Abstract

This paper proposes a thorough theoretical analysis of Stochastic Gradient Descent (SGD) with decreasing step sizes. First, we show that the recursion defining SGD can be provably approximated by solutions of a time inhomogeneous Stochastic Differential Equation (SDE) in a weak and strong sense. Then, motivated by recent analyses of deterministic and stochastic optimization methods by their continuous counterpart, we study the long-time convergence of the continuous processes at hand and establish non-asymptotic bounds. To that purpose, we develop new comparison techniques which we think are of independent interest. This continuous analysis allows us to develop an intuition on the convergence of SGD and, adapting the technique to the discrete setting, we show that the same results hold to the corresponding sequences. In our analysis, we notably obtain non-asymptotic bounds in the convex setting for SGD under weaker assumptions than the ones considered in previous works. Finally, we also establish finite time convergence results under various conditions, including relaxations of the famous Łojasiewicz inequality, which can be applied to a class of non-convex functions.

1 Introduction

This paper consists in a new analysis of the Stochastic Gradient Descent (SGD) algorithm to optimize a continuously differentiable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ given stochastic estimates of its gradient. This problem naturally appears in many applications in statistics and machine learning, see e.g., Berger and Casella (2002); Gentle et al. (2004); Bottou and LeCun (2005); Nemirovski et al. (2009). Nowadays, SGD (Robbins and Monro, 1951), which can be roughly described as a noisy version of the gradient descent algorithm, and its variants (Polyak and Juditsky, 1992; Kingma and Ba, 2014) are very popular due to their simplicity and efficiency.

In the deterministic setting, the gradient descent algorithm defines the same sequence as the Euler discretization of the gradient flow corresponding to $f$, i.e., the Ordinary Differential Equation
(ODE) \( \frac{dx(t)}{dt} = -\nabla f(x(t)) \). It has been observed that the analysis of the long-time behavior of solutions of this gradient flow equation can give insights on the convergence of discrete gradient descent schemes. In particular, this idea has been adapted to the optimal Nesterov acceleration scheme (Nesterov, 1983) by Su et al. (2016) who derived that this algorithm has a limiting continuous flow associated with a second-order ODE. This result then allows for a much more intuitive analysis of this method and this technique has been subsequently applied to prove tighter results (Shi et al., 2018) or to analyze different settings (Krichene et al., 2015; Aujol et al., 2018; Apidopoulos et al., 2019).

Using ordinary differential equations, and in particular the gradient flow equation, to study SGD can also be done (Ljung, 1977; Kushner and Clark, 1978), and has been applied in numerous papers; see e.g., (Méthivier and Priouret, 1984, 1987; Benveniste et al., 1990; Benaim, 1996; Tadić and Doucet, 2017). However, to take into account more precisely the noisy nature of SGD, it has been recently suggested to use Stochastic Differential Equations (SDE) as continuous-time models for the analysis of SGD. Li et al. (2017) introduced Stochastic Modified Equations (SME) and established weak approximations theorems, gaining more intuition on SGD, in particular to obtain new hyper-parameter adjustment policies. In another line of work, Feng et al. (2019) derived uniform in time approximation bounds using ergodic properties of SDE. To our knowledge, these techniques have only been applied to the study of SGD with fixed stepsize.

The first aim and contribution of this paper is to show that SDEs can also be used as continuous-times processes properly modeling SGD with decreasing stepsizes. In particular in Section 2, we establish that there exists a continuous process \( (X_t)_{t \geq 0} \) solution of a time inhomogeneous (in time) SDE which is the limit of SGD with decreasing stepsizes by establishing new weak and strong approximation estimates between the two processes. This result emphasizes the relevance of the continuous dynamics to the analysis of SGD.

However, most of approximation bounds between solutions of SDEs and recursions defined by SGD are derived under a finite time horizon \( T \) and the error (weak or strong) between the discrete and the continuous-time processes does not go to zero as \( T \) goes to infinity, which is a strong limitation to study the long-time behaviour of SGD, see (Li et al., 2017, 2019). Our goal is not to address this problem here, showing uniform in time bounds between the two processes, but to highlight how the long-time behaviour of the continuous process related to SGD can be used to gain more intuition and insight on the convergence of SGD itself. More precisely, in Section 3 we first study the behaviour of \( (t \mapsto \mathbb{E}[f(X_t)] - \min_{d \in \mathbb{R}^d} f) \) which can be quite easily analyzed under different sets of assumptions on \( f \), including a convex and weakly convex setting. Then, we propose a simple adaptation of the main arguments of this analysis to the discrete setting. This allows us to show, under the same conditions, that \( (\mathbb{E}[f(X_n)] - \min_{d \in \mathbb{R}^d} f)_{n \in \mathbb{N}} \) also converges to 0 with explicit rates, where \( (X_n)_{n \in \mathbb{N}} \) is the recursion defined by SGD.

Finally, based on this interpretation, we improve several results on the convergence of SGD obtained in previous works. In the convex setting, we extend the results of (Shamir and Zhang, 2013) to the case where the stepsize is given by \( \gamma/n^\alpha \) with \( \alpha \in (0, 1) \) and relax some of their assumptions. We emphasize that the continuous analysis of this case is entirely new. Finally, we consider a relaxation of the weakly quasi-convex setting introduced in (Hardt et al., 2018). Indeed, since in many applications, and especially in deep learning, the objective function is not convex, studying SGD in non-convex settings has become necessary. A recent work of Orvieto and Lucchi (2019) uses SDEs to analyze SGD and derive convergence rates in some non-convex settings. However the rates they obtained are not optimal and in this paper we show that our analysis leads to better...
rates under weaker assumptions.

2 SGD with Decreasing Stepsizes as a Time Inhomogeneous Diffusion Process

2.1 Problem Setting and Main Assumptions

Throughout the paper we consider an objective function \( f : \mathbb{R}^d \to \mathbb{R} \) satisfying the following condition.

**A 1.** \( f \) is continuously differentiable and \( L \)-smooth with \( L > 0 \), i.e., for any \( x, y \in \mathbb{R}^d \),
\[
\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|.
\]

We consider the general case where we do not have access to \( \nabla f \) but only to an unbiased estimate.

**A 2.** There exists a probability space \((Z, \mathcal{Z}, \mu_Z)\), \( \eta \geq 0 \) and a function \( H : \mathbb{R}^d \times Z \to \mathbb{R}^d \) such that for any \( x \in \mathbb{R}^d \)
\[
\int_Z H(x, z) \, d\mu_Z(z) = \nabla f(x),
\]
\[
\int_Z \|H(x, z) - \nabla f(x)\|^2 \, d\mu_Z(z) \leq \eta.
\]

Under **A 1** and **A 2**, we consider in this paper the sequence \((X_n)_{n \in \mathbb{N}}\) starting from \( X_0 \in \mathbb{R}^d \) corresponding to SGD with decreasing stepsizes and defined for any \( n \in \mathbb{N} \) by
\[
X_{n+1} = X_n - \gamma(n + 1)^{-\alpha} H(X_n, Z_{n+1}),
\] (1)
where \( \gamma > 0 \), \( \alpha \in [0, 1] \) and \((Z_n)_{n \in \mathbb{N}}\) is a sequence of independent random variables on the probability space \((\Omega, \mathcal{F}, P)\) valued in \((Z, \mathcal{Z})\) such that for any \( n \in \mathbb{N}, Z_n \) is distributed according to \( \mu_Z \).

We now turn to the continuous counterpart of (1). Define for any \( x \in \mathbb{R}^d \), the semi-definite positive matrix
\[
\Sigma(x) = \mu_Z(\{H(x, \cdot) - \nabla f(x)\} \{H(x, \cdot) - \nabla f(x)\}^\top),
\]
and, for \( \alpha \in [0, 1] \), consider the time inhomogeneous SDE,
\[
dX_t = -(\gamma_{\alpha} + t)^{-\alpha} \{\nabla f(X_t)\} \, dt + \gamma_{\alpha}^{1/2} \Sigma(X_t)^{1/2} \, dB_t,
\] (2)
where \( \gamma_{\alpha} = \gamma^{1/(1-\alpha)} \) and \((B_t)_{t \geq 0}\) is a \( d \)-dimensional Brownian motion. For solutions of this SDE to exist, we consider the following assumption on \( x \mapsto \Sigma(x)^{1/2} \).

**A 3.** There exists \( M \geq 0 \) such that for any \( x, y \in \mathbb{R}^d \),
\[
\|\Sigma(x)^{1/2} - \Sigma(y)^{1/2}\| \leq M \|x - y\|.
\]

Indeed, using (Karatzas and Shreve, 1991, Chapter 5, Theorem 2.5), solutions \((X_t)_{t \in \mathbb{R}_+}\) exist if **A 1** and **A 3** hold. In the sequel, the process \((X_t)_{t \in \mathbb{R}_+}\) is referred to as the continuous SGD process in contrast to \((X_n)_{n \in \mathbb{N}}\) which is referred to as the discrete SGD process.
2.2 Approximations Results

In this section, we prove that \((X_t)_{t \geq 0}\) solution of (2) is indeed, under some conditions, a continuous counterpart of \((X_n)_{n \in \mathbb{N}}\) given by (1). First, let the ansatz \((\bar{X}_t)_{t \geq 0}\) be a linear interpolation of \((X_n)_{n \in \mathbb{N}}\), such that for any \(n \in \mathbb{N}\), \(\bar{X}_{n/\gamma} = X_n\), with \(\gamma \alpha = \gamma^{(1-\alpha)}\). We have the following approximation, where \(G\) is a \(d\)-dimensional standard Gaussian random variable:

\[
\begin{align*}
\bar{X}_{(n+1)/\gamma} - \bar{X}_{n/\gamma} &= X_{n+1} - X_n \\
&\approx -\gamma(n+1)^{-\alpha} H(\bar{X}_{n/\gamma}, Z_{n+1}) \\
&\approx -\gamma(n+1)^{-\alpha} \{\nabla f(\bar{X}_{n/\gamma}) + \Sigma(\bar{X}_{n/\gamma})^{1/2} G\} \\
&\approx -\gamma \alpha(n\gamma + \gamma \alpha)^{-\alpha} \{\nabla f(\bar{X}_{n/\gamma}) + \Sigma(\bar{X}_{n/\gamma})^{1/2} G\} \\
&\approx -\int_{n/\gamma}^{(n+1)/\gamma} (s + \gamma \alpha)^{-\alpha} \nabla f(\bar{X}_s) ds \\
&- \gamma \alpha^{1/2} \int_{n/\gamma}^{(n+1)/\gamma} (s + \gamma \alpha)^{-\alpha} \Sigma(\bar{X}_s)^{1/2} dB_s. \tag{3}
\end{align*}
\]

The next two results justify the informal derivation (3) and establish strong and weak approximation controls for SGD.

**Proposition 1.** Let \(\gamma > 0\) and \(\alpha \in [0,1)\). Assume A1, A2 and A3.

(a) Then for any \(T \geq 0\), there exists \(C \geq 0\) such that for any \(\gamma \in (0, \gamma]\), \(n \in \mathbb{N}\) with \(\gamma \alpha = \gamma^{(1-\alpha)}\), \(n\gamma \alpha \leq T\) we have

\[
\mathbb{E}^{1/2} \left[\|X_{n/\gamma} - X_n\|^2\right] \leq C \gamma^\delta (1 + \log(\gamma^{-1})) , \tag{4}
\]

with \(\delta = \min(1, (2-2\alpha)^{-1})\).

(b) If in addition, \((Z, \mathcal{Z}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) and for any \(x \in \mathbb{R}^d\), \(z \in \mathbb{R}^d\) and \(n \in \mathbb{N}^*\),

\[
H(x, z) = \nabla f(x) + \Sigma(x)^{1/2} z , \quad Z_n = \gamma^{-1/2} \int_{(n-1)/\gamma}^{n/\gamma} dB_s ,
\]

then (4) holds with \(\delta = 1\).

The proof is postponed to Appendix A3. Note that the logarithmic term in (4) can be omitted if \(\alpha \neq 1/2\). To the best of our knowledge, this strong approximation result is new and illustrates the fundamental difference between SGD and discretization of SDEs such as the Euler-Maruyama (EM) discretization. Consider the SDE

\[
dY_t = b(Y_t) dt + \sigma(Y_t) dB_t , \tag{5}
\]

where \(b: \mathbb{R}^d \to \mathbb{R}^d\), and \(\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times d}\) are Lipschitz functions, so solutions \((Y_t)_{t \geq 0}\) of (5) exist and are pathwise unique, see (Karatzas and Shreve, 1991, Chapter 5, Theorem 2.5). Let \((Y_n)_{n \in \mathbb{N}}\) be the EM discretization of (5) defined for any \(n \in \mathbb{N}\) by

\[
Y_{n+1} = Y_n + \gamma b(Y_n) + \sqrt{\gamma} \sigma(Y_n) G_{n+1} ,\tag{6}
\]

where \(\gamma > 0\) is the stepsize and \((G_n)_{n \in \mathbb{N}}\) is a sequence of i.i.d. \(d\)-dimensional standard Gaussian random variables. Then for any \(T \geq 0\), there exists \(C \geq 0\) such that for any \(\gamma > 0\), \(n \in \mathbb{N}\),
\( n\gamma \leq T, \mathbb{E}^{1/2}[\|Y_{n\gamma} - Y_n\|^2] \leq C\gamma^\delta \) where \( \delta = 1/2 \) if \( \sigma \) is non-constant and \( \delta = 1 \) otherwise; see e.g., (Kloeden and Platen, 2011; Milstein, 1995). Note that the crucial difference between these results and the ones given by Proposition 1 follows from using use decreasing stepsize in the SGD recursion (1) and therefore we obtain as a limiting process a time inhomogeneous process \((X_t)_{t \geq 0}\) for which the coefficient functions in (2) go to 0 as \( t \to +\infty \).

Another difference (for strong approximation) between SGD and the EM discretization scheme is the noise which can be used in these algorithms. Indeed, if \((G_n)_{n \in \mathbb{N}}\) in (5) is no longer a sequence of Gaussian random variables then for \( b \equiv 0, \sigma = I_d \), (but it holds under mild conditions on \( b \) and \( \sigma \)), there exists \( C \geq 0 \) such that for any \( T \geq 0, \gamma > 0, n \in \mathbb{N}, n\gamma \leq T, \mathbb{E}^{1/2}[\|Y_{n\gamma} - Y_n\|^2] \geq C\sqrt{T} \), i.e., no strong approximation holds. The behavior is different for SGD for which we obtain a strong approximation of order 1/2 at least, whatever the noise is in the condition A2.

We also derive weak approximation estimates of order 1. Note that in the case where \( \alpha \geq 1/2 \), these weak results are a direct consequence of Proposition 1. Denote by \( \mathbb{G}_{p,k} \) the set of \( k \)-times continuously differentiable functions \( g \) such that there exists \( K \geq 0 \) such that for any \( x \in \mathbb{R}^d, \max(\|\nabla g(x)\|, \ldots, \|\nabla^k g(x)\|) \leq K(1 + \|x\|^p) \).

**Proposition 2.** Let \( \bar{\gamma} > 0, \alpha \in (0,1) \) and \( p \in \mathbb{N} \). Assume that \( f \in \mathbb{G}_{p,4}, \Sigma^{1/2} \in \mathbb{G}_{p,3}, A_1, A_2 \) and A3. Let \( g \in \mathbb{G}_{p,2} \). In addition, assume that for any \( m \in \mathbb{N} \) and \( x \in \mathbb{R}^d, \mu_Z(\|H(x,\cdot) - \nabla f(x)\|^{2m}) \leq \eta_m \) with \( \eta_m \geq 0 \). Then for any \( T \geq 0 \), there exists \( C \geq 0 \) such that for any \( \gamma \in (0,\bar{\gamma}], n \in \mathbb{N} \) with \( \gamma_\alpha = \gamma^{1/(1-\alpha)}, n\gamma_\alpha \leq T \) we have

\[
|\mathbb{E}[g(X_{n\gamma_\alpha}) - g(X_n)]| \leq C(1 + \log(\gamma^{-1})).
\]  

The proof is postponed to Proposition 24. Note that the logarithmic term in (7) can be omitted if \( \alpha \neq 1/2 \). These results extend (Li et al., 2017, Theorem 1.1 (a)) to the decreasing stepsize case. Once again, the result obtained in Proposition 2 must be compared to similar weak error controls for SDEs. For example, under appropriate conditions, (Talay and Tubaro, 1990) shows that the EM discretization (6) is a weak approximation of order 1 of (5).

### 3 Convergence of the Continuous and Discrete SGD Processes

#### 3.1 Two Basic Comparison Lemmas

We now turn to the convergence of SGD. In the continuous-time setting, in order to derive sharp convergence rates for (2), we will consider appropriate energy functions \( \mathcal{V} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+ \) which will depend on the conditions imposed on the function \( f \). Then, we show that \((t \mapsto v(t) = \mathbb{E}[\mathcal{V}(t, X_t)])\) satisfies an ODE and prove that it is bounded using the following simple lemma.

**Lemma 3.** Let \( F \in C^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}) \) and \( v \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) such that for all \( t \geq 0 \), \( dv(t)/dt \leq F(t, v(t)) \). If there exists \( t_0 > 0 \) and \( A > 0 \) such that for all \( t \geq t_0 \) and for all \( u \geq A, F(t, u) < 0 \), then there exists \( B > 0 \) such that for all \( t \geq 0, v(t) \leq B \), with \( B = \max(\max_{t \in [0,t_0]} v(t), A) \)

**Proof.** Assume that there exists \( t \geq 0 \) such that \( v(t) > B \), and let \( t_1 = \inf \{t \geq 0 : v(t) > B\} \). By definition of \( B \) we have \( t_1 \geq t_0 \), and by continuity of \( v, v(t_1) = B \). By assumption, \( F(t_1, v(t_1)) < 0 \). Therefore \( dv(t_1)/dt < 0 \) and there exists \( t_2 < t_1 \) such that \( v(t_2) > v(t_1) = B \), hence the contradiction.

\[ \square \]
Considering discrete analogues of the energy functions and ODEs found in the study of the continuous SGD process solution of (2), we also derive explicit convergence bounds for the discrete SGD process. To that purpose, we need to establish a discrete counterpart of Lemma 3.

**Lemma 4.** Let \( F : \mathbb{N} \times \mathbb{R} \to \mathbb{R} \) satisfying for any \( n \in \mathbb{N} \), \( F(n, \cdot) \in C^1(\mathbb{R}, \mathbb{R}) \). Let \((u_n)_{n \in \mathbb{N}}\) be a sequence of nonnegative numbers satisfying for all \( n \in \mathbb{N} \), \( u_{n+1} - u_n \leq F(n, u_n) \). Assume that there exist \( n_0 \in \mathbb{N} \) and \( A_1 > 0 \) such that for all \( n \geq n_0 \) and for all \( x \geq A_1 \), \( F(n, x) < 0 \). In addition, assume that there exists \( A_2 > 0 \) such that for all \( n \geq n_0 \) and for all \( x \geq 0 \), \( F(n, x) \leq A_2 \). Then, there exists \( B > 0 \) such that for all \( n \in \mathbb{N} \) \( u_n \leq B \) with \( B = \max(\max_{n \leq n_0+1} u_n, A_1) + A_2 \).

**Proof.** Assume that there exists \( n \in \mathbb{N} \) such that \( u_n > B \), and let \( n_1 = \inf \{ n \geq 0 : u_n > B \} \).

By definition of \( B \) we have \( n_1 \geq n_0 + 1 \). Moreover \( u_{n_1} - u_{n_1-1} \leq F(n_1 - 1, u_{n_1-1}) \).

Since \( n_1 - 1 \geq n_0 \) we get that \( u_{n_1} - u_{n_1-1} \leq A_2 \) and \( u_{n_1-1} \geq u_{n_1} - A_2 \geq A_1 \). Consequently, \( F(n_1 - 1, u_{n_1-1}) < 0 \) and \( u_{n_1} < u_{n_1-1} \), which is a contradiction. \( \square \)

We now state our convergence results under various assumptions on the function \( f \).

### 3.2 Strongly-Convex Case

First, we illustrate the simplicity and effectiveness of our approach establishing sharp convergence rates under the first following assumption.

**F1.** \( f \) is \( \mu \)-strongly convex with \( \mu > 0 \), i.e., for any \( x, y \in \mathbb{R}^d \), \( \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \| x - y \|^2 \).

The results presented below are not new, see (Bach and Moulines, 2011) for the discrete case and (Orvieto and Lucchi, 2019) for the continuous one, but they can be obtained very easily within our framework. We only present our results in the continuous-time setting and gather their discrete counterparts in Appendix B. First, we derive convergence rates on the last iterates. Denote under **F1** by \( x^* \) the unique minimizer of \( f \).

**Theorem 5.** Let \( \alpha, \gamma \in (0, 1) \) and \((X_t)_{t \geq 0}\) be given by (2). Assume **A1, A2, A3** and **F1**. Then there exists \( C \geq 0 \) (explicit in the proof) such that for any \( T \geq 1 \), \( E\|X_T - x^*\|^2 \leq CT^{-\alpha} \).

**Proof.** Let \( \alpha, \gamma \in (0, 1] \) and consider \( \mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+ \) defined for \( t \geq 0 \) by \( \mathcal{E}(t) = E[(t+\gamma)^\alpha \|X_t - x^*\|^2] \), with \( \gamma = \gamma^{1/(1-\alpha)} \). Using Dynkin’s formula, see Lemma 32, we have for any \( t \geq 0 \),

\[
\mathcal{E}(t) = \mathcal{E}(0) + \alpha \int_0^t \mathcal{E}(s)(s + \gamma)^{-\alpha} ds \\
+ \int_0^t \gamma \alpha E[\text{Tr}(\Sigma(X_s))](s + \gamma)^\alpha ds - 2 \int_0^t E[\langle \nabla f(X_s), X_s - x^* \rangle] ds.
\]

We now differentiate this expression with respect to \( t \) and using **F1**, we get for any \( t > 0 \),

\[
d\mathcal{E}(t)/dt = \alpha \mathcal{E}(t)(t + \gamma)^{-\alpha} - 2E[\langle \nabla f(X_t), X_t - x^* \rangle] + \gamma \alpha E[\text{Tr}(\Sigma(X_t))](t + \gamma)^{-\alpha} \\
\leq \alpha \mathcal{E}(t)(t + \gamma) - 2\mu E[\|X_t - x^*\|^2] + \gamma \eta/(t + \gamma)^\alpha \\
\leq F(t, \mathcal{E}(t)) = \alpha \mathcal{E}(t)(t + \gamma)^{-\alpha} - 2\mu \mathcal{E}(t)(t + \gamma)^{-\alpha} + \gamma \eta(t + \gamma)^{-\alpha} ,
\]

where we have used in the penultimate line that \( \text{Tr}(\Sigma(x)) \leq \eta \) for any \( x \in \mathbb{R}^d \) by **A2**. Hence, since \( F \) satisfy the conditions of Lemma 3 with \( t_0 = (\alpha/\mu)^{1/(1-\alpha)} \) and \( A = \gamma \eta \mu \), applying this result we get that, for any \( t \geq 0 \), \( \mathcal{E}(t) \leq B \) with \( B = \max(\max_{s \in [0, t_0]} \mathcal{E}(s), A) \) which concludes the proof. \( \square \)
In Theorem 27, we extend this result to the discrete setting and recover the rates obtained (Bach and Moulines, 2011, Theorem 1) in the case where $\alpha \in (0, 1]$. In particular, if $\alpha = 1$ then we obtain a convergence rate of order $O(T^{-1})$ which matches the minimax lower-bounds established in Nemirovsky and Yudin (1983); Agarwal et al. (2012). In Figure 1 and Figure 2, we experimentally check that the result we obtain are tight in the simple case where $f(x) = \|x\|^2$ and using synthetic data.

![Figure 1: Evolution of (log($\mathbb{E}[f(X_n)] - \min_{\mathbb{R}^d} f$))$_{n \in \mathbb{N}}$ for different values of $\alpha$. $\mathbb{E}[f(X_n)]$ is approximated by Monte-Carlo using $10^4$ SGD trajectories. Note that for high values of $\alpha$, the asymptotic regime $O(T^{-\alpha})$ is achieved for large $T \geq 0$ only.](image)

Note that in Theorem 5 we do not assume any additional regularity assumption other than $f \in C^1(\mathbb{R}^d, \mathbb{R})$. Also we emphasize that the strong convexity assumption can be relaxed if we only assume that $f$ is weakly $\mu$-strongly convex, i.e., for any $x \in \mathbb{R}^d$, $\langle \nabla f(x), x - x^* \rangle \geq \mu \|x - x^*\|^2$. In, Kleinberg et al. (2018), the authors experimentally show that modern neural networks satisfy a relaxation of this last condition and it was proved in Li and Yuan (2017) that two-layer neural networks with ReLU activation functions are weakly $\mu$-strongly convex if the inputs are Gaussian.

Under the additional assumption that $f$ is smooth, Theorem 5 also implies convergence rates for the process {$\mathbb{E}[f(X_t)] - \min_{\mathbb{R}^d} f$}$_{t \geq 0}$, see Corollary 26. Similarly, this result can be extended to the discrete setting, see Corollary 28, and is again optimal as it matches the minimax lower-bounds Nemirovsky and Yudin (1983); Agarwal et al. (2012).

It should be noted that the strong convexity assumption F1 can be replaced by the weaker following Kurdyka-Łojasiewicz condition: there exists $\tau > 0$, such that for any $x \in \mathbb{R}^d$, $\|\nabla f(x)\|^2 \geq \tau(f(x) - \min_{\mathbb{R}^d} f)$, see Proposition 30. In Section 3.4 we derive similar rates of convergence in the case where $f$ satisfies a relaxation of the Kurdyka-Łojasiewicz inequality.

3.3 Convex Case

In this section, we relax the strong-convexity condition.
For each $\alpha$ in Figure 1 we estimate the slope of $(\log(\mathbb{E}[f(X_n)]) - \min_{n \in \mathbb{N}} f))_n \in \mathbb{N}$ using linear regression. The rates we identify match the theoretical rates derived in Theorem 5.

**F 2.** $f$ is convex, i.e., for any $x, y \in \mathbb{R}^d$, $(\nabla f(x) - \nabla f(y), x - y) \geq 0$. In addition, there exists $x^* \in \arg \min_{\mathbb{R}^d} f$.

We start by studying the continuous process as for the strong convex case under this weaker condition. The discrete analog is given in Theorem 8 after.

**Theorem 6.** Let $\alpha, \gamma \in (0, 1)$ and $(X_t)_{t \geq 0}$ be given by (2). Assume $f \in C^2(\mathbb{R}^d, \mathbb{R})$, $A_1$, $A_2$, $A_3$ and F2. Then, there exists $C \geq 0$ (explicit and given in the proof) such that for any $T \geq 1$,

$$
\mathbb{E}[f(X_T)] - \min_{\mathbb{R}^d} f \leq C \left(1 + \log(T)\right)^2 / T^{\alpha \wedge (1 - \alpha)}.
$$

The general proof is postponed to Appendix C.2. The main strategy to show Theorem 6 is to carefully analyze a continuous version of the suffix averaging (Shamir and Zhang, 2013; Harvey et al., 2019), introduced in the discrete case by Zhang (2004).

To the best of our knowledge, these non-asymptotic results are new for the continuous process $(X_t)_{t \geq 0}$ defined by (2). Note that for $\alpha = 1/2$ the convergence rate is of order $O(T^{-1/2} \log^2(T))$ which matches (up to a logarithmic term) the minimax lower-bound (Agarwal et al., 2012) and is in accordance with the tight bounds derived in the discrete case under additional assumptions (Shamir and Zhang, 2013).

We can relax the assumption $f \in C^2(\mathbb{R}^d, \mathbb{R})$ if we assume that the set $\arg \min_{\mathbb{R}^d} f$ is bounded.

**Theorem 7.** Let $\alpha, \gamma \in (0, 1)$ and $(X_t)_{t \geq 0}$ be given by (2). Assume that $\arg \min_{\mathbb{R}^d} f$ is bounded, $A_1$, $A_2$, $A_3$ and F2. Then, there exists $C \geq 0$ (explicit and given in the proof) such that for any $T \geq 1$,

$$
\mathbb{E}[f(X_T)] - \min_{\mathbb{R}^d} f \leq C (1 + \log(T))^2 / T^{\alpha \wedge (1 - \alpha)}.
$$

The proof is postponed to Appendix C.1 and relies on the fact that if $f$ is convex then for any $\varepsilon > 0$, $f \ast g_\varepsilon$ is also convex, where $(g_\varepsilon)_{\varepsilon > 0}$ is a family of non-negative mollifiers.

We now turn to the discrete counterpart of Theorem 6.
Theorem 8. Let $\gamma, \alpha \in (0, 1)$ and $(X_n)_{n \geq 0}$ be given by (1). Assume A1, A2 and F2. Then, there exists $C > 0$ (explicit and given in the proof) such that for any $N \geq 1$,

$$
\mathbb{E}[f(X_N)] - \min_{x \in \mathbb{R}^d} f \leq C(1 + \log(N + 1))^2/(N + 1)^{\alpha(1 - \alpha)}.
$$

The proof is postponed to Appendix C.3. Once again the proof of this discrete result follows the one of its continuous analogue.

Note that in the case $\alpha = 1/2$ we recover (up to a logarithmic term) the rate $O(N^{-1/2} \log(N + 1))$ derived in (Shamir and Zhang, 2013, Theorem 2) which matches the minimax lower-bound Agarwal et al. (2012), up to a logarithmic term. We also extend this result to the case $\alpha \neq 1/2$. Note however that our setting differs from theirs. (Shamir and Zhang, 2013, Theorem 2) established the convergence rate for a projected version of SGD onto a convex compact set of $\mathbb{R}^d$ under the assumption that $f$ is convex (possibly non-smooth) and $(\mathbb{E}[\|H(X_n, Z_{n+1})\|^2])_{n \in \mathbb{N}}$ is bounded.

In addition to these two conditions, one crucial part of the analysis of (Shamir and Zhang, 2013, Theorem 2) uses that $(\mathbb{E}[\|X_n - x^*\|^2])_{n \in \mathbb{N}}$ is bounded which is possible since $(X_n)_{n \in \mathbb{N}}$ in their setting stays in a compact. In Theorem 8, we replace the conditions considered in (Shamir and Zhang, 2013, Theorem 2) by A1. Actually our proof can be very easily adapted to the simpler setting where $(\mathbb{E}[\|H(X_n, Z_{n+1})\|[2]])_{n \in \mathbb{N}}$ is supposed to be bounded instead of A1. We present this result in Corollary 39. To conclude, (Shamir and Zhang, 2013, Theorem 2) and Theorem 8 while having the same conclusions have really different sets of assumptions and therefore complement each other.

On the other hand, the setting we consider is the same as (Bach and Moulines, 2011), but we always obtain better convergence rates and in particular we get an optimal choice for $\alpha (1/2)$ different from their (2/3), see Table 1.

Table 1: Convergence rates for convex SGD under different settings (B: Bounded Gradients, L: Lipschitz Gradient), up to the logarithmic terms

| Ref. Cond. | Thm.8 | (BM’11) | (BM’11) |
|------------|-------|---------|---------|
| $\alpha \in (0, 1/3)$ | $\alpha$ | $\times$ | $\times$ |
| $\alpha \in (1/3, 1/2)$ | $\alpha$ | $(3\alpha - 1)/2$ | $\times$ |
| $\alpha \in (1/2, 2/3)$ | $1 - \alpha$ | $\alpha/2$ | $\alpha/2$ |
| $\alpha \in (2/3, 1)$ | $1 - \alpha$ | $1 - \alpha$ | $1 - \alpha$ |

In Figure 3, we experimentally assess the results of Theorem 8. We perform SGD on the family of functions $(\varphi_p)_{p \in \mathbb{N}^*}$, where for any $x \in \mathbb{R}$, $p \in \mathbb{N}^*$

$$
\varphi_p(x) = \begin{cases} 
  x^{2p}, & \text{if } x \in [-1, 1], \\
  2p(|x| - 1) + 1, & \text{otherwise}.
\end{cases}
$$

(8)

For any $p \in \mathbb{N}$, $\varphi_p$ satisfies and A1 and F2. Denoting $\alpha_p^*$ the decreasing rate $\alpha$ for which the convergence rate $r_p$ is maximum, we experimentally check that $\lim_{p \to +\infty} r_p = 1/2$ and $\lim_{p \to +\infty} \alpha_p^* = 1/2$. Note also that $\alpha_p^*$ decreases as $p$ grows, which is in accordance with the deterministic setting where the optimal rate in this case is given by $p/(p - 2)$, see Bolte et al. (2017); Frankel et al. (2015).

As an immediate consequence of Theorem 8, we can show that $(\mathbb{E}[\|\nabla f(X_n)\|^2])_{n \in \mathbb{N}}$ enjoys the same rates of convergence as $(\mathbb{E}[f(X_n)] - \min_{x \in \mathbb{R}^d} f)_{n \in \mathbb{N}}$, using that $f$ is smooth, and is in particular bounded.
Figure 3: Convergence rates for the functions $\varphi_p$ defined by (8) match the theoretical results of Theorem 8 asymptotically, i.e., when $p$ is large. Note that for large $\alpha$ and large $p$, the discrepancy between the theoretical rates and the experimental ones is due to the fact that we reached machine precision for such decreasing stepsizes.

**Corollary 9.** Let $\gamma, \alpha \in (0, 1)$ and $(X_n)_{n \geq 0}$ be given by (1). Assume A1, A2 and F2. Then, there exists $C \geq 0$ (explicit and given in the proof) such that for any $N \geq 1$,

$$
\mathbb{E} \left[ \|\nabla f(X_N)\|^{2} \right] \leq C(1 + \log(N + 1))^{2}/(N + 1)^{\alpha \wedge (1-\alpha)}.
$$

In particular, $(\mathbb{E}[\|\nabla f(X_n)\|^{2}])_{n \in \mathbb{N}}$ is bounded which is often found as an assumption for the study of the convergence of SGD in the convex setting, Shalev-Shwartz et al. (2011); Nemirovski et al. (2009); Hazan and Kale (2014); Shamir and Zhang (2013); Recht et al. (2011). Relaxing this assumption is one of our main contributions.

### 3.4 Weakly Quasi-Convex Case

In this section, we no longer consider that $f$ is convex but a relaxation of this condition.

**F3.** There exist $r_1 \in (0, 2)$, $r_2 \geq 0$, $\tau > 0$ such that for any $x \in \mathbb{R}^d$

$$
\|\nabla f(x)\|^{r_1} \|x - x^*\|^{r_2} \geq \tau(f(x) - f(x^*)),
$$
where $x^* \in \text{arg min}_{\mathbb{R}^d} f \neq \emptyset$.

This setting is a generalization of the weakly quasi-convex assumption considered in Orvieto and Lucchi (2019) and introduced in Hardt et al. (2018) as follows.

**F3b.** The function $f$ is weakly quasi-convex if there $\tau > 0$ such that for any $x \in \mathbb{R}^d$

$$
\langle \nabla f(x), x - x^* \rangle \geq \tau(f(x) - f(x^*)),
$$
where $x^* \in \text{arg min}_{\mathbb{R}^d} f \neq \emptyset$. 
This last condition itself is a modification of the quasi-convexity assumption Hazan et al. (2015). It was shown in Hardt et al. (2018) that an idealized risk for linear dynamical system identification is weakly quasi-convex, and in Yuan et al. (2019), the authors experimentally check that a residual network (ResNet20) used on CIFAR-10 (with differentiable activation units) satisfy the weakly quasi-convex assumption.

The assumption \( F_3 \) also embeds the setting where \( f \) satisfies some Kurdyka-Łojasiewicz condition Bolte et al. (2017), i.e., if there exist \( r \in (0,2) \) and \( \tau > 0 \) such that for any \( x \in \mathbb{R}^d \),

\[
\|\nabla f(x)\|^r \geq \tau (f(x) - f(x^*))
\]

then \( F_3 \) is satisfied with \( r_1 = r, r_2 = 0 \) and \( r = \tau \). Kurdyka-Łojasiewicz conditions have been often used in the context of non-convex minimization (Attouch et al., 2010; Noll, 2014). Even though the case \( r_1 = 2 \) and \( r_2 = 0 \) is not considered in \( F_3 \), one can still derive convergence of order \( \alpha \) for \( \alpha \in (0,1) \), see Proposition 30, extending the results obtained in the strongly convex setting. We now state the main theorem of this section.

**Theorem 10.** Let \( \alpha, \gamma \in (0,1) \) and \( (X_t)_{t\geq 0} \) be given by (2). Assume \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \), \( A_1, A_2, A_3 \) and \( F_3 \). In addition, assume that there exist \( \beta, \epsilon \geq 0 \) and \( C_{\beta, \epsilon} \geq 0 \) such that for any \( t \geq 0 \),

\[
\mathbb{E}[\|[X_t - x^*]^{r_{x_3}}\|^{r_{x_3}}] \leq C_{\beta, \epsilon}(\gamma_\alpha + t)^\beta (1 + \log(1 + \gamma_\alpha^{-1}t))^\epsilon,
\]

where \( \gamma_\alpha = \gamma^{1/(1-\alpha)} \) and \( r_3 = (1 - r_1/2)^{-1} \). Then, there exists \( C \geq 0 \) (explicit and given in the proof) such that for any \( T \geq 1 \)

\[
\mathbb{E}[f(X_T)] - \min_{\mathbb{R}^d} f \leq CT^{-\delta_1 \wedge \delta_2} [1 + \log(1 + \gamma_\alpha^{-1}T)]^\epsilon,
\]

where

\[
\begin{align*}
\delta_1 &= (r_1/2)(1 - r_1/2)^{-1}(1 - \alpha) - \beta, \\
\delta_2 &= (r_1/2)\alpha - \beta(1 - r_1/2).
\end{align*}
\]

**Proof.** We give the proof for \( \epsilon = 0 \) and \( r_1 = r_2 = 1 \). The proof in the general case can be found in Appendix D. Without loss of generality, we assume that \( \min_{\mathbb{R}^d} f = 0 \). Let \( \alpha, \gamma \in (0,1) \), set \( \delta = \min(\delta_1, \delta_2) = (1 - \alpha - \beta) / (\alpha - \beta/2) \), with \( \delta_1 \) and \( \delta_2 \) given in (11), and define for any \( t \geq 0 \),

\[
a(t) = \gamma_\alpha + t.
\]

Consider \( \mathcal{E} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) defined for any \( t \geq 0 \) by \( \mathcal{E}(t) = \mathbb{E}[f(X_t)a(t)^\delta] \). Using Dynkin’s formula, see Lemma 32, we have for any \( t \geq 0 \),

\[
\mathcal{E}(t) - \mathcal{E}(0) = \int_0^t \{-a(s)^{\delta - \alpha} \mathbb{E}[\|\nabla f(X_s)\|^2] + \delta a(s)^{\delta - 1} \mathbb{E}[f(X_s)] \\
+ \gamma_\alpha a(s)^{\delta - 2\alpha} \mathbb{E} \left[ \text{Tr}(\nabla^2 f(X_s) \Sigma(X_s)) \right] / 2 \} ds,
\]

and \( (t \mapsto \mathcal{E}(t)) \) is differentiable. In addition, by \( A_1 \) and \( A_2 \), using Lemma 31, we have that almost surely, \( \text{Tr}(\nabla^2 f(X_s) \Sigma(X_s)) \leq \eta L \). Therefore, taking the derivative with respect to the time \( t \) in (12), we get

\[
d\mathcal{E}(t)/dt \leq -a(t)^{\delta - \alpha} \mathbb{E}[\|\nabla f(X_t)\|^2] + \delta a(t)^{-1} \mathcal{E}(t) + \gamma_\alpha a(t)^{\delta - 2\alpha} \eta L/2.
\]

Using \( F_3 \) and Cauchy-Schwarz’s inequality we have for any \( t \geq 0 \)

\[
\tau \mathbb{E}[f(X_t)] \leq \mathbb{E}[\|X_t - x^*\|^2]^{1/2} \mathbb{E}[\|\nabla f(X_t)\|^2]^{1/2}.
\]
Hence, we get by (10), since we set $\varepsilon = 0$, $r_2 r_3 = 2$, for any $t \geq 0$
\[ \tau^2 C_{b, \varepsilon}^{-1} \mathcal{E}(t)^2 a(t)^{-2\delta - \beta} \leq \mathbb{E}[\|\nabla f(X_t)\|^2] \, . \]
Therefore by (13), we have for any $t > 0$, $d\mathcal{E}(t)/dt \leq F(t, \mathcal{E}(t))$, where for any $t \in \mathbb{R}_+$ and $u \in \mathbb{R}$,
\[ F(t, u) = -\tau^2 C_{b, \varepsilon}^{-1} a(t)^{-(\delta + \beta + \alpha)} u^2 + \delta a(t)^{-1} u + \gamma a(t)^{\delta - 2\alpha} \ln / 2 \, . \]
Let $C_3 = \max(C_1, C_2) + 1$ with
\[ C_1 = \begin{cases} \delta & C_{b, \varepsilon} \tau^{-2} \gamma \alpha^{\delta + \beta + \alpha - 1}, \\ \frac{\ln C_{b, \varepsilon} \tau^{-2} \gamma \alpha^{\delta + \beta + \alpha + 1}}{2^{1/2}} & \end{cases} \]
Then, it is easy to verify that if $u \geq C_3$ then $F(t, u) < 0$ for any $t \geq 0$.
Setting $C = \max(C_3, \max_{s \in [0, 1]} \mathcal{E}(s))$ and using Lemma 3 we obtain that for any $t \geq 0$, $\mathcal{E}(t) \leq C$, which concludes the proof. □

First, note that if $f$ satisfies a Kurdyka-Łojasiewicz condition of type (9) then F3 is satisfied with $r_1 = r$ and $r_2 = 0$ and the rates in Theorem 10 simplify and we obtain that $\delta = \min((r/2)(1 - r/2)^{-1}(1 - \alpha), (r/2)\alpha)$. The rate is maximized for $\alpha = (2 - r/2)^{-1}$ and in this case, $\delta = r/(4 - r)$. Therefore, if $r \to 2$, then $\delta \to 1$ and we obtain at the limit the same convergence rate that the case where $f$ is strongly convex F1.

In the general case, $r_2 \neq 0$, the convergence rates obtained in Theorem 10 depends on $\beta$ where $(\mathbb{E}[\|X_t - x^*\|^2 (\gamma_\alpha + t)^{-\beta})_{t \geq 0}$ has at most logarithmic growth. If $\beta \neq 0$, then the convergence rates deteriorate. In what follows, we shall consider different scenarios under which $\beta$ can be explicitly controlled. These estimates immediately imply explicit convergence rates for SGD using Theorem 10.

**Corollary 11.** Let $\alpha, \gamma \in (0, 1)$ and $(X_t)_{t \geq 0}$ be given by (2). Assume $f \in C^2(\mathbb{R}^d, \mathbb{R})$, A1, A2 and A3.
(a) If F3b holds, then there exists $C \geq 0$ such that for any $T \geq 1$
\[ \mathbb{E}[f(X_T)] - \min_{x \in \mathbb{R}^d} f \leq C[T^{1 - 3\alpha/2} + T^{-\alpha/2} + T^{\alpha - 1}] \, . \]
(b) If F3b holds and there exist $c, R > 0$ such that for any $x \in \mathbb{R}^d$ with $\|x - x^*\| \geq R$, $f(x) - f(x^*) \geq c\|x - x^*\|$ then there exists $C \geq 0$ such that for any $T \geq 1$
\[ \mathbb{E}[f(X_T)] - \min_{x \in \mathbb{R}^d} f \leq C[T^{-\alpha/2} + T^{\alpha - 1}] \, . \] (14)
(c) If F3 holds and if there exists $R \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x\| \geq R$, $\langle \nabla f(x), x - x^* \rangle \geq \mathfrak{m}\|x - x^*\|^2$, then there exists $C \geq 0$ such that for any $T \geq 1$, (14) holds.

The proof is postponed to Appendix D. The main ingredient of the proof is to control the growth of $t \mapsto \mathbb{E}[\|X_t - x^*\|^2]$ using either the SDE satisfied by $(\|X_t - x^*\|^2)_{t \geq 0}$ in the case of (a) and (c), or the SDE satisfied by $(f(X_t) - \min_{x \in \mathbb{R}^d} f)_{t \geq 0}$ in the case of (b).

Under F3b, we compare the rates we obtain using Corollary 11-(a) with the ones derived by Orvieto and Lucchi (2019) in Table 2. Note that compared to Orvieto and Lucchi (2019), we establish that SGD converges as soon as $\alpha > 1/3$ and not $\alpha > 1/2$. In addition, the convergence rates we obtain are always better than the ones of Orvieto and Lucchi (2019) in the case $\alpha > 1/2$. 

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Table 2: Rates for continuous SGD with non-convex assumptions

| Ref.   | Cor.11-(a) | Cor.11-(b) | (OL’19) |
|--------|------------|------------|---------|
| $\alpha \in (0, 1/3)$ | $\times$ | $\alpha/2$ | $\times$ |
| $\alpha \in (1/3, 1/2)$ | $(3\alpha - 1)/2$ | $\alpha/2$ | $\times$ |
| $\alpha = 1/2$ | $1/4 + \log.$ | $1/4 + \log.$ | $\times$ |
| $\alpha \in (1/2, 2/3)$ | $\alpha/2$ | $1 - \alpha$ | $2\alpha - 1$ |
| $\alpha \in (2/3, 1)$ | $1 - \alpha$ | $1 - \alpha$ | $1 - \alpha$ |

However, note that in both cases, the optimal convergence rate is $1/3$ obtained using $\alpha = 2/3$. Finally, under additional growth conditions on the function $f$, and using Corollary 11-(b)-(c) we show that the convergence of SGD in the weak quasi-convex case occurs as soon as $\alpha > 0$.

As in the previous sections, we extend our results to the discrete setting.

**Theorem 12.** Let $\alpha, \gamma \in (0, 1)$ and $(X_n)_{n \in \mathbb{N}}$ be given by (1). Assume A1, A2 and F3. In addition, assume that there exist $\beta, \varepsilon, C_{\beta, \varepsilon} \geq 0$ such that for any $n \in \mathbb{N}$, $\mathbb{E}[\|X_n - x^*\|^{r_3}] \leq C_{\beta, \varepsilon}(n + 1)^{\beta 1 + \log(1 + n)}/2$}, where $r_3 = (1 - r_1/2)^{-1}$. Then, there exists $C \geq 0$ (explicit and given in the proof) such that for any $N \geq 1$

$$
\mathbb{E}[f(X_N)] - \min_{x \in \mathbb{R}^d} f \leq CN^{-\delta_1 \wedge \delta_2} (1 + \log(1 + N))\varepsilon,
$$

where $\delta_1, \delta_2$ are given in (11).

The proof is postponed to Appendix D. We can conduct the same discussion as the one after Theorem 10 and Corollary 11 can be extended to the discrete case, see Corollary 45 in Appendix D.

4 Conclusion

In this paper we investigated the connection of SGD with solutions of a particular time inhomogeneous SDE. We first proved (strong and weak) approximation bounds between these two processes motivating convergence analysis of continuous SGD. Then, we turned to the convergence behavior of SGD and showed how the continuous process can provide a better understanding of SGD using tools from ODE and stochastic calculus. In particular, we obtained optimal convergence rates in the strongly convex and convex cases. In the non-convex setting, we considered a relaxation of the weakly quasi-convex condition and improved the state-of-the-art convergence rates in both the continuous and discrete-time setting.

In this work, we assumed in our conditions that the variance of stochastic gradients is finite which implicitly implies that the noise of the continuous dynamics follows from a Brownian motion. However, Simsekli et al. (2019) showed empirically that the noise of applications of SGD for neural networks follows from $\alpha$-distributions with $\alpha < 2$ which do not fall within our framework. We would like to pursue in our next analysis the effect of these heavy-tailed distributions on SGD using appropriate Lévy processes.
convex (Thm 6)  strongly convex
deterministic  Cor.11-(a)
Cor.11-(b)  (OL’19)

Figure 4: Comparison of the convergence rates obtained in the continuous case, against the results of (Orvieto and Lucchi, 2019) in the strongly convex, convex and weakly quasi convex settings.

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18
A Approximation Results

In this section, we present the proof of Proposition 1 in Appendix A.3 and the one of Proposition 2 in Appendix A.4. We begin this section by some useful technical lemmas and results on moment bounds. Throughout this section we will denote all the constants by the letter A followed by some subscript.

A.1 Technical Lemmas

Lemma 13. Let \((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}\) and \((w_n)_{n \in \mathbb{N}}\) such that for any \(n \in \mathbb{N}\), \(u_n, v_n, w_n \geq 0\), \(u_0 \geq 0\) and \(u_{n+1} \leq (1 + v_n)u_n + w_n\). Then for any \(n \in \mathbb{N}\)

\[
 u_n \leq \exp \left[ \sum_{k=0}^{n-1} v_k \right] \left( u_0 + \sum_{k=0}^{n-1} w_k \right).
\]

Proof. The proof is a straightforward consequence of the discrete Grönwall’s lemma.

Lemma 14. Let \(r > 0\), \(\gamma > 0\) and \(\alpha \in [0, 1)\). Then for any \(T \geq 0\), there exists \(A_{\alpha, r} \geq 0\) such that for any \(N \in \mathbb{N}\) with \(N \gamma \alpha \leq T\) we have

\[
 \gamma^r \sum_{k=0}^{N-1} (k + 1)^{-\alpha r} \leq \begin{cases} A_{\alpha, r} \gamma^r (1 + \log(\gamma^{-1}))(1 + \log(T)), & \text{if } \alpha \geq 1/r, \\ A_{\alpha, r} \gamma^r \gamma_{\alpha^r}^{-1} T^{1-\alpha r} & \text{otherwise}. \end{cases}
\]

Note that if \(r = 1\) then \(\gamma^r \sum_{k=0}^{N-1} (k + 1)^{-\alpha r} \leq A_{\alpha, 1} T^{1-\alpha}\).

Proof. Let \(r > 0\), \(\gamma > 0\) and \(\alpha \in [0, 1)\). If \(\alpha > 1/r\) then there exists \(A_{\alpha, r} \geq 0\) such that

\[
 \gamma^r \sum_{k=0}^{N-1} (k + 1)^{-\alpha r} \leq A_{\alpha, r} \gamma^r.
\]

If \(\alpha < 1/r\) then there exists \(A_{\alpha, r} \geq 0\) such that

\[
 \gamma^r \sum_{k=0}^{N-1} (k + 1)^{-\alpha r} \leq A_{\alpha, r} \gamma^r N^{-\alpha r+1} \leq A_{\alpha, r} \gamma^r \gamma_{\alpha^r}^{-1} T^{1-\alpha r}.
\]

if \(\alpha = 1/r\) then there exists \(A_{\alpha, r} \geq 0\) such that

\[
 \gamma^r \sum_{k=0}^{N-1} (k + 1)^{-\alpha r} \leq \gamma^r (1 + \log(N)) \leq A_{\alpha, r} \gamma^r (1 + \log(T))(1 + \log(\gamma^{-1})).
\]

A.2 Moment bounds

The following result is well-known in the field of SDE but its proof is given for completeness.
Lemma 15. Let $p \in \mathbb{N}$, $\bar{\gamma} > 0$ and $\alpha \in [0,1)$. Assume $\textbf{A1}$ and $\textbf{A2}$. Then for any $T \geq 0$, there exists $\mathcal{A}_{T,1} \geq 0$, such that for any $s \geq 0$ and $t \in [s,s+T]$, $\gamma \in (0,\bar{\gamma}]$ and $X_0 \in \mathbb{R}^d$, we have

$$\mathbb{E} \left[ 1 + \|X_t\|^{2p} \right] \leq \mathcal{A}_{T,1} (1 + \|X_0\|^{2p}),$$

where $(X_t)_{t \geq 0}$ is the solution of (2) such that $X_s = X_0$.

If in addition, for any $x \in \mathbb{R}^d$, $\mu_Z(\|H(x,\cdot) - \nabla f(x)\|^{2p}) \leq \eta_p$, with $\eta_p \geq 0$, then for any $T \geq 0$, there exists $\mathcal{A}_{T,1} \geq 0$, such that for any $k_0 \geq 0$, $\gamma \in (0,\bar{\gamma}]$ and $k \in \{k_0,\ldots,k_0+N\}$ with $N\gamma_\alpha \leq T$, and $X_0 \in \mathbb{R}^d$, we have

$$\mathbb{E} \left[ 1 + \|X_k\|^{2p} \right] \leq \mathcal{A}_{T,1} (1 + \|X_0\|^{2p}),$$

where $(X_k)_{k \in \mathbb{N}}$ satisfies the recursion (1) with $X_{k_0} = X_0$.

Proof. Let $p \in \mathbb{N}$, $\alpha \in [0,1)$, $s, T \in [0,\infty)$, $t \in [s,s+T]$, $X_0 \in \mathbb{R}^d$ and $g_p \in C^2(\mathbb{R}^d,[0,\infty))$ such that for any $x \in \mathbb{R}^d$, $g_p(x) = 1 + \|x\|^{2p}$. Let $\bar{\gamma} > 0$ and $\gamma \in (0,\bar{\gamma}]$.

We divide the proof into two parts

(a) Let $(X_t)_{t \geq 0}$ be a solution to (2) such that $X_s = X_0$. We have for any $x \in \mathbb{R}^d$

$$\nabla g_p(x) = 2p \|x\|^{2(p-1)} x, \quad \nabla^2 g_p(x) = 4p(p-1) \|x\|^{2(p-2)} xx^\top + 2p \|x\|^{2(p-1)} \text{Id}.$$  \hfill (15)

Let $n \in \mathbb{N}$, and set $\tau_n = \inf\{u \geq 0 : g_p(X_u) > n\}$. Applying Itô’s lemma and using (2) and (15) we get

$$\mathbb{E} [g_p(X_{t\wedge \tau_n})] - \mathbb{E} [g_p(X_{s\wedge \tau_n})] = \mathbb{E} \left[ \int_{s\wedge \tau_n}^{t\wedge \tau_n} - (\gamma_\alpha + u)^{-\alpha} \langle \nabla f(X_u), \nabla g_p(X_u) \rangle \right] \, du \quad (16)$$

Using $\textbf{A1}$, (15) and the Cauchy-Schwarz inequality we get that for any $u \in [s,s+T]$

$$|\langle \nabla f(X_u), \nabla g_p(X_u) \rangle| \leq 2p \|X_u\|^{2(p-1)} \left\{ |\langle \nabla f(X_u) - \nabla f(0), X_u \rangle| + \|\nabla f(0)\| \|X_u\| \right\} \leq 2p(L + \|\nabla f(0)\|)g_p(X_u).$$  \hfill (17)

In addition, using $\textbf{A1}$, $\textbf{A2}$, (15) and the Cauchy-Schwarz inequality we get that for any $u \in [s,s+T]$

$$\langle \Sigma(X_u), \nabla^2 g_p(X_u) \rangle = 2p \|X_u\|^{2(p-1)} \int_Z \|\nabla f(X_u) - H(X_u, z)\|^2 \, d\mu_Z(z)$$

$$+ 4p(p-1) \|X_u\|^{2(p-2)} \int_Z \langle X_u, H(X_u, z) - \nabla f(X_u) \rangle^2 \, d\mu_Z(z)$$

$$\leq 2p(2p-1) \|X_u\|^{2(p-1)} \eta \leq 2p(2p-1) \eta g_p(X_u).$$  \hfill (18)

Combining (17) and (18) in (16) we get for large enough $n \in \mathbb{N}$

$$\mathbb{E} [g_p(X_{t\wedge \tau_n})] - g_p(X_0) \leq 2p(L + \|\nabla f(0)\|) \mathbb{E} \left[ \int_s^{t\wedge \tau_n} g_p(X_u) \, du \right] + \tilde{\gamma}_\alpha p(2p-1) \mathbb{E} \left[ \int_s^{t\wedge \tau_n} g_p(X_u) \, du \right]$$

$$\leq \{2p(L + \|\nabla f(0)\|) + \tilde{\gamma}_\alpha p(2p-1)\} \int_s^t \mathbb{E} [g_p(X_{u\wedge \tau_n})] \, du.$$  \hfill (16)
Using Grönwall’s lemma we obtain

\[ \mathbb{E} \left[ g_p(X_{t\wedge \tau_n}) \right] \leq g_p(X_0) \exp \left[ T \left\{ 2p(L + \|\nabla f(0)\|) + \bar{\gamma}_n p(2p - 1) \right\} \right]. \]

We conclude upon using Fatou’s lemma and remarking that \( \lim_n \tau_n = +\infty \), since \( X_t \) is well-defined for any \( t \geq 0 \).

(b) Let \( (X_k)_{k \in \mathbb{N}} \) be a sequence which satisfies the recursion (1) with \( X_{k_0} = X_0 \). Let \( A_k = X_k - \gamma(k + 1)^{-\alpha} \nabla f(X_k) \) and \( B_k = \gamma(k + 1)^{-\alpha} (\nabla f(X_k) - H(X_k, Z_{k+1})) \). We have, using Cauchy-Schwarz inequality and the binomial formula,

\[
\|X_{k+1}\|^{2p} = \|A_k + B_k\|^{2p} = \left\{ \|A_k\|^2 + 2 \langle A_k, B_k \rangle + \|B_k\|^2 \right\}^p \\
\leq \sum_{i=0}^{p} \sum_{j=0}^{i} \binom{p}{i} \binom{i}{j} \|A_k\|^{2(p-i)+j} \|B_k\|^{2i-j} \times 2^j \\
\leq \|A_k\|^{2p} + 2^p \sum_{i=1}^{p} \sum_{j=0}^{i} \binom{p}{i} \binom{i}{j} \|A_k\|^{2(p-i)+j} \|B_k\|^{2i-j}.
\]

Using \( A_1 \), there exists \( \bar{\alpha}^{(a)}_{T,1}, \bar{\alpha}^{(b)}_{T,1}, \bar{\alpha}^{(c)}_{T,1} \geq 0 \) such that for any \( \ell \in \{0, \ldots, 2p\} \)

\[
\|A_k\|^{\ell} \leq \sum_{m=0}^{\ell} \binom{\ell}{m} (1 + \gamma(k + 1)^{-\alpha} L)^m \|X_k\|^m (\gamma(k + 1)^{-\alpha} \|\nabla f(0)\|)^{\ell-m} \\
\leq (1 + \gamma(k + 1)^{-\alpha} \bar{\alpha}^{(a)}_{T,1}) \|X_k\|^\ell + \gamma(k + 1)^{-\alpha} \bar{\alpha}^{(b)}_{T,1} (1 + \|X_k\|^{2p}) \\
\leq (1 + \gamma(k + 1)^{-\alpha} \bar{\alpha}^{(c)}_{T,1})(1 + \|X_k\|^{2p}).
\]

Combining this result, (19), Jensen’s inequality and that for any \( \ell \in \mathbb{N} \), \( \mathbb{E} \left[ \|B_k\|^{2\ell} \middle| F_k \right] \leq \gamma^{2\ell} (k + 1)^{-2\alpha \ell} \eta_\ell \) we have

\[
\mathbb{E} \left[ \|X_{k+1}\|^{2p} \middle| F_k \right] \leq (1 + \gamma(k + 1)^{-\alpha} \bar{\alpha}^{(a)}_{T,1}) \|X_k\|^\ell + \gamma(k + 1)^{-\alpha} \bar{\alpha}^{(b)}_{T,1} (1 + \|X_k\|^{2p}) \\
+ 2^p (1 + \gamma(k + 1)^{-\alpha} \bar{\alpha}^{(c)}_{T,1})(1 + \|X_k\|^{2p}) \sum_{i=1}^{p} \sum_{j=0}^{i} \binom{p}{i} \binom{i}{j} \eta_{2i-j}^{1/2} \gamma^{2i-j} (k + 1)^{-\alpha(2i-j)}.
\]

Therefore, there exists \( \bar{\alpha}^{(d)}_{T,1} \geq 0 \) such that

\[
\mathbb{E} \left[ 1 + \|X_{k+1}\|^{2p} \right] \leq (1 + \bar{\alpha}^{(d)}_{T,1} \gamma(k + 1)^{-\alpha}) \mathbb{E} \left[ 1 + \|X_k\|^{2p} \right] + \bar{\alpha}^{(d)}_{T,1} \gamma(k + 1)^{-\alpha}.
\]

We conclude combining this result, Lemma 13 and Lemma 14.

\( \square \)

We use the previous result to prove the following lemma.
Lemma 16. Let $p \in \mathbb{N}$, $\gamma > 0$ and $\alpha \in [0,1)$. Assume A1, A2 and that for any $x \in \mathbb{R}^d$, $\mu_T(\|H(x, \cdot) - \nabla f(x)\|^2) \leq \eta_p$, with $\eta_p \geq 0$. Then for any $T \geq 0$, there exists $\kappa_{T,2} \geq 0$ such that for any $\gamma \in (0, \gamma]$, $k \in \mathbb{N}$ with $(k+1)\gamma_\alpha \leq T$, $t \in [k\gamma_\alpha, (k+1)\gamma_\alpha]$ and $X_0 \in \mathbb{R}^d$, we have

$$\max\left\{ \mathbb{E}\left[\|X_{k+1} - X_0\|^{2p}\right], \mathbb{E}\left[\|X_t - X_0\|^{2p}\right] \right\} \leq \kappa_{T,2}(k + 1)^{-2\alpha p} \gamma^{2p}(1 + \|X_0\|^{2p}),$$

where $(X_k)_{k \in \mathbb{N}}$ satisfies the recursion (1) with $X_k = X_0$ and $(X_t)_{t \geq 0}$ is the solution of (2) with $X_{k\gamma_\alpha} = X_0$.

Proof. Let $p \in \mathbb{N}$, $\alpha \in [0,1)$, $\gamma > 0$, $\gamma \in (0, \gamma]$, $k \in \mathbb{N}$, $t \in [k\gamma_\alpha, (k+1)\gamma_\alpha]$ and $X_0 \in \mathbb{R}^d$. We divide the rest of the proof into two parts.

(a) Let $(X_t)_{t \geq 0}$ be a solution to (2) such that $X_{k\gamma_\alpha} = X_0$. Using A1, A2, Jensen’s inequality, Burkholder-Davis-Gundy’s inequality (Rogers and Williams, 2000, Theorem 42.1) and Lemma 15 there exists $B_p \geq 0$ such that

$$\mathbb{E}\left[\|X_t - X_0\|^{2p}\right] \leq 2^{2p-1}\mathbb{E}\left[\int_{k\gamma_\alpha}^t (\gamma_\alpha + s)^{-\alpha}\nabla f(X_s) ds\right]^{2p} + 2^{2p-1}\gamma_\alpha \mathbb{E}\left[\int_{k\gamma_\alpha}^t (\gamma_\alpha + s)^{-\alpha}\Sigma(X_s)^{1/2}dB_s\right]^{2p} \leq 2^{2p-1}\gamma_\alpha \left(\frac{2^{2p-1}}{2}\int_{k\gamma_\alpha}^t (\gamma_\alpha + s)^{-2\alpha p} + \sup s \in [k\gamma_\alpha, t] \mathbb{E}\left[\|X_s\|^{2p}\right] + \eta_p\right) \mathbb{E}\left[\int_{k\gamma_\alpha}^t (\gamma_\alpha + s)^{-\alpha}ds\right].$$

(b) Let $(X_n)_{n \in \mathbb{N}}$ satisfying the recursion (1) with $X_k = X_0$. Using A1 and A2 we get that

$$\mathbb{E}\left[\|X_{k+1} - X_0\|^{2p}\right] = \mathbb{E}\left[\|\nabla f(X_0)\|^2 + \|H(X_0, Z_{k+1}) - \nabla f(X_0)\|^2\right] \leq \gamma_2^p(k + 1)^{-2\alpha p} \left(\frac{2^p}{2^{2p-1}} L^2 p \left(\frac{2^p}{2^{2p-1}} L^2 p \|\nabla f(0)\|^2 + L^2 p \|X_0\|^2\right) + \int Z \nabla f(X_0) d\mu_Z(z)\right) \leq \gamma_2^p(k + 1)^{-2\alpha p} \left(1 + L^2 p\right) \left(\|\nabla f(0)\|^2 + \eta_p\right) \left(1 + \|X_0\|^{2p}\right).$$

Combining (20) and (21) and setting

$$\kappa_{T,2} = 2^{4p-2}(1 + L^2 p) \left(\|\nabla f(0)\|^2 + \eta_p + \max(A_{T,1}, 1)\right),$$

conclude the proof upon remarking that $\eta_p \leq \eta_p$. □
A.3 Mean-square approximation

Now consider the stochastic process \((\mathbf{X}_t)_{t \geq 0}\) defined by \(\mathbf{X}_0 = X_0\) and solution of the following SDE

\[
d\mathbf{X}_t = -\gamma_\alpha^{-1} \sum_{k=0}^{+\infty} \mathbb{1}_{[k\gamma_\alpha, (k+1)\gamma_\alpha)}(t) (1 + k)^{-\alpha} \gamma \left\{ \nabla f(\mathbf{X}_{k\gamma_\alpha}) dt + \gamma_\alpha^{1/2} \Sigma(\mathbf{X}_{k\gamma_\alpha})^{1/2} d\mathbf{B}_t \right\}.
\]

(22)

Note that for any \(k \in \mathbb{N}\), we have

\[
\mathbf{X}_{(k+1)\gamma_\alpha} = \mathbf{X}_{k\gamma_\alpha} - \gamma (k + 1)^{-\alpha} \left\{ \nabla f(\mathbf{X}_{k\gamma_\alpha}) + \Sigma(\mathbf{X}_{k\gamma_\alpha})^{1/2} G_k \right\},
\]

with \(G_k = \gamma_\alpha^{-1/2} \int_{k\gamma_\alpha}^{(k+1)\gamma_\alpha} dB_s\). Hence, for any \(k \in \mathbb{N}\), \(\mathbf{X}_{k\gamma_\alpha}\) has the same distribution as \(X_k\) given by (1) with \(H(x, z) = \nabla f(x) + \Sigma(x)^{1/2} z\), \((\mathcal{Z}, \mathcal{Z}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) and \(\mu Z\) the Gaussian probability distribution with zero mean and covariance matrix identity.

**Lemma 17.** Let \(\bar{\gamma} > 0\) and \(\alpha \in [0, 1)\). Assume **A2**. Then for any \(T \geq 0\), there exists \(k_{T, \beta} \geq 0\) such that for any \(\gamma \in (0, \bar{\gamma}]\), \(k \in \mathbb{N}\) with \((k + 1)\gamma_\alpha \leq T\) and \(X_0 \in \mathbb{R}^d\) we have

\[
\mathbb{E} \left[ \|\mathbf{X}_{(k+1)\gamma_\alpha} - X_{k+1}\|^2 \right] \leq k_{T, \beta} \gamma^2 (k + 1)^{-2\alpha} (1 + \|X_0\|^2),
\]

where \((X_k)_{k \in \mathbb{N}}\) satisfies the recursion (1) with \(X_k = X_0\) and \((\mathbf{X}_t)_{t \geq 0}\) is the solution of (22) with \(\mathbf{X}_{k\gamma_\alpha} = X_0\).

**Proof.** Let \(\alpha \in [0, 1), \bar{\gamma} > 0, \gamma \in (0, \bar{\gamma}], k \in \mathbb{N}, t \in [k\gamma_\alpha, (k+1)\gamma_\alpha)\) and \(X_0 \in \mathbb{R}^d\). Let \((X_k)_{k \in \mathbb{N}}\) satisfy the recursion (1) with \(X_k = X_0\) and \((\mathbf{X}_t)_{t \geq 0}\) be the solution of (22) with \(\mathbf{X}_{k\gamma_\alpha} = X_0\). Using **A2** we have

\[
\mathbb{E} \left[ \|\mathbf{X}_{(k+1)\gamma_\alpha} - X_{k+1}\|^2 \right] = \gamma^2 (k + 1)^{-2\alpha} \mathbb{E} \left[ \|\nabla f(X_0) + \Sigma^{1/2}(X_0)G_k - H(X_0, Z_k)\|^2 \right]
\]

\[
\leq 2\gamma^2 (k + 1)^{-2\alpha} \left\{ \mathbb{E} \left[ \|\nabla f(X_0) - H(X_0, Z_k)\|^2 \right] + \mathbb{E} \left[ \|\Sigma^{1/2}(X_0)G_k\|^2 \right] \right\}
\]

\[
\leq 4\gamma^2 (k + 1)^{-2\alpha} \eta,
\]

which concludes the proof.

**Lemma 18.** Let \(\bar{\gamma} > 0\) and \(\alpha \in [0, 1)\). Assume **A1**, **A2** and **A3**. Then for any \(T \geq 0\), there exists \(k_{T, \delta} \geq 0\) such that for any \(\gamma \in (0, \bar{\gamma}]\), \(k \in \mathbb{N}\) with \((k + 1)\gamma_\alpha \leq T\) and \(X_0 \in \mathbb{R}^d\) we have

\[
\mathbb{E} \left[ \|\mathbf{X}_{(k+1)\gamma_\alpha} - \mathbf{X}_{(k+1)\gamma_\alpha}\|^2 \right] \leq k_{T, \delta} \left\{ \gamma^4 (k + 1)^{-4\alpha} + \gamma^2 (k + 1)^{-2(1 + \alpha)} \right\} (1 + \|X_0\|^2),
\]

where \((X_t)_{t \geq 0}\) be the solution of (2) with \(\mathbf{X}_{k\gamma_\alpha} = X_0\) and \((\mathbf{X}_t)_{t \geq 0}\) be the solution of (22) with \(\mathbf{X}_{k\gamma_\alpha} = X_0\).

**Proof.** Let \(\alpha \in [0, 1), \bar{\gamma} > 0, \gamma \in (0, \bar{\gamma}], k \in \mathbb{N}, t \in [k\gamma_\alpha, (k+1)\gamma_\alpha]\) and \(X_0 \in \mathbb{R}^d\). Let \((X_t)_{t \geq 0}\) the solution of (2) with \(\mathbf{X}_{k\gamma_\alpha} = X_0\) and \((\mathbf{X}_t)_{t \geq 0}\) is the solution of (22) with \(\mathbf{X}_{k\gamma_\alpha} = X_0\). Using Jensen’s inequality and that \(\gamma_\alpha^{-1} = \gamma_\alpha\) we have

\[
\mathbb{E} \left[ \|\mathbf{X}_{(k+1)\gamma_\alpha} - \mathbf{X}_{(k+1)\gamma_\alpha}\|^2 \right]
\]

(23)
\[
\begin{align*}
\leq & \mathbb{E}\left[ \left\| - \int_{k\gamma_0}^{(k+1)\gamma_0} (\gamma_0 + s)^{-\alpha} \nabla f(X_s) ds - \gamma_0^{1/2} \int_{k\gamma_0}^{(k+1)\gamma_0} (\gamma_0 + s)^{-\alpha} \Sigma(X_s)^{1/2} dB_s \right\|^2 \right] \\
& + \mathbb{E}\left[ \left\| - \gamma_0^{1/2} \int_{k\gamma_0}^{(k+1)\gamma_0} (1 + \gamma_0^{-1}s)^{-\alpha} \nabla f(X_s) ds + \gamma (k+1)^{-\alpha} \nabla f(X_0) \right\|^2 \right] \\
& + 2\mathbb{E}\left[ \left\| - \int_{k\gamma_0}^{(k+1)\gamma_0} (1 + \gamma_0^{-1}s)^{-\alpha} \Sigma(X_s)^{1/2} dB_s \right\|^2 \right] \\
& + \mathbb{E}\left[ \left\| \int_{k\gamma_0}^{(k+1)\gamma_0} \{ (k+1)^{-\alpha} \nabla f(X_0) - (1 + \gamma_0^{-1}s)^{-\alpha} \nabla f(X_s) \} ds \right\|^2 \right] \\
& + 2\mathbb{E}\left[ \int_{k\gamma_0}^{(k+1)\gamma_0} \{ (k+1)^{-\alpha} \Sigma(X_0)^{1/2} - (1 + \gamma_0^{-1}s)^{-\alpha} \Sigma(X_s)^{1/2} \} dB_s \right]^2 \\
\leq & 2\gamma_0^{-2\alpha} \mathbb{E}\left[ \left\| \int_{k\gamma_0}^{(k+1)\gamma_0} \{ (k+1)^{-\alpha} \nabla f(X_0) - (1 + \gamma_0^{-1}s)^{-\alpha} \nabla f(X_s) \} ds \right\|^2 \right] \\
& + 2\gamma_0^{-2\alpha} \mathbb{E}\left[ \left\| \int_{k\gamma_0}^{(k+1)\gamma_0} \{ (k+1)^{-\alpha} \Sigma(X_0)^{1/2} - (1 + \gamma_0^{-1}s)^{-\alpha} \Sigma(X_s)^{1/2} \} dB_s \right\|^2 \right].
\end{align*}
\]

We now treat each term separately. Using Jensen’s inequality, Fubini-Tonelli’s theorem, the fact that for any \( u > 0, \ u^{-\alpha} - (u+1)^{-\alpha} \leq \alpha u^{-(\alpha+1)}, \ \textbf{A1} \) and Lemma 16 we get that

\[
\mathbb{E}\left[ \left\| \int_{k\gamma_0}^{(k+1)\gamma_0} \{ (k+1)^{-\alpha} \nabla f(X_0) - (1 + \gamma_0^{-1}s)^{-\alpha} \nabla f(X_s) \} ds \right\|^2 \right] \\
\leq \gamma_0^2 \sup_{s \in [k\gamma_0,(k+1)\gamma_0]} \mathbb{E}\left[ \left\| (k+1)^{-\alpha} \nabla f(X_0) - (1 + \gamma_0^{-1}s)^{-\alpha} \nabla f(X_s) \right\|^2 \right] ds \\
\leq 2\gamma_0^2 \sup_{s \in [k\gamma_0,(k+1)\gamma_0]} \left\{ \| \nabla f(X_0) \|^2 (k+1)^{-\alpha} - (1 + \gamma_0^{-1}s)^{-\alpha} \| \nabla f(X_s) \|^2 \right\} \\
+ (1 + \gamma_0 s^{-1})^{-2\alpha} \mathbb{E}\left[ \| \nabla f(X_s) - \nabla f(X_0) \|^2 \right]
\]

\[
\leq 2\gamma_0^2 \left[ \alpha^2 \| \nabla f(X_0) \|^2 (k+1)^{-2(1+\alpha)} + (k+1)^{-2\alpha} L^2 \sup_{s \in [k\gamma_0,(k+1)\gamma_0]} \mathbb{E}\left[ \| X_s - X_0 \|^2 \right] \right] \\
\leq 2\gamma_0^2 \left[ \alpha^2 \| \nabla f(X_0) \|^2 (k+1)^{-2(1+\alpha)} + (k+1)^{-4\alpha} L^2 \mathcal{A}_{T,2} \gamma^2 (1 + \| X_0 \|^2) \right] \\
\leq 2\gamma_0^2 \left[ \alpha^2 (\| \nabla f(0) \|^2 + L^2)(k+1)^{-2(1+\alpha)} + (k+1)^{-4\alpha} L^2 \mathcal{A}_{T,2} \gamma^2 \right] (1 + \| X_0 \|^2) .
\]

In addition, using Jensen’s inequality, Itô isometry, Fubini-Tonelli’s theorem, \textbf{A1}, \textbf{A3} and Lemma 16 we have

\[
\mathbb{E}\left[ \left\| \int_{k\gamma_0}^{(k+1)\gamma_0} \{ (k+1)^{-\alpha} \Sigma(X_0)^{1/2} - (1 + \gamma_0^{-1}s)^{-\alpha} \Sigma(X_s)^{1/2} \} dB_s \right\|^2 \right] \\
\leq 2 \left[ (k+1)^{-2\alpha} \int_{k\gamma_0}^{(k+1)\gamma_0} \mathbb{E}\left[ \| \Sigma(X_0)^{1/2} - \Sigma(X_s)^{1/2} \|^2 \right] ds \right].
\]
Combining (23), (24) and (25) concludes the proof upon setting
\[
A_{T,4} = 4 \left[ \eta^2 A_{T,2} + \eta \alpha^2 (kX_0^2 + L^2) + L^2 A_{T,2} \right].
\]

\[
\Box
\]

**Proposition 19.** Let \( \bar{\gamma} > 0 \) and \( \alpha \in [0, 1) \). Assume A1, A2 and A3. Then for any \( T \geq 0 \), there exists \( A_{T,5} \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}) \), \( k \in \mathbb{N} \) with \( (k+1)\gamma \leq T \) and \( X_0 \in \mathbb{R}^d \) we have
\[
\mathbb{E} \left[ \|X_{(k+1)\gamma} - X_{k+1}\|^2 \right] \leq A_{T,5} \left\{ \gamma^4 (k+1)^{-4\alpha} + \gamma^2 (k+1)^{-2\alpha} \right\} (1 + \|X_0\|^2),
\]
where \( (X_k)_{k \in \mathbb{N}} \) satisfies the recursion (1) with \( X_k = X_0 \) and \( (X_t)_{t \geq 0} \) is the solution of (2) with \( X_{k\gamma} = X_0 \).

**Proof.** The proof is straightforward upon combining Lemma 17 and Lemma 18. \(\Box\)

We obtain now the following proposition which is a restatement of Proposition 1.

**Proposition 20.** Let \( \bar{\gamma} > 0 \) and \( \alpha \in [0, 1) \). Assume A1, A2 and A3. Then for any \( T \geq 0 \), there exists \( A_1 \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}) \), \( k \in \mathbb{N} \) with \( k\gamma \leq T \) we have
\[
\mathbb{E}^{1/2} \left[ \|X_{k\gamma} - X_k\|^2 \right] \leq A_1 \gamma^\delta (1 + \log(\gamma^{-1})),
\]
with \( \delta = \min(1, (1-\alpha)^{-1}/2) \). If in addition, \( (Z, \mathcal{Z}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) and for any \( x \in \mathbb{R}^d \), \( z \in \mathbb{R}^d \) and \( n \in \mathbb{N} \),
\[
H(x, z) = \nabla f(x) + \Sigma(x)^{1/2} z, \quad Z_{n+1} = \gamma_{(n+1)\gamma}^{-1} \int_{(n+1)\gamma}^{(n+1)\gamma} dB_s,
\]
then \( \delta = 1 \).

**Proof.** Let \( p \in \mathbb{N} \), \( \alpha \in [0, 1) \), \( \bar{\gamma} > 0 \), \( \gamma \in (0, \bar{\gamma}) \), \( k \in \mathbb{N} \), and \( X_0 \in \mathbb{R}^d \). Let \( (E_k)_{k \in \mathbb{N}} \) such that for any \( k \in \mathbb{N} \), \( E_k = \mathbb{E} \left[ \|X_{k\gamma} - X_k\|^2 \right] \). Note that \( E_0 = 0 \). Let \( Y_{(k+1)\gamma} = X_{k\gamma} - \gamma(k+1)^{-\alpha} H(X_{k\gamma}, Z_{k+1}) \). We have
\[
E_{k+1} = \mathbb{E} \left[ \|X_{(k+1)\gamma} - X_{k+1}\|^2 \right]
\]
\[
= \mathbb{E} \left[ \|X_{(k+1)\gamma} - Y_{(k+1)\gamma} + Y_{(k+1)\gamma} - X_{k+1}\|^2 \right]
\]
\[
= \mathbb{E} \left[ \|X_{(k+1)\gamma} - Y_{(k+1)\gamma}\|^2 \right] + 2 \mathbb{E} \left[ \langle X_{(k+1)\gamma} - Y_{(k+1)\gamma}, Y_{(k+1)\gamma} - X_{k+1} \rangle \right]
\]
\[
+ \mathbb{E} \left[ \|Y_{(k+1)\gamma} - X_{k+1}\|^2 \right].
\]

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\[
\begin{align*}
\mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a} \right\|^2 \right] &+ \mathbb{E} \left[ \left\| \mathbf{Y}_{(k+1)\gamma a} - \mathbf{X}_{k+1} \right\|^2 \right] \\
&+ 2\mathbb{E} \left[ \left( \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a}, \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right) \right] \\
&+ 2\gamma(k+1)^{-\alpha} \mathbb{E} \left[ \left( \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a}, H(X_k, Z_{k+1}) - H(X_{k\gamma a}, Z_{k+1}) \right) \right].
\end{align*}
\]

Let \(a_k = \gamma^4(k+1)^{-4\alpha} + \gamma^2(k+1)^{-2\alpha}\). We now bound each of the four terms appearing in (26)

(a) First, we have using Proposition 19 and Lemma 15

\begin{align*}
\mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a} \right\|^2 \right] &= \mathbb{E} \left[ \mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a} \right\|^2 \right| \mathbf{X}_{k\gamma a} \right] \right. \\
&\leq \mathbb{E} \left[ \mathbb{A}_{T,5}(\gamma^4(k+1)^{-4\alpha} + \gamma^2(k+1)^{-2\alpha}) \left( 1 + \left\| \mathbf{X}_{k\gamma a} \right\|^2 \right) \right] \\
&\leq \mathbb{A}_{T,1,5}(\gamma^4(k+1)^{-4\alpha} + \gamma^2(k+1)^{-2\alpha}) \left( 1 + \left\| \mathbf{X}_0 \right\|^2 \right) \leq \mathbb{A}_{T,6}^{(a)} a_k,
\end{align*}

with \(\mathbb{A}_{T,6}^{(a)} \geq 0\) which does not depend on \(\gamma\) and \(k\).

(b) Second, we have using A1, F2 and that for any \(a, b \geq 0\), \((a+b)^2 \leq 2a^2 + 2b^2\)

\begin{align*}
\mathbb{E} \left[ \left\| \mathbf{Y}_{(k+1)\gamma a} - \mathbf{X}_{k+1} \right\|^2 \right] &= \mathbb{E} \left[ \left\| \mathbf{X}_{k\gamma a} - \mathbf{X}_k - \gamma(k+1)^{-\alpha}(H(X_{k\gamma a}, Z_{k+1}) - H(X_k, Z_{k+1})) \right\|^2 \right] \\
&= \mathbb{E} \left[ \left\| \mathbf{X}_{k\gamma a} - \gamma(k+1)^{-\alpha}\nabla f(X_{k\gamma a}) - \mathbf{X}_k + \gamma(k+1)^{-\alpha}\nabla f(X_k) \right\|^2 \right] \\
&+ \gamma^2(k+1)^{-2\alpha} \mathbb{E} \left[ \left\| H(X_{k\gamma a}, Z_{k+1}) - \nabla f(X_{k\gamma a}) \right\|^2 \right] \\
&+ \gamma^2(k+1)^{-2\alpha} \mathbb{E} \left[ \left\| H(X_k, Z_{k+1}) - \nabla f(X_k) \right\|^2 \right] \\
&\leq (1 + \gamma L(k+1)^{-\alpha})^2 \left\| \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right\|^2 + 4\gamma^2(k+1)^{-2\alpha} \\
&\leq (1 + 2\gamma L(k+1)^{-\alpha} + \gamma^2 L^2(k+1)^{-2\alpha})E_k + 4\gamma^2(k+1)^{-2\alpha} \\
&\leq (1 + \mathbb{A}_{T,6}^{(b)} 1/2)E_k + 4a_k,
\end{align*}

with \(\mathbb{A}_{T,6}^{(b)} \geq 0\) which does not depend on \(\gamma\) and \(k\).

(c) Let \(\overline{\mathbf{Y}}_{(k+1)\gamma a} = \mathbf{X}_{k\gamma a} - \gamma(k+1)^{-\alpha} \left\{ \nabla f(X_{k\gamma a}) + \Sigma(X_{k\gamma a})^{1/2}G_k \right\}\), with \(G_k = \gamma^{-1/2} f_{k\gamma a}^{(k+1)\gamma a} dB_s\). Let \(b_k = \gamma^3(k+1)^{-3\alpha} + 2(k+1)^{-2(1+\alpha)/2}\).

Using A2 we have \(\mathbb{E} \left[ \overline{\mathbf{Y}}_{(k+1)\gamma a} \right| \sigma(X_{k\gamma a}) \right] = \mathbb{E} \left[ \mathbf{Y}_{(k+1)\gamma a} \right| \sigma(X_{k\gamma a}) \right].\) Combining this result, the Cauchy-Schwarz inequality, Lemma 18, Lemma 15 and that for any \(a, b \geq 0\), \((a+b)^{1/2} \leq a^{1/2} + b^{1/2}\) and \(2ab \leq a^2 + b^2\) we obtain

\begin{align*}
\mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a}, \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right\| \right] &= \mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a}, \sigma(X_{k\gamma a}, X_k) \right\| \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right] \\
&= \mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a}, \sigma(X_{k\gamma a}, X_k) \right\| \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right] \\
&\leq \mathbb{E} \left[ \left\| \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a} \right\| \sigma(X_{k\gamma a}, X_k) \right] \left\| \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right\| \right] \\
&\leq \mathbb{E} \left[ \mathbb{E}^{1/2} \left[ \mathbf{X}_{(k+1)\gamma a} - \mathbf{Y}_{(k+1)\gamma a} \right] \sigma(X_{k\gamma a}, X_k) \right] \left\| \mathbf{X}_{k\gamma a} - \mathbf{X}_k \right\| \right].
\end{align*}

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there exists $\gamma$ such that $\gamma(k+1) - 2a(k+1) / 2 \leq 0$ which does not depend on $\gamma$ and $k$.

Finally, we have using (27), (28), (29) and (30) in (26)

$$E_{k+1} \leq \left\{ 1 + (2 + A_{T,6}^{(b)})a_k^{1/2} \right\} E_k + (4 + A_{T,6}^{(a)} + A_{T,6}^{(d)}a_k + A_{T,6}^{(c)})a_k^{3/2} + A_{T,6}^{(e)}b_k$$

Using Lemma 14 and that $a_k^{1/2} \leq \gamma(k+1)^{-\alpha} + \gamma^2(k+1)^{-2\alpha}$, there exists $A_{T,6}^{(e)} \geq 0$ which does not depend on $\gamma$ and $k$ such that

$$(2 + A_{T,6}^{(b)}) \sum_{k=0}^{N-1} a_k^{1/2} \leq A_{T,6}^{(e)} .$$

In addition, we have

$$a_k + a_k^{3/2} + b_k$$

$$\leq (1 + 2^{3/2}) \left[ \gamma^2(k+1)^{-2\alpha} + \gamma^3(k+1)^{-3\alpha} + \gamma^4(k+1)^{-4\alpha} + \gamma^6(k+1)^{-6\alpha} + \gamma(k+1)^{-2(1+\alpha)} \right] .$$

Therefore, using that $\gamma \gamma_\alpha = \gamma_\alpha$ and Lemma 14 there exists $A_{T,6}^{(f)} \geq 0$ which does not depend on $\gamma$ and $k$ such that

$$\sum_{k=0}^{N-1} (4 + A_{T,6}^{(a)} + 2A_{T,6}^{(d)} + A_{T,6}^{(c)})(a_k + a_k^{3/2} + b_k) \leq \begin{cases} A_{T,6}^{(f)} \gamma^2(1 + \log(\gamma^{-1})) & \text{if } \alpha \geq 1/2, \\ A_{T,6}^{(f)} \gamma_\alpha & \text{if } \alpha < 1/2 . \end{cases}$$
We denote \( v_k = (2 + k_{T,0}^{(b)}) a_k^{1/2} \) and \( w_k = (4 + k_{T,0}^{(a)} + 2k_{T,0}^{(d)} + k_{T,0}^{(c)}) (a_k + a_k^{3/2} + b_k) \). Using (31) and Lemma 13 we obtain that

\[
E_k \leq \sum_{k=0}^{N-1} w_k + \exp \left[ \sum_{k=0}^{N-1} v_k \right] \sum_{k=0}^{N-1} v_kw_k
\]

\[
\leq \sum_{k=0}^{N-1} w_k + \exp \left[ \sum_{k=0}^{N-1} v_k \right] \left( \sum_{k=0}^{N-1} v_k \right) \left( \sum_{k=0}^{N-1} w_k \right)
\]  \tag{34} 

Combining (32), (33) and (34) concludes the first part of the proof.

For the second part of the proof \( H(x, z) = \nabla f(x) + \sum(x)^{1/2} z \) and for any \( k \in \mathbb{N} \), we have \( Z_{k+1} = \int_k^{(k+1)\gamma \alpha} dB_s \). We denote \( c_k = \gamma^4(k+1)^{-3\alpha} + \gamma^2(k+1)^{-2(1+\alpha)} \). In (27), \( a_k \) is replaced by \( c_k \). The bound in (28) is replaced by \( (1 + k_{T,0}^{(b)} a_k^{1/2}) E_k \). The bound in (29) remains unchanged and in (30) the upper-bound is replaced by \( k_{T,0}^{(c)} a_k^{1/2} c_k + a_k^{1/2} E_k \). The rest of the proof is similar to the general case. \( \square \)

### A.4 Weak approximation

We recall that \( \mathcal{G}_p \) is the set of twice continuously differentiable functions from \( \mathbb{R}^d \) to \( \mathbb{R} \) such that for any \( g \in \mathcal{G}_p \), there exists \( K \geq 0 \) such that for any \( x \in \mathbb{R}^d \)

\[
\max \left\{ \|\nabla g(x)\|, \|\nabla^2 g(x)\| \right\} \leq K(1 + \|x\|^p),
\]

with \( p \in \mathbb{N} \).

The following lemma will be useful.

**Lemma 21.** Let \( p \in \mathbb{N} \), \( g \in \mathcal{G}_p \) and let \( K \geq 0 \) as in (35). Then, for any \( x, y \in \mathbb{R}^d \)

\[
|g(y) - g(x) - \langle \nabla g(x), y - x \rangle| \leq K(1 + \|x\|^p + \|y\|^p) \|x - y\|^2.
\]

**Proof.** Using that for any \( x \mapsto \|x\|^p \) is convex, and Cauchy-Schwarz inequality we get for any \( x, y \in \mathbb{R}^d \)

\[
|g(x) - g(y) - \langle \nabla g(x), y - x \rangle| \leq \int_0^1 |\nabla^2 g(x + t(y - x))(y - x)\|^2 dt
\]

\[
\leq \|x - y\|^2 \int_0^1 |\nabla^2 g(x + t(y - x))(y - x)\|^2 dt
\]

\[
\leq K(1 + \|x\|^p + \|y\|^p) \|x - y\|^2.
\]

\( \square \)

Before giving the proof of Proposition 2, we highlight that the result is straightforward for \( \alpha \in [1/2, 1) \).

**Proposition 22.** Let \( \bar{\gamma} > 0 \) and \( \alpha \in [1/2, 1) \) and \( p \in \mathbb{N} \). Assume A1, A2 and A3. In addition, assume that for any \( x \in \mathbb{R}^d \), \( \mu_2(H(x, \cdot) - \nabla f(x))^{p/2} \leq \eta_p \), with \( \eta_p \geq 0 \). Then for any \( T \geq 0 \) and \( g \in \mathcal{G}_p \), there exists \( k_{T,7} \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}] \), \( k \in \mathbb{N} \) with \( k_{T,7} \leq T \) and \( X_0 \in \mathbb{R}^d \) we have

\[
\mathbb{E} [\|g(X_{k_{T,7}}) - g(X_k)\|] \leq k_{T,7}\gamma(1 + \log(\gamma^{-1}))
\]

where \( (X_k)_{k \in \mathbb{N}} \) satisfies the recursion (1) and \( (X_t)_{t \geq 0} \) is the solution of (2) with \( X_0 = X_0 \).
Proof. Let \( p \in \mathbb{N} \), \( g \in \mathbb{G}_p \), \( \alpha \in [1/2, 1), \gamma > 0, \gamma \in (0, \bar{\gamma}] \), \( k \in \mathbb{N} \), and \( X_0 \in \mathbb{R}^d \). Using that for any \( x \mapsto ||x||^p \) is convex, for any \( x, y \in \mathbb{R}^d \) we get

\[
|g(x) - g(y)| \leq \int_0^1 |\langle \nabla g(x + t(y-x)), y-x \rangle| dt \leq \|y-x\| \int_0^1 \|\nabla g(x + t(y-x))\| dt \\
\leq ||y-x||K(1 + ||x||^p + ||y||^p).
\]

Combining this result, Proposition 20, Lemma 15 and the Cauchy-Schwarz inequality we get that

\[
\mathbb{E} [||g(X_{k+1}) - g(X_k)||] \leq KA_T,6\gamma (1 + \log(\gamma^{-1}))(A_{T,1} + \bar{A}_{T,1})^{1/2}(1 + \|X_0\|^{2p})^{1/2},
\]

which concludes the proof. \( \square \)

Proposition 23. Let \( p \in \mathbb{N} \) and \( g \in \mathbb{G}_p \). Let \( \gamma > 0 \) and \( \alpha \in (0,1) \). Assume A1, A2, A3 and that for any \( x \in \mathbb{R}^d \), \( \mu_\mathcal{Z}(||H(x,\cdot) - \nabla f(x)||^{2p}) \leq \eta_p \), with \( \eta_p \geq 0 \). Then for any \( T \geq 0 \), there exists \( \mathcal{A}_{T,8} \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}], k \in \mathbb{N} \) with \( (k+1)\gamma \leq T \) and \( X_0 \in \mathbb{R}^d \) we have

\[
\mathbb{E} \left[ \|g(X_{(k+1)}\gamma) - g(X_{k+1})\| \right] \leq \mathcal{A}_{T,8} \left\{ \gamma^2 (k+1)^{-2\alpha} + \gamma (k+1)^{-1+\alpha} \right\} (1 + \|X_0\|^{p+2}),
\]

where \( (X_k)_{k \in \mathbb{N}} \) satisfies the recursion (1) with \( X_k = X_0 \) and \( (X_t)_{t \geq 0} \) is the solution of (2) with \( X_{\kappa \gamma} = X_0 \).

Proof. Let \( \mathbf{X}_{(k+1)\gamma} = X_0 - \gamma (k+1)^{-\alpha} \left\{ \nabla f(X_{\kappa \gamma}) + \Sigma(X_0)\frac{1}{2}G_k \right\} \), with \( G_k = \frac{\gamma^{-1/2}}{f^{(k+1)\gamma}} dB_s \). Using A2 we have \( \mathbb{E}[\mathbf{X}_{(k+1)\gamma}] = \mathbb{E}[X_{k+1}] \). Using Lemma 15, Lemma 16, Lemma 18, Lemma 21 and the Cauchy-Schwarz inequality we have

\[
\mathbb{E} \left[ \|g(X_{(k+1)\gamma}) - g(X_{k+1})\| \right] \\
\leq \mathbb{E} \left[ \|\nabla g(X_0), X_{(k+1)\gamma} - X_k \| \right] + K\mathbb{E} \left[ \|X_{(k+1)\gamma} - X_0\|^2 (1 + \|X_0\|^p + \|X_{(k+1)\gamma}\|^p) \right] \\
+ K\mathbb{E} \left[ \|X_{k+1} - X_0\|^2 (1 + \|X_0\|^p + \|X_{k+1}\|^p) \right] \\
\leq \mathbb{E} \left[ \|\nabla g(X_0), X_{(k+1)\gamma} - \mathbf{X}_{k+1}\| \right] \\
+ 3^{1/2}K\mathbb{E} \left[ \|X_{(k+1)\gamma} - X_0\|^4 \right]^{1/2} \mathbb{E} \left[ (1 + \|X_0\|^{2p} + \|X_{k+1}\|^{2p}) \right]^{1/2} \\
+ 3^{1/2}K\mathbb{E} \left[ \|X_{k+1} - X_0\|^4 \right]^{1/2} \mathbb{E} \left[ (1 + \|X_0\|^{2p} + \|X_{(k+1)\gamma}\|^2) \right]^{1/2} \\
\leq K(1 + \|X_0\|^p)\mathbb{E} \left[ \|X_{(k+1)\gamma} - \mathbf{X}_{k+1}\|^2 \right]^{1/2} \\
+ 3^{1/2}K\mathbb{E} \left[ \|X_{(k+1)\gamma} - X_0\|^4 \right]^{1/2} (1 + \mathcal{A}_{T,1})^{1/2}(1 + \|X_0\|^p) \\
+ 3^{1/2}K\mathbb{E} \left[ \|X_{k+1} - X_0\|^4 \right]^{1/2} (1 + \bar{\mathcal{A}}_{T,1})^{1/2}(1 + \|X_0\|^p) \\
\leq (1 + \|X_0\|^p)\mathcal{A}_{T,1}^{1/2} \left\{ \gamma^2 (k+1)^{-2\alpha} + \gamma (k+1)^{-1+\alpha} \right\} (1 + \|X_0\|),
\]

which concludes the proof. \( \square \)
Proposition 24. Let $\tilde{\gamma} > 0$ and $\alpha \in [0, 1)$. Assume that $f \in \mathbb{G}_{p,4}$, $\Sigma^{1/2} \in \mathbb{G}_{p,3}$ A1, A2 and A3. Let $p \in \mathbb{N}$ and $g \in \mathbb{G}_{p,2}$. In addition, assume that for any $m \in \mathbb{N}$ and $x \in \mathbb{R}^d$, $\mu_Z(\|H(x, \cdot) - \nabla f(x)\|^{2m}) \leq \eta_m$ with $\eta_m \geq 0$. Then for any $T \geq 0$, there exists $K_{T,9} \geq 0$ such that for any $\gamma \in (0, \tilde{\gamma})$, $k \in \mathbb{N}$ with $k\gamma \leq T$ and $X_0 \in \mathbb{R}^d$ we have

$$\|\mathbb{E}[g(X_{k\gamma}) - g(X_k)] \| \leq A_{T,9}(1 + \log(\gamma^{-1})),$$

where $(X_k)_{k \in \mathbb{N}}$ satisfies the recursion (1) and $(X_t)_{t \geq 0}$ is the solution of (2) with $X_0 = X_0$.

Proof. For any $k \in \mathbb{N}$ with $k\gamma \leq T$, let $g_k(x) = \mathbb{E}[g(X_{k\gamma})]$ with $X_0 = x$. Since $f \in \mathbb{G}_{p,4}$, $\Sigma^{1/2} \in \mathbb{G}_{p,3}$ and $g \in \mathbb{G}_{p,2}$ one can show, see (Blagovescenskii and Freidlin, 1961) or (Kunita, 1981, Proposition 2.1), that there exists $m \in \mathbb{N}$ and $K \geq 0$ such that for any $k \in \mathbb{N}$ $g_k \in C^m(\mathbb{R}^d, \mathbb{R})$

$$\max \{\|g_k(x)\|, \ldots, \|\nabla^m g_k(x)\| \} \leq K(1 + \|x\|^p).$$

Therefore, $g_k \in \mathbb{G}_{p,m}$ with constants uniform in $k \in \mathbb{N}$. In addition, for any $k \in \mathbb{N}$ with $k\gamma \leq T$, let $h_k^{(1)}(x) = \mathbb{E}[g_k(X_{(k+1)\gamma})]$ with $X_k = x$ and $h_k^{(2)}(x) = \mathbb{E}[g_k(X_{k\gamma})]$ with $X_{k\gamma} = x$. Using Proposition 23 we have for any $k \in \mathbb{N}$, $k\gamma \leq T$

$$\|h_k^{(1)}(x) - h_k^{(2)}(x)\| \leq A_{T,8} \left\{ \gamma^2(k + 1)^{-2\alpha} + \gamma(k + 1)^{-1 - (1 + \alpha)} \right\}(1 + \|x\|^{m+2}).$$

Therefore, using Lemma 15 we have for any $k \in \mathbb{N}$ with $k\gamma \leq T$ and $j \leq k$,

$$\|\mathbb{E}[h_k^{(1)}(X_j) - h_k^{(2)}(X_j)]\| \leq A_{T,8} A_{T,8} \left\{ \gamma^2(k + 1)^{-2\alpha} + \gamma(k + 1)^{-1 - (1 + \alpha)} \right\}(1 + \|X_0\|^{m+2}). \quad (36)$$

Now, let $k \in \mathbb{N}$ with $k\gamma \leq T$ and consider the family $\{(X_j^{(\ell)})_{\ell \in \mathbb{N}} : j = 0, \ldots, N\}$, defined by the following recursion: for any $j \in \{0, \ldots, N\}$ $X_0 = X_0$ and for any $\ell \in \mathbb{N}$:

(a) if $\ell \geq j$,

$$X_{\ell+1}^j = X_{\ell}^j - \gamma(k + 1)^{-1}H(X_{\ell}^j, Z_{\ell + 1}),$$

(b) if $\ell < j$, $X_{\ell+1}^j = X_{j(\ell+1)\gamma}^j$, where $X_{\ell\gamma} = X_j^j$ and for any $\ell \in [\ell\gamma, (\ell + 1)\gamma]$ we have

$$X_{\ell}^j = X_{\ell}^j - \int_{\ell\gamma}^{\ell} (\gamma + s)^{-\alpha} \nabla f(X_s)ds - \gamma_1^{1/2} \int_{\ell\gamma}^{\ell} (\gamma + s)^{-\alpha} \Sigma^{1/2}(X_s)dB_s.$$

We have

$$\|\mathbb{E}[g(X_{k\gamma}) - g(X_k)]\| = \|\mathbb{E}[g(X_k) - g(X_k^0)]\| = \sum_{j=0}^{k-1} \|\mathbb{E}[g(X_{k+1}^j) - g(X_k^j)]\|.$$

Using (36) we get

$$\|\mathbb{E}[g(X_{k+1}^j) - g(X_k^j)]\| = \|\mathbb{E}[g(X_k^j) - g(X_{k+1}^j) - g(X_k^j)]\|$$

$$= \|\mathbb{E}[h_{k+1-j-1}^1(X_j) - h_{k+1-j-1}^2(X_j)]\|$$

$$\leq A_{T,1} A_{T,8} \left\{ \gamma^2(k + 1)^{-2\alpha} + \gamma(k + 1)^{-1 - (1 + \alpha)} \right\}(1 + \|X_0\|^{m+2}).$$

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\[ a_{T,9}(\alpha) \gamma^2 (k+1)^{-2\alpha} + \gamma (k+1)^{-(1+\alpha)} \],

with \( a_{T,9}(\alpha) \geq 0 \) which does not depend on \( k \) or \( \gamma \). In addition, using Lemma 14 there exists \( a_{T,9}(\alpha) \geq 0 \) such that

\[
\sum_{k=0}^{N-1} \left\{ \gamma^2 (k+1)^{-2\alpha} + \gamma (k+1)^{-(1+\alpha)} \right\} \leq a_{T,9}(\alpha) \gamma \]

Combining these last two results concludes the proof. \( \square \)

A.5 Tightness of the mean-square approximation bound

In this section, we show that the upper-bound derived in Proposition 1 is sharp (up to a logarithmic term).

**Proposition 25.** Let \( \gamma > 0 \), \( \alpha \in [0, 1) \), \( (Z, \mathcal{Z}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \), \( (Z_k)_{k \in \mathbb{N}} \) a sequence of independent \( d \)-dimensional Gaussian random variables independent from \( \left( f_{k+1}^{(k+1)\gamma} \right) \), \( (Z_k)_{k \in \mathbb{N}} \), \( H(x, z) = z \) and \( f = 0 \). Then there exists \( \bar{a}_1 \geq 0 \) such that for any \( \gamma \in (0, \bar{\gamma}] \) we have

\[
\mathbb{E}^{1/2} \left[ \|X_{k\gamma} - X_k\| \right] \geq \bar{a}_1 \gamma^\delta ,
\]

with \( N = \lfloor T/\gamma \rfloor \) and \( \delta = \min(1, (1-\alpha)^{-1}/2) \).

**Proof.** First, remark that for any \( x \in \mathbb{R}^d \), \( \Sigma(x) = \text{Id} \). We have using Itô’s isometry

\[
\mathbb{E} \left[ \|X_{k\gamma} - X_k\|^2 \right] = \mathbb{E} \left[ \left\| \gamma^{1/2} \int_0^{N\gamma} (s + \gamma \alpha)^{-\alpha} \, dB_s - \gamma \sum_{k=0}^{N-1} (k+1)^{-\alpha} Z_{k+1} \right\|^2 \right]
\]

\[
= \sum_{k=0}^{N-1} \left\{ \gamma^2 \mathbb{E} \left[ \int_{k\gamma}^{(k+1)\gamma} (s + \gamma \alpha)^{-2\alpha} \, ds \right] + \gamma^2 (k+1)^{-2\alpha} \right\}
\]

\[
\geq \gamma^2 \sum_{k=0}^{N-1} (k+1)^{-2\alpha} \geq \gamma^2 \int_{1/2}^{N+1/2} (s+1)^{-2\alpha} .
\]

We now distinguish three cases.

(a) If \( \alpha = 1/2 \) then

\[
\mathbb{E} \left[ \|X_{k\gamma} - X_k\|^2 \right] \geq \gamma^2 \log(N + 1/2) - \log(1/2) \geq \bar{a}_1^{(a)} \gamma^2,
\]

with \( \bar{a}_1^{(a)} \) which does not depend on \( N \) or \( \gamma \).

(b) If \( \alpha > 1/2 \),

\[
\mathbb{E} \left[ \|X_{k\gamma} - X_k\|^2 \right] \geq \gamma^2 (3/2)^{-2\alpha+1}(2\alpha - 1)^{-1} \geq \bar{a}_1^{(b)} \gamma^2,
\]

with \( \bar{a}_1^{(b)} \) which does not depend on \( N \) or \( \gamma \).

(c) If \( \alpha < 1/2 \),

\[
\mathbb{E} \left[ \|X_{k\gamma} - X_k\|^2 \right] \geq \gamma^2 (N+3/2)^{-2\alpha+1}(1-2\alpha)^{-1} \geq \gamma^2 \gamma^{2\alpha-1}(T+3\gamma^\alpha/2)^{-2\alpha+1}(1-2\alpha)^{-1} \geq \bar{a}_1^{(c)} \gamma^\alpha,
\]

with \( \bar{a}_1^{(c)} \) which does not depend on \( N \) or \( \gamma \). \( \square \)
B Strongly-Convex Case

**Corollary 26.** Let \( \alpha, \gamma \in (0, 1) \) and \((X_t)_{t \geq 0}\) be given by (2). Assume A1, A2 A3 and F1. Then there exists \( C \geq 0 \) such that for any \( T > 0 \), \( \mathbb{E} [|f(X_T)|] - \min_{x \in \mathbb{R}^d} f \leq CT^{-\alpha} \).

**Proof.** The proof is a direct consequence of A1, (Nesterov, 2004, Lemma 1.2.3) and Theorem 5. \( \square \)

We state now a discrete analogous of Theorem 5. Note that the proof is considerably simpler than the one of (Bach and Moulines, 2011).

**Theorem 27.** Let \( \gamma \in (0, 1) \) and \( \alpha \in (0, 1) \). Let \((X_n)_{n \geq 0}\) be given by (1). Assume A2 and F1. Then there exists \( B_3 > 0 \) such that for all \( N \geq 1 \),

\[
\mathbb{E} \left[ \|X_N - x^*\|^2 \right] \leq B_3 N^{-\alpha}.
\]

In the case where \( \alpha = 1 \) we have to assume additionally that \( \gamma > 1/(2\mu) \).

**Proof.** Let \( \gamma \in (0, 1) \) and \( \alpha \in (0, 1) \). Let \((X_n)_{n \geq 0}\) be given by (1). Using F1 we get for all \( n \geq 0 \),

\[
\mathbb{E} \left[ \|X_{n+1} - x^*\|^2 | \mathcal{F}_n \right] = \mathbb{E} \left[ \|X_n - x^* - \gamma(n + 1)^{-\alpha} H(X_n, Z_{n+1})\|^2 \right] 
\]

\[
= \|X_n - x^*\|^2 + \gamma^2(n + 1)^{-2\alpha} \mathbb{E} \left[ \|H(X_n, Z_{n+1})\|^2 | \mathcal{F}_n \right] 
- 2\gamma(n + 1)^{-\alpha} \mathbb{E} \left[ \langle X_n - x^*, H(X_n, Z_{n+1}) \rangle | \mathcal{F}_n \right] 
\]

\[
\leq \|X_n - x^*\|^2 + \gamma^2(n + 1)^{-2\alpha} \left[ \eta + \|\nabla f(X_n)\|^2 \right] 
- 2\gamma(n + 1)^{-\alpha} \|X_n - x^*, \nabla f(X_n)\| 
\]

\[
\mathbb{E} \left[ \|X_{n+1} - x^*\|^2 \right] \leq \mathbb{E} \left[ \|X_n - x^*\|^2 \right] \left[ 1 - 2\gamma(n + 1)^{-\alpha} \mu + \gamma^2(n + 1)^{-2\alpha} \mu L^2 \right] + \eta \gamma^2(n + 1)^{-2\alpha}.
\]

We note now \( u_n = \mathbb{E} \left[ \|X_n - x^*\|^2 \right] \) and \( v_n = n^\alpha u_n \). Using (37) et Bernoulli’s inequality we have, for all \( n \geq 0 \)

\[
v_{n+1} - v_n = (n + 1)^\alpha u_{n+1} - n^\alpha u_n 
= (n + 1)^\alpha(u_{n+1} - u_n) + u_n((n + 1)^\alpha - n^\alpha) 
\leq \left[ -2\gamma\mu + \gamma^2 L^2(n + 1)^{-\alpha} \right] u_n + \eta \gamma^2(n + 1)^{-\alpha} + u_n n^\alpha [(1 + 1/n)^\alpha - 1] 
\leq \left[ -2\gamma\mu + \gamma^2 L^2(n + 1)^{-\alpha} + an^{\alpha-1} \right] u_n + \eta \gamma^2(n + 1)^{-\alpha}.
\]

Therefore, in the case where \( \alpha < 1 \), there exists \( n_0 \geq 0 \) such that for all \( n \geq n_0 \),

\[
v_{n+1} - v_n \leq -\gamma \mu u_n + \eta \gamma^2(n + 1)^{-\alpha} 
\leq -\gamma \mu n^{-\alpha} u_n + \eta \gamma^2(n + 1)^{-\alpha} 
\leq (n + 1)^{-\alpha}(-\gamma \mu u_n + \eta \gamma^2) .
\]

And in the case where \( \alpha = 1 \), if \( \gamma > 1/(2\mu) \) we have the existence of \( n_1 \geq 0 \) such that for all \( n \geq n_1 \),

\[
v_{n+1} - v_n \leq \left[ (1/2 - \gamma\mu) + \gamma^2 L^2(n + 1)^{-\alpha} + an^{\alpha-1} \right] u_n + \eta \gamma^2(n + 1)^{-\alpha} .
\]

This shows that, for \( \alpha \in (0, 1] \), there exists a constant \( B_3 > 0 \) such that for all \( n \geq 0 \), \( v_n \leq B_3 \). This proves the result. \( \square \)
Using A1 and the descent lemma (Nesterov, 2004, Lemma 1.2.3) we have the immediate corollary

**Corollary 28.** Let \( \alpha \in (0, 1) \) and \( \gamma \in (0, 1) \). Let \((X_n)_{n \geq 0}\) be given by (1). Assume A1, A2 and F1. Then there exists \( B_4 > 0 \) such that for all \( N \geq 1 \),

\[
\mathbb{E} [ f(X_N) - f^* ] \leq B_4 N^{-\alpha}.
\]

If \( \alpha = 1 \) we have also assumed that \( \gamma > 1/(2\mu) \).

We state now an equivalent result of Theorem 27 under weaker assumptions, namely the Łojasiewicz inequality with \( r = 2 \), that we restate as it is usually given, with \( c > 0 \).

\[
\forall x \in \mathbb{R}^d, f(x) - f(x^*) \leq c \| \nabla f(x) \|^2.
\]

Note that (38) is verified for all strongly convex functions (Karimi et al., 2016). Under this condition we have the following proposition.

**Proposition 29.** Let \( \alpha, \gamma \in (0, 1) \) and \((X_t)_{t \geq 0}\) be given by (2). Assume A1, A2, A3 and that \( f \) verifies (38). Then there exists \( B_5 > 0 \) such that for any \( T > 0 \),

\[
\mathbb{E} [ f(X_T) - f^* ] \leq B_5 T^{-\alpha}.
\]

**Proof.** Let \( \alpha, \gamma \in (0, 1) \) and \((X_t)_{t \geq 0}\) be given by (2). Without loss of generality we can assume that \( f^* = \min_{x \in \mathbb{R}^d} f(x) = 0 \). We note \( \mathcal{E}(t) = \mathbb{E}[f(X_t)] \) and we apply Lemma 32 to the stochastic process \((t + \gamma_{n})^\alpha f(X_t))_{t \geq 0}\), and using A1, A2, A3, (38) and Lemma 31 this gives, for all \( t > 0 \),

\[
\mathcal{E}(t) - \mathcal{E}(0) = \int_0^t \alpha(s + \gamma_{n})^{\alpha-1} \mathbb{E}[f(X_s)] ds - \int_0^t \mathbb{E}

\[
\mathbb{E} \left[ \frac{\gamma}{2} f(X_s \Sigma(X_s)) \right] ds
\]

\[
\frac{d\mathcal{E}(t)}{dt} \leq \alpha \mathcal{E}(t) \frac{t + \gamma_{n}}{\gamma_{n} - 1} - \frac{1}{c} \mathcal{E}(t) \frac{t + \gamma_{n}}{\gamma_{n} - 1} + L\eta(t + \gamma_{n})^{-\alpha}
\]

We can now apply Lemma 3 to \( F(t, x) = \alpha x(t + \gamma_{n})^{-1} - \frac{1}{c} x(t + \gamma_{n})^{-\alpha} + L\eta(t + \gamma_{n})^{-\alpha} \) with \( t_0 = (2c\alpha)^{1/(1-\alpha)} \) and \( A = 4cL\eta \), which shows the existence of \( B_5 > 0 \) such that for all \( t > 0 \), \( \mathcal{E}(t) \leq B_5 \), concluding the proof.

And we now state its discrete counterpart, which is an equivalent of Corollary 28.

**Proposition 30.** Let \( \alpha \in (0, 1) \) and \( \gamma \in (0, 1) \). Let \((X_n)_{n \geq 0}\) be given by (1). Assume A1, A2 and that \( f \) verifies (38). Then there exists \( B_6 > 0 \) such that for all \( N \geq 1 \),

\[
\mathbb{E} [ f(X_N) - f^* ] \leq B_6 N^{-\alpha}.
\]

In the case where \( \alpha = 1 \) we have to assume additionally that \( \gamma > 2/c \).

**Proof.** Let \( \alpha \in (0, 1) \) and \( \gamma \in (0, 1) \). Let \((X_n)_{n \geq 0}\) be given by (1). Let \( n \geq 0 \). Applying the descent lemma (using A1) gives

\[
\mathbb{E} [ f(X_{n+1})|\mathcal{F}_n ] = \mathbb{E} [ f(X_n - \gamma/(n + 1)^\alpha H(X_n, Z_{n+1})|\mathcal{F}_n ]
\]

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\[ \leq f(X_n) - \gamma/(n + 1)^{\alpha} \mathbb{E} \left[ \langle \nabla f(X_n), H(X_n, Z_{n+1}) \rangle \right] \mathcal{F}_n \]
\[ + \gamma^2/(n + 1)2^\alpha (L/2) \mathbb{E} \left[ \| H(X_n, Z_{n+1}) \|^2 \right] \mathcal{F}_n \]
\[ \leq f(X_n) - \gamma/(n + 1)^{\alpha} \| \nabla f(X_n) \|^2 + (L\gamma^2/2)(n + 1)^{-2\alpha} \left[ \eta + \| \nabla f(X_n) \|^2 \right] \]
\[ \mathbb{E} \left[ f(X_{n+1}) \right] - f^* \leq \mathbb{E} \left[ f(X_n) \right] - f^* - \gamma(n + 1)^{-\alpha} \mathbb{E} \left[ \| \nabla f(X_n) \|^2 \right] \left[ -1 + (L\gamma/2)(n + 1)^{-\alpha} \right] \]
\[ + (L\gamma^2/2)(n + 1)^{-2\alpha} \eta \cdot \]

This shows the existence of \( n_2 \geq 0 \) such that using (38) we have for all \( n \geq n_2 \),
\[ \mathbb{E} \left[ f(X_{n+1}) \right] - f^* \leq \mathbb{E} \left[ f(X_n) \right] - f^* - (\gamma/2)(n + 1)^{-\alpha} \mathbb{E} \left[ \| \nabla f(X_n) \|^2 \right] + (L\gamma^2/2)(n + 1)^{-2\alpha} \eta \]
\[ \leq (\mathbb{E} \left[ f(X_n) \right] - f^*) \left[ 1 - (\gamma c^{-1}/2)(n + 1)^{-\alpha} \right] + (L\gamma^2/2)(n + 1)^{-2\alpha} \eta \cdot \]

We note now for all \( n \geq 0 \), \( u_n = \mathbb{E} \left[ f(X_n) \right] - f^* \) and \( v_n = n^\alpha u_n \). We have
\[ v_{n+1} - v_n = (n + 1)^\alpha u_{n+1} - n^\alpha u_n \]
\[ = (n + 1)^\alpha(u_{n+1} - u_n) + u_n((n + 1)^\alpha - n^\alpha) \]
\[ \leq -(\gamma c^{-1}/2)u_n + (L\gamma^2\eta/2)(n + 1)^{-\alpha} + u_n n^\alpha \left[ (1 + 1/n)^\alpha - 1 \right] \]
\[ \leq u_n(-\gamma c^{-1}/2 + \alpha n^{\alpha-1}) + (L\gamma^2\eta/2)(n + 1)^{-\alpha} \cdot \]

If \( \alpha < 1 \), or if \( 1 - \gamma c^{-1}/2 < 0 \) we have the existence of \( n_3 \geq n_2 \) and \( \bar{B} > 0 \) such that for all \( n \geq n_3 \),
\[ v_{n+1} - v_n \leq \bar{B}u_n + (L\gamma^2\eta/2)(n + 1)^{-\alpha} \]
\[ \leq \left\{ -\bar{B}v_n + (L\gamma^2\eta/2) \right\} (n + 1)^{-\alpha} \cdot \]

This proves the existence of \( B_6 > 0 \) such that for all \( n \geq 0 \),
\[ v_n \leq B_6 \cdot \]

concluding the proof. \( \Box \)

C Convex Case

C.1 Technical Results

Lemma 31. Let \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \). Assume A1 and A2. Then for any \( x \in \mathbb{R}^d \) we have
\[ \left| \langle \nabla^2 f(x), \Sigma(x) \rangle \right| \leq L \eta, \quad \left| \langle \nabla f(x) \nabla f(x)^T, \Sigma(x) \rangle \right| \leq \eta^2 \| \nabla f(x) \|^2 \cdot \]

Proof. Let \( x \in \mathbb{R}^d \). Using Cauchy-Schwarz’s inequality, we have \( \left| \langle \nabla^2 f(x), \Sigma(x) \rangle \right| \leq \| \nabla^2 f(x) \| \| \Sigma(x) \|_* \),
where \( \| \cdot \| \) is the operator norm and \( \| \cdot \|_* \) is the nuclear norm. Using A1 we have \( \| \nabla^2 f(x) \| \leq L \) for all \( x \in \mathbb{R}^d \).

In addition, denoting \( (\lambda_i)_{i \in \{1, \ldots, d\}} \) the eigenvalues of \( \Sigma(x) \), using that \( \Sigma \) is positive semi-definite and A2 we have
\[ \| \Sigma(x) \|_* = \sum_{i=1}^d |\lambda_i| = \sum_{i=1}^d \lambda_i = \text{Tr}(\Sigma(x)) \leq \eta \cdot \]

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This concludes the first part of the proof.

For the second part we have
\[
\left| \langle \nabla f(x) \nabla f(x)^\top, \Sigma(x) \rangle \right| \leq \sup_{i \in \{1, \ldots, d\}} \lambda_i \|\nabla f(x)\|^2 \leq \eta^2 \|\nabla f(x)\|^2 .
\]

\[ \Box \]

The following lemma consists into taking the expectation in Itô’s formula.

**Lemma 32.** Let \( \alpha \in [0, 1) \) and \( \gamma > 0 \). Assume \( f, g \in C^2(\mathbb{R}^d, \mathbb{R}) \), \( A_1, A_2 \) and \( A_3 \) and let \( (X_t)_{t \geq 0} \) solution of (2). Then for any \( \varphi \in C^1([0, +\infty), \mathbb{R}) \), \( Y \in \mathcal{F}_0 \) and \( \mathbb{E} [\|Y\|^2 + |g(Y)|] < +\infty \), we have the following results:

(a) For any \( t \geq 0 \),
\[
\mathbb{E} [\|X_t - Y\|^2 \varphi(t)] = \mathbb{E} [\|X_0 - Y\|^2 \varphi(0)] - 2 \int_0^t (\gamma + s)^{-1} \varphi(s) \mathbb{E} [\langle \nabla f(X_s), X_s - Y \rangle] \, ds \tag{39}
+ \gamma \int_0^t (\gamma + s)^{-2} \varphi(s) \mathbb{E} [\operatorname{Tr}(\Sigma(X_s))] \, ds + \int_0^t \varphi'(s) \mathbb{E} [\|X_s - Y\|^2] \, ds.
\]

(b) For any \( t \geq 0 \)
\[
\mathbb{E} [(f(X_t) - g(Y)) \varphi(t)] = \mathbb{E} [(f(X_0) - g(Y)) \varphi(0)] - \int_0^t (\gamma + s)^{-1} \varphi(s) \mathbb{E} [\|\nabla f(X_s)\|^2] \, ds
+ (\gamma/2) \int_0^t (\gamma + s)^{-2} \varphi(s) \mathbb{E} [(\nabla^2 f(X_s), \Sigma(X_s))] \, ds + \int_0^t \varphi'(s) \mathbb{E} [f(X_s) - g(Y)] \, ds.
\]

(c) If \( \mathbb{E}[\|Y\|^{2p}] < +\infty \), then for any \( t \geq 0 \)
\[
\mathbb{E} [\|X_t - Y\|^{2p} \varphi(t)] = \mathbb{E} [\|X_0 - Y\|^{2p} \varphi(0)]
- 2p \int_0^t (\gamma + s)^{-1} \varphi(s) \mathbb{E} [\langle \nabla f(X_s), X_s - Y \rangle \|X_t - Y\|^{2(p-1)}] \, ds
+ \gamma_p \int_0^t (\gamma + s)^{-2} \varphi(s) \mathbb{E} [\operatorname{Tr}(\Sigma(X_s)) \|X_s - Y\|^{2(p-1)}] \, ds
+ \gamma_p 2p(p-1) \int_0^t (\gamma + s)^{-2} \varphi(s) \mathbb{E} [(\Sigma(X_s), (X_t - Y)(X_t - Y)^\top) \|X_s - Y\|^{2(p-2)}] \, ds
+ \int_0^t \varphi'(s) \mathbb{E} [(f(X_s) - g(Y))^{2p}] \, ds.
\]

(d) If \( \mathbb{E}[|g(Y)|^p] < +\infty \), then for any \( t \geq 0 \)
\[
\mathbb{E} [(f(X_t) - g(Y))^p \varphi(t)] = \mathbb{E} [(f(X_0) - g(Y))^p \varphi(0)]
- p \int_0^t (\gamma + s)^{-1} \varphi(s) \mathbb{E} [\|\nabla f(X_s)\|^2 (f(X_s) - g(Y))^{p-1}] \, ds
+ \gamma_p (p/2) \int_0^t (\gamma + s)^{-2} \varphi(s) \mathbb{E} [\langle \nabla^2 f(X_s), \Sigma(X_s)(f(X_s) - g(Y))^{p-2} \rangle] \, ds
+ \gamma_p p(p - 1)/2 \int_0^t (\gamma + s)^{-2} \varphi(s) \mathbb{E} [\langle \nabla f(X_s) \nabla f(X_s)^\top, \Sigma(X_s)(f(X_s) - g(Y))^{p-2} \rangle]
+ \int_0^t \varphi'(s) \mathbb{E} [(f(X_s) - g(Y))^p] \, ds.
\]

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Proof. Let \( \alpha \in [0,1), \gamma > 0 \) and \((X_t)_{t\geq 0}\) the solution of (2). Note that for any \( t \geq 0 \), we have

\[
\langle X \rangle_t = \gamma \alpha \int_0^t (\gamma \alpha + s)^{-2} \text{Tr}(\Sigma(X_s)) ds .
\]

We divide the rest of the proof into our parts.

(a) First, let \( y \in \mathbb{R}^d \) and \( F_y : [0, +\infty) \times \mathbb{R}^d \) such that for any \( t \in [0, +\infty) \), \( x \in \mathbb{R}^d \), \( F_y(t, x) = \varphi(t)\|x-y\|^2 \). Since \((X_t)_{t \geq 0}\) is a strong solution of (2) we have that \((X_t)_{t \geq 0}\) is a continuous semi-martingale. Using this result, the fact that \( F \in C^{1,2}([0, +\infty), \mathbb{R}^d) \) and Itô’s lemma (Karatzas and Shreve, 1991, Chapter 3, Theorem 3.6) we obtain that for any \( t \geq 0 \) almost surely

\[
F_y(t, X_t) = F_y(0, X_0) + \int_0^t \partial_1 F_y(s, X_s) ds + \int_0^t \langle \partial_2 F_y(s, X_s), dX_s \rangle
\]

\[
+ (1/2) \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\langle X \rangle_s \rangle
\]

\[
= F_y(0, X_0) + \int_0^t \varphi'(s) \|X_s - y\|^2 ds + \int_0^t \langle \partial_2 F_y(s, X_s), dX_s \rangle
\]

\[
+ (1/2) \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\langle X \rangle_s \rangle
\]

\[
= F_y(0, X_0) + \int_0^t \varphi'(s) \|X_s - y\|^2 ds - 2 \int_0^t (\gamma \alpha + s)^{-1} \varphi(s) \langle \nabla f(X_s), X_s - y \rangle ds
\]

\[
+ 2\gamma \alpha^{1/2} \int_0^t (\gamma \alpha + s)^{-1} \varphi(s) (X_s - y, \Sigma(X_s)^{1/2} dB_s) + \gamma \alpha \int_0^t (\gamma \alpha + s)^{-2} \varphi(s) \text{Tr}(\Sigma(X_s)) ds .
\]

Using \( A1 \) have for any \( x \in \mathbb{R}^d \),

\[
|\langle \nabla f(x), x - y \rangle| \leq \|\nabla f(0)\| \|x - y\| + L \|x\| \|x - y\| .
\]

Therefore, using this result Lemma 15, Cauchy-Schwarz’s inequality and that \( \mathbb{E}[\|Y\|^2] < +\infty \), we obtain that for any \( t \geq 0 \) there exists \( \bar{a} \geq 0 \) such that

\[
\sup_{s \in [0,t]} \mathbb{E}[\|X_s - Y\|^2] \leq \bar{a}, \quad \sup_{s \in [0,t]} \mathbb{E}[|\langle \nabla f(X_s), X_s - Y \rangle|] \leq \bar{a} .
\]

(41)

In addition, we have using \( A2 \) that for any \( t \geq 0 \), \( \mathbb{E}[\|\text{Tr}(\Sigma(X_s))\| = \mathbb{E}[\text{Tr}(\Sigma(X_s))] \leq \eta \). Combining this result, (41), (40), that \((\int_0^t (\gamma \alpha + t)^{-1} \varphi(t)\|X_t - Y, \Sigma(X_t)^{1/2} dB_t\|)_{t \geq 0}\) is a martingale and Fubini-Lebesgue’s theorem we obtain for any \( t \geq 0 \)

\[
\mathbb{E}[\varphi(t) \|X_t - Y\|^2] = \mathbb{E}[F_Y(t, X_t) | \mathcal{F}_0]
\]

\[
= \mathbb{E}[\varphi(0) \|X_0 - Y\|^2] + \int_0^t \varphi'(s) \mathbb{E}[\|X_s - Y\|^2] ds
\]

\[
- 2 \int_0^t (\gamma \alpha + s)^{-1} \varphi(s) \mathbb{E}[\|\nabla f(X_s), X_s - Y\|] ds
\]

\[
+ \gamma \alpha \int_0^t (\gamma \alpha + s)^{-2} \varphi(s) \mathbb{E}[\text{Tr}(\Sigma(X_s))] ds ,
\]

which concludes the proof of (39).
(b) Second, let \( y \in \mathbb{R}^d \) and \( F : [0, +\infty) \times \mathbb{R}^d \) such that for any \( t \in [0, +\infty) \), \( x \in \mathbb{R}^d \), \( F_y(t, x) = \varphi(t)(f(x) - g(y)) \). Using that \((X_t)_{t \geq 0}\) is a continuous semi-martingale, the fact that \( F \in C^{1,2}([0, +\infty), \mathbb{R}^d)\) and Itô’s lemma (Karatzas and Shreve, 1991, Chapter 3, Theorem 3.6) we obtain that for any \( t \geq 0 \) almost surely

\[
F_y(t, X_t) = F_y(0, X_0) + \int_0^t \partial_1 F_y(s, X_s) ds + \int_0^t \langle \partial_2 F_y(s, X_s), dX_s \rangle + (1/2) \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\mathbf{X}_s \rangle
\]

Using that \( (\varphi(t))_{t \geq 0} \) is a continuous semi-martingale, the fact that \( \varphi(t) \) is a martingale and Cauchy-Schwarz’s inequality and that \( \mathbb{E}[g(Y)^2] < +\infty \), we obtain that for any \( t \geq 0 \) there exists \( \bar{a} \geq 0 \) such that

\[
\sup_{s \in [0,t]} \mathbb{E} \left[ |f(X_s) - g(Y)| \right] \leq \bar{a}, \quad \sup_{s \in [0,t]} \mathbb{E} \left[ \| \nabla f(X_s) \|^2 \right] \leq \bar{a}.
\]

Combining this result, Lemma 31, the fact that \((\int_0^t \varphi(s)(\nabla f(X_s), \Sigma(X_s)^{1/2} dB_s))_{t \geq 0}\) is a martingale and Fubini-Lebesgue’s theorem we obtain that for any \( t \geq 0 \)

\[
\mathbb{E} \left[ F_y(t, X_t) \right] = \mathbb{E} \left[ \mathbb{E} \left[ F_Y(t, X_t) | \mathcal{F}_0 \right] \right]
\]

\[
= \mathbb{E} \left[ \varphi(0)(f(X_0) - g(Y)) \right] + \int_0^t \varphi'(s) \mathbb{E} \left[ (f(X_s) - g(Y)) \right] ds
\]

\[
- \int_0^t (\gamma_\alpha + s)^{-1} \varphi(s) \mathbb{E} \left[ \| \nabla f(X_s) \|^2 \right] ds
\]

\[
+ (\gamma_\alpha/2) \int_0^t (\gamma_\alpha + s)^{-2} \varphi(s) \mathbb{E} \left[ \| \nabla^2 f(X_s), \Sigma(X_s) \|^2 \right] ds.
\]

(c) Let \( y \in \mathbb{R}^d \) and \( F_y : [0, +\infty) \times \mathbb{R}^d \) such that for any \( t \in [0, +\infty) \), \( x, y \in \mathbb{R}^d \), \( F_y(t, x) = \varphi(t) \| x - y \|^{2p} \). Using that \((X_t)_{t \geq 0}\) is a continuous semi-martingale, the fact that \( F_y \in C^{1,2}([0, +\infty), \mathbb{R}^d)\) and Itô’s lemma (Karatzas and Shreve, 1991, Chapter 3, Theorem 3.6) we obtain that for any \( t \geq 0 \) almost surely

\[
F_y(t, X_t) = F_y(0, X_0) + \int_0^t \partial_1 F_y(s, X_s) ds + \int_0^t \langle \partial_2 F_y(s, X_s), dX_s \rangle + (1/2) \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\mathbf{X}_s \rangle
\]

\[
= F_y(0, X_0) + \int_0^t \varphi'(s) \| X_s - y \|^{2p} ds + \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\mathbf{X}_s \rangle
\]

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(d) Let $y \in \mathbb{R}^d$ and $F : [0, +\infty) \times \mathbb{R}^d$ such that for any $t \in [0, +\infty)$, $x, y \in \mathbb{R}^d$, $F_y(t, x) = \varphi(t)(f(x) - g(y))^{2p}$. Using that $(X_t)_{t \geq 0}$ is a continuous semi-martingale, the fact that $F \in C^{1,2}([0, +\infty) \times \mathbb{R}^d)$ and Itô’s lemma (Karatzas and Shreve, 1991, Chapter 3, Theorem 3.6) we obtain that for any $t \geq 0$ almost surely

\[
F_y(t, X_t) = F_y(0, X_0) + \int_0^t \partial_1 F_y(s, X_s) \, ds + \int_0^t \langle \partial_2 F_y(s, X_s), dX_s \rangle + (1/2) \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\langle X \rangle_s \rangle
\]

\[
= F_y(0, X_0) + \int_0^t \varphi'(s)(f(X_s) - g(y))^{2p} \, ds
\]

\[
+ \int_0^t \langle \partial_2 F_y(s, X_s), dX_s \rangle + (1/2) \int_0^t \langle \partial_{2,2} F_y(s, X_s), d\langle X \rangle_s \rangle
\]

\[
= F_y(0, X_0) + \int_0^t \varphi'(s)(f(X_s) - g(y))^{2p} \, ds
\]

Using A1 and that for any $a, b \geq 0$, $(a + b)^2 \leq 2(a^2 + b^2)$ we have for any $x, y \in \mathbb{R}^d$, Therefore, using this result Lemma 15, Cauchy-Schwarz’s inequality and that $\mathbb{E}[\|Y\|^2] < +\infty$, we obtain that for any $t \geq 0$ there exists $\hat{A} \geq 0$ such that

\[
\sup_{s \in [0, t]} \mathbb{E} \left[ \|X_s - Y\|^{2p} \right] \leq \hat{A}, \quad \sup_{s \in [0, t]} \mathbb{E} \left[ \langle \nabla f(X_s), X_s - Y \rangle \|X_s - Y\|^{2(p-1)} \right] \leq \hat{A}.
\]
\[-2p \int_0^t (\gamma_0 + s)^{-1} \varphi(s) \| \nabla f(X_s) \|^2 (f(X_s) - g(y))^{2(p-1)} ds \]
\[+ 2p \gamma_0 \int_0^t (\gamma_0 + s)^{-1} \varphi(s) \langle \nabla f(X_s), \Sigma(X_s)^{1/2} (f(X_s) - g(y)) \rangle^{2(p-1)} dB_s \]
\[+ \gamma_0 \int_0^t (\gamma_0 + s)^{-1} \varphi(s) \langle \nabla^2 f(X_s), \Sigma(X_s) \rangle (f(X_s) - g(y))^{2(p-1)} ds \]
\[+ 2p(p - 1) \int_0^t (\gamma_0 + s)^{-2} \varphi(s) \langle \nabla f(X_s) \nabla f(X_s)^\top, \Sigma(X_s) \rangle (f(X_s) - g(y))^{2(p-2)} ds \]

Using A1 and that for any \(a, b \geq 0, (a + b)^2 \leq 2(a^2 + b^2)\) we have for any \(x, y \in \mathbb{R}^d\),
\[|f(x) - g(y)|^{2p} \leq 4^{2p-1} |f(0)|^{2p} + 4^{2p-1} \| \nabla f(0) \|^2 \| x \|^2 + (4^{2p-1} L/2) \| x \|^4p + 4^{2p-1} |g(y)|^{2p}, \]
\[\| \nabla f(x) \|^2 \leq 2 \| \nabla f(0) \|^2 + 2L^2 \| x \|^2.\]

Therefore, using this result Lemma 15, Lemma 31, Hölder’s inequality and that \(\mathbb{E}[g(Y)^2] < +\infty\), we obtain that for any \(t \geq 0\) there exists \(\tilde{k} \geq 0\) such that
\[\sup_{s \in [0, t]} \mathbb{E} \left[ |f(X_s) - g(Y)|^{2p} \right] \leq \tilde{k}, \quad \sup_{s \in [0, t]} \mathbb{E} \left[ \| \nabla f(X_s) \|^2 |f(X_s) - g(Y)|^{2(p-1)} \right] \leq \tilde{k}, \]
\[\sup_{s \in [0, t]} \mathbb{E} \left[ \langle \nabla f(X_s) \nabla f(X_s)^\top, \Sigma(X_s) \rangle (f(X_s) - g(Y))^{2(p-2)} \right] \leq \tilde{k}.\]

Combining this result, Lemma 31, the fact that \(\langle 0 \varphi(s) \langle \nabla f(X_s), \Sigma(X_s)^{1/2} (f(X_s) - g(Y)) \rangle dB_s \rangle_{t \geq 0}\) is a martingale and Fubini-Lebesgue’s theorem we obtain that for any \(t \geq 0\)
\[\mathbb{E} [F_y(t, X_t)] = \mathbb{E} [\mathbb{E} [F_Y(t, X_t) | F_0]] \]
\[= \mathbb{E} \left[ \varphi(0) (f(X_0) - g(Y))^{2p} \right] + \int_0^t \varphi'(s) \mathbb{E} \left[ (f(X_s) - g(Y))^{2p} \right] ds \]
\[- 2p \int_0^t (\gamma_0 + s)^{-1} \varphi(s) \mathbb{E} \left[ \| \nabla f(X_s) \|^2 (f(X_s) - g(y))^{2(p-1)} \right] ds \]
\[+ \gamma_0 \int_0^t (\gamma_0 + s)^{-2} \varphi(s) \mathbb{E} \left[ \langle \nabla^2 f(X_s), \Sigma(X_s) \rangle (f(X_s) - g(Y))^{2(p-1)} \right] ds \]
\[+ 2\gamma_0 p(p - 1) \int_0^t (\gamma_0 + s)^{-2} \varphi(s) \mathbb{E} \left[ \langle \nabla f(X_s) \nabla f(X_s)^\top, \Sigma(X_s) \rangle (f(X_s) - g(Y))^{2(p-2)} \right] ds. \]

The following lemma is a useful tool that converts results on \(C^2\) functions to smooth functions.

**Lemma 33.** Assume A1, F2, A3 and that \(\arg\min_{x \in \mathbb{R}^d} f\) is bounded. Then there exists \((f_\varepsilon)_{\varepsilon > 0}\) such that for any \(\varepsilon > 0, f_\varepsilon\) is convex, \(C^2\) with \(L\)-Lipschitz continuous gradient. In addition, there exists \(C \geq 0\) such that the following properties are satisfied.

\[(a)\] For all \(\varepsilon > 0, f_\varepsilon\) admits a minimize \(x_\varepsilon^*\) and \(\limsup_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon^*) \leq f(x^*).\)

\[(b)\] \(\liminf_{\varepsilon \to 0} \| x_\varepsilon^* \| \leq C.\)

\[(c)\] For any \(T \geq 0, \lim_{\varepsilon \to 0} \mathbb{E} [\| f_\varepsilon(X_{T, \varepsilon}) - f(X_T) \|] = 0, \) where \((X_{t, \varepsilon})_{t \geq 0}\) is the solution of (2) replacing \(f\) by \(f_\varepsilon\).
Proof. Let \( \varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+) \) be an even compactly-supported function such that \( \int_{\mathbb{R}^d} \varphi(z)dz = 1 \). For any \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \), let \( \varphi_\varepsilon(x) = \varepsilon^{-d} \varphi(x/\varepsilon) \) and \( f_\varepsilon = \varphi_\varepsilon * f \). Since \( \varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+) \) and is compactly-supported, we have \( f_\varepsilon \in C^\infty(\mathbb{R}^d, \mathbb{R}) \). In addition, we have for any \( \varepsilon > 0 \), \( (\nabla f)_\varepsilon = \nabla f_\varepsilon \).

First, we show that for any \( \varepsilon \), \( f \) is convex and \( \nabla f_\varepsilon \) is \( L \)-Lipschitz continuous. Let \( \varepsilon > 0 \), \( x, y \in \mathbb{R}^d \) and \( t \in [0, 1] \). Using \( \mathbf{F2} \) we have

\[
f_\varepsilon(tx + (1-t)y) = \int_{\mathbb{R}^d} f(tx + (1-t)y - z) \varphi_\varepsilon(z)dz \leq \int_{\mathbb{R}^d} \{tf(x-z) + (1-t)f(y-z)\} \varphi_\varepsilon(z)dz \\
\leq tf_\varepsilon(x) + (1-t)f_\varepsilon(y).
\]

Hence, \( f_\varepsilon \) is convex. In addition, using \( \mathbf{A1} \) and that \( \int_{\mathbb{R}^d} \varphi_\varepsilon(z)dz = 1 \) we have

\[
\|\nabla f_\varepsilon(x) - \nabla f_\varepsilon(y)\| \leq \int_{\mathbb{R}^d} \|\nabla f(x-z) - \nabla f(y-z)\| \varphi_\varepsilon(z)dz \leq L \|x-y\|,
\]

which proves that \( \nabla f_\varepsilon \) is \( L \)-Lipschitz continuous.

Second, we show that \( f_\varepsilon \) and \( \nabla f_\varepsilon \) converge uniformly towards \( f \) and \( \nabla f \). Let \( \varepsilon > 0 \), \( x \in \mathbb{R}^d \). Using the convexity of \( f \) and that \( \varphi_\varepsilon \) is even, we get

\[
f_\varepsilon(x) - f(x) = \int_{\mathbb{R}^d} (f(x-z) - f(x)) \varphi_\varepsilon(z)dz \\
\geq -\int_{\mathbb{R}^d} \langle \nabla f(x), z \rangle \varphi_\varepsilon(z)dz \\
\geq -\langle \nabla f(x), \int_{\mathbb{R}^d} z \varphi_\varepsilon(z)dz \rangle \geq 0,
\]

Conversely, using the descent lemma (Nesterov, 2004, Lemma 1.2.3) and that \( \varphi_\varepsilon \) is even, we have

\[
f_\varepsilon(x) - f(x) = \int_{\mathbb{R}^d} (f(x-z) - f(x)) \varphi_\varepsilon(z)dz \\
\leq \int_{\mathbb{R}^d} (-\langle \nabla f(x), z \rangle + (L/2) \|z\|^2) \varphi_\varepsilon(z)dz \\
\leq (L/2) \int_{\mathbb{R}^d} \varepsilon^2 \|z/\varepsilon\|^2 \varepsilon^{-d} \varphi(z/\varepsilon)dz \leq (L/2)\varepsilon^2 \int_{\mathbb{R}^d} \|u\|^2 \varphi(u)du.
\]

Combining (42) and (43) we get that \( \lim_{\varepsilon \to 0} \|f - f_\varepsilon\|_\infty = 0 \). Using \( \mathbf{A1} \) we have for any \( x \in \mathbb{R}^d \)

\[
\|\nabla f_\varepsilon(x) - \nabla f(x)\| \leq \|(\nabla f)_\varepsilon(x) - \nabla f(x)\| \leq \int_{\mathbb{R}^d} \|\nabla f(x-z) - \nabla f(x)\| \varphi_\varepsilon(z)dz \leq L \varepsilon \int_{\mathbb{R}^d} \|z\| \varphi(z)dz,
\]

Hence, we obtain that \( \lim_{\varepsilon \to 0} \|\nabla f_\varepsilon - \nabla f\|_\infty = 0 \). Finally, since \( f \) is coercive (Bertsekas, 1997, Proposition B.9) and \( (f_\varepsilon)_{\varepsilon > 0} \) converges uniformly towards \( f \) we have that for any \( \varepsilon > 0 \), \( f_\varepsilon \) is coercive.

We divide the rest of the proof into three parts.

(a) Let \( \varepsilon > 0 \). Since \( f_\varepsilon \) is coercive and continuous it admits a minimizer \( x^*_\varepsilon \). In addition, we have

\[
f_\varepsilon(x^*_\varepsilon) \leq f_\varepsilon(x) \leq f(x^*) + \|f_\varepsilon - f\|_\infty.
\]

Therefore, \( \limsup_{\varepsilon \to 0} f_\varepsilon(x^*_\varepsilon) \leq f(x^*) \).

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(b) Let \( \varepsilon \in (0,1) \). Using (44), we obtain that \( |f_{\varepsilon}(x^*)| \leq |f(x^*)| + \sup_{\varepsilon \in (0,1)} \| f_{\varepsilon} - f \|_{\infty} \). Since \( f \) is coercive, we obtain that \( (x^*_\varepsilon)_{\varepsilon \in (0,1)} \) is bounded and therefore there exists \( \mathcal{C} \geq 0 \) such that \( \liminf_{\varepsilon \to 0} \| x^*_\varepsilon \| \leq \mathcal{C} \).

(c) Let \( \varepsilon > 0, T \geq 0 \) and \((X_{t,\varepsilon})_{t \geq 0}\) be the solution of (2) replacing \( f \) by \( f_{\varepsilon} \). Using (2), the fact that \( \lim_{\varepsilon \to 0} \| \nabla f - \nabla f_{\varepsilon} \|_{\infty} = 0 \), A1 and Grönwall’s inequality (Pachpatte, 1998, Theorem 1.2.2) we have

\[
\mathbb{E} \left[ \| X_{T,\varepsilon} - X_T \|^2 \right] \leq \mathbb{E} \left[ \int_0^T (\gamma_\alpha + s)^{-\alpha} \{ -\nabla f_{\varepsilon}(X_{t,\varepsilon}) + \nabla f(X_t) \} \, dt \right]^2 \leq 2\gamma_\alpha^{-2\alpha} T \int_0^T \mathbb{E} \left[ \| \nabla f(X_{t,\varepsilon}) - \nabla f(X_t) \|^2 \right] \, dt + 2\gamma_\alpha^{-2\alpha} T^2 \| \nabla f - \nabla f_{\varepsilon} \|^2_{\infty} \leq 2L\gamma_\alpha^{-2\alpha} T \int_0^T \mathbb{E} \left[ \| X_{t,\varepsilon} - X_t \|^2 \right] \, dt + 2\gamma_\alpha^{-2\alpha} T^2 \| \nabla f - \nabla f_{\varepsilon} \|^2_{\infty} \leq 2\gamma_\alpha^{-2\alpha} T^2 \| \nabla f - \nabla f_{\varepsilon} \|^2_{\infty} \exp \left[ 2L\gamma_\alpha^{-2\alpha} T^2 \right].
\]

Therefore \( \lim_{\varepsilon \to 0} \mathbb{E} \left[ \| X_{T,\varepsilon} - X_T \|^2 \right] = 0 \). In addition, using the Cauchy-Schwarz inequality, A1 and Lemma 15 we have

\[
\mathbb{E} \left[ |f(X_{T,\varepsilon}) - f(X_T)| \right] \leq \mathbb{E} \left[ \left( \int_0^T \| \nabla f(X_T + t(X_{t,\varepsilon} - X_t)) \| \, dt \right) \right] \leq \mathbb{E} \left[ \left( \int_0^T \| X_{t,\varepsilon} - X_t \| \, dt \right) \right] \leq 3^{1/2} (\| x^* \|^2 + 2A_{T,1})^{1/2} \mathbb{E} \left[ \| X_{T,\varepsilon} - X_T \|^2 \right]^{1/2} \leq 3^{1/2} (\| x^* \|^2 + 2A_{T,1})^{1/2} (1 + \| x_0 \|^2)^{1/2} \mathbb{E} \left[ \| X_{T,\varepsilon} - X_T \|^2 \right]^{1/2}.
\]

Therefore, using (45), (46) and the fact that \( \lim_{\varepsilon \to 0} \| f - f_{\varepsilon} \|_{\infty} = 0 \) we obtain that

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ |f_{\varepsilon}(X_{T,\varepsilon}) - f(X_T)| \right] \leq \lim_{\varepsilon \to 0} \mathbb{E} \left[ |f(X_{T,\varepsilon}) - f(X_T)| \right] + \lim_{\varepsilon \to 0} \| f - f_{\varepsilon} \|_{\infty} = 0,
\]
which concludes the proof.

\[ \square \]

\textbf{Lemma 34.} \textit{Let} \( x, y \geq 1 \). \textit{Let} \( \alpha \in (0,1/2) \). \textit{If} \( y < x \) \textit{then} \( x^\alpha - y^\alpha \leq x^{1-\alpha} - y^{1-\alpha} \).

\textit{Proof.} \textit{Let} \( \lambda \in (0,1) \) \textit{such that} \( y = \lambda x \). \textit{Then} \( x^\alpha - y^\alpha = x^\alpha (1 - \lambda^\alpha) \leq x^{1-\alpha} (1 - \lambda^{1-\alpha}) = x^{1-\alpha} - y^{1-\alpha} \) \textit{because} \( x > 1, \lambda < 1 \) \textit{and} \( \alpha \leq 1 - \alpha \). \[ \square \]

The following property is a well-known property of functions with Lipschitz gradient but is recalled here for completeness.

\textbf{Lemma 35.} \textit{Assume A1. Then} \textit{for any} \( x \in \mathbb{R}^d \), \( \| \nabla f(x) \|^2 \leq 2L (f(x) - f^*) \).

\textit{Proof.} \textit{Using A1 and that} \( f^* = \min_{\mathbb{R}^d} f \), \textit{we have for any} \( x \in \mathbb{R}^d \)

\[
f^* - f(x) \leq f(x - \nabla f(x)/L) - f(x) \leq - \| \nabla f(x) \|^2 / L + \| \nabla f(x) \|^2 / (2L) \leq - \| \nabla f(x) \|^2 / (2L),
\]
which concludes the proof. \[ \square \]

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C.2 Proof of Theorem 6

In this section we prove Theorem 6. We begin by a lemma to bound $\mathbb{E} \left[ \|X_t - x^*\|^2 \right]$.

Lemma 36. Assume F2. Let $(X_t)_{t \geq 0}$ be given by (2). Then, for any $\alpha, \gamma \in (0, 1)$, there exists $C_{1, \alpha}^{(c)} \geq 0$ and $C_{2, \alpha}^{(c)} \geq 0$ and a function $\Phi_{\alpha}^{(c)} : \mathbb{R}_{+} \to \mathbb{R}_{+}$ such that, for any $t \geq 0$,

$$\mathbb{E} \left[ \|X_t - x^*\|^2 \right] \leq C_{1, \alpha}^{(c)} \Phi_{\alpha}^{(c)}(t + \gamma \alpha) + C_{2, \alpha}^{(c)} .$$

And we have

$$\Phi_{\alpha}^{(c)}(t) = \begin{cases} t^{1-2\alpha} & \text{if } \alpha < 1/2 , \\ \log(t) & \text{if } \alpha = 1/2 , \\ 0 & \text{if } \alpha > 1/2 . \end{cases}$$

The values of the constants are given by

$$C_{1, \alpha}^{(c)} = \begin{cases} \gamma \alpha (1 - 2\alpha)^{-1} & \text{if } \alpha < 1/2 , \\ \gamma \alpha & \text{if } \alpha = 1/2 , \\ 0 & \text{if } \alpha > 1/2 . \end{cases}$$

$$C_{2, \alpha}^{(c)} = \begin{cases} \|X_0 - x^*\|^2 & \text{if } \alpha < 1/2 , \\ \|X_0 - x^*\|^2 - \gamma \alpha \eta \log(\gamma \alpha) & \text{if } \alpha = 1/2 , \\ \|X_0 - x^*\|^2 + (2\alpha - 1)^{-1} \gamma^2 \cdot 2 - 2\alpha \eta & \text{if } \alpha > 1/2 . \end{cases}$$

Proof. Let $\alpha, \gamma \in (0, 1)$ and $t \geq 0$. Let $(X_t)_{t \geq 0}$ be given by (2). We consider the function $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_{+}$ defined as follows

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^d, F(t, x) = \|x - x^*\|^2 .$$

Applying Lemma 32 to the stochastic process $(F(t, X_t))_{t \geq 0}$ and using F2 and A2 gives that for all $t \geq 0$,

$$\mathbb{E} \left[ \|X_t - x^*\|^2 \right] - \mathbb{E} \left[ \|X_0 - x^*\|^2 \right] = -2 \int_0^T (t + \gamma \alpha)^{-\alpha} \langle X_t - x^*, \nabla f(X_t) \rangle dt$$

$$+ \int_0^T \gamma \alpha (t + \gamma \alpha)^{-2\alpha} \text{Tr}(\Sigma(X_t)) dt$$

$$\leq \gamma \alpha \eta \int_0^T (t + \gamma \alpha)^{-2\alpha} dt .$$

We now distinguish three cases:

(a) **Case where** $\alpha < 1/2$: In that case we have:

$$\mathbb{E} \left[ \|X_t - x^*\|^2 \right] \leq \|X_0 - x^*\|^2 + \gamma \alpha \eta (1 - 2\alpha)^{-1} ((T + \gamma \alpha)^{1-2\alpha} - \gamma^1 - 2\alpha)$$

$$\leq \|X_0 - x^*\|^2 + \gamma \alpha \eta (1 - 2\alpha)^{-1} (T + \gamma \alpha)^{1-2\alpha} .$$

(b) **Case where** $\alpha = 1/2$: In that case we obtain:

$$\mathbb{E} \left[ \|X_t - x^*\|^2 \right] \leq \|X_0 - x^*\|^2 + \gamma \alpha \eta (\log(T + \gamma \alpha) - \log(\gamma \alpha))$$

$$\leq \gamma \alpha \eta \log(T + \gamma \alpha) + \|X_0 - x^*\|^2 - \gamma \alpha \eta \log(\gamma \alpha) .$$

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(c) Case where $\alpha > 1/2$: In that case we have:

$$
\mathbb{E} \left[ \|X_t - x^*\|^2 \right] \leq \|X_0 - x^*\|^2 + \gamma_\alpha \eta(1 - 2\alpha)^{-1}((T + \gamma_\alpha)^{1-2\alpha} - \gamma_\alpha^{1-2\alpha}) \\
\leq \|X_0 - x^*\|^2 + (2\alpha - 1)^{-1}\gamma_\alpha^{2-2\alpha}\eta.
$$

\[\square\]

We now turn to the proof of Theorem 6.

**Proof.** Let $f \in C^2(\mathbb{R}^d, \mathbb{R})$. Let $\gamma \in (0, 1)$ and $\alpha \in (0, 1/2]$ and $T \geq 1$. Let $(X_t)_{t \geq 0}$ be given by (2).

Let $S: [0, T] \to [0, +\infty)$ defined by

$$
S(t) = \begin{cases} 
\int_{t-T}^T \mathbb{E} [f(X_s)] - f^* \, ds, & \text{if } t > 0, \\
\mathbb{E} [f(X_T)], & \text{if } t = 0.
\end{cases}
$$

With this notation we have

$$
\mathbb{E} [f(X_T)] - f^* = S(0) - S(1) + S(1) - S(T) + S(T) - f^*.
$$

We preface the rest of the proof with the following computation. For any $x_0 \in \mathbb{R}^d$ we define the function $F_{x_0} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ by

$$
F_{x_0}(t, x) = (t + \gamma_\alpha)^\alpha \|x - x_0\|^2.
$$

In the following we will choose either $x_0 = x^*$ or $x_0 = X_s$ for $s \in [0, T]$. Using Lemma 36, that $\Phi$ is non-decreasing and that for any $a, b \geq 0$, $(a + b)^2 \leq a^2 + b^2$, we have

$$
\mathbb{E} \left[ \|X_t - x_0\|^2 \right] = \mathbb{E} \left[ \|(X_t - x^*) + (x^* - x_0)\|^2 \right] \\
\leq 2\mathbb{E} \left[ \|X_t - x^*\|^2 \right] + 2\mathbb{E} \left[ \|x_0 - x^*\|^2 \right] \\
\leq 2c_{1,\alpha}(c)(t + \gamma_\alpha) + 4c_{2,\alpha}(c) + 2c_{1,\alpha}(c)(T + \gamma_\alpha) \\
\leq 2c_{1,\alpha}(c)(t + \gamma_\alpha) + 2c_{2,\alpha}(c)(T + \gamma_\alpha) + c_{3,\alpha}.
$$

with $c_{3,\alpha} = 4c_{2,\alpha}$. This gives in particular, for every $t \in [0, T]$,

$$
(t + \gamma_\alpha)^\alpha \mathbb{E} \left[ \|X_t - x_0\|^2 \right] \leq \left[ c_{3,\alpha} + 2c_{2,\alpha}(T + \gamma_\alpha)^{1-2\alpha} \log(T + \gamma_\alpha) \right] (t + \gamma_\alpha)^{\alpha-1} \tag{47}
$$

$$
+ 2c_{1,\alpha}(c) \log(T + \gamma_\alpha)(t + \gamma_\alpha)^{-\alpha},
$$

with $c_{1,\alpha} = 0$ if $\alpha > 1/2$. Notice that the additional $\log(T + \gamma_\alpha)$ term is only needed in the case where $\alpha = 1/2$. For any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have

$$
\partial_t F_{x_0}(t, x) = \alpha(t + \gamma_\alpha)^{\alpha-1} \|x - x_0\|^2, \\
\partial_t F_{x_0}(t, x) = 2(t + \gamma_\alpha)^\alpha (x - x_0), \\
\partial_{xx} F_{x_0}(t, x) = 2(t + \gamma_\alpha)^\alpha.
$$

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Using Lemma 32 on the stochastic process \((F_{x_0}(t, X_t))_{t \geq 0}\), we have that for any \(u \in [0, T] \)
\[
\mathbb{E}[F_{x_0}(T, X_T)] - \mathbb{E}[F_{x_0}(T - u, X_{T-u})] = \int_{T-u}^{T} \alpha(t + \gamma_\alpha)^{\alpha-1} \mathbb{E}[\|X_t - x_0\|^2] \, dt
- 2\int_{T-u}^{T} \mathbb{E}[\langle X_t - x_0, \nabla f(X_t) \rangle] \, dt
+ \int_{T-u}^{T} \gamma_\alpha(t + \gamma_\alpha)^{-\alpha} \mathbb{E}[\text{Tr}(\Sigma(X_t))] \, dt . \tag{48}
\]
Combining this result, F2, A2, (47) and (48) we obtain for any \(u \in [0, T] \)
\[
- (T - u + \gamma_\alpha)^{\alpha} \mathbb{E}[\|X_{T-u} - x_0\|^2]
\leq C_{3, \alpha} \int_{T-u}^{T} \alpha(t + \gamma_\alpha)^{\alpha-1} \, dt + \eta \gamma_\alpha \int_{T-u}^{T} (t + \gamma_\alpha)^{-\alpha} \, dt
+ 2\alpha C_{1, \alpha} \log(T + \gamma_\alpha) \left\{ \int_{T-u}^{T} (t + \gamma_\alpha)^{-\alpha} \, dt + (T + \gamma_\alpha)^{1-2\alpha} \int_{T-u}^{T} (t + \gamma_\alpha)^{\alpha-1} \, dt \right\}
- 2\int_{T-u}^{T} \mathbb{E}[f(X_t) - f(x_0)] \, dt
\leq C_{3, \alpha} ((T + \gamma_\alpha)^{\alpha} - (T - u + \gamma_\alpha)^{\alpha}) - 2\int_{T-u}^{T} \mathbb{E}[f(X_t) - f(x_0)] \, dt
+ (\gamma_\alpha \eta + 2\alpha C_{1, \alpha}^{(e)})(1 - \alpha)^{-1} \left( (T + \gamma_\alpha)^{1-\alpha} - (T - u + \gamma_\alpha)^{1-\alpha} \right) \log(T + \gamma_\alpha)
+ 2C_{1, \alpha}^{(e)} \log(T + \gamma_\alpha) \{ (T + \gamma_\alpha)^{\alpha} - (T - u + \gamma_\alpha)^{\alpha} \} (T + \gamma_\alpha)^{1-2\alpha} .
\]
Therefore, we get for any \(u \in [0, T] \)
\[
\int_{T-u}^{T} \mathbb{E}[f(X_t) - f(x_0)] \, dt \leq (C_1/2) ((T + \gamma_\alpha)^{\alpha} - (T - u + \gamma_\alpha)^{\alpha})
+ (1/2)(T - u + \gamma_\alpha)^{\alpha} \mathbb{E}[\|X_{T-u} - x_0\|^2]
+ (C_1/2) \left( (T + \gamma_\alpha)^{1-\alpha} - (T - u + \gamma_\alpha)^{1-\alpha} \right) \log(T + \gamma_\alpha) , \tag{49}
\]
with \(C_1 = \max(C_{3, \alpha}^{(e)}, (\gamma_\alpha \eta + 4\alpha C_{1, \alpha}^{(e)})(1 - \alpha)^{-1})\). We divide the rest of the proof into three parts, to bound the quantities \(S(1) - S(T)\), \(S(T) - f^*\) and \(S(0) - S(1)\).

(a) **Bounding \(S(1) - S(T)\):**
In the case where \(\alpha \leq 1/2\), Lemma 34 gives that for all \(u \in [0, T]\):
\[
((T + \gamma_\alpha)^{\alpha} - (T - u + \gamma_\alpha)^{\alpha}) \leq \left( (T + \gamma_\alpha)^{1-\alpha} - (T - u + \gamma_\alpha)^{1-\alpha} \right) ,
\]
and we also have, for all \(u \in [0, T]\):
\[
(T + \gamma_\alpha)^{1-\alpha} - (T + \gamma_\alpha - u)^{1-\alpha}
= \left( ((T + \gamma_\alpha)^{1-\alpha} - (T + \gamma_\alpha - u)^{1-\alpha})(T + \gamma_\alpha)^{\alpha} + (T + \gamma_\alpha - u)^{\alpha}) \right)/(((T + \gamma_\alpha)^{\alpha} + (T + \gamma_\alpha - u)^{\alpha})
\leq ((T + \gamma_\alpha) - (T + \gamma_\alpha - u) + (T + \gamma_\alpha)^{1-\alpha}(T + \gamma_\alpha - u)^{\alpha} - (T + \gamma_\alpha)^{\alpha}(T + \gamma_\alpha - u)^{1-\alpha})/(T + \gamma_\alpha)^{\alpha} \tag{50}
\]
\[ \leq 2u/(T + \gamma_\alpha)^\alpha. \]

And in the case where \( \alpha > 1/2 \), for all \( u \in [0, T] \):
\[
\left((T + \gamma_\alpha)^{1-\alpha} - (T - u + \gamma_\alpha)^{1-\alpha}\right) \leq \left((T + \gamma_\alpha)^{\alpha} - (T - u + \gamma_\alpha)^{\alpha}\right),
\]
and we also have, for all \( u \in [0, T] \):
\[
(T + \gamma_\alpha)^{\alpha} - (T + \gamma_\alpha - u)^{\alpha}
= \left(\left((T + \gamma_\alpha)^{\alpha} - (T + \gamma_\alpha - u)^{\alpha}\right)\left((T + \gamma_\alpha)^{1-\alpha} + (T + \gamma_\alpha - u)^{1-\alpha}\right)\right) / \left((T + \gamma_\alpha)^{1-\alpha} + (T + \gamma_\alpha - u)^{1-\alpha}\right)
\leq \left((T + \gamma_\alpha) - (T + \gamma_\alpha - u) + (T + \gamma_\alpha)^{\alpha}(T + \gamma_\alpha - u)^{1-\alpha} - (T + \gamma_\alpha)^{1-\alpha}(T + \gamma_\alpha - u)^{1-\alpha}\right) / (T + \gamma_\alpha)^{1-\alpha}
\leq 2u/(T + \gamma_\alpha)^{1-\alpha}.
\]
Now, plugging \( x_0 = X_{T-u} \) in (49) we obtain, for all \( u \in [0, T] \):
\[
\mathbb{E}\left[\int_{T-u}^T f(X_t) - f(X_{T-u})dt\right] \leq 2C_1 \log(T + \gamma_\alpha)(T + \gamma_\alpha)^{-\min(\alpha,1-\alpha)}u. \tag{51}
\]
Since \( S \) is a differentiable function and using (51), we have for all \( u \in (0, T) \),
\[
S'(u) = -u^{-2} \int_{T-u}^T \mathbb{E}\left[f(X_t)\right] dt + u^{-1} \mathbb{E}\left[f(X_{T-u})\right] = -u^{-1}(S(u) - \mathbb{E}[f(X_{T-u})]). \tag{52}
\]
This last result implies \(-S'(u) \leq 2C_1 \log(T + \gamma_\alpha)/(T + \gamma_\alpha)^{-\min(\alpha,1-\alpha)}u^{-1} \) and integrating we get
\[
S(1) - S(T) \leq 2C_1 \log(T + \gamma_\alpha) \log(T)(T + \gamma_\alpha)^{-\min(\alpha,1-\alpha)}.
\]
(b) **Bounding \( S(T) - f^* \):**

Using (48), with \( u = T \) and \( x_0 = x^* \), and \( \|X_0 - x^*\| \leq C_1 \) we obtain
\[
\int_0^T \mathbb{E}\left[f(X_s)\right] ds - T f^* \leq (C_1/2) \left((T + \gamma_\alpha)^{\alpha} - \gamma_\alpha + \left((T + \gamma_\alpha)^{1-\alpha} - \gamma\right) \log(T + \gamma_\alpha)\right)
+ (1/2) \gamma_\alpha \mathbb{E}\left[\|X_0 - x^*\|^2\right]. \tag{53}
\]
Using this result we have
\[
S(T) - f^* \leq T^{-1} C_1 (T + \gamma_\alpha)^{\max(1-\alpha,\alpha)} \log(T + \gamma_\alpha) + C_1 \gamma_\alpha T^{-1}/2 \leq 2C_1 T^{-\min(\alpha,1-\alpha)} \log(T + \gamma_\alpha).
\]
(c) **Bounding \( S(0) - S(1) \):**

We have
\[
S(0) - S(1) = \mathbb{E}[f(X_T)] - S(1) = \int_{T-1}^T (\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_s)]) ds.
\]
Using Lemma 32 on the stochastic process \( f(X_t) \) and \( A1 \), we have for all \( s \in [T-1,T] \)
\[
\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_s)] = -\int_s^T (\gamma_\alpha + t)^{-\alpha} \|\nabla f(X_t)\|^2 dt + (L/2) \gamma_\alpha \int_s^T (t + \gamma_\alpha)^{-2\alpha} \text{Tr}(\Sigma(X_t)) dt
\]
\[45\]
\[
\leq (\eta L/2)\gamma_0 \int_s^T (t + \gamma_\alpha)^{-2\alpha} dt \\
\leq (C_1 L/2)(s + \gamma_\alpha)^{-2\alpha}(T - s).
\]

Plugging this result into (54) yields
\[
S(0) - S(1) \leq (C_1 L/2) \int_{T-1}^T (T - s)(s + \gamma_\alpha)^{-2\alpha} ds \leq C_1 L(T - 1 + \gamma_\alpha)^{-2\alpha} \leq C_1 L(T - 1)^{-2\alpha}. \tag{55}
\]

Combining (52), (53) and (55) gives the desired result
\[
\mathbb{E}[f(X_T)] - f^* \leq C^{(c)} \left[ \log(T)^2 T^{-\min(\alpha,1-\alpha)} + \log(T)T^{-\min(\alpha,1-\alpha)} + T^{-\min(\alpha,1-\alpha)} + (T - 1)^{-2\alpha} \right],
\]
with \( C^{(c)} = 4C_1(1 + L). \)

We now give the extension of Theorem 6 to the case where the function \( f \) is only continuously differentiable and such that \( \arg \min_{\mathbb{R}^d} f \) is bounded, see Theorem 7.

**Proof.** Let \( \alpha, \gamma \in (0,1] \) and \( T \geq 0 \). \((f_\varepsilon)_{\varepsilon \geq 0}\) be given by Lemma 33. Let \( \delta = \min(\alpha, 1 - \alpha) \). We can apply, Theorem 6 to \( f_\varepsilon \) for each \( \varepsilon > 0 \). Therefore there exists \( C^{(c)}_\varepsilon \) such that
\[
\mathbb{E}[f(X_{T,\varepsilon})] - f(x^*_\varepsilon) \leq C^{(c)}_\varepsilon \left[ \log(T)^2 T^{-\delta} + \log(T)T^{-\delta} + T^{-\delta} + (T - 1)^{-2\alpha} \right], \tag{56}
\]
where \((X_{t,\varepsilon})_{t \geq 0}\) is given by (2) with \( X_t = x_0 \) (upon replacing \( f \) by \( f_\varepsilon \)) and
\[
C^{(c)}_\varepsilon = 4 \max(2C^{(c)}_{2,\alpha} + 2 \|x_0 - x^*_\varepsilon\|^2, (\gamma_\alpha \eta + 2\alpha C^{(c)}_{1,\alpha})(1 - \alpha)^{-1}).
\]
Using (56) and Lemma 33 we have
\[
\mathbb{E}[f(X_T)] - f^* \leq \liminf_{\varepsilon \to 0} \mathbb{E}[f_\varepsilon(X_{t,\varepsilon})] - \limsup_{\varepsilon \to 0} f_\varepsilon(x^*_\varepsilon)
\leq \liminf_{\varepsilon \to 0} \{ \mathbb{E}[f_\varepsilon(X_{t,\varepsilon})] - f_\varepsilon(x^*_\varepsilon) \}
\leq \liminf_{\varepsilon \to 0} C^{(c)}_\varepsilon \left[ \log(T)^2 T^{-\delta} + \log(T)T^{-\delta} + T^{-\delta} + (T - 1)^{-2\alpha} \right]
\leq C^{(c)}_1 \left[ \log(T)^2 T^{-\delta} + \log(T)T^{-\delta} + T^{-\delta} + (T - 1)^{-2\alpha} \right],
\]
with \( C^{(c)}_1 = 3 \max(2C^{(c)}_{2,\alpha} + 4 \|x_0\|^2 + 4C^2, (\gamma_\alpha \eta + 2C^{(c)}_{1,\alpha})(1 - \alpha)^{-1}) \), where \( C = \max_{y \in \arg \min_{\mathbb{R}^d} f} \|y\|. \)

**C.3 Proof of Theorem 8**

In this section we prove Theorem 8. The proof is clearly more involved than the one of Theorem 6. We will follow a similar way as in the proof of Theorem 6, with more technicalities. One of the main argument of the proof is the suffix averaging technique that was introduced in (Shamir and Zhang, 2013).

We begin by the discrete counterpart of Lemma 36.
Lemma 37. Assume A1, F2, A2. Then for any $\alpha, \gamma \in (0, 1)$, there exists $c_{1,\alpha}^{(d)} \geq 0$, $c_{2,\alpha}^{(d)} \geq 0$ and a function $\Phi_{\alpha}^{(d)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for any $n \geq 0$,

$$\mathbb{E} \left[ \|X_n - x^*\|^2 \right] \leq c_{1,\alpha}^{(d)} \Phi_{\alpha}^{(d)} (n + 1) + c_{2,\alpha}^{(d)}.$$

And we have

$$\Phi_{\alpha}^{(d)} (t) = \begin{cases} t^{1-2\alpha} & \text{if } \alpha < 1/2, \\ \log(t) & \text{if } \alpha = 1/2, \\ 0 & \text{if } \alpha > 1/2. \end{cases}$$

The values of the constants are given by

$$c_{1,\alpha}^{(d)} = \begin{cases} 2\gamma^2 \eta (1 - 2\alpha)^{-1} & \text{if } \alpha < 1/2, \\ \gamma^2 \eta & \text{if } \alpha = 1/2, \\ 0 & \text{if } \alpha > 1/2, \end{cases}$$

$$c_{2,\alpha}^{(d)} = \begin{cases} 2 \max_{k \leq (\gamma L/2)^{1/\alpha}} \mathbb{E} \left[ \|X_k - x^*\|^2 \right] & \text{if } \alpha < 1/2, \\ 2 \max_{k \leq (\gamma L/2)^{1/\alpha}} \mathbb{E} \left[ \|X_k - x^*\|^2 \right] + 2\gamma^2 \eta & \text{if } \alpha = 1/2, \\ 2 \max_{k \leq (\gamma L/2)^{1/\alpha}} \mathbb{E} \left[ \|X_k - x^*\|^2 \right] + \gamma^2 \eta (2\alpha - 1)^{-1} & \text{if } \alpha > 1/2, \end{cases}$$

Proof. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ verifying assumptions A1 and F2. We consider $(X_n)_{n \geq 0}$ satisfying (1). Let $x^* \in \mathbb{R}^d$ be given by F2. We have, using (1) and A2 that for all $n \geq (\gamma L/2)^{1/\alpha}$,

$$\mathbb{E} \left[ \|X_{n+1} - x^*\|^2 \bigg| F_n \right] = \mathbb{E} \left[ \|X_n - x^* - (n + 1)^{-\alpha} H(X_n, Z_{n+1})\|^2 \bigg| F_n \right]$$

$$= \|X_n - x^*\|^2 - 2\gamma/(n + 1)^\alpha \langle X_n - x^*, \mathbb{E} [H(X_n, Z_{n+1})] \rangle + \gamma^2 (n + 1)^{-2\alpha} \mathbb{E} \left[ \|H(X_n, Z_{n+1})\|^2 \bigg| F_n \right]$$

$$= \|X_n - x^*\|^2 - 2\gamma/(n + 1)^\alpha \langle X_n - x^*, \nabla f(X_n) \rangle + \gamma^2 (n + 1)^{-2\alpha} \mathbb{E} \left[ \|H(X_n, Z_{n+1}) - \nabla f(X_n) + \nabla f(X_n)\|^2 \bigg| F_n \right]$$

$$= \|X_n - x^*\|^2 - 2\gamma/(n + 1)^\alpha \langle X_n - x^*, \nabla f(X_n) \rangle + \gamma^2 (n + 1)^{-2\alpha} \mathbb{E} \left[ \|\nabla f(X_n)\|^2 \bigg| F_n \right]$$

$$+ 2 \mathbb{E} \left[ \langle H(X_n, Z_{n+1}) - \nabla f(X_n), \nabla f(X_n) \rangle \bigg| F_n \right]$$

$$= \|X_n - x^*\|^2 - 2\gamma/(n + 1)^\alpha \langle X_n - x^*, \nabla f(X_n) \rangle + \gamma^2 (n + 1)^{-2\alpha} \mathbb{E} \left[ \|\nabla f(X_n)\|^2 \bigg| F_n \right]$$

$$\leq \|X_n - x^*\|^2 - 2\gamma/\alpha (n + 1)^{-\alpha} \|\nabla f(X_n)\|^2 + \gamma^2 \eta (n + 1)^{-2\alpha}$$

$$\mathbb{E} \left[ \|X_{n+1} - x^*\|^2 \right] \leq \mathbb{E} \left[ \|X_n - x^*\|^2 \right] + \gamma^2 \eta (n + 1)^{-2\alpha},$$

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where we used the co-coercivity of $f$. Summing the previous inequality leads to

$$
E[\|X_n - x^*\|^2] - E[\|X_0 - x^*\|^2] \leq \gamma^2 \eta \sum_{k=1}^{n} k^{-2\alpha}.
$$

As in the previous proof we now distinguish three cases:

(a) **Case where $\alpha < 1/2$:** In that case we have:

$$
E[\|X_n - x^*\|^2] \leq \|X_0 - x^*\|^2 + \gamma^2 \eta (1 - 2\alpha)^{-1} (n + 1)^{1 - 2\alpha} \leq \|X_0 - x^*\|^2 + 2\gamma^2 \eta (1 - 2\alpha)^{-1} n^{1 - 2\alpha}.
$$

(b) **Case where $\alpha = 1/2$:** In that case we obtain:

$$
E[\|X_n - x^*\|^2] \leq \|X_0 - x^*\|^2 + \gamma^2 \eta (\log(n) + 2).
$$

(c) **Case where $\alpha > 1/2$:** In that case we have:

$$
E[\|X_n - x^*\|^2] \leq \|X_0 - x^*\|^2 + \gamma^2 \eta (2\alpha - 1)^{-1}.
$$

We now turn to the proof of Theorem 8 by stating an intermediate result where we assume a condition bounding $E[\|\nabla f(X_n)\|^2]$. This Proposition provides non-optimal convergence rates for SGD but will be used as a central tool to improve them and obtain optimal convergence rates.

**Proposition 38.** Let $\gamma, \alpha \in (0, 1)$ and $x_0 \in \mathbb{R}^d$ and $(X_n)_{n \geq 0}$ be given by (1). Assume A1, F2, A2. Suppose additionally that there exists $\alpha^* \in [0, 1/2]$, $\beta > 0$ and $C_0 \geq 0$ such that for all $n \in \{0, \ldots, N\}$

$$
E[\|\nabla f(X_n)\|^2] \leq \begin{cases} C_0(n + 1)^\beta \log(n + 1) & \text{if } \alpha \leq \alpha^* , \\ C_0 & \text{if } \alpha > \alpha^* . \end{cases}
$$

(58)

Then there exists $\tilde{C}_\alpha \geq 0$ such that, for all $N \geq 1$,

$$
E[f(X_N)] - f^* \leq \tilde{C}_\alpha \left\{ (1 + \log(N + 1))^2/(N + 1)^{\min(\alpha, 1-\alpha)} \Psi_\alpha(N + 1) + 1/(N + 1) \right\},
$$

with

$$
\Psi_\alpha(n) = \begin{cases} n^\beta(1 + \log(n)) & \text{if } \alpha \leq \alpha^* , \\ 1 & \text{if } \alpha > \alpha^* . \end{cases}
$$

**Proof.** Let $\alpha, \gamma \in (0, 1)$ and $N \geq 1$. Let $(X_n)_{n \geq 0}$ be given by (1).

Let $(S_k)_{k \in \{0, \ldots, N\}}$ defined by

$$
\forall k \in \{0, \ldots, N\}, S_k = (k + 1)^{-1} \sum_{t=N-k}^{N} E[f(X_t)].
$$

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With this notation we have $$\mathbb{E} [f(X_N)] - f^* = (S_0 - S_N) + (S_N - f^*)$$. As in the proof of Theorem 6, we preface the proof with the following computation. Let $$\ell \in \{0, \ldots, N\}$$, let $$k \geq \ell$$, let $$x_0 \in \mathcal{F}_{\ell}$$. Using F2 we have

$$\mathbb{E} \left[ \|X_{k+1} - x_0\|^2 | \mathcal{F}_k \right] = \mathbb{E} \left[ \|X_k - x_0 - \gamma (k+1)^{-\alpha} H(X_k, Z_{k+1})\|^2 | \mathcal{F}_k \right]$$

$$= \|X_k - x_0\|^2 + \gamma^2 (k+1)^{-2\alpha} \mathbb{E} \left[ \|H(X_k, Z_{k+1})\|^2 | \mathcal{F}_k \right]$$

$$- 2\gamma (k+1)^{-\alpha} \langle X_k - x_0, \nabla f(X_k) \rangle$$

$$= \mathbb{E} [f(X_k) - f(x_0)] \leq (2\gamma)^{-1} (k+1)^{\alpha} \left( \mathbb{E} \left[ \|X_k - x_0\|^2 \right] - \mathbb{E} \left[ \|X_{k+1} - x_0\|^2 \right] \right)$$

$$+ (\gamma / 2) (k+1)^{-\alpha} \mathbb{E} \left[ \|H(X_k, Z_{k+1})\|^2 | \mathcal{F}_k \right]$$

Let $$u \in \{0, \ldots, N\}$$. Summing now (59) between $$k = N - u$$ and $$k = N$$ gives

$$\mathbb{E} \left[ \sum_{k=N-u}^{N} f(X_k) - f(x_0) \right] \leq (\gamma \eta / 2) \sum_{k=N-u}^{N} (k+1)^{-\alpha}$$

$$+ (2\gamma)^{-1} \sum_{k=N-u+1}^{N} \mathbb{E} \left[ \|X_k - x_0\|^2 \right] ((k+1)^{\alpha} - k^{\alpha})$$

$$+ (\gamma / 2) \sum_{k=N-u}^{N} \mathbb{E} \left[ \|\nabla f(X_k)\|^2 \right] (k+1)^{-\alpha}$$

$$+ (2\gamma)^{-1} (N - u + 1)^{\alpha} \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right].$$

In the following we will take for $$x_0$$ either $$x^*$$ or $$X_m$$ for $$m \in [0, N]$$. We now have to run separate analyses depending on the value of $$\alpha$$.

(a) **Case** $$\alpha \leq \alpha^*$$:

In that case (58) gives that

$$\mathbb{E} \left[ \|\nabla f(X_k)\|^2 \right] \leq C_0 (N + 1)^\beta \log(N + 1),$$

and Lemma 37 gives that for all $$k \in \{0, \ldots, N\},$$

$$\mathbb{E} \left[ \|X_k - x_0\|^2 \right] \leq 2 \mathbb{E} \left[ \|X_k - x^*\|^2 \right] + 2 \mathbb{E} \left[ \|x_0 - x^*\|^2 \right]$$

$$\leq 2C_{1,\alpha}(d)(k+1)^{1-2\alpha} \log(k+1) + 2C_{1,\alpha}(d)(N+1)^{1-2\alpha} \log(N+1) + 4C_{2,\alpha}(d)$$

$$\leq 4C_{1,\alpha}(d)(N+1)^{1-2\alpha} \log(N+1) + 4C_{2,\alpha}(d).$$

We note $$C_{3,\alpha}(d) = 4C_{2,\alpha}(d)$$. Equation (60) leads therefore to, with $$C(b) = ((\gamma \eta / 2) + (\gamma / 2)C_0)(1 - \alpha)^{-1}$$.

$$\mathbb{E} \left[ \sum_{k=N-u}^{N} f(X_k) - f(x_0) \right] \leq (\gamma \eta / 2) (1 - \alpha)^{-1} \left( (N + 1)^{1-\alpha} - (N - u)^{1-\alpha} \right)$$
+ (2\gamma)^{-1}(N - u + 1)^\alpha \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right] \\
+ (2\gamma)^{-1} \left( C_{3,\alpha}^{(d)} + 4C_{1,\alpha}^{(d)} (N + 1)^{1-2\alpha} \log(N + 1) \right) ((N + 1)^\alpha - (N - u + 1)^\alpha) \\
+ (\gamma/2) C_0 (N + 1)^{1-\alpha} \log(N + 1)(1 - \alpha)^{-1} ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) \\
\leq C^{(b)} (N + 1)^{\beta} (1 + \log(N + 1))^2 \left( (N + 1)^{1-\alpha} - (N - u)^{1-\alpha} \right) \\
+ (2\gamma)^{-1}(N - u + 1)^\alpha \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right] \\
+ (2\gamma)^{-1}C_{3,\alpha}^{(d)} ((N + 1)^\alpha - (N - u)^\alpha) \\
+ (2\gamma)^{-1}4C_{1,\alpha}^{(d)} ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) \\
\leq C^{(d)} (N + 1)^{\beta} (1 + \log(N + 1))^2 \left( (N + 1)^{1-\alpha} - (N - u)^{1-\alpha} \right) \\
+ (2\gamma)^{-1}(N - u + 1)^\alpha \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right] ,

where we used Lemma 34 and where we noted \( C^{(d)} = C^{(b)} + (2\gamma)^{-1}(C_{3,\alpha}^{(d)} + 4C_{1,\alpha}^{(d)}) \).

Notice now that, similarly to Equation (50) we have

\[
(N + 1)^{1-\alpha} - (N - u)^{1-\alpha} \\
= \left\{ ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) ((N + 1)^\alpha + (N - u)^\alpha) \right\} ((N + 1)^\alpha + (N - u)^\alpha)^{-1} \\
\leq 2(u + 1)/(N + 1)^\alpha .
\]

(b) \textbf{Case } \alpha \in (\alpha^*, 1/2):

In that Lemma 37 gives that for all \( k \in \{0, \ldots, N\} \),

\[
\mathbb{E} \left[ \|X_k - x_0\|^2 \right] \leq 2\mathbb{E} \left[ \|X_k - x^*\|^2 \right] + 2\mathbb{E} \left[ \|x_0 - x^*\|^2 \right] \\
\leq 2C_{1,\alpha}^{(d)} (k + 1)^{1-2\alpha} \log(k + 1) + 2C_{1,\alpha}^{(d)} (N + 1)^{1-2\alpha} \log(N + 1) + 4C_{2,\alpha}^{(d)} \\
\leq 4C_{1,\alpha}^{(d)} (N + 1)^{1-2\alpha} \log(N + 1) + 4C_{2,\alpha}^{(d)} .
\]

Using (58), Equation (60) rewrites

\[
\mathbb{E} \left[ \sum_{k=N-u}^{N} f(X_k) - f(x_0) \right] \leq (\gamma\eta/2)(1 - \alpha)^{-1} ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) \\
+ (2\gamma)^{-1}(N - u + 1)^\alpha \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right] \\
+ (2\gamma)^{-1} \left( C_{3,\alpha}^{(d)} + 4C_{1,\alpha}^{(d)} \log(N + 1)(N + 1)^{1-2\alpha} \right) ((N + 1)^\alpha - (N - u + 1)^\alpha) \\
+ (\gamma/2) C_0 (1 - \alpha)^{-1} ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) \\
\leq C^{(b)} ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) + (2\gamma)^{-1}(N - u + 1)^\alpha \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right] \\
+ (2\gamma)^{-1} \left( C_{3,\alpha}^{(d)} + 4C_{1,\alpha}^{(d)} \right) (1 + \log(N + 1)) ((N + 1)^\alpha - (N - u)^\alpha) \\
\leq C^{(d)} (1 + \log(N + 1)) ((N + 1)^{1-\alpha} - (N - u)^{1-\alpha}) \\
+ (2\gamma)^{-1}(N - u + 1)^\alpha \mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right] .
\]
(c) **Case $\alpha > 1/2$:**

In that case, $\alpha > \alpha^*$ and Lemma 37 gives

$$\forall k \in \{0, \ldots, N\}, \mathbb{E} \left[ \|X_k - x_0\|^2 \right] \leq 2\mathbb{E} \left[ \|X_k - x^*\|^2 \right] + 2\mathbb{E} \left[ \|x_0 - x^*\|^2 \right] \leq 4c_{2,\alpha}^{(d)} = c_{3,\alpha}^{(d)}.$$  

Using Lemma 34 and (58) we rewrite Equation (60) as

$$\mathbb{E} \left[ \sum_{k=N-u}^{N} f(X_k) - f(x_0) \right] \leq ((\gamma_\eta/2) + \gamma c_{0}/2)(1 - \alpha)^{-1} \left( (N + 1)^{1-\alpha} - (N - u)^{1-\alpha} \right)$$

$$+ (2\gamma)^{-1}(N - u + 1)^{\alpha}\mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right]$$

$$+ (2\gamma)^{-1}c_{3,\alpha}^{(d)} ((N + 1)^{\alpha} - (N - u + 1)^{\alpha})$$

$$\leq c^{(b)} \left( (N + 1)^{1-\alpha} - (N - u)^{1-\alpha} \right) + (2\gamma)^{-1}(N - u + 1)^{\alpha}\mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right]$$

$$+ (2\gamma)^{-1}c_{3,\alpha}^{(d)} ((N + 1)^{\alpha} - (N - u)^{\alpha})$$

$$\leq c^{(d)} ((N + 1)^{\alpha} - (N - u)^{\alpha}) + (2\gamma)^{-1}(N - u + 1)^{\alpha}\mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right].$$

Notice now that, similarly to Equation (50) we have

$$(N + 1)^{\alpha} - (N - u)^{\alpha}$$

$$= \left\{ ((N + 1)^{\alpha} - (N - u)^{\alpha}) \left( (N + 1)^{1-\alpha} + (N - u)^{1-\alpha} \right) \right\} \left( (N + 1)^{1-\alpha} + (N - u)^{1-\alpha} \right)^{-1}$$

$$\leq 2(u + 1)/(N + 1)^{1-\alpha}.$$  

Finally, putting the three cases above together we obtain

$$\mathbb{E} \left[ \sum_{k=N-u}^{N} f(X_k) - f(X_{N-u}) \right] \leq 2c^{(d)}(u + 1)/(N + 1)^{\min(\alpha,1-\alpha)}(1 + \log(N + 1))\Psi_\alpha(N + 1)(61)$$

$$+ (2\gamma)^{-1}(N - u + 1)^{\alpha}\mathbb{E} \left[ \|X_{N-u} - x_0\|^2 \right],$$

with

$$\Psi_\alpha(n) = \begin{cases} n^{\beta}(1 + \log(n)) & \text{if } \alpha \leq \alpha^*, \\ 1 & \text{if } \alpha > \alpha^*. \end{cases}$$  

Note that the additional $\log(N + 1)$ factor can be removed if $\alpha \neq 1/2$. We bound now the quantities $(S_0 - S_N)$ and $(S_N - f^*)$.

(a) **Bounding $(S_0 - S_N)$:**

Let $u \in \{0, \ldots, N\}$. Equation (61) with the choice $x_0 = X_{N-u}$ gives

$$\mathbb{E} \left[ \sum_{k=N-u}^{N} f(X_k) - f(X_{N-u}) \right] \leq 2c^{(d)}(u + 1)/(N + 1)^{\min(\alpha,1-\alpha)}(1 + \log(N + 1))\Psi_\alpha(N + 1).$$
And then,
\[
S_u = (u + 1)^{-1} \sum_{k=N-u}^{N} \mathbb{E}[f(X_k)]
\leq 2c(d)(N + 1)^{-\min(\alpha, 1 - \alpha)}(1 + \log(N + 1))\Psi_{\alpha}(N + 1) + \mathbb{E}[f(X_{N-u})].
\] (62)

We have now, using (62),
\[
u S_{u-1} = (u + 1)S_u - \mathbb{E}[f(X_{N-u})]
= uS_u + S_u - \mathbb{E}[f(X_{N-u})]
\leq uS_u + 2c(d)(N + 1)^{-\min(\alpha, 1 - \alpha)}(1 + \log(N + 1))\Psi_{\alpha}(N + 1)
\]
\[
S_u - S_u \leq 2c(d)u^{-1}(N + 1)^{-\min(\alpha, 1 - \alpha)} \log(N + 1)
\]
\[
S_0 - S_N \leq 2c(d)(N + 1)^{-\min(\alpha, 1 - \alpha)}(1 + \log(N + 1))\Psi_{\alpha}(N + 1)\sum_{u=1}^{N}(1/u)
\]
\[
S_0 - S_N \leq 2c(d)(N + 1)^{-\min(\alpha, 1 - \alpha)}(1 + \log(N + 1))^2\Psi_{\alpha}(N + 1).
\]

(b) **Bounding** \((S_N - f^*)\):

Equation (61) with the choice \(x_0 = x^*\) and \(u = N\) gives
\[
(N + 1)^{-1}E\left[\sum_{k=0}^{N} f(X_k) - f(x^*)\right] \leq 2c(d)(1 + \log(N + 1))(N + 1)^{-\min(\alpha, 1 - \alpha)}\Psi_{\alpha}(N + 1)
\] (64)
\[+ (2\gamma)^{-1}(N + 1)^{-1}\|X_0 - x^*\|^2\]
\[S_N - f^* \leq 2c(d)(1 + \log(N + 1))^2(N + 1)^{-\min(\alpha, 1 - \alpha)}\Psi_{\alpha}(N + 1)
\] \[+ (2\gamma)^{-1}(N + 1)^{-1}\|X_0 - x^*\|^2.\]

And finally, putting Equations (63) and (64) together gives, for \(\tilde{c}_{\alpha} = 2\max((2\gamma)^{-1}\|X_0 - x^*\|^2, 2c(d)),\)
\[
\mathbb{E}[f(X_N)] - f^* \leq \tilde{c}_{\alpha}\left\{(1 + \log(N + 1))^2/(N + 1)^{\min(\alpha, 1 - \alpha)}\Psi_{\alpha}(N + 1) + 1/(N + 1)\right\}.
\]

\[\square\]

We can finally conclude the proof of Theorem 8.

**Proof.** We begin by proving by induction over \(m \in \mathbb{N}^*\) the following statement \(\mathcal{H}_m:\)

For any \(\alpha > 1/(m+1),\) there exists \(c_{\alpha}^+ > 0\) such that for all \(n \in \{0, \ldots, N\},\) 
\[
\mathbb{E}\left[\|\nabla f(X_n)\|^2\right] \leq c_{\alpha}^+,
\]
and for any \(\alpha \leq 1/(m+1),\) there exists \(c_{\alpha}^- > 0\) such that for all \(n \in \{0, \ldots, N\},\) 
\[
\mathbb{E}\left[\|\nabla f(X_n)\|^2\right] \leq c_{\alpha}^-n^{1-(m+1)\alpha}(1 + \log(n))^3.
\]
For \( m = 1 \), \( \mathcal{H}_1 \) is an immediate consequence of \( \textbf{A1} \) and Lemma 37, with \( C_{\alpha}^+ = L^2C_{2,\alpha}^{(d)} \) and \( C_{\alpha}^- = L^2 \max(C_{1,\alpha}^{(d)}, C_{2,\alpha}^{(d)}) \).

Now, let \( m \in \mathbb{N}^* \) and suppose that \( \mathcal{H}_m \) holds. Let \( \alpha \in (0, 1) \). Setting \( \alpha^* = 1/m + 1 \) we see that (58) is verified with \( \beta = 1 - (m + 1)\alpha \).

Consequently, using \( \textbf{A1}, \textbf{F2}, \textbf{A2} \) we can apply Proposition 38 which shows that, for \( \alpha \leq 1/(m + 1) \):

\[
\mathbb{E}[f(X_N)] - f^* \leq \tilde{C}_\alpha \left( 1 + \log(N + 1) \right)^2/(N + 1)^{\min(\alpha,1-\alpha)} \Psi_\alpha(N + 1) + 1/(N + 1)
\]

(65)
\[
\leq \tilde{C}_\alpha \left( 1 + \log(N + 1) \right)^3(N + 1)\min(\alpha,1-\alpha) + 1/(N + 1)
\]
\[
\leq \tilde{C}_\alpha \left( 1 + \log(N + 1) \right)^1(N + 1)\min(\alpha,1-\alpha) + 1/(N + 1)
\].

In particular, if \( \alpha > 1/(m + 2) \) we have the existence of \( \tilde{C}_\alpha > 0 \) such that for all \( n \in \{0, \cdots, N\} \), \( \mathbb{E}[f(X_n)] - f^* \leq \tilde{C}_\alpha \). And using \( \textbf{A1} \) and Lemma 35 we get that, for all \( n \in \{0, \cdots, N\} \)

\[
\mathbb{E}\left[\left\|\nabla f(X_n)\right\|^2\right] \leq 2\mathbb{E}[f(X_n) - f^*] \leq 2L\tilde{C}_\alpha ,
\]

which proves \( \mathcal{H}_{m+1} \) for \( \alpha > 1/(m + 2) \), with \( C^{(d)}_\alpha = 2L\tilde{C}_\alpha \). And (65) proves \( \mathcal{H}_{m+1} \) for \( \alpha \leq 1/(m + 2) \) with \( C^{(d)}_\alpha = 2\tilde{C}_\alpha \).

Finally this proves that \( \mathcal{H}_m \) is true for any \( n \geq 1 \).

Now, let \( \alpha \in (0, 1) \). Since \( \mathbb{R} \) is archimedean, there exists \( m \in \mathbb{N}^* \) such that \( \alpha > 1/(m + 1) \) and therefore \( \mathcal{H}_m \) shows the existence of \( C_0 > 0 \) such that \( \mathbb{E}\left[\left\|\nabla f(X_n)\right\|^2\right] \leq C_0 \) for all \( n \in \mathbb{N}^* \).

Applying Proposition 38 gives the existence of \( C^{(d)}_\alpha > 0 \) such that for all \( N \geq 1 \)

\[
\mathbb{E}[f(X_N)] - f^* \leq C^{(d)}_\alpha (1 + \log(N + 1))^2/(N + 1)^{\min(\alpha,1-\alpha)} ,
\]

with \( C^{(d)}_\alpha = 2\tilde{C}_\alpha \), concluding the proof.

We present now a corollary of the previous theorem under a different setting. Let us assume, as in (Shamir and Zhang, 2013), that \( \nabla f \) is not Lipschitz-continuous but bounded instead.

**Corollary 39.** Let \( \gamma, \alpha \in (0, 1) \) and \( x_0 \in \mathbb{R}^d \) and \( (X_n)_{n \geq 0} \) be given by (1). Assume \( \textbf{F2}, \textbf{A2} \) and \( \nabla f \) bounded. Then there exists \( C^{(d)}_\gamma \geq 0 \) such that, for all \( N \geq 1 \),

\[
\mathbb{E}[f(X_N)] - f^* \leq C^{(d)}_\gamma (1 + \log(N + 1))^2/(N + 1)^{\min(\alpha,1-\alpha)} .
\]

**Proof.** The proof follows the same lines as the ones of Lemma 37 and Proposition 38. We show that both conclusions hold under the assumption that \( \nabla f \) is bounded instead of being Lipschitz-continuous.

In order to prove that Lemma 37 still holds, let us do the following computation. We consider \( (X_n)_{n \geq 0} \) satisfying (1). We have, using (1), \( \textbf{F2} \) and \( \textbf{A2} \) that for all \( n \geq 0 \),

\[
\mathbb{E}\left[\left\|X_{n+1} - x^*\right\|^2\middle|\mathcal{F}_n\right] = \mathbb{E}\left[\left\|X_n - x^* - \gamma(n + 1)^{-\alpha}H(X_n, Z_{n+1})\right\|^2\middle|\mathcal{F}_n\right]
\]
\[
= \left\|X_n - x^*\right\|^2 - 2\gamma/(n + 1)^{\alpha} \mathbb{E}[H(X_n, Z_{n+1})]\mathbb{E}[X_n - x^*, \mathbb{E}[H(X_n, Z_{n+1})]\mathcal{F}_n]
\]
\[
+ \gamma^2(n + 1)^{-2\alpha} \mathbb{E}\left[\left\|H(X_n, Z_{n+1})\right\|^2\middle|\mathcal{F}_n\right]
\]

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\[ = \|X_n - x^*\|^2 - 2\gamma/(n + 1)^\alpha \langle X_n - x^*, \nabla f(X_n) \rangle + \gamma^2 \eta(n + 1)^{-2\alpha} + \gamma^2(n + 1)^{-2\alpha} \|\nabla f(X_n)\|^2 \]
\[ \mathbb{E}\left[\|X_{n+1} - x^*\|^2\right] \leq \mathbb{E}\left[\|X_n - x^*\|^2\right] + \gamma^2(\eta + \|\nabla f\|_\infty)(n + 1)^{-2\alpha}. \]

And we obtain the same equation as in (57), with a different constant before \((n + 1)^{-2\alpha}\). Hence the conclusions of Lemma 37 still hold, because A1 is never used in the remaining of the proof.

We can now apply safely Proposition 38 (since A1 is only used to use Lemma 37) with \(\alpha^* = 0\). This concludes the proof. \(\square\)

## D Weakly Quasi-Convex Case

In this section we give the proofs of the results presented in Section 3.4.

### D.1 Proof of Theorem 10

Without loss of generality, we assume that \(f^* = 0\). Let \(\alpha, \gamma \in (0, 1)\), \(x_0 \in \mathbb{R}^d\), \(a_t = \gamma \alpha + t\), \(\ell_t = 1 + \log(1 + \gamma^{-\alpha} t)\) for any \(t \geq 0\) and \(\delta = \min(\delta_1, \delta_2)\) with \(\delta_1\) and \(\delta_2\) given in Theorem 10. Using Lemma 32, we have for any \(t \geq 0\)
\[ \mathbb{E}\left[f(X_t)\right] \leq \mathbb{E}\left[\|X_t - x^*\|^{2r_{2r_3}}\right]^{1/2}. \]

Define for any \(t \geq 0\), \(\mathcal{E}(t) = \mathbb{E}[f(X_t)]\). \((t \mapsto \mathcal{E}(t))\) is differentiable and using A1 and A2 we have for any \(t > 0\),
\[ \frac{d\mathcal{E}(t)}{dt} \leq -\ell_t^{-\varepsilon} a_t^{\delta - \alpha} a_t^{2\delta - 2\alpha} \mathbb{E}\left[\|\nabla f(X_t)\|^2\right] + (\gamma \alpha/2)\ell_t^{-\varepsilon} a_t^{-2\alpha} \mathbb{E}\left[\langle \nabla f(X_t), \Sigma(X_t) \rangle\right]. \]

Using, F3 and Hölder’s inequality we have for any \(t \geq 0\)
\[ \tau \mathbb{E}\left[f(X_t)\right] \leq \mathbb{E}\left[\|X_t - x^*\|^{2r_{2r_3}}\right]^{1/2} \mathbb{E}\left[\|\nabla f(X_t)\|^2\right]^{1/2}. \]

Noting that \((r_3r_1)^{-1} = r_1^{-1} - 1/2\), we get for any \(t \geq 0\)
\[ \mathbb{E}\left[\|\nabla f(X_t)\|^2\right] \geq \tau\frac{2r_1}{2r_2} \mathbb{E}\left[f(X_t)\right]^{2r_1} \mathbb{E}\left[\|X_t - x^*\|^{2r_{2r_3}}\right]^{1/2} \]
\[ \geq \tau\frac{2r_1}{2r_2} \mathbb{E}\left[f(X_t)\right]^{2r_1} \]
\[ \geq \tau\frac{2r_1}{2r_2} \mathbb{E}\left[f(X_t)\right]^{2r_1} \]
\[ \geq \tau\frac{2r_1}{2r_2} \mathbb{E}\left[f(X_t)\right]^{2r_1} \]

Therefore, we have for any \(t \geq 0\)
\[ \frac{d\mathcal{E}(t)}{dt} \leq -\tau\frac{2r_1}{2r_2} C_{\beta, \varepsilon}^{1-2r_1} a_t^{(2r_1-1)(\delta + \alpha - 1)} \mathcal{E}(t)^{2r_1^{-1}} + \gamma \alpha \ell_t^{-\varepsilon} a_t^{-2\alpha} \mathbb{E}\left[\|\nabla f\|^2\right] + \delta a_t^{-1} \mathcal{E}(t). \]

Let \(D = \max(D_1, D_2)\) with
\[ D_1 = \left(\delta \mathbb{E}\right) C_{\beta, \varepsilon}^{2r_1^{-1}} a_t^{(2r_1-1)(\delta + \alpha - 1)} \mathcal{E}(t)^{2r_1^{-1}}, \]
\[ D_2 = \left(\mathbb{E}\mathbb{E}\right) C_{\beta, \varepsilon}^{2r_1^{-1}} a_t^{(2r_1-1)(\delta + \alpha - 1)} \mathcal{E}(t)^{2r_1^{-1}}. \]

If \(\mathcal{E}(t) \geq D_3\) then \(d\mathcal{E}(t)/dt \leq 0\). Let \(D = \max(D_3, \mathcal{E}(0))\), then for any \(t \geq 0\), \(\mathcal{E}(t) \leq D\), which concludes the proof.
D.2 Technical lemmas

**Lemma 40.** Assume that $f$ is continuous, that $x^* \in \arg\min_{x \in \mathbb{R}^d} f(x)$ and that there exist $c, R \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x - x^*\| \geq R$ we have $f(x) - f(x^*) \geq c\|x - x^*\|$. Let $p \in \mathbb{N}$, $X$ a $d$-dimensional random variable and $D_4 \geq 1$ such that $\mathbb{E}[(f(X) - f(x^*))^{2p}] \leq D_4$. Then there exists $D_5 \geq 0$ such that
\[
\mathbb{E}\left[\|X - x^*\|^{2p}\right] \leq D_5 D_4.
\]

**Proof.** Since $f$ is continuous there exists $a \geq 0$ such that for any $x \in \mathbb{R}^d$, $f(x) - f(x^*) \geq c\|x - x^*\| - a$. Therefore, using Jensen’s inequality and that $D_4 \geq 1$ we have
\[
\mathbb{E}\left[\|X - x^*\|^{2p}\right] \leq c^{-2p} \sum_{k=0}^{2p} \binom{k}{2p} \mathbb{E}\left[(f(X) - f(x^*))^k\right] a^{2p-k}
\]
\[
\leq c^{-2p} \sum_{k=0}^{2p} \binom{k}{2p} \mathbb{E}\left[(f(X) - f(x^*))^{2p}\right]^{k/(2p)} a^{2p-k}
\]
\[
\leq c^{-2p} \sum_{k=0}^{2p} \binom{k}{2p} (k/(2p)) a^{2p-k} \leq D_5 D_4,
\]
with $D_5 = c^{-2p} \sum_{k=0}^{2p} (k) a^{2p-k}$. □

**Lemma 41.** Assume $F_3$ with $r_1 = r_2 = 1$. Then for any $p \in \mathbb{N}$ with $p \geq 2$ and $d$-dimensional random variable $X$ we have
\[
\mathbb{E}\left[\|\nabla f(X)\|^2 (f(X) - f(x^*))^{p-1}\right] \geq \mathbb{E}\left[(f(X) - f(x^*))^p\right]^{1+1/p} \mathbb{E}\left[\|X - x^*\|^{2p}\right]^{-1/p}.
\]

**Proof.** Let $p \in \mathbb{N}$ with $p \geq 2$ and let $\varpi = 2p/(p+1)$. Using $F_3$ we have for any $x \in \mathbb{R}^d$
\[
\|x - x^*\|^\varpi \|\nabla f(x)\|^\varpi (f(x) - f(x^*))^{\varpi(p-1)/2} \geq (f(x) - f(x^*))^{\varpi(p+1)/2} \geq (f(x) - f(x^*))^p.
\]
Let $\varsigma = 2\varpi^{-1} = 1 + p^{-1}$ and $\varpi$ such that $\varsigma^{-1} + \varpi^{-1} = 1$. Using Hölder’s inequality the fact that $\varpi = 2p$ we have
\[
\mathbb{E}\left[\|X - x^*\|^\varpi \|\nabla f(X)\|^\varpi (f(X) - f(x^*))^{\varpi(p-1)/2}\right] \leq \mathbb{E}\left[\|\nabla f(X)\|^2 (f(X) - f(x^*))^{p-1}\right]^{1/\varsigma} \mathbb{E}\left[\|X - x^*\|^{2p}\right]^{1/\varpi}.
\]
Since, $\varpi^{-1} = (1 + p)^{-1}$ we have
\[
\mathbb{E}\left[\|\nabla f(X)\|^2 (f(X) - f(x^*))^{p-1}\right] \geq \mathbb{E}\left[(f(X) - f(x^*))^p\right]^{1+1/p} \mathbb{E}\left[\|X - x^*\|^{2p}\right]^{-1/p},
\]
which concludes the proof. □

**Lemma 42.** Let $\alpha, \gamma \in (0, 1)$. Assume that $F_3b$ holds then for any $p \in \mathbb{N}$, there exists $D_{p,4} \geq 0$ such that for any $t \geq 0$
\[
\mathbb{E}\left[\|X_t - x^*\|^{2p}\right]^{1/p} \leq D_{p,4} \left\{1 + (\gamma \alpha + t)^{1-2\alpha}\right\}.
\]

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Proof. Let \( \alpha, \gamma \in (0, 1) \) and \( p \in \mathbb{N} \). Let \( \mathcal{E}_{t,p} = \mathbb{E} \left[ \| X_t - x^* \|^{2p} \right] \). Using Lemma 31 and Lemma 32 we have for any \( t > 0 \)

\[
d\mathcal{E}_{t,p}/dt = -2p(\gamma_\alpha + t)^{-\alpha} \mathbb{E} \left[ \langle \nabla f(X_t), X_t - x^* \rangle \right] \| X_t - x^* \|^{2(1+\alpha)} + \frac{p}{2} \mathbb{E} \left[ \left\| \nabla f(X_t) \right\|^{2(1+\alpha)} \right]
\]

(66)

\[
+ 2(2p - 1) \mathbb{E} \left[ \langle (X_t - x^*)^T (X_t - x^*), \Sigma(X_t) \rangle \right] \| X_t - x^* \|^{2(1+\alpha)} - 2p \gamma_\alpha (2p - 1)(\gamma_\alpha + t)^{-2\alpha} \mathbb{E} \left[ \| X_t - x^* \|^{2(\alpha)} \right]
\]

\[
\leq p \gamma_\alpha \eta (2p - 1)(\gamma_\alpha + t)^{-2\alpha} \mathbb{E} \left[ \| X_t - x^* \|^{2(\alpha)} \right].
\]

If \( p = 1 \), the proposition holds and by recursion and using (66) we obtain the result for \( p \in \mathbb{N} \). □

D.3 Control of the norm in the convex case

Proposition 43. Let \( \alpha, \gamma \in (0, 1) \). Let \( m \in [0, 2] \) and \( \varphi > 0 \) such that for any \( \alpha \in \mathbb{N} \) there exists \( D_{p,2} \geq 0 \) such that for any \( t \geq 0 \), \( \mathbb{E}[\| X_t - x^* \|^{2p}]^{1/p} \leq D_{p,1} \{ 1 + (\gamma_\alpha + t)^{\varphi} \}. \) Assume \( A_1 \) and \( F3b \) and that there exist \( R \geq 0 \) and \( c > 0 \) such that for any \( x \in \mathbb{R}^d \), with \( \| x \| \geq R \), \( f(x) - f(x^*) \geq c \| x - x^* \| \). Then, for any \( p \in \mathbb{N} \), there exists \( D_{p,2} \geq 0 \) such that for any \( t \geq 0 \),

\[
\mathbb{E} \left[ \| X_t - x^* \|^{2p} \right]^{1/p} \leq D_{p,2} \{ 1 + (\gamma_\alpha + t)^{m-\varphi} \}.
\]

Proof. If \( \alpha \geq m/\varphi \) the proof is immediate since \( \sup_{t\geq 0} \{ \mathbb{E}[\| X_t - x^* \|^{2p}]^{1/p} \} < +\infty \). Now assume that \( \alpha < m/\varphi \). Let \( p \in \mathbb{N} \), \( \delta_p = p(1 + \varphi/\alpha) - pm \) and \( t \mapsto \mathcal{E}_{t,p} \) such that for any \( t \geq 0 \), \( \mathcal{E}_{t,p} = (f(X_t) - f(x^*))^{2p}(\gamma_\alpha + t)^{\delta_p} \). Using Lemma 32 we have for any \( t > 0 \)

\[
d\mathcal{E}_{t,p}/dt = -2p(\gamma_\alpha + t)^{-\alpha+\delta_p} \mathbb{E} \left[ \| \nabla f(X_t) \|^{2(1+\alpha)} (f(X_t) - f(x^*))^{2p-1} \right] + \frac{p}{2} \mathbb{E} \left[ \left\| \nabla^2 f(X_t) \right\|^{2(1+\alpha)} (f(X_t) - f(x^*))^{2p-1} \right]
\]

(67)

\[
+ (2p - 1) \mathbb{E} \left[ \langle \nabla f(X_t), \nabla f(X_t)^T, \Sigma(X_t) \rangle (f(X_t) - f(x^*))^{2p-2} \right] + \delta_p(\gamma_\alpha + t)^{-1} \mathcal{E}_{t,p}.
\]

Combining (67), Lemma 31, Lemma 35, Lemma 41 and the fact that for any \( t \geq 0 \), \( \mathbb{E}[\| X_t - x^* \|^{4p}]^{1/(2p)} \leq D_{p,1} \{ 1 + (\gamma_\alpha + t)^{m-\varphi} \} \) we get

\[
d\mathcal{E}_{t,p}/dt \leq -2p(\gamma_\alpha + t)^{-\alpha+\delta_p} \mathbb{E} \left[ (f(X_t) - f(x^*))^{2p} \right]^{1+1/(2p)} \mathbb{E} \left[ \| X_t - x^* \|^{4p} \right]^{-1/(2p)}
\]

\[
+ p(2p - 1) \mathbb{E} \left[ \langle \nabla f(X_t), \nabla f(X_t)^T \rangle \Sigma(X_t) (f(X_t) - f(x^*))^{2p-1} \right] + \delta_p(\gamma_\alpha + t)^{-1} \mathcal{E}_{t,p}
\]

\[
\leq -2p(\gamma_\alpha + t)^{-\alpha+\delta_p/(2p)} \mathcal{E}_{t,p}^{1+1/(2p)} \mathbb{E} \left[ \| X_t - x^* \|^{2p} \right]^{-1/(2p)}
\]

\[
+ p(2p - 1) \mathbb{E} \left[ \langle \nabla f(X_t), \nabla f(X_t)^T \rangle \Sigma(X_t) \right] + \delta_p(\gamma_\alpha + t)^{-1} \mathcal{E}_{t,p}
\]

\[
\leq -2p(\gamma_\alpha + t)^{-\alpha+\delta_p/(2p)} \mathcal{E}_{t,p}^{1+1/(2p)} D_{p,1} \{ 1 + (\gamma_\alpha + t)^{m-\varphi} \}^{-1}
\]

\[
+ p(2p - 1) \mathbb{E} \left[ \langle \nabla f(X_t), \nabla f(X_t)^T \rangle \Sigma(X_t) \right] + \delta_p(\gamma_\alpha + t)^{-1} \mathcal{E}_{t,p}
\]

\[
\leq -2p(\gamma_\alpha + t)^{-\alpha+\delta_p/(2p)} \mathcal{E}_{t,p}^{1+1/(2p)} D_{p,1} \{ 1 + (\gamma_\alpha + t)^{m-\varphi} \}^{-1}
\]

\[
\leq -2p(\gamma_\alpha + t)^{-\alpha+\delta_p/(2p)} \mathcal{E}_{t,p}^{1+1/(2p)} D_{p,1} \{ 1 + (\gamma_\alpha + t)^{m-\varphi} \}^{-1}
\]

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and that we obtain that there exists $\tilde{a}, \tilde{b}$ which concludes the proof.

Therefore, using Lemma 3, there exists $D_p(\alpha) \geq 1$ such that for any $t \geq 0$, $E_{t,p} \leq D_p(\alpha)$. Hence, for any $t \geq 0$,

$$ \mathbb{E}\left[(f(X_t) - f(x^*))^{2p}\right] \leq D_p(1 + (\gamma_\alpha + t)^{p\alpha - p(1 + \varphi)\alpha}) . $$

Using Lemma 40, there exists $D_5 \geq 0$ such that

$$ \mathbb{E}\left[\|X_t - x^*\|^{2p}\right] \leq D_5(1 + (\gamma_\alpha + t)^{p\alpha - p(1 + \varphi)\alpha}) , $$

which concludes the proof upon using that for any $a, b \geq 0$, $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$.

The following corollary is of independent interest.

**Corollary 44.** Let $\alpha, \gamma \in (0, 1)$. Assume $F2$ and that arg min$_{\mathbb{R}^d} f$ is bounded. Then, for any $p \geq 0$ and $t \geq 0$,

$$ \mathbb{E}\left[\|X_t - x^*\|^{p}\right] < +\infty . $$

**Proof.** Without loss of generality we assume that $x^* = 0$ and $f(x^*) = 0$. First, since arg min$_{\mathbb{R}^d} f$ is bounded, there exists $\tilde{R} \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x\| \geq \tilde{R}$, $f(x) > 0$. Let $S = \{x \in \mathbb{R}^d, \|x\| = 1\}$ and consider $m : S \rightarrow (0, +\infty)$ such that for any $\theta \in S$, $m(\theta) = f(\tilde{R}\theta)$. $m$ is continuous since $f$ is convex and therefore it attains its minimum and there exists $m^* > 0$ such that for any $\theta \in S$, $m(\theta) \geq m^*$. Let $x \in \mathbb{R}^d$ with $\|x\| \geq 2\tilde{R}$. Since $f_\alpha : [0, +\infty) \rightarrow \mathbb{R}$ such that $f_\alpha(t) = f(tx)$ is convex we have

$$(f(x) - f(\tilde{R}x/\|x\|))(\|x\| - \tilde{R})^{-1} \geq (f(\tilde{R}x/\|x\|))\tilde{R}^{-1} \geq m^*\tilde{R}^{-1} .$$

Therefore, there exists $c > 0$ and $R \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x\| \geq R$, $f(x) \geq c\|x\|$. Let $p \in \mathbb{N}$. Noticing that $F2$ implies that $F3b$ holds we can apply Lemma 42 and Proposition 43 with $m = 1$ and $\alpha = 2$. Applying repeatedly Proposition 43 we obtain that there exists $D_p \geq 0$ such that

$$ \mathbb{E}\left[\|X_t - x^*\|^{2p}\right]^{1/p} \leq D_p\{1 + (\gamma_\alpha + t)^{m-[\alpha^{-1}]\alpha}\} \leq D_p\{1 + (\gamma_\alpha + t)^{m-[\alpha^{-1}]\alpha}\} \leq D_p\{1 + \gamma_\alpha^{m-[\alpha^{-1}]\alpha}\} , $$

which concludes the proof.

**D.4 Proof of Corollary 11**

Let $\alpha, \gamma \in (0, 1)$ and $X_0 \in \mathbb{R}^d$. Using Lemma 32, we have for any $t \geq 0$

$$ \mathbb{E}\left[\|X_t - x^*\|^2\right] = \|X_0 - x^*\|^2 - \int (\gamma_\alpha + s)^{-\alpha}(f(X_s), X_s - x^*)ds - \int (\gamma_\alpha + s)^{-2\alpha}(\Sigma(X_s), \nabla^2 f(X_s))ds . $$
Let $\mathcal{E}_t = \mathbb{E}[\|X_t - x^*\|^2]$. Using (68), we have for any $t \geq 0$,
\begin{equation}
\mathcal{E}'_t \leq -(\gamma_0 + t)^{-\alpha} \mathbb{E} [\langle \nabla f(X_s), X_s - x^* \rangle] + (\gamma_0 \ln(2)(\gamma_0 + t)^{-2\alpha}.
\end{equation}

We divide the proof into three parts.

(a) First, assume that $\textbf{F3b}$ holds. Combining this result and (69), we get that for any $t \geq 0$,
\[\mathcal{E}'_t \leq \gamma_0 \ln(2) d(\gamma_0 + t)^{-2\alpha}.\]
Therefore, there exist $\beta, \varepsilon \geq 0$ and $C_{\beta, \varepsilon} \geq 0$ such that $\mathbb{E}[\|X_t - x^*\|^2] < C_{\beta, \varepsilon}(\gamma_0 + t)^{-\beta}(1 + \log(1 + \gamma_0^{-1}t))^\varepsilon$ with $\beta = 0$ and $\varepsilon = 0$ if $\alpha > 1/2$, $\beta = 1 - 2\alpha$ and $\varepsilon = 0$ if $\alpha < 1/2$ and $\beta = 0$ and $\varepsilon = 1$ if $\alpha = 1/2$. Combining this result and Theorem 10 concludes the proof.

(b) We can apply Lemma 42 and Proposition 43 with $m = 1$ and $\varphi = 2$. Applying repeatedly Proposition 43 we obtain that there exists $D_p \geq 0$ such that
\begin{align*}
\mathbb{E} \left[ \|X_t - x^*\|^{2p} \right]^{1/p} \leq D_p \{1 + (\gamma_0 + t)^{m-\lceil \alpha^{-1} \rceil \alpha} \} \leq D_p \{1 + (\gamma_0 + t)^{m-\lceil m/\alpha \rceil \alpha} \} \leq D_p \{1 + \gamma_0^{m-\lceil m/\alpha \rceil \alpha} \},
\end{align*}
which concludes the proof.

(c) Finally, assume that there exists $R \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x\| \geq R$, $\langle \nabla f(x), x - x^* \rangle \geq m \|x - x^*\|^2$. Therefore, since $\langle x \mapsto \nabla f(x) \rangle$ is continuous, there exists $a \geq 0$ such that for any $x \in \mathbb{R}^d$, $\langle \nabla f(x), x - x^* \rangle \geq m \|x - x^*\|^2 - a$. Combining this result and (69), we get that for any $t \geq 0$,
\begin{align*}
\mathcal{E}'_t \leq -m(\gamma_0 + t)^{-\alpha} \mathcal{E}_t + (\gamma_0 + t)^{-\alpha} a + \gamma_0 \ln(\gamma_0 + t)^{-2\alpha}.
\end{align*}
Hence, if $\mathcal{E}_t \geq \max(a/m, \ln) \mathcal{E}_t \leq 0$ and for any $t \geq 0$, $\mathcal{E}_t \leq \max(a/m, \ln, \mathcal{E}_0)$ and is bounded. Therefore, there exist $\beta, \varepsilon \geq 0$ and $C_{\beta, \varepsilon} \geq 0$ such that $\mathbb{E}[\|X_t - x^*\|^2] < C_{\beta, \varepsilon}(\gamma_0 + t)^{-\beta}(1 + \log(1 + \gamma_0^{-1}t))^\varepsilon$ with $\beta = \varepsilon = 0$, which concludes the proof.

D.5 Proof of Theorem 12

Without loss of generality, we assume that $f^* = 0$. Let $\alpha, \gamma \in (0, 1)$, $x_0 \in \mathbb{R}^d$. Let $\delta = \min(\delta_1, \delta_2)$, with $\delta_1, \delta_2$ given in Theorem 12 and let $(E_k)_{k \in \mathbb{N}}$ such that for any $k \in \mathbb{N}$, $E_k = (k + 1)^{\delta} \mathbb{E} [f(X_k)] (1 + \log(k + 1)^{-\varepsilon}$. There exists $c_\delta \in \mathbb{R}$ such that for any $x \in [0, 1]$, $(1 + x)^{\delta} \leq 1 + c_\delta x$. Hence, for any $n \in \mathbb{N}$ we have
\begin{align*}
(n + 2)^{\delta} - (n + 1)^{\delta} \leq (n + 1)^{\delta} \left\{ (1 + (n + 1)^{-1})^{\delta} - 1 \right\} \leq c_\delta (n + 1)^{\delta - 1}.
\end{align*}
Using (Nesterov, 2004, Lemma 1.2.3) and A2 we have for any $n \in \mathbb{N}$ such that $n \geq (2L\gamma)^{1/\alpha}$
\begin{align}
&\mathbb{E} [f(X_{n+1}) | \mathcal{F}_n] \leq f(X_n) - \gamma(n + 1)^{-\alpha} \mathbb{E} [\|\nabla f(X_n), H(X_n, Z_{n+1}) \| | \mathcal{F}_n]
&\quad + (L/2) \gamma^2 (n + 1)^{-2\alpha} \mathbb{E} [\|H(X_n, Z_{n+1})\|^2 | \mathcal{F}_n]
\end{align}
\begin{align*}
\mathbb{E} [f(X_{n+1})] \leq \mathbb{E} [f(X_n)] - \gamma(n + 1)^{-\alpha} \mathbb{E} [\|\nabla f(X_n)\|^2]
&\quad + L\gamma^2 (n + 1)^{-2\alpha} \mathbb{E} [\|\nabla f(X_n)\|^2] + L\gamma^2 (n + 1)^{-2\alpha} \eta
\end{align*}
\begin{align*}
&\leq \mathbb{E} [f(X_n)] - \gamma(n + 1)^{-\alpha} \{1 - L\gamma(n + 1)^{-\alpha}\} \mathbb{E} [\|\nabla f(X_n)\|^2] + L\gamma^2 (n + 1)^{-2\alpha} \eta
&\leq \mathbb{E} [f(X_n)] - \gamma(n + 1)^{-\alpha} \mathbb{E} [\|\nabla f(X_n)\|^2] / 2 + L\gamma^2 (n + 1)^{-2\alpha} \eta.
\end{align*}
Combining (70) and (71) we get for any $n \in \mathbb{N}$ such that $n \geq (2L\gamma)^{1/2}$

\[
E_{n+1} - E_n = (n + 2)^\delta \mathbb{E} [f(X_{n+1})] (1 + \log(n + 2))^{-\varepsilon} - (n + 1)^\delta \mathbb{E} [f(X_n)] (1 + \log(n + 1))^{-\varepsilon}
\]

\[
\leq (1 + \log(n + 1))^{-\varepsilon} \left\{ (n + 2)^\delta - (n + 1)^\delta \right\} \left( \mathbb{E} [f(X_{n+1})] - \mathbb{E} [f(X_n)] \right)
\]

\[
+ (n + 1)^\delta \left\{ \mathbb{E} [f(X_{n+1})] - \mathbb{E} [f(X_n)] \right\}
\]

\[
\leq (1 + \log(n + 1))^{-\varepsilon} \left\{ (n + 2)^\delta - (n + 1)^\delta \right\} \left( \mathbb{E} [f(X_n)] + L\gamma^2(n + 1)^{-2\alpha\eta} \right)
\]

\[
+ (n + 1)^\delta \left\{ -\gamma(n + 1)^{-\alpha} \mathbb{E} \left[ \|\nabla f(X_n)\|^2 \right] / 2 + L\gamma^2(n + 1)^{-2\alpha\eta} \right\}
\]

\[
\leq (1 + \log(n + 1))^{-\varepsilon} \left\{ c_\delta(n + 1)^{\delta-1}(\mathbb{E} [f(X_n)] + 2\gamma^2(n + 1)^{-2\alpha\eta}) \right\}
\]

\[
+ (n + 1)^\delta \left\{ -\gamma(n + 1)^{-\alpha} \mathbb{E} \left[ \|\nabla f(X_n)\|^2 \right] / 2 + L\gamma^2(n + 1)^{-2\alpha\eta} \right\}
\]

\[
\leq c_\delta E_n + 2L\gamma^2(1 + c_\delta)(n + 1)^{-2\alpha}(1 + \log(n + 1))^{-\varepsilon}\eta
\]

\[- \gamma(n + 1)^{\delta-\alpha}(1 + \log(n + 1))^{-\varepsilon} \mathbb{E} \left[ \|\nabla f(X_n)\|^2 \right] / 2 .
\]

Using (3) and the fact that for any $k \in \mathbb{N}$, $\mathbb{E} [\|X_k - x^*\|^2 r_1 r_2] \leq C_{\beta,\varepsilon}(k + 1)^\delta(1 + \log(1 + k))^{\varepsilon}$ and Hölder’s inequality and that $r_1 r_2 = 2(2r_1^{-1} - 1)^{-1}$, we have for any $k \in \mathbb{N}$

\[
\mathbb{E} \left[ \|\nabla f(X_k)\|^2 \right] \geq \mathbb{E} [f(X_k)]^{2r_1^{-1}} C_{\beta,\varepsilon}^{-2r_1^{-1} - 1} \tau^{2r_1^{-1}} (k + 1)^{-\beta(2r_1^{-1} - 1)(1 + \log(k + 1))^{-\varepsilon(2r_1^{-1} - 1)} . (73)
\]

Combining (72) and (73) we get that for any $n \in \mathbb{N}$ with $n \geq (4\gamma)^{1/\alpha}$

\[
E_{n+1} - E_n \leq c_\delta E_n + 2L\gamma^2(1 + c_\delta)(n + 1)^{-2\alpha}(1 + \log(n + 1))^{-\varepsilon}\eta
\]

\[- \gamma(n + 1)^{\delta-\alpha}(2r_1^{-1} - 1) \mathbb{E} [f(X_n)]^{2r_1^{-1}} C_{\beta,\varepsilon}^{-2r_1^{-1} - 1} \tau^{2r_1^{-1}} (1 + \log(n + 1))^{-\varepsilon} / 2
\]

\[
\leq c_\delta E_n + 2L\gamma^2(1 + c_\delta)(n + 1)^{-2\alpha}(1 + \log(n + 1))^{-\varepsilon}\eta
\]

\[- \gamma(n + 1)^{\alpha-\beta(2r_1^{-1} - 1)} E_n^{2r_1^{-1}} C_{\beta,\varepsilon}^{-2r_1^{-1} - 1} \tau^{2r_1^{-1}} / 2 .
\]

Let $D_3 = \max(D_1, D_2)$ with

\[
\begin{cases}
D_1 = (2c_\delta(C_{\beta,\varepsilon}^{-2r_1^{-1} - 1} \tau^{-2r_1^{-1}})^{2r_1^{-1} - 1} , \\
D_2 = (4L\gamma^2(1 + c_\delta)C_{\beta,\varepsilon}^{-2r_1^{-1} - 1} \tau^{-2r_1^{-1}})^{r_1/2} .
\end{cases}
\]

If $E_n \geq D_3$ and $n \geq (4\gamma)^{1/\alpha}$ then $E_{n+1} \leq E_n$. Therefore, we obtain by recursion that $E_n \leq D$ with

$D = \max(E_0, \ldots, E_{[(2L\gamma)^{1/\alpha}]}, D_3)$.

D.6 Discrete counterpart of Corollary 11

Corollary 45. Let $\alpha, \gamma \in (0, 1)$ and $x_0 \in \mathbb{R}^d$. Assume A1, A2. Then we have:

(a) if F3b holds then, there exists $D \geq 0$ such that for any $N \in \mathbb{N}^*$

\[
\mathbb{E} [f(X_N)] - f^* \leq D \left[ N^{(1-3\alpha)/2} + N^{-\alpha/2} + N^{\alpha-1} \right] ,
\]

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(b) if F3 holds and if there exists $R \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x\| \geq R$, $\langle \nabla f(x), x - x^* \rangle \geq m \|x - x^*\|^2$, then there exists $D \geq 0$ such that for any $N \in \mathbb{N}$

$$E[f(X_N)] - f^* \leq D \left[N^{-\alpha/2} + N^{\alpha - 1}\right].$$

Proof. Let $\alpha, \gamma \in (0, 1)$ and $x_0 \in \mathbb{R}^d$. We have for any $n \in \mathbb{N}$,

$$E\left[\|X_{n+1} - x^*\|^2\right] = E\left[\|X_n - x^*\|^2\right] + 2E\left[\langle X_n - x^*, X_{n+1} - X_n \rangle\right] + E\left[\|X_{n+1} - X_n\|^2\right] \leq E\left[\|X_n - x^*\|^2\right] - 2\gamma(n+1)^{-\alpha}E\left[\langle X_n - x^*, \nabla f(X_n) \rangle\right] + 2\gamma^2(n+1)^{-2\alpha}E\left[\|\nabla f(X_n)\|^2\right] + 2\gamma(n+1)^{-2\alpha} \gamma.$$

We now divide the proof into two parts.

(a) Using F3b and Lemma 35 we have for any $x \in \mathbb{R}^d$,

$$\langle \nabla f(x), x - x^* \rangle \geq \tau(f(x) - f(x^*)) \geq \tau \|\nabla f(x)\|^2 / (2L).$$

Using A1, (74) and (75) we have for any $n \geq (4\gamma L / \tau)^{1/\alpha}$

$$E\left[\|X_{n+1} - x^*\|^2\right] \leq E\left[\|X_n - x^*\|^2\right] + 2\gamma(n+1)^{-\alpha}(-\tau / (2L) + \gamma(n+1)^{-\alpha})E\left[\|\nabla f(X_n)\|^2\right] + 2\gamma(n+1)^{-2\alpha} \gamma.$$

Therefore, there exist $\beta, \varepsilon \geq 0$ and $C_{\beta, \varepsilon} \geq 0$ such that $E[\|X_n - x^*\|^2] < C_{\beta, \varepsilon}(n+1)^{-\beta}(1 + \log(1 + n))^\varepsilon$ with $\beta = 0$ and $\varepsilon = 0$ if $\alpha > 1/2$, $\beta = 1 - 2\alpha$ and $\varepsilon = 0$ if $\alpha < 1/2$ and $\beta = 0$ and $\varepsilon = 1$ if $\alpha = 1/2$. Combining this result and Theorem 12 concludes the proof.

(b) Finally, assume that there exists $R \geq 0$ such that for any $x \in \mathbb{R}^d$ with $\|x\| \geq R$, $\langle \nabla f(x), x - x^* \rangle \geq m \|x - x^*\|^2$. Therefore, since $(x \mapsto \nabla f(x))$ is continuous, there exists $a \geq 0$ such that for any $x \in \mathbb{R}^d$, $\langle \nabla f(x), x - x^* \rangle \geq m \|x - x^*\|^2 - a$. Combining this result and (74) we get that for any $n \in \mathbb{N}$ such that $n \geq (2/\gamma)^{\alpha^{-1}}$

$$E\left[\|X_{n+1} - x^*\|^2\right] \leq (1 - \gamma(n+1)^{-\alpha})E\left[\|X_n - x^*\|^2\right] + 2\gamma(n+1)^{-\alpha}a + 2\gamma^2(n+1)^{-2\alpha} \gamma.$$

Hence, if $n \geq (2/\gamma)^{\alpha^{-1}}$ and $E[\|X_n - x^*\|^2] \geq \max(2a, 2\gamma \eta)$ then $E[\|X_{n+1} - x^*\|^2] \leq E[\|X_n - x^*\|^2]$. Therefore, we obtain by recursion that for any $n \in \mathbb{N}$, that $(E[\|X_n - x^*\|^2])_{n \in \mathbb{N}}$ is bounded which concludes the proof by applying Theorem 12.

$\square$