Classification of the Blow-Up Behavior for a Semilinear Wave Equation with Nonconstant Degenerate Coefficients

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Abstract. We consider a nonlinear wave equation with nonconstant coefficients. In particular, the coefficient in front of the second-order space derivative is degenerate. We give the blow-up behavior and the regularity of the blow-up set. Partial results are given at the origin, where the degeneracy occurs. Some nontrivial obstacles, due to the nonconstant speed of propagation, have to be surmounted.

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1. Introduction

We consider the following nonlinear wave equation with nonconstant coefficients in the radial case:

\[
\begin{aligned}
\partial_t^2 u &= a(x) \left( \partial_x^2 u + \frac{N-1}{x} \partial_x u \right) + b(x) |u|^{p-1} u + f(x, t, \partial_x u, \partial_t u), \\
\partial_x u(x, t) \sqrt{a(x)} &\to 0 \text{ as } x \to 0, \\
u(0) &= u_0 \text{ and } u_t(0) = u_1,
\end{aligned}
\]

where \( u(t) : x \in \mathbb{R}^+ \to u(x, t) \in \mathbb{R} \) and \( N \) is the dimension of the physical space.

We assume that \( a \in C^1(\mathbb{R}_+^*) \) and \( b \in C^1(\mathbb{R}_+) \) satisfy the following conditions for all \( x > 0 \),

\[
\begin{aligned}
a(x) > 0, b(x) > 0, \\
\int_0^1 \frac{dx}{\sqrt{a(x)}} < +\infty, \\
| (N-1) \frac{\sqrt{a(x)}}{x} - \frac{1}{2} \frac{a'(x)}{\sqrt{a(x)}} - d-1 \phi(x) \phi' \phi' | &\leq M,
\end{aligned}
\]

for some \( \epsilon_0 > 0, M > 0 \) and

\[
d \in \mathbb{N}^{*1},
\]

where \( \phi \) is defined by

\[
\phi(x) = \int_0^x \frac{dy}{\sqrt{a(y)}}.
\]

Note that the third estimate in (1.2) is useful only near the origin. However, introducing such a change will induce a lot of pure technical and trivial complications in the proof. For that reason, we don’t make this change and leave it to the interested reader to check by himself that it works straightforwardly.

The exponent \( p \) is superlinear and subcritical (in relation to \( d \)) , in the sense that

\[
p > 1 \text{ and } p < \frac{d+3}{d-1} \text{ if } d \geq 2.
\]

Conditions (1.3) and (1.5) will prove to be meaningful after a change of variables we perform in (1.11).

We assume in addition that \( f \) and \( g \) are \( C^1 \) functions, where \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R}^4 \to \mathbb{R} \) satisfy

\[
\begin{aligned}
|f(u)| \leq M(1 + |u|^q), \text{ for all } u \in \mathbb{R} \text{ with } (q < p, M > 0), \\
|g(x, t, v, z)| \leq M(1 + |v| \sqrt{a(x)} + |z|), \text{ for all } x, t, v, z \in \mathbb{R}.
\end{aligned}
\]

\(^{1}\text{In particular, at some point we will integrate with respect to the weight } (1 - \frac{2}{r^2})^{\frac{2}{p-1}} - \frac{d-1}{2} r^{d-1} \text{ which is in } L^1(0,1) \text{ if (1.3) and (1.5) hold.}\)
A typical example that satisfies (1.2) and which will be discussed in this paper is the following:

$$a(x) = |x|^\alpha$$ with \( \alpha < 2 \) \hspace{1cm} (1.7)

The example (1.7) shows a degeneracy at \( x = 0 \) when \( \alpha \neq 0 \). Note that for \( \alpha < 0 \), the wave speed goes to infinity and for \( \alpha \in (0, 2) \) it goes to zero. For this case, conditions (1.2), (1.3) and (1.6) are fulfilled for \( N \geq 2 \), \( |g(x, t, v, z)| \leq M(1 + |v||x|^{\frac{2}{2-\alpha}} + |z|) \), and \( d = \frac{2(N-\alpha)}{2-\alpha} \).

Note in particular that example (1.7) is not relevant for \( N = 1 \), since the third condition of (1.2) is never fulfilled, for any \( \alpha < 2 \). On the contrary, when \( N = 2 \), example (1.7) is valid for any value \( \alpha < 2 \), and we always have \( d = 2 \).

When \( N \geq 3 \), we need to take \( \alpha \) of the form \( \alpha_k = \frac{2(k-N)}{k-2} \) where \( k \geq 3 \) is an integer, and in this case, \( d = k \).

Initial data \((u_0, u_1)\) will be considered in the space \( H_1 \times H_0 \) defined by

$$H_0 = \{v \in L^2_{loc}(\mathbb{R}^+) \mid V \in L^2_{loc,u,rad}(\mathbb{R}^+)\},$$

$$H_1 = \{v \in L^2_{loc}(\mathbb{R}^+) \mid V \in H^1_{loc,u,rad}(\mathbb{R}^+)\},$$

where \( V(X) = v(x), X = \phi(x) \), \( \phi \) was given in (1.4),

$$L^2_{loc,u,rad}(\mathbb{R}^+) = \{V \in L^2_{loc}(\mathbb{R}^+) \mid \sup_{r_0 \geq 1} \frac{1}{r_0^{d-1}} \int_{r_0}^{r_0+1} v(r)^2r^{d-1}dr < +\infty\},$$

and

$$H^1_{loc,u,rad}(\mathbb{R}^+) = \{V \in L^2_{loc,u,rad}(\mathbb{R}^+) \mid \partial_r V \in L^2_{loc,u,rad}(\mathbb{R}^+)\}.$$

We recall the spaces \( L^2_{loc,u}(\mathbb{R}^d) \) and \( H^1_{loc,u}(\mathbb{R}^d) \) introduced by Antonini and Merle in [2] by the following norms:

$$||v||^2_{L^2_{loc,u}} = \sup_{a \in \mathbb{R}^d} \int_{|x-a|<1} |v(x)|^2dx \quad \text{and} \quad ||v||^2_{H^1_{loc,u}} = ||v||^2_{L^2_{loc,u}} + ||\nabla v||^2_{L^2_{loc,u}},$$

we show in Appendix A that the spaces \( L^2_{loc,u,rad} \) and \( H^1_{loc,u,rad} \) are simply the radial versions of the \( L^2_{loc,u} \) and \( H^1_{loc,u} \) spaces.

Equation (1.1) corresponds to physical situations where the wave propagates in non-homogeneous media (see for example [15]). It appears in models of traveling waves in a non-homogeneous gas with damping that changes with the position. The unknown \( u \) denotes the displacement; the coefficient \( a \), called the bulk modulus, accounts for changes of the temperature depending on the location.

When \( a(x) \equiv 1 \), this equation was considered by Hamza and Zaag in [5] (see also [4] for some related results). Basically, the authors showed that the results previously proved by Merle and Zaag in [10], [11], [13] and [14] for the unperturbed semilinear wave equation

$$\partial_t^2 u = \partial_x^2 u + |u|^{p-1}u \hspace{1cm} (1.10)$$
do extend to the perturbed case. We also mention the work of Alexakis and Shao [1] who study the energy concentration in backward light cones near blow-up points.

In this paper, we want to explore the case where $a(x) \neq 1$. When $a$ is space dependent, we find that although the blow-up results of [5] remain valid, some nontrivial obstacles have to be surmounted, in particular, at the origin where the degeneracy may occur (see for instance the typical example (1.7)). Since the problem does not have a constant speed of propagation, we have to apply an appropriate transformation to obtain the desired estimates. In fact, we remark that we can reduce to the case $a(x) \equiv 1$ thanks to the following change of variables:

$$U(X, t) = u(x, t), \; X = \phi(x)$$

where $\phi$ is given in (1.4).

Applying this transformation to (1.1), we see that $U$ satisfies:

$$\partial_t^2 U = \partial_X^2 U + \left( (N - 1) \frac{\sqrt{a(x)}}{x} - \frac{1}{2} \frac{a'(x)}{\sqrt{a(x)}} \right) \partial_X U + \beta(X) |U|^{p-1} U + f(U) + g(x, t, \frac{\partial_X U}{\sqrt{a(x)}}, \partial_t U)$$

where $\beta(X) = b(x)$ and $U(t) : X \in \mathbb{R}^+ \rightarrow U(X, t) \in \mathbb{R}$.

We rewrite this equation as follows

$$\partial_t^2 U = \partial_X^2 U + \frac{d-1}{X} \partial_X U + \beta(X) |U|^{p-1} U + f(U) + G(X, t, \partial_X U, \partial_t U)$$

(1.12)

with

$$G(X, t, \partial_X U, \partial_t U) = g(x, t, \frac{\partial_x u}{\sqrt{a(x)}}, \partial_t u) + \left( (N - 1) \frac{\sqrt{a(x)}}{x} - \frac{1}{2} \frac{a'(x)}{\sqrt{a(x)}} - \frac{d-1}{X} \right) \partial_X U.$$ 

We see from (1.2) and (1.6) that we have

$$|f(U)| \leq M(1 + |U|^q), \text{ for all } U \in \mathbb{R} \text{ with } (q < p, M > 0),$$

(1.13)

$$|G(X, t, \partial_X U, \partial_t U)| \leq M(1 + |\partial_X U| + |\partial_t U|).$$

(1.14)

Note that we have

$$\partial_X U(0, t) = 0$$

thanks to the condition on the space derivative in (1.1).

As for the Cauchy problem for equation (1.1), we remark that thanks to the change of variables (1.11), we reduce to the formalism of Hamza and Zaag in [6]. Indeed, recalling that $(u_0, u_1) \in H_0 \times H_1$, we derive by definition that $(U(X, 0), \partial_t U(X, 0)) \in L_{loc,u,rad}^2 \times H_{loc,u,rad}^1$ defined in (1.8) and (1.9).

Therefore, as mentioned in [6] we use a modification of the argument by Georgiev and Todorova [16] to derive a solution $(U, \partial_t U) \in C([0, T_0), H_{loc,u,rad}^1 \times H_{loc,u,rad}^1)$.
for some $T_0 > 0$. Thanks to the finite speed of propagation, we extend the definition of $U(X, t)$ to the following domain

$$D_U = \{(X, t); 0 \leq t < T_U(X)\},$$

for some 1–Lipschitz function $T_U$.

Going back to problem (1.1), we see that we have a unique solution $(u, \partial_t u) \in C([0, T_0), H_0 \times H_1)$ which is defined on a larger domain

$$D_u = \{(x, t)|0 \leq t < T(x)\},$$

where

$$T(x) = T_U(\phi(x)).$$

Since $T'(x) = \frac{T'_U(\phi(x))}{\sqrt{a(x)}}$, it follows that $T$ is a Lipschitz function, with $\frac{1}{\sqrt{a(x)}}$ as local Lipschitz constant for $x \neq 0$. Note that $T(x)$ and $\Gamma$ will be referred as the blow-up time and the blow-up curve in the following.

Proceeding as in the case $a(x) \equiv 1$, we introduce the following non-degeneracy condition for $\Gamma$. If we introduce for all $x \in \mathbb{R}$, $t \leq T(x)$ and $\delta > 0$, the generalized cone

$$C_{x,t,\delta} = \{(\xi, \tau) \neq (x, t)|0 \leq \tau \leq t - \delta|\phi(\xi) - \phi(x)|\}, \quad (1.16)$$

then our non-degeneracy condition is the following: $x_0$ is a non-characteristic point if

$$\exists \delta = \delta(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } C_{x_0, T(x_0), \delta_0}. \quad (1.17)$$

If condition (1.17) is not true, then we call $x_0$ a characteristic point.

We denote by $\mathcal{R}$ the set of noncharacteristic points and $\mathcal{S}$ the set of characteristic points.

Note that the set $C_{x,t,\delta}$ defined in (1.16) is a cone in the variables $(X, t)$ (1.11). In the $(x, t)$ variables, its boundary is given by the characteristics associated with the linear problem

$$\partial_t^2 u = a(x) \partial_x^2 u.$$

In order to state our results, we will use similarity variables associated with $U(X, t)$ defined in (1.11), which turn out to be a nonlinear version of the standard similarity variables, when related directly to $u(x, t)$:

$$w_{x_0}(y, s) = (T(x_0) - t)^{-\frac{p-1}{2}} u(x, t), \quad y = \frac{\phi(x) - \phi(x_0)}{T(x_0) - t}, \quad s = -\log(T(x_0) - t).$$

(1.18)
Applying this transformation to (1.12), we see that \( w_{x_0}(y, s) \) satisfies the following equation for all \(|y| < 1\) and \( s \geq -\log T(x_0)\):

\[
\partial_s^2 w = (1 - y^2)\partial_y^2 w_{x_0} - 2\frac{p + 1}{p - 1} y \partial_y w_{x_0} - 2 \frac{p + 1}{p - 1} w + b(x_0)|w|^{p-1}w - \frac{p + 3}{p - 1} \partial_s w - 2y \partial_{y,s} w
\]

\[
e^{-s} \frac{(d-1)}{\phi(x_0)} + ye^{-s} \partial_y w + e^{-\frac{2ps}{\nu-1}} f(e^{\frac{ps}{\nu-1}} w) + (b(x) - b(x_0))|w|^{p-1}w
\]

\[
e^{-\frac{2ps}{\nu-1}} G(\phi(x_0)) + ye^{-s}, T_0 - e^{-s}, e^{\frac{(p+1)s}{\nu-1}} \partial_y w, e^{\frac{(p+1)s}{\nu-1}} (\partial_s w + y \partial_{y} w + \frac{2}{p - 1} w)).
\]

(1.19)

Let us introduce the solitons

\[
\kappa(\hat{d}, y) = \kappa_0 \frac{(1 - \hat{d}^2)^{\frac{1}{p-1}}}{(1 + \hat{d} y)^{\frac{1}{p-1}}}, \quad \kappa_0 = \left( \frac{2(p + 1)}{(p - 1)^2} \right)^{\frac{1}{p-1}}, \quad (\hat{d}, y) \in (-1, 1)^2.
\]

We also introduce

\[
\hat{\xi}_i(s) = \left( i - \frac{k + 1}{2} \right) \frac{p - 1}{2} \log s + \hat{\alpha}_i(p, k)
\]

(1.20)

where the sequence \((\hat{\alpha}_i)_{i=1,\ldots,k}\) is uniquely determined by the fact that \((\hat{\xi}_i(s))_{i=1,\ldots,k}\) is an explicit solution with zero center of mass for this ODE system:

\[
\forall i = 1, \ldots, k, \quad \frac{1}{c_1} \hat{\xi}_i = e^{-\frac{2}{\nu-1}(\xi_i - \xi_{i-1})} - e^{-\frac{2}{\nu-1}(\xi_{i+1} - \xi_i)},
\]

where \(\xi_0(s) \equiv \xi_{k+1}(s) \equiv 0\) and \(c_1 = c_1(p) > 0\) appeared for the first time in Proposition 3.2 page 590 of Merle and Zaag [13].

1.1. Blow-Up Results

We dissociate two cases in this subsection. In fact, equation (1.19) has a different structure according to the position of \(x_0\).

1.1.1. Behavior Outside the Origin. When \(x_0 \neq 0\), by (1.4), we have \(\phi(x_0) \neq 0\); hence, the term \(\frac{e^{-s}}{\phi(x_0)} + ye^{-s} \partial_y w_{x_0}\) in (1.19) is a lower-order term bounded by \(\frac{2e^{-s}}{\phi(x_0)} |\partial_y w_{x_0}|\) for \(s\) large and will be treated as a perturbation, as in Hamza and Zaag [5].

Accordingly, we may write the second and first order space derivatives in equation (1.19) in the following divergence form:

\[
(1 - y^2)\partial_y^2 w_{x_0} - 2\frac{p + 1}{p - 1} y \partial_y w_{x_0} = \frac{1}{\rho(y)} \partial_y (\rho(1 - y^2) \partial_y w_{x_0})
\]

where \(\rho(y) = (1 - y^2)^{\frac{2}{\nu-1}}\) exactly as in the one dimensional case of the standard semilinear wave equation (1.10).
We recall that for the unperturbed case (ignoring line 2 and 3 in (1.19)), the Lyapunov functional is given by

\[ E(w, \partial_s w) = \int_{-1}^{1} \left( \frac{1}{2} |\partial_s w|^2 + \frac{1}{2} |\partial_y w|^2 (1 - y^2) + \frac{p + 1}{(p - 1)^2} |w|^2 - \frac{b(x_0)}{p + 1} |w|^{p+1} \right) \rho dy. \]  

(1.21)

where \((w, \partial_s w) \in H^1_\rho \times L^2_\rho\), with

\[ L^2_\rho = \left\{ v \bigg| \|v\|_{L^2_\rho} = \int_{-1}^{1} |v(x)|^2 \rho dy < +\infty \right\}, \]  

(1.22)

and

\[ H^1_\rho = \left\{ v \bigg| \|v\|_{L^\rho_\rho} + \|\nabla v\|_{L^2_\rho} < +\infty \right\}. \]  

(1.23)

We see that \(E\) is well defined from the fact that the three first terms of its expression in (1.21) are in \(L^1_\rho\); for the last term we need to use the Hardy–Sobolev inequality given by Merle and Zaag in Appendix B page 1163 of [8]:

\[ \|w\|_{L^{p+1}_\rho} \leq C \|w\|_{H^1_\rho}. \]

Now, if \(u\) is a solution of (1.19), with blow-up surface \(\Gamma : \{x \rightarrow T(x)\}\), and if \(x_0 \neq 0\), then we have the following:

**Theorem 1. (Bound in similarity variables outside the origin)**

i) **(Non-characteristic case)**: If \(x_0 \neq 0\) is a non-characteristic point, then, for all \(s\) large enough:

\[ 0 < \epsilon_0(p) \leq \|w_{x_0}(s)\|_{H^1(-1,1)} + \|\partial_s w_{x_0}(s)\|_{L^2(-1,1)} \leq K(x_0). \]

ii) **(Characteristic case)**: If \(x_0 \neq 0\) is a characteristic point, then, for all \(s\) large enough:

\[ \|w_{x_0}(s)\|_{H^1_\rho} + \|\partial_s w_{x_0}(s)\|_{L^2_\rho} \leq K(x_0). \]

Using the bound in Theorem 1, together with the compactness procedure based on the existence of a Lyapunov for equation (1.19) (which is a perturbation of the functional \(E(w, \partial_s w)\) defined in (1.21)), we derive the following:

**Theorem 2. (Blow-up behavior in similarity variables outside the origin)**

i) **(Non-characteristic case)** The set \(R \cap \mathbb{R}^*_+\) is open, and \(T\) is of class \(C^1\) on that set. Moreover, there exist \(\mu_0 > 0\) and \(C_0 > 0\) such that for all \(x_0 \in R \cap \mathbb{R}^*_+\), there exist \(\theta(x_0) = \pm 1\) and \(s_0(x_0) \geq -\log T(x_0)\) such that for all \(s \geq s_0:\)

\[ \left\| \left( \begin{array}{c} w(s) \\ \partial_s w(s) \end{array} \right) - \theta(x_0) \left( \begin{array}{c} \kappa(T'(x_0) \sqrt{a(x_0)}) \\ 0 \end{array} \right) \right\|_{H^1_\rho \times L^2_\rho} \leq C_0 e^{-\mu_0(s-s^*)}, \]  

(1.24)

where \(w = w_{x_0}\). Moreover, \(E(w, \partial_s w) \rightarrow E(\kappa_0, 0)\) as \(s \rightarrow \infty\).
ii) **(Characteristic case)** If \( x_0 \in S \cap \mathbb{R}_+^* \), there is \( \xi_0(x_0) \in \mathbb{R} \) such that:

\[
\left\| \left( \frac{w(s)}{\partial_s w(s)} \right) - \theta_1 \left( \sum_{i=1}^{k(x_0)} (-1)^{i+1} \kappa(\hat{d}_i(s), \cdot) \right) \right\|_{L^1_x \times L^2_t} \to 0, \tag{1.25}
\]

where \( w = w_{x_0} \) and \( E(w, \partial_s w) \to k(x_0)E(\kappa_0, 0) \) as \( s \to \infty \), for some \( k(x_0) \geq 2 \), \( \theta_i = \theta_1(-1)^{i+1} \), \( \theta_1 = \pm 1 \), and continuous \( \hat{d}_i(s) = -\tan \hat{\xi}_i(s) \) with

\[
\hat{\xi}_i(s) = \bar{\xi}_i(s) + \xi_0, \tag{1.26}
\]

where \( \bar{\xi}_i(s) \) is introduced in (1.20).

**Remark.** Estimate (1.24) holds in \( H^1 \times L^2(-1, 1) \), thanks to the covering argument introduced by Merle and Zaag in [9]. From the Sobolev embedding, it holds also in \( L^\infty \times L^2 \).

**Remark.** Following the strategy of Côte and Zaag in [3], refined in [7] by Hamza and Zaag, for every \( k_0 \geq 2 \) and \( \xi_0 \in \mathbb{R} \), we are able to construct examples of solutions to equation (1.1) showing a characteristic-point and obeying the modality described in item ii) of Theorem 2.

Going back to \( u(x,t) \) thanks to (1.18), we have the following corollary:

**Corollary 3.** *(Blow-up profile for equation (1.1) in the non-characteristic case outside the origin)* If \( \mathcal{R} \cap \mathbb{R}_+^* \), then we have

\[
uu(x, t) \sim \frac{\theta(x_0)\kappa_0(1 - a(x_0)|T'(x_0)|^2)|^{\frac{1}{p-1}}}{(T(x_0) - t + T'(x_0)^{\sqrt{a(x_0)}(\phi(x) - \phi(x_0))})^\frac{2}{p-1}} \text{ as } t \to T(x_0)
\]

uniformly for \( x \) such that \( |\phi(x) - \phi(x_0)| < T(x_0) - t \).

We also obtain the regularity of the blow-up set:

**Proposition 4.** *(Regularity of the blow-up set outside the origin)*

i) **(Non-characteristic case)** It holds that \( \mathcal{R} \neq \emptyset \), \( \mathcal{R} \setminus \{0\} \) is an open set, and \( x \mapsto T(x) \) is of class \( C^1 \) on \( \mathcal{R} \setminus \{0\} \) and for all \( x \in \mathcal{R} \setminus \{0\} \), \( |T'(x)| < \frac{1}{\sqrt{a(x)}} \).

ii) **(Characteristic case)** Any \( x_0 \in \mathcal{S} \setminus \{0\} \) is isolated. In addition, if \( x_0 \in \mathcal{S} \setminus \{0\} \) with \( k(x_0) \) solitons and \( \xi_0(x_0) \in \mathbb{R} \) as center of mass of the solitons’ center as shown in (1.25) and (1.26), then

\[
T'(x) + \frac{\theta(x)}{\sqrt{a(x)}} \sim \frac{\theta(x)e^{-2\theta(x)\xi_0(x_0)}}{\sqrt{a(x_0)||x - x_0||^{(k(x_0) - 1)(p-1)}}}, \tag{1.27}
\]

\[
T(x) - T(x_0) + |\phi(x) - \phi(x_0)| \sim \frac{\nu|\phi(x) - \phi(x_0)|e^{-2\theta(x)\xi_0(x_0)}}{\log |x - x_0|^\left(\frac{(k(x_0) - 1)(p-1)}{2}\right)}, \tag{1.28}
\]

as \( x \to x_0 \), where \( \theta(x) = \frac{x - x_0}{|x - x_0|} \) and \( \nu = \nu(p) > 0 \).
Remark. If $a$ is H"older continuous, then we may replace \( \frac{\theta(x)}{\sqrt{a(x)}} \) by \( \frac{\theta(x)}{\sqrt{a(x_0)}} \) in (1.27) and replace (1.28) by
\[
T(x) - T(x_0) + \frac{|x - x_0|}{\sqrt{a(x_0)}} \sim e^{-\theta(x)\xi_0(x_0)} \phi \theta \frac{|x - x_0|}{\sqrt{a(x_0)}} e^{-2\theta(x)\xi_0(x_0)} \begin{bmatrix} \log |x - x_0| |k(x_0) - 1|^{(p - 1)} \end{bmatrix}.
\]

1.1.2. Behavior at the Origin. When $x_0 = 0$, we have \( \phi(x_0) = 0 \); hence, the term \( \frac{e^{-\theta(d-1)}}{\phi(x_0) + y^2} \partial_y w_0 = \frac{d-1}{y} \partial_y w_0 \) in equation (1.19) and can no longer be treated as a perturbation.

Accordingly, we may write the second and first order space derivatives in the following divergence form:
\[
(1 - y^2) \partial_y^2 w_0 - 2p + 1 \frac{1}{p - 1} \partial_y w_0 + \frac{d - 1}{y} \partial_y w_0 = \frac{1}{\rho_0(y)} \partial_y (\rho_0 (1 - y^2) \partial_y w_0)
\]
where
\[
\rho_0(y) = (1 - y^2)^{\frac{p-1}{2}} y^{-d-1}.
\]

For the case where \( (f, g) \equiv (0, 0) \), in one space dimension, we introduce the functional
\[
E_0(w, \partial w) = \int_0^1 \left( \frac{1}{2} |\partial w|^2 + \frac{1}{2} |\partial_y w|^2 (1 - y^2) + \frac{p + 1}{(p - 1)^2} |w|^2 - \frac{\beta(0)}{p + 1} |w|^{p+1} \right) \rho_0 dy.
\]

Note first that $E_0$ is defined if \( (w, \partial w) \in H^1_{\rho_0} \times L^2_{\rho_0} \), where the norms $L^2_{\rho_0}$ and $H^1_{\rho_0}$ are defined by the same way as in (1.22) and (1.23), but only on the domain \( (0, 1) \) and with weight \( \rho_0 \) given in (1.30).

Adapting the techniques introduced by Antonini and Merle (See Section 2 page 1144 in [2]) to our case where \( \rho_0 \) is given by (1.30), we see that
\[
\frac{d}{ds} E_0(w, \partial w) = (d - 1 - \frac{4}{p - 1}) \int_0^1 (\partial w)^2 \rho_0 dy,
\]
as $d$ satisfies (1.3)-(1.5), $E_0$ (1.31) is decreasing and is a Lyapunov functional. Another way to justify this: the functional in (1.31) is simply the radial version of the functional of [2] considered in the space $\mathbb{R}^d$ (which is not the physical space $\mathbb{R}^N$).

Considering $w(y, s)$ as a (nonnecessarily radial) function defined in $\mathbb{R}^d$, we may use the perturbative techniques of Hamza and Zaag in [5] to derive the following:

Theorem 5. (Bound in similarity variables at the origin in the non-characteristic case) If $u$ is a solution of (1.19) with blow-up surface $\Gamma = \{x \to T(x)\}$, and if $0$ is a non-characteristic point, then, for $s$ large enough:
\[
0 < \epsilon_0(p) \leq \|w_0(s)\|_{H^1_{\rho_0}} + \|\partial_s w_0(s)\|_{L^2_{\rho_0}} \leq K,
\]
\[
\|w_0(s)\|_{H^1_{\rho_{d-1}(0, 1)}} + \|\partial_s w_0(s)\|_{L^2_{\rho_{d-1}(0, 1)}} \leq K.
\]
1.2. Strategy of the Proof of the Results

Thanks to the transformation (1.11), we reduce to the case where \( a(x) \equiv 1 \) in the remaining part of the paper. In comparison with the paper by Hamza and Zaag [6], our equation allows a non-constant term in front of the reaction-term \(|u|^{p-1}u\), namely \( \beta(x) \not\equiv 1 \). As in [6], the most delicate point is to obtain a Lyapunov functional in similarity variables defined in (1.18). Thus, in the following section, we focus on the proof of the existence of a Lyapunov functional for equation (1.19) in the first subsection and then we give some hints on how to adapt the strategy of [6] to derive the blow-up behavior outside and at the origin in the second and third subsections.

2. Proof of the Results

We prove the blow-up results for (1.12) which we recall in the following:

\[
\begin{align*}
\partial_t^2 U &= \partial_X^2 U + \frac{d-1}{X} \partial_X U + \beta(X)|U|^{p-1}U + f(U) + G(X,t,\partial_X U, \partial_t U), \text{ for } X > 0 \\
U_X(0,t) &= 0, \\
U(0) &= U_0 \text{ and } U_t(0) = U_1,
\end{align*}
\]

with

\[
|f(U)| \leq M(1 + |U|^q), \text{ for all } U \in \mathbb{R} \text{ with } (q < p, M > 0),
\]

\[
|G(X,t,\partial_X U, \partial_t U)| \leq M(1 + |\partial_X U| + |\partial_t U|).
\]

In fact, this is almost the same equation as in [6] except for the coefficient \( \beta(X) \) in front of \(|U|^{p-1}U\) which was taken identically equal to 1 in [6]. For that reason, we follow the strategy of [6] and focus mainly on the treatment of the term \( \beta(X)|U|^{p-1}U \). Given some \( X_0 = \phi(x_0) \in \mathbb{R}_+ \), where \( \phi \) was defined in (1.4), we introduce the following self-similar change of variables, as in (1.18):

\[
\begin{align*}
w_{X_0}(y,s) &= \left( T_U(X_0) - t \right)^{\frac{2}{p-1}} U(X,t), \quad y = \frac{X - X_0}{T_U(X_0) - t}, \\
s &= -\log(T_U(X_0) - t).
\end{align*}
\]

Note that the curve \( T_U \) of \( U \) is given by the curve \( T \) of \( u \), in fact :

\( T_U(X) = T(x) \) with \( X = \phi(x) \).

This change of variables transforms the backward light cone with vertex \((X_0, T_U(X_0))\) into the infinite cylinder \((y,s) \in (-1,1) \times [-\log T_U(X_0), +\infty)\).
The function $w_{X_0}$ (we write $w$ for simplicity) satisfies the following equation for all $|y| < 1$ and $s \geq -\log T_U(X_0)$:

\[
\begin{align*}
\partial_s^2 w &= (1 - y^2)\partial_y^2 w - 2\frac{p+1}{p-1}y\partial_y w - 2\frac{p+1}{(p-1)^2}w + \beta(X_0)|w|^{p-1}w \\
&\quad - \frac{p+3}{p-1}\partial_s w - 2y\partial_{ys} w \\
+ e^{-s} \frac{(d-1)}{X_0 + ye^{-s}}\partial_y w + e^{-2s\frac{2}{r-1}}f(e^{\frac{2s}{r-1}}w) + (\beta(X_0 + ye^{-s}) - \beta(X_0))|w|^{p-1}w \\
&\quad + e^{-2s\frac{2}{r-1}}G(X_0 + ye^{-s}, T_0 - e^{-s}, e^{\frac{p+1}{r-1}}\partial_y w, e^{\frac{p+1}{r-1}}(\partial_s w + y\partial_y w + \frac{2}{p-1}w)).
\end{align*}
\]

(2.3)

In the whole paper, we use the notation

\[
F(u) = \int_0^u f(v)dv.
\]

(2.4)

### 2.1. A Lyapunov Functional in Similarity Variables Outside the Origin

In this subsection, we prove the existence of a Lyapunov functional and the novelty lays in the new coefficient $\beta(X) \neq 1$. We recall that for the case $(f,G) \equiv (0,0)$ with a constant $\beta$, the Lyapunov functional in one space dimension is

\[
E_0(w, \partial_s w) = \int_{-1}^1 \left(\frac{1}{2}|\partial_s w|^2 + \frac{1}{2}|\partial_y w|^2(1 - y^2) + \frac{p+1}{p+1}w - \frac{\beta(X_0)}{p+1}|w|^{p+1}\right)\rho dy.
\]

(2.5)

In order to find a Lyapunov functional for our equation (2.3), we introduce

\[
E(w, \partial_s w) = E_0(w, \partial_s w) + I(w(s), s) + J(w(s), s) + K(w(s), s),
\]

(2.6)

where

\[
I(w(s), s) = -e^{-\frac{2(p+1)s}{p-1}}\int_{-1}^1 F(e^{\frac{2s}{r-1}}w)\rho dy,
\]

(2.7)

\[
J(w(s), s) = -\frac{1}{p+1}\int_{-1}^1 (\beta(X_0 + ye^{-s}) - \beta(X_0))|w|^{p+1}\rho dy
\]

(2.8)

\[
K(w(s), s) = -e^{-\gamma s}\int_{-1}^1 w\partial_s w\rho dy,
\]

(2.9)

with

\[
\gamma = \min\left(\frac{1}{2}, \frac{p-q}{p-1}\right) > 0.
\]

(2.10)

Then, we claim the following:

**Proposition 2.1. (Energy estimates outside the origin)** (i) There exist $C = C(p, M) > 0$ and $S_0 \in \mathbb{R}$ such that for all $X_0 > 0$ and for all $s \geq \max(-\log T_U(X_0), S_0, -4\log X_0, -\log \frac{X_0}{2})$,

\[
\frac{d}{ds} E(w(s), s) \leq \frac{p+3}{2} e^{-\gamma s} E_0(w(s), s) - \frac{3}{p-1} \int_{-1}^1 (\partial_s w)^2 \frac{p}{1-y^2} \rho dy + Ce^{-2\gamma s}.
\]
(ii) There exists $S_1(p, N, M, q) \in \mathbb{R}$ such that, for all $s \geq \max(s_0, S_1)$, we have $H(w(s), s) \geq 0$.

**Remark.** From (i), we see that $H$ given by

$$H(w(s), s) = E(w(s), s) e^{- \frac{p+3}{2} \gamma s} + \mu e^{-2 \gamma s} (\mu > 0)$$

is a Lyapunov functional for equation (2.3).

**Proof of Proposition 2.1.** In this proof, we use the notation, $x_+ = \max(0, x)$.

(i) We proceed like Hamza and Zaag in [6] (See page 1092) and we deal with the new term coming from (2.3). For that reason, we give the equations, recall the estimates already proved in [6] and focus only on the new term. We multiply equation (2.3) by $\partial_x w$ and integrate for $y \in (-1, 1)$, using (2.7) and (2.8); we have for $X = X_0 + ye^{-s}$:

$$\frac{d}{ds}(E_0(w(s), s) + I(w(s), s) + J(w(s), s))$$

$$= \frac{-4}{p-1} \int_{-1}^{1} \frac{(\partial_x w)^2 \rho dy}{1-y^2} + (N-1) e^{-s} \int_{-1}^{1} \partial_x w \partial_y w \frac{\rho dy}{X}$$

$$+ \frac{2(p+1)}{p-1} \int_{-1}^{1} F(e^{\frac{2s}{p-1}} w) \rho dy + \frac{2}{p-1} e^{-\frac{2p}{p-1}} \int_{-1}^{1} f(e^{\frac{2s}{p-1}} w) w \rho dy$$

$$+ \frac{e^{-s} \int_{-1}^{y} \beta'(X_0 + ye^{-s})|w|^{p+1} \rho dy}{p+1} I_3(s)$$

$$e^{-\frac{2s}{p-1}} \int_{-1}^{1} G(X_0 + ye^{-s}, T_0 - e^{-s}, e^{\frac{p+1}{p-1}} \partial_y w, e^{\frac{p+1}{p-1}} (\partial_x w + y \partial_y w + \frac{2}{p-1} w)) \partial_x w \rho dy$$

(2.11)

The terms $I_1, I_2$ and $I_3$ can be controlled exactly as in page 1092 in [6]. For $I_5$, comparing to the previous work, we see that $G$ involves new terms, but as it satisfies condition (1.14), we can also adapt the study of Hamza and Zaag in [6] to get:

$$|I_1(s)| \leq C e^{-s} \int_{-1}^{1} (\partial_y w)^2 \rho (1-y^2) dy + \frac{C e^{-s}}{X_0} \int_{-1}^{1} (\partial_x w)^2 \frac{\rho}{1-y^2} dy,$$

(2.12)

$$|I_2(s)| + |I_3(s)| \leq C e^{-\frac{2(p-1)}{p-1}} + C e^{-\frac{2(p-1)}{p-1}} \int_{-1}^{1} |w|^{p+1} \rho dy,$$

(2.13)

$$|I_5(s)| \leq C e^{-s} \int_{-1}^{1} \left( (\partial_y w)^2 (1-|y|)^2 + \frac{(\partial_x w)^2}{1-y^2} + w^2 \right) \rho dy + C e^{-s}.$$  

(2.14)

For the new term $I_4$, we use the fact that $\beta$ is of class $C^1$; we get:

$$|I_4(s)| \leq \frac{e^{-s}}{p+1} ||\beta'||_{L^\infty((x_0-T)^+, x_0+T)} \int_{-1}^{1} |w|^{p+1} \rho dy.$$  

(2.15)
Using (2.11), (2.12), (2.13), (2.14) and (2.15), we have

\[
\frac{d}{ds}(E_0(w(s), s) + I(w(s), s) + J(w(s), s)) \leq \left( \frac{-4}{p-1} + C e^{-\gamma s} \right) \int_{-1}^{1} (\partial_s w)^2 \frac{p}{1-y^2} dy \\
+ Ce^{-s} \int_{-1}^{1} ((\partial_y w)^2 (1 - |y|^2) + w^2) \rho dy \\
+ Ce^{-2\gamma s} \int_{-1}^{1} |w|^{p+1} \rho dy + Ce^{-2\gamma s}. \tag{2.16}
\]

Now, we consider \((K(w(s), s)) \) (2.9). Using equation (2.3) and integration by parts, we write:

\[
e^{\gamma s} \frac{d}{ds}(K(w(s), s)) = - \int_{-1}^{1} (\partial_s w)^2 \rho dy + \int_{-1}^{1} (\partial_y w)^2 (1 - y^2) \rho dy + \frac{2p + 2}{(p-1)^2} \int_{-1}^{1} w^2 \rho dy \\
- \beta(X_0) \int_{-1}^{1} |w|^{p+1} \rho dy + (\gamma + \frac{p+3}{p-1} - 2N) \int_{-1}^{1} w \partial_s w \rho dy - 2 \int_{-1}^{1} w \partial_s w \rho' dy \\
- 2 \int_{-1}^{1} \partial_s w \partial_y w \rho dy - e^{-\frac{2\gamma s}{p-1}} \int_{-1}^{1} w f \left( \frac{e^{\gamma s}}{w} \right) \rho dy - (N-1)e^{-s} \int_{-1}^{1} w \partial_y w \frac{\rho}{r} dy \\
- e^{-\frac{2\gamma s}{p-1}} \int_{-1}^{1} w G(X_0 + ye^{-s}, T_0 - e^{-s}, e^{-\frac{(p+1)s}{r}}, (\partial_s w + ye^{-s} \partial_y w + \frac{2}{p-1} w))^\rho dy \\
- \int_{-1}^{1} (\beta(X_0 + ye^{-s}) - \beta(X_0)) |w|^{p+1} \rho dy.
\]

Using (2.7) and (2.8), we get:

\[
e^{\gamma s} \frac{d}{ds}(K(w(s), s)) = \frac{p+3}{2} \left( E_0(w(s)) + I(w(s)) + J(w(s)) \right) - \frac{p-1}{4} \int_{-1}^{1} (\partial_y w)^2 (1 - y^2) \rho dy \\
- \frac{p+1}{2(p-1)} \int_{-1}^{1} w^2 \rho dy - \frac{p-1}{2(p+1)} \beta(X_0) \int_{-1}^{1} |w|^{p+1} \rho dy \\
+ (\gamma + \frac{p+3}{p-1} - 2N + \frac{p+3}{2} e^{-\gamma s}) \int_{-1}^{1} w \partial_s w \rho dy \\
\underbrace{+ \frac{8}{p-1} \int_{-1}^{1} w \partial_s w \frac{y^2}{1-y^2} \rho dy - 2 \int_{-1}^{1} \partial_s w \partial_y w \rho dy - e^{-\frac{2\gamma s}{p-1}} \int_{-1}^{1} w f \left( \frac{e^{\gamma s}}{w} \right) \rho dy}_{K_2(s)} \\
\underbrace{- e^{-\frac{2\gamma s}{p-1}} \int_{-1}^{1} w G(X_0 + ye^{-s}, T_0 - e^{-s}, e^{-\frac{(p+1)s}{r}}, (\partial_s w + ye^{-s} \partial_y w + \frac{2}{p-1} w))^\rho dy}_{K_3(s)} \\
\underbrace{- \frac{1}{(p-1)^2} \int_{-1}^{1} \left( \beta(X_0 + ye^{-s}) - \beta(X_0) \right) |w|^{p+1} \rho dy}_{K_5(s)}
\]
\[
\begin{align*}
&\frac{p}{2} + 3 e^{-\frac{2(p+1)s}{p-1}} \int_{-1}^{1} F(e^{\frac{2}{p-1} s} w) \rho \, dy -(N-1) e^{-s} \int_{-1}^{1} w \partial_y w \rho \, dy \\
&- \frac{p-1}{2(p+1)} \int_{-1}^{1} (\beta(X_0 + ye^{-s}) - \beta(X_0)) |w|^{p+1} \rho \, dy \\
&\leq K_0(s) + K_1(s) + K_2(s) + K_3(s) + K_4(s) + K_5(s)
\end{align*}
\]

(2.17)

Note that all the terms \(K_1, K_2, K_3, K_4, K_5, K_6\) and \(K_7\) have been studied in [6] (for details see page 1094 in [6]). For the reader’s convenience, we recall the following estimates:

\[
\begin{align*}
|K_1(s)| &\leq C e^{\frac{-2s}{p-1}} \int_{-1}^{1} (\partial_s w)^2 \frac{\rho}{1 - y^2} \, dy + C e^{\frac{-2s}{p-1}} \int_{-1}^{1} w^2 \rho \, dy, \\
|K_2(s)| &\leq C e^{\frac{-2s}{p-1}} \int_{-1}^{1} (\partial_s w)^2 \frac{\rho}{1 - y^2} \, dy + C e^{\frac{-2s}{p-1}} \int_{-1}^{1} w^2 \rho \, dy \\
&+ C e^{\frac{-2s}{p-1}} \int_{-1}^{1} (\partial_y w)^2 \rho (1 - y^2) \, dy, \\
|K_3(s)| &\leq C e^{\frac{-2s}{p-1}} \int_{-1}^{1} (\partial_s w)^2 \frac{\rho}{1 - y^2} \, dy + C e^{\frac{-2s}{p-1}} \int_{-1}^{1} (\partial_y w)^2 \rho (1 - y^2) \, dy, \\
|K_4(s)| + |K_6(s)| &\leq C e^{-\gamma s} + C e^{-\gamma s} \int_{-1}^{1} |w|^{p+1} \rho \, dy, \\
|K_5(s)| &\leq C e^{-\gamma s} \int_{-1}^{1} (\partial_s w)^2 \frac{\rho}{1 - y^2} \, dy + C e^{-\gamma s} \int_{-1}^{1} (\partial_y w)^2 \rho (1 - y^2) \, dy \\
&+ C e^{-\gamma s} \int_{-1}^{1} w^2 \rho \, dy + C e^{-\gamma s} \\
|K_7(s)| &\leq C e^{\frac{-s}{2}} \int_{-1}^{1} (\partial_y w)^2 \rho (1 - y^2) \, dy + C e^{\frac{-s}{2}} \int_{-1}^{1} w^2 \rho \, dy.
\end{align*}
\]

(2.18) - (2.23)

For the new term \(K_8\), using the fact that \(\beta\) is of class \(C^1\) we see that:

\[
|K_8(s)| \leq \frac{p-1}{2(p+1)} ||\beta||_{L^\infty(X_0-T,X_0+T)} e^{-s} \int_{-1}^{1} |w|^{p+1} \rho \, dy.
\]

(2.24)

By the same way, using the fact that \(\beta\) is of class \(C^1\), we prove that \(J\) (2.8) satisfies:

\[
|J(w(s))| \leq C e^{-s} \int_{-1}^{1} |w|^{p+1} \rho \, dy.
\]

(2.25)
Using the definitions of $I$ (2.7), $F$ (2.4) and the condition (1.6), we see that:

$$\left| I(w(s)) \right| = \left| Ce^{-\frac{2p+1}{p-1}s} \int_{-1}^{1} \int_{0}^{2|s|} f(v) dv \rho dy \right|$$

$$\leq Ce^{-\frac{2p}{p-1}s} \int_{-1}^{1} |w| \rho dy + Ce^{-\frac{2(p-q)}{p-1}s} \int_{-1}^{1} |w|^{q+1} \rho dy$$

$$\leq Ce^{-\frac{2(p-q)}{p-1}s} + Ce^{-\frac{2p}{p-1}s} \int_{-1}^{1} w^2 \rho dy + Ce^{-\frac{2(p-q)}{p-1}s} \int_{-1}^{1} |w|^{p+1} \rho dy.$$  \hspace{1cm} (2.26)

Using (2.17)-(2.26) and the definition of $\gamma$ (2.10), we deduce that

$$e^{\gamma s} \frac{d}{ds}(K(w(s), s)) \leq \frac{p+3}{2} E_0(w(s))$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{p-1}{4} \right) \int_{-1}^{1} (\partial_y w)^2 (1-y^2) \rho dy$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{p+1}{2(p-1)} \right) \int_{-1}^{1} w^2 \rho dy$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{p+1}{2(p-1)} \beta(X_0) \right) \int_{-1}^{1} |w|^{p+1} \rho dy$$

$$+ Ce^{-\frac{2p}{p-1}} \int_{-1}^{1} (\partial_y w)^2 \rho \frac{1}{1-y^2} dy + Ce^{-\gamma s}. \hspace{1cm} (2.27)$$

Using the definition (2.6) of $E$, (2.16), (2.27), we get (remember from (2.10) that $\gamma \leq \frac{1}{2}$)

$$\frac{d}{ds}(E(w(s), s)) \leq Ce^{-2\gamma s} + \frac{p+3}{2} e^{-\gamma s} E_0(w(s), s)$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{4}{p-1} \right) \int_{-1}^{1} (\partial_y w)^2 \frac{\rho}{1-y^2} dy$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{p+1}{2(p-1)} \right) e^{-\gamma s} \int_{-1}^{1} w^2 \rho dy$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{p-1}{4} \right) e^{-\gamma s} \int_{-1}^{1} (\partial_y w)^2 (1-|y|^2) \rho dy$$

$$+ \left( Ce^{-\frac{2p}{p-1}} - \frac{p+1}{2(p-1)} \beta(X_0) \right) e^{-\gamma s} \int_{-1}^{1} |w|^{p+1} \rho dy.$$  

Then, for $S_0$ well chosen large enough so that $s \geq \max(-\log T(X_0), S_0, -4 \log X_0, - \log \frac{X_0}{2})$, we write

$$\frac{d}{ds} E(w(s), s) \leq \frac{p+3}{2} e^{-\gamma s} E_0(w(s), s) - \frac{3}{p-1} \int_{-1}^{1} (\partial_y w)^2 \frac{\rho}{1-y^2} dy + Ce^{-2\gamma s}.$$  

This yields item $(i)$ of Proposition 2.1.
This follows from the blow-up criterion proved by Antonini and Merle in [2]. In fact, we need to follow the perturbative argument of Hamza and Zaag in [6]. As in [6], it is easy to prove the following identity for large $s$ and for any $w \in H$:

$$H(w) \geq -\frac{2\beta(X_0)}{p+1} \int_{-1}^{1} |w|^{p+1} \rho dy.$$  

For more details, see [6][ii] page 1096 and see page 1147 in [2]. This concludes the proof of Proposition 2.1.

\[\square\]

### 2.2. Blow-Up Results Outside the Origin

In this subsection, we give the main ideas of the proofs of our blow-up results outside the origin (Theorem 1, Theorem 2, Corollary 3 and Proposition 4). However, we will not give the details. In fact, thanks to the transformation (1.11), we obtain the equation (1.12) which is almost the same equation already studied by Hamza and Zaag in [6]. In addition, in Sect. 2.1, we see that a Lyapunov functional is available (see the remark following Proposition 2.1), so with this informations, the reader can easily see that the strategy adapted in [6] from the strategy developed by Merle and Zaag in [8], [9], [10], [11], [13] and [14] together with Côte and Zaag [3] holds with very minor adaptations (See also [12]). For that reason, we will sketch the main steps in the following and explicit only the delicate estimates:

- In Step 1, we show that the solution is bounded in self-similar variables in the energy norm. In particular, we will prove Theorem 1.
- In Step 2, we find the asymptotic behavior and derive the regularity of the blow-up curve. In particular, we will prove Theorem 2, Corollary 3 and Proposition 4.

**Step 1 : Boundedness of the solution in similarity variables**

We derive with no difficulty the following:

**Proposition 2.2.** For all $X_0 > 0$, there is a $C_2(X_0) > 0$ and $S_2(X_0) \in \mathbb{R}$ such that for all $X \in \left[\frac{X_0}{2}, \frac{3X_0}{2}\right]$ and $s \geq S_2(X_0)$,

$$\int_{-1}^{1} \left( \partial_y w^2 (1 - y^2) + w^2 + \partial_s w^2 + \beta(X_0 + ye^{-s}) |w|^{p+1} \right) \rho dy \leq C_2(X_0).$$

**Proof of Proposition 2.2.** The adaptation by Hamza and Zaag in page 1091 in [6] to the perturbed case works in our case ($\beta(X_0 + ye^{-s}) \neq 1$) with no difficulty. As in [6], the adaptation is straightforward from [8] and Proposition 3.5 page 66 in [10].

**Proof of Theorem 1.** \ i) Consider $x_0 > 0$ and $X_0 = \phi(x_0) > 0$. Let us start with the upper bound. Since $b$ is continuous, $\beta$ is continuous too and as $\beta(X_0 + ye^{-s}) > 0$ for $y \in (-1, 1)$ and $s$ large enough., we derive from Proposition 2.2 that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \partial_y w^2 + w^2 + \partial_s w^2 + |w|^{p+1} \right) dy \leq C_3(X_0).$$
Using the covering method of Proposition 3.4 in [9], we recover the desired upper bound. As for the lower bound, it follows exactly as in the unperturbed case in Lemma 3.1 in [9], simply because equation (1.12) is well posed in $H^1 \times L^2$, from [16] as we explained in the introduction.

ii) It is a direct consequence of Proposition 2.2.

\[\square\]

**Step 2 : Dynamics of the solution and properties of the blow-up curve**

We recall form the definition of $T$ (1.15) that

\[T(x) = T_U(\phi(x)) = T_U(X), \text{ and } T'_U(X) = \sqrt{a(x)}T'(x).\] (2.28)

**Proof of Theorem 2.** i) **Non-characteristic case :** As we said before, our equation (1.12) is the same as in [6], except for the coefficient $\beta(X) \not\equiv 1$. Thanks to Proposition 2.2, the adaptation of Hamza and Zaag of the analysis of [10] and [11] works; in particular, this is the case for Theorem 1 page 1097 which gives the following profile of $w$

\[w(y,s) \sim \theta(x_0)\kappa_0 \frac{(1 - T'_U(X_0))^\frac{1}{p-1}}{(1 + T'_U(X_0)y)^\frac{2}{p-1}} \text{ as } s \to +\infty,\] (2.29)

where $X_0 = \phi(x_0)$, in $H^1(-1,1)$. Using (2.28), we get our statement.

ii) **Characteristic case :** The approach of Côte and Zaag in [3] adapted to the perturbed case by Hamza and Zaag in [6] stays valid in our case (for more details see page 1105 in [6]).

\[\square\]

**Proof of Corollary 3.** Applying the transformation (1.18) to the profile given in (2.29) and using the Sobolev embedding, we see that for $X_0 \in \mathcal{R} \cap \mathbb{R}^*_+$, we have

\[U(X,t) \sim \frac{\theta(x_0)\kappa_0(1 - T'_U(X_0))^\frac{1}{p-1}}{(T_U(X_0) - t + T'_U(X_0)(X - X_0))^{\frac{2}{p-1}}} \text{ as } t \to T_U(X_0)\]

uniformly for $X$ such that $|X - X_0| < T_U(X_0) - t$. Applying (2.28), we get the result.

\[\square\]

**Proof of Proposition 4.** i) We can easily see that the strategy developed in the non-perturbed case in [11] and then adapted to the perturbed case in [6] works in the present case ($\beta(X) \not\equiv 1$), with minor adaptations.

ii) Let $x_0 \in \mathcal{S}\setminus\{0\}$ with $k(x_0)$ solitons and $\xi_0(x_0) \in \mathbb{R}$ as center of mass of the solitons as shown in (1.25) and (1.26). Proceeding as in the adaptation by Hamza and Zaag in Theorem 5 in [6] to the perturbed case (see also Theorem 1 and 2 in [14], where this statement was proved with no perturbation), we get

\[T'_U(X) + \theta_U(X) \sim \frac{\theta_U(X)\nu e^{-2\theta_U(X)\xi_0(X_0)}}{|\log |X - X_0||^{(k_U(X_0)-1)(p-1)}},\]

\[T_U(X) - T_U(X_0) + |X - X_0| \sim \frac{\nu e^{-2\theta_U(X)\xi_0(X_0)}|X - X_0|}{|\log |X - X_0||^{(k_U(X_0)-1)(p-1)}},\]
as $X \to X_0$, where $\theta_U(X) = \frac{X - X_0}{|X - X_0|}$ and $\nu = \nu(p) > 0$. Using the correspondence between $x$ and $X$ and also $T_U$ and $T$ shown in (1.11) and (1.15), we recover our conclusion.

2.3. Blow-Up Results at the Origin

Proof of Theorem 5. The proof is done in the framework of similarity variables (1.18), with $x_0 = 0$. Since $d$ is an integer, one clearly sees that the equation satisfied by $w_0$ in (1.19) is simply the radial version of the multi-dimensional equation considered in $\mathbb{R}^d$. Since $p$ is subconformal in relation to $d$, as shown in (1.5), we are in the setting considered by Hamza and Zaag in [6] for perturbed equations, with the exceptions that we have a non-constant coefficient in front of the nonlinear term here. As we have already seen while investigating the Lyapunov functional in Sect. 2.1, that is not an issue, and one can adapt the proof of [6] to the present equation, with no difficulties.

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A $L^2_{loc,u}$ for Radial Functions

Note that we handle only $L^2$-type spaces, since the extension to $H^1$-type spaces is natural. Consider $u$ a radial solution in $L^2_{loc,u}$ in $\mathbb{R}^d$ and introduce $\tilde{u}$ such that $u(x) = \tilde{u}(r)$ with $r = |x|$, $\forall x \in \mathbb{R}^d$.

Let $A = \sup_{x_0 \in \mathbb{R}^d} \int_{B(x_0,1)} |u(x)|^2 dx$ the square of the $L^2_{loc,u}$ norm in $\mathbb{R}^d$ and $B = \sup_{r_0 \geq 1} \frac{1}{r_0^{d-1}} \int_{r_0-1}^{r_0+1} |\tilde{u}(r)|^2 r^{d-1} dr$. We also define for the crown $C(r_0,1)$ by

$\forall r_0 \geq 1, C(r_0,1) = \{x \in \mathbb{R}^d, |r_0 - 1 \leq |x| < r_0 + 1\}$.

We aim at proving that the square root of $B$ is an equivalent norm to the $L^2_{loc,u}$ in the radial setting. More precisely, we have the following:
Lemma A.1.  

i) \( \exists \alpha(d) > 0 \) such that \( A \leq \alpha(d)B \).

ii) \( \exists \beta(d) > 0 \) such that \( B \leq \beta(d)A \).

Proof.  

i) It is enough to show that for any \( x_0 \in \mathbb{R}^d \),

\[
\int_{B(0,2)} |u(x)|^2 dx \leq \bar{\alpha}(d)B, \text{ for some } \bar{\alpha}(d) > 0.
\]

Consider \( x_0 \in \mathbb{R}^d \). If \( |x_0| < 1 \) and \( x \in B(x_0,1) \) then \( |x| < |x_0| + 1 < 2 \). Consequently,

\[
\int_{B(x_0,1)} |u(x)|^2 dx \leq \int_{B(0,2)} |u(x)|^2 dx = \omega_{d-1} \int_0^2 |\bar{u}(r)|^2 r^{d-1} dr \leq \omega_{d-1} B;
\]

where \( \omega_{d-1} \) is the volume of the sphere \( S^{d-1} \). Now, if \( |x_0| \geq 1 \), then we have \( B(x_0,1) \subset C(|x_0|,1) \). Furthermore, for geometric considerations, we know that there exists \( \alpha(d,|x_0|) > 0 \) such that the crown \( C(|x_0|,1) \) contains \( \alpha(d,|x_0|) r_0^{d-1} > 0 \) disjoint copies of \( B(x_0,1) \), with

\[
\alpha(d,|x_0|) \equiv \alpha_0(d) r_0^{d-1} \quad \text{as} \quad r_0 \to +\infty \quad \text{for some} \quad \alpha_0(d) > 0. \tag{A.1}
\]

If we denote by \( x_i \) for \( i \in \{0, \ldots, \alpha - 1\} \) the centers of those balls, then we have

\[
\int_{\bigcup_{i=0}^{\alpha-1} B(x_i,1)} |u(x)|^2 dx \leq \int_{C(|x_0|,1)} u(r)^2 r^{d-1} dr
\]

\[
= \omega_{d-1} \int_{r_0}^{r_0+1} |\bar{u}(r)|^2 r^{d-1} dr \leq \omega_{d-1} B r_0^{d-1}, \tag{A.2}
\]

on the one hand. On the other hand, since the difference between the two crown’s radii is 2 and the balls are of radius 1, it follows that

\[
|x_i| = |x_0|, \quad \forall i \in \{0, \ldots, \alpha - 1\} \tag{A.3}
\]

Since \( u \) is radial and the balls \( B(x_i,1) \) are disjoint, using (A.3) we see that

\[
\int_{\bigcup_{i=0}^{\alpha-1} B(x_i,1)} |u(x)|^2 dx = \alpha(d,r_0) \int_{B(x_0,1)} |u(x)|^2 dx.
\]

Combining this with (A.2) and (A.1), we conclude the proof of item i).

ii) Consider \( r_0 \geq 1 \). From geometric considerations, there exists \( \beta(d,r_0) > 0 \) such that the crown \( C(r_0,1) \) is contained in \( \beta(d,r_0) \) copies of \( B(0,1) \), with

\[
\beta(d,r_0) \equiv \beta_0(d) r_0^{d-1} \quad \text{as} \quad r_0 \to +\infty \quad \text{for some} \quad \beta_0(d) > 0. \tag{A.4}
\]

Denoting by \( y_i \) for \( i \in \{0, \ldots, \beta - 1\} \) the centers of those balls, we have

\[
\frac{1}{r_0^{d-1}} \int_{r_0-1}^{r_0+1} |\bar{u}(r)|^2 r^{d-1} dr = \frac{1}{\omega_{d-1} r_0^{d-1}} \int_{C(|x_0|,1)} |u(x)|^2 dx
\]

\[
\leq \frac{1}{\omega_{d-1} r_0^{d-1}} \sum_{i=0}^{\beta-1} \int_{B(y_i,1)} |u(x)|^2 dx \leq \frac{\beta(d,r_0)}{\omega_{d-1} r_0^{d-1}} A.
\]
Using (A.4), we conclude the proof of item ii). □

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