Easily Testable Necessary and Sufficient Algebraic Criteria for Delay-independent Stability of a Class of Neutral Differential Systems *

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Abstract: This paper analyzes the eigenvalue distribution of neutral differential systems and the corresponding difference systems, and establishes the relationship between the eigenvalue distribution and delay-independent stability of neutral differential systems. By using the “Complete Discrimination System for Polynomials”, easily testable necessary and sufficient algebraic criteria for delay-independent stability of a class of neutral differential systems are established. The algebraic criteria generalize and unify the relevant results in the literature. Moreover, the maximal delay bound guaranteeing stability can be determined if the systems are not delay-independent stable. Some numerical examples are provided to illustrate the effectiveness of our results.

Keywords: Neutral Dynamical Systems, Delay-independent Stability, Global Hyperbolicity, Complete Discrimination System for Polynomials, Algebraic Criteria, Delay Bound.

1 Introduction

Neutral differential system model can often be found in control process, physics, chemical engineering and ecology 2 8 10 21. Stability of neutral differential systems is an important performance index and has been investigated by many authors. From the control-theoretic viewpoint, the delay-independent stability of a delay system ensures robustness and reliability of the system for all delays. It is well known that the asymptotic behavior of the zero solution of a linear autonomous retarded differential system is determined by its eigenvalues. Just as ordinary differential systems, the stability of the zero solution is equivalent to all eigenvalues being with negative real parts 2 20 7 13 16 21 23. However, it is more complex when considering the neutral differential system, i.e., all eigenvalues being with negative real parts are not sufficient for the stability of

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the zero solution [2 3 8 9 10 16 21], and the global hyperbolicity of a corresponding difference system must be taken into account [1 9 10]. In neutral differential system, when characterizing system stability by means of the corresponding characteristic equation, it is usually required that the supremum of the real parts of all eigenvalues is negative [2 3 8 9 10 16, 21], which is notably different from retarded differential system.

Consider the linear autonomous neutral differential system

$$\dot{x}(t) - \sum_{k=1}^{N} B_k \dot{x}(t - \gamma_k \cdot r) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \gamma_k \cdot r)$$

(1)

where \( R^+ = [0, \infty), R = (-\infty, +\infty), x \in R^n, A_0 \in R^{n \times n}, A_k \in R^{n \times n}, B_k \in R^{n \times n}, r = (r_1, \ldots, r_M) \in (R^+)^M, \gamma_k = (\gamma_{k1}, \ldots, \gamma_{kM}), \gamma_{kj} \geq 0 \) are integers, \( \gamma_k \neq 0 \), \( \gamma_k \cdot r = \sum_{j=1}^{M} \gamma_{kj} r_j, k = 1, 2, \ldots, N, j = 1, 2, \ldots, M \). Suppose \( \det[I - \sum_{k=1}^{N} B_k] \neq 0 \) (the notation \( \det[\cdot] \) stands for the determinant of the corresponding matrix). The characteristic equation of System (1) is

$$G(\lambda, r, A, B) = \det[\lambda(I - \sum_{k=1}^{N} B_k e^{-\lambda \gamma_k \cdot r}) - A_0 - \sum_{k=1}^{N} A_k e^{-\lambda \gamma_k \cdot r}] = 0$$

(2)

where \( I \) stands for the \( n \times n \) identity matrix, \( A = (A_0, A_1, \ldots, A_N), B = (B_1, \ldots, B_N) \).

**Definition 1** [9] System (1) is said to be **delay-independent stable**, if \( \forall r \in (R^+)^M \), the supremum of the real part of \( \lambda \) satisfying Equation (2) is negative, that is, there exists a \( \delta > 0 \), such that \( \{ \text{Re}\lambda : G(\lambda, r, A, B) = 0 \} \cap [-\delta, \infty) = \emptyset \).

In the literature, delay-independent stability is also called all-delay stability, unconditional stability or absolute stability.

Consider the following difference system which is closely related to System (1)

$$x(t) - \sum_{k=1}^{N} B_k x(t - \gamma_k \cdot r) = 0$$

(3)

The characteristic equation of System (3) is

$$E(\lambda, r, B) = \det[I - \sum_{k=1}^{N} B_k e^{-\lambda \gamma_k \cdot r}] = 0$$

(4)

**Definition 2** [9] System (3) is said to be **globally hyperbolic** at \( B \), if \( \forall r \in (R^+)^M \), the real part of \( \lambda \) satisfying Equation (4) does not intersect with a neighborhood of zero, that is, there exists a \( \delta > 0 \), such that \( \{ \text{Re}\lambda : E(\lambda, r, B) = 0 \} \cap [-\delta, \delta] = \emptyset \).

In the papers [9 10 16], some elegant theoretical results on the stability of the neutral differential system have been established. One important result is

**Lemma 1** [9] Let

$$p(\lambda, s_1, \ldots, s_M, A, B) = \det[\lambda - (I - \sum_{k=1}^{N} B_k s_{1k}^{\gamma_{1k}} \cdots s_{Mk}^{\gamma_{Mk}})^{-1}(A_0 + \sum_{k=1}^{N} A_k s_{1k}^{\gamma_{1k}} \cdots s_{Mk}^{\gamma_{Mk}})]$$

(5)

then, System (1) is delay-independent stability if and only if

(H1) System \( y(t) - \sum_{k=1}^{N} B_k y(t - \gamma_k \cdot r) = 0 \) is globally hyperbolic at \( B \);

(H2) \( \text{Re}\lambda[I - \sum_{k=1}^{N} B_k]^{-1} \sum_{k=0}^{N} A_k] < 0 \), where \( \text{Re}\lambda[\cdot] \) denotes the real part of eigenvalues of the corresponding matrix;

(H3) \( p(iy, s_1, \ldots, s_M, A, B) \neq 0, \forall y \in R - \{0\}, |s_j| = 1, j = 1, 2, \ldots, M \).


The above lemma gives necessary and sufficient conditions on delay-independent stability of a general linear autonomous neutral differential system, which is the fundamental basis for studying neutral differential systems. However, it is difficult to apply these conditions in practice, because \((H1)\) and \((H3)\) are both transcendental and can’t be explicitly tested in engineering computations. Therefore, many authors in the area of differential systems tried to derive simple easily-testable stability criteria from various viewpoints \([11, 13, 18, 19, 20, 22, 23, 24, 25]\). These papers considered some special systems and just obtained some sufficient conditions, furthermore, most of which have never considered the global hyperbolicity of the corresponding difference systems.

In this paper, we consider a more general linear multi-delay neutral differential system as follows

\[
\dot{x}(t) - \sum_{k=1}^{N} B_k \dot{x}(t - k\tau) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - k\tau)
\]  

(6)

where \(x \in \mathbb{R}^n, A_0 \in \mathbb{R}^{n \times n}, A_k \in \mathbb{R}^{n \times n}, B_k \in \mathbb{R}^{n \times n}, \tau \in \mathbb{R}^+, k = 1, 2, \cdots, N, \) and we assume \(\det[I - \sum_{k=1}^{N} B_k] \neq 0.\) Based on Lemma 1, and discussions on the global hyperbolicity of the corresponding difference system \([1, 9]\), algebraic criteria for delay-independent stability of the above system will be established by means of the “Complete Discrimination System for Polynomials” \([27, 28, 29]\). The algebraic criteria generalize and unify the relevant results in the literature. Moreover, the maximal delay bound guaranteeing stability can also be determined when the system is not delay-independent stable.

This paper is arranged as follows: In section 2, the relevant results of the “Complete Discrimination System for Polynomials” are presented; Section 3 discusses the global hyperbolicity of difference system \(y(t) - \sum_{k=1}^{N} B_k y(t - k\tau) = 0;\) Section 4 analyzes the delay-independent stability of System (6) and establishes the easily testable algebraic criteria; Section 5 presents some corollaries and examples by means of the algebraic criteria provided in section 4; Finally, a simple conclusion is given in section 6.

2 Complete Discrimination System for Polynomials

From Lemma 1 in section 1, the problem of delay-independent stability of System (6), boils down to analyzing the eigenvalues’ distribution of System (6) and the corresponding difference system. In the sequel, we will transform this problem into the real root distribution of some related polynomials. The following is some results of the “Complete Discrimination System for Polynomials”, which can effectively discriminate the distribution of the roots of polynomials by a series of explicit expressions and calculations with polynomial coefficients \([6, 27, 28, 29]\).

Suppose

\[
f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n
\]  

(7)

the Sylvester matrix \([6, 27, 28, 29]\) of \(f(x)\) and its derivative \(f'(x)\) is defined as

\[
\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_n \\
0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} \\
a_0 & a_1 & \cdots & a_{n-1} & a_n \\
0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} \\
& & \cdots & \cdots & \cdots \\
a_0 & a_1 & \cdots & a_n \\
0 & na_0 & \cdots & a_{n-1}
\end{bmatrix}
\]
which is called the 
**discrimination matrix** of \( f(x) \), denoted as \( \text{Discr}(f) \).

\[
[D_1(f), D_2(f), \cdots, D_n(f)]
\]

is a sequence of the determinants of the first \( n \) even-order principal sub-matrixes of \( \text{Discr}(f) \), formed by the first 2\( k \) rows and first 2\( k \) columns, \( k = 1, 2, \ldots, n \), which is called the **discrimination sequence** of \( f(x) \). Furthermore

\[
[\text{sign}(D_1), \text{sign}(D_2), \cdots, \text{sign}(D_n)]
\]

is called the **sign list** of the discrimination sequence \([D_1, D_2, \cdots, D_n]\), where \( \text{sign}(\cdot) \) is the sign function, namely

\[
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]

Construct a **revised sign list** \([\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]\) for a given sign list \([s_1, s_2, \cdots, s_n]\) as follows:

1) If \([s_i, s_{i+1}, \cdots, s_{i+j}]\) is a section of the given list, where \( s_i \neq 0; s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0; s_{i+j} \neq 0 \), then replace the subsection

\[
[s_{i+1}, s_{i+2}, \cdots, s_{i+j-1}]
\]

by

\[
[-s_i, -s_i, s_i, -s_i, -s_i, s_i, -s_i, \cdots],
\]

namely, \( \varepsilon_{i+r} = (-1)^{\frac{r+1}{2}} \cdot s_i, r = 1, 2, \cdots, j-1 \), where \( \lfloor \alpha \rfloor \) denotes the greatest integer equal to or smaller than \( \alpha \).

2) Otherwise, let \( \varepsilon_k = s_k \), i.e. no change is made for other terms.

**Lemma 2** [27, 28, 29] Given a real polynomial \( f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in P^n \), if the number of sign changes in the revised sign list of its discrimination sequence is \( \nu \), and the number of nonzero elements in the revised sign list of its discriminant sequence is \( \mu \), then the number of the distinct real roots of \( f(x) \) is \( \mu - 2\nu \).

**Remark 1** The discrimination sequence of polynomial \( f(x) \) can also be constructed by the principal sub-matrices of Bezout matrix [28, 29] of \( f(x) \) and \( f'(x) \); the number of distinct real roots of the polynomial \( f(x) \) can also be determined by the sign differences of Bezout matrix [28, 29] of \( f(x) \) and \( f'(x) \).

**Remark 2** The original complete discrimination system of polynomial [27, 28, 29] is more general than Lemma 2, which can also be used to determine the number of complex roots and the multiplicities of repeated roots.

### 3 The Global Hyperbolicity of Difference Systems

Consider the difference system

\[
x(t) - \sum_{k=1}^{N} B_k x(t - \gamma_k \cdot r) = 0
\]

and its characteristic equation
E(λ, r, B) = \det[I - \sum_{k=1}^{N} B_k e^{-\lambda k \tau}] = 0 \quad (9)

where all parameters are the same as before. In order to analyze the global hyperbolicity of the
difference system (8), for its characteristic equation (9), by [1], we have

Lemma 3 [1] Suppose \(r_1, r_2, \cdots, r_M\) are commensurable, that is, there exist a \(\beta > 0\) and some
integers \(n_k\) such that \(r_k = n_k \beta, k = 1, \cdots, M\), then there exists an integer \(p\), such that \(E(\lambda, r, B)\)
is a polynomial of some degree \(p\) in \(e^{-\lambda \beta}\). Denote the \(p\)-th-degree coefficient of \(e^{-\lambda \beta}\) as \(A_p\), then

\[ E(\lambda, r, B) = A_p \prod_{\nu=1}^{p} (e^{-\lambda \beta} - s_{\nu}) \]

and

\[ \bar{Z}(B, r) = Z(B, r) = \{ -\frac{1}{\beta} \ln |s_{\nu}|, \nu = 1, 2, \cdots, p \} \]

where \(Z(B, r) = \{ \text{Re} \lambda : E(\lambda, r, B) = 0 \}\), \(\bar{Z}(B, r) = \text{cl}(Z(B, r))\) is the closure of \(Z(B, r)\).

Now, consider the corresponding difference system of System (6)

\[ y(t) - \sum_{k=1}^{N} B_k y(t - k \tau) = 0 \quad (10) \]

Its characteristic equation is

\[ e(\lambda, \tau, B) = \det[I - \sum_{k=1}^{N} B_k e^{-\lambda k \tau}] = 0 \quad (11) \]

Lemma 4 The difference system (10) is globally hyperbolic at \(B\) if and only if

\[ \forall \theta \in [0, 2\pi], \det[I - \sum_{k=1}^{N} B_k e^{ik\theta}] \neq 0. \]

Proof. From Lemma 3, the number of the roots of the characteristic equation (11) is finite, and distribute discretely in \(R \times R\) plane. So the following conclusion comes naturally:

If the real part of all characteristic roots is nonzero, then there exists a \(\delta > 0\), such that
\(\{ \text{Re} \lambda : E(\lambda, r, B) = 0 \}\) \(\cap [-\delta, \delta] = \phi\).

That is, the difference system (10) is globally hyperbolic at \(B\) if and only if

\[ \forall y \in R, e(iy, \tau, B) = \det[I - \sum_{k=1}^{N} B_k e^{-iy k \tau}] \neq 0. \]

Letting \(-y \tau = \theta\), the lemma is proved. \(\square\)

Owing to the above lemma, it may be feasible to transform the problem of global hyperbolicity of the
difference system (10) into the real root existence of some related polynomials.

4 Algebraic Criteria for Delay-independent Stability of Neutral Differential Systems

Consider the linear multi-delay neutral differential system

\[ \dot{x}(t) - \sum_{k=1}^{N} B_k \dot{x}(t - k \tau) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - k \tau) \quad (12) \]
The characteristic equation is

\[ H(\lambda, \tau, A, B) = \det[\lambda(I - \sum_{k=1}^{N} B_k e^{-\lambda k\tau}) - A_0 - \sum_{k=1}^{N} A_k e^{-\lambda k\tau}] = 0 \] (13)

The corresponding difference system is

\[ y(t) - \sum_{k=1}^{N} B_k y(t - k\tau) = 0 \] (14)

and its characteristic equation is

\[ e(\lambda, \tau, B) = \det[I - \sum_{k=1}^{N} B_k e^{-\lambda k\tau}] = 0 \] (15)

**Theorem 1** The neutral differential system (12) is delay-independent stable if and only if

(i) \( \det[I - \sum_{k=1}^{N} B_k e^{ik\theta}] \neq 0, \forall \theta \in [0, 2\pi], \)

(ii) \( \text{Re}\lambda[(I - \sum_{k=1}^{N} B_k)^{-1}\sum_{k=0}^{N} A_k] < 0, \)

(iii) \( \det[iy(I - \sum_{k=1}^{N} B_k e^{ik\theta}) - A_0 - \sum_{k=1}^{N} A_k e^{ik\theta}] \neq 0, \forall \theta \in [0, 2\pi], \forall y \in \mathbb{R}, y \neq 0. \)

**Proof.** By lemma 1 and lemma 4, letting \(-y\tau = \theta\), the theorem is proved. \(\square\)

Condition (i) and condition (iii) are also transcendental, which are still difficult to test numerically. Next, we will give a simple criterion.

The bilinear transform \( \omega = \frac{1 + z}{1 - z} \) is a one-to-one mapping between the set \( \{\omega = e^{i\theta} : \theta \in [0, 2\pi]\} \) and the set \( \{z = iy : y \in \mathbb{R}\}. \) By this transformation, condition (i) in Theorem 1 is equivalent to

\[ \det[I - \sum_{k=1}^{N} B_k (\frac{1 + iz}{1 - iz})^k] \neq 0 \]

that is

\[ \det[(1 - iz)^N I - \sum_{k=1}^{N} B_k (1 + iz)^k (1 - iz)^{N-k}] \neq 0 \]

Let

\[ d(z) = \det[(1 - iz)^N I - \sum_{k=1}^{N} B_k (1 + iz)^k (1 - iz)^{N-k}] \]

and \( f(z) = \text{Re}[d(z)], g(z) = \text{Im}[d(z)], \) where \( f(z), g(z) \) are real polynomials in \( z. \) Suppose

\[ \begin{cases} 
 f(z) = a_0 z^{l_1} + a_1 z^{l_1-1} + \cdots + a_{l_1}, \\
 g(z) = b_0 z^{m_1} + b_1 z^{m_1-1} + \cdots + b_{m_1},
\end{cases} \] (16)

where \( a_0 \neq 0, b_0 \neq 0. \)

Then condition (i) in Theorem 1 is equivalent to the condition that \( f(z) \) has no common real roots.

By the same transformation, condition (iii) in Theorem 1 is equivalent to

\[ \det[iy(I - \sum_{k=1}^{N} B_k (\frac{1 + iz}{1 - iz})^k) - A_0 - \sum_{k=1}^{N} A_k (\frac{1 + iz}{1 - iz})^k] \neq 0 \]
Then condition (iii) in Theorem 1 is equivalent to the condition that (20) has no common real roots, i.e.,

$$\det[(iyI - A_0)(1 - iz)^N - iy \sum_{k=1}^{N} B_k(1 + iz)^k(1 - iz)^{N-k} - \sum_{k=1}^{N} A_k(1 + iz)^k(1 - iz)^{N-k}] \neq 0$$

Let

$$D(z, y) = \det[(iyI - A_0)(1 - iz)^N - iy \sum_{k=1}^{N} B_k(1 + iz)^k(1 - iz)^{N-k} - \sum_{k=1}^{N} A_k(1 + iz)^k(1 - iz)^{N-k}]$$

and $F(z, y) = \text{Re}[D(z, y)], G(z, y) = \text{Im}[D(z, y)],$ where $F(z, y), G(z, y)$ are real polynomials in $(z, y)$. Suppose

$$\begin{cases} F(z, y) = a_0(y)z^l + a_1(y)z^{l-1} + \cdots + a_{l_1}(y), \\ G(z, y) = b_0(y)z^{m_2} + b_1(y)z^{m_2-1} + \cdots + b_{m_2}(y). \end{cases} \quad (17)$$

Then condition (iii) in Theorem 1 is equivalent to the condition that (19) has no common real roots $(z, y), z \in R, y \in R - \{0\}$.

Before presenting the stability criteria, we first introduce two lemmas in algebra theory [6,29].

**Lemma 5** The equation (16) has at least one common root if and only if its resultant equals zero, i.e.,

$$R(f, g) = \det \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{l_1} \\ a_0 & a_1 & \cdots & a_{l_1} \\ \vdots & & & \vdots \\ b_0 & b_1 & b_2 & \cdots & b_{m_1} \\ b_0 & b_1 & \cdots & b_{m_1} \\ \vdots & & & \vdots \end{bmatrix} = 0. \quad (18)$$

**Lemma 6** Suppose $a_0(y) \neq 0$ or $b_0(y) \neq 0$, the equation (16) has at least one common root $(z, y)$ and $y \in R$ if and only if its resultant

$$R(F, G) = \det \begin{bmatrix} a_0(y) & a_1(y) & a_2(y) & \cdots & a_{l_2}(y) \\ a_0(y) & a_1(y) & \cdots & a_{l_2}(y) \\ \vdots & & & \vdots \\ b_0(y) & b_1(y) & b_2(y) & \cdots & b_{m_2}(y) \\ b_0(y) & b_1(y) & \cdots & b_{m_2}(y) \\ \vdots & & & \vdots \end{bmatrix} = 0 \quad (19)$$

has a real root $y \in R$.

Similarly, $F(z, y), G(z, y)$ can be rewritten in the following formula:

$$\begin{cases} F(z, y) = c_0(z)y^{l_3} + c_1(z)y^{l_3-1} + \cdots + c_{l_3}(z), \\ G(z, y) = d_0(z)y^{m_3} + d_1(z)y^{m_3-1} + \cdots + d_{m_3}(z). \end{cases} \quad (20)$$

Then condition (iii) in Theorem 1 is equivalent to the condition that (20) has no common real roots $(z, y), z \in R, y \in R - \{0\}$.

Also from algebra theory [6,29], we have:
Lemma 7 Suppose \( c_0(z) \neq 0 \) or \( d_0(z) \neq 0 \), the equation (20) has at least one common root \((z, y)\) and \( z \in R \) if and only if its resultant

\[
\tilde{R}(F, G) = \begin{vmatrix}
    c_0(z) & c_1(z) & c_2(z) & \cdots & c_{i_3}(z) \\
c_0(z) & c_1(z) & \cdots & c_{i_3}(z) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
d_0(z) & d_1(z) & d_2(z) & \cdots & d_{m_3}(z) \\
d_0(z) & d_1(z) & \cdots & d_{m_3}(z) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
d_0(z) & d_1(z) & \cdots & d_{m_3}(z)
\end{vmatrix}
\]

has a real root \( z \in R \).

From Lemma 6 and Lemma 7, we have:

Lemma 8 If \((z_0, y_0)\) are a pair of real roots of \( F(z, y) = 0, G(z, y) = 0 \), then \( R(F, G)(y_0) = 0, \tilde{R}(F, G)(z_0) = 0 \).

Proof. The lemma is proved by using lemma 6 and lemma 7 directly. \( \square \)

Remark 4 It should be noticed that the converse proposition of Lemma 8 is not correct, which can be tested by some examples easily.

Now we are in position to present our main result. The algebraic criteria for the delay-independent stability of the neutral differential system (12) are as follows:

Theorem 2 The neutral differential system (12) is delay-independent stable if and only if

(i) (10) has no common real roots.
(ii) \( \text{Re} \lambda [(I - \sum_{k=1}^{N} B_k)^{-1} \sum_{k=0}^{N} A_k] < 0 \).
(iii) (17) (or (20)) has no common real roots \((z, y), z \in R, y \in R - \{0\}\).

Proof. From the above discussion, condition (i) in theorem 1 is equivalent to (10) having no common real roots; condition (ii) in theorem 2 is same as condition (ii) in theorem 1; condition (iii) in theorem 1 is equivalent to (17) (or (20)) having no common real roots \((z, y), z \in R, y \in R - \{0\}\).

This completes the proof. \( \square \)

By Lemma 5, the condition (i) above is equivalent to one of the following conditions:

(a1) the determinant \( R(f, g) \neq 0 \).

(a2) \( R(f, g) = 0 \) and \( g(z) \neq 0 \), where \( z \) is the real root of \( f(z) \), and \( R(f, g) \) is the resultant of (10).

(a3) \( R(f, g) = 0 \) and \( f(z) \neq 0 \), where \( z \) is the real root of \( g(z) \), and \( R(f, g) \) is the resultant of (10).

Similarly, by Lemma 6, the condition (iii) above is equivalent to one of the following conditions:

(b1) when \( a_0(y) \neq 0 \) or \( b_0(y) \neq 0 \), \( R(f, g) \) has no nonzero real root, where \( R(f, g) \) is the resultant of (17).

(b2) when \( a_0(y) \neq 0 \) or \( b_0(y) \neq 0 \), \( R(f, g) \) has nonzero real root. But when taking the root back into (17), then (17) has no common real root.

(b3) when there exists a real number \( y \), such that \( a_0(y) = 0 \) and \( b_0(y) = 0 \), then take this real number \( y \) into (17), and (17) has no common real root.

Dually, by Lemma 7, the condition (iii) above is also equivalent to one of the following conditions:

(c1) when \( c_0(z) \neq 0 \) or \( d_0(z) \neq 0 \), \( \tilde{R}(f, g) \) has no nonzero real root, where \( \tilde{R}(f, g) \) is the resultant of (20).
(c2) when \( c_0(z) \neq 0 \) or \( d_0(z) \neq 0 \), \( \tilde{R}(f, g) \) has nonzero real root. But when taking the root back into (20), then \( \tilde{R} \) has no common real root.

(c3) when there exists a real number \( z \), such that \( c_0(z) = 0 \) and \( d_0(z) = 0 \), then take this real number \( z \) into (20), and \( \tilde{R} \) has no common real root.

Furthermore, by Lemma 8, the condition (iii) above can also be checked with the following condition:

\[
\forall y_0 \in \{ y | R(F, G)(y) = 0 \} \cap \{ R - \{0\} \}, \forall z_0 \in \{ z | \tilde{R}(F, G)(z) = 0 \} \cap R, (y_0, z_0) \text{ are not the roots of Equations (17) or Equations (20)}.
\]

Remark 5 The conditions in Theorem 2 are all algebraic conditions. Condition (ii) can be checked by the well-known Hurwitz Criterion [6]. Determining real roots of polynomials in conditions (i) and (iii) in Theorem 2 can be carried out by the “Complete Discrimination System for Polynomials” mentioned before.

Remark 6 The conditions in Theorem 2 are necessary and sufficient. The sign list of the discrimination sequence of polynomials with symbolic coefficients can be obtained easily by computer [27, 28, 29]. Therefore, “on-line” determining the delay-independent stability of neutral differential systems [12] can be realized, namely, an efficient algorithm can be set up by this theorem.

Remark 7 System (12) is very general, which covers various forms of systems studied in [5, 7, 11, 12, 13, 14, 19, 20, 21, 22, 23, 24, 25, 26, 30]. Theorem 1 and Theorem 2 generalize the relevant results in [7, 30]. More specifically, if \( B_k = 0, k = 1, \cdots, N \), in (12), then System (12) degenerates to a retarded differential system. The results in this paper are completely consistent with that in [7, 30].

Just like the discussion in [7], when System (12) is not delay-independent stable, the maximal delay bound guaranteeing stability can be determined as following

\[
T := \min \left\{ \tau | \tau > 0, (z, y) \in \left\{ \begin{array}{l} F(z, y) = 0, \quad z \in R, y \in R - \{0\} \\ G(z, y) = 0 \end{array} \right\}, e^{i\theta} = \frac{1 + iz}{1 - iz}, -y\tau = \theta \right\}. \tag{22}
\]

In the following section, some numerical examples are provided to test the delay-independent stability of delay systems and compute the maximal delay bound when the systems are not delay-independent stable.

5 Corollaries and Examples

In what follows, we will present some explicit algebraic criteria for some simple neutral differential systems.

Corollary 1 The system

\[
\dot{x}(t) + cx(t - \tau) + ax(t) + bx(t - \tau) = 0 \quad (1 + c \neq 0)
\]

(23)

is delay-independent stable if and only if

\[
(1 - c)(b - a) < 0 \text{ and } (1 + c)(b + a) > 0
\]

Proof. Let

\[
\begin{align*}
f(z) &= \text{Re}(1 - iz + c(1 + iz)) = c + 1 \\
g(z) &= \text{Im}(1 - iz + c(1 + iz)) = cz - z \\
[\lambda + (1 + c)^{-1}(a + b)] &= 0, \lambda = -(1 + c)^{-1}(a + b)
\end{align*}
\]
\[ F(z, y) = \text{Re}[(iy + a)(1 - iz) + iyc(1 + iz) + b(1 + iz)] = yz(1 - c) + a + b \]
\[ G(z, y) = \text{Im}[(iy + a)(1 - iz) + iyc(1 + iz) + b(1 + iz)] = y(c + 1) + z(b - a) \]

Obviously, \( f(z) = 0 \) and \( g(z) = 0 \) have no common real root.

By the condition (ii) in theorem 2, we have \( \lambda = -(1 + c)^{-1}(a + b) < 0 \), that is \( (1 + c)(b + a) > 0 \).

Finally, it is easy to see that \( F(z, y) = 0 \) and \( G(z, y) = 0 \) has no common real roots \( (z, y), z \in R, y \in R - \{0\}, \) if and only if \( (1 - c)(b - a) < 0 \). Thus by theorem 2, we complete the proof.

**Corollary 2** The system

\[
\dot{x}(t) + c\dot{x}(t - 2\tau) + ax(t) + ax(t - \tau) = 0 \quad (1 + c \neq 0) \tag{24}
\]

is delay-independent stable if and only if

\[ a > 0 \quad \text{and} \quad -1 < c \leq \frac{1}{3} \]

**Proof.** Let
\[ f(z) = \text{Re}[(1 - iz)^2 + c(1 + iz)^2] = c(1 - z^2) - z^2 + 1 \]
\[ g(z) = \text{Im}[(1 - iz)^2 + c(1 + iz)^2] = 2cz - 2z \]
\[ |A - (1 + c)^{-1}(-a - a)| = 0, \lambda = -2a(1 + c)^{-1} \]
\[ F(z, y) = \text{Re}[(iy + a)(1 - iz)^2 + iyc(1 + iz)^2 + a(1 + iz)(1 - iz)] = a + 2yz - 2cyz + az^2 + a(1 - z^2) \]
\[ G(z, y) = \text{Im}[(iy + a)(1 - iz)^2 + iyc(1 + iz)^2 + a(1 + iz)(1 - iz)] = y(1 - z^2) - 2az + cy(1 - z^2) \]

Obviously, \( f(z) = 0 \) and \( g(z) = 0 \) have no common real root.

By the condition (ii) in theorem 2, we have \( \lambda = -2a(1 + c)^{-1} < 0 \) that is \( a > 0, 1 + c > 0 \); or \( a < 0, 1 + c < 0 \).

Finally, the resultant of \( F(z, y) \) and \( G(z, y) \)
\[ \tilde{R}(F, G)(z) \text{ det} \begin{vmatrix} 2z - 2cz & 2a \\ (1 - z^2)(1 + c) & -2az \end{vmatrix} = 2a(3c - 1)z^2 - 2a(1 + c) \]

It is easy to see that \( F(z, y) = 0 \) and \( G(z, y) = 0 \) has no common real roots \( (z, y), z \in R, y \in R - \{0\}, \) if and only if \( a > 0 \) and \( -1 < c \leq \frac{1}{3} \). Thus by theorem 2, we complete the proof.

**Remark 8** Corollary 1 and Corollary 2 are consistent with the relevant results in [23, 22], respectively. They have also discussed delay-dependent stability of System (23) and System (24).

Next, we will present some numerical examples using the main results of this paper.

**Example 1** Consider the following system

\[ \dot{X}(t) - CX(t - \tau) = AX(t) + BX(t - \tau) \]

where
\[ C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, B = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

and \( \alpha \) is a nonzero constant.

We now consider the maximal stability bound in terms of \( \alpha \). Let \( I \) be the \( 2 \times 2 \) identity matrix.
\[ f(z) := \text{Re}[\det[I(1 - iz) - C(1 + iz)]] = 0.81 - 1.21z^2 \]
\[ g(z) := \text{Im}[\det[I(1 - iz) - C(1 + iz)]] = -1.98z \]
\[ \det[I - (1 - C)^{-1}(A + B)] = \lambda^2 + 3.3333\lambda + 2.4691 + 1.2346\alpha - 1.2346\alpha^2 \]
\[ F(z, y) := \text{Re}[\det[(iyI - A)(1 - iz) - iyC(1 + iz) - B(1 + iz)] = y^2(1.21z^2 - 0.81) + 6.0yz + \alpha - 2z^2 - 2\alpha^2 + 2z^2\alpha + 2\alpha^2 + 2 \]
\[ G(z, y) := \text{Im}[\det[(iyI - A)(1 - iz) - iyC(1 + iz) - B(1 + iz)] = 1.98yz^2 + y(2.7 - 3.3z^2) - 4z - 2z\alpha^2 \]

Obviously, \( f(z) \) and \( g(z) \) have no common real roots.

If $\text{Re}\{[(I - C)^{-1}(A + B)] < 0$, by the Hurwitz criterion, we have $-0.99997 < \alpha < 2.0$.

Finally, the resultant of $F(z, y)$ and $G(z, y)$

$$\bar{R}(F, G)(z) := (-5.9049 \alpha + 5.9049 \alpha^2 - 11.81) + (-5.802 - 24.592 \alpha^2 + 12.96 \alpha^4 - 14.25 \alpha^3 - 11.87) z^2 + (6.336 \alpha - 31.68 \alpha^4 - 81.08 \alpha^2 - 106.01 + 3.168 \alpha^3) z^4 + (25.483 \alpha - 40.608 \alpha^2 - 86.346 + 17.424 \alpha^3 + 19.36 \alpha^4) z^6 + (13.177 \alpha + 13.177 \alpha^2 - 26.354) z^8$$

By a careful calculation, $\bar{R}(F, G)(z) = 0$ has no real root when $\alpha \in (-0.99997, 1]$. In fact, if $\alpha \in (-0.99997, 1]$, all coefficients of $\bar{R}(F, G)(z)$ have the same sign. Thus, $F(z, y) = 0$ and $G(z, y) = 0$ have no common real roots. Hence, by Theorem 2, when $\alpha \in (-0.99997, 1]$, the system is delay-independent stable.

**Remark 9** The criteria in [13] [19] do not work here. [20] concluded that the system is delay-independent stable when $|\alpha| \leq 0.989$. By using the criteria in this paper, we conclude that the system is delay-independent stable when $\alpha \in (-0.99997, 1]$, which gives a less conservative bound of $\alpha$.

**Example 2** Consider the following system

$$\dot{X}(t) - C \dot{X}(t - \tau) = AX(t) + BX(t - \tau)$$

where

$$C = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\alpha$ is a nonzero constant.

We now consider the maximal stability bound in terms of $\alpha$. Let $I$ be the $2 \times 2$ identity matrix.

$$f(z) := \text{Re}\{\text{det}[I(1 - iz) - C(1 + iz)]\} = 0.84 - 0.84 z^2$$

$$g(z) := \text{Im}\{\text{det}[I(1 - iz) - C(1 + iz)]\} = -2.32 z$$

$$\text{det}[I - (I - C)^{-1}(A + B)] = \lambda^2 + 2.381 \lambda - 0.95238 \lambda \alpha + 1.1905 - 1.1905 \alpha^2$$

$$F(z, y) := \text{Re}\{\text{det}[(iyI - A)(1 - iz) - i yC(1 + iz) - B(1 + iz)]\} = -0.84 y^2 + 1 + 4z y + 0.84 z^2 y^2 - z^2 + 1.6 \alpha^2 - 3 \alpha^2 + 2 \alpha^2 z^2$$

$$G(z, y) := \text{Im}\{\text{det}[(iyI - A)(1 - iz) - i yC(1 + iz) - B(1 + iz)]\} = 2 y + 2.32 z y^2 - 2 z - 2 z^2 y - 0.8 y^2 + 0.8 z^2 y - 2 \alpha^2 z$$

Obviously, $f(z)$ and $g(z)$ have no common real roots.

If $\text{Re}\{[(I - C)^{-1}(A + B)] < 0$, by the Hurwitz criterion, we have $-1.0 < \alpha < 1.0$.

Finally, the resultant of $F(z, y)$ and $G(z, y)$

$$R(F, G)(y) := (-4.0 + 8.0 \alpha^4 - 4.0 \alpha^8) + (2.56 \alpha^2 - 2.56 \alpha^6 - 16.0 + 16.0 \alpha^4) y^2 + (7.3856 \alpha^4 + 5.12 \alpha^2 - 23.795) y^4 + (2.4945 \alpha^2 - 15.59) y^6 - 3.7978 y^8$$

It is easy to see that $R(F, G)(y) = 0$ has no real root when $\alpha \in (-1.0, 1.0)$. In fact, if $\alpha \in (-1.0, 1.0)$, all coefficients of $R(F, G)(z)$ have the same sign. Thus, $F(z, y) = 0, G(z, y) = 0$ have no common real roots. Hence, when $\alpha \in (-1.0, 1.0)$, the system is delay-independent stable.

**Remark 10** The criteria in [13] [19] [20] can work here, and they can get $\alpha$’s bounds respectively as: $|\alpha| \leq 0.2$ [13]; $|\alpha| \leq 0.2$ [19]; $|\alpha| \leq 0.9165$ [20]. The result of this paper is $\alpha \in (-1.0, 1.0)$, which obviously gives a less conservative bound of $\alpha$.

**Example 3** Consider the following system

$$\dot{X}(t) - C \dot{X}(t - \tau) = AX(t) + BX(t - \tau)$$

where

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$$

11
We will discuss whether the system is delay-independent stable. Let \( I \) be the \( 2 \times 2 \) identity matrix.

\[
\begin{align*}
  f(z) & := \text{Re}[\det[I(1 - iz) - C(1 + iz)]] = 0.64 - 1.44z^2 \\
  g(z) & := \text{Im}[\det[I(1 - iz) - C(1 + iz)]] = -1.92z \\
  \det[\lambda I - (I - C)^{-1}(A + B)] & = \lambda^2 + 3.75\lambda + 2.7344 = 0 \\
  F(z, y) & := \text{Re}[\det[(iyI - A)(1 - iz) - iyC(1 + iz) - B(1 + iz)]] = -0.64y^2 + 6.0zy + 1.75 + 1.44z^2y^2 - 1.75z^2 \\
  G(z, y) & := \text{Im}[\det[(iyI - A)(1 - iz) - iyC(1 + iz) - B(1 + iz)]] = 2.4y + 1.92zy^2 - 4.5z - 3.6z^2y \\
\end{align*}
\]

Obviously, \( f(z) \) and \( g(z) \) have no common real roots.

By the Hurwitz criterion, it is obvious \( \text{Re}[\det[I - (I - C)^{-1}(A + B)]] < 0 \).

The resultant of \( F(z, y) \) and \( G(z, y) \), \( R(F, G)(y) \), is

\[
R(F, G)(y) := -35.684y^6 - 122.57y^4 - 3.3974y^8 - 152.46y^2 - 62.016 \\
R(F, G)(y) = 0 \text{ has no real roots, that is, } F(z, y) = 0, G(z, y) = 0 \text{ have no common real roots.}
\]

Hence, the system is delay-independent stable.

**Remark 11** By the criteria in [18, 20], the maximum allowable bound guaranteeing asymptotic stability of the system was obtained, they are \( \tau_{\text{max}} = 0.1352 \) in [18] and \( \tau_{\text{max}} = 0.7516 \) in [20], respectively. However, this paper concludes that the system is delay-independent stable. The reason for the different conclusions is that the criterion in this paper is necessary and sufficient, whereas the criteria in [18, 20] are only sufficient conditions.

The above three examples were all discussed in [20]. Sharper stability bounds are obtained by using the algebraic criteria in Theorem 2.

**Example 4** Consider the system

\[
\dot{X}(t) - C\dot{X}(t - \tau) = AX(t) + BX(t - \tau)
\]

where

\[
C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \\
A = \begin{bmatrix} -3 & -2 & -2 \\ 2 & -2 & -2 \\ 0 & 0 & -2 \end{bmatrix}, \\
B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}
\]

We will discuss whether the system is delay-independent stable. Let \( I \) be the \( 3 \times 3 \) identity matrix.

\[
\begin{align*}
  f(z) & := \text{Re}[\det[I(1 - iz) - C(1 + iz)]] = 0.28 - 3.6z^2 \\
  g(z) & := \text{Im}[\det[I(1 - iz) - C(1 + iz)]] = -1.78z + 2.34z^3 \\
  \det[\lambda I - (I - C)^{-1}(A + B)] & = \lambda^3 + 16.036\lambda^2 + 87.857\lambda + 160.71 \\
  F(z, y) & := \text{Re}[\det[(iyI - A)(1 - iz) - iyC(1 + iz) - B(1 + iz)]] = 45 + 72.4zy + 24.75z^2y^2 - 1.78y^3z + 2.34z^3y^3 - 10.4z^3y - 4.49y^2 - 35z^2 \\
  G(z, y) & := \text{Im}[\det[(iyI - A)(1 - iz) - iyC(1 + iz) - B(1 + iz)]] = 19.89y^2z - 52.6z^2y + 3.6y^3z^2 - 6.87z^3y^2 - 0.28y^3 + 5z^3 - 75z + 24.6y \\
\end{align*}
\]

It is easy to see that \( f(z) = 0, g(z) = 0 \) have no common real roots.

By the Hurwitz criterion, all roots of \( \det[\lambda I - (I - C)^{-1}(A + B)] \) have negative real parts.

Furthermore, by a careful calculation using Lemma 8, \( F(z, y) \) and \( G(z, y) \) have no common real roots.

Hence, the system is delay-independent stable.

**Remark 12** Let \( C = 0 \) in this example, then it becomes a retarded differential system. In this case, the conclusion in this paper is still correct and is consistent with the ones in [7, 30].

Finally, we present a more complex example on computing the maximal allowable delay bound.
Example 5 Consider the following system

\[
\dot{X}(t) - C\dot{X}(t - \tau) = AX(t) + B_1X(t - \tau) + B_2X(t - 2\tau) + B_3X(t - 3\tau)
\]

where

\[
C = \begin{bmatrix}
0.02 & 0.03 & 0 \\
0 & 0.01 & 0 \\
0 & 0 & 0.5 \\
0 & 0 & 0.25
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-2 & -3 & -5 & -2
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
-0.05 & 0.005 & 0.25 & 0 \\
0.005 & 0.005 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & -0.5 & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.005 & 0.0025 & 0 & 0 \\
0 & 0 & 0.05 & 0 \\
0 & 0 & 0 & 0.0005 \\
-1 & 0 & -0.5 & -0.5 & 0
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
0.0375 & 0 & 0.075 & 0.125 \\
0.05 & 0.05 & 0 & 0 \\
0.05 & 0.05 & 0 & 0 \\
0 & 0.05 & 0 & 0
\end{bmatrix}
\]

Let \( I \) be the 4 \times 4 identity matrix.

\[
f(z) := \text{Re}[\det[I(1 - iz) - C(1 + iz)]] = 0.36383 - 5.7048z^2 + 1.9316z^4
\]

\[
g(z) := \text{Im}[\det[I(1 - iz) - C(1 + iz)]] = -2.4477z + 5.5521z^3
\]

\[
det[\lambda - (I - C)^{-1}(A + B_1 + B_2 + B_3)] = \lambda^4 + 3.949\lambda^3 + 16.661\lambda^2 + 19.9\lambda + 11.646
\]

\[
F(z, y) := \text{Re}[\det([iyI - A](1 - iz)^3 - iyC(1 + iz)(1 - iz)^2 - B_1(1 + iz)(1 - iz)^2 - B_2(1 + iz)^2(1 - iz) - B_3(1 + iz)^3)] = 365.27y^2z^2 + 4587.5y^2z^6 - 771.21z^3y - 2331.4z^7y + 1.9316z^{12}y^4 - 911.59y^4z^6 + 625.81y^4z^8 - 4.821z^{12}y^2 - 35.473yz^2 - 36.62yz^4 + 26.68z^{11}y^3 - 104.21z^{10}y^4 + 2.008z^{12} - 21.357z^{11}y + 321.27z^{10}y^2 + 1189.5z^9y^3 - 2558.4y^2z^4 + 64.845yz + 2334.2z^{12}y - 6.0618y^2 - 211.53z^2 - 1765.1z^6 - 2396.7z^8y^2 - 555.07z^9y^3 + 621.06z^{10}y - 17.459yz^3z + 335.39yz^3 - 1413.6yz^5 + 1747.8z^7y^3 + 813.91z^8 - 111.95z^{10} + 0.3638yz^4 + 4.2371
\]

\[
G(z, y) := \text{Im}[\det([iyI - A](1 - iz)^3 - iyC(1 + iz)(1 - iz)^2 - B_1(1 + iz)(1 - iz)^2 - B_2(1 + iz)^2(1 - iz) - B_3(1 + iz)^3)] = -5.3583y^4z + 140.1yz^3 + 1835.7y^3z^2 - 311.72y^3z + 97.23zy^2z + 4002.2yz^2 - 805.43yz^4 + 1542.3z^4y - 59.03z^3y^2 - 278.51z^2y - 163.01zy^4 + 7.2403y - 44.703z - 682.09y^2z^3 - 1171.7y^2z^3 - 1457.7z^3y - 1190.1z^3y^3 - 3875.1z^2y^2 + 1062.4z^2y^2 - 2694.5y^5z + 69.44yz^2z + 0.6645z^{12}y + 887.7z^7y^4 - 1.6155z^{12}y^3 + 21.005z^{11}y^4 + 164.45z^{10}y^3 + 22.40z^{11}z - 1.4368yz^3 + 605.17z^3 - 1683.4z^5 + 1381.2z^7 - 356.38z^9
\]

Obviously, \( f(z) = 0, g(z) = 0 \) have no common real roots.

By Hurwitz Criterion, \( \det[\lambda - (I - C)^{-1}(A + B_1 + B_2 + B_3)] \) has only negative-real-part roots.

Further, by Lemma 8, we have the common real roots of \( F(z, y) = 0 \) and \( G(z, y) = 0 \) are \( y = -0.86798, z = 10.823 \), \( y = 0.58902, z = 10.7 \), \( z = -1.6587, y = 1.3876 \), \( y = 1.1338, z = -0.42313 \), \( y = -0.58902, z = -10.7 \), \( y = 0.31594, z = -1.2099 \), \( z = -10.823, y = 0.86798 \), \( y = -1.1338, z = 0.42313 \), \( y = -3.3266, z = 0.24369 \), \( y = 3.3266, z = -0.24369 \), \( y = -1.3876, z = 1.6587 \), \( y = -0.31594, z = 1.2099 \).

Therefore, the system is not delay-independent stable by Theorem 2. The maximal delay bound guaranteeing the stability is \( T = 0.14371 \) (by taking all common real roots of \( F(z, y) = 0 \) and \( G(z, y) = 0 \) into \( \mathbb{R}^2 \)).

Remark 13 Set \( C = 0 \) in Example 5, then the system degenerates to a retar differential system. The maximal delay bound was obtained in \( \mathbb{R}^4 \) as \( T = 0.4777 \). The same conclusion can be obtained by using the results in this paper.
From the above corollaries and examples, it is obvious that our criteria work well on judging the delay-independent stability of neutral differential systems, and its simplicity, accuracy, convenience, and wide applicability greatly facilitate the engineering practice.

6 Conclusion

This paper establishes some algebraic criteria for determining the delay-independent stability of a class of neutral differential systems

\[ \dot{x}(t) - \sum_{k=1}^{N} B_k \dot{x}(t - k\tau) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - k\tau), \]

and presents a method for determining the maximal delay bound guaranteeing stability if the systems are not delay-independent stable. To the best of our knowledge, the algebraic criteria has some noteworthy characters comparing with the related literature.

- **Generality**
  The system \[ \dot{x}(t) - \sum_{k=1}^{N} B_k \dot{x}(t - k\tau) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - k\tau) \] is more general, and covers various forms of systems studied in [5, 7, 11, 12, 13, 14, 19, 20, 22, 23, 24, 25, 26, 30]. The criteria in this paper can be applied to all these systems, and the relevant results are consistent with or better than that of those literature, which have been illustrated by corollaries and examples.

- **Necessary-Sufficient Conditions**
  The criteria for delay-independent stability are necessary and sufficient, and the broader conditions can be got comparing with that of some literature (see example 1-3).

- **Practicality**
  The conditions in the criteria are all algebraic conditions. Condition (ii) can be checked by Hurwitz Criterion [6]. Determining real roots of polynomials in conditions (i) and (iii) can be carried out by the “Complete Discrimination System for Polynomials” mentioned before. The sign list of the discrimination sequence of polynomials with symbolic coefficients can be obtained easily by computer [27, 28, 29]. Therefore, “on-line” operation can be realized.

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