All Ramsey \((C_n, K_6)\) critical graphs for large \(n\)

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Abstract
Let \(G\) and \(H\) be graphs without loops or multiple edges. If for any two-coloring of the edges of a complete graph \(K_n\), there is a copy of \(G\) in the first color, red, or a copy of \(H\) in the second color, blue, we will say \(K_n \rightarrow (G, H)\). The Ramsey number \(r(G, H)\) is defined as the smallest positive integer \(n\) such that \(K_n \rightarrow (G, H)\). Similarly, the star-critical Ramsey number \(r^*(G, H)\) is defined as the smallest positive integer \(k\) such that \(K_{r(G,H)-1} \sqcup K_{1,k} \rightarrow (G, H)\). In this paper, when \(n \geq 17\), we show that there exists exactly sixty seven non-isomorphic Ramsey critical \(r(C_n, K_6)\) graphs. Finally, we will also prove that \(r^*(C_n, K_6) = 4n - 2\) when \(n \geq 17\).

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1 Introduction
The Ramsey number \(r(G, H)\) is the smallest positive integer \(n\) such that \(K_n \rightarrow (G, H)\). Given a graph \(G = (V, E)\), we define the independence number, \(\alpha(G)\), as the size of
the largest independent set. Alternatively, Ramsey number \( r(G, K_m) \) can be viewed as the smallest positive integer \( n \) such that every graph \( H \) of order \( n \) either contains \( G \) as a subgraph or else satisfies \( \alpha(H) \geq m \). The classical Ramsey number \( r(s, t) \) is defined as \( r(K_s, K_t) \) and the diagonal Ramsey number is defined as the \( r(n, n) \). These numbers have been studied extensively in the last five decades. The exact determination of diagonal Ramsey number \( r(n, n) \), progresses rapidly in difficulty from the nearly trivial \( r(3, 3) = 6 \) to stubbornly resistant \( r(5, 5) \) [at present known to be between 43 and 48]. There are many interesting variations of the basic problem of finding classical numbers. One such variation is the star-critical Ramsey numbers introduced by Hook and Isaak in 2010 (see [5, 6]). Since \( K_n \sqcup K_1, n = K_{n+1} \setminus K_{1,n−k} \), the star-critical Ramsey number \( r^*(G, H) \) alternatively can be defined as the smallest positive integer \( k \) such that \( K_n \setminus K_{1,n−k−1} \to (G, H) \) where \( n = r(G, H) \). Several authors have studied \( r^*(G, H) \) for special pairs of graphs. A few such examples are trees versus complete graphs, stripes versus stripes, fans versus complete graphs, \( C_n \) vs \( K_3 \) and others (see [2, 4, 5, 6, 12]).

\section{Notation}

The complete graph on \( n \) vertices, a cycle on \( n \) vertices and a star on \( n + 1 \) vertices are denoted by \( K_n, C_n \) and \( K_{1,n} \) respectively. Given a graph \( G = (V, E) \), we say \( I \subseteq V \) is an independent set if no pair of vertices of \( I \) are adjacent to each other in \( G \). Equivalently, \( I \) forms a clique in \( G^c \). Consequently, \( \alpha(G) = \max\{|I| : I \text{ is an independent set of } G\} \). Given a graph \( G = (V, E) \) and a non-empty subset \( S \) of \( V \), the induced subgraph of \( S \) in \( G \) denoted by \( G[S] \) is defined as the subgraph obtained by deleting all the vertices of \( S^c \) from \( G \). Moreover, \( G \setminus S \) is define as \( G[V(G) \setminus S] \).

\section{Lemmas needed to generate Ramsey \( (C_n, K_6) \) critical graphs for \( n \geq 17 \)}

In an attempt to prove Bondy and Erdős conjecture \( r(C_n, K_m) = (n−1)(m−1) + 1 \) for all \((n, m) \neq (3, 3)\) satisfying \( n \geq m \geq 3 \) under certain restrictions, Schiermeyer has proved that \( r(C_n, K_6) = 5(n−1) + 1 \) for \( n \geq 6 \) (see [11, 10]). Characterizing all \( (C_n, K_6) \) Ramsey critical graphs boils down to finding all (red/blue) colorings of \( K_{r(C_n, K_6)−1} \) such that there is no red \( C_n \) or a blue \( K_6 \). Thus, equivalently it suffices for us to find all \( C_n \)-free graphs on \( K_{r(C_n, K_6)−1} \) vertices such that \( \alpha(G) < 6 \). In order to reach the main goals, we first prove that any \( C_n \)-free graph (where \( n \geq 17 \)) of order \( 5(n−1) \) with \( \alpha(G) \leq 5 \) contains \( 5K_{n−1} \).
Hence the result.

Lemma 1 ([7], Lemma 8). A \( C_n \)-free graph (where \( n \geq 7 \)) of order \( 4(n-1) \) with no independent set of 5 vertices contains an isomorphic copy of \( 4K_{n-1} \).

Lemma 2 is a direct consequence of [3], by Bollabás et al.

Lemma 2 Suppose \( G \) contains the cycle \( (u_1, u_2, ..., u_{n-1}, u_1) \) of length \( n-1 \) but no cycle of length \( n \). Let \( Y = V(G) \setminus \{u_1, u_2, ..., u_{n-1}\} \). If \( \alpha(G) = m-1 \) where \( m \leq \frac{n+2}{2} \) and \( \{x_1, x_2, ..., x_{m-1}\} \subseteq Y \) is an \((m-1)\)-element independent set. Then no member of this set is adjacent to \( m-2 \) or more vertices on the cycle.

The following lemma plays a pivotal role in proving the main results of this paper.

Lemma 3 A \( C_n \)-free graph (where \( n \geq 17 \)) of order \( 5(n-1) \) with no independent set of 6 vertices contains \( 5K_{n-1} \).

Proof. Suppose that \( G \) is a \( C_n \)-free graph on \( 5(n-1) \) vertices with no independent set of size 6. Since \( r(C_{n-1}, K_6) = 5n - 9 \leq 5(n-1) \) (see [3, 10]), there exists a cycle \( (u_1, u_2, ..., u_{n-1}, u_1) \) of length \( n-1 \) in \( G \). Let \( C = \{u_1, u_2, ..., u_{n-1}\} \). Define \( H \) as the induced subgraph of \( G \) not containing the vertices of the cycle. Then, \( |V(C)| = n-1 \) and \( |V(H)| = 4(n-1) \).

Suppose that there exists an independent set \( Y = \{y_1, y_2, y_3, y_4, y_5\} \) of size 5 in \( H \). That is, \( \alpha(G) = 5 \). From Lemma 2 (as \( 5 \leq \frac{n+2}{2} \)), it follows no vertex of \( Y \) is adjacent to four or more vertices of the \( C_{n-1} \). Thus, \( |E(I, V(C))| \leq 15 < n-1 \). That is, there is a vertex \( x \in V(C) \) adjacent to no vertices of \( I \). Therefore, \( I \cup \{x\} \) is a 6 element independent set, a contradiction. Hence, we can assume that \( H \) is a \( C_n \)-free graph of order \( 4(n-1) \) containing no independent set of order 5. By Lemma 1 we get that \( H \) contains a \( 4K_{n-1} \). Now consider any two vertices of \( V(C) \) say \( v \) and \( w \) such that \( (v, w) \notin E(G) \). In order to avoid a \( C_n \) both \( v \) and \( w \) can be adjacent to at most one vertex of each of the four copies of \( K_{n-1} \) in \( H \). Moreover, any vertex of any copy of \( K_{n-1} \) in \( H \) can be adjacent to at most one vertex of another copy of a \( K_{n-1} \) in \( H \). Thus, each copy of a \( K_{n-1} \) will have at most 5 vertices adjacent to some vertex outside that of \( K_{n-1} \), in \( V(H) \setminus \{v, w\} \). Since \( (n-1) - 5 \geq 1 \), we can select \( x_1 \) in the first \( K_{n-1} \), \( x_2 \) in the second \( K_{n-1} \), \( x_3 \) in the third \( K_{n-1} \) and \( x_4 \) in the fourth \( K_{n-1} \) such that \( \{x_1, x_2, x_3, x_4\} \) is an independent set of size four and no vertex of \( \{x_1, x_2, x_3, x_4\} \) is adjacent to any vertex of \( \{v, w\} \). Therefore \( \{x_1, x_2, x_3, x_4, v, w\} \) is an independent set of size 6, a contradiction. That is, \( (v, w) \in E(G) \). Since \( v, w \) are two arbitrary vertices of \( V(C_{n-1}) \) it follows that \( G[V(C_{n-1})] = K_{n-1} \) as required. Hence the result.
All Ramsey \((C_n, K_6)\) critical graphs for \(n \geq 17\)

We have already observed that any Ramsey \((C_n, K_6)\) critical graph will consist of a red graph containing \(5K_{n-1}\), with respect to the red/blue coloring. Label the vertex set of each of the five \(K_{n-1}\)’s by \(V_1, V_2, V_3, V_4\) and \(V_5\). By Lemma 3, there can be two types of Ramsey \((C_n, K_6)\) critical graphs. The first type of Ramsey \((C_n, K_6)\) critical graphs will satisfy the condition that at most one vertex of each \(V_i\) is adjacent to any other vertex in \(V_i^c\). The second type of Ramsey \((C_n, K_6)\) critical graphs will satisfy the condition that there exists a \(V_k\) for some \(1 \leq k \leq 5\) such that at least two vertices of \(V_k\) have neighbors in \(V_k^c\). Moreover, worth noting that such a critical graph is completely determined by the structure of the external edges between \(V_i\)’s and not by the \(\binom{5}{2}\) edges inside each of the five \(V_i\)’s. This fact is taken into consideration when representing the Ramsey \((C_n, K_6)\) critical graphs.

Each subgraph of \(K_5\) generates a unique Ramsey \((C_n, K_6)\) critical graph of type 1. Thus, as illustrated in the following figure, there are 34 critical graphs \((R_i\) where \(1 \leq i \leq 34\)) of type 1 generated by the 34 subgraphs of \(K_5\).
First note that each and every type 2 critical graph is obtained by an appropriate vertex splitting of some type 1 critical graph. As illustrated in the following figure, there are exactly 33 type 2 critical graphs (labeled $S_i$ where $1 \leq i \leq 33$) generated by 18 critical graphs of type 1. Note that exactly sixteen type 1 critical graphs do not generate single type 2 critical graph.
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Therefore, we can conclude that there are a total of 67 Ramsey \((C_n, K_6)\) critical graphs out of which 34 are of type 1 critical graphs (labeled \(R_i\) where \(1 \leq i \leq 34\)) and 33 are of type 2 critical graphs (labeled \(S_i\) where \(1 \leq i \leq 33\)).

5 MAIN RESULT

Theorem 1 If \(n \geq 17\), then \(r_*(C_n, K_6) = 5n - 3\).
Proof. To find a lower bound for $r_s(C_n, K_6)$, color the graph $K_{5(n-1)+1} \setminus K_{1,n-2}$, such that the red graph consists of a $4K_{n-1} \cup (K_{n-1} \cup K_{1,1})$ as illustrated in the following figure.

Figure 3. A red $C_n$-free coloring of $K_{5(n-1)+1} \setminus K_{1,n-2}$ which contains no blue $K_6$.

Hence, $K_{5(n-1)+1} \setminus K_{1,n-2} \not\rightarrow (C_n, K_6)$. Therefore, $r_s(C_n, K_6) \leq 5n - 3$.

Next to show that, $r_s(C_n, K_5) \leq 5n - 3$, assume that there exists a red $C_n$-free red/blue coloring of a graph $G = K_{5(n-1)+1} \setminus K_{1,n-3}$ that contains no blue $K_5$. Let $H$ be the graph obtained by deleting the vertex of degree $5n - 3$ (say $v$) from $G$ (i.e., $H = G \setminus v$).

Then, $H$ is a graph on $5(n - 1)$ vertices such that it contains no red $C_n$ or a blue $K_6$. Therefore, by lemma [3] we get that $H$ contains a red $5K_{n-1}$. Let $V_1, V_2, V_3, V_4$ and $V_5$ denote the sets of vertices of the five connected components of size $n - 1$. In order to avoid a red $C_n$, $v$ is adjacent to at most one red neighbor in each of the four sets $V_1, V_2, V_3, V_4$ and $V_5$. We use the prerogative of assuming that $V_5$ represent the connected component that has the least amount of vertices connected to $v$. Then as $V_5$ have at least 2 vertices adjacent to $v$, we can select $v_5 \in V_5$ such that it is adjacent to $v$ in blue. For each $1 \leq j \leq 4$, let $S_j = \bigcup_{i=1}^{4} V_i \setminus V_j$. Clearly, each vertex of $V_i$ ($1 \leq i \leq 4$) has at most 5 vertices that have at least one red neighbor in $G[S_i \cup \{v, v_5\}]$. As $n > 16$, we can conclude that each $V_i$ ($1 \leq i \leq 4$) will have at least one vertex having no red neighbors in $G[S_i \cup \{v, v_5\}]$. In other words, we can select $v_i \in V_i$ ($1 \leq i \leq 4$)
lying in the blue neighborhoods of \( v_5 \) and \( v \) such that \( \{v_1, v_2, v_3, v_4\} \) induce a blue \( K_4 \). Therefore, \( \{v_1, v_2, v_3, v_4, v_5, v\} \) will induce a blue \( K_6 \), a contradiction.

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