Polynomial-time Construction of Optimal Tree-structured Communication Data Layout Descriptions

Robert Ganian
Algorithms and Complexity Group
Vienna University of Technology
Austria
rganian@gmail.com

Stefan Szeider
Algorithms and Complexity Group
Vienna University of Technology
Austria
stefan@szeider.net

Martin Kalany
Parallel Computing Group
Vienna University of Technology
Austria
kalany@par.tuwien.ac.at

Jesper Larsson Träff
Parallel Computing Group
Vienna University of Technology
Austria
traff@par.tuwien.ac.at

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Abstract

We show that the problem of constructing tree-structured descriptions of data layouts that are optimal with respect to space or other criteria, from given sequences of displacements, can be solved in polynomial time. The problem is relevant for efficient compiler and library support for communication of non-contiguous data, where tree-structured descriptions with low-degree nodes and small index arrays are beneficial for the communication soft- and hardware. An important example is the Message-Passing Interface (MPI) which has a mechanism for describing arbitrary data layouts as trees using a set of increasingly general constructors. Our algorithm shows that the so-called MPI datatype reconstruction problem by trees with the full set of MPI constructors can be solved optimally in polynomial time, refuting previous conjectures that the problem is NP-hard. Our algorithm can handle further, natural constructors, currently not found in MPI.

Our algorithm is based on dynamic programming, and requires the solution of a series of shortest path problems on an incrementally built, directed, acyclic graph. The algorithm runs in \( O(n^4) \) time steps and requires \( O(n^2) \) space for input displacement sequences of length \( n \).

1 Introduction

It is a common situation for instance in parallel, numerical libraries that substructures of large, static data structures have to be communicated among processors, e.g., row- or column vectors or sub-matrices of multi-dimensional matrices, or irregular substructures corresponding to the non-zeros or other special elements of larger structures. This requires efficient access to the typically non-contiguously stored substructure elements in some predefined order, either for the application which “(un)packs” the elements (from) to some structured communication buffer, or for the communication soft- or hardware to handle the non-consecutive communication in a way

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that is transparent to the application. For the latter approach, concise and efficient descriptions of such substructures are needed. For instance, lists of element addresses or displacements are neither concise (space proportional to the number of elements is required) nor efficient (processing time is at least doubled, since also the list has to be traversed). For substructures with some regularities, much better representations are obviously possible. Often, tree representations are used with leaves describing base-types and interior constructor nodes how subtrees are repeated. For example, complex data types in C-like languages can be built recursively using a small number of constructors (like arrays and structs) from given primitive types (ints, chars, doubles, etc.), and the resulting type trees describe to the compiler how data are laid out in memory. The same kind of mechanism could be used to describe substructures of such data types (but is not a part of C). The Message-Passing Interface (MPI) is an important example of a parallel communication interface, indeed often used to implement parallel numerical libraries, which provides a generic, explicit mechanism for describing non-consecutive application data to allow the library implementation to perform non-consecutive communication in an efficient way, possibly by directly exploiting hardware features for, e.g., strided, non-consecutive communication. Given such a tree-structured description of an application data layout, it is a natural question to ask whether this description is optimal under some given cost model reflecting the cost of storing or processing the description. Likewise, given a trivial description of a data layout in the form of a long list of addresses (or offsets, or displacements), it is natural to ask for an algorithm for constructing an efficient, that is, cost-optimal representation as a tree with some given set of constructors. In the MPI community, the former problem is referred to as type normalization, and the latter as type reconstruction. Both problems are eventually important for the implementation of very high-quality MPI libraries. The problems would be similarly important in other parallel interfaces or languages supporting communication of arbitrarily structured, non-consecutive data. Ideally, a compiler would be able to perform the normalization (optimization) of data layout descriptions given more or less explicitly by the application programmer in the code with the constructs available in the parallel language.

In this paper, we investigate primarily the type reconstruction problem for a given set of constructors, that is, the problem of finding the most concise tree representation of a given substructure specified by an explicit list of displacements. As the set of constructors, we use a convenient abstraction of the type constructors found in MPI [Chapter 4]. This is both a natural and powerful set that includes constructors for the case where a single substructure is repeated in a regular or irregular pattern as well as the case where different substructures are concatenated with given displacements. Our main result is to show that an optimally concise tree representation can be found in polynomial time for the whole set of constructors, and thus as a corollary that both type reconstruction and type normalization for the whole set of MPI derived data type constructors can be solved in polynomial time. This is an interesting result since the computational hardness of the problem was not known before. Indeed, the problem was believed not to be in P by parts of the MPI community. Specifically, we give an algorithm that finds an optimal type tree description for a sequence of displacements of length \( n \) in \( O(n^4) \) operations. The algorithm is based on a non-trivial use of dynamic programming requiring the solution of a single-source shortest path problem for each new subproblem solution. Using standard dynamic programming techniques, the space requirement is \( O(n^2) \).

MPI libraries typically employ simple forms of type normalization to derived data types set up by the application programmer (this is folklore, but see [5,6,10] for explicit descriptions). In recent papers [4,13], the problem was more systematically analyzed, and it was shown that when restricted to certain homogeneous constructors (those having a single child) the reconstruction and normalization problems can be solved quite efficiently in low, polynomial time. It was explicitly conjectured that the problems with the full set of MPI derived data type constructors
would be NP-hard \[3,4\]. We stress that when it is allowed to fold the constructed trees into even more concise, directed acyclic graphs (DAGs), the optimality of our construction is no longer guaranteed. We discuss this problem at the end of the paper.

The notion of an optimal tree-like representation of a data layout is of course relative to the way the tree will be used and processed by the parallel programming language or library implementation. Processing typically includes the ability to pack and unpack parts of the layout independently using hardware support for blocked, strided memory access and similar features of the communication subsystem. We do not deal with the problem of efficient datatype-tree processing here, but abstract storage and processing costs with a simple, parameterized cost model, which must be adapted to the concrete situation. The literature on optimization of the processing of tree representations of data layouts in MPI is large; some pointers are given in \[13\].

The paper is structured as follows. We define the set of considered constructors and precisely formulate the type reconstruction problem in Section 2. Our main result is given in Section 3, which describes our dynamic programming algorithm, proves correctness and establishes the complexity bound. In Section 4 we discuss how our approach can be extended to include other convenient and in specific situations more concise constructors, and how the problem changes when trees can be folded into DAGs. Concluding remarks, including a discussion of relevant future work in this area are given in Section 5.

2 The type reconstruction problem

A data layout is an ordered sequence of relative (integer) displacements, each indexing a certain base data type (integer, char, floating point number) relative to some base address. Since the semantics of base-types will not be important for the following, we abstract the problem to consider from here onward displacement sequences which we write as \(D = \langle d_0, d_1, \ldots, d_{n-1} \rangle\) with the displacements \(D[i] = d_i\) being indexed from 0 to \(n - 1\). We point out that the complexity of the problems that we investigate does not change by considering full type maps consisting of sequences of displacements with their associated basetype (and number of bytes occupied), as would have to be done in a concrete implementation of our algorithms for real libraries, although of course the structure of the reconstructed types may look different. A segment of an \(n\)-element displacement sequence from index \(i\) to index \(j\) is denoted by \(D[i, j] = \langle d_i, d_{i+1}, \ldots, d_j \rangle\), \(0 \leq i \leq j < n\). A prefix of length \(c\) is the segment \(D[0, c-1]\). The displacements of the sequence are arbitrary (non-negative, negative) integers, and the same displacement can appear more than once (although this will normally not be the case, and is often disallowed, e.g., for some uses of derived data types in MPI). Thinking of displacements as (Byte) addresses, it is clear that any application data layout can be described by a displacement sequence. The ordering constraint (displacement sequence, not displacement set) implies that data are accessed in a specific order. This is often important for data layouts used in communication operations.

Displacement sequences typically contain regularities and some form of structure, since they can be thought of as arising from a specific application, and this can be exploited to obtain more concise descriptions. We do this by type trees, where interior constructor nodes describe some ordered catenation of the layout(s) described by the child(ren) node(s). It is natural to ask for an efficient, polynomial time algorithm for computing the most concise and efficient representation for a given set of constructors and cost model.

We consider the following set of constructors that subsume constructors found in C-like programming languages, as well as the derived data type constructors found in MPI:
Definition 1 (Basic type constructors) A basic tree may be constructed from the following four basic constructors:

1. A leaf \( \text{con}(c) \) with count \( c \) describes a sequence of \( c \) adjacent relative displacements \( 0, 1, 2, \ldots, c-1 \).

2. A (homogeneous) vector \( \text{vec}(c, d, C) \) with count \( c \) and stride \( d \) describes the catenation of \( c \) sequences \( C \) at relative displacements \( 0, d, 2d, \ldots, (c-1)d \).

3. A (homogeneous) index \( \text{idx}(c, \langle i_0, i_1, \ldots, i_{c-1} \rangle, C) \) with count \( c \) and indices \( \langle i_0, i_1, \ldots, i_{c-1} \rangle \) describes the catenation of \( c \) sequences \( C \) at relative displacements \( i_0, i_1, \ldots, i_{c-1} \).

4. A heterogeneous index, or struct, \( \text{strc}(c, \langle i_0, i_1, \ldots, i_{c-1} \rangle, \langle C_0, C_1, \ldots, C_{c-1} \rangle) \), with count \( c \) and indices \( \langle i_0, i_1, \ldots, i_{c-1} \rangle \) describes the catenation of \( c \) sequences \( C_0, C_1, \ldots, C_{c-1} \) at relative displacements \( i_0, i_1, \ldots, i_{c-1} \).

For example, the displacement sequence \( \langle 3, 5, 7, 9, 11 \rangle \) can be described by \( \text{idx}(1, \langle 3 \rangle, \text{vec}(5, 2, \text{con}(1))) \). A more involved example is shown in Figure 1. Note that any displacement sequence \( D \) of length \( n \) can trivially be represented as \( \text{idx}(n, D, \text{con}(1)) \).

We refer to vertices of type trees as nodes, where each node is one of the constructors.

It can easily be shown that each of the MPI derived data type constructors (for contiguous, vector, index, and structured subtrees) \cite{MPI} Chapter 4] is expressible by the basic constructors of Definition 1 and that the mapping is almost one-to-one. For instance, the \text{MPI}_{\text{Type}}_{\text{vector}} constructor denotes a layout consisting of a strided sequence of blocks, each being a strided sequence of some type \( B \). This is expressed as \( \text{vec}(c, s, \text{vec}(b, e, B)) \) where \( c \) is the number of blocks, \( s \) their stride, \( b \) the number of elements in each block, and \( e \) the stride used within each block. We treat base types as sequences of bytes which can be expressed by leaf nodes, e.g., a 32-bit entity like \text{int} would be expressed by \text{con}(4). The \text{idx} constructor makes it possible to express the repetition of the same layout \( B \) each at some arbitrary displacement; for this only the sequence of start indices (and the size of this sequence) needs to be represented. The most expressive, arbitrary branching constructor \text{strc} can express the catenation of a sequence of possibly different, smaller layouts each starting at an arbitrary displacement. This is the only constructor node with arity greater than one. In contrast to the similar MPI constructor \text{MPI}_{\text{Type}}_{\text{create} \text{struct}}, which also takes a repetition count (blocklength) for each substructure, the \text{strc} constructor saves this extra sequence. If a substructure is indeed a repetition of some even smaller substructure, this information is part of the substructure and not of the \text{strc} node itself. The basic constructors increase in generality and storage cost: an \text{idx} node is a \text{strc} node where all substructures are similar, and therefore does not need to store a sequence of subtypes.
a vec node is an idx node with regularly strided displacements, which can be computed from a single scalar instead of storing an explicit index sequence. As the example in Figure 1 shows, the strc constructor makes unbounded compression possible over the idx constructor.

To make it possible to express further common patterns without redundancy, we also consider a few auxiliary constructors. The patterns that these constructors capture can all be expressed by two-level nestings of basic constructors, but possibly at a higher cost. For practical purposes and depending on the application usage patterns that are intended to be supported, it might therefore make sense to have a richer set of constructors. For instance, MPI has both an MPI_Type_create_indexed_block (which is captured by the idx basic constructor node) and an MPI_Type_indexed constructor which stores also a repetition count for each index. In cases where all substructures are repeated the same number of times, this is strictly redundant, and there are therefore use cases for both constructors. We include the auxiliary constructors to argue informally that our algorithm can handle a large set of reasonable constructors.

Definition 2 (Auxiliary type constructors) An extended tree may contain also the following two auxiliary constructors:

1. A strided bucket, vecbuc\(\langle c, d, e, \langle b_0, b_1, \ldots, b_{c-1} \rangle, C \rangle\) with count \(c\) and strides \(d, e\) describes the catenation of \(c\) sequences at relative displacements \(0, d, 2d, \ldots, (c - 1)d\). The \(i\)-th sequence is the catenation of \(b_i\) sequences \(C\) at relative displacements \(0, e, 2e, \ldots, (b_i - 1)e\).

2. An indexed bucket, idxbuc\(\langle c, e, \langle i_0, i_1, \ldots, i_{c-1} \rangle, \langle b_0, b_1, \ldots, b_{c-1} \rangle, C \rangle\), with count \(c\) and sub-stride \(e\) describes the catenation of \(c\) sequences at relative indices \(i_0, i_1, \ldots, i_{c-1}\). The \(i\)-th sequence is the catenation of \(b_i\) sequences \(C\) at relative displacements \(0, e, 2e, \ldots, (b_i - 1)e\).

As can be seen from the discussion above, the indexed bucket constructor corresponds to the MPI_Type_indexed constructor. There is no MPI counterpart of the other, arguably natural constructor. We discuss these constructors in more detail in Section 4.1.

Each basic or extended tree represents one displacement sequence, obtained by an ordered traversal of the nodes of the type tree. This process is called flattening and is captured by the algorithm in Listing 1 for the basic constructors; the auxiliary constructors can be handled similarly. The converse is not true: a displacement sequence will almost always have several possible type tree representations.

We make no claim that Listing 1 depicts a particularly good way of implementing flattening [14]. Note that the size of the displacement sequence described by a type tree \(T\) could be much larger than the number of nodes in \(T\). Within this paper, we assume that all numbers can be represented by a constant number of bits; otherwise, our main result still holds, but the upper bound on space requirements increases by a logarithmic factor.

By the conciseness of a type tree we mean the space taken by the representation. This is constant for vector and leaf nodes and proportional to the size of the index and type sequences for the other constructors. Processing costs are related to conciseness: the concise vector constructor that describes a strided repetition of a sub-pattern can often be handled by strided memory-copy or strided communication operations, whereas constructors with sequences of displacements or types need at least a traversal of the corresponding sequences and typically entails a more irregular and expensive access to memory. We will therefore first focus on a simple cost model for optimizing conciseness.

The cost of a type node shall be proportional to the number of words that must be stored to process the node. This includes the node type (\(\text{con}, \text{vec}, \text{idx}, \text{strc}\)), count, displacement or pointer to index or type array, pointer to child node(s), and a lookup cost for the elements in
Listing 1: Flattening procedure defining the displacement sequence represented by a given basic tree $T$. The procedure is called with a base offset, which will normally be 0. The procedure can trivially be extended to also cover extended trees.

```plaintext
Function Flatten(T, base)
switch T.nodetype do
  case con /* leaf of consecutive indices */
    for i ← 0; i < T.c; i++ do
      print base + i
  case vec /* strided layout */
    for i ← 0; i < T.c; i++ do
      Flatten(T.subtype, base + i · T.d)
  case idx /* indexed layout */
    for i ← 0; i < T.c; i++ do
      Flatten(T.subtype, base + T.D[i])
  case strc /* indexed layout with subtypes */
    for i ← 0; i < T.c; i++ do
      Flatten(T.subtypes[i], base + T.D[i])
end switch
```

lists of indices or types:

\[
\begin{align*}
\text{cost}(\text{con}(c)) &= K_{\text{con}} \\
\text{cost}(\text{vec}(c, d, C)) &= K_{\text{vec}} \\
\text{cost}(\text{idx}(c, \langle \ldots \rangle, C)) &= K_{\text{idx}} + cK_{\text{lookup}} \\
\text{cost}(\text{strc}(c, \langle \ldots \rangle, \langle \ldots \rangle)) &= K_{\text{strc}} + 2cK_{\text{lookup}}
\end{align*}
\]

The constants can be adjusted to reflect other overheads related to representing and processing a node. We define the cost of a type tree $T$ to be the additive cost of its nodes $T_i$: cost($T$) = $\sum_i$ cost($T_i$).

Listing 2: A possible Typenode structure for representing nodes in type trees or DAGs.

```plaintext
struct {
  enum nodetype = {con, vec, idx, strc}
  int c /* count */
  int d /* stride */
  int D[] /* displacement of subtypes */
  Typenode subtype /* subtype */
  Typenode subtypes[] /* array of subtypes */
} Typenode
```

For the examples given in this paper, we take $K_{\text{con}} = K_{\text{vec}} = K_{\text{idx}} = K_{\text{strc}}$, and $K_{\text{lookup}} = 1$. For instance, with a C-style structure as shown in Listing 2 to represent any of the type constructors, all constructors indeed have the same constant in the cost (which we could take as 6 units). We remark that our algorithm is not dependent on the specific choice of the cost
function, and that our results also hold for other reasonable cost functions where the cost of a
node is a function of the node itself and the costs of its children.

We can now formally define the problem that we will solve in the next section. Recall that
a type tree $T$ represents a displacement sequence $D$ if $\text{Flatten}(T, 0) = D$.

**Basic Type Reconstruction Problem**

**Instance:** A displacement sequence $D$ of length $n$.

**Task:** Find a least-cost (or optimal) basic tree $T$ representing $D$; that is, $\text{cost}(T) \leq \text{cost}(T')$ for any basic tree $T'$ representing $D$.

3 Basic tree reconstruction in polynomial time

We now present our main result, namely that the **Basic Type Reconstruction Problem** can be solved in polynomial time. Subsequently show that extending the set of the auxiliary constructors of Definition 2.

**Theorem 1** For any input displacement sequence $D$ of length $n$, the **Basic Type Reconstruction Problem** can be solved in $O(n^4)$ time and $O(n^2)$ space.

**Proof outline:** We first give a characterization of the structure of optimal basic trees (Lemma 1) which allows for a simple and elegant procedure to solve the special case of displacement sequences in normal form (Definition 0).

The fundamental observation for the proof is that any (non-trivial) displacement sequence can be described by either a catenation of the same kind of shorter displacement sequences (and thus by either a vector or an index constructor) or by a catenation of different, but shorter displacement sequences (and thus by a struct constructor). In both cases, for an optimal description, the description of the shorter sequences must likewise be optimal, and the principle of optimality applies. This intuition is formalized in Lemma 2 and Lemma 3. Lemma 4 proves the claim for the special case of displacement sequences in normal form, with a detailed procedure given in Listing 5.

Finally, Lemma 5 shows how to construct an optimal basic tree for any displacement sequence out of an optimal basic tree representation of its normal form.

**Definition 3 (Repetition, Strided Repetition)** A repetition in a displacement sequence $D$ of length $n$ is a prefix $C = D[0, q - 1]$ of length $q$ s.t. $q$ is a divisor of $n$ and for all $i, j$, $1 \leq i < n/q$, $0 \leq j < q$ we have that $D[j] - D[0] = D[iq + j] - D[iq]$. A strided repetition of length $q$ additionally fulfills $D[(i + 1)q] - D[iq] = D[q] - D[0]$ for all $i$, $0 \leq i < n/q - 1$, where $d = D[q] - D[0]$ is the stride of the repetition.

The intention of the functions `Repeated` and `Strided` (see Listing 3) is to find (strided) repetitions $C$ of a displacement sequence $D$ that can be exploited to represent $D$ via an idx or vec constructor with subsequence $C$. It is easy to see that `Repeated` and `Strided` as outlined both take linear time.

As mentioned above, any displacement sequence $D$ can be described by either a catenation of the same kind of shorter displacement sequences or by a catenation of different, but shorter displacement sequences. Additionally, a representation via a con node is possible if $D$ is a trivial displacement sequence $(0, 1, \ldots, n - 1)$. In terms of type trees, this means that an optimal basic tree $T$ for a displacement sequence $D$ is either
Note that adding some value represented by the basic tree rooted at a node on every root to leaf path. Let \( D \) be a fixed displacement sequence and let \( T \) be a basic tree representing \( D \) with a minimum number of bad nodes. We will show that \( T \) is, in fact, nice.

Assume that a bad index node (the proof is analogous for a bad struct node) \( N_I = \text{idx}(c, \langle i_0, \ldots, i_{c-1}, \ldots \rangle) \) is present in the \( k \)-th subtree of a struct node \( N_S = \text{strc}(c, \langle i'_0, \ldots, i'_{c-1}, \ldots \rangle) \) s.t. there is no other shifted node on the path from \( N_I \) to \( N_S \). We can change \( N_I \)

### Listing 3: Trivial checks for repetitions and strided repetitions.

```plaintext
Function Repeated(D, n, q)
for i ← q; i < n; i ← i + q do
    for j ← 1; j < q; j ← j + 1 do
        if D[j] - D[0] \neq D[i + j] - D[i] then
            return false
return true

Function Strided(D, n)
d ← D[1] - D[0]
for i ← 1; i < n; i ← i + 1 do
    if D[i] - D[i - 1] \neq d then return false
return true
```

1. \( T = \text{con}(n) \), a single \text{con} node with count \( n \); or

2. \( T = \text{vec}(c, d, S) \), where the prefix \( D[0, q - 1] \) of length \( q = n/c \) is a strided repetition in \( D \) with stride \( d \) and \( S \) is an optimal basic tree for the prefix \( \langle D[0], \ldots, D[q - 1] \rangle \); or

3. \( T = \text{idx}(c, \langle i_0, \ldots, i_{c-1} \rangle, S) \), where the prefix \( D[0, q - 1] \) of length \( q = n/c \) is a repetition in \( D \), \( S \) is an optimal basic tree for the sequence \( \langle D[0] - i_0, \ldots, D[q - 1] - i_0 \rangle \) and the indices \( i_0, \ldots, i_{c-1} \) are such that \( \text{Flatten}(T, 0) = D \); or

4. \( T = \text{strc}(c, \langle i_0, \ldots, i_{c-1} \rangle, \langle S_0, \ldots, S_{c-1} \rangle) \), where the \( S_j \) for \( 0 \leq j < c \) are optimal basic trees for some sequences \( C_j \) which together with the indices \( i_0, \ldots, i_{c-1} \) are such that \( \text{Flatten}(T, 0) = D \).

While the first case can be handled with a single scan of \( D \), the others are more involved. In the following, we give a more detailed characterization of (optimal) basic trees to tackle the problem.

**Definition 4 (Shifted node)** We call an index node \( \text{idx}(c, \langle i_0, \ldots \rangle, C) \) or a struct node \( \text{strc}(c, \langle i_0, \ldots \rangle, \langle \ldots \rangle) \) with \( i_0 \neq 0 \) a shifted node; \( s = i_0 \) is called the node’s shift.

Note that adding some value \( s \) to all indices of an \text{idx} or \text{strc} node \( N \) shifts the sequence represented by the basic tree rooted at \( N \) by \( s \).

**Definition 5 (Nice basic tree)** A nice basic tree contains at most one shifted node, which is the first \text{idx} or \text{strc} node on every root to leaf path.

**Lemma 1** For any basic tree \( T \) representing a displacement sequence \( D \), a nice basic tree representation \( \hat{T} \) of \( D \) of equal cost exists.

**Proof:** A node is bad if it is a shifted node and it is not the first \text{idx} or \text{strc} node on every root to leaf path. Let \( D \) be a fixed displacement sequence and let \( T \) be a basic tree representing \( D \) with a minimum number of bad nodes. We will show that \( T \) is, in fact, nice.

Assume that a bad index node (the proof is analogous for a bad struct node) \( N_I = \text{idx}(c, \langle i_0, \ldots, i_{c-1}, \ldots \rangle) \) is present in the \( k \)-th subtree of a struct node \( N_S = \text{strc}(c', \langle i'_0, \ldots, i'_{c-1}, \ldots \rangle) \) s.t. there is no other shifted node on the path from \( N_I \) to \( N_S \). We can change \( N_I \)
to a non-shifted index node by subtracting its shift \( s = i_0 \) from all indices \( i_j \), for \( 0 \leq j < c \) and adding \( s \) to the \( k \)-th index \( i'_k \) of \( N_S \), i.e., \( \tilde{N}_I = \text{idx}(c, (0, i_1 - s, \ldots, i_{c-1} - s), \ldots) \) and \( \tilde{N}_S = \text{strc}(c', (i'_0, \ldots, i'_k + s, \ldots, i'_{c-1}), \ldots) \). Notice that the basic tree obtained in this way still represents the same displacement sequence \( D \) but contains one less bad node, and hence the existence of such a node \( N_I \) would contradict our choice of \( T \).

Hence there is no \text{strc} node on the path from a bad node \( N_I \) to the root node \( R \). If this path contains an index node \( N'_I \neq N_I \), proceed analogously to the previous case: \( \tilde{N}_I = \text{idx}(c, (0, i_1 - s, \ldots, i_{c-1} - s), \ldots) \) and \( \tilde{N}'_S = \text{strc}(c', (i'_0 + s, \ldots, i'_{c-1} + s), \ldots) \). Again, the obtained basic tree also represents \( D \) but contains one less bad node, contradicting our original choice of \( T \). Consequently, \( T \) does not contain any bad nodes and thus must be a nice basic tree. \( \square \)

**Corollary 1** Any optimal basic tree \( T \) contains at most one index node with count 1, i.e., at most one node of the form \( N = \text{idx}(1, (i_0), \ldots) \). Additionally, there is no other \text{idx} or \text{strc} node on the path from \( N \) to the root.

**Proof:** Assume that \( T \) contains two index nodes with count 1. Since \( T \) is a tree, there is an index node \( N \) with count 1 s.t. the path from \( N \) to the root node of \( T \) contains another \text{idx} or \text{strc} node. In a cost-equivalent nice basic tree representation \( \tilde{T} \) (obtained by applying the procedure from the proof of Lemma 1), the corresponding index node is \( \tilde{N} = \text{idx}(1, (0), T') \). Note that the type tree rooted at \( \tilde{N} \) represents exactly the same displacement sequence as its subtype \( T' \). Thus a representation \( T' \) of less cost exists, which contradicts the assumption that \( T \) is optimal. \( \square \)

The following proposition, although not directly required for the analysis, provides some additional insight into the structure of optimal basic trees.

**Proposition 1** The height of an optimal basic tree is \( O(\log n) \).

**Proof:** It is easy to see that an optimal basic tree does not contain two consecutive \text{strc} nodes, as they can always be merged into one while reducing the cost. For any basic tree \( T \) that represents a sequence of length \( n \), a basic tree \( \text{idx}(c, (\ldots), T) \) or \( \text{vec}(c, \ldots, T) \) with \( c \geq 2 \) represents a sequence of length at least \( 2n \). Let \( P \) be a maximum-length path from a leaf to the root of an arbitrary optimal basic tree. Since any optimal basic tree contains at most one \text{idx} node with count \( c = 1 \) (Corollary 1) and no \text{vec} node with \( c = 1 \), the length of the represented sequence at least doubles with at least every other node on \( P \). \( \square \)

**Definition 6** The normal form \( \hat{D} \) of a displacement sequence \( D \) of length \( n \) is defined as \( \hat{D}[i] = D[i] - D[0] \), for all \( i, 0 \leq i < n \).

In other words, the normal form \( \hat{D} \) of a displacement sequence \( D \) is obtained by shifting \( D \) so that its first element is 0.

**Corollary 2** An optimal basic tree \( T \) for a displacement sequence \( \hat{D} \) in normal form does not contain any shifted nodes or any \text{idx}, \text{vec} or \text{strc} node with count 1.

**Proof:** It follows directly from Lemma 1 and Corollary 1 that there exists an optimal basic tree \( T \) for \( \hat{D} \) which does not contain any shifted nodes. Note that a non-shifted \text{idx}, \text{vec} or \text{strc} node with count 1 does not change the represented sequence. Thus, removing such nodes
from a basic tree reduces the cost while not changing the represented displacement sequence. It follows that no such node can be part of an optimal basic tree. □

Observe that since there are no shifted nodes in an optimal basic tree \( T \) for \( \hat{D} \), any subtree of \( T \) represents a segment of \( \hat{D} \) in normal form. In the following, we will use \( T_{i,j} \) to denote an optimal basic tree representation for the normalized segment \( \hat{D}[i,j] \) of \( \hat{D} \).

For convenience, we define the function \( \text{Min}(S,T) \) which, given two basic trees \( S \) and \( T \), returns the one with least cost (if either is null, the other is returned). Note that the cost of a basic tree can trivially be computed by a simple traversal. However, when constructing basic trees from the bottom up (as we will do in this section), we keep for each node the cost of the subtree rooted at that node. This allows for the cost of a basic tree to be queried in constant time and thus for a constant-time implementation of \( \text{Min} \).

\[
\text{Listing 4: Algorithm to find a least-cost representation for a displacement sequence in normal form with an idx or vec node as root node.}
\]

1. \textbf{Function} \( \text{Repetition}(D, n) \)
2. \( T_r \leftarrow \text{null} \)
3. \textbf{foreach} divisor \( q \) of \( n \), \( q < n \) \textbf{do}
4. \( c \leftarrow n/q \)
5. \textbf{if} \( \text{Repeated}(D, n, q) \) \textbf{then}
6. \( \text{for} \ i = 0; i < c; i++ \textbf{ do} \)
7. \( I[i] \leftarrow D[iq] \)
8. \( T_{\text{idx}} \leftarrow \text{idx}(c, I, T_{0,q-1}) \)
9. \( T_r = \text{Min}(T_{\text{idx}}, T_r) \)
10. \textbf{if} \( \text{Strided}(I, c) \) \textbf{then}
11. \( d \leftarrow I[1] - I[0] \)
12. \( T_{\text{vec}} \leftarrow \text{vec}(c, d, T_{0,q-1}) \)
13. \( T_r = \text{Min}(T_{\text{vec}}, T_r) \)
14. \textbf{return} \( T_r \)

\textbf{Lemma 2} Let \( \hat{D} \) be any displacement sequence of length \( n \) in normal form and assume that optimal basic tree representations for all normal form prefixes of length less than or equal to \( \lfloor n/2 \rfloor \) are known. A representation \( T_r \), where the root node of \( T_r \) is either an idx or a vec node and \( T_r \) is of least cost w.r.t. all possible representations of that form, can be computed in \( O(n\sqrt{n}) \) time.

\textbf{Proof:} Listing 4 enumerates all possible representations of the desired form and chooses the one with least cost among them. Note that for the divisor \( q = 1 \), the trivial representation \( \text{idx}(n, \hat{D}, \text{con}(1)) \) (which exists for any displacement sequence \( \hat{D} \)), is generated and thus a valid representation for \( \hat{D} \) is guaranteed to be found. For the same reasons as given in Corollary 2, idx nodes with count 1 cannot be part of a least-cost representation of the desired form and thus need not be considered.

The number of divisors of \( n \) is upper-bounded by \( 2\lfloor \sqrt{n} \rfloor \) and, by assumption, optimal representations for all prefixes of \( \hat{D} \) of length less than or equal to \( \lfloor n/2 \rfloor \) are known, i.e., \( T_{0,j} \) is known for all \( j \), \( O \leq j \leq n/2 \). This implies the claimed runtime bound. □
Lemma 3 Let $\hat{D}$ be any displacement sequence of length $n$ in normal form and assume that optimal basic tree representations are known for all normal form segments of length strictly less than $n$. A representation $T_s$, where the root node of $T_s$ is a $\text{strc}$ node and $T_s$ is of least cost w.r.t. to all possible representations of that form, can be computed in $O(n^2)$ time.

Proof: Construct a weighted, directed acyclic graph $G = (V, E, w)$ with $V = \{v_0, \ldots, v_n\}$, $E = \{(v_i, v_j) \mid 0 \leq i < j \leq n, j - i < n\}$ and the weight function $w$ which is defined for all edges $(v_i, v_j)$ in $E$ as $w(v_i, v_j) = 2K_{\text{lookup}} + \text{cost}(T_{i,j-1})$. The intended meaning of this construction is as follows. A node $v_i$ corresponds to the $i$-th element of $\hat{D}$ ($v_n$ is a special vertex that corresponds to the hypothetical first element after the end of $\hat{D}$) and an edge $(v_i, v_j)$ with $i < j$ corresponds to the segment $\hat{D}[i, j - 1]$ in normal form. The weight of an edge $(v_i, v_j)$ is equal to the cost of the optimal representation $T_{i,j-1}$ of the segment $\hat{D}[i, j - 1]$ (which exists by the assumption) plus a cost of $2K_{\text{lookup}}$ for including this representation as a subtype in a $\text{strc}$ node. The edge $(v_0, v_n)$, which is not part of the constructed graph, can be thought of as corresponding to the type tree $T_{0,n-1}$, i.e., the optimal type tree representation of $\hat{D}$ we want to compute.

Let $P = \langle v_0, u_1, \ldots, u_k, v_n \rangle$ be a shortest path in $G$ from $v_0$ to $v_n$ with $u_i \in V$ for $1 \leq i \leq k$. Then the basic tree $\text{strc}(k + 1, \langle \hat{D}[0], \hat{D}[u_1], \ldots, \hat{D}[u_k] \rangle, (T_{0,u_1-1}, T_{u_1,u_2-1}, \ldots, T_{u_k,n-1}))$ is a valid representation of $\hat{D}$. Note that by construction, for any valid representation of $\hat{D}$ of the desired form, a corresponding path from $v_0$ to $v_n$ exists in $G$ and thus a shortest path represents the desired solution of least cost. Given $P$, this representation can be constructed in linear time, since optimal representations for all required segments are known by the assumption. The resulting graph has $\binom{n}{2}$ edges and the runtime is dominated by the cost of $O(n^2)$ time for finding a shortest path in a DAG. □

We can now give the complete dynamic programming algorithm for constructing optimal basic trees for displacement sequences in normal form, which proves Lemma 4. Due to Lemma 4, it suffices to construct an optimal nice basic tree which according to Corollary 2 cannot contain any shifted nodes nor any $\text{idx}$, $\text{vec}$ or $\text{strc}$ nodes with count 1. The algorithm is shown in Listing 5.

Lemma 4 For any input displacement sequence $\hat{D}$ of length $n$ in normal form, the Basic Type Reconstruction Problem can be solved in $O(n^4)$ time and $O(n^2)$ space.

Proof: The input to the algorithm is an $n$-element displacement sequence $\hat{D}$ in normal form. The algorithm computes an optimal basic tree $T[i,j]$ for each normalized segment $\hat{D}[i,j]$, $0 \leq i \leq j < n$, which is stored with edge $(i,j+1)$ in the constructed graph $G$. Note that the solution for the whole input sequence $\hat{D}$ can be read off of the edge $(v_0, v_n)$.

The algorithm starts with a preprocessing step to find all segments whose normal form is representable with a single $\text{con}$ node. Note that the normal form of any segment of length 1 can trivially be represented as $\text{con}(1)$ and since no other valid representations exist for this particular kind of displacement sequence, this representation is optimal. A straightforward implementation of this preprocessing step as in Listing 5 is clearly feasible in time $O(n^2)$.

The algorithm computes optimal basic tree representations for all normalized segments of $\hat{D}$, via a bottom up dynamic programming approach. The dynamic programming table to be filled in is implicit in the graph $G$, where each segment $\hat{D}[i,j]$ is associated with an edge $(v_i, v_{j+1})$. Note that after the preprocessing step, solutions for all segments of length 1 are known. By incrementally computing optimal representations for all segments of length 2, \ldots, $n$, it is ensured that Lemmas 2 and 3 can be applied to compute an optimal representation for each segment as follows. A basic tree $T_r$, whose root node is either an $\text{idx}$ or a $\text{vec}$ node, and
representations of the prefixes of the argument displacement sequence \( D \). The optimal basic tree for a normalized general case, \( \hat{D} \) (i.e., for a segment \( \hat{D}[i, j] \)), we pass an additional argument \( o \) representing the offset of the segment within the input displacement sequence \( \hat{D} \) (i.e., for a segment \( \hat{D}[i, j] \), we have \( o = i \)), and in lines 10 and 12 replace the argument \( T_{0,q-i} \) with \( T_{o,o+q-i} \).

To compute \( T_s \) in Listing 5, contrary to Lemma 3, we do not construct a new graph for each segment when computing its representation \( T_s \). Instead a single dynamic, incrementally built

```latex
 Listing 5: Algorithm to find a least-cost basic tree representation.

\begin{verbatim}
1 Function Typetree(\( \hat{D}, n \))
2     /* Initialization */
3     \( G = (\{v_0, \ldots, v_n\}, \emptyset) \)
4     /* Preprocessing: find leaf nodes */
5     for \( i \leftarrow 0; i \leq n; i++ \) do
6         \( j \leftarrow i \)
7         \( T_{i,j} \leftarrow \text{con}(j - i + 1) \)
8         \( w_{i,j} \leftarrow 2 + \text{cost}(T_{i,j}) \)
9         Add edge \((v_i, v_{j+1})\) with basic tree \( T_{i,j} \) and weight \( w_{i,j} \) to \( G \)
10        \( j \leftarrow j + 1 \)
11        while \( j \leq n \) and \( \hat{D}[j] - \hat{D}[j-1] == 1 \)
12     /* Find solutions for all segments */
13     for \( l \leftarrow 2; l \leq n; l++ \) do
14         for \( i \leftarrow 0; i \leq n - l; i++ \) do
15             /* Compute optimal basic tree for normalized segment \( \hat{D}[i, i+l-1] \) */
16                 \( j \leftarrow i + l - 1 \)
17             /* Find best representation with idx or vec node as root */
18             \( \hat{D}_{i,j} \) be the normalized segment \( \hat{D}[i, j] \)
19             \( T_r \leftarrow \text{Repetition}(\hat{D}_{i,j}, l, i) \)
20             \( T_{i,j} \leftarrow \text{Min}(T_r, T_{i,j}) \)
21             /* Find best representation with strc node as root */
22             Find shortest path \( P \) from \( v_i \) to \( v_{j+1} \) in \( G \)
23             Assume \( P = (v_i, u_1, \ldots, u_k, v_{j+1}) \)
24             \( I \leftarrow (0, \hat{D}[u_1] - \hat{D}[v_i], \ldots, \hat{D}[u_k] - \hat{D}[v_k], \hat{D}[v_k]) \)
25             \( \text{subtypes} \leftarrow (T_{v_i, u_1-1}, T_{u_1, u_2-1}, \ldots, T_{u_k, v_{j+1}}) \)
26             \( T_s \leftarrow \text{strc}(k + 1, I, \text{subtypes}) \)
27             \( T_{i,j} \leftarrow \text{Min}(T_s, T_{i,j}) \)
28             Add edge \((v_i, v_{j+1})\) with representation \( T_{i,j} \) and weight \( K_{\text{lookup}} + \text{cost}(T_{i,j}) \) to \( G \)
29        return \( T_{0,n-1} \) /* Stored with edge \((v_0, v_n)\) */
\end{verbatim}
```

a basic tree \( T_s \), whose root node is a \text{strc} node, are computed. Both are of least cost w.r.t. all basic tree representations of the desired form. The optimal basic tree for a normalized segment \( \hat{D}[i, i+l-1] \) is necessarily one of \( T_r \), \( T_s \) or a representation via a \text{con} node (if such a representation is possible), which was already computed in the preprocessing step.

To compute \( T_r \), a small, technical extension of procedure \text{Repetition} (Listing 4) for finding representations via \text{idx} or \text{vec} nodes is necessary. The procedure requires access to optimal representations of the prefixes of the argument displacement sequence \( D \). However, in the general case, \( D \) is a segment of \( \hat{D} \), that is, \( D = \hat{D}[i, j] \), and its prefixes therefore start with \( \hat{D}[x] \). To account for this (and avoid copying \( \hat{D}[i, j] \)), we pass an additional argument \( o \) representing the offset of the segment within the input displacement sequence \( \hat{D} \) (i.e., for a segment \( \hat{D}[i, j] \), we have \( o = i \)), and in lines 10 and 12 replace the argument \( T_{0,q-i} \) with \( T_{o,o+q-i} \).
graph $G$ suffices to solve the problem for all segments of $\hat{D}$. By construction, when computing the desired representation of a segment $\hat{D}[i, i + l - 1]$, $G$ contains edges representing optimal representations for all segments of length less than $l$ (and possibly some edges representing solutions of length $l$). A shortest path from node $v_i$ to $v_{i+l}$ in $G$ therefore leads to the same representation as the one constructed by Lemma 4.

To find such a shortest path, for each segment $\hat{D}[i, i + l - 1]$ of length $l$, one single-source shortest path (SSSP) problem on a weighted DAG with $l + 1$ nodes and $O(l^2)$ edges has to be solved. Since $G$ is a topologically sorted DAG by construction, SSSP is solvable in $O(|V| + |E|)$ time, where $|V|$ denotes the number of vertices and $|E|$ denotes the number of edges in $G$. To compute the desired representations for all segments of length $l$, a shortest path has to be computed for each of the $n + 1 - l$ node pairs $(v_i, v_{i+l})$, for $0 \leq i \leq n + 1 - l$. The total runtime is thus upper bounded by $\sum_{l=1}^{n-1} l^2(n + 1 - l)$, which is $O(n^4)$.

The algorithm constructs a graph with $O(n^2)$ edges, where a basic tree $T_{i,j}$, representing the solution for the normalized segment $\hat{D}[i,j]$, is associated with each edge $(v_i,v_j)$. Note that for each edge $(v_i,v_j)$ it suffices to store the root node of the associated basic tree $T_{i,j}$ plus pointers to its child nodes, which are already stored with the respective edges. To meet the desired space bound, only a constant amount of space may be used by each edge and associated basic tree. This is trivially true for con nodes (apart from one word indicating the node’s kind and the cost of the type tree rooted at the node, only the count $c$ needs to be stored) as well as vec nodes (two integer values and one pointer to the child node are required in addition to the node’s kind and the cost of the type tree rooted at this node). However, idx and strc nodes may require $\Omega(n)$ space in the worst case (e.g., if $\text{id}(n,\hat{D},\text{con}(1))$ is the optimal representation of $\hat{D}$). We employ a standard trick often used in dynamic programming algorithms and store for each node only the information required to reconstruct the full solution once the algorithm in Listing 5 has terminated. If for an idx node the count $c$ is known, the full idx node is easily derived as $\text{id}(c,\hat{D}[0],\hat{D}[q],\ldots,\hat{D}[(c-1)q]),T_{0,q-1})$ with $q = n/c$. The parameters of a strc node associated with an edge $(v_i,v_j)$ can be reconstructed by again computing the shortest path from node $v_i$ to $v_j$ and mapping it to a strc node as done in Lemma 4. Note that this reconstruction step does not change the asymptotic runtime bound and that the required space for each node is $O(1)$, from which the claimed upper bound of $O(n^2)$ space follows directly.

The following Corollary 3 and Lemma 5 show how the algorithm of Lemma 4 can be applied to general displacement sequences.

**Corollary 3** For any optimal basic tree with an index node $N$ with count 1, i.e., a node $N = \text{id}(1,\langle i_0 \rangle,\ldots)$, a representation $T'$ of equal cost s.t. $N$ is the root node of $T'$, exists.

**Proof:** Due to Corollary 1 there is no idx or strc node on the path from $N$ to the root and thus $N$ shifts the whole sequence by $i_0$. This shift can be represented equivalently by removing $N$ from the basic tree and adding a new root node to represent the shift, i.e., by letting $T' = \text{id}(1,\langle i_0 \rangle,T \setminus N)$. □

**Lemma 5** Given optimal basic trees $\hat{T}_{i,j}$ for all normalized segments $\hat{D}[i,j]$ of a displacement sequence $\hat{D}$, an optimal basic tree $T$ representing $\hat{D}$ can be computed in $O(n^2)$ time and $O(n)$ space.

**Proof:** By Lemma 4 for any optimal basic tree $T$ a cost-equivalent nice basic tree $\tilde{T}$ representing the same displacement sequence $\hat{D}$ exists and it therefore suffices to find an optimal nice basic tree representation $\tilde{T}$ for $\hat{D}$. By assumption, an optimal nice basic tree representation $\tilde{T} = \hat{T}_{0,n-1}$ for the normalized sequence $\hat{D}$ exists. To construct $\tilde{T}$, find the first node $N$
on any root to leaf path in $\hat{T}$ that is either an $\text{idx}$ or a $\text{strc}$ node and add the displacement sequence’s shift $s = D[0]$ to the indices of this node, i.e., if $N = \text{idx}(c, \langle i_0, \ldots, i_{c-1} \rangle, T')$ in $\hat{T}$, set $\hat{N} = \text{idx}(c, \langle i_0 + s, \ldots, i_{c-1} + s \rangle, T')$ in $\hat{T}$ and analogously for the case of $N$ being a $\text{strc}$ node.

Note that $\hat{T}$ has the same cost as $T$ and thus is an optimal basic tree representation for $D$.

If such a node does not exist, it follows from Lemma 4 and Corollary 3 that the optimal solution is either

- $\hat{T} = \text{idx}(c, \langle \ldots, \hat{T}_{0,n/c-1} \rangle, T_{0,n/c-1})$, for some divisor $c$ of $n$, or
- $\hat{T} = \text{strc}(c, \langle \ldots, \hat{T}_{0,\ldots,\hat{T}_{c-1}} \rangle, \langle \hat{T}_{0}, \ldots, \hat{T}_{c-1} \rangle)$, for some $1 < c < n$.

Note that for $\text{idx}$ nodes, both the trivial representation $\text{idx}(n, D, \text{con}(1))$ as well as the representation $\text{idx}(1, \langle D[0] \rangle, T)$ which only adds a shifted node to $\hat{T}$, need to be checked. Since solutions for all normalized segments are already known, this construction is feasible in $O(n^2)$ time and $O(n)$ space. □

**Proof:** [of Theorem 1] The Basic Type Reconstruction Problem for a displacement sequence $D$ of length $n$ can be solved by computing an optimal basic tree representation for the normalized displacement sequence $\hat{D}$ (Lemma 3) and the post-processing step given in Lemma 5. The claimed space and time bounds follow directly from the given Lemmas. □

### 4 Computing more concise representations

In this section we discuss possibly more space efficient tree representations by allowing a richer set of constructors, exemplified by the auxiliary constructors introduced in Definition 2. We then explain why computing representations by DAGs is an apparently harder problem. Finally, we discuss the applicability of our algorithms to the type normalization problem.

#### 4.1 Handling the auxiliary constructors

The auxiliary constructors of Definition 2 can be handled by slight extensions to our algorithm in a way that polynomial-time type reconstruction is still possible. Basically, only the part that checks for vector or index patterns shown in Listing 4 needs to be extended. Assume that a repeated prefix $C$ of length $q$ has been found in the given displacement sequence $D$, and that $D'$ is the displacement sequence consisting of every $q$th element of $D$, i.e., $D' = [D[0], D[q], D[2q], \ldots]$.

The *strided bucket*, $\text{vecbuc}(c, d, e, \langle b_0, b_1, \ldots, b_{c-1} \rangle, C)$ constructor can concisely describe application data layouts consisting of buckets each with some maximum number of elements (the stride $d$) where each bucket contains some (possibly different) number of elements $b_i$ with bucket stride $e$. This description is likely to be less costly than describing such a layout by a $\text{strc}$ constructor with each subtype describing one bucket. To incorporate the strided bucket it simply has to be checked in Listing 4 whether $D'$ follows the strided bucket pattern, and this can easily be done in linear time. There are two cases to consider. If the first bucket has more than one element, take as bucket stride $e = D'[1] - D'[0]$ and scan the index list for repetitions at stride $e$. The first violation at some position $i$ forces the maximum bucket size to be $d = D'[i] - D'[0]$. Now continue to scan till the end of $D'$, checking that the $e, d$ strided pattern repeats and counting the number of elements $b_i$ in each bucket of $e$-strided displacements. Otherwise, the first bucket has only one element. Take instead as maximum bucket size $d = D'[1] - D'[0]$, and scan for repetitions with stride $d$. The first violation at some position $i$ forces the bucket stride to be $e = D'[i] - D'[i - 1]$. As in the other case, the bucket sizes $b_i$ are counted by scanning...
$D'$ till the end. If an index $i$ is found where $D'[i] - D'[i - 1] \neq e$ and $D'[i] - D'[j] \neq d$ where $j$ is the start of the current bucket in $D'$, then $D'$ is not a displacement sequence of a strided bucket layout.

The strided bucket constructor is in a sense the opposite of the index constructor. Instead of an index sequence it takes a sequence of bucket sizes, and has (roughly) the same cost. Interestingly, there is no such constructor in the MPI standard.

The indexed bucket, \texttt{idxbuc}(e, e, \langle i_0, i_1, \ldots, i_c-1 \rangle, \langle b_0, b_1, \ldots, b_{c-1} \rangle, C), on the other hand corresponds closely to the \texttt{MPI\_Type\_indexed} constructor. For each index, a repetition count $b_i$ gives the number of repeats of $C$ in the bucket starting at that index; all repetitions use the same stride $e$ (the constructor could trivially be extended to the case where each index has its own stride). For each possible bucket stride, the number of buckets that this stride will give rise to has to be counted. The stride $e$ leading to a smallest number of buckets is a candidate for the representation of $D'$ and $C$ as an \texttt{idxbuc} node. We observe that each $i$ with $D'[i + 1] - D'[i] = e$ joins two $e$-strided segments $D'[j, i]$ and $D'[i + 1, k]$ into one bucket starting at index $j$. Therefore, the stride that occurs most often in the stride sequence $S[i] = D'[i + 1] - D'[i], 0 \leq i < n-1$, will lead to the smallest number of buckets. To count the number of occurrences of each stride, we either sort $S$ or count by hashing during the scan of $D'$. Let $e$ be a stride with the most occurrences. A final scan of $D'$ suffices to compute the start indices and sizes of the buckets with stride $e$.

### 4.2 Type reconstruction into DAGs

A type tree describing some given displacement sequence may have multiple instances of the same subtree. Our algorithm in particular constructs nice type trees (Definition 5) in which all displacement sequences in index and struct nodes except perhaps one start at index 0, and it can well happen that the same index or struct node occurs many times. A more concise representation results if such trees are folded into directed acyclic graphs with only one node for each substructure.

Type DAGs represent displacement sequences by the same flattening procedure as shown in Listing 1 for trees. Each path from the root node in the type DAG to a leaf is traversed in order to generate the corresponding displacement sequence. Thus the processing cost of a type DAG would arguably be similar to the processing costs of a tree. By a similar traversal of a DAG an equivalent tree can be constructed, simply by making a new copy each time a node is visited.

The space required for the DAG can be much smaller than the space required for the corresponding tree. One can therefore define also for DAGs our cost model for optimizing conciseness as the additive cost of the nodes in the DAG; and not as the sum of the costs of all paths traversed. The type reconstruction problem into DAGs is now to find the least-cost DAG representing the given displacement sequence.

One crucial difficulty which arises when dealing with such type DAGs is that the best representation for a subsequence no longer needs to be locally optimal, since costs savings can be achieved by reusing other nodes of the DAG. This is illustrated in Figure 2.

In particular, this implies that the type tree constructed by unfolding a cost-optimal DAG is not necessarily a cost-optimal tree, and conversely, that the DAG obtained by folding a given, cost-optimal type tree is not necessarily a cost-optimal DAG. This constitutes a fundamental problem for our general approach for handling type trees, and new ideas are needed to solve the type reconstruction problem into DAGs.
4.3 The type normalization problem

The type normalization problem subsumes the type reconstruction problem that we have considered so far. Type normalization asks to improve the cost of an already given tree description of the data layout. Since any data layout can be represented as a single \texttt{idx} node with the whole displacement sequence as index sequence, type normalization includes type reconstruction as a special case. Type normalization is the problem that compiler or library implementors are typically faced with: application data structures described as trees are given by the programmer as part of the code, and an internal, optimal representation is to be constructed by the programming system.

The trivial solution is to flatten the given type tree and apply the type reconstruction algorithm on the resulting displacement sequence. Since the size of the resulting displacement sequence is not bounded by the size or conciseness of the tree, this is highly undesirable. We would like a procedure where the complexity can be bounded by the conciseness of the type trees, specifically the total size of the index sequences in the tree.

As shown in [13], if the set of basic constructors is restricted to exclude the \texttt{strc} constructor, it is possible to perform type normalization by only rechecking optimality of the \texttt{idx} nodes. In this case, type normalization can be done in time proportional to the conciseness of the given tree. When the \texttt{strc} constructor is allowed, arbitrarily more concise representations can be possible as shown in Figure 1. Optimality of a subtree that does not use the \texttt{strc} constructor does therefore not imply optimality when \texttt{strc} is allowed. It is therefore necessary to flatten the whole tree and apply the tree reconstruction algorithm on the resulting displacement sequence.

5 Conclusion

The main result of this paper is that the type reconstruction problem into trees is actually solvable in polynomial time. However, an $O(n^4)$ algorithm is not useful for larger values of $n$ as might be the case in parallel applications where $n$ could be proportional to the number of processors which in itself could be in the range of tens to hundreds of thousands. We note that our bottom-up dynamic programming algorithm performs a considerable amount of almost redundant checking for (strided) repetitions in displacement sequence segments. An asymptotically more efficient algorithm, perhaps based on a top-down approach, is likely to exist. Whether an exact, practically efficient algorithm for the full problem is possible, we do not know at the point of writing.
Restricting the power of the constructors can permit more efficient algorithms. As shown in [13], if only con, vec and idx nodes are allowed, then the type reconstruction problem for a displacement sequence of length \( n \) can be solved in \( O(n\sqrt{n}) \) time. However, the resulting restricted trees can and often will be much more costly, as shown in Figure 1. The high complexity of our algorithm is caused by the unbounded branching constructor strc node. A slightly better, \( O(n^3) \) time algorithm would result from allowing only bounded branching, for instance a binary struct constructor that concatenates only two subtrees. For such a constructor, the shortest path computation of Lemma 3 could be done in linear time. In some contexts, bounded branching might be sufficiently expressive.

An alternative approach would be to look for low-complexity approximation algorithms with provable approximation guarantees. Or, even weaker, for heuristics that perhaps work well for the intended application cases. This reflects the state in current MPI libraries.

As discussed, type trees can be represented more concisely as directed acyclic graphs (DAGs). To the best of our knowledge, it is still open whether a cost-optimal DAG representation for an arbitrary displacement sequence can likewise be constructed in polynomial time.

A related problem to consider is the following. Given two displacement sequences of the same length, construct a least-cost tree (or DAG) representing a mapping between the two sequences. Such a tree (DAG) has uses when copying between different data layouts; this arises, e.g., in matrix transposition. In the MPI context this operation has been called transpacking [7, 11]. Our dynamic programming algorithm may extend to this case as well.

Our work was specifically inspired by the derived data type mechanism of MPI. We believe that this idea is applicable in a much wider context of (parallel) programming interfaces and languages, and that the type normalization and reconstruction problems as defined here, as well as the associated processing of data layouts represented by trees, have relevance extending beyond the motivating context.

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