A Lyapunov framework for nested dynamical systems on multiple time scales with application to converter-based power systems

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Abstract—In this work, we present a Lyapunov function framework for establishing stability with respect to a compact set of a nested interconnection of nonlinear dynamical systems ordered from slow to fast according to their convergence rates. The proposed approach explicitly considers more than two time scales, and does not require modeling multiple time scales via scalar time constants. Motivated by the technical results, we develop a novel control strategy for a grid-forming power converter that consists of an inner cascaded two-degree of freedom controller and dispatchable virtual oscillator control as a reference model. The resulting closed-loop converter-based AC power system is in the form of a nested system with multiple time scales. We apply our technical results to obtain explicit bounds on the controller set-points, branch powers, and control gains that guarantee almost global asymptotic stability of the multi-converter AC power system with respect to a pre-specified solution of the AC power-flow equations. Finally, we validate the performance of the proposed control structure in a case study using a high-fidelity simulation with detailed hardware validated converter models.

I. INTRODUCTION

TIME-SCALE separation arguments are ubiquitous in control design and analysis of large-scale engineering systems that contain dynamics on multiple time scales from different physical domains. Traditionally, singular perturbation theory has been the standard tool to analyze nonlinear dynamics that evolve on multiple time scales [1]–[3]. Within this framework, stability conditions are typically provided for hyperbolic fixed points of systems with two time scales and a “small” scalar time constant describing the fast time scale. The results can be extended to linear systems with two time scales and a fast time scale modeled by multiple time constants [4], slow-fast control systems with non-hyperbolic fixed points [5], and multiple time scales by successively grouping them into two time scales (see, e.g., [2]). In contrast, our approach explicitly considers multiple time scales, stability with respect to a compact set, does not require modeling time scales via scalar time constants, and exploits the nested structure typically exhibited by systems with multiple time scales such as power systems [6] and biological system [7]. To this end, we develop a general Lyapunov function framework for stability analysis of nested nonlinear dynamical systems that can be ordered from slow to fast in terms of convergence rates to their set of steady-states and only depend on the states of slower systems and the next fastest one.

The analysis in this paper is based on a recently developed Lyapunov characterization of almost global asymptotic stability with respect to a compact set presented in [8] that requires that the set of states that are unstable but attractive has zero Lebesgue measure. The technical contribution is twofold: first, we develop a Lyapunov characterization of unstable hyperbolic fixed points that have a region of attraction of measure zero; then we provide a Lyapunov framework that results in conditions under which the guarantees obtained by applying the aforementioned Lyapunov conditions in [8] to a reduced-order system translate to the full-order nested dynamical system. Our results can be interpreted as an extension of the conditions for systems with two time scales in [9] Ch. 11.5 to multiple nested systems and a more general notion of stability. Moreover, we reduce conservatism by allowing for a wider range of comparison functions.

Motivated by the transition of power systems towards renewable energy sources that are connected to the system via power electronics [10], [11], we apply our technical results to multi-converter AC power systems. The analysis and control of power systems and microgrids is typically based on reduced-order models of various degrees of fidelity that exploit the pronounced time-scale separation between the dynamics of synchronous machines and power converters and the transmission network [3], [6], [12]–[14]. While these ad-hoc model simplifications have proved themselves useful their validity for converter-based systems is questionable. For instance, the assumption that the dynamics of transmission lines can be neglected breaks down for power systems dominated by fast acting power converters [8], [15]–[17]. In this work, we make the time-scale separation argument rigorous by explicitly considering the interaction of dynamics on different time scales (i.e., converter dynamics, inner controls, line dynamics) and quantifying the parameters (e.g., set-points, control gains, transmission line parameters, etc.) for which stability for the overall system can be ensured.

The prevalent approach to grid-forming control is so-called droop-control [17], [18] and synchronous machine emulation [19]–[21]. However, while droop control and machine emulation can provide useful insights, stability guarantees are local and typically don’t extend to line dynamics, detailed converter models, and operating points with non-zero power flows. In contrast, virtual oscillator control (VOC) ensures almost global synchronization [22]–[24] but cannot be dispatched, and almost global asymptotic stability with line dynamics can be ensured for dispatchable virtual oscillator control (dVOC) [25], [26] for appropriate control gains and power converters modeled as controllable voltage sources [8].

Motivated by this result, we consider a converter-based power system model that includes dynamic models of voltage...
source converters as well as transmission network dynamics. We develop a control strategy for the converters that uses dispatchable virtual oscillator control (dVOC) as a reference model for a cascaded two-degree of freedom voltage and current controller in stationary $\alpha\beta$ coordinates instead of local $dq$ coordinates. This allows us to model the whole inverter based power system as a nested interconnection of subsystems (dVOC, the transmission line dynamics, the inner control loops) that evolve on different time-scales and to apply our novel Lyapunov-based stability criterion to obtain conditions on the parameters (i.e., set-points, control gains, and network parameters) that guarantee almost global asymptotic stability of overall system. Finally, we validate the proposed control architecture in a high-fidelity simulation of the hardware setup described in [26].

The remainder of this section recalls some basic notation and results from graph theory. Section II provides definitions and a preliminary technical results on almost global asymptotic stability with respect to sets. The main theoretical contribution is given in Section III. In Section IV we present a detailed model of a multi-converter AC power system and the control objectives. Section V presents a cascaded two-degree of freedom control structure that tracks a reference obtained by dVOC as a reference model and Section VI presents stability conditions for the multi-converter system. The results are illustrated using a high-fidelity simulation study in Section VII and Section VIII provides the conclusions.

Notation

We use $\mathbb{R}$ and $\mathbb{N}$ to denote the set of real and natural numbers and define $\mathbb{R}_{\geq a} := \{x \in \mathbb{R} | x \geq a\}$ and, e.g., $\mathbb{R}_{(a,b)} := \{x \in \mathbb{R} | a \leq x < b\}$. Given $\theta \in [-\pi, \pi]$ the 2D rotation matrix is given by

$$\mathcal{R}(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathbb{R}^{2 \times 2}. $$

Moreover, we define the 90° rotation matrix $J := \mathcal{R}(\pi/2)$ that can be interpreted as an embedding of the complex imaginary unit $\sqrt{-1}$ into $\mathbb{R}^2$. Given a matrix $A$, $A^T$ denotes its transpose. We use $\|A\|$ to indicate the induced 2-norm of $A$. We write $A \succ 0$ ($A \succ 0$) to denote that $A$ is symmetric and positive semidefinite (definite). For column vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ we use $(x,y)^T := [x^T,y^T]^T \in \mathbb{R}^{n+m}$ to denote a stacked vector, and $\|x\|$ denotes the Euclidean norm. The absolute value of a scalar $y \in \mathbb{R}$ is denoted by $|y|$. Furthermore, $I_n$ denotes the identity matrix of dimension $n$, and $\otimes$ denotes the Kronecker product. For any matrix $M$ and any $n \in \mathbb{N}$, we define $M_n = I_n \otimes M$. Matrices of zeros of dimension $n \times m$ are denoted by $0_{n\times m}$ and $0_n$ denotes column vector of zeros of length $n$. We use $\|x\|_C := \min_{x \in C} \|x - z\|$ to denote the distance of a point $x$ to a set $C$. We use $\varphi_f(t,x_0)$ to denote the solution of $\frac{dx}{dt} = f(x)$ at time $t \geq 0$ starting from the initial condition $x(0) = x_0$ at time $t_0 = 0$.

II. ALMOST GLOBAL ASYMPTOTIC STABILITY WITH RESPECT TO A COMPACT SET

Consider the dynamical system

$$\frac{dx}{dt} = f(x),$$

where $x \in \mathbb{R}^n$ denotes the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous function. In order to state the main results of the paper, we require the following definition of almost global asymptotic stability with respect to a set $C$.

**Definition 1 (Almost global asymptotic stability)** The dynamical system (1) is called almost globally asymptotically stable with respect to a compact set $C \subset \mathbb{R}^n$ if

(i) it is almost globally attractive with respect to $C$, i.e.,

$$\lim_{t \to \infty} \|\varphi_f(t,x_0)\|_C = 0$$

holds for all $x_0 \notin C$, and $C$ has zero Lebesgue measure,

(ii) it is Lyapunov stable with respect to $C$, i.e., for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ such that

$$\|x_0\|_C < \delta \implies \|\varphi_f(t,x_0)\|_C < \varepsilon, \quad \forall t \geq 0. $$

Next, we recall the definition of comparison functions used to establish stability properties of dynamical systems [28].

**Definition 2 (Comparison functions)** A function $\chi_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\chi_c(0) = 0$; it is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$-function and $\chi_c(s) \to \infty$ as $s \to \infty$.

Next, consider a set $U \subset \mathbb{R}^n$ that is invariant with respect to (1) (i.e., $\varphi_f(t,x_0) \in U$ for all $t \geq 0$ and all $x_0 \in U$), satisfies $C \cap U = \emptyset$, and corresponds to e.g., undesirable equilibria or limit cycles of (1). In this case global asymptotic stability of (1) with respect to $C$ cannot be established. Instead, the following theorem provides a Lyapunov function characterization of almost global asymptotic stability with respect to $C$ [8] Th. 1.

**Theorem 1 (Lyapunov functions)** Consider a compact set $C \subset \mathbb{R}^n$ and a zero Lebesgue measure set $U \subset \mathbb{R}^n$ that is invariant with respect to (1). Moreover, consider a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ and comparison functions $\chi_1, \chi_2 \in \mathcal{K}_\infty$ and $\chi_3 \in \mathcal{K}$ such that

$$\chi_1(||x||_C) \leq V(x) \leq \chi_2(||x||_C)$$

$$\frac{d}{dt} V(x) = -\chi_3(||x||_{C \cup U})$$

holds for all $x \in \mathbb{R}^n$. Moreover, let

$$Z_{U,f} := \{x_0 \in \mathbb{R}^n | \lim_{t \to \infty} \|\varphi_f(t,x_0)\|_U = 0\}.$$ 

denote the region of attraction of $U$ under (1). If $Z_{U,f}$ has zero Lebesgue measure, the dynamics (1) are almost globally asymptotically stable with respect to $C$.

Besides finding a suitable Lyapunov function, the main difficulty in applying Theorem 1 is to verify that the region of attraction $Z_{U,f}$ of the (unstable) attractive set $U$, has measure zero. To this end, our first contribution is a Lyapunov-like condition that characterizes unstable hyperbolic fixed points with region of attraction that has zero Lebesgue measure.

**Theorem 2 (Unstable hyperbolic fixed point)** Consider a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, its Jacobian $A_x := \frac{\partial f(x)}{\partial x} |_{x = x^*}$ at an equilibrium $x^* \in \mathbb{R}^n$, and the linearized dynamics $\frac{dx}{dt} = A_x x_\delta$, where $x_\delta := x - x^*$. Moreover, consider a quadratic function $V_\delta := \frac{1}{2} x_\delta^T P_\delta x_\delta$. If
\( \frac{d}{dt} \mathcal{V}_s(x_s) := \frac{\partial \mathcal{V}_s}{\partial x_s} A_s^* x_s < 0 \) holds for all \( x_s \neq 0_s \) and there exists \( x^* \in \mathbb{R}^n \) such that \( \mathcal{V}_s(x^*) < \mathcal{V}_s(0_s) \), then \( x^* \) is an unstable hyperbolic fixed point of \( (I) \), and its region of attraction \( \mathcal{Z}_{(x^*,f)} \) has measure zero.

**Proof:** We first establish that \( x^* \) is an unstable equilibrium point. To this end note that \( \mathcal{V}_0(x^*) < \mathcal{V}_0(0_0) \) implies that \( \mathcal{V}_s(x^*) < \mathcal{V}_s(0_s) \) holds for all \( c^* \in \mathbb{R}^n_0 \) and that \( \mathcal{V}_s \) is unbounded from below. Moreover, using \( \frac{d}{dt} \mathcal{V}_s < 0 \) for all \( x_s \neq 0_s \) it follows for all \( c^* \in \mathbb{R}^n_0 \) that \( \mathcal{V}_s(\varphi_{A,s}(t,c^*)x^*)) \rightarrow -\infty \) and \( ||\varphi_{A,s}(t,c^*)x^*)|| \rightarrow \infty \) as \( t \rightarrow \infty \). In other words, \( 0_s \) is an unstable equilibrium of \( \frac{d}{dt} x_s = A_s x_s \). Next, assume that \( x^* \) is not a hyperbolic fixed point, i.e., \( A_{s*} \) has eigenvalues with zero real part. This implies the existence of initial conditions \( x_{\delta,0} \neq 0_o \) such that \( \varphi_{A,s}(t,x_{\delta,0}) \) remains bounded for all \( t \in \mathbb{R}^2 \), but does not converge to the origin \( 0_o \). However, because \( \mathcal{V}_s(\varphi_{A,s}(t,x_{\delta,0})) \) is strictly decreasing in \( t \) for all \( x_{\delta,0} \neq 0_o \) it either holds that \( ||\varphi_{A,s}(t,x_{\delta,0})|| \rightarrow 0 \) as \( t \rightarrow \infty \) or \( \mathcal{V}_s(\varphi_{A,s}(t,x_{\delta,0})) \rightarrow -\infty \) and \( ||\varphi_{A,s}(t,x_{\delta,0})|| \rightarrow \infty \) as \( t \rightarrow \infty \). Therefore, there exists no initial state \( x_{\delta,0} \neq 0_o \) for which \( \varphi_{A,s}(t,x_{\delta,0}) \) remains bounded for all \( t \in \mathbb{R}^2 \), but does not converge to the origin \( 0_o \), i.e., \( A_{s*} \) cannot have eigenvalues with zero real part. Because the equilibrium is unstable, at least one eigenvalue of the linearized system must have positive real part, and the equilibrium is an unstable hyperbolic fixed point. Finally, because the vector field \( f \) is continuously differentiable it directly follows from [29] Prop. 11 that \( \mathcal{Z}_{(x^*,f)} \) has Lebesgue measure zero.

III. STABILITY THEORY FOR NESTED SYSTEMS ON MULTIPLE TIME SCALES

In this section we will present a model of nested dynamical systems on multiple time-scales and extend the results from Theorem [1] and Theorem [2] to this class of systems.

A. Nested systems on multiple time scales

Consider the nested dynamical system on \( N \) time scales shown in Figure 1 given by

\[
\begin{align*}
\frac{d}{dt} x_1 &= f_1(x_1, \ldots, x_{i+1}), \quad \forall i \in \mathbb{N}_{[1,N-1]}, \\
\frac{d}{dt} x_N &= f_N(x_1, \ldots, x_N),
\end{align*}
\]

where \( x_i \in \mathbb{R}^{n_i} \) denotes the state of the subsystem \( i \in \mathbb{N}_{[1,N-1]} \) and \( f_i : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_{i+1}} \rightarrow \mathbb{R}^{n_i} \) and \( f_N : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N} \rightarrow \mathbb{R}^{n_N} \) are Lipschitz continuous functions. Letting \( x = (x_1, \ldots, x_N) \) and \( f = (f_1, \ldots, f_N) \), we obtain the full system dynamics \( (I) \) with \( n = \sum_{i=1}^{N} n_i \). Broadly speaking, we assume that the dynamics are ordered from slow to fast convergence to their set of steady-states, i.e., the outer dynamics are the slowest and the inner right dynamics are the fastest (see Figure 1). To make this argument precise, we recursively define the steady-state maps for each \( f_i, i \in \mathbb{N}_{[2,N]} \).

**Assumption 1 (Steady-state maps)**

Let us define \( f_N^*(x_1, \ldots, x_{N-1}) := f_N(x_1, \ldots, x_N) \). We assume that there exists a unique steady-state map \( x_N^*(x_1, \ldots, x_{N-1}) \)

such that \( f_N(x_1, \ldots, x_{N-1}, x_N^*) = 0 \). Then for \( i \in \mathbb{N}_{[1,N-1]} \) we recursively define

\[
\begin{align*}
f_{N-i}^*(x_1, \ldots, x_{N-i}, x_{N-i+1}^*) = f_{N-i}(x_1, \ldots, x_{N-i}, x_{N-i+1}^*),
\end{align*}
\]

and recursively assume that, for all \( i \in \mathbb{N}_{[1,N-2]} \), there exists a unique steady-state map \( x_{N-i}^*(x_1, \ldots, x_{N-i-1}) \) such that \( f_{N-i}^*(x_1, \ldots, x_{N-i-1}, x_{N-i}^*) = 0 \).

In other words, \( f_i^* \) denotes the vector field corresponding to the dynamics with index \( i \in \mathbb{N}_{[1,N-1]} \) with the state \( x_{i+1} \) restricted to its steady-state map, i.e., \( x_{i+1} = x_{i+1}^* \). Intuitively, if the systems are ordered from slow to fast convergence to their set of steady-states, Assumption \( (I) \) suggests the natural model reduction procedure that successively replaces fast dynamics by their steady-state maps. Given \( r \in \mathbb{N}_I \), this results in the reduced-order dynamics (see Figure 2) with state vector \( (\hat{x}_1, \ldots, \hat{x}_r) \in \mathbb{R}^{\sum_{i=1}^{r} n_i} \) given by

\[
\begin{align*}
\frac{d}{dt} \hat{x}_i &= f_i(\hat{x}_1, \ldots, \hat{x}_{i+1}), \quad \forall i \in \mathbb{N}_{[1,r-1]}, \\
\frac{d}{dt} \hat{x}_r &= f_r^*(\hat{x}_1, \ldots, \hat{x}_{r}),
\end{align*}
\]

where \( \hat{x}_i = x_i^*(x_1, \ldots, x_{i-1}) \), \( \forall i \in \mathbb{N}_{[r+1,N]} \). We emphasize that \( (5) \) defines a reduced-order dynamical system. Next, given a set \( \Omega \subset \mathbb{R}^{n_r} \), we define the mapping of \( \Omega \) under the steady-state maps as

\[
\lambda^*(\Omega) := \left\{ \begin{array}{c}
\begin{bmatrix}
x_1 \\
\vdots \\
x_{N}
\end{bmatrix} \in \mathbb{R}^{n_r} \\
x_2 = x_2^*(x_1) \\
\vdots \\
x_N = x_N^*(x_1, \ldots, x_{N-1})
\end{array} \right\}.
\]

In the remainder of this section we derive conditions that allow us to extend guarantees of the type given in Theorem \( (I) \) and Theorem \( (2) \) for the reduced-order system \( \frac{d}{dt} \hat{x}_i = f_i^*(\hat{x}_i) \) and a set \( C_1 \subset \mathbb{R}^{n_1} \) to the full dynamics \( (I) \) and \( \lambda^*(C_1) \subset \mathbb{R}^{n_r} \).

B. Lyapunov function for nested systems

For all \( i \in \mathbb{N}_{[2,N]} \) we use \( y_i := x_i - x_i^* \in \mathbb{R}^{n_i} \) to denote the difference of \( x_i \) to its steady-state map \( x_i^* \). For all \( \forall i \in \mathbb{N}_{[1,N]} \) we define the continuously differentiable Lyapunov functions \( V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_{\geq 0} \). Given positive constants \( \mu_i \in \mathbb{R}_{\geq 0} \) to be determined, a Lyapunov function candidate \( \nu : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N} \rightarrow \mathbb{R}_{\geq 0} \) for the system \( (I) \) is given by

\[
\nu := \mu_1 V_1(x_1) + \sum_{i=2}^{N} \mu_i V_i(y_i).
\]
For clarity of the exposition, we omit the arguments of $V_i$, $x_i^t$, $f_i$ in the remainder. We require the following assumption that bounds the decrease of the individual Lyapunov functions $V_i$ in (7) for their associated reduced-order models and bounds their increase due to neglecting slower and faster dynamics.

**Assumption 2** Given compact sets $C_1 \subset \mathbb{R}^{n_1}$ and $U_1 \subset \mathbb{R}^{n_1}$, for all $i \in \mathbb{N}_{1,N}$, there exist positive semidefinite functions $\psi_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\psi'_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Functions $\sigma_i$ such that $\sigma_i(||x||_{1,i,N}) \leq \psi_i(x_1) + \psi'_i(x_1)$ and $\sigma_i(||y_i||) \leq \psi_i(y_i) + \psi'_i(y_i)$ for all $i \in \mathbb{N}_{2,N}$, and positive constants $\alpha_i \in \mathbb{R}_{>0}$ such that

$$\frac{\partial V_i}{\partial x_1} \leq -\alpha_i \psi_i(x_1)^2 - \alpha'_i \psi'_i(y_i)^2,$$

holds, and for all $i \in \mathbb{N}_{1,N}$ it holds that

$$\frac{\partial V_i}{\partial y_i} f_i \leq -\alpha_i \psi_i(y_i)^2 - \alpha'_i \psi'_i(y_i)^2.$$

Moreover, there exist positive constants $\beta_{i,i+1} \in \mathbb{R}_{>0}$ such that

$$\frac{\partial V_i}{\partial x_1} (f_i - f'_i) \leq \beta_{i,2} \psi_i(x_1) \psi_2(y_2)$$

holds, and for all $i \in \mathbb{N}_{2,N-1}$ it holds that

$$\frac{\partial V_i}{\partial y_i} (f_i - f'_i) \leq \beta_{i,i+1} \psi_i(y_i) \psi_{i+1}(y_{i+1}).$$

Finally, for all $i \in \mathbb{N}_{2,N}$, $j \in \mathbb{N}_{1,i+1}$, and $k \in \mathbb{N}_{i,i-1}$, there exist $b_{i,j,k} \in \mathbb{R}_{>0}$ such that

$$\frac{\partial V_i}{\partial x_k} f_k \leq \sum_{j=2}^{k+1} b_{i,j,k} \psi_i(y_j) \psi_j(y_j) + b_{i,1,k} \psi_i(y_1) \psi_1(x_1).$$

In particular, the first inequality in Assumption 2 bounds the decrease of the Lyapunov function $V_i$ along the trajectories of the reduced-order model $\frac{dx_i}{dt} = f_i(x_i)$. Moreover, for $i \in \mathbb{N}_{2,N}$ the second inequality bounds the decrease of the Lyapunov function candidates $V_i$ in the error coordinates $y_i$ under the assumption that all slower states are constant (i.e., $\frac{dy_j}{dt} = 0$ for all $j < i$) and all faster states are in their steady state (i.e., $x_j = x_j^\infty$ for all $j > i$). The remaining inequalities bound the additional terms in the time derivative of the Lyapunov function $\nu$ along the full dynamics (4) that arise because the slower states are generally not constant and the faster states are generally not in their steady state.

Note that a Lyapunov-based stability proof requires that the right hand side of the first two inequalities in Assumption 2 can be bounded by appropriate comparison functions. If $\psi' = 0$, then $\psi_i$ needs to be lower bounded by a $\mathcal{K}$-function and satisfy the last three inequalities in Assumption 2. In contrast, we only require that $\psi_i + \psi'_i$ is bounded from below by a suitable $\mathcal{K}$-function $\sigma_i$ (see, e.g., Theorem 1). This allows for additional flexibility in choosing $\psi_i$ to obtain improved bounds (see Section VI-C). For simplicity of notation we define

**Definition 3**

$$\beta_{i,i} := \sum_{k=1}^{i-1} b_{i,1,k} \quad \forall i \in \mathbb{N}_{2,N},$$

$$\beta_{i,j} := \sum_{k=j+1}^{i-1} b_{i,j,k} \quad \forall i \in \mathbb{N}_{3,N}, \forall j \in \mathbb{N}_{2,i-1},$$

$$\gamma_i := b_{i,i-1} \quad \forall i \in \mathbb{N}_{2,N}.$$ 

In other words, $\beta_{i,i-1}$, $\beta_{i-1,i}$, and $\gamma_i$ will be used to bound the difference between the reduced-order model (5) with $r = i - 1$ and the full dynamics $\frac{dx_i}{dt} = f_i$. Moreover, $\alpha_i$ bounds the convergence rate of $y_i$ under the assumption that all slower states are constant all faster states are in their steady state.

Moreover, we define $\mu_i$ used in the Lyapunov function (7) as $\mu_i := \prod_{j=1}^{i-1} \beta_{j,j+1}$ for all $i \in \mathbb{N}_{2,N}$, $\mu_1 = 1$, and we define the symmetric matrix $M$ as follows.

**Definition 4** Starting from $M_1 = \alpha_1$ the symmetric matrix $M \in \mathbb{R}^{N \times N}$ is recursively defined for all $i \in \mathbb{N}_{2,N}$ by its leading principal minors

$$M_i = \left[ \begin{array}{cc} M_{i-1} & -\beta_i \mu_i \\ \star & \alpha_i - \gamma_i \mu_i \end{array} \right],$$

and $\beta_i := (\ldots, \frac{1}{2} \beta_{i-3}, \frac{1}{2} \beta_{i-2}, \beta_{i-1})$.

Note that the constants $\beta_{i,j}$ are given by Assumption 2 for $j = i + 1$ and Definition 3 for $i \neq j$.

In the next section we show that the derivative of the Lyapunov function $\nu$ along the trajectories of (4) is bounded by $\frac{d}{dt} \nu \leq -\psi(\nu) + M(\psi)$, i.e., $\nu$ is decreasing if $M$ is positive definite. The main result of this section are two theorems that exploit this fact to establish almost global asymptotic stability of the nested system (4) using the Lyapunov function $\nu$. Subsequently, we will provide tractable conditions for verifying that $M$ is positive definite (see Section III-D).

C. Almost global asymptotic stability of nested systems

We are now ready to state the main result that establishes almost global asymptotic stability of $\mathbb{X}^s(C_1)$ with respect to a set $\mathbb{X}^s(\mathbb{C}_1)$, where $\mathbb{X}^s(\cdot)$ is defined in (6).

**Theorem 3 (Almost global asymptotic stability of nested systems)** Consider compact sets $C_1 \subset \mathbb{R}^{n_1}$ and $U_1 \subset \mathbb{R}^{n_1}$. Assume that, for all $i \in \mathbb{N}_{1,N}$, there exists $\chi^{\infty}_i \in \mathbb{X}^\infty$ and $\chi^\infty_{x_i} \in \mathbb{X}^\infty$ such that $\chi^{\infty}_i(||x||_{1,i,N}) \leq V_i(x_1) \leq \chi^\infty_{x_i}(||x||_{1,i,N})$ holds and $\chi^{\infty}_i(||y_i||) \leq V_i(y_i) \leq \chi^\infty_{x_i}(||y_i||)$ holds for all $i \in \mathbb{N}_{2,N}$. Suppose Assumption 7 and 2 hold, $M$ is positive definite, and the region of attraction $Z_{X^s(U_1),f}$ of $X^s(U_1)$ has measure zero, then the system (4) is almost globally asymptotically stable with respect to $X^s(C_1)$.

**Proof**: Let us consider the Lyapunov function candidate $\nu$ defined in (7). Using Lemma 1 (given in the appendix) with $\epsilon_i = 1$ for all $i \in \mathbb{N}_{1,N}$, we obtain $M = H$, $M' = H'$, and

$$\frac{d}{dt} \nu \leq -\psi(\nu) + M(\psi),$$

where $M'$ is a positive definite diagonal matrix, and, if $M'$ positive definite, there exists a positive constant $\alpha_M \in \mathbb{R}_{>0}$ such that $M \succ \alpha_M$ and $M' \succ \alpha_M$. It follows that

$$\frac{d}{dt} \nu \leq -\alpha_M \sum_{i=1}^{N_i} \psi_i^2 + \psi'_i^2 \leq -\frac{1}{2} \alpha_M \sum_{i=1}^{N_i} (\psi_i + \psi'_i)^2,$$
where the last inequality follows from the fact that $-\alpha_M(\psi_1^2 + \psi_2^2) \leq -\frac{1}{2}\alpha_M(\psi_1^2 + \psi_2^2)$ is equivalent to $\frac{1}{2}\alpha_M(\psi_1 - \psi_2)^2 \geq 0$ for all $i \in \mathbb{N}_{[1,N]}$. We conclude that

$$\frac{d}{dt} \nu \leq -\frac{1}{2}\alpha_M(\sigma_1(\|x_1\|_{s^1(U)}, U) + \sum_{i=2}^{N_c} \sigma_i(\|y_i\|^2)).$$  \hspace{1cm} (8)

The right hand side of (8) is positive definite and radially unbounded w.r.t. $X^*(C_1) \cup X^*(U_1)$, and $X^*(C_1) \cup X^*(U_1)$ is compact. Using the same steps as in [28, p. 98] there exists a function $\chi_3 \in \mathcal{K}$ such that

$$\frac{d}{dt} \nu \leq -\chi_3(\|x\|_{X^*(C_1) \cup X^*(U_1)}).$$

Moreover, under the hypothesis of the theorem, for all $i \in \mathbb{N}_{[1,N]}$, there exists $\chi_1^{\nu_1} \in \mathcal{K}_\infty$ and $\chi_2^{\nu_1} \in \mathcal{K}_\infty$ such that $\chi_1^{\nu_1}(\|x_1\|_{C_1}) \leq V_1(x_1) \leq \chi_2^{\nu_1}(\|y_1\|)$ holds and $\chi_1^{\nu_1}(\|y_1\|) \leq V_1(y_1) \leq \chi_2^{\nu_1}(\|y_1\|)$ holds for all $i \in \mathbb{N}_{[2,N]}$. For $j \in \mathbb{N}_{[1,2]}$, we define the functions $\tilde{x}_j := \sum_{i=1}^{N_c} \mu_i \chi_{Vi}$ that are positive definite and radially unbounded w.r.t. the compact set $X^*(C_1)$. Since it holds that $\tilde{x}_1 \leq \nu_1 \leq \tilde{x}_2$, following the same steps as in [28, p. 98] there exist $\chi_1 \in \mathcal{K}_\infty$ and $\chi_2 \in \mathcal{K}_\infty$ such that $\chi_1(\|x\|_{X^*(C_1)}) \leq \nu(x) \leq \chi_2(\|x\|_{X^*(C_1)})$. Finally, by the hypothesis of the theorem, the region of attraction $\mathcal{A}(x_0, X_1)$ of $X^*(U_1)$ has measure zero. With $C = X^*(C_1) \cup X^*(U_1)$, and $\nu = \nu$, it follows from Theorem 1 that (2) is almost globally asymptotically stable with respect to $X^*(C_1)$.

Theorem 3 requires that the region of attraction $\mathcal{A}(x_0, X_1)$ has measure zero. This can be verified using the following result that relies on the characterization of an unstable hyperbolic fixed point given in Theorem 2 as well as Assumption 2.

**Theorem 4 (Region of Attraction)** Suppose that $f_i$ in (1) is continuously differentiable with linearized dynamics at an equilibrium $x^* = (x_1, \ldots, x_N)$ given by

$$\frac{d}{dt} x_{\delta,i} = f_{\delta,i}(x_{\delta,1}, \ldots, x_{\delta,i}, \ldots, x_{\delta,N-1}) := \sum_{j=1}^{N} \frac{df_{\delta,i}}{dx_{\delta,j}} |_{x_{\delta,j} = x_{\delta,j}, x_{\delta,j}}$$

where $x_{\delta,j} := x_j - x_j^*$ for all $j \in \mathbb{N}_{[1,N]}$, and

$$\frac{d}{dt} x_{\delta,i} = f_{\delta,i}(x_{\delta,1}, \ldots, x_{\delta,i}, \ldots, x_{\delta,N-1}) := \sum_{j=1}^{i} \frac{df_{\delta,i}}{dx_{\delta,j}} |_{x_{\delta,j} = x_{\delta,j}, x_{\delta,j}}$$

for all $i \in \mathbb{N}_{[1,N-1]}$. Suppose that Assumption 1 and 2 hold for the linearized system (i.e., for $f_{\delta,i}, x_{\delta,1}, y_{\delta,1}$ instead of $f_i, x_i$, and $y_i$), $C_1 = \emptyset$ and $U_1 = \{0,n\}$, and $V_1 = \frac{1}{b} x_1^T P_1 x_1$, $V_{\delta,1} = \frac{1}{b} y_1^T P_1 y_1$ for all $i \in \mathbb{N}_{[2,N]}$. Moreover, assume that $M$ is positive definite. If there exists a function $\chi_3 \in \mathcal{K}$ such that $\chi_3(\|x\|_{X^*(C_1) \cup X^*(U_1)}) < V_1(x_0)$, then the region of attraction $\mathcal{A}(x_0, f)$ has measure zero.

**Proof:** We define the Lyapunov-like function $\nu_3 := \frac{1}{2} x_{\delta,1}^T P_1 x_{\delta,1} + \sum_{i=1}^{N} \frac{1}{2} x_{\delta,i}^T P_1 x_{\delta,i}$. Following the same steps as in the proof of Theorem 3 i.e., using Lemma 1 (given in the appendix) with $\epsilon_i = 1$ for all $i \in \mathbb{N}_{[1,N]}$, Lemma 2 Assumption 2 and Proposition 1 it follows that there exists a function $\chi_3 \in \mathcal{K}$ such that $\frac{d}{dt} \nu_3 \leq -\chi_3(\|x\|)$ holds. Moreover, by the hypothesis of the theorem it holds that $V_1(x_0) < V_1(x_0)$ and $V_{\delta,1}(0) = 0$ for all $i \in \mathbb{N}_{[2,N]}$. Letting $x_{\delta,i} = (x_{\delta,1}, x_{\delta,2}, \ldots, x_{\delta,N})$ it directly follows that $\nu_3(x_{\delta,1}) < \nu_3(0)$ and the theorem follows by noting that $\nu_3$ satisfies the conditions of Theorem 2.

Note that Theorem 3 and Theorem 4 provide a way to extend results obtained for the reduced-order system $\frac{d}{dt} x_1 = f_1(x_1)$ to the full-order system (1). In particular, if there exists a Lyapunov function $V_1$ and Lyapunov-like function $V_{\delta,1}$ for which the conditions of Theorem 1 and Theorem 2 hold, then Theorem 3 and Theorem 4 require to find Lyapunov functions $V_i$ and $V_{\delta,i}$ for all $i \in \mathbb{N}_{[2,N]}$ in the error coordinates $y_i$ that satisfy Assumption 2.

Moreover, Theorem 3 and Theorem 4 require that $M$ is positive definite. In the next section we exploit the structure of $M$ to provide a recursive sufficient condition for $M$ to be positive definite. This simplified recursive condition will be used to provide analytical stability guarantees for the multi-converter power system considered in Section IV.

**D. Necessary and sufficient condition for $M$ to be positive definite.**

The following condition is necessary and sufficient for $M$ to be positive definite.

**Condition 1 (Positivity condition)** For all $i \in \mathbb{N}_{[2,N]}$ it holds that $\gamma_i + \mu_i \beta_i \mu_{i-1} \beta_{i-1} < \alpha_i$, where $M_i$ denotes the $i$-th leading minor of the matrix $M$.

Verifying Condition 1 requires inverting the matrix $M_i$, i.e., the complexity of the inequalities that need to be verified grows considerably with $N$. In contrast, the following recursive sufficient condition avoids this issue by introducing additional variables $c_i \in \mathbb{R}_{>0}$, that allow us to exploit the structure of the problem. In particular, the variable $c_{i-1}$ lower bounds the smallest eigenvalue of $M_{i-1}$ and can be interpreted as bound on the convergence rate of the reduced-order dynamics (5) with $r = i - 1$.

**Condition 2 (Sufficient positivity condition)** For all $i \in \mathbb{N}_{[2,N]}$ it holds that

$$1 < \frac{\alpha_i \beta_{i-1} \beta_{i-1} \beta_{i-1}}{\beta_{i-1} \beta_{i-1} \beta_{i-1} + \gamma_i \beta_{i-1} \beta_{i-1}},$$

where $c_1 := \alpha_1 \in \mathbb{R}_{>0}$, $\alpha_i \in \mathbb{R}_{>0}$, $\gamma_i \in \mathbb{R}_{>0}$, $\beta_{i-1} \in \mathbb{R}_{>0}$ for all $i \in \mathbb{N}_{[2,N]}$ and $j \in \mathbb{N}_{[1,i-1]}$, and

$$c_i := \frac{1}{2} \left( \alpha_i - \gamma_i + \frac{\beta_{i-1} \beta_{i-1} \beta_{i-1}}{\beta_{i-1} \beta_{i-1} \beta_{i-1}} - \sqrt{D_i} \right)$$

for all $i \in \mathbb{N}_{[2,N]}$ and $D_i := (\alpha_i - \gamma_i + \frac{\beta_{i-1} \beta_{i-1} \beta_{i-1}}{\beta_{i-1} \beta_{i-1} \beta_{i-1}})^2 + 4(\beta_{i-1} \beta_{i-1} \beta_{i-1} - \alpha_i - \gamma_i)^2$.

Condition 2 holds if $y_i$ converges fast enough relative to the reduced-order system (5) with $r = i - 1$. Next, we show that Condition 2 $\implies$ Condition 1 $\iff M > 0$.

**Proposition 1 (Positive definite $M$)** If Condition 1 holds, then the matrix $M$ is positive definite. Suppose that Condition 2 holds, then $c_i \in \mathbb{R}_{>0}$ and $M_i \geq \mu_i c_i$ holds for all $i \in \mathbb{N}_{[2,N]}$, and the matrix $M$ is positive definite.

**Proof:** Using Lemma 2 (given in the appendix) it can be verified that $M$ is positive definite if and only if $M_i \geq 0$ (see Definition 4) holds for all $i \in \mathbb{N}_{[1,N]}$, i.e., if and only if
Condition [1] holds. Next, using the partitioning of the matrix $M_i$ from Definition [4] and applying the Schur complement it can be verified that $$M_i \geq \mu_i c_i \quad \text{for all } i \in N_{[2,N]}$$ if and only if $M_i^{-1} \geq \mu_i c_i$ and

$$\alpha_i - \gamma_i - c_i - \beta^T (M_i^{-1} - c_i)^{-1} \beta_i \geq 0$$  \hspace{1cm} (10)

holds for all $i \in N_{[2,N]}$. The remainder of this proof establishes that (10) and $M_i^{-1} \geq \mu_i c_i$ hold for all $i \in N_{[2,N]}$ if Condition [2] holds. To this end, we first show that if $c_{i-1} \in R_{>0}$ and Condition [2] hold, then $c_i \in R_{>0}$ and $\frac{b_i}{b_{i-1}} c_{i-1} - c_i \in R_{>0}$. In particular, $c_i \in R_{>0}$ holds if and only if

$$(\alpha_i - \gamma_i + \frac{b_i}{b_{i-1}} c_{i-1})^2 > D_i$$ \hspace{1cm} (11)

holds. Rewriting $D_i > 0$ as $D_i = (\alpha_i - \gamma_i + \frac{b_i}{b_{i-1}} c_{i-1})^2 + 4 \beta^T \beta_i$ it can be verified that (11) is identical to (9). Moreover, by definition of $c_i$ it follows that $\frac{b_i}{b_{i-1}} c_{i-1} - c_i = \gamma_i + \sqrt{D_i - \alpha_i}$. Therefore, $\frac{b_i}{b_{i-1}} c_{i-1} - c_i \in R_{>0}$ holds if $c_i \in R_{>0}$ and $c_{i-1} \in R_{>0}$ holds using $c_i \in R_{>0}$ and $c_{i-1} \in R_{>0}$ it follows that $\frac{b_i}{b_{i-1}} c_{i-1} - c_i \in R_{>0}$ if (11) holds. This proves the claim.

Next, we show that $M_{i-1} \geq \mu_{i-1} c_{i-1}$, $c_i \in R_{>0}$, and Condition [2] guarantee that $M_i \geq \mu_i c_i$ and (10) hold. In particular, $M_{i-1} \geq \mu_{i-1} c_{i-1}$ implies that

$$\mu_{i-1} M_{i-1} - c_i \geq \mu_{i-1} \frac{b_{i-1}}{b_{i-1}} c_{i-1} - c_i = \frac{b_{i-1}}{b_{i}} \frac{b_{i-1}}{b_{i-1}} c_{i-1} - c_i \geq 0,$$

and it follows that $M_{i-1} - \mu_i c_i \geq 0$ and $\left(\frac{b_{i-1}}{b_{i}} \frac{b_{i-1}}{b_{i-1}} c_{i-1} - c_i\right)^{-1} \geq (M_i^{-1} M_{i-1} - c_i)^{-1}$. Therefore, (10) holds if

$$\alpha_i - \gamma_i - c_i - \beta^T (\frac{b_{i-1}}{b_{i}} \frac{b_{i-1}}{b_{i-1}} c_{i-1} - c_i)^{-1} \beta_i \geq 0$$ \hspace{1cm} (12)

holds. Multiplying (12) by $\frac{b_{i-1}}{b_{i}} \frac{b_{i-1}}{b_{i-1}} c_{i-1} - c_i \in R_{>0}$ results in the equivalent condition $\frac{b_{i-1}}{b_{i}} \frac{b_{i-1}}{b_{i-1}} c_{i-1} - c_i (\alpha_i - \gamma_i - c_i - \beta^T \beta_i \geq 0$ and it can be verified that $c_i$ defined in Condition [2] is a solution of this inequality. Thus, $M_{i-1} \geq \mu_{i-1} c_{i-1}$, $c_i \in R_{>0}$, and Condition [2] guarantee that $M_i \geq \mu_i c_i$ holds for all $i \in N_{[2,N]}$.

Next, noting that $M_1 = \mu_1 c_1 \in R_{>0}$ it follows by induction that $c_i \in R_{>0}$ and $M_i \geq \mu_i c_i$ holds for all $i \in N_{[2,N]}$. Finally, applying the Sylvestor criterion it can be shown that $M_i \geq \mu_i c_i$ for all $i \in N_{[1,N]}$ implies that $M \geq 0$.

IV. MULTI-CONVERTER POWER SYSTEM AND CONTROL OBJECTIVES

A. Model of an converter based power system

The three-phase power system considered in this article consists of $N$ DC/AC converters, interconnected through $M$ resistive-inductive lines. All electrical quantities in the network are assumed to be balanced. This allows us to work in $\alpha \beta$ coordinates obtained by applying the well-known Clarke transformation to the three-phase variables.

In this work we consider an averaged model of a two-level voltage source converter consisting of an averaged switching stage that modulates the DC voltage $v_{DC,k}$ into an AC voltage $v_{m,k} = \frac{1}{2} m_k v_{DC,k}$, where $m_k$ is the modulation signal. The overall setup is depicted in Figure 3. Each converter is connected to the transmission network via an RLC filter with resistance $r_{f,k} \in R_{>0}$, inductance $\ell_{f,k} \in R_{>0}$, capacitance $c_{f,k} \in R_{>0}$, and a conductance $g_{f,k} \in R_{>0}$ to ground. The electrical states of the $k$-th converter (in $\alpha \beta$ coordinates) are the output filter current $I_{f,k} \in R^2$, the terminal voltage $V_k \in R^2$, and the current injected into the transmission network $I_{b,k} \in R^2$. We consider the standard setup in which the DC/AC converter is controlled via cascaded current and voltage control loops (see Section V).

For the purpose of this work we assume that the DC voltage $v_{DC,k}$ is regulated to be constant via the source feeding

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Model of a DC/AC converter with reference model, reference tracking controller, and transmission network.}
\end{figure}
the converter. Under this standard assumption, the open-loop dynamics of a single converter are given by

\begin{align}
L_f \frac{d}{dt} i_f &= -R_f i_f - \psi + \psi_m, \\
C_f \frac{d}{dt} \psi &= -G_f \psi - \omega_0 + L_f,
\end{align}

(14a)

(14b)

where \( R_f := \text{diag}(\{r_{f,k}\}_{k=1}^N) \otimes I_2 \), \( L_f := \text{diag}(\{l_{f,k}\}_{k=1}^N) \otimes I_2 \), \( C_f := \text{diag}(\{c_{f,k}\}_{k=1}^N) \otimes I_2 \), and \( G_f := \text{diag}(\{g_{f,k}\}_{k=1}^N) \otimes I_2 \). Moreover, the vector \( i_f := (i_{f,1}, \ldots, i_{f,N}) \in \mathbb{R}^{2N_c} \) collects the output filter currents and \( \psi_m := (\psi_{m,1}, \ldots, \psi_{m,N}) \in \mathbb{R}^{2N_c} \) collects the modulated AC voltages.

The electrical states of a transmission line are the line currents \( v_{i,j,k} \in \mathbb{R}^2 \). The electrical parameters of the transmission lines are the resistance \( r_{t,j,k} \in \mathbb{R}_{>0} \) and inductance \( \ell_{i,j,k} \in \mathbb{R}_{>0} \). Moreover, for each transmission line, we define the impedance \( Z_{j,k} := I_2 r_{t,j,k} + \omega_0 J_{t,j,k} \) and the admittance \( Y_{j,k} := Z_{j,k}^{-1} \).

The network topology is given by the oriented incidence matrix \( B \in \{-1,0,1\}^{N_t \times N_c} \) of the connected, undirected, weighted graph \( G = (\mathcal{N}, \mathcal{E}, \mathcal{W}) \); where \( \mathcal{N} := \mathbb{N}_{[1,N_c]} \) is the set of nodes, corresponding to the converters; \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \) is the set of edges (with cardinality \(|\mathcal{E}| = N_t\)) corresponding to the transmission lines, and the weights \( \mathcal{W} \) are given by

\[ \|Y_{j,k}\| := \frac{1}{\sqrt{\alpha_{i,j,k} + \omega_0 r_{t,j,k}}}. \]

For the notational simplicity, we associate an index \( l \in \{1, \ldots, N_t\} \) to each of the \( N_t \) transmission lines \((j,k) \in \mathcal{E}\) and define the extended incidence matrix \( B := B \otimes I_2 \).

The transmission network dynamics, in matrix form, are given by

\[ L_T \frac{d}{dt} \dot{I}_T = -R_T I_T + B^T v, \]

(16)

where \( R_T := \text{diag}(\{r_{T,l}\}_{l=1}^{N_t}) \otimes I_2 \), \( L_T := \text{diag}(\{l_{T,l}\}_{l=1}^{N_t}) \otimes I_2 \), and \( \dot{I}_T := (\dot{I}_{1,1}, \ldots, \dot{I}_{1,M}) \in \mathbb{R}^{2N_c} \). Moreover, \( v := (v_1, \ldots, v_N) \in \mathbb{R}^{2N_c} \) denotes the vector of terminal voltages and \( \psi_0 := (\psi_{0,1}, \ldots, \psi_{0,N}) \in \mathbb{R}^{2N_c} \) collects all currents that flow form the converter into the network, and, by Kirchoff’s currents law, is given by

\[ \dot{I}_T = B \psi_0. \]

Finally, we define the quasi steady-state of the network. To this end, we change variables to a coordinate frame rotating at the nominal frequency \( \omega_0 \),

\[ v = R_N(\omega_0 t)v, \quad i = R_M(\omega_0 t)i, \quad \omega_0 = R_N(\omega_0 t)\omega_0, \]

(18)

where \( R_N(\theta) := I_N \otimes R(\theta) \). In this rotating frame the transmission network dynamics are given by

\[ L_T \frac{d}{dt} i_t = -Z_T i_t + B^T v, \]

(19)

where \( Z_T = R_T + \omega_0 J_M L_T \) and \( J_M := I_M \otimes J \). The quasi-steady-state model is obtained by considering the steady-state map of the exponentially stable dynamics \( 19 \).

**Definition 5 (Quasi-steady-state network model)**

The quasi-steady-state model of the transmission line currents \( i_t \) and output currents \( i_o \) is given by the steady-state map \( i_t^*(v) := Z_T^{-1} B^T v \) of \( 19 \) and \( i_o^*(v) := Y_{\text{net}} v \) with \( Y_{\text{net}} := B Z_T^{-1} B^T \). Moreover, we define the Laplacian \( L := B \{\|Y\|\}^{N_t} I_2 \) and the extended Laplacian \( \Lambda := L \otimes I_2 \) in \( \alpha \beta \) coordinates.

In conventional power systems the approximation \( i_t = i_t^*(v) \) is typically justified due to the pronounced time-scale separation between the dynamics of the transmission lines and the dynamics of synchronous machines. However, for converter-based power systems the electromagnetic transients of the lines have a significant influence on the stability boundaries, and the approximation is no-longer valid [8], [13]. In the remainder of this work, we make the time-scale separation argument rigorous and explicitly quantify how large the time-scale separation needs to be between the converter, the network, and the converter control.

In the next section we discuss the control objectives in detail before presenting the control design in Section \[7\].

**B. Control objectives**

Before defining the control objectives we define instantaneous active, reactive, and apparent power as follows.

**Definition 6 (Instantaneous Powers)**

Given the voltage \( v_{k} \) and output current \( i_{o,k} \) at node \( k \in \mathbb{N}_{[1,N_c]} \), we define the instantaneous active power \( p_k := \Re[v_{k}^* i_{o,k}] \in \mathbb{R} \), the instantaneous reactive power \( q_k := \Im[v_{k}^* i_{o,k}] \in \mathbb{R} \), and the instantaneous apparent power \( s_k := \sqrt{p_k^2 + q_k^2} \). Moreover, for all \((j,k) \in \mathcal{E}\), we define the instantaneous active and reactive branch powers \( p_{j,k} := v_{j,k}^* i_{j,k} \) and \( q_{j,k} := v_{j,k}^* J_{j,k} i_{j,k}. \)

The control objective is to stabilize the steady-state behavior specified by:

- **Frequency synchronization**: Given the nominal synchronous frequency \( \omega_0 \), at steady state it holds that:

\[ \frac{d}{dt} \dot{\omega}_k = \omega_0 J_{\dot{\omega}_k}, \quad k \in \mathbb{N}_{[1,N_c]}, \]

(20a)

\[ \frac{d}{dt} \dot{I}_{f,k} = \omega_0 J_{\dot{I}_{f,k}}, \quad k \in \mathbb{N}_{[1,N_c]}, \]

(20b)

\[ \frac{d}{dt} \dot{I}_{o,k} = \omega_0 J_{\dot{I}_{o,k}}, \quad k \in \mathbb{N}_{[1,N_c]}, \]

(20c)

- **Terminal voltage magnitude**: Given the terminal voltage magnitude set-points \( v_{k}^* > 0 \), at steady state it holds that:

\[ \|v_k\| = v_k^*, \quad k \in \mathbb{N}_{[1,N_c]}, \]

(21)

- **Power injection**: At steady state, each converter injects the prescribed active and reactive power i.e.,

\[ v_{k}^T i_{o,k} = p_k^*, \quad v_{k}^T J_{o,k} i_{o,k} = q_k^*, \quad k \in \mathbb{N}_{[1,N_c]}, \]

(22)

Equations \( 20 \) and \( 21 \) ensure that, at steady state, the voltages and currents in the power system evolve as perfect sinusoidal signals, with frequency \( \omega_0 \). Equations \( 21 \) and \( 22 \) specify the power injection and the voltage magnitudes at every node \([8]\). Using the quasi-steady-state network model \( i_t^*(v) = Y_{\text{net}} v \), the local non-linear specification on the power injection of every converter given by \( 22 \) can be expressed as linear, albeit, non-local specification on voltages as follows \([8]\):

- **Phase synchronization**: Given a relative angle set-points \( \theta_{k}^* \in [-\pi,\pi] \), at steady state it holds that:

\[ \frac{v_k}{v_k^*} - R(\theta_{k}^*) \frac{v_1}{v_1^*} = 0, \quad k \in \mathbb{N}_{[1,N_c]} \setminus \{1\}. \]

(23)
Where \( \theta_j^{\ast} : = \theta_j^t - \theta_j^d \) for all \( j, k \in [1, N] \) is the relative steady-state angle between the terminal voltages of the \( j \)-th and \( k \)-th converter, and we specified phase angles relative to the first node. Finally, the instantaneous active power, reactive power, and terminal voltage set-points need to satisfy the power-flow equations [6].

**Condition 3 (Consistent set-points)** The set-points \( p_k^\ast \in \mathbb{R} \), \( q_k^\ast \in \mathbb{R} \), \( v_k^\ast \in \mathbb{R}_{>0} \) for active power, reactive power, and voltage magnitude respectively, are consistent with the power-flow equations, i.e., for all \( (j, k) \in \mathcal{E} \) there exist relative angles \( \theta_{jk}^\ast \in [-\pi, \pi) \) and steady-state branch powers \( p_{jk}^\ast \in \mathbb{R} \) and \( q_{jk}^\ast \in \mathbb{R} \) given by

\[
\begin{align*}
p_{jk}^\ast & := \| Y_{jk} \| \left( v_k^\ast r_{jk} - v_k^\ast v_j^\ast (r_j + \omega_0 \ell_j \sin(\theta_{jk}^\ast)) \right), \\
q_{jk}^\ast & := \| Y_{jk} \| \left( v_k^\ast \omega_0 \ell_j - v_k^\ast v_j^\ast (\omega_0 \ell_j \cos(\theta_{jk}^\ast) - r_j \sin(\theta_{jk}^\ast)) \right),
\end{align*}
\]

such that \( p_k^\ast = \sum_{(j,k) \in \mathcal{E}} p_{jk}^\ast \) and \( q_k^\ast = \sum_{(j,k) \in \mathcal{E}} q_{jk}^\ast \) holds for all \( k \in [1, N] \).

V. DISPARTEABLE VIRTUAL OSCILLATOR CONTROL WITH CASCADED INNER CONTROL LOOPS

In Section IV we introduced a system model for a multi-converter power system and the corresponding control objectives. In this section, we will propose a control law that admits a fully decentralized implementation and consists of two cascaded inner loops that track reference voltage. However, the standard cascaded PI voltage and current control loops are formulated in terms of local rotating reference frames for every converter [31]. Typically this approach severely complicates a nonlinear stability analysis of multi-converter system. In contrast, we propose a cascaded two-degree of freedom inner control structure in stationary (i.e., global non-rotating) coordinates that, under suitable assumptions, preserves the stability guarantees obtained for dispatchable virtual oscillator control obtained under the assumption that the terminal voltage \( v \) can be controlled directly [8], [25]. After presenting the control law, we re-formulate the closed-loop dynamics and control objectives in a global rotating frame, and provide the steady-state maps and reduced-order models required to apply the results developed in Section III.

A. Dispatchable virtual oscillator control as reference model

We require the following assumption that is commonly made in the stability analysis of AC power systems.

**Assumption 3 (Uniform inductance-resistance ratio)** The ratio between the inductance and resistance of every transmission line is constant, i.e., for all \( (j, k) \in \mathcal{E} \) it holds that

\[
\frac{\ell_{jk}}{r_{jk}} = \rho \in \mathbb{R}_{>0}.
\]

This assumption typically holds for transmission lines on the same voltage level and simplifies the analysis while preserving the main salient features of the system. We define the angle \( \kappa := \tan^{-1}(\omega_0 \rho) \). Note that under Assumption 3 it holds that \( \mathcal{L} = \mathcal{R}(\kappa) \mathcal{V}_{\text{net}} \).

The multi-converter system (without internal converter dynamics) is almost globally asymptotically stable with respect to the specifications given in the previous section if the converter terminal voltages \( v_k \) evolve according to (see [8])

\[
\frac{dv_k}{dt} = \omega_0 J_k \dot{v}_k + \eta (K_k \dot{v}_k - \mathcal{R}(\kappa) \dot{\mathcal{V}}_{\text{net},k} + \eta_0 \Phi_k (v_k)) \tag{24},
\]

where \( \eta \in \mathbb{R}_{>0} \) and \( \eta_0 \in \mathbb{R}_{>0} \) are control gains, and \( K_k \) and \( \Phi_k \) are given by

\[
K_k := \frac{1}{v_k^2} \mathcal{R}(\kappa) \left[ \begin{array}{cc} p_k^\ast & q_k^\ast \\ -q_k^\ast & p_k^\ast \end{array} \right], \quad \Phi_k (v_k) := I_2 - \frac{\| v_k \|^2}{v_k^2} I_2. \tag{25}
\]

Note that the term \( \omega_0 J_k \dot{v}_k \) induces a harmonic oscillation of \( v_k \) at the nominal frequency \( \omega_0 \). Moreover, \( \Phi_k (v_k) \dot{v}_k \) can be interpreted as a voltage regulator, i.e., depending on the sign of the normalized quadratic voltage error \( \Phi_k (v_k) \) the voltage vector \( \dot{v}_k \) is scaled up or down. Finally, the term \( K_k \dot{v}_k - \mathcal{R}(\kappa) \dot{\mathcal{V}}_{\text{net},k} \) can be interpreted either in terms of tracking power set-points, i.e., [22], or in terms of phase synchronization, i.e., [23]. For details see [8], [25], [26].

To this end, we use (24) as a reference model, i.e., for each node \( k \in [1, N] \) the reference model is defined as

\[
\frac{dv_k}{dt} := \omega_0 J_k \dot{v}_k + \eta (K_k \dot{v}_k - \mathcal{R}(\kappa) \dot{\mathcal{V}}_{\text{net},k} + \eta_0 \Phi_k (v_k) \dot{\tilde{v}}_k), \quad \tag{26}
\]

with \( K_k \) and \( \Phi_k \) defined as in (25). Next, we design a tracking controller that controls the terminal voltage \( v_k \) to track the reference voltage \( \tilde{v}_k \).

B. Two-degree-of-freedom reference tracking controllers

We propose two local (i.e., decentralized) cascaded two-degree-of-freedom PI controllers for the filter current and terminal voltage. An outer loop provides a reference \( \dot{v}_f,k \) for filter currents \( \dot{i}_{f,k} \) and ensures that terminal voltage \( v_k \) is tracking the voltage reference signal \( \tilde{v}_k \). Furthermore, an inner loop computes the control signal \( \sum_{m,k} \) and guarantees that the filter current \( \dot{i}_{f,k} \) tracks the reference \( \dot{v}_f,k \) provided by the outer loop.

Given the open loop dynamics (14) of the filter; first it is assumed that for the voltage dynamics, given by (14), the filter current \( \dot{i}_{f,k} \) can be used as a control signal. Then, using \( Y_{f,k} := \omega_0 J_{f,k} + g_{f,k} \), the two-degree of freedom voltage PI controller \( \dot{v}_{f,k} \) is defined as

\[
\begin{align}
\frac{d}{dt} \zeta_{f,k} & := \omega_0 J_{\zeta_{f,k}} + \eta_0 \Phi_k (v_k) \dot{\tilde{v}}_k, \tag{27a} \\
\dot{v}_{f,k} & := Y_{f,k} \zeta_{f,k} + \sum_{m,k} - K_{p,v,k} (\zeta_{v,k} - \tilde{v}_k) - K_{i,v,k} \zeta_{v,k}, \tag{27b}
\end{align}
\]

where the term \( Y_{f,k} \zeta_{f,k} \) compensates the filter admittance losses at the synchronous steady-state behaviour \( \mathcal{V}_{f,k} \) with frequency \( \omega_0 \) and \( \zeta_{v,k} \) is an integrator state that rotates at the nominal frequency \( \omega_0 \). We stress that \( \zeta_{f,k}, \tilde{v}_k, \dot{v}_{f,k} \) and \( \dot{i}_{f,k} \) are obtained from local measurements. The proportional and integral gains are denoted with \( K_{p,v,k} \in \mathbb{R}_{>0} \) and \( K_{i,v,k} \in \mathbb{R}_{>0} \), respectively.

However, because \( \dot{i}_{f,k} \) is not an input of the system, the controller (27) cannot be applied directly. Therefore, we use another two-degree of freedom current PI controller

\[
\begin{align}
\frac{d}{dt} \zeta_{f} & := \omega_0 J_{\zeta_{f}} + \dot{i}_{f,k} - \dot{v}_{f,k}, \tag{28a} \\
\zeta_{m,k} & := Z_{f,k} \zeta_{m,k} + \sum_{m,k} - K_{p,i,k} (\dot{i}_{f,k} - \dot{v}_{f,k}) - K_{i,i,k} \zeta_{m,k}, \tag{28b}
\end{align}
\]

that tracks the reference \( \dot{v}_{f,k} \) by acting on the control input \( \zeta_{m,k} \). The term \( Z_{f,k} \zeta_{m,k} \) where \( Z_{f,k} := \omega_0 J_{f,k} + \tau_{f,k} \),
compensates the filter impedance losses at the synchronous steady-state behavior \((20b)\) with frequency \(\omega_0\) and \(\zeta_{fs}\) is an integrator state that rotates at the nominal frequency \(\omega_0\).

Finally, the proportional and integral gains are denoted with \(K_{p,fs} \in \mathbb{R}_{>0}\) and \(K_{i,fs} \in \mathbb{R}_{>0}\), respectively.

### C. Closed-loop dynamics in a rotating frame

In order to simplify the analysis, it is convenient to perform the change of variables to a (global) rotating reference frame rotating at the nominal frequency \(\omega_0\). First, we define vectors \(\hat{\mathbf{u}} \in \mathbb{R}^{2N_c}, \hat{\mathbf{I}}_f \in \mathbb{R}^{2N_c}, \hat{\mathbf{u}}_v \in \mathbb{R}^{2N_c}\), as well as \(\hat{\mathbf{f}} f \in \mathbb{R}^{2N_c}, \hat{\mathbf{C}} \in \mathbb{R}^{2N_c}\), that collect the states of the individual converters and transmission lines (i.e., \(\hat{\mathbf{u}} := (\hat{u}_1, \ldots, \hat{u}_N)\)).

Next, we define the vector of voltage variables as \(\mathbf{x}_v := (\mathbf{u}_v, \mathbf{C}_v)\) and filter current variables as \(\mathbf{x}_f := (\mathbf{I}_f, \mathbf{C}_f)\) and collect all states in the vector \(\mathbf{x} := (\mathbf{u}_v, \mathbf{x}_f) \in \mathbb{R}^n\), with \(n := 10N_c + 2N_t\). The change of variables to a reference frame rotating at the nominal frequency \(\omega_0\) is given by:

\[
\begin{bmatrix}
\mathbf{x} \\
\mathbf{x}_f \\
\mathbf{C}_f
\end{bmatrix} = \mathcal{R}_{\mathbf{T}_N} + N_0(\omega_0 t)
\begin{bmatrix}
\mathbf{x} \\
\mathbf{x}_f \\
\mathbf{C}_f
\end{bmatrix}.
\]  

Next, we define the matrices

\[
\mathcal{K} := \text{diag}(\{K_k\}_{k=1}^{N_c}), \quad \Phi(v) := \text{diag}(\{\Phi(v_k)\}_{k=1}^{N_c}), \quad \mathbf{K}_{p,v} := \text{diag}(\{K_{p,v} I_2\}_{k=1}^{N_c}),
\]

\[
\mathbf{K}_{i} := \text{diag}(\{K_{i} I_2\}_{k=1}^{N_c}), \quad \mathbf{K}_{i,v} := \text{diag}(\{K_{i,v} I_2\}_{k=1}^{N_c}),
\]

and obtain

\[
\frac{d}{dt} \hat{\mathbf{u}} = \eta(\mathcal{K} \hat{\mathbf{u}} - \mathbf{R}(\kappa) \mathbf{B}_i + \eta_\Phi \hat{\mathbf{u}} \hat{\mathbf{v}}),
\]

\[
\frac{d}{dt} \mathbf{i}_f = L_T^{-1}(-Z_T \mathbf{i}_f + \mathbf{B}_f \mathbf{v}),
\]

\[
\frac{d}{dt} \mathbf{v} = \left[ C_f^{-1}(-Y_f \mathbf{v} - \mathbf{B}_i \mathbf{i}_f + \mathbf{i}_f) \right],
\]

\[
\frac{d}{dt} \mathbf{c}_f = \left[ L_T^{-1}(-K_{p,f} \mathbf{i}_f - \mathbf{k}_f \mathbf{c}_f) \right],
\]

where \(Y_f = \text{diag}(\{Y_f k\}_{k=1}^{N_c})\) and \(\mathbf{i}_f\) is given by

\[
\mathbf{i}_f = Y_f \mathbf{v} + \mathbf{B}_i - K_{p,v}(\mathbf{v} - \hat{\mathbf{v}}) - K_{i,v} \mathbf{c}_f.
\]

The dynamics of the overall system \(x\) in the rotating frame are given by \(\frac{d}{dt} x = f_x(x)\), where \(f_x = (f_x, f_f, f_f)\). Additionally, it can be easily noticed that the dynamics \(f_x\) are given in the normal nested form \(4\).

### D. Steady-state maps and reduced-order dynamics

Letting \(N = 4\), \((x_1, x_2, x_3, x_4) := (\hat{\mathbf{u}}, \mathbf{i}_f, \mathbf{v}_f, \mathbf{f}_f)\), and \((f_1, f_2, f_3, f_4) := (f_1, f_3, f_4, f_f)\) we can define the steady-state maps \(x^*_f\), \(x^*_v\), and \(i^*_f\) as in Assumption \(4\). Moreover, we can define the corresponding reduced-order dynamics \(f^*_x, f^*_f, f^*_v\), and \(f^*_f\). The steady-state maps of the closed loop system \((30)\) are given by

\[
\begin{align*}
&x^*_f(\hat{\mathbf{u}}, \mathbf{i}_f, \mathbf{v}_f, \mathbf{f}_f) := (f^*_1(\hat{\mathbf{u}}, \mathbf{i}_f, \mathbf{v}_f, \mathbf{f}_f), 0_{2N_c}), \\
&x^*_v(\hat{\mathbf{u}}) := (\hat{\mathbf{u}}, 0_{2N_c}), \\
&i^*_f(\hat{\mathbf{u}}) := Z^{-1} \mathbf{B}_f \hat{\mathbf{u}}.
\end{align*}
\]

Moreover, we obtain the following reduced-order dynamics

\[
\begin{align*}
&f^*_f(\hat{\mathbf{u}}, \mathbf{i}_f) := \left[-C_f^{-1}(K_{p,v}(\mathbf{v} - \hat{\mathbf{v}}) + K_{i,v} \hat{\mathbf{c}_f})\right], \\
&f^*_v(\hat{\mathbf{u}}, \mathbf{i}_f) := \left[-L_T^{-1}(-Z_T^{-1} \mathbf{i}_f + \mathbf{B}_f \hat{\mathbf{u}})\right], \\
&f^*_c(\hat{\mathbf{u}}) := \eta(\mathcal{K} \hat{\mathbf{u}} - \mathbf{R}(\kappa) \mathbf{v}_f + \eta_\Phi \hat{\mathbf{u}} \hat{\mathbf{v}}).
\end{align*}
\]

Note that \((33a)\) is obtained by assuming that the current control \((28)\) perfectly tracks its reference. Moreover, \((33b)\) is obtained by additionally assuming that the voltage control \((27)\) perfectly tracks its reference. Finally, \((33c)\) is the reduced-order model of the closed-loop system obtained by assuming that the inner controls perfectly track their reference and that the transmission network is in its quasi steady state (see Definition \(5\)). This is the setup considered in \(25\).

### E. Control objectives in the rotating frame

To formalize the control objectives \((20a)\) to \((21)\), we define the sets

\[
S = \left\{ \hat{\mathbf{u}} \in \mathbb{R}^{2N_c} \mid \hat{\mathbf{u}} = \mathbf{R}(\theta^* k) \hat{\mathbf{v}}, \forall k \in \mathbb{N}_{[2, N_c]} \right\},
\]

\[
\mathcal{A} = \left\{ \hat{\mathbf{u}} \in \mathbb{R}^{2N_c} \mid \|\hat{\mathbf{u}}\| = v_k^*, \forall k \in \mathbb{N}_{[1, N_c]} \right\}.
\]

The intersection \(S \cap \mathcal{A}\) of the sets \(S\) (synchronization) and \(\mathcal{A}\) (voltage magnitude) is the desired steady-state behavior of the reference model \((26)\). Moreover, if the voltage \(v\) converges to its steady-state, i.e., \(x_v = x_v^*(v) = (\hat{\mathbf{u}}, 0_{2N_c})\), it follows that \(v \in S \cap \mathcal{A}\) and the voltage \(v\) meets the control objectives \((23)\) and \((24)\). Next, note that the control objectives \((20b)\), \((20a)\), and \((20c)\) in the stationary frame correspond to the steady-state maps \((32a)\), \((32b)\), and \((32c)\) in the rotating frame. Therefore, we can express all specifications stated in Section \(IV-B\) in the rotating frame as \(x \in \mathcal{X}^s(S \cap \mathcal{A})\).

### VI. Stability Analysis

In this section we use the results of Section \(III\) to analyze stability of the multi-converter AC power system \((30)\) with dispatchable virtual converter control as a reference model and cascaded inner tracking controls presented in Section \(V\).

### A. Main result

We use the following condition to establish almost global asymptotic stability of the multi-converter power system.

**Condition 4 (Stability Condition)** The set-points \(p^*_k, q^*_k, v^*_k\) and the steady-state angles \(\theta^*_{jk}\) satisfy Condition \(2\). There exists a maximal steady-state angle \(\theta^* \in [0, \pi]\) such that \(\|\hat{\mathbf{u}}\| \leq \theta^*\) holds for all \((j, k) \in \mathbb{N}_{[1, N_k]} \times \mathbb{N}_{[1, N_c]}\). Furthermore, for all \(k \in \mathbb{N}_{[1, N_c]}\), the line admittances \(\|Y_{jk}\|\), the stability
margin \( c \in \mathbb{R}_{>0} \), the set-points \( p^k, q^k, v^*_k \), and the gains \( \eta \in \mathbb{R}_{>0} \) and \( \eta_a \in \mathbb{R}_{>0} \) satisfy
\[
\sum_{j:(j,k) \in \mathcal{E}} \left| \frac{1}{v_k^*} v^j_k \cos(\theta^j_{jk}) \right| + \eta_a \leq 1 + \cos(\bar{\theta}^*_{jk}) v^2_{\min} \lambda_2(L) - c
\]
\[
\eta \leq \rho \| \bar{Y}_{\text{term}} \| (c + 5\| K - L \|)
\]
with \( v^\min_k := \min_{k \in [n_1,n_c]} v^k \) and \( v^\max_k := \max_{k \in [n_1,n_c]} v^k \) are the smallest and largest magnitude set-points, and \( \lambda_2(L) \) is the second smallest eigenvalue of the graph Laplacian \( L \).

Condition 5 is identical to the stability conditions given in \([8]\) were the converters are modeled as controllable voltage sources. Broadly speaking, the first inequality in Condition 4 ensures that the network is not to heavily loaded and the second inequality ensures that there is a sufficient time-scale separation between the dVOC reference model and the line dynamics. The reader is referred to \([8, \text{Prop. 2}]\) for an insightful interpretation of Condition 4 in terms of power set points, transmission line time constants, and network admittances.

Condition 5 Consider the control gain \( \eta \) and stability margin \( c \) that satisfy Condition 2 and the stability margin \( c_2 \) defined in Condition 4 (with \( \alpha_2 = 1, \gamma_2, \beta_2, \) and \( \beta_{21} \) defined in Propositions 7-8). The control gains \( K_{p,vk} \in \mathbb{R}_{>0} \), \( K_{i,vk} > c_{f,k} \) of the voltage PI controller satisfy
\[
1 + \max_{k \in [n_1,n_c]} \frac{K_{i,vk}}{K_{p,vk}} \min_{c \in [1,\infty]} c_{f,k} - 1 < \frac{4\eta c_2}{\| BR_1^{-1} B^T \| (1 + 4\eta^2)}
\]

Condition 6 Consider the control gains \( \eta \), \( K_{p,vk} \), and \( K_{i,vk} \), and stability margins \( c_1 \), \( c_2 \) such that Condition 3 is satisfied. Given the stability margin \( c_3 \) defined in Condition 4 (with \( \alpha_3, \gamma_3 = 0, \beta_3, \beta_{31}, \) and \( \beta_{31} \) as in Propositions 7-9), the control gains \( K_{f,jk} \in \mathbb{R}_{>0} \), \( K_{f,jk} > c_{f,k} \) of the current PI controller satisfy
\[
1 + \max_{k \in [n_1,n_c]} \frac{K_{f,jk}}{K_{p,vk}} \min_{c \in [1,\infty]} c_{f,k} - 1 < \frac{4c_3 \beta_{34}}{4\eta_2 \beta_{34}}
\]
where \( \beta_{34} = \max_{k \in [n_1,n_c]} \frac{1}{K_{p,vk}} \left( 1 + \frac{K_{i,vk}}{K_{p,vk}} \right) \), \( \beta_{41} = \max_{k \in [n_1,n_c]} \frac{K_{p,vk}}{c_{f,k}} \), \( \beta_{42} = \max_{k \in [n_1,n_c]} \frac{1}{c_{f,k}} \), \( \beta_{43} = \sup_{i \in [1,\infty]} \left| \frac{Y'_f}{\max_{k \in [n_1,n_c]} K_{i,vk}} \right| \), \( \gamma_4 = \left| Y'_f C_j^{-1} \right| \), and \( Y'_f := Y_f - K_{p,v} \).

Proposition 2 (Stabilizing control gains) Consider set-points \( p^k, q^k, v^*_k \), and steady-state angles \( \theta^*\) that satisfy Condition 3 and suppose that there exists a control gain \( \eta_a \) and stability margin \( c \) such that the first inequality of Condition 2 holds. Then, there exists control gains \( \eta \), \( K_{p,vk}, K_{i,vk}, K_{f,jk}, \) and \( K_{i,f} \) such that Conditions 5-6 hold.

The proof is provided in the appendix. An interpretation of the stability conditions is shown in Figure 4. The parameter \( \rho \) is the time constant of the transmission lines that cannot be influenced through control. Moreover, the synchronization gain \( \eta \) is the time constant of the reference model, and the control gains \( K_{p,ij} \) and \( K_{p,v} \) dominantly influence the convergence rate of the filter current and the terminal voltage closed loop dynamics. Hence, a sufficient time-scale separation must be enforced through the control gains. Broadly speaking, the second inequality of Condition 4 requires the reference model to be slow enough compared to the line dynamics, Condition 5 implies that the controlled voltages settle sufficiently fast compared to the transmission line dynamics, and Condition 6 requires the controlled filter current to converge faster than the terminal voltages. Next, the conditions 4 to 6 reflect the nested structure of the system. In particular, given the network parameters, control gains and set-points for the reference model can be computed that satisfy Condition 2. Furthermore, given the network parameters, the control gains and set-points of the reference model, and the filter capacitance, stabilizing voltage control gains can be found using Condition 3. Finally, for fixed voltage control gains, the current control gains can be found using Condition 6.

We are now ready to state the main result of the section.

Theorem 6 (Almost global stability of \( \mathcal{X}^{\text{eq}}(S \cap A) \)) Consider set-points \( p^k, q^k, v^*_k \), steady-state angles \( \theta^* \), a stability margin \( c_1 \), \( c_2 \), and \( c_3 \) and control gains \( \eta \), \( K_{p,vk}, K_{i,vk}, K_{f,jk}, \) \( K_{p,ij} \) and \( K_{p,v} \) such that Conditions 4-6 hold. Then, the dynamics \( (30) \) are almost globally asymptotically stable with respect to \( \mathcal{X}^{\text{eq}}(S \cap A) \), and the origin \( 0_n \) is an exponentially unstable equilibrium.

The proof of Theorem 6 is given in Section VI-G and is obtained by applying Theorem 3 and Theorem 4. Before presenting the proof, we present the individual Lyapunov functions for the reduced-order system \( \hat{v} = f_0(\hat{v}) \), the transmission lines, terminal voltages, and filter currents, and derive bounds as required by Assumption 2.2.

B. Lyapunov function for the reduced-order system

Given the voltage set-points \( v^*_k \) and relative steady-state angles \( \theta^* \) for all \( k \in [n_1,n_c] \), we define the matrix \( S := [v^*_k R(\theta^*_{11})^T \ldots v^*_N R(\theta^*_{1N})^T]^T \) whose null space encodes \( (23) \), and the projector \( P_S := I_{2N} - \frac{1}{\| S \|} SS^T \) onto the nullspace of \( S \). Then, the Lyapunov function candidate \( V_0 : \mathbb{R}^{2Nc} \rightarrow \mathbb{R}_{\geq 0} \) for the reduced-order dynamics \( \dot{\bar{v}} = f_0(\bar{v}) \) is given by
\[
V_0(\bar{v}) := \frac{1}{2} \bar{v}^T P_S \bar{v} + \frac{1}{2} \eta \eta_a \sum_{k=1}^{N_c} \left( \frac{v^2_{\min} - \| \bar{v}^*_k \|}{v^*_k} \right)^2
\]
where $\eta_0 \in \mathbb{R}_{>0}$ is the voltage controller gain and, given $c \in \mathbb{R}_{>0}$, the constant $\alpha_1$ is given by

$$\alpha_1 := \frac{c}{5\eta_0 |\mathcal{K} - \mathcal{L}|^2}. \quad (35)$$

Furthermore, the constants $\eta$ and $c$ cannot be chosen arbitrarily. They must be chosen such that Condition [4] is always satisfied. Moreover, we define the comparison function $\psi_v : \mathbb{R}^{2N_\mathcal{E}} \rightarrow \mathbb{R}_{\geq 0}$ as

$$\psi_v(\hat{v}) := \eta(\|\mathcal{K} - \mathcal{L}\| \|\hat{v}\| S + \eta_0 \|\Phi(\hat{v})\|). \quad (36)$$

**Proposition 3** [8] Prop. 3) (Lyapunov function for the reference model) Consider $V_v(\hat{v})$ defined in (34) and set-points $p_v^r$, $q_v^r$, $\nu_v$, steady-state angles $\theta_{j,k}^v$, $\alpha$, $c$ and $\eta$ such that Condition [4] holds. For any $\eta \in \mathbb{R}_{>0}$, there exists $\chi_{V}^v, \chi_{V}^e \in \mathcal{K}_\infty$ such that

$$\chi_{V}^v(\|\hat{v}\| S_{\mathcal{K}\mathcal{A}}) \leq V_v(\hat{v}) \leq \chi_{V}^e(\|\hat{v}\| S_{\mathcal{K}\mathcal{A}}) \quad (37)$$

holds for all $\hat{v} \in \mathbb{R}^{2N_\mathcal{E}}$. Moreover, for all $\hat{v} \in \mathbb{R}^{2N_\mathcal{E}}$ the derivative of $V_v$ along the trajectories of the reduced-order dynamics $\frac{d}{dt} \hat{v} = f_{\hat{v}}^c(\hat{v})$ defined in (35) satisfies

$$\frac{dV_v}{dt} f_{\hat{v}}^c(\hat{v}) \leq -\alpha_1 \psi_v(\|\hat{v}\| S_{\mathcal{K}\mathcal{A}, \mathcal{S}_{\mathcal{N}, \mathcal{S}}}). \quad (38)$$

**C. Lyapunov function for the transmission lines**

The Lyapunov function candidate $V_I : \mathbb{R}^{2N_i} \rightarrow \mathbb{R}_{\geq 0}$ for the transmission line dynamics is defined using the error coordinates $y_i := i_t - i^*_{t,I}$ as

$$V_I(y_i) := \frac{\rho}{2} y_i^T (B_i^T B + L_T B_n \delta_{t,L}^T L_T) y_i, \quad (39)$$

where $B_n := B_n \otimes I_2 \in \mathbb{R}^{N_i \times N_0}$, and the columns of the matrix $B_n$ span the nullspace of $B$. Because the graph $\mathcal{G}$ is connected it follows from the rank-nullity theorem that $N_{t_0} := N_i - N_c + 1$. Next, consider the functions $\psi_i(y_i) = \|B_i y_i\|$, $\psi_i(y_i) = \|B_n L_T y_i\|$ that exploit the additional degree of freedom given by $\psi_i(y_i)$ in Assumption [2] to reduce conservatism in the stability bounds.

**Proposition 4** [8] Prop. 4) (Lyapunov function for the transmission lines) Consider $V_I(y_i)$ defined in (39), then there exists $\chi_{V}^{i,c}, \chi_{V}^{i,e} \in \mathcal{K}_\infty$ such that

$$\chi_{V}^{i,c}(\|y_i\|) \leq V_I(y_i) \leq \chi_{V}^{i,e}(\|y_i\|) \quad (40)$$

holds for all $y_i \in \mathbb{R}^{2N_i}$. Moreover, for all $\hat{v} \in \mathbb{R}^{2N_e}$ and $i_t \in \mathbb{R}^{2N_i}$ it holds for the reduced order dynamics (35b)

$$\frac{dV_I}{dt} f_{\hat{v}}^c(\hat{v}, y_i + i^*_{t,I}) \leq -\psi_i(y_i)^2 - \psi_i(y_i)^2. \quad (40)$$

**D. Lyapunov function for the terminal voltage**

Using the error coordinates $y_v := x_v - x^*_v$, the Lyapunov function candidate $V_v : \mathbb{R}^{2N_v} \rightarrow \mathbb{R}_{\geq 0}$ for the terminal voltage of the converters is defined as

$$V_v(y_v) := \frac{1}{2} y_v^T \left[ K_v^{-1} C_f \begin{array}{cc} K_v^{-1} C_f & K_v^{-1} C_f \\ K_v^{-1} C_f & K_v^{-1} C_f \end{array} K_v^{-1} C_f \right] y_v. \quad (40)$$

Moreover, consider the function $\psi_v(y_v) := \|y_v\|$.

**Proposition 5** (Lyapunov function for the terminal voltage) Consider Lyapunov function the $V_v(y_v)$ defined in (40). There exist $\chi_{1V}, \chi_{2V} \in \mathcal{K}_\infty$ such that

$$\chi_{1V}(\|y_v\|) \leq V_v(y_v) \leq \chi_{2V}(\|y_v\|) \quad (41)$$

holds for all $y_v \in \mathbb{R}^{2N_v}$. Moreover, for all $\hat{v} \in \mathbb{R}^{2N_v}$, $y_t \in \mathbb{R}^{2N_i}$ and $y_v \in \mathbb{R}^{2N_v}$ it holds for the reduced order dynamics (35a)

$$\frac{\partial V_v}{\partial y_v} f_{\hat{v}}^c(\hat{v}, y_v + x^*_v) \leq -\alpha_3 \psi_v(y_v)^2, \quad (42)$$

where $\alpha_3 := 1 - \max_{k \in N_{1, \mathcal{K}_{\mathcal{S}}}} \left\{ \frac{\delta_{I,k}^v}{K_v^{-1}} \right\}$. The proof is provided in the Appendix.

**E. Lyapunov function for the converter filter current**

Using the error coordinates $y_f := x_f - x^*_f$, the Lyapunov function candidate $V_f : \mathbb{R}^{2N_v} \rightarrow \mathbb{R}_{\geq 0}$ is defined as

$$V_f(y_f) := \frac{1}{2} y_f^T \left[ K_v^{-1} L_2 \begin{array}{cc} K_v^{-1} L_2 & K_v^{-1} L_2 \\ K_v^{-1} L_2 & K_v^{-1} L_2 \end{array} K_v^{-1} L_2 \right] y_f. \quad (43)$$

Moreover, consider the function $\psi_f(y_f) := \|y_f\|$.

**Proposition 6** (Lyapunov function for the filter current) Consider Lyapunov function $V_f(y_f)$ defined in (43), then there exist $\chi_{1F}, \chi_{2F} \in \mathcal{K}_\infty$ such that

$$\chi_{1F}(\|y_f\|) \leq V_f(y_f) \leq \chi_{2F}(\|y_f\|) \quad (44)$$

holds for all $y_f \in \mathbb{R}^{2N_v}$. Moreover, for all $\hat{v} \in \mathbb{R}^{2N_v}$, $i_t \in \mathbb{R}^{2N_i}$ and $x_v \in \mathbb{R}^{2N_v}$ and $y_f \in \mathbb{R}^{2N_v}$ it holds for the reduced order dynamics (35a)

$$\frac{\partial V_f}{\partial y_f} f_{\hat{v}}^c(\hat{v}, y_f + i^*_{t,C}) \leq -\alpha_4 \psi_f(y_f)^2, \quad (45)$$

where $\alpha_4 := 1 - \max_{k \in N_{1, \mathcal{K}_{\mathcal{S}}}} \left\{ \frac{\delta_{I,k}^v}{K_v^{-1}} \right\}$. The proof is given in the Appendix.

**F. Interactions between the time scales**

In addition to establishing that the functions $V_v$, $V_I$, $V_f$, and $V_f$ are Lyapunov functions when considering the reduced-order system and error dynamics of the faster dynamics in isolation, we require bounds on the interactions between the different time-scales according to Assumption [2]. These bounds are given in Propositions [4][12] in the Appendix.

**G. Proof of the main result**

To prove Theorem [6] we combine the Lyapunov function candidates defined in Proposition [3] to [6] as follows

$$\nu := \mu_1 \tilde{V}_v(\hat{v}) + \mu_2 \tilde{V}_v(y_i) + \mu_3 \tilde{V}_v(y_v) + \mu_4 \tilde{V}_f(y_f),$$

where $\mu_i$ is defined as in Theorem [3]. Moreover, to show that the region of attraction of the origin has measure zero, we define the Lyapunov-like function

$$\nu_\delta := V_\delta(v) + \mu_2 \tilde{V}_v(y_i) + \mu_3 \tilde{V}_v(y_v) + \mu_4 \tilde{V}_f(y_f),$$
where $V_{\delta,\ell}(\hat{v}_{3}) := \hat{v}_{3}^T(P_{S} - 2\eta_{s}\alpha_{1}I_{2Nc})\hat{v}_{3}$. Note that the functions $f_{s}^{e}, f_{c}^{e}$, and $f_{s}^{o}$ are linear and the corresponding Lyapunov function candidates $V_{i}(y_{i}), V_{c}(y_{c}), V_{f}(y_{f})$ are quadratic. We now use Theorem 3 and Theorem 4 to prove Theorem 6.

**Proof of Theorem 6** First, observe that the stability conditions ensure that Condition 3 holds and it follows from Proposition 1 that $M$ is positive definite.

Next, we show that the region of attraction of the equilibrium $x^* = 0$ has measure zero. To this end, note that for all $\hat{v}_{3} \in S \setminus \{0_{2Nc}\}$ it holds that $V_{\delta,\ell}(\hat{v}_{3}) < V_{\delta,\ell}(0_{2Nc})$. Moreover, replacing $\psi_{\ell}(\hat{v})$ with $\psi_{\ell}^{0}(\hat{v}_{3}) := \eta_{c}(||K - L||||\hat{v}_{3}||_{S} + \eta_{s}||\hat{v}_{3}||)$ and using the bounds given in Proposition 3 Proposition 8 Proposition 9 and Proposition 12 it can be verified that Assumption 2 holds for the linearized dynamics. The linearized reference voltage dynamics are given by $\frac{d\hat{v}_{3}}{dt} := \eta_{c}(K\hat{v}_{3} - R(\omega)I_{b} + \eta_{s}\hat{v}_{3})$. Next, we use $\psi_{\ell} = 0$, $\psi_{c} = 0$, $\hat{y}_{f} = 0$ and note that $||\hat{v}_{3}|| \leq \eta_{c}^{-1}\psi_{\ell}^{0}(\hat{v}_{3})$, $||\hat{y}_{f}|| \leq ||y_{o}|| + ||y_{n}||$, $\psi_{c}(y_{c}) = ||y_{c}||$, and $\psi_{f}(y_{f}) = ||y_{f}||$. Therefore, the conditions of Theorem 4 are satisfied and the region of attraction of the origin under 30 has measure zero.

Next, we show that the conditions of Theorem 3 are satisfied for the dynamics 30, the Lyapunov functions $V_{e}(\hat{v}), V_{r}(y_{r}), V_{f}(y_{f})$, $C_{1} = S \cap A$, and $U_{1} = \{0_{2Nc}\}$. To this end, we note that the Propositions 8 and Proposition 12 establish that Assumption 2 holds and appropriate $\mathcal{K}_{\infty}$ functions bounding $V_{\ell}(\hat{v}), V_{r}(y_{r}), V_{f}(y_{f})$ from above and below exist. Next, note that because $\psi_{\ell}(\hat{v})$ is positive definite and radially unbounded with respect to $S \cap A \cup 0_{2Nc}$, and $S \cap A \cup 0_{2Nc}$ is compact, the same steps as in [28, p. 98] can be used to show that there exists a $\mathcal{K}$-function $\sigma_{\ell}(||\hat{v}||_{S \cap A \cup 0_{2Nc}}) \leq \psi_{\ell}(t)$. We note that $||y_{f}|| \leq ||y_{o}|| + ||y_{n}||$, $\psi_{c}(y_{c}) = ||y_{c}||$, $\psi_{f}(y_{f}) = ||y_{f}||$, and the region of attraction $\mathcal{Z}(0_{\omega}, f_{s})$ of the origin has measure zero. Therefore, the conditions of Theorem 3 are satisfied and the theorem follows.

**VII. ILLUSTRATIVE EXAMPLE**

As an illustrative example, we use a high-fidelity PLECS model of a single-phase converter-based microgrid with $\omega_{0} = 60$ Hz and a resistive load shown in Figure 5 that has been validated using the hardware testbed described in [26]. In contrast to the theoretical analysis, the switching stage of the voltage source converters in the PLECS model are not averaged and the DC voltage is not constant. Specifically, the simulation uses the model shown in Figure 6 which consists of a two-level DC/AC voltage source converter, an RLC output filter, a DC-link capacitor, and a DC/DC boost converter used to stabilize the DC voltage. Moreover, the converter switches are driven via pulse width modulation (with 30 kHz base frequency). Finally, the controllers are discretized at a frequency of 15 kHz, and the electromagnetic dynamics are simulated using a variable step ODE solver with a maximum step size of 1.66 ns.

**VIII. CONCLUSION AND OUTLOOK**

In this work, we developed a Lyapunov function framework for stability analysis of nested nonlinear dynamical systems on multiple time scales and obtained conditions for (almost) global asymptotic stability with respect to a set. Our approach explicitly considers multiple time scales, convergence rates instead of scalar time constants, and reduces conservatism.
We applied this technical contribution to a multiple-converter power system model that includes transmission lines dynamics, converter dynamics, cascaded current and voltage control loops, and dVOC as a reference model. Finally, we obtain explicit stability conditions on the control gains that enforce the well-known time scale separation between the different dynamics, i.e., the dVOC reference model has to be sufficiently slow relative to the line dynamics and the controlled converter voltage and current have to be sufficiently fast compared to the line dynamics. Moreover, the converter current has to be faster than the terminal voltage. Finally, we used a high-fidelity simulation with detailed converter models (i.e., full switching and DC side dynamics) to validate the performance of the proposed control strategy.

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APPENDIX

Lemma 1 (Lyapunov decrease) For all \( i \in \mathbb{N}_{[1,N]} \) let \( \alpha^{(c)}_i := \epsilon_i^{-1} \alpha_i \), for all \( i \in \mathbb{N}_{[1,N-1]} \) let \( \beta^{(c)}_{i,i+1} := \epsilon_i^{-1} \beta_{i,i+1} \), for all \( i \in \mathbb{N}_{[2,N]} \) let \( \beta^{(c)}_{i,1} := \sum_{k=1}^{i-1} \epsilon_k^{-1} \beta_{i,1,k} \) and \( \gamma^{(c)}_i := \epsilon_{i-1}^{-1} b_{i,i-1} \), and for all \( i \in \mathbb{N}_{[3,N]} \) and all \( j \in \mathbb{N}_{[2,i-1]} \) let \( \rho^{(c)}_{i,j} := \sum_{k=j-1}^{i-1} \epsilon_k^{-1} b_{i,j,k} \). Consider the function \( \nu_c = \mu_1^{(c)} \gamma(x_1) + \sum_{i=2}^{N} \mu_i^{(c)} \psi_i(y_i) \) with \( \mu_i^{(c)} = \prod_{j=1}^{i-1} \beta^{(c)}_{j,j+1} \) for all \( i \in \mathbb{N}_{[1,N]} \) and \( \mu_1^{(c)} = 1 \). Under Assumption 2, the derivative of \( \nu_c \) along the trajectories of the (13) is bounded by

\[
\frac{d}{dt} \nu_c \leq -H' \left[ \psi_1 \psi_2 \cdots \psi_N \right]^T H \left[ \psi_1 \psi_2 \cdots \psi_N \right]
\]

where \( H' := \text{diag}(\{\mu_i^{(c)} \alpha_i^{(c)}\}_{i=1}^{N}) \), and \( H \) is defined recursively for all \( i \in \mathbb{N}_{[2,N]} \) starting from \( H_1 = \alpha_1^{(c)} \) by

\[
H_i = \left[ \frac{H_{i-1}}{\star} \right] \frac{\beta^{(c)}_i (\alpha^*_1 - \gamma_i^{(c)}) \mu_i^{(c)}}{\gamma_i^{(c)} \mu_i^{(c)}},
\]

and \( \beta^{(c)}_i := (\cdots, 1, \beta^{(c)}_{i-3}, \cdots, \beta^{(c)}_{i-2}, \beta^{(c)}_{i-1}) \).

Proof: The time derivative of (7) is given by

\[
\frac{d}{dt} \nu_c := \frac{\mu_1^{(c)}}{\epsilon_1} \frac{\partial \gamma(x_1)}{\partial x_1} \frac{\partial x_1}{\partial t} + \sum_{i=2}^{N-1} \frac{\mu_i^{(c)}}{\epsilon_i} \frac{\partial \psi_i(y_i)}{\partial y_i} \frac{\partial y_i}{\partial t} + \frac{\mu_N^{(c)}}{\epsilon_N} \frac{\partial \psi_N}{\partial y_N} \frac{\partial y_N}{\partial t}
\]
\[-\sum_{i=2}^{N} \sum_{k=1}^{i-1} \mu_i^{(c)} \frac{\partial V_i}{\partial y_i} \frac{\partial^2 y_i}{\partial x_k^2} f_k.\]  

Next, we add and subtract \(\mu_i^{(c)} \frac{\partial V_i}{\partial y_i} f_i^2\) for all \(i \in \mathbb{N}_{[1,N]}\).

Using Assumption 2 and Definition 3, one obtains

\[\mu_i^{(c)} \frac{\partial V_i}{\partial y_i} f_i^2 \leq -\mu_i^{(c)} \alpha_i^{(c)} \psi_i(x_i)^2 - \mu_i^{(c)} \alpha_i^{(c)} \psi_i'(x_i)^2,\]

and for all \(i \in \mathbb{N}_{[2,N]}\) one obtains

\[\mu_i^{(c)} \frac{\partial V_i}{\partial y_i} f_i^2 \leq -\mu_i^{(c)} \alpha_i^{(c)} \psi_i(y_i)^2 - \mu_i^{(c)} \alpha_i^{(c)} \psi_i'(y_i)^2,\]

Next, using Assumption 2 and Definition 3 for all \(i \in \mathbb{N}_{[2,N]}\) it holds that

\[-\sum_{k=1}^{i-1} \mu_i^{(c)} \frac{\partial V_i}{\partial y_i} \frac{\partial^2 y_i}{\partial x_k^2} f_k \leq \sum_{k=1}^{i-1} \mu_i^{(c)} \frac{\partial^2 y_i}{\partial x_k^2}, \]

where \(\psi_i = \psi_i(x_i)\) and \(\psi_i = \psi_i(y_i)\) for all \(i \in \mathbb{N}_{[2,N]}\). Using these bounds, we can bound \(\frac{d}{dt} \nu_i\) by two quadratic forms

\[\frac{d}{dt} \nu_i \leq - \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_N \\ \psi_1' & \psi_2' & \cdots & \psi_N' \\ \end{bmatrix}^T H^{(\mu)} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \end{bmatrix} - \begin{bmatrix} \psi_1' \\ \psi_2' \\ \vdots \\ \psi_N' \\ \end{bmatrix}^T H' \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \end{bmatrix},\]

where \(H := \text{diag}(\{\mu_i^{(c)} \alpha_i^{(c)}\}_{i=1}^{N})\) and \(H\) is defined recursively for all \(i \in \mathbb{N}_{[2,N]}\) starting from \(H^{(\mu)} = \alpha_i^{(c)}\).

The lemma follows by noting that \(\frac{\mu_i^{(c)}}{\mu_j^{(c)}} \beta_i^{(c)} = \beta_i^{(c)}\).

**Lemma 2 (Positive definiteness)** Consider a symmetric matrix \(A = \{a_{i,j}\}_{N \times N}\) with \(a_{i,j} \geq 0\) for all \(i \in \mathbb{N}_{[1,N-1]}\) and \(a_{i,j} = a_{j,i} \leq 0\) for all \(i \in \mathbb{N}_{[1,N]}\), \(j \in \mathbb{N}_{[1,N]}\), and \(i \neq j\). For all \(k \in \mathbb{N}_{[1,N]}\), let \(A_k = \{a_{i,j}\}_{k \times k}\) denote the \(k\)-th leading principal minor of \(A\). \(A\) is positive definite if and only if for all \(k \in \mathbb{N}_{[2,N]}\), \(A_k\) is invertible and satisfies

\[a_{k,k} > \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{k-1,k-1} \end{bmatrix}^T A_k^{-1} \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{k-1,k} \end{bmatrix} \geq 0,\]

where equality holds if \(\begin{bmatrix} a_{1,1} \\ \vdots \\ a_{k-1,k} \end{bmatrix} = \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{k-1,k} \end{bmatrix} = 0_{k-1,1}.

**Proof:** Using the Schur complement, it can be verified that the \(k\)-th minor of \(A\) is positive definite if and only if

\[a_{k,k} - \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{k-1,k} \end{bmatrix}^T A_k^{-1} \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{k-1,k} \end{bmatrix} > 0\]

and \(A_{k-1}\) is positive definite. In particular, \(A_{N-1}\) is positive definite if and only if (47) holds for \(k = N - 1\) and \(A_{N-1}\) is positive definite. The Lemma follows by induction over \(i \in \mathbb{N}_{[0,N-2]}\).

**Proof of Theorem 5** Consider the Lyapunov function candidate \(\nu_i = \mu_i^{(c)} V_i(x_i) + \sum_{k=1}^{N-1} \mu_k^{(c)} V_k(y_k)\) with \(\mu_i^{(c)} = \prod_{j=1}^{i-1} \frac{\beta_j^{(c)}}{\beta_j^{(c)} + 1}\) for all \(i \in \mathbb{N}_{[1,N]}\), and \(\mu_1^{(c)} = 1\). Using Lemma 1 we obtain

\[\frac{d}{dt} \nu_i \leq - \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_N \\ \psi_1' & \psi_2' & \cdots & \psi_N' \\ \end{bmatrix}^T H \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \end{bmatrix} - \begin{bmatrix} \psi_1' \\ \psi_2' \\ \vdots \\ \psi_N' \\ \end{bmatrix}^T H' \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \end{bmatrix},\]

Next, using Lemma 2 it follows that \(H\) is positive definite if

\[\gamma_i^{(c)} + \mu_i^{(c)} \beta_i^{(c)} H_i^{-1} \beta_i^{(c)} < 0,\]

holds for all \(i \in \mathbb{N}_{[2,N]}\). Because the left hand side of (48) only depends on \(\epsilon_j\) for all \(j < i\) it directly follows that, for every \(i \in \mathbb{N}_{[2,N]}\) there exists \(\epsilon_i \in \mathbb{R}_++\) with \(\epsilon_i < \epsilon_{i-1}\) such that (48) holds. Using the same steps as in the proof of Theorem 3 there exists a function \(\chi_i \in \mathcal{K}\) such that \(\frac{d}{dt} \nu_i \leq -\chi_i(\|x\|, x'(C_1)) - \chi_i(\|x\|, x'(\mathcal{C}_1))\). Let \(\mathcal{C} = \chi_i(C_1)\), and \(\mathcal{U} = 0\), it follows from Theorem 1 that the system is almost globally asymptotically stable with respect to \(\mathcal{C} = \chi_i(C_1)\). Since \(\mathcal{U} = 0\), (46) holds for all \(x \in \mathbb{R}^n\), and it follows that the dynamics are globally asymptotically stable.

**Proof of Proposition 2** If the first inequality of Condition 4 holds there exists \(\eta\) such that the second inequality of the condition holds. Next, note that the right-hand side of the inequality in Condition 5 is positive and independent of the gains \(K_{p,v_k}\) and \(K_{i,v_k}\). Moreover, letting \(\xi_p = K_{i,v_k}/K_{p,v_k} \in \mathbb{R}_+\) and \(\xi_i = K_{i,v_k}/c_{i,k} \in \mathbb{R}_+\) for all \(k \in \mathbb{N}_{[1,N]}\), the left-hand side of the inequality in Condition 5 becomes \(1 + \xi_p^{-1} \in \mathbb{R}_+\) and can be made arbitrarily small by choosing \(\xi_p \in \mathbb{R}_+\) small enough and \(\xi_i \in \mathbb{R}_+\) large enough. Using the same arguments it can be verified that there always exists \(K_{p,f_k} \in \mathbb{R}_+\) and \(K_{i,f_k} \in \mathbb{R}_+\) such that Condition 6 holds.

**Proof of Proposition 3** Condition 5 implies that \(1 - \max_{k \in [1,N]} \{c_{j,k} K_{p,v_k}\} > 0\) and it follows that \(V_k(y_k)\) is positive definite. Thus, there exist \(\chi_1 \in \mathcal{K}\) and \(\chi_2 \in \mathcal{K}\) such that (41) holds. Moreover, we obtain

\[\frac{\partial V_k}{\partial y_k} f_k^2(\bar{y}_k, y_k) + x_k^2 = -\gamma_k \begin{bmatrix} I_{2N} - K_{i,v_k} C_f \\ 0_{2N \times 2N} \\ I_{2N} \end{bmatrix} \bar{y}_k\]

and the proposition follows immediately.

**Proof of Proposition 6** Condition 6 implies that \(1 - \sum_{k=1}^{N-1} C_{j,k} K_{i,v_k} = 0\) holds. Thus, there exists \(\chi_1 \in \mathcal{K}\) and \(\chi_2 \in \mathcal{K}\) such that (41) holds. Moreover, we obtain

\[\frac{\partial V_k}{\partial y_k} f_k^2(\bar{y}_k, y_k) + x_k^2 = -\gamma_k \begin{bmatrix} I_{2N} - K_{i,v_k} C_f \\ 0_{2N \times 2N} \\ I_{2N} \end{bmatrix} \bar{y}_k\]

and the proposition follows immediately.
max_{k \in \mathbb{N}_1, \mathbb{N}_c} \left\{ \frac{\ell_f + K_{v,k}}{K_{v,k}} \right\} > 0 \text{ and it follows that } V_f(y) \text{ is positive definite. Thus, there exist } \chi_1^{\nu_1} \in \mathcal{H}_\infty \text{ and } \chi_2^{\nu_2} \in \mathcal{H}_\infty \text{ such that (13) hold. Moreover, we obtain}
\[
\frac{\partial V_f}{\partial y_f} f_f^\alpha(\hat{v}, y_i + i_t^k, y_v + x_v^k, y_f + x_f^k) = -y_f \left[ \begin{array}{c|c}
(2N - K_{v,k}^{-1} I_f) & 0_{2N \times 2N} \\
\hline
0_{2N \times 2N} & I_{2N}
\end{array} \right] y_f
\]
and the proposition follows immediately.

Proposition 7 \[\overset{[8]}{\text{Prop. 5}}\] Let \( \beta_{12} = \|K - \mathcal{L}\|^{-1} \). For all \( \hat{v} \in \mathbb{R}^{2N_v} \) and \( y_i \in \mathbb{R}^{2N_i} \) it holds that
\[
\frac{\partial V_i}{\partial y_i}(f_i - f_i^*) \leq \beta_{12} \psi_i(\hat{v}) \psi_i(y_i).
\]

Proposition 8 \[\overset{[8]}{\text{Prop. 6}}\] Let \( \beta_{21} = \rho \|Y_{\text{net}}\| \) and \( \gamma_2 = \eta \rho \|Y_{\text{net}}\| \). Then for all \( \hat{v} \in \mathbb{R}^{2N_v} \) and \( y_i \in \mathbb{R}^{2N_i} \) it holds that
\[
-\frac{\partial V_i}{\partial y_i}(f_i^y) \leq \beta_{21} \psi_i(\hat{v}) \psi_i(y_i) + \gamma_2 \psi_i(y_i)^2.
\]

Proposition 9 Let \( \beta_{23} := \|BR_T^{-1}B^T\| \). For all \( y_i \in \mathbb{R}^{2N_i} \) and \( y_v \in \mathbb{R}^{2N_v} \) it holds that
\[
\frac{\partial V_i}{\partial y_v}(f_v - f_v^*) \leq \beta_{23} \psi_i(\hat{v}) \psi_v(y_v).
\]

Proof: Since \( f_v \) is separable and linear in both arguments, it holds that
\[
\frac{\partial V_i}{\partial y_v}(f_v(\hat{v}, y_i + i_t^k, y_v + x_v^k) - f_v^y(\hat{v}, y_i + i_t^k)) = \frac{\partial V_i}{\partial y_v} \left[ L_T^{-1} B^T y_v = \rho_y B^T B + y_i L_T B_v B_n L_T^{-1} B^T y_v \right]
\]
and the proposition follows immediately.

Proposition 10 Let \( \beta_{31} := \max_{k \in \mathbb{N}_1, \mathbb{N}_c} \frac{\ell_f + K_{v,k}}{K_{v,k}}(1 + k \in \mathbb{N}_1, \mathbb{N}_c) \) and \( \beta_{32} := \eta \beta_{31} \). Then for all \( \hat{v} \in \mathbb{R}^{2N_v}, \ y_i \in \mathbb{R}^{2N_i} \) and \( y_v \in \mathbb{R}^{2N_v} \) it holds that
\[
-\frac{\partial V_i}{\partial y_v}(f_v - f_v^*) \leq \beta_{31} \psi_v(\hat{v}) \psi_i(y_i) + \beta_{32} \psi_i(\hat{v}) \psi_v(\hat{v}).
\]

Proof: Since \( f_v \) is separable in its arguments and linear in \( i_t \) it holds that \( f_v(\hat{v}, y_i + i_t^k) = f_v(\hat{v}, y_i) + f_v^x(\hat{v}) \) and we obtain
\[
\frac{\partial V_i}{\partial y_v}(f_v - f_v^*) = -y_v \left[ \begin{array}{c|c}
K_{v,k}^{-1} C_{f,v} & K_{v,k}^{-1} C_{f,v} \end{array} \right] \left( f_v^x(\hat{v}) - \eta \mathcal{R}(\hat{v}) B y_i \right)
\]
Next, note that \( \|\mathcal{R}(\hat{v}) B y_i\| = \|B y_i\| \). Moreover, using \( \|\mathcal{L}(\hat{v}) P_S y_i\| \leq \|\mathcal{L}(\hat{v})\| \|y_i\| \) it holds that \( \|f_v^x(\hat{v})\| \leq \psi_v(\hat{v}) \) and the proposition directly follows.

Proposition 11 Let \( \beta_{34} := \max_{k \in \mathbb{N}_1, \mathbb{N}_c} \frac{1}{K_{v,k}} \). For all \( y_i \in \mathbb{R}^{2N_i} \) and \( y_v \in \mathbb{R}^{2N_v} \) it holds that
\[
\frac{\partial V_i}{\partial y_v}(f_v - f_v^*) \leq \beta_{34} \psi_i(y_i) \psi_v(y_v).
\]

Proof: Since \( f_v \) is linear and separable in all its arguments it holds that \( f_v(\hat{v}, y_i + i_t^k, y_v + x_v^k, y_f + x_f^k) = f_v(\hat{v}, y_i, y_v) = f_v(\hat{v}, y_i) + f_v^x(\hat{v}) \) and the proposition follows immediately.

Proposition 12 Let \( \beta_{41} := \beta_{34} \mathcal{L}(\hat{v}) P_S \), \( \beta_{42} := \beta_{43} \mathcal{L}(\hat{v}) P_S \), \( \beta_{43} := \max_{k \in \mathbb{N}_1, \mathbb{N}_c} (1 + k \in \mathbb{N}_1, \mathbb{N}_c) \), \( \beta_{44} := \max_{k \in \mathbb{N}_1, \mathbb{N}_c} (1 + k \in \mathbb{N}_1, \mathbb{N}_c) \), \( \beta_{45} := \|Y_f^T\| \max_{k \in \mathbb{N}_1, \mathbb{N}_c} (1 + k \in \mathbb{N}_1, \mathbb{N}_c) \), \( \beta_{46} := Y_f^T \mathcal{L}(\hat{v}) P_S \), \( \gamma_2 = \beta_{43} \mathcal{L}(\hat{v}) P_S \), \( \gamma_4 = \beta_{44} \mathcal{L}(\hat{v}) P_S \), and \( \gamma_5 := \gamma_4 \mathcal{L}(\hat{v}) P_S \). Then, for all \( \hat{v} \in \mathbb{R}^{2N_v} \) and \( y_i \in \mathbb{R}^{2N_i}, y_v \in \mathbb{R}^{2N_v} \) and \( y_f \in \mathbb{R}^{2N_f} \) it holds that
\[
-\frac{\partial V_i}{\partial y_f}(f_v - f_v^*) \leq \beta_{41} \psi_i(\hat{v}) \psi_f(y_f) + \beta_{42} \psi_i(\hat{v}) \psi_f(y_f) + \beta_{43} \psi_i(\hat{v}) \psi_f(y_f) + \gamma_2 \psi_i(\hat{v}) \psi_f(y_f) + \gamma_4 \psi_i(\hat{v}) \psi_f(y_f).
\]

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