Incentives and Efficiency in Constrained Allocation Mechanisms

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Abstract

We study private-good allocation under general constraints. Several prominent examples are special cases, including house allocation, roommate matching, social choice, and multiple assignment. Every individually strategy-proof and Pareto efficient two-agent mechanism is an “adapted local dictatorship.” Every group strategy-proof $N$-agent mechanism has two-agent marginal mechanisms that are adapted local dictatorships. These results yield new characterizations and unifying insights for known characterizations. We find all group strategy-proof and Pareto efficient mechanisms for the roommates problem. We give a related result for multiple assignment. We prove the Gibbard–Satterthwaite Theorem and give a partial converse.

1 Introduction

Market design often involves constraints. School choice assignments must meet quotas for under-represented students at high-performing schools. Medical residency assignments must place enough doctors in rural areas. The allocation of radio frequency in spectrum auctions must satisfy a variety of complicated engineering conditions to minimize cross-channel interference.

Although successful approaches have been tailored for particular constraints in specific problems, to date there is little general understanding of how constraints affect the two classic economic considerations of efficiency and incentives. Theoretically, a unified approach would enable analytical insights to be shared between contexts. Practically, a flexible theory of constraints for market design would expand applicability. Real-world problems involve many ad hoc considerations that are difficult to anticipate. The tools of market design should be general enough to accommodate such constraints.

We study object allocation with private values for completely general constraints. A finite number of objects are allocated to a finite number of agents and an arbitrary constraint circumscribes the set of feasible social allocations. Each agent has strict preferences over the objects assigned to her, but is indifferent to others’ assignments.

While other agents’ assignments have no direct effect on one’s well-being, others’ assignments do limit the profiles of allocations that are jointly feasible. Obviously, the assignment of a house to

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*We thank Samson Alva, Simon Board, Kim Border, Ben Brooks, Haluk Ergin, Satoshi Fukuda, Thomas Gresik, Yuhta Ishii, Yuichiro Kamada, Timothy Kehoe, Rohit Lamba, Jacob Leshno, Jay Lu, Delong Meng, Moritz Meyer-ter-Vehn, Michèle Müller, Roger Myerson, Farzad Pourbabaei, Wenfeng Qiu, Doron Ravid, Phil Reny, Tomasz Sadzik, Chris Shannon, David Rahman, Ron Siegel, Ran Shorrer, Hugo Sonnenschein, Alexander Westkamp, Bill Zame and various seminar participants for helpful feedback.

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one agent precludes another agents’ consumption of that same house. Even with purely private values, constraints introduce linkage across agents’ allocations. Each agent $i$ is indirectly concerned with any other $j$’s assignment, not because $i$ cares about $j$’s consumption, but rather because $j$’s assignment will limit the set of objects for $i$ that are jointly feasible with $j$’s assignment. Our goal is to study the set of incentive compatible and efficient mechanisms for a fixed, but arbitrary constraint. We study how different features of a constraint make it amenable for implementation, that is, to understand what kinds of constraints yield what kinds of truthful and efficient mechanisms. For any constraint on the set of feasible allocations, our findings characterize the class of mechanisms that are immune to manipulation by any group of agents yet still yield Pareto efficient outcomes.

Allowing for complete generality in the constraint, we characterize all mechanisms that satisfy canonical incentive and efficiency axioms. We start by considering two-agent environments. This case admits a surprisingly parsimonious characterization of the set of individually strategy-proof and Pareto efficient mechanisms for all constraints. We show that all such mechanisms are “local dictatorships” where the set of infeasible allocations is partitioned into two regions and each region is assigned a local dictator. For a given preference profile, the agents’ top choices determine some (possibly infeasible) social allocation. If this allocation is feasible, then there is no conflict of interest and the mechanism assigns each agent their favorite object. Otherwise, the favorite allocation is infeasible, and a local dictator is empowered at that infeasible allocation. The dictator is assigned their top object and the non-dictator is assigned their favorite object compatible with the dictator’s top object. However, not all dictatorship partitions will be strategy-proof. Instead, some structure is required of the partition to ensure these desiderata are maintained. If two infeasible allocations agree on either coordinate, that is, if they assign either agent the same object, then they must share the same local dictator.

With three or more agents, the set of individually strategy-proof and Pareto efficient mechanisms no longer admits a tidy characterization. Indeed, even for the classic house allocation setting, the collection of such mechanisms is still unknown. Nevertheless, we can make substantial progress by strengthening our incentive compatibility notion to group strategy-proofness. In this case, the mechanism is required to give no incentive for any group of agents to misreport their preferences. However, due to a result by Alva (2017), a coalition has a profitable deviation if and only if a pair has a profitable deviation. As a result we can check on incentives for two-agent marginal mechanisms, where all but two agents’ reports are fixed. This is a well-defined two-agent mechanism and we can then bootstrap our two-agent characterization to provide a recursive characterization of group strategy-proofness.

Our study of constrained allocation yields some surprising theoretical insights. Several prominent problems which, at first glance may appear unconstrained and unrelated, can be neatly expressed as special constraints of our model. For example, the classical social choice problem corresponds to the constraint where all agents are required to consume the same object.\footnote{The term “object” is figurative. In social choice, the objects are usually policy choices or political candidates.} From this perspective, the social choice problem presents itself as a special constrained private-goods allocation problem. A simple application of our results gives the Gibbard–Satterthwaite Theorem: that all strategy-proof social choice mechanisms are dictatorial.\footnote{For the social choice constraint, a strategy-proof mechanism is automatically group strategy-proof.} With this novel presentation of social choice as a constraint, we can now sensibly formulate and prove a partial inverse to Gibbard–Satterthwaite: we give general conditions on the constraint which guarantee that there are non dictatorial mechanisms. We get a
possibility result for two-sided matching and college assignment as immediate corollaries.

Another prominent case of our theory is house allocation, where a finite number of indivisible objects must be assigned to agents with unit-demand. Expressed this way, the house allocation problem is almost the opposite of the social choice problem: no two agents can be assigned the same object. Pycia and Ünver (2017) provided a full characterization of the group strategy-proof and Pareto efficient house allocation mechanisms. They showed that all such mechanisms are variants of the hierarchical exchange mechanisms of Papai (2000) where some agents can “broker” objects and when exactly three agents remain a “braid” can form (Bade 2016). We generalize this problem and consider the case of combinatorial assignment where agents can be assigned bundles of up to \( k \) goods. House allocation is the special case where \( k = 1 \). We show that for \( k \geq 2 \) the theory collapses and the only group strategy-proof and Pareto efficient mechanisms are sequential dictatorships.\(^3\)

A third prominent problem that can be expressed as a constraint is the roommates problem, where an even number of agents need to be matched into pairs. In this case, the “objects” are the other agents and the constraint requires that: first, no agent is matched to herself; and second, if \( i \) is assigned to \( j \), then \( j \) is also assigned to \( i \). We use our results to demonstrate that all group strategy-proof and Pareto efficient roommate mechanisms are sequential dictatorships.

These examples illustrate a key conceptual contribution of our paper: to provide a novel framework to unify positive and negative results across these applications, tying together seemingly disparate environments and results by viewing them as different constraints on the image rather than through restrictions of preferences on the domain. Traditionally, positive results in specific environments are seen as escaping the impossibilities of Arrovian social choice by restricting preferences in the domain of the mechanism to convenient special cases, such as assuming single-peaked rankings or quasi-linear preferences. In contrast, our model can provide a different reconciliation of these positive results by interpreting these environments as relaxing constraints in the image of the mechanism: outside of the restrictive social choice constraint, all agents need not consume the same object and instead there is room for compromise to yield mechanisms beyond dictatorship. That is, our model illuminates that the “diagonal” constraint implicit in the social choice problem generates maximal tension between efficiency and incentives, while other constraints allow more scope for their coexistence.

1.1 Literature Review

To our knowledge, this paper is the first to identify the entire set of mechanisms that satisfy criteria regarding incentives and efficiency for general constraints. However, we mentioned that several canonical problems can be parameterized as specific constraints in our model, so we first review findings for these problems. One example is the house allocation problem, where no two agents can share the same object. The two famous group strategy-proof and Pareto efficient mechanisms for house allocation are top trading cycles, attributed to David Gale by Shapley and Scarf (1974) and shown to have these features by Bird (1984), and serial dictatorship.\(^4\) Abdulkadiroğlu and Sönmez (1999) and Papai (2000) construct additional classes of group strategy-proof and Pareto efficient mechanisms

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\(^3\)Sequential dictatorship is a simple variant on the well-known serial dictatorship mechanism. In serial dictatorship, agents are called in a pre-specified order to choose their favorite object compatible with the choices of earlier dictators. In sequential dictatorship, the ordering of dictators can be endogenous to the choices of earlier dictators.

\(^4\)Serial dictatorship is analyzed comprehensively by Svensson (1994) and Svensson (1999).
that include mechanisms beyond top trading cycles and serial dictatorship. However, a complete characteri-
zerization was outstanding until Pycia and Ünver (2017) provided an impressive full description of all group strategy-proof and Pareto efficient mechanisms. Pycia and Ünver (2017) showed that this class includes the hierarchical exchange mechanisms of Papái (2000), the “braid” mechanisms of Bade (2016) and a new variant of these mechanisms with “brokers.” Our paper was inspired in part by the possibility of attaining a characterization for such an important problem.

The roommates, or one-sided matching, problem can be viewed as a further restriction on the house allocation constraint. Here, the objects are the agents, and the allocation must satisfy the constraint that if \( j \) is allocated to \( i \), then \( i \) is also allocated to \( j \). While incentives and efficiency are well-understood for house allocation, similar insights for one-sided matching were yet unknown. This is in large part because the roommates problem may have no stable outcomes, as originally observed in the famous paper by Gale and Shapley (1962). Since then, a voluminous literature in operations research and computer science, starting with Irving (1985), constructs efficient algorithms to find stable matchings when they exist, and the study of stability for one-sided matching is now well-known as the “stable roommates problem.” In contrast, there is little discussion of incentives and efficiency for the roommates problem.\(^5\) As an application of our main results, we find that all group strategy-proof and Pareto efficient mechanisms for the roommates problem are sequential dictatorships. To our knowledge, this observation is novel and establishes a result for one-sided matching akin to the characterization of Gibbard and Satterthwaite for social choice or that of Pycia and Ünver (2017) for house allocation.

A relaxation of house allocation allows agents to own multiple houses, which is sometimes called the combinatorial or multiple assignment problem. Our main finding is that sequential dictatorship is the unique group strategy-proof and Pareto efficient mechanism for the environment where each agent can be assigned at most \( k \) objects. This mirrors related results for multiple assignment. To our knowledge, (Papái 2001) provided the first such result. She proves this for the case where \( k \) is maximal, that is equal to the number of objects, while we consider the general case. Hatfield (2009) provides this characterisation for the case where \( k \) is a lower bound on the number of objects, as opposed to the upper bound we consider here.

A last important constraint is the classic Arrovian social choice model. The celebrated theorem of Gibbard (1973) and Satterthwaite (1975) initiated the field of implementation theory by observing the tension between incentives and efficiency for social choice. In our model, the Arrovian social choice corresponds to the case where all agents must be assigned a common outcome. Viewed this way, the social choice constraint is almost the opposite of the house allocation constraint. We derive the Gibbard-Satterthwaite Theorem as a corollary of our main characterization. This provides a new perspective on the classic result by considering social choice environments as a severe constraint in the allocation problem with purely private values. Correspondingly, our perspective also offers a novel escape from the assumptions of the Gibbard-Satterthwaite Theorem, namely by relaxing the severity of the constraint. In fact, this framing allows us to ask the inverse question of the Gibbard-Satterthwaite Theorem: what constraints admit group strategy-proof and Pareto efficient mechanisms beside dictatorships, even with full preference domains?

\(^5\) An exception is the working paper by Abraham and Manlove (2004) that studies the computational hardness of finding Pareto optimal matches for the roommates problem.
Our general environment with private goods was examined by Barberà, Berga, and Moreno (2016) for social choice. They study implementation across different restrictions on the domain of preference. In contrast, we always consider the full domain of preferences. We complement the insights of Barberà, Berga, and Moreno (2016) by considering different constraints on the image of allocations that are feasible for a mechanism. Related, a series of papers consider constraints in social choice problems where the common consumption space is a product space (Barberà, Massó, and Neme 1997, Barberà, Massó, and Serizawa 1998, Barberà, Massó, and Neme 2005), adding feasibility considerations to the general model of Border and Jordan (1983). These papers also connect the structure of the feasible set to the space of mechanisms satisfying various normative axioms, so our exercise of connecting constraints to mechanisms has precedents. An important difference is that these papers allow preferences to depend on the entire allocation profile, rather than the pure private values we assume in this paper.

Finally, in contemporaneous work Meng (2019) proved a theorem for social choice with exogenous indifference classes which is mathematically equivalent to our two-agent characterization. While indifference classes and constraints can be mapped to each other in a mathematical sense, our aims are different and the contributions of the two papers are independent. Meng (2019) does not observe the usefulness of the result for constrained market design, while we did not observe its usefulness for problems with indifferences. We compare the technical results more specifically when we introduce our two-agent characterization.

2 Model

We begin by introducing primitives. Let \( N \) be a finite set of agents and \( \mathcal{O} \) be a finite set of objects. We use the term “object” because our leading examples are allocation problems, but note that \( \mathcal{O} \) need not be physical objects like houses, but can be political candidates, roommates, and so on. Let \( \mathcal{A} = \mathcal{O}^N \) denote the set of all potential allocations of objects to agents. Equivalently, \( \mathcal{A} \) is the set of maps \( \mu : N \to \mathcal{O} \) and when useful we adapt this perspective. A suballocation is a map \( \sigma : M \to \mathcal{O} \) where \( M \subset N \). Let \( \mathcal{S} \) denote the set of all suballocations. Our task is to assign objects to agents in a way that is consistent with an exogenous constraint that reflects the set of feasible allocations for a particular application. Importantly, the constraint is exogenous to the problem and is given to the mechanism designer as a fixed set of feasible outcomes. Formally, we are given a nonempty constraint \( C \subset \mathcal{A} \) and \( (a_i)_{i \in N} \in C \) means that it is feasible to allocate each agent \( i \) the object \( a_i \) simultaneously. Since the constraint can be arbitrary, it loses no generality to assume a common set of objects for all agents.\footnote{For example, if each agent had their own set of objects \( \mathcal{O}_i \), we could let \( \mathcal{O} = \bigcup_i \mathcal{O}_i \) and add the constraint that \( i \) be assigned an object in \( \mathcal{O}_i \).}

Moving to preferences and types, agents have strict preferences over the objects and are assumed to be indifferent between any two allocations in which they receive the same object. A preference for agent \( i \) will typically be denoted \( \succeq_i \), and we will write \( x \succeq_i y \) to mean that either \( x \) is strictly ranked above \( y \) or that \( x = y \). We assume purely private goods, or selfishness over allocations. That is, the only part of an allocation \( (a_i)_{i \in N} \) that matters to agent \( j \) is her own allocation \( a_j \), and she is indifferent between any two allocations where \( a_j = a'_j \). Thus any other agent’s consumption imposes no direct
externality on agent $j$. This does not mean there is no conflict of interest in this model. By assuming purely private values, all of the tension in our model flows only through the constraint. That is, the issue is purely due to limited “supply” of objects and not due to direct externalities.

We will use $P$ to denote the set of strict preferences (or linear orders) on $O$ and $\mathcal{P} = P^N$ to denote the set of preference profiles.\footnote{A binary relation $B \subseteq O \times O$ is a linear order if it is complete, transitive, and antisymmetric.} Our primary object of interest in this paper is a feasible mechanism, which is simply a map $f : \mathcal{P} \to C$. Our task will be to find feasible mechanisms satisfying desirable conditions regarding incentives and efficiency to be introduced below.

First, observe that the generality of the model embeds several well-known problems as special constraints:

- **House Allocation**: A finite number of houses must be distributed to a finite number of agents. The houses cannot be shared so no two agents can be allocated the same one. This gives rise to the constraint
  \[ C = \{(a_i)_{i \in N} \mid a_i \neq a_j \text{ when } i \neq j\}. \]
  This setting has been the subject of considerable interest since at least Shapley and Scarf (1974). Two prominent mechanisms used in practice are Gale’s top trading cycles algorithm and Gale and Shapley’s deferred acceptance algorithm (with priorities for houses).

- **School Choice**: A finite number of students $N$ need to be assigned to one of a finite number of schools $A$. Each school $a$ has capacity $q_a$. One school $\emptyset$ corresponds to the option to remain unmatched and $q_\emptyset = N$. This gives rise to the constraint
  \[ C = \{\nu : N \to A \mid |\mu^{-1}(a)| \leq q_a \text{ for all } a \in A\}. \]

- **Roommates Problem**: Universities are often tasked with assigning students into shared dormitory rooms. Assuming $N$ is even, this problem can be captured in our environment by setting $O = N$ and imposing the constraint
  \[ C = \{\mu : N \to N \mid \mu \circ \mu = id \text{ and } \mu(i) \neq i \text{ for all } i\} \]
  where $id$ is the identity map $i \mapsto i$. The first condition requires that if $i$ is assigned roommate $j$ then $j$ is also assigned $i$ and the second condition requires that all agents are assigned a roommate.

- **Two-sided Matching**: The set of agents $N$ is composed of two disjoint sets $M$ and $W$ where $|M| = |W|$. Agents need to be matched into pairs with the constraint that $m$’s need to be matched with $w$’s. This gives rise to the constraint
  \[ C = \{\mu : N \to N \mid \mu \circ \mu = id \text{ and } \mu(m) \in W \text{ for all } m\} \]

- **Social Choice**: If the constraint specifies that all agents receive the same object (without specifying ex-ante which object will be chosen) we get the classical version of the social choice problem.\footnote{See Barberà (2001) for a general statement of the social choice problem with restricted domains.}
Specifically, if $$C = \{(a_i)_{i \in N} | a_i = a_j \text{ for all } i, j\}$$

the constraint requires that all agents be given the same social choice, but which outcome is chosen is a function of the preference profile.

- Multiple Assignment: Let $$H$$ be a set of basic items and let $$\mathcal{O} = \{A \subset H : |A| \leq k\}$$. This is analogous to the house allocation problem, except agents can have up to $$k$$ houses and have preferences over bundles of houses. Each item must have a unique owner, so the constraint is

$$C = \{(s_i)_{i \in N} \in \mathcal{O}^N : s_i \cap s_j = \emptyset, \text{ when } i \neq j\}.$$ 

Our model is able to accommodate these examples as special cases because of its generality in admitting arbitrary constraints. We will have more explicit analyses of these examples later in the paper.

Before moving on, we record here some notation used throughout the paper. For any subset $$M \subset N$$, given a preference profile $$\succeq = (\succeq_i)_{i \in N} \in \mathcal{P}$$ and a profile of alternative preferences for agents in $$M$$, $$(\succeq_j^i)_{j \in M}$$, we will write $$(\succeq^i, \succeq^i-M)$$ to refer to the profile in which an agent $$j$$ from $$M$$ reports $$\succeq^i_j$$ and any agent $$i$$ from $$N-M$$ reports $$\succeq^i_i$$. We will often want to consider how a mechanism $$f$$ changes when a few agents change their preferences, that is the difference between $$f(\succeq^i)$$ and $$f(\succeq^i', \succeq^i-M)$$. When the initial preference profile $$\succeq^i$$ is clear, we will sometimes write $$\succeq^i-M$$ instead of $$\succeq^i-M$$. For any set of agents $$M$$, let $$\pi_M : A \to \mathcal{O}$$ be the projection map so that given an allocation $$(a_j)_{j \in M}$$, we have $$\pi_M a := (a_j)_{j \in M}$$. Given a constraint $$C \subset A$$, let $$C^M = \{\mu : M \to \mathcal{O} | \exists b \in C \text{ s.t. } b_i = \mu(i) \forall i \in M\} = \pi_M(C)$$ which we will call the projection of $$C$$ on $$M$$. An element of $$C^M$$ will be referred to as a feasible suballocation for agents in $$M$$. If $$\mu : M \to \mathcal{O}$$ and $$\mu' : M' \to \mathcal{O}$$ are suballocations with $$M \subset M'$$ which agree on their shared domain, $$\mu'$$ is called an extension of $$\mu$$. If $$\mu'$$ is a feasible suballocation (which of course implies that $$\mu$$ is) then $$\mu'$$ is called a complete extension of $$\mu$$. Given a feasible suballocation $$\mu$$, we will let $$C(\mu)$$ denote the set of complete and feasible extensions of $$\mu$$. For $$x \in \mathcal{O}$$ and $$\succeq_i \in P$$, define $$LC_{\succeq_i}(x) = \{y \in \mathcal{O} | y \prec_i x\}$$ be the (strict) lower contour set of $$x$$ at $$\succeq_i$$. Likewise, $$UC_{\succeq_i}(x) = \{y \in \mathcal{O} | y \succ_i x\}$$ is the (strict) upper contour set of $$x$$ at $$\succeq_i$$. For a preference $$\succeq_i$$, define $$\tau_n(\succeq_i)$$ as the nth top choice under $$\succeq_i$$. Likewise, for any preference profile $$\succeq$$, define $$\tau_n(\succeq)$$ as the allocation in which each agent gets their nth top choice. To save on notation, we will often omit the subscript when referring to the top choice (i.e. writing $$\tau(\succeq)$$ to mean $$\tau_1(\succeq))$$. We will use $$\bar{C}$$ to denote the set of infeasible allocations. If $$A$$ and $$B$$ are sets of objects and $$\succeq \in P$$, we say $$A \succeq B$$ if $$a \succeq b$$ for all $$a \in A$$ and $$b \in B$$. For disjoint sets of objects $$A_1, A_2 \ldots A_m$$ we will denote $$P[A_1, A_2 \ldots A_m] = \{\succeq \in P | A_1 \succ A_2 \succ \cdots \succ A_m\}$$ and $$P^\uparrow[A_1, A_2 \ldots A_m] = \{\succeq \in P | A_j \succ \cup_{i=1}^m A_i, \text{ for all } j\}$$ When the $$A_i$$ are singletons, we will abuse notation and drop the curly brackets, writing for example $$P^\uparrow[a]$$ to denote $$P^\uparrow[\{a\}]$$. We will also abuse notation slightly and use $$N$$ to refer both to the set of agents and to the number of agents.

In practice, mechanisms are designed to satisfy efficiency and incentive properties, for which the following are well-known conditions.

**Definition 1.** A mechanism $$f : \mathcal{P} \to C$$ is
1. **strategy-proof** if, for every $i \in N$ and every $\succsim \in \mathcal{P}$,

$$f_i(\succsim) \succsim_i f_i(\succsim'_i, \succsim_{-i})$$

for all $\succsim'_i \in P$. That is, truth-telling is a weakly dominant strategy.

2. **group strategy-proof** if, for every $\succsim \in \mathcal{P}$ and every $M \subset N$, there is no $\succsim'_M$ such that

(a) $f_j(\succsim'_M, \succsim_{-M}) \succsim_j f_j(\succsim)$ for all $j \in M$;
(b) $f_k(\succsim'_M, \succsim_{-M}) \succsim_k f_k(\succsim)$ for at least one $k \in M$.

3. **pairwise strategy-proof** if, for every $\succsim \in \mathcal{P}$ and every pair of agents $i, j$, there is no $\succsim'_i$ and $\succsim'_j$ such that

(a) $f_i(\succsim'_i, \succsim'_j, \succsim_{-(i,j)}) \succsim_i f_i(\succsim)$;
(b) $f_j(\succsim'_i, \succsim'_j, \succsim_{-(i,j)}) \succsim_j f_j(\succsim)$.

4. **Pareto efficient** if there is no allocation $a \in C$ and preference profile $\succsim$ such that $a \neq f(\succsim)$ and $a_j \succsim_j f_j(\succsim)$ for all $j$.

Strategy-proofness requires that no agent can improve her outcome by misreporting her preference. Group strategy-proofness is similar, except requiring that no group can collectively misreport their preferences without hurting anyone while strictly benefiting at least one agent. This is often called “strong group strategy-proofness” to contrast it with a weaker notion requiring that deviating coalitions make all agents strictly better-off. Pareto efficiency is a classic efficiency axiom. It requires that there is no way to make any agent better-off without making another agent worse-off. Group strategy-proofness can be relaxed to pairwise strategy-proofness, which only requires that no pair of agents can profitably deviate. In fact, as we will see when we move to the $N$-agent case, our environment falls under the hypotheses of Theorem 1 by (Alva 2017) so there is no gap between group and pairwise strategy-proofness.

One candidate deviating coalition is the grand coalition. So if $f$ is group strategy-proof and $f(\succsim) = a$ for some profile $\succsim$, then $a$ can never Pareto dominate $f(\succsim')$ for any other profile $\succsim'$, since all agents could collectively report $\succsim$ instead of $\succsim'$ to get an improvement.

**Remark 1.** If $f : \mathcal{P} \rightarrow A$ is group strategy-proof then it is Pareto efficient on its image.\(^9\)

The goal of this paper is to understand the correspondence between the primitives (the set of agents, objects, and the constraint) and the set of group strategy-proof and Pareto efficient mechanisms. We will use $\mathcal{GS}(C)$ to denote the set of feasible group strategy-proof mechanisms.

## 3 Characterization Results

We begin with the two-agent case where we can explicitly construct the class of strategy-proof and Pareto efficient mechanisms for an arbitrary constraint. All such mechanisms turn out to be what we

\(^9\)That is, if an allocation Pareto dominates $f(\succsim)$, then that allocation is outside the image of $f$. 

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call “adapted local dictatorships” where a dictator is chosen as a function of the announced preferences.

We then turn to the $n$-agent case where we introduce the notion of “marginal mechanisms” and show that an $n$-agent mechanism is group strategy-proof if and only if each 2-agent marginal mechanism is an adapted local dictatorship.

### 3.1 Two Agents

For the special but important base case with two agents, strategy-proof and Pareto efficient mechanisms take a particularly simple form. The set of infeasible allocations $\bar{C}$ splits into two subsets $D_1$ and $D_2$ over which the two agents are given dictatorship rights. Specifically, given any preference profile, we consider the allocation $(a, b)$ where $a$ and $b$ are respectively the top choices of agents 1 and 2. If $(a, b)$ is feasible, Pareto efficiency forces this allocation. Otherwise, $(a, b)$ belongs to either $D_1$ or $D_2$. If it belongs to $D_1$, agent 1 receives her top object $a$, while 2 receives her favorite object $b'$ compatible with $a$ (such that $(a, b') \in C$). Likewise if $(a, b)$ belongs to $D_2$, 2 keeps her top choice and 1 must compromise. We call this type of mechanism a “local dictatorship.” We dub the two agents the “local dictator” and the “local compromiser” respectively.

Given that we have allowed for complete generality in the constraint, the procedure above should account for the possibility that the local compromiser may not be able to find a feasible object. For example, we should not specify the local dictator at $(x, y)$ to be agent 1, if for all $y'$, the allocation $(x, y')$ is infeasible, so there is no compromise agent 2 could make to allow agent 1 to consume her favorite object $x$. Notice, however that this is a trivial difficulty as 1 can never feasibly be assigned object $x$. It would seem that we should therefore be able to ignore 1’s ranking of $x$. This turns out to be true, and we can ignore objects that are never assigned to an agent without loss of generality. It is no more difficult to show this for any number of agents, so we include the general result here.

**Lemma 1.** Fix a constraint $C$ for any number of agents. Let $\chi_i = \{ y \in O | \forall y_{-i} (y, y_{-i}) \in \bar{C} \}$. Suppose $f : \mathcal{P} \rightarrow C$ is group strategy-proof and Pareto efficient. If $\succ_i$ and $\succ'_i$ are preference profiles such that, for all $i$, $\succ_i|_{O-\chi_i} = \succ'_i|_{O-\chi_i}$, then $f(\succ_i) = f(\succ'_i)$.

Let $\bar{C}^* = \{(x, y) \mid (x, y) \notin C \text{ and } x \notin \chi_1, y \notin \chi_2 \}$. That is, $\bar{C}^*$ is the set of infeasible allocations excluding the trivial cases described above. We will call $O-\chi_i$ agent $i$’s individually-feasible choices.

As mentioned, Pareto efficiency requires allocating both agents their top choices if doing so is feasible. The main job of a mechanism is to adjudicate the outcome when one agent must give up on her top choice. It turns out that strategy-proofness will demand a local dictator is determined as a function of only the agents’ highest ranked objects. Suppose $(a, b)$ is the allocation that assigns each agent her favorite individually-feasible object. If $(a, b) \in C$, then any efficient mechanism gives this allocation. If not, then a local dictatorship assigns an agent as dictator. That dictator, say it is agent 1, gets her favorite object $a$, while the non-dictator agent 2 must compromise and is assigned her favorite object among those that are mutually feasible with 1 being assigned $a$. Since the identity of the dictator only depends on the agents’ top choices, a local dictatorship partitions the space of infeasible allocations into two groups, one where agent 1 is the dictator and another for agent 2.\(^\text{10}\)

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\(^\text{10}\)Note that this does not imply local dictatorships satisfy “tops-only” conditions from social choice. That is because the non-dictator’s object still depends on her entire rank order to determine her second-best assignment.
Definition 2. A mechanism is a local dictatorship for the constraint $C$ if there is a local dictator assignment $D : \tilde{C}^* \to \{1, 2\}$ such that, for any $(\tilde{z}_1, \tilde{z}_2)$, if $a$ and $b$ are 1 and 2’s top individually-feasible choices,

$$f(\tilde{z}_1, \tilde{z}_2) = \begin{cases} (a, b) & \text{if } (a, b) \in C \\ (a, \arg \max_{z_2} \{b' : (a, b') \in C\}) & \text{if } D(a, b) = 1 \\ (\arg \max_{z_1} \{a' : (a', b) \in C\}, b) & \text{if } D(a, b) = 2 \end{cases} \quad (1)$$

We first demonstrate that local dictatorship is necessary for individual strategy-proofness and Pareto efficiency with two agents.

Lemma 2. For any constraint $C$, if $f : \mathcal{P} \to C$ is strategy-proof and Pareto efficient, then it is a local dictatorship.

Proof. If $\tilde{C}$ is empty, then all allocations are feasible and the unique Pareto efficient mechanism gives each agent her top choices. So assume $\tilde{C}$ is nonempty and fix a strategy-proof and Pareto efficient mechanism $f : P^2 \to C$.\(^{11}\) Let $a$ and $b$ be individually-feasible objects for 1 and 2 respectively. Since each is individually-feasible, there are $a'$ and $b'$ with $(a', b)$ and $(a, b')$ in $C$. Let $\tilde{z}_1 \in P^1[a, a']$ and $\tilde{z}_2 \in P^1[b, b']$. By Pareto efficiency, $f(\tilde{z}_1, \tilde{z}_2) = (a, b')$ or $f(\tilde{z}_1, \tilde{z}_2) = (a', b)$. Assume without loss that $f(\tilde{z}_1, \tilde{z}_2) = (a, b')$. We will show that this implies that 1 is the local dictator at $(a, b)$. That is, for any other preference profile where 1 top-ranks $a$ and 2 top-ranks $b$, 1 gets $a$ while 2 gets her favorite object compatible with $a$. Pick any other $\tilde{z}_2'$ which top-ranks $b$. By 2’s strategy-proofness, $f_2(\tilde{z}_1, \tilde{z}_2') \neq b$, but then from Pareto efficiency, $f_1(\tilde{z}_1, \tilde{z}_2') = a$, since otherwise, the allocation $(a', b)$ would Pareto dominate $f(\tilde{z}_1, \tilde{z}_2')$. Thus $f_1(\tilde{z}_1, \tilde{z}_2') = a$ whenever $\tilde{z}_2' \in P^1[b]$. Then using 1’s strategy-proofness, we have that $f_1(\tilde{z}_1, \tilde{z}_2') = a$ for all $\tilde{z}_1, \tilde{z}_2'$ with $\tau(\tilde{z}_1, \tilde{z}_2') = (a, b)$. Finally, by Pareto efficiency, $f(\tilde{z}_1, \tilde{z}_2') = (a, \arg \max_{z_2} \{b' : (a, b') \in C\})$ whenever $\tau(\tilde{z}_1, \tilde{z}_2') = (a, b)$. Thus we say that 1 is the local dictator at $(a, b)$. Since $(a, b)$ was arbitrary every other infeasible allocation in $\tilde{C}^*$ has a local dictator by a symmetric argument.

However, this only establishes necessity. Not all dictatorship partitions will be strategy-proof. For example, suppose agent 1 is the local dictator when $(a, b)$ are the top choices, while agent 2 is the local dictator at $(a, b')$. Then agent 2 may benefit from gaining local dictatorship rights by misreporting her top choice as $b'$. For this reason, we show that if 1 is the local dictator at $(a, b)$, she must also be the dictator at $(a, b')$. Similarly, if $(a', b')$ is infeasible and 1 is the dictator at $(a, b')$ while 2 is the dictator at $(a', b')$, agent 1 may wish to announce $a'$ as her top choice even when $a$ is truly her top choice, again to gain dictatorship rights. So the same dictator must have control at any two infeasible joint allocations that share an individual allocation. This consistency must also hold indirectly: if we have a sequence of infeasible allocations $(a_1, b_1), (a_1, b_2), (a_2, b_2), \ldots, (a_n, b_n)$ where each allocation shares a coordinate with is predecessor, then the same agent must be dictator for all allocations in the sequence.

Theorem 1. For any two-agent constraint $C$, a mechanism $f$ is strategy-proof and Pareto efficient if

\(^{11}\)Serial dictatorship always is Pareto efficient and strategy-proof, so one exists.
and only if it is a local dictatorship such that
\[
D(a, b) = D(a', b') \text{ whenever } a = a' \text{ or } b = b'.
\]

Any local dictatorship satisfying this condition is an adapted local dictatorship.

**Proof.** First we verify necessity of the axioms. The necessity of local dictatorship was established in Lemma 2. It remains to show that the local dictatorship must be adapted. Suppose that \((a, b)\) and \((a', b')\) are two infeasible allocations in \(C^*\) and either \(a = a'\) or \(b = b'\). Without loss, assume \(a = a'\).

Suppose by way of contradiction that \((a, b)\) and \((a', b')\) have different local dictators. For example, suppose \(D(a, b) = 1\) and \(D(a', b') = 2\). Consider the preference profile \((\succ_1, \succ_2)\) where \(\succ_1 \in P^\uparrow [a]\) and \(\succ_2 \in P^\uparrow [b, b', b'']\) where \(b''\) is such that \((a, b'')\) \(\in C\) (such a \(b''\) exists because \(a\) is individually-feasible). Then since 1 is the local dictator at \((a, b)\), we get \(f(\succ_1, \succ_2) = (a, b'')\). However, if \(\succ_2 \in P^\uparrow [b']\), then \(f_2(\succ_1, \succ_2) = b' >_2 b'' = f_2(\succ_1, \succ_2)\) since 2 is the local dictator at \((a, b')\). This gives a violation of strategy-proofness. Thus either 1 is the local dictator at both \((a, b)\) and \((a', b')\) or 2 is.

Now we turn to sufficiency. Suppose \(f\) is an adapted local dictatorship. Efficiency is immediate.

To verify strategy-proofness, fix a preference profile \((\succ_1, \succ_2)\). If agent 1 is the dictator at \(\tau_1(\succ_1, \succ_2)\), then she clearly has no reason to misreport. So suppose 1 is not the dictator and let \(\succ_1'\) be an alternative preference for agent 1. Either \(\tau_1(\succ_1', \succ_2)\) is feasible, or by adaptedness, agent 2 is still the dictator at this profile because her top choice, say \(b\), is the same as in \((\succ_1, \succ_2)\). Since \(f_2(\succ_1', \succ_2) = b\), it must be that \(f_1(\succ_1', \succ_2) \in \{a': (a', b) \in C\}\) by feasibility. But then \(f_1(\succ_1, \succ_2) = \max_{\succ_1} \{a': (a', b) \in C\}\), verifying strategy-proofness.

Let \(
\Gamma(C)\n\) denote the graph with the infeasible allocations \(C^*\) as the vertices and where \((a, b)\) and \((a', b')\) share an edge if either \(a = a'\) or \(b = b'\). Adaptedness is the same as the requirement that connected components of this graph share the same dictator.

It may be useful to illustrate this construction and the associate graph in an example. Figure 1 gives an example for a specific constraint, chosen to describe the construction. The top-left panel (A) shows the constraint; grey cells are infeasible allocations. Panel (B) permutes \(\chi_1 = \{a_4\}\) and \(\chi_2 = \{a_4, a_6\}\) to the top and left most objects.\(^{12}\) In panel (C), a particular 4-element connected component of \(\Gamma(C)\), namely \(\{(a_2, a_1), (a_2, a_3), (a_6, a_3), (a_6, a_8)\}\) is shaded black. No element of the grey set is connected to any member of \(C^*\) which is shaded black. Again, we may permute the rows and columns to display the connected components of \(\Gamma(C)\) more easily. Hence in panel (D), we again permute the objects. As we can now more easily observe that there are three connected components of \(\Gamma(C)\) which are indicated as \(E_1\), \(E_2\), and \(E_3\). We can then assign a dictator to each component independently as described above. In particular, there are exactly eight \(2^3\) distinct strategy-proof and Pareto efficient mechanisms for this constraint.

In particular, as the example makes clear, the number of strategy-proof and Pareto efficient mechanisms is exponential in the number of connected components in \(\Gamma(C)\).

**Corollary 1.** The number of strategy-proof and Pareto efficient mechanisms is \(2^E\) where \(E\) is the number of connected components of \(\Gamma\).

\(^{12}\)The ordering of rows and columns is arbitrary.
To understand the range of possible mechanisms for some familiar problems, consider figure 2. It presents the house allocation and social choice problems, which are exact complements. On the left is house allocation. Each grey infeasible allocation is disconnected from every other infeasible allocation. Every mechanism corresponds to a labeling of the grey boxes with 1’s and 2’s, which can be done independently. Another way to think about this is that each object is “owned” by one of the agents. If either agent top-ranks an object they own, she will be assigned it. If both agents top-rank the other agents’ object, they trade. This illustrates that other mechanisms can be reparameterized as local dictatorship mechanisms. In the two-agent case, since house allocation has the maximal number of disconnected components, it admits the maximal number of strategy-proof and Pareto efficient mechanisms, namely $2^0$.

On the right of figure 2 is the social choice constraint. With three or more objects, it is possible to move from any grey square to another through a path in the infeasible space where at each step at most one agents’ allocation changes. In other words, the graph $\Gamma$ is connected. Then Theorem 1 immediately yields the two-agent version of the Gibbard–Satterthwaite Theorem, because a single agent is the dictator everywhere. That result famously requires at least three alternatives. Our analysis provides new perspective on this cardinality requirement. With two objects, the constraint would be the top-left $2 \times 2$ constraint. The two infeasible allocations are now disconnected, and therefore each can be assigned a different dictator. If the same agent is dictator in both infeasible squares, this is
simple dictatorship. The only nondictatorial mechanism, up to relabelling, is where agent 1 is the dictator at \((a_1, a_2)\) and agent 2 is the dictator at \((a_2, a_1)\). Then the mechanism is unanimity rule with \(a_1\) as the default option, or equivalently a veto rule where either agent can veto the adoption of \(a_2\). This again illustrates of how well-known rules can be expressed as local dictatorships. Finally, social choice admits the fewest possible strategy-proof and Pareto efficient mechanisms, having the minimal number of connected components, namely one.

![Figure 2: The social choice and house allocation constraints for two agents and 10 objects.](image)

Finally, one theoretical benefit of having a common representation of mechanisms across arbitrary constraints is that this unified language allows for comparisons across constraints. For example, fix an adapted local dictatorship for some constraint \(C_0\). Now suppose the constraint relaxes to some superset \(C_1 \supset C_0\), keeping fixed the individually infeasible objects. Then the complementary set of infeasible allocations \(\overline{C}_0\) contracts to \(\overline{C}_1\). However, we can still import the same local dictator assignment from the original mechanism by using its relativized assignments. That is, if \(D_0 : \overline{C}_0 \rightarrow \{1, 2\}\) is the original local dictator assignment, then its relativization \(D_1 = D_0|_{\overline{C}_1} : \overline{C}_1 \rightarrow \{1, 2\}\) defines a strategy-proof and Pareto efficient mechanism for the relaxed problem. This insight is also practically useful: writing the mechanism as a local dictatorship makes clear how to adapt an existing mechanism when constraints relax in a way that respects existing claims of priority.

In independent and contemporaneous work, Meng (2019) provides a characterization of all strategy-proof and Pareto efficient mechanisms for the two-agent social choice problem when agents are known to be indifferent between classes of alternatives that are fixed a priori. His characterization involves assigning a dictator at all profiles of preferences over announced indifference classes, where the dictator assignment must respect a cell-connected property. The structure of his characterization has similarities to our assignment of local dictators to the infeasible set. In fact, either result can be deduced from the other.\(^\text{13}\) While technically equivalent, these results are cast for very different questions, his for

\(^\text{13}\)Our private values environment can be parameterized in his social choice setting by interpreting feasible allocation vectors as the set of social outcomes then taking the indifference classes to be the allocation vectors that share a common projection for that agent, that is, where her private allocation is the same. His model can also be considered a special case of ours, by interpreting each element of an agent’s partition as an individual object and a product of partition elements as an allocation vector. Then his framework, requiring a common social-choice outcome for all agents, can be parameterized in our framework with the particular constraint that only allocation vectors where the associated partition elements have a nonempty intersection are feasible.
known indifference classes and ours for a known constraint. While the papers should share precedence for the mathematical result, the substantive contributions, interpretations, and applications of the two papers differ.

3.2 N Agents

We now proceed with the characterization for an arbitrary number of agents. The result will require additional machinery, which we introduce now.

Definition 3. Let \( f : \mathcal{P} \to C \) and let \( M \) be a proper subset of \( N \). Let \( \succsim_{-M} \) be a profile of preferences for the agents not in \( M \). The marginal mechanism at \( \succsim_{-M} \) is denoted \( f_{\succsim_{-M}} : P^M \to O^M \) and defined as the function

\[
\succsim_M \mapsto \left[f_j(\succsim_M, \succsim_{-M})\right]_{j \in M}.
\]

We denote \( I(\succsim_{-M}) = f_{\succsim_{-M}}(P^M) \) which will be referred to as \( M \)'s marginal option set.

Thus a marginal mechanism holds fixed some of the agents’ preferences \( \succsim_{-M} \) and defines an \( |M| \)-agent mechanism for the remaining agents, mapping their profile of announcements \( \succsim_{-M} \) to an \( M \)-agent allocation. Clearly, marginal mechanisms inherit the group strategy-proofness of the original mechanism. It turns out that in the other direction, group strategy-proofness of the two-agent marginal mechanisms suffices for groups strategy-proofness of the full mechanism.

Lemma 3 (Alva (2017), Theorem 1). The mechanism \( f : \mathcal{P} \to C \) is group strategy-proof if and only if it is pairwise strategy-proof.

This significantly reduces the number of conditions to verify that a mechanism is group strategy-proof. Rather than verifying incentives for all coalitions, it is sufficient to check that no two agents can profitably misreport their preferences. The equivalence of group and pairwise incentives was originally proved for the house allocation domain by Papái (2000). The most general equivalence was proven by Alva (2017) for a broad set of preference domains. We give a more direct proof for our setting in the appendix.

While this result takes a significant step towards understanding group strategy-proofness, it is especially useful in light of our characterization of two agent strategy-proof and efficient mechanisms. For two-agent mechanisms, there is only one group coalition, namely the grand coalition. Therefore group strategy-proofness of a two-agent mechanism is equivalent to individual strategy-proofness and Pareto efficiency on its image. Combining Lemma 3 with Theorem 1, we get a characterization of all group strategy-proof and Pareto efficient mechanisms.

Theorem 2. A mechanism \( f : \mathcal{P} \to C \) is group strategy-proof and Pareto efficient if and only if \( f_{\succsim_{-ij}} \) is an adapted local dictatorship (using the marginal constraint) for any two agents \( i, j \) and any residual preference profile \( \succsim_{-ij} \).

Compared to Theorem 1, this characterization is considerably less descriptive. Theorem 1 gives an explicit and simple procedure that captures all strategy-proof and Pareto efficient mechanisms, which are also group strategy-proof on their images. Theorem 2 describes such mechanisms more indirectly. Nevertheless, as we will see in our applications, this characterization substantially reduces
the burdens in verifying and constructing group strategy-proof and Pareto efficient mechanisms for many constraints.

4 Existence Results

In this section, we introduce sequential dictatorships, a class of mechanisms which generalize the familiar serial dictatorship mechanisms and which are group strategy-proof and Pareto efficient for any constraint. We then provide a theorem which provides conditions on the constraint $C$ that are sufficient to guarantee that $C$ includes mechanisms which are not sequential dictatorships. As a corollary, we deduce that the two-sided matching and school choice problems admit non-dictatorial mechanisms which are group strategy-proof and Pareto efficient.

4.1 Sequential Dictatorships

We begin by formalizing the sequential dictatorship mechanism. This is always group strategy-proof and Pareto efficient. In the more well-known serial dictatorship, there is a fixed ordering of agents and each agent picks her top choice that forms a feasible suballocation with the choices of earlier dictators. Sequential dictatorship slightly generalizes serial dictatorship by allowing the picking order of remaining agents to depend on the choices of earlier dictators.

A shortcoming of sequential dictatorship is that it concentrates power to the early dictators, so is sometimes considered an unfair mechanism. However, in many situations sequential dictatorship is the only available group strategy-proof and Pareto efficient mechanism. The applications in the sequel of this section demonstrate that characterization for social choice, one-sided matching, and multiple assignment.

We now formally define sequential dictatorship. Recall that $S$ is the set of suballocations (i.e. the maps $\mu : M \to \mathcal{O}$ where $M \subset N$). Let $S'$ be the set of incomplete suballocations. A picking order is a map $\zeta : S' \to N$ such that for any suballocation $\mu$, $\zeta(\mu)$ is an agent not allocated an object under $\mu$. For each picking ordering and for any constraint $C$ we may define the sequential dictatorship for $\zeta$ to be the mechanism whose allocation at any preference profile is determined by the following algorithm:

| Step 1 | The agent $d_1 \equiv \zeta(\emptyset)$ is the first dictator. She chooses her favorite object $a_1$ from $\pi_{d_1}C$. Let $\mu_1$ be the suballocation in which $d_1$ is assigned $a_1$ and all other agents are unassigned. |
|--------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Step $k$ | The agent $d_k \equiv \zeta(\mu_{k-1})$ chooses her favorite object $a_k$ from $\pi_{d_k}C(\mu_{k-1})$. Let $\mu_k$ be the allocation $\mu_{k-1} \cup \{(d_k, a_k)\}$. If all agents have been assigned an object, stop. If not, continue to step $k + 1$. |

$^a$Recall that we think of an allocation both as a map $\mu : N \to \mathcal{O}$ and as a subset of $N \times \mathcal{O}$

$^{14}$Where $M$ is a proper subset of $N$. 

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Standard serial dictatorship, where the order of dictators is fixed and invariant to earlier choices, are a prominent special case of sequential dictatorships where the picking order only depends on the agents identified in a suballocation, that is where \( \zeta(\mu) = \zeta(\mu') \) whenever \( \mu \) and \( \mu' \) are suballocations to the same subcoalition \( M \) of agents.

Notice that a single mechanism can be the result of many picking orders. This is because the picking order \( \zeta \) can be defined in any way off the “algorithm path,” in the sense that suballocations which will never be realized can be assigned any agent as the next dictator. For example, in the serial dictatorship mechanism, any allocation in which a single agent other than the dictator is assigned an object will never be realized, so the picking order there is immaterial. Keeping this redundancy in mind, it is nonetheless convenient to take \( S' \) as the domain of all picking orders. The following obvious remark implies non-emptiness of the set of group strategy-proof and Pareto efficient mechanisms.

**Remark 2.** For any constraint \( C \), sequential dictatorship is group strategy-proof and Pareto efficient.

We say that a constraint \( C \) admits non-dictatorial group strategy-proof and efficient mechanism if there are group strategy-proof and efficient mechanisms which are not sequential dictatorships.

### 4.2 Extending Mechanisms

It will be useful to describe a procedure that can be used to extend group strategy-proof mechanisms defined only for a proper subset of agents. First, fix \( C \) and \( M \), a proper subset of \( N \). Let \( f \) be a group strategy-proof and Pareto efficient mechanism for the agents in \( M \) on the projection \( C^M \). Given an allocation \( \mu \in C^M \), recall that \( C(\mu) \) is the set of all allocations in \( C \) which extend \( \mu \). Define \( C^*(\mu) = \pi_{M^c} C(\mu) \) to be the set of extensions for the agents in \( M^c \). Suppose that for each \( \mu \in C^M \) we have a mechanism \( g_\mu : P^{M^c} \to C^*(\mu) \), then we can extend \( f \) to the mechanism \( h : \mathcal{P} \to C \) where

\[
h_i(\succsim) = \begin{cases} f_i(\succsim_M) & \text{if } i \in M \\ [g_j(\succsim_M)(\succsim_{M^c})]_j & \text{if } j \in M^c \end{cases}
\]

we denote this mechanism \( f + (g_\mu)_{\mu \in C^*(\mu)} \).

**Lemma 4.** If \( f \) and each \( g_\mu \) is group strategy-proof, then \( f + (g_\mu)_{\mu \in C^*(\mu)} \) is group strategy-proof.

### 4.3 Non-dictatorial Mechanisms

We now give a partial converse to the Gibbard-Satterthwaite Theorem. That is, we provide conditions which guarantee that a constraint admits non-dictatorial group strategy-proof and efficient mechanisms.

Fix a constraint \( C \). If there is a pair of agents \( i, j \) such that \( \Gamma(C^{(i,j)}) \) has at least two connected components then by Theorem 1 we can construct a non-dictatorial mechanism \( f \) for \( i \) and \( j \) on the constraint \( C^{(i,j)} \). By Lemma 4 we can extend this mechanism to \( C \). This is summarized by the following theorem.

**Theorem 3.** If a constraint \( C \) is such that for some \( i, j \), the graph \( \Gamma(C^{(i,j)}) \) has more than one component, then \( GS^n(C) \) is strictly larger than the set of sequential dictatorship mechanisms.
This result provides a simple test to verify that non-dictatorial mechanisms exist. Many cases where other mechanisms are available fall under this result.

**Corollary 2.** The following settings admit non-dictatorial mechanisms:

- Two-sided matching with at least two agents on each side
- School choice with at least two schools $s$ and $t$ with capacity $q_s$ and $q_t$ such that $q_s + q_t \leq N$
- House allocation with at least two houses.

As shown by the intricate characterization of Pycia and Ünver (2017), an exact description of the class of mechanisms can be difficult when there are non-dictatorial mechanisms. To our knowledge, a concise description of the group strategy-proof and efficient mechanisms for the two-sided matching and school choice problems is outstanding.

A necessary condition to have only sequential dictatorships is therefore that the two-agent marginal constraints are totally connected. This is obviously true for the social choice problem, and much of the work in the proof of the roommates characterization is establishing this claim for the marginal constraints as a consequence of the structure of the grand constraint.

## 5 Characterizations for Specific Constraints

In this section, we apply our results to specific families of constraints. Specifically, we explore a class of constraints that we call “single-compromising” as well as the the social choice, roommates and multiple assignment problems.

### 5.1 Single-Compromising Constraints

A constraint $C$ is called **single-compromising** if for every infeasible $(a_i)_{i \in N}$, for every $i$ there is a $a'_i$ such that $(a'_i, a_{-i})$ is feasible. Thus, from any infeasible allocation, all agents have the possibility of unilaterally compromising to make the social allocation feasible. As a practical example, imagine a manager who needs to distribute tasks among workers. The requirement is that every task be assigned to at least one worker. In this case, the constraint is single-compromising because any agent can unilaterally take on all unassigned tasks.

In this case, every group strategy-proof and Pareto efficient mechanism can be written in a simple manner analogous to the characterization of the two-agent case. The generalization again colors the space of infeasible allocations.

For any $C$, let $\alpha : \bar{C} \to N$ be such that $\alpha(a) = i \implies \alpha(a'_i, a_{-i}) = i$ whenever $(a'_i, a_{-i}) \in \bar{C}$. We call such an $\alpha$ an **adapted local compromiser assignment**. Given such an $\alpha$, define $f^\alpha$ to be the mechanism such that for any profile $\succsim$ if $\tau_1(\succsim) \in C$ then $f(\succsim) = \tau_1(\succsim)$. Otherwise, if $\tau_1(\succsim) \in \bar{C}$, then let $i = \alpha(\tau_1(\succsim))$. Then $f(\succsim)$ is the allocation where all agents $j \neq i$ get their top-ranked alternative and $i$ gets their top-ranked object among those which make this feasible.

**Proposition 1.** Let $C$ be single-compromising. A mechanism $g$ is group strategy-proof and Pareto efficient if and only if $g = f^\alpha$ for some adapted local compromiser assignment $\alpha$. 

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5.2 Social Choice

We first turn to the social choice problem and prove the celebrated Gibbard-Satterthwaite Theorem (Gibbard 1973), (Satterthwaite 1975). The goal is to give a simple demonstration of how to apply our characterization results.

**Lemma 5.** Let $C$ be the social choice constraint, i.e. $C = \{(a_i)_{i \in N} | a_i = a_j \text{ for all } i, j \in N\}$ then a map $f : \mathcal{P} \rightarrow C$ is group strategy-proof if and only if it is individually strategy-proof.

We can then apply our main characterization results to the special case of the diagonal social choice constraint to derive that all group strategy-proof and onto mechanisms are dictatorships, which by virtue of Lemma 5 is equivalent to the Gibbard–Satterthwaite Theorem.

**Theorem 4** (Gibbard–Satterthwaite). Let $C$ be the social choice constraint. If $|O| > 2$ and $f : \mathcal{P} \rightarrow C$ is surjective and group strategy-proof then it is dictatorial.\(^{15}\)

We leave the formal proof to the appendix but we sketch the argument here. First observe that some two-agent marginal mechanism has at least three outcomes. By way of contradiction, suppose all have at most two. Some marginal mechanism, say for agents 1 and 2 when others report $(\succeq_3, \ldots, \succeq_n)$, has exactly two outcomes since otherwise the whole mechanism is single-valued. A simple consequence of Theorem 1 that we discussed earlier is that the marginal mechanism with two outcomes ($a$ and $b$) is either a dictatorship (say of agent 1) or a “veto” mechanism where one object (say $b$) is the default unless both prefer $a$. Suppose player 2’s type $\succeq_2^*$ has $c \succ_2^* a \succ_2^* b$. Then agent 1’s preference is followed in both dictatorship and the veto mechanism. But the marginal mechanism for agents 1 and 3 when others report $(\succeq_2^*, \succeq_4, \ldots, \succeq_n)$ must also have $a$ and $b$ as outcomes. Repeating the argument across agents, $a$ or $b$ is implemented even though all agents prefer $c$, violating group strategy-proofness.

Any two-agent mechanism (say for agents 1 and 2) with three or more outcomes is dictatorial (say for 1) by visual inspection. Then 2’s type is irrelevant, so there also exists a $(1,3)$-marginal mechanism with at least three outcomes. Agent 1 must still be dictator, so 3’s type is irrelevant. Repeating across agents, no preference matters besides 1’s so she is a dictator.

Interestingly, the group strategy-proof and Pareto efficient mechanisms for the roommates problem and the social choice problem are the same: sequential dictatorships. For the roommates problem, there is enough flexibility in the constraint that a sequential dictatorship still has room for the those who are second or later in the picking order to have nontrivial choices. In the social choice problem, all sequential dictatorships are (simple) dictatorships because agents picking second or later have no choices to make because the first dictator’s choice determines the entire allocation profile. So our model presents a unified view of both problems as allowing only sequential dictatorships.

5.3 The Roommates Problem

We now apply our general results to the canonical roommates problem. In the roommates problem, an even number of agents need to be paired as roommates. Each agent has a strict preference over

\(^{15}\)In fact, we only need that $|im(f)| > 2$ in which case we could drop items never allowed and recover the same statement.
the other agents as roommates. As discussed earlier, we can model this in our environment by letting \( \mathcal{O} = N \) and using the constraint

\[
C = \{ \mu : N \to N \mid \mu(i) \neq i \text{ for all } i \text{ and } \mu^2 = id \}
\]

Any feasible mechanism for this constraint will be called a **roommates mechanism**. As mentioned in the introduction, the literature on the roommates problem has focused on stable matching and there is little known about incentives and efficiency for one-sided matching.

Theorem 5 characterizes all group strategy-proof and Pareto efficient roommates mechanisms. This is akin to the Gibbard–Satterthwaite Theorem that demonstrates all such mechanisms are dictatorships for the social choice problem and akin to the recent result of Pycia and Ünver (2017) that characterizes all such mechanisms for the house allocation problem. The analogous characterization had been yet outstanding for one-sided matching. We settle this question for the roommates problem, and show that all mechanisms with these properties are sequential dictatorships.

**Theorem 5.** A roommates mechanism is group strategy-proof and Pareto efficient if and only if it is a sequential dictatorship.

While the proof of Theorem 5 is involved and therefore only fully described in the Appendix, we illustrate some of the main ideas here. The proof studies whether two-agent marginal mechanisms are local dictatorships, which is also the crucial step in the argument we provided for the Gibbard–Satterthwaite Theorem. Suppose that \( f \) is a group strategy-proof and Pareto efficient roommates mechanism. Consider any pair \( i \) and \( j \) and any preference profile \( \succeq_{ij} \). Let \( X_i \) and \( X_j \) denote the set of objects (partners) with whom \( i \) and \( j \) cannot be matched in the marginal mechanism \( f_{\succeq_{ij}} \). Lemma 1 implies that \( X_i \) and \( X_j \) can be ignored in the marginal mechanism. For this discussion, assume that it is possible for \( i \) and \( j \) to be matched with each other in this marginal mechanism. Since \( i \) and \( j \) can be matched, the infeasible allocations in the marginal constraint is a superset of those shown in the left panel of figure 3.

\[
\begin{align*}
&\text{Figure 3: The marginal constraint } I(\succeq_{ij}) \\
&\text{Notice that if there are any infeasible allocations in the set } (\mathcal{O} - X_i - \{j\}) \times (\mathcal{O} - X_j - \{i\}) \text{ then the graph } \Gamma(f_{\succeq_{ij}}) \text{ of this marginal constraint is totally connected and there must be a single dictator that is common to all infeasible allocations, as illustrated in the right panel of figure 3. Otherwise, if the infeasible space is not connected, then the infeasible space is as illustrated in the left panel.}
\end{align*}
\]
of figure 3. For that case, the picture strongly resembles the two-agent social choice problem with two objects. We discussed the possible mechanisms for that case after figure 2, showing that these are either dictatorship or unanimity/veto mechanisms with a default option. From Theorem 1, we know that each of the two disconnected components must be assigned a dictator, leaving four possible mechanisms as illustrated in figure 4.

![Figure 4: The possible marginal mechanisms](image)

Panels (A) and (B) assign the same dictator to both regions and are therefore standard dictatorships. Panels (C) and (D) are veto mechanisms. In (C), $i$ and $j$ match with each other only if both top-rank each other as their favorite roommates, which is like a veto option with default “not match together.” In (D), they match if even one of them has the other as her favorite, which is like a veto option with the opposite default of “match together.” The rest of the proof is a lengthy induction on the number of agents that every marginal mechanism, including the grand mechanism, is a sequential dictatorship. The specific steps involve verifying that mechanisms (C) and (D) are not possible for the two-agent marginal mechanisms because they are precluded by the structure of the grand constraint across all agents in the roommates problem. We only mention this here as an example of how central the two-agent characterization can be for proving results with many agents. The visual intuition generated by the Theorem 1 connects the marginal constraint of the roommates problem with the social-choice constraint with two objects. This connection ends up being useful in understanding the possible kinds of mechanisms for the roommates problem.

Although the results in this paper are generally unrelated to stability, this one does speak to a stability. In fact, it immediately exposes a tension between incentives and stability. As mentioned,
an important feature of the roommates problem is the lack of stable outcomes. A standard escape to find positive results is to demand something weaker than standard stability. For example, one relaxation only requires that pairs of agents where each ranks the other as her favorite must be matched, substantially reducing the set of relevant blocking pairs. This weaker stability condition is called “mutually best” by Toda (2006) and “pairwise unanimity” by Takagi and Serizawa (2010). However, sequential dictatorships violate even this very weak form of stability. So a corollary of Theorem 5 is that group strategy-proofness and Pareto efficiency are incompatible with even this weak stability notion, exposing a strong tension between incentives and stability for the roommates problem. This negative observation for the roommates problem is not new; in fact, this corollary of our result can also be implicitly derived from Theorem 2 of Takamiya (2013) without an explicit characterization of group strategy-proofness.\footnote{We thank Yuichiro Kamada for pointing this out to us.} We mainly provide this result to show that our results regarding incentives and efficiency can indirectly provide insights into stability.

**Corollary 3** (Takamiya (2013)). *No group strategy-proof and Pareto efficient roommates mechanism can guarantee that mutual top choices are matched.*

### 5.4 Multiple Assignment

Consider the problem where there is a finite set $H$ of “houses” and each agent can own up to $k$ houses. In this case $O$ is the set of subsets of $H$ of size $k$ or less. We allow agents to have arbitrary preferences over these subsets. An allocation $(s_i)_{i \in N}$ is feasible if for every $i$ and $j$, $s_i \cap s_j = \emptyset$. The house allocation problem is the special case where $k = 1$. For four houses $H = \{a, b, c, d\}$ and two agents 1 and 2 the constraint is shown in figure 5. Applying Theorem 1 to this particular constraint, simple visual inspection verifies that the entire infeasible space is connected, so the only strategy-proof and efficient mechanisms are sequential dictatorships, recalling that dictatorship and sequential dictatorship are equivalent with two agents. This turns out to be generally true. We use our prior characterizations to prove following characterization for any number of agents and any $k \geq 2$. 
Theorem 6. If \( k \geq 2 \) all group strategy-proof and Pareto efficient mechanisms are sequential dictators.

Proof. We will prove the result by induction over the number of agents. Our base case begins with just two agents 1 and 2. If \(|H| = 1\), there is a single infeasible allocation, so the result holds immediately by Theorem 1. Now suppose that \(|H| \geq 2\) and that \((s_1, s_2)\) and \((t_1, t_2)\) are infeasible. We need to show that the same local dictator will be assigned to both \((s_1, s_2)\) and \((t_1, t_2)\). Let \( x \) be a house in \( s_1 \cap s_2 \) and \( y \) be a house in \( t_1 \cap t_2 \) (we don’t rule out the case that \( x = y \)). Then every allocation in the sequence \((s_1, s_2), (s_1, \{x, y\}), (\{x, y\}, \{x, y\}), (\{x, y\}, t_2), (t_1, t_2)\) is infeasible (If \( x = y \), the set \( \{x, y\} = \{x\} = \{y\} \)). Furthermore, each infeasible allocation differs from the previous one by a single agent’s allocation. In light of Theorem 1, any Pareto efficient and strategy-proof mechanism must have a single dictator assigned to the entire constraint.

We proceed by induction on the number of agents. Suppose that the theorem holds for \( n \) agents. Fix a group strategy-proof and efficient mechanism \( f \) for the \( n + 1 \) agents \( \{0, 1, 2, \ldots, n\} \). It will be enough to show that a single agent always gets their top choice since after this agent gets their top choice, we have a sub-problem for the remaining agents which is covered by the induction assumption. We establish that at least one agent must get their top choice with a series of facts, each of which rely on the induction assumption:

**Fact 0:** For any preference profile \( \succ \) where some agent \( i \) top-ranks \( \emptyset \), \( i \) is matched with \( \emptyset \) by \( f \) and all other agents are matched via some sequential dictatorship. The picking order is independent of \( i \)’s ranking.

**Proof of fact 0:** \( i \) must be matched with \( \emptyset \) by Pareto efficiency. If not, there is a Pareto improvement where \( i \) is matched with \( \emptyset \) and all other agents’ allocations are unchanged. The marginal mechanism for agents other than \( i \) is now a sequential dictatorship by the induction assumption. By non-bosiness the picking order is independent of \( i \)’s preference. That is, the picking order does not depend on \( i \)’s ranking of alternatives below \( \emptyset \).

**Fact 1** There is an agent \( i \) such that for any \( x \in O \) and any preference profile where all agents top-rank \( x \) and second-rank \( \emptyset \), \( i \) gets \( x \).

**Proof of Fact 1:** By Pareto efficiency at every preference profile \( \succ \) described above exactly one agent gets assigned \( x \) and the other agents are assigned \( \emptyset \). Suppose that \( \succ \) is a profile where all agents top-rank \( x \) and second-rank \( \emptyset \) and \( \succ' \) is a profile where all agents top-rank \( y \) and second-rank \( \emptyset \). Since there are at least three agents, at least one agent \( k \) is assigned \( \emptyset \) at both \( \succ \) and \( \succ' \). However, by group strategy-proofness, if \( k \) pushes \( \emptyset \) to the top of their ranking, no one’s assignment should change. This contradicts Fact 0 since there must be a dictator among the agents other than \( k \).

**Fact 2** Let \( i \) be the agent guaranteed to exist by Fact 1. For any preference profile \( \succ \) where at least one agent gets \( \emptyset \), \( i \) must get their top choice.

**Proof of Fact 2:** Let \( k \) be the agent who gets \( \emptyset \). By non-bosiness, the outcome doesn’t change if \( \succ_k \) is changed to a profile which top-ranks \( \emptyset \). However in this case, the marginal mechanism for the agents other than \( k \) must be a sequential dictatorship by the induction assumption. The only agent who can be the first dictator in this mechanism is agent \( i \) by Fact 1.

**Fact 3** Let \( i \) be the agent guaranteed to exist by Fact 2. Let \( j \) be the agent who is the first dictator in the marginal mechanism when \( i \) top-ranks \( \emptyset \). This agent exists by Fact 0. For any profile
\( \succsim_{-ij} \) the marginal mechanism for \( i \) and \( j \) is a dictatorship.

**Proof of Fact 3:** Let \( g \) be the marginal mechanism for \( i \) and \( j \) at \( \succsim_{-ij} \). Now consider the marginal option set \( I(\succsim_{-ij}) \). By definition, \( I(\succsim_{-ij}) \). Note that for any \( x \in O \) if \( i \) top-ranks \( \emptyset \) and \( j \) top-ranks \( x \) then \( j \) gets \( x \). Likewise if \( i \) top-ranks \( x \) and \( j \) top-ranks \( \emptyset \) then \( i \) gets \( x \) by Fact 2. This implies that all outcomes are individually-feasible for both agents. Take any \((x, y)\) and \((x', y')\) which are both outside of \( I(\succsim_{-ij}) \) (so that both are infeasible for \( i \) and \( j \) in the marginal mechanism \( g \)). Now \((y, y)\) and \((y', y')\) are both in the complement of \( C_{ij} \) so are both in the complement of \( I(\succsim_{-ij}) \). Let \( \Gamma \) be the graph whose vertices are the elements of \( \bar{I}(\succsim_{-ij}) \) and where any two vertices \((u, v)\) and \((u', v')\) are connected if \( u = u' \) or \( v = v' \). We have already established that \((y, y)\) and \((y', y')\) are in the same connected component of \( \Gamma \) in the discussion of the base case. Furthermore, \((x, y)\) is connected to \((y, y)\) and \((x', y')\) is connected to \((y', y')\). Thus \((x, y)\) and \((x', y')\) are in the same connected component of \( \Gamma \). Since both were arbitrary, \( \Gamma \) is connected and, by Theorem 1 we get the desired result.

Now we are ready to finish the proof. Let \( i \) be the agent described in Fact 2 and \( j \) the agent described in Fact 3. Let \( \succsim \) be an arbitrary profile. Let \( \succsim_j \) top-rank \( \emptyset \). Then \( f(\succsim_j, \succsim_{-j})(i) \) is \( i \)'s top-ranked choice by Fact 2. Now since the \( i, j \)-marginal mechanism is a dictatorship by Fact 3, \( j \)'s preference cannot affect \( i \)'s allocation so that \( f(\succsim)(i) \) is also \( i \)'s top-ranked alternative.

\[ \square \]

While to our knowledge this exact result is novel, similar results for have been observed for nearby settings. Papái (2001) proved the result for the special case where \( k = H \), that is, when there is no cap on the number of objects an agent can own. Hatfield (2009) proved the result for the related case where each agent can have exactly \( k \) houses and no fewer, while our model allows agents to have fewer than \( k \) objects.\(^{17}\)

While the claim itself is closely related to known results, the argument we invoke is entirely different. We reach this conclusion by analyzing the structure of specific two-agents mechanisms. Our main point here is to show how sequential dictatorship is linked across three seemingly disparate settings: social choice, one-sided matching, and multiple assignment. What ties the proofs together across settings is the importance of two-agent marginal mechanisms in understanding the grand mechanism and then invoking our two-agent result to understand the structure of the marginal mechanisms. In all the proofs, a key step is to invoke the structure of the constraint to show that the existence of a dictator for particular two-agent marginal mechanisms implies sequential dictatorship for the grand mechanism. This requires work for each setting because generally the desired implication fails: while the existence of mechanisms beyond sequential dictatorship implies every two-agent marginal mechanism is dictatorial, the converse direction is not generally true.

\(^{17}\)To be more precise, both papers show sequential dictatorship is the unique individually strategy-proof, nonbossy, and Pareto efficient mechanism. However, since individual strategy-proofness and nonbossiness are equivalent to group strategy-proofness in our setting, these are the same conditions.
A Appendix

A.1 Preliminary Observations

We start with a simple result which recasts strategy-proofness in terms of option sets.

**Lemma 6** (Barberá (1983)). A mechanism \( f : \mathcal{P} \to C \) is strategy-proof if and only if there exist nonempty correspondences \( g_i : P^{N-1} \to O \) such that, for all agents \( i \), \( f_i(\zeta) = \max_{\zeta_i} g_i(\zeta_{-i}) \)

**Proof.** Define \( g_i(\zeta_{-i}) = f_i(P, \zeta_{-i}) \) then by strategy-proofness, we have \( f_i(\zeta_i, \zeta_{-i}) = \arg \max_{\zeta_i} g_i(\zeta_{-i}) \). Conversely, if \( f \) is defined such that there are \( \{g_i\} \) as in the proposition, then

\[
f(\zeta_i, \zeta_{-i}) = \arg \max_{\zeta_i} g_i(\zeta_{-i}) \quad \Rightarrow \quad f(\zeta_i', \zeta_{-i}) = f(\zeta).
\]

\( \square \)

It will be useful to relate group strategy-proofness with two other notions: nonbossiness and Maskin monotonicity.

**Definition 4.** A mechanism \( f : \mathcal{P} \to C \) is

1. **nonbossy** if, for all \( \zeta \in \mathcal{P} \),
   \[
f_i(\zeta_i', \zeta_{-i}) = f_i(\zeta) \quad \Rightarrow \quad f(\zeta_i', \zeta_{-i}) = f(\zeta).
\]

2. **Maskin monotonic** if, for all \( \zeta, \zeta' \in \mathcal{P} \),
   \[
   LC_{\zeta_i} [f_i(\zeta)] \supset LC_{\zeta_i} [f_i(\zeta)] \quad \text{for all} \quad i \quad \Rightarrow \quad f(\zeta') = f(\zeta).
   \]

**Proposition 2.** If \( f : \mathcal{P} \to A \) the following are equivalent:

1. \( f \) is group strategy-proof.
2. \( f \) is strategy-proof and nonbossy.
3. \( f \) is Maskin monotonic.\(^{18}\)

We can now demonstrate the desired implications for the equivalence in turn:

**Proof.** (1) \( \Rightarrow \) (2): Of course any group strategy-proof mechanism is individually strategy-proof. We will show nonbossiness by contradiction. Suppose there is a profile \( \zeta \) and an agent \( i \) with an alternative announcement \( \zeta_i' \) such that \( f_i(\zeta) = f_i(\zeta_i', \zeta_{-i}) \) but for some \( j \), \( f_j(\zeta) \neq f_j(\zeta_i', \zeta_{-i}) \). Then if \( f_j(\zeta) > f_j(\zeta_i', \zeta_{-i}) \), the coalition \( \{i, j\} \) can improve their outcome at \( (\zeta_i', \zeta_{-i}) \) by announcing \( (\zeta_i, \zeta_j) \). Otherwise, if \( f_j(\zeta) < f_j(\zeta_i', \zeta_{-i}) \), the coalition \( \{i, j\} \) can improve their outcome at \( \zeta \) by announcing \( (\zeta_i', \zeta_j) \).

(2) \( \Rightarrow \) (3): Suppose we have two profiles \( \zeta, \zeta' \in \mathcal{P} \) such that

\[
LC_{\zeta_i} [f_i(\zeta)] \supset LC_{\zeta_i} [f_i(\zeta)] \quad \text{for all} \quad i
\]

then notice that \( f_1(\zeta_1', \zeta_2, \ldots, \zeta_n) = f_1(\zeta) \) by Lemma 6 and by nonbossiness we have \( f(\zeta_1', \zeta_2, \ldots, \zeta_n) = f(\zeta) \). We can proceed, changing one preference at a time, to show that \( f(\zeta') = f(\zeta) \) as desired.

\(^{18}\)Note that the image of \( f \) may be arbitrary, so the claim is true for any constraint \( C \subset A \).
(3) $\implies$ (1): Let $\succsim_i \in \mathcal{P}$ and $\succsim^i_j$ be a candidate deviation for agents in $A$ so that

$$f(\succsim^j_A, \succsim^j - A) \succsim_j f(\succsim_j)$$

we will show that this implies $f(\succsim^j_A, \succsim^j - A) = f(\succsim_j)$. For each $j \in A$ construct $\succsim^*_{j}$ to be identical to $\succsim_j$ except that it ranks $f_j(\succsim^j_A, \succsim^j - A)$ first. For any $j \in A$ we have

$$LC_{\succsim^j} (f_j(\succsim^j_A, \succsim^j - A)) \supset LC_{\succsim^j} (f_j(\succsim^j_A, \succsim^j - A))$$

for all $j$. The first is immediate. To see the second, notice that if $f_j(\succsim^j_A, \succsim^j - A) = f_j(\succsim_j)$ then it holds trivially. If instead, $f_j(\succsim^j_A, \succsim^j - A) \neq f_j(\succsim_j)$, by assumption we have $f_j(\succsim^j_A, \succsim^j - A) \succsim_j f_j(\succsim_j)$ and since $\succsim^*$ only moves up the position of $f_j(\succsim^j_A, \succsim^j - A)$, the second statement holds. However, by Maskin monotonicity, the first statement gives $f(\succsim^j_A, \succsim^j - A) = f(\succsim^j_A, \succsim^j - A)$ and the second gives $f(\succsim^j_A, \succsim^j - A) = f(\succsim_j)$, so putting them together we get

$$f(\succsim^j_A, \succsim^j - A) = f(\succsim^j_A, \succsim^j - A) = f(\succsim_j)$$

as desired.

\[\Box\]

The relationship between group strategy-proofness and Maskin monotonicity was first revealed by the proof of the Muller–Satterthwaite Theorem, which proceeds by showing that either group or individual strategy-proofness is equivalent to Maskin monotonicity for the social choice problem (Muller and Satterthwaite 2017).\(^{19}\)

This equivalence between group strategy-proofness and Maskin monotonicity was then extended to other problems as well, including to house allocation by Svensson (1999) and for two-sided matching by Takamiya (2003). Takamiya (2003) unified these observations in a general statement for all indivisible-good economies without externalities that also applies to our model, and should be credited for the equivalence between (1) and (3) in Proposition 2.

The importance of nonbossiness in relating group and individual incentives was observed for the house allocation problem by Papáí (2000), who proved that individual and group strategy-proofness are equivalent for nonbossy rules. Proposition 2 pushes her observation to more general allocation problems with arbitrary constraints. Thomson (2016) surveys the literature on nonbossiness and its applications, and reviews many specific environments where group and individual incentives coincide. Proposition 2 at this level of abstract generality was also independently proved in the main result of Alva (2017), who makes a more general observation that the three conditions in Proposition 2 are equivalent for a broad class of preference domains, so we do not claim precedence for the proposition. We mainly present the result here to highlight its importance of this social-choice logic towards establishing our main characterization results. Our proof is also different from that in Alva (2017) because we are not interested in general preference domains, so our argument is consequently more direct and more limited.

### A.2 Proof of Remark 1

By way of contradiction, suppose that $f : \mathcal{P} \to im(f)$ is group strategy-proof and that there is a profile $\succsim$ and an allocation $(a_i)_{i \in N} \in im(f)$ such that $a_i \succsim f_i(\succsim)$ for all $i$ with at least one strict. By definition, there is an alternative profile $\succsim'$ such that $f(\succsim') = (a_i)_{i \in N}$ which is a profitable deviartion from $\succsim$.\(^{19}\)

\(^{19}\)Recall the Muller–Satterthwaite Theorem: all Maskin monotonic and surjective social choice functions are dictatorial.
A.3 Proof of Lemma 1

Let \( \{g_l\}_{l \in N} \) be as defined in Lemma 6 of the text. For each \( j \) the preference \( \succeq_j' \) does not change the relative ranking of the objects in \( g_j(\succeq_{-j}) \) hence we have \( f_j(\succeq_{-j}') = f_j(\succeq_{-j}) \) by Lemma 6 so by nonbossiness \( f(\succeq_j', \succeq_{-j}) = f(\succeq_j, \succeq_{-j}) \). Repeating this argument one agent at a time gives the result.

A.4 Proof of Lemma 3 (N-agent characterization)

If \( f \) is group strategy-proof, the marginal mechanisms are group strategy-proof by definition. For the other direction, suppose that every two-agent marginal mechanism is group strategy-proof. By Proposition 2 it suffices to show that \( f \) is individually strategy-proof and nonbossy. Then \( f \) is individually strategy-proof since for any \( i \) and any profile \( \succeq_i \) we can choose \( j \neq i \) and consider the marginal mechanism \( f_{\succeq_{-ij}}^j \). Then in this marginal mechanism \( i \) cannot profit from misreporting, hence she cannot be in \( f \). Now suppose we have \( f_i(\succeq_i', \succeq_{-i}) = f_i(\succeq_i) \) and for some \( j \), \( f_j(\succeq_i', \succeq_{-i}) \neq f_j(\succeq_i) \), either \( f_j(\succeq_i', \succeq_{-i}) \succ_j f_j(\succeq_i) \) or \( f_j(\succeq_i', \succeq_{-i}) \prec_j f_j(\succeq_i) \). However, by assumption the marginal mechanism \( f_{\succeq_{-ij}}^j \) is strategy-proof. From the two-agent characterization, no two-agent group strategy-proof mechanism can have this property.

A.5 Proof of Remark 2

By Proposition 2, it suffices to show that sequential dictatorships are strategy-proof and nonbossy. It is clear that for any \( \zeta \) the sequential dictatorship for \( \zeta \) is individually strategy-proof. Since \( \zeta \) only depends on the allocations of earlier dictators, it is also nonbossy.

To see that it is Pareto efficient, by Remark 1 it is enough to establish that its image is exactly \( C \). By construction, the image is a subset of \( C \). For any feasible allocation \( a \in C \) let \( \succeq_j \) put \( a_i \) first for all \( i \). Then \( f(\succeq_j) = a \) so \( im(f) = C \).

A.6 Proof of Lemma 4

By Proposition 2, it is enough to show that \( h : f + (g_{l})_{\mu \in C^*(\mu)} \) is monotonic. Let \( M \) be the set of agents such that \( im(f) = C^M \). Let \( \succeq \) and \( \succeq' \) be two profiles as in the definition of monotonicity. Since \( f \) is monotonic, \( f(\succeq_M) = f(\succeq'_M) \). Let \( \mu = f(\succeq_M) \). Then for any \( j \in M^c \) we have \( h_j(\succeq) = g_{\mu}(\succeq_M^c)_j = g_{\mu}(\succeq_M^c)_j \).

A.7 Proof of Theorem 3

If \( \Gamma(C^{i,j}) \) has more than one connected component we may assign a different local dictator to each component as in Theorem 1 to get a non-dictatorial marginal mechanism. We can then extend this mechanism using sequential dictatorship as described in Lemma 4.

A.8 Proof of Corollary 2

The projection of the house allocation constraint is shown in Figure 2. If there are at least two houses, this can easily be seen to have a graph \( \Gamma \) with at least two connected components.

For the two-sided matching problem, pick a pair of agents \( m_1 \in M \) and \( w_1 \in W \). Then \( \Gamma(C^{w_1,m_1}) \) has two connected components, namely the infeasible allocations \( (m_1, w_k) \) where \( k \neq 1 \) and the infeasible allocations \( (m_l, w_1) \) with \( l \neq 1 \).

In the school choice problem, with schools \( s \) and \( t \) such that \( q_s + q_t \leq N \). Construct a non-dictatorial mechanism \( f \) as follows. First run serial dictatorship where agents \( 1, 2, \ldots, N - 2 \) pick in order of their index. If the suballocation does not result in exactly one seat remaining at both \( s \) and \( t \), have agents \( N - 1 \) and \( N - 2 \)
pick in order. If $s$ and $t$ both have exactly one seat remaining, use one of the non-dictatorial strategy-proof and efficient mechanisms to match $N - 1$ and $N - 2$. This construction will result in a non-bossy and strategy-proof mechanism which is group strategy-proof and efficient by Proposition 2 and Remark 1.

A.9 Proof of Proposition 1

We show that every group strategy-proof and Pareto efficient mechanism is of the form $f^{\alpha}$ for some adapted local compromiser assignment $\alpha$. Let $C$ be a single-compromising constraint and fix and a group strategy-proof, efficient mechanism $g : \mathcal{O} \to C$. Let $a = (a_i)_{i \in N}$ be infeasible. For every $i$ there is an object $a_i'$ such that $(a', a_{-i}) \in C$. Let $\zeta_i \in P^I[a_i, a'_i]$ for each $i$. Since $g$ always returns feasible allocations, there is at least one agent $k$ who doesn’t get their top choice at the constructed preference profile $\zeta = (\zeta_i)_{i \in N}$. However, Pareto-efficiency then implies that $g_i(\zeta) = a_i$ for all $i \neq k$ and $g_k(\zeta) = a_k'$. By Maskin monotonicity and Lemma 6 we have that for any $\zeta'_{-k}$ with $\max_{\zeta'_{-k}} O = a_j$ for all $j \neq k$, $a_k \notin g_k(\zeta'_{-k})$, so that $k$ always compromises when the top choice is $a$. Define $\alpha(a_i) = k$ (we can do this unambiguously because no other agent always compromises at $a$, e.g. at the profile $\zeta$). Since $a$ was an arbitrary infeasible allocation, we can do the same for any other infeasible allocation to define $\alpha$ on all of $C$. Finally, we establish inductively that $f$ is local priority according to $\alpha$. Pick any preference profile $\zeta'$. Start at $a_1^1 = (\max_{\zeta'_{-1}} O)_{i \in N}$. If this is feasible, then $f$ being Pareto efficient implies $g(\zeta') = a_1$. Otherwise, it is infeasible, and by the previous argument, we have an agent $k = \alpha(a_1)$ who must compromise. Replace $\zeta_k'$ with the same preference, except that it puts $a_k'$ last. By Maskin monotonicity, this cannot affect the outcome of $f$. We therefore repeat the above process at the new profile. This is exactly how the local priority mechanism according to $\alpha$ works, giving the result.

Now we need to show that $\alpha$ has to satisfy the property that if $\alpha(a_i) = i$ then for any $(a_i', a_{-i}) \in \bar{C}$, we have $\alpha(a_i', a_{-i}) = \{i\}$. However this follows from similar reasoning as in the two-agent case. If, instead $k = \alpha(a_i', a_{-i})$ consider the profile $\zeta$ with $\tau_1(\zeta) = a_i$ and $\tau_2(\zeta_{-i}) = a_i'$ and $\tau_2(\zeta_k) = a_k'$ where $(a_i', a_{-i}) \in C$. We get a violation of Pareto efficiency since the local priority algorithm would make both $i$ and $k$ compromise to their second-best choice, which would be Pareto dominated by $(a_k', a_{-i})$.

The fact that this mechanism is group strategy-proof and Pareto efficient is now a simple consequence of Maskin monotonicity and Remark 1.

A.10 Proof of Lemma 5

By Proposition 2, it is enough to show that any strategy-proof mechanism is non-bossy. However, nonbossiness is immediate since all agents are allocated the same object.

A.11 Proof of Theorem 4 (Gibbard–Satterthwaite Theorem)

Let $C$ be the diagonal (i.e. the social choice constraint) and $|O| \geq 3$.

From Proposition 2, it suffices to show that any group strategy-proof mechanism is dictatorial. We will show this in two steps. First, we will show that for some $i, j$ and some profile $\zeta_{-ij} = (\zeta_k)_{k \neq i, j}$ we have $|I^{ij}(\zeta_{-ij})| \geq 3$. From the characterization of two-agent mechanisms, we will see that $f^{ij}_{\zeta_{-ij}}$ is dictatorial. We will then show that this implies the entire mechanism is dictatorial.

1. Suppose by way of contradiction that for all $i, j$ and all $\zeta_{-ij}$ we have $|I^{ij}(\zeta_{-ij})| < 3$. First, note that if for all $i, j$ and all $\zeta_{-ij}$ we have $|I^{ij}_{\zeta_{-ij}}| = 1$ then $f$ is single-valued which contradicts the surjectivity of $f$. Hence there is at least one pair of agents $i, j$ and $\zeta_{-ij}$ such that $|I^{ij}(\zeta_{-ij})| \geq 2$. For simplicity and without loss, let $i = 1$ and $j = 2$. By assumption then $|I^{ij}(\zeta_{-ij})| = 2$ and without loss assume

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20To see that $f(\zeta) = f(\zeta')$, change one preference at a time. No single change can alter $f$, so we get the result.
Likewise, for all $j, i \in \mathcal{N}$, conditional on each of $2$ strategy-proof and Pareto efficient roommates mechanisms are sequential dictatorships. We will show this for $m < n$ since a single agent's match determines the matches of everyone else. In this case, the result follows from the $\succeq_i$ we will first consider the possible two-agent marginal mechanisms. Let $\phi_1$ and $\phi_2$ where

$$\phi_1(\succsim_1, \succsim_2) = \begin{cases} a & \text{if } a \succ_1 b \\ b & \text{if } a \prec_1 b \end{cases}$$

and

$$\phi_2(\succsim_1, \succsim_2) = \begin{cases} a & \text{if } a \succ_1 b \text{ and } a \succ_2 b \\ b & \text{otherwise} \end{cases}$$

In the first, agent 1 is a dictator. In the second, $b$ is chosen by default and $a$ is only chosen if both agents prefer it to $b$. Let $c$ be another object in $\mathcal{O}$. If we let $\succsim_3 \in \mathcal{N}[c, a, b]$ then in either case we have $f(\succsim_1, \succsim_2, \succsim_3) = a$ if $a \succ_1 b$ and $f(\succsim_1, \succsim_2, \succsim_3) = b$ if $b \succ_1 a$. We then have that $a$ and $b$ are in $I^{1,2}(\succsim_2, \succsim_3, \ldots, \succsim_n)$. As before we have two possible mechanisms and in either one, if $\succsim_3 \in \mathcal{N}[c, a, b]$ we have $f(\succsim_1, \succsim_2, \succsim_3, \succsim_4, \ldots, \succsim_n) = a$ if $a \succ_1 b$ and $f(\succsim_1, \succsim_2, \succsim_3, \succsim_4, \ldots, \succsim_n) = b$ if $b \succ_1 a$. Continuing in this way, we get a profile of preferences in which all agents prefer $c$, but $c$ is not chosen. Since any group strategy-proof map is efficient on its image we must either have that $c \notin im(f)$ or $f$ is not group strategy-proof. Either way we have a contradiction.

2. From the characterization of two-agent mechanisms, if $|I^{1,2}(\succsim_1, \succsim_2)| \geq 3$ we have a single dictator in the marginal mechanism $f^{1,2}_{\succsim_1, \succsim_2}$. For simplicity let $i = 1, j = 2$ and assume 1 is the dictator. We will show that for any $\succsim_i, f(\succsim_i) = \max_{\succsim_i} I^{1,2}(\succsim_1, \succsim_2)$. Begin with $f(\succsim_1', \succsim_2', \ldots, \succsim_n)$. The statement holds by assumption. Now since 1 is the marginal dictator, changing $\succsim_2$ to $\succsim_2^f$ cannot change the outcome. Hence the statement holds for $f(\succsim_1', \succsim_2, \ldots, \succsim_n)$. Now we have that $I^{1,2}(\succsim_2', \succsim_4, \ldots, \succsim_n) \in I^{1,2}(\succsim_1, \succsim_2)$ contains $\succsim_2$ as a subset. Hence there either 1 or 3 is a local dictator. Clearly it must be 1. Therefore 3’s announcement cannot change the outcome, so we have $f(\succsim_1', \succsim_2', \succsim_3, \succsim_4, \ldots, \succsim_n) = \max_{\succsim_3} I^{1,2}(\succsim_1, \succsim_2)$. Continuing in this way gives the desired result. The assumption that $f$ is surjective implies that 1 is a dictator.

A.12 Proof of Theorem 5 (Roommates characterization)

The “if” direction follows directly from Remark 2.

We will prove the “only if” direction by mathematical induction. First, by Lemma 1, we may ignore any agents’ ranking for infeasibly matching with herself. If $N = 2$ there is only one feasible allocation, so every mechanism is trivially a sequential dictatorships. If $N = 4$, then the problem is a social choice problem since a single agent’s match determines the matches of everyone else. In this case, the result follows from the Gibbard–Satterthwaite Theorem (Theorem 4). Suppose that for all $m < n$ when there are $2m$ agents, all group strategy-proof and Pareto efficient roommates mechanisms are sequential dictatorships. We will show this for $2n$ agents. It will be enough to show that there is an agent $j$ such that $f_j(\succsim) = \max_{\succsim} N$ for all $\succsim$, since, conditional on each of $j$’s choices, the remaining $2n - 2$ agents need to assigned a roommate and since $f$ is group strategy-proof and Pareto efficient, this marginal mechanism will be group strategy-proof and Pareto efficient so the induction assumption will guarantee that it is a sequential dictatorships.

Let $f$ be a group strategy-proof and Pareto efficient roommates mechanism for $2n$ agents with $n \geq 3$. We will first consider the possible two-agent marginal mechanisms. Let $i \neq j$ and fix a profile $\succsim_{-ij}$ of the other agents. Assume $(j, i) \in I^{1,2}(\succsim_{-ij})$, so that it is possible for $i$ and $j$ to match when the other agents announce $\succsim_{-ij}$. For all $k \neq i, (j, k) \notin I^{1,2}(\succsim_{-ij})$ since $(j, k)$ has $i$ matched to $j$ but $j$ matched to $k$. Likewise, for all $k \neq j$ we have $(k, i) \notin I^{1,2}(\succsim_{-ij})$. Define $\chi_i = \{x \in N \mid (x, y) \notin I^{1}(\succsim_{-ij}) \text{ for all } y \in N\}$ and $\chi_j = \{y \in N \mid (x, y) \notin I^{1}(\succsim_{-ij}) \text{ for all } x \in N\}$. Then after possibly permuting the rows and columns,
we get a marginal constraint as illustrated in the two panels of figure 3 in the main text. As usual, we will ignore agents preferences over objects they can never receive.\footnote{In this case, $\chi_i$ and $\chi_j$ are not possible for $i$ and $j$ to match holding fixed the preferences $\succsim_{-ij}$.} If $[N - \chi_i \cup \{j\}] \times [N - \chi_j \cup \{i\}]$ intersects any infeasible point, then the graph $G(I^j(\succsim_{-ij}))$ is totally connected, as illustrated on the right-hand picture of figure 3.\footnote{Recall the relation graph $G(C)$ was defined for every two-agent constraint in section 3.1.} Therefore there must be a single dictator in the marginal mechanism $I^j(\succsim_{-ij})$ by Theorem 1 and Lemma 3. Otherwise, every allocation in $[N - \chi_i \cup \{j\}] \times [N - \chi_j \cup \{i\}]$ is feasible or the set is empty. In the latter case $I^j(\succsim_{-ij})$ is a singleton, and obviously there is only one marginal mechanism. In the former case, as a consequence of Theorem 1 there are four possible Pareto efficient, strategy-proof marginal mechanisms as illustrated in figure 4 in the main text.

In panel (A), $j$ is the dictator since $i$ must compromise at every infeasible allocation. In panel (B), $i$ is the dictator. In Panel (C), $i$ and $j$ are matched together if either top-ranks the other and are only unmatched if both $i$ prefers someone in $N - \chi_i \cup \{j\}$ and $j$ prefers someone in $N - \chi_j \cup \{i\}$. In panel (D), $i$ and $j$ are matched only if both top-rank the other and are unmatched otherwise.

Summarizing, if $(j, i) \in I^j(\succsim_{-ij})$, there are four possible types of mechanisms $f^ij_{ij}$:

1. $f^ij_{ij}$ is constant and $(j, i)$. In this case, $N - \chi_i = \{j\}$ and $N - \chi_j = \{i\}$.
2. $f^ij_{ij}$ is dictatorial, so $i$ gets their top choice from $N - \chi_i$ or $j$ gets their top choice from or $N - \chi_j$ and the other agent gets their top choice consistent with the dictators’ allocation. Note that in a dictatorial mechanism, the non-dictator cannot affect the option set of the dictator.
3. $i$ and $j$ are matched by default, and are unmatched only if both agree. This is shown in panel (C). In this case, all allocations in $[N - \chi_i \cup \{j\}] \times [N - \chi_j \cup \{i\}]$ are feasible.
4. $i$ and $j$ are unmatched by default and are matched only if both agree. This is shown in Panel (D). In this case, all allocations in $[N - \chi_i \cup \{j\}] \times [N - \chi_j \cup \{i\}]$ are feasible.

In the remainder of the proof, we will often need to show that a given two-agent marginal mechanism is dictatorial. To do that, it will be sufficient to show that it is possible for both agents to match with one another, that it is non-constant (i.e. that there are at least two possible allocations for the two agents holding the other agents’ preferences fixed), and that it is not of the third or fourth types. The third type of mechanism is usually easy to rule out. If we can find a preference where one agent top-ranks the other and they are still not matched, it cannot be of type three. Type (4) is somewhat more subtle, but we can rule it out if an agent can match with a second agent even when that agent bottom-ranks the first agent.

Note that if we partition the set of agents into two nonempty even sets $A$ and $N \setminus A$ and if we restrict attention to preferences where the agents in $A$ rank all agents in $A$ over all agents in $N \setminus A$ and likewise the agents in $N \setminus A$ rank themselves above the agents in $A$, then by Pareto efficiency for all such preferences, agents in $A$ are matched to themselves and agents in $N \setminus A$ are matched within their own group. The induction assumption implies that both groups are matched using a sequential dictatorship. The next lemma (whose validity depends on the induction assumption) says that the dictator on either side retain their dictatorship rights if the other agents on their side switch to an arbitrary preference.

**Lemma 7.** Let $A$ be a nonempty proper subset of $N$ with an even number of agents and $|A| \geq 4$. If $\succsim_{N \setminus A} \in [P^T(N \setminus A)]^{N \setminus A}$, then there is an agent $j \in A$ such that

$$f_j(\succsim_{A \setminus \{j\}}; \succsim_{N \setminus (A \setminus \{j\})}) = \arg \max_{\succsim_{ij}} N$$

whenever $\arg \max_{\succsim_{ij}} N \in A$. Equivalently, $g_j(\succsim_{A \setminus \{j\}}; \succsim_{N \setminus (A \setminus \{j\})}) \supset A \setminus \{j\}$ for all $\succsim_{A \setminus \{j\}}$.
Proof. For notational convenience, let \( A = \{1, 2, \ldots, l\} \) and \( N \setminus A = \{l + 1, \ldots, N\} \). Fix a profile \( \succsim \in [P^*(N \setminus A)]^{N \setminus A} \). For any \( \succsim_{1}, \ldots, \succsim_{l} \in P^*((1, 2, \ldots, l)) \), by Pareto efficiency, \( f(\succsim_{1}, \succsim_{2}, \ldots, \succsim_{l}, \succsim \setminus A) \) will match agents in \( \{1, 2, \ldots, l\} \) with other agents in \( \{1, 2, \ldots, l\} \) and agents in \( \{l + 1, \ldots, N\} \) with other agents in \( \{l + 1, \ldots, N\} \). Thus the marginal mechanism \( f(\cdot, \succsim \setminus A) \) restricted to profiles in \( [P^*((1, 2, \ldots, l))]^{l} \) gives a roommates mechanism for the agents in \( \{1, 2, \ldots, l\} \). By the group strategy-proofness and efficiency of \( f \), the marginal mechanism is also group strategy-proof and efficient. By the induction assumption this marginal mechanism is a sequential dictatorship. Without loss, assume that 1 is the first dictator. Then we have \( g_{1}(\succsim_{2}, \ldots, \succsim_{l}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\} \) for all \( \succsim_{2}, \ldots, \succsim_{l} \in P^*((1, 2, \ldots, l)) \). For any \( \succsim_{3}, \ldots, \succsim_{l} \in P^*((1, 2, \ldots, l)) \), consider the 1,2-marginal mechanism. Since \( g_{1}(\succsim_{2}, \ldots, \succsim_{l}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\} \) for all \( \succsim_{2}, \ldots, \succsim_{l} \in P^*((1, 2, \ldots, l)) \), if 1 top-ranks 2 and 2 announces any preference in \( P^*((1, 2, \ldots, l)) \), 1 and 2 are matched. Thus \( (2, 1) \in P^*(\succsim_{3}, \ldots, \succsim_{l}, \succsim \setminus A) \). From the considerations above, there are four possibilities for this marginal mechanism. Let \( \succsim_{1} \) top rank \( j \neq 2 \) and \( j \leq l \) and \( \succsim_{2} \) in \( P^*((1, 2, \ldots, l)) \) top-rank 1. At this profile, 1 and \( j \) are matched. Hence the 1,2 marginal mechanism is not constant. Furthermore, it cannot be of type (3), since 1 is matched with \( j \), despite 2 top-ranking 1. Let \( \succsim_{2} \) be in \( P^*((1, 2, \ldots, l)) \) and top-rank her match at the profile \( (\succsim_{1} \setminus 1, \succsim_{2}) \). Since 1 and 2 are matched when 1 top-ranks 2 and 2 announces \( \succsim_{2} \), the mechanism also cannot be of type (4) (At \( \succsim_{2} \), agent 2 is top-ranking a feasible match in the 1,2 marginal mechanism, but 1 can still match with her). The only possibility left is that the 1,2-marginal mechanism is dictatorial with agent 1 as the dictator. Since non-dictators cannot affect the option set of dictators, we get that \( g_{1}(\succsim_{2}, \ldots, \succsim_{l}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\} \) for any \( \succsim_{2} \) and any \( \succsim_{3}, \ldots, \succsim_{l} \in P^*((1, 2, \ldots, l)) \). We could have carried out the above argument with any \( i \) in place of 2, so in fact we have

\[
g_{1}(\succsim_{2}, \ldots, \succsim_{i-1}, \succsim_{i}, \succsim_{i+1}, \ldots, \succsim_{l}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\}
\]

for any \( \succsim_{i} \) and any \( \succsim_{2}, \ldots, \succsim_{i-1}, \succsim_{i+1}, \ldots, \succsim_{l} \in P^*((1, 2, \ldots, l)) \).

The goal is to show that

\[
g_{1}(\succsim_{2}, \ldots, \succsim_{l}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\}
\]

for all \( \succsim_{2}, \ldots, \succsim_{l} \). We will do this by induction. Specifically we will show that if for any \( 0 < q - 1 < l - 1 \) and any \( A' \subset A - \{1\} \) with \( |A'| = q - 1 \) we have \( g_{1}(\succsim_{A'}, \succsim_{A' \setminus \{1\}}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\} \) for any \( \succsim_{A'} \) and any \( \succsim_{A' \setminus \{1\}} \) in \( [P^*(A)]^{A' \setminus \{1\}} \) then the same holds for any \( A' \subset A - \{1\} \) with \( q \) agents.

For simplicity, let \( A' = \{2, \ldots, q + 1\} \) and pick any \( \succsim_{2}, \ldots, \succsim_{q+1} \). By the induction assumption, we have \( g_{1}(\succsim_{2}, \ldots, \succsim_{q}, \succsim_{q+1}, \ldots, \succsim_{l}, \succsim \setminus A) \supseteq \{2, 3, \ldots, l\} \) for any \( \succsim_{2}, \ldots, \succsim_{q} \) and any \( \succsim_{q+1}, \ldots, \succsim_{l} \in P^*(A) \). Now by the same arguments as above, the 1, \( q+1 \)-marginal mechanism at this profile is either of type (2) (i.e. dictatorial) or it is of type (4). Suppose, by way of contradiction, that it is of type (4) and let \( \succsim_{q+1} \) bottom-rank 1. Then doing so removes \( q + 1 \) from 1’s option set, but leaves it otherwise the same. Let \( \succsim_{q+1} \) top-rank \( q + 1 \) and second-rank \( q \). From the above discussion, we get that 1 is matched to \( q \) at the marginal profile \( (\succsim_{1}^{*}, \succsim_{q+1}^{*}) \). If we let \( \succsim_{q+1}^{*} \in P^*(A) \) top-rank 1, then by Maskin-monotonicity, we have

\[
f(\succsim_{1}^{*}, \succsim_{2}, \ldots, \succsim_{q}, \succsim_{q+1}, \ldots, \succsim_{l}, \succsim \setminus A) = f(\succsim_{1}^{*}, \succsim_{2}, \ldots, \succsim_{q-1}, \succsim_{q+1}, \ldots, \succsim_{l}, \succsim \setminus A)
\]

but on the left we have 1 is matched to \( q \), her second-top choice. By the induction assumption, on the right we should have \( q + 1 \) in 1’s option set since the agents \( q, q + 2, \ldots, l \) are all announcing a preference in \( P^*(A) \), leaving only \( q - 1 \) agents announcing a possibly different preference. This gives a contradiction so we must have that 1 is the dictator in the 1, \( q+1 \)-marginal mechanism.

We will call agent \( j \) in the lemma above, the \textit{marginal dictator}. Having done this, the idea is to partition the agents in two ways. First we consider the partition \( \{1, 2\} \{3, 4, \ldots, N\} \). By Lemma 7, for \( \succsim_{1} \in P^*(2) \) and
\(\preceq_2^* \in P^T(1)\) there is a marginal dictator among \(\{3, 4, \ldots, N\}\) which we can assume without loss is agent 3. Second, we consider the partition \(\{1, 2, 3, 4\}, \{5, 6, \ldots, N\}\) and again Lemma 7 says that given \(\preceq_5^*, \ldots, \preceq_n^* \in P^T(\{5, \ldots, n\})\), there is a marginal dictator among \(\{1, 2, 3, 4\}\). We show that by comparing these two dictators, we can find a single dictator for the whole mechanism.

As above, let \(\preceq_4^* \in P^T(2), \preceq_5^* \in P^T(1)\) and without loss assume that 3 is the marginal dictator among \(\{3, \ldots, N\}\). By Maskin-monotonicity, it is also without loss to suppose that both \(\preceq_1^*\) and \(\preceq_4^*\) are type (3). Also choose \(\preceq_1^*, \ldots, \preceq_n^* \in P^T(\{5, \ldots, n\})\). By Lemma 7, \(g_3(\preceq_1^*, \ldots, \preceq_4^*, \ldots, \preceq_n^*) \supseteq \{4, \ldots, N\}\) for all \(\preceq_4^*, \ldots, \preceq_n^*\). Likewise, for some \(i \in \{1, 2, 3, 4\}\), we have \(g_i(\preceq_1^*\downarrow_{\{1, 2, 3, 4\}\setminus\{i\}}, \ldots, \preceq_n^*) \supseteq \{1, 2, 3, 4\} - \{i\}\) for all \(\preceq_1^*, \ldots, \preceq_n^*\). This gives four cases, corresponding to the possible identities of \(i\). However, note that \(i\) cannot be 4 since 3 and 4 are matched at the profile \((\preceq_1^*, \ldots, \preceq_n^*)\) where 3 top ranks 4 regardless of \(\preceq_4^*\). Since 1 and 2 are so far symmetric, this leaves two cases: \(i = 1\) (and \(i = 2\) by symmetry) and \(i = 3\).

We will start with the latter case. So we have

\[
g_3(\preceq_1^*, \ldots, \preceq_5^*) \supset \{4, \ldots, N\}\] for all \(\preceq_4^*, \ldots, \preceq_n^*\), and

\[
g_3(\preceq_1^*, \ldots, \preceq_3^*, \ldots, \preceq_n^*) = \{1, 2, 4\}\] for all \(\preceq_4^*, \ldots, \preceq_n^*\) \hspace{1cm} (2)

In particular, \(g_3(\preceq_1^*, \ldots, \preceq_4^*, \ldots, \preceq_n^*) = N - \{3\}\) for all \(\preceq_4^*\). We need to show \(g_3(\preceq_1^*, \ldots, \preceq_4^*, \ldots, \preceq_n^*) = N - \{3\}\) for all \(\preceq_4^*, \ldots, \preceq_5^*, \ldots, \preceq_n^*\). Consider the 3, 5-marginal mechanism at the profile \((\preceq_1^*, \ldots, \preceq_5^*, \ldots, \preceq_n^*)\) for any \(\preceq_4^*\). From equation 2 above, 3 and 5 are matched whenever 3 top ranks 5, regardless of 5's preference. It is also possible for 3 to match with 4 regardless of 5's preference. From the discussion about the possible two-agent marginal mechanisms, the only possibility for this marginal mechanism has 3 as the dictator. In this case, 5's announcement cannot affect 3's option set. Thus we have \(g_3(\preceq_1^*, \ldots, \preceq_4^*, \ldots, \preceq_n^*) = N - \{3\}\) for any \(\preceq_4^*, \ldots, \preceq_5^*, \ldots, \preceq_n^*\). Repeating this argument one agent at a time implies that

\[
g_3(\preceq_1^*, \ldots, \preceq_5^*) = N - \{3\}\] for all \(\preceq_4^*, \ldots, \preceq_n^*\), and

a symmetric argument shows that

\[
g_3(\preceq_1^*, \ldots, \preceq_5^*) = N - \{3\}\] for all \(\preceq_1^*, \ldots, \preceq_5^*, \ldots, \preceq_n^*\). \hspace{1cm} (5)

Now we will use equation 4 to get the desired result. We will do this by looking at the 1, 3 and 2, 3 marginal mechanisms. Equation 4 already implies that these mechanisms cannot be type (4) since 3 can match with 1 and 2 even though both bottom-rank her. The main thing to do is show that the marginal mechanisms are not of type (3). To do this, we will need to show that when 1 or 2 switch to top-ranking 3 they do not force a match.

Let \(\preceq_1^{**}\) be identical to \(\preceq_1^*\), except that 3 is top ranked. Define \(\preceq_2^{**}\) equivalently. Now we want to show that the following three equations hold:

\[
g_3(\preceq_1^{**}, \ldots, \preceq_n^*) = N - \{3\}\] for all \(\preceq_4^*, \ldots, \preceq_n^*\), and

\[
g_3(\preceq_1^{**}, \ldots, \preceq_5^*, \ldots, \preceq_n^*) = N - \{3\}\] for all \(\preceq_4^*, \ldots, \preceq_n^*\), and

\[
g_3(\preceq_1^{**}, \ldots, \preceq_5^*) = N - \{3\}\] for all \(\preceq_4^*, \ldots, \preceq_n^*\). \hspace{1cm} (6)

Since the arguments are all symmetric, we will just show equation 6. From equation 5, we know that \(g_3(\preceq_1^{**}, \preceq_2^{**}\).

\[\text{Let } \preceq_1^* \in P^T(2), \preceq_2^* \in P^T(1), \text{by Lemma 7, we have } g_3(\preceq_1^*, \ldots, \preceq_4^*, \ldots, \preceq_n^*) \supseteq \{4, \ldots, N\}. \text{ Let } \preceq_1^{**} \text{ and } \preceq_2^{**} \text{ be the same as } \preceq_1^* \text{ and } \preceq_2^* \text{ respectively, except both bottom-rank 3. Let } \preceq_3 \text{ top rank } k \in \{4, \ldots, N\}. \text{ Then } g_3(\preceq_1^{**}, \ldots, \preceq_4^*, \ldots, \preceq_n^*) = k \text{ for any } \preceq_4^*, \ldots, \preceq_n^*. \text{ But Maskin-monotonicity then says } g_3(\preceq_1^{**}, \preceq_2^{**}, \preceq_3, \preceq_4^*, \ldots, \preceq_n^*) = k \text{ for any } \preceq_4^*, \ldots, \preceq_n^*. \]
Finally, by comparing equations 9 and 10, we get the desired result that $\{3\}$ has the option to not match with 2, even if 2 top-ranks 3. Thus we can only have $\{3\}$ as the marginal dictator in the full mechanism, which implies that the dictator in the full mechanism is constant and if it is dictatoral, 3 must be the dictator. We must also rule out type (3) and type (4) mechanisms. Let $\{3\}$ be of type (3). Finally, suppose that $\{3\}$ is matched with 4 by equation 4. Thus the 3 is the dictator in the marginal mechanism and 5’s preference does not affect 3’s option set so $f(\{z_1^*, z_2^*, z_4^*, z_6^*\}) = f(\{z_1^*, z_2^*, z_4^*, z_6^*\})$ however, on the right hand side, we have 3 matched with 4 by equation 4. Thus the 3, 5-marginal mechanism cannot be of type (3). Finally, suppose that $\{z_5^\prime\}$ ranks agent 3 last. If the marginal mechanism were type (4), we could not have 3 and 5 matched at $(z_3, z_5^\prime)$. However, in a type (4) mechanism, either agent can only remove themselves from the other agents option set. Hence in this case we would have that 3 is matched to 4 at $(z_3, z_5^\prime)$. For the same reasons as above, Maskin monotonicity implies this cannot happen. Hence 3 is the dictator in the marginal mechanism and 5’s preference does not affect 3’s option set so $g_3(z_1^*, z_2^*, z_4^*, z_6^* \ldots, z_N) = N \{3\}$ for all $z_4^*, z_5^\prime$. Repeating this argument one agent at a time gives us equation 6.

Now we claim that equations 4 and 7, together imply that

$$g_3(z_1^*, z_4^*, z_6^* \ldots, z_N) = N \{3\} \quad (9)$$

Equation 4 says that 3 has the option to match with 2, even though 2 bottom-ranks 3 by assumption. Equation 7 that 3 has the option to not match with 2, even if 2 top ranks her. Thus we can only have 3 as the marginal dictator in the 2, 3-marginal mechanism at any $z_1^*, z_4^*, \ldots, z_N$. Since 2 cannot affect 3’s option set, we get equation 9. Repeating the same arguments with equations 6 and 8 show that

$$g_3(z_1^*, z_2^*, z_4^*, z_6^* \ldots, z_N) = N \{3\} \quad (10)$$

Finally, by comparing equations 9 and 10, we get the desired result that $g_3(z_1^*, z_2^*, z_4^*, \ldots, z_N) = N \{3\}$ for all $z_4^*, z_5^\prime, \ldots, z_N$.

Lastly, we come to the case in which 1 is the marginal dictator among $\{1, 2, 3, 4\}$ at the profile $z_1^*$, $z_2^*$, $z_3^*$, $z_N$. Our strategy will be to reduce this to the previous case by showing that for some $z_3^* \in P^1(4)$, $z_4^* \in P^1(3)$, that 1 is also the marginal dictator among $\{1, 2, 5, \ldots, N\}$.

By Lemma 7, we have

$$g_1(z_2^*, z_3^*, z_4^*, z_6^* \ldots, z_N) \supset \{2, 3, 4\} \quad (11)$$

Let $k \in 5, \ldots, N$. As a first step, we want to show that $k \in g_1(z_2^*, z_3^*, z_4^*, z_6^* \ldots, z_N)$ for all $z_2^*, z_3^*, z_4^*$ and to do this it suffices to demonstrate a single preference profile $(z_2^*, z_3^*, z_4^*, z_6^* \ldots, z_N)$ where this holds, since 1 is the marginal dictator among $\{1, 2, 3, 4\}$. Let $z_3^*$ be top-rank $k$. Then $f$ matches 1 and 2 and also 3 and $k$ at the profile $(z_1^*, z_2^*, z_3^*, z_4^*, z_6^* \ldots, z_N)$ for any $z_2^*, z_3^*, z_4^*$, $z_6^* \ldots, z_N$) where this holds, since 1 is the marginal dictator among $\{1, 2, 3, 4\}$. Let $z_3^*$ be the same as $z_2^*$, except that it top-ranks 3 and let $z_3^*$ be the same as $z_2^*$, except that it top-ranks $k$. Since 1 is the marginal dictator among $\{1, 2, 3, 4\}$, 1 and 2 are still matched at the profile $(z_1^*, z_2^*, z_3^*, z_4, z_6^* \ldots, z_N)$, so by Maskin monotonicity, we have
and in particular, 3 and k are still matched. Now consider the 1,k-marginal mechanism at \((z^*_1, z^*_2, z^*_4, z^*_k, \ldots, z^*_N)\). Let \(z^*_k\) be the same as \(z^*_1\), except that it top-ranks k and let \(z^*_k\) be the same as \(z^*_k\), except that it top-ranks 1. We must have that 3 and k are matched in the marginal mechanism at \((z^*_1, z^*_2, z^*_4, z^*_k, \ldots, z^*_N)\) since otherwise Maskin monotonicity says that \(f\) gives the same result as though they had announced \((z^*_1, z^*_k)\), but in this case, 1 and 2 are matched and 3 and k are matched which is inefficient since we could swap 1 and 3’s matches. Thus \((k, 1)\) is in \(I^{1,k}(z^*_1, z^*_2, z^*_4, z^*_k, \ldots, z^*_k, \ldots, z^*_N)\). From the considerations above, there are four possibilities for this mechanism. However, since both \((2, 3)\) and \((k, 1)\) are in the marginal option set, the marginal mechanism is not constant. Note also that if 1 top ranks 3 and k announces \(z^*_k\), then by equation 11, 1 and 3 are matched. Thus it is possible for both 1 and k to match with 3 in this marginal mechanism. But since both can’t match with 3 at the same time, the marginal constraint is like the one shown on the right of figure 3, and there must be a single dictator. We will show that this dictator must be 1. To do this, we will have to take a detour to the 3,k-marginal mechanism.

By equation 11, \(f_1(z^*_1, z^*_2, z^*_3, z^*_4, \ldots, z^*_N) = 2\) for all \(z^*_1, z^*_4\), so by Maskin monotonicity, we have

\[
f(z^*_1, z^*_2, z^*_3, z^*_4, \ldots, z^*_N) = f(z^*_1, z^*_2, z^*_3, z^*_4, \ldots, z^*_N)
\]

for all \(z^*_1, z^*_4\). In particular, we have \(g_3(z^*_1, z^*_2, z^*_4, z^*_5, \ldots, z^*_N) = \{4, \ldots, N\}\) for all \(z^*_1\) by equation 4. Consider the 3,k-marginal mechanism at this profile. If 3 top ranks k they are matched. If 3 top ranks 4 they are not. In the latter case, k is matched to someone from \(\{4, \ldots, N\}\), which she prefers. Hence the marginal mechanism is either a dictatorship with 3 as the dictator, or it is of the third type in which 3 and k are matched if either top-ranks the other. Let \(z^*_3\) top rank 4 and \(z^*_k\) top rank 3. In the type (3) marginal mechanism, we would have 3 and k matched in

\[
f(z^*_1, z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_N) = f(z^*_1, z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_N)
\]

but then Maskin-monotonicity would imply that we get the same outcome if 2 announced \(z^*_2\), yet at this profile, by equation 4, we would have 3 matched to 4. Hence we have that 3 is the dictator in the 3,k-marginal mechanism at \((z^*_1, z^*_2, z^*_4, z^*_k, \ldots, z^*_k, \ldots, z^*_N)\) for all \(z^*_1\). This implies that \(g_3(z^*_1, z^*_2, z^*_k, \ldots, z^*_k, \ldots, z^*_N) = \{4, \ldots, N\}\) for all \(z^*_1\) and \(z^*_k\). So we have \(f_3(z^*_1, z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_k, \ldots, z^*_N) = k\), and by nonbossiness

\[
f(z^*_1, z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_k, \ldots, z^*_N) = f(z^*_1, z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_k, \ldots, z^*_N)
\]

and on the right hand side we know that 1 and 2 are matched and 3 and k are matched. This implies that if k switches from \(z^*_k\) to \(z^*_k\), 1 and k are not matched in the 1,k-marginal mechanism at \((z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_k, \ldots, z^*_k, \ldots, z^*_N)\). Since either 1 or k must be the dictator in their marginal mechanism by earlier arguments, it must be 1 and we have

\[
k \in g_1(z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_N)
\]

and since 2,3,4 can’t affect 1’s option set we get

\[
k \in g_1(z^*_2, z^*_3, z^*_4, z^*_5, \ldots, z^*_k, \ldots, z^*_N)
\]
for all $\succcurlyeq_i, \succcurlyeq_j, \succcurlyeq_k$. Since $k$ was arbitrary, together with equation 11, we have

$$g_1(\succcurlyeq_2, \succcurlyeq_3, \succcurlyeq_4, \ldots, \succcurlyeq_N) = N - \{1\} \quad (12)$$

for all $\succcurlyeq_2, \succcurlyeq_3, \succcurlyeq_4$. This, however, gets us back to the first case since 1 is the marginal dictator among $\{1, 2, 3, 4\}$ at $\succcurlyeq_2, \ldots, \succcurlyeq_N$ and if $\succcurlyeq_4 \in P^4(2)$, $\succcurlyeq_4 \in P^4(3)$, then we must have a marginal dictator among $\{1, 2, 5, \ldots, N\}$, however the only marginal dictator consistent with equation 12 is agent 1. 

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