Atomic clocks: new prospects in metrology and geodesy

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Abstract

We present the latest developments in the field of atomic clocks and their applications in metrology and fundamental physics. In the light of recent advents in the accuracy of optical clocks, we present an introduction to the relativistic modelization of frequency transfer and a detailed review of chronometric geodesy.

1 Introduction

Atomic clocks went through tremendous evolutions and ameliorations since their invention in the middle of the twentieth century. The constant amelioration of their accuracy (figure 1) and stability permitted numerous applications in the field of metrology and fundamental physics. For a long time cold atom Caesium fountain clocks remained unchallenged in terms of accuracy and stability. However this is no longer

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Figure 1: Accuracy records for microwave and optical clocks. From the first Cs clock by Essen and Parry in the 1950’s, an order of magnitude is gained every ten years. The advent of optical frequency combs boosted the performances of optical clocks, and they recently overcome microwave clocks.
true with the recent development of optical clocks. This new generation of atomic clock opens new possibilities for applications, such as chronometric geodesy, and requires new developments, particularly in the field of frequency transfer. The LNE-SYRTE laboratory (CNRS/LNE/Paris Observatory/UPMC) is involved in many aspects of the development of atomic clocks and their applications.

In section 2 we present the latest developments in the field of atomic clocks: microwave clocks, optical clocks, their relation to international time-scales, means of comparisons and applications. Section 3 is an introduction to relativistic time transfer, the modelization of remote frequency comparisons, which lies at the heart of many applications of atomic clocks, such as the realization of international time-scales and chronometric geodesy. Finally, section 4 is a detailed review of the field of chronometric geodesy, an old idea which could become reality in the near future.

2 Atomic clocks

In 1967, the definition of the SI second was changed from astronomical references to atomic references by setting the frequency of an hyperfine transition in the Cs atom [39]. Since then, the accuracy of atomic clocks has improved by five orders of magnitude, enabling better and better time-keeping. More recently, a new generation of atomic clocks, based on atomic transitions in the optical domain are challenging the well established Cs standard and thus offer opportunities for new applications in fundamental physics and geodesy.

2.1 Microwave clocks

In a microwave atomic frequency standard, a microwave electro-magnetic radiation excites an hyperfine electronic transition in the ground state of an atomic species. Observing the fraction of excited atoms $p$ after this interaction (or transition probability) gives an indicator of the difference between the frequency $\nu$ of the microwave radiation and the frequency $\nu_0$ of the hyperfine atomic transition. This frequency difference $\nu - \nu_0$ (or error signal) is fed in a servo-loop that keeps the microwave radiation resonant with the atomic transition. According to Fourier’s relation, the frequency resolution that can be achieved after such an interrogation procedure grows as the inverse of the interaction time $T$, and since consecutive interrogations are uncorrelated, the frequency resolution further improves as the square root of the total integration time $\tau$. Quantitatively, the residual frequency fluctuation of the microwave radiation locked on the atomic resonance are (in dimension-less fractional units, that is to say divided by the microwave frequency):

$$\sigma_y(\tau) = \frac{\xi}{\nu_0 T \sqrt{N}} \sqrt{\frac{T_c}{\tau}},$$

where $T_c$ is the cycle time (such that $\tau/T_c$ is the number of clock interrogations), and $N$ is the number of simultaneously (and independently) interrogated atoms. $\xi$ is a numerical constant, close to unity, that depends on the physics of the interaction between the radiation and the atoms. This expression is the ultimate frequency (in)stability of an atomic clock, also called the Quantum Projection Noise (QPN) limit, referring to the quantum nature of the interaction between the radiation and the atoms. It is eventually reached if all
other sources of noise in the servo-loop are made negligible.

As seen from eq. (1), an efficient way to improve the clock stability is to increase the interaction time $T$. The first atomic clocks therefore comprised a long tube in which a thermal beam of Cs atoms is travelling while interacting with the microwave radiation.

The advance in the physics of cold atoms enabled to prepare atoms with a smaller velocity and consequently largely increase the interaction time, thus the frequency stability of atomic standards. In cold atoms clocks, the interrogation time is only limited by the time after which the atoms, exposed to gravity, escape the interrogation zone. In the atomic fountain clock (see fig. 2), a set of cold atoms are launched vertically in the interaction zone and are interrogated during their parabolic flight in the Earth gravitational field.

In micro-gravity, atoms are not subject to the Earth gravitational field and thus can be interrogated during a longer time window $T$. This will be the case for the Pharao space microwave clock [7].

The reproducibility of an atomic clock stems from the fact that all atoms of a given species are rigorously indistinguishable. For instance, two independent atomic clocks based on caesium will produce the same frequency (9,192,631,770 Hz exactly in the SI unit system), regardless of their manufacturer or their location in space-time.

However, in laboratory conditions, atoms are surrounded by an experimental environment that perturbs their electronic state and thus slightly modify their resonance frequency in a way that depends on experimental conditions. For instance, atoms will interact with DC or AC electro-magnetic perturbations, will be sensitive to collisions between them in a...
way that depends on the atomic density or to the Doppler effect resulting from their residual motion. Such systematic effects, if not evaluated and corrected for, limit the universality of the atomic standard. Consequently, the accuracy of a clock quantifies the uncertainty on these systematic effects. Currently, the best microwave atomic fountain clocks have a relative accuracy of $2 \times 10^{-16}$, which is presumably their ultimate performances given the many technical obstacles to further improve this accuracy.

2.2 SI second and time scales

About 250 clocks worldwide, connected by time and frequency transfer techniques (mainly through GNSS signals) realize a complete architecture that enable the creation of atomic time scales. For this, local time scales physically generated by metrology laboratories are a posteriori compared and common time-scales are decided upon. The International Bureau for Weights and Measures (BIPM) in Sèvres (France) is responsible for establishing the International Atomic Time (TAI). First a free atomic time-scale is build, the EAL. However, this time-scale is free-running and the participating clocks do not aim at realizing the SI second. The rate of EAL is measured by comparison with a few number of caesium atomic fountains which aim at realizing the SI second, and TAI is then derived from EAL by applying a rate correction, so that the scale unit of TAI is the SI second as realized on the rotating geoid [11].

It necessary to take into account the atomic fountain clocks frequency shift due to relativity (see section 3). Recently, a change of paradigm occurred in the definition of TAI. According to UAI resolutions [36], TAI is a realization of Terrestrial Time (TT), which is defined by applying a constant rate correction to Geocentric Coordinate Time (TCG). This definition has been adopted so that the reference surface of TAI is no longer the geoid, which is not a stable surface (see section 4). Finally, the Coordinated Universal Time (UTC) differs from TAI by an integer number of second in order to follow the irregularities of the Earth rotation.

The publication of such time-scales enables world-wide comparison of Cs fountain clocks [30, 31] and the realization of the SI second.

2.3 Optical clocks

A new generation of atomic clocks have appeared in the last 15 years. These clocks consist in locking an electromagnetic radiation in the optical domain ($\nu = 300$ to $800$ THz) to a narrow electronic transition. As seen from eq. (1), increasing the frequency $\nu_0$ of the clock frequency by several orders of magnitude drastically improves the ultimate clock stability, even though the number of interrogated atoms $N$ is usually smaller in optical clocks. Increasing the clock frequency also improves the clock accuracy since most systematic effects (sensitivity to DC electromagnetic fields, cold collisions...) are of the same order of magnitude in frequency units, and thus decrease in relative units. However, two notable exceptions remain, and both have triggered recent research in optical clocks. First, the sensitivity of the clock transition frequency on the ambient black-body radiation is mostly rejected in microwave clocks but not for optical transitions. Thus, the uncertainty on this effect usually only marginally improves when going to optical clocks (with
Ultra-stable cavity
Detection
Optical lattice
Frequency stability $\sigma(\tau)$

**Figure 3**: Sr optical lattice clock. The clock laser, prestabilized on an ultra-stable optical cavity, probes an optical transition of $10^4$ atoms confined in an optical lattice. The excitation fraction (or transition probability) is detected by a fluorescence imaging and a numerical integrator acts on a frequency shifter (FS) to keep the laser on resonance with the atomic transition. The width of the resonance is Fourier-limited at 3 Hz, which, given the clock frequency $\nu_0 = 429$ THz, yields a resonance quality factor $Q = 1.4 \times 10^{14}$, compared to $10^{10}$ for microwave clocks.

the notable exception of a few atomic species such as Al$^+$ for which the sensitivity is accidentally small). Therefore, extra care has to be taken to control the temperature of the atoms environment. Second, the Doppler frequency shift $\delta\nu$ due to the residual velocity $v$ of the atoms scales as the clock frequency $\nu_0$, such that the relative frequency shift remains constant:

$$\frac{\delta\nu}{\nu_0} = \frac{v}{c}$$

(2)

For this reason, the fountain architecture, for which the Doppler effect is one of the limitations, cannot be applicable to optical clocks, and the atoms have to be tightly confined in a trapping potential to cancel their velocity $v$. To achieve this goal, two different technologies have been developed. First, a single ion is trapped in a RF electric field. These ion optical clocks \[34, 8, 16, 25\] achieve record accuracies at $9 \times 10^{-18}$. However, since only a single ion is trapped ($N = 1$, because of the electrostatic repulsion of ions), the stability of these clocks is limited at the QPN level of $2 \times 10^{-15}$ at 1 s. Second, a more recent technology involves trapping of a few thousands neutral atoms in a powerful laser standing wave (or optical lattice) by the dipolar force. Due to its power, this trapping potential is highly perturbative, but for a given ”magic” wavelength of the trapping light \[38\], the perturbation is equal for the two clock levels, hence cancelled for the clock transition frequency. These optical lattice clocks \[38, 21, 29, 25, 26\] have already reached an accuracy of $1 \times 10^{-16}$ and are rapidly catching up with ion clocks. Furthermore, the large number of interrogated atoms allowed the demonstration of unprecedented stabilities (a few $1 \times 10^{-16}$ at 1 s), heading toward their QPN below $1 \times 10^{-17}$ at 1 s.
2.4 Remote comparison of optical frequencies

The recent breakthrough of performances of optical clocks was permitted by the development of frequency combs [44], which realizes a “ruler” in the frequency domain. These lasers enable the local comparison of different optical frequencies and comparisons between optical and microwave frequencies. However, although optical clocks now largely surpass microwave clocks, a complete architecture has to be established to enable remote comparison of optical frequencies, in order to validate the accuracy of optical clocks, build “optical” time-scales, or enable applications of optical clocks such as geodesic measurements. Indeed, the conventional remote frequency comparison techniques, mainly through GPS links, cannot reach the level of stability and accuracy realized by optical clocks.

Fibre links

To overcome the limitations of remote clock comparisons using GNSS signals, comparison techniques using optical fibres are being developed. For this, a stable and accurate frequency signal produced by an optical clock is sent through a fibre optics that links metrology institutes, directly encoded in the phase of the optical carrier. Because of vibrations and temperature fluctuations, the fibre adds a significant phase noise to the signal. This added noise is measured by comparing the signal after a round trip in the fibre to the original signal, and subsequently cancelled. Such a signal can be transported through a dedicated fibre (dark fibre) when available [33], or, more practically along with the internet communication (dark channel) [24]. These comparison techniques are applicable at a continental scale, such a network will presumably be in operation throughout Europe in the near future. In particular, the REFIMEVE+ project will provide a shared stable optical oscillator between a large number of laboratories in France, with a number of applications beside metrology. This network will be connected to international fibre links (NEAT-FT project[1]) that will enable long distance comparisons between optical frequency standards located in various national metrology laboratories in Europe. The ITOC project[2] aims at collecting these comparisons result to demonstrate high accuracy frequency ratios measurements and geophysical applications.

Space-based links

When considering inter-continental time and frequency comparisons, only satellite link are conceivable. In this aspect, The two-way satellite time and frequency transfer (TW-STFT) involves a satellite that actively relays a frequency signal in a round-trip configuration. Also, the space mission Pharao-ACES [7] that involves an ensemble of clocks on board the International Space Station will comprise a number of ground receiver able to remotely compare optical clocks.

2.5 Towards a new definition of the SI second based on optical clocks

Unlike microwave clocks, for which Cs has been an unquestionable choice over half a cen-
A large number of atomic species seem to be equally matched as a new frequency standard based on an optical transition. Ion clocks with Hg$^+$, Al$^+$, Yb$^+$, Sr$^+$, Ca$^+$ have been demonstrated, as well as optical lattice clocks with Sr, Yb and more prospectively Hg and Mg. As an illustration, four optical transitions have already been approved by the CIPM (International Committee for Weights and Measures) as secondary representation of the SI second. Therefore, although the SI second would already gain in precision with optical clocks, a clear consensus has yet to emerge before the current microwave defined SI second can be replaced.

### 2.6 Applications of optical clocks

The level of accuracy reached by optical clocks opens a new range of applications, through the very precise frequency ratios measurements they enable. In fundamental physics, they enable the tracking of dimension-less fundamental constants such as the fine structure constant $\alpha$, the electron to proton mass ratio $\mu = m_e/m_p$, or the quantum chromodynamics mass scale $m_q/\Lambda_{\text{QCD}}$. Because each clock transition frequency has a different dependence on these constants, their variations imply drifts in clock frequency ratios that can be detected by repeated measurements. Currently, combined optical-to-optical and optical-to-microwave clock comparisons put an upper bound on the relative variation of fundamental constants in the $10^{-17}$/yr range [34] [13] [21] [22]. Other tests of fundamental physics are also possible with atomic clocks. The local position invariance can be tested by comparing frequency ratios in the course of the earth rotation around the earth [13] [21], and the gravitational red-shift will be tested with clocks during the Pharao-ACES mission.

The TAI time scale is created from Cs standards, but recently, the Rb microwave transition started to contribute. In the near future, secondary representations of the SI second based on optical transitions could contribute to TAI, which would thus benefit from their much improved stability.

Clock frequencies, when compared to a coordinate time-scale, are sensitive to the gravitational potential of the Earth (see section 3). Therefore it is necessary to take into account their relative height difference when building atomic time-scales such as TAI. However, the most accurate optical clocks can resolve a height difference below 10 cm, which is a scale at which the global gravitational potential is unknown. Because of this, optical clock could become a tool to precisely measure this potential. They come as a decisive addition to relative and absolute gravimeters which are sensitive to the gravitational field, and to satellite based measurement which lack spatial resolution (see section 4).

### 3 Relativistic frequency transfer

In distant comparisons of frequency standards, we are face with the problem of curvature of space and relative motion of the clocks. These two effects change locally the flow of proper time with respect to a global coordinate time. In this section we describe how to compare distant clock frequencies by means of an electromagnetic signal, and how the comparison is affected by these effects.
3.1 The Einstein Equivalence Principle

![Diagram](image)

**Figure 4:** A photon of frequency $\nu_A$ is emitted at point A toward point B, where the measured frequency is $\nu_B$. a) A and B are two points at rest in an accelerated frame, with acceleration $\vec{a}$ in the same direction as the emitted photon. b) A and B are at rest in a non-accelerated (locally inertial) frame in presence of a gravitational field such that $\vec{g} = -\vec{a}$.

Before we treat the general case, let’s try to understand in simple terms what is the frequency shift effect. Indeed, it can be seen as a direct consequence of the Einstein Equivalence Principle (EEP), one of the pillars of modern physics [27, 41]. Let’s consider a photon emitted at point A in an accelerated reference system, toward a point B which lies in the direction of the acceleration (see fig.4). We assume that both point are separated by a distance $h_0$, as measured in the accelerated frame. The photon time of flight is $\delta t = h_0/c$, and the frame velocity during this time increases by $\delta v = a\delta t = ah_0/c$, where $a$ is the magnitude of the frame acceleration $\vec{a}$. The frequency at point B (reception) is then shifted because of Doppler effect, compared to the frequency at point A (emission), by an amount:

$$\frac{\nu_B}{\nu_A} = 1 - \frac{\delta v}{c} = 1 - \frac{ah_0}{c^2} \quad (3)$$

Now, the EEP postulates that a gravitational field $\vec{g}$ is locally equivalent to an acceleration field $\vec{a} = -\vec{g}$. We deduce that in a non-accelerated (locally inertial) frame in presence of a gravitational field $\vec{g}$:

$$\frac{\nu_B}{\nu_A} = 1 - \frac{gh_0}{c^2} \quad (4)$$

where $g = |\vec{g}|$, $\nu_A$ is the photon frequency at emission (strong gravitational potential) and $\nu_B$ is the photon frequency at reception (weak gravitational field). As $\nu_B < \nu_A$, it is usual to say that the frequency at the point of reception is “red-shifted”. One can consider it in terms of conservation of energy. Intuitively, the photon that goes from A to B has to “work” to be able to escape the gravitational field, then it loses energy and its frequency decreases by virtue of $E = h\nu$, with $h$ the Planck constant.

![Diagram](image)

**Figure 5:** Two clocks A and B are measuring proper time along their trajectory. One signal with phase $S$ is emitted by A at proper time $\tau_A$, and another one with phase $S + dS$ at time $\tau_A + d\tau_A$. They are received by clock B respectively at time $\tau_B$ and $\tau + d\tau_B$. 


3.2 General case

The principle of frequency comparison is to measure the frequency of an electromagnetic signal with the help of the emitting clock, $A$, and then with the receiving clock, $B$. We obtain respectively two measurements $\nu_A$ and $\nu_B$. Let $S(x^\alpha)$ be the phase of the electromagnetic signal emitted by clock $A$. It can be shown that light rays are contained in hypersurfaces of constant phase. The frequency measured by $A/B$ is:

$$\nu_{A/B} = \frac{1}{2\pi} \frac{dS}{d\tau_{A/B}}$$

where $\tau_{A/B}$ is the proper time along the worldline of clock $A/B$ (see fig.5). We introduce the wave vector $k^\alpha_{A/B} = (\partial_\alpha S)_{A/B}$ to obtain:

$$\nu_{A/B} = \frac{1}{2\pi} k^\alpha_{A/B} u^\alpha_{A/B}$$

where $u^\alpha_{A/B} = dx^\alpha_{A/B}/d\tau$ is the four-velocity of clock $A/B$. Finally, we obtain a fundamental relation for frequency transfer:

$$\nu_A = \nu_B k^A_0 \nu_B k^B_0$$

This formula does not depend on a particular theory, and then can be used to perform tests of general relativity. Introducing $v^i = dx^i/dt$ and $\dot{k}_i = k_i/k_0$, it is usually written as:

$$\frac{\nu_A}{\nu_B} = \frac{v_A^0 k^A_0 1 + \dot{k}^A v^i}{v_B^0 k^B_0 1 + \dot{k}^B v^i}$$

From eq. (5) we deduce that:

$$\frac{\nu_A}{\nu_B} = \frac{d\tau_B}{d\tau_A} = \left( \frac{dt}{d\tau} \right)_A \frac{dt_B}{d\tau_B} \left( \frac{d\tau}{dt} \right)_B$$

We suppose that space-time is stationary, i.e. $\partial_0 g_{\alpha\beta} = 0$. Then it can be shown that $k_0^A$ is constant along the light ray, meaning that $k_0^A = k_0^B$. We introduce the time transfer function:

$$T(x_A^i, x_B^i) = t_B - t_A$$

Deriving the time transfer function with respect to $t_A$ one obtains:

$$\frac{dt_B}{dt_A} = \frac{1 + \partial T}{1 - \partial T}$$

Inserting eq. (11) in eq. (9), and comparing with eq. (8), we deduce:

$$\dot{k}_i^A = \frac{c}{k_0^A} \frac{\partial T}{\partial x^i_A}, \quad \dot{k}_i^B = c \frac{\partial T}{\partial x^i_B}$$

General formula for non-stationary spacetimes can be found in [14, 20].

As an exemple, let’s take the simple time transfer function:

$$T(x^i_A, x^i_B) = \frac{R_{AB}}{c} + O\left(\frac{1}{c^3}\right)$$

where $R_{AB} = |R_{AB}^i|$ and $R_{AB}^i = x^i_B - x^i_A$. Then we obtain:

$$\frac{dt_B}{dt_A} = 1 + \frac{N^i_{AB} v_A^i}{c} + O\left(\frac{1}{c^3}\right)$$

where $N^i_{AB} = R_{AB}^i/R_{AB}$. Up to the second order, this term does not depend on the gravitational field but on the relative motion of the two clocks. It is simply the first order Doppler effect.
The two other terms in eq. (9) depend on the relation between proper time and coordinate time. In a metric theory one has \( c^2 \frac{dt}{c} = \sqrt{g_{\alpha \beta} dx^\alpha dx^\beta} \). We deduce that:

\[
\frac{u_A^0}{u_B^0} = \frac{[g_{00} + 2g_{0i} \frac{v_i}{c^2} + g_{ij} v_i v_j]^2]_{B}}{[g_{00} + 2g_{0i} \frac{v_i}{c^2} + g_{ij} v_i v_j]^2]_{A}} \quad (15)
\]

### 3.3 Application to a static, spherically symmetric body

As an example, we apply this formalism for the case of a parametrized post-Minkowskian approximation of metric theories. This metric is a good approximation of the space-time metric around the Earth, for a class of theories which is larger than general relativity. Choosing spatial isotropic coordinates, we assume that the metric components can be written as:

\[
g_{00} = -1 + 2(\alpha + 1) \frac{GM}{rc^2} + \mathcal{O} \left( \frac{1}{c^4} \right)
\]

\[
g_{0i} = 0
\]

\[
g_{ij} = \delta_{ij} \left[ 1 + 2\gamma \frac{GM}{rc^2} + \mathcal{O} \left( \frac{1}{c^4} \right) \right]
\]

(16)

where \( \alpha \) and \( \gamma \) are two parameters of the theory (for general relativity \( \alpha = 0 \) and \( \gamma = 1 \)). Then:

\[
\frac{dt}{dt} = 1 - (\alpha + 1) \frac{GM}{rc^2} - \frac{v^2}{2c^2} + \mathcal{O} \left( \frac{1}{c^4} \right)
\]

(17)

where \( v = \sqrt{\delta_{ij} v^i v^j} \). The time transfer function of such a metric is given in eq. (108) of [20]. Then we calculate:

\[
\frac{dt_B}{dt_A} = \frac{q_A + \mathcal{O} \left( \frac{1}{c^7} \right)}{q_B + \mathcal{O} \left( \frac{1}{c^7} \right)} \quad (18)
\]

where

\[
q_A = 1 - \frac{\vec{N}_{AB} \cdot \vec{v}_A}{c} - \frac{2(\gamma + 1)GM}{c^3} \times \frac{R_{AB}\vec{N}_A \cdot \vec{v}_A + (r_A + r_B)\vec{N}_{AB} \cdot \vec{v}_A}{(r_A + r_B)^2 - R_{AB}^2}
\]

(19)

\[
q_B = 1 - \frac{\vec{N}_{AB} \cdot \vec{v}_A}{c} - \frac{2(\gamma + 1)GM}{c^3} \times \frac{R_{AB}\vec{N}_A \cdot \vec{v}_B - (r_A + r_B)\vec{N}_{AB} \cdot \vec{v}_B}{(r_A + r_B)^2 - R_{AB}^2}
\]

(20)

Let assume that both clocks are at rest with respect to the chosen coordinate system, ie. \( \vec{v}_A = 0 = \vec{v}_B \) and that \( r_A = r_0 \) and \( r_B = r_0 + \delta r \), where \( \delta r \ll r_0 \). Then we find:

\[
\frac{\nu_A - \nu_B}{\nu_B} = (\alpha + 1) \frac{GM}{r_0^2 c^2} \delta r + \mathcal{O} \left( \frac{1}{c^4} \right)
\]

(21)

The same effect can be calculated with a different, not necessarily symmetric gravitational potential \( w(t, x^i) \). The results yields:

\[
\frac{\nu_A - \nu_B}{\nu_B} = \frac{w_A - w_B}{c^2} + \mathcal{O} \left( \frac{1}{c^4} \right)
\]

(22)

This is the classic formula given in textbooks for the "gravitational red-shift". However one should bear in mind that this is valid for clocks at rest with respect to the coordinate system implicitly defined by the space-time metric (one uses usually the Geocentric Celestial Reference System), which is (almost) never the case. Moreover, the separation between a gravitational red-shift and a Doppler effect is specific to the approximation scheme used here. One can read the book by Synge [37] for a different interpretation in terms of relative velocity and Doppler effect only.

We note that the lowest order gravitational term in eq. (21) is a test of the Newtonian limit.
of metric theories. Indeed, if one wants to recover the Newtonian law of gravitation for $GM/rc^2 \ll 1$ and $v \ll 1$ then it is necessary that $\alpha = 0$. Then this test is more fundamental than a test of general relativity, and can be interpreted as a test of Local Position Invariance (which is a part of the Einstein Equivalence Principle). See \cite{41, 42} for more details on this interpretation, and a review of the experiments that tested the parameter $\alpha$.

A more realistic case of the space-time metric is treated in article \cite{3}, in the context of general relativity: all terms from the Earth gravitational potential are considered up to an accuracy of $5.10^{-17}$, specifically in prevision of the ACES mission \cite{7}.

4 Chronometric geodesy

Instead of using our knowledge of the Earth gravitational field to predict frequency shifts between distant clocks, one can revert the problem and ask if the measurement of frequency shifts between distant clocks can improve our knowledge of the gravitational field. To do simple orders of magnitude estimates it is good to have in mind some correspondences calculated thanks to eqs. (21) and (22):

$$1 \text{ meter} \leftrightarrow \frac{\Delta \nu}{\nu} \sim 10^{-16}$$
$$\leftrightarrow \Delta \nu \sim 10 \text{ m}^2 \cdot \text{s}^{-2}$$

(23)

From this correspondence, we can already imagine two direct applications of clocks in geodesy: if we are capable to compare clocks to $10^{-16}$ accuracy, we can determine height differences between clocks with one meter accuracy (levelling), or determine geopotential differences with $10 \text{ m}^2 \cdot \text{s}^{-2}$ accuracy.

4.1 A review of chronometric geodesy

The first article to explore seriously this possibility was written in 1983 by Martin Vermeer \cite{40}. The article is named “chronometric levelling”. The term “chronometric” seems well suited for qualifying the method of using clocks to determine directly gravitational potential differences, as “chronometry” is the science of the measurement of time. However the term “levelling” seems too restrictive with respect to all the applications one could think of using the results of clock comparisons. Therefore we will use the term “chronometric geodesy” to name the scientific discipline that deals with the measurement and representation of the Earth, including its gravitational field, with the help of atomic clocks. It is sometimes named “clock-based geodesy”, or “relativistic geodesy”. However this last designation is improper as relativistic geodesy aims at describing all possible techniques (including e.g. gravimetry and gradiometry) in a relativistic framework \cite{18, 28, 35}.

The natural arena of chronometric geodesy is the four-dimensional space-time. At the lowest order, there is proportionality between relative frequency shift measurements – corrected from the first order Doppler effect – and (Newtonian) geopotential differences (see eq. (23)). To calculate this relation we have seen that we do not need the theory of general relativity, but only to postulate Local Position Invariance. Therefore, it is perfectly possible to use clock comparison measurements – corrected from the first order Doppler effect – as a direct measurement of geopotential differences in the framework of classical geodesy, if the measurement accuracy does not reach the magnitude of the higher order terms.
Comparisons between two clocks on the ground generally use a third clock in space. For the comparison between a clock on the ground and one in space, the terms of order $c^{-3}$ in eqs. (19)-(20) reach a magnitude of $\sim 10^{-15}$ and $\sim 3 \times 10^{-14}$ for respectively the ground and the space clock, if they are separated radially by 1000 km. Terms of order $c^{-4}$ omitted in eqs. (21)-(22) can reach $\sim 5 \times 10^{-19}$ in relative frequency shift, which corresponds to a height difference of $\sim 5 \times 10^{-2}$ m and a geopotential difference of $\sim 5 \times 10^{-2}$ m\(^2\)s\(^{-2}\). Clocks are far from reaching this accuracy today, but it cannot be excluded for the future.

In his article, Martin Vermeer explores the “possibilities for technical realisation of a system for measuring potential differences over intercontinental distances” using clock comparisons [40]. The two main ingredients are of course accurate clocks and a mean to compare them. He considers hydrogen maser clocks. For the links he considers a 2-way satellite link over a geostationary satellite, or GPS receivers in interferometric mode. He has also to consider a mean to compare the proper frequencies of the different hydrogen maser clocks. However today this can be overcome by comparing Primary Frequency Standards (PFS), which have a well defined proper frequency based on a transition of Caesium 133, used for the definition of the second. However, this problem will rise again if one uses Secondary Frequency Standards which are not based on Caesium atoms. Then the proper frequency ratio between two different kinds of atomic clocks has to be determined locally. This is one of the purpose of the European project “International timescales with optical clocks” [1] where optical clocks based on different atoms will be compared one each other locally, and to the PFS. It is planned also to do a proof-of-principle experiment of chronometric geodesy, by comparing two optical clocks separated by a height difference of around 1 km using an optical fibre link.

In the foreseen applications of chronometric geodesy, Martin Vermeer mentions briefly intercontinental levelling, and the measurement of the true geoid (disentangled from geophysical sea surface effects) [40]. Few authors have seriously considered chronometric geodesy. Following Vermeer idea, Brumberg and Groten [5] demonstrated the possibility of using GPS observations to solve the problem of the determination of geoid heights, however leaving aside the practical feasibility of such a technique. Bondarescu et al. [4] discuss the value and future applicability of chronometric geodesy, including direct geoid mapping on continents and joint gravity-geopotential surveying to invert for subsurface density anomalies. They find that a geoid perturbation caused by a 1.5 km radius sphere with 20 per cent density anomaly buried at 2 km depth in the Earth’s crust is already detectable by atomic clocks of achievable accuracy. Finally Chou et al. demonstrated the potentiality of the new generation of atomic clocks, based on optical transitions, to measure heights with a resolution of around 30 cm [9].

4.2 The chronometric geoid

Arne Bjerhammar in 1985 gives a precise definition of the “relativistic geoid” [1]:

“The relativistic geoid is the surface where precise clocks run with the same speed and the surface is nearest to mean sea level”
This is an operational definition. Soffel et al. [35] in 1988 translated this definition in the context of post-Newtonian theory. They also introduce a different operational definition of the relativistic geoid, based on gravimetric measurements: a surface orthogonal everywhere to the direction of the plumb-line and closest to mean sea level. He calls the two surfaces obtained with clocks and gravimetric measurements respectively the “u-geoid” and the “a-geoid”. He proves that these two surfaces coincide in the case of a stationary metric. In order to distinguish the operational definition of the geoid from its theoretical description, it is less ambiguous to give a name based on the particular technique to measure it. Relativistic geoid is too vague as Soffel et al. have defined two different ones. The names chosen by Soffel et al. are not particularly explicit, so instead of “u-geoid” and “a-geoid” one can call them chronometric and gravimetric geoid respectively.

Let two clocks be at rest with respect to the chosen coordinate system ($v^i = 0$) in an arbitrary space-time. From formula (9), (11) and (15) we deduce that:

$$\nu_B = \frac{d\tau_B}{d\tau_A} = \left[\frac{g_{00}}{g_{00}}\right]^{1/2}$$

In this case the chronometric geoid is defined by the condition $g_{00} = \text{cst}$.

We notice that the problem of defining a reference surface is closely related to the problem of realizing Terrestrial Time (TT). TT is defined with respect to Geocentric Coordinate Time (TCG) by the relation [36 32]:

$$\frac{dT}{dTCG} = 1 - L_G$$

where $L_G$ is a defining constant. This constant has been chosen so that TT coincides with the time given by a clock located on the classical geoid. It could be taken as a formal definition of the chronometric geoid [43]. If so, the chronometric geoid will differ in the future from the classical geoid: a level surface of the geopotential closest to the topographic mean sea level. Indeed, the value of the potential on the geoid, $W_0$, depends on the global ocean level which changes with time [6]. With a value of $dW_0/dt \sim 10^{-3}$ m$^2$.s$^{-2}$.y$^{-1}$, the difference in relative frequency between the classical and the chronometric geoid would be around $10^{-18}$ after 100 years (the correspondence is made with the help of relations (23)). However, the rate of change of the global ocean level could change during the next decades. Predictions are highly model dependent [17].

5 Conclusion

We presented recent developments in the field of atomic clocks, as well as an introduction to relativistic frequency transfer and a detailed review of chronometric geodesy. If the control of systematic effects in optical clocks keep their promises, they could become very sensible to the gravitational field, which will ultimately degrade their stability at the surface of the Earth. One solution will be to send very stable and accurate clocks in space, which will become the reference against which the Earth clocks would be compared. Moreover, by sending at least four of these clocks in space,
it will be possible to realize a very stable and accurate four-dimensional reference system in space [10].

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