A Distributionally Robust Optimization Approach to Two-Sided Chance-Constrained Stochastic Model Predictive Control With Unknown Noise Distribution

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Abstract—In this work, we propose a distributionally robust stochastic model predictive control (DR-SMPC) algorithm to address the problem of multiple two-sided chance constrained discrete-time linear systems corrupted by additive noise. The prevalent mechanism to cope with two-sided chance constraints is the so-called risk allocation approach, which conservatively approximates the two-sided chance constraints with two single chance constraints by applying Bool’s inequality. In this proposed DR-SMPC framework, an exact second-order cone approach is adopted to abstract the multiple two-sided chance constraints by considering the first and second moments of the noise. With the proposed DR-SMPC algorithm, the worst-case probability of violating safety constraints is guaranteed to be within a prespecified maximum value. By flexibly adjusting this prespecified maximum probability, the feasible region of the initial state can be increased for the SMPC problem. The recursive feasibility and convergence of the proposed DR-SMPC are rigorously established by introducing a binary initialization strategy for the nominal state. A simulation study of a single spring and double mass system was conducted to demonstrate the effectiveness of the proposed DR-SMPC algorithm.

Index Terms—Distributionally robust optimization, second-order cone, stochastic model predictive control (SMPC), two-sided chance constraints.

I. INTRODUCTION

In almost all practical applications, the behavior of the system is falsified by various uncertainties, e.g., unknown parameters, external disturbances, and process noises. The controller may fail to guarantee safe operation or meet quality specifications when facing uncertainties. If the bound of the uncertainties can be quantified or are known, deterministic robust model predictive control (RMPC) approaches have been developed to address these intractable uncertainty problems [1], [2]. It should be highlighted that the robust constraint satisfaction, recursive feasibility, and stability are all established conservatively in RMPC by solving a min–max optimization problem, that is, minimizing the cost while considering the possible maximum impact (worst-case) of uncertainties. Also, the propagation of uncertainty through system dynamics, safe constraints, optimization, and control loops can lead to a very small feasible region of initial state and even no solutions to the optimization problem.

To tackle these limitations, stochastic model predictive control (SMPC) offers a promising way to reduce the inherent conservatism of RMPC by developing chance constraints that allow the constraints to be violated with a prespecified maximum probability [3], [4], [6]. SMPC provides a suitable method for achieving a tradeoff between closed-loop control performance and constraint satisfaction [7]. The potential of chance constraint quantification can be seen in many practical applications, e.g., concentration control in chemical reactors [4] or comfort control for robotic obstacle avoidance [5]. By applying the chance constraint, SMPC makes the feasible set of initial states larger as compared to RMPC.

In recent years, there have been considerable progresses in the field of SMPC research. SMPC approaches can be classified into two broad categories [4]. The first category is called randomized approaches [8], [9], [10], [11], [12]. The randomized approaches use samples or scenarios of the noise to approximate the SMPC problem. Probabilistic closed-loop guarantees can be established by using scenario optimization tools [9]. However, this requires a large amount of sampled data, posing significant demand on computational resources. Furthermore, it is generally quite hard to rigorously establish the recursive feasibility and closed-loop stability of these approaches [12].

The second category is referred to as analytical approximation approaches [6], [13], [14], [15], [16]. In this category, if the prior distribution of the noise is known, the chance constraint can be reformulated using the inverse of the cumulative probability function [6], [13]. However, it is usually quite difficult to acquire an accurate probability distribution of process noise in practice, which may result in the infeasibility or undesirable behavior of the system under SMPC. A milestone in this area [14], [15] is to use Cantelli's inequality to approximate the chance constraints by taking into account the mean and variance of the noise. Despite the fact that the computational burden of SMPC is greatly reduced, it can only handle one-sided chance constraints. However, in practical applications, two-sided constraints are pervasive, e.g., the acceleration of vehicles [17] and the temperature change of the building heating room [11] can be either positive or negative. Consequently, it is of great practical significance to investigate the multiple two-sided chance-constrained SMPC problem. One possible way of dealing with multiple two-sided opportunity constraints is to develop risk allocation mechanisms that decompose the constraints into a set of one-sided constraints that conservatively approximate the original chance constraints.

The risk allocation mechanisms mainly follow the two strategies below. The first strategy usually uses an uniform allocation strategy to obtain a fixed violation probability value for each one-sided constraint [18]. This approach simplifies the optimization problem, but it may lead to significant conservatism in many situations due to the loss of active risk allocation. To address this limitation, the second strategy uses risk allocation as the decision variable in the optimization
problem [20], [21]. The risk-sharing mechanism added to the SMPC scheme by Bool’s inequalities narrows the feasible set of initial state. To overcome the conservatism of the two-sided chance constraints handled by the existing SMPC approaches, a distributionally robust optimization approach is proposed to solve the multiple two-sided chance-constrained SMPC problem. Distributionally robust chance constraints have a direct connection to the chance constraints incorporated in the classical paradigms of SMPC [22]. In this technique, the actual distribution of noise is assumed to belong to an ambiguity set that contains all distributions with a predefined characterization (e.g., the first or second moments of noise). Given the aforementioned benefits, this technique is adopted in this work, where a less conservative approximation approach for distributionally robust chance constraints is developed by using a second-order cone reconstruction. Consequently, this proposed distributionally robust SMPC (DR-SMPC) can be reformulated as a convex optimization problem. Since assigning the value of the real measurement state to the nominal state may lead to infeasible optimization problems, a binary initialization strategy is further developed to determine the nominal state, ensuring the recursive feasibility and the convergence of the proposed method.

The differences between this article and other related distributionally robust optimization works (e.g., [12], [19], [23]) are summarized as follows. The distributionally robust chance constraints are first introduced into the SMPC problem in [12] and [23]. Only one-sided chance constraints are reformulated into computable expressions, which are incapable of dealing with two-sided chance constraints. In [19], a general second-order conic programming reformulation with robust chance constraints on the two-sided distribution was proposed to solve the optimal power flow problem, which is a normal optimization problem, rather than in the context of a receding horizon optimization paradigm. This article proposes a new mechanism to address multiple two-sided distributionally robust chance-constrained SMPC problems. Furthermore, a specific form of terminal constraint for the state is imposed, and the recursive feasibility and convergence are established. The main contributions are threefold, which are summarized as follows.

1. The proposed approach delivers a tractable conic optimization solution to handle two-sided chance constraints by using an ambiguity set or the first and second moments of noise.
2. A less conservative approach based on a second-order cone has been developed to abstract the distributionally robust chance constraints, which provides larger initial feasible sets compared to existing risk allocation approaches.
3. The recursive feasibility and convergence are rigorously established by introducing a binary initialization strategy to determine the nominal state.

**Notation:** The sets \( \mathbb{N}_0 = \{ k \in \mathbb{N} \mid k = 0, 1, 2, \ldots \} \), \( \mathbb{N}_{[0,H]} = \{ k \mid k = 0, 1, 2, \ldots, H \} \), and \( \mathbb{N} = \{ k \mid k = 1, 2, \ldots \} \), \( x_k, u_k \), and \( d_k, \bar{u}_k \) represent the current state and control input of the real and nominal systems at time step \( k \), respectively. \( x_{tk} \) and \( \bar{x}_{tk} \) are predicted states (\( l \) steps ahead at time step \( k \)) of the real and nominal dynamic systems, respectively. Similarly, \( u_{tk} \) and \( \bar{u}_{tk} \) are future control inputs (\( l \) steps ahead at time step \( k \)) of the real and nominal systems. \( \Delta x_{tk} \) is a state error between the predicted state \( x_{tk} \) and the nominal state \( \bar{x}_{tk} \) at time step \( k \). Specifically, \( x_{0:k} = x_k, \bar{x}_{0:k} = \bar{x}_k, \Delta x_{0:k} = \Delta x_k = \bar{x}_k - x_k, u_{0:k} = u_k, \) and \( \bar{u}_{0:k} = \bar{u}_k \).

### II. Problem Statement

#### A. Stochastic System

We consider a discrete-time linear system with additive stochastic noise

\[ x_{k+1} = Ax_k + Bu_k + w_k \]

where \( k \in \mathbb{N}_0 \) is a set of nonnegative integers is the discrete-time index, \( x_k \in \mathbb{R}^{n_x} \) is the state, and \( u_k \in \mathbb{R}^{n_u} \) is the input. \( w_k \in \mathbb{R}^{n_w} \) is an unknown stochastic noise with known mean \( \mu = 0 \) and covariance matrix \( W \geq 0 \). The known pair \( (A, B) \) is assumed to be stabilizable.

Given the state \( x_k \) at time step \( k \), the predicted state is updated according to

\[ x_{l+1|k} = Ax_{l|k} + Bu_{l|k} + w_{l|k}, x_{0|k} = x_k \]

where \( l \in \mathbb{N}_0 \).

#### B. Multiple Two-Sided Distributionally Robust Chance Constraints

The state and input are subject to chance constraints

\[
\begin{align*}
\Pr_{P} \left[ x_{l|k} \in \mathcal{X} | x_{k} \right] & \geq 1 - p_s, l \in \mathbb{N}_0 \\
\Pr_{P} \left[ u_{l|k} \in \mathcal{U} | x_{k} \right] & \geq 1 - p_s, l \in \mathbb{N}_0
\end{align*}
\]

where \( \Pr_{P}[\cdot] \) denotes the probability under the distribution \( P \) of uncertainties, \( \mathcal{X} = \{ x \in \mathbb{R}^{n_x} \mid H^T x \leq h \} \) and \( \mathcal{U} = \{ u \in \mathbb{R}^{n_u} \mid F^T u \leq f \} \) are convex sets containing the origin in its interior with constant matrices \( H \in \mathbb{R}^{p \times n_x} \), \( F \in \mathbb{R}^{q \times n_u} \), and constant vectors \( h \in \mathbb{R}^p \), \( f \in \mathbb{R}^q \), and \( p_s, q_s \in (0, 1) \) are the prespecified maximum probability that the constraints \( x_{l|k} \in \mathcal{X} \) and \( u_{l|k} \in \mathcal{U} \) are allowed to violate.

The chance constraint, also known as the value-at-risk constraint, is generally formulated as a deterministic nonconvex feasible set in the SMPC approach by a common assumption that the probability distribution \( P \) of the stochastic noise \( w_k \) is known. However, this assumption usually does not hold in reality since the true distribution \( P \) of the noise is unknown. To tackle this technique issue, a distributionally robust version of (3) and (4) in [12] and [23] is introduced as

\[
\begin{align*}
\inf_{P} \Pr_{P} \left[ x_{l|k} \in \mathcal{X} | x_{k} \right] & \geq 1 - p_s, l \in \mathbb{N}_0 \\
\inf_{P} \Pr_{P} \left[ u_{l|k} \in \mathcal{U} | x_{k} \right] & \geq 1 - p_s, l \in \mathbb{N}_0
\end{align*}
\]

with an ambiguity set

\[
\mathcal{P} = \left\{ P : \mathbb{E}[P][w_k] = 0, \mathbb{E}_P[w_k w_k^T] = W \right\}
\]

where \( \mathcal{P} \) is the set of all probability distributions on \( \mathcal{P} \) and \( \mathbb{E}_P[\cdot] \) denotes the expectation under distribution \( P \).

Unlike the existing SMPC methods, chance constraints (3) and (4) are studied in the framework of distributionally robust optimization. As shown in (5) and (6), the idea of the distributionally robust optimization is to optimize the “worst-case” distribution among all possible distributions in \( \mathcal{P} \).

In practical applications, most constraints are presented as multiple two-sided constraints [11], [17]. Therefore, it is of practical significance to study multiple two-sided chance constraints in the context of SMPC. The multiple two-sided distributionally robust chance constraints are defined as follows:

\[
\begin{align*}
\inf_{P \in \mathcal{P}} \Pr_{P} \left[ |a_r^T x_{l|k}| \leq b_r \right] & \geq 1 - p_s \\
\inf_{P \in \mathcal{P}} \Pr_{P} \left[ |c_s^T u_{l|k}| \leq d_s \right] & \geq 1 - p_s
\end{align*}
\]

with constant vectors \( a_r \in \mathbb{R}^{n_x} \) and \( c_s \in \mathbb{R}^{n_u} \), and constants \( b_r \in \mathbb{R} \) and \( d_s \in \mathbb{R} \). The inclusion of the absolute value notation indicates that these are two-sided constraints.
C. Cost Function

The objective function is defined as the sum of quadratic stage costs plus a terminal cost, given by

\[
J(x_k, u_k) = \mathbb{E}[\sum_{l=0}^{N-1} (x_{l+1:k}^T Q x_{l+1:k} + u_{l+1:k}^T R u_{l+1:k}) + x_{N+1:k}^T S x_{N+1:k})]
\]

where \(N \in \mathbb{N}\) is the prediction horizon, \(u_k = [u_{0:k}, u_{1:k}, \ldots, u_{N-1:k}]\) is optimized control sequence, \(Q \in \mathbb{R}^{n \times n}\) and \(R \in \mathbb{R}^{n \times n}\) are two known positive definite weighted matrices, and the terminal weighted matrix \(S \in \mathbb{R}^{n \times n}\) satisfies the following assumption.

Assumption 1: The terminal weighted matrix \(S\) is chosen as the solution of the following Lyapunov equation:

\[
(A + BK)^T S (A + BK) - S = -Q - K^T R K
\]

where \(K\) is a feedback gain to be computed.

D. Optimization Problem

The DR-SMPC optimization problem with multiple two-sided distributionally robust chance constraints over the prediction horizon \(N\) is now formally stated as DR-SMPC-P1:

\[
\min_{u_k} J(x_k, u_k)
\]

subject to \(x_{l+1:k} = Ax_{l:k} + Bu_{l:k} + w_{l:k}, l \in \mathbb{N}_{0,N-1}\)

\[
\inf_{P \in \mathcal{P}} \mathbb{P}[a_{l+1}^T x_{l+1:k} + b_{l+1} \leq c_{l+1}^T u_{l+1:k} + d_{l+1}] \geq 1 - p_r^e, l \in \mathbb{N}_{0,N-1}\)

\[
\inf_{P \in \mathcal{P}} \mathbb{P}[a_{l+1}^T x_{l+1:k} + b_{l+1} \leq c_{l+1}^T u_{l+1:k} + d_{l+1}] \geq 1 - p_r^e, l \in \mathbb{N}_{0,N-1}\)

\[
x_{0:k} = \bar{x}_k, r = 1, 2, \ldots, n_r, s = 1, 2, \ldots, n_s
\]

Unfortunately, it can be observed from (12) to (16) that DR-SMPC-P1 contains several sources of intractability: 1) the acquisition of the expectation in (12) is related to the unknown probability measure; 2) the multiple two-sided distributionally robust chance constraints (14)–(15) are generally intractable and nonconvex.

III. DR-SMPC Algorithm

In this section, we shall reformulate DR-SMPC-P1 into a computationally tractable conic optimization problem. To this end, we will find a state feedback structure.

A. Feedback Structure

We define the predicted nominal state and input as \(\bar{x}_{l+1:k}, \bar{u}_{l+1:k}\), and the nominal dynamics model of (2) is expressed as

\[
\bar{x}_{l+1:k} = A \bar{x}_{l:k} + B \bar{u}_{l:k}.
\]

To obtain the computable form of the cost function and two-sided distributionally robust chance constraints, a state feedback control law [1], [16] is designed as follows:

\[
u_{l+1:k} = K(x_{l+1:k} - \bar{x}_{l+1:k}) + \bar{u}_{l+1:k}
\]

where \(K\) is a selected feedback gain and the term \(\bar{u}_{l+1:k}\) replaces \(u_{l+1:k}\) as the new decision variable in DR-SMPC-P1.

A state error between the real predicted state \(x_{l+1:k}\) and the predicted nominal state \(\bar{x}_{l+1:k}\) at time step \(k\) is commonly denoted as \(\Delta x_{l+1:k} = x_{l+1:k} - \bar{x}_{l+1:k}\). Based on (2), (17), and (18), the state error dynamic system is expressed as

\[
\Delta x_{l+1:k} = (A + BK) \Delta x_{l+1:k} + u_{l+1:k}.
\]

At time step \(k = 0\), a proper initialization is used, i.e., \(\bar{x}_0 = x_0\), and recalling that the noise is zero-mean, the expected value of the stochastic error is \(\mathbb{E}[\Delta x_{l+1:k}] = 0\); thus, the predicted covariance matrix \(\Sigma_{l+1:k}\) is updated as

\[
\Sigma_{l+1:k} = \mathbb{E}[\Delta x_{l+1:k} \Delta x_{l+1:k}^T] = (A + BK) \Sigma_{l+1:k} (A + BK)^T + W.
\]

The equivalent sets of the two-sided distributionally robust chance constraints are formed by simply substituting \(x_{l+1:k} = \bar{x}_{l+1:k} + \Delta x_{l+1:k}\) and \(u_{l+1:k} = \bar{u}_{l+1:k} + K \Delta x_{l+1:k}\) into (14)–(15), which are given by

\[
\begin{align*}
\bar{x}_{l+1:k} &= \left\{ x_{l+1:k} \bigg| \inf_{P \in \mathcal{P}} \mathbb{P}\left[ a_{l+1}^T x_{l+1:k} + b_{l+1} \geq 1 - p_r^e \right] \right\} \\
\bar{u}_{l+1:k} &= \left\{ u_{l+1:k} \bigg| \inf_{P \in \mathcal{P}} \mathbb{P}\left[ c_{l+1}^T (u_{l+1:k} + K \Delta x_{l+1:k}) \geq 1 - p_r^e \right] \right\}
\end{align*}
\]

B. Convex Formulation of DR-SMPC-P1

Based on the results above, we show that DR-SMPC-P1 can be represented as a convex cone program problem and hence is computationally tractable.

Theorem 1: DR-SMPC-P1 is exactly reformulated as the following conic optimization problem, which is referred to as DR-SMPC:

\[
\begin{align*}
\min_{u_k} & \sum_{l=0}^{N-1} (\bar{x}_{l+1:k}^T Q \bar{x}_{l+1:k} + \bar{u}_{l+1:k}^T R \bar{u}_{l+1:k}) + \text{trace} \left( \left( Q + K^T R K \right) \Sigma_{l+1:k} \right) \\
& + \text{trace} \left( S \Sigma_{N+1:k} \right)
\end{align*}
\]

subject to \(x_{l+1:k} = A \bar{x}_{l:k} + B \bar{u}_{l:k}, l \in \mathbb{N}_{0,N-1}\)

\[
x_{l+1:k} = A x_{l+1:k} + B u_{l+1:k} + w_{l+1:k}, l \in \mathbb{N}_{0,N-1}\)

\[
u_{l+1:k} = K(x_{l+1:k} - \bar{x}_{l+1:k}) + \bar{u}_{l+1:k}, l \in \mathbb{N}_{0,N-1}\)

\[
\bar{x}_{l+1:k} \in \mathcal{X}_{l+1:k}, r = 1, 2, \ldots, n_r, l \in \mathbb{N}_{0,N-1}\)

\[
\bar{u}_{l+1:k} \in \mathcal{U}_{l+1:k}, s = 1, 2, \ldots, n_s, l \in \mathbb{N}_{0,N-1}\)

\[
\bar{x}_{l+1:k} = \{ x_k \text{ or } \bar{x}_{l+1:k} \}, \bar{x}_{N+1:k} \in \mathcal{X}_{l+1:k}
\]

where \(u_k = [\bar{u}_{0:k}, \bar{u}_{1:k}, \ldots, \bar{u}_{N-1:k}]\), \(y_{(r,i)}\), \(y_{(s,i)}\), \(\lambda_{(r,i)}\), and \(\lambda_{(s,i)}\) are the newly introduced decision variables, \(\mathcal{X}_{l+1:k}\) is a terminal constraint to guarantee recursive feasibility and convergence of the proposed approach, and \(\bar{X}_{l+1:k}\) and \(\bar{U}_{l+1:k}\) are given by

\[
\bar{X}_{l+1:k} := \left\{ x_{l+1:k} \bigg| \begin{array}{c}
(a_{l+1}^T \Sigma_{l+1:k} a_{l+1} - \lambda_{(r,i)})^2 \\
(a_{l+1}^T \Sigma_{l+1:k} a_{l+1} - \lambda_{(s,i)} - y_{(r,i)})^2
\end{array} \\
y_{(r,i)} - \lambda_{(r,i)} \leq 0 \land \lambda_{(s,i)} \leq 0 \land y_{(r,i)} \geq 0 \land y_{(s,i)} \geq 0 \land r = 1, 2, \ldots, n_r.
\right\}
\]
and
\[
\tilde{U}_{l|k} := \left\{ \tilde{u}_{l|k} \mid y_{l|k}^2 + c_l^T K^T \Sigma_l K c_l \leq p_l^2(d_l - \lambda_{l(s,l)})^2, \right. \\
y_{l|k} \geq 0, 0 \leq \lambda_{l(s,l)} \leq d_l, s = 1, 2, \ldots, n_s. \}
\]  

(31)

Proof: See Appendix A.

Note that DR-SMPC is now converted to a conic optimization problem, which is computationally tractable and can be solved by using standard software packages such as CVX [25]. The optimization is carried out using the optimal nominal control sequence \( \tilde{u}_k \), and the first element \( \tilde{u}_{1|k} \) is applied to the feedback control law \( u_k^* = K(x_k - \tilde{x}_k) + \tilde{u}_k \). This is crucial to solve the two-sided chance-constrained problem. However, the determination of the terminal set as well as the feasibility set of the initial nominal state is rather difficult but particularly important for establishing recursive feasibility and convergence of the proposed approach.

C. Determination of Terminal Constraint

It is well known that the terminal set is closely related to the recursive feasibility and stability of the nominal system. Thus, the following terminal set \( \tilde{X}_f \) is imposed:
\[
\tilde{X}_N|k \in \tilde{X}_f \subseteq \tilde{X}_{l|k}. 
\]  

(32)

To satisfy the above condition, the terminal set is chosen as
\[
\tilde{X}_f = \left\{ x_{N|k} \mid a_r^T x_{N|k} \leq y_r + \lambda, r = 1, 2, \ldots, n_r, \right. \\
\] where \( y_r \) and \( \lambda_r \) are extra design parameters and satisfy \( y_r^2 + a_r^T \Sigma r a_r \leq p_r^2(b_r - \lambda, r) \). Therefore, the additional design parameters \( y_r \) and \( \lambda_r \) satisfy \( 0 \leq y_r \leq y_{(r,l)} + \lambda_{(r,l)} \), which implies \( \tilde{X}_{N|k} \in \tilde{X}_f \). \( \Box \)

D. Determination of Feedback Gain \( K \)

In this section, the feedback gain \( K \) is designed in a systematic way to satisfy the requirements for proofs of recursive feasibility and convergence of the proposed approach in Section IV. To be specific, the feedback gain \( K \) is proposed to be designed by solving the following optimization problem:
\[
\begin{align*}
\min_{y_k, \lambda_k} & y_k \\
\text{s.t.} & y_k^2 + c^T K^T \Sigma K c \leq p^2(d - \lambda)^2, y_k \geq 0 \\
& K^T c_l c_l^T \leq \frac{a_l a_l^T}{(y_l + \lambda_l)^2} \\
& (A + BK)^T a_l a_l^T (A + BK) \geq a_l a_l^T \\
& 0 \leq \lambda_k \leq d_k, k = 1, 2, \ldots, n_k, r = 1, 2, \ldots, n_r.
\end{align*}
\]  

where \( y_k \) and \( \lambda_k \) are extra decision variables, and \( K \) will satisfy the following conditions:

(i) Proof of Condition (a): To begin with, a set satisfying \( \tilde{U}_f \subseteq \tilde{U}_{N|k} \), similar to the chosen terminal set \( \tilde{X}_f \), is chosen as
\[
\tilde{U}_f := \left\{ u_{N|k} \mid y_{N|k}^2 + c^T K^T \Sigma K c \leq p^2(d_k - \lambda_k)^2, \right. \\
y_{N|k} \geq 0, 0 \leq \lambda_k \leq d_k, s = 1, 2, \ldots, n_s. \}
\] Keeping in mind sets \( \tilde{X}_f \) and \( \tilde{U}_f \) defined above, \( |a_r^T x_{N|k}| \leq y_r + \lambda_r \) and \( |c_l^T K x_{N|k}| \leq y_r + \lambda_r \) indicate that
\[
\begin{align*}
\bar{x}^T_{N|k} a_r a_r^T \bar{x}_{N|k} \leq (y_r + \lambda_r)^2 \implies \bar{x}^T_{N|k} \left[ \frac{a_r a_r^T}{(y_r + \lambda_r)^2} \right] \bar{x}_{N|k} \leq 1 \\
\end{align*}
\] and
\[
\bar{x}^T_{N|k} K^T c_l c_l^T \bar{x}_{N|k} \leq (y_r + \lambda_r)^2 \implies \bar{x}^T_{N|k} K^T \left[ \frac{c_l c_l^T}{(y_r + \lambda_r)^2} \right] K \bar{x}_{N|k} \leq 1.
\]

Since the chosen feedback gain \( K \) satisfies (35), one can obtain that \( \bar{x}^T_{N|k} K^T \left[ \frac{c_l c_l^T}{(y_r + \lambda_r)^2} \right] K \bar{x}_{N|k} \leq 1 \) holds for \( \bar{x}_{N|k} \in \{ \bar{x}_{N|k} \mid \bar{x}_{N|k} \subseteq \bar{X}_{N|k} \} \), i.e., \( \bar{X}_{N|k} \subseteq \bar{U}_{N|k} \subseteq \bar{U}_f \). Therefore, the feedback gain \( K \) solved from the optimization problem (33)–(37) satisfies the necessary conditions (a) and (b) for proofs of recursive feasibility and convergence of the closed-loop system.

E. Binary Initialization Strategy

It is quite clear that the initial condition \( x_k \) is critical to the performance index. At each step, the latest information about the actual state should be used to reset the nominal state \( x_k \). Specifically, the selected “optimal” current value of \( x_k \) is set to \( x_k \). However, since the possibility of unbounded noise cannot be completely ruled out, the choice of \( x_k = x_k \) may lead to infeasibility of the optimization problem, and the basic property of recursive feasibility would be lost. Therefore, the following binary initialization strategy is adopted to guarantee the recursive feasibility.

Strategy 1: Using the most recent information available on the measured state at time step \( k \), i.e., \( \bar{x}_k \).

Strategy 2: Using the updated information according to the past optimal solution, i.e., \( \bar{x}_k = x_{k-1} + A \bar{x}_{k-1} + B \bar{u}_{k-1} \).

Remark 1: Note that if Strategy 2 is executed, the input applied on the real system is \( u_k = K(x_k - \bar{x}_k) + \bar{u}_k \), which implies that the real system state \( x_{k+1} \) converges to the nominal system state \( x_{k+1} \) for \( x_k = x_{k+1} \) to \( x_{k+1} = (A + BK)(x_k - \bar{x}_k) + w_k \). Strategy 2, i.e., indirect feedback approach, ensures the recursive feasibility of SMPC [12], [16], but does not guarantee optimality. It should be highlighted that this binary initialization strategy is widely used in stochastic MPC to ensure the recursive feasibility and the convergence, for example, see [15], [23], [26].

It is important to note that DR-SMPC with Strategy 1 must be feasible at time step \( k = 0 \). Thus, we define the feasibility set of the initial state for DR-SMPC as
\[
F = \{ x_0 \mid \bar{x}_0 = x_0 \} \text{DR-SMPC with Strategy 1 is feasible}.
\]
Algorithm 1: DR-SMPC.

Require:

Input: $A, B, a_r, b_r, c_s, d_s, W, Q, R, x_k, \bar{x}_k, p^r_s, p^u_s$ and $N$;

Output: $x_{k+1}, \bar{x}_{k+1}$;

Initialize: $x_k = x_h$ at time step $k = 0$, DR-SMPC is feasible;

Off-line: Compute the feedback gain $K$ and the terminal weight $S$ according to the optimization problem (33)–(37) and Assumption 1;

On-line:

while 1 do

Step 1: Solve DR-SMPC with Strategy 1 and DR-SMPC with Strategy 2;

Step 2: if DR-SMPC with Strategy 1 is feasible then

Compare the optimal cost obtained by executing Strategy 1 and Strategy 2;

if the optimal cost obtained by executing Strategy 1 is lower than Strategy 2 then

DR-SMPC with Strategy 1 is executed;

else

DR-SMPC with Strategy 2 is executed;

end if

else

DR-SMPC with Strategy 2 is executed;

end if

Step 3: Acquire nominal input $\bar{u}^*_k$, and the control input $u^*_k = K(x_k - \bar{x}_k) + \bar{u}^*_k$;

Step 4: $u^*_k$ is applied to the system (1);

Step 5: Compute $\bar{x}_{k+1}$ according to (7), and measure the state $x_{k+1}$;

Step 6: Update $k = k + 1$.

end while

The rules for adopting Strategies 1 and 2 at time step $k > 0$ are stated as follows: DR-SMPC with Strategy 1 and DR-SMPC with Strategy 2 are first solved. If DR-SMPC with Strategy 1 is feasible and the optimal cost obtained by executing Strategy 1 is lower than that of Strategy 2, DR-SMPC with Strategy 1 is executed. Otherwise, DR-SMPC with Strategy 2 should be carried out.

F. DR-SMPC Algorithm

Based on Theorem 1 and the binary initialization strategy of the nominal state, the implementation of the DR-SMPC algorithm is described in Algorithm 1.

IV. RECURSIVE FEASIBILITY AND CONVERGENCE

In this section, the recursive feasibility and the convergence of the proposed DR-SMPC algorithm are established.

A. Recursive Feasibility

The recursive feasibility of the proposed DR-SMPC algorithm is established by Theorem 2 below.

Theorem 2: If DR-SMPC is feasible at time step $k$, and the chosen terminal set $\bar{X}_f$ and feedback gain $K$ satisfy the following conditions: i) $\bar{X}_f \subseteq \bar{X}_{\bar{u}^*_k}$; ii) $u^*_N(k) = K x_N(k) \in \bar{U}_N(k)$, $\forall x_N(k) \in \bar{X}_f$; iii) $x_{N+1}|k = A x_N|k + B u^*_N|k + (A + BK) x_N|k \in \bar{X}_f \forall x_N|k \in \bar{X}_f$. Then, it remains feasible at time step $k + 1$.

Proof: See Appendix B.

Remark 3: Note that if DR-SMPC with Strategy 1 or 2 is feasible at time step $k$, two cases (Strategies 1 and 2) are considered to ensure that DR-SMPC is feasible at time step $k + 1$.

Case 1: If DR-SMPC with Strategy 1 is feasible at time step $k + 1$, which means that the conditions for the recursive feasibility hold.

Case 2: If DR-SMPC with Strategy 1 is not feasible at time step $k + 1$, DR-SMPC with Strategy 2 is executed.

Therefore, we only need to prove that DR-SMPC with Strategy 2 is feasible at time step $k + 1$.

B. Convergence

The main result concerning the convergence properties of the algorithm is established by the following theorem.

Theorem 3: If Theorem 2 holds, let $J(\bar{x}_k, \bar{u}^*_k)$ be the optimal cost obtained by executing DR-SMPC with Strategy 1 or 2 at time step $k$ and $J(\bar{x}_{k+1}, \bar{u}^*_N|k)$ be the optimal cost obtained by executing DR-SMPC with Strategy 2 at time step $k$. The following inequality holds:

$$J(\bar{x}_{k+1}, \bar{u}^*_N|k) - J(\bar{x}_k, \bar{u}^*_k) \leq -E_p[x_i^T Q x_k + u^*_k^T R u_k] + \text{trace}(SW).$$

(38)

Proof: See Appendix C.

Remark 3: The optimal cost function obtained by executing DR-SMPC with Strategy I at $k + 1$ is denoted as $J_1(\bar{x}_{k+1}, \bar{u}^*_N|k)$. If DR-SMPC with Strategy 1 is feasible and the optimal cost obtained by executing Strategy 1 is lower than that of Strategy 2 at time step $k + 1$, i.e., $J_1(\bar{x}_{k+1}, \bar{u}^*_N|k) \leq J(\bar{x}_{k+1}, \bar{u}^*_N|k)$, DR-SMPC with Strategy 1 is executed according to the rules. Thus, the optimal cost obtained by executing DR-SMPC with Strategy 1 is expected to satisfy

$$J_1(\bar{x}_{k+1}, \bar{u}^*_N|k) - J(\bar{x}_k, \bar{u}^*_k) \leq -E_p[x_i^T Q x_k + u^*_k^T R u_k] + \text{trace}(SW).$$

(39)

Keep $J_1(\bar{x}_{k+1}, \bar{u}^*_N|k) \leq J(\bar{x}_{k+1}, \bar{u}^*_N|k)$ in mind, inequality (39) holds if inequality (38) is satisfied, which implies that DR-SMPC with Strategy 2 is sufficient to guarantee convergence of the proposed approach.
According to Theorem 3 and a similar conclusion presented in [7], [8], [11], and [27], we have the following:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \mathbb{E}[P] \left[ x^T_k Q x_k + u^T_k R u_k \right] \leq \text{trace}(SW). \tag{40}
\]

This indicates that the state of the system is driven toward a neighborhood of a steady-state condition. Then, system (1) is convergent under the DR-SMPC control law (18).

V. SIMULATION EXAMPLE

In this section, a simulation example is provided to demonstrate the performance of the proposed DR-SMPC algorithm. The proposed DR-SMPC algorithm is compared with the G-SMPC algorithm in [6] and the P-SMPC algorithm in [20].

We consider a two-mass spring system in the example [11], described by

\[
x^{(1)}_k = x^{(3)}_k \tag{41}
\]
\[
x^{(2)}_k = x^{(4)}_k \tag{42}
\]
\[
m_1 x^{(3)}_k = -k_s (x^{(1)}_k - x^{(2)}_k) + u_k + w^{(1)}_k \tag{43}
\]
\[
m_2 x^{(4)}_k = k_s (x^{(1)}_k - x^{(2)}_k) + w^{(2)}_k \tag{44}
\]

where \(m_1\) and \(m_2\) are the masses of two blocks linked by springs of coefficient \(k_s\), \(u\) is the manipulated variable, \(x^{(1)}\), \(x^{(2)}\) and \(x^{(3)}\), \(x^{(4)}\) are the distances and speeds of the two blocks, and \(w^{(1)}\) and \(w^{(2)}\) are uncertainties acting on masses \(m_1\) and \(m_2\), respectively. The parameters are chosen as \(m_1 = 1\), \(m_2 = 1\), and \(k_s = 1.25\).

The objective of this example is to design a controller to regulate the state to the origin in the presence of noise. The quadratic cost matrices of the proposed approach are selected as \(Q = \text{diag}(1, 1, 4, 6)\) and \(R = 1\). The initial condition is \(x_0 = [0.5, 0.5, 0, 0]^T\) and the prediction horizon is \(N = 7\). To guarantee safe operation, we consider constraints on the state and input as \(\Pr[P]\{ |x^{(3)}_k| \leq 0.12 \} \geq 0.8\), \(\Pr[P]\{ |x^{(4)}_k| \leq 0.12 \} \geq 0.8\), \(\Pr[P]\{ |u_k| \leq 0.5 \} \geq 0.99\).

The initial feasible sets under DR-SMPC, G-SMPC, and P-SMPC algorithms are shown in Fig. 1, where \(w_k\) obeys a Gaussian distribution with zero mean and variance of 0.07\(I_2\) at each time step. Apparently, the initial feasible set of the proposed DR-SMPC algorithm is 1.5 times larger than that of P-SMPC algorithm, while the P-SMPC algorithm is more conservative than the DR-SMPC algorithm due to the limitations of the risk allocation mechanism. On the other hand, as for the dimensions of the obtained initial feasibility set, the use of the distributionally robust approach does not result in any significant performance degradation in comparison to the G-SMPC algorithm, even if the noise distribution assumed by the G-SMPC algorithm is consistent with the actual system.

Both the G-MPC and P-SMPC algorithms are based on a fixed uniform risk allocation. A total of 1000 Monte Carlo runs of simulation is conducted for each of the three algorithms and the results are shown in Fig. 2, where \(w_k\) obeys a Laplace distribution with zero mean and 0.07\(I_2\) variance at each time step. Fig. 3 provides the statistics for the percentage of constraint violations in the experiments, where the black dashed line indicates the maximum percentage of violations allowed. It is shown that the maximum percentage of constraint violation for the G-SMPC, the DR-SMPC, and P-SMPC algorithms in 1000 tracks reached 46.2%, 14.9%, and 2.8%, respectively. When using the P-SMPC algorithm, the probability of violating the constraint is much lower than the set value of 20%. This is due to the fact that
the two-sided chance constraint is approximately decomposed into two one-sided chance constraints, leading to conservatism. However, the probability of constraint violation when using the G-MPC algorithm exceeds the set value of 20% because the deviation from the original distribution leads to undesirable behavior of the system.

As a consequence, we can conclude that the P-SMPC algorithm is more conservative than DR-SMPC owing to the limitations of the risk allocation mechanism. For the G-SMPC algorithm, if the noise is Gaussian distributed, the inverse of the cumulative probability function can be used to reformulate the two-sided chance constraint, which leads to better output performance, such as a larger initial feasible set. If the assumptions of the noise distribution are not consistent with the real world, it may lead to undesirable behavior, such as the probability of violating the constraint exceeding the set value.

VI. CONCLUSION

This article advocates a distributionally robust optimization approach for stochastic systems satisfying two-sided chance constraints under unbounded stochastic noise of unknown distribution. A conic representation of multiple two-sided distributionally robust chance constraints is obtained, and consequently, a tractable formulation of the SMPC problem is developed. The recursive feasibility and convergence constraints is obtained, and consequently, a tractable formulation of the SMPC problem is developed. The recursive feasibility and convergence constraints is obtained, and consequently, a tractable formulation of the SMPC problem is developed.

APPENDIX

A. Proof of Theorem 1

Note that \(\bar{x}_{l|k} = \Delta x_{l|k} + \bar{x}_{l|k}\), \(u_{l|k} = \bar{u}_{l|k} + K \Delta x_{l|k}\), \(E_{[\bar{P}]}[\Delta x_{l|k}] = 0\) in (19), \(E_{[\bar{P}]}[\Delta x_{l|k}^T Q \Delta x_{l|k}] = \text{trace}(Q \Sigma_{l|k})\), and \(E_{[\bar{P}]}[\Delta x_{N|k}^T \Sigma \Delta x_{N|k}] = \text{trace}(S \Sigma_{N|k})\). As consequence, (10) is equivalent to the following equation:

\[
J(\bar{x}_k, \bar{u}_k) = \sum_{l=0}^{N-1} \left[ \bar{x}_{l|k}^T Q \bar{x}_{l|k} + \bar{u}_{l|k}^T R \bar{u}_{l|k} + \bar{x}_{N|k}^T S \bar{x}_{N|k} \right]
+ \sum_{l=0}^{N-1} \text{trace} \left[ \left( Q + K^T R K \right) \Sigma_{l|k} \right]
+ \text{trace} \left( S \Sigma_{N|k} \right). \tag{45}
\]

We will use a distributionally robust optimization approach to reformulate the two-sided chance constraints into the second-order cone form. Moreover, we will show that this reformulation is exact according to the following lemma [19].

Lemma 1: The two-sided distributionally robust chance constraints

\[
\inf_{P \in \mathcal{P}} \text{Pr}_{[\bar{P}]} \left[ \left| a_\epsilon(x)^T w + b_\epsilon(x) \right| \leq T_\epsilon \right] \geq 1 - \epsilon \tag{46}
\]

are reformulated into the following convex second-order cone constraints

\[
y_\epsilon^2 + a_\epsilon(x)^T S a(x) \leq \epsilon(T_\epsilon - \lambda_\epsilon)^2 \tag{47}
\]

\[
|b_\epsilon(x)| \leq y_\epsilon + \lambda_\epsilon, T_\epsilon \geq \lambda_\epsilon \geq 0, y_\epsilon \geq 0 \tag{48}
\]

where \(a_\epsilon(x)\) and \(b_\epsilon(x)\) are the functions of the decision variable \(x\); and \(y_\epsilon\) and \(\lambda_\epsilon\) are two involving additional variables, and uncertainty \(w\) satisfies \(\mathcal{P} = \{ P : E_{[\bar{P}]}[w] = 0, E_{[\bar{P}]}[ww^T] = \Sigma \}\).

Letting \(a_\epsilon(x) = a_\epsilon, b_\epsilon(x) = a_\epsilon^T \bar{x}_{l|k}, w = \Delta x_{l|k}\) and \(T_\epsilon = b_\epsilon\) and keeping Lemma 1 in mind, (21) is reformulated into (30). In a similar manner, (22) is reformulated into (31). This completes the proof.

B. Proof of Theorem 2

According to Section III-C, the chosen terminal set satisfy \(\bar{X}_f \subseteq X_{l|k}\). At time step \(k\), if DR-SMPC is feasible, the corresponding optimal solution is the nominal control sequence \(\bar{u}_k = [\bar{u}_{l|k} = \bar{u}_{l|k}, \bar{u}_{l|k}, \ldots, \bar{u}_{N-1|k}]\) with the current and predicted nominal state trajectory \(\bar{x}_k = [\bar{x}_{0|k} = \bar{x}_k, \bar{x}_{l|k}, \bar{x}_{l|k}, \ldots, \bar{x}_{N-1|k}, \bar{x}_{N|k}]\).

At next time step \(k+1\), the suboptimal control sequence is denoted by \(\bar{u}_{k+1} = [\bar{u}_{l|k+1}, \bar{u}_{l|k+1}, \ldots, \bar{u}_{N-1|k+1}] = [\bar{u}_{l|k+1}, \bar{u}_{l|k+1}, \ldots, \bar{u}_{N-1|k+1}, K \bar{x}_{N|k}]\). According to Section III-D, \(K \bar{x}_{N|k} \in \bar{U}_{N|k} \forall \bar{x}_{N|k} \in \bar{X}_f\), there must be \(\bar{x}_{N|k+1} = \bar{x}_{N|k+1} = (A + BK) \bar{x}_{N|k} \in \bar{X}_f\).

Therefore, DR-SMPC remains feasible at time step \(k+1\).

C. Proof of Corollary 3

Considering the optimality of \(J\), we have

\[
J(\bar{x}_{k+1}, \bar{u}_{k+1}) = J(\bar{x}_{k+1}, \bar{u}_{k+1}). \tag{49}
\]

According to (45), we have

\[
J(\bar{x}_{k+1}, \bar{u}_{k+1})
= J(\bar{x}_k, \bar{u}_k) - \bar{x}_{l|k}^T Q \bar{x}_{l|k} - \bar{u}_{l|k}^T R \bar{u}_{l|k} - \bar{x}_{N|k}^T S \bar{x}_{N|k}
- \text{trace} \left[ \left( Q + K^T R K \right) \Sigma_{l|k} \right] - \text{trace} \left( S \Sigma_{N|k} \right)
+ \bar{x}_{N|k}^T Q \bar{x}_{N|k} + \bar{u}_{N|k}^T R \bar{u}_{N|k}
+ \text{trace} \left[ \left( Q + K^T R K \right) \Sigma_{N|k} \right] + \text{trace}(S \Sigma_{N|k}) . \tag{50}
\]

In view of the definition of \(S\) given in (11), we have

\[
\bar{x}_{N|k}^T Q \bar{x}_{N|k} - \bar{x}_{N|k}^T S \bar{x}_{N|k} + \bar{u}_{N|k}^T R \bar{u}_{N|k}
= \bar{x}_{N|k}^T [(A + BK)^T S(A + BK) + Q + K^T R K - S] \bar{x}_{N|k}
= 0 \tag{51}
\]

and

\[
\text{trace}(S \Sigma_{N|k}) = \text{trace}(S \Sigma_{N|k}) + \text{trace}(S \Sigma_{N|k}) = \text{trace}(SW). \tag{52}
\]

Combining (49)–(52), one has

\[
J(\bar{x}_{k+1}, \bar{u}_{k+1}) - J(\bar{x}_k, \bar{u}_k)
\leq -E_{[\bar{P}]}[\bar{x}_{l|k}^T Q \bar{x}_{l|k} + \bar{u}_{l|k}^T R \bar{u}_{l|k}] + \text{trace}(SW) . \tag{53}
\]

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