FISHER, KÄHLER, WEYL, AND THE QUANTUM POTENTIAL

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ABSTRACT. This is a basically expository article tracing connections of the quantum potential to Fisher information, to Kähler geometry of the projective Hilbert space of a quantum system, and to the Weyl-Ricci scalar curvature of a Riemannian flat spacetime with quantum matter.

CONTENTS

1. INTRODUCTION
2. QUANTUM GEOMETRY
3. PROBABILITY ASPECTS
3.1. FISHER INFORMATION
4. THE SCHRODINGER EQUATION IN WEYL SPACE
References

1. INTRODUCTION

There is a comprehensive outline of quantum geometry in [8] (cf. also [1, 4, 5, 7, 9, 11, 13, 14, 19, 20, 21, 22, 23, 28, 30, 31, 33, 36, 37, 41, 42, 43, 44, 45, 46, 55, 56, 63, 68, 71]). We will develop certain features and formulas in a “hands on” approach following [4, 5, 13, 19, 20, 21, 22, 23, 53, 55, 56, 68, 71] and spell out the nature of the Kähler geometry for the projective Hilbert space of a quantum system along with the relation between the Fisher metric and the Fubini-Study metric. Then we go to [15, 16, 26, 29, 34, 35, 61, 62] for discussion of connections between the quantum potential and Fisher information. Finally following [16, 17, 63] we indicate connections of the quantum potential to the Weyl-Ricci scalar curvature of space time, thus connecting quantum geometry, gravity, and Fisher information. Relations of Fisher information to entropy are also sketched. Roughly the idea is that for H the Hilbert space of a quantum system there is a natural quantum geometry on the projective space \( P(H) \) with inner product (A1) \[ \langle \phi|\psi \rangle = (1/2\hbar)g(\phi,\psi) + (i/2\hbar)\omega(\phi,\psi) \] where \( g(\phi,\psi) = 2\hbar R(\phi|\psi) \) is the natural Fubini-Study (FS) metric and \( g(\phi,\psi) = \omega(\phi,J\psi) \) (\( J^2 = -1 \)). On the other hand the FS metric is proportional to the Fisher information metric of the form (A2) \[ \text{Cos}^{-1}|\phi|\psi > |. \]

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Moreover (in 1-D for simplicity) (A3) \( F \propto \int \rho Qdx \) is a functional form of Fisher information where \( Q \) is the quantum potential and \( \rho = |\psi|^2 \). Finally one can argue that in a Riemannian flat spacetime (with quantum matter and Weyl geometry) the Weyl-Ricci scalar curvature is proportional to \( Q \). We will explain this below and refer to [16] for more details and perspective.

2. QUANTUM GEOMETRY

First we sketch the relevant symbolism for geometrical QM from [8] without much philosophy; the philosophy is eloquently phrased in [8, 5, 14, 19, 20, 41, 46] for example. Thus let \( H \) be the Hilbert space of QM and write it as a real Hilbert space with a complex structure \( J \). The Hermitian inner product is then \( \langle \phi, \psi \rangle = \frac{1}{2\hbar} g(\phi, \psi) + \frac{i}{2\hbar} \omega(\phi, \psi) \) (note \( g(\phi, \psi) = 2\hbar \Re(\phi, \psi) \) is the natural Fubini-Study (FS) metric and this is discussed below - cf. [19, 20, 21, 22, 23]). Here \( g \) is a positive definite real inner product and \( \omega \) is a symplectic form (both strongly nondegenerate). Moreover (B2) \( \langle \phi, J\psi \rangle = i \langle \phi, \psi \rangle \) and (B3) \( g(\phi, \psi) = \omega(\phi, J\psi) \). Thus the triple \((J, g, \omega)\) equips \( H \) with the structure of a Kähler space. Now, from [70], on a real vector space \( V \) with complex structure \( J \) a Hermitian form satisfies \( h(JX, JY) = h(X, Y) \). Then \( V \) becomes a complex vector space via \((a + ib)X = aX + bJX\). A Riemannian metric \( g \) on a manifold \( M \) is Hermitian if \( g(X, Y) = g(JX, JY) \) for \( X, Y \) vector fields on \( M \). Let \( \nabla_X \) be he Levi-Civita connection for \( g \) (i.e. parallel transport preserves inner products and the torsion is zero - see (2.1) below). A manifold \( M \) with \( J \) as above is called almost complex. A complex manifold is a paracompact Hausdorff space with complex analytic patch transformation functions. An almost complex \( M \) with Kähler metric (i.e. \( \nabla_X J = 0 \)) is called an almost Kähler manifold and if in addition the Nijenhuis tensor vanishes it is a Kähler manifold (see (2.1) below). Here the defining equations for the Levi-Civita connection and the Nijenhuis tensor are

\[
(2.1) \Gamma^k_{ij} = \frac{1}{2} g^{hk} [\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ji}]; \quad N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]
\]

Further discussion can be found in [70]. Material on the Fubini-Study metric will be provided later.

Next (following [8]) by use of the canonical identification of the tangent space (at any point of \( H \)) with \( H \) itself, \( \Omega \) is naturally extended to a strongly nondegenerate, closed, differential 2-form on \( H \), denoted also by \( \Omega \). The inverse of \( \Omega \) may be used to define Poisson brackets and Hamiltonian vector fields. Now in QM the observables may be viewed as vector fields, since linear operators associate a vector to each element of the Hilbert space. Moreover the Schrödinger equation, written here as \( \dot{\psi} = -(1/\hbar) J\hat{H}\psi \), motivates one to associate to each quantum observable \( \hat{F} \) the vector field (B4) \( Y_{\hat{F}}(\psi) = -(1/\hbar) J\hat{F}\psi \). The Schrödinger vector field is defined so that the time evolution of the system corresponds to the flow along the Schrödinger vector field and one can show that the vector field \( Y_{\hat{F}} \), being the generator of a one parameter family of unitary mappings on \( H \), preserves both the metric \( G \) and the symplectic form \( \Omega \). Hence it is locally, and indeed globally, Hamiltonian since \( H \) is a linear space. In fact the function which generates this Hamiltonian vector field is simply the expectation value of \( \hat{F} \). To see this write (B5) \( F : H \rightarrow \mathbb{R} \) via
$F(\psi) = \langle \psi, \hat{F}\psi \rangle = \langle \hat{F} \rangle = (1/2\hbar)G(\psi, \hat{F}\psi)$. Now if $\eta$ is any tangent vector at $\psi$

$$ (dF)(\eta) = \left. \frac{d}{dt} \langle \psi + t\eta, \hat{F}(\psi + t\eta) \rangle \right|_{t=0} = \langle \psi, \hat{F}\eta \rangle + \langle \eta, \hat{F}\psi \rangle = \frac{1}{\hbar} G(\hat{F}\psi, \eta) = \Omega(Y_{\hat{F}}, \eta) = (iY_{\hat{F}}\Omega)(\eta) $$

where one uses the selfadjointness of $\hat{F}$ and the definition of $Y_{\hat{F}}$ (recall the Hamiltonian vector field $X_f$ generated by $f$ satisfies the equation $iX_f\Omega = df$ and the Poisson bracket is defined via $\{f, g\} = \Omega(X_f, X_g)$). Thus the time evolution of any quantum mechanical system may be written in terms of Hamilton’s equation of classical mechanics; the Hamiltonian function is simply the expectation value of the Hamiltonian operator. Consequently Schrödinger’s equation is simply Hamilton’s equation in disguise and for Poisson brackets we have

$$ \{F, K\}_\Omega = \Omega(X_F, X_K) = \left\langle \frac{1}{i\hbar} [\hat{F}, \hat{K}] \right\rangle $$

where the right side involves the quantum Lie bracket. Note this is not Dirac’s correspondence principle since the Poisson bracket here is the quantum one determined by the imaginary part of the Hermitian inner product. Now look at the role played by $G$. It enables one to define a real inner product $G(X_F, X_K)$ between any two Hamiltonian vector fields and one expects that this Riemann inner product is related to the Jordan product. Indeed

$$ \{F, K\}_+ = \frac{\hbar}{2} G(X_F, X_K) = \left\langle \frac{1}{2}[\hat{F}, \hat{K}]_+ \right\rangle $$

Since the classical phase space is generally not equipped with a Riemannian metric the Riemann product $G$ does not have a classical analogue; however it does have a physical interpretation. One notes that the uncertainty of the observable $\hat{F}$ at a state with unit norm is (B6) $(\Delta \hat{F})^2 = \langle \hat{F}^2 \rangle - \langle \hat{F} \rangle^2 = \{F, F\}_+ - F^2$. Hence the uncertainty involves the Riemann bracket in a simple manner. In fact Heisenberg’s uncertainty relation has a nice form as seen via

$$ (\Delta \hat{F})^2(\Delta \hat{K})^2 \geq \left\langle \frac{1}{2\hbar} [\hat{F}, \hat{K}] \right\rangle^2 + \left\langle \frac{1}{2}[\hat{F}_\perp, \hat{K}_\perp]_+ \right\rangle^2 $$

where $\hat{F}_\perp$ is the nonlinear operator defined by (B7) $\hat{F}_\perp(\psi) = \hat{F}(\psi) - F(\psi)$. Thus $\hat{F}_\perp(\psi)$ is orthogonal to $\psi$ if $\|\psi\| = 1$. Using this one can write (2.5) in the form

$$ (\Delta \hat{F})^2(\Delta \hat{K})^2 \geq \left( \frac{\hbar}{2} \{F, K\}_\Omega \right)^2 + (\{F, K\}_+ - FK^2)^2 $$

The last expression in (2.6) can be interpreted as the quantum covariance of $\hat{F}$ and $\hat{K}$.

The discussion in [8] continues in this spirit and is eminently worth reading; however we digress here for a more “hands on” approach following [19, 20, 21, 22, 23]. Assume $H$ is separable with a complete orthonormal system $\{u_n\}$ and for any $\psi \in H$ denote by $[\psi]$ the
ray generated by $\psi$ while $\eta_n = (u_n|\psi)$. Define for $k \in \mathbb{N}$
\begin{equation}
U_k = \{ [\psi] \in P(H); \eta_k \neq 0 \}; \phi_k : U_k \to \ell^2(C) : \phi_k([\psi]) = \left( \frac{\eta_1}{\eta_k}, \ldots, \frac{\eta_{k-1}}{\eta_k}, \frac{\eta_{k+1}}{\eta_k}, \ldots \right)
\end{equation}
where $\ell^2(C)$ denotes square summable functions. Evidently $P(H) = \cup_k U_k$ and $\phi_k \circ \phi_j^{-1}$ is biholomorphic. It is easily shown that the structure is independent of the choice of complete orthonormal system. The coordinaes for $[\psi]$ relative to the chart $(U_k, \phi_k)$ are \( \{ z^k_n \} \) given via (B8) $z^k_n = (\eta_n/\eta_k)$ for $n < k$ and $z^k_n = (\eta_{n+1}/\eta_k)$ for $n \geq k$. To convert this to a real manifold one can use $z^k_n = (1/\sqrt{2})(x^k_n + iy^k_n)$ with
\begin{equation}
\frac{\partial}{\partial z^k_n} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^k_n} + i \frac{\partial}{\partial y^k_n} \right); \frac{\partial}{\partial \bar{z}^k_n} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^k_n} - i \frac{\partial}{\partial y^k_n} \right)
\end{equation}
extc. Instead of nondegeneracy as a criterion for a symplectic form inducing a bundle isomorphism between $TM$ and $T^*M$ one assumes here that a symplectic form on $M$ is a closed 2-form which induces at each point $p \in M$ a toplinear isomorphism between the tangent and cotangent spaces at $p$. For $P(H)$ one can do more than simply exhibit such a natural symplectic form; in fact one shows that $P(H)$ is a Kähler manifold (meaning that the fundamental 2-form is closed). Thus one can choose a Hermitian metric (B9) $\mathcal{G} = \sum g^k_{mn} dz^k_m \otimes d\bar{z}^k_n$ with
\begin{equation}
g^k_{mn} = (1 + \sum_i z^k_i \bar{z}^k_i)^{-1} \delta_{mn} - (1 + \sum_i z^k_i \bar{z}^k_i)^{-2} \bar{z}^k_m z^k_n
\end{equation}
relative to the chart $U_k, \phi_k$). The fundamental 2-form of the metric $\mathcal{G}$ is (B10) $\omega = i \sum_{m,n} g^k_{mn} dz^k_m \wedge d\bar{z}^k_n$ and to show that this is closed note that $\omega = i \partial \bar{\partial} f$ where locally (B11) $f = \log(1 + \sum_i z^k_i \bar{z}^k_i)$ (the local Kähler function). Note here that $\partial + \bar{\partial} = d$ and $d^2 = 0$ implies $\partial^2 = \bar{\partial}^2 = 0$ so $d\omega = 0$ and thus $P(H)$ is a K manifold.

Now on $P(H)$ the observables will be represented via a class of real smooth functions on $P(H)$ (projective Hilbert space) called Kählerian functions. Consider a real smooth Banach manifold $M$ with tangent space $TM$, and cotangent space $T^*M$. We remark that the extension of standard differential geometry to the infinite dimensional situation of Banach manifolds etc. is essentially routine modulo some functional analysis; there are a few surprises and some interesting technical machinery but we omit all this here. One should also use bundle terminology at various places but we will not be pedantic about this. One hopes here to simply give a clear picture of what is happening. Thus e.g. $L(T^*_xM, T_xM)$ denotes bounded linear operators $T^*_xM \to T_xM$ and $L_n(T_xM, \mathbb{R})$ denotes bounded n-linear forms on $T_xM$. An almost complex structure is provided by a smooth section $J$ of $L(TM) = \text{vector bundle of bounded linear}$ operators with fibres $L(T_xM)$ such that $J^2 = -1$. Such a $J$ is called integrable if its torsion is zero, i.e. $N(X, Y) = 0$ with $N$ as in (2.1). An almost Kähler (K) manifold is a triple $(M, J, g)$ where $M$ is a real smooth Hilbert manifold, $J$ is an almost complex structure, and $g$ is a K metric, i.e. a Riemannian metric such that

- $g$ is invariant; i.e. (B12) $g_x(J_xX_x, J_xY_x) = g_x(X_x, Y_x)$.
- The fundamental two form of the metric is closed; i.e. (B13) $\omega_x(X_x, Y_x) = g_x(J_xX_x, Y_x)$ is closed (which means $d\omega = 0$).
Note that an almost K manifold is canonically symplectic and if J is integrable one says that M is a K manifold. Now fix an almost K manifold \((M, J, g)\). The form \(\omega\) and the K metric \(g\) induce two top-linear isomorphisms \(I_x\) and \(G_x\) between \(T_x^*M\) and \(T_xM\) via \((B14)\) \(\omega_x(l_xa_x, X_x) = < a_x, X_x > \) and \(g_x(G_xa_x, X_x) = < a_x, X_x >\). Denoting the smooth sections by \(I, G\) one checks that \(G = J \circ I\).

**DEFINITION 2.1.** For \(f, h \in C^\infty(M, R)\) the Poisson and Riemann brackets are defined via \((B15)\) \(\{f, h\} = < df, Idh >\) and \((B16)\) \(((f, h)) = < df, Gdh >\). In view of \((B14)\) one can reformulate this as

\[(2.10) \quad \{f, h\} = \omega(Idf, Idh) = \omega(Gdf, Gdh); \quad ((f, h)) = g(Gdf, Gdh) = g(Idf, Idh)\]

**DEFINITION 2.2.** For \(f, h \in C^\infty(M, C)\) the K bracket is \((B17)\) \(< f, h > = ((f, h)) + i\{f, h\}\) and one defines products \((B18)\) \(f \circ h = (1/2)\nu((f, h)) + fh\) \((\nu\) will be determined to be \(h)\) and \(f \ast h = (1/2)\nu < f, h > + fh\). One observes also that

\[(2.11) \quad f \ast h = f \circ h + (i/2)\nu\{f, h\}; \quad f \circ h = (1/2)(f \ast h + h \ast f); \quad \{f, h\} = (1/i\nu)(f \ast h - h \ast f)\]

**DEFINITION 2.3.** For \(f \in C^\infty(M, R)\) let \(X = Idf\); then \(f\) is called Kählerian (K) if \((B19)\) \(L_Xg = 0\) where \(L_X\) is the Lie derivative along \(X\) (recall \(L_Xf = Xf, L_XY = [X, Y], L_X(\omega(Y)) = (L_X\omega)(Y) + \omega(L_X(Y)), \cdots\)). More generally if \(f \in C^\infty(M, C)\) one says that \(f\) is K if \(\Re f\) and \(\Im f\) are K; the set of K functions is denoted by \(K(M, R)\) or \(K(M, C)\).

**REMARK 2.1.** In the language of symplectic manifolds \(X = df\) is the Hamiltonian vector field corresponding to \(f\) and the condition \(L_Xg = 0\) means that the integral flow of \(X\), or the Hamiltonian flow of \(f\), preserves the metric \(g\). From this follows also \(L_XJ = 0\) (since \(J\) is uniquely determined by \(\omega\) and \(g\) via \((B13)\)). Therefore if \(f\) is K the Hamiltonian flow of \(f\) preserves the whole K structure. Note also that \(K(M, R)\) (resp. \(K(M, C)\)) is a Lie subalgebra of \(C^\infty(M, R)\) (resp. \(C^\infty(M, C)\)).

Now \(P(H)\) is the set of one dimensional subspaces or rays of \(H\); for every \(x \in H/\{0\}, \{x\}\) is the ray through \(x\). If \(H\) is the Hilbert space of a Schrödinger quantum system then \(H\) represents the pure states of the system and \(P(H)\) can be regarded as the state manifold (when provided with the differentiable structure below). One defines the K structure as follows. On \(P(H)\) one has an atlas \(\{(V_h, b_h, C_h)\}\) where \(h \in H\) with \(|h| = 1\). Here \((V_h, b_h, C_h)\) is the chart with domain \(V_h\) and local model the complex Hilbert space \(C_h\) where

\[(2.12) \quad V_h = \{[x] \in P(H); (|h|x) \neq 0\}; \quad C_h = [|h|\} +; \quad b_h : V_h \to C_h; \quad [x] \to b_h([x]) = \frac{x}{(|h|x)\} - h\]

This produces a analytic manifold structure on \(P(H)\). As a real manifold one uses an atlas \(\{(V_h, R \circ b_h, RC_h)\}\) where e.g. \(RC_h\) is the realification of \(C_h\) (the real Hilbert space with \(R\) instead of \(C\) as scalar field) and \(R : C_h \to RC_h; v \to Rv\) is the canonical bijection (note \(Rv \neq \Re Rv\)). Now consider the form of the K metric relative to a chart \((V_h, R \circ b_h, RC_h)\) where the metric \(g\) is a smooth section of \(L_2(TP(H), R)\) with local expression \(g^h : RC_h \to \)
and is given via (2.13) or (X, Y, Z). One proves then (cf. [21, 34]) that the flow of the vector field $\omega^h = L^\omega \omega^h$ given via (2.14)

$$\omega^h_{Rz}(Rv, Rw) = 2\nu^3 \left( \frac{(|v|w)}{1 + ||z||^2} - \frac{(v|z)(z|w)}{(1 + ||z||^2)^2} \right)$$

Then using e.g. (2.13) for the FS metric in $P(H)$ consider a Schrödinger Hilbert space with dynamics determined via (B20) $R \times P(H) \rightarrow P(H) : (t, [x]) \mapsto [\exp(-(i/t)H) x]$ where $H$ is a (typically unbounded) self adjoint operator in $H$. One thinks then of Kähler isomorphisms of $P(H)$ (i.e. smooth diffeomorphisms $\Phi : P(H) \rightarrow P(H)$ with the properties $\Phi^* J = J$ and $\Phi^* g = g$). If $U$ is any unitary operator on $H$ the map $[x] \mapsto [Ux]$ is a K isomorphism of $P(H)$. Conversely (cf. [21]) any K isomorphism of $P(H)$ is induced by a unitary operator $U$ (unique up to phase factor). Further for every self adjoint operator $A$ in $H$ (possibly unbounded) the family of maps $(\Phi_t)_{t \in \mathbb{R}}$ given via (B21) $\Phi_t : [x] \mapsto [\exp(-itA)x]$ is a continuous one parameter group of K isomorphisms of $P(H)$ and vice versa (every K isomorphism of $P(H)$ is induced by a self adjoint operator where boundedness of $A$ corresponds to smoothness of the $\Phi_t$). Thus in the present framework the dynamics of QM is described by a continuous one parameter group of K isomorphisms, which automatically are symplectic isomorphisms (for the structure defined by the fundamental form) and one has a Hamiltonian system. Next ideally one can suppose that every self adjoint operator represents an observable and these will be shown to be in $1 - 1$ correspondence with the real K functions.

**DEFINITION** 2.4. Let $A$ be a bounded linear operator on $H$ and denote by $\langle A \rangle$ the mean value function of $A$ defined via (B22) $\langle A \rangle : P(H) \rightarrow \mathbb{C}$, $[x] \mapsto \langle A \rangle [x] = (x|Ax)/||x||^2$. The square dispersion is defined via (B23) $\Delta^2 A : P(H) \rightarrow \mathbb{C}$, $[x] \mapsto \Delta^2 [x] A = \langle (A - \langle A \rangle [x])^2 \rangle [x]$.

These maps (B22) and (B23) are smooth and if $A$ is self adjoint $\langle A \rangle$ is real, $\Delta^2 A$ is nonnegative, and one can define $\Delta A = \sqrt{\Delta^2 A}$. To obtain local expressions one writes $\langle A \rangle^h$ and (d $\langle A \rangle$)$^h$ via (B24) $\langle A \rangle^h (R) = (z + h)|A(z + h))/\sqrt{1 + ||z||^2}$ and

$$\Delta^2 [x] A = \langle (A - \langle A \rangle [x])^2 \rangle [x]$$

Further the local expressions $X^h : RC_h \rightarrow RC_h$ and $Y^h : RC_h \rightarrow RC_h$ of the vector fields $X = Id < A >$ and $Y = Gd < A >$ are

$$X^h(Rz) = (1/i\nu) R(ih|A(z + h))(z + h) - iA(z + h))$$

$$Y^h(Rz) = (1/i\nu) R(-ih|A(z + h))(z + h) + A(z + h))$$

One proves then (cf. [21, 34]) that the flow of the vector field $X = Id < A >$ is complete and is given via (B25) $\Phi_t([x]) = [\exp(-(i/t\nu)A)x]$. This leads to the statement that if $f$ is
a complex valued function on $P(H)$ then $f$ is Kählerian if and only if there is a bounded operator $A$ such that $f = \langle A \rangle$ (cf. Definition 3.2). From the above it is clear that one should take $\nu = \hbar$ for QM if we want to have $\langle \hat{\mathcal{H}} \rangle$ represent Hamiltonian flow ($\hat{\mathcal{H}} \sim$ a Hamiltonian operator) and this gives a geometrical interpretation of Planck’s constant.

The following formulas are obtained for the Poisson and Riemann brackets

$$(2.17)\quad \{\langle A \rangle, \langle B \rangle\}^h(Rz) = \frac{(z + \hbar[(1/\nu)(AB - BA)](z + \hbar))}{1 + \|z\|^2};\quad (\langle A \rangle, \langle B \rangle)^h(Rz) = \frac{1}{\nu} \frac{(z + \hbar[BA - AB](z + \hbar))}{1 + \|z\|^2} - \frac{2}{\nu} \frac{(z + \hbar[A(z + \hbar)](z + \hbar)[B(z + \hbar)])}{1 + \|z\|^2}$$

This leads to the results

1. $\{\langle A \rangle, \langle B \rangle\} = \langle (1/\nu)[A, B] \rangle$
2. $((\langle A \rangle, \langle B \rangle)) = (1/\nu) \langle AB + BA \rangle - (2/\nu) \langle A \rangle \langle B \rangle$; $((\langle A \rangle \langle A \rangle)) = (2/\nu) \Delta^2 A$
3. $\langle A \rangle \langle B \rangle \rangle = (2/\nu)(\langle AB \rangle - \langle A \rangle \langle B \rangle)$
4. $\langle A \rangle \circ \nu \langle B \rangle \rangle = (1/2) \langle AB + BA \rangle$
5. $\langle A \rangle \ast \nu \langle B \rangle \rangle = \langle AB \rangle$

REMARK 2.2. One notes that (setting $\nu = \hbar$) item 1 gives the relation between Poisson brackets and commutators in QM. Further the Riemann bracket is the operation needed to compute the dispersion of observables. In particular putting $\nu = \hbar$ in item 2 one sees that for every observable $f \in K(P(H), \mathbb{R})$ and every state $[\nu] \in P(H)$ the results of a large number of measurements of $f$ in the state $[\nu]$ are distributed with standard deviation $\sqrt{\hbar/2}((f, f))([\nu])$ around the mean value $f([\nu])$. This explains the role of the Riemann structure in QM, namely it is the structure needed for the probabilistic description of QM. Moreover the $\circ \nu$ product corresponds to the Jordan product between operators (cf. item 5) and item 4 tells us that the $\ast \nu$ product corresponds to the operator product. This allows one to formulate a functional representation for the algebra $L(H)$. Thus put $\|f\|_\nu = \sqrt{\sup_{[\nu]}(\langle f \rangle \ast \nu f)([\nu])}$. Equipped with this norm $K(P(H), \mathbb{C})$ becomes a $W^*$ algebra and the map of $W^*$ algebras between $K(P(H), \mathbb{C})$ and $L(H)$ is an isomorphism. This makes it possible to develop a general functional representation theory for $C^*$ algebras generalizing the classical spectral representation for commutative $C^*$ algebras. The $K$ manifold $P(H)$ is replaced by a topological fibre bundle in which every fibre is a $K$ manifold isomorphic to a projective space. In particular a nonzero vector $x \in H$ is an eigenvector of $A$ if and only if $d_{[x]} \langle A \rangle = 0$ or equivalently if and only if $[x]$ is a fixed point for the vector field $Id < A >$ (in which case the corresponding eigenvalue is $\langle A \rangle_{[x]}$).

3. PROBABILITY ASPECTS

We go here to [8, 4, 5, 13, 14, 19, 21, 22, 28, 30, 31, 34, 35, 43, 47, 58, 53, 55, 56, 61, 68, 71]; some of this will be somewhat disjointed but we will organize it later. First from [13, 71] one defines a (Riemannian) metric (statistical distance) on the space of probability distributions $\mathcal{P}$ of the form $\langle C1 \rangle d_{PD}^2 = \sum (dp_j^2/p_j) = \sum p_j (d\log(p_j))^2$. Here one thinks
of the central limit theorem and a distance between probability distributions distinguished via a Gaussian \(e^{-\frac{1}{2}(\bar{p}_j - p_j)^2/p_j}\) for two nearby distributions (involving \(N\) samples with probabilities \(p_j, \bar{p}_j\)). This can be generalized to quantum mechanical pure states via (note \(\psi \sim \sqrt{P}\exp(i\phi)\) in a generic manner)

\[
(3.1) \quad |\psi\rangle = \sum \sqrt{p_j} e^{i\phi_j} |j\rangle \Rightarrow |\tilde{\psi}\rangle = |\psi\rangle + |d\psi\rangle = \sum \sqrt{p_j} + dp_j e^{i(\phi_j + d\phi_j)} |j\rangle
\]

Normalization requires \(\Re(\langle \psi | d\psi \rangle) = -1/2 < d\psi | d\psi >\) and measurements described by the one dimensional projectors \(|j\rangle < j\rangle\) can distinguish \(|\psi\rangle\) and \(|\tilde{\psi}\rangle\) according to the metric \((C1)\). The maximum (for optimal distinguishability) is given by the Hilbert space angle \((C2)\ cos^{-1}(|\tilde{\psi}|\psi > |)\) and the corresponding line element \((PS \sim unhappy)\)

\[
(3.2) \quad \frac{1}{4} ds_{PS}^2 = [\cos^{-1}(|\tilde{\psi}|\psi > |)]^2 \sim 1 - |\tilde{\psi}|\psi > |^2 = d\psi_\perp |d\psi_\perp >
\]

(called the Fubini-Study (FS) metric) is the natural metric on the manifold of Hilbert space rays. Here \((C3)\) \(d\psi_\perp > = |\psi\rangle - |\psi\rangle\langle\psi| d\psi\) is the projection of \(d\psi\) orthogonal to \(|\psi\rangle\). Note that if \(\cos^{-1}(|\tilde{\psi}|\psi > |) = \theta\) then \(\cos(\theta) = |\tilde{\psi}|\psi > |\) and \(\cos^2(\theta) = |\tilde{\psi}|\psi > |^2 = 1 - \sin^2(\theta) \sim 1 - \theta^2\) for small \(\theta\). Hence \(\theta^2 \sim 1 - \cos^2(\theta) = 1 - |\tilde{\psi}|\psi > |^2\). The term in square brackets (the variance of phase changes) is nonnegative and an appropriate choice of basis makes it zero. In \((13)\) one then goes on to discuss distance formulas in terms of density operators and Fisher information but we omit this here. Generally as in \((21)\) one observes that the angle in Hilbert space is the only Riemannian metric on the set of rays which is invariant under unitary transformations. In any event \((C4)\)

\[
ds^2 = \sum \left( \frac{dp_i^2}{p_i} \right), \quad \sum p_i = 1
\]

is referred to as the Fisher metric (cf. \((21)\)). Note in terms of \(dp_i = \tilde{p}_i - p_i\) one can write \(d\sqrt{p} = (1/2)dp/\sqrt{p}\) with \(d\sqrt{p}^2 = (1/4)(dp^2/p)\) and think of \(\sum(d\sqrt{p_i})\) as a metric. Alternatively from \(\cos^{-1}(|\tilde{\psi}|\psi > |\) one obtains \((C5)\)

\[
ds_{12} = \cos^{-1}(\sum \sqrt{p_1} \sqrt{p_2})
\]

as a distance in \(P\). Note from \((C3)\) that \((C6)\)

\[
ds_{12}^2 = \cos^{-1}(\sum \sqrt{p_1} \sqrt{p_2})
\]

begins to look like a FS metric before passing to projective coordinates. In this direction we observe from \((17)\) that the FS metric as in \((2.9)\) can be expressed also via

\[
(3.3) \quad \partial \bar{\partial} \log(|z|^2) = \phi = \frac{1}{|z|^2} \sum z_i \bar{\partial}_i \bar{z}_i - \frac{1}{|z|^4} \left( \sum \bar{z}_i dz_i \right) \wedge \left( \sum z_i d\bar{z}_i \right)
\]

so for \(v \sim \sum v_i \partial_i + \bar{v}_i \bar{\partial}_i\) and \(w \sim \sum w_i \partial_i + \bar{w}_i \bar{\partial}_i\) and \(|z|^2 = 1\) one has \((C7)\)

\[
\phi(v, w) = (v|w) - (v|z)(z|w) \quad (cf. \quad (2.13)).
\]

3.1. FISHER INFORMATION. We summarized in \((16)\) various results on Fisher information, entropy, and the Schrödinger equation (SE) follow \((26)\).\((27)\).\((28)\).\((31)\).\((35)\).\((38)\).\((61)\).\((62)\).

Thus first recall that the classical Fisher information associated with translations of a 1-D observable \(X\) with probability density \(P(x)\) is

\[
(3.4) \quad F_X = \int dx P(x)(\log(P(x))')^2 > 0
\]
One has a well known Cramer-Rao inequality (C8) \( \text{Var}(X) \geq \frac{1}{F_X^2} \) where \( \text{Var}(X) \sim \text{variance of } X \). A Fisher length for \( X \) is defined via (C9) \( \delta X = F_X^{-1/2} \) and this quantifies the length scale over which \( p(x) \) (or better \( \log(p(x)) \)) varies appreciably. Then the root mean square deviation \( \Delta X \) satisfies (C9) \( \Delta X \geq \delta X \). Let now \( P \) be the momentum observable conjugate to \( X \), and \( P_{cl} \) a classical momentum observable corresponding to the state \( \psi \) given via (C10) \( p_{cl}(x) = (\hbar/2i)[(\psi' / \psi) - (\psi / \psi')] \). One has the identity (C11) \( <p>_{\psi} = <p_{cl}>_{\psi} \), following from (C10) with integration by parts. Now define the nonclassical momentum by \( p_{nc} = p - p_{cl} \) and one shows then (C12) \( \Delta X \Delta p \geq \delta X \Delta p \geq \delta X \Delta p_{nc} = \hbar/2 \). Now go to (C12) now where two proofs are given for the derivation of the SE from the exact uncertainty principle (as in (C12) - cf. [34, 35]). Thus consider a classical ensemble of \( n \)-dimensional particles of mass \( m \) moving under a potential \( V \). The motion can be described via the HJ and continuity equations

\[
\frac{\partial s}{\partial t} + \frac{1}{2m} |\nabla s|^2 + V = 0; \quad \frac{\partial P}{\partial t} + \nabla \cdot \left[ \frac{P}{m} \nabla s \right] = 0
\]

for the momentum potential \( s \) and the position probability density \( P \) (note that we have interchanged \( p \) and \( P \) from [35] - note also there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle \( \delta L = 0 \) with Lagrangian (C13) \( L = \int dt d^n x \, P \left[ (\partial s / \partial t) + (1/2m) |\nabla s|^2 + V \right] \). It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial for (cf. [35] for discussion) and one can assume that the momentum associated with position \( x \) is given by (C14) \( p = \nabla s + N \) where the fluctuation term \( N \) vanishes on average at each point \( x \). Thus \( s \) changes to being an average momentum potential. It follows that the average kinetic energy \( < |\nabla s|^2 > /2m \) appearing in (C13) should be replaced by \( < |\nabla s + N|^2 > /2m \) giving rise to

\[
L' = L + (2m)^{-1} \int dt <N \cdot N> = L + (2m)^{-1} \int dt (\Delta N)^2
\]

where \( \Delta N = <N \cdot N>^{1/2} \) is a measure of the strength of the quantum fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the following three assumptions, namely

1. **Action principle**: \( L' \) is a scalar Lagrangian with respect to the fields \( P \) and \( s \) where the principle \( \delta L' = 0 \) yields causal equations of motion. Thus for some scalar function \( f \) one has \( (\Delta N)^2 = \int d^n x \, p f (P, \nabla P, \partial P / \partial t, s, \nabla s, \partial s / \partial t, x, t) \).
2. **Additivity**: If the system comprises two independent noninteracting subsystems with \( P = P_1 P_2 \) then the Lagrangian decomposes into additive subsystem contributions; thus \( f = f_1 + f_2 \) for \( P = P_1 P_2 \).
3. **Exact uncertainty**: The strength of the momentum fluctuation at any given time is determined by and scales inversely with the uncertainty in position at that time. Thus \( \Delta N \rightarrow k \Delta N \) for \( x \rightarrow x/k \). Moreover since position uncertainty is entirely characterized by the probability density \( P \) at any given time the function \( f \) cannot depend on \( s \), nor explicitly on \( t \), nor on \( \partial P / \partial t \).
This leads to the result that (C15) \((\Delta N)^2 = c \int d^n x P|\nabla \log(P)|^2\) where \(c\) is a positive universal constant (cf. [35]). Further for \(\hbar = 2\sqrt{c}\) and \(\psi = \sqrt{P} \exp(is/\hbar)\) the equations of motion for \(p\) and \(s\) arising from \(\delta L' = 0\) are (C16) \(i\hbar \frac{\partial P}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi\).

**Remark 3.1.** In order to relate this to Fisher information we sketch here for simplicity and clarity another derivation of the SE along similar ideas following [61]. Let \(P(y^i)\) be a probability density and \(P(y^i + \Delta y^i)\) be the density resulting from a small change in the \(y^i\). Calculate the cross entropy via

\[
J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i) \log \frac{P(y^i + \Delta y^i)}{P(y^i)} d^n y \simeq \left[ \frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^i} \frac{\partial P(y^i)}{\partial y^k} d^n y \right] \Delta y^i \Delta y^k
\]

The \(I_{jk}\) are the elements of the Fisher information matrix. The most general expression has the form

\[
I_{jk}(\theta^i) = \frac{1}{2} \int \frac{1}{P(x^i|\theta^i)} \frac{\partial P(x^i|\theta^i)}{\partial \theta^j} \frac{\partial P(x^i|\theta^i)}{\partial \theta^k} d^n x
\]

where \(P(x^i|\theta^i)\) is a probability distribution depending on parameters \(\theta^i\) in addition to the \(x^i\). For (C17) \(P(x^i|\theta^i) = P(x^i + \theta^i)\) one recovers (3.7) (straightforward - cf. [61]). If \(P\) is defined over an \(n\)-dimensional manifold with positive inverse metric \(g^{ik}\) one obtains a natural definition of the information associated with \(P\) via

\[
I = g^{ik} I_{ik} = \frac{g^{ik}}{2} \int \frac{1}{P} \frac{\partial P}{\partial y^i} \frac{\partial P}{\partial y^k} d^n y
\]

Now in the HJ formulation of classical mechanics the equation of motion takes the form

\[
\frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V = 0
\]

where \(g^{\mu\nu} = diag(1/m, \cdots, 1/m)\). The velocity field \(u^\mu\) is given by (C18) \(u^\mu = g^{\mu\nu} (\partial S/\partial x^\nu)\). When the exact coordinates are unknown one can describe the system by means of a probability density \(P(t, x^\mu)\) with \(int \text{P} d^n x = 1\) and (C19) \((\partial P/\partial t) + (\partial/\partial x^\mu) (P g^{\mu\nu} (\partial S/\partial x^\nu)) = 0\).

These equations completely describe the motion and can be derived from the Lagrangian (C20) \(L_{CL} = \int P \left\{ \left( \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \lambda \left( \frac{1}{2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2}{P} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \right) \right) + V \right\} dt d^n x\) using fixed endpoint variation in \(S\) and \(P\). Quantization is obtained by adding a term proportional to the information \(I\) defined in (3.9). This leads to

\[
L_{QM} = L_{CL} + \lambda I = \int P \left\{ \left( \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \lambda \left( \frac{1}{2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2}{P} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \right) \right) \right\} dt d^n x
\]

Fixed endpoint variation in \(S\) leads again to (C19) while variation in \(P\) leads to

\[
\frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \lambda \left( \frac{1}{2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2}{P} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \right) \right] + V = 0
\]

These equations are equivalent to the Schrödinger equation if (C21) \(\psi = \sqrt{P} \exp(iS/\hbar)\) with \(\lambda = (2\hbar)^2\).
**REMARK 3.2.** The SE gives to a probability distribution $\rho = |\psi|^2$ (with suitable normalization) and to this one can associate an information entropy $S(t)$ (actually configuration information entropy) \( \mathcal{C}22 \) \( S = -\int p \log(p) d^3x \) which is typically not a conserved quantity. The rate of change in time of $\mathcal{S}$ can be readily found by using the continuity equation \( \mathcal{C}23 \) \( \partial_t \rho = -\nabla \cdot (\nu \rho) \) where $\nu$ is a current velocity field. Note here (cf. also \( \mathcal{C}24 \)) \( \mathcal{C}23 \) \( \partial \mathcal{S}/\partial t = -\int \rho (1 + \log(p)) dx = \int (1 + \log(p)) \partial(\nu \rho) \). Note that a formal substitution of $\nu = -u$ in \( \mathcal{C}23 \) implies the standard free Brownian motion outcome \( \mathcal{C}24 \) \( \partial \mathcal{S}/\partial t = D \int [(\nabla \rho)^2/\rho] d^3x = D \cdot Tr \tilde{F} \geq 0 \) - use \( \mathcal{C}25 \) \( u = D \nabla \log(\rho) \) with $D = \hbar/2m$ and \( \mathcal{C}23 \) with \( \int (1 + \log(p)) \partial(\nu \rho) = -\int \nu \partial(\log(p) = -\int \nu \rho \partial \log(p) = -\int \nu \rho' \sim \int ((\rho')^2/\rho) \) modulo constants involving $D$ etc. Recall here \( \mathcal{F}_3 \sim -(2/D^2) \int \rho Q dx = \int dx [(\nabla \rho)^2/\rho] \) is a functional form of Fisher information. A high rate of information entropy production corresponds to a rapid spreading (flattening down) of the probability density. This delocalization feature is concomitant with the decay in time property quantifying the time rate at which the far from equilibrium system approaches its stationary state of equilibrium \( \mathcal{C}26 \) \( \partial / \partial t Tr \tilde{F} \leq 0 \).  

**REMARK 3.3.** Comparing now with \( \mathcal{C}1 \equiv \mathcal{C}4 \) or \( \mathcal{C}6 \) as a Fisher metric we can define \( \mathcal{F}3 \) as a Fisher information metric in the present context. This should be positive definite in view of its relation to $\langle \Delta N \rangle^2$ in \( \mathcal{C}15 \) for example. In \( \mathcal{F}3 \) we sketched many ways in which the quantum potential arises in the derivation of Schrödinger equations. For $\psi = R \exp(iS/\hbar)$ one has

\begin{equation}
-\frac{\hbar^2}{2m} R'' \equiv -\frac{\hbar^2}{2m} \frac{\partial^2}{\sqrt{\rho}} = -\frac{\hbar^2}{8m} \left[ \frac{2\rho''}{\rho} - \left( \frac{\rho'}{\rho} \right)^2 \right]
\end{equation}

in 1-D while in more dimensions we have a form \( \rho \sim P \)

\begin{equation}
Q \sim -2\hbar^2 g^{\mu\nu} \left[ \frac{1}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} - \frac{2}{P} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \right]
\end{equation}

as in \( \mathcal{F}3 \) (arising from the Fisher metric I of \( \mathcal{F}3 \) upon variation in $P$ in the Lagrangian). It can also be related to an osmotic velocity field $u = D \nabla \log(\rho)$ (cf. \( \mathcal{F}3 \)) \( \mathcal{F}6 \) \( Q = (1/2)u^2 + D \nabla u \) connected to Brownian motion where $D$ is a diffusion coefficient. We refer also to \( \mathcal{F}3, \mathcal{F}9, \mathcal{F}10, \mathcal{F}50 \) for other connections to diffusion and statistical mechanics and to \( \mathcal{F}3, \mathcal{F}2 \) for origins via a conjectured fractal nature of spacetime (there are also many other references in \( \mathcal{F}3 \)).

4. THE SCHRÖDINGER EQUATION IN WEYL SPACE

A deBroglie-Bohm-Weyl theory has been developed recently by a number of authors (cf. \( \mathcal{F}3 \) for references and a sketch based on the summary article \( \mathcal{F}65 \)). In this theory one constructs a relativistic framework with quantum matter based in part on deBroglie - Bohm (dBB) ideas and Weyl geometry. A Bohmian mass field arises in the associated Dirac-Weyl theory, corresponding to a quantum mass $M$, and the geometric aspects of the evolving spacetime manifold are related to quantum effects. A quantum potential is involved of the form \( \mathcal{D}1 \) \( \Omega = (\hbar^2/m^2c^2)(\Box \Psi/|\Psi|) \) with \( \mathcal{D}2 \) \( M^2 = m^2 \exp(\Omega) \). Evidently probabilistic input to the nonrelativistic SE does not apply for relativistic generalizations such as the
Klein-Gordon (KG) equation and this is eloquently discussed in [51]. However in [63] one deals with a geometric derivation of the nonrelativistic SE in Weyl spaces and it turns out that one can relate the standard quantum potential $Q$ to the Ricci-Weyl scalar curvature of spacetime (see [16] for details). The KG equation is also treated by Santamato in [63] and the whole matter is analyzed incisively by Castro in [17]. Again a relation between the relativistic $Q$ and the Weyl-Ricci curvature exists but without the probabilistic connections. We remark from [16], following [57, 58, 59, 60], that one does not expect or want a quantum mechanical particle to be a free falling trajectory; in the conformal metric the particles do not follow geodesics of the conformal metric alone.

We refer to [16, 17, 63] for philosophy here and to [8, 12, 16, 57, 58, 59, 60, 64, 65] for Weyl geometry. In [63] one begins with a stochastic construction of (averaged) classical type Lagrange equations in generalized coordinates for a differentiable manifold $M$ in which a notion of scalar curvature $R$ is meaningful. It is then shown that a theory equivalent to QM (via a SE) can be constructed where the “quantum force” (arising from a quantum potential $Q$) can be related to (or described by) geometric properties of space. To do this one assumes that a (quantum) Lagrangian can be constructed in the form (D3) $L(q, \dot{q}, t) = L_C(q, \dot{q}, t) + \gamma (h^2/m)R(q, t)$ where (D4) $\gamma = (1/6)(n-2)/(n-1)$ with $n = \dim(M)$ and $R$ is a curvature scalar. Now for a Riemannian geometry $ds^2 = g_{ik}(q) dq^i dq^k$ it is standard that in a transplantation $q^i \rightarrow q^i + \delta q^i$ one has (D5) $\delta A^i = \Gamma^i_{k\ell} dk^i dq^k$. Here moreover it is assumed that for $\ell = (g_{ik} A^i A^k)^{1/2}$ one has (D6) $\delta \ell = \ell \phi_k d\ell^k$ where the $\phi_k$ are covariant components of an arbitrary vector of $M$ (Weyl geometry). For the discussion here we review the material on Weyl geometry in [63]. Thus the actual affine connections $\Gamma^i_{k\ell}$ can be found by comparing (D6) with $\delta \ell^2 = \delta (g_{ik} A^i A^k)$ and using (D5). A little linear algebra gives then

\begin{equation}
\Gamma^i_{k\ell} = - \left\{ \frac{i}{k \ell} \right\} + g^{im} (g_{mk} \phi_{\ell} + g_{m\ell} \phi_k - g_{k\ell} \phi_m)
\end{equation}

Thus we may prescribe the metric tensor $g_{ik}$ and $\phi_i$ and determine via (D1) the connection coefficients. Note that $\Gamma^i_{k\ell} = \Gamma^i_{\ell k}$ and for $\phi_i = 0$ one has Riemannian geometry. Covariant derivatives are defined for contravariant $A^k$ via (D7) $A^k_1 = \partial_i A^k - \Gamma^k_{i\ell} A^\ell$ and for covariant $A_k$ via (D8) $A_{k, i} = \partial_i A_k + \Gamma^i_{k\ell} A_{\ell}$ (where $S_{ij} = \partial_i S$). Note $g_{ik, \ell} \neq 0$ so covariant differentiation and operations of raising or lowering indices do not commute. The curvature tensor $R^i_{k\ell m}$ in Weyl geometry is introduced via $A^l_{k, \ell} - A^l_{\ell, k} = F^l_{m\ell k} A^m$ from which arises the standard formula of Riemannian geometry (D9) $R^i_{m\ell k} = -\partial_i \Gamma^i_{m\ell k} + \partial_k \Gamma^i_{m\ell} - \Gamma^i_{nk} \Gamma^k_{m\ell} - \Gamma^i_{mk} \Gamma^k_{n\ell}$ where (4.1) must be used in place of the Riemannian Christoffel symbols. The tensor $R^i_{m\ell k}$ obeys the same symmetry relations as the curvature tensor of Riemann geometry as well as the Bianchi identity. The Ricci symmetric tensor $R_{ik}$ and the scalar curvature $R$ are defined by the same formulas also, viz. $R_{ik} = R^i_{ik}$ and $R = g^{ik} R_{ik}$. For completeness one derives here (D10) $R = \dot{R} + (n - 1)/2 \phi_i \phi^i - 2(1/\sqrt{g}) \partial_i (\sqrt{g} \phi^i)$ where $\dot{R}$ is the Riemannian curvature built by the Christoffel symbols. Thus from (4.1) one obtains

\begin{equation}
g^{ik} \Gamma^i_{k\ell} = -g^{ik} \left\{ \frac{i}{k \ell} \right\} - (n - 2) \phi^i; \; \Gamma^i_{k\ell} = - \left\{ \frac{i}{k \ell} \right\} + n \phi_k
\end{equation}
Since the form of a scalar is independent of the coordinate system used one may compute $R$ in a geodesic system where the Christoffel symbols and all $\partial_t g_{ik}$ vanish; then (D11) reduces to
\[ \Gamma^i_{kl} = \phi_k \Gamma^i_{kl} + \phi_l \Gamma^i_{kl} - \phi_{kl} \phi^i. \]
Hence (D12) $R = -g^{km} \partial_n \Gamma^i_{kl} + \partial_t (g^{kt} \Gamma^i_{kl}) + g^{mk} \Gamma^i_{nl} \Gamma^l_{mi} - g^{ml} \Gamma^i_{nl} \Gamma^l_{mk}$. Further from (D11) one has (D13) $g^{km} \Gamma^i_{nl} \Gamma^l_{mi} = -(n-2)(\phi_k \phi^k)$ at the point in consideration. Putting all this in (D12) one arrives at (D14) $R = \dot{R} + (n-1)(n-2)(\phi_k \phi^k) - 2(n-1)\partial_t \phi^k$ which becomes (D10) in covariant form. Now the geometry is to be derived from physical principles so the $\phi_i$ cannot be arbitrary but must be obtained by the same (averaged) least action principle giving the motion of the particle. The minimum is to be evaluated now with respect to the class of all Weyl geometries having arbitrarily varied gauge vectors but fixed metric tensor and the only term containing the gauge vector is the curvature term. Then observing that $\gamma > 0$ when $n \geq 3$ the minimization involves only (D10). First a little argument shows that $\dot{\rho}(q,t) = \rho(q,t)/\sqrt{g}$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle. Starting from (D15) $\partial_i \rho + \partial_t (\rho v^i) = 0$ a manifestly covariant equation for $\dot{\rho}$ is found to be (D16) $\partial_t \dot{\rho} + (1/\sqrt{g}) \partial_t (\sqrt{g} g^{ik} \partial_k \sqrt{\rho}) = 0$. Some calculation then yields a minimum for (D17) $\phi_i(q,t) = -[1/(n-2)] \partial_i [\log(\rho(q,t))]$. This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force $f_i = \gamma(\hbar^2/m) \partial_i R$ which according to (D10) depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects.

In this spirit one goes next to a geometrical derivation of the SE. Thus inserting (D17) into (D10) one gets (D18) $R = \dot{R} + (1/2\gamma \sqrt{\rho})[1/\sqrt{g}] \partial_t (\sqrt{g} g^{ik} \partial_k \sqrt{\rho})$ where the value (D4) for $\gamma$ has been used. On the other hand the HJ equation can be written as (D19) $\partial_t S + H_C(q, \nabla S, t) - \gamma(\hbar^2/m) R = 0$ where (D3) has been used. When (D18) is introduced into (D19) the HJ equation and the continuity equation (D16), with velocity field given by (D20) $v^i = (\partial H/\partial p_i)(q, \nabla S, t)$ form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the particle (i.e. random motions compatible with (D12)) are obtained by solving (D16) and (D19) simultaneously. For every pair of solutions $S(q,t, \rho(q,t))$ one gets a possible random motion for the particle whose invariant probability density is $\rho$. The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form (D21) $H_C(q,p,t) = (1/2m)g^{ik}(p_i - A_i)(p_k - A_k) + V$ (which is not very restrictive) leading to
\[
\partial_t S + \frac{1}{2m} g^{ik}(\partial_t S - A_i)(\partial_k S - A_k) + V - \frac{\hbar^2}{m} R = 0
\]
(R in (D18)). The continuity equation (D16) is (D22) $\partial_t \dot{\rho} + (1/m \sqrt{\rho}) \partial_t [\rho \sqrt{g} g^{ik} (\partial_k S - A_k)] = 0$. Owing to (D18) (4.3) and (D22) form a set of two nonlinear PDE which must be solved for the unknown functions $S$ and $\dot{\rho}$. Then a straightforward calculation shows that, setting (D23) $\psi(q,t) = \sqrt{\rho(q,t)} \exp[i/\hbar] S(q,t)$, the quantity $\psi$ obeys a linear PDE
(corrected from [63])

\[
(4.4) \quad \frac{i\hbar}{\rho} \partial_t \psi = \frac{1}{2m} \left\{ \frac{i\hbar \sqrt{\partial^2 \sqrt{\rho}/\sqrt{\rho}}}{\sqrt{\partial^2 \sqrt{\rho}/\sqrt{\rho}}} + A_i \right\} g^{ik}(i\hbar \partial_k + A_k) \psi + \left[ V - \gamma \frac{\hbar^2}{m} \hat{R} \right] \psi = 0
\]

where only the Riemannian curvature \( \hat{R} \) is present (any explicit reference to the gauge vector \( \phi_i \) having disappeared). (4.4) is of course the SE in curvilinear coordinates whose invariance under point transformations is well known. Moreover (D23) shows that \(|\psi|^2 = \rho(q, t)\) is the invariant probability density of finding the particle in the volume element \( d^nq \) at time \( t \). Then following Nelson’s arguments that the SE together with the density formula contains QM the present theory is physically equivalent to traditional nonrelativistic QM.

**REMARK 4.1.** We recall (cf. [16]) that in the nonrelativistic context the quantum potential has the form \( Q = -(\hbar^2/2m)(\partial^2 \sqrt{\rho}/\sqrt{\rho}) \) (\( \rho \sim \rho \) here) and in more dimensions this corresponds to \( Q = -(\hbar^2/2m)(\Delta \sqrt{\rho}/\sqrt{\rho}) \). The continuity equation in (D22) corresponds to \( \partial_t \rho + (1/m \sqrt{\rho}) \partial_i \rho \sqrt{\rho} \) \((\partial_i \rho \sqrt{\rho}) \) = 0 (\( \rho \sim \rho \) here). For \( A_k = 0 \) (13) becomes (D24) \( \partial_t S + (1/2m)g^{ik} \partial_i S \rho \rho S + V - \gamma (\hbar^2/m)R = 0 \). This leads to an identification (D25) \( Q \sim -\gamma (\hbar^2/m)R \) where \( R \) is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via (D10)). Here \( \gamma = (1/6)(n - 2)(n - 2) \) as in (D4) which for \( n = 3 \) becomes \( \gamma = 1/12 \); further by (D17) the Weyl field \( \phi_i = -\partial_i \log(\rho) \). Consequently for the SE (4.1) in Weyl space the quantum potential is \( Q = -(\hbar^2/12m)R \) where \( R \) is the Weyl-Ricci scalar curvature. For Riemannian flat space \( \hat{R} = 0 \) this becomes via (D18)

\[
(4.5) \quad R = \frac{1}{2\gamma \rho} \partial_i g^{ik} \partial_k \sqrt{\rho} \sim \frac{1}{2\gamma \rho} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}
\]

as is should and the SE (4.4) reduces to the standard SE \( i\hbar \partial_t \psi = -(\hbar^2/2m)\Delta \psi + V \psi \).

**REMARK 4.2.** The formulation above from [63] is also developed for a derivation of the Klein-Gordon (KG) equation via an average action principle with the restrictions of Weyl geometry released. The spacetime geometry was then obtained from the action principle to obtain Weyl connections with a gauge field \( \phi_i \). The Riemann scalar curvature \( \hat{R} \) is then related to the Weyl scalar curvature \( R \) via an equation (D26) \( R = \hat{R} - 3(1/2) g^{\mu\nu} \phi_{\mu} \phi_{\nu} + (1/\sqrt{-g}) \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \phi_{\nu}) \). Explicit reference to the underlying Weyl structure disappears in the resulting SE (as in [14]). The HJ equation in [63] has then the form (for \( A_k = 0 \) and \( V = 0 \)) (D27) \( g^{\mu\nu} \partial_{\mu} S \partial_{\nu} S = m^2 - (R/6) \) so in some sense (recall here \( \hbar = c = 1 \)) (D28) \( m^2 - (R/6) \sim m^2 \) (via arguments in [63] - cf. also [14]) where \( m^2 = m^2 \exp(\Omega) \) and \( \Omega = (\hbar^2/m^2 c^2)(\sqrt{\rho}/\sqrt{\rho}) \sim (\sqrt{\rho}/m^2 \sqrt{\rho}) \) (for signature \((-++++)\)). Thus for \( \exp(\Omega) \sim 1 + \Omega \) one has (D29) \( m^2 - (R/6) \sim m^2 (1 + \Omega) \Rightarrow (R/6) \sim -\Omega m^2 \sim -\Omega \sqrt{\rho}/\sqrt{\rho} \).

This agrees also with [17] where the whole matter is analyzed incisively (and we recall the remarks at the beginning of Section 4). In this situation the probabilistic aspects (if any) are hidden and we refer to [51] for discussion of this point.

**REMARK 4.3.** For \( \hat{R} = 0 \) one has as in Remark 4.1 \( Q \sim (\gamma \hbar^2/m)R \) where \( \gamma = 1/12 \) with (D30) \( R = (1/2\gamma \sqrt{\rho}) \partial_i g^{ik} \partial_k \sqrt{\rho} = (1/2\gamma \sqrt{\rho}) g^{ik} \partial_i \partial_k \sqrt{\rho} \) (since \( g^{ik} \) can be taken to be
constant - cf. [3]). Then writing out (4.5) we have

\begin{equation}
Q = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} g^{ik} \partial_i \partial_k \sqrt{\rho} = \frac{\hbar^2 g^{ik}}{8m} \left( \frac{2\partial_i \partial_k \rho}{\rho} - \frac{\partial_i \partial_k \rho}{\rho^2} \right)
\end{equation}

(4.6)

corresponding to (3.14). Thus Q and consequently \( R = -\frac{m}{\gamma \hbar^2} Q \) arise from variation of the Fisher metric I of (3.9) in \( P \sim \rho \). Noting that (as in Remark 3.2) integrals of the form \( \int \partial_i \partial_k \rho d^3x \) could be expected to vanish for distributions \( \rho \) decreasing rapidly with their derivatives at \( \infty \) we could say now that (D31) \( \int \rho Q d^3x \sim -\left( \frac{\hbar^2 g^{ik}}{8m} \right) \int \partial_i \partial_k \rho/\rho d^3x = -\left( \frac{\hbar^2}{8m} \right) I \) via (3.9). This says that (\( \gamma = 1/12 \) (D32) \( I \sim -\frac{\hbar^2}{8m} \int \rho \left[ -\frac{\gamma \hbar^2}{m} R \right] d^3x = \left( \frac{\hbar^4}{96m^2} \right) \int \rho R d^3x \) and presents an explicit connection between the Fisher information metric and the Weyl-Ricci scalar curvature \( R \) (for Riemann flat spaces).

\[ \square \]

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