RELATIVITY AND WAVY MOTIONS

ANGELO LOINGER

Abstract. The conditions under which the undulatory character of field disturbances is physically significant.

Summary. – 1. Introduction. – 2. and 3. Physical meaning of covariance in general relativity (GR) – Einstein v. Fock – Refutation of Fock’s viewpoint; consequences concerning the gravitational waves (GW’s). – 4. Electromagnetic waves and GW’s. – 5. Characteristics of Einstein and Maxwell fields. – 6. An example of Einsteinian characteristic. – 6bis. Singularities in GR. – 7. The propagation speed of gravitation. – 8. GW’s in the linear approximation of GR. – Appendix: Levi-Civita, Einstein and the GW’s. – Parergon.

1. – The theme of the present Note has been already touched by me in previous papers, passim, and treated expressly in my article “Waves and uniformity of space-times” [1]. However, it has not attracted the attention of the concerned scholars and therefore I give here a little different treatment of the problem with some further illustration.

2. – The spacetime of special relativity (SR) is usually described by the simple Minkowskian tensor \( \eta_{jk}, (j,k = 0,1,2,3) \), for which: \( \eta_{rs} = 0 \) if \( r \neq s \), \( \eta_{00} = 1 \), \( \eta_{11} = \eta_{22} = \eta_{33} = -1 \); the theory is Lorentz invariant, the Galilean reference frames are physically privileged. As far back as 1917, it was emphasized by Kretschmann [2] – and in subsequent years by Fock [3] – that any physical theory can be reformulated in general co-ordinates without losing its characteristic physical properties. In the case of SR, this means in particular that the Galilean frames maintain their physical privileges; see e.g. chapt. IV of Fock’s treatise cited in [3]. But in general relativity (GR) things stand otherwise: a pregnant formulation by Erwin Schrödinger [4] tells us that: “The geometric structure of the space-time model envisaged in the 1915 theory [GR] is embodied in the following two principles: (i) equivalence of all four-dimensional systems of coordinates obtained from any one of them by arbitrary (point-)transformations; (ii) the continuum has a metrical connexion impressed on it: that is, at every point a certain quadratic form of the coordinate-differentials, \( g_{ik} dx_i dx_k \), called the ‘square of the interval’ between the two points in question, has a fundamental meaning, invariant in the aforesaid transformations. [...] The first [principle], the principle of general invariance, incarnates the idea of General
Relativity.” In other terms, in GR no system of co-ordinates has physical privileges. Seemingly, this concept has been accepted by the overwhelming majority of physicists (with the remarkable exception represented by Fock). However, I wish to emphasize that not all theoreticians have realized all its implications.

3.- Point (i) of the above Schrödinger’s quotation means that the phrase “General Relativity” must be understood literally: no reference system has physical privileges, no physically sensible result depends on the chosen co-ordinates. In any specific instance one chooses of course the reference frame that is the most adequate and simple for geometric and formal reasons, avoiding the introduction of superfluous “inertial forces” (lato sensu), which would complicate uselessly the computations. The choice of the reference system has only a practical meaning because in GR the co-ordinates are mere labels for the point-events of spacetime.

A first important consequence: no fundamental velocity exists in GR, in particular light loses a privilege that it had in SR: now, its propagation speed depends on the inertial-gravitational forces, and can vary from zero to infinity.

Fock [3] did not share Einstein’s standpoint on reference systems. For him only the special theory (SR) deserves the name of “theory of relativity”. (From a strictly geometrical viewpoint, “relativity” and “uniformity” of the concerned manifold – see infra, sect. 4. – are closely related concepts, as it was emphasized in 1927 by E. Cartan.) Fock tried to demonstrate that in GR – called by him “theory of gravitation” – there exists a set of infinite reference frames endowed with an outstanding significance, the harmonic frames. However, his proof rests on unjustified assumptions and therefore is not a real proof. As a matter of fact, the harmonic references are useful for the practical solution of many problems, but do not possess a particular conceptual value.

As it is well known, the harmonic co-ordinates, say $\xi^j$, are such that the components $g_{jk}$’s, $(j, k = 0, 1, 2, 3)$, of metric tensor satisfy the following four conditions ($k = 0, 1, 2, 3$) [5]:

\[(1) \quad \frac{\partial}{\partial \xi^j} \left( g^{jk} \sqrt{-g} \right) = 0 , \]

where $g := \det |g_{jk}|$. According to Fock [5], the $\xi$’s are especially appropriate for evidencing in various stages of approximation the mathematical properties of the GW’s. In Fock’s conception, the fact that the undulatory character of gravitational wavy motions remains unchanged for all the harmonic frames is sufficient to make certain the physical reality of the GW’s. Now, two fundamental remarks can be opposed to Fock. In primis, no physical “mechanism” exists really in the exact GR for the production of GW’s; in particular, it is easy to prove that the trajectories of the bodies of a system, which interact only gravitationally, are geodesic lines [6]. Accordingly, even if we adopted Fock’s conception, the GW’s would reveal themselves as
mere analytical artefacts. *In secundis*, since the correct interpretation of the formalism of Einstein’s field theory affirms that the phrase “General Relativity” must be taken *au pied de la lettre* [4], we can ascertain immediately that the undulatory character of any GW is generally impaired by a change of general co-ordinates. Further, the GW’s have only a *pseudo* stress-energy-momentum tensor.

4.— It is instructive to compare the e.m. waves with the GW’s. The propagation substrate of the e.m. waves of Maxwell theory is Minkowski spacetime, that is a *uniform* (i.e., homogeneous and isotropic) manifold, for which the infinite class of the Galilean frames is physically privileged. When we re-write Maxwell theory according to the formalism of Riemann-Einstein spacetime, the “physicality” of its concepts remains unchanged, in particular the “physicality” of the e.m. waves. On the contrary, the GW’s are undulations of the metric tensor $g_{jk}$, which is the “substance” of Riemann-Einstein spacetime, i.e. of a *not* “fixed” manifold, that does *not* possess a class of physically privileged reference systems. Emission “mechanism” of the e.m. waves can be simply the acceleration of a charge, whereas the acceleration of a mass does *not* generate any GW [6]. (For the special case of the GW’s in the *linear* approximation of GR, see sect.8 *infra*.)

5.— *Characteristic hypersurfaces* of Einstein field equations: in the current literature there is a physically false interpretation of them. They were first written in 1930 by Levi-Civita [7], who gave the correct interpretation.

In SR the differential equations of the characteristics and of the bicharacteristics of Maxwell field can be written respectively, in a Hamilton-Jacobi form, as follows:

\[
(2) \quad H := \frac{1}{2} \eta^{jk} p_j p_k = 0 , \quad \text{with} \quad p_j := \frac{\partial z(y)}{\partial y^j} ;
\]

\[
(3) \quad \frac{dp_j}{d\sigma} = -\frac{\partial H}{\partial y^j} , \quad \frac{dy^j}{d\sigma} = -\frac{\partial H}{\partial p_j} ,
\]

where: $\sigma$ is an auxiliary parameter, $z(y)$ is the function which defines the characteristic hypersurface $z(y) = 0$, and $\text{d}s^2 = \eta_{jk} dy^j dy^k$ yields the Minkowskian interval. Equation $z(y) = 0$, $[y \equiv (y^0, y^1, y^2, y^3)]$, represents physically the *wave front* of an e.m. wave; the characteristic lines of (2) – the *rays* of wave front $z(y) = 0$, given by eq. (3) – coincide with the *null geodesics* $\text{d}s = 0$. Thus we see that *SR comprises the geometric optics*. (This could be foreseen because the e.m. theory is a basic ingredient for defining space and time in SR.)
Eqs. (2) and (3) can be immediately re-written in a system of general co-ordinates \((x^0, x^1, x^2, x^3) \equiv x:\)

\[(4) \quad H := \frac{1}{2} g^{jk} p_j p_k = 0 \quad \text{with} \quad p_j := \frac{\partial z(x)}{\partial x^j};\]

\[(5) \quad \frac{dp_j}{d\sigma} = -\frac{\partial H}{\partial x^j}, \quad \frac{dx^j}{d\sigma} = -\frac{\partial H}{\partial p_j}.\]

Now, as it was demonstrated by Whittaker [8], eqs. (4) and (5) are formally identical to the equations that yield characteristics and bicharacteristics of Maxwell field in a Riemann-Einstein spacetime with metric tensor \(g_{jk}(x)\). And Levi-Civita [7] proved that eqs. (4) and (5) give also characteristics and bicharacteristics of Einstein field. A not fortuitous coincidence!

Since GR comprehends SR, eqs. (4) and (5) have a unique electromagnetic interpretation for Maxwell and Einstein fields, as it was explicitly emphasized by Levi-Civita [7]: both SR and GR comprise constitutionally the geometric optics.

On the contrary, in the current opinion (see e.g. Fock [3], sect. 53) characteristics and bicharacteristics of Einstein field equations give wave fronts and rays of GW’s: an interpretation vitiated by the wishful thinking concerning the real existence of these fictive undulations.

6. – We have seen that both in SR and in GR equation \(ds^2 = 0\) gives the geodesic null lines that represent the light rays of geometric optics. The propagation velocity of these rays is not a universal constant in GR, because it depends on the inertial-gravitational forces as described by potential \(g_{jk}(x)\). A simple example will render evident this well known fact; see also [9].

Let us consider the \(ds^2\) of Brillouin’s form of solution of Schwarzschild problem (to find the gravitational field generated by a material point of mass \(M\), at rest) [10]:

\[(6) \quad ds^2 = \frac{r}{r + 2m} c^2 dt^2 - \frac{r + 2m}{r} dr^2 - (r + 2m)^2 (d\theta^2 + \sin^2 \theta d\varphi^2),\]

where \(m \equiv GM/c^2\). Remark that this form is maximally extended since holds for \(r > 0\).

For \(d\theta = d\varphi = 0\), the condition \(ds^2 = 0\) implies that

\[(7) \quad \frac{dr}{c dt} = \pm \frac{r}{r + 2m};\]

if \(z := ct - \psi(r)\) is the function of the characteristic \(z = 0\), we have:

\[(8) \quad 0 = 2H = \frac{r + 2m}{r} \cdot 1 - \frac{r}{r + 2m} \left(\frac{d\psi}{dr}\right)^2,\]
and therefore:

\[ \psi(r) = \pm (r + 2m \ln r + \text{const}) \]

thus the characteristic has the equation

\[ ct = \pm (r + 2m \ln r + \text{const}) \]

from which

\[ \frac{dr}{c dt} = \frac{\partial z}{\partial t} \frac{\partial z}{\partial r} = \pm \frac{r}{r + 2m}, \]

i.e. again result (7), which can be also re-obtained by means of eqs. (5), written for \((r, ct), (p_r, p_0)\). – Q.e.d.

If \(m = 0\), \(dr/dt = c\), i.e. the value of SR. A value which is also obtained for \(r \to \infty\). It is not a general limiting value because it depends on the chosen frame; with a convenient transformation of general co-ordinates the light speed can assume any desired value.

It is instructive to compare eq.(7) with eq.(4.2) of [9] (\(\alpha\) in (4.2) coincides with the present \(m\)):

\[ (7') \quad \frac{dr}{c dt} = \pm \frac{r - 2m}{r}, \]

which is obtained from the standard form of solution of Schwarzschild problem (erroneously called “by Schwarzschild”). Brillouin’s form [4] is diffeomorphic to “exterior” part \(r > 2m\) of standard form: a simple fact that makes clear how the notion of black hole pertain to a fairy-tale. (The baroque form of solution by Kruskal and Szekeres is a gift of Barmecide).

6bis. – The singularities in GR: they are classifiable in two mathematical categories: curvature singularities and co-ordinate singularities. However, from the physical standpoint the only essential distinction is between singularities characterized by the presence in them of matter and singularities characterized by the absence in them of matter. The two classifications do not always coincide. For instance, the singularity of the Brillouin’s form [4] and the singularity of the original Schwarzschild’s form (which can be obtained, e.g., from the standard form by means of formal substitution \(r \to [r^3 + (2m)^3]^{1/3}\)) are not curvature singularities, but physical ones. On the contrary, the singularity at \(r = 0\) of the standard form is a curvature singularity, but it cannot represent a material point because it is a space-like locus; and the singularity at \(r = 2m\) of the standard form is a co-ordinate and non-physical singularity.

A widespread “Vulgate” of GR affirms that if Schwarzschild problem is solved using co-ordinate-free methods (as orthonormal bases, etc.) the result is necessarily the standard form of solution. Unfortunately, “Vulgate”’s procedure is impaired by a logical fallacy: indeed, these authors have already
chosen initially the above form because they write, first of all, the simple expression $r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$ for the angular part. *Et de hoc satis.*

7.- There is a widespread and erroneous conviction (see e.g. Fock [3], p.194) according to which in GR gravitation is propagated with the speed of light *in vacuo*, i.e. with the speed of light in empty space of SR.

The supporters of this false opinion claim that it follows, e.g., from eqs. (4) and (5), when interpreted as differential equations of wave fronts and rays of GW’s. Now, this is trivially wrong even from the viewpoint of the believers in the physical existence of GW’s, because eqs. (4) and (5) – quite *independently of their interpretation* – affirm in reality that the concerned wave fronts and rays have a propagation velocity that *depends on the metric tensor* $g_{jk}(x)$, even if this tensor has the form of a mathematical undulation.

The non-existence of physical GW’s has the following consequence: if we displace a mass, its gravitational field and the related curvature of the interested manifold *displace themselves along with the mass*: under this respect Einstein field and Newton field behave in an identical way [11].

For the GW’s in the *linear* approximation of GR, see sect.8.

8.- The case of the GW’s according to *linear* approximation of GR can be quickly dispatched. Indeed, it is here sufficient that I recall the decisive argument of sect.4 of a recent paper of mine [12].

As it is well known, in the linearized version of GR one puts approximately

\[(12) \quad g_{jk} \approx \eta_{jk} + h_{jk},\]

where $\eta_{jk}$ is the customary Minkowskian tensor and the $h_{jk}$’s are *small* deviations, that in our case represent the passage of GW’s. The essential point is: $h_{jk}$ *is a tensor only under Lorentz transformations of Galilean co-ordinates.*

Now: $\alpha$) suitable *finite* transformations of *general* co-ordinates can reduce to *zero* the undulations $h_{jk}$, just because their tensorial character is only Lorentzian; $\beta$) for the specially significant instance of *plane* GW’s, remark $\alpha$) implies the reduction to *zero* of the celebrated transverse-traceless (TT) GW’s; $\gamma$) it is true that *in Minkowski space-time* the $h_{jk}$–waves are propagated with the light velocity $c$ *in vacuo*; unfortunately, by virtue of $\alpha$) and $\beta$), they are only phantom entities.

It is regrettable that various physicists insist on publishing useless considerations and computations on $h_{jk}$–waves [13]. It is time that astrophysical community desist from beating the air – and from squandering the money of the taxpayers.

"Dann zuletzt ist unerläßlich,
Daß der Dichter manches hasse;
Was unleidlich ist und häßlich,
Nicht wie Schönes leben lasse."

J.W. v. Goethe
Appendix: Levi-Civita, Einstein and the GW’s

I don’t doubt that Levi-Civita’s electromagnetic interpretation of the characteristics of Einstein field \([7]\) was propitiated by his fundamental memoir of 1917 on GR \([14]\). Indeed, this paper ends with two basic remarks: \(i\) the proposed pseudo tensor \(t^{jk}\) of stress-energy-momentum of Einstein field, which satisfies the differential conditions (\(T^{jk}\) is the matter tensor)

\[
\frac{\partial}{\partial x^k} \left( (t^k_j + T^k_j) \sqrt{-g} \right) = 0 ,
\]

is just a false tensor, that can be reduced to zero with a suitable change of co-ordinates; \(ii\) from the geometrical and analytical standpoint, it is certain that the left-hand side of Einstein equations \([R^{jk} - (1/2)g_{jk}R]\) represents the true stress-energy-momentum tensor of Einstein field – as it had been pointed out also by Lorentz \([15]\).

Quite independently, point \(i\) and point \(ii\) tell us that the GW’s (which are solutions of \(R^{jk} = 0\) are mere mathematical, not physical, undulations, both in the exact GR and in the linear approximation of it.

There was a gentleman’s disagreement between Einstein and Levi-Civita. Against point \(ii\) Einstein raised an intuitive, “sentimental”, objection, which represents an implicit dissatisfaction with the formal structure of his theory: with the proposal by Levi-Civita and Lorentz, the total energy of a closed system is always zero, and the conservation of this value does not imply the further existence of the physical system under any whatever form.

Point \(i\): in Pauli’s and in Weyl’s treatises \([16]\) we find the various stratagems (by several authors) having the aim to prove that under convenient spatio-temporal asymptotic conditions the integrals

\[
J_j := \int (t^0_j + T^0_j) \sqrt{-g} \, dx^1 \, dx^2 \, dx^3
\]

give the conserved total four-momentum of a closed physical system. Now, one can remark that it is conceptually inappropriate to circumvent a property of the differential formalism by prescribing ad hoc conditions to given integrals; besides, an ineffective procedure, for the following reason. As Lorentz \([15]\) and Klein \([17]\) pointed out, there are other quantities – say \(w^{jk}\), different from the above \(t^{jk}\) –, which satisfy the same conditions, in particular such that

\[
\frac{\partial}{\partial x^k} \left( (w^k_j + T^k_j) \sqrt{-g} \right) = 0 ;
\]

the \(w\)’s by Lorentz and Klein depend also on the second derivatives of \(g_{jk}\), but this does not violate any physical principle. Now, the value of the corresponding total four-momentum does not coincide with the value given by eqs. \([14]\).

Of course, Einstein was perfectly aware that results of the kind \([14]\) are only a provisional way out; and on the other hand he thought that the entire GR is only a provisional theory!
So far as the GW’s are concerned, he was always doubtful about their physical existence; a careful reading of his lucid paper of 1937 with N. Rosen [13] is enlightening. And in his beautiful booklet The Meaning of Relativity no mention is made of the gravitational waves [19].

A last remark on these waves. Considering the total failure of all experimental attempts to reveal them, some physicists have recently revived – in private correspondences – an old conjecture: even if the GW’s really existed, it would be impossible to detect them, because the spatio-temporal deformation induced by the passage of a GW would interest both the apparatuses, i.e. the resonant bar or the Michelson interferometer, and the devices that register resp. the bar vibrations or the geodesic deviation of the suspended mirrors of the interferometer. In this second case it is necessary to take into account the interaction of the GW with the light beams in the interferometric arms [20], because it gives a modification of the displacement of the interference fringes generated by the above geodesic deviation.

Parergon

The readers of Number 27 (Spring 2006) of newsletter Matter of Gravity [21] can see that the astrophysicists of the “main stream” are far from a proper understanding of the repeated experimental failures for detecting GW’s and BH’s.

The research brief “Recent progress in binary black hole simulations” contains a very peculiar assertion: since the BH’s have singularities which can represent a hard obstacle for numerical simulations, Pretorius Group “uses black hole excision, whereby the black hole interior is removed from the computational grid. This is justified since the event horizon disconnects the interior causally from the exterior.” Clearly, these scholars have not realized that the event horizon (the singularity at $r = 2m$) is the essence of the (fictive) notion of BH. BH-excision amounts to substitute the standard form of solution of Schwarzschild problem with, e.g., Brillouin’s form [10] or Schwarzschild’s (original) form, that are diffeomorphic to the exterior part ($r > 2m$) of the standard form – and are maximally extended.

The briefs entitled resp. “What’s new in LIGO” and “LISA Pathfinder” describe the continuous refinements of the apparatuses, and declare implicitly a great optimism about the results of future searches: the detection of the GW’s is now behind the corner.

Stat pro ratione voluntas.

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Dipartimento di Fisica, Università di Milano, Via Celoria, 16 - 20133 Milano (ITALY)
E-mail address: angelo.loinger@mi.infn.it