Abstract. What is the maximum possible value of the lead coefficient of a degree \(d\) polynomial \(Q(x)\) if \(|Q(1)|, |Q(2)|, \ldots, |Q(k)|\) are all less than or equal to one? More generally we write \(L_{d, [x_k]}(x)\) for what we prove to be the unique degree \(d\) polynomial with maximum lead coefficient when bounded between 1 and \(-1\) for \(x \in [x_k] = \{x_1, \ldots, x_k\}\). We calculate explicitly the lead coefficient of \(L_{d, [x_k]}(x)\) when \(d \leq 4\) and the set \([x_k]\) is an arithmetic progression. We give an algorithm to generate \(L_{d, [x_k]}(x)\) for all \(d\) and \([x_k]\).

1. Introduction

The maximum density of sets that avoid arithmetic progressions is a long standing problem [Ro53] [Sz90]. In O’Bryant’s paper [OB10] a technique for constructing...
a lower bound on this density involves building sets which try to avoid having k-term progressions map into an interval in the image of a degree d polynomial. This has lead to the question: what is the maximum possible value of the lead coefficient of a degree d polynomial Q(x) when |Q(1)|, |Q(2)|, …, |Q(k)| are all less than or equal to one?

**Theorem 1.** If M > 0 and Q(x) is a polynomial with degree d such that

\[ |Q(x)| = |a_dx^d + \cdots + a_1x + a_0| \leq M \]

for all x in the k-term arithmetic progression \( x_1, x_1 + \Delta, \ldots, x_1 + (k - 1)\Delta \) with \( k > d \), then the lead coefficient \( a_d \) of \( Q(x) \) satisfies for

\[
\begin{align*}
\text{d = 1, } a_1 & \leq \frac{M}{\Delta} \cdot \frac{2}{k - 1} \\
\text{d = 2, } a_2 & \leq \begin{cases} \\
\frac{M}{8} \Delta^2 (k - 1)^2 & \text{for } k \equiv 1 \mod 2 \\
\frac{M}{8} \Delta^2 k(k - 2) & \text{for } k \equiv 0 \mod 2 \\
\end{cases} \\
\text{d = 3, } a_3 & \leq \begin{cases} \\
\frac{M}{32} \Delta^3 (k - 1)^3 & \text{for } k \equiv 1 \mod 4 \\
\frac{M}{32} \Delta^3 k(k - 1)(k - 2) & \text{for } k \equiv 2 \mod 4 \\
\frac{M}{32} \Delta^3 (k + 1)(k - 1)(k - 3) & \text{for } k \equiv 3 \mod 4 \\
\frac{M}{32} \Delta^3 k(k - 1)(k - 2) & \text{for } k \equiv 0 \mod 4 \\
\end{cases} \\
\text{d = 4, } a_4 & \leq \begin{cases} \\
\frac{M}{32} \min_{x \in I} \left\{ \frac{8}{x^2(k - 1)^2 - 4x^4} \right\} & \text{for } k \equiv 1 \mod 2 \text{ and } I = \{ \left\lceil \frac{k - 1}{2\sqrt{2}} \right\rceil, \left\lfloor \frac{k - 1}{2\sqrt{2}} \right\rfloor \} \\
\frac{M}{32} \min_{x \in H} \left\{ \frac{8}{16x^4 - 4((k - 1)^2 + 1)x^2 + (k - 1)^2} \right\} & \text{for } k \equiv 0 \mod 2 \text{ and } H = \{ \frac{1}{4}, \frac{k - 1}{2\sqrt{2}}, \frac{1}{2} \left\lceil \frac{k - 1}{2\sqrt{2}} \right\rceil \} \\
\end{cases}
\end{align*}
\]

and for \( d \geq 5 \) we can use the algorithm described in section 5.3 to find the upper bound for \( a_d \).

Moreover, the inequalities in Theorem 1 are sharp and there is a unique and computable polynomial for which equality is achieved.

**Theorem 2.** Given a k-term arithmetic progression

\[ [x_k] = \{ x_1, x_1 + \Delta, \ldots, x_1 + (k - 1)\Delta \}, \]

for every \( d < k \) there is a unique and computable polynomial

\[ |L_{d,[x_k]}(x)| = |a_dx^d + \cdots + a_1x + a_0| \]

with maximum lead coefficient \( a_d \) such that for all \( x \in [x_k] \) we have that

\[ |L_{d,[x_k]}(x)| \leq 1. \]

Moreover, we have for
\[ d = 1, \ a_1 = \frac{M}{\Delta} \frac{2}{k-1} \]
\[ d = 2, \ a_2 = \begin{cases} \frac{M}{8} & \text{for } k \equiv 1 \mod 2 \\ \frac{\Delta^2 (k-1)^2}{8} & \text{for } k \equiv 0 \mod 2 \end{cases} \]
\[ d = 2, \ a_2 = \begin{cases} \frac{M}{32} & \text{for } k \equiv 1 \mod 4 \\ \frac{\Delta^3 (k-1)^3}{32} & \text{for } k \equiv 0 \mod 4 \end{cases} \]
\[ d = 3, \ a_3 = \begin{cases} \frac{M}{8} & \text{for } k \equiv 1 \mod 2 \\ \frac{\Delta^2 k(k-2)}{8} & \text{for } k \equiv 0 \mod 2 \\ \frac{\Delta^3 (k+1)(k-1)(k-3)}{32} & \text{for } k \equiv 3 \mod 4 \end{cases} \]
\[ d = 4, \ a_4 = \begin{cases} \frac{M}{\Delta^4} \min_{x \in I} \left\{ \frac{-32}{x^2 k^2 - 4 x^4} \right\} & \text{for } k \equiv 1 \mod 2 \text{ and } I = \left\{ \frac{k-1}{2 \sqrt{2}} \right\} \\ \frac{M}{\Delta^4} \min_{x \in H} \left\{ \frac{-32}{16 x^4 - 4((k-1)^2 + 1)x^2 + (k-1)^2} \right\} & \text{for } k \equiv 0 \mod 2 \text{ and } H = \left\{ \frac{k-1}{2 \sqrt{2}}, \frac{k-1}{4 \sqrt{2}} \right\} \end{cases} \]

and for \( d \geq 5 \) we can use the algorithm described in section 7.5 to find \( a_d \).

In general we will call polynomials \( L \)-polynomials if they have maximum possible lead coefficient while their absolute value is bounded on some finite set. More specifically

**Definition 1.** Given \( k > d \geq 1 \) and \( [x_k] = \{x_1, x_2, \cdots, x_k\} \subset \mathbb{R} \) then \( L_{d,[x_k]}(x) \) denotes the unique degree \( d \) polynomial with maximum lead coefficient such that when \( x \in [x_k] \)

\[ |L_{d,[x_k]}(x)| \leq 1. \]

The uniqueness of this polynomial is proved by Theorem 3 in Section 2. In Sections 5 and 6 we will be concerned with \( L_{d,[k]} \), here \( [k] = \{1, 2, \cdots, k\} \).

1.1. **A brief summary.** In Section 2 we discuss the connection between \( L \)-polynomials and Chebyshev \( T \)-polynomials (\( L \)-polynomials being a discrete analog of Chebyshev \( T \)-polynomials). In Section 3 the problem is recast from the perspective of combinatorial geometry. From this perspective we rule out the existence of \( L \)-polynomials when \( k \leq d \) and prove theorems that will then be used: to prove the uniqueness of all \( L_{d,[x_k]} \), to compute the lead coefficients of some \( L_{d,[k]} \), and to describe an algorithm that generates all \( L_{d,[x_k]} \). In Section 4 more such theorems are proved (but without the problem being recast from the perspective of combinatorial geometry) and the uniqueness of \( L_{d,[x_k]} \) is then proved. In Section 5 the lead coefficients of \( L_{d,[k]} \) (when \( d \leq 4 \) and all \( k > d \)) are calculated and an algorithm that generates all \( L_{d,[x_k]} \) (when \( k > d \geq 1 \)) is described. In Section 6 we write some \( L_{d,[k]} \) in terms of corresponding degree \( d \) Chebyshev \( T \)-polynomials.
Chebyshev Polynomials $T_d(x)$ are defined by

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_{d+1}(x) = 2xT_d(x) - T_{d-1}(x).$$

Alternatively, each $T_d(x)$ is the unique polynomial of degree $d$ satisfying the relationship

$$T_d(x) = \cos(d \arccos(x)).$$
Substituting \( \cos(x) \) for \( x \) this becomes
\[
T_d(\cos(x)) = \cos(dx).
\]
The first few \( T_d(x) \) polynomials are
\[
\begin{align*}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x \\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{align*}
\]
An elementary proof \([\text{Ri74}]\) shows
\[
f(x) = \frac{1}{2^{d-1}} T_d(x)
\]
to be the unique monic polynomial of degree \( d \) with minimum infinity norm on the interval \([-1, 1]\). To wit \( \|f(x)\|_\infty = \frac{1}{2^{d-1}} \). And thus \( T_d \) is the polynomial of degree \( d \) that, while bounded between \(-1\) and \( 1 \) on the interval \([-1, 1]\), has the maximum possible lead coefficient (equal to \( 2^{d-1} \), as indicated above by the previous two equations). And thus our \( L \)-polynomials, which are bounded between \(-1\) and \( 1 \) on a set of \( k \) values, can be viewed as a discrete analog of the continuously bounded \( T \)-polynomials.

We can stretch the Chebyshev polynomial \( T_d \) from the interval \([-1, 1]\) to the interval \([x_1, x_k]\) by composing it with the following bijection from \([x_1, x_k]\) to \([-1, 1]\)
\[
s(x) = \frac{2x - (x_k + x_1)}{x_k - x_1},
\]
and thus from the equations above we get the lead coefficient of \( T_d(s(x)) \) is
\[
a_d = 2^{d-1} \left( \frac{2}{x_k - x_1} \right)^d = \frac{2^{2d-1}}{(x_k - x_1)^d}.
\]
This is a lower bound for the maximum lead-coefficient we are looking for since \( |T(s(x))| \leq 1 \) for \( x \in [x_1, x_k] \) are stronger constraints than \( |L_{d, [k]}(x)| \leq 1 \) for \( x \in \{x_1, x_2, \ldots, x_k\} = [x_k] \).

3. A Combinatorial Geometry Perspective on \( L \)-Polynomials

The problem starts looking purely combinatorial when we remember that polynomials can be factored.

Consider a polynomial written in terms of its factors
\[
Q(x) = a_d \prod_{i \in [d]} (x - r_i)
\]
where \( d = \deg(Q(x)) \) and \( r_i \in \mathbb{C} \).

Now the question is: how big can \( a_d \) be while, for all \( x \in [x_k] \), the following inequality holds
\[
|Q(x)| = \left| \prod_{i \in [d]} (x - r_i) \right| = a_d \prod_{i \in [d]} |(x - r_i)| \leq 1?
\]
Thus, finding the maximum value of $a_d$ is equivalent to finding the minimum value of the maximum product of distances

$$\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} \left| x - r_i \right| \right\}$$

for all multisets $\{r_1, \cdots, r_d\} \subset \mathbb{C}$. Put concisely:

**Problem.** For a given $[x_k] = \{x_1, x_2, \cdots, x_k\} \subset \mathbb{R}$ find the minimum

$$\min_{|R|=d} \max_{x \in [x_k]} \left\{ \prod_{r_i \in R} \left| x - r_i \right| \right\}$$

taken over all multisets $R = \{r_1, \cdots, r_d\} \subset \mathbb{C}$.

This minimum is equal to $\frac{1}{a_d}$ where $a_d$ is the lead coefficient of our $L_{d,[x_k]}$.

We do not necessarily have to find the minimizing set $R$ of roots in order to find this minimum value (i.e. the reciprocal of our maximum lead coefficient). Though, perhaps it will be useful to consider how a minimizing multiset of roots $R$ must look.

We do so and continue by ruling out the existence of a maximum lead coefficient for certain cases of our problem.

**3.1. When $k \leq d$.** Construct a multiset of “roots” $R$ such that $[x_k] \subset R$ and thus all products of distances for $x \in [x_k]$ will be 0. That is, for example, define

$$\frac{|Q(x)|}{a_d} = |x - x_1|^{1+(d-k)}|x - x_2||x - x_3| \cdots |x - x_k|,$$

then for all $x \in [x_k]$

$$|Q(x)| = 0 \leq 1.$$ 

Since $|Q(x)|$ is equal to zero no matter what value is picked for $a_d$, there is no maximum $a_d$. From here on we are only concerned with the cases where $k > d$.

**3.2. Some useful theorems.** Now we prove a few lemmas using this combinatorial geometry view.

**Lemma 1.** If for $k > d \geq 2$

$$Q(x) = a_d \prod_{i \in [d]} (x - r_i)$$

is a degree $d$ polynomial such that for a given

$$[x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

we have that

$$\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} \left| x - r_i \right| \right\}$$

is minimal then the multiset $\{r_1, \cdots, r_d\}$ of all $d$ roots of $Q(x)$ is contained in $\mathbb{R}$. 
Proof. Assume that \( r_i = a_i + (\sqrt{-1})b_i \) with \( b_i \neq 0 \) for some \( i \). This means that for all \( x \in [x_k] \)
\[
|x - r_i| = \sqrt{(x - a_i)^2 + (b_i)^2} > \sqrt{(x - a_i)^2} = |x - a_i|.
\]
But then
\[
\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - r_i| \right\} > \max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - a_i| \right\},
\]
contradicting the assumed minimality of the expression (3.1) on the left-side of this inequality. It follows then that
\[
\{ r_1, \cdots, r_d \} = \{ a_1, \cdots, a_d \} \subset \mathbb{R}.
\]

Thus we have proved that the roots of our \( L \)-polynomials are all real.

**Theorem 3.** If for \( k > d \geq 2 \)
\[
Q(x) = a_d \prod_{i \in [d]} (x - r_i)
\]
is a degree \( d \) polynomial such that for a given
\[
[x_k] = \{ x_1 < x_2 < \cdots < x_k \}
\]
we have that
\[
(3.2) \quad \max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - r_i| \right\}
\]
is minimal then the multiset of the \( d \) real (Lemma 1) roots of \( Q(x) \) contains no duplicates for \( k > d \geq 2 \).

Proof. Assume \( Q(x) \) has a root of multiplicity greater than one. If we make no claim about the the ordering of the \( r_i \) we can, without loss of generality, set
\[
r = r_1 = r_2
\]
and rewrite the product of distances (i.e. the factors in our polynomial) as
\[
|x - r|^2 |x - r_3| \cdots |x - r_d|.
\]
We will show that replacing the two identical \( r \)'s with two distinct roots, \( r + \varepsilon \) and \( r - \varepsilon \), yields a smaller maximum product (3.2), a contradiction. We choose \( \varepsilon \) to be the minimum of a subset formed from \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), each of which is defined for a separate case of \( x \):

**Case I:** Set
\[
\varepsilon_1 = \min \left\{ \frac{x - r}{2} : x \in [x_k] \text{ and } x > r \right\}
\]
and thus when \( x > r \) and \( x \in [x_k] \)
\[
|x - (r + \varepsilon_1)||x - (r - \varepsilon_1)| = (x - r)^2 - \varepsilon_1^2 < |x - r|^2.
\]
Now multiplying both sides of the inequality by the other roots’ distances to \( x \) (i.e. the absolute values of \( Q(x) \)’s factors) yields
\[
|x - (r + \varepsilon_1)||x - (r - \varepsilon_1)||x - r_3| \ldots |x - r_d| < |x - r|^{2} |x - r_3| \ldots |x - r_d|
\]
when \( x > r \) and \( x \in [x_k] \).

**Case II:** Set
\[
\varepsilon_2 = \min \left\{ \frac{r - x}{2} : x \in [x_k] \text{ and } x < r \right\}
\]
and thus when \( x < r \) and \( x \in [x_k] \)
\[
|x - (r + \varepsilon_2)||x - (r - \varepsilon_2)| = (r - x)^2 - \varepsilon_2^2 < |x - r|^2.
\]
Now multiplying both sides of the inequality by the other roots’ distances to \( x \) (i.e. the absolute values of \( Q(x) \)’s factors) yields
\[
|x - (r + \varepsilon_2)||x - (r - \varepsilon_2)||x - r_3| \ldots |x - r_d| < |x - r|^2 |x - r_3| \ldots |x - r_d|
\]
when \( x < r \) and \( x \in [x_k] \).

**Case III:** If \( r \in [x_k] \) and \( r \) is a root of multiplicity two set
\[
\varepsilon_3 = \sqrt{\frac{1}{2} \max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \ldots |x - r_d| \right\}}
\]
and thus when \( x = r \) and \( x \in [x_k] \)
\[
|x - (r + \varepsilon_3)||x - (r - \varepsilon_3)| = \varepsilon_3^2 \leq \frac{1}{2} \max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \ldots |x - r_d| \right\}
\]
\[
\leq \frac{1}{2} \max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \ldots |x - r_d| \right\}
\]
\[
< \max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \ldots |x - r_d| \right\}
\]
Now multiplying both sides of the inequality by the other roots’ distances to \( x = r \) (i.e. the absolute values of \( Q(x) \)’s factors) yields
\[
|x - (r + \varepsilon_3)||x - (r - \varepsilon_3)||x - r_3| \ldots |x - r_d| < \max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \ldots |x - r_d| \right\}
\]
when \( x = r \) and \( x \in [x_k] \)

Now we choose \( \varepsilon \). If \( r \in [x_k] \) and \( r \) is a root of multiplicity two set \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \), otherwise set \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2 \} \). Using \( \varepsilon \) in the final inequality of each of the above three cases yields
\[
\max_{x \in [x_k]} \left\{ |x - (r + \varepsilon)||x - (r - \varepsilon)||x - r_3| \ldots |x - r_d| \right\} < \max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \ldots |x - r_d| \right\},
\]
contradicting the minimality of \( (3.2) \).

Thus we have proved that the roots of our \( L \)-polynomials are distinct.
Lemma 2. If for $k > d \geq 2$

$$Q(x) = a_d \prod_{i \in [d]} (x - r_i)$$

is a degree $d$ polynomial such that for a given

$$[x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

we have that

$$\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - r_i| \right\}$$

is minimal then the set $\{r_1 < r_2 < \cdots < r_d\}$ of $d$ distinct (Theorem 2) roots of $Q(x)$ is contained in the open interval $(x_1, x_k)$.

Proof. Suppose that one of the roots is outside of the open interval $(x_1, x_k)$, that is for some $\delta \geq 0$ we have $r_d = x_k + \delta$. We will show that replacing the root $r_d$ with $x_k - \varepsilon$ for some $\varepsilon > 0$ yields a smaller maximum product (3.3), a contradiction. And we will choose $\varepsilon$ to be the minimum of $\varepsilon_1$ and $\varepsilon_2$ which are now defined for two cases of $x$:

Case I: Set

$$\varepsilon_1 = \frac{x_k - x_{k-1}}{2}$$

and thus for any $\delta \geq 0$ and all $x \in \{x_1, \cdots, x_{k-1}\} = [x_{k-1}]$

$$|x - r_d| = |x - (x_k + \delta)| = (x_k + \delta) - x > \left(\frac{x_k + x_{k-1}}{2}\right) - x =$$

$$\left|x - \left(x_k - \frac{x_k - x_{k-1}}{2}\right)\right| = |x - (x_k - \varepsilon_1)|.$$

Now multiplying both sides of the inequality by the other roots’ distances to $x$ (i.e. the absolute values of $Q(x)$’s factors) yields

$$|x - r_1| \cdots |x - r_{d-1}| |x - r_d| > |x - r_1| \cdots |x - r_{d-1}| |x - (x_k - \varepsilon_1)|$$

for $x \in [x_{k-1}]$.

Case II: Set

$$\varepsilon_2 = \frac{1}{2} \max_{x \in [x_k]} \{ |x - r_1| \cdots |x - r_d| \}$$

and thus for $x = x_k$

$$|x - (x_k - \varepsilon_2)| = \varepsilon_2$$

$$= \frac{1}{2} \max_{x \in [x_k]} \{ |x - r_1| \cdots |x - r_d| \}$$

$$= \frac{1}{2} \frac{\max_{x \in [x_k]} \{ |x - r_1| \cdots |x - r_{d-1}| \}}{|x_k - r_1| \cdots |x_k - r_{d-1}|}$$

$$< \frac{\max_{x \in [x_k]} \{ |x - r_1| \cdots |x - r_d| \}}{|x_k - r_1| \cdots |x_k - r_{d-1}|}.$$

Now multiplying both sides of the inequality by the other roots’ distances to $x = x_k$ (i.e. the absolute values of $Q(x)$’s factors) yields

$$|x - r_1| \cdots |x - r_{d-1}| |x - (x_k - \varepsilon_2)| < \max_{x \in [x_k]} \{ |x - r_1| \cdots |x - r_d| \}$$

for $x = x_k$. 
Now set \( \varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \) and thus using \( \varepsilon \) in the final inequality of each of the above two cases yields
\[
\max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_{d-1}| |x - (x_k - \varepsilon)|\} < \max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_d|\}.
\]
contradicting the minimality of (3.3). The same arguments will work for attempting to place a root to the left of the interval \((x_1, x_k)\) (i.e. if for some \( \delta \geq 0 \) we have \( r_1 = x_1 - \delta \)). □

Thus we have proved that the roots of our \( L \)-polynomials are in the interval \((x_1, x_k)\). We can now say that the roots of our \( L \)-polynomials are distinct real numbers of multiplicity one in the interval \((x_1, x_k)\) by putting Lemma 1, Theorem 3 and Lemma 2 together.

4. Some More Useful Theorems (Not from the Combinatorial Geometry Perspective)

First we show that a degree \( d \) \( L \)-polynomial bounded between \(-1\) and \( 1 \) for \( x \in [x_k] \) must pass through the points \((x_1, (-1)^d)\) and \((x_k, 1)\), on both sides of the set of boundary points \( \{(x, \pm 1) : x \in [x_k]\} \)

**Lemma 3.** If for \( k > d \geq 2 \)
\[
Q(x) = a_d \prod_{i \in [d]} (x - r_i)
\]
is a degree \( d \) polynomial with maximum lead coefficient \( a_d \) such that
\[
|Q(x)| \leq 1
\]
when
\[
x \in [x_k] = \{x_1 < x_2 < \cdots < x_k\}
\]
then
\[
Q(x_1) = (-1)^d
\]
and
\[
Q(x_k) = 1.
\]

**Proof.** Let us suppose that \( Q(x_k) < 1 \). We know that \( Q(x) \) has \( d \) distinct roots and that they are contained in the interval \((x_1, x_k)\), from Theorem 3 and Lemma 2 respectively. In other words, for some
\[
x_1 < r_1 < r_2 < \cdots < r_{d-1} < r_d < x_k,
\]
we have
\[
Q(x) = a_d(x - r_1) \cdots (x - r_d).
\]
Now with some soon to be determined \( \varepsilon \) we define
\[
\hat{Q}(x) = Q(x) + \varepsilon(x - r_1) \cdots (x - r_{d-1}).
\]
So for \( x \neq r_d \) we can write
\[
\hat{Q} = \left(1 + \frac{\varepsilon}{a_d(x - r_d)}\right)Q(x).
\]
We will use this \( \hat{Q}(x) \) to contradict the maximality of \( Q(x) \)'s lead coefficient. We will choose \( \varepsilon \) to be the minimum of a subset of \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), which are now defined for three cases of \( x \):
Case I: Set

\[ \varepsilon_1 = \frac{a_d}{2} \left( \min \{ (r_d - x) : x < r_d \text{ and } x \in [x_k] \} \right) \]

and thus for \( x < r_d \) and \( x \in [x_k] \)

\[ \left| \left( 1 + \frac{\varepsilon_1}{a_d(x - r_d)} \right) Q(x) \right| = \left| \left( 1 - \frac{1}{2} \left( \min \{ (r_d - x) : x < r_d \text{ and } x \in [x_k] \} \right) \right) Q(x) \right| < |Q(x)| \leq 1 \]

Case II: Since \( Q(x_k) > 0 \) by Lemma 2, we can set

\[ \varepsilon_2 = \frac{a_d}{2} \left( 1 - \frac{1}{Q(x_k)} \right) \left( \min \{ (x - r_d) : x > r_d \text{ and } x \in [x_k] \} \right) \]

Recall that we assumed \( Q(x_k) < 1 \) and thus for \( x > r_d \) and \( x \in [x_k] \)

\[ \left| \left( 1 + \frac{\varepsilon_2}{a_d(x - r_d)} \right) Q(x) \right| = \left( 1 + \frac{1}{2} \left( \min \{ (x - r_d) : x > r_d \text{ and } x \in [x_k] \} \right) \left( 1 - \frac{1}{Q(x_k)} \right) \right) |Q(x)| \leq \left( 1 + \frac{1}{2} \left( 1 - \frac{1}{Q(x_k)} \right) \right) |Q(x)| \]

Now since \( Q(x) \) is a polynomial with positive lead coefficient, it increases to the right of its largest root \( r_d \). So continuing the above inequality we have for \( x > r_d \) and \( x \in [x_k] \)

\[ \left( 1 + \frac{1}{2} \left( 1 - \frac{1}{Q(x_k)} \right) \right) |Q(x)| \leq \left( 1 + \frac{1}{2} \left( 1 - \frac{1}{Q(x_0)} \right) \right) Q(x_0) = \frac{Q(x_0) + 1}{2} < 1 \]

Case III: If \( r_d \in [x_k] \) set

\[ \varepsilon_3 = \frac{1}{2} \frac{1}{(r_d - r_1) \cdots (r_d - r_{d-1})} \]

and thus for \( x = r_d \) and \( x \in [x_k] \)

\[ |Q(x) + \varepsilon_3(x - r_1) \cdots (x - r_{d-1})| = |Q(r_d) + \varepsilon_3(r_d - r_1) \cdots (r_d - r_{d-1})| = |Q(r_d) + \frac{1}{2}| = 0 + \frac{1}{2} < 1 \]

Otherwise, if \( r_d \notin [x_k] \), set

\[ \varepsilon_3 = 1 \]

Set \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \). Using \( \varepsilon \) for the inequalities in the above three cases gives us that

\[ |\hat{Q}(x)| < 1 \]
for \( x \in [x_k] \). Clearly the lead coefficients of \( \hat{Q}(x) \) and \( Q(x) \) are equal since we defined \( \hat{Q}(x) \) as \( Q(x) \) plus a degree \( d - 1 \) polynomial. But since \( |\hat{Q}(x)| \) is strictly less than one we can have some \( \lambda > 1 \) such that

\[
|\lambda \hat{Q}(x)| \leq 1
\]

for \( x \in [x_k] \). But then \( \lambda a_d \), the lead coefficient of \( \lambda \hat{Q} \) is greater than \( a_d \). This contradicts the maximality of the lead coefficient \( a_d \) of \( Q(x) \). The same arguments work to show that \( Q(x_1) = (-1)^d \). □

Next we prove that \( L_{d,[x_k]} \) passes through some point \((x_i, \pm 1)\) between any two of its consecutive roots.

**Lemma 4.** If for \( k > d \geq 2 \)

\[
Q(x) = a_d \prod_{i \in [d]} (x - r_i)
\]

is a degree \( d \) polynomial with maximum lead coefficient \( a_d \) such that

\[
|Q(x)| \leq 1
\]

when

\[
x \in [x_k] = \{x_1 < x_2 < \cdots < x_k\}
\]

then for any two of \( Q(x) \)'s consecutive roots \( r_i \) and \( r_{i+1} \) there exists an \( x' \in [x_k] \) such that

\[
r_i < x' < r_{i+1}
\]

and

\[
|Q(x')| = 1.
\]

**Proof.** Suppose that \( |Q(x)| < 1 \) for any and all \( x \in [x_k] \) where \( r_i < x < r_{i+1} \). We know that \( Q(x) \) has \( d \) distinct roots and that they are contained in the interval \((x_1, x_k)\), from Theorem [3] and Lemma [2] respectively. In other words, for some

\[
x_1 < r_1 < r_2 < \cdots < r_{d-1} < r_d < x_k,
\]

we have

\[
Q(x) = a_d(x - r_1) \cdots (x - r_d).
\]

Now with some soon to be determined \( \varepsilon > 0 \) we define

\[
\hat{Q}(x) = Q(x) - \varepsilon(x - r_1) \cdots (x - r_{i-1})(x - r_{i+1}) \cdots (x - r_d).
\]

So for \( x \notin \{r_i, r_{i+1}\} \) and \( x \in [x_k] \) we can write

\[
\hat{Q}(x) = \left(1 - \frac{\varepsilon}{a_d(x - r_i)(x - r_{i+1})}\right) Q(x).
\]

We will use this \( \hat{Q}(x) \) to contradict the maximality of \( Q(x) \)'s lead coefficient. We will choose \( \varepsilon \) to be the minimum of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \varepsilon_4 \), which are now defined for four cases of \( x \):

**Case I:** Set

\[
\varepsilon_1 = \frac{a_d}{2} \min \{(x - r_i)(x - r_{i+1}) : x \in [x_k] \text{ and } (x < r_i \text{ or } r_{i+1} < x)\}
\]
and thus for \((x < r_1 \text{ or } r_{i+1} < x)\) and \(x \in [x_k]\)

\[
\left| \frac{1 - \frac{\varepsilon_1}{a_d(x - r_i)(x - r_{i+1})}}{2} \right| Q(x) = \left( 1 - \frac{1}{2} \frac{\min \{(x - r_i)(x - r_{i+1}) : x \in [x_k] \text{ and } (x < r_i \text{ or } r_{i+1} < x)\}}{Q(x)} \right) \frac{\varepsilon_2}{Q(x)} \leq \frac{1}{Q(x)} \frac{\varepsilon_2}{Q(x)} \leq 1
\]

**Case II:** If there exists an \(x \in [x_k]\) such that \(r_i < x < r_{i+1}\) then first set

\[
Q_{\text{max}} = \max \{|Q(x)| : x \in [x_k] \text{ and } r_i < x < r_{i+1}\}.
\]

Recall that we assumed that \(|Q(x)| < 1\) for any and all \(x \in [x_k]\) where \(r_i < x < r_{i+1}\). This means \(0 < Q_{\text{max}} < 1\). Next set

\[
\varepsilon_2 = \frac{a_d}{2} \left( \frac{\varepsilon}{Q_{\text{max}}} - 1 \right) \left( (x - r_i)(x - r_{i+1}) : x \in [x_k] \text{ and } r_i < x < r_{i+1} \right)
\]

otherwise, if there does not exist an \(x \in [x_k]\) such that \(r_i < x < r_{i+1}\), set \(\varepsilon_2 = 1\)

**Case III:** If \(r_i \in [x_k]\) set

\[
\varepsilon_3 = \frac{1}{2} \frac{1}{(r_i - r_1) \cdots (r_i - r_{i-1})(r_i - r_{i+1}) \cdots (r_i - r_d)}
\]

and thus for \(x = r_i\) and \(x \in [x_k]\)

\[
|Q(x) + \varepsilon_3(x - r_1) \cdots (x - r_{i-1})(x - r_{i+1}) \cdots (x - r_d)| = |Q(r_i) + \varepsilon_3(r_i - r_1) \cdots (r_i - r_{i-1})(r_i - r_{i+1}) \cdots (r_i - r_d)| = |Q(r_i) + \frac{1}{2}| = 0 + \frac{1}{2} < 1.
\]

otherwise, if \(r_i \notin [x_k]\), set \(\varepsilon_3 = 1\)

**Case IV:** If \(r_{i+1} \in [x_k]\) set

\[
\varepsilon_4 = \frac{1}{2} \frac{1}{(r_{i+1} - r_1) \cdots (r_{i+1} - r_{i-1})(r_{i+1} - r_{i+2}) \cdots (r_{i+1} - r_d)}
\]

and thus for \(x = r_{i+1}\) and \(x \in [x_k]\)

\[
|Q(x) + \varepsilon_4(x - r_1) \cdots (x - r_{i-1})(x - r_{i+1}) \cdots (x - r_d)| = |Q(r_{i+1}) + \varepsilon_4(r_{i+1} - r_1) \cdots (r_{i+1} - r_{i-1})(r_{i+1} - r_{i+2}) \cdots (r_{i+1} - r_d)| = |Q(r_{i+1}) + \frac{1}{2}| = 0 + \frac{1}{2} < 1.
\]

otherwise, if \(r_{i+1} \notin [x_k]\), set \(\varepsilon_4 = 1\)
Now set $\varepsilon = \min \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. Using $\varepsilon$ in each of the above four cases yields

$$|\hat{Q}(x)| < 1$$

for $x \in [x_k]$. Clearly the lead coefficients of $\hat{Q}(x)$ and $Q(x)$ are equal since we have defined $\hat{Q}(x)$ as $Q(x)$ plus a degree $d - 2$ polynomial. But since $|\hat{Q}(x)|$ is strictly less than one we can have some $\lambda > 1$ such that

$$|\lambda \hat{Q}(x)| \leq 1$$

for $x \in [x_k]$. But then then $\lambda a_d$, the lead coefficient of $\lambda \hat{Q}$ is greater than $a_d$. This contradicts the maximality of the lead coefficient $a_d$ of $Q(x)$. □

Finally we prove that there is a unique $L_{d,[x_k]}(x)$ for every $[x_k]$ and $d$ whenever $k > d \geq 1$.

**Theorem 4.** If for $k > d \geq 2$

$$L_{d,[x_k]}(x) = a_d \prod_{i \in [d]} (x - r_i)$$

is a degree $d$ polynomial with maximum lead coefficient $a_d$ such that

$$|L_{d,[x_k]}(x)| \leq 1$$

when

$$x \in [x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

then $L_{d,[x_k]}(x)$ is the UNIQUE degree $d$ polynomial with maximum lead coefficient $a_d$ such that for a given

$$[x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

we have that for $x \in [x_k]$

$$|L_{d,[x_k]}| \leq 1.$$

**Proof.** Assume that $\hat{Q}(x)$ and $\hat{Q}(x)$ are two degree $d$ polynomials with the same lead coefficient $a_d > 0$ and satisfying our condition that $a_d$ is the maximum lead coefficient such that both

$$|\hat{Q}(x)| \leq 1$$

and

$$|\hat{Q}(x)| \leq 1$$

when $x \in [x_k]$. Now form the average

$$\hat{Q}(x) = \frac{\hat{Q}(x) + \hat{Q}(x)}{2}.$$ 

This average also clearly has lead coefficient $a_d$ and satisfies our condition since

$$|\hat{Q}(x)| = \left|\frac{\hat{Q}(x) + \hat{Q}(x)}{2}\right| \leq \frac{|\hat{Q}(x)| + |\hat{Q}(x)|}{2} \leq 1$$

when $x \in [x_k]$. By Theorem 3 Lemma 3 and Lemma 4 for some $a_1 < a_2 < \cdots < a_{d-1}$ such that

$$\{a_1, a_2, \cdots, a_{d-1}\} \subset \{x_2, x_3, \cdots, x_{k-1}\}$$

and with

$$\{b_1, b_2, \cdots, b_{d+1}\} = \{x_1, a_1, a_2, \cdots a_{d-1}, x_k\}$$
we have that the polynomial \( \hat{Q}(x) \) passes through the \( d + 1 \) points

\[
(b_i, (-1)^{(d+1)-i}) \text{ for } i \in [d + 1]
\]
or, stated differently, that

\[
\hat{Q}(b_i) = (-1)^{(d+1)-i}
\]

for \( i \in [d + 1] \). But then because of our conditions on \( \hat{Q}(x) \) and \( \hat{Q}(x) \) bounding
them between \(-1\) and 1 for \( x \in [x_k] \) and since \( \hat{Q}(x) \) is an average of these two
polynomials, we must also have that

\[
\hat{Q}(b_i) = \hat{Q}(b_i) = (-1)^{(d+1)-i}
\]

for \( i \in [d + 1] \). But if the two degree \( d \) polynomials \( \hat{Q}(x) \) and \( \hat{Q}(x) \) are equal at
\( d + 1 \) points then they are equal everywhere. Thus

\[
L_{d,[x_k]}(x) = \hat{Q}(x) = \hat{Q}(x) = \hat{Q}(x).
\]

\( \square \)

5. Calculating the Lead Coefficients of \( L_{d,[k]} \) for \( d \leq 4 \) and an
Algorithm for Generating All \( L_{d,[x_k]} \)

This section begins with the algebraic computation of the lead coefficients for
some specific examples of \( L_{d,[x_k]} \). Namely, when \([x_k]\) is the arithmetic progression
\([k] = \{1, \ldots, k\} \). Recall from Definition 3 that \( L_{d,[x_k]} \) is the unique degree \( d \)
polynomial with maximum lead coefficient when bounded between \(-1\) and 1 on the
set \([k]\). This section ends with the description of an algorithm for generating all
\( L_{d,[x_k]} \) (and thus their lead coefficients).

5.1. Lead Coefficients of \( L_{1,[k]} \). This case is simple. Lemma 4 forces \( L_{1,[k]}(1) = -1 \) and \( L_{1,[k]}(k) = 1 \). Since \( L_{1,[k]}(x) \) is a line and we have two points lying on it
we find

\[
L_{1,[k]}(x) = a_1 x + a_0 = \frac{2}{k - 1} x - \frac{k + 1}{k - 1}
\]

Thus the lead coefficients of \( L_{1,[k]}(x) \)

\[
a_1 = \frac{2}{k - 1}.
\]

5.2. Lead Coefficients of \( L_{2,[k]} \). Recall that \( L_{2,[k]}(x) \) is the unique (by Theorem
4) degree 2 polynomial bounded between \(-1\) and 1 for \( x \in [k] = \{1, \ldots, k\} \) with
maximum lead coefficient. We shift \( L_{2,[k]}(x) \) so that the \( k \) consecutive \( x \) values
it is bounded on are centered at zero (that is, instead of being bounded for \( x \in\)
\( \{1, \ldots, k\} = [k] \) it is bounded for \( x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2}\} \)). This alters neither the
shape of the polynomial in general nor its lead coefficient. We set

\[
k' = \left\{ -\frac{k-1}{2}, \ldots, \frac{k-1}{2} \right\}
\]

and thus \( L_{2,k'}(x) \) refers to this shifted polynomial which, we will see, minimizes the
number of unknown coefficients. Observe that since \( \deg(L_{2,k'}(x)) = 2 \) is even, the
difference

\[
\frac{L_{2,k'}(x) + L_{2,k'}(-x)}{2} = a_2 x^2 + a_0
\]
describes an even-function that has the same lead coefficient as \( L_{2,k'}(x) \). Since 
\(|L_{2,k'}(x)| \leq 1\) for \( x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}\) then also 
\( \left| \frac{L_{2,k'}(x)+L_{2,k'}(-x)}{2} \right| \leq 1\) for \( x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}\). But by the uniqueness of \( L_{2,k'}(x) \) (from Theorem 4) we must have
\[
L_{2,k'}(x) = \frac{L_{2,k'}(x)+L_{2,k'}(-x)}{2}
\]
and thus
\[
L_{2,k'}(x) = a_2x^2 + a_0.
\]
Two unknown coefficients are better than three. From Lemma 3 \( L_{2,k'} \) passes through \((\frac{k-1}{2}, 1)\). Thus
\[
L_{2,k'} \left( \frac{k-1}{2} \right) = a_2 \left( \frac{k-1}{2} \right)^2 + a_0 = 1.
\]
Solving for \( a_0 \) gives
\[
a_0 = 1 - a_2 \left( \frac{k-1}{2} \right)^2.
\]
Plugging this into
\[
a_2x^2 + a_0 \leq 1
\]
\[
a_2x^2 + a_0 \geq -1,
\]
the constraining inequalities on \( L_{2,k'}(x) \) for \( x \in \{-1, \cdots, 1\}\), we get
\[
a_2 \left( x^2 - \left( \frac{k-1}{2} \right)^2 \right) \leq 0 \tag{5.1}
\]
\[
a_2 \leq \frac{8}{(k-1)^2 - 4x^2} \tag{5.2}
\]
for \( x \in \{-1, \cdots, 1\}\).

Inequality \((5.1)\) is not useful because it is always true, since \( a_2 > 0 \) and \( x^2 < \left( \frac{k-1}{2} \right)^2 \). Inequality \((5.2)\) however gives an upper bound on \( a_2 \). The right-side of inequality \((5.2)\) is minimum for \( x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}\) when \( x \) is closest or equal to zero. For odd \( k \) it is minimum when \( x = 0 \). For even \( k \) it is minimum when \( x = \pm \frac{1}{2} \). Plugging the minimizing values of \( x \) into inequality \((5.2)\) yields
\[
a_2 = \begin{cases} 
\frac{8}{(k-1)^2} & \text{for } k \equiv 1 \mod 2 \\
\frac{8}{k(k-2)} & \text{for } k \equiv 0 \mod 2 
\end{cases}
\]
as the lead coefficient for \( L_{2,k'}(x) \) and thus also for \( L_{2,[k]}(x) \).

5.3. **Lead coefficients of \( L_{3,[k]} \)**: Recall that \( L_{3,[k]}(x) \) is the unique (by Theorem 4) degree 3 polynomial that while bounded between \(-1\) and \(1\) for \( x \in [k] = \{1, \cdots, k\} \) has maximum possible lead coefficient. We shift \( L_{3,[k]}(x) \) so that, as with \( L_{2,[k]}(x) \) above, the \( k \) consecutive \( x \) values it is bounded on are centered at zero (that is, instead of being bounded for \( x \in \{1, \cdots, k\} \) it is bounded for \( x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}\)).
This alters neither the shape of the polynomial in general nor its lead coefficient. Again
\[ k' = \left\{ -\frac{k-1}{2}, \ldots, \frac{k-1}{2} \right\}. \]

So \( L_{3,k'}(x) \) refers specifically to this shifted polynomial which will minimize the number of unknown coefficients. Observe that since \( \deg(L_{3,k'}(x)) = 3 \) is odd, the difference
\[ \frac{L_{3,k'}(x) - L_{3,k'}(-x)}{2} = a_3x^3 + a_1x \]
describes an odd-function that has the same lead coefficient as \( L_{3,k'}(x) \). Since \( |L_{3,k'}(x)| \leq 1 \) for \( x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2} \} \) then also \( \left| \frac{L_{3,k'}(x) - L_{3,k'}(-x)}{2} \right| \leq 1 \) for \( x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2} \} \). But then by uniqueness (from Theorem 4) we must have
\[ L_{3,k'}(x) = \frac{L_{3,k'}(x) - L_{3,k'}(-x)}{2} \]
and thus
\[ L_{3,k'}(x) = a_3x^3 + a_1x. \]

Two unknown coefficients are better than four.

From Lemma 3, \( L_{3,k'} \) passes through \( (\frac{k-1}{2}, 1) \). Thus
\[ L_{3,k'} \left( \frac{k-1}{2} \right) = a_3 \left( \frac{k-1}{2} \right)^3 + a_1 \left( \frac{k-1}{2} \right) = 1. \]

Solving for \( a_1 \) gives
\[ a_1 = \frac{8 - a_3(k-1)^3}{4(k-1)}. \]

Plugging this into
\[ a_3x^3 + a_1x^1 \leq 1 \]
\[ a_3x^3 + a_1x^1 \geq -1, \]
the constraining inequalities on \( L_{3,k'}(x) \) for \( x \in \{-\frac{k-1}{2} + 1, \ldots, \frac{k-1}{2} - 1\} \), we get
\[ a_3 \geq \frac{-4}{2(k-1)x^2 + (k-1)^2x} \tag{5.3} \]
\[ a_3 \leq \frac{-4}{2(k-1)x^2 - (k-1)^2x} \tag{5.4} \]
for \( x \in \{-\frac{k-1}{2} + 1, \ldots, \frac{k-1}{2} - 1\} \).

Inequality \( \text{(5.3)} \) is not useful. It gives lower bounds for our positive \( a_3 \) but we are looking for the maximum possible value of \( a_3 \). Inequality \( \text{(5.4)} \) however gives us upper bounds for \( a_3 \). The right-side of inequality \( \text{(5.4)} \) is minimum on \( x \in \{-\frac{k-1}{2} + 1, \ldots, \frac{k-1}{2} - 1\} \) for \( x \) closest or equal to \( \frac{k-1}{2} \). Plugging the minimizing
values of $x$ into the inequality (5.4) yields

\[
a_3 = \begin{cases} 
32(k-1)^3 & \text{when } k \equiv 1 \text{ mod } 4, \text{ set } x = \frac{k-1}{4} \\
32k(k-1)(k-2) & \text{when } k \equiv 2 \text{ mod } 4, \text{ set } x = \frac{k}{4} \\
32(k+1)(k-1)(k-3) & \text{when } k \equiv 3 \text{ mod } 4, \text{ set } x = \frac{k-3}{4} \text{ or } x = \frac{k+1}{4} \\
k(k-1)(k-2) & \text{when } k \equiv 0 \text{ mod } 4, \text{ set } x = \frac{k-2}{4} 
\end{cases}
\]

for the lead coefficient of $L_{3,k'}(x)$ and thus also for $L_{3,[k]}(x)$.

5.4. **Lead coefficients of $L_{4,[k]}$.** Using the same arguments as from $L_{2,[k]}$ above we know that the maximum possible lead coefficient $a_4 > 0$ of the polynomial and even-function

\[
L_{4,k'}(x) = a_4 x^4 + a_2 x^2 + a_0,
\]

where $k' = \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$ and $k > 4$, is the same as the maximum possible lead coefficient of $L_{4,[k]}$.

This $d = 4$ case is itself split into two cases, one for odd $k$ and one for even $k$:

**Case I:** For odd $k$. By Theorem 3 and 4 we have the equation

\[
L_{4,k'}(0) = 1
\]

and so

\[
a_0 = 1.
\]

By Lemma 3 we have the equation

\[
L_{4,k'} \left( \frac{k-1}{2} \right) = 1
\]

and thus

\[
a_4 \left( \frac{k-1}{2} \right)^4 + a_2 \left( \frac{k-1}{2} \right)^2 + a_0 = 1
\]

Rewriting this in terms of $a_2$ after substituting 1 for $a_0$ gives us

\[
a_2 = -a_4 \left( \frac{k-1}{2} \right)^2
\]

Recall that the lower bound is

\[
L_{4,k'}(x) \geq -1
\]

for $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$. Plugging the results from above into this inequality gives

\[
a_4 x^4 - a_4 \left( \frac{k-1}{2} \right)^2 x^2 + 1 \geq -1
\]

and thus

\[
a_4 \leq \frac{8}{x^2(k-1)^2 - 4x^4}
\]

for $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$.

Optimization of the right-side of this last inequality on the interval $(0, \frac{k-1}{2})$ gives a minimum at $x = \frac{k-1}{2\sqrt{2}}$. We pick the neighboring integer
point that gives the smallest upper bound: so for odd \( k \) the maximum lead coefficient of \( L_{4,k'}(x) \), and thus also of \( L_{4,[k]}(x) \), is

\[
a_4 = \min_{x \in I} \left\{ \frac{8}{x^2(k-1)^2 - 4x^4} \right\}
\]

where \( I = \left\{ \left\lceil \frac{k-1}{2\sqrt{2}} \right\rceil, \left\lfloor \frac{k-1}{2\sqrt{2}} \right\rfloor \right\} \).

**Case II:** For even \( k \). Again by Theorem 3 and Lemma 4 we have

\[
L_{4,k'}\left(\frac{1}{2}\right) = 1
\]

which yields

\[
L_{4,k'}\left(\frac{1}{2}\right) = a_4\left(\frac{1}{2}\right)^4 + a_2\left(\frac{1}{2}\right)^2 + a_0 = 1
\]

and thus

\[
a_0 = 1 - \frac{a_2}{4} - \frac{a_4}{16}
\]

And again by Lemma 3 we have the second equation

\[
L_{4,k'}\left(\frac{k-1}{2}\right) = 1
\]

and thus

\[
a_4\left(\frac{k-1}{2}\right)^4 + a_2\left(\frac{k-1}{2}\right)^2 + a_0 = 1
\]

and so by combining the two equations (5.5) and (5.6) we have

\[
a_0 = a_4\frac{(k-1)^2}{16} + 1
\]

\[
a_2 = -a_4\frac{(k-1)^2 + 1}{4}.
\]

Recall that the lower bound on our polynomial is

\[
L_{4,[k]}(x) \geq -1
\]

for \( x \in \{-\left(\frac{k-1}{2}\right), \cdots, \frac{k-1}{2}\} \). Plugging the results from above section into this inequality yields

\[
a_4 \leq \frac{-32}{16x^4 - 4((k-1)^2 + 1)x^2 + (k-1)^2}
\]

for \( x \in \{-\left(\frac{k-1}{2}\right), \cdots, \frac{k-1}{2}\} \)

Optimization of the right-side of this last inequality on \( (\frac{1}{2}, \frac{k-1}{2}) \) gives a minimum at \( x = \frac{k-1}{2\sqrt{2}} \). We pick the neighboring HALF-integer point that gives the smallest upper bound. So for even \( k \) the lead coefficients of \( L_{4,k'}(x) \), and thus also of \( L_{4,[k]}(x) \), is

\[
a_4 = \min_{x \in H} \left\{ \frac{-32}{16x^4 - 4((k-1)^2 + 1)x^2 + (k-1)^2} \right\}
\]

where \( H = \left\{ \frac{1}{2} \left\lceil \frac{k-1}{\sqrt{2}} \right\rceil, \frac{1}{2} \left\lfloor \frac{k-1}{\sqrt{2}} \right\rfloor \right\} \).
5.5. An algorithm generating all $L_{d,|x_k|}$. For $d = 5$ only one of the three coefficients, $\{a_5, a_3, a_1\}$, can be eliminated when using the algebraic techniques from the sections on $d \leq 4$. For $d = 6$ only two of the four coefficients, $\{a_6, a_4, a_2, a_0\}$, can be eliminated. Fortunately, putting together some of our theorems and lemmas yields an algorithm that generates $L_{d,|x_k|}(x)$ for all $d \geq 1$ and $k > d$.

We begin similarly to Theorem 4. For a given

$$[x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

if $L_{d,|x_k|}(x)$ is the degree $d$ polynomial with maximum lead coefficient $a_d$ such that when $x \in [x_k]$

$$|L_{d,|x_k|}(x)| \leq 1$$

then by Theorem 3, Lemma 3 and Lemma 4 there is some $A = \{a_1 < a_2 < \cdots < a_{d-1}\} \subset \{x_2, x_3, \cdots, x_{k-1}\}$, and thus

$$\{b_1, b_2, \cdots, b_{d+1}\} = \{x_1, a_1, a_2, \cdots, a_{d-1}, x_k\},$$

such that the polynomial $L_{d,|x_k|}(x)$ passes through the $d + 1$ points

$$(b_i, (-1)^{(d+1)-i}) \text{ for } i \in [d+1].$$

But now since $L_{d,|x_k|}(x)$ is a degree $d$ polynomial we could solve for it explicitly if we knew the set $A$. With a computer this is easily done. For each of the $\binom{k-1}{d-1}$ possible $d-1$ element sets $A \subset \{x_2, x_3, \cdots, x_{k-1}\}$, solve for the polynomial passing through the corresponding points

$$(b_i, (-1)^{(d+1)-i}) \text{ for } i \in [d+1].$$

From the multiset of all $\binom{k-2}{d-1}$ such polynomials choose the subset of polynomials bounded between $-1$ and $1$ for $x \in [x_k]$. From this subset the polynomial with maximum lead coefficient is our $L_{d,|x_k|}(x)$ and its maximum lead coefficient is our $a_d$. In particular, setting

$$\{x_1, x_2, \cdots, x_k\} = \{1, 2, \cdots, k\}$$

yields $L_{d,|k|}(x)$.

6. Some $L_{d,|k|}$ in Terms of Chebyshev Polynomials

Now we write some of the $L_{d,|k|}(x)$ in terms of corresponding Chebyshev $T_d(x)$ polynomials. We compose our $L_{d,|k|}(x)$ with

$$t(x) = \frac{k - 1}{2} x + \frac{k + 1}{2}$$

in order to fit the points $|k| = \{1, ..., k\}$ (on which our $L_{d,|k|}(x)$ are bounded between $-1$ and 1) into the interval $(-1, 1)$ (on which Chebyshev’s $T_d(x)$ are bounded between $-1$ and 1).

6.1. $L_{1,|k|}$ in terms of $T_1$.

$$L_{1,|k|} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = x = T_1(x)$$
6.2. \( L_{2,[k]} \) in terms of \( T_2 \).

\[
L_{2,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_2(x) \quad \text{for } k \equiv 1 \mod 2
\]

\[
L_{2,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_2(x) + \frac{2}{k(k - 2)}(x^2 - 1) \quad \text{for } k \equiv 0 \mod 2
\]

6.3. \( L_{3,[k]} \) in terms of \( T_3 \).

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) \quad \text{for } k \equiv 1 \mod 4
\]

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) + \frac{4}{k(k - 2)}(x^3 - x) \quad \text{for } k \equiv 2 \mod 4
\]

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) + \frac{16}{(k + 1)(k - 3)}(x^3 - x) \quad \text{for } k \equiv 3 \mod 4
\]

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) + \frac{4}{k(k - 2)}(x^3 - x) \quad \text{for } k \equiv 0 \mod 4
\]

6.4. \( L_{4,[k]} \) in terms of \( T_4 \). For the \( d = 4 \) case the method of solving for the lead coefficient \( a_4 \) does not directly yield a modular pattern like it did for the \( d \in \{1, 2, 3\} \) cases. For \( 21 \geq k \geq 5 \) we have

\[
L_{4,[5]} \left( \frac{4}{2} x + \frac{6}{2} \right) = T_4(x) + \frac{8}{3}(x^4 - x^2)
\]

\[
L_{4,[6]} \left( \frac{5}{2} x + \frac{7}{2} \right) = T_4(x) + \frac{1}{64}(x^2 - 1)(113x^2 - 25)
\]

\[
L_{4,[7]} \left( \frac{6}{2} x + \frac{8}{2} \right) = T_4(x) + \frac{1}{10}(x^4 - x^2)
\]

\[
L_{4,[8]} \left( \frac{7}{2} x + \frac{9}{2} \right) = T_4(x) + \frac{1}{288}(x^2 - 1)(97x^2 - 49)
\]

\[
L_{4,[9]} \left( \frac{8}{2} x + \frac{10}{2} \right) = T_4(x) + \frac{8}{63}(x^4 - x^2)
\]

\[
L_{4,[10]} \left( \frac{9}{2} x + \frac{11}{2} \right) = T_4(x) + \frac{1}{256}(x^2 - 1)(139x^2 - 27)
\]

\[
L_{4,[11]} \left( \frac{10}{2} x + \frac{12}{2} \right) = T_4(x) + \frac{49}{72}(x^4 - x^2)
\]

\[
L_{4,[12]} \left( \frac{11}{2} x + \frac{13}{2} \right) = T_4(x) + \frac{1}{1728}(x^2 - 1)(817x^2 - 121)
\]

\[
L_{4,[13]} \left( \frac{12}{2} x + \frac{14}{2} \right) = T_4(x) + \frac{1}{10}(x^4 - x^2)
\]

\[
L_{4,[14]} \left( \frac{13}{2} x + \frac{15}{2} \right) = T_4(x) + \frac{1}{3520}(x^2 - 1)(401x^2 - 169)
\]

\[
L_{4,[15]} \left( \frac{14}{2} x + \frac{16}{2} \right) = T_4(x) + \frac{1}{300}(x^4 - x^2)
\]
\[
L_{4,[16]} \left( \frac{15}{2}x + \frac{17}{2} \right) = T_4(x) + \frac{1}{416}(x^2 - 1)(47x^2 - 15)
\]
\[
L_{4,[17]} \left( \frac{16}{2}x + \frac{18}{2} \right) = T_4(x) + \frac{8}{63}(x^4 - x^2)
\]
\[
L_{4,[18]} \left( \frac{17}{2}x + \frac{19}{2} \right) = T_4(x) + \frac{1}{10080}(x^2 - 1)(2881x^2 - 289)
\]
\[
L_{4,[19]} \left( \frac{18}{2}x + \frac{20}{2} \right) = T_4(x) + \frac{1}{10}(x^4 - x^2)
\]
\[
L_{4,[20]} \left( \frac{19}{2}x + \frac{21}{2} \right) = T_4(x) + \frac{1}{16128}(x^2 - 1)(1297x^2 - 361)
\]
\[
L_{4,[21]} \left( \frac{20}{2}x + \frac{22}{2} \right) = T_4(x) + \frac{8}{2499}(x^4 - x^2).
\]

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E-mail address: Karl.Ethan.Levy@Gmail.com