HOLDERIAN WEAK INVARIANCE PRINCIPLE UNDER A HANNAN TYPE CONDITION

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ABSTRACT. We investigate the invariance principle in Hölder spaces for strictly stationary martingale difference sequences. In particular, we show that the sufficient condition on the tail in the i.i.d. case does not extend to stationary ergodic martingale differences. We provide a sufficient condition on the conditional variance which guarantee the invariance principle in Hölder spaces. We then deduce a condition in the spirit of Hannan one.

1. INTRODUCTION

One of the main problems in probability theory is the understanding of the asymptotic behavior of Birkhoff sums $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, where $(\Omega, \mathcal{F}, \mu, T)$ is a dynamical system and $f$ a map from $\Omega$ to the real line.

One can consider random functions constructed from the Birkhoff sums

$$S_{pl}^{n}(f, t) := S_{\lfloor nt \rfloor}(f) + (nt - \lfloor nt \rfloor)f \circ T^{\lfloor nt \rfloor+1},$$

and investigate the asymptotic behaviour of the sequence $\left(S_{pl}^{n}(f, t)\right)_{n \geq 1}$ seen as an element of a function space. Donsker showed (cf. [Don51]) that the sequence $\left(n^{-1/2}(E(f^2))^{-1/2}S_{pl}^{n}(f)\right)_{n \geq 1}$ converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion $W$ when the sequence $(f \circ T^i)_{i \geq 0}$ is i.i.d. and zero mean. Then an intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [MPU06] for the main theorems in this direction.

Our purpose is to investigate the weak convergence of the sequence $\left(n^{-1/2}S_{pl}^{n}(f)\right)_{n \geq 1}$ in Hölder spaces when $(f \circ T^i)_{i \geq 0}$ is a strictly stationary sequence. A classical method for showing a limit theorem is to use a martingale approximation, which allows to deduce the corresponding result if it holds for martingale differences sequences provided that the approximation is good enough. To the best of our knowledge, no result about the invariance principle in Hölder space for stationary martingale difference sequences is known.

1.1. The Hölder spaces. It is well known that standard Brownian motion’s paths are almost surely Hölder regular of exponent $\alpha$ for each $\alpha \in (0, 1/2)$, hence it is natural to consider the random function defined in (1.1) as an element of $H_\alpha[0, 1]$ and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for $\alpha \in (0, 1)$ the Hölder space $H_\alpha[0, 1]$ of functions $x: [0, 1] \to \mathbb{R}$ such that $\sup_{s \neq t} |x(s) - x(t)| / |s - t|^{\alpha}$ is
Brownian motion in the space $H$ is equivalent to the weak convergence of the sequence $(\xi_n)_{n \geq 1}$, defined by

\begin{equation}
\omega_\alpha(x, \delta) = \sup_{0 < t-s < \delta} \frac{|x(t) - x(s)|}{|t-s|^\alpha}.
\end{equation}

We then define $\mathcal{H}_\alpha^0[0,1]$ by $\mathcal{H}_\alpha^0[0,1] := \{ x \in \mathcal{H}_\alpha[0,1], \lim_{\delta \to 0} \omega_\alpha(x, \delta) = 0 \}$. We shall essentially work with the spaces $\mathcal{H}_\alpha^0[0,1]$ which, endowed with $\|x\|_\alpha := \omega_\alpha(x,1) + |x(0)|$, is a separable Banach space (while $\mathcal{H}_\alpha[0,1]$ is not). Since the canonical embedding $\nu: \mathcal{H}_\alpha^0[0,1] \to \mathcal{H}_\alpha[0,1]$ is continuous, each convergence in distribution in $\mathcal{H}_\alpha^0[0,1]$ also takes place in $\mathcal{H}_\alpha[0,1]$.

Let us denote by $D_j$ the set of dyadic numbers in $[0,1]$ of level $j$, that is,

\begin{equation}
D_0 := \{0,1\}, \quad D_j := \{(2l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}, j \geq 1.
\end{equation}

If $r \in D_j$, for some $j \geq 0$, we define $r^+ := r + 2^{-j}$ and $r^- := r - 2^{-j}$. For $r \in D_j$, $j \geq 1$, let $\Lambda_r$ the function whose graph is the polygonal path joining the points $(0,0), (r^-,0), (r,1), (r^+,0)$ and $(1,0)$. We can decompose each $x \in C[0,1]$ as

\begin{equation}
x = \sum_{r \in D} \lambda_r(x) \Lambda_r = \sum_{j=0}^{+\infty} \sum_{r \in D_j} \lambda_r(x) \Lambda_r,
\end{equation}

and the convergence is uniform on $[0,1]$. The coefficients $\lambda_r(x)$ are given by

\begin{equation}
\lambda_r(x) = x(r) - \frac{x(r^+) - x(r^-)}{2}, \quad r \in D_j, j \geq 1,
\end{equation}

and $\lambda_0(x) = x(0)$, $\lambda_1(x) = x(1)$.

Ciesielski proved (cf. \cite{Cie60}) that $\{\Lambda_r; r \in D\}$ is a Schauder basis of $\mathcal{H}_\alpha^0[0,1]$ and the norms $\|\cdot\|_\alpha$ and the sequential norm defined by

\begin{equation}
\|x\|_{\alpha, \text{seq}} := \sup_{j \geq 0} \max_{r \in D_j} 2^{j\alpha} |\lambda_r(x)|,
\end{equation}

are equivalent.

Considering the sequential norm, we can show (see Theorem 3 in \cite{Suq99}) that a sequence $(\xi_n)_{n \geq 1}$ of random elements of $\mathcal{H}_\alpha^0$ vanishing at 0 is tight if and only if for each positive $\varepsilon$,

\begin{equation}
\lim_{j \to \infty} \limsup_{n \to \infty} \mu \left( \sup_{j \geq j} \max_{r \in D_j} 2^{j\alpha} |\lambda_r(\xi_n)| > \varepsilon \right) = 0.
\end{equation}

**Notation 1.1.** In the sequel, we will denote $r_{k,j} := k2^{-j}$ and $u_{k,j} := \lfloor nr_{k,j} \rfloor$ (or $r_k$ and $u_k$ for short). Notice that $u_{k+1,j} - u_{k,j} = \lfloor nr_{k,j} + n2^{-j} \rfloor - u_{k,j} \leq 2n2^{-j}$ if $j \leq \log n$, where $\log n$ denotes the binary logarithm of $n$ and for a real number $x$, $[x]$ is the unique integer for which $[x] \leq x < [x] + 1$.

Now, we state the result obtained by Račkauskas and Suquet in \cite{RS03}.

**Theorem 1.2.** Let $p > 2$ and let $(f \circ T_j)_{j \geq 0}$ be an i.i.d. centered sequence with unit variance. Then the condition

\begin{equation}
\lim_{t \to \infty} t^p \mu \{|f| > t\} = 0
\end{equation}

is equivalent to the weak convergence of the sequence $(n^{-1/2}S_n^p(f))_{n \geq 1}$ to a standard Brownian motion in the space $\mathcal{H}_1^{1/2-1/p}[0,1]$. 

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**References:**

- Cie60: Ciesielski, C. (1960).
- Suq99: Suquet, A. (1999).
- RS03: Račkauskas, A. and Suquet, A. (2003).

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1.2. Some facts about the $L^{p,\infty}$ spaces. Let $p > 2$. We define the $L^{p,\infty}$ space as the collection of functions $f: \Omega \to \mathbb{R}$ such that the quantity
\begin{equation}
\|f\|_{p,\infty} := \sup_{t>0} t^p \mu \{ |f| > t \} < \infty.
\end{equation}
This quantity is denoted like a norm, while it is not a norm. However, there is a constant $\kappa_p$ such that for each $f$,
\begin{equation}
\|f\|_{p,\infty} \leq \sup_{A: \mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \chi_A] \leq \kappa_p \|f\|_{p,\infty}
\end{equation}
and $N_p(f) := \sup_{A: \mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \chi_A]$ defines a norm.

A function $f$ satisfies (1.8) if and only if it belongs to the closure of bounded functions with respect to $N_p$.

Lemma 1.3. If $\lim_{t \to \infty} t^p \mu \{ |f| > t \} = 0$, then for each sub-$\sigma$-algebra $\mathcal{A}$, we have $\lim_{t \to \infty} t^p \mu \{ \mathbb{E}[|f| \mid \mathcal{A}] > t \} = 0$.

Proof. For simplicity, we assume that $f$ is non-negative. For a fixed $t$, the set $\{ \mathbb{E}[f \mid \mathcal{A}] > t \}$ belongs to the $\sigma$-algebra $\mathcal{A}$, hence
\begin{equation}
\mu \{ \mathbb{E}[f \mid \mathcal{A}] > t \} \leq \mathbb{E} \left[ \mathbb{E}[f \mid \mathcal{A}] \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \} \right] = \mathbb{E}[f \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \}].
\end{equation}
By definition of $N_p$,
\begin{equation}
\mathbb{E}[f \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \}] \leq N_p \left( f \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \} \right) \mu \{ \mathbb{E}[f \mid \mathcal{A}] > t \}^{1-1/p}
\end{equation}
hence
\begin{equation}
t^p \mu \{ \mathbb{E}[f \mid \mathcal{A}] > t \} \leq N_p \left( f \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \} \right)^p.
\end{equation}
Notice that
\begin{equation}
\forall s > 0, \quad N_p \left( f \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \} \right) \leq s \mu \{ \mathbb{E}[f \mid \mathcal{A}] > t \}^{1/p} + N_p \left( f \chi \{ f > s \} \right),
\end{equation}
hence
\begin{equation}
\limsup_{t \to \infty} \mathbb{E}[f \chi \{ \mathbb{E}[f \mid \mathcal{A}] > t \}] \leq N_p \left( f \chi \{ f > s \} \right) \leq \kappa_p \sup_{x \geq s} x^p \mu \{ f > x \}.
\end{equation}
By the assumption on the function $f$, the right hand side goes to 0 as $s$ goes to infinity, which concludes the proof of the lemma.

The next lemma provides an estimation of the $L^{p,\infty}$ norm of a simple function.

Lemma 1.4. Let $f := \sum_{i=0}^N a_i \chi(A_i)$, where the family $(A_i)_{i=0}^N$ is pairwise disjoint and $0 \leq a_N < \cdots < a_0$. Then
\begin{equation}
\|f\|^p_{p,\infty} \leq \max_{0 \leq j \leq N} a_j^p \sum_{i=0}^j \mu(A_i).
\end{equation}
Proof. We have the equality
\begin{equation}
\mu \{ f > t \} = \sum_{j=0}^N \chi(a_{j+1},a_j)(t) \sum_{i=0}^j \mu(A_i),
\end{equation}
where $a_{N+1} := 0$, therefore
\begin{equation}
t^p \mu \{ f > t \} \leq \max_{0 \leq j \leq N} a_j^p \sum_{i=0}^j \mu(A_i).
\end{equation}
2. Main Results

The goal of the paper is to give a sharp sufficient condition on the moments of a strictly stationary martingale difference sequence which guarantees the weak invariance principle in $H^0_{\alpha}[0,1]$ for a fixed $\alpha$.

We first show that Theorem 2.1 does not extend to strictly stationary ergodic martingale difference sequences.

An application of Kolmogorov's continuity criterion shows that if $(m \circ T^i)_{i \geq 0}$ is a martingale difference sequence such that $m \in \mathbb{L}^{p+\delta}$ for some positive $\delta$ and $p > 2$, then the partial sum process $(n^{-1/2}S^m_n(m))_{n \geq 1}$ is tight in $H^0_{1/2-1/p}[0,1]$ (see [KR91]).

We provide a condition on the quadratic variance which improves the previous approach (since the previous condition can be replaced by $m \in \mathbb{L}^p$). Then using martingale approximation we can provide a Hannan type condition which guarantees the weak invariance principle in $H^0_{\alpha}[0,1]$.

**Theorem 2.1.** Let $p > 2$ and $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system with positive entropy. There exists a function $m: \Omega \to \mathbb{R}$ and a $\sigma$-algebra $\mathcal{M}$ for which $T, \mathcal{M} \subset \mathcal{M}$ such that:

- the sequence $(m \circ T^i)_{i \geq 0}$ is a martingale difference sequence with respect to the filtration $(T^{-1}\mathcal{M})_{i \geq 1}$;
- the convergence $\lim_{t \to +\infty} t^p \mu \{|m| > t\} = 0$ takes place;
- the sequence $(n^{-1/2}S^m_n(m))_{n \geq 1}$ is not tight in $H^0_{1/2-1/p}[0,1]$.

**Theorem 2.2.** Let $p > 2$ and let $(m \circ T^i, T^{-1}\mathcal{M})$ be a strictly stationary martingale difference sequence. Assume that $tp \mu \{|m| > t\} \to 0$ and $\mathbb{E}[m^2 \mid T, \mathcal{M}] \in \mathbb{L}^{p/2}$. Then

\[(2.1) \quad n^{-1/2}S^m_n(m) \to \eta \cdot W \text{ in distribution in } H^0_{1/2-1/p}[0,1],\]

where the random variable $\eta$ is given by

\[(2.2) \quad \eta = \lim_{n \to \infty} \mathbb{E}[S^m_n \mid T]/n \text{ in } L^1,\]

and $\eta$ is independent of the process $(W_t)_{t \in [0,1]}$.

In particular, (2.1) takes place if $m$ belongs to $\mathbb{L}^p$.

The key point of the proof of Theorem 2.2 is an inequality in the spirit of Doob’s one, which gives $n^{-1}\mathbb{E} \left[ \max_{1 \leq j \leq n} S_j^m(m)^2 \right] \leq 2\mathbb{E}[m^2]$. It is used in order to establish tightness of the sequence $(n^{-1/2}S^m_n(m))_{n \geq 1}$ in the space $C[0,1]$.

**Proposition 2.3.** Let $p > 2$. There exists a constant $C_p$ depending only on $p$ such that if $(m \circ T^i)_{i \geq 1}$ is a martingale difference sequence, then the following inequality holds:

\[(2.3) \quad \sup_{n \geq 1} \left\| n^{-1/2}S^m_n(m) \right\|_{H^0_{1/2-1/p}}^p \leq C_p \left( \|m\|_{p,\infty}^p + \mathbb{E} \left( [m^2 \mid T, \mathcal{M}] \right)^{p/2} \right).\]

**Remark 2.4.** As Theorem 2.1 shows, the term $\mathbb{E} \left( [m^2 \mid T, \mathcal{M}] \right)^{p/2}$ cannot be omitted in general. For the constructed $m$, the quadratic variance is $\kappa m^2$ for some constant $\kappa$ and $m$ does not belong to the $\mathbb{L}^p$ space.
Since for a function $g$,

\[(2.4) \quad \left\| n^{-1/2} S_n^p (g - g \circ T) \right\|_{\mathcal{H}^p_{1/2-1/p}} = n^{-1/p} \max_{1 \leq i \leq j \leq n} \left| g \circ T^j - g \circ T^i \right| \leq 2n^{-1/p} \max_{1 \leq i \leq n} |g \circ T^i|,
\]

the sequence $\left( \left\| n^{-1/2} S_n^p (g - g \circ T) \right\|_{\mathcal{H}^p_{1/2-1/p}} \right)_{n \geq 1}$ converges to 0 in probability if $g$ belongs to $L^p$. Therefore, we can exploit a martingale-coboundary decomposition in $L^p$.

**Corollary 2.5.** Let $p > 2$ and let $f$ be an $\mathcal{M}$-measurable function which can be written as

\[(2.6) \quad f = m + g - g \circ T,
\]

where $m, g \in L^p$ and $(m \circ T^i)_{i \geq 0}$ is a martingale difference sequence for the filtration $(T^{-1} \mathcal{M})_{i \geq 0}$. Then $n^{-1/2} S_n^p (f) \to \eta W$ in distribution in $\mathcal{H}^0_{1/2-1/p} [0, 1]$, where $\eta$ is given by (2.2) and independent of $W$.

We define for a function $h$ the operators $E_k (h) := \mathbb{E}[h \mid T^k \mathcal{M}]$ and $P_i (h) := E_i (h) - E_{i+1} (h)$. The condition $\sum_{i=0}^{\infty} \|P_i (f)\|_2$ was introduced by Hannan in [Han73] in order to deduce a central limit theorem. It actually implies the weak invariance principle (see Corollary 2 in [DMV07]).

**Theorem 2.6.** Let $p > 2$ and let $f$ be an $\mathcal{M}$-measurable function such that

\[(2.7) \quad \mathbb{E} \left[ f \mid \bigcap_{i \in \mathbb{Z}} T^i \mathcal{M} \right] = 0 \quad \text{and}
\]

\[(2.8) \quad \sum_{i \geq 0} \|P_i (f)\|_p < \infty.
\]

Then $n^{-1/2} S_n^p (m) \to \eta W$ in distribution in $\mathcal{H}^0_{1/2-1/p} [0, 1]$, where $\eta$ is given by (2.2) and independent of $W$.

3. Proofs

3.1. **Proof of Theorem 2.1** We need a result about dynamical systems of positive entropy for the construction of a counter-example.

**Lemma 3.1.** Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic probability measure preserving system of positive entropy. There exists two $T$-invariant sub-$\sigma$-algebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ and a function $g : \Omega \to \mathbb{R}$ such that:

- the $\sigma$-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent;
- the function $g$ is $\mathcal{B}$-measurable, takes the values $-1, 0$ and $1$, has zero mean and the process $(g \circ T^n)$ is independent;
- the dynamical system $(\Omega, \mathcal{C}, \mu, T)$ is aperiodic.

This is Lemma 3.8 from [LV01].
We consider the following four increasing sequences of integers \((I_l)_{l \geq 1}, (J_l)_{l \geq 1}, (n_l)_{l \geq 1}\) and \((L_l)_{l \geq 1}\). We define \(k_l := 2^{I_l + J_l}\) and impose the conditions:

\[
\sum_{l=1}^{\infty} \frac{1}{L_l} < \infty; \tag{3.1}
\]

\[
\lim_{l \to \infty} n_l \sum_{i > l} \frac{k_i}{n_i} = 0; \tag{3.2}
\]

\[
\lim_{l \to \infty} J_l 2^{-l/2} = 0; \tag{3.3}
\]

\[
\lim_{l \to \infty} J_l \cdot \mu \left( \left| N \right| \geq 2^{1/p} \frac{L_l}{\|g\|_2} \right) = 1; \tag{3.4}
\]

\[
\text{for each } l, \sum_{i=1}^{l-1} k_l \left( \frac{n_i}{2l} \right)^{1/p} < \frac{n_l^{1/p}}{2}. \tag{3.5}
\]

Here \(N\) denotes a random variable whose distribution is standard normal.

Using Rokhlin’s lemma, we can find for any integer \(l \geq 1\) a measurable set \(C_l \in \mathcal{C}\) such that the sets \(T^{-i}C_l, i = 0, \ldots, n_l - 1\) are pairwise disjoint and \(\mu \left( \bigcup_{i=0}^{n_l-1} T^{-i}C_l \right) > 1/2\).

For a fixed \(l\), we define

\[
k_{l,j} := 2^{I_l + J_l - j}, \quad 0 \leq j \leq J_l, \tag{3.6}
\]

\[
k_{l,j} := 2^{I_l + J_l - j}, \quad 0 \leq j \leq J_l \text{ and } \tag{3.7}
k_l := \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \chi \left( \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i}C_l \right) + \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l}} \right)^{1/p} \chi \left( \bigcup_{i=0}^{k_{l,J_l}-1} T^{-i}C_l \right),
\]

\[
f := \sum_{l=1}^{\infty} f_l, \quad m := g \cdot f. \tag{3.8}
\]

**Proposition 3.2.** We have the estimate \(\|f_l\|_{p,\infty} \leq k'_p L_l^{-1}\) for some constant \(k'_p\) depending only on \(p\). As a consequence, \(\lim_{t \to \infty} t^p \mu \{ |m| > t \} = 0\).

**Proof.** Notice that

\[
\left\| \frac{1}{L_l} \left( \frac{n_l}{k_{l,J_l}} \right)^{1/p} \chi \left( \bigcup_{i=0}^{k_{l,J_l}-1} T^{-i}C_l \right) \right\|_{p,\infty}^p = \frac{1}{L_l^p} k_{l,J_l} \cdot \mu(C_l) \leq \frac{1}{L_l^p}. \tag{3.9}
\]
Next, using Lemma 1.4 with $N := J_l - 1$, $a_j := \frac{1}{L_l} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p}$ and $A_j := \bigcup_{i=k_{l,j-1}}^{k_{l,j-1}-1} T^{-i}C_l$, we obtain

\[
\left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p} \chi \left( \bigcup_{i=k_{l,j-1}}^{k_{l,j-1}-1} T^{-i}C_l \right) \right\|_p^p \leq \max_{0 \leq j \leq J_l-1} \left( \frac{1}{L_l} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p} \right)^{p_j} \sum_{i=0}^{j} \mu(A_j)
\]

(3.10)

\[
\leq \frac{1}{L_l} \max_{0 \leq j \leq J_l-1} \frac{n_l}{k_{l,j-1}} \sum_{i=0}^{j} \frac{k_{l,j-i}}{n_l}
\]

(3.11)

\[
= \frac{1}{L_l} \max_{0 \leq j \leq J_l-1} \sum_{i=0}^{j} \frac{2^{l+j}}{2^{l+i}}
\]

(3.12)

\[
\leq \frac{2}{L_l}.
\]

(3.13)

hence by (1.10), (3.9) and (3.13),

\[
\|f_l\|_{p,\infty} \leq N_p \left( \frac{1}{L_l} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p} \chi \left( \bigcup_{i=0}^{k_{l,j-1}-1} T^{-i}C_l \right) \right) + 
\]

(3.14)

\[
+ N_p \left( \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p} \chi \left( \bigcup_{i=k_{l,j-1}}^{k_{l,j-1}-1} T^{-i}C_l \right) \right)
\]

(3.15)

\[
\leq \kappa_p \left\| \frac{1}{L_l} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p} \chi \left( \bigcup_{i=0}^{k_{l,j-1}-1} T^{-i}C_l \right) \right\|_{p,\infty} + 
\]

(3.16)

\[
+ \kappa_p \left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left( \frac{n_l}{k_{l,j-1}} \right)^{1/p} \chi \left( \bigcup_{i=k_{l,j-1}}^{k_{l,j-1}-1} T^{-i}C_l \right) \right\|_{p,\infty}
\]

(3.17)

\[
\leq \frac{1}{L_l} \kappa_p \left( 1 + 2^{1/p} \right)
\]

(3.18)

We thus define $\kappa'_p := \kappa_p \left( 1 + 2^{1/p} \right)$. 7
We fix $\varepsilon > 0$; using (3.14), we can find an integer $l_0$ such that $\sum_{l > l_0} 1/L_l < \varepsilon$. Since the function $\sum_{l=1}^{l_0} g_{f_l}$ is bounded, we have,

\[(3.19) \quad \limsup_{t \to \infty} t^p \mu \{ |m| > t \} \leq \limsup_{t \to \infty} t^p \mu \left\{ \left| \sum_{l=1}^{l_0} g_{f_l} \right| > \frac{t}{2} \right\} + 2^p \left\| \sum_{l > l_0} g_{f_l} \right\|_{p,\infty}^p \]

\[(3.20) \quad = 2^p \left\| \sum_{l > l_0} g_{f_l} \right\|_{p,\infty}^p \]

\[(3.21) \quad \leq \left( 2 \sum_{l > l_0} N_p(f_l) \right)^p \]

\[(3.22) \quad \leq \kappa_p' \left( \sum_{l > l_0} \frac{1}{L_l} \right)^p \]

\[(3.23) \quad \leq \kappa_p' \varepsilon^p, \]

where the second inequality comes from inequalities (1.10). Since $\varepsilon$ is arbitrary, the proof of Lemma 3.2 is complete. \hfill $\Box$

We denote by $\mathcal{M}$ the $\sigma$-algebra generated by $C$ and the random variables $g \circ T^k$, $k \leq 0$. It satisfies $\mathcal{M} \subset T^{-1} \mathcal{M}$.

**Proposition 3.3.** The sequence $(m \circ T^i)_{i \geq 0}$ is a (stationary) martingale difference sequence with respect to the filtration $(T^{-1} \mathcal{M})_{i \geq 0}$.

**Proof.** We have to show that $\mathbb{E}[m \mid \mathcal{M}] = 0$. Since the $\sigma$-algebra $C$ is $T$-invariant, we have $T\mathcal{M} = \sigma(C \cup \sigma(g \circ T^k, k \leq -1))$. This implies

\[(3.24) \quad \mathbb{E}[m \mid \mathcal{M}] = \mathbb{E}[g f \mid \mathcal{M}] = f \cdot \mathbb{E}[g \mid \mathcal{M}]. \]

Since $g$ is centered and independent of $T\mathcal{M}$, Proposition 3.3 is proved. \hfill $\Box$

It remains to prove that the process $(n^{-1/2} S_n^h (m))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2 - 1/p}^0 [0, 1]$.

**Proposition 3.4.** Under conditions (3.2), (3.3) and (3.4), there exists an integer $l_0$ such that for $l \geq l_0$

\[(3.25) \quad P_l := \mu \left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l} \frac{|S_u + v(g_{f_l}) - S_u(g_{f_l})|}{v^{1/2 - 1/p}} \geq 1 \right\} \geq \frac{1}{16}. \]

**Proof.** Let us fix an integer $l \geq 1$. Assume that $\omega \in C_l$. Then we have

\[(3.26) \quad (f_i \circ T^i)(\omega) = \begin{cases} \frac{1}{L_l} \left( \frac{n_l}{L_l} \right)^{1/p}, & \text{if } 0 \leq i < k_l; \\ \frac{1}{L_l} \left( \frac{n_l}{L_l} \right)^{1/p}, & \text{if } k_l \leq i < k_{l-1}, \text{ and } 1 \leq j \leq J_l; \\ 0, & \text{if } k_{l-1} \leq i \leq n_l - 1. \end{cases} \]
As a consequence,

\[(3.27) \quad C_l \cap \left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1 - k_l,j)^{1/2 - 1/p}} \geq 1 \right\} \]

\[= C_l \cap \left\{ \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1 - k_l,j)^{1/2 - 1/p}} \geq L_l \right\} \]

Since for a fixed \(s \in \{0, \ldots, n_l - k_l\}\), the inequality

\[(3.28) \quad \chi(T^{-s} C_l) \cdot \max_{1 \leq j \leq j_l} \frac{|S_{s+k_l,j-1} - S_{s+k_l,j}|}{(k_l,j-1 - k_l,j)^{1/2 - 1/p}} \]

\[= \chi(T^{-s}(C_l)) \max_{1 \leq u \leq n_l - k_l} \frac{|S_{u+k_l,j-1} - S_{u,j}|}{(u \cdot k_l,j)^{1/2 - 1/p}} \]

takes place and the sets \((T^{-s} C_l)_{s=0}^{n_l-1}\) are pairwise disjoint, we obtain the lower bound

\[(3.29) \quad P_l \geq \sum_{s=0}^{n_l-k_l} \mu \left( T^{-s}(C_l) \cap \left\{ \max_{1 \leq j \leq j_l} \frac{|S_{s+k_l,j-1} - S_{s+k_l,j}|}{(k_l,j-1 - k_l,j)^{1/2 - 1/p}} \geq 1 \right\} \right). \]

Using the fact that \(T\) is measure-preserving, this becomes

\[(3.30) \quad P_l \geq (n_l - k_l) \cdot \mu \left( C_l \cap \left\{ \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1 - k_l,j)^{1/2 - 1/p}} \geq L_l \right\} \right), \]

and plugging (3.29) in the previous estimate, we get

\[(3.31) \quad P_l \geq (n_l - k_l) \mu \left( C_l \cap \left\{ \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1 - k_l,j)^{1/2 - 1/p}} \geq L_l \right\} \right). \]

The sets \(\left\{ \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1)^{1/2 - 1/p}} \geq L_l \right\}\) and \(C_l\) belong to the independent sub-\(\sigma\)-algebras \(\mathcal{B}\) and \(\mathcal{C}\) respectively, hence

\[(3.32) \quad P_l \geq (n_l - k_l) \mu \left( C_l \cap \left\{ \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1)^{1/2 - 1/p}} \geq L_l \right\} \right). \]

By construction, we have \(n_l \cdot \mu(C_l) = \mu (\bigcup_{s=0}^{n_l-1} T^{-s} C_l) > 1/2\), hence

\[(3.33) \quad P_l \geq \frac{1}{2} \left( 1 - \frac{k_l}{n_l} \right) \mu \left( \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1)^{1/2 - 1/p}} \geq L_l \right). \]

It remains to find a lower bound for

\[(3.34) \quad P'_l := \mu \left( \max_{1 \leq j \leq j_l} \frac{|S_{k_l,j-1} - S_{k_l,j}|}{(k_l,j-1)^{1/2 - 1/p}} \geq L_l \right). \]
Let us define the set

\[(3.35) \quad E_j := \left\{ \left| S_{k_{l,j-1}}(g) - S_{k_{l,j}}(g) \right| \geq L_l \right\} \]

Since the sequence \((g \circ T^i)_{i \geq 0}\) is independent, the family \((E_j)_{1 \leq j \leq J_l}\) is independent, hence

\[(3.36) \quad P'_l \geq 1 - \prod_{j=1}^{J_l} (1 - \mu(E_j)). \]

We define the quantity

\[(3.37) \quad c_j := \mu \left\{ |N| \geq \frac{L_l}{\|g\|_2} \left( \frac{k_{l,j-1}}{k_{l,j} - 1} \right)^{1/p} \right\} \]

(we recall that \(N\) denotes a standard normally distributed random variable). By the Berry-Essen theorem, we have for each \(j \in \{1, \ldots, J_l\}\),

\[(3.38) \quad |\mu(E_j) - c_j| \leq \frac{1}{\|g\|_2^2 (k_{l,j-1} - 1)^{1/2}} \leq \frac{\sqrt{2}}{\|g\|_2^3} 2^{-l/2}. \]

Plugging the estimate \((3.38)\) into \((3.36)\) and noticing that for an integer \(N\) and \((a_n)_{n=1}^N, (b_n)_{n=1}^N\) two families of numbers in the unit interval,

\[(3.39) \quad \left| \prod_{n=1}^{N} a_n - \prod_{n=1}^{N} b_n \right| \leq \sum_{n=1}^{N} |a_n - b_n| , \]

we obtain

\[(3.40) \quad P'_l \geq 1 - \prod_{j=1}^{J_l} (1 - \mu(E_j)) + \prod_{j=1}^{J_l} (1 - c_j) - \prod_{j=1}^{J_l} (1 - c_j) \]

\[(3.41) \quad \geq 1 - \prod_{j=1}^{J_l} (1 - c_j) - \sum_{j=1}^{J_l} |\mu(E_j) - c_j| \]

\[(3.42) \quad \geq 1 - \prod_{j=1}^{J_l} (1 - c_j) - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-l/2}. \]

Notice that

\[(3.43) \quad 1 - \prod_{j=1}^{J_l} (1 - c_j) \geq 1 - \max_{1 \leq j \leq J_l} (1 - c_j)^{J_l} \]

and \(c_j \geq \mu \left\{ |N| \geq 2^{1/p} \frac{L_l}{\|g\|_2} \right\} \) for \(1 \leq j \leq J_l\). We thus have

\[(3.44) \quad P'_l \geq 1 - \left( 1 - \mu \left\{ |N| \geq 2^{1/p} \frac{L_l}{\|g\|_2} \right\} \right)^{J_l} - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-l/2}. \]

Using the elementary inequality

\[(3.45) \quad 1 - (1 - t)^n \geq nt - \frac{n(n-1)}{2} t^2 \]
valid for a positive integer \( n \) and \( t \in [0, 1] \), we obtain

\[
(3.46) \quad P_t' \geq J_t \mu \left\{ |\lambda| \geq 2^{1/p} \frac{L_t}{\|g\|_2} \right\} \geq J_t^2 \left( \mu \left\{ |\lambda| \geq 2^{1/p} \frac{L_t}{\|g\|_2} \right\} \right)^2 - J_t \frac{\sqrt{2}}{\|g\|_2}^{2 - l_i/2}.
\]

By conditions (3.3) and (3.4), there exists an integer \( l'_0 \) such that if \( l \geq l'_0 \), then

\[
(3.47) \quad \mu \left\{ \max_{1 \leq j \leq d_l} \left| S_{k_{l,j} - 1}(g) - S_{k_{l,j}}(g) \right| \geq L_4 \right\} \geq \frac{1}{4}.
\]

Combining (3.33) with (3.37), we obtain for \( l \geq l'_0 \)

\[
(3.48) \quad P_l \geq \frac{1}{8} \left( 1 - \frac{k_l}{n_l} \right).
\]

By condition (3.2), we thus get that \( P_l \geq 1/16 \) for \( l \geq l_0 \), where \( l_0 \geq l'_0 \) and \( k_l/n_l \leq 1/2 \) if \( l \geq l_0 \).

This concludes the proof of Proposition 3.4.

\[ \square \]

**Proposition 3.5.** Under conditions (3.1), (3.2), (3.3), (3.4) and (3.5), we have for \( l \) large enough

\[
(3.49) \quad \mu \left\{ \frac{1}{n_i^{1/p}} \max_{1 \leq u \leq n_i - k_l} \left| S_{u + v}(m) - S_u(m) \right| \geq \frac{1}{2} \right\} \geq \frac{1}{32}.
\]

Since the Hölder modulus of continuity of a piecewise linear function is reached at vertices, we derive the following corollary.

**Corollary 3.6.** If \( l \geq l_0 \), then

\[
(3.50) \quad \mu \left\{ \omega_{1/2 - 1/p} \left( \frac{1}{\sqrt{n_i}} S_{n_i}^p(m), \frac{k_l}{n_l} \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32}.
\]

Therefore, for each positive \( \delta \), we have

\[
(3.51) \quad \lim_{n \to \infty} \mu \left\{ \omega_{1/2 - 1/p} \left( \frac{1}{\sqrt{n}} S_n^p(m), \delta \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32},
\]

and the process \( (n^{-1/2} S_n^p(m))_{n \geq 1} \) is not tight in \( H^0_{1/2 - 1/p}([0, 1]) \).

**Proof of Proposition 3.5.** Let \( l_0 \) be the integer given by Proposition 3.4 and let \( l \geq l_0 \). We define \( m'_l := \sum_{i=1}^{l-1} g f_i \) and \( m''_l := \sum_{i=l+1}^{\infty} g f_i \).

We define for \( i \geq 1 \),

\[
(3.52) \quad M_{l,i} := \frac{1}{n_i^{1/p}} \max_{1 \leq u \leq n_i - k_l} \left| S_{u + v}(g f_i) - S_u(g f_i) \right|.
\]

Let \( i \) be an integer such that \( i < l \). Notice that for \( 1 \leq u \leq n_i - k_l \) and \( v \leq k_l \), we have

\[
(3.53) \quad \left| S_{u + v}(g f_i) - S_u(g f_i) \right| = U^u(|S_v(g f_i)|),
\]

where \( U(h)(\omega) = h(T(\omega)) \) and since

\[
(3.54) \quad |S_v(g f_i)| \leq v \|g f_i\|_\infty \leq \frac{k_l}{L_i} \left( \frac{n_i}{2n} \right)^{1/p},
\]

valid for a positive integer \( n \) and \( t \in [0, 1] \), we obtain
the estimate

$$M_{i,j} \leq \frac{k_i}{L_n l_{n_i}^{1/p}} \left( \frac{n_i}{2^l_n} \right)^{1/p}$$

holds. Since

$$\frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l \leq v \leq k_l} |S_{u+v}(m'_i) - S_u(m'_i)| \leq \sum_{i=1}^{l-1} k_i \left( \frac{n_i}{2^l_n} \right)^{1/p}.$$ 

we have by (3.55),

$$\frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l \leq v \leq k_l} |S_{u+v}(m'_i) - S_u(m'_i)| \leq \sum_{i=1}^{l-1} k_i \left( \frac{n_i}{2^l_n} \right)^{1/p}.$$ 

By (3.5), the following bound takes place:

$$\frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l \leq v \leq k_l} |S_{u+v}(m'_i) - S_u(m'_i)| \leq \frac{1}{2}.$$ 

The following set inclusions hold

$$\left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l \leq v \leq k_l} |S_{u+v}(m'_i) - S_u(m'_i)| \neq 0 \right\} \subset \bigcup_{i>l} \{ M_{i,j} \neq 0 \}$$

$$\subset \bigcup_{i>l} \bigcup_{u=1}^{n_l} \{ U^u(gf_i) \neq 0 \}.$$ 

We thus have

$$\mu \left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l \leq v \leq k_l} |S_{u+v}(m'_i) - S_u(m'_i)| \neq 0 \right\} \leq \sum_{i>l} n_l \mu \{ gf_i \neq 0 \}$$

$$\leq n_l \sum_{i>l} \mu \{ f_i \neq 0 \}$$

$$= n_l \sum_{i>l} (k_i + 1) \mu(C_i)$$

$$\leq 2n_l \sum_{i>l} \frac{k_i}{n_i}.$$ 

and by (3.2), it follows that

$$\mu \left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq u \leq n_l - k_l \leq v \leq k_l} |S_{u+v}(m''_i) - S_u(m''_i)| \neq 0 \right\} \leq \frac{1}{32}.$$
Accounting (3.98), we thus have

\[
(3.66) \quad \mu \left\{ \frac{1}{n_t^{1/p}} \max_{1 \leq u \leq n_t-k_t} \left| S_{u+v}(m) - S_u(m) \right| \geq \frac{1}{2} \right\} \\
\quad \geq \mu \left\{ \frac{1}{n_t^{1/p}} \max_{1 \leq u \leq n_t-k_t} \left| S_{u+v}(g_{f_1} + m''_u) - S_u(g_{f_1} + m''_u) \right| \geq 1 \right\} \\
\quad \geq \mu \left\{ \frac{1}{n_t^{1/p}} \max_{1 \leq u \leq n_t-k_t} \left| S_{u+v}(g_{f_1}) - S_u(g_{f_1}) \right| \geq 1 \right\} \\
\quad \quad - \mu \left\{ \frac{1}{n_t^{1/p}} \max_{1 \leq u \leq n_t-k_t} \left| S_{u+v}(m''_u) - S_u(m''_u) \right| \neq 0 \right\},
\]

hence combining Proposition 3.4 with (3.65), we obtain the conclusion of Proposition 3.3.

Theorem 2.1 follows from Corollary 3.6 and Propositions 3.2 and 3.3.

3.2. Proof of Theorem 2.2 and Proposition 2.3.

Proof of Proposition 2.3. Let us fix a positive \( t \). We have to show that for some constant \( C \) depending only on \( p \) and each integer \( n \geq 1 \),

\[
(3.67) \quad P(n, t) := t^p \mu \left\{ \sup_{j \geq 1} 2^{\alpha_j} n^{-1/2} \max_{1 \leq k < 2^j} \left| S^p_n(m, r_{k+1,j}) - S^p_n(m, r_{k,j}) \right| > t \right\} \leq \\
\quad \quad \leq C \left( \| m \|_{p, \infty}^p + \mathbb{E} [m^2 \mid T \mathcal{M}] \right)^{p/2}
\]

(handling the differences \( |S^p_n(m, r_{k,j}) - S^p_n(m, r_{k-1,j})| \) is completely similar, hence omitted).

In the proof, we shall denote by \( C_p \) a constant depending only on \( p \) which may change from line to line.

We define

\[
(3.68) \quad P_1(n, t) := \mu \left\{ \sup_{1 \leq j \leq \log n} 2^{\alpha_j} n^{-1/2} \max_{1 \leq k < 2^j} \left| S^p_n(m, r_{k+1,j}) - S^p_n(m, r_{k,j}) \right| > t \right\}, \text{ and}
\]

\[
(3.69) \quad P_2(n, t) := \mu \left\{ \sup_{j > \log n} 2^{\alpha_j} n^{-1/2} \max_{1 \leq k < 2^j} \left| S^p_n(m, r_{k+1,j}) - S^p_n(m, r_{k,j}) \right| > t \right\},
\]

hence

\[
(3.70) \quad P(n, t) \leq t^p P_1(n, t/2) + t^p P_2(n, t/2).
\]

We estimate \( P_2(n, t) \). For \( j > \log n \), we have the inequality

\[
(3.71) \quad r_{k+1,j} - r_{k,j} = (k+1)2^{-j} - k2^{-j} = 2^{-j} < 1/n,
\]

hence if \( r_{k,j} \) belongs to the interval \([l/n, (l+1)/n)\) for some \( l \in \{0, \ldots, n-1\} \), then
• either \( r_{k+1,j} \in \left[ l/n, (l+1)/n \right) \), and in this case,

\[
(3.72) \quad \left| S_n^{pl}(m, r_{k+1,j}) - S_n^{pl}(m, r_{k,j}) \right| = \left| m \circ T_{l+1} \right| 2^{-j} n \leq 2^{-j} n \max_{1 \leq l \leq n} \left| U_{l}^t(m) \right| ;
\]

• or \( r_{k+1,j} \) belongs to the interval \( [(l+1)/n, (l+2)/n) \). The estimates

\[
(3.73) \quad \left| S_n^{pl}(m, r_{k+1,j}) - S_n^{pl}(m, r_{k,j}) \right| \leq \left| S_n^{pl}(m, r_{k+1,j}) - S_n^{pl}(m, (l+1)/n) \right| + \\
+ \left| S_n^{pl}(m, (l+1)/n) - S_n^{pl}(m, r_{k,j}) \right| \leq 2^{-j} n \max_{1 \leq l \leq n} \left| U_{l}^t(m) \right|
\]

hold.

Considering these two cases, we obtain

\[
(3.74) \quad P_2(n, t) \leq \mu \left\{ \sup_{j > \log n} 2^{\alpha_j} n^{2^{-j} n^{-1/2}} \max_{1 \leq l \leq n} \left| U_{l}^t(m) \right| > t \right\}
\]

\[
(3.75) \quad \leq \mu \left\{ 2n^{\alpha - 1/2} \max_{1 \leq l \leq n} \left| U_{l}^t(m) \right| > t \right\}
\]

\[
(3.76) \quad \leq n \mu \left\{ 2n^{-1/p} \left| m \right| > t \right\}
\]

\[
(3.77) \quad \leq \frac{2p}{t^p} \sup_{x > 0} x^p \mu \left\{ \left| m \right| > x \right\}.
\]

Therefore, establishing inequality (3.67) reduces to find a constant \( C \) depending only on \( p \) such that

\[
(3.78) \quad \sup_n \sup_t t^p P_1(n, t) \leq C \left( \| m \|^p_{p, \infty} + E \left( E[m^2 \mid T, M] \right)^{p/2} \right)
\]

We define \( u_{k,j} := \lfloor nr_{k,j} \rfloor \) for \( k < 2^j \) and \( j \geq 1 \) (see Notation 1.1).

Notice that the inequalities

\[
(3.79) \quad \left| S_{u_{k,j}}(m) - S_n^{pl}(m, r_{k,j}) \right| \leq \left| U^{u_{k,j}+1}(m) \right| \quad \text{and}
\]

\[
(3.80) \quad \left| S_n^{pl}(m, r_{k+1,j}) - S_{u_{k+1,j}}(m) \right| \leq \left| U^{u_{k+1,j}+1}(m) \right|
\]

take place because if \( j < \log n \), then

\[
(3.81) \quad u_{k,j} \leq nr_{k,j} \leq u_{k,j} + 1 \leq u_{k+1,j} \leq nr_{k+1,j} \leq u_{k+1,j} + 1.
\]

Therefore, \( P_1(n, t) \leq P_{1,1}(n, t) + P_{1,2}(n, t) \), where

\[
(3.82) \quad P_{1,1}(n, t) := \mu \left\{ \max_{1 \leq j \leq \log n} 2^{\alpha_j} n^{-1/2} \max_{1 \leq k < 2^j} \left| S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m) \right| > t/2 \right\},
\]

\[
(3.83) \quad P_{1,2}(n, t) := \mu \left\{ \max_{1 \leq j \leq \log n} 2^{\alpha_j} n^{-1/2} \max_{1 \leq l \leq n} \left| U_{l}^t(m) \right| > t/4 \right\}.
\]

Notice that

\[
(3.84) \quad P_{1,2}(n, t) \leq \mu \left\{ n^{\alpha - 1/2} \max_{1 \leq l \leq n} \left| U_{l}^t(m) \right| > t/4 \right\}
\]

\[
(3.85) \quad \leq n \mu \left\{ \left| m \right| > n^{1/p} t/4 \right\}
\]

\[
(3.86) \quad \leq 4p t^{-p} \sup_{x > 0} x^p \mu \left\{ \left| m \right| > x \right\},
\]

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We define for $P$ such that

$$\sup_n \sup_t P_{1,1}(n, t) \leq C \left( \|m\|_{p, \infty}^p + \mathbb{E} [m^2 | T\mathcal{M}] \right)^{p/2}.$$  

We estimate $P_{1,1}(n, t)$ in the following way:

$$P_{1,1}(n, t) \leq \sum_{j=1}^{\log n} 2^j \max_{1 \leq k < 2^j} \mu \left\{ |S_{u_{k+1}, j}(m) - S_{u_{k}, j}(m)| > tn^{1/2}2^{-1-\alpha j} \right\}$$

We define for $1 \leq j \leq \log n$ and $0 \leq k < 2^j$ the quantity

$$P(n, j, k, t) := \mu \left\{ |S_{u_{k+1}, j}(m) - S_{u_{k}, j}(m)| > tn^{1/2}2^{-1-\alpha j} \right\}.$$ 

If $(f \circ T^j)_{j \geq 0}$ is a strictly stationary sequence, we define

$$Q_{f,n}(u) : = \mu \left\{ \max_{1 \leq j \leq n} |f \circ T^j| > u \right\} + \mu \left\{ \left( \sum_{i=1}^{n} U^i\mathbb{E}[f^2 | T\mathcal{M}] \right)^{1/2} > u \right\}.$$ 

The following inequality is Theorem 1 of \cite{Nag03}. It allows us to express the tail function of a martingale by that of the increments and the quadratic variance.

**Theorem 3.7.** Let $m$ be an $\mathcal{M}$-measurable function such that $\mathbb{E}[m | T\mathcal{M}] = 0$. Then for each positive $y$ and each integer $n$,

$$\mu \left\{ |S_n(m)| > y \right\} \leq c(q, \eta) \int_0^1 Q_{m,n}(\varepsilon_q u \cdot y) u^{q-1} du,$$

where $q > 0$, $\eta > 0$, $\varepsilon_q := \eta/q$ and $c(q, \eta) := q \exp(3q \eta^{q+1} - \eta - 1)/\eta$.

We shall use (3.91) with $q := p + 1$, $\eta = 1$ and $y := n^{1/2}2^{-1-\alpha j} t$ in order to estimate $P(n, j, k, t)$:

$$P(n, j, k, t) \leq C_p \int_0^1 \mu \left\{ \max_{1 \leq i \leq u_{k+1}, j - u_{k}, j} |U^i(m)| > n^{1/2}2^{-1-\alpha j} tu_{\varepsilon_p + 1} \right\} u^p du$$

$$+ C_p \int_0^1 \mu \left\{ \left( \sum_{i=1}^{u_{k+1}, j} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2}2^{-1-\alpha j} tu_{\varepsilon_p + 1} \right\} u^p du.$$ 

Exploiting the inequality $u_{k+1, j} - u_{k, j} \leq 2n2^{-j}$, we get from the previous bound

$$P(n, j, k, t) \leq C_p \int_0^1 \mu \left\{ \max_{1 \leq i \leq 2n2^{-j}} |U^i(m)| > n^{1/2}2^{-1-\alpha j} tu_{\varepsilon_p + 1} \right\} u^p du$$

$$+ C_p \int_0^1 \mu \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2}2^{-1-\alpha j} tu_{\varepsilon_p + 1} \right\} u^p du.$$ 

We define for $j \leq \log n$, $t \geq 0$ and $u \in (0, 1)$,

$$P'(n, j, t, u) := \mu \left\{ \max_{1 \leq i \leq 2n2^{-j}} |U^i(m)| > n^{1/2}2^{-1-\alpha j} tu_{\varepsilon_p + 1} \right\},$$ 

and
\[
(3.95) \quad P''(n, j, t, u) := \mu \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T,M]) \right)^{1/2} > n^{1/2}2^{-1-\alpha_j tu \varepsilon_{p+1}} \right\}.
\]

Using the fact that the random variables \(U^i(m), 1 \leq i \leq 2n2^{-j}\) are identically distributed, we derive the bound
\[
(3.96) \quad P'(n, j, t, u) \leq 2n2^{-j}\mu \left\{ |m| > n^{1/2}2^{-1-\alpha_j tu \varepsilon_{p+1}} \right\},
\]
hence
\[
(3.97) \quad P'(n, j, t, u) \leq 2n2^{-j}(n^{1/2}2^{-1-\alpha_j tu \varepsilon_{p+1}})^{-p} \left\| m \right\|_{p,\infty}^p = 2^{p+1}n^{1-p/2}2^{j(-1+p\alpha)}t^{-pu-p} \left\| m \right\|_{p,\infty}^p.
\]
Since \(\alpha\) and \(p\) are linked by the relationship \(1/2 - 1/p = \alpha\), we have \(p\alpha = p/2 - 1\) hence
\[
(3.98) \quad \int_0^1 P'(n, j, t, u)u^p du \leq C_p t^{-p}n^{1-p/2}2^{j(p/2-2)} \left\| m \right\|_{p,\infty}^p.
\]
Notice the following set equalities:
\[
(3.99) \quad \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T,M]) \right)^{1/2} > \varepsilon_{p+1}un^{1/2}2^{-1-\alpha_j t} \right\}
\]
\[
= \left\{ \frac{1}{2n2^{-j}} \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T,M]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{j/p} t^2 \right\}
\]
and that \(n2^{-j} \geq 1\) (because \(j \leq \log n\)), hence
\[
(3.100) \quad \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T,M]) \right)^{1/2} > \varepsilon_{p+1}un^{1/2}2^{-1-\alpha_j t} \right\} \subseteq
\]
\[
\subseteq \bigcup_{N \geq 2} \left\{ \frac{1}{N} \sum_{i=1}^{N} U^i(\mathbb{E}[m^2 | T,M]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{j/p} t^2 \right\},
\]
from which it follows
\[
(3.101) \quad \mu \left\{ \left( \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T,M]) \right)^{1/2} > \varepsilon_{p+1}un^{1/2}2^{-1-\alpha_j t} \right\} \leq
\]
\[
\leq \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^{N} U^i(\mathbb{E}[m^2 | T,M]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{j/p} t^2 \right\}.
\]
Combining (3.98) and (3.101), we obtain
\[
(3.102) \quad \max_{1 \leq k \leq 2^j} \int_0^1 P(n, j, k, t) \leq C_p t^{-p}n^{1-p/2}2^{j(p/2-2)} \left\| m \right\|_{p,\infty}^p
\]
\[
+ C_p \int_0^1 \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^{N} U^i(\mathbb{E}[m^2 | T,M]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{j/p} t^2 \right\} u^p du,
\]
hence by (3.88) and (3.89),

\[(3.103)\quad P_1(n, t) \leq C_p \epsilon_p \|m\|_{p, \infty}^p + \sum_{j=1}^{\log n} 2^j 2^{j(p/2 - 2)} n^{1 - p/2} \]

\[+ C_p \int_0^1 \sum_{j=1}^{\log n} 2^j \mu \left( \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^{N} U^j(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} 2^{-p/2} u^2 \right) u^p du. \]

From the elementary bounds

\[(3.104)\quad \sum_{j=1}^{\log n} 2^j 2^j(1 - p/2) n^{1 - p/2} \leq (1 - 2^{1 - p/2})^{-1} \]

\[(3.105)\quad \sum_{j=1}^{\log n} 2^j \mu \left( |g| > 2^{2j/p} \right) \leq C_p \mathbb{E} |g|^{p/2}, \quad \text{for any non-negative function } g, \]

with

\[(3.106)\quad g := 2^{j-2} 2^{-p/2} u^{-2} \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^{N} U^j(\mathbb{E}[m^2 | T\mathcal{M}]), u \in (0, 1) \]

we obtain

\[(3.107)\quad P_1(n, t) \leq C_p \epsilon_p \|m\|_{p, \infty}^p + C_p \epsilon_p \mathbb{E}(\mathbb{E}[m^2 | T\mathcal{M}])^{p/2}. \]

As the Koopman operator $U$ is an $L^1$-$L^\infty$ contraction, Theorem 1 of [Ste61] gives the existence of a constant $A_p$ such that for each $h \in L^p$,

\[(3.108)\quad \left\| \sup_{N \geq 1} \frac{1}{N} \sum_{j=1}^{N} U^j(h) \right\|_{p/2} \leq A_p \|h\|_{p/2}. \]

Applying (3.108) with $h := \mathbb{E}[m^2 | T\mathcal{M}]$, we get by (3.107)

\[(3.109)\quad P_1(n, t) \leq C_p \epsilon_p \|m\|_{p, \infty}^p + C_p \epsilon_p \mathbb{E}(\mathbb{E}[m^2 | T\mathcal{M}])^{p/2}, \]

which establishes (3.78). This concludes the proof of Proposition 2.3. \hfill \Box

**Proof of Theorem 2.2.** We deduce Theorem 2.2 from Proposition 2.3 by a truncation argument. For a fixed $R$, we define

\[(3.110)\quad m_R := m \left\{ |m| \leq R \right\} - \mathbb{E}[m \left\{ |m| \leq R \right\} | T\mathcal{M}] \quad \text{and} \quad \]

\[(3.111)\quad m_R' := m \left\{ |m| > R \right\} - \mathbb{E}[m \left\{ |m| > R \right\} | T\mathcal{M}]. \]

In this way, the sequences $(m_R \circ T^i)_{i \geq 0}$ and $(m_R' \circ T^i)_{i \geq 0}$ are martingale differences sequences and $m = m_R + m_R'$. Since $|m_R| \leq 2R$ and $(m_R \circ T^i)_{i \geq 0}$ is a martingale difference sequence, the sequence $(n^{-1/2} S_n^R(m_R))_{n \geq 1}$ is tight in $H_{1/2 - 1/p} [0, 1]$. Consequently, for each positive $\epsilon$, the following convergence takes place:

\[(3.112)\quad \lim_{J \to \infty} \lim_{n \to \infty} \sup_{j \geq J} \mu \left\{ \sup_{r \in D_j} 2^{\alpha_j} \max_{r \in D_j} \lambda_r \left( S_n^R(m_R) \right) > \epsilon n^{1/2} \right\} = 0. \]
Using Proposition 2.3, we derive the following bound, valid for each $\varepsilon$ and each $R$,

$$
\lim_{j \to \infty} \limsup_{n \to \infty} \mu \left\{ \sup_{j \geq J} \max_{r \in \mathcal{L}_j} |\lambda_r \left( S_n^{pl}(m) \right) | > \varepsilon n^{1/2} \right\} \leq 
\leq C_p \varepsilon^{-p} \left( \sup_{t \geq 0} t^p \mu \{ |m| \chi \{ |m| > R \} > t \} + \sup_{t \geq 0} t^p \mu \{ \mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] > t \} \right) + \varepsilon^{-p} C_p \mathbb{E} \left( (\mathbb{E}[m^2 \chi \{ |m| > R \} \mid T\mathcal{M}])^{p/2} \right).
$$

The first term is $\sup_{t \geq R} t^p \mu \{ |m| > t \}$, which goes to 0 as $R$ goes to infinity.

The second term can be bounded by $\sup_{t \geq R} t^p \mu \{ \mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] > t \}$. Indeed, if $t \geq R$, we use the inclusion

$$
\{ \mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] > t \} \subset \{ \mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] > t \},
$$

and if $t < R$, then accounting the fact that the random variable $\mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}]$ is greater than $R$, we get

$$
\mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] = \mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] \chi \{ \mathbb{E}[|m| \mid T\mathcal{M}] > R \} \leq \mathbb{E}[|m| \mid T\mathcal{M}] \chi \{ \mathbb{E}[|m| \mid T\mathcal{M}] > R \},
$$

from which it follows that

$$
t^p \mu \{ \mathbb{E}[|m| \chi \{ |m| > R \} \mid T\mathcal{M}] > t \} \leq R^p \mu \{ \mathbb{E}[|m| \mid T\mathcal{M}] > R \}.
$$

By Lemma 3.3, the convergence

$$
\lim_{R \to \infty} \sup_{t \geq R} t^p \mu \{ \mathbb{E}[|m| \mid T\mathcal{M}] > t \} = 0
$$

takes place.

The third term of (3.113) converges to 0 as $R$ goes to infinity by monotone convergence.

This concludes the proof of Theorem 2.2.$\square$

3.3. Proof of Theorem 2.6. By (2.7), the equality $f = \sum_{i \geq 0} P_i(f)$ holds almost surely. For a fixed integer $K$, we define $f_K := \sum_{i=0}^{K} P_i(f)$. Then $f_K$ satisfies the conditions of Corollary 2.5.

Indeed, we have the equalities

$$
P_i(f) - P_0(U^i f) = \mathbb{E}[f \mid T^i \mathcal{M}] - \mathbb{E}[U^i f \mid \mathcal{M}] - \mathbb{E}[f \mid T^{i+1} \mathcal{M}] + \mathbb{E}[U^i f \mid T^{i+1} \mathcal{M}]
$$

and the latter term can be expressed as a coboundary noticing that $(I - U^i) = (I - U) \sum_{k=0}^{i-1} U^k$. Since $P_i(f)$ belongs to the $L^p$ space, we may write $f_K = \sum_{i=0}^{K} P_0(U^i f)$ as $(I - U)g_K$ where $g_K$ belongs to the $L^p$ space. Defining $m_K := \sum_{i=0}^{K} P_0(U^i f)$, the sequence $(m_K \circ T^i)_{i \geq 0}$ is a martingale difference sequence hence for each positive $\varepsilon$,

$$
\lim_{j \to \infty} \limsup_{n \to \infty} \mu \left\{ \sup_{j \geq J} \max_{r \in \mathcal{L}_j} |\lambda_r \left( S_n^{pl}(f_K) \right) | > \varepsilon n^{1/2} \right\} = 0.
$$
Now, we have shown that the convergence in (3.120) holds if \( f_K \) is replaced by \( f - f_K \). To this aim, we use the inclusion

\[
(3.121) \quad \left\{ \sup_{j \geq J} \max_{r \in D_j} \lambda_r \left( S_n^1(f - f_K) \right) > \varepsilon n^{1/2} \right\} \subseteq \\
\subseteq \left\{ \sup_{j \geq 1} \max_{r \in D_j} \lambda_r \left( S_n^1(f - f_K) \right) > \varepsilon n^{1/2} \right\},
\]

hence

\[
(3.122) \quad \mu \left\{ \sup_{j \geq J} \max_{r \in D_j} \lambda_r \left( S_n^1(f - f_K) \right) > \varepsilon n^{1/2} \right\} \leq \varepsilon^{-p} \left\| \frac{1}{\sqrt{n}} S_n^1(f - f_K) \right\|_{\mathcal{H}_{1/2-1/p}^0}^p,
\]

from which it follows that

\[
(3.123) \quad \mu \left\{ \sup_{j \geq J} \max_{r \in D_j} \lambda_r \left( S_n^1(f - f_K) \right) > \varepsilon n^{1/2} \right\} \leq \varepsilon^{-p} \left( \sum_{i \geq K+1} \left\| \frac{1}{\sqrt{n}} S_n^1(P_i(f)) \right\|_{\mathcal{H}_{1/2-1/p}^0} \right)^p.
\]

Notice that for a fixed \( i \), the sequence \((U^i(P_i(f)))_{i \geq 1}\) is a martingale difference sequence (with respect to the filtration \((T^{-1-i}(\mathcal{M}))_{i \geq 0}\)). Therefore, by Proposition 2.3, we obtain

\[
(3.124) \quad \mu \left\{ \sup_{j \geq J} \max_{r \in D_j} \lambda_r \left( S_n^1(f - f_K) \right) > \varepsilon n^{1/2} \right\} \leq \varepsilon^{-p} \left( \sum_{i \geq K+1} \left\| P_i(f) \right\|_{\mathcal{H}_{1/2-1/p}^0} \right)^p.
\]

Plugging this estimate into (3.124), we obtain that for some constant \( C \) depending only on \( p \),

\[
(3.125) \quad \left\| \frac{1}{\sqrt{n}} S_n^1 (P_i(f)) \right\|_{\mathcal{H}_{1/2-1/p}^0} \leq C_p \left\| P_i(f) \right\|_p.
\]

Combining (3.120) and (3.126), we obtain for each \( K \):

\[
(3.127) \quad \lim_{j \to \infty} \limsup_{n \to \infty} \mu \left\{ \sup_{j \geq J} \max_{r \in D_j} \lambda_r \left( S_n^1(f) \right) > n^{1/2} \varepsilon \right\} \leq \\
\leq C \varepsilon^{-p} \left( \sum_{i \geq K+1} \left\| P_i(f) \right\|_p \right)^p.
\]

Since \( K \) is arbitrary, we conclude the proof of Theorem 2.6 thanks to assumption (2.8).
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References

[Cie60] Z. Ciesielski, *On the isomorphisms of the spaces $H_\alpha$ and $m_\theta$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 8 (1960), 217–222. MR 0132389 (24 #A2234)

[DMV07] Jérôme Dedecker, Florence Merlevède, and Dalibor Volný, *On the weak invariance principle for non-adapted sequences under projective criteria*, J. Theoret. Probab. 20 (2007), no. 4, 971–1004. MR 2359065 (2008g:60088)

[Don51] Monroe D. Donsker, *An invariance principle for certain probability limit theorems*, Mem. Amer. Math. Soc., 1951 (1951), no. 6, 12. MR 0040613 (12,723a)

[Han73] E. J. Hannan, *Central limit theorems for time series regression*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 26 (1973), 157–170. MR 0331683 (48 #10015)

[KR91] Gérard Kerkyacharian and Bernard Roynette, *Une démonstration simple des théorèmes de Kolmogorov, Donsker et Ito-Nisio*, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 11, 877–882. MR 1108512 (92g:60009)

[LV01] Emmanuel Lesigne and Dalibor Volný, *Large deviations for martingales*, Stochastic Process. Appl. 96 (2001), no. 1, 143–159. MR 1856684 (2002k:60080)

[MPU06] Florence Merlevède, Magda Peligrad, and Sergey Utev, *Recent advances in invariance principles for stationary sequences*, Probab. Surv. 3 (2006), 1–36. MR 2206313 (2007a:60025)

[Nag03] S. V. Nagaev, *On probability and moment inequalities for supermartingales and martingales*, Proceedings of the Eighth Vilnius Conference on Probability Theory and Mathematical Statistics, Part II (2002), vol. 79, 2003, pp. 35–46. MR 2021875 (2005f:60098)

[RS03] Alfredas Račkauskas and Charles Suquet, *Necessary and sufficient condition for the Lamperti invariance principle*, Teor. Ímovîr. Mat. Stat. (2003), no. 68, 115–124. MR 2000642 (2004g:60050)

[Ste61] E. M. Stein, *On the maximal ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1894–1897. MR 0131517 (24 #A1367)

[Suq99] Ch. Suquet, *Tightness in Schauder decomposable Banach spaces*, Proceedings of the St. Petersburg Mathematical Society, Vol. V (Providence, RI), Amer. Math. Soc. Transl. Ser. 2, vol. 193, Amer. Math. Soc., 1999, pp. 201–224. MR 1736910 (2000k:60009)

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