk)-\)fuzzy quasi filter of \(S\). Then \(\hat{H}(\Psi)\) is \((\in, \vee qk)\)-fuzzy quasi filter of \(S\).

**Proof:** Proof directly follows from Theorems 4.2 and 4.5.

\[\square\]

5. Approximations of \((\in, \vee qk)\)-fuzzy filters in ordered semigroups

In this section, roughness in \((\in, \vee qk)\)-fuzzy filters is being studied. A fuzzy point \(y_{12}\) of \(S\) is known to “belong to” a fuzzy subset \(\Psi\), denoted as \(y_{12} \in \Psi\) if \(\Psi(y_{12}) \geq t\), and is said to be a “quasi-coincident” to \(\Psi\), denoted as \(y_{12}qk\Psi\), if \(\Psi(y_{12}) + t + k > 1\), where \(k \in [0, 1)\).

**Definition 5.1:** A fuzzy subset \(\Psi\) on \(S\) is known as \((\in, \vee qk)-\)fuzzy left (right) filter of \(S\) where \(k \in [0, 1)\) if it holds the following conditions

\[
(FF11) \quad y_1 \leq y_2, y_{1_{2}} \in \Psi \quad \text{implies} \quad y_{2_{2}} \in \vee qk\Psi, \quad \forall y_1, y_2 \in S \text{ and } \forall t \in (0, 1] \\
(FF12) \quad y_{1_{2}}, y_{2_{2}} \in \Psi \quad \text{implies} \quad (y_{1_{2}}, y_{2_{2}})_{min[1, t]} \in \vee qk\Psi, \quad \forall y_1, y_2 \in S \text{ and } \forall t \in (0, 1] \\
(FF13) \quad y_{1_{2}} \in \Psi \quad \text{implies} \quad (y_{1_{2}}, y_{2_{2}})_{t} \in \vee qk\Psi \text{ (resp. } y_{1_{2}}, y_{2_{2}})_{t} \in \vee qk\Psi) \quad \forall y_1, y_2 \in S \text{ and } \forall t \in (0, 1].
\]

If \(\Psi\) is both \((\in, \vee qk)-fuzzy\) left filter and \((\in, \vee qk)-fuzzy\) right filter of \(S\), then \(\Psi\) is called \((\in, \vee qk)-fuzzy\) filter of \(S\) or equivalently \(\Psi\) is known as \((\in, \vee qk)-fuzzy\) filter of \(S\) if it holds \((FF11), (FF12)\), and

\[
(y_{1_{2}}, y_{2_{2}})_{t} \in \Psi \quad \text{implies} \quad y_{1_{2}}, y_{2_{2}} \in \vee qk\Psi, \quad \forall y_1, y_2 \in S \text{ and } \forall t \in (0, 1].
\]

**Definition 5.2:** A fuzzy subset \(\Psi\) on \(S\) is said to be \((\in, \vee qk)-fuzzy\) bi-filter of \(S\) if it holds \((FF11), (FF12)\) and

\[
(FF14) \quad y_{1_{2}}, y_{2_{2}} \in \Psi \quad \text{implies} \quad (y_{1_{2}}, y_{2_{2}})_{t} \in \vee qk\Psi, \quad \forall y_1, y_2 \in S \text{ and } \forall t \in (0, 1].
\]

**Definition 5.3:** \(\Psi\) is called \((\in, \vee qk)-fuzzy\) quasi filter of \(S\) if it satisfies \((FF11), (FF12)\) and

\[
(FF15) \quad y_{1_{2}}, y_{2_{2}} \in \Psi \quad \text{implies} \quad y_{1_{2}}, y_{2_{2}} \in \vee qk\Psi, \quad \forall y_1, y_2, y_3 \in S \text{ and } \forall t \in (0, 1].
\]
**Lemma 5.1:** A fuzzy subset \( \Psi \) on \( S \) is \((E, \in \forall qk)\)-fuzzy quasi filter of \( S \) if and only if it satisfies the following assertions

\[
\begin{align*}
(FF_{11}) & \quad y_1 \leq y_2, \Psi(y_2) \geq \min \left\{ \Psi(y_1), \frac{1 - k}{2} \right\}, \quad \forall y_1, y_2 \in S \\
(FF_{12}) & \quad \Psi(y_1, y_2) \geq \min \left\{ \Psi(y_1), \Psi(y_2), \frac{1 - k}{2} \right\}, \quad \forall y_1, y_2 \in S \\
(FF_{13}) & \quad \Psi(y_1, y_2) \geq \min \left\{ \Psi(y_1), \frac{1 - k}{2} \right\} \\
& \quad (resp. \Psi(y_1, y_2) \geq \min \{\Psi(y_2), \frac{1 - k}{2}\}), \quad \forall y_1, y_2 \in S.
\end{align*}
\]

Further

\[
\begin{align*}
\check{H}(\Psi)(y_1, y_2) &= \sup_{y_1 \in H(y_1)} \Psi(y_1) \\
&= \sup_{y_1 \in H(y_1)} \min \left\{ \Psi(y_1), \frac{1 - k}{2} \right\} \\
&= \min \left\{ \sup_{y_1 \in H(y_1)} \Psi(y_1), \frac{1 - k}{2} \right\}.
\end{align*}
\]

**Lemma 5.2:** \( \Psi \) is \((E, \in \forall qk)\)-fuzzy bi-filter of \( S \) if and only if it holds \((FF_{11}), (FF_{12})\) of Lemma 5.1, and

\[
(FF_{14}) \quad \Psi(y_1, y_2, y_3) \geq \min \left\{ \Psi(y_1), \frac{1 - k}{2} \right\}, \forall y_1, y_2, y_3 \in S.
\]

**Lemma 5.3:** A fuzzy subset \( \Psi \) is an \((E, \in \forall qk)\)-fuzzy quasi filter of \( S \) if and only if it satisfies \((FF_{11}), (FF_{12})\) of Lemma 5.1, and

\[
(FF_{15}) \quad \Psi(y_1) \geq \min \left\{ \Psi(y_2), \frac{y_1 + y_2 - k}{2} \right\}, \quad \forall y_1, y_2, y_3 \in S.
\]

**Theorem 5.1:** Let \( H : S \rightarrow P^*(S) \) be an SVMH and \( \Psi \) be \((E, \in \forall qk)\)-fuzzy left (right) filter of \( S \). Then \( \check{H}(\Psi) \) is \((E, \in \forall qk)\)-fuzzy left (right) filter of \( S \).

**Proof:** To prove this theorem it is exigent that \( \check{H}(\Psi) \) satisfies \((FF_{11}), (FF_{12})\) and \((FF_{13})\). For each \( y_1, y_2 \in S \), if \( y_1 \leq y_2 \), then \( H(y_1) \subseteq H(y_2) \). Now consider

\[
\begin{align*}
\check{H}(\Psi)(y_2) &= \sup_{y_1 \in H(y_1)} \Psi(y_1) \\supseteq \sup_{y_1 \in H(y_1)} \min \left\{ \Psi(y_1), \frac{1 - k}{2} \right\} \\
&= \min \left\{ \sup_{y_1 \in H(y_1)} \Psi(y_1), \frac{1 - k}{2} \right\} \quad \text{that is} \quad \check{H}(\Psi)(y_2) \geq \min \left\{ \check{H}(\Psi)(y_1), \frac{1 - k}{2} \right\}.
\end{align*}
\]

Hence \( \check{H}(\Psi) \) is an \((E, \in \forall qk)\)-fuzzy left filter of \( S \). Similarly it can be shown that \( \check{H}(\Psi) \) is \((E, \in \forall qk)\)-fuzzy right filter of \( S \).

**Theorem 5.2:** Let \( H : S \rightarrow P^*(S) \) be an SVIH and \( \Psi \) be \((E, \in \forall qk)\)-fuzzy left (right) filter of \( S \). Then \( \check{H}(\Psi) \) is \((E, \in \forall qk)\)-fuzzy left (right) filter of \( S \).
**Proof:** Proof directly follows from Theorem 5.1 (using hints from Theorem 4.2).

**Theorem 5.3:** Suppose $H : S \rightarrow P^*(S)$ is an SVMH and $\Psi$ is $(\in, \in \forall kq)$-fuzzy bi-filter of $S$. Then $H(\Psi)$ is $(\in, \in \forall kq)$-fuzzy bi-filter of $S$.

**Proof:** From Theorem 5.1, it is cleared that $(FF_{15})$ and $(FF_{12})$ hold for $H(\Psi)$. For $(FF_{14})$ therefore consider the following for each $y_1, y_2 \in S$.

$$\begin{align*}
\bar{H}(\Psi)(y_1, y_2) &= y_1 \in H(y_1) \Psi(y_1) \\
&= y_1 \in H(y_1) \Psi(y_1) \\
&= ab \in H(y_1) \Psi(aba) \\
&= a \in H(y_1) \Psi(aba) \\
&\geq a \in H(y_1) \min \left\{ \Psi(a), \frac{1-k}{2} \right\} \\
&\text{(since $\Psi$ is an $(\in, \in \forall kq)$-fuzzy bi-filter, therefore by Lemma 5.2)} \\
&= \min \left\{ y_1 \in H(y_1) \Psi(y_1), \frac{1-k}{2} \right\} \\
&\text{implies $H(\Psi)(y_1, y_2)$} \\
&\geq \min \left\{ H(\Psi)(y_1), \frac{1-k}{2} \right\}.
\end{align*}$$

Hence $\bar{H}(\Psi)$ on $S$ is an $(\in, \in \forall kq)$-fuzzy bi-filter of $S$.

**Theorem 5.4:** Suppose $H : S \rightarrow P^*(S)$ is an SVIH and $\Psi$ is $(\in, \in \forall kq)$-fuzzy bi-filter of $S$. Then $\bar{H}(\Psi)$ is $(\in, \in \forall kq)$-fuzzy bi-filter of $S$.

**Proof:** Straightforward as Theorem 5.3.

**Theorem 5.5:** Suppose $H : S \rightarrow P^*(S)$ is an SVMH and $\Psi$ is $(\in, \in \forall kq)$-fuzzy quasi filter of $S$. Then $\bar{H}(\Psi)$ is $(\in, \in \forall kq)$-fuzzy quasi filter of $S$.

**Proof:** From Theorem 4.1, if $y_1 \leq y_2$, then $H(y_1) \subseteq H(y_2)$, therefore $\bar{H}(\Psi)(y_2) \geq \min(\bar{H}(\Psi)(y_1), 0.5)$ for all $y_1, y_2 \in S$. Also $\bar{H}(\Psi)(y_1, y_2) \geq \min(\bar{H}(\Psi)(y_1), \bar{H}(\Psi)(y_2), 0.5)$. Further it is exigent that $\bar{H}(\Psi)$ satisfies $(FF_{15})$. Therefore if $y_1, y_2 = y_3, y_1$ for each $y_1, y_2, y_3 \in S$, consider the following

$$\begin{align*}
\min \left\{ \bar{H}(\Psi)(y_1, y_2), \frac{1-k}{2} \right\} \\
= \min \left\{ y_1 \in H(y_1) \Psi(y_1), \frac{1-k}{2} \right\} \\
= y_1 \in H(y_1) \min \left\{ \Psi(y_1), \frac{1-k}{2} \right\} \\
= y_1 \in H(y_1) \min \left\{ \Psi(y_1), \frac{1-k}{2} \right\} \\
= \min \left\{ \Psi(ab), \frac{1-k}{2} \right\} \\
\leq a \in H(y_1) \Psi(a) \\
\text{(since $\Psi$ is an $(\in, \in \forall kq)$-fuzzy quasi filter)} \\
\text{therefore by Lemma 5.3)}
\end{align*}$$

Hence it is clear that $H(\Psi)$ on $S$ is an $(\in, \in \forall kq)$-fuzzy quasi filter.

**Theorem 5.6:** Suppose $H : S \rightarrow P^*(S)$ is an SVIH and $\Psi$ is $(\in, \in \forall kq)$-fuzzy quasi filter of $S$. Then $\bar{H}(\Psi)$ is $(\in, \in \forall kq)$-fuzzy quasi filter of $S$.

**Proof:** Straightforward as Theorem 5.5.

6. Conclusion

Ordered semigroups are very important algebraic structures due to their various applications. To study uncertainty in different types of fuzzy filters of ordered semigroups, concept of generalized roughness has been introduced. Set valued maps give rise to binary relations in general. These maps with monotone or isotone order help to study roughness in fuzzy filters of ordered semigroups. The concept of approximations is extended to $(\in, \in q)$-fuzzy filters, fuzzy bi-filters, fuzzy quasi filters and $(\in, \in \forall kq)$-fuzzy filters, fuzzy bi-filters, fuzzy quasi filters in ordered semigroups.

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