A NOVEL APPROACH TO IMPROVE THE ACCURACY OF THE BOX DIMENSION CALCULATIONS: APPLICATIONS TO TRABECULAR BONE QUALITY

M. Fernández-Martínez

University Centre of Defence at the Spanish Air Force Academy
MDE-UPCT
30720 Santiago de la Ribera, Murcia, Spain

Yolanda Guerrero-Sánchez* and Pía López-Jornet

University of Murcia
Department of Dermatology, Stomatology, Radiology and Physical Medine
Morales Messegue General University Hospital
Avda. Marques de los Velez, 30008 Murcia, Spain

Abstract. Fractal dimension and specifically, box-counting dimension, is the main tool applied in many fields such as odontology to detect fractal patterns applied to the study of bone quality. However, the effective computation of such invariant has not been carried out accurately in literature. In this paper, we propose a novel approach to properly calculate the fractal dimension of a plane subset and illustrate it by analysing the box dimension of a trabecular bone through a computed tomography scan.

1. Introduction. There are many systemic diseases that may affect the quality of dental bones. Different studies appeared in the literature try to measure the effect of a particular disease on the trabecular structure. To deal with, the main tool applied in this context is the fractal dimension throughout the box-counting model. Such a quantity allows to detect self-similarity patterns. Thus, if some kind of disease destroys that internal structure, its fractal dimension may vary providing a measure regarding this fact. See for instance [7, 8, 12, 14, 17] and [18].

However, the effective calculation of the box dimension has not been carried out in the most accurate manner in empirical applications. In most cases, the analysed images have been treated by a hand computation inspired by the “Archimedes’ exhalation method”, [1, 9, 10, 13, 15] and [16]. It is worth pointing out that the box dimension may be underestimated for great values of the scale $\delta$ (c.f. [11]).

Thus, the main goal in this paper is to introduce a novel approach to accurately tackle with the calculation of the fractal dimension avoiding the human error.

2010 Mathematics Subject Classification. Primary: 28A80.

Key words and phrases. Fractal pattern, fractal dimension, box-counting dimension, Hausdorff dimension, fractal structure, space-filling curve, computed tomography scan, periodontitis, bone quality.

The first author has been partially supported by Grants No. 19219/PI/14 from Fundación Séneca of Región de Murcia and No. MTM2015-64373-P from Spanish Ministry of Economy and Competitiveness.

* Corresponding author: Yolanda Guerrero-Sánchez.
Our proposal consists of applying the mathematical approach introduced in Section 2 to collect data that can subsequently be analysed in order to establish connections between different pathologies such as periodontitis, gingivitis, bone density, or even cancer. Another advantage of our mathematical method is that it allows us to study a wide range of different types of images such as cone beam, ortopantomograph, and RVG to name a few. In addition, this method can be also used in all radiographic images.

In Section 3, we shall analyze the fractal dimension of a computed tomography scan image from a periodontitis patient. It is worth pointing out that in [1], the effects of that disease in the bone quality were studied through the box dimension although the conclusions reported no connections between periodontitis and bone density, likely due to a large margin of error in the box method.

2. How to accurately calculate the box dimension of plane subsets. In this section, we shall introduce a novel approach to accurately calculate the box dimension of plane subsets. We would like to point out that our key result appears in upcoming Theorem 2.2 with the construction of the curve $\alpha$ involved therein being explicitly described in later Theorem 2.7.

Let $\delta > 0$. Recall that a $\delta-$cube in $\mathbb{R}^2$ is a set of the form $[\delta k_1, \delta (k_1 + 1)] \times [\delta k_2, \delta (k_2 + 1)] : k_1, k_2 \in \mathbb{Z}$. Next, we define the classical box dimension for any plane subset in terms of $\delta-$cubes.

**Definition 2.1.** The box dimension of $F \subseteq \mathbb{R}^2$ is given by the (lower/upper) limit

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

(1)

where $N_\delta(F)$ is the number of $\delta-$cubes that intersect $F$.

It is worth mentioning that the limit in Eq. (1) can be discretized by $\delta = 2^{-n}$ (c.f. [3, Remark 2.5]). Our first key result stands in the following terms.

**Theorem 2.2** ([4]). There exists a parameterization $\alpha : [0, 1] \to \mathbb{R}^2$ of a curve for which the next identity holds regarding the (lower/upper) box dimension of $F \subseteq \mathbb{R}^2$:

$$\dim_B(F) = 2 \cdot \dim_B(\alpha^{-1}(F)).$$

Notice that Theorem 2.2 states that the box dimension of any plane subset $F$ can be calculated in terms of the box dimension of its pre-image, $\alpha^{-1}(F) \subseteq [0, 1]$, which is a Euclidean 1-dimensional set, up to a constant (the Euclidean dimension). In order to use Theorem 2.2 in empirical applications regarding fractal dimension, our next objective is to properly determine how to construct such a curve $\alpha$. To tackle this, the concept of a fractal structure will play a key role. We would like to point out that fractal structures allow a deep study of generalized fractal dimension models (c.f. [2]).

First of all, recall that a covering of a set $X$ is a family $\Gamma$ of subsets such that $X = \cup \{A : A \in \Gamma\}$.

**Definition 2.3.** ([6, Definition 2.1]) A fractal structure on $X$ is a countable family of coverings (called levels), $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$, satisfying the two following conditions:

(i) for all $A \in \Gamma_{n+1}$, there exists some $B \in \Gamma_n$ such that $A \subseteq B$.

(ii) Each $B \in \Gamma_n$ can be written as $B = \cup \{A \in \Gamma_{n+1} : A \subseteq B\}$. 
Fig. 1 illustrates the definition of a fractal structure on a set in terms of coverings. A fractal structure on $X$ is said to be starbase if it satisfies that $\text{St}(x, \Gamma) = \{\text{St}(x, \Gamma_n) : n \in \mathbb{N}\}$ is a neighborhood base for all $x \in X$, where $\text{St}(x, \Gamma_n) = \cup \{A \in \Gamma_n : x \in A\}$. Moreover, $\Gamma$ is said to be Cantor complete if for each decreasing sequence $\{A_n : n \in \mathbb{N}\}$ (i.e., $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$) of subsets of $X$ with $A_n \in \Gamma_n$, it holds that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

It is worth pointing out that each Euclidean set can always be endowed with a fractal structure naturally. Its definition can be stated in the following terms.

**Definition 2.4.** ([5, Definition 3.1]) The natural fractal structure on every Euclidean space $\mathbb{R}^d$ is the countable family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ with levels given by

$$\Gamma_n = \left\{ \left[ \frac{k_1}{2^n}, \frac{k_1 + 1}{2^n} \right] \times \ldots \times \left[ \frac{k_d}{2^n}, \frac{k_d + 1}{2^n} \right] : k_1, \ldots, k_d \in \mathbb{Z} \right\}.$$

**Remark 2.5.** ([5, Remark 3.2]) Natural fractal structures can always be induced on Euclidean subsets from previous Definition 2.4. For instance, Fig. 2 displays the natural fractal structure (induced) on $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, i.e., the family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ with levels defined as

$$\Gamma_n = \left\{ \left[ \frac{k_1}{2^n}, \frac{k_1 + 1}{2^n} \right] \times \left[ \frac{k_2}{2^n}, \frac{k_2 + 1}{2^n} \right] : k_1, k_2 \in \{0, 1, \ldots, 2^n - 1\} \right\}.$$

It holds that the image set of any parameterized Euclidean curve can be always endowed with an induced fractal structure as stated next.

**Definition 2.6.** ([3, Definition 3.1]) Let $\Gamma$ be a fractal structure on $[0, 1]$ and $\alpha : [0, 1] \to \mathbb{R}^2$ be a parameterization of a plane curve. The fractal structure induced by $\Gamma$ on $\alpha([0, 1]) \subseteq \mathbb{R}^2$ is defined as the countable family of coverings $\Delta = \{\Delta_n : n \in \mathbb{N}\}$ with levels given by $\Delta_n = \alpha(\Gamma_n) = \{\alpha(A) : A \in \Gamma_n\}$.

Fig. 3 depicts the first two levels of a fractal structure on the image set of a Brownian motion for illustration purposes. Notice that such a fractal structure is induced by the natural fractal structure on the closed unit interval, $[0, 1]$.

Interestingly, the next result we provide allows us to properly construct a curve $\alpha$ to use Theorem 2.2 in applications.
Figure 2. First two levels of the natural fractal structure on $[0, 1] \times [0, 1]$ as a Euclidean subset.

Figure 3. The two images above show the first two levels, $\Delta_1$ and $\Delta_2$, of a fractal structure induced by $\Gamma$ on the image set of a Brownian motion, where $\Gamma$ is the natural fractal structure on $[0, 1]$.

Theorem 2.7. ([3, Theorem 3.6]) Let $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a starbase fractal structure on a metric space $X$ and $\Delta = \{\Delta_n : n \in \mathbb{N}\}$ a $\Delta$–Cantor complete starbase fractal structure on a complete metric space $Y$. Further, let $\alpha_n : \Gamma_n \to \Delta_n$ be a family of maps satisfying the two following conditions:

- If $A \cap B \neq \emptyset$ with $A, B \in \Gamma_n$ for some $n \in \mathbb{N}$, then $\alpha_n(A) \cap \alpha_n(B) \neq \emptyset$.
- If $A \subseteq B$ with $A \in \Gamma_{n+1}$ and $B \in \Gamma_n$ for some $n \in \mathbb{N}$, then $\alpha_{n+1}(A) \subseteq \alpha_n(B)$.

Then there exists a unique continuous map $\alpha : X \to Y$ such that $\alpha(A) \subseteq \alpha_n(A)$ for all $A \in \Gamma_n$ and $n \in \mathbb{N}$. In addition, if $\Gamma$ is $\Gamma$–Cantor complete and the two following conditions stand:

- $\alpha_n$ is onto.
- $\alpha_n(A) = \cup\{\alpha_{n+1}(B) : B \in \Gamma_{n+1}, B \subseteq A\}$ for each $A \in \Gamma_n$. 
then \( \alpha \) is onto and \( \alpha(A) = \alpha_n(A) \) for all \( A \in \Gamma_n \) and \( n \in \mathbb{N} \).

The accuracy of our approach to deal with the calculation of the box dimension of \( F \subseteq \mathbb{R}^2 \) in terms of the calculation of the box dimension of \( \alpha^{-1}(F) \subseteq [0,1] \) (c.f. Theorem 2.2) stands as a consequence of previous Theorem 2.7.

**Remark 2.8.** Theorem 2.7 establishes a one-to-one correspondence among the elements in the set \( \{ B \in \Delta_n : B \cap F \neq \emptyset \} \) and the elements in \( \{ A \in \Gamma_n : A \cap \alpha^{-1}(F) \neq \emptyset \} \). Hence, \( N_n(F) = \text{Card}(\{ B \in \Delta_n : B \cap F \neq \emptyset \}) \) equals \( N_n(\alpha^{-1}(F)) = \text{Card}(\{ A \in \Gamma_n : A \cap \alpha^{-1}(F) \neq \emptyset \}) \) up to a constant, where Card denotes the cardinal number of the involved set.

It is worth noting that Theorem 2.7 can also be understood as a result allowing the construction of space-filling curves. Next, we illustrate this fact throughout the Hilbert’s curve.

**Example 2.9.** ([3, Example 1]) The construction of the classical Hilbert’s plane-filling curve can be iteratively carried out throughout fractal structures. To deal with this, let \( \Gamma \) be a fractal structure on \([0,1]\) with levels given by

\[
\Gamma_n = \left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] : k \in \{0,1,\ldots,2^{2n}-1\} \right\}.
\]

Observe that \( \Gamma \) derives from the natural fractal structure on \([0,1]\). On the other hand, let \( \Delta \) be the natural fractal structure on \([0,1] \times [0,1]\) with levels defined by

\[
\Delta_n = \left\{ \left[ \frac{k_1}{2^n}, \frac{k_1+1}{2^n} \right] \times \left[ \frac{k_2}{2^n}, \frac{k_2+1}{2^n} \right] : k_1, k_2 \in \{0,1,\ldots,2^n-1\} \right\}.
\]

Next, we shall define the image of each level of \( \Gamma \) through a map \( \alpha : [0,1] \to [0,1] \times [0,1] \) which iteratively approaches the Hilbert’s curve. With this aim, let us construct a sequence of maps \( \{ \alpha_n : n \in \mathbb{N} \} \), where the definition of each \( \alpha_n : \Gamma_n \to \Delta_n \) (illustrated in Fig. 4 for its first two levels) can be stated in the following terms:

\[
\alpha \left( \left[ \frac{1}{4}, \frac{1}{2} \right] \right) = \left[ \frac{0}{2}, \frac{1}{2} \right] \times \left[ \frac{0}{2}, \frac{1}{2} \right], \quad \alpha \left( \left[ \frac{1}{4}, \frac{3}{4} \right] \right) = \left[ \frac{0}{2}, \frac{1}{2} \right] \times \left[ \frac{1}{2}, \frac{3}{4} \right].
\]

This allows to define the whole covering \( \alpha(\Gamma_1) = \{ \alpha(A) : A \in \Gamma_1 \} \). We can proceed similarly with the next levels of \( \Delta \). The polygonal in each image of Fig. 4 illustrates how the unit square is filled by \( \alpha \) in each level of \( \Delta \). Such a recursive procedure allows us to refine the definition of \( \alpha_n \) in each stage of this construction since additional information regarding the curve is provided as deeper levels are reached. Accordingly, if \( A \in \Gamma_n \mapsto B \in \Delta_n \) via \( \alpha_n \), then in the next level, \( A = \cup \{ C \in \Gamma_{n+1} : C \subseteq A \} \), \( B = \cup \{ D \in \Delta_{n+1} : D \subseteq B \} \), and each \( C \) is sent to \( D \) via \( \alpha_{n+1} \).

Letting \( n \to \infty \), the Hilbert’s curve \( \alpha \) stands as the limit of the sequence of maps \( \{ \alpha_n : n \in \mathbb{N} \} \).

The plane-filling nature of the curve provided by Theorem 2.2 allows us to guarantee the accuracy of our box dimension calculations. In fact, since \( \alpha \) crosses all the elements in each level of \( \Delta \) (resp., \( \alpha^{-1} \) crosses all the elements in \( \Gamma \)), then all the elements in the family \( \{ B \in \Delta_n : B \cap F \neq \emptyset \} \) (resp., all the elements in the set \( \{ A \in \Gamma_n : A \cap \alpha^{-1}(F) \neq \emptyset \} \)) are considered to tackle the calculation of the box dimension of \( F \) (resp., of \( \alpha^{-1}(F) \)). Following the results above, next we
summarize our theoretical approach to accurately deal with the calculation of the box dimension of a plane subset.

**Corollary 2.10.** Let $\alpha : [0, 1] \to \mathbb{R}^2$ be a parameterization of a plane curve, $\Gamma$ a fractal structure on $[0, 1]$, $\Delta = \alpha(\Gamma)$ the fractal structure induced by $\Gamma$ on $\alpha([0, 1]) \subseteq \mathbb{R}^2$, and $F \subseteq \mathbb{R}^2$. Then the (lower/upper) box dimension of $F$ can be calculated throughout the following expression:

$$\dim_B(F) = 2 \cdot \lim_{\delta \to 0} \frac{\log N_\delta(\alpha^{-1}(F))}{-\log \delta}. \quad (2)$$

To conclude this section, we pose the following

**Conjecture.** Under the same hypothesis as in Theorem 2.2, there exists a parametrized curve $\alpha : [0, 1] \to \mathbb{R}^2$ for which the next identity holds:

$$\dim_H(F) = 2 \cdot \dim_H(\alpha^{-1}(F)).$$

The implications of the conjecture above from the viewpoint of applications are clear: if such a result is true, then the Hausdorff dimension could be enabled to deal with empirical applications. Recall that such a model of fractal dimension is the most accurate since its definition stands in terms of a measure. However, it may be hard to calculate or estimate empirically. Recall also that Theorem 2.7 gives the explicit construction of $\alpha$, and hence, the Hausdorff dimension of a plane subset $F$ could be calculated by the Hausdorff dimension of $\alpha^{-1}(F) \subseteq [0, 1]$ (up to a constant). Finally, the procedure described in [6, Section 3.1] could be used to calculate the Hausdorff dimension of 1-dimensional Euclidean subsets.

3. Calculating the box dimension of periodontal tissues. In this section, we shall apply Eq. (2) to explore the fractal nature of a dental tissue. It is known
that periodontal tissues do possess a self-similar nature whose structure may vary between patients having periodontitis disease to healthy subjects. (c.f. Fig. 5). For illustration purposes, the box dimension of a trabecular bone from a periodontitis patient was analyzed according to Corollary 2.10. To deal with this, we considered a high-resolution caption from a computed tomography scan (CBCT, Cone beam).

Self-similarity patterns were detected in a range of $1 - 10$ levels by a correlation coefficient equal to $0.97801$. The box dimension of the trabecular bone in Fig. 5 (right) calculated according to Eq. (2) was found to be equal to $1.95251$.

**Acknowledgments.** We would like to thank Prof. M.A. Sánchez-Granero for his insightful comments and remarks regarding this study. He also provided us both Figs. 2 and 4.

**REFERENCES**

[1] A. Estrugo-Devesa, J. Segura-Egea, L. García-Vicente, M. Schemel-Suárez, A. Blanco-Carrío, E. Jané-Salas and J. López-López, Correlation between mandibular bone density and skeletal bone density in a Catalanian postmenopausal population, *Oral Surgery, Oral Medicine, Oral Pathology and Oral Radiology, 125* (2018), 495–502.

[2] M. Fernández-Martínez, A survey on fractal dimension for fractal structures, *Applied Mathematics and Nonlinear Sciences, 2* (2016), 437–472.

[3] M. Fernández-Martínez and M. A. Sánchez-Granero, A new fractal dimension for curves based on fractal structures, *Topology and its Applications, 203* (2016), 108–124.

[4] M. Fernández-Martínez and M. A. Sánchez-Granero, Calculating the fractal dimension in higher dimensional spaces, preprint.

[5] M. Fernández-Martínez and M. A. Sánchez-Granero, Fractal dimension for fractal structures, *Topology and Its Applications, 163* (2014), 93–111.

[6] M. Fernández-Martínez and M. A. Sánchez-Granero, How to calculate the Hausdorff dimension using fractal structures, *Applied Mathematics and Computation, 264* (2015), 116–131.

[7] E. Jagelavičienė and R. Kubilius, The relationship between general osteoporosis of the organism and periodontal diseases, *Medicina (Kaunas), 42* (2006), 613–618.

[8] A. Jordão Camargo, E. Saito Arita, M. C. Cortez de Fernández and P. C. Aranha Watanabe, Comparison of Two Radiological Methods for Evaluation of Bone Density in Postmenopausal Women, *International Journal of Morphology, 33* (2015), 732–736.
[9] A. N. Law, A.-M. Bollen and S.-K. Chen, Detecting osteoporosis using dental radiographs: A comparison of four methods, *The Journal of American Dental Association*, 127 (1996), 1734–1742.

[10] P. L. Lin, P. W. Huang, P. Y. Huang and H. C. Hsu, Alveolar bone-loss area localization in periodontitis radiographs based on threshold segmentation with a hybrid feature fused of intensity and the H-value of fractional Brownian motion model, *Computer Methods and Programs in Biomedicine*, 121 (2015), 117–126.

[11] F. Martínez-López, M. A. Cabrero-Vilchez and R. Hidalgo-Álvarez, An improved method to estimate the fractal dimension of physical fractals based on the Hausdorff definition, *Physica A: Statistical Mechanics and its Applications*, 298 (2001), 387–399.

[12] K. R. Phipps, B. K. S. Chan, T. E. Madden, N. C. Geurs, M. S. Reddy, C. E. Lewis and E. S. Orwoll, Longitudinal Study of Bone Density and Periodontal Disease in Men, *Journal of Dental Research*, 86 (2007), 1110–1114.

[13] E. Sener, S. Cinarcik and B. Guniz Baksı, Use of fractal analysis for the discrimination of trabecular changes between individuals with healthy gingiva or moderate periodontitis, *Journal of Periodontology*, 86 (2015), 1364–1369.

[14] M. Tezal, J. Wactawski-Wende, S. G. Grossi, A. W. Ho, R. Dunford and R. J. Genco, The Relationship Between Bone Mineral Density and Periodontitis in Postmenopausal Women, *Journal of Periodontology*, 71 (2000), 1492–1498.

[15] B. Tolga Suer, Z. Yaman and B. Buyuksarac, Correlation of Fractal Dimension Values with Implant Insertion Torque and Resonance Frequency Values at Implant Recipient Sites, *The International Journal of Oral & Maxillofacial Implants*, 31 (2016), 55–62.

[16] S. X. Updike and H. Nowzari, Fractal analysis of dental radiographs to detect periodontitis-induced trabecular changes, *Journal of Periodontal Research*, 43 (2008), 658–664.

[17] A. Yoshihara, Y. Seida, N. Hanada and H. Miyazaki, A longitudinal study of the relationship between periodontal disease and bone mineral density in community-dwelling older adults, *Journal of Clinical Periodontology*, 31 (2004), 680–684.

[18] A. Yoshihara, Y. Seida, N. Hanada, K. Nakashima and H. Miyazaki, The relationship between bone mineral density and the number of remaining teeth in community-dwelling older adults, *Journal of Oral Rehabilitation*, 32 (2005), 735–740.

Received August 2017; revised January 2018.

E-mail address: manuel.fernandez-martinez@cud.upct.es
E-mail address: yolanda.guerreros@um.es
E-mail address: majornet@um.es