Quantum information distribution without quantum channel

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We investigate the optimal quantum state distribution from cloud to many spatially separated users by measure-broadcast-prepare scheme without the availability of quantum channel. The quantum information equally distributed from cloud to arbitrary number of users is generated at each port by ensemble of known quantum states with assistance of classical information of measurement outcomes by broadcasting. The obtained quantum state for each user is optimal in the sense that the fidelity universally achieves the upper bound. We present the universal quantum state distribution by providing physical realizable measurement bases in the cloud as well as the reconstruction method for each user. The quantum information distribution scheme works for arbitrary many identical pure input states in general dimensional system.

In protocols of quantum information processing, entanglement and quantum channel are in general assumed to be available. However, in a certain scenario we may need to distribute quantum information to arbitrary number of users, who are spatially separated, while neither entanglement nor quantum channel is available. Each user may prepare their own quantum state according to classical information broadcasted from the “cloud” who can perform measurement on quantum states need to be distributed. This protocol can be named as classical quantum information distribution (CQID). It is known that there is no-cloning theorem for quantum information which states that an arbitrary quantum state cannot be cloned perfectly [1,2]. For spatially separated users, approximate copies of a quantum state can also be obtained for a number of users by the combination of the quantum cloning machine and teleportation [3] which needs the resource of entangled states and classical communication [2,4,5,6], differing from CQID. One may notice that CQID can be achieved with the help of quantum estimation by a measure-and-prepare scheme [7,9], but with additional condition that the prepared states should not be entangled [10] and be physical realizable.

For CQID as shown in FIG. 1, the cloud will use universal measurement scheme for arbitrary input states, and broadcast the results of measurement. Each user can prepare the quantum state by using ensemble of known quantum states agreed in advance, which are thus in product forms, with probabilities depending on classical information. Next we generally equate the state estimation with CQID, but bear in mind the difference mentioned above. The well-known estimation of quantum states shows that the mean fidelity for input states which are randomly and isotropically distributed can achieve the upper bound [2]. Here, we focus on the case of universality in the sense that each arbitrarily given input can be optimally distributed with the same fidelity.

We first assume that the input is $M$ independent and identically prepared arbitrary pure states in general $d$-dimension Hilbert space $\mathcal{H}$, $\rho = |\psi\rangle\langle\psi|^\otimes M$. It is known that this state is in the symmetric subspace $\mathcal{H}_+^M$ of $\mathcal{H}^\otimes M$ and has a dimension $d^+_M = C_{M+d-1}^M$, where $C_{M+d-1}^M = \frac{(M+d-1)!}{M!(d-1)!}$. The basis of symmetric subspace $\mathcal{H}_+^M$ can be denoted by $d$-dimension vectors $\tilde{m} = (m_1, m_2 \cdots m_d)$ satisfying $\sum_{r=1}^{d} m_r = M$, where $|\tilde{m}\rangle$ refers to the symmetric state in which there are $m_r$ copies in the state $|i\rangle$, and $\{|i\rangle\}_{i=0}^{d-1}$ is the computational basis of Hilbert space $\mathcal{H}$.

The standard quantum estimation process can be considered as a quantum channel $\mathcal{E}(\rho)$ which maps $\mathcal{H}_+^M$ to itself,

$$\tilde{\rho} = \mathcal{E}(\rho) = \sum_{r=1}^{R} \text{Tr}[\hat{O}_r \rho] |\Phi_r\rangle \langle \Phi_r|$$  \hspace{1cm} (1)

where $\hat{O}_r$ is a set of positive operator valued measurement (POVM), and $|\Phi_r\rangle \langle \Phi_r|$ is the corresponding guess in reconstructing the estimated state and also lies in $\mathcal{H}_+^M$. The case $R$ being finite means physical realizable since measure and broadcast can be implemented finitely. The completeness requires

$$\sum_{r=1}^{R} \hat{O}_r = \mathbb{1}_+^M,$$  \hspace{1cm} (2)

$\mathbb{1}_+^M$ is the identity of symmetric subspace $\mathcal{H}_+^M$, to make sure the estimation is trace preserving.

For CQID protocol, we propose that the POVM is performed in the cloud. Additionally, we need to release the constraint of $M$ users to arbitrary number $N$ of users, meaning that there is no restriction on number of audiences. Based on the measurement result, the users reconstruct the state by using known ensemble of states $\{|\Phi_r\rangle\}$. We remark that state $|\Phi_r\rangle$ is not necessarily the
product state for estimation, however, for spatially separated users in CQID, we will show that $|\Phi_r\rangle$ can be in the product form without diminishing the fidelity.

The figure of merit for CQID can be quantified by the fidelity of single copy of the reconstructed state and a separated users in CQID, we will show that the product state for estimation, however, for spatially separated users in CQID, we will show that $|\Phi_r\rangle$ can be in the product form without diminishing the fidelity.

The proof is straightforward. Taking trace over one Hilbert space denoted as, $\text{Tr}_1$, on both sides of Eq. (4), we find that,

$$
\sum_{r=1}^{M-1} c_r |\phi_r\rangle \langle \phi_r|^{\otimes M-1} = \frac{1}{d_M^+} \text{Tr}_1 |\Phi\rangle^{\otimes M} = \frac{M-1}{d_M^{M-1}},
$$

Here we need the relation,

$$
|\bar{m}\rangle = \frac{1}{\sqrt{C_M^L}} \sum_{\vec{k}} C_{\vec{k}}^{M-L} d \prod_{j=1}^{M} \sqrt{m_j!} |\bar{m} - \vec{k}| \langle \vec{k}|
$$

where we have used the notation $C(\vec{k}) = \sum_{i=1}^{d} k_i$. In the same way we have $\{|\phi_3\rangle\}_r^{R_3}$ and $\{c_r\}_r^{R_3}$ is also the $M-2, M-3, \cdots, 1$-copy CSS.

Obviously the basis of $\mathcal{H}$ can form a 1-copy CSS since $\sum_{r=1}^{M} \frac{1}{d^+} |\bar{m}| \langle \bar{m}| = 1/d$. It is known that the states isomorphically distributed in $\mathcal{H}$ can form an arbitrary $M$-copy CSS, which is also related to the symmetric distribution of information channel [10]. This infinite set takes the following form,

$$\int d\phi |\phi\rangle \langle \phi|^\otimes M = \frac{d_M^+}{d^+}, \quad M = 1, 2, 3 \cdots
$$

where the integral is taken over the Haar measurement, $M$ is an arbitrary natural number. However, we need the number of measurements to be finite such that it is physically realizable.

Now, we present our main result.

**Theorem.** For state distribution to achieve optimal mean fidelity, the POVM must be the form of a $M$-copy CSS. Additionally, to make the fidelity identical for an arbitrary input, this CSS should also be the order of $(M+1)$-copy.

To study the optimal fidelity, it is useful to introduce the following operator,

$$\tilde{F} = \int d\psi |\psi\rangle \langle \psi|^\otimes M \text{Tr}[|\psi\rangle \langle \psi|0\rangle \langle 0|].
$$

It is proved that the optimal mean fidelity $\bar{f}$ is upper bounded by the maximal eigenvalue $\lambda_{max}$ of $\tilde{F}$ multiplying the dimension $d$, i.e., $\bar{f} \leq \frac{d^+}{d} \lambda_{max}$. The corresponding POVM has to be $\hat{O}_r = \tilde{c}_r U_r^{\otimes M} |\psi\rangle \langle \psi|^{\otimes M} U_r^{\dagger}$. $\tilde{c}_r$ is the probability and $|\psi\rangle$ is the eigenstate corresponding to the maximal eigenvalue [7].

By calculations, for dimension $d$, we can find that the operator $\tilde{F}$ defined in Eq. (8) is in the diagonal form,

$$
\tilde{F} = \int d\psi |\psi\rangle \langle \psi|^\otimes M \text{Tr}[|\psi\rangle \langle \psi|0\rangle \langle 0|] = \sum_{\bar{m}, \vec{F}} |\bar{m}\rangle \langle \bar{m}| \int d\psi |\bar{m}\rangle \langle \bar{m}| \langle \bar{m}| \langle \psi| \langle \psi|0\rangle \langle 0|] = \sum_{\bar{m}} \frac{d^+}{d_{M+1}} |\bar{m}\rangle \langle \bar{m}| \frac{m_0 + 1}{M + 1},
$$

The proof is straightforward. Taking trace over one Hilbert space denoted as, $\text{Tr}_1$, on both sides of Eq. (4), we find that,

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\sum_{r=1}^{M-1} c_r |\phi_r\rangle \langle \phi_r|^{\otimes M-1} = \frac{1}{d_M^+} \text{Tr}_1 |\Phi\rangle^{\otimes M} = \frac{M-1}{d_M^{M-1}},
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where we have used the notation $C(\vec{k}) = \sum_{i=1}^{d} k_i$. In the same way we have $\{|\phi_3\rangle\}_r^{R_3}$ and $\{c_r\}_r^{R_3}$ is also the $M-2, M-3, \cdots, 1$-copy CSS.

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It is proved that the optimal mean fidelity $\bar{f}$ is upper bounded by the maximal eigenvalue $\lambda_{max}$ of $\tilde{F}$ multiplying the dimension $d$, i.e., $\bar{f} \leq \frac{d^+}{d} \lambda_{max}$. The corresponding POVM has to be $\hat{O}_r = \tilde{c}_r U_r^{\otimes M} |\psi\rangle \langle \psi|^{\otimes M} U_r^{\dagger}$. $\tilde{c}_r$ is the probability and $|\psi\rangle$ is the eigenstate corresponding to the maximal eigenvalue [7].

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$$

The proof is straightforward. Taking trace over one Hilbert space denoted as, $\text{Tr}_1$, on both sides of Eq. (4), we find that,
where summation is taken over all the basis in $H^M_+$. For this diagonal matrix $\tilde{F}$, the largest eigenvalue corresponds condition $m_0 = M$, $\lambda_{\text{max}} = d^2_{M+1}$, the corresponding eigenstate is $|0\rangle_0 \otimes |\tilde{M}\rangle$. The POVM thus takes the form of $M$-identical copies, $\tilde{O}_r = c_r (U_r |0\rangle_0 \otimes |\tilde{U}_r\rangle_0 \otimes |\tilde{M}\rangle) = c_r d^2_{M+1} |\phi_r\rangle |\phi_r\rangle |\phi_r\rangle \otimes |\tilde{M}\rangle$, where $|\phi_r\rangle = U_r |0\rangle_0$. On the other hand, the completeness relation requires $\{ |\phi_r\rangle \}_{r=1}^M$ to be a $M$-copy CSS. The optimal fidelity of state estimation is $\tilde{f}_{\text{opt}} = \frac{M+1}{M+d}$. This fidelity is the same as the optimal fidelity of a $M \to \infty$ quantum cloning machine. The relationship between the fidelity of state estimation and that of the cloning machine is already known, see $[2]$ and the references therein.

Here we specifically point out that the POVM takes the form as $\tilde{O}_r = c_r (U_r |0\rangle_0 \otimes |\tilde{U}_r\rangle_0 \otimes |\tilde{M}\rangle)$, which simplifies the original result $\tilde{O}_r = c_r d^{-2}_{M+1} |\psi_{\text{max}}\rangle |\psi_{\text{max}}\rangle |\tilde{U}_r\rangle_0 \otimes |\tilde{M}\rangle$, where $|\psi_{\text{max}}\rangle$ is generally unknown and may not necessarily be a product state $[7]$.

However, even if the state estimation achieves optimal mean fidelity, it is still far from enough, because for some input states, the fidelity could be undesirably small, which is an unwanted case. Here we further demand that CQID yields the universal fidelity for any input state. Obviously the universal fidelity is upper bounded by the mean fidelity, namely $\frac{M+1}{M+d}$. We now prove that this upper bound is achievable for a $(M+1)$-copy CSS.

We can consider the input to be $M$-copy pure states $|\psi\rangle \otimes |M\rangle$, which is in the symmetric subspace. Here, we present a more general form for an arbitrary matrix in the symmetric subspace for the input,

$$\rho = \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} |\tilde{m}\rangle \langle \tilde{n}|.$$

(10)

Simply, we know that $|\psi\rangle \otimes |\psi\rangle \otimes |M\rangle \in \rho$, meaning that the form of identical pure states is a special case. After tracing out $M-1$ copies, the single copy state is

$$\rho^{(1)} = \frac{1}{M} \sum_{\alpha, \beta} \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sqrt{m_{\alpha, \beta}} |\alpha\rangle \langle \beta| \delta_{\tilde{m} - \tilde{n}, \alpha - \beta} = 1 \frac{1}{M} \sum_{\tilde{m}, \tilde{n}} \sum_{\alpha, \beta} A_{\tilde{m}, \tilde{n}} |\alpha\rangle \langle \beta| \delta_{\tilde{m} - \tilde{n}, \alpha - \beta}.$$

(11)

here $\alpha$ denotes the vector with its $\alpha$-th entry to be 1 and other entries to be 0. If the POVM is $(M+1)$-copy CSS, $d^2_{M+1} \sum_{r=1}^R c_r |\phi_r\rangle \langle \phi_r| \otimes |\tilde{M}\rangle = 1 \sum_{r=1}^R c_r |\phi_r\rangle \langle \phi_r| \otimes |\tilde{M}\rangle$, after some calculations, we can find that the single copy of the output state takes the form,

$$\tilde{\rho}^{(1)} = \frac{1}{M} d^{2}_{M+1} \sum_{r=1}^R d_{M+1} |\phi_r\rangle \langle \phi_r| \otimes |\tilde{M}\rangle = \frac{1}{M} d^{2}_{M+1}.$$

(12)

The calculation details can be found in supplementary [11]. These results show that in the sense of single copy state, the CQID is equivalent to a polarization channel with a universal fidelity $F = \frac{M+1}{M+d}$. So the single copy output state is written universally as the input state with a shrinking factor and a completely mixed state with a corresponding probability. For identical pure input states $\rho = |\psi\rangle \langle \psi| \otimes |M\rangle$, we have $\tilde{\rho}^{(1)} = |\psi\rangle \langle \psi|$. We emphasize that the fidelity is defined between single input and output states.

For the protocol of CQID, the importance of our results is that we only need to find a $(M+1)$-copy CSS, state $|\psi\rangle \langle \psi| \otimes |M\rangle$ can be optimally distributed to arbitrary number of users, provided each user can reconstruct their quantum state by known ensemble of states based on the classical information broadcasted. It is then crucial that the CSS contains only finite number of states, which is physically realizable. Operationally, by using $(M+1)$-copy CSS with finite number of states, we can optimally distribute quantum state to arbitrary number of spatially separated parties without quantum channel. We remark that the optimal fidelity corresponds to that of universal quantum cloning machine for infinite copies, however, the cloning machine needs quantum channel to achieve the same aim.

In the following we show the protocol of CQID by two insightful examples.

**Example A:** First let us consider the case where a single qudit (state in d-dimension Hilbert space) is measured and broadcasted. Our results suggest that if a 2-copy CSS with finite states is found, a single qudit can be distributed with the optimal fidelity $\frac{1}{\sqrt{d}}$. To construct this CSS set, we introduce the so-called mutually unbiased bases (MUBs), see for example $[12-13]$. For a Hilbert space with dimension $d$, the MUBs contain $d + 1$ sets of orthogonal basis $\{ |\phi_t^k\rangle \}, t = 0, \ldots, d - 1, k = 0, \ldots, d$. Any states belong to different basis $|\phi_t^k\rangle$ and $|\phi_t^{k'}\rangle (k \neq k')$ satisfy the condition, $|\langle \phi_t^k | \phi_t^{k'} \rangle| = 1/\sqrt{d}$, meaning unbiased for all states. The construction of MUBs for the case that $d$ is prime is already well-studied and known to take the following form, $|\phi_t^k\rangle = |t\rangle, |\phi_t^{k'}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} (\omega^j)^{d-j} (\omega^{-k+j}) |j\rangle$, $(k \neq 0), t = 0, \ldots, d - 1$, where $|j\rangle$ is the computational basis, $s_j = j + \cdots + (d - 1)$ and $\omega = \exp(2\pi i/d)$.

We point out that MUBs set constitutes a 2-copy CSS,

$$\frac{1}{d(d+1)} \sum_{k=0}^d \sum_{t=0}^{d-1} |\phi_t^k\rangle \langle \phi_t^k| \otimes |\phi_t^k\rangle \langle \phi_t^k| = \frac{1}{d+1} I_d.$$

(13)

This identity can be proved by direct calculations, see supplementary [11]. According to our results, we know that by measurement corresponding to MUBs, a single qudit can be optimally distributed without the availability of quantum channel,

$$\tilde{\rho} = \frac{1}{d(d+1)} \sum_{k=0}^d \sum_{t=0}^{d-1} \text{Tr}(\rho |\phi_t^k\rangle \langle \phi_t^k| \otimes |\phi_t^k\rangle \langle \phi_t^k|) = \frac{1}{d+1} I_d.$$

(14)

The fidelity is $F = 2/(d+1)$ which is optimal. Explicitly, the state $\rho$ is measured in the cloud by projective mea-
measurement corresponding to MUBs, the results are broadcasted. Based on broadcasting information, each user can construct a quantum state \( \hat{\rho} \) by ensemble states of MUBs with optimal fidelity.

However, the MUBs set is not \((M + 1)\)-copy CSS for \( M \geq 1 \). We consider that the construction of general \((M + 1)\)-copy CSS is an open question.

Example B: Now we consider the qubit situation for case \( M = 2, d = 2 \). The 2-dimension MUBs can also be applied to this problem, where MUBs correspond to the known 6 bases denoted as, see for example [2],

\[
|0\rangle, \quad |+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \quad |\pm\rangle = \frac{1}{\sqrt{2}}\{(1\rangle + i|0\rangle)\}
\]

\[
|1\rangle, \quad |-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle), \quad |--\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|0\rangle).
\]

By straightforward calculation, as presented in supplementary [11], one can find that the 6 states form a 3-copy CSS.

\[
\frac{1}{6}\sum_{\alpha=0,1,+,-,\mp} |\alpha\rangle\langle\alpha|^{\otimes 3} = \frac{N^3}{d^3}, \tag{15}
\]

With these 6 bases, one can estimation two identical qubits \(|\psi\rangle^{\otimes 2}\) with optimal fidelity,

\[
\hat{\rho} = \frac{1}{6} \sum_{\alpha} \text{Tr}(|\alpha\rangle\langle\alpha|^{\otimes 2}|\psi\rangle\langle\psi|^{\otimes 2})|\alpha\rangle\langle\alpha|^{\otimes N}, \tag{16}
\]

where we write explicitly \( N \) in the equation to point out that the number of users \( N \) is arbitrary. One can check that a single qubit output takes the form,

\[
\hat{\rho}^{(1)} = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}I_2. \tag{17}
\]

The fidelity is optimal corresponding to universal quantum cloning machine \( 2 \rightarrow \infty \), which confirms that our method is applicable.

We emphasize here that the MUBs-constructed CSS is only valid for limited cases, and for arbitrary \( M \) and \( d \), the completeness relationship is not fulfilled. On the other hand, we imagine that \( \infty \)-copy CSS could only be realized by infinite sets. If it is true, then any effort to find out a physical realizable finite CSS would be futile, making the construction of CSS of arbitrary dimension and copies a crucial task. However, when given a fixed copy number \( M \) and dimension \( d \), the construction of \( M \)-copy CSS could be achievable. Assume that the POVM \( \{O_r\}_{r=1}^R \), or more specifically, the states \(|\phi_r\rangle\langle\phi_r|\), are randomly given, then one only need to find out a set of positive numbers \( \{c_r\}_{r=1}^R \) to satisfy the completeness relationship [3]. This simplifies the CSS construction to solving \( d^2(d^2 + 1)/2 \) linear equations with \( R \) unknown variables. By increasing \( R \), which is the total number of POVMs contained in CSS, these equations will be heavily under-determined so that there are enough free parameters to make the \( R \) unknown variables all positive. However it remains a complicated task when \( M \) is very large and decreasing the number of equations should be considered. It is proved in [7] that by applying a set of rotations \(|\phi^n_m\rangle = \exp(iX\theta_m)|\phi_r\rangle\), where operator \( X \) and constant \( \theta_m \) are carefully chosen, one can decrease the number of equations to \( d^2R \), that is, as long as the diagonal elements in [3] is satisfied, the off-diagonal elements are satisfied as well.

In conclusion, we have studied the CQID protocol meaning quantum information distribution method in the absence of quantum channel and provided a physical realizable measurement-and-prepare scheme which achieves the optimal mean fidelity. The measurement bases of an optimal CQID must take the form of \( M \)-copy CSS. The universal case is also taken into consideration, and we prove that to make the fidelity uniform for arbitrary input, one only needs to further require the bases to be \((M + 1)\)-copy CSS. Two examples for qudit and qubit are given to show the applicable of our method.

Acknowledgements: This work was supported by the National Key R & D Plan of China (No. 2016YFA0302104, No. 2016YFA0300600), the National Natural Science Foundation of China (Nos. 91536108, 11774406), and the Chinese Academy of Sciences (No. XDB21030300).

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I. THE EXPLICIT FORM OF OPERATOR $\hat{F}$

First we consider the maximum eigenvalue of the operator defined in Eq. (8) in the main text. We have,

$$\hat{F} = \sum_{\vec{m}, \vec{n}} |\vec{m}\rangle \langle \vec{n}| \int d\psi (|\psi\rangle \langle \psi|)^{\otimes M} |\vec{m}\rangle \langle \psi| \psi \rangle \langle 0| \langle 0|$$

$$= \sum_{\vec{m}, \vec{n}} |\vec{m}\rangle \langle \vec{n}| \int d\psi \text{Tr}[|\vec{m}\rangle \langle \psi|^{\otimes M} |\vec{m}\rangle \langle 0| \langle 0|]$$

$$= \sum_{\vec{m}, \vec{n}} |\vec{m}\rangle \langle \vec{n}| \int d\psi \text{Tr}[|\vec{m}| \otimes |0\rangle \langle 0| |\psi|^{\otimes M+1}]$$

$$= \sum_{\vec{m}, \vec{n}} |\vec{m}\rangle \langle \vec{n}| \langle \vec{m}| \otimes |0\rangle \langle 0| \int d\psi |\psi|^{\otimes M+1}$$

$$= \sum_{\vec{m}, \vec{n}} \frac{1}{d_{M+1}^+} |\vec{m}\rangle \langle \vec{n}| \sum_{\vec{r}} \text{Tr}[(|\vec{n}\rangle \langle 0| \otimes |0\rangle \langle 0|)] |\vec{r}\rangle \langle \vec{r}|$$

$$= \sum_{\vec{m}, \vec{n}} \frac{1}{d_{M+1}^+} |\vec{m}\rangle \langle \vec{n}| \sqrt{\frac{m_0 + 1}{M+1}} \sqrt{\frac{n_0 + 1}{M+1}} \delta_{\vec{m}, \vec{n}}$$

$$= \sum_{\vec{m}} \frac{1}{d_{M+1}^+} |\vec{m}\rangle \langle \vec{m}| \frac{m_0 + 1}{M+1}. \quad (18)$$

This result gives Eq. (9) in the main text. The above calculation indicates that operator $\hat{F}$ is diagonal under the bases we selected. The eigenvalue is $\lambda_{\vec{m}} = m_0$, and achieves maximum when $m_0 = M$, i.e., $\vec{m} = (M, 0, 0 \cdots 0)$. The optimal fidelity is thus $\bar{f}_{opt} = d_{M}^+/d_{M+1}^+ = \frac{M+1}{M+d}.$

II. SINGLE COPY OUTCOME IN QUANTUM INFORMATION DISTRIBUTION

Without loss of generality, suppose the input state takes the form as shown in Eq. (10) in the main text,

$$\rho = \sum_{\vec{m}, \vec{n}} A_{\vec{m}, \vec{n}} |\vec{m}\rangle \langle \vec{n}|. \quad (19)$$

After tracing out arbitrary $(M-1)$ copies, the remaining single copy state is,

$$\rho^{(1)} = \sum_{\vec{m}, \vec{n}} \frac{1}{M} A_{\vec{m}, \vec{n}} \sum_{\alpha, \beta} \sqrt{m_\alpha n_\beta} \text{Tr}_{M-1} [\hat{m} - \hat{\alpha} | \hat{m} - \hat{\beta}]$$

$$= \sum_{\vec{m}, \vec{n}} \frac{1}{M} A_{\vec{m}, \vec{n}} \sum_{\alpha, \beta} \sqrt{m_\alpha n_\beta} |\alpha\rangle \langle \beta| \delta_{\vec{m} - \vec{\alpha}, \vec{n} - \vec{\beta}} \quad (20)$$
After the quantum state distribution process, the single copy state of the outcome \( \tilde{\rho} \) is,

\[
\tilde{\rho}^{(1)} = \text{Tr}_{M-1}[\mathcal{E}(\rho)]
\]

\[
= \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sum_{r=1}^R d_M^+ c_r \text{Tr}[^r_\tilde{m}\int_\tilde{m}^M \langle \phi_r \mid |\tilde{m}\rangle \langle \tilde{m} | \phi_r \rangle
\]

\[
= \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sum_{r=1}^R d_M^+ c_r \sum_{\alpha, \beta=0}^{d-1} |\alpha\rangle \langle \beta|
\]

\[
\text{Tr}[^r_\tilde{m}\int_\tilde{m}^M \langle \phi_r \mid |\tilde{m}\rangle \langle \tilde{m} | \phi_r \rangle \langle \beta | |\alpha\rangle]
\]

\[
= \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sum_{\alpha, \beta=0}^{d-1} |\alpha\rangle \langle \beta| \text{Tr}[(\langle m \mid \otimes |\beta\rangle)(\langle \tilde{m} | \otimes |\alpha\rangle)]
\]

\[
= \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sum_{\alpha, \beta=0}^{d-1} |\alpha\rangle \langle \beta|
\]

\[
\sum_{r=1}^R d_M^+ c_r \langle \phi_r \mid |\phi_r \rangle \int_\tilde{m}^{M+1}
\]

Then we take into account the CSS relation (4) in the main text, and note that \( \int_\tilde{m}^{M+1} = \sum_{s}^{C(s)=M+1} |s\rangle \langle s| \), we have,

\[
\tilde{\rho}^{(1)} = \frac{d_M^+}{d_M+1} \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sum_{\alpha, \beta=0}^{d-1} |\alpha\rangle \langle \beta|
\]

\[
\text{Tr}[(\langle m \mid \otimes |\beta\rangle)(\langle \tilde{m} | \otimes |\alpha\rangle)] E^{M+1}_{\tilde{m}, \tilde{n}}
\]

\[
= \frac{d_M^+}{d_M+1} \sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \sum_{\alpha, \beta=0}^{d-1} |\alpha\rangle \langle \beta| \text{Tr}[(\langle m \mid \otimes |\beta\rangle)(\langle \tilde{m} | \otimes |\alpha\rangle)]
\]

\[
\sum_{s}^{C(s)=M+1} |s\rangle \langle s|
\]

\[
= \sum_{\alpha, \beta=0}^{d-1} |\alpha\rangle \langle \beta|
\]

\[
\sum_{\tilde{m}, \tilde{n}} A_{\tilde{m}, \tilde{n}} \delta_{\tilde{m}+\tilde{m}, \tilde{n}+\tilde{n}} \frac{\sqrt{m_\beta + 1} \sqrt{n_\alpha + 1}}{\sqrt{M+1} \sqrt{M+1}} |\alpha\rangle \langle \beta|
\]

The Kronecker-\( \delta \) requires when \( \alpha \neq \beta \), we have \( m_\beta + 1 = n_\alpha \) and \( n_\alpha + 1 = m_\beta \), and when \( \alpha = \beta \), we have \( m_\beta = n_\alpha \). Then the above equation takes a more concise form,

\[
\tilde{\rho}^{(1)} = \sum_{\alpha=0}^{d-1} \frac{m_\alpha + 1}{M+d} A_{\tilde{m} \tilde{m}} |\alpha\rangle \langle \alpha| + \sum_{\alpha \neq \beta} \sum_{\tilde{m}, \tilde{n}} \frac{\sqrt{m_\alpha n_\beta}}{M+d} A_{\tilde{m}, \tilde{n}} |\alpha\rangle \langle \beta|
\]

\[
= \frac{M}{M+d} \left( \sum_{\alpha=0}^{d-1} \frac{m_\alpha}{M+d} A_{\tilde{m} \tilde{m}} |\alpha\rangle \langle \alpha| + \sum_{\alpha \neq \beta} \sum_{\tilde{m}, \tilde{n}} \frac{\sqrt{m_\alpha n_\beta}}{M+d} A_{\tilde{m}, \tilde{n}} |\alpha\rangle \langle \beta| \right)
\]

\[
+ \frac{1}{M+d} \sum_{\alpha=0}^{d-1} |\alpha\rangle \langle \alpha| \sum_{\tilde{m}} A_{\tilde{m} \tilde{m}}
\]

When compared with (20), we obtain equation (12) in the main text,

\[
\tilde{\rho}^{(1)} = \frac{M}{M+d} \rho^{(1)} + \frac{1}{M+d} I
\]
III. 2-COPY CSS FOR D-DIMENSION CASE

Next we will prove that,

\[
\hat{Q} = \frac{1}{d(d+1)} \left( \sum_j |j\rangle \langle j|^2 + \sum_{k=1}^d \sum_{l=0}^{d-1} |\psi_l^{(k)}\rangle \langle \psi_l^{(k)}|^2 \right) = \frac{d^2}{2d^4},
\]

where the mutually unbiased bases take the form,

\[
|\psi_l^{(k)}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} (\omega^t)^{d-j} (\omega^{-k})^{s_j} |j\rangle,
\]

\[
s_j = j + \cdots + (d-1) = \frac{1}{2} (d^2 - j^2 - (d - j)),
\]

\[
\omega = e^{\frac{2\pi i}{d}}.
\]

Note that there are two kinds of basis in \( \mathcal{H}_d^2 \): \( |jj\rangle \) and \( |j_1, j_2\rangle = \frac{1}{\sqrt{d}} (|j_1 j_2\rangle + |j_2 j_1\rangle), j_1 \neq j_2 \). The inner product gives,

\[
\langle jj | \psi_l^{(k)} \rangle \otimes \langle jj | \psi_l^{(k)} \rangle = \frac{1}{d} \omega^{2t(d-j) - 2ks_j},
\]

\[
\langle j_1, j_2 | \psi_l^{(k)} \rangle \otimes \langle j_1, j_2 | \psi_l^{(k)} \rangle = \frac{\sqrt{d}}{d} \omega^{p_2j - d - j_2 - k(s_j + s_{j_2})}
\]

Consider the off-diagonal elements. There are three kinds of off-diagonal elements, and we prove that there are all zeroes.

For \( |j_1, j_1\rangle \langle j_2, j_2| \) type off-diagonal elements,

\[

\langle j_1, j_1 | \hat{Q} | j_2, j_2 \rangle = \frac{1}{d(d+1)} \left( \sum_{k=1}^d \sum_{l=0}^{d-1} \langle j_1, j_1 | \psi_l^{(k)} \rangle \langle \psi_l^{(k)} | j_2, j_2 \rangle \right)
\]

\[
= \frac{1}{d(d+1)} \left( \sum_{k=1}^d \sum_{t=0}^{d-1} \omega^{2t(j_2 - j_1) + k(s_{j_2} - s_{j_1})} \right)
\]

After summing up for all \( t \) and \( k \), the element is non-zero only when

\[
\begin{cases}
  j_1 = j_2, \\
  s_{j_1} = s_{j_2}
\end{cases}
\]

which is the diagonal element case. So arbitrary \( |j_1, j_1\rangle \langle j_2, j_2| \) type off-diagonal element equals to 0.

For \( |j_1, j_2\rangle \langle j_1, j_1| \) type,

\[

\langle j_1, j_2 | \hat{Q} | j_1, j_1 \rangle = \frac{1}{d(d+1)} \left( \sum_{k=1}^d \sum_{l=0}^{d-1} \langle j_1, j_2 | \psi_l^{(k)} \rangle \langle \psi_l^{(k)} | j_1, j_1 \rangle \right)
\]

\[
= \frac{1}{d(d+1)} \left( \sum_{k=1}^d \sum_{t=0}^{d-1} \omega^{t(2j_2 - j_1) - k(s_{j_1} + s_{j_2} - 2s_j)} \right)
\]

The element is non-zero only when

\[
\begin{cases}
  2j = j_1 + j_2 \\
  2s_j = s_{j_1} + s_{j_2}
\end{cases}
\]

Theses equations are fulfilled only when \( j_1 = j_2 = j \), which cannot be fulfilled. So they are also zeroes.

For \( |j_1, j_2\rangle \langle j_3, j_4| \) type,

\[

\langle j_1, j_2 | \hat{Q} | j_3, j_4 \rangle = \frac{1}{d(d+1)} \left( \sum_{k=1}^d \sum_{l=0}^{d-1} \langle j_1, j_2 | \psi_l^{(k)} \rangle \langle \psi_l^{(k)} | j_3, j_4 \rangle \right)
\]

\[
= \frac{1}{d(d+1)} \sqrt{d} \left( \sum_{k=1}^d \sum_{t=0}^{d-1} \omega^{t(j_3 + j_4 - j_1 - j_2) - k(s_{j_1} + s_{j_2} - s_{j_3} - s_{j_4})} \right)
\]
The element is non-zero only when:

\[
\begin{align*}
\begin{cases}
  j_3 + j_4 = j_1 + j_2 \\
  s_{j_3} + s_{j_4} = s_{j_1} + s_{j_2}
\end{cases}
\]

These equations are fulfilled only when \(\{j_1, j_2\} = \{j_3, j_4\}\), which can not be fulfilled either.

Summarizing those three cases concludes that all the off-diagonal elements equal to 0.

Now we consider the two kinds of diagonal elements, \(\langle jj \rangle \langle jj \rangle\) and \(\langle j_1, j_2 \rangle \langle j_1, j_2 \rangle\).

For \(\langle jj \rangle \langle jj \rangle\) type, the element equals to,

\[
\langle jj \rangle \hat{Q} \langle jj \rangle = \frac{1}{d(d+1)} \sum_{j'=0}^{d-1} \langle jj \rangle \langle j' j' \rangle \langle j' j \rangle + \sum_{k=1}^{d} \sum_{t=0}^{d-1} \langle jj \rangle \langle k(k) \rangle \langle j \rangle = \frac{1}{d(d+1)} (1 + d^2) \times \frac{1}{d^2} = \frac{2}{d(d+1)}
\]

For \(\langle j_1, j_2 \rangle \langle j_1, j_2 \rangle\) type, the element equals to,

\[
\langle j_1, j_2 \rangle \hat{Q} \langle j_1, j_2 \rangle = \frac{1}{d(d+1)} \sum_{k=1}^{d} \sum_{t=0}^{d-1} \langle j_1, j_2 \rangle \langle k(k) \rangle \langle j \rangle = \frac{1}{d(d+1)} \times d^2 \times \frac{2}{d^2} = \frac{2}{d(d+1)}
\]

Therefore we have,

\[
\hat{Q} = \frac{\mathbb{1}_2^2}{d_2^2}.
\]

This result gives Eq.(13) in the main text. It is example A.

IV. 3-COPY CSS,2-DIMENSION CASE

For 2-dimension case, the mutually unbiased states writes as \(|1\rangle, |0\rangle, |+\rangle, |−\rangle, |\uparrow\rangle, |\downarrow\rangle\). For 3-copy CSS case, the bases in \(\mathcal{H}_+^3\) are,

\[
\begin{align*}
|0, 3\rangle & = |A\rangle \\
|1, 2\rangle & = |B\rangle \\
|2, 1\rangle & = |C\rangle \\
|3, 0\rangle & = |D\rangle
\end{align*}
\]

Here \(|m_0, m_1\rangle\) implies that this state contains \(m_0\) state \(|1\rangle\) and \(m_1\) state \(|0\rangle\). Therefore,

\[
\begin{align*}
|1\rangle^{\otimes 3} & = |D\rangle \\
|0\rangle^{\otimes 3} & = |A\rangle \\
|+\rangle^{\otimes 3} & = \frac{1}{\sqrt{8}}|A\rangle + \frac{3}{\sqrt{8}}|B\rangle + \frac{3}{\sqrt{8}}|C\rangle + \frac{1}{\sqrt{8}}|D\rangle \\
|−\rangle^{\otimes 3} & = −\frac{1}{\sqrt{8}}|A\rangle + \frac{3}{\sqrt{8}}|B\rangle − \frac{3}{\sqrt{8}}|C\rangle + \frac{1}{\sqrt{8}}|D\rangle \\
|\uparrow\rangle^{\otimes 3} & = −i\frac{1}{\sqrt{8}}|A\rangle − \frac{3}{\sqrt{8}}|B\rangle + i\frac{3}{\sqrt{8}}|C\rangle + \frac{1}{\sqrt{8}}|D\rangle \\
|\downarrow\rangle^{\otimes 3} & = i\frac{1}{\sqrt{8}}|A\rangle − \frac{3}{\sqrt{8}}|B\rangle − i\frac{3}{\sqrt{8}}|C\rangle + \frac{1}{\sqrt{8}}|D\rangle
\end{align*}
\]

By substituting these equations into the following equation, we find that they form a 3-copy CSS,

\[
\frac{1}{6} \sum_{i=1,0,+,-,\uparrow,\downarrow} |i\rangle^{\otimes 3} = \frac{1}{6} \times \frac{3}{2} (|A\rangle \langle A| + |B\rangle \langle B| + |C\rangle \langle C| + |D\rangle \langle D|)
\]

\[
= \frac{\mathbb{1}_2^3}{4}.
\]

This equation gives the Eq.(15) in the main text.
V. NECESSARY CONDITION FOR M-COPY UNIVERSAL OPTIMAL ESTIMATION

Now we prove the necessity for the measurement basis to be (M+1)-copy CSS. Suppose that there exists a set of states which forms an optimal estimation measurement operator, $\rho_{k,\phi} = \sum_{r=1}^{R} c_r |\psi_r\rangle \langle \psi_r|^\otimes M+1 = \mathbb{I}^M_{+}/d^M_{+1} + \hat{P}$

Operator $\hat{P}$ lies in symmetric subspace $\mathcal{H}^M_{+1}$ because the left-hand-side of equation belongs to the symmetric subspace. We prove that there must be $\hat{P} = 0$. The output single copy state is,

$$\rho^{(1)} = \sum_{r=1}^{R} c_r \text{Tr} \left[ \rho |\psi_r\rangle \langle \psi_r|^\otimes M \right]|\psi_r\rangle \langle \psi_r|$$

$$= \sum_{k,l=0}^{d-1} |k\rangle \langle l| \sum_{r=1}^{R} c_r \text{Tr} \left[ \rho |\psi_r\rangle \langle \psi_r|^\otimes M \right] |k\rangle \langle l|$$

$$= \sum_{k,l=0}^{d-1} |k\rangle \langle l| \text{Tr} \left[ (\rho \otimes |l\rangle \langle k|) \sum_{r=1}^{R} c_r |\psi_r\rangle \langle \psi_r|^\otimes M+1 \right]$$

$$= \sum_{k,l=0}^{d-1} |k\rangle \langle l| \text{Tr} \left[ (\rho \otimes |l\rangle \langle k|)(\mathbb{I}^M_{+}/d^M_{+1} + \hat{P}) \right]$$

$$= \frac{M+1}{M+d} \rho^{(1)} + \frac{1}{M+d} \hat{P}$$

To makes sure that for arbitrary input the fidelity is optimal, the second term must always equal to 0. that is,

$$\Delta_{kk} = \text{Tr} \left[ (\rho \otimes |l\rangle \langle k|) \hat{P} \right] = 0, \quad \forall \rho \in \mathcal{H}^M_{+}, |k\rangle, |l\rangle \in \mathcal{H}$$

This condition is satisfied only when $\hat{P} = 0$. The following part gives a detailed proof.

Since $\hat{P} \in \mathcal{H}^M_{+1}$, we apply the following expansion form of the operator:

$$\hat{P} = \sum_{\vec{r},\vec{s}} P_{rs} |\vec{r}\rangle \langle \vec{s}|$$

First consider the diagonal elements $P_{rr}$. Suppose that $r_k \neq 0$, choose $\rho = |\vec{r} - \vec{k}\rangle \langle \vec{r} - \vec{k}|$, \[31\] gives that:

$$0 = \Delta_{kk} = P_{rr} \times \frac{r_k}{M} \Rightarrow P_{rr} = 0$$

That is, the diagonal elements are all zeroes.

Then consider the off-diagonal elements $P_{rs}$, suppose $r_k \neq 0, s_l \neq 0$, and for simplicity, let $\vec{m} = \vec{r} - \vec{k}, \vec{n} = \vec{s} - \vec{l}$. For state $\rho = \frac{1}{\lambda_1^2 + \lambda_2^2} (\lambda_1 |\vec{m}\rangle + \lambda_2 e^{i\phi} |\vec{n}\rangle)(\lambda_1 |\vec{m}\rangle + \lambda_2 e^{-i\phi} |\vec{n}\rangle)$, where $\lambda_1, \lambda_2, \phi$ are non-negative real numbers, $\phi \in [0, 2\pi]$. Then \[31\] gives,

$$\Delta_{kl} = \frac{1}{\lambda_1^2 + \lambda_2^2} (\lambda_1^2 A + \lambda_2^2 B + \lambda_1 \lambda_2 (Ce^{i\phi} + De^{-i\phi})) = 0,$$

which is satisfied for arbitrary $\lambda_1, \lambda_2, \phi$. Here

$$A = \text{Tr}[(|\vec{m}\rangle \langle \vec{m}| \otimes |l\rangle \langle k|) \hat{P}],$$

$$B = \text{Tr}[(|\vec{n}\rangle \langle \vec{n}| \otimes |l\rangle \langle k|) \hat{P}],$$

$$C = \text{Tr}[(|\vec{m}\rangle \langle \vec{m}| \otimes |l\rangle \langle k|) \hat{P}],$$

$$D = \text{Tr}[(|\vec{n}\rangle \langle \vec{n}| \otimes |l\rangle \langle k|) \hat{P}].$$
Then we have $A = B = C = D = 0$, and $C = 0$ gives,

$$\frac{\sqrt{r_k s_l}}{M + 1} P_{rs} = 0 \Rightarrow P_{rs} = 0.$$  \hfill (39)

That is, the off-diagonal elements are also zeroes. Therefore $\hat{P} = 0$, which indicates that the quantum estimation is universal only when its measurement bases are $(M+1)$-copy CSS.