TOPOLOGICAL AND SYMBOLIC DYNAMICS FOR AXIOM A DIFFEOMORPHISMS WITH HOLES

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Abstract. We consider an Axiom A diffeomorphism and the invariant set of orbits which never falls into a fixed hole. We study various aspects of the complexity of the symbolic representation of Ω. Our main result are that each topologically transitive component of Ω is coded and that typically Ω is of finite type.

1. Introduction

Let f be an Axiom A diffeomorphism of a compact s–dimensional Riemannian manifold and Λ ⊂ M the nonwandering set of f. We fix a Markov partition, this yields a natural representation of Λ as a subshift of finite type Σ. We cut out an open hole H out of M and consider the invariant set Ω = ΩH ⊂ Λ of nonwandering points whose orbit (forward and backwards) never falls in the hole. The set Ω corresponds to a subshift Σ = ΣH ⊂ Σ. We call such a subshift Σ an exclusion subshift. We are interested in several questions about the topological structure of exclusion subshifts. More precisely for which holes are exclusion shifts subshifts of finite type (SFT), sofic shifts, or coded systems? As we vary the hole what is the typical type of an exclusion subshift?

Our first result is that an exclusion subshift has at most countably many topologically transitive components and exclusion subshifts are always coded systems on each topological transitive component. Under additional assumptions we can prove the finiteness of the number of transitive components. Next we give a criterion for when the hole leads to a SFT. We then apply this criterion to several different classes of holes (rectangles, polyhedra, holes with continuous boundary) with corresponding topologies/measures to show that the “typical” hole leads
to a SFT. Finally we show that every $\beta$–shift is an exclusion shift, thus in particular there are exclusion shifts which are sofic and ones which are not sofic.

Dynamical systems with holes have been studied extensively by physicists. Cvitanovich and his coworkers have investigated how to characterize the set $\Sigma$ in various settings (see the survey [13]). They have introduced the notion of a pruning front which corresponds to $\partial H$ in our setting. Some aspects of Cvitanovich’s work have been carried out by Carvalho in a mathematical framework for Smale’s horseshoe [3].

Another kind of question about dynamical systems with holes have also been extensively studied by physicists, namely construction of a physical semi–invariant measure and the understanding of the speed of mass disappearance into the holes (the escape rate formula) [15]. A series of mathematical works have confirmed the expectations of the physicists in many settings [6, 7, 12, 17, 18, 19]. Bäcker and Krüger have initiated the study of the topological structure of exclusion subshifts in the one dimensional setting [1].

2. Definitions

Let $M$ be a compact $s$–dimensional ($s \geq 2$) Riemannian manifold and $f : M \to M$ an Axiom A diffeomorphism. Fix a proper, generating Markov partition. Let $(\hat{\Sigma}, \sigma)$ be the resulting SFT and $\pi : \hat{\Sigma} \to M$ the projection map. We cut out an open hole out of $M$ whose boundary $\partial H = \bar{H} \setminus H$ consists of a finite union of topological $(s - 1)$–dimensional spheres and $H = \text{int}(\bar{H})$. Consider the invariant set $\Omega^* = \Omega^*_H$ of points whose orbit (forward and backwards) never falls in the hole and the invariant set $\Omega$ of nonwandering points for $f|_\Omega$. Let $\Sigma^* = \Sigma^*_H = \pi^{-1}\Omega^*$ and $\Sigma = \pi^{-1}\Omega$ with the following convention: if $x \in \Omega$ is on the boundary of the hole and the preimage of $x$ is not unique, then we only consider those preimages of $x$ to be in $\Sigma$ which can be approximated from outside the closure of $H$, i.e. those $s \in \pi^{-1}(x)$ such that $\exists x_j \not\in \bar{H}$ such that $x_j \to x$ and $\exists s_j \in \pi^{-1}(x_j)$ with $s_j \to s$.

We call such a subshift $\Sigma$ an exclusion subshift. Remark: our standard

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1 A Markov partition is called proper if each element of the partition is the closure of its interior.

2 More generally, we can consider a map which is a local diffeomorphisms except on a singularity set which consists of a finite union of codimension one manifolds such that the map admits a finite proper generating Markov partition.

3 Occasionally we will discuss cases when the hole is not open.

4 This convention is not needed in the case that the coding is always unique, for example for horseshoes.
examples will be two dimensional, the $n$–branch horseshoe map and
the $n$–branch Bakers map.

If $M$ is two dimensional, call a hole a rectangle like hole if $\partial H$
consists of a finite number of curves, each of which is a finite length piece
of stable or unstable manifold of $f$. We call the corresponding exclusion
shift a rectangle exclusion shift RES.

We will also consider the one dimensional situation, where $M$ is an
interval or a circle. We will consider continuous maps of the circle
or piecewise continuous maps of the interval which, when considered
as a map of the circle are continuous (for example the doubling map
$f(x) = 2x \mod 1$). Additional we require the existence of a finite gen-
erating Markov partition $\mathcal{P}$ whose elements consist of intervals. Since
the map is not invertible we only require that the forward orbit never
falls into a hole. If the hole consists of a finite union of intervals we call
the corresponding exclusion shift an interval exclusion shift IES. Our
standard example in this framework is the doubling map.

3. SFTs and sofic systems

Consider the alphabet $\{1, 2, \ldots, n\}$ and a finite collection of forbid-
den words on length $m$ ($m$ fixed, usually taken to be 2). The subset
of all sequences in $\{1, \ldots, n\}^\mathbb{Z}$ (resp. $\{1, \ldots, n\}^\mathbb{N}$) where the forbidden
word never appears is called a subshift of finite type (SFT).

**Theorem 3.1.** Every SFT is an exclusion subshift.

**Proof.** Consider a SFT $\Sigma$. Let $n$ be the cardinality of the alphabet of
$\Sigma$ and $m$ the length of the forbidden blocks. Consider a horseshoe map
$f$ with $n$–branches. We consider the standard Markov partition for $f$.
Define $H'$ to be the (finite) union of Markov rectangles (for $f^m$) which
correspond the forbidden $m$–blocks which define $\Sigma$. Markov rectangles
are by definition closed, however since a horseshoe is a Cantor set we
can find an open hole $H$ containing $H'$ such that the intersection of
$H$ with the horseshoe is exactly $H'$. This yields $\Sigma$ as an exclusion
subshift. 

Remark: the same construction works for the $n$-fold Bakers map, ex-
cept that we define the hole $H$ to be the interior of $H'$ and use the
coding convention from section 2.

Let $\Sigma_{even}^+$ be the one sided even shift, that is the set of all 0–1 half
infinite sequences with the constraint that the number of consecutive
ones which occur in between two zeros is always even.
Example 3.2. Let \( M := [0, 1], f(x) := 2x \mod 1, \mathcal{P} = \{ [0, \frac{1}{2}), [\frac{1}{2}, 1] \} \). The even shift \( \Sigma^+_{\text{even}} \) is not an interval exclusion system with respect to \((f, \mathcal{P})\).

Proof. Suppose we have a hole which is a union of intervals and yields \( \Sigma^+_{\text{even}} \) as an exclusion system. Consider the point \( x_n = 01^{2n+1}0^\infty \). The whole orbit of \( x_n \), except \( x_n \) itself is in \( \Sigma^+ \). Thus \( x_n \) must lie in the hole.

Let \( y_n = 01^{2n}0^\infty \). Clearly \( y_n \in \Sigma^+ \) and \( x_{n-1} < y_n < x_n \). Thus the hole must consist of an infinite number of intervals: \( \Sigma^+_{\text{even}} \) can not be an IES.

Remark: the exact same proof shows that \( \Sigma^+_{\text{even}} \) is not an exclusion system with respect to \((f, \mathcal{P})\) for any hole which is a finite union of intervals which are not necessarily open.

Let \( \Sigma_{\text{even}} \) be the two sided even shift, that is the set of all 0–1 bi–infinite sequences with the constraint that the number of consecutive ones which occur in between two zeros is always an even number.

Example 3.3. Let \( M = [0, 1]^2, f : M \to M \) be the Bakers map:

\[
f(x, y) = \begin{cases} 
(2x \mod 1, y/2) & \text{if } x \leq \frac{1}{2} \\
(2x \mod 1, (y + 1)/2) & \text{otherwise.}
\end{cases}
\]

Let \( \mathcal{P} := \{ P_0, P_1 \} \) be the standard Markov partition, i.e. \( P_0 := \{ (x, y) : x \leq 0.5 \} \) and \( P_1 := \{ (x, y) : x \geq 0.5 \} \) Then the even shift \( \Sigma_{\text{even}} \) is not a RES with respect to \((f, \mathcal{P})\).

Proof. Let \( x_n = 0^\infty 1^{2n+1}0^\infty \) (here the decimal point marks the position between the \(-1\)st and \(0\)th elements of the sequence). The point \( x_n \) is not in the even shift, it must fall into the hole under some iteration of \( f \).

We treat several cases, first of all suppose that \( x_n \) falls into the hole at the boundary of \( M \). Then at the instant that \( x_n \) falls into the hole all the 1’s are to the right (or all are to the left) of the decimal point (i.e. \( f^j x_n \in \partial M \) with \( j \leq 0 \) or \( j \geq 2n + 1 \)). Note that the intersection of the rectangular holes with the boundary consists of a finite union of intervals. Thus we can apply the argumentation of the previous example to conclude that it is impossible to have an infinite number of the \( x_n \) fall into the hole when they are on the boundary of \( M \).

In other words all but finitely many \( x_n \) which fall into the hole away from the boundary of \( M \). For such an \( x_n \) consider the code \( a \) at the instance of falling into the hole. It has the form \( 0^\infty 1^p 1^q 0^\infty \) where \( p + q = 2n \).
Consider the sequence \((u_i)_{i \in \mathbb{Z}}\) with \(u_i = 0^\infty 1^2 01^{p-1} 1^q 10^{q+1} 0^\infty\) and the sequence \((v_i)_{i \in \mathbb{Z}}\) with \(v_i = 0^\infty 1^p 1^q 01^{2i} 0^\infty\). These two sequences get arbitrary near from the left and the bottom to \(a\) and all their elements are in the even shift. Thus \(a\) must be a corner of the hole. This contradicts to the assumption that the number of corners of the hole is finite.

Remark: the same proof shows that \(\Sigma_{\text{even}}\) is not a RES with respect to the \(n\)-fold Bakers map, it also shows that \(\Sigma_{\text{even}}\) is not a RES for any hole which is a finite union of closed rectangular like holes. An even simpler proof can be given if \(f\) is a standard horseshoe map with a standard Markov partition.

A closed shift invariant subset of \(\{1, 2, \ldots, n\}^\mathbb{Z}\) (resp. \(\{1, 2, \ldots, n\}^\mathbb{N}\)) is called sofic if it a factor of a SFT. The even shift is an example of a sofic system.

**Theorem 3.4.** 1. In dimension one not every sofic system is an interval exclusion subshift.

2. In dimension two not every sofic system is a rectangular hole exclusion subshift.

**Proof.** In the two examples we showed that the even shift (one sided even shift) can not be represented as a RES (IES) in our standard model: the Bakers map (the doubling map).

To prove part 2 we will show that if the even shift is representable as a RES for some Axiom A diffeomorphism then it is in fact an RES for the Bakers map. The proof of part one is similar.

Let \((f, \mathcal{P})\) be the Bakers map with the standard Markov partition. We suppose by way of contradiction that \(\Sigma_{\text{even}}\) is a RES for some Axiom A diffeomorphism \(g\) with respect to the Markov partition \(\mathcal{Q}\). Consider the SFT \(\Sigma\) defined by \((g, \mathcal{Q})\) and denote the projection \(\pi : \Sigma \to \Lambda_g\). By assumption there is a rectangle like hole \(H\), that is the set \(\Omega_g\) of points which never fall into \(H\) satisfies \(\pi : \Sigma_{\text{even}} \to \Omega_g\).

By the proof of theorem 3.1 we can find an rectangle like hole \(H'\) which yields \(\Sigma\) as a RES for \((f, \mathcal{P})\). Let \(\Lambda_f'\) be the invariant set of points which never fall into \(H'\) and let \(\pi' : \Sigma \to \Lambda_f'\) be the projection.

Consider \(H_{\text{symb}} := \{s \in \Sigma : \pi(s) \in H\}\) and \(H'' := \pi'(H_{\text{symb}})\). Since the maps \(\pi\) and \(\pi'\) preserve local product structure the set \(H''\) is a rectangle like hole. If \(H''\) is open then \(\Sigma_{\text{even}}\) is a RES in the Bakers map, a contradiction. If \(H''\) is not open then the interior of \(H''\) is also a rectangular like hole, which using the convention defined in section 2 shows that \(\Sigma_{\text{even}}\) is an RES for the Bakers map, a contradiction. \(\square\)
We do not know if every sofic system, or even if every coded system is an exclusion subshift. However, if we drop the requirement that the boundary consists of spheres then we get that all subshifts are weak exclusion subshifts, that is they are the complement of an open hole without any further assumptions on the boundary. To see this simply take the compliment of the invariant set which gives rise to the subshift as the hole.

**Theorem 3.5.** Every $\beta$–shift is a RES.

**Proof.** We begin by describing a one dimensional construction from [1]. Consider the dynamical system $h_\beta : [0, 1) \to [0, 1)$ given by $h_\beta x = \beta x \mod 1$. Suppose $\beta \in (1, 2]$ and consider the “partition” $I_0 := [0, \beta^{-1}]$ and $I_1 := [\beta^{-1}, 1]$. Given $\omega, \alpha \in \{0, 1\}^\mathbb{N}$ we say $\omega < \alpha$ if there exists a positive integer $N$ such that $\omega_i = \alpha_i$ for $i = 0, 1, \ldots, N - 1$ and $\omega_N = 0 < \alpha_N = 1$. It is well known that the code $\hat{\beta}$ of the orbit of $\beta$ satisfies $\sigma^n \hat{\beta} < \hat{\beta}$ for all $n > 0$. The set of all sequences which are the code of some orbit is called the one sided $\beta$–shift, it is characterized by

$$\{ x \in \{0, 1\}^\mathbb{N} : \sigma^n x < \beta \text{ for all } n > 0 \}.$$ 

Let $g : [0, 1) \to [0, 1)$ be defined by $g(x) = 2x \mod 1$. Inspired by the above a number $\beta$ is called a $\beta$–number if $g^n \beta < \beta$ for all $n > 0$. If $\beta$ is a $\beta$–number then the set $X_\beta := \{ x \in (0, 1) : g^n x < \beta \ \forall n > 0 \}$ is conjugate to the one sided $\beta$–shift. If $H \subset (0, 1)$ is the interval $(\beta, 1)$ then the set of points whose forward $g$ orbit never falls in $H$ is exactly $X_\beta$.

Now let $f$ be the Bakers map. Fix $\beta$ a $\beta$ number and let $H \subset M$ be the rectangular hole $\{(x, y) : x > \beta, \ 0 \leq y \leq 1\}$. Let $(x_{-i}, y_{-i}) = f^{-1}(x, y)$. Since the hole stretches from the bottom to the top of $M$ it is easy to see that $(x, y) \in \Omega$ if and only if $g^n(x_{-i}) < \beta$ for all $n \geq 0$ and all $i \geq 0$. Thus the exclusion shift in this example is exactly the natural extension of the one sided $\beta$–shift. We remark that the two sided $\beta$–shift is of finite type, sofic, or not sofic if and only if the one sided one has the same property. It is well known that the $\beta$–shift is of finite type if the binary expansion of $\beta$ is finite, it is sofic if it eventually periodic and otherwise it is not sofic [2].

**Remark:** for $\beta \in (n - 1, n]$ we must use the map $g = nx \mod 1$.

4. **Transitive components of $\Sigma$ are coded**

A homeomorphism $T$ of a compact metric space $X$ is called topologically transitive if there exists a dense orbit, or equivalently if every proper closed $T$-invariant subset is nowhere dense. A closed invariant
set \( \hat{X} \) (i.e. \( T^{-1}A = TA = A \)) is called a topological transitive component if \( \hat{X} \) contains a dense orbit and no closed invariant set \( \hat{X}' \supset \hat{X} \) contains a dense orbit.

We remark that the topologically transitive components of a SFT are exactly its irreducible components since, by convention (see section 2) we restrict the map to its nonwandering set (see for example [16]). In particular this implies that the topologically transitive components of a SFT are disjoint. Furthermore periodic points are dense in a topologically transitive component of a SFT or in a topologically transitive component of an Axiom A diffeomorphism.

We call a topologically transitive component \( \hat{X} \) boundary supported (BSC) if periodic points whose orbit avoids \( \overline{H} \) are not dense in \( \hat{X} \) otherwise we call \( \hat{X} \) non boundary supported.

A subshift \( \Sigma \) is called a coded system if it can be represented by an irreducible countable labeled graph [3]. Equivalently, \( \Sigma \) is called coded if \( \Sigma \) contains an increasing sequence of irreducible subshifts of finite type (SFTs) whose union is dense in \( \Sigma \) [4].

**Theorem 4.1.** Every non boundary supported topologically transitive component is coded. There are at most countably many such topologically transitive components.

**Proof.** Let \( \Sigma \) be an exclusion subshift, i.e. \( f \) is an Axiom A map, \( \mathcal{P} \) a fixed proper, generating Markov partition, and \( H \) an open hole with compact boundary. Then \( \mathcal{P}^{(n)} := \bigvee_{i=-n}^{n} f^{i} \mathcal{P} \) is also a proper, generating Markov partition. Let \( H^{(n)} := \text{int}(\bigcup_{\{P \in \mathcal{P}^{(n)}: P \cap \overline{H} \neq \emptyset\}} P) \). Clearly \( H^{(1)} \supset H^{(2)} \supset \cdots \) and \( \overline{H} \subset \bigcap_{n} H^{(n)} \). Furthermore, since \( \mathcal{P} \) is proper we have \( \text{diam}(\mathcal{P}^{(n)}) \to 0 \) as \( n \to \infty \) thus \( \bigcap_{n} H^{(n)} = \overline{H} \). The “exclusion” system \( \Sigma^{(n)} := \Sigma_{H^{(n)}} \) is a SFT. Clearly \( \Sigma^{(1)} \subset \Sigma^{(2)} \subset \cdots \) and

\[
\tilde{\Sigma} := \bigcup_{n} \Sigma^{(n)} \subset \Sigma.
\]  

Clearly any topologically transitive component which is a subset of \( \bigcup_{n} \Sigma^{(n)} \) is non boundary supported since periodic points are dense in SFTs. On the other hand if \( x \in \Sigma \setminus \tilde{\Sigma} \) then \( x \) is not an accumulation point of periodic points \( x_n \) which avoid \( \overline{H} \) since any such periodic point necessarily belongs to \( \Sigma^{(n)} \) for some \( n \). Thus we have shown that every non boundary supported topologically transitive component is contained in \( \tilde{\Sigma} \).

\footnote{We write quotes since the boundary of \( H^{(n)} \) may not be the union of codimension 1 spheres.}
If each $\Sigma^{(n)}$ was topologically transitive then $\tilde{\Sigma}$ would be coded. Let $A_i^{(n)}$ be the topologically transitive components of $\Sigma^{(n)}$. The $A_i^{(n)}$ form a filtration in the sense that for each $A_i^{(n)}$ there exists $A_j^{(n+1)}$ such that $A_i^{(n)} \subset A_j^{(n+1)}$. In other words the transitive components $A_i^{(n)}$ form an at most countable union of directed trees with each node out degree is exactly one. The equality in equation (4.1) implies that each topologically transitive component of $\sigma|\Sigma$ contains a set $\bigcup A_i^{(n)}$ where the union is taken over a path in one of the trees (we call this a path limit). Noticing that such a path is uniquely defined by the root of the tree since the out degree is always one, implies that there are at most countably many such paths and thus $\sigma|\tilde{\Sigma}$ has at most countably many topologically transitive components. This finished the proof of the countability of the claim of the theorem.

Suppose $C \subset \tilde{\Sigma}$ is a topologically transitive component of $\sigma|\tilde{\Sigma}$. To see that $C$ is coded we will define a new filtration. Since $\Sigma^{(n)}$ is a subshift of finite type it has finitely many topologically transitive components $A_i^{(n)}$ which are pairwise disjoint. Consider those components $A_1^{(n)}: i = 1, \ldots, k_n$ which are strictly contained in $C$. We can assume that the $A_i^{(n)}$ are so ordered that $A_1^{(n)} \subset A_1^{(n+1)}$ for all $n$. We only need to show that

$$\bigcup_n A_1^{(n)} = C. \tag{4.2}$$

The rest of the proof is devoted to establishing this equality.

If equation (4.2) is not true there is a $n_0$ such that for all $n \geq n_0$ we can find another $A_1^{(n)} \subset C$ which we denote without loss of generality $A_2^{(n)}$ such that $A_2^{(n)} \subset A_2^{(n+1)}$ for all $n \geq n_0$ but $\bigcup_n A_2^{(n)} \cap \bigcup_n A_1^{(n)} = \emptyset$. In the terminology introduced above this means we can find two disjoint paths in the trees whose path limits are both contained in $C$.

Fix $n \geq n_0$. For $i = 1, 2$ consider a finite word $w_i \in A_i^{(n)}$ where each symbol and each transition which characterize $A_i^{(n)}$ appear in $w_i$. Since $C$ is topologically transitive there is a point $x \in C$ where the words $w_1$ and $w_2$ both appear in $x$. Thus we can find $l$ (which we assume positive without loss of generality) so that $x \in \sigma^l w_1 \cap w_2$.

Consider the point $x \in M$ with symbolic coding $x$. We can assume without loss of generality that $f^i x \notin \partial H$ for $i = 0, \ldots, l$. From our assumptions for any sufficiently large $N$ and for $i = 0, \ldots, l$ the rectangles $P_i \in \mathcal{P}^{(N)}$ defined by $f^i x \in P_i$ are disjoint from $H$. We can form a new SFT, $\hat{\Sigma}$, with alphabet the $P_i$ and all the transitions made by the orbit segment $f^i x : i = 0, \ldots, l$ allowed. This is clearly a topologically
transitive SFT. It also contains $A_i^{(n)}$ for $i = 1, 2$ since any legal transition in these sets is a legal transition in $\hat{\Sigma}$. But, by the construction of $\Sigma^{(m)}$ we have $\hat{\Sigma} \subset \Sigma^{(m)}$ for sufficiently large $m$ and thus $\hat{\Sigma} \subset A_1^{(m)}$ and $A_2^{(m)}$ a contradiction. \qed

Remark: we actually have a stronger property than coded since our subshifts are well approximable from outside as well as inside.

4.1. Finiteness of topological transitive components in dimension 1. Suppose that $M$ is one dimensional, $(f, P)$ is as described in section 2 and $H = \cup_{i=1}^{p} H_i$ where the $H_i$ are disjoint open intervals. The definitions of $\Omega^*, \Omega, \Sigma^*$ and $\Sigma$ are similar to the corresponding definitions in the invertible case with the only difference being that we only require that the forward orbit does not fall into $H$. Let $\Omega^c := I \setminus \Omega$, $\Omega^c$ is open. Let $\hat{H}_i$ be the maximal interval (as subsets of the circle) containing $H_i$ which is a subset of $\Omega^c$ and $\hat{H} = \cup_{i=1}^{r} \hat{H}_i$. It is possible that several $H_i$ amalgamate into one $\hat{H}_j$, thus $r \leq p$.

Since our maps are not invertible we must modify the definition of topologically transitive component. We require the the forward orbit is dense, and that the set is forward invariant. We gather here some facts about topologically transitive components of continuous maps which we will use.

**Proposition 4.2.** 1) If a topologically transitive component contains an isolated point, then this point is periodic and the component coincides with this periodic orbit.

2) Topologically transitive components are finite or uncountable.

3) Any dense orbit is recurrent

*Proof.* 1) Let $X$ denote the topologically transitive component. If $z$ is isolated and the orbit of $x$ is dense then $f^i x = z$ for some $i \geq 0$. The point $x$ is also isolated, for if not then by continuity $z$ is not be isolated. If the orbit of $x$ is not periodic then since $x$ is isolated we have $fX = X \setminus \{x\}$ and $X$ is not forward invariant. This contradiction implies that $x$ is periodic. Since $x$’s orbit is dense it must coincide with the topologically transitive component.

2) A component which is not finite can not contain any isolated points. It is a simple exercise to show that a closed set without isolated points can not be countable.

3) If $X$ is finite then this is clear. Suppose $X$ is uncountable and the forward orbit of $x$ is dense in $X$. Since $x$ is not isolated the forward orbit of $x$ must come arbitrarily close to $x$ to be dense. \qed
Let \( \{X_i : X_i \subset M\} \) be the topologically transitive components of \( f|_\Omega \) and \( \{Y_i : Y_i \subset \Sigma\} \) the topologically transitive components of \( \sigma|_\Sigma \). Let \( \mathcal{I} \) be the index set of the topologically transitive components of \( f|_\Omega \).

**Theorem 4.3.** Every interval exclusion system has a finitely many topologically transitive components, in particular the number is bounded by the equation:

\[
\| \{i \in \mathcal{I} : X_i \text{ at most countable}\} + 2\| \{i \in \mathcal{I} : X_i \text{ uncountable}\} \leq 2r + \|\partial \mathcal{P} \|.
\]

**Proof.** For all \( z \in \Omega^c \) let \( G(z) \) be the maximal interval containing \( z \) which is a subset of \( \Omega^c \), we refer to \( G(z) \) as a gap. Let \( n(z) \) be the smallest positive integer such that \( f^{n(z)}z \in \hat{H} \). Let \( k = k(z) \) be the index such that \( f^{n(z)} \in \hat{H}_k \). Let \( J(z) \) be the set of \( x \in G(z) \) such that the orbit of \( x \) falls into the same component \( \hat{H}_k \) of the maximal hole at the same time, i.e. \( n(x) = n(z) \) and \( k(x) = k(z) \). We claim that the continuity of \( f \) implies that \( J(z) = G(z) \). Indeed, if this is not the case, then \( \partial J(z) \subset f^{-n}\partial \hat{H}_k \). Since \( \partial \hat{H} \subset \Omega \) this contradicts the definition of the gap \( G(z) \).

Until further notice we assume that \( \Sigma \) is a SFT. Each topologically transitive component \( X_i \) is closed, thus we can define \( a_i := \min(X_i) \) and \( b_i := \max(X_i) \). Since \( \Sigma \) is a SFT it has only a finite number of topologically transitive components \( Y_i \) and these components are disjoint. Until further notice we also suppose that all the points \( a_i \) and \( b_i \) have unique coding (i.e. the orbits of \( a_i \) and \( b_i \) do not intersect \( \partial \mathcal{P} \)).

With this additional assumption the disjointness of the \( Y_i \) implies that \( a_i \) and \( b_i \) do not belong to any transitive component other than \( X_i \). This implies that there are gaps \( G_{a_i} (G_{b_i}) \) on the left (right) side of \( a_i \) (\( b_i \)). Fix \( x \in G_{a_i} \). As we saw above all \( z \) between \( x \) and \( a_i \) fall into the same hole at the same time \( n(z) \). By the continuity of \( f \) this implies that \( f^{n(z)}a_i \in \partial \hat{H} \). A similar statement holds for \( G_{b_i} \).

Consider the ordering on \( \Sigma \) which is compatible with the ordering on \( \Omega \). This ordering can always be defined in an inductive manner by simply considering the relative order of the elements of \( \mathcal{P}^{(n)} := \vee_{i=0}^n f^i \mathcal{P} \). (If \( f \) is locally order preserving then this is simply the lexicographical order on \( \Sigma \)). Fix an \( i \in \mathcal{I} \). Consider the symbolic coding of \( a_i \) and \( b_i \). Call these codings \( s = (s_j)_{j \in \mathcal{N}} \) and \( t = (t_j)_{j \in \mathcal{N}} \) (here the dependence on \( i \) is suppressed since \( i \) is fixed). We claim that if \( X_i \) is uncountable then \( s \) is not a preimage of \( t \) and vice versa \( t \) is not a preimage of \( s \). To see this fix a higher block coding which defines Markov transition graph of \( \Sigma \). The fact that \( a_i \) is defined via a minimum implies that if there are several followers of the symbol \( s_j \) in the Markov graph, then
\(s_{j+1}\) is minimal follower in the sense that in the ordering it is smaller than all other followers. In a similar fashion the sequence \(t\) is maximal.

If \(s\) is a preimage of \(t\) or vice versa, then \(s\) and \(t\) are eventually maximal and minimal at the same time. This means that from the point on that they agree there the maximal follower of \(s_j\) is also the minimal follower of \(s_j\), so there is only one follower of \(s_j\). Thus \(s\) and \(t\) are eventually periodic and the \(Y_i\) is simply this periodic orbit. In particular \(Y_i\) is finite, finishing the proof of the claim.

If \(X_i\) is uncountable, then we have just shown the disjointness of the codes of \(a_i\) and \(b_i\). Since we are still assuming that the points \(a_i\) and \(b_i\) have unique coding this implies the disjointness of their \(f\)-orbits. Thus at least two point of \(X_i\) lie on \(\partial \hat{H}\).

On the other hand if \(X_i\) is at most countable, then by proposition \([4.1]\) it is finite and consists of a single periodic orbit. The points \(a_i\) and \(b_i\) lie on this orbit.

Under the above assumptions we have shown that for every uncountable \(X_i\) there are at least two points of \(X_i\) on the border of \(\hat{H}\) and for every at most countable \(X_i\) there is at least one such point. Thus, since the endpoints can only belong to a single \(X_i\) we have shown

\[
\sharp\{i \in I : X_i \text{ at most countable}\} + 2\sharp\{i \in I : X_i \text{ uncountable}\} \leq 2r.
\]

Since \(\pi\) is at most 2 to 1, if we drop the assumption that the \(a_i\) and \(b_i\) have unique coding then (at most) two symbolically disjoint topologically transitive components can share such a point when projected to the interval \(I\). This can happen at most \(\sharp\partial P\) times yielding the formula in the statement of the theorem in the case that \(\Sigma\) is a SFT.

Finally if \(\Sigma\) is not a SFT we approximate \(H\) by Markov holes \(H^{(n)}\) in the same way as in theorem \([4.1]\). Arguing similarly to the proof of theorem \([4.1]\) we have \(\tilde{\Sigma} := \bigcup_j \Sigma_j \subset \Sigma\) and the difference consists of BSCs.

Note that the number of transitive component of \(\tilde{\Sigma}\) is less than or equal to the limsup of the number of components of the approximating sequence. It remains to bound the number of BSCs.

Let \(u \leq 2r\) be the number of points in \(\partial \hat{H}\) which are boundary points of non BSCs. By the definition of a BSC such a point can not belong to a BSC. We claim that the number of BSCs is at most \(2r - u\). Let \(\mathcal{C} := \{c_j\}\) be the set of points in \(\partial H\) which belong to a BSC. By definition of BSCs \(\mathcal{C}\) must be disjoint from \(\pi(\tilde{\Sigma})\) or else they would be approximable by periodic points. Also by definition every \(c_j\) must belong to \(\partial \hat{H}\). If two holes \(H_i\) and \(H_j\) amalgamate to a single \(\hat{H}_k\) then two points in \(\partial H\) belong to \(\hat{H}_k\) and (i.e. their orbit falls into \(H\)) and thus can not belong to \(\mathcal{C}\). Thus the cardinality of \(\mathcal{C}\) is at most
Clearly any BSC $X_i$ contains at least one of the points $c_j$. We will associate with each BSC $X_i$ a subset $I(X_i)$ of $\partial H \subset X_i$ of “insertable boundary points”. We claim that two distinct BSCs must have nonintersecting sets $I$. Once this claim has been established it immediately follows that the number of BSCs is at most $2r - u$ which will complete the proof of the theorem.

We turn to the proof of the claim. Fix $X = X_i$ a BSC. Let $\hat{c}_j = \pi^{-1}c_j \in \Sigma$. Note that $\hat{c}_j$ is uniquely defined by the convention we made in section 2. Fix $x \in X$ such that the orbit of $x$ is dense in $X$. Consider any $\hat{x} \in \Sigma$ such that $\pi(\hat{x}) = x$. For each $k \geq 0$ we define $n_k(\hat{x})$ in the following way: $n_k(\hat{x})$ is the longest initial block of one of the $\hat{c}_j$ which agrees with the initial block of the same length of $\sigma^k\hat{x}$. We call $n_k(\hat{x})$ the flag of $x$ at time $k$. We remark that $n_k(\hat{x}) < \infty$ if $f^kx \notin \partial H$. By proposition 4.1 either $x$ is an isolated periodic point which hits $\partial H$ or it visits $\partial H$ only a finite number of times. In the second case since by the proposition $x$ is recurrent, we can assume by replacing $x$ by $f^i x$ for sufficiently large $i$ that the orbit of $x$ does not visit $\partial H$ at all.

We remind that $x$ is fixed and thus we drop the $x$ dependence of the notations. We say that a flag $n_k$ is covered by another flag if there is a $k' < k$ such that $n_{k'} + k' \geq n_k + k$ (see figure 1). If a flag is not covered by any other flag we say it is uncovered. The motivation for this terminology is the following fact: if the flag of $x$ is uncovered at time $k$ then we can concatenate $\hat{x}$’s initial segment of length $k$ with the next $n_k$ entries of $\sigma^k\hat{x}$ and the concatenated orbit belongs to $\Sigma^*$ (it may be wandering and thus not belong to $\Sigma$).

We claim that there are infinitely many indices $t_l$ such that the flags at these times are uncovered. To see this this consider $n_1 + 1$. Clearly there must be a $k > 1$ such that $n_k + k > n_1 + 1$. Let $t_1$ be the smallest such $k$, this flag must be uncovered. Arguing inductively the flags at times $t_{l+1} := \min\{k > t_l : n_k + k > n_{t_l} + t_l\}$ are uncovered.

For each $k \geq 0$ we define the color(s) of $x$ at time $k$ to be the set of $c_j$ such that the initial blocks of length $n_k(\hat{x})$ of $\hat{c}_j$ and $\sigma^k\hat{x}$ coincide. We assume that we start with a fine enough Markov partition that each $c_j$ belongs to a different element of the time zero partition. This implies that the color of $x$ is unique for each $k$. Let $I(X) := I(X, x)$ be the set of colors which occur infinitely often at uncovered times. Clearly this set is nonempty.

We are now ready to prove the above claim. Consider $I(X, x)$ and $I(Z, z)$ which have nonempty intersection, and let $c = c_j$ belong to this intersection. We will recursively construct a point $w \in \Sigma$ such that the orbit of $\pi(w)$ is dense in $X \cup Z$. To construct $w$ fix a positive sequence
\[ \varepsilon_m \to 0. \] Consider an initial segment of \( \hat{x} \) of length \( t_{i_0} \) such that \( f^i x : i = 0, \ldots, t_{i_0} \) is dense in \( X \). Consider the smallest \( k_1 \) such that \( \sigma^{k_1} \hat{x} \) and \( \hat{c} \) agree at at least \( n_{t_{i_0}}(\hat{x}) \) places. Furthermore consider an initial segment of \( \sigma^{k_1} \hat{x} \) of length \( t_{i_1} \) where \( t_{i_1} \) is the smallest \( t_i(\sigma^k z) \) such that the orbit segment \( f^i z : i = 0, \ldots, t_{i_0} \) is \( \varepsilon_1 \) dense in \( Z \). Consider the smallest \( k_2 \) such that \( \sigma^{k_2} \hat{x} \) and \( \hat{c} \) agree at at least \( n_{t_{i_1}}(\sigma^{k_1} \hat{x}) \) places. These definitions guarantee that the sequence \( \hat{x}(t_{i_0}) \star \sigma^{k_1} \hat{x}(t_{i_1}) \star \sigma^{k_2} \hat{x}(t_{i_2}) \star \cdots \). By construction \( w \in \Sigma^* \) and the orbit of \( \pi(w) \) is clearly dense in \( X \cup Z \). In particular \( \pi(w) \) in nonwandering and thus \( w \in \Sigma \). This finishes the proof of the claim.

\section{5. Genericity results}

5.1. A criterion for SFTs.

**Lemma 5.1.** If for each \( x \in \partial H \) there is an \( i \) such that \( f^i x \in H \) then \( f|_\Omega \) is a uniformly hyperbolic diffeomorphism and \( \Sigma \) is a SFT.

**Proof.** Fix a generating Markov partition \( \mathcal{P} \) and let \( \mathcal{P}^{(n)} := \bigvee_{i=-n}^{n} f^i \mathcal{P}. \) Let \( P_{x}^{(n)} := \bigcup_{\{P \in \mathcal{P}^{(n)} : x \in P\}} P. \) If \( x \in \partial H \) and \( f^i x \in H \) then since \( \mathcal{P} \) is generating by continuity there is a \( n(x) \) such that \( f^i P_{x}^{(n(x))} \subseteq H. \) Since \( \partial H \) is compact we can cover \( \partial H \) by a finite collection of the sets \( P_{x}^{(n(x))} \) to obtain a neighborhood \( N \) of \( \partial H \) such that \( N \cap \Omega = \emptyset. \) Since we used a finite collection of \( P_{x} \), the hole \( H' := N \cup H \) consists of a finite union of element of \( \mathcal{P}(N) \) for some sufficiently large integer \( N \) and thus \( \Sigma \) is a SFT.

5.2. Results in the Hausdorff metric. Let \( s \) denote the dimension of \( M \). Consider the set \( C \) of all holes such that \( \partial H \) is a continuous, i.e. there is a continuous map \( h : S^{s-1} \to M \) whose image is \( \partial H \). For \( H \in C \) let \( H^\varepsilon := \bigcup_{x \in \partial H} B(x, \varepsilon) \). For \( H_1, H_2 \in C \) we define

\[ d(H_1, H_2) := \inf \{ \varepsilon > 0 : H_1 \subset H_2^\varepsilon, H_2 \subset H_1^\varepsilon \}. \]

**Lemma 5.2.** If the set \( \Omega_H \) is totally disconnected then for every \( \varepsilon > 0 \) there is a hole \( H' \in C \) with \( d(H', H) < \varepsilon \) and an open neighborhood \( \mathcal{U} \subset C \) of \( H' \) such that for all holes \( H'' \in \mathcal{U} \) the subshift \( \Sigma \) is a SFT.
Proof. Consider the set \( V := \{ H' \in C : H \subset H', \ d(H, H') < \epsilon \} \).
Since \( \Omega_H \) is totally disconnected there are holes \( H' \in V \) such that \( \partial H' \cap \Omega_H = \emptyset \). Since \( \Omega_H \subset \Omega_{H'} \) such an \( H' \) satisfies the requirements of lemma 5.1 and so defines a SFT.

Next we will show that an open set \( U \) of holes satisfy the requirements of lemma 5.1. By compactness of \( H' \) we can find an open set \( B \subset H' \) with \( d(B, H') > 0 \) and a positive integer \( N \) such that for all \( x \in \partial H \) there exists an integer \( i \), satisfying \( |i| \leq N \) such that \( f^i x \in B \). Then, just as in the proof of lemma 5.1 there is a neighborhood \( U \) of \( \partial H' \) such that for each \( x \in U \) for some \( i \) satisfying \( |i| \leq N \) we have \( f^i x \in B \). This immediately implies that we can choose a small neighborhood \( U \subset C \) of \( H' \) such that for any hole \( H'' \in U \) the boundary of \( H'' \) satisfies the requirements of lemma 5.1 and thus the corresponding shift \( \Sigma'' \) is a SFT.

Theorem 5.3. In the two dimensional case the set of \( H \in C \) for which \( \Sigma \) is a SFT is open and dense.

Proof. Consider an arbitrary hole \( H \in C \) and the associated invariant set \( \Omega_H \). If \( x \) is a generic point in the sense that it visits (in both forward and backwards time) any cylinder set (defined by the Markov partition) with the correct frequency, then \( W^s(x) \) and \( W^u(x) \) completely fall into \( H \) and thus are both disjoint from \( \Omega_H \). Both \( W^s(x) \) and \( W^u(x) \) are curves which are dense in \( M \), thus since \( M \) is two dimensional the complement of their union is totally disconnected (i.e. no two points are in the same connected component). Since \( \Omega_H \) is a subset of this set it is also totally disconnect. Apply Lemma 5.2 finishes the proof.

5.3. Rectangle like holes. In this section we assume that \( M \) is two dimensional. We call a hole a rectangle like hole if \( \partial H \) consists of a finite number of curves, each of which is a piece of a stable or unstable manifold of \( f \). We assume that at each point where two curves meet, that one is stable, the other is unstable. Thus the number of corners is always even, and if we fix an orientation of the boundary we only need to give the coordinates of every other corner point to describe a rectangle like hole. If we fix the number \( 2n \) of corners of a rectangle like hole \( (n \geq 2) \) then we can parameterize the set of all such rectangles by an open subset \( R^{(n)} \) of \( T^{2n} \). We will consider the Lebesgue measure on \( R^{(n)} \).

Theorem 5.4. For every \( n \geq 2 \), the set of rectangle like holes with \( 2n \) corners for which \( \Sigma \) is a SFT is of full measure and contains an open dense subset of \( R^{(n)} \).
**Proof.** Consider the set $G$ of generic points in the same sense as in the proof of theorem 5.3. The set $G$ is of full Lebesgue measure. Suppose $H$ is a rectangle like hole with the property that every other corner point (those which are noted in the description of $H \in R^{(n)}$) is in $G$. Such a hole satisfies the requirements of lemma 5.1 and thus $\Omega$ is a SFT. Clearly the set of such holes is dense and of full measure. The proof of openness is the same as in the proof of the previous theorem.

5.4. **Polyhedral holes.** In this section we suppose that $M$ is $s$–dimensional and has a flat structure (i.e. $M \subset R^s$ or $M \subset T^s$). We consider holes which are the interior of arbitrary polyhedron. Fix the number of corners $n \geq s + 1$. The set of $n$-gons is an open subset of $M^n$ which we denote by $P^{(n)}$.

**Theorem 5.5.** For every $n \geq s+1$ the set of polyhedral holes for which $\Sigma$ is a SFT has positive Lebesgue measure.

**Proof.** We call a polyhedral hole $H \in P^{(n)}$ large if the Hausdorff dimension of the associated invariant set $\Omega$ is less than or equal to one. We note that if $H \subset H'$ then $\Sigma_{H'} \subset \Sigma_H$. Thus if $H$ is a large hole then so is $H'$.

Our proof is local. Fix an open set $B$. Consider the set $\Sigma_B$ of points which never fall into $B$. Suppose $B$ is large enough that $a := \dim(\Omega_B) \leq 1$. This implies that $\dim(\text{proj}_\theta(\Omega_B)) = a \leq 1$ for almost every $\theta$ ([4], Theorem 6.9). Here $\text{proj}_\theta$ denotes the orthogonal projection from $M$ onto $L_\theta$, the line through the origin in the direction $\theta \in S^{s-1}$. The term almost every $\theta$ refers to the Lebesgue measure on the $S^{s-1}$.

Consider the set of $(s - 1)$–dimensional polyhedra contained in a co-dimension one hyperplane with normal direction $\theta$. Such a polyhedron plays the role of a face of a polyhedral hole. Let $t$ be the parameter on $L_\theta$. The projection of any such face onto $L_\theta$ is simply a point $t \in L_\theta$. The pair of parameters $\{\theta, t\}$ determine a family of faces which lie in a common co-dimension one hyperplane.

Suppose $\theta$ is a generic direction, i.e. $\dim(\text{proj}_\theta(\Omega_B)) = a \leq 1$. Then for a.e. $t$, any polyhedral face normal to $\theta$ will be disjoint from $\Omega_B$.

Now consider any polyhedron $H$ which contains $B$ in it’s interior and for which all the faces are generic in the above sense, i.e. they do not intersect the set $\Omega_B$. This immediately implies that the boundary $\partial H$ satisfies the requirements of lemma 5.1. The set of such polyhedra is clearly open, and locally dense, and locally of full measure.

\qed
In fact we can show a bit more. Let $\hat{P}^{(n)} \subset P^{(n)}$ be the set of all large polyhedral holes. If we choose $B \in \hat{P}^{(n)}$ in the above proof then we have shown:

**Theorem 5.6.** For every $n \geq s + 1$ the set of large polyhedral holes for which $\Sigma$ is a SFT is of full measure and contains an open dense subset of $\hat{P}^{(n)}$.

Remark: We believe that the set of polyhedral holes defining a SFT is of full measure and contains an open and dense set. Our strategy of proof can not be used since the projection onto a line $L_\theta$ of a set of dimension greater than one has positive one dimensional measure for a.e. $\theta$.

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