On the first law of holographic complexity

S. Sedigheh Hashemi, Ghadir Jafari, and Ali Naseh
School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM),
P. O. Box 19395-5531, Tehran, Iran
E-mail: hashemiphys@ipm.ir, ghjafari@ipm.ir, naseh@ipm.ir

ABSTRACT: In this paper, we examine the proposed first law of holographic complexity through studying different perturbations around various spacetime backgrounds. We present a general expression for the variation of the holographic complexity on arbitrary backgrounds by an explicit covariant computation. Interestingly, the general expression can be written as a function of gravitational conserved charges reminiscent of the first law of thermodynamics.

KEYWORDS: Holographic Complexity, First law, Black holes
1 Introduction

In recent years, ideas of quantum information theory have played an increasingly important role in shedding light on dark corners of quantum field theory (QFT) and quantum gravity. One fascinating concept from quantum information theory that has been recently discussed in the context of QFT is the quantum circuit complexity. Loosely speaking, the quantum circuit complexity is defined as the size of the optimal unitary transformation, $U_T$, which can prepare a target state $|\Psi_T\rangle$ from a reference state $|\Psi_R\rangle$ by using a set of elementary gates [1, 2].

In the context of AdS/CFT correspondence, studies have aimed at understanding the growth of the Einstein-Rosen bridge in the AdS background in terms of quantum complexity in the dual boundary CFT. There are two independent proposals for the gravitational observables, which will be dual to the complexity of a holographic boundary state [3–7]. One is the complexity=volume (CV) conjecture [3] and the other complexity=action (CA) conjecture [4, 5]. The first proposal, states that the complexity of an state in the boundary CFT is dual to the volume of the extremal surface meeting the asymptotic boundary on the
desired time slice. The second conjecture equates the holographic complexity of a boundary state with the gravitational action evaluated on the Wheeler-deWitt (WdW) patch,

\[ C_A(\Sigma) = \frac{I_{\text{WdW}}}{\pi}, \quad (1.1) \]

where the WdW patch is defined as the domain of dependence of any bulk Cauchy surface approaching asymptotically the time slice \( \Sigma \) on the boundary. It should be noted that the gravitational observables are sensitive to the bulk physics deep inside a black hole and exploring their properties is an active area of research [8–17]. An important limitation of this approach is that the concept of circuit complexity is not yet well-understood in the context of interacting QFTs. Therefore, developing the concept of circuit complexity for QFT states, in particular for states of a strongly coupled CFT [18, 19], would be an important task. Nielsen’s geometric approach [20, 21] gives a framework to describe the complexity of QFT states. Based on this approach, we easily find out that the variation circuit complexity only depends on the end point of the optimal trajectory [22], this feature designated by Bernamonti et. al [22], as the first law of complexity. The authors examined variation of holographic complexity for two nearby target states. Those target states are dual to smooth geometries in the bulk gravitational theory. They considered the variations of the holographic complexity \( \delta C_A \) under changing the target state by perturbing and AdS background by backreaction of a scalar field. The result was that the gravitational contributions to the variation canceled each other, and the final variation came from the scalar field action alone.

In the holographic calculations of [22], the background is taken to be the AdS space and the perturbations were restricted to preserve the spherical symmetry. In this paper, we repeat the calculations of [22] in the black hole background with (instead of pure AdS) and find the corrections to the first law of complexity. The interesting result is that the mass of the black hole appears in the expression for the first law. Then we will generalize our analysis to arbitrary perturbations and general backgrounds and find the most general expression for first law of holographic complexity.

This paper is organized as follows: In section 2, we follow a similar holographic analysis to that of the [22], with the difference that we consider perturbations around a charged AdS black hole. In section 3, we find a general expression for the first law of holographic complexity using the covariant approach for variation of the on-shell action evaluated in the WdW patch. Finally, section 4 includes the discussion and the results.

## 2 First law of holographic complexity for black holes

In [22], Bernamonti et. al. considered four dimensional Einstein-Hilbert gravity coupled to a free massless scalar field in order to check the first law of complexity. The action for this theory is given by

\[ I_{\text{bulk}} = \int d^4x \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi \right], \quad (2.1) \]
where its vacuum AdS$_4$ solution is
\[
d s^2 = \frac{L^2}{\cos^2 \rho} \left( -d\tau^2 + d\rho^2 + \sin^2 \rho \, d\Omega^2 \right). \tag{2.2}
\]
Here, $L$ is the AdS radius of curvature and the coordinate $\rho$ has the range $[0, \pi/2]$. They study the scalar perturbations on AdS background up to second order in perturbations and showed that the variation of the holographic complexity, $\delta C$, is given by
\[
\delta C \sim \int_{\partial \text{WDW}} ds \, d^2 \Omega \sqrt{\gamma} \left( \delta \Phi \partial_s \delta \Phi \right), \tag{2.3}
\]
where $s$ denotes the geodesic parameter of null boundaries of WDW patch. It is worth to note that the changes of complexity came entirely from the scalar field action and the gravitational contributions canceled each other. To further explore their results in the following we study the effect of perturbations on top of black hole backgrounds. In order to that let us consider Einstein-Hilbert-Maxwell gravity coupled to a massive scalar field in a four-dimensional bulk theory. The action is given by
\[
I_{\text{bulk}} = \int d^4 x \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right]. \tag{2.4}
\]
However, as mentioned in the above, we consider the background metric to be a charged AdS-Schwarzschild black hole with a solution of the form
\[
d s^2 = -f(r) \, dt^2 + \frac{1}{f(r)} \, dr^2 + r^2 d\Omega^2. \tag{2.5}
\]
The blackening factor for this solution is given by
\[
f(r) = 1 + \frac{r^2}{L^2} - \frac{2M}{r} + \frac{q^2}{r^2}. \tag{2.6}
\]
Now, we perturb the solution (2.5) by turning on the scalar field on top of that and let it to backreact. By considering the scalar perturbation as $\Phi = \epsilon \delta \Phi$, the equations of motions imply that in general for metric and gauge field perturbations we have respectively $g_{\alpha \beta} = g^{(0)}_{\alpha \beta} + \epsilon^2 \delta g_{\alpha \beta}$, and $A_\alpha = A^{(0)}_\alpha + \epsilon^2 \delta A_\alpha$. Moreover, for simplicity, we assume that the backreaction respects the spherical symmetry, therefore it implies that the metric should be
\[
d s^2 = -\left( f(r) + \epsilon^2 a(t, r) \right) \, dt^2 + \left( \frac{1}{f(r)} - \epsilon^2 \frac{d(t, r)}{f(r)^2} \right) \, dr^2 + r^2 d\Omega^2, \tag{2.7}
\]
where $a(t, r)$ and $d(t, r)$ are general functions of their arguments. Also, for our later convenience, we will use the following redefinition of perturbations
\[
\delta \mu_1 = -\frac{1}{f(r)^2} (a + d), \quad \delta \mu_3 = -\frac{1}{2f(r)} (a - d). \tag{2.8}
\]
In the next section, it becomes evident that $\delta \mu_1$ and $\delta \mu_3$ are perturbations of the metric in null directions.

In order to examine the first law of holographic complexity in the above model, let us find the variation of holographic complexity in the complexity=action (CA) conjecture. The variation of action is given by two classes of terms:

$$\delta C_A = \frac{1}{\pi} \left( \delta I_{\text{WdW}} + I_{\delta \text{WdW}} \right),$$

where $\delta I_{\text{WdW}}$ indicates the variation due to the change of the background fields within the original WdW, and $I_{\delta \text{WdW}}$ is the variation due to the change in the shape of the WdW patch [22]. On the other hand, for having a well defined variational principle, the holographic complexity evaluated by the CA conjecture, needs additional contributions from boundary terms [9], as the following

$$I = I_{\text{bulk}} + I_{\beta} + I_{\kappa} + I_{\text{ct}}$$

$$= \int d^4y \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right]$$

$$+ 2 \int_{\text{WdW}} d^2x \sqrt{\gamma} a + 2 \int_{\text{WdW}} ds d^2\Omega \sqrt{\kappa} + 2 \int_{\text{WdW}} ds d^2\Omega \sqrt{\gamma} \Theta \log(\ell_{\text{ct}} \Theta).$$

In the above expression, the null joint term, $I_{\beta}$, should be evaluated where the null boundaries intersect the regulator surface in the WdW patch. The induced metric on the joint is $\sqrt{\gamma}$, and $a = \epsilon \log |\ell| \ell$ [9]. The next is $I_{\kappa}$, depending on the scalar $\kappa$, which describes that how much the coordinate $s$, which parametrises the null boundary direction fails to be an affine parameter ($\ell^\mu \nabla_\mu \ell_\nu = \kappa \ell_\nu$). Finally, the last term is the counter term action in order to ensure that the action is invariant under the reparameterisations of the null boundary, where $\ell_{\text{ct}}$ is an arbitrary scale and $\Theta = \partial_s \log \sqrt{\gamma}$ is the expansion scalar, with $\gamma$ being the induced metric on the boundary. For our case, $I_{\text{bulk}}$, $I_{\kappa}$, $I_{\beta}$ and $I_{\text{ct}}$ will be relevant in calculating the variation (2.9). Let us now find the variation of each term separately. But, we first need the ingredients for that.

Before applying the perturbations, the null boundaries are defined by hypersurfaces which are determined by

$$\Psi = t \pm r^*(r) = \text{const},$$

where $r^*$ is the tortoise coordinate, $r^*(r) = \int \frac{dr}{f(r)}$, and the $+$ ($-$) sign denote the future (past) directed null geodesics. It can be easily checked that for the null boundaries we have

$$\nabla^\alpha \Psi \nabla_\alpha \Psi = 0. \tag{2.12}$$

However, in the perturbed geometry, the above condition is no longer satisfied. Let us assume that the equation of null hypersurface in the perturbed geometry is given by $\Psi' = \Psi + \delta \Psi = \text{const}$, where $\delta \Psi$ is a small arbitrary function,\(^1\) which is determined by the condition that $\Psi'$ is null in perturbed geometry. Using this condition, we find the following

\(^1\)Here we assume it to be just a function of $t$ and $r$.\hfill – 4 –
relation between $\delta \Psi$ and the metric perturbations

$$\partial_r \delta \Psi = \frac{1}{2} \delta \mu_1 + \frac{1}{f(r)} \partial_t \delta \Psi. \quad (2.13)$$

On the other hand, the null normal to the boundaries in the original and perturbed geometry are given by $\ell_\alpha = \nabla_\alpha \Psi$, and $\ell'_\alpha = \nabla_\alpha \Psi' = \ell_\alpha + \delta \ell_\alpha$.

As we mentioned earlier, the null vector satisfies the geodesics equation $\ell^\alpha \nabla_\alpha \ell_\beta = \kappa \ell_\beta$, where $\kappa$ measures the failure of $s$ to be an affine parameter on the null generators. In the above setup, both $\ell_\alpha$ and $\ell'_\alpha$ define affine geodesics and can be easily calculated to give $\kappa' = \kappa = \delta \kappa = 0$. The expansion scalar, $\Theta$, for the perturbed metric can be calculated and gives

$$\Theta = \nabla_\alpha \ell^\alpha = \frac{2}{r} (1 + \delta \mu_3 + \partial_t \delta \Psi), \quad (2.14)$$

from which $\delta \Theta = \frac{2}{r} (\delta \mu_3 + \partial_t \delta \Psi)$.

Now, given the above expressions, the contributions coming from the action (2.10) can be obtained. It follows from the above expressions that $\delta I_{\kappa}$ will vanish. The other contribution comes from the counterterm, given by

$$\delta I_{\text{ct}} = 2 \epsilon^2 \int_{\partial W} dS d^2 \Omega \sqrt{\gamma} \delta \Theta = 2 \epsilon^2 \int_{\partial W} ds \, d^2 \Omega \sqrt{\gamma} \frac{2}{r} (\delta \mu_3 + \partial_t \delta \Psi). \quad (2.15)$$

There is also a contribution from the joint terms. By using $\ell_\alpha$ and $\bar{\ell}_\alpha$ for normals to the intersecting null boundaries, we find

$$a = \log(|g^{\alpha \beta} \ell_\alpha \bar{\ell}_\beta|) = \log \frac{2}{f(r)} + \delta \mu_3 - \frac{f(r)}{2} (\bar{\ell}^\alpha \nabla_\alpha \delta \Psi + \ell^\alpha \nabla_\alpha \delta \bar{\Psi}). \quad (2.16)$$

As a result

$$\delta I_{\text{jt}} = 2 \epsilon^2 \int_{\beta} d^2 \Omega \sqrt{\gamma} \left[ \delta \mu_3 - \frac{f(r)}{2} (\bar{\ell}^\alpha \nabla_\alpha \delta \Psi + \ell^\alpha \nabla_\alpha \delta \bar{\Psi}) \right]. \quad (2.17)$$

Having the expressions for variations of the boundary and the joint terms, the next step is to find the variation of the bulk action. The contribution from gravitational part of the action is given by

$$\delta I_{\text{bulk, gravity}} = \int_{\partial W} ds \, d^2 \Omega \sqrt{\gamma} \ell_\beta \left( \nabla_\alpha \delta g^{\alpha \beta} - \nabla_\beta \delta g^\alpha \right)$$

$$= \epsilon^2 \int_{\partial W} ds \, d^2 \Omega \sqrt{\gamma} \left[ - \frac{2 \delta \mu_3}{r} + \delta \mu_1 f'(r) + f(r) \left( \frac{\delta \mu_1}{r} + \frac{1}{2} \partial_s \delta \mu_3 \right) + \partial_s \delta \mu_3 + \frac{1}{2} \partial_r \delta \mu_1 - \frac{1}{f(r)} \partial_t \delta \mu_3 \right]. \quad (2.18)$$

Moreover, the contribution from the matter fields is given by

$$\delta I_{\text{bulk, matter}} = \epsilon^2 \int_{E} d^{D-2}x \, ds \, \sqrt{\gamma} (\ell^\alpha \delta A^{\alpha \beta} F_{\alpha \beta} + \frac{1}{2} \delta \Phi \ell^\alpha \nabla_\alpha \Phi). \quad (2.19)$$
In addition, $I_{\delta WdW}$ can be determined by integrating the on-shell, zeroth order, Lagrangian density over the additional spacetime volume closed off by the perturbed WdW patch. The result leads to

$$I_{\text{bulk, } \delta WdW} = \int_{\delta WdW} d^4 y \sqrt{-g_0} \left( R(g_0) + \frac{6}{L^2} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right)$$

$$= -\epsilon^2 \int ds d^2 \Omega \sqrt{\gamma} \left( \frac{Q^2}{r^3} + \frac{r}{L^2} \right),$$

(2.20)

where $R(g_0) = -12/L^2$ for charged AdS black hole and the solution for $A_\alpha$ has been used.

Adding up all the above results, and after some integration by parts and using the expression (2.6) for $f(r)$, we will get the following result

$$\delta I_{\text{bulk}} + I_{\text{bulk}, \delta WdW} + \delta I_{\text{ct}} + \delta I_{\text{jt}} = \epsilon^2 \int_{\partial WdW} ds d^2 \Omega \sqrt{\gamma} \left( \frac{M}{r^2} \delta \mu_1 - \delta A_\alpha \ell_\beta F^{\alpha\beta} + \frac{1}{2} \delta \Phi \ell^\alpha \nabla_\alpha \delta \Phi \right).$$

(2.21)

When $M = 0$ and the Maxwell field is turned off, this result will reduce to the one obtained by [22]. It is worth noting that for charged black hole case the gravitational contributions dose not completely cancel each other in contrast to the AdS background studied in [22]. Actually, those contributions appear proportional to the energy of spacetime. In the next section, we will investigate the general form of the variation of holographic complexity for arbitrary perturbations around generic backgrounds. We will see that the results of the next section confirms the above results and can be generalized for perturbations which do not preserve the spherical symmetry.

### 3 General variations of holographic complexity

In the previous section, we have found the variation of the holographic complexity for a charged AdS black hole under perturbations sourced by an scalar filed which they preserve the spherical symmetry. In this section, we will consider arbitrary background $g_{\alpha\beta}$, which is perturbed by a general perturbation $\delta g_{\alpha\beta}$. The basic point for the variation of action in the WdW-patch is that the previous null boundaries of WdW patch will no longer remain null under a generic metric perturbation. It can be easily seen as follows: before acting the perturbation, let us first consider the boundary, which is determined by a scalar field $\phi = \text{conts}$. The fact that the boundary is null means that

$$\nabla_\alpha \phi \nabla^\alpha \phi = 0.$$  

(3.1)

One can see that under $g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = g_{\alpha\beta} + \delta g_{\alpha\beta}$, the left hand side changes according to

$$\nabla_\alpha \phi \nabla^\alpha \phi \rightarrow \nabla_\alpha \phi \nabla^\alpha \phi - \delta g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi,$$

(3.2)

The other distinction between our work and [22] is that we considered the contributions of the joint terms and these terms automatically cancel the surface integral when doing integration by parts. In contrast, to remove those surface integrals, the authors of [22] used the known asymptotic fall-off for the perturbations.
therefore, \( \phi = \text{conts} \) is no longer a null hypersurface in the deformed metric. The remedy to this problem is that the WdW patch of the deformed metric is different from that of the original one. Suppose that the null boundary of the deformed geometry is specified with the condition \( \phi' = \phi + \delta \phi = \text{const} \). Deforming \( \phi \) alongside \( g_{\alpha \beta} \) in (3.1) we get

\[
\nabla_\alpha \phi \nabla_\alpha \phi' = \nabla_\alpha \phi \nabla_\alpha \phi - \delta g^{\alpha \beta} \nabla_\alpha \phi \nabla_\beta \phi' + 2 \nabla_\alpha \phi \nabla_\alpha \delta \phi.
\]

(3.3)

Thus, by using the nullness condition in perturbed geometry \( g'^{\alpha \beta} \nabla_\alpha \phi' \nabla_\beta \phi' = 0 \), we find

\[
\nabla_\alpha \phi \nabla_\alpha \delta \phi = \frac{1}{2} \delta g^{\alpha \beta} \nabla_\alpha \phi \nabla_\beta \phi.
\]

(3.4)

There will be two contributions from the variation of the action in this region which are the results of the above displacements in the boundaries of the WdW patch. The first one comes from the variations of metric and the second arises from the variation of the boundaries.

In the following, we first introduce the setup for dealing with null hyper-surfaces in order to investigate the variation of on-shell action under a general perturbation. Since the most important contribution comes from the variation of the Einstein-Hilbert term, we will start with finding a general expression for the variation of this term and then arrive at a general rule for the variation of the holographic complexity.

3.1 The set up

A hypersurface \( \mathcal{N} \), characterized by \( \phi_0 = 0 \), is called a null hypersurface if and only if \( \nabla_\alpha \phi_0 \nabla^\alpha \phi_0 = 0 \). This feature of null boundary indicates that the normal vector to the null surface is also tangent to it. This property is the origin of some difficulties when dealing with such hypersurfaces because as a consequence the induced metric becomes degenerate. As a result, constructing a projection to the null surface just from its normal is not possible. One standard remedy to this problem is to introduce an auxiliary null vector \( k^\alpha \), which lays out of the hypersurface and therefore \( \ell^\alpha k^\alpha \neq 0 \) when \( \ell^\alpha \) is the null normal to the boundary e.g. \( \ell^\alpha \propto \nabla_\alpha \phi_0 \) on the boundary.\footnote{For more details about the geometry of null hypersurfaces we refer the interested reader to [23].}

By defining \( \ell_\alpha \) as the normal to the null boundary, we introduce the auxiliary null form \( k_\alpha \) and take the normalization of the null forms to be everywhere as

\[
\ell_\alpha k^\alpha = 0 , \quad k_\alpha k^\alpha = 0 \quad \text{and} \quad \ell_\alpha k^\alpha = -1 .
\]

(3.5)

With the help of \( \ell_\alpha \) and \( k_\alpha \), we can define the projection given by

\[
q^{\alpha \beta} = \delta^{\alpha \beta} + \ell^\alpha k_\beta + k^\alpha \ell_\beta .
\]

(3.6)

This projection is not in fact a projection on the null surface. Instead, it essentially projects spacetime vectors onto the co-dimension two surface \( \mathcal{S} \), to which \( \ell_\alpha \) and \( k_\alpha \) are orthogonal.

By using the covariant differentiation of vectors \( \ell_\alpha \) and \( k_\alpha \), projecting them in different directions, and using \( q^{\alpha \beta} \), \( \ell^\alpha \) and \( k^\alpha \), we can define the following geometric objects from
Thus, the components of metric perturbations have been decomposed into a tensor

$$
\nabla_\alpha \ell_\beta \quad \text{and} \quad \nabla_\alpha \ell_\beta
$$

$$
\begin{align*}
\nabla_\alpha \ell_\beta &= -\Theta_{\alpha\beta} - \omega_\alpha \ell_\beta - \ell_\alpha \eta_\beta - k_\alpha a_\beta + \kappa k_\alpha \ell_\beta - \bar{\kappa} \ell_\alpha \ell_\beta, \\
\nabla_\alpha k_\beta &= -\Xi_{\alpha\beta} + \omega_\alpha k_\beta - k_\alpha \bar{\eta}_\beta - \ell_\alpha \bar{a}_\beta - \kappa k_\alpha k_\beta + \bar{\kappa} \ell_\alpha k_\beta.
\end{align*}
$$

(3.7) (3.8)

These relations are generalizations of the relation $\nabla_\alpha n_\beta = -K_{\alpha\beta} + n_\alpha a_\beta$ to the case in hand, where decomposition has been done with two null vectors. The definitions are as follows

$$
\begin{align*}
\Theta_{\alpha\beta} &= -q^\sigma \alpha q^\delta \beta \nabla_\sigma \ell_\delta, & \Xi_{\alpha\beta} &= -q^\sigma \alpha q^\delta \beta \nabla_\sigma k_\delta, \\
\eta_\alpha &= q^\sigma \alpha k^\beta \nabla_\beta \ell_\sigma, & \bar{\eta}_\beta &= q^\sigma \alpha \ell^\beta \nabla_\beta k_\sigma, \\
\omega_\alpha &= q^\sigma \alpha k^\beta \nabla_\beta \ell_\sigma = -q^\sigma \alpha \ell^\beta \nabla_\sigma k_\beta, & a_\alpha &= q^\sigma \alpha \ell^\beta \nabla_\beta k_\sigma, \\
\kappa &= \ell^\alpha k^\beta \nabla_\alpha \ell_\beta = -\ell^\alpha \ell^\beta \nabla_\alpha k_\beta, & \bar{\kappa} &= k^\alpha \ell^\beta \nabla_\alpha k_\beta = -k^\alpha k^\beta \nabla_\alpha \ell_\beta,
\end{align*}
$$

(3.9)

where $\Theta_{\alpha\beta}$ and $\Xi_{\alpha\beta}$ are extrinsic curvatures of $S$, while $\omega_\alpha$, $\eta_\alpha$ and $\bar{\eta}_\alpha$ are twists. In addition, $a_\alpha$ and $\bar{a}_\alpha$ are tangent accelerations of $\ell^\alpha$ and $k^\alpha$ to $S$, respectively. Moreover, $\kappa$ and $\bar{\kappa}$ are in-affinity parameters.

### 3.2 Variations and their decompositions

If consider general variations in metric given by $\delta g_{\mu\nu}$, we can decompose the variation of metric into $q^\alpha_{\beta}$, $\ell^\alpha$ and $k^\alpha$ directions by

$$
\begin{align*}
\delta q_{\alpha\beta} &= q^\sigma \alpha q^\delta \beta \delta g_{\sigma\delta}, \\
\delta u_{1\alpha} &= -q^\beta \alpha \ell^\sigma \delta g_{\beta\sigma}, & \delta u_{2\alpha} &= -q^\beta \alpha k^\sigma \delta g_{\beta\sigma}, \\
\delta \mu_1 &= \ell^\alpha \ell^\beta \delta q_{\alpha\beta}, & \delta \mu_2 &= k^\alpha k^\beta \delta g_{\alpha\beta}, & \delta \mu_3 &= \ell^\alpha k^\beta \delta g_{\alpha\beta}.
\end{align*}
$$

(3.10)

Thus, the components of metric perturbations have been decomposed into a tensor $\delta q_{\alpha\beta}$, two vectors $(\delta u_{1\alpha}, \delta u_{2\alpha})$, and three scalars $(\delta \mu_1, \delta \mu_2, \delta \mu_3)$. As a result the variation of metric can be expressed as

$$
\delta g_{\alpha\beta} = \delta q_{\alpha\beta} + 2k_\alpha \delta u_{1\beta} + 2\ell_{(\alpha} \delta u_{2\beta)} + k_\alpha k_\beta \delta \mu_1 + \ell_\alpha \ell_\beta \delta \mu_2 + 2\ell_{(\alpha} k_\beta) \delta \mu_3.
$$

(3.11)

Here, following the notation of [9], we have introduced $\delta$ to show that these quantities are not necessarily variation of some function (i.e. $\delta q_{\alpha\beta} \neq \delta (q_{\alpha\beta})$). We also impose that variation keeps the normalization conditions of the frame forms $\ell_\alpha$ and $k_\alpha$ unchanged. Therefore, from the normalization conditions (3.5) and (3.11), we get

$$
\begin{align*}
0 &= \delta (\ell_\alpha \ell^\alpha) = 2\ell^\alpha \delta \ell_\alpha - \ell^\alpha \ell^\beta \delta q_{\alpha\beta} = 2\ell^\alpha \delta \ell_\alpha - \delta \mu_1, \\
0 &= \delta (k_\alpha k^\alpha) = 2k^\alpha \delta k_\alpha - k^\alpha k^\beta \delta g_{\alpha\beta} = 2k^\alpha \delta k_\alpha - \delta \mu_2, \\
0 &= \delta (\ell_\alpha k^\alpha) = \ell^\alpha \delta k_\alpha + k^\alpha \delta \ell_\alpha - \ell^\alpha k^\beta \delta g_{\alpha\beta} = \ell^\alpha \delta k_\alpha + k^\alpha \delta \ell_\alpha - \delta \mu_3.
\end{align*}
$$

(3.12)
By assuming that $\ell_\alpha$ and $k_\alpha$ remain orthogonal to $S$, we can solve the above three equations for $\delta \ell_\alpha$ and $\delta k_\alpha$ to get

$$\delta \ell_\alpha = \delta \beta \ell_\alpha - \frac{1}{2} \delta \mu_1 k_\alpha, \quad (3.13)$$

$$\delta k_\alpha = - (\delta \beta + \delta \mu_3) k_\alpha - \frac{1}{2} \delta \mu_2 \ell_\alpha. \quad (3.14)$$

These equations relate variations of vectors $\ell_\alpha$ and $k_\alpha$ to variations of metric components. Here, $\bar{\delta} \beta$ is an arbitrary function which is related to the rescaling of gauge freedom in definitions of $\ell_\alpha$ and $k_\alpha$. Now, suppose that $\ell_\alpha = \nabla_\alpha \phi$ on the null hypersurface; we have $\delta \ell_\alpha = \nabla_\alpha \delta \phi$. So, by using (3.13), we get

$$\bar{\delta} \mu_1 = 2 \ell^\alpha \nabla_\alpha \delta \phi,$$
$$\bar{\delta} \beta = -k^\alpha \nabla_\alpha \delta \phi. \quad (3.15)$$

Therefore, just the projection of $\nabla_\alpha \delta \phi$ along $\ell_\alpha$ is fixed by metric perturbations and projection along $k_\alpha$, i.e. $k^\alpha \nabla_\alpha \delta \phi$, will remain arbitrary. In fact it is easy to see that these deformations are just related to arbitrariness in definition of null hypersurface. These deformations correspond to $\delta \phi = \delta \phi(\phi)$, consequently we have $\nabla_\alpha \delta \phi(\phi) = \delta \phi'(\phi) \nabla_\alpha \phi = \delta \phi'(\phi) \ell_\alpha$. If we consider the parameterization of null generators in both initial spacetime and after perturbations to be affine, then $\delta \beta = 0$

### 3.3 The surface term on null boundary

By varying the Hilbert-Einstein term, we can calculate the surface term to be

$$\delta S_{EH} = \int_B \sqrt{q} (\ell^\alpha \nabla_\beta \delta g_{\alpha \beta} - \ell^\alpha \nabla_\alpha \delta g^\beta_\beta), \quad (3.16)$$

when we called each null segments in the boundary by $B$. Using the expression (3.11) for $\delta g_{\alpha \beta}$ we have

$$\ell^\alpha \nabla_\beta \delta g_{\alpha \beta} - \ell^\alpha \nabla_\alpha \delta g^\beta_\beta =$$
$$- \ell^\alpha \nabla_\alpha \delta q^\beta_\beta - \nabla_\alpha \delta u^\alpha_1 - k^\alpha \nabla_\alpha \delta \mu_1 + \ell^\alpha \nabla_\alpha \delta \mu_3$$
$$+ (-\delta q_{\alpha \beta} - 2 k(\alpha) \delta u_{1 \beta}) - \ell_\alpha \delta u_{2 \beta} - k_\alpha k_\beta \delta \mu_1 - (\ell_\alpha k_\beta + g_{\alpha \beta}) \delta \mu_3) \nabla^\alpha \ell^\beta$$
$$- \delta \mu_1 \nabla_\alpha k^\alpha. \quad (3.17)$$

In obtaining the above result we have used the relations

$$\ell^\alpha \nabla_\alpha (\delta u_{1 \beta}) = - \delta u^\alpha_1 \nabla_\alpha \ell_\beta,$$
$$\ell^\alpha \nabla_\alpha \delta q_{\beta \mu} = - \delta q_{\beta \mu} \nabla_\alpha \ell^\alpha,$$
$$\ell^\alpha \nabla_\alpha k_\beta = - k^\beta \nabla_\alpha \ell_\beta, \quad (3.18)$$

and similar relations for $\delta u_{2 \beta}$. These are direct consequence of normalization conditions (3.5) and definitions (3.10). Now using the expressions (3.7) and (3.8) for $\nabla_\alpha \ell_\beta$ and $\nabla_\alpha k_\beta$
we get
\[ \ell^\alpha \nabla^\beta g_{\alpha \beta} - \ell^\alpha \nabla_\alpha \sigma^\beta_{\beta} = -a^\alpha \delta u_2 + 2k \delta_{\mu_1} + \delta u_1 (2 \eta_\alpha + \eta_\alpha + \omega_\alpha - \delta q_{\alpha \beta} \Theta_{\alpha \beta} - \delta \mu_3 \Theta_{\alpha \alpha} - \delta \mu_1 \Xi_{\alpha \alpha} - \ell^\alpha \nabla_\alpha \delta q_{\beta} \]
\[ - D_\alpha \delta u_1 - k^\alpha \nabla_\alpha \delta_{\mu_1} + \ell^\alpha \nabla_\alpha \delta_{\mu_3}, \]  
(3.19)

where \( D_\alpha \) is the covariant derivative on the co-dimension two surface \( S \) orthogonal to \( \ell_\alpha \) and \( k_\alpha \), and in the above relation it is defined by \( D_\alpha \delta u_1 = q_{\alpha \beta} \nabla_\alpha \delta u_{1 \beta} \). As it is shown in appendix A, by going through similar procedure, we can find the following relation for \( \delta \Theta \)

\[ \delta \Theta = \delta \mu_3 \Theta + \delta \beta \Theta + \frac{1}{2} \delta_{\mu_1} \Xi + \frac{1}{2} \ell^\alpha \nabla_\alpha \delta q_{\beta} + D_\alpha \delta u_1 - \delta u_1 (\omega_\alpha + \eta_\alpha) + a^\alpha \delta u_2. \]  
(3.20)

Solving the above equation for \( \ell^\alpha \nabla_\alpha \delta q_{\beta} \), and substituting the result in (3.19), we find

\[ \ell^\alpha \nabla^\beta g_{\alpha \beta} - \ell^\alpha \nabla_\alpha \sigma^\beta_{\beta} = -2 \delta \Theta - \delta q_{\alpha \beta} \Theta_{\alpha \beta} \]
\[ - \delta u_1 (2 \eta_\alpha - \eta_\alpha + \omega_\alpha) + 2k \delta_{\mu_1} + \Theta (\delta \mu_3 + 2 \delta \beta) - k^\alpha \nabla_\alpha \delta_{\mu_1} + \ell^\alpha \nabla_\alpha \delta_{\mu_3} + D_\alpha \delta u_1. \]  
(3.21)

This expression has to be integrated on the null boundary. The last term is a total derivative; because we suppose that the surface orthogonal to the null directions, \( S \), is compact, the last term will vanish. To manipulate other terms we use the following integration by parts

\[ \int_B d^d x \nabla_\alpha \phi = - \int_B d^{d-1} x \sqrt{q} \Theta \phi + \int_{\partial B} d^{d-2} x \sqrt{q} \phi, \]  
(3.22)

for every scalar \( \phi \). On the other hand, by using relations (3.15) we can deduce the following identity

\[ k^\alpha \nabla_\alpha \delta_{\mu_1} = \bar{k} \delta_{\mu_1} - 2 \ell^\alpha \nabla_\alpha \delta_{\beta}. \]  
(3.23)

Moreover, if \( \ell = \nabla_\alpha \phi \), and by using the symmetric property of \( \nabla_\alpha \ell_\beta \), one can see that \( \omega_\alpha = \eta_\alpha \). Using this fact and the relations (3.22), (3.23) the final expression for the surface term leads to

\[ \int_B d^d x ds \sqrt{q} (\ell^\alpha \nabla^\beta g_{\alpha \beta} - \ell^\alpha \nabla_\alpha \sigma^\beta_{\beta}) = - \int_B d^{d-2} x ds \left[ 2 \delta (\sqrt{q} \Theta) + (\Theta_{\alpha \beta} - \Theta q_{\alpha \beta}) \delta q_{\alpha \beta} \right] \]
\[ - 2 \eta_\alpha \delta u_1 + \bar{k} \delta_{\mu_1} + \int_{\partial B} d^{d-2} x \sqrt{q} (\delta_{\mu_3} + 2 \delta \beta). \]  
(3.24)

The last term is lying on the boundary of a null surface, or at the joint of intersecting null surfaces. Consider another another null surface defined by \( \phi' = \text{const} \), with normal \( \ell'_\alpha = \nabla_\alpha \phi' \); at the joint \( C = \partial B \), we have \( \ell'_\alpha \ell'^\alpha = e^P \) for some scalar \( P \). As a result

\[ \ell'_\alpha = -e^P k_\alpha, \quad \text{and} \quad k'_\alpha = -e^{-P} \ell_\alpha. \]  
(3.25)
Corresponding to the relations (3.15), we also find
\[ \delta \mu_2 = 2e^{-P} k^\alpha \nabla_\alpha \delta \phi', \]
\[ \delta \beta' = e^{-P} \ell^\alpha \nabla_\alpha \delta \phi'. \]  
(3.26)

Now, by varying both sides of \( \ell_\alpha \ell_\alpha = e^P \), and using the above relations, one can find
\[ \delta P = \delta \mu_3 + \delta \beta + \delta \beta'. \]  
(3.27)

Summing the joint contribution of two neighboring boundaries, we get\(^4\):
\[ (\delta \mu_3 + 2\delta \beta) + (\delta \mu_3 + 2\delta \beta') = 2\delta P. \]  
(3.28)

Having this fact, the expression (3.24) recast into new form
\[
\int_B d^{d-2}x ds \sqrt{q} (\ell^\alpha \nabla_\beta \delta g_{\alpha \beta} - \ell^\alpha \nabla_\alpha \delta g_{\beta \beta}) = -2\delta \int_B d^{d-2}x ds \sqrt{q} (\Theta) + 2\delta (\int_{\partial B} d^{d-2}x \sqrt{q} P)
\]
\[
- \int_B d^{d-2}x ds \sqrt{q} [(\Theta_{\alpha \beta} - \Theta q_{\alpha \beta}) \delta q^{\alpha \beta} + 2\eta^\alpha \delta u_{1 \alpha} - \kappa \delta \mu_1] - \int_{\partial B} d^{d-2}x \sqrt{q} P q_{\alpha \beta} \delta q^{\alpha \beta},
\]  
(3.29)

with \( P = \log(\ell \cdot \ell') \). The terms in the first line are total variation terms and they can be canceled by variation boundary terms in (2.10). Moreover, the terms in the second line have canonical forms. When dealing with variational principle, these terms will vanish by choosing Dirichlet boundary conditions on the null hypersurface. Note that \( \delta q_{\alpha \beta}, \delta u_{1 \alpha}, \) and \( \delta \mu_1 \) are variations of metric in the direction of null boundary\(^5\). When one is just interested in obtaining well-defined action for variational principle (as in [9]), during the above calculation these terms could be omitted, whenever they arise. However, in situations like this, where we are interested in the difference of on-shell action between two configurations, these terms are of critical importance.

Let us note an important point here that was first observed in [25]; that is, when dealing with the variation of action there is no ambiguity corresponding to defining null geodesics, and that the variation of action is invariant with respect to this gauge freedom. The result of this fact is that when dealing with variation of action in WdW patch or the first law of holographic complexity we don’t have to use the counter term \( \Theta \log(l_4 \Theta) \), and we can use the \( \int \sqrt{q} (\Theta + \kappa) \) for the boundary action as originally proposed in [26].

### 3.4 Change from displacement of the boundary

As we have seen earlier, the WdW patch can be displaced by adding a general metric perturbation, so when calculating holographic complexity we must also consider the contribution on this displacement. Let us suppose that bulk action can be integrated in some normalized direction \( n^\alpha \). This is the case for example in Einstein-Hilbert theory in spherical

\(^4\)Note that \( \delta \mu_3 = \delta g_{\alpha \beta} \ell^\alpha k^\beta = \delta g_{\alpha \beta} \ell^\alpha k^\beta = \delta \mu_3 \).

\(^5\)This point is well illustrated in a double foliation formalism analysis of [24].
configurations. Therefore, the on-shell bulk action can be evaluated as
\[ \int d^d x \sqrt{g} \nabla_\alpha (n^\alpha J), \quad (3.30) \]
for some function \( J \). Integrating this term by using Gauss theorem, and when the boundary is specified by \( \phi = \text{const} \), yields
\[ \int_{\partial M} d^{d-1} x \sqrt{g} n^\alpha \nabla_\alpha \phi \ J. \quad (3.31) \]
Now, by displacing the boundary infinitesimally, the change in the action will be
\[ I_{\delta M} = \int_{\partial M} d^{d-1} x \sqrt{g} n^\alpha \nabla_\alpha \delta \phi \ J. \quad (3.32) \]
Note that in the above analysis \( \phi \) is not restricted. So the boundary may be \( \phi = \text{const} \) or \( \phi + \delta \phi = \text{const} \), consequently, the difference between two configurations is given by the above formula. As an example, consider integration of Einstein-Hilbert plus cosmological constant term in spherical background (2.5)
\[ \int d^4 x \sqrt{-g} \left( R - \frac{6}{L^2} \right) = \int r^2 \sin \theta \, dr \, dt \, d\Omega \left( \frac{6}{L^2} + \frac{2f(r)}{r^2} - \frac{4f'(r)}{r} - f''(r) \right). \quad (3.33) \]
In this background the integration in \( r \) can be performed and we can write the integral in the form of (3.30) by
\[ J = \frac{2}{r \sqrt{f(r)}} \left( 1 + \frac{r^2}{L^2} - f(r) - \frac{rf'(r)}{2} \right), \quad (3.34) \]
where here \( n^\alpha \) is normal vector to \( r = \text{const} \) surfaces. The form (3.30) helps us to evaluate integral on surfaces other than \( r = \text{const} \), such as the null one as in (3.31), and therefore to obtain contributions from bulk integral, when changing the boundary as in (3.32). This procedure is the same as doing integration in \( r \) and then using Jacobian matrix to express integral on null surface as in [22] and previous section.

### 3.5 General variation of holographic complexity

In order to study variation of holographic complexity, we consider Einstein-Hilbert-Maxwell-Scalar theory on the D-dimensional spacetime as
\[ I_{\text{bulk}} = \int d^D x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \nabla_\mu \Phi \nabla_\mu \Phi - \frac{1}{2} m^2 \Phi^2 \right]. \quad (3.35) \]
The gravitational part of the bulk action must be complemented with a boundary action in order to get a well defined variational principle. The boundary action for a spacelike or timelike boundary is the well-known Gibbons-Hawking term. For a null case, the boundary action was first proposed in [26] as integral of \( 2 \sqrt{g} (\Theta + \kappa) \). But a well-defined variational
principle also requires some terms on the joint of spacetime segments [27]. The required joint action in the case of null boundaries was found in [9]. For the joint of two null surfaces this term is $2 \log(\ell \cdot \ell')$ integrated on the joint. Therefore, we add the following terms to the action (3.35)

$$I_{\text{Boundary}} = 2 \int_B d^{D-2}x ds \sqrt{q}(\Theta + \kappa) + 2 \int_{\partial B} d^{D-2}x \sqrt{q} \log(\ell \cdot \ell'). \quad (3.36)$$

Now, we consider two solutions of (3.35), and consider the following perturbations

$$g'_\mu\nu = g_{\mu\nu} + \epsilon^2 \delta g_{\mu\nu}, \quad A'_\mu = A_{\mu} + \epsilon^2 \delta A_{\mu}, \quad \Phi' = \Phi_0 + \epsilon \delta \Phi. \quad (3.37)$$

Here we suppose that $\Phi_0 = 0$, and $\epsilon \delta \Phi$ as a source of perturbations. Consequently, equations of motion require perturbations in the metric and the gauge field to be $O(\epsilon^2)$. By varying the action (3.35) under these perturbations and using equations of motion, we get

$$\delta I_{\text{bulk}} = \epsilon^2 \int_B d^{D-2}x ds \sqrt{q} \left( \ell^\alpha \nabla^\beta \delta g_{\alpha\beta} - \ell^\alpha \nabla_\alpha \delta q^{\beta\beta} + \ell^\alpha \delta A^\beta F_{\alpha\beta} + \frac{1}{2} \delta \Phi \ell^\alpha \nabla_\alpha \delta \Phi \right). \quad (3.38)$$

Now, by using equation (3.29), we find the following expression for variation of on-shell action under perturbations

$$\delta I_{\text{WdW}} = -\epsilon^2 \int_B d^{d-2}x ds \sqrt{q} \left[ (\Theta_{\alpha\beta} - \Theta q_{\alpha\beta}) \delta q^{\alpha\beta} + 2q^\alpha \delta u_{1\alpha} - \bar{\kappa} \delta \mu_1 \right] - \epsilon^2 \int_{\partial B} d^{d-2}x \sqrt{q} \bar{P} q_{\alpha\beta} \delta q^{\alpha\beta} + \epsilon^2 \int_B d^{d-2}x ds \sqrt{q} (\ell^\alpha \delta A^\beta F_{\alpha\beta} + \frac{1}{2} \delta \Phi \ell^\alpha \nabla_\alpha \delta \Phi). \quad (3.39)$$

The first line is the contribution from variations of metric, and the second line comes from variations of other fields. Note that in this setup in both configurations $\ell_\alpha = \nabla_\alpha \phi$, $\ell'_\alpha = \nabla_\alpha \phi'$, are affine and so $\kappa = \kappa' = \delta \kappa = 0$.

As we have seen, there is also a contribution from changing the boundaries, which according to (3.32) is given by

$$I_{\delta \text{WdW}} = \int_{\partial \text{WdW}} d^{D-2}x ds \sqrt{q} \, n^\alpha \nabla_\alpha \delta \phi \, J, \quad (3.40)$$

where $n^\alpha$ can be decomposed as $n^\alpha = b \ell^\alpha + \frac{1}{2b} k^\alpha$, such that $n^\alpha$ will normalized to $\pm 1$, and $b$ is a function that depends on the metric. In the case of the black hole metric (2.5) $b = \sqrt{2}$. As a result, (3.40) becomes

$$I_{\delta \text{WdW}} = \int_{\partial \text{WdW}} d^{D-2}x ds \sqrt{q} \, b \, \delta \mu_1 J. \quad (3.41)$$

In this form, it has the same structure as the terms in (3.39). So, accordingly, the variations

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*Here we suppose the null generators in both initial and final spacetimes are parameterized by affine parameters, so $\delta \beta = 0$*
of holographic complexity under arbitrary perturbation is given by

\[
\delta C = \frac{1}{\pi}(\delta I_{WdW} + I_{\delta WdW})
\]

\[
= -\frac{\epsilon^2}{\pi} \int_B d^{d-2}x \sqrt{q}[(\Theta_{\alpha\beta} - \Theta q_{\alpha\beta})\delta q^{\alpha\beta} + 2\eta^{\alpha} \delta u_{1\alpha} - (\bar{\kappa} - bJ)\delta \mu_{1}]
\]

\[
+ \frac{\epsilon^2}{\pi} \int_B d^{d-2}x \sqrt{q}(\ell^{\alpha} \delta A_{\alpha\beta} + \frac{1}{2} \delta \Phi \ell^{\alpha} \nabla_{\alpha} \delta \Phi) - \frac{\epsilon^2}{\pi} \int_{\partial B} d^{d-2}x \sqrt{q} P q_{\alpha\beta} \delta q^{\alpha\beta}.
\] (3.42)

In the following we will calculate the resulting expressions for some backgrounds and perturbations. Note that in this section we have assumed that the gravitational perturbations result from a fluctuation in a scalar field, but the essence of calculation is general and can be extended to any type of perturbations. For example, we can consider gravitational perturbation independently, without considering it as a result of backreaction of other fields. For example we can obtain variations in complexity for two metrics \(g_{\mu\nu}\) and \(g_{\mu\nu} + \epsilon \delta g_{\mu\nu}\), both as solutions of Einstein equation. In that case, the result will be the same as (3.42) without the matter perturbations in the left hand side and the \(\epsilon\) instead of \(\epsilon^2\).

### 3.6 Examples

#### 3.6.1 AdS with non spherical perturbations

As stated previously, the authors of [22] studied perturbations around AdS space and supposed these perturbations to preserve spherical symmetry. Here we will give up this assumption and study variation of holographic complexity under these perturbations. As a simple example of these we consider

\[
\delta (ds^2) = \epsilon^2 s(t, r) d^2 r + \epsilon^2 p(t, r) \sin^2 \theta d\theta^2,
\] (3.43)

where \(s\) and \(p\) are two perturbation functions. Substituting these perturbations in (3.42), for the AdS background we get

\[
\delta C = -\epsilon^2 \int d^3 x \sqrt{\gamma} \left[ \frac{s(t, r) + p(t, r)}{r^3} + \frac{1}{2} \ell^{\alpha} \delta \Phi \nabla_{\alpha} \delta \Phi \right].
\] (3.44)

This result can be obtained using the methods of the previous section. As a result, gravitational perturbations appear in the variation of holographic complexity even in AdS background when we don’t restrict ourselves to spherical preserving perturbations.

#### 3.6.2 Charged AdS black hole

For the second example, we re-drive the results of previous section using the general formula that we have obtained. The background is given by the metric (2.5), and the perturbations are given by

\[
\delta (ds^2) = \epsilon^2 f(r) \left( \delta \mu_3 + \frac{1}{2} \delta \mu_1 f(r) \right) dt^2 - \epsilon^2 \frac{1}{f(r)} \left( \delta \mu_3 - \frac{1}{2} \delta \mu_1 f(r) \right) dr^2.
\] (3.45)
Here, it is supposed that perturbations preserve spherical symmetry, and as a result \( \delta q_{\alpha\beta} = 0 \) and \( \delta u_{1\alpha} = 0 \).

The one form \( \ell_\alpha \) and vector \( k^\alpha \) are give by

\[
\ell_\alpha dx^\alpha = (1 + \partial_t \delta t) dt + \left( f'(r) + \partial_r \delta t \right) dr,
\]

\[
k^{\alpha} \partial_\alpha = -\frac{1}{2} (1 - \partial_t \delta t) \partial_t - \frac{f'(r)}{2} (1 - \partial_t \delta t + \frac{1}{2} f(r) \delta \mu_1) \partial_r.
\] (3.46)

Using definition \( \bar{\kappa} \) in (3.9) we can easily find \( \bar{\kappa} = \frac{1}{2} f'(r) \), and also from (3.34) for \( J \) and substituting in expression (3.42) we exactly find the result of the previous section, e.g., equation (2.21).

### 3.6.3 Slow rotating AdS black hole

In the general law of holographic complexity, (3.42), we have seen that the coefficient of the perturbation \( \delta \mu_1 \) is related to the energy of spacetime. A subsequent question is the physical meaning of other perturbation components. In this subsection, we will show that the coefficient of \( \delta u_{1\alpha} \) is directly related to the angular momentum of spacetime. To examine this statement, we consider the AdS black hole with angular momentum (known as Kerr-AdS black hole) as the background. For the sake of simplicity in our calculations, we suppose the spin of the black hole to be small and work to linear order in angular parameter \( a \). So we consider the background metric as

\[
d s^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + 2a (f(r) - 1) \sin^2 \theta d\phi + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).
\] (3.47)

Also, we assume one component of the perturbations as

\[
\delta(d s^2) = \epsilon^2 \frac{a c(t, r)}{r^2} (f(r) - 1) dt^2 + \frac{1}{2} \epsilon^2 c(t, r) dtd\phi.
\] (3.48)

For these perturbations, we have \( \delta q_{\alpha\beta} = \delta \mu_1 = 0 \), and \( \delta u_{1\alpha} dx^\alpha = -\frac{c(t, r)}{2f(r)} d\phi \). By Using \( \ell_\alpha \) and \( k^\alpha \) for background spacetime, with the following expressions

\[
\ell_\alpha dx^\alpha = dt + \frac{1}{f(r)} dr,
\]

\[
k^{\alpha} \partial_\alpha = -\frac{1}{2} \partial_t - \frac{f'(r)}{f(r)} \partial_r,
\] (3.49)

and the definition of \( \bar{\eta}_\alpha \) in (3.9), we find

\[
\bar{\eta}_\alpha dx^\alpha = \frac{3aM \sin^2 \theta}{r^2}
\]

\[
\delta C = -\epsilon^2 \int d^3 x \sqrt{\gamma} \left[ \frac{3aM c(t, r)}{r^4 f(r)} + \ell^\alpha \delta \Phi \nabla_\alpha \delta \Phi + k^\alpha \delta A_\alpha F^\alpha \beta \right],
\] (3.50)

where the result depends on the angular momentum of the Kerr-AdS black hole \( J = aM \).

This result can be obtained using the method of section 2. We have checked the result with that method and found precise agreement. The fact that \( \bar{\eta}_\alpha \) is related to the angular momentum of spacetime also has been show explicitly in [28] for asymptotically flat spacetimes.
4 Discussion and results

In this paper, we have generalized the first law of holographic complexity, proposed in [22], for arbitrary perturbations and general backgrounds. We observe that mass and angular momentum are the responses for perturbations in the direction of null geodesics, whereas shears tensor $\Theta_{\alpha\beta} - \Theta q_{\alpha\beta}$ is a response to perturbations, which destroy spherical symmetry. To be more concrete, the perturbations, which appear in (3.42) are all tangent to the null hypersurface, following the logic of Brown and York [29], we can interpret their momentum conjugate on the null boundary as components of stress tensor defined on this hypersurface. In other words one can interpret the first law (3.42) as (considering just gravitational perturbations for the moment)

$$\delta C = -\frac{2}{\pi} \int_B d^{d-2}x ds \sqrt{q} T_{nBY}^{\alpha\beta} \delta g_{\alpha\beta},$$

where $T_{nBY}$ is the counterpart of Brown-York stress tensor defined on the null boundaries, and by $\delta g_{\alpha\beta}^\parallel$ we mean components of metric perturbations tangential to the boundary. The explicit relation of $T_{nBY}$ with stress tensor of dual boundary theory is not evident up to now\(^7\). A similar stress tensor for the null hypersurfaces proposed in [28]. In this reference, a general double foliation has been used for description of null hypersurfaces. This general double foliation can be used both for null and non-null boundaries. This general framework, which has been discussed in detail in [24], help us to define an stress tensor similar to Brown-York tensor on null boundaries. In this method, as in the Brown-York procedure, the resulting quasi local energy becomes infinite and the counterterms from reference space time has been used to make the result finite. In this paper we didn’t use the double foliation, and instead variation of boundary defining scalar function has been used in calculations. Furthermore, following [22], we take attention to contributions from bulk action when varying the boundary. In fact, this is exactly these contributions that make the stress tensor finite without needing for counterterms. In this sense our result may be interesting from pure gravitational point of view because it introduces a new method for getting gravitational charges by definition of "quasi-local gravitational stress tensor" on the null hypersurfaces.

Finally, let us comment on the implications of our results for complexity in the field theory. Using Nielsen’s approach to circuit complexity, one can find that the first variation of complexity takes the form of

$$\delta C = p_a \delta x^a \quad \text{with} \quad p_a = \frac{\partial F}{\partial x^a}.$$  

For some cost function $F$. In [22], using their result in holographic complexity, they deduce that the direction along the path $p_a$ is probably orthogonal to the variation of the target state $\delta x^a$, because for the background and perturbations they have considered, first order variations vanish. Our finding reveal that for general target states this is not true, and in general, the first order variations of the complexity will not vanish.

\(^7\)We hope to address this issue in future publications
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A Variation Of $\Theta$

From the definition for $\Theta_{\alpha\beta}$ in (3.9) we have

$$\delta \Theta = q^{\alpha\beta} \delta \Theta_{\alpha\beta} - \Theta^{\alpha\beta} \delta q_{\alpha\beta} = q^{\alpha\beta} \delta (\nabla_\alpha \ell_\beta) - \Theta^{\alpha\beta} \delta q_{\alpha\beta}. \quad (A.1)$$

For the first term we get

$$q^{\alpha\beta} \delta (\nabla_\alpha \ell_\beta) = q^{\alpha\beta} \nabla_\alpha \delta \ell_\beta - \delta \Gamma^\rho_{\alpha\beta} \ell_\rho. \quad (A.2)$$

Using relation (3.13) for $\delta \ell_\alpha$ we find

$$q^{\alpha\beta} \nabla_\alpha \delta \ell_\beta = \delta \mu_4 q^{\alpha\beta} \nabla_\beta \ell_\alpha - \frac{1}{2} \delta \mu_1 q^{\alpha\beta} \nabla_\beta k_\alpha = \delta \mu_4 \Theta - \frac{1}{2} \delta \mu_1 \Xi. \quad (A.3)$$

Also by using the standard expression for $\delta \Gamma^\rho_{\alpha\beta}$

$$\delta \Gamma^\rho_{\alpha\beta} = \frac{1}{2} g^{\rho\sigma} (\nabla_\alpha \delta g_{\beta\sigma} + \nabla_\beta \delta g_{\alpha\sigma} - \nabla_\sigma \delta g_{\alpha\beta}), \quad (A.4)$$

and using the relation (3.11), definitions (3.9) and some identities similar to (3.18) and after some straightforward algebra we get

$$\delta \Theta = \delta \mu_3 \Theta + \delta \mu_4 \Theta + \frac{1}{2} \delta \mu_1 \Xi + \frac{1}{2} \ell^\alpha \nabla_\alpha \delta q^{\beta\gamma} + \mathcal{D}_\alpha \delta u^\alpha_1 - \delta u^\alpha_1 (\omega_\alpha + \bar{\eta}_\alpha) + a^\alpha \delta u_2. \quad (A.5)$$

The last term indeed vanishes when $\ell_\alpha = \nabla_\alpha \phi$ as easily can be seen

$$a_\alpha = q^\sigma_\alpha \ell^\beta \nabla_\beta \ell_\sigma = q^\sigma_\alpha \ell^\beta \nabla_\beta \nabla_\sigma \phi = q^\sigma_\alpha \ell^\beta \nabla_\sigma \nabla_\beta \phi = q^\sigma_\alpha \ell^\beta \nabla_\sigma \ell_\beta = 0.$$

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Details of calculations can also be followed in a supplementary Mathematica notebook which the capability of abstract tensor package xAct [30, 31] has been used for calculations.
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