A Sharp Bound on the $s$-Energy

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Abstract

We derive an (essentially) optimal bound of $O(1/\rho s)^{n-1}$ on the $s$-energy of an $n$-agent averaging system, where $\rho$ is a lower bound on the nonzero weights. The $s$-energy is a generating function with applications to opinion dynamics, synchronization, consensus, bird flocking, inhomogeneous products of stochastic matrices, etc. We discuss a few of the improvements one can derive from the new bounds.

1 Introduction and Results

Let $(g_t)_{t=1}^\infty$ be an infinite sequence of graphs over a fixed vertex set $[n]$. Each $g_t$ is embedded in $[0,1]$, thus forming a graph system. The $s$-energy of the system is defined as $E(s) := \sum_{t>0} \ell_t^s$, where $\ell_t$ is the maximum edge length in the embedding of $g_t$. We focus on a family of multiagent averaging systems where each vertex moves within the convex hull of its neighbors. Each graph $g_t$ is assumed to be undirected and supplied with a self-loop at each vertex. To simplify the notation, fix $t > 0$ and denote by $x_i$ and $y_i$ the positions of vertex $i$ at times $t$ and $t+1$, respectively. For each $i \in [n]$, write $r(i) = \max\{x_j | (i,j) \in g_t\}$ and $l(i) = \min\{x_j | (i,j) \in g_t\}$.

Fix $\rho \in (0,1/2]$ independent of $t$. The move of vertex $i$ from $x_i$ to $y_i$ is subject to:

$$(1-\rho)x_{l(i)} + \rho x_{r(i)} \leq y_i \leq \rho x_{l(i)} + (1-\rho)x_{r(i)}. \tag{1}$$

The inequalities (1) are the only constraints imposed on the motion of vertex $i$ at time $t$. They apply to all the vertices $i \in [n]$ at all discrete times $t > 0$. Note that, given the graph sequence $g_t$, the inequalities (1) are satisfied if and only if $\ell_t \leq s$. The inequalities (1) are satisfied if and only if $\ell_t \leq s$. The inequalities (1) are satisfied if and only if $\ell_t \leq s$. The inequalities (1) are satisfied if and only if $\ell_t \leq s$. The inequalities (1) are satisfied if and only if $\ell_t \leq s$. The inequalities (1) are satisfied if and only if $\ell_t \leq s$.
Even though the 0-energy is unbounded, it may come as a surprise that $\mathcal{E}(s)$ is always finite for any $s > 0$. In particular, the case $s = 1$ shows that it takes only a finite amount of ink to draw the infinite sequence of graphs $g_t$. We state our main result and discuss its consequences and relation to previous work.

**Theorem 1.1.** The $s$-energy satisfies $\mathcal{E}(s) \leq 2^n(1/\rho s)^{n-1}$ for any $0 < s \leq 1$ and any $\rho \in (0, 1/2]$.

The bound is tight up to a constant factor in the exponent. This follows from the existence of graph systems for which $\mathcal{E}(s) \geq 1/(\rho s)^{O(n)}$ for any $0 < s \leq 1$. Our result improves on the previous upper bound of $(1/s)^{n-1}(1/\rho)^{n+O(1)}$ from [1] by reducing the exponent for $1/\rho$ from $n^2$ to the asymptotically optimal $n - 1$.

The $s$-energy is useful for estimating the convergence rate of certain multiagent dynamical systems without the need to make connectivity assumptions. To that end, given any $\varepsilon > 0$, we define the communication count $C_\varepsilon$ as the maximum number of steps $t$ such that $g_t$ has at least one edge of length greater than $\varepsilon$. Obviously, $\mathcal{E}(s) \geq \varepsilon^s C_\varepsilon$ for any $s > 0$. Setting $s = 1$ gives us the bound $C_\varepsilon \leq \frac{2}{\varepsilon} (2/\rho)^{n-1}$. This shows that $C_\varepsilon \leq 2^n(1/\rho)^{2n-1}$ for $\varepsilon \geq \rho^n$, which is known [1] to be the optimal asymptotic exponent for that range of $\varepsilon$. We derive a new bound for small values of $\varepsilon$, which we prove to be essentially optimal.

**Theorem 1.2.** The communication count satisfies $C_\varepsilon = O\left(\frac{1}{\rho} \log \frac{1}{\varepsilon}\right)^{n-1}$ for any $\rho \in (0, 1/2]$ and $\varepsilon \leq 2^{-n}$. Conversely, $C_\varepsilon = \Omega\left(\frac{1}{\rho} \log \frac{1}{\varepsilon}\right)^{n-1}$ for any $\rho \in (0, 1/3]$ and $\varepsilon \leq \rho^{2n}$.

This improves the previous upper bound of $(1/\rho)^{n+O(1)}(\log 1/\varepsilon)^{n-1}$ [1]. The lower bound is also new. We make a few additional observations about the results and their context:

1. The presence of a polylogarithmic factor $(\log 1/\varepsilon)^{n-1}$ in Theorem 1.2 is a distinctive feature of averaging over dynamic networks. If all the graphs $g_t$ were the same and the motion of the vertices obeyed a fixed convex combination, then the dynamics would be described by the products of a single stochastic matrix and the convergence rate would be proportional to $\log 1/\varepsilon$.

2. Our definition of the $s$-energy is a slight variant of the original formulation [1], which introduced the total $s$-energy as $E(s) = \sum_{t \geq 0} \sum_{i,j \in [n]} |x_i(t) - x_j(t)|^s$, where $x_i(t)$ is the position of vertex $i$ in the embedded (undirected) graph $g_t$. Another useful quantity is the kinetic $s$-energy:

$$K(s) = \sum_{t \geq 0} \sum_{i \in [n]} |x_i(t + 1) - x_i(t)|^s.$$ 

In view of the (obvious) relations, $\frac{1}{2} K(s) \leq E(s) \leq \binom{n}{2} E(s)$, Theorem 1.1 is directly applicable all of these variants as well.

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2The notation $l, r$ should not obscure the fact that both functions depend on the graph $g_t$ and its embedding $(x_i)_{t=1}^n$ at time $t$.

2All logarithms are to the base 2.
3. As noted in [3], the $s$-energy can be interpreted as a partition function with $s$ as the inverse temperature. It can also be viewed as a Dirichlet series $\sum_{t>0} e^{-s\ln \ell_t}$. This fact can be used to show that the $s$-energy forms a lossless encoding of the edge lengths that it tallies [1,5]. In some cases, the $s$-energy can be continued meromorphically over the whole complex plane but this cannot be done in general. The strongest results are obtained near the singularity $s = 0$, hence our focus on the interval $(0, 1]$. Sometimes we must content ourselves with higher values of $s$; for example, a recent analysis of inertial Hegselmann-Krause systems entails bounding the kinetic 2-energy [4].

2 Applications

We begin with a straightforward improvement for a simple gossip algorithm: given $\epsilon > 0$ and $n$ agents in $[0, 1]$, pick any two agents $i, j$ such that $x_j - x_i > \epsilon$ and move them closer to each other, specifically to $(1 - \rho)x_i + \rho x_j$ and $\rho x_i + (1 - \rho)x_j$, respectively. Repeat this until all the agents lie within an interval of length $\epsilon$. We conclude immediately from our result that this algorithm always terminates in time $O(\frac{1}{\rho} \log \frac{1}{\epsilon})^{n-1}$.

A number of convergence results published in the literature rely on the previous bound of $\rho^{-\frac{n^2-O(1)}{\rho}}(\log 1/\epsilon)^{n-1}$ for the communication count. Plugging in the new upper bound of $O(\frac{1}{\rho} \log \frac{1}{\epsilon})^{n-1}$ leads to immediate improvements on the convergence of systems in opinion dynamics, synchronization, consensus, bird flocking, inhomogeneous products of stochastic matrices, etc. [1,2,6–10]. These theoretical improvements follow from simply plugging in the new bounds into an existing formula, so there is little point going over them in any technical detail here. For the sake of illustration, however, we mention one of them: in a standard bird flocking model, it is shown in [2] that the flocking network can undergo $n^{O(n^2)}$ switches. Using the new bounds for the $s$-energy, the upper bound becomes $n^{O(n)}$. In general, the improvements we get will cut a bound of the form $(1/\rho)^{n^2}$ down to $(1/\rho)^n$.

3 Proving Theorem 1.1

The proof is algorithmic: we assign credits to the $n$ vertices and set up an economy to exchange credits and pay for the $s$-energy along the way. We fix $t > 0$ and focus the analysis on the transition from time $t$ to $t + 1$, i.e., from the positions $0 \leq x_1 \leq \ldots \leq x_n \leq 1$ to $y_1, \ldots, y_n$; note that the latter might not form a sorted sequence. For each $1 \leq i < j \leq n$, we maintain an account $B_{i,j}$ consisting of $(x_j - x_i)A^j - i$ pennies, where $A := 2/\rho s$ and one penny pays for one unit of $s$-energy.\footnote{You may use your favorite currency but it needs to allow for fractional amounts, such as 1.23 pennies. Throughout this paper, we write the shorthand “for each 1 \leq i < j \leq n” to mean “for each pair $(i, j)$ such that 1 \leq i < j \leq n.”} The account $B_{i,j}$ is always assigned to the ordered pair of vertices of ranks $i$ and $j$. We show that updating the accounts after moving the vertices at time $t$ leaves us with enough unused money to pay for the $s$-energy released at that step; in other words, we prove that the updating releases at least $\ell_t^s$ pennies, where $\ell_t$ is the maximum edge length in
the embedded graph \( g_i \).

No new money is needed past the initial injection at time 1; hence 
\[ E(s) \leq \sum_{i<j} A^{i,j} < \left( \frac{A}{A-1} \right)^{A_{i,j} - 1} < 2^a (1/\rho s)^{a-1}, \]
which proves Theorem 1.1.

It is convenient to break down the charging scheme at time \( t \) into a number of separate stages. Here is how to do that. Each one of the edges of the embedded graph \( g_i \) forms an interval in \([0,1]\). The union of the intervals of positive length consists of disjoint intervals 
\( I_1, \ldots, I_m \) \((m \geq 0)\) within which all the motion takes place: indeed, the vertices within \( I_0 \), for any given \( \alpha \in [m] \), cannot move outside of that interval at time \( t \). We can thus break down the motion into \( m \) stages, each one involving only the vertices within some \( I_0 \). This allows us to focus our analysis on a single stage, for example \( I_1 \). Let \( u < \cdots < v \) be the vertices of \( g_i \) in \( I_1 \). We modify the notation so that the corresponding stage involves the motion of each vertex \( i \in [u,v] \) from \( x_i \) to \( y_i \), where \( y_i = x_i \) for \( i \notin [u,v] \). The next inequality, derived from \( dz^s/dz \geq s \) for \( s, z \in (0,1) \), will prove useful:

\[ 1 - (1 - x)^s \geq sx \quad \text{for any } x \in [0,1]. \] (2)

Let \( \sigma \) be a permutation of \([n]\) such that \( y_{\sigma(1)} \leq \cdots \leq y_{\sigma(n)} \). Intuitively, \( \sigma(i) \) maps the vertex of rank \( i \) prior to stage 1 to the one of the same rank right after (ties can be broken arbitrarily). We prove that, once each account \( B_{i,j} \) has been updated to its new value \( B'_{i,j} \) at the completion of stage 1, we are left with at least \((x_v - x_u)^4 \) pennies, which is enough to cover the \( s\)-energy from the longest edge of \( g_i \) with vertices in \([u,v]\). So as to distinguish the proof from a mysterious magic trick, we provide some intuition via a few simple observations:

- We process the transition for \((i, j)\), i.e., from \( B_{i,j} \) to \( B'_{i,j} \), by considering the pairs \((i, j)\) in the order \((u, v)\), followed by \((u, v-1), (u+1, v)\), then by \((u, v-2), (u+1, v-1), (u+2, v)\), etc., and finally by \((u, u+1), \ldots, (v-1, v)\). Recall that \( B'_{i,j} \) is the account associated with the pair of vertices at positions \( y_{\sigma(i)} \) and \( y_{\sigma(j)} \) after stage 1. For the purposes of this preparatory sketch, we restrict ourselves to the case \( u \leq i < j \leq v \).

- In general, the transition for \((i, j)\) will benefit from money released just for that purpose by the pairs \((i-1, j)\) and \((i, j+1)\), whose accounts will have already been updated. This need will arise when the transition for \((i, j)\) requires an infusion of money because \( y_{\sigma(j)} - y_{\sigma(i)} > x_j - x_i \). Of course, to be consistent, the pair \((i, j)\) will then be expected to provide money for both \((i, j-1)\) and \((i+1, j)\): it will donate in two equal amounts.

- How much money should \((i, j)\) receive? At least \((x_v - x_u)^4 A^{i-1} - (x_j - x_i)^4 A^{j-1} \). This means that, supplied with the money already in \( B_{i,j} \) and what it receives from its donors, the pair \((i, j)\) should be able to stretch from the interval \([x_i, x_j]\) to \([x_u, x_v]\) and pay for it.

The actual interval associated with \( B'_{i,j} \), i.e., \([y_{\sigma(i)}, y_{\sigma(j)}]\) is strictly enclosed in \([x_u, x_v]\) and,

\footnote{It is worth pausing to consider why using a single account per vertex, as we did in \ref{1} for the case \( s = 1 \), would not work for \( s < 1 \). Set \( n = 3 \) and assign \( x_i A^i \) pennies to vertex \( i = 1, 2, 3 \). Initialize the system with \( x_1 = 0 \), \( x_2 = 1 - \varepsilon \), and \( x_3 = 1 \). Assume now that \( y_1 = 0 \) and \( y_2 = y_3 = 1 - \varepsilon/2 \). The account for vertex 3, the only one to release money, gives out only \((1 - (1 - \varepsilon/2)^3)A^1 = \frac{1}{2}s\varepsilon A^1\) pennies. If \( s < 1 \), this is not enough to cover the \( s\)-energy of \( e' \) needed for the first step. Observe that having an account for the pair \((2,3)\) would release \( e'A \) and easily solve that problem. The point is that all length scales matter; hence the need to keep accounts for all pairs \((i, j)\) for \( 1 \leq i < j \leq n \).}
by (1), we can bound how much smaller it must be. This implies that \((i, j)\) is handed too much money, so it can pass some of it down to \((i, j - 1)\) and \((i + 1, j)\), as required. Lemma 3.1 below is the key technical fact that puts numbers behind these ideas. To capture the money transfer from \((i - 1, j)\) to \((i, j)\), it is useful to relate \(y_i\) with \(y_{\sigma(i-1)}\) (and do the same with \(j\)). Since \(y_i\) is not directly accessible, we use a proxy called \(x_i^u\).

- Finally, we pay for the energetic contribution of stage 1 by spending the leftover money from the transition for \((u, u + 1)\), which so happens to be at least \((x_v - x_u)^u\), as desired.

We shall use the following relations without further reminder: \(u \leq l(i) \leq r(i) \leq v\) for all \(u \leq i \leq v\) and \(x_u \leq y_{\sigma(i)} \leq y_{\sigma(j)} \leq x_v\) for all \(u \leq i < j \leq v\). We write \(x_i^u := \rho x_i + (1 - \rho) x_m\).

**Lemma 3.1.** (a) \(x_i^u \leq y_{\sigma(i-1)}\) for all \(u \leq i \leq v\); and (b) \(x_i^u \geq y_{\sigma(i+1)}\) for all \(u \leq i < v\).

**Proof.** By symmetry, we may confine our attention to (a). By (1),
\[y_i \geq (1 - \rho) x_{l(i)} + \rho x_{r(i)} \geq x_{r(i)}^u \geq x_i^u.\]
Because \(x_i \in I_1\), there exists an edge \((j, k)\) such that \(u \leq j < i \leq k\). If \(j = u\) then \(y_j \geq x_{r(j)}^u \geq x_i^u\); else \(y_j \geq (1 - \rho) x_{l(j)} + \rho x_{r(j)} \geq x_k^u \geq x_i^u\). In both cases, \(y_j \geq x_i^u\) for at least one \(j (u \leq j < i)\). Since \(y_i \geq x_i^u \geq x_i^u\) for all \(l (i \leq l \leq v)\), it follows that \(|\{u \leq k \leq v \mid y_k \geq x_i^u\}| > v - i + 1\). □

The update of the accounts \(B_{i,j}\) proceed by downward induction on \(j - i\) beginning at \(n - 1\). To update \(B_{i,j}\), we use a credit worth \(C_{i,j}\) handed down to us inductively from the accounts \(B_{i-1,j}\) and \(B_{i,j+1}\), where, obviously, \(C_{1,0} = 0\). We show how this produces a leftover \(D_{i,j}\), which we can then use for future payments. Here are the details: for all \(1 \leq i < j \leq n\) in descending order of \(j - i = n - 1, \ldots, 1\), apply the following assignments:

\[
\begin{align*}
C_{i,j} & \leftarrow \frac{1}{2}(D_{i-1,j} + D_{i,j+1}) \\
D_{i,j} & \leftarrow B_{i,j} - B_{i,j}' + C_{i,j},
\end{align*}
\]

where \(B_{i,j} = (x_j - x_i)^s A^{j-i}\), \(B_{i,j}' = (y_{\sigma(j)} - y_{\sigma(i)})^s A^{j-i}\), and \(D_{i,j} = 0\) if \(i < 1\) or \(j > n\). The assignments denote transfers of money. This explains the factor of \(1/2\), which is required by the fact that, say, \(D_{i-1,j}\) is assigned to both \(C_{i,j}\) and \(C_{i-1,j-1}\): each one gets half the full allowance.

To prove that all accounts \(B_{i,j}\) can be updated during stage 1, it suffices to show that \(D_{i,j} \geq 0\).

For any \(i \in [n]\), define \(u(i) = u\) and \(v(i) = v\) if \(u \leq i \leq v\); and set \(u(i) = v(i) = i\) otherwise. We prove by induction on \(j - i > 0\) that

\[
\begin{align*}
B_{i,j} + C_{i,j} & \geq (x_{v(j)} - x_{u(i)})^s A^{j-i} \\
D_{i,j} & \geq 0.
\end{align*}
\]
Figure 1: Updating $B_{1,4}$ to its new value of $B'_{1,4}$ releases $D_{1,4}$ pennies, which are passed on evenly to the pairs (1,3) and (2,4). With this scheme in place, updating $B_{2,3}$ to $B'_{2,3}$ can make use of $C_{2,3} = \frac{1}{2}(D_{1,3} + D_{2,4})$ pennies.

- $u \leq i < j \leq v$: By affine invariance, we can always assume that $x_u = x_v = 1 = 0$. We begin with the case $u < i < j < v$ and observe that $v(j) = v$ and $u(i - 1) = u(i) = u$. By Lemma 3.1, $x_i^u \leq y_{\sigma(i-1)}$. Using the lower bound in (2), we derive

$$D_{i-1,j} = B_{i-1,j} + C_{i-1,j} - B'_{i-1,j} \geq ((x_{v(j)} - x_{u(i-1)})^y - (y_{\sigma(j)} - y_{\sigma(i-1)})^x)A^{j+1-i}$$

$$\geq (1 - (1 - x_j^y))A^{j+1-i} \geq x_i^u A^{j+1-i} = \rho s_i A^{j+1-i}.$$

By symmetry, $D_{i,j+1} \geq \rho s(1 - x_j)A^{j+1-i}$. It follows from (3) that $C_{i,j} \geq \frac{1}{2}\rho s(1 - (x_j - x_i))A^{i+1-j} \geq (1 - (j - x_i)^y)A^{j-i}$; therefore,

$$B_{i,j} + C_{i,j} \geq (x_j - x_i)^y A^{j-i} + (1 - (x_j - x_i)^y)A^{j-i} = A^{j-i} = (x_{v(j)} - x_{u(i)})^y A^{j-i},$$

hence (4). Note that the lower bounds on $D_{i-1,j}$ and $D_{i,j+1}$ we just derived still hold if $i = u$ or $j = v$, since they only state nonnegativity, which is known to hold by induction: the same argument can thus be used to show that (4) holds for all $u \leq i < j \leq v$. Since $x_u = x_v = 1 = 0$, this also proves that

$$D_{i,j} = B_{i,j} + C_{i,j} - B'_{i,j} \geq (x_{v(j)} - x_{u(i)})^y A^{j-i} - (y_{\sigma(j)} - y_{\sigma(i)})^x A^{j-i} \geq 0;$$

hence (5).

- $i < u \leq j \leq v$: This time, we set $x_i = 0$ and $x_v = 1$ and note that $u(i) = i$ and $v(j) = v$. We begin with the case $j < v$, which implies that $v(j + 1) = v$. Using, in this order, Lemma 3.1 and (2),

$$D_{i,j+1} = B_{i,j+1} + C_{i,j+1} - B'_{i,j+1} \geq ((x_{v(j+1)} - x_{u(i)})^y - (y_{\sigma(j+1)} - x_i)^y)A^{j+1-i}$$

$$\geq (1 - (x_j^y))A^{j+1-i} \geq (1 - (1 - \rho(1 - x_j)^y))A^{j+1-i} \geq \rho s(1 - x_j)A^{j+1-i}.$$(6)
Since \((x_v(j) - x_{u(i)})^s - (x_j - x_i)^s = 1 - x_j^s \leq 1 - x_j\) and, by induction, \(D_{i-1,j} \geq 0\), it follows that

\[B_{i,j} + C_{i,j} \geq B_{i,j} + \frac{1}{2} D_{i,j+1} \geq (x_j^s + (1 - x_j)) A^{i-j} \geq A^{i-j} = (x_v(j) - x_{u(i)})^s A^{i-j};\]

hence (4). For the case \(j = v\), again note that the lower bounds on \(D_{i-1,j}\) and \(D_{i,j+1}\) we used above still hold, so the rest of the proof can be recycled verbatim. This establishes (4) for all \(i < u \leq j \leq v\). Finally, \(y_{v(j)} \leq x_v\); hence \(x_{u(i)} = x_i = y_{v(i)} \leq y_{v(j)} \leq x_v = x_{v(j)}\), and (5) follows from (4).

The other cases are either trivial or mirror-images of the above. This proves that all the accounts \(B_{i,j}\) can be updated with the money currently in the system. Finally we pay for the energy contribution of stage 1 by tapping into \(D_{u,v+1}\), which is unused. For this to work, it suffices to show that \(D_{u,v+1} \geq (x_v - x_u)^s\). By Lemma 3.1,

\[y_{v(u+1)} - y_{v(u)} \leq x_v^u - x_{u+1}^u \leq \rho(x_v - x_{u+1}) + (1 - \rho)(x_v - x_u) \leq (1 - \rho)(x_v - x_u).\]

Thus it follows from (5), in that order, that

\[D_{u,v+1} = B_{u,v+1} + C_{u,v+1} - B'_{u,v+1} \geq A(x_{v(u+1)} - x_{u(v)})^s - A(y_{v(u+1)} - y_{v(u)})^s \geq (1 - (1 - \rho)^s)A(x_v - x_u)^s \geq \rho s A(x_v - x_u)^s \geq (x_v - x_u)^s,\]

which concludes the proof of Theorem 1.1.

\[\square\]

4 Proving Theorem 1.2

Given \(\rho \in (0, 1/2]\) and \(\varepsilon \leq 2^{-n}\), we can plug in \(s := (n - 1)/\log(1/\varepsilon) < 1\) into Theorem 1.1 and derive \(C_\varepsilon \leq \varepsilon^{-s} O(1/\rho s)^{n-1} = O(1/\rho s)^{n-1} = O(1/\rho s)^{n-1}\), which establishes the upper bound of Theorem 1.2.

To prove the lower bound, we strengthen the conditions a little and assume that \(\varepsilon \leq \rho^{2n}\) and \(\rho \leq 1/3\). We pick 1/3 for simplicity and note that any constant lower than 1/2 would work. The value \(\rho = 1/2\) cannot give us the claimed lower bound, however. To see why, consider the case of two vertices: quite clearly \(C_\varepsilon = 1\), which does not match the claimed bound of \(C_\varepsilon = \Omega(1/\rho s)^{n-1}\).

To prove the lower bound, we revisit an earlier construction of ours [1] and modify its analysis to fit our purposes. If \(n > 1\), the \(n\) vertices of \(g_1\) are positioned at 0, except for \(x_n = 1\). Besides the self-loops, the graph \(g_1\) has the single edge \((n - 1, n)\). At time 2, the vertices are all at 0 except for \(x_{n-1} = \rho\) and \(x_n = 1 - \rho\). The first \(n - 1\) vertices form a system that stays in place if \(n = 2\) and, otherwise, proceeds recursively within \([0, \rho]\) until it converges to the fixed point \(\rho/(n - 1)\): this value is derived from the fact that each step keeps the mass center invariant. After convergence of the vertices labeled 1 through \(n - 1\), the \(n\)-vertex system recurs within \([\rho/(n - 1), 1 - \rho]\). We have \(C(1, \varepsilon) = 0\) and, for \(n > 1\),

\[C(n, \varepsilon) \geq 1 + C\left(n - 1, \frac{\varepsilon}{\rho}\right) + C\left(n, \frac{\varepsilon}{1 - \rho(n/(n - 1))}\right).\]  

\[\text{\footnote{Technically, the \((n - 1)\)-vertex graph system never halts, so the main recursion must wait forever. We can use a limiting argument to terminate it in finite time without changing the conclusion of the analysis.}}\]
Note that the inequality holds only for $\varepsilon < 1$. By expanding the recurrence and using monotonicity,

$$C(n, \varepsilon) \geq k + k C\left(n - 1, \frac{\varepsilon}{\rho(1 - 2\rho)^{k-1}}\right), \quad \text{for } k = \left\lceil \frac{(\log \varepsilon)/n - \log \rho}{2\log(1 - 2\rho)} \right\rceil. \quad (8)$$

Assume now that $\varepsilon \leq \rho^{2n}$. From our choice of $k$, we easily verify that

$$\rho(1 - 2\rho)^{k-1} \geq \varepsilon^{1/n}. \quad (9)$$

Just as (7) necessitates that $\varepsilon < 1$, the recurrence (8) requires that $\varepsilon/(1 - 2\rho)^{k-1} < 1$, which follows from (9). Since $\varepsilon^{1/2n} \leq \rho \leq 1/3$, we have $k \geq \frac{b}{\rho n} \log \frac{1}{\varepsilon}$, for constant $b > 0$. It follows that $C(2, \varepsilon) = \Omega\left(\frac{1}{\rho} \log \frac{1}{\varepsilon}\right)$. For $n > 2$, it follows from (9) that

$$C(n, \varepsilon) \geq \left(\frac{b}{\rho n} \log \frac{1}{\varepsilon}\right) C(n - 1, \varepsilon^{1-1/n}).$$

We verify that the condition $\varepsilon \leq \rho^{2n}$ holds recursively: $\varepsilon^{1-1/n} \leq \rho^{2(n-1)}$. By induction, it follows that $C(n, \varepsilon) \geq \Omega\left(\frac{1}{\rho n} \log \frac{1}{\varepsilon}\right)^{n-1}$, which completes the proof of Theorem 1.2. □

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