QUANTIZATION COEFFICIENTS FOR UNIFORM DISTRIBUTIONS ON THE BOUNDARIES OF REGULAR POLYGONS

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Abstract. In this paper, we give a general formula to determine the quantization coefficients for uniform distributions defined on the boundaries of different regular m-sided polygons inscribed in a circle. The result shows that the quantization coefficient for the uniform distribution on the boundary of a regular m-sided polygon inscribed in a circle is an increasing function of m, and approaches to the quantization coefficient for the uniform distribution on the circle as m tends to infinity.

1. Introduction

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, $\| \cdot \|$ denote the Euclidean norm on $\mathbb{R}^d$ for any $d \geq 1$, and $n \in \mathbb{N}$. For a finite set $\alpha \subset \mathbb{R}^d$, the cost or distortion error for $P$ with respect to the set $\alpha$, denoted by $V(P; \alpha)$, is defined by

$$V(P; \alpha) := \int \min_{a \in \alpha} \| x - a \|^2 dP(x).$$

Then, the nth quantization error for $P$, denoted by $V_n := V_n(P)$, is defined by

$$V_n := V_n(P) = \inf \left\{ V(P; \alpha) : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n \right\}.$$

A set $\alpha$ for which the infimum is achieved and contains no more than $n$ points is called an optimal set of n-means. It is well-known that for a continuous probability measure an optimal set of n-means always contains exactly $n$ elements. If $P$ is the probability distribution, then an optimal set of n-means is denoted by $\alpha_n := \alpha_n(P)$. Optimal sets of n-means for different probability distributions were determined by several authors, for example, see [CR1, CR2, DR1, DR2, GL2, L, R1, R2, R3, R4, R5, R6, RR1, RS]. It has broad applications in engineering and technology (see [GG, GL1, Z]). For any $s \in (0, +\infty)$, the number

$$\lim_{n \to \infty} n^2 V_n(P),$$

if it exists, is called the s-dimensional quantization coefficient for $P$. Bucklew and Wise (see [BW]) showed that for a Borel probability measure $P$ with non-vanishing absolutely continuous part the quantization coefficient exists as a finite positive number. For some more details interested readers can also see [GL1, P]. Let $E(X)$ represent the expected value of a random variable $X$ associated with a probability distribution $P$. Let $\alpha$ be an optimal set of n-means for $P$, and $a \in \alpha$. Then, it is well-known that $a = E(X : X \in M(a|\alpha))$, where $M(a|\alpha)$ is the Voronoi region of $a \in \alpha$, i.e., $M(a|\alpha)$ is the set of all elements $x$ in $\mathbb{R}^d$ which are closest to $a$ among all the elements in $\alpha$ (see [GG, GL1]).

From the work of Rosenblatt and Roychowdhury (see [RR2]), it is known that the quantization coefficient for the uniform distribution on a unit circle is $\pi^2/3$; on the other hand, from the work of Pena et al. (see [PRRSS]), it is known that the quantization coefficient for the uniform distribution on the boundary of a regular hexagon inscribed in a unit circle is 3. Notice that a regular m-sided polygon inscribed in a circle tends to the circle as m tends to infinity. Pena et

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al. conjectured that the quantization coefficient for the uniform distribution on the boundary of a regular $m$-sided polygon inscribed in a circle is an increasing function of $m$ (see [PRRSS]), and approaches to the quantization coefficient for the uniform distribution on the circle as $m$ tends to infinity. In this paper, we prove that the conjecture is true.

The arrangement of the paper is as follows: First, we prove a theorem Theorem 2.1 which gives a technique how to calculate the optimal sets of $n$-means and the $n$th quantization errors for all positive integers $n$ for a uniform distribution defined on any line segment. Next, let $P$ be the uniform distribution defined on the boundary of a regular $m$-sided polygon inscribed in a unit circle. In Proposition 2.3 for $k \geq 2$, we determine the optimal set of $mk$-means and the $mk$th quantization error for the probability distribution $P$. Then, with the help of the proposition, in Theorem 2.4, we have shown that the quantization coefficient for a uniform distribution on the boundary (see [RR2]). Thus, the result in this paper, shows that the conjecture given by Pena et al. in [PRRSS] is true.

2. Main result

In this section, first we give some basic definitions.

Let $i$ and $j$ be the unit vectors in the positive directions of the $x_1$- and $x_2$-axes, respectively. By the position vector $a$ of a point $A$, it is meant that $\overrightarrow{OA} = a$. We will identify the position vector of a point $(a_1, a_2)$ by $(a_1, a_2) := a_1i + a_2j$, and apologize for any abuse in notation. For any two position vectors $a := (a_1, a_2)$ and $b := (b_1, b_2)$, we write $\rho(a, b) := ||(a_1, b_1) - (a_2, b_2)||^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2$, which gives the squared Euclidean distance between the two points $(a_1, a_2)$ and $(b_1, b_2)$. Let $P$ and $Q$ belong to an optimal set of $n$-means for some positive integer $n$, and let $D$ be a point on the boundary of the Voronoi regions of the points $P$ and $Q$. Since the boundary of the Voronoi regions of any two points is the perpendicular bisector of the line segment joining the points, we have $|\overrightarrow{DP}| = |\overrightarrow{DQ}|$, i.e., $(\overrightarrow{DP})^2 = (\overrightarrow{DQ})^2$ implying $(p - d)^2 = (q - d)^2$, i.e., $\rho(d, p) - \rho(d, q) = 0$, where $p, q, d$ are, respectively, the position vectors of the points $P, Q, D$.

We call such an equation a canonical equation.

Let us now give the following theorem.

**Theorem 2.1.** Let $AB$ be a line segment joining the two points $A$ and $B$ given by the position vectors $a := (a_1, b_1)$ and $b := (a_2, b_2)$, respectively. Let $\mu$ be a uniform distribution on $AB$. Let $M(t)$ be the parametric representation of $AB$ for $0 \leq t \leq 1$ such that $M(0) = a$, and $M(1) = b$. Let $D_1$ and $D_2$ be two points on the segment $AB$ at distances $r_1$ and $r_2$ from $A$ and $B$, respectively (see Figure 4). Then, the optimal set of $n$-means for $\mu$ on the segment $D_1D_2$, is given by

$$\alpha_n(\mu, D_1D_2) := \left\{M\left(\frac{r_1}{\ell} + \frac{2j - 1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\right) : 1 \leq j \leq n\right\},$$

with the $n$th quantization error for $\mu$ on the segment $D_1D_2$,

$$V_n(\mu, D_1D_2) := n\int_{\frac{r_1}{\ell}}^{\frac{r_2}{\ell} + \frac{1}{2}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})} \rho(M(t), M(\frac{r_1}{\ell} + \frac{1}{2}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})))d\mu,$$

where $\ell$ is the length of the line segment $AB$.

**Proof.** Since $\ell$ is the length of the line segment $AB$, the probability density function (pdf) $f$ of the uniform distribution $\mu$ on $AB$ is given by $f(x_1, x_2) = \frac{1}{\ell}$ for all $(x_1, x_2) \in AB$, and zero, otherwise. Let $s$ represent the distance of any point on $AB$ from the point $A$. Then, we have $d\mu = d\mu(s) = \mu(ds) = f(x_1, x_2)ds = \frac{1}{\ell}ds$. Notice that $ds = \sqrt{(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2}dt = \ell dt$ yielding
Given, the parametric representation of the line segment $AB$ is $M(t)$ for $0 \leq t \leq 1$ with $M(0) = a$ and $M(1) = b$. Hence, the parameters for the points $D_1$ and $D_2$, which are at distances $r_1$ and $r_2$ from $A$ and $B$ are, respectively, given by $t = \frac{r_1}{\ell}$ and $t = 1 - \frac{r_2}{\ell}$, i.e., if $d_1$ and $d_2$ are the position vectors of the points $D_1$ and $D_2$ (see Figure 1), then we have

$$d_1 = M(\frac{r_1}{\ell}), \quad d_2 = M(1 - \frac{r_2}{\ell}).$$

In fact, we can identify the line segment $D_1D_2$ by its parameters in the closed interval $[\frac{r_1}{\ell}, 1 - \frac{r_2}{\ell}]$. By [RR2], we know that the optimal set of $n$-means with respect to an uniform distribution in the closed interval $[\frac{r_1}{\ell}, 1 - \frac{r_2}{\ell}]$ is given by the set

$$\left\{ \frac{r_1}{\ell} + \frac{j - 1}{n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell}) : 1 \leq j \leq n \right\}.$$

Hence, the optimal set of $n$-means for $\mu$ on the segment $D_1D_2$, is given by

$$\alpha_n(\mu, D_1D_2) := \left\{ M\left(\frac{r_1}{\ell} + \frac{2j - 1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\right) : 1 \leq j \leq n \right\}.$$

If $V_n(\mu, D_1D_2)$ is the corresponding quantization error, we have

$$V_n(\mu, D_1D_2) = n\left(\text{Quantization error due to the point } M\left(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\right)\right).$$

Again, notice that any point on the line segment $D_1D_2$ is given by $M(t)$ for $\frac{r_1}{\ell} \leq t \leq 1 - \frac{r_2}{\ell}$, and the parameters for the points at which the boundary of the Voronoi region of $M\left(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\right)$ cuts the segment $D_1D_2$ are given by $t = \frac{r_1}{\ell}$, and $t = \frac{r_1}{\ell} + \frac{1}{n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})$. Hence, we have

$$V_n(\mu, D_1D_2) = n \int_{\frac{r_1}{\ell}}^{\frac{r_1}{\ell} + \frac{1}{n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})} \rho\left(M(t), M\left(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\right)\right) d\mu.$$

Thus, the proof of the theorem is complete. 

Let the equation of the unit circle be $x_1^2 + x_2^2 = 1$. Let $A_1A_2A_3 \cdots A_m$ be a regular $m$-sided polygon for some $m \geq 3$ inscribed in the circle. Without any loss of generality due to rotational symmetry, we can always assume that the vertex $A_1$ lies on the $x_1$-axis, i.e., the vertex $A_1$ is the point where the circle intersects the positive direction of the $x_1$-axis. Again, notice that each side of the regular $m$-sided polygon subtends a central angle of radian $\frac{2\pi}{m}$. Thus, the position vectors $\tilde{a}_j$ of the vertices $A_j$ are given by $\tilde{a}_j = (\cos \frac{2\pi}{m}(j - 1), \sin \frac{2\pi}{m}(j - 1))$ for $1 \leq j \leq m$. Let $\ell$ be the length of each side of the polygon, then we have

$$\ell = ||\tilde{a}_m - \tilde{a}_{m-1}|| = ||\tilde{a}_{m-1} - \tilde{a}_{m-2}|| = \cdots = ||\tilde{a}_2 - \tilde{a}_1|| = 2\sin \frac{\pi}{m}.$$
Let \( L \) be the boundary of the polygon. Then, we can write
\[
L = \bigcup_{j=1}^{m} L_j,
\]
where \( L_j \) is the side \( A_jA_{j+1} \), and \( A_{m+1} \) is identified as the vertex \( A_1 \). Then, for \( 1 \leq j \leq m \), we can write
\[
L_j := A_jA_{j+1} = \{ M_j : 0 \leq t \leq 1 \}, \quad \text{where } M_j = \tilde{a}_{j+1}t + (1-t)\tilde{a}_j.
\]
Notice that \( M_j \) is a function of \( t \), and any point on the side \( A_jA_{j+1} \) can be represented by \( M_j := M_j(t) \) for \( 0 \leq t \leq 1 \). Thus, we see that \( M_j(0) = \tilde{a}_j \), and \( M_j(1) = \tilde{a}_{j+1} \) for \( 1 \leq j \leq m \). Let \( P \) be the uniform distribution defined on the boundary \( L \) of the polygon. Then, the probability density function (pdf) \( f \) of the uniform distribution \( P \) is given by \( f(x_1, x_2) = \frac{1}{m} \) for all \( (x_1, x_2) \in L \), and zero, otherwise. Let \( s \) represent the distance of any point on \( L \) from the vertex \( A_1 \) tracing along the boundary \( L \) in the counterclockwise direction. Then, we have \( dP = dP(s) = P(ds) = f(x_1, x_2)ds = \frac{1}{m}ds \). For \( 1 \leq j \leq m \), on each \( L_j \), we have
\[
ds = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2}dt = \ell|dt| \text{ yielding } dP(s) = \frac{\ell}{m}|dt| = \frac{1}{m}|dt|.
\]

Remark 2.2. Since \( P \) is uniform, and a regular \( m \)-sided polygon has symmetry of order \( m \), it is not difficult to show that an optimal set \( \alpha_m \) will contain \( m \) points, each from the interior of the \( m \) angles of the regular \( m \)-sided polygon; and for any positive integer \( k \geq 2 \), \( \alpha_{mk} \) will contain \( m \) points, each from the interior of the \( m \) angles, and \( (k-1) \) points from each side of the regular \( m \)-sided polygon. Moreover, the following is true: Let \( A \) be one of the vertices of the regular \( m \)-sided polygon, and for \( k \geq 2 \), let \( a \) be an element of an optimal set of \( mk \)-means that lies in the interior of \( \angle A \). Further, let \( AA_1 \) and \( AA_2 \) be the two adjacent sides of the vertex \( A \). Then, the boundary of the Voronoi region of \( a \) will cut \( AA_1 \) and \( AA_2 \) at two points \( D_1 \) and \( D_2 \) such that \( |AD_1| = |AD_2| = r \) for some real \( r \) such that \( 0 < r \leq \frac{\ell}{2} \), where \( \ell \) is the length of the sides of the polygon.

Proposition 2.3. Let \( \alpha_n \) be an optimal set of \( n \)-means such that \( n = mk \), where \( k \in \mathbb{N} \), and \( k \geq 2 \). Let \( a_j \) be the points that \( \alpha_n \) contains from the interior of the angles \( A_j \) of the regular \( m \)-sided polygon, \( 1 \leq j \leq m \). Then,
\[
\alpha_n = \{ a_j : 1 \leq j \leq m \} \cup \bigcup_{j=1}^{m} \alpha_{j,k-1},
\]
where
\[
a_1 = (1 - \frac{1}{2}r \sin(\frac{\pi}{m}), 0),
\]
\[
a_j = \left( \frac{1}{4} \cos \frac{2\pi(j-1)}{m}(r \cos(\frac{2\pi}{m}) - 1) \csc(\frac{\pi}{m}) + 4, \sin \frac{2\pi(j-1)}{m}(\frac{1}{4}r(\cos(\frac{2\pi}{m}) - 1) \csc(\frac{\pi}{m}) + 1) \right)
\]
for \( 2 \leq j \leq m \), and \( \alpha_{j,k-1} := \{ M_j(\frac{\pi}{2} + \frac{2\pi i - 1}{2(k-1)}(1 - \frac{2\pi}{m})) : 1 \leq i \leq k - 1 \} \) for \( 1 \leq j \leq m \), and
\[
r = \frac{4 \sin(\frac{\pi}{m})}{2(k-1)\sqrt{3 \cos^2(\frac{\pi}{m}) + 1} + 4}.
\]
Moreover, the quantization error for \( n \)-means is given by
\[
V_n = \frac{2 \sin^2(\frac{\pi}{m})(3 \cos(\frac{2\pi}{m}) + 5)}{3 \left(k\sqrt{6 \cos(\frac{2\pi}{m}) + 10} - \sqrt{6 \cos(\frac{2\pi}{m}) + 10 + 4} \right)^2}.
\]

Proof. Let \( \alpha_n \) be an optimal set of \( n \)-means, where \( n = mk \) for some positive integer \( k \geq 2 \). Since \( a_j \) are the points that \( \alpha_n \) contains from the interior of the angles \( A_j \), by Remark 2.2 due to uniform distribution and symmetry, we can say that there exists a real number \( r \), where \( 0 < r \leq \frac{\ell}{2} \), such that the boundary of the Voronoi region of each \( a_j \) will cut the the two adjacent
sides at distances $r$ from the vertex $A_j$. Notice that the two adjacent sides of the vertex $A_1$ are $A_m A_1$ and $A_1 A_2$ in the polygon. Again, by the hypothesis $A_1$ is the point that $\alpha_n$ contains from $\angle A_1$. If the boundary of the Voronoi region of $A_1$ cuts $A_m A_1$ and $A_1 A_2$ at $D_1$ and $D_2$, respectively, we have

$$a_1 = E(X : X \in D_1 A_1 \cup A_1 D_2) = \frac{\int_{D_1 A_1} (x_1, x_2) dP + \int_{A_1 D_2} (x_1, x_2) dP}{\int_{D_1 A_1} 1 dP + \int_{A_1 D_2} 1 dP},$$

which implies

$$a_1 = \frac{\int_0^1 M_n(t) dt + \int_0^\frac{\pi}{\ell} M_1(t) dt}{\int_0^1 1 dt + \int_0^1 \frac{\pi}{\ell} 1 dt} = (1 - \frac{1}{2} r \sin (\frac{\pi}{m}), 0).$$

Similarly, for $2 \leq j \leq m$, we obtain

$$a_j = \frac{\int_0^1 M_{j-1}(t) dt + \int_0^\frac{\pi}{\ell} M_j(t) dt}{\int_0^1 1 dt + \int_0^1 \frac{\pi}{\ell} 1 dt}$$

yielding

$$a_j = (\frac{1}{4} \cos \frac{2\pi(j - 1)}{m} (r (\cos \frac{2\pi}{m}) - 1) \csc (\frac{\pi}{m}) + 4), \sin \frac{2\pi(j - 1)}{m} (\frac{1}{4} r (\cos \frac{2\pi}{m}) - 1) \csc (\frac{\pi}{m}) + 1).$$

Again, by Remark 2.2, $\alpha_n$ contains $(k - 1)$ points from each side of the regular $m$-sided polygon. For $1 \leq j \leq m$, let $\alpha_{j,k-1}$ be the optimal set of $(k - 1)$-means that $\alpha_n$ contains from the side $A_j A_{j+1}$. Recall that the parametric representation of the side $A_j A_{j+1}$ is $M_j(t)$, and the $(k - 1)$ means from each side occur due to an uniform distribution on the segment bounded by the two points represented by the parameters $t = \frac{\pi}{\ell}$ and $t = 1 - \frac{\pi}{\ell}$. Hence, by Theorem 2.1, we have

$$\alpha_{j,k-1} := \left\{ M_j(\frac{r}{\ell} + 2i - 1) (1 - \frac{2r}{\ell}) : 1 \leq i \leq k-1 \right\}.$$

To calculate the quantization error, we proceed as follows: By symmetry, the quantization error contributed by all the points $a_j$ for $1 \leq j \leq m$ is given by

$$m \int_{D_1 A_1 \cup A_1 D_2} \rho((x_1, x_2), a_1) dP = 2m \int_{A_1 D_2} \rho((x_1, x_2), a_1) dP = 2 \int_0^\frac{\pi}{\ell} \rho(M_1(t), a_1) dt,$$

implying

$$m \int_{D_1 A_1 \cup A_1 D_2} \rho((x_1, x_2), a_1) dP = \frac{1}{24} r^3 (3 \cos \frac{2\pi}{m} + 5) \csc (\frac{\pi}{m}).$$

Again, by Theorem 2.1, the quantization error contributed by all the sets $\alpha_{j,k-1}$ for $1 \leq j \leq m$ is given by

$$m V_n(\mu, \alpha_{j,k-1}) := (k - 1) \int_0^\frac{\pi}{\ell} \rho\left(M(t), M(\frac{r}{\ell} + \frac{1}{2(k-1)} (1 - \frac{2r}{\ell}))\right) dt.$$

implying

$$m V_n(\mu, \alpha_{j,k-1}) = \frac{1}{3(k-1)^2} \csc (\frac{\pi}{m}) (\sin (\frac{\pi}{m}) - r)^3.$$

Hence, by (2) and (3), the quantization error for $n$-means is given by

$$V_n = \frac{1}{24} \csc (\frac{\pi}{m}) \left(r^3 \cos \frac{2\pi}{m} + 5 \right) + \frac{8}{(k-1)^2} (\sin (\frac{\pi}{m}) - r)^3.$$
Notice that for a given $k$, the quantization error $V_n$ is a function of $r$. Solving $\frac{\partial V_n}{\partial r} = 0$, we have $r = \frac{4\sin(\frac{\pi}{m})}{2(k-1)\sqrt{3\cos^2(\frac{\pi}{m})} + 1 + 4}$. Putting $r = \frac{4\sin(\frac{\pi}{m})}{2(k-1)\sqrt{3\cos^2(\frac{\pi}{m})} + 1 + 4}$, we have
\[
V_n = \frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m}) + 5)}{3\left(k\sqrt{6\cos(\frac{2\pi}{m})} + 10 - \sqrt{6\cos(\frac{2\pi}{m}) + 10 + 4}\right)}.
\]
Thus, the proof of the theorem is complete. \qed

Let us now prove the following theorem.

**Theorem 2.4.** Let $P$ be the uniform distribution on the boundary of a regular $m$-sided polygon inscribed in a unit circle. Then, the quantization coefficient for $P$ exists as a finite positive number which equals $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$, i.e., $\lim_{n \to \infty} n^2V_n = \frac{1}{3}m^2\sin^2(\frac{\pi}{m})$.

**Proof.** Let $n \in \mathbb{N}$ be such that $n \geq 2m$. Then, there exists a unique positive integer $\ell(n) \geq 2$ such that $m\ell(n) \leq n < m(\ell(n) + 1)$. Then,
\[
(\ell(n))V_{m(\ell(n)+1)} < n^2V_n < (m(\ell(n) + 1))^2V_{m\ell(n)}.
\]
We have
\[
\lim_{n \to \infty} (m\ell(n))^2V_{m(\ell(n)+1)} = \lim_{\ell(n)\to\infty} (m\ell(n))^2 \frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m}) + 5)}{3\left((\ell(n)+1)\sqrt{6\cos(\frac{2\pi}{m})} + 10 - \sqrt{6\cos(\frac{2\pi}{m}) + 10 + 4}\right)}^2 = \frac{1}{3}m^2\sin^2(\frac{\pi}{m}),
\]
and
\[
\lim_{n \to \infty} (m(\ell(n) + 1))^2V_{m\ell(n)} = \lim_{\ell(n)\to\infty} (m(\ell(n) + 1))^2 \frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m}) + 5)}{3\left(\ell(n)\sqrt{6\cos(\frac{2\pi}{m})} + 10 - \sqrt{6\cos(\frac{2\pi}{m}) + 10 + 4}\right)}^2 = \frac{1}{3}m^2\sin^2(\frac{\pi}{m}),
\]
and hence, by (II) we have $\lim_{n \to \infty} n^2V_n = \frac{1}{3}m^2\sin^2(\frac{\pi}{m})$, i.e., the quantization coefficient exists as a finite positive number which equals $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$. Thus, the proof of the theorem is complete. \qed

**Remark 2.5.** Since $\lim_{m \to \infty} 2\sin(\frac{\pi}{m}) = 0$, by (II), we can conclude that when $m$ tends to $\infty$, then the length of each side of the regular $m$-sided polygon becomes zero, i.e., the regular $m$-sided polygon coincides with the circle. Moreover, for $m \geq 3$, we have
\[
\frac{d}{dm}(\frac{1}{3}m^2\sin^2(\frac{\pi}{m})) = \frac{2}{3}\sin(\frac{\pi}{m})(m\sin(\frac{\pi}{m}) - \pi\cos(\frac{\pi}{m})) > 0
\]
yielding the fact that the quantization coefficient $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$ for the uniform distribution on the boundary of the regular $m$-sided polygon is an increasing function of $m$. Again,
\[
\lim_{m \to \infty} \frac{1}{3}m^2\sin^2(\frac{\pi}{m}) = \frac{\pi^2}{3},
\]
i.e., when $m$ tends to infinity, then the quantization coefficient of the regular $m$-sided polygon equals $\frac{\pi^2}{3}$. Recall that $\frac{\pi^2}{3}$ is the quantization coefficient for the uniform distribution on the unit circle. Thus, the result in this paper, proves the conjecture given by Pena et al. in the paper [PRRSS].

**Remark 2.6.** If $m = 6$, we see that $\lim_{n \to \infty} n^2V_n = 3$, which is the quantization coefficient for the uniform distribution on the boundary of a hexagon inscribed in a unit circle. Thus, the result in this paper, also generalizes a result given by Pena et al. in the paper [PRRSS].
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