QCD coherence in the structure function
and associated distributions at small $x$ \footnote{Research supported in part by the Italian MURST and the EC contract CHRX-CT93-0357}

Giuseppe Marchesini
Dipartimento di Fisica, Università di Milano,
INFN, Sezione di Milano, Italy

Abstract

We recall the origin of angular ordering of soft parton emission in the region of small $x$ and show that this coherent structure can be detected in associated distributions. For structure functions at small $x$ and at fixed transverse momentum the angular ordering is masked because of the complete inclusive cancellations of collinear singularities for $x \to 0$. Therefore, in this case the dependence on the hard scale is lost and the angular ordered region becomes equivalent to multi-Regge region in which all transverse momenta are of the same order. In this limit one derives the BFKL equation. In general such a complete cancellation does not hold for the associated distributions at small $x$. The calculation, which requires an analysis without any collinear approximation, is done by extending to small $x$ the soft gluon factorization techniques largely uses in the region of large $x$. Since one finds angular ordering in the both regions of small and large $x$, one can formulate a unified evolution equation for the structure function, a unified coherent branching and jet algorithm which allows the calculation of associated distributions in all $x$ regions. Such a unified formulation valid for all $x$ is presented and compared with usual treatments. In particular we show that the associated distributions at small $x$ are sensitive to coherence. By replacing angular ordering with the multi-Regge region one neglects large singular contributions in the associated distributions.
1 Introduction

Soft gluons emission in perturbative QCD takes place into angular ordered regions [1]-[4]. Since this coherent emission factorizes, one can extend the branching process, which sums collinear singularities to leading (and next-to-leading) order, to include angular ordering. This coherent branching process allows one to compute parton distributions either by numerical methods via Monte Carlo simulations or by analytical methods via “jet calculus” algorithms [1, 5]. Experiments are well compatible with coherence and for a recent review see Ref. [6]. An important case in which soft gluons are involved is deep inelastic scattering (DIS) in the region of large $x$ with $x = Q^2/2pq$ and $Q^2 = -q^2$ (see fig. 1). Coherence in the structure function $F(x, Q)$ for $x \to 1$ can be taken into account in the Altarelli, Parisi, Gribov, Lipatov, Dokshitzer (AP) equation [7] by using as evolution variable the angle of emitted partons. Coherent branching at large $x$ reproduces [8] the leading and next-to-leading contribution of the anomalous dimension $\gamma_\omega(\alpha_S)$

$$\gamma_\omega(\alpha_S) = Q^2 \frac{\partial \ln F_\omega(Q)}{\partial Q^2}, \quad F_\omega(Q) = \int_0^1 dx x^\omega F(x, Q),$$

(1)

in the limit $\omega \to \infty$ which corresponds to the region of large $x$.

The small $x$ region in DIS appears quite different from the one of large $x$ (or finite $x$ in general). For $x \to 0$ one focus the attention on the soft exchanged gluons $k_1, \ldots, k_r$ rather than on the emitted ones $p_1, \ldots, p_r$. The basis for the analysis of the small $x$ structure function is the Balitskii, Fadin, Kuraev and Lipatov (BFKL) equation [9] which was found almost two decades ago in the field of soft physics to discuss the rise of the total cross section. The dominant phase space region considered is the so called ”multi-Regge” region in which subenergies are large and all emitted transverse momenta are of the same order of magnitude. In general, if the ratio of transverse momenta vanishes, one has collinear singularities. However for fully inclusive distributions all these collinear singularities completely cancel, so that the regions of very different transverse momenta are not particularly important.

Later it was realized [9, 10] that this equation could be extended to describe the structure function for $x \to 0$. This equation involves the unintegrated structure function $F(x, k^2)$ related to the structure function by

$$F_\omega(Q) = \int d^2k F_\omega(k^2) \Theta(Q - k), \quad k \equiv |k|,$$

(2)

where $k$ is the total transverse momentum of emitted gluons, $k_r = \sum p_{ti}$. For $\omega \to 0$ one probes the small $x$ region and in this limit the $k$-structure function satisfies the BFKL equation

$$F_\omega(k) = \delta^2(k) + \frac{\bar{\alpha}_S}{\omega} \int \frac{d^2p}{\pi p_t^2} \left[ F_\omega(k + p_t) - \Theta(k - p_t) F_\omega(k) \right], \quad \bar{\alpha}_S = \frac{C_A \alpha_S}{\pi},$$

(3)

with $C_A = 3$. The corresponding $\omega \to 0$ anomalous dimension is given by an implicit equation and has an expansion in $\alpha_S^k/\omega^k$. The first six terms are

$$\gamma_\omega(\alpha_S) = \frac{\bar{\alpha}_S}{\omega} + 2\zeta(3) \frac{\bar{\alpha}_S^4}{\omega^4} + 2\zeta(5) \frac{\bar{\alpha}_S^6}{\omega^6} + \cdots,$$

(4)
with \( \zeta(3) = 1.202 \ldots \) and \( \zeta(5) = 1.037 \ldots \) the Riemann functions. For the expansion up to 14 terms see [3]. As well known \( \gamma_\omega(\alpha_S) \) has a singularity at \( \omega = \omega^* = 4 \ln 2 \alpha_S \) which implies that the structure function rises asymptotically at small \( x \). The two terms in the kernel of (3) have a simple interpretation. The first is positive and is the contribution for the emission of a gluon of transverse momentum \( p_t \). Thus, before the emission, the \( k \)-structure function has momentum \( k + p_t \). The second negative contribution in the kernel corresponds to the virtual correction in which the virtual transverse momenta \( p_t \) is bounded by the total emitted transverse momentum \( k \). There are two related features which make the BFKL equation apparently very different from the AP equation. The first is that there is no ordering \( p \) in the real emission and virtual momenta. The second is that the collinear singularities for \( p_t \to 0 \) present in the real and virtual contributions cancel completely. This last feature can be further understood if we consider the exclusive representation of the \( k \)-structure function in eq. (3). To this end we need to introduce a cutoff \( \mu \) both for the real and the virtual contribution and the BFKL equation becomes equivalent to the exclusive representation (see the kinematical diagram in Fig. 2)

\[
\mathcal{F}_\omega(k) = \delta^2(k) + \sum_{r=1}^{n-1} \int_{Q^2} d^2p_t \prod_{i=1}^r \frac{d^2p_i}{z_i^2} \Delta_R(z_i, k_i) \delta^2(k - k_r), \quad k_f = p_{f\ell} + k_{\ell-1}
\]

\[
\Delta_R(z_i, k_i) = \exp \left\{ -\alpha_S \int \frac{dz}{z} \int \frac{k_i^2}{\mu^2} \frac{dp_i^2}{p_i^2} \right\} = \exp \left\{ -\alpha_S \ln \frac{1}{z_i} \ln \frac{k_i^2}{\mu^2} \Theta(k_i - \mu) \right\}, \quad k_i = |k_i|.
\]

The function \( \Delta_R(z_i, k_i) \) sums all virtual contributions and corresponds to the so-called gluon “Regge trajectory”. In this exclusive representation we have included, together with the collinear cutoff \( \mu \), also the upper limit for the emitted transverse momenta given by the hard scale \( Q \) (see (2)). Due to the complete cancellation of collinear singularities at the fully inclusive level and to the scale invariance of the measure \( d^2p_t/\mu_p^2 \), the \( \mu \) and \( Q \) dependence in the \( k \)-structure function (3) enters only as higher twist corrections, i.e. as powers of \( \mu^2/Q^2 \) and for \( \mu^2/Q^2 \to 0 \) the representation (3) is equivalent to the BFKL equation (3). Since \( Q \) enters only to higher twists one has that for \( x \to 0 \) the hard scale is lost to the present level of accuracy. In DIS for \( x \to 0 \) the hard scale \( Q \) is recovered only through the phase space boundary in the \( k \) dependence of the coefficient function via the high energy (or \( k \)) factorization (see [11]). On the other hand to the present accuracy the hard scale for the argument of \( \alpha_S \) can not be identified. Since only the coefficients of \( \alpha_S^k \omega^k \) are known, a scale change between \( Q \) and \( Q' \)

\[
\frac{\alpha_S(Q')}{\omega} = \frac{\alpha_S(Q)}{\omega} + \frac{\alpha_S^2(Q)}{\omega} 2b \ln \left( \frac{Q}{Q'} \right) + \cdots, \quad b = (11C_A - 2n_f)/12\pi
\]

can not be discriminated. The control of the hard scale can be obtained by the calculation of subleading corrections \( \alpha_S \alpha_S^k \omega^k \) in the anomalous dimension. The first steps in this direction are reported in Ref. [12].

From the above discussion it seems that the dynamical features of the two DIS regions of small and large \( x \) is quite different. However it was shown that the two regions have the same structure of coherence. The calculation of the structure function for \( x \to 0 \) was done [2]-[4] in the framework of hard processes thus considering not only the soft singularities for \( x \to 0 \) but also the collinear singularities when the ration of two emitted transverse momenta vanishes. This region is beyond the multi-Regge region. The result was the discovery of coherence.
for \( x \to 0 \). One finds that for \( x \to 0 \) initial state soft gluons are emitted into the angular ordered region as in the case of large \( x \). In this way the description of DIS is unique for all region of \( x \).

Notice the accuracy required for this calculation. Since in fully inclusive distributions all collinear singularities cancel for \( x \to 0 \), one needs to perform a calculation without making any collinear approximation. In the light-cone expansion infinite leading twists operators contribute and one has to make a calculation to all loop accuracy. Actually, by using the soft gluon factorization techniques for \( x \to 0 \), it was shown \cite{4} that it is possible to attain such an accuracy by appropriate use of soft gluon coherence and of cancellation of soft singularities between real-virtual contributions. One can use soft gluon techniques also for \( x \to 0 \) because this region is dominated by soft exchanged gluons \( k, \ldots, k_r \) and this implies that, a part for the first fast gluon \( p_1 \), all remaining emitted gluons are soft.

Many data are now available from Hera in the small \( x \) region and then the analysis of coherence at small \( x \) can be attempted. To help this analysis, in this paper we present a systematic account of coherence in all \( x \) regions. In particular, by using the coherent branching process discussed above we show how computes the associated distributions via a generalization of jet calculus \cite{1, 5}. This allows one to study quantitatively the rôle of coherence for \( x \to 0 \) and to single out the differences between the present formulation with coherence and the exclusive representation in \cite{3} or the usual AP evolution equation naively extended to small \( x \).

The result of this analysis in the case of small \( x \) \cite{2, 3} can be summarized in terms of very simple extension of the exclusive representation of the BFKL \( k \)-structure function in \cite{2}. It involves the following two modifications:

i) the emission takes place in the angular ordered region. When all exchanged gluons are soft angular ordering can be expressed by including a factor \( \Theta(p_{ti} - z_{i-1}p_{t_{i-1}}) \) in the phase space for each emission in \cite{3}. The collinear cutoff is needed only for the first emission with \( i = 1 \), i.e. \( p_{ti} > z_0p_{t0} = \mu \) (this cutoff is not needed for the \( k \)-structure function);

ii) also the virtual corrections involve angular ordering. The form factors \( \Delta_R(z_i, k_i) \) in \cite{3} are modified by substituting the cutoff \( p_t > \mu \) by the angular ordered constraint \( p_t > zp_{ti} \).

On the basis of this branching structure one deduces \cite{2, 3} an evolution equation for the \( k \)-structure function at small \( x \) which involves as evolution an angular variable. From this equation one observes that there is a complete cancellation of the collinear singularities. In this case angular ordering becomes equivalent to the multi-Regge region \cite{3}. On the contrary we show that in the associated distributions, such as the associated multiplicity, the collinear singularities do not cancel. In this case one can not use the multi-Regge region in \cite{3} but must take into account angular ordering and the above modifications.

The fact that both for small and large \( x \) parton emission and virtual corrections factorizes and have the same structure of coherence, i.e. angular ordering, allows one to deduce a unified branching process for all \( x \) regions. This has been used to construct a Monte Carlo program \cite{13} including coherence in the small \( x \) region which has been used to study the differences with the one loop formulation and to compute numerically DIS and heavy flavour production processes. Moreover one can write \cite{14} a unified evolution equation for the \( k \)-
structure function which extends the AP equation to include the coherence at small \( x \) \cite{2,3}. We show that this equation, which \( x \to 0 \) reduces to the BFKL equation, can be used as the bases for the jet calculus to compute the associated distributions.

In Sec. 2 we recall the results of Refs. \cite{2,3,4} of the DIS analysis for \( x \to 0 \) together with \( x \to 1 \). We describe the accuracy of the approximations in the various region of \( x \). In Sect. 4 we discuss the branching process with coherence for all \( x \) and in Sect. 5 we recall the unified evolution equation \cite{14} for the k-structure function valid in all \( x \) regions. In Se. 6 we discuss the small \( x \) region. At the fully inclusive level we show that from the evolution equation with coherence one deduces the BFKL equation valid in the multi-Regge region. At the semi-inclusive level (associated distributions) we show that the present formulation with coherence can not be approximated by using only the multi-Regge region as in the BFKL equation. In Sect. 7 we describe the extension of jet calculus to include the coherence at small \( x \) and compare with a formulation in the multi-Regge region. Finally Sect. 8 contains some conclusion.

2 Derivation of recurrence relation and solution

We consider deep inelastic scattering represented at parton level in Fig. 1 where \( q \) is the hard colour singlet probe and \( p' \) represents the recoiling system of partons, such as, for instance, a pair of heavy quark-antiquark. For simplicity we assume that the incoming \( p \) and the outgoing partons \( p_1 \ldots p_r \) are gluons. We take \( p = (E,0,0,E) \) and introduce a light-like vector \( \vec{p} = (E,0,0-E) \) in the opposite \( z \)-direction. For the emission momenta we use the Sudakov parametrization

\[
p_i = y_i p + \bar{y}_i \bar{p} + p_\mu, \quad \xi_i \equiv \frac{\bar{y}_i}{y_i} = \frac{p_\mu^2}{s y_i^2}, \quad s = 2p\bar{p},
\]

where \( \ln \xi_i \) is the (pseudo-)rapidity. For the exchanged momentum one has \( k = xp + \bar{x}\bar{p} + k_i \). We assume the frame \( q = \bar{p} - xp \) with \( x = Q^2/2pq \) and consider \( Q^2 = -q^2 \) large. In this limit we are interested in the region where all \( \bar{y}_i \) are small.

The cross section of this process factorizes into the elementary hard scattering distribution and the structure function. For \( x \to 0 \) this factorization can be generalized \cite{11} to include higher order corrections by introducing the structure function at fixed \( k \). To leading twist, in the gauge \( \eta = \vec{p} \equiv q + xp \sim p' \), the amplitude of Fig. 1 can be factorized \cite{3,4} as in Fig. 3 into the elementary vertex \( V^\mu(q,k) \) and the multi-gluon emission subamplitude \( m^\mu(p,k;p_1,\ldots,p_r) \) which is a matrix in the colour indices of gluons \( p, p_1, \ldots, p_r \) and \( k \). In this gauge, for \( x \to 0 \), the elementary vertex \( V^\mu \), is proportional to \( \bar{p}^\mu \) and one defines the multi-gluon subamplitudes \( M_r(p_1,\ldots,p_r) = (p^\mu/2p\bar{p}) m^\mu_r(p,k;p_1,\ldots, p_r) \). One introduces the structure function at fixed total transverse momentum

\[
k^2 \tilde{F}(x,k) = \frac{1}{x} \sum_r \frac{1}{r!} \int \prod_{i=1}^r (dp_i) | M_r(p_1,\ldots,p_r)|^2 \delta(1-x-Y_r)\delta^2(k-k_r),
\]

\[
(dp) = \frac{d^3p}{(2\pi)^32\omega}; \quad Y_r = \sum_{i=1}^r y_i; \quad k_r = -\sum_{i=1}^r p_\mu^i,
\]
with $|M_r(p_1 \ldots p_r)|^2$ the spin and colour average of the multi-gluon squared subamplitude. The structure function is obtained by integrating over $k$

$$F(x, Q) = \int d^2k, \quad \mathcal{F}(x, k, Q), \quad \mathcal{F}(x, k, Q) \equiv \Theta(Q - k) \tilde{F}(x, k), \quad k \equiv |k|. \quad (8)$$

By integrating over $y_i$ in (7) one finds infrared (IR) divergences which are cancelled by virtual contributions. These cancellations are most easily seen in old fashion perturbation theory in which virtual gluons are on-shell. To order $\alpha_s^3$, the distribution $|M_r|^2$ is given by integrals over $\ell = n - r$ on-shell virtual momenta $v_1 \cdots v_\ell$. We introduce the on-shell momenta $\{q_1, \ldots, q_n\}$ which includes both the $r$ real $\{p_1, \ldots, p_r\}$ and the $\ell = n - r$ on-shell virtual $\{v_1, \ldots, v_\ell\}$ momenta and we use the Sudakov parametrization $q_i = y_i p + \tilde{y}_i \bar{p} + q_{\ell i}$ with $\tilde{y}_i = q_{\ell i}^2/y_i s$. One has to consider contributions with various ordering in the $y_i$ variables of the on-shell real and virtual momenta and for a given contribution we denote by $f_{r, \ell}(p_1 \ldots p_r, v_1 \ldots v_\ell)$ the corresponding integrand in which the spin and colour sum is already performed. We then introduce the index $R_n$ which identifies the real $\{p_i\}$ or virtual $\{v_i\}$ momenta in the set $\{q_i\}$ and denote by $f_{R_n}(q_1 \ldots q_n)$ the corresponding integrand. We then have the following representation of the structure function.

$$x \mathcal{F}(x, k, Q) = \sum_{n=1}^{\infty} \frac{2^n!}{n!} k_{\bar{r}} Y_n \sum_{R_n} f_{R_n}(q_1 \ldots q_n)$$

$$\times \Theta(1 - x - Y_{R_n}) \delta^2(k - k_{R_n}), \quad \Theta^Y_{1 \ldots n} \equiv \theta(y_1 - y_2) \cdots \theta(y_{n-1} - y_n), \quad (9)$$

where $Y_{R_n} = Y_r$ and $k_{R_n} = k_r$ are the contributions from the real gluons and are defined in (4). Obviously the integrand $f_{R_n}(q_1 \ldots q_n)$ depends on the specific index $R_n$. The bound $Q > q_{\ell i}$ is coming from $Q^2 > k^2$ in (8) and the ordering of the $y_i$ variables is coming both from the statistical factor $1/r!$ for real momenta and from all possible ordering of the virtual on-shell momenta.

We need now to compute $f_{R_n}(q_1 \ldots q_n)$. We will do this in the strongly $y$-ordered region

$$y_n \ll \cdots \ll y_1, \quad (10)$$

in which we can apply the soft gluon techniques and factorize the contribution of the softest gluon, either real or virtual. This region gives the leading contribution both for $x \to 0$ and for $x \to 1$. The fact that for $x \to 1$ the strongly ordered phase space gives the dominant contribution is well known [1]. The fact that the ordering (10) gives also the dominant contribution for $x \to 0$ is due to the infrared regularities of the inclusive distributions. Since in (9) the various singular contributions for $y_i \to 0$ cancel, the integration regions $y_i < x$ give a contribution to the structure function of order $x$ which can be neglected for $x \to 0$. On the other hand in the remaining regions $y_i > x$ the real-virtual cancellations are inhibited by the presence of $\delta(1 - x - Y_{R_n})$ in the integrand. Therefore $x$ plays the role of an infrared cutoff and logarithmic contributions in $x$ are generated by the lower bound in the $y_i$ integrations.

In the remaining part of this section we shall recall [2, 3, 4] that by factorizing the softest (real or virtual) gluon $q_n$ one deduces the following recurrence relation connecting $f_{R_n}$ and
\[ f_{R_{n-1}} \]

\[
\sum_{R_n} f_{R_n}(q_1 \ldots q_n) g_5^2 (dq_n) \delta(1 - x - Y_{R_n}) \delta^2(k - k_{R_n}) \simeq \sum_{R_{n-1}} f_{R_{n-1}}(q_1 \ldots q_{n-1}) \bar{\alpha}_S \frac{dy_n d^2 q_n}{y_n \pi q_n^2} \\
\times \left\{ \delta(1 - x - Y_{R_{n-1}} - y_n) \delta^2(k - k_{R_{n-1}} + q_n) - \delta(1 - x - Y_{R_{n-1}}) \delta^2(k - k_{R_{n-1}}) [1 - \Theta(y_n - x)\Theta(q_n - k)] \right\}. \tag{11} \]

As we shall recall, this recurrence relation is valid in the infrared limit (i.e. \( y_n \ll y_i \)) for both \( x \to 0 \) (without collinear approximation) and \( x \to 1 \) (in the collinear approximation). The first contribution in the curly bracket corresponds to the case in which the softest gluon is emitted, while the second corresponds to the case in which \( q_n \) is a virtual gluon. Consequently, the momentum of the softest gluon \( q_n \) enters in the the \( \delta \)-function only for the first contribution. This recurrence equation gives rise to an integral equation for the \( k \)-structure function. One needs to fix the total transverse momentum \( k \) because this variable enters in the coefficient of the \( \delta \)-function of the last term of (11).

### 2.1 Recurrence relation for \( f_R \)

We summarize here how to obtain from the soft gluon factorization the recurrence relation (11) and discuss the relative accuracy. First we recall the case of large \( x \) and then we discuss the case of \( x \) both small and large.

**Large \( x \)**. As well known [1], the factorization of the softest (real or virtual) gluon \( q_n \) in this region gives the colour matrix factor

\[
- \left[ J_{R_{n-1}}(q_n) \right]^2 \left\{ \delta(1 - x - Y_{R_{n-1}} - y_n) \delta^2(k - k_{R_{n-1}} + q_n) - \delta(1 - x - Y_{R_{n-1}}) \delta^2(k - k_{R_{n-1}}) \right\}. \tag{12} \]

This result is valid in the soft approximation \( (y_n \ll y_i) \) and no collinear approximations are needed. In the first contribution \( q_n \) is the real emitted gluon thus it contributes to the \( \delta \)-functions in (11). In the second contribution the gluon \( q_n \) is virtual and does not contribute to the \( \delta \)-functions. The soft emission factor is the squared of the eikonal current

\[
J_{R_{n-1}}(q_n) = - \frac{p}{p q_n} T_p + \sum_{i \in R_{n-1}} \frac{q_i}{q_i q_n} T_{q_i} + \frac{p'}{p' q_n} T_{p'}, \quad T_{p'} = T_p - \sum_{i \in R_{n-1}} T_{p_i}, \tag{13} \]

with \( T_{q_i} \) and \( T_{p} \) the colour matrices of external gluons, and \( T_{p'} \) the colour matrix of the recoiling system \( p' \). The emission factor is universal since it depends only on the fastest charges, therefore it does not depends on particular Feynman graph contributions. Moreover the factor is the same for \( q_n \) real and virtual.

The current \( J \) is a matrix in the space of \( \{ p, q_1, \ldots, q_{n-1}, p' \} \) colour indices and in the use of (13) the difficulty is the colour algebra. Since the integrand \( f_{R_n} \) is given by the average over the colour charges, the factorized expression in (12) gives a recurrence relation for \( f_{R_n} \).
only if the colour factor in (12) is approximately proportional to the unit matrix. In this case of large $x$ this is true only in the collinear limit in which the emitted gluons $q_i$ are taken parallel to the incoming one, i.e. $q_i \simeq y_i p$. By using colour conservation in (13) and $T^2_p = C_A = N_C$, in this limit one has

$$- [J_{R_{n-1}}(q_n)]^2 \simeq -C_A \left( \frac{p}{pq_n} - \frac{p'}{p'q_n} \right)^2 \simeq \frac{4C_A}{q^2}. \quad (14)$$

For large $x$ the coherent emission is obtained in this approximation to the leading collinear accuracy. Therefore the analysis in this case is to the leading IR and collinear accuracy.

**Small and large $x$.** This analysis can be extended to include the region of small $x$. Here the analysis seems much more difficult since one can not make any collinear approximation. Moreover soft gluon factorization techniques are typically used in the case in which the softest gluon $q_n$ is softer than all other external $q_i$ and internal lines. In the $x \to 0$ case instead we have that the exchanged gluon $k_n$ could be softer than $q_n$. Actually the soft gluon technique has been generalized to the $x \to 0$ case in Ref. [2, 3] and one finds that the contribution of the gluon $q_n$ can be factorized in $f_{R_n}$ without the need of any collinear approximation [4]. There are three factorized contributions:

i) the softest gluon $q_n$ is a real emitted one. One can generalize the result (12) also to the small $x$ case and factorize the colour matrix $- [J_{R_{n-1}}(q_n)]^2$. It is surprising that also for $x \to 0$ the leading IR contribution for the real emission is given by the eikonal current involving only the external partons. Actually, as shown in [4], for $x \to 0$ the contribution of the emission of $q_n$ from the softest exchanged line $k_n$ is important. This contribution compensate exactly the rescaling $k^2_n/(k_n - q_n)^2$ for the exchanged propagator and, for $x \to 0$, the emission is given by the squared eikonal current (12) without any collinear approximation;

ii) the softest $q_n$ is a virtual on-shell gluon not connected to the exchanged gluon $k$. In this case we have again the squared eikonal current factor in (12);

iii) the softest $q_n$ is a virtual on-shell gluon connected to the exchanged gluon $k$. This factor, which is singular only for $x \ll y_n$, is positive and proportional to the unit colour matrix. We have

$$\frac{4C_A}{q^2_n} \frac{(k + q_n)q_n}{(k + q_n)^2} = \frac{4C_A}{q^2_n} \Theta(q^2_n - k^2) \quad (15)$$

where the last expression is obtained upon azimuthal integration.

It is an important consequence of coherence in the limit $x \to 0$ that if one takes together the first two contributions the colour algebra becomes trivial as in (14) but without any collinear approximation. This can be shown by observing first that in the strongly ordered region (10) and for $x \to 0$ one has $y_1 \to 1$. Thus in this limit the $\delta$ functions in the $y$ variables are given by $\delta(1 - y_1)$ both in the real and in the virtual contribution. Therefore, apart for this common $\delta(1 - y_1)$ function, the sum of the real and the first virtual contribution are given by

$$[J_{R_{n-1}}(q_n)]^2 \left\{ \delta^2(k - k_{R_{n-1}} + q_n) - \delta^2(k - k_{R_{n-1}}) \right\} \simeq \frac{4C_A}{q^2_n} \left\{ \delta^2(k - k_{R_{n-1}} + q_n) - \delta^2(k - k_{R_{n-1}}) \right\}, \quad (16)$$
as in the collinear limit. However this expression is valid in the limit \( y_n \ll y_i \) without requiring any collinear approximation. To show this observe that for \( y_n \ll y_i \), the difference of the two delta functions gives a non-vanishing contribution only in the collinear region with \( \theta_i \sim y_n/y_i \theta_n \). Thus for \( y_n/y_i \to 0 \) we can use the approximate expression (14) for the squared eikonal current. Therefore it is the coherence of the soft radiation for \( x \to 0 \) which allows us to compute the eikonal current in the collinear limit. For finite \( x \) one cannot neglect the differences in the arguments of the \( \delta \) function in \( y_i \) for the real and the virtual contributions and, in this case, the approximation (14) is valid only if one requires the collinear limit.

In conclusion from the factorization results in (15) and (16) we obtain in the soft limit \( y_n \ll y_i \) the recurrence relation in (11) which is valid both for \( x \to 0 \) (without collinear approximation) and for \( x \to 1 \) (in the collinear approximation) The three \( \delta \)-function contributions in the curly bracket of (11) have the following origin: the first corresponds to the real emission; the second and third correspond to the eikonal and non-eikonal virtual correction respectively. The new feature of the last contribution, which is relevant only for \( x \to 0 \), is that it involves the total transverse momentum. This implies that the kernel and then the solution of the recurrence equation must involve \( k \). This is in agreement with the fact that for small \( x \) one introduces (11) the structure function at fixed \( k \).

### 2.2 General solution of the recurrence relation

From the recurrence relation (11) one obtains the multi-gluon distributions \( |M_r|^2 \) in (7). The virtual corrections exponentiate and can be summed to give (see appendix A)

\[
\frac{1}{k^2} \prod_{i=1}^r (dp_i) \ |M_r(p_1, \ldots, p_r)|^2 \Theta_{1 \ldots r}^{Y_{1 \ldots r}} \simeq \tilde{\alpha}_S^r \prod_{i=1}^r \frac{dy_i}{y_i} \frac{d^2 p_{it}}{\pi p_{it}^2} \ V_r(p_1, \ldots, p_r) \Theta_{1 \ldots r}^{Y_{1 \ldots r}}, \tag{17}
\]

where the virtual corrections are given by

\[
V_r(p_1p_2 \ldots p_r) = \exp \left\{-\tilde{\alpha}_S \int_1^1 \frac{dy}{y} \int \frac{Q^2}{q^2} dq^2 \right\} \ \prod_{i=1}^r \exp \left\{\tilde{\alpha}_S \int_{x_i}^{x_{i-1}} \frac{dy}{y} \int \frac{Q^2}{k_i^2} dq^2 \right\}, \tag{18}
\]

with

\[
k_i = k_{i-1} - p_{it}, \quad x_i = x_{i-1} - y_i, \quad k = k_r, \quad x = x_r .
\]

The first exponential in \( V_r \) is the eikonal form factor which sums all virtual corrections corresponding to the second term in the curly bracket of (11). Notice that for \( y \to 0 \) this integral diverges. Similarly all real integrations diverge for \( y_i \to 0 \). These IR singularities cancel in inclusive quantities such as the structure function. However, if we want to keep separately real and virtual contributions, one should regularize these IR singularities by a momentum transverse cutoff. This will be carefully done later.

The remaining factor in (18) is given by the form factors which sum the contribution from the last term in the curly bracket of (11), corresponding to the non-eikonal virtual corrections. These form factors give singular contributions only in the case of soft exchange \( (x_i \ll x_{i-1}) \), i.e. for \( x \to 0 \).
3 Coherence and angular ordering

The expression of the squared amplitudes \( |M|^2 \) in (17) is given in the strongly \( y \)-ordered region (10). In order to obtain an evolution equation one has to exchange \( y \)-ordering with some other ordered variable involving the transverse momentum or angle of emitted gluon. The identification of this variable allows us to exhibit the coherence properties of these distributions. One can show that within the leading IR accuracy one can replace in (17) the ordering in \( y \) with the ordering in the angles between the incoming gluon and the emitted ones, i.e. the ordering in \( \xi \) (see Appendix B). We have

\[
\prod_{i=1}^{r} dy_i \frac{d^2 p_{it}}{\pi p_{it}^2} V_r(p_1 p_2 \ldots p_r) \Theta_{1 \ldots r}^{Y} \Rightarrow \prod_{i=1}^{r} dy_i \frac{d^2 p_{it}}{\pi p_{it}^2} V_r(p_1 p_2 \ldots p_r) \Theta_{r \ldots 1}^{\xi},
\]

where

\[
\Theta_{r \ldots 1}^{\xi} \equiv \Theta(\xi_r - \xi_{r-1}) \ldots \Theta(\xi_2 - \xi_1),
\]

so that the gluons \( \{p_1 \ldots p_r\} \) are now emitted in the angular ordered phase space region

\[
\xi_1 < \ldots < \xi_r.
\]

Since the \( y \)-variables are not any more ordered we introduce the usual \( z_i \) variables

\[
y_i = (1 - z_i)x_{i-1}, \quad x_i = z_1 x_{i-1}, \quad x = x_r = z_1 \cdots z_r,
\]

so that the \( y \)-integration factors in (17) can be written

\[
\bar{\alpha}_s \frac{r}{x} \prod_{i=1}^{r} dy_i \delta(1 - x - \sum_{i=1}^{r} y_i) = \prod_{i=1}^{r} dz_i \\bar{\alpha}_s \left\{ \frac{1}{1 - z_i} + \frac{1}{z_i} \right\} \delta(x - z_1 \cdots z_r).
\]

We recognize the singular parts of the gluon splitting function. For \( z_i \to 1 \) we have soft emission and for \( z_i \to 0 \) soft exchange which is relevant for \( x \to 0 \). The finite contribution to the gluon splitting function \( -2 + z_i(1 - z_i) \) can be included. They can not be obtained within the soft gluon techniques.

The IR singularities \( z_i \to 1 \) present in the real emission distribution (21) are cancelled, at the inclusive level, by the virtual corrections in \( V_r \). If we want to keep separately the real and virtual contributions we need to regularize these IR singularity by introducing a cutoff \( Q_0 \) in the transverse momenta. As usual we introduce the rescaled transverse momenta

\[
q_i \equiv \frac{p_{ti}}{1 - z_i} = x_{i-1} \sqrt{s \xi_i},
\]

so that the angular ordered region (20) becomes

\[
\xi_i > \xi_{i-1}, \quad \xi_1 > \xi_0 \quad \Rightarrow \quad q_i > z_{i-1} q_{i-1}, \quad q_1 > \mu,
\]

where we have introduced also the collinear cutoff \( \xi_0 \) or \( \mu = \sqrt{s \xi_0} \). The IR singularities in (21) for \( y_i \to 0 \) (i.e. \( z_i \to 1 \)) are regularized by limiting the emitted transverse momentum, \( p_{ti} > Q_0 \), giving

\[
1 - z_i > Q_0/q_i.
\]
In terms of these variables, the virtual corrections in (13) can be written, to leading IR order, in the form (see Appendix C)

\[ V_r(p_1 p_2 \ldots p_r) \simeq \Delta S(Q, z_r q_r) \prod_{i=1}^{r} \Delta S(q_i, z_{i-1} q_{i-1}) \Delta(z_i, q_i, k_i) , \]  

(25)

where \( \Delta_S \) is the usual gluon Sudakov form factor in the double logarithmic approximation

\[ \Delta_S(q_i, z_{i-1} q_{i-1}) = \exp \left\{ - \int_0^{q_i^2} \frac{dq^2}{q^2} \int_0^{1-Q_0/q} dz \frac{\bar{\alpha}_S}{1-z} \right\} . \]  

(26)

Notice that the \( q \) integration region \( \bar{q} > z_{i-1} q_{i-1} \) corresponds to the angular ordering constraint (23). Here the regularization of IR singularity \( (1-z) > Q_0/q \) is the same as in the real emission phase space.

The other non-Sudakov form factor is given by

\[ \Delta(z_i, q_i, k_i) = \exp \left\{ -\bar{\alpha}_S \int_z^{1} \frac{dz}{z} \int^{\frac{k_i^2}{p^2}} dp^2 \Theta(p - z q_i) \right\} , \quad k_i \equiv \sum_{\ell=1}^{i} p_{\ell t} . \]  

(27)

The upper limit \( k_i^2 \) in the \( p^2 \) integration comes from the non-eikonal form factor in (13), while the lower limit \( z q_i \) corresponds to the angular ordering \( \xi > \xi_i \) \( (p = z x_{i-1} \sqrt{s} \xi > z q_i) \) and comes from a part of the eikonal form factor not included into \( \Delta_S \) (see Appendix C). This form factor is different from the Sudakov form factor in various respects. The Sudakov form factor is associated to IR singularities for the virtual on-shell gluon thus it regularizes the \( z \to 1 \) singularity in the splitting function of the usual evolution equation. It is a function of the local variables at the vertex such as the two angular variables in (26). The form factor in (27) is associated to a singularity in the soft exchange gluon \( z_i \to 0 \). It depends on the energy fraction \( z_i \), the angular variable \( q_i \) and the exchanged transverse momentum \( k_i \). It is this \( k \)-dependence which gives one of the important feature for coherence in the small \( x \) region. For \( x \) finite (27) gives a subleading correction which is not present in the usual Sudakov form factor even at subleading level (13). This form factor is related to the Regge form factor of the gluon \( \Delta_R(z_i, k_i) \) which is found in the BFKL analysis (1). Here the lower bound integration over \( p \) is bounded for \( z \to 0 \) by a collinear cutoff \( \mu \). Due to this connection we shall call the non-Sudakov form factor \( \Delta(z, q, k) \) also the hard Regge form factor.

4 Unified coherent branching

In conclusion, from (17), (19), (21) and (25) we can write the \( k \)-structure function in eq. (1) in the form (see the kinematical diagram in Fig. 2)

\[
\mathcal{F}(x; k, k_0; Q, \mu) = \delta(1-x) \delta^2(k - k_0) \Delta_S(Q, \mu) \Theta(Q - \mu) + \sum_{r=1}^{\infty} \int_0^{Q^2} \Delta_S(Q, z_r q_r) \times \prod_{i=1}^{r} \left\{ \frac{d^2 q_i}{\pi q_i^2} dz_i \bar{\alpha}_S P(z_i, q_i, k_i) \Delta_S(q_i, z_{i-1} q_{i-1}) \Theta(q_i - z_{i-1} q_{i-1}) \right\} \delta(x - x_r) \delta^2(k - k_r) ,
\]

(28)
where \( x_i = z_i x_{i-1} \), \( k_i = -q_i + k_{i-1} \) and \( k_i = |k_i| \). We have included an incoming transverse momentum \( k_0 \) and the collinear cutoff \( \mu = z_0 q_0 \) which corresponds to the factorization scale for collinear singularities. The splitting function is
\[
P(z_i, q_i, k_i) = \frac{\Theta(1 - z_i - Q_i/q_i)}{1 - z_i} + \frac{\Delta(z_i, q_i, k_i)}{z_i}.
\] (29)

We have included the non-Sudakov form factors \( \Delta \) only in the \( z_i \to 0 \) singular contribution of the gluon splitting function (21) since \( \Delta \) is regular for finite \( z_i \) and gives a non leading correction. It is easy to include in \( P(z, q, k) \) the finite terms \(-2 + z_i(1 - z_i)\) which are relevant in the region of \( x \) not large or small. It is also possible to generalize the equation and the splitting function to include the quark contributions. The variable \( Q \) has been introduced in (8) as the upper bound on the total \( k \) (see (2)). To the present accuracy this is equivalent to set \( Q \) as the upper bound of all emitted transverse momenta.

The main feature of this branching is angular ordering which is present in the real emission phase space \((q_i > z_{i-1} q_{i-1})\) (see (13)), in the Sudakov form factor \((q > z_i q_i)\) (see (24)), and in the non-Sudakov form factor \((p > z q_i)\) (see (27)). The fact that the non-Sudakov form factor \( \Delta(z_i, q_i, k_i) \), relevant only at small \( x \), depends also on the exchanged transverse momentum \( k_i \) is the reason why we need to introduce the unintegrated structure function. The branching distribution in (28) and (24) is the basis for the new Monte Carlo program [13] which simultaneously takes into account coherence for large \( x \) (to double logarithms) and for small \( x \) (to all loops).

In the small \( x \) region eq. (28) sum all leading IR singularities, i.e. powers of \( \alpha_s^2 / x \ln^{n-1} x \). This contributions are obtained when all variable \( z \) vanish. Here the lower boundaries for the transverse momenta vanishes both in the real emission \((q_i > z_{i-1} q_{i-1} \to 0)\) and in the non-Sudakov form factor \((p > z q_i \to 0)\). We have than that the collinear singularities arising from the vanishing of these lower boundaries generate not only logarithms in the transverse momenta, which give powers of \( \ln Q/\mu \), but also logarithms in the variables \( z \), which give further powers of \( \ln x \). Recall that in the present formulations all these logarithms of collinear origin are included without approximations. As we shall discuss later, in the \( k \)-structure function all collinear singularities fully cancel so that angular ordering becomes equivalent to the multi-Regge region. This is why the expansion of the \( k \)-structure function contains only powers of \( \alpha_s^2 / x \ln^{n-1} x \) and the BFKL equation is obtained.

5 Unified evolution equation

The distribution \( f(x, k, k_0; Q, \mu) \) in (28) is given in terms of the angular ordered phase space and therefore one can deduce an integral or a differential equation in the angular variable. See Ref. [2] [3] for the case \( x \to 0 \). Here we consider the extension to the regions of \( x \) both small and large.

For \( x \) not small one has that all \( z_i \) are finite and, to leading collinear logarithm, angular ordering in (21) is equivalent to \( q_i \) ordering \((q_i > z_{i-1} q_{i-1} \simeq q_{i-1})\). The evolution equation is obtained by differentiating (28) with respect to \( Q \) and one obtains the usual AP evolution equation with coherence [1], i.e. the leading \( \alpha_s \ln(1 - x) \) contributions are correctly summed by taking into account the rescaling factor in the angular evolution variable \( q_i = p_i/(1 - z_i) \).
For $x \to 0$ one has that $Q$ can not play the rôle of an angular variable. Indeed the last branching phase space is $Q > q_r > z_{r-1} q_{r-1}$ so that, by differentiating with respect to $Q$, we get $Q = q_r$ while the upper scale for the angular variable must be $z_r q_r$. Therefore, in order to obtain an evolution equation also for $x \to 0$ one needs to introduce an additional variable $\bar{q}$ giving this upper scale for the last angle of the emission

$$\xi_r < \bar{\xi} \Rightarrow z_r q_r < \bar{q} = x \sqrt{s \xi}.$$  

We then introduce the distribution $A(x; k_0, \bar{q}; Q, \mu)$ defined by (28) in which we add the constraints $\bar{q} > z_n q_n$. We have then

$$A(x; k, k_0; \bar{q}; Q, \mu) \equiv \delta(x - 1) \delta^2(k - k_0) \Delta_S(\bar{q}, \mu) \Theta(\bar{q} - \mu) + \sum_r \int_0^{Q^2} \Theta(\bar{q} - z_r q_r) \Delta_S(q_r, \bar{q}, z_r q_r)$$

$$\times \prod_{i=1}^r \left\{ \frac{d^2 q_i}{\pi q_i^2} d z_i \bar{\alpha}_S P(z_i, q_i, k_i) \Delta_S(q_i, z_{i-1} q_{i-1}) \Theta(q_i - z_{i-1} q_{i-1}) \right\} \delta(x - x_r) \delta^2(k - k_r).$$

At $\bar{q} = Q$ we have

$$A(x, k, k_0; \bar{q} = Q; Q, \mu) = F(x, k, k_0; Q, \mu),$$

and for $\bar{q} < x \mu$ the integral vanish thus

$$A(x, k, k_0; \bar{q} = x \mu; Q, \mu) = \delta(1 - x) \delta^2(k - k_0).$$

This distribution satisfies the following integral equation

$$A(x, k, k_0; \bar{q}; Q, \mu) = \delta(x - 1) \delta^2(k - k_0) \Delta_S(\bar{q}, \mu) \Theta(\bar{q} - \mu)$$

$$+ \int_0^{Q^2} \Delta_S(\bar{q}, z q) \Theta(\bar{q} - z q) \frac{d^2 q}{\pi q^2} \frac{d z}{z} \bar{\alpha}_S P(z, q, k) A\left(\frac{x}{z}, k'; k_0; q; Q, \mu\right),$$

with $k' = k + (1 - z) q$. The evolution equation valid for large and small $x$ is obtained by differentiating with respect to the last angular variable

$$\bar{q}^2 \frac{\partial}{\partial \bar{q}^2} A(x, k, k_0; \bar{q}; Q, \mu)$$

$$= \int_x^{1} \frac{d z}{z} \bar{\alpha}_S \tilde{P}(z, \bar{q}, k) \left[ \frac{1}{1 - z} + \frac{\Delta(z, \bar{q}, k)}{z} \right] A\left(\frac{x}{z}, k', k_0; \bar{q}, Q, \mu\right) \Theta(Q - \bar{q}),$$

where $k' \equiv k + \frac{(1 - z) q}{z}$ and the azimuthal integration over the direction of $\bar{q}$ is understood. The $z \to 1$ singularity in the kernel is regularized by the virtual contribution coming by differentiating the Sudakov form factors in (32). Notice that the finite $-2 + z(1 - z)$ should be added if one considers regular contribution for $z$ away from the boundaries $z = 0$ and $z = 1$. This evolution equation can be generalized to include also the quark contributions.

It is important to notice that (33) is actually an evolution equation in the angular variable $\bar{q}/x = \sqrt{s \xi}$ and not in $\bar{q}$ itself. This is seen in the fact that the derivative of the distribution $A(x, k, k_0; \bar{q}; Q, \mu)$ in the l.h.s of (33) at $x$ and $\bar{q}$ is given by the distribution at $x' = x/z$ and $\bar{q}' = \bar{q}/z$. Moreover the boundary condition (11) is given at the minimum angle $\sqrt{s \xi_{\bar{q}}} = \bar{q}/x = \mu$. Of course one can use instead a boundary condition at fixed $\bar{q}$. We have than that the solution of (33) is obtained by increasing $\bar{q}/x$ up to the values $\bar{q} < Q$. 

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Consider the two limits \( x \to 1 \) and \( x \to 0 \). In the first case we recover easily the usual evolution equation for the structure function \( F(x, Q) \) in \([8]\) with coherence included \([1]\). This is seen by observing that for finite \( x \) also \( z \) is finite. Recalling that in this case we required only the leading collinear accuracy, we can make the following approximations in \((33)\): i) take as ordering variable \( \bar{q} = Q \); ii) neglect the non-Sudakov form factor which it is not singular for finite \( z \); iii) replace in the integrand \( \bar{q}/z \) with \( \bar{q} \). With all these simplification we obtain

\[
Q^2 \frac{\partial}{\partial Q^2} F(x, k, k_0; Q, \mu) = \int_x^1 \frac{dz}{z} \bar{\alpha}_S \left[ \frac{1}{1 - z} \right]_{+} \frac{1}{z} F \left( \frac{x}{z}, k', k_0; Q, \mu \right),
\]

with \( k' = k + (1 - z)Qn \) and the integration over the two-dimensional versor \( n \) is understood. Notice that by neglecting the non-Sudakov form factor, there is no \( k \) dependence in the kernel. Therefore we can integrate this equation over \( k \) and obtain the usual AP equation for the structure function (a part for the finite part in \( z \) for the splitting function). The \( k \) dependence of \( F(x, k, k_0; Q, \mu) \) is known \([1]\) and is of a Sudakov form factor type.

We consider now \((33)\) in the small \( x \) region where we can neglect the \( 1/(1 - z)_+ \) term in the kernel. We obtain (see also \([2, 3]\))

\[
\bar{q}^2 \frac{\partial}{\partial \bar{q}^2} A(x, k, k_0; \bar{q}; Q, \mu) = \int_x^1 \frac{dz}{z} \bar{\alpha}_S \Delta(z, \bar{q}/z) \Delta(z, \bar{q}/z) A \left( \frac{x}{z}, k', k_0; \bar{q}/z, Q, \mu \right) \Theta(Q - \bar{q}/z),
\]

with \( k' \equiv k + \bar{q} \). Notice that for \( Q \) and \( xQ < \bar{q} < Q \) the \( z \)-integration is over the range \( \bar{q}/Q < z < 1 \). Therefore this distribution becomes independent of \( \bar{q} \) as \( \bar{q} \) approaches \( Q \) and satisfies the BFKL equation. This is analyzed in the next Section.

### 6 Coherence at small \( x \)

At small \( x \) we can neglect in \((33)\) both the real and virtual contributions which are singular for \( z \to 1 \). The corresponding approximation in \((30)\) gives

\[
A(x, k, k_0; \bar{q}; Q, \mu) = \delta(x - 1) \delta^2(k - k_0) \Theta(\bar{q} - \mu)
+ \sum_r \int_0^Q \Theta(\bar{q} - z_r q_r) \prod_{i=1}^r \left\{ \frac{d^2 q_i}{\pi q_i^2} \int_{z_i}^1 \frac{d\bar{q}}{\bar{q}} \Delta(z_i, q_i, k_i) \Theta(q_i - z_{i-1} q_{i-1}) \right\} \delta(x - x_r) \delta^2(k - k_r),
\]

with \( x_i = z_i x_{i-1} \), \( k_i = k_{i-1} - q_i \) and \( k_i = |k_i| \).

As we shall see, in this fully inclusive distribution all collinear singularities in the real emission phase space \( q_i \to z_{i-1} q_{i-1} \to 0 \) are compensated by the collinear singularities in the non-Sudakov form factors \((27)\) for \( p \to z q_i \to 0 \). Therefore at the inclusive level one can simplify the phase space by substituting angular ordering with the multi-Regge region in which all transverse momenta are independent and by regularizing the exclusive collinear singularities in \((33)\) by the cutoff \( \mu \) in both the real emission and the virtual integrals. After performing the sum the limit \( \mu \to 0 \) is finite. This corresponds to make the following approximations in \((35)\)

\[
\Theta(q_i - z_{i-1} q_{i-1}) \Rightarrow \Theta(q_i - \mu)
\Delta(z_i, q_i, k_i) \Rightarrow \Delta_R(z_i, k_i).
\]
Within this approximation we obtain the following representation of the $k$-structure function
\[ F_0(x, k, k_0; Q, \mu) = \delta(1 - x) \frac{\delta^2(k - k_0)}{\Theta(Q - \mu) +} \]
\[ \sum_{r=1}^{\infty} \int_{\mu^2}^{Q^2} \prod_{i=1}^{r} \left( \frac{d^2q_i}{\pi q_i^2} dz_i \frac{\bar{\alpha}_S}{z_i} \Delta_R(z_i, k_i) \right) \delta(x - x_r) \frac{\delta^2(k - k_r)}{}, \]

in which the phase space is the multi-Regge region (see (3)).

### 6.1 Perturbative expansion

We check explicitly to four loops that the small $x$ structure functions in (35) and (37) are the same for large $Q$. We will see that the real and virtual contributions have different type of singularities. In the coherent formulations (35) we find stronger singularities for $x \rightarrow 0$ than in the approximate form in (37). All these stronger singularities do cancel in the fully inclusive sum ($k$-structure function), but they do not cancel for associated distributions.

Consider the contributions to the structure functions for a fixed number $r$ of emitted initial state gluons
\[ F_\omega(Q) = \int_0^1 dx x^\omega F(x, Q) = 1 + \sum_{r=1}^{\infty} F_\omega^{(r)}(Q). \]

In the formulation with coherence the structure function is obtained by integrating (35) over $k$ and setting $q = Q$. For simplicity we assume $k_0 = 0$ and we obtain
\[ F_\omega(Q, \mu) = \Theta(Q - \mu) + \sum_{r=1}^{\infty} \int_{\mu^2}^{Q^2} \prod_{i=1}^{r} \left( \frac{d^2q_i}{\pi q_i^2} dz_i \frac{\bar{\alpha}_S}{z_i} \Delta(z_i, q_i, k_i) \Theta(q_i - z_{i-1}q_{i-1}) \right), \]

with the collinear cutoff $\mu = z_0q_0$.

In the formulation without coherence in (37) we obtain
\[ F_{0,\omega}(Q, \mu) = \Theta(Q - \mu) + \sum_{r=1}^{\infty} \int_{\mu^2}^{Q^2} \prod_{i=1}^{r} \left( \frac{d^2q_i}{\pi q_i^2} dz_i \frac{\bar{\alpha}_S}{z_i} \Delta_R(z_i, k_i) \right). \]

For the structure function $F_{0,\omega}(Q)$ in (37) without coherence we readily obtain the perturbative expansion. For large $Q$ one obtains to four loops ($T \equiv \ln(Q/\mu)$)
\[ F_{0,\omega}^{(1)}(Q) = 2 \frac{\bar{\alpha}_S}{\omega} T - \frac{1}{2} \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^2 + \frac{1}{3} \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^3 - \frac{1}{4} \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^4 + \cdots, \]
\[ F_{0,\omega}^{(2)}(Q) = \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^2 - \frac{7}{6} \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^3 + \frac{29}{24} \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^4 + 8 \left( \frac{\bar{\alpha}_S}{\omega} T \right)^4 \zeta(3) T + \cdots, \]
\[ F_{0,\omega}^{(3)}(Q) = \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^3 - \frac{23}{12} \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^4 - 4 \left( \frac{\bar{\alpha}_S}{\omega} T \right)^4 \zeta(3) T + \cdots, \]
\[ F_{0,\omega}^{(4)}(Q) = \left( 2 \frac{\bar{\alpha}_S}{\omega} T \right)^4 + \cdots. \]

The perturbative expansion of these contributions is of the form
\[ F_{0,\omega}^{(r)}(Q) = \sum_{n=r}^{\infty} C_{0}^{(r)}(n; T) \frac{\bar{\alpha}_S^n}{\omega^n}. \]
Summing all exclusive emission contribution one finds the perturbative result \[ (4) \] of the BFKL anomalous dimension up to four loops. The first correction to the one loop anomalous dimension is only at the fourth loop. The Riemann zeta functions are obtained from integrals of type
\[
\int_{q^2}^{Q^2} \frac{d^2 q}{\pi q^2} \ln^2 \left( \frac{(q - q')^2}{q^2} \right) \simeq \int_{q^2}^{Q^2} \frac{d^2 k}{\pi (k - q)^2} \ln^2 \left( \frac{k^2}{q'^2} \right) = 4 \zeta(3) + \frac{\ln^3}{\zeta} (Q^2 / q'^2) + O \left( \frac{\mu^2}{Q^2} \right). \quad (40)
\]
We neglected the collinear cutoff \( \mu \) since the integral is regular for \( q \to 0 \). Moreover we have taken \( Q \) as the upper bound both for the emitted and the exchanged transverse momentum which is valid for \( q' \ll Q \).

In the coherent formulation the perturbative contribution to four loops are
\[
F^{(1)}_{\omega}(Q) = \frac{2 \bar{\alpha}_S}{\omega} T - \frac{(2 \bar{\alpha}_S)^2}{\omega^3} T + 3 \frac{(2 \bar{\alpha}_S)^3}{\omega^5} T - 15 \frac{(2 \bar{\alpha}_S)^4}{\omega^7} T + \ldots,
\]
\[
F^{(2)}_{\omega}(Q) = \frac{(2 \bar{\alpha}_S)^2}{\omega^3} \left\{ \frac{T^2}{\omega^2} + \frac{T}{\omega^3} \right\} - 3 \frac{(2 \bar{\alpha}_S)^3}{\omega^5} \left\{ \frac{T^2}{\omega^4} + \frac{5}{2} \frac{T}{\omega^5} \right\} + 5 \frac{(2 \bar{\alpha}_S)^4}{\omega^7} \left\{ \frac{T^2}{\omega^6} + \frac{32 T}{\omega^7} + \frac{7}{2} \zeta(3) \frac{T}{\omega^8} \right\} + \ldots,
\]
\[
F^{(3)}_{\omega}(Q) = \frac{(2 \bar{\alpha}_S)^3}{\omega^3} \left\{ \frac{T^3}{\omega^3} + \frac{T^2}{\omega^4} + \frac{2 T}{\omega^5} \right\} - 4 \frac{(2 \bar{\alpha}_S)^4}{\omega^5} \left\{ \frac{T^3}{\omega^4} + \frac{6}{2} \frac{T^2}{\omega^5} + \frac{22 T}{\omega^6} + \frac{1}{2} \zeta(3) \frac{T}{\omega^7} \right\} + \ldots,
\]
\[
F^{(4)}_{\omega}(Q) = \frac{(2 \bar{\alpha}_S)^4}{\omega^3} \left\{ \frac{T^4}{\omega^4} + \frac{1}{2} \frac{T^3}{\omega^5} + \frac{5}{2} \frac{T^2}{\omega^6} + \frac{5 T}{\omega^7} \right\} + \ldots. \quad (41)
\]
The Riemann zeta functions are obtained from the same integrals in \[ (40) \] with the lower bound \( \mu \) in general substituted by \( z q' \). The perturbative expansion of \( F^{(r)}_{\omega}(Q) \) is of the form
\[
F^{(r)}_{\omega}(Q) = \sum_{n=r}^{\infty} \sum_{m=1}^{n} C^{(r)}(n, m; T) \frac{\bar{\alpha}_S^n}{\omega^{2n-m}}.
\]
The singular terms with \( m < n \) are related to coherence since they are coming from angular ordering, i.e. the \( z \) dependence in the emission phase space \( (q_i > z_{i-1} q_{i-1}) \) and in the virtual integration \( (p > z q_i) \). By summing all the exclusive contributions one finds that all these more singular terms cancel and obtains the same anomalous dimension dimension \[ (4) \] to four loop. This is an example of the complete cancellations of the collinear singularities in the \( k \)-structure function.

At a less inclusive level, such as for the associated distributions, the collinear singular terms \( (\bar{\alpha}_S^n / \omega^{2n-m}) \) with \( m < n \) do not cancel any more. In the approximation \[ (36) \] instead, the perturbative coefficients are given only by powers \( (\bar{\alpha}_S / \omega)^n \). Therefore the approximation of neglecting angular ordering by simply putting the cutoff \( \mu \) (see \[ (36) \]) is not any more valid for the associated distributions.

As an example we consider the average number of gluons emitted in the initial state, i.e. we neglect the final state branching of these gluons. This average is obtained by summing the contributions \[ (39) \] and \[ (41) \] with a weight given by \( r \). For the formulation with and without coherence we find
\[
\sum_{r} r F^{(r)}_{\omega}(Q, \mu) = \frac{2 \bar{\alpha}_S}{\omega} T + \left( \frac{2 \bar{\alpha}_S}{\omega} \right)^2 (T^2 + \frac{T}{\omega}) + \ldots,
\]
\[
\sum_{r} r F^{(r)}_{0, \omega}(Q, \mu) = \frac{2 \bar{\alpha}_S}{\omega} T + \frac{3}{2} \left( \frac{2 \bar{\alpha}_S}{\omega} \right)^2 T^2 + \ldots. \quad (42)
\]
6.2 Relation with the BFKL equation

We want to show now that the structure functions at fixed $k$ satisfy the BFLK equation for $x \to 0$. Consider first the case of no coherence obtained by approximating angular ordering with the multi-Regge region. The unintegrated structure function $F_{0\omega}(k, k_0; Q, \mu)$ is given in (37). We show that the dependence on $Q$ and $\mu$ is of higher twist type and, neglecting this dependence, the unintegrated structure function satisfies the BFKL equation. From (37) we have

$$F_{0\omega}(k, k_0, Q, \mu) = \delta^2(k - k_0) \Theta(Q - \mu) + \int_{\mu}^{Q^2} \frac{d^2q}{\pi q^2} dz \frac{\bar{\alpha}_S}{z} \Delta(z, k) F_{0\omega}(k + q, k_0; Q, \mu), \quad (43)$$

and one readily deduces the following integral equation

$$F_{0\omega}(k, k_0; Q, \mu) = \frac{\bar{\alpha}_S}{\omega} \int_{\mu}^{Q^2} \frac{d^2q}{\pi q^2} \{ F_{0\omega}(k + q, k_0; Q, \mu) - \Theta(k - q) F_{0\omega}(k, k_0; Q, \mu) \} + \delta^2(k - k_0) \Theta(Q - \mu) \left[ 1 + \frac{\bar{\alpha}_S}{\omega} \ln \frac{k_0^2}{\mu^2} \Theta(k_0 - \mu) \right]. \quad (44)$$

The kernel of this equation is regular for the collinear limit $q \to 0$ so that in the integral we do not need the cutoff $\mu$. The cutoff can be removed from the inhomogeneous term by defining the unintegrated structure function for which we can take the limit $\mu/Q \to 0$

$$\tilde{F}_{0\omega}(k, k_0) \equiv F_{0\omega}(k, k_0; Q, \mu) \left[ 1 + \frac{\bar{\alpha}_S}{\omega} \ln \frac{k_0^2}{\mu^2} \Theta(k_0 - \mu) \right]^{-1}, \quad \frac{\mu}{Q} \to 0. \quad (45)$$

This distribution satisfies the BFKL equation in (8) with the inhomogeneous term $\delta^2(k - k_0)$.

We want to show that, due to the cancellation of collinear singularities, the unintegrated distribution $A_{\omega}(k, k_0; \tilde{q}; Q, \mu)$ obtained in the present formulation, satisfies to the present accuracy the BFKL equation. From (33) we obtain (see [2, 3])

$$A_{\omega}(k, k_0; \tilde{q}; Q, \mu) = \delta^2(k - k_0) \Theta(\tilde{q} - \mu) + \int_{\mu}^{Q^2} \frac{d^2q}{\pi q^2} dz \frac{\bar{\alpha}_S}{z} \Delta(z, q, k) \Theta(\tilde{q} - qz) A_{\omega}(k + q, k_0; Q, \mu), \quad (46)$$

Integrating by part

$$\int_{0}^{1} dz \frac{\partial}{\partial z} \Delta(z, q, k) \Theta(\tilde{q} - qz) = \frac{\bar{\alpha}_S}{\omega} \left\{ \Theta(\tilde{q} - q) - \int_{0}^{1} dz \frac{\partial}{\partial z} (\Delta(z, q, k) \Theta(\tilde{q} - qz)) \right\}$$

we obtain (the $Q$ and $\mu$ dependence is understood)

$$A_{\omega}(k, k_0; \tilde{q}) = \frac{\bar{\alpha}_S}{\omega} \int_{\mu}^{Q^2} \frac{d^2q}{\pi q^2} \{ A_{\omega}(k + q, k_0; q) - \Theta(k - q) A_{\omega}(k, k_0; q') \} + \delta_{\omega}(k, \tilde{q}) + \delta^2(k - k_0) \Theta(\tilde{q} - \mu) \left[ 1 + \frac{\bar{\alpha}_S}{\omega} \ln \frac{k_0^2}{\mu^2} \Theta(k_0 - \mu) \right], \quad (47)$$

with $q' \equiv \min(q, \tilde{q})$ and

$$\delta_{\omega}(k, \tilde{q}) \equiv \frac{\bar{\alpha}_S}{\omega} \int_{q}^{Q^2} \frac{d^2q}{\pi q^2} A_{\omega}(k + q, k_0; q) \left[ \left( \frac{\tilde{q}}{q} \right)^{\omega} \Delta(q, q, k) \right] - 1.$$
As before we remove the $\mu$ dependence in the inhomogeneous term by defining the distribution

$$\tilde{A}_\omega(k, k_0; \bar{q}; Q, \mu) \equiv A_\omega(k, k_0; \bar{q}; Q, \mu) \left[ 1 + \frac{\alpha_S}{\omega} \ln \frac{k_0^2}{\mu^2} \Theta(k_0 - \mu) \right]^{-1}.$$  

(48)

In the equation for this distribution we can remove the cutoff $\mu$ and take the limit $\mu/Q \to 0$. The contribution $\delta_\omega(k, \bar{q})$ vanishes for $\bar{q} \to Q$ and does not contains leading contributions given by the powers $\alpha_S^2/\omega^n$. By neglecting $\delta_\omega(k, \bar{q})$ we have that the integral equation for $\tilde{A}$ becomes the BFKL equation and in these limits we can approximate

$$\tilde{A}_\omega(k, k_0; \bar{q}; Q, \mu) \simeq \tilde{F}_\omega(k, k_0).$$  

(49)

7 Associated parton distributions

The coherent formulation can be used to compute not only the structure function but also the associated distributions for any value of $x$ large or small by generalizing the jet calculus \[3\]. As discussed in the previous section the present formulation with coherence is equivalent to the BFKL equation only for the fully inclusive case ($k$-structure function), but different for distributions of the associated radiation.

As illustration we consider the single inclusive parton distribution with a given energy fraction $y$ in the hard process of Fig. 1. The relevant distribution is described in Fig. 4 and is given in terms of the distribution $A(x, k, k_0; Q; \bar{q}, \mu)$ as follows (take $k_0 = 0$ for simplicity)

$$\Sigma(x, y, Q, \mu) = \int dx' d^2k d^2k' \frac{1}{x' z} A \left( \frac{x}{x' z}, k, k'; \bar{q} = Q, zq \right) \frac{1}{x'(1 - z)} D \left( \frac{y}{x'(1 - z)}, p_t \right)$$

$$\times \int_{\mu}^{Q} d^2q \frac{dz}{\pi q^2} \bar{\alpha}_S P(z, q, k') A(x', k' + p_t, k_0; q, Q, \mu),$$  

(50)

where $p_t = (1 - z)q$ is the initial state emitted transverse momentum, $P(z, q, k)$ is the splitting function in (29) which includes the non-Sudakov factor and $D(y/x'' q)$ is the inclusive distribution of the final state gluon of energy fraction $y$ emitted in a jet of energy fraction $x'' = (1 - z)x'$ and the hard scale is $p_t = (1 - z)q$ as required by angular ordering. The maximum angular variable is set at $\bar{q} = Q$ as required by (22). From this expression we can compute for instance the average associated multiplicity. Notice that in these cases one maximises the value of $p_t$.

Consider the region of small $x$ in which all $z$ variables are small. Taking the moments one finds

$$\Sigma_{\omega, \omega'}(Q, \mu) \equiv \int dx \, d\omega \, dy \, d\omega' \Sigma(x, y; Q, \mu) = \int d^2k d^2k' A_\omega(k, k'; \bar{q} = Q, zq) D_\omega(q)$$

$$\times \int_{\mu}^{Q} d^2q \frac{dz}{\pi q^2} (1 - z) \omega' \frac{\alpha_S}{z} \Delta(z, q, k') A_{\omega + \omega'}(k' + q, k_0; q, Q, \mu),$$  

(51)

where for small $x$ we have approximated $p_t \simeq q$.

It is easy to check that this distribution satisfies the energy sum rule, namely one has

$$\Sigma_{\omega, 1}(Q) = \int dx \, \omega' \, (1 - x) F(x, Q) = F_\omega(Q) - F_{\omega + 1}(Q).$$

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We take $\omega' = 1$ in (51) and use the energy sum rule $D_{\omega'=1}(q) = 1$. We have

$$
\Sigma_{\omega,1}(Q) = \int d^2 k \, d^2 k' \, A_{\omega}(k, k'; \bar{q} = Q; Q, zq) \int_{Q^2} d^2 \bar{q} \, \frac{d^2 \bar{q}}{(2\pi)^2} \, dz \left( z^{\omega} - z^{\omega+1} \right) \frac{\bar{\alpha}_S}{z} \Delta(z, q, k') \times A_{\omega+1}(k' + q, k_0; q; Q, \mu).
$$

(52)

For the second term we use the small $x$ evolution equation (34). Changing integration variable from $q$ to $\bar{q} = zq$ we have

$$
\bar{q}^2 \frac{\partial}{\partial \bar{q}^2} A_{\omega+1}(k', k_0; \bar{q}; Q, \mu) = \int_0^1 dz \, z^{\omega+1} \frac{\bar{\alpha}_S}{z} \Delta(z, \bar{q}/z, k') A_{\omega+1}(k' + \bar{q}/z, k_0; \bar{q}/z; Q, \mu) \Theta(Q - \bar{q}/z).
$$

For the first term we use the evolution equation in the minimum angle

$$
\bar{q}^2 \frac{\partial}{\partial \bar{q}^2} A_{\omega}(k, k' + q; \bar{q}; Q, q) = -\int_0^1 dz \, z^{\omega} A_{\omega}(k, k'; \bar{q}; Q, zq) \frac{\bar{\alpha}_S}{z} \Delta(z, q, k') \Theta(Q - q).
$$

Finally, changing variables $k' + q$ to $k'$ in this second contribution we have

$$
\Sigma_{\omega,1}(Q) = -\int d^2 k \, d^2 k' \int d^2 \bar{q} \, \frac{\partial}{\partial \bar{q}^2} \left\{ A_{\omega}(k, k'; \bar{q} = Q; Q, q) \, A_{\omega+1}(k', k_0; q; Q, \mu) \right\}.
$$

By using the boundary condition

$$
A_{\omega}(k, k_0; \bar{q} = \mu; Q, \mu) = \delta^2(k - k_0),
$$

and

$$
F_{\omega}(Q) = \int d^2 k \, A_{\omega}(k, k_0; \bar{q} = Q; Q, \mu),
$$

we find that the sum rule is satisfied.

It is interesting to compare (51) with the corresponding expression obtained neglecting coherence by extending the multi-Regge phase space [16, 18]. As an example consider the case in which we neglect the final state radiation, i.e. we set $D_{\omega'}(q) = 1$ in (51). If we take $\omega' = 0$ we are computing the average number of emitted initial state gluons. This average has been computed perturbatively up to the fourth loop in the previous Section. This show that the high singular terms $\alpha^n_S/\omega^{2n-m}$ with $m < n$, present in the formulation with coherence, do not cancel. Therefore, in this case the approximation of substituting angular ordering with multi-regge region is not valid.

In general we can identify in (51) the origin of these high singular terms. From (48) and (49) we have that the collinear dependence on the cutoff $zq$ in the first distribution $A$ inside the integral of (51) is given by a factor

$$
1 + \frac{2 \bar{\alpha}_S}{\omega} \ln \frac{k'}{zq} \Theta(k' - zq).
$$

This factor is not compensated by the integration over the non-Sudakov form factor due to angular ordering constraints $k > zq$ and this generates $\alpha^n_S/\omega^{2n-m}$ with $m < n$ contributions not present in the multi-Regge region. The conclusion is that for the associated distributions in general angular ordering can not be approximated by multi-Regge regions.
7.1 Jet calculus at small $x$

The expression of the single inclusive distribution in (51) can be generalized by using the jet calculus algorithm. This algorithm is based on the fact that the branching process is factorized. For instance the generating functional of $y$ distributions can be obtained as follow. Consider first the generating functional $G_t[1, q, u]$ of inclusive distribution in a gluon jet of energy fraction 1 and angular variable $q$ which depends on the function $u(y)$. This functional is normalized by $G_t[1, q, 1] = 1$ and the single inclusive $y$ distribution is given by

$$D(y, Q) = \frac{\delta}{\delta u(y)} G_t[1, q, u] |_{u=1}.$$ 

The general $n$-gluon inclusive distributions are obtained by further functional differentiation at $u(y) = 1$. Then we define the corresponding functional for the hard process at scale $Q$ of initial state radiation at small $x$. We define the space like functional $G_s[x, k, k_0; \bar{q}; Q, \mu; u]$ with the previous meaning for the variables. By using the factorization of the branching process this functional satisfies the following evolution equation

$$G_s[x, k, k_0; \bar{q}; Q, \mu; u] = \delta(1 - x) \delta^2(k - k_0) \Theta(\bar{q} - \mu) + \int_0^{Q^2} \Theta(\bar{q} - zq) d^2q \frac{d^2q}{\pi q^2 z}$$

$$\times \left\{ \frac{\bar{\alpha}_S}{z} \Delta(z, q, k) G_t[x', q, u_x] \right\} G_s\left[\frac{x}{z}, \frac{k + q, k_0}{Q, \mu, u}\right],$$

where $x' = (1 - z)x/z$ and $u_x(y) = u(xy)$. From (43) we have that at $u(y) = 1$ this functional is the unintegrated distribution $A(x, k, k_0; \bar{q}, Q, \mu)$. The single inclusive distribution in (50) is obtained by differentiation

$$\Sigma(x, y, Q) = \int d^2k d^2k' \frac{\delta}{\delta u(y)} G_s[x, k, k_0; Q, \mu, u] |_{u=1}.$$ 

To see that this distribution coincides with the one in (51) one proceeds in some formal way exploiting factorization which is the bases of jet calculus. We write (53) in the formal way

$$G_s[u] = 1 + K[u] \cdot G_s[u],$$

where the kernel $K[u]$ corresponds to the distribution in the curly bracket in (53). Taking the functional derivative one has

$$\Sigma = \frac{\delta}{\delta u(y)} G_s[u] |_{u=1} = G_s[u] \cdot \frac{\delta K[u]}{\delta u(y)} \cdot G_s[u] |_{u=1}.$$ 

The expression in (50) is obtained from the fact that $G_s[1]$ is into account that $G_s[1]$ is the distribution $A$.

8 Conclusions

We recall the main features of the analysis here presented for small and large $x$. 

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1) At the level of the fully inclusive distributions at small $x$ ($k$-structure function) the dominant region of phase space is the multi-Regge region with strong $x_i$-ordering and with transverse momenta of the same order. This is due to the complete cancellation of the collinear singularities which arises when two emitted transverse momenta are very different. This property is the basis of the BFKL equation and of the fact that the hard scale associated to collinear singularities is lost.

2) For associated distributions the collinear singularities do not cancel in general and the dominant region of phase space is determined by angular ordering. The analysis of the associated distributions for small $x$ requires the complete structure of collinear singular terms, i.e. no collinear approximations. This level of accuracy can be actually attained for $x \to 0$ by using soft gluon factorization. While for the amplitudes the soft gluon factorization does not require any collinear approximation, for the distributions one is usually forced to introduce collinear approximation in order to perform the multi-gluon colour algebra. For $x \to 0$ one avoids this approximation and the basis of this fact is in eqs. (15) and (16). Due to the real-virtual cancellation of soft singularities and to the coherent structure of the eikonal current, one finds that the distribution of a soft gluon emitted by a jet of partons does not vanish for $x \to 0$ only if the jet of partons are confined into an angular cone with aperture of order $x$. Therefore for $x \to 0$ we can use collinear estimates. Notice that the approximation involved is the soft approximation, not the collinear one.

3) QCD coherence has the same origin for small and large $x$ and in both regions the emission factorizes. Therefore one can write a unified evolution equation which resums all IR singularities which at small $x$ does not involve any collinear approximation while at large $x$ is accurate only to leading collinear level. Contributions from non soft radiation as well as contributions from quarks can be included in a natural way (see also [18]). In this equation, the most important difference between the two regions of small and large $x$ is the presence of the non-Sudakov or hard Regge form factor (27) for $x \to 0$. For finite $x$ this form factor gives a finite correction while for $x \to 0$ this form factor cancels all the collinear singularities in the $k$-structure function. Thus at this fully inclusive level angular ordering becomes equivalent to the multi-Regge phase space. For associated distribution instead the cancellation does not take place and one can not neglect angular ordering. Thus from these distributions one can reveal the coherence of QCD radiation in small $x$ processes. To study these associated distributions we have formulated a unified branching algorithm for DIS processes which allows one to compute associated distributions in all regions of $x$.

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Appendix A

Take $n = m + \ell$ and consider the case in which the $\ell$ softest gluons $q_{m+1}, \ldots, q_{m+\ell}$ are virtual while $q_m$ is real

$$y_{m+\ell} \ll \cdots y_{m+1} \ll y_m.$$  \hfill (54)

Since the quantities $Y_{R_n}$ and $k_{R_n}$ do not involve the $\ell$ softest momenta the two corresponding $\delta-$functions can be factorized and one can sum over these softest momenta and reconstruct
the form factors. From (11) we have in the region (54)
\[
g_s^{2 \ell} \prod_1^\ell (dq_{m+\ell}) f_{R_n}(q_1, \cdots, q_{m+\ell}) = \prod_1^\ell \left\{ -\bar{\alpha}_s \frac{dy_{m+i}}{y_{m+i}} \frac{d^2 q_{m+i}}{\pi q_{m+i}} \left[ 1 - \Theta(y_{m+i} - x) \Theta(q_{m+i} - k) \right] \right\} f_{R_m}(q_1, \cdots, q_m),
\]
Integrating over the region (54) and summing over \( \ell \) one finds
\[
\int \Theta^Y_{1 \cdots n} \sum_{\ell=0}^\infty g_s^{2 \ell} \prod_1^\ell (dq_{m+i}) f_{R_n}(q_1, \cdots, q_n)
= \Theta^Y_{1 \cdots m} f_{R_m}(q_1, \cdots, q_m) \exp \left\{ -\bar{\alpha}_s \int_{y_m}^{y_m} dy \int_y^{Q^2} \frac{dp^2}{p^2} + \bar{\alpha}_s \int_x^{y_m} dy \int_k^{Q^2} \frac{dp^2}{p^2} \right\},
\]
where in (54) we have \( x_m = x + y_{m+\ell} + \cdots + y_{m+1} \sim y_m \). Iterating this procedure to the next to soft emitted gluons we obtain the result in (18).

**Appendix B**

We consider the exchange of \( y \) with \( \xi \) ordering in (17). The factor not obviously symmetric with respect to \( y_i \) and \( \xi_i \) is given by
\[
\Theta^Y_{1 \cdots r} \prod_1^r T(y_i, y_{i+1}, k_i), \quad T(y_i, y_{i-1}, k_i) = \exp \left\{ -\bar{\alpha}_s \int_{y_i}^{y_{i+1}} dy \int_y^{Q^2} \frac{dp^2}{p^2} \right\},
\]
where the \( \Theta \)-function corresponds the strong ordering region (10). Here we have used \( x_i \simeq y_{i+1}, x \equiv y_{n+1}, \) and \( k_i = -q_i + k_{i-1} \). Since the quantity in (57) is integrated over we can exchange variable names as follows
\[
\Theta^Y_{1 \cdots r} \prod_1^r T(y_i, y_{i+1}, k_i) \Rightarrow \Theta^\xi_{r \cdots 1} \sum_{\text{perm.}} \Theta^Y_{\ell_1 \cdots \ell_r} \prod_1^r T(y_{\ell_1}, y_{\ell_1+1}, q_{\ell_1 \cdots \ell_1}),
\]
where \( q_{\ell_1 \cdots \ell_r} \equiv q_{\ell_1} + \cdots + q_{\ell_r} \).

Consider the case with two real gluons. There are two permutations in (58). The first with \( y_2 \ll y_1 \) and \( \xi_1 < \xi_2 \) gives \( y_1 \simeq 1 \) and \( x_1 = x + y_2 \simeq 1 \) so that
\[
\Theta^\xi_2 \Theta^Y_2 T(y_1; y_2; q_1) T(y_2; x; q_{12}) \simeq \Theta^\xi_2 \Theta^Y_2 T(1; x_1; q_1) T(x_1; x; q_{12}).
\]
The second permutation with the ordering \( y_1 \ll y_2 \) and \( \xi_1 < \xi_2 \) gives \( y_2 \simeq 1 \), \( x_1 = x + y_2 \simeq 1 \) and \( q_1 \ll q_2 \) so that we can write
\[
\Theta^\xi_2 \Theta^Y_2 T(y_2; y_1; q_2) T(y_1; x; q_{12}) \simeq \Theta^\xi_2 \Theta^Y_2 T(1; x_1; q_1) T(x_1; x; q_{12}).
\]
The two contributions have the same form within the leading IR accuracy and we conclude that we can substitute \( y \)-ordering with \( \xi \)-ordering
\[
\Theta^Y_2 T(y_1; y_2; q_1) T(y_2; x; q_{12}) \Rightarrow \Theta^\xi_2 T(1; x_1; q_1) T(x_1; x; q_{12})
\]
Consider the case with three emitted gluons \( r = 3 \). After singling out, as in (58), the angular ordering \( \xi_1 < \xi_2 < \xi_3 \) with \( \Theta_{321}^\xi \) we have six permutations of \( y \) ordering configurations. Consider the contribution from the fundamental permutation

\[
\Theta_{321}^\xi \Theta_{123}^Y T(y_1, y_2; y_3) T(y_2, y_3; q_{12}) T(y_3, x; q_{123}).
\]

Since \( \xi_1 < \xi_2 < \xi_3 \) and \( y_3 < y_2 < y_1 \), we have \( y_1 \approx 1, x_2 = x + y_3 \approx y_3 \) and \( x_1 = x_2 + y_2 \approx y_2 \) so that, factorizing \( \Theta_{321} \Theta_{123}^Y \), we have

\[
T(y_1, y_2; q_1) T(y_2, y_3; q_{12}) T(y_3, x; q_{123}) \approx T(1, y_1; q_1) T(x_1, x_2; q_{12}) T(x_2, x; q_{123}).
\]

The same expression is obtained for all other permutations within the leading IR accuracy. Consider for instance the contribution

\[
\Theta_{321}^\xi \Theta_{213}^Y T(y_2, y_1; q_2) T(y_1, y_3; q_{12}) T(y_3, x; q_{123}),
\]

where \( \xi_1 < \xi_2 < \xi_3 \) and \( y_3 < y_1 < y_2 \). We have \( y_2 \approx 1, x_2 = x + y_3 \approx y_3, x_1 = y_1 \approx 1 \) and \( q_1 < q_2 \). After factorizing \( \Theta_{321} \Theta_{213}^Y \), we have

\[
T(y_2, y_1; q_2) T(y_1, y_3; q_{12}) T(y_3, x; q_{123}) \approx T(1, y_1; q_2) T(y_1, x_2; q_{12}) T(x_2, x; q_{123})
\]

\[
\approx T(1, x_2; q_{12}) T(x_2, x; q_{123}) \approx T(1, x_1; q_1) T(x_1, x_2; q_{12}) T(x_2, x; q_{123}).
\]

For the other permutations one proceeds as before and finds the same expression so that one can substitute, to leading IR accuracy,

\[
\Theta_{213}^Y T(y_2, y_1; q_2) T(y_1, y_3; q_{12}) T(y_3, x; q_{123}) \Rightarrow \Theta_{321}^\xi T(1, x_1; q_1) T(x_1, x_2; q_{12}) T(x_2, x; q_{123}).
\]

The proof for the general case can be done by induction. A general physical argument which allows the substitution of \( y \)-ordering with \( \xi \)-ordering is given in [9].

Appendix C

The form in (18) of virtual corrections has been discussed [3] and will be recalled here for completeness. From (18) we have

\[
\ln V_r(p_1 \cdots p_r) = -\bar{\alpha}_S \int_0^1 dy y \int_0^1 d\xi \frac{\xi}{\xi} + \sum_{i=1}^r \bar{\alpha}_S \int_{y_i}^{y_{i-1}} dy \int_{\xi_i}^{\xi_{i+1}} d\xi \Theta(y \sqrt{s\xi - q_0}) + \sum_{i=1}^r \bar{\alpha}_S \int_{y_i}^{y_{i-1}} dy \int_{\xi_i}^{\xi_{i+1}} d\xi \frac{d^2 p^2}{p^2} \Theta(p-y \sqrt{s\xi}).
\]

where in the first term we introduced the angular variable \( \xi \) of virtual gluon. This first term with the singularity for \( y, \xi \to 0 \) can be expressed as sum of Sudakov form factors which regularize the singularities in the emitted phase space for \( y_i, q_i \to 0 \). To this end we introduce in the virtual correction the same IR cutoff as in the real emission, namely for the virtual transverse momenta we set the cutoff \( Q_0 \). In the angular ordered region of the emitted phase space

\[
\xi_0 < \xi_1 < \cdots < \xi_r < 1,
\]

we have \( q_i = x_{i-1} \sqrt{s\xi_i} \) and introduce the virtual momentum \( p = y \sqrt{s\xi} > Q_0 \). We have

\[
\ln V_r(p_1 \cdots p_r) = -\sum_{i=1}^r \bar{\alpha}_S \int_{y_i}^{y_{i-1}} dy \int_{\xi_i}^{\xi_{i+1}} d\xi \Theta(y \sqrt{s\xi - q_0}) + \sum_{i=1}^r \bar{\alpha}_S \int_{y_i}^{y_{i-1}} dy \int_{\xi_i}^{\xi_{i+1}} d\xi \frac{d^2 p^2}{p^2} \Theta(p-y \sqrt{s\xi}).
\]

Introducing in the first sum the variables \( y = (1-z)x_i \) and \( q = x_i \sqrt{s\xi} \) and in the second variable \( y = zx_{i-1} \), we deduce the final result in (25).
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Figure Captions

Figure 1: Deep inelastic scattering at parton level. The dotted line represents the off shell photon $q$. For the dominant small $x$ contribution the incoming $p$ and the outgoing $p_i$ partons are gluons and the recoiling system is a quark-antiquark pair of momentum $p'$.

Figure 2: Kinematical diagram for parton emission.

Figure 3: Graphical representation of the elementary vertex factorization.

Figure 4: Jet calculus diagram for the associated single inclusive distribution of parton $q$ with transverse momentum $p_t$ and energy fraction $y$. The exchanged partons $k$, $k'$ and $k' + q$ have transverse momenta and energy fractions given by $k$, $x$, $k'$, $zx'$ and $k' + p_t$, $x'$ respectively.
Figure 1

Figure 2
Figure 3

\[ p' \]
\[ k \mu \]
\[ q \]
\[ V^\mu(q, k) \]

Figure 4

\[ m^{(\nu)}_\mu(p, k; p_1 \ldots p_r) \]