A sharp lower bound for the lifespan of small solutions to the Schrödinger equation with a subcritical power nonlinearity

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Abstract: Let $T_\varepsilon$ be the lifespan for the solution to the Schrödinger equation on $\mathbb{R}^d$ with a power nonlinearity $\lambda |u|^{2\theta/d} u$ ($\lambda \in \mathbb{C}$, $0 < \theta < 1$) and the initial data in the form $\varepsilon \varphi(x)$. We provide a sharp lower bound estimate for $T_\varepsilon$ as $\varepsilon \to +0$ which can be written explicitly by $\lambda$, $d$, $\theta$, $\varphi$ and $\varepsilon$. This is an improvement of the previous result by H. Sasaki [Adv. Diff. Eq. 14 (2009), 1021–1039].

Key Words: subcritical nonlinear Schrödinger equation, lifespan, detailed lower bound.

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1 Introduction

1.1 Backgrounds

We consider the following initial value problem:

\[
\begin{aligned}
    i\partial_t u + \frac{1}{2} \partial_x^2 u &= \lambda |u|^{p-1} u, \quad t > 0, \quad x \in \mathbb{R}, \\
    u(0, x) &= \varepsilon \varphi(x), \quad x \in \mathbb{R},
\end{aligned}
\]  

(1.1)

where $i = \sqrt{-1}$, $\lambda \in \mathbb{C}$ and $p > 1$. $\varphi$ is a prescribed $\mathbb{C}$-valued function which belongs to a suitable weighted Sobolev space, and $\varepsilon > 0$ is a small parameter which is responsible for the size of the initial data. We are interested in the lifespan $T_\varepsilon$ for the solution $u = u(t, x)$ to (1.1) in the case of $p < 3$ and $\text{Im} \lambda > 0$. Before going into details, let us summarize the backgrounds briefly.

First we consider the simpler case $p > 3$. In this case, small data global existence for (1.1) is well-known. Moreover, the solution behaves like the free solution in the large time. On
the other hand, when \( p \leq 3 \), non-existence of asymptotically free solution has been shown in \([13], [1]\). Roughly speaking, the critical exponent \( p = 3 \) comes from the condition for convergence of the integral

\[
\int_1^\infty \frac{dt}{t^{(p-1)/2}}.
\]

Note that this threshold becomes \( p = 1 + 2/d \) in the \( d \)-dimensional settings. Next let us turn our attention to the case \( p \leq 3 \). In \([6]\), it has been shown that the solution to (1.1) with \( p = 3 \) and \( \lambda \in \mathbb{R} \) behaves like

\[
u(t,x) = \frac{1}{\sqrt{it}} \alpha(x/t) e^{i(x^2/(2t) - \lambda |\alpha(x/t)|^2 \log t)} + o(t^{-1/2}) \text{ in } L^\infty(\mathbb{R}_x)
\]

as \( t \to \infty \) with a suitable \( \mathbb{C} \)-valued function \( \alpha \) satisfying \( \| \alpha \|_{L^\infty} \leq C \varepsilon \). An important consequence of this asymptotic expression is that the solution decays like \( O(t^{-1/2}) \) in \( L^\infty(\mathbb{R}_x) \), while it does not behave like the free solution unless \( \lambda = 0 \). In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. This result has been extended in \([11]\) to the case where \( p \) is less than and sufficiently close to 3. When \( \lambda \in \mathbb{C} \), the situation changes slightly. Indeed, it has been verified in \([14]\) that the small data solution to (1.1) decays like \( O(t^{-1/2}(\log t)^{-1/2}) \) in \( L^\infty(\mathbb{R}_x) \) as \( t \to \infty \) if \( p = 3 \) and \( \text{Im} \lambda < 0 \). This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect (see also \([17]\) for a closely related result for the Klein-Gordon equation). The above-mentioned result has been extended in \([9], [10], [5], [7]\), etc., to the case \( p < 3 \) and \( \text{Im} \lambda < 0 \). However, it should be noted that these results essentially rely on the a priori \( L^2 \)-bound for the solution \( u \) coming from the conservation law

\[
\|u(t,\cdot)\|_{L^2}^2 - 2 \text{Im} \lambda \int_0^t \|u(\tau,\cdot)\|_{L^{p+1}}^{p+1} d\tau = \|u(0,\cdot)\|_{L^2}^2,
\]

which is valid only when \( \text{Im} \lambda \leq 0 \). In what follows, we focus on the remaining case \( p \leq 3 \) and \( \text{Im} \lambda > 0 \). This is the worst situation for global existence because the nonlinearity must be considered as a long-range perturbation and the a priori \( L^2 \)-bound for \( u \) is violated. To the authors’ best knowledge, there is no positive result in that case. As for the lifespan \( T_\varepsilon \), the standard perturbative argument yields a lower estimate in the form

\[
T_\varepsilon \geq \begin{cases} 
\varepsilon^{C/\varepsilon^2} & \text{when } p = 3 \\
C \varepsilon^{-2(p-1)/(3-p)} & \text{when } 1 < p < 3
\end{cases}
\]

with some \( C > 0 \), provided that \( \varepsilon \) is suitably small (see Section \([3]\) below for more detail). In other words, we have

\[
\lim \inf_{\varepsilon \to 0^+} \int_1^{T_\varepsilon} \left( \frac{\varepsilon}{t^{1/2}} \right)^{p-1} dt > 0.
\]

However, this estimate does not tell us the dependence of \( T_\varepsilon \) on \( \text{Im} \lambda \). So we are led to the question: how does \( T_\varepsilon \) depend on \( \text{Im} \lambda \)? In the cubic case, two of the authors have derived
the following more precise estimate for $T_\varepsilon$ in the previous papers [16], [12]:

$$\lim \inf_{\varepsilon \to +0} (\varepsilon^2 \log T_\varepsilon) \geq \frac{1}{2 \text{Im } \lambda} \sup_{\xi \in \mathbb{R}} |\hat{\varphi}(\xi)|^2,$$

where

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} \varphi(y) \, dy, \quad \xi \in \mathbb{R}.$$ 

This gives an answer to the question raised above for the cubic case. In fact, more general cubic nonlinear terms depending also on $\partial_x u$ have been treated in [16], [12] (see also [11] for a related work). When $p < 3$ and $\text{Im } \lambda > 0$, the situation is the most delicate and quite little is known so far. To the authors’ knowledge, there is only one result which concerns the dependence of $T_\varepsilon$ on $\text{Im } \lambda$ in the case of $p < 3$:

**Proposition 1.1** (Sasaki [13]). Assume $2 \leq p < 3$, $\text{Im } \lambda > 0$ and $(1 + x^2)\varphi \in \Sigma$. Let $T_\varepsilon$ be the supremum of $T > 0$ such that (1.1) admits a unique solution $u \in C([0, T); \Sigma)$. Then we have

$$\lim \inf_{\varepsilon \to +0} (\varepsilon^{2(p-1)/(3-p)}T_\varepsilon) \geq \left( \frac{3 - p}{2(p - 1) \text{Im } \lambda} \sup_{\xi \in \mathbb{R}} |\hat{\varphi}(\xi)|^{p-1} \right)^{2/(3-p)},$$

where $\Sigma = \{ f \in L^2(\mathbb{R}) \mid \|f\|_\Sigma < \infty \}$ with $\|f\|_\Sigma = \|f\|_{L^2} + \|\partial_x f\|_{L^2} + \|xf\|_{L^2}$.

The aim of this paper is to improve Proposition 1.1 regarding the following three points:

- to extend the admissible value of $p$ to the full range $1 < p < 3$,
- to relax the decay assumption on $\varphi(x)$ as $|x| \to \infty$,
- to give a higher dimensional generalization.

### 1.2 Main result

In what follows, we consider a $d$-dimensional generalization of (1.1). For the notational convenience, we write the power $p$ of the nonlinearity as $p = 1 + 2\theta/d$ so that the condition $1 < p < 1 + 2/d$ is interpreted as $0 < \theta < 1$. Then we are led to the following initial value problem:

$$\begin{cases}
i\partial_t u + \frac{1}{2} \Delta u = \lambda |u|^{2\theta/d} u, & t > 0, \quad x \in \mathbb{R}^d, \\
u(0, x) = \varepsilon \varphi(x), & x \in \mathbb{R}^d, \end{cases}$$

(1.2)

where $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_d)^2$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. To state the main result, let us introduce some notations. For $s, \sigma \geq 0$, we denote by $H^{s,\sigma}$ the weighted Sobolev spaces

$$H^{s,\sigma} := \{ f \in L^2(\mathbb{R}^d) \mid (1 + |x|^2)^{\sigma/2}(1 - \Delta)^{s/2} f \in L^2(\mathbb{R}^d) \}.$$
equipped with the norm
\[ \|f\|_{H^s,\sigma} := \|(1 + |x|^2)^{s/2}(1 - \Delta)^{s/2}f\|_{L^2}. \]
We also define \( \Sigma^s := H^{s,0} \cap H^{0,s} \) with the norm \( \|f\|_{\Sigma^s} := \|f\|_{H^{s,0}} + \|f\|_{H^{0,s}}. \) We set \( U(t) := \exp(i\frac{t}{2}\Delta) \) so that the solution \( v \) to the free Schrödinger equation
\[ i\partial_t v + \frac{1}{2}\Delta v = 0, \quad v(0, x) = \phi(x) \]
can be written as \( v(t) = U(t)\phi. \) The main result is as follows.

**Theorem 1.2.** Let \( 1 \leq d \leq 3, \) \( 0 < \theta < 1 \) and \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda > 0. \) Assume
\[ \frac{d}{2} < s < \min\{2, 1 + 2\theta/d\} \]
and \( \varphi \in \Sigma^s. \) Let \( T \) be the supremum of \( T > 0 \) such that (1.2) admits a unique solution \( u \) satisfying \( U(\cdot)^{-1}u \in C([0, T); \Sigma^s). \) Then we have
\[ \liminf_{\varepsilon \to 0} (\varepsilon^{2\theta/d}T\varepsilon^{1-\theta}) \geq \frac{(1 - \theta)d}{2\theta\text{Im} \lambda \sup_{\xi \in \mathbb{R}^d} |\hat{\varphi}(\xi)|^{2\theta/d}}, \]
where
\[ \hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} \varphi(y) \, dy, \quad \xi \in \mathbb{R}^d. \]

**Remark 1.1.** The assumption \( (1.3) \) is never satisfied when \( d \geq 4. \) That is the reason why Theorem 1.2 is available only for \( d \leq 3. \) When \( d = 1 \) or 2, \( (1.3) \) is satisfied for any \( 0 < \theta < 1. \) In particular, our result can be viewed as an extension of Proposition 1.1 because it corresponds to the case of \( d = 1, 1/2 \leq \theta < 1 \) and \( s = 1 \) in Theorem 1.2. On the other hand, when \( d = 3, \) \( (1.3) \) is satisfied only if \( \theta > 3/4 \) (or, equivalently, \( 3/2 < p < 5/3 \) with \( p = 1 + 2\theta/3 \)). The authors do not know whether the same assertion holds true or not when \( d \geq 4 \) or \( d = 3 \) with \( \theta \leq 3/4. \)

**Remark 1.2.** The authors do not know whether \( (1.4) \) is optimal or not. An example of blowing-up solution to (1.1) with arbitrarily small \( \varepsilon > 0 \) has been given by Kita [8] under a particular choice of \( \varphi \) and some additional restrictions on \( \lambda \) and \( p. \) However, it seems difficult to specify the lifespan for the blowing-up solution given in [8].

Now, let us explain the differences between the approach of [13] and ours. The method of [13] consists of two steps: the first is to construct a suitable approximate solution \( u_a \) which blows up at the expected time, and the second is to get an a priori estimate not for the solution \( u \) itself but for their difference \( u - u_a \) (see also [16] for the cubic case). Drawbacks of this approach come from the first step. In fact, according to Remark 1.3 in [13], this approach can not be used in the case \( 1 < p < 2. \) Remark that this implies the method of [13] is not suitable for \( d \)-dimensional settings when \( d \geq 2, \) because our main interest is the case of
Also, in view of Proposition 3.1 in [13], the additional decay assumption on \( \varphi \) as \( |x| \to \infty \) (i.e., higher regularity for \( \hat{\varphi} \)) seems essential for the method of [13]. On the other hand, our approach presented below does not rely on approximate solutions at all. Instead, we will reduce the original PDE (1.2) to a simpler ordinary differential equation satisfied by \( A(t, \xi) = \mathcal{F} \left[ U(t)^{-1} u(t, \cdot) \right] (\xi) \) up to a harmless remainder term \( R \) (see (5.1) below). An ODE lemma prepared in Section 4 below will allow us to get an a priori bound for \( u \) directly. Similar idea has been used in [12] for one-dimensional cubic derivative nonlinear Schrödinger equations, but we must be more careful because we are considering the situation in which the degree of the nonlinearity is lower.

We close the introduction with the contents of this paper. In the next section, we state basic lemmas which will be useful in the subsequent sections. In Section 3, we will derive a rough lower estimate for \( T_\varepsilon \), that is, \( \lim_{\varepsilon \to 0} \varepsilon^{2d/|\theta|} \frac{1}{T_\varepsilon - \theta} > 0 \). Section 4 is devoted to an ODE lemma which plays an important role in getting an a priori bound for the solution. After that, the main theorem will be proved in Section 5 by means of the so-called bootstrap argument. Finally, in Section 6 we give a few comments on the critical case \( \theta = 1 \). In what follows, we denote several positive constants by the same letter \( C \), which may vary from one line to another.

### 2 Basic lemmas

In this section, we introduce several lemmas that will be useful in the subsequent sections.

**Lemma 2.1.** Let \( s > d/2 \). There exists a constant \( C \) such that

\[
\| \phi \|_{L^\infty} \leq \frac{C}{(1 + t)^{d/2}} \| U(t)^{-1} \phi \|_{L^2},
\]

for \( t \geq 0 \).

**Proof.** We start with the standard Gagliardo-Nirenberg-Sobolev inequality:

\[
\| \phi \|_{L^\infty} \leq C \| \phi \|^{1-d/2s}_{L^2} \| (-\Delta)^{s/2} \phi \|^{d/2s}_{L^2}.
\]

We also introduce \( \mathcal{M}(t) = \exp \left( \frac{\| |x|^s U(t) \|_{L^2}}{2t} \right) \). Then we can check that

\[
U(t)|x|^s U(t)^{-1} \phi = \mathcal{M}(t)(-t^2 \Delta)^{s/2} \mathcal{M}(t)^{-1} \phi,
\]

from which it follows that

\[
t^{d/2} \| \phi \|_{L^\infty} = t^{d/2} \| \mathcal{M}(t)^{-1} \phi \|_{L^\infty}
\leq C \| \mathcal{M}(t)^{-1} \phi \|^{1-d/2s}_{L^2} \| (-t^2 \Delta)^{s/2} \mathcal{M}(t)^{-1} \phi \|^{d/2s}_{L^2}
\leq C \| \phi \|^{1-d/2s}_{L^2} \| |x|^s U(t)^{-1} \phi \|^{d/2s}_{L^2}.
\]
Combining the two inequalities above, we obtain
\[
(1 + t)^{d/2} \| \phi \|_{L^\infty} \leq \frac{C(1 + t)^{d/2}}{(1 + t^{d/2})} \| \phi \|_{L^2}^{1-d/2d} \big( \| (-\Delta)^{s/2} \phi \|_{L^2}^{d/2s} + \| |x|^s \mathcal{U}(t)^{-1} \phi \|_{L^2}^{d/2s} \big) \\
\leq C \big( \| \phi \|_{H^{s,0}} + \| \mathcal{U}(t)^{-1} \phi \|_{H^{0,s}} \big) \\
= C \| \mathcal{U}(t)^{-1} \phi \|_{\Sigma^s}.
\]

Lemma 2.2. Let \( \gamma \in (0,1] \) and \( s > d/2 + 2\gamma \). There exists a constant \( C \) such that
\[
\| \phi \|_{L^\infty} \leq \frac{1}{t^{d/2}} \| \mathcal{U}(t)^{-1} \phi \|_{L^\infty} + \frac{C}{t^{d/2+\gamma}} \| \mathcal{U}(t)^{-1} \phi \|_{H^{0,s}}
\]
for \( t \geq 1 \).

See Lemma 2.2 in [6] for the proof.

Next we define \( \mathcal{G} : \mathbb{C} \to \mathbb{C} \) with \( p > 1 \) by \( \mathcal{G}(z) = |z|^{p-1}z \) for \( z \in \mathbb{C} \). Note that the nonlinear term in (1.2) is \( \lambda \mathcal{G}_{1+2\theta/d}(u) \) with \( 0 < \theta < 1 \), \( \text{Im} \lambda > 0 \). The following lemmas are concerned with estimates for \( \mathcal{G} \).

Lemma 2.3. For \( z, w \in \mathbb{C} \), we have
\[
|\mathcal{G}(z) - \mathcal{G}(w)| \leq p(|z| + |w|)^{p-1} |z - w|.
\]

Proof. Without loss of generality, we may assume \( |z| > |w| \). For \( \nu > 0 \), we observe the relations
\[
|z|^\nu - |w|^\nu = (|z| - |w|) \int_0^1 \nu (t|z| + (1-t)|w|)^{\nu-1} \, dt
\]
and
\[
\sup_{t \in [0,1]} \left( t|z| + (1-t)|w| \right)^{\nu-1} |w| \leq \left\{ \begin{array}{ll}
( |z| + |w| )^{\nu-1} |w| & \text{(if } \nu \geq 1 \text{)} \\
|w|^\nu & \text{(if } \nu < 1 \text{)}
\end{array} \right\} \leq (|z| + |w|)^\nu.
\]
Then we have
\[
|(|z|^\nu - |w|^\nu)w| \leq ||z| - |w| | \cdot \nu ( |z| + |w| )^\nu \leq \nu ( |z| + |w| )^\nu |z - w|.
\]
We apply the above inequality with \( \nu = p - 1 \) to obtain
\[
|\mathcal{G}(z) - \mathcal{G}(w)| \leq |(|z|^{p-1} - |w|^{p-1})w| + |z|^{p-1} |z - w| \leq p(|z| + |w|)^{p-1} |z - w|.
\]

Lemma 2.4. Let \( 0 \leq s < \min\{2,p\} \). There exists a constant \( C \) such that
\[
\| \mathcal{G}(\phi) \|_{H^{s,0}} \leq C \| \phi \|_{L^\infty}^{p-1} \| \phi \|_{H^{s,0}}
\]
and
\[
\| \mathcal{U}(t)^{-1} \mathcal{G}(\phi) \|_{H^{0,s}} \leq C \| \phi \|_{L^\infty}^{p-1} \| \mathcal{U}(t)^{-1} \phi \|_{H^{0,s}}
\]
for \( t \geq 0 \).
For the proof, see Lemma 3.4 in [3], Lemma 2.3 in [6], etc.

**Corollary 2.5.** Let $d/2 < s < \min\{2, p\}$. There exists a constant $C$ such that

$$\|\mathcal{U}(t)^{-1}G_p(\phi)\|_{\Sigma^s} \leq \frac{C}{(1+t)^{d(p-1)/2}}\|\mathcal{U}(t)^{-1}\phi\|_{\Sigma^s}^p$$

for $t \geq 0$.

**Proof.** By Lemmas 2.4 and 2.1, we have

$$\|\mathcal{U}(t)^{-1}G_p(\phi)\|_{\Sigma^s} = \|G_p(\phi)\|_{H^{s,0}} + \|\mathcal{U}(t)^{-1}G_p(\phi)\|_{H^{0,s}} \leq C\|\phi\|_{L^\infty}^{p-1}(\|\phi\|_{H^{s,0}} + \|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}}) \leq \frac{C}{(1+t)^{d(p-1)/2}}\|\mathcal{U}(t)^{-1}\phi\|_{\Sigma^s}^p.$$  

\[\square\]

**Lemma 2.6.** Let $\gamma \in (0, 1/2]$ and $d/2 + 2\gamma < s < \min\{2, p\}$. Then there exists a constant $C$ such that

$$\left|\mathcal{F}\mathcal{U}(t)^{-1}G_p(\phi) - \frac{1}{t^{d(p-1)/2}}G_p(\mathcal{F}\mathcal{U}(t)^{-1}\phi)\right|_{L^\infty} \leq \frac{C}{t^{d(p-1)/2 + \gamma}}\|\mathcal{U}(t)^{-1}\phi\|_{H^{0,s}}^p$$

for $t \geq 1$.

This lemma can be shown in almost the same way as the derivation of (3.16) and (3.17) in [6] (see also Lemma 2.2 in [9]), so we skip the proof.

### 3 A rough lower estimate for the lifespan

In what follows, we write $N(u) = \lambda|u|^{2d/d}u = \lambda G_{1+2d/d}(u)$ and $\Phi = \|\varphi\|_{\Sigma^s}$, where $s$ satisfies (1.3). The goal of this section is to derive a rough lower estimate for $T_\varepsilon$. The argument of this section is quite standard and any new idea is not needed, so we shall be brief.

**Proposition 3.1.** Let $T_\varepsilon$ be the lifespan defined in the statement of Theorem 1.2. There exists $D_0 > 0$ such that $T_\varepsilon \geq D_0\varepsilon^{-2d/(1-\theta)d}$. Moreover the solution $u$ satisfies

$$\|\mathcal{U}(t)^{-1}u(t)\|_{\Sigma^s} \leq 2\Phi\varepsilon$$

(3.1)

for $t \leq D_0\varepsilon^{-2d/(1-\theta)d}$.

**Proof.** Since the local existence in $\Sigma^s$ is well-known (see e.g., [2] and the references cited therein), what we have to do is to see the solution $u(t)$ stays bounded as long as $t$ is less than the expected value.
Let $T > 0$ and let $u(t)$ be the solution to (1.2) in the time interval $[0, T)$. We set

$$E(T) = \sup_{t \in [0, T)} \|\mathcal{U}(t)^{-1} u(t)\|_{\Sigma^s}.$$ 

Then, it follows from Corollary 2.5 that

$$\|\mathcal{U}(t)^{-1} N(u)\|_{\Sigma^s} \leq C E(T)^{2q/d+1} (1 + t)^\theta$$

for $t < T$. Therefore the standard energy integral method leads to

$$E(T) \leq \|u(0)\|_{\Sigma^s} + C \int_0^T \|\mathcal{U}(t)^{-1} N(u)\|_{\Sigma^s} dt \leq \varepsilon \|\varphi\|_{\Sigma^s} + C E(T)^{2q/d+1} \int_0^T \frac{dt}{(1 + t)^\theta} \leq \Phi \varepsilon + C_* E(T)^{2q/d+1} T^{1-\theta},$$

where the constant $C_*$ is independent of $\varepsilon$ and $T$. With this $C_*$, we choose $D_0 > 0$ so that

$$C_* 3^{1+2q/d} \Phi^{2q/d} D_0^{1-\theta} \leq 1.$$ 

Now we assume $E(T) \leq 3\Phi \varepsilon$. Then the above estimate yields

$$E(T) \leq \Phi \varepsilon + C_* (3\Phi \varepsilon)^{2q/d+1} (D_0 \varepsilon^{-2q/d(1-\theta)})^{1-\theta} \leq 2\Phi \varepsilon$$

if $T \leq D_0 \varepsilon^{-2q/d(1-\theta)}$. This shows that the solution $u(t)$ can exist as long as $t \leq D_0 \varepsilon^{-2q/d(1-\theta)}$. In other words, we have $T_\varepsilon \geq D_0 \varepsilon^{-2q/d(1-\theta)}$. We also have the desired estimate (3.1). \qed

**Remark 3.1.** In the proof of Proposition 3.1, we do not use any information on the sign of $\text{Im} \lambda$. We need something more to clarify the dependence of $T_\varepsilon$ on $\text{Im} \lambda$, that is our main purpose of the present work.

### 4 An ODE Lemma

In this section, we introduce an ODE lemma which will be used effectively in the next section. The argument in this section is a modification of that of §2 in [12] to fit for the present purpose.

Throughout this section, we always suppose $0 < a < 1$, $b > 0$ and $\lambda \in \mathbb{C}$ with $\text{Im} \lambda > 0$. Let $\psi_0 : \mathbb{R}^d \to \mathbb{C}$ be a continuous function satisfying

$$\Psi_0 := \sup_{\xi \in \mathbb{R}^d} |\psi_0(\xi)| < \infty.$$ 

We set $q = \frac{b}{2(1-a)}$ and define $\tau_1 > 0$ by

$$\frac{1}{\tau_1} = (2q \text{ Im} \lambda \Psi_0 b^{1/(1-a)})^{1/(1-\theta)}.$$
For fixed $t_* > 0$, let $\eta_0(t, \xi)$ be the solution to
\begin{equation}
\begin{cases}
dt \eta_0 = \frac{\lambda}{t^a} |\eta_0|^b \eta_0, & t > t_*, \xi \in \mathbb{R}^d, \\
\eta_0(t_*, \xi) = \varepsilon \psi_0(\xi), & \xi \in \mathbb{R}^d,
\end{cases}
\tag{4.1}
\end{equation}
where $\varepsilon > 0$ is a parameter. It is immediate to check that
\begin{equation}
|\eta_0(t, \xi)|^b = \frac{(\varepsilon |\psi_0(\xi)|)^b}{1 + 2q \text{Im} \lambda |\psi_0(\xi)|^b t_1^{1-a} - 2q \text{Im} \lambda |\psi_0(\xi)|^b t_1^{1-a}}
\end{equation}
as long as the denominator is strictly positive. In view of this expression, we see that
\begin{equation}
\sup_{(t, \xi) \in [t_*, \sigma \varepsilon^{-2q} \times \mathbb{R}^d]} |\eta_0(t, \xi)| \leq C_0 \varepsilon
\end{equation}
for $\sigma \in (0, \tau_1)$, where
\begin{equation}
C_0 = \frac{\Psi_0}{(1 - (\sigma / \tau_1)^{1-a})^{1/b}}.
\end{equation}
Next we consider a perturbation of (4.1). Let $T > t_*$ and let $\psi_1 : \mathbb{R}^d \to \mathbb{C}, \rho : [t_*, T] \times \mathbb{R}^d \to \mathbb{C}$ be continuous functions satisfying
\begin{equation}
|\psi_1(\xi)| \leq C_1 \varepsilon^{1+\delta}
\end{equation}
and
\begin{equation}
|\rho(t, \xi)| \leq \frac{C_2 \varepsilon^{1+b+\delta}}{t^a}
\end{equation}
with some positive constants $C_1, C_2$ and $\delta > 0$. Let $\eta(t, \xi)$ be the solution to
\begin{equation}
\begin{cases}
dt \eta = \frac{\lambda}{t^a} |\eta|^b \eta + \rho, & t \in (t_*, T), \xi \in \mathbb{R}^d, \\
\eta(t_*, \xi) = \varepsilon \psi_0(\xi) + \psi_1(\xi), & \xi \in \mathbb{R}^d.
\end{cases}
\end{equation}
The following lemma asserts that an estimate similar to (4.2) remains valid if (4.1) is perturbed by $\rho$ and $\psi_1$:
\begin{lemma}
Let $\sigma \in (0, \tau_1)$ and let $\eta(t, \xi)$ be as above. We set $T_* = \min\{T, \sigma \varepsilon^{-2q}\}$ for $0 < \varepsilon \leq \min\{1, \sigma^{-1/q}, M^{-1/\delta}\}$. We have
\begin{equation}
|\eta(t, \xi)| \leq C_0 \varepsilon + M \varepsilon^{1+\delta} \leq (C_0 + 1) \varepsilon
\end{equation}
for $(t, \xi) \in [t_*, T_*) \times \mathbb{R}^d$, where
\begin{equation}
M = 2 \left( C_1^2 + \frac{C_2^2}{2C_3} \right)^{1/2} \exp \left( \frac{C_3 \sigma^{1-a}}{2(1-a)} \right)
\end{equation}
with
\begin{equation}
C_3 = 2 |\lambda|(b+1)(2C_0 + 1)^b + \frac{1}{2}.
\end{equation}
Proof. We set \( w = \eta - \eta_0 \) and
\[
T_{**} = \sup \{ \tilde{T} \in [t_*, T_*) \mid \sup_{(t, \xi) \in [t_*, \tilde{T}) \times \mathbb{R}^d} |w(t, \xi)| \leq M \varepsilon^{1+\delta} \}.
\]
We observe that
\[
i \partial_t w = \frac{\lambda}{t^a} \left( |\eta_0 + w|^b (\eta_0 + w) - |\eta_0|^b \right) + \rho, \quad w(t_*, \xi) = \psi_1(\xi).
\]
We also note that \( T_{**} > t_* \), because of the estimate
\[
|w(t_*, \xi)| = |\psi_1(\xi)| \leq C_1 \varepsilon^{1+\delta} \leq \frac{M}{2} \varepsilon^{1+\delta}
\]
and the continuity of \( w \). Now we set
\[
f(t, \xi) = |w(t, \xi)|^2 + \frac{C_2^2}{2C_3^2} \varepsilon^{2+2\delta}.
\]
Then it follows from Lemma 2.3 that
\[
\partial_t f(t, \xi) = 2 \text{Im} \left( i \partial_t w \cdot \overline{w} \right)
\leq \frac{2|\lambda|}{t^a \tau^a} \left( 2 |\eta_0| + |w| \right)^b |w|^2 + |\rho||w|
\leq \frac{2|\lambda|(b + 1)}{t^a} \left( 2C_0 \varepsilon + M \varepsilon^{1+\delta} \right)^b |w|^2 + |w| \cdot \frac{C_2 \varepsilon^{1+b+\delta}}{t^a}
\leq \frac{e^b}{t^a} \left\{ \left( C_3 - \frac{1}{2} \right) |w|^2 + |w| \cdot C_2 \varepsilon^{1+\delta} \right\}
\leq \frac{e^b}{t^a} \left( C_3 |w|^2 + \frac{C_2^2}{2} \varepsilon^{2+2\delta} \right)
= \frac{C_3 e^b}{t^a} f(t, \xi)
\]
for \( t \in (t_*, T_{**}) \), as well as
\[
f(t_*, \xi) \leq (C_1 \varepsilon^{1+\delta})^2 + \frac{C_2^2}{2C_3} \varepsilon^{2+2\delta} \leq \left( \frac{C_2^2}{C_1} + \frac{C_2^2}{2C_3} \right) \varepsilon^{2+2\delta}.
\]
These lead to
\[
f(t, \xi) \leq f(t_*, \xi) \exp \left( \int_{t_*}^{\sigma \varepsilon^{-2\theta}} \frac{C_3 e^b}{\tau^a} d\tau \right)
\leq \left( \frac{C_2^2}{C_1} + \frac{C_2^2}{2C_3} \right) \varepsilon^{2+2\delta} \exp \left( \frac{C_3 \sigma^{1-a}}{1-a} \varepsilon^{b-2\theta(1-a)} \right)
\leq \left( \frac{M}{2} \varepsilon^{1+\delta} \right)^2,
\]
whence
\[ |w(t, \xi)| \leq \sqrt{f(t, \xi)} \leq \frac{M}{2} \varepsilon^{1+\delta} \]
for \((t, \xi) \in [t_*, T_*) \times \mathbb{R}^d\). This contradicts the definition of \(T_*\) if \(T_*\) is strictly less than \(T_*\). Therefore we conclude that \(T_* = T_*\). In other words, we have
\[
\sup_{(t, \xi) \in [t_*, T_*) \times \mathbb{R}^d} |w(t, \xi)| \leq \sqrt{f(t, \xi)} \leq M \varepsilon^{1+\delta}.
\]
Going back to the definition of \(w\), we have
\[
|\eta(t, \xi)| \leq |\eta_0(t, \xi)| + |w(t, \xi)| \leq C_0 \varepsilon + M \varepsilon^{1+\delta}
\]
for \((t, \xi) \in [t_*, T_*) \times \mathbb{R}^d\), as desired. \(\square\)

5 Bootstrap argument in the large time

Now we are ready to pursue the behavior of the solution \(u(t)\) of (1.2) for \(t \gtrsim o(\varepsilon^{-2\theta/(1-\theta)})\). For this purpose, we set \(t_* = \varepsilon^{-\theta/(1-\theta)}\), and let \(\varepsilon\) be small enough to satisfy \(\varepsilon^{\theta/(1-\theta)} < D_0\). Then, since \(t_* \leq D_0 \varepsilon^{-2\theta/(1-\theta)}\), Proposition 3.1 gives us \(E(t_*) \leq 2\Phi \varepsilon\). Next we set
\[
\tau_0 := \left( \frac{(1-\theta)d}{2\theta \text{Im} \sup_{\xi \in \mathbb{R}^d} |\hat{\phi}(\xi)|^{2\theta/d}} \right)^{1/(1-\theta)}
\]
and fix \(\sigma \in (0, \tau_0)\), \(T \in (t_*, \sigma \varepsilon^{-2\theta/(1-\theta)})\). Note that the right-hand side in (1.4) is equal to \(\tau_0^{1-\theta}\). For the solution \(u(t)\) in the interval \(t \in [0, T]\), we put
\[
E(T) = \sup_{t \in [0, T]} \|U(t)^{-1}u(t)\|_{\Sigma^*}
\]
as in the proof of Proposition 3.1. The following lemma is the main step toward Theorem 1.2.

**Lemma 5.1.** Let \(\sigma\) and \(T\) be as above. Then there exist constants \(\varepsilon_0 > 0\) and \(K > 4\Phi\), which are independent of \(T\), such that the estimate \(E(T) \leq K \varepsilon\) implies the better estimate \(E(T) \leq K \varepsilon/2\) if \(\varepsilon \in (0, \varepsilon_0]\).

**Proof.** It suffices to consider \(t \in [t_*, T)\), because we already know that \(E(t_*) \leq 2\Phi \varepsilon\). For \(t \in [t_*, T)\), we set \(A(t, \xi) = \mathcal{F}[U(t)^{-1}u(t, \cdot)](\xi)\) and
\[
R(t, \xi) = \mathcal{F}[U(t)^{-1}N(u(t, \cdot))](\xi) - t^{-\theta}N(A(t, \xi))
\]
so that
\[
i \partial_t A = \mathcal{F}U(t)^{-1} \mathcal{L}u = \mathcal{F}U(t)^{-1} N(u) = \frac{\lambda}{v^\theta} |A|^{2\theta/d} A + R. \tag{5.1}
\]
Next we take $\gamma = (2s - d)/8 \in (0, 1/2]$. Note that $s - d/2 = 4\gamma > 2\gamma$. Since $R$ can be written as

$$R(t, \xi) = \lambda \left( F^\perp(t)^{-1} G_{1+2\theta/d}(u) - t^{-\theta} G_{1+2\theta/d}(F^\perp(t)^{-1} u) \right),$$

Lemma 2.6 yields

$$|R(t, \xi)| \leq \frac{C}{t^{\theta+\gamma}} E(T)^{2\theta/d+1} \leq \frac{C\varepsilon^{1+2\theta/d}}{t^\theta} K^{1+2\theta/d} t_s^{\gamma} \leq \frac{C\varepsilon^{1+2\theta/d+\gamma\theta/2d(1-\theta)}}{t^\theta}$$

if $E(T) \leq K\varepsilon$ and $K^{1+2\theta/d\varepsilon^{\theta/2d(1-\theta)}} \leq 1$. Moreover, when we put $\psi(\xi) = A(t_*, \xi) - \varepsilon \hat{\phi}(\xi)$, we have

$$|\psi(\xi)| \leq C\|U(t_*)^{-1} u(t_*, \cdot) - \varepsilon \varphi\|_{H^{0,s}}$$

$$\leq C \int_0^{t_*} \|U(t)^{-1} N(u(t))\|_{H^{0,s}} dt$$

$$\leq C \int_0^{e^{-\theta/(1-\theta)d}} (2\Phi \varepsilon)^{1+2\theta/d} (1+t)^\theta dt$$

$$\leq C\varepsilon^{1+2\theta/d} \int_0^{e^{-\theta/(1-\theta)d}} \frac{dt}{(1+t)^\theta}$$

$$\leq C\varepsilon^{1+\theta/d},$$

where we have used Lemma 2.4, Lemma 2.1 and Proposition 3.1. Therefore we can apply Lemma 4.1 with $\gamma = A$, $a = \theta$, $b = 2\theta/d$, $\delta = \min\{\theta/d, \gamma\theta/2d(1-\theta)\}$, $v_0 = \hat{\phi}$ and $\rho = R$ to obtain

$$|A(t, \xi)| \leq (C_0 + 1)\varepsilon$$

for $(t, \xi) \in [t_*, T] \times \mathbb{R}^d$, where

$$C_0 = \frac{\sup_{\xi \in \mathbb{R}^d} |\hat{\phi}(\xi)|}{(1 - (\sigma/\tau_0)^{1-\theta})^d 2^d}.$$

Note that $C_0$ is independent of $\varepsilon$, $K$ and $T$. By this estimate and Lemma 2.2, we have

$$\|u(t)\|_{L^\infty} \leq t^{-d/2} \|A(t_*, \cdot)\|_{L^\infty} + C t^{-d/2 - \gamma} \|U(t)^{-1} u(t)\|_{\Sigma^*}$$

$$\leq t^{-d/2} \left( (C_0 + 1)\varepsilon + CK\varepsilon t_s^{\gamma} \right)$$

$$\leq t^{-d/2} \left( C\varepsilon + CK\varepsilon^{1+\gamma\theta/d(1-\theta)} \right)$$

$$\leq C\varepsilon t^{-d/2},$$

if $K\varepsilon^{\gamma\theta/d(1-\theta)} \leq 1$. By the standard energy inequality combined with Lemma 2.3, we obtain

$$\sup_{t_* \leq t < T} \|U(t)^{-1} u(t)\|_{\Sigma^*} \leq \|U(t_*)^{-1} u(t_*)\|_{\Sigma^*} \exp \left( \int_{t_*}^T C\|u(t)\|_{L^\infty}^{2\theta/d} dt \right)$$

$$\leq 2\Phi \varepsilon \exp \left( C\varepsilon^{2\theta/d} \int_0^{\sigma_0 - \varepsilon^{2\theta/d(1-\theta)}} \frac{dt}{t^\theta} \right)$$

$$\leq (2\Phi C_0)\varepsilon.$$
for \( t \in [t_*, T) \), where the constant \( C_* \) is independent of \( \varepsilon, K \) and \( T \). Now we set \( K = 4\Phi e^{C_*} \).

Then we arrive at the desired estimate \( E(T) \leq K\varepsilon/2 \).

**Proof of Theorem 1.2.** Let \( T_* \) be the lifespan defined in the statement of Theorem 1.2. We fix \( \sigma \in (0, \tau_0) \) and set

\[
T^* = \sup \{ t \in [0, T_*) \mid E(t) \leq K\varepsilon \},
\]

where \( K \) is given in Lemma 5.1. Now we assume \( T^* \leq \sigma\varepsilon^{-2\theta/d(1-\theta)} \). Then, Lemma 5.1 with \( T = T^* \) implies \( E(T^*) \leq K\varepsilon/2 \) if \( \varepsilon \leq \varepsilon_0 \). By the continuity of \([0, T_\varepsilon) \ni T \mapsto E(T)\), we can choose \( \tilde{\delta} > 0 \) such that \( E(T^* + \tilde{\delta}) \leq K\varepsilon \), which contradicts the definition of \( T^* \). Therefore we must have \( T^* \geq \sigma\varepsilon^{-2\theta/d(1-\theta)} \) if \( \varepsilon \leq \varepsilon_0 \). As a consequence, we obtain

\[
\lim_{\varepsilon \to +0} \varepsilon^{2\theta/d}T^* \geq \sigma^{1-\theta}.
\]

Since \( \sigma \in (0, \tau_0) \) is arbitrary, we arrive at the desired estimate \( (1.4) \). \( \square \)

## 6 The critical case

We conclude this paper with a few comments on the critical case \( \theta = 1 \), that is,

\[
\begin{aligned}
&i\partial_t u + \frac{1}{2} \Delta u = \lambda |u|^{2/d} u, \quad t > 0, \ x \in \mathbb{R}^d, \\
u(0, x) = \varepsilon \varphi(x), \quad x \in \mathbb{R}^d, 
\end{aligned}
\]

with \( \text{Im} \lambda > 0 \). As mentioned in the introduction, one dimensional case \((d = 1)\) has been covered in the previous works \([16], [12]\). Minor modifications of the method in the previous sections allow us to treat the case of \( d = 2, 3 \).

**Theorem 6.1.** Let \( 1 \leq d \leq 3 \) and \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda > 0 \). Assume \( \varphi \in \Sigma^s \) with \( s \) satisfying \((1.3)\). Let \( T_\varepsilon \) be the supremum of \( T > 0 \) such that \((6.1)\) admits a unique solution \( u \) satisfying \( U(\cdot)^{-1}u \in C([0, T); \Sigma^s) \). Then we have

\[
\lim_{\varepsilon \to +0} \left( \varepsilon^{2/d} \log T_\varepsilon \right) \geq \frac{d}{2 \text{Im} \lambda} \sup_{\xi \in \mathbb{R}^d} |\hat{\varphi}(\xi)|^{2/d}.
\]

Since the proof is almost the same as that for Theorem 1.2, we only point out where this lower bound comes from. As in Section 5, we can see that \( A(t, \xi) = \mathcal{F}[U(t)^{-1}u(t, \cdot)](\xi) \) satisfies

\[
i\partial_t A = \lambda t |A|^{2/d} A + R
\]

with \( A(1, \xi) = \varepsilon \hat{\varphi}(\xi) + \psi(\xi) \), where \( R \) and \( \psi \) are regarded as remainder terms. If \( R \) and \( \psi \) could be neglected, then we would have

\[
|A(t, \xi)|^{2/d} = \frac{(\varepsilon |\hat{\varphi}(\xi)|)^{2/d}}{1 - (2/d) \text{Im} \lambda |\hat{\varphi}(\xi)|^{2/d} \log t}.
\]

The desired lower bound is characterized by the time when this denominator vanishes.
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