From condensed matter to QCD: a journey through gauge theories on board of a variational tool

Fabio Siringo

Dipartimento di Fisica e Astronomia dell’Università di Catania,
INFN Sezione di Catania, Via S.Sofia 64, I-95123 Catania, Italy

Abstract

Starting with a review of the thermal fluctuations in superconductors, the Gaussian Effective Potential is shown to be a powerful variational tool for the study of the breaking of symmetry in gauge theories. A novel re-derivation of the massive expansion for QCD is presented, showing its variational nature and its origin from the Gaussian potential that also provides a variational proof for chiral symmetry breaking and dynamical generation of a gluon mass.

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I. INTRODUCTION

Since 1873, when Lord Rayleigh described a variational method for calculating the frequencies of mechanical systems, the Rayleigh-Ritz method has become an important tool for the approximate solution of physical problems in quantum mechanics and quantum field theory. My personal experience on variational methods dates back to 1985, when I was a graduate student of Professor Renato Pucci’s. He proposed to put an hydrogen molecule inside a rigid box and evaluate the energy. His key idea was the insertion of a dielectric constant for simulating the effects of the other molecules as if it were in a very dense phase under high pressure. That very physical idea was successful and since then Renato Pucci has been contributing to the physics of solids under pressure with many model calculations based on remarkable physical ideas. Hydrogen was believed to become a superconductor in its solid phase under pressure and the fascinating Anderson-Higgs mechanism of gauge symmetry breaking was one of the milestones in Professor Pucci’s teaching. That is where my personal journey has begun, going from the scalar $U(1)$ gauge theory of superconductivity, through the $SU(2) \times U(1)$ theory of weak interactions, up to $SU(3)$ theory and QCD. Still collaborating with Renato Pucci in 2003, we found that a variational tool like the Gaussian Effective Potential (GEP) can describe the thermal fluctuations of a superconductor in its broken-symmetry phase. While the same variational tool had been very successful for describing the breaking of symmetry in a scalar theory, its potentiality in the study of gauge theories were not fully explored yet. The idea was then developed through several papers attempting to enlarge the gauge group, introduce fermions and eventually describe other mechanisms of symmetry breaking, like the chiral symmetry breaking of QCD where the gluon and quark masses emerge without any breaking of the gauge symmetry.

In this contribution, after reviewing the use of the GEP for the study of superconductivity, the massive expansion is re-derived from the GEP and shown to be a powerful variational tool for addressing the problem of mass generation in Yang-Mills theories and QCD, even when the gauge symmetry is not broken. While the massive expansion provides an analytical description of the propagators of QCD from first principles and is in remarkable agreement with the data of lattice simulations, its variational nature is hided and disguised to look like a perturbative method. The present
novel and alternative derivation of the massive expansion illustrate its direct origin from
the GEP. Moreover, in chiral QCD the GEP provides a variational proof of chiral symmetry
breaking and dynamical generation of the gluon mass.

II. GEP AND SUPERCONDUCTIVITY

The standard Ginzburg-Landau (GL) effective Lagrangian still provides the best frame-
work for a general description of the phenomenology of $U(1)$ gauge symmetry breaking and
superconductivity (the Anderson-Higgs mechanism). The GL action can be seen as a power
expansion of the exact action around the critical point and is recovered by any microscopic
theory around the transition. The Gaussian fluctuations can be studied by the GEP for
$U(1)$ scalar electrodynamics in three space dimensions where it represents the standard
static GL effective model of superconductivity.

At variance with the approach of Ibanez-Meier et al. who computed the GEP by use
of a general covariant gauge, we work in unitarity gauge, in order to make the physical
content of the theory more evident. It has been shown to be formally equivalent to a full
gauge-invariant method once all the gauge degrees of freedom have been integrated out.
The variational method provides a way to evaluate both the correlation length $\xi$ and the
penetration depth $\lambda$ as a solution of coupled equations. The GL parameter $\kappa_{GL} = \lambda/\xi$ is
found to be temperature dependent in contrast to the simple mean-field description and its
behaviour turns out to be in perfect agreement with many experimental data.

Let us consider the standard static GL action

$$S = \int d^3x \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^\ast (D^\mu \phi) + \frac{1}{2} m_B^2 \phi^\ast \phi + \lambda_B (\phi^\ast \phi)^2 \right]. \quad (1)$$

where $\phi$ is a complex (charged) scalar field, its covariant derivative is defined according to

$$D_\mu \phi = \partial_\mu + i e_B A_\mu \quad (2)$$

and $\mu, \nu = 1, 2, 3$ run over the three space dimensions. The magnetic field components are
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We may assume a transverse gauge $\nabla \cdot A = 0$, and then switch to
unitarity gauge in order to make $\phi$ real.

By a pure variational argument the longitudinal gauge field can be integrated out
yielding the effective action

\[ S = \int d^3x \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{m_B^2}{2} \phi^2 + \lambda_B \phi^4 + \frac{e^2_B \phi^2 A^2}{2} + \frac{1}{2} (\nabla \times A)^2 + \frac{(\nabla \cdot A)^2}{2\epsilon} \right]. \]  

(3)

The partition function is expressed as a functional integral over the real scalar field \( \phi \) and the generic three-dimensional vector field \( A \), with the extra prescription that the parameter \( \epsilon \) is set to zero at the end of the calculation. As usual, the free energy (effective potential) follows by inserting a source term and by a Legendre transformation [3, 4].

The GEP may be evaluated by the \( \delta \) expansion method [24, 27] and is a variational estimate of the exact free energy. We introduce a shifted field

\[ \tilde{\phi} = \phi - \varphi \]  

(4)

then we split the Lagrangian into two parts

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \]  

(5)

where \( \mathcal{L}_0 \) is the sum of two free-field terms describing a vector field \( A_\mu \) with mass \( \Delta \) and a real scalar field \( \tilde{\phi} \) with mass \( \Omega \):

\[ \mathcal{L}_0 = \left[ + \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} \Delta^2 A_\mu A^\mu + \frac{(\nabla \cdot A)^2}{2\epsilon} \right] + \left[ \frac{1}{2} (\nabla \tilde{\phi})^2 + \frac{1}{2} \Omega^2 \tilde{\phi}^2 \right]. \]  

(6)

The interaction then reads

\[ \mathcal{L}_{int} = v_0 + v_1 \tilde{\phi} + v_2 \tilde{\phi}^2 + v_3 \tilde{\phi}^3 + v_4 \tilde{\phi}^4 + \frac{1}{2} \left( e^2_B \varphi^2 - \Delta^2 \right) A_\mu A^\mu + e^2_B \varphi A_\mu A^\mu \tilde{\phi} + \frac{1}{2} e^2_B A_\mu A^\mu \tilde{\phi}^2 \]  

(7)

where

\[
\begin{align*}
    v_0 &= \frac{1}{2} m_B^2 \varphi^2 + \lambda_B \varphi^4, & v_1 &= m_B^2 \varphi + 4 \lambda_B \varphi^3, \\
    v_2 &= \frac{1}{2} m_B^2 + 6 \lambda_B \varphi^2 - \frac{1}{2} \Omega^2, & v_3 &= 4 \lambda_B \varphi, & v_4 &= \lambda_B.
\end{align*}
\]  

(8)

The non-conventional splitting of the Lagrangian has two important effects: arbitrary mass parameters are inserted in the free part; mass counterterms are inserted in the interaction in order to leave the Lagrangian unmodified. Then the standard perturbation theory is used for determining the first-order effective potential. The sum of vacuum graphs up to
first order yields the free-energy density

\[ V_{\text{eff}}[\varphi] = I_1(\Omega) + 2I_1(\Delta) + \]
\[ + \left[ \lambda_B \varphi^4 + \frac{1}{2} m_B^2 \varphi^2 + \frac{1}{2} \{ m_B^2 - \Omega^2 + 12\lambda_B \varphi^2 + 6\lambda_B I_0(\Omega) \} I_0(\Omega) \right] \]
\[ + (e_B^2 \varphi^2 + e_B^2 I_0(\Omega) - \Delta^2) I_0(\Omega) \]  
(9)

where the divergent integrals \( I_n \) are defined according to

\[ I_0(M) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{M^2 + k^2}, \quad I_1(M) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln(M^2 + k^2) \]  
(10)

and must be regularized somehow.

The free energy (9) now depends on the mass parameters \( \Omega \) and \( \Delta \). Since none of them was present in the original GL action of Eq.(3), the free energy should not depend on them, and the minimum sensitivity method\[28] can be adopted in order to fix the masses: the free energy is required to be stationary for variations of \( \Omega \) and \( \Delta \). On the other hand the stationary point can be shown to be a minimum for the free energy and the method is equivalent to a pure variational method\[24]. At the stationary point the masses give the inverse correlation lengths for the fields, the so called coherence length \( \xi = 1/\Omega \) and penetration depth \( \lambda = 1/\Delta \).

The stationary conditions

\[ \frac{\partial V_{\text{eff}}}{\partial \Omega^2} = 0, \quad \frac{\partial V_{\text{eff}}}{\partial \Delta^2} = 0 \]  
(11)

give two coupled gap equations:

\[ \Omega^2 = 12\lambda_B I_0(\Omega) + m_B^2 + 12\lambda_B \varphi^2 + 2e_B^2 I_0(\Delta) \]  
(12)
\[ \Delta^2 = e_B^2 \varphi^2 + e_B^2 I_0(\Omega). \]  
(13)

For any value of \( \varphi \), the equations must be solved numerically and the minimum-point values \( \Omega \) and \( \Delta \) must be inserted back into Eq.(9) in order to get the Gaussian free energy \( V_{\text{eff}}(\varphi) \) as a function of the order parameter \( \varphi \). For a negative and small enough \( m_B^2 \), we find that \( V_{\text{eff}} \) has a minimum at a non zero value of \( \varphi = \varphi_{\text{min}} > 0 \), thus indicating that the system is in the broken-symmetry superconducting phase. Of course the masses \( \Omega, \Delta \) only take their physical value at the minimum of the free energy \( \varphi_{\text{min}} \). That point may be found by requiring that

\[ \frac{\partial V_{\text{eff}}}{\partial \varphi^2} = 0 \]  
(14)
where as usual the partial derivative is allowed as far as the gap equations (12), (13) are satisfied\cite{17}. The condition (14) combined with the gap equation (12) yields the very simple result
\[
\varphi_{\text{min}}^2 = \frac{\Omega^2}{8\lambda_B}.
\] (15)
However, we notice that here the mass $\Omega$ must be found by solution of the coupled gap equations. Thus Eqs. (15), (12) and (13) must be regarded as a set of coupled equations and must be solved together in order to find the physical values for the correlation lengths and the order parameter.

Insertion of Eq. (15) into Eq. (13) yields a simple relation for the GL parameter $\kappa_{\text{GL}}$
\[
\kappa_{\text{GL}}^2 = \left(\frac{\lambda}{\xi}\right)^2 = \kappa_0 \frac{1}{1 + \frac{I_0(\Omega)}{\varphi_{\text{min}}^2}}
\] (16)
where $\kappa_0 = e_B^2/(8\lambda_B)$ is the mean-field GL parameter which does not depend on temperature. Eq. (16) shows that the GL parameter is predicted to be temperature dependent through the non trivial dependence of $\Omega$ and $\varphi_{\text{min}}$. At low temperature, where the order parameter $\varphi_{\text{min}}$ is large, the deviation from the mean-field value $\kappa_0$ is negligible. Conversely, close to the critical point, where the order parameter is vanishing, the correction factor in Eq. (16) becomes very important\cite{3, 4}.

It is instructive to look at the effective potential in the limit $\varphi \to 0$ of the unbroken-symmetry phase and at the *chiral* point $m_B = 0$ where the original Lagrangian is scaleless. In that limit Eq. (9) reads
\[
V_{\text{eff}}[0] = [I_1(\Omega) + 2I_1(\Delta)] - \frac{1}{2} \left[\Omega^2 I_0(\Omega) + 2\Delta^2 I_0(\Delta)\right]
+ 3\lambda_B [I_0(\Omega)]^2 + e_B^2 I_0(\Omega) I_0(\Delta)
\] (17)
and is a function of the mass parameters. Its minimum might fall at a finite set of masses $\Delta_0$, $\Omega_0$ yielding a generation of mass from a scaleless Lagrangian. That property turns out to be useful for addressing the problem of mass generation in chiral QCD where the gauge symmetry is not broken. Moreover, we observe that all the terms in Eq. (17) arise from the sum of the vacuum graphs up to first order, as shown in Fig. 1, where the internal lines are the massive propagators that can be read from the free-particle Lagrangian $\mathcal{L}_0$ of Eq. (6). We obtain a *massive* expansion, with massive free-particle propagators in the loops, from a massless Lagrangian.
III. THE GAUSSIAN EFFECTIVE POTENTIAL REVISITED

The massive expansion can be seen as an expansion around the vacuum of massive particles. The search for the best vacuum is the aim of the GEP that has been studied by several authors, mainly in the context of spontaneous symmetry breaking and scalar theories. While the GEP is a genuine variational method, several extensions to higher orders have been proposed. However, being a first-order approximation, the GEP fails to predict any useful result for the fermions of the standard model, because of the minimal gauge interaction that requires a second order graph at least \cite{12, 19}. Even the idea of an expansion around the optimized vacuum of the GEP is not new \cite{29} but has not been developed further.

In the next section, the pure variational nature of the GEP is used as a tool for demonstrating that the standard vacuum of QCD is unstable towards the vacuum of massive gluons and quarks. Expanding around the optimized vacuum we recover the massive expansion that has been recently developed for pure Yang-Mills theory \cite{16, 20, 21}. Thus, the unconventional massive perturbative expansion can be seen to emerge from the GEP formalism in a natural way.

One of the important merits of the GEP is its paradox of being a pure variational method disguised as a perturbative calculation, making use of the standard graphs of perturbation theory. In this section we set the formalism of the expansion, starting with the simple scalar theory and then moving towards Yang-Mills theory and QCD in the next section.

Let us revise briefly the method for the simple case of a self-interacting scalar theory \cite{17} where the effective potential is given by three vacuum graphs as shown in Fig. 1 (to be

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Vacuum graphs contributing to the GEP for different theories}
\end{figure}
compared with the six graphs of the scalar electrodynamics of Sec II. The Lagrangian density reads

\[ \mathcal{L} = \frac{1}{2} \phi \left( -\partial^2 - m_B^2 \right) \phi - \frac{\lambda}{4!} \phi^4 \]  

(18)

where \( m_B \) is a bare mass. We can then split the total Lagrangian as \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \) where the trial free part is

\[ \mathcal{L}_0 = \frac{1}{2} \phi \left( -\partial^2 - m^2 \right) \phi \]  

(19)

and describes a free scalar particle with a trial mass \( m \neq m_B \). The interaction follows as

\[ \mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4 - \frac{1}{2} \left( m_B^2 - m^2 \right) \phi^2 \]  

(20)

so that the total Lagrangian has not been changed. If we neglect the interaction, then a free Hamiltonian \( \mathcal{H}_0 \) is derived from \( \mathcal{L}_0 \) and its ground state \( \left| m \right\rangle \) satisfies

\[ \mathcal{H}_0 \left| m \right\rangle = E_0(m) \left| m \right\rangle \]  

(21)

and depends on the trial mass \( m \). Restoring the interaction \( \mathcal{L}_{int} \), the full Hamiltonian reads \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int} \) and by standard perturbation theory, the first-order energy of the ground state reads

\[ E_1(m) = E_0(m) + \langle m | \mathcal{H}_{int} | m \rangle = \langle m | \mathcal{H} | m \rangle \]  

(22)

and is equivalent to the first-order effective potential \( V_1(m) \) evaluated by standard perturbation theory with the interaction \( \mathcal{L}_{int} \). Thus, the stationary condition

\[ \frac{\partial V_1(m)}{\partial m} = \frac{\partial E_1(m)}{\partial m} = 0 \]  

(23)

gives the best value of \( m \) that minimizes the vacuum energy of the ground state \( \left| m \right\rangle \). While being a pure variational method, the first-order effective potential \( V_1(m) = E_1(m) \) can be evaluated by the sum of all the vacuum graphs up to first order (the three loop graphs in Fig. 1). The resulting optimized effective potential is the GEP. Usually, the effective potential is evaluated for any value of the average \( \varphi = \langle \phi \rangle \) and the best \( m \) also depends on that average. If the symmetry is not broken, then the minimum of the effective potential is at \( \varphi = 0 \) where \( V_1(m) \) is a function of the trial mass, to be fixed by the stationary condition Eq.(23). We assume that the gauge symmetry is not broken in QCD so that \( V_1(m) \) at \( \varphi = 0 \) is the effective potential we are interested in. The variational nature of the method ensures that the true vacuum energy is smaller than the minimum of \( V_1(m) \). At the minimum, \( \left| m \right\rangle \)
provides an approximation for the vacuum and is given by the vacuum of a free massive scalar particle with mass equal to the optimized mass parameter $m \neq m_B$. Of course, the optimal state $|m\rangle$ is just a first approximation and the actual vacuum is much richer. However, we expect that a perturbative expansion around that approximate vacuum would be the best choice for the Lagrangian $\mathcal{L}$, prompting towards an expansion with an interaction $\mathcal{L}_{\text{int}}$ and a free part $\mathcal{L}_0$ that depend on $m$ and can be optimized by a clever choice of the parameter $m$. Different strategies have been proposed for the optimization, ranging from the stationary condition of the GEP, Eq.(23), to Stevenson’s principle of minimal sensitivity\cite{28}. A method based on the minimal variance has been recently proposed for QCD and other gauge theories\cite{10–14}. In all those approaches, the underlying idea is that an optimal choice of $m$ could minimize the effect of higher orders in the expansion. Since the total Lagrangian does not depend on $m$, the physical observables are expected to be stationary at the optimal $m$, thus suggesting the use of stationary conditions for determining the free parameter. As a matter of fact, if all graphs were summed up exactly, then the dependence on $m$ would cancel in the final result, so that the strength of that dependence measures the weight of the neglected graphs at any order.

Leaving aside the problem of the best choice of $m$, we observe that at $\phi = 0$ the calculation of the first-order effective potential $V_1(m)$ is quite straightforward and follows from the first-order expansion of the effective action $\Gamma(\phi)$

$$e^{\Gamma(\phi)} = \int_{1PI} \mathcal{D}\phi e^{iS_0(\phi+\bar{\phi})+iS_{\text{int}}(\phi+\bar{\phi})}$$

(24)

where the functional integral is the sum of all one-particle irreducible (1PI) graphs and $S = S_0 + S_{\text{int}}$ is the action. The effective potential then follows as $V_1(m) = -\Gamma(0)/\mathcal{V}_4$ where $\mathcal{V}_4$ is a total space-time volume. Moreover, being interested in the chiral limit, let set $m_B = 0$ in the interaction Eq.(20) and study a massless scalar theory.

The vertices of the theory can be read from $\mathcal{L}_{\text{int}}$ in Eq.(20) where we set $m_B = 0$ and are used in Fig.1 in the vacuum graphs up to first order. The usual four-point vertex $-i\lambda$ is accompanied by the counterterm $im^2$ that is denoted by a cross in the graphs. This counterterm must be regarded as part of the interaction so that the expansion is not loopwise and we find one-loop and two-loop graphs summed together in the first-order effective potential. That is where the non-perturbative nature of the method emerges since the expansion is not in powers of $\lambda$ but of the whole interaction $\mathcal{L}_{\text{int}}$. The zeroth order
(massive) propagator $i\Delta_m$ follows from $\mathcal{L}_0$

$$i\Delta_m(p) = \frac{i}{p^2 - m^2}$$

and is shown as a straight line in the vacuum graphs.

The tree term is the classical potential and vanishes in the limit $\varphi \to 0$. The first one-loop graph in Fig.1 gives the standard one-loop effective potential, containing some effects of quantum fluctuations. It must be added to the second one-loop graph in Fig.1, the crossed graph containing one insertion of the counterterm. It is instructive to see that the exact sum of all one-loop graphs with $n$ insertions of the counterterm gives the standard vacuum energy of a massless particle. In other words, if we sum all the crossed one-loop graphs the dependence on $m$ disappears and we are left with the standard one-loop effective potential of Weinberg and Coleman\cite{30}$\ V^0_{1L} = -\Gamma^0_{1L}/\mathcal{V}_4$ where $\Gamma^0_{1L}$ is the standard one-loop effective action at $\varphi = 0$

$$e^{i\Gamma^0_{1L}} = \int \mathcal{D}\phi e^{i\frac{1}{2}\int (-\partial^2)\phi + \int d^4x} \sim \left[\text{Det}(\Delta^{-1}_0)\right]^{-\frac{1}{2}}$$

and $\Delta^{-1}_0 = p^2$ is the free-particle propagator of a massless scalar particle.

Up to an additive constant, not depending on $m$, Eq.(26) can be written as

$$V^0_{1L} = \frac{-i}{2\mathcal{V}_4} \text{Tr} \log(\Delta^{-1}_m + m^2)$$

then expanding the log we obtain a massive expansion

$$V^0_{1L} = \frac{-i}{2\mathcal{V}_4} \text{Tr} \left\{ \log(\Delta^{-1}_m) + m^2\Delta_m - \frac{1}{2}m^2\Delta_m m^2\Delta_m + \cdots \right\}$$

that is shown pictorially in Fig.2 as a sum of crossed one-loop vacuum graphs. While the sum cannot depend on $m$, if we truncate the expansion at any finite order we obtain a function of the mass parameter. As a test of consistency, one can easily check that, once renormalized, the sum of all the crossed one-loop vacuum graphs in Fig.2 gives zero exactly.

The calculation of the GEP requires the sum of only the first two terms of Eq.(28), the two one-loop graphs in Fig.1. We cannot add higher-order terms without spoiling the variational method since the average value of the Hamiltonian in the trial state $\ket{m}$ is $E_1(m) = V_1(m)$, according to Eq.(22). Using the identity

$$\Delta_m = -\frac{\partial}{\partial m^2} \log(\Delta^{-1}_m)$$

\[29\]
the sum of one-loop graphs in Fig.1 can be written as

\[ V_{1L}(m) = \left(1 - \frac{\partial}{\partial m^2}\right) I_1(m) = I_1(m) - \frac{1}{2} m^2 I_0(m) \]  

(30)

where the diverging integrals \( I_1(m), I_0(m) \) generalize Eq.(10) and are defined as

\[ I_1(m) = \frac{1}{2i} \int \frac{d^4p}{(2\pi)^4} \log(-p^2 + m^2), \quad I_0(m) = -i \int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2 + m^2} \]  

(31)

so that

\[ \frac{\partial I_1(m)}{\partial m^2} = \frac{1}{2} I_0(m). \]  

(32)

We recognize \( I_1(m) \) as the standard one-loop effective potential of Weinberg and Coleman for a massive scalar particle in the limit \( \varphi \to 0 \). This term contains the quantum fluctuations at one-loop. The second term is a correction coming from the counterterm and arises because the exact Lagrangian was massless. The calculation of the GEP also requires the two-loop graph in Fig.1 that is first-order in \( \lambda \). A lazy way to evaluate it is by substituting the vertex \( im^2 \) in the crossed one-loop graph with the seagull one-loop self energy graph \(-i\Sigma_{1L}\) that reads

\[ \Sigma_{1L} = \frac{\lambda}{2} I_0(m) \]  

(33)

and adding a 1/2 symmetry factor. The resulting two-loop term is

\[ V_{2L}(m) = \frac{\lambda}{8} [I_0(m)]^2. \]  

(34)

The GEP follows as the sum \( V_{1L} + V_{2L} \)

\[ V_{\text{GEP}}(m) = I_1(m) - \frac{1}{2} m^2 I_0(m) + \frac{\lambda}{8} [I_0(m)]^2. \]  

(35)

At this stage we just recovered the GEP in the limit \( \varphi \to 0 \) and Eq.(35) agrees with the well known GEP in that limit [12, 17, 27, 31] (also compare to Eq.(17) by setting \( 4!\lambda_B = \lambda, \Omega = m \) and neglecting gauge field loops).

\[ \circ + \square + \square + \times + \times + \times + \times \]  

Figure 2: Pictorial display of the right hand side of Eq.(28).
More precisely, \( V_{GEP} \) is the GEP when \( m \) is optimized by the stationary condition Eq. (23) that reads

\[
\frac{\partial V_{GEP}(m)}{\partial m^2} = \frac{1}{2} \left( \frac{\partial I_0(m)}{\partial m^2} \right) \left[ \frac{\lambda I_0(m)}{2} - m^2 \right] = 0
\]

(36)
yielding the usual gap equation of the GEP

\[
m^2 = \frac{\lambda I_0(m)}{2}.
\]

(37)

From a mere formal point of view, the GEP predicts the existence of a mass for the massless scalar theory. That is of special interest because for \( m_B = 0 \) the Lagrangian in Eq. (18) has no energy scale, just like Yang-Mills theory and QCD in the chiral limit. Thus, it can be regarded as a toy model for the more general problem of mass generation and chiral symmetry breaking.

Actually, the integrals \( I_0, I_1 \) are badly diverging, and a mass scale arises from the regulator that must be inserted in order to get a meaningful theory. We can see that, in dimensional regularization, by setting \( d = 4 - \epsilon \), the integral \( I_0 \) is

\[
I_0(m) = -\frac{m^2}{16\pi^2} \left[ \frac{2}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} + 1 + O(\epsilon) \right]
\]

(38)
where \( \bar{\mu} = (2\sqrt{\pi}\mu) \exp(-\gamma/2) \) is an arbitrary scale. Integrating Eq. (32) and neglecting an integration constant (that does not depend on \( m \))

\[
I_1(m) = -\frac{m^4}{64\pi^2} \left[ \frac{2}{\epsilon} + \log \frac{\bar{\mu}^2}{m^2} + \frac{3}{2} + O(\epsilon) \right].
\]

(39)
If we follow the usual approach of Weinberg and Coleman [30], the divergences must be absorbed by the physical renormalized parameters. Thus, let us define a physical renormalized energy scale \( \Lambda \) as

\[
\log \Lambda^2 = \log \bar{\mu}^2 + \frac{2}{\epsilon} + 1
\]

(40)
and write the integrals \( I_1, I_0 \) as simply as

\[
I_0(m) = \frac{m^2}{16\pi^2} \log \frac{m^2}{\Lambda^2}
\]

\[
I_1(m) = \frac{m^4}{64\pi^2} \left[ \log \frac{m^2}{\Lambda^2} - \frac{1}{2} \right].
\]

(41)
This approach is the same that is usually followed in lattice simulations of QCD: the lattice provides a scale that can be changed without affecting the physical scale which remains fixed at a phenomenological value. We assume that when \( \epsilon \rightarrow 0 \) the scale \( \bar{\mu} \) also changes, keeping
Λ fixed at a physical value which cannot be predicted by the theory, but must come from the phenomenology.

First of all, we observe that by our renormalization scheme the standard one-loop effective potential is recovered, since that is equal to $I_1(m)$ in Eqs.[II] and can be recognized as the mass-dependent term of the standard one-loop effective potential in the limit $\varphi \to 0$. That term has a relative maximum at $m = 0$, is negative for $m < \Lambda \exp(1/4)$ and has the absolute minimum at $m = \Lambda$. Thus, the one-loop effective potential would predict a massive vacuum if the symmetry were not broken and the physical vacuum were at $\varphi = 0$.

The full renormalized GEP is finite in terms of the physical scale $\Lambda$ and can be written as

$$V_{GEP}(m) = \frac{\Lambda^4}{128\pi^2} U(\alpha, m^2/\Lambda^2)$$

(42)

where the adimensional potential $U(\alpha, x)$ is

$$U(\alpha, x) = x^2 \left[ \alpha(\log x)^2 - 2 \log x - 1 \right]$$

(43)

and $\alpha$ is the effective coupling $\alpha = \lambda/(16\pi^2)$.

The behavior of the potential $U(\alpha, x)$ is shown in Fig.3. For any coupling $\alpha$ the point $x = 0$ is a relative minimum while the potential has a relative maximum at $x = 1/e$. The

![Figure 3: The adimensional potential $U(\alpha, x)$ of Eq.(43) is shown for different values of the effective coupling $\alpha$.](image)
absolute minimum is at \( x_0 = \exp(2/\alpha) \) where \( U(\alpha, x_0) = -x_0^2 < 0 \). The two stationary points \( x = 1/e \) and \( x = x_0 \) are the points where the first or second factor in Eq.(36) is zero, respectively. Thus the absolute minimum \( m^2/\Lambda^2 = x_0 = \exp(2/\alpha) \) is the solution of the gap equation, Eq.(37). However, since the original theory has no scale, the quantitative value of \( m \) remains arbitrary as it depends on the unknown scale \( \Lambda \). We can only predict that, since the GEP provides a genuine variational approximation for the vacuum energy, the massless vacuum must be unstable towards the vacuum of a massive scalar particle with an exact effective potential \( V_{\text{exact}}(m) \leq V_{\text{GEP}}(m) < 0 \).

In the chiral limit, the GEP can be easily extended to more complex theories by just adding up the graphs of Fig.11. It is instructive to see how Eq.(17) can be recovered by the graphs for scalar \( U(1) \) electrodynamics. In the next section the GEP is evaluated for the chiral limit of QCD.

**IV. QCD IN THE CHIRAL LIMIT**

The full Lagrangian of QCD, including \( N_f \) massless chiral Quarks, can be written as

\[
\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{YM}} + \sum_{i=1}^{N_f} \bar{\Psi}_i \left[ i\partial - g A_a \hat{T}_a \right] \Psi_i
\]  

(44)

where \( \mathcal{L}_{\text{YM}} \) is the full \( SU(N) \) Yang-Mills Lagrangian, including a covariant gauge-fixing term and the ghost terms arising from the Faddeev-Popov determinant. The generators of \( SU(N) \) satisfy the algebra

\[
\left[ \hat{T}_a, \hat{T}_b \right] = i f_{abc} \hat{T}_c
\]  

(45)

with the structure constants normalized according to

\[
f_{abc} f_{dbc} = N \delta_{ad}.
\]  

(46)

In a background field \( \tilde{A}_a^\mu \) the effective action \( \Gamma(\tilde{A}) \) is the sum of 1PI graphs that can be formally given by the functional integral

\[
e^{i\Gamma(\tilde{A})} = \int_{1PI} \mathcal{D}_{\Psi, A, \omega} e^{iS[A + \tilde{A}, \Psi, \omega]} 
\]  

(47)

where \( \omega_a \) are the ghost fields. Assuming that the gauge symmetry is not broken, we are interested in the study of the limit \( \tilde{A} \to 0 \) and write the effective action as

\[
e^{i\Gamma} = \int_{1PI} \mathcal{D}_{A, \omega} e^{iS_{\text{YM}}[A, \omega] + i\Gamma[\Psi][A]} 
\]  

(48)
where $S_{YM}$ is the action of pure Yang-Mills theory and the effective action $\Gamma_\Psi$ is given by a functional integral over quark fields

$$e^{i\Gamma_\Psi[A]} = \int D\psi e^{i \int \bar{\psi} \hat{D}(A) \psi \, dx} \tag{49}$$

with the operator $\hat{D}(A)$ that is given by

$$\hat{D}(A) = i \partial - g A_\mu \hat{T}_a. \tag{50}$$

The quark fields can be integrated exactly, yielding, up to a constant,

$$i \Gamma_\Psi(A) = \log \det \hat{D}(A). \tag{51}$$

While $S_{YM}$ contains the vertices of pure Yang-Mills theory, the expansion of $\Gamma_\Psi$ in powers of $g A_\mu$ provides the standard insertions of quark-gluon vertices, yielding the usual Feynman rules of QCD. Some vacuum graphs, up to second order and two loops, are shown in Fig. 1.

As already noticed for the scalar theory, the calculation of the GEP requires the first-order effective potential that results from the sum of connected vacuum graphs up to first-order. Thus we may focus on the one-loop graphs in Fig. 1 and on the only first-order two-loop graph (the fourth for $SU(N)$ in Fig. 1). All other graphs are second order at least, starting from the other two-loop graphs of Fig. 1. Thus, at first order, the effective potential $V_1$ is just the sum of independent ghost, gluon and quark terms. This is an important limit of the GEP that cannot take in due account the second-order graphs, leaving us with a decoupled description of quarks, gluons and ghosts. We can write the first-order effective potential as

$$V_1 = V_{YM} + V_\Psi \tag{52}$$

where the quark term contains only the one-loop zeroth-order vacuum graph that arises from Eq. (51) at $g = 0$

$$V_\Psi = \frac{i}{V_4} \log \det \hat{D}_0 \tag{53}$$

having defined the zeroth-order operator $\hat{D}_0 = i \partial$. The Yang-Mills term $V_{YM}$ is the first-order effective potential of pure Yang-Mills theory and can be written as

$$V_{YM} = \frac{i}{V_4} \log \int_{1st-order} D_{A,\omega} e^{i S_{YM}[A,\omega]} \tag{54}$$

and is given by the one-loop ghost graph plus the one-loop and two-loop gluon graphs in Fig. 1.
At this stage, the whole calculation might seem to give trivial constant terms. However, we are interested in the change of these terms when a massive zeroth order propagator is taken from the beginning for gluons and quarks. As already seen for the scalar theory, we have the freedom of adding a mass term in the zeroth order Lagrangian provided that we subtract the same mass term in the interaction. The resulting massive expansion contains new two-point vertices (the mass counterterms) and their insertion in a graph does not change the number of loops but increases the order of the graph. Moreover, the first-order vacuum graphs in Fig. 1 remain uncoupled when any number of counterterms is inserted, so that we can study the change induced by the masses on $V_\Psi$ and $V_{YM}$ separately. It is instructive to see how the massive expansion\cite{16,21} of Yang-Mills theory emerges naturally in the calculation of the GEP and can be extended to chiral quarks.

A. Pure Yang-Mills Theory

In a generic linear covariant $\xi$-gauge, the first-order effective potential $V_{YM}$ can be written as the sum of the second and fourth graph in Fig. 1 namely the zeroth order gluon loop and the first-order two-loop graph which contains one insertion of the four-gluon vertex. We may drop the decoupled ghost loop that only gives an additive constant to the effective potential.

By the same notation of Sec.III, we denote by $V_{1L}^0$ the one-loop graph that gives the standard one-loop effective potential in the limit of a vanishing background field

$$V_{1L}^0 = \frac{i}{V_4} \log \int D_A e^{i \int A_\mu \Delta_0^{-1} A_\nu d^4 x}$$

containing the quadratic part of $S_{YM}$ in Eq.(54) written in terms of the gluon propagator

$$\Delta_0^{\mu \nu}(p) = \Delta_0^T(p) t^{\mu \nu}(p) + \Delta_0^L(p) l^{\mu \nu}(p)$$

where $t^{\mu \nu}$, $l^{\mu \nu}$ are the transversal and longitudinal Lorentz projectors

$$t^{\mu \nu}(p) = g^{\mu \nu} - \frac{p^\mu p^\nu}{p^2}, \quad l^{\mu \nu}(p) = \frac{p^\mu p^\nu}{p^2}$$

and the corresponding free-particle scalar functions are

$$\Delta_0^T(p) = \frac{1}{-p^2}, \quad \Delta_0^L(p) = \frac{\xi}{-p^2}.$$
The determinant of \( \Delta_0 \) can then be written as a product of determinants in the orthogonal Lorentz subspaces, \( \text{Det} \Delta_0 = (\text{Det} \Delta_0^T)(\text{Det} \Delta_0^L) \), yielding

\[
V_{1L}^0 = \frac{i}{2V_4} \left[ \text{Tr} \log \Delta_0^T + \text{Tr} \log \Delta_0^L \right].
\] (59)

From now on, we work in the Landau gauge and take the limit \( \xi \to 0 \). In that limit \( \Delta_0^L \to 0 \) and the longitudinal part gives an (infinite) additive constant that we drop. The relevant part we will focus on reads

\[
V_{1L}^0 = \frac{N_A}{2i} \int \frac{d^4p}{(2\pi)^4} \log \left( \Delta_0^{T-1} \right)
\] (60)

where \( N_A \) is a factor arising from the trace over color and Lorentz indices.

Following the same steps that lead to the GEP for a scalar theory, we may modify the quadratic part of the Lagrangian, i.e. \( \Delta_0^{-1} \) in \( S_{YM} \), provided that we add a counterterm to the total Lagrangian in order to leave it unchanged. Thus we add a mass term to the transversal part \( \Delta_0^T \), leaving the longitudinal part unmodified. That would be a reasonable choice in any gauge since the longitudinal part \( \Delta_0^L \) is left unmodified by the interaction at any order of perturbation theory. We define a new massive zeroth-order propagator \( \Delta_m^T \) as

\[
\Delta_m^{T-1} = \Delta_0^{T-1} + m^2 = -p^2 + m^2
\] (61)

and insert the counterterm

\[
\delta L_c = m^2 t^{\mu\nu} A_{\mu A} A_{\nu A}
\] (62)

in the Lagrangian density. Then we look at the change of the first-order effective potential as a function of the mass parameter \( m \), including the counterterm as a vertex of the theory. The result is formally equivalent to that obtained for the massless scalar theory in Eq.(28) and Fig.2. By insertion of the counterterm, the one-loop gluon loop gives rise to an infinite sum of crossed loops where the straight line in Fig.2 is now given by the massive propagator of Eq.(61) and the crosses denote the insertion of a two-point vertex \( -im^2 t^{\mu\nu} \). Even in a generic \( \xi \)-gauge the longitudinal part of the gluon propagator would not add any higher order contribution because of the transversal projector in the counterterm. Since everything is transversal in the Landau gauge, from now on we drop any projector \( t^{\mu\nu} \) and the superscript \( T \) in the transverse propagator. Writing \( \log(\Delta_0^{-1}) = \log(\Delta_m^{-1} - m^2) \) in Eq.(60) and expanding the log, the one-loop graph \( V_{1L}^0 \), that does not depend on \( m \), reads

\[
V_{1L}^0 = \frac{N_A}{2i} \int \frac{d^4p}{(2\pi)^4} \left\{ \log \Delta_m^{-1} - \sum_{n=1}^{\infty} \frac{(m^2 \Delta_m)^n}{n} \right\}.
\] (63)
As before, in order to evaluate the GEP we must truncate the expansion and retain terms up to the first order, namely the zeroth-order gluon loop and the first order crossed loop that are the first two graphs in Fig. 2. Then, at first-order, the one loop effective potential is

$$V_{1L}(m) = N_A \left( 1 - \frac{\partial}{\partial m^2} \right) I_1(m)$$ \hspace{1cm} (64)

that is the same result of Eq.(30) scaled by the trace factor $N_A$.

The GEP also includes the two-loop first-order graph, the fourth graph in Fig. 1 with the propagator replaced by the massive propagator $\Delta_m$ and no insertions of the counterterm that would raise the order of the graph. By the same argument that leads to Eq.(34), the two-loop graph is easily evaluated by substituting the vertex $-i m^2$ in the crossed one-loop graph with the one-loop seagull self energy graph $-i \Pi_{1L}$ that reads\[15\]

$$\Pi_{1L} = -\frac{9 N g^2}{4} I_0(m)$$ \hspace{1cm} (65)

and adding a 1/2 symmetry factor. The resulting two-loop term is

$$V_{2L}(m) = \frac{9 N_A N g^2}{16} [I_0(m)]^2.$$ \hspace{1cm} (66)

Adding the one-loop terms the GEP reads

$$V_{GEP}(m) = N_A \left\{ I_1(m) - \frac{1}{2} m^2 I_0(m) + \frac{9 N g^2}{16} [I_0(m)]^2 \right\}$$ \hspace{1cm} (67)

which is exactly the same result of Eq.(35) for a scalar theory with an effective coupling $\lambda = 9 N g^2 / 2$, scaled by the trace factor $N_A$. Then by dimensional regularization, in the same scheme of Sec. III, the GEP of pure Yang-Mills theory can be written as

$$V_{GEP}(m) = \frac{\Lambda^4 N_A}{128 \pi^2} U(\alpha, m^2 / \Lambda^2)$$ \hspace{1cm} (68)

where the effective coupling $\alpha = \lambda / (16 \pi^2) = 9 N \alpha_s / (8 \pi), \alpha_s = g^2 / (4 \pi)$ and $\Lambda$ is an unknown scale that must be fixed by the phenomenology. The adimensional potential $U(\alpha, x)$ was defined in Eq.(43) and shown in Fig. 3.

B. Including Chiral Fermions

The inclusion of a set of chiral quarks is straightforward. As shown in Fig. 1, up to first order, the fermions are decoupled in the effective potential and we must just add the two
one-loop graphs for the quarks. Let us derive them by the same method of Sec. IV A. For fermions, the standard one-loop effective potential $V_\Psi$ of Eq. (53) can be written as

$$V_\Psi = \frac{i}{V_4} \log \text{Det} \left( \hat{D}_M + M \right)$$

where the massive inverse propagator $\hat{D}_M = \hat{D}_0 - M$ and the parameter $M$ is an arbitrary trial quark mass. The exact expansion of Fig. 2 is recovered again as

$$V_\Psi = \frac{i}{V_4} \text{Tr} \left[ \log \hat{D}_M \right] + \frac{i}{V_4} \text{Tr} \left[ \sum_{n=1}^{\infty} \frac{(\hat{D}_M^{-1} M)^n}{n} (-1)^{n+1} \right].$$

yielding a massive expansion for the fermions. The GEP contains only graphs up to first order and is given by the first two terms, the two fermion loops in Fig. 1. The first term in the expansion, the zeroth-order loop, is

$$V_\Psi^{(0)} = i \text{Tr} \int \frac{d^4 p}{(2\pi)^4} \log(p - M) = -4 I_1(M)$$

while the second term, the crossed first-order loop, by Eq.(32) reads

$$V_\Psi^{(1)} = -M \frac{\partial}{\partial M} V_\Psi^{(0)} = 4 M^2 I_0(M).$$

We observe that, without the crossed graph, the one-loop vacuum energy would be given by $V_\Psi^{(0)} = -4 I_1(M)$ which is unstable and unbounded from below according to Eq.(11). On the other hand, with one counterterm insertion, the first-order crossed graph makes the GEP bounded and yields the total first order effective potential

$$V_\Psi = -4 \left[ I_1(M) - M^2 I_0(M) \right]$$

which is exactly the GEP found in Ref. 32 by a direct variational method, provided that we take the chiral limit and set the external gluon field to zero. By dimensional regularization, inserting Eq.(11), the quark contribution to the GEP reads

$$V_\Psi(M) = \frac{3M^4}{16\pi^2} \left[ \log \frac{M^2}{\Lambda^2} + \frac{1}{6} \right]$$

and has a minimum at $M_0^2 = \Lambda^2 e^{-2/3}$ where $V_\Psi(M_0) = -3 M_0^4/(32\pi^2) < 0.$
V. DISCUSSION

Let us summarize the main findings of the previous sections.

The GEP for the GL model of superconductivity, namely $U(1)$ scalar electrodynamics, is recovered by a more general analysis based on a massive expansion, yielding a mass generation even when the original model is scaleless. The derivation of the GEP for pure $SU(N)$ Yang-Mills theory and chiral QCD also gives an original independent way to introduce the massive expansion: a change of the expansion point with massive propagators in the internal lines of the loops. The expansion acquires an evident variational meaning and emerges from the same variational argument that leads to the GEP. However, while the GEP is limited because of its first-order nature that leaves the fermions decoupled, in the massive expansion higher order terms can be easily included, yielding a powerful analytical tool for the study of QCD in the infrared and providing two-point functions that are in very good agreement with the results of lattice simulations\[16, 21–23\].

That said, the GEP gives a variational proof for chiral symmetry breaking and dynamical mass generation. Even if the actual values of the masses cannot be trusted because the quarks are decoupled, the variational nature of the calculation gives a proof that the vacuum of massless gluons and quarks is not stable. The Yang-Mills effective potential is given by the function $U(\alpha, x)$ of Eq.(43) and is shown in Fig. 3. An interesting feature is the occurrence of an unstable relative minimum at $m = 0$ and a stable minimum at $m > 0$. We could speculate and see an analogy with the double solution that occurs in the Dyson-Schwinger formalism: an unphysical massless scaling solution and a physical massive gluon propagator. Even if decoupled, the quark term of the GEP has an absolute minimum at a finite $M > 0$ according to Eq.(74), predicting the breaking of chiral symmetry of QCD.

We can see the absolute minimum of the GEP as a best expansion point for the massive expansion. In that sense, it is relevant to note that, once the crossed graph is included, the quark term of the GEP is also bounded from below. In fact, the counterterm keeps trace of the scaleless nature of the original Lagrangian and is needed for imposing that the Lagrangian is not modified in the expansion.

We are left with two independent mass parameters, $m$ and $M$, that must be determined by the phenomenology since their explicit expressions depend on the unknown renormalized scale $\Lambda$. Assuming that the scale $\Lambda$ is the same in the gluon and quark sector, which is not
obvious, the minimum of the GEP would give a best ratio of masses by Eqs. (68), (74)

\[
\frac{M_0}{m_0} = e^{\left(\frac{1}{3} + \frac{2}{3\alpha}\right)}
\]

(75)

linking together the dynamical generation of the gluon mass with the chiral symmetry breaking. While highly non-perturbative and non analytic in the limit \( \alpha \to 0 \), the suggested ratio of Eq. (75) suffers the limitations of the quark-gluon decoupling in the GEP and can be only regarded as a starting point for more refined calculations.

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