ON THE NORMALISED $p$-PARABOLIC EQUATION IN ARBITRARY DOMAINS

NIKOLAI UBOSTAD

Abstract. The boundary regularity for the normalised $p$-parabolic equation $u_t = \frac{1}{p}|Du|^{2-p}\Delta_p u$ is studied. Perron’s method is used to construct solutions in arbitrary domains. We classify the regular boundary points in terms of barrier functions, and prove an Exterior Sphere condition. We identify a fundamental solution, and a Petrovsky criterion is established. We examine the convergence of solutions as $p \to \infty$.

1. Introduction

We investigate Perron solutions of the normalised $p$-parabolic equation

\begin{equation}
\begin{aligned}
    u_t &= \frac{1}{p}|Du|^{2-p}\Delta_p u \\
    &= \frac{1}{p}\text{tr}(D^2 u) + \frac{p-2}{p}\left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle,
\end{aligned}
\end{equation}

where $1 < p < \infty$, in general domains $\Omega \subset \mathbb{R}^n \times (-\infty, \infty)$. Here $\Delta_p u$ denotes the $p$-Laplace operator of $u$, $\Delta_p u = \text{div}(|Du|^{p-2}Du)$, and we let

$A_p u := \frac{1}{p}|Du|^{2-p}\Delta_p u$.

The operator $A_p$ is called normalised since it is homogeneous of degree 1, that is $A_p(\alpha u) = \alpha A_p u$. In contrast, the $p$-Laplace operator is homogeneous of degree $p - 1$.

Since the equation cannot be written on divergence form, the distributional weak solutions are not available to us. The correct notion is the viscosity solutions, introduced in [CLS3].

Sternberg [Ste29] observed that Perron’s method for solving the Dirichlet boundary value problem for Laplace’s equation in [Per23] could be extended to the heat equation. We adapt Perron’s method.

Date: September 19, 2018.
to the non-linear equation (1.1) in general domains, not necessarily space-time cylinders.

Equation (1.1) was studied by Does in connection with image processing, see [Doc09]. The existence of viscosity solutions on cylinders $Q_T = Q \times (0, T)$ was established by Perron’s method.

Integral to the potential theory is the regularity of boundary points. A point $\zeta_0 \in \partial \Omega$ is called regular if, for every continuous function $f : \partial \Omega \to \mathbb{R}$, we have

$$\lim_{\eta \to \zeta_0} u(\eta) = f(\zeta_0), \quad \eta \in \Omega.$$  

The boundary values are prescribed as for an elliptic problem. For example, the points on $Q \times \{t = T\}$ are not regular boundary points of the cylinder $Q_T$.

We characterize the regularity of boundary points in terms of barriers. We say that $w$ is a barrier at $\zeta$ if $w$ is a positive supersolution of (1.1) defined on the entire domain, such that $w(\zeta) = 0$ and $w(\eta) > 0$ for $\zeta \neq \eta \in \partial \Omega$.

To keep the presentation within reasonable limits, we investigate only the case where the boundary data $f$ is bounded. This is not a serious restriction.

The related equation

$$(1.2) \quad u_t = |Du|^{2-p} \Delta_p u,$$

without the factor $1/p$ present, was investigated in [BG15]. For $p \geq 2$, it was proved that a boundary point $\zeta$ is regular if, and only if, there exists a barrier at $\zeta$. The authors also showed that in the case of space-time cylinder $Q_T$, $(x, t) \in \partial Q \times (0, T]$, is a regular boundary point if and only if $x \in \partial Q$ is a regular boundary point for the elliptic $p$-Laplacian.

The regularity of a point is a very delicate issue. Using the Petrovsky criterion, one can construct a domain where the origin is regular for the equation $u_t = \Delta u$, while it is irregular for $u_t = \frac{1}{2} \Delta u$, cf [Wat12]. Therefore it is quite remarkable that as $p \to \infty$, the domain in our Petrovsky criterion converges precisely to the domain in the Petrovsky criterion for the normalised $\infty$-parabolic equation (1.6) derived in [Ubo17], namely that the origin is an irregular point for the domain enclosed by the hypersurfaces

$$(1.3) \quad \{(x, t) \in \mathbb{R}^n \times (0, \infty) : |x|^2 = -4t \log |\log |t||\}$$

and $\{t = -c\}$, for $0 < c < 1$. 
We consider (1.1) instead of (1.2) because of the convergence properties as \( p \to \infty \). Results similar to ours could easily be established for (1.2) by the same methods. Indeed, this was recently proven in [BBP17]. The regular, non-normalised \( p \)-parabolic equation
\[
(1.4) \quad u_t = \Delta_p u,
\]
\( 1 < p < \infty \), has an interesting history. The initial value problem was first studied by Barenblatt in connection with the propagation of heat after thermonuclear detonations in the atmosphere, cf. [Bar52]. The equation has several applications, for example in image processing with variable \( p = p(x) \) in [BSA15]. The \( p \)-parabolic equation, together with its stationary counterpart the \( p \)-Laplace equation, also have interesting applications related to game theory and "Tug-of-War" games, see [MPR10] and [PS+08]. For the regularity theory regarding equations of this type, we mention [DiB95].

For \( p = 1 \), the equation is connected to motion by mean curvature, investigated by Evans and Spruck in [ES+91]. The Dirichlet problem in general domains for (1.4) was studied by Kilpeläinen and Lindqvist in [KL96]. See also [JK06, CW03].

Also worth mentioning is the case \( p = \infty \), when we get the \( \infty \)-parabolic equation
\[
(1.5) \quad u_t = \Delta_\infty u,
\]
and the related normalised \( \infty \)-parabolic equation
\[
(1.6) \quad u_t = |Du|^{-2} \Delta_\infty u := \Delta^N_\infty u,
\]
where the \( \infty \)-Laplace operator is given by
\[
\Delta_\infty u = \left< D^2 u \, Du, Du \right>.
\]
Both of these have an interesting theory in their own right, see [JK06] regarding the normalised case and [CW03] for (1.5).

As \( p \to \infty \), (1.1) converges to (1.6) and not to (1.5).

Our first result is a characterization of regular boundary points via exterior spheres:

**Theorem 1.1 (Exterior Sphere).** Let \( \zeta_0 = (t_0, x_0) \in \partial \Omega \), and suppose that there exists a closed ball \( \{ (x, t) : |x - x'|^2 + (t - t')^2 \leq R_0^2 \} \) intersecting \( \overline{\Omega} \) precisely at \( \zeta_0 \). Then \( \zeta_0 \) is regular, if the intersection point is not the south pole, that is \( (x_0, t_0) \neq (x', t' - R_0) \). If the point of intersection is the north pole, we must restrict the radius of the sphere.

We use a barrier function to prove the following Petrovsky criterion:
Theorem 1.2. The origin \((0,0)\) is a regular point for the domain enclosed by the hypersurfaces
\[
\{(x,t) \in \mathbb{R}^n \times (0,\infty) : |x|^2 = -\beta t \log |t| \}\ and \{t = -c\},
\]
for \(0 < c < 1\), where
\[
\beta = 4 \frac{p-1}{p}.
\]

We also have the following irregularity result, showing that Theorem 1.2 is in some sense sharp:

Theorem 1.3. The origin is not a regular point of the domain \(\Omega\) defined by
\[
|x|^2 = -\beta(1+\epsilon)t \log |t|, \ t = -c
\]
for any \(\epsilon > 0\), and \(\beta\) as in Theorem 1.2.

These results are similar to the classical Petrovsky criterion for the heat equation, derived in [Pet35].

The article is structured as follows. In Section 2 we investigate several explicit solutions of (1.1). We also transform it into a heat equation with variable coefficient. Basic facts regarding viscosity solutions, Perron solutions, a comparison principle and the barrier characterization are displayed in Section 3. The exterior sphere condition in Theorem 1.1 is derived in Section 4. As a demonstration of the necessity of eliminating the south pole, we show that the latest moment of the \(p\)-parabolic ball is not regular. Section 5 is dedicated to the Petrovsky criterion, that is the proof of Theorem 1.2 and Theorem 1.3.

1.1. Notation. In what follows \(\Omega\) is an arbitrary domain in \(\mathbb{R}^n \times (-\infty,\infty)\). \(Q_T\) is a space-time cylinder: \(Q_T = Q \times (0,T), \partial \Omega\) is the Euclidean boundary of \(\Omega\) and \(\partial_p Q_T\) is the parabolic boundary of \(Q_T\), i.e. \((Q \times \{0\}) \cup (\partial Q \times (0,T))\). (the ”bottom” and the sides of the cylinder. The top is excluded.) \(\zeta, \eta \in \mathbb{R}^n \times \mathbb{R}\) are points in space-time, that is \(\zeta = (x,t)\).

We denote by \(Du\) the gradient of \(u(x,t)\) taken with respect to the spatial coordinates \(x\), and \(D^2u\) is the spatial Hessian matrix of \(u\). \(\langle a, b \rangle\) is the Euclidean inner product of the vectors \(a, b \in \mathbb{R}^n\). and \(x \otimes y\) denotes the tensor product of the vectors \(x, y\), that is \((x \otimes y)_{i,j} = x_i y_j\).

The space of lower semi-continuous functions from \(\Omega\) to \(\mathbb{R} \cup \{\infty\}\) is denoted by \(\text{LSC}(\Omega)\), while \(\text{USC}(\Omega)\) contains the upper semicontinuous ones.
2. Several solutions

We derive several explicit solutions to (1.1), and identify the fundamental solution.

2.1. Uniform propagation. Assume \( u(x,t) = w(\langle a, x \rangle - bt) \), \( a \in \mathbb{R}^n \), \( b \in \mathbb{R} \). We then get

\[
\begin{align*}
  u_t &= -bw', \\
  u_{x_i} &= a_i w', \\
  u_{x_ix_j} &= a_i a_j w'',
\end{align*}
\]

and hence

\[
\Delta u = \Delta^N_{\infty} = |a|^2 w''.
\]

Inserting this into (1.1), we get that \( w \) must satisfy

\[
w' + w''|a|^2 \frac{p-1}{bp} = 0,
\]

with solution

\[
w(\zeta) = A + Be^{-\frac{\zeta}{m}},
\]

or

\[
u(x,t) = A + Be^{-\frac{1}{m}(\langle a, x \rangle - bt)},
\]

with \( m = |a|\frac{p-1}{bp} \) and \( \zeta = \langle a, x \rangle - bt \).

2.2. Separable solution. Assume \( u(x,t) = f(r) + g(t) \), \( r = |x| \). We get

\[
\begin{align*}
  u_t &= g', \\
  \Delta u &= f'' + \frac{n-1}{r} f', \\
  \Delta^N_{\infty} u &= f''.
\end{align*}
\]

Thus

\[
g'(t) = c,
\]

and

\[
f''(r) + \frac{n-1}{(p-1)r} f'(r) = c.
\]

So

\[
f(r) = \frac{1}{2n+p} \frac{cp}{2n+p} r^2 + c_1 \frac{p-1}{p-n} r^{\frac{p-n}{p-1}} + c_2,
\]

Setting \( c_1 = c_2 = 0 \), we get the solution

\[
u(x,t) = c \left( \frac{p}{n+p} |x|^2 + \frac{2p-1}{p-n} |x|^{\frac{p-n}{p-1}} + 2t \right).
\]
2.3. **Heat Equation transformation.** We search for solutions on the form \( u(x, t) = v(r^\nu, t) \), where \( r = |x| \) and \( \nu \) is a critical exponent to be determined. This gives

\[
\Delta u = \frac{\partial^2}{\partial r^2} v(r^\nu, t) + \frac{n-1}{r} \frac{\partial}{\partial r} v(r^\nu, t),
\]

\[
\Delta_\infty u = \frac{\partial^2}{\partial r^2} v(r^\nu, t).
\]

Calculating, we get

\[
\frac{\partial}{\partial r} v(r^\nu, t) = \nu r^{\nu-1} v',
\]

\[
\frac{\partial^2}{\partial r^2} v(r^\nu, t) = \nu (\nu - 1) r^{\nu-2} v' + \nu^2 r^{2\nu-2} v'',
\]

where

\[
v' = \frac{\partial v}{\partial \rho}, \quad \rho = r^\nu
\]

Hence we get

\[
\Delta u = \nu (\nu - 1) r^{\nu-2} v' + \nu^2 r^{2\nu-2} v'' + \nu r^{\nu-2} v',
\]

\[
\Delta_\infty u = \nu (\nu - 1) r^{\nu-2} v' + \nu^2 r^{2\nu-2} v''.
\]

Inserting this into (1.1) and collecting terms we get

\[
(2.1) \quad v_t = \nu^2 p^{p-1} \frac{1}{p} r^{2\nu-2} v'' + \nu p^{\nu-2}[p-2](\nu - 1) + n + \nu v'.
\]

We want to eliminate the first order terms in (2.1), so we demand

\[(p - 2)(\nu - 1) + n + \nu = 0,
\]

or

\[
\nu = \frac{p - n}{p - 1}.
\]

Then (1.1) reads

\[
v_t = \frac{(p - n)^2}{p(p - 1)} \frac{2 - n}{p - n} \rho^{\nu - n} v''
\]

where

\[
\rho = |x|^{\frac{p - n}{p - 1}}.
\]
2.4. **Similarity I.** We make the ansatz

\[ u(x, t) = F(\zeta), \quad \zeta = \frac{|x|^2}{t}, \]

We calculate

\[ u_t = -\frac{F'(\zeta)\zeta}{t}, \]
\[ Du = \frac{2F'(\zeta)x}{t}, \]
\[ D^2 u = \frac{2F'(\zeta)}{t} I + \frac{4F''(\zeta)}{t^2} (x \otimes x). \]

Hence

\[ \text{tr}(D^2 u) = \frac{2F'(\zeta)}{t} n + \frac{4F''(\zeta)}{t^2} |x|^2, \]
\[ \Delta^N_x u = \frac{2F'(\zeta)}{t} + \frac{4F''(\zeta)}{t^2} |x|^2. \]

So, if \( u \) is a solution then

\[ -\frac{F'(\zeta)\zeta}{t} - \alpha \frac{F'(\zeta)}{t} - \beta \frac{F''(\zeta)\zeta}{t} = 0, \]

with \( \alpha, \beta \) as in (2.3). If \( t \neq 0; \)

\[ \frac{F''(\zeta)}{F'(\zeta)} = \frac{d}{d\zeta} \log F' = -\frac{\alpha}{\beta} \frac{1}{\zeta}. \]

Integrating the above gives

\[ \log F' = -\frac{\zeta}{\beta} - \frac{\alpha}{\beta} \log \zeta, \]

or

\[ F(\zeta) = C \int_0^\zeta s^{-\frac{\alpha}{\beta}} e^{-\frac{\zeta}{\beta}} \, ds. \]

This leads to the solution

\[ (2.2) \quad u(x, t) = C \int_0^{|x|^2/t} s^{-\frac{\alpha}{\beta}} e^{-\frac{\zeta}{\beta}} \, ds. \]

This solution is not differentiable where \( x = 0 \). However, \( (2.2) \) is a solution outside the line \( \{0\} \times (0, \infty) \subset \mathbb{R}^n \times (0, \infty) \), and a subsolution or supersolution depending on the sign of \( C \) in all of \( \mathbb{R}^n \times (0, \infty) \).
2.5. Similarity II. We note that if \( u(x,t) \) is a solution of (1.1), then so is \( v(x,t) = u(Ax, A^2 t) \). We search for solutions on the form

\[
u(x,t) = g(t) f(\zeta), \quad \zeta = \frac{|x|^2}{t}.
\]

Inserting this into (1.1), we get

\[
u_t = g'(t) f(\zeta) - \frac{g(t) f'(\zeta) \zeta}{t},
\]

\[
Du = \frac{2g(t) f'(\zeta)}{t} x,
\]

and

\[
D^2 u = \frac{2g(t) f'(\zeta)}{t} I + \frac{4g(t) f''(\zeta)}{t^2} (x \otimes x).
\]

Hence we see

\[
\text{tr}(D^2 u) = \frac{2g(t) f'(\zeta)}{t} n + \frac{4g(t) f''(\zeta)}{t^2} |x|^2,
\]

and

\[
\Delta_N^\infty u = \frac{2g(t) f'(\zeta)}{t} + \frac{4g(t) f''(\zeta)}{t^2} |x|^2.
\]

Therefore, \( u \) is a solution to (1.1) if

\[
g'(t) f(\zeta) - \frac{g(t) f'(\zeta) \zeta}{t} = \frac{2p + n - 2}{p} g(t) f'(\zeta) + \frac{4p - 1}{p} g(t) f''(\zeta) \zeta.
\]

with \( \alpha = 2 \frac{p+n-2}{p} \) and \( \beta = 4 \frac{p-1}{p} \).

\[
tg'(t) f(\zeta) - \alpha g(t) f'(\zeta) = g(t) \zeta (f'(\zeta) + \beta f''(\zeta)).
\]

for \( t > 0 \). The right hand side of this is zero if \( f(\zeta) = e^{-\frac{\zeta}{\beta}} \). Inserting this back in, we see that

\[
f(\zeta) \left( tg'(t) + \frac{\alpha}{\beta} g(t) \right) = 0,
\]

with solution

\[
g(t) = t^{-\frac{\alpha}{\beta}}.
\]

Together, this gives

\[
u(x,t) = t^{-\frac{\alpha}{\beta}} e^{-\frac{|x|^2}{\beta t}},
\]

(2.3)

\[
\alpha = 2 \frac{p+n-2}{p}, \quad \beta = 4 \frac{p-1}{p}.
\]
This is a solution for \( t > 0 \), and if we replace \( t \) by \(-t\) we get a solution for negative \( t \), as well.\footnote{It has recently come to the author’s attention that this solution also was found in [BG13].}

**Remark 2.1.** As \( p \to \infty \), \( \alpha \to 2 \) and \( \beta \to 4 \). This gives that (2.3) converges to the fundamental solution

\[
W(x, t) = \frac{1}{\sqrt{t}} e^{-\frac{|x|^2}{4t}}.
\]

of the normalised \( \infty \)-parabolic equation found in [JK06]. Compare also (2.3) with the fundamental solution to the heat equation,

\[
H(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.
\]

3. **Comparison and Perron solutions**

In this Section we present several basic facts regarding the existence of solutions to (1.1), and present Perron’s method. We start with the definition of viscosity solutions. If \( Du = 0 \), we replace the operator \( A_p \) with its lower or upper semicontinuous envelope:

**Definition 3.1.** A lower semicontinuous function \( u \in L^\infty(\Omega) \) is a viscosity supersolution of (1.1) provided that, if \( u - \phi \) has a minimum at \( \zeta_0 \in \Omega \) for \( \phi \in C^2(\Omega) \), then

\[
\begin{align*}
\phi_t(\zeta_0) - A_p \phi(\zeta_0) & \geq 0, & \text{if } D\phi(\zeta_0) \neq 0, \\
\phi_t(\zeta_0) - \frac{1}{p} tr(D^2\phi(\zeta_0)) - \frac{p-2}{p} \lambda(D^2\phi(\zeta_0)) & \geq 0, & \text{if } D\phi(\zeta_0) = 0.
\end{align*}
\]

An upper semicontinuous function \( u \in L^\infty(\Omega) \) is a viscosity subsolution of (1.1) provided that, if \( u - \phi \) has a maximum at \( \zeta_0 \in \Omega \) for \( \phi \in C^2(\Omega) \), then

\[
\begin{align*}
\phi_t(\zeta_0) - A_p \phi(\zeta_0) & \leq 0, & \text{if } D\phi(\zeta_0) \neq 0, \\
\phi_t(\zeta_0) - \frac{1}{p} tr(D^2\phi(\zeta_0)) - \frac{p-2}{p} \Lambda(D^2\phi(\zeta_0)) & \leq 0, & \text{if } D\phi(\zeta_0) = 0.
\end{align*}
\]

A function that is both a viscosity sub- and supersolution is called a viscosity solution.

Here \( \lambda(D^2\phi(\zeta_0)), \lambda(D^2\phi(\zeta_0)) \) denotes the smallest and largest eigenvalues of the Hessian matrix \( D^2\phi(\zeta_0) \), and \( tr(D^2\phi(\zeta_0)) \) is its trace.

It turns out that the second condition in Definition 3.1 can be relaxed. This is the Lemma 2 in [MPR10].
Lemma 3.2. An upper semicontinuous function $u \in L^\infty(\Omega)$ is a viscosity subsolution of (1.1) provided that, if $u - \phi$ has a maximum at $\zeta_0 \in \Omega$ for $\phi \in C^2(\Omega)$, then either
\[ \phi_t(\zeta_0) - A_p \phi(\zeta_0) \leq 0, \text{ if } D\phi(\zeta_0) \neq 0, \]
or
\[ \phi_t(\zeta_0) \leq 0, \text{ if } D\phi(\zeta_0) = 0, \quad D^2\phi(\zeta_0) = 0. \]
A similar result holds for viscosity supersolutions.

We define $p$-parabolic functions in $\Omega$ as follows:

Definition 3.3. A function $u \in LSC(\Omega) \cap L^\infty(\Omega)$ is a supersolution to (1.1) if it satisfies the following comparison principle:

On each set of the form $Q_{t_1,t_2} = Q \times (t_1,t_2)$ with closure in $\Omega$, and for each solution $h$ to (1.1) continuous up to the closure of $Q_{t_1,t_2}$, and $h \leq u$ on $\partial_p Q_{t_1,t_2}$, then
\[ h \leq u \text{ in } Q_{t_1,t_2}. \]

In Banerjee–Garofalo [BG15], they use the name generalized super/subsolution instead of super/subparabolic function. They prove that these are the same as the viscosity super/subsolutions in a given domain. Hence we can use the term parabolic interchangeably with viscosity solution.

The assumption that supersolutions are bounded is not needed. See Theorem 2.6 in [BBP17].

We shall improve this to include comparison on general domains $\Sigma$ compactly contained in $\Omega$, see Lemma 4.3.

Using the classical comparison principle for cylindrical domains $Q_T = Q \times (0,T)$ and a covering argument, we can prove Theorem 3.10 from [BG15]. This is the comparison principle, essential for Perron’s method to work.

Theorem 3.4. Suppose $u$ is a supersolution bounded from above and $v$ is a subsolution bounded from below of (1.1) in a bounded open set $\Omega \subset \mathbb{R}^{n+1}$. If at each point $\zeta_0 \in \partial \Omega$ we have
\[ \limsup_{\zeta \to \zeta_0} v(\zeta) \leq \liminf_{\zeta \to \zeta_0} u(\zeta), \]
then $v \leq u$ in $\Omega$. 
3.1. **The Perron Method.** We start with a definition. Let $f : \partial \Omega \to \mathbb{R}$ be a continuous function.

**Definition 3.5.** A function $u$ belongs to the upper class $\mathcal{U}_f$ if $u$ is a viscosity supersolution in $\Omega$ and

$$\liminf_{\eta \to \zeta} u(\eta) \geq f(\zeta)$$

for $\zeta \in \partial \Omega$.

Likewise, a function $v$ belongs to the lower class $\mathcal{L}_f$ if $v$ is a viscosity subsolution in $\Omega$ and

$$\limsup_{\eta \to \zeta} v(\eta) \leq f(\zeta)$$

for all $\zeta \in \partial \Omega$.

We define the upper solution

$$\mathcal{H}_f(\zeta) = \inf\{u(\zeta) : u \in \mathcal{U}_f\},$$

and the lower solution

$$\underline{H}_f(\zeta) = \sup\{v(\zeta) : v \in \mathcal{L}_f\}.$$

Note that at each point the inf and sup are taken over the functions.

**Remark 3.6.** The Comparison Principle, Theorem 3.4, gives immediately that $v \leq u$ in $\Omega$, for $v \in \mathcal{L}_f$ and $u \in \mathcal{U}_f$, and hence

$$H_f \leq \mathcal{H}_f.$$

Whether $\underline{H}_f = \mathcal{H}_f$ holds in general, is a more subtle question.

We need that the Upper and Lower Perron solutions, indeed, are viscosity solutions to (1.1).

**Theorem** (Theorem 3.12 in [BG15]). The upper Perron solution $\mathcal{H}_f$ and the lower Perron solution $\underline{H}_f$ are solutions to (1.1) in $\Omega$.

An integral part of the theory of Perron solutions are the boundary regularity and the barrier functions.

**Definition 3.7.** We say that $\zeta_0 \in \partial \Omega$ is a regular boundary point if

$$\lim_{\zeta \to \zeta_0} H_f(\zeta) = f(\zeta_0).$$

for every continuous function $f : \partial \Omega \to \mathbb{R}$.

Note that we instead could have used $\mathcal{H}_f$ in the above, since $H_f = -\mathcal{H}_{-f}$. 
Remark 3.8. The Petrovsky condition in Theorem 1.3 shows that a point can be regular for $u_t = A_p u$ but not for $u_t = A_q u$, $p < q$. Hence it would be more accurate to use the term $p$-regular, but we use regular where no confusion will arise.

Definition 3.9. A function $w$ is a barrier at $\zeta_0 \in \partial \Omega$ if

1. $w > 0$ and $w$ is $p$-superparabolic in $\Omega$,
2. $\liminf_{\zeta \to \eta} w(\zeta) > 0$ for $\zeta_0 \neq \eta \in \partial \Omega$,
3. $\lim_{\zeta \to \zeta_0} w(\zeta) = 0$.

Using barrier functions, we can prove the following classical result, which is Theorem 4.2 in [BG15]:

Theorem 3.10. A boundary point $\zeta_0$ is regular if and only if there exists a barrier at $\zeta_0$.

The existence of a barrier is a local property in the following sense: Let $\tilde{\Omega}$ be another domain such that

$$\overline{B} \cap \tilde{\Omega} = \overline{B} \cap \Omega$$

for an open ball $B$ centered at $\zeta_0$. Suppose there is a barrier $w$ at $\zeta_0$, and let

$$m = \inf \{ w(\zeta) : \zeta \in \partial B \cup \tilde{\Omega} \}.$$

It now follows that the function

$$v = \begin{cases} 
\min(w, m) & \text{in } B \cup \tilde{\Omega}, \\
m & \text{in } \Omega \setminus B
\end{cases}$$

is a barrier in $\Omega$. Indeed, since $w$ is assumed to be a barrier in $\tilde{\Omega}$, $w|_{\tilde{\Omega}} > 0$ by the definition, and therefore $m > 0$ and $v > 0$. Since $\tilde{\Omega} \cap \overline{B} = \Omega \cap \overline{B}$ we see that $\liminf_{\eta \to \zeta} v(\eta) = \liminf_{\eta \to \zeta} w(\eta) > 0$ on $\partial \tilde{\Omega} \cap \overline{B}$ and $\liminf_{\eta \to \zeta} v(\eta) = m > 0$ elsewhere. At last, $\lim_{\zeta \to \zeta_0} v(\zeta) = \lim_{\zeta \to \zeta_0} w(\zeta) = 0$. From this we get the following useful corollary.

Corollary 3.11. Let $\tilde{\Omega} \subset \Omega$, and let $\zeta_0$ be a common boundary point. If $\zeta_0$ is not a regular point for $\tilde{\Omega}$, then it is not a regular point for $\Omega$.

Proof. Let $\tilde{\Omega} \subset \Omega$, and let $\zeta_0$ be an irregular boundary point for $\tilde{\Omega}$. Assume that $\zeta_0$ is regular for $\Omega$. Then Theorem 3.10 gives that there exists a barrier, $w$, in $\Omega$. The above implies the existence of a barrier in $\tilde{\Omega}$, contradicting the irregularity of $\zeta_0$. $\Box$

A classical application of the theory of viscosity solutions is the following convergence lemma.
Lemma 3.12. Assume that \( \{u_p\}_p \) is a sequence of viscosity solutions of
\[
  u_t - A_p u = 0.
\]
Assume further that \( \{u_p\}_p \) contains a subsequence \( \{u_{p_j}\}_j \) that converges uniformly to a function \( u_\infty \) in \( \Omega \). Then, as \( j \to \infty \), the \( u_{p_j} \) converge to \( u \), the viscosity solution of the normalised \( \infty \)-parabolic equation (1.6), that is
\[
  u_t - \Delta^N u = 0.
\]

Proof. We show that viscosity subsolutions of (1.1) converge to viscosity subsolutions of (1.5). The proof for supersolutions is similar. We say that \( u \in USC(\Omega) \) is a viscosity subsolution to (1.6) if, for every function \( \phi \in C^2(\Omega) \) such that \( u - \psi \) has a maximum at \( \zeta_0 \), we have
\[
  \begin{cases}
  \phi_t(\zeta_\infty) - \Delta^N u(\zeta_\infty) \leq 0, & \text{for } D\phi(\zeta_\infty) \neq 0 \\
  \phi_t(\zeta_\infty) - \Lambda(D^2\psi(\zeta_\infty)) \leq 0, & \text{for } D\phi(\zeta_\infty) = 0.
  \end{cases}
\]
Assume that \( u_\infty - \phi \) has a maximum at \( \zeta_\infty \) for \( \phi \in C^2(Q_T) \).

1. Assume first that \( D\phi(\zeta_{p_j}) \neq 0 \) for \( j \) greater than some number \( N \). By definition of viscosity subsolution, we then have
\[
  \begin{align}
  \phi_t(\zeta_{p_j}) - \frac{1}{p_j} \Delta\phi(\zeta_{p_j}) - \frac{p_j - 2}{p_j} \Delta^N u(\zeta_{p_j}) & \leq 0.
  \end{align}
\]
Since \( u_{p_j} \to u_\infty \) uniformly, standard arguments, cf [?] gives that the maximum points \( \zeta_{p_j} \), converge to a maximum point \( \zeta_\infty \) of \( u_\infty - \phi \). Hence, letting \( j \to \infty \) in (3.1) we see that
\[
  \phi_t(\zeta_\infty) - \Delta^N u(\zeta_\infty) \leq 0.
\]

2. If \( D\phi(\zeta_{p_j}) = 0 \) for \( j > N \), we have
\[
  \phi_t(\zeta_{p_j}) - \frac{1}{p} \text{tr}(D^2\phi(\zeta_{p_j})) - \frac{p - 2}{p} \Lambda(D^2\phi(\zeta_{p_j})) \leq 0
\]
and arguing as in the first case, we get as \( p_j \to \infty \);
\[
  \phi_t(\zeta_\infty) - \Lambda(D^2\phi(\zeta_\infty)) \leq 0.
\]
This shows that \( u_\infty \) is indeed a viscosity subsolution of the normalised \( \infty \)-parabolic equation. \( \square \)

Remark 3.13. The existence of a uniformly convergent subsequence of \( u_p \) is not known to exist in general. Does finds such an example for the initial-boundary value problem with smooth boundary data in \[Doe09\].
4. Exterior Sphere Condition

We use the barrier characterization to prove Theorem 1.1. We repeat the result here for completeness.

**Theorem (Exterior sphere).** Let $\zeta_0 = (t_0, x_0) \in \partial \Omega$, and suppose that there exists a closed ball $\{(x, t) : |x - x'|^2 + (t - t')^2 \leq R_0^2\}$ intersecting $\Omega$ precisely at $\zeta_0$. Then $\zeta_0$ is regular, if the intersection point is not the south pole, that is $(x_0, t_0) \neq (x', t' - R_0)$.

**Proof.** We use the exterior sphere to construct a suitable barrier function at $\zeta_0$. Define

$$w(x, t) = e^{-aR_0^2} - e^{-aR^2}, \quad R^2 = |x - x'|^2 + (t - t')^2,$$

for a constant $a > 0$ to be determined. Clearly $w(x_0, t_0) = 0$, and close to $(x_0, t_0)$ we have

$$\delta < |x - x'|, \quad -2R_0 < t - t'.$$

We prove that $w$ is a viscosity supersolution. Calculating the derivatives, we get

$$Dw(x, t) = 2ae^{-aR^2}(x - x'),$$

$$w_t(x, t) = 2ae^{-aR^2}(t - t'),$$

$$D^2w(x, t) = 2ae^{-aR^2}(I_n - 2a(x - x') \otimes (x - x')).$$

This shows that $Dw = 0$ precisely when $x = x'$. According to Definition 3.1 we need to check the cases $x = x'$ and $x \neq x'$ separately.

1. Assume that $x \neq x'$. Then the point of contact is not the north pole. It suffices to show that $w$ is a classical supersolution. Inserting the derivatives into (4.1) we get

$$w_t - \frac{1}{p} \Delta w + \frac{p - 2}{p} \Delta_N w = 2a(t - t')e^{-aR^2} - \frac{1}{p}e^{-aR^2} \left[ (p - 1)(2a - 4a^2r^2) + 2ar\frac{n - 1}{r} \right]$$

$$= 2a e^{-aR^2} \left[ (t - t') + 2a\frac{p - 1}{p} |x - x'|^2 - \frac{p + n - 2}{p} \right].$$

In light of (4.1), we have

$$w_t - A_p w > 2ae^{-aR^2} \left[ -2R_0 + 2a\frac{p - 1}{p} \delta^2 - \frac{p + n - 2}{p} \right]$$

For the right hand side of this to be positive, we must have

$$-R_0 + a\frac{p - 1}{p} \delta^2 > \frac{p + n - 2}{2p},$$
and choosing $a$ big enough to ensure this, shows that $w$ is superparabolic.

2. If the point of intersection is the north pole, i.e $(x_0, t_0) = (x', t' + R_0)$, we can find points arbitrarily close to the line $x = x'$ such that

$$w_t - A_p w = 2a e^{-aR^2} \left[ R_0 - \frac{p + n - 2}{p} \right] + \epsilon,$$

for any $\epsilon > 0$. We see that we must demand that the radius $R_0$ satisfies

$$R_0 \geq \frac{\alpha}{2}, \quad \alpha = 2\frac{p + n - 2}{p}$$

for $w$ to be a barrier in this case.

Assume now that $x = x'$. We need to verify that for every $\phi \in C^2(\Omega)$ touching $w$ from below at $(x', t)$ we have

$$(4.3) \quad \phi_t(x', t) \geq \frac{1}{p} \text{tr}(D^2 \phi(x', t)) + \frac{p - 2}{p} \lambda(D^2 \phi(x', t)).$$

Assume to the contrary that there is a $\phi$ such that $w - \phi$ has a minimum at $(x', t)$, but that

$$\phi_t(x', t) < \frac{1}{p} \text{tr}(D^2 \phi(x', t)) + \frac{p - 2}{p} \lambda(D^2 \phi(x', t)).$$

Since $w - \phi$ has a minimum, we must have

$$\phi_t(x', t) = u_t(x', t), \; D\phi(x', t) = Du(x', t), \; D^2 u(x', t) \geq D^2 \phi(x', t).$$

This implies, for any $z \in \mathbb{R}^n$

$$\langle D^2 w, z, z \rangle \geq \langle D^2 \phi, z, z \rangle$$

and, since $D^2 w$ is a scalar multiple of the identity matrix,

$$\text{tr}(D^2 w)|z|^2 \geq \text{tr}(D^2 \phi)|z|^2$$

at $(x', t)$. Hence

$$\frac{1}{p} \text{tr}(D^2 w)|z|^2 + \frac{p - 2}{p} \langle D^2 w, z, z \rangle \geq \frac{1}{p} \text{tr}(D^2 \phi)|z|^2 + \frac{p - 2}{p} \langle D^2 \phi, z, z \rangle \geq \frac{1}{p} \text{tr}(D^2 \phi)|z|^2 + \frac{p - 2}{p} \lambda(D^2 \phi)|z|^2 \geq \phi_t|z|^2 = w_t|z|^2.$$

Inserting $x = x'$ in $(4.2)$ and dividing by $|z|^2$ this is

$$\frac{1}{p} 2an e^{-aR^2} + \frac{p - 2}{p} 2a e^{-aR^2} > 2a e^{-aR^2} (t - t'),$$
or

\[
\frac{n + p - 2}{p} > (t - t').
\]

This is a contradiction because of our restriction on the radius, and hence (4.3) must hold, and \( w \) is a supersolution even in this case.

The condition \( R_0 \geq \alpha/2 \) restricts the set of exterior spheres usable in a positive way. The author does not know if this restriction can be circumvented.

The exclusion of the south pole \( (x_0, t_0) = (x', t' - R_0) \) in the above is strictly necessary, since then for \( (x, t) \) close to \( (x_0, t_0) \) we could have

\[
(t - t') < 0 \text{ and } |x - x'| = |x' - x'| = 0,
\]

and so

\[
w_t - A_p w = 2a e^{-aR^2} \left[ (t - t') - \frac{p + n - 2}{p} \right] < 0
\]

for any positive \( a \), since \( p + n \geq 2 \). \( \square \)

Another way to see that it is necessary to exclude the south pole is to consider the Dirichlet problem on the cylinder \( Q_T = Q \times (0, T) \).

**Example 4.1.** Suppose that \( f : \partial Q_T \to \mathbb{R} \) is continuous. Theorem 3.4 and Theorem 2.6 in [BG15] gives the existence of a unique viscosity solution \( h \) in \( Q_T \).

Now construct the upper and lower Perron solutions \( \overline{H}_f \) and \( \underline{H}_f \). Since both are \( p \)-parabolic in \( Q_T \), uniqueness gives that \( \overline{H}_f = \underline{H}_f = h \), regardless of what values we choose at that part of the boundary where \( t = T \). Indeed, \( h \) itself need not be in either the upper or lower class, because we may not have that either \( h > f \) or \( h < f \) on the plane \( t = T \). However, if we define

\[
\tilde{h} = h(x, t) + \frac{\epsilon}{T - t},
\]

we see that

\[
\tilde{h}_t - A_p \tilde{h} = 0 + \frac{\epsilon}{(T - t)^2},
\]

so \( \tilde{h} \) is in \( U_f \) for \( \epsilon > 0 \), and in \( L_f \) for \( \epsilon < 0 \). Therefore, it is possible for every point on the top of the cylinder to be irregular. we can say that \( f \) is *resolutive* in this case.

We provide another example of an irregular boundary point.
Example 4.2 (Latest moment on heat balls). Recall the self-similar solution derived in Section 2.5. We define the fundamental solution to (1.1) as

\[ H_p(x, t) = t^{-\frac{\alpha}{\beta}} e^{-\frac{|x|^2}{\beta t}}. \]

Analogous to the heat equation and the \( p \)-parabolic equation, we define the normalised \( p \)-parabolic balls by the level sets

\[ H_p(x_0 - x, t_0 - t) > c \]

We want to prove that the latest moment, or "centre" \((x_0, t_0)\) of (4.4) is not a regular point. Fix \( c > 0 \). We can assume that \((x_0, t_0) = (0, 0)\), so that (4.4) reads

\[ (-t)^{-\frac{\alpha}{\beta}} e^{-\frac{|x|^2}{\beta (-t)}} > c \]

for \( t < 0 \). But this is equivalent to

\[ |x|^2 < t \left( \frac{\log c}{\beta} + \alpha \log |t| \right), \]

and this inequality defines a domain containing the one in the Petrovsky criterion 1.2. Hence the origin must be irregular.

We prove that it suffices to consider arbitrary domains in the equivalent definition of \( p \)-parabolic functions. The proof follows the same idea as in [KL96].

Lemma 4.3. A function \( u \in LSC(\Omega) \cap L^\infty(\Omega) \) is \( p \)-superparabolic if and only if for each domain \( \Sigma \) with compact closure in \( \Omega \), and for each solution \( h \in C(\Sigma) \) to (1.1), the condition \( h \leq u \) on \( \partial \Sigma \) implies \( h \leq u \) in \( \Sigma \).

Proof. Assume first that Definition 3.3 holds. If \( \Sigma \) is a box or finite union of boxes, the result is clearly true. The case where \( \Sigma \) is arbitrary follows by covering the set \( \{ h \geq u + \epsilon \} \) with finitely many boxes.

For the other direction, let \( Q_{t_1, t_2} \) be a box with closure in \( \Omega \) and let \( h \in C(\overline{Q}_{t_1, t_2}) \) be \( p \)-parabolic, and so that \( h \leq u \) on \( \partial_p Q_{t_1, t_2} \). Assume that

\[ Q = (a_1, b_1) \times \cdots \times (a_n, b_n). \]

Let \( \delta > 0 \) be so that \( \delta < t_2 - t_1 \), and choose a hyperplane \( P_\delta \) such that the points \((x, t_2 - \delta)\) with \( x_1 = a_1 \) and \((y, t_2)\) with \( y_1 = b_1 \) belong to \( P_\delta \). Let \( \Sigma \) be the subset of \( Q_{t_1, t_2} \) that contains all the points below the hyperplane, that is all \((x, t)\) with \( t < s \) and \((x, s) \in P_\delta \).

The Exterior sphere condition Theorem 1.1 immediately gives that every point on \( \partial \Sigma \) is regular. Fix \( \epsilon > 0 \), and choose \( \delta \) so small that

\[ u(x, t) \geq h(x, t) - \frac{\epsilon}{t_2 + \frac{\delta}{2} - t} \]
for \((x,t) \in P_\delta \cap \Sigma\). Let \(\overline{H}_\theta\) be the upper Perron solution in \(\Sigma\) with

\[
\theta = h - \frac{\epsilon}{t_2 + \frac{\delta}{2} - t}
\]

as boundary function. Then \(\overline{H}_\theta\) is continuous up to \(\partial \Sigma\), and we have

\[
u \geq \overline{H}_\theta\]

in all of \(\Sigma\) since the inequality holds on \(\partial \Sigma\). Hence

\[
u(x,t) \geq h(x,t) - \frac{\epsilon}{t_2 + \frac{\delta}{2} - t}
\]

in \(\Sigma\), and letting \(\epsilon, \delta \to 0\), we get

\[
u \geq h
\]

in the box \(Q_{t_1, t_2}\). \(\square\)

5. The Petrovsky Criterion

We provide the proof of the Petrovsky Criterion, repeated here for completeness.

**Theorem.** The origin \((x,t) = (0,0)\) is a regular point for (1.1) in the domain \(\Omega\) enclosed by the hypersurfaces

\[
\{(x,t) \in \mathbb{R}^n \times (-\infty, 0) : |x|^2 = -\beta t \log |\log |t||\} \quad \text{and} \quad \{t = -c\},
\]

for a small constant \(0 < c < 1\). Recall that

\[
\beta = 4^p - 1.
\]

According to Theorem 3.10 it suffices to find a barrier function \(w\) so that

1. \(w\) is a supersolution in \(\Omega\),
2. \(w(x,t) > 0\) for \((x,t) \in \Omega\),
3. \(\liminf_{(y,s) \to (x,t)} w(y,s) > 0\) for \((x,t) \neq (0,0) \in \partial \Omega\),
4. \(\lim_{(x,t) \to (0,0)} w(x,t) = 0\).

Our barrier will be on the form

\[
w(x,t) = f(t)e^{-\frac{|x|^2}{\beta t}} + g(t),
\]

for smooth functions \(f\) and \(g\). Differentiating formally, we get

\[
w_t(x,t) = e^{-\frac{|x|^2}{\beta t}} \left( f'(t) + \frac{|x|^2}{\beta t^2} f(t) \right) + g'(t),
\]

(5.2)

\[
Dw(x,t) = -x \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}},
\]

(5.3)
and
\[ D^2 w(x,t) = -\frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} I_n + \frac{4f(t)}{t^2 \beta^2} e^{-\frac{|x|^2}{\beta t}} x \otimes x \]
(5.4)
\[ = \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -I_n + \frac{2}{\beta t} x \otimes x \right). \]

From (5.4) we see that
\[ \text{tr}(D^2 w(x,t)) = \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -n + \frac{2|x|^2}{\beta t} \right). \]
(5.5)

From (5.3) and (5.4), (or observing that \( w(x,t) = G(r,t) \)), we get
\[ \Delta_{\infty}^N w = G_{rr}, \]
(5.6)
\[ \left\langle D^2 w \left| \frac{Dw}{|Dw|} \right|, \frac{Dw}{|Dw|} \right\rangle = \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -1 + \frac{2|x|^2}{\beta t} \right). \]

From (5.6) and (5.5), we calculate
\[ A_p w = \frac{1}{p} \text{tr}(D^2 w) + \frac{p-2}{p} \left\langle D^2 w \left| \frac{Dw}{|Dw|} \right|, \frac{Dw}{|Dw|} \right\rangle,
\[ = \frac{1}{p} \cdot \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -n + \frac{2|x|^2}{\beta t} \right) + \frac{p-2}{p} \cdot \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -1 + \frac{2|x|^2}{\beta t} \right) \]
\[ = \frac{2f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -\frac{n}{p} - \frac{p-2}{p} + \frac{2|x|^2}{\beta t} \left( \frac{1}{p} + \frac{p-2}{p} \right) \right) \]
\[ = \frac{f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -2 \left( \frac{n+p-2}{p} \right) + \frac{|x|^2}{\beta t} \left( \frac{p-1}{p} \right) \right) \]
\[ = \frac{f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -\alpha + \frac{|x|^2}{t} \right), \]

where
\[ \alpha = \frac{2n + p - 2}{p}. \]

This, together with (5.2), gives
\[ w_t - A_p w \]
(5.7)
\[ = e^{-\frac{|x|^2}{\beta t}} \left( f'(t) + \frac{|x|^2}{\beta t} f(t) \right) + g'(t) - \frac{f(t)}{\beta t} e^{-\frac{|x|^2}{\beta t}} \left( -\alpha + \frac{|x|^2}{t} \right) \]
\[ = e^{-\frac{|x|^2}{\beta t}} \left( f'(t) + \frac{|x|^2}{\beta t} f(t) + \frac{\alpha f(t)}{\beta t} - \frac{|x|^2 f(t)}{\beta t^2} \right) + g'(t) \]
\[ = e^{-\frac{|x|^2}{\beta t}} \left( f'(t) + \frac{\alpha f(t)}{\beta t} + g'(t) e^{-\frac{|x|^2}{\beta t}} \right). \]
Choose
\[ f(t) = -c \frac{1}{|\log|t||^{\delta+1}}, \quad g(t) = \frac{1}{|\log|t||^\delta}, \]
for constants \(0 < c < 1, \delta\) to be determined. We are now in position to prove the following theorem:

**Theorem 5.1.** The smooth function \(w : \Omega \to \mathbb{R}\) given by
\[(5.8)\]
\[ w(x,t) = -c \frac{1}{|\log|t||^{\delta+1}} e^{-\frac{|x|^2}{2|t|}} + \frac{1}{|\log|t||^\delta} \]
is a barrier at \((0, 0)\).

**Proof.** We check the requirements listed in Definition 3.9.

1. We must check that \(w\) is a viscosity supersolution in \(\Omega\). Equation (5.3) shows that \(Dw = 0\) precisely when \(x = 0\), so assume first that \(x \neq 0\). It suffices to show that \(w\) is a classical solution in this case. We first differentiate \(f\) and \(g\):
\[ f'(t) = -c(\delta + 1) \frac{1}{t|\log|t||^{\delta+2}}, \quad g'(t) = \delta \frac{1}{t|\log|t||^{\delta+1}}. \]

Inserting the derivatives into (5.7) gives
\[ w_t - A_p w = e^{-\frac{|x|^2}{2|t|}} \left( -c(\delta + 1) \frac{1}{t|\log|t||^{\delta+2}} - c \frac{\alpha}{\beta} \frac{1}{|\log|t||^{\delta+1}} + \delta \frac{1}{|\log|t||^{\delta+1}} e^{-\frac{|x|^2}{2|t|}} \right) \]
\[ = \frac{1}{t|\log|t||^{\delta+1}} e^{-\frac{|x|^2}{2|t|}} \left( -c(\delta + 1) - c \frac{\alpha}{\beta} + \delta e^{-\frac{|x|^2}{2|t|}} \right), \]
\( t \) is negative, so \( e^{-\frac{|x|^2}{2|t|}} < 1 \), hence
\[ w_t - A_p w > \frac{1}{t|\log|t||^{\delta+1}} e^{-\frac{|x|^2}{2|t|}} \left( -c(\delta + 1) - c \frac{\alpha}{\beta} + \delta \right) \]
For this to be positive, the expression inside the parentheses must be negative. Choosing
\[(5.9)\]
\[ \delta = c \frac{\alpha}{\beta} \]
ensures this, and with this choice \(w\) is superparabolic in this case.

Assume that \(x = 0\) so that \(Dw = 0\). From (5.4) and (5.5) we deduce
\[ \text{tr}(D^2 w(0, t)) = c \frac{2n}{\beta t|\log|t||^{\delta+1}}, \]
and
\[ \lambda(D^2 w(0, t)) = c \frac{2}{\beta t|\log|t||^{\delta+1}}. \]
Since
\[
    w_t(0, t) = f'(t) + g'(t) = \frac{1}{t|\log |t||^{\delta+1}} \left( \frac{-c(\delta + 1)}{|\log |t||} + \delta \right),
\]
Definition 3.1 demands that we verify
\[
    \frac{1}{t|\log |t||^{\delta+1}} \left( \frac{-c(\delta + 1)}{|\log |t||} + \delta \right) \\
    \geq \frac{1}{p} \cdot \frac{c}{\beta t|\log |t||^{\delta+1}} + \frac{2n}{p} \cdot \frac{2}{\beta |\log |t||^{\delta+1}}
\]
for \( t < 0 \). This is equivalent to
\[
    \frac{-c(\delta + 1)}{|\log |t||} + \delta \leq \frac{2cn}{p\beta} + \frac{2c(p-2)}{p\beta} = \frac{2c}{p\beta} + \frac{n-2}{p\beta} = \frac{c\alpha}{\beta}.
\]
Because of our choice of \( \delta \) in (5.9), the above inequality is satisfied for all \( t \), and (5.8) satisfies the second condition in Definition 3.1.

It remains to show that \( w \) is a viscosity supersolution. Let \( \phi \in C^2(\Omega) \) touch \( w \) from below at \((0, t)\). Since \( w - \phi \) has a minimum at \((0, t)\), we have
\[
    w_t = \phi_t, \quad Dw = D\phi, \quad D^2w > D^2\phi
\]
at this point. Since \( D^2w(0, t) = \frac{2f(t)}{\beta t} \mathbb{I}_n \), a scalar multiple of the identity matrix, this implies
\[
    \lambda(D^2w(0, t)) > \Lambda(D^2\phi(0, t)) > \lambda(D^2\phi(0, t)),
\]
where \( \lambda \) is the smallest eigenvalue and \( \Lambda \) is the greatest. Since
\[
    \text{tr}(D^2\phi(0, t)) = \sum_{i=1}^n \lambda_i(D^2\phi(0, t)),
\]
we get
\[
    \phi_t(0, t) = w_t(0, t) \\
    \geq \frac{1}{p} \text{tr}(D^2w(0, t)) - \frac{p-2}{p} \lambda(D^2w(0, t)) \\
    \geq \frac{1}{p} \text{tr}(D^2\phi(0, t)) - \frac{p-2}{p} \lambda(D^2\phi(0, t)),
\]
which implies that \( w \) is indeed a viscosity supersolution.

2. Since (5.1) implies \(-\frac{|x|^2}{\beta t} < \log |\log |t||\), we see
\[
    w(x, t) > -c \frac{1}{|\log |t||^{\delta+1}} e^{\log |\log |t||} + \frac{1}{|\log |t||^{\delta}} = \frac{1-c}{|\log |t||^{\delta}} > 0,
\]
for \( 0 < c < 1 \), as desired, and (2) in the Definition holds.
3. \(w\) is continuous in \(\Omega\), so we only need to check that the restriction of \(w\) to \(\partial\Omega\) is positive. We see
\[
w(x,t)|_{\partial\Omega} = -c \frac{1}{|\log |t||^{\delta+1}} e^{-\frac{\beta t \log |t||}{|\log |t||^{\delta}}} + \frac{1}{|\log |t||^{\delta}} = 1 - c > 0.
\]

4. We see that
\[
\lim_{t \to 0^-} f(t) = \lim_{t \to 0^-} g(t) = 0.
\]
Since \(|x|^2 < -\beta t \log |\log |t|| \to 0\), we see
\[
e^{-\frac{|x|^2}{\beta t}} = \mathcal{O}(|\log |t||)
\]
as \(t \to 0^-\). Therefore
\[
\lim_{(x,t) \to (0,0^-)} w(x,t) = 0,
\]
and (4) in the Definition is satisfied.

Together, these points show that (5.8) is indeed a barrier at \((0,0)\), and hence the origin is a regular point for the domain (5.1). \(\blacksquare\)

**Remark 5.2.** Since \(\beta \to 4\) as \(p \to \infty\), we see that (5.1) converges to the Petrovsky criterion for the \(\infty\)-parabolic equation (1.3). Note also that the result is completely independent of the number of spatial variables \(n\).

We now turn to the proof that Theorem 1.2 is sharp; any constant greater than \(\beta\) in (5.1) will produce domain containing \(\Omega\) where the origin is irregular.

**Theorem.** The origin is not a regular point for the domain \(\Omega\) enclosed by the hypersurfaces
\[
\{(x,t) \in \mathbb{R}^n \times (-\infty,0) : |x|^2 = -\beta(1+\epsilon)t \log |\log |t||\}
\]
and \(\{t = -c\}\),
for any \(\epsilon > 0\).

**Proof.** The proof proceeds by constructing a domain \(\tilde{\Omega}\) contained in \(\Omega\), with the origin as common boundary point. We then show that \((0,0)\) is irregular for \(\tilde{\Omega}\), and Lemma 3.11 then implies that \((0,0)\) regarded as a boundary point of \(\Omega\) is irregular, too. We shall construct a smooth function \(w\) so that

1. \(w\) is subparabolic in \(\tilde{\Omega}\),
2. \(w\) is continuous on \(\bar{\Omega} \setminus \{(0,0)\}\),
3. The upper limit of \(w\) at interior points converging to \((0,0)\) is greater than its upper limit for the points converging to \((0,0)\) along the boundary.
To see why the existence of such a \( w \) implies that the origin is irregular, consider the boundary data \( f : \partial \tilde{\Omega} \to \mathbb{R} \) defined as follows. Let \( f = w \) near \((0,0)\), and set
\[
f(0, 0) = \lim_{\partial \tilde{\Omega} \ni (x,t) \to (0,0)} v(x,t).
\]
As we shall see, this limit exists. For the rest of the boundary, continuously extend \( f \) to a large constant \( b \).

If \( b \) is large enough, the comparison principle implies that every function \( u \in U_f \) which satisfies \( u \geq f \) on \( \partial \tilde{\Omega} \) also satisfies \( u \geq w \) in \( \tilde{\Omega} \) since \( w \) is a subsolution by (1). Taking the infimum over all such \( u \), we see
\[
\lim_{\partial \tilde{\Omega} \ni (y,s) \to (0,0)} \sup w(y,s) = f(0,0),
\]
and so \((0,0)\) is not a regular point for \( \tilde{\Omega} \).

Our function \( w \) will be on the form
\[
(5.11) \quad w(x,t) = f(t)e^{-|x|^2/(2\beta t)} + g(t),
\]
for suitable functions \( f \) and \( g \). Here \( k \in (\frac{1}{2}, 1) \) will be chosen later, and \(-1 < t < 0\). Indeed, we shall choose \( t \) to be very close to 0.

Calculating, we get
\[
w_t(x,t) = f'(t)e^{-|x|^2/(2\beta t)} + f(t)\frac{|x|^2k}{\beta t^2}e^{-|x|^2/(2\beta t)} + g'(t)
\]
and
\[
w_r = -f(t)\frac{2r k e^{-|x|^2/(2\beta t)}}{\beta t},
\]
\[
w_{rr} = f(t)e^{-|x|^2/(2\beta t)} \left( 4r^2 k^2 - \frac{2k}{\beta t} \right)
\]
Inserting this into (1.1), we get
\[
(5.12) \quad w_t - \mathcal{A}_p w = e^{-|x|^2/(2\beta t)} \left[ f'(t) + f(t)\frac{|x|^2(k^2 - k^2)}{\beta t^2} + f(t)\frac{\alpha k}{\beta t} \right] + g'(t),
\]
with \( \alpha \) as in (2.3). Choose
\[
f(t) = \frac{-1}{|\log |t||^{1+\epsilon_1}} \quad \text{and} \quad g(t) = \frac{1}{|\log |\log |t||^\gamma},
\]
where \( \epsilon_1 \) is a positive constant.

The case \( x \neq 0 \). We see that \( Dw = 0 \) precisely when \( x = 0 \). We show that (5.11) is a classical subsolution when \( x \neq 0 \).
Inserting derivatives into (5.12), we get

\[ w_t - \mathcal{A}_p w = e^{-|x|^2/\beta t} \left[ \frac{-(1 + \epsilon_1)}{t |\log |t||^{2+\epsilon_1}} - \frac{|x|^2(k-k^2)}{\beta t^2 |\log |t||^{1+\epsilon_1}} + \frac{\alpha k}{\beta t |\log |t||^{1+\epsilon_1}} + e^{\frac{|x|^2}{\beta t}} \cdot \frac{1}{t \cdot \log^2 |\log |t|| \cdot |\log |t||} \right]. \]

Multiplying by \( t |\log |t||^{1+\epsilon_1} \), we see that the sign of (5.13) coincides with the sign of

\[ -\frac{1 + \epsilon_1}{\log |t|} + \frac{|x|^2}{\beta t} (k - k^2) + \frac{\alpha k}{\beta} - e^{\frac{|x|^2}{\beta t}} \cdot \frac{|\log |t||^{\epsilon_1}}{\log^2 |\log |t||}. \]

We can choose \( |t| \) small enough that

\[ \left| \frac{1 + \epsilon_1}{\log |t|} \right| < \frac{\alpha k}{\beta}, \]

and then

\[ \frac{|x|^2}{\beta t} (k - k^2) + \frac{2\alpha k}{\beta} - e^{\frac{|x|^2}{\beta t}} \cdot \frac{|\log |t||^{\epsilon_1}}{\log^2 |\log |t||} < 0. \]

This inequality is satisfied if \( |x| \) is so small that

\[ \frac{2\alpha k}{\beta} < e^{\frac{|x|^2}{\beta t}} \cdot \frac{|\log |t||^{\epsilon_1}}{\log^2 |\log |t||}. \]

or if \( |x| \) so large that

\[ \frac{|x|^2(k - k^2)}{\beta |t|} > \frac{2\alpha k}{\beta}. \]

We argue that at least one of these inequalities must hold. Indeed, fix \( |t| \) so that

\[ \frac{\epsilon_1}{2} \log |\log |t|| > \frac{\alpha}{\beta}. \]

1. In the case (5.15), we take logarithms to get

\[ \log \frac{2\alpha k}{\beta} < \frac{|x|^2}{\beta t} + \epsilon_1 \log |\log |t|| - 2 \log \log |\log |t||, \]
or

\[
\frac{|x|^2}{\beta |t|} < \epsilon_1 \log |\log |t|| - 2 \log \log |\log |t|| - \log \frac{2\alpha k}{\beta} < \epsilon_1 \log |\log |t|| - \log k < \epsilon_1 \log |\log |t|| - \frac{\epsilon_1}{2} \log |\log |t|| = \frac{\epsilon_1}{2} \log |\log |t|| > 2
\]

for $|t|$ small enough. Hence (5.14) is satisfied.

2. On the other hand, if (5.16) holds, we calculate

\[
\frac{|x|^2}{\beta |t|} > \frac{2\alpha}{\beta(1-k)} > 4 \frac{\alpha}{\beta}.
\]

Since we chose $|t|$ according to (5.17), we have that at least one of the inequalities (5.15) or (5.16) is satisfied for any $x \neq 0$, and $w$ is a subsolution.

**The case $x = 0$.** Then $Dw = 0$, and according to Definition 3.1 we need to show that for every $\phi \in C^2(\Omega)$ so that $w - \phi$ has a maximum at $(0,t)$, we have

\[
(5.18) \quad \phi_t(0, t) \leq \frac{1}{p} \text{tr}(D^2\phi(0, t)) + \frac{p-2}{p} \Lambda(D^2\phi(0, t)).
\]

We show that $w$ itself satisfies this condition. An argument similar to the one in the proof of Theorem 1.2 then shows that $w$ is a viscosity subsolution.

Inserting the derivatives at $(0, t)$, we see that (5.18) reads

\[
f'(t) + g'(t) \leq -\frac{\alpha f(t)}{\beta t} k,
\]

or

\[
\frac{1 + \epsilon_1}{t \cdot |\log |t||^{2+\epsilon_1}} + \frac{1}{\log^2 |\log |t|| \cdot |\log |t|| \cdot t} \leq \frac{\alpha k}{\beta t \cdot |\log |t||^{1+\epsilon_1}}.
\]

This is the same as

\[
\frac{1 + \epsilon_1}{|\log |t||} - \frac{|\log |t||^{\epsilon_1}}{\log^2 |\log |t||} + \frac{\alpha k}{\beta} \leq 0,
\]

but this inequality is the same as the one in (5.14), and because of our choices of $|t|$ and $k$. This shows that the condition (5.18) holds, and $w$ is a subsolution even in this case.
Now we consider the level set \( w(x, t) = m, m < 0 \), and calculate

\[
w(x, t) = \frac{-1}{|\log |t||^{\epsilon_1 + 1}} e^{-\frac{|x|^2 k}{|t|}} + \frac{1}{\log |\log |t||} = m
\]

\[
\iff \frac{-1}{|\log |t||^{\epsilon_1 + 1}} e^{-\frac{|x|^2 k}{|t|}} = m - \frac{1}{\log |\log |t||}
\]

\[
\iff e^{-\frac{|x|^2 k}{|t|}} = |\log |t||^{\epsilon_1 + 1} \left( \frac{1}{\log |\log |t||} - m \right)
\]

\[
\iff -\frac{|x|^2 k}{\beta t} = (\epsilon_1 + 1) \log |\log |t|| + \log \left( \frac{1}{\log |\log |t||} - m \right),
\]

or simply

\[
(5.19) \quad x^2 = -\beta t \left( \frac{\epsilon_1 + 1}{k} \log |\log |t|| + \frac{1}{k} \log \left( \frac{1}{\log |\log |t||} - m \right) \right).
\]

Letting \( \tilde{\Omega} \) denote the domain enclosed by \( (5.19) \) and the hyperplane \( t = c < 0 \), we have that for \( m < 0 \), the function \( v \) \((5.11)\) is negative in \( \tilde{\Omega} \), and \( w(x, 0) = 0 \). This shows that the origin is an irregular boundary point for \( \tilde{\Omega} \).

The inclusion \( \tilde{\Omega} \subset \Omega \) requires that

\[
\frac{\epsilon_1 + 1}{k} \log |\log |t|| + \frac{1}{k} \log \left( \frac{1}{\log |\log |t||} - m \right) < (1 + \epsilon) \log |\log |t||
\]

for small \( |t| \). Fix \( k \) close to 1 and \( \alpha \) close to 0 so that

\[
\frac{\epsilon_1 + 1}{k} < 1 + \frac{\epsilon}{2}.
\]

Thus we have to verify that

\[
\left( \frac{1}{\log |\log |t||} + |m| \right)^{\frac{1}{2}} \leq |\log |t||^{\frac{1}{2}},
\]

but this obviously holds for small \( |t| \) since the left-hand side is bounded.

Hence \( \tilde{\Omega} \subset \Omega \) for \( \epsilon_1 \) and \( c \) close to 0, and \( k \) close to 1, \((0, 0)\) is an irregular boundary point for \( \Omega \) as well. \( \Box \)

**Acknowledgement**

The author would like to thank Jana Björn and Vesa Julin for discovering a flaw in the original proof of the Petrovsky criterion.
ON THE NORMALISED \( p \)-PARABOLIC EQUATION

REFERENCES

[Bar52] Grigory I. Barenblatt. On self-similar motions of compressible fluid in a porous medium. *Prikladnaya Matematika i Mekhanika (Applied Mathematics and Mechanics (PMM))*, 1952.

[BBP17] Anders Björn, Jana Björn, and Mikko Parviainen. The tusk condition and Petrovski criterion for the normalized \( p \)-parabolic equation. *arXiv preprint arXiv:1712.06807*, 2017.

[BG13] Agnid Banerjee and Nicola Garofalo. Gradient bounds and monotonicity of the energy for some nonlinear singular diffusion equations. *Indiana University Mathematics Journal*, pages 699–736, 2013.

[BG15] Agnid Banerjee and Nicola Garofalo. On the Dirichlet boundary value problem for the normalized \( p \)-Laplacian evolution. *Communications on Pure & Applied Analysis*, 14(1), 2015.

[BSA15] George Baravdish, Olof Svensson, and Freddie Aastrom. On backward \( p(x) \)-parabolic equations for image enhancement. *Numerical Functional Analysis and Optimization*, 36(2):147–168, 2015.

[CL83] Michael G. Crandall and Pierre-Louis Lions. Viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[CW03] Michael G. Crandall and Pei-Yong Wang. Another way to say caloric. *Journal of Evolution Equations*, 3(4):653–672, 2003.

[DiB95] Emmanuele DiBenedetto. *Partial Differential Equations*. Birkhäuser, 1995.

[Doe09] Kerstin Does. An evolution equation involving the normalized \( p \)-Laplacian. *Communications on Pure and Applied analysis*, 2009.

[ES+91] Lawrence C. Evans, Joel Spruck, et al. Motion of level sets by mean curvature. i. *Journal of Differential Geometry*, 33(3):635–681, 1991.

[JK06] Petri Juutinen and Bernd Kawohl. On the evolution governed by the Infinity Laplacian. *Mathematische Annalen*, 335(4):819–851, 2006.

[KL96] Tero Kilpeläinen and Peter Lindqvist. On the Dirichlet boundary value problem for a degenerate parabolic equation. *SIAM Journal on Mathematical Analysis*, 27(3):661–683, 1996.

[MPR10] Juan J. Manfredi, Mikko Parviainen, and Julio D. Rossi. An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM Journal on Mathematical Analysis*, 42(5):2058–2081, 2010.

[Per23] Oskar Perron. Eine neue Behandlung der ersten Randwertaufgabe für \( \Delta u = 0 \). *Mathematische Zeitschrift*, 18(1):42–54, 1923.

[Pet35] Ivan G. Petrovsky. Zur ersten Randwertaufgabe der Wärmeleitungsgleichung. *Compositio Math*, 1:383–419, 1935.

[PS+08] Yuval Peres, Scott Sheffield, et al. Tug-of-war with noise: A game-theoretic view of the \( p \)-Laplacian. *Duke Mathematical Journal*, 145(1):91–120, 2008.

[Ste29] Wolfgang Sternberg. Über die Gleichung der Wärmeleitung. *Mathematische Annalen*, 101(1):394–398, 1929.

[Ubo17] N. Ubostad. Boundary Regularity for the \( \infty \)-Heat Equation. *ArXiv e-prints*, September 2017.

[Wat12] Neil A. Watson. *Introduction to Heat Potential Theory*, volume 182 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2012.
