A functional limit theorem for the sine-process

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Abstract

The main result of this paper is a functional limit theorem for the sine-process. In particular, we study the limit distribution, in the space of trajectories, for the number of particles in a growing interval. The sine-process has the Kolmogorov property and satisfies the Central Limit Theorem, but our functional limit theorem is very different from the Donsker Invariance Principle. We show that the time integral of our process can be approximated by the sum of a linear Gaussian process and independent Gaussian fluctuations whose covariance matrix is computed explicitly. We interpret these results in terms of the Gaussian Free Field convergence for the random matrix models. The proof relies on a general form of the multidimensional Central Limit Theorem under the sine-process for linear statistics of two types: those having growing variance and those with bounded variance corresponding to observables of Sobolev regularity 1/2.

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1 Introduction

1.1 Formulation of the main result

In this paper we study the asymptotic behaviour of trajectories of determinantal random point processes; for basic definitions and background concerning determinantal point processes, see Section 2.2 below. Mostly we deal with the sine-process given by the kernel

\[ K_{\text{sine}}(x, y) = \frac{\sin(x - y)}{\pi(x - y)} \text{ if } x \neq y \text{ and } K_{\text{sine}}(x, x) \equiv \frac{1}{\pi}, \quad x, y \in \mathbb{R}. \]  

(1.1)

The sine-process is a strongly chaotic stationary process: it satisfies the Kolmogorov property [27], [13], [31], having, therefore, Lebesgue spectrum and positive entropy, and enjoys an analogue of the Gibbs property [9, 10], namely, the quasi-invariance under the group of diffeomorphisms with compact support. At the same time, the sine-process is rigid in the sense of Ghosh and Peres [18, 19]: the number of particles in a bounded interval is almost surely determined by the configuration in its exterior. The reason for the rigidity is the slow growth of the variance for the sine-process: for instance, the number of particles \( \#_{[0,N]} \) in the interval \( [0, N] \) satisfies

\[ \text{Var } \#_{[0,N]} = \frac{1}{\pi^2} \ln N + O(1), \]  

(1.2)

see e.g. Exercise 4.2.40 from [1]. This slow growth of the variance can be seen from the form \( \rho(\theta) = |\theta| \) of the spectral density for the sine-process, cf. [38], [12]. Costin and Lebowitz [14] showed that the sine-process satisfies the Central Limit Theorem: the random variable

\[ \xi_N := \frac{\#_{[0,N]} - E \#_{[0,N]}}{\sqrt{\text{Var } \#_{[0,N]}}} \]  

(1.3)
converges in distribution to the normal law:
\[
\mathcal{D}(\hat{\xi}_N) \to \mathcal{N}(0, 1) \quad \text{as} \quad N \to \infty.
\] (1.4)

Here \(\mathbb{E}, \text{Var}\) and \(\mathcal{D}\) stand for the expectation, variance and distribution under the sine-process.

The Central Limit Theorem was subsequently proven for arbitrary determinantal processes governed by self-adjoint kernels and arbitrary additive statistics with growing variance (see [38, 39, 40, 41, 20]), in particular, for the Airy and Bessel processes ([39]), for which the variance of the number of particles has logarithmic growth and rigidity holds [11].

Many classical dynamical systems satisfying the Central Limit Theorem also satisfy the Donsker Invariance Principle, which, informally speaking, states that trajectories of the system can be approximated by the Brownian motion, cf. [37]. The main result of this paper is a functional limit theorem for the sine-process. The limit dynamics is completely different from Brownian motion. As far as we know, this is the first example of such behaviour in the theory of dynamical systems. More specifically, we investigate asymptotic behaviour, as \(N \to \infty\), of the piecewise continuous random process
\[
\xi^N_t = \frac{\#[0,tN] - \mathbb{E}\#[0,tN]}{\pi^{-1}\sqrt{\ln N}}, \quad 0 \leq t \leq 1,
\] (1.5)
under the sine-process. Trajectories of the process \(\xi^N_t\) become extremely irregular when \(N\) grows (see Proposition 1.3), so that the sequence of distributions of trajectories \(\mathcal{D}(\xi^N_t)\) does not have a limit in any separable metric space. That is why, instead of the process \(\xi^N_t\) itself we study its time integral \(\int_0^t \xi^N_s \, ds\) in the space of continuous functions \(C([0,1], \mathbb{R})\).

We fix \(0 < \tau \leq 1\) and set
\[
\eta^N := \frac{1}{\tau} \int_0^\tau \xi^N_s \, ds \quad \text{and} \quad z^N_t := \pi^{-1}\sqrt{\ln N}\left(\int_0^t \xi^N_s \, ds - t\eta^N\right),
\] (1.6)
so that
\[
\int_0^t \xi^N_s \, ds = t\eta^N + \frac{z^N_t}{\pi^{-1}\sqrt{\ln N}}.
\] (1.7)

The parameter \(\tau\) is fixed throughout the paper, and we skip it in the notation. Recall that the Gaussian Free Field in the plain \(\mathbb{R}^2\) is a generalized centred Gaussian process in \(\mathbb{R}^2\) given by the covariance function
\[
\mathcal{G}(t, s) = -\frac{1}{2\pi} \ln |t - s|, \quad t, s \in \mathbb{R}.
\] (1.8)

Let
\[
W(t, s) := \pi^{-1}\left(\int_0^t \int_0^s \frac{1}{\tau} \int_0^\tau \int_0^\tau + \frac{ts}{\tau^2} \int_0^\tau \int_0^\tau \left(\mathcal{G}(u, v) - \mathcal{G}(u, 0) - \mathcal{G}(0, v)\right) du dv\right).
\] (1.9)
\[
\int_0^t \xi_s^N \, ds = \eta^N t + \frac{z_t}{\sqrt{\ln N}} + o\left( \frac{1}{\sqrt{\ln N}} \right)
\]

Figure 1: Up to terms of the size \( o\left( (\ln N)^{-1/2} \right) \), the integral \( \int_0^t \xi_s^N \, ds \) decomposes to the sum of the linear in time process \( \eta^N t \) and Gaussian fluctuations \( \frac{z_t}{\sqrt{\ln N}} \). Deviation of the process \( \eta^N t \) from the process \( \eta t \) is of the size \( \frac{C}{\sqrt{\ln N}} \).

Computing (1.9) explicitly, we find the following expression. Set \( \theta(t) := \frac{t^2 \ln |t|}{4\pi^2} \) and \( \theta(0) := 0 \). Then
\[
W(t, s) = w(t, s) + w(s, t),
\]
where
\[
w(t, s) := \frac{1}{2} \theta(t - s) - \left( 1 - \frac{s}{\tau} \right) \theta(t) - \frac{s}{\tau} \theta(t - \tau) + \frac{t}{\tau} \left( 1 - \frac{s}{\tau} \right) \theta(\tau).
\]

Our first main result is

**Theorem 1.1.** For any \( 0 < \tau \leq 1 \), under the sine-process we have the weak convergence of measures
\[
\mathcal{D}(\eta^N, z^N) \rightarrow \mathcal{D}(\eta, z) \quad \text{as} \quad N \rightarrow \infty \quad \text{in} \quad \mathbb{R} \times C([0,1], \mathbb{R}),
\]
where \( \eta \) and \( z \) are independent, \( \eta \sim \mathcal{N}(0, 1/2) \) and \( z_t \) is a centred continuous Gaussian random process with the covariances
\[
\mathbb{E} z_t z_s = W(t, s), \quad 0 \leq t, s \leq 1.
\]

Proof of Theorem 1.1 is given in Section 5.1. Informally, Theorem 1.1 states that, up to terms of the size \( o\left( (\ln N)^{-1/2} \right) \), the process \( \int_0^t \xi_s^N \, ds \) can be decomposed to a linear random process \( \eta^N t \) and small Gaussian fluctuations \( \frac{z_t}{\sqrt{\ln N}} \), see figure (1). Here \( z_t \) is a continuous centred Gaussian process whose covariances (1.11), which we compute explicitly, are governed by the Gaussian Free Field in the plain; for explanation of this phenomenon see Section 1.3. About the linear process \( \eta^N t \) we know that asymptotically it is governed by the process \( \eta t \), where \( \eta \sim \mathcal{N}(0, 1/2) \) is independent from \( z \). For the rate of convergence of \( \eta^N \) to \( \eta \) we have

**Proposition 1.2.** Cumulants \( (A_k^N) \) and \( (A_k) \) of the random variables \( \eta^N \) and \( \eta \) satisfy \( A_1^N = A_1 = 0 \), \( |A_2^N - A_2| \leq C_2 (\ln N)^{-1} \) and
\[
|A_k^N - A_k| \leq \frac{C_k}{(\ln N)^{k/2 - 1}} \quad \text{for all} \quad k \geq 3,
\]
with some constants \( C_k \).
For a short reminding about cumulants see the beginning of Section 3. Proof of Proposition 1.2 is given in Section 5.1. Informally, Proposition 1.2 states that the deviation of the process $\eta^N t$, specifying the linear growth of the integral (1.7), from the process $\eta t$ is of the size $(\ln N)^{-1/2}$. So that, it coincides with the size of the term $z_t/\pi^{-1}\sqrt{\ln N}$, specifying the nonlinear fluctuations.

Theorem 1.1 has the following statistical interpretation. In order to predict behaviour of the process $\int_0^t \xi^N_s \, ds$ on the whole time interval $0 \leq t \leq 1$ it suffices to know its realization at arbitrarily small positive time $\tau$. Indeed, then we determine $\eta^N$ by the formula (1.6) and approximate the integral $\int_0^t \xi^N_s \, ds$ by the sum $\eta^N t + \frac{z_t}{\pi^{-1}\sqrt{\ln N}}$, where $z_t$ is the Gaussian process from Theorem 1.1.

The main order asymptotic $D(\eta^N) \xrightarrow{N \to \infty} D(\eta)$ from Theorem 1.1 only uses the logarithmic growth of the variance and holds for a general determinantal process with logarithmically growing variance (in particular, similar convergence takes place under the Airy and Bessel processes). We show this in Section 6. To prove the asymptotic $D(z^N) \to D(z)$ however we crucially use the form of the sine-kernel (1.1). More specifically, this asymptotic relies on a multidimensional Central Limit Theorem 1.10 discussed in Section 1.5. To establish the latter we analyse the corresponding cumulants using a combinatorial identity (4.38) which is due to Soshnikov [40] and is specific for the sine-process. While we expect the result to hold for the discrete sine-process, additional arguments are needed. It would be interesting to establish the convergence analogous to $D(z^N) \to D(z)$ for a general determinantal process with logarithmically growing variance by using some different method, e.g. that of contour integrals developed in [3] (for discussion of this paper see Section 1.3).

The rest of Section 1 is organized as follows. In the next subsection we describe motivation behind Theorem 1.1. In Section 1.3 we explain why the Gaussian Free Field correlation function appears in (1.9). In Section 1.4 we state Theorem 1.9 which is our second main result. There we show that a large class of observables — ergodic integrals corresponding to a shift operator on the space of configurations, has exactly the same asymptotic behaviour under the sine-process as that described by Theorem 1.1. In Section 1.5 we discuss the multidimensional Central Limit Theorem mentioned above, which is the main ingredient of the proofs of Theorems 1.1 and 1.9. In Section 1.6 we outline the proof of Theorem 1.1.

### 1.2 Finite dimensional distributions and motivation behind Theorem 1.1

We first look at the finite-dimensional distributions of the process $\xi^N_t$. Let $\eta_t$, $0 \leq t \leq 1$, be a family of independent identically distributed Gaussian random variables satisfying $\eta_t \sim N(0,1/2)$.

**Proposition 1.3.** For any $0 \leq t_1 < \ldots < t_d \leq 1$, $d \geq 1$, we have

$$D(\xi^N_{t_1}, \ldots, \xi^N_{t_d}) \to D(\eta_{t_0} - \eta_{t_1}, \ldots, \eta_{t_0} - \eta_{t_d}) \quad \text{as} \quad N \to \infty. \quad (1.13)$$

Proposition 1.3 generalizes convergence (1.4) to many dimensions and is established in Section 5.1. Without a detailed proof a similar result was stated by Soshnikov, see [38],

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1 We are deeply grateful to Leonid Petrov for this remark.
p. 962 and [39], p. 499. Note that the covariance matrix \( (b_{ij})_{1 \leq i,j \leq N} \) of the limiting vector 
\( (\eta_0 - \eta_t, \ldots, \eta_0 - \eta_d) \) has the form

\[
b_{ij} = 1/2 + \delta_{ij}/2,
\]

(1.14)

where \( \delta_{ij} \) is the Kronecker symbol; here we assume \( t_1 \neq 0 \) since otherwise \( b_{ij} = 0 \) due
to the identity \( \eta_0 - \eta_t = 0 \). The covariance matrix \( (b_{ij}) \) is independent from the choice
of times \( t_1, \ldots, t_d \). In particular, this means that if the limit as \( N \to \infty \) of the process
\( \xi_t^N \) exists in some sense, then it can not be a continuous process, so nothing as Brownian
motion can appear. Note that the proof of the convergence (1.13) crucially uses the
logarithmic growth of the variance (1.2).

Remark 1.4. In the sense of finite-dimensional distributions, asymptotic for large \( N \) behaviour of the process \( \xi_t^N \) is close to behaviour of the fractional Brownian motion
\( B^H_t \) with small parameter \( H \ll 1 \). Indeed, in Lemma 4.1 of [5] it is pointed out that
\( \mathcal{D}(B^H_{t_1}, \ldots, B^H_{t_d}) \to \mathcal{D}(\Delta \eta) \) as \( H \to 0 \), where \( \Delta \eta \) is the Gaussian vector from Proposition 1.3.

In view of the convergence (1.13), we expect that the limiting behaviour of the process
\( \xi_t^N \) is governed by the increments \( \eta_0 - \eta_t \) of the process \( \eta_t \). However, the process \( \eta_t \) does
not exist in a classical sense (more precisely, it cannot be defined over a separable metric
space). That is why, in order to regularize the limiting dynamics, instead of the process
\( \xi_t^N \) we study its time integral \( \int_0^t \xi_s^N \, ds \). We expect that when \( N \to \infty \) the latter is governed
by the difference \( \int_0^t \eta_0 - \eta_s \, ds = \eta_0 t - \int_0^t \eta_s \, ds \) where the integral \( \int_0^t \eta_s \, ds \) should be defined
appropriately. Here one can draw an analogy with the white noise, which is not defined in
the classical sense but its time integral gives the Brownian motion. However, this heuristic
idea leads us to the following rigorous result.

**Proposition 1.5.** For any function \( \phi \in L^1[0,1] \) we have

\[
\mathcal{D}\left( \int_0^1 \phi(s) \, \xi_s^N \, ds \right) \to \mathcal{D}(\eta_0 \int_0^1 \phi(s) \, ds) \quad \text{as} \quad N \to \infty,
\]

(1.15)

where \( \eta_0 \sim \mathcal{N}(0,1/2) \).

Remark 1.6. Here and below by the normal law with zero expectation and variance
we understand the Dirac delta-measure at zero \( \delta_0 \). In particular, if \( \int_0^1 \phi(s) \, ds = 0 \) then
\( \mathcal{D}\left( \int_0^1 \phi(s) \, \xi_s^N \, ds \right) \to \delta_0 \).

Choosing \( \phi = \mathbb{I}_{[0,t]} \) we find the leading term of the asymptotic for the process \( \int_0^t \xi_s^N \, ds \),
claimed in Theorem 1.1:

\[
\mathcal{D}\left( \int_0^t \xi_s^N \, ds \right) \to \mathcal{D}(\eta t) \quad \text{as} \quad N \to \infty,
\]

(1.16)
where we set $\eta := \eta_0$. Thus, we do not observe the integral $\int_0^t \eta_s \, ds$. The reason is that the process $\eta_t$ is completely uncorrelated in time and has a bounded variance (in difference with the white noise whose variance is the delta-function). So that, $\eta_t$ oscillates fast with not very large amplitude and averages out under the integration over the interval $[0, t]$. Note that convergence (1.15) takes place even for a very rough observable $\phi$: only integrability of $\phi$ is assumed.

Proposition 1.5 is a particular case of Proposition 6.1, in which we establish a stronger result for an important class of determinantal point processes including those with logarithmically growing variance, as the sine, Airy and Bessel processes; see Section 6.

Proposition 1.5 gives some information about the asymptotic behaviour of the process $\xi^N_t$. But we lose a lot: we do not observe any influence of the process $\eta_t$ which we find at the level of finite dimensional distributions. Our next goal is to catch the process $\eta_t$. The informal identity $\int_0^t \eta_s \, ds = 0$ resembles the law of large numbers. To observe the influence of $\eta_t$ we try to look at the Central Limit Theorem scaling. Since we expect that, informally,

$$\int_0^t \xi^N_s \, ds - \eta_0 t \to - \int_0^t \eta_s \, ds \quad \text{as} \quad N \to \infty,$$

we need to find a sequence $\alpha_N \to \infty$ as $N \to \infty$, such that the random process

$$z^N_t = \alpha_N \left( \int_0^t \xi^N_s \, ds - \eta_0 t \right)$$

converges to a non-trivial limit. However, joint distribution of the process $\xi^N_t$ and the random variable $\eta_0$ is undefined. To overcome this difficulty we note that, due to (1.16), $\mathcal{D}(\eta^N) \overset{N \to \infty}{\to} \mathcal{D}(\eta_0)$ where $\eta^N$ is defined in (1.6), and replace in the definition (1.17) of the process $z^N_t$ the random variable $\eta_0$ by $\eta^N$. Then, setting $\alpha_N = \pi^{-1} \sqrt{\ln N}$ we arrive at Theorem 1.1.

**Remark 1.7.** It could seem that influence of the process $\eta_t$ could be discovered by consideration of some nonlinear functional of the process $\xi^N_t$ such as, for example, the integral

$$\int_0^1 \phi(t)(\xi^N_t)^m \, dt \quad \text{for integer} \quad m \geq 2, \quad \text{where} \quad \phi \in L^1[0, 1].$$

However, this is not the case. Indeed, we expect that $(\xi^N_t)^m \approx (\eta_0 - \eta_t)^m = \sum_{k=0}^m (-1)^k C^k_m \eta^k \eta_0^{m-k}$, if $N$ is large. Since terms $\eta^k_t$ and $\eta^k_s$ are independent for $t \neq s$, the situation here is similar to that of Proposition 1.5: the integral $\int_0^1 \phi(t)\eta^k_t \, dt$ averages the terms $\eta^k_t$, so feels only their means $\mathbf{E} \eta^k_t$. More precisely, one can prove that

$$\mathcal{D}\left( \int_0^1 \phi(t)(\xi^N_t)^m \, dt \right) \overset{N \to \infty}{\to} \mathcal{D}\left( \int_0^1 \phi(s) \, ds \sum_{k=0}^m (-1)^k C^k_m \eta_0^{m-k} \mathbf{E} \eta^k_t \right). \quad (1.18)$$
Comparing with the right-hand side of (1.15), the r.h.s. of (1.18) depends on the moments 
\( E \eta^k_t \), so that now we feel the “noise” \( \eta_t \) but in a trivial way. Indeed, all the randomness is still due to \( \eta_0 \), although modified by the moments of \( \eta_t \).

### 1.3 Connection with the Gaussian Free Field

In this section we explain in heuristic way the appearance of the Gaussian Free Field in the formula (1.9). Let us recall that the Gaussian Free Field in the plane \( \mathbb{R}^2 \) is a generalized centred Gaussian process in \( \mathbb{R}^2 \) whose covariance is given by the Green function \( G \) of the Laplace operator in the plane, \( \Delta G(t, s) = \delta(t - s) \), where \( \delta \) is the Dirac delta-function and \( t, s \in \mathbb{R} \). Explicitly, the function \( G \) takes the form (1.8). The Gaussian Free Field in the upper half-plane \( \mathbb{H} := \{ z \in \mathbb{C} : \Im z > 0 \} \) is defined similarly: its correlation function \( \tilde{G} \) is given by the Green function of the Laplace operator in \( \mathbb{H} \) with the Dirichlet boundary conditions. Explicitly,

\[
\tilde{G}(z, w) = -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \bar{w}} \right|, \quad z, w \in \mathbb{C}.
\]

The Gaussian Free Field is known to describe the asymptotic behaviour of the so-called height function in many models that have the random matrix-type behaviour. In particular, this is the case for a class of random surfaces [43, 23, 16, 3, 24, 32] and random matrices [2, 4]. Below we explain this in more details on the examples provided by [3, 2].

Let \( A^N \) be an \( N \times N \) random matrix from the Gaussian Unitary Ensemble. That is, \( A^N = (a_{ij}^N)_{1 \leq i, j \leq N} \) is Hermitian, where \( a_{ll}^N \sim \mathcal{N}(0, 1) \), \( \Re a_{ij}^N, \Im a_{ij}^N \sim \mathcal{N}(0, 1/2) \) for \( i \neq j \) and \( a_{ll}, \Re a_{ij}, \Im a_{ij} \) are independent for all \( l \) and \( i > j \). Introduce the height function \( h^N_t \) by the formula

\[
h^N_t := \{ \text{the number of eigenvalues of } A^N \text{ that are } \geq \sqrt{Nt} \}.
\]

In [2] it is shown that the limiting behaviour of the random process \( h^N_t \) is governed by the Gaussian Free Field in the upper half-plane. In particular, it is proven that, roughly speaking,

\[
\text{Cov}(h^N_t, h^N_s) \to \text{const } \tilde{G}(y_t, y_s) \quad \text{as } N \to \infty \quad \text{for } t \neq s,
\]

where \( y_t, y_s \in \mathbb{H} \) are defined appropriately.

In [3] a family of stochastic growth models in \( 2 + 1 \) dimensions is considered, that belongs to the anisotropic KPZ class. For an appropriately defined height function \( h^N_t \), where \( t \) is a parameter from \( \mathbb{R}^3 \), the Central Limit Theorem is proven,

\[
\mathcal{D} \left( \frac{h^N_t - \mathbb{E} h^N_t}{\sqrt{\ln N}} \right) \to \mathcal{D}(\eta_t) \quad \text{as } N \to \infty \quad \text{for any } t,
\]

where \( \eta_t \sim \mathcal{N}(0, \sigma^2) \), for some \( \sigma^2 > 0 \) independent from \( t \). In particular, we have \( \text{Var } h^N_t \sim \sigma^2 \ln N \). On the other hand, as in (1.19), it is shown that for \( t \neq s \) the limiting covariance \( \text{Cov}(h^N_t, h^N_s) \) of the height function is governed by the Gaussian Free Field in the upper half-plane, without the normalization factor \((\ln N)^{-1}\) (actually, in [3] a similar result is also proven for higher order moments).

Since the asymptotic behaviour of the spectrum of a large class of Wigner matrices (in particular, of the Gaussian Unitary Ensemble) is governed by the sine-process, the
appearance of the Gaussian Free Field in Theorem 1.1 is not surprising. Indeed, the random process \( \#_{[0,Nt]} \) we are interested in can be represented as

\[
\#_{[0,Nt]} = h_0^N - h_t^N, \tag{1.21}
\]

where we set \( h_s^N := \#_{[s,N \cdot \infty]} \). However, the height function \( h_s^N \) in this case is defined only informally, since we have \( h_s^N = \infty \) almost surely. Then, heuristically, the process \( \xi_t^N \) defined in (1.5) takes the form

\[
\xi_t^N = \frac{h_0^N - h_t^N - \mathbb{E}(h_0^N - h_t^N)}{\pi^{-1} \sqrt{\ln N}}.
\]

Drawing an analogy with (1.19) and (1.20), one could expect the convergence

\[
\mathcal{D}(\xi_t^N) \rightarrow \mathcal{D}(\eta_0 - \eta_t) \quad \text{for any } t,
\]

where \( \eta_0, \eta_t \) are independent identically distributed centred Gaussian random variables. Due to Proposition 1.3, we see that, indeed, this convergence takes place. Thus, representation (1.21), also being informal, gives a correct intuition for the asymptotic behaviour of the process \( \#_{[0,Nt]} \). Let us now informally express the covariance \( \text{Cov}(z_t^N, z_s^N) \) of the process \( z_t^N \), defined in (1.6), through the covariance of the height function \( h_t^N \). We have

\[
\text{Cov}(z_t^N, z_s^N) = \frac{\ln N}{\pi^2} \text{Cov} \left( \int_0^t \xi_u^N \, du - \frac{t}{\tau} \int_0^\tau \xi_u^N \, du, \int_0^s \xi_v^N \, dv - \frac{s}{\tau} \int_0^\tau \xi_v^N \, dv \right) \tag{1.22}
\]

\[
= \left( \int_0^t \int_0^\tau dudv - \frac{t}{\tau} \int_0^\tau \int_0^s dudv - \frac{s}{\tau} \int_0^\tau \int_0^\tau dudv + \frac{ts}{\tau^2} \int_0^\tau \int_0^\tau dudv \right) \text{Cov}(\#_{[0,uN]}, \#_{[0,vN]}).
\]

Due to (1.21),

\[
\text{Cov}(\#_{[0,uN]}, \#_{[0,vN]}) = \text{Var} h_0^N + \text{Cov}(h_u^N, h_v^N) - \text{Cov}(h_0^N, h_u^N) - \text{Cov}(h_0^N, h_v^N). \tag{1.23}
\]

Then

\[
\text{Cov}(z_t^N, z_s^N) = \left( \int_0^t \int_0^\tau dudv - \frac{t}{\tau} \int_0^\tau \int_0^s dudv - \frac{s}{\tau} \int_0^\tau \int_0^\tau dudv + \frac{ts}{\tau^2} \int_0^\tau \int_0^\tau dudv \right)
\]

\[
\times \left( \text{Cov}(h_u^N, h_v^N) - \text{Cov}(h_0^N, h_u^N) - \text{Cov}(h_0^N, h_v^N) \right),
\]

since the term \( \text{Var} h_0^N \) is cancelled by the integration. In view of (1.19), the formula above perfectly agrees with (1.9), (1.11). The only difference is that in our case the covariance \( \text{Cov}(h_t^N, h_s^N) \) is expected to be governed by the Gaussian Free Field in the plane (as., e.g., in [43]) rather than in the half-plane as in (1.19). Namely, we expect the convergence

\[
\text{Cov}(h_t^N, h_s^N) \rightarrow \pi^{-1} \mathcal{G}(t,s) \quad \text{as } \quad N \rightarrow \infty, \quad \text{for } t \neq s. \tag{1.24}
\]

Note that the only term in (1.23) which is expected to grow with \( N \) is \( \text{Var} h_0^N \). In particular, the reason for which the covariance \( \text{Cov}(z_t^N, z_s^N) \) does not grow is that this term is cancelled by the integration in (1.22). This gives an indication to the fact that
this is the fixed end 0 of the interval \([0, Nt]\) who is responsible for the growth of the covariance (1.23).

To make sure that the heuristic argument above is correct, we next consider the random process \(\#_{[aNt, bNt]}\), where for definiteness we assume \(t > 0\) and \(a, b \in \mathbb{R} \setminus \{0\}\). On an intuitive level, we have

\[
\#_{[aNt, bNt]} = h_{at}^N - h_{bt}^N.
\]

Then

\[
\text{Cov}(\#_{[aNt, bNt]}, \#_{[aNs, bNs]}) = \text{Cov}(h_{at}^N, h_{as}^N) + \text{Cov}(h_{bt}^N, h_{bs}^N) - \text{Cov}(h_{at}^N, h_{bs}^N) - \text{Cov}(h_{bt}^N, h_{as}^N).
\]

Consequently, if the ends of the intervals \([at, bt]\) and \([as, bs]\) do not coincide, we expect the covariance \(\text{Cov}(\#_{[aNt, bNt]}, \#_{[aNs, bNs]})\) to be bounded in \(N\) and to be governed by the Gaussian Free Field in the plain accordingly to the convergence (1.24). This turns out to be true, namely in Section 5.3 we prove

**Proposition 1.8.** Assume that \(t, s > 0\) satisfy \(t \neq s\), \(at \neq bs\) and \(bt \neq as\), so that the ends of the intervals \([at, bt]\) and \([as, bs]\) do not coincide. Then, as \(N \to \infty\), we have

\[
\text{Cov}(\#_{[aNt, bNt]}, \#_{[aNs, bNs]}) \to \pi^{-1}(\mathcal{G}(at, as) + \mathcal{G}(bt, bs) - \mathcal{G}(at, bs) - \mathcal{G}(bt, as)).
\]

In particular, one could study directly the integral \(I_t^N := \int_0^t \#_{[aNs, bNs]} - E \#_{[aNs, bNs]} \, ds\), without the normalization \((\ln N)^{-1/2}\) and without extracting the linear term \(\eta^N t\) as in (1.7), which appear because of the fixed left end of the interval \([0, t]\). Indeed, due to Proposition 1.8, the covariances \(\text{Cov}(I_t^N, I_s^N)\) converge as \(N \to \infty\) for any \(t, s > 0\) and are governed by the Gaussian Free Field.

### 1.4 Functional limit theorem for ergodic integrals

In this section we explain that ergodic integrals corresponding to a shift operator acting on the space of configurations possess the same asymptotic behaviour as the number of particles \#_{[0, Nt]}\). Denote by \(\text{Conf}(\mathbb{R})\) the space of locally finite configurations on \(\mathbb{R}\),

\[
\text{Conf}(\mathbb{R}) = \{\mathcal{X} \subset \mathbb{R} \mid \mathcal{X} \text{ does not have limit points in } \mathbb{R}\}.
\]

Let \(T^u, u \geq 0\), be a shift operator acting on \(\text{Conf}(\mathbb{R})\) as

\[
T^u : \text{Conf}(\mathbb{R}) \mapsto \text{Conf}(\mathbb{R}), \quad T^u(\mathcal{X}) = \mathcal{X} - u.
\]

Consider the dynamical system

\[
(\text{Conf}(\mathbb{R}), (T^u)_{u \geq 0}, \mathbf{P}), \quad \text{(1.25)}
\]

where \(\mathbf{P}\) is the probability measure on \(\text{Conf}(\mathbb{R})\), given by the sine-process. Take a bounded measurable function \(\varphi : \mathbb{R} \mapsto \mathbb{R}\) with compact support. The linear statistics \(\mathcal{S}_\varphi\) corresponding to the function \(\varphi\) is introduced by the formula

\[
\mathcal{S}_\varphi : \text{Conf}(\mathbb{R}) \mapsto \mathbb{R}, \quad \mathcal{S}_\varphi(\mathcal{X}) := \sum_{x \in \mathcal{X}} \varphi(x). \quad \text{(1.26)}
\]
In particular, if \( \varphi = \mathbb{1}_{[a,b]} \), we have \( S_\varphi = \#_{[a,b]} \). Assume that the function \( \varphi \) satisfies the normalization requirement
\[
\int_{-\infty}^{\infty} \varphi(u) \, du = 1. \tag{1.27}
\]
Consider the ergodic integral
\[
\int_{0}^{tN} S_\varphi \circ T^u \, du, \quad 0 \leq t \leq 1,
\]
where
\[
S_\varphi \circ T^u(\mathcal{X}) = \sum_{x \in T^u(\mathcal{X})} \varphi(x) = \sum_{x \in \mathcal{X}} \varphi(x-u). \tag{1.28}
\]
Let
\[
\varphi_t^N := \int_{0}^{tN} \varphi(\cdot - u) \, du.
\]
Then, exchanging the integral with the sum, we see that the ergodic integral coincides with the linear statistics \( S_{\varphi_t^N} \),
\[
\int_{0}^{tN} S_\varphi \circ T^u \, du = S_{\varphi_t^N}. \tag{1.29}
\]
Consider the random process
\[
\xi_{\varphi,t}^N := \frac{S_{\varphi_t^N} - \mathbb{E}S_{\varphi_t^N}}{\pi^{-1}\sqrt{\ln N}}.
\]
In the next theorem, which is our second main result, we show that under the sine-process the process \( \xi_{\varphi,t}^N \) possesses exactly the same asymptotic behaviour as the process \( \xi_t^N \) given by (1.5). Fix \( 0 < \tau \leq 1 \) and set
\[
\eta_t^N = \frac{1}{\tau} \int_{0}^{\tau} \xi_{\varphi,s}^N \, ds.
\]
Choose the random process \( z_t^N \) in such a way that
\[
\int_{0}^{t} \xi_{\varphi,s}^N \, ds = t\eta_t^N + \frac{z_t^N}{\pi^{-1}\sqrt{\ln N}}.
\]

**Theorem 1.9.** Under the sine-process we have

1. For any \( t > 0 \),
\[
\text{Var} \, S_{\varphi_t^N} = \pi^{-2} \ln N + O(\sqrt{\ln N}) \quad \text{as} \quad N \to \infty.
\]
Figure 2: Function \( \varphi^N \). Here \( m := \inf \text{supp } \varphi \) and \( M := \sup \text{supp } \varphi \).

2. For any \( 0 < t_1 < \ldots < t_d \leq 1, d \geq 1 \), distribution of the random vector

\[
\xi^N_{\varphi} := (\xi^N_{\varphi,t_1}, \ldots, \xi^N_{\varphi,t_d})
\]

satisfies \( \mathcal{D}(\xi^N_{\varphi}) \to \mathcal{D}(\Delta \eta) \) as \( N \to \infty \), where \( \Delta \eta \) the Gaussian random vector from Proposition 1.3.

3. The distribution \( \mathcal{D}(\eta^N, z^N) \) satisfies \( \mathcal{D}(\eta^N, z^N) \to \mathcal{D}(\eta, z) \) as \( N \to \infty \) in \( \mathbb{R} \times C([0,1], \mathbb{R}) \), where the random variable \( \eta \) and the random process \( z_t \) are as in Theorem 1.1.

4. Cumulants \( (A^N_k) \) and \( (A^k) \) of the random variables \( \eta^N \) and \( \eta \) satisfy \( A^N_1 = A_1 = 0 \), \( |A^N_2 - A_2| \leq C_2(\ln N)^{-1/2} \), and (1.12) for \( k \geq 3 \) and some constants \( C_k \).

Theorem 1.9 is proven in Section 5.3. To see the connection between the processes \( \xi^N_{\varphi,t} \) and \( \xi^N_{\varphi} \), observe that the function \( \varphi^N \) has the form as shown on figure 2: it has a flat part of the length \( \sim N \) where \( \varphi^N_t(x) = \int_{-\infty}^{\infty} \varphi(x) \, dx = 1 \), and «tails» with the length of order one. So that, \( \varphi^N_t \) «almost coincides» with a shifted indicator function \( I_{[0,N]} \), if \( N \) is large. But the linear statistics \( S_{I_{[0,N]}} \) is equal exactly to the number of particles \( \#_{[0,N]} \).

1.5 Central Limit Theorem for linear statistics

Proofs of Theorems 1.1 and 1.9 follow the same pattern and rely on the multidimensional Central Limit Theorem 4.3, which we state below in a simpler form. Recall that the linear statistic \( S_{\varphi} \) of a function \( \varphi \) is defined in (1.26).

**Theorem 1.10.** Let \( f_1, \ldots, f_p, g_1, \ldots, g_q : \mathbb{R} \to \mathbb{R}, p, q \geq 0 \), be measurable bounded functions with compact supports. Set \( f_i^N := f_i(\cdot/N), g_j^N := g_j(\cdot/N) \) and consider the corresponding linear statistics

\[
S^N_{f_1}, \ldots, S^N_{f_p}, S^N_{g_1}, \ldots, S^N_{g_q}
\]

as random variables under the sine-process. Assume that

1. There exists a sequence \( V_N \to \infty \) as \( N \to \infty \) and numbers \( b_{ij}^f \) satisfying \( b_{ii}^f > 0 \), such that for any \( i, j \)

\[
\frac{\text{Cov}(S^N_{f_i}, S^N_{f_j})}{V_N} \to b_{ij}^f \quad \text{as} \quad N \to \infty. \quad (1.30)
\]

2. The functions \( g_i \) belong to the Sobolev space \( H^{1/2}(\mathbb{R}) \).
Let \((\xi_f^N, \xi_g^N)\) be the random vector with components
\[
\xi_{f,i}^N := \frac{S_{f,i}^N - ES_{f,i}^N}{\sqrt{V_N}} \quad \text{and} \quad \xi_{g,j}^N := S_{g,j}^N - ES_{g,j}^N.
\]
Then we have the weak convergence \(\mathcal{D}(\xi_f^N, \xi_g^N) \overset{N \to \infty}{\to} \mathcal{D}(\xi_f, \xi_g)\), where \((\xi_f, \xi_g)\) is a centred Gaussian random vector with the covariance matrix \(\begin{pmatrix} (b_{f,i}^j) & 0 \\ 0 & (b_{g,k}^l) \end{pmatrix}\) and \(b_{g,k}^l = \langle g_k, g_l \rangle^{1/2}\).

Note that under the assumption \(g_i \in H^{1/2}\) the variances \(\text{Var} S_{g,i}^N\) do not grow at all, so that assumption (1.30) can not be satisfied for the functions \(g_i\). Conversely, the inclusion \(f_i \in H^{1/2}\) can not take place once (1.30) holds.

The difference between Theorems 1.10 and 4.3 is that in the latter we admit more general dependence of the functions \(f_i^N, g_j^N\) on \(N\) than in Theorem 1.10. This is needed for the proof of Theorem 1.9.

The marginal convergence \(\mathcal{D}(\xi_{f,i}^N) \overset{N \to \infty}{\to} \mathcal{D}(\xi_f)\) does not use the special structure of the sine-kernel and takes place under a large class of determinantal point processes, once (1.30) holds. We prove this in Theorem 4.1 and use in Section 6, where we establish the main order asymptotic from Theorem 1.1 for a general determinantal process with logarithmically growing variance. To establish the convergence \(\mathcal{D}(\xi_{g,i}^N) \overset{N \to \infty}{\to} \mathcal{D}(\xi_g)\), however, we crucially use the form of the sine-kernel. Indeed, proof of our Central Limit Theorem is based on analysis of cumulants \((A_{k}^N)_{k \in \mathbb{Z}^p+q}\) of the random vector \((\xi_{f,i}^N, \xi_{g,i}^N)\). In particular, we show that \(A_{k}^N \overset{N \to \infty}{\to} 0\) once \(|k| > 2\). For the cumulants corresponding to the component \(\xi_{f,i}^N\) the latter convergence follows from general estimates obtained in Section 3 and decay of the normalization factor \(V_N^{-1}\). For the component \(\xi_{g,i}^N\) such normalization is lacking and the analysis is more delicate. We rely on the combinatorial identity (4.38) obtained by Soshnikov in [40], while application of the latter requires the relation (3.30) which is specific for the sine-process.

The main novelty of Theorem 1.10 is that we study asymptotic behaviour of the joint linear statistics \((S_{f,i}^N, S_{g,j}^N)\), so that we work simultaneously on two different scales, corresponding to the growing and bounded variance. Indeed, the marginal convergence \(\mathcal{D}(\xi_{f,i}^N) \to \mathcal{D}(\xi_f)\) in the generality of Theorem 4.1 generalizes convergences obtained by Costin and Lebowitz [14] and Soshnikov [38, 39, 41], see Section 4.1 for the discussion. The convergence \(\mathcal{D}(\xi_{g,i}^N) \to \mathcal{D}(\xi_g)\) was proven by Spohn [42] and Soshnikov [40, 41]. For further developments see also works [21, 25, 26, 7, 8], where certain Central Limit Theorems were established for linear statistics with bounded variance, related to the marginal convergence \(\mathcal{D}(\xi_{g,i}^N) \to \mathcal{D}(\xi_g)\). More precisely, in [21, 25] and [7] the Central Limit Theorems were proven for linear statistics of various orthogonal polynomial ensembles on mesoscopic scales. In [26] and [8] those were obtained for linear statistics of certain biorthogonal ensembles.
1.6 Outline of the proofs of Theorems 1.1 and 1.9

First we discuss Theorem 1.1. We note that, due to (1.5),
\[ \eta^N = \frac{\tau^{-1} \int_0^\tau \left( \#_{[0,sN]} - \mathbb{E} \#_{[0,sN]} \right) ds}{\sqrt{\ln N}} = \frac{S_{fN} - \mathbb{E} S_{fN}}{\sqrt{\ln N}}, \]
where \( S_{fN} \) is the linear statistics corresponding to the function \( f^N(x) = f(x/N) \) with \( f(x) = \frac{1}{\tau} \int_0^\tau I_{[0,s]}(x) ds \). Similarly, \( z^N_t = \frac{1}{\tau} \int_0^\tau \left( \#_{[0,sN]} - \mathbb{E} \#_{[0,sN]} \right) ds - \frac{t}{\tau} \int_0^\tau \left( \#_{[0,sN]} - \mathbb{E} \#_{[0,sN]} \right) ds = S_{g_t} - \mathbb{E} S_{g_t} \),
\[ z^N_t = \int_0^t \left( \#_{[0,sN]} - \mathbb{E} \#_{[0,sN]} \right) ds - \frac{t}{\tau} \int_0^\tau \left( \#_{[0,sN]} - \mathbb{E} \#_{[0,sN]} \right) ds = S_{g_t} - \mathbb{E} S_{g_t}, \]
where \( g_t^N(x) = g_t(x/N) \) and the functions \( g_t, 0 \leq t \leq 1 \), are given by
\[ g_t(x) := \int_0^t I_{[0,s]}(x) ds - \frac{t}{\tau} \int_0^\tau I_{[0,s]}(x) ds. \] (1.31)

It is easy to see that the functions \( f \) and \( g_t \) have compact support, are piecewise linear, and the functions \( g_t \) are continuous (see (5.9)-(5.10) for the explicit form of \( g_t \)). In particular, \( g_t \in H^1(\mathbb{R}) \) for all \( 0 \leq t \leq 1 \).

Next we show that \( \text{Var} \ S_{fN} \sim \frac{1}{2\pi^2} \ln N \) and that the pairing \( \langle g_t, g_s \rangle_{1/2} \) equals to the right-hand side of (1.11). Thus, for any \( 0 \leq t_1 < \ldots < t_d \leq 1 \) the functions \( (f, g_{t_1}, \ldots, g_{t_d}) \) satisfy assumptions of Theorem 1.10, with \( V_N = \pi^{-2} \ln N, b_{11}^f = 1/2 \) and \( b_{ij}^g = \text{r.h.s. of (1.11)} \). The latter implies the convergence
\[ \mathcal{D}(\eta^N, z^N_{t_1}, \ldots, z^N_{t_d}) \to \mathcal{D}(\eta, z_{t_1}, \ldots, z_{t_d}) \quad \text{as} \quad N \to \infty, \] (1.32)
where the random variable \( \eta \) and the random process \( z_t \) are as in Theorem 1.1. Then, using a compactness argument in a standard way, we show that convergence (1.32) implies assertion of the theorem.

Proof of Theorem 1.9 uses similar argument. Its main difference from the proof of Theorem 1.1 is that the functions \( f^N \) and \( g_t^N \) depend on \( N \) in a more complicated way. That is why instead of Theorem 1.10 we use more general Theorem 4.3.

1.7 Organization of the paper

In Section 2 we first introduce notation which will be used throughout the paper. Then we recall some basic definitions concerning determinantal point processes and establish some simple facts needed in the sequel. In Section 3 we compute and estimate cumulants of linear statistics first under a general determinantal process and then specify our attention on the sine-process. Results obtained there are used in Section 4, where we establish the Central Limit Theorems 4.1 and 4.3, which are discussed Section 1.5. Section 5 is devoted to the proofs of our main results: Theorems 1.1, 1.9 and Propositions 1.2, 1.3, 1.8. In Section 6 we prove an analogue of Proposition 1.5 for an important class of determinantal processes, including those with logarithmically growing variance (in particular, the Airy and Bessel processes).
2 Preliminaries

2.1 Notation

1. By \( C, C_1, \ldots \) we denote various positive constants. By \( C(a), \ldots \) we denote constants depending on a parameter \( a \). Unless otherwise stated, the constants never depend on \( N \).

2. For \( d \geq 1 \) we set \( \mathbb{Z}_d := \{ \mathbb{Z}^d \ni k = (k_1, \ldots, k_d) \neq 0 : k_j \geq 0 \ \forall 1 \leq j \leq d \} \).

3. For \( k \in \mathbb{Z}_d^+ \) and \( z \in \mathbb{C}^d \) we denote \( |k| := k_1 + \cdots + k_d, \ k! := k_1! \cdots k_d! \) and \( z^k := z_1^{k_1} \cdots z_d^{k_d} \).

4. Our convention for the Fourier transform is as follows: \( \hat{h}(t) = \mathcal{F}(h) = \int_{-\infty}^{\infty} h(x)e^{-itx} \, dx \).

For the inverse Fourier transform we write \( \mathcal{F}^{-1}(\hat{h})(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{h}(t)e^{itx} \, dt \).

5. We denote by \( \| \cdot \| \) the usual operator norm, by \( \| \cdot \|_{HS} \) — the Hilbert-Schmidt norm and by \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{L^m}, m \geq 1, \) — the Lebesgue \( L^\infty \) and \( L^m \)-norms. By \( H^n(\mathbb{R}) \), \( n > 0 \), we denote the Sobolev space of order \( n \) and for functions \( f, g \in H^n(\mathbb{R}) \) we set

\[
\|f\|_n^2 := \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |u^{2n}|\hat{f}(u)|^2 \, du, \quad \langle f, g \rangle_n := \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |u|^{2n} \hat{f}(u) \overline{\hat{g}(u)} \, du, \quad (2.1)
\]

and \( \|f\|_{L^2}^2 + 2\pi \|f\|_n^2 \).

6. By \( \text{Conf}(\mathbb{R}^m) \) we denote the space of locally finite configurations of particles in \( \mathbb{R}^m \), \( m \geq 1 \),

\[
\text{Conf}(\mathbb{R}^m) := \{ \mathcal{X} \subset \mathbb{R}^m | \mathcal{X} \text{ does not have limit points in } \mathbb{R}^m \}. \quad (2.2)
\]

7. Let \( \mathcal{X} \in \text{Conf}(\mathbb{R}^m) \). By \( \#_B(\mathcal{X}) := \# \{ B \cap \mathcal{X} \} \) we denote the number of particles from the configuration \( \mathcal{X} \) intersected with the set \( B \).

8. For a bounded compactly supported function \( h : \mathbb{R}^m \mapsto \mathbb{R} \), by \( \mathcal{S}_h \) we denote the corresponding linear statistics,

\[
\mathcal{S}_h : \text{Conf}(\mathbb{R}^m) \mapsto \mathbb{R}, \quad \mathcal{S}_h(\mathcal{X}) = \sum_{x \in \mathcal{X}} h(x).
\]

9. By \( \mathbb{I}_B \) we denote the indicator function of a set \( B \in \mathbb{R}^m \).

2.2 Determinantal point processes

In this section we recall some basic definitions and facts concerning determinantal processes. Determinantal (or fermion) random point processes form a special class of random point processes, which was introduced by Macchi in seventies (see [29, 30, 15]). They play an important role in the random matrix theory, statistical and quantum mechanics, probability, representation and number theory. For detailed background see [38, 34, 35], see also Chapter 4.2 in [1].
Consider on the space of locally finite configurations \( \text{Conf}(\mathbb{R}^m) \), defined in (2.2), a \( \sigma \)-algebra \( \mathcal{F} \) generated by cylinder sets
\[
\mathcal{C}^n_B = \{ \mathcal{X} \in \text{Conf}(\mathbb{R}^m) : \#_B(\mathcal{X}) = n \},
\]
where \( n \) and \( B \) run over natural numbers and bounded Borel subsets of \( \mathbb{R}^m \) correspondingly. The triple \( (\text{Conf}(\mathbb{R}^m), \mathcal{F}, \mathbf{P}) \), where \( \mathbf{P} \) is a probability measure on \( (\text{Conf}(\mathbb{R}^m), \mathcal{F}) \), is called a random point process.

Assume that there exists a family of locally integrable nonnegative functions \( \rho_n : (\mathbb{R}^m)^n \to \mathbb{R}, n \geq 1 \), such that for any \( n \geq 1 \) and any mutually disjoint Borel subsets \( B_1, \ldots, B_n \) of \( \mathbb{R}^m \) we have
\[
\mathbb{E} \#_{B_1} \cdots \#_{B_n} = \int_{B_1 \times \cdots \times B_n} \rho_n(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]
The functions \( \rho_n \) are called correlation functions. Under natural assumptions the family \( (\rho_n)_{n \geq 1} \) determines the probability \( \mathbf{P} \) uniquely, see e.g. [38].

Consider a non-negative integral operator \( K : L^2(\mathbb{R}^m, dx) \to L^2(\mathbb{R}^m, dx) \) with a Hermitian kernel \( K : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}, \)
\[
Kf(x) = \int_{\mathbb{R}^m} K(x, y) f(y) \, dy, \quad K \geq 0. \tag{2.3}
\]
Assume that \( K \) is locally trace class, i.e. for any bounded Borel set \( B \subset \mathbb{R}^m \) the operator \( I_B K I_B \) is trace class. Lemmas 1 and 2 from [38] imply that it is possible to choose the kernel \( K \) in such a way that for any bounded Borel sets \( B_1, \ldots, B_n, n \geq 1 \), we have
\[
\text{tr} I_{B_1} K I_{B_1} K I_{B_2} \cdots K I_{B_n} = \int_{B_1 \times \cdots \times B_n} K(x_1, x_2) K(x_2, x_3) \cdots K(x_n, x_1) \, dx_1 \cdots dx_n. \tag{2.4}
\]
In particular, for \( n = 1 \) we have \( \text{tr} I_{B_1} K I_{B_1} = \int_{B_1} K(x, x) \, dx \). Assume that (2.4) is satisfied.

**Definition 2.1.** A random point process is called determinantal if it has the correlation functions of the form
\[
\rho_n(x_1, \ldots, x_n) \equiv \det \begin{pmatrix}
K(x_1, x_1) & \cdots & K(x_1, x_n) \\
\vdots & \ddots & \vdots \\
K(x_n, x_1) & \cdots & K(x_n, x_n)
\end{pmatrix} \text{ for all } n \geq 1.
\]

Determinantal processes possess the following property, which can be viewed as their equivalent definition. Take any bounded measurable function \( h : \mathbb{R}^m \to \mathbb{R} \) with a compact support \( D := \text{supp} \, h \). Consider the corresponding linear statistics \( S_h \). Then the generating function \( \mathbb{E} z^{S_h}, z \in \mathbb{C} \), takes the form
\[
\mathbb{E} z^{S_h} = \det \left( 1 + (z^h - 1) K I_D \right), \tag{2.5}
\]
where the expectation is taken under the determinantal process and \( \det \) denotes the Fredholm determinant. The latter is well-defined since the operator \( K \) is locally trace class.
2.3 Elementary inequalities for the trace

We will need the following elementary inequalities. Consider a determinantal point process on \( \mathbb{R}^m \), \( m \geq 1 \), given by a Hermitian kernel \( K \). Take a bounded measurable function \( h : \mathbb{R}^m \to \mathbb{R} \) with compact support. Set

\[
D := \text{supp} \, h \quad \text{and} \quad K_D := I_D K I_D,
\]

where the integral operator \( K \) is defined in (2.3).

**Proposition 2.2.** We have

\[
h(K_D - K_D^2)h \geq 0.
\]

**Proof.** It is well-known that \( 0 \leq K \leq \text{Id} \). Consequently, \( 0 \leq K_D \leq \text{Id} \) and \( K_D - K_D^2 = K_D(\text{Id} - K_D) \geq 0 \). Then, denoting by \( \langle \cdot, \cdot \rangle_{L^2} \) the scalar product in \( L^2(\mathbb{R}^m, dx) \), for any function \( f \in L^2(\mathbb{R}^m, dx) \) we obtain

\[
\langle (h(K_D - K_D^2)h) f, f \rangle_{L^2} = \langle (K_D - K_D^2)h f, h f \rangle_{L^2} \geq 0.
\]

\[\square\]

It is well-known that

\[
\text{Var} \, S_h = \int_D h^2(x)K(x, x) \, dx - \int_D \int_D h(x)h(y)|K(x, y)|^2 \, dx \, dy,
\]

\[
= \text{tr} \, h^2 K_D - \text{tr}(hK_D)^2.
\]

The traces above are well-defined since the operator \( K \) is locally trace class, so that the operators \( h^2 K_D \) and \( (hK_D)^2 \) are trace class. Denote by \([\cdot, \cdot]\) the commutator, \([A, B] = AB - BA\). We have

\[
\|[K_D, h]\|_{HS}^2 = 2 \int_D \int_D h^2(x)|K(x, y)|^2 \, dx \, dy - 2 \int_D \int_D h(x)h(y)|K(x, y)|^2 \, dx \, dy
\]

\[
= 2 \left( \text{tr} \, h^2 K_D^2 - \text{tr}(hK_D)^2 \right). \tag{2.7}
\]

**Proposition 2.3.** We have

\[
0 \leq \text{tr} \, h^2(K_D - K_D^2) \leq \text{Var} \, S_h \quad \text{and} \quad \|[K_D, h]\|_{HS}^2 \leq 2 \text{Var} \, S_h. \tag{2.8}
\]

**Proof.** Proposition 2.2 together with cyclicity of the trace implies \( \text{tr} \, h^2(K_D - K_D^2) \geq 0 \). Next, subtracting (2.7) divided by two from (2.6), we get

\[
\text{Var} \, S_h - \frac{1}{2} \|[h, K_D]\|_{HS}^2 = \text{tr} \, h^2(K_D - K_D^2).
\]

Since \( \|[h, K_D]\|_{HS}^2, \text{tr} \, h^2(K_D - K_D^2) \geq 0 \), we obtain (2.8).

\[\square\]

**Proposition 2.4.** For any linear operators \( G_1, \ldots, G_n, F \), \( n \geq 1 \), we have

\[
[G_1 \cdots G_n, F] = \sum_{l=1}^n G_1 \cdots G_{l-1}[G_l, F]G_{l+1} \cdots G_n.
\]

**Proof.** By induction.
3 Cumulants of linear statistics

3.1 Cumulants and traces

In this section we compute cumulants of linear statistics viewed as random variables under a determinantal process and obtain some estimates for them. Despite that in the present paper we mainly work with the sine-process, first in this section we consider a general determinantal process. This is needed for the proof of Theorem 4.1.

Recall that the numbers \((J_k)_{k \in \mathbb{Z}_d^+}\) are called cumulants of a random vector \(\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d\) if for any sufficiently small \(y \in \mathbb{R}^d\) we have

\[
\ln \mathbb{E} e^{iy \cdot \nu} = \sum_{k \in \mathbb{Z}_d^+} J_k \frac{(iy)^k}{k!},
\]

where \(\cdot\) denotes the standard scalar product in \(\mathbb{R}^d\) while \((iy)^k\) and \(k!\) are defined in item 3 of Section 2.1. A cumulant \(J_k\) can be expressed through the moments \((m_l)_{|l| \leq |k|}\) of the random vector \(\nu\) and the other way round. If \((e_1, \ldots, e_d)\) is the standard basis of \(\mathbb{Z}_d^d\) then

\[
J_{e_i} = \mathbb{E} \nu_i \quad \text{and} \quad J_{e_i + e_j} = \text{Cov}(\nu_i, \nu_j) \quad \text{for any} \ 1 \leq i, j \leq d. \tag{3.1}
\]

The vector \(\nu\) is Gaussian iff \(J_k = 0\) for all \(|k| \geq 3\). For more information see e.g. [36], Section 2.12.

Let \(h_1, \ldots, h_d : \mathbb{R}^m \mapsto \mathbb{R}, d \geq 1\), be bounded Borel measurable functions with compact supports and \(h := (h_1, \ldots, h_d)\). Consider the vector of linear statistics

\[
S_h := (S_{h_1}, \ldots, S_{h_d}) \tag{3.2}
\]
as a random vector under a determinantal process given by a Hermitian kernel \(K\). Denote

\[
D := \bigcup_{i=1}^d \text{supp} h_i \quad \text{and} \quad K_D = \mathbb{1}_D K \mathbb{1}_D, \tag{3.3}
\]

where the locally trace class operator \(K\) is given by (2.3). The proofs of the following Lemma 3.1 and Proposition 3.2 are routine (cf. formulas (1.14) and (2.7) from [40]) and we include them for completeness.

**Lemma 3.1.** For any \(k \in \mathbb{Z}_d^d\) satisfying \(|k| \geq 2\) the cumulant \(B_k\) of the random vector (3.2) has the form

\[
B_k = k! \sum_{j=1}^{|k|} \frac{(-1)^{j+1}}{j} \sum_{a_1, \ldots, a_j \in \mathbb{Z}_d^+: \quad a_1 + \cdots + a_j = k} \text{tr} \frac{h^{a_1} K \cdots h^{a_j-1} K h^{a_j} K_D}{a_1! \cdots a_j!}. \tag{3.4}
\]

**Proof.** Due to (2.5) with \(z := e^i\) and \(h := h \cdot y\), we have

\[
\ln \mathbb{E} e^{iS_h \cdot y} = \ln \det (1 + (e^{ih \cdot y} - 1)K_D).
\]

Then, Lemma XIII.17.6 from [33] implies that for a sufficiently small \(y \in \mathbb{R}^d\) we have

\[
\ln \mathbb{E} e^{iS_h \cdot y} = \ln \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \text{tr} \left( (e^{ih \cdot y} - 1)K_D \right)^j \right)
\]

\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{l=1}^{\infty} \frac{\text{tr}(iy \cdot h)^{l_1} K \cdots (iy \cdot h)^{l_j} K_D}{l_1! \cdots l_j!} \tag{3.5}
\]
Note that
\[(iy \cdot h)^n = \sum_{m_1, \ldots, m_n = 1}^d iy_{m_1}h_{m_1} \cdots iy_{m_n}h_{m_n}. \quad (3.6)\]

We have
\[iy_{m_1}h_{m_1} \cdots iy_{m_n}h_{m_n} = (iy_1h_1)^{a_1} \cdots (iy_dh_d)^{a_d} = (iy)^{a_1}h^{a_1}, \]
where
\[a^n := (a^n_1, \ldots, a^n_d) \in \mathbb{Z}_+^d \quad \text{and} \quad a^n_r := \#\{q, 1 \leq q \leq l^n : m_q = r\}. \quad (3.7)\]

Next we replace in (3.6) the summation over \(m_1, \ldots, m^n\) by that over \(a^n \in \mathbb{Z}_+^d\). To this end, we note that \(|a^n| = l^n\) and for a given vector \(a^n\) the number of vectors \((m_1, \ldots, m^n)\) satisfying (3.7) is equal to \(l^n!/a^n!\). Then
\[(iy \cdot h)^n = \sum_{a^n \in \mathbb{Z}_+^d : |a^n| = l^n} \frac{l^n!}{a^n!} (iy)^{a^n}h^{a^n}. \]

Now (3.5) implies
\[\ln \mathbb{E} e^{S_{h,y}} = \sum_{j=1}^\infty (-1)^{j+1} \sum_{\nu^1, \ldots, \nu^j = 1}^\infty \sum_{a_1, \ldots, a^j \in \mathbb{Z}_+^d : |a^1| = \nu^1, |a^j| = \nu^j} \frac{\text{tr}(iy)^{a^1}h^{a^1}K \cdots (iy)^{a^j}h^{a^j}K_D}{l! \cdots l'} \frac{l! \cdots l'}{a^1! \cdots a^j!}, \]
where in the last equality the second sum is taken only over \(j \leq |k|\) since for \(j > |k|\) the relation \(a^1 + \cdots + a^j = k\) with \(a^1, \ldots, a^j \in \mathbb{Z}_+^d\) is impossible. \(\square\)

**Proposition 3.2.** For any \(k \in \mathbb{Z}_+^d\) satisfying \(|k| \geq 2\) we have
\[\sum_{j=1}^{\lfloor k \rfloor} \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}_+^d : a^1 + \cdots + a^j = k} \frac{1}{a^1! \cdots a^j!} = 0. \quad (3.8)\]

**Proof.** Denote the left-hand side of (3.8) by \(T_k\). Represent the function
\[g(x) := x_1 + \ldots + x_d \quad \text{where} \quad x = (x_1, \ldots, x_d), \quad (3.9)\]
in the form \(g(x) = \ln (1 + (e^{x_1 + \cdots + x_d} - 1))\). Developing the logarithm and exponents to the series, we see that \(g(x) = \sum_{k \in \mathbb{Z}_+^d} T_k x^k\). Indeed,
\[g(x) = \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left( \sum_{n_1=0}^\infty \frac{x_1^{n_1}}{n_1!} \cdots \sum_{n_d=0}^\infty \frac{x_d^{n_d}}{n_d!} - 1 \right)^j = \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left( \sum_{n \in \mathbb{Z}_+^d} \frac{x^n}{n!} \right)^j = \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}_+^d} \frac{x^{a^1} \cdots x^{a^j}}{a^1! \cdots a^j!} = \sum_{k \in \mathbb{Z}_+^d} T_k x^k. \]

Thus, due to (3.9), we have \(T_k = 0\) for \(|k| \geq 2\). \(\square\)

Lemma 3.1 together with Proposition 3.2 immediately implies
Corollary 3.3. For any $k \in \mathbb{Z}_+^d$ satisfying $|k| \geq 2$, the cumulants $B_k$ of the random vector (3.2) can be represented in the form

$$B_k = k! \sum_{j=1}^{|k|} \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}_+^d: \sum a^i = k} \frac{\text{tr} h^{a_1} K \ldots h^{a_j} K_D - \text{tr} h^j K_D}{a^1! \ldots a^j!}.$$ (3.10)

In the next lemma we estimate the right-hand side of (3.10).

Lemma 3.4. Let $k \in \mathbb{Z}_+^d$, $|k| \geq 2$, and vectors $a^1, \ldots, a^j \in \mathbb{Z}_+^d$, $j \geq 1$, satisfy $a^1 + \cdots + a^j = k$. Then

$$|\text{tr} h^{a_1} K \ldots h^{a_j} K_D - \text{tr} h^j K_D| \leq C(|k|, d, j) \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{-k} \sum_{j=1}^d \text{Var} S_{h_i}. \quad (3.11)$$

Proof of Lemma 3.4 follows a scheme similar to that used in the proof of Lemma 3.2 from [6]. However, in [6] only the case when $K$ is a projection was considered and the operators $h_i K$, $Kh_i$ were assumed to be of the trace class. We do not impose these restrictions.

Proof. Step 1. We argue by induction. If $j = 1$ then the left-hand side of (3.11) is equal to zero. Consider the case $j = 2$. Using cyclicity of the trace, by a direct computation we get

$$\text{tr} h^{a_1} K h^{a_2} K_D = \text{tr} h^{a_1} K_D h^{a_2} K = \frac{1}{2} \text{tr}[h^{a_1}, K_D][h^{a_2}, K_D] + \text{tr} h^j K_D^2.$$

Then

$$|\text{tr} h^{a_1} K h^{a_2} K_D - \text{tr} h^j K_D| \leq \frac{1}{2} \|h^{a_1}, K_D\|_H \|h^{a_2}, K_D\|_H + |\text{tr}(h^j K_D^2 - h^j K_D)|. \quad (3.12)$$

We estimate the terms of the right-hand side above separately. Set

$$\gamma(x) := \sqrt{h_1^2(x) + \cdots + h_d^2(x)}. \quad (3.13)$$

Using the convention $\frac{0}{0} := 0$, we obtain

$$|\text{tr}(h^j K_D - h^j K_D^2)| = \left| \text{tr} \frac{h^j}{\gamma^2} \gamma(K_D - K_D^2) \gamma \right| \leq \left\|\frac{h^j}{\gamma^2}\right\|_{\infty} \text{tr} \gamma(K_D - K_D^2) \gamma, \quad (3.14)$$

since, due to Proposition 2.2, the operator $\gamma(K_D - K_D^2) \gamma$ is non-negative. Clearly,

$$\left\|\frac{h^j}{\gamma^2}\right\|_{\infty} \leq \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{-k}.$$

On the other hand, due to (2.8), we have

$$\text{tr} \gamma(K_D - K_D^2) \gamma = \gamma^2(K_D - K_D^2) = \sum_{l=1}^d \text{tr} h_l^2(K_D - K_D^2) \leq \sum_{l=1}^d \text{Var} S_{h_l}. \quad (3.15)$$

Thus,

$$|\text{tr}(h^j K_D - h^j K_D^2)| \leq \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{-k} \sum_{l=1}^d \text{Var} S_{h_l}. \quad (3.16)$$
We now estimate the Hilbert-Schmidt norm of the commutators from (3.12). Due to Proposition 2.4, for any $b \in \mathbb{Z}_+^d$ we have
\[
\|[h^b, K_D]\|_{HS} \leq |b| \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{b_i-1} \sum_{l=1}^{d} \|[h_l, K_D]\|_{HS}
\leq C(|b|, d) \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{b_i-1} \left(\sum_{l=1}^{d} \text{Var} S_{h_l}\right)^{1/2},
\] (3.17)
where in the last inequality we have used the second relation from (2.8). Now (3.12) joined with (3.16) and (3.17) implies the desired estimate.

Step 2. Assume that $j \geq 3$. Denote
\[
G := h^{a_1} K_D h^{a_2} K_D \cdots h^{a_j-3} K_D h^{a_j-2},
\] (3.18)
so that $\text{tr} h^{a_1} K \cdots h^{a_j} K_D = \text{tr} h^{a_1} K_D \cdots h^{a_j} K_D = \text{tr} G K_D h^{a_j-1} K_D h^{a_j} K_D$ (in particular, for $j = 3$ we have $G = h^3$). It suffices to show that
\[
|\text{tr} G K_D h^{a_j-1} K_D h^{a_j} K_D - \text{tr} G K_D h^{a_j-1+a_j} K_D| \leq C(|k|, d) \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{2(k-1/2)} \sum_{l=1}^{d} \text{Var} S_{h_l}.
\] (3.19)
A direct computation gives
\[
\text{tr} G K_D h^{a_j-1} K_D h^{a_j} K_D = \text{tr} G K_D [h^{a_j-1}, K_D][h^{a_j}, K_D] + \text{tr} G K_D h^{a_j-1} K_D^2 h^{a_j} - \text{tr} G K_D^2 h^{a_j-1} K_D h^{a_j} + \text{tr} G K_D^2 h^{a_j-1+a_j} K_D.
\] (3.20)
Write
\[
|\text{tr} G K_D h^{a_j-1} K_D h^{a_j} K_D - \text{tr} G K_D h^{a_j-1+a_j} K_D| \leq |\text{tr} G K_D [h^{a_j-1}, K_D][h^{a_j}, K_D]|
+ |\text{tr} G K_D h^{a_j-1} K_D^2 h^{a_j} - \text{tr} G K_D^2 h^{a_j-1} K_D h^{a_j}|
+ |\text{tr} G K_D^2 h^{a_j-1+a_j} K_D - \text{tr} G K_D h^{a_j-1+a_j} K_D| =: I_1 + I_2 + I_3.
\] (3.21)
We estimate the terms $I_1, I_2, I_3$ separately. We have
\[
I_1 \leq \|[G K_D],[h^{a_j-1}, K_D]\|_{HS} \|[h^{a_j}, K_D]\|_{HS}.
\] (3.22)
Recalling that $0 \leq K_D \leq \text{Id}$, we obtain
\[
\|G K_D\| \leq \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{2|-a_j-1|-|a_j|}.
\] (3.23)
Then the relation (3.17) implies
\[
I_1 \leq C(|k|, d) \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{2(k-2)} \sum_{l=1}^{d} \text{Var} S_{h_l}.
\]
Next,
\[
I_2 \leq |\text{tr} G K_D h^{a_j-1} K_D^2 h^{a_j} - \text{tr} G K_D h^{a_j-1} K_D h^{a_j}|
+ |\text{tr} G K_D h^{a_j-1} K_D h^{a_j} - \text{tr} G K_D^2 h^{a_j-1} K_D h^{a_j}| =: I_2' + I_2''.
\]
Due to (3.15) and (3.23),

\[
I'_2 = \left| \text{tr} GK_D \frac{\hat{h}^{a_{i-1}}}{\gamma} (K_D^2 - K_D) \gamma \frac{\hat{h}^{a_{j}}}{\gamma} \right|
\]

\[
\leq \|GK_D\| \left\| \frac{\hat{h}^{a_{i-1}}}{\gamma} \right\| \left\| \frac{\hat{h}^{a_{j}}}{\gamma} \right\| \text{tr} \gamma (K_D - K_D^2) \gamma \leq \max_{1 \leq i \leq d} \|h_i\|_{\infty}^{|k| - 2} \sum_{l=1}^{d} \text{Var} S_{h_l}.
\]

In a similar way we get the same estimate for the terms $I''_2$ and $I_3$. Then (3.21) implies (3.19).

\[\square\]

### 3.2 Cumulants under the sine-process

In this section we assume that $K = K_{\text{sine}}$ is the sine-kernel given by (1.1). Using its special structure we rewrite the traces from (3.4) in an appropriate way, representing them through the Fourier transforms $\hat{h}_i$.

Let $k \in \mathbb{Z}_+^d$, $v = (v_1, \ldots, v_{|k|})$ and $a^1, \ldots, a^j \in \mathbb{Z}_+^d$, $j \geq 1$, satisfy $a^1 + \ldots + a^j = k$. Denote

\[
\hat{h}^{a^1 \ldots a^j}(v) := \hat{h}_1(v_1) \hat{h}_2(v_{a^1} + 1) \hat{h}_2(v_{a^1 + a^2}) \ldots \hat{h}_d(v_{a^1 + \ldots + a^2 + 1}) \ldots \hat{h}_d(v_{a^1 + \ldots + a^j + 1}) \hat{h}_1(v_{a^1 + \ldots + a^j + 1})
\]

We abbreviate the relation above as

\[
\hat{h}^{a^1 \ldots a^j}(v) = \prod_{i=1}^{|k|} \hat{h}_i(v_i),
\]

where $l_i = r_i$, $1 \leq r \leq d$, if

\[
i \in \cup_{s=1}^d [a^1 + \ldots + a^s - 1 + r + 1, a^1 + \ldots + a^s - 1]
\]

Let for $j \geq 2$

\[
J^{a^1 \ldots a^j}(v) := - \max \left( 0, \sum_{i=1}^{a^1} v_i, \sum_{i=1}^{a^1 + a^2} v_i, \ldots, \sum_{i=1}^{a^1 + \ldots + a^j - 1} v_i \right)
\]

\[
- \max \left( 0, - \sum_{i=1}^{a^1} v_i, - \sum_{i=1}^{a^1 + a^2} v_i, \ldots, - \sum_{i=1}^{a^1 + \ldots + a^j - 1} v_i \right)
\]

and for $j = 1$ set $J^{a^1 \ldots a^j} := 0$.

**Proposition 3.5.** Let $K = K_{\text{sine}}$ and vectors $k, a^1, \ldots, a^j \in \mathbb{Z}_+^d$, $|k| \geq 2$, $j \geq 1$, satisfy $a^1 + \ldots + a^j = k$. Then

\[
\text{tr} h^{a^1} K \ldots h^{a^j} K_D = \frac{1}{(2\pi)^{|k|}} \int_{v_1 + \ldots + v_{|k|} = 0} \hat{h}^{a^1 \ldots a^j}(v) \max (2 + J^{a^1 \ldots a^j}(v), 0) dS,
\]

where $dS$ is an elementary volume of the hyperplane $v_1 + \ldots + v_{|k|} = 0$, normalized in such a way that $dS(v_1, \ldots, v_{|k|}) = dv_1 \ldots dv_{|k| - 1}$.
Proof. In this proof we always consider the kernel $K$ as a function of one variable

$$K(x) = \frac{\sin x}{\pi x}.$$  \hfill (3.27)

Denote the trace from the left-hand side of (3.26) by $\text{Tr}$.

**Step 1.** Assume first $j = 1$. We have

$$\text{Tr} = \text{tr} h^k K_D = \int_{-\infty}^{\infty} h^k(x)K(0) \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} h^k(x) \, dx = \frac{1}{\pi} \mathcal{F}(h^k)(0).$$

Denote by $*$ the convolution and set $\hat{h}^k := \hat{h}_{1}^{k_1} \ast \ldots \ast \hat{h}_{d}^{k_d}$. Changing the order of the convolutions, we obtain $\hat{h}^k = \ast_{i=1}^{k} \hat{h}_{i}$, where we recall that the indices $l_i$ are defined below (3.24). Then, using that $\mathcal{F}(fg) = (2\pi)^{-1} \hat{f} \ast \hat{g}$ for all $f,g \in L^2(\mathbb{R})$ we get

$$\text{Tr} = \frac{1}{\pi(2\pi)^{|k|-1}} \hat{h}^k(0)$$

$$= \frac{2}{(2\pi)^{|k|}} \int_{\mathbb{R}^{|k|-1}} \hat{h}_{l_1}(-y_1) \hat{h}_{l_2}(y_1 - y_2) \ldots \hat{h}_{l_{|k|-1}}(y_{|k|-2} - y_{|k|-1}) \hat{h}_{l_{|k|}}(y_{|k|-1}) \, dy_1 \ldots dy_{|k|-1}.$$  

Next we change the variables, $v_1 := -y_1$ and for $2 \leq i \leq |k| - 1$ we set $v_i := y_i - y_{i-1}$. Then, denoting $v_{|k|} := -v_1 - \ldots - v_{|k|-1}$ (so that $y_{|k|-1} = v_{|k|}$) and passing from the integration over $\mathbb{R}^{|k|-1}$ to the integration over the hyperplane $v_1 + \ldots + v_{|k|} = 0$ in $\mathbb{R}^{|k|}$, we arrive at (3.26):

$$\text{Tr} = \frac{2}{(2\pi)^{|k|}} \int_{v_1+\ldots+v_{|k|}=0} \prod_{i=1}^{|k|} \hat{h}_{l_i}(v_i) \, dS.$$

**Step 2.** Let now $j \geq 2$. In this step we show that

$$\text{Tr} = \frac{1}{(2\pi)^{|k|}} \int_{\mathbb{R}^j} \hat{h}^{*a_1}(y_1 - y_2) \hat{K}(y_2) \hat{h}^{*a_2}(y_2 - y_3) \hat{K}(y_3) \ldots \hat{h}^{*a_{j-1}}(y_j - y_1) \hat{K}(y_1) \, dy_1 \ldots dy_{j}.  \hfill (3.28)$$

We have

$$\text{Tr} = \int_{\mathbb{R}^j} h^{a_1}(x_1)K(x_1 - x_2)h^{a_2}(x_2)K(x_2 - x_3)\ldots h^{a_{j-1}}(x_{j-1})K(x_{j-1} - x_j) \, dx_1 \ldots dx_{j}.$$  

Note that $\mathcal{F}(\mathcal{K}(\cdot - b))(y) = \hat{\mathcal{K}}(y) e^{-iyb}$, for all $b \in \mathbb{R}$. Then, using that $(f,g)_L^2 = (2\pi)^{-1} (\hat{f}, \hat{g})_L^2$ and $\mathcal{F}(fg) = (2\pi)^{-1} \hat{f} \ast \hat{g}$ for all $f,g \in L^2(\mathbb{R})$, and that the function $\hat{\mathcal{K}}$ is real, we find

$$\int_{-\infty}^{\infty} \mathcal{K}(x_j - x_1)h^{a_1}(x_1)K(x_1 - x_2) \, dx_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\mathcal{K}(x_j - \cdot))(y) \mathcal{F}(h^{a_1}(\cdot)K(\cdot - x_2))(y) \, dy$$

$$= \frac{1}{(2\pi)^{|a_1|+1}} \int_{\mathbb{R}^2} \hat{\mathcal{K}}(y_1)e^{iy_1x_j} \hat{h}^{*a_1}(y_1 - y_2) \hat{K}(y_2)e^{-iy_2x_2} \, dy_1 dy_2.$$
Thus,

\[
\text{Tr} = \frac{1}{(2\pi)^{|a_1|+1}} \int_{\mathbb{R}^{j+1}} \hat{h}^a(y_1-y_2)\hat{K}(y_2)e^{-iy_2x_2}h^{a_2}(x_2)K(x_2-x_3)\ldots h^{a_j}(x_j)e^{iy_jx_j}K(y_j)\,dy_1dy_2dx_2\ldots dx_j.
\]

Since

\[
\int_{-\infty}^{\infty} e^{-iy_2x_2}h^{a_2}(x_2)K(x_2-x_3)\,dx_2 = \mathcal{F}(h^{a_2}(\cdot)K(\cdot-x_3)) = \frac{1}{(2\pi)^{|a_2|}} \int_{-\infty}^{\infty} \hat{h}^{a_2}(y_2)\hat{K}(y_2)e^{-iy_2x_3}\,dy_2,
\]

we obtain

\[
\text{Tr} = \frac{1}{(2\pi)^{|a_1|+|a_2|+1}} \int_{\mathbb{R}^{j+1}} \hat{h}^a(y_1-y_2)\hat{K}(y_2)\hat{h}^{a_2}(y_2-y_3)\hat{K}(y_3)e^{-iy_3x_3} \hat{h}^{a_3}(x_3)K(x_3-x_4)\ldots h^{a_j}(x_j)e^{iy_jx_j}K(y_j)\,dy_1dy_2dy_3dx_3\ldots dx_j.
\]

Continuing the procedure, finally we arrive at the formula

\[
\text{Tr} = \frac{1}{(2\pi)^{|a_1|+\ldots+|a_{j-1}|+1}} \int_{\mathbb{R}^{j+1}} \hat{h}^a(y_1-y_2)\hat{K}(y_2)\ldots \hat{h}^{a_{j-1}}(y_{j-1}-y_j)\hat{K}(y_j) e^{-iy_jx_j}h^{a_j}(x_j)e^{iy_jx_j}K(y_j)\,dy_1\ldots dy_jdx_j.
\]

Step 2. Writing the convolutions from (3.28) explicitly, we obtain

\[
\text{Tr} = \frac{1}{(2\pi)^{|k|}} \int_{\mathbb{R}^{|k|}} \prod_{i=1}^{|k|} \hat{h}_i(y_i-y_{i+1})\hat{K}(y_{i+1}) \prod_{i=|a_1|+|a_2|} \hat{h}_i(y_i-y_{i+1})\hat{K}(y_{i+1})
\]

\[
\ldots \prod_{i=|a_1|+\ldots+|a_{j-1}|+1} \hat{h}_i(y_i-y_{i+1})\hat{K}(y_1)\,dy_1\ldots dy_k,
\]

where we set \(y_{|k|+1} := y_1\). Introducing the variables \(y := y_1\) and \(v_i := y_i - y_{i+1}, 1 \leq i \leq |k| - 1\), and using the relation \(y_n = y - \sum_{i=1}^{n-1} v_i\), we obtain

\[
\text{Tr} = \frac{1}{(2\pi)^{|k|}} \int_{\mathbb{R}^{|k|-1}} \prod_{i=1}^{|k|-1} \hat{h}_i(v_i)\hat{h}_{|k|}(-v_1-\ldots-v_{|k|-1})
\]

\[
\int_{-\infty}^{\infty} \hat{K}(y)\hat{K}(y - \sum_{i=1}^{|a_1|} v_i)\ldots \hat{K}(y - \sum_{i=1}^{|a_1|+\ldots+|a_{j-1}|} v_i)\,dy\,dv_1\ldots dv_{|k|-1}.
\]

Denoting \(v_{|k|} = -v_1 - \ldots - v_{|k|-1}\) and passing from the integration over \(\mathbb{R}^{|k|-1}\) to that over the hyperplane \(v_1 + \ldots + v_{|k|} = 0\), we find

\[
\text{Tr} = \frac{1}{(2\pi)^{|k|}} \int_{v_1+\ldots+v_{|k|}=0} \hat{h}^{a_1,\ldots,a_j}(v) \int_{-\infty}^{\infty} \hat{K}(y)\hat{K}(y - \sum_{i=1}^{|a_1|} v_i)\ldots \hat{K}(y - \sum_{i=1}^{|a_1|+\ldots+|a_{j-1}|} v_i)\,dy\,dS.
\]

(3.29)
Step 3. Using that the Fourier transform of the sine-kernel \((3.27)\) has the form \(\hat{K} = \mathbb{I}_{[-1,1]}\), by a direct computation we find
\[
\int_{-\infty}^{\infty} \hat{K}(y)\hat{K}(y - \sum_{i=1}^{\lfloor |a| \rfloor} v_i) \cdots \hat{K}(y - \sum_{i=1}^{\lfloor |a| + \cdots + |a| - 1 \rfloor} v_i) \, dy = \max(2 + J^{\lfloor |a| \rfloor, \cdots, |a|}(v), 0),
\]  
where the function \(J^{\lfloor |a| \rfloor, \cdots, |a|}\) is defined in \((3.25)\). Then \((3.29)\) implies \((3.26)\).

**Remark 3.6.** In the proof of Proposition 3.5 we use the special structure of the sine-kernel only in Step 3.

**Corollary 3.7.** For any bounded measurable functions \(h, h_1, h_2 : \mathbb{R} \to \mathbb{R}\) with compact support under the sine-process we have
\[
\Var S_h = \frac{1}{4\pi^2} \left( 2 \int_{|s| \geq 2} |\hat{h}(s)|^2 \, ds + \int_{|s| < 2} |s||\hat{h}(s)|^2 \, ds \right),
\]  
\[
\Cov(S_{h_1}, S_{h_2}) = \frac{1}{4\pi^2} \Re \left( 2 \int_{|s| \geq 2} \hat{h}_1(s)\hat{h}_2(s) \, ds + \int_{|s| < 2} |s|\hat{h}_1(s)\overline{\hat{h}_2(s)} \, ds \right).
\]  
In particular, Corollary 3.7 implies
\[
\Var S_h \leq \|h\|_1^2/2,
\]  
where we recall that the seminorm \(\|\cdot\|_1/2\) is defined in \((2.1)\).

**Proof.** We first prove the formula \((3.31)\). Recall that the variance \(\Var S_h\) is given by \((2.6)\). Since \(K(x, x) = 1/\pi\), we have
\[
\tr h^2 K_D = \int_{-\infty}^{\infty} h^2(x) K(x, x) \, dx = \|h\|_{L^2}^2/\pi = \|\hat{h}\|_{L^2}^2/2\pi^2.
\]  
On the other hand, Proposition 3.5 implies
\[
\tr(hK_D)^2 = \frac{1}{4\pi^2} \int_{v_1 + v_2 = 0} \hat{h}(v_1)\hat{h}(v_2) \max(2 - |v_1|, 0) \, dS.
\]  
Then, using that \(\hat{h}(-s) = \overline{\hat{h}(s)}\) since the function \(h\) is real, we get
\[
\tr(hK_D)^2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |\hat{h}(s)|^2 \max(2 - |s|, 0) \, ds = \frac{1}{4\pi^2} \int_{|s| < 2} |\hat{h}(s)|^2 (2 - |s|) \, ds.
\]  
Inserting \((3.34)\) and \((3.35)\) into \((2.6)\), we find
\[
\Var S_h = \frac{1}{4\pi^2} \left( 2\|\hat{h}\|_{L^2}^2 - \int_{|s| < 2} |\hat{h}(s)|^2 (2 - |s|) \, ds \right) = \frac{1}{4\pi^2} \left( 2 \int_{|s| \geq 2} |\hat{h}(s)|^2 \, ds + \int_{|s| < 2} |s||\hat{h}(s)|^2 \, ds \right).
\]  
To get formula \((3.32)\), we note that
\[
\Cov(S_{h_1}, S_{h_2}) = \frac{1}{2} \left( \Var S_{h_1} + \Var S_{h_2} - \Var(S_{h_1} - S_{h_2}) \right).
\]  
Since \(S_{h_1} - S_{h_2} = S_{h_1 - h_2}\), the formula \((3.32)\) follows from the identity \((3.31)\).
4 Central Limit Theorems for linear statistics

In this section we prove multidimensional Central Limit Theorems 4.1 and 4.3.

4.1 Linear statistics with growing variance: Theorem 4.1

Let \( d \geq 1 \) and \( h_1^N, \ldots, h_d^N : \mathbb{R}^m \to \mathbb{R}, N \in \mathbb{N}, \) be a family of bounded Borel measurable functions with compact supports. Consider the corresponding vector of linear statistics

\[
S_{h^N} := (S_{h_1^N}, \ldots, S_{h_d^N})
\]

as a random vector under a determinantal process given by a Hermitian kernel \( K^N \). Denote by \( E^N, \text{Var}^N \) and \( \text{Cov}^N \) the corresponding expectation, variance and covariance. In this section we prove the Central Limit Theorem for the vector \( S_{h^N} \) under assumption that the variances \( \text{Var}^N S_{h_j^N} \) grow to infinity as \( N \to \infty \).

**Theorem 4.1.** Assume that there exists a sequence \( V_N \to \infty \) as \( N \to \infty \), \( V_N > 0 \), such that the following two conditions hold.

1. For all \( 1 \leq i, j \leq d \) there exist the limits

\[
\frac{\text{Cov}^N(S_{h_i^N}, S_{h_j^N})}{V_N} \to b_{ij}, \quad (4.1)
\]

for some numbers \( b_{ij} \).

2. We have

\[
\max_{1 \leq j \leq d} \| h_j^N \|_\infty = o(\sqrt{V_N}) \quad \text{as} \quad N \to \infty. \quad (4.2)
\]

Let \( \xi^N \in \mathbb{R}^d \) be a random vector with components

\[
\xi_j^N = \frac{S_{h_j^N} - E^N S_{h_j^N}}{\sqrt{V_N}}. \quad (4.3)
\]

Then for the family of distributions \( \mathcal{D}(\xi^N) \) we have the weak convergence \( \mathcal{D}(\xi^N) \to \mathcal{D}(\xi) \) as \( N \to \infty \), where \( \xi \) is a centred Gaussian random vector with the covariance matrix \((b_{ij})\).

Theorem 4.1 generalizes results obtained in works [14, 38, 39, 41], where the Central Limit Theorems for various linear statistics were established, under the assumption that \( \text{Var}^N S_{h_j^N} \to \infty \) as \( N \to \infty \). More precisely, in papers [14] and [38] the Central Limit Theorem was proven in the one-dimensional setting (i.e. \( d = 1 \)) for the linear statistics corresponding to a family of functions \( h^N \) of the form

\[
h^N(x) = \mathbb{1}_A(x/N), \quad (4.4)
\]

where \( A \) is a bounded Borel set, so that \( S_{h^N} = \#_A \). In [39] the author considered the linear statistics of the same form under the Airy and Bessel processes. He showed that their variances \( \text{Var} S_{h^N} \) have the logarithmic growth and proved a multidimensional Central Limit Theorem (i.e. \( d \geq 1 \)). In [41] a one-dimensional Central Limit Theorem
was established for a general family of bounded measurable functions $h^N$ with compact supports, under the assumptions that
\[ \|h^N\|_\infty = o((\text{Var}_N S_{h^N})^\delta) \quad \text{and} \quad E_N S_{h^N} = O((\text{Var}_N S_{h^N})^\delta), \] (4.5)
for any $\varepsilon > 0$ and some $\delta > 0$. This result can not be applied for the linear statistics corresponding to the family of functions (4.4) under the sine, Airy and Bessel processes. Indeed, the variance in these cases has the logarithmic growth while the expectation grows as $N^n$, $n > 0$, so that (4.5) fails. Since in Theorem 4.1 we do not impose assumption (4.5), it covers all the Central Limit Theorems above.

Note that in Theorem 4.1 the multidimensional case $d > 1$ follows from the one-dimensional one $d = 1$ by the linearity of statistics $S_{h^N}$. Proof of this fact literally repeats the proof of Proposition 4.6 below. However, we establish Theorem 4.1 directly in the multidimensional setting because it does not change the proof.

**Proof of Theorem 4.1.** The proof uses a method developed in [14] and [38], and is based on application of Corollary 3.3 and Lemma 3.4. Since the normal law is specified by its moments it suffices to show that the moments of the random vector $\xi^N$ converge to the moments of $\xi$ (see [17], page 269). Denote by $(A^N_k)_{k \in Z^d_+}$ and $(A_k)_{k \in Z^d_+}$ the cumulants of $\xi^N$ and $\xi$ respectively, so that
\[ A_k = \begin{cases} 0 & \text{if } |k| \neq 2, \\ b_{ij} & \text{if } k = e_i + e_j, \end{cases} \]
where $(e_l)$ is the standard base of $Z^d$. Since the moments can be expressed through the cumulants, it suffices to prove that
\[ A^N_k \to A_k \quad \text{as} \quad N \to \infty \quad \text{for any } k \in Z^d_+. \] (4.6)
In the case $|k| \leq 2$ the convergence (4.6) is clear. Indeed, due to (3.1), we have
\[ A^N_{e_i} = 0 \quad \text{and} \quad A^N_{e_i+e_j} = \text{Cov}_N \left( \frac{S_{h_i^N} - E_N S_{h_i^N}}{\sqrt{V_N}}, \frac{S_{h_j^N} - E_N S_{h_j^N}}{\sqrt{V_N}} \right) = \frac{\text{Cov}_N(S_{h_i^N}, S_{h_j^N})}{V_N}, \]
so that (4.6) follows from assumption (4.1). It remains to study the case $|k| \geq 3$. By definition (4.3) of the vector $\xi^N$ we have
\[ A^N_k = \frac{B^N_k}{V_N^{\lfloor |k|/2 \rfloor}}, \] (4.7)
where $B^N_k$ are cumulants of the random vector $S_{h^N}$. Due to Corollary 3.3 joined with Lemma 3.4, we have
\[ |B^N_k| \leq C \max_{1 \leq i \leq d} \|h_i^N\|_{\infty} \sum_{l=1}^{d} \text{Var}_N S_{h_i^N}. \]
Then, assumptions (4.1) and (4.2) imply $B^N_k = o(V_N^{\lfloor |k|/2 \rfloor})$, if $|k| \geq 3$. Now the desired convergence (4.6) follows from (4.7). \[ \square \]
4.2 Joint linear statistics of growing and bounded variances: Theorem 4.3

Consider a family of measurable bounded functions with compact supports \( f_1^N, \ldots, f_p^N, g_1^N, \ldots, g_q^N : \mathbb{R} \mapsto \mathbb{R} \), where \( N \in \mathbb{N} \) and \( p, q \geq 0 \). In this section we prove a multidimensional Central Limit Theorem 4.3 for the vector of the linear statistics

\[
(S_{f_1^N}, \ldots, S_{f_p^N}, S_{g_1^N}, \ldots, S_{g_q^N}), \tag{4.8}
\]

under the sine-process. We assume that the functions \( f_i^N \) are as in Theorem 4.1 while the functions \( g_j^N \) are supposed to be sufficiently regular and for large \( N \) asymptotically behave as \( g_j^\infty(\cdot/N) \), for some functions \( g_j^\infty \) independent from \( N \). This situation is not covered by Theorem 4.1 since under our hypotheses the variances \( \text{Var} S_{g_j^N} \) do not grow at all, so that condition (4.1) fails.

Before formulating our assumptions let us note that all of them except f.1 are automatically satisfied if \( f_i^N(x) = f_i(x/N), g_j^N(x) = g_j(x/N) \), where the functions \( f_i, g_j \) are bounded measurable with compact supports and \( g_j \) belong to the Sobolev space \( H^{1/2}(\mathbb{R}) \).

For the proof of this fact see Example 4.4 in the next section.

We assume that there exist sequences \( V_N, R_N \to \infty \) as \( N \to \infty \), \( V_N, R_N > 0 \), such that for all \( 1 \leq i \leq p, 1 \leq j \leq q \), the following hypotheses hold. Let

\[
f_i^N(x) := f_i^N(R_Nx) \quad \text{and} \quad g_j^N(x) := g_j^N(R_Nx). \tag{4.9}
\]

f.1 Under the sine-process there exist the limits

\[
\frac{\text{Cov}(S_{f_i^N}, S_{f_j^N})}{V_N} \to b_{ij}^f \quad \text{as} \quad N \to \infty, \tag{4.10}
\]

for some numbers \( b_{ij}^f \) and any \( 1 \leq i, j \leq p \).

f.2 We have \( \max_{1 \leq i \leq p} \| f_i^N \|_\infty = o(\sqrt{V_N}) \) as \( N \to \infty \).

f.3 We have \( \max_{1 \leq i \leq p} \| f_i^N \|_{L_2} = o(\sqrt{V_N}) \) as \( N \to \infty \).

Since \( \| f_i^N \|_{L_2} = R_N^{-1/2} \| f_i^N \|_{L_2} \), assumption f.3 just means that the norm \( \| f_i^N \|_{L_2} \) grows slower than \( (R_NV_N)^{1/2} \).

g.1 The functions \( g_j^N \) belong to the Sobolev space \( H^{1/2}(\mathbb{R}) \) and \( g_j^N \to g_j^\infty \) as \( N \to \infty \) in \( H^{1/2}(\mathbb{R}) \), for some functions \( g_j^\infty \) and any \( j \).

g.2 The functions \( g_j^N \) are bounded uniformly in \( N \).

Before stating the theorem let us note that, due to the estimate (3.33) and the following obvious proposition, assumption g.2 implies in particular that

\[
\text{the variances } \text{Var} S_{g_j^N} \text{ are bounded uniformly in } N. \tag{4.11}
\]

**Proposition 4.2.** For any function \( k \in H^{1/2}(\mathbb{R}) \) and any \( \delta \neq 0 \) we have \( \| k \|_{1/2} = \| k_\delta \|_{1/2} \), where \( k_\delta(x) := k(\delta x) \).
Proof. Since $\hat{k}_\delta(x) = \delta^{-1} \hat{k}(\delta^{-1} x)$, we get
\[ 4\pi^2 \|k_\delta\|_1^2 = \delta^{-2} \int_{-\infty}^{\infty} |v||\hat{k}(\delta^{-1} v)|^2 dv = \int_{-\infty}^{\infty} |u||\hat{k}(u)|^2 du = 4\pi^2 \|k\|_1^2, \]
where we set $u = \delta^{-1} v$. \hfill $\square$

**Theorem 4.3.** Let a family of measurable bounded compactly supported functions $f_1^N, \ldots, f_p^N$, $g_1^N, \ldots, g_q^N$, $p,q \geq 0$, satisfies assumptions f.1-f.3, g.1-g.2 above. Consider the vector of linear statistics (4.8) as a random vector under the sine-process. Let $\xi^N = (\xi_f^N, \xi_g^N) \in \mathbb{R}^{p+q}$ be a random vector with components
\[ \xi_{f_j}^N = \frac{S_{f_j}^N - \mathbb{E}S_{f_j}^N}{\sqrt{V_N}}, \quad \xi_{g_i}^N = S_{g_i}^N - \mathbb{E}S_{g_i}^N. \tag{4.12} \]
Then $\mathcal{D}(\xi^N) \to \mathcal{D}(\xi)$ as $N \to \infty$, where $\xi = (\xi_f, \xi_g) \in \mathbb{R}^{p+q}$ is a centred Gaussian random vector with the covariance matrix
\[ \begin{pmatrix} (b^g_{ij}) & 0 \\ 0 & (b^g_{kl}) \end{pmatrix}, \]
where $b^g_{kl} = \langle g_k^\infty, g_l^\infty \rangle_1^2$. In particular, the components $\xi_f$ and $\xi_g$ of the vector $\xi$ are independent.

Theorem 4.3 applied to the functions (4.13) implies Theorem 1.10 stated in Section 1.5. If $q = 0$ then Theorem 4.3 is covered by Theorem 4.1, while in the case $p = 0, q = 1$ it is proven by Spohn [42] and Soshnikov [40, 41] in slightly less generality; see the discussion in Section 1.5. The case $p = q = 1$ was not considered before and is the main novelty of the theorem, while the general situation $p, q \geq 0$ follows from the three just mentioned cases (see Proposition 4.6 below).

Proof of Theorem 4.3 employs a method developed in [40] mixed with that related to the method used in the proof of Theorem 4.1.

Note that the required regularity $H^{1/2}$ of the functions $g_i^N$ is optimal: if we replace $H^{1/2}$ by $H^{1/2-\varepsilon}$ then assertion of the theorem will not be true any more. Indeed, the indicator function $I_{[0,N]}$ belongs to the space $H^{1/2-\varepsilon}$, for all $\varepsilon > 0$. But the linear statistics $S_{I_{[0,N]}} = \#_{[0,N]}$ has (logarithmically) growing variance, so that the indicator $I_{[0,N]}$ belongs to the class of functions $f_i^N$ but not $g_j^N$.

### 4.3 Examples

In this section we present two examples where assumptions f.2-g.2 are satisfied. We will use them in Section 5, when proving our main results, Theorems 1.1 and 1.9.

**Example 4.4.** Let
\[ f_i^N(x) = f_i \left( \frac{x}{N} \right), \quad g_j^N(x) = g_j \left( \frac{x}{N} \right), \quad 1 \leq i \leq p, \quad 1 \leq j \leq q, \tag{4.13} \]
where the functions $f_i, g_j$ are bounded measurable with compact supports and $g_j$ belong to the Sobolev space $H^{1/2}(\mathbb{R})$. Then assumptions f.2-g.2 are fulfilled with $R_N = N$, arbitrary sequence $V_N$ and $g_j^\infty = g_j$.

---

\(^2\)Here and below by f.2-g.2 we mean f.2,f.3,g.1,g.2.
Proof. Assumptions \( f.2 \) and \( g.2 \) are obviously satisfied. Fulfilment of assumptions \( f.3 \) and \( g.1 \) immediately follows from the fact that, due to (4.13), we have \( f^N_i = f_i \) and \( g^N_j = g_j \).

**Example 4.5.** Assume that functions \( f^N_i, g^N_j \) satisfy assumptions \( f.2-g.2 \). Take a bounded measurable function \( \varphi \) with compact support such that \( \int_{-\infty}^{\infty} \varphi(x) \, dx = 1 \). Then the functions

\[
f^N_{\varphi,i} := \varphi \ast f^N_i, \quad g^N_{\varphi,j} := \varphi \ast g^N_j
\]

also satisfy \( f.2-g.2 \) with the same sequences \( V_N, R_N \) and functions \( g^N_j^\infty \).

**Proof.** Assumption \( f.2 \) follows from the identity

\[
\|f^N_{\varphi,i}\|_\infty \leq \|f^N_i\|_\infty \int_{-\infty}^{\infty} |\varphi(x)| \, dx.
\]

Assumptions \( g.2 \) follows in the same way. To get assumption \( f.3 \) we define the functions \( f^N_{\varphi,i} \) as in (4.9) and note that \( f^N_{\varphi,i}(v) = \hat{\varphi}(v/R_N) f^N_i(v) \). Then

\[
\|f^N_{\varphi,i}\|_{L^2} = \frac{1}{\sqrt{2\pi}} \|\hat{\varphi}(\cdot/R_N) f^N_i\|_{L^2} \leq \frac{1}{\sqrt{2\pi}} \|\hat{\varphi}\|_{L^2} \|f^N_i\|_{L^2} = \|\hat{\varphi}\|_{L^2} \|f^N_i\|_{L^2}.
\]

Since \( \varphi \in L^1(\mathbb{R}) \), we have \( \|\hat{\varphi}\|_{L^\infty} < \infty \), so that assumption \( f.3 \) follows.

The fact that the functions \( g^N_{\varphi,j} \) belong to the space \( H^{1/2}(\mathbb{R}) \) is implied by the inequality

\[
\|g^N_{\varphi,j}\|_{H^{1/2}} \leq \|\hat{\varphi}\|_{L^\infty} \|g^N_j\|_{H^{1/2}},
\]

which can be obtained similarly to the argument above. To establish the convergence claimed in assumption \( g.1 \), it suffices to show that \( \|g^N_{\varphi,j} - g^N_j\|_{H^{1/2}} \rightarrow 0 \) as \( N \rightarrow \infty \). Using that

\[
\hat{g}^N_{\varphi,j}(v) = \hat{\varphi}(R_N^{-1} v) \hat{g}^N_j(v)
\]

and \( \hat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x) \, dx = 1 \), we obtain

\[
2\pi \|g^N_{\varphi,j} - g^N_j\|_{H^{1/2}} = \int_{-\infty}^{\infty} (1 + |v|) \left| \hat{\varphi}(R_N^{-1} v) - \hat{\varphi}(0) \right|^2 \|g^N_j(v)\|^2 \, dv
\]

\[
\leq \max_{|v| \leq \sqrt{\pi N}} \left| \hat{\varphi}(R_N^{-1} v) - \hat{\varphi}(0) \right|^2 \int_{|v| \leq \sqrt{\pi N}} (1 + |v|) \|g^N_j(v)\|^2 \, dv
\]

\[
+ 2 \|\hat{\varphi}\|^2 \int_{|v| \geq \sqrt{\pi N}} (1 + |v|) \|g^N_j(v)\|^2 \, dv.
\]

Using assumption \( g.1 \) for the functions \( g^N_j \), the continuity of the function \( \hat{\varphi} \) and the relation \( \|\hat{\varphi}\|_{L^\infty} < \infty \), we see that both of the summands above go to zero as \( N \rightarrow \infty \).

**4.4 Beginning of the proof of Theorem 4.3**

The rest of Section 4 is devoted to the proof of Theorem 4.3. From now on we will skip the upper index \( N \) in the notation \( f^N_i, g^N_j, f^N_i, g^N_j \). We start by noting that, due to the linearity of statistics \( S^*_f, S^*_g \) in \( f_i, g_j \), it suffices to establish the theorem in the case \( p, q \leq 1 \), so when there is at most one function \( f_i \) and one function \( g_j \).
Proposition 4.6. Assume that Theorem 4.3 is established in the case \( p = q = 1 \). Then it holds for any \( p, q \geq 0 \).

Proposition 4.6 is proven in Section 4.6. Below we assume \( p = q = 1 \). This simplification is not crucial: with minor modifications, the proof we present below suits as well for the case of arbitrary \( p, q \). However, under this assumption the notation become simpler.

Further on we skip the lower index 1, so for the functions \( f_1, g_1, f_1, g_1 \) and the numbers \( b_{11}^f, b_{11}^g \) we write \( f, g, f, g, b, b \). To prove the theorem, it suffices to show that the cumulants \( (A_k^N)_{k \in \mathbb{Z}_+^2} \) of the random vector \( \xi^N \) satisfy

\[
A_k^N \to A_k \quad \text{as} \quad N \to \infty, \tag{4.14}
\]

where \( k = (k_f, k_g) \in \mathbb{Z}_+^2 \) and

\[
A_k = \begin{cases} 
  b^f & \text{if } k_f = 2, k_g = 0, \\
  b^g & \text{if } k_f = 0, k_g = 2, \\
  0 & \text{otherwise}.
\end{cases}
\]

By the definition (4.12) of the vector \( \xi^N \), for \( |k| = 1 \) we have \( A_k^N = 0 \) and for \( |k| \geq 2 \)

\[
A_k^N = \frac{B_k^N}{(V_N)^{k_f/2}}, \tag{4.15}
\]

where \( B_k^N \) are cumulants of the random vector \( (S_f, S_g) \). Further on we assume \( |k| \geq 2 \).

We single out four cases: \( k_g = 0; k_g \geq 1 \) and \( k_f \geq 3 \); \( k_g \geq 1 \) and \( k_f = 2 \); \( k_g \geq 1 \) and \( k_f \leq 1 \). The last one turns out to be the most complicated, so we study it separately in the next subsection. The reason is that in this case the denominator in (4.15) grows too slowly or does not grow at all, so that estimates for the cumulants \( B_k^N \) like those we use to study the other cases, do not suffice in this situation to prove the convergence \( A_k^N \to 0 \) for \( |k| \geq 3 \). Instead, we employ combinatorial techniques developed by Soshnikov in [40].

Note that we use the special form of the sine-kernel only in this last case.

Case 1: \( k_g = 0 \). In this situation convergence (4.14) is established in the proof of Theorem 4.1. Indeed, the cumulant \( A_k^N \) in the present case coincides with the cumulant \( A_k^N \) of the random variable \( \xi^N_f \).

Case 2: \( k_g \geq 1 \) and \( k_f \geq 3 \). Set

\[
h = (h_1, h_2) := (f, g). \tag{4.16}
\]

In view of Corollary 3.3, the desired convergence immediately follows from (4.15) joined with the following proposition.

Proposition 4.7. In the case \( k_f \geq 3 \) (while \( k_g \) is arbitrary), for any \( a^1, \ldots, a^j \in \mathbb{Z}_+^2 \), \( j \geq 1 \), satisfying \( a^1 + \ldots + a^j = k \), we have

\[
\text{tr} h^{a^1} K \ldots h^{a^j} K_D - \text{tr} h^k K_D = o(V_N^{k_f/2}) \quad \text{as} \quad N \to \infty.
\]

Proposition 4.7 is obtained as a refinement of Lemma 3.4, adapted for the present situation. Its proof is given in Section 4.6.

Case 3: \( k_g \geq 1 \) and \( k_f = 2 \). Consider a partition \( k = a^1 + \ldots + a^j \) from Corollary 3.3. Let \( a^i = (a^i_1, a^i_2) \in \mathbb{Z}_+^2 \), so that \( k_f = a^{j}_1 + \ldots + a^{j}_f \) and \( k_g = a^1_2 + \ldots + a^j_2 \). Since \( k_f = 2 \), there are only two possible situations:
S1 There is $1 \leq l \leq j$ such that $a^l_f = k_f$ and for all $i \neq l$ we have $a^i_f = 0$.

S2 There are $1 \leq l_1 < l_2 \leq j$ such that $a^{l_1}_f = a^{l_2}_f = 1$, while for all $i \neq l_1, l_2$ we have $a^i_f = 0$.

**Proposition 4.8.** In the situation S1 above we have

$$\text{tr } h^{a^1} K \ldots h^{a^j} K_D - \text{tr } h^k K_D = o(V_N) \quad \text{as} \quad N \to \infty. \quad (4.17)$$

In the situation S2,

$$\text{tr } h^{a^1} K \ldots h^{a^j} K_D - \text{tr } f g^k K f K_D = o(V_N). \quad (4.18)$$

Proof of Proposition 4.8 is given in Section 4.6. Assume that a sequence $(\tilde{B}^N_k)_{N \in \mathbb{N}}$ satisfies

$$B^N_k - \tilde{B}^N_k = o(V_N). \quad (4.19)$$

Then, in view of (4.15) and equality $k_f = 2$, we have

$$\lim_{N \to \infty} A^N_k = \lim_{N \to \infty} \frac{\tilde{B}^N_k}{V_N},$$

in the sense that if one of the limits exists then the other exists as well and the two are equal. Due to Corollary 3.3 joined with Proposition 4.8, the choice

$$\tilde{B}^N_k = k! \left( \text{tr } f g^k K f K_D - \text{tr } h^k K_D \right) \sum_{j=2}^{\lfloor k \rfloor} \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}^2_+ \text{satisfying S2:} \ a^1 + \ldots + a^j = k} \frac{1}{a^1! \ldots a^j!} \quad (4.20)$$

satisfies (4.19). Then, to prove that $A^N_k \to 0$ as $N \to \infty$, it suffices to show that the sum from the right-hand side of (4.20) vanishes, i.e.

$$L_k := \sum_{j=2}^{\lfloor k \rfloor} \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}^2_+ \text{satisfying S2:} \ a^1 + \ldots + a^j = k} \frac{1}{a^1! \ldots a^j!} = 0. \quad (4.21)$$

Let us subtract $L_k$ from the both sides of identity (3.8). Using that $|k| = k_g + k_f = k_g + 2$, we find

$$L_k = - \sum_{j=1}^{k_g+2} \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}^2_+ \text{satisfying S1:} \ a^1 + \ldots + a^j = k} \frac{1}{a^1! \ldots a^j!} = \sum_{j=1}^{k_g+1} \frac{(-1)^{j}}{j} \sum_{l=1}^{j} \sum_{a^1, \ldots, a^l \in \mathbb{Z}^2_+ \text{satisfying S1:} \ a^1 + \ldots + a^l = k, \ a^l_f = k_f} \frac{1}{a^1! \ldots a^l!} \quad (4.21)$$

$$= \sum_{j=1}^{k_g+1} \frac{(-1)^{j}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}^2_+ \text{satisfying S1:} \ a^1 + \ldots + a^j = k, \ a^j_f = k_f} \frac{1}{a^1! \ldots a^j!}.$$
In the last sum from (4.21) the $f$-components $a^1_f, \ldots, a^j_f$ are defined uniquely, $a^1_f = \ldots = a^{j-1}_f = 0$ and $a^j_f = k_f$. Then we can pass from the summation over $a^1_f, \ldots, a^j_f \in \mathbb{Z}_+^2$ to that over $a^1_g, \ldots, a^j_g$, where $a^1_g, \ldots, a^{j-1}_g > 0$ and $a^j_g \geq 0$. Using that $a^1! \ldots a^j! = k_f a^1_g! \ldots a^j_g!$, we obtain

$$L_k = \sum_{j=1}^{k_g+1} (-1)^j \sum_{a^1_g, \ldots, a^{j-1}_g > 0, a^j_g \geq 0: \ a^1_g + \ldots + a^j_g = k_g} \frac{1}{k_f a^1_g! \ldots a^j_g!}.$$  \hspace{1cm} (4.22)

Next we separate the last sum from (4.22) into two parts, over $a^1_g, \ldots, a^j_g$ such that $a^j_g = 0$ and such that $a^j_g \neq 0$. We find

$$L_k = \sum_{j=1}^{k_g+1} (-1)^j \left( \sum_{a^1_g, \ldots, a^{j-1}_g > 0: \ a^1_g + \ldots + a^j_g = k_g} \frac{1}{k_f a^1_g! \ldots a^{j-1}_g!} + \sum_{a^1_g, \ldots, a^{j-1}_g > 0: \ a^1_g + \ldots + a^j_g = k_g} \frac{1}{k_f a^1_g! \ldots a^{j-1}_g!} \right).$$  \hspace{1cm} (4.23)

Denote

$$x_j := \sum_{a^1_g, \ldots, a^{j-1}_g > 0: \ a^1_g + \ldots + a^j_g = k_g} \frac{1}{k_f a^1_g! \ldots a^{j-1}_g!}.$$ 

Since in the case $j = k_g + 1$ the set $\{a^1_g, \ldots, a^j_g > 0: a^1_g + \ldots + a^j_g = k_g\}$ is empty, the relation (4.23) takes the form

$$L_k = -x_1 + \sum_{j=2}^{k_g} (-1)^j (x_{j-1} + x_j) + (-1)^{k_g+1} x_{k_g} = 0.$$ 

This finishes the consideration of Case 3.

### 4.5 Conclusion of the proof of Theorem 4.3

Here we consider the last case, when $|k| \geq 2$,

$$k_g \geq 1 \quad \text{and} \quad k_f \leq 1.$$  \hspace{1cm} (4.24)

We will need the following «smoothing» proposition which is established in Section 4.6.

**Proposition 4.9.** Assume that Theorem 4.3 is proven when the assumption g.1 is replaced by a stronger assumption

**g.1′** The function $g$ belong to the Sobolev space $H^1(\mathbb{R})$ and $g \rightarrow g^\infty$ as $N \rightarrow \infty$ in $H^1(\mathbb{R})$, for some function $g^\infty$.

Then it holds under the assumption g.1 as well.

Further on we assume that the function $g$ satisfies condition g.1′. Recall that the function $h$ is given by (4.16). Set

$$h = (h_1, h_2) := (f, g).$$
Due to Proposition 3.5, for any $a^1, \ldots, a^j \in \mathbb{Z}_+^2$, $j \geq 1$, satisfying $a^1 + \ldots + a^j = k$, the trace $\text{tr} h^{a^1}K \ldots h^{a^j}K_D$ has the form (3.26). Since $h_i(s) = R_N h_i(R_N s)$, the change of variables $u_i := R_N v_i$ transforms (3.26) to

$$\text{tr} h^{a^1}K \ldots h^{a^j}K_D = \frac{1}{(2\pi)^{|k|}} \int_{u_1 + \ldots + u_{|k|} = 0} F_N^{a^1, \ldots, a^j}(u) \, dS,$$

where

$$F_N^{a^1, \ldots, a^j}(u) := \hat{h}^{a^1, \ldots, a^j}(u) \max \left( 0, 2R_N + J^{a^1, \ldots, a^j}(u) \right),$$

and the function $\hat{h}^{a^1, \ldots, a^j}$ is defined as in (3.24), with $\hat{h}_i$ replaced by $\hat{h}_i$. In the present case, when $p = q = 1$ and $k_f \leq 1$, the function $\hat{h}^{a^1, \ldots, a^j}$ has a simplified form. To explain this, assume first $k_f = 1$. Then there exists a unique $1 \leq m \leq j$ such that $a^m = 1$ while for $i \neq m$ we have $a^i = 0$. Thus, $a^m = (1, a^m_q)$ and $a^i = (0, a^i_q)$ for $i \neq m$. Set

$$A_f := |a^1| + \ldots + |a^{m-1}| + 1 = a^1 + \ldots + a^{m-1} + 1.$$

Then we have $\hat{h}_{iA_f} = \hat{f}$ and $\hat{h}_i = \hat{g}$ for any $i \neq A_f$. To cover the case $k_f = 0$ we set

$$\hat{\phi} := \begin{cases} \hat{f} & \text{if } k_f = 1, \\ \hat{g} & \text{if } k_f = 0. \end{cases}$$

Then we obtain

$$\hat{h}^{a^1, \ldots, a^j}(v) = \hat{\phi}(v_{A_f}) \prod_{1 \leq i \leq |k|, i \neq A_f} \hat{g}(v_i),$$

where in the case $k_f = 0$ we choose $A_f$ arbitrary. In particular, we see that if $k_f = 1$ the function $\hat{h}^{a^1, \ldots, a^j}$ depends on the vectors $a^1, \ldots, a^j$ only through the number $A_f$ while in the case $k_f = 0$ it is independent from $a^1, \ldots, a^j$.

Due to Lemma 3.1 joined with (4.25), the cumulant $B^N_k$ takes the form

$$B^N_k = \frac{k!}{(2\pi)^{|k|}} \sum_{j=1}^{|k|} \frac{(-1)^{j+1}}{j} \sum_{a^1, \ldots, a^j \in \mathbb{Z}_+^2 : a^1 + \ldots + a^j = k} \int_{u_1 + \ldots + u_{|k|} = 0} F_N^{a^1, \ldots, a^j}(u) \, dS.$$
To finish the proof of the theorem it suffices to check that under the assumption (4.24) assertions B1 and B2 below are satisfied (note that, in fact, the proof of B1 does not use assumption (4.24)).

**B1.** We have
\[
B_{k,1}^N = 0 \quad \text{if} \quad |k| > 2 \quad \forall N, \tag{4.31}
\]
and if $|k| = 2$,
\[
\frac{B_{k,1}^N}{V_N^{k_f/2}} \xrightarrow{N \to \infty} \begin{cases} b^g & \text{if } k_f = 0, k_g = 2, \\ 0 & \text{if } k_f = k_g = 1. \end{cases} \tag{4.32}
\]

**B2.** We have
\[
\frac{B_{k,2}^N}{V_N^{k_f/2}} \xrightarrow{N \to \infty} 0. \tag{4.33}
\]

**Proof of B1.** For $|u_1| + \ldots + |u_k| \leq R_N$ we have $\max (0, 2R_N + J^{\alpha^1, \ldots, \alpha^j}(u)) = 2R_N + J^{\alpha^1, \ldots, \alpha^j}(u)$. So that,
\[
F_N^{\alpha^1, \ldots, \alpha^j}(u) = \hat{h}^{\alpha^1, \ldots, \alpha^j}(u)(2R_N + J^{\alpha^1, \ldots, \alpha^j}(u)). \tag{4.34}
\]

Then the integral from (4.30) takes the form $I_1 + I_2$, where
\[
I_1 := 2R_N \int_{u_1 + \ldots + u_k = 0, \atop |u_1| + \ldots + |u_k| \leq R_N} \hat{h}^{\alpha^1, \ldots, \alpha^j} dS \quad \text{and} \quad I_2 := \int_{u_1 + \ldots + u_k = 0, \atop |u_1| + \ldots + |u_k| \leq R_N} \hat{h}^{\alpha^1, \ldots, \alpha^j} J^{\alpha^1, \ldots, \alpha^j} dS.
\]

Changing the order in the product (4.28), we obtain
\[
I_1 = 2R_N \int_{u_1 + \ldots + u_k = 0, \atop |u_1| + \ldots + |u_k| \leq R_N} \hat{f}(u_1)\hat{g}(u_2) \ldots \hat{g}(u_k) dS.
\]

Thus, the integral $I_1$ is independent from the choice of the vectors $\alpha^i$, so that in the formula (4.30) it can be put in front of the sums. In view of Proposition 3.2 the sums vanish, so that only the integral $I_2$ has an input to the term $B_{k,1}^N$:
\[
B_{k,1}^N = \frac{k!}{(2\pi)^{k(|k|)}} \sum_{j=1}^{k(|k|)} (-1)^{j+1} \sum_{\substack{\alpha^1, \ldots, \alpha^j \in \mathbb{Z}_+^k: \\ \alpha^1 + \ldots + \alpha^j = k, \atop |\alpha^1| = \ell^1, \ldots, |\alpha^j| = \ell^j}} 1_{\frac{1}{a^1! \ldots a^j!}} \int_{u_1 + \ldots + u_k = 0, \atop |u_1| + \ldots + |u_k| \leq R_N} \hat{h}^{\alpha^1, \ldots, \alpha^j}(u) J^{\alpha^1, \ldots, \alpha^j}(u) dS. \tag{4.35}
\]

Denote by $\Sigma_{|k|}$ the symmetric group of degree $|k|$.

**Proposition 4.10.** Let $l^1, \ldots, l^j \in \mathbb{N} \setminus \{0\}$, $j \geq 1$, satisfy $l^1 + \ldots + l^j = |k|$. Then
\[
\sum_{\substack{\alpha^1, \ldots, \alpha^j \in \mathbb{Z}_+^j: \\ \alpha^1 + \ldots + \alpha^j = k, \atop |\alpha^1| = \ell^1, \ldots, |\alpha^j| = \ell^j}} 1_{\frac{1}{a^1! \ldots a^j!}} \int_{u_1 + \ldots + u_k = 0, \atop |u_1| + \ldots + |u_k| \leq R_N} \hat{h}^{\alpha^1, \ldots, \alpha^j}(u) J^{l^1, \ldots, l^j}(u) dS \tag{4.36}
\]
\[
= \frac{1}{k! l^1! \ldots l^j!} \sum_{\sigma \in \Sigma_{|k|}} \int_{u_1 + \ldots + u_k = 0, \atop |u_1| + \ldots + |u_k| \leq R_N} \hat{f}(u_1)\hat{g}(u_2) \ldots \hat{g}(u_{|k|}) J^{l^1, \ldots, l^j}(u^\sigma) dS,
\]

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where \( u^\sigma := (u_{\sigma(1)}, \ldots, u_{\sigma(|k|)}) \).

Proof of Proposition 4.10 is postponed to Section 4.6. In view of the definition (3.25) of the function \( J^{1 \cdots \nu} \), Proposition 4.10 applied to (4.35) implies

\[
B_{k,1}^N = \frac{1}{(2\pi)^{|k|}} \int_{u_1 + \ldots + u_{|k|} = 0, \ |u_1| + \ldots + |u_{|k|}| \leq R_N} \hat{\phi}(u_1) \hat{\phi}(u_2) \cdots \hat{\phi}(u_{|k|}) \left( G(u) + G(-u) \right) dS, \tag{4.37}
\]

where

\[
G(u) := \sum_{j=1}^{|k|} \left( -1 \right)^j \sum_{l^1, \ldots, l^\nu > 0} \frac{1}{l^1! \cdots l^\nu!} \max(0, \sum_{i=1}^{l^1} u_{\sigma(i)}, \sum_{i=1}^{l^1+l^2} u_{\sigma(i)}, \ldots, \sum_{i=1}^{l^1+\ldots+l^\nu-1} u_{\sigma(i)}). \tag{4.39}
\]

Then, (4.37) implies (4.31). Now it remains only to study the term \( B_{k,1}^N \) in the case \(|k| = 2\).

Case \( k_f = 0 \) and \( k_g = 2 \). Since the function \( g \) is real, we have \( \hat{g}(-s) \equiv \overline{g(s)} \). Then, in view of (4.37) and (4.38), we get

\[
B_{k,1}^N = \frac{1}{2(2\pi)^2} \int_{u_1 + u_2 = 0, \ |u_1| + |u_2| \leq R_N} \hat{g}(u_1) \hat{g}(u_2) |u_1| dS = \frac{1}{4\pi^2} \int_{-R_N/2}^{R_N/2} |s| \overline{\hat{g}(s) \overline{g(s)}} dS. \tag{4.40}
\]

Due to assumption \( g.1' \) (even \( g.1 \) suffices here), the right-hand side of (4.39) converges to \( b^2 \), so that we get (4.32).

Case \( k_f = k_g = 1 \). Relation (4.37) joined with (4.38) implies

\[
A_{k,1}^N := \frac{B_{k,1}^N}{V_N^{k_f/2}} = \frac{1}{4\pi^2 \sqrt{V_N}} \int_{-R_N/2}^{R_N/2} |s| \overline{\hat{f}(s) \overline{g(s)}} dS.
\]

Using the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

\[
|A_{k,1}^N| \leq \frac{1}{4\pi^2} \left( \frac{1}{V_N} \int_{-R_N/2}^{R_N/2} |\hat{f}(s)|^2 |s|^2 \overline{g(s)}^2 dS \right)^{1/2} \left( \int_{-R_N/2}^{R_N/2} |s|^2 |\overline{\hat{g}(s)}|^2 dS \right)^{1/2} \tag{4.42}
\]

Due to assumption \( f.3 \), the first integral above goes to zero as \( N \to \infty \). Since, in view of assumption \( g.1' \), the second one is bounded uniformly in \( N \), the desired convergence follows.

\footnote{It seems that in [40] the factor 2 is omitted: it is written \( G(u) \) instead of \( 2G(u) \).}
**Proof of B2.** Since, by the definition, \( J^{[a_1],...,[a_j]} \leq 0 \), we have \( |F^{[a_1],...,[a_j]}| \leq 2R_N|\hat{h}^{a_1,...,a_j}| \), see (4.26). Thus, it suffices to prove that

\[
V_N^{-k_f/2} R_N \int_{u_1+...+u_{|k|}=0, \ |u_1|+...+|u_{|k|}| \geq R_N} |\hat{h}^{a_1,...,a_j}(u)| \, dS \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \tag{4.41}
\]

for any \( a_1, \ldots, a_j \). Due to (4.28), changing the order in the product \( \hat{h}^{a_1,...,a_j} \) we obtain

\[
|\hat{h}^{a_1,...,a_j}(u)| \leq |\hat{\phi}(u_1)\hat{g}(u_2)\hat{g}(u_3)\ldots\hat{g}(u_{|k|})|.
\]

Excluding the variable \( u_1 \), we get

\[
R_N \int_{u_1+...+u_{|k|}=0, \ |u_1|+...+|u_{|k|}| \geq R_N/2} |\hat{h}^{a_1,...,a_j}(u)| \, dS
\]

\[
\leq R_N \int_{|u_2|+...+|u_{|k|}| \geq R_N/2} |\hat{\phi}(-u_2 - \ldots - u_{|k|})\hat{g}(u_2)\ldots\hat{g}(u_{|k|})| \, du_2 \ldots du_{|k|}
\]

\[
\leq 2 \int_{|u_2|+...+|u_{|k|}| \geq R_N/2} (|u_2| + \ldots + |u_{|k|}|) |\hat{\phi}(-u_2 - \ldots - u_{|k|})\hat{g}(u_2)\ldots\hat{g}(u_{|k|})| \, du_2 \ldots du_{|k|}
\]

\[
= 2(|k| - 1) \int_{|u_2|+...+|u_{|k|}| \geq R_N/2} |u_2| |\hat{\phi}(-u_2 - \ldots - u_{|k|})\hat{g}(u_2)\ldots\hat{g}(u_{|k|})| \, du_2 \ldots du_{|k|}
\]

\[
= 2(|k| - 1)L^N.
\]

Next we separate the cases \(|k| = 2\) and \(|k| > 2\).

**Case** \(|k| = 2\). Applying the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

\[
L^N = \int_{|u_2| \geq R_N/2} |u_2\hat{\phi}(-u_2)\hat{g}(u_2)| \, du_2 \leq \|\hat{\phi}\|_{L_2} \left( \int_{|u_2| \geq R_N/2} |u_2|^2|\hat{g}(u_2)|^2 \, du_2 \right)^{1/2}.
\]

Assumptions \( f.3 \) and \( g.1' \) (or \( g.1 \)) imply that

\[
\|\hat{\phi}\|_{L_2} \leq CV_N^{k_f/2}. \tag{4.42}
\]

Then, using assumption \( g.1' \), we find \( V^{-k_f/2} L^N \rightarrow 0 \) as \( N \rightarrow \infty \). So that, we get (4.41).

**Case** \(|k| > 2\). We have

\[
L^N \leq \int_{|u_3|+...+|u_{|k|}| \geq R_N/4} |\hat{g}(u_3)\ldots\hat{g}(u_{|k|})| \, \int_{|u_2| \leq R_N/4} |u_2\hat{\phi}(-u_2 - \ldots - u_{|k|})\hat{g}(u_2)| \, du_2 \ldots du_{|k|}
\]

\[
+ \int_{|u_2| \geq R_N/4} |\hat{g}(u_3)\ldots\hat{g}(u_{|k|})| \, \int_{|u_2| \geq R_N/4} |u_2\hat{\phi}(-u_2 - \ldots - u_{|k|})\hat{g}(u_2)| \, du_2 \ldots du_{|k|}
\]

\[
=: L_1^N + L_2^N.
\]

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Using the Cauchy-Bunyakovsky-Schwarz inequality, we find
\[ L_1^N \leq C \| \hat{\phi} \|_{L^2} \| \hat{g} \|_1 \int_{|u_3|+\ldots+|u_{|k|}| \geq R_N/4} |\hat{g}(u_3) \ldots \hat{g}(u_{|k|})| \, du_2 \ldots du_{|k|} \quad \text{and} \]
\[ L_2^N \leq \| \hat{g} \|_{L^1} |k|^{-2} \| \hat{\phi} \|_{L^2} \left( \int_{|u_2| \geq R_N/4} |u_2|^2 |\hat{g}(u_2)|^2 \, du_2 \right)^{1/2}. \]

In view of (4.42) and assumption g.1', to see that \( V_N^{-k_f/2} L_2^N \to 0 \) as \( N \to \infty \) it suffices to show that the \( L^1 \)-norm \( \| \hat{g} \|_{L^1} \) is finite and bounded uniformly in \( N \). This follows from the estimate
\[ \int_{\infty}^{\infty} |\hat{g}(s)| \, ds = \int_{\infty}^{\infty} \frac{|s| + 1}{|s| + 1} |\hat{g}(s)| \, ds \leq C \| \hat{g} \|_{H^1} \left( \int_{\infty}^{\infty} \frac{ds}{(|s| + 1)^2} \right)^{1/2} = C_1 \| \hat{g} \|_{H^1}. \quad (4.43) \]

To see that \( V_N^{-k_f/2} L_1^N \to 0 \), we need additionally prove that the integral \( \int_{|s|>M} |\hat{g}(s)| \, ds \) converges to zero as \( M \to \infty \) uniformly in \( N \). This follows similarly.

### 4.6 Proofs of auxiliary results

In this section we establish Propositions 4.6-4.10 used in the proof of Theorem 4.3.

**Proof of Proposition 4.6.** Let the functions \( f_1, \ldots, f_p, g_1, \ldots, g_q \) satisfy assumptions f.1-g.2. Set
\[ B := \begin{pmatrix} (b_{ij}^f) & 0 \\ 0 & (b_{ij}^g) \end{pmatrix}, \]
so that \( B \) is a \((p+q) \times (p+q)\)-matrix. To establish the proposition, it suffices to prove that for any \((t, s) \in \mathbb{R}^{p+q}\) we have the convergence of the characteristic functions
\[ \mathbb{E} e^{i(\xi_f^N \cdot t + \xi_g^N \cdot s)} \to e^{-\frac{1}{2}(t, s)B(t, s)^T} \quad \text{as} \quad N \to \infty, \quad (4.44) \]
where \( T \) stands for the transposition. Note that
\[ \xi_f^N \cdot t = \sum_{j=1}^p t_j S_{f_j} - \mathbb{E} S_{f_j} = \frac{S_{f \cdot t} - \mathbb{E} S_{f \cdot t}}{\sqrt{V_N}} =: \xi_f^N \cdot t. \]

Similarly, we have \( \xi_g^N := \xi_g^N \cdot s \). Thus, it remains to study the "2-dimensional" characteristic function \( \mathbb{E} e^{i(\xi_f^N \cdot t + \xi_g^N \cdot s)} \). We claim that the functions \( f \cdot t \) and \( g \cdot s \) satisfy assumptions of Theorem 4.3. Indeed, assumptions f.2, f.3 and g.2 are obviously fulfilled. Assumption g.1 is also obvious, with the function \( \mathbf{g}^\infty \cdot s = \sum_{j=1}^N \mathbf{g}_j^\infty \cdot s_j \). Thus, it remains to check assumption f.1. We have
\[ \frac{\text{Var} S_{f \cdot t}}{V_N} = \sum_{i,j=1}^p \frac{t_i t_j \text{Cov} (S_{f_i}, S_{f_j})}{V_N} \to \sum_{i,j=1}^p t_i t_j b_{ij}^f = t (b_{ij}^f) t^T, \quad \text{as} \quad N \to \infty, \]
due to assumption \( f.1 \) for the functions \( f_1, \ldots, f_p \). So, the function \( f \cdot t \) also satisfies \( f.1 \).

Here we emphasize that in Theorem 4.3 we do not assume that \( b^T \) are different from zero (cf. Remark 1.6), so that the situation when \( t(b^T) t^T = 0 \) does not pose problems.

Now, applying Theorem 4.3 to the functions \( f \cdot t \) and \( g \cdot s \), we get the convergence \( D(\xi_{t,t}, \xi_{g,s}) \to D(\xi_{t,t}, \xi_{g,s}) \), where \( (\xi_{t,t}, \xi_{g,s}) \) is a centred Gaussian random vector with the covariance matrix

\[
\mathbb{B}_2 := \begin{pmatrix} t(b^T) t^T & 0 \\ 0 & \| \gamma \|_{1/2} \end{pmatrix}.
\]

In particular, we have the convergence of the corresponding characteristic functions at the point \((1, 1)\)

\[
\mathbb{E} e^{i(\xi_{t,t}^N + \xi_{g,s}^N)} \to e^{-\frac{1}{2}(1,1)\mathbb{B}_2(1,1)^T}.
\]

Now, to get (4.44) it remains to note that \((1, 1)\mathbb{B}_2(1,1)^T = (t, s)\mathbb{B}(t, s)^T. \)

**Proof of Proposition 4.7.** We follow the scheme used in the proof of Lemma 3.4. Assume first that \( j = 2 \) (the case \( j = 1 \) is trivial). Then we have estimates (3.12) and (3.14). Assumptions \( f.2 \) and \( g.2 \) state that

\[
\| f \|_\infty = o(\sqrt{V_N}) \text{ and } \| g \|_\infty \leq C.
\]

Then

\[
\| h^{k_f} \|_{1/2} \leq C \| f \|_{k_f}^{k_f-2} = o(V_N^{k_f/2}),
\]

where \( \gamma \) is defined in (3.13) with \( d = 2 \). Then, in view of (3.14) and (3.15), we have

\[
| \text{tr}(h^{k_f} K_D - h^{k_f} K_D^2) | \leq o(V_N^{k_f-2})(\text{Var} \mathcal{S}_f + \text{Var} \mathcal{S}_g) = o(V_N^{k_f/2}).
\]

Here we have used that \( \text{Var} \mathcal{S}_f \leq CV_N \) and \( \text{Var} \mathcal{S}_g \leq C \), accordingly to assumption \( f.1 \) and (4.11).

Take any \( c = (c_f, c_g) \in \mathbb{Z}_+^2 \). If \( c_f = 0 \) then, due to Proposition 2.4 joined with (2.8), we have

\[
\| [h^{c_f}, K_D] \|_{HS} \leq C \| g \|_{\infty}^{c_f-2} \| g, K_D \|_{HS} \leq C_1 \| g \|_{\infty}^{c_f-2}(\text{Var} \mathcal{S}_g)^{1/2} \leq C_2.
\]

If \( c_f = 1 \) then

\[
\| [h^{c_f}, K_D] \|_{HS} \leq C \| f \|_{\infty} \| g \|_{\infty}^{c_f-1}(\text{Var} \mathcal{S}_g)^{1/2} + C \| g \|_{\infty}^{c_f-1}(\text{Var} \mathcal{S}_f)^{1/2}
\]

\[
\leq o(\sqrt{V_N}) + C_1 \sqrt{V_N} \leq C_2 V_N^{c_f/2}.
\]

If \( c_f \geq 1 \), then arguing similarly we find

\[
\| [h^{c_f}, K_D] \|_{HS} = o(V_N^{c_f/2}).
\]

Take \( a^1, a^2 \in \mathbb{Z}_+^2 \) satisfying \( a^1 + a^2 = k \). Since \( k_f \geq 3 \), the situation \( a^1_f, a^2_f \leq 1 \) is impossible. Then, without loss of generality we assume that \( a^2_f > 1 \) and get

\[
\| [h^{a^1_f}, K_D] \|_{HS}\| [h^{a^2_f}, K_D] \|_{HS} \leq CV_N^{a^1_f/2} o(V_N^{a^2_f/2}) = o(V_N^{k_f/2}).
\]

(4.50)
Now estimates (4.46) and (4.50) imply that the right-hand side of (3.12) is bounded by $o(V_N^{k_f/2})$, so that we get the desired inequality. The case $j \geq 3$ can be studied in a similar way, following the scheme of the proof of Lemma 3.4.

**Proof of Proposition 4.8.** To get the desired estimates we revise the proof of Lemma 3.4, additionally using the regularity of the function $g$ and the fact that the operator $K$ corresponds to the sine-kernel, so that $K$ is a projection: $K^2 = K$. The latter relation will be used in estimates analogous to (3.12) and (3.21), to kill there the second and the second and the third terms of the right-hand side correspondingly. The problem here is that $K_D^2 \neq K_D$. Thus, our first aim is to reduce estimates on the operator $K_D$ to estimates on $K$.

Let $m \geq 2$ and $b_1, \ldots, b_m \in Z_+$. Since the supports $\text{supp} h_i$ are compact and the sine-kernel $K$ has the form (1.1), the operators $Kh^{b_i}$ and $h^{b_i}K$ are Hilbert-Schmidt. This implies that the operator $h^{b_i}K \ldots h^{b_m}K$ is of the trace class as a product of Hilbert-Schmidt operators. Jointly with cyclicity of the trace this provides

$$
\text{tr} h^{b_1}K \ldots h^{b_m}K_D = \text{tr} h^{b_1}K \ldots h^{b_m}K\|D = \text{tr} \|D h^{b_1}K \ldots h^{b_m}K = \text{tr} h^{b_1}K \ldots h^{b_m}K.
$$

Thus, (4.51) implies the desired inequality (4.17).

Now estimates (4.46) and (4.50) imply that the right-hand side of (3.12) is bounded by $o(V_N^{k_f/2})$, so that we get the desired inequality. The case $j \geq 3$ can be studied in a similar way, following the scheme of the proof of Lemma 3.4.
Assume now \( j \geq 3 \). Define the operator \( G \) as in (3.18) with \( K_D \) replaced by \( K \). Then, literally repeating (3.20)-(3.22) with \( K_D \) replaced by \( K \) and using the identity \( K^2 = K \), we get

\[
| \text{tr} GKh^{a_j-1}kh^{a_j}K - \text{tr} GKh^{a_j-1+a_j}K | \leq \| GK \| [[h^{a_j-1}, K]]_{\text{HS}} \| [h^{a_j}, K] \|_{\text{HS}}. \tag{4.54}
\]

Without loss of generality we assume that \( a_j^1 = k_j \) and \( a_j^j = 0 \) for \( i \geq 2 \) (in particular, \( a_j^{j-1} = a_j^j = 0 \)). Then, arguing as above, we see that the Hilbert-Schmidt norms from (4.54) are bounded uniformly in \( j \). On the other hand,

\[
\| GK \| \leq \| f \|_{k_j^{j-1}}^{a_j-1} \| g \|_{k_j}^{a_j-1} = o(V_N^{k_j/2}) = o(V_N), \tag{4.55}
\]
due to (4.45). Thus, the right-hand side of (4.54) is bounded by \( o(V_N) \). Now, by the induction axiom, we get the desired inequality (4.17).

**Case S2.** Without loss of generality we assume that \( a_j^1 = a_j^n = 1 \) for some \( n > 1 \), while for \( i \neq 1, n \) we have \( a_j^i = 0 \). Consider first the situation when \( j \geq 3 \). Then \( j, j-1 \neq 1 \). If additionally \( j, j-1 \neq n \), then the norms \( [[h^{a_j}, K]]_{\text{HS}} \) and \( [[h^{a_j-1}, K]]_{\text{HS}} \) are bounded uniformly in \( N \). Moreover, (4.55) is satisfied, so that the right-hand side of inequality (4.54) is bounded by \( o(V_N) \). If one of the numbers \( j - 1 \) or \( j \) is equal to \( n \), then, arguing as in (4.48), we see that the Hilbert-Schmidt norm of the corresponding commutator is majorated by \( \sqrt{N} \). On the other hand, the product from (4.55) is then bounded by \( o(V_N^{k_j-1}) \) in this case. Thus, the right-hand side of (4.54) is majorated by

\[
C\sqrt{N} o(V_N^{k_j-1}) = o(V_N).
\]

Summing up, for \( j \geq 3 \) we obtain

\[
| \text{tr} GKh^{a_j-1}kh^{a_j}K - \text{tr} GKh^{a_j-1+a_j}K | = o(V_N). \tag{4.56}
\]

Let now \( j = 2 \). Then \( \text{tr} h^{a_j}Kh^{a^2}K = \text{tr} fg^{a_j^1}Kfg^{a_j^2}K \), for some \( 1 \leq m_1, m_2 \leq p \). Using cyclicity of the trace, we obtain

\[
| \text{tr} fg^{a_j^1}Kfg^{a_j^2}K - \text{tr} fg^{k_j}KfK | \leq | \text{tr} fg^{a_j^1}Kfg^{a_j^2}K - \text{tr} fg^{a_j^1}Kfg^{a_j^2}KfK | + | \text{tr} fKfg^{a_j^1}Kg^{a_j^2}KfK - \text{tr} Kfg^{k_j}KfK | \leq o(V_N), \tag{4.57}
\]

in view of (4.56). Now the desired estimate (4.18) follows by induction from (4.56) and (4.57).

**Proof of Proposition 4.9.** Let the functions \( f, g \) satisfy assumptions f.1-g.2. Consider a smooth function \( w : \mathbb{R} \rightarrow \mathbb{R} \)

\[
w(x) = \begin{cases} Ce^{\frac{1}{|x|^2 - 1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}
\]

where the constant \( C \) is chosen in such a way that \( \int_{-\infty}^{\infty} w(x) \, dx = 1 \). Set \( w_\varepsilon(x) := \varepsilon^{-1}w(\varepsilon^{-1}x) \), where \( 0 < \varepsilon < 1 \), and let

\[g_\varepsilon := w_\varepsilon * g.\]

**Step 1.** In this step we show that for any \( \varepsilon > 0 \) the function \( g_\varepsilon \) defined through the function \( g_\varepsilon \) as in (4.9) satisfies assumptions g.1’, g.2 with \( g_\varepsilon = w_\varepsilon * g_\varepsilon \). Fulfilment of
g.2 follows from the inequality
\[ \|g_{\epsilon}\|_\infty = \|g_{\epsilon}\|_\infty \leq \|g\|_\infty \int_{-\infty}^{\infty} w_\epsilon(x) \, dx = \|g\|_\infty = \|g\|_\infty. \]
Since the function \( w_\epsilon \) is smooth, the function \( g_\epsilon \) also is, so in particular \( g_\epsilon \) belongs to the space \( H^1(\mathbb{R}) \). Then, to get assumption g.1' it suffices to show that \( g_\epsilon \to g_\epsilon^\infty \) as \( N \to \infty \) in \( H^1(\mathbb{R}) \). We have
\[ \|g_\epsilon - g_\epsilon^\infty\|_{H^1}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + |u|^2) \left| \hat{w}_\epsilon - \hat{g} \right|^2 du \leq \|\hat{\alpha}_\epsilon\|_\infty \|g - g^\infty\|_{H^{1/2}}^2, \tag{4.58} \]
where \( \hat{\alpha}_\epsilon := (1 + |u|^2)(1 + |u|^{-1})|\hat{w}_\epsilon|^2 \). Since the function \( \hat{w}_\epsilon \) is of the Schwartz class, the norm \( \|\hat{\alpha}_\epsilon\|_\infty \) is finite (although dependent from \( \epsilon \)). Then, assumption g.1 for the function \( g \) implies that the right-hand side of (4.58) goes to zero as \( N \to \infty \), for each \( \epsilon > 0 \).

Step 2. It remains to show that assertion of Theorem 4.3 holds for the functions \( f, g \).
By assumption of the proposition, it is satisfied for the functions \( f, g_\epsilon \). So that, for the random vector \( (\xi_f^N, \xi_g^N) \) defined as in (4.12) and any \( t, s \in \mathbb{R} \) we have
\[ \mathbb{E} e^{i(t\xi_f^N + s\xi_g^N)} \to e^{-\frac{1}{2} (t,s)B(t,s)^T} \quad \text{as} \quad N \to \infty, \tag{4.59} \]
where \( B_{\epsilon} := \begin{pmatrix} b_f^\epsilon & 0 \\ 0 & b_g^\epsilon \end{pmatrix} \) and \( b_g^\epsilon := \|g_\epsilon^\infty\|_{1/2}^2 \). Let \( B := \begin{pmatrix} b_f^\epsilon & 0 \\ 0 & b_g^\epsilon \end{pmatrix} \). Then
\[ \left| \mathbb{E} e^{i(t\xi_f^N + s\xi_g^N)} - e^{-\frac{1}{2} (t,s)B(t,s)^T} \right| \leq I_{1,\epsilon}^N + I_{2,\epsilon}^N + I_{3,\epsilon}, \]
where
\[ I_{1,\epsilon}^N = \left| \mathbb{E} e^{i(t\xi_f^N + s\xi_g^N)} - e^{-\frac{1}{2} (t,s)B(t,s)^T} \right|, \quad I_{2,\epsilon}^N = \left| \mathbb{E} e^{i(t\xi_f^N + s\xi_g^N)} - e^{-\frac{1}{2} (t,s)B(t,s)^T} \right|, \]
and
\[ I_{3,\epsilon} = \left| e^{-\frac{1}{2} (t,s)B(t,s)^T} - e^{-\frac{1}{2} (t,s)B(t,s)^T} \right|. \]
In view of (4.59), \( I_{1,\epsilon}^N \to 0 \) as \( N \to \infty \) for any \( \epsilon > 0 \) and any \( t, s \). Thus, to finish the proof of the proposition it remains to show that \( I_{2,\epsilon}^N, I_{3,\epsilon} \to 0 \) as \( \epsilon \to 0 \) uniformly in \( N \), for any \( t, s \). We have
\[ I_{2,\epsilon}^N \leq \mathbb{E} \left| e^{i\xi_g^\epsilon} - e^{i\xi_g^\epsilon} \right| \leq \mathbb{E} |s| \left| \xi_g^N - \xi_g^\epsilon \right| \leq |s| \sqrt{\text{Var}(\xi_g^N - \xi_g^\epsilon)}. \]
Due to (3.33),
\[ \text{Var}(\xi_g^N - \xi_g^\epsilon) = \text{Var} S_{g-g_\epsilon} \leq \|g - g_\epsilon\|_2^2 = \|g - g_\epsilon\|_{1/2}^2, \]
where in the last identity we used Proposition 4.2. For any \( r > 0 \) we have
\[ 4\pi^2 \|g - g_\epsilon\|_{1/2}^2 = \int_{-\infty}^{\infty} |u| |1 - \hat{w}_\epsilon(u)|^2 \hat{g}(u)^2 \, du \]
\[ \leq \sup_{|x| \leq r} \left| 1 - \hat{w}_\epsilon(x) \right|^2 \int_{-r}^{r} |u| |\hat{g}|^2 \, du + (\|\hat{w}_\epsilon\|_\infty + 1)^2 \int_{|u| \geq r} |u| |\hat{g}|^2 \, du. \]
Due to assumption $g.1$ for the function $g$ and the relation $\hat{w}_{\varepsilon}(x) = \hat{w}(\varepsilon x) \to \hat{w}(0) = 1$ as $\varepsilon \to 0$, which holds for any $x$, we see that the first term above goes to zero as $\varepsilon \to 0$, for any $r$ uniformly in $N$. Using assumption $g.1$ again, we find that the second term goes to zero when $r \to \infty$, uniformly in $\varepsilon$ and $N$. Consequently,

$$\|g - g_{\varepsilon}\|_{2,1/2}^2 \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly in } N,$$

(4.60)

so that $I_{2,\varepsilon}^N$ also does. To show that $I_{3,\varepsilon} \to 0$ as $\varepsilon \to 0$, it suffices to prove that $\|g^\infty - g_{\varepsilon}^\infty\|_{1/2} \to 0$ as $\varepsilon \to 0$. This follows by taking the limit $N \to \infty$ in (4.60). □

**Proof of Proposition 4.10.** Set

$$\eta_1 := \hat{\phi} \text{ and } \eta_2 = \ldots = \eta_{|k|} := \hat{g}.$$ 

Then the sum from the right-hand side of (4.36) can be rewritten as

$$\sum_{\sigma \in \mathcal{S}_{|k|}} \int_{u_1 + \ldots + u_{|k|} = 0} \eta_{\sigma(1)}(u_1) \eta_{\sigma(2)}(u_2) \ldots \eta_{\sigma(|k|)}(u_{|k|}) J^{l_1,\ldots,l^j}(u) dS. \tag{4.61}$$

Fix any partition $a^1 + \ldots + a^j = k$, where $|a^i| = l^i$ for all $i$. The function $J^{a^1,\ldots,a^j}(u)$ depends on $u$ only through the unordered sets $\{u_1, \ldots, u_{|a^1|}\}, \{u_{|a^1|+1}, \ldots, u_{|a^1|+|a^2|}\}, \ldots$. Then the integral from the left-hand side of (4.36), corresponding to this partition, coincides with the integral from (4.61), corresponding to a permutation $\sigma$, iff among the functions $\eta_{\sigma(1)}, \ldots, \eta_{\sigma(|a^1|)}$ there are exactly $a^1_j$ functions equal to $\hat{f}$ and $a^1_g$ functions equal to $\hat{g}$; among the functions $\eta_{\sigma(|a^1|+1)}, \ldots, \eta_{\sigma(|a^1|+|a^2|)}$ there are $a^2_j$ functions equal to $\hat{f}$, $a^2_g$ functions equal to $\hat{g}$, and so on. The number of such permutations can be found directly and is equal to

$$\frac{k! |l_1|! \ldots |l^j|!}{a^1! \ldots a^j!}.$$

Thus, the sum (4.61) can be rewritten as

$$\sum_{\substack{a^1, \ldots, a^j \in \mathbb{Z}^2; a^1 + \ldots + a^j = k, \|a^1\| = l^1, \ldots, |a^j| = l^j}} \frac{k! |l_1|! \ldots |l^j|!}{a^1! \ldots a^j!} \int_{u_1 + \ldots + u_{|k|} = 0} \hat{h}^{a^1,\ldots,a^j}(u) J^{l_1,\ldots,l^j}(u) dS.$$

□

5 Proofs of main results

In this section we establish Propositions 1.2, 1.3, 1.8 and Theorems 1.1, 1.9.

5.1 Proofs of Theorem 1.1 and Propositions 1.2, 1.3

Here we prove Propositions 1.3, 1.2 and Theorem 1.1.

**Proof of Proposition 1.3.**
The number of particles \( \#_{[0,t,N]} \) coincides with the linear statistics \( S_{i,N}^{t} \), where \( f_{t}^{N} := \mathbb{I}_{[0,t,N]} \). Then the desired convergence would follow from the Central Limit Theorem 4.1 if we show that

\[
\frac{\operatorname{Cov}(S_{i,N}^{t}, S_{j,N}^{t})}{\pi^{-2} \ln N} \to b_{ij} \quad \text{as} \quad N \to \infty,
\]

(5.1)

where \( b_{ij} \) are given by (1.14). In the case \( i = j \) convergence (5.1) follows from (1.2). Assume that \( i > j \). Since \( S_{i,N}^{t} = S_{i,j}^{N} = \#_{[t,\infty]} \), due to (1.2) we have

\[
\operatorname{Var}(S_{i,N}^{t} - S_{i,j}^{N}) = \pi^{-2} \ln N + O(1).
\]

Then (5.1) follows from (1.2) and the obvious relation

\[
\operatorname{Cov}(S_{i,N}^{t}, S_{j,N}^{t}) = \frac{1}{2} \left( \operatorname{Var} S_{i,N}^{t} + \operatorname{Var} S_{j,N}^{t} - \operatorname{Var}(S_{i,N}^{t} - S_{j,N}^{t}) \right).
\]

(5.2)

**Proof of Theorem 1.1.**

*Step 1.* In this step we show that for any \( 0 \leq t_1 < \ldots < t_d \leq 1 \),

\[
\mathcal{D}(\eta^{N}, z_{t_1}, \ldots, z_{t_d}^{N}) \to \mathcal{D}(\eta, z_{t_1}, \ldots, z_{t_d}) \quad \text{as} \quad N \to \infty.
\]

(5.3)

Note that

\[
\eta^{N} = \frac{S_{i,N}^{t} - \mathbb{E} S_{i,N}^{t}}{\pi^{-1} \sqrt{\ln N}} \quad \text{and} \quad z_{t}^{N} = \frac{S_{g_{t}^{N} - \mathbb{E} S_{g_{t}^{N}}}}{\pi^{-1} \sqrt{\ln N}},
\]

(5.4)

where \( f^{N}(x) = f(x/N), \quad g_{t}^{N}(x) = g_{t}(x/N) \) and

\[
f(x) := \frac{1}{\tau} \int_{0}^{\tau} \mathbb{I}_{[0,s]}(x) \, ds, \quad g_{t}(x) := \int_{0}^{\tau} \mathbb{I}_{[0,s]}(x) \, ds - \frac{t}{\tau} \int_{0}^{\tau} \mathbb{I}_{[0,s]}(x) \, ds.
\]

(5.5)

The following simple result is established in the next section.

**Proposition 5.1.** We have

1. \( \operatorname{Var} S_{i,N}^{t} = \frac{1}{2\pi^{2}} \ln N + O(1) \).

2. \( g_{t} \in H^{1}(\mathbb{R}) \), for any \( t \in [0, 1] \).

3. \( \langle g_{t}, g_{s} \rangle_{1/2} = \text{right-hand side of (1.11)} \), for any \( t, s \in [0, 1] \).

We claim that the family of functions \( f^{N}, g_{1}^{N}, \ldots, g_{d}^{N} \) satisfies assumptions of Theorem 4.3. Indeed, due to Proposition 5.1(1), assumption f.1 is fulfilled with \( V_{N} = \pi^{-2} \ln N \) and \( b_{11}^{t} = 1/2 \). Assumptions f.2-g.2 are fulfilled as well with \( R_{N} = N \) and \( g_{t}^{\infty} = g_{t} \) since we are in the situation of Example 4.4, in view of Proposition 5.1(2). Then, in due to Proposition 5.1(3), Theorem 4.3 implies the convergence (5.3).

*Step 2.* In this step we show that the family of measures \( \{ \mathcal{D}(\eta^{N}, z^{N}), N \in \mathbb{N} \} \) is tight in the space \( \mathbb{R} \times C([0,1], \mathbb{R}) \). To this end, it suffices to prove that the family of measures \( \{ \mathcal{D}(\eta^{N}), N \in \mathbb{N} \} \) is tight in \( \mathbb{R} \) and the family \( \{ \mathcal{D}(z^{N}), N \in \mathbb{N} \} \) is tight in \( C([0,1], \mathbb{R}) \). Indeed, then for any \( \varepsilon > 0 \) we will be able to find compact sets \( K_{\eta} \subset \mathbb{R} \) and \( K_{z} \subset C([0,1], \mathbb{R}) \) such that

\[
\mathbf{P}(\eta^{N} \in K_{\eta}) > 1 - \varepsilon/2 \quad \text{and} \quad \mathbf{P}(z^{N} \in K_{z}) > 1 - \varepsilon/2, \quad \text{for all} \ N.
\]
Then we will have
\[
\mathbf{P}\left((\eta^N, z^N) \in K_\eta \times K_z\right) = \mathbf{P}\left(\eta^N \in K_\eta\right) - \mathbf{P}\left(\eta^N \in K_\eta, z^N \notin K_z\right) \\
\geq \mathbf{P}\left(\eta^N \in K_\eta\right) - \mathbf{P}\left(z^N \notin K_z\right) > 1 - \varepsilon.
\]

Tightness of the family of measures \(\{\mathcal{D}(\eta^N), N \in \mathbb{N}\}\) follows from convergence (5.3) since the weak convergence implies the tightness. To show that the family \(\{\mathcal{D}(z^N), N \in \mathbb{N}\}\) is tight, we first formulate the following proposition.

**Proposition 5.2.** Consider a family of bounded measurable functions with compact supports \(h^N_t : \mathbb{R} \mapsto \mathbb{R}, \ 0 \leq t \leq 1, \ N \in \mathbb{N}\). Assume that for each \(t\) and \(N\) the function \(h^N_t\) belongs to the Sobolev space \(H^{1/2}(\mathbb{R})\). Assume also that there exist constants \(C, \delta > 0\) such that for any \(0 \leq t, s \leq 1\) and \(N \in \mathbb{N}\) we have
\[
\|h_0^N\|_{1/2} \leq C, \quad \|h^N_t - h^N_s\|_{1/2} \leq C(t - s)^{1+\delta}.
\]
Consider the random process
\[
\zeta^N_t := \mathcal{S}h^N_t - \mathbb{E}\mathcal{S}h^N_t, \quad 0 \leq t \leq 1,
\]
under the sine-process. Then there exists a continuous modification \(\tilde{\zeta}^N_t\) of the process \(\zeta^N_t\) such that the family of measures \(\{\mathcal{D}(\tilde{\zeta}^N), N \in \mathbb{N}\}\) is tight in the space of continuous functions \(C([0, 1], \mathbb{R})\).

Proof of Proposition 5.2 is given in the next section. Now to get the desired tightness of the family \(\{\mathcal{D}(z^N)\}\), it remains only to check that assumption (5.6) is satisfied for the functions \(g^N_t\). Its first part is obvious since \(g^N_0 = 0\). Using that \(\hat{g}^N_t(u) = N\hat{g}_t(Nu)\), we find
\[
\langle g^N_t, g^N_s \rangle_{1/2} = \frac{N^2}{4\pi^2} \int_{-\infty}^{\infty} |u|\hat{g}_t(Nu)\overline{g}_s(Nu) \, du = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |v|\hat{g}_t(v)\overline{g}_s(v) \, dv = \langle g_t, g_s \rangle_{1/2}. \tag{5.7}
\]
Then, recalling the notation \(\theta(u) = \frac{u^2}{4\pi^2} \ln|u|, \ \theta(0) = 0\), and using Proposition 5.1(3), we obtain
\[
\frac{1}{2}\|g^N_t - g^N_s\|_{1/2}^2 = \frac{t - s}{\tau} \left(\theta(t) - \theta(s) + \theta(s - \tau) - \theta(t - \tau) - \frac{t - s}{\tau} \theta(\tau)\right) - \theta(t - s) =: \Theta(t, s) - \theta(t - s).
\]

Since the derivative \(\theta'(u)\) is bounded uniformly in \(u \in (0, 1)\), we have \(\Theta(t, s) \leq C(t - s)^2\). Since \(|\theta(t - s)| \leq C(\delta)(t - s)^{1+\delta}\) for any \(0 < \delta < 1\), the second part of assumption (5.6) is satisfied as well.

**Step 3.** In this step we derive the required convergence (1.10) from the first two steps by standard argument. Since the family of measures \(\{\mathcal{D}(\eta^N, z^N), N \in \mathbb{N}\}\) is tight, by the Prokhorov Theorem it is weakly compact. Take a subsequence \(N_k \to \infty\) such that
\[
\mathcal{D}(\eta^N_k, z^N_k) \to \mathcal{D}(\tilde{\eta}, \tilde{z}) \quad \text{as} \quad k \to \infty \quad \text{in} \quad \mathbb{R} \times C([0, 1], \mathbb{R}),
\]
where \(\mathcal{D}(\tilde{\eta}, \tilde{z})\) is a limit point. Due to (5.3), for any \(0 \leq t_1 < \ldots < t_d \leq 1\) we have
\[
\mathcal{D}(\tilde{\eta}, \tilde{z}_{t_1}, \ldots, \tilde{z}_{t_d}) = \mathcal{D}(\eta, z_{t_1}, \ldots, z_{t_d}).
\]
Since finite-dimensional distributions specify a process, all the limit points coincide with \( D(\eta, z) \), so that we get the desired convergence. Proof of the theorem is completed.

**Proof of Proposition 1.2.**

Consider first a cumulant \( A_k^N \) with \( k \geq 3 \). Denote by \( (B_n^N) \) cumulants of the random variable \( S_{fN} \), where the function \( f^N \) is defined above (5.5). Due to Corollary 3.3 joined with Lemma 3.4, we have

\[
|B_k^N| \leq C \|f^N\|_k^{k-2} \text{Var} S_{f^N}.
\]

Since the norm \( \|f^N\|_\infty \) is independent from \( N \), Proposition 5.1 implies \( |B_k^N| \leq C \ln N \).

In view of (5.4), we have

\[
A_k^N = B_k^N \left( \frac{\pi}{2} - \frac{1}{2} \ln N \right) \ln N.
\]

Then

\[
|A_k^N| \leq C \ln N \left( \frac{\ln N}{\ln N} \right)^{k/2 - 1}.
\]

Since for \( k \geq 3 \) cumulants \( A_k \) of the normal distribution vanish, we get the desired estimate (1.12).

For \( k = 2 \) we have

\[
A_2^N = \text{Var} \eta^N = \frac{\text{Var} S_{f^N}}{\pi^2 \ln N}, \quad A_2 = \text{Var} \eta = 1/2.
\]

Then the desired estimate follows from Proposition 5.1. Since \( A_1^N = E \eta^N = 0 \) and \( A_1 = E \eta = 0 \), the proof of the proposition is finished.

**5.2 Proofs of auxiliary propositions**

Here we establish Propositions 5.1 and 5.2 used in the previous section.

**Proof of Proposition 5.1.** Item 1. Since \( f^N = \tau^{-1} \int_0^\tau \mathbb{I}_{[0,s^N]} ds \), we have \( S_{f^N} = \tau^{-1} \int_0^\tau h_s^N ds \), where \( h_s^N = \mathbb{I}_{[0,s^N]} \). Then, using the Fubini theorem, we get

\[
\text{Var} S_{f^N} = E \left( \frac{1}{\tau} \int_0^\tau S_{h_s^N} - E S_{h_s^N} ds \right)^2 = \frac{1}{\tau^2} \int_0^\tau \int_0^\tau \text{Cov}(S_{h_s^N}, S_{h_t^N}) dsdt.
\] (5.8)

Let us represent the covariance \( \text{Cov}(S_{h_s^N}, S_{h_t^N}) \) through the variances \( \text{Var} S_{h_s^N}, \text{Var} S_{h_t^N} \) as in (5.2). Since \( S_{h_t^N} = \#_{[0,s^N]} \), we have \( S_{h_t^N} - S_{h_s^N} = \#_{[s^N,t^N]} \), if \( t > s \). Then, due to the logarithmic grows of the variances (1.2), we obtain

\[
\text{Cov}(S_{h_s^N}, S_{h_t^N}) = \frac{1}{2\pi^2} \ln N + O(1),
\]

for \( t \neq s \). It can be shown that the integral \( \int_0^\tau \int_0^\tau O(1) dsdt \) is bounded uniformly in \( N \).

Now (5.8) implies the desired relation.

Item 2. Calculating the integrals from (5.5) explicitly, we see that the functions \( g_t \) are piecewise linear and continuous, so that \( g_t \in H^1(\mathbb{R}) \). Indeed, for \( 0 \leq t \leq \tau \) we have

\[
g_t(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x \geq \tau, \\ x(\tau^{-1}t - 1) & \text{if } 0 \leq x \leq t, \\ t(\tau^{-1}x - 1) & \text{if } t \leq x \leq \tau. \end{cases}
\] (5.9)
For $\tau \leq t \leq 1$,
\[
g_t(x) = \begin{cases} 
0 & \text{if } x \leq 0 \text{ or } x \geq t, \\
x(\tau^{-1}t - 1) & \text{if } 0 \leq x \leq \tau, \\
t - x & \text{if } \tau \leq x \leq t.
\end{cases} 
\tag{5.10}
\]

Item 3. Since $g_0 = g_\tau = 0$, in the case $t = 0, \tau$ or $s = 0, \tau$ the result is trivial. Assume that $t, s \neq 0, \tau$. By a direct computation we find
\[
\hat{g}_t(y) = \frac{h_t(y)}{y^2}
\quad \text{where} \quad h_t(y) := 1 - e^{-ity} - \frac{t}{\tau}(1 - e^{-ity}).
\]

Then, using that $\hat{g}_t(y) = \hat{g}_t(-y)$, we obtain
\[
\langle g_t, g_s \rangle_{1/2} = \frac{1}{2\pi^2} \Re \int_0^\infty y\hat{g}_t(y)\hat{g}_s(y) \, dy = \frac{1}{2\pi^2} \Re \int_0^\infty \frac{h_t(y)h_s(y)}{y^3} \, dy.
\]
Integrating by parts two times we find
\[
\int_0^\infty \frac{h_t(y)}{y^3} \, dy = -\frac{h_t(0)}{2y^2} \bigg|_0^\infty - \frac{h_t(y)'}{2y} \bigg|_0^\infty + \int_0^\infty \frac{(h_t(y)')''}{2y} \, dy, 
\tag{5.11}
\]
where the prime stands for the derivative with respect to $y$. We have
\[
h_t'(y) = ite^{-ity} - ite^{-ity} \quad \text{and} \quad h_t''(y) = t^2e^{-ity} - \tau te^{-ity}.
\tag{5.12}
\]
Since $h_t(0) = h_t'(0) = 0$ for any $t$, we have $(h_t(y))'(0) = (h_t(y))''(0) = (h_t(y))'''(0) = 0$, so that the boundary terms from (5.11) vanish. Then, using (5.11) and (5.12), by a direct computation we find
\[
\Re \int_0^\infty \frac{h_t(y)}{y^3} \, dy = \Re \int_0^\infty \frac{(h_t(y))''}{2y} \, dy = -\int_0^\infty \frac{v(t, s) + v(s, t)}{2y}, 
\tag{5.13}
\]
where
\[
v(t, s) = \frac{(t - s)^2}{2} \cos((t - s)y) - t^2(1 - \frac{s}{\tau}) \cos(ty) - \frac{s}{\tau}(\tau - t)^2 \cos((t - \tau)y) + t(\tau - s) \cos(\tau y).
\]

**Proposition 5.3.** Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R} \setminus \{0\}$ and $a_1 + \ldots + a_n = 0$. Then
\[
\int_0^\infty \sum_{i=1}^n \frac{a_i \cos(b_i y)}{y} \, dy = -\sum_{i=1}^n a_i \ln |b_i|.
\tag{5.14}
\]

Observe that the last integral from (5.13) has the form (5.14). Then, applying Proposition 5.3 we obtain the desired identity.

**Proof of Proposition 5.3.** Since $a_1 + \ldots + a_n = 0$, the integral under the question converges. Take $\epsilon > 0$ and write
\[
\int_0^\infty \sum_{i=1}^n \frac{a_i \cos(b_i y)}{y} \, dy = \int_0^\epsilon + \int_\epsilon^\infty \sum_{i=1}^n \frac{a_i \cos(b_i y)}{y} \, dy =: I_0^\epsilon + I_\epsilon^\infty.
\]
Clearly, \( I_0^\varepsilon \to 0 \) as \( \varepsilon \to 0 \). On the other hand,

\[
I_\infty = \sum_{i=1}^{n} a_i \int_{\varepsilon} \frac{\cos(b_i y)}{y} \, dy = \sum_{i=1}^{n} a_i \int_{|b_i|\varepsilon} \frac{\cos y}{y} \, dy = \sum_{i=1}^{n} a_i \int_{|b_i|\varepsilon} \frac{\cos y}{y} \, dy + \sum_{i=2}^{n} a_i \int_{|b_i|\varepsilon} \frac{|b_i|\varepsilon}{y} \, dy.
\]

Since \( a_1 + \ldots + a_n = 0 \), this implies \( I_\infty = \sum_{i=2}^{n} a_i \int_{|b_i|\varepsilon} \frac{\cos y}{y} \, dy \). Letting \( \varepsilon \) go to zero, we obtain

\[
I_\infty \sim \sum_{i=2}^{n} a_i \int_{|b_i|\varepsilon} \frac{1}{y} \, dy = -\sum_{i=2}^{n} a_i \ln \frac{|b_i|}{|b_1|} = -\sum_{i=1}^{n} a_i \ln |b_i|.
\]

\[\square\]

**Proof of Proposition 5.2.** Due to the Kolmogorov-Čentsov Theorem (see Theorem 2.8 in [22]) and Problem 2.4.11 from [22], to prove the proposition it suffices to show that

1. \( \sup_{N \in \mathbb{N}} E|\zeta_0|^2 < \infty \)
2. \( \sup_{N \in \mathbb{N}} E(\zeta_N^t - \zeta_N^s)^2 \leq C(t - s)^{1+\delta} \) uniformly in \( 0 \leq s, t \leq 1 \).

We have

\[
E(\zeta_N^t - \zeta_N^s)^2 = \text{Var} S_{h_N^t - h_N^s}.
\]

Due to estimate (3.33) of Corollary 3.7, the right-hand side of (5.15) is bounded by \( \|h_N^t - h_N^s\|_{1/2}^2 \). Then assumption (5.6) implies item (2) above. Assertion of item (1) follows in the same way,

\[
E|\zeta_0|^2 = \text{Var} S_{h_0^t} \leq \|h_0^t\|_{1/2}^2 \leq C.
\]

\[\square\]

### 5.3 Proofs of Theorem 1.9 and Proposition 1.8

**Proof of Theorem 1.9**

**Item 1.** Denote \( m := \inf \text{supp} \varphi \) and \( M := \sup \text{supp} \varphi \). It is easy to see that, in view of (1.27), the function \( \varphi_t^N \) has the form

\[
\varphi_t^N = \mathbb{I}_{[M, m+Nt]} + r_t^N,
\]

where \( |r_t^N| \leq C \) and the Lebesgue measure \( \text{Leb}(\text{supp} r_t^N) \leq C_1 \), with constants \( C, C_1 \) independent from \( N \) (see figure 2). Then

\[
\text{Var} S_{\varphi_t^N} = \text{Var} (S_{\mathbb{I}_{[M, m+Nt]}} + S_{r_t^N}) = \text{Var} S_{\mathbb{I}_{[M, m+Nt]}} + \text{Var} S_{r_t^N} + 2 \text{Cov}(S_{\mathbb{I}_{[M, m+Nt]}}, S_{r_t^N}).
\]

In view of (1.2), \( \text{Var} S_{\mathbb{I}_{[M, m+Nt]}} = \pi^{-2} \ln N + O(1) \). Clearly, \( \text{Var} S_{r_t^N} \leq C \), where \( C \) is independent from \( N \). Then the desired relation follows from (5.16) joined with the Cauchy-Bunyakovsky-Schwartz inequality

\[
|\text{Cov}(S_{\mathbb{I}_{[M, m+Nt]}}, S_{r_t^N})| \leq \sqrt{\text{Var} S_{\mathbb{I}_{[M, m+Nt]}} \sqrt{\text{Var} S_{r_t^N}}},
\]

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Item 2. To get the desired result it suffices to note that assumptions of Theorem 4.1 are satisfied for the family of functions \( \varphi_{t_1}^N, \ldots, \varphi_{t_d}^N \) with \( V_N = \pi^{-2} \ln N \) and the covariance matrix \((b_{ij})\) from (1.14). Indeed, estimate (4.2) is obvious since the functions \( \varphi_{t_i}^N \) are bounded uniformly in \( N \). Assumption (4.1) follows from the logarithmic growth of the variance by the argument similar to that used in the proof of Proposition 1.3.

Item 3. We follow the same strategy as when proving Theorem 1.1. We set

\[
\varphi_{\tau}^N := \frac{1}{\tau} \int_0^\tau \varphi_s^N \, ds \quad \text{and} \quad g_{\varphi,t}^N := \int_0^t \varphi_s^N \, ds - \frac{t}{\tau} \int_0^\tau \varphi_s^N \, ds.
\]

Then we have

\[
\eta_N^N = \frac{S_{f_{\varphi}^N} - \mathbf{E} S_{f_{\varphi}^N}}{\pi^{-1} \sqrt{\ln N}} \quad \text{and} \quad z_t^N = S_{g_{\varphi,t}^N} - \mathbf{E} S_{g_{\varphi,t}^N}.
\]

Take any \( 0 \leq t_1 < \ldots < t_d \leq 1 \). We claim that the functions \( f_{\varphi}^N, g_{\varphi,t_1}^N, \ldots, g_{\varphi,t_d}^N \) satisfy assumptions of Theorem 4.3 with \( V_N = \pi^{-2} \ln N, R_N = N, b_{11}^N = 1/2 \) and the functions \( g_{\varphi,t_i}^\infty = g_{t_i}, \) where the \( g_{t_i} \) are defined in (5.5). Indeed, note that

\[
\varphi_s^N = \varphi \ast \mathbb{I}_{[0,sN]}.
\]

Consider the functions \( f^N \) and \( g_{t}^N \), defined above (5.5). We have

\[
f_{\varphi}^N = \varphi \ast \left( \frac{1}{\tau} \int_0^\tau \mathbb{I}_{[0,sN]} \, ds \right) = \varphi \ast f^N.
\]

Similarly,

\[
g_{\varphi,t}^N = \varphi \ast g_{t}^N. \tag{5.17}
\]

As it was shown in the proof of Theorem 1.1, the functions \( f^N, g_{t}^N \) satisfy assumptions of Theorem 4.3 with \( V_N, R_N, b_{11}^N \) as above and \( g_{\varphi,t}^\infty = g_{t_i} \). Then, due to Example 4.5, the functions \( f_{\varphi}^N, g_{\varphi,t_1}^N, \ldots, g_{\varphi,t_d}^N \) fulfill assumptions f.2-g.2, with the same \( V_N, R_N, b_{11}^N \) and \( g_{\varphi,t_i}^\infty = g_{t_i}^\infty \). To show that assumption f.1 is satisfied as well, it suffices to prove that

\[
\operatorname{Var} S_{f_{\varphi}^N} = \frac{1}{2\pi^2} \ln N + O(\sqrt{\ln N}).
\]

In view of item 1 of the theorem, this can be shown by the argument used in the proof of Proposition 5.1(1). Now Theorem 4.3 joined with Proposition 5.1(3) implies the convergence

\[
\mathcal{D}(\eta_N^N, z_{t_1}^N, \ldots, z_{t_d}^N) \to \mathcal{D}(\eta, z_{t_1}, \ldots, z_{t_d}) \quad \text{as} \quad N \to \infty, \tag{5.18}
\]

where the random variable \( \eta \) and the process \( z_t \) are as in the formulation of Theorem 1.1.

Next we show that the family of measures \( \{\mathcal{D}(\eta_N^N, z_N^N), N \in \mathbb{N}\} \) is tight. To this end, as in Theorem 1.1, it suffices to prove that the family of functions \( g_{\varphi,t}^N \) satisfies assumption (5.6) of Proposition 5.2. The first estimate from (5.6) is obvious since \( g_{\varphi,0}^N = 0 \). In view of the identity \( \tilde{g}_{\varphi,t}^N = \tilde{\varphi} \tilde{g}_{t}^N \) which follows from (5.17), we have

\[
\|g_{\varphi,t}^N - g_{\varphi,s}^N\|_{1/2} \leq \|\tilde{\varphi}\|_{\infty} \|g_{t}^N - g_{s}^N\|_{1/2}.
\]
Since \( \varphi \in L^1(\mathbb{R}) \), the norm \( \| \varphi \|_\infty \) is finite. Then it remains to establish the second estimate from (5.6) for the functions \( g_i^N \). But it was already done in the proof of Theorem 1.1.

Now, literally repeating arguments from Step 3 of the proof of Theorem 1.1, we see that the convergence of finite-dimensional distributions (5.18) together with the tightness of the family of measures \( \mathcal{D}(\eta^N, z^N) \) implies the desired convergence \( \mathcal{D}(\eta, z) \).

Item 4. The proof literally repeats that of Proposition 1.2. The rate of convergence of the cumulants \( A_2^N \) and \( A_2 \) in this case is different with that from Proposition 1.2 because of the correction \( O(\sqrt{\ln N}) \) in item 1 of the theorem (cf. (1.2)).

**Proof of Proposition 1.8**

Set \( f_t := \mathbb{I}_{[at, bt]} \) and \( f_t^N(x) := f_t(x/N) \). Then we have \( \#_{[at, bt]} = \mathcal{S}_{f_t^N} \). Due to Corollary 3.7,

\[
\text{Cov}(\mathcal{S}_{f_t^N}, \mathcal{S}_{f_s^N}) = \frac{1}{4\pi^2} \Re \left( 2 \int_{|u| \geq 2} \hat{f}_t^N(u)\hat{f}_s^N(u) \, du + \int_{|u| < 2} |u|\hat{f}_t^N(u)\hat{f}_s^N(u) \, du \right) =: \frac{1}{4\pi^2} (I_1 + I_2).
\]

Note that \( \hat{f}_t^N(u) = \frac{e^{-iaNt} - e^{-ibNt}}{iu} \). Let us study the integral \( I_1 \). We find

\[
I_1 = 2 \int_{|u| \geq 2} \cos \left( N(at - as)u \right) + \cos \left( N(bt - bs)u \right) - \cos \left( N(at - bs)u \right) - \cos \left( N(bt - as)u \right) \, du.
\]

Due to the assumption \( t \neq s, at \neq bs \) and \( bt \neq as \), the arguments of the cosines do not vanish, so they oscillate fast for \( N \gg 1 \). Since the integral \( I_1 \) converges absolutely for every \( N \), this implies \( I_1 \to 0 \) as \( N \to \infty \). Let us turn to the integral \( I_2 \). We have

\[
I_2 = 2 \int_0^2 \cos \left( N(at - as)v \right) + \cos \left( N(bt - bs)v \right) - \cos \left( N(at - bs)v \right) - \cos \left( N(bt - as)v \right) \, dv.
\]

Changing the variable \( v := Nu \), we find

\[
I_2 = 2 \int_0^{2N} \cos \left( (at - as)v \right) + \cos \left( (bt - bs)v \right) - \cos \left( (at - bs)v \right) - \cos \left( (bt - as)v \right) \, dv.
\]

Due to Proposition 5.3, we obtain

\[
I_2 \to 2(\ln |at - bs| + \ln |bt - as| - \ln |at - as| - \ln |bt - bs|).
\]

\( \Box \)

6 Main order asymptotic for determinantal processes with logarithmically growing variance

In this section we prove a generalized version of Proposition 1.5 for an important class of determinantal processes. The latter includes processes with logarithmically growing variance, e.g. the sine, Bessel and Airy processes.
Let \( h^N : [0, 1] \times \mathbb{R}^m \mapsto \mathbb{R} \) be a family of Borel measurable bounded functions with compact supports. Consider the linear statistics
\[
S^N_{h^N} := \sum_{x \in X} h^N(t, x)
\]
as a random variable under a determinantal process given by a Hermitian kernel \( K^N \). Denote by \( \text{Var}_N, \text{Cov}_N \) and \( \mathbb{E}_N \) the corresponding variance, covariance and expectation.

**Proposition 6.1.** Assume that there exists a sequence \( V_N \to \infty \) as \( N \to \infty \), \( V_N > 0 \), such that the following three conditions hold.

1. There exists a constant \( C \) such that for any \( N \) and almost all \( t \in [0, 1] \) we have
\[
\frac{\text{Var}_N S^N_{h^N}}{V_N} \leq C. \tag{6.1}
\]

2. There exists \( b \in \mathbb{R} \) such that for almost all \( (t, s) \in [0, 1]^2 \) we have
\[
\frac{\text{Cov}_N(S^N_{h^N}, S^N_{h^N})}{V_N} \xrightarrow{N \to \infty} b. \tag{6.2}
\]

3. We have
\[
\|h^N\|_{\infty} = o(\sqrt{V_N}).
\]

Denote
\[
\xi^N_t = \frac{S^N_{h^N} - \mathbb{E}_N S^N_{h^N}}{\sqrt{V_N}}, \quad 0 \leq t \leq 1.
\]

Then for any functions \( \phi_1, \ldots, \phi_n \in L^1[0, 1], n \geq 1 \), we have
\[
\mathcal{D}\left( \int_0^1 \phi_1(t) \xi^N_t \, dt, \ldots, \int_0^1 \phi_n(t) \xi^N_t \, dt \right) \xrightarrow{N \to \infty} \mathcal{D}\left( \eta \int_0^1 \phi_1(t) \, dt, \ldots, \eta \int_0^1 \phi_n(t) \, dt \right), \quad \tag{6.3}
\]
where \( \eta \sim \mathcal{N}(0, b) \).

The principal assumption of Proposition 6.1 is (6.2). In particular, it is satisfied for determinantal processes with the logarithmic growth of the variance. Let us explain this on the following examples. Consider the sine or the Bessel process and the linear statistics corresponding to the function
\[
h^N(t, x) = I_{[0, Nt]}(x), \quad \text{so that} \quad S^N_{h^N} = \#_{[0, Nt]}.
\]
It is known that for \( 0 < a < b \) we have
\[
\text{Var} \#_{[aN,bN]} \sim C \ln N \quad \text{as} \quad N \to \infty,
\]
for the both processes (see (1.2) for the sine-process and [39] for the Bessel process). Then, literally repeating arguments from the proof of the convergence (5.1), we obtain (6.2) with \( b = 1/2 \) and \( V_N = C \ln N \). The same holds for the Airy process, if one puts \( h^N(t, x) = I_{[-Nt,0]}(x) \) (so that \( S^N_{h^N} = \#_{[-tN,0]} \)) and \( a < b < 0 \).
It can be checked that in the examples above the other assumptions of Proposition 6.1 are satisfied as well, so that the convergence (6.3) takes place. In particular, taking \( n = 1 \) and \( \phi_1 = \mathbb{I}_{[0,t]} \), we get the following corollary, which is a version of the main order asymptotic from Theorem 1.1 for the Airy and Bessel processes. Set

\[
\xi_{A,t}^N = \frac{\#[-tN,0] - E \#[-tN,0]}{\sqrt{\text{Var} \ #[-tN,0]}} \quad \text{and} \quad \xi_{B,t}^N = \frac{\#[0,tN] - E \ #[0,tN]}{\sqrt{\text{Var} \ #[0,tN]}}.
\]

**Corollary 6.2.** Under the Airy processes for any \( 0 \leq t \leq 1 \) we have

\[
\mathcal{D}(\int_0^t \xi_{A,s}^N \, ds) \Rightarrow \mathcal{D}(\eta t)
\]

as \( N \to \infty \), where \( \eta \sim \mathcal{N}(0,1/2) \). Under the Bessel process, we have

\[
\mathcal{D}(\int_0^t \xi_{B,s}^N \, ds) \Rightarrow \mathcal{D}(\eta t).
\]

Similar result holds true for the ergodic integrals under the shift operator (studied in Section 1.4). Let \( \varphi : \mathbb{R} \mapsto \mathbb{R} \) be a bounded measurable function with compact support satisfying \( \int_0^1 \varphi(s) \, ds = 1 \). Set

\[
\varphi_{A,t}^N := \int_0^{tN} \varphi(\cdot + u) \, du \quad \text{and} \quad \varphi_{B,t}^N := \int_0^{tN} \varphi(\cdot - u) \, du.
\]

Denote

\[
\xi_{A,\varphi,t}^N := \frac{S_{\varphi_{A,t}^N} - E S_{\varphi_{A,t}^N}}{\sqrt{\text{Var} S_{\varphi_{A,t}^N}}},
\]

and define \( \xi_{B,\varphi,t}^N \) in the same way.

**Corollary 6.3.** Assertion of Corollary 6.2 holds if replace the processes \( \xi_{A,s}^N \) and \( \xi_{B,s}^N \) by the processes \( \xi_{A,\varphi,s}^N \) and \( \xi_{B,\varphi,s}^N \).

**Proof of Proposition 6.1.** Let \( \phi_i^N(x) := \int_0^1 \phi_i(t) h^N(t,x) \, dt \). Using the Fubini theorem, for any \( 1 \leq i \leq n \) we obtain

\[
\int_0^1 \phi_i(t) \xi_t^N \, dt = \frac{S_{\phi_i^N} - E S_{\phi_i^N}}{\sqrt{V_N}}.
\]

We claim that the family of functions \( \phi_i^N \) satisfies assumptions of Theorem 4.1. Indeed, the only condition fulfilment of which is not obvious is (4.1). Let us check it. In view of the Fubini theorem, we have

\[
\text{Cov}_N(S_{\phi_i^N}, S_{\phi_j^N}) = E_N \left( \int_0^1 \phi_i(t) \xi_t^N \, dt \int_0^1 \phi_j(s) \xi_s^N \, ds \right) = \int_0^1 \phi_i(t) \phi_j(s) \frac{\text{Cov}_N(S_{h_i^N}, S_{h_j^N})}{V_N} \, dt \, ds.
\]

Then, using the dominated convergence theorem, (6.1) and (6.2) we get

\[
\frac{\text{Cov}_N(S_{\phi_i^N}, S_{\phi_j^N})}{V_N} \to b \int_0^1 \int_0^1 \phi_i(t) \phi_j(s) \, dt \, ds =: c_{ij} \quad \text{as} \quad N \to \infty,
\]

(6.4)
so that assumption (4.1) is fulfilled with $b_{ij} = c_{ij}$. Now it remains to apply Theorem 4.1. Indeed, since the limiting vector $\xi$ obtained there is Gaussian with the covariance matrix $(c_{ij})$, it coincides in distribution with the random vector from the right-hand side of (6.3).

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