A Padé-aided analysis of nonperturbative $NN$ scattering in $^1S_0$ channel

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We carried out a Padé approximant analysis on a compact factor of the $T$-matrix for $NN$ scattering to explore the nonperturbative renormalization prescription in a universal manner. The utilities and virtues for this Padé analysis were discussed.

INTRODUCTION

Since Weinberg’s seminal work \cite{1}, the effective field theory (EFT) approach to the nuclear forces has been extensively investigated \cite{2}. However, such applications are plagued by severe nonperturbative UV divergence. For the EFT approach to be useful, the regularization and subtraction scheme must be carefully worked out together with a consistent set of power counting rules. In other words, appropriate renormalization prescription is needed in nonperturbative regimes. There have been many contributions to this issue \cite{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16} (More could be found in Ref. \cite{2}), among which there are some controversies and debates. At some points, different approaches could lead to rather disparate predictions \cite{17}.

Recently, a compact parametrization of the $T$-matrix is proposed in Ref. \cite{18}, with which the obstacle for renormalization being identified as the compact form of the $T$-matrix. In a concrete example \cite{19}, it was shown that the $T$-matrix could only be renormalized through ‘endogenous’ counter terms, which result in nontrivial prescription dependence. Such prescription dependence must be removed or fixed by imposing appropriate physical boundary conditions, for instance, through certain procedure of data fitting, as is frequently done in literature. The conventional power counting could be preserved within such procedures.

In this short report, we sketch a Padé approximant analysis basing on the aforementioned parametrization for the $T$-matrix. The paper is organized as follows: In Sec. II, the parametrization proposed in Ref. \cite{18} is briefly described with some related remarks. In Sec. III, the Padé approximant of a factor in the compact parametrization of the $T$-matrix is employed to parametrize the nonperturbative prescription dependence. Then predictions for phase shifts with some related remarks. In Sec. III, the Padé approximant of a factor in the compact parametrization of the $T$-matrix could only be renormalized through ‘endogenous’ counter terms, which result in nontrivial prescription dependence being identified as the compact form of the $T$-matrix \cite{21}. However, such applications are plagued by severe nonperturbative UV divergence. For the EFT approach to be useful, the regularization and subtraction scheme must be carefully worked out together with a consistent set of power counting rules. In other words, appropriate renormalization prescription is needed in nonperturbative regimes. There have been many contributions to this issue \cite{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16} (More could be found in Ref. \cite{2}), among which there are some controversies and debates. At some points, different approaches could lead to rather disparate predictions \cite{17}.

THE COMPACT PARAMETRIZATION

To describe low energy $NN$ scattering, one first constructs the potential $V$ from $\chi$PT \cite{1} up to certain chiral order, then computes the $T$-matrix through Lippmann-Schwinger equation (LSE), which is plagued with severe nonperturbative UV divergences. To appreciate the crucial aspects, a compact form of $T$-matrix in diagonal channels \cite{20} is proposed in Ref. \cite{18} as follows basing on LSE,

\[
\frac{1}{T_{2S+1L_j}(p', p; E)} = \frac{1}{V_{2S+1L_j}(p', p)} - \mathcal{G}_{2S+1L_j}(p', p; E),
\]

\[
\mathcal{G}_{2S+1L_j}(p', p; E) = \int \frac{dkd^2}{(2\pi)^2} \frac{V_{2S+1L_j}(p', k)G_0(k; E)T_{2S+1L_j}(k, p; E)}{V_{2S+1L_j}(p', p)T_{2S+1L_j}(p', p; E)}. \tag{2}
\]

Here $G_0(k; E) \equiv 1/(E - k^2/M + i\epsilon)$, with $M$ being nucleon mass, $p'$ and $p$ being the off-shell external momenta. Using the on-shell relation between $K$- and $T$-matrix \cite{21}:

\[
\frac{1}{T(p)} = \frac{1}{K(p)} + i \frac{M}{4\pi p},
\]

we arrive at the following on-shell relations (from now on, we omit the subscript ‘$2S+1L_j$’)

\[
\mathcal{G}(p) = V^{-1}(p) - K^{-1}(p) - i \frac{M}{4\pi p}; \tag{3}
\]

\[
T^{-1}(p) = V^{-1}(p) - \mathcal{G}(p). \tag{4}
\]

Obviously, $\mathcal{G}$ assumes all the nonperturbative divergences in a compact form. Any approximation to the quantity $\mathcal{G}$ leads to a nonperturbative scheme for $T$. Here we should remind that the power counting is applied in the construction of the potential $V(p, p')$. 

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In perturbation theory, UV divergences are removed order by order before the amplitudes are summed up. While for Eq. (4) or (10) in nonperturbative regime, infinitely many UV divergent amplitudes like $V G_0 V G_0 \cdots V V$ must be lump summed into a compact form. Then one must specify the order for implementing the following two incommutable procedures: subtraction versus nonperturbative summation. So a natural discrimination arises between 'endogenous' and 'exogenous' counter terms (or equivalent operations) that are introduced before and after the summation respectively. The compact form of $T$-matrix fails the 'exogenous' counter terms. In other words, the renormalization through 'endogenous' counter terms is the only sensible procedure in nonperturbative regime. Since the Schrödinger equation approach is intrinsically nonperturbative, any successful subtraction in the Schrödinger equation approach serves as a concrete instance for 'endogenous' renormalization in action. In practice, 'endogenous' subtraction is a formidable task in the $T$-matrix formalism: To complete the subtraction AND summation to ALL orders! That is why various forms of finite cutoff prevail in literature. In whatever approaches, the compact form of $T$-matrix persists and makes nonperturbative prescription dependence strikingly different from the perturbative cases. Physical boundary conditions must be imposed as a nontrivial procedure.

In what follows, all the EFT couplings are collectively denoted by $[C, \ldots]$ and pion mass by $m_\pi$. In general, any prescription could be parametrized by a set of dimensionless constants $[q, \ldots]$ and a dimensional scale $\mu$, including various finite cutoff schemes.

**A PADÉ-AIDED ANALYSIS OF THE EFT FOR $NN$ SCATTERING**

**Motivation**

From the above discussions, it is clear that the nonperturbative renormalization prescription is solely assumed in the factor $G$. Obviously, the $G$ factor could not be perturbative in terms of the EFT couplings, and its nontrivial nonperturbative prescription dependence is parametrized by the parameters to be physically fixed. These points have been demonstrated in Ref. [19]. From the point of view of Eq. (1), the main issue in the EFT for low energy $NN$ scattering is to work out an appropriate procedure or prescription of renormalization in the nonperturbative regime. Since the Schrödinger equation approach is intrinsically nonperturbative, any successful subtraction in the Schrödinger equation approach serves as a concrete instance for 'endogenous' renormalization in action. In practice, 'endogenous' subtraction is a formidable task in the $T$-matrix formalism: To complete the subtraction AND summation to ALL orders! That is why various forms of finite cutoff prevail in literature. In whatever approaches, the compact form of $T$-matrix persists and makes nonperturbative prescription dependence strikingly different from the perturbative cases. Physical boundary conditions must be imposed as a nontrivial procedure.

**Formulation**

The idea is very simple, we parametrize the factor $G$ in terms of Padé approximant. This is reasonable as $G$ is nonperturbative in terms of the EFT couplings and prescription parameters. In formulae, we employ the following parametrization of the $G$ factor ($p = \sqrt{M T}$):

$$G(p) \| \text{padé} \Rightarrow \left\{ \nu_0 + \nu_1 p^2 + \cdots - \frac{M}{4\pi} p \right\} \| \text{Taylor} \Rightarrow g(0) + g(1)p^2 + \cdots - \frac{M}{4\pi} p. \quad (5)$$

Here, the Taylor series is also listed as an expansion in much lower energy regions. Note the significant distinction between the Padé analysis of the $G$ factor here and that of the whole $T$-matrix: In the former case, EFT is indispensable in the systematic construction of the kernel (potential), while in the latter case, EFT plays no role at all.

Obviously $G$ assumes all the prescription dependence through $\nu_i, \delta_j$ or $g(n)$ that are nontrivial functions of the EFT couplings $[C, \ldots, M, m_\pi]$, and the prescription parameters, $[q, \ldots, \mu]$. Then instead of $[q, \ldots, \mu]$, we could use $\nu_i, \delta_j$ or $g(n)$ to parametrize the renormalization prescription of the $T$-matrix within the EFT approach,

$$T^{-1}(p; [C, \ldots, M, m_\pi]; [\nu, \ldots, \delta, \ldots]) \Rightarrow V^{-1}(p; [C, \ldots, M, m_\pi]) - \frac{\nu_0 + \nu_1 p^2 + \cdots}{\delta_0 + \delta_1 p^2 + \cdots} + \frac{M}{4\pi} p. \quad (6)$$
or

\[ T^{-1}(p; [C_{\ldots}, M, m_\pi]; [g_{\ldots}]) = V^{-1}(p; [C_{\ldots}, M, m_\pi]) - (g(0) + g(1)p^2 + \cdots) + \frac{M}{4\pi}ip. \]  

(7)

Intuitively, nonperturbative prescription dependence in \( \nu_i, \delta_j \) or \( g(n) \) could be understood from the rigorous nonperturbative solution of on-shell \( T \)-matrix for \( ^1S_0 \) channel scattering with contact potential at next-to-leading order (Nlo) and next-to-next-to-leading order (Nnlo)

\[
\begin{align*}
\text{Nlo : } V_{(2)} &= C_0 + C_2(p^2 + p'^2); \\
\Rightarrow T^{-1} &= \frac{(1 - C_2J_3)^2}{C_0 + C_2J_5 + C_2(2 - C_2J_3)p^2} + J_0 + \frac{M}{4\pi}ip; \\
\text{Nnlo : } V_{(4)} &= C_0 + C_2(p^2 + p'^2) + C_4p^2p'^2 + C_4(p^4 + p'^4); \\
\Rightarrow T^{-1} &= \frac{N_0 + N_1p^2 + N_2p^4 + D_4p^4}{D_0 + D_1p^2 + D_2p^4 + D_3p^6} + J_0 + \frac{M}{4\pi}ip,
\end{align*}
\]

(8)
(9)

with \([N_i, D_j]\), which correspond to \([\nu_i, \delta_j]\), being polynomials in terms of coupling \([C_{\ldots}]\) and \([J_n]\). Here the constants \([J_n]\) come from divergent loop integrals and hence parametrize the renormalization prescription, like the set \([g_{\ldots}, \mu]^{19}\).

Then \(N_i, D_j\) effectively parametrize a prescription due to their nontrivial dependence on \([J_n]\). In Padé approximant, \(\nu_i, \delta_j\) or \(g(n)\) take over the role of \(N_i, D_j\), and the physical boundary conditions are to be imposed on \(\nu_i, \delta_j\) or \(g(n)\).

**Power counting and nonperturbative prescription**

As is stressed in the previous section, EFT and its power counting rules enter through the potential. Through Eq. (1) the EFT elements and their power counting rules are carried over to the \( T \)-matrix in a nonperturbative manner. Since the kernel of the LSE or the potential is perturbative, its renormalization is still perturbatively implementable within the EFT power counting rules. Therefore, for the \( T \)-matrix, Eq. (1) just provides a concise separation of the nonperturbative renormalization information from other things. Both \( \mathcal{G} \) and \( V \) are in principle EFT objects, the sole and important distinction is that the former exclusively carries the information about the nonperturbative renormalization prescription. Thus, we could append some subscripts to Eq. (1) as follows,

\[
T_{\text{EFT,n.p.t.}}^{-1}(p) = V_{\text{EFT,p.t.}}^{-1}(p) - \mathcal{G}_{\text{EFT,n.p.t.}}(p).
\]

(10)

In general, within a natural EFT, the nonperturbative objects (e.g. \( \mathcal{G} \)) should also exhibit certain degree of naturalness in the sense that, the scales involved in such objects should not deviate very much from the natural sizes. But the renormalization in nonperturbative regimes does allow for other unconventional scenarios, without violating natural power counting rules. As a matter of fact, the Padé analysis of \( \mathcal{G} \) alone according to Eq. (1) does not affect anything of the EFT power counting rules, i.e., the implementation of the EFT power counting rules and the nonperturbative renormalization procedures are disentangled. The possible subtleness in the nonperturbative factor \( \mathcal{G} \) should not be misunderstood as the inconsistency or the unnaturalness of the EFT power counting rules or even as the inapplicability of EFT method at all. That is, the unnaturalness in \( \nu_i, \delta_j \) or \( g(n) \) could have nothing to do with the inconsistency of EFT power counting.

**Fitting and predictions: \(^1S_0\) channel**

With the preceding preparations, we can demonstrate the predictions of phase shifts for the \(^1S_0\) channel NN scattering at different orders of potential using different Padé approximants. For each case, the prescription parameters, i.e., the Padé parameters are fixed through fitting the phase shifts in the low energy ends. The laborious loop integrations and ‘endogenous’ subtractions are naturally avoided.

We will employ the potentials that are worked out in Ref.[4] (denoted as EGM from now on), which contains no energy dependence, and less contact couplings—a favorable aspect for fitting the ‘physical’ values for \( \nu_i, \delta_j \) or \( g(n) \). One could well employ other construction schemes for potentials. In fact, one could compare any pair of potential schemes only through fitting the parameters \( \nu_i, \delta_j \) or \( g(n) \). It is obvious that at any chiral order with any Padé approximant, different Padé parameters would yield rather different phase shifts curves. We will not show the figures for demonstrating such nontrivial prescription dependence due to space limitation. Let us focus on more interesting
figures with the Padé parameters determined through boundary conditions: fitting in the low energy regions: (1) At Lo, \( T_{\text{lab}} \in (0, 3) \) in units of \( \text{Re} V(2) \) at Nlo, while \( T_{\text{lab}} \in (0.2, 13) \); (3) At Nnlo, \( T_{\text{lab}} \in (3, 23) \). The phase shift is obtained from the following formula,

\[
\delta(p) = \arctan \left\{ -\frac{M_p}{4\pi} \left( \frac{1}{V(p)} - \text{Re}(G(p)) \right)^{-1} \right\}.
\]

(11)

The phase shifts predicted at Lo, Nlo and Nnlo are depicted in Fig.1(a), (b), (c) respectively. At each order, three Padé approximants are shown respectively: (1) \( \text{Re}(G) \approx g_{(0)} \) (dotted lines); (2) \( \text{Re}(G) \approx g_{(0)} + g_{(1)} p^2 \) (dashed lines); (3) \( \text{Re}(G) \approx (1 + \frac{\nu}{p_0} p^2)/(\frac{\nu}{p_0} + \frac{\lambda}{\nu} p^2) \) (solid lines). From these diagrams, one could find that, at each order, the prediction improves as more Padé parameters are present, which is a natural tendency. One could also anticipate that with each Padé approximant, the predictions should also improve as higher order terms are present in the potential, which are responsible for the interactions at higher energy. The results are shown in Fig.2. In each figure, the predictions are compared among different orders of potential with a fixed Padé approximant. Note that in Fig.2(b), in the whole range of the figure, the Lo curve almost identically coincides with the Nnlo curve.

Globally, the improvement with chiral orders is obvious: Nlo prediction (solid line) is better than Nl prediction (dashed line), and Nlo prediction is better than Lo (dotted line). There are also some interesting details: From these figures, we could see that in the higher energy region, all the Nlo curves have larger deviation from PWA data than the Nnlo curves. Sometimes, they are even worse than the Lo curves (C.f. Fig.2(b) and (c)).

Here, we note that the solid curve in Fig.1(a) seems puzzling. With only leading order potential \( V_{1s} \sim \frac{\sigma^5_1 \cdot q \cdot \sigma^2 \cdot q}{q^2 + m^2} \) plus a contact term \( V_c = C_0 \), one obtains pretty good predictions for the phase shifts, especially at the higher energies. The reason lies in the Padé approximant of \( G \), \( \text{Re}(G) \approx (1 + \frac{\nu}{p_0} p^2)/(\frac{\nu}{p_0} + \frac{\lambda}{\nu} p^2) \), which is fixed 'physically' and effectively 'induces' higher order interactions upon iteration. This result agrees with the findings in Ref.14, where a nonperturbative of \( T \)-matrix is obtained using a simple potential \( V = V_{1s}(\text{Lo}) + V_{(2)} \) (in Eq.8, Nlo), and the prediction of the phase shifts is surprisingly good in a wider range of energy after the nonperturbative divergences is removed through fitting. Our analysis above provides a simple explanation of this surprise: The nonperturbative renormalization is properly treated! A closer look at the lower energy regions reveals that the higher order predictions still dominate the lower order ones (C.f. Fig.2(c)). Generally, the lower order predictions could 'win' at higher energies only by chance.

**UTILITIES OF PADÉ APPROXIMANT AND DISCUSSIONS**

Now we have seen that after the nonperturbative prescription dependence is properly resolved (here realized through low energy region fitting), the EFT approach facilitates 'physical' prediction for low energy \( NN \) scattering, at least in \( 1S_0 \) channel. Since no specification of regularization and renormalization is needed, the Padé approximant of \( G \) defined in Eq.11 in fact provides a universal parametrization of the renormalization prescription dependence of the \( T \)-matrix in nonperturbative regimes. Note that both the potential and the Padé approximant of \( G \) could be systematically extended to higher orders in EFT.

Comparing with previous results, we find that the Lo prediction with \( \text{Re}(G) \approx (1 + \frac{\nu}{p_0} p^2)/(\frac{\nu}{p_0} + \frac{\lambda}{\nu} p^2) \) (C.f. Fig.2(c)) differs significantly from that given in Ref.2 and looks better. This nontrivial difference in predictions at leading order reflects the importance of nonperturbative renormalization. However, at higher chiral orders, especially at Nnlo, our results show no obvious differences in comparison with Ref.2. That means, including higher order interactions would lessen or tend to remove the nonperturbative renormalization prescription dependence. This is a marvellous fact, since the fundamental requisite in EFT application is that renormalization prescription dependence should decrease as higher order interactions are included. Therefore, in spite of being an approximation approach, the procedures described above substantially proved or ascertained the rationality and applicability of EFT method in nuclear forces in a very general context. This is in sheer contrast to most known approaches where a renormalization prescription must be specified and hence the exploration of the prescription dependence is apparently limited.

So far we determined the Padé parameters through fitting with the potential defined by EGM4 where the contact couplings were determined within a special cutoff scheme. In principle, we should determine the couplings in a way that is more prescription-independent: fitting through the combined space \([C, \mu] \bigcup [g, \lambda, \nu] \), which should lead to a better way for determining the EFT couplings. Now the Padé approximant provides us a convenient approach to do so without really carrying out the formidable task of loop integrations and renormalization to all orders. We will perform the investigations along this line in the future.

We believe other utilities could be derived from this analysis and the parametrization Eq.11.
From the above results, it is also obvious that it is fairly sufficient to employ Padé up to \( \text{Re}(G) \approx (1 + \frac{\delta_1}{\delta_0} p^2)/(\delta_0 + \frac{\delta_1}{\delta_0} p^2) \). For some channels, say \( P \)-wave, it is often sufficient to use \( \text{Re}(G) \approx g(0) + g(1) p^2 \), which will be demonstrated in a separate report.

**SUMMARY**

In summary, we performed a Padé analysis on a compact factor of the \( T \)-matrix for \( NN \) scattering so that the nonperturbative prescription dependence and related effects could be conveniently explored. Such analysis suggests a useful theoretical instrument as well as a general and prescription-independent approach to test the efficiency and rationality of the application of EFT methods in nonperturbative regimes. Some related literature were also explained and discussed in favor of our analysis.

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[1] S. Weinberg, Phys. Lett. **B 251**, 288 (1990); Nucl. Phys. **B 363**, 1 (1991).
[2] See, e.g., P. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. **52**, 339 (2002) [nucl-th/0203055]; Ulf-G. Meissner, [nucl-th/0409028](#).
[3] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. **C 53**, 2086 (1996); U. van Kolck, Nucl. Phys. **A 645**, 327 (1999).
[4] E. Epelbaum, W. Glöckle and U. Meissner, Nucl. Phys. **A 671**, 295 (2000), Eur. Phys. J. **A 15**, 543 (2002), **A 19**, 125,401 (2004).
[5] J.V. Steele and R.J. Furnstahl, Nucl. Phys. **A 637**, 46 (1999).
[6] T.S. Park, K. Kubodera, D.P. Min and M. Rho, Phys. Rev. **C 58**, R637 (1998).
[7] D.B. Kaplan, M.J. Savage and M.B. Wise, Phys. Lett. **B 424**, 390 (1998); Nucl. Phys. **B 534**, 329 (1998); S. Fleming, T. Mehen and I.W. Stewart, Nucl. Phys. **A 677**, 313 (2000); Phys. Rev. **C 61**, 044005 (2000).
[8] T. Frederico, V.S. Timóteo and L. Tomio, Nucl. Phys. **A 653**, 209 (1999).
[9] J. Gegelia, Phys. Lett. **B 463**, 133 (1999); J. Gegelia and G. Japaridze, Phys. Lett. **B 517**, 476 (2001); J. Gegelia and S. Scherer, [nucl-th/0403052](#).
[10] D. Eiras and J. Soto, Eur. Phys. J. **A 17**, 89 (2003) [nucl-th/0107009].
[11] S.R. Beane, P. Bedaque, M.J. Savage and U. van Kolck, Nucl. Phys. **A 700**, 377 (2002).
[12] T. Barford and M.C. Birse, Phys. Rev. **C 67**, 064006 (2003).
[13] J.A. Oller, [arXiv:nucl-th/0207086](#).
[14] J. Nieves, Phys. Lett. **B 568**, 109 (2003) [nucl-th/0301080].
[15] M.P. Valderrama and E. Ruiz Arriola, Phys. Lett. **B 580**, 149 (2004); Phys. Rev. **C 70**, 044006 (2004).
[16] R. Higa, [nucl-th/0411046](#).
[17] S.R. Beane and M.J. Savage, Nucl. Phys. **A 713**, 148 (2003); Nucl. Phys. **A 717**, 91 (2003); E. Epelbaum, W. Glöckle and U.-G. Meissner, Nucl. Phys. **A 714**, 535 (2003).
[18] J.-F. Yang, [nucl-th/0310048v6](#).
[19] J.-F. Yang and J.-H. Huang, Phys. Rev. **C 71**, 034001(2005), *ibid.*, **C 71**, 069901(E) (2005) [nucl-th/0409023-3](#).
[20] The generalization to the coupled channels is straightforward. We will perform such analysis in the coupled channels in the future.
[21] R.G. Newton, *Scattering Theory of Waves and Particles*, 2nd Edition (Springer-Verlag, New York, 1982), p187.
[22] P.M. Stevenson, Phys. Rev. **D 23**, 2016 (1981); G. Grunberg, Phys. Rev. **D 29**, 2315 (1984); S.J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. **D 28**, 228 (1983).
[23] V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, Phys. Rev. **C 48**, 792 (1993).
FIG. 1: Predictions for $^1S_0$ phase shifts with three different Padé approximants: (1) $\text{Re}(\mathcal{G}) \approx g(0)$ (dotted line); (2) $\text{Re}(\mathcal{G}) \approx g(0) + g(1)p^2$ (dashed line); (3) $\text{Re}(\mathcal{G}) \approx (1 + \nu_1 \nu_0 p^2)/(\delta_0 \nu_0 + \delta_1 \nu_0 p^2)$ (solid line). The circles denote the PWA data[23]. (a) for Lo; (b) for Nlo; (c) for Nnlo .

FIG. 2: Predictions for $^1S_0$ phase shifts at different orders of potential: Dotted line for Lo, dashed line for Nlo and solid line for Nnlo. The circles denote the PWA data[23]. (a) for $\text{Re}(\mathcal{G}) \approx g(0)$; (b) for $\text{Re}(\mathcal{G}) \approx g(0) + g(1)p^2$; (c) for $\text{Re}(\mathcal{G}) \approx (1 + \nu_1 \nu_0 p^2)/(\delta_0 \nu_0 + \delta_1 \nu_0 p^2)$.