A GENERALIZATION OF TRIPLE STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

Carlos Granados\(^1\), Ajoy Kanti Das\(^2\)

\(^1\)Estudiante de Doctorado en Matemáticas
Magister en Ciencias Matemáticas
Universidad de Antioquia
Medellín, COLOMBIA

\(^2\)Department of Mathematics
Bir Bikram Memorial College
Agartala-799004
Tripura, INDIA

Abstract: In this paper, we introduce a class of summability methods that can be applied to \(\lambda\)-triple statistical convergence in topological groups and we show some interesting results.

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1. Introduction and definitions preliminaries

Looking through historically at statistical convergence of single sequences, we shall recall that the notion of statistical convergence of sequences was first studied by Fast [3]. The notion of statistical convergence of a sequence \((x_n)\) in a locally convex Hausdorff topological linear space \(X\) was presented recently by Maddox [8], where it was shown that the slow oscillation of \((s_n)\) was a Tauberian condition for the statistical convergence of \((s_n)\). In [7], statistical convergence to normed spaces was extended by Kolk. Further in [1] and [2], Cakalli extended
this notation to topological Hausdorff groups. The study of triple sequence in different fields of sequences spaces has grown in the last decade (see [4, 5, 6]).

By the convergence of a triple sequence, we mean the convergence in Pringshein’s sense [9]. A triple sequence \( x = (x_{asd}) \) is said to be convergent in the Pringshein’s sense if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{asd} - \psi| < \varepsilon \) whenever \( a, s, d \geq N \). \( \psi \) is called the Pringsheim limit of \( x \). Furthermore, A triple sequence \( (x = x_{asd}) \) is said to be Cauchy sequence if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{pql} - x_{asd}| < \varepsilon \) for all \( p \geq a \geq N, q \geq s \geq N \) and \( l \geq d \geq N \). In a topological group \( E \), the above definitions become as in the following: a triple sequence \( x = (x_{asd}) \) in \( E \) is said to be convergent to \( \psi \) in \( E \) in the Pringshein’s sense if for every neighbourhood \( V \) of 0 there exists \( N \in \mathbb{N} \) such that \( x_{asd} - \psi \in V \) whenever \( a, s, d \geq N \). \( \psi \) is called the Pringsheim limit of \( x \). A triple sequence \( x = (x_{asd}) \) is said to be a Cauchy sequence if for every neighbourhood \( V \) of 0 there exists \( N \in \mathbb{N} \) such that \( x_{pql} - x_{asd} \in V \) for all \( p \geq a \geq N, q \geq s \geq N \) and \( l \geq d \geq N \).

By \( E \), we will denote an Abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset \( B \) of \( E \), \( s(B) \) will denote the set of all sequences \( (x_a) \) such that \( (x_a) \) is in \( B \) for \( a = 1, 2, 3, ... \) \( c(E) \) will denote the set of all convergent sequences. On the other hand, a sequence \( (x_a) \) in \( E \) is called statistically convergent to an element \( \psi \) of \( E \) if for each neighbourhood \( V \) of 0, (see [2]) \( \lim_{a \to \infty} \frac{1}{a} \left| z \leq a : x_a - \psi \notin V \right| = 0 \), and is called statistically Cauchy in \( E \) if for each neighbourhood \( V \) of 0 there exists a positive integer \( a_0(V) \), depending on the neighbourhood \( V \), such that \( \lim_{a \to \infty} \frac{1}{a} \left| z \leq a : x_a - x_{a_0(V)} \notin V \right| = 0 \) where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in \( E \) is denoted by \( S(E) \) and the set of all statistically Cauchy sequences in \( E \) is denoted by \( SC(E) \). It is known that \( SC(E) = S(E) \) if \( E \) is complete. Additionally, those notions and the notion \( \lambda \)-statistical convergent were extended for double sequences by Savas [11].

On the other hand, let \( \lambda = (\lambda_p, \mu = (\mu_q) \) and \( \phi = (\phi_l) \) be three non-decreasing sequences of positive real numbers, three of them of which tends to \( \infty \) as \( p, q \) and \( l \) approach \( \infty \), respectively. Besides, let \( \lambda_{p+1} \leq \lambda_p + 1, \mu_1 = 1, \mu_{q+1} \leq \mu_q + 1, \mu_1 = 1 \) and \( \phi_{l+1} \leq \phi_l + 1, \phi_1 = 1 \). The collection of such sequence will be denoted by \( \Delta \). We write the generalized double de la Valée-Poussin mean by

\[
 t_{p,q,l}(x) = \frac{1}{\lambda_p \mu_q \phi_l} \sum_{a \in I_p, s \in J_q, d \in W_l} x_{asd},
\]

where \( I_p = [p - \lambda_p + 1, p], J_Q = [Q - \mu q + 1, q] \) and \( W_l = [l - \phi_l + 1, l] \).
Throughout this paper we shall denote $\lambda_p \mu_q \phi_l$ by $\lambda_{pql}$ and $(a \in I_p, s \in J_q, d \in W_l)$ by $(a, s, d) \in I_{pql}$.

The aim of this paper is to introduce the $\lambda$-triple statistical convergence of triple sequences in topological groups and to prove some useful theorems.

2. $\lambda$-Triplet statistical convergence

Let $J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be a three-dimensional set of positive integers and let $J(q, w, e)$ be the numbers of $(a, s, d)$ in $J$ such that $a \leq q, s \leq w$ and $d \leq e$. Then, the three-dimensional analogue of natural density can be defined as follows. The lower asymptotic density of a set $J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is defined as

$$\delta_3(J) = \lim_{q,w,e} \inf J(q,w,e) / qwe,$$

In case that the sequence $(J(q,w,e) / qwe)$ has a limit in Pringsheim’s sense, then we say that $J$ has a triple natural density and is defined as

$$\delta_3(J) = \lim_{q,w,e} J(q,w,e) / qwe.$$

Sahiner and Tripathy [10] called a real triple sequence $x = (x_{asd})$ statistically convergent to the number $\psi$ if for each $\varepsilon > 0$, the set $\{(a, s, d), a \leq q, s \leq w$ and $d \leq e : |x_{asd} - \psi| \geq \varepsilon\}$ has triple natural density zero. In this case, we write $S_3$-lim$_{a,s,d} x_{asd} = \psi$ and we denote the set of all statistically convergent triple sequences by $S_3$. Now, we define statistical convergence of triple sequences $x = (x_{asd})$ in a topological group in the following.

Definition 1. A triplet sequence $x = (x_{asd})$ is statistically convergent to a point $\psi$ of $E$ if for each neighbourhood $V$ of 0 the set

$$\{(a, s, d), a \leq q, s \leq w \text{ and } d \leq e : x_{asd} - \psi \notin V\}$$

has a triple natural density zero. In this case, we write $S_3(E)$-lim$_{a,s,d} x_{asd} = \psi$ and we write the set of all statistically convergent triple sequences by $S_3(E)$.

Definition 2. A triplet sequence $x = (x_{asd})$ is said to be $S_3^\lambda$-convergent to $\psi$ of $E$ (or $\lambda$-triple statistically convergent to $\psi$ of $E$) of for each neighbourhood $V$ of 0, the set

$$\{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin V\}$$
Δ has triple natural density zero. In this case we write $S^3_{\lambda}\lim_{a,s,d \to \infty} x_{asd} = \psi$ or \( x_{asd} \to \psi(S^3_{\lambda}) \), and we write the set of all $\lambda$-statistically convergent triple sequences by $S^3_{\lambda}(E)$.

**Remark 3.** A $\lambda$-statistically convergent triple sequence has a unique limit, i.e. if \( x \) is $\lambda$-statistically convergent to elements $\psi_1$ and $\psi_2$ of $E$, then $\psi_1 = \psi_2$.

**Theorem 4.** A triple sequence $x = (x_{asd})$ in $E$ is $\lambda$-triple statistically convergent to $\psi$ if and only if there exists a subset $J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta^3_{\lambda}(J) = 1$ and $\lim_{a,s,d \to \infty} x_{asd} = \psi$ where limit is being taken over the set $E$, i.e. $(a, s, d) \in E$.

**Proof.** **Necessity:** Let us suppose that $x$ be $\lambda$-triple statistically convergent to $\psi$, and $(V_r)$ be a base of nested closed neighbourhood of 0. Now, write $J_r = \{(a, s, d) \in I_{pq} : x_{asd} - \psi \notin V_r \}$ and $Q_r = \{(a, s, d) \in I_{pq} : x_{asd} - \psi \in V_r \}$ where $r = 1, 2, 3, \ldots$. Then, $\delta^3_{\lambda}(J_r) = 0$ and

$$Q_1 \supset Q_2 \supset \ldots \supset Q_a \supset Q_{a+1} \supset \ldots \quad (1)$$

and

$$\delta^3_{\lambda}(Q_r) = 1, r = 1, 2, 3, \ldots \quad (2)$$

Now, we have to show that for $(a, s, d) \in Q_r, (x_{asd})$ is $\lambda$-triple statistically convergent to $\psi$. Now, consider that $(x_{asd})$ is not $\lambda$-triple statistically to $\psi$ so that there is a neighbourhood $V$ of 0 such that $x_{asd} - \psi \notin V$ for in finitely many terms. Let $V_r \subset V$ where $r = 1, 2, 3, \ldots$ and $Q_V = \{(a, s, d) : x_{asd} - \psi \in V \}$. Then, $\delta^3_{\lambda}(Q_V) = 0$ and by (1), $Q_r \subset Q_V$. Therefore, $\delta^3_{\lambda}(Q_r) = 0$ which is a contradiction to (2). Hence, $(x_{asd})$ is $\lambda$-triple statistically convergent to $\psi$.

**Sufficiency:** Consider that there exists a subset $J = \{(a, s, d) \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}\}$ such that $\delta^3_{\lambda}(J) = 1$ and $\lim_{a,s,d} x_{asd} = \psi$, i.e. there exists an $r_0 \in \mathbb{N}$ such that for each neighbourhood $V$ of 0, $x_{asd} - \psi \notin V$ for every $a, s, d \geq r_0$. Now,

$$J_V = \{(a, s, d) : x_{asd} - \psi \notin V \} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{(a_{r_0+1}, s_{r_0+1}, d_{r_0+1}), (a_{r_0+2}, s_{r_0+2}, d_{r_0+2}), \ldots\}.$$  

Therefore, $\delta^3_{\lambda}(J_V) \leq 0$. It follows that $x$ is $\lambda$-triple statistically convergent to $\psi$. \qed

**Corollary 5.** If a triple sequence $(x_{asd})$ is $\lambda$-triple statistically convergent to $\psi$. Then, there exists a triple sequence $(y_{asd})$ such that $\lim_{a,s,d} y_{asd} = \psi$ and $\delta^3_{\lambda}\{ (a, s, d) : x_{asd} = y_{asd} \} = 1$, i.e. $x_{asd} = y_{asd}$ for almost all $a, s, d$. 

Definition 6. In a topological group, triple sequence \( x = (x_{asd}) \) is called \( \lambda \)-triple statistically Cauchy if for each neighbourhood \( V \) of 0 there exists \( G = G(V), H = H(V) \) and \( Q = Q(V) \) such that for all \( a, q \geq G, s, w \geq H \) and \( d, e \geq Q \) the set \( \{(a, s, d) \in I_{pql} : x_{asd} - x_{qwe} \notin V\} \) has triple natural density zero. In this case, we denote the set of all statistically Cauchy triple sequences by \( S_3 C(E) \).

Theorem 7. Let \( E \) be complete. A triple sequence \( x = (x_{asd}) \) in \( E \) is \( \lambda \)-triple statistically convergent if and only if \( x \) is \( \lambda \)-triple statistically Cauchy.

Proof. Let \( x = (x_{asd}) \) be \( \lambda \)-triple statistically convergent to \( \psi \). Let \( V \) be any neighbourhood of 0. Then, we can choose a symmetric neighbourhood \( W \) of 0 such that \( W + W \subset V \). Then for this neighbourhood \( W \) of 0, the set \( \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin W\} \) has triple natural \( \lambda \)-density 0. For each neighbourhood \( V \) of 0, the set \( \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin V\} \) has triple natural \( \lambda \)-density zero. Then, we can choose numbers \( G, H \) and \( Q \) such that \( x_{GHQ} - \psi \notin V \). Now, we write \( T_V = \{(a, s, d) \in I_{pql} : x_{asd} - x_{GHQ} \notin V\}, L_W = \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin W\} \) and \( K_W = \{(N, M, J) \in I_{pql} : x_{GHQ} - \psi \notin W\} \). Then, \( T_V \subset L_W \cup K_W \) and hence \( \delta_3^\lambda(T_V) = \delta_3^\lambda(L_W) + \delta_3^\lambda(K_W) = 0 \). Therefore, we obtain that \( x \) is \( \lambda \)-triple statistically Cauchy. To prove the converse suppose that there is a \( \lambda \)-triple statistically Cauchy sequence \( x \) but it is not \( \lambda \)-triple statistically convergent. Then we can find natural numbers \( G, H \) and \( Q \) such that the set \( T_V \) has triple natural \( \lambda \)-density zero. It follows from this that the set \( Z_V = \{(a, s, d) \in I_{pql} : x_{asd} - X_{GHQ} \notin V\} \) has triple natural density 1. Now, we can choose a neighbourhood \( W \) of 0 such that \( W + W \subset V \). Now, take any fixed non-zero element \( \psi \) of \( E \). Let \( x_{asd} - X_{GHQ} = x_{asd} - \psi + \psi - X_{GHQ} \). It follows from this equality that \( x_{asd} - x_{GHQ} \in V \) if \( x_{asd} - \psi \in W \). Since \( x \) is not \( \lambda \)-triple statistically convergent to \( \psi \), the set \( L_W \) has triple natural density 1, i.e. the set \( \{(a, s, d) : a \leq q, s \leq w, d \leq e : x_{asd} - \psi \notin W\} \) has triple natural density 0. Hence the set \( \{(a, s, d) : a \leq q, s \leq w, d \leq e : x_{asd} - x_{GHQ} \in W\} \) has triple natural density 0, i.e. the set \( T_V \) has triple natural density 1 which is a contradiction.

\( \square \)

Taking into account Theorems 4 and 7, we can state the following theorem and the proof is following directly by the previous results.

Theorem 8. If \( E \) is complete, then the following conditions are equivalent:

1. \( x \) is \( \lambda \)-triple statistically convergent to \( \psi \),
2. $x$ is $\lambda$-triple statistically Cauchy,

3. there exists a subsequence $y$ of $x$ such that $\lim_{a,s,d} y_{asd} = \psi$.

References

[1] H. Cakalli, Lacunary statistical convergence in topological groups, *Indian J. Pure Appl. Math.*, 26, No 2 (1995), 113-119.

[2] H. Cakalli, On statistical convergence in topological groups, *Pure and Appl. Math. Sci.*, 43, No 1 & 2 (1996), 27-31.

[3] H. Fast, Sur la convergence statistique, *Colloq. Math.*, 2 (1951), 241-244.

[4] C. Granados and A. Dhital, Statistical convergence of double sequences in neutrosophic normed spaces, *Neutrosophic Sets and Systems*, 42 (2021), 333-344.

[5] C. Granados, New notions of triple sequences on ideal spaces in metric spaces, *Advances in the Theory on Nonlinear Analysis and its Applications*, 5, No 3 (2021), 363-368.

[6] C. Granados, A generalization of the strongly Cesàro ideal convergence through double sequence spaces, *International Journal of Applied Mathematics*, 34, No 3 (2021), 525-533; DOI: 10.12732/ijam.v34i3.8.

[7] E. Kolk, The statistical convergence in Banach spaces, *Tartu Ul. Toime.*, 928 (1991), 41-52.

[8] I.J. Maddox, Statistical convergence in locally convex spaces, *Math. Cambr. Phil. Soc.*, 104 (1988), 141-145.

[9] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann.*, 53 (1900), 289-321.

[10] A. Sahiner and B.C. Tripathy, Some I-related properties of triple sequences, *Selcuk J. Appl. Math.*, 9, No 2 (2008), 9-18.

[11] E. Savas, Generalized double statistical convergence in topological group, *Advances in Intelligent Systems and Computing, Proc. of the Sixth Intern. Conf. on Mathematics and Computing*, 2020 (2020), 461-468.