Enhanced prediction for discrete-time input-delayed systems with unknown disturbances

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Abstract

This paper deals with the problem of predicting the future state of discrete-time input-delayed systems in the presence of unknown disturbances that can affect both the input and the output equations of the plant. Since the disturbance is unknown, an exact prediction of the plant states is not feasible. We propose the use of a high-order extended Luenberger-type observer for the plant states, disturbances, and their finite difference variables. Then, a new method for computation of the prediction is proposed which, under certain assumptions, allows for enhanced prediction and consequently improved attenuation of the unknown disturbances. Detailed analysis of the performance of the proposed scheme is carried out, while linear matrix inequalities (LMIs) are used for the observer design in order to mitigate the prediction errors.

Key words: Discrete time; Prediction; Time delays; Unknown disturbances; Linear matrix inequalities.

1 Introduction

Time delays have been extensively studied over the years due to their harmful impact on the closed loop, which can cause undesired oscillatory behavior or even instability. Due to being infinite-dimensional systems (in the continuous-time case), the analysis and control of time-delayed plants are evolved when compared to non-delayed ones (Fridman, 2014; Gu et al., 2003).

When controlling such systems, a well-established idea is that of predicting the system output or state ahead of the delay and then to feedback such prediction so that the closed-loop system is equivalent to that of a non-delayed system. In such cases, one can say that the delay has been compensated, that is, its undesired effects have been mitigated by means of prediction. This idea was first developed in frequency domain for single-input single-output (SISO) open-loop stable systems by the seminal work of Smith (1957) and was later extended to multiple-input multiple-output (MIMO) open-loop unstable systems with the aid of time-domain analysis (Manitius & Olbrot, 1979; Artstein, 1982). Recently, interest in the problem of prediction has resurfaced in the context of unknown disturbances. In case the system is affected by such disturbances, it becomes impossible to perfectly predict the future states of the system due to the solution being dependent on future values of the disturbance. Some important works in the last years have tackled this problem.

To deal with unknown disturbances, Léchappé et al. (2015) proposed a solution based on a modification to the Artstein predictor (Artstein, 1982). The main idea consisted of adding a term that compares the current state of the plant with the delayed prediction, which leads to some information about the disturbance being feedbacked into the scheme. Both Sanz et al. (2016) and Castillo & García (2021) employ high-order extended observers capable of estimating the disturbance and its derivatives up to some order \( r \). Then, such observations are used to help define new predictive schemes that lead to a decrease in the error caused by the unknown disturbance. In Furtat et al. (2018), the Finite Spectrum Assignment (FSA) idea from Manitius & Olbrot (1979) is used along with a disturbance predictor to propose a new control law for disturbance compensation for input-delayed systems. Prediction for neutral
type systems with input delays has also been investigated in Kharitonov (2015), while Zhou et al. (2017); Yoon & Lin (2015) propose predictor-based controllers for systems with delays also in the state and output of the plant. These last three cited works, however, do not take disturbances into account.

Despite the rich recent literature on modified predictors to deal with input-delayed systems affected by unknown disturbances in the continuous-time domain, it seems that less attention has been given to the discrete-time counterpart of this problem. Liu & Zhou (2016) deals with delay compensation in the case of simultaneous input and state delays, however disturbances are not considered. In Hao et al. (2017), both a predictor and a control law are designed to deal with disturbances. An extended observer capable of estimating the plant state is not assumed to be feasible, which makes it affected by unknown disturbances, which can affect both the state and output equations of the plant. Differently from Wu & Wang (2021), direct measurement to the plant state is not assumed to be feasible, which makes the problem even more evolved. Inspired by the idea from Castillo & García (2021), we employ a high-order extended state observer which is capable of estimating the disturbance and its finite differences up to some order \( r \). The main idea consists then of plugging such estimations into a truncated version of the Newton series (from the calculus of finite differences) to estimate future disturbances, which are then used into the solution of the plant to generate a new predictive scheme. As will be shown in the paper, such prediction is capable of generating enhanced results with respect to both disturbance attenuation and minimization of prediction errors when compared with Wu & Wang (2021). Furthermore, we prove that with the proposed predictive scheme it is possible to find analytical expressions for bounds on the \( l_2 \) norm of the prediction error. Additionally, optimization problems are formulated in terms of linear matrix inequalities (LMIs) to design the high-order extended observer with the aim to minimize such errors. Simulation results are then used to emphasize the effectiveness of the proposal.

The rest of this paper is organized as follows. We start by providing a general formulation and the problem statement in Section 2. Next, in Section 3, we present the employed extended observer and derive the equations for the proposed predictive scheme. Moving forward, in Section 4, we present the main results of the paper concerning convex conditions for the observer design and minimization of prediction errors. A numerical example from the literature is then presented in Section 5. Finally, in Section 6, we end the paper with concluding remarks and future perspectives.

**Notation.** For a matrix \( Y \in \mathbb{R}^{n \times m} \), \( Y^\top \in \mathbb{R}^{m \times n} \) means its transpose. For matrices \( W = W^\top \) and \( Z = Z^\top \) in \( \mathbb{R}^{n \times n} \), \( W \succ Z \) means that \( W - Z \) is positive definite. Likewise, \( W \succeq Z \) means that \( W - Z \) is positive semi-definite. \( Z^n_+ \) stands for the set of symmetric positive definite matrices. \( I \) and \( 0 \) denote identity and null matrices of appropriate dimensions, although their dimensions can be explicitly presented whenever relevant. The symbol \( * \) denotes symmetric blocks in the expression of a matrix. For matrices \( W \) and \( Z \), \( \text{diag}(W, Z) \) corresponds to the block-diagonal matrix. For integers \( a < b \), we use \([a, b]\) to denote the set \( \{a, a + 1, \ldots, b - 1, b\} \). For a discrete function \( f(k) : \mathbb{Z} \to \mathbb{R}^n \), the forward difference operator is defined by \( \Delta f(k) = f(k + 1) - f(k) \). Similarly, \( \Delta^2 f(k) = \Delta f(k + 1) - \Delta f(k), \Delta^3 f(k) = \Delta^2 f(k + 1) - \Delta^2 f(k), \ldots, \Delta^{r+1} f(k) = \Delta^r f(k + 1) - \Delta^r f(k) \). Furthermore, we define \( \Delta^0 f(k) = f(k) \). We use the notation \( ||f(k)|| \) to denote the vector norm \( \sqrt{f^\top(k) f(k)} \) throughout the text. Additionally, \( ||f||_{l_2} \) is used to denote the local \( l_2 \) norm of \( f(k) \), given by \( \sqrt{\sum_{\tau=0}^{k} ||f(\tau)||^2} \).

Finally, \( (\tau)_a = \frac{\Gamma(\tau + 1)}{\Gamma(\tau + a)} \) denotes the binomial coefficient, where \( x^a \) stands for the falling factorial \( x^a = \prod_{j=1}^{a} (x - j) \).

### 2 Problem formulation

#### 2.1 General view

Consider an input-delayed plant given by

\[
\begin{align*}
x(k + 1) &= Ax(k) + B_u u(k - d) + B_w w(k), \\
y(k) &= Cx(k) + D_u u(k), \\
u(k) &= \theta(k), \quad k \in [-d, -1],
\end{align*}
\]

where \( x(k) \in \mathbb{R}^n, \ u(k) \in \mathbb{R}^m, \ w(k) \in \mathbb{R}^q \), and \( y(k) \in \mathbb{R}^{m_y} \) are the plant state, the control input, the unknown disturbance, and the plant output, respectively. Furthermore, \( x(0) \) and \( \theta(k) \) define the system initial condition. The plant input delay \( d \) is assumed to be a known positive constant. One classical idea to control time-delay systems dating back to the 50s (Smith, 1957) consists of feedbacking a prediction of the output of the equivalent delay-free system (i.e. \( d = 0 \) in (1)) so that any controller designed without taking into account the delay would appropriately stabilize the closed-loop system. In the case of state-feedback control and regarding the...
time-domain model (1) this would be equivalent to feedbacking the value of \( x(k + d) \), that is \( u(k) = Kx(k + d) \), so that the following closed-loop equation would be obtained

\[
x(k + 1) = (A + B_uK)x(k) + B_ww(k),
\]

(2)

thus eliminating the delay influence from the closed loop. From recursion in (1), an exact prediction \( x(k + d) \) can be easily found as

\[
x(k + d) = A^d x(k) + \sum_{j=1}^{d} A^{j-1} B_u u(k - j)
\]

(3)

\[
+ \sum_{j=1}^{d} A^{j-1} B_w w(k + d - j).
\]

One can notice, however, that this equation cannot be computed since it depends on the current and future values of the disturbance, that is the values \( w(k + s) \), \( s \in [0, d - 1] \). Having knowledge of such values is unthinkible in almost all real systems, therefore strategies to compute an approximation of (3) have been studied along the years. One classical choice to compute approximated predictions is to ignore the term due to the disturbance in (3), leading to

\[
x_p(k) = A^d x(k) + \sum_{j=1}^{d} A^{j-1} B_u u(k - j).
\]

(4)

Clearly, (4) results in a prediction error given by

\[
x(k + d) - x_p(k) = \sum_{j=1}^{d} A^{j-1} B_w w(k + d - j).
\]

(5)

The recently published strategy in Wu & Wang (2021) showed that such choice is not appropriate due to large prediction errors that lead to poor attenuation of the disturbances. To deal with this, Wu & Wang (2021) proposes two predictors based on the original idea for continuous-time plants presented in Léchappé et al. (2015). In this same vein, in this paper we will also deal with the problem of finding a prediction \( \hat{x}(k + d) \) that leads to smaller errors and consequently better attenuation of disturbances. To this end, we first make the following assumption on \( w(k) \).

**Assumption 1** The disturbance is \((r + 1)\)-times finite differentiable with respect to the \( \Delta \) operator so that for any positive integer \( r \)

\[
\| \Delta^{(r+1)} w(k) \| \leq \delta,
\]

(6)

which implies that \( \| \Delta^{(r+1)} w \|_{l_\infty} \leq \delta \sqrt{k + 1} \), since

\[
\sqrt{\sum_{\tau=0}^{k} \| \Delta^{(r+1)} w(\tau) \|^2} \leq \delta \sqrt{k + 1} < \infty,
\]

(7)

for \( 0 \leq k < \infty \), where relation (7) means that \( \Delta^{(r+1)} w(k) \) belongs to the \( l_\infty^2(\mathbb{Z}) \) space.

### 2.2 Problem statement

The following statement summarizes the problem we intend to solve in this paper.

**Problem 2** Given the plant matrices \( A, B_u, B_w, C, D_w \), the delay \( d \), and taking into account Assumption 1, we would like to compute a prediction \( \hat{x}(k + d) \) with the following characteristics:

(i) For all \( k \in [0, \infty) \) the prediction error is limited and bounds on its \( l_2 \) norm, given by

\[
\sqrt{\sum_{\tau=0}^{k} \| x(\tau + d) - \hat{x}(\tau + d) \|^2},
\]

which can be analytically expressed.

(ii) The prediction error can be minimized through LMI-based design of the prediction scheme.

### 3 Prediction scheme

In this section we present a new predictive scheme for system (1). The main idea consists of employing a high-order extended observer that allows to estimate the disturbances and their finite differences \( \Delta \) up to order \( r \). Such observations are then plugged into a series to approximate the future values of the unknown disturbance, leading to the computation of the predictions. We start the section by defining the high-order extended observer equation, followed by presentation of the proposed predictive scheme.

#### 3.1 High-order extended state observer

Since system (1) does not give direct measurement to the plant state, a first step to compute a prediction is to estimate the value of \( x(k) \), which can be done by means of a state observer. Furthermore, to deal with the disturbance summation error term (5), it is necessary to gather some knowledge about the disturbance as well, which can help decrease the error caused by this term. Such task can be accomplished by means of an extended state observer. In this work we employ observers that allow to estimate not only the plant state and the disturbance signal but also the difference operators (up to order \( r \)) of
the disturbance. The idea of using high-order extended observers is inspired by Castillo & García (2021), which deals with continuous-time systems, where the disturbances and their time derivatives are observed. Herein, we propose the use of the following high-order observer to deal with the case of discrete-time systems

\[
\begin{aligned}
\dot{\eta}(k+1) &= \bar{A}\dot{\eta}(k) + \bar{B}_u w(k-d) + Le_y(k), \\
\hat{y}(k) &= \bar{C}\dot{\eta}(k),
\end{aligned}
\] (8)

where \(\dot{\eta}(k) = [\dot{x}(k)^\top \ \dot{w}^\top(k)]^\top \in \mathbb{R}^n\), with \(n = (r+1)q + n_p\), is the observer state, \(e_y(k) = y(k) - \hat{y}(k)\) is the observation error, \(L \in \mathbb{R}^{n \times m_y}\) is the observer gain to be designed, and

\[
\bar{A} = \begin{bmatrix} A & B_w \ 0_{(r+1)q \times m_u} & I \end{bmatrix}, \quad \Pi = \begin{bmatrix} I_q & I_q & \cdots & 0 \ 0 & \cdots & \cdots & \vdots \ \vdots & \cdots & \cdots & I_q \ 0 & \cdots & 0 & I_q \end{bmatrix},
\]

\[
\bar{B}_u = \begin{bmatrix} B_u \\
0_{(r+1)q \times m_u} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & D_w \ 0_{n_p \times m_u} & I_q \end{bmatrix}.
\]

System (8) is a Luenberger-type observer for the augmented variable \(\dot{\eta}(k) = [x(k)^\top \ \dot{w}^\top(k)]^\top \in \mathbb{R}^n\), with \(\dot{w}^\top(k) = [w^\top(k) \Delta w^\top(k) \cdots \Delta^r w^\top(k)]^\top\), which under Assumption 1 satisfies

\[
\eta(k+1) = \bar{A}\eta(k) + \bar{B}_u w(k-d) + \bar{B}_u \Delta^r w(k),
\] (9)

where \(\bar{B}_w = \left[0_{q \times r \times n_p} \ I_q \right]^\top\). Next, we will present a new strategy that employs the observed variables to compute a prediction to system (1).

3.2 A new expression for the prediction

From the calculus of finite differences, the following expression, known as the Newton series, holds for the operator \(\Delta\) (Jordan & Carver, 1950; Rota & Taylor, 1994)

\[
w(k + s) = \sum_{m=0}^{\infty} \binom{s}{m} \Delta^m w(k).
\] (10)

From equation (10) we can rewrite (3) as

\[
\begin{aligned}
x(k + d) &= A^d x(k) + \sum_{j=1}^{d} A^{j-1} B_u u(k-j) \\
&\quad + \sum_{j=1}^{d} A^{j-1} B_w \left[ \sum_{m=0}^{\infty} \binom{d-j}{m} \Delta^m w(k) \right],
\end{aligned}
\] (11)

By splitting the infinite sum in (11) into two parts, one with terms ranging between 0 and \(r\) and other with terms between \(r + 1\) and \(\infty\), we arrive at the following expression for (3)

\[
\begin{aligned}
x(k + d) &= A^d x(k) + \sum_{j=1}^{d} A^{j-1} B_u u(k-j) \\
&\quad + \sum_{j=1}^{d} A^{j-1} B_w \left[ \sum_{m=0}^{r} \binom{d-j}{m} \Delta^m w(k) \right] \\
&\quad + \sum_{j=1}^{d} A^{j-1} B_w \left[ \sum_{m=r+1}^{\infty} \binom{d-j}{m} \Delta^m w(k) \right],
\end{aligned}
\] (12)

where the second summation term depends on the differences up to order \(r\) of the disturbance \(w(k)\). Then, given any observation of the variable \(\eta(k)\), the following prediction can be defined

\[
\begin{aligned}
\dot{x}(k + d) &= A^d \dot{x}(k) + \sum_{j=1}^{d} A^{j-1} B_u u(k-j) \\
&\quad + \sum_{j=1}^{d} A^{j-1} B_w \left[ \sum_{m=0}^{r} \binom{d-j}{m} \Delta^m \dot{w}(k) \right],
\end{aligned}
\] (13)

where the summation with terms ranging from 0 to \(r\) is a truncation of the infinite Newton series which can yield approximate estimations for the future values of the disturbance \(w(k + s), s \in [1, d - 1]\). By rewriting (13), an expression for the prediction depending on the observer state variable \(\dot{\eta}(k)\) is given by

\[
\begin{aligned}
\dot{x}(k + d) &= \Gamma(d) \dot{\eta}(k) + \sum_{j=1}^{d} A^{j-1} B_u u(k-j) \\
&\quad + \sum_{j=1}^{d} A^{j-1} B_w \left[ \sum_{m=0}^{r} \binom{d-j}{m} \Delta^m \dot{w}(k) \right],
\end{aligned}
\] (14)

where

\[
\Gamma(d) = \begin{bmatrix} A^d & \ T(d) \end{bmatrix},
\] (15)

with

\[
T(d) = \sum_{j=1}^{d} A^{j-1} B_w \left[ I_q(0_{d-j}) \ I_q(0_{d-j}) \ \cdots \ I_q(0_{d-j}) \right].
\] (16)

Therefore, in this work, we propose the utilisation of prediction (14), which can be implemented by using the high-order extended state observer (8). The main advantage of employing (14) in comparison with (4) is that we are able to use information from the disturbance in the prediction even though the disturbance is unknown. This way, the prediction error (5) can be significantly reduced, as discussed in the next subsection. However, let us first state the following remark, which is useful in the special case that \(C = I\) and \(D_w = 0\).
Remark 3 In the case that the plant output matrices are such that $C = I$ and $D_w = 0$, it follows that $y(k) = x(k)$ and thus prediction (14) can be improved by replacing $\hat{x}(k)$ with $y(k) = x(k)$ in (14), yielding the modified prediction given by

$$\hat{x}(k + d) = A^d y(k) + \Gamma(d) \hat{\eta}(k) + \sum_{j=1}^{d} A^{d-j} B_u u(k - j),$$

where, in this case, $\Gamma(d) = \begin{bmatrix} 0 & T(d) \end{bmatrix}$. Note that, however, even in this case observation of $x(k)$ is needed since the error $e_y(k) = y(k) - \hat{y}(k)$ is required in order to properly observe the variable $w(k)$.

3.3 Error analysis

From (12) and (14) (or (17) in case of Remark 3), a unique expression for the prediction error can be found as follows

$$x(k + d) - \hat{x}(k + d) = \mathcal{E}_d(k) + \mathcal{E}_e(k),$$

where

$$\mathcal{E}_d(k) = \Gamma(d)(\eta(k) - \hat{\eta}(k))$$

with $\Gamma(d)$ given by (15) in case of the prediction (14) or $\Gamma(d) = \begin{bmatrix} 0 & T(d) \end{bmatrix}$ in case Remark 3 applies, and

$$\mathcal{E}_e(k) = \sum_{j=1}^{d} A^{d-j} B_w \left( \sum_{m=r+1}^{\infty} \left( \frac{d - j}{m} \right) \Delta^m w(k) \right).$$

From (19), note that $\mathcal{E}_d(k)$ is an error term that depends on the quality of the observation, which can be minimized by proper design of the observer (8), i.e. by design of the gain $L$. Such design will be realized in Section 4. On the other hand, error $\mathcal{E}_e(k)$ in Equation (20) is an error which is inevitable to the prediction. Nonetheless, under Assumption 1, the $l_2$ norm of this error is bounded, as stated in the next Proposition.

Proposition 4 By taking into account Assumption 1, the $l_2$ norm of $\mathcal{E}_e(k)$ is bounded such that

$$\|\mathcal{E}_e\|_{l_2} \leq \sqrt{\sum_{\lambda=0}^{k} \delta^2 \mu^2} = \delta \mu \sqrt{k + 1},$$

where $\mu = \max_{j=1 \ldots d} \sigma_{\max}(Y_j^2) \phi_j$, with

$$\phi_j = \sum_{l=0}^{d-j-r-1} \left( \frac{d-j}{l+r+1} \right)^{2l},$$

$Y_j$ denotes the maximum singular value of $Y_j^2$. Furthermore, in case $r > d - 2$, $\|\mathcal{E}_e\|_{l_2} = 0, \forall k \in [0, \infty)$.

Proof. See Appendix A for proof.

Therefore, even though the disturbance $w(k)$ is unknown, by employing prediction (14) (or (17)) we can guarantee that the error caused by $\mathcal{E}_e(k)$ is limited, as shown by (21). Furthermore, under some special circumstances cited in Proposition 4, its $l_2$ norm might be null. It is worth to recall that according to Problem 2, one of the objectives of this paper is to show that the $l_2$ norm of the prediction error $x(k + d) - \hat{x}(k + d)$ is bounded, which depends on both $\mathcal{E}_e(k)$ and $\mathcal{E}_d(k)$. A complete solution to this problem will be presented within the next section.

4 Main results

In this section, we employ LMI-based design to the observer (8) with the aim to minimize influence of the disturbance in the prediction error. Furthermore, the main results are presented, so that a solution to Problem 2 is provided. Initially, let us present some important theoretical preliminaries in the next subsection.

4.1 Theoretical preliminaries

Let us recall the following definition (see, for example, Khalil (2002)).

Definition 5 A mapping $H : l_2^{loc}(\mathbb{Z}) \rightarrow l_2^{loc}(\mathbb{Z})$ is finite-gain $l_2^{loc}(\mathbb{Z})$-stable if there exist nonnegative constants $\gamma$ and $\beta$ such that

$$\sqrt{\sum_{\tau=0}^{k} \|H(u(\tau))\|^2} \leq \gamma \sqrt{\sum_{\tau=0}^{k} \|u(\tau)\|^2} + \beta$$

for all $u(k) \in l_2^{loc}(\mathbb{Z})$ and $k \in [0, \infty)$.

Definition 5 is important as it will be helpful to establish the main Theorem in this section. To this end, from (8) and (9), and by defining the variable $e_\eta(k) = \eta(k) - \hat{\eta}(k)$, let us consider the following system

$$\begin{cases}
e_\eta(k + 1) = (\tilde{A} - L\tilde{C}) e_\eta(k) + \tilde{B}_w \Delta^{r+1} w(k), \\
\mathcal{E}_d(k) = \Gamma(d) e_\eta(k),
\end{cases}$$

which is a mapping from $\Delta^{r+1} w(k)$ to the prediction error $\mathcal{E}_d(k)$. We are now ready to state the main results of this paper.

4.2 Observer design

In this subsection we present a solution to Problem 2. First, let us consider the following theorem for the design of the observer gain $L$. 

Then, by computing
\[
\begin{bmatrix}
P - \Gamma^T(d)\Gamma(d) & 0 & A^T P - \overline{C}^T W^+ \\
* & \gamma I & B_w^T P \\
* & * & P
\end{bmatrix} > 0. \quad (24)
\]

which, by taking the square roots and using the fact that \(\sqrt{a^2 + b^2} \leq a + b\) for \(a, b \in \mathbb{R}^+\), leads to
\[
\|e_\theta\|_{l_2} < \gamma \|\Delta^{r+1} w\|_{l_2} + \sqrt{V(c_\eta(0))}. \quad (30)
\]

Thus, by Definition 5, the mapping \(\Delta^{r+1} w(k) \mapsto \theta_\delta(k)\) from (23) is \(l_2^{oc}(Z)\)-stable with \(l_2\) gain less than or equal to \(\gamma\) and bias term \(\beta = \sqrt{V(c_\eta(0))} \leq \sqrt{\xi}\), which proves item 1 in Theorem 6. Item 2 comes directly by plugging relation (7) from Assumption 1 into (30). Finally, note that when \(\Delta^{r+1} w(k) = 0\), relation (27) implies \(\Delta V(e_\eta(k)) < -\|\theta_\delta(k)\|^2 < 0\), therefore the trajectories of \(e_\eta(k)\), and consequently of \(\delta_\eta(k) = \Gamma(d)e_\eta(k)\), asymptotically converge to zero. Thus, all items in Theorem 6 are proven and the proof is complete. \(\square\)

4.2.1 The case of \(D\)-Stability

In case the designer wants to have degrees of decision regarding the location of the eigenvalues of the matrix \((\bar{A} - L\bar{C})\), some \(D\)-Stability condition can be solved along with LMI (24). Let \(\lambda(\bar{A} - L\bar{C}) = \{\lambda_1, \ldots, \lambda_n\}\) be the set of \(n\) eigenvalues of the matrix \(\bar{A} - L\bar{C}\). In this work, we apply the particular case of \(D\)-stability such that \(\zeta_0 < \text{Re}(\lambda_i) < \zeta_a, \forall i \in [1,n]\), where \(\text{Re}(\lambda_i)\) refers to the real part of the pole and \(-1 \leq \zeta_0 < \zeta_a \leq 1\) determines the region of the complex plane that contains the poles. In this case, we can say that matrix \(\bar{A} - L\bar{C}\) is \(\mathbb{H}_{\zeta_0, \zeta_a}\)-stable. The following Corollary can be stated.

Corollary 7 For given scalars \(\zeta_0, \zeta_a\), let there exist matrices \(P \in \mathbb{S}^+_{n, m}\), \(W\) in \(\mathbb{R}^{n \times m}\) and a scalar \(\gamma\) such that
\[
\begin{align*}
\bar{A}^T P - \bar{C}^T W^+ + P\bar{A} - W\bar{C} - 2\zeta_0 P & < 0, \\
\bar{A}^T P - \bar{C}^T W^+ + P\bar{A} - W\bar{C} - 2\zeta_a P & > 0,
\end{align*}
\]

are solved along with (24). Then, for any initial condition \(e_\eta(0)\) in the ellipsoid \(e_\eta(0) Pe_\eta(0) \leq \epsilon\), for some \(\epsilon > 0\), and for the observer gain given by \(L = P^{-1}W\), the following is true:

- Items 1, 2, and 3 from Theorem 6 hold.
- The poles of \((\bar{A} - L\bar{C})\) will be contained within a region of the complex plane such that \(\zeta_0 < \text{Re}(\lambda_i) < \zeta_a, \forall i \in [1,n]\).

Proof. Proof of Corollary 7 is straightforward. First, let us remind that satisfaction of (24) guarantees that Items 1, 2, and 3 from Theorem 6 hold. Moreover, matrix \((\bar{A} - L\bar{C})\) is \(\mathbb{H}_{\zeta_0, \zeta_a}\)-stable if and only if there exist a matrix \(P \in \mathbb{S}^+_{n, m}\) such that (see, for example, Duan & Yu (2013))
\[
\begin{align*}
(\bar{A} - L\bar{C})^T P + P(\bar{A} - L\bar{C}) - 2\zeta_0 P & < 0, \\
(\bar{A} - L\bar{C})^T P + P(\bar{A} - L\bar{C}) - 2\zeta_a P & > 0.
\end{align*}
\]
LMI (31) are then obtained after the change of variables $W = PL$. □

It is important to comment that other D-Stability regions could have been chosen for Corollary 7. See, for example, Chilali & Gahinet (1996) for a general revision of such regions, which can be viewed as extra degrees of freedom for the designer. Next subsection elaborates optimization procedures for achieving minimization of the prediction error.

### 4.2.2 Minimization of the prediction error

One way to minimize the $l_2$ gain of the mapping $\Delta^{r+1}w(k) \mapsto \hat{\sigma}_r(k)$ and therefore minimize the energy of the prediction error is to minimize $\tilde{\gamma}$ while solving LMI (24). The following optimization problem can then be formulated in order to achieve better results of the predictor in case of Theorem 6

$$
\min_{\{P, W, \gamma\}} \tilde{\gamma} \quad \text{subject to} \quad (24).
$$

In the case of Corollary 7, for given scalars $\zeta_a$ and $\zeta_b$, the following optimization problem applies

$$
\min_{\{P, W, \gamma\}} \tilde{\gamma} \quad \text{subject to} \quad (24) \text{ and } (31).
$$

### 4.3 Summary of main results

The main goals of this paper have been achieved, that is, we have solved Problem 2. In another words, for a given plant (1) and taking into account Assumption 1, we have shown that the proposed predictive scheme leads to a prediction with the following characteristics

1. For all $k \in [0, \infty)$, the $l_2$ norm of the prediction error $x(k + d) - \hat{x}(k + d)$ is bounded such as

$$
\sqrt{\sum_{\tau=0}^{k} ||x(\tau + d) - \hat{x}(\tau + d)||^2} \leq \|\hat{\sigma}_a\|_{\ell^2_{loc}} + \|\hat{\sigma}_r\|_{\ell^2_{loc}}
$$

which leads to

$$
\sqrt{\sum_{\tau=0}^{k} ||x(\tau + d) - \hat{x}(\tau + d)||^2} \leq (\gamma + d\mu)\delta\sqrt{k + 1 + \sqrt{\epsilon}}.
$$

2. Such an error can be minimized by running either optimization problem (33) or (34).

Moreover, the following proposition holds.

#### Proposition 8

The scheme proposed in this paper guarantees null steady-state prediction error in the case of $\Delta^{r+1}w(k) = 0$.

**Proof.** In case $\Delta^{r+1}w(k) = 0$, error $\hat{\sigma}_r(k)$ in (20) is null for all $k \in [0, \infty)$. Moreover, according to item 3 of Theorem 6, the error due to observation $\hat{\sigma}_o(k)$ converges asymptotically to zero, thus implying that the prediction error (18) also converges asymptotically to zero. □

Proposition 8 implies that constant disturbances are perfectly compensated by the prediction scheme since $\delta = 0$ for all $r \geq 0$ in this case. In the case of ramp-like disturbances, null prediction error is achieved for $r \geq 1$. More generally, let $n_r$ be the order of an unknown time-varying polynomial disturbance $w(k) = w_0 + w_1k + \cdots + w_nk^n$, then null steady-state prediction error is achieved whenever $r \geq n_r$ since $\Delta^{r+1}w(k) = 0$ in this case. Although one can enlarge the family of time-varying disturbances for which null prediction error is achieved at steady-state by increasing the parameter $r$, it should be noted that complexity also increases due to the order of the observer state being augmented, consequently leading to less efficiency in solving optimization problems (33) and (34).

### 5 Numerical example

In this section, we present comparisons of the proposed prediction scheme with the two predictors from Wu & Wang (2021). Table 1 illustrates the differences among compared prediction schemes. First and second predictors from Wu & Wang (2021) do not employ any observer and are based on the idea from Léchappé et al. (2015), where the approximated future state is computed based on the standard prediction that ignores the effect of the disturbance plus addition of a term that takes into account previous prediction errors. In this work, we propose a solution using an observer to estimate states in addition to disturbances and their finite differences ($\Delta^i$) up to the order $i = r$. Then, such observations are used to help predicting the future values of the disturbance, which are in turn used to compute an approximation of the summation $\sum_{j=1}^{d}A^{j-1}B_ww(k + d - j)$, leading to the use of prediction (14) (or (17) in the special case that Remark 3 applies). The proposed solution leads to significant advantages in terms of both disturbance attenuation and minimization of prediction errors, as shown in the next subsections.

In order to evaluate the proposed predictor, let us consider the perturbed input-delayed system recently anal-
Comparison between predictive strategies, where \( x_p(k) = A^d x(k) + \sum_{j=1}^{d} A^{j-1} B_a u(k-j) \).

| Observer order | Observed variables | Computed prediction | Type of disturbances |
|----------------|--------------------|---------------------|----------------------|
| Wu and Wang, 1st | \( \eta = [x^T(k) \ w^T(k)] \) | \( x_{p1} = x_p(k) + x(k) - x_p(k-d) \) | Not specified |
| Wu and Wang, 2nd | - | \( x_{p2} = x_{p1}(k) + x(k) - x_{p1}(k-d) \) | Not specified |

5.1 Constant disturbances

In this case, we consider a constant disturbance \( w(k) = 1.6, k \in [0, \infty) \). Applying optimization problem (34) with \( \zeta_a = 1, \zeta_b = 0 \), we get

\[
L = \begin{bmatrix} -0.3899 & 3.2000 & -0.0000 \\ 1.0000 & -0.7314 & 0.8621 \end{bmatrix}^T
\]

for the observer.

Figure 1 shows the standard Euclidean norm of the prediction error for the compared schemes. From the simulations, we can verify that the prediction error converges to zero in all cases. However, for both predictors from Wu & Wang (2021), the norm of the prediction error reaches much higher values on the transitory phase. On the other hand, the predictor proposed in this work is able to reach much lower values for the norm, thus presenting a great improvement on the prediction mainly during the transitory response.

5.2 Time-varying disturbances

In this case, we consider a sinusoidal disturbance \( w(k) = 0.6 \sin \left( \frac{3\pi}{2} k \right), k \in [0, \infty) \). Applying optimization problem (34) with \( \zeta_a = 1, \zeta_b = 0 \), we get

\[
L = \begin{bmatrix} L_x & L_w \end{bmatrix}^T
\]

for the observer.

Figure 2 shows the norm of the plant states for the compared predictor schemes. We can verify that the norm reaches high values on the transitory response in both predictors from Wu & Wang (2021), while on steady state the disturbance is not properly attenuated. On the other hand, the proposed predictor presents improved performance by attenuating the disturbance at steady state and by presenting much lower values for the norm during the transient phase.
6 Conclusions

This work proposed a new method for computing the prediction of discrete-time input-delayed systems in the presence of unknown disturbances. The method employs extended observers for the disturbances and their finite differences up to order r, which are then employed to estimate future values of the disturbance signal, leading to prediction (14) (or (17) in case Remark 3 applies).

Effectiveness was demonstrated by means of numerical examples that compared the proposed method against two strategies from the recent literature. From the comparisons, it was clear that the attenuation of the unknown disturbance was enhanced. Analytical expressions were also derived to prove that, under Assumption 1, the $l_2$ norm of the prediction error is bounded, as given by (35), for all $k \in [0, \infty)$, while LMI-based design of the observer was developed to successfully minimize such an error by means of either optimization procedure (33) or (34). Moreover, we showed that null steady-state prediction error is achieved for a class of unknown time-varying polynomial disturbances whenever $r \geq n_r$, being $n_r$ the order of the disturbance.

Future research envisages the development of new predictive and LMI-based design methods for the closed loop. For example, the use of LPV filters to improve the estimation of the future values of the disturbance, and an LMI design methodology that takes into account the closed-loop control law rather than just the observer part. Extension of the proposed predictive scheme for systems with multiple input delays is also of interest.

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A Proof of Proposition 4

Consider equation (20). By applying the change of variables $l = m - r - 1$ and the identity

$$\Delta^{l+r+1}w(k) = \sum_{i=0}^{l} (-1)^i \binom{l}{i} \Delta^{r+1}w(k+i)$$  \hspace{1cm} (A.1)

we obtain the expression $\delta_r(k) = \sum_{j=1}^{d} A^{l-1}B_o W_j$, where

$$W_j = \sum_{l=0}^{\infty} \left( \frac{d-j}{l+r+1} \right) \sum_{i=0}^{l} (-1)^i \binom{l}{i} \Delta^{r+1}w(k+i),$$
which leads to $\|\delta_r(k)\|$ being bounded such as

$$\|\delta_r(k)\| \leq \sum_{j=1}^{d} \|Y_j^2 W_j\|,$$  \hspace{1cm} (A.2)

where $Y_j = B_j^T A_j^{-1} A_{j-1} B_w$. Next, by applying

$$\sum_{j=1}^{d} \|Y_j^2 W_j\| \leq d \max_{j=1\ldots d} \|Y_j^2 W_j\|$$

and using the fact that $\|Y_j^2 W_j\| \leq \sigma_{\text{max}}(Y_j^2) \|W_j\|$, where $\sigma_{\text{max}}(Y_j^2)$ denotes the maximum singular value of $Y_j^2$, we obtain

$$\|\delta_r(k)\| \leq d \max_{j=1\ldots d} \left( \sigma_{\text{max}}(Y_j^2) \|W_j\| \right).$$  \hspace{1cm} (A.3)

Now, let us find an expression for $\|W_j\|$. From the definition of $W_j$, it follows that

$$\|W_j\| \leq \sum_{l=0}^{\infty} \left( \frac{d-j}{l+r+1} \right) \sum_{i=0}^{l} (-1)^i \binom{l}{i} \|\Delta^{r+1} w(i)\|,$$

where $|\cdot|$ stands for the absolute value operator and $\Delta^{r+1} w(i)$ is a short notation for $\Delta^{r+1} w(k+i)$. Since $\|\Delta^{r+1} w(k+i)\| \leq \delta$ for all $i$ (from Assumption 1) and

$$\sum_{i=0}^{l} \binom{l}{i} = 2^l,$$

the following expression holds

$$\|W_j\| \leq \delta \sum_{l=0}^{\infty} \left( \frac{d-j}{l+r+1} \right) 2^l.$$  \hspace{1cm} (A.4)

Furthermore, by taking into account the fact that

$$\sum_{l=0}^{\infty} \left( \frac{d-j}{l+r+1} \right) 2^l = 0, \text{ for } l > d - j - r - 1,$$

we find the bound

$$\|W_j\| \leq \delta \varphi_j,$$  \hspace{1cm} (A.5)

where $\varphi_j = \sum_{l=0}^{d-j-r-1} \left( \frac{d-j}{l+r+1} \right) 2^l$ is a finite series. From (A.3) and (A.5), we arrive at the expression

$$\|\delta_r(k)\| \leq \delta d \mu, \forall k \in [0, \infty),$$  \hspace{1cm} (A.6)

where $\mu = \max_{j=1\ldots d} \left( \sigma_{\text{max}}(Y_j^2) \varphi_j \right)$. Finally, summing (A.6) from 0 to $k$ and taking square roots yields

$$\|\delta_r\|_{l^2} \leq \sqrt{\sum_{k=0}^{k} \delta^2 d^2 \mu^2} = \delta d \mu \sqrt{k + 1},$$  \hspace{1cm} (A.7)

$\forall k \in [0, \infty)$, thus completing the demonstration of Equation (21). Moreover, $\varphi_j = 0$ whenever the upper limit of the sum is negative, i.e. $d - j - r - 1 < 0$. Therefore, if $r > d - 2$, which corresponds to $j = 1$, it follows that $\varphi_j = 0$ for all $j \in \{1 \ldots d\}$. Thus, $\mu = 0$ implying $\|\delta_r\|_{l^2} = 0$ in (A.7), which completes the proof of the Proposition.