Computation of the spectrum of dc²-balanced codes

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Abstract—We apply the central limit theorem for deriving approximations to the auto-correlation function and power density function (spectrum) of second-order spectral null (dc²-balanced) codes. We show that the auto-correlation function of dc²-balanced codes can be accurately approximated by a cubic function. We show that the difference between the approximate and exact spectrum is less than 0.04 dB for codeword length \( n = 256 \).

I. INTRODUCTION

Spectral null, or dc-balanced, codes have been applied in cable transmission [11], [12], magnetic recording [13], and optical recording systems [14], [15]. Spectral null codes have recently been advocated in visible light communications (VLC) systems, where light intensity of solid-state light sources, mostly LEDs, are varied [6]. It is desirable that the intensity variation of the light is invisible to the users, that is, annoying flicker should be mitigated [7]. This requirement implies that the frequency, that is, \( H(0) = 0 \). 

Section II commences with background on dc²-balanced codes. In Section III, we derive an approximation to the auto-correlation function and spectrum of dc²-balanced codes for asymptotically large values of the codeword length \( n \) by counting dc²-balanced codewords using the central limit theorem. Approximations for asymptotically large \( n \) will be discussed in Section IV. In Section V, we appraise the spectral performance of dc²-balanced codes. Section VI shows our conclusions.

II. BACKGROUND ON DC²-BALANCED BLOCK CODES

Let the \( n \)-bit codeword \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) over the binary symbol alphabet \( \mathcal{Q} = \{0, 1\} \), be a member of a codebook \( S \). The encoder emits codewords from \( S \) randomly and independently (i.i.d.). The auto-correlation function, \( \rho(i) \), of a sequence of codeword symbols is given by [13], [19], [20]

\[
\rho(i) = \frac{1}{n|S|} \sum_{x \in S} \sum_{j=1}^{n-i} x'_j x'_{j+i}, \quad 0 \leq i \leq n - 1, \tag{1}
\]

where \( |S| \) denotes the cardinality of \( S \) and \( x'_i = 2x_i - 1 \), \( x_i \in \{-1, 1\} \), is the bipolar representation of \( x_i \). If both \( x \) and its inverse \( \bar{x} \) are members of \( S \), then the power spectral density (psd), in short spectrum, versus frequency \( \omega \) of the emitted symbol sequence is

\[
H(\omega) = 1 + 2 \sum_{i=1}^{n-1} \rho(i) \cos(i\omega). \tag{2}
\]

A regular ‘full-set’ dc-balanced block code comprises all possible codewords that have equal numbers of 0’s and 1’s (\( n \) even). Franklin and Pierce [2] showed that the spectrum of a full-set dc-balanced block code has a null at the zero frequency, that is, \( H(0) = 0 \). Dc²-balanced spectral null codes are dc-balanced codes that satisfy a second condition, namely \( H(0) = H(0) \) or \( H(0) = 0 \), where \( H(2)(0) \) denotes the second derivative of \( H(\omega) \) at \( \omega = 0 \). Note that the above frequency domain conditions imply, see [2], that

\[
\sum_{i=1}^{n-1} \rho(i) = -\frac{1}{2} \quad \text{and} \quad \sum_{i=1}^{n-1} i^2 \rho(i) = 0. \tag{4}
\]

A codeword, \( \mathbf{x} \), is dc²-balanced if it satisfies [10], [21]

\[
\sum_{i=1}^{n} x_i = \frac{n}{2} \quad \text{and} \quad \sum_{i=1}^{n} i x_i = \frac{n(n+1)}{4}. \tag{5}
\]

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A block code comprising a full set of $d_{c}^2$-balanced codewords, denoted by $S_2$, is defined by

$$S_2 = \left\{ x \in \mathbb{Q}^n : \sum_{i=1}^{n} x_i = \frac{n}{2}, \sum_{i=1}^{n} i x_i = \frac{n(n+1)}{4} \right\}. \quad (6)$$

The set $S_2$ is empty if $n \mod 4 \neq 0$. Let $x \in S_2$ then its reverse $x_r = (x_1, \ldots, x_n) \in S_2$, since for a $x \in S_2$

$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} (n+1-i)x_i = \frac{n(n+1)}{4}. \quad (7)$$

A useful metric of the low-frequency spectral content, denoted by $\chi$, called Low Frequency Spectral Weight (LFSW) [15], is the first non-zero coefficient of the Taylor expansion of $\chi$, that is,

$$H(\omega) \sim \chi \omega^4, \ \omega \ll 1. \quad (8)$$

We derive from (7) that

$$\chi = \frac{1}{12} \sum_{i=1}^{n-1} i^4 \rho(i). \quad (9)$$

The number of $d_{c}^2$-balanced codewords, denoted by $N_{d_{c}^2} = |S_2|$, for asymptotically large $n$, equals [16], [22]

$$N_{d_{c}^2} \sim \frac{4\sqrt{3}}{\pi n^2} 2^n, \ n \mod 4 = 0, \ n \gg 1. \quad (10)$$

In the range $n < 256$ we have found experimentally that a better approximation is found by applying a small correction term, namely

$$N_{d_{c}^2} \sim \frac{4\sqrt{3}}{\pi n^2} 2^n \left(1 - \frac{1.211}{n}\right), \ n \mod 4 = 0, \ n \gg 1. \quad (11)$$

We consider here the spectral properties of full-set block codes, that is, $S_2$ denotes the set of all possible words, $x$, that satisfy condition (5). Finding an expression of the spectral properties of a full-set $S_2$ for large values of $n$ is an open problem as the computation requires the evaluation of (1) for each $x \in S_2$ [12]. In the next section, our main contribution, we address an alternative method, which is based on statistical analysis, which gives a simple and good approximation to the spectrum.

### III. AUTO-CORRELATION FUNCTION

Let $x$ be a codeword in $S_2$, and let $i_0$ and $i_1$, $i_0 \neq i_1$, $1 \leq i_0, i_1 \leq n$, be two (different) index positions in the codeword $x$. Then, we obtain for the average correlation, denoted by $r(i_0, i_1)$, between the symbols at positions $i_0$ and $i_1$ averaged over all codewords $x \in S_2$,

$$r(i_0, i_1) = \frac{1}{N_{d_{c}^2}} \sum_{x \in S_2} (2x_{i_0} - 1)(2x_{i_1} - 1) = \frac{N_{d_{c}^2}(x_{i_0} = x_{i_1}) - N_{d_{c}^2}(x_{i_0} \neq x_{i_1})}{N_{d_{c}^2}}, \quad (12)$$

where $N_{d_{c}^2}(A)$ denotes the number of $d_{c}^2$-balanced codewords $x$ that satisfy condition $A$. Then, using (11) and (11), we find the auto-correlation function

$$\rho(i) = \frac{1}{n} \sum_{j=1}^{n-i} r(j, j + i), \ 0 \leq i \leq n - 1. \quad (13)$$

For reasons of symmetry, we have

$$r(i_0, i_1) = \frac{2N_{d_{c}^2}(x_{i_0} = x_{i_1}) - 1}{N_{d_{c}^2}} = \frac{4N_{d_{c}^2}(x_{i_0} = x_{i_1}) - 1}{N_{d_{c}^2}}. \quad (14)$$

By using the central limit theorem, we compute below an approximation to the number of $d_{c}^2$-balanced codewords that have a ‘1’ at positions $i_0$ and $i_1$, $N_{d_{c}^2}(x_{i_0} = x_{i_1} = 1)$, for asymptotically large values of $n$.

#### A. Counting of codewords using the central limit theorem

The number of $d_{c}^2$-balanced codewords, $x$, $N_{d_{c}^2}(x_{i_0} = x_{i_1} = 1)$, that is required for computing the auto-correlation function using (13) and (14), can be computed using generating functions. For very large $n$, however, this rapidly becomes an impractically cumbersome exercise, and an efficient alternative method is considered a desideratum.

To that end, we exploit the central limit theorem by regarding the integer variables $x_i \in \{0, 1\}$ as i.i.d. binary random variables whose numerical outcomes ‘0’ or ‘1’ are equally likely. We define the stochastic variables $c$ and $p$ by

$$c = x_1 + x_2 + \cdots + x_n \quad (15)$$

and

$$p = x_1 + 2x_2 + \cdots + nx_n, \quad (16)$$

where $x_{i_0} = x_{i_1} = 1$, $i_0, i_1 \in \{1, \ldots, n\}$.

The central limit theorem [23], Chapter 8, states that for asymptotically large $n$ the distribution of the stochastic variables $c$ and $p$, which are obtained by summing a large number, $n$, of independent stochastic variables, approaches a two-dimensional Gaussian distribution.

Let $E[.]$ denote the expected value operator for all possible codewords in $S_2$. Let the parameters $\mu_c = E[c]$ and $\mu_p = E[p]$ denote the average of $c$ and $p$, and let $\sigma^2_c = E[(c - \mu_c)^2]$ and $\sigma^2_p = E[(p - \mu_p)^2]$ denote the variance of $c$ and $p$. The parameter $r$ denotes the linear correlation coefficient between the random variables $c$ and $p$. Then the bi-variate Gaussian distribution, denoted by $G(c, p)$, is given by

$$G(c, p) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\phi(c, p)}, \quad (17)$$

where

$$\phi(c, p) = \left(\frac{c - \mu_c}{\sigma_c}\right)^2 + \left(\frac{p - \mu_p}{\sigma_p}\right)^2 - \frac{2r(c - \mu_c)(p - \mu_p)}{\sigma_c\sigma_p}. \quad (18)$$

and

$$f(c, p) = \frac{1}{2(1-r^2)} f(c, p), \quad (19)$$

where

$$E[x_i] = E[x_i^2] = \frac{1}{2}, \ \text{and} \ E[x_i x_j] = \frac{1}{4}, \ i \neq i_0, i_1, \quad (20)$$

$$E[x_i] = E[x_i^2] = 1, i = i_0, i_1, \ \text{E}[x_i x_j] = \frac{1}{2}, \ i, j = i_0, i_1,$$
and $E[x_i x_j] = 1$. We may find after a routine computation using \(20\) that
\[
\mu_c = E \left[ \sum_{i=1}^{n} x_i \right] = \sum_{i=1}^{n} E[x_i] = \frac{n - 2}{2} + 2
\]
and similarly
\[
\mu_p = E \left[ \sum_{i=1}^{n} x_i \right] = \sum_{i=1}^{n} E[x_i] = \frac{n(n + 1)}{4} + \frac{i_0 + i_1}{2}.
\]
The variances $\sigma_c^2$, $\sigma_p^2$, and the correlation coefficient $r$ can be found without too much difficulty:
\[
\sigma_c^2 = E \left[ \sum_{i=1}^{n} (x_i - \mu_c)^2 \right] = \frac{n - 2}{4}, \tag{21}
\]
\[
\sigma_p^2 = E \left[ \sum_{i=1}^{n} (ix_i - \mu_p)^2 \right] = \frac{n(n + 1)(2n + 1)}{24} - \frac{i_0^2 + i_1^2}{4}, \tag{22}
\]
and
\[
r^2 = \frac{E[\sum_{i=1}^{n} (x_i - \mu_c) \cdot \sum_{i=1}^{n} (ix_i - \mu_p)]}{\sigma_c^2 \sigma_p^2} = \frac{3}{2(n - 2)} \frac{(n^2 + n - 2(i_0 + i_1))^2}{2n^3 + 3n^2 + n - 6(i_0^2 + i_1^2)}. \tag{23}
\]
The total number of $n$-sequences with $x_{i_0} = x_{i_1} = 1$, equals $2^{n-2}$, so that for asymptotically large $n$, the number of $n$-sequences versus $c$ and $p$, denoted by $N(c, p; x_{i_0} = x_{i_1} = 1)$, can be approximated by
\[
N(c, p; x_{i_0} = x_{i_1} = 1) \sim 2^{n-2} G(c, p) = \frac{2^n}{8\pi \phi_1} e^{-\phi(c, p)}. \tag{24}
\]
A $dc^2$-balanced codeword satisfies, by definition, the conditions, see \(5\), $c = n/2$ and $p = n(n + 1)/4$. Then, $N_{dc^2}(x_{i_0} = x_{i_1} = 1)$ is found after substituting $c = n/2$ and $p = n(n + 1)/4$ into \(24\). We find
\[
N_{dc^2}(x_{i_0} = x_{i_1} = 1) = 2^n \frac{2^n}{8\pi \phi_1} e^{-\phi_2}, \tag{25}
\]
where, see \(18\) and \(19\).
\[
\phi_2 = \phi \left( c = \frac{n}{2}, p = \frac{n(n + 1)}{4} \right) = \frac{1}{2(1 - r^2)} \left( \left( \frac{1}{\sigma_c} \right)^2 + \left( \frac{i_0 + i_1}{\sigma_p} \right)^2 - \frac{r(i_0 + i_1)}{\sigma_c \sigma_p} \right) = \frac{4\sigma_p^2 + \sigma_c^2 (i_0 + i_1)^2 - 4\sigma_c \sigma_p (i_0 + i_1)}{8\phi_1}. \tag{26}
\]
After combining \(10\), \(14\), and \(25\), we obtain
\[
r(i_0, i_1) = \frac{n^2}{\sqrt{192} \phi_1} e^{-\phi_2} - 1. \tag{27}
\]
In order to reduce the clerical work and offer more insight, we define the four (real) variables
\[
\gamma = 12n[(i_0 - n - 1)i_0 + (i_1 - n - 1)i_1],
\]
and $E[x_i x_j] = 1$. We may find after a routine computation using \(20\) that
\[
\delta = (i_0 - i_1)^2,
\]
\[
r_1 = -\frac{1}{n^2} (8n^3 + 13n^2 + 4n + \gamma - 12\delta),
\]
and
\[
r_2 = \frac{1}{8n^2} [12n^2 + 4n + \gamma - 6(n + 2)\delta].
\]
With some effort we find the expressions
\[
\phi_1 = \frac{n^4}{192} (1 + r_1) \tag{28}
\]
and
\[
\phi_2 = \frac{8}{n^2} \frac{1 + r_2}{1 + r_1}. \tag{29}
\]
We finally obtain
\[
r(i_0, i_1) = \frac{n^2}{\sqrt{192} \phi_1} e^{-\phi_2} - 1 = \frac{1}{\sqrt{1 + r_1}} e^{-\frac{n^2}{8(1 + r_1)^2}} - 1, \tag{30}
\]
where we can easily verify, since $x$ and $x_r \in S_2$, see \(7\), that
\[
r(i_0, i_1) = r(n + 1 - i_0, n + 1 - i_1). \tag{31}
\]
The auto-correlation function, $\rho(i)$, is found using \(13\). In the next subsection, we show results of computations.

B. Results of computations

By invoking \(13\) and \(30\) we are now able to compute an estimate of the auto-correlation function $\rho(i)$. Figure 1 shows results of computations for $n = 32, 64, 128$. As a comparison we plotted the exact auto-correlation function of a full set of $dc^2$-balanced sequences, denoted by $\rho(i)$, which was computed using an enumeration technique and generating functions \(10\).

The accuracy of the approximate auto-correlation function, $\rho(i)$, cannot easily be determined from Figure 1 for the larger values of $n$. Figure 2 Curve ‘without correction’, shows $|\rho(i) - \rho(i)|$, the difference between the two auto-correlation functions versus $i/n$ for the selected $n = 128$ and 256. We notice that the difference between the two functions decreases
then we find two linear equations with two unknowns, $\rho \to 0$ (as it should), while
\[ \sum \rho(i) = -1/2 \text{ and } \sum i^2 \rho(i) = 0, \]
which accumulate the small error differences, are not necessarily satisfied. We have observed that with increasing $n$ that $\sum \rho(i) + 1/2$ is converging to zero (as it should), while $\sum i^2 \rho(i)$ is not. As a result, the spectra, computed using $\rho(i)$ do not satisfy the spectral conditions (3).

We propose to add a small correction term to $\rho(i)$ so that both 'checks', $\sum \rho(i) = -\frac{1}{2}$ and $\sum i^2 \rho(i) = 0$, are satisfied. We add to $\rho(i)$ the correction term $a + bi$, where the (real) parameters, $a$ and $b$, are chosen such that $\sum (\rho(i) + a + bi) = -\frac{1}{2}$ and $\sum i^2 (\rho(i) + a + bi) = 0$. Define
\[ a_0 = \sum \rho(i) + \frac{1}{2} \]
and
\[ a_1 = \sum i^2 \rho(i), \]
then we find two linear equations with two unknowns, $a$ and $b$, namely
\[ \sum_{i=1}^{n-1} (\rho(i) + a + bi) = a_0 - \frac{1}{2} + na + b \sum_{i=1}^{n-1} i = -\frac{1}{2} \]
and
\[ \sum_{i=1}^{n-1} i^2 (\rho(i) + a + bi) = a_1 + a \sum_{i=1}^{n-1} i^2 + b \sum_{i=1}^{n-1} i^3 = 0. \]

After solving the above system, where we substitute the well-known expressions for $\sum i^k$, $k = 1, 2, 3$, we obtain
\[ a = -3 \frac{n(n-1)a_0 - 2a_1}{n(n-1)(n-2)} \]
and
\[ b = \frac{2(n^2n-1)a_0 - 6a_1}{n^2(n-1)(n-2)}. \]

For example, for $n = 128$, we find that $a_0 = -0.0156$ and $a_1 = -22.21$. So that $a = 0.0003063$ and $b = -0.000029$. The result of the correction can be seen in Figure 2, the 'checks', for $n = 128$ and 256. We notice in the range $i/n < 0.6$ a significant improvement in the accuracy of the estimate of the auto-correlation function.

C. Further approximations for asymptotically large $n$

With (30) we can straightforwardly compute the auto-correlation function $\rho(i)$ and spectrum $H(\omega)$. In this section, we attempt to approximate $\rho(i)$ for asymptotically large $n$, which might offer more insight in the trade-offs between redundancy and spectral properties. We apply to (30) the well-known series approximations
\[ \frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8} x^2 - \frac{5}{16} x^3 + \cdots \]
and
\[ \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots. \]

We have experimented with the various options available for trading accuracy versus simplicity of the expression, and propose
\[ r(i_0, i_1) \sim -\frac{8}{n} (1 + r_2) - \frac{r_1}{2} n \gg 1. \]

Using (13), we obtain
\[ \rho(i) \sim \frac{n-i}{2n^5} (12i^2 + 4n^2 + 4n^2i - 4n^3 + n^2 + 4n). \]

Then, after deleting the smallest terms, we obtain the simple cubic function
\[ \rho(i) \sim \frac{2}{n^3} (n-i)(i^2 + in - n^2), \]
which can be rewritten as
\[ \rho(i) \sim \frac{2}{n^5} (n-i)(i - c_0 n)(i - c_1 n), \]
where $c_{0,1} = (-1 \mp \sqrt{5})/2$. The checks (4) for the above $\rho(i)$ yield
\[ a_0 = \frac{1}{2} + \sum \rho(i) = \frac{1}{n} - \frac{1}{2n^2} \]
and
\[ a_1 = \sum i^2 \rho(i) = \frac{1}{12} + \frac{1}{6n^2}. \]

In order to satisfy both checks (4), we add to $\rho(i)$ the correction term $a + bi$, and define
\[ \rho'(i) = \frac{2}{n^3} (n-i)(i - c_0 n)(i - c_1 n) + a + bi, \]
where after using (32) and (33), we obtain
\[ a = -\frac{6n^2 - n + 2}{2(n-2)n^3} \sim -\frac{3}{n^2} \]
and
\[ b = \frac{4n^3 - 2n^2 + n - 2}{n^3(n-1)(n-2)} \sim \frac{4}{n^3}. \]

Note that $a$ and $b$ are relatively small terms in (38) for asymptotically large $n$. Figure 3 shows the difference between

![Graph showing the difference between estimated and exact auto-correlation functions](image-url)
exact and estimated auto-correlation function $|\rho'(i) - \hat{\rho}(i)|$ versus $i/n$ for $n = 256$. As a final proof of the pudding, we compare the (exact) spectrum of full set codewords versus the spectrum, denoted by $H'(\omega)$, which is computed using the above approximated auto-correlation function $\rho'(i)$. The difference, $H'(\omega)/\hat{H}(\omega)$ (dB), between the spectrum, $H'(\omega)$, computed using $\rho'(i)$, and the exact spectrum, $\hat{H}(\omega)$, of full set codewords, which was computed using generating functions, is plotted in Figure 4. We may observe that the difference between the two spectra is very small, less than 0.05 dB for $n = 128$ and less than 0.03 dB for $n = 256$.

The LFSW metric, $\chi'$, is, using (39),

$$\chi' = \frac{1}{12} \sum_{i=1}^{n-1} i^4 \rho'(i) \sim \frac{n^4}{720} \left(1 + \frac{4}{n}\right).$$

Table I shows $\chi'$ for selected values of $n$, where as a comparison we have listed the LFSW of full set dc$^2$-balanced codes, denoted by $\hat{\chi}$. We may notice that for $n = 256$ the difference between $\chi$ and $\hat{\chi}$ is less than half a percent.

### D. Comparison with prior art

In [17], it is postulated that the auto-correlation of dc$^2$-balanced spectral null codes, denoted by $\rho_a(i)$, can be modelled by the simple parabola’s equation

$$\rho_a(i) = \beta(i + \alpha)(i - n),$$

where the (real) parameters $\alpha$ and $\beta$ are given by

$$\alpha = \frac{3n^2 - 2}{5n}$$

and

$$\beta = -\frac{15}{(n - 1)(n - 2)(4n + 3)}.$$  

It has been shown in [17] that the parabola’s equation (40) is an accurate approximation to the exact correlation function of full-set dc$^2$-balanced spectral null codes. Figure 3 Curve (b), shows the difference between exact and estimated auto-correlation function $|\rho_a(i) - \hat{\rho}(i)|$ versus $i/n$ for $n = 256$. We notice that the newly developed $\rho'(i)$, Curve (a), is almost an order more accurate than (40) presented in the prior art. Figure 5 Curve b, shows that the quotient of the exact spectrum and the one based on prior art (40), $H(\omega)/H_a(\omega)$, is for $n = 256$ less than 0.7 dB, and also here we notice that the newly developed theory is more than an order more accurate.

### IV. Appraisal of Spectral Performance

A system designer is usually confronted with a restricted redundancy budget, so that with a given redundancy the designer searches for a balanced code that offers the best
rejection of low-frequency components. In this section, we compare the spectral performance of regular dc-balanced codes with that of dc\(^2\)-balanced codes. We start with a summary of properties of dc-balanced codes.

### A. Codes with a first-order spectral null

Let the codeword length of a regular full-set dc-balanced code be denoted by \(n_1, n_1\) even. Each codeword has an equal number of 0’s and 1’s, so that the number of available dc-balanced codewords, denoted by \(N_{dc}\), is simply [24]

\[
N_{dc} = \left(\frac{n_1}{n_1/2}\right) \sim \frac{1}{\sqrt{2\pi n_1^2}} 2^{n_1}, \quad n_1 \gg 1.
\]

(41)

The auto-correlation function, \(\rho_1(i)\), and the spectrum, \(H_1(\omega)\), of dc-balanced codes is [2]

\[
\rho_1(i) = \frac{1}{n_1(n_1 - 1)}(i - n_1) \quad (42)
\]

and

\[
H_1(\omega) = \frac{n_1}{n_1 - 1} \left\{ 1 - \left(\frac{\sin \frac{n_1 \omega}{2} \sin \frac{n_1}{2}}{n_1 \sin \frac{n_1}{2}}\right)^2 \right\}. \quad (43)
\]

At the very low-frequency end, we have [15]

\[
H_1(\omega) \sim \chi_1 \omega^2, \quad \omega \ll 1, \quad (44)
\]

where

\[
\chi_1 = \frac{n_1(n_1 + 1)}{12}. \quad (45)
\]

### B. Performance comparison

We compare the spectral content of dc-balanced versus that of dc\(^2\)-balanced codes, where we assume that both types of codes have the same redundancy. Let \(R\) and \(R_1\) denote the maximum information rate of a dc\(^2\)-balanced code or dc-balanced of length \(n\) and \(n_1\), respectively, then we have, using (10),

\[
R = \frac{1}{n} \log_2 N_{dc} = \frac{1}{n} \log_2 \frac{4\sqrt{3}}{n^2} 2^n = 1 - \frac{1}{n} \log_2 \frac{\pi n^2}{4\sqrt{3}} \quad (46)
\]

and, using (41),

\[
R_1 = 1 - \frac{1}{2n_1} \log_2 \frac{\pi}{2} n_1. \quad (47)
\]

Table II shows a few examples of the codewords length \(n\) and \(n_1\) for which dc-balanced and dc\(^2\)-balanced codes have equal redundancy, respectively, that is, \(R = R_1\). In the range shown in Table II the codeword length \(n\) of a dc\(^2\)-balanced code is approximately a factor of 4.5 larger than the codeword length \(n_1\) of a dc-balanced code for achieving the same rate \(R = R_1\). Figure 6 shows three examples of spectrum pairs of dc-balanced and dc\(^2\)-balanced codes with the same redundancy versus frequency for a) \(R = R_1 = 0.98\), b) \(R = R_1 = 0.94\), and c) \(R = R_1 = 0.90\), see also Table II. We may notice the points of intersection of the spectra of dc-balanced and dc\(^2\)-balanced codes. A further perusal of the diagram reveals that the points of intersection are at around -20 dB, which implies that dc\(^2\)-balanced codes are to be preferred when a low-frequency spectral suppression is required better than around -20 dB. Additional computations show that this ‘20 dB rule’ applies to all codes with a rate larger than 0.75.

### V. Conclusions

By applying the central limit theorem, we have derived an approximate expression for the auto-correlation function and spectrum of full-set dc\(^2\)-balanced codes for asymptotically large values of the codeword length \(n\). We have shown that the auto-correlation function of dc\(^2\)-balanced codes can be accurately approximated by a simple cubic function. We have compared the approximate spectrum with the exact spectrum of full set dc\(^2\)-balanced codes. We have shown that the difference between the approximated and exact spectrum is less than 0.04 dB for \(n = 256\).

### References

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