An Efficient Algorithm for Generalized Polynomial Partitioning and Its Applications*

Pankaj K. Agarwal † Boris Aronov‡ Esther Ezra§ Joshua Zahl¶

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Abstract

Guth [13] showed that given a family $S$ of $n$ $g$-dimensional semi-algebraic sets in $\mathbb{R}^d$ and an integer parameter $D \geq 1$, there is a $d$-variate partitioning polynomial $P$ of degree at most $D$, so that each connected component of $\mathbb{R}^d \setminus Z(P)$ intersects $O(n/D^{d-g})$ sets from $S$. Such a polynomial is called a generalized partitioning polynomial. We present a randomized algorithm that efficiently computes such a polynomial $P$. Specifically, the expected running time of our algorithm is only linear in $|S|$, where the constant of proportionality depends on $d$, $D$, and the complexity of the description of $S$. Our approach exploits the technique of quantifier elimination combined with that of $\varepsilon$-samples.

We present four applications of our result. The first is a data structure for answering point-location queries among a family of semi-algebraic sets in $\mathbb{R}^d$ in $O(\log n)$ time; the second is a data structure for answering range search queries with semi-algebraic ranges in $O(\log n)$ time; the third is a data structure for answering vertical ray-shooting queries among semi-algebraic sets in $\mathbb{R}^d$ in $O(\log^2 n)$ time; and the fourth is an efficient algorithm for cutting algebraic planar curves into pseudo-segments, i.e., into Jordan arcs, each pair of which intersect at most once.

1 Introduction

Background and related work. In 2010, Guth and Katz [14] resolved the Erdős distinct distances problem in the plane. A major ingredient in their proof was a partitioning theorem for points in $\mathbb{R}^d$. Specifically, they proved that, given a set of $n$ points in $\mathbb{R}^d$ and an integer $D \geq 1$, there is a $d$-variate “partitioning polynomial” $P$ of degree at most $D$ so that each connected component of $\mathbb{R}^d \setminus Z(P)$ contains $O(n/D^d)$ points from the set. Their polynomial partitioning theorem has led to a flurry of new results in combinatorial and incidence geometry, harmonic analysis, and theoretical computer science.

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†Department of Computer Science, Duke University, Box 90129, Durham, NC 27708-0129 USA
pankaj@cs.duke.edu

‡Department of Computer Science and Engineering, Tandon School of Engineering, New York University, Brooklyn, NY 11201, USA. boris.aronov@nyu.edu.

§Department of Computer Science, Bar-Ilan University, Ramat Gan, Israel and School of Math, Georgia Institute of Technology, Atlanta, Georgia 30332, USA. eezea3@math.gatech.edu.

¶Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada.
jzahl@math.ubc.ca.
The Guth-Katz result established the existence of a partitioning polynomial, but it did not give an effective way to compute such a polynomial given a set of points. In [3], Agarwal, Matoušek, and Sharir developed an efficient algorithm to compute partitioning polynomials, matching the degree bound obtained in [14] up to a constant factor. They used their algorithm to obtain a linear-size data structure for the problem of range searching with semi-algebraic sets in the “low storage / sublinear query” regime.

In 2015, Guth [13] generalized the Guth-Katz partitioning polynomial result from points to semi-algebraic sets. Recall that a semi-algebraic set in $\mathbb{R}^d$ is the locus of points in $\mathbb{R}^d$ that satisfy a Boolean formula over a set of polynomial inequalities. Informally, he proved that given a collection of $n$ $g$-dimensional semi-algebraic sets in $\mathbb{R}^d$ and an integer $D \geq 1$, there is a $d$-variate partitioning polynomial $P$ of degree at most $D$ so that each connected component of $\mathbb{R}^d \setminus Z(P)$ intersects $O(n/D^{d-g})$ semi-algebraic sets from the collection (the implicit constant in the $O(\cdot)$ notation depends on $d$ and on the degree and number of polynomials required to define each semi-algebraic set). We refer to such a polynomial $P$ as a generalized partitioning polynomial.

To sum up, Guth’s proof established the existence of a generalized partitioning polynomial, but it did not give an effective way to compute such a polynomial given a collection of semi-algebraic sets. In [4], the last three authors developed a computationally efficient way to construct a partitioning polynomial for a set of algebraic curves in $\mathbb{R}^3$. For other settings, however, no effective method for computing a partitioning polynomial was known prior to the present work.

Our results. Our main result is a computationally efficient implementation of Guth’s polynomial partitioning theorem for semi-algebraic sets (Theorem 4). Given a set of $n$ semi-algebraic sets in $\mathbb{R}^d$, our algorithm computes a polynomial partition of degree $D$ in expected running time linear in $n$ and singly-exponential in $D$.

Next, we present four applications of our algorithm in Section 4:

(i) Let $S$ be a family of $n$ semi-algebraic sets in $\mathbb{R}^d$, each of complexity at most $b$ for some constant $b > 0$ (see Section 2 for the definition of the complexity of a semi-algebraic set). Each set in $S$ is assigned a weight that belongs to a semigroup. We present a data structure of size $O(n^{d+\varepsilon})$, for any constant $\varepsilon > 0$, that can compute, in $O(\log n)$ time, the cumulative weight of the sets in $S$ containing a query point. The data structure can be constructed in $O(n^{d+\varepsilon})$ randomized expected time. This is a significant improvement over the best known data structure by Koltun [16], for $d > 4$, that used $O(n^{2d-4+\varepsilon})$ space.

(ii) Let $P$ be a set of $n$ points in $\mathbb{R}^d$, each of which is assigned a weight, and let $\mathcal{R}$ be a (possibly infinite) family of semi-algebraic sets in $\mathbb{R}^d$. Suppose that there exists a positive integer $t$ and an injection $f : \mathcal{R} \to \mathbb{R}^t$ so that for each $p \in P$, the set $f(\{R \in \mathcal{R} \mid p \in R\})$ is a semi-algebraic set in $\mathbb{R}^t$ of complexity at most $b$. We can construct in $O(n^{t+\varepsilon})$ randomized expected time a data structure of size $O(n^{t+\varepsilon})$, for any constant $\varepsilon > 0$, that can compute in $O(\log n)$ time the cumulative weight of $P \cap \gamma$ for a query range $\gamma \in \mathcal{R}$. The previous best known data structure used $O(n^{2t-4+\varepsilon})$ space.

(iii) Given a family $S$ of $n$ semi-algebraic sets in $\mathbb{R}^d$, we present a data structure of size $O(n^{d+\varepsilon})$, for any constant $\varepsilon > 0$, that can answer vertical ray shooting queries in $O(\log^2 n)$ time. The data structure can be constructed in $O(n^{d+\varepsilon})$ randomized expected time.

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1Guth stated his result for the special case where the semi-algebraic sets are real algebraic varieties, but his proof in fact holds in the more general setting of semi-algebraic sets.

2We refer the reader to [9] Chapter 2] for a formal definition of dimension of a semi-algebraic set.
Finally, we follow the technique of Sharir and Zahl \[19\] to cut \(n\) algebraic planar curves into a collection of \(O(n^{3/2+\varepsilon})\) pseudo-segments (that is, a collection of Jordan arcs, each pair of which intersects at most once), where the constant of proportionality depends on the degree of the curves. By exploiting Theorem \[3\] we show that this collection can be constructed in comparable time bound.

2 Preliminaries

In what follows, the complexity of a semi-algebraic set \(S\) in \(\mathbb{R}^d\) is the minimum value \(b\) so that \(S\) can be represented as the locus of points \(x \in \mathbb{R}^d\) satisfying a Boolean formula with at most \(b\) atoms of the form \(P(x) < 0\) or \(P(x) = 0\), with each \(P\) being a \(d\)-variate polynomial of degree at most \(b\).

Hereafter we write \(X(n) = O_{a_1, a_2, \ldots, a_k}(Y(n))\) to mean that there exists a constant \(C\) depending only on \(a_1, \ldots, a_k\) so that \(X(n) \leq CY(n)\), for all positive integers \(n\).

Our analysis makes extensive use of concepts and results from real algebraic geometry and random sampling. We review them below.

2.1 Polynomials, partitioning, and quantifier elimination

**Sign conditions.** Consider polynomials \(P_1, \ldots, P_\ell \in \mathbb{R}[x_1, \ldots, x_d]\). A sign condition on \((P_1, \ldots, P_\ell)\) is an element of \((-1,0,1)^\ell\). A strict sign condition on \((P_1, \ldots, P_\ell)\) is an element of \((-1,1)^\ell\). A sign condition \((\nu_1, \ldots, \nu_\ell) \in \{-1,0,1\}^\ell\) is realizable if the set

\[
\{x \in \mathbb{R}^d \mid \text{sign}(P_j(x)) = \nu_j \text{ for each } j = 1, \ldots, \ell\}
\]

is non-empty. A realizable strict sign condition is defined analogously. The set \([1]\) is called the realization of the sign condition. The set of realizations of sign conditions (resp., realizations of strict sign conditions) corresponding to the tuple \((P_1, \ldots, P_\ell)\) is the collection of all non-empty sets of the above form. These sets are pairwise disjoint and partition \(\mathbb{R}^d\), by definition.

While a tuple of \(\ell\) polynomials has \(3^\ell\) sign conditions and \(2^\ell\) strict sign conditions, not all of them may be realizable. In fact, Milnor and Thom (see, e.g., \[8, 12\]) showed that any \(\ell\) polynomials in \(\mathbb{R}[x_1, \ldots, x_d]\) of degree at most \(s\) (with \(2 \leq d \leq \ell\)) have at most \((50s\ell/d)^d\) realizable sign conditions.

**Polynomials and partitioning.** The set of polynomials in \(\mathbb{R}[x_1, \ldots, x_d]\) of degree at most \(b\) is a real vector space of dimension \(\binom{b+d}{d}\); we identify this vector space with \(\mathbb{R}^{\binom{b+d}{d}}\). For a point \(q \in \mathbb{R}^{\binom{b+d}{d}}\), let \(P_q \in \mathbb{R}[x_1, \ldots, x_d]\) be the corresponding polynomial of degree at most \(b\).

**Remark 1.** Consider the polynomial \(Q(q, x) \in \mathbb{R}[q_1, \ldots, q_{\binom{b+d}{d}}, x_1, \ldots, x_d]\) given by \(Q(q, x) := P_q(x)\). Since we can write \(Q(q, x) = \sum_{i=1}^{\binom{b+d}{d}} q_i H_i(x)\), where \(H_i\) is a monomial of degree at most \(b\), \(Q\) has degree \(b+1\).

For each positive integer \(j\), let \(D_j\) be the smallest positive integer so that \(\binom{D_j+b+d}{d} > 2^{j-1}\); we have \(D_j = O_d(2^{j/d})\). For each \(j = 1, \ldots, k\), pick a \(2^{j-1}\)-dimensional subspace \(\mathbb{V}_j\) of the vector space of polynomials in \(\mathbb{R}[x_1, \ldots, x_d]\) of degree at most \(D_j\). These subspaces \(\mathbb{V}_j\) will be fixed hereafter. For each positive integer \(k\), define the product space

\[
\mathbb{V}_k := \bigotimes_{j=1}^{k} \mathbb{V}_j.
\]
We identify each point \( y \in \mathbb{Y}_k \) with a \( k \)-tuple of polynomials \( \mathbf{P}_y = (P_1, \ldots, P_k) \), where \( P_j \in \mathbb{V}_j \subset \mathbb{R}[x_1, \ldots, x_d] \). For each \( j = 1, \ldots, k \), \( \deg(P_j) = O_d(2^{j/d}) \) and thus \( \deg \left( \prod_{j=1}^k P_j \right) = O_d(2^{k/d}) \).

Let \( (P_1, \ldots, P_k) \in \mathbb{Y}_k \), let \( S \) be a collection of semi-algebraic sets in \( \mathbb{R}^d \), and let \( \alpha \geq 1 \). We say that \( (P_1, \ldots, P_k) \) is a \((k, \alpha)\)-partitioning tuple for \( S \) if \( (P_1, \ldots, P_k) \) has \( 2^k \) realizable strict sign conditions and the realization of each of them intersects at most \(|S|/\alpha \) sets from \( S \).

Guth [13] proved that, if \( \alpha \) is chosen appropriately, then a \((k, \alpha)\)-partitioning tuple is guaranteed to exist:

**Proposition 1** (Generalized Polynomial Partitioning [13]). Let \( S \) be a family of semi-algebraic sets in \( \mathbb{R}^d \), each of dimension at most \( g \) and complexity at most \( b \). For each \( k \geq 1 \), there exists a \((k, \alpha)\)-partitioning tuple for \( S \), with \( \alpha = \Omega_{b,d}(2^{k(1-g/d)}) \).

We also recall Theorem 2.16 from [7]:

**Proposition 2** (Point Location in Semi-Algebraic Sets). Let \( \mathcal{P} \) be a set of at most \( s \) polynomials in \( \mathbb{R}[x_1, \ldots, x_d] \) of degree at most \( t \). Then there is an algorithm that computes a set of points meeting every semi-algebraically connected component of every realizable sign condition on \( \mathcal{P} \) in time \( O(s^d t^{O(d)}) \). There is also an algorithm providing the list of signs of all the polynomials of \( \mathcal{P} \) at each of these points in time \( O(s^{d+1} t^{O(d)}) \).

**Singly exponential quantifier elimination.** Let \( h \) and \( \ell \) be non-negative integers and let \( \mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[x_1, \ldots, x_h, y_1, \ldots, y_\ell] \). Let \( \Phi(y) \) be a first-order formula given by

\[
(\exists x_1, \ldots, x_h) F(P_1(x, y), \ldots, P_s(x, y)), \tag{3}
\]

where \( y = (y_1, \ldots, y_\ell) \) is a block of \( \ell \) free variables; \( x \) is a block of \( h \) variables, and \( F(P_1, \ldots, P_s) \) is a quantifier-free Boolean formula with atomic predicates of the form \( \text{sign}(P_i(x_1, \ldots, x_h, y)) = \sigma \), with \( \sigma \in \{-1, 0, 1\} \).

The Tarski-Seidenberg theorem states that the set of points \( y \in \mathbb{R}^\ell \) satisfying the formula \( \Phi(y) \) is semi-algebraic. The next proposition is a quantitative version of this result that bounds the number and degree of the polynomial equalities and inequalities needed to describe the set of points satisfying \( \Phi(y) \). This proposition is known as a “singly exponential quantifier elimination,” and its more general form (where \( \Phi(y) \) may contain a mix of \( \forall \) and \( \exists \) quantifiers) can be found in [7] Theorem 2.27.

**Proposition 3.** Let \( \mathcal{P} \) be a set of at most \( s \) polynomials, each of degree at most \( t \) in \( h + \ell \) real variables. Given a formula \( \Phi(y) \) of the form (3), there exists an equivalent quantifier-free formula

\[
\Psi(y) = \bigvee_{i=1}^I \bigwedge_{j=1}^{J_i} \left( \bigvee_{n=1}^{N_{ij}} \text{sign}(P_{ijn}(y)) = \sigma_{ijn} \right) , \tag{4}
\]

where \( P_{ijn} \) are polynomials in the variables \( y \), \( \sigma_{ijn} \in \{-1, 0, 1\} \),

\[
I \leq s^{(h+1)(\ell+1)O(h+\ell)}, \\
J_i \leq s^{(h+1)O(h)}, \\
N_{ij} \leq t^{O(h)},
\]

and the degrees of the polynomials \( P_{ijn}(y) \) are bounded by \( t^{O(h)} \).
2.2 Range spaces, VC dimension, and \(\varepsilon\)-samples

We first recall several standard definitions and results from [15, Chapter 5]. A range space is a pair \(\Sigma = (X, \mathcal{R})\), where \(X\) is a set and \(\mathcal{R}\) is a collection of subsets of \(X\). Let \((X, \mathcal{R})\) be a range space and let \(A \subseteq X\) be a set. We define the restriction of \(\Sigma\) to \(A\), denoted by \(\Sigma_A\) to be \((A, \mathcal{R}_A)\), where \(\mathcal{R}_A := \{R \cap A \mid R \in \mathcal{R}\}\). If \(A\) is finite, then \(|\mathcal{R}_A| \leq 2^{|A|}\). If equality holds, then we say \(A\) is shattered.

We define the shatter function by \(\pi_{\mathcal{R}}(z) := \max_{|A| = z} |\mathcal{R}_A|\). The VC dimension of \(\Sigma\) is the largest cardinality of a set shattered by \(\mathcal{R}\). If arbitrarily large (finite) subsets can be shattered, we say that the VC dimension of \(\Sigma\) is infinite.

Let \(\Sigma\) be a range space, \(A\) a finite subset of \(X\), and \(0 \leq \varepsilon \leq 1\). A set \(B \subseteq A\) is an \(\varepsilon\)-sample (also known as \(\varepsilon\)-approximation) of \(\Sigma_A\) if

\[
\left| \frac{|A \cap R|}{|A|} - \frac{|B \cap R|}{|B|} \right| \leq \varepsilon \quad \forall R \in \mathcal{R}.
\]

The following classical theorem of Vapnik and Chervonenkis [20] guarantees that if the VC-dimension of \(\Sigma\) is finite, then for each positive \(\varepsilon > 0\), a sufficiently large random sample of \(A\) is likely to be an \(\varepsilon\)-sample.

**Proposition 4** (\(\varepsilon\)-Sample Theorem). Let \(\Sigma = (X, \mathcal{R})\) be a range space of VC dimension at most \(d\) and let \(A \subseteq X\) be finite. Let \(0 < \varepsilon, \delta < 1\). Then a random subset \(B \subseteq A\) of cardinality \(\delta d \log \frac{1}{\delta}\) is an \(\varepsilon\)-sample for \(\Sigma_A\) with probability at least \(1 - \delta\).

**Proposition 5** ([12] [15]). Let \(\Sigma = (X, \mathcal{R})\) be a range space whose shatter function \(\pi_{\mathcal{R}}(z)\) satisfies the bound \(\pi_{\mathcal{R}}(z) \leq C z^\rho\), for all positive integers \(z\), where \(\rho > 0\) is a real parameter. Then \(\Sigma\) has VC dimension at most \(4\rho \log(C\rho)\).

We next closely follow the arguments in the proof of Corollary 2.3 from [12], and show the following theorem:

**Theorem 1.** Let \(Z \subseteq \mathbb{R}^m \times \mathbb{R}^n\) be a semi-algebraic set of complexity \(b\). For each \(y \in \mathbb{R}^n\), define \(R_y = \{x \in \mathbb{R}^m \mid (x, y) \in Z\}\). Then the range space \((\mathbb{R}^m, \{R_y \mid y \in \mathbb{R}^n\})\) has VC dimension at most \(200n^2 \log b\).

**Proof.** By assumption, there are polynomials \(f_1, \ldots, f_b\) and a Boolean formula \(\Phi\), so that, for \((x, y) \in \mathbb{R}^m \times \mathbb{R}^n\), \((x, y) \in Z\) if and only if \(\Phi(f_1(x, y) \geq 0, \ldots, f_b(x, y) \geq 0) = 1\).

Put \(\mathcal{R} = \{R_y \mid y \in \mathbb{R}^n\}\). Fix a positive integer \(z\) and let \(p_1, \ldots, p_z \in \mathbb{R}^m\). Our goal is to bound

\[
\left| \{R \cap \{p_1, \ldots, p_z\} \mid R \in \mathcal{R}\} \right| = \left| \{R_y \cap \{p_1, \ldots, p_z\} \mid y \in \mathbb{R}^n\} \right|.
\]

For each \(j = 1, \ldots, z\), define

\[
W_j = \{y \in \mathbb{R}^n \mid \Phi(f_1(p_j, y) \geq 0, f_2(p_j, y) \geq 0, \ldots, f_b(p_j, y) \geq 0) = 1\}.
\]

Let \(A = [z] = \{1, \ldots, z\}\) and suppose that there exists \(y \in \mathbb{R}^n\) with \(R_y \cap \{p_1, \ldots, p_z\} = A\). This means \(y \in W_j\) for each \(j \in A\) and \(y \not\in W_j\) for each \(j \in [z] \setminus A\), i.e., the semi-algebraic set \(S_A\) consisting of those points \(y \in \mathbb{R}^n\) satisfying the Boolean formula

\[
\bigwedge_{j \in A} \left( \Phi(f_1(p_j, y) \geq 0, f_2(p_j, y) \geq 0, \ldots, f_b(p_j, y) \geq 0) = 1 \right) \land \bigwedge_{j \in [z] \setminus A} \left( \Phi(f_1(p_j, y) \geq 0, f_2(p_j, y) \geq 0, \ldots, f_b(p_j, y) \geq 0) = 0 \right)
\]

The following bound is not the strongest possible (see, e.g. [15] Chapter 7 for an improved bound), but is sufficient for our purposes.
is non-empty. Observe that if $A$ and $A'$ are distinct subsets of $[z]$, then $S_A$ and $S_{A'}$ are disjoint and, in fact,
\[ |\{R_y \cap \{p_1, \ldots, p_z\} \mid y \in \mathbb{R}^n\}| = |\{A \subset [z] \mid S_A \neq \emptyset\}|. \]
Each of the non-empty sets $S_A$ contains at least one realization of a sign condition of the $bz$ polynomials
\[ \{f_i(p_j, y) \mid i = 1, \ldots, b; j = 1, \ldots, z\}, \]
each of degree at most $b$. By a result of Milnor and Thom stated in Section 2.1, these polynomials determine at most $(50 \cdot b \cdot bz/n)^n \leq (50b^2z)^n$ realizable sign conditions. Thus
\[ |\{R_y \cap \{p_1, \ldots, p_z\} \mid y \in \mathbb{R}^n\}| \leq (50b^2z)^n. \tag{6} \]
Since (6) holds for every choice of $p_1, \ldots, p_z \in \mathbb{R}^n$, we conclude that
\[ \pi_R(z) \leq (50b^2z)^n. \]
By Proposition 5, $(\mathbb{R}^m, \{R_y \mid y \in \mathbb{R}^n\})$ has VC dimension at most $4n\log((50b^2)^n) \leq 200n^2 \log b.$ \hfill \Box

## 3 Computing Generalized Polynomial Partition

In this section we obtain the main result of the paper: given a collection $S$ of semi-algebraic sets in $\mathbb{R}^d$, each of dimension at most $g$ and complexity at most $b$, a $(k, \Omega_b d(D^{d-g}))$-partitioning tuple for $S$ can be computed efficiently. We obtain this result in several steps. First, we represent a semi-algebraic set in $\mathbb{R}^d$ of complexity at most $b$ as a point in a parameter space $(\mathbb{R}^{(b+d)/d})_b$—each point in $(\mathbb{R}^{(b+d)/d})_b$ corresponds to a tuple of $b$ polynomials in $d$ variables, each of degree at most $b$. We use the set $\mathbb{Y}_k$ (defined in Section 2.1) to parameterize the space of sign conditions specified by $k$ tuples $(P_1, \ldots, P_k)$ of polynomials. With these parameterizations in place, the condition that a semi-algebraic set $S$ intersects a given sign condition is represented by a subset $W_G$ of pairs of points from $(\mathbb{R}^{(b+d)/d})_b \times \mathbb{Y}_k$.

In Theorem 2, we prove that $W_G$ is semi-algebraic, and its complexity depends only on $k$, $b$, and $d$. This means that for each semi-algebraic set $S$, the set of tuples $(P_1, \ldots, P_k) \in \mathbb{Y}_k$ whose realization intersects $S$ is semi-algebraic. This in turn implies that if $S_1, \ldots, S_N$ are semi-algebraic sets and if $M \leq N$, then the set of $k$-tuples $(P_1, \ldots, P_k) \in \mathbb{Y}_k$ whose realization intersects at most $M$ of the $N$ sets $S_1, \ldots, S_N$ is semi-algebraic. Unfortunately, however, the complexity of this subset of $\mathbb{Y}_k$ might be very large; in particular, it is likely to be exponential in $N$.

To circumvent this problem, we use the theory of $\varepsilon$-samples. That is, we show that rather than considering a large number of semi-algebraic sets $S_1, \ldots, S_N$, it suffices to select a small number of these sets at random. If a tuple $(P_1, \ldots, P_k) \in \mathbb{Y}_k$ has the property that each of its realizable sign conditions intersect few sets from the random sample, then with high probability each of the realizable sign conditions will intersect few sets from the original collection. This property is shown by applying Theorems 1 and 2.

### 3.1 The parameter space of semi-algebraic sets

Fix positive integers $b$, $d$, $g$ and $k$, and let $D = 2^{k/d}$. Hereafter we assume that $D = \Omega(2^b)$, which can be enforced by choosing $k$ sufficiently large.
As above, we denote by \( S \) a family of semi-algebraic sets in \( \mathbb{R}^d \), each of dimension at most \( g \) and complexity at most \( b \). Let \( G: \{0,1\}^b \to \{0,1\} \) be a Boolean function. Let \( \mathcal{X} = (\mathbb{R}^{(b+d)})^b \). We identify a point \( x = (q_1, \ldots, q_b) \in \mathcal{X} \) with the semi-algebraic set

\[
Z_{x,G} = \{ v \in \mathbb{R}^d | G(P_{q_1}(v) \geq 0, \ldots, P_{q_b}(v) \geq 0) = 1 \} \subset \mathbb{R}^d.
\]

Observe that each semi-algebraic set in \( S \) is of the form \( Z_{x,G} \) for some choice of \( x \in \mathcal{X} \) and a Boolean function \( G \). Let \( y = y_k \). For each \( y \in \mathcal{Y} \), define \( S_y := \{ u \in \mathbb{R}^d | P_1(u) > 0, \ldots, P_k(u) > 0 \} \), where \( (P_1, \ldots, P_k) \) is the tuple associated to \( y \).

Define

\[
W_G := \{ (x,y) \in \mathcal{X} \times \mathcal{Y} | Z_{x,G} \cap S_y \neq \emptyset \}.
\]

**Theorem 2.** The set \( W_G \) is semi-algebraic, defined by \( O_b(e^{\text{Poly}(D)}) \) polynomials, each of degree \( D^{O_b(d)} \).

Before proceeding with the proof of the theorem, we note that the complexity of \( W_G \) is only singly exponential in \( D \), which we will exploit in Section 3.3.

**Proof.** Define \( V = \{ (x,y,v) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^d | v \in Z_{x,G} \cap S_y \} \). The condition \( v \in Z_{x,G} \) is a Boolean condition on \( b \) polynomials. By Remark \( 1 \), each of these polynomials has degree at most \( b + 1 \). Similarly, the condition \( v \in S_y \) consists of \( k \) polynomial inequalities, each of degree at most \( D + 1 \). This means that there exists a set of polynomials \( Q = \{ Q_1, \ldots, Q_{b+k} \} \) of degree \( b + D + 1 \) in the variables \( x, y, v \), and a Boolean function \( F(z_1, \ldots, z_{b+k}) \) so that

\[
V = \{ (x,y,v) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^d | F(Q_1(x,y,v), \ldots, Q_{b+k}(z,y,v)) = 1 \}.
\]

With the above definitions

\[
W_G = \{ (x,y) | \exists (v_1, \ldots, v_d) : F(Q_1(x,y,v), \ldots, Q_{b+k}(x,y,v)) = 1 \}.
\]

We now apply Proposition \( 3 \). We have a set \( Q \) of \( s = b + k \) polynomials, each of degree at most \( t = b + D + 1 \). The variables \( h \) and \( \ell \) from the hypothesis of Proposition \( 3 \) are set to \( h = d \) and \( \ell = O \left( (b+d)^b + D^d \right) = \text{Poly}(D) \), recall that \( D \) is sufficiently larger than \( b \), and thus \( \ell \) is a suitably chosen polynomial function of \( D \). With these assignments, Proposition \( 3 \) says that \( W_G \) can be expressed as a quantifier-free formula of the form

\[
\bigwedge_{i=1}^I \bigwedge_{j=1}^{J_i} \left( \bigvee_{n=1}^{N_{ij}} \text{sign}(P_{ijn}(x,y)) = \sigma_{ijn} \right),
\]

where \( P_{ijn} \) are polynomials in the variables \( (x,y) \), \( \sigma_{ijn} \in \{-1,0,1\} \),

\[
I \leq (b+k)^{\text{Poly}(D)}(b+D)^{\text{Poly}(D)} = O_b(e^{\text{Poly}(D)}),
\]

\[
J_i \leq (b+k)^{d+1}(b+D)^{O(d)} = O_b(D^{O(d)}),
\]

\[
N_{ij} \leq (b+D)^{O(d)} = O_b(D^{O(d)}),
\]

where the degrees of the polynomials \( P_{ijn}(y) \) are bounded by \( \langle b+D \rangle^{O(d)} = D^{O_b(d)} \).

Summarizing, the quantifier-free formula \( 7 \) for \( W_G \) is a Boolean combination of \( O_b(e^{\text{Poly}(D)}) \) polynomial inequalities, each of degree \( D^{O_b(d)} \), as claimed. \( \square \)
Theorem 3. Let $S$ be a family of $m$ semi-algebraic sets in $\mathbb{R}^d$, each of dimension at most $g$ and complexity at most $b$. Let $1 \leq k \leq \log m$ and let $D = 2^{k/d}$. Then a $(k, \Omega_{b,d}(D^{d-g}))$-partitioning tuple for $S$ can be computed in $O_{b,d}(e^{\text{Poly}(m)})$ time.

Proof. Set $Y = Y_k$. As above, we identify points in $Y$ with $k$ tuples $(P_1, \ldots, P_k)$ of polynomials. The argument in Theorem 2 as well as the fact that the class of semi-algebraic sets is closed under the operation of taking a projection, show that, for each $S \in S$ and each $\sigma \in \{-1, 1\}^k$,

$$I_{S,\sigma} := \{ y \in Y \mid S \cap \{ \sigma_1 P_1 > 0, \sigma_2 P_2 > 0, \ldots, \sigma_k P_k > 0 \} \neq \emptyset \}$$

is a semi-algebraic set in $Y$ that can be expressed as a Boolean combination of $O_b(e^{\text{Poly} D})$ polynomials, each of degree $D^{O_b(d)}$, where $N' = |S'|$. Therefore

$$K := \bigcup_{S' \subseteq S, |S'| \geq N} \{ y \in Y \mid y \in I_{S,\sigma} \text{ for every } S \in S' \}$$

is a semi-algebraic set in $Y$ that can be expressed as a Boolean combination of

$$O_{b,d}\left(\binom{m}{N} me^{\text{Poly} D}\right) = O_{b,d}(e^{m+\text{Poly}(D)}) = O_{b,d}(e^{\text{Poly} m})$$

polynomials, each of degree $D^{O_b(d)}$. This and the fact that the class of semi-algebraic sets is closed under the operation of taking complement imply that

$$\text{Good}(\sigma) := Y \setminus K = \{ y \in Y \mid y \in I_{S,\sigma} \text{ for at most } C_{b,d}|S|D^{g-d} \text{ sets } S \in S \}$$

is a semi-algebraic set in $Y$ that can be expressed as a Boolean combination of $O_{b,d}(e^{\text{Poly}(m)})$ polynomials, each of degree $D^{O_b(d)}$. This means that the set

$$\bigcap_{\sigma \in \{-1, 1\}^k} \text{Good}(\sigma)$$

is a semi-algebraic set in $Y$ that can be expressed as a Boolean combination of $O_{b,d}(e^{\text{Poly}(m)})$ polynomials, each of degree $D^{O_b(d)}$. Recall that by assumption $1 \leq k \leq \log m$ and $D = 2^{k/d}$. It thus follows that the degree is bounded by $\text{Poly}(m)$. Similarly, the dimension of the space $Y$ is bounded by $\text{Poly}(m)$ as well.

Proposition 1 guarantees that if $C_{b,d}$ is selected sufficiently large, then the set (10) is non-empty. By Proposition 2 it is possible to locate a point in this set in $O_{b,d}(e^{\text{Poly}(m)})$ time, concluding the proof of the theorem. \qed
3.3 Speeding up the algorithm using $\varepsilon$-sampling

In this section we first state and prove the following lemma:

**Lemma 1.** For every choice of positive integers $b$ and $d$, there is a constant $C = C(b, d)$ so that the following holds. Let $C_0$ be a positive integer. Let $S$ be a finite collection of semi-algebraic sets in $\mathbb{R}^d$, each of dimension at most $g$ and complexity at most $b$. Let $k$ be a positive integer and let $D = 2^{k/d}$. Let $B \subseteq S$ be a randomly chosen subset of $S$ of cardinality at least $CD^C$ and let $(P_1, \ldots, P_k)$ be a $(k, \frac{D^{d-g}}{C_0})$-partitioning tuple for $B$. Then with probability at least $1/2$, each of the $D^d$ realizable sign conditions of $(P_1, \ldots, P_k)$ intersects $O_{b,d}(|S|C_0D^{g-d})$ elements from $S$.

Note that Lemma 1 states that it is sufficient to consider a random subset $B$ of size polynomial in $D$ in order to obtain an appropriate partitioning tuple for the entire collection $S$, with reasonable probability.

**Proof.** Define $\mathcal{X}$ and $\mathcal{Y}$ as above, and let $G : \{0,1\}^b \rightarrow \{0,1\}$. For each $y \in \mathcal{Y}$, define the range

$$R_{y,G} = \{x \in \mathcal{X} \mid \mathcal{Z}_{x,G} \cap S_y \neq \emptyset\}.$$ 

Define $\mathcal{R}_G = \{R_{y,G} \mid y \in \mathcal{Y}\}$. By Theorems 1 and 2, the range space $(\mathcal{X}, \mathcal{R}_G)$ has VC dimension $O_{b,d}(\text{Poly}(D))$. Define $\mathcal{R} = \bigcup_G \mathcal{R}_G$, where the union is taken over the $2^b$ Boolean functions $G : \{0,1\}^b \rightarrow \{0,1\}$. Since the shatter function grows by at most a multiplicative factor of $2^b$, the VC dimension of the range space $(\mathcal{X}, \mathcal{R})$ is $O_{b,d}(\text{Poly}(D))$ as well (this is a standard fact, see, e.g., [15, Chapter 5]).

We are now ready to prove the statement of the lemma. Set $\varepsilon = O_{b,d}(C_0D^{g-d})$. Suppose that $B$ is an $\varepsilon$-sample of $S$ and that $(P_1, \ldots, P_k)$ is a $(k, \frac{D^{d-g}}{C_0})$-partitioning tuple for $B$. Then for each range $R \in \mathcal{R}$, we have $|B \cap R|/|B| \leq O_{b,d}(C_0D^{g-d})$. Combining this with $\varepsilon$-sample properties, we obtain:

$$\left|\frac{|S \cap R|}{|S|} - \frac{|B \cap R|}{|B|}\right| \leq \varepsilon,$$

and by the choice of $\varepsilon$ (with an appropriate constant of proportionality) we have:

$$\left|\frac{|S \cap R|}{|S|}\right| \leq 2\varepsilon,$$

and thus

$$|S \cap R| = O_{b,d}(|S|C_0D^{g-d}).$$

The corresponding cardinality of $B$ is

$$\frac{\text{VC-dim}(\mathcal{R})}{\varepsilon^2} \log(2/\varepsilon) = O_{b,d}(\text{Poly}(D)).$$

We next proceed as follows. We select a random sample of $S$ of cardinality $CD^C$ and use Theorem 3 to compute the corresponding partitioning tuple $(P_1, \ldots, P_k)$. This takes $O_{b,d}(\text{ePoly}(D))$ time. By Lemma 1, this tuple will be a $(k, \Omega_{b,d}(D^{d-g}))$-partitioning tuple for $S$ with probability at least $1/2$. We can verify whether the partitioning tuple works in $O_{b,d}(|S|\text{Poly}(D))$ time. If the tuple does not produce the appropriate partition, we discard it and try again. Specifically, the verification step is done as follows. For each semi-algebraic set $S \in S$ we compute the subset of sign conditions of $(P_1, \ldots, P_k)$, with which it has a non-empty intersection. To this end, we restrict each of the polynomials $P_1, \ldots, P_k$ to $S$ and apply Proposition 2 on this restricted collection, thereby obtaining
Theorem 4. Let $S$ be a finite collection of semi-algebraic sets in $\mathbb{R}^d$, each of which has dimension at most $g$ and complexity at most $b$. Let $k \geq 1$ and let $D = 2^{k/d}$. Then a $(k, \Omega_{b,d}(D^{d-g}))$-partitioning tuple for $S$ can be computed in $O_{b,d}(|S| \text{Poly}(D) + e^{\text{Poly}(D)})$ expected time by a randomized algorithm.

4 Applications

In this section we describe a few applications of Theorem 4, namely, point location amid semi-algebraic sets, semi-algebraic range searching with logarithmic query time, vertical ray shooting amid semi-algebraic sets, and cutting algebraic curves into pseudo-segments.

4.1 Point location

Let $S$ be a set of $n$ semi-algebraic sets in $\mathbb{R}^d$, each of complexity at most $b$. Each set $S$ has a weight $w(S)$. We assume that the weights belong to a semigroup, i.e., subtractions are not allowed, and that the semigroup operation can be performed in constant time. We wish to preprocess $S$ into a data structure so that the cumulative weight of the sets in $S$ that contain a query point can be computed in $O(\log n)$ time. Note that if the weight of each set is 1 and the semigroup operation is Boolean $\lor$, then the point-location query becomes an instance of union-membership query: determine whether the query point lies in $\bigcup S$. We follow a standard hierarchical partitioning scheme of space, e.g., as in \cite{10}, but use Theorem 4 at each stage. Using this hierarchical partition, we construct a tree data structure $T$ of $O(\log n)$ depth, and a query is answered by following a path in $T$.

More precisely, we fix sufficiently large positive constants $D = D(b,d)$ and $n_0 = n_0(D)$. If $n \leq n_0$, $T$ consists of a single node that stores $S$ itself. So assume that $n > n_0$. Using Theorem 4, we construct a tuple $P = (P_1, \ldots, P_k)$ of $d$-variate polynomials of degree at most $D$, which have $2^k = O(D^d)$ realizable sign conditions, each of which with a realization that meets the boundaries of at most $O(|S|/D)$ sets of $S$. For each realizable sign condition $\sigma$, let $S_{\sigma} \subseteq S$ be the family of sets whose boundaries meet the realization of $\sigma$, and let $S_{\sigma}^* \subseteq S$ be the family of sets that contain the realization of $\sigma$. We compute $S_{\sigma}$, $S_{\sigma}^*$, and $W_{\sigma} = w(S_{\sigma}^*)$, as follows: We first apply Proposition 2 to $P$ to compute, in $O(D^{O(d)})$ time, a representative point in each realization of a sign condition.

Next, fix a set $S \in S$ and mark all realizations that meet the boundary of $S$. This step is similar to the one described in the proof of Theorem 4, that is, we restrict each of the polynomials $P_1, \ldots, P_k$ to the algebraic varieties representing the boundary of $S$ and apply Proposition 2 to this restricted collection. Each remaining realization is either contained in $S$ or disjoint from it, which can be determined by testing, for each such realization, whether its representing point (computed earlier using Proposition 2 on the original collection $P$) is contained in $S$. This task can be completed in overall $O(nD^{O(d)})$ time over all sets of $S$.

We create the root $v$ of $T$ and store the tuple $P$ at $v$. We then create a child $z_{\sigma}$ for each realizable sign condition $\sigma$ and store $\sigma$ and $W_{\sigma}$ at $z_{\sigma}$. We recursively construct the data structure for each $S_{\sigma}$ and attach it to $z_{\sigma}$ as its subtree.

Since each node of $T$ has degree at most $O(D^d)$ and the size of the subproblem reduces by a factor of $D$ at each level of the recursion, a standard analysis shows that the total size of the data
structure is \( O(n^{d+\varepsilon}) \), where \( \varepsilon > 0 \) is a constant that can be made arbitrarily small by choosing \( D \) and \( n_0 \) to be sufficiently large. Similarly, the expected preprocessing time is also \( O(n^{d+\varepsilon}) \).

Given a query point \( q \in \mathbb{R}^d \), we compute the cumulative weight of the sets containing \( q \) by traversing a path in the tree in a top-down manner: We start from the root and maintain a partial weight \( W \), which is initially set to 0. At each node \( v \), we find the sign condition \( \sigma \) of the polynomial tuple at \( v \) whose realization contains \( q \), add \( W_{\sigma} \) to \( W \), and recursively query the child \( z_{\sigma} \) of \( v \). The total query time is \( O(\log n) \), where the constant of proportionality depends on \( D \) (and thus on \( \varepsilon \)).

Putting everything together, we obtain the following:

**Theorem 5.** Let \( S \) be a set of \( n \) semi-algebraic sets in \( \mathbb{R}^d \), each of complexity at most \( b \) for some constant \( b > 0 \), and let \( w(S) \) be the weight of each set \( S \in S \) that belongs to a semigroup. Assuming that the semigroup operation can be performed in constant time, \( S \) can be preprocessed in \( O_{b,d}(n^{d+\varepsilon}) \) randomized expected time into a data structure of size \( O_{b,d}(n^{d+\varepsilon}) \), for any constant \( \varepsilon > 0 \), so that the cumulative weight of the sets that contain a query point can be computed in \( O_{b,d,\varepsilon}(\log n) \) time.

### 4.2 Range searching

Next, we consider range searching with semi-algebraic sets: Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \). Each point \( p \in P \) is assigned a weight \( w(p) \) that belongs to a semigroup. Again we assume that the semigroup operation takes constant time. We wish to preprocess \( P \) so that for a query range \( \gamma \), represented as a semi-algebraic set in \( \mathbb{R}^d \), the cumulative weight of \( \gamma \cap P \) can be computed in \( O(\log n) \) time. Here we assume that the query ranges (semi-algebraic sets) are parameterized as described in Section 3.1. That is, we have a fixed \( b \)-variate Boolean function \( G \). A query range is represented as a point \( x \in \mathbb{R}^d \), for some \( t \leq \left( \frac{b+d}{d} \right)^b \), and the underlying semi-algebraic set is \( Z_{x,G} \). We refer to \( t \) as the dimension of the query space, and to the range searching problem in which all query ranges are of the form \( Z_{x,G} \) as \( (G,t) \)-semi-algebraic range searching.

For a point \( p \in \mathbb{R}^d \), let \( S_p \subseteq \mathbb{R} \) denote the set of semi-algebraic sets \( Z_{x,G} \) that contain \( p \), i.e., \( S_p = \{ x \in \mathbb{R} \mid p \in Z_{x,G} \} \). It can be checked that \( S_p \) is a semi-algebraic set whose complexity depends only on \( b,d \), and \( G \). Let \( S = \{ S_p \mid p \in P \} \). For a query range \( Z_{x,G} \), we now wish to compute the cumulative weight of the sets in \( S \) that contain \( x \). This can be done using Theorem 5.

Putting everything together, we obtain the following:

**Theorem 6.** Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \), let \( w(p) \) be the weight of \( p \in P \) that belongs to a semigroup, and let \( G \) be a fixed \( b \)-variate Boolean function for some constant \( b > 0 \). Let \( t \leq \left( \frac{b+d}{d} \right)^b \) be the dimension of the query space. Assuming that the semigroup operation can be performed in constant time, \( P \) can be preprocessed in \( O_{b,d}(n^{t+\varepsilon}) \) randomized expected time into a data structure of size \( O_{b,d}(n^{t+\varepsilon}) \), for any constant \( \varepsilon > 0 \), so that a \( (G,t) \)-semi-algebraic range query can be answered in \( O_{b,d,\varepsilon}(\log n) \) time.

**Remark.** If \( G \) is the conjunction of a set of \( b \) polynomial inequalities, then the size of the data structure can be significantly improved by using a multi-level data structure, with a slight increase in the query time to \( O(\log^b n) \); see, e.g., [11]. Roughly speaking, the value of \( t \) will now be the dimension of the parametric set of each polynomial defining the query semi-algebraic set, rather than the dimension of the parametric space of the entire semi-algebraic set (which is the conjunction of \( b \) such polynomials).
4.3 Vertical ray shooting

We next present an efficient data structure for answering *vertical ray-shooting* queries: Preprocess a collection $S$ of $n$ semi-algebraic sets in $\mathbb{R}^d$, each of complexity at most $b$, into a data structure so that the first set of $S$ hit by $\rho_q$, the ray emanating in the $(+x_d)$-direction from a query point $q$, can be reported quickly. If there are more than one such set, the query procedure returns one of them, arbitrarily.

**The overall data structure.** Our data structure for vertical ray shooting is similar to the one described in Section 4.1 except that we store an auxiliary data structure at each node of the tree to determine which of its children the query procedure should visit recursively.

Again we fix two constants $D$ and $n_0$. If $n \leq n_0$, the tree data structure $\mathcal{T}$ consists of a single node that stores $S$. So assume that $n > n_0$. We compute a partitioning polynomial tuple $\mathcal{P} = (P_1, \ldots, P_k)$ for $S$ of degree $D$. For each realizable sign condition $\sigma$, we compute the set $S_\sigma \subseteq S$ whose boundaries meet the realization of $\sigma$. We create the root node $v$ of $\mathcal{T}$ and create a child $w_\sigma$ for each realizable sign condition $\sigma$. We store two auxiliary data structures $DS_1(v)$ and $DS_2(v)$ at $v$, described below, each of which can be constructed in $O(n^{d+\epsilon})$ randomized expected time and requires $O(n^{d+\epsilon})$ space. Given a query point $q \in \mathbb{R}^d$, $DS_1(v)$, $DS_2(v)$ together determine, in $O(\log n)$ time, the sign condition $\sigma$ whose realization contains the first intersection point of $\rho_q$ with a set of $S$. We recursively construct the data structure for each subset $S_\sigma$ and attach it to $w_\sigma$ as its subtree. These subtrees together with the root node $v$ and the auxiliary data structure stored at $v$ form the overall data structure $\mathcal{T}$.

A standard analysis of multi-level data structure (see e.g., [1]) shows that the total size of $\mathcal{T}$ is $O(n^{d+\epsilon})$, for any constant $\epsilon > 0$, and that it can be constructed in $O(n^{d+\epsilon})$ randomized expected time.

For a query point $q \in \mathbb{R}^d$, the first set hit by $\rho_q$ can be computed by traversing a root-to-leaf path in $\mathcal{T}$. Suppose we are at a node $v$. If $v$ is a leaf, then we naively check all sets in $S_\sigma$ to find the first among them hit by $\rho_q$. Otherwise, we use the auxiliary data structures $DS_1(v)$ and $DS_2(v)$ to determine in $O(\log n)$ time the sign condition $\sigma$ whose realization contains the first intersection point of $\rho_q$ and a set of $S$. We recursively visit the child $w_\sigma$ of $v$. Since the depth of $\mathcal{T}$ is $O(\log n)$, the total query time is $O(\log^2 n)$.

This completes the description of the overall algorithm. What remains is to describe the auxiliary data structures $DS_1, DS_2$.

**Auxiliary data structures.** Recall that the auxiliary data structures are used to determine the sign condition of $\mathcal{P}$ whose realization contains the first intersection point of a vertical ray with a set of $S$. We first refine the realizations of sign conditions of $\mathcal{P}$ into “cylindrical” cells, as follows. Let $f = \prod_{i=1}^k P_i$; by construction, the degree of $f$ is $O(D)$. By Warren [21, Theorem 2], the number of connected components of $\mathbb{R}^d \setminus Z(f)$ is at most $O(D)^d$; from now on we refer to these components as cells. We refine the cells of $\mathbb{R}^d \setminus Z(f)$ using the so-called first-stage CAD (cylindrical algebraic decomposition); see, e.g., [8, Chapter 5] for a detailed overview of standard CAD. That is, this is a simplified version of CAD, presented in [3].

Roughly speaking, the first-stage CAD for $f$ is obtained by constructing a collection $\mathcal{G}$ of polynomials in the variables $x_1, \ldots, x_{d-1}$, whose zero sets contain the projection onto $\mathbb{R}^{d-1}$ of the locus of points in $Z(f)$ of vertical tangency, self-intersection of zeros sets (roots with multiplicity), or a singularity of some other kind. Having constructed $\mathcal{G}$, the first-stage CAD is obtained as the

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4This notion is somewhat different than the notion of realizable sign conditions, where one can have several connected components representing the same sign condition.
arrangement $\mathcal{A}(\{f\} \cup \mathcal{G})$ in $\mathbb{R}^d$, where the polynomials in $\mathcal{G}$ are now regarded as $d$-variate polynomials, that is, we lift them in the $x_d$-direction; the geometric interpretation of the lifting operation is to erect a “vertical wall” in $\mathbb{R}^d$ over each zero set within $\mathbb{R}^{d-1}$ of a $(d-1)$-variate polynomial from $\mathcal{G}$, and the first-stage CAD is the arrangement of these vertical walls plus $Z(f)$. It follows by construction that the cells of $\mathcal{A}(\{f\} \cup \mathcal{G})$ are vertical stacks of “cylindrical” cells. In more detail, for each cell $\tau$ of $\mathcal{A}(\{f\} \cup \mathcal{G})$, there is unique cell $\varphi$ of the $(d-1)$-dimensional arrangement $\mathcal{A}(\mathcal{G})$ in $\mathbb{R}^{d-1}$ such that one of the following two cases occur: (i) $\tau = \{(x, \xi(x)) \mid x \in \varphi\}$, where $\xi: \varphi \rightarrow \mathbb{R}$ is a continuous semi-algebraic function (i.e., $\tau$ is the graph of $\xi$ over $\varphi$); or (ii) $\tau = \{(x, t) \mid x \in \varphi, t \in (\xi_1(x), \xi_2(x))\}$, where $\xi_i$, $i = 1, 2$ is a continuous semi-algebraic function on $\varphi$, the constant function $\varphi \rightarrow \{-\infty\}$, or the constant function $\varphi \rightarrow \{+\infty\}$, and $\xi_1(x) < \xi_2(x)$ for all $x \in \varphi$ (i.e., $\tau$ is a cylindrical cell over $\varphi$ bounded from below (resp. above) by the graph of $\xi_1$ (resp. $\xi_2$)). As stated in [3], the total number of cells in $\mathcal{A}(\{f\} \cup \mathcal{G})$ is $D^{O(d)}$, and each of them is a semi-algebraic set defined by $D^{O(d^4)}$ polynomials of degree $D^{O(d^4)}$ this is, in fact, an application of Proposition [2].

For a cell $\varphi \in \mathcal{A}(\mathcal{G})$, let $V(\varphi)$ be the stack of cells of $\mathcal{A}(\{f\} \cup \mathcal{G})$ over $\varphi$, i.e., the set of cells that project to $\varphi$.

We note that the sign condition of $\mathcal{P}$ is the same for all points in a cell of $\mathcal{A}(\{f\} \cup \mathcal{G})$, i.e., each cell lies in the realization of a single sign condition of $\mathcal{P}$. It thus suffices to find the cell of $\mathcal{A}(\{f\} \cup \mathcal{G})$ that contains the first intersection point of a vertical ray with a set of $\mathcal{S}$ in order to find the sign condition of $\mathcal{P}$ whose realization contains such a point. We construct $\mathcal{D}_{S_1}, \mathcal{D}_{S_2}$ on the cells of $\mathcal{A}(\{f\} \cup \mathcal{G})$ to quickly determine the desired cell.

**The structure $\mathcal{D}_{S_1}$.** Fix a cell $\tau$ of $\mathcal{A}(\{f\} \cup \mathcal{G})$. $\mathcal{D}_{S_1}$ is used to determine whether a query ray $\rho_q$ whose source point lies in $\tau$ intersects any set of $\mathcal{S}$ inside $\tau$.

For each input set $\mathcal{S}$ that intersects $\tau$, let $\uparrow \mathcal{S}$ be the set of points $x$ in $\mathbb{R}^d$ such that the vertical ray $\rho_x$ intersects $\mathcal{S} \cap \tau$, i.e., $\uparrow \mathcal{S}$ is the union of the rays in the $(-x_d)$-direction emanating from the points of $\mathcal{S} \cap \tau$. $\uparrow \mathcal{S}$ is a semi-algebraic set whose complexity depends only on $b, d$, and $D$. Let $\uparrow \mathcal{S}_\tau = \{\uparrow \mathcal{S} \mid \mathcal{S} \in \mathcal{S}, \mathcal{S} \cap \tau \neq \emptyset\}$. $\uparrow \mathcal{S}_\tau$ can be computed in $O(n)$ time. Using Theorem [5] we process $\uparrow \mathcal{S}_\tau$ into a data structure $\mathcal{D}_{S_1}(\tau)$ of size $O(n^{d+\epsilon})$ so that for a query point $q \in \mathbb{R}^d$, we can determine in $O(\log n)$ time whether $q \in \bigcup \uparrow \mathcal{S}_\tau$, i.e., whether $\rho_q$ intersects any set of $\mathcal{S}$ within $\tau$. We construct $\mathcal{D}_{S_1}(\tau)$ for all cells $\tau$ of $\mathcal{A}(\{f\} \cup \mathcal{G})$. The total size of the data structure, summed over all cells of $\mathcal{A}(\{f\} \cup \mathcal{G})$, is $O(n^{d+\epsilon})$, and it can be constructed in $O(n^{d+\epsilon})$ randomized expected time.

**The structure $\mathcal{D}_{S_2}$.** Fix a cell $\tau$ of $\mathcal{A}(\{f\} \cup \mathcal{G})$. $\mathcal{D}_{S_2}$ is used to determine whether a line parallel to the $x_d$-axis intersects any set of $\mathcal{S}$ inside $\tau$.

For each input set $\mathcal{S}$ that intersects $\tau$, let $\downarrow \mathcal{S}$ be the projection of $\mathcal{S} \cap \tau$ onto the hyperplane $x_d = 0$. For a point $q \in \mathbb{R}^d$, the vertical line (parallel to the $x_d$-axis) through $q$ intersects $\mathcal{S}$ inside $\tau$ if and only if $\downarrow q \in \downarrow \mathcal{S}$ (where $\downarrow q$ is the projection of $q$ onto the hyperplane $x_d = 0$). $\downarrow \mathcal{S}$ is a $(d - 1)$-dimensional semi-algebraic set whose complexity depends only on $b$ and $D$. Let $\downarrow \mathcal{S}_\tau = \{\downarrow \mathcal{S} \mid \mathcal{S} \in \mathcal{S}, \mathcal{S} \cap \tau \neq \emptyset\}$. $\downarrow \mathcal{S}_\tau$ can be constructed in $O(n)$ time. Using Theorem [5] we process $\downarrow \mathcal{S}_\tau$ into a data structure $\mathcal{D}_{S_2}(\tau)$ of size $O(n^{d-1+\epsilon})$ so that for a query point $q \in \mathbb{R}^d$, we can determine in $O(\log n)$ time whether $q \in \bigcup \downarrow \mathcal{S}_\tau$, i.e., whether the vertical line through $q$ intersects any set of $\mathcal{S}$ inside $\tau$. We construct $\mathcal{D}_{S_2}(\tau)$ for all cells of $\mathcal{A}(\{f\} \cup \mathcal{G})$. The total size of the data structure, summed over all cells of $\mathcal{A}(\{f\} \cup \mathcal{G})$, is $O(n^{d-1+\epsilon})$, and it can be constructed in $O(n^{d-1+\epsilon})$ randomized expected time.

**Answering a query.** Given a query point $q \in \mathbb{R}^d$, we determine the cell of $\mathcal{A}(\{f\} \cup \mathcal{G})$ that contains the first intersection point of $\rho_q$ with a set of $\mathcal{S}$ as follows. First, we determine the cell $\tau$ of $\mathcal{A}(\{f\} \cup \mathcal{G})$ that contains the query point $q$. Using $\mathcal{D}_{S_1}(\tau)$, we determine in $O(\log n)$ time whether $\rho_q$ intersects $\mathcal{S}$ inside $\tau$. If the answer is yes, then $\tau$ is the desired cell. So assume that the answer is no. Let $\varphi$ be the cell of the $(d-1)$-dimensional arrangement $\mathcal{A}(\mathcal{G})$ such that $\downarrow q \in \varphi$. Let $V(\varphi) = \{\tau_1, \ldots, \tau_r\}$ be the stack of cells over $\varphi$, and let $\tau = \tau_i$ for some $i \leq k$. We visit the cells
of $V(\tau)$ one by one in order, starting from $\tau_{i+1}$ until we find a cell $\tau_j$ such that $q \in \downarrow S_{\tau_j}$. Since $q$ lies below $\tau_j$, $\rho_q$ intersects $S$ inside $\tau_j$ if and only if $q \in \downarrow (S_{\tau_j})$. If there is no such cell, we conclude that $\rho_q$ does not intersect $S$. Otherwise $\tau_j$ is the cell of $A(\{f\} \cup S)$ that contains the first intersection point of $\rho_q$ with a set of $S$.

Putting everything together we obtain the following.

**Theorem 7.** Let $S$ be a collection of $n$ semi-algebraic sets in $\mathbb{R}^d$, each of complexity at most $b$ for some constant $b > 1$. $S$ can be preprocessed, in $O_{b,d}(n^{d+\varepsilon})$ randomized expected time, into a data structure of size $O_{b,d}(n^{d+\varepsilon})$, for any constant $\varepsilon > 0$, so that a vertical ray-shooting query can be answered in $O_{b,d,\varepsilon}(\log^2 n)$ time.

### 4.4 Cutting algebraic curves into pseudo-segments

Sharir and Zahl [19] presented a technique for cutting algebraic plane curves into pseudo-segments, by lifting curves into three dimensions. More precisely, they prove that $n$ non-overlapping algebraic curves of bounded degree $d$ can be cut into $O_d(n^{3/2 + \varepsilon})$ Jordan arcs so that each pair of arcs intersect in at most one point. Their procedure exploits Proposition 11 for algebraic curves in three dimensions; see [19, Theorem 1.1]. Theorem 4 can be used to prove a slightly weaker constructive and efficient variant of the above result. Specifically, we have:

**Theorem 8.** Let $C$ be a set of $n$ algebraic plane curves, each of degree at most $d$, with no two sharing a common component. Then $C$ can be cut into $O_d(n^{3/2 + C_d/\log \log n})$ Jordan arcs, where $C_d$ is a constant that depends on $d$, so that each pair of arcs intersect in at most one point. This cutting can be computed in randomized expected time $O_d(n^{3/2 + C_d/\log \log n})$.

**Proof Sketch.** The proof of Theorem 1.1 in [19] is completely algorithmic, except for the application of Guth’s partitioning for lifted curves in $\mathbb{R}^3$. In [19], Guth’s partitioning for curves is applied using a parameter $D$ that is polynomial in $n$ (the specific value used is $D = n^{1/4}$, though any value $D = n^c$ with $0 < c < 1/4$ would suffice).

Instead of using a partitioning of degree $D = n^{1/4}$, we will use a partitioning of degree $D = C_d \log n$, where $C_d$ is a sufficiently small constant that depends only on $d$. Throughout this argument, we will use $C_d$ and $c_d$ to denote large and small constants that depend only on $d$.

The partitioning step is applied as follows. We begin at stage $0$ with $(C_d D^3)^0 = 1$ sets, each of which contains $n/(D^2)^0 = n$ curves from $C$. At stage $k$, we apply Theorem 4 to each of the $(C_d D^3)^k$ sets of curves to obtain at most $C_d D^3(C_d D^3)^k = (C_d D^3)^{k+1}$ sets, each of which contains at most $n/D^{2(k+1)}$ curves from $C$, in

$$\left(C_d D^3\right)^k \cdot O_d\left((n/D^{2k}) \text{ Poly}(D) + e^{\text{Poly}(D)}\right) = O_d\left(n(C_d D)^k \text{ Poly}(D) + (C_d D^3)^k e^{\text{Poly}(D)}\right)$$

expected time. Note that the implicit constant in the $\text{Poly}(D)$ terms is independent of $k$.

We repeat this process until each set contains $O(1)$ curves from the original collection; this requires $k_0 = O_d(\log n \log D) \leq \log n/(C_d c_d \log \log n)$ repetitions, and $nD^{k_0} \leq n^{3/2}$.

The total expected time to perform all of these steps is

$$O_d\left(C_d^{\log n/(C_d c_d \log \log n)} n^{3/2} \text{ Poly}(c_d \log n) + (C_d(c_d \log n))^{(C_d c_d \log \log n)} e^{\text{Poly}(c_d \log n)}\right)$$

$$= O_d(n^{3/2 + C_d/\log \log n}).$$

A similar analysis shows that the number of Jordan arcs obtained through this cutting procedure is $O_d(n^{3/2 + C_d/\log \log n})$.

By replacing Theorem 1.1 from [19] with our Theorem 8 we obtain effective and efficient version of all of the subsequent results from [19].
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