WELL-POSEDNESS OF THE PLASMA-VACUUM INTERFACE PROBLEM FOR IDEAL INCOMPRESSIBLE MHD

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Abstract. In this paper, we prove the local well-posedness of plasma-vacuum interface problem for ideal incompressible magnetohydrodynamics under the stability condition: the magnetic field \( \mathbf{h} \) and the vacuum magnetic field \( \hat{\mathbf{h}} \) are non-collinear on the interface (i.e., \( |\mathbf{h} \times \hat{\mathbf{h}}| > 0 \)), which was introduced by Trakhinin as a stability condition for the compressible plasma-vacuum interface problem.

1. Introduction

1.1. Presentation of the problem. Magnetohydrodynamic (MHD) models describe macroscopic plasma phenomena, from laboratory research on thermonuclear fusion to plasma-astrophysics of the solar system. In the laboratory research, the main topic is magnetic plasma confinement for energy production by controlled thermonuclear reactions. The plasma-vacuum interface appears as a typical phenomenon when the plasma is separated by a vacuum from outside wall. The total pressure is balanced on the interface, while the normal part of the magnetic field vanishes and the tangent part may jump, thus forms a tangential discontinuity. Mathematically the plasma-vacuum interface is formulated as a free boundary problem for the MHD system, see for example [8]. By ignoring the viscosity, the resistivity and heat conduction, we assume that the plasma fluid is ideal and incompressible. The evolution of the velocity \( \mathbf{u} = (u^1, u^2, u^3) \), the magnetic field \( \mathbf{h} = (h_1, h_2, h_3) \) and the total pressure \( p \) is formulated by the following system of partial differential equations:

\[
\begin{align*}
& \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{h} \cdot \nabla \mathbf{h} + \nabla p = 0, \\
& \text{div} \mathbf{u} = 0, \quad \text{div} \mathbf{h} = 0, \\
& \partial_t \mathbf{h} + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u} = 0.
\end{align*}
\]

Here the total pressure \( p = q + \frac{1}{2} |\mathbf{h}|^2 \) with \( q \) the fluid pressure. For technical reason, we consider the plasma-vacuum interface problem under a simplified configuration. Denote \( \Omega = \mathbb{T}^2 \times (-1, 1) \) with the top/bottom boundary \( \Gamma^\pm = \mathbb{T}^2 \times \{ \pm 1 \} \) and assume that the plasma is initially confined in the domain

\[
\Omega_{f_0}^- = \{ x = (x', x_3) \in \Omega | x_3 < f_0(x') \}, \quad x' = (x_1, x_2) \in \mathbb{T}^2,
\]

where \( f_0(x') \) is a function defined on \( \mathbb{T}^2 \) and

\[
\Gamma_{f_0} := \{ x \in \Omega | x_3 = f_0(x'), x' \in \mathbb{T}^2 \}
\]

is the initial interface. Consequently,

\[
\Omega_{f_0}^+ = \{ x = (x', x_3) \in \Omega | x_3 > f_0(x') \}
\]
is the region of the initial vacuum. After the initial time, the plasma evolves and the interface moves simultaneously. At the time $t > 0$, let us assume the interface is represented as
\[
\Gamma_f = \Gamma_{f(0)} := \{ x \in \Omega | x_3 = f(t, x'), x' \in \mathbb{T}^2 \},
\]
and denote
\[
\Omega_f^- = \Omega_{f(0)}^- = \{ x = (x', x_3) \in \Omega | x_3 < f(t, x') \}, \quad Q_T^- := \cap_{t \in (0, T)} t \times \Omega_f^-,
\]
\[
\Omega_f^+ = \Omega_{f(0)}^+ = \{ x = (x', x_3) \in \Omega | x_3 > f(t, x') \}, \quad Q_T^+ := \cap_{t \in (0, T)} t \times \Omega_f^+.
\]

With these notations, the evolution of the plasma part is formulated as
(1.1) \[ \partial_t u + u \cdot \nabla u - h \cdot \nabla h + \nabla p = 0 \quad \text{in} \quad Q_T^-, \]
(1.2) \[ \text{div } u = 0, \quad \text{div } h = 0 \quad \text{in} \quad Q_T^- \]
(1.3) \[ \partial_t h + u \cdot \nabla h - h \cdot \nabla u = 0 \quad \text{in} \quad Q_T^- \]

In the vacuum domain $\Omega_f^+$, we consider so-called pre-Maxwell dynamics. In such case, the magnetic field $\hat{h} = (\hat{h}_1, \hat{h}_2, \hat{h}_3)$ is determined by the div-curl system:
(1.4) \[ \text{div } \hat{h} = 0, \quad \text{curl } \hat{h} = 0 \quad \text{in} \quad \Omega_f^+. \]

The physical quantities of the plasma and the vacuum region are related by the pressure balance condition on the interface $\Gamma_f$:
(1.5) \[ p(t, x) = \frac{1}{2} |\hat{h}(t, x)|^2, \quad (t, x) \in \Gamma_f \]
as well as
(1.6) \[ h \cdot N_f = 0, \quad \hat{h} \cdot N_f = 0, \quad (t, x) \in \Gamma_f. \]
Here
\[
N_f = (N_1, N_2, N_3) = (-\partial_1 f, -\partial_2 f, 1)
\]
is the normal vector of $\Gamma_f$. As the interface moves with the fluid particles, its normal velocity $\partial_t f$ satisfies
(1.7) \[ \partial_t f = u \cdot N_f. \]
Moreover, on the artificial boundaries $\Gamma^\pm$, we prescribe the following boundary conditions:
(1.8) \[ u_3 = 0, \quad h_3 = 0 \quad \text{on} \quad \Gamma^-, \]
(1.9) \[ \hat{h} \times e_3 = \hat{J} \quad \text{on} \quad \Gamma^+. \]
where $e_3 = (0, 0, 1)$ is the unit normal vector on $\Gamma^+$. Here to avoid trivial solution $\hat{h}$ in the vacuum, a surface current $\hat{J} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ is added as an outer force term to the elliptic system \(1.4\). In real laboratory plasma, this surface current can be produced by a system of coils, see [8] [15]. Finally, the system is supplemented with the initial data
(1.10) \[ u(0, x) = u_0(x), \quad h(0, x) = h_0(x) \quad \text{in} \quad \Omega_{f_0}^- \]
\[
f(0, x_1, x_2) = f_0(x_1, x_2), \]
\[
\text{subject to the boundary conditions} \quad \text{on} \quad \Gamma_0, \quad \text{as well as} \quad \text{initial conditions} \quad \text{on} \quad x_3 = f(0, x_1, x_2). \]
which satisfy the following compatibility conditions:

\[
\begin{align*}
\text{(1.11)} & \quad \begin{cases}
\text{div } u_0 = 0, \text{ div } h_0 = 0 \quad \text{in } \Omega^+_{f_0}, \\
\, h_0 \cdot N_{f_0} = 0 \quad \text{on } \Gamma_{f_0}, \quad u_{30}, h_{30} = 0 \quad \text{on } \Gamma^-.
\end{cases}
\end{align*}
\]

From (1.9) and the fact that curl \( \hat{h} \) = 0, we also need compatibility conditions on the imposed surface current:

\[
\begin{align*}
\text{(1.12)} & \quad \partial_1 \hat{J}_1 + \partial_2 \hat{J}_2 = 0, \quad \hat{J}_3 = 0 \quad \text{on } \Gamma^+.
\end{align*}
\]

Note that the initial magnetic field \( \hat{h}_0 \) in the vacuum region is uniquely determined from \( \Gamma_{f_0}, u_0, h_0 \) and \( \hat{J}_0 = \hat{J}(0, x') \) by solving the following div-curl system:

\[
\begin{align*}
\text{(1.13)} & \quad \begin{cases}
\text{curl } \hat{h}_0 = 0, \quad \text{div } \hat{h}_0 = 0 \quad \text{in } \Omega^+_{f_0}, \\
\, \hat{h}_0 \cdot N_{f_0} = 0 \quad \text{on } \Gamma_{f_0}, \quad \hat{h}_0 \times e_3 = \hat{J}_0 \quad \text{on } \Gamma^+.
\end{cases}
\end{align*}
\]

The solvability of this div-curl system will be shown in Section 4. Also note that since

\[
\partial_t \text{div } h + u \cdot \nabla \text{div } h = 0,
\]

the divergence free restriction on \( h \) is automatically satisfied if it holds for \( h_0 \). Similar argument also yields \( h \cdot N_f = 0 \) provided \( h_0 \cdot N_{f_0} = 0 \).

1.2. Backgrounds. In the absence of the magnetic field, the system is reduced to the incompressible Euler equations with a free boundary, which is so-called water-wave problem. In this case, it is well-known that a sufficient condition ensuring the well-posedness of the water-wave problem is the Taylor’s sign condition:

\[
\text{(1.14)} \quad \frac{\partial p}{\partial N} \leq -\varepsilon < 0 \quad \text{on } \Gamma_f.
\]

See \([1, 6, 12, 13, 16, 23, 24, 25]\) and references therein. In fact, it is also necessary in the absence of surface tension \([7]\).

The fact that the magnetic field has the stabilizing effect for the current-vortex sheet problem was found by physicists long before \([2, 18]\). In past decade, there are many works devoted to the well-posedness of the current-vortex sheet problem under a suitable stability condition \([3, 5, 17, 19, 20, 22]\).

In \([21]\), Trakhinin introduced the following stability condition for the linearized compressible plasma-vacuum interface problem:

\[
\text{(1.15)} \quad |h \times \hat{h}| > 0 \quad \text{on } \Gamma_f.
\]

Under this condition, Secchi and Trakhinin \([15]\) proved the well-posedness of the compressible plasma-vacuum interface problem, and Morando, Trakhinin and Trebeschi \([14]\) proved the well-posedness of the linearized incompressible plasma-vacuum interface problem. However, the well-posedness of nonlinear incompressible problem (i.e., the system (1.1)-(1.10)) is still open under (1.15).

Motivated by works on the water-wave problem, Luo and Hao \([10, 11]\) established \( a \text{ priori } \) estimates for the incompressible plasma-vacuum interface problem under the Taylor’s sign condition (1.14). Recently, Gu and Wang \([9]\) proved the well-posedness of the incompressible plasma-vacuum problem under (1.14). Let us also mention that the well-posedness of the plasma-vacuum interface problem under (1.14) is still...
unknown when the vacuum magnetic field \( \hat{h} \) is non trivial. In [10][11][9], the authors only considered the case of \( \hat{h} = 0 \). This problem is also unsolved in the compressible case [21].

1.3. Main result. The goal of this paper is to prove the well-posedness of the system (1.1)-(1.10) under the stability condition (1.11). This condition implies that \( h \) and \( \hat{h} \) are non-collinear everywhere on \( \Gamma_f \), which means

\[
\inf_{\xi_1, \xi_2 \in [0, 1]} \inf_{\xi_1 + \xi_2 = 1} \left[ (h_1 \xi_1 + h_2 \xi_2)^2 + (\hat{h}_1 \xi_1 + \hat{h}_2 \xi_2)^2 \right] \geq c_1.
\]

It will be shown in Section 5 that (1.16) implies that there exists a positive constant \( c_1 \) such that

\[
\Lambda(h, \hat{h}) \equiv \inf_{\xi_1, \xi_2 \in [0, 1]} \inf_{\xi_1 + \xi_2 = 1} \left[ (h_1 \xi_1 + h_2 \xi_2)^2 + (\hat{h}_1 \xi_1 + \hat{h}_2 \xi_2)^2 \right] \geq c_1.
\]

Our main result is stated as follows.

**Theorem 1.1.** Let \( s \geq 3 \) be an integer. Assume that

\[
f_0 \in H^{s+\frac{1}{2}}(\mathbb{T}^2), \quad u_0, h_0 \in H^s(\Omega_f^-),
\]

\[
\hat{f} \in C([0, T_0]; H^{s-\frac{1}{2}}(\mathbb{T}^2)), \quad \partial_t \hat{f} \in C([0, T_0]; H^{s-\frac{1}{2}}(\mathbb{T}^2)),
\]

which satisfies the compatibility conditions (1.11)-(1.12) and the stability condition:

\[
-(1 - 2c_0) \leq f_0 \leq (1 - 2c_0), \quad \Lambda(h_0, \hat{h}_0) \geq 2c_1
\]

for some \( c_0 \in (0, \frac{1}{2}) \) and \( c_1 > 0 \). Then there exists \( T \in (0, T_0) \) such that the system (1.1)-(1.10) admits a unique solution \((f, u, h, \hat{h})\) in \([0, T]\) such that

\[
f \in L^\infty(0, T; H^{s+\frac{1}{2}}(\mathbb{T}^2)), \quad u, h \in L^\infty(0, T; H^s(\Omega_f^-)), \quad \hat{h} \in L^\infty(0, T; H^s(\Omega_f^+)),
\]

\[
-(1 - c_0) \leq f \leq (1 - c_0), \quad \Lambda(h, \hat{h}) \geq c_1.
\]

The idea of the proof is motivated by our recent work on nonlinear stability of incompressible current-vortex sheet problem [17]. The key ingredient is to derive an evolution equation of the scaled normal velocity \( u \cdot N \) rather than the usual normal component of velocity on the interface. By some tricky observations, we find in present case that the evolution equation of the height function of the interface takes as follows

\[
D_t^2 f = \sum_{i,j=1,2} (h_i h_j + \hat{h}_i \hat{h}_j) \partial_i \partial_j f + \text{L.O.T.},
\]

where \( D_t f = \partial_t f + u_1 \partial_1 f + u_2 \partial_2 f, g \) denotes the trace of \( g \) on the interface, and L.O.T. denotes the lower order terms. Now, the stability condition (1.11) ensures that the equation (1.21) is strictly hyperbolic. Indeed, the principal symbol of the operator \(-\sum_{i,j=1,2} (h_i h_j + \hat{h}_i \hat{h}_j) \partial_i \partial_j\) is

\[
(h_1 \xi_i) + (\hat{h}_1 \xi_i) \geq c_1 |\xi|^2.
\]
The motion of the fluid and the magnetic field will be described by the following vorticity and current system:

\[
\begin{cases}
\partial_t \omega + \mathbf{u} \cdot \nabla \omega - \mathbf{h} \cdot \nabla \hat{\mathbf{\xi}} = \omega \cdot \nabla \mathbf{u} - \mathbf{\xi} \cdot \nabla \mathbf{h} & \text{in } Q_T, \\
\partial_t \hat{\mathbf{\xi}} + \mathbf{u} \cdot \nabla \hat{\mathbf{\xi}} - \mathbf{h} \cdot \nabla \omega = \mathbf{\xi} \cdot \nabla \mathbf{u} - \omega \cdot \nabla \mathbf{h} - 2 \nabla u_i \times \nabla h_i & \text{in } Q_T.
\end{cases}
\]

where \(\omega = \nabla \times \mathbf{u}, \mathbf{\xi} = \nabla \times \mathbf{h}\). With the vorticity and current, the velocity and magnetic field can be recovered by solving two div-curl systems defined on the fluid domain and the vacuum domain respectively. For the fluid domain, we need to solve a div-curl system with given normal component on the boundaries, which is the same as in [17]. For the vacuum domain, we need to solve another div-curl system for \(\hat{\mathbf{h}}\):

\[
\begin{cases}
\text{div } \hat{\mathbf{h}} = 0, & \text{curl } \hat{\mathbf{h}} = 0 \text{ in } \Omega^+_f, \\
\hat{\mathbf{h}} \cdot \mathbf{n}_f = \hat{\mathbf{\theta}} \text{ on } \Gamma_f, & \hat{\mathbf{h}} \times \mathbf{e}_3 = \hat{\mathbf{j}} \text{ on } \Gamma^+.
\end{cases}
\]

The main difference of these two system is the boundary condition on the fixed boundary. We state the solvability and estimates of solutions to the div-curl systems in Section 4.

The construction of the approximate solution is completed by introducing the suitable linearization of the system and the iteration map. It can be proved that the constructed approximate sequence is a Cauchy sequence, and the limit system is equivalent to the origin system.

The rest of this paper is organized as follows. In Section 2, we introduce the harmonic coordinate and preliminary results on the harmonic extensions and Dirichlet to Neumann operators. In Section 3, the system is replaced by a new formulation. In Section 4, two div-curl systems are solved. In Section 5, we establish the uniform estimates for the linearized system. Section 6,8 are devoted to proving existence (and uniqueness) of the solution.

2. Harmonic Coordinate and Dirichlet-Neumann Operator

In this section, we recall some facts and well-known results on the reference domain, the harmonic coordinate and Dirichlet-Neumann operators.

We first introduce some notations used throughout this paper. The coordinate in the fluid region is denoted as \(x = (x_1, x_2, x_3)\) or \(y = (y_1, y_2, y_3)\), while \(x' = (x_1, x_2)\) or \(y' = (y_1, y_2)\) is the natural coordinates on the interface or on the top/bottom boundary \(\Gamma^\pm\). For a function \(g : \Omega \to \mathbb{R}\), we denote \(\nabla g = (\partial_1 g, \partial_2 g, \partial_3 g)\) and for a function \(\eta : \mathbb{T}^2 \to \mathbb{R}, \nabla \eta = (\partial_1 \eta, \partial_2 \eta)\). The trace on \(\Gamma_f\) for a function \(g : \Omega^+_f \to \mathbb{R}\) is denoted by \(g\). Thus, for \(i = 1, 2\),

\[
(\partial_i g)(x') = \partial_i g(x', f(x')) + \partial_3 g(x', f(x'))\partial_i f(x') = \partial_i g + \partial_3 g\partial_i f(x').
\]

Finally, the Sobolev norm in \(\Omega^+_f\) is denoted as \(\| \cdot \|_{H^k(\Omega^+_f)}\) and \(\| \cdot \|_{H^k}\) is the Sobolev norm in \(\mathbb{T}^2\).

To solve the free boundary problem, we introduce a fixed reference domain. Let \(\Gamma^*_f\) be a fixed graph given by

\[
\Gamma^*_f = \{(y_1, y_2, y_3) : y_3 = f_3(y_1, y_2)\}.
\]
The reference domain $\Omega^\pm_\ast$ is given by
\[
\Omega_\ast = \mathbb{T}^2 \times (-1, 1), \quad \Omega^\pm_\ast = \{y \in \Omega_\ast | y_3 \geq f_\ast(y')\}.
\]

We will seek a free boundary lying in a neighborhood of the reference domain. To this end, we define
\[
\Upsilon(\delta, k) \overset{\text{def}}{=} \{f \in H^k(\mathbb{T}^2) : \|f - f_\ast\|_{H^k(\mathbb{T}^2)} \leq \delta\}.
\]
For $f \in \Upsilon(\delta, k)$, the graph $\Gamma_f$ is defined as
\[
\Gamma_f \overset{\text{def}}{=} \{x \in \Omega_\ast | x_3 = f(t, x'), x' \in \mathbb{T}^2\},
\]
and use the notations $\Omega^\pm_f, N_f, \Gamma^\pm_f$, etc., as in Section I.

**Remark 2.1.** Since we intend to solve the plasma-vacuum interface problem locally in time, a natural choice of $\Gamma_\ast$ would be certainly the initial interface $\Gamma_0$.

To handle the plasma-vacuum interface problem, we need to introduce different Dirichlet-Neumann operators on $\Omega^\pm_f$. For a function $\psi(x') = \psi(x_1, x_2) \in H^k(\mathbb{T}^2)$, its harmonic extension from $\Gamma_f$ to $\Omega^\pm_f$ is denoted as $\mathcal{H}^\pm_f \psi$, i.e.,
\[
\begin{cases}
\Delta \mathcal{H}^\pm_f \psi = 0 \text{ in } \Omega^\pm_f, \\
(\mathcal{H}^\pm_f \psi)(x', f(x')) = \psi(x'), \ x' \in \mathbb{T}^2, \\
\partial_3 (\mathcal{H}^\pm_f \psi)(x', \pm 1) = 0, \ x' \in \mathbb{T}^2.
\end{cases}
\]
Moreover, for a function $g$ defined in $\Omega^\pm_f$, we denote $\mathcal{H}^\pm_f g$ as a harmonic function in $\Omega^\pm_f$ such that
\[
\begin{cases}
\Delta \mathcal{H}^\pm_f g = 0 \text{ in } \Omega^\pm_f, \\
(\mathcal{H}^\pm_f g)(x', f(x')) = g(x', f(x')), \ x' \in \mathbb{T}^2, \\
\partial_3 (\mathcal{H}^\pm_f g)(x', \pm 1) = \partial_3 g(x', \pm 1), \ x' \in \mathbb{T}^2.
\end{cases}
\]
The Dirichlet-Neumann (D-N) in the following context) operators are defined as
\[
N^\pm_f \psi = \mp N_f \cdot \nabla \mathcal{H}^\pm_f \psi \|_{\Gamma_f}, \quad \mathcal{N}^\pm_f g = \mp N_f \cdot \nabla \mathcal{H}^\pm_f g \|_{\Gamma_f}.
\]
For a function $g$ defined in $\Omega^\pm_f$, we denote
\[
\mathcal{N}^\pm_f g = \mathcal{N}^+_f g - \mathcal{N}^-_f g.
\]
Given $f \in \Upsilon(\delta, k)$, a map (harmonic coordinate) $\Phi^\pm_f$ from $\Omega^\pm_\ast$ to $\Omega^\pm_f$ is defined by harmonic extension:
\[
\begin{cases}
\Delta \Phi^\pm_f = 0 \text{ in } \Omega^\pm_\ast, \\
\Phi^\pm_f(y', f_\ast(y')) = (y', f(y')), \ y' \in \mathbb{T}^2, \\
\Phi^\pm_f(y', \pm 1) = (y', \pm 1), \ y' \in \mathbb{T}^2.
\end{cases}
\]
Given $\Gamma_*$, there exists $\delta_0 = \delta_0(||f||_{W^{1,\infty}}) > 0$ so that $\Phi_f^\pm$ is a bijection whenever $\delta \leq \delta_0$. Thus, there exists an inverse map $\Phi_f^{\pm-1}$ from $\Omega_f^\pm$ to $\Omega_f^\pm$ such that

$$\Phi_f^{\pm-1} \circ \Phi_f^\pm = \text{Id}_{\Omega_f^\pm}, \Phi_f^\pm \circ \Phi_f^{\pm-1} = \text{Id}_{\Omega_f^\pm}.$$

We have the following properties of harmonic coordinates, see [17] for example.

**Lemma 2.1.** Let $f \in \Upsilon(\delta_0, s-\frac{1}{2})$ for $s \geq 3$. There exists a constant $C$ depending only on $\delta_0$ and $||f||_{H^{s-\frac{1}{2}}}$ so that

1. If $u \in H^s(\Omega_f^\pm)$ for $\sigma \in [0, s]$, then

$$||u \circ \Phi_f^\pm||_{H^\sigma(\Omega_f^\pm)} \leq C||u||_{H^\sigma(\Omega_f^\pm)}.$$

2. If $u \in H^\sigma(\Omega_f^\pm)$ for $\sigma \in [0, s]$, then

$$||u \circ \Phi_f^{\pm-1}||_{H^\sigma(\Omega_f^\pm)} \leq C||u||_{H^\sigma(\Omega_f^\pm)}.$$

3. If $u, v \in H^\sigma(\Omega_f^\pm)$ for $\sigma \in [2, s]$, then

$$||(uv) \circ \Phi_f^{\pm-1}||_{H^\sigma(\Omega_f^\pm)} \leq C||u||_{H^\sigma(\Omega_f^\pm)||v||_{H^\sigma(\Omega_f^\pm)}}.$$

**Proposition 2.1.** Let $s \geq 3$ be an integer. If $f \in H^{s+\frac{1}{2}}(\Upsilon_2)$, then we have

$$||N_f g||_{H^{s-\frac{1}{2}}} \leq C \left(||f||_{H^{s+\frac{1}{2}}}||g||_{H^s(\Omega_f^\pm)}\right).$$

**Proof.** We write

$$\overline{N_f g} = \left(\overline{N^+_f g} - N^+_f g\right) + \left(N^+_f g - \overline{N^-_f g}\right).$$

The corresponding estimate for the second term on the right hand side has been shown in the appendix of [17]. The first term in fact satisfies

$$||\overline{N^+_f g} - N^+_f g||_{H^{s-\frac{1}{2}}} \leq C \left(||f||_{H^{s+\frac{1}{2}}}||g||_{H^s(\Omega_f^\pm)}\right).$$

To prove this, we first note that

$$\overline{H^+_f g} = \overline{H^+_f g} - H^+_f g$$

satisfies

$$\Delta \overline{H^+_f g} = 0 \text{ in } \Omega_f^\pm, \quad \overline{H^+_f g} = 0 \text{ on } \Gamma_f, \quad \partial_3 \overline{H^+_f g} = \partial_3 g \text{ on } \Gamma^+.$$

It follows that

$$\int_{\Omega_f^\pm} \left|\nabla \overline{H^+_f g}\right|^2 dx = \int_{\Gamma_f} \left(\overline{H^+_f g}\right)(\partial_3 g) dx' \leq ||\overline{H^+_f g}||_{L^2}||\partial_3 g||_{L^2}$$

$$\leq C||\overline{H^+_f g}||_{H^s(\Omega_f^\pm)}||g||_{H^s(\Omega_f^\pm)} \leq C||\nabla \overline{H^+_f g}||_{L^2(\Omega_f^\pm)}||g||_{H^s(\Omega_f^\pm)},$$

where in the last inequality we applied Poincaré’s inequality to $\overline{H^+_f g}$. On the other hand, according to standard interior (and boundary near $\Gamma_f$) elliptic estimates, we have

$$||\overline{H^+_f g}||_{H^s(\Omega_f^\pm)} \leq C \left(||f||_{H^{s+\frac{1}{2}}}||g||_{H^s(\Omega_f^\pm)}\right).$$

Here $\Omega_f^\pm$ is any sub-domain of $\Omega_f^\pm$ away from $\Gamma^+$, such as

$$\widetilde{\Omega_f^\pm} = \{x \in \Omega_f^\pm | f(x') < x_3 < 1 - \epsilon_0\} \text{ for sufficiently small } \epsilon_0.$$

We thus conclude the proof of (2.8), and then (2.7) follows. \qed
Finally we introduce a commutator estimates that will be used frequently.

**Lemma 2.2.** For \( s > 2 \), we have
\[
\| [a, \langle \nabla \rangle^s] u \|_{L^2} \leq C \| a \|_{H^s} \| u \|_{H^{s-1}}.
\]
Here \( \langle \nabla \rangle \) is the \( s \)-order derivatives on \( \mathbb{T}^2 \) defined as follows
\[
\langle \nabla \rangle^s f(k) = \left( 1 + |k|^2 \right)^{s/2} \hat{f}(k), \quad k = (k_1, k_2), \quad k_1, k_2 \in \mathbb{Z}.
\]

3. Reformulation of the problem

In this section, we replace the system \( (1.1) - (1.10) \) by an equivalent formulation, which consists of the (evolution) equations of the following quantities:
- the height function of the interface: \( f \);
- the scaled normal velocity on the interface: \( \theta = u \cdot N_f \);
- the vorticity and current in the fluid region: \( \omega = \nabla \times u, \xi = \nabla \times h \);
- the average of tangential part of velocity and magnetic field on the fixed bottom boundary:
\[
\beta_i(t) = \int_{\Omega} u_i(t, x', -1) dx', \quad \gamma_i(t) = \int_{\Omega} h_i(t, x', -1) dx', \quad i = 1, 2.
\]
To simplify notations, from now on we drop the minus superscript “-”. Hence,
\[
\Omega_f = \Omega_f^+, \quad \Gamma = \Gamma^+, \quad N_f = N_f^+, \quad \mathcal{H}_f = \mathcal{H}_f^+, \text{ etc.}
\]

3.1. Evolution of the interface and the scaled normal velocity. Let
\[
\theta(t, x') = u(t, x', f(t, x')) \cdot N_f(t, x').
\]
Then we have
\[
\partial_t f(t, x') = \theta(t, x').
\]
Clearly, \((1 + |\nabla f|^2)^{-1/2} \theta\) is the normal component of the fluid velocity on the interface.

According to \( (2.1) \), for a vector field \( \mathbf{v} = (v_1, v_2, v_3) \) defined in \( \Omega_f^+ \) or \( \Omega_f \) we calculate \( \mathbf{v} \cdot \nabla \mathbf{v} \cdot N_f \) as follows:
\[
(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot N_f = v_1 \partial_1 v_j N_j + v_2 \partial_2 v_j N_j + v_3 \partial_3 v_j N_j
\]
\[
= v_1 \partial_1 v_j N_j + v_2 \partial_2 v_j N_j + (\mathbf{v} \cdot N_f) \left( \partial_3 \mathbf{v} \cdot N_f \right)
\]
\[
= v_1 \partial_1 (\mathbf{v} \cdot N_f) + v_2 \partial_2 (\mathbf{v} \cdot N_f) - v_1 v_j \partial_1 N_j - v_2 v_j \partial_2 N_j + (\mathbf{v} \cdot N_f) \left( \partial_3 \mathbf{v} \cdot N_f \right).
\]
Hereafter we use Einstein’s notation of summation for repeated indices \( i, j = 1, 2, 3 \) as well as summation on \( i, j = 1, 2 \) in case of making no confusion. From the calculations above, we obtain the following lemma.

**Lemma 3.1.** For a vector \( \mathbf{v} = (v_1, v_2, v_3) \) defined in \( \Omega_f^+ \) or \( \Omega_f \),
\[
(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot N_f = \left( \partial_3 \mathbf{v} \cdot N_f \right) \left( \mathbf{v} \cdot N_f \right)
\]
\[
= v_1 \partial_1 (\mathbf{v} \cdot N_f) + v_2 \partial_2 (\mathbf{v} \cdot N_f) + \sum_{i, j = 1, 2} v_i v_j \partial_i \partial_j f.
\]
By restricting the equation (1.1) to $\Gamma_f$ and taking inner product with $N_f$, we deduce from Lemma 3.1 (recall $h \cdot N_f = 0$ on $\Gamma_f$) that
\[
\frac{\partial}{\partial \tau} \vartheta = (\vartheta_u + \partial_3 u \partial_3 f) \cdot N_f + u \cdot \partial_3 N_f \bigg|_{t = \tau_f(x, \xi)}
\]
\[= (u \cdot \nabla u + h \cdot \nabla u - \nabla p + \partial_3 u \partial_3 f) \cdot N_f - u \cdot (\partial_3 \partial_3 f, \partial_2 \partial_3 f, 0) \bigg|_{t = \tau_f(x, \xi)}
\]
\[= (u \cdot \nabla u + \partial_3 u (u \cdot N_f)) \cdot N_f + (h \cdot \nabla h) \cdot N_f
\]
\[= - N_f \cdot \nabla p - u \cdot (\partial_3 \theta, \partial_2 \theta, 0) \bigg|_{t = \tau_f(x, \xi)}
\]
\[= - 2(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - N_f \cdot \nabla p - \sum_{i,j=1,2} u_i u_j \partial_i \partial_j f + \sum_{i,j=1,2} h_i h_j \partial_i \partial_j f.
\]
(3.4)

To give the trace of the pressure $p$ on $\Gamma_f$, we first take divergence to (1.1) to yield
\[
\Delta p = \text{tr}(\nabla h \nabla h) - \text{tr}(\nabla u \nabla u).
\]
(3.5)

Let $p_{v_1,v_2}$ be the solution of the following mixed boundary value problem:
\[
\begin{cases}
\Delta p_{v_1,v_2} = -\text{tr}(\nabla v_1 \nabla v_2) & \text{in } \Omega_f, \\
p_{v_1,v_2} = 0 & \text{on } \Gamma_f, \quad \mathbf{e}_3 \cdot \nabla p_{v_1,v_2} = 0 & \text{on } \Gamma.
\end{cases}
\]
(3.6)

Since $p = p|_{\Gamma_f} = \frac{1}{2} |\hat{h}|^2|_{\Gamma_f}$, we obtain the following representation formula for the pressure $p$:
\[
p = \mathcal{H}_f p + p_{u,u} - p_{h,h} = \frac{1}{2} \mathcal{H}_f |\hat{h}|^2 + p_{u,u} - p_{h,h}.
\]
(3.7)

It follows from (3.4) that
\[
\frac{\partial}{\partial \tau} \vartheta = -2(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - \sum_{i,j=1,2} (u_i u_j - h_i h_j) \partial_i \partial_j f
\]
\[= - \frac{1}{2} N_f |\hat{h}|^2 - N_f \cdot \nabla (p_{u,u} - p_{h,h}).
\]
(3.8)

Note that
\[
-N_f |\hat{h}|^2 = (\mathcal{H}_f - N_f) |\hat{h}|^2 - N_f \cdot \nabla (|\hat{h}|^2 - \mathcal{H}_f |\hat{h}|^2) + N_f \cdot \nabla (|\hat{h}|^2)
\]
\[= N_f |\hat{h}|^2 - N_f \cdot \nabla (|\hat{h}|^2 - \mathcal{H}_f |\hat{h}|^2) + N_f \cdot \nabla (|\hat{h}|^2).
\]
(3.9)

Furthermore, by Lemma 3.1
\[
\sum_{i,j=1,2} \hat{h}_i \hat{h}_j \partial_i \partial_j f = (\hat{h} \cdot \nabla \hat{h}) \cdot N_f = \hat{h}_i \partial_i \hat{h}_j N_j = \hat{h}_i \partial_i \hat{h}_j N_j = \frac{1}{2} N_f \cdot \nabla |\hat{h}|^2,
\]
(3.10)

where we used curl $\hat{h} = 0$. From (3.8)-(3.10), we finally obtain
\[
\frac{\partial}{\partial \tau} \vartheta = -2(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - \sum_{i,j=1,2} (u_i u_j - h_i h_j - \hat{h}_i \hat{h}_j) \partial_i \partial_j f
\]
\[- N_f \cdot \nabla (p_{u,u} - p_{h,h}) - \frac{1}{2} N_f \cdot \nabla (|\hat{h}|^2 - \mathcal{H}_f |\hat{h}|^2) + \frac{1}{2} N_f |\hat{h}|^2.
\]
(3.11)
3.2. The equations for the vorticity and current. Let
\[ \omega = \nabla \times u, \quad \xi = \nabla \times h \]
be the vorticity and current in \( \Omega_f \) respectively. Then \( \omega, \xi \) satisfy
\[
\begin{align*}
\partial_t \omega + u \cdot \nabla \omega - h \cdot \nabla \xi &= \omega \cdot \nabla u - \xi \cdot \nabla h \quad \text{in } Q_T, \\
\partial_t \xi + u \cdot \nabla \xi - h \cdot \nabla \omega &= \xi \cdot \nabla u - \omega \cdot \nabla h - 2 \nabla u_i \times \nabla h_i \quad \text{in } Q_T.
\end{align*}
\]
Here we used the fact that
\[
\begin{align*}
\varepsilon^{ijk} \partial_j u_i \partial_j h_k - \varepsilon^{ijk} \partial_j h_i \partial_j u_k \\
= \varepsilon^{ijk} \partial_j u_i (\partial_j h_k - \partial_k h_i) + \varepsilon^{ijk} \partial_j u_i \partial_k h_i - \varepsilon^{ijk} \partial_j h_i \partial_k u_i + \varepsilon^{ijk} \partial_j h_i (\partial_k u_i - \partial_i u_k) \\
= -\xi \cdot \nabla u + \omega \cdot \nabla h - 2 \nabla u_i \times \nabla h_i.
\end{align*}
\]

As in [17], to uniquely recover a divergence free vector field from its curl and normal component on the bottom boundary in \( \Omega_f \), we need to prescribe the mean value of its tangential components on the bottom boundary. To this end, let
\[
\beta_i(t) = \int_{T^2} u_i(t, x', -1) dx', \quad \gamma_i(t) = \int_{T^2} h_i(t, x', -1) dx', \quad i = 1, 2.
\]
Thanks to \( u_i(t, x', -1) \equiv 0 \), we deduce that for \( i = 1, 2 \),
\[
\partial_t u_i + u_j \partial_j u_i - h_j \partial_j h_i - \partial_i p = 0 \quad \text{on } \Gamma,
\]
which yields
\[
\partial_t \beta_i + \int_{\Gamma} (u_j \partial_j u_i - h_j \partial_j h_i) dx' = 0,
\]
or equivalently
\[
\beta_i(t) = \beta_i(0) - \int_0^t \int_{\Gamma} (u_j \partial_j u_i - h_j \partial_j h_i) dx' dt. \quad (3.14)
\]
Similarly, we have
\[
\gamma_i(t) = \gamma_i(0) - \int_0^t \int_{\Gamma} (u_j \partial_j h_i - h_j \partial_j u_i) dx' dt. \quad (3.15)
\]

Finally, to solve \( \hat{h} \), and to recover \( u, h \) from \( \omega, \xi \) in \( \Omega_f \), one needs to solve two types of div-curl system. We leave it to the next section.

4. Div-curl system

In this section, we consider two div-curl systems, which have also been considered in [4] for the bounded domain. Assume that \( \Gamma_f \) is a given graph with \( f \in H^{s+\frac{1}{2}}(\mathbb{T}^2) \) for \( s \geq 2 \) satisfying
\[ -(1 - c_0) \leq f \leq (1 - c_0). \]
The first system reads as follows
\[
\begin{cases}
\text{div } v = g, & \text{curl } v = \omega \quad \text{in } \Omega_f, \\
v \cdot N_f = \partial \text{ on } \Gamma_f, & v \cdot e_3 = 0, \quad \int_{\Gamma} v_i dx' = \alpha_i \text{ on } \Gamma, \quad i = 1, 2.
\end{cases}
\]
where \( \omega \) and \( \vartheta \) are given functions in \( \Omega_f \) and \( \Gamma_f \) respectively, and \( \alpha_i, i = 1, 2 \) are given real numbers. Assume that \( \omega, \vartheta \) satisfy the following compatibility conditions:

\[
\text{div } \omega = 0 \text{ in } \Omega_f, \quad \int_{\Gamma} \omega \, d\gamma = 0, \quad \int_{\Omega_f} g \, dx = \int_{\Gamma^+} \vartheta \, d\gamma.
\]

The following proposition has been proved in [17].

**Proposition 4.1.** Let \( f \in H^{s+\frac{1}{2}}, s \geq 2 \) and \( \sigma \in [1, s] \). Assume \( g, \omega \in H^{\sigma-1}(\Omega_f), \vartheta \in H^{\sigma-\frac{1}{2}}(\Gamma_f) \) satisfying (4.2). Then there exists a unique \( v \in H^\sigma(\Omega_f) \) of (4.1) so that

\[
\|v\|_{H^\sigma(\Omega_f)} \leq C \left( \|f\|_{H^{s+\frac{1}{2}}} \right) \left( \|\nabla \omega\|_{H^{\sigma-1}(\Omega_f)} + \|\vartheta\|_{H^{\sigma-\frac{1}{2}}(\Gamma_f)} + |\alpha_1| + |\alpha_2| \right).
\]

The second div-curl system is

\[
\begin{aligned}
\text{div } \hat{h} &= \hat{g}, \quad \text{curl } \hat{h} = \hat{\omega} \text{ in } \Omega_f^+, \\
\hat{h} \cdot N_f &= \hat{\vartheta} \text{ on } \Gamma_f, \quad \hat{h} \times e_3 = \hat{J} \text{ on } \Gamma^+.
\end{aligned}
\]

Here \( \hat{J} = (\hat{J}_1(x^\prime), \hat{J}_2(x^\prime), \hat{J}_3(x^\prime)) \) is a given vector on \( \Gamma^+ \). To solve this boundary value problem, we need the following compatibility conditions on \( \hat{\omega} \) and \( \hat{\vartheta} \):

\[
\text{div } \hat{\omega} = 0 \text{ in } \Omega_f^+, \quad \partial_1 \hat{J}_1 + \partial_2 \hat{J}_2 = \hat{\omega}_3, \quad \hat{J}_3 = 0 \text{ on } \Gamma^+.
\]

**Proposition 4.2.** Let \( f \in H^{s+\frac{1}{2}}, s \geq 2 \) and \( \sigma \in [1, s] \). Assume \( \hat{g}, \hat{\omega} \in H^{\sigma-1}(\Omega_f^+), \hat{\vartheta}, \hat{J} \in H^{\sigma-\frac{1}{2}} \) satisfying the compatibility condition (4.4). Then there exists a unique \( \hat{h} \in H^\sigma(\Omega_f^+) \) of the div-curl system (4.3) so that

\[
\|\hat{h}\|_{H^\sigma(\Omega_f^+)} \leq C \left( \|f\|_{H^{s+\frac{1}{2}}} \right) \left( \|\hat{g}\|_{H^{\sigma-1}(\Omega_f^+)} + \|\hat{\omega}\|_{H^{\sigma-1}(\Omega_f^+)} + \|\hat{\vartheta}\|_{H^{\sigma-\frac{1}{2}}} + \|\hat{J}\|_{H^{\sigma-\frac{1}{2}}} \right).
\]

**Proof.** By Proposition 4.1 there is a vector field \( \hat{h} \) satisfies:

\[
\begin{aligned}
\text{div } \hat{h} &= 0, \quad \text{curl } \hat{h} = \hat{\omega} \text{ in } \Omega_f^+, \\
\hat{h} \cdot N_f &= 0 \text{ on } \Gamma_f, \quad \hat{h} \cdot e_3 = 0 \text{ on } \Gamma^+,
\end{aligned}
\]

and \( \|\hat{h}\|_{H^\sigma(\Omega_f^+)} \leq C \left( \|f\|_{H^{s+\frac{1}{2}}} \right) \|\hat{\omega}\|_{H^{\sigma-1}(\Omega_f^+)} \). On the other hand, according to (4.4), \( \partial_1(\hat{J}_1 - \hat{h}_2) + \partial_2(\hat{J}_2 + \hat{h}_1) = \partial_1 \hat{J}_1 + \partial_2 \hat{J}_2 - \hat{\omega}_3 = 0 \), thus, there exists a scalar function \( \tilde{J} \in H^{\sigma+\frac{1}{2}} \) such that

\[
\hat{J}_1 - \hat{h}_2 = \partial_2 \tilde{J}, \quad \hat{J}_2 + \hat{h}_1 = -\partial_1 \tilde{J}.
\]

Let \( \tilde{J} \in H^{\sigma+1}(\Omega_f^+) \) be an extension of \( \tilde{J} \) to \( \Omega_f^+ \) such that

\[
\tilde{J} = 0 \text{ near } \Gamma_f, \quad \|\tilde{J}\|_{H^{\sigma+1}(\Omega_f^+)} \leq C \left( \|f\|_{H^{s+\frac{1}{2}}} \right) \left( \|\hat{\omega}\|_{H^{\sigma-1}(\Omega_f^+)} + \|\hat{J}\|_{H^{\sigma-\frac{1}{2}}} \right).
\]

We consider the following mixed Dirichlet-Neumann problem:

\[
\begin{aligned}
\Delta \phi &= \Delta \tilde{J} - \hat{g} \text{ in } \Omega_f^+, \\
\nabla \phi \cdot N_f &= -\hat{\vartheta} \text{ on } \Gamma_f, \quad \phi = 0 \text{ on } \Gamma^+.
\end{aligned}
\]

The existence, uniqueness and regularity of this mixed boundary value problem are standard [12]. Moreover,

\[
\|\phi\|_{H^{\sigma+1}(\Omega_f^+)} \leq C \left( \|f\|_{H^{s+\frac{1}{2}}} \right) \left( \|\hat{g}\|_{H^{\sigma-1}(\Omega_f^+)} + \|\hat{\omega}\|_{H^{\sigma-1}(\Omega_f^+)} + \|\hat{\vartheta}\|_{H^{\sigma-\frac{1}{2}}} + \|\hat{J}\|_{H^{\sigma-\frac{1}{2}}} \right).
\]
Let
\[ \hat{h} = \hat{\omega} + \nabla \hat{j} - \nabla \phi. \]

Obviously, \( \text{div} \, \hat{h} = \Delta \hat{j} - \Delta \phi = \hat{\omega} \), \( \text{curl} \, \hat{h} = \text{curl} \, \hat{\omega} = 0 \) in \( \Omega_f^+ \). Moreover,
\[ \hat{h} \cdot N_f = \hat{\omega} \cdot N_f + \nabla \hat{j} \cdot N_f - \nabla \phi \cdot N_f = \hat{\omega} \]
due to \( \hat{j} = 0 \) near \( \Gamma_f \). As to the boundary value on \( \Gamma^+ \), we note that since \( \partial_1 \phi, \partial_2 \phi = 0 \) on \( \Gamma^+ \),
\[ \hat{h}_{|\Gamma} \times e_3 = \left( \hat{\omega} + \nabla \hat{j} - \nabla \phi \right)_{|\Gamma} \times e_3 = \left( \hat{h}_2 + \partial_2 \hat{j} - \partial_2 \phi, -\hat{h}_1 - \partial_1 \hat{j} + \partial_1 \phi, 0 \right) = (\hat{j}_1, \hat{j}_2, 0). \]

We conclude the proof of existence of solution to the system (4.3) and the regularity of all, we give the following lemma on a new formulation of the stability condition (5.1).

The proof of the uniqueness is similar to the proof of Lemma 5.4 in [17]. We present it here for completeness. It suffices to consider \( \hat{\omega}, \hat{\omega}, \hat{\omega} = 0 \) and \( \hat{j} = 0 \). We periodically extend \( \Omega_f^+ \) to be an unbounded domain in \( \mathbb{R}^3 \), which is denoted by \( \Omega_{f,p}^+ \). Then \( \hat{h} = \nabla \phi \), where \( \phi \) is a function on \( \Omega_{f,p}^+ \). Let \( \zeta(x) = \phi(x_1 + 2\pi, x_2, x_3) - \phi(x_1, x_2, x_3) \) for \( x \in \Omega_{f,p}^+ \). Then \( \nabla \zeta(x) = 0 \), and thus \( \zeta(x) \) is a constant. The condition \( \partial_1 \phi = \hat{h}_1 = 0 \) on \( \Gamma^+ \) implies \( \zeta(x) \equiv 0 \), and then \( \phi \) is periodic in \( x_1 \) in \( \Omega_{f,\Gamma}^+ \). Similarly, \( \phi \) is periodic in \( x_2 \). Thus, \( \phi \) can be viewed as a function on \( \Omega_f^+ \) and is a constant on \( \Gamma^+ \). Then we can obtain a function \( \phi \) in \( \Omega_f^+ \) satisfies
\[ \Delta \phi = 0 \quad \text{in} \quad \Omega_f^+, \quad N_f \cdot \nabla \phi = 0 \quad \text{on} \quad \Gamma_f, \quad \phi = \text{constant} \quad \text{on} \quad \Gamma^+. \]

Thus, the uniqueness of (4.6) implies \( \phi \equiv \text{constant} \), and then \( \hat{h} \equiv 0 \).

5. Uniform estimates for the linearized system

Given \( f(t, x'), \mathbf{u}(t, x), \mathbf{h}(t, x), \hat{h}(t, x) \), we assume there exist positive constants \( \delta_0, c_0 \) and \( L_0, L_1, L_2 \) such that for \( t \in [0, T] \),
- \( \| (\mathbf{u}, \mathbf{h}, \hat{h})(t) \|_{L^\infty(\Omega_f)} \leq L_0; \)
- \( \| f(t) \|_{H^{l+\frac{3}{2}}(\Omega_f)} + \| \partial_1 f(t) \|_{H^{l+\frac{3}{2}}(\Omega_f)} + \| (\mathbf{u}, \mathbf{h})(t) \|_{H^{l}(\Omega_f)} + \| \hat{h}(t) \|_{H^l(\Omega_f)} + \| \partial_1 \hat{h}(t) \|_{H^{l-1}(\Omega_f)} \leq L_1; \)
- \( \| (\partial_1 \mathbf{u}, \partial_1 \mathbf{h}, \partial_1 \hat{h}) \|_{L^\infty(\Omega_f)} \leq L_2; \)
- \( \| f(t) - f(x') \|_{H^{l+\frac{3}{2}}(\Omega_f)} \leq \delta_0, \quad -(1 - c_0) \leq f(t, x') \leq (1 - c_0), \quad x' \in \mathbb{T}^2; \)
- \( \Lambda(\mathbf{h}, \hat{h})(t) \geq c_1; \)

together with
\[ \text{(5.1)} \quad \begin{cases} \text{div} \mathbf{u} = \text{div} \mathbf{h} = 0 \text{ in } \Omega_f, \\ \mathbf{h} \cdot N_f = 0, \quad \partial_1 f = \mathbf{u} \cdot N_f \text{ on } \Gamma_f, \\ u_3 = h_3 = 0 \text{ on } \Gamma, \\ \text{div} \hat{h} = 0, \quad \text{curl} \hat{h} = 0 \text{ in } \Omega_f^+, \\ \hat{h} \cdot N_f = 0 \text{ on } \Gamma_f, \\ \hat{h} \times \mathbf{e}_3 = \hat{j} \text{ on } \Gamma^+. \end{cases} \]

In this section, we linearize the equivalent system derived in Section 3 around \( (f, \mathbf{u}, \mathbf{h}, \hat{h}) \), and present the uniform energy estimates for the linearized system. First of all, we give the following lemma on a new formulation of the stability condition (1.16).

Lemma 5.1. Under the stability condition (1.16), there exists \( c_1 > 0 \) such that
\[ \Lambda(\mathbf{h}, \hat{h}) \overset{\text{def}}{=} \inf_{\phi_1, \phi_2 \in L^2_0, \phi_1^2 + \phi_2^2 = 1} (h_1 \phi_1 + h_2 \phi_2)^2 + (\hat{h}_1 \phi_1 + \hat{h}_2 \phi_2)^2 \geq c_1. \]
Proof. Let \( \mathbf{q} = (q_1, q_2, q_3) \perp \mathbf{N}_f \) with \( q_3 = q_1 \partial_1 f + q_2 \partial_2 f \) and \( (q_1, q_2) \) determined by
\[
\begin{pmatrix}
1 + (\partial_1 f)^2 & \partial_1 f \partial_2 f \\
\partial_1 f \partial_2 f & 1 + (\partial_2 f)^2
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}
= \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}.
\]
Then by the fact \( \mathbf{h} \cdot \mathbf{N}_f = \mathbf{h} \cdot \mathbf{N}_f = 0 \), we get
\[
h_1 \varphi_1 + h_2 \varphi_2 = \sum_{i=1}^{3} h_i q_i, \quad \hat{h}_1 \varphi_1 + \hat{h}_2 \varphi_2 = \sum_{i=1}^{3} \hat{h}_i q_i,
\]
which along with (1.16) gives
\[
\inf_{\varphi_1^2 + \varphi_2^2 = 1} (h_1 \varphi_1 + h_2 \varphi_2)^2 + (\hat{h}_1 \varphi_1 + \hat{h}_2 \varphi_2)^2 > 0.
\]
Since the inequality above holds for all \( x \in \Gamma_f \), there exists a constant \( c_1 > 0 \) such that
\[
\inf_{\varphi_1^2 + \varphi_2^2 = 1} (h_1 \varphi_1 + h_2 \varphi_2)^2 + (\hat{h}_1 \varphi_1 + \hat{h}_2 \varphi_2)^2 \geq c_1,
\]
which yields (5.2). \( \square \)

For the system (3.2) and (3.11), we introduce the following linearized system:
\[
\begin{aligned}
\partial_t \hat{f} &= \tilde{\vartheta}, \\
\partial_t \hat{\vartheta} &= -2(u_1 \partial_1 \tilde{\vartheta} + u_2 \partial_2 \tilde{\vartheta}) + \sum_{i,j=1,2} (-u_i u_j + h_i h_j + \hat{h}_i \hat{h}_j) \partial_i \partial_j \hat{f} + g,
\end{aligned}
\]
where
\[
g = -\mathbf{N}_f \cdot \nabla(p_{u,u} - p_{h,h}) - \frac{1}{2} \mathbf{N}_f \cdot \nabla(|\mathbf{H}|^2 - |\mathbf{\hat{H}}|^2) + \frac{1}{2} \mathbf{N}_f |\mathbf{h}|^2.
\]
We remark that \( \int_{\mathbb{R}} \tilde{\vartheta} dx' \) may not vanish since we have performed a linearization.

Now we introduce the energy functional \( E_s(t) \) defined by
\[
E_s(t) = \left\| \left( \partial_t + u_i \partial_i \right) (\nabla)^{s-\frac{1}{2}} \hat{f} \right\|_{L^2}^2 + \frac{1}{2} \left\| \left( h_i - \partial_i \right) (\nabla)^{s-\frac{1}{2}} \hat{f} \right\|_{L^2}^2 + \frac{1}{2} \left\| \left( \hat{h}_i - \partial_i \right) (\nabla)^{s-\frac{1}{2}} \hat{f} \right\|_{L^2}^2.
\]
Also we define the standard energy
\[
E_s(t) = \| \hat{f}(t) \|_{H^{s-\frac{1}{2}}}^2 + \| \partial_t \hat{f}(t) \|_{H^{s-\frac{1}{2}}}^2.
\]
It is easy to see that there exists \( C(L_0) > 0 \) so that
\[
E_s(t) \leq C(L_0) E_s(t).
\]
The stability condition guarantees that there exists \( C(c_1, L_0) \) so that
\[
E_s(t) \leq C(c_1, L_0) \left\{ E_s(t) + \left\| \partial_t \hat{f} \right\|_{L^2}^2 + \| \hat{f} \|_{L^2}^2 \right\}.
\]
Before to state the energy estimates, we first give the following lemma concerning \( g \) defined by (5.4).

**Lemma 5.2.** It holds that
\[
\| g \|_{H^{s-\frac{1}{2}}} \leq C(L_1).
\]
Proof. According to the definition of \( p_{u,h} \), we obtain by standard elliptic estimates that
\[
\| \mathbf{N}_f \cdot \nabla (p_{u,u} - p_{h,h}) \|_{H^{s+\frac{1}{2}}} \leq C(L_1) \| \nabla (p_{u,u} - p_{h,h}) \|_{H^{s+\frac{1}{2}}} \leq C(L_1) \| \nabla (p_{u,u} - p_{h,h}) \|_{H^{(\Omega,T)}}
\]
\[
\leq C(L_1) \| (u, h) \|_{H^{2}(\Omega,T)} \leq C(L_1).
\]

Applying similar argument to \( \hat{\rho} = |\hat{h}|^2 - \hat{H}_1^2 |\hat{h}|^2 \) yields the same estimate for the second term in \( g \). Finally, the same estimate for the third term follows from (2.7) in Proposition 2.1.

\[\Box\]

**Proposition 5.1.** Given initial data \( \bar{f}_0 \in H^{s+\frac{1}{2}}, \, \bar{\partial}_0 \in H^{s-\frac{1}{2}}, \) there exists a unique solution \( (\bar{f}, \bar{\partial}) \in C([0,T]; H^{s+\frac{1}{2}} \times H^{s-\frac{1}{2}}) \) to the system (5.3) so that
\[
\sup_{t \in [0,T]} E_s(t) \leq C(c_1, L_0) \left( 1 + \| \bar{\partial}_0 \|^2_{H^{s-\frac{1}{2}}} + \| \bar{f}_0 \|^2_{H^{s+\frac{1}{2}}} \right) e^{C(c_1, L_1, L_2) T}.
\]

**Proof.** We only present the uniform estimates, which ensure the existence and uniqueness of the solution. Using the fact that
\[
\partial_t^2 \bar{f} = -2 \sum_{i=1,2} u_i \bar{\partial}_i \bar{f} + \sum_{i,j=1,2} (-u_i u_j + h_i h_j + \hat{h}_i \hat{h}_j) \partial_i \partial_j \bar{f} + g,
\]
a direct calculation shows that
\[
\frac{1}{2} \frac{d}{dt} \left\| (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f} \right\|_{L^2(T^2)}^2
\]
\[
= \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, \langle \nabla \rangle^{s-\frac{1}{2}} \partial_t^2 \bar{f} + u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f}) + \partial_t u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}) \right\rangle
\]
\[
= \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, \langle \nabla \rangle^{s-\frac{1}{2}} (\partial_t \bar{f} + \partial_t \bar{f}) \right\rangle
\]
\[
+ \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, \langle \nabla \rangle^{s-\frac{1}{2}} g + u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f}) + \partial_t u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}) \right\rangle
\]
\[
= \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, -u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f}) \right\rangle
\]
\[
+ \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, -u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f}) \right\rangle
\]
\[
+ 2 \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, [u_i, \langle \nabla \rangle^{s-\frac{1}{2}}] \partial_t \bar{f} \right\rangle
\]
\[
+ \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, [u_i, \langle \nabla \rangle^{s-\frac{1}{2}}] \partial_t \bar{f} \right\rangle
\]
\[
+ \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, \langle \nabla \rangle^{s-\frac{1}{2}} g + \partial_t u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}) \right\rangle
\]
\[
+ \left\langle (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}, \langle \nabla \rangle^{s-\frac{1}{2}} g + \partial_t u_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}) \right\rangle \equiv I_1 + \cdots + I_5.
\]

It follows from Lemma 2.2 that
\[
I_3 \leq 2 \| (\partial_t + u_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f} \|_{L^2} \| [u_i, \langle \nabla \rangle^{s-\frac{1}{2}}] \partial_t \bar{f} \|_{L^2}
\]
\[
\leq CE_s(t)^2 \| u \|_{H^{s+\frac{1}{2}}} \| \partial_t \bar{f} \|_{H^{s+\frac{1}{2}}},
\]
as well as
\[
I_4 \leq CE_s(t)^2 \left( \| u \|_{H^{s+\frac{1}{2}}}^2 + \| h \|_{H^{s+\frac{1}{2}}}^2 + \| \hat{h} \|_{H^{s+\frac{1}{2}}}^2 \right) \| \bar{f} \|_{H^{s+\frac{1}{2}}}
\]
Also we have
\[
I_5 \leq E_s(t)^2 \left( \| g \|_{H^{s+\frac{1}{2}}} + \| \partial_t u \|_{L^\infty} \| \bar{f} \|_{H^{s+\frac{1}{2}}} \right).
\]
We get by integration by parts that
\[
\langle \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, -u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \partial_t \tilde{f} \rangle \leq \| \partial_t u \|_{L^\infty} \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2,
\]
\[
\langle u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, -u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \partial_t \tilde{f} \rangle + \frac{1}{2} \frac{d}{dt} \| u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[
= \langle u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_t u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle \leq \| u_u \|_{L^\infty} \| \partial_t u \|_{L^\infty} \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2,
\]
which give rise to
\[
I_1 \leq -\frac{1}{2} \frac{d}{dt} \| u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2 + (1 + \| u_u \|_{W^{1,\infty}} + \| \partial_t u \|_{L^\infty})^2 \left( \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 \right).
\]
Similarly, we have
\[
\langle \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, -u_u u_u \partial_t \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle - \frac{1}{2} \frac{d}{dt} \| u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[
= -\langle u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_j u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle + \langle \langle \nabla \rangle^{-\frac{1}{2}} \partial_j \tilde{f}, \partial_j (u_u u_u) \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle
\]
\[
\leq \| u \|_{L^\infty} (\| \partial_t u \|_{L^\infty} + \| \nabla u \|_{L^\infty}) \left( \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 \right),
\]
and
\[
\langle u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, -u_u u_u \partial_t \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle
\]
\[
= \langle \partial_j (u_u u_u u_u) \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle - \langle u_u u_u u_u \partial_t \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle
\]
\[
= \langle \partial_j (u_u u_u u_u) \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle - \langle u_u \partial_t (u_u u_u) \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, u_u \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle
\]
\[
+ \langle u_u (\partial_t u_u) \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, u_u \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle \leq C \| u \|_{L^\infty} \| \nabla u \|_{L^\infty} \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2,
\]
as well as
\[
\langle \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle \leq -\frac{1}{2} \frac{d}{dt} \| \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2 + \| u \|_{L^\infty} (\| \partial_t u \|_{L^\infty} + \| \nabla u \|_{L^\infty}) \left( \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 \right),
\]
\[
\langle \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f}, \partial_t \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \rangle \leq -\frac{1}{2} \frac{d}{dt} \| \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2 + \| u_u \|_{L^\infty} (\| \partial_t u_u \|_{L^\infty} + \| \nabla u_u \|_{L^\infty}) \left( \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 \right).
\]
Thus, we obtain
\[
I_2 \leq \frac{1}{2} \frac{d}{dt} \| u_u \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2 - \frac{1}{4} \frac{d}{dt} \| \partial_j \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2 - \frac{1}{4} \frac{d}{dt} \| \partial_t \langle \nabla \rangle^{-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[
+ C(1 + \| u, h, \tilde{h} \|_{W^{1,\infty}} + \| \partial_t u, \partial_t h, \partial_t \tilde{h} \|_{L^\infty})^3 \left( \| \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 \right).
\]
Collecting these estimates of $I_1, \cdots, I_5$ above, we conclude that
\[
\frac{d}{dt} E_5(t) \leq \| \partial_t f \|_{H^{s-\frac{1}{2}}}^2 + C(L_0)(1 + \| u, h, \tilde{h} \|_{H^{s-\frac{1}{2}}} + \| \partial_t u, \partial_t h, \partial_t \tilde{h} \|_{L^\infty})^3 \cdot E_5(t).
\]
On the other hand, it is obvious that
\[ \frac{d}{dt}(\|\tilde{\partial}_t f\|_{L^2}^2 + \|\tilde{f}\|_{L^2}^2) \leq C(L_0)\mathcal{E}_s(t) + \|g\|_{L^2}^2. \]

Then by (5.6), we deduce that
\[ \mathcal{E}_s(t) \leq C(c_1, L_0)\left(\|\tilde{\partial}_0\|_{H^{s+\frac{1}{2}}}^2 + \|\tilde{f}_0\|_{H^{s+\frac{1}{2}}}^2 + \int_0^t \|g(\tau)\|_{H^{s-\frac{1}{2}}}^2 d\tau \right) 
+ \int_0^t \left(1 + \|u, h, \tilde{h}\|_{H^{s+\frac{1}{2}}} \right) + \|\tilde{\partial}(u, h, \tilde{h})(\tau)\|_{L^\infty}^3 \mathcal{E}_s(\tau) d\tau \right), \]
which together with Lemma 2.1 gives rise to
\[ \mathcal{E}_s(t) \leq C(c_1, L_0)\left(\|\tilde{\partial}_0\|_{H^{s+\frac{1}{2}}}^2 + \|\tilde{f}_0\|_{H^{s+\frac{1}{2}}}^2 + \int_0^t \|g(\tau)\|_{H^{s-\frac{1}{2}}}^2 d\tau + C(L_1, L_2) \int_0^t \mathcal{E}_s(\tau) d\tau \right). \]
The desired estimate (5.7) follows from Lemma 5.2 and Gronwall’s inequality. □

For the vorticity and current system (5.12)-(5.13), we introduce the following linearized system:
\begin{align*}
\partial_t \tilde{\omega} + u \cdot \nabla \tilde{\omega} - h \cdot \nabla \tilde{\xi} &= \tilde{\omega} \cdot \nabla u - \tilde{\xi} \cdot \nabla h \quad \text{in } Q_T, \\
\partial_t \tilde{\xi} + u \cdot \nabla \tilde{\xi} - h \cdot \nabla \tilde{\omega} &= \tilde{\xi} \cdot \nabla u - \tilde{\omega} \cdot \nabla h - 2\nabla u_t \times \nabla h_t \quad \text{in } Q_T,
\end{align*}

(5.8) \hspace{1cm} (5.9)

together with the initial data
\[ \tilde{\omega}(0, x) = \tilde{\omega}_0(x), \quad \tilde{\xi}(0, x) = \tilde{\xi}_0(x), \quad x \in \Omega_f. \]

The following proposition can be proved in a standard way (see [17]).

**Proposition 5.2.** Given \( \tilde{\omega}_0, \tilde{\xi}_0 \in H^{s-1}(\Omega_f) \), there exists a unique solution \( (\tilde{\omega}, \tilde{\xi}) \) to the initial value problem (5.8)-(5.10) such that
\[ \sup_{t \in [0, T]} \left( \|\tilde{\omega}(t)\|^2_{H^{s-1}(\Omega_f)} + \|\tilde{\xi}(t)\|^2_{H^{s-1}(\Omega_f)} \right) \leq \left(1 + \|\tilde{\omega}_0\|^2_{H^{s-1}(\Omega_f)} + \|\tilde{\xi}_0\|^2_{H^{s-1}(\Omega_f)} \right) e^{C(L_1)T}. \]

Moreover, it holds that
\[ \frac{d}{dt} \int_{\Gamma} \tilde{\omega}_3 d\sigma' = 0, \quad \frac{d}{dt} \int_{\Gamma} \tilde{\xi}_3 d\sigma' = 0. \]

Finally, the magnetic field \( \hat{h} \) in the vacuum is considered as a secondary variable computed from \( \Gamma_f \) and \( \tilde{J} \) by solving the following div-curl system:
\begin{align*}
\begin{cases}
\text{curl } \hat{h} = 0, \quad \text{div } \hat{h} = 0 \quad \text{in } \Omega_f^+, \\
\hat{h} \cdot N_f = 0 \quad \text{on } \Gamma_f, \quad \hat{h} \times e_3 = \tilde{J} \quad \text{on } \Gamma^+,
\end{cases}
\end{align*}

(5.11)

for any fixed time \( t \geq 0 \). According to Proposition 4.2
\[ \|\hat{h}(t)\|_{H^s(\Omega_f^+)} \leq C \left(\|f(t)\|_{H^{s+\frac{1}{2}}} \right) \|\tilde{J}(t)\|_{H^{s-\frac{1}{2}}}. \]

(5.12)
6. Construction of the iteration map

Given $f_0 \in H^{s+\frac{1}{2}}(T^2)$, $u_0$, $h_0 \in H^s(\Omega_{f_0})$ and $\hat{\mathbf{j}} \in C^k([0, T_0]; H^{s-\frac{1}{2}})$, $k = 0, 1$, we first solve $\hat{h}_0 \in H^s(\Omega_{f_0}^* )$ from (1.13) as mentioned before, and assume furthermore that there exists $c_0, c_1 > 0$ so that

$$-1 + \frac{c_0}{2} \leq f_0(x') \leq 1 - \frac{c_0}{2}, \quad \Lambda(h_0, \hat{h}_0) \geq 2c_1. \tag{6.1}$$

We then choose $f_s = f_0$ and take $\Omega_s = \Omega_{f_0}$ as the reference region. The initial data $(f_1, (\partial_t f)_I, \omega_s, \xi_s, \beta_1, \gamma_1)$, which satisfy certain evolution equations, while the secondary variable $\hat{h}$ is determined by $(f, \mathbf{j})$ through (5.11).

In addition, we choose a large constant $M_0 > 1$ so that

$$\|f_1\|_{H^{s+\frac{1}{2}}} + \|((\omega_s, \xi_s))\|_{H^{s-1}(\Omega_s)} + \|((\partial_t f)_I)\|_{H^{s-\frac{1}{2}}} + \|\beta_1\| + |\gamma_1| + \|\hat{h}_0\|_{H^s(\Omega_s)}$$
$$+ \|\mathbf{j}(t)\|_{H^{s-\frac{1}{2}}(T^2)} + \|\partial_t \mathbf{j}(t)\|_{H^{s-\frac{1}{2}}(T^2)} \leq M_0. \tag{6.2}$$

The iteration map we constructed is essentially based on iterating the unknowns $f, \omega_s, \xi_s, \beta_1, \gamma_1$, which satisfies certain evolution equations, while the secondary variable $\hat{h}$ is determined by $(f_1, \mathbf{j})$ through (5.11).

Now we introduce the following functional space.

**Definition 6.1.** Given two positive constants $M_1, M_2 > 0$ with $M_1 \geq 2M_0$, we define the space $\mathcal{X} = \mathcal{X}(T, M_1, M_2)$ as the collection of $(f, \omega_s, \xi_s, \beta_1, \gamma_1)$, which satisfies

$$\|f(0)\|_{H^{s+\frac{1}{2}}} + \|((\omega_s, \xi_s))\|_{H^{s-1}(\Omega_s)} + \|((\partial_t f)_I)\|_{H^{s-\frac{1}{2}}} + \|\beta_1\| + |\gamma_1| \leq M_1, \tag{6.3}$$
$$\sup_{t \in [0, T]} \|f(t)\|_{H^{s+\frac{1}{2}}} + \|\partial_t f(t)\|_{H^{s-\frac{1}{2}}} + \|((\omega_s, \xi_s))\|_{H^{s-1}(\Omega_s)} + \|\beta_1(t)\| + |\gamma_1(t)| \leq M_1, \tag{6.4}$$
$$\sup_{t \in [0, T]} \|\partial_t^2 f(t)\|_{H^{s-\frac{1}{2}}} + \|\partial_t \omega_s, \partial_t \xi_s(t)\|_{H^{s-1}(\Omega_s)} + \|\partial_t \beta_1(t)\| + |\partial_t \gamma_1(t)| \leq M_2, \tag{6.5}$$

together with the following compatibility conditions:

$$\int_{T^2} \partial_t f(t, x') dx' = 0, \quad \int_{\Gamma} \omega_s dx' = \int_{\Gamma} \xi_s dx' = 0. \tag{6.7}$$

Given $(f, \omega_s, \xi_s, \beta_1, \gamma_1) \in \mathcal{X}(T, M_1, M_2)$, our goal is to construct an iteration map

$$(\tilde{f}, \tilde{\omega}_s, \tilde{\xi}_s, \tilde{\beta}_1, \tilde{\gamma}_1) = \mathcal{F}(f, \omega_s, \xi_s, \beta_1, \gamma_1) \in \mathcal{X}(T, M_1, M_2)$$

with suitably chosen constants $M_1, M_2$ and $T$. 


6.1. **Recover the bulk region, velocity and magnetic fields.** Recall that
\[ \Omega_f^+ = \{ x \in \Omega | x_3 > f(t, x') \}, \quad \Omega_f^- = \{ x \in \Omega | x_3 < f(t, x') \}, \]
and the harmonic coordinate map \( \Phi_f : \Omega_e \to \Omega_f \) defined in (2.6).

We first define \( \hat{h} \) by solving (5.11). Then \( \partial_t \hat{h} \) satisfies
\[ \begin{aligned}
curl \partial_t \hat{h} &= 0, \quad \text{div} \partial_t \hat{h} = 0 \text{ in } \Omega_f^+, \\
\partial_t \hat{h} \cdot N_f &= -\partial_t f \partial_t \hat{h} \cdot N_f + \hat{h}_1 \partial_t \partial_1 f + \hat{h}_2 \partial_t \partial_2 f \text{ on } \Gamma_f, \\
\partial_t \hat{h} \times e_3 &= \partial_i \hat{J} \text{ on } \Gamma^+.
\end{aligned} \] (6.8)

It follows from Proposition 4.2 that
\[ \| \hat{h}(t) \|_{H^1(\Omega_f^+)} + \| \partial_t \hat{h}(t) \|_{H^{-1}(\Omega_f^+)} \leq C(M_0, M_1). \] (6.9)

To recover \( u, h \) in \( \Omega_f \), we define an operator which projects any vector field in \( \Omega_f \) to its divergence-free part. More precisely, for any \( \omega \in H^1(\Omega_f) \), let \( P^\text{div}_f \omega = \omega - \nabla \phi \) with \( \phi \) solving the following mixed boundary value problem:
\[ \Delta \phi = \text{div} \omega \text{ in } \Omega_f, \quad \partial_3 \phi = 0 \text{ on } \Gamma, \quad \phi = 0 \text{ on } \Gamma_f. \] (6.10)

We denote
\[ \begin{aligned}
\tilde{\omega} &= P^\text{div}_f (\omega \circ \Phi_f^{-1}), \\
\tilde{\xi} &= P^\text{div}_f (\xi \circ \Phi_f^{-1}).
\end{aligned} \]

It follows from Lemma 2.1 for harmonic coordinates and standard elliptic estimates that
\[ \| (\tilde{\omega}, \tilde{\xi}) \|_{H^{-1}(\Omega_f)} \leq C(M_1), \quad \| (\partial_t \tilde{\omega}, \partial_t \tilde{\xi}) \|_{H^{-2}(\Omega_f)} \leq C(M_1, M_2). \] (6.11)

Moreover, since \( \text{div} \tilde{\omega} = 0 \) and \( \tilde{\omega}_3 = e_3 \cdot \tilde{\omega} = e_3 \cdot \omega_s = \omega_{s3} \) on \( \Gamma \), \( \tilde{\omega} \) satisfies compatibility conditions in (4.2) according to (6.7). Similar argument applies to \( \tilde{\xi} \). Then we can define the velocity field \( u \) and magnetic field \( h \) in \( \Omega_f \) by solving the following div-curl system
\[ \begin{aligned}
\text{curl} u &= \tilde{\omega}, \quad \text{div} u = 0 \text{ in } \Omega_f, \\
u \cdot N_f &= \partial_t f \text{ on } \Gamma_f, \quad u \cdot e_3 = 0, \quad \int_{\Gamma} u_i dx' = \beta_i \text{ on } \Gamma, \quad i = 1, 2,
\end{aligned} \] (6.12)

and
\[ \begin{aligned}
\text{curl} h &= \tilde{\xi}, \quad \text{div} h = 0 \text{ in } \Omega_f, \\
h \cdot N_f &= 0 \text{ on } \Gamma_f, \quad h \cdot e_3 = 0, \quad \int_{\Gamma} h_i dx' = \gamma_i \text{ on } \Gamma, \quad i = 1, 2.
\end{aligned} \] (6.13)

It follows from Proposition 4.1 and (6.11) that
\[ \begin{aligned}
\| u \|_{H^1(\Omega_f)} &\leq C(M_1)(\| \tilde{\omega} \|_{H^{-1}(\Omega_f)} + \| \partial_t f \|_{H^{-2}} + |\beta_1| + |\beta_2|) \leq C(M_1), \quad \| h \|_{H^1(\Omega_f)} \leq C(M_1)(\| \tilde{\xi} \|_{H^{-1}(\Omega_f)} + |\gamma_1| + |\gamma_2|) \leq C(M_1).
\end{aligned} \] (6.14)

In addition,
\[ u(0) = u_0, \quad h(0) = h_0. \]
Using the fact that on $\Gamma_f$,
\[
\partial_t (u \cdot N_f) = (\partial_t u + \partial_3 u \partial_t f) \cdot N_f + u \cdot \partial_t N_f,
\]
we deduce that
\[
\begin{cases}
\text{curl} \partial_t u = \partial_t \tilde{\omega}, & \text{div} \partial_t u = 0 \text{ in } \Omega_f, \\
\partial_t u \cdot N_f = \partial_t f - \partial_t f \partial_3 u \cdot N_f + u_1 \partial_1 \partial_t f + u_2 \partial_2 \partial_t f \text{ on } \Gamma_f, \\
\partial_t u \cdot \mathbf{e}_3 = 0, & \int_\Gamma \partial_t u \, dx = \partial_t \beta_i \text{ on } \Gamma, \ i = 1, 2.
\end{cases}
\]
By Proposition 4.1 and 6.11 again,
\[
\|\partial_t u\|_{H^{s-1} (\Omega_f)} \leq C(M_1, M_2),
\]
which implies
\[
\|u(t)\|_{L^\infty(\Gamma_f)} \leq \|u_0\|_{L^\infty(\Gamma_0)} + \int_0^t \|\partial_t u\|_{L^\infty(\Gamma_f)} \, dt \leq M_0 + TC(M_1, M_2).
\]
Similarly,
\[
\|\partial_t h(t)\|_{H^{s-1} (\Omega_f)} \leq C(M_1, M_2), \quad \|h(t)\|_{L^\infty(\Gamma_f)} \leq M_0 + TC(M_1, M_2).
\]
Also, we have
\[
\|f(t) - f_0\|_{L^\infty} \leq \|f(t) - f_0\|_{H^{s-\frac{1}{2}}} \leq T \|\partial_t f\|_{H^{s-\frac{1}{2}}} \leq TM_1,
\]
as well as
\[
|\Lambda (h, \hat{h}) - \Lambda (h_0, \hat{h}_0)| \leq TC (\|\partial_t h\|_{L^\infty (\Gamma_f)}, \|\partial_t \hat{h}\|_{L^\infty (\Gamma_f)}) \leq TC (M_0, M_1, M_2).
\]
Without loss of generality, we assume $M_1 \leq C(M_1) \leq C(M_0, M_1) \leq C(M_1, M_2) \leq C(M_0, M_1, M_2)$ and choose $T \leq \min [1, T_0]$ small enough so that
\[
TC (M_0, M_1, M_2) \leq \min [\delta_0, c_1] (\leq M_0).
\]
Let $L_0 = 2M_0$, $L_1 = 10C(M_0, M_1)$, $L_2 = 10C(M_0, M_1, M_2)$. We conclude that for any $t \in [0, T]$,
\[
\begin{align*}
&\|(u, h, \hat{h})(t)\|_{L^\infty (\Gamma_f)} \leq L_0; \\
&\|f(t)\|_{H^{s-\frac{1}{2}}} + \|\partial_t f(t)\|_{H^{s-\frac{1}{2}}} + \|(u, h)(t)\|_{H^s (\Omega_f)} + \|\hat{h}(t)\|_{H^s (\Omega_f)} + \|\partial_t \hat{h}(t)\|_{H^{s-1} (\Omega_f')} \leq L_1; \\
&\|\partial_t u, \partial_t h\|_{L^\infty (\Gamma_f)} \leq L_2; \\
&-(1 - c_0) \leq f(t, x') \leq (1 - c_0), \quad \|f(t) - f_0\|_{H^{s-\frac{1}{2}}} \leq \delta_0; \\
&\Lambda (h, \hat{h})(t) \geq c_1;
\end{align*}
\]
which are nothing but the bounds listed at the beginning of Section 5.

6.2. Define the iteration map. Given $(f, u, h, \hat{h})$ as above, we define the iteration map. We first solve $\hat{f}_1$ by the linearized system (5.3) and then $(\tilde{\omega}, \tilde{\xi})$ by (5.8) and (5.9) with the initial data
\[
\left( \hat{f}_1(0), \tilde{\omega}(0), \tilde{\xi}(0) \right) = (f_0, (\partial_t f)_t, \omega_{s_1}, \xi_{s_1}).
\]
We define
\[ \tilde{\omega}_s = \tilde{\omega} \circ \Phi_f, \quad \tilde{\xi}_s = \tilde{\xi} \circ \Phi_f, \]
\[ \tilde{\beta}_i(t) = \beta_i(0) - \int_0^t \int \left( u_j \partial_j u_l - h_j \partial_j h_l \right) dx' d\tau, \]
\[ \tilde{\gamma}_i(t) = \gamma_i(0) - \int_0^t \int \left( u_j \partial_j h_i - h_j \partial_j u_i \right) dx' d\tau, \]
\[ \tilde{f}(t, x') = \tilde{f}_1(t, x') - \langle \tilde{f}_1 \rangle + \langle f_0 \rangle. \]

Note that \( \langle \tilde{f} \rangle = \langle f_0 \rangle \) and \( \int_{\Omega^2} \partial_i \tilde{f}(t, x') dx' = 0 \) for \( t \in [0, T] \). Moreover, according to Proposition 5.2,
\[ \int_{\Gamma} \tilde{\omega}_s dx' = \int_{\Gamma} \tilde{\omega}_s dx' = 0, \quad \int_{\Gamma} \tilde{\xi}_s dx' = \int_{\Gamma} \tilde{\xi}_s dx' = 0. \]

The iteration map \( \mathcal{F} \) is defined as follows
(6.16)
\[ \mathcal{F}(\bar{f}, \bar{\omega}_s, \bar{\xi}_s, \bar{\beta}_i, \bar{\gamma}_i) \xrightarrow{\text{def}} (\tilde{f}, \tilde{\omega}_s, \tilde{\xi}_s, \tilde{\beta}_i, \tilde{\gamma}_i). \]

**Proposition 6.1.** There exist \( M_1, M_2, T > 0 \) depending on \( c_0, c_1, \delta_0, M_0 \) so that \( \mathcal{F} \) is a map from \( \mathcal{X}(T, M_1, M_2) \) to itself.

**Proof.** First note that the initial conditions is automatically satisfied. Proposition 5.1 and Proposition 5.2 ensure that for any \( t \in [0, T] \),
\[ \left( \| \tilde{f}(t) \|_{H^{s+\frac{1}{2}}} + \| \partial_i \tilde{f}(t) \|_{H^{s-\frac{1}{2}}} + \| \tilde{\omega}_s(t) \|_{H^{s-1}(\Omega_s)} + \| \tilde{\xi}_s(t) \|_{H^{s-1}(\Omega_s)} \right) \leq C(c_0, M_0) e^{c(M_1, M_2)T}. \]

From the equation (5.3), (5.8) and (5.9) together with (6.9) for \( \hat{h} \), we deduce that
\[ \sup_{t \in [0, T]} \left( \| \partial_i^2 \tilde{f} \|_{H^{s-\frac{1}{2}}} + \| \partial_i \tilde{\omega}_s, \partial_i \tilde{\xi}_s \|_{H^{s-2}(\Omega_s)} \right) \leq C(M_1). \]

Moreover, it is obvious that
\[ |\partial_i \tilde{\beta}_i(t)| + |\partial_i \tilde{\gamma}_i(t)| \leq C(M_1), \]
\[ |\tilde{\beta}_i(t)| + |\tilde{\gamma}_i(t)| \leq M_0 + TC(M_1), \]
\[ \| \tilde{f}(t) - f_0 \|_{H^{s-\frac{1}{2}}} \leq \int_0^t \| \partial_i \tilde{f}(\tau) \|_{H^{s-\frac{1}{2}}} d\tau \leq TC(M_1). \]

We take \( M_1 = 2 \max\{M_0, C(c_0, M_0)\} \) and \( M_2 = C(M_1) \). Finally, let \( T \) be sufficiently small depending only on \( c_0, c_1, \delta_0, M_0 \) so that all other conditions in Definition 6.1 are satisfied.

\[ \square \]

### 7. Contraction of the Iteration Map

7.1. **Contraction.** Let \( (f^A, \omega^A, \xi^A, \beta_i^A, \gamma_i^A) \) and \( (f^B, \omega^B, \xi^B, \beta_i^B, \gamma_i^B) \) be two elements in \( \mathcal{X}(T, M_1, M_2) \), and \( (\hat{f}^C, \hat{\omega}^C, \hat{\xi}^C, \hat{\beta}_i^C, \hat{\gamma}_i^C) = \mathcal{F}(f^C, \omega^C, \xi^C, \beta_i^C, \gamma_i^C) \) for \( C = A, B \). Correspondingly, we have quantities \( \hat{u}^C, \hat{h}^C \) and \( \hat{\tilde{h}}^C \) defined in \( \Omega^C_{f^C} \) and \( \Omega^+_{f^C} \) respectively. For a quantity \( q \), we denote by \( q^D \) the difference \( q^A - q^B \).
Proposition 7.1. There exists $T > 0$ depending on $c_0, \delta_0, M_0$ so that

$$\tilde{E}_s^D := \sup_{t \in [0, T]} \left( \| f^D(t) \|_{H^{1/2}} + \| \partial_t f^D(t) \|_{H^{1/2}} + \| \omega^D(t) \|_{H^{-1}(\Omega)} \right)$$

$$+ \| \tilde{g}^D(t) \|_{H^{-1}(\Omega)} + \| \tilde{\beta}^D(t) \| + |\tilde{\gamma}^D(t)| \right)$$

$$\leq \frac{1}{2} \sup_{t \in [0, T]} \left( \| f^D(t) \|_{H^{1/2}} + \| \partial_t f^D(t) \|_{H^{1/2}} + \| \omega^D(t) \|_{H^{-1}(\Omega)} \right)$$

$$+ \| \tilde{g}^D(t) \|_{H^{-1}(\Omega)} + \| \tilde{\beta}^D(t) \| + |\tilde{\gamma}^D(t)| \right):= E_s^D.$$

Proof. First of all, by the elliptic estimates, we have

$$\| \Phi_{f^A} - \Phi_{f^B} \|_{H^1(\Omega)} \leq C(M_0) \| f^A - f^B \|_{H^{1/2}} \leq CE_s^D.$$  

Due to the fact that $u^A$ and $u^B$ are defined on different regions, one can not estimate their difference directly. To this end, we introduce for $C = A, B$,

$$u^C = u^C \circ \Phi_{f^C}, \quad h^C = h^C \circ \Phi_{f^C}, \quad \hat{h}^C = \hat{h}^C \circ \Phi_{f^C}.$$  

We first show that

$$(7.1) \quad \left\| \left( u^D, h^D \right) \right\|_{H^{-1}(\Omega)} + \left\| \hat{h}^D \right\|_{H^{-1}(\Omega^*)} \leq CE_s^D.$$  

For a vector field $v$, defined on $\Omega^*$, we define

$$\text{curl}_C v_* = (\text{curl}(v_* \circ (\Phi_{f^C})^{-1})) \circ \Phi_{f^C}, \quad \text{div}_C v_* = (\text{div}(v_* \circ (\Phi_{f^C})^{-1})) \circ \Phi_{f^C},$$

for $C = A, B$. Then we find by (6.12) that for $C = A, B$,

$$\left\{ \begin{array}{l}
\text{curl}_C u^C = \tilde{\omega}^C, \\
\text{div}_C u^C = 0 \quad \text{in} \Omega^*, \\
u^C \cdot e_3 = 0, \\
\int_{\Gamma} u^C d\Gamma' = \beta^C_i \quad \text{on} \Gamma.
\end{array} \right.$$  

Thus, we obtain

$$\left\{ \begin{array}{l}
\text{curl}_A u^D = \tilde{\omega}^D + (\text{curl}_B - \text{curl}_A) u^B \
\text{div}_A u^D = (\text{div}_B - \text{div}_A) u^B \\
u^D \cdot N_{f^A} = \partial_t f^D + u^B \cdot (N_{f^B} - N_{f^A}) \
\int_{\Gamma} u^D d\Gamma' = \beta^D_i \quad \text{on} \Gamma.
\end{array} \right.$$  

A tedious but direct calculation shows that

$$\| (\text{curl}_B - \text{curl}_A) u^B \|_{H^{-1}(\Omega^*)} \leq C \| \Phi_{f^A} - \Phi_{f^B} \|_{H^{-1}(\Omega)} \leq C \| f^D \|_{H^{1/2}} \leq CE_s^D.$$  

Similarly,

$$\| (\text{div}_B - \text{div}_A) u^B \|_{H^{-1}(\Omega)} \leq CE_s^D, \quad \| u^B \cdot (N_{f^B} - N_{f^A}) \|_{H^{1/2}} \leq CE_s^D.$$  

We deduce from Proposition 4.1 that

$$\| u^D \|_{H^{-1}(\Omega)} \leq C \left( \| \tilde{\omega}^D \|_{H^{-1}(\Omega^*)} + \| \partial_t f^D \|_{H^{1/2}} + E^D \right) \leq CE_s^D.$$  

By applying similar arguments to $h, \hat{h}$, we have from Proposition 4.1 and 4.2 that

$$\| h^D \|_{H^{-1}(\Omega)} \leq CE_s^D, \quad \| \hat{h}^D \|_{H^{-1}(\Omega^*)} \leq CE_s^D.$$
Thus, we conclude the proof of (7.1).

To estimate $f^D$, we first note that

\begin{equation}
\begin{aligned}
\partial_t j^D_1 &= \tilde{\theta}^D, \\
\partial_t \tilde{\theta}^D &= 2\left(u^A_i \partial_i \tilde{\theta}^D + u^A_j \partial_j \tilde{\theta}^D\right) + \sum_{i,j=1,2} \left(-u^B_i u^A_i + h^B_i h^A_i + \hat{h}^B_i \hat{h}^A_i\right) \partial_i \partial_j f^D + R,
\end{aligned}
\end{equation}

where

\begin{align*}
R &= -2 \left(\left(u^B_i + u^A_i\right) \partial_i \tilde{\theta}^B + \left(u^B_j + u^A_j\right) \partial_j \tilde{\theta}^B\right) \\
&\quad + \sum_{i,j=1,2} \left(\left(-u^B_i u^A_i + h^B_i h^A_i + \hat{h}^B_i \hat{h}^A_i\right) - \left(-u^B_i u^A_i + h^B_i h^A_i + \hat{h}^B_i \hat{h}^A_i\right)\right) \partial_i \partial_j f^D \\
&\quad + g^A - g^B.
\end{align*}

Here for $C = A, B,$

\begin{align*}
g^C &= -\frac{1}{2} N_{f^c} \cdot \nabla(p_{w^c}e^c - p_{h^c}h^c) + \frac{1}{2} N_{f^c} \cdot \nabla(|h^C|^2 - \hat{H}^D_j |\hat{h}^C|^2 + \frac{1}{2} (\hat{N}^+_{f^c} - N_{f^c})|\hat{h}^C|^2).
\end{align*}

It is direct to verify that

\begin{equation}
||R||_{H^{s-\frac{1}{2}}} \leq C E^D_s.
\end{equation}

We denote

\begin{align*}
\hat{F}^D_s(\partial_t f^D_1, f^D_1) &= \|\partial_t + u^A_i \partial_i \|_{L^2}^2 + \frac{1}{2} \|h^B_i \partial_i \|_{L^2}^2 + \frac{1}{2} \|h^B_i \partial_i \|_{L^2}^2, \\
\hat{F}^D_s(\partial_t f^D_1, f^D_1) &= \|\partial_t + u^A_i \partial_i \|_{L^2}^2.
\end{align*}

A similar argument as in Proposition [5.1] gives

\begin{equation}
\frac{d}{dt} \left(\hat{F}^D_s(\partial_t f^D_1, f^D_1) + ||f^D_1||_{L^2}^2 + ||\partial_t f^D_1||_{L^2}^2\right) \leq C(E^D_s + \hat{E}^D_s).
\end{equation}

where

\begin{equation}
\hat{E}^D_s = \sup_{t \in [0, T]} \left(||f^D_1(t)||_{H^{s-\frac{1}{2}}} + ||\partial_t f^D_1(t)||_{H^{s-\frac{1}{2}}}\right).
\end{equation}

Stability condition on $h^A, \hat{h}^A$ implies that

\begin{equation}
\|f^D_1\|_{H^{s-\frac{1}{2}}}^2 + \|\partial_t f^D_1\|_{H^{s-\frac{1}{2}}}^2 \leq C\left(\hat{F}^D_s(\partial_t f^D_1, f^D_1) + ||f^D_1||_{L^2}^2 + ||\partial_t f^D_1||_{L^2}^2\right).
\end{equation}

It follows that

\begin{equation}
\hat{E}_{1s} \leq CTE_s^D,
\end{equation}

which implies

\begin{equation}
\sup_{t \in [0, T]} \left(||f^D(t)||_{H^{s-1}} + ||\partial_t f^D(t)||_{H^{s-\frac{1}{2}}}\right) \leq CTE_s^D.
\end{equation}

Similar to the proof of Proposition 5.2, one can show that

\begin{equation}
\sup_{t \in [0, T]} \left(||\partial_t u^D(t)||_{H^{s-2}(\Omega)} + ||\partial_\gamma u^D(t)||_{H^{s-2}(\Gamma_s)}\right) \leq CTE_s^D.
\end{equation}
Finally, by using the equation
\[ \tilde{p}_i^C(t) = p_i^C(0) + \int_0^t \int_\Gamma u_j^C(\partial_j u_i^C - h_j^C \partial_j h_i^C)dx'd\tau, \]
it is obvious that
\[ |\tilde{p}_i^D(t)| \leq |p_i^D| + CTE_s^D = CTE_s^D. \]
(7.5)

Similarly,
\[ |\tilde{\gamma}_i^D(t)| \leq |\gamma_i^D| + CTE_s^D = CTE_s^D. \]
(7.6)

We deduce from (7.1) and (7.3)–(7.6) that
\[ E_s^D \leq CTE_s^D. \]
The proof is concluded by taking \( T = \frac{1}{2C} \) with \( C \) depending only on \( c_0, \delta_0, M_0 \). \( \square \)

7.2. **The limit system.** Proposition 6.1 and Proposition 7.1 ensure that the map \( F \) has a unique fixed point \((f, \omega, \xi, \beta_i, \gamma_i) \) in \( \mathcal{X}(T, M_1, M_2) \). From the construction of \( F \), we know that \((f, \omega, \xi, \beta_i, \gamma_i) \) satisfies
\[
\begin{aligned}
\begin{cases}
\partial_t f = \theta - \langle \theta \rangle, \\
\partial_t \theta = 2(u_i \partial_1 \theta + u_2 \partial_2 \theta) - \sum_{i,j=1,2} (u_i u_j - h_i h_j - \hat{h}_i \hat{h}_j) \partial_i \partial_j f \\
\quad - N_f \cdot \nabla (p_{u,u} - p_{h,h}) + \frac{1}{2} N_f \cdot \nabla (|\hat{h}|^2 - \hat{H}_f |\hat{h}|^2) + \frac{1}{2} (\hat{N}_f^+ - N_f) |\hat{h}|^2,
\end{cases}
\end{aligned}
\]
where \((u, h)\) solves the div-curl system
\[
\begin{aligned}
\begin{cases}
\text{curl } u = P_f^{\text{div } \omega}, \quad \text{div } u = 0 \text{ in } \Omega_f, \\
u \cdot N_f = \partial_t f \text{ on } \Gamma_f, \quad u_3 = 0 \text{ on } \Gamma, \\
\int_\Gamma u_3 dx' = \beta_i, \quad \partial_3 \beta_i = - \int_\Gamma (u_3 \partial_j u_i - h_3 \partial_j h_i) dx', \quad i = 1, 2,
\end{cases}
\end{aligned}
\]
(7.8)

and
\[
\begin{aligned}
\begin{cases}
\text{curl } h = P_f^{\text{div } \xi}, \quad \text{div } h = 0 \text{ in } \Omega_f, \\
h \cdot N_f = 0 \text{ on } \Gamma_f, \quad h_3 = 0 \text{ on } \Gamma, \\
\int_\Gamma h_3 dx' = \gamma_i, \quad \partial_3 \gamma_i = - \int_\Gamma (u_3 \partial_j h_i - h_3 \partial_j u_i) dx', \quad i = 1, 2,
\end{cases}
\end{aligned}
\]
(7.9)

and \( \hat{h} \) solves the div-curl system
\[
\begin{aligned}
\begin{cases}
\text{curl } \hat{h} = 0, \quad \text{div } \hat{h} = 0 \text{ in } \Omega_f^+, \\
\hat{h} \cdot N_f = 0 \text{ on } \Gamma_f, \quad \hat{h} \times e_3 = J \text{ on } \Gamma^+,
\end{cases}
\end{aligned}
\]
(7.10)

as well as
\[
\begin{aligned}
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega = h \cdot \nabla \xi + \omega \cdot \nabla u - \xi \cdot \nabla h \text{ in } Q_f, \\
\partial_t \xi + u \cdot \nabla \xi = h \cdot \nabla \omega + \xi \cdot \nabla u - \omega \cdot \nabla h - 2 \nabla u_i \times \nabla h_i \text{ in } Q_T.
\end{cases}
\end{aligned}
\]
(7.11)

We also recall that \( p_{v,v} \) for \( v = u, h \) is determined by the elliptic equation
\[
\begin{aligned}
\begin{cases}
\Delta p_{v,v} = -\text{tr}(\nabla v \nabla v) \text{ in } \Omega_f, \\
p_{v,v} = 0 \text{ on } \Gamma_f, \quad e_3 \cdot \nabla p_{v,v} = 0 \text{ on } \Gamma.
\end{cases}
\end{aligned}
\]
(7.12)
8. FROM THE LIMIT SYSTEM TO THE PLASMA-VACUUM INTERFACE SYSTEM

It is not obvious whether the limit system (7.7)-(7.12) is equivalent to the plasma-vacuum interface system (1.1)-(1.10). Following [17], we split the proof into several steps.

**Step 1.** curl $\mathbf{u} = \omega$ and curl $\mathbf{h} = \xi$.
By $\text{div} \, \mathbf{u} = \text{div} \, \mathbf{h} = 0$, it is easy to verify that
\[
\partial_t \text{div} \omega + \mathbf{u} \cdot \nabla \text{div} \omega = \mathbf{h} \cdot \nabla \text{div} \xi \quad \text{in } Q_T,
\]
\[
\partial_t \text{div} \xi + \mathbf{u} \cdot \nabla \text{div} \xi = \mathbf{h} \cdot \nabla \text{div} \omega \quad \text{in } Q_T,
\]
which imply that $\text{div} \omega = \text{div} \xi = 0$ since it is satisfied initially. Hence $\text{curl} \, \mathbf{u} = \omega$, $\text{curl} \, \mathbf{h} = \xi$ according to (7.8) and (7.9).

**Step 2.** Determination of the pressure.
Let the pressure $p$ in the plasma region be given by
\[
(8.1) \quad p = \frac{1}{2} \mathcal{H}(\mathbf{h})^2 + p_{u,u} - p_{h,h}.
\]
From the calculations in Section 3.1,
\[
(8.2) \quad -N_f |\mathbf{h}|^2 = 2 \sum_{i,j=1,2} \mathbf{h} \partial_i \partial_j f - \mathbf{N}_f \cdot \nabla(|\mathbf{h}|^2 - \mathcal{H}_f^2 |\mathbf{h}|^2) + (\mathcal{N}_f - N_f) |\mathbf{h}|^2,
\]
see (3.9) and (3.10).

**Step 3.** The velocity equation.
Let
\[
\mathbf{w} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{h} \cdot \nabla \mathbf{h} + \nabla p.
\]
We will show that $\mathbf{w}$ satisfies the following homogeneous equations:
\[
(8.3) \quad \begin{cases}
\text{div} \, \mathbf{w} = 0, & \text{curl} \, \mathbf{w} = 0 \quad \text{in } \Omega_f, \\
\mathbf{w} \cdot \mathbf{N}_f = 0 \quad \text{on } \Gamma_f, & w_3 = 0 \quad \text{on } \Gamma, & \int_{\Gamma_f} w_i dx = 0, & i = 1, 2,
\end{cases}
\]
which implies $\mathbf{w} \equiv 0$, i.e.,
\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{h} \cdot \nabla \mathbf{h} + \nabla p = 0 \quad \text{in } \Omega_f.
\]

First, by the definition of $p$,
\[
(8.4) \quad \text{div}(-\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{h} \cdot \nabla \mathbf{h} + \nabla p) = 0,
\]
which together with $\text{div} \, \partial_t \mathbf{u} = 0$ yields $\text{div} \, \mathbf{w} = 0$ in $\Omega_f$. On the other hand, a direct computation by using the equation of $\omega$ shows
\[
\text{curl} \, \partial_t \mathbf{u} = \partial_t \text{curl} \, \mathbf{u} = \partial_t \omega = -\mathbf{u} \cdot \nabla \omega + \mathbf{h} \cdot \nabla \xi + \omega \cdot \nabla \mathbf{u} - \xi \cdot \nabla \mathbf{h} = \text{curl}(-\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{h} \cdot \nabla \mathbf{h} + \nabla p).
\]
Thus, we obtain $\text{curl} \, \mathbf{w} = 0$ in $\Omega_f$.

Since $u_3 = 0, h_3 = 0$ on $\Gamma$,
\[
(8.5) \quad w_3 = \partial_t u_3 + u_i \partial_i u_3 - h_i \partial_i h_3 - \partial_3 p = 0 \quad \text{on } \Gamma.
\]
Moreover, according to (7.8),
\[ \int_{\Gamma} w_i \, dx' = \int_{\Gamma} (\partial_i u_i + u_j \partial_i u_j - h_j \partial_j h_j) \, dx' = 0, \quad i = 1, 2. \]

It only leaves the boundary condition of \( w \) on \( \Gamma_f \) to be proved. To this end, we first define the projection operator \( P : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) as
\[ Pg = g - \langle g \rangle. \]

By conversing the computations in Section 3.1, we find
\[
P\left\{ -2(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - N_f \cdot \nabla p - \sum_{i,j=1,2} u_{ij} \partial_i \partial_j f + \sum_{i,j=1,2} h_{ij} \partial_i \partial_j f \right\}
\]
\[= -2P(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - \frac{1}{2} N_f \cdot \hat{h}^2 - P(n_{u,u} - p_{h,h})
- \sum_{i,j=1,2} u_{ij} \partial_i \partial_j f + P \sum_{i,j=1,2} h_{ij} \partial_i \partial_j f
\]
\[= P\left\{ -2(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - \sum_{i,j=1,2} (u_{ij} h_{ij} - h_{ij}) \partial_i \partial_j f
- N_f \cdot \nabla (p_{u,u} - p_{h,h}) - \frac{1}{2} N_f \cdot \nabla (|\hat{h}|^2 - \hat{H}_f^+ |\hat{h}|^2) + \frac{1}{2} (N_f^+ - N_f) |\hat{h}|^2 \right\}
= P \partial_\theta.\]

Recalling that \( \partial_\theta = \partial_f^2 f, \partial_f = u \cdot N_f \), we obtain
\[
P\left\{ \partial_\theta (u \cdot N_f) + 2(u_1 \partial_1 (u \cdot N_f) + u_2 \partial_2 (u \cdot N_f)) + N_f \cdot \nabla p
+ \sum_{i,j=1,2} (h_{ij} u_{ij} - h_{ij}) \partial_i \partial_j f \right\} = 0,
\]
which together with the fact
\[ \partial_i N_f = (-\partial_1 \partial_j f, -\partial_2 \partial_j f, 0) = \left( -\partial_1 (u \cdot N_f), -\partial_2 (u \cdot N_f), 0 \right) \]
implies
\[
P\left\{ (\partial_x u + \partial_y u \cdot \partial_x f) \cdot N_f + (u_1 \partial_1 (u \cdot N_f) + u_2 \partial_2 (u \cdot N_f))
+ \sum_{i,j=1,2} (h_{ij} u_{ij} - h_{ij}) \partial_i \partial_j f + N_f \cdot \nabla p \right\} = 0.
\]

It follows from Lemma 3.1 that

\[ P \left\{ (\partial_\theta u + u \cdot \nabla u - h \cdot \nabla h + \nabla p) \big|_{\Gamma_f} \cdot N_f \right\} = 0. \tag{8.6} \]

On the other hand, (8.4) and (8.5) imply that
\[
\int_{\mathbb{R}^2} (\partial_\theta u + u \cdot \nabla u - h \cdot \nabla h + \nabla p) \big|_{\Gamma_f} \cdot N_f \, dx' = 0,
\]
which together with (8.6) yields
\[ w \cdot N_f = (\partial_\theta u + u \cdot \nabla u - h \cdot \nabla h + \nabla p) \cdot N_f = 0 \text{ on } \Gamma_f. \]

This concludes the proof of (8.3).
Step 4. The magnetic field equation.
Let \( b = \partial_1 h - h \cdot \nabla u + u \cdot \nabla h \). It suffices to show that

\[
\begin{aligned}
\text{div} b &= 0, \quad \text{curl} b = 0 \quad \text{in} \ \Omega_f, \\
\mathbf{b} \cdot N_f &= 0 \quad \text{on} \ \Gamma_f, \quad b_3 = 0 \quad \text{on} \ \Gamma, \quad \int_{\Gamma_f} b_i \, dx' = 0, \quad i = 1, 2.
\end{aligned}
\]

By using \( \mathbf{b} \cdot N_f = 0 \) on \( \Gamma_f \), we get

\[
0 = \partial_1 (\mathbf{b} \cdot N_f) + u_1 \partial_1 (\mathbf{b} \cdot N_f) + u_2 \partial_2 (\mathbf{b} \cdot N_f) \\
= (\partial_1 h + \partial_3 h \partial_3 f) \cdot N_f + h \cdot \partial_1 N_f + (u_1 \partial_1 h + u_1 \partial_3 h \partial_3 f) \cdot N_f \\
+ u_1 h \cdot \partial_1 N_f + (u_2 \partial_2 h + u_2 \partial_3 h \partial_3 f) \cdot N_f + u_2 h \cdot \partial_2 N_f \\
= (\partial_1 h + u \cdot \nabla h) \cdot N_f + \partial_1 f \partial_3 h \cdot N_f + h \cdot \partial_1 N_f \\
+ u_1 h \cdot \partial_1 N_f + u_2 h \cdot \partial_2 N_f + (u_1 \partial_3 h \partial_3 f + u_2 \partial_3 h \partial_3 N_f - u_3 \partial_3 h) \cdot N_f \\
= (\partial_1 h + u \cdot \nabla h) \cdot N_f + h \cdot \partial_1 N_f + u_1 h \cdot \partial_1 N_f + u_2 h \cdot \partial_2 N_f.
\]

On the other hand,

\[
\begin{aligned}
\mathbf{h} \cdot \partial_1 N_f + u \mathbf{h} \cdot \partial_1 N_f + u_3 \mathbf{h} \cdot \partial_2 N_f \\
= -h \partial_1 (u \cdot N_f) - h_2 \partial_2 (u \cdot N_f) + u \mathbf{h} \cdot \partial_1 N_f + u_3 \mathbf{h} \cdot \partial_2 N_f \\
= -h \left( \partial_1 u + \partial_3 u \partial_3 f \right) \cdot N_f - h_2 \left( \partial_2 u + \partial_3 u \partial_3 f \right) \cdot N_f \\
- h \partial_1 u \cdot \partial_1 N_f - h_2 \partial_2 u \cdot \partial_2 N_f - \sum_{i,j=1,2} u_i h \partial_1 \partial_j f \\
= h \left( \partial_1 u + \partial_3 u \partial_3 f \right) \cdot N_f - h_2 \left( \partial_2 u + \partial_3 u \partial_3 f \right) \cdot N_f \\
= -(\mathbf{h} \cdot \nabla u) \cdot N_f - (\partial_3 u \cdot N_f)(\mathbf{h} \cdot N_f) \\
= -(\mathbf{h} \cdot \nabla u) \cdot N_f.
\end{aligned}
\]

Thus, we deduce that

\[
(\partial_1 h - h \cdot \nabla u + u \cdot \nabla h) \cdot N_f = 0 \quad \text{on} \ \Gamma_f.
\]

Moreover,

\[
d\text{iv}(h \cdot \nabla u - u \cdot \nabla h) = 0
\]

together with \( \text{div} \partial_1 \mathbf{h} = 0 \) implies \( \text{div} \mathbf{b} = 0 \). By (7.11), we have

\[
\begin{aligned}
\text{curl}(\partial_1 \mathbf{h}) &= \partial_1 \xi = \text{curl}(-u \cdot \nabla \xi + h \cdot \nabla \omega + \xi \cdot \nabla u - \xi \cdot \nabla h - 2 \nabla u_i \nabla h_i) \\
&= \text{curl}(h \cdot \nabla u - u \cdot \nabla h).
\end{aligned}
\]

Other boundary conditions in (8.7) follows in a similar manner as in Step 3.

We remark that since \( \hat{h} \) is a secondary variable solved out from \( f \) (and \( \hat{J} \)), it automatically satisfies the equations. Furthermore, stability condition (1.20) has been shown at the end of Section 6.1 Hence, Step 1-Step 4 are enough to ensure that \( (u, h, \hat{h}, f, p) \) obtained in Section 7 is a solution of the system (1.1)-(1.10).
Acknowledgment

Yongzhong Sun is supported by NSF of China under Grant No. 11571167 and the PAPD of Jiangsu Higher Education Institutions. Wei Wang is supported by NSF of China under Grant No. 11501502 and the Fundamental Research Funds for the Central Universities. Zhifei Zhang is partially supported by NSF of China under Grant No. 11425103.

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