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Dynamic Evolution of a Quasi-Spherical General Polytropic Magnetofluid with Self-Gravity

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Abstract In various astrophysical contexts, we analyze self-similar behaviours of magnetohydrodynamic (MHD) evolution of a quasi-spherical polytropic magnetized gas under self-gravity with the specific entropy conserved along streamlines. In particular, this MHD model analysis frees the scaling parameter $n$ in the conventional polytropic self-similar transformation from the constraint of $n + \gamma = 2$ with $\gamma$ being the polytropic index and therefore substantially generalizes earlier analysis results on polytropic gas dynamics that has a constant specific entropy everywhere in space at all time. On the basis of the self-similar nonlinear MHD ordinary differential equations, we examine behaviours of the magnetosonic critical curves, the MHD shock conditions, and various asymptotic solutions. We then construct global semi-complete self-similar MHD solutions using a combination of analytical and numerical means and indicate plausible astrophysical applications of these magnetized flow solutions with or without MHD shocks.

Keywords clusters of galaxies · magnetohydrodynamics · stars: formation · stars: neutron · stars: winds, outflows · supernovae: general

1 Introduction

Dynamic evolutions of various astrophysical shock flows under gravity on completely different spatial and temporal scales have been attracting astrophysicists, mainly because of a rich variety of observational phenomenology in different astrophysical systems, such as stellar collapses, supernova explosions and supernova remnants (e.g., Bethe et al. 1979; Goldreich & Weber 1980; Chevalier 1982; Yahil 1983; Lattimer & Prakash 2004; Lou & Wang 2006, 2007; Lou & Cao 2008; Lou & Hu 2008 in preparation), star formation, proto-stellar core formation as well as ‘champagne flows’ in star-forming molecular clouds (e.g., Shu 1977; Shu et al. 1987; Tsai & Shu 1995; Shu et al. 2002; Shen & Lou 2004; Bian & Lou 2005; Lou & Gao 2006), formation and spherical accretions of black holes (e.g., Bondi 1952; Bahcall & Ostriker 1975; Henawi & Ostriker 2002; Cai & Shu 2005; Hu et al. 2006), active galactic nuclei (AGNs) at the galactic level (e.g., Small & Blandford 1992) and the formation of galaxy clusters and voids in the Universe (e.g., Gunn & Gott 1972; Fillmore & Goldreich 1984a, b; Bertschinger 1985) as well as dynamic galaxy cluster winds (Lou, Jiang & Jin 2008; Jiang & Lou 2008 in preparation). This broad class of theoretical models poses challenges for a deeper and simpler theoretical understanding of relevant astrophysical shock flows under consideration. Meanwhile, these problems also involve a diverse range of physical processes and the relevant model investigations often involve numerical hydrodynamic simulations with data input from high-energy physics experiments and/or theories (e.g., Janka & Müller 1996 for supernovae).

In this paper, we mainly focus on basic hydrodynamic and magnetohydrodynamic (MHD) aspects of this class of problems because of the relative simplicity and generality, allowing for various novel and interesting solution features to manifest. We introduce the general polytropic description to subsume several unspecified energetic processes into a few index parameters. There might be certain evolution phases where a simple and direct anal-
ysis works almost as well as numerical simulations do (e.g., Janka & Müller 1996), and in this situation such an analysis would be extremely valuable for physical insight (e.g., Yahil 1983) and for testing numerical codes in construction. In this spirit, we advance a theoretical MHD flow model with this kind of simplification (e.g., self-similarity) and idealization (e.g., a completely random transverse magnetic field and specific entropy conservation along streamlines) to reveal plausible MHD behaviours and shocks of magnetized gas flows of astrophysical interests.

While self-similar evolutions of gas dynamics in spherical geometry have been actively pursued early on (e.g., Sedov 1959; Bodenheimer & Sweigart 1968; Larson 1969a, b; Penston 1969a, b), extensively analyzed by many in diverse astrophysical contexts (e.g., Shu 1977; Hunter 1977; Cheng 1978; Goldreich & Weber 1980; Chevalier 1982; Yahil 1983; Whittworth & Summers 1985; Tsai & Hsu 1995; Shu et al. 2002; Shen & Lou 2004; Fatuzzo et al. 2004; Lou & Gao 2006) and significantly reformulated in various physical aspects (e.g., Terebey, Shu & Cassen 1984; Suto & Silk 1988; Chiueh & Chou 1994; McLaughlin & Putritz 1997; Boley & Lynden-Bell 1995; Cai & Shu 2005), we have found novel features and generalized similarity solutions for this type of classical nonlinear problems in recent years. As examples, our new similarity solution features include envelope expansion with core collapse (EECC) solutions which smoothly pass through the sonic critical line twice (Lou & Shen 2004; Lou & Gao 2006), quasi-static asymptotic solutions (Lou & Wang 2006), quasi-static asymptotic MHD solutions (Lou & Wang 2007; Wang & Lou 2007), quasi-static asymptotic solutions in two gravity-coupled fluids (Lou, Jiang & Jin 2008; Jiang & Lou 2008 in preparation), strong magnetic field solutions (Yu & Lou 2006; Wang & Lou 2007), and various new (MHD) shock solutions (Shen & Lou 2004; Bian & Lou 2005; Yu et al. 2007).

In addition to the known Larson-Penston (LP) type solutions and central free-fall solution (Shu 1977), the existence of several new nonlinear self-similar MHD polytropic flow solutions reveals multiple clues and leads to new concepts.

First of all, knowing these self-similar nonlinear solutions, we realize that a number of self-similar behaviours may possibly occur in seemingly similar flow systems, and the evolution problem of which goes where remains an important open question. Clearly, this cannot be immediately answered within the self-similar framework. While it is generally thought that self-similar behaviour gradually emerges when a dynamical system evolves sufficiently far away from its initial and boundary conditions, we would emphasize that initial and boundary conditions do affect the nonlinear self-similar behaviour in the sense that similarity behaviour emerges eventually. In other words, the final self-similar phase still retains memory of a certain class of initial and boundary conditions. As an example, Tsai & Hsu (1995) introduced a self-similar behaviour that has the outer (initial) configuration the same as that of Shu (1977), but their inner (final) behaviour is qualitatively different from that of Shu (1977): the model of Tsai & Hsu (1995) includes a self-similar isothermal shock and also the central mass accretion rate differs from that of Shu (1977). As suggested by simulation results of Tsai & Hsu (1995), a stronger push at the centre of a static isothermal sphere may result in a shock, while a weaker push will lead to an isothermal expansion-wave collapse solution (EWCS; see their Figs. 1 – 3).

Secondly, although former nonlinear self-similar solutions were often obtained from numerical simulations first (e.g., Larson 1969a, b; Penston 1969a, b), the very existence of different self-similar behaviours in the same physical setting would imply that so far numerical hydrodynamic simulations might have not been sufficiently thorough in exploring the overall dynamic evolution features of nonlinear flows with or without shocks. These semi-analytical solutions are important clues and benchmarks for numerical code development. More importantly, numerical simulations can further tell whether new self-similarity solutions obtained in our analysis can be actually realized under realistic flow situations, by introducing different initial and boundary conditions for a similar flow system and by observing which similarity behaviour does it manifest eventually.

Finally, it is also possible that one flow system may not behave self-similarly in the strict sense, nonetheless its evolution can be qualitatively explained by intuitions gained by examining various self-similar solutions: for example, the central free-fall phase of Shu (1977) is common for a wide range of polytropic index $\gamma$; the existence of other new solutions requires specialized conditions. Physically, our new asymptotic solutions have different features in terms of the balance and competition of the involved forces, which tells us about the possible asymptotic flow behaviours.

As an essential ingredient in our theoretical model development, magnetic fields are important in various astrophysical settings; their presence in ionized plasma systems is ubiquitous and they play dynamical as well as diagnostic roles in many ways: they are responsible for sporadic violent activities and for producing relativistic particles such as cosmic rays through shocks and reconnection. The generation of magnetic fields is related to convective motions, sustained turbulence and differential rotations, via dynamo effects or magnetorotational instabilities (MRI) (e.g., Parker 1979; Thompson & Duncan 1993 for dynamo effects of magnetized pulsars; Chandrasekhar 1961; Balbus & Hawley 1998; Balbus 2003 for disc magnetorotational instabilities). For the purpose of our formulation here, we take the presence of magnetic field for granted and simply presume that a completely random magnetic field permeates in a gas medium with self-gravity. This gas medium can be stellar interior, hot
stellar coronae, interstellar medium (ISM), or intracluster medium (ICM) and so forth.

During the collapse phase of a quasi-spherical system under self-gravity, a highly conducting magnetofluid undergoes a magnetic field enhancement through the magnetic flux conservation, commonly referred to as the frozen-in condition for magnetic field. When rotation is sufficiently slow in an astrophysical system, the overall geometry may remain quasi-spherical and our model analysis with the quasi-spherical random-field approximation (e.g., Zel’dovich & Novikov 1971) can be applicable (Yu & Lou 2005; Yu et al. 2006; Lou & Wang 2007; Wang & Lou 2007; Lou & Hu 2008, in preparation). Chiu et al. & Chou (1994) first studied a quasi-spherical isothermal MHD problem by including the magnetic pressure gradient into the radial momentum equation. However, the magnetic tension force does not average to zero (Yu & Lou 2005; Wang & Lou 2007) and thus should be also included. Although on small scales, magnetic tension force tends to drive inhomogeneous gas motions, we presume that these non-spherical flows may be neglected as compared to the large-scale radial bulk motion of gas. The key point is that the (self-)gravity is strong enough to hold on the entire gas mass and induce core collapse. Hence on large scales in a quasi-spherical geometry, a completely random magnetic field contributes to the dynamics in the form of the average magnetic pressure gradient force and the average magnetic tension force in the radial direction.

In a series of papers during past several years, we have systematically investigated isothermal MHD problems (Yu & Lou 2005; Yu et al. 2006) and polytropic MHD problems in the special case of a constant specific entropy in space at all time (i.e., \(\rho = \kappa \rho^\gamma\) with a constant \(\gamma\), a pressure \(p\), a mass density \(\rho\), and a polytropic index \(\gamma\); Wang & Lou 2007; Lou & Wang 2007; Lou & Hu 2008, in preparation). Under certain circumstances, the isothermality may be a crude yet sensible first approximation for astrophysical fluid dynamics with or without magnetic field (e.g., for star formation processes within a collapsing molecular cloud, Shu 1977; hot bubbles and superbubbles in ISM produced by supernovae, Lou & Zhai 2008, in preparation; or central core collapse region of a globular cluster, Bahcall & Ostriker 1975; Inagaki & Lynden-Bell 1983), even for modeling shocks in dynamical processes (e.g., Courant & Friedrichs 1976; Spitzer 1978). In other situations, a polytropic fluid with a constant specific entropy is invoked for theoretical model investigation (e.g., Goldreich & Weber 1980 for a stellar collapse of a relativistically hot gas prior to the rebound process; Yahil 1983 for gravitational core collapses; Suto & Silk 1988; Lou & Cao 2008 for rebound processes in stellar core collapses). While the polytropic approximation with a constant specific entropy is a useful description for various astrophysical processes, there is however no obvious reason why the entropy should necessarily remain constant in a polytropic gas. In dynamic processes, a more general situation involves the ‘specific entropy’ conservation along streamlines; and this includes the constant entropy as a special case. Therefore, self-similar polytropic gas evolution under this ‘equation of state’ represents a significant generalization of the ‘constant entropy’ problem, and can be more adaptable to various dynamic model applications in astrophysics.

Indeed, several authors have adopted such an ‘equation of state’ and carried out their model analyses to various extents (e.g., Cheng 1978; Chevalier 1982; Faituzzo et al. 2004; Wang & Lou 2007; Lou & Cao 2008; Lou & Hu 2008, in preparation). In technical terms, adopting such an ‘equation of state’ is equivalent to freeing the choice of index parameter \(n\) in the usual self-similar transformation equation (e.g., Zel’dovich & Novikov 1971) can be applicable (Yu & Lou 2005; Yu et al. 2006; Lou & Wang 2007; Wang & Lou 2007; Lou & Hu 2008, in preparation). In technical terms, adopting such an ‘equation of state’ is equivalent to freeing the choice of index parameter \(n\) in the usual self-similar transformation equation (e.g., Zel’dovich & Novikov 1971) can be applicable (Yu & Lou 2005; Yu et al. 2006; Lou & Wang 2007; Wang & Lou 2007; Lou & Hu 2008, in preparation). Under certain circumstances, the isothermality may be a crude yet sensible first approximation for astrophysical fluid dynamics with or without magnetic field (e.g., for star formation processes within a collapsing molecular cloud, Shu 1977; hot bubbles and superbubbles in ISM produced by supernovae, Lou & Zhai 2008, in preparation; or central core collapse region of a globular cluster, Bahcall & Ostriker 1975; Inagaki & Lynden-Bell 1983), even for modeling shocks in dynamical processes (e.g., Courant & Friedrichs 1976; Spitzer 1978). In other situations, a polytropic fluid with a constant specific entropy is invoked for theoretical model investigation (e.g., Goldreich & Weber 1980 for a stellar collapse of a relativistically hot gas prior to the rebound process; Yahil 1983 for gravitational core collapses; Suto & Silk 1988; Lou & Cao 2008 for rebound processes in stellar core collapses). While the polytropic approximation with a constant specific entropy is a useful description for various astrophysical processes, there is however no obvious reason why the entropy should necessarily remain constant in a polytropic gas. In dynamic processes, a more general situation involves the ‘specific entropy’ conservation along streamlines; and this includes the constant entropy as a special case. Therefore, self-similar polytropic gas evolution under this ‘equation of state’ represents a significant generalization of the ‘constant entropy’ problem, and can be more adaptable to various dynamic model applications in astrophysics.

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This paper with special emphasis on theoretical aspects is structured as follows. Our motivation and background information are provided in Section 1 as an introduction. Section 2 describes the basic formalism on nonlinear MHD flows with a completely random transverse magnetic field, including the self-similar transformation and MHD shock conditions. Section 3 presents various analytical asymptotic MHD solutions for large and small \(x\) as well as along the magnetosonic critical curve; these asymptotic MHD solutions are valuable for understanding relevant physics and can be utilized to construct global semi-complete solutions which are valid in the range of \(0 < x < +\infty\). Section 4 shows examples of various possible semi-complete global solutions with or without MHD shocks and outlines astrophysical applications of these solution types. We discuss our results in Section 5. Mathematical details are included in several appendices for the convenience of reference.

2 Formulation of the Model Problem

2.1 Nonlinear Magnetohydrodynamic Equations

In spherical polar coordinates \((r, \theta, \phi)\), the basic nonlinear equations for a quasi-spherical MHD evolution of
a general polytropic gas under self-gravity include the mass conservation equation
\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u) = 0 ,
\]
(1)

\[
\frac{\partial M}{\partial t} + u \frac{\partial M}{\partial r} = 0 ,
\]
(2)

\[
\frac{\partial M}{\partial r} = 4\pi r^2 \rho ,
\]
(3)

where \(\rho(r,t)\) is the gas mass density, \(u(r,t)\) is the bulk radial flow speed, \(M(r,t)\) is the enclosed mass within radius \(r\) at time \(t\); the radial momentum equation
\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = -\frac{\partial \rho}{\partial r} - \frac{GM\rho}{r^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( < B_t^2 > \right) - \frac{< B_t^2 >}{4\pi r} ,
\]
(4)

where \(p\) is the thermal gas pressure, \(< B_t^2 >\) is the ensemble average of the random transverse magnetic field \(B_t\) squared, \(G = 6.67 \times 10^{-8} \text{ dyne cm}^2 \text{ g}^{-2}\) is the gravitational constant; the magnetic induction equation
\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) \left( r^2 < B_t^2 > \right) + 2\nu r^2 < B_t^2 > \frac{\partial u}{\partial r} = 0 ,
\]
(5)

and the conservation equation for ‘specific entropy’ \(\ln \rho^\gamma\) along streamlines.

The Poisson equation relating the mass density and the gravitational potential is consistently satisfied under the approximation of a quasi-spherical symmetry. This set of MHD equations is the same as that of Wang \& Lou (2007) and Lou \& Wang (2007), except for equation (4), which is the key difference between this analysis and our former analyses. In other words, the equation of state \(p = \kappa \rho \gamma\) with \(\kappa\) being a constant is only one special case satisfying equation (4) and is referred to as the ‘usual’ or ‘conventional’ polytropic equation of state. Equation of state (4) has been adopted in various previous works as noted in Section 1. The assumptions on random magnetic field distribution (Zel'dovich \& Novikov 1971) and magnetic induction are the same as those in Yu \& Lou (2005), Yu et al. (2005), Lou \& Wang (2007) and Wang \& Lou (2007). The work of Yu \& Lou (2005) studies the free-fall collapse of an isothermal magnetized gas cloud (Shu 1977; Chien \& Chou 1994) and is relevant to star formation and to formation of hot bubbles as well as superbubbles in ISM.

\footnote{The concept of ‘specific entropy’ is here extended to situations where \(\gamma\), simply regarded as an index parameter, is not necessarily the ratio of actual gas specific heats.}

contexts of shocks and ‘champagne flows’ (Tsai \& Hsu 1995; Shu et al. 2002; Bian \& Lou 2005), the MHD model of Yu et al. (2005) extends the scenario and analysis to magnetized isothermal clouds. The models of Lou \& Wang (2006, 2007) are designed for rebound shock process in supernovae without or with a random magnetic field and explore the origin of intense magnetic fields on compact objects. Given the above approximations, a proper combination of equations (1) – (6) leads to an MHD energy conservation [see equation (7) of Wang \& Lou (2007) and equation (4.2) of Fan \& Lou (1999)].

By performing the time reversal transformation \(t \rightarrow -t, r \rightarrow r, u \rightarrow -u, \rho \rightarrow \rho, p \rightarrow p, M \rightarrow M\) and \(< B_t^2 > \rightarrow < B_t^2 >\), the above MHD equations are invariant. Using this time reversal invariant property one can construct complete global solutions from semi-complete global solutions (e.g., Lou \& Shen 2004).

2.2 MHD Self-Similarity Transformation

In order to transform nonlinear MHD partial differential equations (1) – (6) to a set of nonlinear MHD ordinary differential equations (ODEs), we introduce the following MHD self-similarity transformation, namely
\[
r = k^{1/2} t^n x , \quad u = k^{1/2} t^{n-1} v , \quad \rho = \frac{\alpha}{4\pi Gt^2} ,
\]
\[
p = \frac{k^{2n-4} \beta}{4\pi Gt^2}, \quad M = \frac{k^{3/2} t^{3n-2m}}{(3n-2)G}, \quad < B_t^2 > = \frac{k^{2n-4} \beta}{G} w .
\]
(7)

Here, \(x\) is a dimensionless independent variable, and \(v(x), \alpha(x), \beta(x), m(x)\) and \(w(x)\) are functions of \(x\) only; \(k\) is a dimensional scaling factor to consistently make \(x, v, \alpha, \beta, m\) and \(w\) dimensionless; \(n\) is an important scaling parameter as noted in Section 1, which essentially controls the scaling of dimensional physical quantities with respect to time \(t\) and radius \(r\). As usual, we refer to \(v(x), \alpha(x), \beta(x), m(x)\) and \(w(x)\) as the reduced radial flow speed, mass density, thermal gas pressure, enclosed mass, and magnetic energy density (associated with the completely random transverse magnetic field), respectively.

Under self-similarity transformation (7), equations (2) and (3) together give the reduced enclosed mass \(m(x)\) as
\[
m = \alpha x^2 (nx - v) ,
\]
(8)

where \(nx - v > 0\) is required in order to ensure that the total enclosed mass remains positive (see Appendix A for more details). Also under the same transformation equation (1), mass conservation equation (1) becomes
\[
(nx - v)\alpha' - \alpha v' = -2(x - v)\alpha/x ,
\]
(9)

while magnetic induction equation (3) reads
\[
(nx - v)w' - 2wv' = 2[v - (2 - n)x]w/x .
\]
(10)
Equations (9) and (10) together lead to the first integral
\[ w = h\alpha^2 x^2, \] (11)
where the integration constant \( h \) is a dimensionless parameter representing the strength of random transverse magnetic field \( B_t \). In essence, relation (11) physically corresponds to the magnetic frozen-in condition. The above treatment parallels those of Wang & Lou (2007) and of Yu & Lou (2005), and the results are the same. Combining equations (8) and (9), the ‘specific entropy’ conservation equation (11) along streamlines is equivalent to
\[ \beta = C\alpha^\gamma m^q, \] (12)
where exponent parameter \( q \equiv 2(n + \gamma - 2)/(3n - 2) \) and hence \( \gamma = 2 - n + (3n - 2)q/2 \). In general, equations (2) and (6) imply that \( pp^{-\gamma} \) (directly related to the ‘specific entropy’) can be a fairly arbitrary function of the enclosed mass \( M(r, t) \) along streamlines. It is our requirement of self-similarity solutions that leads to the specific power-law form of relation (12). The key point of this analysis is that \( q \neq 0 \) or \( n + \gamma \neq 2 \) is allowed in general. As noted by Lou & Cao (2008), here we do not need a proportional coefficient in equation (12) for \( \gamma \neq 4/3 \). The reason is simply that if we write
\[ \beta = C\alpha^\gamma m^q, \] (13)
where \( C \) is an arbitrary scaling coefficient, then a change of parameter \( k \) in self-similarity transformation equation (7) to \( C^{1/(1-3q/2)}k \) would make the coefficient \( C \) disappear. We shall not consider the special case of \( \gamma = 4/3 \) or \( q = 2/3 \), in which there exists yet another parameter as an multiplicative scaling factor between \( \beta \) and \( \alpha^\gamma m^q \) (see Lou & Cao 2008 for a more detailed analysis on the special case of \( \gamma = 4/3 \) which substantially generalized the earlier research analysis of Goldreich & Weber 1980 and Yahil 1983). Taking equations (8), (11) and (12) together, we obtain the following MHD ODE
\[ \alpha^2 - n + 3q/2 x^{2q}(nx - v)^q/\alpha - (nx - v)v' + hx^2 \alpha' = -(n - 1)v - 2h\alpha x - (nx - v)\alpha/(3n - 2). \] (14)
Equations (9) and (11) together lead to separate equations for \( \alpha' \) and \( v' \) respectively in the compact form of
\[ X(x, \alpha, v)\alpha' = A(x, \alpha, v), \quad X(x, \alpha, v)v' = V(x, \alpha, v), \] (15)
where the three functional coefficients \( X, A, \) and \( V \) are defined explicitly below, namely
\[ X(x, \alpha, v) = \left( 2 - n + \frac{3n - 2}{2}q \right) \alpha^{1-n+3q/2} x^{2q}(nx - v)^q + h\alpha x^2 - (nx - v)^2, \]
\[ A(x, \alpha, v) = 2\left( \frac{x - v}{x} \right) \alpha\left[ q\alpha^{1-n+3q/2} x^{2q}(nx - v)^q - (nx - v) \right] - \alpha \left[ (n - 1)v + \frac{(nx - v)}{(3n - 2)}\alpha + 2h\alpha x \right] + q\alpha^{1-n+3q/2} x^{2q-1}(nx - v)^q - 1(3nx - 2v), \]
\[ V(x, \alpha, v) = 2\left( \frac{x - v}{x} \right) \alpha \left[ 2 - n + \frac{3n}{2}q \right] + q\alpha^{1-n+3q/2} x^{2q-1}(nx - v)^q + h\alpha x \]
\[ - (nx - v) \left[ (n - 1)v + \frac{(nx - v)}{(3n - 2)}\alpha + 2h\alpha x \right] + q\alpha^{1-n+3q/2} x^{2q-1}(nx - v)^q - 1(3nx - 2v). \] (16)
Equations (15) and (16) are coupled nonlinear MHD ODEs and with specified ‘boundary’ and ‘initial’ conditions (i.e., appropriate asymptotic solutions at large and small \( x \)), they can be integrated numerically by the standard fourth-order Runge-Kutta method (e.g., Press et al. 1986); one needs to pay special attention on the magnetosonic singular surface where \( X(x, \alpha, v) = 0 \) in both ODEs of (15). The solutions of these two coupled nonlinear MHD ODEs cannot smoothly cross the singular surface unless they cross it with a MHD shock, or they satisfy the critical condition on the so-called magnetosonic critical curve where both the denominator \( X(x, \alpha, v) \) and the numerators \( A(x, \alpha, v) \) and \( V(x, \alpha, v) \) of these two equations in (15) vanish simultaneously. The magnetosonic critical curve can be determined numerically by the specific procedure described in Appendix H and the magnetosonic critical condition is derived and solved numerically in Appendix C. As necessary checks, our analysis here reduces to earlier known results (e.g., Shu 1977; Suto & Silk 1988; Lou & Shen 2004; Yu & Lou 2005; Bian & Lou 2005; Lou & Wang 2006; Wang & Lou 2007) if we set the relevant parameters to appropriate values (e.g., \( q = 0, \gamma = n = 1, \) and \( h = 0, \) and so forth).

2.3 MHD Shock Conditions in Self-Similarity Form

An MHD shock is characterized by discontinuities in thermal gas pressure, mass density, temperature, tangential magnetic field and radial flow velocity, caused by steepening of nonlinear MHD flow evolution or by various explosion and jet processes. In our self-similar and quasi-spherical symmetric formulation for a general polytropic gas, a self-similar MHD shock can be readily constructed mathematically, which represents a possible evolution of an MHD shock in such a magnetized flow system (see e.g., Chevalier 1982; Tsai & Hsu 1995; Shu et al. 2002; Shen & Lou 2004; Bian & Lou 2004; Yu et al. 2006; Lou & Wang 2007 for earlier results on self-similar
shocks either with or without a completely random magnetic field).

To characterize an MHD shock specifically, we need to impose several conservation laws across the two sides of a shock front in the comoving framework of reference. Denoting \( u \) as the flow velocity, \( u_s \) as the shock velocity, \( \rho \) as the gas mass density, \( p \) as the thermal gas pressure, and \( < B^2 > \) as the mean square of the random transverse magnetic field, respectively, we have in the shock framework of reference the mass conservation

\[
[\rho (u_s - u)]^2_1 = 0 , \tag{17}
\]

the radial momentum conservation

\[
\left[ p + \rho (u_s - u)^2 + \frac{< B^2 >}{8\pi} \right]^2_1 = 0 , \tag{18}
\]

the MHD energy conservation equation

\[
\left[ \frac{\rho (u_s - u)^3}{2} + \gamma \rho (u_s - u) + \frac{< B^2 >}{4\pi} (u_s - u) \right]^2_1 = 0 , \tag{19}
\]

and the magnetic induction equation

\[
[(u_s - u)^2 < B^2 >]_1^2 = 0 . \tag{20}
\]

This set of jump conditions for an MHD shock is the same as that adopted in Lou & Wang (2007). We here use a pair of square brackets outside each expression enclosed to denote the difference between the upstream (marked by subscript ‘1’) and downstream (marked by subscript ‘2’) quantities, as has been done conventionally for shock analyses (Landau & Lifshitz 1959; Zel’dovich & Raizer 1966, 1967).

There are two parallel sets of MHD self-similar transformation in the upstream and downstream domains respectively, because the parameter \( k \) in self-similarity transformation equation (7) is related to the sound speed and thus can be different in the two flow regions across a shock (see e.g., Lou & Wang 2006 and Lou & Gao 2006). We shall set \( k_2 = \lambda^2 k_1 \) with the scaling ratio \( \lambda \) representing this difference in the upstream \( k_1 \) and the downstream \( k_2 \). The following relations are necessary for consistency

\[
h_1 = h_2 , \quad x_1 = \lambda x_2 , \tag{21}
\]

and we shall simply use \( h \) instead of \( h_1 \) and \( h_2 \) from now on. Using the two relations in (21), MHD shock jump conditions (17) to (20) can be readily cast into the following self-similarity form

\[
\alpha_1 (nx_1 - v_1) = \lambda \alpha_2 (nx_2 - v_2) , \tag{22}
\]

\[
\alpha_1^{2-n+3nq/2} x_1^{q/2} (nx_1 - v_1)^q + \alpha_1 (nx_1 - v_1)^2 + \frac{h \alpha_1^2 x_1^2}{2} = \lambda^2 \left[ \alpha_2^{2-n+3nq/2} x_2^{q/2} (nx_2 - v_2)^q + 2h \alpha_1^2 x_2^2 \right] , \tag{23}
\]

\[
(nx_1 - v_1)^2 + \frac{2\gamma}{(\gamma - 1)} \alpha_1^{1-n+3nq/2} x_1^{q/2} (nx_1 - v_1)^q + 2h \alpha_1 x_1^2 = \lambda^2 \left[ (nx_2 - v_2)^2 + \frac{2\gamma}{(\gamma - 1)} \alpha_2^{1-n+3nq/2} x_2^{q/2} (nx_2 - v_2)^q + 2h \alpha_2 x_2^2 \right] . \tag{24}
\]

These three self-similar MHD shock equations (22) – (24) can be explicitly solved and relevant details of derivation can be found in Appendix D.

3 Asymptotic MHD Solutions

We connect relevant asymptotic MHD solutions by numerical integration to construct semi-complete global solutions for self-similar MHD flows with or without shocks. Asymptotic analytical MHD solutions are extremely valuable and illustrate the balance and dominance of various forces such as gravity, pressure force and Lorentz force. All MHD solutions obtained here are more general with \( q \neq 0 \) (i.e., \( n + \gamma \neq 2 \)) and are consistent with previous results published in the literature (Shu 1977; Suto & Silk 1988; Lou & Shen 2004; Yu & Lou 2005; Wang & Lou 2007; Lou & Wang 2006, 2007; Lou & Li 2007 in preparation; Lou & Cao 2008).

3.1 MHD Solutions of Pressure Dominance

When the thermal pressure becomes significant in an asymptotic MHD solution, it is generally expected that the pressure term will enter the asymptotic ODEs governing this solution. Therefore, parameter \( q \) will appear in the leading order of the asymptotic MHD solution.

3.1.1 Asymptotic MHD Solutions of Finite Density and Velocity at Large \( x \)

For asymptotic MHD solutions at large \( x \), the gravitational force, magnetic force and thermal pressure gradient force are all in the same order of magnitude, and \( v(x) \) and \( \alpha(x) \) become finite or approach zero in the limit of \( x \to +\infty \). In this case of large \( x \), we have to leading order the following coupled nonlinear MHD ODEs

\[
\alpha' = -\frac{2\alpha}{nx} , \quad v' = \frac{(n-1)v}{nx} + \left[ \frac{1}{(3\alpha - 2)} - \frac{2h(n-1)}{n^2} \right] \alpha - 2(2-n)\alpha q^{-2} \alpha^{1-n+3nq/2} x^{-3q-2} , \tag{25}
\]
and the corresponding asymptotic MHD solution is

$$\alpha = Ax^{-2/n} + \cdots, \quad (26)$$

$$v = Bx^{1-1/n} + \left\{-\left[\frac{n}{(3n-2)} + \frac{2h(n-1)}{n}\right]A \right. +2(2-n)n^{q-1}A^{1-n+3q/2}x^{1-2/n} + \cdots, \quad (27)$$

where $A$ and $B$ are two constants of integration. The radial profile of magnetic field can be readily derived from equation (27). We find that inequality $n > 2/3$ (i.e., a positive enclosed mass $M$) is sufficient to warrant the validity of this asymptotic MHD solution. For $2/3 < n < 1$, the two coefficients $A$ and $B$ can be fairly arbitrary. The special isothermal case (Shu 1977) corresponds to $B = 0$, $q = 0$, $n = 1$, $\gamma = 1$, and $h = 0$; and the $B \neq 0$, $q = 0$, $n = 1$, $\gamma = 1$, $h = 0$ version was implicit in Hunter (1977). The constant coefficient $B$ is a free parameter in this solution, because $v = Bx^{1-1/n}$ itself would satisfy $\partial v/\partial t + u\partial v/\partial r = 0$, as if no force is active. For $n = 1$, the flow approaches a constant radial velocity at the outer portion (Whithworth & Summers 1985; Lou & Shen 2004). This asymptotic solution has been applied to various collapse problems (e.g. Shu 1977; Yahil 1983; Lou & Shen 2004; Bian & Lou 2005; Lou & Wang 2006; Lou & Gao 2006), because of the common feature of inflow, outflow, contraction and/or collapse of dynamic evolutions. Note that $n = 1$ case here differs from that of Suto & Silk (1988) because we require specific entropy conservation along streamlines. For $1 < n < 2$, we should set $B = 0$ to avoid the flow speed divergence of the first term in solution (27) at large $x$. With $n = 2$ and $B = 0$, we have a constant asymptotic inflow speed again by solution (27). For $n > 2$, we must require both leading order terms in flow speed solution (27) to vanish to avoid velocity divergence; nevertheless for coefficient $A > 0$, the second term in flow speed solution (27) does not vanish; this second term would vanish for a trivial solution of $A = 0$. Unless other constraints appear, we must require $2/3 < n \leq 2$, corresponding to a range of mass density scaling $\rho \propto r^{-3}$ to $\rho \propto r^{-1}$. For a comparison, Cheng (1978) indicated the more obvious limit of $\rho \propto r^{-3}$ for $n \to 2/3$, while our analysis here also gives another limit of $\rho \propto r^{-1}$ for $n \to 2$ due to the second term in asymptotic solution (27) for the flow speed. At large $x$, the temperature scales as $x^{2(n-1)/n}$. For $n \leq 1$, the temperature remains finite, while for $n > 1$, the temperature diverges as $x \to \infty$ and one needs to invoke a finite size of an astrophysical system in order to make use of this solution.

### 3.1.2 Magnetostatic Solution for a Polytropic Sphere

Setting $v = 0$ for all $x$ in equation (15), one immediately obtains an exact magnetostatic solution

$$\alpha = A_0 x^{-2/n} + N x^{K-1-2/n}, \quad v = L x^K, \quad (29)$$

into coupled nonlinear MHD ODEs (15) and (16), where $A_0$ is explicitly defined by equation (25), and then obtain two nonlinear algebraic equations for the three relevant parameters $N$, $K$ and $L$, namely

$$n(K-1)N = (K + 2 - 2/n)A_0L, \quad (30)$$

$$\left[\frac{QK}{n} + \frac{1}{(3n-2)} + \left(\frac{1}{n} - \frac{4}{n^2}\right)Q\right]A_0L = \left\{\frac{n^2}{2(3n-2)} + nh + \frac{3n}{2}Q\right\}K$$

$$+ \frac{n^2}{2(3n-2)} - (2-n)h + \left(\frac{3n}{2} - 6\right)Q\right\}N. \quad (31)$$

Here we have introduced a handy parameter

$$Q \equiv q \left[\frac{n^2}{2(2-n)(3n-2)} - \frac{(1-n)}{(2-n)h}\right]. \quad (32)$$

for simplicity and for the convenience of mathematical derivation. Equations (30) and (31) together give rise to

---

2 In equation (11) of Lou & Wang (2006), there are two typos; in the coefficients of both $a(x)$ and $m(x)$ there, the exponent should be $-1/n$ instead of $1/n$. 

---
a quadratic equation for $K$ in terms of three parameters $n$, $Q$ and $h$, namely

$$
\left[ \frac{n^2}{2(3n-2)} + nh + \frac{(3n-2)}{2}Q \right] K^2 + \frac{(3n-4)}{n}K 
+ \frac{2(2-n)(1-n)}{n} h + \frac{n^2 + (3n-2)^2(1-4/n)Q}{(3n-2)} = 0 .
$$

(33)

Once proper roots of $K$ are known, parameters $N$ and $L$ are proportional to each other by equation (39) and only one of them is free to choose. The existence of the exact magnetostatic SPS solution (28), as well as the requirement of $\mathcal{R}(K) > 1$ constrain the parameter regime of this quasi-magnetostatic solution. In reference to equations (6) and (7) of Lou & Wang (2007) corresponding to $Q = 0$ here, we have explicitly discussed the parameter regime where two real roots $K > 1$ exist. We therefore expect that at least for a sufficiently small $Q \neq 0$, it would be still possible to have two real roots of $K > 1$ given by quadratic equation (33). Of course, it is also possible to have one root $K > 1$ and the other root $K < 1$. For example, for $\gamma = 1.4$, $n = 1.4$ and $h = 0$, we have a positive $K = 1.056$ and a negative $K = -1.199$; with other parameters the same but $h = 1$, the two $K$ roots are $K = 1.0373$ and $-1.180$, respectively.

### 3.1.4 MHD Thermal Fall Solutions at Small $x$

Lou & Li (2007 in preparation) report a possible asymptotic thermal fall solution for small $x$ without magnetic field; for this solution, the thermal pressure gradient force almost balances the gravitational force, yet the mass density and radial infall speed still approach infinity at small $x$. In the case of asymptotic MHD thermal fall solutions, magnetic force is much smaller than both the pressure gradient force and the gravity at small $x$. The leading nonlinear ODEs for small $x$ read

$$
wv' + \beta'/\alpha - va/(3n-2) = 0 ,
$$

(34)

$$
(\alpha') + 2\alpha/x^2 = 0 ,
$$

(35)

where the effect of magnetic field does not appear explicitly (i.e., the absence of $h$ parameter) in the regime of small $x$. This pair of coupled nonlinear ODEs (34) and (35) at small $x$ are the same as those in Lou & Li (2007 in preparation), and the corresponding asymptotic solution is therefore the same to the leading order, namely

$$
\alpha = \left[ \frac{\gamma(3n-2)}{(\gamma-1)} \right]^{-1/(\gamma-1)} m(0)^{(q-1)/(\gamma-1)} x^{-1/(\gamma-1)} ,
$$

(36)

$$
v = - \left[ \frac{\gamma(3n-2)}{(\gamma-1)} \right]^{1/(\gamma-1)} m(0)^{(q+\gamma-2)/(\gamma-1)} x^{(2\gamma-3)/(\gamma-1)} .
$$

(37)

Here $m(0)$ is the value of the total reduced enclosed mass $m(x)$ as $x$ approaches 0, representing an increasing point mass at the very centre. This asymptotic solution at small $x$ is valid for $3/2 < \gamma < 5/3$. The corresponding radial profile of the random magnetic field can be readily inferred from equation (11).

### 3.1.5 Solutions of MHD Thermal Expansion at Large $x$

The asymptotic MHD thermal fall solutions (30) and (37) at small $x$ in the preceding subsection 3.1.4 can be integrated numerically and connected with the asymptotic MHD thermal expansion solution at large $x$ (see Figure 6). For this latter solution at large $x$, the reduced radial flow velocity $v(x)$ becomes linear in $x$ while the mass density approaches zero for $x \to \infty$. The force that is dominant here is the thermal pressure gradient, driving the gas into expansion. In this case of $v \sim cx$ with $c$ being a constant coefficient, we obtain from coupled nonlinear ODEs (9) and (14) at large $x$

$$
(n-c)x\alpha' - ca = -2(1-c)\alpha ,
$$

(38)

$$
(n-1)cx - (n-c)cx + \beta'/\alpha = 0 .
$$

(39)

By further assuming the power-law form of $\alpha \sim E x^p$ at large $x$ with $E$ and $P$ being two parameters and using equations (8) and (12), we obtain from equations (38) and (39)

$$
P = -(3q-2)/(1 - n + 3nq/2) ,
$$

(40)

$$
E^{1-n+3nq/2}(n-c)^q(2 + P) = c(1-c) ,
$$

(41)

and

$$
P = (3c-2)/(n-c) .
$$

(42)

Parameters $P$, $c$ and $E$ are then readily determined by three algebraic equations (40)–(42) once the values of $q$ and $n$ parameters are specified. One can easily fix $P$ from equation (40) first, then calculate $c$ from equation (12) and finally obtain $E$ by equation (11). This solution for MHD thermal expansion is valid when $\gamma > 4/3$ (i.e., $q > 2/3$), because we require $P < 0$ for a converging $\alpha(x)$ at large $x$. Again, the corresponding radial profile of magnetic field can be readily inferred from equation (11).

Asymptotic solution here should be compared with asymptotic solutions (26) and (27) at large $x$ when $n < 1$. 
3.2 MHD Solutions with Weak Thermal Pressure

For MHD solutions involving weak thermal pressure, parameter \( q \) should not enter the leading order terms of the asymptotic solution.

3.2.1 MHD Free-Fall Solutions at Small \( x \)

For asymptotic central free-fall solutions at small \( x \), first found by Shu (1977) under the isothermal approximation, the gravity force is virtually the only force in action, and the radial velocity and mass density profiles both diverge in the limit of \( x \to 0^+ \). The leading order ODEs at small \( x \) then read

\[
\alpha' = -\frac{2\alpha}{x} - \frac{\alpha^2}{(3n-2)x},
\]

\[
v' = \frac{\alpha}{(3n-2)},
\]

and the corresponding asymptotic solution appears as

\[
\alpha(x) = \left[ \frac{(3n-2)m(0)}{2x^3} \right]^{1/2},
\]

\[
v(x) = -\left[ \frac{2m(0)}{(3n-2)x} \right]^{1/2},
\]

with an integration constant \( m(0) \), representing an increasing point mass at the very centre. The special isothermal case of \( n = 1 \) and \( \gamma = 1 \) was first studied by Shu (1977) in the context of star formation (Shu et al. 1987). It is particularly interesting to note that it is now possible to have \( \gamma = 1 \) and \( n > 2/3 \) case, still corresponding to a non-isothermal gas flow (i.e., \( q \neq 0 \) and a variable sound speed). On the other hand, it is also possible to have \( n = 1 \) and \( \gamma > 1 \) case with \( q \neq 0 \). With various combinations of sensible parameters, we can now readily construct such kind of solutions numerically. In terms of modeling the MHD processes of star or core formation in magnetized clouds (Zhou et al. 1992; Shen & Lou 2004; Fatuzzo et al. 2004; Lou & Gao 2006), our general polytropic MHD model framework is more versatile including the possible role of a completely random transverse magnetic field. In particular, we can model various molecular spectral line profiles in star forming regions; we shall pursue this exploration in separate papers.

To the leading order, this asymptotic MHD free-fall solution does not involve polytropic index \( \gamma \) and the corresponding profile of magnetic field can be readily determined by equation (11). In this case, magnetic field does not play a dynamically important role but may reveal diagnostic features if shock accelerated relativistic electrons are present. We now consider the parameter regime that allows for this asymptotic MHD free-fall solution at small \( x \). Substituting equations (14) and (15) into coupled nonlinear MHD ODEs (16) and (17) and requiring the emergence of equation (13) for small \( x \) values with consistent orders of magnitudes for the higher order terms, we obtain two inequalities

\[
n > 2/3 \quad \text{and} \quad \gamma < 5/3; \tag{46}
\]

these requirements appear to be the same as those of Suto & Silk (1988), but the polytropic equation of state adopted is different between theirs and ours, that is, it is no longer necessary to impose the condition \( n + \gamma = 2 \) or \( q = 0 \) in our more general MHD polytropic model formulation.

3.2.2 Strong-Field Asymptotic MHD Solutions in the Small-\( x \) Regime

The strong-field asymptotic MHD solutions in the small-\( x \) regime (see Wang & Lou 2007 for the \( q = 0 \) counterpart of this asymptotic solution in a conventional polytropic gas) are central or collapse solutions with the magnetic Lorentz force against the self-gravity and with their strengths being comparable in magnitudes; both forces overwhelm the thermal pressure gradient force. For such strong-field asymptotic solutions, the velocity profile remains finite but the mass density profile becomes divergent as \( x \to 0^+ \). More specifically in the regime of \( x \to 0^+ \), we obtain leading-order terms for the three coefficients in equation (16), namely

\[
X(x, \alpha, v) \sim h\alpha x^2,
\]

\[
A(x, \alpha, v) \sim -\frac{(nx - v)}{(3n - 2)}\alpha^2 - 2h\alpha^2 x,
\]

\[
V(x, \alpha, v) \sim 2h\alpha x^2(1 - n) + \frac{(nx - v)^2}{(3n - 2)}\alpha,
\]

such that the two coupled nonlinear MHD ODEs in (16) can be simplified to

\[
v(x) = \left\{ n - \frac{(3n - 2)}{2} \left[ h \pm (h^2 - 4h)^{1/2} \right] \right\} x, \tag{48}
\]

and

\[
\alpha(x) = D_0 x^{-5/2 \mp \sqrt{n - h}/(2h)} \tag{49}
\]

to the leading order of small \( x \), where \( D_0 \) is an integration constant; the corresponding magnetic field strength profile is simply characterized by equation (11). For strong-field asymptotic MHD solutions (18) and (19) to be physically valid, it is necessary to require that \( h > 4 \), corresponding to a sufficiently strong magnetic field regime and hence the name of this type of asymptotic MHD solutions. In a wide range of astrophysical systems, the typical value of \( h \) ranges from \( \sim 10^{-3} \) to \( \sim 10^5 \). The strong-field situations may happen for magnetospheric plasmas surrounding magnetic white dwarfs, radio pulsars, anomalous X-ray pulsars (AXPs), magnetars and so forth. Typical magnetic field strengths are \( \sim 10^7 - 10^9 \)G
Semi-complete MHD solutions with inner free-fall asymptotic solutions and without MHD shocks. All these solutions involve a completely random magnetic field and a general polytropic gas. The relevant parameters are $\gamma = 1.1$, $n = 0.85$, $q = -0.182$, and $h = 0.3$. The two perpendicular dotted straight lines are abscissa and ordinate axes, respectively. The dash-dotted curves are the magnetosonic critical curves. In the upper panel, the MHD solution labelled with 'Shu1' is a polytropic MHD counterpart of the isothermal expansion-wave collapse solution (EWCS; Shu 1977), and is constructed by integrating from large $x$ asymptotic solutions (26) and (27); the MHD solution labelled with 'Hunter1' is constructed in a similar manner for $A = 1.6 < A_0$, and $B = 0$; the MHD solution labelled with 'Shu2' is constructed with two parameters $A$ slightly larger than $A_0 = 1.842$ and $B = 0$; the MHD solution labelled with 'Hunter2' is constructed with $A = 1.8839$ on the magnetosonic critical curve inwards to the same $x_F$ and outwards to large $x$; this solution goes smoothly across the magnetosonic critical curve twice. The other MHD solution labelled with 'LS2' is constructed by integrating from $(x, v, \alpha) = (3.631 \times 10^{-5}, 1.497 \times 10^{0})$ on the magnetosonic critical curve inwards to small $x$ and outwards to $x_F = 0.3$, and by integrating from $(x, v, \alpha) = (2.683, 0.9359, 0.4795)$ on the magnetosonic critical curve inwards to the same $x_F$ and outwards to large $x$; this solution goes smoothly across the magnetosonic critical curve twice. The other MHD solution labelled with 'LS1' is constructed by integrating from $(x, v, \alpha) = (0.8392, -1.581, 56.94)$ on the magnetosonic critical curve inwards to small $x$ and outwards to $x_F = 0.3$, and by integrating from $(x, v, \alpha) = (0.4357, -0.8466, 2.8839)$ on the magnetosonic critical curve inwards to the same $x_F$ and outwards to large $x$ (Lou & Shen 2004). The values of $m(0)$ for the 'Shu1', 'Shu2', 'Hunter1', 'Hunter2', 'Hunter3', 'LS1' and 'LS2' MHD solutions are 1.286, 1.565, 1.777, 1.767, 1.864, 0.08699 and 9.763 $\times$ 10$^{-5}$, respectively. The 'LS1' MHD solution has its large $x$ asymptotic solution with parameters $A = 4.241$ and $B = 0.3843$ in asymptotic solution (26) and (27), and the 'LS2' MHD solution has its large $x$ asymptotic solution with $A = 0.4973$ and $B = 2.431$ in asymptotic solution (26) and (27). As examples of illustration, the 'LS1' and 'LS2' MHD solutions cross the magnetosonic critical curve twice smoothly.

For magnetic white dwarfs, $\sim 10^9 - 10^{10}$G for millisecond radio pulsars, $\sim 10^{11} - 10^{12}$G for radio pulsars, $\sim 10^{12} - 10^{13}$G for AXPs, and $\sim 10^{13} - 10^{15}$G for magnetars.

To the leading order, the polytropic index $\gamma$ does not appear in the strong-field asymptotic solution, because the thermal pressure force is much weaker than both gravity and magnetic force. By requiring $\beta = m^2\alpha^3 \ll \alpha^2x^2$ in the derivation, the parameter regime for these strong-field asymptotic MHD solutions at small $x$ to be valid is

$$\left[-\frac{5}{2} - \frac{(h^2 - 4h)^{1/2}}{2h}\right] \left(-n + \frac{3nq}{2}\right) + 3q - 2 > 0 \quad (50)$$

By setting $q = 0$ and thus $n + \gamma = 2$, inequality (50) reduces to inequality (69) of Wang & Lou (2007) for a conventional polytropic gas permeated with a completely random transverse magnetic field.

Numerical examples of using strong-field asymptotic solutions (48) and (49) with the upper signs are shown in Figure 5. These solutions (labelled by 'WL1' and 'WL2') pass through the magnetosonic curve smoothly and join the asymptotic solutions (26) and (27) at large $x$. One can also construct MHD shock solutions in this scheme numerically.

We note empirically that numerical integrations from very small $x$ outwards using strong-field asymptotic solutions (48) and (49) have a tendency to be unstable, especially for the lower-sign solution. On the other hand, we are unable to obtain the lower-sign solutions in (45) and (49) by numerically integrating from the magnetosonic critical curve to small $x$ so far. This problem might be related to properties of perturbations in a self-similar flow (Lou & Bai 2008 in preparation).

Lou & Wang (2007) proposed a class of self-similar MHD rebound shock models for supernovae based on quasi-static MHD shock solutions with $q = 0$ and focussed on the origin(s) of intense magnetic fields on remnant compact objects such as radio pulsars (magnetic field strengths of $\sim 10^{11} - 10^{12}$G) and magnetic white dwarfs (surface magnetic field strengths of $\sim 10^7 - 10^9$G).

It can be possible that MHD rebound shock models based on our strong-field solutions (see Figure 5) with $q \neq 0$ are relevant to strong magnetic fields of $\sim 10^{13} - 10^{15}$G observationally estimated for magnetars and anomalous X-ray pulsars (AXPs). This would correspond to strong surface magnetic field strengths higher than several thousand gauss on progenitor stars (e.g., magnetic Ap stars). We shall discuss this interesting problem in separate papers.

4 Global Semi-Complete MHD Solutions
4.1 MHD Expansion-Wave Collapse Solutions

Shu (1977) constructed the expansion-wave collapse solution (EWCS) for a self-similar isothermal gas flow and later developed the so-called inside-out collapse scenario for protostar formation (see Shu, Adams & Lizano 1987 and extensive references therein). Cheng (1978) presented the polytropic generalization of EWCS. Chueh & Chou (1994) studied the MHD generalization of an isothermal EWCS. In reference to the work of Yu & Lou (2005), in this context, we can now construct the general polytropic MHD EWCS counterpart here and show a specific example labelled with ‘Shu1’ in Figure 1. In our general polytropic MHD EWCS, the outer portion is the outer part of a magnetized SPS (no singularity is actually involved because $x = 0$ is excluded), while the inner portion approaches an MHD asymptotic free-fall solution with the random magnetic field being advected radially inward. From the perspective of theoretical model development, we can now test thermodynamic properties of magnetized clouds by comparing with observational inferences. Using the isothermal version of this solution, Shu (1977) outlined a physical scenario for protostar formation, which can now be further extended on the basis of our generalized polytropic MHD model solution. At the beginning, the magnetized SPS is static; then at $t = 0$, a disturbance may take place and a magnetized core collapse under self-gravity occurs in an inside-out manner; the ‘bounding surface’ (or the stagnation surface) of the magnetized collapsing sphere travels outward at the magnetosonic speed (not a constant in general) in a self-similar fashion. Shu (1977) utilized the isothermal version of this EWCS solution to model the core collapse of a molecular cloud in protostar formation. Zhou et al. (1993) attempted to test Shu’s isothermal dynamical model by fitting the observed molecular spectral line profiles. However, it should be noted that by incorporating a non-isothermal temperature profile inferred from data, this isothermal model test by fitting molecular spectral line profiles is not fully self-consistent. In contrast, with a general polytropic model and the ideal gas law, the temperature variation is a natural consequence. It would be highly desirable to carry out a parallel analysis similar to that of Zhou et al. (1993) but for a general polytropic gas cloud model so that profiles of temperature, density and flow speed are all self-consistently prescribed. By setting $h = 0$ for the absence of magnetic field, our generalized polytropic EWCSs are the same as those of Cheng (1978). The outer magnetized SPS involves a completely random magnetic field and thus a global quasi-spherical symmetry. As a result, small-scale random flows with a zero mean are expected. In this sense, the outer magnetized SPS portion here provides a large-scale mean profile. The presence of a magnetic field would make a cloud to behave more fluid like on large scales (the solar wind is an example) and can give rise to radiative diagnostics when relativistic electrons are involved.

The main difference between our general polytropic MHD EWCS here and former EWCSs lies in the degree of freedom. For every determined MHD flow with a set of parameters $\{\gamma, n, h\}$, there is at most one EWCS and these three parameters are now allowed to change independently. In the isothermal EWCS case of Shu (1977), we have $\gamma = 1$, $n = 1$ and $h = 0$. In the conventional polytropic EWCS case of Lou & Wang (2006), we have $\gamma + n = 2$ ($q = 0$) and $h = 0$. In the conventional polytropic MHD problem studied by Wang & Lou (2007), $q = 0$ or $n + \gamma = 2$ is required and there are two parameters $\gamma$ and $h$ that are allowed to vary independently. In the present generalized polytropic MHD model, the parameter $n$ is also allowed to change and is the key extension of this paper for this model problem. Therefore, all polytropic MHD solutions in this analysis have one more degree of freedom for fitting observational data.

4.2 Inner Free-Fall with Outer Inflow/Outflow

Shu (1977) also constructed global semi-complete isothermal solutions that approach a central free fall at small $x$, but with $B = 0$, $h = 0$, $n = \gamma = 1$ (or $q = 0$) in asymptotic solution $\{27\}$ and here $A > A_0$ for the outer asymptotic singular isothermal sphere (SIS) solution (n.b., the singularity at $x = 0$ is actually excluded in this construction). In fact, Shu used this sequence of solutions to introduce the EWCS as the limiting case and to suggest an inside-out collapse scenario for star formation. As an example, we show a generalized polytropic MHD extension of such solutions in Figure 1, labelled by ‘Shu2’ with $B = 0$ and $A > A_0$; it is possible and straightforward to construct a sequence of such solutions with $B = 0$ by choosing different $A$ values larger than $A_0$. In addition, there is still the case of $B \neq 0$ in asymptotic solution $\{27\}$, which is qualitatively similar yet has a constant flow speed at large $x$ in the isothermal case for outer envelopes (e.g., Lou & Shen 2004; Shen & Lou 2004; Fatuzzo et al. 2004; Yu & Lou 2005; Yu et al. 2006). There are increasing observational evidence indicating that star forming clouds do have systematic flows far away from the core (e.g., Fatuzzo et al. 2004) and our model can accommodate either inflows or outflows.

In general polytropic cases with $n < 1$, the radial flow speed remains finite and approaches zero at large $x$. With $n = 1$, the radial flow speed remains finite and approaches a constant value $B$ at large $x$. While for $n > 1$, the asymptotic flow term associated with this $B$ coefficient in asymptotic solution $\{27\}$ diverges at large $x$; to model a real system, one then needs to set $B$ equal to zero. This parameter $B$ represents a component of radial flow speed which satisfies the momentum equation as if there is no force in action. Because Hunter (1977) first constructed global semi-complete isothermal solutions with this behaviour, we label ‘Hunter1’, ‘Hunter2’
Fig. 2 Semi-complete MHD shock solutions with free falls as $x \to 0$ with four parameters $\gamma = 1.05$, $n = 0.9$, $q = -0.143$, and $h = 0.1$. In the upper panel, the MHD shock solution labelled with ‘BL1’ is constructed by integrating from a magnetosonic critical point ($x, v, \alpha$) = (0.5, $-0.6520, 3.053$) inwards to small $x$ and outwards to a MHD shock point ($x_s, v_s, \alpha_s$) = (0.7, $-0.2731, 2.191$) downstream; using MHD shock conditions (22) – (24), we obtain the corresponding upstream physical variables and thus integrate from ($x_{s1}, v_{s1}, \alpha_{s1}$) = (0.70015, $-0.6761, 1.515$) outwards to large $x$. The ‘BL2’ MHD shock solution is constructed by the same inner portion with a different MHD shock point at ($x_s, v_s, \alpha_s$) = (1.9061, 0.4034, 0.5728) and then integrate outwards to large $x$. The ‘BL1’ MHD solution has a large $x$ asymptotic solution with $A = 0.6536$ and $B = -1.780$ in expressions (26) and (27), while these parameters for ‘BL2’ MHD shock solution are $A = 2.168$ and $B = 0.4534$ in expressions (26) and (27). In the lower panel, the ‘TH1’ and ‘TH2’ MHD solutions both have the same outer SPS portion with $v = 0$ and $\alpha = A_0 x^{-2/n}$ as described by expression (28); this is the outer part of a magnetized SPS with the radial profile of a random magnetic field given by relation (1). The MHD shock solution labelled with ‘TH1’ has upstream and downstream MHD shock points ($x_{s1}, v_{s1}, \alpha_{s1}$) = (1.7555, 0.5311) and ($x_s, v_s, \alpha_s$) = (1.7514, 0.3858, 1.130) and crosses the magnetosonic critical curve at ($x, v, \alpha$) = (0.1376, $-1.081, 12.20$). The MHD solution labelled with ‘TH2’ has upstream and downstream MHD shock points ($x_{s1}, v_{s1}, \alpha_{s1}$) = (1.4766, 0.7800) and ($x_s, v_s, \alpha_s$) = (1.4760, 0.4623, 0.1963), and crosses the magnetosonic critical curve at ($x, v, \alpha$) = (7.825 $\times$ 10$^{-6}$, $-1.081, 4.972 \times 10^5$). The ‘BL3’ solution has shock points ($x_{s1}, v_{s1}, \alpha_{s1}$) = (4.758, $-1.298, 3.289$) and ($x_s, v_s, \alpha_s$) = (0.4743, $-0.2907, 7.889$), and crosses the magnetosonic critical curve at ($x, v, \alpha$) = (4.885 $\times$ 10$^{-4}$, $-1.945, 4.474 \times 10^5$). The values of $m(0)$ for solutions ‘BL1’, ‘BL2’, ‘TH1’, ‘TH2’ and ‘BL3’ are 0.6987, 0.6987, 0.6987, 0.2665, 8.461 $\times$ 10$^{-5}$ and 2.589 $\times$ 10$^{-3}$. The values of shock parameter $\lambda$ for these solutions are 1.000210, 1.000320, 1.00032, 1.000390 and 1.00303, respectively.

and ‘Hunter3’ in Fig. 1 as examples of this type of flow solutions. Here the ‘Hunter2’ solution is one with a less dense envelope than the SPS yet with an inflow speed; the ‘Hunter3’ solution is one with a more dense envelope yet with an outflow speed. Both involve free-fall solutions in the central core collapse region.

Qualitatively, this type of general polytropic MHD solution has either an inflow or an outflow in the outer envelope and a free-fall inner core (Lou & Shen 2004; Shen & Lou 2004; Yu & Lou 2005; Yu et al. 2006; Lou & Gao 2006). As the presence of inflows is inferred observationally in star-forming regions and molecular cloud cores (see, e.g., Fatuzzo, Adams & Myers 2004 and references therein), our model analysis here is more general in the following four aspects, namely, (i) the inclusion of a completely random magnetic field, (ii) the construction of an MHD shock, (iii) the possibility of a constant ‘specific entropy’ everywhere at all time, and (iv) the possibility of ‘specific entropy’ conservation along streamlines yet with a variable ‘specific entropy’ in space and time. Physically, unless for a ‘black hole’ at the centre, the central free-fall of gas will ultimately lead to a ‘sphere of transient activities’ around the core region; the change of equation of state and/or the ignition of thermal nuclear reaction mark the onset of a star formation process.

In other words, we expect that the mathematical singularity of a free-fall MHD solution as $x \to 0^+$ will be taken care of by other relevant physical processes when a proto star forms.

Apart from the index parameters $n$ and $\gamma$ that are allowed to vary independently for different polytropic MHD flows, there are still two parameters $A$ and $B$ in asymptotic MHD solutions (26) and (27). In a continuous manner, these two parameters can change and be mapped to a central mass point $m(0)$ or central mass accretion rate. This mapping can be readily determined by numerical integration. Fatuzzo et al. (2004) made a semi-analytical estimate of this mapping. In general, larger $A$ values or smaller $B$ values will lead to a larger $m(0)$ as expected on intuitive ground. Observationally, this corresponds to various central mass accretion rates in protostar forming clouds.

4.3 Envelope Expansion with Core Collapse in Magnetized Clouds

Lou & Shen (2004) constructed isothermal EECC solutions using a solution matching technique in the $\alpha - v$ phase diagram (see also Bian & Lou 2005 and Wang & Lou 2007). We constructed such kind of general polytropic MHD solutions and present the first two of these series of MHD solutions in the lower panel of Figure 1. The first EECC solution labelled with ‘LS1’ is a MHD solution with an inner free fall (i.e., magnetized core collapse) and an outer outflow (i.e., magnetized envelope expansion). This kind of evolution behaviour may be qualitatively applicable in asymptotic giant branch (AGB)
stars, post-AGB stars, and/or protoplanetary nebulae (PPNe); along this line, Lou & Shen (2004) proposed to utilize EECC solutions (including the second ‘LS2’ solution which is qualitatively different) to grossly catch key features of MHD collapses and flows. Also in the context of a collapsing magnetized cloud to form a protostellar core, general polytropic MHD EECC solutions with $q \neq 0$ exemplified here appear more general to account for various plausible situations (Shen & Lou 2004; Lou & Gao 2006). In all these considerations, the inner singularity as $x \to 0^+$ can be removed once a departure from self-similarity occurs and/or a different equation of state is adopted.

Except for the additional scaling parameter $n$, for every fixed MHD flow profile there exists at most a series of discrete solutions, each with a qualitatively different manner (e.g., different times to cross the $v = 0$ axis). It is common to this kind of flows (see also Hunter 1977; Lou & Shen 2004; Bian & Lou 2005) that the $\alpha$ versus $v$ phase diagram obtained by integrating from the inner portion (from $x$ values smaller than the meeting point $x_F$) tends to spiral into a specific point. This point is on the phase curve obtained by integrating from the outer portion (from $x$ values larger than the meeting point $x_F$).

4.4 Expansion of MHD Shocks into an Outer Static Envelope Configuration

Tsai & Hsu (1995) constructed an isothermal self-similar shock solution which connects an outer static configuration to a central free-fall solution with a shock. Bian & Lou (2005) further constructed various possible isothermal shock solutions in a more comprehensive manner. We present here the first three of such MHD shock solutions in the lower panel of Figure 2. Based on extensive numerical exploration, Tsai & Hsu (1995) also suggested that in a star-forming cloud, when a strong burst of thermal nuclear energy is released from the core, instead of a smooth evolution in the form of EWCS as discussed by Shu (1977), a shock can gradually emerge and travels outward in a self-similar manner. Due to a sharp density profile of the singular isothermal sphere (the counterpart of which is SPS in our model analysis here), the shock travels long and evolves into a self-similar shock. They provided numerical simulations to support this scenario of shock initiation and evolution.

The degree of freedom of this series of generalized polytropic MHD shock solutions is like that of the MHD EECC solutions. Because the outer SPS is prescribed, the MHD shock solutions need to be properly matched in the $\alpha$ versus $v$ phase diagram. Similar spiral-in features in the $\alpha$ versus $v$ phase diagram (e.g., Lou & Shen 2004) may exist, leading to a series of discrete MHD shock solutions. Corresponding to each of such MHD shock solutions, there is a specific central mass accretion rate.

Fig. 3 Semi-complete self-similar polytropic solutions with twin MHD shocks for parameters $\gamma = 1.05$, $n = 0.9$, $q = -0.143$, $h = 0.1$. The dash-dotted curve is the magnetosonic critical curve. The inner portion is the same for the two MHD solutions labelled with ‘BL4’ and ‘BL5’, i.e., the MHD shock points for the inner shock are $(x_{s1}, v_{s1}, \alpha_{s1}) = (0.4129, -0.4585, 6.949)$ for the downstream side and $(x_{s1}, v_{s1}, \alpha_{s1}) = (0.4133, -1.147, 3.743)$ for the upstream side, and both MHD solutions also smoothly cross the magnetosonic critical curve twice at the same two points at $(x, v, \alpha) = (1.425 \times 10^{-4}, -2.164, 2.126 \times 10^{4})$ and at $(x, v, \alpha) = (0.7, -0.4545, 2.146)$. The common inner portion of both MHD solutions has two nodes (i.e., $v = 0$) for $x < 0.3$ and diverges as $x \to 0$. Parameter $\lambda$ for the inner MHD shock is 1.00100 and parameter $m(0)$ for the central free-fall solution is $9.338 \times 10^{-4}$. The MHD solution labelled with ‘BL4’ has the outer MHD shock at $(x_{s2}, v_{s2}, \alpha_{s2}) = (1.0000961, -0.3316, 1.144)$ for the upstream side; while the ‘BL5’ MHD solution has the outer MHD shock at $(x_{s2}, v_{s2}, \alpha_{s2}) = (1.9, 0.7693, 0.8311)$ for the downstream side and $(x_{s1}, v_{s1}, \alpha_{s1}) = (1.90019, 0.4803, 0.6358)$ for the upstream side. The values of $\lambda$ for the two outer MHD shocks of ‘BL4’ and ‘BL5’ are 1.0000961 and 1.000102. The ‘BL4’ MHD solution has a large $x$ asymptotic solutions [20] and [27] with $A = 1.076$ and $B = -0.9545$ (inflow), while for the ‘BL5’ MHD solution, the two corresponding parameters are $A = 2.403$ and $B = 0.6354$ (outflow).

4.5 Generalized Polytropic MHD Shock Solutions with Inner Free Fall

Bian & Lou (2005) explored various isothermal hydrodynamic shock solutions and noted astrophysical applications to AGB, PNe, protostar formation, quasars and supernova explosions and so forth. We here present two generalized polytropic MHD shock solutions ‘BL1’ and ‘BL2’ as illustrative examples in Figure 2. Within the framework of our model analysis, it is possible to con-
struct a variety of general polytropic MHD shocks adapted to various astrophysical flow situations.

On the basis of a conventional polytropic hydrodynamic shock model with \( n + \gamma = 2 \) and thus \( q = 0 \), Lou & Gao (2006) examined observationally inferred information for star-forming cloud cores. They noted that star-forming regions are well studied in the inner core and outer edge regions, which may be utilized to constrain or test self-similar core collapse scenario with or without inflows and/or outflows (Shen & Lou 2004; Fatuzzo et al. 2004). While globally smooth solutions such as EECC solutions have less degrees of freedom when they encounter the sonic critical curve (which occurs often), the possible existence of shocks is physically plausible and mathematically convenient to join different inner and outer asymptotic solutions into a global shock flow solution (i.e., by crossing the sonic critical curve with a shock). Lou & Gao (2006) also compared polytropic model with observations. By taking into account of radiative transfer, we shall further fit various observed molecular spectral line profiles for given underlying general polytropic MHD shock flows with \( q \neq 0 \). Thus shock solutions provide a simple and direct model scenario for star-forming cloud cores that appear grossly quasi-spherical on large scales.

Once parameters \( \gamma \), \( n \) and \( h \) are prescribed, a global semi-complete MHD shock solution has two degrees of freedom, namely, one can choose the location where the solution smoothly crosses the magnetosonic critical curve, and one can also choose the MHD shock point; this is illustrated in the two panels of Figure 2 where two MHD shock solutions have different outer envelopes. The location of smoothly crossing the magnetosonic critical curve determines the parameter \( n(0) \) (related to the central mass point and mass accretion rate) in the core and the complete choice corresponds to a specific set of parameters \( A \) and \( B \) for the outer large \( x \) asymptotic MHD solution (26) and (27). The plethora of this type of MHD shock solutions can accommodate various astrophysical situations including protostar-forming cloud cores.

4.6 Twin Polytropic MHD Shock Solutions

Bian & Lou (2005) also constructed isothermal hydrodynamic twin shock models, in which two shocks appear in a self-similar flow and the solutions cross the sonic critical curve thrice, namely, once smoothly and twice with shocks. For general polytropic MHD flows, we present the counterpart solutions in Figure 3. This type of general polytropic MHD shock solutions further expand the solution space, providing more plausible models of quasi-spherical processes involving random magnetic field, gravitational core collapse and far-away inflows/outflows. We note that this general polytropic MHD twin shock model differs from the so-called forward-reverse shock pair (see, e.g., Chevalier 1982), because both shocks here are ‘forward shocks’ in the sense that the shock moves forward relative to the local MHD flow. Conceptually, we emphasize the possibility of twin or multiple general polytropic MHD shocks in a magnetized flow system with \( q \neq 0 \) (see Yu et al. 2006 for the isothermal case).

When three parameters \( \gamma \), \( n \) and \( h \) are chosen, one can still adjust the place where the MHD flow crosses the magnetosonic critical curve smoothly, and then choose the place where an outer MHD shock appears; the former choice gives the central mass point \( m(0) \) or central mass accretion rate, while for the latter choice, the shock location \( x \) also corresponds to the outward travel speed of this MHD shock. Figure 3 shows that different choices of outer MHD shocks correspond to very different outer MHD flow profiles in the envelope.

4.7 Inner Quasi-Magnetostatic Solutions with Outer Inflows

Using a conventional polytropic hydrodynamic formulation with \( q = 0 \), Lou & Wang (2006) first presented the quasi-static asymptotic solution in the regime of small \( x \) and then constructed a rebound shock model to catch certain gross features of a class of supernovae. We attempted to relate the mass of a progenitor star and the type of a remnant compact object (such as a white dwarf, a neutron star or a black hole) left behind after a gravitational core collapse and a subsequent emergence of rebound shock. The above model can be extended to include a completely random transverse magnetic field with quasi-spherical symmetry (Lou & Wang 2007). In this MHD rebound shock model, we explore the origin(s) of intense magnetic fields on compact objects from the perspective of fossil field associated with massive progenitor stars. As an important part of further model development, we have also constructed generalized polytropic MHD solutions that smoothly cross the magnetosonic critical curve and approach this quasi-magnetostatic asymptotic solution (29) as shown in Figure 4 and labeled with ‘LW1’. In the regime of small \( x \) and with complex coefficients \( K \) and \( L \), the MHD flow velocity actually oscillates with a decreasing magnitude as \( x \rightarrow 0^+ \) (not easily seen here; see Lou & Wang 2006 for a similar example in detail). Qualitatively speaking, this generalised polytropic MHD solution represents a far-away or initial inflow leading to an eventual quasi-magnetostatic inner core and carries a desirable feature of forming a central magnetostatic configuration instead of a divergent free fall towards a central mass point. Therefore, this type of generalized polytropic MHD solutions should also provide a physically plausible model framework to study protostar formation processes (Shen & Lou 2004; Lou & Gao 2006); we shall discuss this important application in separate papers.

This type of general polytropic MHD solution has one degree of freedom in construction, namely, the choice
of the point where a solution crosses the magnetosonic critical curve.

4.8 Inner Quasi-Magnetostatic Solutions with Rebound MHD Shocks

Lou & Wang (2006) and subsequently Lou & Wang (2007) proposed a conventional polytropic gas model (i.e., \( q = -0.5 \) and thus \( n + \gamma = 2 \)) for rebound shock process in class of supernova explosions. In particular, Lou & Wang (2007) have incorporated a random magnetic field (e.g. Zel’dovich & Novikov 1971) and attempted to address the fundamental issue on the physical origin of strong magnetic field of a compact object left behind a rebound MHD shock. Based on our estimates, we proposed that for a progenitor star of initial mass in the range of \( 6 \sim 8 M_\odot \), an MHD rebound shock initiated during the core collapse may eventually leave behind a magnetized white dwarf with magnetic field strengths in the range of \( \sim 10^{16} \sim 10^{18} G \) depending on surface magnetic field strengths of the progenitor star. Similarly, for a progenitor star of initial mass in the range of \( \gtrsim 8 M_\odot \), a stronger MHD rebound shock initiated during the gravitational core collapse may ultimately leave behind a neutron star with magnetic field strengths in the range of \( \sim 10^{11} \sim 10^{13} G \) depending on surface magnetic field strengths of progenitor stars. In reference to the key result of Lou & Wang (2007), the generalized polytropic MHD model here allows \( n + \gamma \neq 2 \) and thus \( q \neq 0 \). The major physical consequence is that ‘specific entropy’ is conserved along streamlines but ‘specific entropy’ is not uniformly distributed in space and time during the gravity-induced core collapse and rebound within a progenitor star. As an example of illustration, we display our generalized polytropic rebound MHD shock solution labelled with ‘LW2’ in Figure 4. This solution carries a feature that a strong MHD shock emerges surrounding an inner magnetized core, ploughs through an infalling magnetized outer envelope (either with a stellar wind or just with an inflow) and leaves behind a quasi-magnetostatic compact configuration of high density. The ‘rebound’ MHD shock here is a physical simplification of a neutrino-driven shock, as opposed to the earlier concept of a ‘prompt’ shock. One important flexibility here is that the constraint \( n + \gamma = 2 \) is now relaxed such that relevant coefficients allow for various plausible combinations. In the analysis of Lou & Wang (2007), we have already pointed out that the asymptotic scaling laws of the mass density profile and magnetic field profile in terms of radius \( r \) is independent of the equation of state, which is apparent from the analysis of this paper. This is the crucial difference between this model under the present formulation and those of former analyses. For example, in our former model (Lou & Wang 2007), it appears that

\[
<B_t^2>^{1/2} \sim r^{3-2/n},
\]

(51)
and we need \( n \) to approach \( 2/3 \) for the strongest magnetic field amplification or the fastest field variation from the stellar surface to the central core; this appears to constrain the value of polytropic index \( \gamma \) in our former analysis with \( q = 0 \). However, in the current model frame work with \( q \neq 0 \), variations of \( n \) and \( \gamma \) are no longer constrained in this regard. We can still let \( n \) approach \( 2/3 \) but the \( \gamma \) value is not necessarily close to \( 4/3 \).

In constructing such general polytropic rebound MHD shock solution, we can choose the point where the solution would cross the magnetosonic critical curve smoothl (see Fig. 4 caption for the solution construction procedure; the solution portion between the magnetosonic critical point and the downstream MHD shock point is actually ignored later) and the MHD shock point. There fore except for the choice of parameters \( \gamma \), \( n \) and \( h \), we still have two degrees of freedom for constructing such type of semi-complete solutions with \( q \neq 0 \).

To describe a collapsing stellar core prior to the emergence of a rebound shock for a supernova explosion, Goldreich & Weber (1980) derived an exact homologous solution using a polytropic model with \( \gamma = 4/3 \) and constant specific entropy everywhere within the stellar core, compared with numerical simulations (Van Riper & Arnett 1978; Bethe et al. 1979), and identified parameter range allowing for a homologous core collapse. In the light of current model analysis and complementary to the analysis of Goldreich & Weber (1980), Lou & Cao (2008) realized that it is not necessary to impose the condition of a constant specific entropy throughout the stellar core (also unlikely in reality) and one can still derive homologous solutions for a collapsing stellar core. Besides, we further obtain a broad class of self-similar solutions for \( \gamma = 4/3 \) with or without shocks.

4.9 Inner Magnetoaccretion with Outer Inflows

Wang & Lou (2007) report a novel magnetoaccretion solution for a conventional polytropic gas \( (q = 0) \) with a completely random magnetic field. The isothermal MHD counterpart of this magnetoaccretion solution was described in Yu et al. (2006). We here construct the counterpart solutions \( [48] \) and \( [49] \) with the upper signs in a generalized polytropic \( (q \neq 0) \) MHD formulation and display two examples of these results in Figure 5. These solutions involve fairly strong magnetic field \( (i.e., h > 4) \). Especially at small \( x \), we would expect a ‘sphere’ of magnetic activities and transients within which the self-similarity and quasi-spherical symmetry are destroyed. However, the physical scenario corresponding to this solution, i.e., the possibility that the gravitational energy can be effectively converted into magnetic energy via magnetoaccretion processes, is tantalizing. We shall further analyze this solution and its astrophysical applications in separate papers.

In constructing this type of general polytropic MHD solution, we can choose the point where the solution crosses the magnetosonic critical curve smoothly and therefore we have only one degree of freedom. Numerical integrations with asymptotic solutions \( [48] \) and \( [49] \) for the lower signs appear unstable in our experiment. This problem remains to be investigated further (Lou & Bai 2007 in preparation).

4.10 MHD Solutions of Inner Thermal Fall and Outer Thermal Expansion

The hydrodynamic thermal fall asymptotic solution at small \( x \) reported by Lou & Li (2007 in preparation) is in a parameter regime quite different from earlier analyses (i.e., \( \gamma > 4/3 \) here). In our polytropic MHD generalization, this asymptotic MHD solution \( [36] \) and \( [37] \) seems...
to directly connect with the MHD thermal expansion solution \(\text{[42]}\) at large \(x\) as presented in Figure \(\text{[6]}\). While not being a proof, our extensive numerical exploration seems to indicate that this type of MHD solutions does not encounter the magnetosonic critical curve.

There is one degree of freedom for constructing this type of MHD solutions, namely, the choice of inner mass point \(m(0)\), once parameters \(\gamma, n\) and \(h\) have been chosen.

5 Discussion

In this paper, we have shown that more general polytropic self-similar MHD shock flow solutions can be obtained under various conditions and they can be applied or adapted to various astrophysics systems by examples. In particular, we focus on the case of \(q \neq 0\), that is, the specific entropy is conserved along streamlines but is not necessarily constant in space \(r\) and time \(t\). More specifically, the specific entropy is related to the enclosed mass that varies in both space and time in general. While this more general polytropic MHD formalism extends the parameter regimes of several known classes of self-similar solutions, this same condition of \(q \neq 0\) (or \(n + \gamma \neq 2\)) also excludes certain solutions, notably the exact global MHD solution (Wang & Lou 2007). In astrophysical applications, it would be informative and important to know these generalizations and constraints for relevant theoretical model development.

In this paper, we mainly studied quasi-spherical polytropic MHD gas flows as models for various stellar level dynamical processes. However, with various sensible extensions and adaptations (such as the two-fluid problem with dark matter-baryon matter coupling by gravity at the galaxy cluster level; Lou 2005; Lou, Jiang & Jin 2008; Jiang & Lou 2008 in preparation), similarity solutions can be applied to different plasma flow systems with their own characteristic spatial and temporal scales. Another example is the attempt of Suto & Silk (1988) to relate self-similar solutions to galaxy formation. In principle, if an MHD system experiences a long dynamic evolution, it may behave self-similarly and thus be described by our model. We emphasize that in Suto & Silk (1988), their cases of \(n = 1\) with different values of \(\gamma \neq 1\) do not correspond to specific entropy conservation along streamlines but correspond to specific entropy variations in time \(t\) only. In our cases of \(n = 1\) with different values of \(\gamma \neq 1\), the specific entropy is a function of both \(r\) and \(t\), and in particular, is conserved along streamlines.

In the preceding section \(\text{[4]}\), we have systematically presented various astrophysical applications and implications of our theoretical model development in reference to earlier observations and theoretical models in relevant astrophysical contexts. These include the following cases.

(1) The isothermal EWCS (Shu 1977; Shu et al. 1987) for the inside-out collapse scenario of star formation can now be generalized by our model here in several aspects, namely, the inclusion of a completely random magnetic field in the collapsing cloud and the more general polytropic ISM without the restriction of an isothermal gas. These two aspects provide the modelling basis for massive star formation, radio synchrotron and/or x-ray emission diagnostics and radiative synthesis of molecular line profiles in star-forming clouds. In particular, a more realistic non-isothermal temperature profile can be determined within a self-consistent polytropic model framework. This is extremely important for modelling protostellar cores embedded in molecular clouds.

(2) The isothermal EECC solutions (Lou & Shen 2004; Shen & Lou 2004) for the envelope expansion and core collapse scenario of star formation in molecular clouds can now be generalized by our model here in several aspects, namely, the inclusion of a completely random magnetic field in the collapsing cloud and the more general polytropic ISM without the restriction of an isothermal gas. In addition to the implications already noted immediately above in item (1), we can now take into account of inflows (Fatuzzo et al. 2004), outflows of ISM in clouds as well as MHD shocks in a more realistic manner. There are growing observational evidence for inflows, outflows and shocks around star-forming cores in ISM clouds. Our MHD shock flow solutions are necessary for certain observed line profiles (Lou & Gao 2008 in preparation).

(3) The isothermal magnetized EECC solutions (Yu & Lou 2005; Yu et al. 2006) is now replaced by a more general polytropic plasma flow which is permeated with a completely random magnetic field. By removing the isothermal constraint, the radial scaling of mass density profiles in a cloud may vary from \(\sim r^{-1}\) to \(r^{-3}\); observationally inferred radial density profiles of molecular clouds do fall within this range. For the hot magnetized intracluster medium (ICM) in the context of clusters of galaxies on Mpc scales (e.g., Lou et al. 2008; Jiang & Lou 2008 in preparation), we predict the same mass density profile range.

(4) The isothermal shock models (Tsai & Hsu 1996; Shu et al. 2002; Shen & Lou 2004) are substantially extended by our MHD model here in several aspects, namely, the inclusion of a completely random magnetic field, the specific entropy conservation along streamlines, and the inclusion of inflows and outflows. Our models are more versatile in modelling the so-called “champagne flows” in star-forming \(\text{H}_2\) regions which involve ionization fronts and shocks.

(5) Various hydrodynamic isothermal shocks have been extensively explored by Bian & Lou (2005) for possible asymptotic flows at large and small radii. For modelling “champagne flows” in star-forming \(\text{H}_2\) regions, Shu et al. (2002) considered only central isothermal Larson-Penston type solutions. We can now further construct “champagne flows” with central free falls, with a completely random magnetic field and with MHD shocks. It is also possible to construct MHD shock flows associated with
different central solutions. In general, relativistic cosmic-ray particles (electrons in particular) can be abundantly produced in either MHD wind shocks or accretion shocks. (6) The quasi-static model solutions first obtained in a conventional polytropic gas (Lou & Wang 2006, 2007; Wang & Lou 2007) are now generalized to more general polytropic magnetized gas flows in this paper. Based on the new quasi-static asymptotic solutions, Lou & Wang (2006, 2007) proposed a new class of rebound shocks for supernovae. Among other things, we would like to understand the ejection of stellar materials of the progenitor star, the mass of a compact object left behind, as well as the origin of intense magnetic fields of compact objects. For a rebound shock process in a progenitor star with the specific entropy conserved along streamlines, the entropy distribution in association with the enclosed mass becomes an important aspect for various possible solutions (see also Lou & Cao 2008 for the case of $\gamma = 4/3$). This can give rise to a variety of rebound MHD shocks in supernova explosions. Of course, we still need to learn about the specific energetic corresponding to any chosen entropy distribution at a certain stage.

(7) The asymptotic magneto-accretion solutions with magnetic parameter $h > 4$ were first studied by Yu & Lou (2006) for the case of an isothermal gas and by Wang & Lou (2007) for a magnetized conventional polytropic gas. This asymptotic MHD solution has now been extended to a general polytropic MHD gas flow and may be relevant to extremely intense magnetic fields inferred for magnetars (e.g., $10^{13} - 10^{15}$ G). A strong surface magnetic field of the progenitor star is implied. Magnetic Ap stars have surface magnetic fields of several to ten thousand gauss and give a typical $h > 4$ by estimates.

While our model results are mainly restricted to $n > 2/3$ and $\gamma < 4/3$ for most situations (except that the thermal fall and thermal expansion solutions have $3/2 < \gamma < 5/3$), the special case of $\gamma = 4/3$ contains a variety of substantially new solutions (Lou & Cao 2008) that differ from those homologous core collapse solutions explored earlier by Goldreich & Weber (1980) in the context of supernovae. Lou & Cao (2008) describe this new theoretical model development in the absence of a completely random magnetic field and obtain various novel results as compared to our analyses here. In particular, the constant coefficient $C$ in equation (12) cannot be absorbed by a rescaling transformation, and features of hydrodynamics as well as MHD systems will depend upon the choice of this constant coefficient $C$ in a nontrivial manner.

Parallel to the analysis of Zhou et al. (1993), in which the isothermal model of Shu (1977) was claimed to be supported by observational inferences, we are investigating radiative diagnostics of the velocity, density and temperature profiles in collapsing cores of clouds from molecular spectral line profiles (Lou & Gao 2008 in preparation). Meanwhile, magnetic fields in proto-stellar clouds not only affect their dynamic evolution, but also provide diagnostic signals, such as synchrotron radiation and shock acceleration of relativistic electrons etc. By including radiative transfer processes, our model is capable of predicting such diagnostic signals.

We also note earlier theoretical analyses on the instability of known self-similar solutions (e.g., Ori & Piran 1988; Hanawa & Nakayama 1997; Hanawa & Matsumoto 1999, 2000; Semelin, Sanchez & de Vega 2001; Lou & Bai 2008 in preparation). Instability analysis helps to test whether certain self-similar flows will actually occur in nature; or if they do occur, whether they will last long enough. Our model provides a variety of new polytropic MHD self-similar solutions with or without shocks, all ready to be examined for such an instability analysis. At this stage, these stability questions remain completely open.

Finally, numerical simulations are powerful and necessary in determining how likely our various self-similar MHD solutions will eventually emerge in a sensible way for plausible initial and boundary conditions based upon astrophysical input. Moreover, nonlinear instability analysis also needs to employ numerical simulations. We hope that the results presented here will trigger extensive numerical explorations for transient behaviours leading to these self-similar MHD solutions with specific entropy conserved along streamlines.

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A Comments on the Positive Definiteness of $nx - v$

For a self-similar gas flow where the self-gravity may or may not play a significant role and for the self-similar transformation in the form of

$$r = k^{1/2} t^{\alpha} x, \quad u = k^{1/2} t^{\alpha - 1} v, \quad \rho = \alpha/(4\pi Dt^\alpha),$$

(52)

where $D$ and $k$ are two constant positive parameters and $n$ and $s$ are two exponents with $3n - s > 0$, we then derive from mass conservation equation (13)

$$M(r_2) - M(r_1) = \frac{k^{3/2} t^{3n-s}}{(3n-s)D^\alpha} \int_{r_1}^{r_2} (nx - v)^{\gamma_2} dx,$$

(53)

where $x_i = r_i/(k^{1/2} t^{\gamma_0})$ with subscript $i = 1, 2$ at a given time $t$. The condition for equation (53) to be valid is that the gas flow within the radial range between $r_2$ and $r_1$ evolves in

---

3 For $n > 2/3$ and $\gamma > 4/3$, we should proceed with care about magnetosonic critical curves with $n + \gamma > 2$. 

In contrast to the above case, we now consider the situation where the self-gravity is indeed important. If we are to establish a self-similar evolution, the total enclosed mass \( M(r, t) \) would enter the radial momentum equation and thus be a self-similar variable under consideration. From a dimensional analysis, if we take only \( k \) and \( D \) as dimensional parameters, the sensible transformation for total enclosed mass \( M(r, t) \) in the power law form should be defined as

\[
M = \frac{k^{3/2} t^{3n-s}}{(3n - s) D} m(x),
\]

where the reduced enclosed mass \( m(x) \) depends on \( x \) only. Comparing equations (54) and (55), we find (reducing the overall scaling factor)

\[
m(x) = \alpha x^2 (nx - v)|_{r=0}^2,
\]

or we have

\[
m(x) = \alpha x^2 (nx - v) + \text{constant}.
\]

However, since equation (11) only takes care of regions where \( r > 0 \) while the mass at the origin \( r = 0 \) is not included, this constant is not arbitrary. A difference in this constant means a difference in the total enclosed mass that changes with time in a scaling of \( t^{3n-s} \). The equation that includes the mass at \( r = 0 \) is equation (2), and this fixes the constant in equation (57) to be zero. Thus, in order to introduce the total enclosed mass into the radial momentum equation as a self-similar variable, \( nx - v > 0 \) is required for \( 3n - s > 0 \). Otherwise for \( 3n - s < 0 \), we should require \( nx - v < 0 \) accordingly.

**B Determination of the Magnetosonic Critical Curve**

The magnetosonic critical curve of equation (15) is defined as the curve along which both the numerators and denominators \( A(x, \alpha, v), X(x, \alpha, v) \) and \( V(x, \alpha, v) \) of equation (15) vanish. In fact, only two of these three vanishing conditions are independent and we shall simply use the intersection of the two surfaces \( A(x, \alpha, v) = 0 \) and \( X(x, \alpha, v) = 0 \) to determine the magnetosonic critical curve.

In order to obtain the magnetosonic critical curve numerically, we denote \( \theta \equiv nx - v \) and \( \theta > 0 \) is required for a positive total enclosed mass (see Appendix A). We readily obtain from \( A(x, \alpha, v) = X(x, \alpha, v) = 0 \) the following two equations, namely

\[
\alpha = \left\{ \begin{array}{c} 0 \quad \left[ 2 - n + \frac{3n - 2}{2} + \frac{n - 1}{2} \right] \left( n - 1 \right) \left( nx - \theta \right) \theta - \frac{2\theta^2}{x} \\ \left[ 2 - n + \frac{3n - 2}{2} + \frac{n - 1}{2} \left( 2nx + \frac{2\theta^2}{3n - 2} \right) \\ \left( 2 - n + \frac{3n - 2}{2} + \frac{n - 1}{2} \right) \left[ \frac{2n + 3n - 2}{2} \right] \right] \right. 
\]

and

\[
\left( 2 - n + \frac{3n - 2}{2} \right) \alpha^{1-n+3nq/2} x^{2nq\theta^2} + \theta^2 = \theta^2.\]

A direct substitution of equation (58) into equation (59) to replace \( \alpha \) yields a nonlinear equation for \( x \) and \( \theta \) which leads to a curve \( \theta = \theta(x) \) for the magnetosonic critical curve. Using this curve, the definition of \( \theta \) and equation (58), the magnetosonic critical curve is then determined in the form of \( \theta(x, \alpha, v) \). In our analysis, the relation \( \theta = \theta(x) \) is determined numerically for a set of three specified parameters \( (n, q, h) \); the definition of \( q \) contains the polytropic index \( \gamma \) parameter.
C Eigensolutions crossing the Magnetosonic Critical Curve

Using the l'Hôpital rule and with a superscript prime 'r' indicating a differentiation with respect to x, we obtain the following differential relation along the magnetosonic critical curve for \( v' \) and \( \alpha' \) from equation \( X(x, \alpha, v)v' = V(x, \alpha, v) \), namely

\[
[f_1(x, \alpha, v)\alpha' + f_2(x, \alpha, v)v' + f_3(x, \alpha, v)]v' = f_4(x, \alpha, v)\alpha' + f_5(x, \alpha, v)v' + f_6(x, \alpha, v),
\]

where functional coefficients \( f_i(x, \alpha, v) \) with \( i = 1, \ldots, 6 \) are defined by

\[
f_1(x, \alpha, v) \equiv \alpha x^2 + (2 - n + \frac{3n - 2}{2}q)(1 - n + \frac{3nq}{2}),
\]

\[
f_2(x, \alpha, v) \equiv 2\theta - q(2 - n + \frac{3n - 2}{2}q)
\]

\[
x_1^{-1-n+3nq/2}x_2^{3q-1}\theta^{q+1},
\]

\[
f_3(x, \alpha, v) \equiv 2\alpha \theta - q(2 - n)x^2 - 2nq - \frac{3n - 2}{2}q
\]

\[
x_1^{-1-n+3nq/2}x_2^{3q-1}\theta^{q+1},
\]

\[
f_4(x, \alpha, v) \equiv 2(1 - n)x^2 - \frac{\theta^2}{(3n - 2)}
\]

\[
+ [2(1 - n)(2 - n) - n(3n - 2)q] (1 - n + 3nq/2)
\]

\[
x_1^{-1-n+3nq/2}x_2^{3q-1}\theta^{q+1},
\]

\[
f_5(x, \alpha, v) \equiv (n(2\alpha - 2\theta) + 2\alpha q)
\]

\[
+ 2q(2(n + 1)x + n(n - 1)v + 2n(n - 1)\theta - 2n\alpha\theta/(3n - 2)
\]

\[
+ 2nq(2(n + 1)x - n(3n - 2)q) \alpha^{-1-n+3nq/2}x_2^{3q-1}\theta^{q+1}
\]

\[
+ nq(2(n + 1)x - n(3n - 2)q) \alpha^{-1-n+3nq/2}x_2^{3q-1}\theta^{q+1}
\]

\[
+ (2q - 1)(2n + 3nq) \alpha^{-1-n+3nq/2}x_2^{3q-1}\theta^{q+1},
\]

\[\text{[see equations (15) and (16)]. Together with the differential relation}
\]

\[
\alpha' = \frac{\alpha v' - 2(x - v)\alpha/\left(nx - v\right)}{(nx - v)},
\]

\[\text{we obtain an algebraic quadratic equation for } v' \equiv dv/dx \text{ at the}
\]

\[\text{magnetosonic critical curve, namely}
\]

\[
\left[\frac{\alpha}{(nx - v)} f_1(x, \alpha, v) + f_2(x, \alpha, v)\right] (v')^2
\]

\[
+ \left[f_3(x, \alpha, v) - \frac{2(x - v)\alpha}{x(nx - v)} f_1(x, \alpha, v)
\]

\[
- \frac{\alpha}{(nx - v)} f_4(x, \alpha, v) - f_5(x, \alpha, v)\right] v'
\]

\[
+ \frac{2(x - v)\alpha}{x(nx - v)} f_4(x, \alpha, v) - f_6(x, \alpha, v) = 0.
\]

The two real roots \( v' \) of quadratic equation \( (63) \) on the magnetosonic critical curve represents two eigensolutions which can go across the magneto sonic critical curve smoothly.

D Solution to MHD Shock Conditions

In reference to MHD shock jump equations \( (22) \rightarrow (24) \), we may denote \( \Gamma_i \equiv n - v_i/\chi_i \) and rearrange these MHD shock equations into the following equivalent form

\[
\alpha_1 \Gamma_1 = \alpha_2 \Gamma_2,
\]

\[
\alpha_1^2 - n + 3nq/2 x_1^{3q-2} \Gamma_1^q + \Gamma_1^2 + \frac{h \alpha_1^2}{2} = \alpha_2^2 - n + 3nq/2 x_2^{3q-2} \Gamma_2^q + \alpha_2 \Gamma_2^2 + \frac{h \alpha_2^2}{2},
\]

\[
\frac{2\gamma}{(\gamma - 1)} \alpha_1^{1-n+3nq/2} x_1^{3q-2} \Gamma_1^q + \Gamma_1^2 + 2h \alpha_1 = \frac{2\gamma}{(\gamma - 1)} \alpha_2^{1-n+3nq/2} x_2^{3q-2} \Gamma_2^q + \Gamma_2^2 + 2h \alpha_2.
\]

Using the first mass conservation equation in \( (64) \), the downstream reduced density \( \alpha_2 \) can be immediately replaced to obtain

\[
\alpha_1^{2-n+3nq/2} x_1^{3q-2} \Gamma_1^q + \alpha_1 \Gamma_1^2 + \frac{h \alpha_1^2}{2} = \alpha_2^{2-n+3nq/2} x_1^{3q-2} \Gamma_2^q + \alpha_1 \Gamma_2^2 + \frac{h \alpha_1^2}{2} + \gamma \alpha_1 \Gamma_2^2 + 2h \alpha_1
\]

\[
\frac{2\gamma}{(\gamma - 1)} \alpha_1^{1-n+3nq/2} x_1^{3q-2} \Gamma_1^q + \Gamma_1^2 + 2h \alpha_1 = \frac{2\gamma}{(\gamma - 1)} \alpha_2^{1-n+3nq/2} x_2^{3q-2} \Gamma_2^q + \Gamma_2^2 + 2h \alpha_2
\]

\[
\Gamma_1^2 + 2h \alpha_1 \alpha_1 \Gamma_1^2 = \frac{2\gamma}{(\gamma - 1)} \frac{1}{\Gamma_2^{1-n+3nq/2}} x_2^{3q-2} \Gamma_2^q + \Gamma_2^2 + 2h \alpha_2
\]

Eliminating \( x_2^{(3q-2)} \) term in both equations of \( (65) \), we obtain a cubic equation for \( \Gamma_2 \) of the downstream side (subscript 2) in terms of upstream variables (subscript 1)

\[
\left(\Gamma_1 + \frac{1}{\Gamma_2}\right) = \left(\alpha_1^{1-n+3nq/2} x_1^{3q-2} \Gamma_1^q + \Gamma_1^2 + \frac{h \alpha_1^2}{2 \Gamma_1^2}ight) + \left(\alpha_2^{1-n+3nq/2} x_2^{3q-2} \Gamma_2^q + \Gamma_2^2 + \frac{h \alpha_2^2}{2 \Gamma_2^2}\right)
\]

\[
+ \left(\alpha_1^{1-n+3nq/2} x_1^{3q-2} \Gamma_1^q + \frac{\gamma - 1}{2\gamma} \Gamma_1^2 + \frac{\gamma - 1}{\gamma} \frac{h \alpha_1}{2 \Gamma_1^2}\right) \Gamma_2
\]

\[
+ \frac{(2 - \gamma)}{2\gamma} h \alpha_1 \Gamma_1 = 0.
\]

\(^{4}\) Please note that in defining similar new variables above equation (46) of Lou & Wang (2006), there was a typo in this regard.
Finally, because $I_2 = I_1$ is a trivial and unphysical solution of equation (65) for an MHD shock, we then obtain a quadratic equation of $I_2$ for possible physical MHD shock solutions

$$
\frac{(\gamma + 1)I_2^2}{2\gamma} - \left( \frac{\alpha_1 - 3\alpha_2/2}{3^{3/2}} \right) \frac{1}{x_1} \frac{\gamma - 1}{2\gamma} I_1 + \frac{h\alpha_1}{2I_1} I_2
$$

and the positive $I_2$ root of this quadratic equation represents a proper MHD shock solution.

For $I_1 > 0$ and $1 < \gamma < 2$, quadratic equation (67) always has two real roots for $I_2$ with the plus-sign root being positive and the minus-sign root being negative. The first relation of equation (65) gives the corresponding $\alpha_2$. One can usually determine a sensible $x_2$ and thus $v_2$ and $\lambda$ values. Occasionally, we encounter the unrealizable situation of a complex $x_2$ caused by the presence of a random magnetic field (i.e., $h \neq 0$). For $h = 0$, the situation of a complex $x_2$ would not arise. This suggests that self-similar MHD shocks cannot form within a certain radius and time for a given set of parameters.

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\[ \gamma = 1.65 \]
\[ n = 0.8 \]
\[ h = 0.5 \]