BABUŠKA-OSBORN TECHNIQUES IN DISCONTINUOUS GALERKIN METHODS: L₂-NORM ERROR ESTIMATES FOR UNSTRUCTURED MESHES

EMMANUIL H. GEORGOULIS, CHARALAMBOS G. MAKRIDAKIS, AND TRISTAN PRYER

Abstract. We prove the inf-sup stability of the interior penalty class of discontinuous Galerkin schemes in unbalanced mesh-dependent norms, under a mesh condition allowing for a general class of meshes, which includes many examples of geometrically graded element neighbourhoods. The inf-sup condition results in the stability of the interior penalty Ritz projection in L₂ as well as, for the first time, quasi-best approximations in the L₂-norm which in turn imply a priori error estimates that do not depend on the global maximum meshsize in that norm. Some numerical experiments are also given.

1. Introduction

Discontinuous Galerkin (dG) methods are a popular family of non-conforming finite element-type approximation schemes for partial differential equations (PDEs) involving discontinuous approximation spaces. In the context of elliptic problems their inception can be traced back to the 1970s [16, 5, 1]; see also [2] for an overview and history of these methods for second order problems. For higher order problems, for example the (nonlinear) biharmonic problem, dG methods are a useful alternative to using C¹-conforming elements [5, 19, 8, 9, 18], whose implementation (especially in the context of hp-version finite elements) can become complicated.

The derivation of L₂-norm a priori error estimates is standard in the literature: for a standard dG method (e.g., symmetric interior penalty), for the Poisson problem with standard boundary conditions, and for piecewise linear finite elements, a combination of H¹ bounds and a duality approach yield the bound

\[ \|u - u_h\|_{L_2(\Omega)} \leq C \max_{K \in \mathcal{T}_h} h_K \left( \sum_{K \in \mathcal{T}_h} h_K^2 \left\| D^2 u \right\|_{L_2(K)}^2 \right)^{1/2}, \]

i.e., the bound is identical to the respective bound for conforming finite element methods. It is well known that such a bound is not sharp: it is often desirable to use non-quasiuniform meshes generated, for instance, through an adaptive mesh refinement algorithm. In [14], it was shown that this bound can be improved under some assumptions on the mesh to

\[ \|u - u_h\|_{L_2(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}_h} h_K^4 \left\| D^2 u \right\|_{L_2(K)}^2 \right)^{1/2}, \]

for the conforming finite element method. In this work, we prove (1.2) for the symmetric interior penalty dG method, thereby extending the results from [14] into the dG setting, under similar mesh assumptions. The mesh assumption, informally speaking, reads \( \|h\|/\|h\|_{L_\infty(\Gamma)} \leq \alpha \), for some \( 0 \leq \alpha < 1 \), with \( h \) denoting an element-wise constant function characterising the local meshsize and \( \{\cdot\} \) the jump and average across the mesh skeleton \( \Gamma \). This effectively restricts the level of grading allowed on the underlying mesh, nonetheless allowing for geometrically graded meshes arising from adaptive mesh refinement procedures for example.

The proof of (1.2) relies on a new inf-sup condition shown for unbalanced L₂ and H²-like mesh-dependent norms like those used in [7], however builds on this making use of new localisation techniques developed in [14] for the conforming finite element method and resolves a number of technical difficulties specific to the dG setting. In particular, in contrast to the conforming case, local bounds for the interface terms arising in the interior penalty dG bilinear form have to be also treated using non-standard “bubble”-function techniques. At the same time, contrary to the respective conforming results in [14], the proof for the interior penalty dG
method presented below does not make use of superapproximation results \[17\], thereby, allowing for greater control of the value of the mesh-grading constant $\alpha$.

This is in keeping with the spirit of the seminal work of Babuška and Osborn \[3\], see also \[4\], where the respective result to \[1\] for continuous finite element methods in one spatial dimension for second and fourth order problems are first proven. The present approach, however, is quite different on the technical level and results in inf-sup stability for $L^2$- and $H^2$-like mesh-dependent norms under the aforementioned mesh assumption. Other potential applications of the analysis presented below include the development of convergent adaptive dG schemes for the $L^2$-norm error, which would follow the respective developments of \[6\] for conforming finite element methods, and quasi-best approximation results for nonconforming methods for elliptic \[20\] and for evolution problems \[15\].

We also take the opportunity to extend the ideas of \[10\] into the $L^2$ setting. This allows us to circumvent regularity restrictions that would require $u \in H^s$ for $s > 3/2$. Our analysis is quite general and holds for functions $u \in H^1$ only. The tools used to prove this result include an $H^2$-conforming reconstruction operator used in the a posteriori analysis of fourth order problems \[9\].

2. Model problem and discretisation

To assist the exposition of the key ideas, we shall consider the Poisson problem with homogeneous Dirichlet boundary conditions as model problem. The results presented in this work can be also proven with straightforward modifications for more general elliptic problems, such as ones with variable diffusivity and/or non-homogeneous boundary conditions.

More specifically, let $\Omega \subset \mathbb{R}^d$ be an open Lipschitz domain and consider the problem: find $u \in H^1_0(\Omega)$, such that

\[
\mathcal{A}(u, v) = \langle f, v \rangle \quad \forall \, v \in H^1_0(\Omega),
\]

where $\langle \cdot , \cdot \rangle$ denotes the $L^2$ inner product and the bilinear form $\mathcal{A} : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ is given by

\[
\mathcal{A}(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx.
\]

Now, if $\Omega$ is such that $\Delta u \in L^2(\Omega)$, we can also consider the unbalanced bilinear form $\mathcal{A} : H^2(\mathcal{T}) \cap H^1_0(\Omega) \times L^2(\Omega) \to \mathbb{R}$ given by

\[
\mathcal{A}(u, v) := -\int_{\Omega} \Delta u v \, dx,
\]

whose stability can be inferred via an inf-sup condition.

2.1. Proposition (inf-sup stability of the Laplacian). With $\mathcal{A}$ defined as in (2.3) we have that

\[
\sup_{v \in L^2(\Omega)} \frac{\mathcal{A}(u, v)}{\|v\|_{L^2(\Omega)}} = \|\Delta u\|_{L^2(\Omega)}.
\]

Also, assuming that $\Omega$ is convex, then the Miranda-Talenti inequality $\|u\|_{H^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)}$ holds for some $C > 0$ independent of $u$, $f$ and we have the a priori bound

\[
|u|_{H^2(\Omega)} \leq C_{reg} \|f\|_{L^2(\Omega)},
\]

for some $C_{reg} > 0$ also independent of $u$ and of $f$.

Proof The proof is immediate upon application of the Cauchy-Schwarz inequality on (2.3). \qed

2.2. Discretisation. Let $\mathcal{T}$ be a conforming mesh of $\Omega \subset \mathbb{R}^d$ into simplicial and/or box-type elements, namely, $\mathcal{T}$ is a finite family of sets such that

1. $\mathcal{T}$ implies $K$ is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$) or an open box (quadrilateral for $d = 2$, hexahedron for $d = 3$),
2. for any $K, J \in \mathcal{T}$ we have that $\overline{K} \cap \overline{J}$ is either empty or a complete $(d-r)$-dimensional simplex/box (i.e., it is either a vertex for $r = d$, an edge for $r = d-1$, a face for $r = d-2$ when $d = 3$, or the whole of $\overline{K}$ and $\overline{J}$) of both $\overline{K}$ and $\overline{J}$ and
3. $\bigcup_{K \in \mathcal{T}} K = \overline{\Omega}$.
The shape regularity constant of $\mathcal{T}$ is defined as
\begin{equation}
\mu(\mathcal{T}) := \inf_{K \in \mathcal{T}} \frac{\rho_K}{h_K},
\end{equation}
where $\rho_K$ is the radius of the largest inscribed ball of $K$ and $h_K$ is its diameter. An indexed family of triangulations $\{\mathcal{T}^n\}_n$ is called shape regular if
\begin{equation}
\mu := \inf_n \mu(\mathcal{T}^n) > 0.
\end{equation}

For $s > 0$, we define the broken Sobolev space $H^s(\mathcal{T})$, by
\begin{equation}
H^s(\mathcal{T}) := \{ w \in L^2(\Omega) : w|_K \in H^s(K), K \in \mathcal{T} \},
\end{equation}
along with the broken gradient and Laplacian $\nabla_h \equiv \nabla_h(\mathcal{T})$ and $\Delta_h \equiv \Delta_h(\mathcal{T})$, i.e., the element-wise gradient and Laplacian operators.

We consider the finite element space
\begin{equation}
\mathbb{V} := \{ \phi \in L^2(\Omega) : \phi|_K \in \mathbb{P}^k(K) \}
\end{equation}
where $\mathbb{P}^k(K)$ is the space of polynomials of total degree $k$. Alternatively, when $K \in \mathcal{T}$ is a box-type element, we can also consider polynomials of degree $k$ in each variable, typically mapped from a reference hypercube. The fine properties of the respective finite element spaces are of no essential consequence to the results below, as long as standard best approximation bounds are available for the elements considered.

Let also $\Gamma = \cup_{K \in \mathcal{T}} \partial K$ denote the skeleton of the mesh $\mathcal{T}$ and set $\Gamma_{\text{int}} := \Gamma \setminus \partial \Omega$ to denote the skeleton interior to $\Omega$.

2.3. Definition (jumps and averages). We define average and jump operators for arbitrary scalar $v \in H^s(\mathcal{T})$ and vector $\mathbf{v} \in [H^s(\mathcal{T})]^d$ functions, with $s > 3/2$, as
\begin{equation}
\{ v \} = \frac{1}{2} (v|_{K_1} + v|_{K_2}), \quad \langle v \rangle = \frac{1}{2} (v|_{K_1} + v|_{K_2}),
\end{equation}
\begin{equation}
\langle v \rangle = v|_{K_1} n_{K_1} + v|_{K_2} n_{K_2}, \quad \langle \mathbf{v} \rangle = (v|_{K_1}) \mathbf{n}_{K_1} + (v|_{K_2}) \mathbf{n}_{K_2}.
\end{equation}

Note that on the boundary of the domain $\partial \Omega$ the jump and average operators are defined as
\begin{equation}
\{ v \} \big|_{\partial \Omega} := v, \quad \langle v \rangle \big|_{\partial \Omega} := v,
\end{equation}
\begin{equation}
\langle v \rangle \big|_{\partial \Omega} := v \mathbf{n}, \quad \langle \mathbf{v} \rangle \big|_{\partial \Omega} := v^\top \mathbf{n}.
\end{equation}

Further, we define $h : \Omega \to \mathbb{R}_+$ to be the piecewise constant meshsize function of $\mathcal{T}$ given by $h|_K := h_K$, $K \in \mathcal{T}$ and $h|_{\Gamma} := \langle h \rangle$. The conformity assumption of the mesh, along with shape regularity imply the equivalence
\begin{equation}
C_{qu}^{-1} h_K \leq h(x) \leq C_{qu} h_K,
\end{equation}
for all $x \in \omega_K := \cup_{K' \in \mathcal{T} : K \cap K' \neq \emptyset} K'$, for some $C_{qu} > 0$ depending only on $\mu$.

2.4. Interior penalty discontinuous Galerkin method. We consider the interior penalty (IP) discontinuous Galerkin discretisation of (2.2), reading: find $u_h \in \mathbb{V}$ such that
\begin{equation}
\mathcal{A}_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in \mathbb{V},
\end{equation}
where
\begin{equation}
\mathcal{A}_h(u_h, v_h) = \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\Gamma} \langle v_h \rangle \cdot \langle \nabla u_h \rangle + \theta \| u_h \| \cdot \| \nabla v_h \| - \sigma \| u_h \| \cdot \| v_h \| \, ds,
\end{equation}
where $\sigma > 0$ is the, so-called, discontinuity penalisation parameter given by
\begin{equation}
\sigma := C_\sigma \frac{k^2}{h},
\end{equation}
and $\theta \in [-1, 1]$ a (global) constant used to select between the symmetric IP dG method ($\theta = 1$) and its non-symmetric variant ($\theta = -1$). As expected optimal results are obtained when $\theta = 1$. For completeness we also discuss the case of $\theta \neq 1$ (see Remark 3.2). The constant $C_\sigma > 0$ is also typically chosen globally:
when \( \theta = 1 \) it should be chosen large enough so as to counteract a constant of an inverse estimate to achieve coercivity, while it can be chosen freely when \( \theta = -1 \). Numerical evidence suggests that the choice \( \theta = -1 \) results in dG methods which converge suboptimally with respect to the meshsize \( h \) for even polynomial degrees, when the error is measured in the \( L_2 \)-norm.

2.5. Definition (mesh dependent norms). We introduce the mesh dependent \( L_2 \), \( H^1 \) and \( H^2 \) norms to be

\[
\|w\|_{0,h}^2 := \|w\|_{L_2(\Omega)}^2 + \left( h^{3/2} \|\nabla w\|_{L_2(\Gamma)}^2 + h^{1/2} \|w\|_{L_2(\Gamma)}^2 \right) + \left( h^{1/2} \|w\|_{L_2(\Gamma)}^2 \right)
\]

\[
\|w\|_{1,h}^2 := \|\nabla w\|_{L_2(\Omega)}^2 + \|\sqrt{\sigma} \|w\|_{L_2(\Gamma)}^2
\]

\[
\|w\|_{2,h}^2 := \|\nabla w\|_{L_2(\Omega)}^2 + \|\nabla_h \|w\|_{L_2(\Omega)}^2 + \left( h^{-1/2} \|\nabla w\|_{L_2(\Gamma)}^2 \right) + \left( h^{-3/2} \|w\|_{L_2(\Gamma)}^2 \right).
\]

2.6. Remark (motivation and properties of mesh dependent norms). The motivation for the norms given in Definition 2.5 is that, upon integration by parts, the IP dG bilinear form becomes

\[
\mathcal{A}_h(u_h, v_h) = -\int_\Omega \Delta_h u_h \ v h \ dx + \int_{\Gamma_{\text{int}}} \left( \nabla u_h \cdot \nabla v_h \right) \ ds - \int_{\Gamma} \theta \ |u_h| \cdot \left| \nabla v_h \right| - \sigma \ |u_h| \cdot \left| v_h \right| \ ds,
\]

whence, for \( w, v \in H^2(\mathcal{F}) \),

\[
|\mathcal{A}_h(w, v)| \leq C \|w\|_{2,h} \|v\|_{0,h}.
\]

Notice that the norm \( \|\cdot\|_{2,h} \) includes \( \|\cdot\|_{1,h} \) to ensure it is, indeed, a norm.

The norm \( \|\cdot\|_{0,h} \) is also equivalent to the \( L_2 \) norm over \( \mathcal{V} \) in view of standard inverse inequalities, that is, for any \( w_h \in \mathcal{V} \) there exists a \( C > 0 \) such that

\[
C^{-1} \|w_h\|_{0,h} \leq \|w_h\|_{L_2(\Omega)} \leq \|w_h\|_{0,h}.
\]

2.7. Proposition (continuity and coercivity of \( \mathcal{A}_h(\cdot, \cdot) \) in \( \|\cdot\|_{1,h} \)). For \( C_\sigma \) large enough, the bilinear form \( \mathcal{A}_h(\cdot, \cdot) \) satisfies

\[
\mathcal{A}_h(u_h, u_h) \geq c_0 \|u_h\|_{1,h}^2,
\]

\[
\mathcal{A}_h(u_h, v_h) \leq C_0 \|u_h\|_{1,h} \|v_h\|_{1,h},
\]

for \( c_0, C_0 > 0 \) independent of \( h \), \( C_\sigma \), \( u_h \), and \( v_h \).

Lax-Milgram Theorem guarantees a unique solution to the problem (2.15).

The main result of this work is the following theorem, the proof of which we shall dedicate Section 3 to.

2.8. Theorem (Inf-sup stability of the dG method). Let \( \mathcal{A}_h(\cdot, \cdot) \) be the bilinear form given in (2.15) with \( \theta = 1 \), and assume that the penalty parameter \( \sigma \) is chosen large enough to ensure the validity of (2.23).

Suppose that the underlying mesh of the finite element space satisfies

\[
\|\|h\|/\|h\|\|_{L_\infty(\Gamma)} \leq \alpha, \quad \text{for some } 0 \leq \alpha < 1 \text{ small enough.}
\]

Then, there exists a constant \( \gamma > 0 \), independent of \( h \), \( w_h \) and \( v_h \), such that

\[
\sup_{0 \neq v_h \in \mathcal{V}} \frac{\mathcal{A}_h(w_h, v_h)}{\|v_h\|_{2,h}} \geq \gamma \|w_h\|_{0,h} \quad \forall \ w_h \in \mathcal{V}.
\]

2.9. Remark (Interpreting the condition \( \|\|h\|/\|h\|\|_{L_\infty(\Gamma)} \leq \alpha \)). Figure 1 shows three different classes of mesh, one being generated through a newest vertex bisection adaptive refinement procedure another being an artificially graded mesh and a third being of Shishkin type. In all cases the values of \( \|\|h\|/\|h\|\|_{L_\infty(\Gamma)} \) are computed. As expected, standard, shape-regular locally adapted meshes generated through newest vertex bisection refinement satisfy \( \|\|h\|/\|h\|\|_{L_\infty(\Gamma)} < 1 \).

We note the difference in approach between the analysis presented in this work and the related works of [14, 6]. Indeed, the mesh functions and the underlying assumptions on the mesh proposed here is different than [14, 6] and, although related, it appears that the condition (2.25) allows the validity of the dG-a priori estimates for more general meshes. For instance, for \( \bar{h} \) denoting the piecewise linear continuous
mesh function defined in \cite{14,6}, then for the graded mesh in Figure 1, we have \( \| h \| / \| h \|_{L_\infty(\Gamma)} \approx 2.5 \) and \( \| h \| / \| h \|_{L_\infty(\Gamma)} \approx 0.61 \). Similar behaviour has been observed in other examples of meshes.

Equipped with Theorem 2.8, we have the following result, stating the \( L^2 \)-norm error optimality of the interior penalty dG method under the mesh assumption (2.25).

2.10. Corollary. Let \( R : H^2(\mathcal{T}) \to \mathcal{V} \) be the dG-Ritz-projection operator defined for \( u \in H^2(\mathcal{T}) \) by

\[ A_h(R u, v_h) = A_h(u, v_h) \quad \forall v_h \in \mathcal{V}. \]

Then under the hypotheses of Theorem 2.8 we have:

1. \( R \) is stable in \( \| \cdot \|_{0,h} \), that is:

\[ \| R u \|_{0,h} \leq \gamma^{-1} \| u \|_{0,h}. \]

2. \( R \) satisfies quasi-optimal error bounds in \( \| \cdot \|_{0,h} \), that is:

\[ \| u - R u \|_{0,h} \leq (1 + \gamma^{-1}) \inf_{v_h \in \mathcal{V}} \| u - v_h \|_{0,h}. \]

3. If \( u \in H^{k+1}(\Omega) \) solves (2.1) and \( u_h \in \mathcal{V} \) solves (2.14), then

\[ \| u - u_h \|_{0,h} \leq C \sum_{K \in \mathcal{T}} \left( \| h^{k+1} D^{k+1} u \|_{L_2(K)}^2 \right)^{1/2}. \]

Proof For \( 1 \), Theorem 2.8, the definition of \( R \) (2.27) and the continuity bound (2.21), imply

\[ \gamma \| R u \|_{0,h} \leq \sup_{0 \neq v_h \in \mathcal{V}} \frac{A_h(R u, v_h)}{\| v_h \|_{2,h}} \leq \sup_{0 \neq v_h \in \mathcal{V}} \frac{A_h(u, v_h)}{\| v_h \|_{2,h}} \leq \| u \|_{0,h}. \]

For \( 2 \), note that for any \( v_h \in \mathcal{V} \)

\[ \| u - R u \|_{0,h} \leq \| u - v_h \|_{0,h} + \| R(v_h - u) \|_{0,h} \leq (1 + \gamma^{-1}) \| u - v_h \|_{0,h}, \]

due to (1). Finally, (3) follows by choosing \( v_h \) to be an appropriate interpolant and using respective best approximation bounds. \( \square \)

3. Proof of Theorem 2.8

We begin by proving a crucial technical result regarding the stability of the dG-Ritz-projection operator in the \( L_2 \)-norm.

3.1. Lemma (\( L_2 \)-stability of \( R \)). Let \( w \in H^2(\Omega) \) and assume that the mesh satisfies (2.25). Then, for \( \theta = 1 \), its dG-Ritz-projection \( R w \) satisfies the bound

\[ \| R w \|_{L_2(\Omega)} \leq C \left( \| h^2 \nabla w \|_{L_2(\Gamma)} + \| w \|_{L_2(\Omega)} + \| h^{1/2} \nabla w \|_{L_2(\Gamma)} \right). \]
Figure 2. In this experiment we test the $L_2$ convergence of the interior penalty method and demonstrate that even for the worst class of mesh given in Figure 1 optimal $L_2$ convergence is achieved. Here we chose $C_\sigma = 20$, smaller values of $C_\sigma$ resulted in a suboptimal convergence in $L_2$ norm.

(a) After 1 global refinement.  
(b) After 2 global refinements.  
(c) After 3 global refinements.  
(d) After 4 global refinements.  
(e) After 5 global refinements.  
(f) Convergence rates of the approximation.

Proof Let $g \in H^2(\Omega)$ be the solution to

$$-\Delta g = Rw \quad \text{in } \Omega, \quad g = 0 \text{ on } \partial\Omega,$$

(3.2)
for which we assume the a priori bound (2.5). Since $\mathcal{A}_h(\cdot, \cdot)$ is consistent, we have

$$
\|Rw\|^2_{L^2(\Omega)} = -\int_{\Omega} \Delta gRw \, dx = -\int_{\Omega} \Delta g \, dx - \int_{\Omega} \Delta g(Rw-w) \, dx
$$

(3.3)

$$
= \int_{\Omega} Rw \, dx + \int_{\Gamma} \nabla g \cdot \nabla h(Rw-w) \, dx - \int_{\Gamma} \|\nabla g\| \cdot \|Rw\| \, ds.
$$

Let $\Pi : H^1(\Omega) \to \mathcal{V} \cap H_h^0(\Omega)$ a suitable conforming projection with optimal approximation properties. Then, from the elliptic projection definition, we have

$$
\int_{\Omega} \nabla_h Rw \cdot \nabla \Pi g \, dx - \int_{\Gamma} \|\nabla \Pi g\| \cdot \|Rw\| \, ds = \mathcal{A}_h(Rw, \Pi g) = \mathcal{A}_h(w, \Pi g) = \int_{\Omega} \nabla w \cdot \nabla \Pi g \, dx.
$$

Combining (3.3) with (3.4), we arrive at

$$
\|Rw\|^2_{L^2(\Omega)} = \int_{\Omega} Rw \, dx + \int_{\Omega} \nabla (g - \Pi g) \cdot \nabla h(Rw-w) \, dx - \int_{\Gamma} \|\nabla (g - \Pi g)\| \cdot \|Rw\| \, ds
$$

(3.5)

$$
\leq \|Rw\|^2_{L^2(\Omega)} \|u\|_{H_h^2(\Omega)} + \|h^{-1}\nabla (g - \Pi g)\|_{L^2(\Omega)} \|h \nabla h(Rw-w)\|_{L^2(\Omega)}
$$

$$
+ \left\{h^{-1/2} \|\nabla (g - \Pi g)\|_{L^2(\Gamma)} \right\} \left\{h^{1/2} \|Rw\|_{L^2(\Gamma)} \right\}.
$$

From the optimal approximation properties of the projection/interpolant $\Pi$, we have

$$
\|h^{-1}\nabla h(g - \Pi g)\|_{L^2(\Omega)} + \left\{h^{-1/2} \|\nabla (g - \Pi g)\|_{L^2(\Gamma)} \right\} \leq C_{ap} \|g\|_{H^2(\Omega)} \leq C_{ap} C_{reg} \|Rw\|_{L^2(\Omega)}.
$$

Setting $\tilde{c} := C_{ap} C_{reg}$ and combining the above, therefore, we arrive at

$$
\|Rw\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \tilde{c} \left( \|h \nabla h(Rw-w)\|_{L^2(\Omega)} + \left\{h^{1/2} \|Rw\|_{L^2(\Gamma)} \right\} \right).
$$

It remains to show the bound

$$
\|h \nabla h Rw\|^2_{L^2(\Omega)} + \left\{h^{1/2} \|Rw\| \right\}^2_{L^2(\Gamma)} \leq C \left( \|h \nabla w\|^2_{L^2(\Gamma)} + \left\{h^{3/2} \|\nabla w\| \right\}^2_{L^2(\Gamma)} \right) + \frac{1}{4\tilde{c}} \|Rw\|^2_{L^2(\Omega)},
$$

to conclude the proof. To that end, (2.27) with $v_h = h^2 R w$ implies

$$
\|h \nabla h Rw\|^2_{L^2(\Omega)} = \int_{\Omega} \nabla_h Rw \cdot \nabla_h (h^2 Rw) \, dx
$$

$$
= \int_{\Omega} \nabla w \cdot \nabla_h (h^2 Rw) \, dx
$$

$$
+ \int_{\Gamma} \left\{\|\nabla (Rw-w)\| \cdot \|h^2 Rw\| + \|\nabla (h^2 Rw)\| \cdot \|Rw\| - \|h\| \|\nabla h\| \cdot \|Rw\| \right\} \, ds.
$$

Using now the elementary identities $\|h^2 Rw\| = \|h\|^2 \|Rw\| + \|h^2\| \|Rw\|$ and $\|h^2\| = 2 \|h\| \|h\|$, which are valid on each internal face $e \in \Gamma_{\text{int}}$, we arrive at

$$
\|h \nabla h Rw\|^2_{L^2(\Omega)} + \int_{\Gamma} \sigma \|\nabla h\|^2 \|Rw\|^2 \, ds = \int_{\Omega} \langle h \nabla w \rangle \cdot \langle h \nabla h Rw \rangle \, dx + \int_{\Gamma} \|\nabla (Rw-w)\| \cdot \|Rw\| \|h^2\| \, ds
$$

$$
+ 2 \int_{\Gamma_{\text{int}}} h \|\nabla (Rw-w)\| \cdot \|h\| \|Rw\| \, ds
$$

$$
+ \int_{\Gamma} \|\nabla (h^2 Rw)\| \cdot \|Rw\| \, ds - 2 \int_{\Gamma_{\text{int}}} \sigma h \|Rw\| \cdot \|h\| \|Rw\| \, ds,
$$

recalling that $h := \|h\|$ on $\Gamma_{\text{int}}$. Using (2.25), we proceed to bound each skeletal term on the right hand side of (3.6). To that end let $C_{\text{inv}} > 0$ denote the constant of the trace-inverse estimate, that is, $C_{\text{inv}}$ satisfies

$$
\|v\|^2_{L^2(\Omega)} \leq C_{\text{inv}} h^2 K^{-1} \|v\|_{L^2(K)},
$$
for \( e \subset \partial K \), and \( v \in P^k(K) \), and recall \( C_{qu} > 0 \) is the local quasi-uniformity constant from \((2.13)\). Then, in view of the definition of the penalty parameter \((3.12)\), Cauchy-Schwarz and Young’s inequalities, we see

\[
\begin{aligned}
&\int_{\Gamma} \left\| \nabla (Rw - w) \right\| \cdot \left\| Rw \right\| \cdot h^2 \, ds \\
&\quad \leq \frac{2C_{inv}C_{qu}}{k^2C_{\sigma}} \| h \nabla_h Rw \|^2_{L^2(\Omega)} + \frac{2C_{qu}}{k^2C_{\sigma}} \| h^{3/2} \nabla w \|^2_{L^2(\Gamma)}/4.
\end{aligned}
\]  

(3.7)

for the first skeletal term. Splitting the second up we have

\[
\begin{aligned}
&2 \int_{\Gamma_{int}} \left\{ \nabla Rw \right\} \cdot \left\{ h \right\} \| h \| \| Rw \| \, ds \\
&\quad \leq 2 \alpha \int_{\Gamma_{int}} h^2 \| \nabla w \| \| Rw \| \, ds \\
&\quad \leq \alpha \| h^{3/2} \nabla Rw \|^2_{L^2(\Gamma_{int})} + \alpha \| h^{1/2} Rw \|^2_{L^2(\Gamma_{int})} \\
&\quad \leq \alpha C_{inv}C_{qu} \left( \| h \nabla_h Rw \|^2_{L^2(\Omega)} + \| Rw \|^2_{L^2(\Omega)} \right).
\end{aligned}
\]  

(3.8)

A similar argument to \((3.7)\) shows

\[
\begin{aligned}
&\int_{\Gamma} \left\{ \nabla (h^2 Rw) \right\} \cdot \left\{ h \right\} \| h \| \| Rw \| \, ds \\
&\quad \leq \left( \| \sigma \| h^2 \right)^{-1/2} \left\| h^{2} \nabla Rw \right\| ^2_{L^2(\Gamma)} + \frac{1}{4} \left\| \sqrt{\sigma} h^2 \right\| \| Rw \|^2_{L^2(\Gamma)} \\
&\quad \leq \frac{C_{inv}C_{qu}}{k^2C_{\sigma}} \| h \nabla_h Rw \|^2_{L^2(\Omega)} + \frac{1}{4} \left\| \sqrt{\sigma} h^2 \right\| \| Rw \|^2_{L^2(\Gamma)}.
\end{aligned}
\]  

(3.9)

and

\[
\begin{aligned}
&2 \int_{\Gamma_{int}} \sigma \| h \| \| Rw \| \cdot \left\{ h \right\} \| h \| \| Rw \| \, ds \\
&\quad \leq 2 \alpha \int_{\Gamma_{int}} \sigma h^2 \| Rw \| \| h \| \| Rw \| \, ds \\
&\quad \leq 4 \alpha \left\| \sigma h^{1/2} \| h^{2} \nabla Rw \right\| ^2_{L^2(\Gamma_{int})} + \frac{4}{4} \left\| \sqrt{\sigma} h^2 \right\| \| Rw \|^2_{L^2(\Gamma_{int})} \\
&\quad \leq 4 \alpha C_{inv}C_{qu} k^2C_{\sigma} \| Rw \|^2_{L^2(\Omega)} + \frac{4}{4} \left\| \sqrt{\sigma} h^2 \right\| \| Rw \|^2_{L^2(\Gamma_{int})}.
\end{aligned}
\]  

(3.10)

Substituting the above estimates \((3.7)\)–\((3.11)\) into \((3.6)\), we deduce

\[
\begin{aligned}
\left( \frac{1}{2} - \alpha C_{inv}C_{qu} - \frac{3C_{inv}C_{qu}}{k^2C_{\sigma}} \right) \| h \nabla_h Rw \|^2_{L^2(\Omega)} + \frac{2 - \alpha}{4} \int_{\Gamma} \sigma \| h^2 \| \| Rw \| \, ds \\
&\quad \leq \frac{1}{2} \| h \nabla w \|^2_{L^2(\Omega)} + \left( \alpha + \frac{2C_{qu}}{k^2C_{\sigma}} \right) \| h^{3/2} \nabla w \|^2_{L^2(\Gamma)} \\
&\quad \quad + 2 \alpha C_{inv}C_{qu} \left( 1 + 2k^2C_{\sigma} \right) \| Rw \|^2_{L^2(\Gamma)}.
\end{aligned}
\]  

(3.11)

Therefore, assuming that the discontinuity-penalisation constant \( C_{\sigma} \) is chosen so that

\[
\begin{aligned}
&\left( \frac{1}{2} - \alpha C_{inv}C_{qu} - \frac{3C_{inv}C_{qu}}{k^2C_{\sigma}} \right) \| h \nabla_h Rw \|^2_{L^2(\Omega)} + \frac{2 - \alpha}{4} \int_{\Gamma} \sigma \| h^2 \| \| Rw \| \, ds \\
&\quad \leq \frac{1}{2} \| h \nabla w \|^2_{L^2(\Omega)} + C \| h^{3/2} \nabla w \|^2_{L^2(\Gamma)} \\
&\quad \quad + \frac{c}{16} \| Rw \|^2_{L^2(\Omega)}.
\end{aligned}
\]  

(3.12)
Notice that this is not the only choice of $C_\sigma$ and $\alpha$. There is a subtle dependency between the two values in that they are coupled such that choosing a larger $C_\sigma$ allows more flexibility on the selection of $\alpha$.

The result already follows by combining the above bounds. \hfill \Box

3.2. **Remark** (Non-symmetric interior penalty methods). For $\theta \in [-1, 1)$, (3.5) becomes

\[
\|Rw\|_{L^2(\Omega)}^2 = \int_\Omega Rw w \, dx + \int_\Omega \nabla (g - \Pi g) \cdot \nabla_h (Rw - w) \, dx
\]

\[
- \int_\Gamma \|\nabla (g - \Pi g)\| \cdot \|Rw\| \, ds - \int_\Gamma (1 - \theta) \|\nabla \Pi g\| \cdot \|Rw\| \, ds,
\]

which, in turn, implies

\[
\|Rw\|_{L^2(\Omega)} \leq \|w\|_{L^2(\Omega)} + \bar{c} \left( \|h \nabla_h (Rw - w)\|_{L^2(\Omega)} + \|h^{1/2} \|Rw\|\|_{L^2(\Gamma)} \right) - (1 - \theta) \int_\Gamma \|\nabla \Pi g\| \cdot \|Rw\| \, ds.
\]

Estimating the last term on the right-hand side of the above bound gives

\[
\int_\Gamma \|\nabla \Pi g\| \cdot \|Rw\| \, ds \leq \left( \|\sigma \|_{L^2} \right)^{-1/2} \left( \|\nabla (g - \Pi g)\|_{L^2(\Gamma)} \right) \left( \|\nabla \Pi g\| \cdot \|Rw\|\|_{L^2(\Gamma)} \right)
\]

\[
+ \left( \|\sigma \|_{L^2} \right)^{-1/2} \left( \|\nabla g\|\right)_{L^2(\Gamma)} \left( \|\nabla \Pi g\| \cdot \|Rw\|\|_{L^2(\Gamma)} \right) \leq \frac{2C_{\text{reg}}}{k \sqrt{\min \{h\} C_\sigma}} \|Rw\|_{L^2(\Omega)} \left( \|\nabla \Pi g\| \cdot \|Rw\|\|_{L^2(\Gamma)} \right),
\]

which yields (3.1) only with values of $\theta > 1 - \min \{h\}$ when $C_\sigma$ is chosen independent of $h$. When $C_\sigma$ is chosen to depend on negative powers of $h$, i.e., in the case of super-penalisation, we can retrieve (3.1) for non-symmetric versions of the interior penalty dG method.

3.3. **Proof of Theorem 2.8** We give the proof of Theorem 2.8 for $\theta = 1$. Our goal is for fixed $v_h \in V$ to construct a $w_h \in V$ such that

\[
\mathcal{A}_h (w_h, v_h) \geq \|v_h\|_{L^2(\Omega)}^2
\]

and then to show one can find a constant $C > 0$ such that

\[
\|w_h\|_{2, \Gamma} \leq C \|v_h\|_{L^2(\Omega)}.
\]

It is the case that each of the four components of the $\|\cdot\|_{2, \Gamma}$-norm must be controlled; we shall bound each separately.

**Step 1**: For fixed $v_h$, let $\Phi \in V$ be the solution of the dual problem

\[
\mathcal{A}_h (\Phi, \Psi) = \langle v_h, \Psi \rangle \quad \forall \Psi \in V.
\]

To control $\|\cdot\|_{1, \Gamma}$, coercivity (2.28) yields

\[
\|\Phi\|_{1, \Gamma}^2 \leq \frac{1}{c_0} \mathcal{A}_h (\Phi, \Phi) = \frac{1}{c_0} \langle v_h, \Phi \rangle \leq \frac{1}{c_0} \|v_h\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)} \leq \frac{C_P}{c_0} \|v_h\|_{L^2(\Omega)} \|\Phi\|_{1, \Gamma},
\]

through a discrete Poincaré inequality and hence

\[
\|\Phi\|_{1, \Gamma} \leq \frac{C_P}{c_0} \|v_h\|_{L^2(\Omega)}.
\]

**Step 2**: Let $w_1 = -b_K^2 \Delta \Phi$, for with $b_K$ denoting the standard polynomial bubble function vanishing on $\partial K$. We, then, have

\[
\mathcal{A}_h (w_1, \Phi) = \sum_{K \in \mathcal{T}} \int_K -\Delta \Phi w_1 \, dx = \sum_{K \in \mathcal{T}} \int_K |\Delta \Phi|^2 b_K^2 \, dx,
\]

9
since $w_1|_e = 0$ and $\nabla w_1|_e = 0$ for all $e \in \mathcal{E}$. Due to the equivalence of norms on finite dimensional linear spaces

$$\frac{1}{C_1} \sum_{K \in \mathcal{T}} \| \Delta \Phi \|_{L^2(K)}^2 \leq \int_K |\Delta \Phi|^2 \, d^2 x = \mathcal{A}_h(w_1, \Phi) = \mathcal{A}_h(\Phi, w_1)$$

(3.21) 

using the symmetry of the bilinear form $\mathcal{A}_h$. Recalling that $w_1$ is discrete, making use of Lemma 3.1 and of local inverse inequalities, we have

$$\|R w_1\|_{L^2(\Omega)} \leq C \left( \|h \nabla w_1\|_{L^2(\Omega)} + \|w_1\|_{L^2(\Omega)} + \left\| h^{3/2} \nabla w_1 \right\|_{L^2(\Gamma)} \right)$$

(3.22)

Combining (3.21) and (3.22), we see

$$\left( \sum_{K \in \mathcal{T}} \| \Delta \Phi \|_{L^2(K)}^2 \right)^{1/2} \leq C \|v_h\|_{L^2(\Omega)}.$$

(3.23)

Step 3: Let $e$ be an internal edge of two neighbouring elements $K$ and $K'$ and let $b_e$ be a polynomial bubble function vanishing on the boundary of $K \cup K' \cup e$, so as to have by construction that $\nabla b_{e} \cdot n_e|_e = 0$; the simplest such bubble function is of degree four when $d = 2$, is a bubble function on the largest rhombus $\tilde{K}_e$ contained fully in $K \cup K' \cup e$ and having $e$ as one of its diagonals $e$ (see, e.g., [9] for details of such a construction). A completely analogous construction when $d = 3$ yields the same properties. The mesh regularity assumed implies that $\text{diam}(\tilde{K}_e)$ is uniformly bounded above and below by the mesh-function $h$.

Let also $v_e : K \cup K' \cup e \rightarrow \mathbb{R}$ given by $v_e := h^{-1} \nabla \Phi$ on the face $e$ and extended as a constant on the direction of the normal to $e$. Setting

$$w_2 := \sum_{e \in \mathcal{E}} b_e^2 v_e,$$

we have $w_2 \in H^1_0(\Omega)$ and that $\nabla w_2 \cdot n_e|_e = 0$ for all $e \in \mathcal{E}$. Therefore,

$$\mathcal{A}_h(w_2, \Phi) = \sum_{K \in \mathcal{T}} \int_K -\Delta \Phi w_2 \, dx + \int_{\Gamma_{\text{int}}} [\nabla \Phi] \cdot \{ w_2 \} \, ds$$

(3.24)

$$= \sum_{K \in \mathcal{T}} \int_K -\Delta \Phi w_2 \, dx + \int_{\Gamma_{\text{int}}} h^{-1} \| \nabla \Phi \|^2 b_e^2 \, ds.$$

The equivalence of all norms of a finite dimensional linear space implies that there exists a constant $C_2 > 0$, independent of $\Phi$ and $h$, such that

$$\frac{1}{C_2} \int_{\Gamma_{\text{int}}} h^{-1} \| \nabla \Phi \|^2 \, ds \leq \int_{\Gamma_{\text{int}}} h^{-1} \| \nabla \Phi \|^2 b_e^2 \, ds \leq \mathcal{A}_h(w_2, \Phi) - \sum_{K \in \mathcal{T}} \int_K \Delta \Phi w_2 \, dx$$

(3.25)

$$= \mathcal{A}_h(R w_2, \Phi) - \sum_{K \in \mathcal{T}} \int_K \Delta \Phi w_2 \, dx = \langle v_h, R w_2 \rangle - \sum_{K \in \mathcal{T}} \int_K \Delta \Phi w_2 \, dx$$

$$\leq C \| v_h \|_{L^2(\Omega)} \left( \|R w_2\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)} + \left\| h^{3/2} \nabla w_2 \right\|_{L^2(\Gamma)} \right)$$

$$\leq C \| v_h \|_{L^2(\Omega)} \| w_2 \|_{L^2(\Omega)},$$

making use of (3.23), (3.1) and of standard inverse estimates, respectively. To finish, we observe the bound

$$\| w_2 \|_{L^2(\Omega)} \leq C \left\| \sqrt{h} w_2 \right\|_{L^2(\Gamma_{\text{int}})} \leq C \left\| h^{-1/2} \nabla \Phi \right\|_{L^2(\Gamma_{\text{int}})}$$

(3.26)

which, in turn, implies

$$\left\| h^{-1/2} \nabla \Phi \right\|_{L^2(\Gamma_{\text{int}})} \leq C \| v_h \|_{L^2(\Omega)}.$$
Step 4: As before, let $e$ be an internal edge of two neighbouring elements $K$ and $K'$ and let $b_e$ as in Step 3. Let also $p'_e : K \cup K' \cup e \to \mathbb{R}$ be the plane passing through $e$ with slope equal to $h^{-3}$. Then, upon defining the function $z_{e|e} := (\Phi|_{\partial K \cap e} - \Phi|_{\partial K' \cap e})$ extended as a constant in the direction normal to $e$, we set $w_3 : H^1_0(\Omega) \to \mathbb{R}$ given by

$$w_3 := \sum_{e \in \mathcal{E}} z_{e|e} b_e^2 p'_e,$$

where $z_{e|e} := \Phi|_{\partial K \cap e}$ is on the boundary faces $e \subset \partial \Omega$. We note that the sign of the jump of $\Phi$ in the definition of $z_{e|e}$ is of no significance in what follows, so no effort is made in determining it exactly. With these definitions, we have $w_3 = 0$ on $\Gamma_{\text{int}}$ and

$$\nabla w_3 \cdot n_e|e = h^{-3} \|\Phi\|_e b_e^2,$$

on each $e \in \mathcal{E}$. Therefore, we have

$$\mathcal{A}_h(w_3, \Phi) = \sum_{K \in \mathcal{T}} \int_K -\Delta \Phi w_3 \, dx + \int_{\Gamma} \frac{b_e^2}{h^3} \|\Phi\|^2 \, ds.$$

As before, there exists a constant $C_2 > 0$, independent of $\Phi$ and of $h$, such that

$$\frac{1}{C_2} \int_{\Gamma} h^{-3} \|\Phi\|^2 \, ds \leq \int_{\Gamma} \frac{b_e^2}{h^3} \|\Phi\|^2 \, ds = \mathcal{A}_h(w_3, \Phi) + \sum_{K \in \mathcal{T}} \int_K \Delta \Phi w_3 \, dx$$

$$\leq \|v_h\|_{L^2(\Omega)} \|Rw_3\|_{L^2(\Omega)} + \left( \sum_{K \in \mathcal{T}} \int_K |\Delta \Phi|^2 \, dx \right)^{1/2} \|w_3\|_{L^2(\Omega)}$$

$$\leq C \|v_h\|_{L^2(\Omega)} \left( \|Rw_3\|_{L^2(\Omega)} + \|w_3\|_{L^2(\Omega)} + \|h^{-3/2} \nabla w_3\|_{L^2(\Gamma)} \right)$$

$$\leq C \|v_h\|_{L^2(\Omega)} \|w_3\|_{L^2(\Omega)},$$

from (3.23), (3.1) and standard inverse estimates.

Also, we have

$$\|w_3\|^2_{L^2(\Omega)} \leq C \sum_{K \in \mathcal{T}, e \subset \partial K} \|p'_e\|^2_{L^\infty(K)} \|z_{e|e}\|^2_{L^2(K)} \leq C \sum_{e \subset \partial \mathcal{E}} h_K^{-4} \|\sqrt{h} \Phi\|^2_{L^2(e)} \leq C \|h^{-3/2} \Phi\|^2_{L^2(\Gamma)},$$

which, finally, implies

$$\|h^{-3/2} \Phi\|_{L^2(\Gamma)} \leq C \|v_h\|_{L^2(\Omega)},$$

which, in turn, already proves the result.

4. Relaxation of regularity requirements

In the above discussion, we assumed for clarity of presentation that for the exact solution we have $u \in H^2(\Omega)$; the analysis presented also holds if $u \in H^s(\Omega)$ for $s > 3/2$. In this section we shall deduce a useful a priori bound for the interior penalty method with $\theta = 1$ for the case $u \in H^1(\Omega)$ also, by showing that

$$\|u - u_h\|_{L^2(\Omega)} \leq C \left( \inf_{w_h \in \mathcal{V}} \|u - w_h\|_{L^2(\Omega)} + \|h^2(f - P_k f)\|_{L^2(\Omega)} \right),$$

where $P_k$ is the $L^2$-orthogonal projection operator into element-wise polynomials of degree $k$. To do so, we shall use ideas from [1], extended to the present setting through the following Lemmata. The main result of the section is stated in Theorem 4.5.

4.1. Lemma. For $w \in H^1(\Omega), v \in H^2(\Omega)$ and $w_h \in \mathcal{V}$ it holds that

$$|\mathcal{A}(w, v) - \mathcal{A}_h(w_h, v)| \leq C \|w - w_h\|_{L^2(\Omega)} \|v\|_{2,h}.$$

Proof Since $v \in H^2(\Omega)$ we have, through the consistency of the scheme, that

$$|\mathcal{A}(w, v) - \mathcal{A}_h(w_h, v)| = -\int_\Omega (w - w_h) \Delta v \leq \|w - w_h\|_{L^2(\Omega)} \|v\|_{2,h}.$$
4.2. **Lemma** (Reconstruction operator). Let \( HCT(k + 2) \) denote the Hsieh-Clough-Tocher macro-element space, then there exists an operator \( E : V \to HCT(k + 2) \subset H^2(\Omega) \) such that for \( \alpha = 0, 1, 2 \)
\[
\sum_{K \in \mathcal{T}} \| E(w_h) - w_h \|^2_{H^\alpha(K)} \leq C \left( \| h^{1/2 - \alpha} [w_h] \|^2_{L^2(\Gamma)} + \| h^{3/2 - \alpha} [\nabla w_h] \|^2_{L^2(\Gamma)} \right) \quad \forall \ w_h \in V.
\]

**Proof** The proof of this is given in [9] Lemma 3.1.

4.3. **Remark.** The use of Lemma 4.1 will be crucial subsequently and, for this reason, the use of an \( H^2 \) reconstruction operator is necessary. This is in contrast to the argument in [10] where an \( H^1 \) conforming reconstruction was used.

4.4. **Lemma** (A posteriori lower bound). Let \( u \in H^s(\Omega) \) be the weak solution of (2.1) and \( w_h \in V \) be an arbitrary finite element function. Then,
\[
\sup_{v_h \in V} \frac{\langle f, v_h - E(v_h) \rangle - \mathcal{A}_h(w_h, v_h - E(v_h))}{\| v_h \|_{2,h}} \leq C \left( \sum_{K \in \mathcal{T}} \| u - w_h \|^2_{L^2(K)} + \| h^2 (f - P_k f) \|^2_{L^2(K)} \right)^{1/2}.
\]

**Proof** We begin by noting that
\[
\langle f, v_h - E(v_h) \rangle - \mathcal{A}_h(w_h, v_h - E(v_h)) = \sum_{K \in \mathcal{T}} \int_K (f + \Delta w_h)(v_h - E(v_h)) \, dx
\]
\[
- \int_{\Gamma_{\text{int}}} [\nabla w_h] \cdot [v_h - E(v_h)] \, ds
\]
\[
+ \int_{\Gamma} (\| w_h \| \cdot [\nabla v_h - \nabla E(v_h)] - \sigma [w_h] \cdot [v_h]) \, ds
\]
\[
=: \sum_{K \in \mathcal{T}} \mathcal{J}_1,K + \sum_{e \in \mathcal{E}} \mathcal{J}_{2,e} + \mathcal{J}_{3,e} + \mathcal{J}_{4,e}.
\]

We proceed to control each term separately. Firstly,
\[
\mathcal{J}_{1,K} \leq \| h^2 (f + \Delta w_h) \|^2_{L^2(K)} \| h^{-2} (v_h - E(v_h)) \|^2_{L^2(K)}
\]
\[
\leq \left( \| h^2 (P_k f + \Delta w_h) \|^2_{L^2(K)} + \| h^2 (f - P_k f) \|^2_{L^2(K)} \right) \| h^{-2} (v_h - E(v_h)) \|^2_{L^2(K)}.
\]

Now, making use of the interior bubble function \( b_K \), we have
\[
\| h^2 (P_k f + \Delta w_h) \|^2_{L^2(K)} \leq C \int_K h^4 (P_k f + \Delta w_h) b^2_K (P_k f + \Delta w_h) \, dx
\]
\[
= C \int_K h^4 (P_k f - f) + (f + \Delta w_h) b^2_K (P_k f + \Delta w_h) \, dx
\]
\[
\leq C \| h^2 (P_k f - f) \|^2_{L^2(K)} \| h^2 b^2_K (P_k f + \Delta w_h) \|^2_{L^2(K)}
\]
\[
+ C \int_K h^4 (f + \Delta w_h) b^2_K (P_k f + \Delta w_h) \, dx.
\]

Since \( b^2_K = 0 \) and \( \nabla b^2_K = 0 \) on \( \partial K \), we have
\[
\int_K h^4 (f + \Delta w_h) b^2_K (P_k f + \Delta w_h) \, dx = \int_K h^4 (u - w_h) \cdot \Delta (b^2_K (P_k f + \Delta w_h)) \, dx
\]
\[
\leq C \| u - w_h \|^2_{L^2(K)} \| h^2 (P_k f + \Delta w_h) \|^2_{L^2(K)}
\]
making use of inverse inequalities. Hence combining (4.7) and (4.8) with (4.9) we see
\[
\mathcal{J}_{1,K} \leq C \left( \| u - w_h \|^2_{L^2(K)} + \| h^2 (P_k f - f) \|^2_{L^2(K)} \right) \| h^{-2} (v_h - E(v_h)) \|^2_{L^2(K)}.
\]

Secondly,
\[
\mathcal{J}_{2,e} \leq \left( \| h^{3/2} [\nabla w_h] \|^2_{L^2(e)} \right) \| h^{-3/2} ([v_h - E(v_h)] \|^2_{L^2(e)}.
\]
Now
\[ \left\| h^{3/2} \| \nabla w_h \|_{L^2(e)} \right\|_{L^2(e)}^2 \leq C \int_{\Omega} h^3 \| \nabla w_h \| b^2_h \| \nabla w_h \| \, ds \]
\[ \leq C \int_{\Omega} h^4 \| \nabla w_h - \nabla u \| b^2_e v_e \, ds, \]
with \( v_e \) defined in Step 3 of the Proof of Theorem 2.8. Now since \( v_e b^2_h \) vanishes over the \( \partial(K \cup K^c) \) and \( \nabla b^2_e \cdot n = 0 \) we see
\[ \left\| h^{3/2} \| \nabla w_h \|_{L^2(e)} \right\|_{L^2(e)}^2 \leq C \int_{\Omega} h^4 \left( \| u - w_h \| + \| f + \Delta_h w_h \| \right) \| \nabla w_h \| \, dx \]
\[ \leq C \left( \| u - w_h \|_{L^2(K \cup K^c)} + \| f + \Delta_h w_h \|_{L^2(K \cup K^c)} \right) \| h^2 v_e b^2_e \|_{L^2(K \cup K^c)}, \]
though inverse inequalities. Now note that in view of the properties of \( b_e \) there exists a constant such that
\[ \left\| h^2 v_e b^2_e \right\|_{L^2(K \cup K^c)} \leq C \left\| h^{3/2} \| \nabla w_h \|_{L^2(e)} \right\|_{L^2(e)} \]
to see
\[ J_{2,e} \leq C \left( \| u - w_h \|_{L^2(K \cup K^c)} + \| h^2 (f + \Delta_h w_h) \|_{L^2(K \cup K^c)} \right) \left\| h^{-3/2} \| v_h - E(v_h) \| \right\|_{L^2(e)}. \]
The third term
\[ J_{3,e} \leq \left\| h^{1/2} \| w_h - u \|_{L^2(e)} \right\|_{L^2(e)} \left\| h^{-1/2} \| \nabla w_h - \nabla E(v_h) \| \right\|_{L^2(e)} \]
\[ \leq C_{\text{inco}} C_{\text{qu}} \| w_h - u \|_{L^2(K \cup K^c)} \left\| h^{-1/2} \| \nabla w_h - \nabla E(v_h) \| \right\|_{L^2(e)}. \]
Finally the fourth term,
\[ J_{4,e} \leq C_{\sigma} k^2 \left( h^{1/2} \| w_h - u \|_{L^2(e)} \right) \left( h^{-3/2} \| v_h - E(v_h) \|_{L^2(e)} \right) \]
\[ \leq C_{\sigma} C_{\text{inco}} C_{\text{qu}} k^2 \| w_h - u \|_{L^2(e)} \left( h^{-3/2} \| v_h - E(v_h) \|_{L^2(e)} \right). \]
Collecting all the information thusfar from (4.10), (4.15), (4.16) and (4.17) we can conclude that
\[ \left( f, v_h - E(v_h) \right) - \mathcal{A}(w_h, v_h - E(v_h)) \leq C \sum_{K \in \mathcal{F}} \left( \| u - w_h \|_{L^2(K)} + \| h^2 (P_k f - f) \|_{L^2(K)} \right) \eta_K (v_h - E(v_h)), \]
where
\[ \eta_K (z) = \max \left( \| h^{-2} z \|_{L^2(K)}, \max_{e \in \partial K} \| h^{-3/2} z \|_{L^2(e)}, \max_{e \in \partial K} \| h^{-1/2} \nabla z \|_{L^2(e)} \right). \]
Using the approximability properties of \( E \) given in Lemma 4.2 we see
\[ \sum_{K \in \mathcal{F}} \eta_K (v_h - E(v_h)) \leq C \| v_h \|_{L^2(e)}, \]
and hence the result follows from a discrete Cauchy-Schwarz inequality. \( \square \)

4.5. **Theorem** (Optimal convergence for weak solutions). Let \( u \in H^1(\Omega) \) be the weak solution of (2.2) and \( w_h \in \mathcal{V} \) be the dG solution (2.15) and that the conditions of Theorem 2.8 hold. Then there exists a constant such that
\[ \left\| u - w_h \right\|_{L^2(\Omega)} \leq C \left( \inf_{w_h \in \mathcal{V}} \left\| u - w_h \right\|_{L^2(\Omega)} + \left( \sum_{K \in \mathcal{F}} \| h^2 (f - P_k f) \|_{L^2(K)} \right)^{1/2} \right) \]
where \( P_k \) is the \( L^2 \) orthogonal projector into piecewise polynomials of degree \( k \).
Proof of Theorem 4.5 Note that from Theorem 2.8 we have that for any \( w_h \in V \)

\[
(4.22) \quad \| u_h - w_h \|_{0,h} \leq \sup_{v_h \in V} \mathcal{A}_h(u_h - w_h, v_h).
\]

By adding and substracting appropriate terms we see

\[
(4.23) \quad \mathcal{A}_h(u_h - w_h, v_h) = \mathcal{A}(u, E(v_h)) - \mathcal{A}_h(w_h, E(v_h)) + (f, v_h - E(v_h)) - \mathcal{A}_h(w_h, v_h - E(v_h))
\]

and by Lemma 4.1

\[
(4.24) \quad \mathcal{A}(u, E(v_h)) - \mathcal{A}_h(w_h, E(v_h)) \leq C \| u - w_h \|_{L^2(\Omega)} \| E(v_h) \|_{2,h}^2
\]

\[
\leq C \| u - w_h \|_{L^2(\Omega)} \| v_h \|_{2,h}^2,
\]

by Lemma 4.2 Hence

\[
(4.25) \quad \| u_h - w_h \|_{0,h} \leq C \left( \| u - w_h \|_{L^2(\Omega)} + \sup_{v_h \in V} (f, v_h - E(v_h)) - \mathcal{A}_h(w_h, v_h - E(v_h)) \right).
\]

So clearly,

\[
(4.26) \quad \| u - u_h \|_{L^2(\Omega)} \leq \| u - w_h \|_{L^2(\Omega)} + \| w_h - w_h \|_{L^2(\Omega)}
\]

\[
\leq C \left( \| u - w_h \|_{L^2(\Omega)} + \| u_h - w_h \|_{0,h} \right)
\]

\[
\leq C \left( \| u - u_h \|_{L^2(\Omega)} + \sup_{v_h \in V} (f, v_h - E(v_h)) - \mathcal{A}_h(w_h, v_h - E(v_h)) \right).
\]

The result follows from Lemma 4.4.

\[\square\]

\[\text{References}\]

[1] D. N. Arnold, An interior penalty finite element method with discontinuous elements, SIAM J. Numer. Anal., 19 (1982), pp. 742–760.
[2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal., 39 (2001/02), pp. 1749–1779.
[3] I. Babuška and J. Osborn, Analysis of finite element methods for second order boundary value problems using mesh dependent norms, Numer. Math., 34 (1980), pp. 41–62.
[4] I. Babuška, J. Osborn, and J. Pitkärinta, Analysis of mixed methods using mesh dependent norms, Math. Comp., 35 (1980), pp. 1039–1062.
[5] G. A. Baker, Finite element methods for elliptic equations using nonconforming elements, Math. Comp., 31 (1977), pp. 45–59.
[6] A. Demlow and R. Stevenson, Convergence and quasi-optimality of an adaptive finite element method for controlling \( L^2 \) errors, Numer. Math., 117 (2011), pp. 185–218.
[7] E. Georgoulis and T. Pryer, Analysis of discontinuous galerkin methods using mesh-dependent norms and applications to problems with rough data, ArXiv preprint, https://arxiv.org/pdf/1610.04994v1, (2016).
[8] E. H. Georgoulis and P. Houston, Discontinuous Galerkin methods for the biharmonic problem, IMA J. Numer. Anal., 29 (2009), pp. 573–594.
[9] E. H. Georgoulis, P. Houston, and J. Virtanen, An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems, IMA J. Numer. Anal., 31 (2011), pp. 281–298.
[10] T. Gudi, A new error analysis for discontinuous finite element methods for linear elliptic problems, Math. Comp., 79 (2010), pp. 2169–2189.
[11] K. Harriman, P. Houston, B. Senior, and E. Süli, hp-version discontinuous Galerkin methods with interior penalty for partial differential equations with nonnegative characteristic form, in Recent advances in scientific computing and partial differential equations (Hong Kong, 2002), vol. 330 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2003, pp. 89–119.
[12] R. Hartmann, Adjoint consistency analysis of discontinuous Galerkin discretizations, SIAM J. Numer. Anal., 45 (2007), pp. 2671–2696.
[13] P. Houston, C. Schwab, and E. Süli, Discontinuous hp-finite element methods for advection-diffusion-reaction problems, SIAM J. Numer. Anal., 39 (2002), pp. 2133–2163.
[14] C. Makridakis, On the babuska-osborn approach to finite element analysis: \( l^2 \) estimates for unstructured meshes, Preprint, (2016).
[15] C. G. Makridakis and I. Babuška, On the stability of the discontinuous Galerkin method for the heat equation, SIAM J. Numer. Anal., 34 (1997), pp. 389–401.
[16] J. Nitsche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, Abh. Math. Sem. Univ. Hamburg, 36 (1971), pp. 9–15. Collection of articles dedicated to Lothar Collatz on his sixtieth birthday.
[17] J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp., 28 (1974), pp. 937–958.
[18] T. Pryer, *Discontinuous Galerkin methods for the p-biharmonic equation from a discrete variational perspective*, Electron. Trans. Numer. Anal., 41 (2014), pp. 328–349.
[19] E. Suli and I. Mozolevski, *hp-version interior penalty DGFEMs for the biharmonic equation*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1851–1863.
[20] A. Veeser and P. Zanotti, *Quasi-optimality of nonconforming methods for linear variational problems*, In preparation, 2017.

Emmanuil H. Georgoulis, Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK and School of Applied Mathematical and Physical Sciences, National Technical University of Athens, Zografou 15780, Greece. Emmanuil.Georgoulis@le.ac.uk

Charalambos G. Makridakis, Institute for Applied and Computational Mathematics-FORTH, Heraklion-Crete, Greece, GR 70013 and Department of Mathematics, University of Sussex, Brighton BN1 9QH, UK.
C.Makridakis@sussex.ac.uk

Tristan Pryer, Department of Mathematics and Statistics, University of Reading, Whiteknights, PO Box 220, Reading RG6 6AX, UK. T.Pryer@reading.ac.uk