Least Squares Finite Element Methods for Sea Ice Dynamics

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Abstract
A first-order system least squares formulation for the sea-ice dynamics is presented. In addition to the displacement field, the stress tensor is used as a variable. As finite element spaces, standard conforming piecewise polynomials for the displacement approximation are combined with Raviart-Thomas elements for the rows in the stress tensor. Computational results for a test problem illustrate the least-squares approach.

1 Introduction

Ice and snow covered surfaces reflect more than half of the solar radiation they are receiving and play therefore a major role in climate modelling. Each year, Antarctic sea ice extent reaches its maximum (17-20 million square kilometers) in September and its minimum (3-4 million square kilometers) in February. These important oscillations make the current predictive models of Antarctic sea ice require an accurate knowledge and understanding of the processes. Developing computational sea-ice modelling based on observed and measured data to study and predict the break-up and fracture evolution of sea-ice during the Antarctic spring was one of main scientific aims of the Winter 2017 cruise (Voyage 25) of the S.A. Agulhas II. This was funded by DST/NRF and took place from 28 June to 13 July 2017.

Sea ice is a complex material which is formed by the freezing of sea water. Since the ice stress is a source in the other equations of the climate models, its approximation plays an important role in the simulations of the ice. They can be computed from the velocity in a post-processing step, but the loss of accuracy due to the reconstruction step can lead to non-physical solutions. An alternative approach consists in the use of variational formulations involving the stress $\sigma \in H(\text{div}, \Omega)$ as an independent variable. Appropriate finite element spaces based on a triangulation $T$ are the $H(\text{div}, \Omega)$-conforming spaces, e.g. the Raviart-Thomas Space.

2 Problem Formulation

As most sea ice dynamic models currently used, our model is based on the viscous-plastic formulation introduced by Hibler [5]. There, sea ice is modeled by its velocity $\mathbf{u}$, the ice concentration $A$ and the average ice height $H$ over a domain $\Omega$. The model consists in a momentum equation for the velocity $\mathbf{u}$ and the balance laws for ice concentration $A$ and the average ice height $H$. Neglecting the thermodynamical effects,
i.e. the source terms in these balance laws, the model can be written as

$$\rho_{ice} H \frac{\partial u}{\partial t} + F(u) - \text{div} \sigma(u, A, H) = 0,$$

where the force term involving the ice, air and water densities $\rho_{ice}, \rho_a$ and $\rho_w$, the air and water drag coefficients $C_a$ and $C_o$, the coriolis parameter $f_c$, the radial unit vector $e_r$ and the velocity fields $v_o$ and $v_a$ of ocean and atmospheric flow is given by

$$F(v) = f_c e_r \times (v - v_o) - \rho_a C_a \|v_a\|^2 v_a - \rho_o C_o \|v_o - v\|^2 (v_o - v)$$

and the stress-strain relation involving the ice strength parameter $P^*$ and the ice concentration parameter $C$ is given by

$$\sigma = P \frac{\left( \text{dev} \epsilon(u) + 2 \text{tr} \epsilon(u) I \right)}{\Delta(u)} - I$$

with $P = P^* H e^{-C(1-A) - 1}$

$$\Delta(u) = \sqrt{\text{dev} \epsilon(u) : \text{dev} \epsilon(u) + 4 \text{tr} (\epsilon(u))^2 + \Delta_{min}^2}$$

where $\Delta_{min} = 2 \cdot 10^{-9} \text{ s}^{-1}$ is a limitation for $\Delta(u)$. In [7], the authors propose a variational formulation where $(u, p)$ with $p = (A, H)$ is sought in $(H^1(\Omega))^2 \times (L^2(\Omega))^2$ such that

$$\left( \rho_{ice} H \frac{\partial u}{\partial t}, v \right) + (F(u), v) + (\sigma(u, H, A), \nabla v) = 0,$$

$$\left( \frac{\partial p}{\partial t} + \nabla p \cdot u + \text{div}(u)p, q \right) = 0$$

holds for all $(v, q) \in (H^1(\Omega))^2 \times (L^2(\Omega))^2$. The constraints $H \geq 0$ and $A \in [0, 1]$ are embedded in the trial-spaces and are realized by a projection of the solution.

### 3 A Least-Squares Method

The Least-Squares Method (see [2]) consists in minimizing the $L^2$-residuals in the partial differential equations. Therefore, we insert define a new variable $\sigma$ for the
stress and consider the stress-strain relationship \[3\] as an additional equation in order to obtain the following first order system for \((\sigma, u, A, H)\):

\[
\rho_{\text{ice}} \frac{\partial u}{\partial t} + F(u) - \text{div} \sigma = 0 \\
P(A, H) \left( \frac{\text{dev} \varepsilon(u)}{\Delta(u)} + \frac{2\text{tr} \varepsilon(u)}{\Delta(u)} I - I \right) = \sigma \\
\frac{\partial A}{\partial t} + \text{div}(vA) = 0 \\
\frac{\partial H}{\partial t} + \text{div}(vH) = 0
\]

The least-squares functionals then reads

\[
\mathcal{F}(\sigma, u, H) = \mathcal{F}_m(\sigma, u, A, H) + \mathcal{F}_c(\sigma, u, A, H) + \mathcal{F}_e(\sigma, u, A, H) \quad \text{(5)}
\]

with

\[
\mathcal{F}_m(\sigma, u, A, H) = \left\| \rho_{\text{ice}} \frac{\partial u}{\partial t} + F(u) - \text{div} \sigma \right\|_0^2,
\]

\[
\mathcal{F}_c(\sigma, u, A, H) = \left\| \frac{\partial H}{\partial t} + \text{div}(uH) \right\|_0^2 + \left\| \frac{\partial A}{\partial t} + \text{div}(uA) \right\|_0^2,
\]

\[
\mathcal{F}_e(\sigma, u, A, H) = \left\| \sigma - \frac{P(A, H)}{2} \left( \frac{\text{dev} \varepsilon(u)}{\Delta(u)} + \frac{\text{tr} \varepsilon(u)}{\Delta(u)} I - I \right) \right\|_0^2.
\]

The time discretization can be realised using a \(\theta\)-scheme and decoupling the advection equations from the rest of the system such that for each time step \(n+1\), the linear functional

\[
\mathcal{G}^{n+1}(A^{n+1}, H^{n+1}; u^n, H^n, A^n) = \left\| \frac{H^{n+1} - H^n}{t} + \text{div}(u^n H^{n+1}) \right\|_0^2 \\
+ \left\| \frac{A^{n+1} - A^n}{t} + \text{div}(u^n A^{n+1}) \right\|_0^2 \quad \text{(6)}
\]

is first minimized over all \((A^{n+1}, H^{n+1}) \in (L^2(\Omega))^2\), and then the functional

\[
\mathcal{F}^{n+1}(\sigma^{n+1}, u^{n+1}; \sigma^n, u^n, A^{n+1}, H^{n+1}) \\
= \left\| \rho_{\text{ice}} \frac{H^{n+1}}{t} \frac{u^{n+1} - u^n}{t} + F(u^{n+1}) - \text{div} \sigma^{n+1} \right\|_0^2 \\
+ \mathcal{F}_c(\sigma^{n+1}, u^{n+1}, A^{n+1}, H^{n+1}) \quad \text{(7)}
\]

with the time discretized variables

\[
u^{n+1} = \theta u^n + (1 - \theta) u^n
\]

and

\[
\sigma^{n+\theta} = \theta \sigma^{n+1} + (1 - \theta) \sigma^n,
\]

is minimized over all \((\sigma^{n+1}, u^{n+1}) \in (H_{\text{div}}(\Omega))^2 \times (H_{1,0}^1(\Omega))^2\). For the spatial discretization, a conforming subspace \(W_h\) of \((H_{\text{div}}(\Omega))^2 \times (H_{1,0}^1(\Omega))^2 \times (L^2(\Omega))^2\). Therefore, a triangulation \(T_h\) of the domain \(\Omega\) is considered. In this work, we choose \(W_h = (RT_2^2(T_h) \times P_2^2(T_h)) \times P_1^2(T_h))\) in order to have appropriate convergence properties.
For the minimization of the nonlinear Functional $F^{n+1}$ in each time step, the Least-Squares Functional is linearized around a given approximation $(\sigma^k, u^k, A^k, H^k)$ and the minimization is then carried out iteratively solving a sequence of linearized least squares problems. Additionally, the Least-Squares Functional is minimized with respect to the linear inequality constraints $A \in [0,1]$ and $H \geq 0$, that leads to a constraint optimization problem that we solved with an active set strategy. Since the variables $A$ and $H$ are now decoupled from $u$ and $\sigma$, we can define the stress-strain relationship by

$$C(u; A, H) := \sigma(u; A, H) = \frac{P(A, H)}{2} \left( \frac{\text{dev } \epsilon(u) + 2\text{tr } \epsilon(u)}{\Delta(u)} \right) - 1 \right) \quad (8)$$

The Gateaux derivative of $C(u; A, H)$ in direction $v$ is denoted by $C(u; A, H)[v]$ and given by

$$J_C(u; A, H)[v] = \frac{P(A, H)}{2} \left( \frac{\text{dev } \epsilon(u) + 2\text{tr } \epsilon(v)}{\Delta(u)} + J_{\Delta^{-1}}(u)[v] \langle \text{dev } \epsilon(u) + 2\text{tr } \epsilon(u) \rangle \right) \quad (9)$$

with $J_{\Delta^{-1}}(u)[v] = -\Delta(u)^{-3} \langle \text{dev } \epsilon(u) : \text{dev } \epsilon(v) + 4\text{tr } \epsilon(u) \rangle \epsilon(\epsilon(v))$.

The first variation of the minimization of $F^{n+1}$ is then given by

$$B(\sigma^{n+1}, u^{n+1}, \tau, v; \sigma^n, u^n, A^n, H^{n+1}) = \left. \frac{\partial F^{n+1}(\sigma^{n+1} + \tau \tau, u^{n+1} + \tau v; \sigma^n, u^n, A^n, H^n)}{\partial \tau} \right|_{\tau=0}$$

$$= \frac{\partial}{\partial \tau} \left( \sigma^{n+1} + \tau \tau - C(u^{n+1} + \tau v; A^{n+1}, H^{n+1}), \sigma^{n+1} + \tau \tau - C(u^{n+1} + \tau v; A^{n+1}, H^{n+1}) \right)_{\tau=0}$$

$$+ 2 \left( \rho_{\text{ice}} H^{n+1} u^{n+1} - u^n \right) \frac{\partial}{\partial \tau} \left( \frac{\text{dev } \epsilon(u^{n+1} + \tau v; \sigma^n, u^n, A^n, H^n)}{\Delta(u^n)} \right)_{\tau=0}$$

$$+ 2 \left. \left( \rho_{\text{ice}} H^{n+1} u^{n+1} + \tau v - u^n \right) \frac{\partial}{\partial \tau} \left( \text{dev } \epsilon(u^{n+1} + \tau v; \sigma^n, u^n, A^n, H^n) \right)_{\tau=0}$$

$$+ 2 \left( \rho_{\text{ice}} H^{n+1} u^{n+1} + \tau v - u^n \right) \frac{\partial}{\partial \tau} \left( \text{dev } \epsilon(u^{n+1} + \tau v; \sigma^n, u^n, A^n, H^n) \right)_{\tau=0}$$

$$= 2 \left( \sigma^{n+1} - C(u^{n+1}; A^{n+1}, H^{n+1}) \right) + 2 \rho_{\text{ice}} H^{n+1} u^{n+1} + \tau v - u^n \frac{\partial}{\partial \tau} \left( \text{dev } \epsilon(u^{n+1} + \tau v; \sigma^n, u^n, A^n, H^n) \right)_{\tau=0}$$

Setting this first variation to zero leads to a necessary condition such that the Gauss-Newton Method in each time step consists in setting iteratively $(\sigma^{n+1,k+1}, u^{n+1,k+1}) = (\sigma^{n+1,k}, u^{n+1,k}) + (\delta \sigma, \delta u)$ where $(\delta \sigma, \delta u) \in (RT_0^3(\mathcal{T}_h) \times P_1^2(\mathcal{T}_h))$ is the solution of

$$B(\sigma^{n+1,k}, u^{n+1,k}; \tau, v; \sigma^n, u^n, A^n, H^n + 1) = \left( \sigma^{n+1,k} - J_C(u^{n+1,k}; A^{n+1}, H^{n+1}) \delta u, \tau - J_C(u^{n+1,k}; A^{n+1}, H^{n+1})[v] \right)$$

$$+ 2 \rho_{\text{ice}} H^{n+1} \frac{\partial u}{\partial \tau} \left( \text{dev } \epsilon(u^{n+1,k} + (1 - \theta) u^n; \sigma^{n+1,k}, u^{n+1,k}, A^n, H^n) \right) - \theta \text{div } \delta \sigma, \rho_{\text{ice}} H^{n+1} \frac{\partial v}{\partial \tau} \left( \text{dev } \epsilon(u^{n+1,k} + (1 - \theta) u^n; \sigma^{n+1,k}, u^{n+1,k}, A^n, H^n) \right) - \theta \text{div } \tau$$

for all $(\tau, \sigma) \in (RT_0^3(\mathcal{T}_h) \times P_1^2(\mathcal{T}_h))$.

### 4 Test Case

In order to investigate the approximation properties of the Least-Squares method, we consider the same test case as in [7], involving a quadratic domain (see also [8]), and
maximal ocean velocity $v_{mo}$ 0.01 ms$^{-1}$
maximal ocean velocity $v_{mo}$ 15 ms$^{-1}$
sea ice density $\rho_{ice}$ 900 kg m$^{-3}$
air density $\rho_a$ 1.3 kg m$^{-3}$
water density $\rho_o$ 1026 kg m$^{-3}$
air drag coefficient $C_a$ 1.2 · 10$^{-3}$
water $C_o$ 5.5 · 10$^{-3}$
coriolis parameter $f_c$ 1.46 · 10$^{-4}$ s$^{-1}$

Figure 2: Parameter used in the simulation

simulating the sea ice dynamics for $T = 8$ days. Since the Least-Squares Method approximates all the residuals of the partial differential equation simultaneously, we scale the domain to the unit square $\Omega = [0,1]^2$. Since the wind field is a cyclone from the midpoint of the computational domain to the edge followed by an anticyclone diagonally passing from the edge to the midpoint, we define the time $t_m = t - 4$ measured in days with respect to the time when the wind forcing alternates from cyclonic to anticyclonic. Further, let $\mathbf{x}(t) = \mathbf{x} - \mathbf{x}^m(t)$ denote the position with respect to the center of the cyclone $\mathbf{x}^m(t) = \mathbf{x}^m(t)(e_1 + e_2)$ with $x^m(t) = 0.1(9 - |t_m|)$. Then, the prescribed wind field is given by

$$v_a = 10v_{ma} \left( 1 - \frac{2}{e^{t_m/\epsilon}e^{-|t_m|} + 1} \right) e^{-\frac{18|t_m|}{40}} R \left( \frac{17}{40} \pi + \frac{t_m}{40|t_m|} \pi \right) \bar{x},$$

(10)

with $R(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}$, (11)

and a maximal wind velocity $v_{ma}^m$, while the circular steady ocean current is

$$v_o = v_{mo} \left( \frac{2y - 1}{1 - 2x} \right),$$

(12)

with a maximal ocean velocity $v_{mo}^m$. Finally, the initial conditions are given by zero velocity, constant ice concentration $A = 1$ and $H^0(x,y) = 0.3 + 0.005(\sin(250x) + \sin(250y))$. All simulations are executed with Fenics, using the inherent Newton solver. The velocity results at $t = 2, 4, 6, 8$ days are shown in the figure. Further investigations are needed, in particular regarding the ellipticity of the Least-Squares Functional, the possibility of considering domain with curved boundaries (see [1]) and the relation to others standard or mixed methods (as in [4]).

References

[1] F. Bertrand, S. Münzenmaier, and G. Starke First-order System Least Squares on Curved Boundaries: Higher-order Raviart–Thomas Elements. SIAM J. Numer. Anal. (2014) 52, 3165-3180.

[2] P. Bochev and M. Gunzburger, Least-Squares Finite Element Methods, Springer, New York, 2009.
Figure 3: Sea-ice velocity at $t = 2, 4, 6, 8$.

[3] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer, Heidelberg, 2013.

[4] J. Brandts, Y. Chen and J. Yang, *A note on least-squares mixed finite elements in relation to standard and mixed finite elements*. IMA J. Numer. Anal. (2006) 26: 779-789.

[5] W.D. Hibler, *A dynamic thermodynamic sea ice model*. J. Phys. Oceanogr (1979) 566 9(4):815-846.

[6] E.C. Hunke, *Viscous-plastic sea ice dynamics with the EVP model: linearization issues*. J. Comp. Phys. (2001) 170:18-38.

[7] C. Mehlmann und T. Richter, *A modified global Newton solver for viscous-plastic sea ice models*, Ocean Modeling, Vol. 116, p.96:107, 2017.