Covariant and Equivariant Formality Theorems.

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Abstract

We give a proof of Kontsevich’s formality theorem for a general manifold using Fedosov resolutions of algebras of polydifferential operators and polyvector fields. The main advantage of our construction of the formality quasi-isomorphism is that it is based on the use of covariant tensors unlike Kontsevich’s original proof, which is based on \( \infty \)-jets of polydifferential operators and polyvector fields. Using our construction we prove that if a group \( G \) acts smoothly on a manifold \( M \) and \( M \) admits a \( G \)-invariant affine connection then there exists a \( G \)-equivariant quasi-isomorphism of formality. This result implies that if a manifold \( M \) is equipped with a smooth action of a finite or compact group \( G \) or equipped with a free action of a Lie group \( G \) then \( M \) admits a \( G \)-equivariant formality quasi-isomorphism. In particular, this gives a solution of the deformation quantization problem for an arbitrary Poisson orbifold.

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1 Introduction

Preserving symmetries in quantization procedures is one of the most important problems in mathematical physics. In this paper we discuss the problem for Kontsevich’s formality quasi-isomorphism [1].

The purpose of this paper is twofold. First, we propose a manifestly covariant construction of Kontsevich’s formality quasi-isomorphism for a general smooth manifold \( M \) using the Fedosov resolutions of the algebras of polydifferential operators and polyvector fields and Kontsevich’s quasi-isomorphism of formality for the space \( \mathbb{R}^d \). The quasi-isomorphism obtained by our procedure depends on an affine torsion free connection on \( M \). Second, we consider a manifold \( M \) with a smooth action of a group \( G \) and show that if \( M \) admits a \( G \)-invariant affine connection then the formality quasi-isomorphism corresponding to the connection is \( G \)-equivariant.

Our method may be regarded as a generalization of the work [2], in which a covariant construction of a star-product on an arbitrary Poisson manifold is presented. As in [2], we use the Fedosov resolution\(^2\) of the algebra of functions on \( M \) in the algebra of sections of the formally completed symmetric algebra of the cotangent bundle \( T^*M \) and a fiberwise formality quasi-isomorphism. However, we use a fixed Fedosov differential and instead modify the fiberwise formality quasi-isomorphism unlike the authors of [2]. It is this modification that allows us to get a more general result, namely, to construct a formality quasi-isomorphism for an arbitrary smooth manifold.

\(^1\)On leave of absence from: Univ. Center of JINR (Dubna) and ITEP (Moscow)
\(^2\)In work [3] construction of the resolution is called Weinstein’s exponential map [4].
A sketchy proof for the formality of the algebra of polydifferential operators on an arbitrary smooth manifold is originally given in [1] (see section 7). A more detailed explanation of the proof can be found in the appendix of paper [5]. In this context it is also worth mentioning paper [6], in which a covariant deformation quantization for an arbitrary Poisson manifold is proposed via a path integral approach.

Our construction of the formality quasi-isomorphism for a general manifold $M$ may be regarded as a modification of Kontsevich’s approach [1], [5] in which the $\infty$-jets are replaced by infinite collections of symmetric covariant tensors and the flat connection on the jet bundle is replaced by Fedosov differential, constructed with the help of some torsion free connection on $M$. This modification turns out to be much simpler for applications because at all the stages of our construction of the quasi-isomorphism we deal with manifestly covariant objects.

We use our construction of the formality quasi-isomorphism in order to prove an equivariant formality theorem (see theorem 5 in section 5), which is the main result of the paper. This theorem allows us to get several interesting corollaries. Namely, it turns out that if a finite or compact group $G$ acts smoothly on a manifold $M$ or a Lie group $G$ acts freely on $M$ then $M$ admits a $G$-equivariant formality quasi-isomorphism. In particular, this gives us a solution of the deformation quantization problem for an arbitrary Poisson orbifold (see corollary 3 in section 5).

The structure of this paper is as follows. The next section is devoted to basic notations we use throughout the paper. In this section we recall some required notions of homotopy theory of differential graded Lie algebras, we review necessary properties of Kontsevich’s formality quasi-isomorphism [1] for the space $\mathbb{R}^d$, and finally we introduce some notions required for a construction of the Fedosov resolutions of algebras of polyvector fields and polydifferential operators. In the third section we present the Fedosov resolutions of the algebras of functions, polydifferential operators, and polyvector fields on a smooth manifold. Section 4 is devoted to our construction of a formality quasi-isomorphism for a general smooth manifold. In section 5 we prove an equivariant formality theorem and present its corollaries. Finally, in the concluding section of the paper we discuss possible applications and generalizations of the equivariant formality theorem.

Throughout the paper the summation over repeated indices is assumed. Sometimes we omit the prefix “super-” referring to super-algebras, Lie super-brackets, and super(co)commutative (co)multiplications. We assume that $M$ is a smooth real manifold of dimension $d$. We omit symbol $\wedge$ referring to a local basis of exterior forms as if we thought of $dx^i$’s as anti-commuting variables. Finally, we always assume that a nilpotent linear operator is the one whose second power is vanishing.

2 Preliminaries.

This section is devoted to basic notations we use throughout the paper. Although the objects we introduce here are well known we assemble in this section all basic definitions and results we need since notations and terminology vary from source to another.

We start with a sketchy introduction of some required notions of homotopy theory of differential graded Lie algebras (DGLA). A more detailed discussion of the theory can be found in paper [7].
Let \((\mathfrak{h}, d_\mathfrak{h}, \{,\})\) be a DGLA. We assume that the differential \(d_\mathfrak{h}\) is of degree one and the Lie super-bracket \(\{,\}\) is of degree zero. To include \((\mathfrak{h}, d_\mathfrak{h}, \{,\})\) in the context of strong homotopy Lie algebras we associate to \(\mathfrak{h}\) a coassociative cocommutative coalgebra \(C_\bullet(\mathfrak{h}[1])\) cofreely cogenerated by the vector space \(\mathfrak{h}\) with a shifted parity. The DGLA structure \((d_\mathfrak{h}, \{,\}\)) induces on \(C_\bullet(\mathfrak{h}[1])\) a coderivation \(Q\) with two non-vanishing structure maps

\[
Q_1 = d_\mathfrak{h} : \mathfrak{h} \mapsto \mathfrak{h}[1],
\]

\[
Q_2 = \{,\} : \wedge^2 \mathfrak{h} \mapsto \mathfrak{h}.
\]

With this definition the set of axioms of the DGLA \((\mathfrak{h}, d_\mathfrak{h}, \{,\}\)) is equivalent to nilpotency of coderivation (1)

\[
Q^2 = 0.
\]

Given two DG Lie algebras \((\mathfrak{h}_1, d_1, [\cdot, \cdot]_1)\) and \((\mathfrak{h}_2, d_2, [\cdot, \cdot]_2)\) we shall be interested in their morphisms in the category of strong homotopy Lie algebras, namely

**Definition 1** An \(L_\infty\)-morphism \(F\) from the DGLA \((\mathfrak{h}_1, d_1, [\cdot, \cdot]_1)\) to the DGLA \((\mathfrak{h}_2, d_2, [\cdot, \cdot]_2)\) is a homomorphism of the coassociative cocommutative coalgebras

\[
F : C_\bullet(\mathfrak{h}_1[1]) \mapsto C_\bullet(\mathfrak{h}_2[1])
\]

compatible with the nilpotent coderivations \(Q_1\) and \(Q_2\) corresponding to the DGLA structures \((d_1, [\cdot, \cdot]_1)\) and \((d_2, [\cdot, \cdot]_2)\), respectively

\[
Q_2 F(X) = F(Q_1 X), \quad \forall X \in C_\bullet(\mathfrak{h}_1[1]).
\]

Furthermore,

**Definition 2** A quasi-isomorphism \(F\) from the DGLA \((\mathfrak{h}_1, d_1, [\cdot, \cdot]_1)\) to the DGLA \((\mathfrak{h}_2, d_2, [\cdot, \cdot]_2)\) is an \(L_\infty\)-morphism from \(\mathfrak{h}_1\) to \(\mathfrak{h}_2\) whose first structure map

\[
F_1 : \mathfrak{h}_1 \mapsto \mathfrak{h}_2
\]

gives an isomorphism of the spaces of cohomologies \(H^\bullet(\mathfrak{h}_1, d_1)\) and \(H^\bullet(\mathfrak{h}_2, d_2)\).

In what follows the notation

\[
F : (\mathfrak{h}_1, d_1, [\cdot, \cdot]_1) \sim (\mathfrak{h}_2, d_2, [\cdot, \cdot]_2)
\]

means that \(F\) is a quasi-isomorphism form the DGLA \((\mathfrak{h}_1, d_1, [\cdot, \cdot]_1)\) to the DGLA \((\mathfrak{h}_2, d_2, [\cdot, \cdot]_2)\).

By unfolding the formal definition we see that a homomorphism \(F\) of coassociative cocommutative coalgebras \(C_\bullet(\mathfrak{h}_1[1])\) and \(C_\bullet(\mathfrak{h}_2[1])\) is uniquely defined by a semi-infinite collection of polylinear maps

\[
F_n : \wedge^n \mathfrak{h}_1 \mapsto \mathfrak{h}_2[1 - n], \quad n \geq 1
\]

and the compatibility of \(F\) with the coderivations \(Q_1\) and \(Q_2\) on \(C_\bullet(\mathfrak{h}_1[1])\) and \(C_\bullet(\mathfrak{h}_2[1])\), respectively, is equivalent to the following semi-infinite collection of equations

\[
d_2 F_n(\gamma_1, \gamma_2, \ldots, \gamma_n) = \sum_{i=1}^{n} (-)^{k_1 + \ldots + k_{i-1} + 1 - n} F_n(\gamma_1, \ldots, d_1 \gamma_i, \ldots, \gamma_n) = 0.
\]
\[ \frac{1}{2} \sum_{k,l \geq 1, k+l=n} \frac{1}{k!l!} \sum_{\varepsilon \in S_n} \pm [F_k(\gamma_{\xi_1}, \ldots, \gamma_{\xi_k}), F_l(\gamma_{\xi_{k+1}}, \ldots, \gamma_{\xi_{k+l}})]_2 - \]

\[ - \sum_{i \neq j} \pm F_{n-1}(\gamma_i, \gamma_j, \gamma_1, \ldots, \gamma_i, \gamma_j, \ldots, \gamma_n), \quad \gamma_i \in \mathfrak{b}_i^{k_i}. \]

**Remark.** Notice that in order to define the signs in formulas (4) one should use a rather complicated rule. For example, the signs that stand before the terms of the first sum at the right hand side depend on permutations \( \varepsilon \in S_n \), on degrees of \( \gamma_i \), and on the numbers \( k \) and \( l \). The simplest way to check that all the signs are correct is to show that the right hand side of equation (4) is closed with respect to the following differential acting on the space of graded polylinear maps

\[ d_{\text{Hom}} : \text{Hom}(\wedge^n \mathfrak{h}_1, \mathfrak{h}_2[k]) \mapsto \text{Hom}(\wedge^n \mathfrak{h}_1, \mathfrak{h}_2[k+1]), \]

\[ d_{\text{Hom}} \Psi(\gamma_1, \gamma_2, \ldots, \gamma_n) = d_2 \Psi(\gamma_1, \gamma_2, \ldots, \gamma_n) - \]

\[ - \sum_{i=1}^n (-)^{k_1 + \ldots + k_{i-1} + k} \psi(\gamma_1, \ldots, d_1 \gamma_i, \ldots, \gamma_n), \quad \gamma_i \in \mathfrak{h}_i^{k_i}, \]

where \( \Psi \in \text{Hom}(\wedge^n \mathfrak{h}_1, \mathfrak{h}_2[k]) \).

**Example.** An important example of a quasi-isomorphism from a DGLA \( \mathfrak{h}_1 \) to a DGLA \( \mathfrak{h}_2 \) is provided by a DGLA-homomorphism

\[ H : \mathfrak{h}_1 \mapsto \mathfrak{h}_2, \]

which induces an isomorphism on the spaces of cohomologies \( H^\bullet(\mathfrak{h}_1, d_1) \) and \( H^\bullet(\mathfrak{h}_2, d_2) \). In this case the quasi-isomorphism has the only non-vanishing structure map

\[ F_1 = H. \]

To conclude the introductory part on the homotopy theory we recall that a DGLA algebra \( (\mathfrak{h}, d, [\cdot, \cdot]) \) is called formal\(^3\) if there is a quasi-isomorphism from the graded Lie algebra \( H^\bullet(\mathfrak{h}, d) \) to the DGLA \( \mathfrak{h} \).

Let now \( M \) be a smooth manifold of dimension \( d \) and \( D_{\text{poly}}(M) \) be a vector space of polydifferential operators on \( M \)

\[ D_{\text{poly}}(M) = \bigoplus_{k=-1}^{\infty} D^k_{\text{poly}}(M), \]

where the \( D^k_{\text{poly}}(M) \) consists of polydifferential operators of rank \( k + 1 \)

\[ \Phi : C^\infty(M) \otimes^{k+1} \mapsto C^\infty(M). \]

The space \( D_{\text{poly}}(M) \) can be endowed with the so-called Gerstenhaber bracket which is defined between homogeneous elements \( \Phi_1 \in D^k_{\text{poly}}(M) \) and \( \Phi_2 \in D^k_{\text{poly}}(M) \) as follows

\(^3\)Our definition of formality is slightly different from the conventional one. However, if \( \mathfrak{h} \) is formal in the sense of our definition then it is conventionally formal.
\[
[\Phi_1, \Phi_2](a_0, \ldots, a_{k_1+k_2}) = \sum_{i=0}^{k_1} (-)^{ik_2} \Phi_1(a_0, \ldots, a_{i-1}, \Phi_2(a_i, \ldots, a_{i+k_2}), \ldots, a_{k_1+k_2})
\]

(7)

Direct computation shows that (7) is a Lie (super)bracket and therefore \(D_{poly}\) is Lie (super)algebra. The multiplication operator \(m_0 \in D^1_{poly}(M)\)

\[
m_0 : C^\infty(M) \otimes C^\infty(M) \mapsto C^\infty(M)
\]

satisfies the associativity condition, which can be written in terms of bracket (7) as

\[
[m_0, m_0] = 0.
\]

(8)

Thus \(m_0\) defines a nilpotent interior derivation of \(D_{poly}(M)\)

\[
\partial \Phi = [m_0, \Phi] : D^k_{poly}(M) \mapsto D^{k+1}_{poly}(M), \quad \partial^2 = 0,
\]

(9)

which turns the Lie algebra \(D_{poly}(M)\) into a DGLA.

A DGLA of cohomologies of \(D_{poly}(M)\) is described by the Hochschild-Kostant-Rosenberg theorem, which says that

\[
H^\bullet(D_{poly}(M), \partial) = T_{poly}(M),
\]

where \(T_{poly}(M)\) is a DGLA of the polyvector fields with a vanishing differential \(d_T = 0 : T^k_{poly}(M) \mapsto T^{k+1}_{poly}(M)\) and a Lie bracket being the standard Schouten-Nijenhuis bracket. Namely,

\[
T_{poly}(M) = \bigoplus_{k=-1}^{\infty} T^k_{poly}(M), \quad T^k_{poly}(M) = \Gamma(\wedge^{k+1} TM),
\]

(10)

and the Schouten-Nijenhuis bracket \([\cdot, \cdot]_{SN}\) is defined as an ordinary Lie bracket between vector fields and then extended by Leibniz rule with respect to the \(\wedge\)-product to an arbitrary pair of polyvector fields.

The formality theorem by Kontsevich [1] says that for any smooth manifold \(M\) there exists a quasi-isomorphism from the DGLA \(T_{poly}(M)\) of polyvector fields on \(M\) to the DGLA \(D_{poly}(M)\) of polydifferential operators on \(M\). In our paper we use this result for the case when \(M = \mathbb{R}^d\).

In paper [1] Kontsevich proposed an interesting technique for computing structure maps of a quasi-isomorphism \(U\) from the DGLA \(T_{poly}(\mathbb{R}^d)\) of polyvector fields on \(\mathbb{R}^d\) to the DGLA \(D_{poly}(\mathbb{R}^d)\) of polydifferential operators on \(\mathbb{R}^d\). Although the existence of the formality quasi-isomorphism for \(\mathbb{R}^d\) has been also proved by Tamarkin [8] here we need explicit Kontsevich’s construction because the quasi-isomorphism \(U\) given in [1] satisfy certain peculiar properties\(^4\), which we use in our construction of the formality quasi-isomorphism for a general manifold. We assemble the properties of \(U\) in the following

\(^4\)see the beginning of section 7 in [1]
Theorem 1 (Kontsevich, [1]) There exists a quasi-isomorphism $U$

\[ U : T_{\text{poly}}(\mathbb{R}^d) \sim D_{\text{poly}}(\mathbb{R}^d) \]  

(11)

from the DGLA $T_{\text{poly}}(\mathbb{R}^d)$ of polyvector fields to the DGLA $D_{\text{poly}}(\mathbb{R}^d)$ of polydifferential operators on the space $\mathbb{R}^d$ such that

1. One can replace $\mathbb{R}^d$ in (11) by its formal completion $\mathbb{R}^{d}_{\text{formal}}$ at the origin.

2. The quasi-isomorphism $U$ is equivariant with respect to linear transformations of the coordinates on $\mathbb{R}^{d}_{\text{formal}}$.

3. If $n > 1$ then

\[ U_n(v_1, v_2, \ldots, v_n) = 0 \]  

(12)

for any set of vector fields $v_1, v_2, \ldots, v_n \in T^0_{\text{poly}}(\mathbb{R}^{d}_{\text{formal}})$.

4. If $n \geq 2$ and $v \in T^0_{\text{poly}}(\mathbb{R}^{d}_{\text{formal}})$ is linear in the coordinates on $\mathbb{R}^{d}_{\text{formal}}$ then for any set of polyvector fields $\gamma_2, \ldots, \gamma_n \in T_{\text{poly}}(\mathbb{R}^{d}_{\text{formal}})$

\[ U_n(v, \gamma_2, \ldots, \gamma_n) = 0 \]  

(13)

We now turn to some definitions required for constructing Fedosov resolutions of algebras of polyvector fields and polydifferential operators. First, we give a definition of a bundle $S^\Lambda M$ of the formally completed symmetric algebra of the cotangent bundle $T^*M$. This bundle is a natural analogue of the Weyl algebra bundle used in paper [9] by Fedosov.

Definition 3 The bundle $S^\Lambda M$ of formally completed symmetric algebra of the cotangent bundle $T^*M$ is defined as a bundle over the manifold $M$ whose sections are infinite collections of symmetric covariant tensors $a_{i_1 \ldots i_p}(x)$, where $x^i$ are local coordinates, $p$ runs from 0 to $\infty$, and the indices $i_1, \ldots, i_p$ run from 1 to $d$.

It is convenient to introduce auxiliary variables $y^i$, which transform as contravariant vectors. This allows us to rewrite any section $a \in \Gamma(SM)$ in the form of the formal power series

\[ a = a(x, y) = \sum_{p=0}^{\infty} a_{i_1 \ldots i_p}(x) y^{i_1} \ldots y^{i_p}. \]  

(14)

In this way the variables $y^i$ may be thought of as formal coordinates on the fibers of the tangent bundle $TM$.

It is easy to observe that the vector space $\Gamma(SM)$ is naturally endowed with the commutative product which is induced by a fiberwise multiplication of formal power series in $y^i$. This product makes $\Gamma(SM)$ into a commutative algebra with a unit.

Now we turn to definitions of formal fiberwise polyvector fields and formal fiberwise polydifferential operators on $SM$. 
Definition 4 A bundle $T_{\text{poly}}^k$ of formal fiberwise polyvector fields of degree $k$ is a bundle over $M$ whose sections are $C^\infty(M)$-linear operators $v : \Lambda^{k+1}\Gamma(SM) \mapsto \Gamma(SM)$ of the form
\[
v = \sum_{p=0}^{\infty} \psi^{i_0...i_k}_{i_1...i_p}(x)y^{i_1} \ldots y^{i_p} \frac{\partial}{\partial y^{j_0}} \wedge \ldots \wedge \frac{\partial}{\partial y^{j_k}},
\]
where we assume that the infinite sum in $y$’s is formal and $\psi^{i_0...i_k}_{i_1...i_p}(x)$ are tensors symmetric in indices $i_1, \ldots, i_p$ and antisymmetric in indices $j_0, \ldots, j_k$.

Extending the definition of the formal fiberwise polyvector field by allowing the fields to be inhomogeneous we define the total bundle $T_{\text{poly}}$ of formal fiberwise polyvector fields
\[
T_{\text{poly}} = \bigoplus_{k=-1}^{\infty} T_{\text{poly}}^k, \quad T^{-1}_{\text{poly}} = SM.
\]
We mention that the fibers of the bundle $T_{\text{poly}}$ form a DGLA $T_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ of polyvector fields on the formal completion $\mathbb{R}^d_{\text{formal}}$ of $\mathbb{R}^d$ at the origin.

Definition 5 A bundle $D_{\text{poly}}^k$ of formal fiberwise polydifferential operator of degree $k$ is a bundle over $M$ whose sections are $C^\infty(M)$-polylinear maps $\Psi : \otimes^{k+1}\Gamma(SM) \mapsto \Gamma(SM)$ of the form
\[
\Psi = \sum_{\alpha_0...\alpha_k} \sum_{p=0}^{\infty} \Psi^{\alpha_0...\alpha_k}_{i_1...i_p}(x)y^{i_1} \ldots y^{i_p} \frac{\partial}{\partial y^{\alpha_0}} \otimes \ldots \otimes \frac{\partial}{\partial y^{\alpha_k}},
\]
where $\alpha$’s are a multi-indices $\alpha = j_1 \ldots j_l$ and
\[
\frac{\partial}{\partial y^{\alpha}} = \frac{\partial}{\partial y^{j_1}} \ldots \frac{\partial}{\partial y^{j_l}},
\]
the infinite sum in $y$’s is formal, and the sum in the orders of derivatives $\partial/\partial y$ is finite.

Notice that the tensors $\Psi^{\alpha_0...\alpha_k}_{i_1...i_p}(x)$ are symmetric in covariant indices $i_1, \ldots, i_p$.

As well as for polyvector fields we define the total bundle $D_{\text{poly}}$ of formal fiberwise polydifferential operators as the direct sum
\[
D_{\text{poly}} = \bigoplus_{k=-1}^{\infty} D_{\text{poly}}^k, \quad D^{-1}_{\text{poly}} = SM.
\]
We mention that the fibers of the bundle $D_{\text{poly}}$ form a DGLA $D_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ of polydifferential operators on $\mathbb{R}^d_{\text{formal}}$.

For our purposes we need to tensor the bundles we introduced with the exterior algebra bundle $\Lambda T^*M$. Namely, instead of the commutative algebra $\Gamma(SM)$ of formal power series in the fiber coordinates $y^i$ of the tangent bundle $TM$ we will need a super-commutative algebra $\Omega(M, SM)$ of exterior forms on $M$ with values in $SM$, namely
\[
\Omega(M, SM) = \{ a(x, y, dx) = \sum_{p,q \geq 0} a_{i_1...i_p,j_1...j_q}(x)y^{i_1} \ldots y^{i_p} dx^{j_1} \ldots dx^{j_q} \},
\]
where \( a_{i_1 \ldots i_p}^{j_1 \ldots j_q}(x) \) are covariant tensors symmetric in indices \( i_1, \ldots, i_p \) and antisymmetric in indices \( j_1, \ldots, j_q \).

Algebra \( \Omega(M, SM) \) is \( \mathbb{Z} \)-graded with respect to degree \( p \) in “\( y \)” and \( \mathbb{Z} \)-graded with respect to the ordinary exterior degree \( q \).

\[
\Omega(M, SM) = \bigoplus_{p, q \geq 0} \Omega^q(M, S^p M).
\]

Next, we introduce vector spaces \( \Omega(M, T_{\text{poly}}) \) and \( \Omega(M, D_{\text{poly}}) \) of smooth exterior forms on \( M \) with values in \( T_{\text{poly}} \) and \( D_{\text{poly}} \) respectively. It is easy to see that both \( \Omega(M, T_{\text{poly}}) \) and \( \Omega(M, D_{\text{poly}}) \) are naturally endowed with DGLA structures induced by the respective fiberwise DGLA structures on \( T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and \( D_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \). We denote the differential and the Lie bracket in \( \Omega(M, D_{\text{poly}}) \) by \( \partial \) and \( [\cdot, \cdot] \), respectively, and denote the Lie bracket in \( \Omega(M, T_{\text{poly}}) \) by \( [\cdot, \cdot]_{SN} \).

Although \( \Omega(M, SM) \) is a commutative subalgebra of the DGLA \( \Omega(M, T_{\text{poly}}) \) (\( \Omega(M, D_{\text{poly}}) \)) of polyvector fields (polydifferential operators) of degree \(-1\) we consider \( \Omega(M, SM) \) separately since we refer to \( \Omega(M, SM) \) not only as to a Lie subalgebra of the DGLA \( \Omega(M, T_{\text{poly}}) \) (\( \Omega(M, D_{\text{poly}}) \)) but also as to a super-commutative algebra with an ordinary multiplication.

The DG Lie algebras \( \Omega(M, T_{\text{poly}}) \) and \( \Omega(M, D_{\text{poly}}) \) are \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \)-graded. Namely, they have an obvious grading of the exterior forms, grading with respect to the degree of polyvector field or polydifferential operator, and finally grading with respect to a difference of degrees in \( y \) and in \( \partial/\partial y \). While the parity of elements of \( \Omega(M, SM) \) is defined by the exterior degree \( q \), the parity of the elements of \( \Omega(M, T_{\text{poly}}) \) (\( \Omega(M, D_{\text{poly}}) \)) is defined by the sum \( r = q + k \) of the exterior degree \( q \) and the degree \( k \) of a polyvector field (polydifferential operator).

Due to properties 1 and 2 in theorem 1 we have a fiberwise quasi-isomorphism

\[
U^f : (\Omega(M, T_{\text{poly}}), 0, [\cdot, \cdot]_{SN}) \sim (\Omega(M, D_{\text{poly}}), \partial, [\cdot, \cdot]).
\]

from the DGLA \( (\Omega(M, T_{\text{poly}}), 0, [\cdot, \cdot]_{SN}) \) to the DGLA \( (\Omega(M, D_{\text{poly}}), \partial, [\cdot, \cdot]) \). We use the quasi-isomorphism (20) in section 4 in order to prove the formality theorem for a general smooth manifold. Now we turn to another important ingredient of our construction.

### 3 Fedosov resolutions of \( T_{\text{poly}}(M) \) and \( D_{\text{poly}}(M) \)

In this section we construct resolutions for the DGLA \( D_{\text{poly}}(M) \) of polydifferential operators and the DGLA \( T_{\text{poly}}(M) \) of polyvector fields on an arbitrary smooth manifold \( M \) using the DGLA \( \Omega(M, D_{\text{poly}}) \) of exterior forms with values in fiberwise polydifferential operators and the DGLA \( \Omega(M, T_{\text{poly}}) \) of exterior forms with values in fiberwise polyvector fields on \( SM \). These resolutions may be regarded as classical analogs of the construction of the so-called quantum exponential Fedosov map from the algebra of functions on a symplectic manifold to an algebra of sections of the Weyl bundle [9]. In this sense the resolutions are more reminiscent of what is called generalized formal exponential map used in work [2] and discussed in papers [4] and [10].

We will proceed with the DG Lie algebras \( D_{\text{poly}}(M) \) and \( T_{\text{poly}}(M) \) and the algebra of functions \( C^\infty(M) \) simultaneously and denote the same operations on different algebras
\(\Omega(M, T\text{poly}), \Omega(M, D\text{poly})\) and \(\Omega(M, SM)\) by the same letters. In what follows it does not lead to any confusion.

The differential
\[
\delta = dx^i \frac{\partial}{\partial y^i} : \Omega^q(M, SM) \mapsto \Omega^{q+1}(M, SM), \quad \delta^2 = 0
\]  

on the algebra \(\Omega(M, SM)\) obviously extends to differentials on \(\Omega(M, T\text{poly})\) and \(\Omega(M, D\text{poly})\).

Namely,
\[
\delta = [dx^i \frac{\partial}{\partial y^i}, \cdot]_{SN} : \Omega^q(M, T\text{poly}) \mapsto \Omega^{q+1}(M, T\text{poly}), \quad \delta^2 = 0,
\]
and
\[
\delta = [dx^i \frac{\partial}{\partial y^i}, \cdot] : \Omega^q(M, D\text{poly}) \mapsto \Omega^{q+1}(M, D\text{poly}), \quad \delta^2 = 0.
\]

Since the multiplication \(m \in \Gamma(D^1\text{poly})\) in \(\Gamma(SM)\) is \(\delta\)-closed
\[\delta m = 0\]
\(\delta\) (anti)commutes with the differential \(\partial\) in \(\Omega(M, D\text{poly})\). By definition \(\delta\) is a derivation of the Lie algebras \(\Omega(M, T\text{poly})\) and \(\Omega(M, D\text{poly})\). Thus \(\delta\) is compatible with DGLA structures on \(\Omega(M, T\text{poly})\) and \(\Omega(M, D\text{poly})\).

Due to the Poincare lemma for the space \(\mathbb{R}^d_{\text{formal}}\) the complexes \((\Omega(M, SM), \delta)\), \((\Omega(M, T\text{poly}), \delta)\), \((\Omega(M, D\text{poly}), \delta)\) are acyclic and their zero cohomologies can be computed easily, namely

\[
H^0(\Omega(M, SM), \delta) = C^\infty(M),
\]
\[
H^0(\Omega(M, T\text{poly}), \delta) = \mathcal{F}^0T_{\text{poly}},
\]
and
\[
H^0(\Omega(M, D\text{poly}), \delta) = \mathcal{F}^0D_{\text{poly}},
\]
where \(\mathcal{F}^0T_{\text{poly}}\) (\(\mathcal{F}^0D_{\text{poly}}\)) denotes the vector space of all fiberwise polyvector fields (15) (fiberwise polydifferential operators (17)) with constant coefficients in \(y\)'s.

Therefore a natural projection \(\sigma\) from \(\Omega(M, SM)\) (\(\Omega(M, T\text{poly}), \Omega(M, D\text{poly})\)) to \(C^\infty(M)\) \((\mathcal{F}^0T_{\text{poly}}, \mathcal{F}^0D_{\text{poly}})\)
\[
\sigma a = a \bigg|_{y^i = dx^i = 0}
\]
gives a morphism of complex \((\Omega(M, SM), \delta)\) \((\Omega(M, T\text{poly}), \delta)\), \((\Omega(M, D\text{poly}), \delta)\)) into itself and this morphism is homotopic to the identity map. One can easily guess the respective homotopy operator in the from
\[
\delta^{-1}a = y^k i \left( \frac{\partial}{\partial x^k} \right) \int_0^1 a(x, ty, tdx) \frac{dt}{t},
\]
where \(a\) is an element of \(\Omega(M, SM)\) (\(\Omega(M, T\text{poly}), \Omega(M, D\text{poly})\)), \(i(\partial/\partial x^k)\) stands for the interior derivative of exterior forms by the vector field \(\partial/\partial x^k\), and \(\delta^{-1}\) is extended to \(C^\infty(M)\) \((\mathcal{F}^0T_{\text{poly}}, \mathcal{F}^0D_{\text{poly}})\) by zero.
We will use the property of the homotopy operator $\delta^{-1}$ in the following form

$$a = \sigma(a) + \delta\delta^{-1}a + \delta^{-1}\delta a, \quad \forall \ a \in \Omega(M, \mathcal{B}),$$

where $\mathcal{B}$ is either $SM$ or $T_{\text{poly}}$ or $D_{\text{poly}}$.

Thus we have already got a resolution $(\Omega(M, SM), \delta)$ of the commutative algebra $C^\infty(M)$.

Now we need to deform this resolution in order to get $F^0T_{\text{poly}}$ and $F^0D_{\text{poly}}$ to be identified with $T_{\text{poly}}(M)$ and $D_{\text{poly}}(M)$ respectively. In this way we will get resolutions of the DG Lie algebras $T_{\text{poly}}(M)$ and $D_{\text{poly}}(M)$.

We consider an affine torsion free connection $\nabla_i$ on $M$ and associate to it the following derivation of $\Omega(M, SM)$

$$\nabla = dx^i \frac{\partial}{\partial x^i} + \Gamma : \Omega^q(M, SM) \mapsto \Omega^{q+1}(M, SM),$$

where

$$\Gamma = -dx^i \Gamma^k_{ij}(x)y^j \frac{\partial}{\partial y^k},$$

with $\Gamma^k_{ij}(x)$ being Christoffel symbols of $\nabla_i$.

The derivation $\nabla$ obviously extends to derivations of the DG Lie algebras $\Omega(M, T_{\text{poly}})$ and $\Omega(M, D_{\text{poly}})$

$$\nabla = dx^i \frac{\partial}{\partial x^i} + [\Gamma, \bullet] \ : \Omega^q(M, T_{\text{poly}}) \mapsto \Omega^{q+1}(M, T_{\text{poly}}),$$

$$\nabla = dx^i \frac{\partial}{\partial x^i} + [\Gamma, \bullet] \ : \Omega^q(M, D_{\text{poly}}) \mapsto \Omega^{q+1}(M, D_{\text{poly}}).$$

It is clear by definition that $\nabla$ is indeed a derivation of Lie algebra structures of $\Omega(M, T_{\text{poly}})$ and $\Omega(M, D_{\text{poly}})$. On the other hand the multiplication $m \in \Gamma(D^1_{\text{poly}})$ in $\Gamma(SM)$ is "covariantly constant" $dm + [\Gamma, m] = 0$ and hence the derivation $\nabla$ commutes with the differential $\partial$ in $\Omega(M, D_{\text{poly}})$.

In general derivation (27) is not nilpotent as $\delta$. Instead we have the following expression for $\nabla^2$

$$\nabla^2 a = Ra : \Omega^q(M, SM) \mapsto \Omega^{q+2}(M, SM),$$

where

$$R = -\frac{1}{2}dx^i dx^j (R_{ij})^k_l(x)y^l \frac{\partial}{\partial y^k},$$

and $(R_{ij})^k_l(x)$ is the standard Riemann curvature tensor of the connection $\nabla_i$.

Analogously, for $\Omega(M, T_{\text{poly}})$ and $\Omega(M, D_{\text{poly}})$ we have

$$\nabla^2 a = [R, a]_{SN} : \Omega^q(M, T_{\text{poly}}) \mapsto \Omega^{q+2}(M, T_{\text{poly}}),$$

$$\nabla^2 a = [R, a] : \Omega^q(M, D_{\text{poly}}) \mapsto \Omega^{q+2}(M, D_{\text{poly}}).$$

Notice that since the connection $\nabla_i$ is torsion free derivations $\nabla$ and $\delta$ (anti)commute.
\( \delta \nabla + \nabla \delta = 0. \) (34)

We use the derivation (27) in order to deform the nilpotent differential \( \delta \) on \( \Omega(M, SM) \), \( \Omega(M, T_{poly}) \), and \( \Omega(M, D_{poly}) \).

\[
D = \nabla - \delta + A : \Omega^q(M, SM) \mapsto \Omega^{q+1}(M, SM),
\]

\[
D = \nabla - \delta + [A, \bullet]_{SN} : \Omega^q(M, T_{poly}) \mapsto \Omega^{q+1}(M, T_{poly}),
\]

\[
D = \nabla - \delta + [A, \bullet] : \Omega^q(M, D_{poly}) \mapsto \Omega^{q+1}(M, D_{poly}),
\]

where

\[
A = \sum_{p=2}^{\infty} dx^k A^j_{ki_1...i_p}(x)y^{i_1}...y^{i_p} \frac{\partial}{\partial y^j}
\]

is viewed as an element of \( \Omega^1(M, T_{poly}^0) \) and an element of \( \Omega^1(M, D_{poly}^0) \).

Due to the following theorem one can explicitly construct a nilpotent differential \( D \) in the framework of ansatz (35)

**Theorem 2** *Iterating the equation*

\[
A = \delta^{-1} R + \delta^{-1}(\nabla A + \frac{1}{2}[A, A])
\]

(36)

*in degrees in \( y \) one constructs \( A \in \Omega^1(M, T_{poly}^0) \subset \Omega^1(M, D_{poly}^0) \) such that \( \delta^{-1} A = 0 \) and the derivation \( D \) (35) is nilpotent

\[
D^2 = 0.
\]

**Proof.** The proof of the theorem is analogous to the proof of theorem 3.2 in [9].

First, we observe that the equation

\[
\delta A = R + \nabla A + \frac{1}{2}[A, A]
\]

(37)

implies that \( D^2 = 0 \).

Recurrent procedure (36) converges to some element

\[
A \in \Omega^1(M, T_{poly}^0) \subset \Omega^1(M, D_{poly}^0),
\]

since the operator \( \delta^{-1} \) raises the degree in \( y \). An obvious identity \((\delta^{-1})^2 = 0\) implies that for such \( A \)

\[
\delta^{-1} A = 0
\]

and hence due to homotopy operator property (26) we get that the element \( A \) satisfies the following consequence of equation (37)

\[
\delta^{-1} \delta A = \delta^{-1} R + \delta^{-1}(\nabla A + \frac{1}{2}[A, A]).
\]

(38)

We define

\[
C = -\delta A + R + \nabla A + \frac{1}{2}[A, A].
\]
Using the Bianchi identities for the Riemann curvature tensor we get that
\[ \delta R = 0, \quad \nabla R = 0. \]

The latter equations imply that \( C \) satisfies the following condition
\[ \nabla C - \delta C + [A, C] = 0. \tag{39} \]

Due to (38) \( \delta^{-1} C = 0 \). Thus applying \( \delta^{-1} \) to (39) and using homotopy operator property (26) once again we get that
\[ C = \delta^{-1} (\nabla C + [A, C]). \]

Since the operator \( \delta^{-1} \) raises the degree in \( y \) the latter equation has a unique zero solution.
Thus the theorem is proved. \( \Box \)

In what follows we refer to the nilpotent differential \( D \) (35) as Fedosov differential.

Now we are going to prove that the complexes \( (\Omega(M, SM), D), (\Omega(M, T_{poly}), D) \) and \( (\Omega(M, D_{poly}), D) \) are acyclic and their zero cohomologies are isomorphic to \( C^\infty(M), F^0 T_{poly} \) and \( F^0 D_{poly} \), respectively.

**Theorem 3** For a bundle \( B \) that is either \( SM \) or \( T_{poly} \) or \( D_{poly} \) we have that
\[ H^*(\Omega(M, B), D) = H^0(\Omega(M, B), D). \tag{40} \]

Furthermore,
\[ H^0(\Omega(M, SM), D) \cong C^\infty(M) \]

as commutative algebras and
\[ H^0(\Omega(M, T_{poly}), D) \cong F^0 T_{poly}, \]
\[ H^0(\Omega(M, D_{poly}), D) \cong F^0 D_{poly} \]

as vector spaces.

**Proof.** This is a generalization of the proof of [9](Theorem 3.3). Let \( a \) be an element of \( \Omega^q(M, B) \) for \( q > 0 \) and \( Da = 0 \). Our purpose is to solve the equation
\[ a = Db. \tag{41} \]

We claim that the following recurrent procedure\(^5\)
\[ b = -\delta^{-1} a + \delta^{-1} (\nabla b + [A, b]) \tag{42} \]
converges to an element \( b \in \Omega^{q-1}(M, B) \) such that \( \delta^{-1} b = 0, \sigma b = 0, \) and \( Db = a \). All the claims besides the last one are obvious by construction. Let us prove that \( Db = a \).

We denote by \( h \) the element
\[ h = a - Db \in \Omega^q(M, B) \]
and mention that \( Dh = 0 \) or equivalently
\(^5\)For \( B = SM \) one may use \( A \bullet \) instead of \([A, \bullet]\).
\[
\delta h = \nabla h + [A, h]. \tag{43}
\]

In virtue of equation (42)
\[
\delta^{-1} h = 0.
\]
Furthermore, since \( q > 0 \)
\[
\sigma h = 0,
\]
and hence applying homotopy property (26) we get
\[
h = \delta^{-1}(\nabla h + [A, h]).
\]
The latter equation has a unique vanishing solution since \( \delta^{-1} \) raises the degree in \( y \). Thus we have proved (40).

We will give the proof only for the isomorphism of the vector spaces
\[
H^0(\Omega(M, D_{poly}), D) \cong F^0 D_{poly}
\]
since the analogous statement for \( T_{poly} \) is proved in the same way and the isomorphism of commutative algebras \( H^0(\Omega(M, SM), D) \) and \( C^\infty(M) \) is proved in ([4], sect. 6).

As in [9] we give a constructive proof. Namely, we will define a bijective map \( \tau \) from \( F^0 D_{poly} \) to the subspace \( Z^0(\Omega(M, D_{poly}), D) \) of \( D \)-closed forms of degree zero such that for any \( a_0 \in F^0 D_{poly} \)
\[
\sigma(\tau a_0) = \tau a_0 \bigg|_{y=0} = a_0. \tag{44}
\]
Since \( Z^0(\Omega(M, D_{poly}), D) = H^0(\Omega(M, D_{poly}), D) \) this would prove the statement.

For any \( a_0 \in F^0 D_{poly} \) we define an element \( a = \tau a_0 \in \Omega^0(M, D_{poly}) \) by the following recurrent procedure
\[
a = a_0 + \delta^{-1}(\nabla a + [A, a]). \tag{45}
\]
This procedure converges since \( \delta^{-1} \) raises the degree in \( y \).

First, we prove that \( Da = 0 \) and \( \sigma a = a_0 \). While the latter statement is obvious the former one requires some work. Let \( f = Da \) then \( Df = 0 \), \( \sigma f = 0 \), and \( \delta^{-1} f = 0 \) by (45). Hence due to (26) we have
\[
f = \delta^{-1}(\nabla f + [A, f]).
\]
This equation has a unique vanishing solution since \( \delta^{-1} \) raises the degree in \( y \).

Thus we have an \( \mathbb{R} \)-linear map \( \tau \) from \( F^0 D_{poly} \) to \( Z^0(\Omega(M, D_{poly}), D) \) which is obviously injective. Furthermore, if \( b \in Z^0(\Omega(M, D_{poly}), D) \) and \( \sigma b = 0 \) then due to (26)
\[
b = \delta^{-1}(\nabla b + [A, b])
\]
and hence \( b = 0 \) since \( \delta^{-1} \) raises the degree in \( y \). Therefore the map \( \tau \) is also surjective and the theorem is proved. \( \Box \)

**Remark.** The ordinary multiplication \( m \) in \( \Gamma(SM) \) viewed as an element of \( \Gamma(D_{poly}^1) \) turns out to be \( D \)-closed and
\[
\sigma m = m \in F^0 D_{poly}^1.
\]
where $\mathcal{F}^0\mathcal{D}^1_{\text{poly}}$ is a vector space of fiberwise bidifferential operators on $SM$ with constant coefficients in $y$. Since $m$ is $D$-closed the Fedosov differential $D$ (anti)commutes with the differential $\partial$ in $\Omega(M, \mathcal{D}_{\text{poly}})$. Thus $D$ respects the DGLA structures both on $\Omega(M, \mathcal{T}_{\text{poly}})$ and $\Omega(M, \mathcal{D}_{\text{poly}})$.

It turns out that $\mathcal{F}^0\mathcal{T}_{\text{poly}} (\mathcal{F}^0\mathcal{D}_{\text{poly}})$ can be identified with the vector space $T_{\text{poly}}(M)$ ($D_{\text{poly}}(M)$) of polyvector fields (of polydifferential operators) on $M$. Namely, we have the following

**Proposition 1** Given the Fedosov differential (35) one can construct an isomorphism of vector spaces

$$\mu : \mathcal{F}^0\mathcal{D}_{\text{poly}} \mapsto D_{\text{poly}}(M),$$

whose restriction\(^6\) to $\mathcal{F}^0\mathcal{T}_{\text{poly}}$ gives an isomorphism from the vector space $\mathcal{F}^0\mathcal{T}_{\text{poly}}$ to the vector space $T_{\text{poly}}(M)$. The map $\mu$ preserves degrees of polydifferential operators (polyvector fields).

**Proof.** We give a proof for polydifferential operators since a proof for polyvector fields is completely analogous. First, we restrict ourselves to the case of degree 0 polydifferential operators, that is to ordinary differential operators.

A construction of the desired isomorphism is based on the observation that for any function $a_0 \in C^\infty(M)$ and for any integer $p \geq 0$

$$\frac{\partial}{\partial y^i} \ldots \frac{\partial}{\partial y^p} \tau(a_0)\bigg|_{y=0} = \partial_{x^i} \ldots \partial_{x^p} a_0(x) + \text{lower order derivatives of } a_0. \quad (47)$$

Due to this observation an isomorphism from $\mathcal{F}^0\mathcal{D}_{\text{poly}}^0$ to $D_{\text{poly}}^0$ is defined with the help of the identification between the space of functions $C^\infty(M)$ and the space of $D$-closed sections in $\Gamma(SM)$.

Namely, the following map

$$\mu : \mathcal{F}^0\mathcal{D}_{\text{poly}}^0 \mapsto D_{\text{poly}}^0(M)$$

from the space $\mathcal{F}^0\mathcal{D}_{\text{poly}}^0$ of fiberwise differential operators on $SM$ with constant coefficients in $y$ to the space $D^0_{\text{poly}}(M)$ of differential operators on $M$

$$\mu(\Psi)(a_0) = \sigma \Psi(\tau(a_0)) = \Psi(\tau(a_0))\bigg|_{y=0}, \quad \Psi \in \mathcal{F}^0\mathcal{D}_{\text{poly}}^0, \quad a_0 \in C^\infty(M) \quad (48)$$

gives an isomorphism of the respective vector spaces. Then one can obviously extend the map $\mu$ to the isomorphism from $\mathcal{F}^0\mathcal{D}_{\text{poly}}$ to $D_{\text{poly}}(M)$.

Due to the above proposition $\mathcal{F}^0\mathcal{T}_{\text{poly}} (\mathcal{F}^0\mathcal{D}_{\text{poly}})$ automatically acquires a structure of DGLA, induced via the map $\mu^{-1}$ from the vector space $T_{\text{poly}}(M)$ ($D_{\text{poly}}(M)$). On the other hand the Fedosov differential $D$ respects the structure of DGLA on $\Omega(M, \mathcal{T}_{\text{poly}})$ ($\Omega(M, \mathcal{D}_{\text{poly}})$). Thus, the natural question is whether $(\Omega(M, \mathcal{T}_{\text{poly}}), D)$ ($\Omega(M, \mathcal{D}_{\text{poly}}), D$) is a resolution of $T_{\text{poly}}(M)$ ($D_{\text{poly}}(M)$) as a DGLA or not. The following proposition gives a positive answer to the question.

**Proposition 2** A DGLA structure induced on cohomologies of the complex $(\Omega(M, \mathcal{T}_{\text{poly}}), D)$ ($(\Omega(M, \mathcal{D}_{\text{poly}}), D)$) coincides with the DGLA structure induced from $T_{\text{poly}}(M)$ ($D_{\text{poly}}(M)$) via the map $\mu^{-1}$ (46)

$$H^\bullet(\Omega(M, \mathcal{T}_{\text{poly}}), D) \cong T_{\text{poly}}(M), \quad H^\bullet(\Omega(M, \mathcal{D}_{\text{poly}}), D) \cong D_{\text{poly}}(M).$$

\(^6\)Abusing notations we denote the isomorphism from $\mathcal{F}^0\mathcal{T}_{\text{poly}}$ to $T_{\text{poly}}(M)$ by the same letter $\mu$. 

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Proof. As in the previous proof we restrict ourselves to the case of polydifferential operators since the proof for polyvector fields is completely analogous.

First, we prove that the composition of the maps \( \mu \) and \( \sigma \)

\[
\mu \circ \sigma : \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) \mapsto \mathcal{D}_\text{poly}(M) \tag{49}
\]

respects Lie brackets in \( \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) \) and \( \mathcal{D}_\text{poly}(M) \). Since definitions of the Lie brackets in both Lie algebras are based on composition of operators (see (7)) it suffices to prove that a restriction of the map (49) to the subalgebra \( \mathcal{Z}^0(\Omega(M, \mathcal{D}_0^\text{poly}), D) \) of \( \mathcal{D}_\text{poly}(M) \) respects Lie algebra structures.

To prove this we observe that for any \( a_0 \in C^\infty(M) \) and for any \( \mathcal{D}_\text{poly} \)-closed operator \( P \in \Gamma(\mathcal{D}_0^\text{poly}) \) we denote \( P_1 = \mu \circ \sigma(P_1) \) and \( P_2 = \mu \circ \sigma(P_2) \). Due to (50) we have

\[
\tau(P_1 P_2 a_0) = P_1(\tau(P_2 a_0)) = P_1 P_2 \tau(a_0) .
\]

Hence

\[
P_1 P_2 a_0 = \sigma(P_1 P_2 \tau(a_0)) = \sigma(\sigma(P_1 P_2) \tau a_0) = \mu \circ \sigma(P_1 P_2) a_0 .
\]

Thus the map (49) respects Lie algebra structures in \( \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) \) and \( \mathcal{D}_\text{poly}(M) \).

Next, we mention that under the map \( \mu \) the multiplication \( m \in \mathcal{F}^0 \mathcal{D}^1_\text{poly} \) in the algebra \( \Gamma(SM) \) turns to the multiplication \( m_0 \in \mathcal{D}^1_\text{poly}(M) \) in the algebra \( C^\infty(M) \).

\[
\mu(m) = m_0 .
\]

Due to this observation and the remark made after the proof of theorem 3 we have that for \( m \in \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) \)

\[
\mu \circ \sigma(m) = m_0 \in \mathcal{D}^1_\text{poly}(M) .
\]

Therefore since the differential on \( \mathcal{D}_\text{poly}(M) \) is an interior derivation by the element \( m_0 \) and the differential on \( \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) \) is an interior derivation by the element \( m \) the map (49) is an isomorphism of the DG Lie algebras \( \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) \) and \( \mathcal{D}_\text{poly}(M) \). On the other hand we know from theorem 3 that

\[
H^\bullet(\Omega(M, \mathcal{D}_\text{poly}), D) \cong \mathcal{Z}^0(\Omega(M, \mathcal{D}_\text{poly}), D) .
\]

Hence the desired statement is proved. \( \Box \)

4 Formality theorem for a general manifold via Fedosov resolutions

In the previous section we construct Fedosov resolutions of the algebras of polydifferential operators and polyvector fields using the DG Lie algebras \((\Omega(M, \mathcal{D}_\text{poly}), D + \partial, [, ])\) and
In terms of strong homotopy Lie algebras this means that we have two quasi-isomorphisms of DG Lie algebras

\[ U_T : T_{\text{poly}}(M) \cong (\Omega(M, T_{\text{poly}}), D, [,])_{SN}, \]

\[ U_D : D_{\text{poly}}(M) \cong (\Omega(M, D_{\text{poly}}), D + \partial, [,]), \]

induced by the homomorphism \( \tau \circ \mu^{-1} \). On the other hand we have the fiberwise quasi-isomorphism (20) from the DGLA \( (\Omega(M, T_{\text{poly}}), 0, [,])_{SN} \) to the DGLA \( (\Omega(M, D_{\text{poly}}), \partial, [,]) \).

In this section we use the quasi-isomorphism (20) and the Fedosov resolutions (51), (52) in order to prove that

**Theorem 4 (Kontsevich, [1])** For any smooth manifold \( M \) there exists a quasi-isomorphism \( \mathcal{U} \) from the DGLA \( T_{\text{poly}}(M) \) of polyvector fields on \( M \) to the DGLA \( D_{\text{poly}}(M) \) of polydifferential operators on \( M \).

**Proof.** We propose an explicit construction of the desired quasi-isomorphism from the DGLA \( T_{\text{poly}}(M) \) to the DGLA \( D_{\text{poly}}(M) \). This construction consists of two steps.

First, we observe that the Fedosov differential (35) provides us with a Maurer-Cartan element in the DGLA \( \Omega(M, T_{\text{poly}}) \). By twisting (20) with the help of this Maurer-Cartan element we get the quasi-isomorphism

\[ \mathcal{U} : (\Omega(M, T_{\text{poly}}), D, [,])_{SN} \cong (\Omega(M, D_{\text{poly}}), D + \partial, [,]), \]

which readily gives us the quasi-isomorphism

\[ \mathcal{U} : T_{\text{poly}}(M) \cong (\Omega(M, D_{\text{poly}}), D + \partial, [,]) \]

due to the presence (51). Second, we contract (54) to the quasi-isomorphism

\[ \mathcal{U} : T_{\text{poly}}(M) \cong Z^0_D(\Omega(M, D_{\text{poly}}), \partial, [,]), \]

which yields the desired quasi-isomorphism \( \mathcal{U} \) from \( T_{\text{poly}}(M) \) to \( D_{\text{poly}}(M) \) since the DG Lie algebras \( D_{\text{poly}}(M) \) and \( Z^0_D(\Omega(M, D_{\text{poly}}), \partial, [,]) \) are isomorphic via the map \( \tau \circ \mu^{-1} \).

In the remaining part of this section we complete the proof of theorem 4 following the two steps outlined above.

### 4.1 Construction of the quasi-isomorphism from \( (\Omega(M, T_{\text{poly}}), D, [,])_{SN} \) to \( (\Omega(M, D_{\text{poly}}), D + \partial, [,]) \)

We present the Fedosov differential (35) in the form

\[
D = d + [B, \bullet]_{SN} : \Omega^q(M, T_{\text{poly}}) \mapsto \Omega^{q+1}(M, T_{\text{poly}}),
\]

\[
D = d + [B, \bullet] : \Omega^q(M, D_{\text{poly}}) \mapsto \Omega^{q+1}(M, D_{\text{poly}}),
\]

where

\[
d = dx^i \frac{\partial}{\partial x^i},
\]
and

\[ B = -dx^i \frac{\partial}{\partial y^i} - dx^i \Gamma^k_{ij}(x)y^j \frac{\partial}{\partial y^k} + \sum_{p \geq 2} dx^i A_{ij_1...j_p}(x)y^{j_1}...y^{j_p} \frac{\partial}{\partial y^k}. \] (57)

Notice that \( B \) is only locally viewed as a fiberwise vector field or a fiberwise differential operator. Namely, if \( W \) is a coordinate disk on \( M \) a restriction of \( B \) to \( W \) gives an element of \( \Omega^1(W, T^0_{\text{poly}}) \) which can be also viewed as an element of \( \Omega^1(W, D^0_{\text{poly}}) \). The perhaps surprising thing is that the transformation law for \( B \) upon a change of coordinates takes a very simple form

\[ \tilde{B} \big|_{W \cap W} = B \big|_{W \cap W} + dx^i H^k_{ij}(x)y^j \frac{\partial}{\partial y^k}. \] (58)

Concrete expression for \( H^k_{ij}(x) \) is not important. The main observation we are going to use is that the additional term is locally a fiberwise polyvector field, which is linear in \( y \)'s.

Let us now restrict ourselves to a coordinate disk \( W \). On \( W \) both the differential \( d \) and the element \( B \in \Omega^1(W, T^0_{\text{poly}}) \subset \Omega^1(W, T^0_{\text{poly}}) \) are well defined separately. It is easy to see that \( d \) commutes with the fiberwise DGLA structures on \( \Omega(W, T^0_{\text{poly}}) \) and \( \Omega(W, D^0_{\text{poly}}) \). Moreover, since and the quasi-isomorphism \( U^f \) (20) of the DG Lie algebras \( \Omega(W, T^0_{\text{poly}}) \), \( \Omega(W, D^0_{\text{poly}}) \) is also fiberwise and since \( W \) is contractible \( U^f \) gives a quasi-isomorphism of DG Lie algebras

\[ U^f : (\Omega(W, T^0_{\text{poly}}), d, [,]_{SN}) \sim (\Omega(W, D^0_{\text{poly}}), d + \partial, [\cdot,\cdot]). \] (59)

Due to nilpotency of derivations (56) \( B \in \Omega(W, T^0_{\text{poly}}) \subset \Omega(W, D^0_{\text{poly}}) \) is a Maurer-Cartan element in the DG Lie algebras \( (\Omega(W, T^0_{\text{poly}}), d, [,]_{SN}) \) and \( (\Omega(W, D^0_{\text{poly}}), d + \partial, [\cdot,\cdot]) \). Furthermore, using the terminology of strong homotopy Lie algebras one may say that the DG Lie algebras \( (\Omega(W, T^0_{\text{poly}}), D, [,]_{SN}) \) and \( (\Omega(W, D^0_{\text{poly}}), D + \partial, [\cdot,\cdot]) \) are obtained from the DG Lie algebras \( (\Omega(W, T^0_{\text{poly}}), d, [,]_{SN}) \) and \( (\Omega(W, D^0_{\text{poly}}), d + \partial, [\cdot,\cdot]) \), respectively with the help of a twisting\(^7\) by the Maurer-Cartan element \( B \). Namely, the nilpotent coderivations \( Q_T \) and \( Q_D \) on coassociative cocommutative coalgebras \( C_\bullet(\Omega(W, T^0_{\text{poly}})[1]) \) and \( C_\bullet(\Omega(W, D^0_{\text{poly}})[1]) \) corresponding to the DGLA structures \( (D, [,]_{NS}) \) and \( (D + \partial, [\cdot,\cdot]) \) on \( \Omega(W, T^0_{\text{poly}}) \) and \( \Omega(W, D^0_{\text{poly}}) \) are related to the nilpotent coderivations \( Q_T \) and \( Q_D \), corresponding to the DGLA structures \( (d, [,]_{NS}) \) and \( (d + \partial, [\cdot,\cdot]) \) as follows

\[ Q_T(X) = \exp((-B)\wedge)Q_T(\exp(B\wedge)X), \]
\[ Q_D(Y) = \exp((-B)\wedge)Q_D(\exp(B\wedge)Y), \] (60)

\[ X \in C_\bullet(\Omega(W, T^0_{\text{poly}})[1]), \quad Y \in C_\bullet(\Omega(W, D^0_{\text{poly}})[1]), \]

where the sums of the form

\[ \exp(B\wedge) = B \wedge + \frac{1}{2!} B \wedge B \wedge + \ldots \]

are finite since \( B \wedge B \wedge \ldots \wedge B = 0 \) for \( p > d = \text{dim} M \).

\(^7\)This terminology is borrowed from [11] (see App. B 5.3). However, the twisting by a Maurer-Cartan element we use here is different from the one in [11].
Property 3 in theorem 1 implies that the quasi-isomorphism $U^f$ (59) maps the Maurer-Cartan element $B$ of $(\Omega(W, T_{\text{poly}}), d, [\cdot]_{SN})$ to $B$, which is viewed as a Maurer-Cartan element of the DGLA $(\Omega(W, D_{\text{poly}}), d + \partial, [\cdot])$. Therefore a quasi-isomorphism from the DGLA $(\Omega(W, T_{\text{poly}}), D, [\cdot]_{SN})$ to the DGLA $(\Omega(W, D_{\text{poly}}), D + \partial, [\cdot])$ can be obtained by a twisting the quasi-isomorphism $U^f$ with the help of the Maurer-Cartan element $B$. Namely, the formula

$$U(X) = \exp((-B) \wedge U^f(\exp(B \wedge)X), \quad \forall X \in C_\bullet(\Omega(W, T_{\text{poly}})[1])$$

(61)

gives a quasi-isomorphism from the DGLA $(\Omega(W, T_{\text{poly}}), D, [\cdot]_{SN})$ to the DGLA $(\Omega(W, D_{\text{poly}}), D + \partial, [\cdot])$. Notice that $B$ which stands to the right of $U^f$ is viewed as an element of $\Omega(W, T_{\text{poly}})$ and $B$ which stands to the left of $U^f$ is viewed as an element of $\Omega(W, D_{\text{poly}})$.

Thus we have constructed a quasi-isomorphism $U$ from the DGLA $(\Omega(W, T_{\text{poly}}), D, [\cdot]_{SN})$ to the DGLA $(\Omega(W, D_{\text{poly}}), D + \partial, [\cdot])$ for an arbitrary coordinate disk $W$ on the manifold $M$. Remarkably, it turns out that the quasi-isomorphism $U$ does not depend on a choice of local coordinates on $W$ and hence we have the following proposition.

**Proposition 3** Formula (61) defines a quasi-isomorphism from the DGLA $(\Omega(M, T_{\text{poly}}), D, [\cdot]_{SN})$ to the DGLA $(\Omega(M, D_{\text{poly}}), D + \partial, [\cdot])$.\]

**Proof.** To prove the assertion we observe that the $n$-th structure map $U_n$ of the quasi-isomorphism (61) looks as follows

$$U_n(v_1, \ldots, v_n) = U^f_n(v_1, \ldots, v_n) + \sum_{m \geq 1} \frac{1}{m!} U^f_{n+m}(B, \ldots, B, v_1, \ldots, v_n), \quad (62)$$

where the sum over $m$ is finite since $B \wedge B \wedge \ldots \wedge B = 0$ for $p > d = \dim M$.

Due to property 4 in theorem 1 and transformation law (58) for $B$ the map $U_n$ does not depend on a choice of local coordinates and therefore $U$ is indeed defined as a quasi-isomorphism from the DGLA $(\Omega(M, T_{\text{poly}}), D, [\cdot]_{SN})$ to DGLA $(\Omega(M, D_{\text{poly}}), D + \partial, [\cdot])$. \hfill $\Box$

Composing (51) with $U$ we get a quasi-isomorphism

$$U : T_{\text{poly}}(M) \sim (\Omega(M, D_{\text{poly}}), D + \partial, [\cdot]). \quad (63)$$

In the following subsection we use (63) in order to get a quasi-isomorphism from $T_{\text{poly}}(M)$ to $D_{\text{poly}}(M)$.

### 4.2 Contraction of $U$ to a quasi-isomorphism from $T_{\text{poly}}(M)$ to $D_{\text{poly}}(M)$

In the previous subsection we used a twisting of two DG Lie algebras and a quasi-isomorphism between them. Here we are going to modify the quasi-isomorphism (63) between the DG Lie algebras $T_{\text{poly}}(M)$ and $(\Omega(M, D_{\text{poly}}), D + \partial, [\cdot])$ without changing the algebras themselves. We start with a description of the modification in a general setting.

Let $(\mathfrak{h}_1, d_1, [\cdot], _1)$ and $(\mathfrak{h}_2, d_2, [\cdot], _2)$ be two DG Lie algebras. As in section 2 we associate to $\mathfrak{h}_1$ and $\mathfrak{h}_2$ coassociative cocommutative coalgebras $C_\bullet(\mathfrak{h}_1[1])$ and $C_\bullet(\mathfrak{h}_2[1])$ with nilpotent
coderivations $Q_1$ and $Q_2$, induced by the DGLA structures of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Let $F$ be a quasi-isomorphism from $\mathfrak{h}_1$ to $\mathfrak{h}_2$.

The statement we are going to use here can be formulated as

**Proposition 4** Let

$$F_m : \wedge^m \mathfrak{h}_1 \mapsto \mathfrak{h}_2[1-m]$$

be the structure maps of the quasi-isomorphism $F$ and $n$ be any natural number $n \geq 1$. Then it is possible to construct a quasi-isomorphism $\tilde{F}$ from $\mathfrak{h}_1$ to $\mathfrak{h}_2$ whose structure maps are

$$
\begin{align*}
\tilde{F}_m(\gamma_1, \ldots, \gamma_m) &= F_m(\gamma_1, \ldots, \gamma_m), & \text{if } 1 \leq m < n, \\
\tilde{F}_n(\gamma_1, \ldots, \gamma_n) &= F_n(\gamma_1, \ldots, \gamma_n) + d_2 V_n(\gamma_1, \ldots, \gamma_n) - \\
\sum_{i=1}^n (-1)^{l_1 + \ldots + l_{i-1}} V_n(\gamma_1, \ldots, d_1 \gamma_i, \ldots, \gamma_n) \\
\tilde{F}_m(\gamma_1, \ldots, \gamma_m) &= F_m(\gamma_1, \ldots, \gamma_m) + W_m(\gamma_1, \ldots, \gamma_m), & \text{otherwise,}
\end{align*}
$$

(64)

where $\gamma_i \in \mathfrak{h}_1^{k_i}$, $V_n$ is an arbitrary polylinear map

$$V_n : \wedge^n \mathfrak{h}_1 \mapsto \mathfrak{h}_2[-n]$$

and the maps

$$W_m : \wedge^m \mathfrak{h}_1 \mapsto \mathfrak{h}_2[1-m]$$

are expressed in terms of $V_n$ and structure maps $F_m$ via the differentials $d_1$, $d_2$ and the brackets $[,]_1$ and $[,]_2$.

**Proof.** We claim that the desired quasi-isomorphism $\tilde{F}$ can be found in the following form

$$\tilde{F}(X) = F(X) + Q_2 V(X) + V(Q_1 X), \quad \forall X \in C_\bullet(\mathfrak{h}_1[1]),$$

(65)

where $V$ is a linear map from $C_\bullet(\mathfrak{h}_1[1])$ to $C_\bullet(\mathfrak{h}_2[1])$ which satisfies the following relation

$$\Delta_2 V(X) = (F \otimes V + V \otimes F + \frac{1}{2}(V \otimes Q_2 V + Q_2 V \otimes V + \frac{1}{2}(V \otimes V Q_1 + V Q_1 \otimes V))(\Delta_1 X),$$

(66)

$$\forall X \in C_\bullet(\mathfrak{h}_1[1]),$$

$\Delta_1$ and $\Delta_2$ denote comultiplications in $C_\bullet(\mathfrak{h}_1[1])$ and $C_\bullet(\mathfrak{h}_2[1])$, respectively.

The compatibility of the homomorphism (65) with the nilpotent coderivations $Q_1$ and $Q_2$ follows directly from the definition while the compatibility with the comultiplications $\Delta_1$ and $\Delta_2$ follows from equations (66).

Notice that equations (66) simply mean that the linear map $V$ is defined by structure maps $V_m$ ($m \geq 1$), which are arbitrary polylinear maps

$$V_m : \wedge^m \mathfrak{h}_1 \mapsto \mathfrak{h}_2[-m].$$

(67)

For example, for any $\gamma \in \mathfrak{h}_1$

$$V(\gamma) = V_1(\gamma),$$

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and for any pair $\gamma_1 \in \mathfrak{h}_1^{k_1}, \gamma_2 \in \mathfrak{h}_1^{k_2}$

$$V(\gamma_1 \wedge \gamma_2) = V_2(\gamma_1, \gamma_2)$$

$$+(F_1(\gamma_1) \wedge V_1(\gamma_2) + \frac{1}{2} V_1(\gamma_1) \wedge d_2 V_1(\gamma_2) + \frac{1}{2} V_1(\gamma_1) \wedge V_1(d_1 \gamma_2) - (-)^{k_1 k_2}(\gamma_1 \leftrightarrow \gamma_2)) .$$

Using (66) in this way one can easily express the linear map $V$ in terms of the structure maps (67).

Thus the first two structure maps of the shifted homomorphism $\tilde{F}$ takes the form

$$\tilde{F}_1(\gamma) = F_1(\gamma) + d_2 V_1(\gamma) + V_1(d_1 \gamma) ,$$

$$\tilde{F}_2(\gamma_1, \gamma_2) = F_2(\gamma_1, \gamma_2) + ([F_1(\gamma_1), V_1(\gamma_2)]_2 + [V_1(\gamma_1), d_2 V_1(\gamma_2)]_2$$

$$+[V_1(\gamma_1), V_1(d_1 \gamma_2)]_2 - (-)^{k_1 k_2}(\gamma_1 \leftrightarrow \gamma_2)) - V_1([\gamma_1, \gamma_2]_1) ,$$

where $\gamma \in \mathfrak{h}_1, \gamma_1 \in \mathfrak{h}_1^{k_1}$ and $\gamma_2 \in \mathfrak{h}_1^{k_2}$. The map $\tilde{F}_1$ obviously establishes an isomorphism of the spaces of cohomologies $H^*(\mathfrak{h}_1, d_1)$ and $H^*(\mathfrak{h}_2, d_2)$ since the map $F_1$ does.

It is clear that the linear map $V$ with a single non-vanishing structure map

$$V_n : \wedge^n \mathfrak{h}_1 \mapsto \mathfrak{h}_2[-n]$$

gives the desired quasi-isomorphism $\tilde{F}$ and the proposition follows. \(\square\)

Now we are going apply this proposition to the quasi-isomorphism $\mathcal{U} = U \circ U_T$ from the DGLA $(T_{poly}(M), 0, [], | \cdot |_{SN})$ to the DGLA $(\Omega(M, D_{poly}), D + \partial, [\cdot, \cdot])$ in order to contract it to a quasi-isomorphism $\mathcal{U}$ from the DGLA $(T_{poly}(M), 0, [], | \cdot |_{SN})$ to the DGLA $Z_D^0(\Omega(M, D_{poly}), \partial, [], [])$.

We formulate a precise statement as

**Proposition 5** One can construct a quasi-isomorphism $\mathcal{U}$ from the DGLA $(T_{poly}(M), 0, [], | \cdot |_{SN})$ to the DGLA $(\Omega(M, D_{poly}), D + \partial, [], [])$ such the structure maps $\mathcal{U}_n$ of $\mathcal{U}$ take values in $Z_D^0(\Omega(M, D_{poly}), \partial, [], [])$.

The proof of the proposition is based on the well-known technique of spectral sequences. Although the spectral sequence that appears here is of the simplest type some more care is needed in using this language since the maps we deal with should respect the Lie algebra structures as well. For this reason we give here a detailed proof.

**Proof.** We will proceed by induction in $n$.

**Base of induction.** For $n = 1$ we have

$$(D + \partial)\mathcal{U}_1(\gamma) = 0, \quad \forall \gamma \in T_{poly}(M) .$$

Let

$$\mathcal{U}_1(\gamma) = \sum_{q=0}^{d} \mathcal{U}_1^q(\gamma)$$

be a decomposition of $\mathcal{U}_1(\gamma)$ with respect to the exterior degree. Then due to (69) (or simply due to the top degree argument) the component $\mathcal{U}_1^d(\gamma)$ of the maximal exterior degree $d = \text{dim} M$ is $D$-closed

$$D\mathcal{U}_1^d(\gamma) = 0 .$$
Hence $U_1^d(\gamma)$ is $D$-exact and there exists a linear map $V_1^d : T_{\text{poly}}(M) \mapsto \Omega^{d-1}(M, D_{\text{poly}})$ such that

$$U(\gamma) + (D + \partial)V_1^d(\gamma)$$

has the maximal exterior degree $q_{\text{max}} < d$. Proceeding in this way we can construct a linear map $V_1 : T_{\text{poly}}(M) \mapsto \Omega(M, D_{\text{poly}})$ such that

$$U_1(\gamma) + (D + \partial)V_1(\gamma)$$

is of exterior degree zero. Using proposition 4 we perform the shift

$$U \mapsto \tilde{U} = U + Q_{\Omega(M, D_{\text{poly}})} \circ V + V \circ Q_{T_{\text{poly}}(M)},$$

where the linear map $V : C_\bullet(T_{\text{poly}}(M)[1]) \mapsto C_\bullet(\Omega(M, D_{\text{poly}})[1])$ has the only non-vanishing structure map $V_1$ constructed above. This yields a quasi-isomorphism $\tilde{U}$ from $T_{\text{poly}}(M)$ to $(\Omega(M, D_{\text{poly}}), D + \partial, [\cdot, \cdot])$ such that for any $\gamma \in T_{\text{poly}}(M)$ the element $\tilde{U}_1(\gamma)$ is of exterior degree zero. Moreover, due to the equation

$$(D + \partial)\tilde{U}_1(\gamma) = 0$$

$\tilde{U}_1(\gamma)$ belongs to $Z^0(\Omega(M, D_{\text{poly}}), D)$. Thus the base statement of the induction is proved.

**Step of induction.** Let us assume that for all $m < n$ the maps $U_m$ take values in $Z^0(\Omega(M, D_{\text{poly}}), D)$. For the structure map $U_n$ we have

$$(D + \partial)U_n(\gamma_1, \gamma_2, \ldots, \gamma_n) =$$

$$= \frac{1}{2} \sum_{k,l \geq 1, k+l=n} \frac{1}{k!l!} \sum_{\varepsilon \in \mathcal{S}_n} \pm [U_k(\gamma_{\varepsilon_1}, \ldots, \gamma_{\varepsilon_k}), U_l(\gamma_{\varepsilon_{k+1}}, \ldots, \gamma_{\varepsilon_{k+l}})] -$$

$$- \sum_{i \neq j} \pm U_{n-1}([\gamma_i, \gamma_j]_{\mathcal{S}_n}; \gamma_1, \ldots, \hat{\gamma_i}, \ldots, \hat{\gamma_j}, \ldots, \gamma_n), \quad \gamma_i \in T_{\text{poly}}^k(M).$$

By the assumption of induction the right hand side of equation (70) is of exterior degree zero. Hence, by reasoning as above, we conclude that there exists a polylinear map

$$V_n : \wedge^n T_{\text{poly}}(M) \mapsto \Omega(M, D_{\text{poly}})[-n],$$

such that for any $\gamma_1, \ldots, \gamma_n \in T_{\text{poly}}(M)$

$$U_n(\gamma_1, \ldots, \gamma_n) + (D + \partial)V_n(\gamma_1, \ldots, \gamma_n)$$

is of exterior degree zero. Therefore, due to proposition 4 $U$ can be shifted to a quasi-isomorphism $\tilde{U}$, whose structure map $\tilde{U}_n$ takes values in $\Omega^0(M, D_{\text{poly}})$. On the other hand $\tilde{U}_n$ should satisfy the same cohomological equation (70) as $U_n$ and hence $\tilde{U}_n$, in fact, takes values in $Z^0(\Omega(M, D_{\text{poly}}), D)$. This completes the proof of the proposition and the proof for theorem 4. The desired quasi-isomorphism

$$\mathcal{U} : T_{\text{poly}}(M) \sim D_{\text{poly}}(M)$$

is a composition of the quasi-isomorphism $\tilde{U}$ and the DGLA-isomorphism

$$\mu \circ \sigma : Z^0_D(\Omega(M, D_{\text{poly}}), \partial, [\cdot, \cdot]) \mapsto D_{\text{poly}}(M). \quad \Box$$
As one easily sees, the constructed quasi-isomorphism $\mathcal{U}$ from the DGLA $T_{\text{poly}}(M)$ to the DGLA $D_{\text{poly}}(M)$ depends on a choice of the affine torsion free connection $\nabla_i$. In the following section we will see that symmetries of the connection $\nabla_i$ determine symmetries of the respective quasi-isomorphism $\mathcal{U}$.

5 Equivariant formality theorem

In this section we consider a manifold $M$ equipped with a smooth action of a group $G$. Given this action on $M$ we can canonically extend it to an action of $G$ on the DGLA $T_{\text{poly}}(M)$ of polyvector fields and the DGLA $D_{\text{poly}}(M)$ of polydifferential operators. It naturally raises the question as to whether, there exists a $G$-equivariant formality quasi-isomorphism from the DGLA $T_{\text{poly}}(M)$ to the DGLA $D_{\text{poly}}(M)$. Using our construction of the quasi-isomorphism of formality we prove that

**Theorem 5** If a manifold $M$ is equipped with a smooth action of a group $G$ and $M$ admits a $G$-invariant torsion free connection $\nabla_i$ then one can construct a $G$-equivariant quasi-isomorphism from the DGLA $T_{\text{poly}}(M)$ to the DGLA $D_{\text{poly}}(M)$.

Before giving a proof for the theorem we mention some interesting corollaries.

First, provided the conditions of theorem 5 are satisfied we have a quasi-isomorphism between the respective DG Lie algebras of $G$-invariants

**Corollary 1** If a manifold $M$ is equipped with a smooth action of a group $G$ and $M$ admits a $G$-invariant torsion free connection $\nabla_i$ then one can construct a quasi-isomorphism from the DGLA $(T_{\text{poly}}(M))^G$ to the DGLA $(D_{\text{poly}}(M))^G$.

Second, given a smooth action of a finite or compact group $G$ one can always construct a $G$-invariant torsion free connection using the standard averaging procedure. Hence,

**Corollary 2** If a manifold $M$ is equipped with a smooth action of a finite or compact group $G$ then one can construct a $G$-equivariant quasi-isomorphism from the DGLA $T_{\text{poly}}(M)$ to the DGLA $D_{\text{poly}}(M)$.

Since a quasi-isomorphism between DG Lie algebras provides a one-to-one correspondence between the moduli spaces of Maurer-Cartan elements of the DG Lie algebras our procedure gives a solution for the deformation quantization problem of an arbitrary Poisson orbifold. Namely,

**Corollary 3** Given a smooth action of a finite group $G$ on a manifold $M$ and a $G$-invariant Poisson structure $\alpha \in (\wedge^2TM)^G$ one can always construct a $G$-invariant star-product $\ast$, corresponding to $\alpha$. Furthermore, $G$-invariant star-products on $M$ corresponding to the Poisson bracket $\alpha$ are classified up to equivalence by non-trivial $G$-invariant deformations of $\alpha$.

Notice that an existence of a star-product on an arbitrary Poisson orbifold follows in principle from the results of paper [2]. However, in order to get the above classification of star-products one has to use equivariant formality theorem 5.

Finally, if the group $G$ is neither compact nor finite the averaging procedure is not applicable. However, one can still get a $G$-equivariant quasi-isomorphism of formality provided the action of $G$ is “nice” enough, namely

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Corollary 4 If a manifold $M$ is equipped with a free action of a Lie group $G$ then one can construct a $G$-equivariant quasi-isomorphism from the DGLA $T_{poly}(M)$ to the DGLA $D_{poly}(M)$.

Proof of theorem 5. We are going to prove that if one uses a $G$-invariant torsion free connection $\nabla_i$ in the above construction of the quasi-isomorphism from the algebra $T_{poly}(M)$ of polyvector fields to the algebra $D_{poly}(M)$ of polydifferential operators then the resulting quasi-isomorphism is $G$-equivariant.

First, given a smooth action of a group $G$ on $M$ one can canonically extend the action to the spaces $\Omega(M, SM)$, $\Omega(M, T_{poly})$ and $\Omega(M, D_{poly})$ in such a way that the (super)commutative product and DGLA structures $(0, [\cdot, \cdot]_{SN})$ and $(\partial, [\cdot, \cdot])$ of $\Omega(M, SM)$, $\Omega(M, T_{poly})$ and $\Omega(M, D_{poly})$, respectively, are $G$-invariant. Second, since $G$ acts on fiber variables $y$ by linear transformations property 2 in theorem 1 implies that the fiberwise quasi-isomorphism (20) is $G$-equivariant.

Next, the differential $\delta$ (21) (22), (23) the projection $\sigma$ (24) as well as the homotopy operator $\delta^{-1}$ (25) are obviously $G$-invariant. Moreover, $G$-invariance of the torsion free connection $\nabla_i$ implies $G$-invariance of the derivation $\nabla$ (27) (29), (30) and $G$-invariance of the respective Riemann curvature tensor $(R_{ij})^{k}_{l}(x)$. Therefore the recurrent procedure (36) converges to a $G$-invariant element $A \in \Omega^1(M, T_{poly}) \subset \Omega^1(M, D_{poly})$ and hence the respective Fedosov differential (35) turns out to be $G$-invariant as well.

Since the map $\tau$ from $\mathcal{F}^0 D_{poly} (\mathcal{F}^0 T_{poly})$ to $\mathcal{Z}^0 (\Omega(M, D_{poly}), D) (\mathcal{Z}^0 (\Omega(M, T_{poly}), D))$ is defined by $G$-equivariant recurrent procedure (45) it is obviously $G$-invariant. By an analogous line of arguments we see that the map $\mu$ (46) from $\mathcal{F}^0 D_{poly} (\mathcal{F}^0 T_{poly})$ to $D_{poly}(M)$ ($T_{poly}(M)$) is also $G$-invariant and therefore quasi-isomorphisms (51) and (52) are both $G$-equivariant.

Thus it suffices to prove that twisting procedure (61) and the procedure of contraction respect the action of $G$. Since recurrent procedure (42) for solving cohomological equation (41) is $G$-equivariant there is nothing to do with the procedure of contraction. However, some work is required to prove the $G$-equivariance of twisting procedure (61).

To show that twisting procedure (61) is $G$-equivariant we consider the action $\pi(g)$ of $g \in G$ on the element

$$\mathfrak{P} = U_{n+m}^{f}(B, \ldots, B, \mathfrak{v}_1, \ldots, \mathfrak{v}_n) \in \Omega(M, D_{poly}),$$

(71)

where $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ are some elements of $\Omega(M, T_{poly})$. The quasi-isomorphism $\textsf{U}$ is $G$-equivariant if for any $g \in G$ we have

$$\pi(g)U_{n+m}^{f}(B, \ldots, B, \mathfrak{v}_1, \ldots, \mathfrak{v}_n) =$$

$$U_{n+m}^{f}(\pi(g)B, \ldots, \pi(g)B, \pi(g)\mathfrak{v}_1, \ldots, \pi(g)\mathfrak{v}_n),$$

(72)

where $\pi(g)$ acts on all the terms of sum (57) as on elements of $\Omega^1(M, T_{poly}^0)$ besides the term

$$\Gamma = -dx^i \Gamma^k_{ij}(x)y^j \frac{\partial}{\partial y^k},$$

(73)

on which $\pi(g)$ acts by transforming the Christoffel symbols $\Gamma^k_{ij}(x)$. 

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On the other hand $G$-equivariance of $U^f$ implies that

$$
\pi(g)U_{n+m}^f(B, \ldots, B, v_1, \ldots, v_n) = \sum_{m} U_{n+m}^f(\pi^{\text{tensor}}(g)B, \ldots, \pi^{\text{tensor}}(g)B, \pi(g)v_1, \ldots, \pi(g)v_n),
$$

(74)

where $\pi^{\text{tensor}}(g)$ acts on the whole sum (57) as on the element of $\Omega^1(M, T_{poly}^0)$. But the difference between $\pi^{\text{tensor}}(g)B$ and $\pi(g)B$

$$
\pi B - \pi^{\text{tensor}}(g)B = H(g).
$$

(75)

is a fiberwise polyvector field linear in the fiber variables $y$’s. Hence due to property 4 in theorem 1

$$
\sum_{m} U_{n+m}^f(\pi^{\text{tensor}}(g)B, \ldots, \pi^{\text{tensor}}(g)B, \pi(g)v_1, \ldots, \pi(g)v_n) = \sum_{m} U_{n+m}^f(\pi(g)B, \ldots, \pi(g)B, \pi(g)v_1, \ldots, \pi(g)v_n)
$$

and the theorem follows. □

6 Concluding remarks.

We conclude the paper by discussing possible generalizations and applications of the equivariant formality theorem.

Notice that if a Lie group $G$ does not act freely on the manifold $M$ the existence of a $G$-equivariant formality quasi-isomorphism cannot be guaranteed. However, using the formality quasi-isomorphism $\mathcal{U}$ one can construct a quasi-isomorphism $\mathcal{U}_g$ from the DGLA of co-chains $C^\bullet(g, T_{poly}(M))$ to the DGLA of co-chains $C^\bullet(g, D_{poly}(M))$, where $g$ is the Lie algebra of $G$. This quasi-isomorphism $\mathcal{U}_g$ is a natural generalization of the quasi-isomorphism of DG Lie algebras $(T_{poly}(M))^G$ and $(D_{poly}(M))^G$ of invariants. It is interesting to find relaxed conditions on the action of the Lie group $G$ that would allow us to contract the quasi-isomorphism $\mathcal{U}_g$ to a quasi-isomorphism of the algebras $(T_{poly}(M))^G$ and $(D_{poly}(M))^G$ of invariants.

An important application of equivariant quasi-isomorphisms of formality is related to quantization [12], [13], [14], [15], [16], [17] of classical dynamical $r$-matrices [18]. Despite numerous attempts that have been undertaken recently in this direction, the quantization problem of dynamical $r$-matrices remains unsolved even for the case of triangular dynamical $r$-matrices over an abelian base. At the moment only certain classes of dynamical $r$-matrices are known to admit quantization. Paper [12] shows that non-degenerate triangular dynamical $r$-matrices over an abelian base can be quantized with the help of the Fedosov method\(^8\) [9]. A class of the so-called completely degenerate dynamical $r$-matrices has been quantized in

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\(^8\)An interesting modification of the Fedosov method that leads to quantization of constant triangular $r$-matrices has been proposed in the paper [19]
paper [13] via the vertex-IRF transformation [20]. Interesting examples of dynamical r-
matrices over a non-abelian base found in works [18] and [21] have been recently quantized
in papers [16], [17].

In recent paper [15] Kontsevich’s formality theorem has been applied to quantization of
triangular dynamical r-matrices over an abelian base. Namely, in [15] it is shown that if a
triangular dynamical r-matrix over an abelian base satisfies a so-called affinization condition
then one may prove a version of equivariant formality theorem that gives a solution for the
quantization problem of this r-matrix.

We would like to mention that quantization of an arbitrary dynamical r-matrix can
be reduced to a version of a problem of invariant deformation quantization. However, in
general the action of the respective symmetry group is not free (in case of non-abelian base
it is not even regular). For this reason we suspect that there are even examples of triangular
dynamical r-matrices over an abelian base which cannot be quantized.

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