A Survey of Qualitative Spatial and Temporal Calculi: Algebraic and Computational Properties

FRANK DYLLA and JAE HEE LEE, University of Bremen
TILL MOSSAKOWSKI, University of Magdeburg
THOMAS SCHNEIDER, ANDRÉ VAN DELDEN, and JASPER VAN DE VEN, University of Bremen
DIEDRICH WOLTER, University of Bamberg

Qualitative spatial and temporal reasoning (QSTR) is concerned with symbolic knowledge representation, typically over infinite domains. The motivations for employing QSTR techniques include exploiting computational properties that allow efficient reasoning to capture human cognitive concepts in a computational framework. The notion of a qualitative calculus is one of the most prominent QSTR formalisms. This article presents the first overview of all qualitative calculi developed to date and their computational properties, together with generalized definitions of the fundamental concepts and methods that now encompass all existing calculi. Moreover, we provide a classification of calculi according to their algebraic properties.

CCS Concepts: • General and reference → Surveys and overviews; • Computing methodologies → Symbolic calculus algorithms; Temporal reasoning; Spatial and physical reasoning;

Additional Key Words and Phrases: Qualitative reasoning, knowledge representation, relation algebra

ACM Reference Format:
Frank Dylla, Jae Hee Lee, Till Mossakowski, Thomas Schneider, André van Delden, Jasper van de Ven, and Diedrich Wolter. 2017. A survey of qualitative spatial and temporal calculi: Algebraic and computational properties. ACM Comput. Surv. 50, 1, Article 7 (April 2017), 39 pages.
DOI: http://dx.doi.org/10.1145/3038927

1. INTRODUCTION

Knowledge about our world is densely interwoven with spatial and temporal facts. Nearly every knowledge-based system contains means for representation of, and possibly reasoning about, spatial or temporal knowledge. Among the different options available to a system designer, ranging from domain-level data structures to highly abstract logics, qualitative approaches stand out for their ability to mediate between the domain level and the conceptual level. Qualitative representations explicate relational knowledge between (spatial or temporal) domain entities, allowing individual statements to be evaluated by truth values. The aim of qualitative representations is

This work was supported by the DFG-funded SFB/TR 8 “Spatial Cognition,” projects QShape and LogoSpace. Author names appear in alphabetic order.
Authors' addresses: F. Dylla, T. Schneider, A. van Delden, and J. van de Ven, Faculty of Computer Science, University of Bremen, Postfach 330440, 28334 Bremen, Germany; email: {dylla, ts, schola, jasper}@cs.uni-bremen.de; J. H. Lee, Centre for Quantum Software and Information, University of Technology Sydney, 15 Broadway, Ultimo NSW 2007, Australia; email: jaehee.lee@uts.edu.au; T. Mossakowski, Faculty of Computer Science, Otto von Guericke University of Magdeburg, Universitätsplatz 2, 39106, Magdeburg, Germany; email: till@iks.cs.ovgu.de; D. Wolter, Faculty of Information Systems and Applied Computer Sciences, University of Bamberg, 96045 Bamberg, Germany; email: diedrich.wolter@uni-bamberg.de.
Permission to make digital or hard copies of part or all of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies show this notice on the first page or initial screen of a display along with the full citation. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, to republish, to post on servers, to redistribute to lists, or to use any component of this work in other works requires prior specific permission and/or a fee. Permissions may be requested from Publications Dept., ACM, Inc., 2 Penn Plaza, Suite 701, New York, NY 10121-0701 USA, fax +1 (212) 869-0481, or permissions@acm.org.
© 2017 ACM 0360-0300/2017/04-ART7 $15.00
DOI: http://dx.doi.org/10.1145/3038927
to focus on the aspects that are essential for a task at hand by abstracting away from other, unimportant aspects. As a result, a wide range of representations have been applied, using various kinds of knowledge representation languages. The most fundamental principles for representing knowledge qualitatively that are at the heart of virtually every representation language are captured by a construct called qualitative (spatial or temporal) calculus. In past decades, a great variety of qualitative calculi have been developed, each tailored to specific aspects of spatial or temporal knowledge. They share common principles but differ in formal and computational properties.

This article presents an up-to-date comprehensive overview of qualitative spatial and temporal reasoning (QSTR). We provide a general definition of QSTR (Section 2), give a uniform account of a calculus that is more integrative than existing ones (Section 3), identify and differentiate algebraic properties of calculi (Section 4), and discuss their role within other knowledge representation paradigms (Section 5) as well as alternative approaches (Section 6). Besides the survey character, the article provides a taxonomy of the most prominent reasoning problems, a survey of all existing calculi proposed so far (to the best of our knowledge), and the first comprehensive overview of their computational properties.

This article is accompanied by an electronic appendix that contains additional examples, observations, proofs, and detailed experimental results, marked “◁” in the text.

Demarcation of Scope and Contribution

This article addresses researchers and engineers working with knowledge about space or time and wishing to employ reasoning on a symbolic level. We supply a thorough overview of the wealth of qualitative calculi available, many of which have emerged from concrete application scenarios, for example, proposed for geographical information systems (GISs) [Egenhofer 1991; Frank 1991] and now readily employed in current systems; for applications in general, see also the overview given in Ligozat [2011]. Our survey focuses on the calculi themselves (Tables I and II) and their computational and algebraic properties, that is, characteristics relevant for reasoning and symbolic manipulation (Table IV, Figure 6). To this end, we also categorize reasoning tasks involving qualitative representations (Figure 2).

We exclusively consider qualitative formalisms for reasoning on the basis of finite sets of relations over an infinite spatial or temporal domain. As such, the mere use of symbolic labels is not surveyed. We also disregard approaches augmenting qualitative formalisms with an additional interpretation such as fuzzy sets or probability theory.

This article significantly advances from previous publications with a survey character in several regards. Ligozat [2011] describes in the course of the book “the main” qualitative calculi and describes their relations, complexity issues, and selected techniques. Although an algebraic perspective is taken as well, we integrate this in a more general context. In addition to mentioning general axioms in the context of relation algebras, we present a thorough investigation of calculi regarding these axioms. He also gives references to applications that employ QSTR techniques in a broad sense. Our survey supplements precise definitions of the underlying formal aspects, which will then be general enough to encompass all existing calculi that we are aware of. Chen et al. [2013] summarize the progress in QSTR by presenting selected key calculi for important spatial aspects. They give a brief introduction to basic properties of calculi but do not detail formal properties or provide a picture of the entire variety of formalisms achieved so far, as provided by this article. Algebra-based methods for reasoning with qualitative constraint calculi have been covered by Renz and Nebel [2007]. Their description applies to calculi that satisfy rather strong properties, which we relax. We
present revised definitions and an algebraic closure algorithm that generalizes to all existing calculi, and, to the best of our knowledge, we give the first comprehensive overview on computational properties. Cohn and Renz [2008] present an introduction to the field that extends the earlier article of Cohn and Hazarika [2001] by a more detailed discussion of logic theories for mereotopology and by presenting efficient reasoning algorithms.

2. WHAT IS QUALITATIVE SPATIAL AND TEMPORAL REASONING

We characterize QSTR by considering the reasoning problems it is concerned with. Generally speaking, reasoning is a process to generate new knowledge from existing knowledge. Knowledge primarily refers to facts given explicitly, possibly implicating implicit ones. Sound reasoning is involved with explicating the implicit, allowing it to be processed further. Thus, sound reasoning is crucial for many applications. In QSTR, it is a key characteristic and the applied reasoning methods are largely shaped by the specifics of qualitative knowledge about spatial or temporal domains as provided within the qualitative domain representation.

2.1. A General Definition of QSTR

Qualitative domain representations employ symbols to represent semantically meaningful properties of a perceived domain, abstracting away any details not regarded relevant to the context at hand. The perceived domain comprises the available raw information about objects. By qualitative abstraction, the perceived domain is mapped to the qualitative domain representation, called domain representation from now on. Various aims motivate research on qualitative abstractions, most importantly the desire to develop formal models of common sense relying on coarse concepts [Williams and de Kleer 1991; Bredeweg and Struss 2004] and to capture the catalog of concepts and inference patterns in human cognition [Kuipers 1978; Knauff et al. 2004], which in combination enables intuitive approaches to designing intelligent systems [Davis 1990] or human-centered computing [Frank 1992]. Within QSTR, it is required that qualitative abstraction yields a finite set of elementary concepts. The following definition aims to encompass all contexts in which QSTR is studied in the literature.

**Definition 2.1.** QSTR is the study of techniques for representing and manipulating spatial and temporal knowledge by means of relational languages that use a finite set of symbols. These symbols stand for classes of semantically meaningful properties of the represented domain (positions, directions, etc.).

Spatial and temporal domains are typically infinite and exhibit complex structures. Due to their richness and diversity, QSTR is confronted with unique theoretic and computational challenges. Consequently, there is a high variety of domain representations, each focusing on specific aspects relevant to specific tasks. To achieve qualitative abstraction, QSTR uses a relational language to formulate domain representations. It turns out that binary relations can capture the most relevant facets of space and time—this class also received the most attention by the research community. Expressive power is purely based on these predefined relations; no conjuncts or quantifiers are considered. Thus, the associated reasoning methods can be regarded as variants of constraint-based reasoning. Additionally, constraint-based reasoning techniques can be used to empower other methods, for example, to assess the similarity of represented entities or logic inference.

Finally, to map a domain representation to the perceived domain, a realization process is applied. This process instantiates entities in the perceived domain that are based on entities provided in the domain representation.
Figure 1 depicts the overall view on knowledge representation and aligns with the well-known view on intelligent agents considered in AI, which connects the environment to the agent and its internal representation by means of perception (which is an abstraction process as well) and, vice versa, by actions (see, e.g., Russell and Norvig [2009, Chapter 2]).

2.2. Taxonomy of Constraint-Based Reasoning Tasks

Figure 2 depicts an overview of constraint-based reasoning tasks in the context of QSTR. We now briefly describe these tasks and highlight some associated literature. The description is deliberately provided at an abstract level: each task may come in different flavors, depending on specific (application) contexts. Also, applicability of specific algorithms largely depends on the qualitative representation at hand. The following taxonomy is loosely based on the overview by Wallgrün et al. [2013].

In the following, we refer to the set of objects received from the perceived domain by applying qualitative abstraction as domain entities. These are, for example, geometric entities such as points, lines, or polygons. In general, domain entities can be of any type regarding spatial or temporal aspects.

We further use the notion of a qualitative constraint network (QCN), a special form of domain representation. Commonly, a QCN \( Q \) is depicted as a directed labeled graph, whose nodes represent abstract domain entities (i.e., with no specific values from the domain assigned), and whose edges are labeled with constraints (i.e., symbols representing relationships required to hold between these entities—see Figure 3(b)). An
assignment of concrete domain entities to the nodes in $Q$ is called a solution of $Q$ if the assigned entities satisfy all constraints in $Q$. Section 3.2 has precise definitions.

**Constraint Network Generation.** This task determines relational statements that describe given domain entities regarding specific aspects, using a predetermined qualitative language fulfilling certain properties, that is, in our case provided by a qualitative spatial calculus. For example, Figure 3(b) could be the QCN derived from the scene shown in Figure 3(a). Techniques for solving this task are described (e.g., in Cohn et al. [1997], Worboys and Duckham [2004], Forbus et al. [2004], and Dylla and Wallgrün [2007]).

**Consistency Checking.** This decision problem is considered the fundamental QSTR task [Renz and Nebel 2007]: given an input QCN $Q$, decide whether a solution exists. Applicable algorithms depend on the kinds of constraints that occur in $Q$ and are addressed in Sections 3.2 and 3.4.

**Model Generation.** This task determines a solution for a QCN $Q$, that is, a concrete assignment of a domain entity for each node in $Q$. This may be computationally harder than merely deciding the existence of a solution. For instance, Figure 3(a) could be the result of the model generation for the QCN shown in Figure 3(c). Typically, a single QCN has infinitely many solutions, due to the abstract nature of qualitative representations. Implementations of model generation may thus choose to introduce further parameters for controlling the kind of solution determined. Techniques for solving this task are described (e.g., in Schultz and Bhatt [2012], Kreutzmann and Wolter [2014], Schockaert and Li [2015]).

**Equivalence Transformation.** Taking a QCN $Q$ as input, equivalence transformation methods determine a QCN $Q'$ that has exactly the same solutions but meets additional criteria. Two variants are commonly considered.

- **Smallest equivalent network representation** determines the strongest refinement of the input $Q$ by modifying its constraints in order to remove redundant information. Figure 3(b) depicts a refinement of Figure 3(c) since in Figure 3(c) the relation between $A$ and $C$ is not constrained at all (i.e., being “<, =, >”), whereas Figure 3(b) involves the tighter constraint “<”. Thus, the QCN $Q$ in Figure 3(c) contains five base relations, whereas the QCN $Q'$ in Figure 3(b) contains only three. Methods for this task are addressed, for example, by van Beek [1991] and Amaneddine and Condotta [2013].

- **Most compact equivalent network representation** determines a QCN $Q'$ with a minimal number of constraints: it removes whole constraints that are redundant. In that sense, Figure 3(c) shows a more compact network than Figure 3(b). This task is addressed, for example, by Wallgrün [2012] and Duckham et al. [2014].

With this taxonomy in mind, the next section studies properties of qualitative representations and their reasoning operations.

3. **QUALITATIVE SPATIAL AND TEMPORAL CALCULI FOR DOMAIN REPRESENTATIONS**

The notion of a qualitative (spatial or temporal) calculus is a formal construct that, in one form or another, underlies virtually every language for qualitative domain representations. In this section, we survey this fundamental construct, formulate minimal requirements to a qualitative calculus, discuss their relevance to spatial and temporal representation and reasoning, and list calculi described in the literature. As mentioned in Section 2.2, domain entities can be of any type representing spatial or temporal aspects. The notion of a qualitative calculus has been devised to deal with any entities; thus, we omit an exhaustive list. Instead, we refer to Table I listing entities covered by known calculi.
Existing calculi are entirely based on binary or ternary relations between entities, which comprise, for example, points, lines, intervals, or regions. Binary relations are used to represent the location or moving direction of two entities relative to one another without referring to a third entity as a reference object. Examples of relations are “overlaps with” (for intervals or regions) or “move toward each other” (for dynamic objects). Additionally, a binary calculus is equipped with a converse operation acting on single relation symbols and a binary composition operation acting on pairs of relation symbols, representing the natural converse and composition operations on the domain relations, respectively. Converse and composition play a crucial role for symbolic reasoning: from the knowledge that the pair \((x, y)\) of entities is in relation \(r\), a symbolic reasoner can conclude that \((y, x)\) is in the converse of \(r\); and if it is additionally known that the pair \((y, z)\) is in \(s\), then the reasoner can conclude that \((x, z)\) is in the relation resulting in composing \(r\) and \(s\). In addition, most calculi provide an identity relation that allows representing the (explicit or derived) knowledge that, for example, \(x\) and \(y\) represent the same entity.

Depending on the properties postulated for converse and composition, notions of a calculus of varying strengths exist [Nebel and Scivos 2002; Ligozat and Renz 2004]. The algebraic properties of binary calculi are well understood; see Section 4.

The main motivation for using ternary relations is the requirement of directly capturing relative frames of reference that occur in natural language semantics [Levinson 2003]. In these frames of reference, the location of a target object is described from the perspective of an observer with respect to a reference object. For example, a hiker may describe a mountain peak to be to the left of a lake with respect to his or her own point of view. Another important motivation is the ability to express that an object is located between two others. Thus, ternary calculi typically contain projective relations for describing relative orientation and/or betweenness. The commitment to ternary (or \(n\)-ary) relations complicates matters significantly: instead of a single converse operation, there are now five (or \(n! - 1\)) nontrivial permutation operations, and there is no longer a unique choice for a natural composition operation. For capturing the algebraic structure of \(n\)-ary relations, Condotta et al. [2006] proposed an algebra but there are other arguably natural choices, and they lead to different algebraic properties, as shown in Section 4. These difficulties may be the main reason algebraic properties of ternary calculi are not as deeply studied as for binary calculi. Fortunately, this will not prevent us from establishing our general notion of a qualitative spatial (or temporal) calculus with relation symbols of arbitrary arity. However, we will then restrict our algebraic study to binary calculi; a unifying algebraic framework for \(n\)-ary calculi has yet to be established.

### 3.1. Requirements to Qualitative Spatial and Temporal Calculi

We start with minimal requirements used in the literature. We use the following standard notation. A **universe** is a nonempty set \(\mathcal{U}\). With \(X^n\), we denote the set of all \(n\)-tuples with elements from \(X\). An **\(n\)-ary domain relation** is a subset \(r \subseteq \mathcal{U}^n\). We use the prefix notation \(r(x_1, \ldots, x_n)\) to express \((x_1, \ldots, x_n) \in r\); in the binary case, we will often use the infix notation \(xry\) instead of \(r(x, y)\).

**Abstract Partition Schemes.** Ligozat and Renz [2004] note that most spatial and temporal calculi are based on a set of JEPD (jointly exhaustive and pairwise disjoint) domain relations. The following definition is predominant in the QSTR literature [Ligozat and Renz 2004; Cohn and Renz 2008].

**Definition 3.1.** Let \(\mathcal{U}\) be a universe and \(\mathcal{R}\) a set of nonempty domain relations of the same arity \(n\). \(\mathcal{R}\) is called a set of **JEPD relations** over \(\mathcal{U}\) if the relations in \(\mathcal{R}\) are jointly exhaustive, that is, \(\mathcal{U}^n = \bigcup_{r \in \mathcal{R}} r\), and pairwise disjoint.
An \(n\)-ary abstract partition scheme is a pair \((\mathcal{U}, \mathcal{R})\) where \(\mathcal{R}\) is a set of JEPD relations over the universe \(\mathcal{U}\). The relations in \(\mathcal{R}\) are called base relations.  

In Definition 3.1, the universe \(\mathcal{U}\) represents the set of all (spatial or temporal) entities. The main ingredients of a calculus will be relation symbols representing the base relations in the underlying partition scheme. A constraint linking an \(n\)-tuple \(t\) of entities via a relation symbol will thus represent complete information (modulo the qualitative abstraction underlying the partition scheme) about \(t\). Incomplete information is modeled by \(t\) being in a composite relation, which is a set of relation symbols representing the union of the corresponding base relations. The set of all relation symbols represents the universal relation (the union of all base relations) and indicates that no information is available.

The requirement that all base relations are JEPD ensures that every \(n\)-tuple of entities belongs to exactly one base relation. Thanks to PD (pairwise disjointness), there is a unique way to represent any composite relation using relation symbols and, due to JE (joint exhaustiveness), the empty relation can never occur in a consistent set of constraints, which is relevant for reasoning. See Section 3.2.

Partition Schemes, Identity, and Converse. Ligozat and Renz [2004] base their definition of a (binary) qualitative calculus on the notion of a partition scheme, which imposes additional requirements on an abstract partition scheme. In particular, it requires that the set of base relations contains the identity relation and is closed under the converse operation. The analogous definition by Condotta et al. [2006] captures relations of arbitrary arity. Before we define the notion of a partition scheme, we discuss the generalization of identity and converse to the \(n\)-ary case.

The binary identity relation is given as usual by

\[
id^2 = \{(u, u) \mid u \in \mathcal{U}\}.
\]

\(\triangleleft \text{Ex. A.7}\)

The most inclusive way to generalize Equation (1) to the \(n\)-ary case is to fix a set \(M\) of numbers of all positions where tuples in \(\text{id}^n\) are required to agree. Thus, an \(n\)-ary identity relation is a domain relation \(\text{id}_M^n\) with \(M \subseteq \{1, \ldots, n\}\) and \(|M| \geq 2\), which is defined by

\[
\text{id}_M^n = \{(u_1, \ldots, u_n) \in \mathcal{U}^n \mid u_i = u_j \text{ for all } i, j \in M\}.
\]

This definition subsumes the “diagonal elements” \(\Delta_{ij}\) of Condotta et al. [2006] for the case \(|M| = 2\). However, it is not enough to restrict attention to \(|M| = 2\) because there are ternary calculi that contain all identities \(\text{id}_{1,2}^2\), \(\text{id}_{1,3}^3\), \(\text{id}_{2,3}^3\), and \(\text{id}_{1,2,3}^3\), an example being the LR calculus, which was described as “the finest of its class” [Scivos and Nebel 2005]. Since the relations in an \(n\)-ary abstract partition scheme are JEPD, all identities \(\text{id}_M^n\) are either base relations or subsumed by those. The stronger notion of a partition scheme should thus require that all identities be made explicit.

For binary relations, \(\text{id}^2\) from Equation (1) is the unique identity relation \(\text{id}^2_{[1,2]}\). The standard definition for the converse operation \(^c\) on binary relations is

\[
r^c = \{(v, u) \mid (u, v) \in r\}.
\]

\(\triangleleft \text{Ex. A.8}\)

In order to generalize the reversal of the pairs \((u, v)\) in Equation (2) to \(n\)-ary tuples, we consider arbitrary permutations of \(n\)-tuples. An \(n\)-ary permutation is a bijection \(\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}\). We use the notation \(\pi: (1, \ldots, n) \mapsto (i_1, \ldots, i_n)\) as an abbreviation for “\(\pi(1) = i_1, \ldots, \pi(n) = i_n\)” The identity permutation \(\iota: (1, \ldots, n) \mapsto (1, \ldots, n)\) is called trivial; all other permutations are nontrivial.
A finite set $P$ of $n$-ary permutations is called generating if each $n$-ary permutation is a composition of permutations from $P$. For example, the following two permutations form a (minimal) generating set:

$\text{sc} : (1, \ldots, n) \mapsto (2, \ldots, n, 1)$ \hspace{1cm} \text{(shortcut)}

$\text{hm} : (1, \ldots, n) \mapsto (1, \ldots, n-2, n, n-1)$ \hspace{1cm} \text{(homing)}

The names have been introduced in Freksa and Zimmermann [1992] for ternary permutations, together with a name for a third distinguished permutation:

$\text{inv} : (1, \ldots, n) \mapsto (2, 1, 3, \ldots, n)$ \hspace{1cm} \text{(inversion)}

Condotta et al. [2006] call shortcut “rotation” ($r\curvearrowright$) and homing “permutation” ($r\leftrightarrow$).

For $n = 2$, sc, hm, and inv coincide; indeed, there is a unique minimal generating set, which consists of the single permutation $\text{sc} : (1, 2) \mapsto (2, 1)$. For $n \geq 3$, there are several generating sets, for example, $\{\text{sc, hm}\}$ and $\{\text{sc, inv}\}$.

Now an $n$-ary permutation operation is a map $\cdot_\pi$ that assigns to each $n$-ary domain relation $r$ an $n$-ary domain relation denoted by $r^\pi$, where $\pi$ is an $n$-ary permutation and the following holds: $r^\pi = \{(u_{\pi(1)}, \ldots, u_{\pi(n)}) | (u_1, \ldots, u_n) \in r\}$.

We are now ready to give our definition of a partition scheme, lifting Ligozat and Renz’s binary version to the $n$-ary case, and generalizing Condotta et al.’s $n$-ary version to arbitrary generating sets.

**Definition 3.2.** An $n$-ary partition scheme $(U, R)$ is an $n$-ary abstract partition scheme with the following two additional properties.

1. $R$ contains all identity relations $\text{id}_M^\pi$, $M \subseteq \{1, \ldots, n\}$, $|M| \geq 2$.
2. There is a generating set $P$ of permutations such that, for every $r \in R$ and every $\pi \in P$, there is some $s \in R$ with $r^\pi = s$.

It is important to note that violations of Definition 3.2 (e.g., depicted in Example A.11) are not necessarily bugs in the design of the respective calculi—in fact, they are often a feature of the corresponding representation language, which is deliberately designed to be just as granular as necessary, and may thus omit some identity relations or converses/compositions of base relations.

Thus, violations of Definition 3.2 are unavoidable, and we adopt the more general notion of an abstract partition scheme.

**Calculi.** Intuitively, a qualitative (spatial or temporal) calculus is a symbolic representation of an abstract partition scheme and additionally represents the composition operation on the relations involved. As before, we need to discuss the generalization of binary composition to the $n$-ary case before we can define it precisely.

For binary domain relations, the standard definition of composition is

$$r \circ s = \{(u, w) | \exists v \in U : (u, v) \in r \text{ and } (v, w) \in s\}. \quad (3)$$

We are aware of three ways to generalize Equation (3) to higher arities. The first is a binary operation on the ternary relations of the calculus double-cross (2-cross) [Freksa
1992; Freksa and Zimmermann 1992] (see also Figure 8 in the appendix):
\[ r \circ^3_{E^I} s = \{(u, v, w) \mid \exists x : (u, v, x) \in r \text{ and } (v, x, w) \in s\}. \]

A second alternative results in \(n(n-1)\) binary operations \(i \circ^j_{E} \) [Isli and Cohn 2000; Scivos and Nebel 2005]: the composition of \(r\) and \(s\) consists of those \(n\)-tuples that belong to \(r\) (\(s\), respectively) if the \(i\)th (\(j\)th, respectively) component is replaced by some uniform element \(v\):
\[ r \circ^3_{E^I} s = \{(u_1, \ldots, u_n) \mid \exists v : (u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_n) \in r \text{ and } (u_1, \ldots, u_{j-1}, v, u_{j+1}, \ldots, u_n) \in s\}. \]

In the ternary case, this yields, for example:
\[ r \circ^3_{E^I} s = \{(u, v, w) \mid \exists x : (u, v, x) \in r \text{ and } (u, x, w) \in s\}. \]...

If we assume, for example, that the underlying partition scheme speaks about the relative position of points, we can consider Equation (4) to say: if the position of \(x\) relative to \(u\) and \(v\) is determined by the relation \(r\) (as given by \((u, v, x) \in r\)) and the position of \(w\) relative to \(u\) and \(x\) is determined by the relation \(s\) (as given by \((u, x, w) \in s\)), then the position of \(w\) relative to \(u\) and \(v\) can be inferred to be determined by \(r \circ^3_{E^I} s\).

The third is perhaps the most general, resulting in an \(n\)-ary operation [Condotta et al. 2006]: \(\circ(r_1, \ldots, r_n)\) consists of those \(n\)-tuples that, for every \(i = 1, \ldots, n\), belong to the relation \(r_i\) whenever their \(i\)th component is replaced by some uniform \(v\):
\[ \circ(r_1, \ldots, r_n) = \{(u_1, \ldots, u_n) \mid \exists v \in U : (u_1, \ldots, u_{n-1}, v) \in r_1 \text{ and } (u_1, \ldots, u_{n-2}, v, u_n) \in r_2 \text{ and } \ldots \text{ and } (v, u_2, \ldots, u_n) \in r_n\}. \]

For binary domain relations, all these alternative approaches collapse to Equation (3).

In light of the diverse views on composition, we define a composition operation on \(n\)-ary domain relations to be an operation of arity \(2 \leq m \leq n\) on \(n\)-ary domain relations, without imposing additional requirements. Those are not necessary for the following definitions, which are independent of the particular choice of composition.

We now define our minimal notion of a calculus, which provides a set of symbols for the relations in an abstract partition scheme (Rel), and for some choice of nontrivial permutation operations \(\{^1, \ldots, ^k\}\) and some composition operation \(\circ\).

**Definition 3.3**. An \(n\)-ary qualitative calculus is a tuple \((\text{Rel}, \text{Int}, ^1, \ldots, ^k, \circ)\) with \(k \geq 1\) and the following properties:

- Rel is a finite, nonempty set of \(n\)-ary relation symbols (denoted \(r, s, t, \ldots\)). The subsets of Rel, including singletons, are called composite relations (denoted \(R, S, T, \ldots\)).
- \(\text{Int} = (U, \varphi, ^1, \ldots, ^k, \circ)\) is an interpretation with the following properties:
  - \(U\) is a universe.
  - \(\varphi : \text{Rel} \to 2^U\) is an injective map assigning an \(n\)-ary relation over \(U\) to each relation symbol, such that \((U, \varphi(r) \mid r \in \text{Rel})\) is an abstract partition scheme. The map \(\varphi\) is extended to composite relations \(R \subseteq \text{Rel}\) by setting \(\varphi(R) = \bigcup_{r \in R} \varphi(r)\).
  - \(^1, \ldots, ^k\) is a set of \(n\)-ary nontrivial permutation operations.
  - \(\circ\) is a composition operation on \(n\)-ary domain relations that has arity \(2 \leq m \leq n\).
- Every permutation operation \(^d\) is a map \(^d : \text{Rel} \to 2^{\text{Rel}}\) that satisfies
\[ \varphi(r^{d}) \supseteq \varphi(r)^{d_i} \]
for every $r \in \text{Rel}$. The operation $^{\varphi}i$ is extended to composite relations $R \subseteq \text{Rel}$ by setting $R^{\varphi}_{^i} = \bigcup_{r \in R} r^{\varphi}_{^i}$.

—The composition operation $\circ$ is a map $\circ: \text{Rel}^m \rightarrow 2^{\text{Rel}}$ that satisfies

$$\varphi(\circ(r_1, \ldots, r_m)) \supseteq \circ(\varphi(r_1), \ldots, \varphi(r_m))$$

(7)

for all $r_1, \ldots, r_m \in \text{Rel}$. The operation $\circ$ is extended to composite relations $R_1, \ldots, R_m \subseteq \text{Rel}$ by setting $\circ(R_1, \ldots, R_m) = \bigcup_{r_1 \in R_1} \cdots \bigcup_{r_m \in R_m} \circ(r_1, \ldots, r_m)$.

In the special case of binary relations, the natural converse is the only nontrivial permutation operation, that is, $k = 1$. 

\begin{footnotesize}
\begin{itemize}
\item \textcircled{Ex. A.16, A.17, A.18}
\end{itemize}
\end{footnotesize}

Abstract Versus Weak and Strong Operations. We call permutation and composition operations with Properties (6) and (7) abstract permutation and abstract composition, following Ligozat’s naming in the binary case [Ligozat 2005]. For reasons explained further later, our notion of a qualitative calculus imposes weaker requirements on the permutation operation than Ligozat and Renz’s notions of a weak (binary) representation [Ligozat 2005; Ligozat and Renz 2004] or the notion of a (binary) constraint algebra [Nebel and Scivos 2002]. The following definition specifies those stronger variants (see, e.g., Ligozat and Renz [2004]).

\begin{footnotesize}
\begin{itemize}
\item \textcircled{Ob. B.2}
\end{itemize}
\end{footnotesize}

Definition 3.4. Let $(\text{Rel}, \text{Int}, ^{-1}, \ldots, ^{-k}, \circ)$ be a qualitative calculus based on the interpretation $\text{Int} = (U, \varphi, ^{r_1}, \ldots, ^{r_k}, \circ)$. The permutation operation $^{\varphi}i$ is a weak permutation if, for all $r \in \text{Rel}$:

$$r^{\varphi}_{^i} = \bigcap \{S \subseteq \text{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\varphi}_{^i}\}.$$  

(8)

The permutation operation $^{\varphi}i$ is a strong permutation if, for all $r \in \text{Rel}$:

$$\varphi(r^{\varphi}_{^i}) = \varphi(r)^{\varphi}_{^i}.$$  

(9)

The composition operation $\circ$ is a weak composition if, for all $r_1, \ldots, r_m \in \text{Rel}$:

$$\varphi(\circ(r_1, \ldots, r_m)) = \bigcap \{S \subseteq \text{Rel} \mid \varphi(S) \supseteq \circ(\varphi(r_1), \ldots, \varphi(r_m))\}.$$  

(10)

The composition $\circ$ is a strong composition if, for all $r_1, \ldots, r_m \in \text{Rel}$:

$$\varphi(\circ(r_1, \ldots, r_m)) = \circ(\varphi(r_1), \ldots, \varphi(r_m)).$$  

(11)

In the literature, the equivalent variant $r^{\varphi}_{^i} = \{s \in \text{Rel} \mid \varphi(s) \cap \varphi(r)^{\varphi}_{^i} \neq \emptyset\}$ of Equation (8) is sometimes found, and analogously for Equation (10). 

\begin{footnotesize}
\begin{itemize}
\item \textcircled{Ex. A.19, A.20}
\end{itemize}
\end{footnotesize}

In terms of composition tables, abstract composition requires that each cell corresponding to $\circ(r_1, \ldots, r_m)$ contain at least those relation symbols $t$ whose interpretation intersects with $\circ(\varphi(r_1), \ldots, \varphi(r_m))$. Weak composition additionally requires that each cell contains exactly those $t$. Strong composition, in contrast, imposes a requirement on the underlying partition scheme: whenever $\varphi(t)$ intersects with $\circ(\varphi(r_1), \ldots, \varphi(r_m))$, it has to be contained in $\circ(\varphi(r_1), \ldots, \varphi(r_m))$, and analogously for permutation.

The previous “at least” is a crucial requirement: if some cell did not contain any relation symbol $t$ earlier, then the composition table would give rise to unsound inferences, (e.g., described in Example A.20). Abstractness as in Properties (6) and (7) thus captures minimal requirements to the operations in a qualitative calculus that ensure soundness of reasoning, as described in Section 3.2.

Along the same lines, adding unnecessary relations to a cell in the table leads to weaker inferences and thus amounts to a loss of knowledge. Weakness (Properties (8)
and (10)) ensures that this loss is kept to the unavoidable minimum. This last observation is presumably the reason that existing calculi (see Section 3.4) typically have at least weak operations—we are not aware of any calculus with only abstract operations. In Section 3.2, we will see that abstract composition is a minimal requirement for ensuring soundness of the most common reasoning algorithm, a-closure, and review the impact of the various strengths of the operations on reasoning algorithms.

The three notions form a hierarchy:

**Fact 3.5.** Every strong permutation (composition) is weak, and every weak permutation (composition) is abstract.

It suffices to postulate the properties’ weakness and strongness with respect to relation symbols only: they carry over to composite relations as shown in Fact 3.6 can go to appendix, too.

**Fact 3.6.** Given a qualitative calculus \((\text{Rel}, \text{Int}, \_\_1, \ldots, \_\_k, \circ)\), the following holds: for all composite relations \(R \subseteq \text{Rel}\) and \(i = 1, \ldots, k\):

\[
\varphi(R^i) \supseteq \varphi(R)^{\pi i}.
\]

(12)

For all composite relations \(R_1, \ldots, R_m \subseteq \text{Rel}\):

\[
\varphi(\circ(R_1, \ldots, R_m)) \supseteq \circ(\varphi(R_1), \ldots, \varphi(R_m)).
\]

(13)

*If \(\_\_i\) is a weak permutation, then, for all \(R \subseteq \text{Rel}\):

\[
R^i = \bigcap \{S \subseteq \text{Rel} \mid \varphi(S) \supseteq \varphi(R)^{\pi i}\}.
\]

If \(\_\_i\) is a strong permutation, then, for all \(R \subseteq \text{Rel}\):

\[
\varphi(R^i) = \varphi(R)^{\pi i}.
\]

If \(\circ\) is a weak composition, then, for all \(R_1, \ldots, R_m \subseteq \text{Rel}\):

\[
\circ(R_1, \ldots, R_m) = \bigcap \{S \subseteq \text{Rel} \mid \varphi(S) \supseteq \circ(\varphi(R_1), \ldots, \varphi(R_m))\}.
\]

If \(\circ\) is a strong composition, then, for all \(R_1, \ldots, R_m \subseteq \text{Rel}\):

\[
\varphi(\circ(R_1, \ldots, R_m)) = \circ(\varphi(R_1), \ldots, \varphi(R_m)).
\]

Suppose we want to achieve that the symbolic permutation operations provided by a calculus \(C\) capture all permutations at the domain level. Then \(C\) needs to be permutation-complete in the sense that at least weak permutation operations for all \(n!\) nontrivial permutations can be derived uniquely by composing the ones defined.

In the binary case, where the converse is the unique nontrivial (and generating) permutation, every calculus is permutation-complete. However, as noted earlier, the converse is not strong for the binary CDR [Skiadopoulos and Koubarakis 2005] and RCD [Navarrete et al. 2013] calculi (cf. Definition 3.2 ff.). There are also ternary calculi whose permutations are not strong: for example, the shortcut, homing, and inversion operations in the single-cross and double-cross calculi [Freksa 1992b; Freksa and Zimmermann 1992] are only weak. Since these calculi provide no further permutation operations, they are not permutation-complete. However, it is easy to compute the two missing permutations and thus make both calculi permutation-complete.

Ligozat and Renz’s [2004] basic notion of a binary qualitative calculus is based on a weak representation, which requires an identity relation, an abstract composition, and the converse being strong, thus excluding, for example, CDR and RCD. A representation is a weak representation with a strong composition and an injective map \(\varphi\). Our basic notion of a qualitative calculus is more general than a weak representation by not
requiring an identity relation, and by only requiring abstract permutations and composition, thus including CDR and RCD. On the other hand, it is slightly more restrictive by requiring the map \( \varphi \) to be injective. However, since base relations are JEPD, the only way for \( \varphi \) to violate injectivity is to give multiple names to the same relation, which is not really intuitive. It is even problematic because it leads to unintended behavior of the notion of weak composition (or permutation): if there are two relation symbols for every domain relation, then the intersections in Equations (8) and (10) will range over disjoint composite relations \( S \) and thus become empty.

Recently, Westphal et al. [2014] gave a new definition of a qualitative calculus that does not explicitly use a map—in our case the interpretation \( \text{Int} \)—that connects the symbols with their semantics. Instead, they employ the “notion of consistency” [Westphal et al. 2014, p. 211] for generating a weak algebra from the Boolean algebra of relation symbols. As with Ligozat and Renz [2004], their definition of a qualitative calculus is confined to binary relations only.

### 3.2. Spatial and Temporal Reasoning

As in the area of classical constraint satisfaction problems (CSPs), we are given a set of variables and constraints: a constraint network or a qualitative CSP.\(^1\) The task of constraint satisfaction is to decide whether there exists a valuation of all variables that satisfies the constraints. In calculi for spatial and temporal reasoning, all variables range over the entities of the specific domain of a qualitative calculus. The relation symbols defined by the calculus serve to express constraints between the entities. More formally, we have:

**Definition 3.7 (QCSP).** Let \( C = (\text{Rel, Int, } ^{-1}, \ldots, ^{-k}, \circ) \) be an \( n \)-ary qualitative calculus with \( \text{Int} = (\mathcal{U}, \varphi, ^{-1}, \ldots, ^{-k}, \circ) \), and let \( X \) be a set of variables ranging over \( \mathcal{U} \). An \( n \)-ary qualitative constraint in \( C \) is a formula \( R(x_1, \ldots, x_n) \) with variables \( x_1, \ldots, x_n \in X \) and a relation \( R \subseteq \text{Rel} \). We say that a valuation \( \psi : X \rightarrow \mathcal{U} \) satisfies \( R(x_1, \ldots, x_n) \) if \((\psi(x_1), \ldots, \psi(x_n)) \in \varphi(R)\) holds.

A qualitative constraint satisfaction problem (QCSP) is the task to decide whether there is a valuation \( \psi \) for a set of variables satisfying a set of constraints.

\(\langle\text{Ex. A.21}\rangle\)

For simplicity and without loss of generality, we assume that every set of constraints contains exactly one constraint per set of \( n \) variables. Thus, of binary constraints, either \( r_{x_1,x_2} \) or \( r'_{x_2,x_1} \) is assumed to be given—the other can be derived using the converse; multiple constraints regarding variables \( x_1, x_2 \) can be integrated via intersection. In the following, \( r_{x_1,\ldots,x_n} \) stands for the unique constraint between the variables \( x_1, \ldots, x_n \).

Several techniques originally developed for finite-domain CSPs can be adapted to spatial and temporal QCSPs. Since deciding CSP instances is already NP-complete for search problems with finite domains, heuristics are important. One particularly valuable technique is constraint propagation, which aims at making implicit constraints explicit in order to identify variable assignments that would violate some constraint. By pruning away these variable assignments, a consistent valuation can be searched more efficiently. A common approach is to enforce \( k \)-consistency; the following definition is standard in the CSP literature [Dechter 2003].

**Definition 3.8.** A QCSP with variables \( X \) is \( k \)-consistent if, for all subsets \( X' \subseteq X \) of size \( k - 1 \), we can extend any valuation of \( X' \) that satisfies the constraints to a valuation of \( X' \cup \{z\} \) also satisfying the constraints, for any additional variable \( z \in X \setminus X' \).

\(^1\)In the CSP domain, “CSP” usually refers to a single instance, not the decision or computation problem. This is the same as a qualitative constraint network (QCN) as introduced in Section 2.2.
QCSPs are naturally 1-consistent as universes are nonempty and there are no unary constraints. An $n$-ary QCSP is $n$-consistent if $r^{x_1} = r_{x_1}(x_1, \ldots, x_k)$ for all $i$ and $r_{x_1, \ldots, x_k} \neq \emptyset$: domain relations are typically serial; that is, for any $r$ and $x_1, \ldots, x_{k-1}$, there is some $x_k$ with $r(x_1, \ldots, x_k)$. In the case of binary relations, this means that 2-consistency is guaranteed in calculi with a strong converse by $r^{x_1, x_2} = r_{x_1, x_2}(x_1, x_2)$ and $r_{x_1, x_2} \neq \emptyset$, and seriality of $r$ means that, for every $x$, there is a $y$ with $r(x, y)$.

Already examining $(n + 1)$-consistency may provide very useful information. The following is best explained for binary relations and then generalized to higher arities. A 3-consistent binary QCSP is called path consistent, and Definition 3.8 can be rewritten using binary composition as

$$\forall x, y \in X \quad r_{x, y} \subseteq \bigcap_{z \in X} r_{x, z} \circ r_{z, y}. \tag{14}$$

We can enforce 3-consistency by computing the fixpoint of the refinement operation

$$r_{x, y} \leftarrow r_{x, y} \cap (r_{x, z} \circ r_{z, y}), \tag{15}$$

applied to all variables $x, y, z \in X$. In finite CSPs with variables ranging over finite domains, composition is also finite and the procedure always terminates since the refinement operation is monotone and there can thus only be finitely many steps until reaching the fixpoint. Such procedures are called path consistency algorithms and require $O(|X|^3)$ time [Dechter 2003].

Enforcing path consistency with QCSPs may not be possible using a symbolic algorithm since Equation (15) may lead to relations not expressible in 2Cons. This problem occurs when composition in a qualitative calculus is not strong. It is, however, straightforward to weaken Equation (15) using weak composition:

$$r_{x, y} \leftarrow r_{x, y} \cap (r_{x, z} \circ r_{z, y}). \tag{16}$$

The resulting procedure is called enforcing algebraic closure or a-closure for short. The QCSP obtained as a fixpoint of the iteration is called algebraically closed.

If composition in a qualitative calculus is strong, a-closure and path consistency coincide. Since there are finitely many relations in a qualitative calculus, a-closure shares all computational properties with the finite CSP case.

A natural generalization from binary to $n$-ary relations can be achieved by considering $(n + 1)$-consistency (recall that path consistency is 3-consistency). In the context of symbolic computation with qualitative calculi, we thus need to lift Equations (14) and (15) to the particular composition operation available. For composition as defined by Equation (5), one obtains

$$\forall x_1, \ldots, x_n \in X \quad r_{x_1, \ldots, x_n} \subseteq \bigcap_{y \in X} \circ(r_{x_1, \ldots, x_{n-1}, y}, r_{x_1, \ldots, x_{n-2}, y, x_n}, \ldots, r_{y, x_2, \ldots, x_n}),$$

and the symbolic refinement operation in Equation (16) becomes

$$r_{x_1, \ldots, x_n} \leftarrow r_{x_1, \ldots, x_n} \cap \circ(r_{x_1, \ldots, x_{n-1}, y}, r_{x_1, \ldots, x_{n-2}, y, x_n}, \ldots, r_{y, x_2, \ldots, x_n}). \tag{17}$$

The reason that, in Definition 3.3, we require composition to be at least abstract is that Inclusion (7) guarantees that reasoning via a-closure is sound: enforcing $k$-consistency or a-closure does not change the solutions of a CSP, as only impossible valuations are locally removed. If application of a-closure results in the empty relation, then the QCSP is known to be inconsistent. By contrast, an algebraically closed QCSP may not be consistent. However, for several qualitative calculi (or at least subalgebras thereof), a-closure and consistency coincide. See also Section 3.4.
Since domain relations are JEPD, deciding QCSPs with arbitrary composite relations can be reduced to deciding QCSPs with only atomic relations (i.e., relation symbols) by means of search (cf. Renz and Nebel [2007]). The approach to reason in a full algebra is thus to refine a composite relation $R \cup S$ to either $R$ or $S$ in a backtracking search fashion, until a dedicated decision procedure becomes applicable. Computationally, reasoning with the complete algebra is typically NP-hard due to the exponential number of possible refinements to atomic relations. For investigating reasoning algorithms, one is thus interested in the complexity of reasoning with atomic relations. If they can be handled in polynomial time, maximal tractable subalgebras that extend the set of atomic relations are of interest too. Efficient reasoning algorithms for atomic relations and the existence of large tractable subalgebras suggest efficiency in handling practical problems. The search for maximal tractable subalgebras can be significantly eased by applying the automated methods proposed by Renz [2007]. These exploit algebraic operations to derive tractable composite relations and, complementary, search for embeddings of NP-hard problems. Using a-closure plus refinement search has been regarded as the prevailing reasoning method. Certainly, a-closure provides an efficient cubic time method for constraint propagation, but Table IV clearly shows that the majority of calculi require further methods as decision procedures.

3.3. Tools to Facilitate Qualitative Reasoning

There are several tools that facilitate one or more of the reasoning tasks. The most prominent plain-QSTR tools are GQR [Westphal et al. 2009], a constraint-based reasoning system for checking consistency using a-closure and refinement search, and the SparQ reasoning toolbox [Wolter and Wallgrun 2012],\(^2\) which addresses various tasks from constraint- and similarity-based reasoning. Besides general tools, there are implementations addressing specific aspects (e.g., reasoning with CDR [Liu et al. 2010]) or tailored to specific problems (e.g., Phalanx for sparse RCC-8 QCSPs [Sioutis and Condotta 2014]). In the contact area of qualitative and logical reasoning, the DL reasoners Racer [Haarslev et al. 2012] and PelletSpatial [Stocker and Sirin 2009] offer support for handling a selection of qualitative formalisms. For logical reasoning about qualitative domain representations, the tools Hets [Mossakowski et al. 2007], SPASS [Weidenbach et al. 2002], and Isabelle [Nipkow et al. 2002] have been applied, supporting the first-order Common Algebraic Specification Language (CASL) [Astesiano et al. 2002] as well as its higher-order variant, HasCASL (see Wollf et al. [2007]).

3.4. Existing Qualitative Spatial and Temporal Calculi

In the following, we present an overview of existing calculi obtained from a systematic literature survey, covering publications in the relevant conferences and journals in the past 25 years, and following their citation graphs. To be included in our overview, a qualitative calculus has to be based on a spatial and/or temporal domain, fall under our general definition of a qualitative calculus (Definition 3.3: provide symbolic relations, the required symbolic operations, and semantics based on an abstract partition scheme), and be described in the literature either with explicit composition/converse tables or with instructions for computing them. These selection criteria exclude sets of qualitative relations that have been axiomatized in the context of logical theories (see Section 5.2) or qualitative calculi designed for other domains, such as ontology alignment [Inants and Euzenat 2015].

Tables I and II list, to the best of our knowledge, all calculi satisfying these criteria. Table I lists the names of families of calculi and their domains. Table II lists all variants of these families with original references, arity, and number of their base relations

---

\(^2\)Available at https://github.com/dwolter/sparq.
### Table I. Existing Families of Spatial and Temporal Calculi

| Abbrev. | Name | Domain | Aspect |
|---------|------|--------|--------|
| 1,-2-cross | Single/Double Cross Calculus | points in 2D | relative location |
| 9-int | Nine-Intersection Model | simple n-d regions | topology |
| 9̂-int | 9- and 9-h Intersection Calculi | 9-int & bodies, lines, points in 2D/3D | |
| ABAₙₖ | Alg. of Bipartite Arrangements | 1D intervals in 2D | rel. loc./orientation |
| BA | Block Algebra (aka Rectangle Algebra or Rectangle Calculus) | n-d blocks | order |
| CBM | Calculus Based Method | 2D regions, lines, and points | topology |
| CDA | Closed Disk Algebra | 2D closed disks | topology |
| CDC | Cardinal Direction Calculus | points in 2D | cardinal directions |
| CDR | Cardinal Direction Relations | 2D regions | cardinal directions |
| CI | Algebra of Cyclic Intervals | intvls. on closed curves | cyclic order |
| CYC | Cyclic Ordering (CYC₅ aka Geometric Orientation) | oriented lines in 2D | relative orientation |
| DepCalc | Dependency Calculus | partially ordered points | partial order |
| DIA | Directed Intervals Algebra | directed 1D intvls. in 1D | order/orientation |
| DRA | Dipole Calculus | oriented line segms. in $\mathbb{R}^2$ | rel. loc/orientation |
| DRA-conn | Dipole Connectivity | connectivity of the above | connectivity |
| EIA | Extended Interval Algebra | 1D intervals in 1D | order |
| EOPRA | Elevated Oriented Point Rel. Alg. | OPRA & local distance | |
| EPRA | Elevated Point Relation Algebra | CDC & local distance | |
| GenInt | Generalized Intervals | unions of 1D intvls. | order |
| IA | (Allen's) Interval Algebra | 1D intervals in 1D | order |
| INDU | Intl. and Duration Network | IA & relative duration | |
| LOS | Lines of Sight | 2D regions in 3D | obscuration |
| LR | LR Calculus (aka Flip-Flop) | points in 2D | relative location |
| MC-4 | MC-4 | regions in 2D | congruence |
| OCC | Occlusion Calculus | 2D regions in 3D | obscuration |
| OM-3D | 3D Orientation Model | points in 3D | relative location |
| OPRRA | Oriented Point Rel. Algebra | oriented points in 2D | rel. loc/orientation |
| PC | Point Calculus (aka Point Algebra) | points in n-d | total order |
| QRPC | Qualitative Rectilinear Projection Calculus | oriented points in 2D | relative motion |
| QTC | Qualitative Trajectory Calculus | moving points in 1D/2D | relative motion |
| RCC | Region Connection Calculus | general regions | topology |
| RCD | Rectang. Card. Dir. Calculus | bounding boxes in 2D | cardinal directions |
| RL-DL-3-12 | Region-in-the-Frame-of-Directed-Line | regions & paths in 2D | relative motion |
| ROC | Region Occlusion Calculus | 2D regions in 3D | obscuration |
| SIC | Semi-Interval Calculus | 1D intervals in 1D | order |
| STAR | Star Calculi | points in 2D | direction |
| SV | StarVars | oriented points in 2D | relative direction |
| TPCC | Ternary Point Config. Calc. | points in 2D | relative location |
| TPR | Ternary Projective Relations | points or regions in 2D | relative location |
| VR | Visibility Relations | convex regions | obscuration |
Table II. Overview of Existing Spatial and Temporal Calculi, Legend in Table III

| Variant | Specifics | Reference(s) | Params | St |
|---------|-----------|--------------|--------|----|
| 1-, 2-cross | Freksa and Zimmermann [1992] | t 8, 15 | 8 |
| 3-int | Egenhofer [1991] | b 8 |
| 9/1-int | 10 variants* | Kurata [2010] | b \(<233\) | 1 |
| ABA_3 | Gottfried [2004] | b 125 | 1 |
| BAN | n dimensions | Balbiani et al. [1998; 1999] | b 13 | 1, 2 |
| CBM | Clementini et al. [1993] | b 7 |
| CDA | Egenhofer and Sharma [1993] | b 8 |
| CDC | Frank 1991; Ligozat [1998] | b 9 |
| CDR | original version | Skiadopoulos and Keubarakis [2004] | b 511 | 3 |
| eCDR | connected variant | Skiadopoulos and Keubarakis [2005] | b 289 | 3 |
| CI | Balbiani and Osmari [2000] | b 16 |
| CYC_2 | binary | Ili and Cohn [2006] | b 4 | 1 |
| CYC_3 | ternary | ibid. | t 24 |
| DepCalc | Ragni and Scivos [2005] | b 5 |
| DIA | Renz [2001] | b 26 |
| DRA_1 | coarse-grained* | Moratz et al. [2000] | b 24 |
| DRA_2 | fine-grained | ibid. | b 72 |
| DRA_3 | f+parallelism | Moratz et al. [2011] | b 80 |
| DRA_conn | Wailgrun et al. [2010] | b 7 |
| EIA | Zhang and Renz [2014] | b 27 |
| EOPRA_m | granularity n | Moratz and Wailgrun [2012] | b \(O(n^2)\) |
| EPRA | granularity n | Moratz and Wailgrun [2012] | b \(O(n^2)\) |
| EIA×EIA | coarse variant | Zhang and Renz [2014] | b 351 |
| EIA×EIA | finer variant | ibid. | b 729 |
| GenInt | Condotta [2000] | b 13 |
| IA | Allen [1983] | b 13 |
| INDU | Pujari et al. [1999] | b 25 |
| LOS-14 | convex regions | Galten [1994] | b 14 |
| LR | Scivos and Nobel [2005]; Ligozat [1993] | t 9 |
| MC-4 | Cristani [1999] | b 4 |
| OGC | convex regions | Kähler [2002] | b 8 |
| OM-3D | Pacheco et al. [2001] | t 75 |
| OPRA_m | granularity n | Moratz [2006]; Mokssazoki & M. [2012] | b \(O(n^2)\) |
| OPRA_3 | plus alignment | Dylla and Lee [2010] | b \(O(n^2)\) |
| PC_n | n dimensions | Vilain and Kautz [1986] | b 3 |
| PC_n | binaries | Balbiani and Condotta [2002] | b 3 |
| QRPC | Glez-Cabrera et al. [2013] | b 48 |
| QTC-B1_1, 2D variants | Van de Weghe et al. [2006] | b 39, 27 |
| QTC-B2_1, 2D variants | ibid. | b 9, 305 |
| QTC-N | network variant | Delafontaine et al. [2011] | b 17 |
| RCC-5 | without tangentiality | Randell et al. [1992] | b 5 |
| RCC-8 | with tangentiality | ibid. | b 8 |
| RCC-15, 23 | concave regions | Cohn et al. [1997] | b 15, 23 |
| RCC-62 | concave regions | OuYang et al. [2007] | b 62 |
| RCC*-7, 9 | + lower-dim. features | Clementini and Cohn [2014] | b 7, 9 |
| (V)RCC-3D+ | with occlusion | Sabharwal and Leepold [2014] | b 13, 47 |
| RCD | Navarrete et al. [2013] | b 36 |
| RIDL-3-12 | Kurata and Shi [2006] | b 1772 |
| ROC-20 | Randell et al. [2001] | b 20 |
| SIC | Freksa [1992a] | b 13 |
| STAR_m | granularity n | Ranz and Mtrita [2004] | b \(O(n)\) |
| STAR_2 | revised variants | ibid. | b \(O(n)\) |
| SV_n | granularity n | Lee et al. [2013] | b \(O(n)\) |
| TPCC | Moratz and Ragni [2008] | t 25 |
| TPR_p | for points | Clementini et al. [2006; 2010] | t 7 |
| TPR_r | for regions | ibid. | t 34 |
| VR | Tarquini et al. [2007] | t 7 |

ACM Computing Surveys, Vol. 50, No. 1, Article 7, Publication date: April 2017.
Table III. Legend for Table II

| Params | Arity – (b)inary, (t)ernary – and number of relation symbols |
|---|---|
| St | Status of availability: ☐ base relations, ☐ composition table, ☐ complexity results |
|   | ☐ table and complexity, ☐ SparQ implementation, https://github.com/dwolter/sparq |
| a | 2 variants over 5 domains each |
| b | Not based on abstract partition scheme (violates JEPD over \(U \times U\)) |
| c | Original work describes how to compute the composition table |
| 1 | For \(n = 1\) |
| 2 | For \(n = 2\) |
| 4 | For \(n = 4\), regular version only |

(which is an indicator for the level of granularity offered and for the average branching factor to expect in standard reasoning procedures). Additionally, we indicate which calculi are implemented in SparQ and can be obtained from there.

Representational aspects of calculi are shown in Figure 4, grouping calculi by the type of their basic entities and the key aspects captured. For all temporal and selected spatial calculi, we iconographically show one exemplary base relation to illustrate the kind of statements it permits. For a complete understanding of the respective calculus, the interested reader is referred to the original research papers cited in Table II. We sometimes use a more descriptive relation name than the original work.

Figure 5 shows the known relations between the expressivity of existing calculi. There are several ways to measure these via the existence of faithful translations, not only between base relations over the same domain, but also between representations of related domains or between representations concerned with a different domain. For example, the dependency calculus \(\text{DepCalc}\) representing dependency between points is isomorphic to \(\text{RCC-5}\), representing topology of regions. Both calculi feature the same algebraic structure representing partial-order relationships in the domain.

Since expressivity of qualitative representations solely relies on how relations are defined, there are distinct calculi that exhibit the same expressivity when Boolean combinations of constraints are considered [Wolter and Lee 2016]. These connections are particularly interesting, not only from the perspective of selecting an appropriate representation, but also in view of computational properties. For example, deciding consistency of atomic constraint networks over the point calculus \(\text{PC}\) is polynomial. Using Boolean combinations of \(\text{PC}\) relations, one can simulate Allen interval relations. Nebel and Bürckert [1995] have exploited this relationship to lift a tractable subset to Allen. In Figure 5, we give an overview of these expressivity relations. An arrow \(A \rightarrow B\) indicates that sets of constraints over relations from calculus \(A\) can be expressed by Boolean formulas of constraints over relations from calculus \(B\). For clarity, we only show direct relations, not their transitive closure. Calculi in a joint box are of equivalent expressivity. For those expressivity relations that do not follow directly from the original papers defining the respective calculi, proof sketches are provided by Wolter and Lee [2016] and in Appendix D.

Computational aspects of calculi are shown in Table IV, as far as they have already been identified. Some fairly straightforward supplements have been made while compiling this table; their proofs are in Appendix E. According to the discussion in the previous section, we give the computational complexity for deciding consistency with atomic QCSPs and the best-known complete decision procedure, which is different from a-closure in those cases where a-closure is incomplete. We only indicate the type of algorithm applicable (e.g., linear programming), not its most efficient realization. We furthermore list tractable subalgebras that cover at least all atomic relations—these subalgebras allow for reasoning in the full algebra via combining the named decision
Fig. 4. Classification of qualitative calculi by representable statements with selected example relations.
Table IV. Overview of the Known Complexity Landscape of Deciding Consistency for Existing Spatial and Temporal Calculi (Legend: See Table V)

| Abbrev. | Complexity\(^2\) (Atomic QCSP) | Decision Procedure\(^2\) (Atomic QCSP) | Largest Known Tractable Subalgebra\(^3\) | And Its Coverage\(^4\) |
|---------|-------------------------------|--------------------------------------|---------------------------------|-----------------|
| 1.2-cross | NPh [WL10] | PS | – | – |
| 9-int | recognizing string graphs [SSD03] | – | – | – |
| BA\(_m\) | O\((n^3)\) [BCC02] | AC | Strongly preconvex relations [BCF99] | – |
| CDC | O\((n^3)\) [Lig98] | AC | pre-convex relations \(\geq 25\%\) | – |
| CDR | O\((n^3)\) [LZLY10] dedicated [LZLY10] | – | – | – |
| cCDR | NP\(_c\) [LL11] dedicated [LZLY10] | – | – | – |
| CI | O\((n^3)\) [BO00] | AC | nice relations \(0.75\%\) | – |
| CYC\(_t\) | O\((n^3)\) [IC00] | strong 4-consistency \(CT\) | 0.01\% | – |
| DepCalc | O\((n^3)\) [RS05] | AC | \(\tau_{28}\) [RS05] | 87.5\% [RS05] |
| DIA | O\((n^3)\) [Ren01] | AC | \(\mathcal{H}^4\) (M) (ORD-Horn) | – |
| DRA\(_{copt}\) | NPh [WL10] | PS | – | – |
| DRA-conn | O\((n^3)\) \(< E.1\) | AC | DRA-conn | 100\% |
| ElA | P \(< E.2\) translation to INDU | – | – | – |
| GenInt | P [Con00] | AC | strongly preconvex general relations \(< 1\%\) for 3-intvls \(< E.3\) | – |
| IA | O\((n^3)\) [VKvB89] | AC | ORD-Horn \([NB95, KJJ03]\) | 10.6\% |
| INDU | P [BCL06] | translation to Horn-ORD SAT | strongly preconvex relations | 13.6\% |
| LR | NPh [WL10] | PS | – | – |
| MC-4 | P dedicated [Cri99] | M-99 | 75.0\% | – |
| OM-3D | NPh \(< E.4\) | PS | – | – |
| OPRA\(_{1}\)\(^*\) | NPh [WL10] | PS | – | – |
| PC\(_m\) | O\((n^2)\) [vB92] dedicated | PC\(_m\) | 100\% [VK86] | – |
| RCC-5\(^a\) | O\((n^4)\) [Ren02] | AC [JD97] | \(K_s^{28}\) [JD97] | 87.5\% [JD97] |
| RCC-8\(^a\) | O\((n^4)\) [Ren02] | AC [Ren02] | \(\mathcal{H}_s\) [Ren99] | 62.6\% [Ren99] |
| RCD | O\((n^3)\) [NMSC13] | transit. to IA; AC | convex relations | \(< 0.001\%\) |
| STAR\(_m\) | P [LRW13] | LP | convex relations \(< E.5\) | \(m = 4: < 1\%\) |
| STAR\(_m\)\(^b\) | O\((n^4)\) [RM04] | AC | convex relations | \(m = 3: 28\%\) |
| STAR\(_m\)\(^c\) | O\((n^4)\) [RM04] | – | 4-consistency | \(m = 4: 12.5\%\) \(m = 8: < 1\%\) |
| SV\(_m\) | NP\(_c\) [LRW13] | LP with search | – | – |
| TPCC | NPh [WL10] | PS | – | – |

\(^1\) Complexity of deciding consistency (atomic relations plus universal relation).
\(^2\) Best-known algorithm.
\(^3\) Name of largest known tractable subalgebra that includes all base relations (LKTS).
\(^4\) Percentage of LKTS compared to the complete algebra.
\(^a\) For unconstrained regions; connectedness constraints can increase complexity up to PSpace [KPWZ10].
\(^b\) For \(m < 3\).
\(^c\) For \(m \geq 3\).
Table V. Legend for Table IV

| AC    | Algebraic closure                        |
|-------|------------------------------------------|
| ACS   | Algebraic closure plus search            |
| PS    | (Multivariate) polynomial systems solving [Basu et al. 2006] |
| LP    | Reducible to linear programming and thus polynomial |
| NPC; NPh | NP-complete; NP-hard (NP-membership unknown) |
| P; PSpace | In polynomial time; in polynomial space |

[BCC02] Balbiani et al. [2002] [LZLY10] Liu et al. [2010]
[BCF99] Balbiani et al. [1999] [NB95] Nebel and Bürckert [1995]
[BCL06] Balbiani et al. [2006] [NMSC13] Navarrete et al. [2013]
[BOO00] Balbiani and Osmani [2000] [Ren99] Renz [1999]
[Con00] Condotta [2000] [Ren01] Renz [2001]
[Cri99] Cristani [1999] [Ren02] Renz [2002]
[GPP95] Grigni et al. [1995] [RM04] Renz and Mitra [2004]
[IC00] Isli and Cohn [2000] [RS05] Ragni and Scivos [2005]
[JD97] Jonsson and Drakensen [1997] [SSD03] Schaefer et al. [2003]
[KJJ03] Krokhin et al. [2003] [vB92] van Beek [1992]
[KPWW10] Kontchakov et al. [2010] [VK86] Vilain and Kautz [1986]
[Lig98] Ligozat [1998] [VKB99] Vilain et al. [1990]
[LL11] Liu and Li [2011] [WL10] Wolter and Lee [2010]
[LRW13] Lee et al. [2013]

Fig. 5. Expressivity relations between calculi.

procedure with a search for a refinement. The complexity is given as “P” (in polynomial time), “NPC” (NP-complete), and “NPh” (NP-hard, NP-membership unknown).

4. ALGEBRAIC PROPERTIES OF SPATIAL AND TEMPORAL CALCULI

Algebraic properties have been recognized as a formal tool for measuring the information preservation properties of a calculus and for providing the theoretical underpinnings for vital optimizations to reasoning procedures [Isli and Cohn 2000; Ligozat and Renz 2004; Düntsch 2005; Dylla et al. 2013].

To start with information preservation, it is important to distinguish two sources for a loss of information: one is qualitative abstraction, which maps the perceived,
continuous domain to a symbolic, discrete representation using $n$-ary domain relations and operations on them (such as composition and permutation operations). The loss of information associated with this mapping is mostly intended. To understand the other, we recall that a qualitative calculus consists of symbolic relations and operations, representing the domain relations and operations. While the domain operations are known to satisfy strong algebraic properties, those do not necessarily carry over to the symbolic operations—for example, if the operation $\cdot_{hm}$ representing homing (Section 3.1) is only abstract or weak, then there will be symbolic relations $r$ with $(r_{hm})_{hm} \neq r$, although, at the domain level, $(R_{hm})_{hm} = R$ holds for any $n$-ary relation $R$, including the interpretation $\varphi(r)$ of $r$. This loss of information indicates an unintended structural misalignment between the domain level and the symbolic level. Having its roots in the abstraction step, where the set of domain relations and operations is determined, the information loss becomes noticeable only with the symbolic representation.

If we want to measure how well the symbolic operations in a calculus preserve information, we can compare their algebraic properties with those of their domain-level counterparts. If they share all algebraic properties, this indicates that they maximally preserve information. In addition, algebraic properties seem to supply a finer-grained measure than the mere distinction between abstract, weak, and strong operations: there are 14 axioms for binary relation algebras and variants, each containing two inclusions or implications that may or may not hold independently.

Several algebraic properties can be exploited to justify and implement optimizations in constraint reasoners. For example, associativity of the composition operation $\odot$ for binary symbolic relations ensures that, if the reasoner encounters a path $ArBsClD$ of length 3, then the relationship between $A$ and $D$ can be computed “from left to right.” Without associativity, it may be necessary to compute $(r \odot s) \odot t$ as well as $r \odot (s \odot t)$.

In order to study the algebraic properties of spatial and temporal calculi, the classical notion of a relation algebra (RA) [Maddux 2006] plays a central role [Isli and Cohn 2000; Ligozat and Renz 2004; Dünstsch 2005; Mossakowski 2007]. The axioms in the definition of an RA reflect the algebraic properties of the relevant operations on binary domain relations—the operations are union, intersection, complement, converse, and binary compositions; the properties include commutativity, several variants of associativity, and distributivity. These properties have been postulated for binary calculi [Ligozat and Renz 2004; Dünstsch 2005], but it has been shown that not all existing calculi satisfy these strong properties [Mossakowski 2007]. It is the main aim of this subsection to study the algebraic properties of existing binary calculi and derive from the results a taxonomy of calculus algebras.

Unfortunately, it is far from straightforward to extend this study to arity 3 or higher: while algebraic properties of ternary and $n$-ary calculi have been studied [Isli and Cohn 2000; Scivos and Nebel 2005; Condotta et al. 2006], we are aware of only one axiomatization for a ternary RA [Isli and Cohn 2000], based on one particular choice of permutation (homing and shortcut) and composition (the binary variant in Equation (4)). However, existing calculi are based on different choices of these operations, and each choice comes with different algebraic properties at the domain level, for example:

—Not all permutations are involutive: for example, in the ternary case, we do not have $(R_{sc})_{sc} = R$ for all domain relations $R$, but rather $((R_{sc})_{sc})_{sc} = R$.
—Each variant of the composition operation has its own neutral element, that is, a relation $E$ such that $R \odot E = E \odot R = R$ for all relations $R$: for example, in the ternary case, $3 \odot 3$ (Section 3.1) has $id_{[1,2,3]}^3$ as the neutral element while $\odot_{FZ}$ has $id_{[1,2]}^3$.
—Some variants of the composition operation have stronger properties than others: for example, $3 \odot 3$ is associative while $\odot_{FZ}$ is not.
Establishing a unifying algebraic framework for \( n \)-ary qualitative calculi and determining the algebraic properties of existing calculi would require a whole new research program. In the remainder of this section, we will therefore restrict our attention to the binary case.

### 4.1. The Notion of a Relation Algebra

The notion of an (abstract) RA is defined in Maddux [2006] and makes use of the axioms listed in Table VI.

**Definition 4.1.** Let \( \mathcal{R} \) be a set of relation symbols containing \( \text{id} \) and \( 1 \) (the symbols for the identity and universal relation), and let \( \cup, \cdot \) be binary and \( \circlearrowleft, \circlearrowright \) unary operations on \( \mathcal{R} \). The tuple \((\mathcal{R}, \cup, \circlearrowleft, 1, \cdot, \circlearrowright, \text{id})\) is a

—nonassociative relation algebra (NA) if it satisfies Axioms \( R_1 - R_3, R_5 - R_{10} \);
—semiassociative relation algebra (SA) if it is an NA and satisfies Axiom \( S \);
—weakly associative relation algebra (WA) if it is an NA and satisfies \( W \); and
—relation algebra (RA) if it satisfies \( R_1 - R_{10} \),

for all \( r, s, t \in \mathcal{R} \).

Clearly, every RA is a WA; every WA is an SA; every SA is an NA.

In the literature, a different axiomatization is sometimes used, for example, in Ligozat and Renz [2004]. The most prominent difference is that \( R_{10} \) is replaced by \( PL \), “a more intuitive and useful form, known as the Peircean law or De Morgan’s Theorem K” [Hirsch and Hodkinson 2002]. It is shown in Hirsch and Hodkinson [2002, Section 3.3.2] that, given \( R_1 - R_3, R_5, R_7 - R_9 \), the axioms \( R_{10} \) and \( PL \) are equivalent. The implication \( PL \Rightarrow R_{10} \) does not need \( R_5 \) and \( R_8 \).

All axioms except \( PL \) can be weakened to only one of two inclusions, which we denote by a superscript \( \supseteq \) or \( \subseteq \). For example, \( R_7^\supseteq \) denotes \( (r)^{\supseteq} \geq r \). Likewise, we use \( PL \Rightarrow \) and \( PL \Leftarrow \). Furthermore, Table VI contains the redundant axiom \( R_{6l} \) because it may be satisfied when some of the other axioms are violated. It is straightforward to establish that \( R_6 \) and \( R_{6l} \) are equivalent given \( R_7 \) and \( R_9 \).

Thanks to Definition 3.3, certain axioms are satisfied by every calculus:  

**Fact 4.2.** Every qualitative calculus (Definition 3.3) satisfies \( R_1 - R_9, R_5, R_7^\supseteq, R_8, W^\supseteq, S^\supseteq \) for all (atomic and composite) relations. This axiom set is maximal: each of the remaining axioms in Table VI is not satisfied by some qualitative calculus.

### 4.2. Discussion of the Axioms

We will now discuss the relevance of the aforementioned axioms for spatial and temporal representation and reasoning. Due to Fact 4.2, we only need to consider axioms \( R_4, R_6, R_7, R_9, R_{10} \) (or \( PL \)) and their weakenings \( R_{6l}, S, W \).

**R_4** (and \( S, W \)). Axiom \( R_4 \) is helpful for modeling since it allows parentheses in chains of compositions to be omitted. For example, consider the following statement in natural language about the relative length and location of two intervals A and D. *Interval A is before some equally long interval that is contained in some longer interval that meets the shorter interval D.* This statement is just a conjunction of relations between A and the unnamed intermediary intervals B, C, and D. Although it intuitively does not matter whether we give priority to the composition of the relations between A, B and B, C or to the composition of the relations between B, C and C, D, there are calculi such as INDU that do not satisfy Axiom \( R_4 \)—then the example statement needs to be interpreted as a Boolean formula consisting of a conjunction over both alternatives.
Table VI. Axioms for Relation Algebras and Weaker Variants [Maddux 2006]

| Axiom | Description |
|-------|-------------|
| R₁    | \( r \cup s = s \cup r \) \text{ U-commutativity} |
| R₂    | \( r \cup (s \cup t) = (r \cup s) \cup t \) \text{ U-associativity} |
| R₃    | \( F \cup s \cup F \cup s = r \) \text{ Huntington's axiom} |
| R₄    | \( r \circ (s \circ t) = (r \circ s) \circ t \) \text{ o-associativity} |
| R₅    | \( (r \cup s) \circ t = (r \circ t) \cup (s \circ t) \) \text{ o-distributivity} |
| R₆    | \( r \circ id = r \) \text{ identity law} |
| R₇    | \( (r')^" = r \) \text{ "-involution} |
| R₈    | \( (r \cup s)^" = r^" \cup s^" \) \text{ "-distributivity} |
| R₉    | \( (r \circ s)^" = s^" \circ r^" \) \text{ "-involutive distributivity} |
| R₁₀   | \( r^\circ s \circ s \cup \bar{s} = \bar{s} \) \text{ Tarski/de Morgan axiom} |
| W     | \( (r \cap id) \circ 1 \circ 1 = (r \cap id) \circ 1 \) \text{ weak o-associativity} |
| S     | \( (r \circ 1) \circ 1 = r \circ 1 \) \text{ o semi-associativity} |
| R₆₁   | \( id \circ r = r \) \text{ left-identity law} |
| PL    | \( (r \circ s) \cap t^\circ = \emptyset \iff (s \circ t) \cap r^\circ = \emptyset \) \text{ Peircean law} |

We note that violation of \( R₄ \) is independent of composition not being strong, as shown in Section 4.4. The presence of strong composition, however, implies \( R₄ \) since composition of binary domain relations over \( \mathcal{U} \) is associative:

**FACT 4.3.** Every qualitative calculus where composition is strong satisfies \( R₄ \).

Furthermore, already a weakening \( R₄^\circ \) or \( R₄^≤ \) is useful for optimizing reasoning algorithms, allowing the “finer” composition—say, \( r \circ (s \circ t) \) in case of \( R₄^≤ \)—to be computed when a chain of compositions needs to be evaluated.

**R₆ and R₆₁.** Presence of an \( id \) relation allows the standard reduction from the correspondence problem to satisfiability: to test whether a constraint system admits the equality of two variables \( x, y \), one can add an \( id \)-constraint between \( x, y \) and test the extended system for satisfiability.

**R₇ and R₉.** These axioms allow for certain optimizations in symbolic reasoning, in particular algebraic closure. If a relation \( r \) satisfies \( R₇ \), then reasoning systems do not need to store both constraints \( A \circ B \) and \( B \circ A \), since \( r' \) can be reconstructed as \( r^\circ \) if needed. Similarly, \( R₉ \) grants that, when enforcing algebraic closure by using Equation (16) to refine constraints between variable \( A \) and \( B \), it is sufficient to compute composition once and, after applying the converse, reuse it to refine the constraint between \( B \) and \( A \) too. Current reasoning algorithms and their implementations use the described optimizations; they produce incorrect results for calculi violating \( R₇ \) or \( R₉ \).

**R₁₀ and PL.** These axioms reflect that the relation symbols of a calculus indeed represent binary domain relations (i.e., pairs of elements of a universe). This can be explained from two different points of view.

1. If binary domain relations are considered as sets, \( R₁₀ \) is equivalent to \( r^\circ \circ \circ s \subseteq \bar{s} \).
   If we further assume the usual set-theoretic interpretation of the composition of two domain relations, the previous inclusion reads as: *For any \( X, Y \), if \( Z \circ r \) for some \( Z \) and, \( Z \circ U \) implies not \( U \circ s \) \( Y \) for any \( U \), then not \( X \circ s \) \( X \). This is certainly true because \( X \) is one such \( U \).*
2. Under the same assumptions, each side of \( PL \) says (in a different order) that there can be no triangle \( X \circ Y, Y \circ s \) \( Z, Z \circ t \) \( X \). The equality then means that the
“reading direction” does not matter (see also Düntsch [2005]). This allows for reducing nondeterminism in the a-closure procedure, as well as for efficient refinement and enumeration of consistent scenarios.

4.3. Prerequisites for Being a Relation Algebra

The following correspondence between properties of a calculus and notions of a relation algebra is due to Ligozat and Renz [2004]: every calculus $C$ based on a partition scheme is an NA. If, in addition, the interpretations of the relation symbols are serial base relations, then $C$ is an SA. Furthermore, $R_7$ is equivalent to the requirement that the converse operation is strong. This is captured by the following lemma.

**Lemma 4.4.** Let $C = (\text{Rel}, \text{Int}, \cdot, \diamond)$ be a qualitative calculus. Then the following properties are equivalent:

1. $C$ has a strong converse.
2. Axiom $R_7$ is satisfied for all relation symbols $r \in \text{Rel}$.
3. Axiom $R_7$ is satisfied for all composite relations $R \subseteq \text{Rel}$.

**Proof.** Items (2) and (3) are equivalent due to distributivity of $\cdot$ over $\cup$, which is introduced with the cases for composite relations in Definition 3.3.

For “(1) $\implies$ (2),” the following chain of equalities, for any $r \in \text{Rel}$, is due to $C$ having a strong converse: $\varphi(r^{-}) = \varphi(r)^\cdot = \varphi(r)^{-} = \varphi(r)$. Since $\text{Rel}$ is based on JEPD relations and $\varphi$ is injective, this implies that $r^{-} = r$.

For “(2) $\implies$ (1),” we show the contrapositive. Assume that $C$ does not have a strong converse. Then $\varphi(r^{-}) \supseteq \varphi(r)^\cdot$, for some $r \in \text{Rel}$; hence, $\varphi(r)^{-} \supseteq \varphi(r)^{-}$. We can now modify the previous chain of equalities replacing the first two equalities with inequalities, the first of which is due to Requirement (6) in the definition of the converse (Definition 3.3): $\varphi(r^{-}) \supseteq \varphi(r)^{-} \supseteq \varphi(r)^{-} = \varphi(r)$. Since $\varphi(r^{-}) \neq \varphi(r)$, we have that $r^{-} \neq r$. □

4.4. Algebraic Properties of Existing Spatial and Temporal Calculi

We study the algebraic properties of individual calculi, aiming to find those that are abstract relation algebras and identifying relevant weaker algebraic properties. We have analyzed the calculi listed in Table II, restricting our selection to the 31 calculi with (1) binary relations, because the notion of a relation algebra is best understood for binary relations, and (b) available SparQ implementations (marked $\S$).

We have written a CASL specification of the axioms listed in Table VI along with weakenings thereof. These have been used with Hets to determine congruence of calculus and axioms. Additionally, SparQ and its built-in analysis tools have been employed to double-check results. Due to Fact 4.2, it suffices to test Axioms $R_4$, $R_6$, $R_7$, $R_9$, and $R_{10}$ (or PL) and, if necessary, the weakenings $S$, $W$, and $R_{6l}$.

Figure 6 shows the results of our tests; for further details, see Appendix G. Figure 6 arranges the analyzed calculi as a hierarchy, with the strongest notion (relation algebra) at the top and the weakest (weakly associative Boolean algebra) at the bottom. Arrows represent the is-a relation; that is, every relation algebra (RA) is an “RA minus id law” as well as a semiassociative RA and a weakly associative Boolean algebra.

With the exceptions of RCD, CCDR, and all QTC variants, all tested calculi are at least semiassociative relation algebras; most of them are even relation algebras. Hence, only these calculi enjoy all advantages for representation and reasoning optimizations discussed in Section 4.2. For other groups of calculi, special care in implementations of

---

3For the parameterized calculi DRA, OPRA, and QTC, we count every variant separately.
reasoning procedures needs to be taken. In Section 4.5, we present a revised algorithm to compute algebraic closure that respects all eventualities.

The three groups of calculi that are SAs but not RAs are the Dipole Calculus variant \( \text{DRA}_f \) (refined \( \text{DRA}_{fp} \) and coarsened \( \text{DRA-conn} \) are even RAs!), as well as \( \text{INDU} \) and \( \text{OPRA}_m \) for at least \( m = 1, \ldots, 8 \). These calculi do not even satisfy one of the inclusions \( R_f \supseteq 4 \) and \( R_f \subseteq 4 \), which implies that the reasoning optimizations described in Section 4.2 for Axiom \( R_4 \) cannot be applied. As a side note, our observations suggest that the meaning of the letter combination “RA” in the abbreviations “DRA” and “OPRA” should stand for “Reasoning Algebra,” not for “Relation Algebra.”

In principle, one cannot completely rule out that the violations of associativity are due to errors in the published operation tables or in the experimental setup. This applies to nonviolations too, but systematic nonviolations are less likely to be caused by errors than sporadic violations. In the case of \( \text{DRA}_f \), \( \text{INDU} \), and \( \text{OPRA}_m \), \( m = 1, \ldots, 8 \), the relatively high percentage of violations seems to rule out implementation errors. However, to be certain that these calculi indeed violate \( R_4 \), one has to find counterexamples and verify them using the original definition of the calculus. For \( \text{DRA}_f \) and \( \text{INDU} \), this was done by Moratz et al. [2011] and Balbiani et al. [2006]. Interestingly, the violation of associativity was attributed to the converse or composition not being strong. We remark, however, that composition cannot be the culprit as, for example, \( \text{DRA}_{fp} \) has an associative, but only weak, composition operation. While \( \text{DRA}_{fp} \) is associative due to strong composition [Moratz et al. 2011], none of the \( \text{OPRA}_m \) calculi are associative [Mossakowski and Moratz 2015].

The B-variants of QTC violate only the identity laws \( R_6, R_{6l} \). As observed in Mossakowski [2007], it is possible to add a new \( \text{id} \) relation symbol, modify the interpretation of the remaining relation symbols such that they become JEPD, and adapt the converse and composition tables accordingly, thus obtaining relation algebras.

The C-variants of QTC additionally violate \( R_4, R_9, R_{10}, \) and PL. Consequently, most of the reasoning optimizations described in Section 4.2 cannot be applied to the C-variants of QTC. The remarkably few violations of \( R_9, R_{10}, \) and PL might be due to errors in the composition table, but the nontrivial verification is part of future work.
cCDR and RCD are the only calculi with a weak converse in our tests. cCDR satisfies only W in addition to the axioms that are always satisfied by a Boolean algebra with distributivity. Hence, cCDR enjoys none of the advantages for representation and reasoning discussed before. Similarly to the C-variants of QTC, the relatively small number of violations of PL may be due to errors in the tables published. RCD additionally satisfies \( R_4 \). Since both calculi satisfy neither \( R_7 \) nor \( R_9 \), current reasoning algorithms and their implementations yield incorrect results for them, as seen in Section 4.2.

4.5. Universal Procedure for Algebraic Closure

We noted in Section 4.2 that existing descriptions and implementations of a-closure (e.g., in GQR and SparQ) use optimizations based on certain relation algebra axioms. Our analysis in Section 4.4 reveals that there are calculi that violate some of these axioms, for example, \( R_9 \); hence, those optimizations lead to incorrect results. In Algorithm 1, we present a universal algorithm that computes a-closure correctly for all calculi and uses optimizations only when they are justified. Its input is a graph \((V, C)\) representing a constraint network, and \( C_{i,j} \) denotes the relation between the \( i \)th and \( j \)th node \((r_{x,y} \text{ in Equation (15)})\). Its main control structure is that of the popular path consistency algorithm PC-2 [Mackworth 1977]. Algorithm 1 enforces 2- and 3-consistency and relies on its input being 1-consistent by implicitly assuming all \( C_{i,i} \) to cover identity.

Algorithm 1’s main function is \( \text{A-CLOSURE} \), which employs a queue to store constraint relations that may give rise to an application of the refinement operation according to Equation (15). The function \( \text{REVISE} \) implements Equation (15). If \( R_9 \) is violated (the converse is not distributive over composition), the refinement from \( C_{j,i} \) needs to be computed in addition to \( C_{i,j} \). In addition, both \( \text{A-CLOSURE} \) and \( \text{REVISE} \) exploit conformance of a calculus with \( R_7 \) (strong converse) to halve the space for storing the constraints. Flag \( s \) indicates whether full storage is required. If \( R_7 \) is satisfied \((s \text{ is false})\), then \( C_{i,j} \) can be obtained by computing \( C_{j,i} \); this is done in the auxiliary function \( \text{LOOKUP} \).

5. COMBINATION AND INTEGRATION

Although qualitative calculi and constraint-based reasoning are predominant features of qualitative knowledge representation languages, they are rarely used by themselves in applications. For example, many applications involve several aspects of spatial and temporal knowledge simultaneously (e.g., topology and orientation of spatial objects). Others require additional forms of symbolic reasoning, such as logical reasoning. These requirements can best be solved by combining calculi or integrating them with other symbolic formalisms. In this section, we review the interaction of qualitative calculi with other components of knowledge representation languages.

5.1. Qualitative Calculi in Constraint-Based Knowledge Representation Languages

The simplest case of a qualitative knowledge representation language is a single qualitative calculus. Sometimes further elements of constraint languages are used in addition, for example, constants and difference operators as in the case of PIDN [Pujari and Sattar 1999], or a restricted form of disjunction [Li et al. 2013].

To model several aspects of spatial and temporal knowledge and their interdependencies, combinations of calculi are studied. Wölfli and Westphal [2009] identify two general approaches to such combinations and reasoning therein: loose integration is based on the simple cross-product of the base relations plus interdependency constraints [Gerevini and Renz 2002; Westphal and Wölfli 2008]; tight integration designs a new calculus internalizing the interdependencies [Wölfli and Westphal 2009]. For example, INDU combines IA and PC\(_1\) tightly, reducing the 13 \( \times \) 3 pairs of relations to the 25
ALGORITHM 1: Universal Algebraic Closure Algorithm \(\text{A-CLOSURE}\)

1. **Function** \(\text{LOOKUP}(C, i, j, s)\):
   
   \[
   \text{if } s \lor (i < j) \text{ then} \\
   \quad \text{return } C_{i,j}^{-} \\
   \text{else} \\
   \quad \text{return } C_{j,i}^{-} 
   \]

2. **Function** \(\text{REVISE}(C, i, j, k, s)\):
   
   \[
   \text{if } ***\text{RA9 does not hold} \lor s \text{ then} \\
   \quad r' \leftarrow \text{LOOKUP}(C, j, i, s) \cap \text{LOOKUP}(C, j, k, s) \\
   \quad r' \leftarrow r' \cap r^{-} \\
   \quad \text{if } r' \neq C_{j,i}^{-} \text{ then} \\
   \quad \quad \text{assert } r' \neq \emptyset \\
   \quad \quad u \leftarrow \text{true} \\
   \quad \quad C_{j,i} \leftarrow r' \\
   \quad \text{if } r \neq C_{i,j}^{-} \text{ then} \\
   \quad \quad \text{assert } r \neq \emptyset \\
   \quad \quad u \leftarrow \text{true} \\
   \quad \quad C_{i,j} \leftarrow r \\
   \quad \text{return } (C, u) 
   \]

3. **Function** \(\text{A-CLOSURE}(V, C = \{C_{i,j} | i, j \in V\})\):
   
   \[
   \text{for } i, j \in V \text{ do} \\
   \quad C_{i,j} \leftarrow C_{i,j} \cap C_{j,i}^{-} \\
   \quad \text{if } R_7 \text{ does not hold} \text{ then} \\
   \quad \quad s \leftarrow \text{True} \\
   \quad \quad Q \leftarrow \text{queue with elements } \{(i, j) | i, j \in V\} \\
   \quad \text{else} \\
   \quad \quad s \leftarrow \text{False} \\
   \quad \quad Q \leftarrow \text{queue with elements } \{(i, j) | i, j \in V, i < j\} \\
   \while Q \text{ not empty do} \\
   \quad \text{dequeue } (i, j) \text{ from } Q \\
   \quad \text{for } k \in V, k \neq i, k \neq j \text{ do} \\
   \quad \quad (C, u) \leftarrow \text{REVISE}(C, i, k, j, s) \\
   \quad \text{if } u \text{ then} \\
   \quad \quad \text{if } s \text{ then} \\
   \quad \quad \quad \text{enqueue } (i, k) \text{ in } Q \text{ unless already in queue} \\
   \quad \quad \text{else} \quad R_7 \Rightarrow \text{ only one of } (i, k) \text{ and } (k, i) \text{ is required} \\
   \quad \quad \quad \text{enqueue } (\min\{i, k\}, \max\{i, k\}) \text{ in } Q \text{ unless already in queue} \\
   \quad \text{else} \\
   \quad \quad \text{enqueue } (k, j) \text{ in } Q \text{ unless already in queue} \\
   \quad \text{if } u \text{ then} \\
   \quad \quad \text{if } s \text{ then} \\
   \quad \quad \quad \text{enqueue } (k, j) \text{ in } Q \text{ unless already in queue} \\
   \quad \quad \text{else} \quad R_7 \Rightarrow \text{ only one of } (i, k) \text{ and } (k, i) \text{ is required} \\
   \quad \quad \quad \text{enqueue } (\min\{k, j\}, \max\{k, j\}) \text{ in } Q \text{ unless already in queue} \\
   \quad \text{return } C 
   \]
semantically possible. A combination of RCC-8 with IA was introduced in Gerevini and Nebel [2002]; several combinations of RCC-8 with direction calculi were analyzed [Liu et al. 2009; Cohn et al. 2014]. In general, combinations do not inherit algebraic and reasoning properties from their constituent calculi (cf. Figure 5 and 6 for INDU).

Hernández [1994] describes the use of topological and orientation relations, which does not result in a dedicated calculus, but reveals the effects of constraining one aspect on reasoning in the other.

Alternative ways to solve the combination problem include formalizing the domain and qualitative relations in an abstract logic—which typically are computationally more expensive—or applying the efficient paradigm of linear programming to qualitative calculi over real-valued domains [Kreutzmann and Wolter 2014].

5.2. Qualitative Relations and Classical Logics: Spatial Logics

There are several developments to enrich qualitative representation with concepts found in classical logics or to combine the two strands. Domain representations purely based on qualitative relations can be viewed as quantifier-free formulae with variables ranging over a certain spatial or temporal domain. QCSP instances can be posed as satisfiability problems of conjunctive constraint formulae with existentially quantified variables. Adopting this logic view for QCSPs leads to the field of spatial logics [Aiello et al. 2007], which is involved with combinations of qualitative calculi and logics. Already in the 1930s topological statements such as those expressible in RCC were found to constitute a fragment of the modal logic S4 plus the universal modality (S4_u), comprehensively described by Bennett [1997]. The Cartesian product of S4_u with linear temporal logic captures topological relationships changing over time [Bennett et al. 2002]. Qualitative relations and their interrelations can also be described by axiomatic systems; this approach was argued to be composed of the composition-table approach and support the construction of composition tables [Eschenbach 2001]. Axiomatic systems are given (e.g., in Eschenbach and Kulik [1997], Gotts [1996], and Hahmann and Grüninger [2011]). The field of spatial logics can thus be viewed as a continuum between purely qualitative knowledge representation languages and logics. Current work studies the computational complexity of increasing expressivity of qualitative relations, for example, by introducing Boolean expressions of spatial variables PO(A∩B,C) [Wolter and Zakharyaschev 2000], introducing a temporal modality [Kontchakov et al. 2007], or even combining spatial and temporal logics [Gabelaia et al. 2005].

5.3. Qualitative Calculi and Description Logics

Description logics (DLs) are a successful family of knowledge representation languages tailored to capturing conceptual knowledge in ontologies and reasoning over it [Baader et al. 2007]. The most prominent DL-based ontology language is the W3C standard OWL. Several approaches to combining DLs and qualitative calculi have evolved, aiming at describing spatial and temporal qualities of application domains. A principal approach developed by Lutz and Miličić [2007] allows adding qualitative calculi that satisfy certain admissibility conditions to ALC, the basic DL, incorporating spatial/temporal reasoning into a standard DL reasoning procedure. According to the authors, a practical implementation would be challenging. Stocker and Sirin [2009] describe PelletSpatial, an extension of the DL reasoner Pellet [Sirin et al. 2007], for query answering over nonspatial (DL) and spatial (RCC-8) knowledge. Batsakis and Petrakis [2011] describe SOWL, an OWL ontology capturing static, spatial, and temporal information, using a DL axiomatization of spatial relations from the calculi CDC and RCC-8. Temporal and spatial reasoning are separated (a-closure and Pellet, respectively). Ben Hmida

http://www.w3.org/TR/owl2-overview.
et al. [2012] sketch an implementation of logic programming that combines 9-int with OWL ontologies and constructive solid geometry.

5.4. Qualitative Calculi and Situation Calculus
The situation calculus is a popular framework for reasoning about action and change; runtime systems such as DTGolog [Ferrein et al. 2004] and ReadyLog [Ferrein and Lakemeyer 2008] are used in robotic applications. Qualitative relations are relevant to world modeling and underlie high-level behavior specifications [Schiffer et al. 2012].

Bhatt et al. [2006] aim at general integration of QSTR into reasoning about action and change, that is, a general domain-independent theory, in order to reason about dynamic and causal aspects of spatial change. With a naive characterization of objects based on their physical properties, they particularly investigate key aspects of a topological theory of space on the basis of RCC-8 [Bhatt and Loke 2008].

6. ALTERNATIVE APPROACHES
This section presents an overview of reasoning techniques that have also been used to address QSTR reasoning problems but are not based on QSTR techniques. Since spatial reasoning connects to fields in mathematics related to geometry or topology, there are manifold possible connections to make. In the following, we only hint at fields that have already proven to provide impulses to QSTR research.

6.1. Algebraic Topology
Fundamental concepts of algebraic topology resemble expressivity of topological QSTR calculi such as RCC-8. For example, Euler's well-known polyhedron formula “vertices - edges + faces = 2” is a representative of Euler characteristics that characterize topological invariants of a space or body. The PLCA framework [Takahashi 2012] exploits the Euler characteristics to reason about topological space by invariants.

6.2. Combinatorial Geometry
A set of Jordan curves (i.e., sets that are homeomorphic to the interval [0, 1] in the plane) induce an intersection graph. The string graph problem poses the question of whether a given graph can be an intersection graph of a set of curves in the plane. While the problem itself already is of a spatial nature, Schaefer and Štefankovič [2004] reduced reasoning about topological relations in RCC-8 about planar regions to the string graph problem and later proved the string graph problem to be NP-complete [Schaefer et al. 2003], directly contributing to QSTR research.

An alternative approach to reasoning with directional relations can be found in oriented matroid theory, which consists of several equivalent combinatorial structures such as directed graphs, point and vector configurations, pseudoline arrangements, and arrangements of hyperplanes [Björner et al. 1999]. Knuth [1992] points out the importance of oriented matroids for qualitative spatial reasoning. In the context of LR constraint networks, a connection to the oriented matroid axiomatization of so-called chirotopes leads to complexity results in QSTR [Wolter and Lee 2010; Lee 2014].

6.3. Graph Theoretical Approaches
Worboys [2013] describes topological configurations through their representation as labeled trees, called map trees. Graph edit operations on map trees can be defined to correspond to spatial change of the topological configuration, providing an efficient approach to reason about spatial change.

A different way to represent qualitative spatial change consists of describing the change on two levels of detail. Stell [2013] represents a scene of regions via a bipartite graph \((U, V, E)\), where the elements of \(U\) (\(V\)) represent regions that can be seen as
connected at a coarse level of detail (when accounting for finer details). This way it is possible to describe the splitting, connecting, and changing of distance of regions, as well as the creation, deletion, and change of size of a (part of a) region.

6.4. Logic Frameworks

Viewing vectors in a vector space as abstract arrows, Aiello and Ottens [2007] introduce a hybrid modal logic (arrow logic) for capturing mereotopological relations between sets of vectors. Inversion and composition of arrows are modeled by morphological operators such as dilation, erosion, and difference. A resolution calculus allows for automated reasoning about topological relations and relative size.

6.5. Model-Theoretic and Constraint Reasoning Methods

Qualitative constraint satisfaction problems can be reformulated as general constraint satisfaction problems. Then, the consistency problem can be tackled using model-theoretic methods [Bodirsky and Wölf 2011; Westphal 2015] or using SAT solving or datalog programs [Westphal 2015], leading to greater flexibility.

6.6. Quantitative Methods

Linear programming (LP) techniques have been used to decide constraint problems posed as linear inequalities, allowing polyhedral regions, lines, and points to be represented. LP can mix free-ranging variables with concrete values (e.g., points at known positions) and, beyond consistency checking, determine a model in polynomial time. By posing QCSP instances as LPs, constraints originating in distinct calculi can easily be mixed. While some QSTR problems can almost directly be posed as LPs [Jonsson and Bäckström 1998; Ligozat 2011; Lee et al. 2013], disjunctive LP formulae allow several QSTR calculi to be handled simultaneously [Kreutzmann and Wolter 2014]. In a similar fashion, Schockaert et al. [2011] combine qualitative and quantitative reasoning of relations about different spatial aspects by using genetic optimization. Techniques for deciding satisfiability of equations yield advancements on the inherent problem of consistency checking for directional constraints such as those present in the LR calculus, as (disjunctions of) linear equations can capture relevant geometric invariances [Lücke and Mossakowski 2010; van Delden and Mossakowski 2013].

7. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

Qualitative spatial and temporal reasoning explores potentially interesting domain conceptualizations and their computational effects. As a consequence, QSTR is connected to various research areas in and around artificial intelligence, such as knowledge representation, linguistics, and spatial cognition. Thus, QSTR plays the role of a hub for connecting symbolic techniques to real-world applications. The notion of a qualitative calculus attests to this role by representing knowledge about spatial and temporal domains as an abstract algebra that provides the semantics to knowledge representation languages. Reasoning with qualitative representations occurs in several forms, with deductive forms of inference, such as deciding consistency, being in a central position. This is captured in the qualitative constraint satisfaction problem, which is decidable for all qualitative calculi (in the strict sense of Definition 3.3), ranging from low-order polynomial time complexity to within PSPACE (cf. Table IV). With this survey, we present the first comprehensive overview of the known computational properties of all qualitative calculi proposed so far.

7.1. Beneficiaries of This Survey

This survey addresses a broad range of researchers and engineers from different research communities and application areas. We expect three groups of beneficiaries.
The first group comprises researchers and engineers who apply QSTR and build systems for their applications. Our survey provides them with a comprehensive and concise overview of the formalisms available, allowing objective design choices.

The second group consists of researchers contributing to QSTR to whom we provide revised definitions that are general enough to address all formalisms proposed so far. The overview of domain conceptualizations studied so far fosters identification of interesting new conceptualizations to be studied. Moreover, the summary of algebraic and computational properties of existing formalisms reveals open research questions: for calculi not listed in Table V, reasoning properties have still to be analyzed.

Last but not least, the third group benefiting from this presentation consists of developers of reasoning tools. In order to accrete the position of QSTR as a hub, sophisticated tools are necessary that disseminate formalisms and algorithms, linking basic research to application development. On the one hand, we provide pointers to all formalisms proposed and the decision methods necessary to perform reasoning. This also reveals commonalities between formalisms, hopefully gearing tools toward becoming universal in the sense that they allow many variants of representations to be handled. On the other hand—and related to the discrepancy between the amount of formalisms proposed and those fully analyzed discussed before—the most efficient algorithms to decide QCSP instances have often not yet been identified and solid algorithm engineering can likely yield a great leap ahead for QSTR.

### 7.2. Open Problem Areas in QSTR

**Combining Qualitative Abstractions.** Despite the work reported in Section 5.1, generally applicable methods for combining existing abstractions for different spatial and temporal aspects are missing—a potential threat to the applicability of qualitative methods. It is clearly not feasible to identify all potentially useful combinations individually: there are infinitely many abstractions that give rise to a qualitative calculus.

**Integration with Other Symbolic Methods.** In addition to the previous observation that an application may need to handle more than one calculus at the same time, expressivity provided by domain-independent knowledge representation techniques may be important too. There are first contributions (e.g., combining description logic with QSTR), but these are limited to specific combinations using specific methods. A promising approach is the integration of a variety of QSTR formalisms into a first-order framework [Bhatt et al. 2011]—the challenge being the development of efficient reasoning methods. We expect that this will result in a combination of first-order methods, constraint-solving methods, relation-algebraic methods, and specialized methods for the existential theory over the reals. See van Delden and Mossakowski [2013] for some first steps.

**Integration with Quantitative Approaches.** Qualitative approaches link metric data and symbolic reasoning, but consistent interpretation of sensor data considering its inevitable uncertainty is a recurring and challenging task. An algorithmic understanding of this problem has, to the best of our knowledge, not been developed yet. Conversely, it can also be helpful to link qualitative inference with quantitative or other kinds of constraints. As Liu and Li [2012] recently discovered, constraint-based qualitative reasoning with information partially grounded in data can differ significantly from classic qualitative reasoning and thus calls for further exploration.

**Algebras for Higher-Arity Qualitative Calculi.** Abstract algebras provide the foundations for symbolic knowledge manipulation and enable optimizations to reasoning procedures. Our study gives an extensive account of algebraic properties of existing
binary calculi, but we have also seen that it is highly nontrivial to extend this study to ternary calculi. The main problem is a missing notion of relation algebra already for ternary relations that is general enough to encompass the variety of existing calculi.

*Practical Reasoning Algorithms.* Few of the various methods required in qualitative reasoning (see Table V) have been studied rigorously in a practical context. In light of continuously growing databases, identifying best-practice algorithms, evaluating the scaling behavior, and potentially developing heuristic approximations will be crucial to foster the relevance of QSTR methods.

By completing the picture of computational complexity and identifying practical solutions to reasoning with all individual calculi, either individually or in combination with one another or even other KR techniques, it will be possible to realize truly universal QSTR tools. These tools will foster the position of QSTR as a hub, not only conceptually, but also implemented in almost all knowledge-based systems.

**ELECTRONIC APPENDIX**

The electronic appendix for this article can be accessed in the ACM Digital Library. It contains additional examples, observations, proofs, and details for Sections 3 and 4.

**ACKNOWLEDGMENTS**

This article has benefited greatly from discussions with Immo Colonius, Arne Kreutzmann, and Jan-Oliver Wallgrün and particularly from the profound and constructive comments by the anonymous reviewers. This work has been supported by the DFG-funded SFB/TR 8 “Spatial Cognition,” projects R3-[QShape], and R4-[LogoSpace]. Special thanks go to Erwin R. Catesbeiana for the provision of his sitting area.

**REFERENCES**

Marco Aiello and Brammert Ottens. 2007. The Mathematical Morpho-Logical View on Reasoning about Space. In *Proc. of the 20th International Joint Conference on Artificial Intelligence (IJCAI’07)*. Morgan Kaufmann, 205–211.

Marco Aiello, Ian E. Pratt-Hartmann, and Johan F.A.K. van Benthem (Eds.). 2007. *Handbook of Spatial Logics*. Springer.

James F. Allen. 1983. Maintaining knowledge about temporal intervals. *Commun. ACM* 26, 11 (1983), 832–843.

Nouhad Amaneddine and Jean-François Condotta. 2013. On the Minimal Labeling Problem of Temporal and Spatial Qualitative Constraints. In *Proc. of the Twenty-Sixth International Florida Artificial Intelligence Research Society Conference (FLAIRS’13)*. AAAI Press, 16–21.

Egidio Astesiano, Michel Bidoit, Hélène Kirchner, Bernd Krieg-Brückner, Peter D. Mosses, Donald Sannella, and Andrzej Tarlecki. 2002. CASL: the Common Algebraic Specification Language. *Theor. Comput. Sci.* 286, 2 (2002), 153–196.

Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider (Eds.). 2007. *The Description Logic Handbook: Theory, Implementation, and Applications* (2nd ed.). Cambridge University Press.

Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. 2002. Tractability Results in the Block Algebra. *J. Log. Comput.* 12, 5 (2002), 885–909.

Philippe Balbiani, Jean-François Condotta, and Gérard Ligozat. 2006. On the consistency problem for the INDU calculus. *J. Applied Logic* 4, 2 (2006), 119–140.

Philippe Balbiani and Jean-François Condotta. 2002. Spatial reasoning about points in a multidimensional setting. *Appl. Intell.* 17, 3 (2002), 221–238.

Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. 1998. A model for reasoning about bidimensional temporal relations. In *Proc. of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR’98)*. Morgan Kaufmann, 124–130.

Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. 1999. A tractable subclass of the block algebra: constraint propagation and preconvex relations. In *Proc. of the 9th Portuguese Conference on Artificial Intelligence (EPIA’99)* (LNCS), Vol. 1695. Springer, 75–89.
Philippine Balbiani and Aomar Osmani. 2000. A model for reasoning about topologic relations between cyclic intervals. In Proc. of Principles of Knowledge Representation and Reasoning Proceedings of the Seventh International Conference (KR’00). Morgan Kaufmann, 378–385.

Saugata Basu, Richard Pollack, and Marie-Françoise Roy. 2006. Algorithms in Real Algebraic Geometry. Springer.

Sotiris Batsakis and Euripides G. M. Petrakis. 2011. SOWL: A framework for handling spatio-temporal information in OWL 2.0. In Proc. of the 5th International Symposium on Rule-Based Programming and Applications (RuleML’11) (LNCS), Vol. 6826. Springer, 242–249.

Brandon Bennett. 1997. Logical Representations for automated reasoning about spatial relationships. Ph.D. Dissertation. The University of Leeds, School of Computer Studies, UK.

Brandon Bennett, Anthony G. Cohn, Frank Wolter, and Michael Zakharyaschev. 2002. Multi-Dimensional Modal Logic as a Framework for Spatio-Temporal Reasoning. Appl. Intell. 17, 3 (2002), 239–251.

Mehul Bhatt, Jae Hee Lee, and Carl P. L. Schultz. 2011. CLP(QS): A Declarative Spatial Reasoning Framework. In Proc. of the 10th International Conference on Spatial Information Theory (COSIT’11) (LNCS), Max J. Egenhofer et al. (Ed.), Vol. 6899. Springer, 210–230.

Mehul Bhatt and Seng W. Loke. 2008. Modelling Dynamic Spatial Systems in the Situation Calculus. Spatial Cognition & Computation 8, 1–2 (2008), 86–130.

Mehul Bhatt, J. Wenny Rahayu, and Gerald Sterling. 2006. Qualitative Spatial Reasoning with Topological Relationships in the Situation Calculus. In Proc. of the Nineteenth International Florida Artificial Intelligence Research Society Conference (FLAIRS’06). AAAI Press, 713–718.

Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. 1999. Oriented Matroids. Cambridge University Press.

Manuel Bodirsky and Stefan Wölf. 2011. RCC8 Is Polynomial on Networks of Bounded Treewidth. In Proc. of the 22nd International Joint Conference on Artificial Intelligence (IJCAI’11). AAAI Press, 756–761.

Bert Bredeweg and Peter Struss. 2004. Current Topics in Qualitative Reasoning. AI Mag. 24, 4 (2004), 13–16.

Juan Chen, Anthony G. Cohn, Dayou Liu, Shengsheng Wang, Jihong Ouyang, and Qianguan Yu. 2013. A survey of qualitative spatial representations. Knowledge Eng. Review 30, 1 (2013), 106–136.

Eliseo Clementini and Roland Billen. 2006. Modeling and Computing Ternary Projective Relations between Regions. IEEE TKDE 18, 6 (2006), 799–814.

Eliseo Clementini and Anthony G. Cohn. 2014. RCC*-9 and CBM. In Proc. of the 8th International Conference on Geographic Information Science (GIScience’14) (LNCS), Vol. 8728. Springer, 349–365.

Eliseo Clementini, Paolino Di Felice, and Peter van Oosterom. 1993. A Small Set of Formal Topological Relationships Suitable for End-User Interaction. In Proc. of Advances in Spatial Databases, Third International Symposium, (SSD’93) (LNCS), Vol. 692. Springer, 277–295.

Eliseo Clementini, Spiros Skiadopoulos, Roland Billen, and Francesco Tarquini. 2010. A Reasoning System of Ternary Projective Relations. IEEE TKDE 22, 2 (2010), 161–178.

Anthony G. Cohn and Shyamanta M. Hazarika. 2001. Qualitative spatial representation and reasoning: an overview. Fund. Inform. 46 (2001), 1–29.

Anthony G. Cohn, Brandon Bennett, John Gooday, and Nicholas Mark Gotts. 1997. Qualitative Spatial Representation and Reasoning with the Region Connection Calculus. GeoInformatica 1, 3 (1997), 275–316.

Anthony G. Cohn, Sanjiang Li, Weiming Liu, and Jochen Renz. 2014. Reasoning about Topological and Cardinal Direction Relations between 2-Dimensional Spatial Objects. J. Artif. Intell. Res. (JAIR) 51 (2014), 493–532.

Anthony G. Cohn and Jochen Renz. 2008. Qualitative Spatial Representation and Reasoning. Foundations of Artificial Intelligence 3, Handbook of Knowledge Representation and Reasoning (2008), 551–596.

Jean-François Condotta. 2000. Tractable Sets of the Generalized Interval Algebra. In Proc. of the 14th European Conference on Artificial Intelligence (ECAI’00). IOS Press, 78–82.

Jean-François Condotta, Gérard Ligozat, and Mahmoud Saade. 2006. A Generic Toolkit for n-ary Qualitative Temporal and Spatial Calculi. In Proc. of the 13th International Symposium on Temporal Representation and Reasoning (TIME’06). IEEE Computer Society, 78–86.

Matteo Cristani. 1999. The Complexity of Reasoning about Spatial Congruence. J. Artif. Intell. Res. (JAIR) 11 (1999), 361–390.

Ernest Davis. 1990. Representations of Commonsense Knowledge. Morgan Kaufmann.

Rina Dechter. 2003. Constraint processing. Morgan Kaufmann.

Matthias Delafontaine, Peter Bogaert, Anthony G. Cohn, Frank Witlox, Philippe De Maeyer, and Nico Van de Weghe. 2011. Inferring additional knowledge from QTCN relations. Inf. Sci. 181, 9 (2011), 1573–1590.

ACM Computing Surveys, Vol. 50, No. 1, Article 7, Publication date: April 2017.
Matt Duckham, Sanjiang Li, Weiming Liu, and Zhiguo Long. 2014. On Redundant Topological Constraints. In Proc. of the 14th International Conference on the Principles of Knowledge Representation and Reasoning (KR'14). AAAI Press, 618–621.

Ivo Düntsch. 2005. Relation Algebras and their Application in Temporal and Spatial Reasoning. Artif. Intell. Rev. 23, 4 (2005), 315–357.

Frank Dylla and Jae Hee Lee. 2010. A Combined Calculus on Orientation with Composition Based on Geometric Properties. In Proc. of the 19th European Conference on Artificial Intelligence (ECAI’10) (FAIA), Vol. 215. IOS Press, 1087–1088.

Frank Dylla, Till Mossakowski, Thomas Schneider, and Diedrich Wolter. 2013. Algebraic Properties of Qualitative Spatio-temporal Calculi. In Proc. of the 11th International Conference on Spatial Information Theory (COSIT’13) (LNCS), Vol. 8116. Springer, 516–536.

Frank Dylla and Jan Oliver Wallgrun. 2007. Qualitative Spatial Reasoning with Conceptual Neighborhoods for Agent Control. Journal of Intelligent & Robotic Systems 48, 1 (2007), 55–78.

Max J. Egenhofer. 1991. Reasoning about Binary Topological Relations. In Proc. of Advances in Spatial Databases, Second International Symposium (SSD’91) (LNCS 525). 143–160.

Max J. Egenhofer and Jayant Sharma. 1993. Assessing the Consistency of Complete and Incomplete Topological Information. Geographical Systems 1, 1 (1993), 47–68.

Carola Eschenbach. 2001. Viewing composition tables as axiomatic systems. In Proc. of the 2nd International Conference on Formal Ontology in Information Systems (FOIS’01). ACM Press, 93–104.

Carola Eschenbach and Lars Kulik. 1997. An Axiomatic Approach to the Spatial Relations Underlying Left-Right and in Front of-Behind. In Proc. of the 21st Annual German Conference on Artificial Intelligence (KI’97) (LNCS), Vol. 1303. Springer, 207–218.

Alexander Ferrein, Christian Fritz, and Gerhard Lakemeyer. 2004. On-Line Decision-Theoretic Golog for Unpredictable Domains. In Proc. of the 27th Annual German Conference on Artificial Intelligence (KI’04) (LNCS), Vol. 3238. Springer, 322–336.

Alexander Ferrein and Gerhard Lakemeyer. 2008. Logic-based Robot Control in Highly Dynamic Domains. Robotics and Autonomous Systems 56, 11 (2008), 980–991.

Kenneth D. Forbus, Jeffrey M. Usher, and Vernell Chapman. 2004. Qualitative spatial reasoning about sketch maps. AI Mag. 25, 3 (2004), 61–72.

Andrew U. Frank. 1991. Qualitative Spatial Reasoning with Cardinal Directions. In Proc. of the 7th Austrian Conference on Artificial Intelligence (ÖGAI’91). 157–167.

Andrew U. Frank. 1992. Qualitative spatial reasoning about distances and directions in geographic space. J. Vis. Lang. Comput. 3, 4 (1992), 343–371.

Christian Freksa. 1992a. Temporal Reasoning Based on Semi-Intervals. Artif. Intell. 54, 1 (1992), 199–227.

Christian Freksa. 1992b. Using Orientation Information for Qualitative Spatial Reasoning. In Spatio-Temporal Reasoning (LNCS), Vol. 639. Springer, 162–178.

Christian Freksa and Kai Zimmermann. 1992. On the utilization of spatial structures for cognitively plausible and efficient reasoning. In Proc. of IEEE International Conference on Systems, Man and Cybernetics (ICSMC’92). IEEE, 261–266.

David Gabelaia, Roman Kontchakov, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. 2005. Combining Spatial and Temporal Logics: Expressiveness vs. Complexity. J. Artif. Intell. Res. (JAIR) 23 (2005), 167–243.

Antony P. Galton. 1994. Lines of Sight. In Proc. of the Seventh Annual Irish Conference on AI and Cognitive Science (AICS’94). 103–113.

Alfonso Gerevini and Bernhard Nebel. 2002. Qualitative Spatio-Temporal Reasoning with RCC-8 and Allen’s Interval Calculus: Computational Complexity. In Proc. of the 15th European Conference on Artificial Intelligence (ECAI’02). IOS Press, 312–316.

Alfonso Gerevini and Jochen Renz. 2002. Combining topological and size information for spatial reasoning. Artif. Intell. 137, 1–2 (2002), 1–42.

Francisco Jose Glez-Cabrera, Jose Vicente Alvarez-Bravo, and Fernando Diaz. 2013. QRPC: A new qualitative model for representing motion patterns. Expert Syst. Appl. 40, 11 (2013), 4547–4561.

Björn Gottfried. 2004. Reasoning about intervals in two dimensions. In Proc. of the IEEE International Conference on Systems, Man & Cybernetics (ICSMC’04). IEEE, 5324–5332.

Nicholas Mark Gotts. 1996. Formalizing Commonsense Topology: The INCH Calculus. In Proc. of the 4th International Symposium on Artificial Intelligence and Mathematics (ISAIM’96). 72–75.

Michelangelo Grigni, Dimitris Papadias, and Christos H. Papadimitriou. 1995. Topological Inference. In Proc. of the Fourteenth International Joint Conference on Artificial Intelligence (IJCAI’95). 901–907.
A Survey of Qualitative Spatial and Temporal Calculi

Volker Haarslev, Kay Hidde, Ralf Möller, and Michael Wessel. 2012. The RacerPro knowledge representation and reasoning system. J. Web Sem. 3, 3 (2012), 267–277.

Torsten Hahmann and Michael Grüninger. 2011. Multidimensional Mereotopology with Betweenness. In Proc. of the 22nd International Joint Conference on Artificial Intelligence (IJCAI’11). AAAI Press, 906–911.

Daniel Hernández. 1994. Qualitative representation of spatial knowledge. LNCS, Vol. 804. Springer, Berlin.

Robin Hirsch and Ian Hodkinson. 2002. Relation algebras by games. Studies in logic and the foundations of mathematics, Vol. 147. Elsevier.

Helmi Ben Hmida, Frank Boocks, Christophe Cruz, and Christophe Nicolle. 2012. From quantitative spatial operator to qualitative spatial relation using Constructive Solid Geometry, logic rules and optimized 9-1M model: A semantic based approach. In Proc. of IEEE International Conference on Computer Science and Automation Engineering (CSAE’12), Vol. 3. IEEE, 453–458.

Armen Inants and Jérôme Euzenat. 2015. An Algebra of Qualitative Taxonomical Relations for Ontology Alignments. In Proc. of the 14th International Semantic Web (ISWC’15) (LNCS), Vol. 9366. Springer, 253–268.

Amar Isli and Anthony G. Cohn. 2000. A new approach to cyclic ordering of 2D orientations using ternary relation algebras. Artif. Intell. 122, 1–2 (2000), 137–187.

Peter Jonsson and Christer Bäckström. 1998. A unifying approach to temporal constraint reasoning. Artif. Intell. 102, 1 (1998), 143–155.

Peter Jonsson and Thomas Drakengren. 1997. A Complete Classification of Tractability in RCC-5. J. Artif. Intell. Res. (JAIR) 6 (1997), 211–221.

Markus Knauff, Gerhard Strube, Corinne Jola, Reinhold Rauh, and Christoph Schlieder. 2004. The Psychological Validity of Qualitative Spatial Reasoning in One Dimension. Spatial Cognition & Computation 4, 2 (2004), 167–188.

Amar Isli and Anthony G. Cohn. 2000. A new approach to cyclic ordering of 2D orientations using ternary relation algebras. Artif. Intell. 122, 1–2 (2000), 137–187.

Peter Jonsson and Christer Bäckström. 1998. A unifying approach to temporal constraint reasoning. Artif. Intell. 102, 1 (1998), 143–155.

Peter Jonsson and Thomas Drakengren. 1997. A Complete Classification of Tractability in RCC-5. J. Artif. Intell. Res. (JAIR) 6 (1997), 211–221.

Markus Knauff, Gerhard Strube, Corinne Jola, Reinhold Rauh, and Christoph Schlieder. 2004. The Psychological Validity of Qualitative Spatial Reasoning in One Dimension. Spatial Cognition & Computation 4, 2 (2004), 167–188.

Donald E. Knuth. 1992. Axioms and Hulls. LNCS, Vol. 606. Springer.

Christian Köhler. 2002. The Occlusion Calculus. In Proc. of Workshop on Cognitive Vision. Zurich, Switzerland.

Roman Kontchakov, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. 2007. Spatial Logic + Temporal Logic = ? In Handbook of Spatial Logics. Springer, 497–564.

Roman Kontchakov, Ian Pratt-Hartmann, Frank Wolter, and Michael Zakharyaschev. 2010. Spatial logics with connectedness predicates. Log. Meth. Comp. Sci. 6, 3, Article 7 (2010), 43 pages.

Arne Kreutzmann and Diedrich Wolter. 2014. Qualitative Spatial and Temporal Reasoning with AND/OR Linear Programming. In Proc. of the 21st European Conference on Artificial Intelligence (ECAI’14) (FAIA), Vol. 263. IOS Press, 495–500.

Andrei Krokhin, Peter Jeavons, and Peter Jonsson. 2003. Reasoning about temporal relations: The tractable subalgebras of Allen’s interval algebra. J. ACM 50, 5 (2003), 591–640.

Benjamin Kuipers. 1978. Modeling Spatial Knowledge. Cogn. Sci. 2, 2 (1978), 129–153.

Yohei Kurata. 2010. 9+ intersection calculi for spatial reasoning on the topological relations between heterogeneous objects. In Proc. of the 18th ACM SIGSPATIAL International Symposium on Advances in Geographic Information Systems, (ACM-GIS’10). ACM Press, 390–393.

Yohei Kurata and Hui Shi. 2008. Interpreting Motion Expressions in Route Instructions Using Two Projection-Based Spatial Models. In Proc. of the 31st Annual German Conference on AI (KI’08) (LNCS), Vol. 5243. Springer, 258–266.

Jae Hee Lee. 2014. The Complexity of Reasoning with Relative Directions. In Proc. of the 21st European Conference on Artificial Intelligence (ECAI’14) (FAIA), Vol. 263. IOS Press, 507–512.

Jae Hee Lee, Jochen Renz, and Diedrich Wolter. 2013. StarVars—Effective Reasoning about Relative Directions. In Proc. of the 23rd International Joint Conference on Artificial Intelligence (IJCAI’13). AAAI Press, 976–982.

Stephen C. Levinson. 2003. Space in Language and Cognition: Explorations in Cognitive Diversity. Cambridge University Press, Cambridge.

Sanjiang Li, Weiming Liu, and Shengsheng Wang. 2013. Qualitative constraint satisfaction problems: An extended framework with landmarks. Artif. Intell. 201 (2013), 32–58.

Gérard Ligozat. 1993. Qualitative Triangulation for Spatial Reasoning. In Proc. of the International Conference on Spatial Information Theory (COSIT’93). Springer, 54–68.

Gérard Ligozat. 1998. Reasoning about Cardinal Directions. J. Vis. Lang. Comput. 9, 1 (1998), 23–44.

Gérard Ligozat. 2005. Categorical Methods in Qualitative Reasoning: The Case for Weak Representations. In Proc. of the International Conference on Spatial Information Theory (COSIT’05) (LNCS), Vol. 3693. Springer, 265–282.
Algebraische Eigenschaften qualitativer Constraint-Kalküle

Isabel Navarrete, Antonio Morales, Guido Sciavicco, and M. Antonia Cardenas-Viedma. 2013. Spatial reasoning with rectangular cardinal relations – The convex tractable subalgebra. Ann. Math. Artif. Intell. 67, 1 (2013), 31–70.

Bernhard Nebel and Hans-Jürgen Bürckert. 1995. Reasoning about temporal relations: A maximal tractable subclass of Allen’s interval algebra. J. ACM 42, 1 (1995), 43–66.

Bernhard Nebel and Alexander Scivos. 2002. Formal Properties of Constraint Calculi for Qualitative Spatial Reasoning. KI 16, 4 (2002), 14–18.

Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. 2002. Isabelle/HOL, A Proof Assistant for Higher-Order Logic. LNCS, Vol. 2283. Springer.

Jihong OuYang, Qian Fu, and Dayou Liu. 2007. A Model for Representing Topological Relations Between Simple Concave Regions. In Proc. of the 7th International Conference on Computational Science (ICCS’07) (LNCS), Vol. 4487. Springer, 160–167.

Julio Pacheco, M. Teresa Escrig, and Francisco Toledo. 2001. Representing and Reasoning on Three-Dimensional Qualitative Orientation Point Objects. In Proc. of the 10th Portuguese Conference on Artificial Intelligence (EPIA’01) (LNCS), Vol. 2258. Springer, 298–305.

Arun K. Pujari, G. Vijaya Kumari, and Abdul Sattar. 1999. INDU: An Interval and Duration Network. In Proc. of the 12th Australian Joint Conference on Artificial Intelligence (AI’99) (LNCS), Vol. 1747. Springer, 291–303.

Algebraische Eigenschaften qualitativer Constraint-Kalküle

Isabel Navarrete, Antonio Morales, Guido Sciavicco, and M. Antonia Cardenas-Viedma. 2013. Spatial reasoning with rectangular cardinal relations – The convex tractable subalgebra. Ann. Math. Artif. Intell. 67, 1 (2013), 31–70.

Bernhard Nebel and Hans-Jürgen Bürckert. 1995. Reasoning about temporal relations: A maximal tractable subclass of Allen’s interval algebra. J. ACM 42, 1 (1995), 43–66.

Bernhard Nebel and Alexander Scivos. 2002. Formal Properties of Constraint Calculi for Qualitative Spatial Reasoning. KI 16, 4 (2002), 14–18.

Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. 2002. Isabelle/HOL, A Proof Assistant for Higher-Order Logic. LNCS, Vol. 2283. Springer.

Jihong OuYang, Qian Fu, and Dayou Liu. 2007. A Model for Representing Topological Relations Between Simple Concave Regions. In Proc. of the 7th International Conference on Computational Science (ICCS’07) (LNCS), Vol. 4487. Springer, 160–167.

Julio Pacheco, M. Teresa Escrig, and Francisco Toledo. 2001. Representing and Reasoning on Three-Dimensional Qualitative Orientation Point Objects. In Proc. of the 10th Portuguese Conference on Artificial Intelligence (EPIA’01) (LNCS), Vol. 2258. Springer, 298–305.

Arun K. Pujari, G. Vijaya Kumari, and Abdul Sattar. 1999. INDU: An Interval and Duration Network. In Proc. of the 12th Australian Joint Conference on Artificial Intelligence (AI’99) (LNCS), Vol. 1747. Springer, 291–303.
A Survey of Qualitative Spatial and Temporal Calculi

Arun K. Pujari and Abdul Sattar. 1999. A new framework for reasoning about points, intervals and durations. In Proc. of the Sixteenth International Joint Conference on Artificial Intelligence (IJCAI’99). Morgan Kaufmann, 1259–1267.

Marco Ragni and Alexander Scivos. 2005. Dependency calculus: Reasoning in a general point relation algebra. In Proc. of the 28th Annual German Conference on AI (KI’05) (LNCS), Vol. 3698. Springer, 49–63.

David A. Randell, Zhan Cui, and Anthony G. Cohn. 1992. A Spatial Logic based on Regions and “Connection”. In Proc. of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR’92). Morgan Kaufmann, 165–176.

David A. Randell, Mark Witkowski, and Murray Shanahan. 2001. From Images to Bodies: Modelling and Exploiting Spatial Occlusion and Motion Parallax. In Proc. of the 17th International Joint Conference on Artificial Intelligence (IJCAI’01). Morgan Kaufmann, 57–66.

Jochen Renz. 1999. Maximal Tractable Fragments of the Region Connection Calculus: A Complete Analysis. In Proc. of the Sixteenth International Joint Conference on Artificial Intelligence (IJCAI’99). Morgan Kaufmann, 448–455.

Jochen Renz. 2001. A Spatial Odyssey of the Interval Algebra: 1. Directed Intervals. In Proc. of the 17th International Joint Conference on Artificial Intelligence (IJCAI’01). Morgan Kaufmann, 51–56.

Jochen Renz. 2002. Qualitative Spatial Reasoning with Topological Information. LNCS, Vol. 2293. Springer.

Jochen Renz. 2007. Qualitative spatial and temporal reasoning: Efficient algorithms for everyone. In Proc. of the 20th International Joint Conference on Artificial Intelligence (IJCAI’07). Morgan Kaufmann, 526–531.

Jochen Renz and Debasis Mitra. 2004. Qualitative Direction Calculi with Arbitrary Granularity. In Proc. of the 8th Pacific Rim International Conference on Artificial Intelligence (PRICAI’04) (LNCS), Vol. 3157. Springer, 65–74.

Jochen Renz and Bernhard Nebel. 2007. Qualitative spatial reasoning using constraint calculi. See Aiello et al. [2007], 161–215.

Stuart Russell and Peter Norvig. 2009. Artificial Intelligence: A Modern Approach (3rd ed.). Prentice Hall.

Chaman L. Sabharwal and Jennifer L. Leopold. 2014. Evolution of Region Connection Calculus to VRCC-3D+. New Math. Natural Comput. 10 (2014), 1–39. Issue 20.

Marcus Schaefer, Eric Sedgwick, and Daniel Štefankovič. 2003. Recognizing String Graphs in NP. J. Comput. Syst. Sci. 67, 2 (2003), 365–380. STOC 2002 special issue.

Marcus Schaefer and Daniel Štefankovič. 2004. Decidability of String Graphs. J. Comput. Syst. Sci. 68, 2 (2004), 319–334. STOC 2001 special issue.

Stefan Schiffer, Alexander Ferrerin, and Gerhard Lakemeyer. 2012. Reasoning with Qualitative Positional Information for Domestic Domains in the Situation Calculus. J. Intell. and Robotic Systems 66, 1–2 (2012), 273–300.

Steven Schockaert and Sanjiang Li. 2015. Realizing RCC8 networks using convex regions. Artif. Intell. 218 (2015), 74–105.

Steven Schockaert, Philip D. Smart, and Florian A. Twaarch. 2011. Generating approximate region boundaries from heterogeneous spatial information: An evolutionary approach. Inf. Sci. 181, 2 (2011), 257–283.

Carl Schultz and Mehul Bhatt. 2012. Towards a Declarative Spatial Reasoning System. In Proc. of the 20th European Conference on Artificial Intelligence (ECAI’12) (FAIA), Vol. 242. IOS Press, 925–926.

Alexander Scivos and Bernhard Nebel. 2005. The Finest of its Class: The Natural Point-Based Ternary Calculus for Qualitative Spatial Reasoning. In Proc. of Spatial Cognition 2004 (LNCS), Vol. 3343. Springer, 283–303.

Michael Sioutis and Jean-François Condotta. 2014. Tackling Large Qualitative Spatial Network of Scale-Free-Like Structure. In Proceedings of the 8th Hellenic Conference on Artificial Intelligence (SETN’14) (LNCS), Vol. 8445. Springer, 178–191.

Evren Sirin, Bijan Parsia, Bernardo Cuenca Grau, Aditya Kalyanpur, and Yarden Katz. 2007. Pellet: A practical OWL-DL reasoner. J. Web Sem. 5, 2 (2007), 51–53.

Sпирос Сkiadopoulos and Manolis Koubarakis. 2004. Composing cardinal direction relations. Artif. Intell. 152, 2 (2004), 143–171.

Sпирос Сkiadopoulos and Manolis Koubarakis. 2005. On the consistency of cardinal direction constraints. Artif. Intell. 163, 1 (2005), 91–135.

John G. Stell. 2013. Granular Description of Qualitative Change. In Proc. of the 23rd International Joint Conference on Artificial Intelligence (IJCAI’13). AAAI Press, 1111–1117.

Markus Stocker and Evren Sirin. 2009. PelletSpatial: A Hybrid RCC-8 and RDF/OWL Reasoning and Query Engine. In Proc. of the 5th International Workshop on OWL: Experiences and Directions (OWLED’09) (CEUR Workshop Proceedings), Vol. 529. CEUR-WS.org, 2–31.
Kazuko Takahashi. 2012. PLCA: A Framework for Qualitative Spatial Reasoning Based on Connection Patterns of Regions. In *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions*. IGI Global, Chapter 2, 63–96.

Francesco Tarquini, Giorgio De Felice, Paolo Fogliaroni, and Eliseo Clementini. 2007. A Qualitative Model for Visibility Relations. In *Proc. of the 30th Annual German Conference on AI (KI’07)* (LNCS), Vol. 4667. Springer, 510–515.

Peter van Beek. 1991. Temporal query processing with indefinite information. *Artif. Intell. Med.* 3, 6 (1991), 325–339.

Peter van Beek. 1992. Reasoning about qualitative temporal information. *Artif. Intell.* 58 (1992), 297–326.

Nico Van de Weghe, Bart Kuipers, Peter Bogaert, and Philippe De Maeyer. 2005. A Qualitative Trajectory Calculus and the Composition of Its Relations. In *Proc. of First International Conference on GeoSpatial Semantics (GeoS’05)* (LNCS), Vol. 3799. Springer, 60–76.

André van Delden and Till Mossakowski. 2013. Mastering Left and Right – Different Approaches to a Problem That Is Not Straight Forward. In *Proc. of 36th Annual German Conference on AI (KI’13)* (LNCS), Vol. 8077. Springer, 248–259.

Marc B. Vilain and Henry A. Kautz. 1986. Constraint propagation algorithms for temporal reasoning. In *Proc. of the 5th National Conference on Artificial Intelligence (AAAI’86)*. Morgan Kaufmann, 377–382.

Marc B. Vilain, Henry A. Kautz, and Peter van Beek. 1990. Constraint Propagation Algorithms for Temporal Reasoning: A Revised Report. In *Readings in Qualitative Reasoning About Physical Systems*. 373–381.

Jan Oliver Wallgrün. 2012. Exploiting Qualitative Spatial Reasoning for Topological Adjustment of Spatial Data. In *Proc. of the SIGSPATIAL 2012 International Conference on Advances in Geographic Information Systems* (formerly known as GIS), *SIGSPATIAL’12*. ACM Press, 229–238.

Jan Oliver Wallgrün, Frank Dylla, Alexander Klippel, and Jinlong Yang. 2013. Understanding Human Spatial Conceptualizations to Improve Applications of Qualitative Spatial Calculi. In *Proc. of the 27th International Workshop on Qualitative Reasoning (QR’13)*. 131–137.

Jan Oliver Wallgrün, Diedrich Wolter, and Kai-Florian Richter. 2010. Qualitative Matching of Spatial Information. In *Proc. of the 18th ACM SIGSPATIAL International Symposium on Advances in Geographic Information Systems*. (ACM-GIS’10). ACM Press, 300–309.

Christoph Weidenbach, Uwe Brehm, Thomas Hillenbrand, Enno Keen, Christian Theobald, and Dalibor Topić. 2002. SPASS Version 2.0. In *Proc. of the 18th International Conference on Automated Deduction (CADE’18)* (LNCS), Vol. 2392. Springer, 275–279.

Matthias Westphal. 2015. *Qualitative constraint-based reasoning: methods and applications*. Ph.D. Dissertation. University of Freiburg, Germany.

Matthias Westphal, Julien Hüe, and Stefan Wölll. 2014. On the Scope of Qualitative Constraint Calculi. In *Proc. of the 37th Annual German Conference on AI (KI’14)* (LNCS), Vol. 8736. Springer, 207–218.

Matthias Westphal and Stefan Wölll. 2008. Bipath Consistency Revisited. In *Proc. of ECAI 2008 Workshop on Spatial and Temporal Reasoning*. IOS Press, Amsterdam, 36–40.

Matthias Westphal, Stefan Wölll, and Zeno Gantner. 2009. GQR: A Fast Solver for Binary Qualitative Constraint Networks. In *Proc. of the AAAI Spring Symposium on Benchmarking of Qualitative Spatial and Temporal Reasoning Systems (TR SS-09-02)*. AAAI Press, 51–52.

Brian C. Williams and Johan de Kleer. 1991. Qualitative Reasoning About Physical Systems: a Return to Roots. *Artif. Intell.* 51, 1–3 (1991), 1–9.

Stefan Wölll, Till Mossakowski, and Lutz Schröder. 2007. Qualitative Constraint Calculi: Heterogeneous Verification of Composition Tables. In *Proc. of the Twentieth International Florida Artificial Intelligence Research Society Conference (FLAIRS’07)*. AAAI Press, 665–670.

Stefan Wölll and Matthias Westphal. 2009. On Combinations of Binary Qualitative Constraint Calculi. In *Proc. of the 22nd International Joint Conference on Artificial Intelligence (IJCAI’11)*. AAAI Press, 967–973.

Diedrich Wolter and Jae Hee Lee. 2010. Qualitative reasoning with directional relations. *Artif. Intell.* 174, 18 (2010), 1498–1507.

Diedrich Wolter and Jae Hee Lee. 2016. Connecting Qualitative Spatial and Temporal Representations by Propositional Closure. In *Proc. of the 25th International Joint Conference on Artificial Intelligence (IJCAI’16)*. 1308–1314.

Diedrich Wolter and Jan Oliver Wallgrün. 2012. Qualitative Spatial Reasoning for Applications: New Challenges and the SparQ Toolbox. In *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions*. IGI Global, Chapter 11, 336–362.

Frank Wolter and Michael Zakharyaschev. 2000. Spatial reasoning in RCC–8 with Boolean region terms. In *Proc. of the 14th European Conference on Artificial Intelligence (ECAI’00)*. IOS Press, 244–248.
Michael Worboys. 2013. Using Maptrees to Characterize Topological Change. In Proc. of the 11th International Conference on Spatial Information Theory (COSIT’13) (LNCS), Vol. 8116. Springer, 74–90.
Michael Worboys and M. Duckham. 2004. GIS: A Computing Perspective (2nd ed.). CRC Press, Boca Raton FL.
Peng Zhang and Jochen Renz. 2014. Qualitative Spatial Representation and Reasoning in Angry Birds: The Extended Rectangle Algebra. In Proc. of the 14th International Conference on the Principles of Knowledge Representation and Reasoning (KR’14). AAAI Press.

Received April 2015; revised January 2016; accepted December 2016