Optimality guarantees for distributed statistical estimation

John C. Duchi\textsuperscript{1}  Michael I. Jordan\textsuperscript{1,2}  Martin J. Wainwright\textsuperscript{1,2}  Yuchen Zhang\textsuperscript{1}

\textsuperscript{1}Department of Electrical Engineering and Computer Science
and \textsuperscript{2}Department of Statistics
University of California, Berkeley
\{jduchi,jordan,wainwrig,yuczhang\}@eecs.berkeley.edu

June 2014

Abstract

Large data sets often require performing distributed statistical estimation, with a full data set split across multiple machines and limited communication between machines. To study such scenarios, we define and study some refinements of the classical minimax risk that apply to distributed settings, comparing to the performance of estimators with access to the entire data. Lower bounds on these quantities provide a precise characterization of the minimum amount of communication required to achieve the centralized minimax risk. We study two classes of distributed protocols: one in which machines send messages independently over channels without feedback, and a second allowing for interactive communication, in which a central server broadcasts the messages from a given machine to all other machines. We establish lower bounds for a variety of problems, including location estimation in several families and parameter estimation in different types of regression models. Our results include a novel class of quantitative data-processing inequalities used to characterize the effects of limited communication.

1 Introduction

Rapid growth in the size and scale of datasets has fueled increasing interest in statistical estimation in distributed settings (for instance, see the papers \cite{8,35,11,13,26,2} and references therein). Modern data sets are often too large to be stored on a single computer, and so it is natural to consider methods that involve multiple computers, each assigned a smaller subset of the full dataset. Yet communication between machines or processors is often expensive, slow, or power-intensive; as noted by Fuller and Millett in a survey of the future of computing, “there is no known alternative to parallel systems for sustaining growth in computing performance,” yet the power consumption and latency of communication is often relatively high \cite{16}. Indeed, bandwidth limitations on network and inter-chip communication often impose significant bottlenecks on algorithmic efficiency. It is thus important to study the amount of communication required between machines or chips in algorithmic development, especially as we scale to larger and larger datasets.

The focus of the current paper is the communication complexity of a few classes of statistical estimation problems. Suppose we are interested in estimating some parameter $\theta(P)$ of an unknown distribution $P$, based on a dataset of $N$ i.i.d. observations. In the classical setting, one considers

\footnote{An extended abstract of this manuscript \cite{34} appeared in the 2013 conference on Advances in Neural Information Processing Systems.}
centralized estimators that have access to all \( N \) observations. One standard way to evaluate estimators is by studying minimax rates of convergence, which characterize the optimal (worst-case) performance over all centralized schemes. By way of contrast, in the distributed setting, one is given \( m \) different machines, and each machine is assigned a subset of the sample of size \( n = \lfloor \frac{N}{m} \rfloor \). Each machine may perform arbitrary operations on its own subset of data, and it then communicates results of these intermediate computations to the other processors or to a central fusion node. In this paper, we try to answer the following question: what is the minimal number of bits that must be exchanged in order to achieve (up to constant factors) the optimal estimation error realized by a centralized scheme?

While there is a rich literature on statistical minimax theory (e.g. [19, 33, 31, 30]), little of it characterizes the effects of limiting communication. In other areas, ranging from theoretical computer science [32, 1, 20], decentralized detection and estimation (e.g., [28, 27]), to information theory (e.g., [17, 15]), there is of course a substantial literature on communication complexity. Though related to these bodies of work, our problem formulation and results differ in several ways.

- In theoretical computer science [32, 1, 20], the prototypical problem is the distributed computation of a bivariate function \( \theta : \mathcal{X} \times \mathcal{Y} \to \Theta \), defined on two discrete sets \( \mathcal{X} \) and \( \mathcal{Y} \), using a protocol that exchanges bits between processors. The most classical problem is to find a protocol that computes \( \theta(x, y) \) correctly for all \( (x, y) \in \mathcal{X} \times \mathcal{Y} \), and exchanges the smallest number of bits to do so. More recent work studies randomization and introduces information-theoretic measures for communication complexity [3, 4, 5], where the problem is to guarantee that \( \theta(x, y) \) is computed correctly with high probability under a given (known) distribution \( P \) on \( x \) and \( y \). In contrast, our goal is to recover characteristics of an unknown distribution \( P \) based on observations drawn from \( P \). Though this difference is somewhat subtle, it makes work on communication complexity difficult to apply in our settings. However, lower bounds on the estimation of population quantities \( \theta(P) \) based on communication-constrained observations—including those we present here—do imply lower bounds in classical communication complexity settings.

- Work in decentralized detection and estimation also studies limits of communication. For example, Tsitsiklis and Luo [29] provide lower bounds on the difficulty of distributed convex optimization, and Luo and Tsitsiklis [24] study limits on certain distributed algebraic computations. In these problems, as in other early work in communication complexity, data held by the distributed parties may be chosen adversarially, which precludes conclusions about statistical estimation. Other work in distributed control provides lower bounds on consensus and averaging, but in settings where messages sent are restricted to be of particular smooth forms [27]. Study of communication complexity has also given rise to interesting algorithmic schemes; for example, Luo [23] considers architectures in which machines may send only a single bit to a centralized processor; for certain problems, he shows that if each machine receives a single one-dimensional sample, it is possible to achieve the optimal centralized rate to within constant factors.

- Information theorists have also studied problems of distributed estimation; for instance, see the paper of Han and Amari [17] for an overview. In particular, this body of work focuses on the problem of testing a hypothesis or estimating a parameter from samples \( \{(x_i, y_i)\}_{i=1}^{n} \) where \( \{(x_i)\}_{i=1}^{n} \) and \( \{(y_i)\}_{i=1}^{n} \) are correlated but stored separately in two machines. Han and Amari [17] study estimation error for encoding rates \( R > 0 \), or with sequences of rates \( R_n \) converging
to zero as the sample size $n$ increases. In contrast to these asymptotic formulations—which often allow more communication than is required to attain centralized (unconstrained) minimax rates in our settings—we study fixed bounds on rates (say of the form $R_n \leq c/n$) for finite sample sizes $n$, and we ask when it is possible to achieve the minimax statistical rate.

In this paper, we formulate and study two decentralized variants of the centralized statistical minimax risk, one based on protocols that engage in only a single round of message-passing, and the other based on interactive protocols that can use multiple rounds of communication. The main question of interest is the following: how must the communication budget $B$ scale as a function of the sample size $n$ at each machine, the total number of machines $m$, and the problem dimension $d$ so that the decentralized minimax risk matches the centralized version up to constant factors? For some problems, we exhibit an exponential gap between this communication requirement and the number of bits required to describe the problem or communicate its solution. To exhibit these gaps, we provide lower bounds using information-theoretic techniques, with the main novel ingredient being certain forms of quantitative data processing inequalities. We also establish (nearly) sharp upper bounds, some of which are based on recent work by a subset of current authors on practical schemes for distributed estimation (see Zhang et al. [35]).

The remainder of this paper is organized as follows. We begin in Section 2 with background on the classical minimax formalism, and then introduce two distributed variants of the minimax risk. Section 3 is devoted to the statement of our main results, as well as the discussion of their consequences in specific settings. We turn to the proofs of our main results in Section 4, deferring more technical results to the appendices, and we conclude in Section 5 with a discussion.

**Notation:** For a random variable $X$, we let $P_X$ denote the probability measure on $X$, so that $P_X(S) = P(X \in S)$, and we abuse notation by writing $p_X$ for the probability mass function or density of $X$, depending on the situation, so that $p_X(x) = P(X = x)$ in the discrete case and denotes the density of $X$ at $x$ when $p_X$ is a density. We use $\log$ to denote log-base $e$ and $\log_2$ for log in base 2. For discrete random variable $X$, we let $H(X) = -\sum_x p_X(x) \log p_X(x)$ denote the (Shannon) entropy (in ents), and for probability distributions $P, Q$ on a set $\mathcal{X}$, with densities $p, q$ with respect to a base measure $\mu$, we write the KL-divergence as

$$D_{\text{kl}}(P|Q) := \int_{\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} d\mu(x).$$

The mutual information $I(X;Y)$ between random variables $X$ and $Y$ where $Y$ has distribution $P_Y$ is defined as

$$I(X;Y) := \mathbb{E}_{P_X} \left[ D_{\text{kl}}(P_Y(\cdot|X)||P_Y(\cdot)) \right] = \int D_{\text{kl}}(P_Y(\cdot|X = x)||P_Y(\cdot)) dP_X(x).$$

We let $\lor$ and $\land$ denote maximum and minimum, respectively, so that $a \lor b = \max\{a, b\}$. For an integer $k \geq 1$, we use $[k]$ as shorthand for the set $\{1, \ldots, k\}$. We let $a_{1:n}$ be shorthand for a sequence $a_1, \ldots, a_n$, and the notation $a_n \geq b_n$ means there is a numerical constant $c > 0$ such that $a_n \geq cb_n$ for all $n$. Given a set $A$, we let $\sigma(A)$ denote the Borel $\sigma$-field on $A$. 3
2 Background and problem formulation

In this section, we begin by giving background on the classical notion of minimax risk in statistics. We then introduce two distributed variants of the minimax risk based on the notions of independent and interactive protocols, respectively.

2.1 Classical minimax risk

For a family of probability distributions \( \mathcal{P} \), consider a function \( \theta : \mathcal{P} \rightarrow \Theta \subseteq \mathbb{R}^d \). A canonical example throughout the paper is the mean function, namely \( \theta(P) = \mathbb{E}_P[X] \). Another simple example is the median \( \theta(P) = \text{med}_P(X) \), or more generally, quantiles of the distribution \( P \). Now suppose that we are given a collection of \( N \) observations, say \( X_{1:N} := \{X_1, \ldots, X_N\} \), drawn i.i.d. from some unknown member \( P \) of \( \mathcal{P} \). Based on the sample \( X_{1:N} \), our goal is to estimate the parameter \( \theta(P) \), and an estimator \( \hat{\theta} \) is a measurable function of the \( N \)-vector \( X_{1:N} \in \mathcal{X}^N \) into \( \Theta \).

We assess the quality of an estimator \( \hat{\theta} = \hat{\theta}(X_{1:N}) \) via its mean-squared error

\[
R(\hat{\theta}, \theta(P)) := \mathbb{E}_P[\| \hat{\theta}(X_{1:N}) - \theta(P) \|_2^2],
\]

where the expectation is taken over the sample \( X_{1:N} \). For an estimator \( \hat{\theta} \), the function \( P \rightarrow R(\hat{\theta}, \theta(P)) \) defines the risk function of \( \hat{\theta} \) over the family \( \mathcal{P} \). Taking the supremum all \( P \in \mathcal{P} \) yields the worst-case risk of the estimator. The minimax rate for the family \( \mathcal{P} \) is defined in terms of the best possible estimator for this worst-case criterion, namely via the saddle point criterion

\[
\mathcal{M}_N(\theta, \mathcal{P}) := \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} R(\hat{\theta}, \theta(P)), \tag{1}
\]

where the infimum ranges over all measurable functions of the data \( X_{1:N} \). Many papers in mathematical statistics study the classical minimax risk \( \mathcal{M}_N \), and its behavior is precisely characterized for a range of problems \[19, 33, 31, 30\]. We consider a few instances of such problems in the sequel.

2.2 Distributed protocols

The classical minimax risk \( \mathcal{M}_N \) imposes no constraints on the choice of estimator \( \hat{\theta} \). In this section, we introduce a refinement of the minimax risk that calibrates the effect of communication constraints. Suppose we have a collection of \( m \) distinct computers or processing units. Assuming for simplicity\(^2\) that \( N \) is a multiple of \( m \), we can then divide our full data set \( X_{1:N} \) into a family of \( m \) subsets, each containing \( n = \frac{N}{m} \) distinct observations, with \( X^{(i)} \) denoting the subset assigned to machine \( i \in [m] = \{1, \ldots, m\} \). With this set-up, our goal is to estimate \( \theta(P) \) via local operations at each machine \( i \) on the data subset \( X^{(i)} \) while performing a limited amount of communication between machines.

More precisely, our focus is a class of distributed protocols \( \Pi \), in which at each round \( t = 1, 2, \ldots \), machine \( i \) sends a message \( Y_{t,i} \) that is a measurable function of the local data \( X^{(i)} \) and potentially of past messages. It is convenient to model this message as being sent to a central fusion center. Let \( \bar{Y}_t = \{Y_{t,i}\}_{i \in [m]} \) denote the collection of all messages sent at round \( t \). Given a total of \( T \) rounds, the protocol \( \Pi \) collects the sequence \((\bar{Y}_1, \ldots, \bar{Y}_T)\), and constructs an estimator \( \hat{\theta} := \hat{\theta}(\bar{Y}_1, \ldots, \bar{Y}_T) \).

\(^2\)Although we assume in this paper that every machine has the same amount of data, our techniques are sufficiently general to allow for different sized subsets for each machine.
The length $L_{t,i}$ of message $Y_{t,i}$ is the minimal number of bits required to encode it, and the total length $L = \sum_{t=1}^{T} \sum_{i=1}^{m} L_{t,i}$ of all messages sent corresponds to the total communication cost of the protocol. Note that the communication cost is a random variable, since the length of the messages may depend on the data, and the protocol may introduce auxiliary randomness.

It is useful to distinguish two different protocol classes, namely independent versus interactive. An independent protocol $\Pi$ is based on a single round ($T = 1$) of communication in which machine $i$ sends a single message $Y_{1,i}$ to the fusion center. Since there are no past messages, the message $Y_{1,i}$ can depend only on the local sample $X^{(i)}$. Given a family $\mathcal{P}$, the class of independent protocols with budget $B \geq 0$ is

$$A_{\text{ind}}(B, \mathcal{P}) = \left\{ \text{independent protocols } \Pi \text{ such that } \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ \sum_{i=1}^{m} L_i \right] \leq B \right\}.$$ (For simplicity, we use $Y_{i}$ to indicate the message sent from processor $i$ and $L_{i}$ to denote its length in the independent case.) It can be useful in some situations to have more granular control on the amount of communication, in particular by enforcing budgets on a per-machine basis. In such cases, we introduce the shorthand $B_{1:m} = (B_1, \ldots, B_m)$ and define

$$A_{\text{ind}}(B_{1:m}, \mathcal{P}) = \left\{ \text{independent protocols } \Pi \text{ such that } \sup_{P \in \mathcal{P}} \mathbb{E}_P [L_i] \leq B_i \text{ for } i \in [m] \right\}.$$

In contrast to independent protocols, the class of interactive protocols allows for interaction at different stages of the message passing process. In particular, suppose that machine $i$ sends message $Y_{t,i}$ to the fusion center at time $t$, which then posts it on a “public blackboard,” where all machines may read $Y_{t,i}$ (this posting and reading incurs no communication cost). We think of this as a global broadcast system, which may be natural in settings in which processors have limited power or upstream capacity, but the centralized fusion center can send messages without limit. In the interactive setting, the message $Y_{t,i}$ is a measurable function of the local data $X^{(i)}$ and the past messages $\bar{Y}_{1:t-1}$. The family of interactive protocols with budget $B \geq 0$ is

$$A_{\text{inter}}(B, \mathcal{P}) = \left\{ \text{interactive protocols } \Pi \text{ such that } \sup_{P \in \mathcal{P}} \mathbb{E}_P [L] \leq B \right\}.$$

### 2.3 Distributed minimax risks

We can now define the distributed minimax risks that are the central objects of study in this paper. Our goal is to characterize the best achievable performance of estimators $\hat{\theta}$ that are functions of the vector of messages $\bar{Y}_{1:T} := (\bar{Y}_1, \ldots, \bar{Y}_T)$. As in the classical minimax setting, we measure the quality of a protocol $\Pi$ and estimator $\hat{\theta}$ by the mean-squared error

$$R(\hat{\theta}, \theta(P)) := \mathbb{E}_{P,\Pi} \left[ \| \hat{\theta}(\bar{Y}_{1:T}) - \theta(P) \|^2 \right],$$

where the expectation is now taken over the randomness in the messages, which is due to both their dependence on the underlying data as well as possible randomness in the protocol. Given a communication budget $B$, the minimax risk for independent protocols is

$$\mathfrak{M}^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B) := \inf_{\Pi \in A_{\text{ind}}(B, \mathcal{P})} \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} R(\hat{\theta}, \theta(P)). \quad (2)$$
Here, the infimum is taken jointly over all independent protocols \( \Pi \) that satisfy the budget constraint \( B \), and over all estimators \( \hat{\theta} \) that are measurable functions of the messages in the protocol. The minimax risk should also be understood to depend on both the number of machines \( m \) and the individual sample size \( n \) (we leave this implicit on the right hand side of definition \( 2 \)). We define the minimax risk for interactive protocols, denoted by \( M^{\text{inter}}_{n,m} \), analogously, where we instead take the infimum over the class of interactive protocols. These communication-dependent minimax risks are the central objects in this paper: they provide a sharp characterization of the optimal estimation rate as a function of the communication budget \( B \).

3 Main results and their consequences

We now turn to the statement of our main results, along with some discussion of their consequences. We begin with a rather simple bound based on the metric entropy of the parameter space; it confirms the natural intuition that any procedure must communicate at least as many bits as are required to describe a problem solution. We show that this bound is tight for certain problems, but our subsequent more refined techniques allow substantially sharper guarantees.

3.1 Lower bound based on metric entropy

We begin with a general but relatively naive lower bound that depends only on the geometric structure of the parameter space, as captured by its metric entropy. In particular, given a subset \( \Theta \subset \mathbb{R}^d \), we say \( \{\theta_1, \ldots, \theta^K\} \) are \( \delta \)-separated if \( \|\theta^i - \theta^j\|_2 \geq \delta \) for \( i \neq j \). We then define the packing entropy of \( \Theta \) as

\[
E_{\Theta}(\delta) := \log_2 \max \left\{ M \in \mathbb{N} \mid \{\theta_1, \ldots, \theta_M\} \subset \Theta \text{ are } \delta\text{-separated} \right\}.
\]

It is straightforward to see that the packing entropy is continuous from the right and non-increasing in \( \delta \), so that the inverse function \( E_{\Theta}^{-1}(B) := \sup\{\delta \mid E_{\Theta}(\delta) \geq B\} \) is well-defined. With this definition, we have the following claim:

**Proposition 1.** For any family of distributions \( \mathcal{P} \) and parameter set \( \Theta = \theta(\mathcal{P}) \), the interactive minimax risk is lower bounded as

\[
M^{\text{inter}}_{n,m}(\theta, \mathcal{P}, B) \geq \frac{1}{8} \left( E_{\Theta}^{-1}(2B + 2) \right)^2.
\]

We prove this proposition in Section 4.1. The same lower bound trivially holds for \( M^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B) \), as any independent protocol is a special case of an interactive protocol. Although Proposition 1 is a relatively generic statement, not exploiting any particular structure of the problem, it is in general unimprovable by more than constant factors, as the following example illustrates.

**Example 1** (Bounded mean estimation). Suppose our goal is to estimate the mean \( \theta = \theta(\mathcal{P}) \) of a class of distributions \( \mathcal{P} \) supported on the interval \([0, 1]\), so that \( \Theta = \theta(\mathcal{P}) = [0, 1] \). Suppose that a single machine \( (m = 1) \) receives \( n \) i.i.d. observations \( X_i \) according to \( \mathcal{P} \). The packing entropy has lower bound \( E_{\Theta}(\delta) \geq \log_2(1/\delta) \), and consequently, Proposition 1 implies that the distributed minimax risk is lower bounded as

\[
M^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B) \geq M^{\text{inter}}_{n,m}(\theta, \mathcal{P}, B) \geq \frac{1}{8} \left( 2^{-2B-2} \right)^2.
\]
Setting $B = \frac{1}{4}\log_2 n$ yields the lower bound $\mathfrak{M}^{\text{ind}}_{n,m}(\theta, \mathcal{P}([0,1]), B) \geq \frac{1}{128} n$.

This lower bound is sharp up to the constant pre-factor; it can be achieved by the following simple method. Given its $n$ observations, the single machine computes the sample mean $\bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$. The sample mean must lie in the interval $[0,1]$, and so can be quantized to accuracy $\frac{1}{n}$ using $\log_2 n$ bits, and this quantized version $\hat{\theta}$ can be transmitted. A straightforward calculation shows that $\mathbb{E}[(\hat{\theta} - \theta)^2] \leq \frac{2}{n}$, and Proposition I yields an order-optimal bound. 

### 3.2 Independent protocols in multi-machine settings

We would like to study how the budget $B$—the number of bits required to achieve the minimax rate—scales with the number of machines $m$. For our first set of results in this setting, we consider the non-interactive case, where each machine $i$ sends messages $Y_i$ independently of all the other machines. These results serve as pre-cursors to our later results on interactive protocols.

We first provide lower bounds for mean estimation in the $d$-dimensional normal location family model:

$$\mathcal{N}_d := \{ \mathcal{N}(\theta, \sigma^2 I_{d \times d}) \mid \theta \in \Theta = [-1,1]^d \}. \tag{3}$$

Here each machine receives an i.i.d. sample of size $n$ from a normal distribution $\mathcal{N}(\theta, \sigma^2 I_{d \times d})$ with unknown mean $\theta$. The following result provides a lower bound on the distributed minimax risk with independent communication:

**Theorem 1.** Given a communication budget $B_i$ for each machine $i = 1, \ldots, m$, there exists a universal (numerical) constant $c$ such that

$$\mathfrak{M}^{\text{ind}}_{n,m}(\theta, \mathcal{N}_d, B_1:m) \geq c \sigma^2 d \min \left\{ \frac{mn}{\sigma^2}, \frac{m \log m}{\log m \log m \log m}, \frac{m}{\left( \sum_{i=1}^{m} \min \{1, \frac{1}{2^i} \} \right) \log m} \right\}. \tag{4}$$

See Section 4.4 for the proof of this claim.

Given centralized access to the full $mn$-sized sample, the minimax rate for the mean-squared error is $\frac{\sigma^2 d}{mn}$ (e.g. Lehmann and Casella [22]). This optimal rate is achieved by the sample mean. Consequently, the lower bound (4) shows that each machine individually must communicate at least $\frac{d}{\log m}$ bits for a decentralized procedure to match the centralized rate. If we ignore logarithmic factors, this lower bound is achievable by the following simple procedure:

(i) First, each machine computes the sample mean of its local data and truncates it to the interval $[-1 - \frac{\sigma}{\sqrt{n}}, 1 + \frac{\sigma}{\sqrt{n}}]$.

(ii) Next, each machine quantizes each coordinate of the resulting estimate to precision $\frac{\sigma^2}{mn}$, using $O(1) d \log \frac{mn}{\sigma^2}$ bits to do so.

(iii) The machines send these quantized averages to the fusion center using $B = O(1) d m \log \frac{n}{\sigma^2}$ total bits.

(iv) Finally, the fusion center averages them, obtaining an estimate with mean-squared error of the order $\frac{\sigma^2 d}{mn}$. 

7
The techniques we develop also apply to other families of probability distributions, and we finish our discussion of independent communication protocols by presenting a result that gives lower bounds sharp to numerical constant prefactors. In particular, we consider mean estimation for the family $P_d$ of distributions supported on the compact set $[-1,1]^d$. One instance of such a distribution is the Bernoulli family taking values on the Boolean hypercube $\{-1,1\}^d$.

Proposition 2. Assume that each of $m$ machines receives a single observation ($n = 1$) from a distribution in $P_d$. There exists a universal constant $c > 0$ such that

$$\minind_{m,m}(\theta, P_{\mathcal{D}}, B_{1:m}) \geq c \frac{d}{m} \min \left \{ m, \frac{m}{\sum_{i=1}^{m} \min \{ 1, \frac{B_i}{d} \} } \right \},$$

where $B_i$ is the budget for machine $i$.

See Section 4.3 for the proof.

The standard minimax rate for $d$-dimensional mean estimation on $P_d$ scales as $d/m$, which is achieved by the sample mean. Proposition 2 shows that to achieve this scaling, we must have $\sum_{i=1}^{m} \min \{ 1, \frac{B_i}{d} \} \gtrsim m$, showing that each machine must send $B_i \gtrsim d$ bits. This lower bound is also achieved by a simple scheme:

(i) Each machine $i$ receives an observation $X_i \in [-1,1]^d$. Based on this observation, it generates a Bernoulli random vector $Z_i = (Z_{i1}, \ldots, Z_{id})$ with $Z_{ij} \in \{0,1\}$ taking the value 1 with probability $(1 + X_{ij})/2$, independently across coordinates.

(ii) Machine $i$ uses $d$ bits to send the vector $Z_i \in \{0,1\}^d$ to the fusion center.

(iii) The fusion center then computes the average $\hat{\theta} = \frac{1}{m} \sum_{i=1}^{m} (2Z_i - 1)$. This average is unbiased, and its expected squared error is bounded by $d/m$.

Note that for both the normal location family of Theorem 1 and the simpler bounded single observation model in Proposition 2 there is an exponential gap between the information required to describe the problem to the minimax mean squared error of $\frac{d}{mn}$—which scales as $\mathcal{O}(1)d \log(mn)$—and the number of bits that must be communicated, which scales nearly linearly in $m$. See also our discussion following Theorem 2.

3.3 Interactive protocols in multi-machine settings

Having provided results on mean estimation in the non-interactive setting, we now turn to the substantially harder setting of distributed statistical inference where feedback is permitted. As described in Section 2.2, in the interactive setting the fusion center may freely broadcast every message received to all other machines in the network. This freedom allows more powerful algorithms, rendering the task of proving lower bounds more challenging.

Let us begin by considering the uniform location family $U_d = \{ P_{\theta}, \theta \in [-1,1]^d \}$, where $P_{\theta}$ is the uniform distribution on the rectangle $[\theta_1 - 1, \theta_1 + 1] \times \cdots \times [\theta_d - 1, \theta_d + 1]$. For this problem, a direct application of Proposition 1 gives a nearly sharp result:

Proposition 3. Consider the uniform location family $U_d$ with $n$ i.i.d. observations per machine:
(a) There are universal (numerical) constants $c_1, c_2 > 0$ such that

$$M_{n,m}^{\text{inter}}(\theta, U, B) \geq c_1 \max \left\{ \exp \left( -c_2 \frac{B}{d} \right), \frac{d}{(mn)^2} \right\}.$$ 

(b) Conversely, given a budget of $B = d \left[ 2 \log_2(2mn) + \log(m)\left( \lceil \log_2 d \rceil + 2 \log_2(2mn) \right) \right]$ bits, there is a universal constant $c$ such that

$$M_{n,m}^{\text{inter}}(\theta, U, B) \leq c \frac{d}{(mn)^2}.$$ 

See Section 4.5 for the proof of this claim.

If each of the $m$ machines receives $n$ observations, we have a total sample size of $mn$, so the minimax rate over all centralized procedures scales as $d/(mn)^2$ (for instance, see Lehmann and Casella [22]). Consequently, Proposition 3(b) shows that the number of bits required to achieve the centralized rate has only logarithmic dependence on the number $m$ of machines. Part (a) shows that this logarithmic dependence on $m$ is unavoidable: at least $B \gtrsim d \log(mn)$ bits are necessary to attain the optimal rate of $d/(mn)^2$.

It is natural to wonder whether such logarithmic dependence holds more generally. The following result shows that it does not: for some problems, the dependence on $m$ must be (nearly) linear. In particular, we reconsider estimation in the normal location family model (3), showing a lower bound that is nearly identical to that of Theorem 1.

**Theorem 2.** For $i = 1, \ldots, m$, assume that each machine receives an i.i.d. sample of size $n$ from a normal location model (3), and that there is a total communication budget $B$. Then there exists a universal (numerical) constant $c$ such that

$$M_{n,m}^{\text{inter}}(\theta, N_d, B) \geq c \frac{\sigma^2 d}{mn} \min \left\{ \frac{mn}{\sigma^2}, \frac{m}{B/d + 1} \log m \lor 1 \right\}. \quad (5)$$

See Section 4.6 for the proof of this claim.

Theorem 2 is analogous to, but slightly weaker than, the corresponding lower bound from Theorem 1 for the non-interactive setting. In particular, the lower bound (5) shows that at least $B \gtrsim \frac{dm}{\log m}$ bits are required for any distributed procedure—even allowing fully interactive communication—to attain the centralized minimax rate. Thus, in order to achieve the minimax rate up to logarithmic factors, the total number of bits communicated must scale (nearly) linearly with the product of the dimension $d$ and number of machines $m$.

Moreover, these two theorems show that there is an exponential gap between the number of bits required to communicate the problem solution and the number required to compute it in a distributed manner. More specifically, assuming (for simplicity) that $\sigma^2 = 1$, describing a solution of the normal mean estimation problem to accuracy $\frac{d}{mn}$ in squared $\ell_2$-error requires at most $O(1)d \log(mn)$ bits. On the other hand, these two theorems show that nearly $dm$ bits must be communicated. This linear scaling in $m$ is dramatically different from—exponentially worse than—the logarithmic scaling for the uniform family. Establishing sharp communication-based lower bounds thus requires careful study of the underlying family of distributions.
Note that in both Theorems 1 and 2, the upper and lower bounds differ by logarithmic factors in the sample size $n$ and number of machines $m$. It would be interesting to close this minor gap. Another open question is whether the distributed minimax rates for the independent and interactive settings are the same up to constant factors, or whether their scaling actually differs in terms of these logarithmic factors.

3.4 Consequences for regression

The problems of mean estimation studied in the previous section, though simple in appearance, are closely related to other, more complex problems. In this section, we show how lower bounds on mean estimation can be used to establish lower bounds for distributed estimation in two standard but important generalized linear models [18]: linear regression and probit regression.

3.4.1 Linear regression

Let us begin with a distributed instantiation of linear regression with fixed design matrices. Concretely, suppose that each of $m$ machines has stored a fixed design matrix $A^{(i)} \in \mathbb{R}^{n \times d}$ and then observes a response vector $b^{(i)} \in \mathbb{R}^d$ from the standard linear regression model

$$b^{(i)} = A^{(i)} \theta + \varepsilon^{(i)}, \quad (6)$$

where $\varepsilon^{(i)} \sim N(0, \sigma^2 I_{n \times n})$ are independent noise vectors. Our goal is to estimate the unknown regression vector $\theta \in \Theta = [-1,1]^d$, identical for each machine. Our result involves the smallest and largest eigenvalues of the rescaled design matrices via the quantities

$$\lambda_{\max}^2 := \max_{i \in \{1, \ldots, m\}} \frac{\lambda_{\max}(A^{(i)})^T A^{(i)}}{n}, \quad \text{and} \quad \lambda_{\min}^2 := \min_{i \in \{1, \ldots, m\}} \frac{\lambda_{\min}(A^{(i)})^T A^{(i)}}{n} > 0. \quad (7)$$

**Corollary 1.** Given the linear regression model (6), there is a universal positive constant $c$ such that

$$\mathfrak{M}_{n,m}^{\text{inter}}(\theta, \mathcal{P}, B) \geq c \frac{\sigma^2}{\lambda_{\max}^2 mn} \min \left\{ \frac{\lambda_{\max}^2 mn}{\sigma^2}, \frac{m}{(B/d + 1) \log m} \lor 1 \right\}. \quad (8a)$$

Conversely, given a budget $B \geq dm \log(mn)$, there is a universal constant $c'$ such that

$$\mathfrak{M}_{n,m}^{\text{ind}}(\theta, \mathcal{P}, B_{1:m}) \leq c' \frac{\sigma^2}{\lambda_{\min}^2 mn}. \quad (8b)$$

It is a classical fact (e.g. [22]) that the minimax rate for $d$-dimensional linear regression scales as $d \sigma^2/(nm)$. Part (a) of Corollary 1 shows this optimal rate is attainable only if the total budget $B$ grows as $\frac{dm \log(m)}{\log m}$. Part (b) of the corollary shows that the minimax rate is achievable—even using an independent protocol—with budgets that match the lower bound to within logarithmic factors.

**Proof.** The upper bound [83] follows from the results of Zhang et al. [35]. Their results imply that the upper bound can be achieved by solving each regression problem separately, quantizing the (local) solution vectors $\hat{\theta}^{(i)} \in [-1,1]^d$ to accuracy $\frac{1}{mn}$ using $B_i = \lceil d \log_2(mn) \rceil$ bits and performing a form of approximate averaging.
In order to prove the lower bound [11], we show that solving an arbitrary Gaussian mean estimation problem can be reduced to solving a specially constructed linear regression problem. This reduction allows us to apply the lower bound from Theorem [2]. Given \( \theta \in \Theta \), consider the Gaussian mean model

\[
X^{(i)} = \theta + w^{(i)}, \quad \text{where} \quad w^{(i)} \sim N\left(0, \frac{\sigma^2}{\lambda_{\text{max}}^2 n} I_{d \times d}\right).
\]

Each machine \( i \) has its own design matrix \( A^{(i)} \), and we use it to construct a response vector \( b^{(i)} \in \mathbb{R}^n \). Since \( \lambda_{\text{max}}(A^{(i)} A^{(i)\top}/n) \leq \lambda_{\text{max}}^2 \), the matrix \( \Sigma^{(i)} := \sigma^2 I_{n \times n} - \frac{\sigma^2}{\lambda_{\text{max}}^2 n} A^{(i)}(A^{(i)})^\top \) is positive semidefinite. Consequently, we may form a response vector via

\[
b^{(i)} = A^{(i)} X^{(i)} + z^{(i)} = A^{(i)} \theta + A^{(i)} w^{(i)} + z^{(i)}, \quad z^{(i)} \sim N(0, \Sigma^{(i)}) \text{ independent of } w^{(i)}.
\]

The independence of \( w^{(i)} \) and \( z^{(i)} \) guarantees that \( b^{(i)} \sim N(A^{(i)} \theta, \sigma^2 I_{n \times n}) \), so the pair \((b^{(i)}, A^{(i)})\) is faithful to the regression model [6].

Now consider a protocol \( \Pi \in \mathcal{A}_{\text{inter}}(B, \mathcal{P}) \) that can solve any regression problem to within accuracy \( \delta \), so that \( \mathbb{E}[\|\hat{\theta} - \theta\|_2^2] \leq \delta^2 \). By the previously described reduction, the protocol \( \Pi \) can also solve the mean estimation problem to accuracy \( \delta \), in particular via the pair \((A^{(i)}, b^{(i)})\) described in expression [11]. Combined with this reduction, the corollary thus follows from Theorem [2].

### 3.4.2 Probit regression

We now turn to the problem of binary classification, in particular considering the probit regression model. As in the previous section, each of \( m \) machines has a fixed design matrix \( A^{(i)} \in \mathbb{R}^{n \times d} \), where \( A^{(i,k)} \) denotes the \( k \)-th row of \( A^{(i)} \). Machine \( i \) receives \( n \) binary responses \( Z^{(i)} = (Z^{(i,1)}, \ldots, Z^{(i,n)}) \), drawn from the conditional distribution

\[
\mathbb{P}(Z^{(i,k)} = 1 \mid A^{(i,k)}, \theta) = \Phi(A^{(i,k)} \theta) \quad \text{for some fixed } \theta \in \Theta = [-1, 1]^d,
\]

where \( \Phi(\cdot) \) denotes the standard normal CDF. The log-likelihood of the probit model [10] is concave (c.f. [7], Exercise 3.54). Under condition [7] on the design matrices, we have:

**Corollary 2.** Given the probit model [10], there is a universal constant \( c > 0 \) such that

\[
\mathcal{M}^{\text{inter}}_{n,m}(\theta, \mathcal{P}, B_{1:m}) \geq c \frac{d}{\lambda_{\text{max}}^2 mn} \min \left\{ \frac{\lambda_{\text{max}}^2 mn}{m}, \frac{m}{(B/d + 1) \log m} \right\}.
\]

Conversely, given a budgets \( B_i \geq d \log(mn) \), there is a universal constant \( c' \) such that

\[
\mathcal{M}^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B_{1:m}) \leq c' \frac{d}{\lambda_{\text{min}}^2 mn}.
\]

**Proof.** As in Corollary [1], the upper bound [11b] follows from the results of Zhang et al. [35].

Turning to the lower bound [11a], our strategy is to show that probit regression is at least as hard as linear regression, in particular by demonstrating that any linear regression problem can be solved via estimation in a specially constructed probit model. Given an arbitrary regression vector
θ ∈ Θ, consider a linear regression problem (6) with noise variance σ² = 1. We construct the binary responses for our probit regression (Z(1,1), . . . , Z(1,n)) by

\[ Z(i,k) = \begin{cases} 1 & \text{if } b(i,k) \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (12) \]

By construction, we have \( P(Z(i,k) = 1 | A(i), \theta) = \Phi(A(i,k)\theta) \) as desired for our model (10). By inspection, any protocol Π ∈ A_inter(B, P) solving the probit regression problem provides an estimator with the same mean-squared error as the original linear regression problem via the construction (12). Consequently, the lower bound (11a) follows from Corollary 1.

4 Proofs

We now turn to the proofs of our main results, deferring more technical results to the appendices.

4.1 Proof of Proposition 1

This result is based on the classical reduction from estimation to testing (e.g., [21, 19, 6]). For a given δ > 0, introduce the shorthand \( M = 2E_\Theta(2\delta) \) for the 2\( \delta \) packing number, and form a collection of points \{θ₁, . . . , θ₉\} that form a maximal 2\( \delta \)-packing of Θ. Now consider any family of conditional distributions \{P(. | ν), ν ∈ [M]\} such that θ(P(· | ν)) = θν.

Suppose that we sample an index \( V \) uniformly at random from \([M]\), and then draw a sample \( X \sim P(· | V) \). The associated testing problem is to determine the underlying instantiation of the randomly chosen index. Let \( Y = (Y₁, . . . , Y_T) \) denote the messages sent by the protocol Π, and let \( \hat{θ}(Y) \) denote any estimator of \( θ \) based on \( Y \). Any such estimator defines a testing function via

\[ \hat{V} := \arg\min_{ν \in V} \| \hat{θ}(Y) - θν \|₂. \]

Since \{θν\}ν∈V is a 2\( \delta \)-packing, we are guaranteed that \( \| \hat{θ}(Y) - θν \|₂ \geq \delta \) whenever \( \hat{V} \neq V \), whence

\[ \max_{ν \in V} E \left[ \| \hat{θ}(Y) - θν \|₂^2 \right] \geq \sum_{ν \in V} P(V = ν)E \left[ \| \hat{θ}(Y) - θV \|₂^2 | V = ν \right] \]

\[ \geq \sum_{ν \in V} δ^2 P(V = ν)P(\hat{V} \neq V | V = ν) = δ^2 P(\hat{V} \neq V). \quad (13) \]

It remains to lower bound the testing error \( P(\hat{V} \neq V) \). Fano’s inequality [10, Chapter 2] yields

\[ P(\hat{V} \neq V) \geq 1 - \frac{I(V; Y) + 1}{E_Θ(2\delta)}. \]

Finally, the mutual information can be upper bounded as

\[ I(V; Y) \leq H(Y) \leq B, \]

where inequality (i) is an immediate consequence of the definition of mutual information, and inequality (ii) follows from Shannon’s source coding theorem [10]. Combining inequalities (13) and (14) yields

\[ \gamma_{\min, m}(θ, P, B) \geq δ^2 \left\{ 1 - \frac{B + 1}{E_Θ(2\delta)} \right\} \quad \text{for any } δ > 0. \]
Because \(1 - \frac{B+1}{2^{|2B+2|}} \geq \frac{1}{2}\) for any any choice of \(\delta\) such that \(2\delta \leq \mathcal{E}_\Theta^{-1}(2B+2)\), setting \(\delta = \frac{1}{2} \mathcal{E}_\Theta^{-1}(2B+2)\) yields the claim.

4.2 A slight refinement

We now describe a slight refinement of the classical reduction from estimation to testing that underlies many of the remaining proofs. It is somewhat more general, since we no longer map the original estimation problem to a strict test, but rather a test that allows errors. We then leverage some variants of Fano’s inequality developed by a subset of the current authors [12].

Defining \(V = \{-1, +1\}^d\), consider an indexed family of probability distributions \(\{P(\cdot \mid \nu)\}_{\nu \in V} \subset P\). Each member of this family defines the parameter \(\theta_\nu := \theta(P(\cdot \mid \nu)) \in \Theta\). In particular, suppose that we construct the distributions such that \(\theta_\nu = \delta \nu\), where \(\delta > 0\) is a fixed quantity that we control.

For any \(\nu \neq \nu'\), we are then guaranteed that \(\|\theta_\nu - \theta_{\nu'}\|_2 = 2\delta \sqrt{d_{\text{ham}}(\nu, \nu')} \geq 2\delta\) where \(d_{\text{ham}}(\nu, \nu')\) is the Hamming distance between \(\nu, \nu' \in V\). This lower bound shows that \(\{\theta_\nu\}_{\nu \in V}\) is a special type of \(2\delta\)-packing, in that the squared \(\ell_2\)-distance grows proportionally to the Hamming distance between the indices \(\nu\) and \(\nu'\).

Now suppose that we draw an index \(V\) from \(V\) uniformly at random, then drawing a sample \(X\) from the distribution \(P(\cdot \mid V)\). Fixing \(t \geq 0\), the following lemma [12] reduces the problem of estimating \(\theta\) to finding a point \(\nu \in V\) within distance \(t\) of the random variable \(V\).

**Lemma 1.** Let \(V\) be uniformly sampled from \(V\). For any estimator \(\hat{\theta}\) and any \(t \geq 0\), we have

\[
\sup_{P \in P} \mathbb{E}[\|\hat{\theta} - \theta(P)\|_2^2] \geq \delta^2 (\lceil t \rceil + 1) \inf_{\hat{\nu}} P(d_{\text{ham}}(\hat{\nu}, V) > t),
\]

where the infimum ranges over all testing functions \(\hat{\nu}\) mapping the observations \(X\) to \(V\).

Setting \(t = 0\), we recover the standard reduction from estimation to testing as used in the proof of Proposition 1. The lemma allows for some additional flexibility in that it suffices to show that, for some \(t > 0\) to be chosen, it is difficult to identify \(V\) within a Hamming radius of \(t\). The following variant [12] of Fano’s inequality controls this type of error probability:

**Lemma 2.** Let \(V \to X \to \hat{V}\) be a Markov chain, where \(V\) is uniform on \(V\). For any \(t \geq 0\), we have

\[
P(d_{\text{ham}}(\hat{V}, V) > t) \geq 1 - \frac{I(V; X) + \log 2}{\log |V|},
\]

where \(N_t := \max_{\nu \in V} |\{\nu' \in V : d_{\text{ham}}(\nu, \nu') \leq t\}|\) is the size of the largest \(t\)-neighborhood in \(V\).

We thus have a clear avenue for obtaining lower bounds: constructing a large packing set \(V\) with (1) relatively small \(t\)-neighborhoods, and (2) such that the mutual information \(I(V; X)\) can be controlled. Given this set-up, the remaining technical challenge is the development of quantitative data processing inequalities, which allow us to characterize the effect of bit-constraints on the mutual information \(I(V; X)\). In general, these bounds are significantly tighter than the trivial upper bound used in the proof of Proposition 1. Examples of such inequalities in the sequel include Lemmas 3, 6, and 9.
4.3 Proof of Proposition 2

Given an index $\nu \in V$, suppose that each machine $i$ receives a $d$-dimensional sample $X^{(i)}$ with coordinates independently sampled according to

$$P(X_j = \nu_j \mid \nu) = \frac{1 + \delta \nu_j}{2} \quad \text{and} \quad P(X_j = -\nu_j \mid \nu) = \frac{1 - \delta \nu_j}{2}.$$ 

Note that by construction, we have $\theta_\nu = \delta \nu = E_\nu[X]$, as well as

$$\max_{x_j} \frac{P(x_j \mid \nu)}{P(x_j \mid \nu')} \leq \frac{1 + \delta}{1 - \delta} = e^\alpha \quad \text{where} \quad \alpha := \log \frac{1 + \delta}{1 - \delta}. \quad (15)$$

Moreover, note that for any pair $(i, j)$, the sample $X^{(i)}_j$, when conditioned on $V_j$, is independent of the variables $\{X^{(i)}_{j'} : j' \neq j\} \cup \{V_{j'} : j' \neq j\}$.

Recalling that $Y_i$ denotes the message sent by machine $i$, consider the Markov chain $V \rightarrow X^{(i)} \rightarrow Y_i$. By the usual data processing inequality \cite{10}, we have $I(V; Y_i) \leq I(X^{(i)}; Y_i)$. The following result is a quantitative form of this statement, showing how the likelihood ratio bound (15) causes a contraction in the mutual information.

Lemma 3. Under the preceding conditions, we have

$$I(V; Y_i) \leq 2(e^{2\alpha} - 1)^2 I(X^{(i)}; Y_i).$$

See Appendix B.1 for the proof of this result. It is similar in spirit to recent results of Duchi et al. \cite{14, Theorems 1–3}, who establish quantitative data processing inequalities in the context of privacy-preserving data analysis. Our proof, however, is different, as we have the Markov chain $V \rightarrow X \rightarrow Y$, and instead of a likelihood ratio bound on the channel $X \rightarrow Y$ as in the paper \cite{14}, we place a likelihood ratio bound on $V \rightarrow X$.

Next we require a certain tensorization property of the mutual information, valid in the case of independent protocols:

Lemma 4. When $Y_i$ is a function only of $X^{(i)}$, then

$$I(V; Y_{1:m}) \leq \sum_{i=1}^m I(V; Y_i).$$

See Appendix B.2 for a proof of this claim.

We can now complete the proof of the proposition. Using Lemma 3 we have

$$I(V; Y_i) \leq 2 \left( e^{2\log \frac{1 + \delta}{1 - \delta}} - 1 \right)^2 I(X^{(i)}; Y_i) = 2 \left( \frac{(1 + \delta)^2}{(1 - \delta)^2} - 1 \right)^2 \leq 80\delta^2 I(X^{(i)}; Y_i),$$

valid for $\delta \in [0, 1/5]$. Applying Lemma 4 yields

$$I(V; Y_{1:m}) \leq \sum_{i=1}^m I(V; Y_i) \leq 80\delta^2 \sum_{i=1}^m I(Y_i; X^{(i)}).$$

The remainder of the proof is broken into two cases, namely $d \geq 10$ and $d < 10$. 

14
Case \(d \geq 10\): By the definition of mutual information, we have
\[
I(Y_i; X^{(i)}) \leq \min \{H(Y_i), H(X^{(i)})\} \leq \min \{B_i, d\},
\]
where the final step follows since \(H(X^{(i)}) \leq d\) and \(H(Y_i) \leq B_i\), the latter inequality following from Shannon’s source coding theorem [10]. Putting together the pieces, we have
\[
I(V; Y_1:m) \leq 80\delta^2 \sum_{i=1}^m \min \{B_i, d\}.
\]
Combining this upper bound on mutual information with Lemmas 1 and 2 yields the lower bound
\[
\mathcal{M}_{n,m}^{\text{ind}}(\theta, \mathcal{P}, B_{1:m}) \geq \delta^2 (\lfloor d/6 \rfloor + 1) \left(1 - \frac{80\delta^2 \sum_{i=1}^m \min \{B_i, d\} + \log 2}{d/6}\right).
\]
The choice \(\delta^2 = \min \{1/25, d/960 \sum_{i=1}^m \min \{B_i, d\}\}\) guarantees that the expression inside parentheses in the previous display is lower bounded by 2/25, which completes the proof for \(d \geq 10\).

Case \(d < 10\): In this case, we make use of Le Cam’s method instead of Fano’s method. More precisely, by reducing to a smaller dimensional problem, we may assume without loss of generality that \(d = 1\), and we set \(V = \{-1, 1\}\). Letting \(V\) be uniformly distributed on \(V\), the Bayes error for binary hypothesis testing is (e.g. [32, 30, Chapter 2])
\[
\inf_{\hat{\nu}} \mathbb{P}(\hat{\nu} \neq V) = \frac{1}{2} - \frac{1}{2} \|P_1 - P_{-1}\|_{TV}.
\]
As \(\theta_\nu = \delta \nu\) by construction, the reduction from estimation to testing in Lemma 2 implies
\[
\inf_{\widehat{\theta}} \max_{P \in \{P_1, P_{-1}\}} \mathbb{E}[\|\hat{\theta} - \theta(P)\|_2^2] \geq \delta^2 \left(\frac{1}{2} - \frac{1}{2} \|P_1 - P_{-1}\|_{TV}\right).
\]
Finally, as we show in Appendix B.3 we have the following consequence of Pinsker’s inequality:
\[
\|P_Y(\cdot | V = \nu) - P_Y(\cdot | V = \nu')\|_{TV}^2 \leq 2I(Y; V).
\]
Thus
\[
\mathcal{M}_{n,m}^{\text{ind}}(\theta, \mathcal{P}, B_{1:m}) \geq \delta^2 \left(\frac{1}{2} - \frac{1}{2} \sqrt{2I(V; Y_1:m)}\right).
\]
Arguing as in the previous case \((d \geq 10)\), we have the upper bound \(I(X^{(i)}; Y_i) \leq \min \{B_i, 1\}\), and hence
\[
\mathcal{M}_{n,m}^{\text{ind}}(\theta, \mathcal{P}, B_{1:m}) \geq \delta^2 \left[\frac{1}{2} - 7 \left(\delta^2 \sum_{i=1}^m \min \{B_i, 1\}\right)^{1/2}\right].
\]
Setting \(\delta^2 = \min \left\{\frac{1}{25}, \frac{1}{400 \sum_{i=1}^m \min \{B_i, 1\}}\right\}\) completes the proof.
4.4 Proof of Theorem 1

This proof follows a similar outline to that of Proposition 2. We assume that the sample $X^{(i)}$ at machine $i$ contains $n_i$ independent observations from the multivariate normal distribution, and we will use the fact that $n_i \equiv n$ at the end of the proof, demonstrating that the proof technique is sufficiently general to allow for different sized subsets in each machine. We represent the $i$th as a $d \times n_i$ matrix $X^{(i)} \in \mathbb{R}^{d \times n_i}$. We use $X^{(i,k)}$ and $X^{(i)}_{j}$ to denote, respectively, the $k$th column and $j$th row of this matrix. Throughout this argument, we assume that $m \geq 5$; otherwise, Proposition 1 provides a stronger result.

As in the previous section, we consider a testing problem in which the index $V \in \{-1, +1\}^d$ is drawn uniformly at random. Our first step is to provide a quantitative data processing inequality analogous to Lemma 3, but which applies in somewhat more general settings. To that end, we abstract a bit from our current setting, and consider a model such that for any $(i, j)$, we assume that given $V_j$, the $j$th row $X^{(i)}_j$ is conditionally independent of all other rows $\{X^{(i)}_{j'} : j' \neq j\}$ and all other packing indices $\{V_{j'} : j' \neq j\}$. In addition, letting $P_{X_j}$ denote the probability measure of $X^{(i)}_j$, we assume that there exist measurable sets $G_j \subset \text{range}(X^{(i)}_j)$ such that

$$\sup_{S \in \sigma(G_j)} \frac{P_{X_j}(S \mid V = \nu)}{P_{X_j}(S \mid V = \nu')} \leq \exp(\alpha),$$

Let $E_j$ be a $\{0, 1\}$-valued indicator variable for the event $X^{(i)}_j \in G_j$ (i.e. $E_j = 1$ iff $X^{(i)}_j \in G_j$, and we leave the indexing on $i$ implicit). We have the following bound:

**Lemma 5.** Under the conditions stated in the preceding paragraph, we have

$$I(V; Y_i) \leq 2(e^{4\alpha} - 1)^2 I(X^{(i)}; Y_i) + \sum_{j=1}^d H(E_j) + \sum_{j=1}^d P(E_j = 0).$$

See Appendix C.1 for the proof of this claim.

Our next step is to bound the terms involving the indicator variables $E_j$. Fixing some $\delta > 0$, for each $\nu \in \{-1, 1\}^d$ define $\theta_\nu = \delta \nu$, and conditional on $V = \nu \in \{-1, 1\}^d$, let $X^{(i,k)}$, $k = 1, \ldots, n_i$, be drawn i.i.d. from a $N(\theta_\nu, \sigma^2 I_{d \times d})$ distribution. The following lemma applies to any pair of non-negative numbers $(a, \delta)$ such that

$$\max_{i \in [m]} \frac{\sqrt{n_i}a\delta}{\sigma^2} \leq \frac{1}{4} \quad \text{and} \quad a \geq \delta \max_{i \in [m]} \sqrt{n_i}. \quad (18)$$

It also involves the binary entropy function $h_2(p) := -p \log_2(p) - (1 - p) \log_2(1 - p)$.

**Lemma 6.** For any pair $(a, \delta)$ satisfying condition (18), we have

$$I(V; Y_i) \leq \frac{dn_i\delta^2}{\sigma^2}, \quad \text{and} \quad (19a)$$

$$I(V; Y_i) \leq 128\frac{\delta^2a^2}{\sigma^4}n_iH(Y_i) + d h_2(p^*_i) + dp^*_i, \quad (19b)$$

where $p^*_i := \min \left\{ 2 \exp \left( -\frac{(a-\sqrt{m})^2}{2\sigma^2} \right), \frac{1}{2} \right\}$.

With the bounds (19a) and (19b) on the mutual information $I(Y_i; V)$, we may now divide our proof into two cases: when $d < 10$ and $d \geq 10$. 

16
Case $d \geq 10$: In this case, we require an additional auxiliary result, which we prove via Lemma 6. (See Appendix C.3 for the proof of this claim.)

**Lemma 7.** For all $\delta \in \left[0, \frac{\sigma^2}{16} (\log m \max_i n_i)^{-\frac{1}{2}} \right]$, we have

$$\sum_{i=1}^{m} I(V; Y_i) \leq \delta^2 \sum_{i=1}^{m} \frac{n_i}{\sigma^2} \min \left\{ 128 \cdot 16 \log m \cdot H(Y_i), d \right\} + d \left( \frac{2}{49} + 2 \cdot 10^{-5} \right). \quad (20)$$

Combining the upper bound (20) on the mutual information with the minimax lower bounds in Lemmas 1 and 2, and noting that $6(2/49 + 2 \cdot 10^{-5}) + 6 \log 2/d \leq 2/3$ when $d \geq 10$ yields the following minimax bound:

$$\mathfrak{m}^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B_{1:m}) \geq \frac{\delta^2}{6} \left( \frac{1}{2} - \frac{6\delta^2 \sum_{i=1}^{m} n_i \min \{128 \cdot 16 \log m \cdot H(Y_i), d\}}{d\sigma^2} \right). \quad (21)$$

Using this result, we now complete the proof of the theorem. By Shannon’s source coding theorem, we have $H(Y_i) \leq B_i$, whence the minimax bound (21) becomes

$$\delta^2 \left( \frac{1}{2} + 1 \right) \left( \frac{1}{3} - \frac{6\delta^2 \sum_{i=1}^{m} n_i \min \{128 \cdot 16B_i \log m, d\}}{d\sigma^2} \right).$$

In particular, if we choose

$$\delta^2 = \min \left\{ 1, \frac{\sigma^2}{16^2 \max_i n_i \log m \cdot 36 \sum_{i=1}^{m} n_i \min \{128 \cdot 16B_i \log m, d\}} \right\}, \quad (22)$$

we obtain

$$\frac{1}{3} - \delta^2 \frac{6\sum_{i=1}^{m} n_i \min \{128 \cdot 16B_i \log m, d\}}{d\sigma^2} \geq \frac{1}{6},$$

which yields the minimax lower bound

$$\mathfrak{m}^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B_{1:m}) \geq \frac{1}{6} \left( \frac{1}{2} + 1 \right) \min \left\{ 1, \frac{\sigma^2}{16^2 \max_i n_i \log m \cdot 36 \sum_{i=1}^{m} n_i \min \{128 \cdot 16B_i \log m, d\}} \right\}.$$  

To obtain inequality (11), we simplify by assuming that $n_i \equiv n$ for all $i$ and perform simple algebraic manipulations, noting that the minimax lower bound $d\sigma^2/(nm)$ holds independently of any communication budget.

Case $d < 10$: As in the proof of Proposition 2, we cover this case by reducing to dimension $d = 1$ and applying Le Cam’s method, in particular via the lower bound (17). Substituting in the $\delta^2$ assignment (22) and the relation $H(Y_i) \leq B_i$ into Lemmas 4 and 7, we find that

$$I(V; Y_{1:m}) \leq \sum_{i=1}^{m} I(V; Y_i) \leq \frac{1}{36} + \frac{2}{49} + 2 \cdot 10^{-5} < \frac{1}{8}.$$  

Applying Le Cam’s method to this upper bound implies the lower bound $\mathfrak{m}^{\text{ind}}_{n,m}(\theta, \mathcal{P}, B_{1:m}) \geq \delta^2/4$, which completes the proof.
4.5 Proof of Proposition 3

Proposition 3 involves both a lower and upper bound. We prove the upper bound by exhibiting a specific interactive protocol \( \Pi^* \), and the lower bound via an application of Proposition 1.

Proof of lower bound: Applying Proposition 1 requires a lower bound on the packing entropy of \( \Theta = [-1,1]^d \). By a standard volume argument, the 2\( \delta \)-packing entropy has lower bound

\[
\mathcal{E}_\Theta(2\delta) \geq \log_2 \frac{\text{Volume}(\Theta)}{\text{Volume}(\{ x \in \mathbb{R}^d : \|x\|_2 \leq 2\delta \})} \geq d \log \left( \frac{1}{2\delta} \right).
\]

Inverting the relation \( B = \mathcal{E}_\Theta(\delta) = \mathcal{E}_\Theta(1/(mn)) \) yields the lower bound.

Proof of upper bound: Consider the following communication protocol \( \Pi^* \in \mathcal{A}_{\text{inter}}(B, \mathcal{P}) \):

(i) Each machine \( i \in [m] \) computes its local minimum \( a_j^{(i)} = \min\{X_j^{(i,k)} : k \in [n]\} \) for each coordinate \( j \in [d] \).

(ii) Machine 1 broadcasts the vector \( a^{(1)} \), where each of its components is quantized to accuracy \((mn)^{-2}\) in \([-2,2]\), rounding down, using \( 2d \log_2(2mn) \) bits. Upon receiving the broadcast, all machines initialize global minimum variables \( s_j \leftarrow a_j^{(1)} \) for \( j = 1,\ldots,d \).

(iii) In the order \( i = 2,3,\ldots,m \), machine \( i \) performs the following operations:

   (i) Find all indices \( j \) such that \( a_j^{(i)} < s_j \), calling this set \( J_i \). For each index \( j \in J_i \), machine \( i \) updates \( s_j \leftarrow a_j^{(i)} \), and then broadcasts the list of indices \( J_i \) (which requires \( |J_i| \log_2 d \) bits) and the associated values \( s_j \), using a total of \( |J_i| \log_2 d + 2|J_i| \log(2mn) \) bits.

   (ii) All other machines update their local vectors \( s \) after receiving machine \( i \)'s update.

(iv) One machine outputs \( \hat{\theta} = s + 1 \).

Using the protocol \( \Pi^* \) above, it is clear that for each \( j \in [d] \) we have computed the global minimum

\[
s_j^* = \min \{ X_j^{(i,k)} : i \in [m], k \in [n] \}
\]

to within accuracy \( 1/(mn)^2 \) (because of quantization). As a consequence, classical convergence analyses (e.g. [22]) yield that the estimator \( \hat{\theta} = s + 1 \) achieves the minimax optimal convergence rate \( \mathbb{E}[\|\hat{\theta} - \theta\|_2^2] \leq c \frac{d}{(mn)^2} \), where \( c > 0 \) is a numerical constant.

It remains to understand the communication complexity of the protocol \( \Pi^* \). To do so, we study steps 2 and 3. In Step 2, machine 1 sends a \( 2d \log_2(2mn) \)-bit message as \( Y_1 \). In Step 3, machine \( i \) sends \( |J_i|([\log_2 d] + 2 \log_2(2mn)) \) bits, that is, at most

\[
\sum_{j=1}^{d} 1 \left\{ a_j^{(i)} < \min\{a_j^{(1)}, \ldots, a_j^{(i-1)}\} \right\} ([\log_2 d] + 2 \log_2(2mn))
\]

bits, as no message is sent for index \( j \) if \( a_j^{(i)} \geq \min\{a_j^{(1)}, \ldots, a_j^{(i-1)}\} \). By inspection, this event happens with probability bounded by \( 1/i \), so we find that the expected length of message \( Y_i \) is

\[
\mathbb{E}[L_i] \leq \frac{d([\log_2 d] + 2 \log_2(2mn))}{i}.
\]
Putting all pieces together, we obtain that

$$
E[L] = \sum_{i=1}^{m} E[L_i] \leq 2d \log(2mn) + \sum_{i=2}^{m} \frac{d([\log_2 d] + 2 \log_2(2mn))}{i} \leq d[2 \log_2(2mn) + \log(m)(\log d + 2 \log_2(2mn))].
$$

4.6 Proof of Theorem 2

As in the proof of Theorem 1, we choose $V \in \{-1,1\}^d$ uniformly at random, and for some $\delta > 0$ to be chosen, we define the parameter vector $\theta := \delta V$. Suppose that machine $i$ draws a sample $X^{(i)} \in \mathbb{R}^{d \times n}$ of size $n$ i.i.d. according to a $N(\theta, \sigma^2 I_{d \times d})$ distribution. We denote the full sample—along dimension $j$ by $X_j$. In addition, for each $j \in [d]$, we let $V_j$ denote the coordinates of $V \in \{-1,1\}^d$ except the $j$th coordinate.

Although the local samples are independent, since we now allow for interactive protocols, the messages can be dependent: the sequence of random variables $Y = (Y_1, \ldots, Y_T)$ is generated in such a way that the distribution of $Y_t$ is $(X^{(i_t)}, Y_{t\neq t-1})$-measurable, where $i_t \in \{1, \ldots, m\}$ is the machine index upon which the message sending message $Y_t$. We assume without loss of generality that the sequence $\{i_1, i_2, \ldots, \}$ is fixed in advance: if the choice of index $i_t$ is not fixed but chosen based on $Y_{t-1}$ and $X$, we simply say there exists a default value (say no communication or $Y_t = -1$) that indicates “nothing” and has no associated bit cost.

To prove our result, we require an analogue of Lemma 5 (cf. the proof of Theorem 1). Assuming temporarily that $d = 1$, we prove our analogue for one-dimensional interactive protocols, and in the sequel, we show how it is possible to reduce multi-dimension problems to this statement. As in the proof of Theorem 1, we abstract a bit from our specific setting, instead assuming a likelihood ratio constraint, and provide a data processing inequality for our setting. Let $V$ be a Bernoulli variable uniformly distributed on $\{-1,1\}$, and let $P_{X^{(i)}}$ denote the probability measure of the $i$th sample $X^{(i)} \in \mathbb{R}^n$. Suppose there is a (measurable) set $G$ such that for any $\nu, \nu' \in \{-1,1\}$, we have

$$
\sup_{S \in \sigma(G)} \frac{P_{X^{(i)}}(S \mid \nu)}{P_{X^{(i)}}(S \mid \nu')} \leq e^\alpha.
$$

Finally, let $E$ be a $\{0,1\}$-valued indicator variable for the event $\cap_{i=1}^{m} \{X^{(i)} \in G\}$.

**Lemma 8.** Under the previously stated conditions, we have

$$
I(V; Y) \leq 2(e^{4\alpha} - 1)^2 I(X; Y) + H(E) + P(E = 0).
$$

See Appendix [D.1] for the proof.

Using this lemma as a building block, we turn to the case that $X^{(i)}$ is $d$-dimensional. Making an explicit choice of the set $G$, we obtain the following concrete bound on the mutual information. The lemma applies to any pair $(\alpha, \delta)$ of non-negative reals such that

$$
\frac{\sqrt{n}a\delta}{\sigma^2} \leq \frac{1}{4} \quad \text{and} \quad a \geq \delta \sqrt{n},
$$

and, as in Lemma 6, involves the binary entropy function $h_2(p) := -p \log(p) - (1 - p) \log(1 - p)$.
Lemma 9. Under the preceding conditions, we have

\[ I(V_j; Y \mid V_{\neg j}) \leq 128 \frac{\delta^2 n a^2}{\sigma^4} I(X_j; Y \mid V_{\neg j}) + m h_2(p^*) + mp^* \]

where \( p^* := \min \left\{ 2 \exp \left( -\frac{(a - \sqrt{n})^2}{2\sigma^2} \right), \frac{1}{2} \right\} \).

We prove the lemma in Section D.2.

To apply Lemma 9, we require two further intermediate bounds on mutual information terms. By the chain rule for mutual information, we have

\[ I(V; Y) = \sum_{j=1}^{d} I(V_j; Y \mid V_{1:j-1}) = \sum_{j=1}^{d} [H(V_j \mid V_{1:j-1}) - H(V_j \mid Y, V_{1:j-1})] = \sum_{j=1}^{d} [H(V_j \mid V_{\neg j}) - H(V_j \mid Y, V_{\neg j})], \]

where equality (i) follows since the variable \( V_j \) is independent of \( V_{\neg j} \). Since conditioning can only reduce entropy, we have \( H(V_j \mid Y, V_{\neg j}) \geq H(V_j \mid Y, V_{\neg j}) \), and hence

\[ I(V; Y) \leq \sum_{j=1}^{d} [H(V_j \mid V_{\neg j}) - H(V_j \mid Y, V_{\neg j})] = \sum_{j=1}^{d} I(V_j; Y \mid V_{\neg j}). \tag{24} \]

Turning to our second intermediate bound, by the definition of the conditional mutual information, we have

\[ \sum_{j=1}^{d} I(X_j; Y \mid V_{\neg j}) = \sum_{j=1}^{d} [H(X_j \mid V_{\neg j}) - H(X_j \mid Y, V_{\neg j})] \leq H(X) - \sum_{j=1}^{d} H(X_j \mid Y, V_{\neg j}) \leq H(X) - \sum_{j=1}^{d} H(X_j \mid Y, V) \leq H(X) - H(X \mid Y, V) = I(X; Y, V), \]

where equality (i) follows by the independence of \( X_j \) and \( V_{\neg j} \), inequality (ii) because conditioning reduces entropy, and inequality (iii) because \( H(X \mid Y, V) \leq \sum_j H(X_j \mid Y, V) \). Noting that \( I(X; V, Y) \leq H(V, Y) \leq H(Y) + d \), we conclude that

\[ \sum_{j=1}^{d} I(X_j; Y \mid V_{\neg j}) \leq I(X; V, Y) \leq H(Y) + d. \tag{25} \]

We can now complete the proof of the theorem. Combining inequalities (24) and (25) with Lemma 9 yields

\[ I(V; Y) \leq 128 \frac{\delta^2 n a^2}{\sigma^4} (H(Y) + d) + md h_2(p^*) + mdp^*, \tag{26} \]

20
where we recall that $p^* = \{2 \exp\left(-\frac{(a - \sqrt{n} \delta)^2}{2 \sigma^2}\right), \frac{1}{2}\}$.

Inequality (26) is the analog of inequality (19b) in the proof of Theorem 1; accordingly, we may follow the same steps to complete the proof. The case $d < 10$ is entirely analogous; the case $d \geq 10$ involves a few minor differences that we describe here.

Setting $a = 4\sigma \sqrt{\log m}$, choosing some $\delta$ in the interval $[0, \frac{\sigma}{16 \sqrt{n \log m}}]$, and then applying the bound (26), we find that

$$I(V; Y) \leq \frac{128 \cdot 16n \log m}{\sigma^2} (H(Y) + d) + d \left(\frac{2}{49} + 2 \cdot 10^{-5}\right).$$

Combining this upper bound on the mutual information with Lemmas 1 and 2, we find that

$$M_{inter}^{\infty}(\theta, P, B) \geq \delta^2 \left[\frac{d}{6} + 1\right] \frac{d \sigma^2}{2048 \cdot 36 \cdot n(B + d) \log m},$$

where the second step follows since $H(Y) \leq B$, by the source coding theorem [10]. Setting

$$\delta^2 = \min\left\{1, \frac{\sigma^2}{256 n \log m}, \frac{d \sigma^2}{2048 \cdot 36 \cdot n(B + d) \log m}\right\} = \min\left\{1, \frac{d \sigma^2}{2048 \cdot 36 \cdot n(B + d) \log m}\right\},$$

we obtain

$$M_{inter}^{\infty}(\theta, P, B) \geq \delta^2 \frac{d}{6} + 1 = \min\left\{1, \frac{d \sigma^2}{2048 \cdot 36 \cdot n(B + d) \log m}\right\} \frac{d}{6} + 1$$

Noting that $M_{inter}^{\infty}(\theta, P, B) \geq M_{inter}^{\infty}(\theta, P, \infty) \geq \frac{\sigma^2 d}{nm}$ completes the proof.

5 Discussion

In this paper, we have established lower bounds on the amount of communication required for several statistical estimation problems. Our lower bounds are information-theoretic in nature, based on variants of Fano’s and Le Cam’s methods. In particular, they rely on novel types of quantitative data processing inequalities that characterize the effect of bit constraints on the mutual information between parameters and messages. Several open questions remain. Our arguments are somewhat complex, and our upper and lower bounds differ by logarithmic factors. It would be interesting to understand which of our bounds can be sharpened; tightening the upper bounds would lead to interesting new distributed inference protocols, while improving the lower bounds could require new technical insights. We believe it will also be interesting to explore the application and extension of our results and techniques to other—perhaps more complex—problems in statistical estimation.

Acknowledgements

This work was supported in part by the U.S. Army Research Laboratory, U.S. Army Research Office grant W911NF-11-1-0391, Office of Naval Research MURI grant N00014-11-1-0688, and National Science Foundation grant CIF-31712-23800. In addition, JCD was supported in part by a Facebook Graduate Fellowship.
A Contractions in total variation distance

As noted in the main body of the paper, our results rely on certain quantitative data processing inequalities. They are inspired by results on information contraction under privacy constraints developed by a subset of the current authors (Duchi et al. [14]). In this appendix, we present a technical result—a contraction in total variation distance—that underlies many of our proofs of the data processing inequalities (Lemmas 3, 5, and 8).

Consider a random vector \((A, B, C, D)\) with joint distribution \(P_{A,B,C,D}\), where \(A\), \(C\) and \(D\) take on discrete values. Denoting the conditional distribution of \(A\) given \(B\) by \(P_{A|B}\), suppose that \((A, B, C, D)\) respect the conditional independence properties defined by the directed graphical model in Figure 1. In analytical terms, we have

\[
P_{A,B,C,D} = P_A P_{B|A} P_{C|A,B} P_{D|B,C}. \tag{27}
\]

In addition, we assume that there exist functions \(\Psi_1 : A \times \sigma(C) \to \mathbb{R}_+\) and \(\Psi_2 : B \times \sigma(C) \to \mathbb{R}_+\) such that

\[
P_C(S \mid A, B) = \Psi_1(A, S) \Psi_2(B, S) \tag{28}
\]

for any (measurable) set \(S\) in the range \(C\) of \(C\). Since \(C\) is assumed discrete, we abuse notation and write \(P(C = c \mid A, B) = \Psi_1(A, c) \Psi_2(B, c)\). Lastly, suppose that

\[
\sup_{S \in \sigma(B)} \frac{P_B(S \mid A = a)}{P_B(S \mid A = a')} \leq \exp(\alpha) \quad \text{for all } a, a' \in A. \tag{29}
\]

The following lemma applies to the absolute difference

\[
\Delta(a, C, D) := \left| P(A = a \mid C, D) - P(A = a \mid C) \right|.
\]

**Lemma 10.** Under conditions (27), (28), and (29), we have

\[
\Delta(a, C, D) \leq 2(e^{2\alpha} - 1) \min \left\{ P(A = a \mid C), P(A = a \mid C, D) \right\} \|P_B(\cdot \mid C, D) - P_B(\cdot \mid C)\|_{TV}.
\]
Proof. By assumption, $A$ is independent of $D$ given $\{B, C\}$. Thus we may write

$$
\Delta(a, C, D) = \left| \int P(A = a \mid B = b, C) (dP_B(b \mid C, D) - dP_B(b \mid C)) \right|.
$$

Combining this equation with the relation $\int_B P(A = a \mid C) (dP_B(b \mid C, D) - dP_B(b \mid C)) = 0$, we find that

$$
\Delta(a, C, D) = \left| \int_B (P(A = a \mid B = b, C) - P(A = a \mid C)) (dP_B(b \mid C, D) - dP_B(b \mid C)) \right|.
$$

Using the fact that $\left| \int f(b) d\mu(b) \right| \leq \sup_B \{ |f(b)| \} \int |d\mu(b)|$ for any signed measure $\mu$ on $B$, we conclude from the previous equality that for any version $P_A(\cdot \mid B, C)$ of the conditional probability of $A$ given $\{B, C\}$ that since $\int |d\mu| = \|\mu\|_{TV}$,

$$
\Delta(a, C, D) \leq 2 \sup_{b \in B} \{|P(A = a \mid B = b, C) - P(A = a \mid C)| \} \|P_B(\cdot \mid C, D) - P_B(\cdot \mid C)\|_{TV}.
$$

Thus, to prove the lemma, it is sufficient to show\(^3\) that for any $b \in B$

$$
|P(A = a \mid B = b, C) - P(A = a \mid C)| \leq (e^{2\alpha} - 1) \min\{P(A = a \mid C), P(A = a \mid C, D)\}. \quad (30)
$$

To prove this upper bound, we consider the joint distribution \((27)\) and likelihood ratio bound \((29)\). The distributions $\{P_B(\cdot \mid A = a)\}_{a \in A}$ are all absolutely continuous with respect to one another by assumption \((29)\), so it is no loss of generality to assume that there exists a density $p_B(\cdot \mid A = a)$ for which $P(B \in S \mid A = a) = \int p_B(b \mid A = a) d\mu(b)$, for some fixed measure $\mu$ and for which the ratio $p_B(b \mid A = a)/p_B(b \mid A = a')$ is independent of $B$, $A$, $C$.

By elementary conditioning we have for any $S_B \in \sigma(B)$ and $c \in C$ that

$$
P(A = a \mid B \in S_B, C = c) = \frac{P(A = a, B \in S_B, C = c)}{P(B \in S_B, C = c)} = \frac{P(B \in S_B, C = c \mid A = a)P(A = a)}{\sum_{a' \in A} P(A = a')P(B \in S_B, C = c \mid A = a')} = \frac{P(A = a) \int_{S_B} P(C = c \mid B = b, A = a)p_B(b \mid A = a) d\mu(b)}{\sum_{a' \in A} P(A = a') \int_{S_B} P(C = c \mid B = b, A = a')p_B(b \mid A = a') d\mu(b)},
$$

where for the last equality we used the conditional independence assumptions \((27)\). But now we recall the decomposition formula \((28)\), and we can express the likelihood functions by

$$
P(A = a \mid B \in S_B, C = c) = \frac{P(A = a) \int_{S_B} \Psi_1(a, c) \Psi_2(b, c) p_B(b \mid A = a) d\mu(b)}{\sum_{a'} P(A = a') \int_{S_B} \Psi_1(a', c) \Psi_2(b, c) p_B(b \mid A = a') d\mu(b)}.
$$

As a consequence, there is a version of the conditional distribution of $A$ given $B$ and $C$ such that

$$
P(A = a \mid B = b, C = c) = \frac{P(A = a) \Psi_1(a, c) p_B(b \mid A = a)}{\sum_{a'} P(A = a') \Psi_1(a', c) p_B(b \mid A = a')}.
$$

\(^3\) If $P(A = a \mid C)$ is undefined, we simply set it to have value 1 and assign $P(A = a \mid B, C) = 1$ as well.
Define the shorthand
\[ \beta = \frac{P(A = a)\Psi_1(a, c)}{\sum_{a' \in A} P(A = a')\Psi_1(a', c)}. \]

We claim that
\[ e^{-\alpha}\beta \leq P(A = a \mid B = b, C = c) \leq e^{\alpha}\beta. \tag{32} \]

Assuming the correctness of bound (32), we establish inequality (30). Indeed, \( P(A = a \mid C = c) \) is a weighted average of \( P(A = a \mid B = b, C = c) \), so we also have the same upper and lower bound for \( P(A = a \mid C) \), that is
\[ e^{-\alpha}\beta \leq P(A = a \mid C) \leq e^{\alpha}\beta. \]

The conditional independence assumption that \( A \) is independent of \( D \) given \( B, C \) (recall Figure 1 and the product (27)) implies
\[ P(A = a \mid C = c, D = d) = \frac{\int_B P(A = a \mid B = b, C = c, D = d)dP_B(b \mid C = c, D = d)}{\int_B P(A = a \mid B = b, C = c)dP_B(b \mid C = c, D = d)}, \]
and the final integrand belongs to \( \beta[e^{-\alpha}, e^{\alpha}] \). Combining the preceding three displayed expressions, we find that
\[ |P(A = a \mid B = b, C) - P(A = a \mid C)| \leq (e^{\alpha} - e^{-\alpha})\beta \leq (e^{\alpha} - e^{-\alpha})e^{\alpha} \min \{ P(A = a \mid C), P(A = a \mid C, D) \}. \]

This completes the proof of the upper bound (30).

It remains to prove inequality (32). We observe from expression (31) that
\[ P(A = a \mid B = b, C) = \frac{P(A = a)\Psi_1(a, c)}{\sum_{a' \in A} P(A = a')\Psi_1(a', c)\frac{p_B(b \mid A = a')}{p_B(b)}}, \]
By the likelihood ratio bound (29), we have \( p_B(b \mid A = a')/p_B(b \mid A = a) \in [e^{-\alpha}, e^{\alpha}] \), and combining this with the above equation yields inequality (32). \( \square \)

### B Auxiliary results for Proposition 2

In this appendix, we collect the proofs of auxiliary results involved in the proof of Proposition 2.

#### B.1 Proof of Lemma 3

Let \( Y = Y_i \); throughout the proof we suppress the dependence on the index \( i \) (and similarly let \( X = X^{(i)} \) denote a single fixed sample). We begin with the observation that by the chain rule for mutual information,
\[ I(V; Y) = \sum_{j=1}^{d} I(V_j; Y \mid V_{1:j-1}). \]
Using the definition of mutual information and non-negativity of the KL-divergence, we have

\[ I(V_j; Y \mid V_{1:j-1}) = \mathbb{E}_{V_{1:j-1}} \left[ \mathbb{E}_Y \left[ D_{\text{KL}} \left( P_{V_j} (\cdot \mid Y, V_{1:j-1}) \parallel P_{V_j} (\cdot \mid V_{1:j-1}) \right) \mid V_{1:j-1} \right] \right] \]

\[ \leq \mathbb{E}_{V_{1:j-1}} \left[ \mathbb{E}_Y \left[ D_{\text{KL}} \left( P_{V_j} (\cdot \mid Y, V_{1:j-1}) \parallel P_{V_j} (\cdot \mid V_{1:j-1}) \right) \right] + D_{\text{KL}} \left( P_{V_j} (\cdot \mid V_{1:j-1}) \parallel P_{V_j} (\cdot \mid Y, V_{1:j-1}) \mid V_{1:j-1} \right) \right]. \]

Now, we require an argument that builds off of our technical Lemma 10. We claim that Lemma 10 implies that

\[ |P(V_j = v_j \mid V_{1:j-1}, Y) - P(V_j = v_j \mid V_{1:j-1})| \leq 2(e^{2\alpha} - 1) \min \{P(V_j = v_j \mid V_{1:j-1}, Y), P(V_j = v_j \mid V_{1:j-1})\} \times \left\| P_{X_j} (\cdot \mid V_{1:j-1}, Y) - P_{X_j} (\cdot \mid V_{1:j-1}) \right\|_{\text{TV}}. \]  \hspace{1cm} (33)

Indeed, making the identification

\[ V_j \to A, \quad X_j \to B, \quad V_{1:j-1} \to C, \quad Y \to D, \]

the random variables satisfy the condition 27 clearly, condition 28 because \( V_{1:j-1} \) is independent of \( V_j \) and \( X_j \), and condition 29 by construction. This gives inequality 33 by our independence assumptions. Expanding our KL divergence bound, we have

\[ D_{\text{KL}} \left( P_{V_j} (\cdot \mid Y, V_{1:j-1}) \parallel P_{V_j} (\cdot \mid V_{1:j-1}) \right) + D_{\text{KL}} \left( P_{V_j} (\cdot \mid V_{1:j-1}) \parallel P_{V_j} (\cdot \mid Y, V_{1:j-1}) \right) \]

\[ = \sum_{v_j} \left( P_{V_j} (v_j \mid Y, V_{1:j-1}) - P_{V_j} (v_j \mid V_{1:j-1}) \right) \log \frac{P_{V_j} (v_j \mid Y, V_{1:j-1})}{P_{V_j} (v_j \mid V_{1:j-1})}. \]

Now, using the elementary inequality for \( a, b \geq 0 \) that

\[ \left| \log \frac{a}{b} \right| \leq \frac{|a - b|}{\min\{a, b\}}, \]

inequality 33 implies that

\[ \left( P_{V_j} (v_j \mid Y, V_{1:j-1}) - P_{V_j} (v_j \mid V_{1:j-1}) \right) \log \frac{P_{V_j} (v_j \mid Y, V_{1:j-1})}{P_{V_j} (v_j \mid V_{1:j-1})} \]

\[ \leq \left( P_{V_j} (v_j \mid Y, V_{1:j-1}) - P_{V_j} (v_j \mid V_{1:j-1}) \right)^2 \]

\[ \leq 4(e^{2\alpha} - 1)^2 \min \{P_{V_j} (v_j \mid Y, V_{1:j-1}), P_{V_j} (v_j \mid V_{1:j-1})\} \left\| P_{X_j} (\cdot \mid V_{1:j-1}, Y) - P_{X_j} (\cdot \mid V_{1:j-1}) \right\|_{\text{TV}}^2. \]

Substituting this into our bound on KL-divergence, we obtain

\[ I(V_j; Y \mid V_{1:j-1}) = \mathbb{E}_{V_{1:j-1}} \left[ \mathbb{E}_Y \left[ D_{\text{KL}} \left( P_{V_j} (\cdot \mid Y, V_{1:j-1}) \parallel P_{V_j} (\cdot \mid V_{1:j-1}) \right) \mid V_{1:j-1} \right] \right] \]

\[ \leq 4(e^{2\alpha} - 1)^2 \mathbb{E}_{V_{1:j-1}} \left[ \mathbb{E}_Y \left[ \left\| P_{X_j} (\cdot \mid V_{1:j-1}, Y) - P_{X_j} (\cdot \mid V_{1:j-1}) \right\|_{\text{TV}}^2 \mid V_{1:j-1} \right] \right]. \]

Using Pinsker’s inequality, we then find that

\[ \mathbb{E}_{V_{1:j-1}} \left[ \mathbb{E}_Y \left[ \left\| P_{X_j} (\cdot \mid V_{1:j-1}, Y) - P_{X_j} (\cdot \mid V_{1:j-1}) \right\|_{\text{TV}}^2 \mid V_{1:j-1} \right] \right] \]

\[ \leq \frac{1}{2} \mathbb{E}_{V_{1:j-1}} \left[ \mathbb{E}_Y \left[ D_{\text{KL}} \left( P_{X_j} (\cdot \mid Y, V_{1:j-1}) \parallel P_{X_j} (\cdot \mid V_{1:j-1}) \right) \mid V_{1:j-1} \right] \right] = \frac{1}{2} I(X_j; Y \mid V_{1:j-1}). \]
In particular, we have
\[ I(V_j; Y \mid V_{1:j-1}) \leq 2 \left( e^{2\alpha} - 1 \right)^2 I(X_j; Y \mid V_{1:j-1}). \] (34)

Lastly, we argue that \( I(X_j; Y \mid V_{1:j-1}) \leq I(X_j; Y \mid X_{1:j-1}) \). Indeed, we have by definition that
\[
I(X_j; Y \mid V_{1:j-1}) \overset{(i)}{=} H(X_j) - H(X_j \mid Y, V_{1:j-1}) \\
\overset{(ii)}{\leq} H(X_j) - H(X_j \mid Y, V_{1:j-1}, X_{1:j-1}) \\
\overset{(iii)}{=} H(X_j \mid X_{1:j-1}) - H(X_j \mid Y, X_{1:j-1}) = I(X_j; Y \mid X_{1:j-1}).
\]

Here, equality (i) follows since \( X_j \) is independent of \( V_{1:j-1} \), inequality (ii) because conditioning reduces entropy, and equality (iii) because \( X_j \) is independent of \( X_{1:j-1} \). Thus
\[
I(V; Y) = \sum_{j=1}^{d} I(V_j; Y \mid V_{1:j-1}) \leq 2(e^{2\alpha} - 1)^2 \sum_{j=1}^{d} I(X_j; Y \mid X_{1:j-1}) = 2(e^{2\alpha} - 1)^2 I(X; Y),
\]
which completes the proof.

### B.2 Proof of Lemma 4

By assumption, the message \( Y_i \) is constructed based only on \( X^{(i)} \). Therefore, we have
\[
I(V; Y_{1:m}) = \sum_{i=1}^{m} I(V; Y_i \mid Y_{1:i-1}) = \sum_{i=1}^{m} H(Y_i \mid Y_{1:i-1}) - H(Y_i \mid V, Y_{1:i-1}) \\
\leq \sum_{i=1}^{m} H(Y_i) - H(Y_i \mid V, Y_{1:i-1}) \\
= \sum_{i=1}^{m} H(Y_i) - H(Y_i \mid V) = \sum_{i=1}^{m} I(V; Y_i)
\]
where we have used that conditioning reduces entropy and \( Y_i \) is conditionally independent of \( Y_{1:i-1} \) given \( V \).

### B.3 Proof of inequality 16

Let \( P_\nu \) be shorthand for \( P_Y(\cdot \mid V = \nu) \). The triangle inequality implies that
\[
\|P_\nu - P_{\nu'}\|_{TV} \leq \|P_\nu - (1/2)(P_\nu + P_{\nu'})\|_{TV} + \frac{1}{2} \|P_\nu - P_{\nu'}\|_{TV},
\]
and similarly swapping the roles of \( \nu' \) and \( \nu \), whence
\[
\|P_\nu - P_{\nu'}\|_{TV} \leq 2 \min\{\|P_\nu - (1/2)(P_\nu + P_{\nu'})\|_{TV},\|P_{\nu'} - (1/2)(P_\nu + P_{\nu'})\|_{TV}\}.
\]

By Pinsker’s inequality, we thus have the upper bound
\[
\|P_\nu - P_{\nu'}\|_{TV}^2 \leq 2 \min\{D_{kl}(P_\nu(1/2)(P_\nu + P_{\nu'})), D_{kl}(P_{\nu'}(1/2)(P_\nu + P_{\nu'}))\} \\
\leq D_{kl}(P_\nu(1/2)(P_\nu + P_{\nu'})) + D_{kl}(P_{\nu'}(1/2)(P_\nu + P_{\nu'})) = 2I(Y; V)
\]
by the definition of mutual information.
C Auxiliary results for Theorem 1

In this appendix, we collect the proofs of auxiliary results involved in the proof of Theorem 1.

C.1 Proof of Lemma 5

This proof is similar to that of Lemma 3 but we must be careful when conditioning on events of the form \( X_j^{(i)} \in G_j \). For notational simplicity, we again suppress all dependence of \( X \) and \( Y \) on the machine index \( i \). Our goal is to prove that

\[
I(V_j; Y \mid V_{1:j-1}) \leq H(E_j) + P(E_j = 0) + 2(e^{4\alpha} - 1)^2 I(X_j; Y \mid V_{1:j-1}). \tag{35}
\]

Up to the additive terms, this is equivalent to the earlier bound in the proof of Lemma 3, so that proceeding \textit{mutatis mutandis} completes the proof. We now turn to proving inequality (35).

We begin by noting that \( I(X; Y \mid Z) \leq I(X; W; Y \mid Z) \) for any random variables \( W, X, Y, Z \), because conditioning reduces entropy:

\[
I(X; Y \mid Z) = H(Y \mid Z) - H(Y \mid X, Z) \leq H(Y \mid Z) - H(Y \mid W, X, Z) = I(X; W; Y \mid Z). \tag{36}
\]

As a consequence, recalling the random variable \( E_j \) (the indicator of \( X_j \in G_j \)), we have

\[
I(V_j; Y \mid V_{1:j-1}) \leq I(V_j; Y, E_j \mid V_{1:j-1}) = I(V_j; Y \mid E_j, V_{1:j-1}) + I(V_j; E_j \mid V_{1:j-1})
\]

\[
\leq I(V_j; Y \mid E_j, V_{1:j-1}) + H(E_j \mid V_{1:j-1}) = I(V_j; Y \mid E_j, V_{1:j-1}) + H(E_j), \tag{37}
\]

where the final equality follows because \( E_j \) is independent of \( V_{1:j-1} \). Comparing to inequality (35), we need only control the first term in the bound (37).

To that end, note that given \( E_j \), the variable \( V_j \) is independent of \( V_{1:j-1}, X_{1:j-1}, V_{j+1:d}, \) and \( X_{j+1:d} \). Moreover, by the assumption in the lemma we have for any \( S \in \sigma(G_j) \) that

\[
\frac{P_{X_j}(S \mid V = \nu, E_j = 1)}{P_{X_j}(S \mid V = \nu', E_j = 1)} = \frac{P_{X_j}(S \mid V = \nu)}{P_{X_j}(S \mid V = \nu')} \leq \exp(2\alpha).
\]

Applying Lemma 10 yields that the difference

\[
\Delta_j := P(V_j = \nu_j \mid V_{1:j-1}, Y, E_j = 1) - P(V_j = \nu_j \mid V_{1:j-1}, E_j = 1)
\]

is bounded as

\[
|\Delta_j| \leq 2(e^{4\alpha} - 1) \|P_{X_j}(\cdot \mid V_{1:j-1}, Y, E_j = 1) - P_{X_j}(\cdot \mid V_{1:j-1}, E_j = 1)\|_{TV}
\]

\[
\times \min \{ P(V_j = \nu_j \mid V_{1:j-1}, Y, E_j = 1), P(V_j = \nu_j \mid V_{1:j-1}, E_j = 1) \},
\]

(cf. the inequality in the proof of Lemma 3). Proceeding as in the proof of Lemma 3, this expression leads to the bound

\[
I(V_j; Y \mid V_{1:j-1}, E_j = 1) \leq 2(e^{4\alpha} - 1)^2 I(X_j; Y \mid V_{1:j-1}, E_j = 1). \tag{38}
\]

By the definition of conditional mutual information,

\[
I(V_j; Y \mid E_j, V_{1:j-1}) = P(E_j = 1)I(V_j; Y \mid V_{1:j-1}, E_j = 1) + P(E_j = 0)I(V_j; Y \mid V_{1:j-1}, E_j = 0)
\]

\[
\leq I(V_j; Y \mid V_{1:j-1}, E_j = 1) + P(E_j = 0) \log 2,
\]

where the inequality follows because \( V_j \in \{-1, 1\} \). But combining this inequality with the bounds (38) and (37) gives the desired result (35).
C.2 Proof of Lemma 6

In order to prove inequality (19a), we note that \( V \rightarrow X^{(i)} \rightarrow Y_i \) forms a Markov chain. Thus, the classical data-processing inequality \([10]\) implies that

\[
I(V; Y_i) \leq I(V; X^{(i)}) \leq \sum_{k=1}^{n_i} I(V; X^{(i,k)}).
\]

Let \( P_\nu \) denote the conditional distribution of \( X^{(i,k)} \) given \( V = \nu \). Then the convexity of the KL-divergence establishes inequality (19a) via

\[
I(V; X^{(i,k)}) \leq \frac{1}{|\nu|^2} \sum_{\nu, \nu' \in V} D_{\text{kl}}(P_\nu \| P_{\nu'}) = \frac{\delta^2}{2\sigma^2 |\nu|^2} \sum_{\nu, \nu' \in V} \| \nu - \nu' \|^2 = \frac{d \delta^2}{\sigma^2}.
\]

To prove inequality (19b), we apply Lemma 5. First, consider two one-dimensional normal distributions, each with \( n_i \) independent observations and variance \( \sigma^2 \), but where one has mean \( \delta \) and the other mean \( -\delta \). For fixed \( \alpha \geq 0 \), the ratio of their densities is

\[
\frac{\exp\left(-\frac{1}{2\sigma^2} \sum_{l=1}^{n_i} (x_l - \delta)^2\right)}{\exp\left(-\frac{1}{2\sigma^2} \sum_{l=1}^{n_i} (x_l + \delta)^2\right)} = \exp\left(\frac{\delta}{\sigma^2} \sum_{l=1}^{n_i} x_l\right) \leq \exp\left(\frac{\sqrt{n_i} \delta a}{\sigma^2}\right)
\]
wherever \( |\sum_l x_l| \leq \sqrt{n_i} a \). As a consequence, we see that by taking the sets

\[
G_j = \left\{ x \in \mathbb{R}^{n_i} : \left| \sum_{l=1}^{n_i} x_l \right| \leq \sqrt{n_i} a \right\},
\]
we satisfy the conditions of Lemma 5 with the quantity \( \alpha \) defined as \( \alpha = \sqrt{n_i} \delta a / \sigma^2 \). In addition, when \( \alpha \leq 1.2564 \), we have \( \exp(\alpha) - 1 \leq 2\alpha \), so under the conditions of the lemma, \( \exp(4\alpha) - 1 \leq 8\sqrt{n_i} \delta a / \sigma^2 \). Recalling the definition of the indicator random variable \( E_j = 1\{X_j^{(i)} \in G_j\} \) from Lemma 5, we obtain

\[
I(V; Y_i) \leq 128 \frac{\delta^2 a^2}{\sigma^2} n_i I(X^{(i)}; Y_i) + \sum_{j=1}^{d} H(E_j) + \sum_{j=1}^{d} P(E_j = 0). \tag{39}
\]

Comparing this inequality with inequality (19b), we see that we must bound the probability of the event \( E_j = 0 \).

Bounding \( P(E_j = 0) \) is not challenging, however. From standard Gaussian tail bounds, we have for \( Z_l \) distributed i.i.d. according to \( N(\delta, \sigma^2) \) that

\[
P(E_j = 0) = P\left( \left| \sum_{l=1}^{n_i} Z_l \right| \geq \sqrt{n_i} a \right) = P\left( \sum_{l=1}^{n_i} (Z_l - \delta) \geq \sqrt{n_i} a - n\delta \right) + P\left( \sum_{l=1}^{n_i} (Z_l - \delta) \leq -\sqrt{n_i} a - n\delta \right)
\leq 2 \exp\left( -\frac{(a - \sqrt{n_i} \delta)^2}{2\sigma^2} \right). \tag{40}
\]

Since \( h_2(p) \leq h_2(\frac{1}{2}) \) and \( I(V; Y_i) \leq d \log 2 \) regardless, this provides the bounds on the entropy and probability terms in inequality (39) to yield the result (19b).
Combining inequalities (19a) and (19b) yields
\[
I(V; Y_i) \leq \frac{n_i \delta^2}{\sigma^2} \min \left\{ 128 \frac{a^2}{\sigma^2} H(Y_i), d \right\} + d h_2 \left( \min \left\{ 2 \exp \left( -\frac{(a - \sqrt{n_i \delta})^2}{2\sigma^2} \right), \frac{1}{2} \right\} \right)
\]
\[+ 2d \exp \left( -\frac{(a - \sqrt{n_i \delta})^2}{2\sigma^2} \right),
\]
true for all \(a, \delta \geq 0\) and \(n_i, \sigma^2\) such that \(\sqrt{n_i a \delta} \leq 1.2564a \sigma^2 / 4\) and \(a \geq \delta \sqrt{n_i}\).

Now, we consider each of the terms in the bound in inequality (41) in turn, finding settings of \(\delta\) and \(a\) so that each term is small. Let us set \(a = 4\sigma \sqrt{\log m}\). We begin with the third term in the bound (41), where we note that by defining \(\delta_3\) as the positive root of
\[
\delta_3^2 := \frac{\sigma^2}{16 \cdot 16 \log(m) \max_i n_i},
\]
then for \(0 \leq \delta \leq \delta_3\) the conditions \(\frac{\sqrt{n_i a \delta}}{\sigma^2} \leq 1.2564\) and \(\sqrt{n_i \delta} \leq a\) in Lemma 6 are satisfied. In addition, we have \((a - \sqrt{n_i \delta})^2 \geq (4 - 1/256)^2 \sigma^2 \log m \geq 15\sigma^2 \log m\) for \(0 \leq \delta \leq \delta_3\), so for such \(\delta\)
\[
\sum_{i=1}^m 2 \exp \left( -\frac{(a - \sqrt{n_i \delta})^2}{2\sigma^2} \right) \leq 2m \exp(-15/2 \log m) = \frac{2}{m^{15/2}} < 2 \cdot 10^{-5}.
\]
Secondly, we have \(h_2(q) \leq (6/5)\sqrt{q}\) for \(q \geq 0\). As a consequence, we see that for \(\delta_2\) chosen identically to the choice (42) for \(\delta_3\), we have
\[
\sum_{i=1}^m 2h_2 \left( 2 \exp \left( -\frac{(a - \sqrt{n_i \delta_2})^2}{2\sigma^2} \right) \right) \leq \frac{12m}{5} \sqrt{2} \exp(-15/4 \log m) < \frac{2}{49}.
\]
In particular, with the choice \(a = 4\sigma \sqrt{\log m}\) and for all \(0 \leq \delta \leq \delta_3\), inequality (41) implies the desired bound (20).

D Auxiliary results for Theorem 2

In this appendix, we collect the proofs of auxiliary results for Theorem 2.

D.1 Proof of Lemma 8

We state an intermediate claim from which Lemma 8 follows quickly. Let us temporarily assume that the set \(G\) in the statement of the lemma is \(G = \text{range}(X(i))\), so that there is no restriction on the distributions \(P_{X(i)}\), that is, the likelihood ratio bound (23) holds for all measurable sets \(S\). We claim that in this case,
\[
I(V; Y) \leq 2(e^{2\alpha} - 1)^2 I(X; Y).
\]
Assuming that we have established inequality (23), the proof of Lemma 8 follows, mutatis mutandis, as in the proof of Lemma 5 from Lemma 8.
Let us now prove the claim \[43\]. By the chain-rule for mutual information, we have

\[ I(V; Y) = \sum_{t=1}^{T} I(V; Y_t \mid Y_{1:t-1}). \]

Let \( P_{Y_t} (\cdot \mid Y_{1:t-1}) \) denote the (marginal) distribution of \( Y_t \) given \( Y_{1:t-1} \) and define \( P_V (\cdot \mid Y_{1:t}) \) to be the distribution of \( V \) conditional on \( Y_{1:t} \). Then we have by marginalization that

\[ P_V (\cdot \mid Y_{1:t-1}) = \int P_V (\cdot \mid Y_{1:t-1}, y_t) dP_{Y_t} (y_t \mid Y_{1:t-1}) \]

and thus

\[ I(V; Y_t \mid Y_{1:t-1}) = \mathbb{E}_{Y_t \mid Y_{1:t-1}} \left[ \mathbb{E}_{Y_t} \left[ D_{\text{KL}} (P_V (\cdot \mid Y_{1:t}) \| P_V (\cdot \mid Y_{1:t-1})) \mid Y_{1:t-1} \right] \right]. \]

We now bound the above KL divergence using the assumed likelihood ratio bound on \( P_X \) in the lemma (when \( G = \mathcal{X} \), the entire sample space).

By the nonnegativity of the KL divergence, we have

\[ D_{\text{KL}} (P_V (\cdot \mid Y_{1:t}) \| P_V (\cdot \mid Y_{1:t-1})) \leq D_{\text{KL}} (P_V (\cdot \mid Y_{1:t}) \| P_V (\cdot \mid Y_{1:t-1})) + D_{\text{KL}} (P_V (\cdot \mid Y_{1:t-1}) \| P_V (\cdot \mid Y_{1:t})) \]

\[ = \sum_{\nu \in \mathcal{V}} (p_V (\nu \mid Y_{1:t-1}) - p_V (\nu \mid Y_{1:t})) \log \frac{p_V (\nu \mid Y_{1:t-1})}{p_V (\nu \mid Y_{1:t})} \]

where \( p_V \) denotes the p.m.f. of \( V \).

Next we claim that the difference \( \Delta_t := |p_V (\nu \mid Y_{1:t-1}) - p_V (\nu \mid Y_{1:t})| \) is upper bounded as

\[ |\Delta_t| \leq 2(e^{2n} - 1) \min \{ p_V (\nu \mid Y_{1:t-1}), p_V (\nu \mid Y_{1:t}) \} \| P_X (\cdot \mid Y_{1:t}) - P_X (\cdot \mid Y_{1:t-1}) \|^2_{\text{TV}}. \quad (44) \]

Deferring the proof of this claim to the end of this section, we give the remainder of the proof. First, by a first-order convexity argument, we have

\[ |\log a - \log b| \leq \frac{|a - b|}{\min \{a, b\}} \quad \text{for any} \ a, b > 0. \]

Combining this bound with inequality \[44\] yields

\[ \Delta_t \log \frac{p_V (\nu \mid Y_{1:t-1})}{p_V (\nu \mid Y_{1:t})} \leq \frac{\Delta_t^2}{\min \{ p_V (\nu \mid Y_{1:t-1}), p_V (\nu \mid Y_{1:t}) \}} \]

\[ \leq 4(e^{2n} - 1)^2 \min \{ p_V (\nu \mid Y_{1:t-1}), p_V (\nu \mid Y_{1:t}) \} \| P_X (\cdot \mid Y_{1:t}) - P_X (\cdot \mid Y_{1:t-1}) \|^2_{\text{TV}}. \]

Since \( p_V \) is a p.m.f., we have the following upper bound on the symmetrized KL divergence between \( P_V (\cdot \mid Y_{1:t}) \) and \( P_V (\cdot \mid Y_{1:t-1}) \):

\[ D_{\text{KL}} (P_V (\cdot \mid Y_{1:t}) \| P_V (\cdot \mid Y_{1:t-1})) + D_{\text{KL}} (P_V (\cdot \mid Y_{1:t-1}) \| P_V (\cdot \mid Y_{1:t})) \]

\[ \leq 4(e^{2n} - 1)^2 \| P_X (\cdot \mid Y_{1:t}) - P_X (\cdot \mid Y_{1:t-1}) \|^2_{\text{TV}} \sum_{\nu \in \mathcal{V}} \min \{ p_V (\nu \mid Y_{1:t-1}), p_V (\nu \mid Y_{1:t}) \} \]

\[ \leq 4(e^{2n} - 1)^2 \| P_X (\cdot \mid Y_{1:t}) - P_X (\cdot \mid Y_{1:t-1}) \|^2_{\text{TV}} \]

\[ \leq \frac{1}{2} D_{\text{KL}} (P_X (\cdot \mid Y_{1:t}) \| P_X (\cdot \mid Y_{1:t-1})). \]
where the final step follows from Pinsker’s inequality. Taking expectations, we have
\[ \frac{1}{2} \mathbb{E}_{Y_{1:t-1}} \left[ \mathbb{E}_{Y_t} \left[ D_{KL}(P_{X^{(i_t)}(\cdot \mid Y_{1:t})} \| P_{X^{(i_t)}(\cdot \mid Y_{1:t-1})} \mid Y_{1:t-1}) \right] \right] = \frac{1}{2} I(X^{(i_t)}; Y_t \mid Y_{1:t-1}). \]
Finally, because conditioning reduces entropy (recall inequality (36)), we have
\[ I(X^{(i_t)}; Y_t \mid Y_{1:t-1}) \leq I(X; Y_t \mid Y_{1:t-1}). \]
By the chain rule for mutual information, we have \( \sum_{t=1}^T I(X; Y_t \mid Y_{1:t-1}) = I(X; Y) \), so the proof is complete.

**Proof of inequality (44)** It remains to prove inequality (44); in order to do so, we establish a one-to-one correspondence between the variables in Lemma 10 and the variables in inequality (44).

Let us begin by making the identifications
\[ V \rightarrow A \quad X^{(i_t)} \rightarrow B \quad Y_{1:t-1} \rightarrow C \quad Y_t \rightarrow D. \]

For Lemma 10 to hold, we must verify conditions (27), (28), and (29). For condition (27) to hold, \( Y_t \) must be independent of \( V \) given \( \{Y_{1:t-1}, X^{(i_t)}\} \). Since the distribution of \( P_{Y_t}(\cdot \mid Y_{1:t-1}, X^{(i_t)}) \) is measurable-\( \{Y_{1:t-1}, X^{(i_t)}\} \), condition (29) is satisfied by the assumption in Lemma 8. Finally, for condition (28) to hold, we must be able to factor the conditional probability of \( Y_{1:t-1} \) given \( \{V, X^{(i_t)}\} \) as
\[ P(Y_{1:t-1} = y_{1:t-1} \mid V, X^{(i_t)}) = \Psi_1(V, y_{1:t-1})\Psi_2(X^{(i_t)}, y_{1:t-1}). \]

To prove this decomposition, notice that
\[ P(Y_{1:t-1} = y_{1:t-1} \mid V, X^{(i_t)}) = \prod_{k=1}^{t-1} P(Y_k = y_k \mid Y_{1:k-1}, V, X^{(i_t)}). \]

For any \( k \in \{1, \ldots, t-1\} \), if \( i_k = i_t \)—that is, the message \( Y_k \) is generated based on sample \( X^{(i_t)} = X^{(i_k)} \)—then \( Y_k \) is independent of \( V \) given \( \{X^{(i_t)}, Y_{1:k-1}\} \). Thus, \( P_{Y_k}(\cdot \mid Y_{1:k-1}, V, X^{(i_t)}) \) is measurable-\( \{X^{(i_t)}, Y_{1:k-1}\} \). If the \( k \)th index \( i_k \neq i_t \), then \( Y_k \) is independent of \( X^{(i_t)} \) given \( \{Y_{1:k-1}, V\} \) by construction, which means \( P_{Y_k}(\cdot \mid Y_{1:k-1}, V, X^{(i_t)}) = P_{Y_k}(\cdot \mid Y_{1:k-1}, V) \), thereby verifying the decomposition (45). Thus, we have verified that each of the conditions of Lemma 10 holds, so that inequality (44) follows.

**D.2 Proof of Lemma 9**

To prove Lemma 9, fix an arbitrary realization \( \nu_j \in \{-1, 1\}^{d-1} \) of \( V_j \). Conditioning on \( V_j = \nu_j \), note that \( \nu_j \in \{-1, 1\} \), and consider the distributions of the \( j \)th coordinate of each (local) sample \( X_j^{(i)} \in \mathbb{R}^n \),
\[ P_{X_j^{(i)}}(\cdot \mid V_j = \nu_j, V_{\cdot j} = \nu_{\cdot j}) \quad \text{and} \quad P_{X_j^{(i)}}(\cdot \mid V_j = -\nu_j, V_{\cdot j} = \nu_{\cdot j}). \]

We claim that these distributions—with appropriate constants—satisfy the conditions of Lemma 8. Indeed, fix \( a \geq 0 \), take the set \( G = \{x \in \mathbb{R}^n \mid \|x\|_1 \leq \sqrt{na}\} \), and set the log-likelihood ratio parameter \( \alpha = \sqrt{na}/\sigma^2 \). Then the random variable \( E_j = 1 \) if \( X_j^{(i)} \in G \) for all \( i = 1, \ldots, m \), and
we may apply precisely proof of Lemma 8 (we still obtain the factorization by conditioning everything on $V_j = \nu_j$. Thus we obtain

$$I(V_j; Y | V_j = \nu_j) \leq 2(e^{4a} - 1)^2 I(X_j; Y | V_j = \nu_j) + H(E_j | V_j = \nu_j) + P(E_j = 0 | V_j = \nu_j).$$

(46)

Of course, the event $E_j$ is independent of $V_j$ by construction, so that $P(E_j = 0 | V_j) = P(E_j = 0)$ and $H(E_j | V_j = \nu_j) = H(E_j)$, and standard Gaussian tail bounds (cf. the proof of Lemma 6 and inequality (40)) imply that

$$H(E_j) \leq mh_2 \left( 2 \exp \left( -\frac{(a - \sqrt{n}\delta)^2}{2\sigma^2} \right) \right) \quad \text{and} \quad P(E_j = 0) \leq 2m \exp \left( -\frac{(a - \sqrt{n}\delta)^2}{2\sigma^2} \right).$$

Thus by integrating over $V_j = \nu_j$, inequality (46) implies the lemma.

References

[1] H. Abelson. Lower bounds on information transfer in distributed computations. *Journal of the ACM*, 27(2):384–392, 1980.

[2] M.-F. Balcan, A. Blum, S. Fine, and Y. Mansour. Distributed learning, communication complexity and privacy. In *Proceedings of the Twenty Fifth Annual Conference on Computational Learning Theory*, 2012. URL [http://arxiv.org/abs/1204.3514](http://arxiv.org/abs/1204.3514).

[3] K. Ball. An elementary introduction to modern convex geometry. In S. Levy, editor, *Flavors of Geometry*, pages 1–58. MSRI Publications, 1997.

[4] Z. Bar-Yossef, T. S. Jayram, R. Kumar, and D. Sivakumar. An information statistics approach to data stream and communication complexity. *Journal of Computer and System Sciences*, 68(4):702–732, 2004.

[5] B. Barak, M. Braverman, X. Chen, and A. Rao. How to compress interactive communication. In *Proceedings of the Fourty-Second Annual ACM Symposium on the Theory of Computing*, 2010.

[6] L. Birgé. Approximation dans les espaces métriques et théorie de l’estimation. *Zeitschrift für Wahrscheinlichkeitstheorie und verwebte Gebiet*, 65:181–238, 1983.

[7] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[8] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1), 2011.

[9] A. Chakrabarti, Y. Shi, A. Wirth, and A. Yao. Informational complexity and the direct sum problem for simultaneous message complexity. In *42nd Annual IEEE Symposium on Foundations of Computer Science*, pages 270–278, 2001.

[10] T. M. Cover and J. A. Thomas. *Elements of Information Theory, Second Edition*. Wiley, 2006.
[11] O. Dekel, R. Gilad-Bachrach, O. Shamir, and L. Xiao. Optimal distributed online prediction using mini-batches. *Journal of Machine Learning Research*, 13:165–202, 2012.

[12] J. C. Duchi and M. J. Wainwright. Distance-based and continuum Fano inequalities with applications to statistical estimation. *arXiv:1311.2669 [cs.IT]*, 2013. URL http://arxiv.org/abs/1311.2669.

[13] J. C. Duchi, A. Agarwal, and M. J. Wainwright. Dual averaging for distributed optimization: convergence analysis and network scaling. *IEEE Transactions on Automatic Control*, 57(3):592–606, 2012.

[14] J. C. Duchi, M. I. Jordan, and M. J. Wainwright. Local privacy and statistical minimax rates. *arXiv:1302.3203 [math.ST]*, 2013. URL http://arxiv.org/abs/1302.3203.

[15] A. El Gamal and Y.-H. Kim. *Network Information Theory*. Cambridge University Press, 2011.

[16] S. Fuller and L. Millett. *The Future of Computing Performance: Game Over or Next Level?* National Academies Press, 2011.

[17] S. Han and S. Amari. Statistical inference under multiterminal data compression. *IEEE Transactions on Information Theory*, 44(6):2300–2324, 1998.

[18] T. Hastie and R. Tibshirani. *Generalized additive models*. Chapman & Hall, 1995.

[19] I. A. Ibragimov and R. Z. Has’minskii. *Statistical Estimation: Asymptotic Theory*. Springer-Verlag, 1981.

[20] E. Kushilevitz and N. Nisan. *Communication Complexity*. Cambridge University Press, 1997.

[21] L. Le Cam. Convergence of estimates under dimensionality restrictions. *Annals of Statistics*, 1(1):38–53, 1973.

[22] E. L. Lehmann and G. Casella. *Theory of Point Estimation, Second Edition*. Springer, 1998.

[23] Z.-Q. Luo. Universal decentralized estimation in a bandwidth constrained sensor network. *IEEE Transactions on Information Theory*, 51(6):2210–2219, 2005.

[24] Z.-Q. Luo and J. N. Tsitsiklis. On the communication complexity of distributed algebraic computation. *Journal of the ACM (JACM)*, 40(5):1019–1047, 1993.

[25] Z.-Q. Luo and J. N. Tsitsiklis. Data fusion with minimal communication. *IEEE Transactions on Information Theory*, 40(5):1551–1563, 1994.

[26] R. McDonald, K. Hall, and G. Mann. Distributed training strategies for the structured perceptron. In *North American Chapter of the Association for Computational Linguistics (NAACL)*, 2010.

[27] A. Olshevsky and J. N. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 48(1):33–55, 2009.

[28] J. N. Tsitsiklis. Decentralized detection. In *Advances in Signal Processing, Vol. 2*, pages 297–344. JAI Press, 1993.
[29] J. N. Tsitsiklis and Z.-Q. Luo. Communication complexity of convex optimization. *Journal of Complexity*, 3(3):231–243, 1987.

[30] A. B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer, 2009.

[31] Y. Yang and A. Barron. Information-theoretic determination of minimax rates of convergence. *Annals of Statistics*, 27(5):1564–1599, 1999.

[32] A. C.-C. Yao. Some complexity questions related to distributive computing (preliminary report). In *Proceedings of the Eleventh Annual ACM Symposium on the Theory of Computing*, pages 209–213. ACM, 1979.

[33] B. Yu. Assouad, Fano, and Le Cam. In *Festschrift for Lucien Le Cam*, pages 423–435. Springer-Verlag, 1997.

[34] Y. Zhang, J. C. Duchi, M. I. Jordan, and M. J. Wainwright. Information-theoretic lower bounds for distributed statistical estimation with communication constraints. In *Advances in Neural Information Processing Systems 27*, 2013.

[35] Y. Zhang, J. C. Duchi, and M. J. Wainwright. Communication-efficient algorithms for statistical optimization. *Journal of Machine Learning Research*, 14:3321–3363, November 2013.