Research Article

Resolution of the Min-Max Optimization Problem Applied in the Agricultural Sector with the Estimation of Yields by Nonparametric Statistical Approaches

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The ultimate objective of the problem under study is to apply the min-max tool, thus making it possible to optimize the default risks linked to several areas: the agricultural sector, for example, which requires the optimization of the default risk using the following elements: silage crops, annual consumption requirements, and crops produced for a given year. To minimize the default risk in the future, we start, in the first step, by forecasting the total budget of agriculture investment for the next 20 years, then distribute this budget efficiently between the irrigation and construction of silos. To do this, Bangladesh was chosen as an empirical case study given the availability of its data on the FAO website; it is considered a large agricultural country in South Asia. In this article, we give a detailed and original in-depth study of the agricultural planning model through a calculating algorithm suggested to be coded on the R software thereafter. Our approach is based on an original statistical modeling using nonparametric statistics and considering an example of a simulation involving agricultural data from the country of Bangladesh. We also consider a new pollution model, which leads to a vector optimization problem. Graphs illustrate our quantitative analysis.

1. Introduction

The study is based on the quantification of the risk of having the need exceeding the production and the quantity of the production already ensiled; the same quantification will be applied in the case where the production of a given year exceeds both the need as well as the capacity of the silos for the same year; the idea is to calculate these risks of faults over 20 years in the future, based on the total investment amounts allocated for irrigation and the construction of the silos planned over these 20 years, via 3 calculation methods simulated in N iterations (N distributions of the amounts of irrigation and construction of the silos), and optimize them via a vector Pareto optimization algorithm for faults calculated by considering the pollution and those calculated without the assumption of the pollution; the strategies (belonging to the Pareto front) said optimal strategies will be linked to their amount of investment in irrigation and construction of silos affected and based on the total of these planned investment amounts. Therefore, it is up to the decision-maker to choose a strategy for allocating these investment amounts, among all optimal strategies retained (strategies belong to the Pareto front).

2. The Problem Schematization

2.1. The General Idea of the Work. Minimize the difference between the production $P(n)$, the requirement $\phi(n)$, and the quantity to be removed or ensiled $Q(n)$ after having maximized the risk through 3 scenarios that will be well explained after.

The variables

$$S_{tot} = S(n) + s(n),$$

$$P(n) = p(n) \times S(n) + q(n) \times s(n),$$

with $S_{tot}$ as the total cultivated area in year $n$, $S(n)$ the surface area of irrigated land with yield $p$ in year $n$, $s(n)$ the surface...
of nonirrigated land with yields $q$ in year $n$, $P(n)$ the total production in year $n$, $p(n)$ the yield of irrigated land, and $q(n)$ the yield of nonirrigated land.

Knowing the requirement $\varphi(n)$ of year $n$, the difference between the production $P(n)$ of year $n$ and the requirements can be explained as follows:

$$(1) \quad \text{Initial conditions: } G(2001), R(2001), S_p(2001), p(2001), q(2001), s(2001), P_{\text{irrig}}(2000), P_{\text{cons}}(2000), \text{ and } \varphi(2001): \text{ to fix by assumptions}
$$

(2) Total agricultural areas, areas of irrigated and nonirrigated land

The data was collected via the FAO site, selected country: Bangladesh (total agricultural land = 8.549 million ha and irrigated land areas) [1, 2].

The surface area of nonirrigated lands has been deducted as shown in Table 1.

(3) Irrigation and silo construction budgets: based on FAO data (if we consider that the unit costs of irrigation and silo construction are, respectively $c_y = 622$ $$/ha [3] \text{ and } c_c = 255$/tonne [4] (for the case of tank stores, source FAO)), and based on the total production of the crops and the irrigated land area, we will have Table 2.

It is also assumed that 75% of the total production is devoted to cereals; therefore, the total production can be approximated to the production of cereals.

(4) Irrigated and nonirrigated lands yields: based on FAO publications [5], the yields of irrigated land are recorded between 5 tonnes/ha and 13 tonnes/ha from an irrigation of $4500$ m$^3$/ha and from 0 tonne/ha to 5 tonnes/ha for nonirrigated lands; therefore, we go through a numerical simulation of the respective yield values of irrigated and nonirrigated lands to deduce the simulated productions, while assuming these yields, do not take into account the factor of pollution (the yields of the supposedly polluted land is to be deduced later) (see Table 3).

In order to determine the distribution function of the returns $p$ and $q$ in order to deduce the expectations, nonparametric methods will be the subject of the next part of this study.

4. Nonparametric Estimation of Yield Density: Theoretical Study

4.1. The Histogram Method. The basic idea is to segment the observations belonging to the interval $[a,b]$ into $m$ intervals with length $h = (b-a)/m$.

In addition, for $\forall x_i \in C_k = [L_{k-1}, L_k] [i \in \{1, \ldots, n\}, k \in \{1, \ldots, m\}$ and $L_k = L_{k-1} + h$.

$n$ is the number of the observed values in the interval $[a,b]$, and $x$ is a continued value that belongs to the same interval that we are looking for its probability formula.

The density function [6] associated to $x$ is in the following form:

$$f_H(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{nh} I_{[L_{k-1}, L_k]}(x_i), x \in C_k = \frac{\bar{p}_k}{h},$$

Such that $\bar{p}_k(x) = \sum_{i=1}^{n} \frac{1}{n} I_{[L_{k-1}, L_k]}(x_i), x \in C_k$.  

Therefore, the problem comes down to estimating the probability vector:

$$\hat{p} = \{\hat{p}_1, \ldots, \hat{p}_k, \ldots, \hat{p}_m\}. \quad (9)$$

While the question that arises is how to choose $h$, this comes down to choosing the number of intervals to have a well-smoothed distribution (do not fall in the case of a histogram called "oversmoothing" when $h$ is larger or the opposite case of a histogram called "undersmoothing" when $h$ tends towards zero).

(1) The quadratic risk of $\tilde{f}_H(x)$

The solution proposed in this sense is to establish a risk function $R(h)$ and minimize it as a function of the window $h$; the function adequate to this problem is that expressing the expectation of the quadratic deviation between $\tilde{f}_H(x)$ and $f(x)$:

$$\text{EQM}(x, h) = E\left(\left(\tilde{f}_H(x) - f(x)\right)^2\right)$$

$$= \left(E\left(\tilde{f}_H(x)\right) - f(x)\right)^2 + E\left(\tilde{f}_H(x) - E\left(\tilde{f}_H(x)\right)\right)^2$$

$$= \text{bias}^2(\tilde{f}_H(x)) + \text{variance}(\tilde{f}_H(x)). \quad (10)$$
such that
\[ E(\hat{f}_H(x)) = E\left( \frac{1}{nh} \sum_{i=1}^{n} I_{\frac{x_i}{nh} - 1 < \frac{x}{nh} < \frac{x_i}{nh} + 1} \right) = \frac{1}{nh} E\left( \sum_{i=1}^{n} I_{\frac{x_i}{nh} - 1 < \frac{x}{nh} < \frac{x_i}{nh} + 1} \right). \]  
(11)

However, \( \sum_{i=1}^{n} I_{\frac{x_i}{nh} - 1 < \frac{x}{nh} < \frac{x_i}{nh} + 1} = n_k \) and \( \sum n_k = n \); this says that \( n_k \sim \mathcal{B}(p_k, n) \): the law of succeeding an experiment with a probability \( p_k = n_i/n \), repeating \( n \) times.

Therefore, we will have
\[ E(\hat{f}_H(x)) = \frac{n p_k}{nh} = \frac{p_k}{h}, \]  
(12)
which implies
\[ \text{bias}^2(\hat{f}_H(x)) = \left( E(\hat{f}_H(x)) - f(x) \right)^2 = \left( \frac{p_k}{h} - f(x) \right)^2, \]  
(13)
and that
\[ \text{variance}(\hat{f}_H(x)) = \frac{1}{(nh)^2} \text{variance}\left( \sum_{i=1}^{n} I_{\frac{x_i}{nh} - 1 < \frac{x}{nh} < \frac{x_i}{nh} + 1} \right) \]  
\[ = \frac{n p_k(1-p_k)}{(nh)^2} = \frac{p_k(1-p_k)}{nh}. \]  
(14)

Finally, the expectation of the quadratic deviation of \( \hat{f}_H(x) \) is written as follows:
\[ \text{EQM}(x, h) = \left( \frac{p_k}{h} - f(x) \right)^2 + \frac{p_k(1-p_k)}{nh}. \]  
(15)

(2) The quadratic risk of \( \hat{f}_H(x) \) integrated

For the function to be as a function of \( h \), the \( \text{EQM}(h, x) \) just mentioned, it will be convenient to integrate the function over the interval \([a, b] \):

\[ \text{IntegratedRisk}(h) = R(h) = \int_{a}^{b} \text{EQM}(x)dx \]  
\[ = \int_{a}^{b} \left( \frac{p_k}{h} - f(x) \right)^2 dx + \frac{p_k(1-p_k)}{nh} \int_{a}^{b} dx, \]  
(16)

It seems that the speed of convergence of \( R(h) \) depends on \( nh \), which leads us to reformulate the \( R(h) \) and going through the limited expansion of \( f(x) \):
\[ E(\hat{f}_H(x)) - f(x) = \frac{p_k}{h} - f(x) = \int_{\xi}^{\eta} f(t)dt \]  
\[ = \frac{1}{h} \int_{\xi}^{\eta} (f(t) - f(x))dt. \]  
(17)

Passing through the limited development of the first Taylor degree at point \( x \), we will have
\[ f(t) - f(x) = (t-x)f'(x) + o(t-x). \]  
(18)

Therefore,
\[ \int_{\xi}^{\eta} (f(t) - f(x))dt = \int_{\xi}^{\eta} (t-x)f'(x)dt + o(h^2). \]  
(19)

**Proof of Equation 19.** We consider that \( v(t) = (f(t) - f(x)) - (t-x)f'(x) \).

Moreover, based on the definition of a negligible function \[7\] on a point that says the following: let a function \( f : \mathbb{I} \rightarrow \mathbb{R} \) and \( f \) not vanish at \( I\backslash\{a\} \); \( f \) is said to be negligible in the neighborhood of \( [a] \) in front of \( g \) if and only if
\[ \forall \varepsilon > 0 \exists \alpha > 0, \|x - a\| < \alpha \Rightarrow \|f(x)| < \varepsilon |g(x)|. \]  
(20)

Since \( v(t) \) is negligible in front of \( (t-x) \), then
\[ \forall \varepsilon > 0 \exists \alpha > 0, \|t-a\| < \alpha \Rightarrow \|v(t)| < \varepsilon |t-x| \].  
(21)

\[ \forall \varepsilon > 0 \exists \alpha > 0 \text{ and } |t-a| < \alpha \Rightarrow |v(t)| < \varepsilon |(t-x)| < \varepsilon h \]  
because \((t, x) \in \mathbb{I}_k\).

Therefore,
\[ \int_{\xi}^{\eta} |v(t)|dt < \int_{\xi}^{\eta} \varepsilon dt = \frac{\varepsilon h^2}{2} = \varepsilon 'h^2. \]  
(22)

However, we have
\[ \int_{\xi}^{\eta} v(x)dx < \int_{\xi}^{\eta} |v(t)|dt. \]  
(23)

Therefore,
\[ \forall \varepsilon > 0 \exists \alpha > 0, \]  
\[ |t-a| < \alpha \Rightarrow \int_{\xi}^{\eta} v(t)dt < \varepsilon h^2. \]  
(24)
Finally,\[ \int v(t)\,dt = \int f(t) - f(x) - (t - x)f'(x)\,dt = o(h^2), \tag{25} \]

which implies
\[ \int (f(t) - f(x))\,dt = \int (t - x)f'(x)\,dt + o(h^2). \tag{26} \]

After the calculation is done, we will have
\[ \int (t - x)f'(x)\,dt = \left(h^2 \left(k - \frac{1}{2}\right) - xh\right)f'(x) = h\left(h \left(k - \frac{1}{2}\right) - x\right)f'(x), \tag{27} \]
\[ \int (t - x)f'(x)\,dt = h^2f'(x). \]

Finally, we will have
\[ E\left( f_H(x) - f(x) \right) = \frac{1}{n} \int f(t) - f(x)\,dt = \frac{1}{n}\left(h^2f'(x) + o(h^2)\right), \]
\[ E\left( f_H(x) - f(x) \right) = \left(hf'(x) + o(h)\right). \tag{28} \]

For its part, the variance is given by
\[ \text{variance}(f_H(x)) = \frac{p_k(1 - p_k)}{nh^2}, \]
\[ \text{variance}(f_H(x)) = \frac{1}{nh^2} \int f(t)\,dt \left(1 - \int f(t)\,dt\right). \tag{29} \]

If we approach the quantity \(1 - \int f(t)\,dt\) to 1, the variance can be expressed in the following form:
\[ \text{variance}(f_H(x)) = \frac{1}{nh^2}\left(hf'(x) + h^2f''(x) + o(h^2)\right) = \frac{f(x)}{nh} + \frac{f'(x)}{n} + o\left(\frac{1}{n}\right). \tag{30} \]

Finally,
\[ \text{EQM}(x, h) = \left(hf'(x) + o(h)\right)^2 + \frac{f(x)}{nh} + \frac{f'(x)}{n} + o\left(\frac{1}{n}\right) = \left(hf'(x)\right)^2 + 2f'(x)\,o(h^2) + o(h^2) + \frac{f(x)}{nh} + \frac{f'(x)}{n} + o\left(\frac{1}{n}\right) + o(h^2) + 2f'(x)\,o(h^2). \tag{31} \]

However, the quantification of this risk function, in other words, the determination of optimal \(h\), must be carried out over the entire interval \([a, b]\); therefore, the integral is applied only to the nonnegligible part of \(\text{EQM}(x, h)\):
\[ \text{IntegratedRisk}(h) = R(h) = \int_a^b \text{EQM}(x, h)\,dx = \int_a^b \left(hf'(x)\right)^2 + f(x)\,dx + 1/nh + \frac{f(x)}{nh} + \frac{f'(x)}{n} + f(b) - f(a). \tag{32} \]

The optimal \(h\) corresponds to finding
\[ \text{min}(R(h)). \tag{33} \]

This leads us to look for the argument of the null derivative of \(R(h)\).
\[ \frac{\partial}{\partial h}(R(h)) = \frac{\partial}{\partial h} \left( \int_a^b \left(hf'(x)\right)^2\,dx + (1/nh) + f(b) - f(a)/n \right), \]
\[ \frac{\partial}{\partial h}(h_{\text{opt}}) = 2h_{\text{opt}} \int_a^b \left(f'(x)\right)^2\,dx - \frac{1}{nh_{\text{opt}}} = 0, \tag{34} \]
\[ 2h_{\text{opt}} \int_a^b \left(f'(x)\right)^2\,dx - \frac{1}{n} = 0. \]

Therefore,
\[ h_{\text{opt}} = \left( \frac{1}{2n \int_a^b \left(f'(x)\right)^2\,dx} \right)^{1/3}. \tag{35} \]

However, we do not have the distribution of \(f\), to deduce \(\int_a^b \left(f'(x)\right)^2\,dx\).
To do this, it will be necessary to go through the “cross-validation” estimator.
(3) The “cross-validation” formula of the estimation

It seems that, according to the formulation of Integrate-dRisk, the calculation of \( \int_0^1 (f'_{\hat{t}}(x))^2 \, dx \) turns out to be difficult in practice; hence, the idea is to calculate \( d \) \( R(h) \) \( \rightarrow \) \( f^2(t) \, dt \); this amounts to calculating the formula “cross-validation” (see Figure 1):

\[
\text{CV}(h) = IR(h) - \int f^2(t) \, dt = \int E(\hat{f}_H(x) - f(x))^2 \, dx - \int f^2(x) \, dx \\
= \int E(\hat{f}_H^2(x)) \, dx - 2 \int E(\hat{f}_H(x))f(x) \, dx \\
+ \int f^2(x) \, dx - \int f^2(x) \, dx \\
= \left( \int \left( E(\hat{f}_H^2(x)) + \text{var} (\hat{f}_H(x)) \right) \, dx \right) \\
- 2 \int E(\hat{f}_H(x))f(x) \, dx \\
= \sum_k \left( \int \left( E(\hat{f}_H^2(x)) + \text{var} (\hat{f}_H(x)) \right) \, dx \right) \\
- 2 \int E(\hat{f}_H(x))f(x) \, dx \right). \\
(36)
\]

If we estimate \( E(\hat{f}_H(x)) = p_k/h \), we will have

\[
\text{CV}(h) = \sum_k \left( \int \left( \frac{p_k}{h} \right)^2 + \frac{p_k(1-p_k)}{nh^2} \right) \, dx - 2 \int \frac{p_k}{h} f(x) \, dx \\
= \sum_k \left( \frac{p_k^2}{h} \right) + \frac{p_k(1-p_k)}{nh} - 2 \frac{p_k^2}{h} \right) \\
= \sum_k \left( \frac{p_k(1-p_k)}{nh} - \frac{p_k^2}{h} \right) = \frac{1}{nh} - \frac{\sum_k p_k^2 (1+n)}{nh}. \\
(37)
\]

Finally, we have

\[
h_{\text{CV}} = \text{argmin} (\text{CV}(h)). \\
(38)
\]

4.2. The Kernel Method. The kernel method is applied if the continuity of the distribution function is ensured (the distribution function belongs to class functions \( C^1 \)). Let us consider in a first step the following estimate: \( F_M(x) = 1/M \sum_{i=1}^M I(X_i \leq x) \), \( \forall X_i \in [a, b] \) and the \( X_i \) are i.i.d., \( d \forall i = 1, \cdots, M. \)

Let \( X_i^I = I(X_i \leq x) \).

The strong law of large numbers indicates that \( (1/M) \sum_{i=1}^M X_i^I \to E(X^I) \) almost certainly; \( E(X^I) \) calculation:

\[
E_X^X = E(1(X \leq x)) = \int_0^x \frac{1}{\int_{-\infty}^t f(t) \, dt} \\
= P(X \leq x) = F(x). \\
(39)
\]

And finally, \( F_M(x) \to F(x) \) almost certainly. The derivative of the function \( F(x) \) is given by

\[
f(x) = \frac{F(x+h) - F(x-h)}{2h}, \quad \text{with } h \to 0^+. \\
(40)
\]

This implies

\[
f_M^R(x) = \frac{F_M(x+h) - F_M(x-h)}{2h}. \\
(41)
\]

Such an estimator is known as the Rosenblatt estimator.

\[
f_M^R(x) = \frac{F_M(x+h) - F_M(x-h)}{2h} = \frac{\sum_{i=1}^M I(x-h < X_i < x-h)}{2hM} \\
= \frac{1}{2hM} \sum_{i=1}^M I(x-h < X_i < x-h) = \frac{1}{hM} \sum_{i=1}^M K_{\beta}(X_i - x/h). \\
(42)
\]

Generally, we put \( [8] \)

\[
\hat{f}_K^R(x) = \frac{1}{hM} \sum_{i=1}^M K_{\beta}(X_i - x/h). \\
(43)
\]

\( K \) is called the kernel of the estimate; generally, this kernel can be illustrated following different paces \([9]\):

(i) The triangular kernel: \( K(u) = (u+1)I_{[0,1]}(u) \)

(ii) The parabolic kernel: \( K(u) = (3/4)(1-u^2)I_{[0,1]}(u) \)

(iii) The Gaussian kernel: \( K(u) = (1/\sqrt{2\pi}) \exp(-u^2/2) \)

(1) The quadratic risk of \( \hat{f}_K^R(x) \)

The mean square deviation as mentioned in the histogram approach is defined as follows:

\[
\text{EIQ}_{X}(x,h) = \text{bias}^2 (\hat{f}_K^R(x)) + \text{variance} (\hat{f}_K^R(x)). \\
(44)
\]

Definition 1. Let \( T \) be an interval in \( R \) and the pair \( (\beta, L) \in R^2 \),
the class Hölder \( \mathcal{H}(\beta, L) \) defined on \( T \) is the set of differentiable functions \( l = \lfloor \beta \rfloor \) which satisfy
\[
\left| f'(x) - f'(x') \right| \leq L |x - x'|^{\beta-1} \forall x, x' \in T. \tag{45}
\]

**Definition 2.** Let \( l \geq 1 \) be an integer; we would say that \( K : R \rightarrow R \) is a kernel function of order \( l \) [10] if the functions \( v \rightarrow v^l K(u), j = 0, \ldots, l \), are integrable and which satisfy
\[
\int K(v) dv = 1,
\int v^l K(v) dv = K, \forall j = 1, \ldots, l. \tag{46}
\]

In the following, we will focus on the kernel function of order 2.

**Definition 3 (Cauchy-Schwarz inequality).** If \( f \) and \( g \) are two integral functions on \( T \), then the absolute value of the integral of their product satisfies the following inequality:
\[
\int_T f(x) \ast g(x) dx \leq \sqrt{\int_T f^2(x) dx} \ast \sqrt{\int_T g^2(x) dx}. \tag{47}
\]

Let us calculate the upper bound of the bias [11, 12]:
\[
\text{bias} \left( \tilde{f}_K(x) \right) = E \left( \tilde{f}_K(x) \right) - f(x) = \frac{1}{h} \int K \left( \frac{y-x}{h} \right) f(y) dy - f(x)
= \int K(u)(uh + x) du - f(x).
\]

(48)

Let \( p(\beta, L) = \{ f \in \mathcal{H}(\beta, L) / f \geq 0, \int f(y) dy = 1 \forall y \in R \} \).

Going through a Taylor expansion of order 2 of \( p(uh + x) \), we have
\[
f(uh + x) = f(x) + uh \ast f'(x) + \left( \frac{uh}{2} \right)^2 f''(x + \tau uh), \tag{49}
\]

with \( 0 < \tau < 1 \).

\[
\int K(u)(uh + x) du = \int K(u) f(x) du + \int K(u) uh \ast f'(x) du + \int K(u) \left( \frac{uh}{2} \right)^2 f''(x + \tau uh) du.
\]

(50)

However, we previously assumed that the kernel function
is of order 2, and therefore,

\[
\int K(u)(uh + x)du - p(x) = \int K(u)\left(\frac{(uh)^2}{2}f(x + uh)du - \frac{(uh)^2}{2}f(x)du + \frac{(uh)^2}{2}p^{-}(x)du\right).
\]

(51)

The hypothesis of \( p \in \mathcal{P}(\beta, L) \) implies

\[
K(u)\left| f^{-}(x + uh) - f^{-}(x) \right| \leq |K(u)| |f^{-}(x + uh) - f^{-}(x)| \leq L|K(u)||uh|^{\beta-2}.
\]

(52)

So

\[
\text{bias}^{2}\left(\hat{f}_{K}(x)\right) \leq \int |K(u)|L|uh|^{\beta-2}du + \int K(u)\frac{(uh)^2}{2}f^{-}(x)du.
\]

(53)

Following Cauchy-Schwartz’s theorem, we will have the following inequality:

\[
\left(\int |K(u)|L|uh|^{\beta-2}du\right)^2 \leq \int |K(u)|^2du \int (L|uh|^{\beta-2})^2du \leq \int |K(u)|^2du \int L^2(h)^{2\beta-4} |u|^{2\beta-4}du.
\]

(54)

Knowing that the calculation of the expectation is done on the interval \([a, b]: u > 0 \) if \( y > x \) and vice versa.

\[
\int_{\frac{a+b}{2}+y}^{\frac{a+b}{2}+x} |u|^{2\beta-4}du = \int_{\frac{a+b}{2}+y}^{\frac{a+b}{2}+x} \left(-\frac{(uh)^2}{2}f(x + uh)du \right) + \int_{\frac{a+b}{2}+x}^{\frac{a+b}{2}+y} \left(-\frac{(uh)^2}{2}f(x)du \right) + \int_{\frac{a+b}{2}+x}^{\frac{a+b}{2}+y} \left(-\frac{(uh)^2}{2}p^{-}(x)du \right).
\]

(55)

Finally,

\[
\text{bias}^{2}\left(\hat{f}_{K}(x)\right) \leq \int |K(u)|^2du \int L^2(h)^{2\beta-4} \left\{ -\frac{1}{2\beta-5} \left(\frac{a-x}{h}\right)^{2\beta-5} \right\} + \frac{1}{2\beta-5} \left(\frac{b-x}{h}\right)^{2\beta-5} \right\} + \int K(u)\frac{(uh)^2}{2}f^{-}(x)du.
\]

(56)

Likewise, the variance of \( \hat{f}_{K}(x) \) is calculated as follows:

\[
\text{variance}(\hat{f}_{K}(x)) = E(\hat{f}_{K}(x)) - E^{2}(\hat{f}_{K}(x)).
\]

(57)

The maximum of a probability function is 1:

\[
\text{variance}(\hat{f}_{K}(x)) \leq \frac{1}{nh} \int K^2(u)f(x + hu)du.
\]

(58)

Put \( \int K^2(u)du = c_1 \).

Finally,

\[
\text{EQM}(x, h) \leq c_1 \left(\frac{1}{nh} + L^2(h)^{2\beta-4} \left\{ -\frac{1}{2\beta-5} \left(\frac{a-x}{h}\right)^{2\beta-5} \right\} + \frac{1}{2\beta-5} \left(\frac{b-x}{h}\right)^{2\beta-5} \right\} + \int K(u)\frac{(uh)^2}{2}f^{-}(x)du.
\]

(59)

Computing \( h_{\text{opt}} \) amounts to minimizing the upper bound of \( \text{EQM}(x, h) \).

\[
(2) \text{ The quadratic risk of } \hat{f}_{K}(x) \text{ integrated}
\]

Computing \( h_{\text{opt}} \) from \( R(h) \) amounts to deducing it from the following integral:

\[
\int c_1 \left(\frac{1}{nh} + L^2(h)^{2\beta-4} \left\{ -\frac{1}{2\beta-5} \left(\frac{a-x}{h}\right)^{2\beta-5} \right\} + \frac{1}{2\beta-5} \left(\frac{b-x}{h}\right)^{2\beta-5} \right\} + \int K(u)\frac{(uh)^2}{2}f^{-}(x)du \right\} dx,
\]

(60)
Figure 2: Algorithm of the calculation of strategies ($J_i, J_i^*$).
because

\[
R(h) = \int_{\mathbb{R}^d} EQM(x,h) \, dx \\
\leq \sum_{i} \left( \frac{1}{nh} + L^2(h)^{2\beta-4} \cdot \left( - \frac{1}{2\beta-5} \left( \frac{a-x}{h} \right)^{2\beta-5} \right) + \frac{1}{2\beta-5} \left( \frac{b-x}{h} \right)^{2\beta-5} \right) + \int K(u) \frac{(uh)^2}{2} f^2(x) \, du \, dx.
\]

(61)

(3) The "cross-validation" formula of the estimation

As mentioned previously, the cross-validation formula "CV (h)" of the estimation of distributions is defined by

\[
CV(h) = IR(h) - \int f^2(x) \, dx,
\]

(62)
which implies that

\[ CV(h) = E\left( \int \tilde{f}_K(x)^2 dx \right) - 2E\left( \int \tilde{f}_K(x)f(x) dx \right). \quad (63) \]

Let us say

\[ \tilde{f}_K(x) = \tilde{p}_{M-1} = \frac{1}{(M - 1)h} \sum_{j=1}^{M} K\left( \frac{x_j - x}{h} \right), \quad (64) \]
and show that

\[ \hat{\beta} = \frac{1}{M} \sum_{p_{M-i}} \]

is an unbiased estimator of \( E(\int f_K(x)f(x)dx) \).

This amounts to demonstrating that

\[ \beta = E\left( \int f_K(x)f(x)dx \right) = E\left( \frac{1}{M} \sum_{p_{M-i}} \right) = E\left( \hat{\beta} \right). \]  

Under the assumption that \( X_j \) are i.i.d., therefore we have

\[ E\left( \frac{1}{M} \sum_{p_{M-i}} \right) = E\left( \hat{\beta} \right) = E\left( \hat{P}_{M-i} \right) \]

\[ = E\left( \int \frac{1}{(M-1)h} \sum_{i=1}^{M} K \left( \frac{x_i - z}{h} \right) f(z)dz \right) \]

\[ = \int \left( \int \frac{1}{h} K \left( \frac{x - z}{h} \right) f(z)dz \right) f(x)dx. \]

Likewise,

\[ \beta = E\left( \int f_K(x)f(x)dx \right) \]

\[ = E\left( \int \frac{1}{(M-1)h} \sum_{i=1}^{M} K \left( \frac{x_i - z}{h} \right) f(z)dz \right) \]

\[ = \int \left( \int \frac{1}{h} K \left( \frac{x - z}{h} \right) f(z)dz \right) f(x)dx. \]

5. Theoretical Calculation of Strategies

Recall that the default risk \( \Delta(n) \), for a given year \( n \), "as discussed by Moiseev [13]," is calculated as follows:

\[ \Delta(n) = (p(n) + s(n) + q(n) - \psi(n)) - Q(n). \]

\[ Q(k) = \begin{cases} \min \{ P(n) - \psi(n), G(n) - R(n-1) \}, & \text{if } P(n) \geq \psi(n), \\ \max \{ P(n) - \psi(n), -R(n) \}, & \text{if } P(n) < \psi(n). \end{cases} \]

The nonparametric estimate of yield expectations will allow the calculation of strategies for the distribution of budgets between irrigation and the construction of silos over the next \( N \) years; these strategies are given by

\[ J_1 = \max \{ E(\Delta(n)) \} \]
The project strategy can be calculated in another way:

$$J_2 = \sum_{n=1}^{N} |E(\Delta(n))| \text{ with } \Delta(n) \neq 0.$$  \hspace{1cm} (71)

Strictly positive or strictly negative defaults do not have the same meaning. This leads us to calculate the budget allocation strategy using the following formula:

$$J_3 = \sum_{n=1}^{N} |E(\Delta(n))| \text{ with } \Delta(n) < 0.$$  \hspace{1cm} (72)

The problem of optimizing the min-max comes down to choosing the optimal strategy for the allocation of budgets that minimizes the risk of default; this amounts to finding the following scalar minimum:

$$\min \{ J_1, J_2, J_3 \}.$$  \hspace{1cm} (73)

In the case where the pollution factor is considered, the new default risk values will be obtained from the formula below:

$$\Delta^*(n) = (p^*(n) \ast S(n) + q^*(n) \ast \varphi(n)) - Q^*(n),$$
\[
Q^* (k) = \begin{cases} 
\min (P^*(n) - \varphi(n), G(n) - R^*(n - 1)), & \text{if } P^*(n) \geq \varphi(n), \\
\max (P^*(n) - \varphi(n), -R^*(n - 1)), & \text{if } P^*(n) < \varphi(n).
\end{cases}
\]

\[\text{Figure 13: Outcome of Figure 2 calculation of strategies } (J_1, J_2^*) \text{ under the R software.}\]
Figure 14: The outcome of Pareto optimization Figure 3 under the R software.
function of $\Delta(n)$ and that calculated as a function of $\Delta^*(n)$; this leads us to calculate:

$$J_1^* = \max \{E(\Delta^*(n))\}_{1 \leq n \leq N},$$

$$J_2^* = \frac{1}{N} \sum_{n=1}^{N} |E(\Delta^*(n))| \text{ with } \Delta^*(n) \neq 0,$$

$$J_3^* = \frac{1}{N} \sum_{n=1}^{N} |E(\Delta^*(n))| \text{ with } \Delta^*(n) < 0.$$  \hfill (75)

Then, minimize the following vector:

$$\min (J_i, J_i^*) \text{ with } i = 1, 2, 3.$$  \hfill (76)

### 6. Multiobjective Pareto Optimization (Pareto Front)

A multiobjective optimization problem presents itself, in general, as follows [14, 15]:

$$P = \begin{cases} 
\min (F(X) = (F_1(X), F_2(X), \cdots, F_n(X))), \\
\text{under constraints } g_j(x) \geq 0 \text{ for } j = 1, \cdots, m, \\
h_k(x), k = 1, \cdots, J.
\end{cases}$$  \hfill (77)

#### 6.1. Dominance Principle and Pareto Optimality

For a given order, relation $\prec_p$ in a space of dimension $p$ and let $Y = (Y_1, Y_2, \cdots, Y_p)$ and let $Z = (Z_1, Z_2, \cdots, Z_p)$ be two vectors in the decision space and $U(Y) = (U_1, U_2, \cdots, U_m), U$ ($Z) = (U_1', U_2', \cdots, U_m')$. We will say that $Y$ dominates $Z$ and we write $Y \prec_p Z$, if and only if [17]

$$\forall i \in 1, \cdots, p, \quad U_i \leq U_i'$$

$$\exists j \in 1, \cdots, p, \quad U_j < U_j'.$$  \hfill (78)

A solution $X' = (X_1', X_2', \cdots, X_n')$ of the multiobjective problem $P$ is said to be Pareto optimal [18] if there is no point of the decision space $X^* = (X_1^*, X_2^*, \cdots, X_n^*)$ which dominates $X'$ such that $F_i(X^*) \leq F_i(X)$ for each $i$ and $j$ which verifies that $F_j(X^*) < F_j(X)$. All of these non-dominated points in the Pareto sense constitute the Pareto front [19]. In our example, the vector of irrigation prices over the next 20 years will be the vector of decisions and the vector of objectives is the vector $(J_i, J_i^*)$.

Notice that “yes” means the corresponding identity holds and “no” means the identity does not hold.

Once the strategies are simulated, we will have a data frame of two columns and of length $\text{Nrows}_{\text{sim}}$ of $(J_i, J_i^*)$ (see Figure 2). Therefore and by applying the principle of dominance, the efficient strategies (Pareto Front strategies) are calculated by the algorithm of the Figure 3.

#### 7. Empirical Part

Following a Shapiro-Wilk test [20] of the Gaussian law, the $p$ value of the test ($>5\%$) indicates that the yields of irrigated and nonirrigated land follow a Gaussian law (Figure 4).

Under the assumption of the normal distribution of these returns, we choose to calculate their expectations through a nonparametric estimate of the density by the kernel method by the choice of a Gaussian kernel, through the following program in Figures 5 and 6.

Since the hypothesis of the normality of yields of nonirrigated and those of irrigated land is verified, the median and the mean must be equal, given the Gaussian law is a symmetrical law.

Through the “summary” command, we notice that the arithmetic mean is not significant for estimating the expectation because it is different from the median.

By using the kernel estimate, the average obtained corresponds perfectly to the median; this means that the estimate of the density by this method turns out to be robust for the calculation of the expected returns (mean = median = $E(q) = 2,215$ tonnes/ha for nonirrigated land and mean = median = $E(p) = 9,38$ tonne/ha for irrigated land).

We assume that the pollution factor can reduce the yields of irrigated and nonirrigated land by 12% and 20%, respectively. The average of the returns by the kernel

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**Figure 15:** The optimal Pareto values of $(J_i, J_i^*)$ simulated.
method makes it possible to deduce that $E(q^*(n)) = 1.77$ tonnes/ha and $E(p^*(n)) = 8.25$ tonnes/ha.

The yields $q(n)$ and $p(n)$ were simulated by the numerical “Box-Müller” method taking into account the intervals of the values of $q(n)$ and $p(n)$ according to the FAO site (from 0 to 5 tonnes/ha for nonirrigated land and more than 5 tonnes/ha for irrigated land).

The nonsimulated data series cover the period (area of irrigated land) 2001-2016 (source: FAO site); thereafter, the areas of irrigated land will be forecast over the period 2017-2036.

Total agricultural land = 8,549 million ha (to deduct the surface areas of nonirrigated land).

The total budget to be invested will be planned based on the ARIMA time series (after these values, the PACF and AFC functions practically cancel each other out).

To ensure the choice of the appropriate model, we proceed to build the AIC matrix (Akaike Information Criterion) of the ARIMA models $(p, d, q)$ with $i$ ranging from 1 to 3 and $j$ ranging from 1 to 3 and retain the model which corresponds to the minimum AIC index (Figure 10).

The AIC matrix confirms that the model to be retained is ARIMA $(2, 2, 1)$.

The forecast of this series under the software over the period 2017-2036 is represented by the graph in Figure 11.

The construction of the budgets allocated to irrigation and the construction of silos over the period 2001-2016 is based, respectively, on the unit prices $c_{x} = 622$/ha and $c_{x} = 258$/tonne; then, their forecasts (2017-2036) will be a distribution simulation of the series of investment budgets between irrigation and the construction of silos by digital simulation methods (we go through 4 distribution simulations of the total investment budget).

The product quantity requirements for the year over the period 2017-2036 will be evaluated; after the calculation of the “delta” faults under the software, we move on to calculate the $(J_i, J_j^*)$ with $i = 1, 2, 3$ over the period 2017-2036.

An example of calculation of $\Delta^*$ and $\Delta$ for the $Nrows_{ab} = 249$; the number of simulation of $(J_i, J_j^*)$ with $i = 1, 2, 3$ is...
\[ N_{\text{simule}} = \frac{249}{3} = 83 \]  
\text{(tab, Nrows,tab and N_{\text{simule}} are mentioned in Figure 1), (Figure 12).}

The table of the values (tab) of the strategies \((J, J^*_{\text{ij}}) = (X3, X4)\) simulated (in each iteration of simulation, we obtain a distribution of the total investment budget between the construction of the silos and the irrigation over 20 future years) follows Figure 2 (Figure 13).

The strategies belonging to the so-called efficient Pareto front calculated following Figure 1 are presented as follows (the strategies which do not belong to the front register a missing value NA as indicating the algorithm) (Figure 14).

After eliminating the missing values, NA was obtained by the Pareto front algorithm (Figure 15).

By going through the program encoded on R [23], we can plot the graphs the strategies obtained following Figure 1 and the Pareto front following Figure 3 (Figure 16).

8. Conclusion

Therefore, each point belonging to the Pareto front corresponds to its vector of irrigation and construction of silos amounts and it is up to the decision-maker to choose the distribution of the total investment budget between irrigation and the construction of silos, which gives a strategy belonging to the Pareto front and minimizing the agricultural risk defaults in the case of Bangladesh in future. In this article, the min-max tool was suggested to be an appropriate formula to determine strategies said “optimal” especially when we use the dimension rather than \(d = 1\) and to give an exhaustive study, other mathematics principles are invited to apply non-parametric statistics, ARIMA time series, etc.

Data Availability

All data are available on the FAO website.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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