Harnack inequality for non-local Schrödinger operators.

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Abstract

Let $x \in \mathbb{R}^d$, $d \geq 3$, and $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function with all second partial derivatives being continuous. For $1 \leq i, j \leq d$, let $a_{ij} : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function with all partial derivatives being continuous and bounded. We shall consider the Schrödinger operator associated to

$$L_f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(\cdot) \frac{\partial f}{\partial x_j} \right)(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(y) - f(x) \right] J(x,y) dy$$

where $J : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a symmetric measurable function. Let $q : \mathbb{R}^d \to \mathbb{R}$. We specify assumptions on $a, q$, and $J$ so that non-negative bounded solutions to $L_f + qf = 0$ satisfy a Harnack inequality. As tools we also prove a Carleson estimate, a uniform Boundary Harnack Principle and a 3G inequality for solutions to $L_f = 0$.

1 Introduction

In this paper we study the Harnack inequality for solutions to the Schrödinger operator associated to a specific class of non-local operators. Let $x \in \mathbb{R}^d$, $d \geq 3$, $i, j \in \{1, \ldots, d\}$ and $q, a_{ij} : \mathbb{R}^d \to \mathbb{R}$ and $J : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We consider positive bounded solutions to the Schrödinger equation, $Lu + qu = 0$ where

$$L_f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(\cdot) \frac{\partial f}{\partial x_j} \right)(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(y) - f(x) \right] J(x,y) dy \quad (1.1)$$

We show that when $q$ is in the Kato class, positive bounded solutions to $Lu + qu = 0$ satisfy a Harnack inequality (See Theorem 1.1 for the precise statement). For proving our main result we require several tools from the Potential theory of $L$. In particular we prove a Carleson estimate (Theorem 3.5), a uniform Boundary Harnack Principle (Theorem 3.9) and a 3G inequality (Proposition 4.5) for solutions to $Lu = 0$. In keeping with our objective we have proved these results only for balls, the same proof should go through in $C^{1,1}$ domains (in particular those domains as in Notation 3.1). To prove them we borrow techniques developed in [CS02], [CS03], [CSKV12a] and [CSKV12b] for an operator closely related to $L$ (see 1.3).
The importance of the Harnack inequality and its implications to the theory of elliptic and parabolic partial differential equations are well known. We refer the interested reader to the survey article by Kassmann \[KAS07\], and the references therein for a comprehensive review of the classical Harnack inequality (i.e. for elliptic and parabolic operators).

The Harnack inequality for solutions to (local) Schrödinger operators was first proved in \[ASS2\], where they considered the operator \(\Delta + q\), where \(q\) is a given potential belonging to an appropriate function space. It was shown in \[ASS2\] that non-negative solutions to \(\Delta u + qu = 0\) satisfy a Harnack inequality. We refer the reader to \[CZ94\] for a detailed account of the same. When \(\Delta\) was replaced by a second order elliptic operator in divergence form, the Harnack inequality was established using analytic methods by \[CFG86\] and probabilistic methods in \[CFZ88\]. These would correspond to the case \(J \equiv 0\), \(a\) is uniformly elliptic and bounded in \(\{1,1\}\), and \(q\) being in Kato class (see Assumption (A) and (Q) below).

The simplest and most well studied pure jump process is the symmetric stable process of index \(\alpha\) (i.e. in our notation \(J(x,y) = \frac{c(d,\alpha)}{|x-y|^{d+\alpha}}\), and \(a \equiv 0\), \(\mathcal{L}\), reduces to \(\Delta_f^\alpha\)). This is a Lévy process whose infinitesimal generator is the fractional Laplacian \(\Delta^{\alpha/2}\), defined by

\[
\Delta^{\alpha/2} u(x) = c(d,\alpha) \lim_{\epsilon \to 0} \int_{|y-x| > \epsilon} \frac{u(y) - u(x)}{|y - x|^{d+\alpha}} dy,
\]

where \(0 < \alpha \leq 2\), and \(c(d,\alpha)\) is an appropriate constant. The above limit exists if \(u\) is \(C^\alpha_b(\mathbb{R}^d)\) (the set of all bounded continuously twice differentiable functions). The Harnack inequality for non-negative \(\alpha\)-harmonic functions was proved in the 1930’s by M. Riesz by using the corresponding Poisson Kernel representation (See \[LAN72\]). We refer the reader to \[B09\] for a thorough introduction to the potential analysis of the fractional Laplacian and other related operators. \[BL02\] consider general non-local operators \(A\) defined by

\[
Au(x) = \int_{\mathbb{R}^d\setminus\{0\}} \left[ u(x+h) - u(x) - \frac{|h|}{|h|^{d+\alpha}} \right] \frac{J(x,x+h)}{|h|^{d+\alpha}} dh,
\]

where \(J\) is symmetric in \(h\), uniformly bounded above and below from \(0\). They prove a Harnack inequality holds for non-negative \(A\)-harmonic functions.

The Harnack inequality for positive solutions to the Schrödinger operator associated to \(\Delta_f^\alpha\) has been proved in \[BB00\] when \(q\) is in the Kato-class associated with the pure-jump process. To the best of our knowledge the Harnack inequality for positive solutions to the Schrödinger operator corresponding to the general framework as in \(\{1,1\}\) is not known. In areas such as risk-sensitive control theory Harnack inequality for the Schrödinger operator associated with \(\{1,1\}\) is needed (see \[KP13\]). To establish this inequality is the main purpose of this article.

A Harnack inequality was established in \[F109\] for positive solutions to \(Bu = 0\),

\[
Bf(x) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x)
\]

\[
+ \int_{\mathbb{R}^d\setminus\{0\}} \left[ f(x+h) - f(x) - \frac{|h|}{|h|^{d+\alpha}} \right] J(x,x+h) dh,
\]

(1.2)

with suitable assumptions on \(a_{ij}\), \(b_i\) and the kernel \(J\). When \(B\) is in divergence form, (it is the same as \(\mathcal{L}\) in \(\{1,1\}\)), a parabolic Harnack inequality (which implies the elliptic Harnack inequality) was established in \[CK10\] for positive bounded solutions to \(Lu = 0\) (See Proposition \(1.6\) below). The non-local operator \(\mathcal{L}\) can be written as a sum \(\mathcal{L}_c + \mathcal{L}_j\), where \(\mathcal{L}_c\) is a differential operator, corresponding to the diffusion part of the process and \(\mathcal{L}_j\) an integral operator, corresponding to the jump part of the same process. The absence of scaling makes the...
study of such processes difficult. When \( J(x, y) = \frac{c(d, \alpha)}{|x - y|^{d+\alpha}} \) and \( a(\cdot) \) is the constant function \( a \), the operator reduces to \( L \) of the form

\[
L = a \Delta + \Delta^{\frac{\alpha}{2}}. \tag{1.3}
\]

Recently [BKK15] have proved a boundary Harnack inequality for jump-type Markov processes on metric measure state spaces, under comparability estimates of the jump kernel and Urysohn-type property of the domain of the generator of the process. The result holds for a very general class of Markov process but does not include generators in the general form given by (1.1).

The Markov processes, harmonic functions and Green function associated with (1.3) have been well studied in a series of works by several authors in [CS02], [CS03], [CSKV12a] and [CSKV12b]. We shall use several of the results from these papers and techniques inspired by these works.

Throughout this paper all constants will be denoted by \( c_1, c_2 \ldots \). They are all positive valued and their values are not important. Their dependencies on parameters if needed will be mentioned inside bracket for e.g \( c_1(d) \). We will begin with numbering afresh in each new result and proof.

1.1 Preliminaries and Main result

For any \( x \in \mathbb{R}^d \) and \( r > 0 \), we set \( |x| = \sqrt{\sum_{i=1}^{d} x_i^2} \) and \( B(x, r) = \{ y \in \mathbb{R}^d : |x - y| < r \} \).

**Assumption (A) : (Uniform ellipticity and boundedness)** For all \( i,j \in \{1, 2, \ldots d\} \) the functions \( a_{ij} \) are bounded and have continuous bounded partial derivatives. Furthermore, there exists \( \lambda > 0 \) such that for every \( \xi, x \) in \( \mathbb{R}^d \)

\[
\lambda |\xi|^2 \leq \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \frac{1}{\lambda} |\xi|^2
\]

**Assumption (J) :** The function \( J(\cdot, \cdot) \) is a non-negative, symmetric, measurable function such that there exist \( \kappa > 0 \) and \( \alpha \in (0, 2) \) such that:

\[
\kappa \frac{|x - y|^{d+\alpha}}{|x - y|^{d+\alpha}} \leq J(x, y) \leq \frac{\kappa^{-1}}{|x - y|^{d+\alpha}} \tag{1.4}
\]

for \( x, y \in \mathbb{R}^d, x \neq y \).

We shall work with the conditional gauge to prove the main result. To ensure that the conditional gauge is bounded (Proposition 1.12) we shall follow [CS03] and assume the following about our function \( q \).

**Assumption (Q) (Kato class):** \( q : \mathbb{R}^d \rightarrow \mathbb{R} \) is measurable, and satisfies:

\[
\limsup_{r_j \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|y-x|<r} dy \frac{|q(y)|}{|x - y|^{d-2}} = 0 \tag{1.5}
\]

We are now ready to state our main result.

**Theorem 1.1.** Let \( R \in (0, \frac{1}{2}] \), \( x_0 \in \mathbb{R}^d \). Let \( L \) be the nonlocal operator defined in (1.1) and \( q : \mathbb{R}^d \rightarrow \mathbb{R} \). Assume (A), (J), and (Q). Suppose that \( u \in C^2_b(\mathbb{R}^d) \) is strictly positive and satisfies

\[
Lu(x) + q(x)u(x) = 0
\]
for $x \in B(x_0, R)$. Then, there exists a positive constant $c_1 \equiv c_1(R, q, \kappa, d, \alpha, \lambda)$ such that

$$u(x) \leq c_1 u(y), \quad (1.6)$$

for all $x, y \in B(x_0, R/2)$.

**Remark 1.2.** Despite the operator in (1.1) being inhomogeneous in space, the constant $c_1 \equiv c_1(R, q, \kappa, d, \alpha, \lambda)$ is independent of $x_0$ (See Remark 1.13 after proof of Theorem 1.1). By the standard chain of balls argument, it is easy to see that the Harnack inequality also holds in any ball $B(x_0, R)$ for all $R > 0$ with the appropriate constant $c_1$ depending on $q$ and $R$.

**Remark 1.3.** We did not consider the Schrödinger operator associated to $B$ as in (1.2). It is perhaps possible to state a version of Theorem 1.1 for $B$, by assuming that $q$ satisfies an abstract condition involving the Green function along with the additional assumptions in [F109]. We wanted a verifiable condition like the one mentioned in Assumption (A) so we restricted our attention to $\mathcal{L}$.

Fix $0 < R < \frac{1}{7}$, $x_0 \in \mathbb{R}^d$ and let $B$ denote $B(x_0, R)$ for the rest of the paper.

### 1.2 Proof of Theorem 1.1

For the case $J \equiv 0$, Theorem 1.1 was proved in [CFZ88] (Theorem 5.1). We will follow the ideas in [CFZ88] for proving Theorem 1.1. Below we state several propositions whose proof we shall provide in subsequent sections. Assuming these we shall first prove Theorem 1.1.

Under assumptions (A) and (J) there is a symmetric strong Markov process, $(P^x, X_t)$, with càdlàg paths associated with $\mathcal{L}$ (See [CK10]). We will denote the Green function of the process by $G$, i.e. a Borel function $G(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$E^x \left( \int_0^\infty f(X_s) ds \right) = \int_{\mathbb{R}^d} f(y) G(x, y) dy \quad (1.7)$$

for all bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$.

**Notation 1.4.** For any Borel set $A$,

$$T_A = \inf\{t \geq 0 : X_t \in A\} \quad \text{and} \quad \tau_A = \inf\{t \geq 0 : X_t \notin A\}$$

denote the hitting time and exit time of set $A$.

It is standard to note that if $u \in C^2_b(\mathbb{R}^d)$ and satisfies $\mathcal{L} u = 0$ then via Ito’s formula $u(X_t)$ is a martingale.

**Definition 1.5.** Let $D$ be a bounded domain in $\mathbb{R}^d$. We say that a measurable function $h : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{L}$-harmonic in $D$ if for every open set $U$ such that $U \subset \overline{U} \subset D$,

$$h(x) = E^x(h(X_{\tau_U})), \quad (1.8)$$

for $x \in U$. We say that $h$ is regular harmonic in $D$ if it is harmonic in $D$ and in addition, the relation (1.8) holds with $D$ replacing $U$.

For the rest of the paper we write $\mathcal{L} u = 0$, to mean that $u$ is $\mathcal{L}$-harmonic. In [CK10], a parabolic Harnack inequality was proven for solutions to $\mathcal{L} u = 0$. This will imply the elliptic Harnack inequality (alternative references are [F109], [F209]). We state this next.
Proposition 1.6. Assume (A) and (J). Let \( x_0 \in \mathbb{R}^d \) and \( R \in (0,1/2) \). Suppose \( v \) is nonnegative and bounded on \( \mathbb{R}^d \) and \( \mathcal{L}v = 0 \) in \( B \). Then there exists a positive constant \( c_1 = c_1(R,q,\kappa,d,\alpha,\lambda) \) (in particular independent of \( x_0 \) and \( v \)) such that
\[
v(x) \leq c_1 v(y),
\]
whenever \( x, y \in B(x_0, R) \).

In what follows below we always assume that \( q \) satisfies assumption \((Q)\). To work with probabilistic solutions for the Schrödinger equation we will need to show that the ball is gaugeable, whose definition we now give. For \( t \geq 0 \), let
\[
e_q(t) = \exp \left( \int_0^t q(X_s)ds \right).
\]

Definition 1.7. Let \( U \) be an open subset of the ball \( B \). The function \( H: U \to \mathbb{R} \cup \{\infty\} \) given by
\[
H(x) = \mathbb{E}^x(e_q(\tau_U))
\]
is called the gauge for \((U,q)\). If the gauge \( H \) is bounded in \( U \) we will call \((U,q)\) gaugeable.

We now state the Feynman-Kac representation for solution \( u \in C^2_b(\mathbb{R}^d) \) of \( \mathcal{L}u + qu = 0 \) provided the ball is gaugeable.

Proposition 1.8. Let \( q \) satisfy assumption \((Q)\) and suppose \((B,q)\) is gaugeable. If \( u \in C^2_b(\mathbb{R}^d) \) solves \( \mathcal{L}u + qu = 0 \) in \( B \), then for all \( x \in B \)
\[
u(x) = \mathbb{E}^x(e_q(\tau_B)u(X_{\tau_B}))
\]
We provide a sufficient condition for the ball \( B \) to be gaugeable

Proposition 1.9. Let \( q \) satisfy assumption \((Q)\). Let \( u \) be a bounded solution of \( \mathcal{L}u + qu = 0 \) in \( B \), with \( \inf_B u > 0 \). Then \((B,q)\) is gaugeable.

The next ingredient required in the proof will be to condition on the exit measure and to employ the conditional gauge theory from the literature. For \( x,y \in B \), let \( \mathbb{P}^x \) and \( \mathbb{E}^x \) denote the probability and expectation for the Doob's \( h \)-transformed process of \( X \) with \( h(\cdot) = G_B(\cdot, y) \), where \( G_B \) denotes the Green function of the process \( X \) killed on exiting \( B \). More precisely, \( G_B \) is defined by
\[
\mathbb{E}^x \left( \int_0^{\tau_B} f(X_s)ds \right) = \int_B f(y)G_B(x,y)dy
\]
for all bounded measurable \( f: \mathbb{R}^d \to \mathbb{R} \).

Proposition 1.10. Let \( \phi \) be a non-negative \( \mathcal{F}_{\tau_B} \)- measurable function and \( A \) be a Borel subset of \( B^c \). Then,
\[
\mathbb{E}^x(\phi; X_{\tau_B} \in A) = \mathbb{E}^x(\mathbb{E}^x_X(\phi); X_{\tau_B} \in A)
\]
When \( \mathcal{L} = \Delta \), [CZ94] contains a proof of the above and it explicitly uses the density of the harmonic measure of the process. When \( \mathcal{L} = \Delta^\alpha \), [BB00] contains a proof of the above by using the joint density of \( (X_{\tau_-}, X_{\tau_B}) \). We had to combine both these aspects: when the process exits the ball via the boundary of the ball we show existence of a Martin Kernel and prove a density for the Harmonic measure in Theorem 5.12; and when the process exits the ball into the complement via jump we use the Levy system formula (see (1.4) in [CSKV12a]), thus establishing Proposition 1.10.

Another key step required is to verify the boundedness of the conditional gauge.
Definition 1.11. For any \( x \in B, y \in \bar{B}, x \neq y \), the conditional gauge is defined to be

\[
F(x, y) = \mathbb{E}_y^x \left( e_q(\tau_{B \setminus \{y\}}) \right).
\]

We shall establish the following result about the conditional gauge.

Proposition 1.12. Let \( q \) satisfy assumption (Q). Then, either \( F \equiv \infty \), or there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \leq F(x, y) \leq c_2, \quad x \in B, y \in \bar{B}, x \neq y.
\]

We now have all the ingredients to prove the main result.

Proof of Theorem 1.1: From the hypothesis and Proposition 1.9 we know that \((B, q)\) is gaugeable. So,

\[
\mathbb{E}_x^x \left( e_q(\tau_B) \right) < \infty.
\]

This implies that the conditional gauge \( F(\cdot, \cdot) \) cannot be identically infinity. Therefore by Proposition 1.12 \( F \) satisfies (1.14). From Proposition 1.8 we know that the solution to the Schrodinger equation \( u \) satisfies

\[
u(x) = \mathbb{E}_x^x \left( \left( e_q(\tau_B) \right) u(X_{\tau_B}) \right),
\]

for all \( x \in B \). For \( x, y \in B(x_0, R/2) \), we have

\[
u(x) = \mathbb{E}_x^x \left( \left( e_q(\tau_B) \right) u(X_{\tau_B}) \right) = \mathbb{E}_x^x \left( \left( e_q(\tau_B) \right) u(X_{\tau_B}) \right).
\]

by Proposition 1.10 Then using the upper bound for the conditional gauge from (1.14), in (1.15) we have

\[
u(x) \leq c_2 \mathbb{E}_x^x \left( u(X_{\tau_B}) \right)
\]

In (1.16), applying the Harnack inequality for the \( L \) harmonic function \( v(x) = \mathbb{E}_x^x \left( u(X_{\tau_B}) \right) \), (see Proposition 1.6) we have

\[
u(x) \leq c_3 \mathbb{E}_y^y \left( u(X_{\tau_B}) \right)
\]

We now reverse the estimate, using the lower bound from (1.14) and Proposition 1.10 in (1.17) we have

\[
u(x) \leq \frac{c_3}{c_4} \mathbb{E}_y^y \left( \left( e_q(\tau_B) \right) u(X_{\tau_B}) \right) = c_5 \mathbb{E}_y^y \left( e_q(\tau_B) \right) u(X_{\tau_B}) = c_5 u(y).
\]

This finishes the proof of the theorem.

Remark 1.13. We note that in the proof above the constants in (1.15), (1.16), and (1.17) do not depend on \( x_0 \). Consequently the constant \( c_1 \equiv c_1(R, q, \kappa, d, \lambda) \) in the statement of Theorem 1.1 does not depend \( x_0 \).

Layout: The rest of the paper is organized as follows. In the next section, Section 2, we prove Proposition 1.8 and Proposition 1.9. The remainder of the paper is devoted to proving Proposition 1.10 (proved in Section 3) and Proposition 1.12 (proved in Section 4). These propositions require three key results from the potential theory of \( L \). For this, in Section 3 we prove a Carelson estimate (Theorem 3.5) and a uniform Boundary Harnack Principle (Theorem 3.9) followed by the 3G inequality (Proposition 4.5) in Section 4. Results on the Martin kernel and the Martin Boundary along with the density of the Harmonic measure are contained in Section 5. The results in these sections are of independent interest as well.
2 Gauge and Feynman-Kac Representation

In this section we prove Proposition 1.8 and Proposition 1.9. For any Borel set $A$, let

$$T_A = \inf\{ t : X_t \in A \} \text{ and } \tau_A = \inf\{ t : X_t \in A^c \}$$

be the hitting time and the exits times from $A$ respectively. From Lemma 2.6 in \([\text{CK10}]\), we know that for all $x \in B$ there exists $c_1 > 0$ such that

$$\mathbb{E}^x(\tau_B) \leq c_1 R^2. \quad (2.1)$$

Hence for any Borel set $A \subset B$, $\tau_A$ is finite almost surely $\mathbb{P}^x$ for $x \in A$.

**Definition 2.1.** The function $H : B \to \mathbb{R} \cup \{\infty\}$ given by

$$H(x) = \mathbb{E}^x \left( \exp \left( \int_0^{\tau_B} q(X_s) \, ds \right) \right) \quad (2.2)$$

is called the gauge for $(B, q)$. If the gauge $H$ is bounded in $B$, we will call $(B, q)$ gaugeable.

We shall first prove that if $q$ satisfies (Q) then every sufficiently ‘small’ set is gaugeable. Let $m$ denote the Lebesgue measure on $\mathbb{R}^d$.

**Lemma 2.2.** (Gaugeable sets) Let $q$ satisfy assumption (Q). Then, there exists $\delta > 0$ such that for every ball $C \subset B$, with $m(C) < \delta$, we have $(C, q)$ is gaugeable.

**Proof.** Let $C$ be a ball with $m(C) < \delta$ and $x \in C$. Then using the definition of $G_B$ and the upper bound (4.1) given by Lemma 4.2 below, we have

$$\mathbb{E}^x \left( \int_0^{\tau_C} | q(X_s) | \, ds \right) = \int_C G_B(x, y) | q(y) | \, dy \leq c_1 \int_C |x - y|^{2-d} | q(y) | \, dy = \eta < 1,$$

if $\delta$ is small enough, by assumption (Q). By a standard application of Khasminki’s lemma we have

$$\mathbb{E}^x \left( \exp \left( \int_0^{\tau_C} | q(X_s) | \, ds \right) \right) \leq \frac{1}{1 - \eta} < \infty. \quad (2.3)$$

This proves that $(C, q)$ is gaugeable. \[\square\]

We next prove a Feynman-Kac representation for solutions $u$ to $Lu + qu = 0$.

**Proof of Proposition 1.8:** For $t > 0$, let $Y_t = \int_0^t q(X_s) \, ds$, $V_t = \exp(Y_t)$ and $W_t = u(X_t)$. Note that $[V, W]_t = 0$ (since $V_t$ is continuous). Applying Ito’s product formula to $V, W$ (see [B11]) and taking expectations we have

$$\mathbb{E}^x e^{Y_t} u(X_t) = u(x) + \mathbb{E}^x \int_0^{t \wedge T_B} u(X_s) e^{Y_s} q(X_s) \, ds + \mathbb{E}^x \int_0^{t \wedge T_B} e^{Y_s} L u(X_s) \, ds. \quad (2.4)$$

Since $Lu + qu = 0$, this implies

$$\mathbb{E}^x e^{Y_t} u(X_t) = u(x). \quad (2.5)$$

As $(B, q)$ is gaugeable, allowing $t \to \infty$ in \((2.5)\), the dominated convergence theorem implies \((1.11)\). \[\square\]
When $X_t$ is a Brownian motion, it is known that the union of gaugeable balls is gaugeable (see Lemma 4.16 in [CZ94]). The result is true for solutions to the martingale problem as well. The proof is similar and requires only minor modification. For the sake of completeness we state the result and prove it below.

**Proposition 2.3.** Let $C_1, C_2 \subset B$ be balls with $C_1 \cap C_2 \neq \emptyset$ and suppose $(C_i, q)$ is gaugeable. Let $C = C_1 \cup C_2$. Suppose $q$ satisfies assumption $(Q)$, and there exists a bounded solution $u$ satisfying $Lu + qu = 0$ in $C$, with $\inf_C u > 0$. Then $(C, q)$ is gaugeable.

**Proof.** Define for any $t > 0$, $e_q(t) = \exp(\int_0^t q(X_s) ds)$. Since each $C_i, i = 1, 2$ is gaugeable, we can apply Proposition 1.8 to observe that for $i = 1, 2$ and $x \in C_i$

$$u(x) = \mathbb{E}^x \left( e_q(\tau_{C_i}) u(X_{\tau_{C_i}}) \right). \tag{2.6}$$

We will show that equation (2.6) holds when $C_i$ is replaced by $C$. Without loss of generality we may assume $x \in C_1$. Let $T_0 = 0$, and for $n \geq 1$

$$T_{2n-1} = T_{2n-2} + \tau_{C_1} \circ \theta_{T_{2n-2}} \quad T_{2n} = T_{2n-1} + \tau_{C_2} \circ \theta_{T_{2n-1}}, \tag{2.7}$$

where $\theta$ is the canonical time-shift operator. We will show that

$$u(x) = \mathbb{E}^x \{e_q(T_m)u(X_{T_m})\}, \ m \geq 0. \tag{2.8}$$

This is true for $m = 0$. Suppose that $u(x) = \mathbb{E}^x (e_q(T_{2n})u(X_{T_{2n}}))$. Then on $\{T_{2n} < \tau_C\}$, we have $X_{T_{2n}} \in C_1$. Therefore by (2.6),

$$u(x) = \mathbb{E}^x (T_{2n} = \tau_C, e_q(T_{2n})u(X_{T_{2n}})) + \mathbb{E}^x (T_{2n} < \tau_C, e_q(T_{2n})u(X_{T_{2n}})) \tag{2.9}$$

On $\{T_{2n} = \tau_C\}$, we have $T_{2n} = T_{2n+1}$. Hence the first term on the right side of equation (2.9) is equal to $\mathbb{E}^x (T_{2n} = \tau_C, e_q(T_{2n+1})u(X_{T_{2n+1}}))$. By the definition of $T_{2n+1}$ and the strong Markov property, the second term on the right side of equation (2.9) is equal to $\mathbb{E}^x (T_{2n} < \tau_C, e_q(T_{2n+1})u(X_{T_{2n+1}}))$. Now adding the two terms we obtain (2.8) with $m = 2n + 1$. In a similar manner, one can prove that if (2.8) holds for $m = 2n + 1$, then it holds for $m = 2n + 2$. Hence (2.8) holds for all $m \geq 0$ by induction and it implies that

$$\inf_{x \in C} u(x) \mathbb{E}^x(e_q(T_m)) \leq \sup_{x \in C} u(x). \tag{2.10}$$

We now establish that almost surely, $\lim_{m \to \infty} T_m = \tau_C$. By (2.1) above, we first note that $\tau_C < \infty$ with probability 1. As $T_m$ is increasing in $m$ and $T_m \leq \tau_C$ we have $\lim_{m \to \infty} T_m = T \leq \tau_C$. If $T < \tau_C$, then since $X_{T_{2n-1}} \in \partial C_1$ and $X_{T_{2n}} \in \partial C_2$ for all $n$, we can use the fact that the process has left limits and conclude that

$$X_T \in \partial C_1 \cap \partial C_2 \subset (C_1 \cup C_2)^c.$$ 

But this implies that $T \geq \tau_C$, which is a contradiction. Therefore almost surely $\lim_{m \to \infty} T_m = \tau_C$. Using Fatou’s lemma and the fact that $u$ is strictly positive, from (2.10) we have that

$$\mathbb{E}^x(e_q(\tau_C)) \leq \sup_{x \in C} \frac{u(x)}{\inf_{x \in C} u(x)} < \infty.$$ 

So $(C, q)$ is gaugeable.  \qed
We have seen that every ball of sufficiently small radius is gaugeable. Now we prove Proposition 1.9 which states that the ball $B$ is gaugeable.

**Proof of Proposition 1.9** There exists a a sequence of bounded domains $\{D_n\}$ such that $\overline{D}_n \subset B$, $D_n \uparrow B$. Recall that $m$ denotes the Lebesgue measure, so it has no atoms. Therefore for each $n$, $D_n$ can be written as the finite union of balls $C$, with $m(C) < \delta$ so that each $(C, q)$ is gaugeable. Then by repeated application of Proposition 2.3 $(D_n, q)$ is gaugeable for each $n$, and by Proposition 1.8

$$u(x) = \mathbb{E}^x \left( e_q(\tau_{D_n}) u(X_{\tau_{D_n}}) \right), \quad x \in D_n.$$  

As before, we have

$$\mathbb{E}^x (e_q(\tau_{D_n})) \leq \frac{\sup_{x \in B} u(x)}{\inf_{x \in B} u(x)}$$

Since $\tau_{D_n} \uparrow \tau_B \leq \infty$ a.s, we obtain by Fatou’s lemma that

$$\mathbb{E}^x (e_q(\tau_B)) \leq \liminf_{n \to \infty} \mathbb{E}^x (e_q(\tau_{D_n})) \leq \frac{\sup_{x \in B} u(x)}{\inf_{x \in B} u(x)} < \infty.$$  

\[ \square \]

3 Potential Theory for $\mathcal{L}$

In this section we state and prove a uniform Boundary Harnack principle for $\mathcal{L}$-harmonic functions (Theorem 3.9). The classical version of Boundary Harnack principle follows from this result. A key ingredient to prove the Boundary Harnack principle is the Carleson Estimate (Theorem 3.3), which we prove first. We begin by fixing some notation.

**Notation 3.1.** The ball $B$ is a smooth domain. So there exists a localization radius $R_1 < R/4$ and a constant $M_1$ such that for every $Q \in \partial B$, there exist a smooth function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla \phi(0) = 0$, $|\nabla \phi(x) - \nabla \phi(y)| \leq M_1 |x - y|$, and a coordinate system $CS_Q$ with $y = (\tilde{y}, y_d)$ with its origin at $Q$ such that

$$\begin{align*}
B(Q, R_1) \cap B &= \{ (\tilde{y}, y_d) \in B(0, R_1) : y_d > \phi(\tilde{y}) \}. \\
\text{We define } \rho_Q(y) &= y_d - \phi(\tilde{y}), \text{ denote } \delta_B(x) = \text{dist}(x, \partial B), \quad r_0 = \frac{R_1}{4(1+M_1^2)}, \quad R_0 = \frac{R_1}{\sqrt{1+M_1^2}} \text{ and } \\
D_Q(r_1, r_2) &= \{ y \in B : 0 < \rho_Q(y) < r_1, |\tilde{y}| < r_2 \}.
\end{align*}$$

An important ingredient in the proofs will be the Levy system formula (see (1.4) in [CSKV12a]) associated with the jump process $X$ given by a jump kernel $J$. For any non-negative measurable function $g : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, with $g(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time $T$ (with respect to filtration of $X$), and any $x \in \mathbb{R}^d$

$$\mathbb{E}^x \left( \sum_{s \leq T} g(s, X_s, X_{s-}) \right) = \mathbb{E}^x \left( \int_0^T \left( \int_{\mathbb{R}^d} g(s, X_s, y) J(X_s, y) dy \right) ds \right). \quad (3.1)$$

We remark at this point that the proofs of the results in this entire section follow the notation, ideas, and techniques in [CSKV12a]. Instead of citing the results without proof when required, we reproduce the proof here for the reader’s convenience.
3.1 Carleson Estimate

We begin with some technical lemmas from the literature to understand the behavior of exit distributions of the process $X_t$.

**Lemma 3.2.** There exists positive constants $c_1$ and $c_2$ such that

$$\mathbb{E}^x(\tau_B) \leq c_1 R^2, \quad x \in B,$$

and

$$\mathbb{E}^x(\tau_B) \geq c_2 R^2, \quad x \in B(x_0, \frac{R}{2}).$$

**Proof.** See Lemma 3.4 in [F109]. \qed

**Lemma 3.3.** There exists a non-decreasing function $\psi : (0,1) \to (0,1)$, such that if $C \subset B(x_0, r)$, $\gamma > 0$, $r \in (0, 1)$, and $x \in B(x_0, r/2)$, then

$$\mathbb{P}^x(T_C \leq \tau_{B(0,r)}) \geq \psi\left(\frac{|C|}{r^d}\right).$$

**Proof.** See Corollary 4.9 in [F109]. \qed

**Lemma 3.4.** Let $R_1$ and $\rho_Q$ be defined as above. Then, there exists a constant $\delta = \delta(R_1, M_1) > 0$, such that for all $Q \subset \partial B, x \in B$ with $\rho_Q(x) < R_1/2$,

$$\mathbb{P}^x(X_{\tau(x)} \in B^c) \geq \delta,$$

where $\tau(x) = \inf\{t > 0 : X_t \not\in B \cap B(x, 2\rho_Q(x))\}$.

**Proof.** Denote $\bar{B} = B(x,2\rho_Q(x))$, and $C = \bar{B} \cap B^c$. Recall that $2\rho_Q(x) < R_1 < 1$. Observe that $\{T_C < \tau_{B}\} \subset \{X_{\tau(x)} \in B^c\}$. Therefore using Lemma 3.3 (with $r = 2\rho_Q(x)$), we obtain

$$\mathbb{P}^x(X_{\tau(x)} \in B^c) \geq \mathbb{P}^x(T_C < \tau_{\bar{B}}) \geq \psi(1/2) > 0. \quad (3.4)$$

This shows that we can take $\delta = \psi(1/2)$, and finishes the proof of the lemma. \qed

We are now ready to prove the Carleson estimate.

**Theorem 3.5.** Let $Q \subset \partial B$. Let $u$ be a non-negative function in $\mathbb{R}^d$ that is $L$ harmonic in $B \cap B(Q, r)$, with $r < R_1/2$ and suppose that $u$ vanishes continuously on $B^c \cap B(Q, r)$. Then there exists a positive constant $c = c(\alpha, R_1, M_1)$ such that

$$u(x) \leq cu(x_0), \quad \text{for} \quad x \in B \cap B(Q, r/2). \quad (3.5)$$

where $x_0 \in B \cap B(Q, r)$, with $\rho_Q(x_0) = r/2$.

**Proof.** Since $r < R_1/2$, by the Harnack inequality and a chain argument, it is sufficient to prove (3.5) for $x \in B \cap B(Q, r/12)$ and $\tilde{x}_0 = \tilde{Q}$. We may also normalize so that $u(x_0) = 1$. In the following proof, all the constants $\delta, \beta, \eta$, and $c_i$ are always independent of $r$. First choose $0 < \gamma < \alpha/(d + \alpha)$ and let

$$B_0 = B \cap B(x, 2\rho_Q(x)), \quad B_1 = B(x, r^{1-\gamma}\rho_Q(x)^\gamma).$$

We then set

$$B_2 = B(x_0, \rho_Q(x)/3), \quad B_3 = B(x_0, 2\rho_Q(x)/3)$$

and finishes the proof of the lemma. \qed
and 
\[ \tau_0 = \inf \{t > 0 : X_t \notin B_0 \}, \quad \tau_2 = \inf \{t > 0 : X_t \notin B_2 \}. \]

By Lemma 3.4 there exists \( \delta = \delta(R_1, M_1) \) such that
\[ \mathbb{P}^x(X_{\tau_0} \in B^c) \geq \delta, \quad x \in B(Q, r/4). \] (3.6)

By the Harnack inequality and a chain argument, there exists \( \beta \) such that
\[ u(x) < \left( \frac{\rho Q(x)}{r} \right)^{-\beta} u(x_0), \quad x \in B(Q, r/4). \] (3.7)

Since \( u \) is \( L \)-harmonic in \( B_0 \), we may write
\[ u(x) = \mathbb{E}^x (u(X_{\tau_0}); X_{\tau_0} \in B_1) + \mathbb{E}^x (u(X_{\tau_0}); X_{\tau_0} \notin B_1), \quad x \in B(Q, r/4). \] (3.8)

We will assume the following Lemma and complete the proof of the Theorem.

**Lemma 3.6.** There exists \( \eta > 0 \) such that
\[ \mathbb{E}^x (u(X_{\tau_0}); X_{\tau_0} \notin B_1) \leq u(x_0) \quad \text{if} \quad x \in B \cap B(Q, r/12), \quad \text{and} \quad \rho Q(x) < \eta r. \] (3.9)

We will prove the Carleson estimate by contradiction. Recall that \( u(x_0) = 1 \). Suppose that there exists \( x_1 \in B(x, r/12) \) such that
\[ u(x_1) \geq K > \eta^{-\beta} \vee (1 + \delta^{-1}), \] (3.10)

where \( K \) is a constant that will be specified later. By (3.7) and the assumption that \( u(x_1) \geq K > \eta^{-\beta} \), we have \( \rho Q(x_1)/r)^{-\beta} > u(x_1) \geq K > \eta^{-\beta} \). Hence \( \rho Q(x_1) < \eta r \). Let \( B_0, B_1 \) and \( \tau_0 \) now be defined with respect to the point \( x_1 \) instead of \( x \). Then by (3.8), (3.9) and \( K > (1 + \delta^{-1}) \), we have
\[ K \leq u(x_1) \leq \mathbb{E}^{x_1} (u(X_{\tau_0}); X_{\tau_0} \in B_1) + 1, \]

and hence, using (3.10),
\[ \mathbb{E}^{x_1} (u(X_{\tau_0}); X_{\tau_0} \in B_1) \geq u(x_1) - 1 - \frac{1}{1 + \delta} u(x_1). \]

If \( K \geq 2\eta \), then \( B^c \cap B_1 \subset B^c \cap B(Q, r) \). By using the assumption that \( u = 0 \) on \( B^c \cap B(Q, r) \) and (3.6) we have
\[ \mathbb{E}^{x_1} (u(X_{\tau_0}); X_{\tau_0} \in B_1) = \mathbb{E}^{x_1} (u(X_{\tau_0}); X_{\tau_0} \in B_1 \cap B) \leq \mathbb{P}^x (X_{\tau_0} \in B) \quad \sup_{x \in B \cap B_1} u \]
\[ \leq (1 - \delta) \quad \sup_{x \in B \cap B_1} u. \] (3.11)

Therefore, \( \sup_{x \in B \cap B_1} u(x) > u(x_1)/((1 + \delta)(1 - \delta)) \), i.e. there exists a point \( x_2 \in B \) such that
\[ |x_2 - x_1| \leq r^{-1} \eta \rho Q(x_1) \gamma \] and \( u(x_2) > \frac{1}{1 - \delta^2} u(x_1) \geq \frac{1}{1 - \delta^2} K. \)

By induction, if \( x_k \in B \cap B(Q, r/12) \) with \( u(x_k) \geq K/(1 - \delta^2)^{k-1} \) for \( k \geq 2 \), then there exists \( x_{k+1} \in B \) such that
\[ |x_k - x_{k+1}| \leq r^{-1} \eta \rho Q(x_1) \gamma \] and \( u(x_{k+1}) > \frac{1}{1 - \delta^2} u(x_k) \geq \frac{1}{(1 - \delta^2)^k} K. \] (3.12)
From (3.7) and (5.12), it follows that $\rho_Q(x_k)/r \leq (1 - \delta^2)^{(k-1)/\beta} K^{-1/\beta}$ for every $k \geq 1$. Therefore,

$$|x_k - Q| \leq |x_1 - Q| + \sum_{j=1}^{k-1} |x_{j+1} - x_j| \leq r/12 + \sum_{j=1}^{\infty} r^{1-\gamma} \rho_Q(x_j)^\gamma$$

$$\leq r/12 + r^{1-\gamma} \sum_{j=1}^{\infty} (1 - \delta^2) (j-1)^{\gamma/\beta} K^{-\gamma/\beta} r^\gamma$$

$$= r/12 + r^{1-\gamma} K^{-\gamma/\beta} \sum_{j=1}^{\infty} (1 - \delta^2) (j-1)^{\gamma/\beta}$$

$$= r/12 + r K^{-\gamma/\beta} \frac{1}{1 - (1 - \delta^2)^{\gamma/\beta}}. \quad (3.13)$$

Choose

$$K = \eta \vee (1 + \delta^{-1}) \vee 12^{\beta/\gamma} (1 - (1 - \delta^2)^{\gamma/\beta})^{-\beta/\gamma}.$$ 

Then $K^{-\gamma/\beta} (1 - (1 - \delta^2)^{\gamma/\beta})^{-1} \leq 1/12$, and hence $x_k \in B \cap B(Q, r/6)$ for every $k \geq 1$. Since $\lim_{k \to \infty} u(x_k) = +\infty$, this contradicts the fact that $u$ is bounded on $B(Q, r/2)$. This proves that $u(x) < K$ for every $x \in B \cap B(Q, r/12)$ and completes the proof of the theorem. \square

We now provide the proof of the lemma.

**Proof of Lemma 3.6.** Let $\eta_0 = 2^{-2(d+\alpha)/d}$. Then for $\rho_Q(x) < \eta_0 r$,

$$(\rho_Q(x))^{d/(\alpha+d)} < 1/4, \text{ and } 2\rho_Q(x) \leq r^{1-\gamma} \rho_Q(x)^\gamma - 2\rho_Q(x).$$

Thus if $x \in B \cap B(Q, r/12)$ with $\rho_Q(x) < \eta_0 r$, then $|x - y| < 2 |z - y|$ for $z \in B_0, y \notin B_1$. Now using the Levy system formula from equation (3.11), and the upper bound on expected exit time from (3.2), we have

$$\mathbb{E}^x(u(X_{\tau_0}); X_{\tau_0} \notin B_1) = c_1 \int_{B_0} G_{B_0}(x, z) \int_{|y-x|>r^{1-\gamma} \rho_Q(x)^\gamma} J(z, y) u(y) dy dz$$

$$\leq c_2 \int_{B_0} G_{B_0}(x, z) dz \int_{|y-x|>r^{1-\gamma} \rho_Q(x)^\gamma} |z - y|^{-d-\alpha} u(y) dy$$

$$\leq 2^{d+\alpha} c_3 \int_{B_0} G_{B_0}(x, z) dz \int_{|y-x|>r^{1-\gamma} \rho_Q(x)^\gamma} |x - y|^{-d-\alpha} u(y) dy dz$$

$$\leq c_4 \mathbb{E}^x \left( \tau_B(x, 2\rho_Q(x)) \right) \int_{|y-x|>r^{1-\gamma} \rho_Q(x)^\gamma} |x - y|^{-d-\alpha} u(y) dy$$

$$\leq c_5 \rho_Q(x)^2 (I_1 + I_2), \quad (3.14)$$

with

$$I_1 = \int_{|y-x|>r^{1-\gamma} \rho_Q(x)^\gamma; z-x_0>2\rho_Q(x_0)/3} |x - y|^{-d-\alpha} u(y) dy$$

and

$$I_2 \int_{|y-x|\leq2\rho_Q(x_0)/3} |x - y|^{-d-\alpha} u(y) dy.$$ 

On the other hand, for $z \in B_2, y \notin B_3$, we have

$$|z - y| \leq |z - x_0| + |x_0 - y| \leq \rho_Q(x_0)/3 + |x_0 - y| \leq 2 |x_0 - y|.$$
Now we apply the Levy system formula and the bound in equation (3.3) to obtain

\[ u(x_0) \geq \mathbb{E}^x(u(X_{t_2}), X_{t_2} \notin B_3) \]
\[ \geq c_6 \int_{B_2} G_{B_2}(x_0, z) \int_{|y-x_0| > 2\rho_Q(x_0)/3} |z-y|^{-d-\alpha} u(y) dy dz \]
\[ \geq 2^{-d-\alpha} c_7 \int_{B_2} G_{B_2}(x_0, z) dz \int_{|y-x_0| > 2\rho_Q(x_0)/3} |x_0-y|^{-d-\alpha} u(y) dy \]
\[ \geq 2^{-d-\alpha} c_8 (\rho_Q(x_0)/3)^2 \int_{|y-x_0| > 2\rho_Q(x_0)/3} |x_0-y|^{-d-\alpha} u(y) dy \]
\[ = c_9 \rho_Q(x_0)^2 \int_{|y-x_0| > 2\rho_Q(x_0)/3} |x_0-y|^{-d-\alpha} u(y) dy. \quad (3.15) \]

We shall use (3.15) to estimate \( I_1 \). Now suppose that \(|y-x| > r^{1-\gamma}\rho_Q(x)^\gamma\) and \(x \in B(Q, r/4)\). Then,

\[ |y-x_0| \leq |y-x| + r \leq |y-x| + r^\gamma \rho_Q(x)^{-\gamma} \leq 2r^\gamma \rho_Q(x)^{-\gamma} |y-x| . \]

Therefore,

\[ I_1 = \int_{|y-x| > r^{1-\gamma}\rho_Q(x)^\gamma, |y-x_0| > 2\rho_Q(x_0)/3} |x-y|^{-d-\alpha} u(y) dy \]
\[ \leq \int_{|y-x_0| > 2\rho_Q(x_0)/3} (2^{-1}(\rho_Q(x)/r)^\gamma)^{-d-\alpha} |y-x|^{-d-\alpha} u(y) dy \]
\[ = 2^{d+\alpha}(\rho_Q(x)/r)^{-\gamma(d+\alpha)} \int_{|y-x_0| > 2\rho_Q(x_0)/3} |y-x|^{-d-\alpha} u(y) dy \]
\[ \leq c_{10} 2^{d+\alpha}(\rho_Q(x)/r)^{-\gamma(d+\alpha)} \rho_Q(x_0)^{-2} u(x_0) \]
\[ = c_{11}(\rho_Q(x)/r)^{-\gamma(d+\alpha)} \rho_Q(x_0)^{-2} u(x_0), \quad (3.16) \]

where the last inequality above is due to (3.15). If \(|y-x_0| < 2\rho_Q(x_0)/3\), then

\[ |y-x| \geq |x_0 - Q| - |x - Q| - |y - x_0| > \rho_Q(x_0)/6. \]

This combined with the Harnack inequality (Proposition 1.6) gives

\[ I_2 = \int_{|y-x| \leq 2\rho_Q(x_0)/3} |x-y|^{-d-\alpha} u(y) dy \]
\[ \leq c_{12} \int_{|y-x| \leq 2\rho_Q(x_0)/3} |x-y|^{-d-\alpha} u(x_0) dy \]
\[ \leq c_{13} u(x_0) \int_{|y-x| \geq 2\rho_Q(x_0)/6} |x-y|^{-d-\alpha} dy \]
\[ = c_{14}(\rho_Q(x_0))^{-\alpha} u(x_0). \quad (3.17) \]

Combining (3.14), (3.16), and (3.17) we obtain

\[ \mathbb{E}^x(u(X_{t_0}); X_{t_0} \notin B_1) \]
\[ \leq c_{15} \rho_Q(x_0)^2 \left( c_{16}(\rho_Q(x)/r)^{-\gamma(d+\alpha)} \rho_Q(x_0)^{-2} u(x_0) + c_{17}(\rho_Q(x_0))^{-\alpha} u(x_0) \right) \]
\[ \leq c_{18} u(x_0) \left( (\rho_Q(x)/r)^{2-\gamma(d+\alpha)} + \rho_Q(x)^2 \rho^{-\alpha} \right), \quad (3.18) \]
where in the last inequality we used the fact that \( \rho_Q(x_0) = r/2 \). Now we choose \( \eta \in (0, \eta_0) \) so that
\[
c_{18} \left( \eta^{2-\gamma(d-\alpha)} + \eta^2 \right) \leq 1.
\]
Then for \( x \in B \cap B(Q, r/12) \) with \( \rho_Q(x) < \eta r \), we have by \[3.18\]
\[
\mathbb{E}^x \left( u(X_{\tau_0}) : X_{\tau_0} \notin B_1 \right) \leq c_{18} u(x_0) \left( \eta^{2-\gamma(d-\alpha)} + \eta^2 r^{2-\alpha} \right),
\]
\[
\leq c_{18} \left( \eta^{2-\gamma(d-\alpha)} + \eta^2 \right) u(x_0) \leq u(x_0),
\]
which proves the result. \( \Box \)

### 3.2 Boundary Harnack Principle

We shall first provide estimates for exit probabilities near the boundary. For this we shall consider the same for a truncated process. We define the truncated process \( \hat{X} \) to be the process with the same diffusion component as \( X \) but with the jump kernel to be \( J(x, y)1_{\{|x-y|<1\}} \). That is the jump sizes are restricted to be strictly smaller than 1. The corresponding exit times will be denoted by \( \hat{\tau} \).

**Lemma 3.7.** There exist positive constants \( \delta_0 = \delta_0(R_1, M_1, \alpha) \), \( c_1 = c_1(R_1, M_1, \alpha) \) and, \( c_2 = c_2(R_1, M_1, \alpha) \) such that for every \( Q \in \partial B \), and \( x \in D_Q(2\delta_0, r_0) \) with \( \hat{\tau} = 0 \),
\[
\mathbb{P}^x \left( \hat{X}_{\hat{\tau}D_{Q_0}(\delta_0, r_0)} \in D_Q(2\delta_0, r_0) \right) \geq c_1 \delta_B(x), \text{ and} \tag{3.19}
\]
\[
\mathbb{P}^x \left( \hat{X}_{\hat{\tau}D_{Q_0}(\delta_0, r_0)} \in B \right) \leq c_2 \delta_B(x). \tag{3.20}
\]

**Proof.** Without loss of generality, assume that \( Q = 0 \). Let \( p > 0 \) be such that \( p \neq \alpha \) and \( 1 < p < (2 \wedge 3 - \alpha) \). Recall from Notation 3.1 that for \( y \in B \), \( \rho(y) = y_d - \phi(\tilde{y}) \) and \( D(r_1, r_2) = \{ y \in B : 0 < \rho(y) < r_1, |\tilde{y}| < r_2 \} \). Define for \( y \in B \),
\[
h(y) = \rho(y)1_{B(0,R_0)\cap B}, \quad h_p(y) = h(y)^p,
\]
Since \( \rho(y) \leq \sqrt{1 + M_1^2} \delta_B(y) \), we have \( 0 \leq h(y) \leq 1 \). Also observe that \( D(r_1, r_2) \) is contained in \( B \cap B(0, R_1/4) \) for every \( r_1, r_2 \leq r_0 \). Let \( \hat{L}_d \) denote the integral term in the operator \( L \) but with the truncated kernel \( J(x, y)1_{\{|x-y|<1\}} \). Let
\[
L_c f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial f}{\partial x_j} \right)(x)
\]
denote the diffusion part of the operator \( L \). Applying the product rule in \( L_c \), we will get two kinds of terms, one involving first order derivatives of \( a_{ij} \) and \( f \) and the other involving \( a_{ij} \) and second order derivatives of \( f \). We denote these by \( L^1_c \) and \( L^2_c \) respectively. For every \( y \in B(0, R_0) \cap B \), we have
\[
L_c h(y) = -L_c \phi(\tilde{y}), \tag{3.21}
\]
Next, a routine but tedious computation gives \( L^2_c \mathcal{H}_p(y) = I - II \), where
\[
I = p(p-1)h^{p-2}(y)\left[ 1 + \sum_{i=1}^{d-1} a_{ii}(y) (\partial_i \phi(\tilde{y}))^2 \right]
\]
\[
+ \sum_{1 \leq i,j \leq d-1, i \neq j} a_{ij}(y) \partial_i \phi(\tilde{y}) \partial_j \phi(\tilde{y}) - 2 \sum_{i=1}^{d-1} a_{id}(y) \partial_d \phi(\tilde{y}) \right] \tag{3.22}
\]
and
\[
II = ph^{p-1}(y) \left[ \sum_{i=1}^{d-1} a_{ii}(y) \partial_i \phi(\tilde{y}) + \sum_{1 \leq i,j \leq d-1, i \neq j} a_{ij}(y) \partial_{ij} \phi(\tilde{y}) \right]
\] (3.23)

Using Assumption (A), the term inside the square brackets in \( I \) can be bounded from below by \( \lambda(1 + |\nabla \phi(\tilde{y})|^2) \). Also observe that the term inside the square brackets in \( II \) is just \( \lambda^2 \phi(\tilde{y}) \), so that \( II = ph^{p-1}(y) \lambda^2 \phi(\tilde{y}) \). Combining the previous two observations, we obtain
\[
\lambda^2 h_p(y) \geq \lambda (p-1) h^{p-2}(y) (1 + |\nabla \phi(\tilde{y})|^2) - ph^{p-1}(y) \lambda^2 \phi(\tilde{y}).
\] (3.24)

Now we take care of the first order term \( L^1_c \). Another routine computation gives
\[
L^1_c h_p(y) = -ph^{p-1}(y) \left[ \sum_{1 \leq i,j \leq d-1} (\partial_i a_{ij}(y)) (\partial_j \phi(\tilde{y})) - \sum_{i=1}^{d-1} \partial_i a_{id}(y) + \sum_{j=1}^{d} \partial_d a_{dj}(y) \partial_j \phi(\tilde{y}) \right].
\]

The first term in the square bracket above is nothing but \( L^1_c \phi(\tilde{y}) \), and also note that the second and third terms are bounded because \( a_{ij} \) and \( \phi \) have bounded first derivatives. So we may now write
\[
L^1_c h_p(y) = -ph^{p-1}(y) L^1_c \phi(\tilde{y}) + ph^{p-1}(y) O(1)
\] (3.25)

Adding equations (3.24) and (3.25), we get
\[
L_c h_p(y) = L^2_c h_p(y) + L^1_c h_p(y) \geq \lambda p(p-1) h^{p-2}(y) (1 + (\nabla \phi(\tilde{y}))^2) - ph^{p-1}(y) L_c \phi(\tilde{y}) + ph^{p-1}(y) O(1). \] (3.26)

It now follows that \( \delta_1 \) may be chosen sufficiently small that
\[
L_c h_p(y) \geq c_1 (\rho(y))^{p-2} > 0
\] (3.28)

for \( y \in D(\delta_1, r_0) \) and appropriate constant \( c_1 > 0 \). We will divide the rest of the proof into three steps.

**Step 1.** Constructing suitable superharmonic and subharmonic functions with respect to \( L_c + \hat{L}_d \). Let \( \psi \) be a smooth positive function in \( \mathbb{R}^d \) with bounded first and second order partial derivatives such that \( \psi(y) = \frac{2^{p+1} |y|^2}{r_0^2} \) for \( |y| < r_0/4 \) and \( 2^{p+1} \leq \psi(y) \leq 2^{p+2} \), for \( |y| > r_0/2 \). We now define
\[
u_1(y) = h(y) + h_p(y)
\]
and
\[
u_2(y) = h(y) + \psi(y) - h_p(y)
\]

Note that both \( \nu_1 \) and \( \nu_2 \) are non-negative because \( 0 \leq h \leq 1 \) and \( p \geq 1 \). By a Taylor expansion with remainder of order 2,
\[
| (L_c + \hat{L}_d) \psi(y) | \leq | L_c \psi(y) | + | \hat{L}_d \psi(y) | \leq c_2(\alpha) < \infty.
\] (3.29)

Note that our jump kernel \( J(x, y) \) is uniformly bounded above and below by up to a constant by \( |x - y|^{d+\alpha} \), so we have by [CSKV12a Corollary 3.3], that there exist \( c_3 = c_3(R_1, M_1) \) and \( \delta_2 = \delta_2(R_1, M_1) \in (0, \delta_1) \) such that
\[
\hat{L}_d h_p(y) \geq -c_3 \text{ for } y \in D(\delta_2, r_0).
\]
Then using (3.28), the fact that \( p < 2 \), and the above inequality, we obtain (choosing \( \delta_2 \) smaller if need be)
\[
(L_c + \hat{L}_d)h_\mu(y) \geq c_1 \rho(y)^{p-2} - c_3 \geq \frac{c_1}{2} \rho(y)^{p-2},
\]
(3.30)
for \( y \in D(\delta_2, r_0) \). Making use of [CSKV12a Corollary 3.3] and (3.21), there exist \( c_4 \) and \( \delta_3 \in (0, \delta_2) \) such that for all \( y \in D(\delta_3, r_0) \)
\[
| (L_c + \hat{L}_d)h(y) | \leq c_4 (1 + \rho(y)^{(1-\alpha)^\wedge 0} + 1_{\{\alpha = 1\}} | \log \rho(y) | )
\]
(3.31)
Thus by (3.29) and (3.30) and the fact that \( p < 2 \wedge (3-\alpha) \), there exists \( \delta_4 \in (0, \delta_3) \) such that
\[
(L_c + \hat{L}_d)u_2(y) \leq c_2 + c_4 (1 + | \log \rho(y) | + \rho(y)^{(1-\alpha)^\wedge 0}) - \frac{c_1}{2} \rho(y)^{p-2} \leq -1
\]
(3.32)
for \( y \in D(\delta_4, r_0) \). On the other hand, the lower bound from [CSKV12a Corollary 3.3] gives
\[
(L_c + \hat{L}_d)h(y) \geq - || Lc\phi ||_{\infty} - c_5 (1 + | \log \rho(y) | )
\]
(3.33)
for all \( y \in D(\delta_4, r_0) \). Combining (3.33) with (3.30) and choosing \( \delta_4 \) smaller if necessary, we obtain that for \( y \in D(\delta_4, r_0) \),
\[
(L_c + \hat{L}_d)u_1(y) \geq - || Lc\phi ||_{\infty} - c_5 (1 + | \log \rho(y) | ) + \frac{c_1}{2} \rho(y)^{p-2} \geq 0.
\]
(3.34)

**Step 2.** From sub/super harmonic functions to sub/super-martingale properties

We claim that the estimates (3.32) and (3.34) imply that
\[
t \to u_2(\hat{X}_{t\wedge \tilde{\tau}_D(\delta_4, r_0)}) \text{ is a bounded supermartingale},
\]
(3.35)
\[
\mathbb{E}^x(\tilde{\tau}_D(\delta_4, r_0)) \leq \rho(x),
\]
(3.36)
and
\[
t \to u_1(\hat{X}_{t\wedge \tilde{\tau}_D(\delta_4, r_0)}) \text{ is a bounded submartingale.}
\]
(3.37)

Note that if \( v \) is a bounded \( C^2 \) function in \( \mathbb{R}^d \) with bounded first and second order partial derivatives, then an application of Ito’s formula and the Levy system (3.1) gives
\[
M_t^v = v(\hat{X}_t) - v(\hat{X}_0) - \int_0^t (L_c + \hat{L}_d)v(\hat{X}_s)ds
\]
(3.38)
is a martingale. If the functions \( u_1 \) and \( u_2 \) were \( C^2 \) with bounded derivatives, then the above claims would just follow from (3.31), (3.32), and (3.34). Since they are truncated outside of \( B(0, R_0) \cap B \) the functions are not in \( C^2 \). So we will proceed by using a mollifier. Let \( g \) be a non-negative smooth radial function with compact support in \( \mathbb{R}^d \) such that \( g(x) = 0 \) for \( |x| > 1 \), and \( \int_{\mathbb{R}^d} g(x)dx = 1 \). For \( k \geq 1 \), denote \( g_k(x) = 2^{kd}g(2^kx) \). Define for \( i = 1, 2 \),
\[
u_{ik}(z) = g_k * u_i(z) := \int_{\mathbb{R}^d} g_k(y)u_i(z-y)dy.
\]
Since \( (L_c + \hat{L}_d)u_{ik} = g_k * (L_c + \hat{L}_d)u_i \), we have by (3.32) and (3.34) that \( (L_c + \hat{L}_d)u_{ik} \geq 0 \) and \( (L_c + \hat{L}_d)u_{ik} \leq -1 \), on \( D_k(\delta_4, r_0) = \{ y : 2^{-k} < \rho(y) < \delta_4 - 2^{-k}, |y| < r_0 - 2^{-k} \} \). Since \( u_{ik} \)
\( i = 1, 2 \) are bounded smooth functions with bounded first and second order partials bounded, equation (3.36) tells us that
\[
t \to u_{ik}(\hat{X}_{t\wedge \tilde{\tau}_{D_k(\delta_4, r_0)}}) + t \wedge \tilde{\tau}_{D_k(\delta_4, r_0)}
\]
is a positive supermartingale and similarly that
\[ t \to u_i^k(\hat{X}_{t \wedge \hat{\tau}_{D_k(\delta_4, r_0)}}) \]
is a bounded submartingale. Since \( u_i \) are bounded and continuous, \( u_i^k \) converge uniformly to \( u_i \). Thus
\[ t \to u_1(\hat{X}_{t \wedge \hat{\tau}_{D_k(\delta_4, r_0)}}) + t \wedge \hat{\tau}_{D_k(\delta_4, r_0)} \]
is a positive supermartingale (3.39) and \( t \to u_2(\hat{X}_{t \wedge \hat{\tau}_{D_k(\delta_4, r_0)}}) \) is a bounded submartingale. Since \( D_k(\delta_4, r_0) \) increases to \( D(\delta_4, r_0) \) we see that (3.35) and (3.37) hold. In addition, for each fixed \( k \geq 1 \), and \( t > 0 \), we have from (3.39) that
\[ \E^x \left( u_2(\hat{X}_{t \wedge \hat{\tau}_{D_k(\delta_4, r_0)}}) + t \wedge \hat{\tau}_{D_k(\delta_4, r_0)} \right) \leq u_2(x) \]
Since \( u_2 \geq 0 \), by first letting \( k \to \infty \) and then \( t \to \infty \), we get \( \E^x \left( \hat{\tau}_{D(\delta_4, r_0)} \right) \leq u_2(x) \). Since \( \hat{x} = 0, \psi(x) = 0 \) and therefore \( u_2(x) \leq \rho(x) \). This proves (3.36).

**Step 3.** Deriving exit distribution estimates using sub/super-martingale property

Recall that \( \psi \geq 2^{p+1} \) on \( |y| \geq r_0 \) and \( \psi(x) = 0 \). Therefore by (3.35),
\[ \rho(x) \geq u_2(x) \]
\[ \geq \E^x \left( u_2(\hat{X}_{\hat{\tau}_{D(\delta_4, r_0)}}); X_{\hat{\tau}_{D(\delta_4, r_0)}} \in B \setminus D(\infty, r_0) \right) \]
\[ \geq (2^{p+1} - 1)\P^x \left( X_{\hat{\tau}_{D(\delta_4, r_0)}} \in B \setminus D(\infty, r_0) \right) \]
From (3.37), we have
\[ \rho(x) \leq \rho(x) + \rho(x)^p = u_1(x) \]
\[ \leq \E^x \left( u_1(\hat{X}_{\hat{\tau}_{D(\delta_4, r_0)}}) \right) \]
\[ \leq 2 \P^x \left( \hat{X}_{\hat{\tau}_{D(\delta_4, r_0)}} \in B \right). \]

Combining (3.40) and (3.41), we obtain
\[ \P^x \left( \hat{X}_{\hat{\tau}_{D(\delta_4, r_0)}} \in D(\infty, r_0) \right) = \P^x \left( \hat{X}_{\hat{\tau}_{D(\delta_4, r_0)}} \in B \right) - \P^x \left( X_{\hat{\tau}_{D(\delta_4, r_0)}} \in B \setminus D(\infty, r_0) \right) \]
\[ \geq \frac{2^{p+1} - 3}{2(2^{p+1} - 1)} \rho(x). \]
Now we use the Levy system (3.1) for $\tilde{X}$ to get
\[
\mathbb{P}^x \left( \tilde{X}_{\tau_{D(\delta_4, r_0)}} \in D(\infty, r_0) \setminus D(2\delta_4, r_0) \right)
\leq c\mathbb{E}^x \left( \int_0^{\tau_{D(\delta_4, r_0)}} \mathbb{1}_{\{|\tilde{X}_s-y|<1\}} \frac{1}{\tilde{X}_s-y |d+\alpha|} dyds \right)
\leq c\mathbb{E}^x \left( \int_0^{\tau_{D(\delta_4, r_0)}} \frac{1}{\tilde{X}_s-y |d+\alpha|} dyds \right)
\leq c_6 \left( \int_{D(\infty, r_0) \setminus D(2\delta_4, r_0)} |y|^{-d-\alpha} dy \right) \mathbb{E}^x \left( \tilde{\tau}_{D(\delta_4, r_0)} \right)
\leq c_7 \left( \int_{D(2\delta_4, r_0) \setminus D(3\delta_4/2, r_0)} |y|^{-d-\alpha} dy \right) \mathbb{E}^x \left( \tilde{\tau}_{D(\delta_4, r_0)} \right)
\leq c_8 \mathbb{E}^x \left( \int_0^{\tau_{D(\delta_4, r_0)}} \mathbb{1}_{\{|\tilde{X}_s-y|<1\}} \frac{1}{\tilde{X}_s-y |d+\alpha|} dyds \right)
= c_9 \mathbb{P}^x \left( \tilde{X}_{\tau_{D(\delta_4, r_0)}} \in D(2\delta_4, r_0) \setminus D(3\delta_4/2, r_0) \right). \tag{3.43}
\]
Thus from (3.42) and (3.43),
\[
\mathbb{P}^x \left( \tilde{X}_{\tau_{D(\delta_4, r_0)}} \in D(2\delta_4, r_0) \right) \geq c_{10} \rho(x) \geq c_{11} \delta_B(x). \tag{3.44}
\]
Taking $\delta_0 = \delta_4$ and $c_1 = c_{11}$ gives the estimate (3.19). To obtain (3.20), first recall that $0 \leq h_p \leq 1$. If $|y| > r_0/2$, then $\psi(y) \geq 2^{p+1} \geq 1$. Therefore
\[
u_2(y) = \psi(y) + h(y) - h_p(y) \geq \psi(y) - h_p(y) \geq 1, \text{ for } y \in B(0, r_0/2)^c.
\]
Note also that for $y \in B(0, R_0)$ such that $\delta_4 \leq \rho(y) < R_0$,
\[
u_2(y) = \psi(y) + h(y) - h_p(y) \geq \rho(y) - \rho(y)^p \geq c_{12},
\]
where $c_{12}$ depends on $\delta_4$ and $R$. Therefore using the subharmonicity, we obtain
\[
\rho(x) \geq \nu_2(x) \geq \mathbb{E}^x \left( \nu_2(\tilde{X}_{\tau_{D(\delta_4, r_0)}}) \right) \geq c_{12} \mathbb{P}^x \left( \tilde{X}_{\tau_{D(\delta_4, r_0)}} \in B \right) \tag{3.45}
\]
Since $\rho(x)$ is comparable to $\delta_B(x)$ from above and below, we infer from (3.45) that (3.20) is true (once again with $\delta_0 = \delta_4$). This finishes the proof of the lemma. \qed

We now state and prove the proposition that provides estimates for exit probabilities near the boundary.

**Proposition 3.8.** There exist positive constants $\delta_0 = \delta_0(R_1, M_1, \alpha)$, $c_1 \equiv c_1(R_1, M_1, \alpha)$ and $c_2 \equiv c_2(R_1, M_1, \alpha)$ such that for every $Q \in \partial B$, and $x \in D_Q(2\delta_0, r_0)$ with $\tilde{x} = 0$,
\[
\mathbb{P}^x \left( X_{\tau_{D(\delta_0, r_0)}} \in D(2\delta_0, r_0) \right) \geq c_1 \delta_B(x), \tag{3.46}
\]
\[
\mathbb{P}^x \left( X_{\tau_{D(\delta_0, r_0)}} \in B \right) \leq c_2 \delta_B(x). \tag{3.47}
\]

**Proof.** The estimate (3.46) follows from Lemma 3.7 and the fact that
\[
\mathbb{P}^x \left( X_{\tau_{D(\delta_0, r_0)}} \in D(2\delta_0, r_0) \right) = \mathbb{P}^x \left( \tilde{X}_{\tau_{D(\delta_0, r_0)}} \in D(2\delta_0, r_0) \right) \geq c_{11} \delta_B(x). \tag{3.48}
\]

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To get \((3.47)\) we will once again use Lemma \(3.7\). From that lemma, we know that

\[
\mathbb{P}^x \left( \hat{X}_{D(\delta_0, r_0)} \in B \right) \leq c \delta_B(x).
\]

We would like to obtain a similar bound for the process \(X\). Using the Levy system formula, one has

\[
\mathbb{P}^x \left( X_{D(\delta_0, r_0)} \in B \setminus D(2\delta_0, 2r_0) \right) = \mathbb{E}^x \left[ \int_0^{\tau_{D(\delta_0, r_0)}} \int_{B \setminus D(2\delta_0, 2r_0)} J(x, y) dy ds \right] \leq c_1 \mathbb{E}^x \left[ \int_0^{\tau_{D(\delta_0, r_0)}} \int_{B \setminus D(2\delta_0, 2r_0)} \frac{1}{|x - y|^{d+\alpha}} dy ds \right] \leq c_2 \left( \int_{B \setminus D(2\delta_0, 2r_0)} |y|^{\frac{d}{d-\alpha}} dy \right) \mathbb{E}^x (\tau_{D(\delta_0, r_0)}) \leq c_3 \left( \int_{D(2\delta_0, r_0) \setminus D(3\delta_0/2, 2r_0)} |y|^{-\alpha} dy \right) \mathbb{E}^x (\tau_{D(\delta_0, r_0)}) \leq c_4 \mathbb{E}^x \left[ \int_0^{\tau_{D(\delta_0, r_0)}} \int_{D(2\delta_0, r_0) \setminus D(3\delta_0/2, 2r_0)} J(x, y) dy ds \right] \leq c_5 \mathbb{E}^x (\tau_{D(\delta_0, r_0)}) \leq c_5 \mathbb{P}^x \left( \hat{X}_{D(\delta_0, r_0)} \in D(2\delta_0, r_0) \setminus D(3\delta_0/2, r_0) \right) (3.49)
\]

Therefore we have that

\[
\mathbb{P}^x \left( X_{\tau_{D(\delta_0, r_0)}} \in B \right) = \mathbb{P}^x \left( X_{\tau_{D(\delta_0, r_0)}} \in B \setminus D(2\delta_0, 2r_0) \right) + \mathbb{P}^x \left( \hat{X}_{\tau_{D(\delta_0, r_0)}} \in D(2\delta_0, 2r_0) \right) \leq c_5 \mathbb{P}^x \left( X_{\tau_{D(\delta_0, r_0)}} \in D(2\delta_0, r_0) \setminus D(3\delta_0/2, r_0) \right) + \mathbb{P}^x \left( \hat{X}_{\tau_{D(\delta_0, r_0)}} \in B \right) \leq c_5 \mathbb{P}^x \left( \hat{X}_{\tau_{D(\delta_0, r_0)}} \in D(2\delta_0, r_0) \setminus D(3\delta_0/2, r_0) \right) + \mathbb{P}^x \left( \hat{X}_{\tau_{D(\delta_0, r_0)}} \in B \right) \leq (c_5 + 1) \mathbb{P}^x \left( \hat{X}_{\tau_{D(\delta_0, r_0)}} \in B \right) \leq c_0 \delta_B(x) (3.50)
\]

In the second line above, we have used \((3.49)\) and in the last but one line we have used \((3.20)\). This finishes the proof of the proposition.

We now have all the ingredients to state and prove the following Uniform Boundary Harnack Principle (BHP).

**Theorem 3.9.** Let \(B\) be a fixed ball with characteristics \(R_1\), and \(M_1\) as above. There exists a positive constant \(c_1 \equiv c_1(\alpha, d, R_1, M_1)\) such that for \(Q \in \partial B, r \in (0, R_1)\), and any non-negative function \(u\) on \(\mathbb{R}^d\), that is \(L\)-harmonic in \(B \cap B(Q, r)\), and vanishing continuously on \(B^c \cap B(Q, r)\), we have

\[
\frac{u(x)}{\delta_B(x)} \leq c_1 \frac{u(y)}{\delta_B(y)}, \quad \text{for every } x, y \in B \cap B(Q, r/2).
\]

**Proof.** By the Harnack principle and a chain argument it is sufficient to prove the inequality for \(x, y \in B \cap B(Q, rr_0/8)\). We recall that \(r_0 = \frac{R}{4(1+M_1^2)}\). For any \(r \in (0, R_1)\) and \(y \in \mathbb{R}^d\),
$B \cap B(Q, r_{0}/8)$, let $Q_{y}$ be the point so that $| y - Q_{y} | = \delta_{B}(y)$ and let $y_{0} = Q_{y} + \frac{r}{8}(y - Q_{y})$.

Choose a smooth function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla \phi(0) = 0$, and $| \nabla \phi(x) - \nabla \phi(y) | \leq M_{1} | x - y |$ and an orthonormal coordinate system $CS \equiv CS_{Q}$ with its origins at $Q_{y}$ so that $B(Q_{y}, R_{1}) \cap B = \{ y = (y_{d}, \tilde{y}) \in B(0, R_{1}) \text{ in } CS : y_{d} > \phi(\tilde{y}) \}$. In the above coordinate system $\tilde{y} = 0$, and $y_{0} = (0, r/8)$. For $a_{1}, a_{2} > 0$ define

$$D(a_{1}, a_{2}) = \{ y \in CS : 0 < y_{d} - \phi(\tilde{y}) < a_{1} \frac{r\delta_{0}}{8}, | \tilde{y} | < a_{2} \frac{r_{0}}{8} \}.$$ 

Then it is easy to see that $D(2, 2) \subset B \cap B(Q, r/2)$. Since $u$ is a harmonic function in $B \cap B(Q, r)$ and vanishes continuously in $B \cap B(Q, r)$, it is regular harmonic in $B \cap B(Q, r/2)$ and hence also in $D(2, 2)$ (c.f. Lemma 4.2). Thus by the Harnack inequality we have

$$u(x) = E^{x} \left( u(X_{T_{D(1,1)}}) \right) \geq E^{x} \left( u(X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \in D(2, 1) \right) \geq c_{1} u(x_{0}) E^{x} \left( X_{T_{D(1,1)}} \in D(2, 1) \right) \geq c_{2} u(x_{0}) \delta_{B}(x)/r, \tag{3.51}$$

where in the last line, we used (3.44). Let $w$ be the point with coordinates $(0, r_{0}/16)$. Then observe that there is a positive number $\eta \equiv \eta(M_{1}, \delta_{0}, r_{0}) \in (0, 1)$ such that $B(w, \eta r_{0}/16) \in D(1, 1)$. By the Levy system formula,

$$u(w) \geq E^{w} \left( u(X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \notin D(2, 2) \right) \leq \int_{D(1,1)} G_{D(1,1)}(w, z) \int_{\mathbb{R}^{d} \setminus D(2, 2)} u(y) J(y, z) dy dz \geq c_{3} E^{w} \left( \tau_{B(w, \eta r_{0}/16)} \right) \int_{\mathbb{R}^{d} \setminus D(2, 2)} \frac{u(y)}{|w - y|^{d+\alpha}} dy \geq c_{4} r^{2} \int_{\mathbb{R}^{d} \setminus D(2, 2)} \frac{u(y)}{|w - y|^{d+\alpha}} dy. \tag{3.52}$$

Hence

$$E^{x} \left( u(X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \notin D(2, 2) \right) \leq \int_{D(1,1)} G_{D(1,1)}(x, z) \int_{\mathbb{R}^{d} \setminus D(2, 2)} u(y) J(y, z) dy dz \leq c_{5} E^{x} \left( \tau_{D(1,1)} \right) \int_{\mathbb{R}^{d} \setminus D(2, 2)} \frac{u(y)}{|w - y|^{d+\alpha}} dy \leq c_{6} \delta_{B}(x) r \int_{\mathbb{R}^{d} \setminus D(2, 2)} \frac{u(y)}{|w - y|^{d+\alpha}} dy \leq \frac{c_{6} \delta_{B}(x)}{c_{4} r} u(w). \tag{3.53}$$

On the other hand by the Harnack inequality and the Carleson estimate, we have

$$E^{x} \left( u(X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \in D(2, 2) \right) \leq c_{7} u(x_{0}) E^{x} \left( X_{T_{D(1,1)}} \in D(2, 2) \right) \leq c_{8} u(x_{0}) \delta_{B}(x)/r \tag{3.54}$$

where we used (3.44). Combining the two inequalities, we obtain

$$u(x) = E^{x} \left( u(X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \notin D(2, 2) \right) + E^{x} \left( u(X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \in D(2, 2) \right) \leq \frac{c_{6} \delta_{B}(x)}{c_{4} r} u(w) + \frac{c_{8}}{r} u(x_{0}) \delta_{B}(x) \leq \frac{c_{9}}{r} \delta_{B}(x) (u(x_{0}) + u(w)) \leq \frac{c_{10}}{r} \delta_{B}(x) u(x_{0}) \tag{3.55}$$
Thus by \((3.51)\) and \((3.55)\), we observe that for every \(x, y \in B \cap B(Q, r_0/8)\), we have
\[
\frac{u(x)}{u(y)} \leq \frac{c_{10} \delta_B(x)}{c_2 \delta_B(y)},
\]
which finishes the proof of the theorem.

\section{Green Function Estimates}

Our aim in this section is to prove a 3G-inequality (Proposition \ref{prop:3G}), using which we shall prove Proposition \ref{prop:1.12}. We closely follow the idea of the proofs from \cite{CSKV12} and \cite{CFZ88}. The basic ingredients in the proof of the the 3G estimate are the Carleson estimate (see Theorem \ref{thm:carleson}) and the Boundary Harnack Principle (Theorem \ref{thm:boundary_harnack}). Recall that \(J\) satisfies \((1.4)\). We would like to get upper and lower bound estimates for the Green function \(G_B(x, y)\) for the ball \(B\). We use estimates on the transition density to get these bounds. Let \(p(t, x, y)\) be the transition density for our process \(X_t\). For \(r > 0\) let
\[
p_c(t, r) = e^{-\frac{r^2}{4t^2}} \quad \text{and} \quad p_j(t, r) = t^{-\frac{d}{2}} \wedge \frac{t}{r^{d+\alpha}}.
\]

\textbf{Lemma 4.1.} The transition density \(p(t, x, y)\) satisfies
\[
p(t, x, y) \leq c_1 (t^{\frac{d}{2}} \wedge t^{\frac{d}{2}}) \wedge (p_c(t, c_2 |x - y|) + p_j(t, |x - y|)),
\]
and
\[
p(t, x, y) \geq c_3 t^{\frac{d}{2}}, \quad \text{for} \quad |x - y|^2 \leq t < 1
\]

\textbf{Proof.} See Theorem 1.4 in \cite{CK10}.

\textbf{Lemma 4.2.} Let \(G\) be the Green function for the process \(X_t\). Then,
\begin{enumerate}[(a)]
\item for all \(x, y \in \mathbb{R}^d\), there exist \(c_1 > 0\) such that
\[
G(x, y) \leq \frac{c_1}{|x - y|^{d-2}}. \tag{4.1}
\]
\item for all \(x, y \in \mathbb{R}^d\), with \(|x - y| \leq 7R/8\), there exist \(c_2 > 0\) such that
\[
G(x, y) \geq \frac{c_2}{|x - y|^{d-2}}. \tag{4.2}
\]
\item for all \(x, y\) such that \(|x - y| \leq 7R/8\), there exists \(L \geq 2\) and \(c > 0\) such that
\[
G(x, y) - G(Lx, Ly) \geq \frac{c_3}{|x - y|^{d-2}}. \tag{4.3}
\]
\end{enumerate}

\textbf{Proof.} (a) By definition,
\[
G(x, y) = \int_0^\infty p(t, x, y) dt = \int_0^{[x-y]^2} p(t, x, y) dt + \int_{[x-y]^2}^\infty p(t, x, y) dt = I_1 + I_2 \tag{4.4}
\]
Now, \( I_2 = \int_{|x-y|^2}^{\infty} p(t, x, y) dt \leq c \int_{|x-y|^2}^{\infty} t^{-d} \leq \frac{c_1}{d-2} |x-y|^{2-d}, \)

where we used the bound \( p(t, x, y) \leq c t^{-\alpha} \) from Lemma 4.1. To estimate \( I_1 \) we proceed along the following lines. Suppose first \(|x-y|<1\). Then using Lemma 4.1 again,

\[
I_1 = \int_0^{|x-y|^2} p(t, x, y) dt 
\leq \int_0^{|x-y|^2} p_\epsilon(t, x, y) dt + \int_0^{|x-y|^2} p_j(t, x, y) dt 
\leq c_2 |x-y|^{2-d} + c_3 \int_0^{|x-y|^2} \frac{t}{|x-y|^{d+\alpha}} dt 
= c_2 |x-y|^{2-d} + \frac{c_3}{2} |x-y|^{-d+4} 
\]

(4.5)

In the third line above we have used the fact that \( p_\epsilon \) is a Gaussian density, the fact that \( p_j(t, x, y) = \frac{t}{|x-y|^{d+\alpha}} \) for \( t \leq |x-y|^\alpha \) and that \(|x-y|^2 \leq |x-y|^\alpha \) for \(|x-y|<1\). Since \( \alpha < 2 \) and \(|x-y|<1\), the last line in the above chain is bounded above by \( c_4 |x-y|^{2-d} \). A similar estimate can be proved for \( I_1 \) when \(|x-y|>1\). Combining the two estimates for \( I_1 \) and \( I_2 \) proves (4.1).

(b) By Theorem 3.1 in [CK10], one has \( p(t, x, y) \geq c_1 t^{-\frac{d}{2}} \), for \(|x-y|^{\frac{2}{t}} \leq t \leq 1\). Therefore

\[
G(x, y) = \int_0^\infty p(t, x, y) dt 
\geq \int_{|x-y|^2}^1 c_1 t^{-\frac{d}{2}} dt 
\geq c_2 |x-y|^{2-d} .
\]

(c) Using the bounds in (4.1) and (4.2), we can write

\[
G(x, y) - G(Lx, Ly) \geq (c_2 - \frac{c_1}{L^{d-2}}) |x-y|^{2-d},
\]

for \(|x-y| \leq 7R/8\). Now choose \( L \geq 2 \) such that \( c_2 - \frac{c_1}{L^{d-2}} = c_3 > 0 \). This finishes the proof.

We are now ready to prove the required estimates for \( G_B \). For any \( x \in B \), let \( \delta_B(x) = \text{dist}(x, \partial B) \).

**Lemma 4.3.** Let \( G_B \) denote the Green function for the killed process \( X_t^B \). Then,

\[
G_B(x, y) \leq \frac{c_1}{|x-y|^{d-2}} \text{ for all } x, y \in B \tag{4.6}
\]

and

\[
G_B(x, y) \geq \frac{c_2}{|x-y|^{d-2}} \text{ when } 2 |x-y| \leq \delta_B(x) \wedge \delta_B(y). \tag{4.7}
\]

**Proof.** Since \( G_B(x, y) \leq G(x, y) \) the first part follows from (4.1). Without loss of generality assume that \( \delta_B(y) \leq \delta_B(x) \). For the second part, we divide the proof into two cases. Let \( \tilde{r}_0 = R/8 \) (any positive \( \tilde{r}_0 \) strictly less than \( R \) will also work) and let \( L \) be such that Lemma 4.2 holds.
Case 1: $L \mid x - y \mid \leq \delta_B(y)$ We consider three subcases.

(a) $\delta_B(y) \leq \bar{r}_0$. First observe that $B(y, \delta_B(y)) \subset B$. Then note that since $L \mid x - y \mid \leq \delta_B(y)$, we have $\mid X_{\tau_B(y, \delta(y))} - y \mid \geq \delta_B(y) \geq L \mid x - y \mid$, and therefore

$$G_B(x, y) \geq G_B(y, \delta_B(y))(x, y) = G(x, y) - E^x \left[ G(X_{\tau_B(y, \delta(y))}, y) \right] \geq G(x, y) - \frac{c_1}{L^{d-2}} \mid x - y \mid^{2-d} \geq (c_2 - \frac{c_1}{L^{d-2}}) \mid x - y \mid^{2-d} = c_3 \mid x - y \mid^{2-d}.$$  

(b) $\delta_B(y) > \bar{r}_0$ and $L \mid x - y \mid \leq \bar{r}_0$. In this case, $\mid X_{\tau_B(y, \bar{r}_0)} - y \mid \geq \bar{r}_0 \geq L \mid x - y \mid$. Then,

$$G_B(x, y) \geq G_B(y, \bar{r}_0)(x, y) = G(x, y) - E^x \left[ G(X_{\tau_B(y, \bar{r}_0)}, y) \right] \geq G(x, y) - \frac{c_1}{L^{d-2}} \mid x - y \mid^{2-d} \geq (c_2 - \frac{c_1}{L^{d-2}}) \mid x - y \mid^{2-d} = c_3 \mid x - y \mid^{2-d}.$$  

(c) $\delta_B(y) > \bar{r}_0$ and $L \mid x - y \mid > \bar{r}_0$. In this case, we have $\delta_B(x) \geq \delta_B(y) \geq L \mid x - y \mid > \bar{r}_0$. Choose a point $w \in \partial B(y, \frac{\bar{r}_0}{L})$. Then from the argument in (b) we get

$$G_B(w, y) \geq c_4 \frac{1}{\left( \frac{\bar{r}_0}{L} \right)^{d-2}}.$$  

Now $B$ is connected, Lipschitz, and $\mid x - w \mid \leq \mid x - y \mid + \mid y - w \mid \leq \delta_B(y)/L + \frac{\bar{r}_0}{L}$. Therefore by the Harnack inequality (Proposition 1.6) and a chain argument, we have

$$G_B(x, y) \geq c_5 G_B(w, y) \geq c_6 \frac{1}{\left( \frac{r_0}{2\bar{r}} \right)^{d-2}} \geq c_7 2^{d-2} \frac{1}{\left( \frac{r_0}{2\bar{r}} \right)^{d-2}} \geq c_8 \mid x - y \mid^{2-d},$$  

where in the last inequality, we used that $\mid x - y \mid > \frac{\bar{r}_0}{L}$.

Case 2: $2 \mid x - y \mid \leq \delta_B(y) < L \mid x - y \mid$

Take $x_0 \in \partial B(y, \frac{\delta_B(y)}{L+1})$. Then,

$$\mid x - y \mid \leq \frac{1}{2} \delta_B(y) \leq L \mid x_0 - y \mid = \frac{L}{L + 1} \delta_B(y) \leq \delta_B(x) \wedge \delta_B(y).$$

We also have $\mid x_0 - x \mid \leq \mid x_0 - y \mid + \mid x - y \mid \leq \left( \frac{1}{L+1} + \frac{1}{2} \right) \delta_B(y)$. Therefore, using Harnack inequality (Proposition 1.6) for the $L$ harmonic function $G_B$, and the argument in the Case 1, we obtain

$$G_B(x, y) \geq c_9 G_B(x_0, y) \geq c_{10} \mid x_0 - y \mid^{2-d} \geq c_{11} \mid x - y \mid^{2-d}.$$  

This finishes the proof of the lemma.
Recall Notation 3.1 where φ was the smooth function describing the boundary ∂B. For every $Q \in \partial B$ and $x \in B(Q, R_1) \cap B$, we define

$$\rho_Q(x) = x_d - \phi_Q(\tilde{x}),$$

where $(\tilde{x}, x_d)$ are the coordinates of $x$ in $CS_Q$.

**Notation 4.4.** For a point $x \in B$, let $x^*$ be such that $\delta_B(x) = |x - x^*|$. Let $A_r(x^*)$ be such that $\delta_B(A_r(x^*)) \geq \Lambda r$ and $|A_r(x^*) - x^*| = r$. The constant $\Lambda$ depends only on the Lipschitz nature of $B$. For $0 < r < R_1$, define

$$\tilde{x}_r = \begin{cases} x, & \delta_B(x) \geq \delta_B(A_r(x^*)), \\ A_r(x^*), & \delta_B(x) < \delta_B(A_r(x^*)). \end{cases}$$

(4.10)

We remark that in the case of a ball, there is a canonical way to choose $A_r(x^*)$.

**Proposition 4.5.** Let $G_B$ denote the Green function for the process $X_t$ killed on exiting $B$. Then, there exists a positive constant $c_1$ such that

$$\frac{G_B(x, y)G_B(y, z)}{G_B(x, z)} \leq c_1 \left[ \frac{1}{|x - y|^{d-2}} + \frac{1}{|y - z|^{d-2}} \right] \quad x, y, z \in B.$$

(4.11)

**Proof.** We first prove the following claim.

**Claim:** For $0 < r < R_1$ suppose $|x - y| > 3r$ and $|x - z| > 3r$. If (4.11) holds for $(\tilde{x}_r, y, z)$, then it holds for $(x, y, z)$.

**Proof of Claim:** If $\delta_B(x) \geq \delta_B(A_r(x^*))$, $\tilde{x}_r = x$, so the claim holds trivially. So assume that $\delta_B(x) < \delta_B(A_r(x^*))$. In this case, $c_1 \leq \frac{\delta_B(\tilde{x}_r)}{r} \leq c_2$ and by Theorem 3.9

$$\frac{G_B(\tilde{x}_r, y)}{G_B(\tilde{x}_r, z)} \leq c_3 \frac{G_B(x, y)}{G_B(x, z)} \leq c_4 \frac{G_B(\tilde{x}_r, y)}{G_B(\tilde{x}_r, z)}.$$

However

$$|\tilde{x}_r - y| \geq |x - y| - |x - \tilde{x}_r| \geq |x - y| - r \geq \frac{2}{3} |x - y|.$$

This implies that (4.11) holds for $(x, y, z)$.

Without loss of generality we may assume that $\delta_B(x) \leq \delta_B(z)$. Let $c_1 > 0$ be such that $c_1 |x - z| < R_1$ for all $x, z \in B$. Set $c_2 = (\frac{1}{c_1} + 2)$.

**Case 1.** $|x - z| \leq \frac{\delta_B(x)}{2}$.

If $|x - z| \leq \frac{\delta_B(x)}{2}$, then by (4.2) we have $G_B(x, z) \geq c_3 |x - z|^{2-d}$.

Then using (4.1) we conclude that

$$\frac{G_B(x, y)G_B(y, z)}{G_B(x, z)} \leq c_4 \frac{|x - y|^{2-d} |y - z|^{2-d}}{|x - z|^{2-d}},$$

which implies (4.11). On the other hand, when $|x - z| > \frac{\delta_B(x)}{2}$, select a point $\tilde{z}$ with $|x - \tilde{z}| = \delta_B(x)/2$. Then by the Harnack inequality (Proposition 1.6)

$$c_5 \leq \frac{G_B(x, z)}{G_B(x, \tilde{z})} \leq \frac{1}{c_5},$$

and the lower bound in (4.2) we have $G_B(x, \tilde{z}) \geq c_6 |x - \tilde{z}|^{2-d}$. Now $\delta_B(x)/2 < |x - z| < c_7 \delta_B(x)$, and $|x - \tilde{z}| = \delta_B(x)/2$. Thus

$$|x - \tilde{z}| \leq |x - z| \leq c_8 |x - \tilde{z}|$$

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and (4.11) holds.

**Case 3.** $|x-y| < \frac{c_{1}+1}{A_{c_{1}}} \delta_B(x)$, $|x-z| > \frac{c_1}{A} \delta_B(x)$.

In this case, $|z-x^*| \geq (\frac{A}{A_{c_{1}}} - 1) \delta_B(x)$ and $|y-x^*| < (1 + \frac{c_{1}+1}{A_{c_{1}}}) \delta_B(x)$. Now when $\delta_B(y) < \delta_B(x)$, we use the Carleson estimate (3.3) to conclude $G_B(y, z) \leq c_{9}G_B(A_{\delta_B(x)}(y^*), z)$. However using a standard chain of balls argument and the Harnack inequality (Proposition 1.6) in $B \setminus \{z\}$ from $A_{\delta_B(x)}(y^*)$ to $x$ (with length of chain independent of $\delta_B(x)$), we obtain

$$c_{10} \leq \frac{G_B(x, z)}{G_B(A_{\delta_B(x)}(y^*), z)} \leq c_{11}.$$  

Thus $G_B(y, z) \leq c_{12}G_B(x, z)$ and with upper bound (4.11) applied to $G_B(x, y)$ we obtain (4.11).

**Case 3.** $|x-z| > \frac{c}{A} \delta_B(x)$, $|y-z| > 2 |x-z|$, $|x-y| > \frac{c_{1}+1}{A_{c_{1}}} \delta_B(x)$.

Set $r = c_{1} |x-z|$ so that $r < R_{1}$. Then $|y-z| \geq 3r$, $|x-z| \geq 3r$ and by earlier claim, it suffices to consider $(x, y, \tilde{z}_r)$. Observing that

$$|x-y| \geq |y-z| - |x-z| \geq |x-z| \geq 3r$$

and

$$|x-\tilde{z}_r| \geq |x-z| - r \geq (9-1)r \geq 3r$$

Again the claim proved earlier will imply that it suffices to consider $(\tilde{x}_r, y, \tilde{z}_r)$. But

$$|\tilde{x}_r - \tilde{z}_r| \leq |x-z| + 2r \leq (c_{1}^{-1} + 2)r \leq \frac{c_{2}}{c_{1}} \delta_B(\tilde{x}_r) \wedge \delta_B(\tilde{z}_r),$$

that is, we are back in Case 1 and (4.11) holds.

**Case 4.** $|x-z| > \frac{c}{A} \delta_B(x)$, $|x-y| > \frac{c_{1}+1}{A_{c_{1}}} \delta_B(x)$, $|y-z| < 2 |x-z|$.  

Set $r = c_{1} |x-y|$ and

$$|x-z| \geq |x-y| - |y-z| \geq |x-y| - 2 |x-z|$$

which implies that $|x-z| \geq \frac{1}{4} |x-y| \geq 3r$, we see that Claim applies again. Having switched to $(\tilde{x}_r, y, z)$, we note that

$$|\tilde{x}_r - y| \leq |x-y| + r = \frac{r}{c_{1}^{-1}} + r \leq \frac{c_{1}+1}{A_{c_{1}}} \delta_B(\tilde{x}_r).$$

If $\delta_B(\tilde{x}_r) < \delta_B(z)$ we are in Case 1 or Case 2 and we are done. If $\delta_B(\tilde{x}_r) > \delta_B(z)$ we would be done if $(z, y, \tilde{x}_r)$ satisfies the conditions of either of the first three cases. So we may assume the worst case scenario, that $(z, y, \tilde{x}_r)$ falls into Case 4. However in that case, we may set $s = c_{1} |y-z|$ and use the same argument as the first part of Case 4 and it will follow that $(\tilde{z}_s, y, \tilde{x}_r)$ or $(\tilde{x}_r, y, \tilde{z}_s)$ satisfies Case 1 or 2 and this completes the proof.

We are now ready to prove Proposition 4.12.

**Proof of Proposition 4.12:** Theorem 3.1 in [CS03] contains a proof of the proposition provided $q$ satisfies

**Assumption (Q1):** There exists a Borel set $K \subset B$ of finite measure and a $\delta > 0$ such that

$$\beta(q) := \sup_{F \subset K, m(F) < \delta} \left( \sup_{(x,z) \in (B \times B) \setminus F} \int_{(B \setminus K) \setminus F} \frac{G_B(x, y)G_B(y, z)}{G_B(x, z)} \ |q(y)| \ dy \right) < 1,$$

where $D$ denotes the diagonal $D = \{(x,x) : x \in B\}$.
Using Proposition 4.5 there is a $c_3 > 0$ such that
\[ \frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \leq c_3 \left( |x-y|^{2-d} + |y-z|^{2-d} \right) \] (4.12)
whenever $x, y, z \in B$. Therefore if $q : \mathbb{R}^d \to \mathbb{R}$ is such that it satisfies (Q) then $q$ satisfies (Q1) thereby finishing the proof.

We conclude the section with an estimate on the boundary behavior of the Green Function. We shall prove that the Green function $G_B(x,y)$ decays at least like $\delta_B(y)$ as $y$ approaches the boundary $\partial B$.

**Proposition 4.6.** Let $x$ be a fixed point in $B$. Then, there exists a positive constant $c_1 \equiv c_1(\alpha, R_1, M_1)$ such that for $r \in (0, R_1/2]$, and for $Q \in \partial B$,
\[ G_B(x,y) \geq c_1 \frac{\delta_B(y)}{r}, \]
for $y \in B \cap B(Q, rr_0/8)$.

**Proof.** For any $r \in (0, R_1/2]$ and $y \in B \cap B(Q, rr_0/8)$, let $Q_y$ be the point so that $|y - Q_y| = \delta_B(y)$ and let $y_0 = Q_y + \frac{r(Q_y - y)}{|y - Q_y|}$. Choose a smooth function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla \phi(0) = 0$, and $|\nabla \phi(x) - \nabla \phi(y)| \leq M_1 |x-y|$ and an orthonormal coordinate system $CS \equiv CS_Q$ with its origins at $Q_y$ so that $B(Q_y, R_1) \cap B = \{ y = (y_d, \tilde{y}) \in B(0, R_1) \text{ in } CS : y_d > \phi(\tilde{y}) \}$. In the above coordinate system $\tilde{y} = 0$, and $y_0 = (0, r/8)$. For $a_1, a_2 > 0$ define
\[ D(a_1, a_2) = \{ y \in CS : 0 < y_d - \phi(\tilde{y}) < a_1 \frac{r_0}{8}, |\tilde{y}| < a_2 \frac{rr_0}{8} \}. \]
Then it is easy to see that $D(2, 2) \subset B \cap B(Q, r/2)$. Since $G_B(x, \cdot)$ is a harmonic function in $B \cap B(Q, r)$ and vanishes continuously in $B \cap B(Q, r)$, it is regular harmonic in $B \cap B(Q, r/2)$ and hence also in $D(2, 2)$ (can be shown as in Lemma 4.2 [CSKV12a]). Thus by the Harnack inequality
\[
G_B(x,y) = \mathbb{E}_y \left( G_B(x, X_{T_{D(1,1)}}) \right) \\
\geq \mathbb{E}_y \left( G_B(x, X_{T_{D(1,1)}}); X_{T_{D(1,1)}} \in D(2,1) \right) \\
\geq c_1 G_B(x, y_0) \mathbb{P}_y \left( X_{T_{D(1,1)}} \in D(2,1) \right) \\
\geq c_1 G_B(x, y_0) \frac{\delta_B(y)}{r},
\] (4.13)
where the last inequality is a consequence of Proposition 3.8. This concludes the proof. \qed

## 5 Martin Boundary, Density of Harmonic measure

Our aim in this section is to prove Proposition 1.10. For this we need to understand the exit distribution of the process $X$. This requires us to introduce the notion of Martin boundary and show that it can be identified with the Euclidean boundary in the case of a ball. We also prove that the Martin Kernel gives the density of harmonic measure.

The notations and techniques are borrowed entirely from the literature ([B98], [CSKV12a], [BB00]). Fix $x_0 \in B$. Define
\[ M(x,y) = \frac{G_B(x,y)}{G_B(x_0,y)}, x, y \in B. \]
Note that $M(x_0, y) = 1$ for $y \in B$, $y \neq x_0$. 

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**Notation 5.1.** We define the oscillation of a function \( f : \mathbb{R}^d \to \mathbb{R} \) on a set \( A \subset \mathbb{R}^d \) by
\[
Osc_A f = \sup_{x \in A} f(x) - \inf_{x \in A} f(x).
\]

Let \( x \in \partial B \). Define the box
\[
Q(x, a, b) = \{ y \in B : \rho(y) < a, |(y_1, y_2, \ldots, y_d) - (x_1, x_2, \ldots, x_d)| < b \},
\]
for \( a > 0, b > 0 \) and upper side of the box
\[
U(x, a, b) = \{ y \in \partial Q(x, a, b) : \rho(y) = a \}.
\]

Our first result concerns the regularity of \( M(x, \cdot) \) in a neighborhood of the boundary \( \partial B \).

**Proposition 5.2.** \( M(x, y) \) is uniformly continuous in \( y \) for \( y \) in a neighborhood of \( \partial B \).

**Proof.** Pick \( w \in \partial B \) and choose \( r \) small enough so that \( B \cap B(w, r) \) is the intersection of \( B(w, r) \) with the region above the graph of a Lipschitz function. We note that \( r \) depends on \( B \), but can be chosen independently of \( w \). Fix a coordinate system as in Notation 3.1 and pick \( k_0 \) large enough that \( Q(w, 2^{-k_0}, 2^{-k_0}) \subset B(w, r) \) and \( x, x_0 \notin Q(w, 2^{-k_0}, 2^{-k_0}) \).

Write \( Q_k = Q(w, 2^{-k}, 2^{-k}) \) and \( h(y) = G_B(x_0, y) \). We will show that the oscillation of \( \frac{G_B(x, \cdot)}{h} \) on \( Q_{k+1} \) is controlled by the oscillation of \( \frac{G_B(x, \cdot)}{h} \) on \( Q_k \). Both \( h \) and \( G_B(x, \cdot) \) are harmonic functions on \( B - \{(x, x_0)\} \) that vanish on \( \partial B \). By the Harnack inequality (Proposition 1.6) they both are nonzero and finite for any \( y \in Q_{k_0} \). The Boundary Harnack Principle (Theorem 3.9) in \( Q_k \), now gives that \( \frac{G_B(x, \cdot)}{h} \) is bounded above and below by positive constants.

Fix \( k \geq k_0 \). For \( y \in B \), define \( u(y) = \alpha G_B(x, y) + \beta h(y) \), where \( \alpha \) and \( \beta \) are real numbers so that
\[
\sup_{Q_k} \left( \frac{u}{h} \right) = 1, \quad \inf_{Q_k \setminus \{h \}} \left( \frac{u}{h} \right) = 0, \quad \text{implying} \quad Osc_{Q_k} \left( \frac{u}{h} \right) = 1.
\]
Clearly \( u \) is harmonic in \( Q_k \). Now pick \( z_k \in U(w, 2^{-k}, 2^{-k}) \). If \( \frac{u(z_k)}{h(z_k)} \leq 1/2 \), replace \( u \) by \( h - u \).

So we may assume \( \frac{u(z_k)}{h(z_k)} \geq 1/2 \) without changing the supremum and infimum of \( u/h \) on \( Q_k \). By Theorem 3.9 if \( y \in Q_{k+1} \)
\[
\left( \frac{u}{h} \right)(y) \geq c_1 \left( \frac{u}{h} \right)(z_k) \geq c_1/2.
\]

So
\[
Osc_{Q_{k+1}} \left( \frac{u}{h} \right) \leq 1 - c_1/2 := \gamma
\]
Undoing the algebra, we have
\[
Osc_{Q_{k+1}} \left( \frac{G_B(x, \cdot)}{h(\cdot)} \right) \leq \gamma Osc_{Q_k} \left( \frac{G_B(x, \cdot)}{h(\cdot)} \right)
\]
and \( \gamma < 1 \). Now \( \frac{G_B(x, \cdot)}{h(\cdot)} \) is bounded by \( c_2 \) on \( Q_{k_0} \) by Theorem 3.9.

So
\[
Osc_{Q_{k+1}} \left( \frac{G_B(x, \cdot)}{h(\cdot)} \right) \leq c_2 \gamma^{k-k_0},
\]

or in other words,
\[
\left| \frac{G_B(x, y)}{h(y)} - \frac{G_B(x, y')}{h(y')} \right| \leq c_3 \gamma^k
\]
if \( y, y' \in Q(w, 2^{-k}, 2^{-k}) \). This implies the Hölder continuity of \( \frac{G_B(x, y)}{h(y)} \) which in turn implies uniform continuity of \( M(x, \cdot) \). \( \square \)
One consequence of Proposition [3.2] is that, if \( y \to z \in \partial B \), then \( \frac{G_B(x,y)}{G_B(x_0,y)} = M(x,y) \) converges. We denote this limit by \( M(x,z) \) and refer to it as the Martin Kernel. The next order of business is the so called Martin Boundary. For the existence of the Martin Boundary, we refer the reader to [KW65]. We summarize the result below as a lemma.

**Lemma 5.3.** [Kunita-Watanabe] There is a compactification \( B_M \) of \( B \), unique up to homeomorphism, such that \( M(x,y) \) has a continuous extension to \( B \times (B_M \setminus \{x_0\}) \) and \( M(\cdot, z_1) = M(\cdot, z_2) \) iff \( z_1 = z_2 \).

The set \( \partial B_M = B_M \setminus B \) is called the Martin Boundary for \( X^B \). For \( z \in \partial B_M \), we set \( M(\cdot, z) = 0 \) on \( B^c \). Now \( B_M \) is the smallest compact set for which \( M(x,y) \) is continuous in the extended sense in \( y \). By Proposition [3.2] \( M(x,\cdot) \) is uniformly continuous in a neighborhood of the boundary \( \partial B \), and so this implies that \( B_M \subset \bar{B} \), and we can identify the Martin boundary with a subset of the Euclidean boundary.

**Lemma 5.4.** Let \( z_1, z_2 \in \partial B \). If \( M(\cdot, z_1) \equiv M(\cdot, z_2) \), then \( z_1 = z_2 \).

**Proof.** First we will show that \( M(\cdot, z_1) \) vanishes on \( \partial B - \{z_1\} \). Fix \( y_0 \in B \), with \( y_0 \neq x_0 \). Let \( w \in \partial B, w \neq z_1 \). Note that since \( G_B(x,y_0) \to 0 \) as \( x \to w \), given \( \epsilon > 0 \) there is a \( \delta < |w - z_1|/4 \) such that \( G_B(x,y_0) \leq \epsilon \) if \( |x - w| \leq \delta \). Now applying Theorem [3.9] in \( B \setminus (B(x,\delta/2) \cup B(x_0,\delta/2)) \),

\[
\frac{G_B(x,y)}{G_B(x_0,y)} \leq c_1 \frac{G_B(x_0,y)}{G_B(x_0,y_0)} \leq \frac{c_1 \epsilon}{G_B(x_0,y_0)},
\]

if \( |x - w| \leq \delta \). Letting \( y \to z_1 \), we see that \( M(x,z_1) \to 0 \) as \( x \to w \neq z_1 \). Furthermore the convergence is uniform on compact subsets of \( \partial B - \{z_1\} \).

We next show that there is exactly one measure \( \mu = \delta_{z_1} \) supported on \( \partial B \) such that \( M(x,z_1) = \int M(x,w)\mu(dw) \). If we show this, then \( M(x,z_1) = M(x,z_2) \) will imply that \( \delta_{z_1} = \mu = \delta_{z_2} \), in other words, \( z_1 = z_2 \).

So suppose that \( M(x,z_1) = \int \mu \omega(z_1,w)\mu(dw) \). Then \( M(x_0,z_1) = 1 = \mu(\partial B) \). If \( \mu \neq \delta_{z_1} \), there exists \( \epsilon > 0 \) such that \( \mu_t = \mu |_{B(z_1,\epsilon)} \neq 0 \). Then \( \psi(x) = \int M(x,w)\mu_t(dw) \) is a harmonic function (see lemma [3.11] below), bounded above by \( M(x,z_1) \).

In the first paragraph of this lemma, we showed that if \( |w - z_1| \geq \epsilon \), then \( M(x,w) \to 0 \) as \( x \to w' \in \partial B \cap B(z_1,\epsilon/2) \). On the other hand, if \( x \to w' \in \partial B \cap B(z_1,\epsilon/2) \), then

\[
\int M(x,w)\mu_t(dw) \leq \int M(x,w)\mu(dw) = M(x,z_1) \to 0
\]

We have shown so far that \( \lim_{x \to z} v(x) = 0 \) for all \( z \in \partial B \). We also know from the definition of the Martin kernel that \( v = 0 \) in \( B^c \). We shall need the following Lemma

**Lemma 5.5.** Let \( h : \mathbb{R}^d \to \mathbb{R} \) be a bounded \( \mathcal{L} \) harmonic function in \( B \). Suppose that \( h = 0 \) in \( B^c \) and \( \lim_{x \to z} h(x) = 0 \) for all \( z \in \partial B \). Then, \( h \) is identically 0.

Assuming this fact, let us continue with the argument. Applying Lemma [3.3] we infer that \( v(x) = \int M(x,w)\mu_t(dw) = 0 \). An application of Theorem [3.9] lets us conclude that \( \frac{G_B(x,y)}{G_B(x_0,y)} \) stays bounded below by a positive constant as \( y \to \partial B \). Therefore \( M(x,w) \) is positive for all \( w \). But this must mean that \( \mu_t = 0 \). Since \( \epsilon \) was arbitrary, \( \mu(\{z_1\}) = 0 \). Recalling that \( \mu(\partial B) = 1 \), we arrive at \( \mu = \delta_{z_1} \) as was to be shown.
Proof of Lemma 5.5: Let $D_n$ be an increasing sequence of open sets such that $D_n \subset D_{n+1}$ and $B = \bigcup_n D_n$. Set $\tau_{D_n} = \tau_n$ for simplicity of notation. Note that $\tau_n \uparrow \tau_B$. Now for any $x \in B$, using harmonicity of $h$ and the bounded convergence theorem we have

$$h(x) = \lim_{n \to \infty} E^x (h(X_{\tau_n}); \tau_n < \tau_B) = E^x \left( \lim_{n \to \infty} h(X_{\tau_n}); \tau_n < \tau_B \right) = 0 \quad (5.1)$$

which proves Lemma 12.

Combining the remark following Lemma 5.3 and Lemma 5.4, we obtain the following result

**Theorem 5.6.** There is a one-to-one mapping of the Martin boundary $\partial B_M$ onto the Euclidean boundary $\partial B$, i.e the Martin boundary is the same as the Euclidean boundary.

We next need the Martin representation for positive $L$-harmonic functions, for which we need to quote an abstract result from the general theory of Markov processes.

**Lemma 5.7.** Every positive $L$-harmonic function $h$ in the ball $B$ can be represented as

$$h(x) = \int_{\partial B} M(x, y)\nu(dy),$$

for some positive measure $\nu$ concentrated on the boundary $\partial B$.

*Proof.* We first check that our process $X_t$ satisfies condition (C) in [KW65]. Then we apply Theorem 1 in [KW65] in combination with Theorem 5.6 above to obtain our required result.

The next three lemmas are proved in [CSKV12a]. For the reader’s convenience we present the proof here as well.

**Lemma 5.8.** For every $z \in \partial B$ and every open set $U \subset \bar{U} \subset B$, $M(X_{\tau_U}, z)$ is $\mathbb{P}^x$ integrable.

*Proof.* Take a sequence $z_n$ in $B \setminus \bar{U}$ such that $z_n \to z$. Then using the fact that $M(\cdot, z_n)$ is regular harmonic in $U$, and Fatou’s lemma, we have

$$M(x, z) = \lim_{n \to \infty} M(x, z_n) = \lim_{n \to \infty} \mathbb{E}^x(M(X_{\tau_U}, z_n)) \geq \mathbb{E}^x(M(X_{\tau_U}, z)) \quad (5.2)$$

Since $M(x, z)$ is finite, this concludes the proof of the lemma.

**Lemma 5.9.** For every $z \in \partial B$ and $x \in B$,

$$M(x, z) = \mathbb{E}^x \left( M(X_{\tau_{B(x,r)}}, z) \right),$$

for every $0 < r < \frac{1}{2}(R_1 \wedge \delta_B(x)).$

*Proof.* Fix $z \in \partial B$, $x \in B$, and $r < \frac{1}{2}(R_1 \wedge \delta_B(x))$. From the hypothesis on $r$ we have that $B(z, r) \cap B \subset B \setminus B(x, r)$. Set $\eta_m = \frac{1}{2}r$ and let $z_m \in B$ be the point on the radial line joining $z$ to the center $x_0$ of $B$, such that $d(z_m, z) = \frac{3}{4} \eta_m$. Then $z_m \in B(z, \eta_m)$. Let $\tau_r = \tau_{B(x,r)}$ be exit time from $B(x, r)$. By the harmonicity of $M(\cdot, z_m)$, we have that

$$M(x, z_m) = \mathbb{E}^x (M(X_{\tau_r}, z_m)) \quad (5.3)$$
If we let \( m \to \infty \) in (5.3), we obtain \( M(x, z) \) on the left hand side. We will show that the random variables \( M(X_{\tau}, z_m) \) are uniformly integrable, so that we can take the limit inside the expectation in the right side of (5.3).

As a preliminary, observe that from Theorem 3.9 we know there exists \( c_1 > 0 \) and \( m_0 \in \mathbb{N} \) such that for \( m \geq m_0 \)

\[
M(w, z_m) = \frac{G_B(w, z_m)}{G_B(x_0, z_m)} \leq c_1 \frac{G_B(w, y)}{G_B(x_0, y)} = c_1 M(w, y),
\]

for \( w \in B \setminus B(z, \eta_m), \ y \in B(z, \eta_{m+1}) \). Letting \( y \to z \in \partial B \), we obtain

\[
M(w, z_m) \leq c_1 M(w, z), \ m \geq m_0,
\]

for \( w \in B \setminus B(z, \eta_m) \). Next, from Lemma 5.8, we know that \( M(X_{\tau}, z) \) is \( \mathbb{P}^x \) integrable. Therefore given \( \epsilon > 0 \), there exists \( N_0 > 1 \) such that for \( n \geq N_0 \),

\[
\mathbb{E}^x \left( M(X_{\tau}, z); M(X_{\tau}, z) > \frac{N_0}{c_1} \right) \leq \frac{\epsilon}{4c_1}.
\]

By (5.5) and (5.6),

\[
\mathbb{E}^x (M(X_{\tau}, z_m); M(X_{\tau}, z_m) > N_0, \text{ and } X_{\tau} \in B \setminus B(z, \eta_m)) \\
\leq c \mathbb{E}^x (M(X_{\tau}, z_m); c_1 M(X_{\tau}, z) > N_0) \\
\leq c_1 \frac{\epsilon}{4c_1} = \frac{\epsilon}{4}.
\]

To deal with the other term, we use the Levy system formula (5.11) and write

\[
\mathbb{E}^x (M(X_{\tau}, z_m); X_{\tau} \in B \cap B(z, \eta_m)) \\
= \int_{B \cap B(z, \eta_m)} M(w, z_m) \int_{B(x, r)} G_B(x, y) J(y, w) dy dw \\
\leq c_2 \int_{B \cap B(z, \eta_m)} M(w, z_m) \int_{B(x, r)} |x - y|^{2-d} J(y, w) dy dw.
\]

Recall that \( J(y, w) \leq c_3 |y - w|^{-\alpha - d} \). We have \( |x - w| \geq |x - z| - |z - w| \geq 2r - \frac{r^d}{4} = \frac{3}{4} r \). Ultimately we can estimate \( |y - w| \geq |x - w| - |x - y| \geq \frac{3}{4} r - r = c_4 r \). Plugging this last information into (5.8), and using spherical coordinates to compute \( \int_{B(x, r)} |x - y|^{2-d} = r^2 \), we obtain

\[
\mathbb{E}^x (M(X_{\tau}, z_m); X_{\tau} \in B \cap B(z, \eta_m)) \leq c_5 r^{2-\alpha - d} \int_{B(z, \eta_m)} M(w, z_m) dw
\]

Now using the definition of \( M \) we rewrite the integral as

\[
\int_{B(z, \eta_m)} M(w, z_m) dw = \frac{\int_{B(z, \eta_m)} G_B(w, z_m) dw}{G_B(x_0, z_m)}.
\]

Note that \( B(z, \eta_m) \subset B(z_m, 2\eta_m) \). Therefore \( \int_{B(z, \eta_m)} G_B(w, z_m) dw \leq \int_{B(z_m, 2\eta_m)} G_B(w, z_m) dw \). Using the upper bound \( G_B(w, z_m) \leq c_6 |w - z_m|^{2-d} \) and converting the integral into polar coordinates, we obtain

\[
\int_{B(z, \eta_m)} G_B(w, z_m) dw \leq \int_{B(z_m, 2\eta_m)} G_B(w, z_m) dw \leq c_6 \eta_m^2.
\]
On the other hand, from Proposition 4.6, we know that
\[ G_B(x_0, z_m) \geq c_7 \delta_B(z_m) \geq c_8 \eta_m. \]
Plugging the last two displays into equation (4.9), we infer that
\[ \mathbb{E}^x (M(X_{\tau_U}, z_m); X_{\tau_U} \in B \cap B(z, \eta_m)) \leq c_9 \eta_m \rightarrow 0 \]
as \( m \rightarrow \infty \). Together with (5.8), this shows that the family \( M(X_{\tau_U}, z_m) \) is uniformly integrable and concludes the proof of the lemma.

Lemma 5.10. Let \( z \in \partial B \). Then, the function \( x \rightarrow M(x, z) \) is harmonic in \( B \) with respect to the process \( X \).

Proof. Fix \( z \in \partial B \). Denote \( h(x) = M(x, z) \). Let \( U \) be an open set with \( U \subset \bar{U} \subset B \). For \( x \in U \), let us define
\[
r(x) = \frac{1}{2} (R_1 \wedge \delta_B(x)), \quad \text{and} \quad B(x) = B(x, r(x)).
\]
Now define a sequence of stopping times \( \{T_n\} \) as follows.
\[
T_1 = \inf \{ t > 0 : X_t \notin B(X_0) \},
\]
and for \( n \geq 2 \),
\[
T_n = \begin{cases} 
T_{n-1} + \tau_B(X_{T_{n-1}}) \circ \theta_{T_{n-1}} - \tau_U, & \text{if } X_{T_{n-1}} \in U, \\
\tau_U, & \text{otherwise}
\end{cases}
\]
(5.11)
Observe that \( X_{\tau_U} \in \partial U \) on the set \( \bigcap_n \{ T_n < \tau_U \} \). Therefore, since \( T_n \rightarrow \tau_U \) as \( n \rightarrow \infty \), \( \mathbb{P}^x \) almost surely, and \( h \) is continuous in \( B \), we have
\[
\lim_n h(X_{T_n}) = h(X_{\tau_U}) \text{ on } \bigcap_n \{ T_n < \tau_U \}.
\]
Next using that \( h \) is bounded in \( \bar{U} \), and the dominated convergence theorem, we obtain
\[
\lim_n \mathbb{E}^x \left( h(X_{T_n}); \bigcap_n \{ T_n < \tau_U \} \right) = \mathbb{E}^x \left( h(X_{\tau_U}); \bigcap_n \{ T_n < \tau_U \} \right).
\]
Therefore, using Lemma 5.9
\[
h(x) = \lim_n \mathbb{E}^x \left( h(X_{T_n}) \right)
= \lim_n \mathbb{E}^x \left( h(X_{T_n}); \bigcup_n \{ T_n = \tau_U \} \right) + \lim_n \mathbb{E}^x \left( h(X_{T_n}); \bigcap_n \{ T_n < \tau_U \} \right)
= \mathbb{E}^x \left( h(X_{\tau_U}); \bigcup_n \{ T_n = \tau_U \} \right) + \mathbb{E}^x \left( h(X_{\tau_U}); \bigcap_n \{ T_n < \tau_U \} \right)
= \mathbb{E}^x \left( h(X_{\tau_U}) \right).
\]
(5.12)
This shows that \( h \) is harmonic and finishes the proof of the Lemma.

We will need another simple lemma that follows from Lemma 5.10.

Lemma 5.11. Let \( \mu \) be a finite measure supported in \( \partial B \). Then the function \( h(x) = \int_{\partial B} M(x, w) \mu(dw) \) is harmonic in \( B \).
Proof. Let $U \subset \bar{U} \subset B$. We have to show that $h(x) = \mathbb{E}^x(h(X_{\tau_U}))$ for $x \in U$. To start with, we recall that $M(\cdot, z)$ is defined to be $0$ in $B^c$. Then for $x \in U$, we can write

$$
\mathbb{E}^x(h(X_{\tau_U})) = \mathbb{E}^x(h(X_{\tau_U}); X_{\tau_U} \in B) + \mathbb{E}^x(h(X_{\tau_U}); X_{\tau_U} \notin B)
$$

$$
= \mathbb{E}^x(h(X_{\tau_U}); X_{\tau_U} \in B) + 0
$$

$$
= \mathbb{E}^x \left( \int_{\partial B} M(X_{\tau_U}, w)\mu(dw) \right)
$$

$$
= \int_{\partial B} \mathbb{E}^x(M(X_{\tau_U}, w))\mu(dw)
$$

$$
= \int_{\partial B} M(x, w)\mu(dw)
$$

$$
= h(x),
$$

(5.13)

where we used lemma 5.10 to go from the fourth line to the fifth. This shows that $h$ is harmonic.

Next, we will introduce the notion of harmonic measure and derive the density. Let $\omega^x(\cdot, A) = \mathbb{E}^x(1_A; \tau_B = \infty)$, where $A \subset \partial B$ is a Borel set. For fixed $A \subset \partial B$, $\omega(\cdot, A)$ is a positive harmonic function in $B$ vanishing on the complement of $B$. On the other hand, for fixed $x$, $\omega(x, \cdot)$ is a measure supported on $B^c$. In general the harmonic measure will be supported on $B^c$, but we will restrict our attention to the measure on Borel subsets of the boundary $\partial B$. If $x, x_0 \in B$, the Harnack inequality for $L^2$ harmonic functions says $\omega^x$ is absolutely continuous with respect to $\omega^{x_0}$. Hence we know that a density exists, $\omega^x(dy) = f(x, y)\omega^{x_0}(dy)$. The following theorem identifies this density. We recall that $M$ denotes the Martin kernel defined by

$$
M(x, y) = \frac{G_B(x, y)}{G_B(x_0, y)},
$$

where $x_0$ is a fixed reference point and $x, y \in B$. By the remark following Proposition 5.2 we know that $\lim_{y \to z} M(x, y)$ exists for $z \in \partial B$. We denote this limit by $M(x, z)$. Now we state our theorem.

**Theorem 5.12.** Let $A \subset \partial B$ be a Borel set. Then,

$$
\omega(x, A) = \int_A M(x, z)\omega(x_0, dz).
$$

Proof. Fix $y_0 \in \partial B$. For each $k \in \mathbb{N}$, let $Q_{y_0k}$ denote the cube of the form $[\frac{j_1}{2}, \frac{j_1+1}{2}] \times [\frac{j_2}{2}, \frac{j_2+1}{2}] \times \cdots \times [\frac{j_d}{2}, \frac{j_d+1}{2}]$ that contains $y_0$ and where $j_1, \ldots, j_d$ are integers. Define

$$
h_k(x) = \frac{\omega(x, Q_{y_0k} \cap \partial B)}{\omega(x_0, Q_{y_0k} \cap \partial B)}.
$$

Note that $h_k$ is harmonic in $B$ and $h_k(x_0) = 1$. From the Martin representation theorem, we know that there exists a finite measure $\nu_k$ such that

$$
h_k(x) = \int_{\partial B} M(x, z)\nu_k(dz).
$$

Since $h_k(x_0) = 1$ and $M(x_0, z) = 1$ for all $z \in \partial B$, we can infer that $\nu_k(\partial B) = 1$. Next, we know from the Lebesgue theorem for Radon measures, that the density $f$ can be expressed as

$$
\lim_{k \to \infty} h_k(x) = \lim_{k \to \infty} \frac{\omega(x, Q_{y_0k} \cap \partial B)}{\omega(x_0, Q_{y_0k} \cap \partial B)} = f(x, y_0).
$$

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We will now show that this limit is also \( M(x, y_0) \). Indeed, first observe that since \( \nu_k(\partial B) = 1 \), we can write
\[
h_k(x) - M(x, y_0) = \int_{\partial B} M(x, z)\nu_k(dz) - \int_{\partial B} M(x, y_0)\nu_k(dz).
\]

Therefore
\[
| h_k(x) - M(x, y_0) | \leq \int_{\partial B \cap B(y_0, \delta_0)} | M(x, z) - M(x, y_0) | \nu_k(dz) + \int_{\partial B \cap B(y_0, \delta_0)^c} | M(x, z) - M(x, y_0) | \nu_k(dz) \quad (5.14)
\]
where \( \delta_0 > 0 \) will be specified momentarily. Recall that \( M(x, \cdot) \) is uniformly continuous in \( \partial B \). Therefore given \( \epsilon > 0 \), there is a \( \delta > 0 \), such that if \( |z - y_0| < \delta \),
\[
| M(x, z) - M(x, y_0) | < \epsilon.
\]

Taking \( \delta_0 = \delta \) in (5.14), we obtain
\[
| h_k(x) - M(x, y_0) | \leq \epsilon + 2 \| M(x, \cdot) \|_{\infty} \nu_k(\partial B \cap B(y_0, \delta)^c). \quad (5.15)
\]

We claim that for \( k \) such that \( Q_{y_0k} \subset B(y_0, \delta/2) \), \( \nu_k(\partial B \cap B(y_0, \delta)^c) = 0 \). Combined with equation (5.14), this will finish the proof of the theorem. To prove this, take \( k \) large as specified before and write \( \mu_k = \nu_k |_{\partial B \cap B(y_0, \delta)^c} \). Let us define
\[
u_k(x) = \int M(x, z)\mu_k(dz).
\]

Then \( u_k \leq h_k \) and \( u_k \) is harmonic in \( B \).

We now recall the fact that if \( |w - z_1| > \eta \), then \( M(x, w) \to 0 \) uniformly as \( x \to w' \in B(z_1, \eta/2) \). Therefore using this, we have
\[
limit_{x \to w'} u_k(x) = 0
\]
for \( w' \in B(y_0, \delta/2) \cap \partial B \). Now let \( w' \in B(y_0, \delta/2)^c \cap \partial B \). We know that \( h_k \) vanishes continuously on \( \partial B \setminus Q_{y_0(k-1)} \). Therefore by the Carleson estimate (c.f Theorem 3.5),
\[
limit_{x \to w'} h_k(x) = 0, \text{ for } w' \in \partial B \setminus Q_{y_0(k-1)}.
\]

Since \( u_k(x) \leq h_k(x) \) the same is true of \( u_k \) as well.

The upshot of all this is that \( u_k \) tends to 0 on the boundary \( \partial B \). We already know that \( u_k \) is 0 on the complement of \( B \). Since \( u_k \) is harmonic, we have \( u_k(x) = \mathbb{E}^x(u_k(X_{\tau_B})) \). The boundary values are all 0, so \( u_k(x) = 0 \) in the ball \( B \). But \( u_k(x) = \int M(x, z)\mu_k(dz) \) and \( M(x, z) \) is positive for all \( z \in \partial B \) (from Theorem 3.9). So this means that \( \mu_k = \nu_k |_{\partial B \cap B(y_0, \delta)^c} = 0 \). This finishes the proof of the theorem.

We are now ready to prove Proposition 1.10.

**Proof of Proposition 1.10** By the Monotone class theorem, it is enough to prove the lemma for \( \phi \) of the form \( \phi = 1_{\{t < \tau_B\}}\phi_t \), where \( \phi_t \) is a \( \mathcal{F}_t \) measurable function. Next, we split the right hand side as
\[
\mathbb{E}^x \left( \mathbb{E}^x_{X_{\tau_B}}(\phi); X_{\tau_B} \in A \right) = \mathbb{E}^x \left( \mathbb{E}^x_{X_{\tau_B}}(\phi); X_{\tau_B} \in A, X_{\tau_B} = X_{\tau_B^-} \right) + \mathbb{E}^x \left( \mathbb{E}^x_{X_{\tau_B}}(\phi); X_{\tau_B} \in A, X_{\tau_B} \neq X_{\tau_B^-} \right) \quad (5.16)
\]
Let us label the two terms in equation (5.16) as I and II respectively. We may write I in the following way

\[ I = \int_{A \cap \partial B} \mathbb{E}_x^z(\phi)\omega(x, dz) = \int_{A \cap \partial B} \mathbb{E}_x^z(\phi)M(x, z)\omega(x_0, dz) \]

\[ = \int_{A \cap \partial B} \mathbb{E}_x^z(t < \tau_B; \phi_t M(X_t, z))\omega(x_0, dz) \]

\[ = \mathbb{E}_x^z \left( t < \tau_B; \phi_t \int_{A \cap \partial B} M(X_t, z)\omega(x_0, dz) \right) \]

\[ = \mathbb{E}_x^z(t < \tau_B; \phi_t \omega(X_t, A \cap \partial B)) \]

\[ = \mathbb{E}_x^z(t < \tau_B; \phi_t \mathbb{E}_x^z(X_{\tau_B} \in A \cap \partial B)) \]

\[ = \mathbb{E}_x^z(1_{\{t < \tau_B\}}\phi_t; X_{\tau_B} \in A \cap \partial B) \]

\[ = \mathbb{E}_x^z(\phi; X_{\tau_B} \in A; X_{\tau_B} = X_{\tau_B}^-). \quad (5.17) \]

To deal with II, we first recall the Levy system formula, equation (3.1) which gives us the joint density of \((X_{\tau_B}^-, X_{\tau_B})\) and so we can write

\[ II = \mathbb{E}_x^z(1_{\{t < \tau_B\}}\phi_t; X_{\tau_B} \in A, X_{\tau_B} \neq X_{\tau_B}^-) \]

\[ = \int_A \int_B \mathbb{E}_x^z(\phi)G_B(x, z)J(z, y)dzdy \]

\[ = \int_A \int_B \mathbb{E}_x^z(t < \tau_B; \phi_t G_B(X_t, z))J(z, y)dzdy \]

\[ = \mathbb{E}_x^z \left( t < \tau_B; \phi_t \int_A \int_B G_B(X_t, z)J(z, y)dzdy \right) \]

\[ = \mathbb{E}_x^z(t < \tau_B; \phi_t \mathbb{E}_x^z(X_{\tau_B} \in A; X_{\tau_B} \neq X_{\tau_B}^-)) \]

\[ = \mathbb{E}_x^z(\phi; X_{\tau_B} \in A; X_{\tau_B} \neq X_{\tau_B}^-). \quad (5.18) \]

Adding the two equations (5.17) and (5.18) gives the required result. The lemma is then proved.

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