Varieties of Modules for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

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Abstract

Let $k$ be an algebraically closed field of characteristic 2. We prove that the restricted nilpotent commuting variety $C$, that is the set of pairs of $(n \times n)$-matrices $(A, B)$ such that $A^2 = B^2 = [A, B] = 0$, is equidimensional. $C$ can be identified with the ‘variety of $n$-dimensional modules’ for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or equivalently, for $k[X, Y]/(X^2, Y^2)$. On the other hand, we provide an example showing that the restricted nilpotent commuting variety is not equidimensional for fields of characteristic $> 2$. We also prove that if $e^2 = 0$ then the set of elements of the centralizer of $e$ whose square is zero is equidimensional. Finally, we express each irreducible component of $C$ as a direct sum of indecomposable components of varieties of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-modules.

0 Introduction

Let $G = \text{GL}(n, k)$, where $k$ is an algebraically closed field of characteristic $p > 0$, and let $\mathfrak{g}$ be the Lie algebra of $G$. We denote the $p$-th power of matrices on $\mathfrak{g}$ by $x \mapsto x^{[p]}$, and its iteration $m$ times by $x \mapsto x^{[p]^m}$. (This is the standard notation in the theory of restricted Lie algebras.) Clearly $x$ is nilpotent if and only if $x^{[p]^N} = 0$ for $N \gg 0$. Denote by $\mathcal{N}$ the set of nilpotent elements of $\mathfrak{g}$ and by $\mathcal{N}_1$ the subset of elements satisfying $x^{[p]} = 0$ (the restricted nullcone). It was proved in [10] that $\mathcal{N}_1$ is irreducible. (An explicit description was given in [1].) In [15], Premet proved that the nilpotent commuting variety $C_{\text{nil}}(\mathfrak{g}) := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid x, y \in \mathcal{N}, [x, y] = 0\}$ is irreducible and of dimension $(n^2 - 1)$. More specifically, $C_{\text{nil}}(\mathfrak{g}) = G \cdot (e, u)$ where $e$ is a regular nilpotent element of $\mathfrak{g}$ and $u = ke \oplus ke^2 \oplus \ldots \oplus ke^{n-1}$.

(Here and in what follows we use the dot to denote the action of $G$ by conjugation on $\mathfrak{g}$ or the induced diagonal action on $\mathfrak{g} \times \mathfrak{g}$, and the notation $\overline{V}$ for the Zariski closure of a subset $V$ of an arbitrary affine vector space, where the context is clear.) In fact, Premet proved
that the nilpotent commuting variety of $\text{Lie}(G)$ is equidimensional for any reductive group $G$ over an algebraically closed field of good characteristic.

The nilpotent commuting variety, or more accurately the restricted nilpotent commuting variety $C^\text{nil}_1(g) = \{(x, y) \in N_1 \times N_1 : [x, y] = 0\}$ is related to the cohomology of $G$ by work of Suslin, Friedlander & Bendel. It was proved in [17] that $C^\text{nil}_1(g)$ is homeomorphic to the spectrum of the cohomology ring $\oplus_{i \geq 0} H^2i(G_2, k)$, where $G_2$ is the second Frobenius kernel of $G$. More generally, the restricted nullcone $N_1$ plays an important role in the representation theory of $g$ due to the theory of support varieties of (reduced enveloping algebras of) restricted Lie algebras (studied for the restricted enveloping algebra in [4, 5, 9] and for general reduced enveloping algebras in [6]; see also [13, 14]).

Another perspective is that of varieties of modules (see for example [2]): $C^\text{nil}_1(g)$ can be identified with the variety of $n$-dimensional modules for the truncated polynomial ring $k[X, Y]/(X^p, Y^p)$. There is an isomorphism $k[X, Y]/(X^p, Y^p) \rightarrow k\Gamma$, where $\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Specifically, if $\sigma$ (resp. $\tau$) is a generator for the first (resp. second) copy of $\mathbb{Z}/p\mathbb{Z}$ in $\Gamma$, then $X + 1 \mapsto \sigma$ and $Y + 1 \mapsto \tau$.

This paper began as a preliminary investigation of the restricted nilpotent commuting variety in the simplest possible case: hence we assume $p = 2$. In Section 1 we show that projection onto the first coordinate maps any irreducible component of $C^\text{nil}_1(g)$ onto $N_1$ (that is, the components are ‘determined’ by the dense orbit in $N_1$). Equidimensionality then follows by equidimensionality of $\mathfrak{g}(e) \cap N_1$, which we prove for any $e \in N_1$.

**Proposition.** Let $k$ be of characteristic 2 and let $C^\text{nil}_1$ be the restricted nilpotent commuting variety.

(a) If $n = 2m$, then $C^\text{nil}_1$ has $[m/2] + 1$ irreducible components, each of dimension $3m^2$.
(b) If $n = 2m + 1$, then $C^\text{nil}_1$ has $(m + 1)$ irreducible components, each of dimension $3m(m + 1)$.

This result for $C^\text{nil}_1$ might be expected to indicate that the restricted nilpotent commuting variety is equidimensional for general $p$. However, we show that this is not the case (Remark 2.4). On the other hand, we observe that $\mathfrak{g}(e) \cap N_1$ is equidimensional for many choices of $G$ and $e$. We conjecture that this is true for reductive $G$ in good characteristic. We remark that the intersection $\mathfrak{g}(e) \cap N_1$ can be identified with the support variety $V_{\mathfrak{g}(e)}(k)$, where $k$ is the trivial $\mathfrak{g}(e)$-module. In the final section we express each irreducible component of $C^\text{nil}_1(g)$ as a direct sum of indecomposable components of modules.

Our method for obtaining the above results is a rather crude direct approach. Such a strategy will clearly be inappropriate in general.

**Notation.** We denote by $\text{Mat}_{r \times s}$ the vector space of all $r \times s$ matrices over $k$. If $x \in g$ then the centralizer of $x$ in $g$ (resp. $G$) will be denoted $\mathfrak{z}(e)$ (resp. $Z_G(e)$). We will sometimes abuse notation and use $N_1$ to refer to the set of $p$-nilpotent elements in an arbitrary Lie algebra. This will cause no confusion. We denote by $e_{ij}$ the matrix with 1 in the $(i, j)$-th position, and zeros everywhere else. (The dimension of $e_{ij}$ will always be specified or clear from the context.) Our convention is that all modules are left modules. We denote by $[m/r]$ the integer part of the fraction $m/r$. 

2
1 Centralizers

Let $G = \text{GL}(n, k)$, let $\mathfrak{g} = \text{Lie}(G)$ and let $e_0, e_1, \ldots, e_m$ be a set of representatives for the orbits in $\mathcal{N}_1 = \mathcal{N}_1(\mathfrak{g})$. Clearly $\mathcal{C}^{\text{nil}}_i = \bigcup_{i=0}^m G \cdot (e_i, \mathfrak{z}_\mathfrak{g}(e_i)) \cap \mathcal{N}_1$. In general the set $\mathfrak{z}_\mathfrak{g}(e_i) \cap \mathcal{N}_1$ is not irreducible. For each $i$ let $V_i^{(1)}, V_i^{(2)}, \ldots, V_i^{(r_i)}$ be the irreducible components of $\mathfrak{z}_\mathfrak{g}(e_i) \cap \mathcal{N}_1$. The following Lemma is adapted from [15, Prop. 2.1]. The argument works for arbitrary $G$ and $p$. (The only requirement is that the number of orbits in $\mathcal{N}_1$ is finite. This is well-known if $p$ is good (see [16]) but is true even if $p$ is bad [7].)

Lemma 1.1. Let $X$ be an irreducible component of $\mathcal{C}^{\text{nil}}_1$. Then there is some $i$, $0 \leq i \leq m$, and some $j$, $1 \leq j \leq r_i$, such that $X = G \cdot (e_i, V_i^{(j)})$. Moreover, $V_i^{(j)} \subseteq G \cdot e_i$.

Proof. Since there are finitely many of the sets $G \cdot (e_i, V_i^{(j)})$ and they cover $\mathcal{C}^{\text{nil}}_1$, the first statement is obvious. For the second statement, define an action of $\text{GL}(2)$ on $\mathfrak{g} \times \mathfrak{g}$ by the morphism $\text{GL}(2) \times (\mathfrak{g} \times \mathfrak{g}) \to \mathfrak{g} \times \mathfrak{g}$, $\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x, y) \right) \mapsto (ax + by, cx + dy)$. Clearly any element of $\text{GL}(2)$ preserves $\mathcal{C}^{\text{nil}}_1$. Hence $\text{GL}(2)$ preserves each irreducible component of $\mathcal{C}^{\text{nil}}_1$. In particular, $\tau(X) = X$, where $\tau : (x, y) \mapsto (y, x)$. Suppose therefore that $X = G \cdot (e_i, V_i^{(j)})$ is an irreducible component of $\mathcal{C}^{\text{nil}}_1$. Let $\pi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denote the first projection. Then $
abla(X) = G \cdot e_i$. But $X = \tau(X)$, hence $V_i^{(j)} \subseteq \pi(X)$.

Suppose from now on that $p = 2$. For each $i$, $0 \leq i \leq m = [n/2]$, let $e_i = \begin{pmatrix} 0 & 0 & I_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$, where $I_i$ is the $i \times i$ identity matrix. Here the top left, top right, bottom left and bottom right submatrices are $i \times i$, the top middle and bottom middle submatrices are $i \times (n - 2i)$, the centre left and centre right submatrices are $(n - 2i) \times n$, and the central submatrix is $(n - 2i) \times (n - 2i)$. Then $\{e_0, e_1, \ldots, e_m\}$ is a set of representatives for the conjugacy classes in $\mathcal{N}_1$. It is easy to see, with the standard description of nilpotent orbits via partitions of $n$, that $e_i$ corresponds to the partition $2^i \cdot 1^{n-2i}$. Moreover, we have the following inclusions: $\{0\} = G \cdot e_0 \subseteq G \cdot e_1 \subseteq \cdots \subseteq G \cdot e_M = \mathcal{N}_1$. The condition $V_i^{(j)} \subseteq G \cdot e_i$ is clearly equivalent to the inequality: $\text{rk}(y) \leq i$ for all $y \in V_i^{(j)}$.

Fix $i$ until further notice and let $x$ be an element of the centralizer $\mathfrak{z}_\mathfrak{g}(e_i)$, which must have the form

$$
\begin{pmatrix}
A & B & C \\
0 & E & F \\
0 & 0 & A
\end{pmatrix} : A, C \in \text{Mat}_{i \times i}, E \in \text{Mat}_{(n-2i) \times (n-2i)}, B \in \text{Mat}_{i \times (n-2i)}, F \in \text{Mat}_{(n-2i) \times i}.
$$

The requirement $x \in \mathcal{N}_1$ is equivalent, with this notation, to the conditions $A^2 = BF + [A, C] = 0$, $E^2 = 0$, $AB = BE$ and $EF = FA$. We will frequently use $A, B, C, E, F$ to refer to these submatrices of (an arbitrary) $x \in \mathfrak{z}_\mathfrak{g}(e_i)$ where the element $x$ is clear from the context. Assume for the rest of this section that $n - 2i \geq 2$, and let $V$ be an irreducible component of $\mathfrak{z}_\mathfrak{g}(e_i) \cap \mathcal{N}_1$. We shall prove that $V \nsubseteq G \cdot e_i$. We begin with the following lemma.
Lemma 1.2. Suppose there is some element \( x \in V \) such that \( E \neq 0 \). Then \( V \) is not contained in \( G \cdot e_1 \).

**Proof.** Let the one-dimensional torus \( \lambda : k^\times \to G, t \mapsto \begin{pmatrix} tI_i & 0 & 0 \\ 0 & I_{n-2i} & 0 \\ 0 & 0 & t^{-1}I_i \end{pmatrix} \) act on \( g \) by conjugation. Since \( \lambda(t)e_i \lambda(t^{-1}) = t^2 e_i \), \( \lambda(k^\times) \) preserves \( z_0(e_i) \), and therefore preserves each irreducible component of \( z_0(e_i) \cap \mathcal{N}_1 \). Thus if \( x \in V \) then \( x_0 = \lim_{t \to 0} (\text{Ad} \, \lambda(t)) x = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & F \end{pmatrix} \in V \). For any \( y \in z_0(e) \cap \mathcal{N}_1, y + ke_i \subset z_0(e_i) \cap \mathcal{N}_1 \); hence there is an action of the additive group \( \mathbb{G}_a \) on \( z_0(e_i) \cap \mathcal{N}_1 \) by \( \xi \cdot y = y + \xi e_i \). It follows that each irreducible component of \( z_0(e_i) \cap \mathcal{N}_1 \) is stable under this action of \( \mathbb{G}_a \), hence that \( x_0 + e_i \in V \). But \( \text{rk}(x_0 + e_i) = i + \text{rk}(E) > i \) if \( E \) is non-zero.

We therefore consider the subset \( Y \) of \( z_0(e_i) \cap \mathcal{N}_1 \) consisting of all \( x \) with \( E = 0 \). The conditions for \( x \in \mathcal{N}_1 \) then reduce to: \( A^2 = [A, C] + BF = 0, AB = 0, FA = 0 \). For each \( j, 0 \leq j < [i/2] \), let \( A_j \) be the \((i \times i)\) matrix \( \begin{pmatrix} 0 & 0 & I_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), where the top left, top right, bottom left and bottom right submatrices are \( j \times j \), the top middle and bottom middle submatrices are \( j \times (i-2j) \), the centre left and centre right submatrices are \((i-2j) \times i\), and the central submatrix is \((i-2j) \times (i-2j)\). Since \( Z_G(e_i) \) contains all elements of the form \( \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{pmatrix} \) with \( g \in GL(i) \), it is clear that \( Y = \bigcup Y_j \), where

\[
Y_j = Z_G(e_i) \cdot \left\{ \begin{pmatrix} A_j & B & C \\ 0 & 0 & F \\ 0 & 0 & A_j \end{pmatrix} : A_jB = 0, FA_j = 0, [A_j, C] + BF = 0 \right\}.
\]

Moreover, since this is a finite union, each irreducible component of \( Y \) is an irreducible component of one of the \( Y_j \). Clearly the conditions \( A_jB = FA_j = 0 \) imply that \( B \) and \( F \) can be written respectively as \( \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & F_2 & F_3 \end{pmatrix} \) (where \( B_1 \) (resp. \( B_2 \)) has \( j \) (resp. \( i-2j \)) rows and \( F_2 \) (resp. \( F_3 \)) has \( (i-2j) \) (resp. \( j \)) columns). But \( A_j \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ 0 \end{pmatrix} \), hence if \( x = \begin{pmatrix} 0 \\ 0 \\ B_1 \end{pmatrix} \) then:

\[
\begin{pmatrix} I & x & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_j & B & C \\ 0 & 0 & F \\ 0 & 0 & A_j \end{pmatrix} \begin{pmatrix} I & x & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} A_j & B + A_jx & C + xF \\ 0 & 0 & F \\ 0 & 0 & A_j \end{pmatrix}
\]

4
and \( B + A_j x = \begin{pmatrix} 0 \\ B_2 \\ 0 \end{pmatrix} \).

Similarly, \( (y_1 \ y_2 \ y_3 \ ) A_j = \begin{pmatrix} 0 & 0 & y_1 \end{pmatrix} \). Hence after a further conjugation we may assume in addition that \( F_3 = 0 \). In other words, any element of \( Y_j \) is \( Z_G(e_i) \)-conjugate to one of the form \( \begin{pmatrix} A_j & B & C \\ 0 & 0 & F \\ 0 & 0 & A_j \end{pmatrix} \) such that \( B = \begin{pmatrix} 0 \\ B_2 \\ 0 \end{pmatrix} \) and \( F = \begin{pmatrix} 0 & F_2 & 0 \end{pmatrix} \). The equality

\[
[A_j, C] + BF = 0
\]

now implies that \( [A_j, C] = 0, B_2 F_2 = 0 \). But if

\[
C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}
\]

where \( C_{11}, C_{13}, C_{31} \) and \( C_{33} \) (resp. \( C_{12} \) and \( C_{32} \)) are \( j \times j \) (resp. \( j \times (i-2j) \)), \((i-2j) \times j, (i-2j) \times (i-2j))\), then \( [A_j, C] = \begin{pmatrix} C_{31} & C_{32} & C_{11} + C_{33} \\ 0 & 0 & C_{21} \\ 0 & 0 & C_{31} \end{pmatrix} \). Hence, conjugating further by \( \begin{pmatrix} I_i & 0 & z \\ 0 & I_{n-2i} & 0 \\ 0 & 0 & I_i \end{pmatrix} \), where \( z = \begin{pmatrix} C_{13} & 0 & 0 \\ C_{23} & 0 & 0 \\ C_{11} & C_{12} & 0 \end{pmatrix} \), we may assume that \( C \) is of the form \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

We therefore introduce the subset \( Y_j' \) of \( Y_j \) consisting of all \( x \) of the form \( \begin{pmatrix} A_j & B & C \\ 0 & 0 & F \\ 0 & 0 & A_j \end{pmatrix} \) with \( B = \begin{pmatrix} 0 \\ B_2 \\ 0 \end{pmatrix} \), \( F = \begin{pmatrix} 0 & F_2 & 0 \end{pmatrix} \), \( C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \), and \( B_2 F_2 = 0 \). We have proved that \( Z_G(e_i) \cdot Y_j = Z_G(e_i) \cdot Y_j' \). Note that the \((n-4j) \times (n-4j)\) matrix \( \begin{pmatrix} 0 & B_2 & C_{22} \\ 0 & 0 & F_2 \\ 0 & 0 & 0 \end{pmatrix} \) is an element of \( \mathcal{N}_1(\mathfrak{gl}(n-4j)) \) which commutes with \( \begin{pmatrix} 0 & 0 & I_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

**Remark 1.3.** If we consider pairs \((x, y) \in \mathcal{C}_{n_1}^{n_1} \) as modules for \( k[X, Y]/(X^2, Y^2) \), then we have shown that for any \( y \in Y_j \) the module \( M \) corresponding to \((e_i, y)\) can be expressed as \( M = W \oplus M' \), where \( W \) is a free \( k[X, Y]/(X^2, Y^2) \)-module of rank \( j \) and \( M' \) is a submodule of \( M \) which is annihilated by \( XY \).

**Lemma 1.4.** Each irreducible component of \( Y \) is properly contained in a closed irreducible subset of \( \mathfrak{g}(e_i) \cap \mathcal{N}_1 \).

**Proof.** We prove the lemma by induction on \( n \). There is nothing to prove if \( n = 2 \) or \( n = 3 \) (since we assume \( n-2i \geq 2 \)). By the above remarks, each component of \( Y \) is contained in one of the sets \( \overline{Y}_j \).
We note that $Y_0 = u \cap \mathcal{N}_1$, where $u$ is the Lie algebra of the unipotent radical of $Z_G(e_i)$. (Hence $Y_0$ is already closed.) On the other hand let $j > 0$. Let $e'$ be a nilpotent element of $\mathfrak{gl}(n - 4j)$ of partition type $2^{i-2j}.1^{n-2i}$ (in a form as described after Lemma 1.1), let $u'$ be the Lie algebra of the unipotent radical of $Z_{\mathfrak{gl}(n-4j)}(e')$ and let $a = \begin{pmatrix} A_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_j \end{pmatrix} \in \mathfrak{z}_\mathfrak{gl}(e_i)$.

We define an injective homomorphism of restricted Lie algebras $\mu : \mathfrak{z}_\mathfrak{gl}(n-4j)(e') \to \mathfrak{z}_\mathfrak{gl}(e)$ by $\begin{pmatrix} A' & B' & C' \\ 0 & E' & F' \\ 0 & 0 & A' \end{pmatrix} \mapsto \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & A \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & C' & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & B' \\ 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & F' & 0 \end{pmatrix}$. Here the zero submatrices on the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre middle) in $A$ and $C$ are $j \times j$ (resp. $j \times (i-2j)$, $(i-2j) \times j$); those on the top and bottom in $B$ are $j \times (n-2i)$; and those on the left and right in $F$ are $(n-2i) \times j$. Clearly $a$ commutes with the image of $\mu$ and by the remarks above $a + \mu(u' \cap \mathcal{N}_1) = Y'_j$. But the lemma now follows for $Y'_j$ (and therefore for $\mathfrak{Y} = \mathfrak{Z}_G(e_i) \cdot \mathfrak{Y}_j$) by the induction hypothesis. Hence we have only to prove that the statement of the lemma is true for $Y_0$.

Note that $Y_0$ is the set of $x \in \mathfrak{z}_\mathfrak{gl}(e_i)$ such that $A = 0$, $E = 0$ and $BF = 0$. For each $l$ with $0 \leq l \leq \min{i, n-2i}$, let $b_l$ be the $i \times (n-2i)$ matrix of the form $\begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}$. Here the left (resp. right) column is of width $l$ (resp. $n-2i-l$), and the top (resp. bottom) row is of height $l$ (resp. $i-l$). Since $Z_G(e_i)$ contains all elements of the form $\begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & g \end{pmatrix}$, it is easy to see that $Y_0 = \bigcup_l \mathfrak{Z}_G(e_i) \cdot Z_l$, where

$$Z_l = \left\{ \begin{pmatrix} 0 & b_l & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} : b_l F = 0 \right\}.$$  

Moreover, the sets $\mathfrak{Z}_G(e_i) \cdot Z_l$ are clearly irreducible closed subsets of $Y_0$, and $b_l F = 0$ if and only if $F$ can be written in the form $\begin{pmatrix} 0 \\ f \end{pmatrix}$, where the top part has $l$ rows, and the bottom has $(n-2i-l)$ rows.

Hence it will be enough to prove that, for each $l$, there is a closed irreducible subset of $\mathfrak{z}_\mathfrak{gl}(e_i) \cap \mathcal{N}_1$ which contains $Z_l$ and is not contained in $Y$. Suppose $0 < l < n-2i$, and let $E_0$ be the $(n-2i) \times (n-2i)$ matrix with 1 in the $((n-2i),1)$ position and 0 elsewhere. Then $b_l E_0 = 0$ and $E_0 \begin{pmatrix} 0 \\ f \end{pmatrix} = 0$, hence the set $\left\{ \begin{pmatrix} 0 & b_l & C \\ 0 & \xi E_0 & F \\ 0 & 0 & 0 \end{pmatrix} : \xi \in k, b_l F = 0 \right\}$ is a closed irreducible subset of $\mathfrak{z}_\mathfrak{gl}(e_i) \cap \mathcal{N}_1$ which properly contains $Z_l$. This proves the lemma in this
case. Let \( \theta \) be the automorphism of \( \mathfrak{g} \) given by \( x \mapsto -J(x)J^{-1} \), where \( J \) is the element of \( \text{GL}(n, k) \) with 1 on the antidiagonal, and 0 elsewhere. Then \( \theta(e) = -e \) (hence \( \theta \) stabilizes \( \mathfrak{g}(e) \cap \mathcal{N}_1 \)) and \( \theta \) sends \( \mathbb{Z}_G(e) \cdot Z_0 \) into \( \mathbb{Z}_G(e) \cdot Z_r \), where \( r = \min\{n - 2i, i\} \). Hence we have only to prove the statement of the lemma for \( Z_{n-2i} \) (assuming therefore that \( n - 2i \leq i \)).

Let \( b = b_{n-2i} \); we note that left matrix multiplication by \( b \) is injective. Consider the set \( U \) of all \( x \in \mathfrak{g}(e_i) \cap \mathcal{N}_1 \) of the form \( \begin{pmatrix} A & b & C \\ 0 & E & F \\ 0 & 0 & A \end{pmatrix} \). Then the conditions for \( x \) to be in \( \mathcal{N}_1 \) can be written as: \( A^2 = [A, C] + bF = 0, \ E^2 = 0, \ Ab = bE, \) and \( EF = FA \). But if \( Ab = bE \) then \( EF = FA \) if and only if \( AbF = bFA \). Since \( (\text{ad} A)^2 = \text{ad}(A^2) \) it follows that the condition \( EF = FA \) is redundant.

If \( Ab = bE \) then we can write \( A \) in the form \( \begin{pmatrix} E & A_{12} \\ 0 & A_{22} \end{pmatrix} \), where \( A_{12} \in \text{Mat}((n - 2i) \times (3i - n)) \) and \( A_{22} \in \text{Mat}(3i - n) \times (3i - n) \). Write \( C \) and \( F \) respectively in the following forms:

\[
C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 & F_2 \\ 0 & 0 \end{pmatrix}
\]

where the left- (resp. right-) hand columns are of width \((n - 2i)\) (resp. \((3i - n)\)) and the top (resp. bottom) row of \( C \) is of height \((n - 2i)\) (resp. \((3i - n)\)). Then \( x \in \mathcal{N}_1 \) if and only if \( E^2 = 0, \ A_{22}^2 = 0, \ EA_{12} = A_{12}A_{22} \) and

\[
\begin{pmatrix} F_1 & F_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [E, C_{11}] + A_{12}C_{21} & EC_{12} + C_{12}A_{22} + A_{12}C_{22} + C_{11}A_{12} \\ A_{22}C_{21} + C_{21}E & [A_{22}, C_{22}] + C_{21}A_{12} \end{pmatrix}
\]

These equalities can be restated as an expression for \( F \) in terms of \( A, C, E \) together with the conditions:

\[
A_{22}^2 = [A_{22}, C_{22}] + C_{21}A_{12} = 0, \ A_{22}C_{21} + C_{21}E = 0, \ E = 0, \ EA_{12} + A_{12}A_{22} = 0
\]

But these conditions are equivalent to:

\[
\begin{pmatrix} A_{22} & C_{21} & C_{22} \\ 0 & E & A_{12} \\ 0 & 0 & A_{22} \end{pmatrix}^2 = 0.
\]

Let \( e' \) be a nilpotent element of \( \mathfrak{gl}(4i - n) \) of type \( 2^{4i-n}1^{n-2i} \), in the form described after Lemma [1.1]. Then we have proved that there is an isomorphism of affine varieties \( U \rightarrow (\mathfrak{g}(4i - n)(e') \cap \mathcal{N}_1) \times \text{Mat}_{(n-2i)\times i} \). Specifically, an element \( x \) of the above form is sent to the pair

\[
\begin{pmatrix} A_{22} & C_{21} & C_{22} \\ 0 & E & A_{12} \\ 0 & 0 & A_{22} \end{pmatrix}, \begin{pmatrix} C_{11} & C_{12} \end{pmatrix}.
\]

Notice that \( 4i - n < n \), and that \((4i - n) - 2(3i - n) = n - 2i \geq 2 \). Hence it follows by the induction hypothesis that each irreducible component of \( \mathfrak{g}(4i - n)(e') \cap \mathcal{N}_1 \) contains an element such that \( E \neq 0 \). Thus the same is true for each irreducible component of \( U \). But \( Z_{n-2i} \) is clearly contained in some irreducible component of \( U \). This completes the proof of the lemma.

\[\Box\]
We therefore have the required result of this section:

**Lemma 1.5.** The condition $V_i^{(j)} \subseteq \overline{G \cdot e_i}$ holds if and only if $i = [n/2]$, that is, if and only if $G \cdot e_i = \mathcal{N}_1$.

*Proof.* This follows immediately from Lemmas [1.2] and [1.4] \hfill \square

## 2 Equidimensionality

We proved in the previous section that $C^{nil}_1 = \overline{G \cdot \mathfrak{g}(e_m)} \cap \mathcal{N}_1$, where $m = [n/2]$. In this section we will show that $\mathfrak{z}_0(e) \cap \mathcal{N}_1$ is equidimensional, hence so is $C^{nil}_1$. In fact, we prove equidimensionality of $\mathfrak{z}_0(e) \cap \mathcal{N}_1$ for an arbitrary $e \in \mathcal{N}_1$.

**Lemma 2.1.** Let $e \in \mathcal{N}_1$ be of partition type $2^t. 1^{n-t}$.

(a) If $n$ is even then $\mathfrak{z}_0(e) \cap \mathcal{N}_1$ has $([i/2] + 1)$ irreducible components, each of dimension $(n^2 + (n - 2i)^2)/4 = \dim \mathfrak{z}_0(e)/2$.

(b) If $n$ is odd, then $\mathfrak{z}_0(e) \cap \mathcal{N}_1$ has $(i + 1)$ irreducible components, each of dimension $(n^2 + (n - 2i)^2 - 2)/4 = (\dim \mathfrak{z}_0(e) - 1)/2$.

*Proof.* As in Sect. 1 we choose $e = \begin{pmatrix} 0 & 0 & I_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where the top and bottom rows are of height $i$, the left and right columns are of width $i$, and the central row (resp. middle column) is of height (resp. width) $(n - 2i)$. For $0 \leq j \leq [i/2]$ let $A_j$ be the $i \times i$ matrix of the form $\begin{pmatrix} 0 & 0 & I_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre right, central) submatrices are $j \times j$ (resp. $j \times (i - 2j)$, $(i - 2j) \times j$, $(i - 2j) \times (i - 2j)$). Similarly, for $0 \leq l \leq [(n - 2i)/2]$, let $E_l$ be the matrix of the form $\begin{pmatrix} 0 & 0 & I_l \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre right, central) submatrices are $l \times l$ (resp. $l \times (n - 2(i + l))$, $(n - 2(i + l)) \times l$, $(n - 2(i + l)) \times (n - 2(i + l))$).

We denote by $V_{j,l}$ ($0 \leq j \leq [i/2], 0 \leq l \leq [(n - 2i)/2]$) the set of all $x \in \mathfrak{z}_0(e) \cap \mathcal{N}_1$ of the form $\begin{pmatrix} A_j & B & C \\ 0 & E_l & F \\ 0 & 0 & A_j \end{pmatrix}$. Clearly $\mathfrak{z}_0(e) \cap \mathcal{N}_1 = \bigcup_{j,l} \overline{Z_G(e) \cdot V_{j,l}}$, hence each irreducible component of $\mathfrak{z}_0(e) \cap \mathcal{N}_1$ is equal to $\overline{Z_G(e) \cdot V_{j,l}^{nil}}$ for some irreducible component $V_{j,l}^{nil}$ of (some) $V_{j,l}$. We claim that if $(n - 2i) - 2l \geq 2$ then no such set $\overline{Z_G(e) \cdot V_{j,l}^{nil}}$ is an irreducible component.

Let $x = \begin{pmatrix} A_j & B & C \\ 0 & E_l & F \\ 0 & 0 & A_j \end{pmatrix} \in V_{j,l}$, that is, we assume $A_jB = BE_l$, $E_lF = FA_j$ and
\[ [A_j, C] = BF. \] Write \( B \) as \( \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \), where \( B_{11}, B_{13}, B_{31} \) and \( B_{33} \) (resp. \( B_{12} \) and \( B_{21}, B_{23}, B_{22} \)) are \( j \times l \) (resp. \( j \times (n-2(i+l)), (i-2j) \times l, (i-2j) \times (n-2(i+l)) \)) matrices. Then \( A_j B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & 0 & 0 \\ B_{31} & 0 & 0 \end{pmatrix} \) and \( B E_l = \begin{pmatrix} 0 & 0 & B_{11} \\ 0 & 0 & B_{31} \end{pmatrix} \). Hence \( A_j B = B E_l \) if and only if \( B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{11} \end{pmatrix} \). Similarly, \( F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & 0 & F_{11} \end{pmatrix} \), where \( F_{ij} \) has the same dimensions as \( B_{ij}^t \).

But

\[
\begin{pmatrix} I & y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_j & B & C \\ 0 & E_l & F \\ 0 & 0 & A_j \end{pmatrix} \begin{pmatrix} I & y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} A_j & B + A_j y + y E_l & C + y F \\ 0 & E_l & F \\ 0 & 0 & A_j \end{pmatrix}.
\]

Setting \( y = \begin{pmatrix} B_{13} & 0 & 0 \\ B_{23} & 0 & 0 \\ B_{11} & B_{12} & 0 \end{pmatrix} \), we see that after conjugating by a suitable element of \( Z_G(e) \) we may assume that \( B \) is of the form \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Similarly, conjugating further by a suitable element of the form \( \begin{pmatrix} I & 0 & 0 \\ 0 & I & y \\ 0 & 0 & I \end{pmatrix} \) we may assume that \( F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Under these assumptions it is clear that the condition \( [A_j, C] = BF \) implies that \( B_{22} F_{22} = 0 \) and \( [A_j, C] = 0 \). Finally, conjugating further by an element of the form \( \begin{pmatrix} I & 0 & y \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \), we may assume that \( C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \), where the zero submatrices on the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre right) are \( j \times j \) (resp. \( j \times (i-2j), (i-2j) \times j \)), and \( C_{22} \) is \( (i-2j) \times (i-2j) \). Let \( U_{j,l} \) be the set of all \( x \in \mathfrak{g}(e) \cap \mathcal{N}_1 \) with \( B, F, \) and \( C \) of this form. Then we have proved that \( Z_G(e) \cdot V_{j,l} = Z_G(e) \cdot U_{j,l} \).

Let \( a = \begin{pmatrix} A_j & E_l \\ E_l & A_j \end{pmatrix} \in \mathfrak{gl}(n, k) \) and let \( e' \) be a nilpotent element of \( \mathfrak{gl}(n - 4j) \) of type \( 2^{i-2j}1^{n-2i} \), in a form analogous to the \( e_i \) defined after Lemma 1.4. Let \( u' \) be the Lie algebra of the unipotent radical of \( Z_{GL(n-4j)}(e') \). There is an injective restricted Lie algebra

9
homomorphism $\mu : \mathfrak{gl}(n-4j)(e') \to \mathfrak{gl}(e)$ given by 
\[
\begin{pmatrix}
A' & B' & C' \\
0 & E' & F' \\
0 & 0 & A'
\end{pmatrix}
\mapsto
\begin{pmatrix}
A & B & C \\
0 & E & F \\
0 & 0 & A
\end{pmatrix},
\]

where $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A' & 0 \\ 0 & 0 & 0 \end{pmatrix}$ such that the zero submatrices on the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre right) are $j \times j$ (resp. $j \times (i - 2j)$, $(i - 2j) \times j$), and similarly for $B, C, E, F$. (The dimensions for the corresponding submatrices of $B$ are $j \times l$, $j \times (n - 2(i + l))$ and $(i - 2j) \times l$; for $C$ are the same as for $A$; for $E$ are $l \times l$, $l \times (n - 2(i + l))$ and $(n - 2(i + l)) \times l$; and for $F$ are $l \times j$, $l \times (i - 2j)$ and $(n - 2(i + l)) \times j$.)

Clearly $a$ commutes with the image of $\mu$, $a + \mu(\mathfrak{gl}(n-4j)(e') \cap \mathcal{N}_1) \subset \mathfrak{gl}(e) \cap \mathcal{N}_1$ and $a + u' \cap \mathcal{N}_1 = U_{j,l}$. But by Lemma [1.4] $u' \cap \mathcal{N}_1$ is properly contained in a closed irreducible subset of $\mathfrak{gl}(n-4j)(e') \cap \mathcal{N}_1$. It follows that $U_{j,l}$ is not an irreducible component of $\mathfrak{gl}(e) \cap \mathcal{N}_1$ unless $l = [(n - 2i)/2]$.

We can now prove equidimensionality. If $e$ is of type $2^i.1^2(m-i)$ then the only sets $Z_G(e) \cdot V_{j,l}^0$ which can be irreducible components of $\mathfrak{gl}(e) \cap \mathcal{N}_1$ are those with $l = (m - i)$. But now, with the above description of $x \in V_{j,l}$ the possible $B$ and $F$ are:

\[
B = \begin{pmatrix}
B_{11} & B_{13} \\
0 & B_{23} \\
0 & 0 \\
\end{pmatrix},
\quad F = \begin{pmatrix}
F_{11} & F_{12} & F_{13} \\
0 & 0 & F_{11} \\
\end{pmatrix}.
\]

Here $B_{11}$ and $B_{13}$ are $j \times (m - i)$ matrices, $B_{23}$ is $(i - 2j) \times (m - i)$, $F_{11}$ and $F_{13}$ are $(m - i) \times j$, and $F_{12}$ is $(m - i) \times (i - 2j)$. The condition $BF = [A_j, C]$ then simply specifies a unique possible $C$ modulo $\mathfrak{gl}(i)(A_j)$. It follows that $V_{j,(m-i)}$ is irreducible of dimension $2i(m - i) + \dim \mathfrak{gl}(i)(A_j)$. Let $W_{j,(m-i)} = Z_G(e) \cdot V_{j,(m-i)}$ and consider the morphism $\pi : W_{j,(m-i)} \to \mathfrak{gl}(i) \times \mathfrak{gl}(n-2i)$ given by

\[
\begin{pmatrix}
A & B & C \\
0 & E & F \\
0 & 0 & A
\end{pmatrix} \mapsto (A, E).
\]

It is easy to see that the image of $\pi$ is $\text{GL}(i) \cdot A_j \times \mathcal{N}_1(\mathfrak{gl}(n-2i))$. Moreover, $(\text{GL}(i) \cdot A_j) \times (\text{GL}(n-2i) \cdot E_{m-i})$ is an open subset of $\pi(W_{j,(m-i)})$ and the fibre over each such point is isomorphic to (in fact, conjugate to) $V_{j,(m-i)}$. It follows by the standard theorem on dimensions (see for example [8, Thm. 4.3]) that $\dim W_{j,(m-i)} = i^2 - \dim \mathfrak{gl}(i)(A_j) + 2(m - i)^2 + 2i(m - i) + \dim \mathfrak{gl}(i)(A_j) = 2m^2 - 2mi + i^2$.

Since this dimension is independent of $j$, the sets $W_{j,(m-i)}$, $0 \leq j \leq |i/2|$ are the irreducible components of $\mathfrak{gl}(e) \cap \mathcal{N}_1$. Moreover, $\{0\} = \text{GL}(i) \cdot A_0 \subset \text{GL}(i) \cdot A_1 \subset \ldots \subset \text{GL}(i) \cdot A_{|i/2|}$, hence $W_{j,(m-i)}$ cannot be contained in $W_{j',(m-i)}$ for $j' < j$. By equality of dimensions, the $W_{j,(m-i)}$ are distinct.

Similarly, if $e$ is of type $2^j.1^2(m-i)+1$ then any irreducible component is of the form $Z_G(e) \cdot V_{j,(m-i)}^0$ for some $j$ and some irreducible component $V_{j,(m-i)}^0$ of $V_{j,(m-i)}$. If $2j < i$ then the possible $B, F$ are of the form:
\[
B = \begin{pmatrix}
B_{11} & b_1 & B_{13} \\
0 & b_2 & B_{23} \\
0 & 0 & B_{11}
\end{pmatrix}, \\
F = \begin{pmatrix}
F_{11} & F_{12} & F_{13} \\
0 & f_2 & f_3 \\
0 & 0 & F_{11}
\end{pmatrix}
\]

where \(B_{11}\) and \(B_{13}\) are \(j \times (m - i)\) matrices, \(F_{11}\) and \(F_{13}\) are \((m - i) \times j\) matrices, \(b_1\) (resp. \(b_2\)) is a column vector of dimension \(j\) (resp. \(i - 2j\)), \(f_2\) (resp. \(f_3\)) is a row vector of dimension \(i - 2j\) (resp. \(j\)), \(B_{23}\) is an \((i - 2j) \times (m - i)\) matrix, and \(F_{12}\) is \((m - i) \times (i - 2j)\).

In this case the condition \(BF = [A_j, C]\) specifies a unique value of \(C\) modulo \(\mathfrak{g}(i, j)\), and in addition the requirement that \(b_2f_3 = 0\). But \(b_2\) is a column vector and \(f_2\) a row vector, hence \(b_2f_2 = 0\) implies that either \(b_2 = 0\) or \(f_2 = 0\). It follows that \(V_{j, (m-i)}^+\) has two irreducible components of equal dimension. Denote by \(V_{j, (m-i)}^+\) the irreducible component defined by \(f_3 = 0\), and by \(V_{j, (m-i)}^-\) the irreducible component satisfying \(b_2 = 0\). Therefore, \(\dim V_{j, (m-i)}^+ = \dim V_{j, (m-i)}^- = 2i(m-i) + i + \dim \mathfrak{g}(i, j)\).

Let \(W_{j, (m-i)}^+ = Z_G(e) \cdot V_{j, (m-i)}^+\) and let \(W_{j, (m-i)}^- = Z_G(e) \cdot V_{j, (m-i)}^-\). Then by the argument used above, \(W_{j, (m-i)}^+\) and \(W_{j, (m-i)}^-\) are (irreducible) of dimension \(2m^2 - 2mi + i^2 + 2m - i = (\dim \mathfrak{g}(e) - 1)/2\). If \(i\) is even and \(j = i/2\), then the possible \(B, F\) are of the form:

\[
B = \begin{pmatrix}
B_{11} & b_1 & B_{13} \\
0 & b_2 & B_{23} \\
0 & 0 & B_{11}
\end{pmatrix}, \\
F = \begin{pmatrix}
F_{11} & F_{12} & F_{13} \\
0 & f_2 & f_3 \\
0 & 0 & F_{11}
\end{pmatrix}
\]

where \(B_{11}\) and \(B_{13}\) are \(j \times (m - i)\) matrices, \(F_{11}\) and \(F_{13}\) are \((m - i) \times j\) matrices, and \(b_1\) (resp. \(f_3\)) is a column (resp. row) vector of dimension \(j\). Here the condition \(BF = [A_j, C]\) merely specifies a unique value of \(C\) modulo \(\mathfrak{g}(i, j)\). It follows that \(V_{j, (m-i)}^+\) is irreducible of dimension \(2i(m-i) + i + \dim \mathfrak{g}(i, j)\).

Let \(W_{i/2, (m-i)}^+ = Z_G(e) \cdot V_{j, (m-i)}^+\). Then, by exactly the same argument used for the case \(j < i/2\), we can see that \(\dim W_{i/2, (m-i)}^+\) is irreducible of dimension \((\dim \mathfrak{g}(e) - 1)/2\).

By equality of dimensions, each irreducible component of \(\mathfrak{g}(e) \cap N_i\) is equal to one of the \(W_{j, (m-i)}^+\) \((0 \leq j \leq (i-1)/2)\) if \(i\) is odd (resp. one of the \(W_{j, (m-i)}^-\) \((0 \leq j \leq i/2 - 1)\) or \(W_{i/2, (m-i)}^-\) if \(i\) is even). Note that there are \((i+1)\) possible choices in either case. The argument used above for the case where \(n\) is even shows that we cannot have \(W_{j, (m-i)}^+ = W_{j, (m-i)}^-\) if \(j \neq j'\).

Similarly, if \(i\) is even then \(W_{i/2, (m-i)}^- \neq W_{j, (m-i)}^+\) for \(j < i/2\). Hence it remains to show that \(W_{j, (m-i)}^+ \neq W_{j, (m-i)}^-\). By definition \(W_{j, (m-i)}^+ = Z_G(e) \cdot V_{j, (m-i)}^+\). It is easy to see that \(W_{j, (m-i)}^+ = W_{j, (m-i)}^-\) is equivalent to: \(V_{j, (m-i)}^+ \subset Z_G(e) \cdot V_{j, (m-i)}^+\). Let \(x \in V_{j, (m-i)}^+\) and suppose that \(g \in Z_G(e)\) satisfies \(g x g^{-1} \in V_{j, (m-i)}^+\). Then clearly \(g\) is of the form \(\begin{pmatrix} h_1 & y_1 & y_2 \\ 0 & h_2 & y_3 \\ 0 & 0 & h_1 \end{pmatrix}\) for some \(h_1 \in Z_{GL(i, j)}\) and \(h_2 \in Z_{GL(2(m-i)+1)}\). Moreover, any \(g\) of this form normalizes \(V_{j, (m-i)}\). Let \(L\) be the subgroup of \(Z_{GL(e)}\) of all elements of this form: \(L\) is isomorphic to a product \(Z_{GL(i, j)} \times Z_{GL(2m-i)+1}\). But therefore \(L\) is connected, and hence preserves \(V_{j, (m-i)}^+\). It follows that \(W_{j, (m-i)}^+ \neq W_{j, (m-i)}^-\).

This completes the proof. \(\square\)
Remark 2.2. Note that for any irreducible component \( V \) of \( \mathfrak{z}_g(e) \cap \mathcal{N}_1 \) the intersection with the open orbit in \( \mathcal{N}_1 \) is non-empty, therefore open. This is not true for general \( p \).

We now have our result.

Proposition 2.3. Let \( k \) be an algebraically closed field of characteristic 2 and let \( g = \mathfrak{gl}(n, k) \). Then the restricted nilpotent commuting variety \( \mathcal{C} \) of \( g \) is equidimensional.

(a) If \( n = 2m \) then there are \([m/2] + 1\) irreducible components of \( \mathcal{C} \) of dimension \( 3m^2 \).

(b) If \( n = 2m + 1 \) then there are \((m + 1)\) irreducible components of \( \mathcal{C} \) of dimension \( 3m(m + 1) \).

Proof. By Lemma 1.5 each irreducible component of \( \mathcal{C} \) is of the form \( \mathcal{C}(e, V) \) where \( V \) is an irreducible component of \( \mathfrak{z}_g(e) \cap \mathcal{N}_1 \). Let \( V \) be such an irreducible component, let \( X = \mathcal{C}(e, V) \) and let \( \pi : X \to \mathcal{N}_1 \) be the restriction to \( X \) of the first projection. Since \( g \cdot (e, V) \) contains an open subset of its closure, there is an open subset \( U \) of \( \mathcal{N}_1 \) such that \( \dim \pi^{-1}(x) = \dim V \) for any \( x \in U \). It follows by the standard theorem on dimensions [8, Thm. 4.3] that \( \dim X = \dim \mathcal{N}_1 + \dim V = \dim g - r \), where \( r \) is the codimension of \( X \) in \( \mathfrak{z}_g(e) \). By Lemma 2.1, \( \mathfrak{z}_g(e) \cap \mathcal{N}_1 \) is equidimensional of dimension \( m^2 \) (resp. \( m^2 + m \)) if \( n = 2m \) (resp. \( n = 2m + 1 \)). But hence each of the sets \( G \cdot (e, V) \) is an irreducible component of \( \mathcal{C} \), and is of the dimension stated in the proposition. Moreover, if \( X_1 = \mathcal{C}(e, V_1) = G \cdot (e, V_2) = X_2 \) then, since \( G \cdot (e, V_1) \) contains an open subset of \( X_1 \), \( Z_G(e) \cdot V_1 \) contains an open subset of \( V_2 \), and therefore \( V_1 = V_2 \). This completes the proof of the proposition.

For later use we now label the components of \( \mathcal{C} \). Recall from the proof of Lemma 2.1 that if \( n = 2m \) (resp. \( n = 2m + 1 \)) then the irreducible components of \( \mathfrak{z}_g(e) \) are the sets of the form \( Z_G(e) \cdot V_{j,0} \) (resp. \( Z_G(e) \cdot V_{j,0}^+ \) and \( Z_G(e) \cdot V_{m/2,0} \)) if \( m \) is even) for \( 0 \leq j \leq [m/2] \) (resp. \( 0 \leq j < m/2 \)). If \( n = 2m \) then let \( X_j = G \cdot (e, V_j) \). If \( n = 2m + 1 \) then let \( X_j^+ = G \cdot (e, V_{j,0}^+) \), \( X_j^- = G \cdot (e, V_{j,0}^-) \) and \( X_{m/2} = G \cdot (e, V_{m/2,0}) \) if \( m \) is even.

Remark 2.4. One might ask whether Lemma 1.5 or Prop. 2.3 is true for fields of arbitrary characteristic. In fact they both fail. For example, let \( k \) be of characteristic 7 and let \( g = \mathfrak{gl}(14) \). Let \( e \) be a nilpotent element of \( g \) of type \( 7^2 \). Hence \( G \cdot e = \mathcal{N}_1(\mathfrak{g}) \). It is easy to see that \( \mathfrak{z}_g(e) \equiv \mathfrak{gl}(2, k[t]/(t^7)) \). Identify \( \mathfrak{z}_g(e) \) with \( \mathfrak{gl}(2, k[t]/(t^7)) \) and write an element \( A \in \mathfrak{z}_g(e) \) as \( A_0 + A_1 t + \ldots + A_6 t^6 \) where \( A_i \in \mathfrak{gl}(2, k) \). We have \( \sum_{i=0}^6 A_i t^i \) is the sum of all ordered monomials \( A_{i_1} A_{i_2} \ldots A_{i_j} \) such that \( i_1 + i_2 + \ldots + i_j = i \). Clearly \( x^7 = 0 \) if \( A_0 = 0 \). Hence \( t \mathfrak{gl}(2, k[t]/(t^7)) \) is a closed irreducible subset of \( \mathfrak{z}_g(e) \cap \mathcal{N}_1 \) of dimension 24.

On the other hand, if \( A_0 \neq 0 \), then up to conjugacy there is only one possibility such that \( A_0^7 = 0 \), namely \( A_0 = e_{12} \). Suppose therefore that \( A_0 = e_{12} \). We will determine the conditions on the matrices \( A_i \) such that \( A^7 = 0 \). We remark first of all that \( A_0^7 = 0 \), from which it follows that \( p_1(A) = A_0^6 A_1 + \ldots + A_1 A_0^6 = 0 \), and \( p_2(A) = A_0^6 A_2 + \ldots + A_2 A_0^6 + A_0^6 A_2^4 + \ldots + A_1^2 A_0^4 = 0 \). The expression for \( p_3(A) \) reduces to \( (A_0 A_1)^3 A_0 \). It follows that \( A_1 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) for some \( a, b, d \in k \). Hence \( A_0 A_1^j A_0 = 0 \) for any \( j \geq 0 \). Inspection of
the possible non-zero terms of $p_4$ reveals that $p_4(A) = 0$. Similarly, $p_5(A)$ reduces easily to $A_6^3 A_2 A_2 + \ldots + A_2 A_2 A_6 + A_6 A_2^3 + \ldots + A_2 A_2 A_6$. But each term here either contains $A_6^3$ or it contains $A_2 A_2 A_6$. Therefore $p_5(A) = 0$ also. Finally, we have $p_6(A) = (A_0 A_2)^5 A_0 + A_6^3 A_2 A_2 + \ldots + A_2 A_2 A_6^3 + A_6 A_2^3 A_2 + \ldots + A_2 A_2 A_6$. Hence the set $U$ of $A \in \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1$ such that $A_0 = e_{12}$ is an irreducible Zariski closed subset of $\mathfrak{z}_{\mathfrak{g}}(e)$ of dimension 22. Thus (by the standard theorem on dimensions) $\mathbb{Z}_G(e) \cdot \bar{U}$ is an irreducible subset of $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1$ of dimension 24. We have proved that $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1$ has two irreducible components $V_1$ and $V_2$, both of codimension 4 in $\mathfrak{z}_{\mathfrak{g}}(e)$. We deduce that the closures $G \cdot (e; V_1)$ and $G \cdot (e; V_2)$ are irreducible components of $C_{\mathfrak{z}_{\mathfrak{g}}(e)}(\mathfrak{g})$ (since $G \cdot e$ is not contained in the closure of any other orbit in $\mathcal{N}_1$) and are of dimension $(\dim \mathfrak{g} - 4) = 192$.

On the other hand, let $e'$ be an element of $\mathcal{N}_1(\mathfrak{g})$ of type $7^1.5^1.2^1$, which we may choose to be in Jordan normal form. Let $T$ be the group of invertible diagonal matrices in $G$ and let $\lambda : k^\times \to T$ be the cocharacter such that $e' \in \mathfrak{g}(2; \lambda)$ and the component of $\lambda(t)$ in each Jordan block has determinant 1. (Hence $\lambda$ is an associated cocharacter for $e'$ in the sense of Pommereunen [11, 12].) In particular, $\mathfrak{z}_{\mathfrak{g}}(e') \subset \sum_{i \geq 0} \mathfrak{g}(i; \lambda_i).$ Let $\mathfrak{g}(i) = \mathfrak{g}(i; \lambda)$ for each $i \in \mathbb{Z}$. Recall that a *toral algebra* $\mathfrak{h}$ is a commutative restricted Lie algebra which has a basis $\{h_1, \ldots, h_s\}$ of elements such that $h_i h_j = h_j h_i$. It is easily seen that $\mathfrak{z}_{\mathfrak{g}}(e') \cap \mathfrak{g}(0)$ is a toral algebra of dimension 3 and that $\mathfrak{z}_{\mathfrak{g}}(e') \cap \sum_{i > 0} \mathfrak{g}(i) \subset \sum_{i \geq 2} \mathfrak{g}(i)$. Hence $\mathfrak{z}_{\mathfrak{g}}(e') \cap \mathcal{N}_1 = \mathfrak{z}_{\mathfrak{g}}(e') \cap \sum_{i \geq 0} \mathfrak{g}(i)$ is irreducible of codimension 3 in $\mathfrak{z}_{\mathfrak{g}}(e')$. It follows that $G \cdot (e', \mathfrak{z}_{\mathfrak{g}}(e') \cap \mathcal{N}_1)$ is irreducible of dimension $\dim \mathfrak{g} - 3 = 193$. In particular, it is not contained in either irreducible component of $G \cdot (e, \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1)$ (and vice versa, neither irreducible component of $G \cdot (e, \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1)$ is contained in $G \cdot (e', \mathfrak{z}_{\mathfrak{g}}(e') \cap \mathcal{N}_1)$). Hence in this case neither Lemma 1.5 nor equidimensionality of $C_{\mathfrak{z}_{\mathfrak{g}}(e)}(\mathfrak{g})$ hold.

Although Lemma 1.4 and Prop. 2.3 fail in the above example, we note that the intersection $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1$ is nevertheless equidimensional. Indeed, this result appears to be true in general.

**Conjecture.** Let $G$ be a reductive group over $k$, and suppose the characteristic of $k$ is good for $G$. Let $e \in \mathcal{N}_1$. Then $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1$ is equidimensional.

Some laborious but generally straightforward case-checking establishes that the conjecture is true for the cases $G = \text{GL}(4), \text{GL}(5), \text{SO}(5), \text{GL}(6)$. To illustrate the conjecture and the apparent unpredictability of the number of irreducible components of the intersection $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}_1$, we give a few examples. To determine the irreducible components directly one can use a similar approach to that employed above, that is, consider case-by-case the orbits in $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(0) \cap \mathcal{N}_1$ where $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is the grading of $\mathfrak{g}$ induced by an associated cocharacter for $e$.

(a) $G = \text{GL}(5, k), p = 3$.

(i) $e$ of type $3^1.2^1$. Here $\dim \mathfrak{z}_{\mathfrak{g}}(e) = 9$. There are two irreducible components, both of dimension 6.
(ii) e of type $3^1,1^2$. Then $\dim \mathfrak{z}(e) = 11$, and $\mathfrak{z}(e) \cap \mathcal{N}_1$ has two irreducible components, both of dimension 7.

(iii) e of type $2^2,1^1$. Then $\dim \mathfrak{z}(e) = 13$, and the intersection with $\mathcal{N}_1$ has three components of dimension 8.

(iv) e of type $2^1,1^3$. Then $\dim \mathfrak{z}(e) = 17$, and $\mathfrak{z}(e) \cap \mathcal{N}_1$ is irreducible of dimension 11.

(b) $G = \text{GL}(6, k)$, $p = 3$.

(i) e of type $3^2$. Then $\dim \mathfrak{z}(e) = 12$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ has two components of dimension 8.

(ii) e of type $3^1,2^1,1^1$. Then $\dim \mathfrak{z}(e) = 14$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ has five irreducible components of dimension 12.

(iii) e of type $3^1,1^3$. Then $\dim \mathfrak{z}(e) = 18$ and the intersection with $\mathcal{N}_1$ is irreducible of dimension 12.

(iv) e of type $2^3$. Then $\dim \mathfrak{z}(e) = 18$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ has two irreducible components of dimension 12.

For the remaining cases $\mathfrak{z}(e) \cap \mathcal{N}_1$ is irreducible.

(c) $G = \text{Sp}(8, k)$, $p = 3$.

(i) e of type $3^2,2^2$. Then $\dim \mathfrak{z}(e) = 12$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ is irreducible of dimension 9.

(ii) e of type $3^1,1^2$. Then $\dim \mathfrak{z}(e) = 14$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ is irreducible of dimension 10.

(iii) e of type $2^4$. In this case $\dim \mathfrak{z}(e) = 16$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ is irreducible of dimension 11.

(iv) e of type $2^3,1^2$. In this case $\dim \mathfrak{z}(e) = 18$ and $\mathfrak{z}(e) \cap \mathcal{N}_1$ has two irreducible components, both of dimension 12.

(v) e of type $2^2,1^2$. Then $\mathfrak{z}(e)$ is of dimension 22 and $\mathfrak{z}(e) \cap \mathcal{N}_1$ has two irreducible components, both of dimension 15.

(vi) e of type $2^1,1^6$. Then $\mathfrak{z}(e)$ is of dimension 28 and $\mathfrak{z}(e) \cap \mathcal{N}_1$ is irreducible of dimension 19.

Remark 2.5. This conjecture is not true if the characteristic is bad. Indeed, if $k$ is of characteristic 2, $G$ is simply-connected of type $B_2$ and $e = e_{\alpha_1 + \alpha_2}$ (where $\{\alpha_1, \alpha_2\}$ is a basis for the root system of $G$), then $\mathfrak{z}(e) \cap \mathcal{N}_1$ has one irreducible component of dimension 3 and one of dimension 4. On the other hand, the intersection $\text{Lie}(\mathcal{Z}(e)) \cap \mathcal{N}_1$ is in this case irreducible. Finally, we remark that the conjecture is not true for arbitrary nilpotent elements. For example, if $p = 2$ and $e$ is an element of $\mathfrak{sl}(6)$ of type $3^2$, then $\mathfrak{z}(e) \cap \mathcal{N}_1$ has two components, one of dimension 8 and one of dimension 6.

3 Indecomposable components

We remarked in the introduction that the variety $C = C_{\text{nil}}^n$ can be considered as the variety of $n$-dimensional modules for the group algebra $k\Gamma$, where $\Gamma$ is a product of two cyclic groups of order 2. More generally, let $A$ be any finitely generated associative algebra. To give an $r$-dimensional vector space $V$ the structure of an $A$-module is simply to give a homomorphism $A \to \text{End}(V)$. Such a homomorphism is determined by the values on a set of generators for $A$. Hence, on choosing a basis for $V$, the set $\text{mod}_A^r(k)$ of possible $A$-module structures on
$k^r$ embeds as a Zariski closed subset of the product of a finite number of copies of $\text{Mat}_r(k)$. The general linear group $\text{GL}(r,k)$ acts on $\text{mod}^r_A(k)$ by simultaneous conjugation on the coordinates. It is clear that two points of $\text{mod}^r_A(k)$ are $\text{GL}(r,k)$-conjugate if and only if the corresponding modules are isomorphic.

An irreducible component of $\text{mod}^r_A(k)$ is called **indecomposable** if all points in an open subset correspond to indecomposable modules. A version of the Krull-Remak-Schmidt Theorem holds for irreducible components of $\text{mod}^r_A(k)$. Let $C_i : 1 \leq i \leq l$ be irreducible components of varieties of $A$-modules $\text{mod}^r_A(k)$ and assume that $r_1 + r_2 + \ldots + r_l = r$. There is a morphism $\text{GL}(r,k) \times C_1 \times C_2 \times \ldots \times C_l \to \text{mod}^r_A(k)$. Denote the closure of the image by $\overline{C_1 + C_2 + \ldots + C_l}$. Then any irreducible component of $\text{mod}^r_A(k)$ can be expressed in an essentially unique way as a direct sum $\overline{C_1 + C_2 + \ldots + C_l}$ of indecomposable components $C_i$ of module varieties $\text{mod}^r_A(k)$ (originally proved in [3]; see also [2]. Such a direct sum is not always an irreducible component for arbitrary $C_i$; see [2] Thm. 1.2.) Here we express each component of $C$ as a direct sum of indecomposable components.

Recall that the irreducible components of $C$ are labelled $X_j$ if $n = 2m$ with $0 \leq j \leq [m/2]$ (resp. $X_j^\pm$ $(0 \leq j < m/2)$ if $n = 2m + 1$ and $m$ is odd, $X_j^\pm$ $(0 \leq j < m/2)$ and $X_m/2$ if $m$ is even). For arbitrary $r$, we denote by $X_j(g(r))$ or $X_j^\pm(g(r))$ the irreducible components of $C^{\text{nil}}_1(g(r))$ described in this way. Let $W$ be the irreducible component of $C^{\text{nil}}_1(g(4))$ given by $W = \text{GL}(4) \cdot (e_{12} + e_{34}, e_{13} + e_{24})$ (the “free component of rank 1”), and let $U$ be the irreducible component of $C^{\text{nil}}_1(g(2))$ given by $U = \{(ae, be) : e[2] = 0, a, b \in k\}$. It is easy to see that $W$ (resp. $U$) is an indecomposable component of $C^{\text{nil}}_1(g(4))$ (resp. $C^{\text{nil}}_1(g(2))$).

If $A = kT$ for some group $T$ and $M$ is any left $A$-module, then the dual vector space $M^*$ has the structure of a left $A$-module with $g \in T$ acting via $(g \cdot \chi)(m) = \chi(g^{-1} \cdot m)$ for each $\chi \in M^*, m \in M$. In these circumstances, if $V$ is an irreducible component of $\text{mod}^r_A$, then we denote by $V^*$ the dual component $\{M^* : M \in V\}$. Let $\text{triv} = C^{\text{nil}}_1(g(1))$ denote the variety of one-dimensional $A$-modules. Clearly $\text{triv}$ clearly consists of a single point. We have the following:

**Proposition 3.1.** (a) Suppose $n = 2m$. Then $X_j \cong W_j \oplus U^{n-4j}$ and $X_j^* = X_j$.

(b) Suppose $n = 2m + 1$. Then $X_j^\pm$ are indecomposable components of $C$. Moreover, $X_j^+ = W_j \oplus X_0^+(g(n - 4j)), X_j^- = W_j \oplus X_0^-(g(n - 4j))$ and $(X_j^\pm)^* = X_j^-$. If $m$ is even, then $X_m/2 = W^{m/2} \oplus \text{triv}$. Moreover, $X_m^* = X_m/2$.

**Proof.** Let $e = e_m$ and let $\theta$ be the automorphism of $g$ given by $x \mapsto -J^t(x)J^{-1}$, where $J$ is the matrix with 1 on the anti-diagonal, and 0 elsewhere. Let $G$ (resp. $\theta$) act diagonally on $g \times g$, hence on each irreducible component of $C$. Since $\theta$ is a restricted Lie algebra automorphism of $g$, its induced action on $C$ permutes the irreducible components of $C$. Clearly $\theta(X_j) = X_j^\pm$ (resp. $\theta(X_j^\pm) = (X_j^\pm)^*$). If $n$ is even, recall that $X_j = G \cdot (e, V_{j,0})$, where $V_{j,0}$ is the set defined in the proof of Lemma 2.1. Since the zero here is superfluous, we will write $V_j$ for $V_{j,0}$. Similarly, we will write $V_j^\pm$ for $V_{j,0}^\pm$ in the case $n$ odd below and $V_m/2$ for $V_{m/2,0}$ if $m$ is even. Clearly $\theta(e) = -e$ and $\theta(\text{diag}(A_1, A_2)) = -\text{diag}(A_1, A_2)$. It is easy to choose $g$ such that $\text{Ad} g(e) = -e$ and $\text{Ad} g(\text{diag}(A_1, A_2)) = -\text{diag}(A_1, A_2)$. Hence $\text{Ad} g \circ \theta(X_j) = X_j$, and thus $X_j^* = X_j$. This proves the second statement of (a). We proved
in Lemma 2.1 that $Z_G(e) \cdot V_j = Z_G(e) \cdot U_j$, where $U_j$ is the set of elements of $\mathfrak{z}_G(e)$ of the form
\[
\begin{pmatrix}
    A_j & B \\
    0 & A_j
\end{pmatrix}, \quad B = \begin{pmatrix}
    0 & 0 & 0 \\
    0 & B_{22} & 0 \\
    0 & 0 & 0
\end{pmatrix}.
\]
Here the zero submatrices on the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre right) of $B$ are $j \times j$ (resp. $j \times (m-2j)$, $(m-2j) \times j$) and $B_{22} = (m-2j) \times (m-2j)$. It follows at once that $X_j = W^j \oplus X_0(\mathfrak{gl}(n-4j))$. Hence we have only to prove (a) for the case $j = 0$. Thus consider $V_0 = \left\{ \begin{pmatrix}
    0 & B \\
    0 & 0
\end{pmatrix} : B \in \text{Mat}_{m \times m}(k) \right\}$. The set of semisimple elements is dense in $\mathfrak{gl}(m)$, hence the subset of $V_0$ of elements such that $B$ is semisimple is dense. But any such element is $Z_G(e)$-conjugate to one such that $B$ is diagonal. Hence $X_0$ is the closure of $G \cdot \left\{ \begin{pmatrix}
    0 & I \\
    0 & 0
\end{pmatrix}, \begin{pmatrix}
    0 & B \\
    0 & 0
\end{pmatrix} : B \text{ diagonal} \right\}$. This proves (a).

For (b), suppose first of all that $m$ is even. Recall from the proof of Lemma 2.1 that $X_{m/2} = Z_G(e) \cdot U_{m/2}$, where $U_{m/2}$ is the set of $x \in \mathfrak{z}_G(e)$ of the form
\[
\begin{pmatrix}
    A_j & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & A_j
\end{pmatrix}.
\]
Here the top and bottom (resp. middle) rows are of height $m$ (resp. 1) and the left and right (resp. central) columns are of width $m$ (resp. 1). But then clearly $X_{m/2} = W^m \oplus \text{triv}$. Since $W = W^*$ and clearly $\text{triv} = \text{triv}^*$, we have also that $X_{m/2} = X_{m/2}^*$.

Consider therefore the components $X_j^+$ (for arbitrary $m$). It is easy to see that
\[
\theta(e) = -e \quad \text{and} \quad \theta\left( \begin{pmatrix}
    A_j & b & C \\
    0 & 0 & f \\
    0 & 0 & A_j
\end{pmatrix} \right) = \begin{pmatrix}
    -A_j & -J(t, f) & -J(C, J) \\
    0 & 0 & -bJ \\
    0 & 0 & -A_j
\end{pmatrix}.
\]
We recall from the proof of Lemma 2.1 that $Z_G(e) \cdot V_j^+ = Z_G(e) \cdot U_j^+$, where $U_j^+$ is the set of all $x \in \mathfrak{z}_G(e)$ of the form
\[
\begin{pmatrix}
    A_j & b & C \\
    0 & 0 & f \\
    0 & 0 & A_j
\end{pmatrix} : b \text{ is a column vector such that the first } j \text{ and the last } j \text{ entries are zero,}
\]
and $C$ is of the form
\[
\begin{pmatrix}
    0 & 0 & 0 \\
    0 & C_{22} & 0 \\
    0 & 0 & 0
\end{pmatrix}, \text{ where zero submatrices at the top left, top right, bottom left and bottom right (resp. top middle and bottom middle, centre left and centre right) are of dimension } j \times j \text{ (resp. } (i - 2j) \times j, (i - 2j) \times (i - 2j) \text{) and } C_{22} \text{ is } (i - 2j) \times (i - 2j).
\]
Similarly,
\[
Z_G(e) \cdot V_j^- = Z_G(e) \cdot U_j^-, \text{ where } U_j^- \text{ is the set of all } x \in \mathfrak{z}_G(e) \text{ of the form } \begin{pmatrix}
    A_j & 0 & C \\
    0 & 0 & f \\
    0 & 0 & A_j
\end{pmatrix}.
\]
Here $f$ is a row vector of dimension $i$ such that the first and last $j$ entries are zero. Hence, after applying conjugation by a suitably chosen element $g$, $\text{Ad}_g \circ \theta(V_j^+) = V_j^-$. It follows that $(X_j^+)^* = X_j^-$ and $(X_j^-)^* = X_j^+$. As above, the argument in the proof of Lemma 2.1 shows that $X_j^+ = W^j \oplus X_0^+(\mathfrak{gl}(n-4j))$, and similarly for $X_j^-$. Hence we have only to prove that $X_0^-$ is indecomposable (since the result for $X_0^+$ follows on taking the dual).

Let $x = e_{1,m+1} + e_{2,m+2} + \ldots + e_{m,2m}$. Clearly $x \in V_0^+$. Moreover, $Z_G(x) \cap Z_G(e)$ is the set
of all elements of the form $\begin{pmatrix} aI_m & y & z \\ 0 & a & 0 \\ 0 & 0 & aI_m \end{pmatrix}$. In particular, $\dim Z_G(x) \cap Z_G(e) = m^2 + m + 1$. It follows that $\dim G \cdot (e, x) = 3(m^2 + m) = \dim X_0^+$, hence $G \cdot (e, x) = X_0^+$. To show that $X_0^+$ is indecomposable, it will therefore suffice to show that the module corresponding to $(e, x)$ is indecomposable. But $Z_G(e) \cap Z_G(x)$ contains no non-trivial idempotents. This completes the proof.

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17
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