1. Introduction

Given a function \( f : \mathbb{R} \to \mathbb{R} \) and a matrix \( A := (a_{ij})_{i,j=1}^n \), denote by \( f[A] := (f(a_{ij}))_{i,j=1}^n \) the matrix obtained by applying \( f \) to \( A \) entrywise. Such entrywise functions of matrices have been well-studied in the literature (see e.g. \cite{24, 23, 10, 18, 7, 26, 9, 10, 17, 11}), and have recently received renewed attention due to their application in the regularization of high-dimensional covariance matrices (see e.g. \cite{5, 14, 15} and the references therein). A natural and important question consists of classifying the functions \( f \) for which \( f[A] \) is positive semidefinite for every \( n \times n \) positive semidefinite matrix \( A \). This question was previously studied by Schoenberg \cite{24} and Rudin \cite{23}, who showed that functions preserving Loewner positivity for all dimensions \( n \geq 2 \) are automatically analytic with nonnegative Taylor coefficients. When the dimension \( n \) is fixed, it can be shown \cite{26, 18} that the function \( f \) has to satisfy certain continuity and differentiability assumptions on \((0, \infty)\).

A natural refinement of this problem consists of classifying functions preserving positivity 1) under rank constraints and 2) on arbitrary domains in \( \mathbb{R} \). Given \( I \subset \mathbb{R} \), denote by \( \mathbb{P}_n^k(I) \) the cone of positive semidefinite \( n \times n \) matrices having rank at most \( k \). As we will show later, given \( 1 \leq k \leq n \), there exist non-measurable entrywise functions mapping \( \mathbb{P}_n^1(I) \) into \( \mathbb{P}_n^k(\mathbb{R}) \). This is in sharp contrast to the case where no rank constraint is imposed, where \( f \) is automatically continuous on \((0, \infty)\) (see \cite{26} Theorem 2).

In this paper, we characterize entrywise functions mapping \( \mathbb{P}_n^1(\bar{I}) \) into \( \mathbb{P}_n^1(\mathbb{R}) \) under minimal additional hypotheses, and for arbitrary intervals \( \bar{I} \subset \mathbb{R} \). We demonstrate how a weak assumption of measurability implies that every \( f : \bar{I} \to \mathbb{R} \) mapping \( \mathbb{P}_n^1(\mathbb{R}) \) into \( \mathbb{P}_n^1(\mathbb{R}) \) is smooth on the whole interval \( \bar{I} \) except maybe at the origin. To state our main result, we introduce some notation. Given
\( \alpha \in \mathbb{R}, \) define
\[ \phi_\alpha(x) := |x|^{\alpha}, \quad \psi_\alpha(x) := \text{sgn}(x)|x|^{\alpha}, \quad x \in \mathbb{R} \setminus \{0\}, \]
and let \( \phi_0(0) = \psi_0(0) = 0. \) The following theorem is the main result of the paper.

**Theorem** (Main result). Let \( I \subset \bar{I} \subset \mathbb{R} \) be intervals containing 1 as an interior point. Suppose \( \bar{I} \cap (0, \infty) \) is open, \( \pm \sup \bar{I} \notin I, \) and let \( n \geq 3. \) Then the following are equivalent for a function \( K : \bar{I} \to \mathbb{R} \) which is Lebesgue measurable on \( I. \)

1. \( K \neq 0 \) on \( \bar{I} \) and \( K[A] := (K(a_{ij}))_{i,j=1}^n \in \mathbb{P}_n^1(\mathbb{R}) \) for all \( A \in \mathbb{P}_n^1(\bar{I}). \)
2. \( K(1) \neq 0 \) and \( K[A] \in \mathbb{P}_n^1(\mathbb{R}) \) for all \( A \in \mathbb{P}_n^1(\bar{I}). \)
3. Either \( K \) is a positive constant, or \( K \) is a positive scalar multiple of \( \phi_\alpha \) or \( \psi_\alpha \) on \( \bar{I} \) for some \( \alpha \in \mathbb{R}. \)

Moreover, the distinct maps among \( \{\phi_\alpha, \psi_\alpha : \alpha \in \mathbb{R}\} \) and the constant map \( K \equiv 1 \) are linearly independent on \( \bar{I}. \)

Note that when \( I = \bar{I} = (0, \infty), \) the rank constraint in the Main Theorem is equivalent to the vanishing of all \( 2 \times 2 \) minors of \( f[A]. \) Thus, in that case, the functions mapping \( \mathbb{P}_n^1(\bar{I}) \) into \( \mathbb{P}_n^1(\mathbb{R}) \) are precisely the measurable solutions of Cauchy’s power functional equation \( K(xy) = K(x)K(y) \) on \( (0, \infty), \) which have been classified by Sierpiński \( [25], \) Banach \( [1], \) Alexiewicz and Orlicz \( [2], \) and others. The analysis is much more involved for a general interval \( \bar{I}. \) In proving the Main Theorem of the paper, our strategy is as follows: in Section 2 we classify the solutions of Cauchy’s additive and multiplicative functional equations 1) on a general interval \( \bar{I} \subset \mathbb{R}, \) and 2) assuming Lebesgue measurability only on an arbitrary subinterval \( I \subset \bar{I}. \) In doing so, we generalize previous work on the additive Cauchy functional equation by Sierpiński \( [25] \) and Banach \( [1] \) to this more general setting. We then prove the Main Theorem in Section 3 by showing that every entrywise function mapping \( \mathbb{P}_n^1(\bar{I}) \) into \( \mathbb{P}_n^1(\mathbb{R}) \) satisfies Cauchy’s multiplicative functional equation on \( \bar{I}. \) Although this result is expected, its proof is intricate and requires carefully examining the minors of various matrices. Next, in Section 4 we demonstrate how our main result can be extended to the case where matrices are also allowed to have complex entries. Our characterization involves an interesting family of complex power functions \( \Psi_{\alpha,\beta} : re^{i\theta} \mapsto r^\alpha e^{i\beta \theta} \) for \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{Z}. \) We conclude the paper by showing how our techniques can also be applied to classify the solutions of other functional equations, under domain and measurability constraints.

**Notation:** Given a subset \( S \subset \mathbb{C} \) and integers \( 1 \leq k \leq n \in \mathbb{N}, \) denote by \( \mathbb{P}_n^k(S) \) the set of \( n \times n \) Hermitian positive semidefinite matrices with entries in \( S \) and rank at most \( k. \) Also define \( \mathbb{P}_n(S) := \mathbb{P}_n^0(S), \) and \( S_+ := S \cap (0, \infty), S_- := S \cap (-\infty, 0). \) We denote the complex disc centered at \( a \in \mathbb{C} \) and of radius \( 0 < R \leq \infty \) by \( D(a, R). \) We write \( A \geq 0 \) to denote that \( A \in \mathbb{P}_n^0(\mathbb{C}), \) and write \( A \geq B \) when \( A - B \in \mathbb{P}_n^0(\mathbb{C}). \) We denote by \( I_n \) the \( n \times n \) identity matrix, and by \( 0_{n,n} \) and \( 1_{n,n} \) the \( n \times n \) matrices with every entry equal to 0 and 1 respectively. Finally, we denote the conjugate transpose of a vector or matrix \( A \) by \( A^\dagger. \)

**2. Cauchy functional equations**

The four *Cauchy functional equations*

(a) \( K(x + y) = K(x) + K(y), \) \quad (b) \( K(xy) = K(x)K(y), \)
(c) \( K(x + y) = K(x)K(y), \) \quad (d) \( K(xy) = K(x) + K(y) \)

(for all \( x, y \in \mathbb{R} \)) have been well-studied in the literature \( [1, 8, 19]. \) It is not difficult to show that the following families of nonconstant continuous functions are solutions to these equations: (a) linear maps \( K(x) = \alpha x; \) (b) power functions \( K(x) = x^\alpha; \) (c) exponential maps \( K(x) = e^{\alpha x}; \) (d) logarithm maps \( K(x) = \ln(x^\alpha) \) respectively. More generally, as was shown independently by Sierpiński \( [25] \)
and Banach [4] (see also Alexiewicz and Orlicz [2]), the linear maps are also the only solutions of (a) when \( K \) is only Lebesgue measurable. The Cauchy multiplicative functional equation (b) has also been studied in other related settings such as the Loewner and Lorentz cones [6] [27].

In order to prove our Main Theorem we begin by studying the nonconstant solutions to Equation (2.1) (a) on (1) general intervals \( I \subset \mathbb{R} \) and (2) assuming Lebesgue measurability only on a subinterval \( I \subset \widetilde{I} \). Our main result extends previous work by Sierpiński [25] and Banach [4], where they consider the case \( I = \widetilde{I} = \mathbb{R} \).

**Theorem 1** (Additive Cauchy functional equation). Let \( I \subset \widetilde{I} \subset \mathbb{R} \) be intervals containing 0 as an interior point. Then the following are equivalent for \( g : \widetilde{I} \rightarrow \mathbb{R} \).

1. \( g \) is Lebesgue measurable on \( I \) and additive on \( \widetilde{I} \).
2. \( g \) is continuous on \( \widetilde{I} \) - i.e., \( g(x) = \beta x = \beta \psi_1(x) \) for some \( \beta \in \mathbb{R} \) and all \( x \in \widetilde{I} \).
3. \( g \) is linear on \( \widetilde{I} \) - i.e., \( g(x) = \beta x = \beta \psi_1(x) \) for some \( \beta \in \mathbb{R} \) and all \( x \in \widetilde{I} \).

**Proof.** Clearly, (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1). In order to show that (1) \( \Rightarrow \) (3), we first adapt the arguments in [25] to the interval \( I \) as follows. Set \( R := \min(\inf I, \sup I) \), and fix a rational number \( a_0 \in (0, R) \). Suppose \( g(a_0) = \beta a_0 \) for some \( \beta \). One now shows easily that \( g(a_0/m) = \beta (a_0/m) \) for all \( m \in \mathbb{N} \). We then claim that \( g(a) - g(b) = \beta (a-b) \) for all \( a, b \in I \) with \( a - b \) rational. Indeed, suppose \( a_0 = p/q \) and \( a - b = r/s > 0 \) for integers \( p, q, r, s \in \mathbb{N} \). Now for any integer \( N > \frac{R}{qs} \),

\[
g(a) - g(b) = \sum_{i=1}^{Nq/s} g(a - i/(Nqs)) - g(a - i/(Nqs)) = Nq/s \cdot g \left( \frac{1}{Nqs} \right) = \beta \frac{Nq/s}{Nqs} = \beta(a - b).
\]

Moreover, \( g \) is clearly odd, so the sets

\[
E_\pm := \{ x \in (-R, R) : \pm(g(x) - \beta x) > 0 \}
\]

satisfy: \( E_- = -E_+ \). Now if \( E_\pm \) have positive Lebesgue measure, then by a classical result [25] Lemma 2], there exist \( e_\pm \in E_\pm \) such that \( e_+ - e_- \in \mathbb{Q} \). This is a contradiction since \( g(e_\pm) - \beta e_\pm \) are of different signs.

We next claim that \( g(x) = \beta x \) for all \( x \in (-R, R) \). Suppose this is false, and \( g(a) \neq \beta a \) for some \( a \in (-R, R) \). Now since \( E_\pm \) have Lebesgue measure zero, the set \( G := \{ x \in (2|a| - R, R) : g(x) = \beta x \} \) has positive measure, whence so does \( G' := \{ x \in (|a| - R, R - |a|) : g(x + a) = \beta(x + a) \} \). But since \( g(a) \neq \beta a \), hence \( G' \subset E_- \bigcup E_+ \) both have Lebesgue measure zero, which is a contradiction. We conclude that \( g(x) = \beta x \) for all \( x \in (-R, R) \). By additivity, \( g(x) = \beta x \) for all \( x \in \widetilde{I} \), proving (3). \( \square \)

Using Theorem 1 we can now classify the solution of Cauchy’s multiplicative functional equations to general intervals, under our local measurability assumption. The following result plays a crucial role in the proof of the Main Theorem.

**Theorem 2.** Let \( I \subset \widetilde{I} \subset \mathbb{R} \) be intervals containing 1 as an interior point. Then the following are equivalent for a function \( K : \widetilde{I} \rightarrow \mathbb{R} \) that is not identically zero on \( I \).

1. \( K(1) \neq 0 \), \( K \) is Lebesgue measurable on \( I \), and \( K/K(1) \) is multiplicative on \( \widetilde{I} \).
2. \( K(1) \neq 0 \), \( K \) is monotone on \( I_\pm \), and \( K/K(1) \) is multiplicative on \( \widetilde{I} \).
3. \( K(1) \neq 0 \), \( K \) is continuous on \( I \setminus \{0\} \), and \( K/K(1) \) is multiplicative on \( \widetilde{I} \).
4. \( K(1) \neq 0 \), \( K \) is differentiable on \( I \setminus \{0\} \), and \( K/K(1) \) is multiplicative on \( \widetilde{I} \).
5. Either \( K \) is a nonzero constant on \( \widetilde{I} \), or \( K \) is a scalar multiple of \( \phi_\alpha \) or \( \psi_\alpha \) for some \( \alpha \in \mathbb{R} \).

**Proof.** That (5) \( \Rightarrow \) (4) \( \Rightarrow \) (3) \( \Rightarrow \) (1) and (5) \( \Rightarrow \) (2) \( \Rightarrow \) (1) are standard. It therefore suffices to show that (1) \( \Rightarrow \) (5). Assume that (1) holds and \( K \) is nonconstant on \( \widetilde{I} \). We claim that \( K \) does not change sign on \( \widetilde{I}_+ \). Indeed, since \( 1 \in \widetilde{I} \), the interval \( \widetilde{I}_+ \) is closed
under taking square roots. Now \( K(x)/K(1) = (K(\sqrt{x})/K(1))^2 \geq 0 \) for all \( x \in \tilde{I}_+ \). We next claim that \( K \) does not vanish on \( \tilde{I}_+ \) if \( K(1) \neq 0 \). Indeed, if \( K(x) = 0 \) for any \( x \in \tilde{I}_+ \), then \( K(x^{1/2m}) = 0 \) for all \( m \in \mathbb{N} \) by multiplicativity. Now since \( x^{-1/2m} \in \tilde{I} \) for large enough \( m \), hence \( K(1)^2 = K(x^{1/2m})K(x^{-1/2m}) = 0 \), which is false. We conclude that \( K/K(1) \) is positive on \( \tilde{I}_+ \).

Now define \( g : \tilde{I}_+ \to \mathbb{R} \) via: \( g(x) := \ln(K(e^x)/K(1)) \). Also choose any compact subinterval \( I_0 := [a, b] \) of \( I \), with \( \max(0, \inf I) < a < b < \sup I \). We first claim that \( x \mapsto K(e^x) \) is Lebesgue measurable on \( \ln I_0 \), whence the restriction \( g : I_0 \to \mathbb{R} \) is also Lebesgue measurable. To see the claim, given a Borel subset \( S \) in the image of \( K \circ \exp : [\ln a, \ln b] \to \mathbb{R} \), the set \( K^{-1}(S) \subset [a, b] \) is Lebesgue measurable by assumption. Therefore by the Borel regularity of Lebesgue measure, there exists a Borel set \( S' \) and a null Lebesgue set \( N \) such that \( K^{-1}(S) = S' \Delta N \). But then \( \ln K^{-1}(S) = (\ln S') \Delta (\ln N) \). Since \( \ln : [a, b] \to \mathbb{R} \) is absolutely continuous, it follows that \( \ln K^{-1}(S) \) is also Lebesgue measurable, whence \( K \circ \exp \) is Lebesgue measurable on \( \ln I_0 \). Thus \( g \) is Lebesgue measurable on \( \ln I_0 \). Since \( K/K(1) \) is multiplicative, it follows that \( g \) is also additive on \( \ln I_+ \supset \ln I_0 \supset \{0\} \). Therefore \( g \) is linear on \( \ln \tilde{I}_+ \) by Theorem 2, say \( g(x) = \alpha x \). Reformulating, \( K(x) = K(1)x^\alpha \) for all \( 0 < x \in \tilde{I} \), for some \( \alpha \in \mathbb{R} \).

Next, if \( 0 \in \tilde{I} \), then since \( K \) is nonconstant on \( \tilde{I} \), choose \( x_0 \in \tilde{I} \) such that \( K(x_0) \neq K(1) \). Then,

\[
\frac{K(0)}{K(1)} = \frac{K(0 \cdot x_0)}{K(1)} = \frac{K(0)}{K(1)} \cdot \frac{K(x_0)}{K(1)},
\]

which implies that \( K(0) = 0 \). Finally, if \( \tilde{I} \not\subset [0, \infty) \), define \( \tilde{I}_- := \tilde{I} \cap (-\sqrt{\sup I}, 0) \). Then \( x^2 \in \tilde{I} \) whenever \( x \in \tilde{I}_- \), so we compute:

\[
\frac{K(x^2)}{K(1)} = \frac{K(x^2)}{K(1)} = |x|^{2\alpha}.
\]

Therefore \( K(x)/K(1) = \varepsilon(x)|x|^{\alpha} \), where \( \varepsilon(x) = \pm 1 \). We now show that \( \varepsilon \) is constant on \( \tilde{I}_- \). Indeed, if \( x < y < 0 \) are in \( \tilde{I} \), then compute using that \( y/x \in (0, 1) \subset \tilde{I} \):

\[
\varepsilon(y)|y|^{\alpha} = \frac{K(y)}{K(1)} = \frac{K(x)}{K(1)} \frac{K(x)K(y/x)}{K(1)} = \varepsilon(x)|x|^{\alpha} \frac{y}{x}^{\alpha} = \varepsilon(x)|y|^{\alpha}.
\]

This shows that \( \varepsilon \) is constant on \( \tilde{I}_- \), which in turn implies the assertion (5) on all of \( \tilde{I}_- \). We now show that (5) holds on all of \( \tilde{I}_- \). Indeed, given any \( x \in \tilde{I}_- \) and \( a_0 \in \tilde{I} \cap (1, \infty) \), there exists \( m \in \mathbb{N} \) and \( y \in \tilde{I}_- \) such that \( ya_0^m = x \). Hence by multiplicativity of \( K/K(1) \) on \( \tilde{I} \),

\[
\frac{K(x)}{K(1)} = \frac{K(y)}{K(1)} \left( \frac{K(a_0)}{K(1)} \right)^m = \varepsilon(y)|y|^{\alpha} \cdot (|a_0|^{\alpha})^m = \varepsilon(y)|x|^{\alpha}.
\]

This proves that \( K(x) = c\phi_\alpha \) or \( c\psi_\alpha \) for some \( \alpha \in \mathbb{R} \) and \( c = K(1) \), on all of \( \tilde{I} \). We conclude that \( (1) \implies (5) \).

As a consequence of Theorem 2, we immediately obtain the following corollary.

**Corollary 3** (Multiplicative Cauchy functional equation). Let \( I \subset \tilde{I} \subset \mathbb{R} \) be intervals containing 1 as an interior point. Then the following are equivalent for a function \( K : \tilde{I} \to \mathbb{R} \) that is not identically zero on \( \tilde{I} \):

1. \( K \) is multiplicative on \( \tilde{I} \) and Lebesgue measurable on \( I \).
2. Either \( K \equiv 1 \) on \( \tilde{I} \), or \( K \equiv \phi_\alpha \) or \( \psi_\alpha \) on \( \tilde{I} \) for some \( \alpha \in \mathbb{R} \).

**Proof.** Clearly (2) \( \implies \) (1). Conversely, if \( K(1) = 0 \), then \( K \equiv 0 \) on \( \tilde{I} \) by multiplicativity. Thus if \( K \neq 0 \) then \( K(1) = K(1)^2 \neq 0 \). But then \( K(1) = 1 \), whence \( K = K/K(1) \) is multiplicative on \( \tilde{I} \). Now (1) follows by Theorem 2. \( \Box \)

**Remark 4.** Theorem 2 also classifies all maps \( K : \tilde{I} \to \mathbb{R} \) such that \( K/K(1) \) is multiplicative, and which are (a) Borel measurable, (b) monotone, (c) continuous, or (d) differentiable, \( C^n \) for some
$n \in \mathbb{N}$, or smooth on $I_\pm$ for some interval $1 \in I \subset \bar{I}$. The reason these conditions are equivalent is that they are all satisfied by the functions listed in (5) and imply Lebesgue measurability (1). Moreover, we classify all multiplicative maps which are continuous or $C^m$ on $\mathbb{R}$; see Corollary 10. Similar results can also be proved under other hypotheses; see e.g. [24, Lemma 4.3].

3. Proof of the Main Theorem

We now show the main result of this paper. In order to prove the linear independence part of the result, we need the following generalization to semigroups of the Dedekind Independence Theorem (see [3, Chapter II, Theorem 12]). We include a short proof of this result for convenience.

**Lemma 5.** Suppose $(G, \cdot)$ is any semigroup and $F$ any field. Let $n \geq 1$ and let $\chi_1, \ldots, \chi_n : (G, \cdot) \to F$ denote pairwise distinct multiplicative maps that are not identically zero on $F$. Then $\chi_1, \ldots, \chi_n$ are $F$-linearly independent.

**Proof.** The proof is by induction on $n$. For $n = 1$ the result is clear. Suppose it holds for $n - 1 \geq 1$, and suppose $T := \sum_{j=1}^{n} c_j \chi_j$ is identically zero as a function on $G$. Now choose $g_n \in G$ such that $\chi_n(g_n) \neq \chi_1(g_n)$. Then we obtain:

$$0 = T(g_n) - T(g)\chi_n(g_n) = \sum_{j=1}^{n} c_j (\chi_j(g)\chi_j(g_n) - \chi_j(g)\chi_n(g_n)) = \sum_{j=1}^{n-1} c_j (\chi_j(g_n) - \chi_n(g_n))\chi_j(g).$$

Since $\chi_1(g_n) \neq \chi_n(g_n)$, the above sum is a nontrivial linear combination of the (pairwise distinct) characters $\chi_1, \ldots, \chi_{n-1}$. Hence by the induction hypothesis, $c_1 = \cdots = c_{n-1} = 0$, which then implies that $c_n = 0$ as well. This concludes the proof. $\square$

We can now prove our main result. Note that when there is no constraint on the domain of the function $K$ in our main theorem, i.e., when $\bar{I} = \mathbb{R}$, the function automatically satisfies Cauchy’s multiplicative functional equation [27](b), and the Main Theorem follows immediately from Theorem 2. However, as we now show, proving that $K$ satisfies Equation [27](b) on $\bar{I}$ under the domain constraint is much more involved.

**Proof of the Main Theorem.** For ease of exposition, we write out the proof in several steps.

**Step 1.** We first show the equivalence for $n \geq 3$. It is clear that (3) $\implies$ (2) $\implies$ (1). We now show that (1) $\implies$ (2). Suppose to the contrary that $K(1) = 0$. Define $u_x := (1, \ldots, 1, x)^T$ for $x \in (\sqrt{\inf \bar{I}_+, \sup \bar{I}_+})$. Now examine the minor of $K[u_x u_x^T] \in \mathbb{P}_n^1(\mathbb{R})$, formed by the last two rows and columns. We conclude that $K \equiv 0$ on $(\sqrt{\inf \bar{I}_+, \sup \bar{I}_+})$. Note here that $\inf \bar{I}_+ < 1 < \sup \bar{I}_+$ by assumption.

We now claim that $K \equiv 0$ on $\bar{I}$, which shows by contradiction that (1) $\implies$ (2). The first step is to show that $K \equiv 0$ on $(1, (\sup \bar{I})^{1-2^{-m}})$ for all $m \geq 1$. This was shown in the preceding paragraph for $m = 1$, and we show it for all $m$ by induction. Thus, given $x \in (1, (\sup \bar{I})^{1-2^{-m}})$ for $m \geq 2$,

$$x_m := x^{2(2^{m-1}-1)/(2^{m-1})} \in (1, (\sup \bar{I})^{1-2^{-m}}).$$

Now set $y_m := (\sup \bar{I})^{1/(2^{m-1})}$, and define $u_m := \sqrt{x_m(1, \ldots, 1, y_m)^T} \in \mathbb{R}^n$. Then it is easily verified that $u_m u_m^T \in \mathbb{P}_n^1(\bar{I}_+)$, whence the lower rightmost $2 \times 2$ minor of $K[u_m u_m^T]$ is zero. Since $K(x_m) = 0$ by the induction hypothesis, we conclude that $0 = K(x_m y_m) = K(x)$. This shows that $K \equiv 0$ on $[1, \sup \bar{I})$. A similar argument shows that $K \equiv 0$ on $(\inf \bar{I}_+, 1]$. Thus, $K \equiv 0$ on $\bar{I}_+$. If $0 \in \bar{I}$ then evaluating $K$ entrywise on $1_{1 \times 1} \oplus 0_{(n-1) \times (n-1)} \in \mathbb{P}_n^1(\bar{I})$ shows that $K(0) = 0$ as well. Finally, if $\bar{I}_-$ is nonempty, then the assumptions on $\bar{I}$ imply that $\bar{I}_- \subset -\bar{I}_+$. Thus, given $x \in \bar{I}_-$, evaluating
Step 2. We next prove that \((2) \implies (3)\) for \(n \geq 3\). Suppose \((2)\) holds and \(K\) is nonconstant on \(\tilde{I}\). Define \(I' := (\sqrt{\inf \tilde{I}}, \sup \tilde{I}) \subset \tilde{I}\). Then given \(x, y \in I'\), it is clear that \(uu^T \in \mathbb{P}^1_n(\tilde{I})\), where \(u := (1, 1, \ldots, 1, x, y)^T \in (I')^n\). Considering the \(2 \times 2\) minor corresponding to the \(n - 2\) and \(n\)th rows and the last two columns shows that \(K/K(1)\) is multiplicative on \(I'\). Since \(K/K(1)\) is also Lebesgue measurable on \(I \cap I'\), Theorem 2 implies that \(K\) satisfies \((3)\) on \(I'\). Now set \(u := (1, 1, \ldots, 1, x)^T\) and \(A := uu^T\), with \(x \in I'\). Applying \((2)\) to \(A\), we conclude that \((3)\) holds on \(\tilde{I}_+\).

It remains to show that \((3)\) holds even when \(\tilde{I} \not\subset (0, \infty)\). First if \(\tilde{I} = [0, \infty)\) then choose \(a > 0\) such that \(K(a) \neq K(0)\). Applying \(K\) entrywise to the matrix \(a1_{1 \times 1} \oplus 0_{(n-2) \times (n-2)} \in \mathbb{P}^1_n(\tilde{I})\) yields \(K(0) = 0\), which proves \((3)\). Finally, suppose \(\tilde{I}_-\) is nonempty. There are then two cases:

1. \(K(0) \neq 0\). We then claim that \(K\) is constant on \(\tilde{I}\). Indeed, choose \(a \in \tilde{I}_-\), define \(u := \sqrt{|a|}(1, -1, 0, \ldots, 0)^T \in \mathbb{R}^n\), and \(A := uu^T \in \mathbb{P}^1_n(\tilde{I})\). Now the minor corresponding to the first and third rows and columns of \(K[A]\) is zero, whence \(K(0) = K([a])\). Considering the minor corresponding to the first and second rows, and second and third columns, shows that \(K(a) = K([a]) = K(0)\). This implies that \(K\) is constant on \((\inf \tilde{I}, \inf \tilde{I})\). Repeating the same argument with \(u := (\sqrt{a}, 0, \ldots, 0)\) for all \(0 < a \in \tilde{I}\) shows that \(K\) is constant on \(\tilde{I}\).

2. \(K(0) = 0\). Since \(K\) is nonconstant, it follows by the above analysis that \((3)\) holds on \(\tilde{I}_+\). Thus for \(a \in \tilde{I}_-\), \(K(a) = \pm K([a]) = \pm K(1)|a|^\alpha\) for some \(\alpha \in \mathbb{R}\). In other words, there exists \(\varepsilon : \tilde{I}_- \to \{\pm 1\}\) such that \(K(a) = K([a])\varepsilon(a)\) for all \(a \in \tilde{I}_-\). It remains to show that \(\varepsilon\) is constant. Indeed, if \(x, y \in \tilde{I}_-\) such that \(-\sqrt{|\inf \tilde{I}|} < x < y < 0\), then setting \(u := (1, \ldots, 1, y/x, x)^T\), it follows that \(K[|u|^T] \in \mathbb{P}^1_n(\mathbb{R})\). Since the minor corresponding to the \(n - 2\) and \(n\)th rows, and the \((n - 2)\) and \((n - 1)\)st columns is zero, we compute:

\[
\varepsilon(y)|y|^\alpha = \frac{K(x)K(y/x)}{K(1)^2} = \varepsilon(x)|x|^\alpha|y/x|^\alpha,
\]

from which it follows that \(\varepsilon(y) = \varepsilon(x)\) whenever \(x, y \in \tilde{I}_-\) and \(-\sqrt{|\inf \tilde{I}|} < x < y < 0\). Finally if \(\inf \tilde{I} \leq -1\), then we show that \(\varepsilon\) is constant on all of \(\tilde{I}_-\), which proves \((3)\) on all of \(\tilde{I}\). Choose \(y, a, x \in \tilde{I}_+\) such that \(\inf \tilde{I} \leq y \leq -\sqrt{|\inf \tilde{I}|} < a < 0 < |y| < x < \sup \tilde{I}\), and define \(u := \sqrt{x(a/x, y/x, 1, \ldots, 1)^T} \in \mathbb{R}^n\). One then verifies that \(uu^T \in \mathbb{P}^1_n(\tilde{I})\); now since the minor of \(K[|u|^T]\) corresponding to the first and third columns and the first two rows is zero, we obtain:

\[
\varepsilon(a)|a|^\alpha \cdot (ay/x)^\alpha = \varepsilon(y)|y|^\alpha \cdot (a^2/x)^\alpha,
\]

which shows that \(\varepsilon(y) = \varepsilon(a)\). Therefore \(\varepsilon\) is constant on all of \(\tilde{I}_-\), as desired.

To conclude the proof, the linear independence claim is immediate from Lemma 5 which we apply to the semigroup \(G := (\max(\inf \tilde{I}, -1), 1) \subset \tilde{I}\). \(\square\)

Remark 6. The assumptions on \(\tilde{I}\) in the Main Theorem are necessary in order to obtain the above characterizations. For instance, if \(S := \tilde{I} \cap (-\infty, -\sup \tilde{I})\) is nonempty, then every map : \(S \to \mathbb{R}\) can be extended to a map \(K : \tilde{I} \to \mathbb{R}\) preserving positivity on \(\mathbb{P}^1_n(\tilde{I})\), since no element of \(S\) can occur in any matrix in \(\mathbb{P}^1_n(\tilde{I})\). Even if \(\tilde{I} \subset (0, \infty)\), there can exist other solutions if \(\tilde{I}_+\) is not open. For
instance, if sup $\tilde{I}$ or inf $\tilde{I}$ belongs to $\tilde{I}$, then the Kronecker delta functions $\delta_{x,\text{sup} \tilde{I}}$ or $\delta_{x,\text{inf} \tilde{I}}$ preserve Loewner positivity on $\mathbb{P}_n^1(\tilde{I})$.

The main theorem of the paper characterizes functions mapping $\mathbb{P}_n^1(\tilde{I})$ into itself when $n \geq 3$. It is natural to ask if the same result holds when $n = 2$. We now show that this is not the case.

**Proposition 7.** Let $I \subset \tilde{I} \subset \mathbb{R}$ be intervals containing 1 as an interior point. Suppose $\tilde{I} \cap (0, \infty)$ is open, and $\pm \text{sup} \tilde{I} \notin \tilde{I}$. Then the following are equivalent for a function $K : \tilde{I} \to \mathbb{R}$ which is Lebesgue measurable on $I$ and not identically 0 on $I$:

1. $K[-]$ maps $\mathbb{P}_2^1(\tilde{I})$ to $\mathbb{P}_2^1(\mathbb{R})$.

2. There exists a function $\varepsilon : \tilde{I}_- = \tilde{I} \cap (-\infty, 0) \to \{\pm 1\}$ such that:
   - $\varepsilon$ is Lebesgue measurable when restricted to $I \cap \tilde{I}_-$.
   - Either $K$ is a positive constant, or $K$ is a positive scalar multiple of $\phi_{\alpha}$ or $\psi_{\alpha}$ on $I \cap [0, \infty)$ for some $\alpha \in \mathbb{R}$; and
   - $K(x) \equiv \varepsilon(x)K(|x|)$ for all $x \in \tilde{I}_-$. 

**Proof.** First observe that for all $\varepsilon : \tilde{I}_- \to \{\pm 1\}$ and $A \in \mathbb{P}_2^1(\tilde{I})$, $K[A] \in \mathbb{P}_2^1(\mathbb{R})$ if and only if $(K \cdot \varepsilon)[A] \in \mathbb{P}_2^1(\mathbb{R})$, by the Schur product theorem. Thus, every $K[-]$ satisfying (2) preserves positivity on $\mathbb{P}_2^1(\tilde{I})$. Conversely, we compute:

$$a, b, ab \in \tilde{I}_+ \implies \left( \frac{a}{\sqrt{ab}} \frac{\sqrt{ab}}{b} \right), \left( \frac{ab}{\sqrt{ab}} \frac{\sqrt{ab}}{1} \right) \in \mathbb{P}_n^1(\tilde{I}),$$

from which it follows using the hypotheses that

$$K(a)K(b) = K[(\sqrt{ab})^2] = K(ab)K(1) = K(ab).$$

In other words, $K$ is multiplicative on $\tilde{I}_+$. Now note that if $K(1) = 0$ then $K \equiv 0$ on $\tilde{I}_+$ by multiplicativity; moreover, applying $K$ entrywise to the matrices $\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ and $\left( \begin{array}{cc} |x| & x \\ x & |x| \end{array} \right)$ for $x \in \tilde{I}_-$ shows that $K \equiv 0$ on $\tilde{I}$. We conclude by the hypotheses that $K(1) \neq 0$. Now applying Theorem 2 to $\tilde{I}_+$, we conclude that on $\tilde{I} \cap [0, \infty)$, $K/K(1)$ equals either $\phi_{\alpha} \equiv \psi_{\alpha}$ or a constant.

Next, the only matrices in $\mathbb{P}_2^1(\tilde{I})$ with a zero entry are of the form $0_{2 \times 2}$, $\left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right)$, or $\left( \begin{array}{cc} 0 & 0 \\ 0 & a \end{array} \right)$, with $a \in \tilde{I}_+$. Considering the two cases $K(0) = 0$ and $K(0) \neq 0$, one verifies that the result holds if $\tilde{I} \subset [0, \infty)$. Finally, applying $K$ entrywise to the matrix $\left( \begin{array}{cc} |x| & x \\ x & |x| \end{array} \right)$ in $\mathbb{P}_2^1(\tilde{I})$ for $x \in \tilde{I}_-$ yields: $K(x) = \pm K(|x|)$. In other words, $K(x) = \varepsilon(x)K(|x|)$ for some $\varepsilon : \tilde{I}_- \to \{\pm 1\}$. Moreover, $\varepsilon(x) = K(|x|)/K(x)$ is Lebesgue measurable on $I \cap \tilde{I}_-$, which concludes the proof.

4. Preserving positivity on complex rank one matrices

It is natural to ask if the Main Theorem in this paper has an analogue for matrices with complex entries. We now provide a positive answer to this question. Although we are interested mainly in matrices with entries in $D(0, R)$ for $R > 1$, we will prove a characterization result for more general regions $G \subset \mathbb{C}$, which include sets such as $G = \overline{D}(0, 1) \cup (-1 - \epsilon, 1 + \epsilon)$, $G = S^1 \cup (-1 - \epsilon, 1 + \epsilon)$, and $G = (-R, R) \cup (D(0, R) \setminus D(0, r))$ for $0 \leq r < 1 < R$ and $\epsilon > 0$. In order to state this result, we first introduce a family of complex power functions $\Psi_{\alpha, \beta}$ for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$ by

$$\Psi_{\alpha, \beta}(re^{i\theta}) := r^\alpha e^{i\beta \theta}, \quad r > 0, \ \theta \in (-\pi, \pi], \quad \Psi_{\alpha, \beta}(0) := 0.$$  \quad (4.1)

Then $\Psi_{n, n}(z) = z^n$ and $\Psi_{n, -n}(z) = \overline{z}^n$ for $n \in \mathbb{Z}$ and $z \neq 0$. Moreover, $\Psi_{\alpha, \beta}$ restricted to $\mathbb{R}$ equals $\phi_{\alpha}$ if $\beta$ is even and $\psi_{\alpha}$ if $\beta$ is odd. Additionally, $\Psi_{\alpha, \beta}$ is continuous on $\mathbb{C}^\times$ and multiplicative on $\mathbb{C}$ for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$.
In a forthcoming paper [12] we explore which of the maps \( \Psi_{\alpha,\beta} \) preserve Loewner positivity when applied entrywise to Hermitian positive semidefinite matrices of a fixed order. The following result provides an answer when additional rank constraints are imposed.

**Theorem 8.** Suppose \( n \geq 3 \) and \( G \triangleq \mathbb{C} \) satisfies:

- For all \( z \in S^1 \), the set \( I_z := \{ a \in (0, \infty) : az \in G \} \) is an interval containing 1, but not its supremum if \( \sup I_z > 1 \).
- \( G \) is closed under the conjugation and modulus maps \((z \mapsto \overline{z}, |z|)\).
- \( \overline{I} := G \cap \mathbb{R} \) is an interval with 1 as an interior point, such that \( \pm \sup \overline{I} \notin \overline{I} \).

Suppose \( K : G \to \mathbb{C} \) is Lebesgue measurable on a sub-interval \( I \subset \overline{I} \) containing 1 as an interior point, and either Baire measurable or universally measurable [22, Section 2] when restricted to (the topological group) \( S^1 \). Then the following are equivalent for a function \( K : G \to \mathbb{C} \):

1. \( K \not\equiv 0 \) on \( G \), and \( K[\cdot] \) maps \( P^1_n(G) \to P^1_n(\mathbb{C}) \).
2. \( K(1) > 0 \), and \( K/K(1) : G \to \mathbb{C} \) is multiplicative and conjugation-equivariant.
3. \( K(1) > 0 \), and either \( K \equiv K(1) \) on \( G \) or there exist \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{Z} \) such that \( K = K(1) \cdot \Psi_{\alpha,\beta} \) on \( G \).

Moreover, the maps \( \{ \Psi_{\alpha,\beta} : \alpha \in \mathbb{R}, \beta \in \mathbb{Z} \} \cup \{ K \equiv 1 \} \) are linearly independent on \( D(0, r) \) for any \( 0 < r \leq \infty \).

Note that \( G \) needs to be closed under conjugation in Theorem 8 because if \( z \in G \) but \( \overline{z} \notin G \), then \( z \) can never occur as an entry of a matrix in \( P_n(G) \). Similarly, if \( z \in G \setminus D(0, \sup(G \cap \mathbb{R})) \), then \( z \) can never occur as an entry of a matrix in \( P_n(G) \).

**Proof of Theorem 8.** We prove a cyclic chain of implications. That (3) \( \implies \) (1) is easily verified. We next show that (2) \( \implies \) (3). First cyclic that since \( K/K(1) \) is multiplicative and conjugation-equivariant on \( S^1 \), it is a group endomorphism of \( S^1 \) into itself. Moreover, \( K \) is Baire/universally measurable on \( S^1 \subset \mathbb{C} \). It follows by results by Banach-Pettis (see [21] or [22, Theorem 2.2]) and by Steinhaus-Weil (see [22, Corollary 2.4]) that \( K \mid_{S^1} \) is continuous. Therefore there exists \( \beta \in \mathbb{Z} \) such that \( K(z) = K(1)z^\beta \) for \( z \in S^1 \). There are now two cases to consider. First suppose \( K \) is constant on \( G \cap \mathbb{R} \). If \( \beta \neq 0 \), then

\[
K(0) = K(0 \cdot \exp(i \pi/\beta)) = K(0) \exp(i \pi/\beta)^\beta = -K(0),
\]

which implies that \( K(0) = 0 = K(1) \). This contradicts (2), so \( \beta = 0 \). Therefore by multiplicativity and the assumptions on \( G \),

\[
K(z) = K(|z|)K(z/|z|)/K(1) = K(1)(z/|z|)^0 = K(1), \quad \forall z \in G \setminus \{0\}.
\]

Therefore \( K \) is constant on \( G \), proving (3). The other case is if \( K \) is nonconstant on \( G \cap \mathbb{R} \). Then by the Main Theorem, \( K \equiv K(1)\phi_{\alpha} \equiv K(1)\psi_{\alpha} \) on \( G \cap [0, \infty) \) for some \( \alpha \in \mathbb{R} \). Now compute for \( z \in G \setminus \{0\} \):

\[
K(z) = K(|z| \cdot z/|z|) = \frac{1}{K(1)}K(|z|)K(z/|z|) = K(1)|z|\alpha(z/|z|)^\beta = K(1)\Psi_{\alpha,\beta}(z),
\]

which shows (3).

Finally, we show that (1) \( \implies \) (2). As the proof is intricate, we divide it into four steps for ease of exposition.

**Step 1.** We first claim that if \( K(0) \neq 0 \), then \( K \equiv K(0) \) on \( G \) (from which (2) follows). Indeed, if \( K(0) \neq 0 \), then for all \( 0 < a \in G \), we have that \( K(a) = K(0) \) by considering the minor formed by the first two rows and columns of \( K[a1_{1 \times 1} \oplus 0_{(n-1)\times (n-1)}] \in P^1_n(\mathbb{C}) \). Now given \( z \in G \setminus \{0\} \), define \( u \equiv |z|^{-1/2}(z, |z|, 0, \ldots, 0)^T \); then \( uu^* \in P^1_n(G) \). Now the vanishing of the minor formed by the first two columns and the first and third rows of \( K[uu^*] \in P^1_n(\mathbb{C}) \) implies that \( K \equiv K(0) \) on \( G \).
Step 2. Given the previous step, we will assume that \( K(0) = 0 \) for the remainder of the proof. We next claim that \( K(1) > 0 \), and \( K(K(1)) \) is conjugation-equivariant on \( G \) and multiplicative on \( G \cap \mathbb{R} \). Indeed, note that \( K(1) \geq 0 \) since \( K(1_{n \times n}) \in \mathbb{P}^1_n(\mathbb{C}) \). If \( K(1) = 0 \) then \( K(G \cap [0, \infty)) = 0 \) by the Main Theorem. Now given \( z \in G \setminus \{0\} \), define the vector \( u := \sqrt{|z| \{1, \ldots, 1, \overline{z}/|z|\}}^T \in \mathbb{C}^n \). Applying \( K \) entrywise to the matrix \( uu^* \in \mathbb{P}^1_n(G) \), we conclude that \( K(K(1)) \) is conjugation-equivariant on \( G \), and therefore \( K \equiv 0 \) on \( G \). Since \( K \not\equiv 0 \) by hypothesis, it follows that \( K(1) > 0 \). Now use the Main Theorem to infer that \( K(K(1)) \) is multiplicative on \( G \cap \mathbb{R} \).

Step 3. Next, we show that \( K(K(1)) \) is multiplicative on \( S^1 \subset G \). Indeed, given \( z, z' \in S^1 \), define \( u := (z, z', 1, \ldots, 1)^T \in \mathbb{C}^n \); then \( uu^* \in \mathbb{P}^1_n(G) \), so \( K[uu^*] \in \mathbb{P}^1_n(\mathbb{C}) \). We conclude from (1) that \( K(K(1)) : S^1 \to S^1 \), by considering the vanishing of the minor formed by the first and third rows and columns of \( K[uu^*] \in \mathbb{P}^1_n(\mathbb{C}) \). Now consider the minor formed by the first two columns and the first and third rows. The vanishing of this minor implies that

\[
K(K(1)) = K(\overline{z}K(zz')) = \overline{K(z)K(zz')} = K(z)^{-1}K(zz').
\]

It follows that \( K(K(1)) \) is multiplicative on \( S^1 \subset G \).

Step 4. We now claim that \( K(z) = K(|z|)K(z/|z|)/K(1) \) for all \( z \in G \setminus \{0\} \). The claim would imply that \( (1) \implies (2) \) (recall that \( K(0) = 0 \)), because if \( z, z', z'' \in G \setminus \{0\} \), then by the hypotheses and the conclusions of the previous two steps,

\[
\frac{K(zz'')}{K(1)} = \frac{K(|zz''|)K(zz''/|zz''|)}{K(1)^2} = \frac{K(|z||z'||z''/|z''|)K(1)}{K(1)^4} = \frac{K(z)K(z'')}{K(1)}.
\]

Thus it suffices to prove the claim on all of \( G \setminus \{0\} \). By the previous step, the claim holds on \( S^1 \) and on \( G \cap \mathbb{R} \). Now suppose \( z \in S^1 \) and \( 0 < x < \sqrt{\text{sup } I_z} \). Define \( u := (x, z, 1, \ldots, 1)^T \); then \( uu^* \in \mathbb{P}^1_n(G) \). Now consider the minor formed by the first and third columns, and second and third rows, of \( K[uu^*] \in \mathbb{P}^1_n(\mathbb{C}) \). The vanishing of this minor yields:

\[
\frac{K(xz)}{K(x)} = \frac{K(z)}{K(1)}, \quad \forall z \in S^1, \quad 0 < x < \sqrt{\text{sup } I_z}.
\]

(4.2)

It remains to show that Equation (4.2) also holds for \( z \in S^1 \) and \( x \in [\sqrt{\text{sup } I_z}, \text{sup } I_z] \), assuming that \( \text{sup } I_z > 1 \). This is proved similarly to Step 1 in the proof of the Main Theorem. Namely, given \( x \in (1, (\text{sup } I_z)^{1-2^{-m}}) \) for \( m \in \mathbb{N} \), we claim that \( K(xz) = K(x)K(z)/K(1) \). The proof is by induction on \( m \); the case \( m = 1 \) was shown in the previous paragraph. Now suppose \( m > 1 \); then

\[
x_m := x^{2((2m-1)/(2m-1)-1))} \in (1, (\text{sup } I_z)^{1-2^{-(m-1)}}).
\]

Now set \( y_m := (\text{sup } I_z)^{1/(2m-1)} \), and define \( u_m := \sqrt{x_m}(y_m, z, 1, \ldots, 1)^T \in \mathbb{R}^n \). Then it is easily verified \( x = x_my_m \) and \( u_mu_m^* \in \mathbb{P}^1_n(I_z) \). Therefore the minor formed by the first and third columns, and second and third rows of \( K[uu^*] \) vanishes. This yields:

\[
K(x_my_mz) = \frac{K(x_my_m)K(x_m z)}{K(x_m)} = \frac{K(y_m)K(x_m)z}{K(1)},
\]

since \( K/K(1) \) is multiplicative on \( G \cap \mathbb{R} \) by Step 2. Now \( K(x_my_mz) = K(x_m)K(z)/K(1) \) by the induction hypothesis, since \( x_m < (\text{sup } I_z)^{1-2^{-(m-1)}} \). Therefore,

\[
K(xz) = K(x_my_mz) = \frac{K(y_m)K(x_m z)}{K(1)} = \frac{K(y_m)K(x_m)z}{K(1)^2} = \frac{K(x_m y_m)z}{K(1)} = \frac{K(x)K(z)}{K(1)},
\]

which proves the claim, and with it, the equivalence of the three assertions.

Finally, note that for any \( 0 < r \leq \infty \), the set \( G' := D(0, \min(1, r)) \) is a semigroup under multiplication, and the maps \( \{ \Psi_{\alpha, \beta} : \alpha \in \mathbb{R}, \beta \in \mathbb{Z} \} \cup \{ K \equiv 1 \} \) are (not necessarily \( \mathbb{C}^\infty \)-valued) pairwise distinct characters of \( G \). Therefore by Lemma 5 they are linearly independent on \( G' \), and hence on \( D(0, r) \).
Remark 9. Note as in Remark 3 that Theorem 8 also classifies the Borel/ Haar-measurable maps which preserve Loewner positivity. On the other hand, if we do not make any measurability assumptions about $K|_{S^1}$ in Theorem 8, there exist non-measurable solutions satisfying Theorem 1). For instance, consider any Hamel basis $B := \{x_\gamma\}$ of $\mathbb{R}$ over $\mathbb{Q}$ containing 1 and contained in $(0,2)$, and let $\mathcal{F}$ denote the set of functions $f : B \to \mathbb{R}$ such that $f(1) = 0$. Now given $f \in \mathcal{F}$, define the function $K_f : S^1 \to S^1$ as follows: write $x \in \mathbb{R}$ as a finite sum $\sum_\gamma c_\gamma x_\gamma$, and define

$$K_f : \exp(i\pi x) \mapsto \exp(i\pi \sum_\gamma c_\gamma f(x_\gamma)), \quad \forall x \in \mathbb{R}.$$ 

Note that $K_f$ is well-defined and multiplicative on $S^1$, and hence preserves $\mathbb{P}_n^1(\mathbb{C})$ for all $n \in \mathbb{N}$ and $f \in \mathcal{F}$. However, since $f$ is allowed to vary over all of $\mathcal{F}$, the function $K_f$ is not necessarily Haar measurable.

5. Measurable solutions of Cauchy functional equations

We conclude this paper by using our methods to complete the classification of functions satisfying the four Cauchy functional equations (2.1), under local Borel/Lebesgue measurability assumptions and on general intervals. To begin, it is natural to ask for characterizations of the multiplicative functions $K : I \to \mathbb{R}$ that are also $C^n$ or smooth. The following result is an immediate consequence of Theorem 2

Corollary 10. Suppose $\tilde{I} \subset \mathbb{R}$ is an interval containing 1 as an interior point. Given $K : \tilde{I} \to \mathbb{R}$ and an integer $n \geq 0$, the following are equivalent:

1. $K$ is multiplicative on $\tilde{I}$ and $n$ times differentiable on $I$.
2. $K$ is multiplicative on $\tilde{I}$ and $C^n$ on $I$.
3. Either $K \equiv 0$ or $K \equiv 1$ on $\tilde{I}$, or $K(x) = x^\alpha$ for some integer $\alpha \in (0,n]$, or $K = \phi_\alpha$ or $\psi_\alpha$ for some $\alpha > n$.

We now show that the other two Cauchy functional equations (2.1) can be solved using the aforementioned classifications of all additive and multiplicative measurable maps.

Theorem 11. Let $I \subset \tilde{I} \subset \mathbb{R}$ be intervals containing 0 as an interior point. Then the following are equivalent for $K : \tilde{I} \to \mathbb{R}$.

1. $K$ is Lebesgue measurable on $\tilde{I}$ and satisfies: $K(x+y) = K(x)K(y)$ whenever $x,y,x+y \in \tilde{I}$.
2. $K$ is continuous on $\tilde{I}$ and satisfies: $K(x+y) = K(x)K(y)$ whenever $x,y,x+y \in \tilde{I}$.
3. Either $K \equiv 0$ on $\tilde{I}$, or $K(x) = \exp(\beta x)$ for some $\beta \in \mathbb{R}$ and all $x \in \tilde{I}$.

Suppose instead that $I \subset \tilde{I} \subset \mathbb{R}$ are intervals containing 1 as an interior point. Then the following are equivalent for $K : \tilde{I} \to \mathbb{R}$.

1. $K$ is Lebesgue measurable on $\tilde{I}$ and satisfies: $K(xy) = K(x) + K(y)$ whenever $x,y,xy \in \tilde{I}$.
2. $K$ is continuous on $\tilde{I}$ and satisfies: $K(xy) = K(x) + K(y)$ whenever $x,y,xy \in \tilde{I}$.
3. Either $0 \in \tilde{I}$ and $K \equiv 0$ on $\tilde{I}$, or $0 \not\in \tilde{I}$ and $K(x) = \beta \ln(x)$ for some $\beta \in \mathbb{R}$ and all $x \in \tilde{I}$.

As in Theorem 2, one can replace the continuity assumption in either condition (2) by other constraints, such as $K$ being Borel measurable, monotone, differentiable, $C^n$ for some $n$, or smooth on $I$.

Proof. $K(x+y) = K(x)K(y) :$ For the first set of equivalences, clearly (3) $\implies$ (2) $\implies$ (1). We now assume (1) and first show that if $K(x) = 0$ for some $x \in \tilde{I}$, then $K \equiv 0$ on $I$. Indeed, if $K(x) = 0$, then $K(x/n)^n = K(x) = 0$ for all $n \in \mathbb{N}$. Now if $n$ is large enough, then $\pm x/n \in \tilde{I}$, whence $K(0) = K(x/n)K(-x/n) = 0$. But then $K(x) = K(x+0) = K(x)K(0) = 0$ for all $x \in \tilde{I}$, and $K \equiv 0$. 


Now assume that \( K \not\equiv 0 \) on \( \tilde{I} \); then \( K \) never vanishes on \( \tilde{I} \). Moreover, given \( x \in \tilde{I} \), \( K(x) = K(x/2)^2 \geq 0 \), so it must be positive. Then \( g(x) := \ln K(x) : \tilde{I} \to \mathbb{R} \) is additive on \( \tilde{I} \) and Lebesgue measurable on \( I \). Hence by Theorem \( \Box \) \( g(x) = \beta x \) for some \( \beta \in \mathbb{R} \), whence \( K(x) = e^{g(x)} = e^{\beta x} \) for some \( \beta \in \mathbb{R} \). This shows (3) as desired.

\[
K(xy) = K(x) + K(y) : \text{For the second set of equivalences, once again (3) } \implies (2) \implies (1).
\]

Now if \( K \) satisfies (1) and \( 0 \in \tilde{I} \), then for all \( x \in \tilde{I} \),

\[
K(0) = K(x \cdot 0) = K(x) + K(0) \implies K(x) \equiv 0 \forall x \in \tilde{I}.
\]

Otherwise suppose \( 0 \not\in \tilde{I} \); then \( K_1(x) := e^{K(x)} : \tilde{I} \to \mathbb{R} \) is multiplicative and positive on \( \tilde{I} \) and Lebesgue measurable on \( I \). Hence by Corollary \( \Box \) \( K_1 \equiv 1 \) (whence \( K \equiv 0 \)) or \( K_1(x) = x^\beta \) is positive on \( I \), in which case \( K(x) = \beta \ln(x) \) for \( x \in \tilde{I} \subset (0, \infty) \). This shows (3) and concludes the proof. \( \Box \)

**Concluding remarks.** The main result of this paper characterizes functions \( K \) mapping \( P_n^k(\tilde{I}) \) into \( \mathbb{P}_n(K) \) under weak measurability assumptions. A natural question that now arises is to classify entrywise functions mapping \( P_n^k(\tilde{I}) \) into \( \mathbb{P}_n(K) \) for \( 1 \leq k, l \leq n \) under suitable assumptions. These maps will be explored in detail in future work \[13\].

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