SIGMOIDAL APPROXIMATIONS OF A DELAY NEURAL LATTICE MODEL WITH HEAVISIDE FUNCTIONS

Dedicated to Professor Tomás Caraballo on occasion of his 60th Birthday

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Abstract. The approximation of Heaviside coefficient functions in delay neural lattice models with delays by sigmoidal functions is investigated. The solutions of the delay sigmoidal models are shown to converge to a solution of the delay differential inclusion as the sigmoidal parameter goes to zero. In addition, the existence of global attractors is established and compared for the various systems.

1. Introduction. Han & Kloeden [7] introduced and investigated the following infinite dimensional lattice version of the Amari neural field model [1]:

\[
\frac{d}{dt} u_i(t) = f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{ij} H(u_j(t) - \theta) + g_i(t), \quad i \in \mathbb{Z}^d,
\]

where \( \theta > 0 \) is a given threshold and \( H : \mathbb{R} \to \mathbb{R} \) is the Heaviside function defined by

\[
H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad x \in \mathbb{R}.
\]

To avoid difficulties with the Heaviside function, in [8] they replaced the Heaviside function with a sigmoidal function characterised by a small positive parameter \( \varepsilon \) such as

\[
\sigma_\varepsilon(x) = \frac{1}{1 + e^{-x/\varepsilon}}, \quad x \in \mathbb{R}, \quad 0 < \varepsilon < 1,
\]

and showed that the solutions of the corresponding sigmoidal lattice system converge to a solution of the lattice inclusion system with the Heaviside functions when parameter \( \varepsilon \to 0 \).

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Delays are often included in neural field models to account for the transmission time of signals between neurons. Systems with delay has been investigated extensively, e.g., [4, 14]. Here we are interested in autonomous versions of the above neural lattice systems considered by Han & Kloeden with the inclusion of delays:

$$\frac{d}{dt} u_i(t) = f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j} H(u_j(t - \tau_{i,j}) - \theta) + g_i, \quad i \in \mathbb{Z}^d.$$  \hspace{1cm} (2)

and the corresponding system with a sigmoidal function instead of the Heaviside function, i.e.,

$$\frac{d}{dt} u_i(t) = f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_{\epsilon}(u_j(t - \tau_{i,j}) - \theta) + g_i, \quad i \in \mathbb{Z}^d.$$  \hspace{1cm} (3)

Wang, Kloeden & Yang [14] established the existence and uniqueness of solutions of the delay sigmoidal lattice system (3) as well as the existence of a global attractor. The main goal of this paper is to show that the solutions and the attractors of the sigmoidal lattice system (3) converge to a solution and attractor of the lattice system with Heaviside function (2) when \( \epsilon \to 0 \). This also establishes the existence of a solution of the system with Heaviside function (2) for the same initial value.

The paper is organized as follows. In section 2 some necessary preliminaries and assumptions are provided. Results from [14] for the sigmoidal lattice system (3) are recalled in section 3. The Heaviside lattice system (2) is formulated as differential inclusion with delay (7) and the inflated differential inclusion (10) is established in section 4. Then in section 5, after first giving some important Lemmas, the convergence of a subsequence of solutions of the delay sigmoidal solutions (3) to a solution of the differential inclusion (6) as \( \epsilon \to 0 \) is shown. Finally, in section 6, the existence of global attractors is established for the inflated and sigmoidal systems and these attractors are compared with that of the attractor for the Heaviside system.

2. Assumptions. We follow Han & Kloeden [7, 8] and consider a weighted space of bi-infinite real valued sequences with vectorial indices \( i = (i_1, \ldots, i_d) \in \mathbb{Z}^d \). In particular, given a positive sequence of weights \( (\rho_i)_{i \in \mathbb{Z}^d} \), we consider the separable Hilbert space

$$\ell^2_{\rho} := \left\{ u = (u_i)_{i \in \mathbb{Z}^d} : \sum_{i \in \mathbb{Z}^d} \rho_i u_i^2 < \infty \right\}$$

with the inner product

$$\langle u, v \rangle := \sum_{i \in \mathbb{Z}^d} \rho_i u_i v_i \quad \text{for} \quad u = (u_i)_{i \in \mathbb{Z}^d}, v = (v_i)_{i \in \mathbb{Z}^d} \in \ell^2_{\rho}$$

and norm

$$\| u \|_{\rho} := \sqrt{\sum_{i \in \mathbb{Z}^d} \rho_i u_i^2}.$$

We assume that the delays \( \tau_{i,j} \) are uniformly bounded, i.e., satisfy

**Assumption 1.** There exists a constant \( h \in (0, \infty) \) that \( 0 \leq \tau_{i,j} \leq h \) for all \( i, j \in \mathbb{Z}^d \) and that the weights \( \rho_i \) satisfy

**Assumption 2.** \( \rho_i > 0 \) for all \( i \in \mathbb{Z}^d \) and \( \rho_{\Sigma} := \sum_{i \in \mathbb{Z}^d} \rho_i < \infty \).

The components of the interconnection matrix \( (k_{i,j})_{i,j \in \mathbb{Z}^d} \) are assumed to satisfy
Assumption 3. There exists a constant $\kappa > 0$ such that $\sum_{j \in \mathbb{Z}^d} \frac{k_{i,j}}{\rho_j} \leq \kappa$ for each $i \in \mathbb{Z}^d$.

In addition, the functions $f_i$ in the interconnected terms are assumed to satisfy

Assumption 4. The functions $f_i : \mathbb{R} \to \mathbb{R}$ are continuously differentiable with equi-locally bounded derivatives, i.e., there exists a non-decreasing function $L(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\max_{s \in [-r, r]} |f'_i(s)| \leq L(\rho r), \quad \forall r \in \mathbb{R}^+, i \in \mathbb{Z}^d;$$

Assumption 5. $f_i(0) = 0$ for all $i \in \mathbb{Z}^d$.

Assumption 6. There exist constants $\alpha > 0$ and $\beta_i$ with $\beta = (\beta_i)_{i \in \mathbb{Z}^d} \in \ell_2^2$ satisfying

$$sf_i(s) \leq -\alpha|s|^2 + \beta^2, \quad \forall s \in \mathbb{R}, \quad \forall i \in \mathbb{Z}^d.$$

Finally, we suppose that the constant forcing terms $g := (g_i)_{i \in \mathbb{Z}^d}$ satisfy

Assumption 7. $g \in \ell_2^2$.

2.1. Useful consequences. Assumption 4 implies that $f_i$ is locally Lipschitz with

$$|f_i(x) - f_i(y)| \leq L(\rho_i(|x| + |y|)) \cdot |x - y|, \quad \forall i \in \mathbb{Z}^d, x, y \in \mathbb{R}.$$

By Assumptions 4 – 6 and Lemma 1 in [7] the mapping $f : \ell_2^2 \to \ell_2^2$ defined by

$$f(u) = (f_i(u))_{i \in \mathbb{Z}^d}, \quad u = (u_i)_{i \in \mathbb{Z}^d} \in \ell_2^2,$$

is locally Lipschitz and satisfies the dissipativity condition

$$\langle f(u), u \rangle \leq -\alpha\|u\|_\rho^2 + \|\beta\|_\rho^2.$$

It also follows from Assumptions 2 and 3 that

$$\sum_{j \in \mathbb{Z}^d} |k_{i,j}| \leq \sum_{j \in \mathbb{Z}^d} \frac{|k_{i,j}|}{\rho_j} \sqrt{\rho_j} \leq \left( \sum_{j \in \mathbb{Z}^d} \frac{k_{i,j}^2}{\rho_j} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}^d} \rho_j \right)^{1/2} \leq \sqrt{\kappa \rho \Sigma}.$$ (4)

3. The delay sigmoidal lattice system. The appropriate function space for the solutions of the lattice systems with delays (2) and (3) is the Banach space $C([-h, 0], \ell_2^2)$ of continuous functions $v : [-h, 0] \to \ell_2^2$ with the norm

$$\|v\|_{C([-h, 0], \ell_2^2)} = \max_{s \in [-h, 0]} \|v(s)\|_\rho.$$

For a continuous function $u : [-h, \infty) \to \ell_2^2$ denote by $u_t$ for $t \geq 0$ the segment of the solution in $C([-h, 0], \ell_2^2)$ defined by $u_t(s) = u(t + s)$ for each $s \in [-h, 0]$.

In this section we recall the result of [14] on the existence and uniqueness of the solution of the delay sigmoidal lattice system (3). To this end, we first define the mapping $\Sigma^\tau : C([-h, 0], \ell_2^2) \to \mathbb{R}$ for $i \in \mathbb{Z}^d$ by

$$\Sigma^\tau_i(u) := \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_{\varepsilon}(u_j(-\tau_{i,j} - \theta), \quad u \in C([-h, 0], \ell_2^2).$$

and then we define the mapping $\Sigma^\varepsilon : C([-h, 0], \ell_2^2) \to \ell_2^2$ by $\Sigma^\varepsilon(u) := (\Sigma^\tau_i(u))_{i \in \mathbb{Z}^d}$. Then we can write the delay sigmoidal lattice system (3) as the following differential delay equation on $\ell_2^2$:

$$\frac{du(t)}{dt} = g^\varepsilon(u_t) := f(u(t)) + \Sigma^\varepsilon(u_t) + g.$$ (5)
The coefficient functions satisfy the following properties.

**Lemma 3.1.** The function $\Sigma^\varepsilon(u)$ is uniformly bounded with

$$\|\Sigma^\varepsilon(u)\|_\rho \leq \sqrt{\kappa \rho^\Sigma}.$$

**Proof.** For each $i \in \mathbb{Z}^d$, by inequality (4),

$$|\Sigma^\varepsilon_i(u)| \leq \sum_{j \in \mathbb{Z}^d} |k_{i,j}||\sigma_\varepsilon(u_j(-\tau_{ij}) - \theta)| \leq \sum_{j \in \mathbb{Z}^d} |k_{i,j}| \leq \sqrt{\kappa \rho^\Sigma}$$

Hence

$$\|\Sigma^\varepsilon(u)\|_\rho^2 \leq \sum_{i \in \mathbb{Z}^d} \rho_i \|\Sigma^\varepsilon_i(u)\|^2 \leq \sum_{i \in \mathbb{Z}^d} \rho_i \kappa \rho^\Sigma \leq \kappa \rho^\Sigma^2.$$ 

\[\square\]

**Lemma 3.2.** The function $\Sigma^\varepsilon$ is globally Lipschitz with the Lipschitz constant $\frac{\sqrt{\kappa \rho^\Sigma}}{\varepsilon}$, i.e.,

$$\|\Sigma^\varepsilon(u) - \Sigma^\varepsilon(v)\|_\rho \leq \frac{\sqrt{\kappa \rho^\Sigma}}{\varepsilon} \|u - v\|_{C([-h,0],\ell^2)}$$

**Proof.** First notice that the sigmoidal function $\sigma_\varepsilon$ is differentiable with a uniformly bounded derivative

$$\frac{d}{dx}\sigma_\varepsilon(x) \leq \frac{1}{\varepsilon} \quad \text{for all } x \in \mathbb{R}.$$ 

Hence, for each $i \in \mathbb{Z}^d$, the function $\Sigma^\varepsilon_i$ is globally Lipschitz with the Lipschitz constant $\frac{\sqrt{\kappa \rho^\Sigma}}{\varepsilon}$:

$$|\Sigma^\varepsilon_i(u) - \Sigma^\varepsilon_i(v)| \leq \sum_{j \in \mathbb{Z}^d} k_{i,j} |\sigma_\varepsilon(u_j(-\tau_{ij}) - \theta) - \sigma_\varepsilon(v_j(-\tau_{ij}) - \theta)|$$

$$\leq \frac{1}{\varepsilon} \sum_{j \in \mathbb{Z}^d} k_{i,j} |u_j(-\tau_{ij}) - v_j(-\tau_{ij})|$$

$$\leq \frac{1}{\varepsilon} \sum_{j \in \mathbb{Z}^d} \sqrt{k_{i,j}} \sqrt{\rho_j} |u_j(-\tau_{ij}) - v_j(-\tau_{ij})|$$

$$\leq \frac{1}{\varepsilon} \left( \sum_{j \in \mathbb{Z}^d} \frac{k^2_{i,j}}{\rho_j} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}^d} \rho_j |u_j(-\tau_{ij}) - v_j(-\tau_{ij})|^2 \right)^{1/2}$$

$$\leq \frac{\sqrt{\kappa}}{\varepsilon} \max_{-h \leq s \leq 0} \|u(s) - v(s)\|_\rho.$$ 

Hence

$$\sum_{i \in \mathbb{Z}^d} \rho_i |\Sigma^\varepsilon_i(u) - \Sigma^\varepsilon_i(v)|^2 \leq \frac{\kappa}{\varepsilon^2} \max_{-h \leq s \leq 0} \|u(s) - v(s)\|_\rho^2 \sum_{i \in \mathbb{Z}^d} \rho_i$$

$$\leq \frac{\kappa \rho^\Sigma}{\varepsilon^2} \max_{-h \leq s \leq 0} \|u(s) - v(s)\|_\rho^2.$$ 

\[\square\]

It thus follows that the right hand side coefficient $g^\varepsilon$ is locally Lipschitz. It will be shown in the proof of Theorem 5.5 that it also satisfies the dissipativity condition. The following results were proved in Wang, Kloeden & Yang [14].
Theorem 3.3. Suppose that Assumptions 1–7 hold. Then the sigmoidal lattice delay equation (5) has a unique solution \( u_t \in C([\tau, T], \ell^2_\rho) \), \( t \geq 0 \), for every initial value \( \psi \in C([\tau, T], \ell^2_\rho) \).

Moreover, these solutions generate a semi-dynamical system \( \phi^\varepsilon \) on \( C([\tau, T], \ell^2_\rho) \), which has a global attractor \( \mathcal{A}^\varepsilon \) in \( C([\tau, T], \ell^2_\rho) \).

4. Formulation of the Heaviside system as a lattice inclusion. The lattice system (2) with the Heaviside functions needs to be reformulated mathematically as an inclusion system on the state space \( C([-h, 0], \ell^2_\rho) \). This requires the set-valued mapping \( \chi \) on \( \mathbb{R} \) by

\[
\chi(s) = \begin{cases} 
0, & s < 0, \\
[0, 1], & s = 0, \quad s \in \mathbb{R}, \\
1, & s > 0,
\end{cases}
\]

We define the set-valued operator \( K_i : C([-h, 0], \ell^2_\rho) \to 2^\mathbb{R} \) for each \( i \in \mathbb{Z}^d \) and \( u_t \in C([-h, 0], \ell^2_\rho) \) by

\[
K_i(u_t) = \sum_{j \in \mathbb{Z}^d} k_{i,j} \chi(u_j(t-\tau_{i,j}) - \theta) = \sum_{j \in \mathbb{Z}^d} k_{i,j} \begin{cases} 
0, & u_j(t-\tau_{i,j}) < \theta, \\
[0, 1], & u_j(t-\tau_{i,j}) = \theta, \quad i \in \mathbb{Z}^d, \\
1, & u_j(t-\tau_{i,j}) > \theta,
\end{cases}
\]

Then the delay lattice differential equation system (2) can be reformulated as the lattice differential delay inclusion system:

\[
\frac{d u_i}{dt} \in G_i(u_t) := f_i(u_t) + K_i(u_t) + g_i, \quad i \in \mathbb{Z}^d. \tag{6}
\]

Now define the set-valued operator \( K \) on \( C([-h, 0], \ell^2_\rho) \) by:

\[
K(u_t) := (K_i(u_t))_{i \in \mathbb{Z}^d}, \quad \forall \ u_t \in C([-h, 0], \ell^2_\rho).
\]

Similarly to the proof of Lemma 3.1 it can be shown that \( K \) takes values in \( \ell^2_\rho \) and is uniformly bounded. The system (6) can then be reformulated as a differential delay inclusion

\[
\frac{d u(t)}{dt} \in G(u_t) := f(u(t)) + K(u_t) + g \tag{7}
\]

on the space \( C([-h, 0], \ell^2_\rho) \).

A solution to the differential delay inclusion (7) is defined component-wise as

**Definition 4.1.** A function \( u = (u_i)_{i \in \mathbb{Z}^d} : [-h, T] \to \ell^2_\rho \) is called a solution to the differential inclusion (7) if it is an absolutely continuous function \( u : [-h, T] \to \ell^2_\rho \) such that

\[
\dot{u}_i(t) \in f_i(u_i(t)) + K_i(u_t) + g_i, \quad \forall \ i \in \mathbb{Z}^d, \ \text{a.e.}
\]

**Theorem 4.2.** Let \( T > 0 \) and suppose that Assumptions 1–7 hold. Then for any initial data \( \psi \in C([-h, 0], \ell^2_\rho) \), the differential delay inclusion (7) admits a solution \( u_T(\psi) \in C([-h, 0], \ell^2_\rho) \) for \([0, T]\) with \( u_0(\psi) = \psi \).

We will prove this theorem by showing that a sequence of solutions to the sigmoidal system has a convergent subsequence and its limit is a solution of (7). For this we will use the inflation of the differential delay inclusion (7).
4.1. **Inflated inclusion systems.** Inflated systems are used to compare perturbed or approximated systems with the original system, see Kloeden & Kozyakin [11, 12]. In particular, we will see that the solutions of the delay sigmoidal system (3) are solutions of an inflated system based on the differential inclusion (6). We inflate the set-valued mapping \( \chi \) to give a new set-valued function \( \chi_{\varepsilon} \) defined on \( \mathbb{R} \) by

\[
\chi_{\varepsilon}(s) = \begin{cases} 
[0, \varepsilon], & s < -b(\varepsilon), \\
[0, 1], & -b(\varepsilon) \leq s \leq b(\varepsilon), \\
[1 - \varepsilon, 1], & s > b(\varepsilon),
\end{cases}
\]

where \( b(\varepsilon) \) solves the algebraic equation

\[
\sigma(-b(\varepsilon)) = \varepsilon \quad \text{and} \quad \sigma(b(\varepsilon)) = 1 - \varepsilon.
\]

It is straightforward to check that

\[
\frac{b(\varepsilon)}{\varepsilon} = \ln \frac{1 - \varepsilon}{\varepsilon} \to \infty \quad \text{and} \quad b(\varepsilon) = \varepsilon \ln \frac{1 - \varepsilon}{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Then we inflate the lattice inclusion (6) to obtain the inflated lattice inclusion

\[
\frac{d\mathbf{u}_i(t)}{dt} \in f_i(\mathbf{u}_i) + K_i^\varepsilon(\mathbf{u}_i) + g_i, \quad i \in \mathbb{Z}^d, \quad (8)
\]

where

\[
K_i^\varepsilon(\mathbf{u}_i) := \sum_{j \in \mathbb{Z}^d} k_{i,j} \chi_{\varepsilon}(u_j(t - \tau_{i,j}) - \theta). \quad (9)
\]

Now define the set-valued mapping \( K_i^\varepsilon(\mathbf{u}_i) = (K_i^\varepsilon(\mathbf{u}_i))_{i \in \mathbb{Z}^d} \) where \( K_i^\varepsilon(\mathbf{u}_i) \) is defined in (9). Then \( K_i^\varepsilon(\mathbf{u}_i) \) takes values in \( \ell_\rho^2 \), and the inflated delay inclusion can be written as the following differential delay inclusion on space \( \ell_\rho^2 \):

\[
\frac{d\mathbf{u}(t)}{dt} \in f(\mathbf{u}(t)) + K^\varepsilon(\mathbf{u}_t) + \mathbf{g}. \quad (10)
\]

It is clear that for any \( \mathbf{u}_t = (\mathbf{u}_{i,t})_{i \in \mathbb{Z}^d} \in C([-h, 0], \ell_\rho^2) \), we have

\[
\Sigma_i^\varepsilon(\mathbf{u}_i) = \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_{\varepsilon}(u_j(t - \tau_{i,j}) - \theta) \in K_i^\varepsilon(\mathbf{u}_i).
\]

Thus the solutions of the delay sigmoidal system (3) are solutions of the inflated inclusion system (10).

5. **Convergence of the delay sigmoidal solutions.** Our goal in this section is to show the convergence of a subsequence of solutions of the delay sigmoidal solutions (3) to a solution of the differential inclusion (6) as \( \varepsilon_n \to 0 \). We first give some important lemmas.

5.1. **Some lemmas.** Let \( \mathbb{Z}_N^d := \{i = (i_1, \ldots, i_d) \in \mathbb{Z}^d : |i_1|, \ldots, |i_d| \leq N \} \). For each \( N \in \mathbb{N} \) and \( i \in \mathbb{Z}^d \). We introduce the truncated operator

\[
K_i^{\varepsilon,N}(\mathbf{u}^N_i) = \sum_{j \in \mathbb{Z}_N^d} k_{i,j} \chi_{\varepsilon}(u_j(t - \tau_{i,j}) - \theta), \quad i \in \mathbb{Z}_N^d, \quad (11)
\]

where \( \mathbf{u}_i^N = (u_{i,h})_{h \in \mathbb{Z}_N^d} \) and define

\[
\mathbf{e}_i^{\varepsilon,N}(\mathbf{u}_i) := \sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}_N^d} k_{i,j} \chi_{\varepsilon}(u_j(t - \tau_{i,j}) - \theta), \quad \mathbf{u}_t = (u_{t,i})_{i \in \mathbb{Z}^d} \in C([-h, 0], \ell_\rho^2), \varepsilon \in [0, 1].
\]
Lemma 5.1. For every fixed $N \in \mathbb{N}$, $i \in \mathbb{Z}^d$ and $\epsilon \in [0, 1]$, the set-valued mapping $u_t \mapsto K_i^\epsilon N(u_t)$ is upper semi continuous from $C([-h, 0]; l_\rho^2)$ into the space of nonempty compact convex subsets of $\mathbb{R}^1$, i.e.,

$$\text{dist}_{\mathbb{R}^1}(K_i^\epsilon N(u_t^m), K_i^\epsilon N(u_t^*) \to 0 \text{ for } u_t^m \to u_t^* \text{ in } C([-h, 0]; l_\rho^2), \text{ as } m \to \infty.}$$

Proof. The proof uses the inequality

$$\text{dist}_{\mathbb{R}^d}(A_1 + A_2, B_1 + B_2) \leq \text{dist}_{\mathbb{R}^d}(A_1, A_2) + \text{dist}_{\mathbb{R}^d}(B_1, B_2) \quad (12)$$

for any nonempty compact subsets $A_1, A_2, B_1, B_2$ of $\mathbb{R}^d$, where $A + B := \{a + b : a \in A, b \in B\}$. Hence $u_t \mapsto K_i^\epsilon N(u_t)$ is upper semicontinuous in $u_t \in C([-h, 0]; l_\rho^2)$ since it is the finite sum of the upper semi continuous function $\chi_\epsilon$. □

Lemma 5.2. For every $i \in \mathbb{Z}^d$ and $\epsilon \in [0, 1]$, the set-valued mapping $u_t \mapsto K_i^\epsilon(u_t)$ is upper semi continuous from $C([-h, 0]; l_\rho^2)$ into the nonempty compact convex subsets of $\mathbb{R}^1$, i.e.,

$$\text{dist}_{\mathbb{R}^1}(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*)) \to 0 \text{ for } u_t^m \to u_t^* \text{ in } C([-h, 0]; l_\rho^2) \text{ as } m \to \infty.}$$

Proof. Let $u_t^m \to u_t^*$ as $m \to \infty$ in $C([-h, 0]; l_\rho^2)$. Then for each $i \in \mathbb{Z}^d$ by repeatedly using the inequality (12) we obtain

$$\text{dist}_{\mathbb{R}^1}(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*)) = \text{dist}_{\mathbb{R}^1}
\left(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*) + K_i^\epsilon(u_t^*)\right)$$

$$\leq \text{dist}_{\mathbb{R}^1}
\left(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*)\right) + \text{dist}_{\mathbb{R}^1}
\left(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*)\right) + \text{dist}_{\mathbb{R}^1}(\{0\}, K_i^\epsilon(u_t^*)) + \text{dist}_{\mathbb{R}^1}(\{0\}, K_i^\epsilon(u_t^*))$$

Since the set valued mapping $\chi_\epsilon$ is uniformly bounded by 1, we have

$$\text{dist}_{\mathbb{R}^1}(\{0\}, K_i^\epsilon(u_t^*)) \leq \sum_{j \in \mathbb{Z}^d \setminus Z_N} |k_{i,j}| \text{ and } \text{dist}_{\mathbb{R}^1}(\{0\}, K_i^\epsilon(u_t^*)) \leq \sum_{j \in \mathbb{Z}^d \setminus Z_N} |k_{i,j}|.$$

Thus

$$\text{dist}_{\mathbb{R}^1}(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*)) \leq \text{dist}_{\mathbb{R}^1}(K_i^\epsilon(u_t^m), K_i^\epsilon(u_t^*)) + 2 \sum_{j \in \mathbb{Z}^d \setminus Z_N} |k_{i,j}|.$$

Recall from inequality (4) that $\sum_{j \in \mathbb{Z}^d} |k_{i,j}| \leq \sqrt{N_2}$. Hence for any $\epsilon > 0$ there exists $N(\gamma) \in \mathbb{N}$ such that

$$\sum_{j \in \mathbb{Z}^d \setminus Z_N} |k_{i,j}| \leq \frac{\gamma}{4}, \quad \forall \; N \geq N(\gamma), \; i \in \mathbb{Z}^d. \quad (13)$$

The result then follows. □

Lemma 5.3. For every $u_t \in C([-h, 0]; l_\rho^2)$ and $i \in \mathbb{Z}^d$

$$\lim_{\epsilon \to 0} \text{dist}_{\mathbb{R}^1}(K_i^\epsilon(u_t), K_i(u_t)) = 0.$$
Proof. Fix \( u_t \in C([-h,0]; \mathbb{L}_p^2) \). Then for each \( i \in \mathbb{Z}^d \) and \( N \in \mathbb{N} \), similarly to the proof of Lemma 5.2, we obtain

\[
\text{dist}_{\mathbb{R}^1}\left( \mathcal{K}_i^\varepsilon(u_t), \mathcal{K}_i(u_t) \right) = \text{dist}_{\mathbb{R}^1}\left( K_i^{\varepsilon,N}(u_t), \mathcal{K}_i^N(u_t) \right)
\]

By inequality (4), for any \( \delta > 0 \) there exists \( N(\delta) \in \mathbb{N} \) such that

\[
\sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}_N^d} |k_{i,j}| \leq \frac{\delta}{2}. \tag{14}
\]

Next, we estimate \( \text{dist}_{\mathbb{R}^1}\left( \mathcal{K}_i^\varepsilon(u_t), \mathcal{K}_i(u_t) \right) \) and it follows the conclusion below.

**Lemma 5.4.** For any \( \delta > 0 \), there exists \( \delta' > 0 \), when \( 0 < \varepsilon < \delta' \), we can obtain

\[
\text{dist}_{\mathbb{R}^1}\left( \mathcal{K}_i^\varepsilon(u_t), \mathcal{K}_i(u_t) \right) < \frac{1}{\sqrt{\kappa_{\rho_2}}} \delta.
\]

Proof. When \( u_t(t-\tau_{i,j}) - \theta > 0 \), since \( b(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), there exists \( \varepsilon_1(u_t, N(\delta)) \) such that \( \min\{u_t(t-\tau_{i,j}) - \theta\} > b(\varepsilon) \) when \( \varepsilon \leq \varepsilon_1 \) for each \( j \in \mathbb{Z}_N^d \). So when \( \varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\} \), we have

\[
\text{dist}_{\mathbb{R}^1}\left( \mathcal{K}_i^\varepsilon(u_t), \mathcal{K}_i(u_t) \right) < \varepsilon.
\]

Hence, for any \( \delta > 0 \), there exists \( \delta' = \min\{\frac{1}{\sqrt{\kappa_{\rho_2}}} \delta, \varepsilon_1, \varepsilon_2\} > 0 \), when \( 0 < \varepsilon < \delta' \), we have

\[
\text{dist}_{\mathbb{R}^1}\left( \mathcal{K}_i^\varepsilon(u_t), \mathcal{K}_i(u_t) \right) \leq \text{dist}_{\mathbb{R}^1}\left( K_i^{\varepsilon,N}(u_t), \mathcal{K}_i^N(u_t) \right) + 2 \sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}_N^d} |k_{i,j}|
\]

\[
\leq \delta + \delta
\]

\[
\leq 2\delta, \ i \in \mathbb{Z}^d.
\]

**Remark 1.** Note that \( \mathcal{K}_i(u_t) \subset K_i^{\varepsilon_1}(u_t) \subset K_i^{\varepsilon_2}(u_t) \) if \( \varepsilon_1 \leq \varepsilon_2 \).
5.2. The convergence result. With the above preparation, we are now ready to show the convergence of a subsequence of sigmoidal solutions (5) to a solution of the differential delay inclusion system (7) as $\varepsilon \to 0$.

Consider a sequence $\varepsilon_m \to 0$ as $m \to \infty$. For any given $\psi(\cdot) \in C([-h, 0], \ell_\rho^2)$. Let $u_i^{\varepsilon_m} (\psi)$ be the unique solution to the $\varepsilon_m$-sigmoidal delay lattice system, i.e., its components satisfy the delay equation:

$$\frac{d}{dt} u_i^{\varepsilon_m} (t) = f_i (u_i^{\varepsilon_m} (t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_{\varepsilon_m} (u_j^{\varepsilon_m} (t - \tau_{i,j}) - \theta) + g_i,$$

(15)

$$u_i^{\varepsilon_m} (s) = \psi_i(s), \quad \forall \ s \in [-h, 0].$$

(16)

**Theorem 5.5.** Let $T > 0$, $\psi \in C([-h, 0], \ell_\rho^2)$ and suppose that the Assumptions 1 - 7 hold. Then for any sequence $u_i^{\varepsilon_m} (\psi)$ of the solutions of the sigmoidal lattice systems (5) for $\varepsilon_m \to 0$ as $m \to \infty$ there is a subsequence $u_i^{\varepsilon_k} (\psi) \to u_i^* \in C([-h, 0], \ell_\rho^2)$ for $t \in [0, T]$ and $\varepsilon_m \to 0$ as $k \to \infty$, where $u_i^* \in C([-h, 0], \ell_\rho^2)$ is a solution of the delay inclusion system (7) on the interval $[0, T]$.

**Proof.** The proof is divided into four parts.

I. Componentwise convergent subsequence. First we multiply the equation (15) with $u_i^{\varepsilon_m} (t)$ and use Assumption 6 to obtain

$$\frac{1}{2} \frac{d}{dt} |u_i^{\varepsilon_m} (t)|^2 = u_i^{\varepsilon_m} (t) f_i (u_i^{\varepsilon_m} (t)) + u_i^{\varepsilon_m} (t) \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_{\varepsilon_m} (u_j^{\varepsilon_m} (t - \tau_{i,j}) - \theta) + g_i u_i^{\varepsilon_m} (t)$$

\[\leq - \alpha |u_i^{\varepsilon_m} (t)|^2 + \beta_i^2 + \frac{\alpha}{4} |u_i^{\varepsilon_m} (t)|^2 + \frac{1}{\alpha} \left( \sum_{j \in \mathbb{Z}^d} k_{i,j} \right)^2 + \frac{\alpha}{4} |u_i^{\varepsilon_m} (t)|^2 + \frac{1}{\alpha} g_i^2,\]

where we have used inequality (4). Hence

$$\frac{d}{dt} |u_i^{\varepsilon_m} (t)|^2 \leq - \alpha |u_i^{\varepsilon_m} (t)|^2 + 2\beta_i^2 + \frac{1}{\alpha} (\kappa \rho \Sigma + g_i^2).$$

Integrating the differential inequality (17) then gives

$$|u_i^{\varepsilon_m} (t)|^2 \leq |u_i^{\varepsilon_m} (0)|^2 e^{-\alpha t} + \frac{2}{\alpha} \left( \beta_i^2 + \frac{1}{\alpha} (\kappa \rho \Sigma + g_i^2) \right) (1 - e^{-\alpha t}).$$

Replacing $t$ with $t + s$, $s \in (-h, 0)$, we have

$$|u_i^{\varepsilon_m} (t + s)|^2 \leq |\psi_i|_{C([-h, 0], \mathbb{R})}^2 e^{-\alpha (t + s)} + \frac{2}{\alpha} \left( \beta_i^2 + \frac{1}{\alpha} (\kappa \rho \Sigma + g_i^2) \right) (1 - e^{-\alpha (t + s)}).$$

Thus

$$\left| u_i^{\varepsilon_m} (t, s) \right| \leq |\psi_i|_{C([-h, 0], \mathbb{R})}^2 e^{-\alpha h} e^{-\alpha t} + \frac{2}{\alpha} \left( \beta_i^2 + \frac{1}{\alpha} (\kappa \rho \Sigma + g_i^2) \right) (1 - e^{-\alpha t})$$

$$\leq |\psi_i|_{C([-h, 0], \mathbb{R})}^2 e^{-\alpha h} + \frac{2}{\alpha} \left( \beta_i^2 + \frac{1}{\alpha} (\kappa \rho \Sigma + g_i^2) \right) =: \mu_i^2.$$
Replacing II. Convergent subsequence in $\mathbb{C}$

Actually, we can easily obtain the limit derivative is $\frac{d}{dt}u^\varepsilon_i(t) \leq |f_i(u^\varepsilon_i(t))| + \sum_{j \in \mathbb{Z}^d} |k_{i,j}| |\sigma_{\varepsilon_m}(u^\varepsilon_j(t - \tau_{i,j}) - \theta)| + |g_i|

\leq L(\sqrt{\rho}u^\varepsilon_i(t))|u^\varepsilon_i(t)| + \sum_{j \in \mathbb{Z}^d} |k_{i,j}| + |g_i|

\leq L(\sqrt{\rho}u_i)|u_i| + \sqrt{\epsilon \rho} + |g_i|.

This gives the uniform boundedness and equi-continuity of the sequence $u^\varepsilon_i$ in $[-h, T]$. Hence by the Ascoli-Arzelà Theorem for each $i \in \mathbb{Z}^d$, there is a $u^\varepsilon_i(\cdot) \in \mathcal{C}([-h, T], \mathbb{R})$ and a convergent subsequence $\{u^\varepsilon_{i,m_k}(\cdot)\}_{k \in \mathbb{N}}$ such that $u^\varepsilon_{i,m_k}(\cdot) \to u^\varepsilon_i(\cdot)$ in $\mathcal{C}([-h, T], \mathbb{R})$.

**Remark 2.** We have shown that the derivatives sequence is uniformly bounded, which we combine with the Lipschitz continuity of $f_i$ and sigmoidal function to show that the derivatives are equicontinuous on $[-h, T]$. From these, it follows that there exists a subsequence of derivatives converging to a function in the space $\mathcal{C}$. Actually, we can easily obtain the limit derivative is $\frac{d}{dt}u^\varepsilon_i(\cdot)$.

**II. Convergent subsequence in $\mathcal{C}([-h, 0], \ell_2^0)$.** Similar to the above, multiplying the equation (15) by $\rho_i u^\varepsilon_i(t)$ and summing over $i \in \mathbb{Z}^d$, we obtain

$$\|u^\varepsilon(t)\|_\rho^2 = \sum_{i \in \mathbb{Z}^d} \rho_i |u^\varepsilon_i(t)|^2 \leq e^{-at} \|u^\varepsilon(0)\|_\rho^2 + \frac{2}{\alpha} \left( \|\beta\|_\rho^2 + \frac{1}{\alpha} \left( \rho^2 + \|g\|_\rho^2 \right) \right) (1 - e^{-at}).$$

Replacing $t$ with $t + s$ as part I, then

$$\|u^\varepsilon(t + s)\|_\rho^2 \leq e^{\alpha h} e^{-a t} \|\psi\|_{\mathcal{C}([-h, 0], \ell_2^0)}^2 + \frac{2}{\alpha} \left( \|\beta\|_\rho^2 + \frac{1}{\alpha} \left( \rho^2 + \|g\|_\rho^2 \right) \right) (1 - e^{-at}) \leq e^{\alpha h} \|\psi\|_{\mathcal{C}([-h, 0], \ell_2^0)}^2 + \frac{2}{\alpha} \left( \|\beta\|_\rho^2 + \frac{1}{\alpha} \left( \rho^2 + \|g\|_\rho^2 \right) \right) \leq \nu^2$$

( independent of $m$).

It follows that $\|u^\varepsilon\|_{\mathcal{C}([-h, 0], \ell_2^0)} \leq \nu$.

Now squaring the equation (15), then multiplying by $\rho_i$ and summing over $i \in \mathbb{Z}^d$, we have

$$\left\| \frac{d}{dt}u^\varepsilon_i(t) \right\|_\rho^2 \leq 3 \sum_{i \in \mathbb{Z}^d} \rho_i |f_i(u^\varepsilon_i(t))|^2 + 3 \sum_{i \in \mathbb{Z}^d} \rho_i \left( \sum_{j \in \mathbb{Z}^d} |k_{i,j}| |\sigma_{\varepsilon_m}(u^\varepsilon_j(t - \tau_{i,j}) - \theta)| \right)^2 \leq 3 \sum_{i \in \mathbb{Z}^d} \rho_i |g_i|^2 \leq 3 (L^2(\sqrt{\rho}u) + \rho^2 + \|g\|_\rho^2).$$

Thus the sequence $u^\varepsilon_i$ is uniformly bounded and equi-continuous in $\mathcal{C}([-h, 0], \ell_2^0)$ for $t \in [0, T]$, so by the Ascoli-Arzelà Theorem there is a $u(\cdot) \in \mathcal{C}([-h, T], \ell_2^0)$ and
III. Equivalence of limit points. Fix \( \xi \in [-h, T] \). Since \( u^{\varepsilon_{mk}}(\cdot) \to \hat{u}(\cdot) \) in \( C([-h, T], \ell_p^d) \), for every \( \varepsilon > 0 \) there exists \( M(\varepsilon) \) such that

\[
\|u^{\varepsilon_{mk}}(\xi) - \hat{u}(\xi)\|_p = \sum_{i \in Z^d} \rho_i|u_i^{\varepsilon_{mk}}(\xi) - \hat{u}_i(\xi)|^2 < \varepsilon^2, \quad k \geq M(\varepsilon).
\]

Hence for each fixed \( i \in Z^d \),

\[
|u_i^{\varepsilon_{mk}}(\xi) - \hat{u}_i(\xi)| < \varepsilon/\sqrt{\rho_i}, \quad k \geq M(\varepsilon).
\]

On the other hand, since \( u_i^{\varepsilon_{mk}}(\xi) \to u_i^*(\xi) \) for each \( i \in Z^d \), then for every \( \varepsilon > 0 \) there exist \( M_i(\varepsilon) \) such that

\[
|u_i^{\varepsilon_{mk}}(\xi) - u_i^*(\xi)| < \varepsilon, \quad k \geq M_i(\varepsilon), \quad \forall \ i \in Z^d.
\]

Therefore for every fixed \( i \in Z^d \)

\[
|\hat{u}_i(\xi) - u_i^*(\xi)| \leq |u_i^{\varepsilon_{mk}}(\xi) - \hat{u}_i(\xi)| + |u_i^{\varepsilon_{mk}}(\xi) - u_i^*(\xi)| \leq \varepsilon/\sqrt{\rho_i} + \varepsilon
\]

for every \( k \geq \max\{M_i(\varepsilon), M(\varepsilon)\} \). Notice that the left side of the above inequality is independent of \( k \), so \( \hat{u}_i(\xi) = u_i^*(\xi) \) for every \( i \in Z^d \) and \( \xi \in [-h, T] \), i.e.,

\[
\hat{u}_i(t) = u_i^*(t) \quad \text{on} \quad [-h, T], \quad \forall \ i \in Z^d.
\]

IV. Limit as solution of the delay lattice inclusion. Rearranging the sigmoidal differential equation (15) for the above convergent subsequence \( \{u^{\varepsilon_{mk}}(\cdot)\} \) we obtain

\[
\Sigma_i^{\varepsilon_{mk}}(u_i^{\varepsilon_{mk}}) = \frac{d}{dt}u_i^{\varepsilon_{mk}}(t) - f_i(u_i^{\varepsilon_{mk}}(t)) - g_i, \quad i \in Z^d, \quad a.e..
\]

Define

\[
\Sigma_i^*(u_i^*) = \frac{d}{dt}u_i^*(t) - f_i(u_i^*(t)) - g_i, \quad i \in Z^d, \quad a.e. \tag{19}
\]

From part I, \( u_i^{\varepsilon_{mk}}(\cdot) \) converges uniformly to \( u_i^*(\cdot) \) on \([0, T]\) for each \( i \in Z^d \). Thus \( f_i(u_i^{\varepsilon_{mk}}(\cdot)) \) converges to \( f_i(u_i^*(\cdot)) \) in \( L^1([-h, T], \mathbb{R}) \) for each \( i \in Z^d \). In addition, by Remark 2, \( u_i^{\varepsilon_{mk}}(\cdot) \) converges to \( \frac{d}{dt}u_i^*(\cdot) \) in \( L^1([-h, T], \mathbb{R}) \) for each \( i \in Z^d \). Therefore,

\[
\Sigma_i^{\varepsilon_{mk}}(u_i^{\varepsilon_{mk}}) \ \text{converges to} \ \Sigma_i^*(u_i^*) \ \text{in} \ L^1([0, T], \mathbb{R}) \tag{20}
\]

as \( k \to \infty \) for each \( i \in Z^d \). We need to show \( \Sigma_i^*(u_i^*) \in K_i(u_i^*) \) for each \( i \in Z^d \) and almost every \( t \in [0, T] \). To this end, we estimate

\[
\int_{-h}^T \text{dist}_R(\Sigma_i^*(u_i^*), K_i(u_i^*)) \, dt.
\]

Notice that \( \varepsilon_{mk} \to 0 \). Fix an arbitrary \( \varepsilon \in [0, 1] \), then we can assume without loss of generality that \( \varepsilon_{mk} \leq \varepsilon \) for all \( k \in \mathbb{N} \). Given any \( T > 0 \),

\[
\int_{-h}^T \text{dist}_R(\Sigma_i^*(u_i^*), K_i(u_i^*)) \, dt
\]

\[
\leq \int_{-h}^T |\Sigma_i^*(u_i^*) - \Sigma_i^{\varepsilon_{mk}}(u_i^{\varepsilon_{mk}})| \, dt + \int_{-h}^T \text{dist}_R(\Sigma_i^{\varepsilon_{mk}}(u_i^{\varepsilon_{mk}}), K_i^*(u_i^{\varepsilon_{mk}})) \, dt
\]

\[
+ \int_{-h}^T \text{dist}_R(K_i^*(u_i^{\varepsilon_{mk}}), K_i^*(u_i^*)) \, dt + \int_{-h}^T \text{dist}_R(K_i^*(u_i^*), K_i(u_i^*)) \, dt.
\]
First we see that (i) → 0 by (20). Second, (ii) = 0 because
\[ \Sigma_{m_k}^{\infty}(u^{m_k}) \in K_{m_k}^{\infty}(u^{m_k}) \subset K^i_{m_k}(u^{m_k}). \]
In addition, (iii) → 0 due to the upper semi continuity of \( K^i \) in Lemma 5.2 and (iv) → 0 by Lemma 5.3.

Furthermore, notice that the left side of the above inequality is independent of \( k \) and \( m \), so
\[ \int_{-h}^T \text{dist}_R(\Sigma^i(u^*_i), K_i(u^*_i)) \, dt = 0. \]
Hence
\[ \text{dist}_R(\Sigma^i(u^*_i), K_i(u^*_i)) = 0, \text{ a.e.,} \]
which implies that
\[ \Sigma^i(u^*_i) \in K_i(u^*_i), \text{ a.e..} \]
Finally, equation (19) can be rewritten as
\[ \frac{d}{dt} u^*_i(t) = f_i(u^*_i(t)) + \Sigma^i(u^*_i) + g_i, \text{ a.e..} \]
with \( \Sigma^i(u^*_i) \in K_i(u^*_i) \), i.e., \( \Sigma^i(u^*_i) \) is a selector of the delay differential inclusion (6). Therefore \( u^*_i = (u^*_i,t)_{t \in \mathbb{Z}^d} \) is a solution of the delay differential inclusion (6) in \( C([-h,0], \ell^2_\rho) \) with the initial data \( \psi \in C([-h,0], \ell^2_\rho) \).

6. Existence and comparison of global attractors. This section consists of two parts. In the first part it is shown that the lattice inclusion system generates a set-valued semi-dynamical system with compact values on the Hilbert space \( C([-h,0], \ell^2_\rho) \), while in the second part the existence of global attractors for the various systems and the relationship between them are established.

6.1. Set-valued dynamical systems with compact values. Autonomous set-valued semi-dynamical systems, frequently called set-valued semi-groups or general dynamical systems, see e.g., Szegö & Trecanni [13], are often generated by differential inclusions or differential equations without uniqueness, cf. [2]. The extensive theory for them was mainly developed on the locally compact state space \( \mathbb{R}^d \), but much of it holds here with modifications in the Hilbert space \( \ell^2_\rho \), when the system takes compact values.

For each \( \psi \in C([-h,0], \ell^2_\rho) \), we define the attainability set for (7)
\[ \Phi(\tau, \psi) := \{ v \in C([-h,0], \ell^2_\rho) : \text{there exists a solution } u_i(\psi) \text{ of (7) with } u_i(\psi) = v \}. \]
We will show that these attainability sets generate a set-valued semi-dynamical system. Let \( D(C([-h,0], \ell^2_\rho)) \) denote the family of all nonempty compact subsets of \( C([-h,0], \ell^2_\rho) \).

Definition 6.1. A set-valued semi-dynamical system on \( C([-h,0], \ell^2_\rho) \) with compact attainability sets is given by a mapping \( (t, \psi) \rightarrow \Phi(t, \psi) \in D(C([-h,0], \ell^2_\rho)) \) defined on \( \mathbb{R}^+ \times C([-h,0], \ell^2_\rho) \) such that
1. \( \Phi(0, \psi) = \{ \psi \} \) for all \( \psi \in C([-h,0], \ell^2_\rho) \);
2. \( \Phi(s + t, \psi) = \Phi(s, \Phi(t, \psi)) \) for all \( t, s \in \mathbb{R}^+ \) and all \( \psi \in C([-h,0], \ell^2_\rho) \);
3. \( (t, \psi) \rightarrow \Phi(t, \psi) \) is upper semi continuous in \( (t, \psi) \in \mathbb{R}^+ \times C([-h,0], \ell^2_\rho) \) with respect to the Hausdorff semi-distance \( \text{dist}_{C([-h,0], \ell^2_\rho)} \), i.e.,
\[ \text{dist}_{C([-h,0], \ell^2_\rho)}(\Phi(t, \psi), \Phi(t_0, \psi_0)) \rightarrow 0 \text{ as } (t, \psi) \rightarrow (t_0, \psi_0) \text{ in } \mathbb{R}^+ \times C([-h,0], \ell^2_\rho); \]
(4) $t \to \Phi(t, \psi)$ is continuous in $t \in \mathbb{R}_+$ with respect to the Hausdorff metric \( \text{Dist}_{C([-h,0], \ell^p)} \) uniformly in $\psi$ in compact subsets $B \in \mathcal{D}(C([-h,0], \ell^p_\rho))$, i.e.,

$$\sup_{\psi \in B} \text{Dist}_{C([-h,0], \ell^p_\rho)}(\Phi(t, \psi), \Phi(t_0, \psi)) \to 0 \text{ as } t \to t_0 \text{ in } \mathbb{R}_+.$$ 

**Theorem 6.2.** The attainability sets for (7) generate a set-valued semi-dynamical system $\Phi$ on $C([-h,0], \ell^p_\rho)$ with compact attainability sets.

The proof is similar to that in [8] for the case without delay, but now in the state space $C([-h,0], \ell^p_\rho)$ rather than $\ell^p_\rho$. It is given here for the reader’s convenience.

**Proof.** (i) Firstly, it is straightforward to check that $\Phi(0, \psi) = \psi$.

(ii) We will now show the set-valued semi-group property of $\Phi(t, \psi)$.

Recall that an element $u_i$ in $\Phi(t, \psi)$ is the unique solution to the differential equation

$$\frac{du(t)}{dt} = f(u(t)) + \sigma(t) + g,$$  \hspace{1cm} (21)

for a selector $\sigma(t) \in \mathcal{K}(u_i)$ and initial value $\psi \in C([-h,0], \ell^p_\rho)$.

By definition, for every $v \in \Phi(t_2, \Phi(t_1, \psi))$, there exists $w \in \Phi(t_1, \psi)$ and a solution $u^1_2$ of (7) such that $v = u^1_2(w)$. At the same time, since $w \in \Phi(t_1, \psi)$, there exists a solution $u^2_1$ of (7) such that $w = u^2_1(\psi)$. Let $u^1_2$ be the concatenation of $u^1_2$ and $u^2_1$ and their corresponding selections $\sigma^2(t) \in \mathcal{K}(u^2_1)$ on $[0, t_1]$ and $\sigma^1(t) \in \mathcal{K}(u^1_2)$ on $[t_1, t_2]$. Then $u^1_2(\psi)$ is also a solution to (21) with the corresponding selection $\sigma^*(t)$ given by the concatenation of $\sigma^2(t)$ and $\sigma^1(t)$. This implies that $v = u^*_2(\psi) \in \Phi(t_2, \psi)$. Hence

$$\Phi(t_2, \Phi(t_1, \psi)) \subseteq \Phi(t_2, \psi).$$

On the other hand, for every $v \in \Phi(t_2, \psi)$, there exists a solution $u_i$ of (7) such that $v = u_{t_2}(\psi)$. There also exists a solution $u_i$ of (7) such that $u_i$ is a unique solution of (21). Define $\sigma^1(t) = \sigma(t)$, $u^1_1 = u_i$ on $t \in [0, t_1]$ and $\sigma^2(t) = \sigma(t)$, $u^2_1 = u_i$ on $t \in [t_1, t_2]$. Thus $u^1_2$ and $u^2_1$ are the unique solutions to (21) corresponding to the initial values $u^1_0 = \psi$ and $u^2_0 = u^1_1 = w$ and $u^2_2 = v$. This implies

$$v \in \Phi(t_2, u^1(\psi)) \subseteq \Phi(t_2, \Phi(t_1, \psi)).$$

Hence $\Phi(t_2, \psi) \subseteq \Phi(t_2, \Phi(t_1, \psi))$.

(iii) Next, we show the attainability set $\Phi(t, \psi)$ is compact.

Let $\{v_m\}_{m \in \mathbb{N}}$ with $v_m = (v_m,i)_{i \in \mathbb{Z}^d}$ be an arbitrary bounded sequence with values in the attainability set $\Phi(t, \psi)$. Then for each $m \in \mathbb{N}$ there exists a solution $u_m(t, \psi)$ of the differential delay inclusion satisfying $u_m(t, \psi) = \psi$ and $u_m(t, \psi) = v_m$. For each solution $u_m(t, \psi) = (u_m,i(t, \psi))_{i \in \mathbb{Z}^d}$ given above, define

$$w^N_{m,t} = (u^N_{m,i,t})_{i \in \mathbb{Z}^d} \quad \text{with} \quad u^N_{m,i,t} = \begin{cases} u_{m,i,t}, & i \in \mathbb{Z}_N^d, \\ 0, & \text{otherwise.} \end{cases}$$

For each $m \in \mathbb{N}$, the sequence $\{w^N_{m,t}\}$ converges uniformly for $t \in [0, T]$ to $\{u_{m,t}\}$ in $C([-h,0], \ell^p_\rho)$ as $N \to \infty$. Thus for every $\varepsilon > 0$, there exists $N(\varepsilon, m) > 0$ such that

$$\|w^N_{m,t} - u_{m,t}\|_{C([-h,0], \ell^p_\rho)} < \varepsilon \quad \text{for all } N \geq N(\varepsilon, m), \; m \in \mathbb{N}.$$ 

In particular, for $t = \tau$, we have

$$\|w^N_{m,\tau} - u_{m,\tau}\|_{C([-h,0], \ell^p_\rho)} = \|w^N_{m,\tau} - v_m\|_{C([-h,0], \ell^p_\rho)} < \varepsilon \quad \text{for all } N \geq N(\varepsilon, m), \; m \in \mathbb{N}.$$
A subsequence \( \{w_{m,t}\}_{m \in \mathbb{N}} \) of \( \{w_{m,t}\}_{N \in \mathbb{N}} \) can be constructed so that

\[
\|w_{m,t} - v_m\|_{C([(-h,0], \ell^2_2)} \leq \frac{1}{m} \text{ for } m \text{ large} \quad (22)
\]

Denote \( \eta_{m,t} = w_{m,t}^N \). Then by the above uniform convergence and using the same arguments as in the proof of the existence theorem in [7], there exists a subsequence \( \{\eta_{m,t}\}_{t \in \mathbb{N}} \) of \( \{\eta_{m,t}\}_{m \in \mathbb{N}} \) and a solution \( \eta^*_t \) of the differential delay inclusion with \( \eta^*_0 = \psi \) such that

\[
\|\eta_{m,t} - \eta^*_t\|_{C([(-h,0], \ell^2_2)} \leq \frac{1}{m_l} \text{ for } l \text{ large.} \quad (23)
\]

It follows immediately from (22) and (23) that

\[
\|v_m - \eta^*_t\|_{C([(-h,0], \ell^2_2)} \leq \|\eta_{m,t} - v_m\|_{C([(-h,0], \ell^2_2)} + \|\eta_{m,t} - \eta^*_t\|_{C([(-h,0], \ell^2_2)} \leq \frac{2}{m_l}.
\]

We conclude that every sequence \( \{v_m\}_{m \in \mathbb{N}} \) in \( \Phi(t, \psi) \) has a convergent subsequence \( \{v_{m_l}\}_{l \in \mathbb{N}} \) in \( \Phi(t, \psi) \). Notice that \( \eta^*_t \in \Phi(t, \psi) \). Hence \( \Phi(t, \psi) \) is a compact subset of \( \mathcal{C}([-h,0]; \ell^2_2) \).

(iv) We show the set-valued mapping \( (t, \psi) \to \Phi(t, \psi) \) is upper semi-continuous in \((t, \psi) \) in \( \mathbb{R}^+ \times \mathcal{C}([-h,0]; \ell^2_2) \) for any \( t \geq 0 \).

Consider a sequence \( \{\psi^m\}_{m \in \mathbb{N}} \) such that \( \psi^m \to \psi \) in \( \mathcal{C}([-h,0]; \ell^2_2) \) as \( m \to \infty \). It can be assumed with loss of generality that

\[
\|\psi^m\|_{\mathcal{C}([-h,0]; \ell^2_2)} \leq \|\psi\|_{\mathcal{C}([-h,0]; \ell^2_2)} + 1, \quad \forall m \in \mathbb{N}.
\]

Suppose that \((t, \psi) \to \Phi(t, \psi) \) is not upper semi-continuous. Then there exist an \( \varepsilon_0 > 0 \) and sequences \( t_m \to t \) in \( \mathbb{R}^+ \) and \( \psi^m \to \psi \) in \( \mathcal{C}([-h,0]; \ell^2_2) \) such that

\[
\text{dist}_{\mathcal{C}([-h,0]; \ell^2_2)}(\Phi(t_m, \psi^m), \Phi(t, \psi)) \geq \varepsilon_0, \quad \forall m \in \mathbb{N}. \quad (24)
\]

Since for each \( m \in \mathbb{N} \), the set \( \Phi(t_m, \psi^m) \) is compact, there exists \( v_m \in \Phi(t_m, \psi^m) \) such that

\[
\text{dist}_{\mathcal{C}([-h,0]; \ell^2_2)}(v_m, \Phi(t, \psi)) = \text{dist}_{\mathcal{C}([-h,0]; \ell^2_2)}(\Phi(t_m, \psi^m), \Phi(t, \psi)) \geq \varepsilon_0, \quad \forall m \in \mathbb{N}.
\]

Since \( v_m \in \Phi(t_m, \psi^m) \), there exists a solution \( u_{m,t} \) of the differential inclusion (7) such that

\[
u_{m,0} = \psi^m \quad \text{and} \quad u_{m,t} = v_m.
\]

Closely following the proof for the compactness above and using the fact that the estimates in the proof of Theorem 1 in [7] are uniformly bounded in the initial conditions \( \|\psi^m\|_{\mathcal{C}([-h,0]; \ell^2_2)} \leq \|\psi\|_{\mathcal{C}([-h,0]; \ell^2_2)} + 1 \), we can construct a sequence of "truncated solutions" \( \eta_{m,t} = w_{m,t}^N \) such that

\[
\|\eta_{m,t}(\psi^m) - u_{m,t}(\psi^m)\|_{\mathcal{C}([-h,0]; \ell^2_2)} = \|w_{m,t}^N(\psi^m) - v_m\|_{\mathcal{C}([-h,0]; \ell^2_2)} \leq \frac{1}{m} \text{ for } m \text{ large.} \quad (25)
\]

The terms \( \eta_{m,t} \) of this sequence all satisfy the same estimates for the \( u_{m,t}^N \) as in the proof of Theorem 1 in [7]. Thus by the same arguments there, there exist a subsequence \( \{\eta_{m,t}\}_{t \in \mathbb{N}} \) and a solution \( \eta^*_t \) of the differential inclusion (7) such that

\[
\|\eta_{m,t}(\psi^m) - \eta^*_t(\psi)\|_{\mathcal{C}([-h,0]; \ell^2_2)} \leq \frac{1}{m_l} \text{ for } l \text{ large.} \quad (26)
\]
Hence
\[ \|u_{m,t}(\psi^{m}) - \eta^{*}(\psi)\|_{C([-h,0];\mathbb{E}^{2})} \leq \|\eta_{m,t}(\psi^{m}) - u_{m,t}(\psi^{m})\|_{C([-h,0];\mathbb{E}^{2})} + \|\eta_{m,t}(\psi^{m}) - \eta^{*}(\psi)\|_{C([-h,0];\mathbb{E}^{2})} \]

\[ \leq \frac{1}{m_{t}} + \|\eta_{m,t}(\psi^{m}) - \eta_{m,t}(\psi^{m})\|_{C([-h,0];\mathbb{E}^{2})} + \|\eta_{m,t}(\psi^{m}) - \eta^{*}(\psi)\|_{C([-h,0];\mathbb{E}^{2})} \]

\[ \leq \frac{2}{m_{t}} + \|\eta_{m,t}(\psi^{m}) - \eta_{m,t}(\psi^{m})\|_{C([-h,0];\mathbb{E}^{2})} \leq \|\eta^{*}(\psi)\|_{C([-h,0];\mathbb{E}^{2})} \]

\[ \leq \frac{3}{m_{t}} \]

where we used the equi-continuity in t of the solutions of the differential delay inclusion and \( t_{m,t} \rightarrow t \). In addition, \( \eta^{*}(\psi) \in \Phi(t,\psi) \). This means that

\[ \varepsilon_{0} \leq \text{dist}_{C([-h,0];\mathbb{E}^{2})}(v_{m,t} , \Phi(t,\psi)) \leq \|v_{m} - \eta^{*}(\psi)\|_{C([-h,0];\mathbb{E}^{2})} \rightarrow 0 \text{ as } t \rightarrow \infty, \]

which is impossible. Thus, \( (t,\psi) \rightarrow \Phi(t,\psi) \) is upper semi-continuous in \( (t,\psi) \) in \( \mathbb{R}^{+} \times C([-h,0];\mathbb{E}^{2}) \) for any \( t \geq 0 \).

(v) We show that the set-valued mapping \( t \rightarrow \Phi(t,\psi) \) is continuous in \( t \in \mathbb{R}^{+} \).

We know by (iv) above that \( t \rightarrow \Phi(t,\psi) \) is upper semi-continuous for each fixed \( \psi \in C([-h,0];\mathbb{E}^{2}) \), so let us suppose that \( t \rightarrow \Phi(t,\psi) \) is not lower semi-continuous. Then there exist an \( \varepsilon_{0} > 0 \) and a sequence \( t_{n} \rightarrow t \) in \( \mathbb{R}^{+} \) such that

\[ \text{dist}_{C([-h,0];\mathbb{E}^{2})}(\Phi(t_{n},\psi),\Phi(t_{n},\psi)) \geq \varepsilon_{0} \text{ for all } n \in \mathbb{N}. \]

Since \( \Phi(t,\psi) \) is compact, there exists \( v_{n} \in \Phi(t,\psi) \) such that

\[ \text{dist}_{C([-h,0];\mathbb{E}^{2})}(\Phi(t,\psi),v_{n},\Phi(t_{n},\psi)) \]

Thus

\[ \varepsilon_{0} \leq \text{dist}_{C([-h,0];\mathbb{E}^{2})}(v_{n},\Phi(t_{n},\psi)). \]

Since \( v_{n} \in \Phi(t,\psi) \), there exists a solution \( u_{n} \) of differential delay inclusion such that \( u_{n,0} = \psi \), and \( u_{n,t} = v_{n} \). We denote \( u_{n,t} \) with \( \bar{v}_{n} \). It is obvious that \( \bar{v}_{n} \in \Phi(t_{n},\psi) \).

Hence

\[ \varepsilon_{0} \leq \text{dist}_{C([-h,0];\mathbb{E}^{2})}(v_{n},\Phi(t_{n},\psi)) \leq \text{dist}_{C([-h,0];\mathbb{E}^{2})}(v_{n},\Phi(t_{n},\psi)) \]

By the compactness of \( \Phi(t,\psi) \), there is a convergent subsequence \( v_{n'} \rightarrow \bar{v} \) in \( \Phi(t,\psi) \). Then there exists a (possibly further) subsequence \( t_{n'} \rightarrow t \geq 0 \) with either \( t_{n'} \leq t \) or \( t_{n'} \geq t \) for all \( n' \).

Without loss of generality, we assume the case \( t_{n'} \geq t \) for all \( n' \). \( \Phi(t_{n'},\psi) = \Phi(t_{n'} - t,\Phi(t,\psi)) \). Pick \( \bar{v}_{n'} \in \Phi(t_{n'},\psi) \) with \( \bar{v}_{n'} \in \Phi(t_{n'} - t,v_{n'}) \), which is compact, so there exists a convergent subsequence \( \bar{v}_{n''} \rightarrow v^{*} \). In addition,

\[ \|v_{n''} - v^{*}\|_{C([-h,0];\mathbb{E}^{2})} \leq \|v_{n''} - \bar{v}_{n''}\|_{C([-h,0];\mathbb{E}^{2})} + \|ar{v}_{n''} - v^{*}\|_{C([-h,0];\mathbb{E}^{2})} \]

\[ \leq \|u_{n''}(t) - u_{n''}(t_{n''})\| + \varepsilon \]

\[ \leq 2\varepsilon, \varepsilon \text{ is small}, \]

so \( v_{n''} \rightarrow v^{*} \). Here we have used the equi-continuity in \( t \) of the solutions of the differential delay inclusion for the same initial value (which was proved in Part II of the proof of Theorem 5.5) and \( t_{n''} \rightarrow t \). It follows that \( \bar{v} = v^{*} \). Hence

\[ \varepsilon_{0} \leq \text{dist}_{C([-h,0];\mathbb{E}^{2})}(v_{n''},\bar{v}_{n''}) \rightarrow 0, \]
which is impossible. Thus the set-valued mapping \( t \to \Phi(t, \psi) \) is continuous in \( t \in \mathbb{R}^+ \) with respect to the Hausdorff metric. \( \square \)

It also follows from in the same way as in the proof of Theorem 6.2 that \( \varepsilon \)-inflated lattice differential inclusion (10) generates a set-valued dynamical systems \( \Phi^\varepsilon \) with values in \( D(C([-h, 0], \ell^2_\rho)) \) for every \( \varepsilon \in [0, 1] \).

6.2. Existence and comparison of attractors. To show the existence of the attractor we first need to establish the existence of the absorbing set and to show that the dynamical system is asymptotically compact in an appropriate sense.

We showed in part II of the proof of Theorem 5.5 that the solutions of the various systems satisfy the inequality

\[
\|u^\varepsilon_m(t)\|_\rho \leq e^{-\alpha t} \|\psi\|_\rho^2 C([-h, 0], \ell^2_\rho) + \frac{2}{\alpha} \left( \|\beta\|_\rho^2 + \frac{1}{\alpha} \left( \rho^2_\Sigma \kappa + \|g\|_\rho^2 \right) \right).
\]

This holds for solutions of the sigmoidal systems and the inflated systems as well as the original delay inclusion system since the the sigmoidal functions and the set-valued mappings \( \chi_\varepsilon \) and \( \chi \) all take values in the unit interval. Hence the closed and bounded set

\[
B = \left\{ \psi \in C([-h, 0]; \ell^2_p) : \|\psi\|_\rho^2 \leq 1 + \frac{2}{\alpha} \left( \|\beta\|_\rho^2 + \frac{1}{\alpha} \left( \rho^2_\Sigma \kappa + \|g\|_\rho^2 \right) \right) \right\}
\]

is a positive invariant absorbing set for the inflated systems \( \Phi^\varepsilon \), for every \( \varepsilon \in [0, 1] \).

The following definition is taken from [7]

**Definition 6.3.** A set-valued semi-dynamical system \( \Phi \) on a Banach space \( X \) is said to be **asymptotically upper-semi-compact** in \( X \) if every sequence \( y_n \in \Phi(t_n, x_n) \) has a convergent subsequence in \( X \) whenever \( t_n \to \infty \) as \( n \to \infty \) and \( x_n \in D \), where \( D \) is an arbitrary bounded subset of \( X \).

The asymptotic upper-semi-compactness of the inflated set-valued semi-dynamical system \( \Phi^\varepsilon \) on \( C([-h, 0]; \ell^2_p) \) can be shown by a very similar argument for the lattice inclusions without delay used in [7] and for the sigmoidal systems with delays (5) in [14], so the details are omitted.

Hence \( \Phi^\varepsilon \) has a unique global attractor

\[
\mathcal{A}^\varepsilon = \bigcap_{t \geq 0} \Phi^\varepsilon(t, B)
\]

for every \( \varepsilon \in [0, 1] \); see, e.g., [8].

Since \( \Phi(t, \psi) = \Phi^0(t, \psi) \subset \Phi^\varepsilon(t, \psi) \), the absorbing set for \( \Phi^\varepsilon(t, \psi) \) is also positive invariant and absorbing for \( \Phi(t, \psi) \) and the omega-limit points of \( \Phi(t, \psi) \) are omega-limit points of \( \Phi^\varepsilon(t, \psi) \). Hence the global attractor \( \mathcal{A} := \mathcal{A}^0 \) of \( \Phi \) satisfies \( \mathcal{A} \subset \mathcal{A}^\varepsilon \).

Similarly, the semi-dynamical system generated by the solutions of the \( \varepsilon \)-sigmoidal lattice system satisfies \( \phi^\varepsilon(t, \psi) \subset \Phi^\varepsilon(t, \psi) \), so its global attractor \( \mathcal{A}^\varepsilon \) satisfies \( \mathcal{A}^\varepsilon \subset \mathcal{A}^\varepsilon \) for every \( \varepsilon \in (0, 1] \).

Summarizing,

\[
\mathcal{A} = \mathcal{A}^0 \subset \mathcal{A}^\varepsilon, \quad \mathcal{A}^\varepsilon \subset \mathcal{A}^\varepsilon \text{ for all } \varepsilon \in (0, 1].
\]

In fact, stronger properties hold.
Theorem 6.4. The attractors $\mathcal{A}^\varepsilon$ of the inflated lattice inclusion systems converge to the attractor $\mathcal{A}$ of the lattice inclusion system continuously in $C([-h, 0]; l^2_\rho)$ in the Hausdorff metric $\text{Dist}_{C([-h, 0]; l^2_\rho)}$, i.e.,

$$\text{Dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A}) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$ 

In particular, $\mathcal{A} = \bigcap_{\varepsilon > 0} \mathcal{A}^\varepsilon$.

Proof. The inclusion $\mathcal{A} \subset \mathcal{A}^\varepsilon$ implies that

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}, \mathcal{A}^\varepsilon) = 0, \quad \text{for all} \quad \varepsilon > 0,$$

so it remains to show that

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A}) \to 0 \quad \text{as} \quad \varepsilon \to 0. \quad (29)$$

The global attractor $\mathcal{A}^\varepsilon$ of the set-valued dynamical system $\Phi^\varepsilon$ consists of entire trajectories, i.e., continuous functions $\mathbb{R} \ni t \mapsto u^\varepsilon_t \in C([-h, 0]; l^2_\rho)$ such that $u^\varepsilon_t \in \Phi^\varepsilon(t - s, u^\varepsilon_s)$ for all $s \leq t$ in $\mathbb{R}$, where $u^\varepsilon_t \in \mathcal{A}^\varepsilon$ for all $t \in \mathbb{R}$.

Suppose that the convergence (29) does not hold. Then there are $\eta_0$ and $\varepsilon_n \leq \varepsilon$ with $\varepsilon_n \to 0$ such that

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^{\varepsilon_n}, \mathcal{A}) \geq 2\eta_0 \quad \text{for all} \quad n \in \mathbb{N}. \quad (30)$$

Moreover, since the $\mathcal{A}^{\varepsilon_n}$ is compact there is an $a_n \in \mathcal{A}^{\varepsilon_n}$ such that

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (a_n, \mathcal{A}) = \text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^{\varepsilon_n}, \mathcal{A}) \geq 2\eta_0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Choose an arbitrary sequence of entire solutions $u^\varepsilon_n$ of $\Phi^\varepsilon$ with $u^\varepsilon_n = a_n$. These are also entire solutions of $\Phi^\varepsilon$. Applying a similar argument to that used in the proof of the corresponding theorem without delays in [8], there exists a convergent subsequence $u^\varepsilon_{n_r}$ converging uniformly on any closed and bounded time interval to a continuous function $u^\varepsilon_* : \mathbb{R} \to C([-h, 0]; l^2_\rho)$, which is also an entire trajectory of $\Phi$. In particular, $u^\varepsilon_* \in \mathcal{A}$. Moreover, $u^\varepsilon_{n_r} \to u^\varepsilon_*$, so

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (a_{n_r}, \mathcal{A}) \leq \eta_0,$$

which is a contradiction. \hfill \Box

In addition, the following corollary holds.

Corollary 1. The attractors $\mathcal{A}^\varepsilon$ of the sigmoidal lattice systems converge to the attractor $\mathcal{A}$ of the lattice inclusion system upper semi continuously in $C([-h, 0]; l^2_\rho)$, i.e.,

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A}) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$ 

Proof. This follows immediately from the triangle inequality

$$\text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A}) \leq \text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A}^\varepsilon) + \text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A}) \leq 0 + \text{dist}_{C([-h, 0]; l^2_\rho)} (\mathcal{A}^\varepsilon, \mathcal{A})$$

and Theorem 6.4 above. \hfill \Box
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