Towards a homotopy domain theory

Daniel O. Martínez-Rivillas · Ruy J. G. B. de Queiroz

Received: 28 September 2021 / Accepted: 26 October 2022 / Published online: 4 November 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
An appropriate framework is put forward for the construction of \( \lambda \)-models with \( \infty \)-groupoid structure, which we call homotopic \( \lambda \)-models, through the use of an \( \infty \)-category with cartesian closure and enough points. With this, we establish the start of a project of generalization of Domain Theory and \( \lambda \)-calculus, in the sense that the concept of proof (path) of equality of \( \lambda \)-terms is raised to higher proof (homotopy).

Keywords
Lambda calculus · Homotopic lambda model · Kan complex · Infinity groupoid · Infinity category

Mathematics Subject Classification 03B70

1 Introduction

The purpose of this paper is to give a framework for building a lambda model endowed with a topology, such that any proof of \( \beta \)-equality between \( \lambda \)-terms is not represented by equality between points (extensional equality), rather by the existence of a continuous path between the terms (intensional equality), where the interpretation of these terms corresponds to two points in the space. As an example, given the \( \beta \)-equality between different \( \lambda \)-terms

\[(\lambda x. (\lambda y. yx)z)v \> \beta zv,
\]

these terms are taken to be \( \beta \)-equal because there is a proof \( p_1 \) determined by a finite sequence of \( \beta \)-contractions (\( \succ_1 \beta \)) or inverse \( \beta \)-contractions (\( \prec_1 \beta \)), possibly with \( \alpha \)-conversions, which allows for connecting the terms \( \lambda x. (\lambda y. yx)zv \) and \( zv \), hence

\[(\lambda x. (\lambda y. yx)z)v \succ_1 \beta (\lambda y. yv)z \succ_1 \beta zv.
\]
Now, the problem is to build a topological model, such that the interpretations of \( \lambda \)-terms \((\lambda x \cdot (\lambda y \cdot yx)z)v \) and \( zv \) are different points, and the proof \( p_1 \) is a continuous path which connects both points. This will be used to establish when two proofs (two continuous paths) of a \( \beta \)-equality between different terms (different points) are “equal” (homotopic). Hence, in the example, given an second proof \( p_2 \) which correspond to finite sequence

\[
(\lambda x \cdot (\lambda y \cdot yx)z)v \triangleright_1 \beta (\lambda xzx)v \triangleright_1 \beta zv,
\]

one has that \( p_1 \) and \( p_2 \) are two different proofs, so in the model these interpretations should be two different continuous paths. But, would these proof interpretations be homotopically equal? If in the \( \lambda \)-calculus we call \( \beta \)-homotopy any homotopy of the model, when are two different \( \beta \)-homotopies to be declared “equal”? This can be iterated, and by answering these questions, we could define in the \( \lambda \)-calculus a theory of higher \( \beta \)-equality, with the help of higher homotopies in the \( \lambda \)-model.

Therefore, this aforementioned theory of higher \( \beta \)-equality has a structure of a non-trivial \( \infty \)-groupoid, which extends \( \lambda \)-calculus to a type-free version of the Homotopy Type Theory (HoTT) \([1]\), but with equality relations based on (type-free) computational paths.\(^1\) Whose advantage is that the \( \beta\eta \)-conversions are not equalities of judgment \((a = b : A)\), as in HoTT, but those are intentional equalities \((a =_s b : A)\), which could better preserve the information than HoTT does.

The initiative to search for \( \lambda \)-models with a \( \infty \)-groupoid structure emerged in \([4]\) (called homotopic \( \lambda \)-models), which studied the geometry of any complete partial order (c.p.o) (e.g., \( D_\infty \)), and found that the topology inherent in these models generated trivial higher-order groups. From that moment on, the need arose to look for a type of model that could present a rich geometric structure, where their higher-order fundamental groups would not collapse. In this sense, we will gain the semantics of a type-free theory from a version of HoTT based on computational paths, which can distinguish different proofs of equality of \( \lambda \)-terms.

According to Quillen’s Theorem, each CW complex topological space is homotopically equivalent to a Kan complex\(^2\) (\( \infty \)-groupoid), and, conversely, each Kan complex is homotopically equivalent to a CW complex. Then, instead of working directly with topological spaces, we are going to work with Kan complexes, which are \( \infty \)-categories \([8]\) whose 1-simplexes or edges are weakly invertible. Or, in other words:

**Definition 1.1** \([8]\) A simplicial set \( K \) is a Kan complex if for any \( 0 \leq i \leq n \), any map \( f_0 : \Lambda_i^n \rightarrow K \) admits an extension \( f : \Delta^n \rightarrow K \).

Where the simplicial set \( K \) is defined as a presheaf \( \Delta^{op} \rightarrow Set \), with \( \Delta \) being the simplicial indexing category, whose objects are finite ordinals \([n] = \{0, 1, \ldots, n\}\), and morphisms are the (non strictly) order preserving maps. \( \Delta^n \) is the standard \( n \)-simplex defined for each \( n \geq 0 \) as the simplicial set \( \Delta^n := \Delta(\cdot, [n]) \). And \( \Lambda_i^n \) is a

\(^1\) If \( a, b \) are terms of type \( A \), a computational path \( s \) from \( a \) to \( b \) is a composition of rewrites (each rewrite is an application of the inference rules of the equality theory of Martin-Löf’s type theory). One denotes that by \( a =_s b \) (see \([2, 3]\)).

\(^2\) To ensure the consistency of HoTT, Voevodsky \([5]\) (see \([6]\) for higher inductive types) proved that Homotopy Type Theory (HoTT) has a model in the category of Kan complexes. (See \([7]\), p.11)
horn defined as largest subobject of $\Delta^n$ that does not include the face opposing the $i$-th vertex (see Sect. 1.1).

Finally, to find Kan complexes that model $\lambda$-calculus, the strategy would be to generalize the procedure used in [9], where to show a way to find categories that model $\lambda$-calculus, through the possible solution of domain equations, which are posed on a bicategory with desirable properties of cartesian closure and enough points.

But, before proposing an $\infty$-category with the properties of cartesian closure and having enough points, we first explore in Sect. 2 some consequences of the homotopic $\lambda$-models introduced in [4]. In Sect. 3, we define the homotopy $\lambda$-model on a cartesian closed $\infty$-category. In Sect. 4, we adopt the notion of Kleisli structure to the case of the $\infty$-categories and we define the Kleisli $\infty$-category of a structure. And, finally, in Sect. 5, we propose an $\infty$-category and we prove that it is closed cartesian and has enough points.

1.1 Simplicial sets

For a better understanding of the definitions and basic results on $\infty$-categories which are necessary for the development of this work, we present some notions on simplicial sets [10].

Definition 1.2 (Simplical indexing category) Let $\Delta$ be the category as follows. The objects are finite ordinals $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and morphisms are the (non strictly) order preserving maps. Morphisms in $\Delta$ are often called simplicial operators.

Remark 1.1 There are a coface operator $d^i : [n-1] \to [n]$, which skips the $i$-th element and a codegeneracy operator $s^i : [n+1] \to [n]$, which maps $i$ and $i + 1$ to the same element. All operator $f^*\in\Delta$ can be obtained as a cocomposition of coface and codegeneracy operators.

Definition 1.3 (Simplicial set) A simplicial set $X$ is a functor $X : \Delta^{op} \to \text{Set}$ (or presheaf). A simplicial morphism is just a natural transformation of functors. The category of the simplicial sets $\text{Fun}(\Delta^{op}, \text{Set})$ will be denoted by $s\text{Set}$ or $\text{Set}_\Delta$.

It is typical to write $X_n$ for $X([n])$, and call it the set of $n$-simplexes in $X$.

Remark 1.2 Given a simplex $a \in X_n$ and a simplicial operator $f^* : [m] \to [n]$, the function $f : X_n \to X_m$ is given by $f(a) := X(f^*)(a)$. In this explicit language, a simplicial set consists of

- A sequence of sets $X_0, X_1, X_2, \ldots$,
- Functions $f : X_n \to X_m$ for each simplicial operator $f^* : [m] \to [n]$.

For the coface operator $d^i : [n-1] \to [n]$, the face map is denoted by $d_i : X_n \to X_{n+1}$, $0 \leq i \leq n$. For the codegeneracy operator $s^i : [n+1] \to [n]$, the degeneracy map is written by $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$.

Definition 1.4 (Product of simplicial sets [11]) Let $X$ and $Y$ be simplicial sets. Their product $X \times Y$ is defined by
1. \((X \times Y)_n = X_n \times Y_n = \{(x, y) \mid x \in X_n, y \in Y_n\}\),
2. if \((x, y) \in (X \times Y)_n\), then \(d_i(x, y) = (d_i x, d_i y)\),
3. if \((x, y) \in (X \times Y)_n\), then \(s_i(x, y) = (s_i x, s_i y)\).

Notice that there are evident projection maps \(\pi_1 : X \times Y \to X\) and \(\pi_2 : X \times Y \to Y\) given by \(\pi_1(x, y) = x\) and \(\pi_2(x, y) = y\). These maps are clearly simplicial morphisms.

**Definition 1.5** \((\text{Standard } n\text{-simplex})\) The standard \(n\)-simplex \(\Delta^n\) is the simplicial set defined by

\[ \Delta^n := \Delta(-, [n]). \]

That is, the standard \(n\)-simplex is exactly the functor represented by the object \([n]\).

The standard 0-simplex \(\Delta^0\) is the terminal object in \(sSet\); i.e., for every simplicial set \(X\) there is a unique map \(X \to \Delta^0\). Sometimes we write \(*\) instead of \(\Delta^0\) for this object. The empty simplicial set \(\emptyset\) is the functor \(\Delta^{op} \to \text{Set}\) sending each \([n]\) to the empty set. It is the initial object in \(sSet\), i.e., for every simplicial set \(X\) there is a unique map \(\emptyset \to X\). Besides, there is a bijection \(sSet(\Delta^n, X) \cong X_n\); applying the Yoneda Lemma to category \(\Delta\) [12].

A graphical representation of the convex hull of \(\Delta^n\) is made up of the by \(n+1\) vertices \(0, 1, \ldots, n\) and the faces are the injective simplicial operators \(d^i : [n-1] \to [n]\), which are called non-degenerated. As seen in the Fig. 1 for the first four dimensions.

**Definition 1.6** \(((10))\) For two simplicial sets \(X, Y\) we have a mapping simplicial set, \(\text{Map}(X, Y)\) defined as:

\[ \text{Map}(X, Y)_n := sSet(X \times \Delta^n, Y). \]

Note that in particular \(\text{Map}(X, Y)_0 = sSet(X \times \Delta^0, Y) \cong sSet(X, Y)\) (bijection of sets). Sometimes to simplify notation, the simplicial set \(\text{Map}(X, Y)\) will be written as \(X^Y\) or \([X \to Y]\).

Next, a collection of subobjects of the standard simplexes, called “horns” is defined.

**Definition 1.7** \((\text{Horns})\) For each \(n \geq 1\), there are subcomplexes \(\Lambda^n_i \subset \Delta^n\) for each \(0 \leq i \leq n\). The horn \(\Lambda^n_i\) is the subcomplex of \(\Delta^n\) such that this is the largest subobject that does not include the face opposing the \(i\)-th vertex.

When \(0 < i < n\) one says that \(\Lambda^n_i \subset \Delta^n\) is an inner horn. One also says that it is a left horn if \(i < n\) and a right horn if \(0 < i\).
For example, the horns inside $\Delta^1$ are just the vertices: the left horn, the right horn $\Lambda^1_0 = \{0\} \subset \Delta^1$ and $\Lambda^1_1 = \{1\} \subset \Delta^1$. Neither is an inner horn.

Other example. $\Delta^2$ have three horns: The left horn $\Lambda^2_0$, the internal horn $\Lambda^2_1$ and the right horn $\Lambda^2_2$, see Fig. 2.

1.2 Definition of $\infty$-category and Kan complex

**Definition 1.8** ($\infty$-category [8]) An $\infty$-category is a simplicial set $X$ which has the following property: for any $0 < i < n$, any map $f_0 : \Lambda^n_i \to X$ admits an extension $f : \Delta^n \to X$.

**Definition 1.9** From the definition above, we have the following special cases:

- $X$ is a Kan complex if there is extension for each $0 \leq i \leq n$.
- $X$ is a category if the extension exists uniquely.
- $X$ is a groupoid if the extension exists for all $0 \leq i \leq n$ and is unique.

Next is the definition of cartesian product of $\infty$-categories, which generalizes the cartesian product of categories.

**Definition 1.10** (Cartesian product) A product of $\infty$-categories is the product of the underlying simplicial sets. Thus, $(X \times Y)_n = X_n \times Y_n$ for each $n \geq 0$.

By the bijective correspondence between the set $sSet(K, X \times Y)$ and $sSet(K \to X, K \to Y)$ one has the following proposition (see [13]).

**Proposition 1.1** The product of two $\infty$-categories (as simplicial sets) is an $\infty$-category.

1.3 Categorical constructions in $\infty$-categories

Next, one approaches the $\infty$-categories from the basic notions of the classical categories.

**Definition 1.11** A functor of $\infty$-categories $X \to Y$ is exactly a morphism of simplicial sets. Thus, $Fun(X, Y) = Map(X, Y)$ must be a simplicial set of the functors from $X$ to $Y$. 
Notation 1.1 The notation $\text{Map}(X, Y)$ is normally used for simplicial sets, while $\text{Fun}(X, Y)$ is for \(\infty\)-categories. One will refer to morphisms in $\text{Fun}(X, Y)$ as natural transformations of functors, and equivalences in $\text{Fun}(X, Y)$ as natural equivalences.

The composition of $n$-simplex $f : \Delta^n \to X$ (or $f \in X_n$) with a functor $F : X \to Y$, will be denoted as the image $F(f) \in Y_n$, where $n \geq 0$.

A 0-simplex or vertex $x : \Delta^0 \to X$ will be denoted as an object $x \in X$ in the $\infty$-category $X$.

A 1-simplex $f : \Delta^1 \to X$, such that $f(0) = x$ and $f(1) = y$ will be denoted as a morphism $f : x \to y$ in the $\infty$-category $X$.

An inner horn $\Lambda^2_1 \to X$, which corresponds to composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in the $\infty$-category $X$, will be denoted by $(g, -, f)$ or in some cases to simplify notation it will be denoted by $g \cdot f$.

Proposition 1.2 ([8, 12]) For every $\infty$-category $Y$, the simplicial set $\text{Fun}(X, Y)$ is an $\infty$-category.

The proof of the following theorem can be found in [8, 12].

Theorem 1.1 (Joyal) A simplicial set $X$ is an $\infty$-category if and only if the canonical morphism

$$\text{Fun}(\Delta^2, X) \to \text{Fun}(\Lambda^2_1, X),$$

is a trivial fibration. Thus, each fibre of this morphism is contractible.

The above theorem guarantees the laws of coherence of the composition of 1-simplexes or morphisms of an $\infty$-category. This means that the composition of morphisms is unique up to homotopy, i.e., the composition is well-defined up to a space of choices is contractible (equivalent to $\Delta^0$).

With respect to the Kan complexes, we have the following equivalence.

Proposition 1.3 (Homotopy extension lifting property [8]) The simplicial set $X$ is a Kan complex if and only if the induced map

$$\text{Fun}(\Delta^1, X) \to \text{Fun}([0], X)$$

is a trivial fibration of simplicial sets.

Definition 1.12 (Space of morphisms [13]) For two vertices $x, y$ in an $\infty$-category $X$, define the space of morphisms $X(x, y)$ by the following pullback diagram:

$$\begin{array}{ccc}
X(x, y) & \longrightarrow & \text{Fun}(\Delta^1, X) \\
\downarrow & & \downarrow_{(s,t)} \\
\Delta^0_{(x,y)} & \longrightarrow & X \times X
\end{array}$$

Proposition 1.4 ([8, 13]) The morphism spaces $X(x, y)$ are Kan complexes.

Springer
1.4 Equivalences in $\infty$-categories

In category theory, we have the concept of isomorphism of objects. For the case of the $\infty$-categories, we will have the equivalence of objects (vertices) in the following sense.

**Definition 1.13** (*Equivalent vertices*) A morphism (1-simplex) $f : x \to y$ in an $\infty$-category $X$ is invertible (an equivalence) if there is a morphism $g : y \to x$ in $X$, a pair of 2-simplexes $\alpha, \beta \in X_2$ such that $(g, -, f) \xrightarrow{\alpha} 1_x$ and $(f, -, g) \xrightarrow{\beta} 1_y$, i.e., if the diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
1_x & 1_y \\
x \xleftarrow{g} & y \\
f & \\
\end{array}
\]

commutes under the 2-simplexes $\alpha$ and $\beta$.

**Theorem 1.2** ([8, 12]) Let $X$ be an $\infty$-category. The following are equivalent

1. Every morphism (1-simplex) in $X$ is an equivalence.
2. $X$ is a Kan complex.

1.5 Natural transformations and natural equivalence

**Definition 1.14** ([12, 13]) If $X$ and $Y$ are $\infty$-categories, and if $F, G : X \to Y$ are two functors, a natural transformation from $F$ to $G$ is a map $H : X \times \Delta^1 \to Y$ such that

\[
H(x, 0) = F(x), \quad H(x, 1) = G(x),
\]

for each vertex $x \in X$. Such a natural transformation is invertible or it is a natural equivalence if for any vertex $x \in X$, the induced morphism $F(x) \to G(x)$ (corresponding to the restriction of $H$ to $\Delta^1 \cong \{x\} \times \Delta^1$) is invertible in $Y$. If there is a natural equivalence from $F$ to $G$, we write $F \simeq G$. 
Remark 1.3 This means that for each vertex $x \in X$, one chooses a morphism $H_x : F(x) \to G(x)$ such that the following diagram

$$
\begin{array}{ccc}
F(x) & \xrightarrow{H_x} & G(x) \\
\downarrow{F(f)} & & \downarrow{G(f)} \\
F(x') & \xrightarrow{H_{x'}} & G(x')
\end{array}
$$

commutes under the 2-simplices $\alpha : g \to (G(f), -, H_x)$ and $\beta : (H_{x'}, -, F(f)) \to g$.

1.6 Categorical equivalences and homotopy equivalences

Definition 1.15 (Categorical equivalence [8, 13]) A functor of $\infty$-categories $F : X \to Y$ is a categorical equivalence if there is another functor $G : Y \to X$, such that $GF \simeq 1_X$ and $FG \simeq 1_Y$.

Remark 1.4 From the definition above, if $F : X \to Y$ is a functor of Kan complexes, we say that $F$ is a homotopy equivalence.

Lemma 1.1 ([12, 13]) A functor of $\infty$-categories $F : X \to Y$ is a categorical equivalence if it satisfies the following two conditions:

- Fully Faithful (Embedding): For two objects $x, y \in X$ the induced functor of Kan complexes
  
  $$X(x, y) \to Y(Fx, Fy),$$

  is a homotopy equivalence.

- Essentially Surjective: For every object $y \in Y$ there exists an object $x \in Y$ such that $Fx$ is equivalent to $y$.

1.7 The join of $\infty$-categories

Next, the extension from join of categories to $\infty$-categories. This will enable us to define limit and colimit in an $\infty$-category.

Definition 1.16 (Join [8]) Let $K$ and $L$ be simplicial sets. The join $K \star L$ is the simplicial set defined by

$$(K \star L)_n := K_n \cup L_n \cup \bigcup_{i+1+j=n} K_i \times L_j, \quad n \geq 0.$$
**Example 1.1** ([14])

1. If \( K \in s\text{Set} \) and \( L = \Delta^0 \), then \( K^{\triangleright} = K \star \Delta^0 \) is the cocone or the right cone on \( K \).

   Dually, if \( L \in s\text{Set} \) then \( L^{\triangleleft} = \Delta^0 \star L \) is the cone or the left cone on \( L \).

2. Let \( K = \Lambda^2_0 \). If we see this left horn as a pushout, the cocone \( (\Lambda^2_0)^{\triangleright} \) is isomorphic to the square \( \square = \Delta^1 \times \Delta^1 \), that is, to the filled in diagram

\[
\begin{array}{ccc}
(0,0) & \xrightarrow{(1,0)} & (1,0) \\
\downarrow & & \downarrow \\
(0,1) & \xrightarrow{(1,1)} & (1,1)
\end{array}
\]

**Proposition 1.5** ([8])

(i) For the standard simplexes one has an isomorphism \( \Delta^i \star \Delta^j \cong \Delta^{i+1+j} \), \( i, j \geq 0 \), and these isomorphisms are with the obvious inclusions of \( \Delta^i \) and \( \Delta^j \).

(ii) If \( X \) and \( Y \) are \( \infty \)-categories, then the join \( X \star Y \) is an \( \infty \)-category.

1.8 **The slice \( \infty \)-category**

In the case of the classical categories, if \( A, B \) are categories and \( p : A \to B \) is any functor, one can form the slice category \( B/p \) of the object over \( p \) or cones on \( p \). The following propositions allow us to define the slice \( \infty \)-category.

**Proposition 1.6** ([15]) Let \( K \) and \( S \) be simplicial sets, and \( p : K \to S \) be an arbitrary map. There is a simplicial set \( S/p \) such that there exists a natural bijection

\[
s\text{Set}(Y, S/p) \cong s\text{Set}_p(Y \star K, S),
\]

where the subscript on the righthand side indicates that we consider only those morphisms \( f : Y \star K \to S \) such that \( f|K = p \).

**Proposition 1.7** (Joyal) Let \( X \) be an \( \infty \)-category and \( K \) be a simplicial set. If \( p : K \to X \) is a map of simplicial sets, then \( X/p \) is an \( \infty \)-category. Moreover, if \( q : X \to Y \) is a categorical equivalence, then the induced map \( X/p \to Y_{qp} \) is a categorical equivalence as well.

**Definition 1.17** (Slice \( \infty \)-Categorical [8]) Let \( X \) be an \( \infty \)-category, \( K \) be a simplicial set and \( p : K \to X \) be a map of simplicial sets. Define the slice \( \infty \)-category \( X/p \) of the objects over \( p \) or cones on \( p \). Dually, \( X_{p/} \) is the \( \infty \)-category of objects under \( p \) or cocones on \( p \).

**Example 1.2** Let \( X \) be an \( \infty \)-category and \( x \in X \) be an object, which corresponds to map \( x : \Delta^0 \to X \). The objects of the \( \infty \)-category \( X/x \) of cones on \( x \) are morphisms
$y \to x$ in $X$, and the morphism from $y \to x$ to $z \to x$ in $X/x$, are the 2-simplexes

\begin{center}
\begin{tikzpicture}
    \node (y) at (0,0) {$y$};
    \node (z) at (0,-2) {$z$};
    \node (x) at (2,0) {$x$};
    \draw[->] (y) to (x);
    \draw[->] (z) to (x);
    \draw[->] (y) to (z);
\end{tikzpicture}
\end{center}

in the $\infty$-category $X$.

### 1.9 Limits and colimits

An object $t$ of a category is final if for each object $x$ in this category, there is a unique morphism $x \to t$. Next, one defines the final objects at $\infty$-categories, under a contractible space of morphisms.

**Definition 1.18** (Final object [8]) An object $\omega \in X$ in an $\infty$-category $X$ is a final object if for any object $x \in X$, the Kan complex of morphisms $X(x, \omega)$ is contractible.

**Theorem 1.3** (Joyal) Take an object $\omega \in X$ in an $\infty$-category $X$ and $\pi : X/\omega \to X$ the canonical projection. The following conditions are equivalent.

(i) The object $\omega \in X$ is final.
(ii) The map $\pi : X/\omega \to X$ is a trivial fibration.
(iii) The map $\pi : X/\omega \to X$ is a categorical equivalence.
(iv) The map $\pi : X/\omega \to X$ has a section which sends $\omega$ to $1_{\omega}$.
(v) Any map $f_0 : \partial \Delta^n \to X$, such that $n > 0$ and $f(n) = \omega$, has an extension $f : \Delta^n \to X$.

**Corollary 1.1** ([12]) The final objects of an $\infty$-category $X$ form a Kan complex which is either empty or equivalent to the point.

**Corollary 1.2** ([12]) Let $x$ be a final object in an $\infty$-category $X$. For any simplicial set $A$, the constant map $A \to X$ with value $x$ is a final object in $\text{Map}(A, X)$.

**Definition 1.19** (Limit and colimit [15]) Let $X$ be an $\infty$-category and let $p : K \to X$ be a map of simplicial sets. A colimit for $p$ is an initial object of $X/p$, and a limit for $p$ is a final object of $X/p$.

By the dual of Corollary 1.1, if the colimit exists, then the Kan complex of initial objects is contractible, i.e., the initial object is unique up to contractible choice.

### 1.10 $\infty$-categories of presheaves

**Definition 1.20** ($\infty$-categories of presheaves [8]) Let $S$ be a simplicial set. One lets $P(S)$ or $PS$ denote simplicial set $\text{Fun}(S^{op}, S)$, where $S$ denotes the $\infty$-category of the small Kan complexes or $\infty$-groupoids, also called the $\infty$-category of the spaces. One will say that $P(S)$ is the $\infty$-category of the presheaves on $S$. 
**Proposition 1.8** ([8]) Let $S$ be a simplicial set. The $\infty$-category $P(S)$ of the presheaves on $S$ admits all small limits and colimits.

**Proposition 1.9** ($\infty$-Categorical Yoneda Lemma [8]) Let $S$ be a simplicial set. Then the Yoneda embedding $j : S \to PS$ is fully faithful.

**Notation 1.2** Let $X$ be an $\infty$-category and $S$ be a simplicial set. One lets $\text{Fun}^L(PS, X)$ denote the full subcategory of $\text{Fun}(PS, X)$ spanning those functors $PS \to X$ which preserve small colimits.

The motivation for this notation stems from Adjoint Functor Theorem (will be seen later), where $\text{Fun}^L(PS, X)$ also denotes the full subcategory of $\text{Fun}(PS, X)$ spanning those functors which are left adjoints.

**Theorem 1.4** ([8]) Let $S$ be a small simplicial set and let $X$ be an $\infty$-category which admits small colimits. The composition with the Yoneda embedding $j : S \to PS$ induces an equivalence of $\infty$-categories

$$\text{Fun}^L(PS, X) \to \text{Fun}(S, X).$$

1.11 Some $\infty$-categories of presheaves

In the literature, such as can be seen in [8, 12], one finds the definition of the $\infty$-category of presheaves on a small $\infty$-category $B$ as $PB = [B^{op}, S]$ which sets the categorical equivalence

$$\text{Fun}(A, PB) \simeq \text{Fun}(A \times B^{op}, S),$$

where $S$ is the $\infty$-category of all small Kan complexes. The $\infty$-category $PB$ and this equivalence are defined on the $\infty$-category of all $\infty$-categories $\text{CAT}_\infty$.

Another fundamental result is the following: given a collection of simplicial sets $\mathcal{K}, \mathcal{R} \subseteq \mathcal{K}$ and $A$ an $\infty$-category, there exists an $\infty$-category $P^\mathcal{K}_\mathcal{R}A$ and a functor $j : A \to P^\mathcal{K}_\mathcal{R}A$ with the following properties:

1. $P^\mathcal{K}_\mathcal{R}A$ admits $\mathcal{K}$-indexed colimits, i.e., admits $K$-indexed colimits for each $K \in \mathcal{K}$.
2. For every $\infty$-category $B$ which admits $\mathcal{K}$-indexed colimits, composition with $j$ induces an equivalence of $\infty$-categories

$$\text{Fun}_\mathcal{K}(P^\mathcal{K}_\mathcal{R}A, B) \simeq \text{Fun}_\mathcal{R}(A, B).$$

If $A$ admits all the $\mathcal{R}$-indexed colimits, we also have

3. The functor $j$ is fully faithful.

where $\text{Fun}_\mathcal{R}(A, B)$ is the full subcategory of $\text{Fun}(A, B)$ spanned by those functors which preserve $\mathcal{R}$-indexed colimits, i.e., which preserve $K$-indexed colimits for each $K \in \mathcal{R}$; the same applies to $\text{Fun}_\mathcal{K}(P^\mathcal{K}_\mathcal{R}A, B)$.

**Example 1.3** Let $\mathcal{R} = \emptyset$ and $\mathcal{K}$ be the class of all small simplicial sets. If $A$ is a small $\infty$-category, then $P^\mathcal{K}_\mathcal{R}A \simeq PA$. 

\[\text{Springer}\]
Example 1.4 Let \( \mathcal{R} = \emptyset \) and \( \mathcal{K} \) be the class of all small \( \kappa \)-filtered simplicial sets for some regular cardinal \( \kappa \). If \( A \) is a small \( \infty \)-category, then \( P^K_A \cong \text{Ind}_\kappa A \).

Example 1.5 Let \( \mathcal{R} = \emptyset \) and \( \mathcal{K} \) be the class of all \( \kappa \)-small simplicial sets for some regular cardinal \( \kappa \). If \( A \) is a small \( \infty \)-category, then \( P^K_A \cong P^K A \), where \( P^K \) is the full subcategory of all \( \kappa \)-compact elements of \( PA \).

Example 1.6 Let \( \mathcal{R} \) be the class of all \( \kappa \)-small simplicial sets for some regular cardinal \( \kappa \) and let \( \mathcal{K} \) be the collection of all small simplicial sets. Let \( A \) be a small \( \infty \)-category which admits \( \kappa \)-small colimits, then \( P^K_A \cong \text{Ind}_\kappa A \). Also, we have \( A \cong P^K C \) for some small \( \infty \)-category \( C \) which does not necessarily admit \( \kappa \)-small colimits.

2 Arbitrary syntactical homotopic \( \lambda \)-models

In this section, we discuss some consequences of the arbitrary syntactic homotopic lambda models introduced in [4], which correspond to a direct generalization (2-dimensional) of the traditional structured set models of a cartesian closed category (1-dimensional) as can be seen in [16, 17].

Notation 2.1 For \( K \) being a Kan simplex and \( n \geq 0 \), let \( K_n = \text{Fun}(\Delta^n, K) \) be the Kan complex of the \( n \)-simplexes.

Denote by \( \Omega_n(K, a) \) the class of all the spheres \( \partial \Delta^n \to K \) with initial vertex \( a \in K \).

Let \( \text{Var} \) be the set of all variables of \( \lambda \)-calculus, for all \( n \geq 0 \), each assignment \( \rho : \text{Var} \to K_n \) is an \( n \)-simplex of \( K \), for each \( t \in \text{Var} \), \( x \in \text{Var} \) and \( f \in K_m \).

Denote by \( [f/x] \rho : \text{Var} \to K \) the assignment

\[
(f/x)(t) = \begin{cases} f & \text{if } t = x \\ \rho(t) & \text{if } t \neq x. \end{cases}
\]

Definition 2.1 (Syntactic Homotopic \( \lambda \)-model) A homotopic \( \lambda \)-model is a triple \( (K, \cdot, [] \cdot) \), where \( K \) is a Kan complex, \( \cdot : K \times K \to K \) is a functor, and \( [] \cdot \) is a mapping which assigns to \( \lambda \)-term \( M \) and each assignment \( \rho : \text{Var} \to K_n \), an \( n \)-simplex \([P] \rho \) in \( K \) for each \( n \geq 0 \) such that

1. \( [x] \rho = \rho(x) \);
2. \( [MN] \rho = [M] \rho \cdot [N] \rho \);
3. For each \( f \in K_n \), there is a limit \( \beta_f : [\lambda x.M] \rho \cdot f \to [M]_{[f/x] \rho} \) from \( [M]_{[f/x] \rho} \in K_n \);
4. \( [M] \rho = [M] \sigma \) if \( \rho(x) = \sigma(x) \) for \( x \in \text{FV}(M) \);
5. \( [\lambda x.M(x)] \rho = [\lambda y.M(y)] \rho \) if \( y \notin \text{FV}(M) \);
6. if \( (\forall a \in \Omega_0)(\forall n \geq 1)(\forall \omega \in \Omega_n(K, a)) \) \( ([M]_{[\omega/x] \rho} = [N]_{[\omega/x] \rho}) \), then \( [\lambda x.M] \rho = [\lambda x.N] \rho \).

The homotopic model \( (K, \cdot, [\cdot]) \) is an extensional syntactic homotopic model if it satisfies the additional property: there is a colimit \( \eta : [M] \rho \to [\lambda x.Mx] \rho \) from \( [M] \rho \in K_n \) with \( x \notin \text{FV}(M) \).
Remark 2.1 Note that the condition (3) of the Definition 2.1, by the Homotopy Extension Lifting Property [8], if \( x \in FV(P) \) any cone \( \beta_f : [\lambda x.P]_\rho \cdot f \to [P]_{[f/x]} \rho \) in \( K/[f/x]\rho \) is a limit of \( n \)-simplex \([P]_{[f/x]} \rho \). Since \( K_n \) is a Kan complex, by the theorem mentioned above, the induced functor \( Fun(\Delta^1, K_n) \to Fun(\Delta^0, K_n) \) is a trivial fibration, hence the fibre \((K_n)/[f/x] \rho \) is contractible, that is \( K/[f/x] \rho \) is contractible. Thus the condition (3) is reduced to the existence of a cone \( \beta_f : [\lambda x.P]_\rho \cdot f \to [P]_{[f/x]} \rho \) in \((K_n)/[f/x] \rho \).

Definition 2.2 Let \( \mathfrak{M} = \langle K, \cdot, [\cdot] \rangle \) be a syntactic homotopic \( \lambda \)-model. The notion of satisfaction in \( \mathfrak{M} \) is defined as

\[
\mathfrak{M}, \rho \models M = N \iff [M]_\rho \sim [N]_\rho
\]

\[
\mathfrak{M} \models M = N \iff \forall \rho (\mathfrak{M}, \rho \models M = N)
\]

Lemma 2.1 Let \( \mathfrak{M} = \langle K, \cdot, [\cdot] \rangle \) be a syntactic homotopic \( \lambda \)-model. Then, for all \( M, N, x, n \geq 0 \) and \( \rho : Var \to K_n \),

(i) \( [[z/x]M]_\rho = [M]_{[\rho(z)/x]} \rho \),

(ii) if \( [[N/x]M]_\rho = [M]_{[N]_\rho/x} \rho \), then \( [\lambda y.[N/x]M]_\rho = [\lambda y.M]_{[N]_\rho/x} \rho \),

(iii) \( [[N/x]M]_\rho = [M]_{[N]_\rho/x} \rho \).

Proof (i) One has that,

\[
[[z/x]M]_\rho = [[z/x]M]_{[\rho(z)/x]} \rho \overset{\beta_{\rho(z)}}{\leftarrow} [\lambda z.[z/x]M]_\rho \cdot \rho(z) = [\lambda x.M]_\rho \cdot \rho(z);
\]

\[
[M]_{[\rho(z)/x]} \rho \overset{\beta_{\rho(z)}}{\leftarrow} [\lambda x.M]_\rho \cdot \rho(z)
\]

That is, \( [[z/x]M]_\rho = [M]_{[\rho(z)/x]} \rho \).

(ii) First suppose \( x \notin FV(N) \). Let \( y \neq x \) and \( y \notin FV(N) \). For \( \rho' = [[N]_\rho/x] \rho \) and any \( \omega \in \Omega_n(K, a) \), with an arbitrary vertex \( a \in K \) and any \( n \geq 1 \), one has

\[
[[N/x]M]_{[\omega/y]} \rho' = [[N/x]M]_{[\omega/y]} \rho
\]

\[
= [M]_{[\omega/y][N]_\rho/x} \rho; \quad \text{by hypothesis}
\]

\[
= [M]_{[\omega/y]} \rho'.
\]

By Definition 2.1 (6), \( [\lambda y.[N/x]M]_{\rho'} = [\lambda y.M]_{\rho'} \), hence

\[
[[\lambda y.[N/x]M]_\rho = [\lambda y.[N/x]M]_{\rho'} = [\lambda y.M]_{[N]_\rho/x} \rho'.
\]

If \( x \in FV(N) \), the proof is identical to [16, p.103].

(iii) Follows easy by induction on the \( \lambda \)-term \( M \).

\[\square\]

Theorem 2.1 Let \( \mathfrak{M} = \langle K, \cdot, [\cdot] \rangle \) be a syntactic homotopic \( \lambda \)-model. Then

\[
\lambda \beta \vdash M = N \implies \mathfrak{M} \models M = N.
\]
Proof  By induction on the length of proof. For the axiom \((\lambda x.M)N = [N/x]M\) we proceed

\[
[(\lambda x.M)N]_\rho = [\lambda x.M]_\rho \cdot [N]_\rho
\]

\[
\xrightarrow{\beta [N]_\rho} [M][[N]_\rho/x]_\rho
\]

\[
= [[N/x]M]_\rho
\]

The rule \(M = N \implies \lambda x.M = \lambda x.N\) follows from Definition 2.1 (6). The other rules are trivial. \(\square\)

**Definition 2.3 (h.p.o)** Let \(\hat{K}\) be an \(\infty\)-category. The largest Kan complex \(K \subseteq \hat{K}\) is a homotopy partial order (h.p.o), if for every \(x, y \in K\) one has that \(\hat{K}(x, y)\) is contractible or empty. Hence, the Kan complex \(K\) admits a relation of h.p.o \(\preceq\) defined for each \(x, y \in K\) as follows: \(x \preceq y\) if \(\hat{K}(x, y) \neq \emptyset\), hence the pair \((K, \preceq)\) is a h.p.o. (we denote simply by \(K\)). The \(\infty\)-category \(\hat{K}\) is also called a h.p.o.

**Definition 2.4 (c.h.p.o)** Let \(K\) be an h.p.o.

1. An h.p.o \(X \subseteq K\) is directed if \(X \neq \emptyset\) and for each \(x, y \in X\), there exists \(z \in X\) such that \(x \preceq z\) and \(y \preceq z\).
2. \(K\) is a complete homotopy partial order (c.h.p.o) if
   a. There are initial objects, i.e., \(\bot \in K\) is an initial object if for each \(x \in K\), \(\bot \preceq x\).
   b. For each directed \(X \subseteq K\) the supremum (or colimit) \(\bigvee X \in K\) exists.

**Definition 2.5 (Reflexive and Extensional Kan complex)** Let \(K\) be a c.h.p.o. The Kan complex \(K\) is called reflexive if the full subcategory \([K \to K] \subseteq Fun(K, K)\) of the continuous functors is a retract of \(K\), i.e., there are continuous functors \(F : K \to [K \to K],\ G : [K \to K] \to K\) such that there is a natural equivalence \(\varepsilon : FG \to id_{[K \to K]}\).

If there is a natural equivalence \(\eta : id_K \to GF\), we call \(K\) an extensional Kan complex.

In [18], we proved the existence of extensional Kan complexes.

**Definition 2.6** Let \(K\) be a reflexive Kan complex (via \(F, G\) and \(\varepsilon\)).

1. For \(f, g : \Delta^n \to K\) (or also \(f, g \in K_n\)) define the \(n\)-simplex

\[
\sigma : \Delta^n \to K, \quad \sigma = F(f)(g).
\]

In particular for vertices \(a, b \in K\),

\[
a \cdot b = a \cdot \Delta^0 b = F(a)(b),
\]
besides, \( F(a) \cdot (-) = a \cdot (-) \) and \( F(-)(b) = (-) \cdot b \) are functors on \( K \), then for \( f \in K_n \) one defines the \( n \)-simplexes
\[
a \cdot f = F(a)(f), \quad f \cdot b = F(f)(b).
\]

2. For each \( n \geq 0 \), let \( \rho \) be a valuation at \( K_n \). Define the interpretation \([\ ]_\rho : \Lambda \to K_n\) by induction as follows

(a) \([x]_\rho = \rho(x)\),

(b) \([MN]_\rho = [M]_\rho \cdot [N]_\rho\),

(c) \([\lambda x . M]_\rho = G(\lambda f . [M]_{f/x})_\rho\), where \( \lambda f . [M]_{f/x} = [M]_{-/x}\).

**Lemma 2.2** If \( n \geq 0 \) and \( \rho : \text{Var} \to K_n \), then \( \lambda f . [M]_{f/x} \) defines a functor \( \Delta^n \to [K \to K] \); hence \( \lambda x . M \) is well-defined in Definition 2.6 (2.c).

Proof. By induction on \( P \) we show that \( \lambda f . [P]_{f/(x(i))} \) defines a functor \( K \times \Delta^n \to K \) for each vertex \( i \in \Delta^n \) and all \( \rho \) in \( K_n \), where the map \( [P]_{-/(x(i))} : K \times \Delta^n \to K \) (with \( \rho(x)(-): \Delta^n \to K \)) depicts to \( \lambda f . [P]_{f/x} : K \to K_n \).

For each \( f : \Delta^m \to K \) one has:

(a) \([x]_{f/(x(i))} = f \in K_m \). So \( \lambda f . [x]_{f/(x(i))} = I_K \) (Identity functor), which is continuous.

(b) \([x]_{f/(x(y))} = s^m(\rho(x)(i)) \in K_m \), with \( s^m \) the degeneration operator applied \( m \)-times to vertex \( \rho(x)(i) \). Then \( \lambda f . [x]_{f/(x(y))} \) is the constant functor in the vertex \( \rho(x)(i) \), which is continuous.

(c) \([MN]_{f/(x(i))} = [M]_{f/(x(i))} \cdot \Delta^n[N]_{f/(x(i))} \in K_m \); since by I.H (Induction Hypothesis) \([M]_{f/(x(i))} \cdot \Delta^n[N]_{f/(x(i))} \) are \( m \)-simplices (can be degenerates), hence \([MN]_{f/(x(i))} \) is an \( m \)-simplex. Besides, the functor \([MN]_{-/(x(i))} = F([M]_{-/(x(i))}) \) is continuous by I.H and continuity of \( F \).

(d) \([\lambda y . M]_{f/(x(i))} = G(\lambda g . [M]_{(g/y)f/(x(i))}) \in K_m \); by I.H the map \( \lambda f . \lambda g . [M]_{(g/y)f/(x(i))} : K \to [K \to K] \) is a continuous functor in \( f \) and \( g \) separately, so is continuous [18]. Thus, \( \lambda g . [M]_{(g/y)f/(x(i))} \) is an \( m \)-simplex at \([K \to K] \), applying the continuous functor \( G : [K \to K] \) on it, one has an \( m \)-simplex in \( K \), and hence the functor \( \lambda y . M \) is continuous.

For the proof of Theorem 2.2, we make the following remark.

**Remark 2.2** Just as the category \( Set \) has enough points, the \( \infty \)-category \( S \) has enough points in the following sense: Let \( f, g : X \to Y \) be functors between Kan complexes. If for each \( x \in X \), \( n \geq 0 \) one has \( f^n_x = g^n_x \), with \( f^n_x : \pi_n(X, x) \to \pi_n(Y, f(x)) \) and \( g^n_x : \pi_n(X, x) \to \pi_n(Y, g(x)) \) as maps induced by \( f \) and \( g \) respectively, then one has functorial equivalence \( f \simeq g \). The property ‘\( S \) has enough points’ can also be interpreted as: given a morphism \( f : X \to Y \) in \( S \), if the induced map \( f^n_x : \pi_n(X, x) \to \pi_n(Y, f(x)) \) is an isomorphism of groups for each \( n \geq 0 \) and \( x \in X \), then \( f \) is a homotopy equivalence.

**Theorem 2.2** Let \( K \) be a reflexive Kan complex via the morphisms \( F, G \), and let \( \mathcal{M} = (K, \cdot, [\ ]_\rho) \). Then
1. $\mathcal{M}$ is a syntactic homotopic $\lambda$-model.
2. $\mathcal{M}$ is extensional iff there is a natural equivalence $\eta : \text{id}_K \to GF$.

**Proof**

1. The conditions in Definition 2.1 (1), (2) are trivial. As to (3), given $g \in K_n$,

\[
\begin{align*}
\mathcal{M}_{\rho} : \downarrow \rho &\to \downarrow \rho \\
\rho \cdot g &\equiv F(G(\lambda f.[M]_{f/x}\rho))(g) \\
\rho \cdot g &\equiv [M]_{g/x}\rho,
\end{align*}
\]

where $\varepsilon_{\lambda f.[M]_{f/x}\rho}$ is the natural equivalence, induced by $\varepsilon$, between the functors $F(G(\lambda f.[M]_{f/x}\rho)), \lambda f.[M]_{f/x}\rho : K \to K_n$. Hence $(\varepsilon_{\lambda f.[M]_{f/x}\rho})_g$ is the equivalence induced by the $n$-simplex $g$ in $K$.

The condition (4) is trivial, since if $[M]_\rho \neq [M]_\sigma$ so there is $x \in FV(M)$ such that $\rho(x) \neq \sigma(x)$. The condition (5), given any vertex $a \in K$ and $y \notin FV(M)$,

\[
\lambda f.[M(y)]_{f/y}\rho = \lambda f.[M(x)]_{f/x}\rho.
\]

Applying $G$ and by Definition 2.6 (c), it follows that

\[
\begin{align*}
\mathcal{M}_{\rho} : \downarrow \rho &\to \downarrow \rho \\
\rho \cdot g &\equiv G(\lambda f.[M(y)]_{f/y}\rho) \\
\rho \cdot g &\equiv [M(y)]_{g/y}\rho \\
\rho \cdot g &\equiv [\lambda y.M(y)]_{\rho} \\
\rho \cdot g &\equiv G(\lambda f.[M(x)]_{f/x}\rho) \\
\rho \cdot g &\equiv [M(x)]_{g/x}\rho \\
\rho \cdot g &\equiv [\lambda x.M(x)]_{\rho}.
\end{align*}
\]

Condition (6). By hypothesis, for every vertex $a \in K$, $n \geq 1$ and $\omega \in \Omega_n(K, a)$,

\[
(\lambda f.[M]_{f/x}\rho)(\omega) = [M]_{\omega/x}\rho = [N]_{\omega/x}\rho = (\lambda f.[N]_{f/x}\rho)(\omega),
\]

since $K$ does have enough points, then

\[
\lambda f.[M]_{f/x}\rho = \lambda f.[N]_{f/x}\rho
\]

applying $G$ and by Definition 2.6 (c),

\[
\begin{align*}
\mathcal{M}_{\rho} : \downarrow \rho &\to \downarrow \rho \\
\rho \cdot g &\equiv G(\lambda f.[M]_{f/x}\rho) \\
\rho \cdot g &\equiv [M]_{\rho} \\
\rho \cdot g &\equiv [\lambda x.M(x)]_{\rho} \\
\rho \cdot g &\equiv G(\lambda f.[N]_{f/x}\rho) \\
\rho \cdot g &\equiv [N]_{\rho} \\
\rho \cdot g &\equiv [\lambda x.N(x)]_{\rho}.
\end{align*}
\]

2. Suppose that $\mathcal{M}$ is extensional. Let $\omega \in \Omega_n(K, a)$. Then for all $a \in K$

\[
(GF(\omega)) \cdot a = F(GF(\omega))(a) = ((FG)F(\omega))(a) \xrightarrow{(e_{\omega F})_a} F(\omega)(a) = \omega \cdot a,
\]

$\square$ Springer
Towards a homotopy domain theory

by extensionality

\[ GF(\omega) \xrightarrow{(s_a F)} \omega = id_K(\omega), \]

since \( K \) does have enough points, hence

\[ GF \xrightarrow{s F} Id_K. \]

If \( GF \xrightarrow{\eta} Id_K \). For all \( \omega \in \Omega_n(K, a) \) by hypothesis and Definition 2.6

\[ F(a)(\omega) = a \cdot \omega \xrightarrow{\sigma} a' \cdot \omega = F(a')(\omega), \]

since \( K \) does have enough points, \( Fa \xrightarrow{\sigma} Fa' \). Applying \( G \), it follows that

\[ a \xrightarrow{\eta_a} GF(a) \xrightarrow{G\sigma} GF(a') \xrightarrow{\tilde{\eta}_{a'}} a', \]

where \( \tilde{\eta}_{a'} \) is an inverse of \( \eta_{a'} \).

3 Homotopic \( \lambda \)-models

Next, let us define cartesian closed \( \infty \)-category, points and paths of an \( \infty \)-category and \( \infty \)-homotopic \( \lambda \)-model.

Definition 3.1 (Cartesian closed \( \infty \)-category) Let \( \mathcal{C} \) be an \( \infty \)-category whose objects are small \( \infty \)-categories. We say that \( \mathcal{C} \) is a cartesian closed \( \infty \)-category (c.c.i.) if:

1. \( \mathcal{C} \) has a terminal object \( T \), i.e., \( \mathcal{C}(X, T) \) is contractible for each \( X \in \mathcal{C} \).
2. For \( X, Y \in \mathcal{C} \), there is the cartesian product \( X \times Y \), and this belongs to \( \mathcal{C} \),
3. For \( X, Y, Z \in \mathcal{C} \), there exists an internal morphism spaces \( Y \Rightarrow Z \) in \( \mathcal{C} \) such that sets the natural equivalence

\[ \mathcal{C}(X \times Y, Z) \simeq \mathcal{C}(X, Y \Rightarrow Z). \]

Definition 3.2 (Enough points and n-paths) Take an \( \infty \)-category \( \mathcal{C} \) with a terminal object \( T \). A point of an object \( X \) is a morphism \( x : T \to X \). The class of points of \( X \) is denoted by \( |X|_0 \).

1. We say that \( \mathcal{C} \) does have enough points if for each pair of morphisms \( f, g : X \to Y \) of \( \mathcal{C} \) such that for each point \( x : T \to X \) there is an equivalence \( \sigma_x : f \circ x \simeq g \circ x \) in \( \mathcal{C}(T, Y) \), then there is an equivalence \( \sigma : f \simeq g \) in \( \mathcal{C}(X, Y) \).
2. An object \( X \in \mathcal{C} \) does have enough points if one has (1) in the case that \( Y = X \).
3. Take the points \( x, y \in |X|_0 = \mathcal{C}(T, X) \). A 1-path \( p : x \to y \) in \( X \) is a 1-simplex at \( \mathcal{C}(T, X) \). The class of the 1-paths of \( X \) is denoted by \( |X|_1 = (\mathcal{C}(T, X))_1 \). In general, for each \( n \geq 1 \), the class of \( n \)-paths of \( X \) corresponds to \( |X|_n = (\mathcal{C}(T, X))_n \).
Remark 3.1 Note that in Definition 3.2 (3), since $C$ is an $\infty$-category, all the 1-paths in $X$ (2-simplexes in $X$) are invertible. Then we say that $X \in C$ has a ‘homotopic structure’. If $C$ is an $\infty$-bicategory with a terminal object, we say that an object $X \in C$ has ‘$\infty$-categorical structure’. 

Definition 3.3 (Reflexive object) Let $C$ be a c.c.i. An object $K \in C$ is called reflexive if $(K \Rightarrow K)$ is a weak retract of $K$ i.e., there are morphisms $F : K \to (K \Rightarrow K)$, $G : (K \Rightarrow K) \to K$ such that there is a natural equivalence $\varepsilon : FG \to id_{(K \Rightarrow K)}$.

If there is a natural equivalence $\eta : id_K \to GF$, then $K$ is an extensional object.

Definition 3.4 (Homotopic $\lambda$-model) A homotopic $\lambda$-model of a c.c.i. $C$ is a quadruple $K = \langle K, F, G, \varepsilon \rangle$ where $K \in C$ is a reflexive object via $F$, $G$ and $\varepsilon$ of the definition above. The quintuple $K = \langle K, F, G, \varepsilon, \eta \rangle$ is an extensional homotopic $\lambda$-model, with $\eta$ being the natural equivalence of the same definition.

Remark 3.2 By virtue of the Remark 3.1, if $U$ is a reflexive object in a cartesian closed $\infty$-bicategory $C$, we say that $K$ is an ‘$\infty$-categorical $\lambda$-model’.

4 Kleisli $\infty$-categories

Next, we define the Kleisli structures on the $\infty$-categories, a general and direct version of those initially introduced by [19] for the case of bicategories.

Definition 4.1 (Kleisli structure) Let $\mathcal{K}$ be an $\infty$-category and $\mathcal{A}$ be an $\infty$-category contained in $\mathcal{K}$. A Kleisli structure $P$ on $\mathcal{A} \subseteq \mathcal{K}$ is the following.

- For each vertex $a \in \mathcal{A}$ an arrow $y_a : a \to Pa$ in $\mathcal{K}$.
- For each $a, b \in \mathcal{A}$ a functor $\mathcal{K}(a, Pb) \to \mathcal{K}(Pa, Pb)$, $f \mapsto f^\#$.
- A subcategory $\mathcal{K}^L \subseteq \mathcal{K}$ such that for all the vertices $a, b \in \mathcal{A}$, the homotopy equivalence $\mathcal{K}(a, Pb) \xrightarrow{(\_)^\#} \mathcal{K}^L(Pa, Pb)$.

such for each horn $(g^\#, \_ , f^\#) : \Lambda^2_1 \to \mathcal{K}^L$ one has the equality of fibres $F_{g^\#, f^\#}^L = F_{g^\#, f^\#}$, where $F_{g^\#, f^\#}^L$ and $F_{g^\#, f^\#}$ are the fibres of the canonical maps $Fun(\Delta^2, \mathcal{K}^L) \to Fun(\Lambda^2_1, \mathcal{K}^L)$ and $Fun(\Delta^2, \mathcal{K}) \to Fun(\Lambda^2_1, \mathcal{K})$ respectively, with $f : a \to Pb$ and $g : b \to Pc$ being edges in $\mathcal{K}$. 

Springer
It is clear that $P$ is a functor from $\mathcal{A}$ to $\mathcal{K}$ such that for each 1-simplex $f : a \to b$ of $\mathcal{A}$, sets $Pf = (y_b f)^\# : Pa \to Pb$.

**Example 4.1** The functor $P : \mathcal{C}at_\infty \to \mathcal{C}AT_\infty$, given by $PA = [A^{op}, S]$, is a Kleisli structure on $\mathcal{C}at_\infty \subseteq \mathcal{C}AT_\infty$.

For each small $\infty$-category $A$, the arrow $y_A : A \to PA$ is the Yoneda embedding. For every functor $f : A \to PB$ there exists the functor $f^\# : PA \to PB$ which preserves small colimits. Besides, one has the categorical equivalence

$$Fun(A, PB) \xleftarrow{(-)\#} Fun^L(PA, PB),$$

where $Fun^L(PA, PB)$ is the $\infty$-category of functors which preserve small colimits [8]. If we take the subcategory $Pr^L \subseteq \mathcal{C}AT_\infty$, whose objects are presentable $\infty$-categories and morphisms are functors which preserve small colimits, the categorical equivalence above is restricted to the homotopy equivalence

$$\mathcal{C}AT_\infty(A, PB) \xleftarrow{(-)\#} \mathcal{P}r^L(PA, PB).$$

**Definition 4.2** ($\infty$-Kleisli category) Take a Kleisli structure $P$ on $A \subseteq \mathcal{K}$. Define the $\infty$-category $\mathcal{K}l(P)$ as follows. The objects of $\mathcal{K}l(P)$ are the objects of $A$ and the morphism spaces is defined by

$$\mathcal{K}l(P)(a, b) := \mathcal{K}(a, Pb).$$

for the all objects $a, b \in A$.

**Remark 4.1** By Definition 4.1, the homotopy equivalence $\mathcal{K}l(P)(a, b) \to \mathcal{K}^L(Pa, Pb)$, gets to establish that $\mathcal{K}L(P)$ is an $\infty$-category embedded in $\mathcal{K}^L$. Another interesting way would be to define $\mathcal{K}l(P)$ as a weighted colimit or the pushout of diagram

$$(\mathcal{C}at_\infty \subseteq \mathcal{C}AT_\infty \xleftarrow{P} \mathcal{C}at_\infty)$$

in the category of simplicial sets $\mathcal{S}et_\Delta = Fun(\Delta^{op}, \mathcal{S}et)$ (complete and cocomplete) and so $\mathcal{K}l(P)$ would be an $\infty$-category.

**Proposition 4.1** $P(A \times B) \simeq PA \otimes PB$ in the $\infty$-category $Pr^L$. 

**Proof**

$$Fun^L(P(A \times B), C) \simeq Fun(A \times B, C)$$

$$\simeq Fun(A, C^B)$$

$$\simeq Fun^L(PA, Fun^L(PB, C))$$

Thus, $Pr^L(P(A \times B), C) \simeq Pr^L(PA, PB \to C) \simeq Pr^L(PA \otimes PB, C)$.  

\[\square\]
5 A cartesian closed $\infty$-category (c.c.i) with enough points

In this section we prove that the Kleisli $\infty$-category $Kl(P)$ generated by the structure $P$ on $\text{Cat}_\infty \subseteq \text{CAT}_\infty$ is cartesian closed and has enough points. Thus $Kl(P)$ is a candidate for a higher $\lambda$-model.

Lemma 5.1 The $\infty$-category $Kl(P)$ is cartesian closed.

Proof

\[ \text{Fun}(A \times B, PC) \cong \text{Fun}(A, [B, PC]) \]
\[ = \text{Fun}(A, [B, SC^{\text{op}}]) \]
\[ \cong \text{Fun}(A, [B \times C^{\text{op}}, S]) \]
\[ = \text{Fun}(A, P(B^{\text{op}} \times C)). \]

Thus $Kl(P)(A \times B, C) \cong Kl(P)(A, B^{\text{op}} \times C) = Kl(P)(A, B \Rightarrow C)$. □

Theorem 5.1 The $\infty$-category $Kl(P)$ does have enough points.

Proof A morphism $A \to B$ in $Kl(P)$ corresponds to a functor $A \to PB$. Since $PA$ is a closure of $A$ under small colimits, such a functor corresponds to a small colimit preserving functor $PA \to PB$. Since $PA$ and $PB$ are weakly contractible, the functor $PA \to PB$ is sufficiently determined by all the vertices of $PA$ (points of $A$). □

In [18], one can find some examples of reflexive objects in the category $KL(P)$, which are provided by methods of solving domain equations on arbitrary cartesian closed $\infty$-categories, where these types of equations are called Homotopy Domain Equations.

6 Conclusion

What we have done here is a beginning for the construction of a Homotopy Domain Theory (HoDT) which provides techniques to build homotopy $\lambda$-models that allow for the generalization of Church-like conversion relations (such as, e.g., $\beta$-equality, $\eta$-equality) to higher term-contraction induced equivalences.

Besides, we generalized the Kleisli bicategory to a Kleisli $\infty$-category, and we show that it is closed cartesian with enough points. For future work, we could apply the techniques of HoDT to this Kleisli $\infty$-category and thus obtain a reflexive Kan complex (homotopic $\lambda$-model) with relevant information.

On the other hand, we define the interpretation of the $\beta\eta$-contractions in a reflexive Kan complex, whose $\infty$-groupoid structure induces higher $\beta\eta$-contractions which would inhabit a type of identity (based on computational paths). This work could be seen as the beginning of the semantics of another version, based on computational paths, of the Theory of Homotopy Types.
References

1. Martínez-Rivillas, D., de Queiroz, R.: The theory of an arbitrary higher \(\lambda\)-model, arXiv:2111.07092
2. de Queiroz, R., de Oliveira, A., Ramos, A.: Propositional equality, identity types, and direct computational paths. South Am. J. Logic 2(2), 245–296 (2016)
3. Ramos, A., de Queiroz, R., de Oliveira, A.: On the identity type as the type of computational paths. Logic J. IGPL 25(4), 562–584 (2017)
4. Martínez-Rivillas, D., de Queiroz, R., de Oliveira, A.: The \(\infty\)-groupoid generated by an arbitrary topological \(\lambda\)-model. Logic J. IGPL 30(3), 465–488 (2022). https://doi.org/10.1093/jigpal/jzab015. arXiv:1906.05729
5. Kapulkin, C., Lumsdaine, P., Voevodsky, V.: The simplicial model of univalent foundations, arXiv:1211.2851
6. Lumsdaine, P., Shulman, M.: Semantics of higher inductive types. Math. Proc. Cambridge Philos. Soc. 169, 159–208 (2020)
7. Program, T.U.F.: Homotopy Type Theory: Univalent Foundations of Mathematics, Princeton. Institute for Advanced Study, Princeton (2013)
8. Lurie, J.: Higher Topos Theory. Princeton University Press, Princeton and Oxford (2009)
9. Hyland, M.: Some reasons for generalizing domain theory. Math. Struct. Comput. Sci. 20, 239–265 (2010)
10. Goerss, P., Jardine, J.: Simplicial Homotopy Theory. Birkhäuser Basel, Springer, Switzerland (2009)
11. Friedman, G.: An elementary illustrated introduction to simplicial sets. Rocky Mountain J. Math. 42(2), 353–423 (2012)
12. Cisinski, D.-C.: Higher Categories and Homotopical Algebra. Cambridge University Press, Cambridge (2019)
13. Rezk, C.: Stuff about quasicategories. In Lecture Notes for course at University of Illinois at Urbana-Champaign (2017)
14. Groth, M.: A short course on \(\infty\)-categories, (2015). https://arxiv.org/abs/1007.2925
15. Joyal, A.: Quasi-categories and kan complexes. J. Pure Appl. Algebra 175(1), 207–222 (2002)
16. Barendregt, H.: The Lambda Calculus, its Syntax and Semantics. North-Holland Co., Amsterdam (1984)
17. Hindley, J., Seldin, J.: Lambda-Calculus and Combinators: An Introduction. Cambridge University Press, New York (2008)
18. Martínez-Rivillas, D., de Queiroz, R.: Solving homotopy domain equations. arXiv:2104.01195
19. Hyland, M.: Elements of a theory of algebraic theories. Theoret. Comput. Sci. 546, 132–144 (2014)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s). Author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.