A NOTE ON \(k\)-JET AMPLENESS ON SURFACES

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Abstract. We prove Reider type criterions for \(k\)-jet spannedness and \(k\)-jet ampleness of adjoint bundles for surfaces with at most rational singularities. Moreover, we prove that on smooth surfaces \([n(n+4)/4]\)-very ampleness implies \(n\)-jet ampleness.

Introduction

Let \(L\) be a Cartier divisor on a normal projective surface \(X\) and \(k\) a non-negative integer. \(L\) generates \(k\)-jets at a point \(x \in X\) if the restriction map \(H^0(L) \to H^0(L \otimes \mathcal{O}_X/m_x^{k+1})\) is onto. \(L\) is \(k\)-jet generated (or \(k\)-jet spanned) if it generates \(k\)-jets at each point of \(X\). \(L\) is \(k\)-jet ample if for any distinct points \(x_1, \ldots, x_r\) in \(X\) and positive integers \(k_1, \ldots, k_r\) with \(k_1 + \ldots + k_r = k + 1\) the restriction map \(H^0(L) \to H^0(L \otimes \mathcal{O}_X/m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r})\) is onto. Note that \(L\) is 0-jet ample if and only if it is 0-jet generated and if and only if it is spanned by global sections.

The main aim of this note is to establish Reider type criterions for \(k\)-jet spannedness and \(k\)-jet ampleness of an adjoint bundle. Up to now there were a few trials to find such criterions (see, e.g., [BS2] and [Laz], Section 7) but the optimal results were not known. A Reider type criterion is well known in the case of \(k\)-very ampleness (see [BS1, Theorem 2.1]) and it implies a weak form of the criterion for \(k\)-jet ampleness (see [BS2, Proposition 2.1]). A different version of the criterion for \(k\)-jet spannedness with better bounds was proved in [Laz, Theorem 7.4].

The paper is divided into 4 sections. In the first section we recall some results used in the paper. In Section 2 we prove that on smooth surfaces already \([n(n+4)/4]\)-very ampleness implies \(n\)-jet ampleness. This together with [BS1, Theorem 2.1] gives much better Reider type criterion for \(k\)-jet ampleness than those mentioned above. In Section 3 we give a direct proof of even better criterion for \(k\)-jet spannedness and \(k\)-jet ampleness (without using results of [BS1]) describing also the boundary case in terms of the Seshadri constant. This version of the Reider type criterion is new even in the case of 0-jet spannedness (i.e., in the classically known globally generated case). Our proof works also in a larger category of normal surfaces with at most rational singularities whereas the results of [BS1], [BS2] and [Laz] are known only for smooth surfaces. In particular, our theorem holds for canonical surfaces, where it was used to prove that \(|2K_X|\) has no base components.
for surfaces with $K_X^2 = 4$, $p_g = q = 0$ (see [La2], Theorem 0.1). In the last section we try to explain (after [Laz]) how Seshadri constants appear in the study of adjoint bundles and we give an example of 1-jet spanned but not 1-jet ample line bundle.

1. Preliminaries

1.1. Let $L$ be a line bundle on a normal surface $X$. It is very natural to consider the following definition: $L$ is called $k$-point generated (or $k$-point spanned) at a point $x$ if the restriction map $H^0(O_X(L)) \to H^0(O_x(L))$ is onto for any cluster $\zeta$ supported on $x$ and of degree $\leq k + 1$. In fact it would be more natural to call it $k$-generated, but this notion is reserved for something slightly different. Recall also that $L$ is called $k$-very ample if the restriction map $H^0(O_X(L)) \to H^0(O_x(L))$ is onto for any degree $\leq k + 1$ cluster $\zeta$ in $X$.

We will use those notions in Section 2.

Lemma 1.2. Let $D$ be a Weil divisor on a normal surface. If $D^2 \geq 0$ and $DL > 0$ for some nef divisor $L$, then $D$ is pseudoeffective. Moreover, $D$ is big unless $D^2 = 0$ and $D$ is nef.

Sketch of the proof. The first part of the lemma follows easily from the Hodge index theorem. The second one follows from the Zariski decomposition for $D$, Q.E.D.

The following lemma is a slightly modified version of Corollary 3.7, [La1], and can be obtained similarly as in [La1] by using Lemma 1.2.

Lemma 1.3. Let $X$ be a normal projective surface and $L$ a pseudoeffective Weil divisor on $X$. If $L^2 > 0$ or $L^2 = 0$ and $L$ is not nef then every nontrivial extension $E \in \text{Ext}^1(O(K_X + L), \omega_X)$ is Bogomolov unstable.

Let us also recall the definition of the Seshadri constant.

Definition 1.4. Let $X$ be a normal surface with at most rational singularity at a point $x$ and let $f : Y \to X$ be the minimal resolution of the singularity at $x$ (or the blow up of $X$ at $x$ if $x$ is smooth). Let $Z$ denote the fundamental cycle (respectively: the exceptional divisor). A Seshadri constant of a divisor $L$ at $x$ is defined as

$$\epsilon(L, x) = \sup\{\epsilon \geq 0| f^*L - \epsilon \cdot Z \text{ is nef }\},$$

whenever it is a well defined real number.

2. Relations with $k$-very ampleness

Let $x$ be a smooth point of a surface $X$. Our first problem will be to determine the number

$$l_n = \max\{\deg \zeta: \zeta \text{ is Gorenstein and } m_x^{n+1} \subset I_\zeta \text{ in } O_{X,x}\}.$$

Clearly, $l_0 = 1$ and $l_n < \binom{n+2}{2}$ for $n \geq 1$ since $m_x^{n+1}$ is not locally generated by two elements and hence the corresponding cluster is not Gorenstein. In fact, we have the following theorem:
Theorem 2.1.

\[ l_n = \left\lceil \frac{(n+2)^2}{4} \right\rceil \]

Proof. Note that \( l_n \geq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor \) since if \( z_1, z_2 \) are local parameters and \( \mathcal{I}_\zeta = (z_1^{k+1}, z_2^{k+1}) \) for \( n = 2k \), or \( \mathcal{I}_\zeta = (z_1^{k}, z_2^{k+1}) \) for \( n = 2k - 1 \), then \( \zeta \) is Gorenstein and \( \deg \zeta = \left\lfloor \frac{(n+2)^2}{4} \right\rfloor \).

Let \((X, x)\) be a germ of a smooth surface. By the above it is sufficient to prove that if for a cluster \( \zeta \) supported on the point \( x \), the ideal \( \mathcal{I}_\zeta \) is generated by two elements \( f \) and \( g \) and it contains \( m_x^{n+1} \) then \( \deg \zeta \leq \frac{(n+2)^2}{4} \) (let us recall that in the smooth case Gorenstein cluster on a surface means locally complete intersection).

Let \( V_1 \) and \( V_2 \) be curves defined by \( f \) and \( g \), respectively. It is easy to see that there exists a sequence of blow ups \( \pi: X_k \to \pi^k X_{k-1} \to \ldots \to X_1 \to \pi^1 X \) such that

\[ \pi^* V_i = \tilde{V}_i + \sum e_{Q_j}(V_i) E_j, \]

where \( \tilde{V}_i \) is a strict transform of \( V_i \), \( E_j \) denotes (a pull back of) an exceptional divisor of \( p_j \), \( Q_j = p_j(E_j) \), \( e_{Q_j}(V_i) \) denotes a multiplicity of \( V_i \) along \( Q_j \), \( \tilde{V}_i \) and \( \tilde{V}_2 \) are disjoint and the divisor \( \dot{V}_1 + \tilde{V}_2 + \sum E_j \) is normal crossing.

Now note that the scheme-theoretical preimage of \( I_\zeta \), i.e., \((\pi^* f, \pi^* g) = \mathcal{O}(-\tilde{V}_1 - \sum e_{Q_j}(V_1) E_j) + \mathcal{O}(\tilde{V}_2 - \sum e_{Q_j}(V_2) E_j) \) does not contain \( J = \mathcal{O}(-\sum(e_{Q_j}(V_1) + e_{Q_j}(V_2))E_j - 2E_k) \) and \( \pi_* J \subset m_x^{\sum(e_{Q_j}(V_1) + e_{Q_j}(V_2)) - 2} \).

Therefore if \( m_x^{n+1} \subset (f, g) \) then \( n \geq \sum(e_{Q_j}(V_1) + e_{Q_j}(V_2)) - 2 \).

By the definition the degree of \( \zeta \) is a local intersection multiplicity of \( V_1 \) and \( V_2 \) and from the above it is easy to see that

\[ \deg \zeta = \sum e_{Q}(V_1)e_{Q}(V_2), \]

where we sum over all infinitely near points \( Q \) of \( X \) (this is the formula of M. Noether; see [Fu], Example 12.4.2). Now the theorem follows from the obvious inequality

\[ \sum e_{Q}(V_1)e_{Q}(V_2) \leq \frac{(\sum e_{Q}(V_1) + e_{Q}(V_2))^2}{4} \leq \frac{(n+2)^2}{4}. \]

Theorem 2.2. Let \( L \) be a Cartier divisor on a normal surface \( X \) and \( x \) a smooth point of \( X \).

1. If \( L \) is \((l_k - 1)\)-point generated at a point \( x \) then \( L \) is \( k \)-jet generated at \( x \).
2. If \( X \) is smooth and \( L \) is \((l_k - 1)\)-very ample then \( L \) is \( k \)-jet ample.

Proof. By the cohomology exact sequence

\[ H^0(\mathcal{O}(L)) \to H^0(\mathcal{O}(L)/m_x^{k+1}) \to H^1(m_x^{k+1}\mathcal{O}(L)) \to H^1(\mathcal{O}(L)) \]

and \( \text{Ext}^1(m_x^{k+1}\mathcal{O}(L), \omega_X) \) (\( \text{Ext}^1(\mathcal{O}(L), \omega_X) \)) is dual to \( H^1(m_x^{k+1}\mathcal{O}(L)) \) (\( H^1(\mathcal{O}(L)) \)), respectively) by the Serre duality theorem. If \( L \) is not \( k \)-jet generated at \( x \) then using this one can see that there exists an extension \( \mathcal{E} \in \text{Ext}^1(m_x^{k+1}\mathcal{O}(L), \omega_X) \) not coming from \( \text{Ext}^1(\mathcal{O}(L), \omega_X) \). Set \( \mathcal{F} = \mathcal{E}^* \). Then \( \mathcal{F} \in \text{Ext}^1(\mathcal{I}_\zeta \mathcal{O}(L), \omega_X) \) for some cluster \( \zeta \) contained in the cluster \( \mathcal{O}_x/m_x^{k+1} \).
Since $F$ is locally free, $\zeta$ is a locally complete intersection and hence $\deg \zeta \leq l_k$. But $L$ is $(l_k - 1)$-point generated at $x$, so by the similar arguments as above we prove that every extension in $\text{Ext}^1(I_\zeta \mathcal{O}(L), \omega_X)$ comes from $\text{Ext}^1(\mathcal{O}(L), \omega_X)$, a contradiction.

The proof in the ample case is very similar once we know that

\[(*) \quad l_{k_1-1} + \ldots + l_{k_r-1} \leq l_k\]

for any positive integers $k_1, \ldots, k_r$ with $k_1 + \ldots + k_r = k + 1$. But we have

\[
\sum (k_i + 1)^2 = \sum k_i^2 + 2(k + 1) + r \leq \sum k_i^2 + \sum_{i \neq j} k_i k_j + 1 + 2(k + 1) = \\
= (k_1 + \ldots + k_r)^2 + 2(k + 1) + 1 = (k + 2)^2
\]

from which the inequality $(*)$ follows. This finishes the proof of the theorem.

Remarks.

(1) Similar in vein but weaker theorem was proved in any dimension by Beltrametti and Sommese (see [BS2, Proposition 2.1]). It would be interesting to know whether our theorem can be also generalized to the higher dimensional case.

(2) Note that already the trivial bound $l_n < \binom{n+1}{2}$ for $n > 1$ implies in the surface case better theorem than Proposition 2.1, [BS2]. In fact, Theorems 2.1 and 2.2 imply a part of Corollary 3.2 (Theorem 3.4) by using Reider type theorem for $k$-point spannedness ($k$-very ampleness, respectively; see [BS1]).

3. MAIN THEOREMS

**Theorem 3.1.** Let $L$ be a pseudoeffective Weil divisor on a normal projective surface $X$. Assume that $K_X + L$ is Cartier and generates $(k - 1)$-jets but not $k$-jets at a smooth point $x$.

(1) If $L^2 > (k + 2)^2$ then there exists a curve $D$ containing a Gorenstein cluster $\zeta$ such that $I_\zeta$ contains $m^{k+1}_x$ but it does not contain $m^k_x$ and such that the map $H^0(\mathcal{O}_D(K_X + L)) \to H^0(I_\zeta \mathcal{O}_D(K_X + L))$ is not onto. In particular, $|\mathcal{O}_D(K_X + L)|$ does not generate $k$-jets at the point $x$. Moreover, $L - 2D$ is pseudoeffective, numerically nontrivial and the following inequalities hold:

\[LD - \deg \zeta \leq 2p_a D - 2 - K_X D\]

and

\[LD - \frac{1}{4}(k + 2)^2 \leq D^2.\]

(2) If $L^2 = (k + 2)^2$, then either there exists a curve $D$ as in (1) or $\epsilon(x, L) = k + 2$.

**Proof.** Since $K_X + L$ does not generate $k$-jets at $x$ and generates $(k - 1)$-jets, by the Serre duality theorem there exists a nontrivial extension $F \in \text{Ext}^1(m^{k+1}_x \mathcal{O}(K_X + L), \omega_X)$ not lying in the image of the natural map $\text{Ext}^1(m^k_x \mathcal{O}(K_X + L), \omega_X) \to \text{Ext}^1(m^{k+1}_x \mathcal{O}(K_X + L), \omega_X)$. Let $E$ be a reflexivisation of $F$. Then the cokernel of the natural map $\omega_X \to E$ twisted by $\mathcal{O}(-K_X - L)$ defines an ideal of a cluster $\zeta$. Clearly, $m^{k+1}_x \subset I_\zeta$ but $m^k_x \not\subset I_\zeta$. Since $E \in \text{Ext}^1(I_\zeta \mathcal{O}(K_X + L), \omega_X)$ is reflexive, $\zeta$ is a Gorenstein scheme.
Let $f: Y \to X$ be the blow up of $X$ at $x$ and let us denote by $E$ the exceptional divisor. Then $f_*\mathcal{O}(-(k+1)E) = m_x^{k+1}$ and therefore there exists a rank 2 reflexive sheaf $\mathcal{F}' \in \text{Ext}^1(\mathcal{O}_Y(f^*(K_X + L) - (k+1)E), \omega_Y)$ such that $f_*\mathcal{F}' = \mathcal{F}$. By Lemma 1.2 and the assumption that $L^2 = (k + 2)^2$, the sheaf $\mathcal{F}'$ is Bogomolov unstable unless $f^*L - (k + 2)E$ is nef and $L^2 = (k + 2)^2$ in which case $\epsilon(L, x) = k + 2$. Therefore $\mathcal{E} = (f_*\mathcal{F}')^*\omega$ is also Bogomolov unstable reflexive sheaf. Let $\mathcal{O}(K_X + A)$ be a maximal destabilizing subsheaf of $\mathcal{E}$. Then the composition map $\mathcal{O}(K_X + A) \to \mathcal{E} \to {\mathcal{I}}_\varsigma\mathcal{O}(K_X + L)$ is nonzero and its image twisted by $\mathcal{O}(-K_X - L)$ defines an ideal of effective divisor $D$ containing $\varsigma$ and linearly equivalent to $L - A$. By the instability of $\mathcal{E}$ the divisor $L - 2D$ is pseudoeffective and numerically nontrivial.

Using rather lengthy arguments with diagram chasing one can prove that $\varsigma$ is in a (very) special position with respect to $\mathcal{O}_D(K_X + L)$; see Lemma 4.4.1, [La3], or the proof of Theorem 4.7, [La1]. In particular, it follows that there is an injection

$${\mathcal{I}}_\varsigma\mathcal{O}(K_X + L) \hookrightarrow \omega_D$$

and hence $2p_aD - 2 \geq D(K_X + L) - \deg \varsigma$ which is equivalent to $DL - \deg \varsigma \leq D^2$. It also shows that $\mathcal{O}_D(K_X + L)$ is not $k$-jet generated at $x$. On the other hand it is $(k - 1)$-jet generated at $x$, since $K_X + L$ is $(k - 1)$-jet generated at $x$.

The other inequality can be proven in much the same way as a similar inequality in Theorem 4.7, [La1], Q.E.D.

Remark. There are some variants of the theorem which seems to be worth of pointing out:

1. If under the assumptions of Theorem 3.1 the surface $X$ is Gorenstein one can get also that $LD - \deg \varsigma \leq D^2$ which is better than the second inequality of Theorem 3.1 (and it is not equivalent to the first one unless $X$ is smooth). This inequality is equivalent to $c_2(\mathcal{E}) \geq (K_X + A)(K_X + D)$ which follows from an exact sequence $0 \to \mathcal{O}(K_X + A) \to A \to \mathcal{G} \to 0$, where $\mathcal{G}^* = \mathcal{O}(K_X + D)$ (note that this is more difficult than it seems since $K_X + A$ and $K_X + D$ are not necessarily Cartier; nevertheless it is still true; see, e.g., [La3], Proposition 2.1.5).

2. If $X$ is smooth and $L^2 > 4l_x$ then $\mathcal{E}$ is Bogomolov unstable by the Bogomolov instability theorem. In this case we also get the curve $D$ satisfying assertions of 3.1.(1). This gives better result for $s$ odd and worse for $s$ even.

Now we state the criterion for jet spannedness at singular points of $X$ indicating necessary changes in the proof. In the theorem $Z = Z_\varsigma$ denotes a fundamental cycle of a singularity $(X, x)$ and $\Delta = \Delta_x = f^*K_X - KY$ where $f$ is a minimal resolution of the singularity $(X, x)$. Set $\delta_k = -((k + 1)Z + \Delta)^2$.

**Theorem 3.1**. Let $L$ be a pseudoeffective Weil divisor on a normal projective surface $X$. Assume that $K_X + L$ is Cartier and generates $(k - 1)$-jets but not $k$-jets at a rational singularity $x$.

1. If $L^2 > \delta_k$, then there exists a curve $D$ containing a Gorenstein cluster $\varsigma$ such that $\mathcal{I}_\varsigma$ contains $m_x^{k+1}$ but it does not contain $m_x^k$ and such that the map $H^0(\mathcal{O}_D(K_X + L)) \to H^0(\mathcal{I}_\varsigma\mathcal{O}_D(K_X + L))$ is not onto. In particular, $|\mathcal{O}_D(K_X + L)|$ does not generate $k$-jets at the point $x$. Moreover, $L - 2D$ is pseudoeffective, numerically nontrivial,

$$LD - \deg \varsigma \leq 2p_aD - 2 - K_XD$$

and

$$LD - \frac{1}{2}\delta_k \leq D^2.$$
(2) If $L^2 = \delta_k$, then either there exists a curve $D$ as in (1) or $\epsilon(x, L) = k + 1$.

Proof. Instead of the blow up $f$ we use a minimal resolution of singularity at $x$ and then $E$ is replaced by $Z$. The cluster $\zeta$ is Gorenstein by Theorem 1.5.7, [La3]. The rest of the proof is the same.

Remarks.
(1) Recall that if $X$ has only Du Val singularities then $2p_aD - 2 \leq K_X D + D^2$ (see, e.g., [La3], Corollary 1.3.3; this is just a simple corollary to the Riemann–Roch theorem as written in [La1], Theorem 2.1). Hence in this case the first inequality in Theorem 3.1’, (1) is stronger than $LD - \deg \zeta \leq D^2$. Moreover, in any case this inequality bounds discrete invariants of the curve since $p_aD$ and $D(K_X + L)$ are integers.

(2) Note that we have a trivial bound $\deg \zeta \leq \deg \mathcal{O}_X/m_x^{k+1} = 1/2(k+1)(-kZ^2 + 2)$ (this should be read keeping in mind that $-Z^2 = \text{emb dim}_X X - 1$). Similarly as in Section 2 one can ask about $\max\{\deg \zeta; \zeta$ is Gorenstein and $m_x^{n+1} \subseteq I_\zeta \text{ in } \mathcal{O}_{X,x}\}$ but it seems to be quite difficult question.

The following corollary is a Reider type theorem for $k$-jet spannedness of adjoint divisor for nef $L$ and it considerably improves Corollary 7.5, [Laz] (it should be also compared with Proposition 5.7, [Laz]; see Proposition 4.1):

**Corollary 3.2.** Let $L$ be a Weil divisor on a normal projective surface $X$ such that $K_X + L$ is Cartier and $x$ a smooth point of $X$. If $L$ is nef and $L^2 \geq (k+2)^2$ then one of the following holds:

1. $\epsilon(x, L) = k + 2$ and $L^2 = (k+2)^2$,
2. $K_X + L$ generates $k$-jets at the point $x$,
3. there exists a curve $D$ passing through the point $x$ such that the complete linear system $|\mathcal{O}_D(K_X + L)|$ does not generate $k$-jets at the point $x$ and such that $$LD - l_k \leq D^2 < 1/2LD < l_k.$$ 

Proof. By Theorem 3.1 it is sufficient to prove that if $L$ is nef, $LD - l_k \leq D^2$, $L - 2D$ is pseudoeffective and numerically nontrivial then $LD < 2l_k$ and $(L - 2D)D > 0$.

We know that $(L - 2D)D \geq 0$, i.e., $LD/L^2 \leq 1/2$, and hence by the Hodge index theorem $$D^2 \leq \frac{(LD)^2}{L^2} \leq \frac{1}{2}LD.$$ 

If we have an equality $D^2 = 1/2LD$ then $(L - 2D)L = 0$ and $L$ and $D$ are numerically proportional again by the Hodge index theorem. Using this two facts we see that $L - 2D$ is numerically trivial, a contradiction.

Therefore $LD - l_k \leq D^2 < 1/2LD$, which implies $LD < 2l_k$. Q.E.D.

In the following theorem $\delta(x, k)$, where $x$ is a point of $X$ and $k$ an integer, stands for $(k+1)^2$ if $x$ is smooth and $-(kZ_x + \Delta_x)^2$ if $X$ has a rational singularity at $x$ (and is singular at $x$).

**Theorem 3.3.** Let $L$ be a pseudoeffective Weil divisor on a normal projective surface $X$ such that $K_X + L$ is Cartier and let $k_1, \ldots, k_r$ be positive integers. Assume that $X$ has at most rational singularities at distinct points $x_1, \ldots, x_r$ and the restriction map $$H^0(\mathcal{O}_X(K_X + L)) \to H^0(\mathcal{O}_{x_i}(K_{x_i} + L)/m_{x_i}^{k_r} \otimes \cdots \otimes m_{x_i}^{k_r})$$
is not onto. If $L^2 > \sum_{i=1}^r \delta(x_i, k_i)$, then there exists a curve $D$ passing through $x_1, \ldots, x_r$ and such that

$$H^0(O_D(K_X + L)) \to H^0(O_D(K_X + L)/m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})$$

is not onto. Moreover, $L - 2D$ is pseudoeffective, numerically nontrivial,

$$LD - \sum_{i=1}^r \deg O_X/m_{x_i}^{k_i} \leq 2p_a D - 2 - K_X D$$

and

$$LD - \frac{1}{4} \sum_{i=1}^r \delta(x_i, k_i) \leq D^2.$$ 

The proof of this theorem is analogous to the proof of Theorems 3.1 and 3.1’ and therefore we skip it. Note that we can also treat the case $L^2 = \sum_{i=1}^r \delta(x_i, k_i)$ but the statement of the theorem would be more complicated. Similarly as before one can also write down a more complicated version with better inequality and some “bad” Gorenstein cluster.

Theorem 3.3 can be used to check $k$-jet ampleness of the adjoint line bundle on surfaces with at most rational singularities. If the surface is smooth we have the following theorem (in which we treat also the boundary case):

**Theorem 3.4.** Let $L$ be a pseudoeffective divisor on a smooth surface $X$. Assume that $K_X + L$ is $(k-1)$-jet ample but not $k$-jet ample.

(1) If $L^2 > (k+2)^2$ then there exists a curve $D$ such that $O_D(K_X + L)$ is $(k-1)$-jet ample but not $k$-jet ample, $L - 2D$ is pseudoeffective, numerically nontrivial and $LD - l_k \leq D^2$.

(2) If $L^2 = (k+2)^2$ then either there exists a curve $D$ as in (1) or $k$ is even and there exists a point $x$ such that $K_X + L$ is not $k$-jet generated at $x$ and $\epsilon(x, L) = k+2$.

**Proof.** This is just a simple generalization of Theorem 3.1 (and the second point of the remark after this theorem). The only thing we need to complete the proof is the inequality

$$\sum_{i=1}^r l_{k_i - 1} < l_k$$

if $r \geq 2$ and $\sum_{i=1}^r k_i = k+1$. This is implicitly proved in the proof of Theorem 2.2.

Similarly as before one can get slightly better theorem if $L$ is nef (cf. Corollary 3.2).

4. FURTHER REMARKS ON JET AMPLENESS

In this section we assume that $L$ is a divisor on a smooth surface $X$.

The following proposition is a generalization of Proposition 5.7, [Laz]. The method of the proof is almost the same but we show it for the convenience of the reader. We think that this proposition explains appearing of the Seshadri constants in theorems from Section 3.
Proposition 4.1. Let $x_1, \ldots, x_r$ be distinct points and $k_1, \ldots, k_r$ positive integers.

1) If $\sum_{i=1}^{r} \frac{k_i + 1}{\epsilon(x_i, L)} < 1$ then $H^0(K_X + L) \to H^0(\mathcal{O}(K_X + L)/m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})$ is onto.

2) If $\sum_{i=1}^{r} \frac{k_i + 1}{\epsilon(x_i, L)} = 1$ and $L^2 \geq \sum_{i=1}^{r} (k_i + 1)^2$ then $H^0(K_X + L) \to H^0(\mathcal{O}(K_X + L)/m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})$ is onto unless $r = 1$ and $L^2 = (k_1 + 1)^2$.

Proof. Let $f: Y \to X$ be a blow up of $X$ at $x_1, \ldots, x_r$ and $E_1, \ldots, E_r$ respective exceptional divisors. It is sufficient to prove that $L' = f^*L - (k_1 + 1)E_1 - \cdots - (k_r + 1)E_r$ is nef and big since then $H^1(\mathcal{O}(K_X + L)m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r}) = H^1(K_Y + L') = 0$ by the Kawamata–Viehweg vanishing theorem. Note that

$$L' = \sum_{i=1}^{r} (f^*L - \epsilon(x_i, L)E_i) + \left(1 - \sum_{i=1}^{r} \frac{k_i + 1}{\epsilon(x_i, L)}\right)f^*L.$$ 

Now (1) follows since all the terms on the right are nef and the last one is big. (2) is clear if $(L')^2 = L^2 - \sum_{i=1}^{r} (k_i + 1)^2 > 0$. Otherwise $(L')^2 = 0$ and therefore

$$(f^*L - \epsilon(x_i, L)E_i)(f^*L - \epsilon(x_j, L)E_j) = 0$$

for each pair $i, j$. This is impossible if $i \neq j$. Therefore $r = 1$ and $(f^*L - \epsilon(x_1, L)E_1)^2 = L^2 - (k_1 + 1)^2 = 0$, Q.E.D.

Corollary 4.2. Let $A$ be an ample line bundle on a smooth surface $X$. Let $x_1, x_2, \ldots, x_r$ denote $r$ distinct points on $X$ and $k_1, \ldots, k_r$ be some integers such that $k_1 + \cdots + k_r = k + 1$. If $\epsilon(x_i, A) \geq 1$ for $i = 1, \ldots, r$, then $H^0(K_X + nA) \to H^0(\mathcal{O}(K_X + nA)/m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})$ is onto for $n \geq k + 2 + r$ or for $n \geq k + 1 + r$ if $A^2 > 1$.

Corollary 4.3. If $A$ is ample, globally generated and $(X, A) \not\cong (\mathbb{P}^2, \mathcal{O}(1))$ then $K_X + nA$ is $k$-jet generated at each point of $X$ for $n \geq k + 2$ and $k$-jet ample for $n \geq 2(k + 1)$.

Proof. If $A$ is ample and globally generated then $\epsilon(x, A) \geq 1$ for any point $x$ of $X$. If $A^2 = 1$ then the morphism $\varphi$ defined by $A$ is finite and $\deg \varphi \cdot \deg \mathcal{O}_{\varphi(X)}(1) = A^2 = 1$, so $\deg \varphi = \deg \mathcal{O}_{\varphi(X)}(1) = 1$. Hence $(X, A) \not\cong (\mathbb{P}^2, \mathcal{O}(1))$, a contradiction.

Therefore $A^2 > 1$ and the corollary follows from Corollary 4.2, Q.E.D.

Remark. Corollary 4.3 is analogous to Corollary 3.3.(2), [BS2], which says, in particular, that if $A$ is very ample then $K_X + nA$ is $k$-jet ample for $n \geq k + 2$.

The following proposition follows from Corollary 3.2:

Proposition 4.4. Let $X$ be a minimal surface of the Kodaira dimension $0$. Let $A$ be an ample line bundle on $X$ such that $A^2 \geq 4$. If $\epsilon(A) < 1$ then there exists an irreducible curve $D$ such that $AD = 1$ and $p_a D = 1$. Moreover, the curve $D$ has at most double points as its singularities and if $D$ is singular then $\epsilon(A) = 1/2$.

Remark. From the first part of Proposition 4.4 applied for $2A$ it follows that

$$\inf \{\epsilon(A): A \text{ is ample} \} \geq \frac{1}{2}. \quad (\text{Now the second part of Proposition 4.4 follows from the inequality } \epsilon(A) \leq AD/mult_x D \text{ for any } x \in D.)$$

It can be also seen by applying usual Reider’s theorem since $K_X + 2A$ is globally generated (note that $A^2 \geq 2$ by the Riemann–Roch theorem) and $2\epsilon(A) = \epsilon(K_X + 2A) \geq 1$.

This simple remark is related to Problem 3.2, [EL].
Example 4.5. Seshadri constants on K3 surfaces.

The aim of this example is to give simple proofs of some results obtained in [BDS].

Let $A$ be an ample divisor on a K3 surface $X$. If $\epsilon(x, A) < 1$ for a point $x$ then $L$ is not globally generated. By the results of Saint–Donat it follows that $L$ is not ample effective divisor passing through $x$ point $A$. By Proposition 4.1 we get the following:

By definition $\epsilon(x, A) \leq AF/\text{mult}_xF = 1/\text{mult}_xF$. Since $\epsilon(x, A) \geq 1/2$ by the remark above, it follows that $\text{mult}_xF = 2$ and $\epsilon(x, A) = 1/2$.

Let $S$ denote a set of singular points of the fibres of $f$. As a corollary to the above by Proposition 4.1 we get the following:

1. If $A$ is globally generated then $nA$ is $k$-jet generated for $n \geq k + 2$.
2. Otherwise $A$ is of the form $aE + \Gamma$ described above and $nA$ generates $k$-jets at each point $x \notin S$ for $n \geq k + 2$ and at each point $x \in S$ for $n \geq 2k + 4$.

Clearly, 0-jet spannedness and 0-jet ampleness are the same thing but 1-jet spannedness should not imply 1-jet ampleness. However, finding an explicit example of 1-jet spanned but not 1-jet ample line bundle seems to be nontrivial. Here we provide such an example:

Example 4.6. 1-jet spanned but not 1-jet ample line bundle.

Let $X$ be a numerical Campedelli surface with ample $K_X$ and $\pi^{alg}_1(X) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Theorem 0.3, [La2] together with Proposition 5.5, [La2] say that there are only 4 degree 2 clusters which are contracted by \( 3K_X \) and all of them are scheme-theoretical intersections of unique curves from $|K_X - \tau|$ and $|K_X + \tau|$, $\tau \in \text{Tors}X - \{0\}$. We will prove that for any surface $X$ all those clusters consist of 2 distinct points. This can be proved explicitly by using Xiao’s construction of such surfaces. We recall this construction since we need it to do explicit calculations.

Let us choose homogeneous coordinates $([x_0, x_1, x_2], [y_0, y_1, y_2])$ in $\mathbb{P}^2 \times \mathbb{P}^2$ and let $\tilde{X}_\lambda$ be the complete intersection of two hypersurfaces

\[
\sum_{i=0}^2 x_iy_i = 0 \quad \text{and} \quad (\sum_{i=0}^2 x_i^3)(\sum_{i=0}^2 y_i^3) - \lambda \prod_{i=0}^2 x_iy_i = 0.
\]

For general $\lambda$ this surface is smooth and it is invariant under the action of group $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ with generators acting by

$([x_0, x_1, x_2], [y_0, y_1, y_2]) \rightarrow ([x_1, x_2, x_0], [y_1, y_2, y_0])$

and

$([x_0, x_1, x_2], [y_0, y_1, y_2]) \rightarrow ([x_0, \epsilon x_1, \epsilon^2 x_2], [y_0, \epsilon^2 y_1, \epsilon y_2]),$

where $\epsilon$ is a primitive cube root of 1. The quotient $X_\lambda$ of $\tilde{X}_\lambda$ by $G$ is a required surface. Note that $H^0(K_{\tilde{X}_\lambda}) = \bigoplus_{\tau \in \text{Tors}X} H^0(K_{X_\lambda} + \tau)$ and therefore clusters contracted by $3K_X$ are images of the points defined on $\tilde{X}_\lambda$ by pairs of equations

\[
\sum_{i=0}^2 \epsilon^{ai} x_iy_{i+b} = 0 \quad \text{and} \quad \sum_{i=0}^2 \epsilon^{-ai} x_iy_{i-b} = 0
\]
for \((a, b) = (0, 1), (1, 0), (1, 2), (1, 1)\) (the numeration is cyclic modulo 3).

To prove that those clusters consist of distinct points we should only show that for any \(\lambda\) system of equations (*) and (**) have two solutions lying in different orbits of \(G\).

We have the following solutions of (*) and (**) lying in different orbits of \(G\):
1. \([(0,0,1),[1,-1,0])\) and \([(1,-1,0),[0,0,1])\) for \((a, b) = (1, 0),\)
2. \([(0,1,-1),[1,1,1])\) and \([(1,1,1),[0,1,-1])\) for \((a, b) = (0, 1),\)
3. \([(0,1,-\epsilon^2),[1,\epsilon^2,1])\) and \([(1,\epsilon,1),[0,1,-\epsilon])\) for \((a, b) = (1, 1),\)
4. \([(1,\epsilon^2,1),[0,1,-\epsilon^2])\) and \([(0,1,-\epsilon),[1,\epsilon,1])\) for \((a, b) = (1, 2).\)

This shows that the line bundle \(O_X(3K_X)\) is 1-jet generated but not 1-jet ample.

By Theorem 0.2, [La2] one can expect that for a general surface \(X\) in the moduli space of Godeaux surfaces with \(H_2(X,\mathbb{Z}) = \mathbb{Z}_3\) the line bundle \(O(4K_X)\) is 1-jet spanned but not 1-jet ample. Although an explicit construction of such surfaces is known (see [Rd, Section 3]) it is very complicated and the calculation it involves seems discouragingly large.

References

[BDS] T. Bauer, S. Di Rocco, T. Szemberg, *Generation of jets on K3 surface*, preprint (1996).

[BS1] M. Beltrametti, A. Sommese, *Zero cycles and k-th order embeddings of smooth projective surfaces*, in Problems of the theory of surfaces and their classification, Sympos. Math. **32** (1992), 33–48.

[BS2] M. Beltrametti, A. Sommese, *On k-jet ampleness*, Complex analysis and geometry, ed. by V. Ancona and A. Silva, Plenum Press, New York, (1993), 355–376.

[EL] L. Ein, R. Lazarsfeld, *Seshadri constants on smooth surfaces*, Astérisque **218** (1993), 177–186.

[Fu] W. Fulton, *Intersection theory*, Ergeb. Math. Grenzgeb. (3) **2** (1984), Springer–Verlag.

[La1] A. Langer, *Adjoint linear systems on normal surfaces*, to appear in J. Algebraic Geom.

[La2] ———, *Pluricanonical systems on surfaces with small \(K^2\)*, preprint (1997).

[La3] ———, *Adjoint maps of algebraic surfaces*, Ph. D. Thesis (in Polish), Warsaw University (1998).

[Laz] R. Lazarsfeld, *Lectures on linear series*, Complex algebraic geometry, IAS/Park City Mathematics Series **3** (1997), 163–219.

[Rd] M. Reid, *Surfaces with \(p_g = 0, K^2 = 1\)*, J. Fac. Sci. **25** (1978), Univ. Tokyo, Sect. IA, 75–92.

[Rdr] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127** (1988), 309–316.

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