Constraining power of open likelihoods, made prior-independent

S. Gariazzo
Instituto de Física Corpuscular (CSIC-Universitat de València)
Parc Científic UV, C Catedrático José Beltrán, 2, E-46980 Paterna (Valencia), Spain

Abstract

One of the most criticized features of Bayesian statistics is the fact that credible intervals, especially when open likelihoods are involved, may strongly depend on the prior shape and range. Many analyses involving open likelihoods are affected by the eternal dilemma of choosing between linear and logarithmic prior, and in particular in the latter case the situation is worsened by the dependence on the prior range under consideration. In this letter, using the tools of Bayesian model comparison, we propose a simple method to obtain constraints that depend neither on the prior shape nor range. An application to the case of cosmological bounds on the sum of the neutrino masses is discussed as an example.

Keywords: Bayesian statistics, Neutrino masses, Cosmology

1. Introduction

In several cases, physics experiments try to measure unknown quantities: the mass of some particle, a new coupling constant, the scale of new physics. Most of the times, the absolute scale of such new quantities is completely unknown, and the analyses of experimental data require to scan a very wide range of values for the parameter under consideration, to finally end up with a lower or upper bound when data are compatible with the null hypothesis.

In the context of Bayesian analysis, performing this kind of analysis implies a profound discussion on the choice of the considered priors, which may be logarithmic when many orders of magnitude are involved. A robust analysis usually shows what happens when more than one type of prior is considered, but the calculation of credible intervals always require also a precise definition of the prior range. Especially in the case of logarithmic priors, a choice of the range can be difficult even when physical boundaries (e.g. a mass or coupling must be positive) exist, with the consequence that the selected allowed range for the parameter can influence the available prior volume and as a consequence the bound itself.

Let us consider for example the case of neutrino masses and their cosmological constraints. Current data are sensitive basically only on the sum of the neutrino masses and not on the single mass eigenstates (see e.g. [1][2]). There are therefore good reasons to describe the physics by means of $\Sigma m_\nu$, and to consider a linear prior on it, as the parameter range is limited from below by oscillation experiments [3][4][5] and from above by KATRIN [6]. Even given these considerations, however, one can decide to perform the analysis considering a lower limit $\Sigma m_\nu > 0$ [7], instead of enforcing the oscillation-driven one, $\Sigma m_\nu \gtrsim 60 (100) \text{ meV}$ (respectively for normal and inverted ordering of the neutrino masses, see e.g. [8][9]): the obtained upper bounds will differ in the various cases.

In order to overcome these problems, in this letter we revisit a simple way to use Bayesian model comparison techniques to obtain prior-independent constraints, which can be useful for an easier comparison of the constraining power of various experimental results, not only in the context of cosmology, but in all Bayesian analyses in general.

2. Prior-free Bayesian constraints

The foundation of Bayesian statistics is represented by the Bayes theorem:

$$p(\theta|d, \mathcal{M}_i) = \frac{\pi(\theta|\mathcal{M}_i) \mathcal{L}_{\mathcal{M}_i}(\theta)}{Z_i},$$

where $\pi(\theta|\mathcal{M}_i)$ and $p(\theta|d, \mathcal{M}_i)$ are the prior and posterior probabilities for the parameters $\theta$ given a model $\mathcal{M}_i$, $\mathcal{L}_{\mathcal{M}_i}(\theta)$ is the likelihood function, depending on the parameters $\theta$, given the data $d$ and the model $\mathcal{M}_i$, and

$$Z_i = \int_{\Omega_\theta} d\theta \pi(\theta|\mathcal{M}_i) \mathcal{L}_{\mathcal{M}_i}(\theta)$$

is the Bayesian evidence of $\mathcal{M}_i$ [10], the integral of prior times likelihood over the entire parameter space $\Omega_\theta$.

While the Bayes theorem indicates how to obtain the posterior probability as a function of all the model parameters $\theta$, when presenting results we are typically interested in the marginalized posterior probability as a function of one parameter (or two), which we can generally indicate with $x$. The marginalization is performed over the remaining parameters, which we can indicate with $\psi$:

$$p(x|d, \mathcal{M}_i) = \int_{\Omega_\psi} d\psi p(x, \psi|\mathcal{M}_i, d).$$
Let us now assume that the prior is separable and we can write
\[ p(x|d, M_i) = \pi(x|M_i) \cdot \pi_\psi|d, M_i) \].
Under such hypothesis, Eq. (3) can be written as:
\[ p(x|d, M_i) = \frac{\pi(x|M_i)}{Z_i} \int_{\Omega_p} d\psi \pi_\psi|d, M_i) L_M(x, \psi). \quad (4) \]

Let us consider the marginalized posterior as written in Eq. (2). The prior dependence is only present explicitly outside the integral, and therefore we can obtain a prior-independent quantity \[ \text{just dividing the posterior by the prior. The right-hand side of Eq. (2), however, has an explicit dependence on the value of } x \text{ through the likelihood that appears in the integral. We can note that such integral resembles the definition of the Bayesian evidence in Eq. (2), not anymore for model } M_i, \text{ but for a sub-case of } M_i \text{ which contains } x \text{ as a fixed parameter. Let us label this model with } M_i^0 \text{ and, for sake of brevity, let us define its Bayesian evidence}
\[ Z_i^0 \equiv \int_{\Omega_p} d\psi \pi\psi|M_i^0) L_M(x, \psi), \quad (5) \]
which is independent of the prior \[ \pi(x) \], but still depends on the parameter value \[ x \], now fixed.

Now, let us consider two models \[ M_i^0 \] and \[ M_i^2 \]. We can use the Bayes theorem applied to such models and compare their posteriors:
\[ \frac{p(M_i^0)}{p(M_i^2)} = B_{x_1 x_2} \frac{\pi(M_i^0)}{\pi(M_i^2)}, \quad (6) \]
where we indicate with \[ \pi(M_i^j) \], \[ p(M_i^j) \] the prior and posterior of model \[ M_i^j \], and as usual the Bayes factor is:
\[ B_{x_1 x_2} \equiv \frac{Z_i^0}{Z_i^2}. \quad (7) \]

Few considerations are needed. Since the underlying model \[ M_i \] is the same and only the value of \[ x \] changes, the prior \[ \pi(M_i^0) \] is proportional to the prior of \[ x \] within model \[ M_i \], \[ \pi(x|M_i) \]. In the same way, the model posterior \[ p(M_i^0) \] can be written as the model posterior of \[ M_i \], \[ p(M_i) \], multiplied by the parameter posterior \[ p(x|d, M_i) \]. When taking the ratio, therefore, we get:
\[ \frac{p(x_1|d, M_i)}{p(x_2|d, M_i)} = B_{x_1 x_2} \frac{\pi(x_1|M_i)}{\pi(x_2|M_i)}. \quad (8) \]

Notice that the same equation can be obtained by rewriting Eq. (4) using the definition in Eq. (5), once for \[ x_1 \] and once for \[ x_2 \], and dividing \[ p(x_1|d, M_i) \) by \[ p(x_2|d, M_i) \].

For reasons that will be clear in a moment, let us now rewrite this last equation using \[ x = x_1 \] and \[ x = x_2 \], and move some of the terms between the two sides. The quantity \[ B_{x_1 x_2} \] indicated with \[ R(x, x_0|d) \], has been named “relative belief updating ratio” or “shape distortion function” in the past [11][12][13].
\[ R(x, x_0|d) \equiv B_{x_1 x_2} = \frac{p(x|d, M_i)}{\pi(x|M_i)} / \frac{p(x_0|d, M_i)}{\pi(x_0|M_i)} = \frac{Z_i^0}{Z_i^2}. \quad (9) \]

As we can see from the last equality, which is useful to understand the meaning of \[ R \] but unpractical to compute it, and since \[ \pi(x|M_i) \) does not enter Eq. (5), \[ R(x, x_0|d) \) is completely independent of the prior on the parameter \[ x \]. This quantity therefore represents a prior-independent way to compare some results concerning two values of some parameter \[ x \]. At the practical level, \[ R \] is particularly useful when dealing with open likelihoods, i.e. when data only constrain the value of some parameter from above or from below. In such case, the likelihood becomes insensitive to the parameter variations below (or above) a certain threshold. Let us consider for example the absolute scale of neutrino masses, on which data (either cosmological or at laboratory experiments) only put an upper limit: the data are insensitive to the value of \[ x \] when \[ x \] goes towards 0, so we can consider \[ x_0 = 0 \] as a reference value. Regardless of the prior, when \[ x \] is sufficiently close to \[ x_0 \] the likelihoods in \[ x \] and \[ x_0 \] are essentially the same in all the points of the parameter space \[ \Omega_p \], so \[ Z_i^0 \approx Z_i^0 \] and \[ R(x, x_0) \to 1 \]. In the same way, when \[ x \] is sufficiently far from \[ x_0 \], the data penalize its value \[ (Z_i^0 \ll Z_i^0) \] and we have \[ R(x, x_0) \to 0 \]. In the middle, the function \[ R \] indicates how much \[ x \] is favored/disfavored with respect to \[ x_0 \] in each point, in the same way a Bayes factor indicates how much a model is preferred with respect to another one.

While \[ R \] can define the general behavior of the posterior as a function of \[ x \], any probabilistic limit one can compute will always depend on the prior shape and range, which is an unavoidable ingredient of Bayesian statistic. The description of the results through the \[ R \] function, however, allows to use the data to define a region above which the parameter values are disfavored, regardless of the prior assumptions, and also to guarantee an easier comparison of two experimental results. A good standard could be to provide a (non-probabilistic) “sensitivity bound”, defined as the value of \[ x \] at which \[ R \] drops below some level, for example \[ \ln |R| = 1, 3 \text{ or } 5 \text{ in accordance to the Jeffreys’ scale (see e.g. [2], [10])}. \]

Let us consider \[ x_0 = 0 \] as above: we could say, for example, that we consider as “moderately (strongly) disfavored” the region \[ x > x_0 \] for which \[ \ln R < s \], with \[ s = -3 \text{ (or } -5 \text{) } \], and then use the different values \[ s \], to compare the strengths of different data combinations \[ d \] in constraining the parameter \[ x \]. This will not represent an upper bound at some given confidence level, since it is not a probabilistic bound, but rather a hedge “which separates the region in which we are, and where we see nothing, from the the region we cannot see” [12].

Notice also that, once \[ R \] is known, anyone can obtain credible intervals with any prior of choice: the posterior \[ p(x|d, M_i) \] can easily be computed using Eq. (5) and normalizing to a total probability of 1 within the prior.

3. A simple example with Planck 2018 chains

To demonstrate a simple example with recent cosmological data, we provide in Fig. 1 the function \[ R(\Sigma m_\nu, 0) \] computed in few cases, obtained from the publicly available Planck 2018...
the quantity \( x \) according to Eq. (9). Apart from the normalization constant, \( p \) is present in plots where cosmology, since the results for 1-dimensional marginalized posteriors are often presented in plots where \( \Sigma \) prior on \( x \) for bounds calculations as if a linear prior on \( x \) was considered, or as a shape distortion function, therefore not suitable to compute limits unless some prior is assumed.

Regardless of considering a cut at \( R = e^{-3} \) or \( R = e^{-5} \), indeed, the value of the sensitivity bound only depends on the inclusion of the BAO data. A comparison of the CMB dataset without (P18) or with (P18+BAO) the BAO constraints, therefore, can be summarized by two numbers, considering for example \( s = -5 \):

\[
\begin{align*}
\Sigma m_{\nu,-5} &= 0.4 \text{ eV } \text{(P18)}, \\
\Sigma m_{\nu,-5} &= 0.2 \text{ eV } \text{(P18+BAO)}.
\end{align*}
\]

4. The case of multiple models

In the previous sections we discussed the case when dealing with only one model. The situation is slightly different when more models are considered, for example if one wants to study and take into account several extensions of the same minimal scenario, as in Ref. [22]. It is not difficult to rewrite the definition of \( R \) to deal with multiple models, if we assume that the prior for the parameter \( x \) under consideration is the same in all of them, i.e. that \( \pi(x) \equiv \pi(x|M_i) \) does not depend on \( M_i \).

Let us now recall the method proposed in [22]. The model-marginalized posterior distribution of the parameter \( x \) is obtained as

\[
p(x|d) = \sum_i p(x|M_i,d) p(M_i|d),
\]

where \( p(M_i|d) \) is the posterior probability of the model \( M_i \), which can be computed using [23]

\[
p(M_i|d) = \frac{Z_i \pi(M_i)}{\sum_j Z_j \pi(M_j)}.
\]

In both cases the sum runs over all the studied models. Coming back to Eq. (12) and using Eqs. (4) and (13), we obtain the fully (prior- and model-) marginalized posterior probability of \( x \):

\[
p(x|d) = \sum_i \frac{\pi(x|M_i) Z_i^0 \pi(M_i)}{\sum_j Z_j \pi(M_j)}.
\]

Remembering that we assumed \( \pi(x) \equiv \pi(x|M_0) \) to be independent of \( M_i \), the ratio between prior and marginalized posterior probabilities for the parameter \( x \) is:

\[
\frac{p(x|d)}{\pi(x)} = \frac{\sum_i Z_i^0 \pi(M_i)}{\sum_j Z_j \pi(M_j)}.
\]

If we use this result to define \( R \) again as in equation (9), we have:

\[
R(x_0|d) \equiv \frac{p(x|d)/\pi(x)}{p(x_0|d)/\pi(x_0)} = \frac{\sum_i Z_i^0 \pi(M_i)}{\sum_j Z_j \pi(M_j)}.
\]

The marginalization over the parameters \( \phi \) is not necessarily the same in all the models. As we are not assuming anything on \( \phi \), they can be not the shared ones among the various models and can vary in number. In any case, the marginalization works inside each model independently, using for each \( M_i \) the appropriate parameter space and priors: the differences remain hidden in the definition of \( Z_i^0 \).
which has exactly the same meaning as before, apart for the fact that in this case $R$ has been marginalized over several models.

From the computational point of view, in the model-marginalized case obtaining $R$ is as simple as when only one model is considered. One just has to select a prior $\pi(x)$ and a sufficiently broad range, obtain the marginalized posterior probability as in [22], then divide it by the considered prior and normalize appropriately.

As an example, we provide in Fig. 2 the $R$ function obtained from the vary same posteriors studied in Ref. [22]. Such cases are computed considering the full Planck 2015 (P15) CMB data [24,25], together with the lensing likelihood [26] and the BAO observations by from the SDSS BOSS [24,25], together with the lensing likelihood [26] and the BAO [24,25].

The considered models are the same extensions of the $\Lambda$CDM+$\Sigma m_\nu$, case adopted by the Planck collaboration for the 2015 public release, but with a prior $\Sigma m_\nu > 60$ meV. Also in this case we can see how the $R$ function is very close to one below 0.1 eV and always goes to zero above $\sim 0.7$ eV. In the middle, the various models (dashed lines) have different constraining powers, whose weighted average is represented by the solid line. The model-marginalized, prior-independent result corresponds to

$$\Sigma m_{\nu \leq 5} = 0.6 \ eV \quad \text{(P15+lensing+BAO)}. \quad (17)$$

5. Discussion and conclusions

In this letter we discussed a possible way to show prior-independent results in the context of Bayesian analysis. The method uses Bayesian model comparison techniques to compare the constraining power of the data at different values of the considered parameter, and is particularly useful when open likelihoods are involved in the analysis. We applied the simple formulas to the case of neutrino mass constraints from cosmology, discussing the case of several datasets analyzed with one single cosmological model, and the case where we have only one dataset but multiple models. In the latter case, Bayesian model comparison also allows to take into account the constraints from the different models to obtain a prior-independent and model-marginalized bound. An extended application of this method is left for a separate work.

Acknowledgments

The author receives support from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie individual grant agreement No. 796941, and by the Spanish grants FPA2017-90566-RED (Red Consolider MultiDark), FPA2018-85216-P, SEV-2014-0398 (AEI/1FEDER, UE, MINECO), PROMETEO/2018/165 and GV2016-142 (Generalitat Valenciana).

References

[1] M. Lattanzi, M. Gerbino, Status of neutrino properties and future prospects - Cosmological and astrophysical constraints, Front.in Phys. 5 (2018) 70. arXiv:1712.07109 [doi:10.3389/fphy.2017.00070]
[2] P. De Salas, S. Gariazzo, O. Mena, C. Ternes, M. Tórtola, Neutrino Mass Ordering from Oscillations and Beyond: 2018 Status and Future Prospects, Front.Astron.Space Sci. 5 (2018) 36. arXiv:1806.11051 [doi:10.3389/fspas.2018.00036]
[3] F. Capozzi, E. Lisi, A. Marrone, A. Palazzo, Current unknowns in the three neutrino framework, Prog.Part.Nucl.Phys. 102 (2018) 48–72. arXiv:1804.09678 [doi:10.1016/j.ppnp.2018.05.005]
[4] P. de Salas, D. Forero, C. Ternes, M. Tórtola, J. Valle, Status of neutrino oscillations 2019: J $\sigma$ hint for nonadiabatic mass ordering and improved CP sensitivity, Phys.Lett. B 795 (2019) 109. arXiv:1904.03880 [doi:10.1016/j.physletb.2019.03.018]
URL https://doi.org/10.1016/j.physletb.2018.06.019
[5] I. Esteban, M. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni, T. Schwetz, Global analysis of three-flavour neutrino oscillations: synergies and tensions in the determination of $\theta_{23}$, $\delta_{CP}$, and the mass ordering, JHEP 1901 (2019) 106. arXiv:1811.05487 [doi:10.1007/JHEP01(2019)106]
[6] M. Aker, et al., An improved upper limit on the neutrino mass from a direct kinematic method by KATRIN, arXiv:1909.06048
[7] N. Aghanim, et al., Planck 2018 results. VI. Cosmological parameters, A&A 641 (2020) A6
[8] S. Wang, Y.-F. Wang, D.-M. Xia, Constraints on the sum of neutrino masses using cosmological data including the latest Baryon Oscillation Spectroscopic Survey DR14 quasar sample, Chin.Phys.C42 (2018) 060513.
[9] S. Wang, Y.-F. Wang, D.-M. Xia, Z. Zhang, Impacts of dark energy on weighing neutrinos: mass hierarchies considered, Phys.Rev.D 94 (8) (2016) 083519. arXiv:1608.00672 [doi:10.1103/PhysRevD.94.083519]
[10] R. Trotta, Bayes in the sky: Bayesian inference and model selection in cosmology, Contemp.Phys. 49 (2008) 71–104. arXiv:0803.4089 [doi:10.1080/00107510802066753]
[11] P. Astone, G. D’Agostini, Inferring the intensity of Poisson processes at the limit of the detector sensitivity (with a case study on gravitational wave burst search), Annals Phys.
[12] G. D’Agostini, Confidence limits: What is the problem? Is there the solution?, in: Workshop on confidence limits, CERN, Geneva, Switzerland, 17-18 Jan 2000: Proceedings, 2000, pp. 3–27. arXiv:hep-ex/0002055
[13] G. D’Agostini, Bayesian Reasoning in Data Analysis, WORLD SCIENTIFIC, 2003. doi:10.1142/9789812778506
URL https://doi.org/10.1142/9789812778506
[14] N. Aghanim, et al., Planck 2018 results. V. CMB power spectra and likelihood, arXiv:1807.12078
[15] N. Aghanim, et al., Planck 2018 results. VIII. Gravitational lensing, arXiv:1807.06210

Figure 2: The $R(\Sigma m_\nu, 0)$ function in Eq. 9 obtained considering different models (dashed) together with the model-marginalized one from Eq. 16 (solid), using the full dataset adopted in Ref. [22] (see text for details). The horizontal lines show the levels in $R = 0, -1, -3, -5$, respectively.
[16] S. Alam, et al., The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample, Mon.Not.Roy.Astron.Soc. 470 (2017) 2617–2652. arXiv:1607.03155 doi:10.1093/mnras/stx721

[17] F. Beutler, et al., The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: Baryon Acoustic Oscillations in Fourier-space, Mon.Not.Roy.Astron.Soc. 464 (2017) 3409–3430. arXiv:1607.03149 doi:10.1093/mnras/stv2373

[18] A. J. Ross, et al., The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: Observational systematics and baryon acoustic oscillations in the correlation function, Mon.Not.Roy.Astron.Soc. 464 (2017) 1168–1191. arXiv:1607.03145 doi:10.1093/mnras/stx721

[19] M. Vargas-Magaña, et al., The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: theoretical systematics and Baryon Acoustic Oscillations in the galaxy correlation function, Mon.Not.Roy.Astron.Soc. 477 (2018) 1153–1188. arXiv:1610.03506 doi:10.1093/mnras/sty571

[20] F. Beutler, C. Blake, M. Colless, D. H. Jones, L. Staveley-Smith, L. Campbell, Q. Parker, W. Saunders, F. Watson, The 6dF Galaxy Survey: Baryon Acoustic Oscillations and the Local Hubble Constant, Mon.Not.Roy.Astron.Soc. 416 (2011) 3017–3032. arXiv:1106.3366 doi:10.1111/j.1365-2966.2011.19250.x

[21] A. J. Ross, L. Samushia, C. Howlett, W. J. Percival, A. Burden, M. Mandrera, The clustering of the SDSS DR7 main Galaxy sample - I. A 4 per cent distance measure at $z = 0.15$, Mon.Not.Roy.Astron.Soc. 449 (1) (2015) 835–847. arXiv:1409.3242 doi:10.1093/mnras/stv154

[22] S. Gariazzo, O. Mena, Cosmology-marginalized approaches in Bayesian model comparison: the neutrino mass as a case study arXiv:1812.05449

[23] W. J. Handley, M. P. Hobson, A. N. Lasenby, PolyChord: next-generation nested sampling, Monthly Notices of the Royal Astronomical Society 453 (4) (2015) 4384. arXiv:1506.00171 doi:10.1093/mnras/stv1911

[24] R. Adam, et al., Planck 2015 results. I. Overview of products and scientific results, Astron.Astrophys. 594 (2016) A1. arXiv:1502.01582 doi:10.1051/0004-6361/201527101

[25] P. A. R. Ade, et al., Planck 2015 results. XIII. Cosmological parameters, Astron.Astrophys. 594 (2016) A13. arXiv:1502.01589 doi:10.1051/0004-6361/201525830

[26] P. A. R. Ade, et al., Planck 2015 results. XV. Gravitational lensing, Astron.Astrophys. 594 (2016) A15. arXiv:1502.01591 doi:10.1051/0004-6361/201525941

[27] L. Anderson, et al., The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: baryon acoustic oscillations in the Data Releases 10 and 11 Galaxy samples, Mon.Not.Roy.Astron.Soc. 441 (1) (2014) 24–62. arXiv:1312.4877 doi:10.1093/mnras/stu523