PAIRS OF PYTHAGOREAN TRIANGLES WITH GIVEN CATHETI RATIOS

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Abstract. In this note we investigate the problem of finding pairs of Pythagorean triangles \((a, b, c), (A, B, C)\), with given catheti ratios \(A/a, B/b\). In particular, we prove that there are infinitely many essentially different ("non-similar") pairs of Pythagorean triangles \((a, b, c), (A, B, C)\) satisfying given proportions, provided that \(Aa \neq Bb\).

1. Introduction

In [2] the author investigated the following problem stated by Leech (as was remarked by Smyth in [3]):

"Find two rational right-angled triangles on the same base whose heights are in the ratio \(n : 1\) for \(n\) an integer greater than 1."

By considering an equivalent elliptic curve problem, he find parametric solutions for certain values of \(n\) and present results of a numerical search for solutions with \(n \in \{1, \ldots, 999\}\). Motivated by his findings we deal with the general problem of finding pairs of Pythagorean triangles with given catheti ratios. More precisely, throughout the paper \((a, b, c)\) will denote a Pythagorean triple, i.e. a triple of positive integers satisfying the condition

\[a^2 + b^2 = c^2 \quad \text{and} \quad a, b, c \in \mathbb{N} = \{1, 2, \ldots\}.
\]

Given two such triples \((a, b, c)\) and \((A, B, C)\) we are interested whether there exist essentially different pairs \((a', b', c'), (A', B', C')\) satisfying

\[
\frac{A'}{a'} = \frac{A}{a}, \quad \frac{B'}{b'} = \frac{B}{b}.
\]

As we will see in the sequel under some mild conditions on the pair of triples \((a, b, c), (A, B, C)\), the problem has a positive answer. More precisely, in Section 2 we prove that there are infinitely many essentially different pairs \((a', b', c'), (A', B', C')\) solving (1) provided that \(Aa \neq Bb\). In Section 3 we investigate the case \(Aa = Bb\) and prove

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that there are infinitely many pairs \((a, b, c), (A, B, C)\) such that the system \([1]\) has infinitely many solutions \((a', b', c'), (A', B', C')\). We also present results of our numerical search and observe that for certain pairs \((a, b, c), (A, B, C)\) there is no non-trivial solution. Finally, in the last section we investigate the general problem of finding pairs of Pythagorean triangles \((a, b, c), (A, B, C)\) such that \(A/a = \mu, B/b = \nu\), where \(\mu, \nu\) are given rational numbers. We reduce the problem to the case when \(\mu = 1\) and prove that the set of those \(\nu\) is dense in the Euclidean topology in \(\mathbb{Q}^+\).

2. The case \(Aa \neq Bb\)

In this section we are interested in finding the solutions of system \([1]\) under the assumption \(Aa \neq Bb\). More precisely, together with given triples \((a, b, c)\) and \((A, B, C)\) we will consider the following elliptic curve

\[
H : v^2 = u(a^2u + b^2)(A^2u + B^2)
\]

The basic observation is that each point \(Q = (u, v) \in H(\mathbb{Q})\) of the form \(2P\) \((P \in H(\mathbb{Q}))\) gives the desired pair of triangles \((a', b', c'), (A', B', C')\). Namely, if \((u, v) = P + P\) then \(u = s^2, a^2u + b^2 = t^2\) and \(A^2u + B^2 = r^2\) with some \(s, t, r \in \mathbb{Q}\). Hence

\[
\begin{align*}
(as)^2 + b^2 &= t^2 \\
(As)^2 + B^2 &= r^2
\end{align*}
\]

and \((a', b', c') = (as, b, t), (A', B', C') = (As, B, r)\) do work. The point \(Q' = (1, cC)\) corresponds to the initial situation \((a', b', c') = (a, b, c), (A', B', C') = (A, B, C)\). Hence the natural question in this context is whether the point \(Q'\) is of infinite order or (at least) is the rank of \(H\) positive? We will investigate both problems in the sequel.

After change of variables \(x = a^2A^2u, y = a^2A^2v\) we get a standard form

\[
E : y^2 = x(x + A^2b^2)(x + a^2B^2),
\]

and the image of the point \(Q'\) takes the form \(Q^* = (a^2A^2, a^2A^2cC)\). The group \(E(\mathbb{Q})\) contains exactly 3 points of order 2, hence the torsion part \(\text{Tor}(E(\mathbb{Q}))\) cannot be cyclic. By the Mazur theorem \(\text{Tor}(E(\mathbb{Q})) = \mathbb{Z}_2 \times \mathbb{Z}_{2n}\) for some \(n \in \{1, 2, 3, 4\}\). If \(P + P = (0, 0)\) then \(|P| = 4\), hence \(n = 2\) or \(n = 4\). It follows that if \(Q^*\) has a finite order then \(|Q^*| = 4\) and \(|2Q^*| = 2\). By duplication formula

\[
x(2Q^*) = \frac{(a^2A^2 - B^2b^2)^2}{4c^2C^2}
\]
we infer that $|2Q^*| = 2$ if and only if $Aa = Bb$. Geometrically this means that $(a,b,c)$ and $(A,B,C)$ are "skew-similar". In other cases the point $Q^*$ has infinite order. In this way we have proved

**Theorem 2.1.** If $Aa \neq Bb$ then there exist infinitely many essentially different ("non-similar") pairs of Pythagorean triangles $(a',b',c')$, $(A',B',C')$ satisfying given proportions (1).

3. The case $Aa = Bb$

In previous section we have proved that provided $Aa \neq Bb$, then there are infinitely many Pythagorean triangles $(a',b',c')$, $(A',B',C')$ satisfying given proportions (1).

What is going on in the case $Aa = Bb$? First of all, let us note that primitivity of triangles $(a,b,c)$, $(A,B,C)$ immediately implies that $A = b, B = a$. Then $C = c$ and we deal with the case of cross-similarity of pairs of Pythagorean triangles. In other words, we are interested in the existence of rational points on the curve

$$E_{a,b}: y^2 = x(x + a^4)(x + b^4).$$

We prove

**Theorem 3.1.** There are infinitely many primitive Pythagorean triples $(a,b,c)$ such that the elliptic curve

$$E_{a,b}: y^2 = x(x + a^4)(x + b^4)$$

has positive rank.

*Proof.* Let $a = 2uv, b = u^2 - v^2, c = u^2 + v^2$ be a Pythagorean triple and write $f(x) = x(x + a^4)(x + b^4)$. We note the crucial identity

$$f(-a^3b) = (a^3b(b - a))^2 (u^4 + 2u^3v + 2u^2v^2 - 2uv^3 + v^4).$$

In other words $f(-a^3b)$ is a square if and only if there is a rational point on the quartic curve

$$C : W^2 = U^4 + 2U^3 + 2U^2 - 2U + 1.$$  

The curve $C$ is birationally equivalent with the elliptic curve

$$E' : Y^2 = (X - 24)(X - 6)(X + 30),$$

via the mapping

$$\phi : C \ni (U,W) \mapsto (X,Y) \in E',$$

where

$$(X,Y) = (6(3U^2 + 3U + 1 - 3W), 54(2U^3 + 3U^2 + 2U - 1 - (1 + 2U)W)).$$
and the inverse $\phi^{-1} : E' \ni (X, Y) \mapsto (U, W) \in C$ is given by

$$(U, W) = \left( \frac{72 + Y - 3X}{6(X + 3)}, \frac{Y^2 + 162Y + 6588 - 2X^3 - 9X^2}{36(X + 3)^2} \right).$$

A quick computation with Magma \cite{1}, reveals that rank of $E'(\mathbb{Q})$ is equal to 1. One can check that the generator of the infinite part is $P = (42, -216)$.

Now we compute multiplies $nP = (X_n, Y_n)$ for $n = 1, 2, \ldots$ and then consider corresponding points $(U_n, W_n) = \phi^{-1}(nP)$. By writing $U_n = u_n/v_n, W_n = w_n/v_n^2$ with $\gcd(u_n, w_n, v_n) = 1$ we get the triple $(a_n, b_n, c_n) = (2u_nv_n, |u_n^2 - v_n^2|, u_n^2 + v_n^2)$ which leads to the curve $E_{a_n, b_n}$ with point of infinite order

$$(x_n, y_n) = (-a_n^3b_n, 4a_nb_n(b_n - a_n)w_n).$$

In order to finish the proof we need to know that for a given pair $(a_n, b_n)$ there are only finitely many pairs $(a_m, b_m)$ with $m > n$ such that the curves $E_{a_n, b_n}, E_{a_m, b_m}$ are isomorphic. This can be seen as follows: by standard properties of elliptic curves we know that the curves $E_{a, b}, E_{A, B}$ are isomorphic if and only there is a linear isomorphism $\psi : E_{a, b} : (x, y) \mapsto (p^2x + q, p^3y) \in E_{A, B}$ for some $p, q \in \mathbb{Q}$. Let $f_{a, b}(x, y)$ be the polynomial defining the curve $E_{a, b}$. We see that $E_{a, b} \simeq E_{A, B}$ if and only if

$$f_{A, B}(p^2x + q, p^3y) = C f_{a, b}(x, y),$$

for some $C \in \mathbb{Q}$. By comparing the coefficients on both sides we get that $C = p^6$ and need to consider the following system of equations:

$$\begin{cases} 
q(A^4 + q)(B^4 + q) & = 0 \\
p^2(A^4B^4 - a^4b^4p^4 + 2(A^4 + B^4)q + 3q^2) & = 0 \\
p^4(A^4 + B^4 - (a^4 + b^4)p^2 + 3q) & = 0
\end{cases}$$

Let $G$ be the Gröbner basis of the polynomials defining the above system in the ring $\mathbb{Q}[a, b, A, B][p, q]$. In other words we treat $a, b, A, B$ as constants. We have $|G| = 14$ and note that $G \cap \mathbb{Q}[a, b, A, B] = \{ F \}$, where

$$F = (Ab - aB)(Ab + aB)(aA - bB)(aA + bB) \times$$

$$(A^2b^2 + a^2B^2)(a^2A^2 + b^2B^2) \times$$

$$(a^4A^4 - A^4b^4 - a^4B^4)(A^4b^4 + a^4B^4 - b^4B^4) \times$$

$$(a^4A^4 - A^4b^4 + b^4B^4)(a^4A^4 - a^4B^4 + b^4B^4).$$

Assuming that $a, b, A, B$ are positive we see that the only possibility for vanishing of $F(a, b, A, B)$ is the condition $(Ab - aB)(aA - bB) =$
0. In other words, if \(a_n, b_n\) and \(a_m, b_m\) with \(n \neq m\) come from our construction we need to have \(a_n a_m = b_n b_m\) or \(a_n b_m = a_m b_n\). However, let us note that for given \(n \in \mathbb{N}\) the system of equations
\[
2uva_n = |u^2 - v^2|b_n, \quad u^2 = u^4 + 2u^3v + 2u^2v^2 - 2uv^3 + v^4
\]
has only finitely many solutions in integers \(u, v\). This means that there are only finitely many values of \(m > n\) such that \(a_n a_m = b_n b_m\). Similar reasoning works in the case of the equality \(a_n b_m = a_m b_n\). Thus, in both cases, for only finitely many values of \(m > n\) we have that the curves \(E_{a_n, b_n}, E_{a_m, b_m}\) are isomorphic. In consequence we can construct an infinite set \(A \subset \mathbb{N}\) such that for each \(s_1, s_2 \in A, s_1 \neq s_2\) we have non-isomorphic curves corresponding to \(s_1, s_2\).

\[\Box\]

**Example 3.2.** Let \(P = (42, -216)\) be the generator of the free part of \(E'(\mathbb{Q})\), where \(E'\) is constructed in the proof of the theorem above. We have \(2P = (105/4, 405/8)\) and
\[
\phi^{-1}(2P) = (1/4, -13/16),
\]
i.e., \(u_2 = 1, v_2 = 4\), with \(2u_2v_2 = 8, u_2^2 - v_2^2 = -15\). The corresponding triple is \((a_2, b_2, c_2) = (8, 15, 17)\). On the curve \(E_{8,15}\) we get the point \((8^3 \cdot 15, 2296320)\) of infinite order.

In case of \(3P = (10698/49, 1097928/343)\) we get
\[
\phi^{-1}(3P) = (69/35, -7274/1225)
\]
and the corresponding triple is \((a_3, b_3, c_3) = (4830, 3536, 5986)\). The point of infinite order on \(E_{a_3, b_3}\) is \((-4830^3 \cdot 3536, 3750258651849283392000)\).

We believe that the following is true.

**Conjecture 3.3.** There are infinitely many primitive Pythagorean triples \((a, b, c)\) such that the elliptic curve
\[
E_{a,b} : y^2 = x(x + a^4)(x + b^4)
\]
has rank zero.

**Remark 3.4.** We performed a small numerical search for primitive triples \((a, b, c)\) such that the rank of \(E_{a,b}\) is zero. We used the MAGMA computational package and checked that in the range \(a < b < 10^4\) (in this range there are exactly 890 primitive Pythagorean triples) there are at least 45 triples \((a, b, c)\) for which the rank of \(E_{a,b}\) is equal to zero. In order to get the data we were interested in, first we generated all triples \((a, b, c)\) in the considered range and dealt with the elliptic curve \(E_{a,b}\). Next, we used the procedure \texttt{RankBound(E)}, implemented in MAGMA, which gives an upper bound for the rank of the elliptic
curve $E$. If the computed bound were equal to 0, we printed the triple $(a, b, c)$. In the table below we collect the data we obtained. However, it is very likely that in the range $a < b < 10^4$ there are many more triples $(a, b, c)$ such that the rank of $E_{a,b}$ is 0.

| $(a, b, c)$ | $(a, b, c)$ | $(a, b, c)$ |
|------------|------------|------------|
| (3, 4, 5)  | (581, 3420, 3469) | (2380, 4611, 5189) |
| (5, 12, 13)| (588, 2365, 2437) | (2436, 2923, 3805) |
| (12, 35, 37)| (612, 1075, 1237) | (2725, 5628, 6253) |
| (13, 84, 85)| (660, 2989, 3061) | (3267, 6956, 7685) |
| (19, 180, 181)| (685, 9372, 9397) | (3612, 6955, 7837) |
| (24, 143, 145)| (780, 6059, 6109) | (3751, 6840, 7801) |
| (33, 56, 65)| (913, 3384, 3505) | (4180, 8541, 9509) |
| (64, 1023, 1025)| (924, 5893, 5965) | (4251, 5180, 6701) |
| (69, 2380, 2381)| (949, 2580, 2749) | (4469, 5100, 6781) |
| (115, 252, 277)| (1403, 1596, 2125) | (4740, 5341, 7141) |
| (180, 299, 349)| (1507, 9324, 9445) | (5365, 9828, 11197) |
| (319, 360, 481)| (1820, 8181, 8381) | (5633, 7656, 9505) |
| (339, 6380, 6389)| (2059, 2100, 2941) | (6125, 6612, 9013) |
| (473, 864, 985)| (2147, 6204, 6565) | (6204, 7747, 9925) |
| (540, 629, 829)| (2299, 7140, 7501) | (6811, 8460, 10861) |

Table. Primitive Pythagorean triples $(a, b, c)$ such that the curve $E_{a,b}$ has rank 0.

4. A GENERAL QUESTION

Now we ask a more difficult question: Does there exist a pair of Pythagorean triples $(a, b, c)$ and $(A, B, C)$ satisfying

\[ \frac{A}{a} = \mu, \quad \frac{B}{b} = \nu \]

where $\mu, \nu$ are given positive rationals. If at least one Pythagorean triple can be non-primitive it is easy to see that the answer depends only on the ratio $\nu/\mu$. Indeed, if triples $(a, b, c), (A, B, C)$ solve the equation (1) then the triples $(\mu a, \mu b, \mu c), (A, B, C)$ solve the equation (1) with $\mu = 1$ and $\nu$ replaced by $\nu/\mu$. Therefore in the sequel we confine ourselves to the pairs with $\mu = 1$.

However, before we concentrate on the system (1) we state an easier question concerning characterization of all possible pairs $r_1, r_2$ of positive rational numbers such that there are Pythagorean triples $(a, b, c)$
and $(A, B, C)$ satisfying

$$\frac{A}{a} = r_0, \quad \frac{B}{b} = r_1, \quad \frac{C}{c} = r_2. \quad \text{(5)}$$

As we already noted, without loss of generality, we can assume that $r_0 = 1$. Thus, our problem is equivalent with characterization of solutions of the system

$$a^2 + b^2 = c^2, \quad a^2 + r_1^2b^2 = r_2^2c^2. \quad \text{By writing} \quad a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2 \quad \text{we are interested in solutions of the Diophantine equation}$$

$$\frac{(u^2 - v^2)^2r_1^2 - (u^2 + v^2)^2r_2^2}{(2uv)^2} = -1 \quad \text{with respect to} \quad r_1, r_2. \quad \text{However, this is simple because our equation is quadratic in} \quad r_1, r_2 \quad \text{and we have the rational point at infinity} \quad (u^2 + v^2 : u^2 - v^2 : 0).$$



In consequence, we obtain the parametrization in the following form

$$r_1 = \frac{(u^4 - v^4)w + 2uv((u^4 - u^2)w + 2uv)}{2(u^2 - v^2)^2(u^2 + v^2)w}, \quad r_2 = \frac{(u^4 - v^4)^2w^2 + 4u^2v^2}{2(u^2 - v^2)(u^2 + v^2)^2w}. \quad \text{A quick computation reveals that our ratios} \quad r_1, r_2 \quad \text{are positive (with fixed values of} \quad u, v) \quad \text{if and only if the condition} \quad 0 < w < \frac{2uv}{u^4 - v^4} \quad \text{is satisfied.}$$

Let us return now to our initial question. Of course finding the solutions of the system $\text{(4)}$ is more difficult. Indeed, for a concrete $\nu$ we are confronted with the problem of positivity of the rank of the elliptic curve

$$E_\nu : y^2 = x(x + 1)(x + \nu^2)$$

(compare the paper [2] for the case $\nu \in \mathbb{N}$). We prove a general but weaker theorem.

**Theorem 4.1.** For each $\bar{\nu} \in \mathbb{Q}^+$ there exist $s \in \mathbb{Q}^+$ such that for $\nu = \bar{\nu}s^2$ there exist $(a, b, c)$ and $(A, B, C)$ satisfying $A = a, B = \nu b$.

**Proof.** In order to get the result we consider the following Pythagorean triples

$$a = 2x, \quad b = x^2 - 1, \quad c = x^2 + 1, \quad A = 2x, \quad B = \frac{(y^2 - 1)x}{y}, \quad C = \frac{(y^2 + 1)x}{y}. \quad \text{Thus, we consider the equation}$$

$$\frac{B}{b} = \frac{(y^2 - 1)x}{(x^2 - 1)y} = \bar{\nu}s^2.$$
Equivalently, putting \( S = \frac{x^2 - 1}{x} s \), we are interested in the surface

\[
H : \bar{\nu} x y S^2 = (x^2 - 1)(y^2 - 1)
\]

defined over the rational function field \( \mathbb{Q}(\bar{\nu}) \). First we construct a rational curve lying on \( H \). We are looking for a rational curve of the form

\[
x = T + 1, \quad y = pT + 1, \quad S = tT,
\]

where \( p, T \) need to be determined. A quick computation with the form of our substitution and the equation defining \( H \), reveals that

\[
p = \frac{1}{4} vt^2, \quad T = -\frac{2(\bar{\nu}v^2 + 4)}{3vt^2}
\]

does the job. In consequence, we get that the curve \( L \):

\[
\begin{align*}
x &= \frac{vt^2 - 8}{3vt^2} =: f(t), \\
y &= \frac{2 - \bar{\nu}t^2}{6}, \\
S &= -\frac{2(t^2 + 4)}{3vt}.
\end{align*}
\]

lies on the surface \( H \). We show that there are infinitely many rational curves lying on \( H \). In order to do this we treat the surface \( H \) as a genus one curve, say \( \mathcal{E} \), defined over the rational function field \( \mathbb{Q}(\bar{\nu}, t) \) in the \((y, S)\) plane. Here we use the base change \( x = f(t) \). The curve \( \mathcal{E} \) is written in the Weierstrass form

\[
\mathcal{E} : Y^2 = X(X - \bar{\nu}(f(t)^3 - f(t)))(X + \bar{\nu}(f(t)^3 - f(t))),
\]

where the map \( \psi : H \ni (y, S) \mapsto (X, Y) \in \mathcal{E} \) is given by

\[
X = vf(t)(f(t)^2 - 1)y, \quad Y = Sv^3f(t)^2(f(t)^2 - 1)y.
\]

On the curve \( \mathcal{E} \) we have the point \( P \) coming from the rational curve \( L \) lying on \( H \). More precisely, \( P = (X, Y) \), where

\[
\begin{align*}
X &= \frac{4(\bar{\nu}t^2 - 8)(\bar{\nu}t^2 - 2)^2 (\bar{\nu}t^2 + 4)}{81\bar{\nu}^2t^6}, \\
Y &= -\frac{8(\bar{\nu}t^2 - 8)^2 (\bar{\nu}t^2 - 2)^2 (\bar{\nu}t^2 + 4)^2}{729\bar{\nu}^3t^9}.
\end{align*}
\]

Because for any given \( \bar{\nu} \) positive rational we have \( X, Y \in \mathbb{Q}(t) \setminus \mathbb{Q}[t] \) we immediately get (via the analogue of the Nagell-Lutz theorem for a rational function field) that the point \( P \) is of infinite order on \( \mathcal{E} \). This implies the existence of infinitely many rational curves which lie on \( H \) (coming from the points \( \psi^{-1}(mP) \), where \( m = 1, 2, \ldots \)), with fixed \( x = f(t) \). Finally, we see that, for any given \( \bar{\nu} \) there are infinitely many parametric families of pairs of Pythagorean triples \((a, b, c), (A, B, C)\) satisfying \( a = A, B = vb = \bar{\nu}b \) and hence the result.
Corollary 4.2. The set
\[ \Gamma = \{ \nu \in \mathbb{Q}_{>0} : \text{the elliptic curve } E_\nu \text{ has positive rank} \} \]
is dense in the Euclidean topology in the set \( \mathbb{R}_{>0} \).

Proof. In order to get the result we could use the parametric curve constructed in the proof of Theorem 4.1. However, instead we will treat the equation
\[ y^2 = x(x+1)(x+\nu^2) \]
defining the curve \( E_\nu \) as an equation in the \((y, \nu)\) plane, i.e.,
\[ y^2 = x(x+1)\nu^2 + x^2(x+1), \]
which is a genus 0 curve defined over rational function field \( \mathbb{Q}(x) \). After rational base change \( x = t^2 - 1 \) we see that the curve
\[ y^2 = (t^2 - 1)t^2\nu^2 + (t(t^2 - 1))^2 \]
has \( \mathbb{Q}(t) \)-rational point \((\nu, y) = (0, t(t^2 - 1))\). Thus we obtain the parametric solution in the following form
\[ \nu = \frac{2t(t^2 - 1)u}{t^2(t^2 - 1) - u^2}, \quad y = \frac{t(t^2 - 1)((t^2 - 1)t^2 + u^2)}{t^2(t^2 - 1) - u^2}. \]
We thus see that, for given \( t \in \mathbb{Q} \), on the elliptic curve \( E_{\nu(u)} \), treated over rational function field \( \mathbb{Q}(u) \), we have the point \( P = (t^2 - 1, y(u)) \). It is clear that the point \( P \) is of infinite order on \( E_{\nu(u)} \).

To finish the proof, note that for any given positive \( t \in \mathbb{Q} \) satisfying the property \( t^2(t^2 - 1) > 0 \), the rational map \( \nu : \mathbb{R} \ni u \mapsto \frac{2t(t^2 - 1)u}{t^2(t^2 - 1) - u^2} \in \mathbb{R} \) is continuous and has the obvious property
\[ \nu(0) = 0, \quad \lim_{u \to u_0} \nu(u) = +\infty, \]
where \( u_0 = \sqrt{t^2(t^2 - 1)} \). The density of \( \mathbb{Q} \) in \( \mathbb{R} \) together with the above property of the map \( \nu = \nu(u) \), immediately implies that the set \( \nu(\mathbb{Q}) \cap \mathbb{R}_{>0} \) is dense in \( \mathbb{R}_{>0} \) in the Euclidean topology. The theorem follows. \( \square \)

Remark 4.3. Parameters \( \nu \) for which the rank of \( E_\nu \) is positive arise from a geometric problem, similar to that of congruent numbers. But our Theorem 4.1 illustrates a difference. Whereas being congruent depends only on the square-class of a considered number the situation with numbers \( \nu \) is completely different: by our theorem each square-class contains relevant numbers \( \nu! \).
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