Rotating deformations of $AdS_3 \times S^3$, the orbifold CFT and strings in the pp-wave limit

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Abstract

We construct an exact metric which at short distances is the metric of massless particles in 5+1 spacetime (moving along a diameter of the sphere) and is $AdS_3 \times S^3$ at infinity. We also consider a set of a conical defect spacetimes which are locally $AdS_3 \times S^3$ and have the masses and charges of a special set of chiral primaries of the dual orbifold CFT. We find that excitation energies for a scalar field in the latter geometries agree exactly with the excitations in the corresponding CFT state created by twist operators: redshift in the geometry reproduces ‘long circle’ physics in the CFT. We propose a map of string states in $AdS_3 \times S^3 \times T^4$ to states in the orbifold CFT, analogous to the recently discovered map for $AdS_5 \times S^5$. The vibrations of the string can be pictured as oscillations of a Fermi sea in the CFT.
1 Introduction.

String theory on $AdS_5 \times S^5$ is believed to be dual to $N = 4$ Super-Yang-Mills gauge theory on the boundary of $AdS_5$ [1]. Weakly coupled gauge theory is dual to a string background that is not well described by perturbative supergravity. Nevertheless it is possible to demonstrate several relations between the two dual theories.

String theory on $AdS_3 \times S^3 \times M_4$ is expected to be dual to a 2-dimensional CFT on the boundary of $AdS_3$. It has been conjectured that this CFT can be represented as a deformation of a 2-dimensional sigma model with the orbifold target space $M^N_4/S_N$ [2]. (Here $S_N$ is the permutation group for $N$ variables.) The analogue of free Yang-Mills would be the ‘orbifold point’ where we have the orbifold with no deformation. This orbifold model has given several exact agreements with the dual theory for both BPS and near-BPS quantities.

An essential feature of the orbifold theory is the existence of ‘twist operators’ $\sigma_n$. The CFT is described by $N$ copies of a $c = 6$ CFT. The twist $\sigma_n$ links together $n$ of these copies to yield a $c = 6$ CFT living on a circle that is $n$ times longer than the circle on which the initial CFT was defined. The excitations of the CFT on this long circle can have very low energy if $n$ is large.

In this paper we study two related questions that arise in this duality map. A chiral primary in the CFT is described by a twist $\sigma_n$ which also carries a charge that makes $h = j$. We call the resulting chiral primary $\sigma_n^{--}$. Consider the state where all the $N$ copies of the CFT are twisted so as to make the state $(\sigma_n^{--})^{N/n}|0\rangle_{NS}$. Since all copies of the CFT now live on ‘long circles’ we must have low energy excitations of the CFT, and we must find that in the dual gravity theory the supergravity fields must have corresponding low energy eigenmodes.

For the CFT in the Ramond (R) sector this issue was studied in [3], where supergravity duals were constructed for all the R ground states – these states are in one-to-one correspondence with the chiral primaries of the NS sector which in turn are of the form $\sigma_{n_1}^{--}\sigma_{n_2}^{--}\ldots \sigma_{n_k}^{--}$. It was found that the length of the CFT circles was described in the supergravity dual by the depth of the ‘throat’ of the near horizon geometry. The simplest geometries were dual to configurations of the type $(\sigma_n^{--})^{N/n}$, and for these geometries the travel time around the CFT ‘long circle’ agreed exactly with the travel time down the throat of the gravity solution.

In our present study of the NS sector we therefore look for the gravity duals of the CFT states $(\sigma_n^{--})^{N/n}|0\rangle_{NS}$. We solve the wave-equation in the geometries, and find energy levels that agree exactly with the energy levels for the CFT with ‘long circles’.

The other question relates to the recent proposal by Berenstein, Maldacena and Nastase [4] that strings moving fast on the $S^5$ in $AdS_5 \times S^5$ can be described easily in the dual gauge theory. A supergravity quantum moving fast on the $S^3$ in $AdS_3 \times S^3 \times M_4$ is described by a chiral primary $\sigma_n^{--}$ (or its SUSY descendent) with a large value of $n$. We note that $\sigma_n^{--}$ can be written in the form $(\sigma_2^{--})^n$, and use the ‘bits’ $\sigma_2^{--}, J_0^-, \tilde{J}_0^-, J_0^+, \tilde{J}_0^+$ as analogs of the $Z, X_i$ of [4] to construct the low lying string excitations of the quantum.
In more detail we do the following:

(a) The chiral primary $\sigma_n^-$ in the CFT is described by a supergravity quantum having angular momentum $n$ on the $S^3$. Thus the state $(\sigma_n^-)^{N/n}|0\rangle_{NS}$ should be described by a collection of such quanta. If $n \gg 1$ the quanta will be confined to a narrow width around the circle of rotation, and if we also have $N/n \gg 1$ then the number of quanta will be large and the resulting geometry should be well described by a supergravity solution.

In flat space the metric produced by a massless particle is the Aichelburg–Sexl metric $A$, and we can readily extend this to describe a set of particles uniformly distributed along the line of motion. We construct an exact solution that goes over to the analogue of the Aichelburg–Sexl solution in 5+1 spacetime near the moving particles, and to $AdS_3 \times S^3$ at infinity. Just as in the Aichelburg–Sexl case, the solution at the linearized level turns out to be exact.

(b) We do not however find any obvious way to separate the scalar wave equation in the above metric, in contrast to the separability found in the R sector solutions $R$. Further the travel time for quanta in this geometry from infinity to the center does not agree with the expected time for travel around the ‘long circles’ of the CFT state $(\sigma_n^-)^{N/n}|0\rangle_{NS}$. We thus look for a different metric that could be dual to these chiral primaries. We consider a set of metrics with a conical defect singularity along a circle; such conical defect singularities arose in the R sector solutions and were studied in $R$. For this class of metrics we find that the scalar wave equation separates, and the energy levels for the $l = 0$ harmonic of the scalar match exactly the corresponding energy levels in the dual CFT on the ‘long circle’. The travel time for a quantum from infinity to the center and back also agrees exactly with the travel time around the long circle of the CFT.

These results indicate that for the massless quanta describing the chiral primary $(\sigma_n^-)^{N/n}|0\rangle_{NS}$ the lowest energy solution is not the one with an Aichelburg–Sexl type singularity but rather the one with the conical defect singularity.

(c) We next consider a single quantum with high angular momentum in the geometry $AdS_3 \times S^3 \times M_4$. The quantum is described by a chiral primary $\sigma_n^-$, but the remaining $N - n$ copies of the $c = 6$ CFT are not twisted into ‘long circles’. The chiral primary $\sigma_n^-$ can be written as $(\sigma_2^-)^N$, where $\sigma_2^-$ is a chiral primary that can be regarded as a building block to make all higher chiral primaries. We argue that $\sigma_2^-$ plays the role of the field $Z$ in $R$, and that $J^-_0 \sigma_2^-$, $J^0 \sigma_2^-$ are the analogs of the variables $X_i$ that describe stringy excitations around the supergravity quantum. We find that the CFT description of these string excitations can be interpreted as low energy vibrations of a ‘Fermi fluid’ which arises from the fermionic excitations that give $\sigma_n^-$ its charge.

While we were working on this paper there appeared the paper $R$ which overlaps partly with our discussion in section 5.
2 The Aichelburg–Sexl type solution in $AdS_3 \times S^3$

We begin by recalling the metric produced by a massless point particle moving along the $z$ direction in $3+1$ dimensions $[5]$.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + 8\delta(t-z)\log(x^2 + y^2)(dt - dz)^2$$  \hspace{1cm} (2.1)

If we consider instead a set of such particles distributed uniformly along the $z$-axis then we get a time-independent metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + q\log(x^2 + y^2)(dt - dz)^2.$$ \hspace{1cm} (2.2)

Note that the function $\log(x^2 + y^2)$ appearing in the metric is a harmonic function of the transverse coordinates $x, y$ with $\delta$-function source at the singular line $x = y = 0$. In $5+1$ dimensions the corresponding metric will be $(x_i = x_1 \ldots x_4)$

$$ds^2 = -dt^2 + dz^2 + dx_idx_i + \frac{q}{(x_i x_i)}(dt - dz)^2,$$ \hspace{1cm} (2.3)

We wish to construct solutions that describe a set of massless particles rotating along a diameter of $S^3$ in the space $AdS_3 \times S^3$. This metric arises in IIB string theory after compactification on a 4-manifold $M_4$ which can be $T^4$ or $K3$. We will take the case of $T^4$ for concreteness in this paper, though there is no fundamental difference if we were to take $K3$ instead. The curvature is produced by the 2-form gauge field $B^{NSNS}_{\mu\nu}$; we will just call this field $B_{\mu\nu}$ hereafter. For unperturbed $AdS_3 \times S^3 \times T^4$ the 10-d Einstein metric is $[6]$

$$ds^2 = -(1 + \frac{r^2}{L^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2d\chi^2 + L^2\left\{d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right\} + \sum_{i=1}^{4}dz_idz_i$$ \hspace{1cm} (2.4)

and the gauge field is $[6]$\footnote{The action for the relevant fields is $\int \sqrt{-g} \left[R - \frac{1}{4}H^2\right]$ and $H_{\mu\nu\lambda} = \partial\mu B_{\nu\lambda} + \partial\nu B_{\lambda\mu} + \partial\lambda B_{\mu\nu}$.}$2$

$$B = L^2 \cos^2\theta d\phi \wedge d\psi + \frac{r^2}{L}dt \wedge d\chi.$$ \hspace{1cm} (2.5)

We consider a set of massless particles rotating along the diameter $\theta = 0$ of the $S^3$ at the location $r = 0$ in the $AdS_3$. (We will ignore the $T^4$ in what follows – one may assume that the particle wavefunctions are uniformly smeared along the $T^4$.) We thus look for a solution of the IIB field equations that behave as (2.3) near $\theta = 0, r = 0$, and go over to $AdS_3 \times S^3$ at large $r$.

We first construct the linear solution corresponding to small strength of the source at $\theta = 0, r = 0$. It turns out that just as is the case for the Aichelburg–Sexl metric in 3+1

$$B_{\phi\psi} = L^2 \cos^2\theta \text{ etc.}$$
dimensions, the linear solution is exact. The full 10-d metric is
\[ ds^2 = -(1 + \frac{r^2}{L^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\chi^2 + L^2 \left\{ d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2 \right\} + \sum_{i=1}^4 dz_i dz_i \]
\[ + \frac{q}{r^2 + L^2 \sin^2 \theta} \left\{ \left(1 + \frac{r^2}{L^2}\right) dt + L \cos^2 \theta d\psi \right\}^2 - L^2 \left\{ \frac{r^2}{L^2} d\chi - \sin^2 \theta d\phi \right\}^2 \] (2.6)

and the NS–NS two–form field is\[ B = \frac{r^2}{L} dt \wedge d\chi + \frac{L^2}{L^2} \cos^2 \theta (d\psi - \gamma d\chi) \wedge (d\phi - \gamma dt) \] (3.2)

3 Metrics with conical defects

In [8], [9] a set of metrics was considered which could be obtained as special cases of metrics for rotating charged holes found in [12]. The metrics describe the D1-D5 bound state carrying angular momentum, and go over to flat space at spatial infinity. In the near horizon region (the ‘AdS region’), the solutions have the form
\[ ds^2 = -(\frac{r^2}{L^2} + \gamma^2)dt^2 + r^2 d\chi^2 + \frac{dr^2}{\frac{r^2}{L^2} + \gamma^2} \]
\[ + L^2 \left\{ d\theta^2 + \cos^2 \theta (d\psi - \gamma d\chi)^2 + \sin^2 \theta (d\phi - \gamma \frac{dt}{L})^2 \right\} , \] (3.1)
\[ B = \frac{r^2}{L} dt \wedge d\chi + \cos^2 \theta (d\psi - \gamma d\chi) \wedge (d\phi - \gamma \frac{dt}{L}) \] (3.2)

It was argued in [8], [9] that these solutions describe ground states of the D1-D5 system in the Ramond sector. It was found in [8] that only a special class of Ramond ground states are described by these solutions. The solution for the general ground state was then constructed, and it was found that the singularity had the shape of a complicated curve for a generic metric.

On the other hand we can write down a larger class of metrics similar to the simple form (3.1):
\[ ds^2 = -(\frac{r^2}{L^2} + \gamma^2)dt^2 + r^2 d\chi^2 + \frac{dr^2}{\frac{r^2}{L^2} + \gamma^2} \]
\[ + L^2 \left\{ d\theta^2 + \cos^2 \theta (d\psi - \tilde{\beta} \frac{dt}{L} - \alpha d\chi)^2 + \sin^2 \theta (d\phi - \tilde{\alpha} \frac{dt}{L} - \beta d\chi)^2 \right\} , \] (3.3)
\[ B = \frac{r^2}{L} dt \wedge d\chi + \cos^2 \theta (d\psi - \tilde{\beta} \frac{dt}{L} - \alpha d\chi) \wedge (d\phi - \tilde{\alpha} \frac{dt}{L} - \beta d\chi) \] (3.4)

Such a class of metrics was studied in [10]
All these metrics possess a conical defect at $\theta = 0, r = 0$, and are locally $AdS_3 \times S^3$ elsewhere. Thus they are quite different from the metric found in the previous section.

Due to the presence of the conical singularity we do not know a priori that all these metrics are solutions of Type IIB string theory. We will however try to identify a subclass of such metrics with the dual of chiral primaries of the NS sector, by studying the values of conserved charges and the spectrum of low energy excitations around these solutions.

Let us briefly discuss the charges associated with metric (3.3). The 2+1 gravity theory obtained by reduction of IIB on $S^3 \times M_4$ has a gauge field $A_\mu$, where $\alpha$ is an index on $S^3$ and $\mu$ is an index in the $AdS_3$. The field equation for $A$ is

$$d * dA + dA = * j$$

(3.5)

The conserved charge is thus given by the line integral over the circle $t = t_0, r = r_0$ of the quantity $* dA + A$. The first part is the usual flux of electrodynamics; it vanishes at infinity for the metrics we consider. The second part is a Wilson line, and gives the angular momentum of the solution. The angular momenta corresponding to translations in $\phi$ and $\psi$ directions are:

$$j(\psi) = \frac{\beta L}{4 G_3}, \quad j(\phi) = \frac{\alpha L}{4 G_3}, \quad \bar{j}(\phi) = \frac{\gamma^2}{8 G_3}, \quad \bar{j}(\psi) = \frac{\gamma^2}{8 G_3}, \quad \bar{j}_0 = \frac{\beta L}{2 G_3}, \quad \bar{j}_0 = \frac{\alpha L}{2 G_3},$$

(3.6)

where $G_3$ is three dimensional Newton’s constant. The mass of the solution is

$$M = -\frac{\gamma^2}{8 G_3}$$

(3.7)

The quantities in the gravity theory can be related to quantities in the boundary CFT by the following relations [13, 8, 9]

$$c = \frac{3L}{2G_3}, \quad j_0 = \frac{1}{2}(j(\phi) - j(\psi)), \quad \bar{j}_0 = -\frac{1}{2}(j(\phi) + j(\psi))$$

(3.9)

Let $l_0 + \bar{l}_0$ give the energy and $l_0 - \bar{l}_0$ give the $AdS_3$ momentum of a configuration. Then to map to CFT levels is achieved by addition of a Sugawara term and a constant shift:

$$L_0 \equiv l_0 + \frac{c}{24} + \frac{6(j_0)^2}{c}, \quad \bar{L}_0 \equiv \bar{l}_0 + \frac{c}{24} + \frac{6(\bar{j}_0)^2}{c}$$

(3.10)

For the solutions (3.3)

$$l_0 = \bar{l}_0 = \frac{LM}{2} = \frac{L}{16 G_3}, \quad j_0 = \frac{c}{12}(\alpha - \beta), \quad \bar{j}_0 = -\frac{c}{12}(\alpha + \beta)$$

(3.11)
and

\[ L_0 = \frac{c}{24} (-\gamma^2 + 1 + (\alpha - \beta)^2) \] (3.12)

\[ \bar{L}_0 = \frac{c}{24} (-\gamma^2 + 1 + (\alpha + \beta)^2) \] (3.13)

The Ramond vacuum of the CFT is dual to the geometry (3.1) with

\[ \alpha = \gamma, \quad \tilde{\alpha} = \gamma, \quad \beta = \tilde{\beta} = 0 \] (3.14)

From the charge and dimensions of the above solutions one infers that the NS vacuum is given by

\[ \alpha = \tilde{\alpha} = 0, \quad \beta = \tilde{\beta} = \gamma = 1 \] (3.15)

Chiral primaries in the NS sector have \( L_0 = j_0, \bar{L}_0 = \bar{j}_0 \) which corresponds to

\[ \alpha = \tilde{\alpha} = 0, \quad \beta = \tilde{\beta} = \gamma - 1. \] (3.16)

For these chiral primaries the geometry (3.3) becomes:

\[
\begin{align*}
ds^2 & = -(r^2 + \gamma^2)dt^2 + r^2d\chi^2 + \frac{dr^2}{r^2 + \gamma^2} \\
& \quad + L^2 \left\{ d\theta^2 + \cos^2 \theta (d\psi - \beta \frac{dt}{L})^2 + \sin^2 \theta (d\phi - \beta d\chi)^2 \right\}, \\
B & = \frac{r^2}{L} dt \wedge d\chi + \cos^2 \theta (d\psi - \beta \frac{dt}{L}) \wedge (d\phi - \beta d\chi), \quad \gamma = \beta + 1
\end{align*}
\] (3.17)

For these solutions the parameter \( \gamma \) is related to \( j_0 \) in the following way by eqn. (3.19).

\[ \gamma = 1 - \frac{12}{c} j_0 = 1 - \frac{2j_0}{N} \] (3.19)

Let us see which chiral primaries in the CFT would be dual to these geometries. We review the structure of chiral primaries of the \( N = 4 \) orbifold CFT in Appendix [B]. Consider the chiral primary \( \sigma_n^- \). This has \( h = j = \frac{n-1}{2} \). The states we are interested in are generated by a set of such chiral primaries:

\[ [\sigma_n^-]^{N/n} : \quad L_0 = j_0 = \bar{L}_0 = \bar{j}_0 = \frac{N n - 1}{n} \] (3.20)

We then find from (3.19) the value of the parameter \( \gamma \) in the metrics (3.17) dual to the state \([\sigma_n^-]^{N/n}|0\rangle_{NS} : \)

\[ \gamma = \frac{1}{n} \] (3.21)

\textsuperscript{4} The values of \( \tilde{\alpha}, \tilde{\beta} \) are determined by the spectral flow parameters of the CFT \([8, 9]\).
4 Energy levels and travel time for scalar fields.

The state in the CFT created by the chiral primary $\sigma_{-n}^{-}|0\rangle_{NS}$ is described in the dual theory by a supergravity particle in $AdS_3 \times S^3 \times M_4$ [14]. Thus the state $(\sigma_{-n}^{-})^{N/n}|0\rangle_{NS}$ should be described by a collection of $N/n$ such particles. If $n \gg 1$ the particles have a high angular momentum, and their wavefunctions will be confined to a very narrow width around the diameter on the $S^3$ on which they rotate. Thus they would be pointlike particles for supergravity. If we also have $N/n \gg 1$ then we expect to describe the resulting physics by a classical supergravity solution.

The configuration constructed in the above way will be naturally smeared uniformly along the diameter of rotation. We assume that all wavefunctions are constant over the $M_4$, and so can restrict attention to $AdS_3 \times S^3$. We thus have a line of massless particles in 5+1 dimensional spacetime.

A first guess would therefore be that near to this line of particles the solution behaves like (2.3), the Aichelburg–Sexl solution in flat 5+1 spacetime. We therefore consider the exact solution (2.6).

At this point we are led to compare this solution with the metrics found in [8, 9, 15] for the states of the CFT in the R sector. We expect to have somewhat similar properties for the R and NS cases, since one is just a spectral flow of the other. In the R sector the wave equation for a scalar separated between the angular and radial variables. Such a separation is not obvious for the geometry (2.6). Further, in [3] the travel time around the ‘long circles’ in the CFT agreed exactly with the travel time in the throat of the dual supergravity solution. We do not find such an agreement for the solution (2.6). We are therefore led to consider other possible solutions that could describe the state $(\sigma_n)^{N/n}|0\rangle_{NS}$.

The solutions (3.17), (3.18) have conical defects that are similar to those found for the R sector metrics in [8, 9]. If we assume the relations (3.8)–(3.10) postulated in [13, 9] between quantities in the gravity theory and quantities in the CFT, then we find that $L_0 = j_0, \bar{L}_0 = \bar{j}_0$ for these solutions, so they have the right structure to be identified as the states of the string theory dual to the states $(\sigma_n)^{N/n}|0\rangle_{NS}$ created by chiral primaries in the CFT.

We will now study the energy levels of some supergravity excitations around the metrics (3.17) and see that they agree with the excitation levels expected from the dual CFT.

4.1 Energy levels for $l = 0$ scalar excitations

There are several scalars in the 5+1 dimensional theory obtained by reduction on $M_4$. For concreteness we take $M_4 = T^4$. Let the $T^4$ be described by coordinates $x_1 \ldots x_4$.

\footnote{Note that the $B$ field (2.7) has a divergent quartic invariant $H_{\mu\nu\lambda}H^{\nu\lambda\sigma}H_{\sigma\tau\rho}H^{\tau\rho\mu}$ near the singular line, though the divergence of the invariant is much weaker than would be naively expected from the magnitude of the $H$ field near the singularity.}
Then the graviton $h_{12}$ is a minimally coupled scalar in the remaining 5+1 dimensions

$$\Phi \equiv h_{12}, \quad \Box \Phi = 0 \quad (4.1)$$

In Appendix A we solve this wave equation in the metric (3.17). The wave equation is separable, and we have solutions of the form

$$\Phi(t, r, \chi, \theta, \psi, \phi) = \exp(-i\omega t + ip\psi + iq\phi + i\lambda \chi)H(r)\Theta(\theta) \quad (4.2)$$

We get normalizable solutions for the frequencies (A.15)

$$\omega_k = \beta p L \pm \left\{ \frac{2\gamma}{L} (k + 1 + \frac{l}{2}) + \frac{\lambda + \beta q}{L} \right\} \quad k = 0, 1, 2, \ldots \quad (4.3)$$

Consider the case $l = 0, \lambda = 0$. The spectrum is

$$\omega_k = \frac{2}{nL}(k + 1), \quad k = 0, 1, 2, \ldots \quad (4.4)$$

where we have used the relation $\gamma = 1/n$ for chiral primaries (see (3.20)).

Let us relate the energy of such quanta with the expected changes in quantities in the CFT. For $\delta j_0 = \delta \bar{j}_0 = 0$ the relations (3.10) give

$$\delta (L_0 + \bar{L}_0) = \delta (l_0 + \bar{l}_0) = L\omega_k = \frac{2(k + 1)}{n} \quad (4.5)$$

Now let us consider the CFT itself. The graviton $h_{12}$ is known to be described by a pair of excitations [16]

$$h_{12} \rightarrow \frac{1}{\sqrt{2}}[\partial X^1 \bar{\partial} X^2 + \partial X^2 \bar{\partial} X^1] \quad (4.6)$$

There is one left mover and one right mover on the ‘long circle’ of length $2\pi n$. We can have fractional levels for $L_0, \bar{L}_0$ but $L_0 - \bar{L}_0$ must be an integer [17, 18]. The energy levels for $L_0 - \bar{L}_0 = 0$ are given by

$$\delta L_0 = \delta \bar{L}_0 = \frac{k'}{n}, \quad \delta (L_0 + \bar{L}_0) = \frac{2k'}{n}, \quad k' = 1, 2, \ldots \quad (4.7)$$

so that we get exact agreement with (4.5).

### 4.2 Travel time

In [3] it was found that the time $\Delta t_{CFT}$ for excitations to travel around the ‘long circle’ of the CFT was exactly equal to the time $\Delta t_{SUGRA}$ taken for a supergravity quantum to travel once up and down the ‘throat’ of the dual geometry. In the present case the geometry at infinity is $AdS$ rather than flat space.
We observe from (4.3) and the fact that $\gamma = \frac{1}{n}$, $\beta = \frac{1}{n} - 1$ that the frequencies $\omega_k$ have the form

$$\omega = \frac{1}{L} \left[ \frac{m_1}{n} + m_2 \right], \quad m_1, m_2 \text{ integral} \quad (4.8)$$

Consider any wavefunction for the scalar particle made by an arbitrary superposition of such frequencies. Then the wavefunction returns to its initial form after a time

$$\Delta t_{\text{SUGRA}} = n2\pi L \quad (4.9)$$

Now consider the dual CFT. The left and right movers are massless excitations, and travel at the speed of light around the spatial circle of the cylinder on which the CFT lives. This gives $L\delta \chi = \delta t$ for the right movers and $L\delta \chi = -\delta t$ for the left movers. But the twist $\sigma_n$ makes the circle on which the CFT excitations live a ‘long circle’ of length $2\pi n$ times the initial circle size of the CFT. The state $(\sigma_n^{-})^{N/n}|0\rangle_{NS}$ that we have chosen has all copies of the CFT arranged into long circles of this length. Thus any excitation of this CFT state will return to its initial form after a time

$$\Delta t_{\text{CFT}} = 2\pi nL \quad (4.10)$$

which is in exact agreement with (4.9).

To summarize, we have seen that the CFT state $(\sigma_n^{-})^{N/n}|0\rangle_{NS}$ has all copies of the $c = 6$ CFT joined into ‘long circles’ that wind $n$ times the spatial direction before closing. The low energy excitations in the CFT are thus of energy $\sim 1/n$. In the dual supergravity solution we do not see any ‘multiwinding’ around the $\chi$ direction. What we have instead is a deep ‘throat’ like structure near $r = 0$ that leads to a large redshift between $r = \infty$ and $r \approx 0$. Particle wavefunctions trapped near $r = 0$ have energies that are low ($\sim 1/n$) due to this redshift. This relation ‘multiwinding in CFT $\rightarrow$ redshift in gravity’ is expected to have general validity and was found also in the R sector computations of [3].

5 CFT states representing a single string

5.1 String states in $AdS_5 \times S^5$

Let us review the idea proposed in [4]. We consider a supergravity particle rotating on the diameter of $S^5$ in $AdS_5 \times S^5$, at the origin of $AdS_5$. The $S^5$ is described by coordinates $X^1 \ldots X^6$, and the rotation is in the plane $X^5, X^6$. We write $Z = X^5 + iX^6$, and let the other four $X$ coordinates be called $X^i, i = 1 \ldots 4$. It was noted in [4] that (for appropriate choice of supergravity quantum) the state in the dual CFT is created by the operator $\text{tr}[Z^J]$, where $J$ is the angular momentum of the quantum. It was then argued that for large angular momentum $J$ we can easily describe the excitations of the quantum that change it to a stringy state. There are 8 bosonic excitations possible, and in a light-cone gauge 4 are along the directions $X^i$ while 4 are directions in the $AdS_5$. 


Stringy excitations in the $X^i$ directions are described in the dual theory by operators of the form
\[
\text{tr}[ZZ \ldots ZZX^i ZZ \ldots ZZX^i ZZ \ldots ZZ] \tag{5.1}
\]
Excitations in the $AdS_5$ directions are given by derivatives $\partial_{\mu}, \mu = 1 \ldots 4$.

Each of these insertions $X^i, \partial_{\mu}$ have $\Delta - J = 1$. We do not put insertions of $\bar{Z}$, which has $\Delta - J = 2$. While the OPE of $X^i$ with $Z$ is regular in the free CFT, the OPE of $\bar{Z}$ with $Z$ gives
\[
\langle \bar{Z}(x_1)Z(x_2) \rangle \sim \frac{1}{(x_1 - x_2)^2} \tag{5.2}
\]
The expectation is that the operators with $\bar{Z}$ insertions would be renormalized to high dimensions when we go to the values of coupling appropriate to the supergravity description, and so these would not be visible in the string vibration spectrum.

### 5.2 String spectrum and energy scales for $AdS_3 \times S^3 \times T^4$

Let us now consider the analogue of this problem for $AdS_3 \times S^3 \times T^4$. We take a supergravity quantum rotating on the $S^3$ while staying at the origin in $AdS_3$. The spectrum of string states was found in [13, 4, 20]

\[
\Delta = J + \sum_{m=-\infty}^{\infty} \left\{ \tilde{\omega}_m a_m^{(+)\dagger} a_m^{(+)} + \tilde{\omega}_m a_m^{(-)\dagger} a_m^{(-)} + \tilde{\omega}_m \tilde{a}_m^{(+)\dagger} \tilde{a}_m^{(+)} + \tilde{\omega}_m \tilde{a}_m^{(-)\dagger} \tilde{a}_m^{(-)} \right\}
+ \sum_{m=-\infty}^{\infty} \sum_{i=1}^{4} \left\{ \frac{L^2}{\alpha' \Delta + J} c_m^{(i)\dagger} c_m^{(i)} \right\} + \text{ferm} \tag{5.3}
\]
Here $a$ give oscillators in the the $AdS$ directions, $\tilde{a}$ give oscillations in the sphere directions, and $c_i$ give oscillations in the $T^4$. Further,
\[
\tilde{\omega}_m = \sqrt{\sin^2 \hat{\alpha} + \left( \cos \hat{\alpha} + \frac{L^2}{\alpha'} \frac{2m}{\Delta + J} \right)^2} \approx 1 + \cos \hat{\alpha} \left( \frac{L^2 T_s}{2\pi}(\frac{m}{J}) \right) \tag{5.4}
\]
where $L$ is a radius of the $AdS_3$, $T_s = \frac{1}{2\pi \alpha'}$ is the tension of the elementary string and the approximation in the last step is for small mode numbers $m \ll J$.

At leading order in $m/J$ we have the frequencies $\tilde{\omega}_m \approx 1$ for the oscillators in the $S^3$ and $AdS_3$ directions, which implies that these excitations have
\[
\Delta - J \approx 1 \tag{5.5}
\]
The oscillators in the $T^4$ direction have $\tilde{\omega} \approx 0$ which implies
\[
\Delta - J \approx 0 \tag{5.6}
\]
The correction of order $m$ in (5.4) depends on the value of $\hat{\alpha}$. But the sigma model CFT at the orbifold point is equally far in moduli space from the D1–D5 background.
(\hat{\alpha} = \pi/2) and the NS1-NS5 background (\hat{\alpha} = 0). Any effects involving \hat{\alpha} can only be obtained by studying deformations of the orbifold. We will study the CFT only at the orbifold point, and it is important to note that in such a study we should try to reproduce only the values (5.3), (5.6) and not the subleading corrections.

5.3 CFT operators dual to string oscillators

Consider a supergravity quantum rotating on the $S^3$ while staying at the origin in $AdS_3$. Letting this quantum be made from a suitable combination of $h_{ab}, B_{ab}$ ($a, b$ are indices on $S^3$) we obtain a state with $\h = j = \frac{n - 1}{2}$, $\bar{\h} = \bar{j} = \frac{n - 1}{2}$, which is a created in the dual CFT by a chiral primary

$$\sigma_{n}^{-} |0\rangle_{NS}$$

Let us write explicitly the permutation involved in the twist operator $\sigma_n$. We must finally sum over all choices of the indices involved in the permutation $[21, 22, 23, 24]$, but to do this we must first start by choosing definite values of the permuted indices. Thus let the operator be

$$\sigma_{n}^{-} = \sigma_{(12...n)}$$

where $(12 \ldots n)$ is the permutation in the twist created by $\sigma_{n}^{-}$. Then we can write

$$\sigma_{(12...n)} = \sigma_{(12)} \sigma_{(23)} \ldots \sigma_{(n-1,n)}$$

So the chiral primary $\sigma_2^{-}$ which has a twist of order 2 acts like a building block for other chiral primaries, and is thus similar to the operator $Z$ in the CFT dual to $AdS_5 \times S^5$.

The operators $X_i, \bar{Z}$ in the $AdS_5 \times S^5$ case were $SO(6)$ rotates of the operator $Z$. In the present case the corresponding rotation group is $SO(4) \approx SU(2)_L \times SU(2)_R$. There are two vibrations on the sphere, which suggests that the string oscillators in the sphere directions map to the CFT operators

$$J_0 \sigma_2^{-}, \bar{J}_0 \sigma_2^{-}$$

The analogue of $\bar{Z}$ is the operator with $SO(4)$ spin opposite to the spin of $\sigma_2^{-}$, which is

$$J_0 \bar{J}_0 \sigma_2^{-}$$

Note that the operators (5.10) have $\Delta - J = 1$ while the operator (5.11) has $\Delta - J = 2$; these are similar to the $\Delta - J$ values of the $X_i$ and $\bar{Z}$ respectively.

For the identification between string oscillators and CFT operators to make sense we need to check in addition that the OPE of the operators (5.10) with $\sigma_2^{-}$ is regular, while the OPE of (5.11) with $\sigma_2^{-}$ is singular.

In Appendix B we review the structure of twist operators. Let us note some of the structure of chiral primaries and their excitations. A twist operator $\sigma_n$ just cyclically permutes $n$ copies of the $c = 6$ CFT into each other. To make this into a chiral primary
with $h = j$ we must add charge, which can be done by adding fractional modes of currents $J^+_{k/n}$.

Now consider the OPE of $J^-_0 \sigma^-_2$ with $\sigma^-_2$. Working independently in the holomorphic and antiholomorphic sectors we find the apparent singularity

$$[J^-_0 \sigma^-_2](z)\sigma^-_2(w) \sim \frac{1}{(z-w)^{1/3}}[J^-_3 \sigma^-_3](w) + \ldots, \quad (5.12)$$

The operator $[J^-_3 \sigma^-_3](w)$ has $h - \bar{h}$ nonintegral. But recall that while fractional modes are allowed for the left and right sectors, the allowed operators and states in the CFT are only those with $h - \bar{h}$ integral \[^7^[17, 18]. Thus the singular operator on the RHS of (5.12) does not exist in the CFT, and the OPE is in fact regular

$$[J^-_0 \sigma^-_2](z)\sigma^-_2(w) \sim [J^-_0 \sigma^-_3](w) + \ldots \quad (5.13)$$

Thus $J^-_0 \sigma^-_2$, $\bar{J}^-_0 \sigma^-_2$ are good candidates for the ‘defects’ that can be included in the chain (5.3) to represent string oscillations:

$$\ldots \sigma^-_2 \ldots |J^-_0 \sigma_2|\sigma^-_2 \ldots \sigma^-_2 [\bar{J}^-_0 \sigma^-_2]|\sigma^-_2 \ldots \quad (5.14)$$

Now consider the OPE of $J^-_0 \bar{J}^-_0 \sigma^-_2$ with $\sigma^-_2$. This time we get the leading singularity

$$[\bar{J}^-_0 J^-_0 \sigma^-_2](z)\sigma^-_2(w) \sim \frac{1}{|z-w|^23}[J^-_3 \bar{J}^-_3 \sigma^-_3](w) + \ldots \quad (5.15)$$

The operator on the RHS has $h - \bar{h} = 0$ and is an allowed operator in the theory. Thus we should not include $J^-_0 \bar{J}^-_0 \sigma^-_2$ in the chain (5.14), which is analogous to not allowing $Z$ in the case of $AdS_5 \times S^5$. The combination $ZZ$ can be contracted away to the identity; similarly $(J^-_0 \bar{J}^-_0 \sigma^-_2)(\sigma^-_2)$ can be contracted to the identity and thus removed from the chain.

In Appendix [the] we discuss other OPEs that help support the picture developed here.

### 5.4 Mapping string oscillators to CFT operators

The string vibrations in the sphere directions were mapped above to $J^-_0 \sigma^-_2$, $\bar{J}^-_0 \sigma^-_2$. Following the ideas of [the] the oscillators in the $AdS_3$ directions should be mapped to $\partial_\zeta, \partial_{\bar{\zeta}}$, which upon acting on $\sigma^-_2$ can be written as

$$L_{-1}\sigma^-_2, \bar{L}_{-1}\sigma^-_2 \quad (5.16)$$

These operators also have $\Delta - J = 1$ [the].

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[^7]: See [the] for an alternative interpretation of the $AdS$ oscillators in $AdS_5 \times S^5$. 

There are 4 fermionic vibrations that have have $\Delta - J = 1$, and following the above pattern it is natural to identify these as

$$G_{1/2}^{-1/2} \sigma_{2}^{-}, \ G_{1/2}^{-2/2} \sigma_{2}^{-}, \ G_{1/2}^{-1/2} \sigma_{2}^{-}, \ G_{1/2}^{-2/2} \sigma_{2}^{-}$$  \hspace{1cm} (5.17)

We can place the above ‘modified $\sigma^{2}$’ operators (5.10), (5.16), (5.17) at various points along the chain (5.9) and consider the linear combinations of these possibilities that correspond to definite Fourier harmonics along the chain [4]. This gives the map from string oscillators to CFT operators:

$$a_{m}^{(+)\dagger}|J\rangle \rightarrow \sum_{k=1}^{J} e^{2\pi i mk/J} (\sigma_{2}^{--})^{k-1} [J_{0}^{-} \sigma_{2}^{--}] (\sigma_{2}^{--})^{J-k-1}$$

$$a_{m}^{(-)\dagger}|J\rangle \rightarrow \sum_{k=1}^{J} e^{2\pi i mk/J} (\sigma_{2}^{--})^{k-1} [\bar{J}_{0}^{-} \sigma_{2}^{--}] (\sigma_{2}^{--})^{J-k-1}$$

$$b_{m}^{(+)\dagger}|J\rangle \rightarrow \sum_{k=1}^{J} e^{2\pi i mk/J} (\sigma_{2}^{--})^{k-1} [G_{1/2}^{-1/2} \sigma_{2}^{--}] (\sigma_{2}^{--})^{J-k-1}, \ i = 1, 2$$

$$b_{m}^{(-)\dagger}|J\rangle \rightarrow \sum_{k=1}^{J} e^{2\pi i mk/J} (\sigma_{2}^{--})^{k-1} [\bar{G}_{1/2}^{-1/2} \sigma_{2}^{--}] (\sigma_{2}^{--})^{J-k-1}, \ i = 1, 2$$

$$a_{m}^{(+)\dagger}|J\rangle \rightarrow \sum_{k=1}^{J} e^{2\pi i mk/J} (\sigma_{2}^{--})^{k-1} [L_{-1} \sigma_{2}^{--}] (\sigma_{2}^{--})^{J-k-1}$$

$$a_{m}^{(-)\dagger}|J\rangle \rightarrow \sum_{k=1}^{J} e^{2\pi i mk/J} (\sigma_{2}^{--})^{k-1} [\bar{L}_{-1} \sigma_{2}^{--}] (\sigma_{2}^{--})^{J-k-1}$$  \hspace{1cm} (5.18)

After constructing the above state of the twist operators we must symmetrize by summing over permutations of the $N$ copies of the CFT. The cyclic permutations of the indices $(1 \ldots n)$ involved in the above operators yield the same state, and so these permutations must be added with uniform weight. This forces the total ‘momentum’ of the Fourier modes $m$ to add up to zero in the CFT, which agrees with the similar condition that arises for the string oscillators from the fact that the total momentum along the string world sheet must be zero.

### 5.5 CFT excitations as oscillations of a ‘Fermi fluid’

We can obtain some understanding of the CFT states created by the operators (5.18) by taking the specific case $M_{4} = T^{4}$ and writing the currents $J^{a}$ in terms of the fermions of the CFT. We have 4 real fermions in the holomorphic sector, and 4 real fermions in the antiholomorphic sector. The holomorphic fermions are grouped into a doublet of complex fermions $(\psi^{+}, \psi^{-})$ which forms a spinor of $SU(2)_{L}$. The currents are

$$J^{+} = \psi^{+}(\psi^{-})^{\dagger}$$

$$J^{-} = (\psi^{+})^{\dagger}\psi^{-}$$

$$J^{3} = \frac{1}{2}(\psi^{+}(\psi^{+})^{\dagger} + (\psi^{-})^{\dagger}\psi^{-})$$  \hspace{1cm} (5.19)
Consider for simplicity the chiral primary \( \sigma_n^{--} \) for \( n \) odd. The charge that must be added to the twist operator \( \sigma_n \) to make it a chiral primary is carried by fermions with fractional moding:
\[
\sigma_n^{--} = \left[ \psi^+_{-\frac{1}{2n}} \psi^+_{-\frac{3}{2n}} \ldots \psi^+_{-\frac{2n}{2n}} \right] \left[ (\psi^-)^\dagger_{-\frac{1}{2n}} (\psi^-)^\dagger_{-\frac{3}{2n}} \ldots (\psi^-)^\dagger_{-\frac{2n}{2n}} \right] \sigma_n
\]
(Note that \( \psi^+ \) and \( (\psi^-)^\dagger \) both carry the positive \( SU(2)_L \) spin.)

Now consider a state created by some Fourier mode \( m \) of the insertion \( J_0^- \sigma_2^- \) in the manner shown in (5.18). The Fourier function is approximately constant over a large number of \( \sigma_2 \) operators in the chain, since we seek to describe low energy oscillations of the chain of length \( \sim n \). We thus picture the modes (5.18) as follows. The fermions given in (5.20) form a ‘Fermi sea’ over the long circle of length \( 2\pi n \). The string vibrations on the torus have \( \Delta - J \approx 0 \). For small \( \Delta - J \) we have the following bosonic and fermionic modes in the CFT
\[
\begin{align*}
\partial X_i^\dagger_{\frac{k}{n}}, & \quad \psi^+_{-\frac{1}{2n}} \psi^+_{-\frac{3}{2n}} \ldots \psi^+_{-\frac{2n}{2n}}, & \quad (\psi^+)^\dagger_{\frac{1}{2n}} (\psi^+)^\dagger_{\frac{3}{2n}} \ldots (\psi^+)^\dagger_{\frac{2n}{2n}}, & \quad (\psi^-)^\dagger_{-\frac{1}{2n}} (\psi^-)^\dagger_{-\frac{3}{2n}} \ldots (\psi^-)^\dagger_{-\frac{2n}{2n}}, & \quad \psi^-_{\frac{1}{2n}} \psi^-_{\frac{3}{2n}} \ldots \psi^-_{\frac{2n}{2n}}.
\end{align*}
\]

The modes \( (\psi^+)^\dagger_{\frac{k}{n}} \) and their antiholomorphic counterparts act as annihilation operators for \( k < \frac{n}{2} \), and they become creation operators for \( k \geq \frac{n}{2} \). (Note that since we
have restricted attention to $k \ll n$ we will not reach this point in the excitation spectrum under consideration.

The energy levels implicit in the mode number $-k/n$ in (5.22) are valid only for the CFT at the orbifold point. In $[2]$ it was argued that the orbifold point has $n_5 = 1, n_1 = N$. Looking at the string frequencies (5.3) we note that if we set $n_5 = 1$ and also $\hat{\alpha} = 0$, then $L = \sqrt{\alpha' n_5}$ and for the $T^4$ oscillators we get

$$\tilde{\omega}_n = \frac{2m}{\Delta + J} \approx \frac{m}{J}$$

(5.23)

which agrees with the energy levels in (5.22) in the CFT. Similarly, the string modes in the $AdS_3$ and $S^3$ directions have energies

$$\tilde{\omega}_n \approx 1 + \frac{m}{J}$$

(5.24)

In the CFT at the orbifold point we find from the behavior of free fermion energy levels that

$$\sum_{k=1}^{n} e^{2\pi i mk/n} J_0 - \sigma_{(k,k+1)} \rightarrow J_{-\frac{m}{n}} \sigma_{n}$$

(5.25)

More generally the Fourier modes in (5.18) give operators $J_{-\frac{k}{n}}, G_{-\frac{2}{n}}, L_{-\frac{4}{n}}$ and their antihomomorphic partners, so that we get an exact agreement of energy levels. This agreement of energies cannot be taken too seriously though since at the point $n_5 = 1, n_1 = N$ it is unlikely that the string is well described by its free oscillation frequencies (5.4).

In general it appears to be more useful to think of the CFT excitations as generated by Fourier modes of local insertions as in (5.18), rather than as modes like $J_{-\frac{k}{n}}$ applied to the twist $\sigma_{n}$. As we deform the CFT away from the orbifold point we expect the picture in terms of collective vibrations to be a robust one, since it just uses the fact that locally along the chain we can apply a symmetry transformation to the spin of the chain. The exact energy of the state generated by such oscillations can on the other hand depend on the moduli of the CFT, and so we should use the orbifold point to reproduce just the leading values ($\Delta - J \approx 1, \Delta - J \approx 0$) for the various oscillators of the string.

6 Discussion

Let us collect together the above results (and results in [3]) to get a picture of the relation between rotation in $AdS_3 \times S^3$ and twist operators in the dual CFT.

The geometry $AdS_3 \times S^3$ is dual to the NS vacuum $|0\rangle_{NS}$ of the CFT. In this state the CFT (at the orbifold point) has $N$ copies of a $c = 6$ CFT with each copy living on a

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8In $[1]$ the energy levels of the CFT at the orbifold point were compared to the string levels at $\hat{\alpha} = 0$, and it was noted that only those levels in the CFT were found in the string spectrum which were multiples of $n_5$. 

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spatial circle on length $2\pi$. The operator $\sigma_n^{--}$ joins $n$ of these copies to give a $c = 6$ CFT living on a long circle of length $2\pi n$, and adds charge to give $h = j, \bar{h} = \bar{j}$. The dual of the state $\sigma_n^{--}|0\rangle_{NS}$ is expected to be a massless particle at the center of $AdS_3$ rotating on the $S^3$ with $SU(2) \times SU(2)$ angular momentum $(j, \bar{j})$ [4].

But in the CFT state $\sigma_n^{--}|0\rangle_{NS}$ we have circles of length $2\pi$ and also circles of length $2\pi n$. In [3] dual geometries were constructed for the Ramond sector states which arise from spectral flows of general chiral primary states $\sigma_n^{++}\sigma_n^{--}\ldots\sigma_n^{++}|0\rangle_{NS}$; these states in general have circles of several different lengths. Each metric had a singular curve with a shape depending on the set $\{k_i\}$ and the choices of signs $\pm$.

In our present study of the NS sector we have two cases where simplifications occur:

(a) We take the CFT state $(\sigma_n^{--})^{N/n}|0\rangle_{NS}$ where all circles are of the same length. Since we have a large number $N/n$ of particles in the dual geometry we have to consider their back-reaction, and find the appropriate new geometry.

(b) We take the CFT state $\sigma_n^{--}|0\rangle_{NS}$ where $n \ll N$. In this case since we have just one light particle in the geometry we can ignore its back-reaction and keep the geometry $AdS_3 \times S^3$. In this approximation we can look at the stringy excitations of this quantum, following the ideas of [4].

We started by looking for the geometries required in (a). For $n \gg 1$ the massless particles are pointlike. We first considered the assumption that the metric will take the Aichelburg—Sexl form (2.3) near the singular line of massless particles — this metric is isotropic to leading order near the singularity. It is interesting that the solution (2.6)–(2.7) with this limiting behavior can be found in closed form, and that the linear order solution turns out to be exact.

But the solution (2.6) does not appear to have the right properties to be the dual of the CFT state $(\sigma_n^{--})^{N/n}|0\rangle_{NS}$. Following the analysis of [3] we note that in the CFT all copies of the CFT live on circles of length $2\pi n$, and so low lying excitation energies should come in multiples of $\frac{2}{n^2}$. Further, since all excitations in the CFT travel at the speed of light around the spatial circle we expect that any excitation in the dual geometry will be periodic with period $2\pi n L$. But in the geometry (2.6) the scalar wave equation did not separate so that the energy levels had no simple form; further the time for a particle to start from infinity and return to infinity was not order $\sim 2\pi n L$.

We note that the space $AdS_3 \times S^3$ is not invariant under rotations in all 5 spatial directions, and so it is possible that the lowest energy configuration for a given angular momentum might not be isotropic even near the singular line of particles. In the R sector study of [3] it was found that the metrics dual to the spectral flow of the states $(\sigma_n^{--})^{N/n}|0\rangle_{NS}$ had the simple form found in [8, 9]: the geometries are locally $AdS_3 \times S^3$ and the singular curve is a circle. We thus consider the solution (3.17)–(3.18) which is a simple generalization of the solutions of [8, 9], and has the quantum numbers to be dual.
to the state \((\sigma_{-}^{-})^{N/n}|0\rangle_{NS}\). The conical defect makes the solution non-isotropic around the singular circle.

For the geometries (3.17) we find that the scalar wave equation separates, and we found the energy eigenvalues (4.3). For the simple case \(l = 0\) (particle with no angular momentum) we found that the energy levels (4.4) were multiples of \(\frac{2\pi}{nL}\). This agreed exactly with the CFT expectation, where the scalar considered is described by one left mover and one right mover, and each has energy \(\frac{k}{n}\). We see here the phenomenon that was also seen in [3]. Low energies and long travel times arise in the CFT from long circles. In the dual geometries we do not have long circles but instead deep throats which create a large redshift between infinity and \(r \sim 0\) so that we again get very low energy modes.

We now consider problem (b). In [4] stringy excitations in \(AdS_5 \times S^5\) were described in the dual CFT by adding ‘defects’ \(X^i\) in a chain of operators \(Z\). We have found a similar map in the case \(AdS_3 \times S^3\), where the basic member of the chain is the chiral primary \(\sigma_2^{-}\) and the ‘defects’ are given by symmetry operators like \(J_0^-\) acting on \(\sigma_2^{-}\).

The excitation energies for a string with high \(J\) are to leading order \(\omega \approx 1\) in units of the \(AdS\) scale (in the \(AdS_3\) and \(S^3\) directions). The corrections to these frequencies depend on the tension of the string, and thus on the point in moduli space that we choose for the geometry. The orbifold point can describe at best only one point in moduli space, so we do not expect to reproduce these corrections by working at the orbifold point.

The chiral primary \(\sigma_2^{-}\) has a large number of fermions that carry its charge, and these form a ‘Fermi sea’ in the \(c = 6\) CFT living on the long circle of length \(2\pi k\). The operator \(J_0^-\) rotates the spin direction of these fermions, and thus Fourier modes of \(J_0^-\) along the chain generate low energy oscillations of the Fermi sea. When we deform the theory away from the orbifold point we expect that this concept of collective oscillations will persist and yield the stringy oscillations of the chiral primary.

We can also combine problems (a) and (b) to consider stringy excitations of a quantum moving in the background dual to the state \((\sigma_{-}^{-})^{N/n}|0\rangle_{NS}\). We can give the string a small center of mass energy which takes it away from the singular circle; this lets its low energy excitations be those of a string in \(AdS_3 \times S^3\) without the singularity. We see that the redshift between infinity and \(r \sim 0\) lowers all string excitation energies by a factor \(n\), when these energies are measured from the boundary of \(AdS_3\). In the dual CFT we start with circles of length \(2\pi n\) instead of \(n\), and thus find energies that are also lower by a factor \(n\).

\(^9\)We have not proved directly that this geometry with its conical singularity is a solution of IIB string theory. By contrast in [3] we did prove that all solutions (which also had singular curves) were true solutions in string theory: we started with a string having momentum and winding, found its geometry, and then dualized to obtain D1-D5 metrics with a singular curve.
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A Energy levels for scalar excitations.

Here we solve the wave equation for a massless scalar. We look for the solution of the Klein–Gordon equation

$$\Box \Phi = 0$$  \hfill (A.1)

in the metric (3.17).

We write

$$\Phi(t, r, \chi, \theta, \psi, \phi) = \exp(-i\omega t + ip\psi + iq\phi + i\lambda\chi)H(r)\Theta(\theta),$$ \hfill (A.2)

Then we get the following equations for the two functions $H$ and $\Theta$ (see [7] for details):

$$\frac{1}{r} \frac{d}{dr} \left( r \left( \frac{r^2}{L^2} + \gamma^2 \right) \frac{dH}{dr} \right) + \left\{ \frac{(\omega - \beta p/L)^2}{r^2} - \frac{(\lambda + \beta q)^2}{r^2} \right\} H - \Lambda H = 0 \hfill (A.3)$$

$$\frac{1}{\sin 2\theta} \frac{d}{d\theta} \left( \sin 2\theta \frac{d\Theta}{d\theta} \right) - \left\{ \frac{q^2}{\sin^2 \theta} + \frac{p^2}{\cos^2 \theta} \right\} \Theta = -\Lambda \Theta, \hfill (A.4)$$

The angular equation involves the usual Laplacian on $S^3$, so we get $\Lambda = l(l+2)$ with $l = 0, 1, 2, \ldots$. Assuming that $\lambda + \beta q \geq 0$, we get the solution of the radial equation regular at $r = 0$ [7]:

$$H(x) = x^{(\lambda + \beta q)/2\gamma} (x + \gamma^2)^{(\omega L - \beta p)/2\gamma} F(a, a + l + 1; c; -\frac{x}{\gamma^2}), \hfill (A.5)$$

where $a = -\frac{l}{2} + \frac{\lambda + \beta q + \omega L - \beta p}{2\gamma}$, $c = 1 + \frac{\lambda + \beta q}{\gamma}$. \hfill (A.6)

For large $z$ we have

$$F(a, a + m; c; z) \approx \frac{\Gamma(c)(-z)^{-a-m}}{\Gamma(a+m)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_{n+m}(1-c+a)_{m+n}}{n!(m+n)!} z^{-n} (\log(-z) + h_n)$$

$$+ \frac{\Gamma(c)(-z)^{-a}}{\Gamma(a+m)} \sum_{n=0}^{m-1} \frac{(a)_n \Gamma(m-n)}{n! \Gamma(c-a-n)} z^{-n} \hfill (A.7)$$

We will not need an explicit form of the coefficients $h_n$, they can be found for example in [27]. We then find that normalizability at infinity requires that the finite sum disappears from the last expression, which happens if either one of the conditions:

$$c - a = -k \quad \text{or} \quad a = -l - k - 1 \quad \text{where} \quad k = 0, 1, 2, \ldots$$ \hfill (A.8)
is satisfied. This gives

\[
\frac{\omega L - \beta p}{2\gamma} = \pm \left\{ \frac{l}{2} + 1 + \frac{\lambda + \beta q}{2\gamma} + k \right\}, \quad (A.9)
\]

We thus get the frequencies:

\[
\omega_k = \frac{\beta p}{L} \pm \left\{ \frac{2\gamma}{L}(k + 1 + \frac{l}{2}) + \frac{\lambda + \beta q}{L} \right\}, \quad k = 0, 1, 2, \ldots \quad (A.10)
\]

For the discrete set of frequencies with positive sign in (A.10) we can rewrite the solutions of the wave equation in more convenient form:

\[
H(x) = x^{c-1}(x + \gamma^2)^{k+l/2+1} F(c + k, c + k + l + 1; c; -\frac{x}{\gamma^2})
\]

\[
= x^{c-1}(x + \gamma^2)^{-c-k-l/2}(\gamma^2)^{c+2k+l+1} F(-k, -(k + l + 1); c; -\frac{x}{\gamma^2}) \quad (A.11)
\]

The hypergeometric function in the above expression becomes a finite polynomial of degree \(k\) and \(H(x) \sim x^{-1-l/2}\) as \(x \to \infty\). For the frequencies with negative sign in (A.10) the hypergeometric function in (A.3) is also a finite polynomial.

If \(\lambda + \beta q \leq 0\), then instead of (A.3) we get:

\[
H(x) = x^{-(\lambda + \beta q)/2\gamma}(x + \gamma^2)^{(\omega L - \beta p)/2\gamma} F(a', a' + l + 1; c'; -\frac{x}{\gamma^2}), \quad (A.12)
\]

\[
a' = -\frac{l}{2} - \frac{\lambda + \beta q}{2\gamma} + \frac{\omega L - \beta p}{2\gamma}, \quad c' = 1 - \frac{\lambda + \beta q}{\gamma}. \quad (A.13)
\]

and we get the frequencies

\[
\omega_k = \frac{\beta p}{L} \pm \left\{ \frac{2\gamma}{L}(k + 1 + \frac{l}{2}) - \frac{\lambda + \beta q}{L} \right\} \quad k = 0, 1, 2, \ldots \quad (A.14)
\]

Combining this with (A.10), we can write the spectrum for a general case as

\[
\omega_k = \frac{\beta p}{L} \pm \left\{ \frac{2\gamma}{L}(k + 1 + \frac{l}{2}) + \left| \frac{\lambda + \beta q}{L} \right| \right\} \quad k = 0, 1, 2, \ldots \quad (A.15)
\]

B Chiral primaries in the orbifold CFT

We briefly summarize the nature of chiral primaries in the orbifold CFT and introduce the notation that we use in this paper. Details can be found in [23, 24].

We consider the bound states of \(n_1\) D1 branes and \(n_5\) D5 branes in IIB string theory. We set

\[
N = n_1 n_5 \quad (B.1)
\]
The D5 branes are wrapped on a 4-manifold $M$, and thus appear as effective strings in the remaining 6 spacetime dimensions. $M$ can be $T^4$ or K3. The D1 branes and the effective strings from the D5 branes extend along a common spatial direction $x_5 \equiv y$, and $y$ is compactified on a circle of length $2\pi R$. The low energy dynamics of this system is a $\text{N}=\text{(4,4)}$ supersymmetric 1+1 dimensional conformal field theory (CFT). The CFT has an internal R-symmetry $SU(2)_L \times SU(2)_R \approx SO(4)$. This symmetry arises from the rotational symmetry of the brane configuration in the noncompact spatial directions $x_1, x_2, x_3, x_4$. The group $SU(2)_L$ is carried by the left movers in the CFT and the group $SU(2)_R$ is carried by the right movers.

Consider this CFT at the ‘orbifold point’ [2]. Then the CFT is a 1+1 dimensional sigma model where the target space is the orbifold $M^N/S_N$, the symmetric product of $N$ copies of the 4-manifold $M$.

The $M^N/S_N$ orbifold CFT and its states can be understood in the following way. We take $N$ copies of the supersymmetric $c=6$ CFT which arises from the sigma model with target space $M$. The vacuum of the theory is just the product of the vacuum in each copy of the CFT. In the orbifold theory we find twist operators $\sigma_n$ [21, 22]. The copies $1, 2, \ldots n$ of the CFT permute cyclically into each other $1 \rightarrow 2 \rightarrow \ldots \rightarrow n \rightarrow 1$ as we circle the point of insertion of $\sigma_n$. (The other copies are not touched, and we ignore them for the moment.) In this given twist sector there are operators with various values of $j_3$.

For odd $n$ we start with $\sigma_n$, which is just a twist operator that permutes the copies of $M$ around its insertion point. The dimension of $\sigma_n$ is

$$h_n = \bar{h}_n = \frac{c}{24}(n - \frac{1}{n}) = \frac{6}{24}(n - \frac{1}{n}) = \frac{1}{4}(n - \frac{1}{n})$$

To raise the charge of the operator with minimum increase in dimension consider the application of $J^+_n$. The charge goes up by one unit, while the dimension increases by only $\frac{1}{n}$. The next cheapest operator is $J^+_\frac{n}{2}$, and continuing to apply the cheapest possible operator at each stage we construct the chiral operator in this twist sector with lowest dimension and charge. We will call it $\sigma_n^{-}$ [24]:

$$\sigma_n^- \equiv J^+_\frac{n}{n-2} J^+_\frac{n-4}{n} \ldots J^+_n \bar{J}^+_\frac{n-2}{n} \bar{J}^+_\frac{n-4}{n} \ldots \bar{J}^+_1 \sigma_n$$

and it has

$$h = j_3 = \frac{n - 1}{2}, \quad \bar{h} = \bar{j}_3 = \frac{n - 1}{2}$$

In the case of even $n$, the lightest operator $\sigma_n$ in a given twisted sector has dimension $h_n = \bar{h}_n = \frac{n}{2}$ (see [24] for details), and the chiral primary with lowest dimension is

$$\sigma_n^- \equiv J^+_\frac{n}{n-2} J^+_\frac{n-4}{n} \ldots J^+_0 \bar{J}^+_\frac{n-2}{n} \bar{J}^+_\frac{n-4}{n} \ldots \bar{J}^+_0 \sigma_n$$

Its dimension and charge are given by (B.4).
Each copy of the CFT has the SU(2) currents $J^{(i)a}$, $\bar{J}^{(i)a}$, where the index $i$ labels the copies. Define

$$J^a = \sum_{i=1}^{n} J^{(i)a}, \quad \bar{J}^a = \sum_{i=1}^{n} \bar{J}^{(i)a} \quad \text{(B.6)}$$

Then we can make three additional chiral primaries from $\sigma^{--}$:

$$\sigma^{+ -}_n = J^+_n \sigma^{--}_n, \quad h = j_3 = \frac{n + 1}{2}, \quad \bar{h} = \bar{j}_3 = \frac{n - 1}{2}$$

$$\sigma^{- +}_n = \bar{J}^+_n \sigma^{--}_n, \quad h = j_3 = \frac{n - 1}{2}, \quad \bar{h} = \bar{j}_3 = \frac{n + 1}{2}$$

$$\sigma^{++}_n = J^+_1 \bar{J}^+_n \sigma^{--}_n, \quad h = j_3 = \frac{n + 1}{2}, \quad \bar{h} = \bar{j}_3 = \frac{n + 1}{2} \quad \text{(B.7)}$$

The chiral primaries $\sigma^{--}_n, \sigma^{+ -}_n, \sigma^{- +}_n, \sigma^{++}_n$ correspond respectively to the (0, 0), (2, 0), (0, 2), (2, 2) forms from the cohomology of $M$. Both $T^4$ and K3 have one form of each of these degrees.

The operator $\sigma^{--}_1$ is just the identity operator in one copy of the $c = 6$ CFT. Thus for the complete CFT made from $N$ copies we can write the above chiral operators as

$$\sigma^\pm \pm [\sigma^{--}_1]^{N-n} \quad \text{(B.8)}$$

It is understood here that we must symmetrize the above expression among all permutations of the $N$ copies of the CFT; we will not explicitly mention this symmetrization in what follows.

More generally we can make the chiral operators

$$\prod_{i=1}^{k} [\sigma^{s_i, \bar{s}_i}_{n_i}]^{m_i}, \quad \sum_{i=1}^{k} n_im_i = N \quad \text{(B.9)}$$

where $s_i, \bar{s}_i$ can be $+, -$. This gives the complete set of chiral primaries that result if we restrict ourselves to the above mentioned cohomology of $M$.

C Some computations in the CFT

In [4] strings in $AdS_5 \times S^5$ were argued to be dual to to traces like (5.1). We have argued that strings in $AdS_3 \times S^3 \times T^4$ are described by similar ‘chains’ of $\sigma^{--}_2$ operators with sparsely placed ‘defects’ (eqn. (5.14)); these defects correspond to vibrating the string in the 2 transverse sphere directions and 2 transverse $AdS$ directions.

For this identification to be valid we must ensure that there are no other ‘defects’ that can be included in the chain, since every type of defect corresponds to a string oscillator in the gravity theory. Note that we allow only those operators to appear as defects that have a regular OPE with the basic member of the chain $\sigma^{--}_2$. In computing this OPE we have to be careful that only operators with $h - \bar{h}$ integral are allowed operators in
the CFT, so we discard apparently singular terms in the OPE which result in operators with fractional \( h - \bar{h} \).

We have allowed \( J_0^-, G_{-\frac{1}{2}}^-, L_{-1} \) (and their right moving analogs) to act on \( \sigma_{2^-}^- \) to produce the allowed defects. We start by noting that higher modes of these operators produce singular OPEs with \( \sigma_{2^-}^- \) and so do not give allowed defects:

\[
[J_{-n}^- \sigma_{2^-}^-](z) \sigma_{2^-}^- (w) = -\frac{1}{(w-z)^n} \sigma_{2^-}^-(z) [J_0^- \sigma_{2^-}^-](w) + \ldots, \quad n > 0, \quad (C.1)
\]

\[
[G_{-(\frac{1}{2}+n)}^- \sigma_{2^-}^-](z) \sigma_{2^-}^- (w) = -\frac{1}{(w-z)^n} \sigma_{2^-}^-(z) [G_{-\frac{1}{2}}^- \sigma_{2^-}^-](w) + \ldots, \quad n > 0, \quad (C.2)
\]

\[
[L_{-(n+1)}^- \sigma_{2^-}^-](z) \sigma_{2^-}^- (w) = -\frac{1}{(w-z)^n} \partial_w \sigma_{3^-}^- (w) + \ldots, \quad n > 0 \quad (C.3)
\]

Note that these operators \( J_{-n}^-, G_{-(\frac{1}{2}+n)}^-, L_{-(n+1)}^- \) (with \( n > 0 \)) all have \( \Delta - J > 1 \).

The operators that give allowed defects have \( \Delta - J = 1 \). We also have some other operators with \( \Delta - J = 1 \) but which do not give regular OPEs with \( \sigma_{2^-}^- \) and which therefore do not give new allowed defects. An example is \( J_{-1}^3^3 \):

\[
[J_{-1}^3 \sigma_{2^-}^-](z) \sigma_{2^-}^- (w) \sim \frac{1}{z-w} \sigma_{3^-}^- (w) + \ldots \quad (C.4)
\]

Next we consider the issue of zero modes. In the string theory the chiral primary is represented by a particular member of the multiplet of massless particles. By application of fermionic zero modes \( d_0^+ \) to the string we can change this particle to one of the other massless particles. To see these zero modes in the dual CFT take the case \( M_4 = T^4 \) for concreteness. There are 4 fermionic modes in the \( c = 6 \) CFT with \( \Delta = J = 0: \psi^+_{-\frac{1}{2}}, \bar{\psi}^+_{-\frac{1}{2}}, (\psi^-_{-\frac{1}{2}})^\dagger, (\bar{\psi}^-_{-\frac{1}{2}})^\dagger \) and we can insert these in a chain of \( \sigma_{2^-}^- \) operators. But

\[
(\sigma_{2^-}^-)^k [\psi^+_{-\frac{1}{2}} \sigma_{2^-}^-] (\sigma_{2^-}^-)^J \sim \psi^+_{-\frac{1}{2}} \sigma_{J+1}^- \quad (C.5)
\]

So these operators can be applied to the chain only in the zeroth Fourier harmonic: they do not give additional types of local defects but only give a global change to the entire operator. Thus the string zero modes map to the CFT zero modes (we write only the bosonic operators)

\[
|J\rangle \rightarrow \sigma_{J^-}^-
\]

\[
\{d_0^+ d_0^+ |J\rangle\} \rightarrow \{\sigma_{J-1}^-, \sigma_{J-1}^+, \psi^+ \bar{\psi}^+_{-\frac{1}{2}} \sigma_{J-1}^-\}
\]

\[
d_0^{11}_0 d_0^{11}_0 |J\rangle \rightarrow \sigma_{J-2}^{++} \quad (C.6)
\]

Taking supersymmetry descendents of the above set gives the other massless particles of the gravity multiplet.
Finally we note that we also cannot get new ‘defects’ by applying the above zero mode operators to the allowed defects. Note that $\psi_+ (\psi_-)^\dagger = J_1$. As an example we can check that $J_0^- J_{-1}^+ \sigma_2^-$ has a singular OPE with $\sigma_2^-$

$$J_0^- J_{-1}^+ \sigma_2^-(z) \sigma_2^-(w) \sim \frac{1}{z-w} J_0^- \sigma_3^+(w) + \ldots \quad \text{(C.7)}$$

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