A refinement of Betti numbers in the presence of a continuous function. I

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Abstract

We propose a refinement of the Betti numbers of a compact ANR $X$ in the presence of a continuous function $f : X \to \mathbb{R}$. The refinement consists of finite configurations of points with specified multiplicity located in the plane $\mathbb{R}^2 = \mathbb{C}$, of cardinality (counted with multiplicity) the Betti numbers, equivalently of monic polynomials with complex coefficients $P_f(z)$ of degree the Betti numbers of $X$.

A number of properties are discussed (Theorems 4.1, 4.2 and 4.3) as well as the realization of each such configurations as dimensions of mutually orthogonal subspaces of the homology vector spaces, when equipped with a Hilbert space structure, indexed by the points of the configuration.

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1 Introduction

This paper, and its subsequent part II, provides a short version of a some parts of paper [2] still unpublished, which is joint work with Stefan Haller.

In this paper we propose a refinement of the Betti numbers $b_r(X)$ of a compact ANR $X$, with respect to a field $\kappa$, in the presence of a continuous function $f : X \to \mathbb{R}$. The refinement consists of finite configurations

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of points with multiplicity located in the plane \( \mathbb{R}^2 = \mathbb{C} \), denoted by \( \delta^\ell_r \), equivalently of monic polynomials with complex coefficients \( P^\ell_r(z) \), whose zeros are the points of the configuration \( \delta^\ell_r \), of degree the Betti numbers \( b_r(X) \). The points of the configurations \( \delta^\ell_r \), or the zeros of the polynomials \( P^\ell_r(z) \), are complex numbers \( z = x + iy \in \mathbb{C} \) with both \( x, y \) critical values.

We show that:

1. The assignment \( f \rightsquigarrow P^\ell_r(z) \) is continuous when \( f \) varies in the space of continuous maps equipped with the compact open topology cf. Theorem 4.2.

2. For an open and dense subset of continuous maps the points of the configurations or the zeros of the polynomials \( P^\ell_r(z) \) have multiplicity one, cf. Theorem 4.1.

3. When \( X \) is a closed topological manifold the Poincaré duality between Betti numbers gets refined to a Poincaré Duality between configuration, cf. Theorem 4.3.

We also show that to each zero \( z \) of the polynomial \( P^\ell_r(z) \) one can naturally assign a vector space \( \mathbb{H}^\ell_r(z) \), of dimension the multiplicity of \( z \), which is a quotient \( \mathbb{H}_r(z) = \mathbb{F}_r(z)/\mathbb{F}_r'(z) \), \( \mathbb{F}_r(z) \subset \mathbb{F}_r(z) \subset H_r(X; \kappa) \).

When \( \kappa = \mathbb{R} \) or \( \mathbb{C} \) and \( H_r(X; \kappa) \) is equipped with a Hilbert space structure \( \mathbb{H}_r(z) \) identifies canonically to a subspace \( H_r(z) \) of \( H_r(X; \kappa) \) s.t. \( H_r(z) \perp H_r(z') \) for \( z \neq z' \) and \( \oplus_z H_r(z) = H_r(X; \kappa) \).

We refer to the system \( (H_r(X; \kappa), P^\ell_r(z), \{\mathbb{H}_r(z)\}) \) as the \( r \)–homology spectral package of \((X, f)\) in analogy with the spectral package of \((V, T)\), \( V \) a vector space \( T \) a linear endomorphism, which consists of the characteristic polynomial \( P^T(z) \) with its roots \( z_i \), the eigenvalues of \( T \), and with their corresponding generalized eigenspaces \( V_{z_i} \).

In case \( X \) is the underlying space of a closed Riemannian manifold \((M^n, g)\) and \( \kappa = \mathbb{R} \) or \( \mathbb{C} \) the vector space \( H_r(M^n; \kappa) \), via the identification with the harmonic \((n - r)\)–forms with coefficients in the orientation bundle, has a structure of Hilbert space. It might be of interest to explore this \( r \)–homology spectral package in case \( f \) is a Morse function. In this case this homology spectral package is invariant to the homotopy class of Morse function. The homotopy is considered in the subspace of Morse functions, which is not contractible and has many connected components.

All these results are collected in the main theorems below, Theorems 4.1, 4.2 and 4.3.

It is worth to note that the points of the configurations \( \delta^\ell_r \) located above and on the diagonal in the plane \( \mathbb{R}^2 \) determine and are determined by the closed \( r \)--bar codes in the level persistence of \( f \) while those below diagonal are determined and determine the open \((r - 1)\)--bar codes in the level persistence as observed in [2]. The algorithms proposed in [5] and in [11] can be used for their calculation, however significant simplification of such algorithms can be derived from the present presentation.

Similar refinements hold for angle valued maps. In this case the homology has to be replaced by either the Novikov homology of \((X, \xi(f))\) which is a vector space over the Novikov field of Laurent power series \( \kappa[t^{-1}, t] \) or. in case \( \kappa = \mathbb{R} \) or \( \mathbb{C} \) by the \( L_2 \) homology of the infinite cyclic cover defined by the homotopy class of \( \xi(f) \) of \( f \). In this case the \( L_2 \)--homology is regarded as a Hilbert module over the von-Neumann algebra associated to the group \( \mathbb{Z} \). In this case \( H_r(z) \) are Hilbert submodules, and \( \delta^\ell_r(x) \) is the von Neumann dimension of \( H_r(z) \). Note that the \( L_2 \)--Betti numbers are actually the Novikov–Betti numbers of \((X, \xi(f))\).

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1see section 2.2. below for the definition of regular and critical value
2 Preliminary definitions

2.1 Configurations

let $X$ be a topological space. Let $V$ be a finite dimensional Hilbert space over $\kappa = \mathbb{R}$ or $\kappa = \mathbb{C}$ and $\mathcal{P}(V)$ be the set of subspaces of $V$. Denote by

$$\mathcal{C}_N(X) := \{\delta : X \to \mathbb{Z}_{\geq 0} | \sum_{x \in X} \delta(x) = N\}$$

the set of finite configurations with $N$ points of $X$ and by

$$\mathcal{C}_V(X) := \{\delta : X \to \mathcal{P}(V) | \delta(x) \perp \delta(y), \bigoplus_{x \in X} \delta(x) = V\}$$

the set of finite configurations of weighted vector space with total weight $V$.

One also defines the map $e : \mathcal{C}_V(X) \to \mathcal{C}_{\dim V}(X)$ by

$$e(\delta)(x) = \dim(\delta(x)).$$

One can weaken the requirement of orthogonality to $\delta(x) \cap \delta(y) = 0$. In case $\kappa = \mathbb{R}$ or $\mathbb{C}$ the space of such configurations is homotopy equivalent to $\mathcal{C}_V(X)$.

The sets $\mathcal{C}_N(X)$ and $\mathcal{C}_V(X)$ carry natural topologies, referred to as the collision topology, which make $e$ continuous.

One way to describe these topologies is to specify for each $\delta$ or $\hat{\delta}$ a fundamental system of neighborhoods.

If $\delta$ has as support the set of points $\{x_1, x_2, \ldots, x_k\}$ a fundamental neighborhood $U$ is specified by a collection of $k$ disjoint open neighborhoods $U_1, U_2, \ldots, U_k$ of $x_1, \ldots, x_k$, and consists of $\{\delta' \in \mathcal{C}_N(X) | \sum_{x \in U_i} \delta'(x) = \delta(x_i)\}$.

Similarly if $\hat{\delta}$ has as support the set of points $\{x_1, x_2, \ldots, x_k\}$ with $\hat{\delta}(x_i) = V_i \subseteq V$, a fundamental neighborhood $U$ of $\hat{\delta}$ is specified by a collection of $k$ disjoint open neighborhoods $U_1, U_2, \ldots, U_k$ of $x_1, \ldots, x_k$, and open neighborhoods $O_1, O_2, \ldots, O_k$ of $V_i$ in $G_{\dim V_i}(V)$ and consists of

$$\{\delta' \in \mathcal{C}_V(X) | \bigoplus x \in U_i \delta'(x) \in O_i\}.$$

Here $G_k(V)$ denotes the Grassmanian of $k$-dimensional subspaces of $V$.

Clearly $e$ is continuous, surjective and proper, with fiber above $\delta$, the subset of $G_{n_1}(V) \times G_{n_2}(V) \times \cdots \times G_{n_k}(V)$ consisting of $(V_1', V_2', \ldots, V_k')$, $V_i' \in G_{n_i}(V)$ mutually orthogonal in $V$. Here $n_i = \dim V_i$. Clearly this set is compact and is actually an algebraic variety.

Note that

1. $\mathcal{C}_N(X) = X^N/\Sigma_N$ is the so called $N$-symmetric product and if $X$ is a metric space with distance $D$ the collision topology is the topology defined by the induced metric $\overline{D}$ on $X^N/\Sigma_N$.

2. If $X = \mathbb{R}^2 = \mathbb{C}$ then $\mathcal{C}_N(X)$ identifies to the set of monic polynomials with complex coefficients. To the configuration $\delta$ whose support consists of the points $z_1, z_2, \ldots, z_k$ with $\delta(z_i) = n_i$ one associates the monic polynomial $P(\delta)(z) = \prod_i (z - z_i)^{n_i}$. Then $\mathcal{C}_N(X) = \mathbb{C}^N$ as metric spaces.

3. The space $\mathcal{C}_V(X)$ and then $\mathcal{C}_V(\mathbb{R}^2)$ can be equipped with a complete metric $\overline{D}$ which induces the collision topology but it will not be used here.
2.2 Tame maps

All maps \( f : X \rightarrow \mathbb{R} \) in this paper are continuous proper maps defined on \( X \) an ANR. Since \( f \) is proper \( X \) is locally compact.

1. A map \( f : X \rightarrow \mathbb{R} \) is weakly tame if for any \( t \in \mathbb{R} \), the level \( f^{-1}(t) \) is an ANR. Therefore for any bounded or unbounded closed interval \( I = [a, b] \), \( a, b \in \mathbb{R} \cup \{\infty, -\infty\} \) \( f^{-1}(I) \) is an ANR.

2. The number \( t \in \mathbb{R} \) is a regular value if there exists \( \epsilon > 0 \) small s.t. for any \( t' \in (t - \epsilon, t + \epsilon) \) the inclusion \( f^{-1}(t') \subset f^{-1}(t - \epsilon, t + \epsilon) \) is a homotopy equivalence. A number \( t \) which is not regular value is a critical value. Informally, the critical values are the values \( t \) for which the topology of the level (= homotopy type) changes. One denotes by \( Cr(f) \) the collection of critical values of \( f \).

3. The map \( f \) is called tame if weakly tame and in addition:
   (a) The set of critical values \( Cr(f) \subset \mathbb{R} \) is discrete,
   (b) \( \epsilon(f) := \inf |c - c'|, c \neq c' \) satisfies \( \epsilon(f) > 0 \). (If \( X \) compact then (a) implies (b).)
   There are compact ANR with no weakly tame maps cf [7].

4. An ANR which has the weakly tame maps dense in the set of all maps w.r. to fine \( C_0 \)— topology is called a good ANR.

Note that all spaces homeomorphic to locally finite dimensional simplicial complexes, or finite dimensional manifolds or the Hilbert cube manifolds are good ANR’s (see the Appendix for definition and explanations).

The reader should be aware of the following rather obvious facts.

Observation 2.1

1. If \( f \) is weakly tame then \( f^{-1}([a, b]) \) as a compact ANR has the homotopy type of a finite simplicial complex and therefore finite dimensional homology w.r. to any field \( \kappa \).

2. If \( X \) is a finite simplicial complex and \( f \) a simplicial map then \( f \) is tame with the set of critical values finite. Critical values are among the values of \( f \) on vertices.

3. If \( X \) is homeomorphic to a compact simplicial complex or to a compact topological manifold the set of tame maps is dense in the set of all continuous maps. The same remain true if \( X \) is a Hilbert cube manifold. In particular all these spaces are good ANR’s.\(^2\)

For two subspaces \( V_1 \subseteq V_2 \) of a vector space \( H \) we call the quotient \( V_2/V_1 \) a quotient of subspaces. In case \( H \) is a Hilbert space over \( \kappa = \mathbb{R} \) or \( \mathbb{C} \), any finite dimensional quotient of subspaces \( V_2/V_2 \) had a canonical realization as a subspace of \( H \), precisely the orthogonal complement of the closure of \( V_1 \) inside the closure of \( V_2 \).

3 The configurations \( \delta^f_r \) and \( \hat{\delta}^f_r \)

In this paper we fix a field \( \kappa \), and for a space \( X \) denote by \( H_r(X) \) the homology of \( X \) with coefficients in the field \( \kappa \).

Let \( f : X \rightarrow \mathbb{R} \) be a map. Denote by :

\(^2\)for Hilbert cube manifolds see the appendix, Proposition
1. the sub level \( X_a := f^{-1}(-\infty, a] \),
2. the super level \( X^b := f^{-1}([b, \infty)) \),
3. \( \mathbb{I}_f^a(r) := \text{img}(H_r(X_a) \to H_r(X)) \subseteq H_r(X) \),
4. \( \mathbb{I}_f^b(r) := \text{img}(H_r(X^b) \to H_r(X)) \subseteq H_r(X) \),
5. \( \mathbb{F}_f^b(a, b) = \mathbb{I}_f^a(r) \cap \mathbb{I}_f^b(r) \subseteq H_r(X) \)

We also define \( \mathbb{I}_f^- := 0 \), \( \mathbb{I}_f^- := H_r(X) \) and \( \mathbb{I}_f^+ := H_r(X) \), \( \mathbb{I}_f^\infty := 0 \).

Clearly one has:

Observation 3.1

1. For \( a' \leq a \) and \( b \leq b' \) one has \( \mathbb{F}_f^b(a', b') \subseteq \mathbb{F}_f^b(a, b) \).
2. For \( a \leq a \) and \( b \leq b' \) one has \( \mathbb{F}_f^b(a', b) \cap \mathbb{F}_f^b(a, b') = \mathbb{F}_f^b(a', b') \).
3. \( \sup_{x \in X} |f(x) - g(x)| < \epsilon \) implies \( \mathbb{F}_f^g(a - \epsilon, b + \epsilon) \subseteq \mathbb{F}_f^f(a, b) \).
4. If the number \( a \in \mathbb{R} \) is regular value then there exists \( \epsilon > 0 \) so that for any \( 0 \leq t, t' < \epsilon \) the inclusions \( \mathbb{I}_f((a-t)) \subseteq \mathbb{I}_f((a+t)) \) and \( \mathbb{I}_f((a-t')) \supseteq \mathbb{I}_f((a+t')) \) are isomorphisms for all \( r \).

Note that we also have:

Proposition 3.2 For any \( a \leq b \) \( \dim \mathbb{F}_f^b(a, b) < \infty \).

This is obvious if \( X \) is compact since \( (X_a^b \cap X^b) \) is a compact ANR. For \( X \) not compact one argues as follows:

It is well known that for any locally compact ANR \( X \) there exists \( K \) locally finite simplicial complex and \( \pi : K \to X \) a proper map which is a homotopy equivalence. Since \( \pi \) is proper then for any \( a, b \in \mathbb{R} \) there exists \( K_a \) and \( K_b \), sub complexes of \( K \), s.t.

\( K_a \cap K_b \) is a finite simplicial complex,
\( \pi(K_a) \supseteq f^{-1}(X_a^b) \) and
\( \pi(K_b) \subseteq f^{-1}(X_a^b) \).

It follows from the Meyer Vietoris long exact sequence in homology and from the finite dimensionality of \( H_r(K_a \cap K_b) \) that \( (\text{img}(H_r(K_a) \to H_r(K)) \cap \text{img}(H_r(K_b) \to H_r(K))) \) is finite dimensional. Therefore, since \( \pi \) induces the isomorphism \( \pi_* \) in homology,

\( \mathbb{F}_f^b(a, b) \subseteq \pi_*((\text{img}(H_r(K_a) \to H_r(K)) \cap \text{img}(H_r(K_b) \to H_r(K))) \)

is finite dimensional.

q.e.d.

Consider the sets of the form \( B = (a', a] \times [b, b') \) \( a' < a, b < b' \) and refer to a set of this form as box. We refer to the point \( (a, b) \) as the principal corner of the box \( B \). For \( \alpha > 0 \) and \( B = (a', a] \times [b, b') \) we write \( B + \alpha \) for the box \( (a' + \alpha, a + \alpha] \times [b + \alpha, b' + \alpha) \). We also consider call box the set \( (-\infty, a] \times [b, \infty) \).

To a box as above we assign the quotient of subspaces

\( \mathbb{F}_f^b(B) := \mathbb{F}_f^b(a, b) / \mathbb{F}_f^b(a', b) + \mathbb{F}_f^b(a, b'). \)
Note that \( \mathbb{F}_r^f(-\infty, a] \times [b, \infty) = \mathbb{F}_r^f(a, b) \).

It will be convenient to denote by
\[
\begin{align*}
F_r^f(a, b) & := \dim \mathbb{F}_r^f(a, b) \\
F_r^f(B) & := \dim \mathbb{F}_r^f(B)
\end{align*}
\]
in which case in view of Observation 3.1 item 2. one has
\[
F_r^f(B) := F_r^f(a, b) + F_r^f(a', b') - F_r^f(a', b) - F_r^f(a, b').
\]

It will also be convenient for future use to denote by
\[
(\mathbb{F}_r^f)'(B) := \mathbb{F}_r^f(a, b) + \mathbb{F}_r^f(a', b') \subseteq \mathbb{F}_r^f(a, b)
\]
hence \( \mathbb{F}_r(B) = \mathbb{F}_r(a, b)/(\mathbb{F}_r^f)'(B) \).

To simplify the notation and there is no risk of ambiguity one writes \( \mathbb{F} \cdots \cdots \) resp. \( \mathbb{F}' \cdots \cdots \) instead of \( \mathbb{F} \cdots \cdots \) resp. \( \mathbb{F}' \cdots \cdots \)'.

If \( H_r(X) \) is a Hilbert space (i.e. \( \kappa = \mathbb{R} \) or \( \mathbb{C} \)) one denotes by \( H_r(B) \) the orthogonal complement of \( \mathbb{F}_r(B) = (\mathbb{F}_r(a', b) + \mathbb{F}(a, b')) \) inside \( \mathbb{F}_r(a, b) \), so we have
\[
H_r(B) \subseteq \mathbb{F}_r(a, b) \subseteq H_r(X).
\]

and
\[
i(B) : H_r(B) \to \mathbb{F}_r(B),
\]
the canonical linear map induced by the inclusion of \( H_r(B) \) in \( \mathbb{F}_r(a, b) \), an isomorphism.

**Proposition 3.3** Let \( a' < a < a'' \) and \( b < b'' \) and \( B_1, B_2, \) and \( B \) be the boxes \( B_1 = (a', a] \times [b, b'') \) \( B_2 = (a, a''] \times [b, b'') \) and \( B = (a', a''] \times [b, b'') \) (see Figure 2).

1. The inclusions \( B_1 \subset B \) and \( B_2 \subset B \) induce the linear maps \( i : \mathbb{F}_r(B_1) \to \mathbb{F}_r(B) \) and \( \mathbb{F}_r(B) \to \mathbb{F}_r(B_2) \) such that the following sequence is exact
\[
0 \to \mathbb{F}_r(B_1) \xrightarrow{i} \mathbb{F}_r(B) \xrightarrow{p} \mathbb{F}_r(B_2) \to 0.
\]

2. If \( H_r(X) \) is equipped with a scalar product then
\[
H_r(B_1) \perp H_r(B_2)
\]
and
\[
H_r(B) = H_r(B_1) + H_r(B_2).
\]

**Proposition 3.4** Let \( a' < a \) and \( b' < b < b'' \) and \( B_1, B_2, \) and \( B \) be the boxes \( B_1 = (a', a] \times [b, b'') \) \( B_2 = (a, a''] \times [b', b'') \) and \( B = (a', a'] \times [b', b'') \) (see Figure 3).

1. The inclusions \( B_1 \subset B \) and \( B_2 \subset B \) induce the linear maps \( i : \mathbb{F}_r(B_1) \to \mathbb{F}_r(B) \) and \( \mathbb{F}_r(B) \to \mathbb{F}_r(B_2) \) such that the following sequence is exact.
\[
0 \to \mathbb{F}_r(B_1) \xrightarrow{i} \mathbb{F}_r(B) \xrightarrow{p} \mathbb{F}_r(B_2) \to 0.
\]

2. If \( H_r(X) \) is equipped with a scalar product then
\[
H_r(B_1) \perp H_r(B_2)
\]
and
\[
H_r(B) = H_r(B_1) + H_r(B_2).
\]
Figure 1: The box $B := (a', a] \times [b, b') \subset \mathbb{R}^2$

Figure 2: pair of boxes for Proposition 3.3

Figure 3: pair of boxes for Proposition 3.4
In both Propositions (3.3) and (3.4) item 1. follows from Observation 3.1 item 1. and 2. Concerning item 2. note that $\mathcal{H}_r(B_2)$ as a subspace of $\mathbb{F}_r(a'', b)$ in Proposition 3.3 and as a subspace of $\mathbb{F}_r(a, b'')$ in Proposition 3.4 is orthogonal a subspace which contains $\mathcal{H}_r(B_1)$.

In view of Propositions (3.3) and (3.4) above one has the following:

**Observation 3.5**

1. If $B'$ and $B''$ are two boxes with $B' \subseteq B''$ and $B'$ is located in the upper left corner of $B''$ then the inclusion induces the canonical injective linear maps $i : \mathbb{F}_r(B') \to \mathbb{F}_r(B'')$. (see picture Figure 4)

2. If $B'$ and $B''$ are two boxes with $B' \subseteq B''$ and $B'$ is located in the lower right corner of $B''$ then the inclusion induces the canonical surjective linear maps $i : \mathbb{F}_r(B') \to \mathbb{F}_r(B'')$. (see picture Figure 5)

3. If $B$ is a finite disjoint union of boxes $B = \bigsqcup B_i$ then $\mathbb{F}_r(B)$ is isomorphic to $\bigoplus \mathbb{F}_r(B_i)$; the isomorphism is not canonical. If in addition $H_r(X)$ is a Hilbert space then $\mathcal{H}_r(B) = \bigoplus \mathcal{H}_r(B_i)$.

In view of the above observation denote by $B(a, b : \epsilon) = (a - \epsilon, a] \times [b, b + \epsilon)$ and define

$$\hat{\delta}_r^f(a, b) := \lim_{\epsilon \to 0} \mathbb{F}_r(B(a, b; \epsilon)),$$

(the limit refers to the inverse system $\mathbb{F}_r(B(a, b; \epsilon')) \to \mathbb{F}_r(B(a, b : \epsilon''))$ for $\epsilon' < \epsilon''$.)
and

\[ \delta^f_\epsilon(x, b) := \lim_{\epsilon \to 0} F_\epsilon(B(a, b; \epsilon)), \]

hence \( \dim \hat{\delta}^f_\epsilon(a, b) = \delta^f_\epsilon(a, b) \).

Note that if \( f \) is tame then, for \( \epsilon \) small enough, the inverse system stabilizes and \( \delta^f_\epsilon(a, b) \) can view as a quotient of subspaces of \( H_r(X) \) and, if \( H_r(X) \) is a Hilbert space, as a subspace of \( H_r(X) \).

**Proposition 3.6**

1. If \( B \) is a box then \( \mathbb{F}_r(B) \) is isomorphic to \( \bigoplus_{(a, b) \in B} \hat{\delta}^f_\epsilon(a, b) \), in particular \( F_\epsilon(B) = \sum_{(a, b) \in B} \hat{\delta}^f_\epsilon(a, b) \).

2. If \( L_r(X) \) is a Hilbert space then

   \[ \mathbf{H}(B) = \bigoplus_{(a, b) \in B} \hat{\delta}^f_\epsilon(a, b) \]

   with \( \hat{\delta}^f_\epsilon(a, b) \) and \( \hat{\delta}^f_\epsilon(a', b') \) for \( (a, b) \neq (a', b') \) mutually orthogonal.

In different words, if \( H_r(X) \) is a Hilbert space, the configuration \( \hat{\delta}^f_\epsilon \in C_{H_r(X)}(\mathbb{R}^2) \) realizes the configuration \( \delta^f_\epsilon \in C^{|\dim H_r(X)} \) in the sense that \( \hat{\delta}^f_\epsilon = e(\delta^f_\epsilon) \) equivalently \( \dim \hat{\delta}^f_\epsilon(x) = \delta^f_\epsilon(x) \).

Note that in case \( X \) is not compact \( \text{supp} \delta^f_\epsilon \) can be infinite however the intersection \( B(a, b; \infty) \cap \text{supp} \delta^f_\epsilon \) is a finite set for any \( a, b \in \mathbb{R} \).

Then we have

**Proposition 3.7** If \( a \) or \( b \) is a regular value then \( \hat{\delta}^f_\epsilon(a, b) = 0 \).

Proof: Suppose at least one of \( a, b \) say \( a \) is homologically regular. Then for \( \epsilon \) small \( \mathbb{I}_{a-\epsilon}(r) = \mathbb{I}_a(r) \) hence \( \mathbb{F}_r(a-\epsilon, b) = \mathbb{F}_r(a, b) \) hence \( \mathbb{F}_r(a-\epsilon, a] \times [b, b+\epsilon]) = 0 \) hence \( \hat{\delta}^f_\epsilon(a, b) = 0 \). Similar argument for \( b \) homologically regular.

q.e.d.

Let \( D(a, b; \epsilon) := (a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon] \). If \( x = (a, b) \) one also writes \( D(x; \epsilon) \) for \( D(a, b; \epsilon) \).

**Proposition 3.8** (cf [2] Proposition 5.6) Let \( f : X \to \mathbb{R} \) be a tame map and \( \epsilon < \epsilon(f)/3 \). For any continuous map \( g : X \to \mathbb{R} \) which satisfies \( ||f - g||_\infty < \epsilon \) and \( a, b \in C_r(f) \) critical values then \( g \) satisfies:

\[
\sum_{x \in D(a, b; 2\epsilon)} \delta^g_\epsilon(x) = \delta^f_\epsilon(a, b), \tag{2}
\]

\[
\text{supp } \delta^g_\epsilon \subset \bigcup_{(a, b) \in \text{supp } \delta^f_\epsilon} D(a, b; 2\epsilon) \tag{3}
\]

If in addition \( H_r(X) \) is equipped with a Hilbert space structure the above statement can be strengthen to

\[
\bigoplus_{x \in D(a, b; 2\epsilon)} \hat{\delta}^g_\epsilon(x) = \hat{\delta}^f_\epsilon(a, b) \tag{4}
\]

with \( \hat{\delta}^f_\epsilon(x) \perp \hat{\delta}^f_\epsilon(y) \).
Proposition (3.8) implies that in an \( \epsilon \)-neighborhood of a tame map \( f \) (w.r. to the \( \| \cdot \|_{\infty} \) norm) any other continuous map \( g \) has the support of \( \delta_f^g \), \( 2\epsilon \)-closed from the support of the \( \delta_f^f \) and of the same algebraic cardinality, in case \( X \) compact.

**Proof of Proposition (3.8)** (cf [2]).

First, one orders the set of critical values, \( c_i \), \( \cdots \), \( c_i < c_{i+1} < c_{i+2} \cdots \).

Next, one establishes two intermediate results, Lemma 3.9 and 3.10 below.

**Lemma 3.9** For \( f \) as in Proposition 3.8 one has:

\[
\delta_f^f(c_i, c_j) = F_f^f((c_{i-1}, c_i) \times [c_j, c_{j+1})) = \frac{F_f^f(c_i, c_j)}{F_f^f(c_{i-1}, c_j)} + F_f^f(c_i, c_{j+1}).
\]  

(5)

and therefore

\[
\delta_f^f(c_i, c_j) = F_f^f((c_{i-1}, c_i) \times [c_j, c_{j+1})) = F_f^f(c_{i-1}, c_{j+1}) + F_f^f(c_i, c_j) - F_f^f(c_{i-1}, c_j) - F_f^f(c_i, c_{j+1})
\]  

(6)

**Proof:** Observe that in view of tameness for any \( 0 < \epsilon', \epsilon'' < \epsilon(f) \) one has

\[
\begin{align*}
F_f^f(c_i, c_j) &= F_f^f(c_i + \epsilon', c_j) = F_f(c_{i+1} - \epsilon'', c_j) = F_f(c_{i+1} - \epsilon'', c_{j-1} + \epsilon'') \\
F_f^f(c_i, c_j) &= F_f(c_i + \epsilon', c_{j-1} + \epsilon'') = F_f(c_i - \epsilon', c_{j-1} + \epsilon'')
\end{align*}
\]  

(7)

Since \( \epsilon < \epsilon(f) \) in view of the definition of \( \delta_f^f \), one has

\[
\begin{align*}
\delta_f^f(c_i, c_j) &= F_f^f((c_i - \epsilon, c_i) \times [c_j, c_j + \epsilon)) = \\
&= F_f^f(c_i, c_j) / F_f^f(c_i - \epsilon, c_j) + F_f^f(c_i, c_j + \epsilon)
\end{align*}
\]  

(8)

Combining (8) with (7) one obtains the equality (5)

\[
\delta_f^f(c_i, c_j) = \frac{F_f^f(c_{i-1}, c_j)}{F_f^f(c_i, c_{j+1})} + F_f^f(c_i, c_{j+1})
\]

.

Since \( \dim(F_f^f(c_{i-1}, c_j) + F_f^f(c_i, c_{j+1})) = \dim(F_f^f(c_{i-1}, c_j)) + \dim(F_f^f(c_i, c_{j+1}) - \dim(F_f^f(c_{i-1}, c_{j+1})) \)

equality (5) follows.

To simplify the notation in the following Lemma the index \( r \) will be dropped off.

**Lemma 3.10**

Suppose \( f \) is tame Let \( a = c_i, b = c_j, c_i, c_j \in Cr(f) \) and \( \epsilon < \epsilon(f)/3 \). If \( g \) is a continuous map with \( \|f - g\|_{\infty} < \epsilon \) then

\[
\begin{align*}
F_g^g(a - 2\epsilon, b + 2\epsilon) &= F_f^f(c_{i-1}, c_{j+1}) \\
F_g^g(a + 2\epsilon, b - 2\epsilon) &= F_f^f(c_i, c_j) \\
F_g^g(a + 2\epsilon, b + 2\epsilon) &= F_f^f(c_i, c_{j+1}) \\
F_g^g(a - 2\epsilon, b - 2\epsilon) &= F_f^f(c_{i-1}, c_j)
\end{align*}
\]  

(9)
Proof: Since $||f - g||_\infty < \epsilon$, in view of Observation 3.1, item 3. one has

\[
F^f(a - 3\epsilon, b + 3\epsilon) \subseteq F^g(a - 2\epsilon, b + 2\epsilon) \subseteq F^f(a - \epsilon, b + \epsilon),
\]
\[
F^f(a + \epsilon, b - \epsilon) \subseteq F^g(a + 2\epsilon, b - 2\epsilon) \subseteq F^f(a + 3\epsilon, b - 3\epsilon),
\]
\[
F^f(a + \epsilon, b + 3\epsilon) \subseteq F^g(a + 2\epsilon, b + 2\epsilon) \subseteq F^f(a + 3\epsilon, b + \epsilon),
\]
\[
F^f(a - 3\epsilon, b - \epsilon) \subseteq F^g(a - 2\epsilon, b - 2\epsilon) \subseteq F^f(a - \epsilon, b - 3\epsilon).
\]

(10)

Since $3\epsilon < \epsilon(f)$ one has

\[
F^f(a - 3\epsilon, b + 3\epsilon) = F^f(a - \epsilon, b + \epsilon),
\]
\[
F^f(a + \epsilon, b - \epsilon) = F^f(a + 3\epsilon, b - 3\epsilon),
\]
\[
F^f(a + \epsilon, b + 3\epsilon) = F^f(a + 3\epsilon, b + \epsilon),
\]
\[
F^f(a - 3\epsilon, b - \epsilon) = F^f(a - \epsilon, b - 3\epsilon).
\]

(11)

which imply that in the equation (10) "\subseteq" is actually "=".

Note that in view equalities (7) and for $\epsilon', \epsilon'' < \epsilon(f)$ one has

\[
F^f(c_{i-1}, c_{j+1}) = F^f(a - \epsilon', b + \epsilon'')
\]
\[
F^f(c_i, c_j) = F^f(a + \epsilon', b - \epsilon'')
\]
\[
F^f(c_i, c_{j+1}) = F^f(a + \epsilon', b + \epsilon'')
\]
\[
F^f(c_{i-1}, c_j) = F^f(a - \epsilon', b - \epsilon'').
\]

(12)

Then (10) and (12) imply equalities (9).

Next, Lemma (3.10) implies that

\[
F^g((a - 2\epsilon, a + 2\epsilon) \times [b - 2\epsilon, b + 2\epsilon]) = F^f((c_{i-1}, c_i) \times [c_j, c_{j+1})
\]

which combined with Lemma 3.9 implies that

\[
F^g((a - 2\epsilon, a + 2\epsilon) \times [b - 2\epsilon, b + 2\epsilon]) = \delta^f(a, b)
\]

which combined with Proposition (3.6) implies the inclusion (2) and the equality (4).

To check inclusion (3) observe that:

1. $||f - g||_\infty < \epsilon$ implies $X^f_a \subset X^f_{a+\epsilon} \subset X^f_{a+2\epsilon}$ and $X^b_f \subset X^b_{f-\epsilon} \subset X^b_{f-2\epsilon}$

and when $a, b \in C\tau(f)

\[
F^f(a, b) \subseteq F^g(a + \epsilon, b - \epsilon) \subseteq F^f(a - 2\epsilon, b - 2\epsilon).
\]

(13)

2. When $\epsilon < \epsilon(f)/3$ inclusions (13) imply

\[
F^f(a, b) = F^g(a + \epsilon, b - \epsilon) = F^f(a - 2\epsilon, b - 2\epsilon)
\]

which in view of Proposition 3.6, item 1. one has

\[
\sum_{x \in (-\infty, a] \times (b, \infty)} \delta^f(x) = \sum_{y \in (-\infty, a-\epsilon] \times (b-\epsilon, \infty)} \delta^g(y) = \sum_{x \in (-\infty, a+2\epsilon] \times (b-2\epsilon, \infty)} \delta^f(x).
\]

(14)

Equalities (14) rule out the existence of $x \in sup\delta^g \cap (-\infty, a + 2\epsilon] \times (b - 2\epsilon, \infty)$ with $x$ away from $\cup_{x \in sup\delta^f} D(x; 2\epsilon)$. Since this holds for any two critical values $a, b$ the inclusion (3) holds as stated.

11
4 The main results

We keep the conventions from the previous section and we suppose $X$ is compact.

The theorems below but item (3) in Theorem 4.1 were essentially established in [2], not yet in print.

**Theorem 4.1** (Topological invariance) Suppose $f : X \to \mathbb{R}$ is a map. One has:

1. $\sum_{x \in \mathbb{R}^2} \delta^f_r(x) = \dim H_r(X)$ and if $H_r(X)$ is equipped with a Hilbert space structure then
   $$\oplus_{x \in \mathbb{R}^2} \delta^f_r(x) = H_r(X)$$

2. If $f$ is weakly tame then $\delta^f_r(x) \neq 0$, $x = (a, b)$, implies that both $a, b \in Cr(f)$.

3. If $X$ is homeomorphic to a finite simplicial complex or a Hilbert cube manifold for an open and dense set of maps in the space all maps with compact open topology then $\delta^f_r(x) = 0$ or 1.

If $X = M^n$ and $(M^n, g)$ is a closed Riemannian manifold $H_r(M^n)$ is equipped with an obvious scalar product via identification of $H_r(M^n)$ with the harmonic $(n - r)$ differential forms (w. coefficients in the orientation bundle). Proposition 3.3 implies that the direct sum decomposition induced by a Morse functions $f, H_r(M) = \oplus_{x \in \mathbb{R}^2} \delta^f_r(x)$ remains the same for Morse functions which are homotopic by a homotopy of Morse functions.

Item 3 in the above theorem insures that for an open and dense set of Morse function (containing the set of Morse functions with different values on different critical points) the decomposition has all components of dimension one, providing a canonical base (up to sign) for $H_r(M^n)$ and for the $(n - r)$–harmonic forms.

**Theorem 4.2** (Stability) Suppose $X$ is a good compact ANR. The assignment $f \sim \delta^f_r$ provides a continuous map from the space of real valued maps $C(X; \mathbb{R})$ equipped with the compact open topology to the space of configurations $C_{b_r}(\mathbb{R}^2) = \mathbb{C}^{b_r}$, $b_r = \dim H_r(X)$, equipped with the collision topology (also regarded as the space of monic polynomials of degree $b_r$). Moreover with respect to the canonical metric $D$ on the space of configurations, which induces the collision topology one has

$$D(\delta^f_r, \delta^g_r) < 2D(f, g).$$

A slightly improved statement about the continuity of the assignment $f \sim \delta^f_r$ in case $H_r(X)$ is a Hilbert space can be established with the same arguments.

**Theorem 4.3** (Poincaré Duality) Suppose $X$ is a closed topological manifold of dimension $n$ which is $\kappa$–orientable and $f$ a continuous map. Then $\delta_n^f(a, b) = \delta_n^{−f}(-b, −a)$.

4.1 Proof of Theorem 4.1

1. Since $X$ is a compact ANR then $H_r(X)$ has finite dimension and is equal to $\mathbb{F}_r(M, m)$ where $M > \sup_{x \in X} f(x)$ and $m < \inf_{x \in X} f(x)$. The equality follows from Proposition 3.6. If $f$ is weakly tame and $x = (a, b)$ and at least one of $a$ or $b$ is a regular value, the result is contained in Proposition 3.7.

---

3For the purpose of future work it will be convenient to regard $x = (a, b) \in \sup \delta^f_r(r)$ as a complex number $z = a + ib$ and call the complex number $z = (a + ib) \in \sup \delta^f_r \subset \mathbb{R}$ a homological eigenvalue of the pair $(X, f)$ and to the vector space $\delta^f_r(z)$ a homological eigenspace corresponding to $z$.

4equivalently, the topology induced by the metric $D(f, g) = \sup_{x \in X} |f(x) - g(x)|$
2. The statement is contained in Proposition 3.6 item 3.
3. Observe first that if \( \cdots c_1 < c_2 < \cdots c_N < \cdots \) is the sequence of critical values and \( H_*(X_{c_{i-1}}) \rightarrow H_*(X_{c_i}) \) has co-kernel of dimension at most one then for any \( j \) the inclusion \( \mathbb{F}_r(c_{i-1}, c_j) \subseteq \mathbb{F}_r(c_{i-1}, c_j) \) has co-kernel of dimension at most one and then, by equality (5), \( \delta^j(c_i, c_j) = 0 \) or 1.

If \( X \) is a simplicial complex and \( f : X \rightarrow \mathbb{R} \) a simplicial map which takes different values on vertices the above condition holds. This because \( X^f_{c_i} \) is obtained from \( X^f_{c_i} \) by adding the star of the vertex (on which \( f \) takes the value \( c_i \)) which is contractible. Note these maps form a residual set in the space of all maps from \( X \) to \( \mathbb{R} \) w.r.t. to the compact open topology. Each simplicial map is tame and, in view of Proposition 3.8, an entire neighborhood of such simplicial map consists of maps \( g \) whose \( \delta^g_r(x) \) has values only 0 and 1. This establishes the statement in item 3.

The proof of Theorem 4.2 is given in section (5).

The proof of Theorem 4.3 is given in section (6).

5 Stability

Stability theorem follows from more or less straightforward manner from Proposition 3.8. In order to explain this we begin with a few observations.

1. Consider the space of maps \( C(X, R) \), \( X \) a compact ANR, equipped with the compact open topology.
   - Recall that this topology is induced from the metric \( D(f, g) := \sup_{x \in X} |f(x) - g(x)| \) which is complete.

2. Observe that if \( f, g \in C(X, \mathbb{R}) \) then for any \( t \in [0, 1] \) \( h_t := h_t(f, g) = tf(x) + (1-t)g(x) \in C(X; \mathbb{R}) \) is continuous, hence a ”map” and for any \( 0 = t_0 < t_1 \cdots < t_{N-1} < t_N = 1 \) one has
   \[
   D(f, g) = \sum_{0 \leq i < N} D(h_{t_{i+1}}, h_{t_i}). \tag{15}
   \]

3. If \( X \) is a simplicial complex and \( \mathcal{U} \subset C(X, \mathbb{R}) \) denotes the subset of p.l.-maps then:
   i. \( \mathcal{U} \) is a dense subset in \( C(X, \mathbb{R}) \).
   ii. \( f, g \in \mathcal{U} \) implies \( h_t \in \mathcal{U} \) hence \( \epsilon(h_t) > 0 \), hence for any \( t \in [0, 1] \) there exists \( \delta(t) > 0 \) so that \( |t' - t| < \delta(t) \) implies \( D(h_{t'}, h_t) < \epsilon(h_t)/3 \).
   Recall that \( f \) is p.l. on \( X \) if with respect to some subdivision is simplicial (i.e. the restriction of \( f \) to each simplex is linear) and for any two p.l. maps \( f, g \) there exists a common subdivision of \( X \) which makes \( f \) and \( g \) simultaneously simplicial, hence any \( h_t \) is a simplicial map. Item (i.) follows from the fact that continuous maps can be approximated with arbitrary accuracy by p.l. maps and item (ii.) follows from the continuity in \( t \) of the family \( h_t \) and of the compacity of \( X \).

4. Consider \( C_N(\mathbb{R}^2) = \mathbb{C}^N \), \( N = n_r \), with the canonical metric \( D \) which is also complete. Since any map in \( \mathcal{U} \) is tame, then in view Proposition (3.8), \( f, g \in \mathcal{U} \) with \( D(f, g) < \epsilon(f)/3 \) imply
   \[
   D(\delta^f_r, \delta^g_r) < 2D(f, g). \tag{16}
   \]
   Observe that the inequality (16) extend to all \( f, g \in \mathcal{U} \).
   Indeed, start with \( f, g \in \mathcal{U} \) and consider \( h_t, t \in [0, 1] \) the homotopy defined above. Choose a sequence \( 0 = t_0 < t_2 < t_4, \cdots t_{2N-2} < t_{2N} = 1 \) so that the open intervals \( I_{2i} = (t_{2i} - \delta(t_{2i}), t_{2i} + \delta(t_{2i})) \) cover
[0, 1]. Here \( \delta(t) \) is the positive number specified by item 3. (ii) above. The compacity of \([0, 1]\) makes this possible.

By removing (if necessary) some of the points \( t_{2i} \) and decreasing \( \delta(t_{2i}) \) one can make \( I_{2i} \cap I_{2i+2} \neq \emptyset \) and \( t_{2i-2}, t_{2i+2} \notin I_{2i} \). Choose \( t_1 < t_3 < \cdots < t_{2N-1} \) with \( t_{2i} < t_{2i+1} < t_{2i+2} \) and \( t_{2i+1} \in I_{2i} \cap I_{2i+2} \). We have then \( |t_{2i+1} - t_{2i}| < \delta(t_{2i}) \) and \( |t_{2i+2} - t_{2i+1}| < \delta(t_{2i+2}) \).

In view of item 3. (ii) above |\( t_{2i+1} - t_{2i} | < \delta(t_{2i}) \) implies \( D(h_{t_{2i}}, h_{t_{2i+1}}) < \epsilon(h_{t_{2i}})/3 \) and \( |t_{2i+2} - t_{2i+1}| < \delta(t_{2i+2}) \) implies \( D(h_{t_{2i+2}}, h_{t_{2i+1}}) < \epsilon(h_{t_{2i+2}})/3 \).

In view of item 4. the last inequalities imply \( D(\delta_r h_{t_{2i+1}}, \delta_r h_{t_{2i}}) < 2D(h_{t_{2i}}, h_{t_{2i+1}}) \) and \( D(\delta_r h_{t_{2i+2}}, \delta_r h_{t_{2i+1}}) < 2D(h_{t_{2i+2}}, h_{t_{2i+1}}) \).

Therefore, for any \( 0 \leq k \leq 2N - 1 \) one has \( D(\delta_r h_{t_{k+1}}, \delta_r h_{t_k}) < 2D(h_{t_{k+1}}, h_{t_k}) \). Then

\[
D(\delta^f, \delta^g) \leq \sum_{0 \leq i < 2N-1} D(\delta^h(t_{i+1}), \delta^h(t_i)) \leq 2 \sum_{0 \leq i < 2N-1} D(h_{t_{i+1}}, h_{t_i}).
\]

which by (15) is exactly \( D(f, g) \).

In view of the density of \( U \) and the completeness of the metrics on \( C(X; \mathbb{R}) \) and \( C_b(U, \mathbb{R}^2) \) the inequality (16) extends to the entire \( C(X; \mathbb{R}) \) and establishes the result.

q.e.d.

### 6 Poincaré Duality

For \( f : X \to \mathbb{R} \) as above introduce the function of two variable \( G_r : \mathbb{R}^2 \to \mathbb{Z}_{\geq 0} \) defined by

\[
G^f_r(a, b) := \text{dim} H_r(X)/(\mathbb{I}_a^f(r) + \mathbb{I}_b^f(r)).
\]

For a a box \( B := (a', a] \times [b, b') \subset \mathbb{R}^2 \) as above introduce

\[
G^f_r(B) := -G^f_r(a', b') - G^f_r(a, b) + G_r(a', b) + G_r(a, b').
\]

**Proposition 6.1** For any map \( f : X \to \mathbb{R} \) and any box \( B \) one has \( F^f_r(B) = G^f_r(B) \)

**Proof:** The following picture can help in conceptualizing the proof.

![Diagram](image.png)

One introduces the following notations:

1. \( I_1 := \text{dim}(I_a \cap \mathbb{I}^d) \)
2. \( I_2 := \dim(\mathbb{I}_a \cap \mathbb{I}^c/\mathbb{I}_a \cap \mathbb{I}^d) \)

3. \( I_3 := \dim(\mathbb{I}_b \cap \mathbb{I}^d/\mathbb{I}_a \cap \mathbb{I}^d) \)

4. \( I_4 := \dim(\mathbb{I}_b \cap \mathbb{I}^c/\mathbb{I}_a \cap \mathbb{I}^c + \mathbb{I}_b \cap \mathbb{I}^d) \)

5. \( I_5 := \dim \mathbb{I}_b/\mathbb{I}_a + \mathbb{I}_b \cap \mathbb{I}^c \)

6. \( I_6 := \dim \mathbb{I}^c/\mathbb{I}_a \cup \mathbb{I}^d + \mathbb{I}^d \)

7. \( I_7 := \dim H/\mathbb{I}_b + \mathbb{I}^c \) with \( H = H_r(X) \).

For simplicity in writing we will drop the index \( r \) from notation.

Using the picture above is not hard to notice that:

\[
\begin{align*}
F(a, d) &= I_1 \\
F(b, c) &= (I_1 + I_2 + I_3 + I_4) \\
F(a, c) &= (I_1 + I_2) \\
F(b, d) &= (I_1 + I_3) \\
G(a, d) &= (I_7 + I_6 + I_5 + I_4) \\
G(b, c) &= I_7 \\
G(a, c) &= (I_7 + I_5) \\
G(b, d) &= (I_7 + I_6)
\end{align*}
\]

Then we have:

\[
\begin{align*}
F(a, d) + F(b, c) - F(a, c) - F(b, d) &= I_1 + (I_1 + I_2 + I_3 + I_4) - (I_1 + I_2) - (I_1 + I_3) = I_4 \\
G(a, d) + G(b, c) - G(a, c) - G(b, d) &= (I_7 + I_6 + I_5 + I_4) + I_7 - (I_7 + I_5) - (I_7 + I_6) = I_4.
\end{align*}
\]

Let \( f : M^n \to \mathbb{R} \) be a map, \( M^n \) a \( \kappa \)-orientable closed topological manifold, and \( a, b \) regular values. Let \( i_a : M_a \to M \), \( i^b : M^b \to M \), \( j_a : M \to (M, M_a) \), \( j^b : M \to (M, M^b) \) be the obvious inclusions with \( i_a(k), i^b(k), j_a(k), j^b(k) \) the inclusion induced linear maps for homology in degree \( k \) and \( r_a(k), r^b(k), s_a(k), s^b(k) \) the inclusion induced linear maps in cohomology, (with coefficients in the field \( \kappa \)), as indicated in the diagrams [17] and [18] below. Poincaré Duality provides the following commutative diagrams [17] and [18] with all vertical arrows isomorphisms.
\[ H_r(M^b) \xrightarrow{j^b(r)} H_r(M) \xrightarrow{j^b(r)} H_r(M, M^b) \]

(18)

\[ H^{n-r}(M, M^b) \xrightarrow{s_b(r-n-r)} H^{n-r}(M) \xrightarrow{r_b(n-r)} H^{n-r}(M_b) \]

\[ (H_{n-r}(M, M_b))^{(j_b(n-r))^*} \rightarrow (H_{n-r}(M))^{(i_b(n-r))^*} \rightarrow (H_{n-r}(M_b))^* \]

Observe that:

1. \( F_r(a, b) \equiv \ker(j_a(r), j_b^b(r)) \) by the exactness of the first rows and the diagrams [17] and [18]. Precisely \( \ker_a(r) \cap j_b^b(r) = \beta_a(r) \cap \beta_b(r) \).

2. \( \ker(j_a(r), j_b^b(r)) \equiv \ker(r^a(n-r), r_b(n-r)) \) by the isomorphism of the upper vertical arrows in the same diagrams.

3. \( \ker(r^a(n-r), r_b(n-r)) \equiv \ker((r^a(n-r))^*, (i_b(n-r))^*) \) by the isomorphism of the lower vertical arrow in the same diagrams.

4. \( \ker((r^a(n-r))^*, (i_b(n-r))^*)) = (\text{coker}(i^a(n-r) + i_b(n-r)) \) by standard finite dimensional linear algebra duality.

5. \( (\text{coker}(i^a(n-r) + i_b(n-r)) = \mathcal{G}_r^{−f}(-b, -a) \).

This implies \( \delta^f_r(a, b) = \delta_{n-r}^{−f}(-b, -a) \).

7 Appendix

Recall that:

- The Hilbert cube \( Q \) is the infinite product \( Q = \prod_{i \in \mathbb{Z}_{\geq 0}} I_i \) with \( I_i = [0, 1] \). The topology of \( Q \) is given by the metric \( d_{\overline{u}, \overline{v}} = \sum_i |u_i - v_i|/2^i \) with \( \overline{u} = \{ u_i \in I, i \in \mathbb{Z}_{\geq 0} \} \) and \( \overline{v} = \{ v_i \in I, i \in \mathbb{Z}_{\geq 0} \} \).

- The space \( Q \) is a compact ANR and so is any \( X \times Q \) for any \( X \) compact ANR.

Let \( p_k : X \times Q \rightarrow X \times Q \) be given by

\[ p_k(x, \overline{u}) = (x, u_0, u_1, \ldots, u_k, 0, 0 \ldots 0). \]

For \( f : X \rightarrow \mathbb{R} \) denote by \( \overline{f} := f \cdot \pi_X \) where \( \pi_X : X \times Q \rightarrow X \) the canonical projection on \( X \).

For \( F : X \times Q \rightarrow \mathbb{R} \) denote by \( F_k := F \cdot p_k \).

In view of the definition of \( \overline{f} \) and of the metric on \( Q \) observe that:

**Observation 7.1**

1. If \( f : X \rightarrow \mathbb{R} \) is a tame map so is \( \overline{f} \) and \( \mathbb{F}_{\overline{f}}(a, b) = \mathbb{F}_{\overline{f}}(a, b) \), and then \( \delta^f_r(x) = \delta^{\overline{f}}(x) \).

2. The sequence of maps \( F_n \) is uniformly convergent to the map \( F \).
Recall that a compact Hilbert cube manifold is a compact Hausdorff space locally homeomorphic to the Hilbert cube. The following are two results about Hilbert cube manifolds whose proof can be found in [3].

**Theorem 7.2**

1. (R Edwards) If $X$ is a compact ANR then $X \times Q$ is a Hilbert cube manifold.

2. (T Chapman) Any Hilbert cube manifolds is homeomorphic to $K \times Q$ for some finite simplicial complex $K$.

As a consequence we have

**Proposition 7.3**

1. Any compact Hilbert manifold $M$ is a god ANR.

2. The set of maps $f : M \to \mathbb{R}$ with $\delta f(x) = 0$ or $1$ is open and dense.

**Proof:** A map $f : M \to \mathbb{R}$ is a compact Hilbert cube manifold is called special if there exist a finite simplicial complex $K$, map $g : K \to \mathbb{R}$ and homeomorphism $\theta : M \to K \times Q$ s.t. $\overline{g} \cdot \theta = f$ and a special map is p.l if in addition $g$ is p.l. map.

Since on a simplicial complex any map is $\epsilon$ closed to a p.l. map the same remains true for special maps. By Observation 7.1(1) any special p.l.map is tame because so is any simplicial map.

By Observation 7.1(2) any map in $C(X; \mathbb{R})$ is $\epsilon$ close to a special map and then to a p.l. special map hence a tame map. This proves (1.)

If one replaces special p.l maps with special p.l. maps which takes different values on vertices the same argument shows the density. Proposition 3.8 implies that the set of these maps is open in compact open topology. q.e.d.

**NOTE:**

It is possible to remove the hypothesis of good ANR in Theorem 4.2 using the following definitions.

**Definition 7.4** A number $a \in \mathbb{R}$ is called a homologically regular value if there exists $\epsilon > 0$ so that for any $0 \leq t, t' < \epsilon$ the inclusions $I_{(a-t)}(r) \subseteq I_{(a-t')}^f(r)$ and $I_{(a-t)}^f(r) \supseteq I_{(a+t)}^f(r)$ are isomorphisms for all $r$.

A number $c \in \mathbb{R}$ is a homological critical value if not regular value.

Indeed if the word regular and critical value is replaced by homological regular value and critical value one notices that when $X$ is compact ANR any map is homologically tame, and Propositions 3.7 and 3.8 and Theorem 4.2 continue to hold based on exactly the same arguments.

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