GAMMA INTEGRAL STRUCTURE FOR AN INVERTIBLE POLYNOMIAL OF CHAIN TYPE

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Abstract. The notion of the Gamma integral structure for the quantum cohomology of an algebraic variety was introduced by Iritani, Katzarkov–Kontsevich–Pantev. In this paper, we define the Gamma integral structure for an invertible polynomial of chain type. Based on the $\Gamma$-conjecture by Iritani, we prove that the Gamma integral structure is identified with the natural integral structure for the Berglund–Hübsch transposed polynomial by the mirror isomorphism.

1. Introduction

For a holomorphic function $f: \mathbb{C}^n \rightarrow \mathbb{C}$ with at most an isolated critical point at the origin, K. Saito developed a theory of primitive forms which yields a Frobenius structure on the base space of the universal unfolding $[S-K, ST]$. In order further to study the (exponential) periods of a primitive form, one needs careful analysis of the structure connections of the Frobenius manifold, especially, their integral structures.

A Frobenius manifold can be associated to a smooth projective variety (or orbifold) $X$ by the genus zero Gromov–Witten invariants of $X$. The first structure connection of the Frobenius manifold $M_X$, often called the quantum connection in this setting, is a flat meromorphic connection on $\mathbb{P}^1 \times M_X$. If the ring structure on the tangent space $T_p M_X$ over a point $p \in M_X$ is semi-simple, then the meromorphic connection restricted to $\mathbb{P}^1 = \mathbb{P}^1 \times \{p\}$ has an irregular singular point and a regular singular point. One can define the monodromy data (at the point $p$) of the Frobenius manifold $M_X$, consisting of the monodromy matrix at the regular singular point, the Stokes matrix $S$ at the irregular singular point, and the central connection matrix $C$ between these two singular points. Here, it is very important to note that the monodromy data is locally constant [D3, Isomonodromicity Theorem]. In [D2], Dubrovin formulated a conjecture on a close relationship between the monodromy data of the Frobenius manifold $M_X$ and the structure of the bounded derived category $D^b(X)$ of coherent sheaves on $X$ when $X$ is Fano. Its refined version is given in [CDG] (see Conjecture 3.5 below).

The notion of the Gamma integral structure on the space of multi-valued flat sections of the quantum connection was introduced by Iritani [I] (see also [KKP]). This structure
is defined to be a framing induced by the Grothendieck group of the derived category $\mathcal{D}^b(X)$. To be more precise, suppose that $\mathcal{D}^b(X)$ admits a full exceptional collection $(\mathcal{E}_1, \ldots, \mathcal{E}_{\tilde{\mu}})$. Then the framing is given by the free $\mathbb{Z}$-module spanned by $\hat{\Gamma}_X \text{Ch}(\mathcal{E}_j)$, $j = 1, \ldots, \tilde{\mu}$, where $\hat{\Gamma}_X$ is the Gamma class of $X$ and $\text{Ch}(\mathcal{E}_j)$ is the (modified) Chern character of $\mathcal{E}_j$. For a weak Fano orbifold $X$ such that $\mathcal{D}^b(X)$ admits a full exceptional collection, Galkin–Golyshev–Iritani [GGI] conjectured that the Gamma integral structure describes the central connection matrix $C$ of the Frobenius manifold $M_X$. This conjecture is called the $\Gamma$-conjecture, which is equivalent to a part of the refined Dubrovin’s conjecture (see [I, CDG]).

Mirror symmetry is an equivalence of seemingly different two objects in mathematics, algebraic one and geometric one, which was discovered by physicists in their study of string theory. Consider a polynomial $f_n \in S_n := \mathbb{C}[z_1, \ldots, z_n]$ of the form

$$f_n = f_n(z_1, \ldots, z_n) := z_1^{a_1}z_2 + z_2^{a_2}z_3 + \cdots + z_{n-1}^{a_{n-1}}z_n + z_n^{a_n}, \quad a_i \geq 2,$$

which is called an invertible polynomial of chain type. Let

$$G_{f_n} := \{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \mid f_n(\lambda_1z_1, \ldots, \lambda_nz_n) = f_n(z_1, \ldots, z_n) \} ,$$

which is called the group of maximal diagonal symmetries of $f$. It is expected by Berglund–Hübsch [BH] that a mirror dual object corresponding to the pair $(f_n, G_{f_n})$ is given by the polynomial $\tilde{f}_n \in \mathbb{C}[x_1, \ldots, x_n]$, called the Berglund–Hübsch transpose of the polynomial $f_n$, defined by

$$\tilde{f}_n = \tilde{f}_n(x_1, \ldots, x_n) := x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n}.$$ 

For the pair $(f_n, G_{f_n})$, one can associate a triangulated category $\text{HMF}^{L_{f_n}}_{S_n}(f_n)$ of maximally graded matrix factorizations, which is considered as an analogue of $\mathcal{D}^b(X)$ for a smooth projective variety. On the other hand, a (counter-clockwise) distinguished basis of vanishing cycles in the Milnor fiber of $\tilde{f}_n$ can be categorified to an $A_\infty$-category $\text{Fuk}^{-\tau}(\tilde{f}_n)$ called the directed Fukaya category, whose derived category $\mathcal{D}^b\text{Fuk}^{-\tau}(\tilde{f}_n)$ is an invariant of the polynomial $\tilde{f}_n$ as a triangulated category. Then the homological mirror symmetry conjecture for invertible polynomials predicts an equivalence of the category $\text{HMF}^{L_{f_n}}_{S_n}(f_n)$ and the derived directed Fukaya category $\mathcal{D}^b\text{Fuk}^{-\tau}(\tilde{f}_n)$ (cf. [ET, T2]). There are many evidences of the above conjecture which follow from related results by several authors (cf. [AT, FU1, FU2, Hab, KST, KST2, Kra, LP, Se, T1, T2, U]).

It is known that from the pair $(f, G_f)$ of an invertible polynomial $f$ and the group of maximal diagonal symmetries $G_f$ a Frobenius manifold is constructed by the FJRW theory [FJR] which is analogous to the Gromov-Witten theory in many ways. On the other side, one can associate a Frobenius manifold to the Berglund–Hübsch transpose $\tilde{f}$ of
f by a choice of a primitive form (cf. [S-K, ST]). The classical mirror symmetry conjecture for invertible polynomials predicts an isomorphism of these Frobenius manifolds, which is proven for many cases (cf. [FJR, HLSW, KriSh, LLSS, MS]).

One can define \(\mathbb{Q}\)-graded \(\mathbb{C}\)-vector spaces \(\Omega_{f_n,G_{f_n}}\) and \(\Omega_{\tilde{f}_n}\) respectively as an analogue of the total Hodge cohomology group \(\bigoplus_{p,q} H^{p,q}(X, \Omega^q_X)\) of a smooth projective variety \(X\) where two \(\mathbb{Q}\)-gradings are introduced by exponents (see Definition 2.5). We can associate a \(\mathbb{C}\)-bilinear form \(J_{f_n,G_{f_n}}\) (resp., \(J_{\tilde{f}_n}\)) on \(\Omega_{f_n,G_{f_n}}\) (resp., \(\Omega_{\tilde{f}_n}\)). It is known that there exists an isomorphism \(\Omega_{\tilde{f}_n} \cong \Omega_{f_n,G_{f_n}}\) of \(\mathbb{Q}\)-graded \(\mathbb{C}\)-vector spaces called the mirror isomorphism (cf. [Kre, EGT]). We choose a mirror isomorphism \(\text{mir} : \Omega_{\tilde{f}_n} \cong \Omega_{f_n,G_{f_n}}\) so that it is compatible with the \(\mathbb{C}\)-bilinear forms \(J_{f_n,G_{f_n}}\) and \(J_{\tilde{f}_n}\) (Proposition 2.19). Therefore, we shall denote later by \(\eta(n)\) the matrix representation of the bilinear form \(J_{f_n,G_{f_n}}\) with respect to a specified homogeneous basis on \(\Omega_{f_n,G_{f_n}}\), which is equal to the matrix representation of the bilinear form \(J_{\tilde{f}_n}\) with respect to a specified homogeneous basis on \(\Omega_{\tilde{f}_n}\). The homogeneous basis defines a diagonal matrix \(\tilde{Q}^{(n)}\) whose entries are exponents of \(\tilde{f}_n\) shifted by \(-n/2\).

In [AT], Aramaki–Takahashi showed the existence of a full exceptional collection \((E_1, \ldots, E_{\tilde{\mu}_n})\) on \(HMF_{S_n}^{L_{f_n}}(f_n)\) which gives a Lefschetz decomposition on \(HMF_{S_n}^{L_{f_n}}(f_n)\) in the sense of [KuSm] where \(\tilde{\mu}_n\) is the Milnor number of \(\tilde{f}_n\). Based on this result, we associate a matrix \(\text{ch}_\Gamma^{(n)}\) to the full exceptional collection whose \(j\)-th column is an analogue of \(\Gamma_X \text{Ch}(\mathcal{E}_j)\) for a Fano orbifold \(X\) (see Definition 3.2). The following theorem proves the necessary conditions for the matrix \((2\pi)^{-\frac{n}{2}} \text{ch}_\Gamma^{(n)}\) to give a central connection matrix of a Frobenius manifold whose non-degenerate bilinear form on the tangent space, the grading matrix and the Stokes matrix are given by \(\eta(n), \tilde{Q}^{(n)}\) and \(\chi^{(n)}\), respectively:

**Theorem 1.1** (Theorem 3.4). The matrix \(\text{ch}_\Gamma^{(n)}\) satisfies following equalities:

\[
\left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_\Gamma^{(n)} \right)^{-1} e \left[ \tilde{Q}^{(n)} \right] \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_\Gamma^{(n)} \right) = S^{(n)},
\]

\[
\left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_\Gamma^{(n)} \right)^T e \left[ \frac{1}{2} \tilde{Q}^{(n)} \right] \eta^{(n)} \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_\Gamma^{(n)} \right) = \chi^{(n)},
\]

where \(S^{(n)}\) is the matrix representation of the automorphism with respect to the exceptional basis \([\{E_i\}]_{i=1}^{\tilde{\mu}_n}\) induced by the Serre functor on \(HMF_{S_n}^{L_{f_n}}(f_n)\) and \(\chi^{(n)}\) is the Euler matrix with respect to the exceptional basis.

In the present paper, we study the Gamma integral structure for \((f, G_f)\). Theorem 1.1 enables us to define the Gamma integral structure \(\Omega_{f_n,G_{f_n}} \mathbb{Z}\) of \(\Omega_{f_n,G_{f_n}}\) as the framing of a map defined by the matrix \((2\pi \sqrt{-1})^{-n} \text{ch}_\Gamma^{(n)}\) with respect to a certain basis of \(\Omega_{f_n,G_{f_n}}\) (see Definition 3.6). On the other hand, there is a natural integral structure
on $\Omega_{\tilde{f}_n}$ given by the relative homology group $H_n(C^n, \text{Re}(\tilde{f}_n) \gg 0; \mathbb{Z})$ via the isomorphism $H^n(C^n, \text{Re}(\tilde{f}_n) \gg 0; \mathbb{C}) \cong \mathbb{H}^n(\Omega^*_{\tilde{f}_n}, d - d\tilde{f}_n\wedge) \cong \Omega_{\tilde{f}_n}$. It is known that the middle homology group $H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z})$ of the Milnor fiber of $\tilde{f}_n$ is a free abelian group of rank $\tilde{\mu}_n$ generated by vanishing cycles and there is an isomorphism $H_n(C^n, \text{Re}(\tilde{f}_n) \gg 0; \mathbb{Z}) \cong H_n(C^n, \tilde{f}_n^{-1}(1); \mathbb{Z}) \cong H_n(\tilde{f}_n^{-1}(1); \mathbb{Z})$. Therefore, the integral structure $\Omega_{\tilde{f}_n, \mathbb{Z}}$ is also considered as the image of the free $\mathbb{Z}$-module $H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z})$ in $\Omega_{\tilde{f}_n}$. Now, we can state the following theorem as an analogue of $\Gamma$-conjecture:

**Theorem 1.2** (Theorem 3.8). The mirror isomorphism $\text{mir} : \Omega_{\tilde{f}_n} \cong \Omega_{f_n, G_{f_n}}$ can be chosen so that it induces an isomorphism of free $\mathbb{Z}$-modules $\Omega_{f_n, \mathbb{Z}} \cong \Omega_{f_n, G_{f_n}, \mathbb{Z}}$ and it is compatible with the $\mathbb{Z}/d_n\mathbb{Z}$-action on both hands side. Here, $\mathbb{Z}/d_n\mathbb{Z}$-action on $\Omega_{f_n, \mathbb{Z}}$ is induced by its action on $C^n$ given by $x_i \mapsto e((-1)^{i-1}/d_i)x_i$, $i = 1, \ldots, n$ where $d_i := a_1 \cdots a_i$, $e[-] = e^{2\pi \sqrt{-1}(-)}$ and the action on $\Omega_{f_n, G_{f_n}, \mathbb{Z}}$ is induced by the grading shift $(z_1)$ on $\text{HMF}^L_{S_n}(f_n)$.

Milanov–Zha [MZ] worked on the Gamma integral structure for the case of type ADE, which motivates us to study the one for general invertible polynomials of chain type. Indeed, their result shows that the theorem holds for $A_l$, $D_l$ and $E_7$ cases (and also $E_6$ and $E_8$ cases after generalizing suitably the conjecture for Thom–Sebastiani sum of invertible polynomials of chain type).

It is an important problem to show the existence of Bridgeland stability conditions [B] on a given triangulated category. In [T1, KST], it is shown that there is a stability condition on $\text{HMF}^L_{S_n}(f_n)$ for an invertible polynomial of type ADE and, in terms of the derived directed Fukaya category $\mathcal{D}^b\text{Fuk}^{-}(\tilde{f}) \cong \text{HMF}^L_{S_n}(f_n)$ (see [Sc] for the equivalence $\mathcal{D}^b\text{Fuk}^{-}(\tilde{f}) \cong \mathcal{D}^b(C\tilde{\Delta})$ where $\tilde{\Delta}$ is the Dynkin quiver of the corresponding type), it is expected that the oscillatory integral for a primitive form induces a stability condition (cf. [B2, T1]). The view point of the homological mirror symmetry naturally leads the following based on the above Theorem.

**Conjecture 1.3** (Conjecture 3.9). Let $(\omega_1, \ldots, \omega_n)$ be positive rational numbers such that $f_n(e[\omega_1]z_1, \ldots, e[\omega_n]z_n) = f_n(z_1, \ldots, z_n)$. There exists a Gepner type stability condition $\sigma$ on $\text{HMF}^L_{S_n}(f_n)$ with respect to the auto-equivalence $(z_1)$ and $e[1/d_n] \in \mathbb{C}$ in the sense of Toda [To, Definition 2.3] such that $(z_1), \sigma = \sigma, e\left[\frac{1}{d_n}\right]$ and its stability function $K_0(\text{HMF}^L_{S_n}(f_n)) \rightarrow \mathbb{C}$ is given by

$$Z_\sigma(E_j) := \begin{cases} 
\frac{1}{(2\pi \sqrt{-1})^n} e^{\left[-\frac{j-1}{d_n}\right]} \prod_{i=1}^m (1 - e[-\omega_{2i-1}]) \cdot \int_{(\mathbb{R}_{\geq 0})^n} e^{-\tilde{f}_n(x)} dx, & \text{if } n = 2m-1, \\
\frac{1}{(2\pi \sqrt{-1})^n} e^{\left[-\frac{j-1}{d_n}\right]} \prod_{i=1}^m (1 - e[-\omega_{2i}]) \cdot \int_{(\mathbb{R}_{\geq 0})^n} e^{-\tilde{f}_n(x)} dx, & \text{if } n = 2m,
\end{cases}$$
where \( dx \) denotes the standard volume form \( dx_1 \wedge \cdots \wedge dx_n \).

This conjecture holds for the case of \( n = 1 \) and polynomials of ADE type \([T1, KST]\) (see Proposition 3.10 below).

We briefly outline the contents of the paper. In Section 2, we recall some notations and terminologies of invertible polynomials, and then discuss the mirror isomorphism \( \text{mir} : \Omega_{\tilde{f}_n} \cong \Omega_{f_n, Gf_n} \) for an invertible polynomial of chain type. Section 3 is the main part of this paper. After recalling some facts on \( \text{HMF}_{S_n}^{L_{f_n}}(f_n) \), we define the matrix \( \text{ch}^{(n)} \Gamma \) and state Theorem 3.4. We also formulate a Gamma integral structure on \( \Omega_{f_n, Gf_n} \) and state Theorem 3.8. Section 4 and Section 5 are devoted to proving Theorem 3.4 and Theorem 3.8, respectively. In Section 6, we study the Stokes matrix \( S \) and the central connection matrix \( C \) of the Frobenius manifold associated to \( \tilde{f}_n \) and a certain primitive form \( \zeta \). We describe the relation of the monodromy data \((S, C)\) and the data \((\chi^{(n)}, \text{ch}^{(n)} \Gamma)\) of the category \( \text{HMF}_{S_n}^{L_{f_n}}(f_n) \).

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2. Preliminaries

2.1. Invertible polynomials. A polynomial \( f = f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n] \) is called a weighted homogeneous polynomial if there are positive integers \( w_1, \ldots, w_n \) and \( d \) such that \( f(\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_1, \ldots, z_n) \) for all \( \lambda \in \mathbb{C}^\ast \). A weighted homogeneous polynomial \( f \) is called non-degenerate if it has at most an isolated critical point at the origin in \( \mathbb{C}^n \).

Definition 2.1. A non-degenerate weighted homogeneous polynomial \( f \in \mathbb{C}[z_1, \ldots, z_n] \) is called invertible if the following conditions are satisfied.

- The number of variables coincides with the number of monomials in \( f \):

\[
  f(z_1, \ldots, z_n) = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} z_j^{E_{ij}}
\]

for some coefficients \( c_i \in \mathbb{C}^\ast \) and non-negative integers \( E_{ij} \) for \( i, j = 1, \ldots, n \).

- The matrix \( \mathbb{E} := (E_{ij}) \) is invertible over \( \mathbb{Q} \).

Let \( f = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} z_j^{E_{ij}} \) be an invertible polynomial. Define positive rational numbers \( \omega_1, \ldots, \omega_n \) called the rational weights of \( f \) by the unique solution of the equation

\[
  (\omega_1, \ldots, \omega_n) := (1, \ldots, 1) \mathbb{E}^{-T}
\]
where $E^{-T} := (E^{-1})^T$. Note that $f(\lambda^\omega z_1, \ldots, \lambda^\omega z_n) = \lambda f(z_1, \ldots, z_n)$ for all $\lambda \in \mathbb{C}^*$.

Without loss of generality one may assume that $c_i = 1$ for all $i$ by rescaling the variables. An invertible polynomial $f$ can be written as a Thom–Sebastiani sum $f = f_1 \oplus \cdots \oplus f_p$ of invertible ones (in groups of different variables) $f_\nu$, $\nu = 1, \ldots, p$ of the following types (cf. [KrSk]):

1. $z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_{m-1}^{a_{m-1}} z_m + z_m^{a_m}$ (chain type, $m \geq 1$);
2. $z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_{m-1}^{a_{m-1}} z_m + z_m^{a_m} z_1$ (loop type, $m \geq 2$).

For an invertible polynomial $f = f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$, define a $\mathbb{C}$-algebra called the *Jacobian algebra* $\text{Jac}(f)$ as

$$\text{Jac}(f) := \mathbb{C}[z_1, \ldots, z_n] / \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right).$$

(2.2)

Since $f$ is non-degenerate, the Jacobian algebra $\text{Jac}(f)$ is finite-dimensional. Set $\mu_f := \dim_\mathbb{C} \text{Jac}(f)$ and call it the *Milnor number* of $f$. We sometimes include the case $n = 0$. If $n = 0$, set $\text{Jac}(f) := \mathbb{C}$, naturally considered as the $\mathbb{C}$-algebra of constant functions on a point, and we have $\mu_f = 1$.

Let $\Omega^p(\mathbb{C}^n)$ be the complex vector space of regular $p$-forms on $\mathbb{C}^n$. Consider the complex vector space

$$\Omega_f := \Omega^n(\mathbb{C}^n)/df \wedge \Omega^{n-1}(\mathbb{C}^n).$$

(2.3)

If $n = 0$, then set $\Omega_f := \mathbb{C}$, the complex vector space of constant functions on a point. Note that $\Omega_f$ is naturally a free $\text{Jac}(f)$-module of rank one, namely, by choosing a nowhere vanishing $n$-form $dz := dz_1 \wedge \cdots \wedge dz_n$ we have the following isomorphism

$$\text{Jac}(f) \xrightarrow{\cong} \Omega_f, \quad [\phi(z)] \mapsto [\phi(z)dz].$$

(2.4)

**Proposition 2.2** (cf. [Har]). Define a $\mathbb{C}$-bilinear form $J_f : \Omega_f \times \Omega_f \rightarrow \mathbb{C}$ by

$$J_f([\phi_1(z)dz], [\phi_2(z)dz]) := \text{Res}_{\mathbb{C}^n} \left[ \phi_1(z)\phi_2(z)dz_1 \wedge \cdots \wedge dz_n \right].$$

(2.5)

Then, the bilinear form $J_f$ on $\Omega_f$ is non-degenerate.

**Definition 2.3.** Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be an invertible polynomial. The *group of maximal diagonal symmetries* $G_f$ of $f$ is defined as

$$G_f := \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \mid f(\lambda_1 z_1, \ldots, \lambda_n z_n) = f(z_1, \ldots, z_n)\}.$$

(2.6)

Each element $g \in G_f$ has a unique expression of the form $g = (e[\alpha_1], \ldots, e[\alpha_n])$ with $0 \leq \alpha_i < 1$, where $e[\alpha] := \exp(2\pi \sqrt{-1}\alpha)$. The *age* of $g$ is defined as the rational number $\text{age}(g) := \sum_{i=1}^n \alpha_i$. For each $g \in G_f$, denote by $\text{Fix}(g) := \{z \in \mathbb{C}^n \mid g \cdot z = z\}$ the fixed
locus of $g$, by $n_g := \dim \text{Fix}(g)$ its dimension and by $f^g := f|_{\text{Fix}(g)}$ the restriction of $f$ to the fixed locus of $g$.

Note that the function $f^g$ is an invertible polynomial. Indeed, for an invertible polynomial of chain type, which is of our main interest in this paper, we have the following

**Proposition 2.4.** Let $f_n = z_1 + z_2 + \cdots + z_{n-1} + z^n$ be an invertible polynomial of chain type. For each $g \in G_{f_n} \setminus \{\text{id}\}$, there exists $1 \leq k \leq n$ such that $\text{Fix}(g) = \{z_i = 0 \mid 1 \leq i \leq k\}$. □

We shall use the fact that, for each $g \in G_f$, $\Omega_{f^g}$ admits a natural $G_f$-action by restricting the $G_f$-action on $\mathbb{C}^n$ to $\text{Fix}(g)$ since $G_f$ acts diagonally on $\mathbb{C}^n$. Since $\Omega_f$ has a structure of $\mathbb{Q}$-graded complex vector space with respect to the rational weights $\omega_1, \ldots, \omega_n$, we can associate the following $\mathbb{Q}$-graded $\mathbb{C}$-vector space:

**Definition 2.5.** Define $\mathbb{Q}$-graded complex vector space $\Omega_{f,G_f}$ by

$$\Omega_{f,G_f} := \bigoplus_{g \in G_f} \Omega_{f,g}, \quad \Omega_{f,g} := (\Omega_{f^g})^{G_f}(-\text{age}(g)),$$

where $(\Omega_{f^g})^{G_f}$ denotes the $G_f$-invariant subspace of $\Omega_{f^g}$ and $(\Omega_{f^g})^{G_f}(-\text{age}(g))$ denotes the $\mathbb{Q}$-graded complex vector space $\Omega_{f^g}$ shifted by $-\text{age}(g) \in \mathbb{Q}$.

**Definition 2.6.** Define a non-degenerate $\mathbb{C}$-bilinear form $J_{f,G_f} : \Omega_{f,G_f} \times \Omega_{f,G_f} \to \mathbb{C}$ by

$$J_{f,G_f} := \bigoplus_{g \in G_f} J_{f,g},$$

where $J_{f,g}$ is a perfect $\mathbb{C}$-bilinear form $J_{f,g} : \Omega_{f,g} \times \Omega_{f,g^{-1}} \to \mathbb{C}$ defined by

$$J_{f,g}(\xi_1, \xi_2) := \frac{1}{|G_f|} \cdot \text{Res}_{\text{Fix}(g)} \left[ \phi_1 \phi_2 d\zeta_{n-n_g+1} \wedge \cdots \wedge d\zeta_n \right]$$

for $\xi_1 = [\phi_1 d\zeta_{n-n_g+1} \wedge \cdots \wedge d\zeta_n] \in \Omega_{f,g}$ and $\xi_2 = [\phi_2 d\zeta_{n-n_g-1} \wedge \cdots \wedge d\zeta_n] \in \Omega_{f,g^{-1}}$ (note that $n_g = n_{g^{-1}}$). In particular for each $g \in G_f$ with $n_g = 0$, we define

$$J_{f,g}(1_g, 1_g^{-1}) := \frac{1}{|G_f|},$$

where $1_g$ denotes the constant function 1 on $\text{Fix}(g) = \{0\}$.

We have the *mirror isomorphism* between $\Omega_f$ and $\Omega_{f,G_f}$.

**Proposition 2.7** (cf. [Kre, EGT]). Let $f = \sum_{i=1}^n \prod_{j=1}^n z_j^{e_{ij}}$ be an invertible polynomial. There exists an isomorphism of $\mathbb{Q}$-graded $\mathbb{C}$-vector spaces

$$\text{mir} : \Omega_f \cong \Omega_{f,G_f},$$

where...
Definition 2.11. For each \( \mu \)

\[
\tilde{f} := \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{\mu_j}
\]

is the Berglund–Hübsch transpose of \( f \).

See Proposition 2.19 below for the details of the mirror isomorphism \textbf{mir} when \( f \) is of chain type.

2.2. Invertible polynomials of chain type. From now on, we shall only consider invertible polynomials of chain type. Set

\[
f_n := f_n(z_1, \ldots, z_n) := z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n},
\]

\[
\tilde{f}_n := \tilde{f}_n(x_1, \ldots, x_n) := x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n}.
\]

For simplicity, assume that \( a_i \geq 2 \) for all \( i = 1, \ldots, n \).

Definition 2.8. For each non-negative integer \( n \), define sets \( B_{f_0}' \), \( B_{\tilde{f}_n}' \) of monomials in \( \mathbb{C}[x_1, \ldots, x_n] \) inductively as follows: Let \( B_{f_0}' := \{1\} \) and

\[
B_{f_n}' := \{ x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \mid 0 \leq k_i \leq a_i - 1 \ (i = 1, \ldots, n - 1), \ 0 \leq k_n \leq a_n - 2 \}, \quad n \geq 1.
\]

For \( n = 0, 1 \), let \( B_{\tilde{f}_0}' := B_{f_0}' = \{1\} \) and \( B_{\tilde{f}_1}' := B_{f_1}' = \{ x_1^{k_1} \mid 0 \leq k_1 \leq a_1 - 2 \} \).

For \( n \geq 2 \), let

\[
B_{\tilde{f}_n}' := B_{\tilde{f}_{n-1}}' \cup \{ \phi^{(n-2)}(x_1, \ldots, x_{n-2}) x_n^{a_n - 1} \mid \phi^{(n-2)}(x_1, \ldots, x_{n-2}) \in B_{f_{n-2}}' \}.
\]

Proposition 2.9 (Kre). The set \( B_{\tilde{f}_n}' \) defines a \( \mathbb{C} \)-basis of the Jacobian algebra \( \text{Jac}(\tilde{f}_n) \).

Namely, we have \( \text{Jac}(\tilde{f}_n) = \langle \{ \phi^{(n)}(x) \mid \phi^{(n)}(x) \in B_{f_n}' \} \rangle \).

Set \( d_0 := 1 \) and \( d_i := a_1 \cdots a_i \) for \( i = 1, \ldots, n \). Define a positive integer \( \tilde{\mu}_n \) by

\[
\tilde{\mu}_n := \sum_{i=0}^{n} (-1)^{n-i} d_i,
\]

which satisfies \( \tilde{\mu}_n = d_n - d_{n-1} + \tilde{\mu}_{n-2} \).

Corollary 2.10. The Milnor number \( \mu_{\tilde{f}_n} = \dim_{\mathbb{C}} \text{Jac}(\tilde{f}_n) \) is given by \( \tilde{\mu}_n \).

Denote by \( E \) the invertible matrix associated to \( f_n \), which is given by

\[
E = (a_i \delta_{i,j} + \delta_{i+1,j}) =
\begin{pmatrix}
    a_1 & 1 & 0 & \cdots & 0 \\
    0 & a_2 & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & 0 \\
    0 & \cdots & \cdots & \cdots & a_n
\end{pmatrix}.
\]

Definition 2.11. For each \( k = (k_1, \ldots, k_n) \), define rational numbers \( \omega_{k,1}^{(n)}, \ldots, \omega_{k,n}^{(n)} \) by

\[
(\omega_{k,1}^{(n)}, \ldots, \omega_{k,n}^{(n)}) := (k_1 + 1, \ldots, k_n + 1) E^{-T}.
\]

(2.9)
In particular, \( \omega_{0,1}^{(n)}, \ldots, \omega_{0,n}^{(n)} \) are nothing but the rational weights of \( f_n \) and will be denoted simply by \( \omega_1^{(n)}, \ldots, \omega_n^{(n)} \). It is easy to see that

\[
\omega_i^{(n)} = \sum_{l=i}^{n} (-1)^{i-l} \frac{d_{l-1}}{d_l}, \quad i = 1, \ldots, n. \tag{2.10}
\]

**Proposition 2.12.** For each \( k = (k_1, \ldots, k_n) \) such that \( 0 \leq k_i \leq a_i - 1 \) (\( i = 1, \ldots, n - 1 \)), \( 0 \leq k_n \leq a_n - 2 \), we have \( 0 < \omega_{k,i}^{(n)} \leq 1 \) for all \( i = 1, \ldots, n \). Moreover, for \( k^* := (a_1 - 1, \ldots, a_{n-1} - 1, a_n - 2) - k \), we have the duality property

\[
\omega_{k^*,i}^{(n)} = 1 - \omega_{k,i}^{(n)}, \quad i = 1, \ldots, n. \tag{2.11}
\]

**Proof.** Since \( \omega_{k,n}^{(n)} = (k_n + 1)/a_n \), we have \( 0 < \omega_{k,n}^{(n)} < 1 \). For \( i = 1, \ldots, n - 1 \), we have \( \omega_{k,i}^{(n)} = (k_i + 1 - \omega_{k,i+1}^{(n)})/a_n - 1 \), and hence we obtain \( 0 < \omega_{k,i}^{(n)} < 1 \) inductively.

By the definition of \( k^* \), we have

\[
\left( \omega_{k,1}^{(n)}, \ldots, \omega_{k,n}^{(n)} \right) + \left( \omega_{k^*,1}^{(n)}, \ldots, \omega_{k^*,n}^{(n)} \right) = (a_1 - 1, \ldots, a_{n-1} - 1, a_n - 2)E^{-T} = (1, \ldots, 1). \]

\(\square\)

On the structure of the group \( G_{f_n} \), we have the following

**Proposition 2.13.** The group \( G_{f_n} \) is a cyclic group of order \( d_n \) generated by the element

\[
\left( e \left[ \frac{1}{d_n} \right], \ldots, e \left[ (-1)^{i-1} \frac{d_{i-1}}{d_n} \right], \ldots, e \left[ (-1)^{n-1} \frac{d_{n-1}}{d_n} \right] \right).
\]

\(\square\)

For each non-negative integer \( n \), define sets \( I_n, I_n' \) inductively as follows: Let \( I_0' := \{1\} \) and \( I_0 := \{\kappa \in \{1, \ldots, d_n\} \mid a_n \not\mid \kappa\} \), \( n \geq 1 \). For \( n = 0, 1 \), set \( I_0 := I_0' = \{1\} \) and \( I_1 := I_1' = \{1, 2, \ldots, a_1 - 1\} \). For \( n \geq 2 \), set \( I_n := I_n' \sqcup I_{n-2} \). Note that \( |I_n'| = d_n - d_{n-1} \) and \( |I_n| = \bar{\mu}_n \).

For \( \kappa \in I_n' \) and \( i = 1, \ldots, n \), set

\[
\omega_{\kappa,i}^{(n)} := \left(-1\right)^{i-1} \frac{d_{i-1}}{d_n} \cdot \kappa - \left(-1\right)^{i-1} \frac{d_{i-1}}{d_n} \cdot \kappa.
\]

**Proposition 2.14.** For \( \kappa \in I_n' \) and \( i = 1, \ldots, n \), we have the following duality property;

\[
\omega_{d_n - \kappa, i}^{(n)} = 1 - \omega_{\kappa, i}^{(n)}.
\]

**Proof.** It follows from a direct calculation:

\[
\omega_{d_n - \kappa, i}^{(n)} = \left(-1\right)^{i-1} \frac{d_{i-1}}{d_n} \cdot (d_n - \kappa) - \left(-1\right)^{i-1} \frac{d_{i-1}}{d_n} \cdot (d_n - \kappa)
\]

\[
= 1 - \left(-1\right)^{i-1} \frac{d_{i-1}}{d_n} \cdot \kappa - \left(-1\right)^{i-1} \frac{d_{i-1}}{d_n} \cdot \kappa
\]

\[
= 1 - \omega_{\kappa, i}^{(n)}.
\]
The following correspondence $\psi$ plays an important role in this paper.

**Proposition 2.15.** The map

$$\psi : \{ k = (k_1, \ldots, k_n) | 0 \leq k_i \leq a_i - 1 \ (i = 1, \ldots, n - 1), \ 0 \leq k_n \leq a_n - 2 \} \xrightarrow{\sim} I'_n$$

defined by

$$\psi(k) = \sum_{l=1}^{n} (-1)^{l-1} \frac{d_n}{d_l}(k_l + 1)$$

is a bijection of sets of $d_n - d_{n-1}$ elements and satisfies $\omega^{(n)}_{\psi(k),i} = \omega^{(n)}_{k,i}$ for $i = 1, \ldots, n$.

In particular, we have $\psi(0) = \psi(0, \ldots, 0) = \sum_{l=1}^{n} (-1)^{l-1} \frac{d_n}{d_l}$.

**Proof.** One can check easily by a direct calculation. \hfill \square

**Corollary 2.16.** For each $k = (k_1, \ldots, k_n)$ such that $x^k := x_1^{k_1} \cdots x_n^{k_n} \in B'_{f_n}$, we have

$$\int_{(\mathbb{R}_+)^n} e^{-\tilde{f}_n(x)} x^k dx = \frac{1}{d_n} \prod_{l=1}^{n} \Gamma(1 - \omega^{(n)}_{d_n - \psi(k),i})$$

where $x^k dx := x_1^{k_1} \cdots x_n^{k_n} dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{C}^n)$ and $\Gamma(s)$ denotes the Gamma function

$$\Gamma(s) := \int_{0}^{\infty} y^{s-1} e^{-y} dy.$$

**Proof.** By using the change of variables $y_i := \prod_{j=1}^{n} x_j^{E_{ij}}, \ i = 1, \ldots, n$ and the definition of $\omega^{(n)}_{k,1}, \ldots, \omega^{(n)}_{k,n}$, we have

$$\int_{(\mathbb{R}_+)^n} e^{-\tilde{f}_n(x)} x^k dx = \frac{1}{d_n} \prod_{l=1}^{n} \Gamma(\omega^{(n)}_{k,i}).$$

Hence, we obtain the statement by Proposition 2.14 and Proposition 2.15. \hfill \square

**Remark 2.17.** For any invertible polynomial, one can define rational numbers $\omega_{k,i}$, $\omega_{k,i}$ and a map $\psi$. Hence, we obtain the similar statement of Corollary 2.16 in the same way.

For each $\kappa \in I'_n$ define an element $g_\kappa \in G_{f_n}$ of order $d_n$ by

$$g_\kappa := \left(e \left[\frac{1}{d_n} \kappa \right], \ldots, e \left[(-1)^{i-1} \frac{d_{i-1}}{d_n} \kappa \right], \ldots, e \left[(-1)^{n-1} \frac{d_{n-1}}{d_n} \kappa \right] \right).$$

We will define a basis $\{ \zeta_{\kappa}^{(n)} \}_{\kappa \in I_n}$ (resp., $\{ \xi_{\kappa}^{(n)} \}_{\kappa \in I_n}$) on $\Omega^{\tilde{f}_n}$ (resp., $\Omega_{f_n,G_{f_n}}$) as follows:
(1) Let \( \zeta_1^{(0)} := \phi^{(0)} = 1 \) for \( n = 0 \) and, for \( n = 1 \),
\[
\zeta_\kappa^{(1)} := [\phi_\kappa^{(1)}(x_1)dx_1] = [x_1^{\kappa-1}dx_1], \quad \kappa \in I_1.
\]
For \( n \geq 2 \), let
\[
\zeta_\kappa^{(n)} := \begin{cases} 
[x^kdx] = [x_1^{k_1}\ldots x_n^{k_n}dx_1 \wedge \ldots \wedge dx_n], & \kappa \in I'_n, \\
-[\phi_\kappa^{(n-2)}(x_1,\ldots,x_{n-2})x_n^{\kappa_n-1}dx_1 \wedge \ldots \wedge dx_n], & \kappa \in I_{n-2},
\end{cases}
\]
where \( k = (k_1,\ldots,k_n) = \psi^{-1}(\kappa) \) for \( \kappa \in I'_n \).

(2) Let \( \xi_1^{(0)} := 1 \) for \( n = 0 \) and, for \( n = 1 \),
\[
\xi_\kappa^{(1)} := 1_{g_\kappa}, \quad \kappa \in I_1,
\]
where \( 1_{g_\kappa} \) denotes the generator of \( \Omega_{f_1^{g_\kappa}} \), namely, the constant function 1 on \( \text{Fix}(g_\kappa) = \{0\} \).

For \( n \geq 2 \), note that the natural inclusion map
\[
G_{f_{n-2}} \hookrightarrow G_{f_n}, \quad (e_1[\alpha_1],\ldots,e_{n-2}[\alpha_{n-2}]) \mapsto (e_1[\alpha_1],\ldots,e_{n-2}[\alpha_{n-2}],1,1)
\]
defines an subspace of \( \Omega_{f_{n-2}}G_{f_{n}} \) spanned by
\[
[\xi \wedge d(z_n^{\alpha_{n-1}}) \wedge dz_n], \quad [\xi] \in (\Omega_{f_{n-2}}^{g_\kappa})^{G_{f_{n-2}}}, \quad g \in G_{f_{n-2}}.
\]

Let
\[
\xi_\kappa^{(n)} := \begin{cases} 
1_{g_\kappa} & \kappa \in I'_n, \\
[\xi_\kappa^{(n-2)} \wedge d(z_n^{\alpha_{n-1}}) \wedge dz_n], & \kappa \in I_{n-2},
\end{cases}
\]
where \( 1_{g_\kappa} \) denotes the generator of \( \Omega_{f_{n-2}}^{g_\kappa} \), namely, the constant function 1 on \( \text{Fix}(g_\kappa) = \{0\} \) and \( \xi_\kappa^{(n-2)} \) does a differential form representing \( \xi_\kappa^{(n-2)} \).

**Remark 2.18.** In this paper, it is important that matrices with respect to the basis \( \{\zeta_\kappa^{(n)}\}_{\kappa \in I_n} \) (resp., \( \{\xi_\kappa^{(n)}\}_{\kappa \in I_n} \)) on \( \Omega_{f_n,G_{f_n}} \) (resp., \( \Omega_{f_n} \)) are labelled by the set \( I_n \).

We have the isomorphism of \( \mathbb{Q} \)-graded complex vector spaces
\[
\Omega_{f_n} \cong \bigoplus_{\kappa \in I_n} \mathbb{C} \cdot \zeta_\kappa^{(n)} \bigoplus \Omega_{f_{n-2}}(-1), \quad n \geq 2, \quad (2.12a)
\]
where \( \Omega_{f_{n-2}}(-1) \) is identified with the subspace of \( \Omega_{f_n} \) spanned by \( \zeta_\kappa^{(n)}, \kappa \in I_{n-2} \). On the other hand, we also have the isomorphism of \( \mathbb{Q} \)-graded complex vector spaces
\[
\Omega_{f_n,G_{f_n}} \cong \bigoplus_{\kappa \in I_n} \mathbb{C} \cdot \xi_\kappa^{(n)} \bigoplus \Omega_{f_{n-2},G_{f_{n-2}}(-1)}, \quad n \geq 2, \quad (2.12b)
\]
where \( \Omega_{f_{n-2},G_{f_{n-2}}(-1)} \) is identified with the subspace of \( \Omega_{f_n,G_{f_n}} \) spanned by \( \xi_\kappa^{(n)}, \kappa \in I_{n-2} \).
Denote by $\eta^{(n)}$ the matrix representation of $J_{f_n,G_{f_n}}$ with respect to the basis $\{\zeta^{(n)}_\kappa\}_{\kappa \in I_n}$ of $\Omega_{f_n,G_{f_n}}$. By the following proposition, it turns out that $\eta^{(n)}$ is also the matrix representation of $\widetilde{J}_f$ with respect to the basis $\{\zeta^{(n)}_\kappa\}_{\kappa \in I_n}$ of $\Omega_{\widetilde{f}_n}$.

**Proposition 2.19.** The bijection $\psi$ in Proposition 2.15 induces the following bijection

$$\Psi : \{x_1^{k_1} \cdots x_n^{k_n} dx_1 \wedge \cdots \wedge dx_n \mid x_1^{k_1} \cdots x_n^{k_n} \in \tilde{B}_n\} \overset{\cong}{\longrightarrow} \{1_{g_\kappa} \mid \kappa \in I'_n\},$$

which yields, via (2.12), the mirror isomorphism $\text{mir}$. In particular, the map $\Psi$ is chosen so that $\text{mir}$ is an isomorphism of pairs of $\mathbb{Q}$-graded complex vector spaces and non-degenerate bilinear forms:

$$\text{mir} : (\Omega_{\widetilde{f}_n}, J_{\widetilde{f}_n}) \cong (\Omega_{f_n,G_{f_n}}, J_{f_n,G_{f_n}}), \quad \zeta^{(n)}_\kappa \mapsto \xi^{(n)}_\kappa. \quad (2.13)$$

Moreover, $\eta^{(n)}$ is given by

$$\eta^{(0)} = (1), \quad \eta^{(1)} = \left(\frac{1}{a_1} \delta_{\kappa + \lambda, a_1}\right),$$

and, for $n \geq 2$,

$$\eta^{(n)} = \begin{pmatrix} \frac{1}{d_n} \delta_{\kappa + \lambda, d_n} & 0 \\ 0 & - \frac{1}{a_n} \eta^{(n-2)} \end{pmatrix}, \quad (2.14)$$

where $\kappa$ and $\lambda$ run the set $I'_n$.

**Proof.** First part is shown by [EGT, Proposition 12]. That is, the bijection $\Psi$ defined by

$$\Psi(x_1^{k_1} \cdots x_n^{k_n} dx_1 \wedge \cdots \wedge dx_n) = 1_{g_{\psi(k)}};$$

for $k = (k_1, \ldots, k_n)$ satisfying $x_1^{k_1} \cdots x_n^{k_n} \in \tilde{B}_n'$, induces the isomorphism $\text{mir} : \Omega_{\widetilde{f}_n} \cong \Omega_{f_n,G_{f_n}}$ by the decompositions (2.12).

Here, we shall show by induction that the mirror isomorphism $\text{mir}$ preserves non-degenerate bilinear forms $J_{\widetilde{f}_n}$ and $J_{f_n,G_{f_n}}$. It is clear for the case of $n = 0$. If $n = 1$, then for $k, k' = 0, \ldots, a_1 - 2$ we have

$$J_{\widetilde{f}_n}([x_1^k dx_1], [x_1^{k'} dx_1]) = \text{Res}_x \left[ x_1^{k+k'} dx_1 \right] = \begin{cases} \frac{1}{a_1}, & \text{if } k + k' = a_1 - 2 \\ 0, & \text{otherwise} \end{cases}.$$  

On the other hand, we have a decomposition $G_{f_1} \cong \bigoplus_{\kappa = 1}^{a_1-1} \mathbb{C} \cdot 1_{g_\kappa}$. Hence, they coincide via the mirror isomorphism.
We shall show the general case \( n \geq 2 \). For \( k = (k_1, \ldots, k_n) \) and \( k' = (k'_1, \ldots, k'_n) \) such that \( x_1^{k_1} \cdots x_n^{k_n}, x_1^{k'_1} \cdots x_n^{k'_n} \in B_{f_n}', \) we have

\[
J_{f'_n}([x^k dx], [x^{k'} dx]) = \text{Res}_{C_{\eta}} \left[ \frac{\partial f'_n}{\partial x_1}, \ldots, \frac{\partial f'_n}{\partial x_n} \right]
\]

\[
= \begin{cases} \frac{1}{a_n} & (k + k' = (a_1 - 1, \ldots, a_{n-1} - 1, a_n - 2)) \\ 0 & \text{(otherwise)} \end{cases}
\]

and since \( a_{n-1}x_{n-2}^{a_{n-1} - 1} + x_n^{a_n} = 0 \) in \( \text{Jac}(f'_n) \) for \( \phi, \phi' \in B_{f_n}' \) we have

\[
J_{f'_n}([\phi x_n^{a-1} dx], [\phi' x_n^{a-1} dx]) = \text{Res}_{C_{\eta}} \left[ \frac{\partial f'_n}{\partial x_1}, \ldots, \frac{\partial f'_n}{\partial x_n} \right]
\]

\[
= -\frac{1}{a_n} \text{Res}_{\eta_{n-2}} \left[ \frac{\partial f'_n}{\partial x_1}, \ldots, \frac{\partial f'_n}{\partial x_n} \right]
\]

On the other side, since \( |G_{f_n}| = d_n \) we have \( J_{f_n,G_{f_n}}(1_{g_n}, 1_{g_n}) = 1/d_n \) for \( \kappa \in I' \). For any \( \xi = [\phi dz_{n-g_n+1}^1 \cdots dz_{n-2}^1] \in (\Omega_{f_{n-2}^g})^{G_{f_{n-2}}}, \xi' = [\phi' dz_{n-g_n+1}^1 \cdots dz_{n-2}^1] \in (\Omega_{f_{n-2}^g})^{G_{f_{n-2}}}, \) we have

\[
J_{f_n,G_{f_n}}([\xi^1 \wedge d(z_n^{a_n-1}) \wedge dz_n], [\xi'^1 \wedge d(z_n^{a_n-1}) \wedge dz_n])
\]

\[
= \frac{1}{d_n} \cdot \text{Res}_{\text{Fix}(g)} \left[ \frac{\partial f_n^g}{\partial z_{n-g_n+1}^1}, \ldots, \frac{\partial f_n^g}{\partial z_n^1} \right]
\]

\[
= -\frac{1}{a_n} \cdot \frac{1}{d_{n-2}} \cdot \text{Res}_{\text{Fix}(g)} \left[ \frac{\partial f_n^g}{\partial z_{n-g_n+1}^1}, \ldots, \frac{\partial f_n^g}{\partial z_n^1} \right] = -\frac{1}{a_n} J_{f_{n-2},G_{f_{n-2}}}(\xi, \xi'),
\]

since \( z_n^{a_n-1} + a_n z_n^{a_n-1} = 0 \) in \( \text{Jac}(f_n) \).

Therefore, we obtain the statement by the decompositions (2,12). \( \square \)

### 3. Main results

First, we recall some notations and terminologies on the category of matrix factorizations. See [AT] for details. Consider the maximal grading \( L_{f_n} \) of \( f_n \), which is the
abelian group defined by the quotient

\[ L_{f_n} := \left( \bigoplus_{i=1}^{n} \mathbb{Z} \bar{z}_i \oplus \mathbb{Z} \bar{f}_n \right) \bigg/ \left( \bar{f}_n - \sum_{j=1}^{n} E_{ij} \bar{z}_j; \ i = 1, \ldots, n \right). \]

Then \( S_n := \mathbb{C}[z_1, \ldots, z_n] \) has the natural \( L_{f_n} \)-graded \( \mathbb{C} \)-algebra structure and one can define a triangulated category \( \text{HMF}^{L_{f_n}}(f_n) \) associated to \( f_n \), the homotopy category of \( L_{f_n} \)-graded matrix factorizations. Denote by \( K_0(\text{HMF}^{L_{f_n}}(f_n)) \) the Grothendieck group of \( \text{HMF}^{L_{f_n}}(f_n) \). For each \( \bar{l} \in L_{f_n} \), there is an auto-equivalence \( (\bar{l}) \) on \( \text{HMF}^{L_{f_n}}(f_n) \) defined by grading shift. In particular, we have \( T^2 = (\bar{f}_n) \), where \( T \) is the translation functor on \( \text{HMF}^{L_{f_n}}(f_n) \).

For each \( j \in \mathbb{Z} \), set

\[ E_j := \begin{cases} (S_n/(z_1, z_3, \ldots, z_{2m-1}))^{\text{stab}}_{\text{stab}} ((j-1) \bar{z}_1) & \text{if } n = 2m - 1, \\ (S_n/(z_2, z_4, \ldots, z_{2m}))^{\text{stab}}_{\text{stab}} ((j-1) \bar{z}_1) & \text{if } n = 2m, \end{cases} \tag{3.1} \]

where \((-)^{\text{stab}}_{\text{stab}}\) is the stabilization defined as follows:

- For \( n = 2m - 1 \), the Koszul resolution of \( S_n/(z_1, z_3, \ldots, z_{2m-1}) \) yields an \( L_{f_n} \)-graded matrix factorization \( \mathcal{F} = (F_0 \xrightarrow{f_0} F_1) \) of \( f_n \) such that

\[ F_0 := \bigoplus_k \left( \bigwedge^k \bigoplus_{i=1}^{m} S_n(-\bar{z}_{2i-1}) \right) (k \bar{f}_n), \quad F_1 := \bigoplus_k \left( \bigwedge^{k-1} \bigoplus_{i=1}^{m} S_n(-\bar{z}_{2i-1}) \right) (k \bar{f}_n), \tag{3.2} \]

and denote by \( (S_n/(z_1, z_3, \ldots, z_{2m-1}))^{\text{stab}}_{\text{stab}} \) the \( L_{f_n} \)-graded matrix factorization \( \mathcal{F} \).

- For \( n = 2m \), the Koszul resolution of \( S_n/(z_2, z_4, \ldots, z_{2m}) \) yields an \( L_{f_n} \)-graded matrix factorization \( \mathcal{F} = (F_0 \xrightarrow{f_0} F_1) \) of \( f_n \) such that

\[ F_0 := \bigoplus_k \left( \bigwedge^k \bigoplus_{i=1}^{m} S_n(-\bar{z}_{2i}) \right) (k \bar{f}_n), \quad F_1 := \bigoplus_k \left( \bigwedge^{k-1} \bigoplus_{i=1}^{m} S_n(-\bar{z}_{2i}) \right) (k \bar{f}_n), \tag{3.3} \]

and denote by \( (S_n/(z_2, z_4, \ldots, z_{2m}))^{\text{stab}}_{\text{stab}} \) the \( L_{f_n} \)-graded matrix factorization \( \mathcal{F} \).

Motivated by the homological mirror symmetry, it is proven that \( \text{HMF}^{L_{f_n}}(f_n) \) admits a full exceptional collection which is a Lefschetz decomposition in the sense of Kuznetsov–Smirnov [KuSm].

**Proposition 3.1** ([AT Theorem 1.3]). The sequence \((E_1, \ldots, E_{\bar{p}_n})\) is a full exceptional collection such that \( \chi(E_i, E_j) = \chi_{ij}^{(n)} \) where

\[ \chi(E_i, E_j) := \sum_{p \in \mathbb{Z}} (-1)^p \dim_{\mathbb{C}} \text{HMF}^{L_{f_n}}(f_n)(E_i, T^p E_j), \tag{3.4} \]
and χ^{(n)} is a matrix of size $\tilde{\mu}_n$ defined by $\chi^{(n)} = 1/\varphi_n(N)$, $N := (\delta_{i+1,j})$, and

$$\varphi_n(t) := \prod_{i=0}^{n} (1 - t^d_i)^{(-1)^{n-i}} \quad (n \geq 1), \quad \varphi_0(t) := 1 - t.$$

It is known that there exists the Serre functor $S^{(n)}$ on $\text{HMF}_{S_n}^L(f_n(n))$, which is given by $T^n(-\tilde{z}_1 - \cdots - \tilde{z}_n)$. Denote by $S^{(n)}$ the matrix representation of the automorphism induced by the Serre functor $S^{(n)}$ with respect to $\{[E_j]\}_{j=1}^{\tilde{\mu}_n}$, which satisfies

$$S^{(n)} = (\chi^{(n)})^{-1}(\chi^{(n)})^T. \quad (3.5)$$

Define a diagonal matrix $\tilde{Q}^{(n)}$ of size $\tilde{\mu}_n$ inductively as follows:

$$\tilde{Q}^{(0)} := (0), \quad \tilde{Q}^{(1)} := \left(\left(\omega^{(1)}_{\kappa,l} - \frac{1}{2}\right) \delta_{\kappa,\lambda}\right),$$

and, for $n \geq 2$,

$$\tilde{Q}^{(n)} := \begin{pmatrix} \tilde{P}^{(n)} & 0 \\ 0 & \tilde{Q}^{(n-2)} \end{pmatrix},$$

where $\tilde{P}^{(n)} = (\tilde{P}^{(n)}_{\kappa,\lambda})$ is a matrix of size $(d_n - d_{n-1})$ given by

$$\tilde{P}^{(n)}_{\kappa,\lambda} := \sum_{l=1}^{n} \left(\omega^{(n)}_{\kappa,l} - \frac{1}{2}\right) \delta_{\kappa,\lambda}, \quad \kappa, \lambda \in I'_n.$$

Now we are ready to introduce objects of our main interest.

**Definition 3.2.** For $\kappa \in I'_n$, define $c^{(n)}_\kappa \in \mathbb{C}$ by

$$c^{(n)}_\kappa := \begin{cases} \prod_{l=1}^{n} \Gamma \left(1 - \omega^{(n)}_{\kappa,l}\right) \cdot \prod_{i=1}^{m} \left(1 - \mathbf{e}^{\left(\omega^{(2m-1)}_{\kappa,2i-1}\right)}\right), & \text{if } n = 2m - 1, \\ \prod_{l=1}^{n} \Gamma \left(1 - \omega^{(n)}_{\kappa,l}\right) \cdot \prod_{i=1}^{m} \left(1 - \mathbf{e}^{\left(\omega^{(2m)}_{\kappa,2i}\right)}\right), & \text{if } n = 2m. \end{cases} \quad (3.6)$$

For $j \in \mathbb{Z}$, define $\text{ch}^{(n)}_{\Gamma,j} \in \mathbb{C}^{\tilde{\mu}_n} = \mathbb{C}^{(d_n - d_{n-1}) + \tilde{\mu}_{n-2}}$ inductively by

$$\text{ch}^{(0)}_{\Gamma,j} := (1), \quad \left(\text{ch}^{(1)}_{\Gamma,j}\right)_\kappa := c^{(1)}_\kappa \mathbf{e}^{\left(\omega^{(1)}_{\kappa,j-1}\right)}(j - 1), \quad \kappa \in I_1 = I'_1, \quad (3.7a)$$

$$\left(\text{ch}^{(n)}_{\Gamma,j}\right)_\kappa := \begin{cases} c^{(n)}_\kappa \mathbf{e}^{\left((-1)^{n-1} \omega^{(n)}_{\kappa,j-1}(j - 1)\right)}, & \text{if } \kappa \in I'_n, \\ 2\pi \sqrt{-1} \left(\text{ch}^{(n-2)}_{\Gamma,j}\right)_\kappa, & \text{if } \kappa \in I_{n-2} = I_n \setminus I'_n. \end{cases} \quad (3.7b)$$

Define a matrix $\text{ch}^{(n)}_{\Gamma}$ of size $\tilde{\mu}_n$ by

$$\text{ch}^{(n)}_{\Gamma} := \left(\text{ch}^{(n)}_{\Gamma,1}, \ldots, \text{ch}^{(n)}_{\Gamma,\tilde{\mu}_n}\right). \quad (3.8)$$

Note that the matrix $\text{ch}^{(n)}_{\Gamma}$ is invertible.
Remark 3.3. The first part \( \prod_{\nu=1}^{n} \Gamma(1 - \omega_{\kappa,\nu}(n)) \) of \( c_{\kappa}^{(n)} \) can be considered as the \( \Gamma \)-class on the \( \kappa \)-sector \( \Omega_{f_{n},g_{n}} \) (see [CIR, Definition 2.17]) and the last part of \( c_{\kappa}^{(n)} \) can be considered as the Chern character of \( E_{1} \) (see [PV, PV2] for Chern characters for \( L_{f_{n}} \)-graded matrix factorizations). The part \( e \left[ (-1)^{n-1} \omega_{\kappa,1}^{(n)}(j - 1) \right] \) comes from the automorphism on the Grothendieck group \( K_{0}(\text{HMF}_{S_{n}}(f_{n})) \) induced by the auto-equivalence \( ((-1)^{n}j\vec{z})_{1} \) whose matrix representation is given by

\[
\begin{pmatrix}
e \left[ (-1)^{n-1} \omega_{\kappa,1}^{(n)}(j - 1) \right] \delta_{\kappa,\lambda} & 0 \\
0 & (1)
\end{pmatrix},
\]

which acts on the vector \( \text{ch}^{(n)}_{\Gamma,1} \) to get \( \text{ch}^{(n)}_{\Gamma,j} \). Therefore, \( \text{ch}^{(n)}_{\Gamma,j} \) can be considered as the matrix representation of \( \widehat{\Gamma}_{f_{n},G_{f_{n}}} \text{Ch}(E_{j}) \), the Chern character of \( E_{j} \) multiplied by the Gamma class of the pair \((f_{n}, G_{f_{n}})\), with respect to the basis \( \{ \xi_{\kappa}^{(n)} \}_{\kappa \in \mathcal{I}_{n}} \). See below for the usual Gamma class on a smooth projective variety.

The following theorem is the analogue in our setting of results of [I, Proposition 2.10], namely, the necessary conditions for the matrix \((2\pi)^{-\frac{n}{2}} \text{ch}_{\Gamma}^{(n)}\) to give a central connection matrix of a Frobenius manifold whose non-degenerate bilinear form on the tangent space, the grading matrix and the Stokes matrix are given by \( \eta^{(n)}, \widetilde{Q}^{(n)} \) and \( \chi^{(n)} \), respectively.

**Theorem 3.4.** We have the following equalities:

\[
\begin{align*}
\left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_{\Gamma}^{(n)} \right)^{-1} e \left[ \widetilde{Q}^{(n)} \right] \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_{\Gamma}^{(n)} \right) &= S^{(n)}, \quad (3.9a) \\
\left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_{\Gamma}^{(n)} \right)^{T} e \left[ \frac{1}{2} \widetilde{Q}^{(n)} \right] \eta^{(n)} \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \text{ch}_{\Gamma}^{(n)} \right) &= \chi^{(n)}. \quad (3.9b)
\end{align*}
\]

We prove Theorem 3.4 in Section 4.

Here, we recall (refined) Dubrovin’s conjecture and \( \Gamma \)-conjecture II. Given an algebraic variety \( X \) and \( \tilde{\mu} := \dim_{\mathbb{C}} \bigoplus_{p \in \mathbb{Z}} H^{p}(X; \mathbb{C}) \), the Gamma class \( \widehat{\Gamma}_{X} \) of \( X \) is defined to be

\[
\widehat{\Gamma}_{X} := \prod_{i=1}^{\tilde{\mu}} \Gamma(1 + \delta_{i}),
\]

where \( \delta_{1}, \ldots, \delta_{\tilde{\mu}} \) are the Chern roots of the tangent bundle \( TX \).

We refer the reader to [D2, D3, CDG] for the monodromy data of quantum cohomologies (see also Section 6).

**Conjecture 3.5** ([D2, CDG, GGI]). Let \( X \) be an \( n \)-dimensional smooth Fano variety satisfying \( h^{p,q}(X) = 0 \) for any \( p \neq q \).
(1) The quantum cohomology of $X$ is semi-simple if and only if there exists a full exceptional collection in the derived category $D^b(X)$.

(2) If the quantum cohomology of $X$ is semi-simple, then for any oriented line $\ell$ (with $\phi \in [0,1]$) there is a correspondence between the monodromy data $(S^\phi, C^\phi)$ and full exceptional collections $(E^\phi_1, \ldots, E^\phi_\mu)$ in $D^b(X)$.

(3) The monodromy data $(S^\phi, C^\phi)$ are related to the following geometric data of the corresponding full exceptional collection $(E^\phi_1, \ldots, E^\phi_\mu)$:

(a) the Stokes matrix is equal to the Euler form on $K_0(X) \cong K_0(D^b(X)) \otimes \mathbb{C}$, computed with respect to the exceptional basis $\{[E^\phi_j]\}_{j=1}^\mu$:

$$S^\phi_{ij} = \chi(E^\phi_i, E^\phi_j). \quad (3.11)$$

(b) (Γ-conjecture II) the $j$-th column of the central connection matrix $C^\phi$ on a homogeneous basis $\{T_j\}_{j=1}^\mu$ of the $H^*(X; \mathbb{C})$ is given by

$$C^\phi_j = \frac{1}{(2\pi)^2} \Gamma_X \text{Ch}(E^\phi_j), \quad (3.12)$$

where $\text{Ch}(E) = \sum_{k=1}^{\text{rank } E} e[\delta_k(E)]$ for the Chern roots $\delta_1(E), \ldots, \delta_{\text{rank } E}(E)$ of $E$.

Let us return to our setting.

**Definition 3.6.** We define the $K$-group framing $\text{ch}_n^\Gamma : K_0(\text{HMF}^{L_{S_n}(f_n)}) \to \Omega_{f_n,G_{f_n}}$ of the $\mathbb{C}$-vector space $\Omega_{f_n,G_{f_n}}$ by

$$\text{ch}_n^\Gamma : K_0(\text{HMF}^{L_{S_n}(f_n)}) \to \Omega_{f_n,G_{f_n}}, \quad [E_j] \mapsto \sum_{\kappa \in L_n} \text{ch}_n^\Gamma \xi_{n,\kappa}, \quad (3.13)$$

where $E_j$ is the exceptional object given by (3.1). We call the image

$$\Omega_{f_n,G_{f_n};Z} := \frac{1}{(2\pi \sqrt{-1})^n} \text{ch}_n^\Gamma \left( K_0(\text{HMF}^{L_{S_n}(f_n)}) \right). \quad (3.14)$$

See [W] Definition 2.9 and [CIR] Definition 2.19] for the related notion, the Gamma integral structure of the quantum D-modules.

**Remark 3.7.** The constant factor $(2\pi \sqrt{-1})^{-n}$ is due to the fact that $\Omega_{f_n,G_{f_n}}$ (and $\Omega_f$) has the natural weight $n$ from the view point of the Hodge theory.

The Gamma integral structure $\Omega_{f_n,G_{f_n};Z}$ admits a $\mathbb{Z}/d_n\mathbb{Z}$-action. This action is induced by the grading shift functor $(\vec{z}_1)$ of $\text{HMF}^{L_{S_n}(f_n)}$. 
Conjecture 3.9. There exists a Gepner type stability condition \( \sigma \) on \( \text{HMF}^L_{S_n}(f_n) \) with respect to the auto-equivalence \( (\tilde{z}_1) \) and \( e[1/d_n] \in \mathbb{C} \) in the sense of Toda \cite[Definition 2.3]{Toda} such that \( (\tilde{z}_1).\sigma = \sigma.e \left[ \frac{1}{d_n} \right] \) and its stability function \( Z_\sigma : K_0(\text{HMF}^L_{S_n}(f_n)) \rightarrow \mathbb{C} \) is

Under the isomorphism \( H_n(\mathbb{C}^n, \text{Re}(\tilde{f}_n)) \cong H_n(\mathbb{C}^n, \tilde{f}_n^{-1}(1); \mathbb{Z}) \cong H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \), define a “Poincaré Duality” map \( \mathbb{D} : H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \rightarrow \Omega_{\tilde{f}_n} \) by

\[
\mathbb{D}(L) := \frac{1}{(2\pi i)^n} \sum_{\kappa,k \in I_n} \left( \sum_{\lambda \in I_n} \eta^{\lambda \kappa} \int \int \int e^{-\tilde{f}_n} \zeta^{(n)}_\lambda \right) e^{(n)}, \quad L \in H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z})
\]

where \( \Gamma \) is the element of \( H_n(\mathbb{C}^n, \text{Re}(\tilde{f}_n)) \) corresponding to \( L \in H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \), \( \zeta^{(n)}_\lambda \) in the integrand denotes the class in \( \mathbb{H}^n(\Omega_{\mathbb{C}^n}, d - d\tilde{f}_n \wedge) \) corresponding to the class \( \zeta^{(n)}_\lambda \in \Omega_{\tilde{f}_n} \) under the isomorphism \( \mathbb{H}^n(\Omega_{\mathbb{C}^n}, d - d\tilde{f}_n \wedge) \cong \Omega_{\tilde{f}_n} \), and \( (\eta^{\lambda \kappa}) \) is the inverse matrix of \( (\eta^{(n)}) \). We shall denote by \( \Omega_{\tilde{f}_n;\mathbb{Z}} \) the natural integral structure induced by the Milnor homology, that is,

\[
\Omega_{\tilde{f}_n;\mathbb{Z}} := \mathbb{D} \left( H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \right).
\]

The \( \mathbb{Z}/d_n\mathbb{Z} \)-action on the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \) given by

\[
(x_1, \ldots, x_k, \ldots, x_n) \mapsto \left( e^{\frac{1}{d_1}} x_1, \ldots, e^{\frac{1}{d_k}} (-1)^{k-1} \frac{d_k}{d_n} x_k, \ldots, e^{\frac{1}{d_n}} (-1)^{n-1} \frac{d_n}{d_n} x_n \right)
\]

induces the one on the integral structure \( \Omega_{\tilde{f}_n;\mathbb{Z}} \).

We obtain the following theorem as an analogue of the \( \Gamma \)-conjecture, which is proven for weak Fano toric orbifolds \cite[Theorem 1.1]{Toda} and \cite[Theorem 31]{CCIT}.

**Theorem 3.8** (See Milanov–Zha \cite[Theorem 1.3]{MZ} for ADE cases). The mirror isomorphism \( \text{mir} : \Omega_{\tilde{f}_n} \cong \Omega_{f_n} \) induces an isomorphism of free \( \mathbb{Z} \)-modules \( \Omega_{\tilde{f}_n;\mathbb{Z}} \cong \Omega_{f_n;\mathbb{Z}} \) and the isomorphism is \( \mathbb{Z}/d_n\mathbb{Z} \)-equivariant.

We prove the theorem in Section 5. Theorem 3.8 can naturally be generalized to the Thom–Sebastiani sum of invertible polynomials of chain type. We omit and leave it to the readers.

It is expected that the oscillatory integral \( \int e^{-\tilde{f}_n(x)} dx \) defines a stability condition as a central charge on \( \mathcal{D}^b\text{Fuk}^\ast(\tilde{f}_n) \) (see \cite[11]{Toda} \cite[2]{B2}). From the view point of the homological mirror symmetry conjecture for invertible polynomials, the mirror object dual to \( \int_{\Gamma_j} e^{-\tilde{f}_n(x)} dx \) is given by \( \sum_{\lambda \in I_n} \eta_{\psi(0),\lambda} \text{ch}^{(n)}_{R,\lambda j} \). Based on Corollary 2.16 and Theorem 3.8, we expect the following

**Conjecture 3.9.** There exists a Gepner type stability condition \( \sigma \) on \( \text{HMF}^L_{S_n}(f_n) \) with respect to the auto-equivalence \( (\tilde{z}_1) \) and \( e[1/d_n] \in \mathbb{C} \) in the sense of Toda \cite[Definition 2.3]{Toda} such that \( (\tilde{z}_1).\sigma = \sigma.e \left[ \frac{1}{d_n} \right] \) and its stability function \( Z_\sigma : K_0(\text{HMF}^L_{S_n}(f_n)) \rightarrow \mathbb{C} \) is
given by
\[
Z_\sigma([E_j]) := \begin{cases} 
\frac{1}{(2\pi \sqrt{-1})^n} e^{\left[-\frac{j-1}{d_n}\right]} \prod_{i=1}^{m} \left(1 - e^{-\omega_{2i-1}^{(n)}} \right) \cdot \int_{(\mathbb{R}_{\geq 0})^n} e^{-\tilde{f}_n(x)} dx, & \text{if } n = 2m - 1, \\
\frac{1}{(2\pi \sqrt{-1})^n} e^{\left[-\frac{j-1}{d_n}\right]} \prod_{i=1}^{m} \left(1 - e^{-\omega_{2i}^{(n)}} \right) \cdot \int_{(\mathbb{R}_{\geq 0})^n} e^{-\tilde{f}_n(x)} dx, & \text{if } n = 2m,
\end{cases}
\]
where \(E_j\) is the exceptional object given by (3.1).

Note that this conjecture can also be generalized to the Thom–Sebastiani sum of invertible polynomials of chain type.

A stability condition naturally associated to \(\text{HMF}_{S_n}^L(f_n)\) is constructed in [T1] for \(n = 1\) and in [KST] for the cases when \(f_3\) is of ADE type and \(L_{f_3} \cong \mathbb{Z}\). For this stability condition, indecomposable objects of \(\text{HMF}_{S_n}^L(f_n)\) are semi-stable and each irreducible morphism between two indecomposable objects has phase \(1/h\), where \(h\) is the Coxeter number of the root system of the type of \(\tilde{f}_3\). Since the space of stability conditions has the \(\mathbb{C}\)-action, we have the following result.

**Proposition 3.10.** Conjecture 3.9 holds for \(n = 1\) and for invertible polynomials of ADE type in two and three variables which is the Thom–Sebastiani sum of invertible polynomials of chain type.

**Proof.** Note that the Berglund–Hübsch transpose respects the set of invertible polynomials of ADE type. One can easily show by the same argument as in [KST2, AT] that for any invertible polynomial \(f\) of ADE type in two or three variables, there exists an equivalence of triangulated categories \(\text{HMF}_{S_n}^L(f) \cong \mathcal{D}^b(\mathbb{C}\tilde{\Delta})\) where \(\tilde{\Delta}\) is the Dynkin quiver of the type of \(\tilde{f}\). In particular, there are finitely many indecomposable objects up to isomorphism and translation, any indecomposable object in \(\text{HMF}_{S_n}^L(f)\) can be constructed from one indecomposable object (for example, the one corresponding to the edge vertex of the longest arm of \(\tilde{\Delta}\)) by using Auslander–Reiten triangles and any morphism is a composition of irreducible ones. By requiring that all indecomposable objects are semi-stable and the auto-equivalences \((\tilde{z}_i)\) respect semi-stable objects which increase those phases by \(2\omega_i\), one obtains the expected stability condition (up to \(\mathbb{C}\)-action) similarly to the case when \(L_f \cong \mathbb{Z}\) in [KST] since the Serre functor increases the phase \(\sum_{i=1}^{n}(1 - 2\omega_i) = 1 - 2/h\) where \(h\) is the Coxeter number of the root system associated to \(\tilde{\Delta}\). \(\square\)

4. **Proof of Theorem 3.4**

Since we have
\[
\tilde{Q}^{(n)} \eta^{(n)} = -\eta^{(n)} \tilde{Q}^{(n)},
\]
by Proposition 2.14, it is easy to see that (3.9a) follows from (3.9b). Therefore, only the proof for (3.9b) is necessary, which will be done by induction.

We prepare some lemmas. Let

\[ p_n(t) := \frac{1}{\varphi_n(t)} \cdot (1 - t^{d_n}) = \prod_{i=1}^{n} (1 - t^{d_{i-1}})^{(-1)^{n-i}} \quad (n \geq 1), \quad p_0(t) := 1. \]

Note that \( p_n(t) \) is a polynomial in \( t \) since \( p_1(t) = 1 - t \) and

\[ p_n(t) = p_{n-2}(t) \cdot \frac{1 - t^{d_{n-1}}}{1 - t^{d_{n-2}}}, \quad d_{n-1} = d_{n-2} \cdot a_{n-1}. \]

**Lemma 4.1.** We have

\[
\frac{1}{(2\pi)^n} c_{\kappa}^{(n)} e \left[ \frac{1}{2} \sum_{l=1}^{n} (\omega_{\kappa,l}^{(n)} - \frac{1}{2}) \right] \frac{1}{d_n} c_{d_n - \kappa}^{(n)} = \frac{1}{d_n} p_n \left( e \left[ (-1)^{n-1} \omega_{\kappa,1}^{(n)} \right] \right).
\]

**Proof.** By the Euler’s reflection formula and the definition of \( \omega_{\kappa,l}^{(n)} \), we have

\[
\Gamma \left( 1 - \omega_{\kappa,l}^{(n)} \right) \Gamma \left( 1 - \omega_{d_n - \kappa,l}^{(n)} \right) e \left[ \frac{1}{2} \left( \omega_{\kappa,l}^{(n)} - \frac{1}{2} \right) \right] = \frac{2\pi}{1 - e^{-\omega_{\kappa,l}^{(n)}}}.
\]

Therefore, for \( n = 2m - 1 \) we have

\[
\frac{1}{(2\pi)^{2m-1}} c_{\kappa}^{(2m-1)} e \left[ \frac{1}{2} \sum_{l=1}^{2m-1} (\omega_{\kappa,l}^{(2m-1)} - \frac{1}{2}) \right] \frac{1}{d_{2m-1}} c_{d_{2m-1} - \kappa}^{(2m-1)}
\]

\[
= \frac{1}{d_{2m-1}} \prod_{i=1}^{2m-1} \left( 1 - e^{\omega_{\kappa,2i-1}^{(2m-1)}} \right) \left( 1 - e^{\omega_{d_{2m-1} - \kappa,2i-1}^{(2m-1)}} \right)
\]

\[
\cdot \frac{1}{(2\pi)^{2m-1}} \prod_{l=1}^{2m-1} \Gamma \left( 1 - \omega_{\kappa,l}^{(2m-1)} \right) \Gamma \left( 1 - \omega_{d_{2m-1} - \kappa,l}^{(2m-1)} \right) e \left[ \frac{1}{2} \left( \omega_{\kappa,l}^{(2m-1)} - \frac{1}{2} \right) \right]
\]

\[
= \frac{1}{d_{2m-1}} \prod_{i=1}^{m} \left( 1 - e^{\omega_{\kappa,2i-1}^{(2m-1)}} \right) \cdot \prod_{i=1}^{m} \left( 1 - e^{-\omega_{\kappa,2i}^{(2m-1)}} \right)^{-1}
\]

\[
= \frac{1}{d_{2m-1}} p_{2m-1} \left( e^{\omega_{\kappa,1}^{(2m-1)}} \right),
\]
and for \( n = 2m \) we have
\[
\frac{1}{(2\pi)^{2m}c_{2m}^{(2m)}} e \left[ \frac{1}{2} \sum_{l=1}^{2m} \left( \omega_{\kappa,l}^{(2m)} - \frac{1}{2} \right) \right] \frac{1}{d_{2m}^{(2m)}} c_{d_{2m} - \kappa}^{(2m)} \\
= \frac{1}{d_{2m}} \prod_{i=1}^{m} \left( 1 - e^{\left( \omega_{\kappa,2i}^{(2m)} \right)} \right) \left( 1 - e^{\left( \omega_{d_{2m} - \kappa,2i}^{(2m)} \right)} \right) \\
\cdot \frac{1}{(2\pi)^{2m}} \prod_{l=1}^{2m} \Gamma \left( 1 - \omega_{\kappa,l}^{(2m)} \right) \Gamma \left( 1 - \omega_{d_{2m} - \kappa,l}^{(2m)} \right) e^{\left( \frac{1}{2} \left( \omega_{\kappa,l}^{(2m)} - \frac{1}{2} \right) \right)} \\
= \frac{1}{d_{2m}} \prod_{i=1}^{m} \left( 1 - e^{\left( \omega_{\kappa,2i}^{(2m)} \right)} \right) \cdot \prod_{i=1}^{m} \left( 1 - e^{\left( -\omega_{\kappa,2i-1}^{(2m)} \right)} \right)^{-1} \\
= \frac{1}{d_{2m}} \left. p_{2m} \left( e^{\left( -\omega_{\kappa,1}^{(2m)} \right)} \right) \right. .
\]

\( \square \)

**Lemma 4.2.** The \((i, j)\)-entry \( \chi_{i,j}^{(n)} \) of the matrix \( \chi^{(n)} \) is given by
\[
\chi_{i,j}^{(n)} = \frac{1}{d_n} \sum_{a=1}^{d_n} p_n \left( e^{\left( a / d_n \right)} \right) e^{\left( a / d_n \right)}(i - j). \quad (4.1)
\]

**Proof.** Suppose that \( n = 2m - 1, \ m \in \mathbb{Z}_{\geq 1} \). Since the Poincaré polynomial \( p_n'(t) \) of the graded ring \( \mathbb{C}[z_2, z_4, \ldots, z_{2m-2}]/(z_2^{a_2}, z_4^{a_4}, \ldots, z_{2m-2}^{a_{2m-2}}) \) with respect to the degrees \( \deg(z_{2i}) = d_{2i-1}, \ i = 1, \ldots, m \) is given by
\[
p_n'(t) = \prod_{i=1}^{m-1} \frac{1 - t^{d_{2i}}}{1 - t^{d_{2i-1}}},
\]
we have
\[
\dim_{\mathbb{C}} \left( \mathbb{C}[z_2, z_4, \ldots, z_{2m-2}]/(z_2^{a_2}, z_4^{a_4}, \ldots, z_{2m-2}^{a_{2m-2}}) \right)_l = \frac{1}{d_n} \sum_{a=1}^{d_n} p_n' \left( e^{\left( a / d_n \right)} \right) e^{\left( -a / d_n \right)}.
\]

Since \( p_n(t) = (1 - t)p_n'(t) \), we obtain the statement.

Next, suppose that \( n = 2m, \ m \in \mathbb{Z}_{\geq 1} \). The Poincaré polynomial of the graded ring \( \mathbb{C}[z_1, z_3, \ldots, z_{2m-1}]/(z_1^{a_1}, z_3^{a_3}, \ldots, z_{2m-1}^{a_{2m-1}}) \) with respect to the degrees \( \deg(z_{2i-1}) = d_{2i-2}, \ i = 1, \ldots, m \) is given by \( p_n(t) \). We obtain the statement since
\[
\dim_{\mathbb{C}} \left( \mathbb{C}[z_1, z_3, \ldots, z_{2m-1}]/(z_1^{a_1}, z_3^{a_3}, \ldots, z_{2m-1}^{a_{2m-1}}) \right)_l = \frac{1}{d_n} \sum_{a=1}^{d_n} p_n \left( e^{\left( a / d_n \right)} \right) e^{\left( -a / d_n \right)}.
\]

\( \square \)

Now, we give a proof of the equality \((3.9b)\). For \( n = 0 \), it is clear since \( ch_1^{(0)} = 1, \ \tilde{Q}^{(0)} = 0, \eta^{(0)} = 1 \) and \( \chi^{(0)} = 1 \).
Lemma 4.3. The equality \((3.9b)\) holds for \(n = 1\).

Proof. The \((i, j)\)-entry of the LHS of \((3.9b)\) is given by
\[
\frac{1}{2\pi} \sum_{\kappa=1}^{a_1-1} c^{(1)}_{\kappa} \left[ \frac{1}{2} \left( \omega^{(1)}_{\kappa, i} - \frac{1}{2} \right) \right] \frac{1}{d_1} c^{(1)}_{d_1 - \kappa} e \left[ \omega^{(1)}_{\kappa, 1}(i - j) \right] = \frac{1}{d_1} \sum_{\kappa=1}^{a_1-1} p_1 \left( e \left[ \omega^{(1)}_{\kappa, 1} \right] \right) e \left[ \omega^{(1)}_{\kappa, 1}(i - j) \right],
\]
which is equal to \(\chi^{(1)}_{i, j}\) by Lemma 4.2. \(\square\)

For each \(n \in \mathbb{Z}_{\geq 0}\) and \(l \in \mathbb{Z}\), define \(X^{(n)}_l \in \mathbb{C}\) by
\[
X^{(n)}_l := \frac{1}{d_n} \sum_{a=1}^{d_n} p_n \left( e \left[ \frac{a}{d_n} \right] \right) e \left[ \frac{a}{d_n} l \right] \quad (n \geq 1), \quad X^{(0)}_l := 0. \tag{4.2}
\]

Lemma 4.4. Suppose that \(i \leq j\) for \(i, j = 1, \ldots, \tilde{\mu}_n\). Then, we have
\[
X^{(n)}_l = \chi^{(n)}_{i, j},
\]
where \(l \equiv i - j \pmod{d_n}\).

Proof. The statement follows from Lemma 4.2 and the definition of \(X^{(n)}_l\). \(\square\)

Lemma 4.5. For \(n \geq 2\) and \(l \in \mathbb{Z}\), we have
\[
X^{(n)}_l = \frac{1}{d_n} \sum_{\kappa \in I'_n} p_n \left( e \left[ \frac{\kappa}{d_n} \right] \right) e \left[ \frac{\kappa}{d_n} l \right] + \frac{1}{a_n} X^{(n-2)}_l. \tag{4.3}
\]

Proof. Note that
\[
p_n(t) = p_{n-2}(t) \cdot \frac{1 - t^{dn-1}}{1 - t^{dn-2}} = p_{n-2}(t) \cdot \sum_{b=1}^{d_{n-2}} (t^{dn-2})^b. \tag{4.4}
\]

Therefore, if \(a_n \mid a, a_{n-1} \nmid a\) we have \(p_n(e[a/d_n]) = 0\) and
\[
X^{(n)}_l = \frac{1}{d_n} \sum_{a=1}^{d_n} p_n \left( e \left[ \frac{a}{d_n} \right] \right) e \left[ - \frac{a}{d_n} l \right]
= \frac{1}{d_n} \sum_{\kappa \in I'_n} p_n \left( e \left[ \frac{\kappa}{d_n} \right] \right) e \left[ \frac{\kappa}{d_n} l \right] + \frac{1}{d_{n-2}} \sum_{a_{n-1}=1}^{d_{n-2}} p_{n-2} \left( e \left[ \frac{a'}{d_{n-2}} \right] \right) e \left[ \frac{a'}{d_{n-2}} l \right]
= \frac{1}{d_n} \sum_{\kappa \in I'_n} p_n \left( e \left[ \omega^{(n)}_{\kappa, 1} \right] \right) e \left[ \omega^{(n)}_{\kappa, 1} \cdot l \right] + \frac{1}{a_n} X^{(n-2)}_l.
\]
By Lemma 4.1, Lemma 4.4, and Lemma 4.5, the (i, j)-entry of the LHS of (3.9b) is given by
\[
\sum_{\kappa \in I_n} \frac{1}{(2\pi i)^n} c^{(n)}_{\kappa} e \left[ \frac{1}{2} \sum_{l=1}^{n} \left( \omega^{(n)}_{\kappa,l} - \frac{1}{2} \right) \right] \frac{1}{d_n} c^{(n)}_{d_n-\kappa} e \left[ (-1)^{n-1} \omega^{(n)}_{\kappa,1}(i - j) \right] + \frac{1}{a_n} X^{(n-2)}_{i-j}
\]
\[
= \frac{1}{d_n} \sum_{\kappa \in I_n} p_n \left( e \left[ (-1)^{n-1} \omega^{(n)}_{\kappa,1} \right] \right) e \left[ (-1)^{n-1} \omega^{(n)}_{\kappa,1}(i - j) \right] + \frac{1}{a_n} X^{(n-2)}_{i-j}
\]
\[
= \frac{1}{d_n} \sum_{\kappa \in I_n} p_n \left( e \left[ \omega^{(n)}_{\kappa,1} \right] \right) e \left[ \omega^{(n)}_{\kappa,1}(i - j) \right] + \frac{1}{a_n} X^{(n-2)}_{i-j} = \chi^{(n)}_{i,j}.
\]
Hence, we have finished the proof of Theorem 3.4.

5. Proof of Theorem 3.8

We shall prove Theorem 3.8 by induction on n.

Lemma 5.1. Theorem 3.8 holds for n = 1.

Proof. For j = 1, …, \(a_1 - 1\), set
\[
\gamma_j : (-\infty, \infty) \rightarrow \mathbb{C}, \quad \gamma_j(t) : = \begin{cases} e \left[ \frac{-t-1}{a_1} \right] \cdot t, & t \geq 0, \\
\left[ \frac{-t}{a_1} \right] \cdot t, & t \leq 0. \end{cases}
\]
and denote by \(\Gamma^{(1)}_j\) the image of the map \(\gamma_j : (-\infty, \infty) \rightarrow \mathbb{C}\). Then, \(\Gamma^{(1)}_j\) defines a relative homology class of \(H_1(\mathbb{C}^1, \text{Re}(\bar{f}_1) \gg 0; \mathbb{Z})\) and \(\{\Gamma^{(1)}_1, \ldots, \Gamma^{(1)}_{a_1-1}\}\) generates \(H_1(\mathbb{C}^1, \text{Re}(\bar{f}_1) \gg 0; \mathbb{Z})\). For \(\kappa \in I_1\), namely, for \(\kappa = 1, \ldots, a_1 - 1\), we have
\[
\text{ch}^{(1)}_{\Gamma^{(1)}_j} = \Gamma \left( 1 - \omega^{(1)}_{\kappa,1} \right) \left( 1 - e[\omega^{(1)}_{\kappa,1}] \right) e[\omega^{(1)}_{\kappa,1}(j - 1)]
\]
\[
= a_1 \left( 1 - e \left[ \frac{\kappa}{a_1} \right] \right) e \left[ \frac{\kappa}{a_1}(j - 1) \right] \int_{0}^{\infty} e^{-x_1^a} x_1^{a_1-1-\kappa} dx_1
\]
\[
= \eta^{a_1-\kappa} \int_{\Gamma^{(1)}_j} e^{-\tilde{f}_{\kappa}^{(1)} x_1}.
\]
By the definition of \(\Gamma^{(1)}_j\), the action \(x_1 \mapsto e[1/a_1]x_1\) yields \(\Gamma^{(1)}_j \mapsto \Gamma^{(1)}_{j-1}\). On the other side, since \(E_{j-1} = E_j(\bar{z}_1)\) on \(\text{HM}^{L_{fi}}_{S_1}(f_1)\), the action induced by \((\bar{z}_1)\) is \(\text{ch}^{(1)}_{\Gamma^{(1)}_j} \mapsto \text{ch}^{(1)}_{\Gamma^{(1)}_{j-1}}\).

Lemma 5.2. Theorem 3.8 holds for n = 2.

Proof. Consider the projection
\[
\tilde{f}_{2}^{-1}(1) = \{(x_1, x_2) \mid x_1^{a_1} + x_1 x_2^{a_2} = 1\} \rightarrow \mathbb{C}, \quad (x_1, x_2) \mapsto x_1,
\]
and denote by \(\Gamma^{(2)}_j\) the image of the map \(\gamma_j : (-\infty, \infty) \rightarrow \mathbb{C}\). Then, \(\Gamma^{(2)}_j\) defines a relative homology class of \(H_1(\mathbb{C}^2, \text{Re}(\bar{f}_2) \gg 0; \mathbb{Z})\) and \(\{\Gamma^{(2)}_1, \ldots, \Gamma^{(2)}_{a_2-1}\}\) generates \(H_1(\mathbb{C}^2, \text{Re}(\bar{f}_2) \gg 0; \mathbb{Z})\). For \(\kappa \in I_2\), namely, for \(\kappa = 1, \ldots, a_2 - 1\), we have
\[
\text{ch}^{(2)}_{\Gamma^{(2)}_j} = \Gamma \left( 1 - \omega^{(2)}_{\kappa,1} \right) \left( 1 - \omega^{(2)}_{\kappa,2} \right) e[\omega^{(2)}_{\kappa,1}(j - 1)]
\]
which is a branched covering with branching points \( e[k/a_1], k = 0, \ldots, a_1 - 1 \). Let \( \epsilon \) be a sufficiently small positive number and put

\[
\begin{align*}
l'_0 &:= \{ (x_1, x_2) \in \mathbb{C}^2 \mid x_1 \in [\epsilon, 1], \ x_2 = \left( \frac{1 - x_1^{a_1}}{a_2} \right)^{\frac{1}{a_2}} \}, \\
l'_1 &:= \{ (x_1, x_2) \in \mathbb{C}^2 \mid x_1 \in [\epsilon, 1], \ x_2 = e \left[ -\frac{1}{a_2} \right] \left( \frac{1 - x_1^{a_1}}{\epsilon} \right)^{\frac{1}{a_2}} \}.
\end{align*}
\]

There exists a counter-clockwise loop \( C_\epsilon = \{ x_1(\theta) \in \mathbb{C} \mid 0 \leq \theta \leq 1 \} \) around \( x_1 = 0 \) with starting and end point \( x_1 = \epsilon \) such that its lift in \( \tilde{f}_2^{-1}(1) \) is given by

\[
C'_\epsilon := \left\{ \left( x_1(\theta), e \left[ -\frac{1}{a_2} \right] \left( \frac{1 - x_1^{a_1}}{\epsilon} \right)^{\frac{1}{a_2}} \right) \in \mathbb{C}^2 \right\} 0 \leq \theta \leq 1,
\]

and \( \lim_{\epsilon \to 0} x_1(\theta) = 0 \) for all \( 0 \leq \theta \leq 1 \).

Then the cycle \( S'_1 := l'_1 \circ C'_\epsilon \circ (l'_0)^{-1} \) in \( \tilde{f}_2^{-1}(1) \) defines a homology class \( L^{(2)}_1 \) in \( H_1(\tilde{f}_2^{-1}(1); \mathbb{Z}) \) and for the corresponding relative homology class \( \Gamma^{(2)}_1 \in H_2(\mathbb{C}^2, \text{Re}(\tilde{f}_2) \gg 0; \mathbb{Z}) \) we have

\[
\int e^{-\tilde{f}_2(x)} x^k dx = \lim_{\epsilon \to 0} \left( \int_0^\infty \int_0^\infty e^{-1/a_2} + \int_{D_\epsilon} \right) e^{-\tilde{f}_2(x)} x^k dx,
\]

where \( D_\epsilon := \{ (x_1(\theta), re[-\theta/a_2]) \in \mathbb{C}^2 \mid 0 \leq \theta \leq 1, \ r \in \mathbb{R}_{\geq 0} \} \). By a direct calculation, it follows that

\[
\int e^{-\tilde{f}_2(x_1,x_2)} x_1^{k_1} x_2^{k_2} dx_1 dx_2 = (1 - e^{-\omega^{(2)}_{k,2}}) \int e^{-\tilde{f}_2(x_1,x_2)} x_1^{k_1} x_2^{k_2} dx_1 dx_2
\]
Therefore, by Corollary 2.16 for \( \kappa \in I_2 \) we obtain the equality
\[
\text{ch}^{(2)}_{\Gamma_{1,\kappa}} = \sum_{\lambda \in I_2} \eta^{\lambda \kappa} \int_{\Gamma_{1}} e^{-\tilde{f}_2 \zeta^{(2)}_{\lambda}}.
\]

Define a homology class \( L_j^{(2)} \in H_1(\tilde{f}_2^{-1}(1); \mathbb{Z}) \) as the image of \( L_1^{(2)} \) by the action \((x_1, x_2) \mapsto \left( e \left[ \frac{(j-1)}{a_1} \right] x_1, e \left[ \frac{(j-1)}{a_1 a_2} \right] x_2 \right)\) on \( \tilde{f}_2^{-1}(1) \) and \( \Gamma_j^{(2)} \in H_2(\mathbb{C}^2; \text{Re}(\tilde{f}_2) \gg 0; \mathbb{Z}) \) as the corresponding class of \( L_j^{(2)} \). Hence, we have the equality
\[
\text{ch}^{(2)}_{\Gamma_{1,\kappa}} = \sum_{\lambda \in I_2} \eta^{\lambda \kappa} \int_{\Gamma_j^{(2)}} e^{-\tilde{f}_2 \zeta^{(2)}_{\lambda}}
\]
for \( j = 1, \ldots, \tilde{\mu}_2 \). The statement follows.

Next, we show the general case. For simplicity, we put \( x'_n := (x_2, x_3, \ldots, x_n) \) and \( x''_n := (x_3, x_4, \ldots, x_n) \) for \( x = x_n = (x_1, \ldots, x_n) \), respectively.

Remark 5.3. Let \( S_n^{n-1} \) be an \((n - 1)\)-dimensional sphere in \( \tilde{f}_n^{-1}(1) \) whose homology class defines \( L_1^{(n)} \in H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \). Since \( \tilde{f}_n \) is weighted homogeneous, for each \( w \in \mathbb{R}_{>0} \) there exists an \((n - 1)\)-dimensional sphere \( S_w^{n-1} \) in \( \tilde{f}_n^{-1}(w) \) whose homology class defines \( L_1^{(n)}(w) \in H_{n-1}(\tilde{f}_n^{-1}(w); \mathbb{Z}) \) which is the image of \( L_1^{(n)} \) by the isomorphism \( H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \cong H_{n-1}(\tilde{f}_n^{-1}(w); \mathbb{Z}) \) induced by the parallel transformation.

We prove the case of \( n = 2m - 1 \), \( m \in \mathbb{Z}_{\geq 2} \). If \( x_1 \neq 0 \), by using the change of variables
\[
y_2^{a_2} = x_1 x_2^{a_2}, \quad y_{i-1} y_i^{a_i} = x_{i-1} x_i^{a_i}, \quad i = 3, 4, \ldots, 2m - 1,
\]
we have \( \tilde{f}_{2m-1}(x_{2m-1}) = x_1^{a_1} + \tilde{f}_{2m-2}(y_{2m-1}^{a_{2m-1}}) \).

Lemma 5.4. Suppose that there exists a \((2m - 3)\)-dimensional sphere \( S_1^{2m-3} \) in \( \tilde{f}_n^{-1}(1) \) such that its homology class defines \( L_1^{(2m-2)} \in H_{2m-3}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \) and for the corresponding class \( \Gamma_1^{(2m-2)} \) we have
\[
\text{ch}_{\Gamma_{1,\kappa}}^{(2m-2)} = \sum_{\lambda \in I_{2m-2}} \eta^{\lambda \kappa} \int_{\Gamma_1^{(2m-2)}} e^{-\tilde{f}_{2m-2} \zeta^{(2m-2)}_{\lambda}}.
\]
Then there exists an element
\[ L_1^{(2m-1)} \in H_{2m-2}(\tilde{f}_{2m-1}(1); \mathbb{Z}) \]
whose corresponding class \( \Gamma_1^{(2m-1)} \in H_{2m-1}(\mathbb{C}^{2m-1}, \text{Re}(\tilde{f}_{2m-1}) \gg 0; \mathbb{Z}) \) satisfies that
\[
\int_{\tilde{f}_{2m-1}^{-1}(2m-1)} e^{-\tilde{f}_{2m-1}(x_{2m-1})} x_1^{k_1} \phi(x_{2m-1}) dx_{2m-1} = \int_{\tilde{f}_{2m-2}^{-1}(1)} e^{-\tilde{f}_{2m-2}(y_{2m-1})} \phi(y_{2m-1}) dy_{2m-1},
\]
\[
\text{(5.1)}
\]
for \( x_1^{k_1} \cdot \phi(x_{2m-1}) \in B_{\tilde{f}_{2m-1}}, \) where \( \bar{\phi} \) is a rational number given by \( \phi(x_{2m-1}) = x_1^{\bar{\phi}} \cdot \phi(y_{2m-1}). \)

**Proof.** We construct a \((2m-2)\)-dimensional sphere so that it gives a homology class \( L_1 \in H_{2m-2}(\tilde{f}_{2m-1}(1); \mathbb{Z}) \) in the statement. Consider the projection
\[
\tilde{f}_{2m-1}^{-1}(1) \to \mathbb{C}, \quad (x_1, \ldots, x_{2m-1}) \mapsto x_1.
\]
Let \( \epsilon \) be a sufficiently small positive number, \( l_0 := \{ t \in \mathbb{C} \mid t \in [\epsilon, 1] \}, \)
\( l_1 := \{ e[-1/a_1] t \in \mathbb{C} \mid t \in [\epsilon, 1] \} \) line segments, and \( C_{\epsilon} := \{ e[\theta/a_1] \in \mathbb{C} \mid 0 \leq \theta \leq 1 \}. \) Set \( A_1^{(2m-1)} := l_0 \circ C_{\epsilon} \circ l_1^{-1} \) (see Figure 2).

**Figure 2.** Domain \( A_1^{(2m-1)}. \)

Since \( x_1 \in A_1^{(2m-1)} \) implies \( x_1 \neq 0, \) we have \( \tilde{f}_{2m-1}(x_{2m-1}) = x_1^{a_1} + \tilde{f}_{2m-2}(y_{2m-1}). \)
Since \( S_{1-x_1}^{2m-3} \) vanishes on the fiber over \( x_1 = 1, e[-1/a_1], \)
\( S_1^{2m-2} := \bigcup_{x_1 \in A_1^{(2m-1)}} S_{1-x_1}^{2m-3} \) is homeomorphic to the \((2m-2)\)-dimensional sphere by the assumption and Remark 5.3.
Define $L^{(2m-1)}_1 \in H_{2m-2}(\tilde{f}_{2m-1}(1); \mathbb{Z})$ by the homology class of $\bigcup_{x_1 \in A_1} S^{2m-3}_{1-x_1}$. For the corresponding class $\Gamma^{(2m-1)}_1$, we have

$$\int_{\Gamma^{(2m-1)}_1} e^{-\tilde{f}_{2m-1}(x_{2m-1})} x_1^{k_1} \phi(x'_{2m-1})dx_{2m-1}$$

$$= \lim_{\epsilon \to 0} \left( \int_{1}^{\infty} - \int_{1}^{-[\epsilon/a_1]} + \int_{C} \right) e^{-x_1^{a_1}} x_1^{k_1 + \phi} dx_1 \cdot \int_{\Gamma^{(2m-2)}_1} e^{-\tilde{f}_{2m-2}(y'_{2m-1})} \phi(y'_{2m-1})dy'_{2m-1},$$

It gives the equation (5.1).

By Lemma 5.4 for $k = (k_1, \ldots, k_{2m-1})$ satisfying $x^{k}_{2m-1} \in B^c_{f_{2m-1}}$ we obtain

$$\int_{\Gamma^{(2m-1)}_1} e^{-\tilde{f}_{2m-1}(x_{2m-1})} x_1^{k_1} \phi(x'_{2m-1})dx_{2m-1}$$

$$= \int_{\Gamma^{(1)}_1} e^{-x_1^{a_1}} x_1^{k_1 + \phi} dx_1 \cdot \int_{\Gamma^{(2m-2)}_1} e^{-\tilde{f}_{2m-2}(y'_{2m-1})} \phi(y'_{2m-1})dy'_{2m-1}$$

$$= \frac{1}{a_1} \Gamma^{(2m-1)}(1 - e^{-[\omega_{k,1}]}) \int_{\Gamma^{(2m-2)}_1} e^{-\tilde{f}_{2m-2}(y'_{2m-1})} \phi(y'_{2m-1})dy'_{2m-1},$$

where $k' := (k_2, \ldots, k_{2m-1})$ and $(y'_{2m-1})^{k'} := y_2^{k_2} y_3^{k_3} \ldots y_{2m-1}^{k_{2m-1}}$. For each $x_1^{k_1} \cdot \phi(x'_{2m-3}) \in B^c_{\tilde{f}_{2m-3}}$, it inductively follows that

$$\int_{\Gamma^{(2m-1)}_1} e^{-\tilde{f}_{2m-1}(x_{2m-1})} x_1^{k_1} \phi(x'_{2m-3})dx_{2m-1}$$

$$= \int_{\Gamma^{(1)}_1} e^{-x_1^{a_1}} x_1^{k_1 + \phi} dx_1 \cdot \int_{\Gamma^{(2m-2)}_1} e^{-\tilde{f}_{2m-2}(y'_{2m-1})} \phi(y'_{2m-3})dy'_{2m-1}$$

$$= \int_{\Gamma^{(1)}_1} e^{-x_1^{a_1}} x_1^{k_1 + \phi} dx_1 \cdot \left( -\frac{2\pi \sqrt{-1}}{a_{2m-1}} \right) \int_{\Gamma^{(2m-4)}_1} e^{-\tilde{f}_{2m-4}(y'_{2m-3})} \phi(y'_{2m-3})dy'_{2m-3}$$

$$= -\frac{2\pi \sqrt{-1}}{a_{2m-1}} \int_{\Gamma^{(2m-3)}_1} e^{-\tilde{f}_{2m-3}(x_{2m-3})} x_1^{k_1} \phi(x'_{2m-3})dx_{2m-3}.$$

Therefore, for $k \in I_{2m-1}$ we obtain

$$\text{ch}^{(2m-1)}_{\Gamma, k_1} = \sum_{\lambda \in I_{2m-1}} \eta^{\lambda} \int_{\Gamma^{(2m-1)}_1} e^{-\tilde{f}_{2m-1}(x^{(2m-1)}_\lambda)}.$$

Define a homology class $L^{(2m-1)}_{j} \in H_{2m-2}(\tilde{f}_{2m-1}(1); \mathbb{Z})$ as the image of $L^{(2m-1)}_1$ by the action $(x_1, \ldots, x_k, \ldots, x_{2m-1}) \mapsto \left( e^{-\frac{(j-1)}{a_1}} x_1, \ldots, e^{-\frac{(j-1)}{a_{2m-1}}} x_{2m-1} \right)$ on $\tilde{f}_{2m-1}(1)$ and $\Gamma^{(2m-1)}_j \in H_{2m-1}(\mathbb{C}^{2m-1}, \mathbb{R} ; \mathbb{Z})$ as the corresponding class of $L^{(2m-1)}_{j}$. For $j = 1, \ldots, \mu_{2m-1}$, we have the equality

$$\text{ch}^{(2m-1)}_{\Gamma, k_j} = \sum_{\lambda \in I_{2m-1}} \eta^{\lambda} \int_{\Gamma^{(2m-1)}_j} e^{-\tilde{f}_{2m-1}(x^{(2m-1)}_\lambda)}.$$
Finally, we prove the case of \( n = 2m, m \in \mathbb{Z}_{\geq 2} \). If \( x_2 \neq 0 \), by using the change of variables
\[
y_3^a = x_2x_3^{a_3}, \quad y_i = x_{i-1}x_i^{a_i}, \quad i = 4, 5, \ldots, 2m,
\]
we have \( \tilde{f}_{2m}(x_{2m}) = x_1^{a_1} + x_1x_2^{a_2} + \tilde{f}_{2m-2}(y_{2m}^{a_2}) \).

**Lemma 5.5.** If the assumption in Lemma 5.4 holds, then there exists an element \( L_1^{(2m)} \in H_{2m-1}(\tilde{f}_{2m}^{-1}(1); \mathbb{Z}) \) whose corresponding class \( \Gamma_1^{(2m)} \in H_{2m}(\mathbb{C}^{2m}, \mathrm{Re}(\tilde{f}_{2m}) \gg 0; \mathbb{Z}) \) satisfies that
\[
\int_{\Gamma_1^{(2m)}} e^{-\tilde{f}_{2m}(x_{2m})} x_1^{k_1}x_2^{k_2} \phi(x_{2m}^{a_2}) dx_{2m} = \int_{\Gamma_1^{(2m)}} e^{-(x_1^{a_1} + x_1x_2^{a_2})} x_1^{k_1}x_2^{k_2+\theta} dx_1 dx_2 \cdot \int_{\Gamma_1^{(2m-2)}} e^{-\tilde{f}_{2m-2}(y_{2m})} \phi(y_{2m}^{a_2}) dy_{2m}^{a_2},
\]
for \( x_1^{k_1}x_2^{k_2} \cdot \phi(x_{2m}^{a_2}) \in B_{f_{2m}} \), where \( \theta \) is a rational number given by \( \phi(x_{2m}^{a_2}) = x_2^{\theta} \cdot \phi(y_{2m}^{a_2}) \).

**Proof.** We construct a \((2m-1)\)-dimensional sphere \( S^{2m-1} \) whose homology class \( L_1 \in H_{2m-1}(\tilde{f}_{2m}^{-1}(1); \mathbb{Z}) \) in the statement by gluing two \((2m-1)\)-dimensional disks \( D_1^{2m-1} \) and \( D_2^{2m-1} \) along the boundary.

First, we construct a disk \( D_1^{2m-1} \) as follows. Consider the projection \( \tilde{f}_{2m}^{-1}(1) \rightarrow \mathbb{C}, \quad (x_1, \ldots, x_{2m}) \mapsto x_1 \).

Let \( \epsilon_1 \) be a sufficiently small positive number, \( l_0 := \{ t \in \mathbb{C} \mid t \in [\epsilon_1, 1] \} \) a line segment. Since \( x_1 \in l_0 \) implies \( x_1 \neq 0 \), we have \( \tilde{f}_{2m}(x_{2m}) = x_1^{a_1} + \tilde{f}_{2m-1}(y_{2m}) \). Therefore, since \( S_{1-x_1^{a_1}}^{2m-2} \) vanishes on the fiber over \( x_1 = 1 \), by Remark 5.3 and Lemma 5.4 \( D_1^{2m-1} := \bigcup_{x_1 \in l_0} S_{1-x_1^{a_1}}^{2m-2} \) is homeomorphic to the \((2m-1)\)-dimensional disk.

Next, we construct a disk \( D_2^{2m-1} \) as follows. Consider the projection \( \tilde{f}_{2m}^{-1}(1) \rightarrow \mathbb{C}^2, \quad (x_1, \ldots, x_{2m}) \mapsto (x_1, x_2) \).

Let \( \epsilon_2 \) be a sufficiently small positive number. There exists a counter-clockwise loop \( C_{\epsilon_1} = \{ x_1(\theta) \in \mathbb{C} \mid 0 \leq \theta \leq 1 \} \) around \( x_1 = 0 \) with starting and end point \( x_1 = \epsilon_1 \) such that for each \( 0 \leq \theta \leq 1 \), \( \lim_{\epsilon_1 \rightarrow 0} x_1(\theta) = 0 \) and the set \( D_{\theta}^{2m-2} := \bigcup_{(x_1, x_2) \in \Gamma_0} S_{1-(x_1^{a_1}+x_1^{a_2})}^{2m-3} \) is a subset of \( \tilde{f}_{2m}^{-1}(1) \), where \( \Gamma_0 \) is given by
\[
l_0' := \left\{ \left( x_1(\theta), t \epsilon \left[ \frac{-\theta}{a_2} \right] \right) \in \mathbb{C}^2 \mid t \in \left[ \epsilon_2, \left( \frac{1 - \epsilon_1^{a_1}}{\epsilon_1^{a_2}} \right) \right] \right\}.
\]

Then, for each \( 0 \leq \theta \leq 1 \), \( D_{\theta}^{2m-2} \) is homeomorphic to the \((2m-2)\)-dimensional disk. Therefore, \( D_2^{2m-1} := \bigcup_{0 \leq \theta \leq 1} D_{\theta}^{2m-2} \) is homeomorphic to the \((2m-1)\)-dimensional disk.
By taking $\epsilon_2$ as $\epsilon$ in Lemma 5.4 for $w = 1 - \epsilon_1^{a_2}$, the boundary of $D_2^{2m-1}$ coincides with the one of $D_1^{2m-1}$. Hence, we obtain a $(2m - 1)$-dimensional sphere $S^{2m-1}$ by gluing $D_1^{2m-1}$ and $D_2^{2m-1}$ on their common boundaries. Define $L_1^{(2m)} \in H_{2m-1}(\tilde{f}_{2m}^{-1}(1); \mathbb{Z})$ by the homology class of this $(2m - 1)$-dimensional sphere $S^{2m-1}$. For the corresponding class $\Gamma_1^{(2m)}$, we have

$$\int_{\Gamma_1^{(2m)}} e^{-\tilde{f}_{2m}(x_{2m})} x_1^{k_1} x_2^{k_2} \phi(x_{2m}) dx_{2m}$$

$$= \lim_{\epsilon_1 \to 0} \left( \int_{\epsilon_1}^{\infty} \int_{\epsilon_2}^{\infty} e^{[-1/a_2]_\infty} + \int_{D_{\epsilon_1,\epsilon_2}} e^{-(x_1^{a_1} + x_1^{a_2})} x_1^{k_1} x_2^{k_2 + \theta} dx_1 dx_2 \right)$$

$$\cdot \int_{\Gamma_1^{(2m-2)}} e^{-\tilde{f}_{2m-2}(y_{2m})} \phi(y_{2m}) dy_{2m},$$

where $D_{\epsilon_1,\epsilon_2} := \{(x_1(\theta), re[-\theta/a_2]) | 0 \leq \theta \leq 1, \ r \geq \epsilon_2 \}$. It gives the equation (5.2). □

By Lemma 5.5 for $k = (k_1, \ldots, k_{2m})$ satisfying $x_{2m}^{k} \in B_{f_{2m}}'$ we obtain

$$\int_{\Gamma_1^{(2m)}} e^{-\tilde{f}_{2m}(x_{2m})} x_{2m}^{k} dx_{2m}$$

$$= \int_{\Gamma_1^{(2)}} e^{-\tilde{f}_{2}(x_1, x_2)} x_1^{a_2} x_2^{a_2} dx_1 dx_2 \cdot \int_{\Gamma_1^{(2m-2)}} e^{-\tilde{f}_{2m-2}(y_{2m})} (y_{2m})^k dy_{2m}$$

$$= \frac{1}{d_2} \Gamma^{(2m)}(\omega_{k,1}) \Gamma^{(2m)}(\omega_{k,2})(1 - e^{-\omega_{k,2}^{(2m)}}) \int_{\Gamma_1^{(2m-2)}} e^{-\tilde{f}_{2m-2}(y_{2m})} (y_{2m})^k dy_{2m},$$
where \( k'' := (k_3, \ldots, k_{2m}) \) and \( (y_{2m}')^{k''} := y_3^{k_3} y_4^{k_4} \ldots y_{2m}^{k_{2m}} \). For each \( x_1^{k_1} x_2^{k_2} \cdot \phi(x_{2m-2}'') \in B_{f_{2m-2}}' \), it inductively follows that

\[
\int_{I_1^{(2m)}} e^{-\tilde{f}_2(x_1, x_2)} x_1^{k_1} x_2^{k_2} \phi(x_{2m-2}'') x_{2m-1}^{a_{2m}} \, dx_{2m} \quad = \int_{I_1^{(2)}} e^{-\tilde{f}_2(x_1, x_2)} x_1^{k_1} x_2^{k_2+\bar{\phi}} \, dx_1 \, dx_2 \cdot \int_{I_1^{(2m-2)}} e^{-\tilde{f}_2 - \tilde{f}_4(y_{2m})} \phi(y_{2m}'') y_{2m-1}^{a_{2m-1}} \, dy_{2m} \quad = \int_{I_1^{(2)}} e^{-\tilde{f}_2(x_1, x_2)} x_1^{k_1} x_2^{k_2+\bar{\phi}} \, dx_1 \, dx_2 \cdot \left(-2\frac{\sqrt{-1}}{\alpha_{2m}}\right) \int_{I_1^{(2m-4)}} e^{-\tilde{f}_2 - \tilde{f}_4(y_{2m})} \phi(y_{2m}'') dy_{2m-2} \quad = -\frac{2\pi \sqrt{-1}}{\alpha_{2m}} \int_{I_1^{(2m-2)}} e^{-\tilde{f}_2(x_1, x_2)} x_1^{k_1} x_2^{k_2+\bar{\phi}} \, dx_1 \, dx_2 \phi(x_{2m-2}'') \, dx_{2m-2}.
\]

Therefore, for \( \kappa \in I_{2m} \), we obtain

\[
\text{ch}^{(2m)}_{\Gamma, \kappa} = \sum_{\lambda \in I_{2m}} \eta^{\lambda\kappa} \int_{I_1^{(2m)}} e^{-\tilde{f}_2} \phi^{(2m)}_{\lambda}.
\]

Define a homology class \( L_j^{(2m)} \in H_{2m-1}(\tilde{f}_{2m-1}(1); \mathbb{Z}) \) as the image of \( L_1^{(2m)} \) by the action \( (x_1, \ldots, x_k, \ldots, x_{2m}) \mapsto (e^{(j-1)/a_1} x_1, \ldots, e^{(-1)^{k-1}(j-1)/d_k} x_k, \ldots, e^{-(j-1)/d_{2m}} x_{2m}) \) on \( \tilde{f}_2^{-1}(1) \) and \( \Gamma_2^{(2m)} \in H_{2m}(\mathbb{C}^{2m}, \text{Re}(\tilde{f}_2) \gg 0; \mathbb{Z}) \) as the corresponding class of \( L_j^{(2m)} \). For \( j = 1, \ldots, \mu_{2m} \), we have the equality

\[
\text{ch}^{(2m)}_{\Gamma, \kappa_j} = \sum_{\lambda \in I_{2m}} \eta^{\lambda\kappa} \int_{I_1^{(2m)}} e^{-\tilde{f}_2} \phi^{(2m)}_{\lambda}.
\]

Hence, we have finished the proof of Theorem 6.3.

6. Stokes and Central Connection Matrices for \( \tilde{f}_n \)

In this section, we shall consider a Frobenius manifold associated with \( \tilde{f}_n \). Denote by \( \mathbb{C}_w \) the complex plane whose coordinate is \( w \), which we call \( w \)-plane for simplicity. Define \( p : \mathbb{C}^n \times \mathbb{C}^{\bar{n}} \to \mathbb{C}^{\bar{n}} \) by the natural projection and a function \( \tilde{F}_n : \mathbb{C}^n \times \mathbb{C}^{\bar{n}} \to \mathbb{C}_w \) by

\[
\tilde{F}_n(x; s) := \tilde{f}_n(x) + \sum_{\kappa \in I_n} s_\kappa \cdot \phi^{(n)}_\kappa(x), \quad (6.1)
\]

where \( \{\phi^{(n)}_\kappa(x)\}_{\kappa \in I_n} \) is the basis of \( \text{Jac}(\tilde{f}_n) \) given in Definition 2.8. Let \( (\tilde{\omega}^{(n)}_1, \ldots, \tilde{\omega}^{(n)}_n) \) be the rational weights of \( \tilde{f}_n \). By a direct calculation, we have

\[
\tilde{\omega}^{(n)}_i = \sum_{l=1}^i (-1)^{i-l} \frac{d_{i-1}}{d_i}, \quad i = 1, \ldots, n. \quad (6.2)
\]

Set

\[
q_\kappa := \deg \phi^{(n)}_\kappa(x) = \sum_{i=1}^n k_i \tilde{\omega}^{(n)}_i, \quad \kappa \in I'_n, \quad (6.3)
\]
where \((k_1, \ldots, k_n) = \psi^{-1}(\kappa)\).

**Proposition 6.1.** The pair \((\tilde{F}_n, p)\) is a universal unfolding of \(\tilde{f}_n\), namely it satisfies that

1. \(\tilde{F}_n|_{p^{-1}(0)} = \tilde{f}_n\) in a neighborhood of the origin in \(p^{-1}(0) \cong \mathbb{C}^{\tilde{\mu}_n}\),
2. there exists an \(\mathcal{O}_{\mathbb{C}^{\tilde{\mu}_n}, 0}\)-isomorphism

\[
\rho : \mathcal{T}_{\mathbb{C}^{\tilde{\mu}_n}, 0} \longrightarrow p_* \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^{\tilde{\mu}_n}, 0} \left/ \left(\frac{\partial \tilde{F}_n}{\partial x_1}, \ldots, \frac{\partial \tilde{F}_n}{\partial x_n}\right)\right., \quad \delta \mapsto \hat{\delta} \tilde{F}_n,
\]

where \(\hat{\delta}\) is a lifting on \(\mathbb{C}^n \times \mathbb{C}^{\tilde{\mu}_n}\) of a vector field \(\delta \in \mathcal{T}_{\mathbb{C}^{\tilde{\mu}_n}, 0}\) with respect to the projection \(p\).

Let us fix Euclidean norms \(\|\cdot\|\) on \(\mathbb{C}^n \times \mathbb{C}^{\tilde{\mu}_n}, \mathbb{C}^{\tilde{\mu}_n}\) and \(\mathbb{C}_w\). For positive real numbers \(\varepsilon_M, \varepsilon_\Delta\) and \(\varepsilon_X\), put

\[
M := \{s \in \mathbb{C}^{\tilde{\mu}_n} \mid \|s\| < \varepsilon_M\}, \quad (6.5a)
\]
\[
\Delta := \{w \in \mathbb{C}_w \mid \|w\| < \varepsilon_\Delta\} \quad (6.5b)
\]
\[
\mathcal{X} := \{(x, s) \in \mathbb{C}^n \times \mathbb{C}^{\tilde{\mu}_n} \mid \|(x, s)\| < \varepsilon_X, \|\tilde{F}_n(x; s)\| < \varepsilon_\Delta\} \cap p^{-1}(M). \quad (6.5c)
\]

Denote the fiber of \(\mathcal{X}\) with respect to \(p\) over \(s \in M\) by \(\mathcal{X}_s\) and set

\[
Y_{w; s} := \mathcal{X}_s \cap \tilde{F}_n^{-1}(w), \quad w \in \mathbb{C}_w, \ s \in M. \quad (6.5d)
\]

For the choice \(1 \gg \varepsilon_X \gg \varepsilon_\Delta \gg \varepsilon_M > 0\) of radius, the map \((\tilde{F}_n, p) : \mathcal{X} \longrightarrow \Delta \times M\) defines a fibration which the fiber over 0 is isomorphic to the singularity \(Y_{0; 0} = \{\tilde{f}_n = 0\}\) and generic fiber is homotopic to a bouquet of \(\tilde{\mu}_n\) copies of \((n - 1)\)-dimensional sphere due to Milnor [M]. In particular, for a regular value \(w_0 \in \partial \Delta\) we have

\[
H_n(\mathcal{X}_s, Y_{w_0; s}; \mathbb{Z}) \cong H_{n-1}(Y_{w_0; s}; \mathbb{Z}) \cong \mathbb{Z}^{\tilde{\mu}_n}.
\]

For simplicity, we fix the constants \(\varepsilon_M, \varepsilon_\Delta\) and \(\varepsilon_X\) suitably.

**Definition 6.2.** We shall denote by \(\circ\) the induced product structure on \(\mathcal{T}_M\) by the \(\mathcal{O}_{\mathbb{C}^{\tilde{\mu}_n}, 0}\)-isomorphism \(\rho\). Namely, we have

\[
(\delta \circ \delta') \tilde{F}_n = \hat{\delta} \hat{\delta'} \tilde{F}_n, \quad \delta, \delta' \in \mathcal{T}_M. \quad (6.6)
\]

Define two vector fields \(e\) and \(E\) as follows:

1. The vector field \(e \in \mathcal{T}_M\) corresponding to the unit 1 by the \(\mathcal{O}_{\mathbb{C}^{\tilde{\mu}_n}, 0}\)-isomorphism \(\rho\)
   is called the **unit vector field**. That is,
   \[
   \hat{e} \tilde{F}_n = 1.
   \]
The vector field $E \in T_M$ corresponding to $\tilde{F}_n$ by the $O_{\tilde{\mu}_n,0}$-isomorphism $\rho$ is called the Euler vector field. That is,

$$\hat{E} \tilde{F}_n = \tilde{F}_n.$$

Note that the unit vector field and the Euler vector field are given explicitly by

$$e = \frac{\partial}{\partial s_{\psi(0)}}, \quad E = \sum_{\kappa \in I_n} (1 - q_\kappa) \frac{\partial}{\partial s_\kappa}. \quad (6.7)$$

Let $\mathbb{P}^1_u$ be the complex projective line whose coordinate is $u$. In order to define a notion of a primitive form, it is necessary to define a Saito structure associated with the universal unfolding $\tilde{F}_n$. This structure is given as a tuple consisting of the filtered de Rham cohomology group $H_{\tilde{F}_n}$ (whose increasing filtration is denoted by $H^{(k)}_{\tilde{F}_n}$ ($k \in \mathbb{Z}$)), the Gauß–Manin connection $\nabla$ on $H_{\tilde{F}_n}$ and the higher residue pairing $K_{\tilde{F}_n}$ on $H_{\tilde{F}_n}$. In the paper, we omit the details about those objects and refer the interested reader to [ST].

Since $\tilde{f}_n$ is a weighted homogeneous polynomial, there exists a canonical primitive form defined by exponents of $\tilde{f}_n$. We will use this primitive form in the paper.

**Proposition 6.3** ([S-K], [S-M]). There exists a unique primitive form $\zeta \in \Gamma(M, H^{(0)}_{\tilde{F}_n})$ for the tuple $(H^{(0)}_{\tilde{F}_n}, \nabla, K_{\tilde{F}_n})$ with the minimal exponent $r = \sum_{i=1}^n \tilde{\omega}^{(n)}_i = \sum_{i=1}^n \omega^{(n)}_i$ and the normalization

$$\zeta|_{s=0} = [dx_1 \wedge \cdots \wedge dx_n]. \quad (6.8)$$

The higher residue pairing $K_{\tilde{F}_n}$ and the primitive form $\zeta$ induce a symmetric non-degenerate $O_M$-bilinear form $\eta_\zeta : T_M \times T_M \rightarrow O_M$ defined by

$$\eta_\zeta(\delta, \delta') := K_{\tilde{F}_n} \left( r^{(0)}(u \nabla \delta \zeta), r^{(0)}(u \nabla \delta' \zeta) \right), \quad \delta, \delta' \in T_M. \quad (6.9)$$

Hence, we obtain the following

**Proposition 6.4** (cf. [ST]). The tuple $(M, \eta_\zeta, \circ, e, E)$ is a Frobenius manifold of rank $\tilde{\mu}_n$ and dimension $d_{\tilde{f}_n} := \sum_{i=1}^n (1 - 2\tilde{\omega}^{(n)}_i) = \sum_{i=1}^n (1 - 2\omega^{(n)}_i)$. Namely, it satisfies the following properties:

1. The product $\circ$ is self-adjoint with respect to $\eta$: that is,

$$\eta_\zeta(\delta \circ \delta', \delta'') = \eta_\zeta(\delta, \delta' \circ \delta''), \quad \delta, \delta', \delta'' \in T_M.$$

2. The Levi–Civita connection $\nabla : T_M \otimes_{O_M} T_M \rightarrow T_M$ with respect to $\eta_\zeta$ is flat: that is,

$$[\nabla_\delta, \nabla_{\delta'}] = \nabla_{[\delta, \delta']}, \quad \delta, \delta' \in T_M.$$
The \( O_M \)-linear morphism \( C_\delta : T_M \rightarrow T_M, \delta \in T_M \) defined by \( C_\delta(\cdot) := - \circ \delta \) satisfies
\[
\nabla_\delta(C_\delta'\delta'') - C_\delta'(\nabla_\delta\delta'') - C_\delta(\nabla_\delta'\delta'') = C_\delta'(\nabla_\delta\delta'') - C_\delta(\nabla_\delta'\delta'') - C_\delta'\delta''.
\]

The unit element \( e \) of the \( \circ \)-algebra is a \( \nabla/ \)-flat holomorphic vector field: that is,
\[
\nabla/e = 0.
\]

The bilinear form \( \eta \zeta \) and the product \( \circ \) are homogeneous of degree \( 2 - d, 1 \), respectively with respect to the Lie derivative \( \text{Lie}_E \) of the Euler vector field \( E \): that is,
\[
\text{Lie}_E(\eta\zeta) = (2 - d\tilde{f}_n)\eta\zeta, \quad \text{Lie}_E(\circ) = \circ.
\]

For the Frobenius manifold \( M \), a connection \( \hat{\nabla} \) on \( T_{\mathbb{P}^1 \times M} \), called the first structure connection, is defined as follows:
\[
\begin{align*}
\hat{\nabla}_{\delta'} \delta &= \nabla_{\delta'} \delta - \frac{1}{u} \delta \circ \delta', \quad \delta, \delta' \in T_M, \\
\hat{\nabla}_{u \frac{d}{du}} \delta &= \left( \frac{1}{u} C_E - \tilde{Q} \right) \delta, \quad \delta \in T_M, \\
\hat{\nabla}_{\delta} \frac{d}{du} \frac{d}{du} &= \hat{\nabla}_{\delta} \frac{d}{du} \frac{d}{du} = 0, \quad \delta \in T_M,
\end{align*}
\]
where \( \tilde{Q} \in \text{End}_{O_M}(T_M) \) is defined by
\[
\tilde{Q}(\delta) = \frac{2 - d\tilde{f}_n}{2} \delta - \nabla_\delta E, \quad \delta \in T_M.
\]

Remark 6.5. Note that the parameter \( z \) in [D1, CDG] is \( u^{-1} \) in this paper.

Proposition 6.6 ([D1, Proposition 3.1]). The connection \( \hat{\nabla} \) is flat.

Remark 6.7. In fact, the primitive form \( \zeta \) identifies the first structure connection \( \hat{\nabla} \) with the Gauss–Manin connection on the filtered de Rham cohomology (see [ST]), which is flat.

Define an \( O_M \)-endomorphism \( \mathcal{N} \in \text{End}_{O_M}(T_M) \) by
\[
\mathcal{N}(\delta) := \tilde{Q}(\delta) + \frac{n}{2} \delta, \quad \delta \in T_M.
\]

The eigenvalues of \( \mathcal{N} \) are nothing but exponents of \( \tilde{f}_n \).

There exist local coordinates \( t = (t_\kappa)_{\kappa \in I_n} \), called flat coordinates, such that the unit vector field \( e \) is given by \( \partial_{\psi(0)} \) and \( \text{Ker}\nabla \) is spanned by \( \partial_\kappa \), \( \kappa \in I_n \), where we denote \( \partial/\partial t_\kappa \) by \( \partial_\kappa \). Moreover, one can choose flat coordinates \( t = (t_\kappa)_{\kappa \in I_n} \) satisfying
\[
t_\kappa(0) = 0, \quad \frac{\partial t_\kappa}{\partial s_\lambda}(0) = \delta_{\kappa\lambda}, \quad \kappa, \lambda \in I_n.
\]
In particular, we have
\[ \eta_\kappa (\partial_\kappa, \partial_\lambda) = \eta_\kappa \left( \frac{\partial}{\partial s_\kappa}, \frac{\partial}{\partial s_\lambda} \right) \bigg|_{s=0} = \eta^{(n)}_{\kappa\lambda}, \]
due to (6.13), and
\[ \tilde{Q}(\partial_\kappa) = \tilde{Q}^{(n)}_{\kappa\kappa} \partial_\kappa, \]
since for each \( \kappa \in I_n \) we have \( \text{Lie}_E (\partial_\kappa) = (1 - q_\kappa) \partial_\kappa \) and
\[
q_\kappa - \frac{d_{1n}}{2} = \sum_{i=1}^n k_i \tilde{\omega}^{(n)}_i + \sum_{i=1}^n \tilde{\omega}^{(n)}_i - \frac{n}{2} = \sum_{i=1}^n \omega^{(n)}_{i\kappa,i} - \frac{n}{2} = \tilde{Q}^{(n)}_{\kappa\kappa},
\]
where \( (k_1, \ldots, k_n) = \psi^{-1}(\kappa) \). To summarize, we obtain the following

**Proposition 6.8.** Matrix representations of \( \tilde{Q} \) and \( \eta_\kappa \) with respect to \( \{ \partial_\kappa \}_{\kappa \in I_n} \) are given by \( \tilde{Q}^{(n)} \) and \( \eta^{(n)} \), respectively. Moreover, the diagonal matrix \( N^{(n)} = (n_\kappa) \) defined by
\[
n_\kappa := q_\kappa + \frac{n - d_{1n}}{2} = \deg \zeta^{(n)}_\kappa, \tag{6.14}
\]
is the matrix representing the \( \mathcal{O}_M \)-endomorphism \( N \).

Let \( Z = Z(u, t) \) be a function on an open subset in \( \mathbb{P}^1_u \times M \) and \( (\eta^{\kappa\lambda}) \) the inverse matrix of \( \eta^{(n)} \). We say that \( Z \) is \( \hat{\nabla} \)-flat if it satisfies
\[
\hat{\nabla} \left( \sum_{\kappa, \lambda \in I_n} \eta^{\kappa\lambda} u \frac{\partial Z}{\partial t_\lambda} \frac{\partial}{\partial t_\kappa} \right) = 0,
\]
which is equivalent to the following differential system: for \( \mathcal{I} = (\sum_{\lambda \in I_n} \eta^{\kappa\lambda} u \partial_\lambda Z) \)
\[
\frac{\partial}{\partial t_\kappa} \mathcal{I} = -\frac{1}{u} C_{\partial_\kappa} \mathcal{I}, \quad \kappa \in I_n, \tag{6.15a}
\]
\[
u \frac{d}{du} \mathcal{I} = \left( 1 - \frac{1}{u} C_E - \tilde{Q}^{(n)} \right) \mathcal{I}, \tag{6.15b}
\]
where we denote by the same letters \( C_{\partial_\kappa} \) and \( C_E \) the matrix representations with respect to the basis \( \{ \partial_\kappa \}_{\kappa \in I_n} \). The differential system (6.15) can be considered as a family of meromorphic differential equations on \( \mathbb{P}^1_u \) parametrized by points on \( M \), and the flatness of \( \hat{\nabla} \) means that this family is isomonodromic. The equation has a regular singular point at \( u = \infty \) and an irregular singular point of Poincaré rank one at \( u = 0 \).

Let us consider a fundamental solution of \( \mathcal{I} \) at \( u = \infty \).

**Proposition 6.9 ([D1]).** There exists a unique matrix fundamental solution \( Y_\infty(u, t) = \Phi(u, t)u^{-\tilde{Q}^{(n)}} \) such that \( \Phi(u, t) = 1 + \Phi_1(t)u^{-1} + \Phi_2(t)u^{-2} + \cdots \) is a matrix-valued convergent power series in \( u^{-1} \) satisfying
\[
\Phi(u, 0) = 1, \quad \Phi(-u, t)^T \eta^{(n)} \Phi(u, t) = \eta^{(n)}. \tag{6.16}
\]
In particular, the monodromy around infinity is given by

\[ Y_\infty(e^{-2\pi \sqrt{-1}u}, t) = Y_\infty(u, t)e^{\tilde{Q}(u)}. \]  \hspace{1cm} (6.17)

Denote by \( \mathcal{B} \subset M \) the closed subset, called the *bifurcation set*, consisting of points where the values of canonical coordinates \((w_1, \ldots, w_{\tilde{\mu}_n})\), namely, the critical values of \( \tilde{F}_n(x; s) \), coincide. It is known by [D3, Corollary 3.1] that the product \( \circ \) on the tangent space \( T_sM \) for \( s \in M \setminus \mathcal{B} \) is semi-simple and hence the Frobenius manifold \( M \) is semi-simple.

Fix a point \( s \in M \setminus \mathcal{B} \) and define the matrix \( \Theta = (\Theta_{i\lambda}) \) of size \( \tilde{\mu}_n \) by

\[ \Theta_{i\lambda} := \frac{\partial w_i}{\partial \lambda} \cdot \eta \left( \frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j} \right)^{\frac{1}{2}}, \quad \lambda \in I_n, \ i = 1, \ldots, \tilde{\mu}_n. \]  \hspace{1cm} (6.18)

We can construct a formal solution of (6.15) near \( u = 0 \) as follows.

**Proposition 6.10 ([D3, Lemma 4.3]).** There exists a unique formal matrix fundamental solution \( Y_{\text{formal}}(u) \) of the differential equation (6.15) in the form

\[ Y_{\text{formal}}(u) = \Theta^{-1}G(u)\exp(-W/u), \]  \hspace{1cm} (6.19)

where \( W = \text{diag}(w_1, \ldots, w_{\tilde{\mu}_n}) \) and \( G(u) = 1 + G_1u + G_2u^2 + \cdots \) is a matrix-valued formal power series satisfying

\[ G(-u)^T G(u) = 1. \]

Note that such \( G(u) \) is a divergent power series in general since \( u = 0 \) is an irregular singular point of Poincaré rank one.

Denote by \( \mathbb{C}_u \) the complex plane whose coordinate is \( u \). For \( 0 \leq \phi < 1 \), a line \( \ell = e^{\pi \sqrt{-1}(\phi+1/2)} \mathbb{R} \setminus \{0\} \subset \mathbb{C}_u \) is called *admissible* if

\[ \frac{w_a - w_b}{u} \notin \mathbb{R}, \quad u \in e^{\pi \sqrt{-1}\phi} \mathbb{R}_{>0}, \quad a, b = 1, \ldots, \tilde{\mu}_n. \]

For a small positive number \( \varepsilon \), define sectors in \( \mathbb{C}_u \) by

\[ \Pi_{\text{right}} = \left\{ u \in \mathbb{C}_u \setminus \{0\} \left| \phi - \frac{1}{2} - \varepsilon < \arg \frac{u}{\pi} < \phi + \frac{1}{2} + \varepsilon \right. \right\}, \]

\[ \Pi_{\text{left}} = \left\{ u \in \mathbb{C}_u \setminus \{0\} \left| \phi + \frac{1}{2} - \varepsilon < \arg \frac{u}{\pi} < \phi + \frac{3}{2} + \varepsilon \right. \right\}. \]
It is known by the general theory for ordinary differential equations (cf. [BJL, Theorem A]) that there exist unique solutions $Y_{\text{right}}$ and $Y_{\text{left}}$ of (6.15b) analytic in $u$ in the sectors $\Pi_{\text{right}}$ and $\Pi_{\text{left}}$ satisfying the following asymptotic properties:

$$Y_{\text{right}}(u) \sim Y_{\text{formal}}(u) \quad \text{as} \quad u \to 0, \quad u \in \Pi_{\text{right}},$$

$$Y_{\text{left}}(u) \sim Y_{\text{formal}}(u) \quad \text{as} \quad u \to 0, \quad u \in \Pi_{\text{left}}.$$  

In the sectors $\Pi_+$ and $\Pi_-$ in $\Pi_{\text{right}} \cap \Pi_{\text{left}}$ defined by

$$\Pi_+ := \left\{ u \in \mathbb{C}_u \setminus \{0\} \mid \phi + \frac{1}{2} - \varepsilon < \frac{\arg u}{\pi} < \phi + \frac{1}{2} + \varepsilon \right\},$$

$$\Pi_- := \left\{ u \in \mathbb{C}_u \setminus \{0\} \mid \phi + \frac{3}{2} - \varepsilon < \frac{\arg u}{\pi} < \phi + \frac{3}{2} + \varepsilon \right\},$$

we have two analytic solutions $Y_{\text{right}}(u)$ and $Y_{\text{left}}(u)$. They are related by

$$Y_{\text{right}}(u) = Y_{\text{left}}(u)S^\phi, \quad u \in \Pi_+,  \quad (6.20)$$

$$Y_{\text{right}}(e^{-2\pi\sqrt{-1}u}) = Y_{\text{left}}(u)(S^\phi)^T, \quad u \in \Pi_-, \quad (6.21)$$

with a matrix $S^\phi$ independent of $u$.

**Definition 6.11.** The matrix $S^\phi$ is called the *Stokes matrix* of the first structure connection of $M$ (for the admissible line $\ell$ and the chosen point on $M \setminus \mathcal{B}$).
The fundamental solution $Y_\infty(u) := Y_\infty(u, s)$ around $u = \infty$ is related to the analytic solution $Y_{\text{right}}(u)$ by
\begin{equation}
Y_{\text{right}}(u) = Y_\infty(u) C^\phi, \quad u \in \Pi_{\text{right}}, \quad |u| \gg 0, \tag{6.22}
\end{equation}
with a matrix $C^\phi$ independent of $u$. The matrix $C^\phi$ is called the central connection matrix (for the admissible line $\ell$ and the chosen point $s \in M \setminus \mathcal{B}$). The following proposition is a consequence of the isomonodromic property of the differential equation (6.15b).

**Proposition 6.12** ([D3, Isomonodromicity Theorem]). The Stokes matrix $S^\phi$ and the central connection matrix $C^\phi$ are locally constant as a function on $M \setminus \mathcal{B}$. \hfill \Box

**Proposition 6.13** ([D3]). We have the following relations;
\begin{align}
(C^\phi)^{-1} e[\bar{Q}^{(n)}] C^\phi &= (S^\phi)^{-1} (S^\phi)^T, \tag{6.23a} \\
(C^\phi)^T e \left[ \frac{1}{2} \bar{Q}^{(n)} \right] \eta^{(n)} C^\phi &= S^\phi. \tag{6.23b}
\end{align}

\hfill \Box

Consider an unfolding
\begin{equation}
\tilde{f}_{n; s}(x) := \tilde{f}_n(x) + s \cdot x_n, \tag{6.24}
\end{equation}
with one parameter $s \in \mathbb{C}$ with $|s| \ll 1$. Since we have $0 < \deg s < 1$ where $[\frac{\partial}{\partial s}, E] = (\deg s) \frac{\partial}{\partial s}$, the restriction of the domain of $\tilde{f}_{n; s}$ to a small neighborhood of the origin in
Let \( \varphi \) along this half-line. Let \( \gamma \) so that all lines of the form (cf. [V, Appendix A])

\[
\text{Proposition 6.14}
\]

\[
\text{Remark 6.15. Note that our ordering of vanishing cycles is inverse to the usual one. Since}
\]

\[
\text{we only use this counter-clockwise ordering of vanishing cycles in this paper, we just call it a}
\]

\[
\text{distinguished basis for simplicity.}
\]

\[
\text{Since we have the natural isomorphism}
\]

\[
\text{6.14. Let } \varphi \text{ along this half-line.}
\]

\[
\text{Y \wedge \varphi \text{ we number so that all lines of the form (cf. [V, Appendix A])}
\]

\[
\text{Proposition 6.14}
\]

\[
\text{for each } |s| \ll 1. \text{ Moreover, we have the following}
\]

\[
\text{Proposition 6.14 (cf. [V, Appendix A]). For each } s, \text{ the point on the base space } M \text{ of}
\]

\[
\text{the universal unfolding } \tilde{F}_n \text{ corresponding to } \tilde{f}_{n,s} \text{ belongs to } M \setminus \mathcal{B}.
\]

\[
\text{In what follows, we determine the monodromy data at the point in Proposition 6.14. Let } \phi = 0 \text{ if the line } \sqrt{-1} \mathbb{R} \setminus \{0\} \text{ is admissible, if not, be a sufficiently small positive number so that all lines of the form } e^{\pi \sqrt{-1}(\phi' + 1/2)} \mathbb{R} \setminus \{0\}, \, 0 < \phi' \leq 2 \phi \text{ are admissible. Let } p_1, \ldots, p_{\tilde{\mu}_n} \text{ be the critical points of the holomorphic function } \tilde{f}_{n,s}(x) : \mathbb{C}^n \rightarrow \mathbb{C}_w
\]

\[
\text{and } w_j := \tilde{f}_{n,s}(p_j). \text{ For } u \in e^{\pi \sqrt{-1}\phi} \mathbb{R}_> 0 \text{ and each critical point } p_j, \text{ we can define a relative } n\text{-cycle}
\]

\[
\Gamma_j(u) \in H_n(\mathbb{C}^n, \text{Re}(\tilde{f}_{n,s}(x)/u) \gg 0; \mathbb{Z}),
\]

\[
\text{called the Lefschetz thimble for } p_j \text{ as follows. The image of } \Gamma_j(u) \text{ by } \tilde{f}_{n,s}(x) \text{ is a half-line}
\]

\[
w_j + e^{\pi \sqrt{-1}\phi} \mathbb{R}_> 0 \subset \mathbb{C}_w \text{ and the fiber over a point } w \text{ on this half-line is the } (n - 1)\text{-cycle in}
\]

\[
Y_{w,s} := \{ \tilde{f}_{n,s}(x) = w \} \subset \mathbb{C}^n \text{ which shrinks to the critical point } p_j \text{ by the parallel transport along this half-line.}
\]

\[
\text{Choose } w_0 \in \mathbb{C}_w \text{ so that } \text{Re}(w_0/u) \gg 0 \text{ and we have}
\]

\[
H_n(\mathbb{C}^n, \text{Re}(\tilde{f}_{n,s}(x)/u) \gg 0; \mathbb{Z}) \cong H_n(\mathbb{C}^n, Y_{w_0,s}; \mathbb{Z}).
\]

\[
\text{Take paths } \gamma_j : [0, 1] \rightarrow \mathbb{C}_w \text{ with } \gamma_j(0) = w_j \text{ and } \gamma_j(1) = w_0 \text{ for } j = 1, \ldots, \tilde{\mu}_n \text{ satisfying the following conditions:}
\]

- For each } j = 1, \ldots, \tilde{\mu}_n, \text{ the path } \gamma_j \text{ has no self-intersection.
- For } i \neq j, \gamma_i \text{ and } \gamma_j \text{ have a unique common point } w_0.
- The paths } \gamma_1, \ldots, \gamma_{\tilde{\mu}_n} \text{ are ordered counter-clockwise, namely, arg } \gamma_i'(1) < \text{arg } \gamma_j'(1)
\]

\[
\text{if } i < j.
\]

\[
\text{We obtain an } (n - 1)\text{-sphere in } Y_{w_0,s} \text{ which shrinks to a point at } w_j \text{ by the parallel transport along the path } \gamma_j, \text{ whose homology class } L_j \in H_{n-1}(Y_{w_0,s}; \mathbb{Z}) \text{ is called a vanishing cycle along } \gamma_j. \text{ The ordered set } (L_1, \ldots, L_{\tilde{\mu}_n}) \text{ of vanishing cycles is called a (counter-clockwise) distinguished basis of vanishing cycles.}
\]

\[
\text{Remark 6.15. Note that our ordering of vanishing cycles is inverse to the usual one. Since}
\]

\[
\text{we only use this counter-clockwise ordering of vanishing cycles in this paper, we just call it a distinguished basis for simplicity.}
\]

\[
\text{Since we have the natural isomorphism}
\]

\[
H_n(\mathbb{C}^n, Y_{w_0,s}; \mathbb{Z}) \cong H_{n-1}(Y_{w_0,s}; \mathbb{Z}),
\]
it turns out that homology class represented by the Lefschetz thimble $\Gamma_j(u)$ for $p_j$ are uniquely determined by the vanishing cycle $L_j$.

The primitive form $\zeta$ is given by $[d\tilde{x}]$ over the subset of $M$ where $\tilde{F}_n = \tilde{f}_{n,s}$ since $0 < \deg s < 1$ and the normalization (6.8). Therefore, it follows from the definition of the primitive form that

$$I_j := (I_{\kappa j}), \quad I_{\kappa j} := \frac{1}{(2\pi u)^{\frac{n}{2}}} \sum_{\lambda \in I_n} \eta^{\lambda k} \int_{\Gamma_j(u)} e^{-\tilde{f}_{n,s}(x)/u} \zeta^{(n)},$$

satisfies

$$\frac{\partial}{\partial s} f_j = -\frac{1}{u} \sum_{\lambda \in I_n} \eta^{\lambda k} \int_{\Gamma_j(u)} e^{-\tilde{f}_{n,s}(x)/u} \zeta^{(n)}, \quad (6.28a)$$

$$u \frac{d}{du} I_j = \left( \frac{1}{u} (\deg s) \frac{\partial}{\partial s} - \tilde{Q}^{(n)} \right) I_j. \quad (6.28b)$$

where $\zeta^{(n)}$ is the class in $\mathbb{H}^n(\Omega_{\mathbb{C}^n}^*, d - d\tilde{f}_n \wedge)$ corresponding to the class $\zeta^{(n)}(\pi) \in \Omega_{\tilde{f}_n}^*$ under the isomorphism $\mathbb{H}^n(\Omega_{\mathbb{C}^n}^*, d - d\tilde{f}_n \wedge) \cong \Omega_{\tilde{f}_n}^*$.

By the saddle-point method, the oscillatory integral

$$Z_j(u) := \frac{1}{(2\pi u)^{\frac{n}{2}}} \int_{\Gamma_j(u)} e^{-\tilde{f}_{n,s}(x)/u} \tilde{x}, \quad j = 1, \ldots, \tilde{\mu}_n$$

has an asymptotic expansion

$$Z_j(u) = e^{-w_j/s}(1 + O(u)), \quad u \to 0, \quad j = 1, \ldots, \tilde{\mu}_n,$$

where $\Delta_j$ is the Hessian at $p_j$.

$\Delta_j := \det \left( \frac{\partial^2 \tilde{f}_{n,s}(x)}{\partial x_k \partial x_l} (p_j) \right)_{k,l=1,\ldots,n}, \quad j = 1, \ldots, \tilde{\mu}_n.$

Since $(w_1, \ldots, w_n)$ are canonical coordinates, we have

$$\frac{\partial F_n}{\partial w_i}(p_j) = \delta_{ij}, \quad \eta \left( \frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j} \right) = \frac{\delta_{ij}}{\Delta_i}, \quad i, j = 1, \ldots, \tilde{\mu}_n.$$
Hence, it turns out that the asymptotic expansion of $I(u) := (I_{\kappa j}(u))$ coincides with $Y_{\text{formal}}(u)$ if a suitable branch of the square root is chosen.

The Lefschetz thimble $\Gamma_j(u)$ discontinuously changes when $u$ cross a half-line of the form $e^{\sqrt{-1}\arg(u_+-u_-)}\mathbb{R}_{>0} \subset \mathbb{C}_u$. The discontinuity causes the Stokes phenomena for the oscillatory integrals. In order to determine the Stokes matrix $S^\phi$ of the Frobenius manifold $M$ for the admissible line $\ell$, we establish the correspondence between the analytic solutions $Y_{\text{right/left}}$ and $I$.

Order critical values $\{u_j\}_{j=1}^{\tilde{\mu}_n}$ so that

$$|e^{-w_1/u}| \ll |e^{-w_2/u}| \ll \cdots \ll |e^{-w_{\tilde{\mu}_n}/u}|$$

holds as $u \to 0$ along the line $e^{\pi\sqrt{-1}(\phi+1/2)}\mathbb{R}_{>0}$.

Consider the local system on $\mathbb{C}_u \setminus \{0\}$ whose fiber over $u \in \mathbb{C}_u \setminus \{0\}$ is the relative homology group $H_n(\mathbb{C}^n, \text{Re}(\bar{\tau}_{n,s}(x))/u) \ni 0; \mathbb{Z}$, and let $\Gamma_{\text{right/left},j}(u)$ be a section of the local system on $\Pi_{\text{right/left}}$ satisfying the following condition:

(A) for $u \in \{u \in \mathbb{C}_u \setminus \{0\} \mid 0 < \arg u/\pi < 2\phi\} \subset \Pi_{\text{right}}$ (resp., $u \in \{u \in \mathbb{C}_u \setminus \{0\} \mid 1 < \arg u/\pi < 2\phi+1\} \subset \Pi_{\text{left}}$), $\Gamma_{\text{right},j}(u)$ (resp., $\Gamma_{\text{left},j}(u)$) coincides with the relative homology class represented by the Lefschetz thimble $\Gamma_j(u)$ for the $j$-th critical point $p_j$.

Since $\Pi_{\text{right/left}}$ is simply-connected, the condition (A) determines the section $\Gamma_{\text{right/left},j}(u)$ uniquely.

Let

$$Y_{\text{right}}(u)_{\kappa j} := \frac{1}{(2\pi u)^{\frac{3}{2}}} \sum_{\lambda \in I_n} \eta^{\varphi \kappa} \int_{\Gamma_{\text{right},j}(u)} e^{-\frac{\bar{\tau}_{n,s}(x)}{u}} f^{(n)} \varphi^{(n)}, \quad (6.29a)$$

$$Y_{\text{left}}(u)_{\kappa j} := \frac{1}{(2\pi u)^{\frac{3}{2}}} \sum_{\lambda \in I_n} \eta^{\varphi \kappa} \int_{\Gamma_{\text{left},j}(u)} e^{-\frac{\bar{\tau}_{n,s}(x)}{u}} f^{(n)} \varphi^{(n)}, \quad (6.29b)$$

for $\kappa \in I_n$, $j = 1, \ldots, \tilde{\mu}_n$. The matrix valued function $Y_{\text{right/left}}(u) = (Y_{\text{right/left}}(u)_{\kappa j})$ in $u$ is a fundamental solution of (6.15) which is asymptotic to $Y_{\text{formal}}(u)$ as $u \to 0$ in the whole sector $\Pi_{\text{right/left}}$ by the condition (A). Therefore, the Stokes matrix $S^\phi$ can be read off from the monodromy of integration cycles since the integrand is single-valued. That is, for $u \in \Pi_+$, $S^\phi = (S^\phi_{ij})_{i,j=1, \ldots, \tilde{\mu}_n}$ satisfies

$$\Gamma_{\text{right},j}(u) = \sum_{i=1}^{\tilde{\mu}_n} S^\phi_{ij} \Gamma_{\text{left},i}(u), \quad j = 1, \ldots, \tilde{\mu}_n. \quad (6.30)$$

By the Picard-Lefschetz formula, it turns out that the Stokes matrix coincides with the Seifert matrix.
Proposition 6.16. Let the notations be as above. Let $S$ be the Seifert form on $H_{n-1}(Y_{w_0};\mathbb{Z})$ whose matrix representation is given by

$$S(L_i, L_j) := \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ (-1)^{(n-1)(n-2)/2}I(L_i, L_j) & \text{if } i < j, \end{cases} \quad (6.31)$$

where $I$ is the intersection form on $H_{n-1}(Y_{w_0};\mathbb{Z})$. We have

$$S_{ij} = S(L_i, L_j), \quad i, j = 1, \ldots, \tilde{\mu}_n.$$  

Proof. For $u \in \Pi_+$, choose $w_0 \in \mathbb{C}_w$ so that $\text{Re}(w_0/u) \gg 0$ and the isomorphism (6.26) holds. For each $j$, let $L_{\text{right/left},j} \in H_{n-1}(Y_{w_0};\mathbb{Z})$ be the vanishing cycle corresponding to the Lefschetz thimble $\Gamma_{\text{right/left},j}(u)$. Note that $L_j = L_{\text{right},j}$. By the ordering of $\gamma_1, \ldots, \gamma_{\tilde{\mu}_n}$, we have $L_{\text{left},j} = h_{L_{\text{right},1}}^{-1} \circ \cdots \circ h_{L_{\text{right},j-1}}^{-1}(L_{\text{right},j})$, where $h_{L_{\text{right},k}}$ is the Picard-Lefschetz transformation associated to the vanishing cycle $L_{\text{right},k}$ (see Figure 8). By the Picard-Lefschetz formula

$$h_{L_{\text{right},i}}^{-1}(L) = L - (-1)^{(n-1)(n-2)/2}I(L_{\text{right},i}, L)L_{\text{right},i},$$

we obtain the statement. \qed

![Figure 7](image7.png)

![Figure 8](image8.png)

Figure 7. $h_{L_{\text{right},k}}$ is the monodromy operator along $\tau_k$.

Figure 8. $h_{L_{\text{right},k}}$ is the monodromy operator along $\tau_k$.

Proposition 6.17. For each $j = 1, \ldots, \tilde{\mu}_n$, let $\Gamma_j^0(u) := \lim_{s \to 0} \Gamma_{\text{right},j}(u)$ be the element in $H_n(\mathbb{C}^n, \text{Re}(\tilde{f}_n/u) \gg 0; \mathbb{Z})$. The central connection matrix $C^\phi$ is given by

$$C^\phi = \left( C^\phi_{\kappa j} \right), \quad C^\phi_{\kappa j} = \frac{1}{(2\pi i)^2} \sum_{\lambda \in \mathcal{F}_n} \eta^{\lambda \kappa} \int_{\Gamma_j^0(1)} \tilde{f}_n \zeta^{(n)}_\lambda.$$  \quad (6.32)
Proof. Obviously, we have $Y_{\infty}(u)|_{s=0} = u^{-\bar{Q}(n)}$. Therefore, we have
\[
u^{-\bar{Q}(n)} \cdot C^\phi = Y_{\infty}(u)|_{s=0} \cdot C^\phi = Y_{\text{right}}(u)|_{s=0} = \left(\frac{1}{(2\pi u)^{\frac{n}{2}}} \sum_{\lambda \in I_n} \eta^{\lambda_n} \int_{\Gamma_j(u)} e^{-\frac{f_n(x)}{u}} \xi_{\lambda}^{(n)}\right).
\]
Since $\tilde{f}_n$ is weighted homogeneous, we have
\[
u^{-\frac{n}{2}} \int_{\Gamma_j(u)} e^{-\frac{f_n(x)}{u}} \xi_{\lambda}^{(n)} = \nu^{\tilde{Q}(n)} \int_{\Gamma_j(1)} e^{-\tilde{f}_n(x)} \xi_{\lambda}^{(n)},
\]
and hence the statement follows.
\[\square\]

Corollary 6.18. Let $\{L_i^{(n)}\}_{i=1}^{\tilde{\mu}_n}$ be the cycles constructed in Section 5. We have
\[\mathcal{S}(L_i^{(n)}, L_j^{(n)}) = \chi_{ij}^{(n)}, \quad i,j = 1, \ldots, \tilde{\mu}_n.
\]
Proof. Proposition 6.16 implies that $\mathcal{S}(L_i^{(n)}, L_j^{(n)}) = (U^T S^0 U)_{ij}$ at the limit $\phi \to 0$, where $U = (U_{ij})$ is the change of basis matrix with respect to $\{L_1^{(n)}, \ldots, L_{\tilde{\mu}_n}\}$ and $\{L_1, \ldots, L_{\tilde{\mu}_n}\}$. By Theorem 3.8 and Proposition 6.17, we have $C^0U = (2\pi)^{-\frac{1}{2}}\chi_{1}^{(n)}$. Hence, it follows from Proposition 6.13 and Theorem 3.4 that
\[U^T S^0 U = U^T (C^0)^T \mathbf{e} \left[\frac{1}{2} \bar{Q}(n)\right] \eta^{(n)} C^0 U
\]
\[= \left(\frac{1}{(2\pi)^{\frac{n}{2}}} \chi_{1}^{(n)}\right)^T \mathbf{e} \left[\frac{1}{2} \tilde{Q}(n)\right] \eta^{(n)} \left(\frac{1}{(2\pi)^{\frac{n}{2}}} \chi_{1}^{(n)}\right)
\]
\[= \chi^{(n)}.
\]
\[\square\]

Therefore, we obtain the commutativity of the following diagram:
\[\begin{array}{c}
(K_0(\text{HMF}_{S_{\tilde{n}}}^{L_{\tilde{n}}}(f_n)), \chi) \xrightarrow{\cong} (H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}), \mathcal{S}), \\
(\Omega_{f_n, G_{f_n}}; (2\pi \sqrt{-1})^{-n} \chi_{1}^{(n)}) \downarrow \cong \\
(\Omega_{f_n, G_{f_n}}; (2\pi \sqrt{-1})^{-n} \mathcal{S}_{f_n, G_{f_n}}) \xrightarrow{\text{mir}} (\Omega_{\tilde{f}_n}; (2\pi \sqrt{-1})^{-n} \mathcal{S}_{\tilde{f}_n})
\end{array}
\]
where $\mathcal{S}$ denotes the Seifert form, and the $\mathbb{C}$-bilinear forms $\mathcal{S}_{f_n, G_{f_n}}$ and $\mathcal{S}_{\tilde{f}_n}$ are defined by
\[\mathcal{S}_{f_n, G_{f_n}}(\xi_1, \xi_2) = J_{f_n, G_{f_n}} \left(\mathbf{e} \left[\frac{1}{2} N^{(n)}\right] \xi_1, \xi_2\right), \quad \xi_1, \xi_2 \in \Omega_{f_n, G_{f_n}},
\]
\[\mathcal{S}_{\tilde{f}_n}(\xi_1, \xi_2) = J_{f_n} \left(\mathbf{e} \left[\frac{1}{2} N\right] \xi_1, \xi_2\right), \quad \xi_1, \xi_2 \in \Omega_{f_n},
\]
$N^{(n)}$ and $N$ are regarded as elements of $\text{End}_{\mathbb{C}}(\Omega_{f_n, G_{f_n}})$ and $\text{End}_{\mathbb{C}}(\Omega_{\tilde{f}_n})$, respectively.

Since the intersection form is the symmetrized Seifert form, we have $I = S^\phi + (S^\phi)^T$ by Proposition 6.16. Hence, we obtain the formula between the intersection form and periods introduced by K. Saito.
**Proposition 6.19.** For each \( w \in \mathbb{R}_{>0} \), we have the Saito’s formula of the intersection form (see [S-K] Section 3.4, Theorem) :

\[
I(L_i, L_j) = \frac{(-1)^{\frac{n(n+1)}{2}}}{(2\pi\sqrt{-1})^{n-1}} \sum_{\lambda, \kappa \in I_n} \int_{I_i(w)} \frac{\zeta^{(n)}_{\lambda}}{df_n} \cdot \eta^{\lambda} \cdot w^{\partial_{w}^{n-1}} \int_{I_j(w)} \frac{\zeta^{(n)}_{\kappa}}{df_n}, \quad i, j = 1, \ldots, \tilde{\mu}_n,
\]

where \( L_i(w), i = 1, \ldots, \tilde{\mu}_n \) is the image of \( L_i \) by the isomorphism \( H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \cong H_{n-1}(\tilde{f}_n^{-1}(w); \mathbb{Z}) \) induced by the parallel transformation and \( \frac{\zeta^{(n)}}{df_n} \) denotes the Gelfand–Leary form defined by

\[
\zeta^{(n)}_{\lambda} = df_n \wedge \frac{\zeta^{(n)}_{\lambda}}{df_n}.
\]

**Proof.** By the definition, the intersection matrix \( I \) is given by

\[
I = (-1)^{\frac{n(n+1)}{2}} \left( S^\phi + (-1)^{n-1}(S^\phi)^T \right)
= (-1)^{\frac{n(n+1)}{2}} \left( (C^\phi)^T e^{\left[ \frac{1}{2} \tilde{Q}^{(n)} \right]} \eta^{(n)} C^\phi + (-1)^{n-1}(C^\phi)^T \eta^{(n)} e^{\left[ \frac{1}{2} \tilde{Q}^{(n)} \right]} C^\phi \right)
= (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^n C^T \eta^{(n)} \left( e^{\left[ \frac{1}{2} N^{(n)} \right]} - e^{\left[ \frac{1}{2} N^{(n)} \right]} \right) C^\phi
= (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^n \cdot 2\pi (C^\phi)^T \eta^{(n)} \frac{\sin \pi N^{(n)}}{\pi} C^\phi.
\]

Therefore, by the Euler’s reflection formula and the inverse Laplace transforms, we have

\[
I(L_i, L_j) = \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi\sqrt{-1})^{n-1}} \sum_{\lambda, \kappa \in I_n} \int_{I_i(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\lambda}} \cdot \eta^{\lambda} \cdot \frac{w^{n_{\lambda}-n+1}}{\Gamma(n_{\lambda} - n + 1)} \int_{I_j(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\kappa}}
= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi\sqrt{-1})^{n-1}} \sum_{\lambda, \kappa \in I_n} \frac{1}{2\pi\sqrt{-1}} \int_{c - \sqrt{-1} \infty}^{c + \sqrt{-1} \infty} e^{\frac{u}{\sqrt{\pi}} u^{\lambda}} \eta^{\lambda} \cdot \frac{1}{u} \cdot \int_{I_j(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\kappa}}
= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi\sqrt{-1})^{n-1}} \sum_{\lambda, \kappa \in I_n} \frac{1}{2\pi\sqrt{-1}} \int_{c - \sqrt{-1} \infty}^{c + \sqrt{-1} \infty} e^{\frac{u}{\sqrt{\pi}} u^{\lambda}} \left( \int_{I_j(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\kappa}} \right) \left( \int_{I_j(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\lambda}} \right)
= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi\sqrt{-1})^{n-1}} \sum_{\lambda, \kappa \in I_n} \frac{1}{2\pi\sqrt{-1}} \int_{c - \sqrt{-1} \infty}^{c + \sqrt{-1} \infty} e^{\frac{u}{\sqrt{\pi}} u^{\lambda}} \left( \int_{I_j(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\kappa}} \right) \left( \int_{I_j(1)} e^{-\tilde{f}_n \zeta^{(n)}_{\lambda}} \right)
= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi\sqrt{-1})^{n-1}} \sum_{\lambda, \kappa \in I_n} \frac{1}{2\pi\sqrt{-1}} \int_{I_i(w)} \frac{\zeta^{(n)}_{\lambda}}{df_n} \cdot \eta^{\lambda} \cdot w^{\partial_{w}^{n-1}} \int_{I_j(w)} \frac{\zeta^{(n)}_{\kappa}}{df_n}.
\]
Here, since $\tilde{f}_n$ is weighted homogeneous, note also that
\[
\int_{\Gamma^n_i(u)} e^{-\frac{i}{w} \zeta^{(n)}} = u^n \int_{\Gamma^n_i(1)} e^{-\frac{i}{w} \zeta^{(n)}} = \int_0^{\infty} e^{-\frac{u}{w}} \left( \int_{L_i(w)} \frac{\zeta^{(n)}}{dw} \right) dw
\]
for $\Re(u) > 0$. 

A distinguished basis can be chosen in other ways. The Artin’s braid group $B_{\tilde{\mu}_n}$ on $\tilde{\mu}_n$-stands is a group presented by the following generators and relations:

**Generators:** $\{b_i \mid i = 1, \ldots, \tilde{\mu}_n - 1\}$

**Relations:** $b_i b_j = b_j b_i$ for $|i - j| \geq 2$, $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for $i = 1, \ldots, \tilde{\mu}_n - 2$.

Consider the group $B_{\tilde{\mu}_n} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$, the semi-direct product of the braid group $B_{\tilde{\mu}_n}$ and the abelian group $(\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$, defined by the group homomorphism $B_{\tilde{\mu}_n} \to \mathfrak{S}_{\tilde{\mu}_n} \to \text{Aut}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$, where the first homomorphism is $b_i \mapsto (i, i + 1)$ and the second one is induced by the natural actions of the symmetric group $\mathfrak{S}_{\tilde{\mu}_n}$ on $(\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$.

The group $B_{\tilde{\mu}_n} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$ acts on the set of distinguished basis of vanishing cycles by

\[
b_i(L_1, \ldots, L_{\tilde{\mu}_n}) := (L_1, \ldots, L_{i-1}, L_{i+1}, h_{L_{i+1}}(L_i), L_{i+2}, \ldots, L_{\tilde{\mu}_n}),
\]

\[
b_i^{-1}(L_1, \ldots, L_{\tilde{\mu}_n}) := (L_1, \ldots, L_{i-1}, h^{-1}_{L_i}(L_{i+1}), L_i, L_{i+2}, \ldots, L_{\tilde{\mu}_n}),
\]

\[
p_i(L_1, \ldots, L_{\tilde{\mu}_n}) := (L_1, \ldots, L_{i-1}, -L_i, L_{i+1}, \ldots, L_{\tilde{\mu}_n}),
\]

where

\[
h_{L_{i+1}}(L_i) = L_i - (-1)^{\frac{(n-1)(n-2)}{2}} I(L_i, L_{i+1}) L_{i+1},
\]

\[
h^{-1}_{L_i}(L_{i+1}) = L_{i+1} - (-1)^{\frac{(n-1)(n-2)}{2}} I(L_i, L_{i+1}) L_i.
\]

On the other hand, the group $B_{\tilde{\mu}_n} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$ acts on the set of sequences in $K_0(\text{HMF}_{S_n}^f(f_n))$

\[
\{([\mathcal{E}_1], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]) \mid (\mathcal{E}_1, \ldots, \mathcal{E}_{\tilde{\mu}_n}) \text{ is a full exceptional collection in HMF}_{S_n}^f(f_n)\}
\]

by mutations and parity transformations (cf. [BP] Proposition 2.1):

\[
b_i([\mathcal{E}_1], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]) := ([\mathcal{E}_1], \ldots, [\mathcal{E}_{i-1}], [\mathcal{E}_{i+1}], \mathbb{R}_{\mathcal{E}_{i+1}}\mathcal{E}_i, [\mathcal{E}_{i+2}], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]),
\]

\[
b_i^{-1}([\mathcal{E}_1], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]) := ([\mathcal{E}_1], \ldots, [\mathcal{E}_{i-1}], [\mathbb{L}_\mathcal{E}_{\mathcal{E}_{i+1}}\mathcal{E}_i], [\mathcal{E}_{i+2}], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]),
\]

\[
p_i([\mathcal{E}_1], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]) := ([\mathcal{E}_1], \ldots, [\mathcal{E}_{i-1}], -[\mathcal{E}_i], [\mathcal{E}_{i+1}], \ldots, [\mathcal{E}_{\tilde{\mu}_n}]),
\]

where

\[
[\mathbb{R}_{\mathcal{E}_{i+1}}\mathcal{E}_i] = [\mathcal{E}_i] - \chi(\mathcal{E}_i, \mathcal{E}_{i+1})[\mathcal{E}_{i+1}], \quad [\mathbb{L}_\mathcal{E}_{\mathcal{E}_{i+1}}\mathcal{E}_i] = [\mathcal{E}_{i+1}] - \chi(\mathcal{E}_i, \mathcal{E}_{i+1})[\mathcal{E}_i],
\]

and $p_i$ is the $i$-th generator of $(\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n}$.
Therefore, if a distinguished basis of vanishing cycles \( (L_1, \ldots, L_{\tilde{\mu}_n}) \) is homological mirror dual to a full exceptional collection \( ([E_1], \ldots, [E_{\tilde{\mu}_n}]) \), then we should have \( S(L_i, L_j) = \chi(E_i, E_j) \) in order for \( B_{\tilde{\mu}_n} \rtimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n} \)-actions to be compatible.

**Proposition 6.20 ([G]).** The group \( B_{\tilde{\mu}_n} \rtimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n} \) acts transitively on the set of distinguished bases of vanishing cycles. □

Varolgunes shows that there exist cycles \( L_1, \ldots, L_{\tilde{\mu}_n} \in H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) = H_{n-1}(Y_{1,0}; \mathbb{Z}) \), which is isomorphic to \( H_{n-1}(Y_{\omega_0}; \mathbb{Z}) \) by the parallel transport, such that

\[
(-1)^{\frac{n(n-1)}{2}} I(L_j, L_i) = \chi_{ij}^{(n)}, \quad i < j,
\]

and \( (L_{\tilde{\mu}_n}, \ldots, L_1) \) forms a usual (clockwise) distinguished basis (see [V, Theorem 1.3]).

**Corollary 6.21.** There exists an element \( B \in B_{\tilde{\mu}_n} \rtimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n} \) such that

\[
B^T S^0 B = \chi^{(n)}, \quad C^0 B = \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\lambda \in I_n} \eta_{\lambda}^{n} \int_{\Gamma_j^0(1)} e^{-\tilde{f}_n \gamma^{(n)}_{\lambda}} \right), \tag{6.33}
\]

where \( \Gamma_j^0(1) \) is the image of \( L_j \) under the isomorphism \( H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \cong H_n(\mathbb{C}^n, \text{Re}(\tilde{f}_n) \gg 0; \mathbb{Z}) \).

**Remark 6.22.** \( S := B^T S^0 B \) and \( C := C^0 B \) are also the Stokes matrix and the central connection matrix of the Frobenius manifold with respect to some chosen data.

It is natural to expect that the cycles \( \{L_1^{(n)}, \ldots, L_{\tilde{\mu}_n}^{(n)}\} \) on \( H_{n-1}(\tilde{f}_n^{-1}(1); \mathbb{Z}) \) constructed in Section 5 can be obtained from those given by Varolgunes by the action of an automorphism of \( \tilde{f}_n^{-1}(1) \) and \( B_{\tilde{\mu}_n} \rtimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}_n} \). In particular, we expect that a set of cycles \( \{L_1^{(n)}, \ldots, L_{\tilde{\mu}_n}^{(n)}\} \) forms a distinguished basis of vanishing cycles. In fact, it is obvious for the case of \( n = 1 \) (see Lemma 5.1).

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