The Universal Kummer Threefold

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Abstract

The universal Kummer threefold is a 9-dimensional variety that represents the total space of the 6-dimensional family of Kummer threefolds in $\mathbb{P}^7$. We compute defining polynomials for three versions of this family, over the Satake hypersurface, over the Göpel variety, and over the reflection representation of type $E_7$. We develop classical themes such as theta functions and Coble’s quartic hypersurface using current tools from combinatorics, geometry, and commutative algebra. Symbolic and numerical computations for genus 3 moduli spaces appear alongside toric and tropical methods.

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1 Introduction

Kummer varieties are quotients of abelian varieties by their involution $x \mapsto -x$. Each $g$-dimensional Kummer variety has a natural embedding into $\mathbb{P}^{2g-1}$ by second order theta functions. The moduli space is recorded in a second copy of $\mathbb{P}^{2g-1}$ by the corresponding theta constants. By taking the closure, this construction defines the universal Kummer variety $\mathcal{K}_g$. This is an irreducible projective variety of dimension $\binom{g+1}{2} + g$, defined over the integers, and naturally embedded in $\mathbb{P}^{2g-1} \times \mathbb{P}^{2g-1}$. The image of the projection of $\mathcal{K}_g$ onto the $\mathbb{P}^{2g-1}$ of theta constants is the Satake compactification of the $\binom{g+1}{2}$-dimensional moduli space $A_g(2, 4)$. The various Kummer varieties of dimension $g$ appear as fibers of this map.

Our object of interest is the universal Kummer ideal $\mathcal{I}_g$. This is the bihomogeneous prime ideal in a polynomial ring in $2^{g+1}$ unknowns that defines $\mathcal{K}_g$ as a subvariety of $\mathbb{P}^{2g-1} \times \mathbb{P}^{2g-1}$. 
What motivated this project was our desire to understand the ideal \( I_3 \) of the universal Kummer threefold in \( \mathbb{P}^7 \times \mathbb{P}^7 \). Grushevsky and Salvati Manni write in [GS1, §6] that the ideal \( I_3 \) “is known quite explicitly”, and we were wondering how to communicate its generators to Macaulay 2 [M2]. Our current state of knowledge about this question is presented in Section 8 of this paper. The example of \( I_2 \) is worked out in Example 1.1 below.

In our attempts to understand \( I_3 \), we also studied a variant of \( K_3 \), over a base \( G \) that is a moduli space for plane quartics. We discovered a beautiful mathematical story that connects classical algebraic geometry topics, such as Coble’s quartic hypersurface, Göpel functions, and the type \( E_7 \) reflection arrangement, with more modern topics, such as toric geometry, tropical methods, and numerical computation. Our study of this variation was originally inspired by Vinberg’s theory of \( \theta \)-representations [Vin], but can be defined without any reference to this theory. However, as we shall see in Section 8, this point of view gives the construction of a large number of equations that we cannot see how to construct otherwise.

Moduli of plane quartics have been studied extensively, notably in [Cob] and [DO, §IX]. A key feature of Kummer varieties of the Jacobians of plane quartics is this: in their embedding in \( \mathbb{P}^7 \), there is a unique quartic hypersurface, called the Coble quartic, whose singular locus is the Kummer variety. We are interested in the moduli space of Coble quartics. This moduli space can be parametrized by the Göpel functions mentioned in [DO, §IX]. These functions embed it as a subvariety \( G \) of \( \mathbb{P}^{134} \), so we call it the Göpel variety. In fact, it sits in a linear \( \mathbb{P}^{14} \subset \mathbb{P}^{134} \), and it can be alternatively parametrized by a Macdonald representation of the Weyl group of type \( E_7 \). Hence the rich combinatorics of reflection arrangements is embodied in \( G \). Colombo, van Geemen and Looijenga [CGL] studied the Göpel variety from this perspective, and realized it as a moduli space of marked del Pezzo surfaces of degree 2.

It is natural to ask for the prime ideal of \( G \) embedded in \( \mathbb{P}^{14} \). We solve this problem and calculate its graded Betti numbers. A delightful picture emerges in the embedding in \( \mathbb{P}^{134} \). Together with trinomial linear equations that define \( \mathbb{P}^{14} \) in \( \mathbb{P}^{134} \), all equations are written as binomials. This shows that there is a larger toric variety \( T \) sitting in \( \mathbb{P}^{134} \) that contains \( G \) as a linear section. In fact, there are 35 cubics and 35 quartics that cut out \( G \). The cubics are genus 3 analogues of the toric Segre cubic relations of Howard et al. in [HMSV, (1.2)].

We obtain a range of results on \( T \) and the corresponding 35-dimensional polytope \( A \). It has 135 vertices and 63 distinguished facets, and their incidence relations admit a compact description in terms of the finite symplectic space \( (\mathbb{F}_2)^6 \). Furthermore, it exhibits symmetry under the Weyl group of type \( E_7 \). It should be very fruitful to further study the combinatorics of the polytope \( A \), such as its face lattice and Ehrhart polynomial. Even more important, this also paves the way for studying the tropicalization of \( G \), which we hope will provide a useful model for genus 3 curves over fields with a non-trivial, non-archimedean valuation.

All relevant basics regarding theta functions and abelian varieties will be reviewed in the next two sections, along with pointers to the literature and to software. As a warm-up, we first present the solution to our motivating problem for the much easier case of genus \( g = 2 \).

**Example 1.1** (The Universal Kummer surface). The five-dimensional variety \( K_2 \) is a hypersurface of degree \((12, 4)\) in \( \mathbb{P}^3 \times \mathbb{P}^3 \). Its principal prime ideal \( I_2 \) in the polynomial ring...
For fixed $u_{ij}$, this $5 \times 5$-determinant defines Kummer’s quartic as a surface in $\mathbb{P}^3$, with coordinates $(x_{00}:x_{01}:x_{10}:x_{11})$, written as in [BL, Exercise 3, page 204] or [Mum, page 354].

Note that the last four rows in (1.1) represent the Jacobian matrix of the first row. The matrix derives from the fact that $(u_{00}:u_{01}:u_{10}:u_{11})$ must be a singular point on the Kummer surface. The surface and its 16 nodes are invariant under the group $(\mathbb{Z}/2\mathbb{Z})^4$ which acts by sign changes and permutations [Mum, page 353]. The 15 other nodes are easily found:

$$(u_{00} : -u_{10} : u_{01} : -u_{11}), (u_{00} : u_{10} : -u_{01} : -u_{11}), \ldots, (u_{11} : -u_{01} : -u_{10} : u_{00}).$$

A convenient representation of the 16₆ configuration of nodes is the matrix product

$$(u_{00} u_{10} u_{01} u_{11}) (u_{00} u_{10} u_{10} - u_{00}) (u_{01} u_{11} - u_{00} u_{10}) (u_{11} u_{00} u_{11} - u_{01})$$

whose 16 entries are the linear forms whose coefficients are the 16 nodes. As explained in Hudson’s book [Hud, §16], the combinatorial structure of the 16 nodes can be read off from this $4 \times 4$-matrix, and it leads to various alternate forms of the defining quartic polynomial in [Hud, §19]. In [Hud, §102] it is shown that the product (1.2) expresses quadratic monomials in theta functions with characteristics in terms of the second order theta functions.

To illustrate how our coordinates can express geometric properties, we note that

$$(u_{00} u_{11} + u_{10} u_{01})(u_{00} u_{01} + u_{10} u_{11})(u_{00} u_{10} + u_{01} u_{11})(u_{00} u_{10} - u_{01} u_{11})$$

vanishes if and only if the given abelian surface is a product of two elliptic curves. If this happens, then the 16₆ configuration of all nodes degenerates to a more special matroid, and the quartic Kummer surface in $\mathbb{P}^3$ degenerates to a double quadric. The 10 factors in (1.3) are the non-zero entries left in the matrix product (1.2) after replacing each $x_{ij}$ by $u_{ij}$. □

This article is organized as follows. In Section 2 we review classical material that can be mostly found in the books of Coble [Cob] and Dolgachev–Ortland [DO]. We define a Kummer threefold in $\mathbb{P}^7$ as the image of a transcendental map whose coordinates are second-order theta functions. Each Kummer threefold is the singular locus of the associated Coble quartic hypersurface in $\mathbb{P}^7$, which is a natural genus three analogue of the Kummer surface in $\mathbb{P}^3$.

In Section 3 we focus on the moduli space $A_3(2, 4)$ of polarized abelian threefolds with suitable level structure. That space is a quotient of the Siegel upper halfspace. It is embedded
into $\mathbb{P}^7$ by second-order theta constants [GG]. The resulting Satake hypersurface $\mathcal{S}$ has degree 16. Its defining polynomial has 471 terms with integer coefficients, displayed in Proposition 3.1. We examine how the combinatorics of this polynomial expresses the geometry of $\mathcal{A}_3(2, 4)$, notably its hyperelliptic locus, its Torelli boundary, and its Satake boundary. Later in Table 3, this is refined to the stratification of $\mathcal{S}$ that was found by Glass [Gla, Theorem 3.1].

The theta series in Sections 2 and 3 are defined over the field of complex numbers $\mathbb{C}$, and we use floating point approximations for computing them. The resulting algebraic objects, however, do not require the complex numbers. This is, of course, well-known to the experts in abelian varieties [BL]. All the ideals we feature in this paper can be generated by polynomials with integer coefficients, and their projective varieties are thus defined over any field.

In Section 4 we express the Göpel variety $\mathcal{G}$ as the Zariski closure of the image of an explicit rational map $\mathbb{P}^6 \dasharrow \mathbb{P}^{14}$ of degree 24. Its 15 coordinates are polynomials of degree 7 that span an irreducible representation of the Weyl group of type $E_7$. The root system and its reflection arrangement play a prominent role, as do configurations of 7 points in $\mathbb{P}^2$. After completion of our work, we learned that our parametrization of the Göpel variety had already been studied in [CGL], under the name Coble linear system. The connection is made explicit in Theorem 4.6 where we give an affirmative answer to [CGL, Question 4.19].

In Section 5 we study the defining prime ideal of the Göpel variety $\mathcal{G}$. We show that it is minimally generated by 35 cubics and 35 quartics in 15 variables. This ideal is Gorenstein, it has degree 175, and we determine its Hilbert series and its minimal free resolution.

Section 6 is concerned with a beautiful re-embedding of the Göpel variety $\mathcal{G}$ into a projective space of dimension 134. The 135 coordinates are the Göpel functions of [DO, §9.7]. The Weyl group $W(E_7)$ acts on these by signed permutations. We construct a toric variety $\mathcal{T}$ of dimension 35 in $\mathbb{P}^{134}$ whose intersection with $\mathbb{P}^{14}$ is $\mathcal{G}$. Algebraically, the ideal of $\mathcal{G}$ is now generated by binomials and linear trinomials. We study the combinatorics of the toric ideal of $\mathcal{T}$ and its associated convex polytope $\mathcal{A}$. Its 63 distinguished facets and its 135 vertices are indexed by the lines and the Lagrangians in the finite symplectic space $(\mathbb{F}_2)^6$.

In Section 7 we return to the embedding of $\mathcal{K}_3$ defined by theta constants and theta functions. The universal Coble quartic is an irreducible subvariety of codimension two in $\mathbb{P}^7 \times \mathbb{P}^7$. It is the complete intersection of the Satake hypersurface $\mathcal{S}$ of degree $(16, 0)$ and one other hypersurface $\mathcal{C}$ of degree $(28, 4)$. The latter is given by an explicit polynomial with 372060 terms in 16 variables. Together, the two polynomials generate a prime ideal. In response to the preprint version of this article, Grushevsky and Salvati Manni [GS2] found a shorter representation of the same polynomial, using a more conceptual geometric approach. This was further developed by Dalla Piazza and Salvati Manni in [DPSM]; see Remark 8.7.

The universal Kummer variety $\mathcal{K}_3$ lives in $\mathcal{S} \times \mathbb{P}^7 \subset \mathbb{P}^7 \times \mathbb{P}^7$. Other variants of this variety can be defined in $\mathbb{P}^6 \times \mathbb{P}^7$, via the parametrization of Section 4, or in $\mathcal{G} \times \mathbb{P}^7 \subset \mathbb{P}^{14} \times \mathbb{P}^7$, via the Göpel embeddings of Sections 5 and 6. In Section 8 we study the bihomogeneous prime ideals for these three variants of the universal Kummer variety. We derive lists of minimal generators for two of these ideals. Conjectures 8.1 and 8.6 state that these lists are complete.

Section 9 examines all of our constructions from the perspective of tropical geometry. That perspective was further developed by three of us in the subsequent article [RSS].
Supplementary materials

We have created supplementary files so that the reader can reproduce many of the calculations that are claimed throughout the text. Most of them are in the format of Macaulay 2 [M2]. These files can be found at [http://math.berkeley.edu/~svs/supp/univ_kummer/](http://math.berkeley.edu/~svs/supp/univ_kummer/). We have also included the supplementary files in version 3 of the arXiv submission of this paper, and they can be obtained by downloading the source.

A word on the symbolic computations we present: many of the calculations are significantly faster (almost all finishing within a few seconds) if the field of coefficients chosen is finite; one that we have used is $\mathbb{Z}/101$. However, we have made no attempt to verify which primes give bad reduction. All calculations can also be performed over the rational numbers. In this setting, many calculations take at most a few minutes, and the hardest one in Theorem 5.1 took approximately 15 minutes on a modern mid-level performance computer.

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2 From theta functions to Coble quartics

In this section, we review Riemann’s theta function and its relatives in the case of genus 3, and we use this to give a definition of Kummer threefolds as subvarieties of $\mathbb{P}^7$. We also discuss background material on the action of the 2-torsion of the associated abelian threefold on $\mathbb{P}^7$ and introduce the Coble quartic associated to a non-hyperelliptic Kummer threefold.

Let $\tau$ be a symmetric $3 \times 3$ matrix with complex entries whose imaginary part is positive definite. The set $\mathcal{H}_3$ of all such matrices is a six-dimensional complex manifold, called the Siegel upper halfspace. Each matrix $\tau \in \mathcal{H}_3$ determines a lattice $\Lambda = \mathbb{Z}^3 + \tau \mathbb{Z}^3$ of rank 6 in $\mathbb{C}^3$, and a three-dimensional abelian variety $A_\tau = \mathbb{C}^3/\Lambda$. The Riemann theta function corresponding to a matrix $\tau \in \mathcal{H}_3$ is the function $\theta: \mathbb{C}^3 \to \mathbb{C}$ defined by the Fourier series

$$
\theta(\tau; z) = \sum_{n \in \mathbb{Z}^3} \exp \left[ \pi i n^t \tau n + 2\pi i n^t z \right].
$$

(2.1)

This series converges for all $z \in \mathbb{C}^3$ and $\tau \in \mathcal{H}_3$, and it satisfies the functional equation

$$
\theta(\tau; z + a + \tau b) = \theta(\tau; z) \cdot \exp \left[ 2\pi i (-b^t z - \frac{1}{2} b^t \tau b) \right] \quad \text{for } a, b \in \mathbb{Z}^3.
$$

(2.2)

Deconinck et al. [DHBHS] gave a careful convergence analysis and they implemented the numerical evaluation of $\theta(\tau; z)$ in Maple. Their work has been extended by Swierczewski and Deconinck [SD] who implemented the evaluation of abelian functions in Sage [Sage].
The Riemann theta function $\theta$ can now be called in Sage as $\text{RiemannTheta}(\tau)(z)$, where $\tau$ is a Riemann matrix and $z$ is a complex vector. We used that Sage code extensively.

Every pair of binary vectors $\epsilon, \epsilon' \in \{0,1\}^3$ defines a theta function with characteristics

$$\theta[\epsilon|\epsilon'](\tau; z) = \sum_{n \in \mathbb{Z}^3} \exp \left[ \pi i (n + \frac{\epsilon}{2})^t \tau (n + \frac{\epsilon}{2}) + 2\pi i (n + \frac{\epsilon}{2})^t (z + \frac{\epsilon'}{2}) \right]. \quad (2.3)$$

From inspection of this Fourier series, one can see that

$$\theta[\epsilon|\epsilon'](\tau; -z) = (-1)^{\epsilon \cdot \epsilon'} \cdot \theta[\epsilon|\epsilon'](\tau; z). \quad (2.4)$$

Therefore, of the $2^{2^3} = 64$ theta functions with characteristics, precisely $2^{3-1}(2^3 + 1) = 36$ are even functions of the argument $z \in \mathbb{C}^3$, and the other $2^{3-1}(2^3 - 1) = 28$ are odd functions of $z$. We shall refer to these as even (or odd) theta functions.

Finally, for any binary vector $\sigma \in \{0,1\}^3$, we consider the second order theta function

$$\Theta_2[\sigma](\tau; z) = \theta(2\tau; 2z + \tau \sigma) \cdot \exp \left[ \pi i \left( \frac{\sigma^t \tau \sigma}{2} + 2\sigma^t z \right) \right] = \sum_{n \in \mathbb{Z}^3} \exp \left[ 2\pi i (n + \frac{\sigma}{2})^t \tau (n + \frac{\sigma}{2}) + 4\pi i (n + \frac{\sigma}{2})^t z \right]. \quad (2.5)$$

The second order theta functions are related to the theta functions with characteristics of an isogenous abelian threefold by the formula

$$\Theta_2[\sigma](\tau; z) = \theta[\sigma|0](2\tau; 2z). \quad (2.6)$$

Further relations between first and second order theta functions are the addition theorem

$$\theta[\epsilon|\epsilon'](\tau; z + w) \cdot \theta[\epsilon|\epsilon'](\tau; z - w) = \sum_{\sigma \in \mathbb{F}_2^3} (-1)^{\sigma \cdot \epsilon \cdot \epsilon'} \cdot \Theta_2[\sigma](\tau; w) \cdot \Theta_2[\sigma + \epsilon](\tau; z) \quad (2.7)$$

and its inversion

$$8 \cdot \Theta_2[\sigma](\tau; w) \cdot \Theta_2[\sigma + \epsilon](\tau; z) = \sum_{\epsilon' \in \mathbb{F}_2^3} (-1)^{\sigma \cdot \epsilon \cdot \epsilon'} \cdot \theta[\epsilon|\epsilon'](\tau; z + w) \cdot \theta[\epsilon|\epsilon'](\tau; z - w). \quad (2.8)$$

For a fixed matrix $\tau \in S_3$, the eight second order theta functions define the Kummer map

$$\kappa_\tau : \mathbb{C}^3 \to \mathbb{P}^7, \ z \mapsto (\Theta_2[000](\tau; z) : \Theta_2[001](\tau; z) : \cdots : \Theta_2[111](\tau; z)). \quad (2.9)$$

The identity (2.2) implies that $\kappa_\tau$ factors through a map $A_\tau \to \mathbb{P}^7$ from the abelian threefold $A_\tau = \mathbb{C}^3 / \Lambda$. This map, which we also denote by $\kappa_\tau$, has the following geometric description. The equation $\theta(\tau; z) = 0$ defines the theta divisor $\Theta$ on $A_\tau$. The divisor $2\Theta$ is ample but not very ample. The eight functions (2.5) form a basis for its space of sections. These are known as the Schrödinger coordinates, and we denote them by $x_{000}, x_{001}, \ldots, x_{111}$. The morphism $A_\tau \to \mathbb{P}(H^0(A_\tau, 2\Theta)) \simeq \mathbb{P}^7$ is given in coordinates by (2.9). The image of the Kummer map $\kappa_\tau$ in $\mathbb{P}^7$ is isomorphic to the quotient $A_\tau / \{z = -z\}$. We call this the Kummer threefold of $\tau$. 6
Let $A_\tau[2]$ denote the subgroup of two-torsion points in the abelian threefold $A_\tau$. This is a group of order 64 which we identify with $\Lambda/2\Lambda \simeq (F_2)^6$. The abelian group $A_\tau[2] \simeq (F_2)^6$ acts naturally on the set of second order theta functions via
\[
\Theta_2[\sigma](\tau; z) \mapsto \Theta_2[\sigma](\tau; z + \delta) \quad \text{where } \delta \in A_\tau[2].
\] (2.10)
This defines an action on $P(H^0(A_\tau, 2\Theta)) \simeq P_7$ by permuting the coordinates $x_{ijk}$ up to sign.

The action lifts to a linear representation of the Heisenberg group $H$ (see [BL, Chapter 6]). This mildly non-abelian group is a certain central extension
\[
1 \to F_2 \to H \to A_\tau[2] \to 1.
\]
This can be made explicit in terms of the Schrödinger coordinates $x_{000}, x_{001}, \ldots, x_{111}$ we are using on $P^7$. The Heisenberg group $H$ is generated by the following six operators:
\[
x_{i,j,k} \mapsto x_{i+1,j,k}, \quad x_{i,j,k} \mapsto x_{i,j+1,k}, \quad x_{i,j,k} \mapsto x_{i,j,k+1}, \quad x_{i,j,k} \mapsto (-1)^i x_{i,j,k}, \quad x_{i,j,k} \mapsto (-1)^j x_{i,j,k}, \quad x_{i,j,k} \mapsto (-1)^k x_{i,j,k}.
\] (2.11)
These operators commute up to sign and give the projective representation of $A_\tau[2]$.

For any given matrix $\tau \in \mathfrak{H}_3$, the Kummer threefold has degree 24 in $P^7$. When $\tau$ is the period matrix of a smooth non-hyperelliptic curve of genus three, the prime ideal of the corresponding Kummer threefold is minimally generated by 8 cubics and 6 quartics in the variables $x$. The Hilbert polynomial of such a Kummer threefold, which is $4n^3 + 4$, agrees with the Hilbert function for $n \geq 1$. This was first derived by Wirtinger in [Wir, §21].

**Remark 2.1.** In commutative algebra [Eis95], it is customary to also look at higher syzygies. Numerical invariants are read off from the Betti table. For a general Kummer threefold, it is

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| total: | 1 | 14 | 48 | 56 | 24 | 3 |
| 0: | 1 | . | . | . | . | . |
| 1: | . | . | . | . | . | . |
| 2: | . | 8 | . | . | . | . |
| 3: | . | 6 | 48 | 56 | 24 | 3 |

This Betti table shows that the Kummer threefold $\kappa_\tau(A_\tau)$ is not arithmetically Cohen–Macaulay. Except for Kummer surfaces in $P^3$, failure to be arithmetically Cohen–Macaulay is a general phenomenon for abelian varieties of dimension $\geq 2$. In fact, this holds for varieties whose structure sheaf has intermediate cohomology via [Eis05, Cor. A1.12, Prop. A1.16].

To compute the above Betti table (in Macaulay 2 [M2]), we used the geometric representation of the Kummer threefold as the singular locus of a certain quartic hypersurface $C_\tau$ in $P^7$. That hypersurface was characterized by Arthur Coble in his seminal book [Cob]. We follow Dolgachev and Ortland [DO] in our discussion of the Coble quartic $C_\tau$.

**Proposition 2.2** ([Bea03, Proposition 2.2], [Cob, §33], [DO, §IX.5, Proposition 7]). *Let $A_\tau$ be the Jacobian of a smooth non-hyperelliptic curve of genus three. There exists a unique quartic hypersurface $C_\tau$ in $P^7$ whose singular locus equals the Kummer threefold $\kappa_\tau(A_\tau)$. The eight partial derivatives of the defining polynomial of $C_\tau$ span the space of cubics containing $\kappa_\tau(A_\tau)$ and they generate the prime ideal of $\kappa_\tau(A_\tau)$ up to saturation.*
The quartic \( C_\tau \) is invariant under the action of the Heisenberg group \( H \). The space of \( H \)-invariant quartics is 15-dimensional, and the defining polynomial \( F_\tau \) of the Coble quartic hypersurface can be written as a linear combination of a basis for this space of \( H \)-invariants.

Using the Schrödinger coordinates \( x \) on \( \mathbb{P}^7 \), we write

\[
F_\tau = r \cdot (x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4) + s_{01} \cdot (x_{00}^2 x_{01} + x_{01}^2 x_{10} + x_{10}^2 x_{11} + x_{11}^2 x_{00}) + s_{10} \cdot (x_{00}^2 x_{10} + x_{10}^2 x_{11} + x_{11}^2 x_{00} + x_{00}^2 x_{01}) + s_{11} \cdot (x_{00}^2 x_{11} + x_{10}^2 x_{11} + x_{01}^2 x_{11} + x_{00}^2 x_{01})
\]

(2.12)

This representation appears in [DO, §IX.5, Proposition 8] and after Theorem 3.2 in [Bea06, Section 3]. The 15 coefficients \( r, s_{\bullet}, t_{\bullet} \) are parameters. This notation is used throughout this paper.

**Remark 2.3.** The monomials in \( x \) that appear in the equation (2.12) of the Coble quartic can be understood combinatorially via the affine geometry of the eight-point vector space \((\mathbb{F}_2)^3\). Namely, the four terms multiplied by \( s_{ijk} \) are the four affine lines parallel to the vector \((i, j, k)\), and the two terms multiplied by \( t_{ijk} \) are the two affine planes perpendicular to the vector \((i, j, k)\), with respect to the usual dot product in \((\mathbb{F}_2)^3\). □

In this paper, each non-hyperelliptic Kummer threefold will be represented as the variety in \( \mathbb{P}^7 \) cut out by the eight partial derivatives \( \partial F_\tau / \partial x_{ijk} \). The adjective “universal” in our title means that we are working over the six-dimensional base of all Coble polynomials \( F_\tau \). This family is obtained by letting \( \tau \) run over the Siegel upper halfspace \( \mathcal{H}_3 \). In particular, \( r, s_{\bullet}, t_{\bullet} \) depend analytically on \( \tau \). We shall review the relevant moduli space in Section 3.

The Coble quartic is closely related to the Kummer surface in Example 1.1. Indeed, the Kummer quartic is the expansion of the determinant (1.1) along the first row:

\[
r \cdot (x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4) + t \cdot (x_{00} x_{01} x_{10} x_{11}) + s_{01} \cdot (x_{00}^2 x_{01} + x_{01}^2 x_{10} + x_{10}^2 x_{11} + x_{11}^2 x_{00}) + s_{10} \cdot (x_{00}^2 x_{10} + x_{10}^2 x_{11} + x_{11}^2 x_{00} + x_{00}^2 x_{01}) + s_{11} \cdot (x_{00}^2 x_{11} + x_{10}^2 x_{11} + x_{01}^2 x_{11} + x_{00}^2 x_{01})
\]

(2.13)

The monomials in \( x_{00}, x_{01}, x_{10}, x_{11} \) seen here correspond to the affine subspaces of \( (\mathbb{F}_2)^2 \). The coefficients are polynomials of degree 12 in the theta constants \( u_{ij} \). They are obtained as the 4 \( \times \) 4-minors of the last four rows in (1.1). These minors satisfy the cubic equation

\[
16r^3 + rt^2 + 4(s_{01}s_{10} - s_{01}^2 - r_{01}^2 - r_{00}^2) = 0.
\]

(2.14)
This cubic defines a hypersurface in $\mathbb{P}^4$ which is known as Segre’s primal cubic. In Section 5 we shall derive the analogous relations for the fifteen coefficients of (2.12).

Remark 2.4. Both the Kummer surface in $\mathbb{P}^3$ and the Coble quartic in $\mathbb{P}^7$ are self-dual hypersurfaces [Pau]. The role of the 16 singular points on the Kummer surface is now played by the 64 singular points on the Kummer threefold. These are the images of the 2-torsion points of $A_\tau$ under the Kummer map $\kappa_\tau$. In analogy to the 16 entries of the matrix (1.2), we consider the 64 linear forms in (3.10) below. Their coefficients give the 64 special singular points of $\kappa_\tau(A_\tau)$ in $\mathbb{P}^7$. These points lie on 64 special hyperplanes. Self-duality gives a $64_{28}$ configuration consisting of points and hyperplanes in $\mathbb{P}^7$. If $A_\tau$ is the Jacobian of a genus three hyperelliptic curve, then the Coble quartic $F_\tau$ becomes the square of a quadric, as we shall see in (8.9), and the Kummer threefold fails to be projectively normal [Kha, §2.9.3].

3 The Satake hypersurface

In order to understand the family of all Kummer threefolds, we now vary the matrix $\tau$ throughout the Siegel upper halfspace $\mathcal{H}_3$. The modular group $\text{Sp}_6(\mathbb{Z})$ consists of block matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d$ are $3 \times 3$ matrices with integer entries such that $\gamma J \gamma^t = J$, where $J = \begin{pmatrix} 0 & -\text{Id}_3 \\ \text{Id}_3 & 0 \end{pmatrix}$. Following [BL, DO, Mum], the modular group $\text{Sp}_6(\mathbb{Z})$ acts on $\mathcal{H}_3$ by

$$
\gamma \circ \tau = (\tau c + d)^{-1}(\tau a + b).
$$

The quotient is the moduli space of principally polarized abelian threefolds:

$$
\mathcal{A}_3 = \mathcal{H}_3 / \text{Sp}_6(\mathbb{Z}).
$$

We will also consider certain level covers of $\mathcal{A}_3$, which can be constructed by taking quotients of $\mathcal{H}_3$ by appropriate normal congruence subgroups of $\text{Sp}_6(\mathbb{Z})$. The subgroup

$$
\Gamma_3(2) = \left\{ \gamma \in \text{Sp}_6(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} \text{Id}_3 & 0 \\ 0 & \text{Id}_3 \end{pmatrix} \pmod{2} \right\}
$$

has index $|\text{Sp}_6(\mathbb{F}_2)| = 1451520$; see (6.2). We also define

$$
\Gamma_3(2,4) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_3(2) : \text{diag}(a^T b) \equiv \text{diag}(c^T d) \equiv 0 \pmod{4} \right\}.
$$

This group has index 64 in $\Gamma_3(2)$, and the quotient group $\Gamma_3(2)/\Gamma_3(2,4)$ is isomorphic to $(\mathbb{F}_2)^6$. These subgroups determine moduli spaces

$$
\mathcal{A}_3(2) = \mathcal{H}_3 / \Gamma_3(2) \quad \text{and} \quad \mathcal{A}_3(2,4) = \mathcal{H}_3 / \Gamma_3(2,4).
$$

We record that the induced quotient map of moduli spaces is a 64-to-1 cover:

$$
\mathcal{A}_3(2,4) \xrightarrow{64:1} \mathcal{A}_3(2).
$$
The Torelli map gives an embedding of the moduli space $\mathcal{M}_3$ of smooth genus three curves into $\mathcal{A}_3$. We can thus regard $\mathcal{M}_3$ as a subset of $\mathcal{A}_3$. (For the experts: this would not be correct if we considered these spaces as stacks, but we will not do this here.) The inverse images of $\mathcal{M}_3$ in $\mathcal{A}_3(2)$ and in $\mathcal{A}_3(2, 4)$ are denoted by $\mathcal{M}_3(2)$ and $\mathcal{M}_3(2, 4)$ respectively.

The moduli spaces above are six-dimensional quasi-projective varieties, and we are interested in the homogeneous prime ideals of certain embeddings. In this section we focus on $\mathcal{A}_3(2, 4)$. Our point of departure is the theta constant map $\vartheta : \mathcal{A}_3 \to \mathbb{P}^7$, which is defined by

$$\vartheta : \tau \mapsto (\Theta_2[000](\tau ; 0) : \Theta_2[001](\tau ; 0) : \cdots : \Theta_2[111](\tau ; 0)).$$  \hfill (3.3)

This map is not injective, since the second order theta constants

$$\Theta_2[\sigma](\tau ; 0) = \sum_{n \in \mathbb{Z}} \exp \left[ 2\pi i (n + \frac{\sigma}{2})^t \tau (n + \frac{\sigma}{2}) \right]$$  \hfill (3.4)

are modular forms of weight $1/2$ with respect to the subgroup $\Gamma_3(2, 4)$ of $\text{Sp}_6(\mathbb{Z})$. Hence the theta constant map $\vartheta$ can be regarded as a morphism from the level cover $\mathcal{A}_3(2, 4) = \mathcal{A}_3/\Gamma_3(2, 4)$ into the projective space $\mathbb{P}^7$. This map is an embedding by [Gla, Theorem 4.1].

The closure $S = \overline{\vartheta(\mathcal{A}_3(2, 4))}$ of the image of the theta map in $\mathbb{P}^7$ is a six-dimensional hypersurface. It is isomorphic to the Satake compactification $\mathcal{A}_3(2, 4)$ of the moduli space $\mathcal{A}_3(2, 4)$. This is a degree 16 hypersurface which we call the Satake hypersurface. Its defining polynomial, which we also denote by $S$, was found by Runge [Run, §6] and we now describe it. Write $G$ for the permutation group of order 1344 that is generated by the four transpositions

$$(u_{001}u_{010})(u_{101}u_{110}), \, (u_{010}u_{101})(u_{011}u_{101}), \, (u_{000}u_{001})(u_{010}u_{011}), \, \text{and} \, (u_{100}u_{101})(u_{110}u_{111}).$$

As an abstract group, $G = \text{SL}_3(\mathbb{F}_2) \times (\mathbb{F}_2)^3$. In our context, it is precisely the subgroup of $\text{Sp}_6(\mathbb{F}_2)$ that acts on the Schrödinger variables $u$ by coordinate permutations. Let $[u^*]_m$ denote the sum over the $G$-orbit of a monomial $u^*$. The index $m$ is the size of this orbit.

**Proposition 3.1** (Runge). The Satake hypersurface $S = \overline{\vartheta(\mathcal{A}_3(2, 4))}$ is an irreducible hypersurface of degree 16 in $\mathbb{P}^7$. Its defining polynomial is the following sum of 471 monomials:

$$[u_{000}^2u_{001}^2u_{010}^2u_{011}^2]_{56} - 2[u_{000}^2u_{001}u_{010}u_{011}u_{101}u_{110}u_{111}]_{8} + 2[u_{000}^2u_{001}u_{010}^2u_{100}u_{011}u_{101}]_{84} + [u_{000}^2u_{001}u_{010}^2u_{100}u_{011}]_{56} - [u_{000}^2u_{001}u_{010}u_{011}u_{100}u_{101}]_{224} + 4[u_{000}^2u_{001}u_{010}u_{100}u_{011}u_{011}]_{28} - 16[u_{000}^2u_{001}u_{010}u_{011}u_{101}u_{110}u_{111}]_{14} + 72[u_{000}^2u_{001}u_{010}^2u_{100}u_{011}u_{110}u_{111}]_{28} + 72[u_{000}^2u_{001}^2u_{010}u_{110}u_{111}]_{14} + 72[u_{000}^2u_{001}^2u_{010}u_{110}u_{111}]_{28}.$$

A more compact representation of the Satake hypersurface $S$, in terms of second order theta constants, appears in [GG, Example 1.4]. Our 471 monomials can be derived from these.

We next discuss a beautiful direct relationship between the polynomials in Example 1.1 and Proposition 3.1. We learned this from Sam Grushevsky and Riccardo Salvati Manni. Both polynomials have total degree 16 in eight unknowns, and the former appears as an initial form in the latter. Let $q$ denote a deformation parameter and set

$$u_{000} = u_{00} + O(q^4), \quad u_{001} = u_{01} + O(q^4), \quad u_{010} = u_{10} + O(q^4), \quad u_{011} = u_{11} + O(q^4), \quad u_{100} = 2qx_{00} + O(q^3), \quad u_{101} = 2qx_{01} + O(q^3), \quad u_{110} = 2qx_{10} + O(q^3), \quad u_{111} = 2qx_{11} + O(q^3).$$  \hfill (3.5)
Under this substitution, the Satake polynomial in Proposition 3.1 takes the form

\[ S = \det[\bullet] \cdot q^4 + O(q^8), \]

where \([\bullet]\) is the 5x5-matrix of Example 1.1 that defines the universal Kummer surface \(K_2\).

This identity can be derived from the Fourier–Jacobi expansion of second-order theta constants. Namely, in that expansion we write the 3\times3-matrix \(\tau \in \mathfrak{H}_3\) in the form

\[ \tau = \begin{pmatrix} s & z \\ 2z & \tau' \end{pmatrix}, \quad \text{where } \tau' \in \mathfrak{H}_2, \ s \in \mathbb{C}, \text{ and } z \in \mathbb{C}^2. \]

If we set \(q = e^{\pi i s/2}\), then each of the eight genus three second order theta constants \(u_\sigma = \Theta_2(\tau; 0)[\sigma]\) has a Taylor series expansion in \(q\). The leading coefficient in these Taylor series is either a theta constant or a theta function of genus two. These expansions are given in (3.5).

The Satake hypersurface \(S = \vartheta(A_3(2,4))\) in \(\mathbb{P}^7\) contains several loci of geometric interest:

- The hyperelliptic locus has codimension one in \(S\), and hence codimension two in \(\mathbb{P}^7\).
- The Torelli boundary \(S \setminus \vartheta(M_3(2,4))\) has codimension two in \(S\).
- The Satake boundary \(S \setminus \vartheta(A_3(2,4))\) has codimension three in \(S\).

For each of these loci, we shall describe its irreducible components and defining polynomials. We begin with the Satake boundary. By [Gee, Lemma 3.5], it consists of 126 three-dimensional subspaces \(\mathbb{P}^3\) in \(\mathbb{P}^7\): each of the 63 non-zero half-periods \(\epsilon \in \Lambda/2\Lambda\) induces a linear involution on \(\mathbb{P}^7\) via the action on second order theta constants given in (2.10), and the fixed point set of this involution on \(\mathbb{P}^7\) is the union of two \(\mathbb{P}^3\)s. For instance, for \(\epsilon = (1/2, 0, 0)\), the involution (2.10) fixes the coordinates \(u_{000}, u_{001}, u_{100}, u_{101}\), switches the sign on \(u_{110}, u_{111}\), and the two \(\mathbb{P}^3\)s are obtained by setting either of these two groups of four variables to zero.

The hyperelliptic locus in \(S\) is the closure of the set of all points \(\vartheta(\tau)\) where \(A_\tau\) is the Jacobian of a smooth hyperelliptic curve of genus 3. It is known (see e.g. [Gla]) that a genus 3 curve is hyperelliptic if and only if one of its 36 first order theta constants \(\theta[\epsilon][\epsilon'](\tau; 0)\) vanishes. We write these 36 divisors in our eight coordinates \(u_{000}, \ldots, u_{111}\) using the formula

\[ \theta[\epsilon][\epsilon'](\tau; 0)^2 = \sum_{\sigma \in \mathbb{F}_2^3} (-1)^{\sigma \cdot \epsilon'} u_\sigma \cdot u_{\sigma + \epsilon}, \quad (3.6) \]

which is obtained by setting \(w = z = 0\) in the addition theorem (2.7). Hence the hyperelliptic locus in \(S\) can be defined set-theoretically by the equation of degree 72 obtained by taking the product of the quadrics (3.6) where \((\epsilon, \epsilon')\) runs over the 36 even theta characteristics.

We will see in Section 7 that the product \(\prod \theta[\epsilon][\epsilon'](\tau; 0)\) of the 36 first order theta constants is in fact a polynomial in \(u\), so the scheme defined by the degree 72 polynomial is not reduced. Modulo each of the quadrics (3.6), the Satake polynomial becomes the square of an octic. In the supplementary materials, we give this octic for one of the 36 components, as well as a verification that the subscheme of this component defined by the octic is reduced. So each component of the hyperelliptic locus is a complete intersection of degree 16. Hence as
a subvariety of \( \mathbb{P}^7 \), the hyperelliptic locus has codimension two, degree \( 576 = 16 \cdot 36 \), and 36 irreducible components. Glass analyzed in [Gla, Theorem 3.1] how many of these 36 theta constants can simultaneously vanish. We shall refine his results in our Table 3.

The Torelli boundary is the closure of the image under \( \vartheta \) of the set of all polarized abelian threefolds that decompose as the product, as a polarized variety, of an elliptic curve and an abelian surface. This means that \( \tau \) can be transformed into block form under \( \text{Sp}_6(\mathbb{Z}) \).

**Proposition 3.2.** The Torelli boundary coincides with the singular locus of the Satake hypersurface \( S \). It is the union of 336 irreducible four-dimensional subvarieties of \( \mathbb{P}^7 \), each of which is defined by the \( 2 \times 2 \)-minors of a \( 2 \times 4 \)-matrix of linear forms in \( u_{000}, u_{001}, \ldots, u_{111} \).

**Proof.** By [GG, Lemma 3.2], the Satake hypersurface is non-singular at each point that represents an indecomposable polarized abelian threefold. It follows from [Gla, Theorem 3.1] that a polarized abelian threefold decomposes as a product if and only if at least two of its first order theta constants vanish. Moreover, the vanishing of two first theta constants implies the vanishing of at least six of the quadrics in (3.6). The relevant 6-tuples \( I \) of even theta characteristics \( m = [\epsilon | \epsilon'] \) have the property that, for any three \( m_1, m_2, m_3 \) in \( I \), the sum \( m_1 + m_2 + m_3 \) is an odd characteristic. There are 336 such 6-tuples \( I \). They correspond to azygetic triads of Steiner complexes, and hence to non-isotropic planes in the symplectic vector space \( (\mathbb{F}_2)^6 \), and hence also to root subsystems \( A_2 \) in \( E_7 \); see [Man, Prop. 1(2)].

Now, if \( \vartheta(\tau) \) is in the Torelli boundary then the \( 3 \times 3 \)-matrix \( \tau \) is in the \( \text{Sp}_6(\mathbb{Z}) \)-orbit of a matrix \( \tau_0 \) that decomposes into two blocks given by matrices in \( H_1 \) and \( H_2 \). Since theta constants behave multiplicatively under this decomposition, the image of the locus of \( \tau_0 \) admitting such a decomposition is a Segre variety \( \mathbb{P}^1 \times \mathbb{P}^3 \) in \( \mathbb{P}^7 \). That Segre variety is defined by the \( 2 \times 2 \)-minors of the \( 2 \times 4 \)-matrix

\[
\begin{pmatrix}
  u_{000} & u_{001} & u_{010} & u_{011} \\
  u_{100} & u_{101} & u_{110} & u_{111}
\end{pmatrix}.
\]

(3.7)

It is easy to check in Macaulay 2 that the ideal of \( 2 \times 2 \) minors of (3.7) contains the partial derivatives of the Satake polynomial \( S \). In particular, this component consists of singular points. Since the other 335 components \( \mathbb{P}^1 \times \mathbb{P}^3 \) are obtained by applying the action of the modular group, we see that all components are in the singular locus. Putting everything together, we conclude that the singular locus of \( S \) coincides with the Torelli boundary. \( \square \)

The 336 irreducible components of the Torelli boundary in \( S \) are a direct generalization of the 10 irreducible factors in (1.3). In both cases, the components are defined by certain quadrics in the \( u \)-coordinates that can be written as \( 2 \times 2 \)-determinants of linear forms. In Section 7 we shall return to the equations of the Torelli boundary when we study the universal Coble quartic in \( \mathbb{P}^7 \times \mathbb{P}^7 \). Let us now define our universal objects in precise terms.

We combine the Kummer map (2.9) and the theta constant map (3.3) by setting

\[
\kappa: H_3 \times \mathbb{C}^3 \to \mathbb{P}^7 \times \mathbb{P}^7, \quad (\tau, z) \mapsto (\vartheta(\tau), \kappa_\tau(z)).
\]

(3.8)

This is the universal Kummer map in genus \( g = 3 \). The closure of its image in \( \mathbb{P}^7 \times \mathbb{P}^7 \) is the universal Kummer threefold. This irreducible variety of dimension nine is denoted

\[
\mathcal{K}_3 = \mathcal{K}_3(2, 4) := \overline{\kappa(H_3 \times \mathbb{C}^3)}.
\]
We fix the coordinates \( u = (u_{000} : u_{001} : \cdots : u_{111}) \) on the first copy of \( \mathbb{P}^7 \), and we fix the coordinates \( x = (x_{000} : x_{001} : \cdots : x_{111}) \) on the second \( \mathbb{P}^7 \). Thus the \( u_{ijk} \) represent the theta constants which can be regarded as coordinates for the moduli space \( \mathcal{A}_3(2, 4) \) of polarized abelian threefolds with level structure, while the \( x_{ijk} \) represent the second order theta functions which are coordinates for the individual abelian threefolds \( \mathcal{A}_\tau \) themselves.

The following two formulas relate these coordinates on our \( \mathbb{P}^7 \times \mathbb{P}^7 \) to the theta functions with characteristics. First, by specializing \( w = z \) in (2.7) we obtain the formula

\[
\theta[\epsilon|\epsilon'](\tau; 2z) \cdot \theta[\epsilon|\epsilon'](\tau; 0) = \sum_{\sigma \in \mathbb{P}^3} (-1)^{\sigma \cdot \epsilon} \cdot x_\sigma \cdot x_{\sigma + \epsilon}.
\] (3.9)

Second, the specialization \( w = 0 \) in (2.7) yields

\[
\theta[\epsilon|\epsilon'](\tau; z)^2 = \sum_{\sigma \in \mathbb{P}^3} (-1)^{\sigma \cdot \epsilon} \cdot u_\sigma \cdot x_{\sigma + \epsilon}.
\] (3.10)

The identity (3.6) arises from either of these by setting \( z = 0 \). Our formulas admit \( \mathbb{Z}[1/8] \)-linear inversions, derived from (2.8), that express the various quadratic monomials on \( \mathbb{P}^7 \times \mathbb{P}^7 \) in terms of thetas with characteristics. Equation (3.9) expresses first order theta functions with doubled argument \( 2z \) as quadratic polynomials in the second order theta functions.

It is sometimes advantageous to embed the Kummer threefold not in \( \mathbb{P}^7 \), but in the larger space \( \mathbb{P}^{35} \) whose coordinates are indexed by even pairs \((\epsilon, \epsilon')\). Likewise, it makes sense to re-embed the universal Kummer variety \( K_3(2, 4) \) from \( \mathbb{P}^7 \times \mathbb{P}^7 \) into \( \mathbb{P}^{35} \times \mathbb{P}^{35} \). Using the addition theorem for theta functions (2.7), this can be accomplished in two different ways. First, we can use the formulas (3.10) and (3.6) to map \( K_3(2, 4) \) into \( \mathbb{P}^{35} \times \mathbb{P}^{35} \) with coordinates

\[
\left( \theta[\epsilon|\epsilon'](\tau; 0)^2, \theta[\epsilon|\epsilon'](\tau; z)^2 \right).
\] (3.11)

Second, we can use the formulas (3.9) and (3.6) to map \( K_3(2, 4) \) into \( \mathbb{P}^{35} \times \mathbb{P}^{35} \) with coordinates

\[
\left( \theta[\epsilon|\epsilon'](\tau; 0)^2, \theta[\epsilon|\epsilon'](\tau; 2z)^2 \right).
\] (3.12)

In both cases, the re-embedding is given by a certain Veronese map \( \mathbb{P}^7 \times \mathbb{P}^7 \to \mathbb{P}^{35} \times \mathbb{P}^{35} \).

**Remark 3.3.** Our motivating problem was to determine the bihomogeneous prime ideal

\[
\mathcal{I}_3 \subset \mathbb{Q}[u, x]
\]

of the universal Kummer threefold \( K_3(2, 4) \). One of the minimal generators of \( \mathcal{I}_3 \) is the Satake polynomial of degree \((16, 0)\). To find others, one might try the following approach. For any two non-negative integers \( r \) and \( s \), consider the space \( \mathbb{Q}[u, x]_{r,s} \) of polynomials that are bihomogeneous of degree \((r, s)\). This space has dimension \( \binom{7}{r} \binom{7}{s} \). We seek to identify the subspace \( \mathcal{I}_3(r,s) \) of polynomials that lie in our ideal \( \mathcal{I}_3 \). That subspace can be computed using (numerical) linear algebra. The basic idea is simple: using Swierczewski’s code for the Riemann theta function \( \theta \), we implemented pieces of Sage code for the second order theta functions (2.5), for the Kummer map (2.9), for the theta constant map (3.3), and for the universal Kummer map (3.8). For each point \((\tau, z) \in \mathcal{H}_3 \times \mathbb{C}^3\), we can thus compute one.
linear constraint on polynomials in \((I_3)_{(r,s)}\), as these vanish at \(\kappa(t, z)\). By plugging in enough points, we get linear equations in \(\binom{7+r}{7}\binom{7+s}{7}\) unknowns whose solution space equals \((I_3)_{(r,s)}\).

In practice, however, this approach does not work at all. The primary reason is that the size of our linear systems is too large, even with the use of state-of-the-art software for numerical linear algebra. The key to any success would be the identification of subspaces in which equations for \(I_3\) can lie. Methods from representation theory are essential here.

For instance, suppose we are told that the Satake hypersurface has degree 16, and we are asked to find its polynomial \(\mathcal{S}\). Solving the naive system of linear equations in \(\binom{7+16}{7}\) unknowns is an impossible task. However, all but 471 of the unknown coefficients are points, we get linear equations in which equations for \(I_3\) can lie. Methods from representation theory are essential here.

We now present an utterly explicit polynomial parametrization \(\gamma\) of the G"{o}pel variety \(\mathcal{G}\). Let \(\mathbb{P}^6\) be the projective space with coordinates \((c_1 : c_2 : c_3 : c_4 : c_5 : c_6 : c_7)\). Our map

\[
\gamma: \mathbb{P}^6 \rightarrow \mathbb{P}^{14}
\]

is defined by the following 15 homogeneous polynomials of degree 7 in \(c_1, c_2, \ldots, c_7\):

\[
r = 4c_1c_2c_3c_4c_5c_6c_7
\]

\[
s_{001} = c_1c_2c_7(c_3^4 - 2c_3^2c_4^2 + c_4^4 - 2c_3^2c_5^2 - 2c_4^2c_5^2 + c_5^4 - 2c_3^2c_6^2 - 2c_4^2c_6^2 - 2c_5^2c_6^2 + c_6^4)
\]

\[
s_{010} = -c_1c_3c_5(c_2^4 - 2c_2^2c_4^2 + c_4^4 - 2c_2^2c_5^2 - 2c_4^2c_5^2 + c_5^4 - 2c_2^2c_6^2 - 2c_4^2c_6^2 - 2c_5^2c_6^2 + c_6^4)
\]

\[
s_{011} = c_1c_4c_6(c_2^4 - 2c_2^2c_3^2 + c_3^4 - 2c_2^2c_5^2 - 2c_3^2c_5^2 + c_5^4 - 2c_2^2c_6^2 - 2c_3^2c_6^2 - 2c_5^2c_6^2 + c_6^4)
\]

\[
s_{100} = -c_2c_3c_6(c_1^4 - 2c_1^2c_4^2 + c_4^4 - 2c_1^2c_5^2 - 2c_4^2c_5^2 + c_5^4 - 2c_1^2c_6^2 - 2c_4^2c_6^2 - 2c_5^2c_6^2 + c_6^4)
\]

\[
s_{101} = c_2c_4c_5(c_1^4 - 2c_1^2c_3^2 + c_3^4 - 2c_1^2c_5^2 - 2c_3^2c_5^2 + c_5^4 - 2c_1^2c_6^2 - 2c_3^2c_6^2 - 2c_5^2c_6^2 + c_6^4)
\]

\[
s_{110} = -c_3c_4c_7(c_1^4 - 2c_1^2c_2^2 + c_2^4 - 2c_1^2c_5^2 - 2c_2^2c_5^2 + c_5^4 - 2c_1^2c_6^2 - 2c_2^2c_6^2 - 2c_5^2c_6^2 + c_6^4)
\]

\[
s_{111} = c_5c_6c_7(c_1^4 - 2c_1^2c_2^2 + c_2^4 - 2c_1^2c_3^2 - 2c_2^2c_3^2 + c_3^4 - 2c_1^2c_4^2 - 2c_2^2c_4^2 - 2c_3^2c_4^2 + c_4^4)
\]
The Göpel variety $\mathcal{G}$ is the closure of the image of the rational map $\gamma: \mathbb{P}^6 \dashrightarrow \mathbb{P}^{14}$ given by the polynomials above. This map defines a 24-to-1 cover of $\mathcal{G}$.

The rest of this section is devoted to a conceptual derivation and geometric explanation of the map $\gamma$. This will then furnish the proof of Theorem 4.1. In Section 6 we shall see that $\gamma$ is equivalent to the embedding via Göpel functions described in [DO, §IX.8, Theorem 5]. Our construction here is based on the representation-theoretic approach developed in [GSW].

Consider the Heisenberg group $H$ that is generated by the six operators in (2.11). Let $H'$ be the subgroup of $\text{GL}_8(\mathbb{C})$ generated by $H$ and the scalar matrices $\mathbb{C}^*$. Let $N(H')$ be its normalizer. Then $N(H')/H' \cong \text{Sp}_6(\mathbb{F}_2)$ by [BL, Exercise 6.14]. Since every element of
$H'$ acts on the 15 parenthesized polynomials in (2.12) by scalar multiplication, the space spanned by these is preserved by the action of $N(H')$. This action factors through $H$, so we get a 15-dimensional representation of $N(H')/H$. This contains $\text{Sp}_6(\mathbb{F}_2)$ as a subgroup, and this 15-dimensional representation of $\text{Sp}_6(\mathbb{F}_2)$ is irreducible. There is a unique such irreducible representation, so we call it $U^{15}$. For our computations we shall use the larger group $W(E_7) = \text{Sp}_6(\mathbb{F}_2) \times \{\pm 1\}$. This is the Weyl group of the root system of type $E_7$. We use the same symbol $U^{15}$ to denote the corresponding irreducible representation of $W(E_7)$.

Let $\mathfrak{h} \cong \mathbb{C}^7$ be the reflection representation of $W(E_7)$. We will work in two different coordinate systems of $\mathfrak{h}$. The first one we call $c_1, \ldots, c_7$ (with the standard quadratic form $\sum_i c_i^2$), which has the property that each $c_i = 0$ is a reflection hyperplane. Explicitly, the following two matrices are a pair of generators for $W(E_7)$ in terms of this basis:

$$
\mu = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix} \quad \nu = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & -1/2 & 0 & 1/2 \\
1/2 & 0 & 0 & 1/2 & 1/2 & 0 & -1/2 \\
1/2 & 0 & 0 & -1/2 & 1/2 & 0 & 1/2 \\
1/2 & 0 & 0 & -1/2 & -1/2 & 0 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

(4.2)

One can verify with GAP [Gap] that they generate a group of the correct size, and it contains seven reflections that satisfy the Coxeter relations of type $E_7$. See our supplementary files.

All representations of the Weyl group $W(E_7)$ are isomorphic to their duals, as is true for any real reflection group. In what follows we shall distinguish between representations and their duals to avoid confusion. The coordinate ring of $\mathfrak{h}$ is $\text{Sym}(\mathfrak{h}^*) = \mathbb{C}[c_1, c_2, c_3, c_4, c_5, c_6, c_7]$. There is a unique copy of the 15-dimensional irreducible representation $U^{15}$ in $\text{Sym}^7(\mathfrak{h}^*)$. We call this subspace $U^{15}_c$. It can be realized explicitly by taking the $W(E_7)$-submodule spanned by the monomial $c_1c_2c_3c_4c_5c_6c_7$. This is an example of a Macdonald representation [Mac].

Since $U^{15} \cong (U^{15}_c)^*$, there exists a unique basis in $U^{15}_c$ which is dual to the basis of the 15 Heisenberg-invariant quartics of (2.12) in $U^{15}$. We computed that dual basis of $U^{15}_c$. It was found to consist of the above 15 polynomials $r, s_{001}, \ldots, t_{111}$ of degree 7 in $c_1, c_2, \ldots, c_7$. These elements of $\text{Sym}^7(\mathfrak{h}^*)$ are the coordinates of our rational map $\gamma : \mathbb{P}^6 \dashrightarrow \mathbb{P}^{14}$.

By multiplying the dual bases and summing, we get a $W(E_7)$-invariant in $U^{15} \otimes U^{15}_c$. This is the defining polynomial $F_r$ of the Coble quartic, with coefficients as above. This concludes our derivation of the map $\gamma$. Here is an alternative way of getting our formulas.

Remark 4.2. Let $A$ be an 8-dimensional complex vector space with basis

$$a_1 = x_{000}, \ a_2 = x_{100}, \ a_3 = x_{010}, \ a_4 = x_{110}, \ a_5 = x_{001}, \ a_6 = x_{101}, \ a_7 = x_{011}, \ a_8 = x_{111}.$$  

The Heisenberg group $H$ acts on $\bigwedge^4 A$ and the space of invariants has the following basis:

$$h_1 = a_1 \wedge a_2 \wedge a_3 \wedge a_4 + a_5 \wedge a_6 \wedge a_7 \wedge a_8,$$

$$h_2 = a_1 \wedge a_2 \wedge a_5 \wedge a_6 + a_3 \wedge a_4 \wedge a_7 \wedge a_8,$$

$$h_3 = a_1 \wedge a_3 \wedge a_5 \wedge a_7 + a_2 \wedge a_4 \wedge a_6 \wedge a_8,$$

$$h_4 = a_1 \wedge a_4 \wedge a_6 \wedge a_7 + a_2 \wedge a_3 \wedge a_5 \wedge a_8,$$

$$h_5 = a_1 \wedge a_3 \wedge a_6 \wedge a_8 + a_2 \wedge a_4 \wedge a_5 \wedge a_7,$$
$$h_6 = a_1 \wedge a_4 \wedge a_5 \wedge a_8 + a_2 \wedge a_3 \wedge a_6 \wedge a_7,$$
$$h_7 = a_1 \wedge a_2 \wedge a_7 \wedge a_8 + a_3 \wedge a_4 \wedge a_5 \wedge a_6.$$  

The techniques in [GSW, §6.2, §6.5] show how to start with any vector in $\bigwedge^4 A$ and produce the equations for a Kummer variety in the projective space of lines in $A^*$ along with the equation for its Coble quartic. If we apply these techniques to the above formula for $F$, then we get the above formula for $F_r$. The derivation of Conjecture 8.1 will discuss this in detail. 

We now come to the proof that our parametrization of the Göpel variety is correct.

**Proof of Theorem 4.1.** The Macdonald representation $U^15_c$ in $\text{Sym}^7(\mathfrak{h}^*)$ is unique. It defines a unique $W(E_7)$-equivariant rational map

$$\gamma : \mathbb{P}^6 \dashrightarrow \mathbb{P}^{14}.$$  

Hence, up to choosing coordinates on the two projective spaces, the map $\gamma$ agrees with the one described in [DO, §IX.7, Remark 7]. The closure of its image must be the Göpel variety $G$.

To see this more explicitly, we now come to our second coordinate system, denoted $d = (d_1, d_2, d_3, d_4, d_5, d_6, d_7)$, for the reflection representation $\mathfrak{h}$ of $W(E_7)$. It is defined by

$$c_1 = d_2 + d_4 + d_5, \quad c_2 = d_1 + d_4 + d_7,$$
$$c_3 = d_2 + d_3 + d_7, \quad c_4 = d_1 + d_3 + d_5,$$
$$c_5 = d_1 + d_2 + d_6, \quad c_6 = d_5 + d_6 + d_7,$$
$$c_7 = d_3 + d_4 + d_6.$$  

The labels on the right hand side define a Fano configuration like (6.3). In these coordinates, the 63 reflection hyperplanes of $W(E_7)$ are given by the 21 linear forms $d_i - d_j$, the 35 linear forms $d_i + d_j + d_k$, and the 7 linear forms $d_i + d_j + d_k + d_l + d_m + d_n$. Here the indices are distinct. As a warning to the reader, we note that the $d$ variables do not represent orthogonal coordinates for $E_7$ since the quadratic form $\sum_i c_i^2$ is not mapped to $\sum_i d_i^2$.

The Weyl group $W(E_7)$ acts by Cremona transformations on the space of configurations of seven points in $\mathbb{P}^2$. Following Dolgachev and Ortland [DO, §IX.7, Remark 7], there exists a canonical $W(E_7)$-equivariant birational map from $\mathbb{P}(\mathfrak{h}) = \mathbb{P}^6$ to that space. In coordinates, this canonical map takes $(d_1 : d_2 : \cdots : d_7)$ to the column configuration of the matrix

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 \\ d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & d_6^2 & d_7^2 \end{pmatrix}.$$  

The rational map $\gamma$ is the composition of that birational morphism and the Göpel map of [DO, §IX.7, Thm. 5], written explicitly in Section 6, that takes 7-tuples in $\mathbb{P}^2$ to points in $G$.

Our claim about $\gamma$ being a 24-to-1 cover follows from the fact that a general net of cubics in $\mathbb{P}^2$ contains 24 cuspidal cubics [Bra, §IV]. Indeed, the ring of invariants for plane cubics is generated by two polynomials of degrees 4 and 6, and the condition for a plane cubic to have a cusp is that both of these invariants vanish [Dol, §10.3]. This implies that, given seven general points in $\mathbb{P}^2$, there exist precisely 24 representations (4.4) of that configuration up to
projective transformations. The net of cubics through these seven points defines a morphism from the corresponding del Pezzo surface of degree 2 onto $\mathbb{P}^2$. The branch locus is a quartic curve, and the 24 cuspidal cubics in that net correspond to the 24 flexes of that quartic.

**Remark 4.3.** The coordinate system $d$ and the connection to cuspidal cubics are due to Bramble [Bra]. Bramble also gives an explicit formula for the plane quartic in terms of $W(E_7)$-invariant polynomials in $d_1, \ldots, d_7$ [Bra, §VI] (there the coordinates are called $t_1, \ldots, t_7$). We first learned about the matrix $D$ from [HKT, §6].

**Remark 4.4.** Here is another way to see that the degree of $\gamma$ is 24. Form the $2 \times 15$-matrix

$$
\begin{pmatrix}
\gamma(c) \\
\gamma(C)
\end{pmatrix} = \begin{pmatrix}
r & s_{001} & s_{010} & \cdots & s_{110} & s_{111} & t_{001} & t_{010} & \cdots & t_{111} \\
R & S_{001} & S_{010} & \cdots & S_{110} & S_{111} & T_{001} & T_{010} & \cdots & T_{111}
\end{pmatrix},
$$

The first row consists of our degree 7 polynomials in $c_1, \ldots, c_7$, and the second row is the image under $\gamma$ of a random point $C \in \mathbb{P}^6$. One needs to verify that its scheme is reduced and consists of 24 points in $\mathbb{P}^6$. In practice, this is done as follows. With probability one, we have $R \neq 0$, and we can consider the affine scheme defined by $r = 1, s_{001} = S_{001}/R, \ldots, t_{111} = T_{111}/R$. Using Macaulay 2 we verify that this is a 0-dimensional scheme of degree 168. This gives the number of solutions of $(c_1, \ldots, c_7) \in \mathbb{A}^7$ that belong to the fiber of this point. But if $\lambda c_1, \ldots, \lambda c_7$ were also a solution, we would need $\lambda^7 = 1$ because all of the functions used to define $\gamma$ are of degree 7. Hence the degree of the fiber of $\gamma$ is $168/7 = 24$, as desired.

**Remark 4.5.** As indicated in the proof of Theorem 4.1, the map $\gamma$ is closely related to a map from the moduli space of 7 points in $\mathbb{P}^2$ to the moduli space of plane quartics. Here, the reflection hyperplanes correspond to the 63 discriminantal conditions which give singular plane quartics. It follows that the non-hyperelliptic locus of the moduli space $\mathcal{M}_3(2)$ is the image in the Göpel variety $\mathcal{G}$ of the complement of the 63 reflection hyperplanes in $\mathbb{P}^6$. The fiber of size 24 corresponds to a choice of flex point on the plane quartic.

In Section 6 we shall define an embedding of the Göpel variety $\mathcal{G}$ into a certain 35-dimensional toric variety $\mathcal{T}$. It lives in $\mathbb{P}^{134}$ and $W(E_7)$ acts on this embedding by permuting coordinates. That toric variety $\mathcal{T}$ and its polytope elucidate the combinatorial structure of our varieties. The image of the 63 reflection hyperplanes gives the 63 boundary divisors as the complement of the non-hyperelliptic locus of $\mathcal{M}_3(2)$ in $\mathcal{G}$. This determines the 63 distinguished facets of the polytope of $\mathcal{T}$ mentioned in Theorem 6.1.

After completing the present work, we learned that our parametrization of the Göpel variety, as well as some of the material in Section 6, had already been found by Colombo, van Geemen and Looijenga in [CGL]. In particular, the indeterminacy locus of our map $\gamma: \mathbb{P}^6 \to \mathcal{G}$ is determined in [CGL, Proposition 4.18]. See also [CGL, §1, Quartic curves] for an interpretation of this result in terms of the geometric invariant theory of plane quartics. We now state their result, and we strengthen it by including the answer to [CGL, Question 4.19]. This underlines the benefit of combining theoretical studies of moduli spaces with hands-on computations.

Recall that a flat of a hyperplane arrangement is the intersection of some subset of the hyperplanes. The dimension of a flat will refer to the dimension of its projectivization.
Theorem 4.6. (cf. [CGL]) The indeterminacy locus of $\gamma$ is the reduced union of 315 two-dimensional flats and 336 one-dimensional flats of the reflection arrangement of $E_7$ in $\mathbb{P}^6$. They correspond to the root subsystems of type $D_4$ and $A_5$ listed in row 9 and row 15 of Table 2 in Section 9. The ideal $\langle r, s_*, t_* \rangle$ is the intersection of these 651 linear ideals.

Proof. The set-theoretic version of this theorem was proved in [CGL, Proposition 4.18]. We established the ideal-theoretic statement with a Macaulay 2 computation that is posted in our supplementary files. The following remark offers a more detailed explanation.

Remark 4.7. The indeterminacy locus of $\gamma$ is the zero set in $\mathbb{P}^6$ of the ideal $\langle r, s_*, t_* \rangle = \langle r, s_{001}, \ldots, s_{111}, t_{001}, \ldots, t_{111} \rangle$ in the polynomial ring $\mathbb{Q}[c_1, c_2, \ldots, c_7] = \mathbb{Q}[d_1, d_2, \ldots, d_7]$. Theorem 4.6 states that this ideal is radical, and that it is the intersection of $651 = 315 + 336$ ideals which are generated by linear forms. One of these components is the height 4 prime ideal

$$\langle d_1-d_2, d_1-d_3, d_1+d_4+d_5, d_2+d_4+d_5, d_3+d_4+d_5, d_1+d_6+d_7, d_2+d_6+d_7, d_3+d_6+d_7, d_2-d_3, d_2+d_4+d_5+d_6+d_7, d_1+d_4+d_5+d_6+d_7, d_1+d_3+d_4+d_5+d_6+d_7 \rangle. \quad (4.5)$$

The 12 listed generators form a root subsystem of type $D_4$ in the root system $E_7$. There are 315 such subsystems in $E_7$. By [Man, §4], they are in bijection with the syzygetic triples of Steiner complexes, and also with the isotropic planes in $(\mathbb{F}_2)^6$. The latter will be discussed in Section 6, and a census of all root subsystems will be given in Table 2. Each root subsystem $D_4$ consists of 12 of the 63 reflection hyperplanes for $E_7$, and these intersect in a $\mathbb{P}^2$ inside $\mathbb{P}^6 = \mathbb{P}(h)$. The rank 4 matroid given by (4.5) is the Reye configuration in [Man, Figure 2].

The other 336 components of $\langle r, s_*, t_* \rangle$ correspond to an orbit of root subsystems of type $A_5$. Each such prime ideal is linear of height 5, and it contains 15 of the 63 linear forms.

To verify the claim, we intersect all 651 linear ideals in Macaulay 2. We note that a naive intersection may run for several hours and still not terminate, but a careful choice of ordering will finish in a matter of seconds. First, we use the $c$-coordinates rather than the $d$-coordinates. Each of our ideals contains at least one of the variables $c_i$. We separate them into seven groups of 93 based on which $c_i$ they contain (there is some redundancy). Then we intersect each of these groups. Finally, we intersect the seven resulting ideals, and the result is $\langle r, s_*, t_* \rangle$. We also used this procedure to get the ideals of the $D_4$ flats and the $A_5$ flats. We list the graded Betti tables for the indeterminacy locus of $\gamma$, the reduced union of the 315 flats of type $D_4$, and the reduced union of the 336 flats of type $A_5$, respectively:

|       | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| total | 1 | 15| 84|168|126|28|
| 0:    | 1 | . | . | . | . | . |
|       | ... | | | | | |
| 6:    | . | 15| . | . | . | . |
| 7:    | . | . | . | . | . | . |
| 8:    | . | . | . | . | . | . |
| 9:    | . | 84| . | . | . | . |
| 10:   | . | . | 168|105|21 | . |
| 11:   | . | . | . | . | . | . |
| 12:   | . | . | . | 21| . | . |
| 13:   | . | . | . | . | 7 | . |

|       | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| total | 1 | 85| 210|315|210|21|
| 0:    | 1 | . | . | . | . | . |
|       | ... | | | | | |
| 6:    | . | 15| . | . | . | . |
| 7:    | . | . | . | . | . | . |
| 8:    | . | . | . | . | . | . |
| 9:    | . | 70| 210| . | . | . |
| 10:   | . | . | . | 315|210| . |
| 11:   | . | . | . | . | . | . |
| 12:   | . | . | . | 21| . | . |
| 13:   | . | . | . | . | 7 | . |

|       | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| total | 1 | 36| 315|595|420|105|
| 0:    | 1 | . | . | . | . | . |
|       | ... | | | | | |
| 6:    | . | 15| . | . | . | . |
| 7:    | . | . | . | . | . | . |
| 8:    | . | . | . | . | . | . |
| 9:    | . | 84| . | . | . | . |
| 10:   | . | 168|105|21 | . | . |
| 11:   | . | . | . | . | . | . |
| 12:   | . | . | . | 21| . | . |
| 13:   | . | . | . | . | 7 | . |
At this point, it is important to note that \( \langle r, s, t \rangle \) has an alternative generating set, consisting of 135 products of linear forms defining reflection hyperplanes. We shall present them in Section 6. This fact makes it obvious that the variety consists of flats of \( E_7 \). Colombo et al. used these 135 generators to prove the set-theoretic result in [CGL, Prop. 4.18]. 

5 Equations defining the Göpel variety

In this section we first determine the equations and graded Betti numbers of the Göpel variety \( G \). Subsequently, we compute the prime ideal of the universal Coble quartic over \( G \). This is the 12-dimensional subvariety in \( \mathbb{P}^{14} \times \mathbb{P}^7 \) whose fibers over \( G \) are the quartics (2.12).

**Theorem 5.1.** The six-dimensional Göpel variety \( G \) has degree 175 in \( \mathbb{P}^{14} \). The homogeneous coordinate ring of \( G \) is Gorenstein, it has the Hilbert series

\[
1 + 8z + 36z^2 + 85z^3 + 36z^4 + 8z^5 + z^6 \over (1 - z)^7,
\]

and its defining prime ideal is minimally generated by 35 cubics and 35 quartics. The graded Betti table of this ideal in the polynomial ring \( \mathbb{Q}[r, s_{001}, \ldots, t_{111}] \) in 15 variables equals

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{total:} & 1 & 70 & 609 & 1715 & 2350 & 1715 & 609 & 70 & 1 \\
0: & 1 & & & & & & & & \\
1: & & & & & & & & & \\
2: & & 35 & 21 & & & & & & \\
3: & & 35 & 588 & 1715 & 2350 & 1715 & 588 & 35 & . \\
4: & & . & . & . & . & . & 21 & 35 & . \\
5: & & . & . & . & . & . & . & . & . \\
6: & & . & . & . & . & . & . & . & 1
\end{array}
\]

**Remark 5.2.** Before giving the proof, let us point out some geometric consequences of this theorem. Since the degree of the numerator is less than the degree of the denominator in the Hilbert series, we see that the Hilbert function agrees with the Hilbert polynomial for all nonnegative inputs. One can calculate directly that the Hilbert polynomial of \( G \) is

\[
\frac{35}{144}t^6 + \frac{35}{48}t^5 + \frac{287}{144}t^4 + \frac{133}{48}t^3 + \frac{343}{72}t^2 + \frac{7}{2}t + 1.
\]

We see from the self-duality of the Betti table that the canonical bundle of \( G \) is

\[
\omega_G = \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^{14}}}^8 (\mathcal{O}_G, \mathcal{O}_{\mathbb{P}^{14}}(-15)) = \mathcal{O}_G(-1).
\]

Hence \( G \) is a Fano variety. Since \( G \) is arithmetically Cohen–Macaulay, and the Hilbert polynomial and the Hilbert function agree, this implies \( H^i(G, \mathcal{O}_G(d)) = 0 \) if either \( d \geq 0 \) and \( i > 0 \) or if \( d < 0 \) and \( i < 6 \) [Eis05, Cor. A1.15].

**Proof of Theorem 5.1.** Our point of departure for the derivation of Theorem 5.1 is [DO, §IX.5, Proposition 8] and the subsequent corollary which gives a list of 63 cubics. We
now review this construction. For each of the 63 non-zero half-periods $\epsilon/2 \in \Lambda/2\Lambda$, we consider the linear involution on $\mathbb{P}^7$ that is induced from the action on second order theta functions seen in (2.10). The fixed point set of this involution on $\mathbb{P}^7$ is the union of two three-dimensional planes $H_+^+$ and $H_-^-$, and each of these planes intersects the Coble quartic in a Kummer surface. The relations (2.14) on the coefficients of these two Kummer surfaces give the same cubic relation on $r, s_{ijk}, t_{ijk}$. This cubic lies in the ideal of the Göpel variety $G$.

In this manner we obtain 63 cubic equations for the Göpel variety $G$, as stated in [DO, Corollary on p. 186]. However, only 35 of these are linearly independent. A vector space basis consists of those cubics that come from the 35 nonzero even theta characteristics.

In addition to the 35 cubics constructed from Kummer surfaces as above, the ideal of $G$ has 35 minimal generators of degree 4. Here is an example of such a quartic generator:

$$48r^2s_{101}s_{110} - 12s_{011}s_{101}s_{110} - 12s_{100}s_{101}s_{110} - 4s_{011}s_{100}s_{111} - 4s_{100}s_{101}s_{111} + 8s_{011}s_{100}s_{111} + 4s_{100}^2s_{111} - 16rs_{010}s_{101}s_{111} + 8s_{010}s_{011}s_{101}s_{111} + 8s_{100}^2s_{101}s_{111} - 16rs_{010}s_{110}s_{111} + 8s_{011}s_{110}s_{110}s_{111} + 8s_{100}s_{110}s_{111} + 8s_{010}s_{110}^2s_{111} - 16rs_{010}s_{110}s_{111} - 4s_{101}s_{110}^2s_{111} - 4s_{100}s_{111}s_{111} - s_{100}s_{111}t_{010}^2 - 2s_{010}s_{101}t_{011} + 4rs_{011}t_{011} + s_{101}s_{110}t_{011} + s_{100}s_{111}^2t_{011} + s_{100}s_{111}t_{010}^2 + s_{100}s_{111}t_{110}^2 - s_{100}t_{010}t_{110} + s_{110}t_{010}t_{111} + s_{101}t_{010}t_{110}t_{111} + 2s_{010}t_{110}t_{111} - 4s_{011}t_{111}^2 - 2s_{101}s_{110}t_{111}$$

The explanation for the derivation of such quartics will come in the proof of Theorem 6.2. From the 70 minimal generators, a Gröbner basis for the ideal of $G$ can be computed in Macaulay 2, and from this one finds the degree 175 and the Hilbert series in Theorem 5.1.

The group $Sp_6(\mathbb{F}_2)$ has exactly two distinct irreducible representations of dimension 35, and these two representations are given respectively by the cubic generators and the quartic generators of our ideal $G$. In particular, we can obtain all 70 generators by lifting the two 6x6-matrices $\mu$ and $\nu$ in Section 4 to $\mathbb{Q}[r, s_{01}, \ldots, t_{111}]$ and applying these to one representative cubic and one representative quartic, for instance those displayed above. In Section 6 we shall
present an alternative model for these two $Sp_6(F_2)$-submodules $Q[r, s_{001}, \ldots, t_{111}]$, namely in terms of binomials in a polynomial ring in 135 unknowns.

It remains to be proved that the 35 cubics and the 35 quartics actually generate the prime ideal of $G$. This is what we shall explain next. The 70 generators, as well as explicit matrix representations for $\mu$ and $\nu$ on the 15 Coble coefficients, are available in our supplementary materials, and the reader can use these to verify all of our calculations in Macaulay 2.

The main idea is to show that the scheme defined by this ideal is generically reduced and satisfies Serre’s criterion ($S_1$). These conditions ensure that our ideal is radical by [Eis95, Exercise 11.10]. By the earlier results of Coble [Cob, §49] and Dolgachev–Ortland [DO, §IX.7, Theorem 5], we already know that the cubics alone cut out $G$ set-theoretically, and hence the variety of our ideal is irreducible of codimension eight. So, to show ‘generically reduced’ only requires showing that at some point on $G$, the Jacobian matrix has rank 8. This is easily done by substituting randomly chosen points from $\mathbb{P}^6$ into the map $\gamma$.

To prove Serre’s condition ($S_1$), we show that our ideal defines an arithmetically Cohen–Macaulay scheme. This is done by adding seven random linear forms to the ideal generated by the cubics and quartics. The resulting ideal defines an artinian quotient of $Q[r, s_{001}, \ldots, t_{111}]$, and we verified that its Hilbert series is the numerator seen in (5.1). The computation shows that the seven linear forms are a regular sequence modulo our ideal, and hence we get that the scheme defined by the 35 cubics and 35 quartics is arithmetically Cohen–Macaulay. We remark that the regular sequence computation is intensive and took approximately 15 minutes to perform. An alternative method is to work over the finite field $\mathbb{Z}/101$. The Hilbert series remains unchanged, but the regular sequence computation takes only a few seconds to perform. This implies the result via standard flatness and semicontinuity arguments.

At this point we have shown that the ideal generated by our cubics and quartics is prime. To infer the Gorenstein property, we use Stanley’s result [Sta, Theorem 4.4], which states that a Cohen–Macaulay domain whose Hilbert series has a palindromic numerator is Gorenstein.

We finally derive the Betti table asserted in Theorem 5.1. We have already shown that the Göpel variety $G$ is arithmetically Gorenstein, which implies that the Betti table is symmetric [Eis95, Corollary 21.16]. Multiplying the numerator of the Hilbert series (5.1) by $(1-z)^8$ gives the graded Euler characteristic (or $K$-polynomial) of the minimal free resolution of $G$. By the Cohen–Macaulay property, the Castelnuovo–Mumford regularity of $G$ is the degree of the numerator of the Hilbert series, which is 6. Therefore, we have the following partial information about the Betti numbers:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| total: | 1 | 70 | ? | ? | ? | ? | ? | 70 | 1 |
| 0: | 1 | . | . | . | . | . | . | . | . |
| 1: | . | . | . | . | . | . | . | . | . |
| 2: | . | 35 | 21 | c | f | e | b | . | . |
| 3: | . | 35 | a | d | g | d | a | 35 | . |
| 4: | . | b | e | f | c | 21 | 35 | . | . |
| 5: | . | . | . | . | . | . | . | . | . |
| 6: | . | . | . | . | . | . | . | 1 | . |

The unknown entries satisfy the linear equations

$$a - c = 588, \ b - d + f = -1715, \ -2e + g = 2350.$$
Now, there are 21 linear syzygies on the 35 cubic generators, and we compute the corresponding $35 \times 21$-matrix whose entries are linear forms in $\mathbb{Q}[r, s_{001}, \ldots, t_{111}]$. Evaluating this matrix at a random point in $\mathbb{Q}^{15}$ shows that it has full rank 21 over the rational function field $\mathbb{Q}(r, s_{001}, \ldots, t_{111})$. Hence its columns are linearly independent. This implies $c = 0$, and consequently $f = e = b = 0$. The above linear equations then imply $a = 588, d = 1715, g = 2350$, so the Betti table is as claimed. This concludes our proof of Theorem 5.1.

Now we study the universal Coble quartic $C$ in $G \times \mathbb{P}^7$. This twelve-dimensional variety is defined as the closure in $\mathbb{P}^{14} \times \mathbb{P}^7$ of the set of pairs $((r : s_{001} : \cdots : t_{111}), (x_{000} : \cdots : x_{111}))$ such that $(r : s_{001} : \cdots : t_{111})$ is a point in the non-hyperelliptic locus of $M_3(2)$ in $G$, and the pair is a point on the hypersurface of bidegree $(1, 4)$ that is given by the polynomial (2.12).

**Corollary 5.3.** The bihomogeneous prime ideal of the universal Coble quartic in $\mathbb{P}^{14} \times \mathbb{P}^7$ is minimally generated by 71 polynomials, namely the $35 + 35$ equations of bidegrees $(3, 0)$ and $(4, 0)$ for the Göpel variety as in Theorem 5.1, along with the bidegree $(1, 4)$ equation (2.12).

**Proof.** The equation (2.12) cuts out a codimension one subscheme $C'$ of $G \times \mathbb{P}^7$. Since $G$ is arithmetically Cohen–Macaulay in its embedding in $\mathbb{P}^{14}$, this implies that (2.12) is a non-zerodivisor on $G \times \mathbb{P}^7$, and that the scheme $C'$ is arithmetically Cohen–Macaulay as well. To show that our ideal is prime, it suffices to show that it is irreducible and generically reduced (as explained in the proof of Theorem 5.1). This will imply $C' = C$.

Consider the projection $\pi : C' \to G$. The fibers of $\pi$ have constant dimension 6 since the Coble coefficients cannot simultaneously vanish on any point of $G$. Let $U \subset M_3(2)$ denote the non-hyperelliptic locus. Over the subvariety $U \subset G$, the fibers of $\pi$ are Coble quartics, and hence irreducible. Since $C'$ is arithmetically Cohen–Macaulay, all of its irreducible components have the same dimension, namely 12. The closure of $\pi^{-1}(U)$ gives an irreducible component. The complement of $\pi^{-1}(U)$ in $C'$ is the preimage of $G \setminus U$. But dim $\pi^{-1}(G \setminus U) \leq 11$, so it cannot be an irreducible component. Hence the variety $C'$ is irreducible.

Now consider the Jacobian matrix of the 71 equations defining $C'$. Since the 70 equations for $G$ involve no $x$ coordinate, this matrix can be put in block triangular form. Choose any point $((r : s_{001} : \cdots : t_{111}), (x_{000} : \cdots : x_{111}))$, where $(r : s_{001} : \cdots : t_{111}) \in U$ is a non-singular point of $G$, and $(x_{000} : \cdots : x_{111})$ is a non-singular point on its Coble quartic. Then this is a non-singular point of $C'$. Hence the scheme $C'$ defined by the 71 proposed equations is also generically reduced. We conclude that they generate the prime ideal of $C = C'$.

6 A toric variety for seven points in $\mathbb{P}^2$

In this section we present an alternative embedding of the Göpel variety, with beautiful combinatorics. We learned this from Dolgachev and Ortland [DO, §IX.7]. Here, the Göpel variety sits in the high-dimensional projective space $\mathbb{P}^{134}$ whose coordinates are the 135 Göpel functions. The corresponding ideal is generated by linear trinomials, cubic binomials and quartic binomials. We will first describe the combinatorial structure of the Göpel functions. The paper [RSS] features the analogous toric varieties for smaller Macdonald representations.

Consider the six-dimensional vector space $(\mathbb{F}_2)^6$ over the two-element field $\mathbb{F}_2$. We fix the following non-degenerate symplectic form on this 64-element vector space:

$$\langle x, y \rangle = x_1y_4 + x_2y_5 + x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3. \quad (6.1)$$
A linear subspace $V \subset (\mathbb{F}_2)^6$ is isotropic if $\langle x, y \rangle = 0$ for all $x, y \in V$. Clearly, all subspaces of dimension $\leq 1$ are isotropic, and there are no isotropic subspaces of dimension $\geq 4$. There are precisely 315 isotropic subspaces of dimension two, and 135 isotropic subspaces of dimension three. The former are called isotropic planes, and the latter are called Lagrangians. There are 63 non-zero vectors in $(\mathbb{F}_2)^6$, and each Lagrangian contains seven of these. Each isotropic plane contains three of these, and is contained in precisely three Lagrangians.

The root system $E_7$ consists of 126 vectors in a seven-dimensional inner product space. If we take this space to be the hyperplane in $\mathbb{R}^8$ given by coordinate sum zero, then the 63 positive roots are $e_i - e_j$ for $1 \leq i < j \leq 7$, and each Lagrangian contains seven of these. Each isotropic plane contains three of these, and is contained in precisely three Lagrangians.

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We copied Table 1 from Cayley’s paper [Cay]. It fixes a particular bijection between the positive roots of $E_7$ and the vectors in $(\mathbb{F}_2)^6 \setminus \{0\}$. Its rows represent the first three coordinates and its columns represent the last three coordinates of a vector in $(\mathbb{F}_2)^6$. For instance, the triple 247 corresponds to the vector $(0, 0, 1, 1, 1, 0)$, and the pair 24 corresponds to $(1, 1, 1, 1, 0, 0)$. This bijection has the property in [DO, Lemma IX.8]: two positive roots in $E_7$ are perpendicular in $\mathbb{R}^8$ if and only if the corresponding vectors $x, y \in (\mathbb{F}_2)^6$ satisfy $\langle x, y \rangle = 0$ in $\mathbb{F}_2$. Cayley had constructed his table in the 1870s to have precisely this property.

The orthogonality preserving bijection between $(\mathbb{F}_2)^6 \setminus \{0\}$ and the positive roots of $E_7$ shows that the Weyl group of $E_7$ (modulo its two-element center) is isomorphic to the group $Sp_6(\mathbb{F}_2)$ of $6 \times 6$-invertible matrices over $\mathbb{F}_2$ that preserve the symplectic form in (6.1) (see [Bou, §VI.4, Exercice 3]). The order of that group is easily seen (cf. [Cob, §II.23, (7)]) to be

$$|Sp_6(\mathbb{F}_2)| = (2^6 - 1) \cdot 2^5 \cdot (2^4 - 1) \cdot 2^3 \cdot (2^2 - 1) \cdot 2^1 = 36 \cdot 8! = 1451520. \quad (6.2)$$

With the relabeling given by Cayley’s table, the 135 Lagrangians fall into two classes with respect to permutations of the set $\{1, 2, 3, 4, 5, 6, 7\}$. First, there are 30 Fano configurations

$$\{124, 235, 346, 457, 561, 672, 713\}. \quad (6.3)$$

We denote this configuration by $f_{1234567}$. Second, there are 105 Pascal configurations like

$$\{12, 34, 56, 78, 127, 347, 567\}. \quad (6.4)$$

Table 1: Cayley’s bijection between lines in $(\mathbb{F}_2)^6$ and positive roots of $E_7$.  

|   | 000 | 100 | 010 | 110 | 001 | 101 | 011 | 111 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 000| 236 | 345 | 137 | 467 | 156 | 124 | 257 |
| 100| 237 | 67  | 136 | 12  | 157 | 48  | 256 |
| 010| 245 | 127 | 23  | 68  | 134 | 357 | 15  |
| 110| 126 | 13  | 78  | 145 | 356 | 25  | 46  |
| 001| 567 | 146 | 125 | 247 | 45  | 17  | 38  |
| 101| 147 | 58  | 246 | 34  | 16  | 123 | 27  |
| 011| 135 | 347 | 14  | 57  | 28  | 36  | 167 |
| 111| 346 | 24  | 56  | 235 | 37  | 267 | 457 | 18 |
We denote this configuration by \( g_{1234567} \). If the configurations are changed by a permutation of \( \{1, 2, 3, 4, 5, 6, 7\} \) then the indices of the label \( f_\bullet \) or \( g_\bullet \) are permuted accordingly. Hence the labeling is not unique. For example, the Fano configuration \( f_{1234567} \) in (6.3) can also be labeled \( f_{2345671} \), and the Pascal configuration \( g_{1234567} \) in (6.4) can also be labeled \( g_{7345612} \). Some permutations give rise to a sign change in \( f_\bullet \) or \( g_\bullet \).

The 315 isotropic planes in \( (\mathbb{F}_2)^6 \) fall into three classes. Each class has cardinality 105. Using the labeling given by Cayley’s Table 1, representatives of these three classes are

\[
\{12, 123, 38\} \quad \text{and} \quad \{123, 145, 167\} \quad \text{and} \quad \{12, 34, 567\}. \tag{6.5}
\]

Consider the six-dimensional variety of unlabeled configurations of seven points in the projective plane \( \mathbb{P}^2 \). There are natural correspondences, described in [DO, §IX], that take such configurations to Cayley octads, hence to plane quartic curves, and hence to abelian threefolds. Here is how to make this completely explicit. We associate with each Lagrangian in \( (\mathbb{F}_2)^6 \) a homogeneous symmetric polynomial in the brackets \([ijk]\), which are the \(3 \times 3\)-minors of the \(3 \times 7\)-matrix of homogeneous coordinates on \( (\mathbb{P}^2)^7 \). These bracket polynomials are called Göpel functions in [Cob, DO]. Each of the 30 Fano configurations translates verbatim into a bracket monomial. For instance, the Fano configuration (6.3) translates into

\[
f_{1234567} = [124][235][346][457][561][672][713]. \tag{6.6}
\]

The 105 Pascal configurations represent six points lying on a conic, e.g. let \( Q_7 \) denote the quartic bracket polynomial that vanishes when the points 1, 2, 3, 4, 5, 6 lie on a conic:

\[
Q_7 = [134][156][235][246] - [135][146][234][256]. \tag{6.7}
\]

Then the Pascal configuration (6.4) translates into the bracket binomial

\[
g_{1234567} = [127][347][567] \cdot Q_7. \tag{6.8}
\]

We have thus defined a coordinate system on \( \mathbb{P}^{134} \), consisting of 30 coordinates \( f_\bullet \) and 105 coordinates \( g_\bullet \), and we have defined a rational map

\[
(\mathbb{P}^2)^7 \dashrightarrow \mathbb{P}^{134} \tag{6.9}
\]

whose coordinates are multi-homogeneous bracket polynomials of degree \((3, 3, 3, 3, 3, 3, 3)\). This map factors through the six-dimensional space of configurations of seven points in \( \mathbb{P}^2 \).

One desirable property of this coordinate system is that it fits well with \( E_7 \). Consider the action of the symplectic group \( Sp_6(\mathbb{F}_2) \) on the 135 Lagrangians in \( (\mathbb{F}_2)^6 \). Then the induced action on our 135 Göpel coordinates \( f_\bullet \) and \( g_\bullet \) is realized by signed permutations.

Dolgachev and Ortland [DO, §IX.7, Proposition 9] describe the following 315 linear relations among the Göpel functions. For each of the 315 isotropic planes, there is a linear relation among the three Göpel coordinates indexed by the three Lagrangians containing that plane. For instance, the three isotropic planes in (6.5) determine the linear relations

\[
\begin{align*}
g_{3124567} - g_{3124657} + g_{3124756} & \quad \text{and} \quad g_{1234567} - f_{1243675} + f_{1243675} & \quad \text{and} \quad g_{5123467} - g_{6123457} + g_{7123456} \\
\{12, 38, 123, 45, 67, 345, 367\} & \quad \text{and} \quad \{123, 145, 167, 18, 23, 45, 67\} & \quad \text{and} \quad \{12, 34, 567, 58, 67, 125, 345\} \\
\{12, 38, 123, 46, 57, 346, 357\} & \quad \text{and} \quad \{123,145,167,247,256,346,357\} & \quad \text{and} \quad \{12, 34, 567, 68, 57, 126, 346\} \\
\{12, 38, 123, 47, 56, 347, 356\} & \quad \text{and} \quad \{123,145,167,246,257,347,356\} & \quad \text{and} \quad \{12, 34, 567, 78, 56, 127, 347\}
\end{align*}
\]
In each column we list the corresponding triple of Lagrangians. These linear forms arise from the three-term quadratic Plücker relations among the brackets $[ijk]$. These linear forms span a 120-dimensional space, so they cut out a 14-dimensional linear subvariety $L \subset \mathbb{P}^{134}$. Thus, the rational map (6.9) factors through this linear space:

$$(\mathbb{P}^2)^7 \dashrightarrow L \subset \mathbb{P}^{134}. \tag{6.10}$$

Returning to the Coble quartic $F_6$ in (2.12), its 15 coefficients span the same irreducible $\text{Sp}_6(\mathbb{F}_2)$-module as the 135 Göpel functions. Coble states in [Cob, §IV.49,(16)] that

$$r, s_{001}, s_{010}, s_{011}, s_{100}, s_{101}, s_{110}, s_{111}, t_{001}, t_{010}, t_{100}, t_{101}, t_{110}, t_{111}$$

can be expressed as integer linear combinations of the Göpel functions $f_\bullet$ and $g_\bullet$.

Constructing such linear combinations turned out to be a non-trivial undertaking, but we succeeded in obtaining formulas for writing $r, s, t$ in the Göpel coordinates $f_\bullet, g_\bullet$ by following exactly the derivation described by Coble [Cob, §IV.49,(16)]. Combining with 120 independent linear trinomial relations among the Göpel functions described above, we inverted the linear relations and found an explicit list of 135 transformation formulas such as

$$f_{1234657} = -4r - 2s_{001} - 2s_{010} - 2s_{011} - 2s_{100} - 2s_{101} - 2s_{110} - 2s_{111} - t_{001} - t_{010} - t_{011} - t_{100} - t_{101} - t_{110} - t_{111}$$

$$f_{1237654} = -4s_{101} - 8r - 4s_{001} - 4s_{100} - 2t_{010}$$

... ... ... ...

(6.11)

Of course, these 135 linear forms in the 15 coefficients of $F$ satisfy the 120 linear trinomials above. The complete list of transformation formulas is posted on our supplementary materials website.

It is important to note that the change of basis in (6.11) is unique up to scalar multiple, since we require it to be $\text{Sp}_6(\mathbb{F}_2)$-equivariant, and the 15-dimensional $\text{Sp}_6(\mathbb{F}_2)$-module appears with multiplicity 1 in the permutation representation given by the Göpel functions. Consequently, the expressions in (6.11) are not just some random transition maps. In finding them explicitly, we resolved an issue that was left open by Coble in [Cob, §IV.49, (16)].

We now present a completely explicit formula for the map $\mathbb{P}^6 \dashrightarrow \mathcal{G} \subset \mathbb{P}^{134}$. It factors through the configuration space of seven points in $\mathbb{P}^2$ as follows. A point $(d_1 : d_2 : \cdots : d_7)$ in $\mathbb{P}^6$ is sent to the configuration given by the columns of the matrix $D$ in (4.4). We specialize the 35 Plücker coordinates $[ijk]$ to the $3 \times 3$-minors of $D$. The result is the product

$$[ijk] = (d_i - d_j)(d_i - d_k)(d_j - d_k)(d_i + d_j + d_k).$$

Under this specialization, the quartic bracket polynomial $Q_i$ for the condition (6.7) that six points lie on a conic maps to a product of 16 linear forms. Substituting these expressions into the Göpel functions (6.6) and (6.8), we obtain a list of 135 polynomials in $d_1, d_2, \ldots, d_7$, each of which is a product of 28 linear forms. All 135 such products share a common factor of degree 21, namely, $\prod_{i<j}(d_i - d_j)$. Removing that common factor, we recover the formulas for the Göpel coordinates as products of seven linear forms. For instance, we find

$$f_{1234567} = (d_1 + d_2 + d_4)(d_2 + d_3 + d_5)(d_3 + d_4 + d_6)(d_4 + d_5 + d_7)$$

$$(d_5 + d_6 + d_7)(d_6 + d_7 + d_2)(d_7 + d_1 + d_3),$$

$$g_{7123456} = (d_1 + d_2 + d_7)(d_3 + d_4 + d_7)(d_5 + d_6 + d_7)$$

$$(d_1 - d_2)(d_3 - d_4)(d_5 - d_6)(-d_1 - d_2 - d_3 - d_4 - d_5 - d_6). \tag{6.12}$$
These formulas also appear in [CGL, §3].

We now introduce variables for the 63 reflection hyperplanes of $E_7$. Denote

$$x_i = d_i - (d_1 + \cdots + d_7),$$
$$x_{ij} = d_i - d_j,$$
$$x_{ijk} = d_i + d_j + d_k.$$ (6.13)

Then, we obtain

$$f_{1234567} = x_{124}x_{137}x_{156}x_{235}x_{267}x_{346}x_{457}, \quad f_{1234576} = x_{124}x_{136}x_{157}x_{235}x_{267}x_{347}x_{456},$$
$$f_{1234657} = x_{124}x_{137}x_{156}x_{236}x_{257}x_{345}x_{467}, \quad f_{1234675} = x_{124}x_{135}x_{167}x_{236}x_{257}x_{347}x_{456},$$
$$\cdots$$

$$g_{1234567} = x_1 x_{23} x_{45} x_{67} x_{123} x_{145} x_{167}, \quad g_{1234576} = x_1 x_{24} x_{35} x_{67} x_{123} x_{135} x_{167},$$
$$g_{1234657} = x_1 x_{23} x_{46} x_{57} x_{123} x_{146} x_{157}, \quad g_{1234675} = x_1 x_{24} x_{35} x_{67} x_{123} x_{135} x_{167},$$
$$\cdots$$

(6.14)

Each of these $x$-monomials is squarefree of degree 7, and represents one of the 30 Fano configurations like (6.3), or one of the 105 Pascal configurations like (6.4).

This is the moment when we come to the object promised in the title of this section. If we regard $x_i$, $x_{ij}$, $x_{ijk}$ as formal variables, the formulas in (6.14) define a monomial map

$$m: \mathbb{P}^{62} \to \mathbb{P}^{134}.$$  

The closure of the image of $m$ is a toric variety $\mathcal{T}$ in $\mathbb{P}^{134}$. We call $\mathcal{T}$ the Göpel toric variety.

Let $\mathcal{A}$ denote the image of the $63 \times 135$-matrix with entries in $\{0, 1\}$ representing the monomial map $m$. The rows of $\mathcal{A}$ are labeled by the 63 parameters $x_i, x_{ij}, x_{ijk}$, or, using Cayley’s bijection, by the 63 vectors in $(\mathbb{F}_2)^6 \setminus \{0\}$. Here we erase all occurrences of the index “8” in Table 1 so as to identify $(\mathbb{F}_2)^6 \setminus \{0\}$ with the subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ that have cardinality 1, 2 or 3. The columns of the matrix $\mathcal{A}$ are labeled by the 135 Göpel coordinates $f_\bullet, g_\bullet$, and thus by the 135 Lagrangians in $(\mathbb{F}_2)^6$. Each row of $\mathcal{A}$ has 15 entries 1; the other 120 entries are 0. Each column of $\mathcal{A}$ has 7 entries 1; the other 56 entries are 0. By computing the Smith normal form, one checks that our $63 \times 135$-matrix $\mathcal{A}$ has rank 36 over any field.

The convex hull of the columns of $\mathcal{A}$ is a convex polytope of dimension 35 in $\mathbb{R}^{63}$. We denote this polytope also by $\mathcal{A}$ and we call it the Göpel polytope. The Göpel polytope $\mathcal{A}$ has 135 vertices and 63 distinguished facets, each given by the vanishing of one of the coordinates in the ambient $\mathbb{R}^{63}$. However, these are not all of the facets of $\mathcal{A}$. In fact, it is a difficult computational problem, which we could not solve, to determine the number of facets of $\mathcal{A}$. Each distinguished facet contains $135 - 15 = 120$ of the vertices, and is indexed by a vector $v \in (\mathbb{F}_2)^6 \setminus \{0\}$. Thus each distinguished facet of $\mathcal{A}$ corresponds combinatorially to the complement of a Lagrangian in $(\mathbb{F}_2)^6 \setminus \{0\}$. These facets are indexed by the 63 $x$-variables.

By construction, the Göpel polytope $\mathcal{A}$ is the polytope associated with the projective toric variety $\mathcal{T}$. We summarize the above discussion:

**Theorem 6.1.** The Göpel toric variety $\mathcal{T}$ has dimension 35 in $\mathbb{P}^{134}$. It has 63 distinguished boundary divisors, each given by the vanishing of one parameter $x_i, x_{ij}$, or $x_{ijk}$. 

27
The ideal for the embedding $\mathcal{T} \subset \mathbb{P}^{134}$ is the toric ideal associated with the matrix $A$. We computed all minimal generators of this ideal of degree at most 4 by writing all products of at most 4 Göpel functions in terms of $x_i$, $x_{ij}$, or $x_{ijk}$ and comparing them. There are no linear forms or quadrics in the toric ideal of $A$. There are 630 cubic binomials of the form

$$g_{132467}g_{6123457}f_{1235746} - f_{1236745}g_{132467}g_{6132457}. \quad (6.15)$$

They form a single orbit under the $\text{Sp}_6(\mathbb{F}_2)$-action by permuting Göpel coordinates. Further, there are precisely 12285 quartic binomials that are not in the ideal generated by the 630 cubics. These form two $\text{Sp}_6(\mathbb{F}_2)$-orbits of sizes 945 and 11340. Representatives for these two orbits are, respectively,

$$f_{1234567}f_{1234675}g_{1243657}g_{2143756} - f_{1234576}f_{1234765}g_{1243756}g_{2143657}, \quad (6.16)$$

It would be desirable to find a Markov basis of $\mathcal{T}$, i.e. a set of minimal generators of the toric ideal $I_A$ of the matrix $A$. This turns out to be nontrivial. In fact, we need binomials of up to degree at least 6 to generate the toric ideal. We constructed examples of binomials of degree 5 and 6 that lie in $I_A$ but are not generated by elements of lower degrees:

$$f_{1234576}g_{2163745}g_{4123657}g_{6142537}g_{7132645} - f_{1236574}g_{2143756}g_{4162537}g_{6132457}g_{7132645}$$

Both of these binomials are indispensable: they represent two-element fibers of the semigroup map $A$: $\mathbb{N}^{135} \rightarrow \mathbb{N}^{63}$. This implies that they must appear in every Markov basis of $I_A$.

We also do not know whether $\mathcal{T}$ is projectively normal, or arithmetically Cohen–Macaulay. If this holds then $\mathcal{T}$ is the toric variety of the polytope $A$ in the strict sense of [CLS, §2.3]. Also we do not know the Hilbert polynomial (or Ehrhart polynomial). At the present time, our $63 \times 135$-matrix $A$ seems too big for the packages 4ti2 (for computing Markov bases), polymake (for computing facets), and normaliz (which is discussed in [CLS, Appendix B.3]).

Now we are ready to connect the map (6.9) with the map (4.1).

**Theorem 6.2.** The Göpel variety $G$ in $\mathbb{P}^{14}$ is linearly isomorphic to the closure of the image of the map $(\mathbb{P}^2)^7 \rightarrow \mathbb{P}^{134}$ described prior to (6.9), which is the (ideal-theoretic) intersection of $\mathcal{T}$ with the linear space $L \simeq \mathbb{P}^{14}$ in (6.10). Its prime ideal in the polynomial ring in 135 unknowns $f_*$ and $g_*$ is minimally generated by 120 linear trinomials, 35 cubic binomials, and 35 quartic binomials.

**Proof.** The composition of the maps (4.3), (4.1) and (6.11) is exactly to (6.12). Thus, the transformation (6.11) defines an isomorphism $L \simeq \mathbb{P}^{14}$ that is compatible with the rational map $(\mathbb{P}^2)^7 \rightarrow L$ in (6.10) and the rational map $(\mathbb{P}^2)^7 \rightarrow G \subset \mathbb{P}^{14}$ in (4.1). Therefore, the image of our map $(\mathbb{P}^2)^7 \rightarrow \mathbb{P}^{134}$ in (6.9) is linearly isomorphic to the Göpel variety $G$. Moreover, under this projective transformation, the ideal of the image of the rational map $(\mathbb{P}^2)^7 \rightarrow \mathbb{P}^{134}$ in $L$ is mapped to the ideal of $G$ in $\mathbb{P}^{14}$, which is the Gorenstein prime ideal generated by the 35 cubics and 35 quartics in Theorem 5.1.

It remains to be shown that the cubic and quartic generators can be represented by binomials modulo the 120 trinomials. Indeed, if we take the 630 cubic binomials described...
above and write them in terms of \( r, s, t \) using (6.11), they span exactly the 35-dimensional vector space of cubics in the ideal of \( G \subset \mathbb{P}^{14} \). Moreover, if we take the quartics generated by the 630 cubic binomials and the 12285 quartic binomials described above and write them in terms of \( r, s, t \), they span exactly the vector space of quartics in the ideal of \( G \subset \mathbb{P}^{14} \). Therefore, the image under (6.11) of the ideal they generate coincides with the ideal of \( G \subset \mathbb{P}^{14} \) in Theorem 5.1.

\[ \square \]

7 The universal Coble quartic in \( \mathbb{P}^7 \times \mathbb{P}^7 \)

Recall that for each \( \tau \in S_3 \) corresponding to a smooth non-hyperelliptic curve of genus three, the Coble quartic \( C_\tau \) is the unique hypersurface of degree 4 in \( \mathbb{P}^7 \) whose singular locus is the Kummer threefold corresponding to that curve. The \textit{universal Coble quartic} \( C(2,4) \) is the Zariski closure in \( \mathbb{P}^7 \times \mathbb{P}^7 \) of the set of pairs \((u,x)\) where \( u \in \vartheta(A_3(2,4)) \) corresponds to a non-hyperelliptic curve and \( x \) lies in the Coble quartic hypersurface corresponding to \( u \). In this section we derive generators for the ideal of \( C(2,4) \) in the theta coordinates \( u \) and \( x \).

We shall derive formulas for the coefficients \( r, s, t \) of the quartic polynomial \( F \) in (2.12) in terms of the second order theta constants \( u \). Our formulas are fairly big, and they define an explicit rational map of degree 64 from the Satake hypersurface to the Göpel variety:

\[ S \overset{64:1}{\longrightarrow} G. \quad (7.1) \]

This is realized concretely as a map \( \mathbb{P}^7 \longrightarrow \mathbb{P}^{14} \) by listing 15 polynomials \( r(u), s(u), t(u) \) in the variables \( u = (u_{000}, \ldots, u_{111}) \). It agrees with the abstract map (3.2) over the non-hyperelliptic locus of \( M_3(2,4) \) and collapses each of the 36 components of the hyperelliptic locus to a single point (see, for example, (8.9)). We also determine the zero set of these polynomials, which is the indeterminacy locus of (7.1) in the hypersurface \( S = \vartheta(A_3(2,4)) \).

The following theorem summarizes the results to be proved in this section:

\textbf{Theorem 7.1.} (a) The coefficients \( r, s, t \) of the Coble quartic can be expressed as polynomials of degree 28 in the eight theta constants \( u \). The resulting polynomial \( F \) from (2.12) is the sum of 372060 monomials of bidegree (28,4) in \((u,x)\).

(b) The locus in the Satake hypersurface \( S \) that is cut out by the coefficients \( r, s, t \) equals the Torelli boundary \( \text{Sing}(S) = \vartheta(A_3(2,4)) \setminus \vartheta(M_3(2,4)) \) (cf. Proposition 3.2).

(c) The prime ideal of the universal Coble quartic equals \( \langle F, S \rangle \), where \( S \) is the Satake polynomial of bidegree (16,0). Hence \( C(2,4) \) is a complete intersection of codimension 2 in \( \mathbb{P}^7 \times \mathbb{P}^7 \), and its bidegree equals \( 16U(28U + 4X) \). Here we write the cohomology ring of \( \mathbb{P}^7 \times \mathbb{P}^7 \) as \( \mathbb{Z}[U,X]/(U^7, X^7) \), with \( U \) and \( X \) representing the two hyperplane classes.

We note that, in response to this theorem, Grushevsky and Salvati Manni [GS2] developed a conceptual geometric approach that yields a shorter representation of the same polynomial.

\textbf{Proof.} (a) We will construct 15 polynomials of degree 28 in the eight unknowns \( u_{000}, \ldots, u_{111} \). The polynomial \( r(u) \) has 5360 terms, each polynomial \( s_{ijk}(u) \) has 7564 terms when \( i+j+k \neq 2 \) and has 7880 terms otherwise, and each polynomial \( t_{ijk}(u) \) has 8114 terms. They are available on our supplementary materials website.
We begin by outlining the derivation of these polynomials. Consider the Coble quartic $C_\tau$ corresponding to some non-hyperelliptic $\tau$. Let $\epsilon$ be a non-zero length 3 binary vector, so $\epsilon/2 \in A_\tau[2]$ represents a 2-torsion point on the abelian threefold $A_\tau$. As in the proof of Theorem 5.1, we write $H_\epsilon^\pm$ for the two three-dimensional fixed spaces of the involution (2.10) on $\mathbb{P}^7$ defined by $\Theta_2[\sigma](\tau; z) \mapsto \Theta_2[\sigma](\tau; z + \epsilon/2)$. The fixed space $H_\epsilon^+$ is given by

$$x_\delta = 0, \quad \delta \notin \epsilon^\perp, \quad (7.2)$$

where the orthogonal complements are taken with respect to the usual Euclidean form on $(\mathbb{F}_2)^3$. On the subspace $H_\epsilon^+ \simeq \mathbb{P}^3$ we use the homogeneous coordinates

$$(x_{000} : x_{0\delta_1} : x_{0\delta_2} : x_{0\delta_3}), \quad \delta_i \in \epsilon^\perp.$$ 

Substituting (7.2) into $F_\tau$, we get the equation of a Kummer surface $K_2^{\epsilon,+} = C \cap H_\epsilon^+$:

$$r \cdot \sum_{\sigma \in \epsilon^\perp} x_\sigma^4 + \sum_{\sigma \in \epsilon^\perp} s_\sigma \cdot m_\sigma + t_\epsilon \prod_{\sigma \in \epsilon^\perp} x_\sigma = 0 \quad \text{where} \quad m_\sigma = \frac{1}{2} \sum_{\nu \in \epsilon^\perp} x_\nu^2 x_{\nu+\sigma}. \quad (7.3)$$

Its singular locus is $H_\epsilon^+ \cap \text{Sing}(C_\tau)$. We claim that one of the 16 singular points of $K_2^{\epsilon,+}$ is

$$p_\epsilon = \kappa_\tau(\frac{\epsilon}{4}) = (\Theta_2[000](\tau; \frac{\epsilon}{4}) : \Theta_2[001](\tau; \frac{\epsilon}{4}) : \cdots : \Theta_2[111](\tau; \frac{\epsilon}{4})). \quad (7.4)$$

Since this point lies in the image of the Kummer map $\kappa_\tau$, it is by definition in the Kummer threefold $K_3 = \kappa_\tau(A_\tau) = \text{Sing}(C_\tau)$. That $p_\epsilon$ is in the fixed space $H_\epsilon^+$ can be seen from the transformation properties of the second order theta functions under $A_\tau[2]$. Indeed,

$$\Theta_2[\delta](\tau; \frac{\epsilon}{4}) = \theta(2\tau; \frac{\epsilon}{2} + \tau \delta) \cdot \exp \left[ \pi i \left( \frac{\delta^t \tau \delta}{2} + 2\delta^t z \right) \right] \quad \text{for any} \ \delta \in (\mathbb{F}_2)^3. \quad (7.5)$$

Furthermore,

$$\theta(2\tau; \frac{\epsilon}{2} + \tau \delta) = \theta(2\tau; -(\frac{\epsilon}{2} + \tau \delta)) = \theta(2\tau; \frac{\epsilon}{2} + \tau \delta) - \epsilon - (2\tau) \delta)$$

$$= \theta(2\tau; \frac{\epsilon}{2} + \tau \delta) \cdot \exp \left[ 2\pi i (\delta^t \frac{\epsilon}{2} + \tau \delta) - (1/2) \delta^t (2\tau) \delta \right]$$

$$= \theta(2\tau; \frac{\epsilon}{2} + \tau \delta) \cdot \exp \left[ \pi i \delta^t \epsilon \right].$$

If $\delta \notin \epsilon^\perp$, then $\exp[\pi i \delta^t \epsilon] = -1$. So $\theta(2\tau; \frac{\epsilon}{2} + \tau \delta) = 0$. Thus, $p_\epsilon$ is in $H_\epsilon^+ \cap K_3$.

Now recall (from Example 1.1) that the equation of a Kummer surface can be expressed in terms of the coordinates of any of its 16 singular points. We apply this to the Kummer surface $K_2^{\epsilon,+}$. Namely, let $L_i$ denote the signed $4 \times 4$-minor of the corresponding matrix (1.1) which is obtained by deleting row 1 and column $i+1$. Then the coefficients of (7.3) are proportional to $L_0, L_1, L_2, L_3, L_4$. Explicitly, we have the relations

$$r : s_{001} : s_{010} : s_{011} : t_{000} = (L_0^{100} : L_1^{100} : L_2^{100} : L_3^{100} : L_4^{100}),$$

$$r : s_{001} : s_{100} : s_{101} : t_{010} = (L_0^{010} : L_1^{010} : L_2^{010} : L_3^{010} : L_4^{010}),$$

$$r : s_{010} : s_{100} : s_{110} : t_{001} = (L_0^{001} : L_1^{001} : L_2^{001} : L_3^{001} : L_4^{001}).$$
(r : s_{001} : s_{110} : s_{111} : t_{110}) = (L_1^{10} : L_2^{10} : L_3^{10} : L_4^{10}),
(r : s_{010} : s_{101} : s_{111} : t_{101}) = (L_0^{10} : L_1^{10} : L_2^{10} : L_3^{10} : L_4^{10}),
(r : s_{011} : s_{100} : s_{111} : t_{011}) = (L_0^{11} : L_1^{11} : L_2^{11} : L_3^{11} : L_4^{11}),
(r : s_{011} : s_{101} : s_{110} : t_{111}) = (L_0^{11} : L_1^{11} : L_2^{11} : L_3^{11} : L_4^{11}).

Combining these formulas, we see that with respect to the normalization \( r = 1 \), the coefficients of the Coble quartic (2.12) are given by

\[
\begin{align*}
  s_{001} &= \frac{L_0^{10}}{L_0^{10}}, \quad s_{010} = \frac{L_2^{10}}{L_0^{10}}, \quad s_{101} = \frac{L_3^{10}}{L_0^{10}}, \quad s_{100} = \frac{L_2^{10}}{L_0^{10}}, \quad s_{111} = \frac{L_3^{10}}{L_0^{10}}, \\
  s_{110} &= \frac{L_3^{01}}{L_0^{01}}, \quad s_{111} = \frac{L_2^{01}}{L_0^{01}}, \quad \text{and} \quad t_\sigma = \frac{L_0^{10}}{L_0^{10}} \quad \text{for} \quad \sigma \in \mathbb{P}^3 \setminus \{0\}.
\end{align*}
\]

The \( L_i \)'s are polynomials of degree twelve in the eight coordinates of the point \( p \), in (7.4). The next step is to relate these coordinates \( \Theta_2[\sigma](\tau; \frac{e}{4}) \) to the variables \( u \) on our \( \mathbb{P}^7 \).

We introduce the following notation for the first order theta constants:

\[
T_\epsilon^\prime = \theta[\epsilon | e'](\tau; 0). \tag{7.6}
\]

By the inverse addition formula (2.8), we have

\[
8\Theta_2[\sigma](\tau; \frac{e}{4})\Theta_2[\sigma + \delta](\tau; \frac{e}{4}) = \sum_{\epsilon' \in (\mathbb{Z}/2)^6} (-1)^{\sigma \cdot \epsilon'} T_{\epsilon'} T_{\epsilon' + \epsilon}. \tag{7.7}
\]

In the special case \( \delta = 0 \), the left hand side becomes a square:

\[
8\Theta_2^2[\sigma](\tau; \frac{e}{4}) = \sum_{\epsilon' \in (\mathbb{Z}/2)^6} (-1)^{\sigma \cdot \epsilon'} T_{\epsilon'}^0 T_{\epsilon' + \epsilon}. \tag{7.8}
\]

Now, for \( 0 \leq i \leq 3 \), the quantity \( L_i^\epsilon \) is actually a polynomial of degree 6 in \( \Theta_2^3[\ast](\tau; \frac{e}{4}) \). Hence formula (7.8) allows us to write these \( L_i^\epsilon \) as polynomials in the variables \( T_{\epsilon'}^0 \). For example,

\[
L_0^{10} = -\frac{1}{64} (T_{000}^0 T_{010}^0 T_{100}^0 T_{110}^0 - T_{001}^0 T_{011}^0 T_{101}^0 T_{111}^0) (T_{000}^0 T_{001}^0 T_{100}^0 T_{101}^0 T_{010}^0 T_{011}^0 T_{110}^0 T_{111}^0) \\
\cdot (T_{000}^0 T_{011}^0 T_{100}^0 T_{111}^0 - T_{001}^0 T_{010}^0 T_{101}^0 T_{110}^0 T_{111}^0),
\]

\[
L_1^{10} = \frac{1}{32} (T_{000}^0 T_{010}^0 T_{100}^0 T_{110}^0 + T_{001}^0 T_{011}^0 T_{101}^0 T_{111}^0) (T_{000}^0 T_{001}^0 T_{100}^0 T_{101}^0 T_{010}^0 T_{011}^0 T_{110}^0 T_{111}^0) \\
\cdot (T_{000}^0 T_{011}^0 T_{100}^0 T_{111}^0 - T_{001}^0 T_{010}^0 T_{101}^0 T_{110}^0 T_{111}^0),
\]

\[
L_2^{10} = \frac{1}{32} (T_{000}^0 T_{010}^0 T_{100}^0 T_{110}^0 - T_{001}^0 T_{011}^0 T_{101}^0 T_{111}^0) (T_{000}^0 T_{001}^0 T_{100}^0 T_{101}^0 T_{010}^0 T_{011}^0 T_{110}^0 T_{111}^0) \\
\cdot (T_{000}^0 T_{011}^0 T_{100}^0 T_{111}^0 - T_{001}^0 T_{010}^0 T_{101}^0 T_{110}^0 T_{111}^0),
\]

\[
L_3^{10} = \frac{1}{32} (T_{000}^0 T_{010}^0 T_{100}^0 T_{110}^0 - T_{001}^0 T_{011}^0 T_{101}^0 T_{111}^0) (T_{000}^0 T_{001}^0 T_{100}^0 T_{101}^0 T_{010}^0 T_{011}^0 T_{110}^0 T_{111}^0) \\
\cdot (T_{000}^0 T_{011}^0 T_{100}^0 T_{111}^0 + T_{001}^0 T_{010}^0 T_{101}^0 T_{110}^0 T_{111}^0).
\]
The quantity \( L_4^t \) can be written in the form

\[
L_4^t = \prod_{\delta \in \epsilon^*} \Theta_2[\delta](\frac{\epsilon}{4}) \cdot M^t
\]  

(7.9)

where \( M^t \) is a polynomial of degree 4 in \( \Theta_2[\delta](\frac{\epsilon}{4}) \). Therefore, we can apply (7.8) to write \( M^t \) as a polynomial of degree 8 in the \( T^t_\epsilon \). To deal with expressions of the form \( \prod_{\delta \in \epsilon^*} \Theta_2[\delta](\frac{\epsilon}{4}) \), we group them into two products of pairs of theta functions and apply (7.7). Note that some of the \( T^t_\epsilon \) on the right hand side of (7.7), namely the ones such that \( \epsilon' \in \epsilon^* \), correspond to odd characteristics and thus vanish. Hence we can write, for example,

\[
L_4^{100} = \frac{1}{8} \cdot \left( \prod_{\epsilon' \in \mathbb{F}_2^3} T^0_{\epsilon'} \right) \cdot \left( (T^{010}_{001})^2 (T^{010}_{100})^2 - (T^{001}_{001})^2 (T^{001}_{110})^2 \right).
\]

In this manner, we express all of the coefficients of the Coble quartic in terms of the \( T^t_\epsilon \). Clearing the denominators, we get the following expressions of degree 28 for \( r, s_\sigma, t_\sigma \):

\[
r = \prod_{\epsilon \in \mathbb{F}_2^3, \{0\}} \left( \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} - \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} \right),
\]

(7.10)

\[
s_\sigma = (-2) \cdot \left( \prod_{\epsilon' \in \sigma^* \setminus \{0\}} T^{00}_{\epsilon'} + \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} \right) \cdot \prod_{\epsilon \in \mathbb{F}_2^3, \{0\}} \left( \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} - \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} \right),
\]

(7.11)

\[
t_\sigma = 8 \cdot \prod_{\epsilon' \in \mathbb{F}_2^3} T^{00}_{\epsilon'} \cdot \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} \left( \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} - \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} \right) \cdot W_\sigma,
\]

(7.12)

where \( W_\sigma \) is a polynomial of degree 4 in the \( T^t_\epsilon \) of the form \( (T^t_\sigma)^2 (T^t_\epsilon)^2 - (T^t_\sigma)^2 (T^t_\epsilon)^2 \).

To write these coefficients as polynomials in the \( u_\epsilon \), we’d like to use the addition formula

\[
(T^t_\epsilon)^2 = \sum_{\sigma \in \mathbb{F}_2^3} (-1)^{\sigma, \epsilon'} u_\sigma \cdot u_{\sigma+\epsilon}.
\]

(7.13)

In order to do this, we must write the coefficients of the Coble quartic as polynomials in the squares of the \( T^t_\epsilon \). The expressions for \( r, s_\sigma, t_\sigma \) above are not yet in this form. However, by expanding the products we observe that these expressions can be written as polynomials in the squares of the \( T^t_\epsilon \) together with \( \prod_{\epsilon \in \mathbb{F}_2^3} T^{00}_{\epsilon'} \). Thus, it remains to express the quantity \( \prod_{\epsilon \in \mathbb{F}_2^3} T^{00}_{\epsilon'} \) as a polynomial in the squares of the \( T^t_\epsilon \). To do this, we make use of the following special case of Riemann’s theta relations [BL, Exercise 7.9]:

\[
\prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} - \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} T^{00}_{\epsilon'} = \prod_{\epsilon \in \epsilon^* \setminus \{0\}} T^t_{\epsilon'}.
\]

(7.14)

Squaring this formula, we get

\[
\prod_{\epsilon' \in \epsilon^* \setminus \{0\}} (T^{00}_{\epsilon'})^2 \prod_{\epsilon' \in \epsilon^* \setminus \{0\}} (T^{00}_{\epsilon'})^2 - 2 \prod_{\epsilon' \in \mathbb{F}_2^3} T^{00}_{\epsilon'} = \prod_{\epsilon \in \epsilon^* \setminus \{0\}} (T^t_{\epsilon'})^2.
\]

(7.15)
Thus, we get a formula for $\prod_{\epsilon \in F_2} T^0_{\epsilon}$ as a polynomial in the squares of the $T$ variables. Then, we apply the formula (7.13) to the expressions (7.10), (7.11), (7.12) for $r, s_\sigma, t_\sigma$. We obtain the 15 polynomials $r(u), s_\sigma(u), t_\sigma(u)$ of degree 28 in the variables $u$ that had been promised.

(b) We next argue that, set-theoretically, the variety of the ideal $\langle r, s_\sigma, t_\sigma \rangle$ equals the Torelli boundary in the Satake hypersurface $S$. First, suppose that $p \in S$ lies in the zero locus of $\langle r, s_\sigma, t_\sigma \rangle$. By (7.10) and Riemann’s theta relations (7.14), the quantity

$$\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} = \prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} - \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'} \quad (7.16)$$

vanishes at $p$ for at least one $\epsilon \in F_2 \setminus \{0\}$. By the proof of Proposition 3.2 or [Gla, Theorem 3.1], in order to show that $p$ lies in the Torelli boundary, it suffices to show that at least 2 first order even theta constants $T^0_{\epsilon'}$ vanish at $p$. We distinguish the following two cases:

Case 1. Suppose $\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'}$ vanishes for two distinct $\epsilon = \epsilon_1, \epsilon_2 \in F_2 \setminus \{0\}$. Since, these monomials have disjoint sets of variables, at least two first order even theta constants vanish.

Case 2. Suppose that $\prod_{\epsilon' \in \epsilon} (T^0_{\epsilon'} - \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'})$ vanishes for exactly one $\epsilon \in F_2 \setminus \{0\}$. Then by (7.11), $s_\sigma$ is a product of $\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} + \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'}$ with some nonzero factors. Thus, $\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} + \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'}$ vanishes. But then the hypothesis of Case 2 implies

$$\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} = \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'} = 0 \quad (7.17)$$

and again we conclude that at least two first order even theta constants vanish. This shows that the variety cut out by the polynomials $r, s_\sigma, t_\sigma$ is contained in the Torelli boundary.

We now show the reverse containment. Suppose a point $p$ lies in the Torelli boundary. By [Gla, Theorem 3.1] at least 6 first order even theta constants $T^0_{\epsilon'}$ vanish at $p$. In order to show that all 15 polynomials $r, s_\sigma, t_\sigma$ vanish at $p$, we divide our analysis into three cases:

Case 1. Suppose that at least two $T^0_{\epsilon_1}, T^0_{\epsilon_2}$ vanish. Then there exists $\epsilon \in F_2 \setminus \{0\}$ such that one of $\epsilon_1, \epsilon_2$ is in $\epsilon^\perp$ and the other is not. For this choice of $\epsilon$, equation (7.17) holds. Since the expressions (7.10) and (7.11) for $r$ and the $s_\sigma$ each contain a factor of each $\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} + \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'}$ or $\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} - \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'}$, we see that $r$ and all $s_\sigma$ vanish. Also, each $t_\sigma$ vanishes because the expression (7.12) has a factor $\prod_{\epsilon' \in F_2} T^0_{\epsilon'}$.

Case 2. Suppose that exactly one $T^0_\epsilon$ vanishes. Since at least six theta constants vanish, at least 5 of the $T^0_{\epsilon'}$ with $\epsilon \neq 0$ vanish. So there exist two vanishing $T^0_{\epsilon_1}, T^0_{\epsilon_2}$ with $\epsilon_1, \epsilon_2 \neq 0$ and $\epsilon_1 \neq \epsilon_2$. By Riemann’s theta relations (7.14), $\prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} - \prod_{\epsilon' \notin \epsilon} T^0_{\epsilon'}$ vanishes for $\epsilon = \epsilon_1, \epsilon_2$. Since the expression (7.10) for $r$ contains all seven such expressions as factors and each $s_\sigma$ contains all but one of these as factors, we see that $r$ and all the $s_\sigma$ vanish. The $t_\sigma$ vanish for the same reason as in Case 1.

Case 3. Suppose that none of the $T^0_{\epsilon'}$ vanishes. Then $r$ and the $s_\sigma$ vanish by the argument in Case 2. Using (7.16), we rewrite equation (7.12) for $t_\sigma$ as follows:

$$t_\sigma = C \cdot \prod_{\epsilon' \notin \epsilon} \prod_{\epsilon' \in \epsilon} T^0_{\epsilon'} \cdot W_\sigma \quad (7.18)$$

where $C$ is nonzero. Let $T^0_{\epsilon_j}, j = 1, \ldots, 6$ be vanishing first order even theta constants whose characteristics form an azygetic 6-set. The vanishing of each of the $T^0_{\epsilon_j}$ implies the vanishing
of 4 of the \( t_\sigma \), namely those with \( \sigma \not\in \epsilon_j^\perp \). If the \( \mathbb{F}_2 \)-vector space spanned by the \( \epsilon_j \) is the whole space \( \mathbb{F}_2^3 \), then all 7 \( t_\sigma \) vanish in this way. If not, without loss of generality, we may assume that \{\( \epsilon_1, \ldots, \epsilon_6 \)\} \( \subset \{001, 010, 011\} \). In this case, it suffices to show that \( t_{100} \) vanishes.

We note that there are many ways to write \( W_\tau \) as a polynomial in the \( T_\sigma \), coming from different ways of grouping the 4 theta constants in (7.9) into two pairs. For \( W_{100} \), we have

\[
W_{100} = (T_{001}^{001})^2(T_{100}^{010})^2 - (T_{010}^{001})^2(T_{001}^{110})^2 \\
= (T_{000}^{010})^2(T_{100}^{010})^2 - (T_{001}^{010})^2(T_{010}^{101})^2 \\
= (T_{000}^{011})^2(T_{100}^{011})^2 - (T_{011}^{011})^2(T_{011}^{111})^2.
\]

By our assumption, six of the twelve \( T \)-variables in the expression for \( W_{100} \) vanish. The only way \( W_{100} \) can be nonzero is for the following three conditions to hold:

\[
T_{000}^{001} = T_{100}^{001} = 0 \quad \text{or} \quad T_{001}^{001} = T_{000}^{110} = 0; \\
T_{010}^{010} = T_{100}^{010} = 0 \quad \text{or} \quad T_{001}^{010} = T_{010}^{101} = 0; \\
T_{011}^{011} = T_{100}^{011} = 0 \quad \text{or} \quad T_{011}^{011} = T_{011}^{111} = 0.
\]

This gives eight cases in total. It can be verified that in each of these eight cases, none of the characteristics of the six vanishing first order even theta constants are azygetic. We conclude that all \( t_\sigma \) vanish, and this completes our proof that the Torelli boundary is contained in the common zero set of our coefficient polynomials \( r(u), s_\sigma(u), t_\sigma(u) \) for the Coble quartic.

(c) Our next goal is to show that \( \langle F, S \rangle \) is a prime ideal. Let \( C' \) denote the subscheme of \( \mathbb{P}^7 \times \mathbb{P}^7 \) defined by this ideal. We begin by proving that \( C' \) is reduced and irreducible. Consider \( C' \) as a family \( \pi: C' \rightarrow \vartheta(\mathcal{A}_3(2, 4)) \). Let \( U \) denote the non-hyperelliptic locus in \( \vartheta(\mathcal{A}_3(2, 4)) \). The fiber over each closed point in \( U \) is an irreducible Coble quartic hypersurface. Thus, \( \pi^{-1}(U) \) is a family over an irreducible base whose fibers are irreducible and have the same dimension. Therefore, \( \pi^{-1}(U) \) is irreducible by [Eis95, Exercise 14.3]. Since \( F \) and \( S \) have no common factor, the ideal \( \langle F, S \rangle \) is a complete intersection. Hence it is Cohen–Macaulay, and all of its minimal primes have the same dimension 12.

We claim that \( C' \) is the closure of \( \pi^{-1}(U) \) in \( \mathbb{P}^7 \times \mathbb{P}^7 \). Suppose it is not. Then it has a twelve-dimensional component contained in \( \overline{\vartheta(\mathcal{A}_3(2, 4)) \setminus U} \times \mathbb{P}^7 \). Since \( \dim(\overline{\vartheta(\mathcal{A}_3(2, 4)) \setminus U}) = 5 \), this can only happen if there is a five-dimensional subvariety \( Z \) of \( \overline{\vartheta(\mathcal{A}_3(2, 4)) \setminus U} \) such that \( Z \times \mathbb{P}^7 \subset C' \), i.e. all of the \( r, s_\sigma, t_\sigma \) vanish on \( Z \). This is impossible because the zero locus of \( r, s_\sigma, t_\sigma \) is the Torelli boundary, which has dimension four. Therefore, \( C' \) is irreducible and \( C = C' \). We now know that the radical of \( \langle F, S \rangle \) is a prime ideal.

The Cohen–Macaulay property implies that \( \langle F, S \rangle \) satisfies Serre’s criterion \( (S_1) \). Since the variables \( x \) do not appear in \( S \), the Jacobian matrix of \( \langle F, S \rangle \) is a \( 2 \times 16 \)-matrix in block triangular form. If we pick any point \( (u, x) \) with \( u \in S \) non-hyperelliptic and \( x \) non-singular on its Coble quartic, then the Jacobian matrix has rank 2 at this point. So, this point is non-singular on the scheme defined by \( F = S = 0 \), and hence \( \langle F, S \rangle \) is generically reduced. Therefore, \( \langle F, S \rangle \) is radical, and it is the prime ideal defining the universal Coble quartic \( C \) in \( \mathbb{P}^7 \times \mathbb{P}^7 \). The variety \( C \) has bidegree \( 16U(28U + 4X) \) because it is a complete intersection defined by two polynomials of bidegree \((16, 0)\) and \((28, 4)\). This completes the proof. \( \square \)
Remark 7.2. The Satake ring $\mathbb{C}[u]/\langle S \rangle$ is not a unique factorization domain (UFD). Our degree 28 polynomial $r(u)$ has distinct factorizations. For any $\sigma \in \mathbb{P}_2^3$, we have $r = c_\sigma q_\sigma$, where

$$c_\sigma = \prod_{\epsilon \in \sigma^+ \setminus 0} \left( \prod_{\epsilon' \in \epsilon^+} T_{\epsilon'}^0 - \prod_{\epsilon' \notin \epsilon^+} T_{\epsilon'}^0 \right) \quad \text{and} \quad q_\sigma = \prod_{\epsilon \notin \sigma^+} \left( \prod_{\epsilon' \in \epsilon^+} T_{\epsilon'}^0 - \prod_{\epsilon' \notin \epsilon^+} T_{\epsilon'}^0 \right)$$

Both $c_\sigma$ and $q_\sigma$ are polynomials in the square of the $T_{\epsilon'}^0$ and $\prod_{\epsilon \in \mathbb{P}_2^3} T_{\epsilon'}^0$. As before, we can rewrite them as polynomials in $u$. Therefore, as polynomials in $u$, we have $r = c_\sigma q_\sigma \mod \langle S \rangle$. Hence $c_\sigma$ is a factor of $r$ in $\mathbb{C}[u]/\langle S \rangle$. It can also be verified that these $c_\sigma$ are pairwise relatively prime in $\mathbb{C}[u]$, thus pairwise relatively prime in $\mathbb{C}[u]/\langle S \rangle$ since $\deg(c_\sigma) = 12 < 16 = \deg(S)$. This is impossible for a UFD. Compare this to results of Tsuyume [Tsu] and others on the UFD property for coordinate rings representing the moduli space $\mathcal{A}_3$. □

We close with a remark that highlights the utility of the formulas derived in this paper.

Remark 7.3. Our supplementary files enable the reader to write geometric properties of plane quartic curves explicitly in terms of the theta constants (3.4). For example, consider the condition that a ternary quartic is the sum of five reciprocals of linear forms. Such quartics are known as Lüroth quartics. A classical result of Morley states that Lüroth quartics form a hypersurface of degree 54 in the $\mathbb{P}^{14}$ of all quartics. See [OtSe] for a modern exposition.

Lüroth quartics are also characterized by the vanishing of the following Morley invariant:

$$f_{1234567} + f_{1234576} + f_{1234657} + f_{1234756} + f_{1234765} + f_{1235467} + f_{1235476} + f_{1235647} + f_{1235674} + f_{1236457} + f_{1236475} + f_{1236547} + f_{1236574} + f_{1236754} + f_{1237456} + f_{1237465} + f_{1237546} + f_{1237564} + f_{1237654} + f_{1237645} + f_{1243567} + f_{1243576} + f_{1243657} + f_{1243756} + f_{1243765}$$

This expression is found in [OtSe, page 379, after Figure 1]. Using the transformation derived in (6.11), the Morley invariant translates into the following linear form in Coble coefficients:

$$6r + s_{001} + s_{010} + s_{100} + s_{011} + s_{101} + s_{110} + s_{111}.$$ We note that, for $S_7 \subset W(E_7)$ embedded as a parabolic subgroup, the 15-dimensional space of Coble coefficients decomposes as a 14-dimensional irreducible $S_7$-module plus the trivial representation. The latter is spanned by the Morley invariant. Now, substituting the polynomials $r(u)$ and $s_{ijk}(u)$ from Theorem 7.1 into this linear form, we obtain a polynomial of degree 28 that has 59256 terms. That expression in $u_{000}, u_{001}, \ldots, u_{111}$ represents the condition that a matrix $\tau$ in the Siegel upper halfspace $\mathcal{H}_3$ comes from a Lüroth quartic. □

8 Equations for universal Kummer threefolds

We now turn to the object that gave our paper its title. The Kummer threefold is the singular locus of the Coble quartic. In the past sections we found the ideals for three variants of the universal Coble quartic. Each is a twelve-dimensional projective variety, over a six-dimensional base. First, in Section 4, the base was $\mathbb{P}^6$. Next, in Corollary 5.3, the base was the Göpel variety $\mathcal{G}$. Finally, in Theorem 7.1, the base was the Satake hypersurface $\mathcal{S}$.  

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For each of the 3 versions of the universal Coble quartic we have a universal Kummer threefold. Each of them is a 9-dimensional irreducible variety. Their respective ambient spaces are $\mathbb{P}^6 \times \mathbb{P}^7$, $\mathbb{P}^{14} \times \mathbb{P}^7$, and $\mathbb{P}^7 \times \mathbb{P}^7$. We discuss their defining equations in this order.

First, consider the parametrization of the Coble quartic (2.12) in terms of $c_1, c_2, \ldots, c_7$ given in (4.1). This defines what we call the flex version of the universal Kummer variety:

$$K_3^{\text{flex}} \subset \mathbb{P}^6 \times \mathbb{P}^7.$$ 

For $(c_1 : \cdots : c_7)$ not lying on any of the reflection hyperplanes of $E_7$, we consider all points $(c, x) = ((c_1 : \cdots : c_7), (x_{000} : \cdots : x_{111}))$ such that $x$ lies in the Kummer variety defined by $c$. The variety $K_3^{\text{flex}}$ is the Zariski closure of the set of all such points $(c, x)$ in $\mathbb{P}^6 \times \mathbb{P}^7$.

The label ‘flex’ refers to the fact that general points $(c, x)$ give a unique parametrization of what we call the flex version of the Coble quartic. The varieties corresponding affine open charts on $\mathbb{P}^7$ are the degree 7 polynomials in $c_1, \ldots, c_7$ listed in (4.1). Each of these Coble derivatives is the sum of 323 terms of bidegree $(7, 3)$ in $(c, x)$.

Our second set of equations for $K_3^{\text{flex}}$ consists of 70 polynomials of bidegree $(6, 4)$ in $(c, x)$. Their construction is considerably more difficult, and we shall now explain it. The idea is to use the methods for degeneracy loci arising from Vinberg’s $\theta$-groups, as developed in [GSW].

We start with $\Omega^3(4)$, the third exterior power of the cotangent bundle of $\mathbb{P}^7$ twisted by $O(4)$. Its global sections are homogeneous differential 3-forms of degree 4 on $\mathbb{P}^7$. The space of global sections $H^0(\mathbb{P}^7, \Omega^3(4))$ is isomorphic to $\Lambda^4 \mathbb{C}^8$ via the map

$$\Lambda^4 \mathbb{C}^8 \to H^0(\mathbb{P}^7, \Omega^3(4)), \quad a_i \wedge a_j \wedge a_k \wedge a_\ell \mapsto a_\ell (da_i \wedge da_j \wedge da_k) - a_k (da_i \wedge da_j \wedge da_\ell) + a_j (da_i \wedge da_k \wedge da_\ell) - a_i (da_j \wedge da_k \wedge da_\ell)$$

Here the basis $\{a_1, a_2, \ldots, a_8\}$ is denoted as in Remark 4.2. Let $U_1, U_2, \ldots, U_8$ be the corresponding affine open charts on $\mathbb{P}^7$. We write $U_i = \text{Spec}(\mathbb{C}[z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_8])$, where $z_s = a_s/a_i$. The restriction $\Omega^3(4)|_{U_i}$ is generated by the 35 forms $dz_j \wedge dz_k \wedge dz_\ell$ where $j, k, \ell \neq i$. The fiber of the rank 35 bundle $\Omega^3(4)$ over any point $x \in U_i$ is identified with $\Lambda^3 \mathbb{C}^7$. The value of an element $\alpha \in \Omega^3(4)$ on that fiber can be written as a linear combination of the forms $dz_j \wedge dz_k \wedge dz_\ell$. The 35 coefficients are homogeneous linear expressions in $x$, or non-homogeneous linear expressions in $z$. We regard these as the coordinates of $\alpha$.

Consider the action of $\text{GL}_7(\mathbb{C})$ on $\Lambda^3 \mathbb{C}^7$. As explained in [GSW, §2.6], this action has unique orbits of codimensions 1, 4, and 7, respectively. We denote the closures of these orbits by $O_1, O_4, O_7$, to indicate their codimension in $\Lambda^3 \mathbb{C}^7$. The orbit closure $O_1$ is a hypersurface of degree 7. Its defining polynomial $f$ generates the ring of $\text{SL}_7(\mathbb{C})$-invariant polynomial functions on $\Lambda^3 \mathbb{C}^7$. We constructed the invariant $f$ using the method in [Kim, Remark 4.4]. Its expansion into monomials has 10680 terms. It is presented in our supplementary files. The ideal for $O_4$ is generated by the 35 partial derivatives of $f$. These are polynomials of degree 6 in 35 unknowns, and we shall now introduce a certain specialization of these.

Consider a section $v = c_1 h_1 + \cdots + c_7 h_7 \in \Lambda^4 \mathbb{C}^8$ where $h_1, \ldots, h_7$ are the tensors in Remark 4.2, and $(c_1 : \cdots : c_7) \in \mathbb{P}^6$ is generic. The section $v$ is a linear combination of the 56 forms $da_i \wedge da_j \wedge da_k$ where each coefficient is a monomial of bidegree $(1, 1)$ in $(c, x)$. On $U_i$ we set $a_i = 1$ and $a_j = z_j$ for $j \neq i$. Then 21 of the 56 summands vanish since $da_i = 0$ and
\[ da_j = dz_j \text{ for } j \neq i. \] What is left is a linear combination of the 35 forms \( dz_j \wedge dz_k \wedge dz_\ell \), where each coefficient is a monomial of bidegree \((1, 1)\) or \((1, 0)\) in \((c, z)\). These 35 coefficients are the coordinates of \( v \). Shortly, we shall plug them into the derivatives of the polynomial \( f \).

The following degeneracy locus has codimension 4 in the 7-dimensional affine space \( U_i \):

\[
\{ x \in U_i \mid v(x) \in O_4 \}. \tag{8.1}
\]

It was shown in \([\text{GSW}, \S 6.2]\) that its closure in \( \mathbb{P}^7 \) is precisely the Kummer threefold over \( c \). Here the coordinates on \( \mathbb{P}^7 \) need to be relabeled \( a_1 = x_{000}, \ldots, a_8 = x_{111} \) as in Remark 4.2. For completeness, we mention that the corresponding degeneracy locus for \( O_7 \) is the set of 64 singular points of the Kummer threefold, but we won’t need to use this.

By plugging the 35 coordinates of \( v \) into the partial derivatives of \( f \), we obtain 35 polynomials in \((c, z)\). These polynomials are homogeneous of degree 6 in \( c \), and they are non-homogeneous of degree 4 in \( z \). For generic \( c \), these equations define the affine Kummer threefold in \( U_i = \{ a_i \neq 0 \} \cong \mathbb{A}^7 \). If we homogenize these equations, then we obtain 35 bihomogeneous polynomials of degree \((6, 4)\) in \((c, x)\). These all lie in the prime ideal of \( \mathcal{K}_3^{\text{gen}} \).

We now repeat this process for the seven other affine charts \( U_j \). This leads to \( 8 \cdot 35 = 280 \) polynomials of bidegree \((6, 4)\) in \((c, x)\), but only 120 of them are distinct. Some of these polynomials have 332 terms, while the others have 362. They span a 70-dimensional vector space over \( \mathbb{Q} \), and we select a basis for that space. We conjecture that this basis suffices:

**Conjecture 8.1.** The prime ideal of the universal Kummer threefold in \( \mathbb{P}^6 \times \mathbb{P}^7 \) is minimally generated by the 78 polynomials above, namely, 8 of bidegree \((7, 3)\) and 70 of bidegree \((6, 4)\).

Second, let us consider the Göpel version of the universal Kummer variety:

\[
\mathcal{K}_3^{\text{göpel}} \subset \mathcal{G} \times \mathbb{P}^7 \subset \mathbb{P}^{14} \times \mathbb{P}^7.
\]

Here, the universal Coble quartic gives an equation of bidegree \((1, 4)\). Its eight partial derivatives are polynomials of bidegree \((1, 3)\). These polynomials are not sufficient to generate the Kummer ideal, even over a general point in \( \mathcal{G} \), because we need 70 quartics. The eight derivatives only give 64 equations of bidegree \((1, 4)\), so we need at least six more polynomials of bidegree \((?, 4)\). We do not know how to produce these extra quartics. In other words, we do not know how to lift the degeneracy locus construction of \((8.1)\) to the Göpel variety \( \mathcal{G} \).

In light of the beautiful combinatorics in Section 6, it is desirable to study this further.

Last but not least, we return to the object that started this project. The theta version of our variety is the Zariski closure of the image of the universal Kummer map \( \kappa \) in \( (3.8) \):

\[
\mathcal{K}_3(2, 4) \subset \mathcal{S} \times \mathbb{P}^7 \subset \mathbb{P}^7 \times \mathbb{P}^7. \tag{8.2}
\]

As before, we use the coordinates \( u = (u_{000} : u_{001} : \cdots : u_{111}) \) on the first \( \mathbb{P}^7 \) to parameterize the moduli of Kummer threefolds, and the coordinates \( x = (x_{000} : x_{001} : \cdots : x_{111}) \) on the second copy of \( \mathbb{P}^7 \) to parameterize points of a particular Kummer threefold. The theta version of the universal Kummer threefold has codimension five, and we have already constructed several polynomials in its defining bihomogeneous prime ideal \( \mathcal{I}_3 \). First, the ideal \( \mathcal{I}_3 \) contains the bidegree \((16, 0)\) polynomial of the Satake hypersurface \( \mathcal{S} \). Second, since the Kummer
threefold is the singular locus of the Coble quartic hypersurface, one has the eight partial
derivatives of $F$ as in Theorem 7.1 with respect to $x_{ijk}$. These have bidegree $(28,3)$.

Third and most important, there are additional generators of bidegree $(16,4)$ in $(u, x)$.
These play the same role as the 70 equations of bidegree $(6, 4)$ in $(c, x)$ for the flex version
in Conjecture 8.1. We will show that there are 882 such additional minimal generators.

**Lemma 8.2.** There exists a polynomial $f$ of bidegree $(16, 4)$ in the universal Kummer ideal
$I_3$, having the same form (2.12) as the Coble quartic, but now $r, s_*, t_*$ are polynomials of
degree 16 in $u$. These can be given explicitly by the formulas

\[
\begin{align*}
    r &= s_{001} = s_{100} = s_{101} = s_{110} = s_{111} = t_{010} = t_{110} = 0 \\
    s_{010} &= (u_{000}u_{011} + u_{001}u_{010} - u_{100}u_{111} - u_{101}u_{110})(u_{000}u_{011} + u_{001}u_{010} + u_{100}u_{111} + u_{101}u_{110}) \\
             &+ (u_{000}u_{011} - u_{001}u_{010} + u_{100}u_{111} - u_{101}u_{110})(u_{000}u_{011} - u_{001}u_{010} - u_{100}u_{111} + u_{101}u_{110}) \\
             &+ (u_{000}u_{010}u_{101}u_{111} - u_{001}u_{010}u_{101}u_{110})(u_{000}u_{010}u_{100}u_{111} - u_{001}u_{010}u_{100}u_{110}) \\
    s_{011} &= -(u_{000}u_{010} + u_{001}u_{011} + u_{100}u_{110} + u_{101}u_{111})(u_{000}u_{010} + u_{001}u_{011} - u_{100}u_{110} - u_{101}u_{111}) \\
             &+ (u_{000}u_{010} + u_{001}u_{011} + u_{100}u_{110} - u_{101}u_{111})(u_{000}u_{010} - u_{001}u_{011} - u_{100}u_{110} + u_{101}u_{111}) \\
             &+ (u_{000}u_{011}u_{100}u_{111} - u_{001}u_{011}u_{100}u_{110})(u_{000}u_{011}u_{101}u_{111} - u_{001}u_{011}u_{101}u_{110}) \\
    t_{001} &= -(u_{000}u_{011} + u_{001}u_{010} - u_{100}u_{111} + u_{101}u_{110})(u_{000}u_{011} - u_{001}u_{010} + u_{100}u_{111} - u_{101}u_{110}) \\
             &+ (u_{000}u_{011} + u_{001}u_{010} + u_{100}u_{111} + u_{101}u_{110})(u_{000}u_{011} + u_{001}u_{010} - u_{100}u_{111} + u_{101}u_{110}) \\
             &+ (u_{000}u_{010}u_{101}u_{111} - u_{001}u_{010}u_{101}u_{110})(u_{000}u_{010}u_{100}u_{111} - u_{001}u_{010}u_{100}u_{110}) \\
    t_{011} &= (u_{000}u_{010} - u_{001}u_{011} - u_{100}u_{110} - u_{101}u_{111})(u_{000}u_{010} - u_{001}u_{011} + u_{100}u_{110} - u_{101}u_{111}) \\
             &+ (u_{001}u_{010} - u_{000}u_{011} - u_{010}u_{110} + u_{101}u_{111})(u_{000}u_{011} + u_{001}u_{010} + u_{100}u_{111} + u_{101}u_{110}) \\
             &+ (u_{000}u_{011}u_{101}u_{110} - u_{001}u_{011}u_{100}u_{111})(u_{000}u_{011}u_{100}u_{111} - u_{001}u_{011}u_{100}u_{110}) \\
    t_{101} &= -(u_{000}u_{011} + u_{001}u_{010} + u_{100}u_{111} + u_{101}u_{110})(u_{000}u_{011} - u_{001}u_{010} + u_{100}u_{111} - u_{101}u_{110}) \\
             &+ (u_{000}u_{011} + u_{001}u_{010} - u_{100}u_{111} - u_{101}u_{110})(u_{000}u_{011} - u_{001}u_{010} - u_{100}u_{111} + u_{101}u_{110}) \\
             &+ (u_{000}u_{010}u_{100}u_{110} - u_{001}u_{010}u_{101}u_{111})(u_{000}u_{010}u_{101}u_{110} - u_{001}u_{010}u_{100}u_{111}) \\
    t_{111} &= (u_{000}u_{010} - u_{001}u_{011} - u_{100}u_{110} - u_{101}u_{111})(u_{000}u_{010} - u_{001}u_{011} + u_{100}u_{110} + u_{101}u_{111}) \\
             &+ (u_{001}u_{010} + u_{000}u_{011} + u_{010}u_{110} + u_{101}u_{111})(u_{000}u_{011} + u_{001}u_{010} + u_{100}u_{111} + u_{101}u_{110}) \\
             &+ (u_{000}u_{011}u_{100}u_{111} - u_{001}u_{011}u_{101}u_{110})(u_{000}u_{011}u_{100}u_{111} - u_{001}u_{011}u_{100}u_{110}) \\
    t_{100} &= [u_{000}u_{010}^4u_{111}^4u_{110}^4] + [u_{000}u_{010}^4u_{111}^4u_{100}^2u_{110}^2u_{111}^2] + [u_{000}u_{010}^4u_{111}^4u_{100}^2u_{110}^2u_{111}^2] + [u_{000}u_{010}^4u_{111}^4u_{100}^2u_{110}^2u_{111}^2] + [u_{000}u_{010}^4u_{111}^4u_{100}^2u_{110}^2u_{111}^2]
\end{align*}
\]

The last coefficient $t_{100}(u)$ is irreducible and has 72 terms. In particular, $f$ is the sum of
1168 monomials of degree $(16, 4)$ in $(u, x)$.

Bert van Geemen informed us that the polynomial $f$ above admits a determinantal rep-
resentation similar to (1.1). That determinant will be derived in a forthcoming paper of his.
Proof. We start with the following quartic relation among the theta constants of genus four:

\[
\begin{align*}
\theta[0010|0001](\tau; 0) & \cdot \theta[0010|1001](\tau; 0) \cdot \theta[0010|0101](\tau; 0) \cdot \theta[0010|1101](\tau; 0) \\
- \theta[0011|0011](\tau; 0) & \cdot \theta[0011|1011](\tau; 0) \cdot \theta[0011|0111](\tau; 0) \cdot \theta[0011|1111](\tau; 0) \\
- \theta[0010|0000](\tau; 0) & \cdot \theta[0010|1000](\tau; 0) \cdot \theta[0010|0100](\tau; 0) \cdot \theta[0010|1100](\tau; 0) \\
+ \theta[0011|0000](\tau; 0) & \cdot \theta[0011|1000](\tau; 0) \cdot \theta[0011|0100](\tau; 0) \cdot \theta[0011|1100](\tau; 0) = 0.
\end{align*}
\] (8.3)

Genus four identities of this form can be derived from genus two theta relations using [Gee, Proposition 4.18]. Such relations hold identically in \( \tau \in \mathcal{H}_4 \), not just on the Schottky locus.

The main computation is to express the relation (8.3) in terms of the genus four moduli variables \( u \). To do this, we first turn (8.3) into a relation between squares of the \( \theta[\epsilon|\epsilon'](\tau; 0) \). This can be done by taking the product of the relation (8.3) with its seven conjugates, where all possible signs in front of each term are chosen. This gives a polynomial of degree 16 in the \( \theta^2[\epsilon|\epsilon'](\tau; 0) \). We next apply the genus four version of (3.6). This writes the squares of the first order theta constants as quadrics in the 16 moduli variables \( u_{0000}, u_{0001}, \ldots, u_{1111} \). The result is a huge homogeneous polynomial of degree 32 in these 16 variables.

To this polynomial we now apply the Fourier–Jacobi expansion technique as in (3.5). This replaces the polynomial above by one of its initial forms, but now in the 16 unknowns \( u_{ijk}, x_{ijk} \). The result is a polynomial of degree \((28, 4)\) having 12268 terms. It lies in the universal Kummer ideal \( \mathcal{I}_3 \) and is expressible as a \( \mathbb{Q}[u] \)-linear combination of the 15 degree four invariants. Remarkably, this degree \((28, 4)\) polynomial turns out to be reducible. It has an extraneous factor of degree \((12, 0)\). That factor is a polynomial in \( u \) which cannot vanish on \( \mathcal{S} \). Dividing out this factor gives the desired bidegree \((16, 4)\) polynomial \( f \in \mathcal{I}_3 \). \( \square \)

The action by the modular group \( \text{Sp}_6(\mathbb{Z}) \) on \( \mathbb{C}[u, x] \) induced by the action on \( \mathbb{P}^7 \times \mathbb{P}^7 \) preserves the ideal \( \mathcal{I}_3 \). On the space of Heisenberg invariant polynomials, the principal congruence subgroup \( \Gamma_3(2) \) acts trivially, so there is a well-defined action of the quotient group \( \text{Sp}_6(\mathbb{F}_2) = \text{Sp}_6(\mathbb{Z})/\Gamma_3(2) \), cf. (3.1). For a detailed discussion of the relevant representation theory, see [DG]. However, the polynomial \( f \) in Lemma 8.2 is not invariant under this action. Therefore, we obtain more polynomials in \( \mathcal{I}_3 \) by applying this action to \( f \).

**Lemma 8.3.** The orbit of the polynomial \( f \) in Lemma 8.2 under the action of \( \text{Sp}_6(\mathbb{F}_2) \) contains exactly 945 elements. They span a vector space over \( \mathbb{C} \) of dimension 882.

**Proof.** In what follows, we will use a right action of \( \text{Sp}_6(\mathbb{F}_2) \). Recall that on the projective coordinates \( (u, x) \), generators of \( \text{Sp}_6(\mathbb{F}_2) \) of the form \( \gamma_1(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \) act by

\[ \gamma_1(A) \circ a_\sigma = a_{A_\sigma} \quad \text{for } a \in \{u, x\}. \]

Generators \( \gamma_2(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \) with \( B^t = B \) act by

\[ \gamma_2(B) \circ a_\sigma = e^{\pi x_\sigma^t B_\sigma} a_\sigma \quad \text{for } a \in \{u, x\}, \]

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and the Weyl element \( \gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) acts by the finite Fourier transform

\[
\gamma_3 \circ a_\sigma = \sum_{\rho \in \mathbb{F}_2^3} (-1)^{\rho \cdot \sigma} a_\rho \quad \text{for } a \in \{u, x\}.
\]

Alternatively, the group \( \text{Sp}_6(\mathbb{F}_2) \) is generated by the following two elements \( \mu' \) and \( \nu' \):

\[
\mu' = \gamma_2(B)\gamma_1(A) \quad \text{for } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8.4)
\]

\[
\nu' = \gamma_1(\tilde{A})\gamma_2(\tilde{B})\gamma_3\gamma_2(\tilde{B}) \quad \text{for } \tilde{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (8.5)
\]

These two generators correspond to \( \mu \) and \( \nu \) in (4.2). They act on \( \mathbb{C}[u, x] \) by the substitutions

\[
\mu' : \quad a_{000} \mapsto a_{000}, a_{001} \mapsto a_{001}, a_{100} \mapsto i \cdot a_{100}, a_{101} \mapsto i \cdot a_{101}, a_{010} \mapsto a_{010}, a_{011} \mapsto a_{011}, a_{110} \mapsto i \cdot a_{110}, a_{111} \mapsto i \cdot a_{111} \quad \text{for } a \in \{u, x\}, \quad i = \sqrt{-1},
\]

\[
\nu' : \quad a_{000} \mapsto a_{000} + a_{100}, a_{001} \mapsto a_{000} - a_{100}, a_{010} \mapsto a_{010}, a_{011} \mapsto a_{011}, a_{101} \mapsto a_{011} + a_{101}, a_{110} \mapsto a_{111} - a_{011}, a_{111} \mapsto a_{011} - a_{111} \quad \text{for } a \in \{u, x\}.
\]

We can try to generate the orbit by applying the generators \( \mu' \) and \( \nu' \) successively. This is challenging because some of the polynomials in the orbit are very large. Instead, for some \( N \geq 2 \), we choose random vectors \((u_j, x_j)\) over a finite field, and we evaluate

\[
(f(\alpha(u_1, x_1)) : \cdots : f(\alpha(u_N, x_N))), \quad (8.6)
\]

where \( \alpha \) is the product of a sequence of \( \mu' \)s and \( \nu' \)s. In this way, we get 945 distinct points in \( \mathbb{P}^{N-1} \). Therefore, the orbit of \( f \) under \( \text{Sp}_6(\mathbb{F}_2) \) has at least 945 elements.

To prove that the number 945 is exact, we study the stabilizer of \( f \) in \( \text{Sp}_6(\mathbb{F}_2) \). It suffices to show that the stabilizer has at least \( |\text{Sp}_6(\mathbb{F}_2)|/945 = 1536 \) elements. It can be seen from the explicit form of \( f \) in Lemma 8.2 that the stabilizer contains \( \gamma_1(A) \) for the 8 unipotent lower triangular \( 3 \times 3 \)-matrices \( A \), and all of the 64 elements \( \gamma_2(B) \). These elements generate a group of order \( 8 \cdot 64 = 512 \). Note that all of these elements act on \( \mathbb{P}^7 \times \mathbb{P}^7 \) by signed permutations on the coordinates. We found another element in the stabilizer, namely

\[
\nu'\gamma_1(A) \quad \text{for } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (8.7)
\]

This element of order 3 does not act by signed permutations on the coordinates and thus is not in our subgroup of order 512. Since the size of the stabilizer divides \( |\text{Sp}_6(\mathbb{F}_2)| = 1451520 \), it must be at least \( 512 \cdot 3 = 1536 \). This shows the orbit has size 945. Alternatively, we can do explicit calculations with these matrices in \text{GAP} and establish the exact size of the subgroup.

We generated the 945 polynomials of degree \((16, 4)\) in implicit form as in \((8.6)\). Due to the size of the problem, it is not feasible to compute the dimension of the vector space spanned
by the orbit of $f$ directly over $\mathbb{Q}(i)$. Therefore, we evaluated them at random configurations of $N \geq 945$ points over various finite fields. In each experiment, the resulting $945 \times N$-matrix has rank 882. Hence the $\mathbb{C}$-vector space spanned by the 945 orbit elements has dimension at least 882.

To prove that the number 882 is exact, we identified exactly $945 - 882 = 63$ linearly independent relations among the 945 orbit elements. Fortunately, the relations are rather simple. There exist 15 distinct group elements $g_1, \ldots, g_{15} \in \text{Sp}_6(\mathbb{F}_2)$ such that

$$\sum_{j=1}^{15} \pm f \circ g_j = 0. \quad (8.8)$$

This relation is found in the following way: first take a set $B$ of 882 linearly independent elements of the form $f \circ g$. The complement $B^c$ contains 63 elements. Take any $g_1 \in B^c$. For each $f \circ g \in B$, test by computation in a finite field if the 882 elements in $B\{f \circ g\} \cup \{f \circ g_1\}$ are still linearly independent. It turns out that only 14 out of the 882 elements satisfy this property. These elements are $f \circ g_2, \ldots, f \circ g_{15}$. Then, it is computationally feasible to find and verify the relation (8.8) over $\mathbb{Q}(i)$ due to the reduced size of the problem. The complete list is found in our supplementary materials. Applying $\text{Sp}_6(\mathbb{F}_2)$ to (8.8) gives 63 linearly independent relations since no two relations involve the same conjugate of $f$.

**Remark 8.4.** We can verify that our polynomials lie in $I_3$ using direct numerical computations. Indeed, by running Swierczewski’s code for theta functions in Sage, we can generate arbitrarily many points $(u, x)$ on the universal Kummer variety $K_3(2, 4)$. The polynomial with 1168 terms was shown to vanish on all of them. Likewise, we can check numerically that this vanishing property is preserved under the two substitutions $\mu', \nu'$.

**Remark 8.5.** The space spanned by the $\text{Sp}_6(\mathbb{F}_2)$-orbit of $f$ is a representation of $\text{Sp}_6(\mathbb{F}_2)$. It is equivalent to a subrepresentation of the representation induced from the trivial representation of the stabilizer subgroup of the polynomial $f$ from Lemma 8.2. According to a GAP calculation, the irreducible subrepresentations of that 945-dimensional induced $\text{Sp}_6(\mathbb{F}_2)$-representation have the following dimensions: 1, 27, 27, 35, 35, 84, 120, 168, 168, 280.

Here the two occurrences of 27 and 35 refer to an irreducible representation that appears with multiplicity 2. Note that $945 - 882 = 63 = 1 + 27 + 35$, and this is the unique way to build 63 from the dimensions of these irreducible summands. So we can use this to determine the structure of these 882 equations as a representation of $\text{Sp}_6(\mathbb{F}_2)$.

We do not completely understand what happens to the universal Kummer threefold $K_3(2, 4)$ and the generators found above when we restrict the base to one of the 36 hyperelliptic divisors in $S$. The issue is that, for fixed $\tau$ in the hyperelliptic locus, the ideal of the Kummer threefold has a different structure. It can be seen from equations (7.10), (7.11) and (7.12) that the vanishing of a theta constant $T_{\tau}$ causes the universal Coble quartic to become the product of a monomial in the remaining non-zero theta constants times the square of a quadric with integer coefficients in the $x$ variables. For instance, if $T_0 = 0$ then

$$F_\tau = (x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2 + x_{11}^2 + x_{12}^2) \cdot \prod_{\epsilon \in \mathbb{F}_2 \setminus \{0\}} (T_{\epsilon})^4. \quad (8.9)$$

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Hence over the hyperelliptic locus, the prime ideal of a Kummer threefold contains a quadric. This can also be seen from the vanishing of a theta constant \( \theta[\epsilon|\epsilon'](\tau;0) \) in equation (3.9). That quadric generates 8 linearly independent cubics and 36 linearly independent quartics. The additional 34 quartic relations were identified recently by Müller [Mul, Theorem 4.2].

We optimistically conjecture that the equations we have found so far are sufficient:

**Conjecture 8.6.** The prime ideal \( \mathcal{I}_3 \) of the universal Kummer variety (8.2) is generated by 891 bihomogeneous polynomials in \((u,x)\): the Satake polynomial of degree \((16,0)\), the eight Coble derivatives of degree \((28,3)\), and the 882 polynomials of degree \((16,4)\).

**Remark 8.7.** In their recent work [DPSM], Dalla Piazza and Salvati Manni have obtained the defining polynomial of the universal Coble quartic directly from the Fourier–Jacobi expansion of a certain relation among theta constants in genus four. They also obtain another polynomial in the defining ideal of \( \mathcal{K}_3(2,4) \). At present, we have not determined whether this latter polynomial is contained in the ideal described in Conjecture 8.6. 

What remains to be done is to better integrate the generic case and the hyperelliptic case. This will be crucial for understanding the relationship between our 70 extra quartics over \( \mathbb{P}^6 \) and our 882 extra quartics over \( \mathcal{S} \). How can we construct these over \( \mathcal{G} \)? How can we lift the map (7.1) to the universal Kummer threefolds? We expect that the study of hyperelliptic moduli in [FS] and its Macdonald representation of \( S_8 \) will be relevant here.

In closing we remark that the Fourier–Jacobi method was the key to success in Lemma 8.2. Here the systematic passage to a non-trivial initial form was driven by a toric (or sagbi) degeneration as in (3.5). This is fundamental also for tropical geometry, our next topic.

## 9 Next steps in tropical geometry

We now take a look at Kummer threefolds and their moduli through the lens of tropical algebraic geometry [Cha, CMV, DFS, HKT, HJJS, MZ, SW]. Each of our ideals defines a tropical variety, which is a balanced polyhedral fan. These fans represent compactifications of our varieties and moduli spaces, and they allow us to understand what happens when the field \( \mathbb{C} \) of complex numbers gets replaced by a field \( K \) with a non-trivial non-archimedean valuation. This section serves as a manifesto in favor of explicit polynomial equations. They are essential for understanding the combinatorics that links classical and tropical moduli spaces of curves. This perspective is developed further in the subsequent article [RSS].

We begin our discussion with the six-dimensional Göpel variety \( \mathcal{G} \). Recall from Theorem 5.1 that \( \mathcal{G} \) sits inside \( \mathbb{P}^{14} \) where it is defined by 35 cubics and 35 quartics. The tropicalization of this irreducible variety is a pure six-dimensional polyhedral fan in \( \mathbb{T}\mathbb{P}^{14} = \mathbb{R}^{15}/\mathbb{R}(1,1,\ldots,1) \). This fan can be computed, at least in principle, with \( \text{Gfan} \) [Gfan]. However, it does not have good combinatorial properties. For tropical geometers, it is much better to pass to the modification arising from the re-embedding, seen in Section 6, of the Göpel variety \( \mathcal{G} \) in \( \mathbb{P}^{134} \). Its ideal is generated by binomials and linear trinomials. We define the tropical Göpel variety to be the tropical variety of that ideal. This is a six-dimensional fan which lives in \( \mathbb{T}\mathbb{P}^{134} \). The Weyl group \( W(E_7) \) acts on \( \text{trop}(\mathcal{G}) \) by permuting coordinates.
We shall construct the tropical Göpel variety \( \text{trop}(\mathcal{G}) \) combinatorially, not from its defining ideal via \texttt{Gfan} [Gfan], but directly from the parametrization given by (6.13) and (6.14). This is equivalent to the parametrization in Section 4, but we have factored it as follows:

\[
\mathbb{P}^6 \xrightarrow{\ell} \mathbb{P}^{62} \xrightarrow{m} \mathbb{P}^{134}. \tag{9.1}
\]

The first map \( \ell \) is given by evaluating the 63 linear forms (6.13) that represent \( E_7 \). The second map \( m \) is the monomial map (6.14). It corresponds to the \( 63 \times 135 \)-matrix \( A \) that encodes incidences of vectors and Lagrangians in \( (\mathbb{F}_2)^6 \setminus \{0\} \). The tropicalization of the monomial map \( m \) is given by the classically-linear map \( \text{TP}^6 \to \text{TP}^{134} \) defined by its transpose \( A^t \). The tropicalization of the linear space image(\( \ell \)) is the Bergman fan of the matroid \( M(E_7) \) of the reflection arrangement of type \( E_7 \), as defined in [ARW]. This fan lives in \( \text{TP}^{62} \), and we have:

**Proposition 9.1.** *The tropical Göpel variety \( \text{trop}(\mathcal{G}) \) coincides with the image of the Bergman fan of \( M(E_7) \) under the linear map \( \text{TP}^{62} \to \text{TP}^{134} \) given by the matrix \( A^t \) from Section 6.*

**Proof.** The factorization (9.1) shows that \( \mathcal{G} \) is the image of a map whose coordinates are products of linear forms. The result then follows immediately from [DFS, Theorem 3.1]. \( \square \)

The first step towards the tropical Göpel variety \( \text{trop}(\mathcal{G}) \) is to list the flats of the matroid \( M(E_7) \). In Table 2, we present the classification of all proper flats of the matroid \( M(E_7) \). There are 30 orbits of flats under the \( W(E_7) \)-action. For each \( W(E_7) \)-orbit we list the rank, the size of the orbit, and a set of linear forms that serves as a representative. These linear forms define an intersection of the hyperplanes in the reflection arrangement, of codimension equal to the rank, and the flat consists of all hyperplanes that contain that linear space.

In Table 2, we are using the bijection between flats and parabolic subgroups of the Weyl group, which can be found in [BI, Theorem 3.1]. These parabolic subgroups correspond to root subsystems in the finite type Dynkin diagram. We calculated Table 2 from scratch. Similar information for the lattice of parabolic subgroups can be found in [GP, Table A.2]. This can also be taken as an independent verification of the correctness of Table 2.

Eleven of the 30 orbits consist of irreducible root subsystems. Their numbers are marked in bold face. The total number of proper irreducible flats of the matroid \( M(E_7) \) is therefore

\[
6091 = 63 + 336 + 1260 + 2016 + 315 + 1008 + 336 + 378 + 288 + 63 + 28.
\]

The Bergman fan of \( M(E_7) \) is a six-dimensional fan in \( \text{TP}^{62} \). It has \( f_1 = 6091 \) rays, one for each irreducible flat \( F \). The rays are generated by their 0-1-incidence vectors \( e_F = \sum_{i \in F} e_i \). A collection \( \mathcal{F} \) of flats is *nested* if, for every antichain \( \{F_1, F_2, \ldots, F_r\} \) in \( \mathcal{F} \) with \( r \geq 2 \), the flat \( F_1 \vee F_2 \vee \cdots \vee F_r \) is not irreducible. (This flat represents the subspace obtained by intersecting the given subspaces). For any nested set \( \mathcal{F} \), we consider the convex cone \( C_\mathcal{F} \) spanned by the linearly independent vectors \( e_F, F \in \mathcal{F} \). The Bergman fan of \( M(E_7) \) is the collection of all cones \( C_\mathcal{F} \) where \( \mathcal{F} \) runs over all nested sets of irreducible flats of \( M(E_7) \).

By results of Ardila, Reiner and Williams in [ARW, §7], this simplicial fan is the coarsest fan structure on its support. Further, if \( f_i \) is the number of \( i \)-dimensional cones \( C_\mathcal{F} \) then

\[
f_6 - f_5 + f_4 - f_3 + f_2 - f_1 + 1 = 1 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 17 = 765765. \tag{9.2}
\]
| Family # | Rank | Size  | Root subsystem | Equations of a representative flat |
|----------|------|-------|----------------|-----------------------------------|
| 1        | 1    | 63    | $A_1$         | $c_7$                             |
| 2        | 2    | 336   | $A_2$         | $c_7, c_1 + c_3 - c_6$            |
| 3        | 2    | 945   | $A_1 \times A_1$ | $c_4, c_7$                       |
| 4        | 3    | 1260  | $A_3$         | $c_7, c_4, c_1 - c_5$             |
| 5        | 3    | 5040  | $A_1 \times A_2$ | $c_7, c_4, c_3 + c_5 - c_6$       |
| 6        | 3    | 3780  | $A_1^3$       | $c_6, c_3 - c_5, c_2 - c_7$      |
| 7        | 3    | 315   | $A_1^3$       | $c_7, c_4, c_3$                  |
| 8        | 4    | 2016  | $A_4$         | $c_7, c_4, c_3 + c_5 - c_6, c_1 - c_5$ |
| 9        | 4    | 315   | $D_4$         | $c_7, c_5, c_4, c_1$             |
| 10       | 4    | 7560  | $A_1 \times A_3$ | $c_7, c_5, c_4, c_3 - c_6$       |
| 11       | 4    | 3360  | $A_2 \times A_2$ | $c_7, c_6, c_3 - c_4 + c_5, c_1 + c_4 + c_5$ |
| 12       | 4    | 1260  | $A_1 \times A_3$ | $c_7, c_5 - c_6, c_4, c_3$       |
| 13       | 4    | 15120 | $A_1^{x^2} \times A_2$ | $c_7, c_6, c_2 + c_3 + 2c_4 + c_5, c_1 + c_4 + c_5$ |
| 14       | 4    | 3780  | $A_1^{x^4}$   | $c_7, c_6, c_5, c_4$             |
| 15       | 5    | 336   | $A_5$         | $c_7, c_6, c_5, c_2 - c_4, c_1 + c_4$ |
| 16       | 5    | 1008  | $A_5$         | $c_7, c_6, c_4, c_3 + c_5, c_1 + c_5$ |
| 17       | 5    | 378   | $D_5$         | $c_7, c_6, c_3, c_2 - c_4, c_1$   |
| 18       | 5    | 6048  | $A_1 \times A_4$ | $c_7, c_6, c_4, c_2 - c_3 - c_5, c_1 + c_5$ |
| 19       | 5    | 945   | $A_1 \times D_4$ | $c_7, c_6, c_5, c_4, c_3$       |
| 20       | 5    | 5040  | $A_2 \times A_3$ | $c_7, c_6, c_4, c_2 - 2c_5, c_1 + c_5$ |
| 21       | 5    | 7560  | $A_1^{x^2} \times A_3$ | $c_7, c_6, c_5, c_4, c_1 + c_2$ |
| 22       | 5    | 10080 | $A_1 \times A_5^{x^2}$ | $c_7, c_6, c_4, c_2 - c_3 - c_5, c_1 - c_3 - 2c_5$ |
| 23       | 5    | 5040  | $A_1^{x^3} \times A_2$ | $c_7, c_6, c_4, c_3, c_1 + c_2 - c_5$ |
| 24       | 6    | 288   | $A_6$         | $c_7, c_6, c_4, c_3 + c_5, c_2 - 2c_5, c_1 + c_5$ |
| 25       | 6    | 63    | $D_6$         | $c_7, c_6, c_5, c_4, c_3, c_2$    |
| 26       | 6    | 28    | $E_6$         | $c_7, c_6, c_4 - c_5, c_3, c_2 - c_5, c_1$ |
| 27       | 6    | 1008  | $A_1 \times A_5$ | $c_7, c_6, c_5, c_4, c_2 + c_3, c_1 - c_3$ |
| 28       | 6    | 378   | $A_1 \times D_5$ | $c_7, c_6, c_5, c_4, c_2, c_1 + c_3$ |
| 29       | 6    | 2016  | $A_2 \times A_4$ | $c_7, c_6, c_4, c_3 + c_5, c_2 - 2c_5, c_1 - 3c_5$ |
| 30       | 6    | 5040  | $A_1 \times A_2 \times A_3$ | $c_7, c_6, c_4, c_3, c_2 - 2c_5, c_1 + c_5$ |

Table 2: The flats of the $E_7$ reflection arrangement.
Equation (9.2) rests on two non-trivial facts from matroid theory. First, the simplicial complex underlying the Bergman fan is a wedge of $\mu$ spheres, where $\mu$ is the Möbius number of the matroid (see [AK, p.42, Corollary]). Second, for the matroid of a finite Coxeter system (as in [ARW]), the Möbius number $\mu$ is the product of the exponents of that group [OrSo, (1.1)]. We note that Bramble [Bra] was the first to determine the fundamental invariants of $W(\mathbf{E}_7)$. As for any reflection group, their degrees $2, 6, 8, 10, 12, 14, 18$ are the exponents plus one.

**Corollary 9.2.** The tropical Göpel variety $\text{trop}(\mathcal{G})$ in $\mathbb{T}^{134}$ is the union of the convex polyhedral cones $\mathcal{A}^iC_F$ where $F$ runs over all nested sets of irreducible root subsystems of $\mathbf{E}_7$.

**Proof.** This follows from Proposition 9.1 and the construction of the Bergman fan in [ARW].

Another approach to the tropical Göpel variety $\text{trop}(\mathcal{G})$ is to use the formulas (6.6) and (6.8) for the Göpel functions $f_\bullet, g_\bullet$ as polynomials of degree 7 in the brackets $[ijk]$. This defines a rational map from the Grassmannian $\text{Gr}(3, 7) \subset \mathbb{P}^{34}$ to the Göpel variety $\mathcal{G} \subset \mathbb{P}^{134}$. The tropicalization of this map is a piecewise linear map $\mu: \mathbb{T}\mathbb{P}^{34} \to \mathbb{T}\mathbb{P}^{134}$. Note that the map $\mu$ is not linear because the expression for $g_\bullet$ in the brackets $[ijk]$ is a binomial and not a monomial. We expect that this rational map does not commute with tropicalization.

The image of the tropical Grassmannian $\text{trop}(\text{Gr}(3, 7))$ under the piecewise linear map $\mu$ is a subfan of the tropical Göpel variety $\text{trop}(\mathcal{G})$. It would be interesting to identify that subfan. Note that, by [HJJS, Theorem 2.1], that tropical Grassmannian has the face numbers

$$f(\text{trop}(\text{Gr}(3, 7))) = (721, 16800, 124180, 386155, 522585, 252000).$$

The tropical Göpel variety $\text{trop}(\mathcal{G})$ is a modification and its face numbers are even larger.

The positive part of the tropical Grassmannian governs the combinatorics of the cluster algebra structure on the coordinate ring of $\text{Gr}(3, 7)$. Interestingly, the Weyl group relevant for this is of type $\mathbf{E}_6$ and not $\mathbf{E}_7$. Namely, it is shown in [SW, §7] that the normal fan of the $\mathbf{E}_6$-associahedron defines a simplicial fan structure on the positive part of $\text{trop}(\text{Gr}(3, 7))$, with f-vector $(42, 399, 1547, 2856, 2499, 833)$. The relationship to cluster algebras is explained in [SW, §8]. In light of this, it would be interesting to examine the positive part of $\text{trop}(\mathcal{G})$.

The Satake hypersurface $\mathcal{S}$ lives in $\mathbb{P}^7$, and it is parametrized by $\mathfrak{H}_3$ via the theta constant map $\vartheta$. The Newton polytope of its defining polynomial of degree 16 has face numbers

$$f(\text{Newton}(\mathcal{S})) = (344, 2016, 3584, 2828, 1120, 224, 22).$$

The corresponding tropical hypersurface $\text{trop}(\mathcal{S})$ in $\mathbb{T}\mathbb{P}^7$ is a six-dimensional fan with 22 rays and 2016 maximal cones. Now we shall explain and derive the following result:

**Proposition 9.3.** The image of the tropical Siegel space $\text{trop}(\mathfrak{H}_3)$ under the piecewise-linear map $\text{trop}(\vartheta)$ is the intersection of $\text{trop}(\mathcal{S})$ with the normal cone of $\text{Newton}(\mathcal{S})$ at the vertex

$$M = -2u_{000}u_{001}u_{010}u_{011}u_{100}u_{101}u_{110}u_{111}.$$  (9.3)

The map $\text{trop}(\vartheta)$ induces the level structure on $\text{trop}(\mathcal{A}_3)$ described by Chan in [Cha, §7.1].
Proof. We first define the terms in Proposition 9.3. Following [CMV, MZ], the tropical Siegel space \( \text{trop}(\mathfrak{F}_3) \) is the cone \( \text{PD}_3 \) of positive definite real symmetric \( 3 \times 3 \) matrices. We use tropical theta functions as described by Mikhalkin and Zharkov in [MZ]. The coordinates of the tropical theta constant map \( \text{trop}(\vartheta) \) are indexed by \( \sigma \in \{0, 1\}^3 \), and they are defined by

\[
\text{trop}(\vartheta)_\sigma(T) = \min \{(n + \sigma/2)^t \cdot T \cdot (n + \sigma/2) : n \in \mathbb{Z}^3\} \quad \text{for} \quad T \in \text{PD}_3. \tag{9.4}
\]

The function \( \text{trop}(\vartheta)_\sigma \) from \( \text{PD}_3 \) to \( \mathbb{R} \) is well-defined and takes non-negative values since \( T \) is positive definite. It is zero for all \( T \) when \( \sigma = (0, 0, 0) \). Hence we have a well-defined map

\[
\text{trop}(\vartheta) : \text{PD}_3 \to \text{trop}(\mathcal{S}) \subset \mathbb{TP}^7.
\]

To show that the image lands in the tropical Satake hypersurface, we set \( \tau = i\rho T \) in (3.3) and (3.4), where \( \rho \to \infty \) is a real parameter. Write \( \Theta_2[\sigma](\tau; 0) \) as a series in \( \epsilon = \exp(-\rho) \), with (9.4) as the exponent of the lowest term. Then each coordinate of \( \vartheta(\tau) \) is a series in \( \epsilon \). Plugging these eight series into the Satake polynomial \( \mathcal{S} \), we obtain zero. Compare the 471 monomials of \( \mathcal{S} \) according to their order in \( \epsilon \) after this substitution. Two (or more) of the monomials must have the same lowest order in \( \epsilon \). This means that \( \text{trop}(\vartheta(T)) \) lies in \( \text{trop}(\mathcal{S}) \).

The vertex of Newton(\( \mathcal{S} \)) given by the monomial \( M \) in (9.3) is simple, i.e. it has precisely seven adjacent edges. These edges are the Newton segments of the seven binomials

\[
\begin{align*}
&u_8^8u_{001}^2u_{010}^2u_{100}^2u_{111}^2 + M, \quad u_8^8u_{000}^2u_{010}^2u_{101}^2u_{110}^2 + M, \quad u_8^8u_{000}^2u_{011}^2u_{100}^2u_{110}^2 + M, \\
u_8^8u_{000}^2u_{011}^2u_{101}^2u_{110}^2 + M, \quad u_8^8u_{000}^2u_{010}^2u_{101}^2u_{111}^2 + M, \quad u_8^8u_{000}^2u_{011}^2u_{101}^2u_{111}^2 + M, \quad u_8^8u_{100}^2u_{101}^2u_{110}^2u_{111}^2 + M.
\end{align*}
\tag{9.5}
\]

Our computations revealed that these seven are all the \( \epsilon \)-leading forms of \( \mathcal{S} \) selected by generic matrices \( T \in \text{PD}_3 \). Hence the image of \( \text{trop}(\vartheta) \) consists of the seven outer normal cones at these edges. This is precisely what we had claimed in the first statement in Proposition 9.3.

The correspondence with the level structure described in [Cha, §7.1] is seen as follows. The domains of linearity of \( \text{trop}(\vartheta) \) define a subdivision of \( \text{PD}_3 \) into infinitely many convex polyhedral cones. This subdivision is a coarsening of the second Voronoi decomposition. We group the cones into seven classes, according to which binomial in (9.5) gets selected by \( T \).

The seven classes in \( \text{PD}_3 \) are naturally labeled by the seven lines in the Fano plane \( \mathbb{P}^2(\mathbb{F}_2) \). These lines are given by the three variables missing in the respective leading monomials in (9.5). For instance, the first binomial in (9.5) determines the line \( \{(1:1:0), (1:0:1), (0:1:1)\} \) because \( u_{110}, u_{101}, u_{011} \) are missing in the monomial prior to \( M \). This subdivision of \( \text{PD}_3 \) into seven classes is precisely the level structure that was discovered by Chan in [Cha, §7.1].

We expect an even more beautiful structure when tropicalizing the re-embedding \( \mathcal{S}' \) of the Satake hypersurface \( \mathcal{S} \) into \( \mathbb{P}^{35} \) given by the quadrics \( (T_\epsilon')^2 \) in (7.13). To see this, we revisit the combinatorial result in [Gla, Theorem 3.1]. Glass classified the points in \( \mathcal{S}' \subset \mathbb{P}^{35} \) according to which of the 36 coordinates are zero at that point. Up to symmetry, there are seven Glass strata in \( \mathcal{S}' \). Table 3 identifies the irreducible components of these strata and their geometric meaning. Glass# is the number of coordinates \( (T_\epsilon')^2 \) that are zero.

Table 3 refines our analysis of the Satake hypersurface in Section 3. For each irreducible component we know their defining polynomials in the unknowns \( u_{000}, u_{001}, \ldots, u_{111} \). These
Table 3: Glass strata in the Satake hypersurface

| Codim | Glass# | Geometric description | Components | How Many |
|-------|--------|-----------------------|------------|----------|
| 1     | 1      | Hyperelliptic locus   | quadric ∩ S | 36       |
| 2     | 6      | Torelli boundary      | P^3 × P^1  | 336      |
| 3     | 9      | Product of three elliptic curves | P^1 × P^1 × P^1 | 1120 |
| 3     | 16     | Satake boundary A_2(2, 4) | P^3       | 126      |
| 4     | 18     | Torelli boundary of A_2(2, 4) | quadric in P^3 | 1260 |
| 5     | 24     | Satake stratum A_1(2, 4) | P^1       | 1260     |
| 6     | 28     | Satake stratum A_0(2, 4) | point     | 1080     |

played an important role in this paper. For instance, the components for Glass# 16 are the spaces $H_i^{±} \simeq \mathbb{P}^3$ in (7.2), and the components for Glass# 6 are the Segre varieties in (3.7).

Now consider the tropical variety trop($S'$) in $\mathbb{T}\mathbb{P}^{35}$. This is a modification of the tropical hypersurface in $\mathbb{T}\mathbb{P}^7$ given by Newton($S$). It has much better properties since the coordinate functions now have a geometric meaning: they are the 36 divisors of the hyperelliptic locus. These appear as distinguished rays in trop($S'$). By the principle of geometric tropicalization [HKT, §2], they span a distinguished subfan of trop($S'$). Here, a collection of coordinate rays spans a cone if and only if the corresponding intersection of divisors appears in Table 3.

The poset of Glass strata seems to embed naturally into the poset of cones in the tropical moduli space trop($A_3$). Compare [Gla, Theorem 3.1] with [Cha, Figure 8]. This deserves to be studied in more detail. Can the table above be lifted to a tropical level structure on $A_3$? The finite groups of Section 3 act on the fans we described. Taking quotients by these groups leads to objects known as stacky fans. The tropical Torelli map of [Cha, CMV] is a morphism of stacky fans from trop($M_3$) onto trop($A_3$). It would be desirable to express the combinatorics of the tropical Torelli map directly in terms of the polynomials and ideals in this paper.

In the first version of this article we asked the following question: What is the relationship between the G"opel variety $G$ and the moduli space $Y(\Delta)$ of degree 2 del Pezzo surfaces constructed by Hacking, Keel, and Tevelev in [HKT]? The latter is a tropical compactification of the configuration space of seven points in $\mathbb{P}^2$. In the meantime, we were able answer this question. It turns out that trop($G$) has the same support as their fan $\mathcal{F}(\Delta)$ in [HKT, Corollary 5.3]. This is explained in [RSS]. That paper features calculations with explicit moduli spaces, mostly in genus 2, that are considerably smaller than the ones we studied here. Our readers may enjoy looking at [RSS] as a point of entry also to the present work.

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