Spherically symmetric solutions of Einstein + non-polynomial gravities

S. Deser · Özgür Sarıoğlu · Bayram Tekin

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Abstract We obtain the static spherically symmetric solutions of a class of gravitational models whose additions to the General Relativity (GR) action forbid Ricci-flat, in particular, Schwarzschild geometries. These theories are selected to maintain the (first) derivative order of the Einstein equations in Schwarzschild gauge. Generically, the solutions exhibit both horizons and a singularity at the origin, except for one model that forbids spherical symmetry altogether. Extensions to arbitrary dimension with a cosmological constant, Maxwell source and Gauss-Bonnet terms are also considered.

1 Introduction

To better appreciate the nature of Ricci-flat geometries, particularly the fundamental exterior Schwarzschild solution of GR, it is useful to explore some alternatives, keeping as much as possible of the GR physics. The study of alternate theories is of course an enormous industry, but our interest here is to probe as simply as possible how the Schwarzschild geometry is deformed by additional terms. This will give a
little more insight into the necessity for and variety of horizons, on which one could
test the universality of our present black hole ideas. To be sure, we will only look
under “lampposts”— models for which spherically symmetric solutions can be found
explicitly. Our deviations from GR consist of those terms, non-polynomial in the Weyl
tensor, whose virtue is to preserve the (first) derivative order of the GR equations in
Schwarzschild gauge and to provide the simplest non-Ricci flat extensions of GR.

Our solution technique (introduced by Weyl [1] for pure GR but justified later
[2, 3]) is to insert in the action a gauge fixed metric already endowed with the desired
symmetries. Gauge fixing means only that the Bianchi identities become implicit,
while ansatzing a spherically symmetric $g_{\mu\nu}$ will enormously simplify the labor of
obtaining the field equations for the two functions on which it depends. We are assured
by [2] that all solutions are obtained thereby. We will also include a cosmological
constant, a Maxwell source and—in $D > 4$—Gauss-Bonnet terms.

2 The models

In this section, we keep to $D = 4$ for ease of notation. The most general spherically
symmetric metric in Schwarzschild coordinates is usefully written as

$$ds^2 = -a(r, t) b^2(r, t) dt^2 + \frac{dr^2}{a(r, t)} + r^2 d\Omega_2.$$  \hspace{1cm} (1)

All nonvanishing components of the mixed Weyl tensor, $C_{\mu\nu}^{\alpha\beta}$, are proportional to
the single function $X$

$$X(r, t) \equiv \frac{1}{r^2} \left(2(a - 1) - 2ra' + r^2a'' \right) + \frac{1}{rb} \left(3ra'b' - 2a(b' - rb'') \right)$$
$$+ \frac{1}{b} \partial_t \left( \frac{1}{a^2b} \partial_t a \right);$$  \hspace{1cm} (2)

here primes denote radial derivatives. This means that any scalar of order $n$ in the Weyl
tensor $C$ is proportional to $X^n$. Indeed, this fact is part of a general classification [4]
of all algebraic curvature invariants of spherical geometries. This classification also
informs us that the local actions that maintain the derivative order of the GR equations
for the metric (1) are non-polynomial terms of the form $(\text{tr} C^n)^{1/n}$, $(\det C^n)^{1/n}$, etc.

Its Ricci scalar is

$$R = -\frac{1}{r^2} \left(2(a - 1) + 4ra' + r^2a'' \right) - \frac{1}{rb} \left(3ra'b' + 2a(2b' + rb'') \right)$$
$$+ \frac{1}{b} \partial_t \left( \frac{1}{a^2b} \partial_t a \right).$$  \hspace{1cm} (3)

The actions we consider then are, in units of $\kappa = 1$,

$$I = \frac{1}{2} \int d^4x \sqrt{-g} \left( R + \beta_n \left| \text{tr} C^n \right|^{1/n} \right),$$  \hspace{1cm} (4)