Network Composition from Multi-layer Data

Kristina Lerman¹, Shang-Hua Teng², and Xiaoran Yan³

¹Information Sciences Institute, University of Southern California
²Computer Science, University of Southern California
³Network Science Institute, Indiana University

Abstract

It is common for people to access multiple social networks, for example, using phone, email, and social media. Together, the multi-layer social interactions form a “integrated social network.” How can we extend well developed knowledge about single-layer networks, including vertex centrality and community structure, to such heterogeneous structures? In this paper, we approach these challenges by proposing a principled framework of network composition based on a unified dynamical process. Mathematically, we consider the following abstract problem: Given multi-layer network data, \((G₁, \ldots, Gₗ)\) over a vertex set \(V\) and additional parameters for intra and inter-layer dynamics, construct a (single) weighted network \(G\) that best integrates the joint process. We use transformations of dynamics to unify heterogeneous layers under a common dynamics. For inter-layer compositions, we will consider several cases as the inter-layer dynamics plays different roles in various social or technological networks. Empirically, we provide examples to highlight the usefulness of this framework for network analysis and network design.

1 Introduction

As a powerful representation for many complex systems, networks model entities and their interactions as vertices and edges. Studies of network structures, including those of vertex centrality and community structure have lead to fundamental insights into the organization and function of social, biological and technological systems [30, 7, 15]. On top of these network structures, different dynamical processes unfold [8, 18, 19]. Our ability to model and predict dynamic network phenomena has led to new applications ranging from ranking web pages to maximizing social influence and controlling epidemics [31, 11, 23].

Traditionally, most research has focused on the simple graph representation where all vertices and edges are of a single type. More recently, there has been great interest in going beyond such a homogeneous model to investigate networks that are capable of capturing multiple types of connections. Extensive efforts towards such heterogeneous models came from both social [34, 35] and computational disciplines [25, 1], as the simple graph abstraction are often too crude a description of reality. For example, it is very common for people to have interactions across multiple social networks, including neighbors, coworkers, and also online interactions through email and social platforms, such as Facebook and Twitter. Each of these networks underlies a different type of social interactions. Because of the different origins and motivations, many names have been given to such heterogeneous models including but not limited to multiplex network, multi-relational networks, and networks of networks. For a comprehensive review see [24]. In this paper, we adopt their terminology and use the general model of multi-layer networks, with multiplex network being a special case when inter-layer structures are absent.

Structure and dynamics of multi-layer networks have been explored in both theoretical graphs and real world data [4, 10, 12, 20, 16], with several re-
searchers generalizing community structure and centrality measures to multiple edge types [29, 28, 22, 6, 32, 33]. However, it remains an open research question as how to build a multi-layer network in the first place. Such networks are often constructed simply by stacking or projecting layers into a single network. When inter-layer edges are explicitly modeled, they usually appear as tunable parameters, despite the fact that general theoretical framework allows much richer representations [24]. One challenge for modeling inter-layer structures is that they are empirically difficult to measure in most cases [16, 17].

In this paper, instead of proposing multi-layer generalizations of network measures, we approach the problem by constructing a single composed network which integrates the multi-layer data. We propose a two-stage framework for multi-layer network composition based on a unified dynamical process, as illustrated by Figure 1. Specifically, the first stage address the layer heterogeneity through layer transformations. In Section 3, we discuss how to transform the layers into homogeneous Markov processes using the framework of the parameterized Laplacian [19]. In Section 4, we will discuss the second stage, which consists of several ways of combining layers in commonly seen social and technological networks. They include how to construct multiplex networks as well as multi-layer structures with observed inter-layer dynamics. The multi-layer case, we choose to model a vertex’s inter-layer transitions, i.e., its participation in the different network layers, as another Markov process. Therefore, we can treat the combined layers of interlinked dynamical processes as a joint Markov process itself. We will prove that under this view a unique composition of a multi-layer network exists. In practice, however, it is difficult to fully observe inter-layer transitions. Hence, we will also consider the problem of network composition with partial information, for example, knowing only the stationary distribution of a inter-layer Markov process. Together, these dynamical process based transformations and compositions capture the heterogeneous structure while leaving a unified underlying topology. As a result, we can directly apply existing network algorithms of vertex centrality and community detection to the correctly composed joint structure. Some applications of multi-layer formalism to real-world data are discussed in Section 5.

2 Preliminaries

In this section, we introduce some basic notations for multi-layer networks and dynamical processes.

**Single-layer data**: A standard network is represented by weighted directed graph \( G = (V, E, A) \), where \( V = \{1, ..., n\} \) and for \( u, v \in V \), \( a_{uv} \geq 0 \) assigns an affinity weight to edge \( (u, v) \in E \). We follow the convention that \( a_{uv} = 0 \) if and only if \( (u, v) \notin E \). \( G \) may have self-loops, and edges in \( G \) are assumed to be directed. In other words, the weighted adjacency matrix \( A \) can be asymmetric and can have none-zero entries on the diagonal. For \( u \in V \), let \( d_{out}^u = \sum_{v=1}^n a_{u,v} \) denote the out-degree of vertex \( u \). Similarly, let \( d_{in}^u = \sum_{v=1}^n a_{v,u} \) denote the in-degree of vertex \( u \). In this paper, we use \( D_A \) (or \( D \) when the context is clear) to denote the diagonal matrix whose entries are out-degrees.

**Multi-layer data**: We consider vertex-aligned multi-layer networks [24]. We usually use \( l \) to denote number of layers, and use \( G^l = (V, E^l, A^l) \) to denote the network at \( l^{th} \) layer. For clarity, we will use superscripts \( i, j, r \) for the layers and subscripts \( u, v, w \) for vertices. Note that the vertex set \( V \) is the same across the layers. Figure 2 is an example of a three-layer network, consisting of (hypothetical) phone contacts, email exchanges and Facebook friendships of four users. In the figure, users appear in multiple layers, connected by a dashed line.
In this paper we take a dynamical view of the network structure. The simplest dynamical process on graphs $G$ is the discrete time unbiased random walk (URW), represented by the transition matrix $M$. For their connections we have the following lemma:

**Lemma 1** For every directed network $G = (V, E, A)$, there is a unique transition matrix $M_A = AD_A^{-1}$, that captures the URW Markov process on $G$. Conversely, given a transition matrix $M$, there is in fact an infinite family of adjacency matrices whose random walk Markov process is consistent with $M$:

$$A_M = \{M\Gamma : \Gamma \text{ is a positive diagonal matrix}\}$$

In other words, every directed network uniquely defines a random walk process. However, given a transition matrix $M$, there remains $n$ degrees of freedom to specify the underlying network. Intuitively, they are vertex scaling factors because each random walk distribution remains the same as long as the whole column is multiplied together. We will use this fact in some of our construction.

Recall that an $n$-dimensional probability vector $\pi$ is the stationary distribution of $M$ if $M\pi = \pi$. The Markov process defined by $M$ is detailed-balanced if for all $u, v \in V$, $\pi_u m_{uv} = \pi_v m_{vu}$. It is well known that [3]:

**Lemma 2** Suppose $M$ is a detailed-balanced transition matrix with stationary $\pi$. Let $\Pi$ be the diagonal matrix defined by $\pi$. Then, for all $\alpha > 0$, $\alpha \cdot M\Pi$ is symmetric. Namely, $A_M = \{\alpha \cdot M\Pi\}$ contains symmetric adjacency matrices if and only if $M$ is detailed-balanced.

Therefore, for undirected graphs, there is only one degree of freedom to specify the underlying network for a given detailed-balanced transition matrix, which can be interpreted as the global scaling factor.

3  Layer Transformation

In [19], Ghosh et al. argued that perceived network structure is a result of the interplay between the network topology and the dynamical process on top of it. We believe this interplay is even more pronounced in multilayer networks, with each layer representing a different type of connection. It is essential to account for the different intra-layer dynamics before the composition.

Taking Figure 2 for example, if we want to trace a message in the combined network, one should take into account the different propagating patterns in each layer. We might weigh the edges in the phone
layer much heavier if it is a business message. Similarly, each user may also has its own habits in terms of how often they check their email and Facebook accounts.

In Section 2, we showed the mathematical mapping from the adjacency matrices to the transition matrices representing simple URWs. The parametrized Laplacian operator introduced in [19] can model a richer family of dynamical processes. As a conservative operator, it has a dominate eigenvalue 0, and models continuous time random random walks or consensus processes with various biases and delays at vertices. In this paper, we shall focus on the random walk formulation:

\[ L = (D' - BA)(D'T)^{-1}. \]

where \( A \) represents the adjacency matrix of the and \( D' \) is the reweighed diagonal degree matrix.

Compared with traditional Laplacians, the parametrized Laplacian has two additional parameters: \( T \) and \( B \). The diagonal matrix \( T \) controls the time delay factors, or local clock rate, at each vertex. In the toy example Figure 2, \( T \) can be used to capture the checking frequency of email and Facebook accounts, or the limited attention a user has in face of information overload [21]. It models user activities as Poison processes where the waiting time between logins are exponentially distributed with means specified by \( T \) [20]. Without loss of generality, we constrain all the entries in \( T \) with \( \tau_u \geq 1 \).

The bias factors form the other diagonal matrix \( B \). It changes the trajectory by giving random walk targets different weights. Note that the degree matrix \( D' \) is now defined as: \( d'_u = \sum_v |BA|_{uv} \). In Figure 2, \( B \) can be used to model different routing strategies in each layer. Such routing biases can be based on structural properties like vertex degree or some external attributes specified by \( B \). Entries \( b_i \) of \( B \) can be quite general, as long as the entries of \( BA \) remains non-negative.

To reach homogeneity across the layers, we need to transform each input layer to equivalent graphs with URWs as the unifying dynamics. Using the two graph transformations under the parametrized Laplacian framework, we have

**Theorem 1** For a directed network \( G = (V, E, A) \), the dynamics \( L = (D' - BA)(D'T)^{-1} \) is equivalent to a URW on another transformed graph.

**Proof** The first transformation under the parametrized Laplacian framework is the bias transformation, which is already defined in the parentheses of Equation (1). A biased random walk from vertex \( u \) to \( v \) with transition probability \( P_{vu} \propto b_i a_{uv} \) is equivalent to an unbiased random walk on a reweighed adjacency matrix: \( A' = BA \), because

\[ P'_{vu} = \frac{a'_{vu}}{\sum_v a'_{vu}} \propto b_i a_{uv} \propto P_{vu}. \]

The second transformation is to view the delay factors \( T \) as self-loops. Under the parametrized Laplacian framework, delays can be understood as rescaling the mean waiting time of random walks at vertices. They can be absorbed in to the scaled adjacency matrix \( W \), which we call the interaction matrix. On top of a bias transformed random walk adjacency \( A' \), we apply the delay factors:

\[ (D' - A')D^{-1}T^{-1} = IT^{-1} - A'D^{-1}T^{-1} \]
\[ = I - (I - T^{-1}) - A'D^{-1}T^{-1} \]
\[ = I - (T - I + A'D^{-1}T^{-1})^{-1} \]
\[ = (D_w - (T - I)D_wT^{-1} - A'D_wD^{-1}T^{-1})D_w^{-1}I \]
\[ = (D_w - (T - I)(D' - A')D_{w}^{-1}I \]
\[ = (D_w - W)D_{w}^{-1}I, \]

where the interaction matrix \( W \) is the reweighed \( A' \) plus the self loops represented by the diagonal matrix \( (T - I)D' \), with \( D_w \) represents its diagonal degree matrix. Delay transformation allows us to rescale different \( T \) to \( I \). A simple special case is when \( T = \alpha I \) is a scalar matrix. It can be understood as rescaling the global time of that layer.
Remark: Giving the dynamical parameters $B$ and $T$, we can always find the interaction matrix $W$ by
$$ W = BA + (T - I)D' . $$

Although by Lemma 1 it is not unique, the interaction matrix $W$ serves the purpose of unifying dynamics across the layers.

Theorem 2 For an undirected network $G = (V, E, A)$, the dynamics $L = (D'BA)(D'T)^{-1}$ is equivalent to a URW on another transformed graph.

Proof The bias transform in the undirected case is discussed in [27, 19]. A biased random walk with transition probability $P_{ij} \propto b_{ij}a_{ij}$ is equivalent to an unbiased random walk on a reweighed adjacency matrix: $a_{ij}' = b_{ij}a_{ij}$. Notice that the matrix products on both sides ensure that the resulting random walk is detailed balanced. The delay transformation remains the same as in Theorem 1.

By Lemma 2 the layer transformations of an undirected graphs produce a unique interaction matrix

Corollary 1 For an undirected network $G = (V, E, A)$, and the dynamical parameters $B, T$, the interaction matrix
$$ W = BAB + (T - I)D' . $$
is unique up to a global scaling factor.

With the transformed layers $W^1, W^2, \ldots W^l$ now all underly the same simple URW, we are ready to discuss the second stage of the framework in Figure 1 how to combine them into a joint structure.

4 Inter-layer composition

Inter-layer composition is the problem of constructing inter-layer edges that connects the transformed layers into a coherent structure. We base our composition on the observed inter-layer dynamics. Depending on the data source and problem of interest, the way layers interact with each other differs. We will consider several commonly seen situations in social and technological networks.

4.1 Multiplex composition

We start with the simplest case of all, when inter-layer structures are absent. In this case, simple matrix addition does the trick, and we have the following algorithm

Algorithm 1 Multiplex network composition

Input: weighted network layers: $G^1 = (V, E^1, A^1), G^2 = (V, E^2, A^2), \ldots, G^l = (V, E^l, A^l)$, parameters of the dynamics: $T^1, T^2, B^2, \ldots, T^l, B^l$, $A$.

Output the $n \times n$ adjacency matrix $W = \sum_{i=1}^{l} W^i$.

4.2 Multi-layer composition

While we recommend Algorithm 1 for purely multiplex networks, we need a more general framework when inter-layer structures do matter. Consider the following mathematical problem:

Formulation 1 (Super-adjacency Composition)

Given $l$ transformed layers $G^1 = (V, E^1, W^1), \ldots, G^l = (V, E^l, W^l)$, and egocentric inter-layer dynamics $(M_v : v \in V)$, compose a $(n \times n)$ weighted super-adjacency matrix, where $n = |V|$,

$$ W = \begin{bmatrix}
W^1 & W^{12} & \ldots & W^{1l'} \\
W^{21} & W^2 & \ldots & W^{2l'} \\
\vdots & \vdots & \ddots & \vdots \\
W^{l1} & W^{l2} & \ldots & W^l
\end{bmatrix}$$

to integrate the multi-layer network data. In addition, we require all off-diagonal blocks of $W$ are diagonal matrices. In other words, $W$ represent a diagonal multi-layer networks, as defined in [24], which means that all inter-layer edges are between the same vertex at different layers.
Remark: Here, in $\mathbb{W}$, the $l$ diagonal $(n \times n)$-blocks are directly fed from the first stage $W^1, W^2, ..., W^l$.

We have used the model of egocentric inter-layer dynamics for each vertex $(M_v : v \in V)$, with $M_v$ being the stochastic transition matrix for the inter-layer instances of the same vertex $v$. Such egocentric models are considered to be fundamental in the formation of social structures[14, 9], and might be readily available from existing social studies. They are also easy to crawl in social networks that provide cross-platform interfaces.

Together with the traditional horizontal perspective in Figure 2, egocentric inter-layer dynamics form a vertical perspective of the same joint system, where a unified dynamical process unfolds. For illustration, consider our toy example of Figure 3. Suppose when Alice receives a message from a phone call, she might pass on the message directly by calling with probability $0.6 = 0.4 + 0.2$, or relay the message through emails with probability 0.3, or post it on a Facebook wall with probability 0.1.

Our plan is to first formulate the network composition problem in the complete-information setting. The mathematical characterizations of this ideal setting can then be used to find feasible solution spaces when only partial-information is available. The additional degrees of freedom will also allow us to optimize the design space of inter-layer edges, or predict missing links. In this section, we will first show that a “Dynamic view of network composition” leads a feasible and unique formulation of $\mathbb{W}$.

Which composed super-adjacency matrix $\mathbb{W}$ properly integrates the multi-layer network data with egocentric inter-layer dynamics?

To answer this questions, recall that the input is specified by $l$ transformed adjacencies $W^1, ..., W^l$, and $n$ egocentric Markov models $(M_v : v \in V)$. In the “dynamical view”, by Lemma 1 each layer $W^i$ also uniquely defines a Markov model, $M^i_W$. Together, these $l + n$ Markov models define a joint Markov model, whose adjacency structure is the desired super-composition. Thus, we aim to identify a weighted $(ln \times ln)$-adjacency matrix $\mathbb{W}$, whose random-walk Markov model, $M^W$, satisfies the following two basic conditions:

Figure 3: Inter-layer dynamics of the toy example.
1. **Layer Consistency**: The random-walk Markov model of each layer, $M_{A_i}$, $i \in [1, 2, \ldots, l]$, is the projection of $M_W$ to that layer, and

2. **Ego Consistency**: The egocentric inter-layer dynamics, $M_v$, of vertex $v \in V$, is the layer marginals of $M_W$ at vertex $v$.

Recall that the projection of a Markov model $M$ onto a subset is simply the stochastic normalization of corresponding principal submatrix of $M$. Thus, Condition 1 is automatically achieved by setting diagonal blocks of $\bar{W}$ as $W^1, \ldots, W^l$ in Formulation 1.

Condition 2 addresses egocentric inter-layer dynamics. Notice that for each $v \in V$, $\bar{W}$ defines an $l \times l$ interlayer adjacency matrix $W_v$. The random-walk process, $M_{W_v}$, is the projection of the joint Markov process $M_W$ to the vertical slice consists of instances of $v$ in different layers. Condition 2 then requires that $M_{W_v}$ should be consistent with $v$’s egocentric inter-layer dynamics $M_v$.

To be more specific, let $q_{v,i}$ denote the transition probability according to $M_{W_v}$ for going from vertex $v$ in the $i^{th}$ layer to some $u$ in the same layer. Let $Q_v$ be the $l \times l$ diagonal matrix of $[q_{v,i}: i \in [l]]$. Then, $Q_v + M_{W_v} \cdot (I - Q_v)$ denote the layer marginals of the joint Markov model $M_W$ at vertex $v$. Consequently, Condition 2 requires that layer marginals $M_v = Q_v + M_{W_v} \cdot (I - Q_v)$. Intuitively, egocentric inter-layer dynamics $M_v$ bridges between the orthogonal projections by including $Q_v$ as well as $M_{W_v}$.

Now we ready to present the main theorem of this paper:

**Theorem 3** For any multi-layer data $(A_i : i \in [l], M_v : v \in V)$, there exists a unique and feasible super-composition $\bar{W}$ that satisfies both Layer Consistency and Ego Consistency.

**Proof** Because Formulation 1 requires that all off-diagonal blocks of $\bar{W}$ are diagonal matrices, we have $(l^2 - l)n$ degrees of freedom after meeting Condition 1.

Notice that $(M_v : v \in V)$ are $n$ stochastic $l \times l$ matrices. Thus, Condition 2 represents $(l^2 - l)n$ dimensional constraints, which matches perfectly with the remaining degrees of freedom. Uniqueness proven.

To prove the feasibility of the unique solution, we introduce the algorithmic framework Algorithm 2.

**Algorithm 2 Multilayer network composition**

**Input**: weighted network layers: $G^1 = (V, E^1, A^1), G^2 = (V, E^2, A^2), \ldots, G^l = (V, E^l, A^l)$, parameters of the dynamics: $T^1, B^1, T^2, B^2, \ldots, T^l, B^l$, and $n \times l$ egocentric inter-layer Markovian matrix $M_u$ for each vertex $u \in V$.

**Algorithm**

- Apply the reweighing transformation $A'^i = B'^i A^i$ ($A'^i = B'^i A^i B'^i$ for undirected graphs)
- Apply the scaling transformation $W^i = A'^i + (T^i - I) D^i$
- Create a $ln \times ln$ empty matrix $\bar{W}$
- Fill the $l$ diagonal blocks (each of size $n \times n$) with $W^1, W^2, \ldots, W^l$
- Construct the off diagonal blocks $W^{ij}$ (each of size $n \times n$) for all layer pairs $i$ and $j$ based on Algorithm 3 with $W^1, W^2, \ldots, W^l$ as inputs

**Output** The super adjacency matrix

$$\bar{W} = \begin{pmatrix}
W^1 & W^{12} & \cdots & W^{1l} \\
W^{21} & W^2 & \cdots & W^{2l} \\
\vdots & \vdots & \ddots & \vdots \\
W^{l1} & W^{l2} & \cdots & W^l
\end{pmatrix}$$

We need a subroutine Algorithm 3 to satisfy inter-layer constraints at each node, we rearrange the row and column of $\bar{W}$ so that the counterparts of the same vertex are grouped together. The rearrangement express $\bar{W}$ with the following block structures:

$$\bar{W} = \begin{pmatrix}
W_1 & W_{12} & \cdots & W_{1n} \\
W_{21} & W_2 & \cdots & W_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
W_{n1} & W_{n2} & \cdots & W_n
\end{pmatrix}$$

where $W_{n,v}$ are $l \times l$ matrices that have already been fixed by Condition 1. The $n$ matrices, $W_v : v \in V$
V on the diagonal blocks, contains all entries that we will need to set using Condition 2. Because the rearrangement of W preserves the diagonal entries up to reordering, the diagonal entries of W_e, v ∈ V, are also set by Condition 1. The rest n(l^2−l) entries lead to the same degrees of freedom we discussed earlier.

The reordered W_u blocks are closely related to the egocentric adjacencies X_u underlying the egocentric inter-layer dynamics M_u. The vertical slice in Figure 3 demonstrates such a X_u, where intra-layer transitions are captured using self-loops. Subroutine Algorithm 3 can now be specified as

**Algorithm 3 Building inter-layer blocks**

**Input:** transformed layers: G_1 = (V, E^1, W^1), G_2 = (V, E^2, W^2), ..., G_t = (V, E^t, W^t), and a l × l egocentric inter-layer transition matrix M_u for vertex u ∈ V.

**Algorithm**

- Create a l × l empty matrix X_u
- Fill the diagonal elements with X_u^{ii} = d_u^{(out)}
- Construct the off diagonal elements
  \[ X_u^{ij} = \frac{M_u^{ij}}{M_u^{ii}} d_u^{(out)} \]

**Output** Block X_u and repeat for each u ∈ V

Using Lemma 1 we can rewrite the steps in Algorithm 3 as X_u = M_u Γ, by setting the i_th entry of Γ uniquely as d_u^{(out)}/M_u^{ii}. Intuitively, we are simply respecting the layer inputs and using the intra-layer dynamics to determine the vertex scaling factor.

From Figure 3 it is clear that the off-diagonal parts of X_u is exactly what we are looking for in W_u blocks. Or W_u = X_u − D_u^{(out)}, where the diagonal matrix D_u^{(out)} is composed of d_u^{(out)} entries. With the uniquely solvable X_u blocks, we can now complete the output W by filling its off diagonal blocks W^{(ij)} with reordered W_u blocks. On top of that, Algorithm 3 will always lead to feasible solutions with the constrains X_u^{(ij)} ≥ 0, provided that M_u entries are well defined. Uniqueness and feasibility proven.

### 4.3 Overdetermined Composition

Based on Theorem 3 we have a fully determined system for consistent network composition when we have complete information about the personalized inter-layer dynamics. In practice, depending on the inputs and constrains, we might have underdetermined, overdetermined or even mixed systems.

If the network is undirected, the Markov process is under the detailed balance condition. They also become reversible, leading to a additional dependency for each independent loop in X_u because of Kolmogorov’s criterion. To count the number of independent loops, we simply subtract the number of edges in a connected tree (l−1) from the total number of edges, as each additional edge on the tree will introduce an independent loop. In our case, we consider all possible inter-layer connections (a complete graph X_u). Then the undirected Markov matrix would lead to

\[ l(l-1) - \frac{1}{2} \left( \frac{l(l-1)}{2} - (l-1) \right) = \frac{l(l+1)}{2} - 1 \]

constrains. The degree of freedom of an undirected marginalized adjacency X_u is \( \frac{l(l-1)}{2} \). There are l − 1 more constrains than variables. Leading to:

**Conjecture 1** In undirected graphs, the existence of a Layer Consistent and Ego Consistent solution to Formulation 2 depends on the inputs.

In our example Figure 3 we have d_u^{(out)} = 3 and

\[ X_u^{(pe)} = \frac{M_u^{(pe)}}{M_u^{(pp)}} d_u^{(p)} = \frac{0.1}{0.2} \times 3 = \frac{1}{2}. \]

If X_u is undirected as shown in Figure 3 we have

\[ M_u^{(pe)} = \frac{X_u^{(pe)}}{\sum_r X_u^{(er)}} = \frac{0.5}{\sum_r X_u^{(er)}}. \]

If the inputs does not satisfy the above constrain, there will not be any feasible solution to Formulation 2.
Other overdetermined systems can arise when we 
have some direct measures of the inter-layer struc-
tures, and approximate solutions can be found by 
minimizing some error terms. In real applications, 
however, it is much more likely that we have less 
empirical measures, and we will be facing systems 
with additional degrees of freedoms. Such underde-
termined systems leave spaces for other considera-
tions, and are often associated with network design 
and other optimization Formulations.

4.4 Underdetermined Compositions

In our combined social networks example Figure 2, 
we might not know each user’s message routing 
strategies, but we can can track the marginal distribution 
among the layers of how the messages are propagated 
for each user. In mixed membership community mod-

dles, however, it is much more likely that we have less 

can track the marginal distribution 

Algorithm 4 Building inter-layer block \( X_u \) with sta-
tionary distributions

**Input:** weighted network layers: \( G_1 = (V,E^1,A^1), G_2 = (V,E^2,A^2), \ldots, G_2 = (V,E^l,A^l) \), 
and \( n \times 1 \) stationary distribution vector \( \pi_u \) for each vertex \( u \in V \).

**Algorithm**

- Create a \( l \times l \) matrix \( X_u \) with \( l^2 \) free variables
- Constrains the diagonal elements with \( X_u^{ii} = d_u^{(\text{out})} \)
- Solve for the off diagonal elements with the constrains

\[
\forall i \neq j, \frac{\pi_u^{(i)}}{\pi_u} = \sum_{\nu} \frac{X_u^{ir}}{X_u^{\nu r}}
\]

**Output** Block \( X_u \) and repeat for each \( u \in V \)

With \( l = 2 \), we recover a fully determined system. In 
Algorithm 4, the solution will be

\[
X_u^{12} = \frac{\pi_u^1 d_u^2 - (1 - \pi_u^1) d_u^{1 \text{(out)}}}{(1 - 2 \pi_u^1)} = \frac{\pi_u^1 (d_u^2 + d_u^{1 \text{(out)}}) - d_u^{1 \text{(out)}}}{(1 - 2 \pi_u^1)}
\]

For it to be feasible, we need \( X_u^{12} \geq 0 \) or

\[
0.5 \leq \pi_u^1 \leq \frac{d_u^{1 \text{(out)}}}{d_u^1 + d_u^{1 \text{(out)}}} \quad \text{or} \quad 0.5 \geq \pi_u^1 \geq \frac{d_u^{1 \text{(out)}}}{d_u^1 + d_u^{1 \text{(out)}}}
\]

For underdetermined systems in general, we can 
specify an optimization objective function and use 
the additional degrees of freedom for network designs. 
Assuming the feasible solutions to Algorithm 4 form 
a family of \( \{W\} \) whose random-walk Markov process 
is consistent with the constraints, we may have

- Minimum network volume: 
  \[
  \min_{W \in \{W\}} \sum_{u,v,j} W_{uj}^v
  \]

- Maximum conductance: 
  \[
  \max_{W \in \{W\}} \min_{S \subseteq V} \frac{\text{cut}(S)}{\sum_{u \in S} \pi_u}
  \]

as potential objective functions.

Another common scenario leading to underdetermined 
systems is when we have inter-layer distance...
measures. Fully specified pair-wise layer distances amounts to $l(l - 1) - 1$ constrains, leaving a single degree of freedom which can also be interpreted as the scaling of the inter-layer edge weights relative to their intra-layer counterparts.

This construction is particularly suitable for combining time series of networks into multilayer structures. The temporal structure forms a one dimensional line. In the case when all vertices in the same layer shares the same time stamp, and we measure the distance between layers simply by the time difference, inter-layer adjacencies $X_u$ become the same for all vertices. Similarly, the degree of freedom for each vertex combines into a global parameter of inter-layer strengths. Such “layer coupled” multilayer structures have appeared in many previous studies as we discussed in Section 1.

We will demonstrate how Algorithm 4 and inter-layer distance models might be applied to real data set in Section 5.

5 Empirical examples

We apply the framework to study real world data sets. We demonstrate that community structure in a multi-layer network is sensitive to details of the inter-layer and intra-layer dynamics. Community structure is produced through graph bisections using the sweeping algorithm in [19].

5.1 Impact of layer transformations

To illustrate how layer transformation affect the structure of the resulting multi-layer network, we use the road network in the city of Washington DC. As shown in Figure 4 we modeled local roads and highways as two layers, with inter-layer edges representing highway entrances and exits. The data are based on the attributed undirected network provided by the 9th DIMACS implementation challenge–Shortest Paths [13]. The top highway layer combines road category $A1, A2, A3$. Inter-layer edges are constructed by matching vertex labels at both layers. After removing the disconnected components, we have a

Figure 4: Bisection of road networks in DC with different composition rules
multi-layer network with 10834 vertices and 28137 edges.

For comparison, we first used a conventional construction. Each highway connection received a weight of 2.0 while a local road or a highway entrances and exits received weights of 1.0. This is based on the average speed estimates listed by road categories [13]. Applying the graph bisection algorithm, we identified the traffic bottleneck along the Anacostia River as demonstrated on the top of Figure 4.

In contrast, when we applied our general composition framework Algorithm 2 to the dataset, a very different picture emerged. Using the total degree as a measure, the conventional construction leads to a 12.7% traffic load on the highways. On the bottom of Figure 4, we scaled the high way layer by a constant factor of 3.14, leading to a 20% highway traffic load. We also introduced traffic delays at each intersection, with a delay factor $\tau$ proportional to its degree. We lack empirical observations for inter-layer Markovian matrices $M_u$. Here we made a simple assumption that vertices with both highway and local road accesses all follow a 20% - 80% inter-layer stationary distribution with 4-times the traffic on the highway layer. This allows us to use Algorithm 4 in the last step of Algorithm 2 and recover a fully determined system.

Using the same graph bisection algorithm, the new composition finds the traffic bottleneck at the center of the city, as demonstrated on the bottom of Figure 4. Based on more realistic traffic patterns, our composition framework puts more weight on the highway layer whose structure is less bottlenecked by the Anacostia River.

### 5.2 Impact of inter-layer edges

We illustrate the impact of inter-layer compositions using a multilayer coauthorship networks. Specifically, we represent coauthorship networks over time as a multi-layer network, where each layer corresponds to a snapshot of the coauthorship network at some time. As in other real world applications, however, the inter-layer dynamics is difficult to specify. Here we use the interlayer distance approach introduced in Section 4.3.

Figure 5 presents a collaboration networks centered around four authors: Shang-hua Teng, Daniel Spielman, Gary Miller and Kristina Lerman, as well as their coauthors on papers appearing in the ACM Digital Library. Each layer represents a separate time period: from bottom to top, it is 1985-1994, 1995-2004, and 2005-2014. The weight of an intra-layer edge represents the number of times two authors collaborated during that time period, while inter-layer edges connect the same author between neighboring decades, with weights reflecting the relative time distances.

Since the distance between layers is uniform, all inter-layer edges have the same weight. Changing the global parameter of inter-layer edge strength, we get three different bisections of the network into communities using the normalized Laplacian [19]. In Figure 5(a), the weight of each inter-layer edge 0.5 (half of the weight of a collaboration), resulting in a mostly horizontal bisection. In Figure 5(c), the weight of inter-layer edges is 5.0, resulting in a vertical bisection, as separating layers is much more expensive. The most interesting case is when we set the weight on inter-layer edges to 1.0. For the earlier two decades, authors surrounding Shang-hua Teng and his Ph.D. advisor Gary Miller forms the red community largely consist of theoretical computer scientists. In the latest decade, however, the algorithm put Shang-hua Teng into the cyan group together with Kristina Lerman where the focus has switched to graph mining and modeling. The algorithm has consistently put Shang-hua Teng and Daniel Spielman in the same community as they collaborated extensively throughout the years.

### 6 Conclusion

In this work, we proposed a mathematically principled framework for multilayer network composition based on a unified dynamical process. We developed theorems and algorithms to construct a joint structure that can reflect the different intra and inter-layer dynamics in the inputs.

We also discussed and demonstrated a few practical situations when the system is not fully deter-
Figure 5: Bisection of coauthor networks using different inter-layer strengths

mined. In future works, we plan to explore approximate solutions for overdetermined systems and investigate in greater details of the associated network design and optimization problems for overdetermined systems.

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