PROPERTIES OF POINCARÉ HALF-MAPS FOR PLANAR LINEAR SYSTEMS VIA AN INTEGRAL CHARACTERIZATION.

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Abstract. This paper deals with fundamental properties of Poincaré half-maps for planar linear systems, which is essential to understand the dynamic behavior of planar piecewise linear systems. In the literature, the study of these Poincaré half-maps is based on the explicit integration of the differential systems in the linearity zones, which leads to the appearance of multiple cases due to the different spectra of the matrices. This flaw is avoided by using a novel integral characterization of Poincaré half-maps, and present the analysis of their properties without the need to resort to a large case-by-case study. Concretely, we focus on the analyticity of the Poincaré half-maps, their series expansions (Taylor and Newton-Puiseux) at the tangency point and at infinity, the relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant, and the sign of their second derivatives.

1. Introduction

The study of the qualitative properties of distinguished solutions of piecewise linear differential systems rests mainly on the analysis of the features of Poincaré maps, which are defined as composition of transition maps between the separation manifolds. Sometimes these transition maps are called Poincaré half-maps. The linearity of the system in each zone invites to its integration, which automatically causes the emergence of a wide range of cases due to the nature of the different spectra of the matrices of the linear systems and the relative position between the equilibria, if any, and the separation manifolds. The number of cases to be studied is high even for two-dimensional systems with two zones of linearity. Moreover, the direct integration leads to different nonlinear equations where the flight time appears as a new variable.

Since the publication of the seminal work by Freire et al. [7], a large number of interesting papers have appeared in order to establish the dynamical behavior in two-dimensional piecewise linear systems with two zones of linearity and,
in particular, to give conditions for the existence and stability of limit cycles and to provide an optimal bound for the number of coexisting limit cycles (see for instance, [9, 12, 13, 14, 15, 16, 17, 18]). None of these papers considers all the possible cases. Moreover, they are forced to use individualized approaches to study the different kind of functions that arise due to the distinct spectra of the matrices. This causes that a same result is usually expressed in different terms and, sometimes, it may be a hard task to obtain a common and brief statement for it. Thus, the use of individualized techniques for each case does not allow a unified view of several properties of the Poincaré half-maps and, when it does, more effort is required to complete the case-by-case study and to achieve independent statements of these cases.

This paper relies on a novel characterization of Poincaré half-maps for planar linear systems [2] which allows us to see the properties of these maps from a common point of view and to prove the results in a simple way, without the need of making particularized case-by-case studies. Accordingly, we will not have any of the disadvantages mentioned in the previous paragraphs because this novel characterization does not require integration of the systems and, therefore, the distinction of the spectra of the matrices is not needed. The strength of this approach can be seen in [3], where the uniqueness of limit cycles for continuous piecewise linear systems was provided in a simple and synthesized way.

In the framework of the study of Poincaré half-maps for planar linear systems, the most relevant properties are those related to the local behavior at tangency points between the flow and the Poincaré section, the behavior at infinity (obviously, in the case of the focus or center), and the sign of the derivatives, in particular, the second one. Some of these properties have been proved just for concrete cases. Even those which are valid for all situations have been proved in a large case-by-case study. This manuscript is devoted to simplifying and unifying the proofs of these properties by considering all possible scenarios simultaneously. In addition, it will be stated a new and interesting property about the relative position between the graphs of Poincaré half-maps and the bisector of the fourth quadrant. Among other things, from this property it is direct that Poincaré half-maps inherit the expansion/compression behavior of the flow of the planar linear system.

The paper is organized as follows. Section 2 presents the integral characterization of Poincaré half-maps for two-dimensional linear systems given in [2]. Two basic consequences of this characterization for the Poincaré half-maps are their analyticity and their understanding as solutions of a differential equation. In Section 3 we summarize the results on analyticity of the Poincaré half-maps given in [2] and obtain the Taylor and Newton-Puiseux series expansions at tangency points and infinity by means of the differential equation. Section 4 studies the relationship between the graphs of Poincaré half-maps and the
bisection of the fourth quadrant, which is used to establish, in Section 5, the
sign of the second derivatives of the Poincaré half-maps.

2. Integral characterization for the Poincaré half-maps

Let us consider, for \( x = (x_1, x_2)^T \), the autonomous linear system

\[
\dot{x} = Ax + b
\]

where \( A = (a_{ij})_{i,j=1,2} \) is a real matrix and \( b = (b_1, b_2)^T \in \mathbb{R}^2 \). Let us choose, without loss of generality, the Poincaré section \( \Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\} \).

Notice that if the coefficient \( a_{12} \) vanishes, system (1) is uncoupled and a
Poincaré half-map on section \( \Sigma \) can not be defined. Hence, let us assume that
\( a_{12} \neq 0 \) (observability condition [4]). Under this assumption, the linear change
of variable \( x = x_1, \quad y = a_{22}x_1 - a_{12}x_2 - b_1, \) with \( a = a_{12}b_2 - a_{22}b_1 \), allows to
write system (1) into the generalized Liénard form,

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
T & -1 \\
D & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} -
\begin{pmatrix}
0 \\
a
\end{pmatrix},
\]

where \( T \) and \( D \) stand for the trace and the determinant of matrix \( A \), respectively. In the new coordinates, since \( x_1 = x \), Poincaré section \( \Sigma \) remains the same.

The first equation of system (2) evaluated on section \( \Sigma \) is reduced to \( \dot{x}|_{\Sigma} = -y \). Therefore, the flow of the system crosses \( \Sigma \) from the half-plane \( \Sigma^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\} \) to the half-plane \( \Sigma^- = \{(x, y) \in \mathbb{R}^2 : x < 0\} \) when \( y > 0 \), from \( \Sigma^- \) to \( \Sigma^+ \) when \( y < 0 \), and it is tangent to \( \Sigma \) at the origin.

Since the system (2) is linear, the definition of its Poincaré half-maps is usually
given in the literature by using the explicit integrated flow (what results
in many case-by-case studies) and the intersection points of their orbits with
the Poincaré section \( \Sigma \) (what forces the nonlinear implicit appearance of the
flight time). Here, we will use a characterization that avoids the explicit com-
putation of the flow (and so the previous flaws), as it is done in [2]. For the
sake of completeness, we give a brief summary of the main results and ideas
of [2] that are going to be used in this paper.

Throughout this work, we suppose that \( a^2 + D^2 \neq 0 \). Notice that this
assumption is a necessary condition to define a Poincaré half-map for linear
system (2) associated to \( \Sigma \). Otherwise (that is, when \( a = D = 0 \)), there is no
possible return to section \( \Sigma \).

The left Poincaré half-map \( P \) and its definition interval \( I \) are given in The-
orem 19 and Corollary 21 of [2]. By using the quadratic polynomial function

\[
W(y) = Dy^2 - aTy + a^2,
\]
the left Poincaré half-map is the unique function \( P : I \subset [0, +\infty) \rightarrow (-\infty, 0] \) that, for every \( y_0 \in I \), satisfies

\[
(4) \quad \text{PV} \left\{ \int_{y_0}^{y_0} \frac{-y}{W(y)} \, dy \right\} = cT,
\]

where \( c \) is given, in terms of the parameters, as follows: (i) \( c = 0 \) if \( a > 0 \), (ii) \( c = \pi \left( D \sqrt{4D - T^2} \right)^{-1} \in \mathbb{R} \) if \( a = 0 \), and (iii) \( c = 2\pi \left( D \sqrt{4D - T^2} \right)^{-1} \in \mathbb{R} \) if \( a < 0 \). Here, \( \text{PV} \{ \cdot \} \) stands for the Cauchy Principal Value at the origin (see, for instance, [11]), which is defined as

\[
\text{PV} \left\{ \int_{y_1}^{y_0} \frac{-y}{W(y)} \, dy \right\} = \lim_{\varepsilon \to 0} \left( \int_{y_1}^{-\varepsilon} \frac{-y}{W(y)} \, dy + \int_{\varepsilon}^{y_0} \frac{-y}{W(y)} \, dy \right),
\]

for \( y_1 < 0 < y_0 \).

Following [2], we are going to focus on the left Poincaré Half-Map because, since system (2) is invariant under the change \((x, y, a) \leftrightarrow (-x, -y, -a)\), the definition of right Poincaré Half-Map and the corresponding results may be translated immediately. As emphasized in [2], the interval \( I \) is essentially related with the roots of the quadratic polynomial function \( W \). In the next remark, we shall briefly comment some of those relationships and other interesting properties of \( P \) which are proved in [2].

**Remark 1.** When \( I \subset [0, +\infty) \) is bounded, then the right endpoint of \( I \) is a real root of the quadratic polynomial function \( W \) given in (3) (see Fig. 1 (a)). In the same way, if \( P(I) \) is bounded, then the left endpoint of \( P(I) \) is also a real root of the quadratic polynomial function \( W \) (see Fig. 1 (a) and Fig. 1 (b)).

The interval \( I \) can be unbounded. For instance, if \( 4D - T^2 > 0 \), then the equilibrium point of system (2) is a focus or a center, the intervals \( I \) and \( P(I) \) are unbounded, and, obviously, \( P(y_0) \) tends to \(-\infty\) as \( y_0 \to +\infty \). In this case, the intervals are \( I = [0, +\infty) \) and \( P(I) = (-\infty, 0] \), except when the equilibrium is a focus (i.e. \( T \neq 0 \)) and it is located in the left half-plane \( \{(x, y) \in \mathbb{R}^2 : x < 0\} \) (i.e. \( a < 0 \)). In fact, when \( T > 0 \), the interval \( P(I) \) is reduced to \((-\infty, \hat{y}_1)\), where \( \hat{y}_1 = P(0) \) (see Fig. 1 (c)). Analogously, for \( T < 0 \), \( I = [\hat{y}_0, +\infty) \) with \( \hat{y}_0 = P^{-1}(0) \) (see Fig. 1 (d)).

Finally, the polynomial function \( W \) is strictly positive in each set \([P(y_0), 0) \cup (0, y_0] \), with \( y_0 \in I \). Besides that, since \( W(0) = a^2 \), then \( W(0) > 0 \) for \( a \neq 0 \) and \( W(0) = 0 \) for \( a = 0 \).

It is worth mentioning that the integral given in (4) diverges when \( a = 0 \) and the Cauchy principal value is necessary to overcome this difficulty. Moreover, in this case, for \( y_1 < 0 < y_0 \), the Cauchy principal value at the origin is given
Figure 1. The left Poincaré half-map $P$ and its interval of definition $I$ for the cases: (a) saddle, (b) node, (c) unstable focus, and (d) stable focus.

by

$$\text{PV} \left\{ \int_{y_1}^{y_0} -\frac{y}{Dy^2} \, dy \right\} = \lim_{\varepsilon \to 0} \left( \int_{y_1}^{-\varepsilon} -\frac{y}{Dy^2} \, dy + \int_{\varepsilon}^{y_0} -\frac{y}{Dy^2} \, dy \right) = \frac{1}{D} \log \left| \frac{y_1}{y_0} \right| .$$

When $a \neq 0$, the integrating function $h(y) = -y/W(y)$ is continuous and, consequently, the Cauchy principal value just takes the value of the integral.
3. ANALYTICITY AND SERIES EXPANSIONS OF POINCARE HALF-MAPS AT THE TANGENCY POINT AND ITS PREIMAGE, AND AT INFINITY

In this section, by means of the integral characterization and a subsequent differential equation, we shall compute the first coefficients of the Taylor expansion of the left Poincaré half-map \( P \). Obviously, the used method does not depend on the spectrum of the matrix of the system. Before obtaining these coefficients it is necessary to determine the analyticity of the left Poincaré half-map \( P \).

When \( P(y_0) \neq 0 \), it is well-known (see, for example, [5]) that the transversality between the flow of the system and the separation line \( \Sigma \) ensures the analyticity of \( P \) at \( y_0 \). The analyticity for the tangency point between the flow and \( \Sigma \) (that is, the origin) is more intricate and, in the literature, it has been approached with a case-by-case study (see some partial results at [Coll, Gasull, Prohens]). However, as follows from Corollary 24 of [2], the maps \( P \) and \( P^{-1} \) are real analytic functions in the open intervals \( \text{int}(I) \) and \( P(\text{int}(I)) \), respectively, and at least one of the following statements is true:

(i) the map \( P \) is a real analytic function at the left endpoint of its domain,
(ii) the map \( P^{-1} \) is a real analytic function at the right endpoint of its domain.

When the equilibrium of system (2) is a center or a focus, the left Poincaré half-map \( P \) can be considered also at infinity. In addition, we shall obtain the first coefficients of the Taylor expansion of \( P \) around the infinity.

A first consequence from the definition of the left Poincaré half-map given in the integral form (4) is easily deduced by computing the derivative with respect to variable \( y_0 \) (see Remark 16 of [2]). Hence, one can see that the graph of the left Poincaré half-map \( P \) and its inverse function \( P^{-1} \), oriented according to increasing \( y_0 \), are particular orbits of the cubic vector field

\[
X(y_0, y_1) = -(y_1 W(y_0), y_0 W(y_1)) =
-\left( y_1 (Dy_0^2 - aTy_0 + a^2), y_0 (Dy_1^2 - aTy_1 + a^2) \right).
\]

In fact, the left Poincaré half-map \( P \) and its inverse function \( P^{-1} \) are solutions of the differential equation

\[
y_1 W(y_0) dy_1 - y_0 W(y_1) dy_0 = 0.
\]

The next proposition is a direct consequence of the results in [2] and allows to obtain the Taylor expansion of \( P \) around the origin when \( a \neq 0 \) and \( P(0) = 0 \). Notice that for \( a = 0 \), the existence of the left Poincaré half-map \( P \) implies \( 4D - T^2 > 0 \). From Remark [4] the interval of definition of \( P \) is \( I = [0, +\infty) \).
and, for \( y_0 \geq 0 \), expression (4) can be written as

\[
\text{PV} \left\{ \int_{y_0}^{y_0} \frac{-y}{Dy^2} \, dy \right\} = \frac{\pi T}{D\sqrt{4D-T^2}}.
\]

Hence, by using the value for \( \text{PV} \) given in (5), the left Poincaré half-map \( P \) is given by

\[
P(y_0) = -\exp \left( \frac{\pi T}{\sqrt{4D-T^2}} \right) y_0, \quad \text{for } y_0 \geq 0.
\]

**Proposition 1.** Assume that \( a \neq 0 \) and \( 0 \in I \). If \( P(0) = 0 \), then left Poincaré half-map \( P \) is a real analytic function in \( I \), it is an involution and its Taylor expansion around the origin writes as

\[
P(y_0) = -y_0 - \frac{2Ty_0^2}{3a} - \frac{4T^2y_0^3}{9a^2} + \frac{2(9DT - 22T^3)y_0^4}{135a^3} + \frac{4(27DT^2 - 26T^4)y_0^5}{405a^4} - \frac{2(27D^2T - 176DT^3 + 100T^5)y_0^6}{945a^5} + \mathcal{O}(y_0^7).
\]

**Proof.** From the hypotheses of the proposition and by means of Theorem 14 and Corollary 24 of [2], it is deduced that left Poincaré half-map \( P \) is a real analytic function in \( I \) and it is an involution. Hence, the derivative of \( P \) at the origin is \( P'(0) = -1 \).

Now, taking into account that \( P \) is a solution of the differential equation given in (3), it is easy to obtain, via undetermined coefficients, the Taylor expansion given in (8) and so the proof is concluded. \( \square \)

Notice that the Taylor expansion around the origin given in Proposition 1 was already obtained in [19]. Although the calculations are not fully detailed in that work, the authors rely on the results given in [6], where the study requires different techniques depending on the situations. Before [19], the same series expansion was obtained in [8], by means of an inversion of the flight time, but only for the focus case.

When \( 0 \in I \) and \( P(0) \neq 0 \), the function \( P \) is a real analytic function at the origin and it is possible to obtain its Taylor expansion of \( P \) around the origin.

**Proposition 2.** Assume that \( 0 \in I \). If \( P(0) = \hat{y}_1 < 0 \), then \( a < 0, T > 0, 4D-T^2 > 0, \hat{y}_1 \) is the right endpoint of the interval \( P(I) \), and the left Poincaré half-map \( P \) is a real analytic function in \( I \) and its Taylor expansion around
the origin writes as
\begin{equation}
\label{eq:9}
P(y_0) = \hat{y}_1 + \frac{W(\hat{y}_1)y_0^2}{2a^2\hat{y}_1} + \frac{TW(\hat{y}_1)y_0^3}{3a^3\hat{y}_1} - \frac{(a^2 + (D - 2T^2)\hat{y}_1^2)W(\hat{y}_1)y_0^4}{8a^4\hat{y}_1^3}
\end{equation}
\begin{align*}
&\quad - \frac{T(5a^2 + (7D - 6T^2)\hat{y}_1^2W(\hat{y}_1))y_0^5}{30a^5\hat{y}_1^5} \\
&\quad + \frac{(9a^4 - 6a^3T\hat{y}_1 + 2a^2(9D - 13T^2)\hat{y}_1^2 + (9D^2 - 46DT^2 + 24T^4)\hat{y}_1^4)W(\hat{y}_1)y_0^6}{144a^6\hat{y}_1^5} \\
&\quad + O(y_0^7).
\end{align*}

**Proof.** The expression given in (7) provides the left Poincaré half-map for the case \(a = 0\). From there, one obtains \(P(0) = 0\) when \(a = 0\).

Suppose that \(0 \in I\) and \(P(0) = \hat{y}_1 < 0\). Then \(a \neq 0\) and expression (4) leads us to
\[\int_{\hat{y}_1}^{y_0} -\frac{y}{W(y)}\,dy = cT.\]
From Remark 1, the polynomial \(W\) is strictly positive and, therefore, the left-hand term of the last expression is also strictly positive. If \(a > 0\), from expression (4), \(c = 0\) and this is impossible. Thus, it is deduced that \(a < 0, T > 0, \) and \(4D - T^2 > 0\). Now, from Remark 1 again, the intervals \(I\) and \(P(I)\) are unbounded and \(P(y_0)\) tends to \(-\infty\) as \(y_0 \to +\infty\).

Next, let us prove that \(\hat{y}_1\) is the right endpoint of the interval \(I\). Let us consider \(y_0 \geq 0\) and \(y_1 \in (\hat{y}_1, 0]\). From the inequalities
\[\int_{\hat{y}_1}^{y_0} -\frac{y}{W(y)}\,dy < \int_{\hat{y}_1}^{y_0} -\frac{y}{W(y)}\,dy < \int_{\hat{y}_1}^{y_0} -\frac{y}{W(y)}\,dy = cT\]
one can see that no point in the interval \(y_1 \in (\hat{y}_1, 0]\) belongs to the interval \(P(I)\) and so the right endpoint of interval \(P(I)\) is \(\hat{y}_1\).

The analyticity of \(P\) is a direct consequence of Theorem 14 and Corollary 21 of [2] and the Taylor expansion around the origin given in (9) follows from the method of undetermined coefficients applied to the differential equation (6).

**Remark 2.** Note that the condition \(P(0) = \hat{y}_1 < 0\) together with the linearity of the system implies that there exists a unstable focus equilibrium in the left half-plane \(\Sigma^-\) (see Fig. 1(c)) and so it is immediate that \(a < 0, T > 0, \) and \(4D - T^2 > 0\). This is an alternative proof for the inequalities of Proposition 2.

On the other hand, the endpoints of intervals \(I\) and \(P(I)\) were also determined in Corollary 21 of [2] in a more generic way. For the sake of completeness, in the previous proof, we have included a different and specific reasoning for this case.
When there exists a point $\hat{y}_0 \in \text{int}(I)$ such that $P(\hat{y}_0) = 0$, then left Poincaré half-map $P$ is a non-analytic function at $\hat{y}_0$. However, in [2] it is proven that the inverse function $P^{-1}$ is analytic at the origin and so it is possible, by means of an inversion, to get a Newton-Puiseux serie expansion for the left Poincaré half-map $P$ around $\hat{y}_0$. Some results about series inversion and Newton-Puiseux series can be found in [10] and the references therein. Also of interest are the results included in [1] concerning the expression of the solutions of differential equations as Newton-Puiseux series expansion and its convergence.

**Proposition 3.** Assume that there exists a value $\hat{y}_0 > 0$ such that $P(\hat{y}_0) = 0$. Then, $a < 0$, $T < 0$, $4D - T^2 > 0$, $\hat{y}_0$ is the left endpoint of the interval $I$, the inverse function $P^{-1}$ is a real analytic function, and the left Poincaré half-map $P$ admits the Newton-Puiseux serie expansion around the point $\hat{y}_0$ given by

$$P(y_0) = a \sqrt{\frac{2\hat{y}_0}{W(\hat{y}_0)}} (y_0 - \hat{y}_0)^{1/2} - \frac{aT}{3} \frac{2\hat{y}_0}{W(\hat{y}_0)} (y_0 - \hat{y}_0) +$$

$$\frac{a^3}{72} \left( \frac{9D + 2T^2}{a^2} + \frac{9}{\hat{y}_0^2} \right) \left( \sqrt{\frac{2\hat{y}_0}{W(\hat{y}_0)}} \right)^3 (y_0 - \hat{y}_0)^{3/2} + O((y_0 - \hat{y}_0)^4),$$

which is valid for $y_0 \geq \hat{y}_0$.

**Proof.** Suppose that there exists a point $\hat{y}_0 > 0$ such that $P(\hat{y}_0) = 0$. An analogous reasoning to the first part of the proof of Proposition 2 leads to the inequalities $a < 0$, $T < 0$, and $4D - T^2 > 0$ and to the fact that $\hat{y}_0$ is the left endpoint of the interval $I$.

The inverse function $P^{-1}$ satisfies $P^{-1}(0) = \hat{y}_0$ and, from differential equation (6), it follows that its derivative at the origin vanishes. This implies that $P$ is a non-analytic function at $\hat{y}_0$. From Theorem 14 and Corollary 21 of [2], it follows that the inverse function $P^{-1}$ is an analytic function at the origin and $P^{-1}$ admits the Taylor expansion (9) by changing $\hat{y}_1$ by $\hat{y}_0$.

Now, the Newton-Puiseux series expansion of $P$ is obtained by the inversion of the Taylor expansion of $P^{-1}$. Note that the direct inversion provides two possible series expansions but, since $P(y_0) \leq 0$ for all $y_0 \in I$, the valid one is that given in (10) and the proof is finished. \qed

An analogous comment to Remark 2 can be made about the inequalities of Proposition 3 and the left endpoint of $I$. The scenario described by the hypothesis stated in Proposition 3 is illustrated in Fig. 1(d).

**Remark 3.** The inversion used to obtain the Newton-Puiseux series expansion of $P$ is equivalent to the computation of the Taylor expansion of $Q(z_0) := P(\hat{y}_0 + z_0^2)$ around $z_0 = 0$ and the subsequent change of $z_0$ by $\sqrt{y_0 - \hat{y}_0}$. In order to get this Taylor expansion it is enough to make the change of variable
\( y_0 \to \dot{y}_0 + z_0^2 \) in the differential equation (6) to achieve a differential equation for the function \( Q \).

Let us recall from Remark 1 that when \( 4D - T^2 > 0 \) the domain \( I \) is unbounded with \( P(y_0) \) tending to \(-\infty\) as \( y_0 \to +\infty \). Thus, the study of the left Poincaré half-map around the infinity is feasible. In fact, the first two terms of the Taylor expansion of left Poincaré half-map \( P \) around the infinity were already obtained in [7] by means of an expression of \( P \) parameterized by the flight time. In the following proposition, we present a simple method to get these and others terms.

**Proposition 4.** Assume that \( 4D - T^2 > 0 \). Then, the Taylor expansion of left Poincaré half-map \( P \) around the infinity writes as

\[
P(y_0) = -\exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right) y_0 + \frac{aT}{D}\left(1 + \exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right)\right) - \frac{a^2}{D} \sinh\left(\frac{\pi T}{\sqrt{4D - T^2}}\right) \cdot \frac{1}{y_0} - \frac{a^3 e^{-\frac{2\pi T}{\sqrt{4D - T^2}}} \left(1 + e^{\frac{\pi T}{\sqrt{4D - T^2}}}\right)^2 T}{6D^2} \cdot \frac{1}{y_0^3} + \mathcal{O}\left(\frac{1}{y_0^3}\right).
\]

**Proof.** Firstly, we shall prove the equality

\[
\lim_{y_0 \to +\infty} \frac{P(y_0)}{y_0} = -\exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right).
\]

If \( a = 0 \), then expression (11) leads us directly to equality (11).

If \( a \neq 0 \), taking into account that \( W(y) > 0 \) for \( y \in \mathbb{R} \) (see Remark 1), then relationship (11) can be written as

\[
\int_{-y_0}^{-y_0'} \frac{-y}{W(y)} dy + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT,
\]

for \( y_0 \in I \), being \( c = 0 \) for \( a > 0 \) and \( c = 2\pi\left(D \sqrt{4D - T^2}\right)^{-1} \) for \( a < 0 \).

The change of variable \( Y = 1/y \) applied to the first integral in expression (12) transforms it into

\[
\int_{1/P(y_0)}^{-1/y_0} \frac{1}{a^2Y^3 - aTY^2 + DY} dY + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT
\]

or, equivalently, into the expression

\[
\int_{1/P(y_0)}^{-1/y_0} \frac{1}{DY} dY + \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{D(a^2Y^2 - aTY + D)} dY + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT.
\]
That is,
\[
\frac{P(y_0)}{y_0} = -\exp \left( DcT - D \int_{-y_0}^{y_0} \frac{-y}{W(y)} \, dy - \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{a^2Y^2 - aTY + D} \, dY \right).
\]

Now, a direct integration provides
\[
\lim_{y_0 \to +\infty} \int_{-1/y_0}^{1/y_0} \frac{-y}{W(y)} \, dy = -\frac{\pi T \text{sign}(a)}{D\sqrt{4D - T^2}}
\]
and taking into account that
\[
\lim_{y_0 \to +\infty} \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{a^2Y^2 - aTY + D} \, dY = 0,
\]
the equality (11) follows.

Thus, the function \(\tilde{P}\), defined by
\[
\tilde{P}(Y_0) = \begin{cases} 
1 & \text{if } Y_0 \neq 0 \text{ and } 1/Y_0 \in I, \\
\frac{1}{P(1/Y_0)} & \text{if } Y_0 = 0,
\end{cases}
\]
has derivative on the right at the origin and its value is
\[
\alpha_1 := \frac{d\tilde{P}}{dY_0}(0^+) = -\exp \left( \frac{-\pi T}{\sqrt{4D - T^2}} \right).
\]

Moreover, it is immediate to see that the function \(\tilde{P}\) is a solution of differential equation
\[
(a^2Y_0^2 - aTY_0 + D) Y_0 dY_1 - (a^2Y_1^2 - aTY_1 + D) Y_1 dY_0 = 0,
\]
obtained from the differential equation (8) by means of the change of variables \((Y_0, Y_1) = (1/y_0, 1/y_1)\) (defined for \(y_0y_1 \neq 0\)). From here, it is deduced that the function \(\tilde{P}\) has derivatives on the right of all orders at \(Y_0 = 0\) and, after a direct computation, one finds
\[
\alpha_2 := \frac{d^2\tilde{P}}{dY_0^2}(0^+) = -\frac{2aT}{D} e^{\frac{2\pi T}{\sqrt{4D - T^2}}} \left( e^{\frac{2\pi T}{\sqrt{4D - T^2}}} + 1 \right),
\]
\[
\alpha_3 := \frac{d^3\tilde{P}}{dY_0^3}(0^+) = \frac{3a^2}{D^2} e^{-\frac{3\pi T}{\sqrt{4D - T^2}}} \left( e^{\frac{\pi T}{\sqrt{4D - T^2}}} + 1 \right) \left( -2T^2 e^{\frac{\pi T}{\sqrt{4D - T^2}}} + De^{\frac{\pi T}{\sqrt{4D - T^2}}} - D - 2T^2 \right),
\]
\[ \alpha_4 := \frac{d^4 \tilde{P}}{dY_0^4} (0^+) = \]
\[ \frac{4\alpha^3 T}{D^3} e^{-\frac{xT}{\sqrt{4D-T^2}}} \left( 1 + e^{\frac{xT}{\sqrt{4D-T^2}}} \right)^2 \left( -8D + 7De^{\frac{xT}{\sqrt{4D-T^2}}} - 6T^2 - 6T^2 e^{\frac{xT}{\sqrt{4D-T^2}}} \right). \]

Since the Taylor expansion of the left Poincaré half-map \( P \) around the infinity is given by

\[ P(y_0) = \frac{1}{\alpha_1} y_0 - \frac{\alpha_2}{2\alpha_1^2} + \frac{3\alpha_2^2 - 2\alpha_1\alpha_3}{12\alpha_1^3} \cdot \frac{1}{y_0} - \frac{3\alpha_2^3 - 4\alpha_1\alpha_2\alpha_3 + \alpha_1^2\alpha_4}{24\alpha_1^4} \cdot \frac{1}{y_0^2} + O \left( \frac{1}{y_0^3} \right), \]

the proof concludes by substituting expressions (14)–(15) into (16).

\[ \Box \]

4. The relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant

To study the relative position between the graph of the left Poincaré half-map and the bisector of the fourth quadrant, it is natural to analyze the sign of the difference \( y_0 - (-P(y_0)) \). In the next proposition, we show the relationship between this difference and the trace \( T \).

**Proposition 5.** The left Poincaré half-map \( P \) satisfies the relationship

\[ \text{sign} \left( y_0 + P(y_0) \right) = -\text{sign}(T) \quad \text{for} \quad y_0 \in I \setminus \{0\}. \]

In addition, when \( 0 \in I \) and \( P(0) \neq 0 \) or when \( T = 0 \), the relationship also holds for \( y_0 = 0 \).

**Proof.** We will prove this proposition by distinguishing the cases \( T = 0 \) and \( T \neq 0 \).

For \( T = 0 \), the integral equation given in (14) is reduced to

\[ \text{PV} \left\{ \int_{P(y_0)}^{y_0} \frac{-y}{Dy^2 + a^2} dy \right\} = 0, \quad \text{for} \quad y_0 \in I. \]

By taking into account that the integrating function is an odd function, it is direct to see that \( P(y_0) = -y_0 \) for all \( y_0 \in I \) and so the proposition is true for \( T = 0 \).

Now, we focus on the proof for the case \( T \neq 0 \) and we will consider the situations \( a = 0 \) and \( a \neq 0 \).

When \( a = 0 \), the left Poincaré half-map \( P \) is given by expression (17) and so the equality \( \text{sign} \left( y_0 + P(y_0) \right) = -\text{sign}(T) \) holds for every \( y_0 \in I \).

When \( a \neq 0 \), let us consider the interval

\[ J = \{ u \in \mathbb{R} : W(y) > 0, \forall y \in [-|u|, |u|] \} \]
and function $g : J \rightarrow \mathbb{R}$ defined by
\[ g(u) = \int_{-u}^{u} \frac{-y}{W(y)} dy, \]
where $W$ is the polynomial function defined in (3).

Notice that function $g$ satisfies $g(0) = 0$, its derivative is
\[ g'(u) = \frac{-2aTu^2}{W(u)W(-u)} \]
and so $\text{sign}(g'(u)) = -\text{sign}(aT)$ for every $u \in J \setminus \{0\}$. Thus, $\text{sign}(g(u)) = -\text{sign}(aT)$ for every $u \in J \cap (0, +\infty)$ and $\text{sign}(g(u)) = \text{sign}(aT)$ for every $u \in J \cap (-\infty, 0)$.

Moreover, if $J = \mathbb{R}$ (i.e., when $4D - T^2 > 0$), then, from (13),
\[ \lim_{u \to +\infty} g(u) = -\frac{\pi T \text{sign}(a)}{D\sqrt{4D - T^2}}. \]

The existence of the forward Poincaré half-map $P$ for the case $a \neq 0$ implies $a > 0$ and $c = 0$ or $a < 0$ and $c = 2\pi \left( D\sqrt{4D - T^2} \right)^{-1} \in \mathbb{R}$. It is straightforward to see that these conditions together with the properties of function $g$ lead to the equality
\[ \text{sign}(cT - g(u)) = \text{sign}(T). \]

Let us consider $y_0 \in \text{int}(I) \cap J$. From equality (14), one gets
\[ cT = \int_{P(y_0)}^{y_0} \frac{-y}{W(y)} dy = \int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy, \]
that is,
\[ \int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy = cT - g(y_0). \]

Thus, from (17),
\[ \text{sign} \left( \int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy \right) = \text{sign}(T) \neq 0 \]
and, taking into account that $-y_0 \cdot P(y_0) \geq 0$, equality $\text{sign}(y_0 + P(y_0)) = -\text{sign}(T)$ holds for every $y_0 \in \text{int}(I) \cap J$. Therefore, the conclusion follows by using the continuity of the left Poincaré half-map and the function $y_1(y_0) = -y_0$.

The next result establishes, as a direct consequence of Proposition 5, the relationship between the graph of the left Poincaré half-map and the bisector of the fourth quadrant.

**Corollary 1.** The following items are true.
(1) If $T = 0$, then the graph of the left Poincaré half-map $P$ of system (2) associated to section $\Sigma \equiv \{x = 0\}$, if it exists, is included in the bisector of the fourth quadrant.

(2) If $T > 0$ (resp. $T < 0$), then the graph of left Poincaré half-map $P$ of system (2) associated to section $\Sigma \equiv \{x = 0\}$, if it exists, is located below (resp. above) the bisector of the fourth quadrant except perhaps at the origin.

5. The sign of the second derivative of Poincaré half-maps

From the differential equation given in (6), it is easy to obtain explicit expressions for the derivatives of $P$ with respect to $y_0$. The first and second derivatives are shown in the next result. Its proof is a simple computation and so it is omitted.

**Proposition 6.** The first and second derivatives of the left Poincaré half-map $P$ with respect to $y_0$, in the interval $\text{int}(I)$, are given by

$$\frac{dP}{dy_0}(y_0) = \frac{y_0W(P(y_0))}{P(y_0)W(y_0)}$$

and

$$\frac{d^2P}{dy_0^2}(y_0) = -\frac{a^2 \left(y_0^2 - (P(y_0))^2\right)W(P(y_0))}{(P(y_0))^3(W(y_0))^2}.$$  

(18)

The sign of the first derivative is obvious because $y_0P(y_0) < 0$ for $y_0 \in \text{int}(I)$ and the polynomial $W$ is positive (see Remark [1]). The sign of the second derivative of left Poincaré half-map $P$ is an immediate consequence of expression given in (18) and Proposition 5.

**Proposition 7.** The sign of the second derivative of left Poincaré half-map $P$ is given by

$$\text{sign} \left(\frac{d^2P}{dy_0^2}(y_0)\right) = -\text{sign}(a^2T) \quad \text{for} \quad y_0 \in \text{int}(I).$$

Note that $a^2$ is written in the previous expression to include the case $a = 0$.

Acknowledgements

VC and EGM are partially supported by the Ministerio de Ciencia, Innovación y Universidades, Plan Nacional I+D+I cofinanced with FEDER funds, in the frame of the project PGC2018-096265-B-I00. FFS is partially supported by the Ministerio de Economía y Competitividad, Plan Nacional I+D+I cofinanced with FEDER funds, in the frame of the project MTM2017-87915-C2-1-P. VC, FFS, and EGM are partially supported by the Consejería de Educación y Ciencia de la Junta de Andalucía (TIC-0130, P12-FQM-1658). DDN
is partially supported by São Paulo Research Foundation (FAPESP) grants 2018/16430-8, 2018/13481-0, and 2019/10269-3, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 306649/2018-7 and 438975/2018-9.

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