POSITIVE SOLUTIONS TO A SUPERCRITICAL ELLIPTIC PROBLEM THAT CONCENTRATE ALONG A THIN SPHERICAL HOLE

By

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Abstract. We consider the supercritical problem

\[-\Delta v = |v|^{p-2} v \quad \text{in } \Theta_\epsilon, \quad v = 0 \quad \text{on } \partial \Theta_\epsilon,\]

where \( \Theta \) is a bounded smooth domain in \( \mathbb{R}^N, N \geq 3, p > 2^* := 2N/(N - 2), \) and \( \Theta_\epsilon \) is obtained by deleting the \( \epsilon \)-neighborhood of some sphere which is embedded in \( \Theta. \) We show that in some particular situations, for small enough \( \epsilon > 0, \) this problem has a positive solution \( v_\epsilon \) and that this solution concentrates and blows up along the sphere as \( \epsilon \to 0. \) Our approach is to reduce this problem by means of a Hopf map to a critical problem of the form

\[-\Delta u = Q(x)|u|^{4/n-2} u \quad \text{in } \Omega_\epsilon, \quad u = 0 \quad \text{on } \partial \Omega_\epsilon,\]

in a punctured domain \( \Omega_\epsilon := \{ x \in \Omega : |x - \xi_0| > \epsilon \} \) of lower dimension. We show that if \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n, n \geq 3, \xi_0 \in \Omega, Q \in C^2(\overline{\Omega}) \) is positive, and \( \nabla Q(\xi_0) \neq 0, \) then for small enough \( \epsilon > 0, \) this problem has a positive solution \( u_\epsilon \) which concentrates and blows up at \( \xi_0 \) as \( \epsilon \to 0. \)

1 Introduction

We are interested in the supercritical problem

[1.1] \[-\Delta v = |v|^{p-2} v \quad \text{in } \mathcal{D}, \quad v = 0 \quad \text{on } \partial \mathcal{D},\]

where \( \mathcal{D} \) is a bounded smooth domain in \( \mathbb{R}^N, N \geq 3, \) and \( p > 2^*, \) with \( 2^* := 2N/(N - 2) \) the critical Sobolev exponent.

Existence of a solution to this problem is a delicate issue. Pohozaev’s identity [20] implies that (1.1) does not have a nontrivial solution if \( \mathcal{D} \) is strictly starshaped and \( p \geq 2^*. \) On the other hand, Kazdan and Warner [10] showed that infinitely

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many radial solutions exist for every $p \in (2, \infty)$ if $\mathcal{D}$ is an annulus. For $p = 2^*$, Bahri and Coron [2] established the existence of at least one positive solution to problem (1.1) in every domain $\mathcal{D}$ having nontrivial reduced homology with $\mathbb{Z}/2$-coefficients. However, this is not enough to guarantee existence in the supercritical case. In fact, for each $1 \leq k \leq N - 3$, Passaseo [18, 19] exhibited domains having the homotopy type of a $k$-dimensional sphere in which problem (1.1) does not have a nontrivial solution for $p \geq 2^*_{N,k} := 2(N - k)/(N - k - 2)$. Existence may fail even in domains with richer topology, as shown in [4].

The first nontrivial existence result for $p > 2^*$ was obtained by del Pino, Felmer and Musso [5] in the slightly supercritical case, i.e. for $p > 2^*$ close to $2^*$. For $p$ slightly below $2^*_{N,1}$, solutions in certain domains concentrating at a boundary geodesic as $p \to 2^*_{N,1}$ were constructed in [7].

A fruitful approach for producing solutions to the supercritical problem (1.1) is to reduce it to some critical or subcritical problem in a domain of lower dimension, either by considering rotational symmetries, by means of maps which preserve the Laplacian, or by a combination of both. This approach has been recently taken in [1, 4, 11, 12, 15, 22] to produce solutions of (1.1) in different types of domains. We also follow this approach to obtain a new type of solution in domains with thin spherical perforations.

We start with some notation. Let $O(N)$ be the group of linear isometries of $\mathbb{R}^N$. For a closed subgroup $\Gamma$ of $O(N)$, we let $\Gamma x := \{ gx : g \in \Gamma \}$ denote the $\Gamma$-orbit of $x \in \mathbb{R}^N$. A domain $\mathcal{D}$ in $\mathbb{R}^N$ is called $\Gamma$-invariant if $\Gamma x \subset \mathcal{D}$ for all $x \in \mathcal{D}$, and a function $u : \mathcal{D} \to \mathbb{R}$ is called $\Gamma$-invariant if it is constant on every $\Gamma x$. We denote by $\mathcal{D}^\Gamma := \{ x \in \mathcal{D} : gx = x \text{ for all } g \in \Gamma \}$ the set of $\Gamma$-fixed points in $\mathcal{D}$.

We consider the problem

\[ -\Delta u = Q(x)u^{(n+2)/(n-2)} \quad \text{in } \Omega, \\
\left(\Phi_{Q,\epsilon}^*\right) \\
\quad u > 0 \quad \text{in } \Omega, \\
\quad u = 0 \quad \text{on } \partial\Omega, \]

in $\Omega := \{ x \in \Omega : |x - \xi_0| > \epsilon \}$, where $n \geq 3$, $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$ which is invariant under the action of some closed subgroup $\Gamma$ of $O(n)$, $\xi_0 \in \Omega^\Gamma$, and the function $Q \in C^2(\overline{\Omega})$ is $\Gamma$-invariant and satisfies $\min_{x \in \Omega} Q(x) > 0$. Note that since $\xi_0 \in \Omega^\Gamma$, $\Omega$ is also $\Gamma$-invariant.

We prove the following result.

**Theorem 1.1.** Assume that $\nabla Q(\xi_0) \neq 0$. Then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, problem $\left(\Phi_{Q,\epsilon}^*\right)$ has a $\Gamma$-invariant solution $u_\epsilon$ which concentrates and blows up at the point $\xi_0$ as $\epsilon \to 0$. 
In [21], Rey considered the autonomous case $Q \equiv 1$. He showed that for each $\xi_0 \in \Omega$, problem $(\rho^*_Q, \epsilon)$ has a solution which concentrates and blows up at $\xi_0$ as $\epsilon \to 0$. Our result requires $\xi_0$ to be a noncritical point of $Q$. It would be interesting to investigate whether a similar statement is true when $\xi_0$ is a possibly nondegenerate critical point of $Q$. This is, however, not the main concern of the present work, since the potential $Q$ which arises from the reductions given below does not have critical points.

Next, we describe two situations where one can apply Theorem 1.1 to obtain solutions of supercritical problems that concentrate and blow up at a sphere.

For $N = 2, 4, 8, 16$, we write $\mathbb{R}^N = \mathbb{K} \times \mathbb{K}$, where $\mathbb{K}$ is either the set of real numbers $\mathbb{R}$, the set of complex numbers $\mathbb{C}$, the set of quaternions $\mathbb{H}$, or the set of Cayley numbers $\mathbb{O}$. The set of units $S_\mathbb{K} := \{ \vartheta \in \mathbb{K} : |\vartheta| = 1 \}$, which is a group if $\mathbb{K} = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{H}$ and a quasigroup with unit if $\mathbb{K} = \mathbb{O}$, acts on $\mathbb{R}^N$ by multiplication on each coordinate, i.e., $\vartheta(z_1, z_2) := (\vartheta z_1, \vartheta z_2)$. The orbit space of $\mathbb{R}^N$ with respect to this action turns out to be $\mathbb{R}^{\dim \mathbb{K} + 1}$, and the projection onto the orbit space is the Hopf map $h_\mathbb{K} : \mathbb{R}^N = \mathbb{K} \times \mathbb{K} \to \mathbb{R} \times \mathbb{K} = \mathbb{R}^{\dim \mathbb{K} + 1}$, given by

$$h_\mathbb{K}(z_1, z_2) := (|z_1|^2 - |z_2|^2, 2z_1z_2).$$

What makes this map special is that it preserves the Laplacian. Maps with this property are called harmonic morphisms [3, 8, 24]. More precisely, the following statement holds true. It can be derived by straightforward computation (cf. Proposition 4.1) or from the general theory of harmonic morphisms, as in [4].

**Proposition 1.2.** Let $N = 2, 4, 8, 16$ and let $\mathcal{D}$ be an $S_\mathbb{K}$-invariant bounded smooth domain in $\mathbb{R}^N = \mathbb{K}^2$ such that $0 \notin \overline{\mathcal{D}}$. Set $\mathcal{U} := h_\mathbb{K}(\mathcal{D})$. If $u$ is a solution to the problem

$$-\Delta u = \frac{1}{2|x|} |u|^{p-2} u \quad \text{in } \mathcal{U},$$

$$u = 0 \quad \text{on } \partial \mathcal{U},$$

(1.2)

then $v := u \circ h_\mathbb{K}$ is an $S_\mathbb{K}$-invariant solution of problem (1.1). Conversely, if $v$ is an $S_\mathbb{K}$-invariant solution of problem (1.1) and $v = u \circ h_\mathbb{K}$, then $u$ solves (1.2).

We apply this result as follows. Let $N = 4, 8, 16$, and let $\Theta$ be an $S_\mathbb{K}$-invariant bounded smooth domain in $\mathbb{R}^N = \mathbb{K}^2$ such that $0 \notin \overline{\Theta}$. Fix a point $z_0 \in \Theta$ and for sufficiently small $\epsilon > 0$, let $\Theta_\epsilon := \{ z \in \Theta : \text{dist}(z, S_\mathbb{K}z_0) > \epsilon \}$, where $S_\mathbb{K}z_0 := \{ \vartheta z_0 : \vartheta \in S_\mathbb{K} \}$. This is also an $S_\mathbb{K}$-invariant bounded smooth domain in $\mathbb{K}^2$. We
consider the supercritical problem

\[-\Delta v = v^{\dim K + 3/(\dim K - 1)} \quad \text{in } \Theta_\epsilon,\]

\[(\phi_\epsilon) \quad v > 0 \quad \text{in } \Theta_\epsilon,\]

\[v = 0 \quad \text{on } \partial \Theta_\epsilon.\]

Then Theorem 1.1 with \( n := \dim K + 1 \), \( \Gamma = \{1\} \), \( \Omega := \mathcal{H}(\Theta) \), \( \zeta_0 := \mathcal{H}(z_0) \) and \( Q(x) := 1/(2|x|) \), together with Proposition 1.2, immediately yields the following result.

**Theorem 1.3.** There exists \( \epsilon_0 > 0 \) such that, for each \( \epsilon \in (0, \epsilon_0) \), the supercritical problem \( (\phi_\epsilon) \) has an \( S_K \)-invariant solution \( v_\epsilon \) which concentrates and blows up along the sphere \( S_K z_0 \) as \( \epsilon \to 0 \).

Now let \( O(m) \times O(m) \) act on \( \mathbb{R}^{2m} \equiv \mathbb{R}^m \times \mathbb{R}^m \) in the obvious way and \( O(m) \) act on the last \( m \) coordinates of \( \mathbb{R}^{m+1} \equiv \mathbb{R} \times \mathbb{R}^m \). We write the elements of \( \mathbb{R}^{2m} \) as \((y_1, y_2)\) with \( y_i \in \mathbb{R}^m \) and the elements of \( \mathbb{R}^{m+1} \) as \( x = (t, \zeta) \) with \( t \in \mathbb{R}, \zeta \in \mathbb{R}^m \).

Recently, Pacella and Srikanth showed that the real Hopf map provides a one-to-one correspondence between \([O(m) \times O(m)]\)-invariant solutions of a supercritical problem in a domain in \( \mathbb{R}^{2m} \) and \( O(m) \)-invariant solutions of a critical problem in some domain in \( \mathbb{R}^{m+1} \). In [16], they proved the following result.

**Proposition 1.4.** Let \( N = 2m, m \geq 2, \) and \( \mathcal{D} \) be an \([O(m) \times O(m)]\)-invariant bounded smooth domain in \( \mathbb{R}^{2m} \) such that \( 0 \notin \mathcal{D} \). Set

\[\mathcal{U} := \{(t, \zeta) \in \mathbb{R} \times \mathbb{R}^m : \mathcal{H}(|y_1|, |y_2|) = (t, |\zeta|) \text{ for some } (y_1, y_2) \in \mathcal{D}\}.

If \( u(t, \zeta) = u(t, |\zeta|) \) is an \( O(m) \)-invariant solution of the problem

\[-\Delta u = \frac{1}{2|x|} |u|^{p-2} u \quad \text{in } \mathcal{U},\]

\[u = 0 \quad \text{on } \partial \mathcal{U},\]

then \( v(y_1, y_2) := u(\mathcal{H}(|y_1|, |y_2|)) \) is an \([O(m) \times O(m)]\)-invariant solution of the problem (1.1).

Conversely, if \( v(y_1, y_2) = v(|y_1|, |y_2|) \) is an \([O(m) \times O(m)]\)-invariant solution of problem (1.1) and \( v = u \circ \mathcal{H} \), then \( u(t, \zeta) = u(t, |\zeta|) \) is an \( O(m) \)-invariant solution of problem (1.3).

We apply this result as follows. Let \( \Phi \) be an \([O(m) \times O(m)]\)-invariant bounded smooth domain in \( \mathbb{R}^{2m} \) such that \( 0 \notin \Phi \) and \((y_0, 0) \in \Phi \). We write

\[S_0^{m-1} := \{(y, 0) : |y| = |y_0|\} \]
for the \([O(m) \times O(m)]\)-orbit of \((y_0, 0)\); and for each small enough \(\epsilon > 0\), we set
\[
\Phi_\epsilon := \{ x \in \Phi : \text{dist}(x, S_0^{m-1}) > \epsilon \}.
\]

This is again an \([O(m) \times O(m)]\)-invariant bounded smooth domain in \(\mathbb{R}^{2m}\). We consider the supercritical problem
\[
-\Delta v = \nu^{(m+3)/(m-1)} \quad \text{in } \Phi_\epsilon,
\]
\[
\nu > 0 \quad \text{in } \Phi_\epsilon,
\]
\[
\nu = 0 \quad \text{on } \partial \Phi_\epsilon.
\]

Then Theorem 1.1 with \(n = m + 1\), \(\Gamma = O(m)\),
\[
\Omega := \{ (t, \zeta) \in \mathbb{R}^m : h_\mathbb{R}(|y_1|, |y_2|) = (t, |\zeta|) \text{ for some } (y_1, y_2) \in \Phi \},
\]
\[
\zeta_0 := (|y_0|, 0, \ldots, 0) \text{ and } Q(x) = 1/(2 |x|),
\]

Together with Proposition 1.4, immediately yields the following result.

**Theorem 1.5.** There exists \(\epsilon_0 > 0\) such that, for each \(\epsilon \in (0, \epsilon_0)\), problem \((\varphi_\epsilon^2)\) has an \([O(m) \times O(m)]\)-invariant solution \(v_\epsilon\) which concentrates and blows up along the \((m - 1)\)-dimensional sphere \(S_0^{m-1}\) as \(\epsilon \to 0\).

The proof of Theorem 1.1 uses the well-known Ljapunov-Schmidt reduction, adapted to the symmetric case. In the following section, we sketch this reduction, highlighting the places where the symmetries play a role. In Section 3, we give an expansion of the reduced energy functional and use it to prove Theorem 1.1. We conclude with some remarks concerning Proposition 1.4.

## 2 The finite dimensional reduction

For every bounded domain \(U\) in \(\mathbb{R}^n\), we take
\[
(u, v) := \int_U \nabla u \cdot \nabla v, \quad ||u|| := \left( \int_U |\nabla u|^2 \right)^{1/2},
\]
as the inner product and its corresponding norm in \(H_0^1(U)\). If we replace \(U\) by \(\mathbb{R}^n\), these are the inner product and the norm in \(D^{1,2}(\mathbb{R}^n)\). We write
\[
||u||_r := \left( \int_U |u|^r \right)^{1/r}
\]
for the norm in \(L^r(U), r \in [1, \infty)\).

If \(U\) is \(\Gamma\)-invariant for some closed subgroup \(\Gamma\) of \(O(n)\), we set
\[
H_0^1(U)^\Gamma := \{ u \in H_0^1(U) : u \text{ is } \Gamma\text{-invariant} \}.
and, similarly, for $D^{1,2}(\mathbb{R}^n)\Gamma$ and $L^r(\Omega)^\Gamma$.

It is well known that the standard bubbles

$$U_{\delta,\xi}(x) = [n(n-2)]^{(n-2)/4} \frac{\delta^{(n-2)/2}}{(\delta^2 + |x-\xi|^2)^{(n-2)/2}}, \quad \delta \in (0, \infty), \quad \xi \in \mathbb{R}^n,$$

are the only positive solutions of the equation

$$-\Delta U = U^p \quad \text{in} \quad \mathbb{R}^n,$$

where $p := (n + 2)/(n - 2)$. Thus, the function $W_{\delta,\xi} := \gamma_0 U_{\delta,\xi}$, with $\gamma_0 := [Q(\xi_0)]^{-1/(p-1)}$, solves the equation

$$-\Delta W = Q(\xi_0)W^p \quad \text{in} \quad \mathbb{R}^n. \quad (2.1)$$

Let

$$\psi_{\delta,\xi}^0 := \frac{\partial U_{\delta,\xi}}{\partial \delta} = \alpha_n \frac{n-2}{2} \delta^{(n-4)/2} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{n/2}},$$

$$\psi_{\delta,\xi}^j := \frac{\partial U_{\delta,\xi}}{\partial \xi_j} = \alpha_n(n-2)\delta^{(n-2)/2} \frac{x_j - \xi_j}{(\delta^2 + |x-\xi|^2)^{n/2}}, \quad j = 1, \ldots, n. \quad (2.2)$$

The space generated by these $n+1$ functions is the space of solutions to the problem

$$-\Delta \psi = pU_{\delta,\xi}^{\psi-1} \psi, \quad \psi \in D^{1,2}(\mathbb{R}^n). \quad (2.3)$$

Note that

$$U_{\delta,\xi} \in D^{1,2}(\mathbb{R}^n)\Gamma \quad \text{if and only if} \quad \xi \in (\mathbb{R}^n)\Gamma$$

and, similarly, for every $j = 0, 1, \ldots, n$,

$$\psi_{\delta,\xi}^j \in D^{1,2}(\mathbb{R}^n)\Gamma \quad \text{if and only if} \quad \xi \in (\mathbb{R}^n)\Gamma.$$

Let $\Omega$ be a $\Gamma$-invariant bounded smooth domain in $\mathbb{R}^n$, $Q \in C^2(\overline{\Omega})$ be positive and $\Gamma$-invariant, and $\xi_0 \in \Omega\Gamma$. For small enough $\epsilon > 0$, set

$$\Omega_\epsilon := \{x \in \Omega : |x-\xi_0| > \epsilon\}.$$

Consider the orthogonal projection $P_\epsilon : D^{1,2}(\mathbb{R}^n) \to H^1_0(\Omega_\epsilon)$; i.e., if $W \in D^{1,2}(\mathbb{R}^n)$, then $P_\epsilon W$ is the unique solution to the problem

$$-\Delta (P_\epsilon W) = -\Delta W \quad \text{in} \quad \Omega_\epsilon, \quad P_\epsilon W = 0 \quad \text{on} \quad \partial \Omega_\epsilon. \quad (2.4)$$

A consequence of the uniqueness is that $P_\epsilon W \in H^1_0(\Omega_\epsilon)\Gamma$ if $W \in D^{1,2}(\mathbb{R}^n)\Gamma$. 


We denote by $G(x, y)$ the Green function of the Laplace operator in $\Omega$ with zero Dirichlet boundary condition and by $H(x, y)$ its regular part, i.e.,
\[
G(x, y) = \beta_n \left( \frac{1}{|x - y|^{n-2}} - H(x, y) \right),
\]
where $\beta_n$ is a positive constant depending only on $n$. The following estimates play a crucial role in the proof of Theorem 1.1.

**Lemma 2.1** ([9, Lemma 3.1]). Assume that $\delta \to 0$ as $\epsilon \to 0$ and $\epsilon = o(\delta)$ as $\epsilon \to 0$. Fix $\eta \in \mathbb{R}^n$, set $\xi := \xi_0 + \delta \eta$, and define
\[
R(x) := P_{\epsilon} U_{\delta, \xi}(x) - U_{\delta, \xi}(x) + \alpha_n \delta^{(n-2)/2} H(x, \xi) + \frac{\alpha_n}{\delta^{(n-2)/2}(1 + |\eta|^2)(n-2)/2} \epsilon^{n-2} \left| \frac{1}{|x - \xi_0|^{n-2}} - H(x, \xi_0) \right|.
\]
Then there exists a positive constant $c$ such that the following estimates hold for every $x \in \Omega \setminus B(\xi_0, \epsilon)$:
\[
|R(x)| \leq c \delta^{(n-4)/2} \left[ \frac{\epsilon^{n-2} (1 + \epsilon \delta^{-n+1})}{|x - \xi_0|^{n-2}} + \delta^2 + \left( \frac{\epsilon}{\delta} \right)^{n-2} \right],
\]
\[
|\partial_\delta R(x)| \leq c \delta^{(n-4)/2} \left[ \frac{\epsilon^{n-2} (1 + \epsilon \delta^{-n+1})}{|x - \xi_0|^{n-2}} + \delta^2 + \left( \frac{\epsilon}{\delta} \right)^{n-2} \right],
\]
\[
|\partial_\xi R(x)| \leq c \delta^{n/2} \left[ \frac{\epsilon^{n-2} (1 + \epsilon \delta^{-n})}{|x - \xi_0|^{n-2}} + \delta^2 + \left( \frac{\epsilon}{\delta} \right)^{n-1} \right].
\]

For each $\epsilon > 0$ and $(d, \eta) \in \Lambda^\Gamma := (0, \infty) \times (\mathbb{R}^n)^\Gamma$, set (see (2.1))
\[
V_{d, \eta} := P_{\epsilon} W_{d, \xi} = \gamma_0 P_{\epsilon} U_{\delta, \xi} \quad \text{with} \quad \delta := d \epsilon^{(n-2)/(n-1)}, \quad \xi := \xi_0 + \delta \eta.
\]
The map $(d, \eta) \mapsto V_{d, \eta}$ is a $C^2$-embedding of $\Lambda^\Gamma$ as a submanifold of $H_0^1(\Omega_\epsilon)^\Gamma$, whose tangent space at $V_{d, \eta}$ is $K_{d, \eta}^\epsilon := \text{span} \{ P_{\epsilon} \psi_{d, \xi}^j : j = 0, 1, \ldots, n \}$. Note that since $\xi_0, \eta \in (\mathbb{R}^n)^\Gamma$, also $\xi \in (\mathbb{R}^n)^\Gamma$; and therefore, $K_{d, \eta}^\epsilon \subset H_0^1(\Omega_\epsilon)^\Gamma$. We write
\[
K_{d, \eta}^{\epsilon, \perp} := \{ \phi \in H_0^1(\Omega_\epsilon)^\Gamma : (\phi, P_{\epsilon} \psi_{d, \xi}^j) = 0 \quad \text{for} \quad j = 0, 1, \ldots, n \}
\]
for the orthogonal complement of $K_{d, \eta}^\epsilon$ in $H_0^1(\Omega_\epsilon)^\Gamma$, and $\Pi_{d, \eta}^\epsilon : H_0^1(\Omega_\epsilon)^\Gamma \to K_{d, \eta}^\epsilon$ and $\Pi_{d, \eta}^{\epsilon, \perp} : H_0^1(\Omega_\epsilon)^\Gamma \to K_{d, \eta}^{\epsilon, \perp}$ for the orthogonal projections, i.e.,
\[
\Pi_{d, \eta}^\epsilon(u) := \sum_{j=0}^n (u, P_{\epsilon} \psi_{d, \xi}^j) P_{\epsilon} \psi_{d, \xi}^j, \quad \Pi_{d, \eta}^{\epsilon, \perp}(u) := u - \Pi_{d, \eta}^\epsilon(u).
\]
Let $i_{\epsilon}^* : L^{2n/(n+2)}(\Omega_\epsilon) \to H_0^1(\Omega_\epsilon)$ be the adjoint operator to the embedding $i_{\epsilon} : H_0^1(\Omega_\epsilon) \to L^{2n/(n-2)}(\Omega_\epsilon)$, i.e.,
\[
v = i_{\epsilon}^*(u) \quad \text{if and only if} \quad (u, \varphi) = \int_{\Omega_\epsilon} u \varphi \quad \text{for all} \quad \varphi \in C_c^\infty(\Omega_\epsilon)
\]
if and only if
\[(2.5) \quad -\Delta v = u \text{ in } \Omega_\epsilon, \quad v = 0 \text{ on } \partial \Omega_\epsilon.\]

Sobolev’s inequality yields a constant $c > 0$, independent of $\epsilon$, such that
\[(2.6) \quad \|i_\epsilon^*(u)\| \leq c \|u\|_{2n/(n+2)} \quad \text{for all } u \in L^{2n/(n+2)}(\Omega_\epsilon), \text{ for all } \epsilon > 0.
\]

Note again that $i_\epsilon^*(u) \in H^1_0(\Omega_\epsilon)$ if $u \in L^{2n/(n-2)}(\Omega_\epsilon)$.

We rewrite problem $(\wp^*_\epsilon, \Omega_\epsilon)$ in the following equivalent way:
\[(2.7) \quad u = i_\epsilon^* [Q(x)f(u)], \quad u \in H^1_0(\Omega_\epsilon),\]

where $f(s) : = (s^+)^p$ and $p : = (n+2)/(n-2)$.

We seek a solution to problem (2.7) of the form
\[(2.8) \quad u_\epsilon = V_{d,\eta} + \phi \quad \text{with } (d, \eta) \in \Lambda^\Gamma \text{ and } \phi \in K^\epsilon_{d,\eta}.\]

As usual, our goal is to find $(d, \eta) \in \Lambda^\Gamma$ and $\phi \in K^\epsilon_{d,\eta}$ such that for small enough $\epsilon$,
\[(2.9) \quad \Pi^\epsilon_{d,\eta}[V_{d,\eta} + \phi - i_\epsilon^*(Qf(V_{d,\eta} + \phi))] = 0\]

and
\[(2.10) \quad \Pi^\epsilon_{d,\eta}[V_{d,\eta} + \phi - i_\epsilon^*(Qf(V_{d,\eta} + \phi))] = 0.\]

First we show that for every $(d, \eta) \in \Lambda^\Gamma$ and small enough $\epsilon$, there exists a unique $\phi \in K^\epsilon_{d,\eta}$ which satisfies (2.9). To this end, we consider the linear operator $L^\epsilon_{d,\eta} : K^\epsilon_{d,\eta} \to K^\epsilon_{d,\eta}$ defined by
\[L^\epsilon_{d,\eta}(\phi) : = \phi - \Pi^\epsilon_{d,\eta} i_\epsilon^*[Qf'(V_{d,\eta})\phi].\]

It has the following properties.

**Proposition 2.2.** For every compact subset $D$ of $\Lambda^\Gamma$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ and each $(d, \eta) \in D$,
\[(2.11) \quad \|L^\epsilon_{d,\eta}(\phi)\| \geq c \|\phi\| \quad \text{for all } \phi \in K^\epsilon_{d,\eta},\]

and the operator $L^\epsilon_{d,\eta}$ is invertible.

The argument given in [9, Lemma 5.1] carries over with minor changes to our situation. Hence, the proof is omitted.
Lemma 2.3 ([13]). For each $a, b, q \in \mathbb{R}$ with $a \geq 0$ and $q \geq 1$, there exists a positive constant $c$ such that

$$\left| |a + b|^q - a^q \right| \leq \begin{cases} 
    c \min\{ |b|^q, a^{q-1} |b| \} & \text{if } 0 < q < 1, \\
    c(|a|^{q-1} |b| + |b|^q) & \text{if } q \geq 1.
\end{cases}$$

Again, the argument given to prove similar results in the literature carries over with minor changes to prove the following result. As some of the estimates are used later, we include the proof.

Proposition 2.4. For every compact subset $D$ of $\Lambda^\Gamma$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ and for each $(d, \eta) \in D$, there exists a unique $\phi_{d, \eta}^\epsilon \in K_{d, \eta}^\epsilon \subset H_0^1(\Omega_{\epsilon})^\Gamma$ which solves equation (2.9) and satisfies

$$(2.12) \quad \left\| \phi_{d, \eta}^\epsilon \right\| \leq c\epsilon^{(n-2)/(n-1)}.$$ 

Moreover, the function $(d, \eta) \mapsto \phi_{d, \eta}^\epsilon$ is a $C^1$-map.

Proof. Note that $\phi \in K_{d, \eta}^{\epsilon, \perp}$ solves equation (2.9) if and only if $\phi$ is a fixed point of the operator $T_{d, \eta}^\epsilon : K_{d, \eta}^{\epsilon, \perp} \rightarrow K_{d, \eta}^{\epsilon, \perp}$ defined by

$$T_{d, \eta}^\epsilon (\phi) = (L_{d, \eta})^{-1} \Pi_{d, \eta}^{\epsilon, \perp} \phi - Qf(V_{d, \eta} + \phi) - f'(V_{d, \eta})\phi - Q(\xi_0)(\gamma_0 U_{\delta, \xi})^p.$$

We prove that $T_{d, \eta}^\epsilon$ is a contraction on a suitable ball.

We first show that there exist $\epsilon_0 > 0$ and $c > 0$ such that for each $\epsilon \in (0, \epsilon_0)$,

$$(2.13) \quad \left\| \phi \right\| \leq c\epsilon^{(n-2)/(n-1)} \quad \text{implies} \quad \left\| T_{d, \eta}^\epsilon (\phi) \right\| \leq c\epsilon^{(n-2)/(n-1)}.$$ 

From Proposition 2.2, we have that for some $c > 0$ and all small enough $\epsilon > 0$,

$$\left\| (L_{d, \eta})^{-1} \right\| \leq c \quad \text{for all } (d, \eta) \in D.$$

Using (2.6), we obtain

$$\left\| T_{d, \eta}^\epsilon (\phi) \right\| \leq c \left\| Qf(V_{d, \eta} + \phi) - f'(V_{d, \eta})\phi \right\|_{2n/(n+2)} + c \left\| \xi \right\|^p_{2n/(n+2)} + c\gamma_0^p \left\| Q(\xi_0) U_{\delta, \xi} \right\|_{2n/(n+2)}.$$
The Mean Value Theorem, Lemma 2.3, and Hölder’s inequality then give

\[ \|Q[f(V_d, \eta + \phi) - f(V_d, \eta)]\|_{2n/(n+2)} \leq c \|f'(V_d, \eta + t\phi) - f'(V_d, \eta)\|_{2n/(n+2)} \]

\[ \leq c \|f'(V_d, \eta + t\phi) - f'(V_d, \eta)\|_{n/2} \|\phi\|_{2^*} \]

\[ \leq c \left( \|\phi\|_{2^*}^2 + \|\phi\|_{2^*}^{4/(n-2)} \right) \|\phi\|_{2^*} \]

\[ \leq c \left( \|\phi\|_{2^*} + \|\phi\|_{2^*}^2 \right). \]

for some \( t \in (0, 1) \). Moreover, using Lemma 2.1, one can show that

\[ \|Qf(V_d, \eta) - Q(\gamma_0 U_{\delta, \xi})^p\|_{2n/(n+2)} \leq c \left( (P_\epsilon U_{\delta, \xi})^p - U_{\delta, \xi}^p \right) \|\phi\|_{2n/(n+2)} \]

\[ \leq c \delta; \]

see [9, inequality (6.4)]. Finally, setting \( y = (x - \xi)/\delta = ((x - \xi_0)/\delta) - \eta \), and \( \Omega_\epsilon := \{ y \in \mathbb{R}^n : \delta y + \xi \in \Omega_\epsilon \} \), and again invoking the Mean Value Theorem, we obtain

\[ \|Q - Q(\xi_0)U_{\delta, \xi}^p\|_{2n/(n+2)} \]

\[ = \left( \int_{\Omega_\epsilon} |Q(\delta \tilde{y} + \delta \eta + \xi_0) - Q(\xi_0)|^{2n/(n+2)} U_f^{p+1}(y) dy \right)^{(n+2)/2n} \]

\[ = \delta \left( \int_{\Omega_\epsilon} |\nabla Q(t \delta \tilde{y} + t \delta \eta + \xi_0), y + \eta)|^{2n/(n+2)} U_f^{p+1}(y) dy \right)^{(n+2)/2n} \]

\[ \leq c \delta. \]

for some \( t \in (0, 1) \). This proves statement (2.13).

Next we show that \( \epsilon_0 > 0 \) can be chosen such that for each \( \epsilon \in (0, \epsilon_0) \), the operator

\[ T_{d, \eta}^\epsilon : \{ \phi \in K_{d, \eta}^{\epsilon, \perp} : \|\phi\| \leq c \epsilon^{(n-2)/(n-1)} \} \rightarrow \{ \phi \in K_{d, \eta}^{\epsilon, \perp} : \|\phi\| \leq c \epsilon^{(n-2)/(n-1)} \} \]

is a contraction and, therefore, has a unique fixed point, as claimed.

For \( \phi_1, \phi_2 \in \{ \phi \in K_{d, \eta}^{\epsilon, \perp} : \|\phi\| \leq c \epsilon^{(n-2)/(n-1)} \} \), the Mean Value Theorem gives

\[ \|T_{d, \eta}(\phi_1) - T_{d, \eta}(\phi_2)\| \leq c \|f(V_d, \eta + \phi_1) - f(V_d, \eta + \phi_2) - f'(V_d, \eta)(\phi_1 - \phi_2)\|_{2n/(n+2)} \]

\[ = c \|[f'(V_d, \eta + (1 - t)\phi_1 + \phi_2) - f'(V_d, \eta)](\phi_1 - \phi_2)\|_{2n/(n+2)} \]

\[ \leq c \|f'(V_d, \eta + (1 - t)\phi_1 + \phi_2) - f'(V_d, \eta)\|_{2} \|\phi_1 - \phi_2\|_{2}. \]

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for some \( t \in [0, 1] \); and, arguing as before, we conclude that

\[
\|f'(V_{d,\eta} + (1-t)\phi_1 + \phi_2) - f'(V_{d,\eta})\|_{n/2}
\leq c \left( \| (1-t)\phi_1 + \phi_2 \|_{2^*} + \| (1-t)\phi_1 + \phi_2 \|_{2^*}^{4/(n-2)} \right)
\leq c \left( \| \phi_1 \|_{2^*} + \| \phi_2 \|_{2^*} + \| \phi_1 \|_{2^*}^{4/(n-2)} + \| \phi_2 \|_{2^*}^{4/(n-2)} \right).
\]

Hence, it follows that for sufficiently small \( \epsilon > 0 \),

\[
\|T_{d,\eta}^\epsilon(\phi_1) - T_{d,\eta}^\epsilon(\phi_2)\| \leq \kappa \| \phi_1 - \phi_2 \|
\]

with \( \kappa \in (0, 1) \).

Finally, a standard argument shows that \((d, \eta) \mapsto \phi_{d,\eta}^\epsilon\) is a \( C^1 \)-map. \( \square \)

Consider the functional \( J_\epsilon : H^1_0(\Omega_\epsilon) \to \mathbb{R} \) defined by

\[
J_\epsilon(u) := \frac{1}{2} \int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega_\epsilon} Q|u|^{p+1}.
\]

It is well known that the critical points of \( J_\epsilon \) are the solutions of problem (2.7). We define the reduced energy functional \( \tilde{J}_\epsilon^\Gamma : \Lambda^\Gamma \to \mathbb{R} \) by

\[
(2.16) \quad \tilde{J}_\epsilon^\Gamma(d, \eta) := J_\epsilon(V_{d,\eta} + \phi_{d,\eta}^\epsilon).
\]

If \( \Gamma = \{1\} \) is the trivial group, we simply write \( \tilde{J}_\epsilon \) instead of \( \tilde{J}_\epsilon^\Gamma \) and \( \Lambda \) instead of \( \Lambda^\Gamma \).

Next we show that the critical points of \( \tilde{J}_\epsilon^\Gamma \) are \( \Gamma \)-invariant solutions of problem (2.7).

**Proposition 2.5.** If \((d, \eta) \in \Lambda^\Gamma\) is a critical point of the function \( \tilde{J}_\epsilon^\Gamma \), then \( V_{d,\eta} + \phi_{d,\eta}^\epsilon \in H^1_0(\Omega_\epsilon)^\Gamma \) is a critical point of the functional \( J_\epsilon \) and, therefore, a \( \Gamma \)-invariant solution of problem (2.7).

**Proof.** Assume first that \( \Gamma \) is the trivial group. Then \( \Lambda = (0, \infty) \times \mathbb{R}^n \), and the statement follows from arguments similar to those used to prove [6, Lemma 6.1] or [9, Proposition 2.2].

If \( \Gamma \) is an arbitrary closed subgroup of \( O(n) \), then \( \Lambda^\Gamma \) is the set of \( \Gamma \)-fixed points in \( \Lambda \) of the action of \( \Gamma \) on the space \( \mathbb{R} \times \mathbb{R}^n \) given by \( g(t, x) := (t, gx) \) for \( g \in \Gamma \), \( t \in \mathbb{R} \), \( x \in \mathbb{R}^n \). By the principle of symmetric criticality [17, 23], if \((d, \eta) \in \Lambda^\Gamma\) is a critical point of the function \( \tilde{J}_\epsilon^\Gamma \), then \((d, \eta)\) is a critical point of \( \tilde{J}_\epsilon : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} \), and the result follows from the previous case. \( \square \)
3 The asymptotic expansion of the reduced energy functional

In order to find a critical point of \( \tilde{J}_\epsilon^{\Gamma_1} \), we use the following asymptotic expansion of the functional \( \tilde{J}_\epsilon^{\Gamma_1} : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} \).

**Proposition 3.1.** The asymptotic expansion

\[
\tilde{J}_\epsilon(d, \eta) = c_0 + Q(\xi_0)^{-2/(p-1)} F(d, \eta) \epsilon^{(n-2)/(n-1)} + o(\epsilon^{(n-2)/(n-1)})
\]

holds \( C^1 \)-uniformly on compact subsets of \( \Lambda_1 \), where \( F : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) is given by

\[
F(d, \eta) := \begin{cases} 
\alpha d + \beta/(1 + |\eta|^2) d - \gamma \langle \nabla Q(\xi_0)/Q(\xi_0), \eta \rangle d & \text{if } n = 3, \\
\beta(1 + |\eta|^2) d^{2-n} - \gamma \langle \nabla Q(\xi_0)/Q(\xi_0), \eta \rangle d & \text{if } n \geq 4.
\end{cases}
\]

for some positive constants \( c_0, \alpha, \beta \) and \( \gamma \).

**Proof.** Write

\[
J_\epsilon(V_{d,\eta} + \phi_{d,\eta}^\epsilon) = \frac{1}{2} \left\| V_{d,\eta} + \phi_{d,\eta}^\epsilon \right\|^2 - \frac{1}{p+1} \int_{\Omega} Q \left| V_{d,\eta} + \phi_{d,\eta}^\epsilon \right|^{p+1}
\]
\[
= J_\epsilon(V_{d,\eta}) + \gamma_0 \int_{\Omega_d} (U_{\delta,\xi}^p - (P \epsilon U_{\delta,\xi})^p) \phi_{d,\eta}^\epsilon
\]
\[
- \gamma_0 \int_{\Omega_d} \left[ Q - Q(\xi_0) \right] (P \epsilon U_{\delta,\xi})^p \phi_{d,\eta}^\epsilon + \frac{1}{2} \left\| \phi_{d,\eta}^\epsilon \right\|^2
\]
\[
- \frac{1}{p+1} \int_{\Omega_d} Q \left( \left| V_{d,\eta} + \phi_{d,\eta}^\epsilon \right|^{p+1} - \left| V_{d,\eta} \right|^{p+1} - (p+1) V_{d,\eta}^p \phi_{d,\eta}^\epsilon \right).
\]

Then Hölder’s inequality and inequalities (2.12), (2.14), and (2.15) yield

\[
J_\epsilon(V_{d,\eta} + \phi_{d,\eta}^\epsilon) = J_\epsilon(V_{d,\eta}) + O \left( \epsilon^{\frac{2(n-2)}{n+1}} \right)
\]

\[
= \gamma_0^2 \left[ \frac{1}{2} \int_{\Omega} U_{\delta,\xi}^p (P \epsilon U_{\delta,\xi}) - \frac{1}{p+1} \int_{\Omega_d} \left| P \epsilon U_{\delta,\xi} \right|^{p+1} \right]
\]
\[
- \frac{1}{p+1} \gamma_0^{p+1} \int_{\Omega_d} \left[ Q - Q(\xi_0) \right] \left| P \epsilon U_{\delta,\xi} \right|^{p+1} + O \left( \epsilon^{\frac{2(n-2)}{n+1}} \right).
\]
Next, compute the first summand on the right-hand side of (3.2). Using Lemma 2.1, we have
\[
\frac{1}{2} \int_{\Omega_{\alpha}} U^p_{\delta,\xi}(P_\epsilon U_{\delta,\xi}) - \frac{1}{p+1} \int_{\Omega_{\alpha}} |P_\epsilon U_{\delta,\xi}|^{p+1}
\]
\[= \frac{p-1}{2(p+1)} \int_{\Omega_{\alpha}} U^{p+1}_{\delta,\xi} - \frac{1}{2} \int_{\Omega_{\alpha}} U^p_{\delta,\xi} (P_\epsilon U_{\delta,\xi} - U_{\delta,\xi})
\]
\[= \frac{p-1}{2(p+1)} \int_{\Omega_{\alpha}} U^{p+1}_{\delta,\xi} - \frac{1}{2} \int_{\Omega_{\alpha}} U^p_{\delta,\xi} (P_\epsilon U_{\delta,\xi} - U_{\delta,\xi})
\]
\[= \frac{p-1}{2(p+1)} \int_{\Omega_{\alpha}} U^{p+1}_{\delta,\xi} + \frac{1}{2} \int_{\Omega_{\alpha}} U^p_{\delta,\xi} \Gamma^\epsilon_{\delta,\xi} + o\left(\frac{\epsilon^{n-2}}{\delta}\right),
\]
where
\[(3.3) \quad \Gamma^\epsilon_{\delta,\xi}(x) := \alpha_n \delta^{(n-2)/2} H(x, \xi) + \alpha_n \frac{1}{\delta^{(n-2)/2}(1 + |\eta|^2(2n-2)/2} |x - \zeta_0|^{n-2}.
\]
Setting \(x = \xi + \delta y\) yields
\[a_n \int_{\Omega_{\alpha}} U^p_{\delta,\xi} \Gamma^\epsilon_{\delta,\xi} = a_n \int_{\Omega_{\alpha}} U^p_{\delta,\xi}(x)(\delta^{(n-2)/2} H(x, \xi))dx
\]
\[+ a_n \int_{\Omega_{\alpha}} U^p_{\delta,\xi}(x) \left(\frac{1}{\delta^{(n-2)/2}(1 + |\eta|^2(2n-2)/2} |x - \zeta_0|^{n-2}\right) dx
\]
\[= a_n \delta^{n-2} \int_{\Omega_{\alpha}} U_{1,0}^p(y) H(\delta y + \delta \eta + \zeta_0, \delta \eta + \zeta_0)dy
\]
\[+ a_n \frac{1}{(1 + |\eta|^2(2n-2)/2} \int_{\Omega_{\alpha}} U_{1,0}^p(y) \left(\frac{1}{\delta^{n-2} |y - \eta|^{n-2}}\right) dy
\]
\[= a_n \left(\int_{\mathbb{R}^n} U^p_{1,0} \right) H(\zeta_0, \zeta_0) \delta^{n-2} (1 + o(1)) + \alpha_n g(\eta) \frac{1}{\delta^{n-2}} \epsilon^{n-2} (1 + o(1)),
\]
where the function \(g : \mathbb{R}^n \to \mathbb{R}\) is defined by
\[g(\eta) := \frac{1}{(1 + |\eta|^2(2n-2)/2} \int_{\mathbb{R}^n} \frac{1}{|y - \eta|^{n-2}} U_{1,0}^p(y)dy.
\]
Since \(-\Delta U = U^p\) in \(\mathbb{R}^n\), an easy computation shows that
\[g(\eta) = \frac{1}{(1 + |\eta|^2(2n-2)/2} U_{1,0}(\eta) = \alpha_n \frac{1}{(1 + |\eta|^2)^{n-2}}.
\]
To compute the second summand on the right-hand side of (3.2), use the Taylor expansion
\[Q(\delta y + \zeta_0 + \delta \eta) = Q(\zeta_0) + \delta \langle \nabla Q(\zeta_0), y + \eta \rangle + O(\delta^2 (1 + |y|^2))
\]
such that $F(d, \eta) = 0$. Moreover, $F_{\eta\eta}(d, \eta) > 0$ for all $\eta \in \mathbb{R}$. Consider the function $F: \mathbb{R} \to \mathbb{R}$ defined by

$$F(\eta) := F(d, \eta) = \frac{1}{2} \left( (1 + |\eta|)^2 |\alpha - \gamma(\xi, \eta)| \right) \quad \text{if } n \geq 4,$$

and consider the half space

$$\mathcal{H} := \{ \eta \in \mathbb{R} : \alpha - \gamma(\xi, \eta) > 0 \}.$$

For each $\eta \in \mathcal{H}$, there exists a unique $\tilde{d} = \tilde{d}(\eta)$, given by

$$\tilde{d}(\eta) = \sqrt{\frac{1 + |\eta|}{\alpha - \gamma(\xi, \eta)}} \quad \text{if } n \geq 4.$$

Proof of Theorem 1.1. We first show that the function $F$ defined in (3.1) has a critical point $d_0$, $\eta_0 \in (0, \infty)$, which is stable under $C^1$-perturbations. Then we deduce from Proposition 3.1 that the functional $\tilde{F}$ has a critical point in $\mathcal{H}$ for small enough $\varepsilon > 0$. The result then follows from Proposition 2.5.

Let $n = 3$. Set $\gamma_0 = \gamma(\xi, \eta)$, $\eta_0 = \eta$, and consider the half space

$$\mathcal{H} := \{ \eta \in \mathbb{R} : \alpha - \gamma(\xi, \eta) > 0 \}.$$
is a strict maximum point of $\bar{F}$. Setting $d_0 := d(\eta_0)$, we deduce from [14, Lemma 5.7] that $(d_0, \eta_0)$ is a $C^1$-stable critical point of the function $F$. Note that since $\zeta_0 \in \Omega^\Gamma$ and $Q$ is $\Gamma$-invariant, $\nabla Q(\zeta_0) \in (\mathbb{R}^n)^\Gamma$. Hence, $(d_0, \eta_0) \in \Lambda^\Gamma$.

For $n \geq 4$, arguing as in the previous case, we easily conclude that if

$$\eta_0 := -\frac{\nabla Q(\zeta_0)}{|\nabla Q(\zeta_0)|}, \quad d_0 := \left(\frac{(n-2)\beta}{2n-2\gamma} \frac{Q(\zeta_0)}{|\nabla Q(\zeta_0)|}\right)^{1/(n-1)},$$

then $(d_0, \eta_0)$ is a $C^1$-stable critical point of the function $F$ and $(d_0, \eta_0) \in \Lambda^\Gamma$. □

4 Final remarks

One may wonder whether the conclusion of Proposition 1.4 is also true in other dimensions. We show that it is not.

For $N = k_1 + k_2$, write the elements of $\mathbb{R}^N$ as $(y_1, y_2)$ with $y_i \in \mathbb{R}^{k_i}$ and the elements of $\mathbb{R}^{m+1}$ as $(t, \zeta)$ with $t \in \mathbb{R}$, $\zeta \in \mathbb{R}^m$.

Proposition 4.1. Let $N = k_1 + k_2$, $\mathcal{D}$ be an $[O(k_1) \times O(k_2)]$-invariant bounded smooth domain in $\mathbb{R}^N$ such that $0 \notin \mathcal{D}$, and $f \in C^0(\mathbb{R})$. Set

$$\mathcal{U} := \{(t, \zeta) \in \mathbb{R} \times \mathbb{R}^m : h_{\mathbb{R}}(|y_1|, |y_2|) = (t, |\zeta|) \text{ for some } (y_1, y_2) \in \mathcal{D}\}$$

and let $u \in C^2(\mathcal{U})$, $u(t, \zeta) = u(t, |\zeta|)$, be an $O(m)$-invariant solution of

$$-\Delta u = \frac{1}{2|x|} f(u)$$

(4.1)

in $\mathcal{U}$. Then $v(y_1, y_2) := u(h_{\mathbb{R}}(|y_1|, |y_2|))$ is an $[O(k_1) \times O(k_2)]$-invariant solution of equation

$$-\Delta v = f(v)$$

(4.2)

in $\mathcal{D}$ if and only if $k_1 = k_2 = m$.

Proof. A straightforward computation shows that a function $v(y_1, y_2) = v(|y_1|, |y_2|)$ solves equation (4.2) in $\{(y_1, y_2) \in \mathcal{D} : y_1 \neq 0, y_2 \neq 0\}$ if and only if $v$ solves

$$-\Delta v = \frac{k_1 - 1}{z_1} \frac{\partial v}{\partial z_1} - \frac{k_2 - 1}{z_2} \frac{\partial v}{\partial z_2} = f(v)$$

(4.3)

in $\mathcal{D}_0 := \{z = (z_1, z_2) \in \mathbb{R}^2 : z_1, z_2 > 0, z_1 = |y_1|, z_2 = |y_2|, (y_1, y_2) \in \mathcal{D}\}$. Similarly, a function $u(t, \zeta) = u(t, |\zeta|)$ solves equation (4.1) in $\{(t, \zeta) \in \mathcal{U} : \zeta \neq 0\}$ if and only if $u$ solves

$$-\Delta u = \frac{m - 1}{x_2} \frac{\partial u}{\partial x_2} = \frac{1}{2|x|} f(u)$$

(4.4)
in \( U_0 := \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0, x_2 = |\zeta|, (x_1, \zeta) \in U \} \).

Assuming that \( u \) solves equation (4.4) in \( U_0 \), we show next that \( v := u \circ h_{\mathbb{R}} \) solves equation (4.3) in \( D_0 \) if and only if \( k_1 = k_2 = m \). A straightforward computation yields

\[
- \Delta v - \frac{k_1 - 1}{z_1} \frac{\partial v}{\partial z_1} - \frac{k_2 - 1}{z_2} \frac{\partial v}{\partial z_2} = 2 |z|^2 \left( - \Delta u - \left[ \frac{k_1 - 1}{|z|^2} - \frac{k_2 - 1}{|z|^2} \right] \frac{\partial u}{\partial x_1} - \left[ \frac{k_1 - 1}{|z|^2} \frac{z_2}{z_1} + \frac{k_2 - 1}{|z|^2} \frac{z_1}{z_2} \right] \frac{\partial u}{\partial x_2} \right).
\]

Note that \( |z|^2 = |h_{\mathbb{R}}(z)| \). So if \( u \) solves equation (4.4), then

\[
- \Delta v - \frac{k_1 - 1}{z_1} \frac{\partial v}{\partial z_1} - \frac{k_2 - 1}{z_2} \frac{\partial v}{\partial z_2} = f(v)
\]

if and only if

\[
\frac{k_1 - 1}{|z|^2} = \frac{k_2 - 1}{|z|^2} \quad \text{and} \quad \frac{k_1 - 1}{|z|^2} \frac{z_2}{z_1} + \frac{k_2 - 1}{|z|^2} \frac{z_1}{z_2} = \frac{m - 1}{z_1 z_2},
\]

i.e., if and only if \( k_1 = k_2 = m \). \( \Box \)

The argument given in [16] to prove Proposition 1.4 uses polar coordinates. Note that if we write \( z_1 = r \cos \theta, z_2 = r \sin \theta, x_1 = \rho \cos \varphi, x_2 = \rho \sin \varphi, \) then the Hopf map \( x = \frac{1}{2} h_{\mathbb{R}}(z) \) becomes

\[
\rho = \frac{1}{2} r^2, \quad \varphi = 2 \theta,
\]

which is the map considered in [16].

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