EDGE CONNECTIVITY OF GEOMETRIC AND CUBICAL LATTICES

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Abstract. The class of k-Cohen-Macaulay posets was introduced and first studied by Baclawski in 1982. These are Cohen-Macaulay posets whose rank is not reduced nor is the Cohen-Macaulay property lost when k − 1 elements are removed from their ground set. In this paper, an edge analogue of this class of posets is presented, and it is proven that geometric and cubical lattices belong to this new class. Finally, it is shown that Baclawski’s characterization for the semimodular lattices which are 2-Cohen-Macaulay also holds for the edge analogue of 2-Cohen-Macaulay posets.

1. Introduction

In the present paper, we introduce a new class of shellable and Cohen-Macaulay partially ordered sets, and show that geometric lattices as well as cubical lattices belong to this class. We recall that a partially ordered set (poset for short) \( P \) is called shellable if there exists a linear ordering of the facets of its order complex (i.e. of the abstract simplicial complex formed by the chains of \( P \)), say \( F_1, F_2, \ldots, F_n \), such that \( F_l \cap (\cup_{i=1}^{l-1} F_i) \) is a nonempty union of maximal proper faces of \( F_l \), for \( l = 2, 3, \ldots, n \). Shellability has remarkable algebraic and topological consequences. For example, if \( P \) is shellable, then its order complex has the homotopy type of a wedge of spheres and it is Cohen-Macaulay over \( \mathbb{Z} \) and all fields (see Section 2 for the precise definitions). Cohen-Macaulayness, like shellability, is essentially a sophisticated connectivity property of a poset, thus it is natural to consider higher connectivity in this setting, in a sense analogous to the higher vertex connectivity of a graph. In [3], Baclawski dealt with exactly this problem. In particular, he introduced the concept of Cohen-Macaulay connectivity, as follows: for \( k \in \mathbb{N} \), a \( k \)-Cohen-Macaulay poset \( P = (P, \leq) \) is a Cohen-Macaulay poset such that \( (P \setminus S, \leq) \) is also Cohen-Macaulay of the same rank as \( P \), for every \((k − 1)\)-element subset \( S \) of the proper part of \( P \). The maximum value of \( k \) for which a poset \( P \) is \( k \)-Cohen-Macaulay, is the Cohen-Macaulay connectivity of \( P \). Analogously, one can define the notion of \( k \)-shellability. When \( k = 2 \), \( k \)-Cohen-Macaulayness reduces to the so-called double Cohen-Macaulayness, a property of great importance, which has been extensively studied. We mention, for instance, that it is a topological property [15], and that a bounded poset \( P \) is 2-Cohen-Macaulay if and only if the type of its Stanley-Reisner ring is given by the absolute value of its Möbius function [2, Corollary 4.7]. Moreover, it is believed that double Cohen-Macaulayness is related to the \( g \)-conjecture, see for instance [9]. For more information and examples of posets enjoying such properties we refer the reader to [12, Section III.3], [13, 15, 17]. We refer also to [5] for a brief discussion related to the Cohen-Macaulay property.

An analogous concept to the vertex connectivity of a graph is the edge connectivity, where, instead of removing vertices from the graph, we remove edges. Since in a poset an edge corresponds to a
cover relation, the following question, which motivated the present work, is raised: What happens if, instead of omitting elements from the ground set of a Cohen-Macaulay or shellable poset, we omit cover relations? For a poset $P$ we denote by $E(P)$ its set of cover relations. Let $e \in E(\bar{P})$, where $\bar{P}$ is the proper part of $P$. We set $P \ominus e$ to be the poset obtained from $P$ by removing the cover relation $e$. That is, the posets $P$ and $P \ominus e$ have the same ground set and their sets of cover relations satisfy $E(P \ominus e) = E(P) \setminus \{e\}$. In this paper we introduce an analogue property of the classical 2-Cohen-Macaulayness (respectively of the 2-shellability) as follows: A Cohen-Macaulay (respectively shellable) poset $P$ is called $2^*$-Cohen-Macaulay (respectively $2^*$-shellable) if for every $e \in E(\bar{P})$ the poset $P \ominus e$ is Cohen-Macaulay (respectively shellable) of the same rank as $P$. Figure 1 illustrates on the left the poset $P = (\{\hat{0}, a_1, a_2, b_1, b_2, \hat{1}\}, \leq)$, where for $i, j = 1, 2$ we have: $\hat{0} \lessdot a_i \lessdot b_j \lessdot \hat{1}$, and on the right the poset $P \ominus a_2 \lessdot b_1$. Remark that $P$ is a $2^*$-shellable poset, and thus $2^*$-Cohen-Macaulay, while $P \ominus a_2 \lessdot b_1$ is neither $2^*$-shellable nor $2^*$-Cohen-Macaulay.

Furthermore, we generalize the above definition by introducing the concept of an edge analogue of $k$-Cohen-Macaulayness and $k$-shellability, called respectively $k^*$-Cohen-Macaulayness and $k^*$-shellability, in a non-straightforward way: We do not remove any set of $k - 1$ cover relations from a Cohen-Macaulay (respectively shellable) poset, but all the cover relations of any closed interval $I$ of $\bar{P}$ with rank$(I) \leq k - 1$, and the resulting poset has to be of the same rank as $P$ and Cohen-Macaulay (respectively shellable). We refer to Section 3 for the precise definition.

Subsequently, we justify the above definition by showing that certain families of posets are $k^*$-shellable. We recall that Baclawski proved that a geometric lattice $L$ is $k$-Cohen-Macaulay if and only if $k \leq \min \{ \# \{x \in L \mid \text{rank}(x) = i\} \}$ [3, Theorem 3.3]. Here we show that a geometric lattice $L$ is $k^*$-shellable, if and only if $k \leq \text{rank}(L) - 1$. Athanasiadis (personal communication) asked whether the face lattices of convex polytopes satisfy similar properties. Here, we focus on the cubical lattice, which is the face lattice of the hypercube, and show that it enjoys such properties. In particular we have the following results.

**Theorem 1.1.** Let $L$ be a geometric lattice of rank $n$. Then $L$ is $(n - 1)^*$-shellable, and thus $(n - 1)^*$-Cohen-Macaulay.

**Theorem 1.2.** The cubical lattice $C_n$ is $n^*$-shellable, and thus $n^*$-Cohen-Macaulay.
Finally, we recall that Baclawski in [3] additionally characterized the semimodular lattices which are $k$-Cohen-Macaulay. In particular, for $k = 2$ he proved a semimodular lattice $\mathcal{L}$ is geometric if and only if $\overline{\mathcal{L}}$ is 2-Cohen-Macaulay [3, Theorem 3.1]. Interestingly, a similar characterization holds for the semimodular lattices which are $2^*$-Cohen-Macaulay. More precisely, it is not hard to prove the following.

**Theorem 1.3.** Let $\mathcal{L}$ be a semimodular lattice. The following are equivalent.

1. $\mathcal{L}$ is geometric;
2. $\mathcal{L}$ is $2^*$-Cohen-Macaulay.

The latter result suggests that there might be a connection between 2-Cohen-Macaulay and $2^*$-Cohen-Macaulay posets. It would be of utmost interest to find such a connection in a more general setting.

Besides the general case of convex polytopes (which is currently under investigation), there are other families of posets which could satisfy such properties. For instance we mention the poset of injective words, noncrossing partitions, and the supersolvable lattices for which the Möbius function on every closed interval is non-zero. We remark that the first two families of posets are shown to be 2-Cohen-Macaulay by Kubitzke and the author [8, Theorem 1.3 and Theorem 1.4], while Welker [17, Theorem 3.1] has proved that a supersolvable lattice $\mathcal{L}$ is 2-Cohen-Macaulay if and only if the Möbius function on every closed interval of $\mathcal{L}$ is non-zero.

This paper is organized as follows. In Section 2, we fix notation related to partially ordered sets and recall the necessary background. In Section 3, we introduce $k^*$-shellable and $k^*$-Cohen-Macaulay posets and present some examples. Finally, in Section 4 we prove Theorems 1.1, 1.2 and 1.3. To achieve our goal, we use for the proofs of the first two theorems suitable edge labelings (EL-labelings), while for the third one we follow the steps of the proof of [3, Theorem 3.1].

### 2. Preliminaries

In this section we fix notation and terminology related to partially ordered sets and recall some facts which are used in the proofs of Theorems 1.1 and 1.2. For more information on these topics we refer the reader to [11, Chapter 3] and [14].

Let $\mathcal{P} = (P, \leq)^1$ be a finite partially ordered set (poset for short) and $x, y \in P$. We say that $y$ covers $x$, and write $x \lessdot y$, if $x < y$ and there is no $z \in P$ such that $x < z < y$. In this case, $x$ is called a lower cover of $y$, while $y$ is called an upper cover of $x$. The set of cover relations of $\mathcal{P}$ is denoted by $E(\mathcal{P})$. The poset $\mathcal{P}$ is called bounded if there exist elements $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leq x \leq \hat{1}$ for every $x \in P$. The proper part of a poset $\mathcal{P}$, denoted by $\mathcal{P}$, is the partially ordered set obtained by the removal of $\hat{0}$ and $\hat{1}$ (if existent). The upper covers of $\hat{0}$ are called atoms of $\mathcal{P}$, while the lower covers of $\hat{1}$ are called coatoms of $\mathcal{P}$. A subset $C$ of a poset $\mathcal{P}$ is called a chain if any two elements of $C$ are comparable in $\mathcal{P}$. The set of maximal chains of $\mathcal{P}$ is denoted by $C(\mathcal{P})$. The length of a (finite) chain $C$ is equal to $|C| - 1$. We say that $\mathcal{P}$ is graded if all maximal chains of $\mathcal{P}$ have the same length. In that case, the common length of all maximal chains of $\mathcal{P}$ is called the rank of $\mathcal{P}$. Moreover, assuming $\mathcal{P}$ has

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$^1$Throughout this paper we use the 'mathcal' font to denote a poset, and the corresponding regular font to denote its ground set.
a minimum element $\hat{0}$, there exists a unique function $\rho : P \to \mathbb{N}$, called the rank function of $P$, such that

$$\rho(y) = \begin{cases} 0 & \text{if } y = \hat{0}, \\ \rho(x) + 1 & \text{if } x \to y. \end{cases}$$

We say that $x$ has rank $i$ if $\rho(x) = i$. For $x \leq y$ in $P$, we denote by $[x, y]$ the closed interval $\{z \in P : x \leq z \leq y\}$ of $P$, endowed with the partial order induced from $P$. The Möbius function $\mu$ of $P$ is the map $\mu : P \times P \to \mathbb{Z}$ defined recursively by

$$\mu(x, y) = \begin{cases} 1, & x = y \\ -\sum_{x \leq z < y} \mu(x, z), & x < y \\ 0, & \text{otherwise.} \end{cases}$$

To every poset $P = (P, \leq)$, one can associate an abstract simplicial complex $\Delta(P)$, called the order complex of $P$. The vertices of $\Delta(P)$ are the elements of $P$ and its $i$-dimensional faces are the chains of $P$ of length $i$. A graded poset $P$ is called Cohen-Macaulay over $k$, where $k$ is a field or $k = \mathbb{Z}$, if for each open interval $(x, y)$ in $P$ the $i$-th reduced homology group $\tilde{H}_i((\Delta(x, y)), k)$ is trivial for $i \neq \text{rank}(x, y) - 2$. The concept of a Cohen-Macaulay poset can also be formulated in terms of a certain commutative ring associated to the poset, and it is a basic result of Reisner [10] that the two definitions are equivalent.

A finite graded poset $P$ is called shellable if the facets of its order complex can be ordered $F_1, F_2, \ldots, F_n$ in such a way that $F_i \cap \left(\bigcup_{i=1}^{l-1} F_i\right)$ is a nonempty union of maximal proper faces of $F_i$ for $l = 2, 3, \ldots, n$.\footnote{The concept of shellability of posets is generalized to non-graded posets [6]. However all posets that we consider here are graded.} A standard method for showing that a poset is shellable, is to show that it is EL-shellable (Björner [4]), which means that the set of its cover relations admits a certain nice labeling. Assume that $P$ is bounded and graded. An edge-labeling of $P$ is a map $\lambda : E(P) \to \Lambda$, where $\Lambda$ is some poset. Let $[x, y]$ be a closed interval of $P$ of rank $n$. To each maximal chain $c : x < x_1 < \cdots < x_{n-1} < y$ of $[x, y]$ we associate the sequence $\lambda(c) = (\lambda(x, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, y))$. We say that $c$ is strictly increasing if the sequence $\lambda(c)$ is strictly increasing in the order of $\Lambda$. The maximal chains of $[x, y]$ can be totally ordered by using the lexicographic order on the corresponding sequences. An edge-lexicographic labeling (EL-labeling) of $P$ is an edge labeling such that in each closed interval $[x, y]$ of $P$ there is a unique strictly increasing maximal chain and this chain lexicographically precedes all other maximal chains of $[x, y]$. The poset $P$ is called EL-shellable if it admits an EL-labeling. An EL-shellable poset is shellable and Cohen-Macaulay over $\mathbb{Z}$ and any field $k$.

A finite lattice $L$ is called:

- semimodular if for all $x, y \in L$, the join $x \lor y$ covers $y$ whenever $x$ covers the meet $x \land y$,
- atomic if every element of $L$ is the join of atoms of $L$,
- complemented if for every $x \in L$ there exists a $y \in L$ such that $x \land y = \hat{0}$ and $x \lor y = \hat{1}$,
- relatively complemented if every closed interval of $L$ is itself complemented,
- geometric if it is semimodular and atomic.

Geometric lattices enjoy many important properties. For instance, they are relative complemented and EL-shellable, and thus Cohen-Macaulay (see [11, Chapter 3.3] and [14, Section 3.2.3]). In fact, the order complex of a geometric lattice was one of the first examples of a Cohen-Macaulay complex.
Moreover, it is known that in geometric lattices, both Cohen-Macaulayness and rank are preserved by the removal of any chain, by the removal of certain antichains \([3, 4]\) as well as by the removal of any principal filter \([16]\). Recently it was shown that the so-called filtered geometric lattices are also Cohen-Macaulay \([1]\), see also \([5]\). A classical EL-labeling for a geometric lattice \(L\) is given as follows. Fix a linear order \(\gamma: a_1, a_2, \ldots, a_l\) of the atoms of \(L\) and for every \(x < y \in E(L)\) set \(\lambda_\gamma(x, y) = \min_i \{x \lor a_i = y\}\). Björner showed that \(\lambda_\gamma\) defines an EL-labeling for \(L\) which is called minimal labeling with respect to the order \(\gamma\), see \([4]\). Interestingly, very recently Davidson and Hersh proved that the minimal labeling leads to a characterization of the geometric lattices. In particular, they proved that a finite atomic lattice \(L\) is geometric if and only if every atom ordering induces a minimal labeling that it is an EL-labeling for \(L\) \([7, \text{Theorem 6}]\).

3. Edge connectivity of posets

In this section we introduce an edge analogue of \(k\)-shellable and \(k\)-Cohen-Macaulay posets. Let \(P\) be a bounded graded poset of rank \(n\), \(\hat{P}\) be its proper part, and \(I\) be a closed interval of \(\hat{P}\) of rank at least 1 (that is, \(I\) is not a singleton). We denote by \(P \ominus I\) the partially ordered set obtained from \(P\) by omitting the cover relations from \(E(I)\). In other words, \(P \ominus I\) has the same ground set as \(P\), and its set of cover relation satisfies \(E(P \ominus I) = E(P) \setminus E(I)\). Clearly, the elements of \(I\) are no longer comparable in \(P \ominus I\). See Figures 2 and 3 for an illustration of the operation \(\ominus\).

![Figure 2](image.png)

**Figure 2.** The posets \(B_4\) and \(B_4 \ominus \{\{4\}, \{2, 3, 4\}\}\), with an EL-labeling. The thick lines represent the unique rising chain of each poset.

We recall that a Cohen-Macaulay poset \(P\) is said to be 2-Cohen-Macaulay (respectively 2-shellable), if for every \(x \in \hat{P}\), the poset \(P \setminus \{x\}\) is Cohen-Macaulay (respectively shellable) of the same rank as \(P\). Dually, if instead of removing points, we remove cover relations from \(P\), we derive the following definition.

**Definition 3.1.** We say that a Cohen-Macaulay (respectively a shellable) poset \(P\) is \(2^*\)-Cohen-Macaulay (respectively \(2^*\)-shellable), if for every cover relation \(e \in E(\hat{P})\), the poset \(P \ominus e\) is Cohen-Macaulay (respectively shellable) of the same rank as \(P\).
The definition of 2-Cohen-Macaulay posets can be generalized to \( k \)-Cohen-Macaulay posets, if instead of removing only one element, we remove any set of at most \( k - 1 \) elements. Similarly we can extend the notion of \( 2^* \)-Cohen-Macaulay as follows.

**Definition 3.2.** We say that a Cohen-Macaulay (respectively a shellable) poset \( P \) is \( k^* \)-Cohen-Macaulay (respectively \( k^* \)-shellable) if for every closed interval \( I \) of \( P \) of rank at most \( k - 1 \), the poset \( P \ominus I \) is Cohen-Macaulay (respectively shellable) of the same rank as \( P \).

**Example 3.3.** The Boolean lattice \( B_4 \) (i.e. the lattice on the subsets of the set \( \{1, 2, 3, 4\} \) ordered by inclusion) is \( 3^* \)-shellable (and thus \( 3^* \)-Cohen-Macaulay), since for every closed interval \( I \) of \( B_4 \) that has rank 1 or 2, the poset \( B_4 \ominus I \) is still shellable of rank 4. Figure 2 illustrates the lattice \( B_4 \) and the poset \( B_4 \ominus \{1\} \), together with an EL-labeling for each poset.

**Example 3.4.** Consider the poset \( P = (\{\hat{0}, a_1, a_2, b_1, b_2, c_1, c_2, \hat{1}\}, \leq) \), where for \( i, j, h = 1, 2 \) we have: \( \hat{0} < a_i < b_j < c_h < \hat{1} \). Then \( P \) is \( 2^* \)-shellable poset but not \( 3^* \)-shellable nor \( 3^* \)-Cohen-Macaulay. Figure 3 illustrates the posets \( P, P \ominus (a_2 \leq b_1) \) and \( P \ominus [a_2, c_2] \).

![Image of posets](image.png)

**Figure 3.** A \( 2^* \)-shellable poset \( P \) which is not \( 3^* \)-shellable.

4. **Proof of Theorems 1.1 – 1.3**

In this section we prove Theorems 1.1 – 1.3. For the proofs of Theorems 1.1 and 1.2 we will follow the same strategy: We will show that when \( L \) is a geometric or a cubical lattice, and \( I = [u, v] \) a closed interval of \( L \), then the poset \( L \ominus I \) admits an EL-labeling.

4.1. **Proof of Theorem 1.1.** We focus on Theorem 1.1. We will need the following simple lemmas.

**Lemma 4.1.** Let \( L = (L, \leq) \) be a geometric lattice, a an atom of \( L \), and \( y \in L \) with \( a < y \). Then there exists an element \( \tilde{y} \in L \) with \( a \not\leq \tilde{y} \leq y \).

**Proof.** Since \( L \) is geometric, it follows that the interval \([0, y]\) is complemented. Thus, for the element \( a \in [\hat{0}, y] \), there exists an element \( \tilde{y} \in [\hat{0}, y] \) such that \( \tilde{y} \wedge a = \emptyset \) and \( \tilde{y} \vee a = y \). Moreover, the identity \( \tilde{y} \wedge a = \emptyset \) implies that \( a \not\leq \tilde{y} \). We recall now that \( L \) is semimodular, thus the identity \( \tilde{y} \wedge a = \emptyset \) implies also that the join \( \tilde{y} \vee a = y \) covers \( \tilde{y} \). This proves the lemma. □
Lemma 4.2. Let $\mathcal{L} = (L, \leq)$ be a geometric lattice, with set of atoms $A$, and let $x, y \in L$ with $x \leq y$. Let $S \subseteq A$ such that $x = \bigvee S$. Then there exists a set $S' \subseteq S \subseteq L$ and $y = \bigvee S'$.

Proof. It is known that every closed interval in a geometric lattice is itself geometric, thus $[x, y]$ is geometric as well. Therefore we can write $y = \bigvee_{i=1}^{n} x_i$ where each $x_i$ covers $x$. Now, for $i = 1, 2, \ldots, n$ consider the interval $I_i = [0, x_i]$ which is also geometric. Since $\mathcal{L}$ is relatively complemented, it follows that for the element $x \in I_i$ there exists an element $z_i$ such that $x \wedge z_i = 0$ and $x \vee z_i = x_i$. Moreover, since $x$ is a coatom of $I_i$, we may assume that $z_i$ is an atom of $I_i$, and thus of $\mathcal{L}$ as well. (If not, there is an atom $a \in A$ with $a \leq z_i$. Then $a \wedge x = 0$, and $a \vee x = z_i$, since $x$ is a coatom of $I_i$.) Let $S$ be a set of atoms whose join is the element $x$. We consider the set $S' = \bigcup \{z_1, z_2, \ldots, z_n\}$ and remark that $S'$ has the required properties. □

Since EL-shellability implies shellability, it suffices to prove the following.

Theorem 4.3. Let $\mathcal{L}$ be a geometric lattice and $\mathcal{I}$ be a closed interval of the proper part of $\mathcal{L}$. Then the poset $\mathcal{L} \cap \mathcal{I}$ is EL-shellable of the same rank as $\mathcal{L}$.

Proof. Let $\mathcal{I} = [u, v]$, where $u, v \in L$. We set $\mathcal{L}' = \mathcal{L} \cap \mathcal{I}$ and denote by $\leq'$ the partial order of $\mathcal{L}'$. First we will show that $\mathcal{L}'$ has the same rank as $\mathcal{L}$. For this it suffices to prove that for every $x \in I$ there is an upper cover $y \in L$ such that $x \leq y \notin E(\mathcal{I})$ and, dually, that for every $y \in I$, there is a lower cover $x \in L$ such that $x \leq y \notin E(\mathcal{I})$.

For the first part: Since $v \neq 1$ and $\mathcal{L}$ is atomic, there exists at least one atom, say $a \in L$ such that $a \leq v$. Moreover $x \leq v$, therefore it follows that $a \leq x$ as well, thus $a \wedge x = 0$. Furthermore, semimodularity of $\mathcal{L}$ implies that the join $a \vee x$ covers $x$. We set $y = a \vee x$ and remark that $y \notin I$, hence the element $y$ has the required properties. The second part is a direct consequence of Lemma 4.1.

We will show now that $\mathcal{L}'$ is shellable. Let $a_1, a_2, \ldots, a_n$ be the atoms of $\mathcal{L}$. Without loss of generality, we may assume that there exist indices $1 \leq l < m < n$ such that $a_l \leq v$ if and only if $l < i \leq n$ and $a_i \leq u$ if and only if $m < i \leq n$. We consider now the minimal EL-labeling $\lambda$ of $\mathcal{L}$ (see Section 2) with respect to the order: $a_1 < a_2 < \cdots < a_l < a_{l+1} < \cdots < a_m < a_{m+1} < \cdots < a_n$. We will prove that this is an EL-labeling for the poset $\mathcal{L}'$ as well.

Let $x, y \in \mathcal{L}'$ with $x \leq y$ (and thus $x \leq y$ in $\mathcal{L}$). Let $J'$ and $J$ denote respectively the interval $[x, y]_{\mathcal{L}'}$ of $\mathcal{L}'$ and $[x, y]$ of $\mathcal{L}$. We will show that in $\mathcal{L}'$ there exists a unique rising chain which is lexicographically first. If $J' = J$, the result follows from the EL-shellability of $\mathcal{L}$ (see Figure 4). If $J' \neq J$, then $E(J) \cap E(\mathcal{I}) \neq \emptyset$. Since $C(J') \subset C(J)$, and $\mathcal{I}$ is EL-shellable, it follows that $J'$ has at most one rising chain. Clearly if such chain does exist, then it has to be lexicographically first (otherwise, $\lambda$ would not even be an EL-labeling for $\mathcal{L}$). Thus it suffices to show that there exists a rising chain in the interval $J'$. Note that $E(J) \cap E(\mathcal{I}) \neq \emptyset$ implies $u \leq y$ and $a_i \leq x$ for every $i \leq l$. Indeed, if $E(J) \cap E(\mathcal{I}) \neq \emptyset$, then there exists $x', y' \in J \cap I$ such that $a_i \leq x' < y' \leq y$ for every $i \geq m + 1$, and $x \leq x' < y' < v$, which implies that $a_i \leq x$ for every $i \leq l$. We distinguish further two cases (see Figure 5).

(i) Let $u \leq x$. Let $i_0$ be the minimum index such that $i_0 \leq l$ and $a_{i_0} \leq y$. Remark that such index exists, because otherwise we have $x, y \in I$, which means that in $\mathcal{L}'$ the elements $x$ and $y$ are incomparable, a contradiction. Consider the element $\hat{x} := x \vee a_{i_0}$. Recall that $a_{i_0} \not\leq x$, since
The case $J' = J$. The thick line represents the unique rising chain in $J' = I$.

(i) Let $x \leq a_0$. Thus $x \land a_0 = \hat{0}$ and since $L$ is semimodular, it follows that $x \leq \tilde{x} \leq y$, thus $x \leq \tilde{x} \leq y$ and $\lambda(x, \tilde{x}) = i_0$. Moreover $E([\tilde{x}, y]) \cap E(I) = \emptyset$, thus by definition of the index $i_0$, the unique rising chain of the interval $[\tilde{x}, y]$ is extended using the edge $x \leq \tilde{x}$ to a unique rising chain of $J'$.

(ii) Let $u \not\leq x$. Then $x \land a_0 = \emptyset$, for some maximum index $m < i_0 \leq n$. Semimodulararity of $L$ implies that $x \lor a_0$ covers $x$. Moreover, since $a_0 \leq x \lor a_0$, it follows from Lemma 4.2 that there is a minimal set of atoms $S$ of $L$ with $a_0 \in S$, such that $x \lor a_0 = \lor S$. Since $x \lor a_0 \leq y$, Lemma 4.2 implies now that there is a minimal set $S'$ of atoms of $L$ with $S \subseteq S'$ and $y = \lor S'$. We set $\tilde{y} = \lor (S' \setminus \{a_0\})$ and remark that $x \leq \tilde{y}$ and that minimality of $S'$ implies $a_0 \not\leq \tilde{y}$. Furthermore, $\tilde{y} \lor a_0 = y$ and $\tilde{y} \ll y$, which imply that $\lambda(\tilde{y}, y) = i_0$. Note that $E([x, \tilde{y}]) \cap E(I) = \emptyset$, therefore, by definition of the index $i_0$, the unique rising chain of it extended using the edge $\tilde{y} \ll y$ to a unique rising chain of $J'$. This completes the proof. □

A standard example of a geometric lattice is the Boolean algebra $B_n$, which is the set of subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. It is a direct consequence of Theorem 1.1 that $B_n$ is $(n - 1)^*$-shellable. In particular, we have the following corollary.

**Corollary 4.4.** The Boolean lattice $B_n$ is $(n - 1)^*$-shellable and thus $(n - 1)^*$-Cohen-Macaulay. Moreover, for every closed interval $I$ of $B_n$ the M"obius function of the poset $B_n \ominus I$ is zero.

**Proof.** Since the lattice $B_n$ is geometric, the first part of the corollary follows directly from Theorem 4.3. Without loss of generality, we may assume that $I = [u, v]$, where $u = \{m, m + 1, \ldots, n\}$ and $v = \{l, l + 1, \ldots, n\}$. Then, from the proof of Theorem 4.3 it follows that the usual labeling for $B_n$, which is $\lambda(S, T) = T \setminus S$ for $S \ll T$ in $B_n$, is an EL-labeling for the poset $B_n \ominus I$ as well. However, by omitting the cover relations from $E(I)$ we do not have anymore a decreasing chain. Then, [6, Proposition 5.7] implies that $\mu(B_n \ominus I) = 0$. □

Figure 5. The case $\mathcal{J}' \neq \mathcal{J}$: $u \leq x$ on the left and $u \not\leq x$ on the right. The thick lines compose the unique rising chain of $\mathcal{J}'$.

4.2. Proof of Theorem 1.2. We now proceed to the proof of Theorem 1.2 using similar techniques. First let us describe how we realize the cubical poset $C_n = (C_n, \leq)$. The ground set $C_n$ consists of all words with $n$ letters on the set $\{0, 1, \Box\}$, together with the empty word which corresponds to the minimum element of $C_n$. The cover relation is defined as follows. Let $x, y \in C_n$. If $x \neq \Box$, we can write $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$, where $x_i, y_i \in \{0, 1, \Box\}$. Then we have $x \lessdot y$ if and only if there exists exactly one index $0 \leq i_0 \leq n$ with $x_{i_0} \neq y_{i_0} = \Box$. If $x = \Box$, then $x \lessdot y$ if and only if $y_i \neq \Box$ for every $i$. For instance, $\Box < 101 < \Box 01 < \Box 1 < \Box \Box$. The elements of rank $i \geq 1$ are those words which contain exactly $(i - 1)$-many $\Box$. For example, the coatoms are those words that contain only one letter different than $\Box$ and have rank $n$, while the maximum element is the word $\Box \Box \cdots \Box$ and has rank $n + 1$. See Figure 6 (left) for an illustration of the poset $C_2$.

Remark 4.5. Let $a$ be an atom and $c$ a coatom of $C_n$, and let $\mathcal{I}$ be a maximal interval of $\overline{C}_n$. Then $[\Box, c] \cong C_{n-1}$, $[a, \overline{1}] \cong B_n$, and $\mathcal{I} \cong B_{n-1}$. Moreover, the poset $\overline{C}_n \cap \mathcal{I}$ is graded of rank $n + 1$. Indeed, without loss of generality, assume that $\mathcal{I} = [11 \cdots 1, \Box \cdots \Box]$. It suffices to show that for every $y \in I$, there exist $x, z \in P \setminus I$ such that $x \lessdot y \lessdot z$. Let $y = y_1 y_2 \cdots y_n$, where $y_1 = 1$, and $y_j \in \{0, 1, \Box\}$ for every $j \geq 2$. Then, the element $z := \Box y_2 \cdots y_n$ does not lie in $I$ and covers $y$. Suppose now that there exists an index $i \geq 2$ such that $y_i = \Box$ (if this is not the case then $y = 11 \cdots 1$, and we set $x = \Box$). We now set $x = y_1 y_2 \cdots y_{i-1} 0 y_{i+1} \cdots y_n$ and notice that $x$ satisfies the required properties. Similarly it follows that $\overline{C}_n \cap \mathcal{I}$ is graded for every closed interval $\mathcal{J}$ of $\overline{C}_n$ (not necessarily maximal).

We will describe an edge labeling for $C_n$ and consequently we will show that it is an EL-labeling (Lemma 4.6). Let $x, y \in C_n$, $x \neq \Box$, with $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$, where $x_i, y_i \in \{0, 1, \Box\}$. Then $x \lessdot y$ in $C_n$ if and only if $x$ and $y$ differ only at the position $i_0$, and we have $y_{i_0} = \Box$. In this case
we label the edge \( x \preceq y \) by the sequence \( \bullet \cdots \cdot x_0 \cdots \bullet \). If \( x = \emptyset \), then \( x \preceq y \) if and only if \( y \) contains no \( \square \). In this case, we label the edge \( x \preceq y \) by the sequence \( y_1 \cdots y_n \). Now we linearly order all these sequences using the following steps:

1. First come the sequences \( 1 \bullet \cdots \bullet \) \( < \) \( \bullet 1 \bullet \cdots \bullet \) \( < \) \( \cdots \) \( < \) \( \bullet \cdots \bullet 1 \).
2. Then we order the ones that do not contain any \( \bullet \) lexicographically, and set \( \bullet \cdots \bullet 1 < 00 \cdots 0 \).
3. Finally we order the sequences \( \bullet \cdots \bullet 0 < \bullet \cdots \bullet 0 < \cdots < 0 \bullet \cdots \bullet \), and set \( 11 \cdots 1 < \) \( \bullet \cdots \bullet 0 \).

For instance, if \( n = 2 \) we get the following linear order:

\[ 1 \bullet < \bullet 1 < 00 < 01 < 10 < 11 < \bullet 0 < \bullet \bullet \).

**Lemma 4.6.** The map \( \lambda \) defined by the above labeling is an EL-labeling for the lattice \( C_n \).

**Proof.** First note that there is a unique maximal chain, which is lexicographically first, namely the chain

\( \emptyset < 00 \cdots 0 < 00 \cdots 0 \square < \cdots < 0 \square \cdots 0 \square \).

with corresponding sequence of labels:

\[ 00 \cdots 0 < \bullet \cdots \bullet 0 < \cdots < \bullet \cdots \bullet \).

Now, consider a closed interval \([x, y]\) of \( C_n \) of rank \( r \leq n \). If \( x \neq \emptyset \), then \([x, y]\) is isomorphic to the algebra Boole \( B_r \), and clearly then the labeling \( \lambda \) restricts to a one for \( B_r \). If on the other hand \( x = \emptyset \), then \( y \neq \square \cdots \square \), therefore \([x, y]\) is isomorphic to the cubical lattice \( C_{r-1} \), and the result follows by induction. \( \square \)

**Example 4.7.** Figure 6 (left) illustrates the lattice \( C_2 \) together with the EL-labeling defined above.

![Figure 6](image-url)

**Figure 6.** The lattice \( C_2 \) and the poset \( C_2 \ominus 11 \ll 1 \square \) with their EL-labelings. The thick lines represent the unique rising chain of each poset.

Since EL-shellability implies shellability, it suffices to prove the following.

**Theorem 4.8.** Let \( I \) be a closed interval of the proper part of \( C_n \). Then \( C_n \ominus I \) is EL-shellable of rank \( n \).
Proof. Without loss of generality, we may assume that \( I = [u, v] \), where for the elements \( u = u_1u_2 \cdots u_n \) and \( v = v_1v_2 \cdots v_n \) there exist indices \( 1 \leq l < m \leq n + 1 \) respectively, such that

- \( u_i = 1 \) for every \( i \leq m \), and \( u_i = \Box \) for every \( i > m \), and
- \( v_i = 1 \) for every \( i \leq l \), and \( v_i = \Box \) for every \( i > l \).

In other words, we may assume that both \( u \) and \( v \) contain only the letters \( 1, \Box \), and the 1s appear consecutively in the beginning of each word.

We set \( C'_n = C_n \oplus I \) and denote by \( \leq ' \) the partial order of \( C'_n \). We have already seen that \( C'_n \) has rank \( n + 1 \). It suffices to show that for every closed interval \( J \) of \( C'_n \) there is a unique increasing maximal chain which is lexicographically first. We will follow the same steps as the ones it the proof of Theorem 4.3. Let \( x, y \in C'_n \) with \( x \leq ' y \). Let \( J' \) and \( J \) denote respectively the interval \( [x, y]_{C'_n} \) of \( C'_n \) and \( [x, y] \) of \( C_n \). If \( J' = J \), the result follows from the EL-shellability of \( C_n \). If \( J' \neq J \), then \( E(J) \cap E(I) \neq \emptyset \). Clearly, \( C(J') \subset C(J) \), and since \( J \) is EL-shellable, it is enough to show that there exists a rising chain in the interval \( J' \). Suppose first that \( x = x_1x_2 \cdots x_n \neq \emptyset \). The fact that \( E(J) \cap E(I) \neq \emptyset \) implies then that \( u \leq y \) and \( x \leq v \), which imply respectively that \( y \) is a word on the set \( \{1, \Box \} \) and, in particular, \( y_i = \Box \) for every \( i \geq m + 1 \), and that \( x_i = 1 \) for every \( i \leq l \). We further distinguish two cases.

(i) Let \( u \leq x \). Then \( x \) is also a word on the set \( \{1, \Box \} \). Moreover, for the element \( y \) we have that \( y_{i_0} = \Box \), for some minimum index \( i_0 \leq l \), since otherwise \( x, y \in I \), which means that in \( C'_n \) the elements \( x \) and \( y \) are incomparable, a contradiction. Notice that for the index \( i_0 \) we have \( x_{i_0} = 1 \). Consider now the element \( \tilde{x} \) obtained from the word \( x \) by replacing the letter \( x_{i_0} = 1 \) with the letter \( \Box \). Then \( x \leq ' \tilde{x} \leq ' y \), and \( \lambda(x, \tilde{x}) = \Box \cdots \Box 1 \cdots \Box \), where 1 appears in the \( i_0 \)-th position. Moreover \( E([\tilde{x}, y]) \cap E(I) = \emptyset \), thus the unique rising chain of \( \tilde{x} \) is extended using the edge \( x \leq ' \tilde{x} \) to a unique rising chain of \( J' \), which is by construction lexicographically first.

(ii) Let \( u \nleq ' x \). Recall that since \( u \leq y \), we have that \( y_i = \Box \) for every \( i \geq m + 1 \). On the other hand, since \( x \leq y \), then \( x_i = 1 \) for every \( i \) such that \( y_i = 1 \). Then either \( x_i = 0 \) for some \( 1 \leq i \leq n \), or \( x_i \neq 0 \) for every \( 1 \leq i \leq n \). Suppose that the first condition holds, and let \( i_0 \) be the minimum such index. Then \( y_{i_0} = \Box \), because \( y \) is a word on the set \( \{1, \Box \} \). We consider the word \( \tilde{y} \) obtained from \( y \) by replacing the letter \( y_{i_0} = \Box \) with the letter 0. Then \( x \leq ' \tilde{y} \leq ' y \), and \( \lambda(\tilde{y}, y) = \Box \cdots \Box 0 \cdots \Box \), where 0 appears in the \( i_0 \)-th position. Furthermore \( E([x, \tilde{y}]) \cap E(I) = \emptyset \), thus the unique rising chain of \( \tilde{y} \) is extended using the edge \( \tilde{y} \leq ' y \) to a unique rising chain of \( J' \), which is by construction lexicographically first. Suppose now that the second condition holds, that is \( x_i \neq 0 \) for every \( 1 \leq i \leq n \). Then both \( x, y \) are words on the set \( \{1, \Box \} \). Let \( i_0 \) be the maximum index such that \( i_0 \geq m + 1 \), \( x_{i_0} = 1 \), while \( y_{i_0} = \Box \) (such index exists because \( u \nleq ' x \) and \( u \leq y \)). Consider the element \( \tilde{y} \) obtained from \( y \) by replacing the letter \( y_{i_0} = \Box \) with the letter 1. Then \( \tilde{y} \leq ' y \), and \( \lambda(\tilde{y}, y) = \Box \cdots \Box 1 \cdots \Box \), where 1 appears in the \( i_0 \)-th position, and \( E([x, \tilde{y}]) \cap E(I) = \emptyset \). Thus the unique rising chain of \( [x, \tilde{y}] \) is extended using the edge \( \tilde{y} \leq ' y \) to a unique rising chain of \( J' \), which is lexicographically first.

Finally, let \( x = \emptyset \). We consider the atom \( \bar{x} = x_1x_2 \cdots x_n \), where for \( 1 \leq i \leq n \) we have \( x_i = 1 \) if and only if \( y_i = 1 \), otherwise \( x_i = 0 \). Clearly then, \( x \leq ' \bar{x} \leq ' y \), \( \lambda(x, \bar{x}) = x_1x_2 \cdots x_n \), and \( E([\bar{x}, y]) \cap E(I) = \emptyset \), and as before the unique rising chain of \( [\bar{x}, y] \) is extended using the edge \( \emptyset \leq ' \bar{x} \) to the unique rising chain of the interval \( J' \). This completes the proof of the theorem.

Corollary 4.9. For every closed interval \( I \) of \( C_n \) the Möbius function of the poset \( C_n \oplus I \) is zero.
Proof. Without loss of generality, we may assume that \( I = [u, v] \), where \( u \) and \( v \) are as in the proof of Theorem 4.8. Then, from the proof of Theorem 4.8, it follows that the EL-labeling for \( C_n \) given in Lemma 4.6, is an EL-labeling for the poset \( C_n \ominus I \) as well. However, by omitting the cover relations from \( E(I) \) we do not have anymore a decreasing chain. Then, \([6, Proposition 5.7]\) implies that \( \mu(C_n \ominus I) = 0 \). □

4.3. Proof of Theorem 1.3. Let \( L \) be a semimodular lattice. The implication \((i) \Rightarrow (ii)\) is a direct consequence of Theorem 4.3. For the implication \((ii) \Rightarrow (i)\) we follow the same steps as those in the proof of [3, Theorem 3.1]. Suppose that \( L \) is a \( 2^* \)-Cohen-Macaulay semimodular lattice which is not geometric. Then there exists an element \( y \in L \), which is not an atom and such that it has a unique lower cover, say \( x \). Suppose first that \( y \neq \hat{1} \). Then \( x < y \in E(\hat{L}) \). We consider the poset \( L \ominus x < y \), which is by hypothesis Cohen-Macaulay of the same rank as \( L \). Remark that since \( y \) has no other lower cover, it is a minimal element in \( L \ominus x < y \) and therefore there exists a maximal chain in \( L \ominus x < y \) of length strictly less than \( \text{rank}(L \ominus x < y) \), thus \( L \ominus x < y \) is not graded and hence not Cohen-Macaulay, a contradiction. Now suppose that \( y = \hat{1} \), and let \( z \) be a lower cover of \( x \). Clearly then \( z < x \in E(\hat{L}) \). We consider the poset \( L \ominus z < x \) is Cohen-Macaulay of the same rank as \( L \). However now, the element \( z \) is a maximal element in \( L \ominus z < x \) and thus the poset \( L \ominus z < x \) contains a maximal chain of length \( \text{rank}(L) - 2 \), which is again a contradiction. This completes the proof of the theorem. □

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