The osteology of spiral structure generated by major planets in proto-planetary disks

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ABSTRACT

Here I describe numerical calculations of the motion of particles in a disk about a solar-mass object perturbed by a planet on a circular orbit with mass greater than 0.001 of the stellar mass. A simple algorithm for simulating bulk viscosity is added to the ensemble of particles and the response of the disk is followed for several planet rotation periods. Spiral structure forms near the inner Lindblad resonance (2-1) and extends to the planetary orbit radius (co-rotation). As for gaseous disks on a galactic scale perturbed by a weak rotating bar-like distortion, this is shown to be related to the appearance of two perpendicular families of periodic orbits near the resonance combined with dissipation which inhibits the crossing of streamlines. Spiral density enhancements result from the crowding of streamlines due to the gradual shift between families. The results, such as the dependence of pitch-angle on radius and the asymmetry of the spiral features, resemble those of sophisticated calculations that include more physical effects. This illustrates that the fundamental process of spiral formation via interaction with planets in such disks is due to orbital motion in a perturbed Keplerian field combined with dissipation.

Key words: Planetary systems: proto-planetary disks – planet-disk interactions – methods: hydrodynamics

1 INTRODUCTION

A striking result of recent high resolution near-infrared and sub-millimeter observations of disks around young stars is the discovery of a wealth of structure (Flock et al. 2015, Perez et al. 2016) – rings, gaps and spirals. Such disks, on the scale of tens to hundreds of astronomical units, are the sites for nascent planetary systems, and the observed structure probably reflects, either as cause or effect, the formation of planets within the disk. The near-infrared radiation is thermal emission of solid particles while the millimeter wave observations – line and continuum – reflect the distribution of the gaseous component of these proto-planetary disks (PPDs). The detection of structure in both these wavelength regimes (Huang et al., 2018) demonstrates that the observed structure is present in both the solid and gaseous components and that this is not only surface phenomena but arises in the underlying disk itself.

For decades it has been appreciated that satellites can sculpt the structure of planetary ring systems (as in that of Saturn): clearing gaps, forming rings, and exciting spiral structure (Goldreich & Tremaine 1979, 1980; Shu 1984). The new observations of PPDs maybe evidence of a similar phenomenon on the scale of forming planetary systems as explored in the work of Goodman & Rafikov (2001) and Ogilvie & Lubow (2002). Significantly a non-trivial fraction of the structure observed in these disks, perhaps one-fifth to one-fourth, is of spiral form, in some cases even grand design two-arm spirals as in galaxy disks (Dong et al. 2018). This suggests that the mechanisms for the generation of structure in PPDs are similar to those in galaxy disks: gravitational instability in the gaseous disk or the effect of pre-existing non-axisymmetric structure in the gravitational field (or some combination of the two). It is the second mechanism that I will consider here.

In galaxies, detailed structure is more evident in gas-rich disk systems than in spheroidal systems or early-type disks that are relatively deficient in gas. An aspect of this difference is due to star-formation which, in effect, lights up the existing structure in the disk. But the argument has been made that the basic structure itself is due to the presence of a dissipative medium that responds in a highly non-linear way to weak perturbations in the axisymmetric potential of the dominant stellar
component – a spiral or a bar form in the underlying stellar distribution and hence in the gravitational field. For example, a weak bar distortion in the potential can excite conspicuous rings or spirals in the gaseous component (Sanders & Huntley 1976, Schwarzschild 1981). In this phenomenon periodic orbits play a primary role (van Albada & Sanders 1982) in forming the basis for gas streamlines.

The same must be true in proto-planetary disks, but in this case the non-axisymmetric distortion is due to forming or formed giant planets. As in the case of galaxy disks it is near the inner-Lindblad resonance (a test particle orbits twice as the planet orbits once) where periodic orbits far from the planet itself have the largest deviations from circular motion and have their dominant effect in initiating structure formation. However there are obvious differences with general galactic disks: in the case of a planetary system the perturbation is one-sided, corresponding to an m=1 distortion in a Fourier description, but the response of orbits at the inner resonance is bisymmetric, m=2. Thus the gas response near the resonance would be expected to reflect this symmetry. I argue here that the overlapping of two families of periodic orbits in a dissipative medium is the basic driver of two-arm spiral structure formation excited by planets in proto-planetary disks, although one would expect m=1 variations about this bisymmetric structure near corotation where the planet is no longer a weak perturbation.

The formation of spirals in disks perturbed by massive planets has been demonstrated in earlier numerical hydrodynamical calculations, so this is not a new result (Dong et al. 2015; Zhu et al. 2015). These previous calculations are sophisticated, including effects such as three dimensional structure, multi-components, and radiative transfer, whereas the simulations employed here are simple in comparison: These are two dimensional, where the zeroth order axisymmetric force is the inverse square law due to the central star (on the order of one solar mass). This symmetry is broken by a planet of one to several thousandths of the stellar mass (one to several Jupiter masses) on a circular orbit within the disk at a distance of several tens of astronomical units. The disk is represented by an ensemble of several thousand particles, and the effects of dissipation are simulated by giving each particle an interaction size on the order of two astronomical units – an interaction which reduces the velocity differences between particles over this distance.

The goal is not to model specific systems but to reduce the problem to its bare bones. The only physics entering the calculation is two-dimensional motion in a perturbed Keplerian potential including a crude mechanism for simulating viscosity. None-the-less with this brutalized approach the spiral structure revealed in more elaborate simulations is re-produced, and the basic anatomy of spiral structure generation in proto-planetary disks can be understood in the terms of the response of sticky test particles at resonance. I do not mean to imply that this is the only, or even the primary, mechanism underlying the observed spirals in the astrophysical environment, but it may be a fundamental driver of this phenomenon.

These spiral arms are, of course, kinematic spirals; they are not dynamic self-gravitating structures which would certainly be expected in cases where the disk mass is a substantial fraction of that of the central star (Dong et al. 2018). None-the-less, the fact remains that massive planets do form in such disks and will, inevitably, have the effect of generating spiral structure via this mechanism within a sufficiently viscous surrounding gaseous disk.

2 RESPONSE OF GAS DISKS TO A WEAK BISYMMETRIC PERTURBATION.

The equation of hydrodynamics in Lagrangian form, neglecting pressure gradient forces and viscous stresses, is of course the equation of motion of particle in the given gravitational potential. That is to say, in the absence of thermal, turbulent, viscous or magnetic pressure forces, the motion of an element of gas is described by an orbit, and in steady state gas flow lines correspond to simple non-intersecting periodic orbits. But a stronger statement can be made: Given an ensemble of particles moving in a potential, filling the entire volume of phase space permitted by the Hamiltonian constraint, and allowing these particles to be “sticky” in the sense that over some finite interaction distance the random velocities are reduced (in effect adding a viscous force that resists compression of a fluid element), then we find that such viscous dissipation forces the motion onto periodic orbits – i.e., the periodic orbits arise as attractors in the phase space of the system and each orbit family has its basin (Lake & Norman 1984), the region of phase space within which a particle will inevitably move toward the attractor.

Only a subset of such periodic orbits can represent streamlines: those that do not cross other period orbits (i.e., separate attractors with non-overlapping basins) and that are not self-looping. Otherwise hydrodynamical effects must intervene and the nature of the gas flow is altered, but often in a way that is clearly related to the original periodic orbits. This was demonstrated more than 40 years ago in Eulerian hydrodynamical simulations which followed the gas response in a galactic potential perturbed by a non-axisymmetric (cos(2θ)) term (Sanders & Huntley 1976). In these calculations the non-axisymmetric forcing is given a figure rotation such that all three principal resonances – inner-Lindblad (ILR), corotation, outer-Lindblad – (OLR) are present within the numerical grid. The assumed axisymmetric force law (r−1.5) assures that these resonances are equally spaced.

The principal families of simple periodic orbits (most nearly circular) near a principal resonance are elongated but differ in orientation by by 90 degrees. These two stable families, are designated X1 and X2 (Contopoulos & Mertzanides 1977) where X1 is dominant between the ILR and corotation and elongated parallel to the major axis of the bar-like distortion. X2 is elongated perpendicular to the distortion and is dominant within the ILR or between two ILRs. For particle orbits the
amplitude of the deviation from circular motion increases near the resonance and the two perpendicular families intersect. But the hydrodynamical simulations demonstrate that the transition from X2 to X1 is gradual and results in a rotation in gas streamlines which now crowd in the locus of a trailing two-arm spiral.

As a test, I have repeated this experiment using the Lagrangian sticky particle technique employed here. The details of this method have been described previously (Sanders 1998) and are briefly summarized below: The particles, in two dimensions, are disks all with radius \( \sigma \). A particle \( i \) may be influenced by a second particle \( j \) at a separation \( r_{ij} < \sigma \) where that influence is weighted by a third order polynomial \( w(x) = r_{ij}/\sigma \) with coefficients chosen such that the peak is at \( w(0) = 1.0 \) and \( w(x) \) falls smoothly to zero at \( x = 1 \). (\( w(x) = 1 - 3x^2 + 2x^3 \)) At every timestep \( \Delta t \), particle \( i \) adjusts its velocity so as to reduce the velocity difference with each neighbor \( j \) but only for approaching neighbors. If \( \mathbf{V}_{ij} \) is the component of the velocity difference along the line joining the two particles (\( \mathbf{V}_{ij} \) is a vector at the position of \( i \) pointing away from \( j \)) then over this time step particle \( i \) changes its velocity by an amount given by the vector sum

\[
\Delta \mathbf{v}_i = \alpha_i \sum_j \mathbf{w}(r_{ij}) \mathbf{V}_{ij}
\]

with

\[
\alpha_i = \Delta t_i / t_s
\]

where \( t_s \) is a dissipation timescale typically taken to be on the order of a characteristic orbit timescale.

This method provides an explicit bulk viscosity that is proportional to the local velocity divergence but only if that divergence is negative (converging flow). The method manifestly conserves linear momentum (the effect of particle \( j \) on particle \( i \) is equal but opposite that of particle \( i \) on particle \( j \)), angular momentum is also conserved to high precision in rotating viscous fluid in an axisymmetric potential. There is no explicit shear viscosity in this scheme, but, because it is dissipational, the method does not conserve energy; consequently there is some radial re-arrangement of the density of particles in the axisymmetric rotating fluid.

In repeating the Sanders-Huntley experiment I took 6000 finite size particles initially uniformly covering a circular area with a radius of 40 distance units (one unit could be taken as 10 pc). For each particle \( \sigma = 3.5 \) units so that on average one particle is overlaps 45 neighboring particles. The strength of the non-axisymmetric perturbation is initially zero but grows to a maximum over one pattern rotation period corresponding to a azimuthal force of 0.05 of the axisymmetric force. The pattern speed is such that the three principal resonances are located at radii 15, 30 and 45. The setup and the results are shown in Figure 1.

Figure 1a shows the principal periodic orbits in the perturbed potential near the inner resonance: the X1 family elongated parallel to the major axis of the bar distortion (along the X-axis), and the X2 family perpendicular to the bar distortion. These two families intersect near the resonance and therefore cannot represent gas streamlines. Figure 1b illustrates the distribution of 6000 particles after five pattern rotation periods in the fully perturbed potential but with no dissipation (this is a pure orbit calculation). We see the pattern of periodic orbits in the crowding of particles which gives rise to rings and gaps. Fig. 1c shows that paths of several particles near the resonance when the viscous force via eq. 1 is in place. These are, in effect, the gas streamlines; the gradual rotation of the streamlines by 90 degrees between X2 and X1 is evident and results in a trailing spiral pattern. Fig. 1d is the distribution of 6000 dissipative particles in the complete non-axisymmetric potential after five bar rotation periods. The trailing spiral structure is present throughout the radial range of the simulation and is similar (but not identical) to that found by Sanders-Huntley in the Eulerian scheme on a 40X40 cartesian grid; in the earlier simulation there is even a trace of the double spiral structure seen here. Overall, the method applied here provides an acceptable solution for the gas distribution and flow in the perturbed potential in the highly supersonic limit.

3 GAS FLOW IN A PROTO-PLANETARY DISK PERTURBED BY A GIANT PLANET

The proposal is that the generation of spiral structure in a PPD containing a Jupiter mass-scale planet is fundamentally the same as in the perturbed galaxy disk considered above with respect to the role of periodic orbits. The origin of the inverse square force is at the center and the source can be scaled to one solar mass. The planet is on a circular orbit at a radius of 32 units where one unit can be scaled to one AU. In the Keplerian potential, this places the ILR and OLR at 20 and 42 astronomical units respectively. Combined with a velocity unit of 1 km/s, the distance scaling of one AU and a mass scale of one solar mass, the time unit is 4.75 years (scaling is straightforward in this gravitational central force problem with no additional physics: keeping \( GM \) fixed while scaling distance by a factor \( f \) is equivalent to scaling the time unit by a factor of \( f^{1.5} \)).

Fig. 2 illustrates the structure of several periodic orbits in the rotating frame of the planetary orbit near the ILR with the different panels corresponding to four different values of the planetary mass: 0.001, 0.002, 0.004 and 0.006 in units of the central mass (1, 2, 4, and 6 Jupiter masses or \( M_J \)). The two perpendicular orbit families are evident in the perturbed
Figure 1. Particles and gas in weak rotating bar potential

Keplerian potential. In Fig. 2a (one Jupiter mass) these periodic orbits are seen to barely overlap. But for 2, 4, and 6 $M_J$ the overlapping becomes more pronounced as do deviations from $m=2$ symmetry. By the above arguments this would imply that the spiral structure should be weak for the 1 $M_J$ perturber, but increasingly conspicuous for the larger mass perturber, and indeed this is evident in the dissipational simulations.

Fig. 3 illustrates the response of the gas disk corresponding to these same planetary masses in the rotating frame. Again I take 6000 particles initially distributed with constant density in a disk with a radius of 40 AU. For every particle $\sigma = 1.8$, so initially the average number of interacting neighbors is 12. To avoid high accelerations an inner boundary is taken at 5 AU; particles crossing the inner boundary are reflected. The panels are snapshots showing the distribution of the ensemble of dissipational particles after four planet orbital periods (780 years). In all cases, a quasi-steady state is reached after two rotation periods.

In the case of a one Jupiter mass perturber, a faint spiral structure is visible as would be expected from the slight overlapping of periodic orbits, but as the mass or the perturber increase, the spiral response becomes stronger. There are two principal arms which wrap through roughly 180 degrees. The breakdown from bisymmetry due to the planet is especially evident in the primary arm which connects to the planet at corotation. There is a cluster of particles about the planet in all cases; this would correspond to the “Hill” sphere within which particles are permanently trapped by the planet (radius of about 3.5 AU for the 4 $M_J$ case)

The amplitude of the spiral response appears to be roughly proportional to the mass of the perturber over this range of planetary masses. This is illustrated in Fig. 4 which is a plot of the gas surface density (smearing particle mass by the
Figure 2. Periodic orbits in the rotating frame near inner resonance with various planet masses. Corotation is at $r = 32$ and the inner resonance at $r = 20$.

The smoothing function $w(x)$ over $\sigma$ as a function of azimuthal angle at a radius of 20 AU for a planet masses of $2 M_J$ (solid curve) and $6 M_J$ (dashed curve). Not only does the amplitude of the response increase with the mass of the perturber, but also the asymmetry between the two principle arms increases with perturber mass. This is expected because of the increasing $m=1$ signal with planet mass in the perturbed potential. This figure also illustrates an additional marker of the basic spiral structure – the angular separation between the primary and secondary arms. At 20 AU ($r/r_p = 0.62$) this varies from 140 degrees ($2 M_J$) to 165 degrees ($6 M_J$) which is broadly consistent with the results given by Bae & Zhu (2018).

For both principal arms the the pitch angle decreases moving inward from co-rotation; thus, the geometry of the arms is not well-described by a logarithmic spiral (a better description, although not perfect, is provided by a generalized hyperbolic spiral). For estimating the run of pitch angle with radius it is best to consider the density distribution displayed in a polar map as in Fig. 5 for the $M_p = 6 M_J$ case (density contours shown in the $\theta$-$r$ space). Here, the two principal arms (and a faint third arm) are evident as well as the deviation, most conspicuous for the primary arm, from a constant pitch angle.

With this display one can readily calculate the pitch angle via the relation $\psi = d \ln(r)/d \theta$; the results are shown in Fig.6 where the run of pitch angle is plotted against radius in terms of the semi-major axis of the $6 M_J$ planet. The solid points show the results for the primary inner spiral arm, the triangle for the secondary arm, the square points for the outer spiral arm (beyond the planet). The X points are values from the numerical calculations of Zhu et al. (2015) ranging over planetary masses from $1 M_J$ to $6 M_J$. The form and range of the pitch angle dependence do not vary greatly with planetary mass.
Figure 3. Snap-shots of the quasi-steady-state density distribution after 4.5 planet rotation periods in the fully dissipative gas of 6000 particles reflecting the fact that streamline rotation of 90 degrees is the maximum possible over the range between ILR and corotation independent of planetary mass.

It is also clear that the results on pitch angle, asymmetry and arm separation are generally consistent with the sophisticated calculations of Zhu et al. This is significant because the only physics in the present calculation is orbital motion in the perturbed Newtonian potential with a crude algorithm for viscosity.

4 TWO FINAL POINTS

In all of the examples considered above, the radius of the circular planetary orbit was near the edge of the gaseous disk ($r_p = 32$, $R_d = 40$). But what is the response of the gas when the planet is more deeply embedded within the disk? Such a case is shown in Fig. 7 where $R_p = 28$ and $R_d = 50$ where $M_p = 4$; here the disk extends beyond the outer-Lindblad resonance at a distance of 37.

In the first panel (Fig. 7a) we see the response of the ensemble of 6000 particles with no dissipation (pure dynamical orbits) after five planetary rotation periods. This reflects the basic structure of the periodic orbits; in particular we see a set of rings and gaps, particularly obvious at the location of the OLR. Fig. 7b illustrates the response when dissipation is added via eq. 1. Within the orbit of the planet the spiral pattern is that seen in the previous simulation (Fig. 3c)), but beyond corotation a second almost disconnected two-arm spiral is evident; the principal arm is the extension of the planetary wake.
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Figure 4. Azimuthal distribution of gas surface density for $2 \, M_J$ (solid curve) and $4 \, M_J$ (dashed curve).

beyond co-rotation and the secondary arm appears completely disconnected from the inner structure. This could be viewed as a prediction: For a disk extending well beyond the circular planetary orbit of a major planet, one should observe two separate spiral patterns.

The properties of the double spiral system are more obvious on a polar density map as shown in Fig. 8. It is particularly clear on this plot that the run of pitch angles for arms beyond the planet orbit radius differs from that within corotation in the sense that the pitch angles become smaller with increasing radius. Moving away from corotation in either direction, the arms wrap up. This could be useful in identifying the location of the planet with respect to the spirals.

A second issue concerns deviations from a pure circular orbit. What is the form of the spiral pattern excited by a planet on a more general elliptical orbit? Figure 9 illustrates the effect of such deviations for the $4 \, M_J$ example. In both both cases the the elliptical orbit is broken down into the circular motion of the guiding center and the epicycle. In panel 9a the radius of the guiding center is 35 AU and the eccentricity is 0.1. In panel 9b the guiding center is at 38.7 AU and the eccentricity is 0.21.

The basic spiral pattern is essentially identical to that excited by the planet on the circular orbit at the guiding center, but the structure of the primary arm that connects to the planet differs and is transient. This also generally true for more eccentric orbits (eccentricity of 0.4 for example) but additional transient structures appear at corotation. I have not pursued this further because of the certainty of secular evolution of the planetary orbit due to gravitational interactions with the excited structure (orbital decay). But the conclusion is that the basic spiral pattern generated is quite independent of deviations from circular motion, expected because the basic pattern of underlying periodic orbits depends on the radius of the guiding center.

5 CONCLUSIONS

In an axisymmetric potential perturbed by a weak rotating bisymmetric distortion two families of periodic orbits elongated perpendicular to one another appear near the inner-Lindblad resonance. These orbits intersect and therefore cannot represent gas stream lines; viscosity forces gas flow onto gradually rotating stream lines and the resulting density distribution is of a spiral form – hence the appearance of spiral structure in a pure bar-like potential.

The same effect occurs in a disk rotating in a Keplerian potential perturbed by a planet even though the perturbation is not bisymmetric. This is because the same two orbit families arise near the inner resonance (2-1 resonance) and the overlapping of these families becomes significant when the mass of the satellite is more than 1/1000 of the central mass. Hence appearance
Figure 5. Gas density distribution for 6 $M_J$ case in a polar projection corresponding to panel 3d.

Figure 6. Pitch angles as a function of radius scaled to planet radius for 6 $M_J$: round-square solid points from the present calculations (solid triangles are secondary spiral); crosses are from the simulations of Zhu et al. 2016.
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Figure 7. Particle and gas distribution for disks extending beyond the outer resonance (r=42).

Figure 8. Gas surface density distributions in polar projection for disk extending beyond the outer resonance (Fig. 7a).

Figure 9. Gas density distribution for 4 $M_J$ planet on non-circular orbits. The epicycle about the guiding center is shown in both cases.
of spiral structure is inevitable in a sufficiently dissipative gas and dust disk about a solar mass star perturbed by a planet of one Jupiter mass or larger.

In calculations described here the only physical effects included are two-dimensional orbital motion in a perturbed Keplerian potential combined with a simple mechanism for bulk viscosity. This is all that is necessary for the generation of spiral structure in the disk environment of the forming planetary system. There is no radiative transfer, no third dimension, no two-fluid hydrodynamics, no shadowing, no self-gravity of the disk. That is not to say that such effects are absent or play no role, but that the form of the periodic orbits near resonance provides the essential ingredient for the spiral response of a dissipative medium in a perturbed Keplerian field. This sort of insight can be lost in more complicated calculations which do include these additional effects.

I have not compared the calculations here directly to recent observations because, with respect to the observable characteristic of the generated spiral structure, these results are generally consistent the more detailed simulations which include more physical effects. Such comparisons have been made (eg, Zhu et al. 2015, Bae & Zhu 2018) and in so far as these agree with the observations, so do the present calculations. The principal point of the present work has been to gain some understanding of the underlying mechanism of spiral structure generation in planet-disk interactions – to elucidate periodic orbits as the skeleton of such structure.

I cannot claim that all spiral structure observed in PPDs is solely due to this mechanism of streamline crowding at resonances. The maximum pitch angle is limited to roughly 25 degrees which may be exceeded in some observed systems. The total range of wrapping is restricted to be about 180 degrees and there will inevitably be a breaking of bisymmetry in the observed spiral structure particularly near the perturbing planet. Certainly in some systems, particularly those with disk masses approaching that of the planet, a non-Keplerian potential as well as self-gravity in the disk will play a role in exciting and shaping the observed structure, with or without the presence of a massive planet. But massive planets do form in such disks and they will have the effect of exciting spiral structure. When the extent of the disk exceeds the outer-Lindblad resonance, there may even be the appearance of a second set of apparently disconnected spiral arms as in Fig. 8. But the acid test for this mechanism would be the actual observation of the perturbing (proto-) planet at a radius permitting an inner resonance at the location of the dominant spiral structure (Ren et al. 2018). Thus far, no such telltale object has been definitively detected.

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