ON DECOMPOSITION FOR PAIRS OF TWISTED CONTRACTIONS

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ABSTRACT. This paper presents Wold-type decomposition for various pairs of twisted contractions on Hilbert spaces. As a consequence, we obtain Wold-type decomposition for pairs of doubly twisted isometries and in particular, new and simple proof of Słoński’s theorem for pairs of doubly commuting isometries are provided. We also achieve an explicit decomposition for pairs of twisted contractions such that the c.n.u. parts of the contractions are in $C_{00}$. It is shown that for a pair $(T, V^*)$ of twisted operators with $T$ as a contraction and $V$ as an isometry, there exists a unique (upto unitary equivalence) pair of doubly twisted isometries on the minimal isometric dilation space of $T$. As an application, we prove that pairs of twisted operators consisting of an isometry and a co-isometry are doubly twisted. Finally, we have given a characterization for pairs of doubly twisted isometries.

1. Introduction

Operator theory on Hilbert spaces has two essential parts: the theory of normal operators and the theory of non-normal operators. The structure of normal operators is well known due to spectral decomposition. On the other hand, the complete structure of non-normal operators is unknown to the literature and an important class of non-normal operators is isometries. Indeed, one of the important problems in operator theory, analytic function theory and operator algebras is the classification and representation of $n$-tuples ($n > 1$) of commuting isometries on a Hilbert space.

In a probabilistic language, Wold [27] first established a notable decomposition for stationary stochastic processes. Later, von Neumann, Kolmogorov, and Halmos coined the abstraction of Wold’s result for isometries on Hilbert spaces: Every isometry can be uniquely written as a direct sum of a unitary operator and copies of the unilateral shifts. This is called Wold decomposition or Wold-von Neumann decomposition (see details in Section 2). This decomposition plays a vital role in many areas of operator algebras and operator theory, namely, dilation theory, invariant subspace theory, operator interpolation problem etc. It is now a natural question: Does there exist Wold-type decomposition for pairs of commuting isometries (in general for contractions) on Hilbert spaces?

There has been a lot of research in this direction for the last few decades and many important and interesting results have been obtained in many directions. Still a complete and explicit structure for pairs of commuting isometries, or, in general, an $n$-tuple of commuting isometries on a Hilbert space, is unknown. Many researchers have investigated Wold-type
decomposition for a pair of commuting isometries/commuting partial isometries/commuting contractions. For instance, Suciu [24] developed a structure theory for a semigroup of isometries. Later, Słońcinski [23] obtained a Wold-type decomposition for pairs of doubly commuting isometries from Suciu’s decomposition of the semigroup of isometries. Burdek, Kosiek and Słońcinski [8] developed the canonical Wold decomposition considering the finite-dimensional wandering space for commuting isometries. Popovici [19] studied the Wold-type decomposition for pairs of commuting isometries. Later, Sarkar [22] generalized Słońcinski’s result for n-tuple of doubly commuting isometries. Many important results have been obtained by many researchers, like Burdak, Kosiek, Pagacz and Słońcinski ([6], [7]), Bercovici, Douglas and Foiaș ([4], [3]), Maji, Sarkar, and Sankar [18], and references therein etc. On the other hand, Halmos and Wallen [14] studied decomposition for power partial isometry in 1970. Burdak [5] developed a new characterization for a pair of commuting (not necessarily doubly commuting) contractions and obtained decomposition results in the case of power partial isometries.

Jeu and Pinto [15] established Wold decomposition for n-tuple (n > 1) of doubly non-commuting isometries. Later, Rakshit, Sarkar, and Suryawanshi [20] generalized those results by introducing $\mathcal{U}_n$-twisted isometries. Recently, we introduced n-tuple of doubly non-commuting and $\mathcal{U}_n$-twisted contractions (see section 3 and section 5 in [17]) and then established a complete description of those tuples.

Dilation theory is one of the most effective tools to study the structure of contractions. One of the striking results is the Sz.-Nagy’s dilation result (see [21]): Every contraction has an isometric dilation on a larger Hilbert space. Later, Andô [1] proved that a commuting pair of contractions dilates to a commuting pair of isometries. Other special classes of operators are $C_{\alpha\beta} = C_{\alpha} \cap C_{\beta}$ ($\alpha, \beta = 0, 1$), which plays a significant role in the study of general contractions on Hilbert spaces. Indeed, every contraction on a Hilbert space has a canonical triangulations of the following types (see [21]):

$$\begin{bmatrix} C_0 & \ast \\ O & C_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_1 & \ast \\ O & C_0 \end{bmatrix}.$$  

The reducibility of an operator means deciding whether the operator has a nontrivial reducing subspace or not, and the classification of invariant subspaces and reducing subspaces of various operators on function spaces has proved to be very challenging research problems in analysis. The reducibility of general $C_0(N)$ operators is complicated. However, Gu [12] studied the reducibility of any power of a $C_0(1)$ operator (see also [9]). Uchiyama [25], [26] discussed hyperinvariant subspaces for contractions of class $C_0$ and also found double commutants for $C_0$ contractions with finite defect indices. Recently, Benhida et al. [2] obtained several equivalent conditions for the reducibility of a contraction in the class $C_{00}$ in terms of a minimal unitary dilation and the characteristic function.

The main aim of this paper is to investigate decomposition of various pairs of twisted operators (in particular, for commuting) as well as the structure of different kinds of operators (contraction, isometry, partial isometry) on Hilbert spaces. The geometry of Hilbert spaces, canonical decomposition for a contraction, classical Wold-von Neumann decomposition for isometry and dilation theory are the essential tools used in this article. One of the main
results is Theorem 3.1 (see Section 3): It says that if a pair of doubly twisted operators \((T, V)\) on a Hilbert space \(\mathcal{H}\), where \(T\) is a contraction, then the canonical decomposition for \(T\) reduces \(V\). Secondly is Theorem 5.3 (see Section 5 for details): Let \((T, V^*)\) be a pair of twisted operators on a Hilbert space \(\mathcal{H}\) such that \(T\) is a contraction and \(V\) is an isometry. Let \(S\) on \(K\) be the minimal isometric dilation for \(T\). If \(\tilde{V}\) on \(K\) is an extension of the isometry \(V\), then \((S, \tilde{V})\) is a pair of doubly twisted isometries on the minimal space \(K\). Moreover, the pair \((S, \tilde{V})\) on the minimal space \(K\) is unique up to unitary equivalence.

The plan for the paper is as follows. In Section 2, we discuss some basic definitions and the canonical decomposition for a contraction as well as classical Wold-von Neumann decomposition theorem for isometry. In Section 3, we investigate decomposition for various pairs of twisted operators. In Section 4, we study the structure of a twisted commutant of a power partial isometry. Section 5 is devoted to dilation of a contraction and we have shown that the Wold-type decomposition for a certain pair of operators holds on the minimal isometric dilation space. Finally, we have given a characterization for doubly twisted isometries in Section 6.

2. Preliminaries

Throughout this paper, \(\mathcal{H}\) denotes a complex separable Hilbert space, \(\mathcal{B}(\mathcal{H})\) as the algebra of all bounded linear operators (operators for short) on \(\mathcal{H}\), and \(P_L\) is the orthogonal projection of \(\mathcal{H}\) onto \(L\). For \(X \in \mathcal{B}(\mathcal{H})\), \(\{X\}'\) denotes the commutant of \(X\). A closed subspace \(\mathcal{M}\) of \(\mathcal{H}\) is invariant under \(T \in \mathcal{B}(\mathcal{H})\) if \(T(\mathcal{M}) \subseteq \mathcal{M}\) and subspace \(\mathcal{M}\) reduces \(T\) if \(T(\mathcal{M}) \subseteq \mathcal{M}\) and \(T(\mathcal{M}^\perp) \subseteq \mathcal{M}^\perp\). A contraction \(T\) on \(\mathcal{H}\) (that is, \(\|T h\| \leq \|h\|\) for all \(h \in \mathcal{H}\)) is said to be a pure contraction if \(T^m \to 0\) as \(m \to \infty\) in the strong operator topology. A contraction \(T\) on \(\mathcal{H}\) is called completely non-unitary (c.n.u. for short) if there does not exist any nonzero \(T\)-reducing subspace \(L\) of \(\mathcal{H}\) such that \(T|_L\) is a unitary operator. A closed subspace \(\mathcal{W} \subseteq \mathcal{H}\) is said to be a wandering subspace of an isometry \(V\) (that is, \(V^* V = I_{\mathcal{H}}\)) if

\[
V^k \mathcal{W} \perp V^\ell \mathcal{W} \quad \text{for all} \quad k, \ell \in \mathbb{Z}_+ \quad \text{with} \quad k \neq \ell.
\]

An isometry \(V\) on \(\mathcal{H}\) is called a unilateral shift or shift if \(\mathcal{H} = \bigoplus_{m \geq 0} V^m \mathcal{W}\) for some wandering subspace \(\mathcal{W}\) of \(V\). Equivalently, an isometry \(V\) on \(\mathcal{H}\) is said to be a pure isometry or shift if \(V^m \to 0\) as \(m \to \infty\) in the strong operator topology (see Halmos [13]). It is noted that if \(\mathcal{W}\) is a wandering subspace of a shift \(V\) on \(\mathcal{H}\), then

\[
\mathcal{W} = \mathcal{H} \ominus V \mathcal{H}
\]

and the dimension of \(\mathcal{W}\) is called the multiplicity of the shift \(V\) (see [21]).

One of the fundamental results in dilation theory is that every contraction on Hilbert spaces can be decomposed into direct sum of unitary and c.n.u. We refer this as canonical decomposition theorem for a contraction ([21]).

**Theorem 2.1.** Every contraction \(T\) on a Hilbert space \(\mathcal{H}\) corresponds a unique decomposition of \(\mathcal{H}\) into an orthogonal sum of two \(T\)-reducing subspaces \(\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{-u}\) such that \(T|_{\mathcal{H}_u}\) is
unitary and $T|_{\mathcal{H}_{-u}}$ is c.n.u. ($\mathcal{H}_u$ or $\mathcal{H}_{-u}$ may equal to $\{0\}$). Moreover

$$\mathcal{H}_u = \{ h \in \mathcal{H} : \| T^nh \| = \| h \| = \| T^{*n}h \| \text{ for } n = 1, 2, \ldots \}.$$ 

Here $T_u = T|_{\mathcal{H}_u}$ and $T_{-u} = T|_{\mathcal{H}_{-u}}$ are called unitary part and c.n.u. part of $T$, respectively and $T = T_u \oplus T_{-u}$ is called the canonical decomposition for $T$.

In particular, canonical decomposition theorem for an isometry coincides with the classical Wold-von Neumann decomposition \((21)\).

**Theorem 2.2.** Let $V$ be an isometry on a Hilbert space $\mathcal{H}$ and $\mathcal{W} = \mathcal{H} \ominus VH$. Then $\mathcal{H}$ decomposes uniquely as a direct sum of two $V$-reducing subspaces $\mathcal{H}_s = \bigoplus_{m=0}^{\infty} V^m \mathcal{W}$ and $\mathcal{H}_u = \mathcal{H} \ominus \mathcal{H}_s$ and

$$V = \begin{bmatrix} V_s & O \\ O & V_u \end{bmatrix} \in \mathcal{B}(\mathcal{H}_s \oplus \mathcal{H}_u),$$

where $V_s = V|_{\mathcal{H}_s}$ is shift and $V_u = V|_{\mathcal{H}_u}$ is unitary.

There are some certain classes of contractions which are important in understanding the structure of a contraction (see \((21)\)). We say that a contraction $T \in \mathcal{B}(\mathcal{H})$ belongs to the class $C_{0}$ if

$$\lim_{n \to \infty} \| T^{*n}h \| = 0 \text{ for all } h \in \mathcal{H},$$

and $T$ belongs to the class $C_{1}$ if

$$\inf_{n} \| T^{*n}h \| > 0 \text{ for all non zero } h \in \mathcal{H}.$$ 

Also a contraction $T$ belongs to the class $C_{0}$ (or $C_{1}$ ) if $T^*$ belongs to $C_{0}$ (or $C_{1}$ ). The class $C_{0} \cap C_{0}$ is denoted by $C_{00}$, that is, a contraction $T \in \mathcal{B}(\mathcal{H})$ belongs to the class $C_{00}$ if

$$\lim_{n \to \infty} \| T^{n}h \| = 0 = \lim_{n \to \infty} \| T^{*n}h \| \text{ for all } h \in \mathcal{H}. $$

The class $C_{11}$ is defined as $C_{1} \cap C_{1}$. For example, an operator $T \in \mathcal{B}(\ell^2(\mathbb{Z}^+))$ is defined by

$$T e_n = \frac{1}{n + 2} e_n \quad (n \in \mathbb{Z}^+, \{ e_n \} \text{ standard orthonormal basis for } \ell^2(\mathbb{Z}^+))$$

and belongs to the class $C_{00}$.

An operator $T$ on $\mathcal{H}$ is called a partial isometry if $\| Tx \| = \| x \|$ for every $x \in (\ker T)^\perp$. The space $(\ker T)^\perp$ is called the initial space of $T$, and ran $T$ is called its final space. We say that $T$ is a power partial isometry if each $T^n$ is a partial isometry for $n \geq 1$. It is easy to see that isometry, co-isometry are examples of power partial isometry.

Let us turn to the definition of tuples of twisted and doubly twisted operators. The notion of $\mathcal{U}_n$-twisted contractions was introduced in our earlier paper \((17)\). We shall here call it as doubly twisted contractions.

Let $n > 1$ and for $1 \leq i < j \leq n$, $\{ U_{ij} \}$ be $\binom{n}{2}$ commuting unitaries on a Hilbert space $\mathcal{H}$ such that $U_{ji} := U_{ij}^*$. Then we refer $\{ U_{ij} \}_{i<j}$ as a twist on $\mathcal{H}$.
Definition 2.3. (Twisted) An $n$-tuple $(T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ is said to be twisted with respect to a twist $\{U_{ij}\}_{i<j}$ if

$$T_iT_j = U_{ij}T_jT_i \quad \text{and} \quad T_kU_{ij} = U_{ij}T_k$$

for all $i, j, k = 1, \ldots, n$ and $i \neq j$.

Definition 2.4. (Doubly Twisted) An $n$-tuple $(T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ is said to be doubly twisted with respect to a twist $\{U_{ij}\}_{i<j}$ if

$$T_iT_j = U_{ij}T_jT_i \quad \text{and} \quad T_kU_{ij} = U_{ij}T_k$$

for all $i, j, k = 1, \ldots, n$ and $i \neq j$.

In particular, if $U_{ij} = I_{\mathcal{H}}$ for $1 \leq i < j \leq n$, then $n$-tuple $(T_1, \ldots, T_n)$ of twisted (or doubly twisted) becomes $n$-tuple of commuting (or doubly commuting) operators. In particular, a pair $(T_1, T_2)$ of operators on $\mathcal{H}$ is called twisted pair with respect to a twist $\{U\}$ if $T_1T_2 = UT_2T_1$ and $T_1U = UT_1$ for $i = 1, 2$. A pair $(T_1, T_2)$ of operators on $\mathcal{H}$ is called doubly twisted pair with respect to a twist $\{U\}$ if $T_1T_2 = UT_2T_1$, $T_1^*T_2 = U^*T_2T_1^*$ and $T_1U = UT_1$ for $i = 1, 2$. We shall call $(T_1, T_2)$ as twisted operators or a doubly twisted operators without referring a twist $\{U\}$. A commuting pair $(T_1, T_2)$ of operators on $\mathcal{H}$ is said to be doubly commuting if $T_1T_2 = T_2^*T_1$.

3. Decomposition for pairs of operators

In this section, we studied decomposition results for pairs of doubly twisted as well as twisted contractions. The following result, which has been shown in our earlier paper [17] will be used frequently in the sequel. However, we present new proof for the sake of completeness.

Theorem 3.1. Let $(T, V)$ be a pair of doubly twisted operators on a Hilbert space $\mathcal{H}$ such that $T$ is a contraction. Let $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{-u}$ be the canonical decomposition for the contraction $T$. Then the decomposition reduces the operator $V$.

Proof. Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a contraction. Then $T^*T \leq I_{\mathcal{H}}$ and also $TT^* \leq I_{\mathcal{H}}$. Now define the defect operators

$$D_T = (I_{\mathcal{H}} - T^*T)^{1/2} \quad \text{and} \quad D_{T^*} = (I_{\mathcal{H}} - TT^*)^{1/2}$$

which are positive operators and bounded by 0 and 1. Since $(T, V)$ is a pair of doubly twisted operators, there is a unitary operator (twist) $U \in \mathcal{B}(\mathcal{H})$ with $T, V \in \{U\}'$ such that

$$TV = UVT \quad \text{and} \quad VT^* = UT^*V.$$ 

Thus

$$VD_T^2 = V - VT^*T = V - UT^*VT = V - UT^*U^*TV = (I_{\mathcal{H}} - T^*T)V = D_T^2V.$$ 

Now using iteration

$$V(D_T^2)^n = (D_T^2)^nV \quad \text{for } n = 0, 1, 2, \ldots.$$ 

Therefore

$$Vp(D_T^2) = p(D_T^2)V \quad \text{(3.1)}$$
for every polynomial \( p(\lambda) = \alpha_0 + \alpha_1 \lambda + \cdots + \alpha_k \lambda^k \). Thus we can choose a sequence of polynomials \( p_n(\lambda) \) that tends to the function \( \lambda^{1/2} \) uniformly in the interval \( 0 \leq \lambda \leq 1 \). Since \( D_T^2 \) is a positive operator also bounded by 0 and 1, by spectral representation of \( D_T^2 \) there is a sequence of operators \( p_n(D_T^2) \) converges to \( D_T \) in the operator norm. Letting \( n \to \infty \) from \( (3.1) \), we get 

\[
VD_T = D_TV.
\]

Similarly, we obtain 

\[
VD_{T*} = D_{T*}V.
\]

Suppose \( T \in \mathcal{B}(\mathcal{H}) \) and \( h \in \mathcal{H} \) such that \( \|Th\| = \|h\| \). Then 

\[
<h, h> = \|h\|^2 = \|Th\|^2 = <T^*Th, h>.
\]

Thus \( <D_T^2h, h> = 0 \) if and only if \( \|D_T h\| = 0 \). Hence the set \( \{h \in \mathcal{H} : \|Th\| = \|h\|\} \) coincides with \( N_{D_T} = \{h \in \mathcal{H} : D_TH = 0\} \) (or \( N_{D_T^*} = \ker(D_T) \)) which is a subspace of \( \mathcal{H} \). Consider \( T(m) = T^m \) \((m \geq 1)\), \( T(0) = I\), \( T(m) = T^m \) \((m \leq -1)\). For fixed integer \( m \in \mathbb{Z}, T(m) \) is a contraction on \( \mathcal{H} \). Therefore, the set \( \{h \in \mathcal{H} : \|T(m)h\| = \|h\|\} \) is same as \( N_{D_T(m)} = \{h \in \mathcal{H} : D_{T(m)}h = 0\} \) which is a subspace of \( \mathcal{H} \). Again consider \( \mathcal{H}_u = \{h \in \mathcal{H} : \|T(m)h\| = \|h\|, m \in \mathbb{Z}\} \). Then \( \mathcal{H}_u \) can be expressed as

\[
\mathcal{H}_u = \bigcap_{m=-\infty}^{\infty} N_{D_T(m)}, \text{ where } D_T(m) = \begin{cases} (I - T^*mT^m)^{\frac{1}{2}} & \forall m \geq 0, \\ (I - T^{|m|}T^{|m|})^{\frac{1}{2}} & \forall m \leq -1. \end{cases}
\]

Since \( D_T \) and \( D_{T*} \) are self-adjoint operators and the pairs \((V, D_T)\) and \((V, D_{T*})\) are commuting, the pairs \((V, D_{T(m)})\) and \((V, D_{T^*m})\) are doubly commuting. Let \( h \in \mathcal{H}_u = \bigcap_{m=-\infty}^{\infty} N_{D_T(m)} \). Then for each fixed \( m \geq 0 \)

\[
D_{T(m)}Vh = VD_{T(m)}h = 0 \text{ and } D_{T(m)}V_*h = V_*D_{T(m)}h = 0.
\]

Therefore \( Vh \) and \( V_*h \in N_{D_{T(m)}} \). Similarly \( Vh \) and \( V_*h \in N_{D_{T^*m}} \) for each fixed \( m \leq -1 \). It says that \( V(\mathcal{H}_u), V_*(\mathcal{H}_u) \subseteq \mathcal{H}_u \). Again \( \mathcal{H}_u \) and \( \mathcal{H}_{-u} \) are orthogonal subspaces of \( \mathcal{H} \). Hence the canonical decomposition \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{-u} \) reduces \( V \).

The immediate consequences of the above result are as follows:

**Remark 3.2.** Let \((T_1, T_2)\) be a pair of doubly twisted contractions on \( \mathcal{H} \). Let \( \mathcal{H} = \mathcal{H}_{u1} \oplus \mathcal{H}_{-u1} \) be the canonical decomposition for the contraction \( T_1 \), where \( T_1|_{\mathcal{H}_{u1}} \) is unitary and \( T_1|_{\mathcal{H}_{-u1}} \) is c.n.u. Then by the above Theorem 3.1, the decomposition reduces the other contraction \( T_2 \). Indeed, the contraction \( T_2 \) breaks into two contractions \( T_2|_{\mathcal{H}_{u1}} \) and \( T_2|_{\mathcal{H}_{-u1}} \) with respect to the decomposition \( \mathcal{H} = \mathcal{H}_{u1} \oplus \mathcal{H}_{-u1} \). Now repeating the process for the contractions \( T_2|_{\mathcal{H}_{u1}} \) and \( T_2|_{\mathcal{H}_{-u1}} \), we have

\[
\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{u-u} \oplus \mathcal{H}_{-uu} \oplus \mathcal{H}_{-u-u},
\]

where

- \( T_1|_{\mathcal{H}_{uu}} \) and \( T_2|_{\mathcal{H}_{uu}} \) are both unitaries,
- \( T_1|_{\mathcal{H}_{u-u}} \) is unitary and \( T_2|_{\mathcal{H}_{u-u}} \) is c.n.u.,
- \( T_1|_{\mathcal{H}_{-uu}} \) is c.n.u. and \( T_2|_{\mathcal{H}_{-uu}} \) is unitary,
- \( T_1|_{\mathcal{H}_{-u-u}} \) and \( T_2|_{\mathcal{H}_{-u-u}} \) are both c.n.u.
Suppose that \( h \in H \) is a contraction. Therefore, from Theorem 3.1

\[ \text{Corollary 3.3.} \text{ Let } (T, V) \text{ be a pair of doubly commuting operators on a Hilbert space } H \text{ such that } T \text{ is a contraction. Then the canonical decomposition } H = H_u \oplus H_{-u} \text{ for } T \text{ reduces the operator } V. \]

**Remark 3.4.** Since \( T \in \{U\}' \) and \( U \) is unitary, the pair \((T, U)\) is doubly commuting and hence the canonical decomposition \( H = H_u \oplus H_{-u} \) for \( T \) reduces \( U \).

The following decomposition result holds for certain pairs of twisted operators.

**Theorem 3.5.** Let \((T, V)\) be a pair of twisted operators on a Hilbert space \( H \) and let \( T \) be a contraction. If \( H = H_u \oplus H_{-u} \) is the canonical decomposition for \( T \) and \( T|_{H_u} \) is in the class \( C_{00} \), then the decomposition \( H = H_u \oplus H_{-u} \) reduces \( V \).

**Proof.** Suppose that \( H = H_u \oplus H_{-u} \) is the canonical decomposition for the contraction \( T \in \mathcal{B}(H) \) and \( T|_{H_u} \) is in the class \( C_{00} \). Then the matricial representation for \( T \) with the canonical decomposition \( H = H_u \oplus H_{-u} \) is \( \begin{bmatrix} T_u & O \\ O & T_{-u} \end{bmatrix} \), where \( T_u = T|_{H_u} \) and \( T_{-u} = T|_{H_{-u}} \).

Since the pair \((T, V)\) is twisted, there is a unitary (twist) \( U \in \mathcal{B}(H) \) such that

\[
TU = UT, \quad VU = UV \quad \text{and} \quad TV = UV T. 
\]

Now the pair \((T, U)\) is doubly commuting on \( H \) as \( U \) is unitary. Therefore, from Corollary 3.3 the canonical decomposition \( H = H_u \oplus H_{-u} \) reduces \( U \). Hence the matricial representation of \( U \) on \( H = H_u \oplus H_{-u} \) is \( \begin{bmatrix} U_1 & O \\ O & U_2 \end{bmatrix} \), where \( U_1 = U|_{H_u} \) and \( U_2 = U|_{H_{-u}} \) are unitaries on \( H_u \) and \( H_{-u} \), respectively. Suppose that the matricial representation for \( V \) is \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) with respect to the decomposition \( H = H_u \oplus H_{-u} \). Then the condition \( TV = UV T \) implies that

\[
\begin{bmatrix} T_u A & T_u B \\ T_{-u} C & T_{-u} D \end{bmatrix} = \begin{bmatrix} U_1 & O \\ O & U_2 \end{bmatrix} \begin{bmatrix} A T_u & B T_{-u} \\ C T_u & D T_{-u} \end{bmatrix} = \begin{bmatrix} U_1 A T_u & U_1 B T_{-u} \\ U_2 C T_u & U_2 D T_{-u} \end{bmatrix}.
\]

Therefore,

\[
T_u B = U_1 B T_{-u} \quad \text{and} \quad T_{-u} C = U_2 C T_u.
\]

Thus

\[
T_u^n B = U_1^n B T_{-u}^n \quad \text{and} \quad T_{-u}^n C = U_2^n C T_u^n \quad \text{for all } n \geq 1.
\]

Now for any \( h \in H_{-u} \),

\[
\|B h\| = \|T_u^n B h\| = \|U_1^n B T_{-u}^n h\| = \|B T_{-u}^n h\| \leq \|B\|\|T_{-u}^n h\| \to 0
\]
as \( n \to \infty \). Therefore, \( Bh = 0 \) for all \( h \in H_{-u} \). This implies that \( B = 0 \). Again for any \( h \in H_{-u} \),

\[
\|C^* h\| = \|T_u^n C^* h\| = \|U_2^n C^* T_u^n h\| = \|C^* T_u^n h\| \leq \|C^*\|\|T_u^n h\| \to 0
\]
as \( n \to \infty \). This implies that \( C^*h = 0 \) for all \( h \in \mathcal{H}_{-u} \). Therefore \( C = 0 \). Hence \( \mathcal{H}_u \) reduces \( V \). Therefore, the decomposition \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{-u} \) reduces \( V \).

This completes the proof. \( \blacksquare \)

There are few remarks and consequences of the above theorem:

**Remark 3.6.** One of the important facts in the above theorem is that the decomposition for the pair of operators does not require the doubly twisted condition.

**Remark 3.7.** If the c.n.u. parts for a pair of twisted contractions are in \( C_{00} \), then one can find decomposition for the twisted pairs which need not be doubly twisted (see Example 3.10).

Now we will give some concrete examples of pairs of twisted operators satisfying certain properties.

**Example 3.8.** Let \( \Lambda = \{(i, j) : i \geq 0 \text{ or } j \geq 0\} \) and \( \{e_{i,j}\}_{(i,j)\in \Lambda} \) be a sequence of orthonormal vectors in some Hilbert space \( \mathcal{K} \). Let \( \mathcal{H} = \text{span}\{e_{i,j}\}_{(i,j)\in \Lambda} \). Suppose that \( T_1, T_2 \) are two operators on \( \mathcal{H} \) such that
\[
T_1e_{i,j} = e_{i+1,j} \quad \text{and} \quad T_2e_{i,j} = e_{ij+1} \quad \text{for } (i, j) \in \Lambda.
\]
Clearly, \( (T_1, T_2) \) is a pair of commuting isometries, that is, twisted isometries on \( \mathcal{H} \) with respect to the twist \( I_\mathcal{H} \). Suppose that \( \mathcal{H} = \mathcal{H}_{u2} \oplus \mathcal{H}_{s2} \) is the Wold decomposition for \( T_2 \) on \( \mathcal{H} \). Then it is easy to see that
\[
\mathcal{H}_{u2} = \bigcap_{n=0}^\infty T_2^n \mathcal{H} = \text{span}\{e_{i,j} : i \geq 0\}.
\]
Now \( e_{0,1} \in \mathcal{H}_{u2} \) but \( e_{-1,1} \notin \mathcal{H}_{u2} \). Again \( T_1^* e_{0,1} = e_{-1,1} \). Therefore, \( \mathcal{H}_{u2} \) does not reduce the operator \( T_1 \) and hence, by Theorem 3.1, the pair \( (T_1, T_2) \) is not doubly twisted. Also \( T_2|_{\mathcal{H}_{s2}} \) does not belong to \( C_{00} \). Similarly, \( T_1|_{\mathcal{H}_{s2}} \) does not belong to \( C_{00} \). Hence the pair \( (T_1, T_2) \) of twisted operators is neither doubly twisted nor \( T_1|_{\mathcal{H}_{u1}}, T_2|_{\mathcal{H}_{s2}} \) belong to \( C_{00} \).

**Example 3.9.** Let \( H^2(\mathbb{D}) \) denotes the Hardy space over the unit disc \( \mathbb{D} \). Now for fixed \( r \) with \( |r| = 1 \), define a weighted shift operator \( S_r \) on \( H^2(\mathbb{D}) \) as
\[
S_rz^n = r^{n+1}z^{n+1} \quad (n \in \mathbb{Z}_+)
\]
where \( \{1, z, z^2, \ldots \} \) is an orthonormal basis for \( H^2(\mathbb{D}) \). Let \( M_z \) be the multiplication operator on \( H^2(\mathbb{D}) \) by the coordinate function \( z \). Then
\[
(M_zS_r)(z^n) = r^{n+1}z^{n+2} \quad \text{and} \quad (S_rM_z)(z^n) = r^{n+2}z^{n+2} \quad \text{for } n \in \mathbb{Z}_+.
\]
Again,
\[
(M_z^*S_r)(z^n) = r^{n+1}z^n \quad \forall \ n \geq 0
\]
and
\[
(S_rM_z^*)(z^n) = \begin{cases} r^n z^n & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases}
\]
Hence \( S_r M_z = r M_z S_r \) but \( M_z^*S_r \neq r S_r M_z^* \). Now define a unitary \( U \) on \( H^2(\mathbb{D}) \) as \( U(z^n) = rz^n \) for \( n \geq 0 \). Then it is easy to see that the pair \( (S_r, M_z) \) is a twisted contractions with the
twist $U$ on $H^2(\mathbb{D})$, but not doubly twisted. Also the c.n.u. parts of both the operators are $H^2(\mathbb{D})$, and do not belong to $C_{00}$.

**Example 3.10.** Let $\mathcal{K}$ be a Hilbert space and $k$ be a fixed positive integer. Define an operator $T$ on the $k$-fold direct sum $\mathcal{H} = \mathcal{K} \oplus \cdots \oplus \mathcal{K}$ as
\[
T(h_1, h_2, \ldots, h_k) = (0, h_1, \ldots, h_{k-1}) \quad \text{for } h_i \in \mathcal{K}.
\]
Then $(T, T)$ is a twisted pair with a twist $I_\mathcal{H}$ (or commuting pair) of truncated shifts of index $k$ on $\mathcal{H}$. Clearly, the adjoint of $T$ on $\mathcal{H}$, denoted as $T^*$, is defined by
\[
T^*(h_1, h_2, \ldots, h_k) = (h_2, \ldots, h_k, 0).
\]
It is easy to see that
\[
TT^* \neq T^*T.
\]
Hence the pair $(T, T)$ is not doubly twisted with a twist $I_\mathcal{H}$ (or not doubly commuting pair).

Now from the canonical decomposition for the contraction $T$, the c.n.u. part is $\mathcal{H}$ (as unitary part is absent). Further, $T$ is in $C_{00}$, in general.

**Example 3.11.** The weighted shift $M_\alpha^0$ on the Hardy space $H^2(\mathbb{D})$ is defined by $M_\alpha^0(f) = \alpha zf$ for all $f \in H^2(\mathbb{D})$, where $z$ is the co-ordinate function and $|\alpha| \leq 1$. For each fixed $r$ with $|r| = 1$, define an operator $A_r$ on $H^2(\mathbb{D})$ as
\[
A_r z^n = r^n z^n \quad (n \in \mathbb{Z}_+),
\]
where $\{1, z, z^2, \ldots\}$ is an orthonormal basis for $H^2(\mathbb{D})$. Then
\[
(M_\alpha^0 A_r)(z^n) = \alpha r^n z^{n+1} \quad \text{and} \quad (A_r M_\alpha^0)(z^n) = \alpha r^{n+1} z^{n+1} \quad \text{for } n \in \mathbb{Z}_+.
\]
Again
\[
([M_\alpha^0] A_r)(z^n) = \begin{cases}
\bar{\alpha} r^n z^{n-1}, & \text{if } n \geq 1 \\
0 & \text{if } n = 0,
\end{cases}
\]
and
\[
(A_r [M_\alpha^0]^*)(z^n) = \begin{cases}
\bar{\alpha} r^{n-1} z^{n-1} & \text{if } n \geq 1 \\
0 & \text{if } n = 0.
\end{cases}
\]
Hence $A_r M_\alpha^0 = r M_\alpha^0 A_r$ and $[M_\alpha^0]^* A_r = r A_r [M_\alpha^0]^*$. Clearly, $(A_r, M_\alpha^0)$ is a pair of doubly twisted operators with respect to the twist $U = rI$ on $H^2(\mathbb{D})$. Here c.n.u. part of $A_r, M_\alpha^0$ is $\{0\}$ and $H^2(\mathbb{D})$, respectively. Moreover, $M_\alpha^0 \notin C_{00}$.

**Example 3.12.** Let $T_1, T_2$ be the contractions on $\ell^2(\mathbb{Z})$ defined as
\[
T_1(e_n) = \frac{r^n}{4} e_{n+1} \quad \text{and} \quad T_2(e_n) = \lambda e_{n+1}
\]
with $|\lambda| < 1$, $|r| = 1$ and $\{e_n\}$ is the standard orthonormal basis for $\ell^2(\mathbb{Z})$. Let $U = rI$ be a unitary operator on $\ell^2(\mathbb{Z})$. Then
\[
T_1 T_2(e_n) = \frac{\lambda r^{n+1}}{4} e_{n+2}, \quad T_2 T_1(e_n) = \frac{\lambda r^n}{4} e_{n+2}.
\]
Again
\[ T_1T_2^*(e_n) = \frac{\lambda r^n - 1}{4} e_n, \quad T_2^*T_1(e_n) = \frac{\lambda r^n}{4} e_n. \]
Clearly, \( T_1T_2^* = UT_2T_1 \) and \( T_2^*T_1 = U^*T_2T_1^* \). Therefore, \((T_1, T_2)\) is a pair of doubly twisted contractions on \( \ell^2(\mathbb{Z}) \) with a twist \( U \). Moreover, the c.n.u. part for \( T_1, T_2 \) is \( \ell^2(\mathbb{Z}) \) and both are in \( C_{\infty} \).

The following particular result follows from the above Theorem 3.13.

**Corollary 3.13.** Let \((T, V)\) be a pair of twisted operators on a Hilbert space \( \mathcal{H} \) such that \( T \) is a contraction. Let \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \) be the canonical decomposition for \( T \). If \( T|_{\mathcal{H}_u} \in C_0 \), (or \( T|_{\mathcal{H}_s} \in C_0 \)), then \( \mathcal{H}_u \) (or \( \mathcal{H}_s \)) is invariant under \( V \).

Now suppose that \((T, V)\) is a pair of twisted operators on a Hilbert space \( \mathcal{H} \) such that \( V \) is isometry. Then from the Wold decomposition for \( V \), we have \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \), where \( V|_{\mathcal{H}_u} \) is unitary and \( V|_{\mathcal{H}_s} \) is shift, that is, \( V|_{\mathcal{H}_s} \) is in \( C_0 \). If \( V \) is a co-isometry, then we have \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \), where \( V|_{\mathcal{H}_u} \) is unitary and \( V|_{\mathcal{H}_s} \) is co-shift, i.e., \( V|_{\mathcal{H}_s} \) is in \( C_0 \). Therefore, we have the following results from the above Corollary.

**Lemma 3.14.** Let \((T, V)\) be a pair of twisted operators on a Hilbert space \( \mathcal{H} \) such that \( V \) is isometry (or \( V \) is co-isometry). Let \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \) be the Wold decomposition for \( V \). Then \( \mathcal{H}_u \) (or \( \mathcal{H}_s \)) is invariant under \( T \).

We now present the the well-known result of Słońcinski [23] for a pair of doubly twisted isometries by using the above results. One can see our recent paper [17] for more results.

**Theorem 3.15.** Let \((V_1, V_2)\) be a pair of doubly twisted isometries on a Hilbert space \( \mathcal{H} \). Then there is a unique decomposition
\[ \mathcal{H} = \mathcal{H}_uu \oplus \mathcal{H}_us \oplus \mathcal{H}_su \oplus \mathcal{H}_{ss}, \]
where \( \mathcal{H}_uu, \mathcal{H}_us, \mathcal{H}_su, \) and \( \mathcal{H}_{ss} \) are the subspaces reducing \( V_1 \) and \( V_2 \) such that
- \( V_1|_{\mathcal{H}_uu}, V_2|_{\mathcal{H}_uu} \) are unitary operators,
- \( V_1|_{\mathcal{H}_us} \) is unitary, \( V_2|_{\mathcal{H}_us} \) is shift,
- \( V_1|_{\mathcal{H}_su} \) is shift, \( V_2|_{\mathcal{H}_su} \) is unitary,
- \( V_1|_{\mathcal{H}_{ss}}, V_2|_{\mathcal{H}_{ss}} \) are shifts.

**Proof.** Suppose \((V_1, V_2)\) is a pair of doubly twisted isometries with respect to a twist \( U \) on \( \mathcal{H} \). Then the Wold decomposition for \( V_1 \) gives
\[ \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s, \]
where \( \mathcal{H}_u \) reduces \( V_1; V_1|_{\mathcal{H}_u} \) is unitary and \( V_1|_{\mathcal{H}_s} \) is shift. Since \((V_1, V_2)\) is a doubly twisted pair on \( \mathcal{H} \), by Theorem 3.11 the subspaces \( \mathcal{H}_u \) and \( \mathcal{H}_s \) reduce the isometry \( V_2 \) and the unitary \( U \). Consequently, \( U|_{\mathcal{H}_u} \) and \( U|_{\mathcal{H}_s} \) are unitary operators on \( \mathcal{H}_u \) and \( \mathcal{H}_s \), respectively. Now Wold-von Neumann decomposition for the isometries \( V_2|_{\mathcal{H}_u} \) on \( \mathcal{H}_u \) and \( V_2|_{\mathcal{H}_s} \) on \( \mathcal{H}_s \) yield
\[ \mathcal{H}_u = \mathcal{H}_uu \oplus \mathcal{H}_{us}, \quad \text{and} \quad \mathcal{H}_s = \mathcal{H}_{su} \oplus \mathcal{H}_{ss}, \]
where $\mathcal{H}_{uu}$ and $\mathcal{H}_{su}$ reduce $V_2$ to unitary operators and $\mathcal{H}_{us}, \mathcal{H}_{ss}$ reduce $V_2$ to unilateral shifts. Again the pairs $(V_1|_{\mathcal{H}_{ua}}, V_2|_{\mathcal{H}_{ua}})$ and $(V_1|_{\mathcal{H}_{us}}, V_2|_{\mathcal{H}_{us}})$ are doubly twisted with respect to twist $U|_{\mathcal{H}_{u}}$ and $U|_{\mathcal{H}_{s}}$ on $\mathcal{H}_{u}$ and $\mathcal{H}_{s}$, respectively. Therefore, by Theorem 3.11 the subspaces $\mathcal{H}_{uu}$ and $\mathcal{H}_{us}$ reduce the unitary $V_1|_{\mathcal{H}_{u}}$ as well as subspaces $\mathcal{H}_{su}$ and $\mathcal{H}_{ss}$ reduce the shift $V_1|_{\mathcal{H}_{s}}$.

This completes the proof.

Every completely non-unitary co-isometry is a co-shift (that is, adjoint of shift) and hence it is in $C_0$. If $T \in \mathcal{B}(\mathcal{H})$ is co-isometry, then the canonical decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ reduce $T$, where $T|_{\mathcal{H}_0}$ is unitary and $T|_{\mathcal{H}_1}$ is in $C_0$. So in particular, if $(V, W)$ is a twisted pair consisting of an isometry and a co-isometry, then it is doubly twisted (For proof see Section 5) and hence we have the following decomposition result. The proof is omitted as it is similar to the above proof.

**Corollary 3.16.** Let $(V, W)$ be a pair of twisted operators on a Hilbert space $\mathcal{H}$ such that $V$ is isometry and $W$ is co-isometry. Then there is a unique decomposition

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{ub} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{sb}$$

where $\mathcal{H}_{uu}, \mathcal{H}_{ub}, \mathcal{H}_{su},$ and $\mathcal{H}_{sb}$ are the subspaces reducing $V, W$ such that

- $V|_{\mathcal{H}_{uu}}$ and $W|_{\mathcal{H}_{uu}}$ are unitary operators,
- $V|_{\mathcal{H}_{ub}}$ is unitary and $W|_{\mathcal{H}_{ub}}$ is co-shift,
- $V|_{\mathcal{H}_{su}}$ is shift and $W|_{\mathcal{H}_{su}}$ is unitary,
- $V|_{\mathcal{H}_{sb}}$ is shift and $W|_{\mathcal{H}_{sb}}$ is co-shift.

In the next result, we obtain an explicit Wold-type decomposition for a pair of twisted contractions satisfying certain conditions.

**Theorem 3.17.** Let $(T_1, T_2)$ be a pair of twisted contractions on a Hilbert space $\mathcal{H}$. Let $\mathcal{H} = \mathcal{H}_{ui} \oplus \mathcal{H}_{-ui}$ be the canonical decomposition for $T_i$ for $i = 1, 2$. If $T_i|_{\mathcal{H}_{-ui}}$ is in $C_{00}$ for $i = 1, 2$, then there is a unique decomposition

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{u-u} \oplus \mathcal{H}_{-uu} \oplus \mathcal{H}_{-u-u},$$

where $\mathcal{H}_{uu}, \mathcal{H}_{u-u}, \mathcal{H}_{-uu},$ and $\mathcal{H}_{-u-u}$ are $(T_1, T_2)$ reducing subspaces of $\mathcal{H}$ such that

- $T_1|_{\mathcal{H}_{uu}}, T_2|_{\mathcal{H}_{uu}}$ are unitary operators,
- $T_1|_{\mathcal{H}_{u-u}}$ is unitary, $T_2|_{\mathcal{H}_{u-u}}$ is completely non-unitary,
- $T_1|_{\mathcal{H}_{-uu}}$ is completely non-unitary, $T_2|_{\mathcal{H}_{-uu}}$ is unitary,
- $T_1|_{\mathcal{H}_{-u-u}}, T_2|_{\mathcal{H}_{-u-u}}$ are completely non-unitary operators.
Furthermore, the decomposition spaces are as follows

\[ \mathcal{H}_{uu} = \bigcap_{m_2 \in \mathbb{Z}_+} \ker((I - T_2^{m_2}T_2^{m_2})|_{\mathcal{H}_{u_1}}) \cap \ker((I - T_2^{m_2}T_2^{m_2})|_{\mathcal{H}_{u_1}}), \]

\[ \mathcal{H}_{u-u} = \bigvee_{m_2 \in \mathbb{Z}_+} \{(I - T_2^{m_2}T_2^{m_2})\mathcal{H}_{u_1} \cup (I - T_2^{m_2}T_2^{m_2})\mathcal{H}_{u_1} \}, \]

\[ \mathcal{H}_{u-u} = \bigcap_{m_2 \in \mathbb{Z}_+} \ker((I - T_2^{m_2}T_2^{m_2})|_{\mathcal{H}_{u-u}}) \cap \ker((I - T_2^{m_2}T_2^{m_2})|_{\mathcal{H}_{u-u}}), \]

\[ \mathcal{H}_{u-u} = \bigvee_{m_2 \in \mathbb{Z}_+} \{(I - T_2^{m_2}T_2^{m_2})\mathcal{H}_{u-u} \cup (I - T_2^{m_2}T_2^{m_2})\mathcal{H}_{u-u} \}, \]

and

\[ \mathcal{H}_{u_1} = \bigcap_{m_1 \in \mathbb{Z}_+} \ker(I - T_1^{m_1}T_1^{m_1}) \cap \ker(I - T_1^{m_1}T_1^{m_1}), \]

\[ \mathcal{H}_{u-u_1} = \bigvee_{m_1 \in \mathbb{Z}_+} \{(I - T_1^{m_1}T_1^{m_1})\mathcal{H} \cup (I - T_1^{m_1}T_1^{m_1})\mathcal{H} \}. \]

**Proof.** Suppose that \((T_1, T_2)\) is a pair of twisted contractions with respect to a twist \(U\) on a Hilbert space \(\mathcal{H}\). Then the canonical decomposition for \(T_1\) gives

\[ \mathcal{H} = \mathcal{H}_{u_1} \oplus \mathcal{H}_{u-u_1} \]

where \(T_1|_{\mathcal{H}_{u_1}}, T_1|_{\mathcal{H}_{u-u_1}}\) are unitary and c.n.u., respectively. Also, for the contraction \(T_1\) on \(\mathcal{H}\), we have the decomposition spaces \(\mathcal{H}_{u_1}\) and \(\mathcal{H}_{u-u_1}\) (see our recent work [17] for more details) as

\[ \mathcal{H}_{u_1} = \bigcap_{m_1 \in \mathbb{Z}_+} \ker(I - T_1^{m_1}T_1^{m_1}) \cap \ker(I - T_1^{m_1}T_1^{m_1}), \]

\[ \mathcal{H}_{u-u_1} = \bigvee_{m_1 \in \mathbb{Z}_+} \{(I - T_1^{m_1}T_1^{m_1})\mathcal{H} \cup (I - T_1^{m_1}T_1^{m_1})\mathcal{H} \}. \]

Since by hypothesis \(T_1|_{\mathcal{H}_{u-u_1}} \in C_{00}\), by Theorem 3.3 the subspaces \(\mathcal{H}_{u_1}, \mathcal{H}_{u-u_1}\) reduce the contraction \(T_2\) as well as the unitary \(U\). Thus, \(U|_{\mathcal{H}_{u_1}}\) and \(U|_{\mathcal{H}_{u-u_1}}\) are unitaries on \(\mathcal{H}_{u_1}\) and \(\mathcal{H}_{u-u_1}\), respectively. Therefore, \((T_1|_{\mathcal{H}_{u_1}}, T_2|_{\mathcal{H}_{u_1}})\) and \((T_1|_{\mathcal{H}_{u-u_1}}, T_2|_{\mathcal{H}_{u-u_1}})\) are pairs of twisted contractions with respect to the twists \(U|_{\mathcal{H}_{u_1}}\) and \(U|_{\mathcal{H}_{u-u_1}}\) on \(\mathcal{H}_{u_1}\) and \(\mathcal{H}_{u-u_1}\), respectively. Again the canonical decomposition for the contractions \(T_2|_{\mathcal{H}_{u_1}}\) and \(T_2|_{\mathcal{H}_{u-u_1}}\) yield

\[ \mathcal{H}_{u_1} = \mathcal{H}_{uu} \oplus \mathcal{H}_{u-u_1}, \quad \text{and} \quad \mathcal{H}_{u-u_1} = \mathcal{H}_{uu} \oplus \mathcal{H}_{u-u_1}, \]

where \(T_2|_{\mathcal{H}_{uu}}, T_2|_{\mathcal{H}_{u-u}}\) are unitaries and \(T_2|_{\mathcal{H}_{u_u}}, T_2|_{\mathcal{H}_{u-u}}\) are c.n.u. By the assumption \(T_2|_{\mathcal{H}_{uu}} \in C_{00}\), that means, \(T_2|_{\mathcal{H}_{u_u}}, T_2|_{\mathcal{H}_{u_u}} \in C_{00}\) as \(\mathcal{H}_{uu} = \mathcal{H}_{u_u} \oplus \mathcal{H}_{u_u}\). Therefore, by Theorem 3.3 the subspaces \(\mathcal{H}_{uu}\) and \(\mathcal{H}_{u_u}\) reduces the unitary operator \(T_1|_{\mathcal{H}_{u_1}}\) and the subspaces \(\mathcal{H}_{u-u}\) and \(\mathcal{H}_{u-u}\) reduces the completely non-unitary \(T_1|_{\mathcal{H}_{u-u}}\). Now using the decomposition formula for the canonical decomposition of the contractions \(T_2|_{\mathcal{H}_{u_1}}\) and \(T_2|_{\mathcal{H}_{u-u}}\),
we have
\[
\mathcal{H}_{uu} = \bigcap_{m_2 \in \mathbb{Z}_+} \ker(I_{\mathcal{H}_{u_1}} - T_2^{m_2}|_{\mathcal{H}_{u_1} T_2^{m_2}|_{\mathcal{H}_{u_1}}}) \cap \ker(I_{\mathcal{H}_{u_1}} - T_2^{m_2}|_{\mathcal{H}_{u_1} T_2^{m_2}|_{\mathcal{H}_{u_1}}})
\]
\[
= \bigcap_{m_2 \in \mathbb{Z}_+} \ker((I - T_2^{m_2} T_2^{m_2})|_{\mathcal{H}_{u_1}}) \cap \ker((I - T_2^{m_2} T_2^{m_2})|_{\mathcal{H}_{u_1}}),
\]
\[
\mathcal{H}_{u-u} = \bigcup_{m_2 \in \mathbb{Z}_+} (I - T_2^{m_2} T_2^{m_2})\mathcal{H}_{u_1} \cup (I - T_2^{m_2} T_2^{m_2})\mathcal{H}_{u_1},
\]
\[
\mathcal{H}_{-u} = \bigcap_{m_2 \in \mathbb{Z}_+} \ker(I_{\mathcal{H}_{-u_1} - T_2^{m_2}|_{\mathcal{H}_{-u_1} T_2^{m_2}|_{\mathcal{H}_{-u_1}}}) \cap \ker(I_{\mathcal{H}_{-u_1} - T_2^{m_2}|_{\mathcal{H}_{-u_1} T_2^{m_2}|_{\mathcal{H}_{-u_1}}})
\]
\[
= \bigcap_{m_2 \in \mathbb{Z}_+} \ker((I - T_2^{m_2} T_2^{m_2})|_{\mathcal{H}_{-u_1}}) \cap \ker((I - T_2^{m_2} T_2^{m_2})|_{\mathcal{H}_{-u_1}}),
\]
\[
\mathcal{H}_{-u-u} = \bigcup_{m_2 \in \mathbb{Z}_+} (I - T_2^{m_2} T_2^{m_2})\mathcal{H}_{-u_1} \cup (I - T_2^{m_2} T_2^{m_2})\mathcal{H}_{-u_1}.
\]

The uniqueness of the decomposition follows from the uniqueness of the canonical decomposition of a single contraction. This finishes the proof of the theorem.

4. Decomposition for Partial Isometry

Let \( \mathcal{H} = \bigoplus_k \mathcal{H}' \), where \( \mathcal{H}' \) is a Hilbert space. A truncated shift \( R' \) of index \( k \) is defined on the \( k \)-fold direct sum \( \mathcal{H} \) as
\[
R'(h_1, h_2, ..., h_k) = (0, h_1, ..., h_{k-1}) \quad \text{for } k \in \mathbb{Z}_+.
\]

Therefore, the matrix representation of the truncated shift operator \( R' \) of index \( k \) is of the form
\[
R' = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
I_{\mathcal{H}'} & 0 & 0 & \cdots & 0 & 0 \\
0 & I_{\mathcal{H}'} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & \cdots & I_{\mathcal{H}'} & 0
\end{bmatrix}_{k \times k}.
\]

Clearly, it is easy to see that \( R' \in \mathcal{C}_0 \) and \( \ker R' = \mathcal{H}' \), \( R'^* R' = P_{(\ker R')^\perp} \). For an example, we consider an operator \( R \) on \( \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \) as
\[
R = \begin{bmatrix}
0 & 0 & 1 & 0 \\
a & 0 & 0 & \sqrt{1 - |a|^2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \text{where } 0 < |a| < 1.
\]

Then one can check that \( R, R^n (n \geq 3) \) are partial isometries but \( R^2 \) is not a partial isometry. Hence \( R \) is not a power partial isometry.

Now we recall the decomposition theorem of Halmos and Wallen \([14]\) on power partial isometries.
Theorem 4.1. Let $R \in \mathcal{B}(\mathcal{H})$ be a power partial isometry. Then there is a unique decomposition
\[ \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \oplus \mathcal{H}_b \oplus (\oplus_{k \geq 1} \mathcal{H}_k), \]
where $\mathcal{H}_u, \mathcal{H}_s, \mathcal{H}_b$ and $\mathcal{H}_k$, $k \geq 1$ are subspaces of $\mathcal{H}$ reducing $R$ such that
- $R_u = R|_{\mathcal{H}_u}$ is a unitary operator,
- $R_s = R|_{\mathcal{H}_s}$ is a unilateral shift,
- $R_b = R|_{\mathcal{H}_b}$ is a backward shift,
- $R_t = R|_{\mathcal{H}_k}$ is a truncated shift of index $k$.

Moreover,
- $\mathcal{H}_u = \cap_{n \geq 0} R^n \mathcal{H}$, and $\mathcal{H}_b = \cap_{n \geq 0} R^n (\ker R^*)$, and $\mathcal{H}_t = \oplus_{k \geq 1} \mathcal{H}_k = \cap_{n \geq 0} R^n (\ker R^*) \cap \oplus_{n \geq 0} R^n (\ker R)$.

Following Halmos and Wallen [14], one can conclude that for a power partial isometry $R$, $R_u = R|_{\mathcal{H}_u} \in C_{11}$, $R_s = R|_{\mathcal{H}_s} \in C_{10}$, $R_b = R|_{\mathcal{H}_b} \in C_{01}$, and $R_k = R|_{\mathcal{H}_k} \in C_{00}$. Thus any power partial isometry $R$ on $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t$ is of following type:
\[
\begin{pmatrix}
\mathcal{H}_u & \mathcal{H}_b & \mathcal{H}_s & \mathcal{H}_t \\
C_{11} & O & O & O \\
O & C_{01} & O & O \\
O & O & C_{10} & O \\
O & O & O & C_{00}
\end{pmatrix}.
\]

Firstly, we find matricial representation for a doubly commutant of a power partial isometry.

Proposition 4.2. Let $(R, B)$ be a pair of doubly commuting operators on $\mathcal{H}$ such that $R$ is a power partial isometry. If $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t$ is the decomposition for $R$, then this decomposition reduces also the operator $B$.

Proof. Suppose that $(R, B)$ is a pair of operators such that $R$ is a power partial isometry and $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t$ is the decomposition for $R$.

Let $\mathcal{K}_1 = \mathcal{H}_u \oplus \mathcal{H}_b$ and $\mathcal{K}_2 = \mathcal{H}_s \oplus \mathcal{H}_t$. If $RB = BR$, then one can prove (similar to proof of the Theorem 3.5), $\mathcal{K}_1$ is an invariant subspace for $B$. So the matrix representation of $B$ with respect to the decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$ is of the form
\[ B = \begin{bmatrix} B_1 & * \\ O & B_2 \end{bmatrix}, \]
where $B_1 = B|_{\mathcal{K}_1}$, and $B_2 = P_{\mathcal{K}_2}B|_{\mathcal{K}_2}$.

Now $B_1$ is a bounded operator on $\mathcal{K}_1 = \mathcal{H}_u \oplus \mathcal{H}_b$ and let $B_1 = \begin{bmatrix} B_{11} & B_{21} \\ B_{31} & B_{41} \end{bmatrix}$ on $\mathcal{K}_1$. Then
\[ R|_{\mathcal{K}_1}B|_{\mathcal{K}_1} = B|_{\mathcal{K}_1}R|_{\mathcal{K}_1}. \]


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yields
\[
\begin{bmatrix}
    R_u B_{11} & R_u B_{21} \\
    R_b B_{31} & R_b B_{41}
\end{bmatrix}
= \begin{bmatrix}
    B_{11} R_u & B_{21} R_b \\
    B_{31} R_u & B_{41} R_b
\end{bmatrix}.
\]

Therefore, \( R_u B_{21} = B_{21} R_b \). Since \( R_u \) is unitary and \( R_b \) is in \( C_{0,1} \), by Corollary 3.13 we have \( B_{21} = 0 \). Therefore, the matrix form of \( B_1 \) on \( K_1 = \mathcal{H}_u \oplus \mathcal{H}_b \) is of the form
\[
\begin{bmatrix}
    B_{11} & O \\
    B_{31} & B_{41}
\end{bmatrix}.
\]

With the similar arguments, we can find the matrix form of \( B_2 \) on \( K_2 = \mathcal{H}_s \oplus \mathcal{H}_t \). Let
\[
B_2 = \begin{bmatrix}
    B_{12} & B_{22} \\
    B_{32} & B_{42}
\end{bmatrix}
\]
on \( K_2 = \mathcal{H}_s \oplus \mathcal{H}_t \). Since \( R_s \) is an isometry and \( R_t \) is in \( C_{0,0} \), we have \( B_2 \) is of the form
\[
\begin{bmatrix}
    B_{33} & O \\
    B_{33} & B_{44}
\end{bmatrix}.
\]

Hence the matrix form of the operator \( B \), commutant of a power partial isometry on \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t \) is of the form
\[
B = \begin{pmatrix}
    \mathcal{H}_u & \mathcal{H}_b & \mathcal{H}_s & \mathcal{H}_t \\
    B_{11} & O & \ast & \ast \\
    \ast & B_{22} & \ast & \ast \\
    O & O & B_{33} & O \\
    O & O & \ast & B_{44}
\end{pmatrix}.
\]

Now if \((R, B)\) is doubly commuting, i.e, \(RB = BR\) and \(R^*B = BR^*\), then \( B \) is of the form
\[
\begin{pmatrix}
    \mathcal{H}_u & \mathcal{H}_b & \mathcal{H}_s & \mathcal{H}_t \\
    B_{11} & O & O & O \\
    O & B_{22} & O & O \\
    O & O & B_{33} & O \\
    O & O & O & B_{44}
\end{pmatrix}.
\]

This shows that the decomposition for a power partial isometry reduces \( B \). \hfill \blacksquare

Now we are in a position to state the main result of this section.

**Theorem 4.3.** Let \((R, Q)\) be a pair of doubly twisted operators on \( \mathcal{H} \) such that \( R \) is a power partial isometry. If \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t \) is the decomposition for \( R \), then this decomposition reduces also the operator \( Q \).

**Proof.** Suppose \((R, Q)\) is a pair of doubly twisted operators with respect to a twist \( U \) on \( \mathcal{H} \). Let \( R \) be a power partial isometry and \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t \) be the decomposition for \( R \).

Consider \( K_1 = \mathcal{H}_u \oplus \mathcal{H}_b \) and \( K_2 = \mathcal{H}_s \oplus \mathcal{H}_t \). Since \( RQ = UQR \), from the proof of Theorem 3.5 \( K_1 \) is an invariant subspace for \( Q \). So the matrix representation of \( Q \) with respect to the
decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$ is of the form

$$Q = \begin{bmatrix} Q_1 & * \\ O & Q_2 \end{bmatrix},$$

where $Q_1 = Q|_{\mathcal{K}_1}$, and $Q_2 = P_{\mathcal{K}_2}Q|_{\mathcal{K}_2}$.

Now the pair $(R, U)$ is doubly commuting on $\mathcal{H}$, and hence from Proposition 4.2 the structure of the unitary $U$ is of the form

$$\mathcal{H}_u \quad \mathcal{H}_b \quad \mathcal{H}_s \quad \mathcal{H}_t$$

$$\begin{pmatrix} U_{11} & O & O & O \\ O & U_{22} & O & O \\ O & O & U_{33} & O \\ O & O & O & U_{44} \end{pmatrix},$$

or

$$\begin{pmatrix} U_1 & O \\ O & U_2 \end{pmatrix},$$

where $U_1 = \begin{bmatrix} U_{11} & O \\ O & U_{22} \end{bmatrix}$ on $\mathcal{K}_1$ and $U_2 = \begin{bmatrix} U_{33} & O \\ O & U_{44} \end{bmatrix}$ on $\mathcal{K}_2$. Then

$$R|_{K_i}Q|_{K_i} = U_iQ|_{K_i}R|_{K_i} \quad (i = 1, 2).$$

Let $Q_i = \begin{bmatrix} Q_{1i} & Q_{2i} \\ Q_{3i} & Q_{4i} \end{bmatrix}$ on $\mathcal{K}_i$ for $i = 1, 2$. Now

$$R|_{K_i}Q|_{K_i} = U_iQ|_{K_i}R|_{K_i}$$

on $\mathcal{K}_1 = \mathcal{H}_u \oplus \mathcal{H}_b$ yields

$$\begin{bmatrix} R_uQ_{11} & R_uQ_{21} \\ R_bQ_{31} & R_bQ_{41} \end{bmatrix} = \begin{bmatrix} U_{11}Q_{11}R_u & U_{11}Q_{21}R_b \\ U_{22}Q_{31}R_u & U_{22}Q_{41}R_b \end{bmatrix}. $$

Therefore, $R_uQ_{21} = U_{11}Q_{21}R_b$. Consequently, $(R|_{K_1}, Q_1)$ is a pair of twisted contractions with respect the twist $U_1$ on $\mathcal{K}_1$. Since $R_u$ is unitary and $R_b$ is in $C_0$, by Corollary 3.13 we have $Q_{21} = 0$. Therefore, the matrix form of $Q_1$ on $\mathcal{K}_1 = \mathcal{H}_u \oplus \mathcal{H}_b$ is of the form

$$\begin{bmatrix} Q_{11} & O \\ Q_{31} & Q_{41} \end{bmatrix}. $$

Again $R_s$ is an isometry and $R_t$ is in $C_{00}$ Theorem 3.5 we have $Q_2$ on $\mathcal{K}_2 = \mathcal{H}_s \oplus \mathcal{H}_t$ is of the form

$$\begin{bmatrix} Q_{33} & O \\ Q_{43} & Q_{44} \end{bmatrix}. $$

Hence the matrix form of the twisted commutant $Q$ with a power partial isometry on $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_b \oplus \mathcal{H}_s \oplus \mathcal{H}_t$ is of the form

$$Q = \begin{pmatrix} \mathcal{H}_u & \mathcal{H}_b & \mathcal{H}_s & \mathcal{H}_t \\ Q_{11} & O & * & * \\ * & Q_{22} & * & * \\ O & O & Q_{33} & O \\ O & O & * & Q_{44} \end{pmatrix}. $$
Since $(R, Q)$ is a doubly twisted pair on $H$. Hence $Q$ is of the form

$$
\begin{pmatrix}
H_u & H_b & H_s & H_t \\
Q_{11} & O & O & O \\
O & Q_{22} & O & O \\
O & O & Q_{33} & O \\
O & O & O & Q_{44}
\end{pmatrix}.
$$

This shows that the decomposition for a power partial isometry reduces $Q$. 

\[\square\]

**Remark 4.4.** In the above proof we use the structure theory of a power partial isometry and hence the proof does not require the explicit orthogonal decomposition spaces for power partial isometry. However, one can easily prove the above result with the help of the explicit decomposition spaces.

5. **Dilation and Wold type decomposition**

In general, the structure of a contraction on a Hilbert space is very difficult to study. Using dilation theory, we show here that certain pairs of contractions are doubly twisted on the minimal isometric dilation space.

**Definition 5.1.** Let $H \subset K$ be two Hilbert spaces. Suppose $T \in \mathcal{B}(H)$ and $S \in \mathcal{B}(K)$ are two bounded operators. Then $S$ is called a dilation of $T$ if

$$T^n h = P_H S^n h$$

for all $h \in H$ and $n \in \mathbb{Z}_+$, where $P_H$ is the orthogonal projection of $K$ onto $H$. A dilation $S$ of $T$ is called minimal if

$$\text{span} \{ S^n h : h \in H, n \in \mathbb{Z}_+ \} = K.$$

An isometric dilation of $T$ is a dilation $S$ which is an isometry.

Now we recall one of the striking results on dilation theory (cf. [21], [11]).

**Theorem 5.2.** For every contraction $T$ on a Hilbert space $H$ there exists an isometric dilation $S$ on some Hilbert space $K (\supset H)$, which is moreover minimal in the sense that

$$K = \bigvee_{n=0}^{\infty} S^n H, \ i.e., \ K = \overline{\text{span}} \{ S^n h : h \in H, n \in \mathbb{Z}_+ \}.$$

This minimal isometric dilation $(S, K)$ of $(T, H)$ is determined up to isomorphism. The space $H$ is invariant for $S^*$ and we have

$$TP_H = P_H S \ \text{and} \ \ T^* = S^*|_H,$$

where $P_H$ denotes the orthogonal projection from $K$ onto $H$.

One of the main results of this section is as follows:
Theorem 5.3. Let \((T, V)\) be a pair of operators on a Hilbert space \(\mathcal{H}\) such that \(T\) is a contraction and \(V\) is an isometry and also \(T^*V = UV T^*\), where \(U\) is unitary and \(T, V \in \{U\}'\). Let \(S\) on \(\mathcal{K}\) be the minimal isometric dilation for \(T\). If \(\tilde{V}\) on \(\mathcal{K}\) is an isometric extension of \(V\), then \((S, \tilde{V})\) is a pair of doubly twisted isometries on \(\mathcal{K}\) and hence there is a Wold-type decomposition for the pair \((S, \tilde{V})\) on the minimal space \(\mathcal{K}\). Moreover, the pair \((S, \tilde{V})\) on the minimal space \(\mathcal{K}\) is unique up to unitary equivalence.

Proof. Suppose that \((T, V)\) is a pair consisting of a contraction \(T\) and an isometry \(V\) on \(\mathcal{H}\). Also \(T^*V = UV T^*\), where \(U\) is unitary and \(T, V \in \{U\}'\). Thus the pairs \((T, U)\) and \((V, U)\) are doubly commuting on \(\mathcal{H}\).

Let \(S\) on \(\mathcal{K}\) be the minimal isometric dilation of \(T\). Then

\[
\mathcal{K} = \overline{\text{span}} \{S^n h : h \in \mathcal{H}, n \in \mathbb{Z}_+\},
\]

\(S^*(\mathcal{H}) \subseteq \mathcal{H}\) and \(T^* = S^*|_{\mathcal{K}}\). Also \(TP_{\mathcal{H}} = P_{\mathcal{K}} S\) where \(P_{\mathcal{K}}\) is the orthogonal projection from \(\mathcal{K}\) onto \(\mathcal{H}\). Consider

\[
\mathcal{L} = \text{span}\{S^n h_n : n \in \mathbb{Z}_+, h_n \in \mathcal{H}\}.
\]

Define \(\tilde{V}\) on \(\mathcal{L}\) as

\[
\tilde{V} \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right) = \sum_{n=0}^{N} \alpha_n S^n U^m V h_n,
\]

where \(\alpha_n \in \mathbb{C}\), \(h_n \in \mathcal{H}\). We shall firstly show that the map \(\tilde{V}\) is well defined on \(\mathcal{L}\). To do that we consider the following.

Let \(\alpha_n, \beta_m \in \mathbb{C}\), and \(h_n, g_m \in \mathcal{H}\). Then

\[
(5.1) \quad \left\langle \sum_{n=0}^{N} \alpha_n S^n U^m V h_n, \sum_{m=0}^{M} \beta_m S^m U^m V g_m \right\rangle = \sum_{n=0}^{N} \sum_{m=0}^{M} \alpha_n \beta_m \langle S^n U^n V h_n, S^m U^m V g_m \rangle.
\]

Suppose \(n \geq m\) for fixed \(m, n\). Using the fact that \(V\) is an isometry on \(\mathcal{H}\), \(S\) is an isometric dilation of \(T\), \(T^*U = UT^*\) and \(T^*V = UV T^*\), we obtain

\[
\langle S^n U^n V h_n, S^m U^m V g_m \rangle = \langle S^{n-m} U^n V h_n, U^m V g_m \rangle
= \langle U^n V h_n, S^{(n-m)} U^m V g_m \rangle
= \langle U^n V h_n, U^m T^{(n-m)} V g_m \rangle
= \langle U^{m-n} V h_n, U^n V T^{(n-m)} g_m \rangle
= \langle V h_n, VT^{(n-m)} g_m \rangle
= \langle h_n, T^{(n-m)} g_m \rangle
= \langle h_n, S^{(n-m)} g_m \rangle
= \langle S^{n-m} h_n, g_m \rangle
= \langle S^n h_n, S^m g_m \rangle.
\]
Thus from the above equation (5.1), we have
\[
\langle \sum_{n=0}^{N} \alpha_n S^n U^n h_n, \sum_{m=0}^{M} \beta_m S^m U^m g_m \rangle = \langle \sum_{n=0}^{N} \alpha_n S^n h_n, \sum_{m=0}^{M} \beta_m S^m g_m \rangle.
\]
In particular, we have
\[
\| \sum_{n=0}^{N} \alpha_n S^n U^n h_n \|^2 = \| \sum_{n=0}^{N} \alpha_n S^n h_n \|^2.
\]
Suppose that
\[
\sum_{n=0}^{N} \alpha_n S^n h_n = \sum_{m=0}^{M} \beta_m S^m g_m.
\]
Then from the above equality (5.2), it is easy to see that
\[
\| \sum_{n=0}^{N} \alpha_n S^n U^n h_n - \sum_{m=0}^{M} \beta_m S^m U^m g_m \|^2 = \| \sum_{n=0}^{N} \alpha_n S^n h_n - \sum_{m=0}^{M} \beta_m S^m g_m \|^2 = 0.
\]
Therefore,
\[
\sum_{n=0}^{N} \alpha_n S^n U^n h_n = \sum_{m=0}^{M} \beta_m S^m U^m g_m.
\]
Hence from the definition of \( \tilde{V} \), we have
\[
\tilde{V} \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right) = \sum_{n=0}^{N} \alpha_n S^n U^n h_n.
\]
This proves that \( \tilde{V} \) is well defined on \( \mathcal{L} \). From equation (5.2), we can conclude that \( \tilde{V} \) is bounded as well as norm preserving linear operator on \( \mathcal{L} \). Again \( \tilde{L} = \mathcal{K} \) follows that \( \tilde{V} \) is an isometry on \( \mathcal{K} \). Also the definition of \( \tilde{V} \) on \( \mathcal{K} \) implies that \( \tilde{V}(h_0) = V(h_0) \) for all \( h_0 \in \mathcal{H} \), i.e., \( \tilde{V}|_{\mathcal{H}} = V \). So \( \tilde{V} \) on \( \mathcal{K} \) is an extension of \( V \).

Now define \( \tilde{U} \) on \( \mathcal{L} \) as
\[
\tilde{U} \left( \sum_{n=0}^{N} a_n S^n h_n \right) = \sum_{n=0}^{N} a_n S^n U h_n,
\]
where \( a_n \in \mathbb{C}, h_n \in \mathcal{H} \). In the similar way as above, we can prove that \( \tilde{U} \) is well-defined and bounded as well as norm preserving operator on \( \mathcal{L} \). Since \( U \) is unitary, \( \tilde{U} \) is an isometric operator from \( \mathcal{L} \) onto \( \mathcal{L} \). Again \( \tilde{L} = \mathcal{K} \). Now using continuity we can show that \( \tilde{U} \) can be extended to a unitary operator (denoted by same \( \tilde{U} \)) on \( \mathcal{K} \). From the definition of \( \tilde{U} \) on \( \mathcal{K} \), we show that \( \tilde{U}(h_0) = U(h_0) \) for all \( h_0 \in \mathcal{H} \), i.e., \( \tilde{U}|_{\mathcal{H}} = U \).
We shall now show that $S$ and $\tilde{V}$ are doubly twisted with respect to the twist $\tilde{U}$ on $\mathcal{L}$. Firstly,

\[
S\tilde{U}\left(\sum_{n=0}^{N} \alpha_n S^n h_n \right) = S\left(\sum_{n=0}^{N} \alpha_n S^n U h_n \right) = \sum_{n=0}^{N} \alpha_n S^{n+1} U h_n = \tilde{U}\left(\sum_{n=0}^{N} \alpha_n S^{n+1} h_n \right) = \tilde{U}S\left(\sum_{n=0}^{N} \alpha_n S^n h_n \right).
\]

Again,

\[
\tilde{V}\tilde{U}\left(\sum_{n=0}^{N} \alpha_n S^n h_n \right) = \tilde{V}\left(\sum_{n=0}^{N} \alpha_n S^n U h_n \right) = \sum_{n=0}^{N} \alpha_n S^n U^n V U h_n = \sum_{n=0}^{N} \alpha_n S^n U^n V h_n = \tilde{U}\left(\sum_{n=0}^{N} \alpha_n S^n U^n V h_n \right) = \tilde{U}\tilde{V}\left(\sum_{n=0}^{N} \alpha_n S^n h_n \right).
\]

Using the definition of $\tilde{U}$ and $\tilde{V}$, we have

\[
\tilde{V}S\left(\sum_{n=0}^{N} \alpha_n S^n h_n \right) = \tilde{V}\left(\sum_{n=0}^{N} \alpha_n S^{n+1} h_n \right) = \sum_{n=0}^{N} \alpha_n S^{n+1} U^{n+1} V h_n = \tilde{U}\left(\sum_{n=0}^{N} \alpha_n S^{n+1} U^n V h_n \right) = \tilde{U}S\tilde{V}\left(\sum_{n=0}^{N} \alpha_n S^n h_n \right).
\]
Also
\[
\tilde{U}\tilde{V}S^*\left(\sum_{n=0}^{N} \alpha_n S^n h_n\right) = \tilde{U}\tilde{V}\left(\sum_{n=0}^{N} \alpha_n S^n S^* h_n\right) \\
= \tilde{U}\tilde{V}\left(\alpha_0 S^* h_0 + \sum_{n=1}^{N} \alpha_n S^{n-1} h_n\right) \\
= \tilde{U}\tilde{V} (\alpha_0 T^* h_0) + \tilde{U}\tilde{V}\left(\sum_{n=1}^{N} \alpha_n S^{n-1} h_n\right) \\
= \alpha_0 \tilde{U}(VT^* h_0) + \tilde{U}\left(\sum_{n=1}^{N} \alpha_n S^{n-1} U^{n-1} V h_n\right) \\
= \alpha_0 UVT^* h_0 + \left(\sum_{n=1}^{N} \alpha_n S^{n-1} U U^{n-1} V h_n\right) \\
= \alpha_0 T^* V h_0 + S^* \left(\sum_{n=1}^{N} \alpha_n S^n U^n V h_n\right) \\
= S^*[\alpha_0 V h_0 + \left(\sum_{n=1}^{N} \alpha_n S^n U^n V h_n\right)] \\
= S^* \tilde{V}[\alpha_0 S^0 h_0 + \left(\sum_{n=1}^{N} \alpha_n S^n h_n\right)] \\
= S^* \tilde{V}\left(\sum_{n=0}^{N} \alpha_n S^n h_n\right).
\]

This implies that
\[
S\tilde{U} = \tilde{U} S, \quad \tilde{V}\tilde{U} = \tilde{U}\tilde{V}, \quad \tilde{V} S = \tilde{U} S \tilde{V} \quad \text{and} \quad S^* \tilde{V} = \tilde{U}\tilde{V} S^* \quad \text{on} \quad \mathcal{L}.
\]

Now the norm preserving operator \(\tilde{V}, \tilde{U}\) on \(\mathcal{L}\) can be extended uniquely by continuity (again denoted by same \(\tilde{V}, \tilde{U}\)) to the closure of \(\mathcal{L}\) (i.e. \(\tilde{\mathcal{L}} = \mathcal{K}\)) such that
\[
S\tilde{U} = \tilde{U} S, \quad \tilde{V}\tilde{U} = \tilde{U}\tilde{V}, \quad \tilde{V} S = \tilde{U} S \tilde{V} \quad \text{and} \quad S^* \tilde{V} = \tilde{U}\tilde{V} S^*.
\]

Therefore, the pair \((S, \tilde{V})\) is a doubly twisted isometries with respect to the twist \(\tilde{U}\) on the minimal space \(\mathcal{K}\). Hence, by Theorem 3.15 the doubly twisted pair of isometries \((S, \tilde{V})\) admits Wold-type decomposition on \(\mathcal{K}\).

Let \(S_1\) and \(S_2\) be two minimal isometric dilations on the space \(\mathcal{K}_1\) and \(\mathcal{K}_2\) for the contraction \(T\), respectively. Therefore,
\[
\mathcal{K}_i = \overline{\text{span}} \{S_i^n h : h \in \mathcal{H}, n \in \mathbb{Z}_+\}, S_i^*(\mathcal{H}) \subseteq \mathcal{H} \quad \text{and} \quad T^* = S_i^*|_{\mathcal{H}} \quad (i = 1, 2).
\]
Now, for \( i = 1, 2 \), consider \( L_i = \text{span}\{S^n_i h_n : n \in \mathbb{Z}_+, h_n \in \mathcal{H}\} \) and define \( \tilde{V}_i \) on \( L_i \) as
\[
\tilde{V}_i \left( \sum_{n=0}^{N_0} \alpha_n S^n_i h_n \right) = \sum_{n=0}^{N_0} \alpha_n S^n_i V h_n,
\]
where \( \alpha_n \in \mathbb{C}, h_n \in \mathcal{H} \). Clearly, the maps \( \tilde{V}_i \) are bounded as well as norm preserving operators on \( L_i \). Define a map \( \hat{U} : L_1 \to L_2 \) by
\[
\hat{U} \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right) = \sum_{n=0}^{N} \alpha_n S^n h_n.
\]
Using the fact that \( T^* = S^*_i |_\mathcal{H} \), it is easy to prove that the map \( \hat{U} \) is well-defined and isometric linear map from \( L_1 \) onto \( L_2 \). Also
\[
\hat{U} \tilde{V}_1 \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right) = \hat{U} \left( \sum_{n=0}^{N} \alpha_n S^n U^n V h_n \right) = \sum_{n=0}^{N} \alpha_n S^n U^n V h_n = \tilde{V}_2 \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right).
\]
Therefore, \( \hat{U} \tilde{V}_1 = \tilde{V}_2 \hat{U} \). Since \( \mathcal{L}_i = \mathcal{K}_i \) for \( i = 1, 2 \); then by continuity \( \hat{U} \) can be extended to a unitary operator (again denoted by same \( \hat{U} \)) from \( \mathcal{K}_1 \) to \( \mathcal{K}_2 \). It says that the isometric extensions \( \tilde{V}_1 \) and \( \tilde{V}_2 \) of \( V \) are isomorphic. Hence the pair \( (S, \tilde{V}) \) is unique up to unitary equivalence on the minimal dilation space. This completes the proof.

The following results are immediate applications of the above Theorem 5.3. The result is known [16] but here we provide a new and simple proof.

**Corollary 5.4.** Let \( V \) be an isometry and \( W \) be a coisometry on a Hilbert space \( \mathcal{H} \). If \((V, W)\) is twisted, then the pair \((V, W)\) is doubly twisted.

**Proof.** Suppose that \( V \) is an isometry, \( W \) is a co-isometry and \( U \) is a unitary on a Hilbert space \( \mathcal{H} \) satisfying \( VW = UWV \) and \( V, W \in \{U\}' \). Let \( S \) on \( \mathcal{K} \supseteq \mathcal{H} \) be the minimal isometric dilation of the co-isometry \( W \), that is,
\[
\mathcal{K} = \overline{\text{span}} \{S^n h : h \in \mathcal{H}, n \in \mathbb{Z}_+ \},
\]
\( S^*(\mathcal{H}) \subseteq \mathcal{H} \) and \( W^* = S^*_i |_\mathcal{H} \). Now define \( \tilde{V} \) on \( L = \text{span}\{S^n h_n : n \in \mathbb{Z}_+, h_n \in \mathcal{H}\} \) as
\[
\tilde{V} \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right) = \sum_{n=0}^{N} \alpha_n S^n V h_n \quad \text{and} \quad \tilde{U} \left( \sum_{n=0}^{N} \beta_n S^n h_n \right) = \sum_{n=0}^{N} \beta_n S^n U h_n,
\]
where $\alpha_n, \beta_n \in \mathbb{C}, h_n \in \mathcal{H}$. Now in similar lines as Theorem 5.3, we can prove that $\tilde{U}$ is unitary and $\tilde{V}$ is isometry on $\mathcal{K}$ with $\tilde{V}S = \tilde{U}S\tilde{V}$ and $\tilde{V}, S \in \{\tilde{U}\}^\prime$. Hence the pair $(\tilde{V}, S)$ is a twisted isometries. We only state the result as the proof is similar lines.

Since $W$ is a co-isometry on $\mathcal{H}$, the minimal isometric dilation $S$ of $W$ is unitary on $\mathcal{K}$. Consequently, $S^*\tilde{V} = \tilde{U}VS^*$ which is same as $\tilde{V}^*S = \tilde{U}^*S\tilde{V}^*$. Using the facts $S^*|_\mathcal{H} = W^*$, $\tilde{U}|_\mathcal{H} = U$ and $\tilde{V}|_\mathcal{H} = V$, we have

\[ W^*Vh = S^*Vh = S^*\tilde{V}h = \tilde{U}\tilde{V}Sh = \tilde{U}\tilde{V}W^*h = \tilde{U}VW^*h = UVW^*h \quad (h \in \mathcal{H}). \]

This implies $W^*V = UVW^*$ on $\mathcal{H}$. Hence the pair $(V, W)$ is doubly twisted.

\begin{proof}
\end{proof}

**Remark 5.5.** It is note worthy to mention that we have used Theorem 5.3 to prove the above result. Alternatively, we provide a simple proof of the result as follows: Suppose $(V, W)$ is a twisted pair with a twist $U$ on $\mathcal{H}$ such that $V$ is an isometry and $W$ is a coisometry. Then we obtain

\[
(W^*V - UVW^*)(W^*V - UVW^*) = (V^*W - WV^*U^*)(W^*V - UVW^*)
\]

\[
= V^*WV^*V - U^*WV^*V^*V - UV^*WVW^* + U^*UWV^*VW^*
\]

\[
= I_{\mathcal{H}} - U^*UWV^*V^*V - UU^*V^*VW^* + I_{\mathcal{H}}
\]

\[
= 0.
\]

Consequently, $W^*V - UVW^* = 0$, that is, $V^*W = U^*WV^*$.

Hence the pair $(V, W)$ is doubly twisted on $\mathcal{H}$.

In the following, we record the above result.

**Lemma 5.6.** Let $(V_1, V_2)$ be a pair of isometries and $U$ be unitary on $\mathcal{H}$ such that $V_1^*V_2 = U^*V_2V_1^*$ with $V_1, V_2 \in \{U\}^\prime$. Then $(V_1, V_2)$ is doubly twisted isometries with a twist $U$.

Using Theorem 5.3, we can generalize the result Theorem 2.3 in [8] to a pair of twisted isometries. We only state the result as the proof is similar lines.

**Theorem 5.7.** Let $(V_1, V_2)$ be a pair of twisted isometries on $\mathcal{H}$ and $\dim \ker V_2^*$ is finite. If $\mathcal{H} = \mathcal{H}_{u1} \oplus \mathcal{H}_{s1}$ is the Wold decomposition for $V_1$, then the decomposition reduces $V_2$.

Suppose that $S$ on $\mathcal{K}$ is the minimal isometric dilation of a contraction $T$ on $\mathcal{H}$. Then the Wold decomposition of $S$ is $S_u \oplus S_s$ on $\mathcal{K} = \mathcal{K}_u \oplus \mathcal{K}_s$ such that $S_u = S|_{\mathcal{K}_u}$ is unitary and $S_s = S|_{\mathcal{K}_s}$ is unilateral shift. Moreover,

\[
\mathcal{K}_u = \bigcap_{n \geq 0} S^n\mathcal{K}, \quad \mathcal{K}_s = \bigoplus_{n \geq 0} S^n\mathcal{W}_s,
\]

where $\mathcal{W}_s = \ker S^*$ is the wandering subspace for $S$ defined by $(I - ST^*)\mathcal{H}$ (cf. [10]). Also defect index $\sigma_T^*$ for a contraction $T$ on $\mathcal{H}$ is defined as $\dim D_T^*\mathcal{H}$, where $D_T^* = (I - TT^*)^{\frac{1}{2}}$. If $\sigma_T^*$ is finite, then we have the following result:
Proposition 5.8. Let \((T, V)\) be a pair of twisted operators on \(\mathcal{H}\) such that \(T\) is a contraction and \(V\) is an isometry. Let \(S\) on \(\mathcal{K}\) be the minimal isometric dilation for \(T\) and \(\tilde{V}\) on \(\mathcal{K}\) is an isometric extension of \(V\). If \(\sigma_{T^*}\) is finite and \(\tilde{V}_u \oplus \tilde{V}_s\) on \(\tilde{\mathcal{K}}_u \oplus \tilde{\mathcal{K}}_s\) is the Wold decomposition for isometry \(\tilde{V}\) on \(\mathcal{K}\), then both \(\tilde{\mathcal{K}}_u\) and \(\tilde{\mathcal{K}}_s\) reduce \(S\).

Proof. Assume that \((T, V)\) is a pair of twisted operators with a twist \(U\) on \(\mathcal{H}\) such that \(T\) is a contraction and \(V\) is an isometry.

Let \(S\) on \(\mathcal{K}\) be the minimal isometric dilation of \(T\). Therefore,
\[
\mathcal{K} = \overline{\text{span}} \left\{ S^n h : h \in \mathcal{H}, n \in \mathbb{Z}_+ \right\},
\]
\(S^*(\mathcal{H}) \subseteq \mathcal{H}\) and \(T^* = S^*|_{\mathcal{H}}\). Consider \(\mathcal{L} = \text{span}\{S^n h_n : n \in \mathbb{Z}_+, h_n \in \mathcal{H}\}\). Now define \(\tilde{V}\) and \(\tilde{U}\) on \(\mathcal{L}\) as
\[
\tilde{V} \left( \sum_{n=0}^{N} \alpha_n S^n h_n \right) = \sum_{n=0}^{N} \alpha_n S^n U^n V h_n \quad \text{and} \quad \tilde{U} \left( \sum_{n=0}^{N} \beta_n S^n h_n \right) = \sum_{n=0}^{N} \beta_n S^n U h_n,
\]
where \(\alpha_n, \beta_n \in \mathbb{C}, h_n \in \mathcal{H}\). It is now easy to check that the map \(\tilde{V}, \tilde{U}\) is well-defined and bounded as well as norm preserving linear operator on \(\mathcal{L}\). Again by continuity \(\tilde{V}\) and the unitary \(\tilde{U}\) can be extended to \(\tilde{\mathcal{L}} = \mathcal{K}\) (write same \(\tilde{V}, \tilde{U}\) on \(\mathcal{K}\)). Applying Theorem 5.3 we have the pair \((S, \tilde{V})\) is a twisted isometries with a twist \(\tilde{U}\) on \(\mathcal{K}\).

Now for every \(h \in \mathcal{H}\)
\[
\|(I - ST^*)h\|^2 = \|h\|^2 - 2 \text{ Re } < h, ST^* h > + \|T^* h\|^2 = \|h\|^2 - \|T^* h\|^2 = \|D_{T^*} h\|^2.
\]
Define \(\hat{U} : D_{T^*} \mathcal{H} \to (I - ST^*)\mathcal{H}\) as
\[
\hat{U}(D_{T^*} h) = (I - ST^*) h.
\]
Then \(\hat{U}\) is an isometry from \(D_{T^*} \mathcal{H}\) onto \((I - ST^*)\mathcal{H}\). Hence \((I - ST^*)\mathcal{H}\) is finite dimensional as by hypothesis \(\sigma_{T^*} = \dim D_{T^*} \mathcal{H}\) is finite. Therefore, \(\hat{U}\) is unitary and \(\sigma_{T^*} = \dim (I - ST^*) \mathcal{H} = \dim \ker S^*\).

Suppose \(\tilde{V}_u \oplus \tilde{V}_s\) on \(\tilde{\mathcal{K}}_u \oplus \tilde{\mathcal{K}}_s\) is the Wold decomposition for isometry \(\tilde{V}\) on \(\mathcal{K}\). Since \((\tilde{V}, S)\) is a pair of twisted isometries on \(\mathcal{K}\) and \(\dim \ker S^*\) is finite dimensional, from Theorem 5.7 we conclude that the decomposition reduces the isometric dilation \(S\).

This completes the proof.

6. Doubly Twisted Isometries

In [18] the second author, Sarkar and Sankar characterize the pair of doubly commuting isometries and also studied defect operator. In this section, we characterize pair of doubly twisted isometries.
Let \((V_1, V_2)\) be a pair of twisted isometries on a Hilbert space \(\mathcal{H}\). Throughout this section we will use the following notations:

\[
V = V_1 V_2 \\
\mathcal{W} = \mathcal{W}(V) = \mathcal{H} \oplus V_1 V_2 \mathcal{H},
\]

and \(\mathcal{W}_i = \mathcal{W}(V_i) = \mathcal{H} \oplus V_i \mathcal{H}\) for \(i = 1, 2\).

The following relations are useful to our discussion.

**Lemma 6.1.** Let \((V_1, V_2)\) be a pair of twisted isometries with respect to a twist \(U\) on a Hilbert space \(\mathcal{H}\) and let \(V = V_1 V_2\). Then for \(n \in \mathbb{N}\), we have the following:

1. \(V_1 V_2^n = U^n V_2^n V_1\) and \(V_2 V_1^n = U^n V_1^n V_2\),
2. \(V_1 V_2^n = U^n V_2^n V_1\) and \(V_2 V_1^n = U^n V_1^n V_2\),
3. \(V_1^* V^n = U^{s(n-1)} V^{n-1} V_2\) and \(V_2^* V^n = U^n V^{n-1} V_1\),
4. \(V^n = U^n V_2^n V_1^n = U^{s(n-1)} V_1^n V_2^n\).

**Proof.** Suppose \((V_1, V_2)\) is a pair of twisted isometries with respect to a twist \(U\) on \(\mathcal{H}\). Then \(V_1 V_2 = U V_2 V_1\), \(U V_1 = V_1 U\) and \(U V_2 = V_2 U\), where \(U\) is unitary on \(\mathcal{H}\). Using this relation, we can prove all the equations. \(\blacksquare\)

**Lemma 6.2.** Let \((V_1, V_2)\) be a pair of twisted isometries on a Hilbert space \(\mathcal{H}\). Then

\[
\mathcal{W} = \mathcal{W}_1 \oplus V_1 \mathcal{W}_2 = V_2 \mathcal{W}_1 \oplus \mathcal{W}_2,
\]

and the operator \(\hat{U}\) on \(\mathcal{W}\) defined by

\[
\hat{U}(\zeta_1 \oplus V_1 \zeta_2) = V_2 \zeta_1 \oplus \zeta_2
\]

for \(\zeta_1 \in \mathcal{W}_1\) and \(\zeta_2 \in \mathcal{W}_2\), is a unitary operator. Moreover,

\[
V_i^* \mathcal{W} = \mathcal{W}_j \quad \text{for} \quad 1 \leq i, j \leq 2 \quad \text{and} \quad i \neq j.
\]

**Proof.** Suppose \((V_1, V_2)\) is a pair of twisted isometries with respect to a twist \(U\) on \(\mathcal{H}\). Then \(V_1 V_2 = U V_2 V_1\) and \(V_1, V_2 \in \{U\}'\). Since \(U\) is unitary, the pairs \((V_i, U)\) are doubly commuting on \(\mathcal{H}\) for \(i = 1, 2\). Let \(V = V_1 V_2\). Then

\[
I - V V^* = (I - V_1 V_1^*) \oplus V_1 (I - V_2 V_2^*) V_1^* = V_2 (I - V_1 V_1^*) V_2^* \oplus (I - V_2 V_2^*).
\]

This implies that

\[
\mathcal{W} = \mathcal{W}_1 \oplus V_1 \mathcal{W}_2 = V_2 \mathcal{W}_1 \oplus \mathcal{W}_2.
\]

Now the operator \(\hat{U}\) is unitary directly follows from the above line.

For the last part, we have

\[
V_i^* \mathcal{W} = V_i^* (V_i \mathcal{W}_j \oplus \mathcal{W}_i) = \mathcal{W}_j
\]

for \(i \neq j\). \(\blacksquare\)

We present an important result and will be used to characterize the doubly twisted isometries.
Lemma 6.3. Let \((V_1, V_2)\) be a pair of twisted isometries with respect to a twist \(U\) on \(\mathcal{H}\) and let \(V = V_1V_2\). Then \(\mathcal{H}_s(V)\) and \(\mathcal{H}_u(V)\) are joint \((V_1, V_2)\)-reducing subspaces of \(\mathcal{H}\). Moreover, \(\mathcal{H}_u(V) \subseteq \mathcal{H}_u(V_i)\) and \(\mathcal{H}_s(V_i) \subseteq \mathcal{H}_u(V)\) for \(i = 1, 2\).

Proof. Suppose that \((V_1, V_2)\) is a pair of twisted isometries with respect to a twist \(U\) on \(\mathcal{H}\). Then \(V_1V_2 = UV_2V_1\), and \(UV_i = V_iU\) for \(i = 1, 2\). From Lemma 6.2, we have

\[
V_1\mathcal{W} = V_1\mathcal{W}_2 \oplus V\mathcal{W}_1 = V_1(\mathcal{H} \ominus V_2\mathcal{H}) \oplus V(\mathcal{H} \ominus V\mathcal{H}) 
\subseteq (\mathcal{H} \ominus V\mathcal{H}) \oplus V\mathcal{W} 
= V\mathcal{W} \oplus V\mathcal{W}.
\]

Now \((U, V_i)\) is a doubly commuting pair and hence \(U\mathcal{W}_i = \mathcal{W}_i\) for \(i = 1, 2\). Therefore, by Lemma 6.1, for each \(n \geq 0\) we have

\[
(6.1) \quad V_1V^n\mathcal{W} = U^nV^nV_1\mathcal{W} \subseteq V^n(\mathcal{W} \oplus V\mathcal{W}) \subseteq \mathcal{H}_s(V).
\]

Similarly,

\[
V_2\mathcal{W} = V_2\mathcal{W}_1 \oplus UV\mathcal{W}_2 = V_2\mathcal{W}_1 \oplus V\mathcal{W}_2 \subseteq \mathcal{W} \oplus V\mathcal{W},
\]

which follows that, for \(n \geq 0\),

\[
(6.2) \quad V_2V^n\mathcal{W} = V^nV_2U^n\mathcal{W} = V^nV_2\mathcal{W} \subseteq V^n(\mathcal{W} \oplus V\mathcal{W}) \subseteq \mathcal{H}_s(V).
\]

From the equations (6.1) and (6.2), it follows that \(\mathcal{H}_s(V)\) is joint \((V_1, V_2)\)-invariant subspace. Again using Lemma 6.1, we have

\[
V_1^*V^n\mathcal{W} = U^{n-1}V^{n-1}V_2\mathcal{W} = V^{n-1}(V_2\mathcal{W}) \subseteq V^{n-1}(\mathcal{W} \oplus V\mathcal{W}),
\]

and

\[
V_2^*V^n\mathcal{W} = U^nV^{n-1}(V_1\mathcal{W}) = V^{n-1}(V_1\mathcal{W}) = V^{n-1}(\mathcal{W} \oplus V\mathcal{W}).
\]

Thus \(\mathcal{H}_s(V)\) is \((V_1^*, V_2^*)\)-invariant subspace. Hence \(\mathcal{H}_s(V)\) is a joint \((V_1, V_2)\) reducing subspace of \(\mathcal{H}\).

Now \(\mathcal{H}_u(V) = \mathcal{H}_s(V)^\perp\), and hence \(\mathcal{H}_u(V)\) and \(\mathcal{H}_s(V)\) are both joint \((V_1, V_2)\)-reducing subspaces of \(\mathcal{H}\). From Lemma 6.1, we have for \(n \geq 0\),

\[
K_n\mathcal{H} = U_nK_nV_1\mathcal{H} = V_2^n(V_1^n\mathcal{H})
= U_nK_nV_2^n\mathcal{H} = V_1^n(V_2^n\mathcal{H}),
\]

which implies

\[
V^n\mathcal{H} \subseteq V_1^n\mathcal{H}, V_2^n\mathcal{H}.
\]

Consequently, \(\mathcal{H}_u(V) \subset \mathcal{H}_u(V_i)\) and hence \(\mathcal{H}_u(V_i)^\perp \subseteq \mathcal{H}_u(V)^\perp\). Therefore \(\mathcal{H}_s(V_i) \subseteq \mathcal{H}_u(V)\) for \(i = 1, 2\). This completes the proof.

The following result is a characterization for doubly twisted isometries.

Theorem 6.4. Let \((V_1, V_2)\) be a pair of twisted isometries with respect to a twist \(U\) on a Hilbert space \(\mathcal{H}\). Then the followings are equivalent:
Proof. Suppose that \((V_1, V_2)\) is a pair of doubly twisted with respect to a twist \(U\) on \(\mathcal{H}\). Then
\[
V_1 V_2 = U V_2 V_1, \quad V_1^* V_2 = U^* V_2 V_1^*, \quad \text{and} \quad V_1, V_2 \in \{U\}'.
\]
Now
\[
V_1 (I - V_2 V_2^*) V_1^* = (V_1 - U V_2 V_2^*) V_1^* = (V_1 - U V_2 U^* V_2^* V_1^*) V_1^* = (I - V_2 V_2^*) V_1 V_1^*.
\]
It follows \(V_1 W_2 \subseteq W_2\), i.e., (1) implies (2). Similarly, we can prove (1) implies (3).

Now we shall only prove that (2) implies (1). Since the pair \((V_1, V_2)\) is twisted isometries with respect to a twist \(U\),
\[
V_1 V_2 = U V_2 V_1, \quad U V_1 = V_1 U, \quad \text{and} \quad U V_2 = V_2 U.
\]
Consider the Wold-von Neumann decomposition for the isometry \(V = V_1 V_2\):
\[
\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V).
\]
Now Lemma \ref{6.3} gives that \(\mathcal{H}_s(V), \mathcal{H}_u(V)\) are \(V_1, V_2\)-reducing subspaces and hence \(V_i|_{\mathcal{H}_u(V)}\) is unitary for \(i = 1, 2\). Again \((U, V)\) is doubly commuting, so by Corollary \ref{3.3} both the subspaces \(\mathcal{H}_u(V)\) and \(\mathcal{H}_s(V)\) reduce \(U\). Therefore, \((V_i|_{\mathcal{H}_u(V)}), V_2|_{\mathcal{H}_u(V)})\) on \(\mathcal{H}_u\) is doubly twisted with respect to the twist \(U|_{\mathcal{H}_u(V)}\). It remains to prove that \(V_2^* V_1 = U V_1 V_2^*\) on \(\mathcal{H}_s(V)\). Now
\[
(V_2^* V_1 - U V_1 V_2^*) V^n = V_2^* V_1 V^n - U V_1 V_2^* V^n
= V_2^* V_1 V_2 V^{n-1} - U V_1 V_2^* V_2 V^{n-1}
= U^2 V_1^2 V^{n-1} - U^2 V_1^2 V^{n-1}
= 0,
\]
and hence
\[
V_2^* V_1 - U V_1 V_2^* = 0 \quad \text{on} \quad V^n \mathcal{W} \quad \forall \quad n \geq 1.
\]
To complete the proof we shall show that
\[
V_2^* V_1 = U V_1 V_2^* \quad \text{on} \quad \mathcal{W}.
\]
Let \(\zeta \in \mathcal{W}\). Then By Lemma \ref{6.2}
\[
\zeta = V_2 \zeta_1 \oplus \zeta_2 \quad \text{for some} \quad \zeta_1 \in \mathcal{W}_1 = \ker V_1^* \quad \text{and} \quad \zeta_2 \in \mathcal{W}_2 = \ker V_2^*.
\]
Since \(V_1 \mathcal{W}_2 \subseteq \mathcal{W}_2\),
\[
V_2^* V_1 \zeta = V_2^* V_1 (V_2 \zeta_1 \oplus \zeta_2) = V_2^* V_1 V_2 \zeta_1 + V_2^* (V_1 \zeta_2)
= U V_2^* V_1 \zeta_1
= UV_1 \zeta_1.
\]
Again
\[ UV_1 V_2^* \zeta = UV_1 V_2^* (V_2 \zeta_1 \oplus \zeta_2) = UV_1 V_2^* V_2 \zeta_1 + UV_1 V_2^* \zeta_2 \]
\[ = UV_1 \zeta_1. \]

Hence \((V_1, V_2)\) is a pair of doubly twisted isometries on \(\mathcal{H}\).

**Remark 6.5.** It is noted that we can characterize doubly twisted isometries by studying defect operators. Characterization for doubly twisted isometries and more on defect operators will be carried out in our future papers.

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