A new Map between Quantum Gauge Theories defined on a Quantum hyperplane and ordinary Gauge Theories: $q$-deformed QED\(^*\)

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Abstract

We introduce a new map between a $q$-deformed gauge theory defined on a general $GL_q(N)$-covariant quantum hyperplane and an ordinary gauge theory in a full analogy with Seiberg-Witten map. Perturbative analysis of the $q$-deformed QED at the classical level is presented and gauge fixing à la BRST is discussed.

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1 Introduction

Motivated by the need to control the divergences in quantum electrodynamics, Snyder [1] proposed that one may use a noncommutative structure for space-time coordinates. Although its great success, this suggestion has been swiftly forsaken. This is partly due to a growing development in the renormalization program which captivates all the attention of the leading physicists. The renormalization prescription solves the quantum inconsistencies without making any ad hoc assumptions on the space-time structure. Thanks to the seminal paper of Connes [2] the interest in noncommutativity [3] has been revived. Natural candidates for noncommutativity are provided by quantum groups [4]. A special role is played by the quantum Yang-Baxter equations which express the hidden symmetry of integrable systems. Nowadays many applications have appeared. One can mention conformal field theories [5, 6], as well as in the vertex and spin models [7, 8], in quantum optics [9] and quantum gauge theories [11, 12, 13].

In a recent paper [14] we have constructed a new map which relates a $q$-deformed gauge field defined on the Manin plane $\hat{x}\hat{y} = q\hat{y}\hat{x}$ and the ordinary gauge field. This map is the analogue of the Seiberg-Witten map [15]. We have found this map using the Gerstenhaber product [16] instead of the Groenewold-Moyal star product [17]. In the present letter we extend our analysis to the general $GL_q(N)$-covariant quantum hyperplane [18] generated by the coordinates $\hat{x}^1,\ldots,\hat{x}^N$ and defined by $\hat{x}^i\hat{x}^j = R_{kl}^{ij}\hat{x}^k\hat{x}^l$, where $R$ is the braiding matrix.

This letter is organized as follows. In Sec. 2, we construct a new map relating $q$-deformed and ordinary gauge fields. In Sec. 3, we present the perturbative $q$-deformed QED at the classical level and we introduce the $q$-deformed BRST and anti-BRST transformations. This study prepare the quantization of the model at hand.

2 $q$-deformed gauge symmetry versus ordinary gauge symmetry

The undeformed QED action is given by

$$S = \int d^4x \left[ \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where

$$D_\mu \psi = (\partial_\mu - iA_\mu) \psi,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2)$$
$S$ is invariant with respect to infinitesimal gauge transformations:

\[
\begin{align*}
\delta_\alpha A_\mu &= \partial_\mu \alpha, \\
\delta_\alpha \psi &= i\alpha \psi, \\
\delta_\alpha \bar{\psi} &= -i\bar{\psi}\alpha.
\end{align*}
\]

(3)

Let us now consider QED defined on a $GL_q(4)$-covariant quantum hyperplane $\tilde{x}^i\tilde{x}^j = q^{ij}\tilde{x}^i$, $i < j$, $q \in C$.

This relation is governed by the braiding $R$ matrix which is explicitly given as:

\[
R^{ij}_{kl} = \delta^i_l \delta^j_k \left((1 - q^{-1}) \delta^{ij} + q^{-1}\right) + (1 - q^{-2}) \delta^i_k \delta^j_l \Theta_{ji}.
\]

(4)

In the deformed case one replaces the ordinary product by the Gerstenhaber star product [16] defined by

\[
f \star g = \mu \circ e^{i\eta x^i \frac{\partial}{\partial x^i} \otimes x^j \frac{\partial}{\partial x^j}} (f \otimes g),
\]

(5)

where the undeformed product $\mu$ is given by

\[
\mu (f \otimes g) = fg.
\]

(6)

A straightforward computation gives then the following commutation relations

\[
x^i \star x^j = \sum_{r=0}^\infty \frac{(i\eta)^r}{r!} x^i x^j = e^{i\eta x^i x^j}, \quad x^j \star x^i = x^j x^i, \quad i < j
\]

(7)

whence

\[
x^i \star x^j = qx^j \star x^i, \quad q = e^{i\eta}.
\]

(8)

Thus we recover the commutation relations for the quantum hyperplane. Let us illustrate by two examples:
The case \( n = 2 \) (the Manin plane):

\[
f \star g = \mu \circ e^{i\eta x \frac{\partial}{\partial x} \otimes y \frac{\partial}{\partial y}} (f \otimes g).
\]  

Using this product it is not difficult to find the usual Manin plane commutation relations: \( x \star y = qy \star x \).

The case \( n = 3 \), we have

\[
f \star g = \mu \circ e^{i\eta (x \frac{\partial}{\partial x} \otimes y \frac{\partial}{\partial y} + x \frac{\partial}{\partial x} \otimes z \frac{\partial}{\partial z} + y \frac{\partial}{\partial y} \otimes z \frac{\partial}{\partial z})} (f \otimes g).
\]  

We can easily prove the commutation relations: \( x \star y = qy \star x, \ x \star z = qz \star x \) and \( y \star z = qz \star x \).

If the spacetime dimension of the quantum hyperplane is \( n \) we have \( \frac{n(n-1)}{2} \) terms present in the tensor product.

Let us take \( \theta^{ij}(x) = \eta x^i x^j \) with \( i < j \). Using the expansion of (5) in \( \eta \) we find

\[
f \star g = fg + i\theta^{ij}(x) \frac{\partial}{\partial x^i} f \frac{\partial}{\partial x^j} g + o(\eta^2), \quad i < j.
\]  

The \( q \)-deformed infinitesimal gauge transformations are defined by

\[
\delta_\alpha \widehat{A}_\mu = \frac{\partial}{\partial x^\mu} \widehat{A}_\mu + \frac{\partial}{\partial x^\mu} \widehat{A}_\mu - i \widehat{A}_\mu \alpha, \\
\delta_\alpha \widehat{\psi} = i \widehat{\psi} \alpha, \\
\delta_\alpha \widehat{\psi} = -i \widehat{\psi} \alpha, \\
\delta_\alpha \widehat{F}_{\mu\nu} = i \widehat{F}_{\mu\nu} - i \widehat{F}_{\mu\nu} \alpha.
\]  

To first order in \( \eta \), the above formulas for the gauge transformations read

\[
\delta_\alpha \widehat{A}_\mu = \partial_\mu \widehat{A}_\mu - \theta^{\rho\sigma} (x) (\partial_\rho \alpha \partial_\sigma A_\mu - \partial_\rho A_\mu \partial_\sigma \alpha) + o(\eta^2), \\
\delta_\alpha \widehat{\psi} = i \widehat{\psi} \alpha - \theta^{\rho\sigma} (x) \partial_\rho \alpha \partial_\sigma \psi + o(\eta^2), \\
\delta_\alpha \widehat{\psi} = -i \widehat{\psi} \alpha + \theta^{\rho} (x) \partial_\rho \psi \partial_\sigma \alpha + o(\eta^2), \\
\delta_\alpha \widehat{F}_{\mu\nu} = -\theta^{\rho\sigma}(x) (\partial_\rho \alpha \partial_\sigma F_{\mu\nu} - \partial_\rho F_{\mu\nu} \partial_\sigma \alpha) + o(\eta^2).
\]
The solutions are given by

\[ \hat{A}_\mu = A_\mu + \theta^{\rho\sigma}(x) (A_\sigma F_{\rho\mu} - A_\rho A_\sigma A_\mu) - \partial_\mu \theta^{\rho\sigma}(x) A_\sigma A_\rho + o(\eta^2), \]

\[ \hat{\psi} = \psi - \theta^{\rho\sigma}(x) A_\rho \partial_\sigma \psi + o(\eta^2), \]

\[ \hat{\alpha} = \alpha - \theta^{\rho\sigma}(x) A_\rho \partial_\sigma \alpha + o(\eta^2). \] (14)

An additional term proportional to \( \partial_\mu \theta^{\rho\sigma}(x) \) appears. In the case where \( \theta^{\rho\sigma} \) is real and antisymmetric we find exactly the Seiberg Witten map.

The \( q \)-deformed curvature \( \hat{F}_{\mu\nu} \) is given by

\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu] \]

\[ = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \hat{A}_\mu \ast \hat{A}_\nu + i \hat{A}_\nu \ast \hat{A}_\mu. \] (15)

Using (11) and (14) we find

\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma}(x) (A_\rho \partial_\sigma F_{\mu\nu} - A_\rho A_\sigma F_{\mu\nu} + F_{\rho\mu} F_{\sigma\nu} + F_{\rho\sigma} F_{\mu\nu}) + \partial_\mu \theta^{\rho\sigma}(x) (A_\rho F_{\mu\nu} - A_\rho A_\sigma A_\nu) - \partial_\nu \theta^{\rho\sigma}(x) (A_\rho F_{\rho\mu} - A_\rho A_\sigma A_\mu) + o(\eta^2), \] (16)

which we can write as

\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} + o(\eta^2), \] (17)

where \( f_{\mu\nu} \) is the quantum correction linear in \( \eta \). The quantum analogue of the action (1) is given by

\[ \hat{S} = \int d^4x \left[ \hat{\psi} \ast (i\gamma^\mu \hat{D}_\mu - m) \hat{\psi} - \frac{1}{4} \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu} \right]. \] (18)

Even for functions \( f, g \) that vanish rapidly enough at infinity, we have

\[ \int Tr \hat{f} \ast \hat{g} \neq \int Tr \hat{f} \hat{g}. \] (19)

This situation is in contrast with noncommutative geometry where the equality holds.
3 Perturbative $q$-deformed QED and gauge fixing à la BRST

The $q$-deformed action (18)

$$\hat{S} = S + \hat{S}_q$$

where $S$ is the undeformed action (1) and $\hat{S}_q$ is the correction linear in $\eta$ and given by

$$\hat{S}_q = \int d^4x \, \theta^{\rho\sigma} (x) \left[ \bar{\psi} \gamma^\mu (A_\sigma F_{\rho\mu} - A_\rho \partial_\sigma A_\mu) \psi 
- \bar{\psi} \gamma^\mu A_\mu \partial_\sigma \psi - \partial_\rho \bar{\psi} \gamma^\mu \partial_\sigma \psi - A_\rho \partial_\sigma \bar{\psi} \gamma^\mu A_\mu \psi 
- m A_\rho \partial_\sigma \bar{\psi} \psi - m \bar{\psi} A_\rho \partial_\sigma \psi - \frac{1}{4} (A_\sigma \partial_\rho F_{\mu\nu} F^{\mu\nu} - A_\rho \partial_\sigma F_{\mu\nu} F^{\mu\nu} + F_{\sigma \rho} F_{\mu\nu} F^{\mu\nu} + F_{\mu \sigma} F_{\rho \nu} F^{\mu\nu}) 
- i (\bar{\psi} \gamma^\mu \partial_\mu (A_\rho \partial_\sigma \psi) + A_\rho \partial_\sigma \bar{\psi} \gamma^\mu \partial_\mu \psi 
- \bar{\psi} \gamma^\mu \partial_\mu A_\rho \partial_\sigma \psi - \partial_\rho \bar{\psi} \partial_\sigma (A_\mu \psi) 
- m \partial_\rho \bar{\psi} \partial_\sigma \psi - \frac{1}{4} \partial_\rho F_{\mu\nu} F^{\mu\nu} \right] 
- \partial_\rho \theta^{\rho\sigma} (x) \left[ \frac{1}{2} (A_\sigma F_{\rho\nu} F^{\mu\nu} - A_\rho \partial_\sigma A_\mu F^{\mu\nu}) 
- i \gamma^\mu A_\rho \partial_\sigma \bar{\psi} \psi \right] 
+ \frac{1}{2} \partial_\nu \theta^{\rho\sigma} (x) (A_\sigma F_{\rho\mu} F^{\mu\nu} - A_\rho \partial_\sigma A_\mu F^{\mu\nu}) + o (\eta^2).$$

A gauge fixing term is needed in order to quantize the system. This is done in the BRST and anti-BRST formalism. The quantum BRST transformations are given by [19]:

$$\hat{s} A_\mu = \partial_\rho \hat{c} - \theta^{\rho\sigma} (x) (\partial_\rho c \partial_\sigma A_\mu - \partial_\rho A_\mu \partial_\sigma c) + o (\eta^2),$$
$$\hat{s} \bar{\psi} = i \hat{c} \bar{\psi} - \theta^{\rho\sigma} (x) \partial_\rho c \partial_\sigma \psi + o (\eta^2),$$
$$\hat{s} \psi = -i \bar{\psi} \hat{c} + \theta^\rho (x) \partial_\rho \bar{\psi} c + c o (\eta^2),$$
$$\hat{s} \bar{F}_{\mu\nu} = -\theta^{\rho\sigma} (x) (\partial_\rho c \partial_\sigma F_{\mu\nu} - \partial_\rho F_{\mu\nu} \partial_\sigma c) + c o (\eta^2),$$
$$\hat{s} \hat{c} = b, \quad \hat{s} c = 0, \quad \hat{s} b = 0.$$
where \( \hat{c}, \hat{\bar{c}} \) are the quantum Faddeev-Popov ghost and anti-ghost fields, \( \hat{b} \) a scalar field (sometimes called the Nielsen-Lautrup auxiliary field) and \( \hat{s} \) the quantum BRST operator. The gauge-fixing term action is introduced as

\[
\hat{S}_{gf} = \int d^4x \left( \hat{\bar{c}} \star \left( \frac{\alpha}{2} \hat{b} - \partial_\mu \hat{A}^\mu \right) \right) (x). \tag{23}
\]

An expansion in \( \eta \) leads to

\[
\hat{S}_{gf} = \int d^4x \left( \frac{\alpha}{2} b^2 + \tau \partial^2 c - b \partial^\mu A_\mu \\
- \theta^{\rho\sigma} (x) [b \partial^\mu (A_\sigma F_{\rho\mu} - A_\rho \partial_\sigma A_\mu) \\
+ A_\rho \partial_\sigma (\tau c) + \tau \partial_\rho c \partial_\sigma A_\mu - \tau \partial_\rho A_\mu \partial_\sigma c \\
+ i (\partial_\rho b \partial_\sigma \partial^\mu A_\mu - \partial_\rho \tau \partial_\sigma \partial^2 c) ] \\
- \partial^\mu \theta^{\rho\sigma} (x) b (A_\sigma F_{\rho\mu} - A_\rho \partial_\sigma A_\mu) \right). \tag{24}
\]

This action corresponds to a highly nonlinear gauge.

The external field contribution is given by

\[
\hat{S}_{ext} = \int d^4x \left( \hat{A}^\mu \star \hat{s} \hat{A}_\mu + \hat{\bar{c}}^\mu \star \hat{s} \hat{c} \right) (x), \tag{25}
\]

where \( \hat{A}^* \), \( \hat{\bar{c}}^* \) are external fields (called antifields in the Batalin-Vilkovisky formalism) and play the role of sources for the BRST-variation of the fields \( \hat{A}, \hat{\bar{c}} \).

The \( \hat{c} \) and \( \hat{\bar{c}} \) play quite asymmetric roles, they cannot be related by Hermitian conjugation. The anti-BRST transformations \([20, 21, 22]\) are given by

\[
\begin{align*}
\hat{\bar{c}} \hat{A}_\mu &= \partial_\mu \hat{\bar{c}} - \theta^{\rho\sigma} (x) (\partial_\rho \tau \partial_\sigma A_\mu - \partial_\rho A_\mu \partial_\sigma \tau) + o (\eta^2), \\
\hat{\bar{c}} \hat{\psi} &= \hat{\bar{c}} \tau \hat{\psi} - \theta^{\rho\sigma} (x) \partial_\rho \tau \partial_\sigma \hat{\psi} + o (\eta^2), \\
\hat{\bar{c}} \hat{\psi} &= - i \sqrt{\tau} \hat{\bar{c}} \hat{\psi} \hat{\bar{c}} \tau + \theta^\rho (x) \partial_\rho \tau \partial_\sigma \hat{\psi} + o (\eta^2), \\
\hat{\bar{c}} \hat{F}_{\mu\nu} &= - \theta^{\rho\sigma} (x) (\partial_\rho \tau \partial_\sigma F_{\mu\nu} - \partial_\rho F_{\mu\nu} \partial_\sigma \tau) + o (\eta^2), \\
\hat{\bar{c}} \hat{c} &= 0, \quad \hat{s} \hat{c} = - b, \quad \hat{\bar{b}} = 0. \tag{26}
\end{align*}
\]

Here \( \hat{\bar{s}} \) is the quantum anti-BRST operator. The complete tree-level action is given by:

\[
\Sigma \left( \hat{A}_\mu, \hat{c}, \hat{\bar{c}}, \hat{b}, \hat{A}^*_\mu, \hat{\bar{c}}^* \right) = \hat{S} + \hat{S}_{gf} + \hat{S}_{ext}. \tag{27}
\]
4 Concluding Remarks

We have defined a $q$-deformed QED at the classical level. Like in noncommu-
tative geometry [23] we have found that the $q$-deformed QED action contains
non-renormalizable vertices of dimension six. It is worthwhile to study the
quantization of the $\eta$-expanded noncommutative $U(1)$ Yang-Mills action and
$q$-deformed BF Yang-Mills theory [12]. We postpone these investigations to a
future work.

Finally, we can claim, in bona fides, that the method developed in this letter
can be applied to various quantum $q$-deformed models as well as to $h$-deformed
model (the so-called Jordanian models) [24, 25].

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