The dyadic and the continuous Hilbert transforms with values in Banach spaces. Part 2.

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Abstract

We show that if the dyadic Hilbert transform with values in a Banach space is \( L^p \) bounded, then so is the Hilbert transform, with a linear relation of the bounds. This result is the counterpart of [2] where the opposite bound was proven.

1 Introduction

Let \( X \) be a Banach space. Let \( D \) the unit disc and \( \partial D \) the unit circle. Let \( f \in L^p(\partial D, X) \) be a \( L^p \) integrable \( X \)–valued function, that is
\[
\|f\|_{L^p(\partial D, X)} := \left( \int_{\partial D} \|f\|_X^p \, dx \right)^{1/p} < \infty.
\]
We also write in short \( L^p(\partial D) \) instead of \( L^p(\partial D, X) \). It is well known that the Hilbert transform \( \mathcal{H} : L^p(\partial D) \to L^p(\partial D) \) is bounded if and only if \( X \) is UMD.

We wish to compare this operator with the so–called dyadic Hilbert transform. Let \( \Omega := [0, 1), D \) the set of dyadic intervals, and
\[
f(x) = (f)_{I_0} + \sum_{I \in D} (f, h_I)h_I
\]
the Haar decomposition of the \( X \)–valued function \( f \). The dyadic Hilbert transform \( \mathcal{S} \) is the operator sending
\[
(f)_{I_0} \mapsto 0, \quad h_{I_0} \mapsto 0, \quad h_{I_±} \mapsto ±h_{I_±}.
\]
It is well known that \( \mathcal{S} : L^p(\Omega) \to L^p(\Omega) \) is bounded in the UMD Banach space \( X \), and moreover we know from the recent preprint [2] that its operator norm is bounded by the operator norm of \( \mathcal{H} : L^p(\partial D) \to L^p(\partial D) \) with a linear dependence. In this part of our study, we prove the converse.

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Theorem 1 Let $H : L^p(\partial D) \to L^p(\partial D)$ the Hilbert transform on the disc with values in $X$, and $S : L^p((0,1)) \to L^p((0,1))$ the dyadic Hilbert transform with values in $X$. We have
\[ \|H\|_{p \to p} \leq \|S\|_{p \to p}. \]

The connection established by the second author in [5] between the Hilbert transform $H_{\mathbb{R}}$ on the real line and the classical Haar shift $S_{cl} : h_I \to (h_{I+} - h_{I-})/\sqrt{2}$ states that the Hilbert transform is, up to a universal multiplicative constant, an average of translated and dilated classical Haar shifts $S_{\alpha,r}^{cl}$ of the form
\[ E_\alpha E_r S_{\alpha,r}^{cl} = H_{\mathbb{R}}. \]
Here $\alpha$ denotes the translation parameter, $r$ the dilation parameter, $E_\alpha$ the averaging operator with respect to translations, and $E_r$ the averaging operator with respect to dilations. Since $\|S_{\alpha,r}^{cl}\|_{p \to p} = \|S_{cl}\|_{p \to p}$, the upper bound $\|H_{\mathbb{R}}\|_{p \to p} \leq \|S_{cl}\|_{p \to p}$ follows. A natural question is to ask if a similar strategy allows one to prove Theorem 1. Unfortunately, we prove in Section 6 that this is not the case since this average procedure yields for the dyadic Hilbert transform $S$
\[ E_\alpha E_r S_{\alpha,r}^{cl} = 0 \]
However, the comparison of the two operators can be achieved through stochastic representations of the Hilbert transform. It will reveal a profound connection between $H$ and $S$, which was an important motivation to us for bringing this dyadic Hilbert transform into play.

Stochastic representation of the Hilbert transform Let $f \in L^p(\partial \mathbb{D})$ given and (with the same notation) $f \in L^p(\mathbb{D})$ its harmonic extension on the unit disc $\mathbb{D}$. Let further $g = Hf$ the Hilbert transform of $f$, and $g$ its harmonic extension. Introduce $W$ the standard two-dimensional Brownian motion started at the origin. The Itô formula ensures that almost surely,
\[ f(W_t) = f(0,0) + \int_0^t \nabla f(W_s) \cdot dW_s + \frac{1}{2} \int_0^t \Delta f(W_s) \, ds = f(0,0) + \int_0^t \nabla f(W_s) \cdot dW_s, \]
since $f$ is harmonic. The formula above is valid for all times $t$ such that $(W_s)_{0 \leq s \leq t}$ remains in the unit disc. Let $\tau$ the stopping time
\[ \tau := \inf\{t > 0; W_t \not\in \mathbb{D}\}. \]
This is the random variable equal to the first instant when the continuous random walk hits $\partial \mathbb{D}$. We have again thanks to Itô’s formula,
\[ f(W_\tau) = f(0,0) + \int_0^\tau \nabla f(W_s) \cdot dW_s. \]
In particular if $W_\tau = z$, i.e. the random walk hits $\partial \mathbb{D}$ at $z$, then
\[ f(z) = f(0,0) + \int_0^\tau \nabla f(W_s) \cdot dW_s. \]
Similarly, owing to the Cauchy–Riemann relations for the conjugate function $g$ of $f$, we have

$$\forall t \leq \tau, \quad g(W_t) = \int_0^t \nabla g(W_s) \cdot dW_s = \int_0^t \nabla f(W_s) \cdot dW_s,$$

where

$$\nabla^\perp f(W_s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla f(W_s)$$

denotes the vector clockwise orthogonal to $\nabla f(W_s)$. In other words, $g(W_t)$ is a martingale transform of $f(W_t)$ with predictable multiplier the rotation matrix above. Finally if $W_\tau = z \in \partial \mathbb{D}$, then

$$g(z) = \int_0^\tau \nabla g(W_s) \cdot dW_s = \int_0^\tau \nabla f(W_s) \cdot dW_s,$$

In the next sections, we will be dealing with the martingale $M^f_t := f(W_t)$ and its martingale transform $M^g_t := g(W_t)$ defined as

$$\forall t \leq \tau, \quad M^f_t := f(0,0) + \int_0^t \nabla f(W_s) \cdot dW_s, \quad M^g_t := \int_0^t \nabla^\perp f(W_s) \cdot dW_s.$$  \hfill (1)

In the case where $f$ and therefore $g$ are Hilbert–valued, the martingales above are on the one hand differentially subordinate, i.e.

$$d[M^f, M^f]_t = \|\nabla f(W_s)\|^2 dt = \|\nabla f(W_s)\|^2 dt = d[M^g, M^g]_t,$$

and on the other hand are orthogonal, i.e.

$$d[M^f, M^g]_t = (\nabla f(W_s), \nabla^\perp f(W_s)) = 0.$$

Those were studied in the work of BAÑUELOS and WANG \cite{1} to which we refer for more details. They prove the sharp martingale $L^p$ bound $\|M^g\|_p \leq c_p \|M^f\|_p$ using special functions, and where $c_p := \|\mathcal{H}\|_{p \to p}$.

In this study, we do not aim at providing directly an absolute constant for the $L^p$ norm of the Hilbert transform, but rather look for a direct connection between the Hilbert and dyadic Hilbert transforms that holds also in the Banach valued case. We establish a tight connection through the use of stochastic integrals.

**Strategy and motivation** Due to the discrete nature of the dyadic shift, we first aim at approximating the two stochastic integrals in (1) by using two–dimensional discrete random walks $(B_k)_{k \geq 0}$ built on top of the dyadic system. We expect discrete martingales $M^f_k$ and $M^g_k$ that should be discrete counterparts of (1), namely

$$M^f_k(x) := f(0,0) + \sum_{l=1}^k \nabla f(B_{l-1}(x)) \cdot dB_l(x), \quad M^g_k(x) := \sum_{l=1}^k \nabla^\perp f(B_{l-1}(x)) \cdot dB_l(x).$$  \hfill (2)
where $x$ spans the dyadic probability space $\Omega := [0, 1)$ with uniform probability density $dP(x) = dx$. In order to relate the dyadic operator $S$ to the Hilbert transform $H$, the discrete random walk $(B_k)_{k \geq 0}$ has to be crafted in a very unique manner, as we discuss now. We want to build this discrete random walk so as to obtain the following diagram, where “convergence” is meant in a weak sense defined later.

\[
\begin{array}{ccc}
 f & \xrightarrow{H} & g \\
 M^f_k & \xrightarrow{S} & M^g_k \\
 \end{array}
\]

With the very specific process $B$ to be chosen, we want $M^g = SM^f$, where

\[(SM^f)_k := \sum_{l=1}^{k} \nabla f(B_{l-1}(x)) \cdot SdB_l(x).	ag{3}\]

But rewriting $M^g$ in (2) as

\[M^g_k(x) = \sum_{l=1}^{k} \nabla^\perp f(B_{l-1}(x)) \cdot dB_l(x) = \sum_{l=1}^{k} \nabla f(B_{l-1}(x)) \cdot dB^\top_l(x),\tag{4}\]

where $dB^\top_l$ denotes the vector anticlockwise orthogonal to $dB_l$, and comparing (3) with (4) suggests that the discrete random walk should have the property

\[SdB_l = dB^\top_l.	ag{5}\]

Finally, notice that (4) is the discrete counterpart of

\[\mathcal{M}^g_t := \int_0^t \nabla f(W_s)^\perp \cdot dW_s = \int_0^t \nabla f(W_s) \cdot dW^\top_s\]

where $dW^\top_s$ denotes the vector anticlockwise orthogonal to $dW_s$.

**Remark 1** A fundamental difference between the dyadic Hilbert transform $S$ and the classical Haar shift $S_{cl}$ is revealed by equation (5). The classical shift transforms the values of a given function $f$ in a way similar to the Hilbert transform on trigonometric functions. Indeed, it transforms a Haar function $h_I$, that we can understand as a dyadic counterpart of the sinus function, into a dyadic counterpart of the cosinus function.

Equation (5) for the dyadic Hilbert transform $S$ has a more abstract interpretation, namely that $S$ acts on the probability space by reordering random walks and sending a random walk $(B_k)_{k \geq 0}$ to $(B^\top_k)_{k \geq 0}$. Similarly, the Hilbert transform can be interpreted as sending an occurrence $(W_t)_{t \geq 0}$ of the Brownian motion to its rotated version $(W^\top_t)_{t \geq 0}$.
2 random walks, stopping times and scalings

Construction of the discrete random walk The set of dyadic intervals $\mathcal{D}$ in the interval $I_0 := [0, 1)$ is

$$\mathcal{D} := \{[m2^{-k}, (m + 1)2^{-k}); \ 0 \leq k, 0 \leq m \leq 2^k - 1\} = \bigcup_{k \geq 0} \mathcal{D}_k$$

where $\mathcal{D}_k := \{[m2^{-k}, (m + 1)2^{-k}); \ 0 \leq m \leq 2^k - 1\}$ is the set of dyadic intervals in generation $k$ and having length $2^{-k}$. We note $\mathcal{D}^- \subset \mathcal{D}$ the subset of left children and $\mathcal{D}^+ \subset \mathcal{D}$ the subset of right children. We have therefore $\mathcal{D} = \{I_0\} \cup \mathcal{D}^- \cup \mathcal{D}^+$. We also note $\mathcal{D}_k^- := \mathcal{D}_k \cap \mathcal{D}^-$ the left children of intervals of $\mathcal{D}_k$, and $\mathcal{D}_k^+ := \mathcal{D}_k \cap \mathcal{D}^+$ the right children of intervals of $\mathcal{D}_k$. Let the sequence $(\varepsilon_k)_{k \geq 0}$ of random variables be defined as

$$\forall k \geq 0, \forall x \in [0, 1), \quad \varepsilon_k(x) = \begin{cases} -1, & x \in \mathcal{D}_{k+1}^- \\ 1, & x \in \mathcal{D}_{k+1}^+ \end{cases}$$

In other words, if $k \geq 0$ and $I \in \mathcal{D}_k$, the random variable $\varepsilon_k$ takes values $\pm 1$ on $I$ with equal probability. Equivalently in terms of the Haar functions

$$\forall k \geq 0, \forall x \in [0, 1), \quad \varepsilon_k(x) = \sum_{l \in \mathcal{D}_k} \sqrt{|I|} h_l(x).$$

Let now $\delta > 0$. We build a twodimensional random walk $B_k(x) = (B_k^1(x), B_k^2(x))$ defined as

$$\{ \\
B_k^1(x) := \sum_{l=1}^k 1(\varepsilon_{l-1}(x) = -1)\varepsilon_l(x)\sqrt{2\delta} =: \sum_{l=1}^k dB_l^1(x) \\
B_k^2(x) := \sum_{l=1}^k 1(\varepsilon_{l-1}(x) = +1)\varepsilon_l(x)\sqrt{2\delta} =: \sum_{l=1}^k dB_l^2(x),
\}$$

with $B_0(x) = (0, 0)$ for all $x$. It follows successively, for any $l \geq 1$,

$$dB_l^1(x) := 1(\varepsilon_{l-1}(x) = -1)\varepsilon_l(x)\sqrt{2\delta} = \sum_{l \in \mathcal{D}_l^-} \sqrt{|I|} h_l(x)$$

$$SdB_l^1(x) := \sqrt{2\delta} \sum_{l \in \mathcal{D}_l^-} \sqrt{|I|} h_l(x) = \sqrt{2\delta} \sum_{l \in \mathcal{D}_l^+} \sqrt{|I|}(-h_l(x)) = -dB_l^2(x)$$

$$dB_l^2(x) := 1(\varepsilon_{l-1}(x) = +1)\varepsilon_l(x)\sqrt{2\delta} = \sum_{l \in \mathcal{D}_l^+} \sqrt{|I|} h_l(x)$$

$$SdB_l^2(x) := \sqrt{2\delta} \sum_{l \in \mathcal{D}_l^+} \sqrt{|I|} h_l(x) = \sqrt{2\delta} \sum_{l \in \mathcal{D}_l^-} \sqrt{|I|}(+h_l(x)) = +dB_l^1(x).$$

As a conclusion, denoting $dB_l := (dB_l^1, dB_l^2)$ a twodimensional increment, we have $SdB_l = dB_l^2$ as desired. We define now the discrete martingales associated
to $f$ and $g = Hf$ respectively, as

$$
\forall k, \quad M^f_k := f(B_0) + \sum_{l=1}^{k} \nabla f(B_{l-1}) \cdot dB_l, \quad M^g_k := g(B_0) + \sum_{l=1}^{k} \nabla g(B_{l-1}) \cdot dB_l,
$$

and observe:

**Lemma 1** Let $M^f$ and $M^g$ as above. We have $SM^f = M^g$ and therefore also

$$
\forall k, \quad \|M^g_k\|_p \leq \|\nabla\|_{p \to p} \|M^f_k\|_p.
$$

**Proof** Since $SdB_t = dB_t^\top$, $\nabla g = \nabla^\perp f$ and $g(B_0) = 0$, we have successively, for all $k \geq 1$,

$$
(SM^f)_k := S \left( f(B_0) + \sum_{l=1}^{k} \nabla f(B_{l-1}) \cdot dB_l \right) := 0 + \sum_{l=1}^{k} \nabla f(B_{l-1}) \cdot SdB_l
$$

$$
= \sum_{l=1}^{k} \nabla f(B_{l-1}) \cdot dB_l^\top = \sum_{l=1}^{k} \nabla f(B_{l-1})^\perp \cdot dB_l
$$

$$
= M^g_k.
$$

Hence the result.

In order to obtain from the inequality above an inequality for the continuous martingales $M^f$ and $M^g$, we need to prove in a suitable sense the convergence of the discrete martingales $M^f$ and $M^g$ towards their continuous counterparts $M^f$ and $M^g$. Since we are only interested in norms of those processes, what we need to obtain is some so-called weak convergence estimate, see e.g. Talay or Kloeden–Platen [6, 3]. Unfortunately, our situation does not exactly fit those expositions, for the following reasons:

- We consider randomly stopped processes as opposed to processes running on a prescribed, deterministic, interval of time.
- The discrete random walk is not a Markov process, rather a so-called Markov process with memory.
- The quadratic covariations of the discrete process are never converging towards that of their continuous counterparts, since they can be null half of the time.
- We do not compare only discrete random walks and their continuous counterparts but also their martingale transforms $M^f$ and $M^g$.

We define in the next section some auxiliary processes and suitable stopped random walks. Only then will we be able to state in a precise manner at the end of this section the convergence result Theorem 2 suited to our needs.
**Scalings and stopped random walks**  Let $W := (W_t)_{t \geq 0}$ the standard two dimensional Brownian process started at the origin. Let again $\tau$ the stopping time

$$\tau := \inf\{t > 0; W_t \notin \mathbb{D}\},$$

that is the first time of exit of the unit disc $\mathbb{D}$. We denote by $W^\tau = (W^\tau_t)_{t \in [0, \infty)} := (W_{(\tau \wedge \tau^\prime)}_{t \in [0, \infty)})$ the corresponding stopped process. Finally, given $t \in \mathbb{R}$, $x \in \mathbb{D}$, we note $(W^\tau_{t+s})_{s \geq 0}$ the process $W^\tau$ started at $x$ at time $t$.

Let $T > 0$ a fixed time, and $N \in \mathbb{N}$ large. Define $\delta$ a small time–step such that $T = N^5 \delta$. In order to denote the corresponding discrete times, we will use the indices $k, l \in [0, N^5]$, typically $t_k := k \delta$, $t_l := l \delta$. We introduce a larger time–step $\theta$ defined as $\theta := N \delta$. Notice that $T = N^4 \theta$, so that also $\theta$ tends to zero as $N$ goes to infinity for fixed $T$. The discrete times corresponding to this time–step will use indices $n, m \in [0, N^4]$, typically $t_n := n \theta$, $t_m := m \theta$. Given the discrete random walk $B$ defined above, we define a new discrete random walk $X$ by sampling $B$ at times that are multiples of $\theta = N \delta$, therefore with indices that are multiples of $N$,

$$\forall n \geq 0, \quad X_n := B_{nN}.$$

Notice further that

$$dX_n := X_n - X_{n-1} = B_{nN} - B_{(n-1)N} = \sum_{l=1}^{N} dB_{(n-1)N+l}.$$

In order to stop $X$ before it leaves the unit disc, we set $\varepsilon = 1/N$ and define the discrete stopping time

$$\tau_\varepsilon := \inf\{t_n; X_n \notin (1 - \varepsilon)\mathbb{D}\}.$$

We will denote by $n_\varepsilon$ the random index such that $\tau_\varepsilon := n_\varepsilon \theta$. From the definition of $X$ we have for all $n$,

$$|X_n - X_{n-1}| := |dX_n| \leq \sum_{l=1}^{N} |dB_{(n-1)N+l}| \leq N \sqrt{2 \delta} = \sqrt{2T} N^{-3/2}.$$

Therefore $|dX_n| \leq \varepsilon$ for $N$ large enough. In particular, before a stopping time we have $X_{\tau_\varepsilon - \theta} \in (1 - \varepsilon)\mathbb{D}$ which implies that the stopped process $X^{\tau_\varepsilon}$ always remains in the unit disc $\mathbb{D}$. Moreover at stopping time, the random walk $X_{\tau_\varepsilon}$ lies in the band $\mathbb{D} \setminus (1 - \varepsilon)\mathbb{D}$ of width $\varepsilon$ near the interior boundary $\partial \mathbb{D}$ of the unit disc. We note $X^{\tau_\varepsilon}$ the corresponding stopped process. Finally we will denote by $k_\varepsilon$ the random index such that $\tau_\varepsilon := k_\varepsilon \delta$, or equivalently $k_\varepsilon := N n_\varepsilon$. We note $B^{\tau_\varepsilon}$ the process $B$ stopped at $\tau_\varepsilon$.

Notice that the discrete martingales $(M^{f_\varepsilon}_k)_{k \in [0, N^4]}$ and $(M^{g_\varepsilon}_k)_{k \in [0, N^4]}$ are martingale transforms of the discrete random walk $(B_k)_{k \in [0, N^4]}$. We will aslo consider a sampled version of $(M^{f}_n)_{n \in [0, N^4]}$, namely $(M^{f}_n)_{n \in [0, N^4]}$, where $M^{f}_n := M^{f}_n$. 


for $k = nN$. We use the same notation for both processes. Which one is meant will be clear from the context. Similarly we note $(\mathcal{F}_k)_{k \in [0,N^q]}$ the filtration associated to the discrete random walk $(B_k)_{k \in [0,N^q]}$ and $(\bar{\mathcal{F}}_n)_{n \in [0,N^q]}$, with $\bar{\mathcal{F}}_n := \mathcal{F}_{nN}$, the filtration associated to the coarse random walk $(X_n)_{n \in [0,N^q]}$.

**Notations** We assume without loss of generality that $f \in \text{L}^p(\partial \mathbb{D}, X)$ and its harmonic extension (also noted $f$) $f \in \text{L}^p(\mathbb{D}, X)$ are smooth Frechet differentiable functions, that is $f \in \mathcal{C}^k$ for all $k \geq 0$. We note $|f(x)| := |f(x)|_X$ the Banach space norm of $f(x)$. The $\text{L}^p$–norm of $f$ is defined on the circle in the usual way

$$||f||_p := ||f||_{\text{L}^p(\partial \mathbb{D}, X)} := \left( \int_{\partial \mathbb{D}} |f(x)|^p \, dx \right)^{1/p}.$$ 

However if $Y$ is an $\text{L}^p$ integrable random variable on the probability space $(\Omega, \mathcal{P})$, the stochastic $\text{L}^p$ norm is defined as

$$||Y||_p := \mathbb{E}(|Y|^p)^{1/p}.$$ 

If $Y$ is real valued, then $|Y|$ denotes the absolute value of $Y$ whereas if $Y$ is $X$–valued, then $|Y| := |Y|_X$ denotes the Banach space norm of $Y$.

Now, if $f := f(x_1, \ldots, x_m)$ is a $X$–valued function of $m$ variables defined on the open set $U \subset \mathbb{R}^m$, we note $Df := (\partial_1 f, \ldots, \partial_m f)$ its derivatives in the Frechet sense, where $Df : U \times \mathbb{R}^m \to X$ is continuous. Further, given a $m$–multiindex $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $|\alpha| = k$ we note as usual $D^\alpha f := \partial^{\alpha_1} \ldots \partial^{\alpha_m} f$ its partial derivatives of order $k$ in the Frechet sense, where $D^\alpha f : U \times (\mathbb{R}^m)^k \to X$ is continuous. Finally, using again multindices, monomials of the form $x^\alpha$, where $x := (x_1, \ldots, x_m)$, are a shorthand for $x^\alpha := x_1^{\alpha_1} \ldots x_m^{\alpha_m}$.

The dual space of $X$ is noted $X^*$, and the dual space of $\text{L}^p(U, X)$ is noted $\text{L}^q(U, X^*)$ with $1/p + 1/q = 1$.

For the convergence results, our main parameters are $\varepsilon > 0$, a small number, and $T > 0$, a large number. We note $c(\varepsilon)$ a generic function tending to 0 uniformly in $T$ when $\varepsilon$ goes to zero, $c_T(\varepsilon)$ a generic function tending to 0 when $\varepsilon$ goes to zero for any fixed $T$, $c(T)$ a generic function tending to 0 when $T$ goes to infinity, uniformly in $\varepsilon$. It is implicit that those functions all depend on the fixed function $f$ and its derivatives. Further dependences will be mentionned when needed.

**Convergence result** In order to prove Theorem 1 we need the following convergence result of the discrete martingales towards their continuous counterparts.

**Theorem 2 (Convergence of $\text{L}^p$ norms of martingales)** Let $f$ as above. We have

$$\lim_{T \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}|\mathcal{M}_T^f|^p = \mathbb{E}|\mathcal{M}_{\infty}^f|^p$$

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A key ingredient is the notion of weak consistency presented in the next Section 3. This allows us to prove auxiliary convergence results in Section 4. Finally Section 5 is devoted to the proofs of the main results, namely Theorem 2 and Theorem 4.

3 Weak consistency and moments estimates.

Convergence results will be obtained by proving the weak consistency of the (sampled) stopped discrete random walk $X^{\tau\epsilon}$ with the continuous stopped two-dimensional Brownian process $W^\tau$. Due to the stopping process, we cannot rely on the standard definition of weak consistency based on discrete and continuous stochastic equations with prescribed coefficients depending smoothly on the process alone as used in e.g. [6, 3]. The definition adapted to our situation simply reads:

Definition 1 We say that $X^{\tau\epsilon}$ is weakly consistent with $W^\tau$, iff there exists a function $c := c(\epsilon)$ tending to zero when $\epsilon$ goes to zero, such that for all $n$, all 2–multiindex $\alpha := (\alpha_1, \alpha_2)$ with $|\alpha| = 2$, there holds

\[
\left| \mathbb{E}(X_{n+1}^{\tau\epsilon} - X_n^{\tau\epsilon} | \tilde{F}_n) - \mathbb{E}(W_{t_{n+1}}^{\tau,\tau\epsilon} - W_t^{\tau,\tau\epsilon} | \tilde{F}_n) \right| \leq \theta c(\epsilon),
\]

\[
\left| \mathbb{E}((X_{n+1}^{\tau\epsilon} - X_n^{\tau\epsilon})^\alpha | \tilde{F}_n) - \mathbb{E}\left(\left(W_{t_{n+1}}^{\tau,\tau\epsilon} - W_t^{\tau,\tau\epsilon}\right)^\alpha | \tilde{F}_n\right) \right| \leq \theta c(\epsilon).
\]

We can now state

Lemma 2 The discrete stopped process $X^{\tau\epsilon}$ is weakly consistent with the continuous stopped process $W^\tau$.

Proof The weak consistency is a straightforward consequence of the next two moments lemmas.

Lemma 3 (Discrete Moments) Let $X^{\tau\epsilon}$ as above. There exists a function $c := c(\epsilon)$ tending to zero when $\epsilon$ goes to zero, such that for all $n < n_\epsilon$, all 2–multiindex $\alpha := (\alpha_1, \alpha_2)$ with $|\alpha| = 2$, there holds

\[
\mathbb{E}(dX_{n+1}^{\tau\epsilon} | \tilde{F}_n) = 0
\]

\[
\mathbb{E}((dX_{n+1}^{\tau\epsilon})^\alpha | \tilde{F}_n) = 1(\alpha_1 \neq \alpha_2)\theta(1 + c(\epsilon)),
\]

Moreover for all $p \geq 2$, all $i = 1, 2$, there holds

\[
\mathbb{E}(dX_{n+1}^{\tau\epsilon,i} | \tilde{F}_n) \lesssim \theta^{p/2}.
\]

Notice that $1(\alpha_1 \neq \alpha_2) = 1$ if $\alpha \in \{(2, 0), (0, 2)\}$ and $1(\alpha_1 \neq \alpha_2) = 0$ if $\alpha = (1, 1)$. 
Lemma 4 (Continuous moments) Let $W^\tau$ and $W^{\tau,t,x}$ as above. There exists a function $c := c_T(e)$ tending to zero when $\varepsilon$ goes to zero for all fixed $T$, such that for all $2$-multiindex $\alpha := (\alpha_1, \alpha_2)$ with $|\alpha| = 2$, there holds

$$\forall t \geq 0, \forall x \in \mathbb{D}, \quad \mathbb{E}(W^{\tau,t,x}_{t+\theta} - W^{\tau,t,x}_t) = 0,$$

$$\forall t \geq 0, \forall x \in (1-\varepsilon)\mathbb{D}, \quad \mathbb{E}(W^{\tau,t,x}_{t+\theta} - W^{\tau,t,x}_t)^\alpha = 1(\alpha_1 \neq \alpha_2)\theta(1 + c_T(\varepsilon)).$$

$$\forall t \geq 0, \forall x \in \mathbb{D}, \forall p \geq 2, \quad \mathbb{E}(|W^{\tau,t,x}_{t+\theta} - W^{\tau,t,x}_t|^p) \lesssim \varepsilon^{p/2}.$$

Proof of Lemma 3 (Discrete Moments) For simplicity, we omit the superfix $\tau_\varepsilon$ in the discrete stopped processes $X^{\tau_\varepsilon}$ and $B^{\tau_\varepsilon}$. Recall that $(\mathcal{F}_k)_{k \in [0,N]}$ is the filtration associated to the discrete random walk $(B_k)_{k \in [0,N]}$, and $(\mathcal{F}_n)_{n \in [0,N]}$, with $\mathcal{F}_n := \mathcal{F}_N$, the filtration associated to the coarse random walk $(X_n)_{n \in [0,N]}$. Recall finally that $X_n := B_{nN}$, and therefore $dX_{n+1} := X_{n+1} - X_n = \sum_{l=1}^N dB_{nN+l}$. We are only interested in increments occuring before stopping, otherwise the estimate is trivial, all increments being zero after stopping. We have for all $n \leq N$,

$$\mathbb{E}(dX_{n+1} | \mathcal{F}_n) = \sum_{l=1}^N \mathbb{E}(dB_{nN+l} | \mathcal{F}_n) = 0$$

since $B$ is a martingale. Now for the second order moments, let $\alpha = (\alpha_1, \alpha_2)$ a multiindex with $|\alpha| = 2$. We estimate for example the variance of the first coordinate

$$\mathbb{E}((dX_{n+1}^1)^2 | \mathcal{F}_n) := \mathbb{E} \left( \sum_{l,l'=1}^N dB_{nN+l}^1 dB_{nN+l'}^1 | \mathcal{F}_n \right) = \sum_{l=1}^N \mathbb{E}((dB_{nN+l}^1)^2 | \mathcal{F}_n)$$

where we used that $B$ is a martingale. We have in the sum above for $l = 1$,

$$\mathbb{E}((dB_{nN+1}^1)^2 | \mathcal{F}_n) = \mathbb{E} \left( 1(\varepsilon_N = -1)^2 \varepsilon_{nN+1}^2 \left( \sqrt{2\delta} \right)^2 | \mathcal{F}_n \right) = 1(\varepsilon_N = -1) \cdot 2\delta.$$

For the other summands, with $l \geq 2$, we have

$$\mathbb{E}((dB_{nN+l}^1)^2 | \mathcal{F}_n) = \mathbb{E} \left( 1(\varepsilon_{nN+l-1} = -1)^2 \varepsilon_{nN+l}^2 \left( \sqrt{2\delta} \right)^2 | \mathcal{F}_n \right) = 2\delta \mathbb{E}(1(\varepsilon_{nN+l-1} = -1) | \mathcal{F}_n) = 2\delta \left[ \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 \right] = \delta$$

Since $\theta = N\delta$, we have, for $\alpha = (2,0)$,

$$\mathbb{E}((dX_{n+1}^1)^2 | \mathcal{F}_n) = 1(\varepsilon_N = -1) \cdot 2\delta + (N-1)\delta = \theta(1 + c(\varepsilon)),$$

and the same estimate holds for the second coordinate. Finally since $dB_{n}^1(x)dB_{n+1}^2(x) = 0$ for all $n$, all $x$, it follows $\mathbb{E}(dX_{n+1}^1 dB_{n+1}^2 | \mathcal{F}_n) = 0$ for all $n$.  

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For the higher order moments, if $|\alpha| = p$, we have

$$E(\langle dX_{n+1}^\alpha \rangle)^p \leq E(\langle X_{n+1}^1 \rangle^p) + E(\langle dX_{n+1}^2 \rangle^p).$$

For integer moments, estimate for example

$$E(\langle dX_{n+1}^1 \rangle^{2p}) = \sum_{l_1, \ldots, l_{2p}} E(\langle dB_{nN+l_1} \cdots dB_{nN+l_{2p}} \rangle) \leq \frac{(2p)!}{2} \sum_{l_1, l_3, \ldots, l_{2p-1}} E(\langle (dB_{nN+l_1})^2 (dB_{nN+l_3})^2 \cdots (dB_{nN+l_{2p-1}})^2 \rangle) \lesssim N^p \delta^p \lesssim \theta^p.$$

By Hölder, we get for any $p \geq 2$, $E(\langle dX_{n+1}^\alpha \rangle^p) \lesssim \theta^p$.

**Proof of Lemma 4 (Continuous Moments)** The first statement is obvious. For the second statement, notice that $x \in (1 - \epsilon)\mathbb{D}$ implies $\text{dist}(x, \partial\mathbb{D}) \geq \epsilon$. We now take advantage of the fact that the standard mean deviation of the non-stopped Brownian motion $W_{t+\theta}^{t,x} - W_{t}^{t,x}$ over one time step $\theta$ is $\sigma_\theta = \sqrt{T/N} = \epsilon \sqrt{T}/N$. Therefore $\sigma_\theta \ll \epsilon$, i.e. $\sigma_\theta = c_T(\epsilon)\epsilon$. This means that for $x \in (1 - \epsilon)\mathbb{D}$, we have $\text{dist}(x, \partial\mathbb{D}) \geq \epsilon$. In other words the point $x$ is “very far away” from the boundary as compared to the distance the Brownian motion can diffuse. More precisely, as a consequence of standard estimates of first hitting times of the Brownian motion, we have for such a $x$ and the corresponding stopped process $W_{t}^{t,x,\theta}$, that

$$P(t + \theta > \tau) = c_\theta(\epsilon) = c(c_T(\epsilon)) = c_T(\epsilon).$$

By Hölder, we get for any $p \geq 2$, $E(\langle dX_{n+1}^\alpha \rangle^p) \lesssim \theta^p$.

Note now $(W_{t}^{t,x,\theta})_{i=1,2}$ the two components of the Brownian motion. Consider for example the second moment of the first coordinate. Using Itô formula and taking expectation yields

$$E(W_{t+\theta}^{t,x,1} - W_{t}^{t,x,1})^2 = \frac{1}{2} E \int_t^{(t+\theta) \wedge \tau} d[W_{t}^{t,x,1}, W_{t}^{t,x,1}]_s = \frac{1}{2} E \left( \int_t^{(t+\theta)} d[W_{t}^{t,x,1}, W_{t}^{t,x,1}]_s [t + \theta \leq \tau] \right) P(t + \theta \leq \tau) + \frac{1}{2} E \left( \int_t^{(t+\theta)} d[W_{t}^{t,x,1}, W_{t}^{t,x,1}]_s [t + \theta > \tau] \right) P(t + \theta > \tau) =: \theta \delta_{\alpha_1 \alpha_2} P(t + \theta \leq \tau) + A(t, x, \theta) P(t + \theta > \tau),$$

where clearly $0 \leq A(t, x, \theta) \leq \theta$ for all $(t, x)$. This yields the second statement since $P(t + \theta > \tau) = c_T(\epsilon)$ and $P(t + \theta \leq \tau) = 1 - c_T(\epsilon)$ in the case $\alpha = (2, 0)$. The case $\alpha = (0, 2)$ is similar. The case $\alpha = (1, 1)$ is trivial since $d[W_{t}^{t,x,1}, W_{t}^{t,x,2}]_s = 0$. The third statement of the Lemma follows the same lines using the known higher moments for the Brownian motion.
4 Auxiliary convergence results

Our goal is to prove the following two convergence results:

**Lemma 5 (Weak convergence)** Let $T > 0$. Let $\psi$ harmonic on $\mathbb{D}$ with $\psi$ smooth on $\partial \mathbb{D}$. Assume weak consistency. Then we have
\[
\mathbb{E}\psi(X_T^\tau) = \mathbb{E}\psi(W_T^\tau) + c_{\psi,T}(\varepsilon) + c(T)
\]

**Lemma 6 (Weak convergence of discrete martingale transforms)** Let $T > 0$. Let $f$ as above.
\[
\|f(X_T^\tau) - M_T^f\|_p = c_T(\varepsilon)
\]

**Proof of Lemma 5** Let $\psi$ harmonic on $\mathbb{D}$, with $\psi$ smooth on $\partial \mathbb{D}$. We first split \[
\mathbb{E}\psi(X_T^\tau) = \mathbb{E}(\psi(X_T^\tau)|\tau_\varepsilon \leq T)\mathbb{P}(\tau_\varepsilon \leq T) + \mathbb{E}(\psi(X_T^\tau)|\tau_\varepsilon > T)\mathbb{P}(\tau_\varepsilon > T)
\]

We claim that the second term is small uniformly w.r.t. $\varepsilon$ when $T$ is large. Indeed, by definition of $\tau_\varepsilon$, this term collects the contribution of those trajectories that remained in the disc $(1 - \varepsilon)\mathbb{D}$ during the whole interval of time $[0, T]$. We claim that this is small for $T$ large. Indeed, let $\tilde{B} = (\tilde{B}_1, \tilde{B}_2)$ the rotation of angle $\pi/4$ of $B$, i.e. $\tilde{B}_1 := (B_1 + B_2)/\sqrt{2}$, and $\tilde{B}_2 := (B_2 - B_1)/\sqrt{2}$. It follows
\[
\tilde{B}_1^k(x) = \sum_{l=1}^k \varepsilon_l(x)\sqrt{3}, \quad \tilde{B}_2^k(x) = \sum_{l=1}^k \varepsilon_{l-1}(x)\varepsilon_l(x)\sqrt{3},
\]
that is both $\tilde{B}_1^k$ and $\tilde{B}_2^k$ are (non independent) standard centered discrete random walks. Let $\tilde{\tau}_1$ and $\tilde{\tau}_2$ the first exit times
\[
\tilde{\tau}_1 := \inf\{t_k; \|\tilde{B}_1^k\| > 1\}, \quad \tilde{\tau}_2 := \inf\{t_k; \|\tilde{B}_2^k\| > 1\}.
\]

But
\[
\tau_\varepsilon > T \iff X_n \in (1 - \varepsilon)\mathbb{D}, \quad n \in [0, N^4]
\]
\[
\Rightarrow B_{nN} \in (1 - \varepsilon)\mathbb{D}, \quad n \in [0, N^4]
\]
\[
\Rightarrow B_k \in \mathbb{D}, \quad k \in [0, N^5]
\]
\[
\Rightarrow \tilde{\tau}_1 > T \text{ and } \tilde{\tau}_2 > T,
\]
where we have used that $|B_k - B_{nN}| \leq N\sqrt{\delta} \ll \varepsilon$ for $k \in [(n-1)N+1, \ldots, nN]$. In particular,
\[
\mathbb{P}(\tau_\varepsilon > T) \leq \mathbb{P}(\tilde{\tau}_1 > T) = c(T).
\]
The last equality is a consequence of first hitting time estimates of standard centered discrete random walks, see e.g. Lawler [4]. The function $\psi$ being bounded on $\mathbb{D}$, we have also $\mathbb{E}(\psi(X_T^\tau)|\tau_\varepsilon > T)\mathbb{P}(\tau_\varepsilon > T) = c(T)$. Similarly for
the second term, $E(\psi(X^\tau_T)|\tau_\varepsilon \leq T)P(\tau_\varepsilon \leq T) = E(\psi(X^\tau_T)|\tau_\varepsilon \leq T) + c_\psi(T)$. On the other hand, since $\psi$ is harmonic, we have immediately

$$E\psi(W_T^\tau) = \psi(W_0^\tau) = \psi(0,0),$$

so that

$$E\psi(X^\tau_T) - E\psi(W_T^\tau) = E(\psi(X^\tau_T)|\tau_\varepsilon \leq T) - \psi(0,0) + c(T) = E(\psi(X^\tau_T)|\tau_\varepsilon \leq T) + c_\psi(T).$$

Now since $E(\psi(W_{t_{n+1}}^{\tau,t_n,x})) = \psi(W_{t_n}^{\tau,t_n,x}) = \psi(x)$ for all $x \in D$, we have

$$E(\psi(X^\tau_T) - \psi(X^\tau_0)) = E\sum_{n=1}^{n} (\psi(X^\tau_{t_n}) - \psi(X^\tau_{t_{n-1}}))$$

where $R(x, y)$ is the Taylor rest

$$R(x, y) := \frac{1}{3!} \sum_{|\alpha|=3} D^\alpha \psi(x + \theta_{x,y}(y - x)) \cdot (y - x)^\alpha, \quad x, y \in D, \theta_{x,y} \in [0, 1].$$

Using the weak consistency of $X^\tau_T$ with $W^\tau$, we get, recalling $T = N^4 \theta$,

$$|A| \leq E\sum_{n=1}^{n} |D\psi(X^\tau_{t_{n-1}})| \cdot \left( E(|X^\tau_{t_n} - X^\tau_{t_{n-1}}|_{F_{n-1}}) - E\left( |W_{t_n}^{\tau,t_n-1,X^\tau_{t_{n-1}} - X^\tau_{t_{n-1}}} - X^\tau_{t_{n-1}}|_{F_{n-1}} \right) \right)$$

Using the weak consistency of $X^\tau_T$ with $W^\tau$, we get, recalling $T = N^4 \theta$,

$$|B| \leq E\sum_{n=1}^{n} \sum_{|\alpha|=2} |D^\alpha \psi(X^\tau_{t_{n-1}})| \times$$

$$\left| E((X^\tau_{t_n} - X^\tau_{t_{n-1}})^\alpha |F_{n-1}) - E\left( (W_{t_n}^{\tau,\xi_{t_n-1}X^\tau_{t_{n-1}} - X^\tau_{t_{n-1}}} - W_{t_n}^{\tau,\xi_{t_{n-1}}})^\alpha |F_{n-1} \right) \right|$$

Similarly for the second moments, since $X^\tau_{t_{n-1}} \in (1 - \varepsilon)D$,
Finally, since third order moments are at most of order \( \theta^{3/2} \), we deduce

\[
|C| = \left| \mathbb{E} \sum_{n=1}^{n_{\varepsilon}} \mathbb{E}(R(X_{t_{n-1}}^{\tau^*}, X_{t_n}^{\tau^*}) | \mathcal{F}_{n-1}) + \mathbb{E} \left( R \left( X_{t_{n-1}}^{\tau^*}, W_{t_n}^{\tau, X_{t_{n-1}}^{\tau^*}, t_n} \right) | \mathcal{F}_{n-1} \right) \right|
\]
\[
\lesssim \|D^3 \psi\|_{\infty} \sum_{n=1}^{n_{\varepsilon}} \mathbb{E}(|X_{t_{n-1}}^{\tau^*} - X_{t_n}^{\tau^*}|^3 | \mathcal{F}_{n-1}) + \mathbb{E} \left( |W_{t_n}^{\tau, X_{t_{n-1}}^{\tau^*}, t_n} - X_{t_{n-1}}^{\tau^*}|^3 | \mathcal{F}_{n-1} \right)
\]
\[
\lesssim \|D^3 \psi\|_{\infty} \sum_{n=1}^{n_{\varepsilon}} \theta^{3/2} \lesssim \|D^3 \psi\|_{\infty} \theta^{1/2} = c_{\psi,T}(\varepsilon).
\]

This concludes the proof of the weak convergence. \( \square \)

**Proof of Lemma 6** We aim at estimating \( \|f(X_{t_n}^{\tau^*}) - M^f_T\|_p := (\mathbb{E}|f(X_{t_n}^{\tau^*}) - M^f_T|^p)^{1/p} \). Split first

\[
f(X_{t_n}^{\tau^*}) - f(X_{0}^{\tau^*}) = f(B_{T_n}^{\tau^*}) - f(B_{0}^{\tau^*}) = \sum_{k=1}^{k_{\varepsilon}} [f(B_{k_{\varepsilon}}^{\tau^*}) - f(B_{k_{\varepsilon}-1}^{\tau^*})]
\]
\[
= \sum_{k=1}^{k_{\varepsilon}} \sum_{i=1,2} \partial_i f(B_{k-1}) dB_{k, i} + \frac{1}{2} \sum_{k=1}^{k_{\varepsilon}} \sum_{i,j=1,2} \partial_{i,j}^2 f(B_{k-1}) dB_{k, i} dB_{k, j}
\]
\[
+ \sum_{k=1}^{k_{\varepsilon}} R^f_k(B_{k-1}, dB, k),
\]

and on the other hand

\[
M^f_T - M^f_0 := \sum_{k=1}^{k_{\varepsilon}} \sum_{i=1,2} \partial_i f(B_{k-1}) dB_{k, i}.
\]

Since \( f(X_{0}^{\tau^*}) = M^f_0 = f(0) \), it follows simply

\[
f(X_{t_n}^{\tau^*}) - M^f_T = \frac{1}{2} \sum_{k=1}^{k_{\varepsilon}} \sum_{i,j=1,2} \partial_{i,j}^2 f(B_{k-1}) dB_{k, i} dB_{k, j} + \sum_{k=1}^{k_{\varepsilon}} R^f_k(B_{k-1}, dB, k) =: A + B
\]

For the second term above, we observe that for all \( k, \) all \( x, \)

\[
|R^f_k(B_{k-1}(x), dB(x))| \lesssim \|D^3 f\|_{\infty} \delta^{3/2},
\]

and therefore

\[
\|B\|_p \lesssim \sum_{k=1}^{N_{\varepsilon}} \|D^3 f\|_{\infty} \delta^{3/2} = \|D^3 f\|_{\infty} (N_{\varepsilon} \delta)^{1/2} = c_T(\varepsilon).
\]

Recalling that \( \tau^* = n_{\varepsilon} \delta = k_{\varepsilon} \theta \) or equivalently \( k_{\varepsilon} = N_{\varepsilon} n_{\varepsilon} \), we split the sum \( A \)
into blocks of size $N$, namely $A = \sum_{n=1}^{N} A_n$, with
\[
A_n = \frac{1}{2} \sum_{l=1}^{N} \partial_{l1}^2 f(B_{(n-1)N+l-1})(dB_{(n-1)N+l}^1)^2 + \partial_{22}^2 f(B_{(n-1)N+l-1})(dB_{(n-1)N+l}^2)^2
\]
\[
= \delta \sum_{l=1}^{N} \partial_{l1}^2 f(B_{(n-1)N+l-1})1(\varepsilon_{(n-1)N+l-1} = -1) + \partial_{22}^2 f(B_{(n-1)N+l-1})1(\varepsilon_{(n-1)N+l-1} = +1)
\]
\[
= \delta \sum_{l=1}^{N} \partial_{l1}^2 f(B_{(n-1)N+l-1}) + \partial_{22}^2 f(B_{(n-1)N+l-1})
\]
\[
+ \delta \sum_{l=1}^{N} \partial_{22}^2 f(B_{(n-1)N+l-1}) - \partial_{l1}^2 f(B_{(n-1)N+l-1})\varepsilon_{(n-1)N+l-1}
\]
\[
= \delta \sum_{l=1}^{N} \partial_{22}^2 f(B_{(n-1)N+l-1})\varepsilon_{(n-1)N+l-1},
\]
where we used $1(\varepsilon = \pm 1) = \frac{1}{2} + (1(\varepsilon = \pm 1) - \frac{1}{2}) = \frac{1}{2} \pm \frac{e}{2}$ and the harmonicity of $f$. We split further
\[
A_n = \delta \sum_{l=1}^{N} [\partial_{22}^2 f(B_{(n-1)N+l-1}) - \partial_{22}^2 f(B_{(n-1)N})]\varepsilon_{(n-1)N+l-1}
\]
\[
+ \delta \partial_{22}^2 f(B_{(n-1)N}) \sum_{l=1}^{N} \varepsilon_{(n-1)N+l-1}
\]
\[
=: B_n + C_n
\]
For $B_n$, we observe $|\partial_{22}^2 f(B_{(n-1)N+l-1}) - \partial_{22}^2 f(B_{(n-1)N})| \lesssim \|D^3 f\|_{\infty} t \sqrt{\delta}$, therefore $\|B_n\|_p \lesssim N^2 \delta^{3/2}$ and
\[
\left\| \sum_{n=1}^{N} B_n \right\|_p \lesssim N^6 \delta^{3/2} = (N^5 \delta)N(TN^{-5})^{1/2} = c_3(T).
\]
Next the norm of $C_n$ is estimated
\[
\|C_n\|_p \lesssim \delta \|D^2 f\|_{\infty} \left( \mathbb{E} \left\| \sum_{l=1}^{N} \varepsilon_{(n-1)N+l-1} \right\|_p^p \right)^{1/p}.
\]
Notice that the sum above is $\sum_{l=1}^{N} \varepsilon_{(n-1)N+l-1} = dX_{n}^1 + dX_{n}^2$, and we know from the moment estimates that $\|dX_{n}^i\|_p \lesssim \theta^{1/2}$, $i = 1, 2$. We conclude
\[
\left\| \sum_{n=1}^{N} C_n \right\|_p \lesssim N^4 \delta \|D^2 f\|_{\infty} (N\delta)^{1/2} = (N^5 \delta)\|D^2 f\|_{\infty} N^{-1/2} \delta^{1/2} = c_T(\varepsilon)
\]
This concludes the proof of Lemma 6.

5 Proofs of the main results

Proof of Theorem 2 (Convergence of $L^p$ norms of martingales) Recall that we want to prove

$$
\lim_{T \to \infty} \lim_{\epsilon \to 0} \frac{E[|M^f_T|^p]}{E[f(W^\tau_\infty)]^p} = 1
$$

or equivalently in terms of norms

$$
\lim_{T \to \infty} \lim_{\epsilon \to 0} \|M^f_T\|_p = \|f(W^\tau_\infty)\|_p.
$$

Split first as the sum of three differences

$$
\|f(W^\tau_\infty)\|_p - \|M^f_T\|_p
= \|f(W^\tau_\infty)\|_p - \|f(W^\tau_T)\|_p + \|f(W^\tau_T)\|_p - \|f(X^\tau_T)\|_p + \|f(X^\tau_T)\|_p - \|M^f_T\|_p
=: A + B + C
$$

As seen before, we have $P(\tau > T) = c(T)$, therefore $|A| = c(T)$. For the third term, we have simply $|C| \leq \|f(W^\tau_T) - f(X^\tau_T)\|_p = c_T(\epsilon)$ thanks to Lemma 6.

For the second term, define successively on the boundary $\partial \mathbb{D}$, $\psi(x) := \frac{1}{|f(x)|^p_X}$ and $\psi_\eta := \psi * \rho_\eta$ a mollified version of $\psi$ tending to $\psi$ when $\eta$ goes to zero. We also denote $\psi$ (resp. $\psi_\eta$) defined on $\mathbb{D}$ the Poisson extension of $\psi|_{\partial \mathbb{D}}$ (resp. $\psi_\eta|_{\partial \mathbb{D}}$). Since $f \in L^\infty(\partial \mathbb{D}; X)$, it follows that $\psi$ and $\psi_\eta$ are bounded in $\mathbb{D}$, and $\psi_\eta = \psi + c(\eta)$ in $L^\infty(\mathbb{D})$. Finally, notice that if $|x| < 1$, then we have for the Poisson extensions $\psi(x) = |f(x)|^p_X$. However $\psi(x) = |f(x)|^p_X + c(\epsilon)$ for those $x$'s next to the boundary, i.e. $1 - \epsilon < |x| \leq 1$. We can now estimate the first term of $B$:

$$
\|f(W^\tau_T)\|_p = \mathbb{E}(|f(W^\tau_T)|^p)^{1/p} = \mathbb{E}(|f(W^\tau_T)|^p|T > \tau)^{1/p} + c(T)
= \mathbb{E}(\psi(W^\tau_T)|T > \tau)^{1/p} + c(T) = \mathbb{E}(\psi(W^\tau_T))^{1/p} + c(T)
= \mathbb{E}(\psi_\eta(W^\tau_T))^{1/p} + c(\eta) + c(T)
$$

where we have used that $W^\tau_T \in \partial \mathbb{D}$ when $T > \tau$. Similarly, since $P(\tau_\epsilon > T) = c(T)$ and $1 - \epsilon \leq X^\tau_T \leq 1$ for $T > \tau_\epsilon$, we have

$$
\|f(X^\tau_T)\|_p = \mathbb{E}(|f(X^\tau_T)|^p)^{1/p} = \mathbb{E}(|f(X^\tau_T)|^p|T > \tau_\epsilon)^{1/p} + c(T)
= \mathbb{E}(\psi(X^\tau_T)|T > \tau_\epsilon)^{1/p} + c(\epsilon) + c(T)
= \mathbb{E}(\psi_\eta(X^\tau_T))^{1/p} + c(\epsilon) + c(T)
= \mathbb{E}(\psi_\eta(W^\tau_T))^{1/p} + c(\eta) + c(\epsilon) + c(T)
$$

It follows,

$$
|B| \leq |\mathbb{E}(\psi_\eta(W^\tau_T))^{1/p} - \mathbb{E}(\psi_\eta(X^\tau_T))^{1/p}| + c(\eta) + c(\epsilon) + c(T)
= c_{\eta,T}(\epsilon) + c(\eta) + c(\epsilon) + c(T)
$$

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where we used Lemma 5 for the second line. Finally
\[ \| f(W_\infty^T) \|_p - \| M_T^f \|_p = A + B + C = c_\eta,T(\varepsilon) + c(\eta) + c(\varepsilon) + c(T). \]

Fix any small \( \eta > 0 \), choose \( T > 0 \) large enough so that \( c(T) \leq \eta \), then \( \varepsilon > 0 \) small enough so that \( c_\eta,T(\varepsilon) + c(\varepsilon) \leq \eta \). Hence
\[
\lim_{T \to \infty} \lim_{\varepsilon \to 0} \| f(W_\infty^T) \|_p - \| M_T^f \|_p \leq c(\eta),
\]
therefore
\[
\lim_{T \to \infty} \lim_{\varepsilon \to 0} \| M_T^f \|_p = \| f(W_\infty^T) \|_p
\]
as desired. This concludes the proof of Theorem 2. \( \square \)

**Proof of Theorem 1.** This is now a direct consequence of Theorem 2. Let \( f \in L^p(\partial \Omega) \). Its \( L^p \) norm is directly related to the stochastic \( L^p \) norm
\[
\| f(W_\infty^T) \|_p := (\mathbb{E} |f(W_\infty^T)|_p^n)^{1/n} = \left( \int_{\partial \Omega} |f(x)|_p^n \frac{dz}{2\pi} \right)^{1/n} = \frac{1}{(2\pi)^{1/n}} \| f \|_{L^p(\partial \Omega)},
\]
and the same relation holds for the smooth function \( g := \mathcal{H}f \). From Theorem 2 we know
\[
\| f(W_\infty^T) \|_p = \lim_{T \to \infty} \lim_{\varepsilon \to 0} \| M_T^f \|_p, \quad \| g(W_\infty^T) \|_p = \lim_{T \to \infty} \lim_{\varepsilon \to 0} \| M_T^g \|_p,
\]
and from Lemma 1 we know that for all \( T > 0 \), \( \varepsilon > 0 \),
\[ \| M_T^f \|_p \leq \| \mathcal{S} \|_{p \to p} \| M_T^g \|_p. \]
It follows that for any \( f \in L^p(\partial \Omega) \),
\[ \| g \|_{L^p(\partial \Omega)} = \| \mathcal{H}f \|_{L^p(\partial \Omega)} \leq \| \mathcal{S} \|_{p \to p} \| f \|_{L^p(\partial \Omega)}, \]
that is \( \| \mathcal{H} \|_{p \to p} \leq \| \mathcal{S} \|_{p \to p}. \) \( \square \)

### 6 Averaging of the dyadic Hilbert transform

We prove that the average of the dyadic Hilbert transform \( \mathcal{S} \) in the sense of \( \mathcal{S} \) is null. For that, let \( D^{\alpha,r} = \{ 2^r \mathcal{I} + \alpha : \mathcal{I} \in \mathcal{D} \} \) the dilated and translated dyadic grid. Here, let \( 1 \leq r < 2 \) and \( \alpha \in \mathbb{R} \). Denote by \( h^{\alpha,r}_j \) the corresponding \( L^2 \)-normalized Haar functions and by \( S^{\alpha,r} \) the dyadic Hilbert transform associated to \( D^{\alpha,r} \). Since in the usual sense,
\[
\mathcal{S}^{\alpha,r} : f(x) \mapsto \sum_{I \in D^{\alpha,r}} [-\langle f, h^{\alpha,r}_I \rangle h^{\alpha,r}_I(x) + \langle f, h^{\alpha,r}_{I^-} \rangle h^{\alpha,r}_{I^-}(x)],
\]
the kernel of \( \mathcal{S}^{\alpha,r} \) is
\[
K^{\alpha,r}(t,x) = \sum_{I \in D^{\alpha,r}} [-h^{\alpha,r}_I(t)h^{\alpha,r}_I(x) + h^{\alpha,r}_{I^-}(t)h^{\alpha,r}_{I^-}(x)].
\]

Here is the illustration of the sign distribution of a single term for a given interval \( I \).
Following the strategy of the second author in [5], the average of the kernel by dilation and translation we consider is:

\[
\mathbb{E}_r \mathbb{E}_\alpha K^{\alpha,r}(t,x) = \frac{1}{\log 2} \int_1^2 \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x) \, d\alpha \frac{dr}{r}.
\]

Lemma 7 We have \( \mathbb{E}_r \mathbb{E}_\alpha K^{\alpha,r}(t,x) = 0 \).

Proof While these averages can be calculated explicitly, one can also argue as follows. The kernel \( K^{\alpha,r}(t,x) \) can be split into a sum of partial kernels

\[
K^{\alpha,r}_-(t,x) = \sum_{I \in D^{\alpha,r}} h^{\alpha,r}_I(t) h^{\alpha,r}_I(x)
\]

and

\[
K^{\alpha,r}_+(t,x) = \sum_{I \in D^{\alpha,r}} -h^{\alpha,r}_{I_{-}}(t) h^{\alpha,r}_{I_{+}}(x).
\]

\( K^{\alpha,r}_-(t,x) \) is supported under the diagonal, \( t > x \) while \( K^{\alpha,r}_+(t,x) \) is supported over the diagonal, \( t < x \). Clearly the averages produce no mass on the diagonal. Observe now that the average of the even kernel \( K^{\alpha,r}_-(t,x) - K^{\alpha,r}_+(t,x) \) is dilation invariant, translation invariant and symmetric. It therefore represents the zero operator. We may then conclude that the averages of the partial kernels themselves are 0. The claim follows. \( \square \)

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