Detailed Stability Analysis of Electroweak Strings

MARGARET JAMES

D.A.M.T.P., Silver Street,
University of Cambridge, Cambridge, CB39EW, U.K.

LEANDROS PERIVOLAROPOULOS

Division of Theoretical Astrophysics,
Harvard-Smithsonian Center for Astrophysics,
60 Garden Street, Cambridge, MA 02138.

TANMAY VACHASPATI

Tufts Institute of Cosmology, Department of Physics and Astronomy,
Tufts University, Medford, MA 02155.

ABSTRACT:

We give a detailed stability analysis of the Z-string in the standard electroweak model. We identify the mode that determines the stability of the string and numerically map the region of parameter space where the string is stable. For $\sin^2 \theta_W = 0.23$, we find that the strings are unstable for a Higgs mass larger than 23GeV. Given the latest constraints on the Higgs mass from LEP, this shows that, if the standard electroweak model is realized in Nature, the existing vortex solutions are unstable.
1. Introduction

Recent studies have shown that the stability of topological defects may persist when they are embedded in theories where they are not topologically stable. Such “embedded defects” are exact solutions of the equations of motion in the theories where they are embedded and can be dynamically stable.

A typical example of such a defect is the semilocal string [1,2], an embedding of the Nielsen-Olesen vortex in a “semilocal” model with $SU(2)_{\text{global}} \times U(1)_{\text{local}} \rightarrow U(1)_{\text{global}}$ symmetry breaking. The semilocal string has been shown to be stable for a finite parameter sector [3,4] even though the vacuum manifold in the semilocal model is $S^3$. In fact, since $\pi_1(S^3) = 1$, the stability of the semilocal string is dynamical rather than topological. The crucial reason for this stability is that for a certain parameter region, the increase in gradient energy necessary for the string to decay is more than the corresponding decrease in potential energy. Therefore, for that parameter region, the decay would increase the total energy and is not favored energetically.

The obvious generalization of the semilocal model is the electroweak model in which the $SU(2)$ symmetry becomes gauged: $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{em}}$. It is possible to show that the Nielsen-Olesen vortex may be embedded in the electroweak model i.e. the electroweak equations of motion with a vortex-like ansatz reduce to the Nielsen-Olesen equations [5,6]. Two such embedded vortex solutions - the so-called $\tau-$ and $Z-$ strings - are known [7]. Here we shall only consider the $Z-$string as the $\tau-$string is expected to be unstable for all values of the parameters [7].

The phenomenological successes of the standard electroweak model [8] make the question of stability of the electroweak string a very interesting and important
one. A stable electroweak string would for the first time in particle physics provide a macroscopic, stable coherent state at low enough energies to be accessible to particle physics experiments. It is this question of stability [9] that we are addressing in detail in this paper. In particular, we construct a map of the parameter space of the standard electroweak model showing the range of parameters where the \(Z\)-string is stable and where it is unstable. Here we give the details of the calculation as well as the physical reasoning for the simplifications that occur in this originally highly complicated problem. A brief exposition of the main results we derive here may be found in Ref.[15].

The structure of the paper is the following: in the next section we give a review of the electroweak string showing that the electroweak equations of motion reduce to the Nielsen-Olesen equations for a particular vortex ansatz. In section 3 we show how can the stability problem of the electroweak string be reduced to the eigenvalue problem of a single Schroedinger-like eigenvalue problem. This is a non-trivial simplification since the initial system of coupled perturbations involves twenty coupled degrees of freedom which after tedious manipulations not only decouple but also reduce to a single eigenvalue equation. Finally in section 4 we solve this eigenvalue problem and construct a map showing the parameter sector corresponding to stability.
2. Review of Electroweak Strings

We consider static bosonic field configurations in the Weinberg-Salam model: there is no time dependence and we choose a gauge where the zero components of the gauge fields are set to zero. The energy functional is given by

\[
E = \int d^3x \left[ \frac{1}{4} G_{ij}^a G_{ij}^a + \frac{1}{4} F_{Bij} F_{Bij} + (D_j \phi) \dagger (D_j \phi) + \lambda (\phi \dagger \phi - \eta^2/2)^2 \right] \tag{2.1}
\]

The Weinberg angle is given by \( \tan \theta_w = g'/g \). The masses of the W-boson, the Z-boson and Higgs boson are, respectively,

\[
M_W = \frac{1}{2} g \eta, \quad M_Z = \frac{1}{2} \alpha \eta, \quad M_H = \sqrt{2} \lambda \eta, \tag{2.2}
\]

where

\[
\alpha \equiv \sqrt{g^2 + g'^2} \tag{2.3}
\]

The time-independent field equations are

\[
D_j F_{ij}^a = -\frac{1}{2} ig (\phi \dagger \tau^a D_i \phi - (D_i \phi) \dagger \tau^a \phi) \tag{2.4}
\]

\[
\partial_j f_{ij} = -\frac{1}{2} ig' \left( \phi \dagger D_i \phi - (D_i \phi) \dagger \phi \right) \tag{2.5}
\]

\[
D_i D_i \phi = 2 \lambda (\phi \dagger \phi - \frac{1}{2} \eta^2) \phi, \tag{2.6}
\]

The symbols are in the standard notation defined in Ref. 10. In addition, we recall the usual mixing formula:

\[
Z^\mu \equiv \cos \theta_W W^{\mu^3} - \sin \theta_W B^\mu, \quad A^\mu \equiv \sin \theta_W W^{\mu^3} + \cos \theta_W B^\mu, \tag{2.7}
\]

The vortex solution extremising (2.1) is given by [5,6]:

\[
W^{\mu^1} = 0 = W^{\mu^2} = A^\mu, \quad Z^\mu = [A^\mu]_{NO} = \frac{\nu_{NO}(r)}{r} e_\theta \phi = f_{NO}(r) e^{im\theta} \Phi, \quad \Phi \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.8}
\]
where, the coordinates \( r \) and \( \theta \) are polar coordinates in the \( xy \)-plane. The integer \( m \) is the winding number of the vortex and, here, we shall restrict ourselves to the case \( m = 1 \). The subscript \( NO \) on the functions \( f \) and \( A^\mu \) means that they are identical to the corresponding functions found by Nielsen and Olesen [11] for the usual Abelian-Higgs string. On substituting eq. (2.8) into the equations of motion they reduce to

\[
f'' + \frac{f'}{r} - \left(1 - \frac{\alpha}{2} v\right)^2 \frac{f}{r^2} - 2\lambda \left(f^2 - \frac{\eta^2}{2}\right) f = 0 \tag{2.9}
\]

\[
v'' - \frac{v'}{r} + \alpha \left(1 - \frac{\alpha}{2} v\right) f^2 = 0 \tag{2.10}
\]

where primes denote differentiation with respect to \( r \) and the subscripts \( NO \) have been dropped for convenience. These are solved together with the boundary conditions:

\[
f(0) = 0 = v(0), \quad f(\infty) = \frac{\eta}{\sqrt{2}}, \quad v(\infty) = \frac{2}{\alpha} \tag{2.11}
\]

The string solutions resulting from these equations have been studied previously by several authors in a lot of detail. A sample of these papers may be found in the collection of Ref. 12.

At this point, it is useful to note the symmetries of the string configuration. Firstly it is axially symmetric i.e. it is invariant under the action of the symmetry operator generated by the generalised angular momentum operator

\[
K_z = L_z + S_z + I_z . \tag{2.12}
\]

\( L_z \) and \( S_z \) are the usual orbital and spin pieces respectively of the spatial angular momentum operator. Explicitly

\[
L_z = -i \frac{\partial}{\partial \theta} \mathbf{1} \quad (S_z \vec{a})_j = -i \epsilon_{3jk} \vec{a}_k \mathbf{1} , \tag{2.13}
\]
where $\vec{a}$ is any vector field and $1$ is the $2 \times 2$ unit matrix. Note that $S_z$ annihilates the scalar Higgs field. $I_z$ is composed of a $U(1)$ generator, $Y$ and an $SU(2)$ generator, $T^3$

$$I_z = -\frac{1}{2}(Y - T^3). \quad (2.14)$$

$Y$ and $T^3$ act on the Higgs field on the left. $Y$ annihilates the gauge field and $T^3$ acts via a commutator bracket.

The configuration has two further symmetries. It is invariant under the combination of reflection in the $x$-axis and complex conjugation. Also it is invariant under the action of the global $U(2)$ gauge transformation given by

$$\overline{U} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.15)$$

Our eventual expansion of the perturbations will be in Fourier modes but we shall point out some connections between these and the eigenfunctions of the above commuting symmetry operators.
3. Stability of Electroweak Strings

The vortex solution given in eq. (2.8) is not topologically stable. This means that any field configuration can be continuously deformed to the vacuum. Hence we are investigating the metastability of the vortex solution i.e. whether it is a \textit{local} maximum or minimum in configuration space. We consider infinitesimal perturbations of the vortex configuration and ask if the variation in the energy is positive or negative.

Let us write

\[
\phi = \left( \phi_1, \phi_{NO} + \phi_2 \right) \quad (3.1)
\]

\[
Z^\mu = Z_{NO}^\mu + \delta Z^\mu \quad (3.2)
\]

\[
T^1 \equiv \text{diag}(-\cos2\theta_W, 1), \quad (3.3)
\]

and,

\[
d_j \equiv (\partial_j \mathbf{1} + i \frac{1}{2} \alpha T^1 Z_j). \quad (3.4)
\]

The perturbations can depend on the \(z\)-coordinate and the \(z\)-components of the vector fields can be non-zero also. However, since the vortex solution has translational invariance along the \(z\)-direction, it is easy to see that the \(z\)-dependence in the perturbations can be ignored and the \(z\)-components of the gauge fields can be set to zero. This follows from (2.1) where the relevant \(z\)-dependent terms in the integrand are:

\[
\frac{1}{2} G^a_{i3} G^a_{i3} + \frac{1}{4} F_{Bi3} F_{Bi3} + (D_3\phi)^\dagger (D_3\phi) \quad (3.5)
\]

This contribution to the energy is strictly non-negative and is minimized (that is, made to vanish) by setting the \(z\)-components of the gauge fields to zero and also considering the perturbations to be independent of the \(z\)-coordinate. For this reason, we shall drop all reference to the \(z\)-coordinate in the calculations below.
and it will be understood that the energy is actually the energy per unit length of the string.

Now we calculate the energy of the perturbed configuration discarding terms of cubic and higher order in the infinitesimal perturbations. We find,

\[ E = (E_{NO} + \delta E_{NO}) + E_1 + E_c + E_W \]  

(3.6)

where, \( E_{NO} \) is the energy of the Nielsen-Olesen string and \( \delta E_{NO} \) is the energy variation due to the perturbations \( \phi_2 \) and \( \delta Z^\mu \). The variation \( E_1 \) is due to the perturbation \( \phi_1 \) in the upper component of the Higgs field:

\[ E_1 = \int d^2x \left[ |\bar{d}_j \phi_1|^2 + 2\lambda(f^2 - \eta^2/2)|\phi_1|^2 \right], \]

(3.7)

where,

\[ \bar{d}_j \equiv \partial_j - i\frac{\alpha}{2}\cos(2\theta_W)Z_j. \]

(3.8)

The contribution from the \( \phi \) and \( \bar{W}^\vec{a} \) [13] interaction is:

\[ E_c = \cos\theta_W \int d^2x J^\vec{a}_j \bar{W}^\vec{a} \]  

(3.9)

\[ J^\vec{a}_j \equiv \frac{1}{2} i\alpha \left[ \phi^{\dagger} \tau^\vec{a} d_j \phi - (d_j \phi)^{\dagger} \tau^\vec{a} \phi \right]. \]

(3.10)

and the energy in the \( \bar{W}^\vec{a} \) and \( \bar{A} \) bosons is

\[ E_W \equiv \int d^2x \left[ \gamma \bar{W}^1 \times \bar{W}^2 \cdot \vec{\nabla} \times \vec{Z} + \frac{1}{2}|\vec{\nabla} \times \bar{W}^1 + \gamma \bar{W}^2 \times \vec{Z}|^2 \right. \]

\[ + \left. \frac{1}{2}|\vec{\nabla} \times \bar{W}^2 + \gamma \vec{Z} \times \bar{W}^1|^2 + \frac{1}{4}g^2 f^2(\bar{W}^\vec{a})^2 + \frac{1}{2}(\vec{\nabla} \times \vec{A})^2 \right]. \]

(3.11)

where, \( \gamma \equiv g \cos \theta_W \). It may be remarked that the \( f \) and \( \vec{Z} \) fields in eqs. (3.7)-(3.11) are the unperturbed fields of the string since we are only keeping up to quadratic terms in the infinitesimal quantities.

Firstly we note that the perturbations of the fields that make up the string do not couple to the other available perturbations. i.e. the perturbations in the
fields \( f \) and \( v \) only occur inside the variation \( \delta E_{NO} \). We can understand this as follows: the perturbation of the string solution has \( U = 1 \), where \( U \) is given by eq. (2.15), whereas the other perturbations have \( U = -1 \). Now, since we know that the Nielsen-Olesen string with unit winding number is stable to perturbations for any values of the parameters then necessarily, \( \delta E_{NO} \geq 0 \) and the perturbations \( \phi_2 \) and \( \delta Z^\mu \) cannot destabilize the vortex. Then, we are justified in ignoring these perturbations and setting \( \delta E_{NO} = 0 \). Also we note that the \( \vec{A} \) boson only appears in the last term of eq. (3.11) and obviously makes a positive contribution so we can set \( \vec{A} \) to zero.

We now consider an expansion of the remaining perturbations in Fourier modes. This gives,

\[
\phi_1 = \chi(r)e^{im\theta}
\]  

for the \( m \)th mode where \( m \) is any integer. For the gauge fields we have,

\[
\vec{W}^1 = \left\{ f_1(r)\cos(n\theta) + f_1\sin(n\theta) \right\} \hat{e}_r + \frac{1}{r} \left\{ -h_1\sin(n\theta) + h_1\cos(n\theta) \right\} \hat{e}_\theta
\]

(3.13)

\[
\vec{W}^2 = \left\{ -\bar{f}_2(r)\sin(n\theta) + f_2\cos(n\theta) \right\} \hat{e}_r + \frac{1}{r} \left\{ \bar{h}_2\cos(n\theta) + h_2\sin(n\theta) \right\} \hat{e}_\theta
\]

(3.14)

for the \( n \)th mode where \( n \) is a non-negative integer. Inserting the expressions for the \( m \)th mode of \( \phi_1 \) and the \( n \)th mode of the \( \vec{W}^\alpha \) fields in the energy functional gives:

\[
E_1 = 2\pi \int dr \ r \left[ \chi^2 + \left\{ \frac{1}{r^2} \left( m + \frac{\alpha}{2} \cos 2\theta_W \right) + 2\lambda \left( f_2^2 - \frac{\eta^2}{2} \right) \right\} \chi^2 \right]
\]

(3.15)

\[
E_c = \delta \pm n, 1 - m \pi \alpha \cos \theta_W \int dr \ r \left[ -(f \chi' - \chi f') (f_2 \mp f_1) \right. \\
\left. - \frac{f}{r^2} \chi \left\{ \mp n + 2 - \frac{\alpha}{2} (1 - \cos 2\theta_W) v \right\} (h_2 \pm h_1) \right]
\]

(3.16)

for \( n \neq 0 \). In the case when \( n = 0 \), the expression for \( E_c \) is equal to the above
expression multiplied by a factor of 2 and with \( f_1 \) and \( h_2 \) set equal to zero. Finally,

\[
E_W = \pi \int \frac{dr}{r} \left[ \gamma (f_2 h_1 - f_1 h_2) v' + \frac{1}{2} \left| -n f_1 + h_1' - \gamma v f_2 \right|^2 + \frac{1}{2} \left| n f_2 + h_2' + \gamma v f_1 \right|^2 + \frac{1}{4} g^2 f^2 [r^2 (f_1^2 + f_2^2) + h_1^2 + h_2^2] \right] + (f_{\bar{a}} \to \bar{f}_{\bar{a}}, h_{\bar{a}} \to \bar{h}_{\bar{a}}) \tag{3.17}
\]

for \( n \neq 0 \). In the case when \( n = 0 \), \( E_W \) is given by (3.17) multiplied by a factor of 2 and with \( f_1 \) and \( h_2 \) set equal to zero.

First let us inspect \( E_1 \). Here the negative contributions can come from the term proportional to \( \chi^2 \). The coefficient of \( \chi^2 \) is composed of two terms: the second term comes from the potential part and is negative while the first term comes from the kinetic part and is always positive and is smallest when \( m = 0 \), at least in the region near the center of the string where an instability is most likely to develop. That is, the \( m = 0 \) mode is the “most dangerous” mode.

Next we inspect \( E_W \). Here the analysis is less obvious. Yet one can see that the only term that can be negative is the first term that arises from the term \( \vec{W}_1 \times \vec{W}_2 \cdot \vec{\nabla} \times \vec{Z} \) in eq. (3.11). For this to contribute at the center of the string, the vector fields must not vanish there. But the only mode that need not vanish at the center is the \( n = 1 \) mode; all other modes have to vanish at the center if they are to be single-valued and finite. Therefore the only mode that can give negative contributions at the center of the string is the \( n = 1 \) mode. Hence, we restrict ourselves to considering the \( m = 0, n = 1 \) mode.

There is also a generally accepted idea which leads us to expect the least stable mode to be the \( m = 0, n = 1 \) mode. As noted, the original configuration has \( K_z = 0 \) and \( \mathcal{U} = 1 \). The \( m = 0 \) mode is also \( K_z = 0 \) with \( \mathcal{U} = -1 \). Turning our attention to the gauge fields since

\[
K_z e^{i n \phi} \tau_\pm = (n \pm 1) e^{i n \phi} \tau_\pm \tag{3.18}
\]
where \(\tau_{\pm} = \tau_1 \pm i \tau_2\), we see that taking the combinations \(f_1 - f_2\) and \(h_1 + h_2\) we get \(K_z = 0\) and \(U = -1\). Intuitively we expect the least stable mode to have \(K_z = 0\) since as \(|K_z|\) increases one gets an increasing centrifugal barrier. The other combinations \(f_1 + f_2\) and \(h_1 - h_2\) are a superposition of \(K_z = \pm 2\) and we will see they decouple and drop out of our analysis.

Also note that the barred and unbarred perturbations decouple. (Under the combined operation of reflection in the x-axis and complex conjugation the barred and unbarred variables have opposite sign.) Furthermore, the stability problem in the barred variables is contained within the problem of the unbarred variables (all we need to do is to set \(\phi_1 = 0\)). Therefore, it is sufficient to consider only the unbarred functions.

We will now systematically simplify the expression for the energy variation. After a lot of algebra, we obtain the first step:

\[
\frac{\delta E}{2\pi} = \int dr \, r \left[ \left( 1 - \frac{(grf)^2}{2P_+} \right) \chi^2 + M^2 \chi^2 \right]
\]

\[
+ \int \frac{dr}{2r} \left[ A_{\pm} \xi_{\pm}'^2 + S_{\pm} \xi_{\pm}^2 \right]
\]

\[
+ \alpha \cos \theta_W \int dr \, \sqrt{P_+} \left[ \frac{(1 - \gamma v)\xi_+ + \gamma v'\xi_+}{P_+} \right] \left( f\chi' - f'\chi \right) - \frac{f\chi}{r^2} \left( 1 - \alpha \sin^2 \theta_W v \right) \xi_+
\]

\[
+ T_+(F_-, \chi, \xi_+) + T_-(F_+, \xi_-)
\]

where,

\[
T_+(F_-, \chi, \xi_+) = \int \frac{dr}{2r} \left[ \sqrt{P_+} F_- + \frac{(1 - \gamma v)\xi_+ + \gamma v'\xi_+}{\sqrt{P_+}} \right] - \alpha \cos \theta_W r^2 \frac{(f\chi' - f'\chi)^2}{\sqrt{P_+}}
\]

(3.19)

(3.20)
\[
T_-(F_+, \xi-) = \int \frac{dr}{2r} \left[ \sqrt{P_- F_+} + \frac{(1 + \gamma v)\xi'_- - \gamma v'\xi_-}{\sqrt{P_-}} \right]^2
\]  
(3.21)

\[
P_\pm = (1 \mp \gamma v)^2 + \frac{g^2 r^2 f^2}{2}
\]  
(3.22)

\[
F_\pm = \frac{f_2 \pm f_1}{2}
\]  
(3.23)

\[
\xi_\pm = \frac{h_2 \pm h_1}{2}
\]  
(3.24)

\[
A_\pm (r) = \frac{g^2 r^2 f^2}{2P_\pm (r)}
\]  
(3.25)

\[
M^2 = \frac{\alpha^2}{4r^2} \cos^2 2\theta_W v^2 + 2\lambda \left( f^2 - \eta^2 \right) - \frac{(g f')^2}{2P_+} - \frac{1}{r} \frac{d}{dr} \left( \frac{g^2 r^3 f'}{2P_+} \right)
\]  
(3.26)

\[
S_\pm (r) = \frac{g^2 f^2}{2} - \frac{\gamma^2 v'^2}{P_\pm (r)} \pm r \frac{d}{dr} \left[ \frac{v'}{r} \left( 1 \mp \gamma v \right) \right] .
\]  
(3.27)

As expected the problem in \( \chi, F_- \) and \( \xi_+ \) has decoupled from the problem in \( F_+ \) and \( \xi_- \). Furthermore the only terms containing \( F_- \) and \( F_+ \) are \( T_- \) and \( T_+ \) respectively and since these are whole squares they can be set to zero. This then leaves us with a problem in \( \chi \) and \( \xi_+ \) and a problem in just \( \xi_- \).

We first discuss the \( \xi_- \) problem. We conjecture that the relevant potential, \( S_- \) is positive for any values of the parameters \( \beta \) and \( \cos \theta_W \) (where \( \beta = 8\lambda / g^2 \)). We motivate this by considering the asymptotic behaviour as \( r \to 0 \) and as \( r \to \infty \): in both these limits \( S_- \) is always positive. Our conjecture is therefore reasonable since the function is basically exponential, modulated by polynomials. (We have checked it numerically for many pairs of parameters.)

Hence our analysis is reduced to considering perturbations in \( \xi_+ \) and \( \chi \) alone. We express the change in the energy in the form of an eigenvalue problem:

\[
\delta E[\chi, \xi_+] = 2\pi \int dr \ r (\chi, \xi_+) O \left( \frac{\chi}{\xi_+} \right)
\]  
(3.28)

where, \( O \) is a \( 2 \times 2 \) matrix differential operator.
It is now useful to identify the form of the perturbations $\chi$ and $\xi_+$ that are pure gauge transformations of the string configuration. It is easy to see that perturbations of the form

$$
\delta \phi = ig\psi \phi_0, \quad \delta W_i = -iD_0i\psi, \quad (3.29)
$$

where $\psi$ is a real $L(SU(2))$ valued function and the 0 subscript denotes the un-perturbed fields, is an infinitesimal gauge transformation of the original vortex solution. (In (3.29), $W_i$ represents $\vec{r} \cdot \vec{W}_i$.) If we now require that these purely gauge perturbations do not affect the string configuration itself, then we can only have

$$
\psi = s(r)\begin{pmatrix} 0 & ie^{-i\phi} \\ -ie^{i\phi} & 0 \end{pmatrix} \quad (3.30)
$$

where $s(r)$ is any smooth function. This means that perturbations given by

$$
\begin{pmatrix} \chi \\ \xi_+ \end{pmatrix} = s(r)\begin{pmatrix} -gf \\ 2(1 - \gamma v) \end{pmatrix} \quad (3.31)
$$

are pure gauge perturbations that do not affect the string configuration. Therefore, such perturbations cannot contribute to the energy variation and must be annihilated by $O$. Then, in the two-dimensional space of $(\chi, \xi_+)$ perturbations, we can choose a basis in which one direction is pure gauge and is given by (3.31) and the other orthogonal direction is the direction of physical perturbations. The physical mode is,

$$
\zeta = (1 - \gamma v)\chi + \frac{gf}{2}\xi_+. \quad (3.32)
$$

It was a good check on our algebra that on eliminating $\xi_+$ in terms of $\zeta$ and $\chi$ in eq. (3.28) the functional reduces to one depending only on $\zeta$:

$$
\delta E[\zeta] = 2\pi \int dr \ r\zeta \overline{O}\zeta \quad (3.33)
$$

where $\overline{O}$ is the differential operator

$$
\overline{O} = -\frac{1}{r} \frac{d}{dr} \left( \frac{r}{P_+} \frac{d}{dr} \right) + U(r) \quad (3.34)
$$
and

\[ U(r) = \frac{f''}{P f'^2} + \frac{2S_+}{g^2 r^2 f^2} + \frac{1}{r} \frac{d}{dr} \left( \frac{r f'}{P f} \right). \]  

(3.35)

To summarise, the question of stability reduces to asking if the operator \( \overline{O} \) has negative eigenvalues in its spectrum. That is, whether the eigenvalue \( \omega \) of the Schrödinger equation,

\[ \overline{O} \zeta = \omega \zeta, \]  

(3.36)

can be negative. The eigenfunction \( \zeta \) must also satisfy the boundary conditions \( \zeta(r = 0) = 1 \) and \( \zeta \to c \) (\( c \) is some constant) as \( r \to \infty \).

In this way we have reduced the stability analysis down to one Schrödinger equation which we will solve numerically.

4. Numerical Analysis

We solve eq. (3.36) together with the Nielsen-Olesen eqs. (2.9) and (2.10). However, it is convenient to work with rescaled dimensionless variables. Hence, we define

\[ P \equiv \frac{\sqrt{2}}{\eta} f, \quad V \equiv \frac{\alpha}{2} v, \quad R \equiv \frac{\alpha \eta}{2\sqrt{2}} r. \]  

(4.1)

In terms of these dimensionless variables, the Nielsen-Olesen equations (2.9) and (2.10) become,

\[ P'' + \frac{P'}{R} - (1 - V)^2 \frac{P}{R^2} + \beta(1 - P^2)P = 0 \]  

(4.2)

\[ V'' - \frac{V'}{R} + 2(1 - V)P^2 = 0 \]  

(4.3)

where primes now denote differentiation with respect to \( R \). The functions \( P \) and \( V \) also satisfy the boundary conditions:

\[ P(0) = 0 = V(0), \quad P(\infty) = 1, \quad V(\infty) = 1 \]  

(4.4)
The problem now has only two free parameters: \( \cos \theta_W \) and \( \beta \). This may be seen by rescaling fields and coordinates in the operator \( \hat{O} \) as in (4.1). With these rescalings the quantities \( S_+ \) and \( P_+ \) in the eigenvalue problem (3.36) get replaced by:

\[
S_+^* = \frac{P^2}{2} - \cos^2 \theta_W \frac{V'^2}{P_+^*} - \frac{\beta}{2} \frac{d}{dR} \left\{ \frac{V'}{R} \left\{ \frac{n - 2 \cos^2 \theta_W V}{P_+^*} \right\} \right\}
\]

where,

\[
P_+^* = (1 - 2 \cos^2 \theta_W V)^2 + 2 \cos^2 \theta_W R^2 P^2
\]

The rescaled eigenvalue problem was solved by using a fifth order Runge-Kutta algorithm. We kept \( \beta \) fixed and found \( \theta_w \) for which the lowest eigenvalue changes sign. We repeated this procedure for several values of \( \beta \) and found the corresponding values of critical parameters \((\sqrt{\beta}, \sin^2 \theta_w)\). The above method was used to scan the range \( 0.07 \leq \beta \leq 1.0 \). Lower values of \( \beta \) make the numerical analysis fairly intensive since then there are two widely different scales in the problem corresponding to the two widely different masses. Our results are shown in Fig. 1 where we plot the critical values of \( \sqrt{\beta} \) (the ratio of the Higgs mass to the Z mass) versus the corresponding values of \( \sin^2 \theta_w \). In sector III, on the right-hand side of the data line, equation (3.36) had no negative eigenvalues implying string stability. Thus we may distinguish three sectors in Fig. 1: sector I where the electroweak strings are unstable, sector III where strings are stable, and, the presently unexplored region shown as sector II \((\beta < 0.07 \text{ or } m_H < 24 \text{GeV})\). It is evident that the physically realized values: \( \sin^2 \theta_w = 0.23 \) and \( \sqrt{\beta} = m_H/m_Z > 0.62 \) (see Ref. 14) lie entirely inside sector I. This brings us to the main result of this paper: if the standard electroweak model is the physically realized model, then the existing vortex solutions in the bare model are unstable.
5. Outlook

We have rigorously established that the electroweak model admits stable vortex solutions for a certain range of parameters. This result is exciting in that it makes the possibility of observing coherent states in particle physics closer to reality. On the other hand, it is somewhat disappointing that the values of the parameters that Nature has actually chosen are such that the vortex is unstable. However, this still does not mean that the vortex will be unstable in the real world since our analysis only applies to the bare electroweak model. In the context of the early universe, for example, we must do a stability analysis at high temperatures. One should also consider the possibility that Nature has chosen an extension of the standard model where the Higgs potential is more complicated. In such circumstances, the stability issue would have to be readdressed.

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References

1. T. Vachaspati and A. Achúcarro, Phys. Rev. D 44, 3067 (1991).

2. G. W. Gibbons, M. Ortiz, F. Ruiz-Ruiz and T. Samols, DAMTP preprint (1992); J. Preskill, Cal Tech preprint (1992).

3. M. Hindmarsh, Phys. Rev. Lett. 68, 1263 (1992).

4. A. Achúcarro, K. Kuijken, L. Perivolaropoulos and T. Vachaspati, Nucl. Phys. B, to be published.

5. T. Vachaspati, Phys. Rev. Lett. 68, 1977 (1992).

6. T. Vachaspati, Tufts preprint (1992).

7. T. Vachaspati and M. Barriola, Tufts preprint (1992).

8. S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam in “Elementary Particle Theory”, ed. N. Svarthholm, Stockholm: Almqvist, Forlag AB, pg 367.

9. We are only considering the stability under small perturbations. Hence, the word “stability” should always be understood as “meta-stability”.

10. J. C. Taylor, “Gauge Theories of Weak Interactions”, Cambridge University Press, 1976.

11. H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).

12. “Solitons and Particles”, ed. C. Rebbi and G. Soliani, World Scientific, 1984.

13. Barred indices always range from 1 to 2.

14. J. Steinberger, Phys. Rep. 203, 345 (1991).

15. M. James, L. Perivolaropoulos and T. Vachaspati, to appear in Phys. Rev. D (rapid communication).
FIGURE CAPTIONS

A map of parameter space showing the results of the stability analysis. Sector I contains unstable strings, sector III contains stable strings and we have not explored sector II. We also indicate the physically allowed range of parameter space. The data from LEP constrains the Higgs mass to $M_H > 53 GeV$ which implies $\sqrt{\beta} > 0.62 GeV$ and the observed $\sin^2 \theta_W$ is 0.23.