Witnessing quantum memory in non-Markovian processes

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We present a method to detect a quantum memory in a non-Markovian process. We call a process Markovian when the environment does not provide a memory that retains correlations across consecutive system-environment interactions. We define two types of non-Markovian processes, depending on the required memory being classical or quantum. We formalise this distinction using the process matrix formalism, through which a process is represented as a multipartite state. Within this formalism, a test for entanglement in a state can be mapped to a test for quantum memory in the corresponding process. This allows us to apply separability criteria and entanglement witnesses to the detection of a quantum memory. We apply the method to a physically motivated example of a process with quantum memory: two initially entangled qubits, representing system and environment respectively, coupled according to the Ising model with a transverse field. We prove Markovianity and classical memory for the special cases of vanishing coupling and transverse field, respectively. Using the Positive Partial Transpose criterion, we find that for almost all other parameters the process has a quantum memory. As with entanglement witnesses, our method of witnessing a quantum memory provides a versatile experimental tool for open quantum systems.

Introduction.— A challenge present in every experimental setup is to monitor the interaction between some desired system and the environment at many different time steps of an experiment. A common simplification is to assume that the environment does not have a memory, namely that the evolution of the system during a certain time interval is not correlated with the evolution at a different time, or with the state of the environment before the experiment started. A process with such a memory-less environment is generally called Markovian [1] and can still have noise: for example, stray light or temperature, as long as they do not affect different parts of the experiment in a correlated way.

Several methods that account for noise, such as error correction for quantum computation, typically rely on the assumption of Markovianity [2]. However, it is often the case that system-environment interactions at different times are correlated and/or depend on the initial state through some memory in the environment. An important distinction is whether the memory can or cannot be represented classically. A classical memory is typically easier to simulate and correct for, whereas a quantum memory implies more intricate coherences between the system and the environment. On the other hand, there are situations where detecting a quantum memory is a desired result: one may may regard a system of interest as ‘environment’, due to having limited access to it. In such cases, verifying the existence of a quantum memory through a probe certifies that the process preserves the required coherences.

Here, we give a rigorous definition of processes with classical memory and present a technique to detect a quantum memory using witnesses—a set of measurements that verify the property without the need of full tomography. We use the process matrix formalism [3, 4] and in this language the process can be represented as a multipartite state. For a specific partition of the state into subsystems, entanglement of the state corresponds to the existence of a quantum memory. Therefore, given a process we can employ all the known techniques that verify entanglement, in order to verify that the process is non-Markovian with quantum memory. We use the method that involves entanglement witnesses which maps them to witnesses for quantum memory.

To illustrate our method, we apply it to a physically motivated example of a process: the system and environment are qubits jointly prepared in a maximally entangled state and later interact according to the Ising model, in between two measurement stations $A$ and $B$ for the system. A quantum memory witness corresponds to the operations of $A$ and $B$ to verify the property. As separability criteria for the search of witnesses we use the positive partial transpose (PPT) applied on the state [5] and on symmetric extensions of the state [6]. We find that, in our example, the two methods are equivalent, fully characterising the process apart from a small set of parameters. We present our findings in a phase diagram of the parameters of the model with areas of Markovianity, classical and quantum memory.

Quantum processes.— A typical setup can be seen as follows: in a single run of the experiment a system goes...
through a number measurements stations from which we
extract our data, while in-between it may interact with
the environment. We call the process Markovian when
the environment does not provide a memory to correlate
either the initial state of the system and the various inter-
actions with the environment, or the various interactions
themselves which occur across the experiment.

There is a considerable amount of literature to propose
various definitions of markovianity for quantum systems
[7–15], and most focus on the properties of the time-
dependent reduced state for the system, which cannot
capture genuine multipartite temporal correlations. Here
we consider the notion of markovianity originally intro-
duced by [16, 17], and recently reformulated within the
comb formalism [18]. An example of a Markovian process
is shown in Figure 2 for a setup with three measurement
stations, \( A, B, C \), and two channels \( T_1, T_2 \). The channels
may allow an interaction of the system with the environ-
ment but is such that each is independent of the other
and the initial state. The output of \( C \) is discarded.

![Figure 2. A typical representation of a Markovian process where an initial state is going through measurements stations \( A, B, C \) and channels \( T_1, T_2 \) that may allow a local interaction with the environment.](image)

The process matrix formalism [3, 4] is a convenient way
to write down quantum processes. It assumes a number
of measurement stations where arbitrary quantum oper-
ations can be performed. An arbitrary quantum oper-
ation can be represented by a completely positive map
(CP) \( \mathcal{M}^{A_i \rightarrow A_o} \) that maps the input system \( A_i \) to
the output system \( A_o \) of the operation. The map might cor-
correspond to the transformation of the system resulting
from the detection of a particular measurement outcome.
The collection of these maps \( \mathcal{J}^A = \{ \mathcal{M}^A \} \), correspond-
ting to all possible outcomes of a measurement, repres-
ts all possible operations of a given instrument, with
\( \sum_{A \in \mathcal{J}^A} \mathcal{M}^A \) being a CP and trace preserving (CPTP)
map. The correlations of these operations can be represent-
ed as the joint probability of their maps given their
input system and the initial state. The output of
\( C \) is discarded.

Figure 2. A typical representation of a Markovian process
where an initial state is going through measurements stations
\( A, B, C \) and channels \( T_1, T_2 \) that may allow a local interaction with the environment.

to a CP map \( \mathcal{M}^A : \mathcal{L}(\mathcal{H}^{A_i}) \rightarrow \mathcal{L}(\mathcal{H}^{A_o}) \) is defined as
\( \mathcal{M}^{A_i A_o} := [\mathcal{I} \otimes \mathcal{M}(|I\rangle\langle I|)]^T \), where \( \mathcal{I} \) is the identity
map, \( |I\rangle = \sum_{j=1}^{d_{A_i}} |jj\rangle \in \mathcal{H}^{A_i} \otimes \mathcal{H}^{A_i} \), \( \{ |jj\rangle \}_{j=1}^{d_{A_i}} \) is an or-
thonormal basis on \( \mathcal{H}^{A_i} \) and \( T \) denotes matrix transposi-
tion in that basis and some basis on \( \mathcal{H}^{A_o} \).

In this language, it was found [21, 22] that the process matrix of a Markovian process must have the following form

\[
\mathcal{W}^{AB\cdots}_{M} = \rho^{A_i} \otimes \mathcal{T}^{A_o B_i} \otimes \cdots , \tag{2}
\]

where \( \mathcal{T}^{A_o B_i} \) is the Choi matrix, defined as above but
without the transposition, of the channel \( \mathcal{T}^{A \rightarrow B} \) describ-
ing the evolution from station \( A \) to station \( B \), while \( \rho \)
represents the state before the first measurement.

Now let us see what happens when the process is non-
Markovian, that is, when it cannot be written in the
above form. One specific case is when the different time
evolutions between stations are classically correlated, and
possibly correlated with the initial state. Concretely, we
can imagine that, in each run of the experiment, the first
station receives a state \( \rho_j \) from a set. The evolution
from the first to the second step can depend probabilis-
tically on \( j \). Furthermore, during the evolution some
classical information might go back into the environment,
and later get correlated with the future time evolution.
This case is generally represented by letting the evolution
between time steps be represented by a CP map, with the
various CP maps correlated with each other and with the
initial state, and satisfying the condition that the entire
process takes place with probability 1.

Such a process can be written as

\[
\mathcal{W}^{AB\cdots}_{M} = \sum_j q_j \rho_j^{A_i} \otimes \mathcal{T}^{A_o B_i} \otimes \cdots , \tag{3}
\]

where \( \rho_j, T_j \) are positive semidefinite matrices and \( W \)
is a ‘comb’, i.e. it satisfies a set of linear constraints that en-
sure normalization of probabilities [23]. This process can
be simulated with a classical memory that performs the
switching between the different Markovian ones; hence
we define a process with classical memory to be one that
has the above form.

If however the process cannot be written in either of
the above forms, we call it process with quantum memory.
For example, a beam splitter that behaves differently on the
polarisation of the incoming photon will entangle the
path degree of freedom with polarisation. If we look at the
evolution of the path degree of freedom, this can only be
simulated with a quantum memory that carries the pola-
risation degree of freedom and interacts coherently with
the path. A more general situation would be that the
external system—the quantum memory—can be jointly
prepared with the initial state and can later interact with
the system in a coherent way, as shown in Figure 3.

Detecting a quantum memory. — From Equation 2 we
observe that a Markovian process is a tensor product of
a state with Choi matrices of the channels. Choi’s theorem [20] ensures that these matrices are positive semi-definite; hence the process matrix can be viewed as a product state, up to normalization. In what follows we focus on scenarios with two measurement stations, A before B for which the output of B can be ignored since the only contribution to non-Markovianity stems from the initial system-environment correlations and their later interaction. Therefore, a process with classical memory $W = \sum_j q_j \rho_j^{AO} \otimes T_j^{AOB_i}$ can be written as a bipartite separable state $\rho_{sep}^{AB}$, where $A = A_j$, $B = A_O B_1$. We can now see that detecting entanglement of the state translates to detecting a quantum memory for the process; this is our main result.

One approach to detecting entanglement is through entanglement witnesses. An entanglement witness is a hermitian operator $Z$ whose expectation value is positive for all separable states, $\langle Z \rangle = \text{Tr}(ZW_{CI}) \geq 0$. A negative expectation value guarantees entanglement of the state, whereas a positive value provides no guarantee for separability. In the case of processes, we define a quantum memory witness to be a hermitian operator $Z$ whose expectation value is positive for all process matrices with classical memory

$$\langle Z \rangle = \text{Tr}(ZW_{CI}) \geq 0. \quad (4)$$

We note that an entanglement witness is also a quantum memory witness for the associated process, but the reverse is not necessarily true. This is because the space of process matrices is smaller than the space of valid process matrices as defined in [24], hence $\text{Tr}(ZW) = \text{Tr}(ZL(W)) = \text{Tr}(ZL(W)) - L$ is self adjoint—and so $L(Z)$ is also a quantum memory witness, although not necessarily an entanglement witness.

A quantum memory witness is a hermitian matrix, hence it can always be decomposed into combinations of CP maps [24]. From Equation 1 we see that the witness corresponds to the different maps of the measurement stations; for example for a bipartite process

$$Z = \sum_{i,j} \alpha_{i,j} M_i^{A1A_O} \otimes M_j^{B1_i}. \quad (5)$$

Separability criteria— An entanglement witness is obtained through various separability criteria: a property that is proved to hold for all separable states. These criteria provide necessary (but not sufficient) conditions for a state to be separable. The most notable criterion is based on partial transposition: if a state is separable, it must have a positive partial transpose (PPT) [5] with respect to any subsystem. For example, for a bipartite state $\rho^{AB}$ it is separable if (but not only if) $\rho^{TA} > 0$, where $\rho^{TA}$ denotes partial transposition with respect to subsystem A (transposition with respect to B provides an equivalent condition). An entanglement witness is obtained through the eigenvector corresponding to the negative eigenvalue of $\rho^{TA}$. If $\rho^{TA} < 0$, there exist $|\psi\rangle$ such that $\langle \psi | \rho^{TA} | \psi \rangle < 0$. Then $Z = |\psi\rangle\langle\psi|^{TA}$ is an entanglement witness as its expectation value is negative for the state $\rho$ and positive for any separable state [5]

$$\text{Tr}(Z \rho^A \otimes \rho^B) = \text{Tr}(\langle \psi | (\rho^{A})^{TA} \otimes \rho^B \rangle) \geq 0. \quad (6)$$

A different family of separability criteria was introduced by Doherty et al. [6]. These criteria are based on the PPT criterion applied on symmetric extensions of the state. For example, for $\rho^{AB}$, one can apply the PPT criterion to the extended state $\tilde{\rho}^{ABA}$ and obtain the following necessary conditions for the state to be separable:

$$\tilde{\rho} > 0, \quad \tilde{\rho}^{TA} > 0, \quad \tilde{\rho}^{TB} > 0. \quad (7)$$

One can keep extending the state (i.e. $\rho^{ABA}$) and obtain new families of conditions, each at least as strong as the previous ones. This creates a hierarchy of families of separability criteria which was proven to be complete—for an entangled state it is guaranteed that some level of the hierarchy will prove non-separability. The first step of the hierarchy is the PPT criterion, the first symmetric extension is the second step, and so on. As we discussed earlier, these results directly extend to define a complete family of criteria for classical memory.

Example— We use the above separability criteria to detect a quantum memory on the process of Figure 3. System and environment are two qubits prepared in the state $|\phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$. The system interacts with the environment according to the Ising model for nearest neighbor interaction between two dipoles [25], in-between two measurement stations, A and B. We consider the following Hamiltonian

$$H = -J \sigma_x \sigma_x - h(\sigma_z 1 + 1 \sigma_z), \quad (8)$$

where $J$ is the coupling strength with which the dipoles are aligned along the $x$ direction and $h$ is the strength of the external magnetic field along the $z$ direction. The operator that describes the evolution of system and environment is $U(J,h,t) = e^{-iH(J,h)t}$ acting on $A_O E_1$. The Choi matrix of the interaction is

$$[[U(J,h,t)]] = [[e^{-iH(J,h)t}]]^{A_O E_1 B_1 E_O}, \quad (9)$$
where, for a linear operator \( x \), we define \( \langle x \rangle = 1 \otimes x^\dagger \). The process matrix, as we obtain in the Appendix, is

\[
W^{A_i A_0 B_i}(J, h, t) = \text{Tr}_{E_0}[[U(J, h, t)]].
\] (10)

To detect a quantum memory for this process, we start with the PPT criterion [5]. As stated above, we can simply check if the partial transpose of the state is negative, \( \rho^{T_A} = W^{T_A i} < 0 \). If so, the state is entangled and an entanglement witness can be constructed by taking the projector of the negative eigenvector of \( W^{T_A i} \).

We apply the criterion to our process for various values of the parameters of the Hamiltonian. We detect a quantum memory for any combination of non-zero parameters \((J, h)\), fixing \( t = 1 \) and present our results in the next section. Note that although the problem is a simple check of negativity of the partial transpose \( W^{T_A i} \), it can be cast as the following SDP

\[
\text{maximize} \quad -\text{Tr}(ZW^{T_A i})
\]

\[
\text{subject to} \quad \text{Tr} Z = 1.
\] (11)

For a positive value of the maximizing quantity (over the hermitian semidefinite \( Z \)), the quantum memory witness is \( Z_l = Z^{T_A} \). Although this way of finding a witness is slower than to simply check if \( W^{T_A i} < 0 \), the advantage of the SDP is that we can add restrictions to the witness we want to find, in the form of linear constraints. For example, one might wish to obtain a witness of the form \( Z = \sum_{i,j} \psi^{A_j} \otimes \psi^{A_0} \otimes E_j \), where Alison only has to perform projective measurements, where the re-prepared state coincides with the measured one, while Ben is measuring with some projector \( E_j \). In this case, the linear constraint is \( PZP = Z \), where \( P \) is a swap operator that swaps the systems \( A_i \) and \( A_0 \) in the witness \( Z \).

The second step of the hierarchy developed by Doherty et al. [6] provides stronger criteria for separability. The main idea is that the PPT criterion is applied on a symmetric extension of the state, i.e. on \( W^{A_i A_0 B_i A_j} \). We present the associated SDPs in the Appendix.

**Results.**— We run our code using the PPT criterion for various combinations of the parameters \((J, h, t)\). As \( t \) is the time of interaction between the system and the memory, for time \( t = 0 \) the process is trivially Markovian for every nonzero \((J, h)\). This is also the case for \( J = 0 \) for every nonzero \((t, h)\) because the \( h^- \) term is affecting both systems separately. In the Appendix, we find all the other parameter values for which the process is Markovian. For \( h = 0 \) and every nonzero \((J, t)\) the process is with classical memory (or Markovian), as we prove in the Appendix. For a fixed time \( t = 1 \) we run the code for \((J, h) \in [0, 10] \) and present our findings in Figure 4. The value \( \text{Tr}(ZW) \) becomes negative when the PPT criterion has detected a quantum memory while in all other areas, a value close to zero is in general inconclusive. For this reason, we run the SDP testing the second step of the hierarchy. We find the same results, suggesting that the areas of not detecting a quantum memory were probably areas with classical memory. However, as we present in the Appendix, the points in the Figure where the lines are crossing correspond to Markovian processes. Additionally, we verify that these are the only Markovian points by calculating the norm distance \( ||W - W_M|| \) of each process with an associated Markovian one, as constructed in [22], which becomes zero when the process is Markovian.

![Figure 4. Heatmap of the value Tr(ZW) across the parameter space (J, h) ∈ [0, 10] with steps of 1/15, for t = 1, where Z is the witness obtained through the PPT criterion and W is the process matrix of our example.](image)

Our method expands considerably the investigation of open quantum systems. Our definition of non-Markovianity with classical memory, which leads to mapping entanglement to a quantum memory, provides an experimentally accessible tool and paves the way for further research. It can be used in experimental situations to detect that an external quantum system (environment) retains quantum correlations over time—whether this is desired or not. Furthermore, entanglement leads to many properties for states, which can be mapped to interesting properties for processes, for example multipartite entanglement could correspond to novel types of quantum non-Markovianity in multi-time-step processes. Hence, investigating other properties of entanglement could unveil more information about quantum memory in open quantum systems.

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For the second level of the hierarchy, the problem is no longer a simple computational task—we need to find if there exist a symmetric extension $\tilde{\rho}^{ABA}$ of a state $\rho^{AB}$ (for a process matrix $W^{AB}$), such the state and all of its partial transposes with respect to $A$ and $B$ are positive. The latter conditions are linear constraints on $\tilde{\rho}^{ABA}$ and so the problem can be cast as an SemiDefinite Program which can be solved efficiently [6]. We develop this method as it is a stronger test of non-separability, although in our example it turns out that the first level completely characterizes the type of non-Markovinity.

The SDP has a primal and a dual form. The primal form is whether it is possible to find an extension of the state such that the PPT criterion is satisfied. This is called the feasibility problem and is formulated as follows

$$\text{minimize } 0$$
$$\text{subject to } \tilde{\rho} \oplus \tilde{\rho}^{TA} \oplus \tilde{\rho}^{TB} > 0,$$

where the variable is $\tilde{\rho}$, defined to be a semidefinite matrix. Note that a block diagonal matrix is positive if and only if each of its block matrices is positive.

The primal SDP detects entanglement when its output is that the problem is infeasible. The dual SDP finds an explicit extension that yields one of the block matrices negative; hence it finds a witness. A witness is a Hermitian operator that yields a negative eigenvalue when applied on the initial state, i.e. Tr$(Z\rho) < 0$. The form of the dual SDP is asking to minimize the value Tr$(Z\tilde{\rho})$.

The actual implementation of the SDP is asking to minimize a slightly different value but the concept remains the same. To explicitly write the dual form we need some definitions. Due to symmetries on the extension $\tilde{\rho}$, it can be written as follows

$$\tilde{\rho} = G(x) = G_0 + \sum J x_J G_J$$

where $G_0$ is a fixed part that depends only on the given $\rho$ and $x_J$ is the coefficients for the rest of the basis elements $G_J$. The dual form of the SDP writes

$$\text{maximize } -\text{Tr}(F_0 Z)$$
$$\text{subject to } Z > 0$$
$$\text{Tr}(F_i Z) = 0,$$

where $F_0 = G_0 \oplus G_0^{Ta} \oplus G_0^{Tb}$, $G_0 = \Lambda(\rho)$, and $\Lambda$ is a linear map from $\mathcal{H}^A \otimes \mathcal{H}^B$ to $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^A$. Each $F_i$ is a block diagonal matrix composed of $G_J, G_J^{Tb}, G_J^{Ta}$. The SDP is maximizes the above value over the variable $Z$, defined to be hermitian semidefinite, living on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^A$. The witness is obtained by $Z = \Lambda^*(Z)$, where $\Lambda^*$ is the adjoint map of $\Lambda$.

**APPENDIX**

**higher-order criteria**

For the second level of the hierarchy, the problem is no longer a simple computational task—we need to find if there exist a symmetric extension $\tilde{\rho}^{ABA}$ of a state $\rho^{AB}$ (for a process matrix $W^{AB}$), such the state and all of its partial transposes with respect to $A$ and $B$ are positive. The latter conditions are linear constraints on $\tilde{\rho}^{ABA}$ and so the problem can be cast as an SemiDefinite Program which can be solved efficiently [6]. We develop this method as it is a stronger test of non-separability, although in our example it turns out that the first level completely characterizes the type of non-Markovinity.

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where $G_0$ is a fixed part that depends only on the given $\rho$ and $x_J$ is the coefficients for the rest of the basis elements $G_J$. The dual form of the SDP writes

$$\text{maximize } -\text{Tr}(F_0 Z)$$
$$\text{subject to } Z > 0$$
$$\text{Tr}(F_i Z) = 0,$$

where $F_0 = G_0 \oplus G_0^{Ta} \oplus G_0^{Tb}$, $G_0 = \Lambda(\rho)$, and $\Lambda$ is a linear map from $\mathcal{H}^A \otimes \mathcal{H}^B$ to $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^A$. Each $F_i$ is a block diagonal matrix composed of $G_J, G_J^{Tb}, G_J^{Ta}$. The SDP is maximizes the above value over the variable $Z$, defined to be hermitian semidefinite, living on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^A$. The witness is obtained by $Z = \Lambda^*(Z)$, where $\Lambda^*$ is the adjoint map of $\Lambda$.

**Derivation of the process matrix**

To construct the process matrix for our example, we can use the rules to construct combs by sequential composition of processes [23]. For the bipartite scenario considered here, the rules are as follows: we start from an initial state $\rho^{A_i E_i}$, while the joint system-environment
evolution is given, in Choi form, by a matrix $T^{A_O E_O B_I}$.

The comb for $A_I$, $A_O$, $B_I$, and $E_O$ is obtained combining $\rho$ and $T$ through the link product:

$$\rho * T = \text{tr}_{E_I} \left[ (\rho^{T_{E_I}} \otimes 1^{A_O B_I E_O}) (1^{A_I} \otimes T) \right],$$  

(15)

with the partial transpose acting on the common subsystem that is traced over. In our example, $T = [U]$ and, omitting the tensor product symbols,

$$[[U]] = \sum_{j'j'k'} |j'k'|^{A_O E_I} (U|jk|j'k'|U^\dagger)^{B_I E_O},$$  

(16)

while the initial state is $\rho = |\phi^+\rangle \langle \phi^+|$, so that $\rho^{T_{E_I}} = \frac{1}{2} \sum_i |i\rangle \langle i|^{A_I} |i\rangle^{E_I}$. The link product then gives

$$|\phi^+\rangle \langle \phi^+| * [[U]] = \frac{1}{2} \sum_{i,j} |j\rangle \langle j|^{A_I} |j\rangle \langle j|^{A_O} (U|jk|j'k'|U^\dagger)^{B_I E_O},$$  

(17)

which is, up to normalisation, the same expression as (16), with the first two factors swapped and $E_I$ substituted with $A_I$. The process matrix describing the first qubit is obtained by tracing out $E_O$

**Classical memory in the $h = 0$ case**

In this case the Hamiltonian is

$$H(J) = -J \sigma_x \otimes \sigma_x,$$  

(18)

so it has product eigenstates, built from the Pauli matrix’s eigenstates $\sigma_x |\mu\rangle = |\mu\rangle$, $\mu = \pm 1$:

$$H(J)|\mu\rangle \langle \nu| = -J \mu \nu |\mu\rangle \langle \nu|,$$  

(19)

$$U(J,t)|\mu\rangle \langle \nu| = e^{-J \mu \nu t} |\mu\rangle \langle \nu|,$$  

(20)

$$\mu, \nu = \pm 1.$$

Using the eigenstates of $H(J)$ in the definition of the Choi isomorphism, and taking the partial trace over $E_O$ of expression (17), we obtain for the process matrix

$$W^{A_I A_O B_I}(J,t) = \frac{1}{2} \sum_{\mu_1, \mu_2} |\mu_1 \mu_1\rangle \langle \mu_2 \mu_2|^{A_I A_O} \text{tr}_{E_O} \left( U|\mu_1 \nu_1\rangle \langle \mu_2 \nu_2|U^\dagger \right)^{B_I E_O}$$  

$$= \frac{1}{2} \sum_{\mu_1, \mu_2, \nu} e^{iJ(\mu_1 - \mu_2) \mu t} |\nu\rangle \langle \nu|^{A_I} |\mu_1\rangle |\mu_2\rangle^{A_O} |\mu_1\rangle |\mu_2\rangle^{B_I}.$$

We see that this matrix is an equal mixture of two product matrices:

$$W^{A_I A_O B_I}(J,t) = \frac{1}{2} \sum_{\nu = \pm 1} |\nu\rangle |\nu|^{A_I} [e^{i\nu J \sigma_z \cdot t}]^{A_O B_I}.$$

Therefore, the process we are describing is the equal mixture of two Markovian processes: for $\nu = \pm 1$ respectively, these correspond to processes where the system is initially in state $|\nu\rangle$, while from $A$ to $B$ it evolves according to the (single-qubit) Hamiltonian $H_\nu = -\nu \sigma_x$. Thus, even though in our model the initial system-environment state is entangled, the process for the system alone can be reproduced by substituting the environment with a classical variable, which samples with equal probabilities the values $\nu = \pm 1$, and determines both the initial state and the evolution from first to second time step.

**Markovian points**

Here we find the parameters $J$, $h$, for which the process is Markovian. A sufficient condition is that the unitary matrix that represents the system-environment evolution factorises: $U(J,h,t) = e^{-JH(J,h)t} = U_e \otimes U_\sigma$. As $U(J,h,t) = U(J_t,h_t,1)$, it is sufficient to consider the $t = 1$ case.

We find that the unitary factorises for the parameter values

$$J = \pi k_1, \quad k_1 \in \mathbb{Z},$$

$$h = \pm \frac{\pi}{2} \sqrt{k_2^2 - k_1^2}, \quad k_2 \in \mathbb{Z}, \quad k_2^2 \geq k_1^2,$$

plus an additional point $h = 0, J = \pi/2$, for which $U(J,h,1) = i \sigma_z \otimes \sigma_x$.

In principle, there could be parameter values for which the evolution is not trivial but the process is Markovian. We verified that this is not the case numerically: for a range of parameters, we calculated the norm distance of $W$ from the corresponding Markovian process $W_M = \rho^{A_I} \otimes T^{A_O B_I}$, where $\rho^{A_I}$ and $T^{A_O B_I}$ are obtained as partial traces of $W$ (this is the Markovianity test used in Ref. [22]). We found that the norm distance vanishes (and hence the process is Markovian) only for the parameter values found above. The values of the norm distance are shown in Figure 5.

![Figure 5. The norm distance $||W - W_M||$ across the parameter space $(J,h) \in [0,10]$ for $t = 1.$](image-url)