Stochastic ecological kinetics of regime shifts in a time-delayed lake eutrophication ecosystem

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Abstract. Lake eutrophication ecosystems tend to exhibit bistable kinetics with two preferential configurations of oligotrophic and eutrophic states, and perturbations in the environmental parameters can cause regime shifts between two alternative stable states, yielding catastrophic regime shifts. In this article, we study stochastic kinetics of regime shifts in a time-delayed lake eutrophication ecological system, where the environmental parameters are assumed to be disturbed by both input and loss external noises. The dynamical behavior of stochastic delay on the regime shifts between oligotrophic and eutrophic states is analyzed using numerical simulations. Time delays in loss and recycling processes are considered. It is shown from time series and probability distribution that noises and delays can induce regime shifts between two stable states. Theoretical analyses are used to support our numerical simulations. By using the mean first passage time (MFPT) technique, we also discuss the escape time problem between two stable states in time-delayed ecosystems. A detailed study of the MFPT depending on various noise characteristics is performed, and the main results are compared for different cases of time delays.

Key words: environmental noise; lake eutrophication ecosystem; mean first passage time; probability distribution; time delays.

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INTRODUCTION

As a fundamental behavior of an ecological system, bistability has been studied extensively through theoretical analyses and experiments (May 1977, Scheffer et al. 2009). Studies indicate that ecological systems occasionally undergo rapid regime shifts from one stable state to an alternative stable state with dramatically different properties, accompanied by degradation of ecosystem services and attendant economic losses (Carpenter et al. 2011). These regime shifts are fascinating from the point of view of understanding how basic ecology and feedback processes of these systems can lead to their abrupt dynamics and hysteresis. They are of critical importance in the context of management of ecological resources and in maintaining services that these ecosystems provide to humans, because a regime shift may be irreversible due to the effects of hysteresis. Most of the previous investigations focused on the ecological systems with deterministic models (May 1976). However, for the real ecosystems, external conditions are never constant, and stochastic events such as weather extremes, fires, or pest outbreaks can cause fluctuations in the conditioning factors, but often affect the regime shifts from one stable state.
to an alternative stable state directly (Scheffer et al. 2001). Such abrupt transitions that occur without large external shocks but only with gradual changes in external conditions have been termed catastrophic regime shifts (Scheffer and Carpenter 2003). In the last decade, environmental perturbation has been considered as an important factor that should be included in the modeling of ecological systems (Vilar and Solé 1998, Gallagher and Appenzeller 1999, Zimmer 1999, Bjornstad and Grenfell 2001, Spagnolo and La Barbera 2002, Beisner et al. 2003, Carpenter et al. 2008, Guttal and Jayaprakash 2008) and biological systems (Spagnolo et al. 2004b, Chichigina et al. 2005, Valenti et al. 2006, Fiasconaro et al. 2008). Holling (1973) has emphasized the importance of noises in ecological dynamics and resilience. Investigation of a vegetation model showed that noises could actually stabilize the system through a statistically stable intermediate state between two alternative stable states (D’Odorico et al. 2005). The role of environmental noise on phytoplankton dynamics has been discussed in Denaro et al. (2013) and Valenti et al. (2015), and the noise-enhanced stability (NES) of the metastable states has also been studied (Agudov and Spagnolo 2001, Dubkov et al. 2004, Valenti et al. 2004, Fiasconaro and Spagnolo 2009).

It is generally accepted that lake eutrophication ecosystems tend to exhibit two preferential states, corresponding to oligotrophic and eutrophic lakes (Carpenter et al. 1999, Scheffer et al. 2000). The existence of these two alternative stable states is usually associated with recycling of phosphorus between sediment and water (Carpenter et al. 1999). The results of van Nes et al. (2003) showed that under certain conditions, the ecosystem could appear highly uncertain in the parameter values due to the existence of alternative attractors. A study based on noisy environmental variables for a lake eutrophication model indicates that rising variance of the dynamical variable could indicate impending catastrophic regime shifts in advance (Carpenter and Brock 2006). Guttal and Jayaprakash (2007) considered the impact of random fluctuations in environmental parameter on the bistable ecosystems for lake eutrophication. As predicted by their theoretical model, frequent regime shifts could be a more likely scenario in real ecosystems. Recently, Wang et al. (2012) investigated phosphorus dynamics in a lake approaching eutrophication under a regime of strong external perturbations. They showed that the flickering would give early warning signals of a critical transition to a eutrophic lake state (Dakos et al. 2013).

All of the above works on lake eutrophication ecosystems are only for a single noise case, but real ecosystems are always disturbed by the internal and external random noises simultaneously, and the two noises may be independent of each other. However, in certain situations (Wu et al. 1994, Madureira et al. 1996, Jia and Li 1997, Tessone et al. 2000, Zeng et al. 2012, 2015), both noises may have a common origin and thus not be independent. On the level of a Langevin-type description of a dynamical system, the action of the cross-correlation between two noises can change the dynamics of the system. The microscopic realization of cross-correlated noise processes has been investigated in Shit et al. (2010) and shown in a wide range of applications, such as in the cases of biological systems (Ai et al. 2003, Liu and Jia 2004, Zhang et al. 2011, Ghosh et al. 2012), optical systems (Zhang et al. 2008, Wei et al. 2013), laser systems (Zhu 1993, Jia et al. 1995), and linear systems (Berdichevsky and Gitterman 1999, Bag et al. 2001). Therefore, although we have not found an ecological rationale for the cross-correlation between the input noise (i.e., the fluctuation of the rate of phosphorus input) and the loss noise (i.e., the fluctuation of the rate of phosphorus loss), we were interested in determining what effects would result from such a correlation between the two noises in a lake eutrophication ecosystem. In addition, previous work on lake eutrophication dynamics neglected the possible effects induced by time delays. Time-delayed ecological models offer an interesting class of models both for incorporating more ecological realism and for mathematical sophistication. From an ecological point of view, time-delayed models capture ecological feedbacks that do not always occur instantaneously. For example, a high population density in the current year will leave exploited foraging grounds that may take years to recover, thus affecting populations several years into the future. In other words, the density-dependent effects are not instantaneous; rather, they have a
time lag (D’Odorico et al. 2013, Gutta et al. 2013). In the framework of differential equations, such processes can be modeled as a time delay (Yang et al. 2014a, Zeng et al. 2014). Delay and bistability have been extensively studied (Masoller 2001, Bratsun et al. 2005, Gupta et al. 2013), but the impacts of time delays on bistable lake eutrophication ecosystems are not clear. In general, the interaction between delay and stochasticity is complex (Tsimring and Pikovsky 2001, Masoller 2003, Tian et al. 2009). The effect of time-delayed feedback on the bistable system from the point of view of regime shift and kinetics has been studied recently (Mondal et al. 2016). Meanwhile, it appears that delay can significantly affect the stochastic properties of the system and play an important role in the dynamics, such as in the cases of stochastic delay-accelerated signaling in genetic pathways (Josić et al. 2011), catastrophic regime shifts in vegetation ecosystems (Zeng and Wang 2012, Zeng et al. 2013), delayed feedback as a mean of control of noise-induced motion (Janson et al. 2004), and noise- and delay-enhanced stability in a chemical reaction system (Yang et al. 2014b). In order to understand further the mechanisms of the regime shifts between oligotrophic and eutrophic lakes in natural ecosystems, we use nonequilibrium statistical physics to study the impact of the combination of noises and different time delays on lake eutrophication ecosystems. It is shown that noises and time delays can be used as control parameters of the regime shifts between the eutrophic and oligotrophic lakes.

This article is organized as follows. In the section Basic Model, the model with noise sources and time delays is presented. The time series and probability distribution of regime shifts between two states are analyzed, and the mechanisms for fluctuation- and delay-induced regime shifts are explained from the point of view of the effective potential. In the section The MFPT, the effects of noises and time delays on mean first passage time (MFPT) are discussed by numerical simulations. Finally, we end with results and discussion in the section Conclusion.

Basic Model

It is well known that lake eutrophication ecosystems are able to exhibit two alternative stable states (Carpenter et al. 1999, Scheffer et al. 2000) of oligotrophy or eutrophy for the phosphorus cycle. Let \( p(t) \) denote the concentration of phosphorus in the water column, and \( p \geq 0 \) since the concentration of phosphorus cannot be negative. In general, the macroscopic rate equation for \( p(t) \) can be expressed as

\[
\frac{dp}{dt} = c - sp + rg(p). \tag{1}
\]

For lakes, the parameter \( c \) denotes the rate of phosphorus input from the watershed and \( s \) denotes the rate of phosphorus loss, such as sedimentation, outflow, and sequestration in biomass of consumers or benthic plants. The parameter \( r \) is the maximum rate of phosphorus recycling, and phosphorus can be recycled from sediments or by consumers. For \( r = 0 \), the model has a single equilibrium at \( c/s \). The last term, however, can cause alternative stable states. For example, \( g(p) \) is a function that increases steeply at a threshold \( m \), as in the case of the Hill function:

\[
g(p) = \frac{p^m}{m^m + p^m}, \tag{2}
\]

where the parameter \( m \) denotes the phosphorus value at which the half-maximum rate of recycling occurs, and the exponent \( q \) determines the steepness of the switch occurring around \( m \). Notice that Eq. 1 can have multiple stable states only if the maximum \( \{rg'(p)\} > s \). Thus, a steeper Hill function (resulting from higher \( q \) values) creates stronger hysteresis. One important property of Eq. 1 is its bistability, with three fixed points, denoted by \( p_{s1}, p_{u1} \) and \( p_{s2} \) with \( p_{s1} < p_u < p_{s2} \). For the stability of \( p_{s1}, p_{s2} \) and \( p_{u} \), it is easy to see that both \( p_{s1} \) and \( p_{s2} \) are locally asymptotically stable and \( p_u \) is unstable. In ecology, \( p_{s1} \) corresponds to a low-nutrient regime that corresponds to the oligotrophic state, and \( p_{s2} \) corresponds to a high nutrient (phosphorus) concentration that is identified with the eutrophic state. Here, we are more interested in how the dynamic properties of Eq. 1 are affected by the maximum rate \( r \) of phosphorus recycling. The relationship between bistability and \( r \) is plotted in Fig. 1 (i.e., \( p \) vs. \( c \) or \( s \) for \( dp/dt = 0 \)), where, following Carpenter et al. (1999), the parameters are taken as \( m = 1.0 \) and \( q = 8.0 \) (see also Wang et al. 2012; in this study, we keep these parameters fixed). Clearly, for small \( r \), bistability does not exist. As the value of
increases, bistability occurs, and the range of values of the parameter over which bistability occurs becomes higher with increasing \( r \). Therefore, recycling of phosphorus can lead to ecological bistability, and this increases in the lake eutrophication ecosystem. Several observations indicated that the transition in Erhai Lake in ca. 2001 corresponds to the classic development of a bistable system, that is, the diatom state responses to varying concentrations of total dissolved phosphorus, which shows evidence of alternative states and hysteresis (i.e., bistability; Wang et al. 2012).

Analyzing dynamical behaviors is an important task in both theoretical studies and practical applications on lake ecosystems. More sophisticated theoretical models and general techniques are needed to examine dynamical behaviors of lake ecosystems, taking into consideration the loss or recycling process of phosphorus in water. Both the loss (due to sedimentation) and the recycling represent complex processes, that is, depending on whether the lake is anoxic, on the biota, or on iron compounds present. In other words, the loss or recycling process of phosphorus is not instantaneous, rather, they have a time lag. In the framework of differential equations, such processes can be modeled as a time delay. For simplicity, we chose to include possible time delays in the loss process (Fig. 2a), that is, the loss term \(-sp(t - \tau)\); time delays (\( \beta \)) in the recycling process (Fig. 2b), that is, the recycling term \( \frac{rp(t - \beta)}{m + p(t - \beta)} \) and global delays (\( \theta \)) appearing in both loss and recycling processes, that is, the global term

\[
\left( c - sp(t - \theta) + \frac{rp(t - \theta)}{m + p(t - \theta)} \right).
\]

In addition, changes in variance due to impending regime shifts may be difficult to distinguish from other drivers of variance such as exogenous noise that affects the ecosystem (Scheffer et al. 2001, Ditlevsen and Johnsen 2010). To describe the lake eutrophication ecological model more accurately, we must take into consideration an environmental perturbation influence. Environmental perturbation has long been considered as an important factor that should be included in the modeling of ecological systems. One is inherent uncertainty in an ecological system, and the other is considerable anthropogenic disturbances that exacerbate the

![Fig. 1. Bifurcation diagram for the amount \( p \) of phosphorus as a function of \( c \) and \( s \) for different maximum rates \( r \) of recycling of phosphorus, respectively. (a) \( s = 1.0 \) and (b) \( c = 0.5 \).](image)

![Fig. 2. Phosphorus flows in the model.](image)
uncertainty in the way an ecosystem responds (Folke et al. 2004). In lakes, Carpenter and Brock (2006) have pointed out that variance due to fluctuations in recycling may be difficult to discriminate from variance due to noise nutrient input. On the other hand, they also pointed out that the true model for ecosystem dynamics was unknown and inferences about changing variance must be drawn from approximate models and that environmental parameters (i.e., the rate of phosphorus loss or input) are likely to be affected by internal or external fluctuations. This implies that phosphorus input (or loss) can also be considered as a random variable. In other words, the environmental parameters (c and s) from Eq. 1 are not constants, but fluctuate in time.

In our model, as shown in Fig. 2, the input noise (additive noise), denoted by \( \xi(t) \) (\( j = 1, 2, \) and \( 3 \)) represent the three cases of time delays, respectively, alters the nutrient input; that is, the \( c \) in Eq. 1 is replaced by \( c = c + \xi(t) \), where \( \xi(t) \) is defined as a Gaussian white noise with \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t)\xi(t') \rangle = 2\alpha_j\delta(t-t') \), and \( \alpha_j \) denotes the input noise intensity for three cases of time delays, respectively. The loss noise (multiplicative noise) alters the rate of phosphorus loss (sedimentation, etc.); that is, the \( s \) in Eq. 1 is replaced by \( s = s + \eta(t) \), where \( \eta(t) \) is also a Gaussian white noise with \( \langle \eta(t) \rangle = 0 \) and \( \langle \eta(t)\eta(t') \rangle = 2\beta_j\delta(t-t') \), and \( \beta_j \) denotes the loss noise intensity for three cases of time delays. Guttal and Jayaprakash (2007) and Wang et al. (2012) only considered that the input noise (or loss noise) altered the rate of phosphorus input (or loss). Here, we consider simultaneously the impacts of input and loss noises. A natural question is whether the two noises are statistically correlated. However, we could imagine the correlation arising from the recycling of phosphorus (or physically it would mean that the origin of the two noise processes is the same); that is, the nutrient input (affected by noise) is ecologically coupled to the nutrient loss (also affected by noise). The correlation form between the two noises is assumed to be as follows (Wu et al. 1994, Madureira et al. 1996, Jia and Li 1997):

\[
\langle \xi_j(t)\eta(t') \rangle = \langle \eta_j(t)\xi(t') \rangle = 2\lambda_j\sqrt{D_j}\alpha_j\delta(t-t'),
\]

\( j = 1, 2, \) and \( 3 \) (3)

where \( \lambda_j \) is the cross-correlation intensity between \( \xi_j(t) \) and \( \eta_j(t) \) for three cases of time delays; that is, \( \lambda_1 \) denotes the cross-correlation intensity between \( \xi_1(t) \) and \( \eta_1(t) \) under a time delay in the loss process, \( \lambda_2 \) denotes the cross-correlation intensity between \( \xi_2(t) \) and \( \eta_2(t) \) under a time delay in the recycling process, and \( \lambda_3 \) denotes the cross-correlation intensity between \( \xi_3(t) \) and \( \eta_3(t) \) under a time delay in both loss and recycling processes. For the cross-correlation between \( \xi(t) \) and \( \eta(t) \), a positive correlation (\( \lambda_j > 0.0 \)) implies that the probability for input noise \( \xi(t) \) is greater than \( \eta(t) \), and a negative correlation (\( \lambda_j < 0.0 \)) implies that the probability for input noise \( \xi(t) \) is smaller than \( \eta(t) \) (Goodman 1985). Since both \( \xi(t) \) and \( \eta(t) \) are Gaussian white noises (e.g., two noises fluctuate on the same timescale), a positive correlation (or negative correlation) indicates that two noises enhance the nutrient input (or loss) of each other statistically. In a previous model developed by Guttal and Jayaprakash (2007), we also noticed that the effect of the loss noise (or input noise) on the ecological systems is ignored. In fact, for the cross-correlation between \( \xi(t) \) and \( \eta(t) \), we have no experimental evidence that indicates that the parameter \( \lambda_j \) should be positive or negative. Therefore, for the effect of \( \lambda_j \) we will only provide some theoretical and numerical considerations.

Next, we investigated the effects of environmental perturbations in this lake ecosystem for three cases of time delays, namely time delay (\( \tau \)) in the loss process, time delay (\( \beta \)) in the recycling process, and global delay (\( \theta \)) appearing in both loss and recycling processes.

**Time delay in the loss process**

First, we consider a local time delay in the phosphorus loss process. Such a time delay \( \tau \) in the loss process can affect the concentration \( p(t) \) of phosphorus. Therefore, the Langevin equation corresponding to Eq. 1 with Eq. 2 is given by

\[
\frac{dp}{dt} = c - sp_t + \frac{np_t}{mt} - p_t \eta_1(t) + \xi_1(t),
\]

\( \tau \) and \( \beta \) appearing in both loss and recycling processes.

where \( \eta_1(t) \) and \( \xi_1(t) \) are the Gaussian white noises with zero mean, and the correlation form between the two noises can satisfy the statistical property of Eq. 3, namely

\[
\langle \xi_j(t)\eta(t') \rangle = \langle \eta_j(t)\xi(t') \rangle = 2\lambda_j\sqrt{D_j}\alpha_j\delta(t-t'),
\]

\( j = 1, 2, \) and \( 3 \)
\begin{align}
\langle \xi_1(t) \eta_1(t') \rangle &= \langle \eta_1(t) \xi_1(t') \rangle \\
&= 2\lambda_1 \sqrt{D_1 \alpha_1 \delta(t-t')}
\end{align}

(5)

where \(\alpha_1\) and \(D_1\) describe the intensity of white noise \(\xi_1(t)\) (input) and \(\eta_1(t)\) (loss), respectively, and \(\lambda_1\) denotes the intensity of cross-correlation between \(\xi_1(t)\) and \(\eta_1(t)\). The \(p_\tau\) (time delay) is previous to the time when \(dp/dt\) is computed. Since \(p_\tau \eta_1(t)\) is linearly dependent on the phosphorus concentration, for simplicity we call this form a linear time delay. From Eqs. 4 and 5, for fixed loss noise intensity, by increasing the intensity of the input noise, it is clear that the noise can give rise to negative fluctuations of the population density at \(t = 0\) or in some time instance if the input noise is strong enough (also see Mikhailov and Loskutov 1991). Thus, here we assume that the intensity of the input (additive) noise is much lower (\(\alpha_j \leq 0.25, j = 1, 2, \text{and} 3\)) by comparison with the intensity of the loss (multiplicative) noise.

The numerical simulations are performed by integrating Langevin Eq. 4 with Eq. 5 directly. The Box-Muller algorithm is used to generate Gaussian noise (Ramirez-Piscina et al. 1993). The numerical data of time series are obtained by using the modified Euler procedure with a time step of \(h = 0.01\), and the data of probability distribution are saved over 500 different trajectories. We performed 100,000 simulations to determine this limit of stability. The impacts of the noise intensities \(\alpha_1\) and \(D_1\), the cross-correlation intensity \(\lambda_1\), and linear time delay \(\tau\) on the time series and probability distribution of the

\[\text{Fig. 3. The time series (top) and probability distribution (bottom) of } p(t) \text{ for different input noise intensities } \alpha_1.\]

The other parameter values are \(m = 1.0, q = 8.0, \gamma = 1.0, c = 0.5, s = 1.0, D_1 = 0.1, \lambda_1 = 0.5, \text{and} \tau = 0.5\).
nutrient concentration are shown in Figs. 3–6, respectively.

In Figs. 3 and 4, the time series of the phosphorus concentration \( p(t) \) and the probability distribution \( Q(p) \) are plotted by simulating the Langevin Eq. 4 with Eq. 5 directly for different noise intensities \( \alpha_1 \) of the rate of phosphorus input and for different noise intensities \( D_1 \) of the rate of phosphorus loss, respectively. For the time series (top panels), it is shown in Fig. 3 that the phosphorus concentration \( p(t) \) concentrates on the oligotrophic state when the intensity of the input noise \( \alpha_1 \) is small (see \( \alpha_1 = 0.01 \), which corresponds to the low nutrient concentration by tuning the \( \alpha_1 \) to a very low value. However, the increase in \( \alpha_1 \) causes the high-nutrient-concentration state (eutrophic state) to become populated, corresponding to an increase in the phosphorus concentration. This result indicates that regime shifts from the oligotrophic state to the eutrophic state can be induced by the fluctuation of the rate of phosphorus input (i.e., input noise) in the lake ecological system for a time delay in the loss process. In contrast, it should be pointed out from Fig. 4 that the fluctuation of the rate of phosphorus loss (i.e., loss noise) can cause regime shifts from the eutrophic state to the oligotrophic state. It is shown in Fig. 3 (bottom panels) that at smaller \( \alpha_1 \) (see \( \alpha_1 = 0.01 \)), the probability distribution as a function of \( p \) shows a single peak at small \( p \) value, which corresponds to the oligotrophic state (low nutrient). As the values of \( \alpha_1 \) increase (see \( \alpha_1 = 0.25 \)), the peak for the small value of \( p \) decreases. Simultaneously, for large value of \( p \), a new peak appears, which
corresponds to the eutrophic state (high nutrient). Fig. 4 illustrates what occurs when $\alpha_1$ is fixed and $D_1$ is changed. For small $D_1$ (see $D_1 = 0.1$), the probability distribution $Q(p)$ as a function of $p$ exhibits two peaks. As the value of $D_1$ increases (see $D_1 = 0.9$), the peak for the large value of $p$ disappears and the small value of $p$ increases. Thus, the fluctuation of the rate of phosphorus input can induce regime shifts from the oligotrophic state to the eutrophic state and vice versa. We draw a comparison between our results and previous investigations (Guttal and Jayaprakash 2007), which consider the effect of noise on a bistable eutrophication system. For large intensity of loss noise (Fig. 4), we have verified that the lake eutrophication model of Guttal and Jayaprakash (2007) shows a reduced and eliminated bistable region in agreement with our results. However, for large intensity of input noise (Fig. 3), ecological bistability in the ecosystem can be induced, and we believe that bistability is a more likely scenario in ecological systems.

In Figs. 5 and 6, the time series of the phosphorus concentration $p(t)$ and the probability distribution $Q(p)$ are plotted by simulating the Langevin Eq. 4 with Eq. 3 directly for different cross-correlation intensities $\lambda_1$ between two noises and for different linear time delays $\tau$. For a negative $\lambda_1$ value (see $\lambda_1 = -0.9$ in Fig. 5), the phosphorus concentration $p(t)$ concentrates on the oligotrophic state, which corresponds to the probability distribution $Q(p)$ showing a single peak at a smaller $p$ value (oligotrophic state). However, when the value of $\lambda_1$ is increased (see $\lambda_1 = 0.5$ in Fig. 5), a
second peak appears at a larger value of $p$ (eutrophic state). As the value of $\lambda_1$ continues to increase, the second peak becomes higher, and the first peak will disappear when $\lambda_1$ is larger (see $\lambda_1 = 0.9$ in Fig. 5), suggesting that the increase in $\lambda_1$ causes the high-nutrient-concentration state (eutrophic state) to become populated, and concentrates on eutrophic state thoroughly (see $\lambda_1 = 0.9$). On the other hand, it is shown in Fig. 5 that $Q(p)$ is changed from one peak to two peaks, and then to one peak when $\lambda_1$ is increased; that is, the cross-correlation between two noises can induce a succession of two transitions. This type of nonequilibrium transition has been called the reentrance phenomenon (Castro et al. 1995); it is induced by one colored noise source. However, here the reentrance phenomenon exists for the cross-correlation noise source. Fig. 6 shows that the height of two peaks is decreased when the value of $\tau$ is increased, and the position of the left peak is slightly shifted to a small value of $p$, while the position of the right peak is shifted to a large value of $p$. Namely, regime shifts from the oligotrophic state to the eutrophic state can be induced by the positive cross-correlation between two noises and linear time delay, while regime shifts from the eutrophic state to the oligotrophic state can be induced by the negative cross-correlation between two noises. Therefore, the cross-correlation intensity between two noises and linear time delay can be used as a control parameter of regime shifts in lake eutrophication ecosystems.

Fig. 6. The time series (top) and probability distribution (bottom) of $p(t)$ for different linear time delays of $\tau$. The other parameter values are $m = 1.0$, $q = 8.0$, $\gamma = 1.0$, $\epsilon = 0.5$, $s = 1.0$, $\alpha_1 = 0.25$, $D_1 = 0.1$, and $\lambda_1 = 0.5$. 
In the lake eutrophication model, we have observed from Figs. 3–5 that ecological bistability can be induced by noises and from Figs. 3–6 that the regime shifts between two states are for some values of the noises and time delay. Regime shifts between two alternative states (oligotrophic and eutrophic states) have been observed in various lakes. For example, Schelske and Brenzonik (1992) reported on the large, shallow Lake Apopka in Florida, USA, where a shallow lake in southwestern China (Gao et al. 2005), around 2001. Dianchi Lake is also a eutrophic lake in southwestern China (Gao et al. 2005), which implies the phosphorus fractions in sediment profiles and their potential contribution to eutrophication. With the introduction of fluctuations in the parameters of the model, we can relate these observed shifts between two regimes similar to those shown in Figs. 3 and 5, where the input noise (or cross-correlation between two noises) causes regime shifts from the oligotrophic state to the eutrophic state. While a serious comparison requires a much more detailed model, it is interesting to note how exogenous noise in simple models can produce the observed behavior. It is well known that eutrophication of lakes has become a global problem that is likely to intensify in the coming decades due to the increases in human population, demand for food, land conversion, fertilizer use, and nitrogen deposition. However, our results in Figs. 4 and 6 show that the loss noise and linear time delay can induce regime shifts from the eutrophic state to the oligotrophic state, that is, the loss noise- and linear time delay-induced oligotrophy.

Next, we provide a theoretical analysis for the probability distribution in a time-delayed lake eutrophication ecosystem with noises. Let \( Q(p, t) \) denote the probability density distribution that the phosphorus concentration exactly equals \( p \) at time \( t \). Using the small time delay approximation of the probability density approach (Guillouzic et al. 1999, 2000, Frank 2005), Eq. 4 can be rewritten as

\[
\frac{dp}{dt} = f_{\text{eff}}(p) + g_{\text{eff}}(p)\eta_1(t) + \xi_1(t),
\]

where

\[
f_{\text{eff}}(p) = \int_{-\infty}^{+\infty} \left( c + \frac{\gamma p^d}{m^d + p^d} - sp \right) \times P_1(p, t - \tau|p, t)dp, \quad (7)
\]

\[
g_{\text{eff}}(p) = \int_{-\infty}^{+\infty} (-p)P_2(p, t - \tau|p, t)dp. \quad (8)
\]

In Eqs. 7 and 8, \( P_1(p, t - \tau|p, t) \) and \( P_2(p, t - \tau|p, t) \) denote the conditional distributions of \( p(t) \) in the deterministic part and stochastic part, respectively, which are given by

\[
P_1(p, t - \tau|p, t) = \sqrt{\frac{1}{2\pi G(p)\tau}} \times \exp \left( -\frac{(p - p - f(p)\tau)^2}{2G(p)\tau} \right), \quad (9)
\]

\[
P_2(p, t - \tau|p, t) = \sqrt{\frac{1}{2\pi G(p)\tau}} \times \exp \left( -\frac{(p - p - g(p)\tau)^2}{2G(p)\tau} \right), \quad (10)
\]

where

\[
f(p) = c + \frac{\gamma p^d}{m^d + p^d} - sp
\]

and \( g(p) = -p \). Thus, the stochastic-delayed differential equation can be approximately reduced to the ordinary stochastic equation. Substituting Eqs. 9 and 10 into Eqs. 7 and 8, we obtain
where \(\Phi\) calculated for Gaussian noises.

The average which remains in Eq. 13 may be calculated for Gaussian noises \(\xi_i(t)\) and \(\eta_i(t)\) by a functional formula, using the Novikov theorem (Novikov 1964, 1965):

\[
\langle \Phi[\xi_1, \xi_2] \rangle = \int_0^t dt' \gamma_{kl} \frac{\delta \Phi[\xi_1, \xi_2]}{\delta \xi_{kl}}, \quad (k,l = 1,2)
\]

where \(\Phi[\xi_1, \xi_2]\) is a function of \(\xi_1\) and \(\xi_2\) and \(\gamma_{kl} = \langle \xi_k(t) \xi_l(t') \rangle\) are their correlation functions. Now, using the Novikov theorem to calculate the averages \(\langle \eta_1(t) \delta(p(t) - p) \rangle\) in Eq. 13, we have

\[
\langle \eta_1(t) \delta(p(t) - p) \rangle = \int_0^t dt' \gamma_{11}(t, t') \frac{\delta \delta(p(t) - p)}{\delta \eta_1(t')} + \int_0^t dt' \gamma_{12}(t, t') \frac{\delta \delta(p(t) - p)}{\delta \xi_1(t')}.
\]

(15)

Applying the Fox approach (Fox 1986a) and the Hänggi Ansatz (Hänggi et al. 1985) to Eq. 15, we obtain

\[
\langle \eta_1(t) \delta(p(t) - p) \rangle = -D_1 \frac{\partial}{\partial p} g_{\text{eff}}(p) Q(p, t) - \lambda_1 \sqrt{D_1 \alpha_1} \frac{\partial}{\partial p} g_{\text{eff}}(p) Q(p, t),
\]

(16)

Similarly, the average \(\langle \xi_1(t) \delta(p(t) - p) \rangle\) in Eq. 13 is given by

\[
\langle \xi_1(t) \delta(p(t) - p) \rangle = -\alpha_1 \frac{\partial}{\partial p} g_{\text{eff}}(p) Q(p, t) - \lambda_1 \sqrt{D_1 \alpha_1} \frac{\partial}{\partial p} g_{\text{eff}}(p) Q(p, t).
\]

(17)

Substituting Eqs. 16 and 17 into Eq. 13, we ultimately obtain the delayed Fokker–Planck equation corresponding to Eq. 6:

\[
\frac{\partial Q(p, t)}{\partial t} = -\frac{\partial}{\partial p} f_{\text{eff}}(p) Q(p, t)
+ D_1 \frac{\partial}{\partial p} g_{\text{eff}}(p) \frac{\partial}{\partial p} Q(p, t)
+ \lambda_1 \sqrt{D_1 \alpha_1} \frac{\partial}{\partial p} g_{\text{eff}}(p) \frac{\partial}{\partial p} Q(p, t)
+ \alpha_1 \frac{\partial^2}{\partial p^2} Q(p, t)
+ \lambda_1 \sqrt{D_1 \alpha_1} \frac{\partial^2}{\partial p^2} g_{\text{eff}}(p) Q(p, t).
\]

(18)

Simplifying the right-hand side of Eq. 18, we have

\[
\frac{\partial Q(p, t)}{\partial t} = -\frac{\partial}{\partial p} A(p) Q(p, t) + \frac{\partial^2}{\partial p^2} B(p) Q(p, t),
\]

(19)

where

\[
A(p) = f_{\text{eff}} + \frac{1}{2} B'(p),
\]

(20)

\[
B(p) = D_1 g_{\text{eff}}^2 + 2 \lambda_1 \sqrt{D_1 \alpha_1} g_{\text{eff}} + \alpha_1.
\]

(21)

Eqs. 19 and 21 may be written in the form of

\[
\frac{\partial Q(p, t)}{\partial t} + \frac{\partial S(p, t)}{\partial p} = 0,
\]

(22)

and

\[
S(p, t) = \left[ A(p) - \frac{\partial}{\partial p} B(p) \right] Q(p, t).
\]

(23)

Because Eq. 22 is a continuity equation for a probability distribution, \(S(p, t)\) has to be interpreted as a probability current. If this probability current vanishes at the boundaries \(p = p_{\text{min}}\) and \(p = p_{\text{max}}\), Eq. 22 then guarantees that the normalization is preserved

\[
\int_{p_{\text{min}}}^{p_{\text{max}}} Q(p, t) dp = \text{const.}
\]

(24)

For natural (reflecting or absorbing) boundary conditions \((p_{\text{min}} = 0, p_{\text{max}} = +\infty)\), \(Q(p, t)\) and \(S(p, t)\) also vanish at \(p = 0\) or \(p = +\infty\).

For a stationary process, the probability current must be constant. With natural boundary conditions, the probability current must be zero.
and from Eq. 22, we have $\partial Q(p, t)/\partial t = 0$; that is, the probability distribution function $Q(p, t)$ is independent of time $t$. From Eq. 23, the stationary probability distribution (SPD) under natural boundary conditions obtains

$$
\left[ A(p) - \frac{\partial}{\partial p} B(p) \right] Q_{st}(p) = 0. \quad (25)
$$

Solving Eq. 25, the SPD $Q_{st}(p)$ is given by

$$
Q_{st}(p) = N \exp[-U_{FP}(p)], \quad (26)
$$

where $N$ is a normalization constant, and

$$
U_{FP}(p) = \ln B(p) - \int^p \frac{A(p')}{B(p')} \, dp'. \quad (27)
$$

is called the effective potential of the Fokker–Planck equation.

The impacts of the noise intensities $\alpha_1$ and $D_1$, cross-correlation intensity $\lambda_1$ between two noises, and linear time delay $\tau$ on the SPD are shown in Fig. 7a–d through theoretical analysis. From Fig. 7a–d, we can clearly see that $Q_{st}(p)$ is changed from one peak to two peaks when $\alpha_1$ is increased (or from two peaks to one peak when $D_1$ is increased), and from one peak to two peaks, and then to one peak when $\lambda_1$ is increased, and two peaks are low and broad when $\tau$ is increased. Therefore for a time delay in the loss process, the input or loss noise can only induce a transition, while the cross-

Fig. 7. The stationary probability distribution of the amount $p$ of phosphorus for different input noise intensities $\alpha_1$, different loss noise intensities $D_1$, different cross-correlation intensities $\lambda_1$, and different time delays of $\tau$. The other parameter values are $m = 1.0$, $q = 8.0$, $\gamma = 1.0$, $c = 0.5$, and $s = 1.0$. (a) $D_1 = 0.1$, $\lambda_1 = 0.5$, and $\tau = 0.5$; (b) $\alpha_1 = 0.25$, $\lambda_1 = 0.5$, and $\tau = 0.5$; (c) $\alpha_1 = 0.25$, $D_1 = 0.1$, and $\tau = 0.5$; and (d) $\alpha_1 = 0.25$, $D_1 = 0.1$, and $\lambda_1 = 0.5$. 
A correlation between two noises can induce a succession of two transitions, that is, the reentrance phenomenon. This is consistent with the numerical simulations of Figs. 3–6; the theoretical analysis supports our numerical simulations. From the above analyses, it is found that the phenomena of noise- and delay-induced regime shifts between oligotrophic and eutrophic states are the physical examples of noise- and delay-induced transitions. In addition, it is also shown that not only the presence of the recycling of phosphorus can lead to ecological bistability, but also the presence of the noises can induce ecological bistability (Figs. 3–5).

We provide a graphical representation of some of the fluctuation-induced regime shifts for a time delay in the loss process. From the physics point of view, it is well known that generalized force determines the dynamics of the phosphorus concentration $p$, and then, force can be written as the (negative) derivative of a function that can be thought of as the effective potential of the model; that is, $F = -dU_F (p)/dp$. The minima and maxima of the potential correspond to stable points.

Fig. 8. The effective potential for the lake eutrophication model at three different input noise intensities $\alpha_1$. The other parameter values are $D_1 = 0.1$, $k_1 = 0.5$, $\tau = 0.5$, $m = 1.0$, $q = 8.0$, $\gamma = 1.0$, $c = 0.5$, and $s = 1.0$. 

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$p_{si}$ ($i = 1, 2$) and the unstable point $p_u$; that is, the SPD has two peaks corresponding to the two stable points $p_{s1}$ and $p_{s2}$, and one valley corresponding to the unstable point $p_u$. This is an instructive way to understand the dynamics of the ecosystem (Scheffer and Carpenter 2003). In physics, the minimum of $U_{FP}(p)$ corresponding to $p_{s1}$ (or $p_{s2}$) is called the potential well, and the maximum of $U_{FP}(p)$ corresponding to $p_u$ is called the potential barrier (Risken 1992). Thus, for convenience, we call the potential well corresponding to $p_{s1}$ (or $p_{s2}$) as the left (or right) well.

Fig. 9. The effective potential for the lake eutrophication model at three different cross-correlation intensities $\lambda_1$. The other parameter values are $D_1 = 0.1$, $a_1 = 0.25$, $\tau = 0.5$, $m = 1.0$, $q = 8.0$, $\gamma = 1.0$, $c = 0.5$, and $s = 1.0$.

Figs. 8–11 show the effective potential $U_{FP}(p)$ as a function of $p$ at three different noise levels ($\sigma_1$, $D_1$, and $\lambda_1$) and linear time delay $\tau$, respectively. In the deterministic model, the system state will end up in one of the potential minima depending on the initial conditions. Now we can understand what noises do to the system: The key quantity is the magnitude of the maximum slope of the potential (i.e., the maximum force) that acts on the system around either of the minima in the direction of the other minimum. Clearly, the maximum slope of the potential in
the left (or right) well decreases (or increases) with the increases in the input noise intensity \( \alpha_1 \), cross-correlation intensity \( \kappa_1 \), and linear time delay \( \tau \), shown in Figs. 8–10, respectively. However, Fig. 11 shows that the maximum slope of the potential in the left (or right) well increases (or decreases) with the increase in the loss noise intensity \( D_1 \). Accordingly, the increase in \( \alpha_1 \) or \( \kappa_1 \) (or \( \tau \)) makes the net force in the noisy problem large enough to kick it out of the left well, while the increase in \( D_1 \) makes the net force large enough to kick it out of the right well; that is, the increase in \( \alpha_1 \) or \( \kappa_1 \) (or \( \tau \)) causes regime shifts from the oligotrophic state to the eutrophic state, while the increase in \( D_1 \) causes regime shifts from the eutrophic state to the oligotrophic state.

The depth of a potential well can be related to its resilience. Here, we define the depth of the left well as \( d_\left( p_u \right) - U(p_u) \) and similarly the depth of the right well as \( d_\left( p_\mu \right) - U(p_\mu) \). Figs. 8–11 show that the depth of the potential well varies with the increase in noise parameters \( \alpha_1 \), \( \lambda_1 \), \( D_1 \), and \( \tau \). In Figs. 8–10, for smaller \( \alpha_1 \) or negative \( \lambda_1 \), the \( U(p) \) as a function of \( p \) exhibits only the left well, at which it corresponds to the oligotrophic state \( (p_{\mu}) \). As the values of \( \alpha_1 \) or \( \lambda_1 \)

![Graph](image-url)

**Fig. 10.** The effective potential for the lake eutrophication model at three different loss noise intensities \( D_1 \). The other parameter values are \( D_1 = 0.1, \alpha_1 = 0.25, \lambda_1 = 0.5, m = 1.0, q = 8.0, \gamma = 1.0, c = 0.5, \) and \( s = 1.0 \).
(or τ) increase, the depth of the left well decreases and the depth of the right well increases. As the value of λ1 continues to increase (Fig. 9), the depth of the left well disappears and the depth of the right well increases further. Namely, the system state should be more easily attracted by the right well (p1) with the increase in α1 or λ1 (or τ), and the input noise or positive cross-correlation noise (or linear time delay) can induce eutrophication of the lake ecological system. In other words, the system state should be more easily attracted by the left well (p3) with the increase in D1, and the loss noise can induce oligotrophy of the lake ecological system. In short, the mechanisms for fluctuation- and delay-induced regime shifts between oligotrophic and eutrophic states are explained from the point of view of the effective potential of the ecological model for a time delay in the loss process.

**Time delay in the recycling process**

Secondly, we consider a time delay in the recycling process, where β reflects the time taken for the recycling of phosphorus from sediments or by consumers. Therefore, the Langevin equation corresponding to Eq. 1 with Eq. 2 becomes
\[
\frac{dp}{dt} = c - sp + \frac{rp_p^q}{m^p + p_p^q} - p\eta_2(t) + \xi_2(t),
\]
(28)

in which \(\eta_2(t)\) and \(\xi_2(t)\) are the Gaussian white noises with zero mean, and the correlation form between the two noises can satisfy the statistical property of Eq. 3, namely

\[
\langle \xi_2(t)\eta_2(t') \rangle = \langle \eta_2(t)\xi_2(t') \rangle = 2\lambda_2 \sqrt{D_2} \alpha_2 \delta(t - t'),
\]
(29)

where \(\alpha_2\) and \(D_2\) describe the intensity of white noises \(\xi_2(t)\) (input) and \(\eta_2(t)\) (loss), respectively, and \(\lambda_2\) denotes the intensity of cross-correlation between \(\xi_2(t)\) and \(\eta_2(t)\). The \(p_p\) appears in a non-linear term. For time delay in recycling process, the impacts of the noise intensities \(\alpha_2\) and \(D_2\), the cross-correlation intensity \(\lambda_2\) and nonlinear time delay \(\beta\) on the time series and probability distribution of the nutrient concentration are shown in Figs. 12–15, respectively.

In Figs. 12 and 13, the time series of the phosphorus concentration \(p(t)\) and the probability distribution are plotted by simulating Langevin Eq. 28 directly with Eq. 29, for different noise intensities \(\alpha_2\) of the rate of phosphorus input and different noise intensities \(D_2\) of the rate of phosphorus loss, respectively. For the time series (top panels), it is shown in Fig. 12 that the phosphorus concentration \(p(t)\) concentrates on the oligotrophic state when the intensity of the input noise \(\alpha_2\) is small (see \(\alpha_2 = 0.01\)), which corresponds to the

![Fig. 12. The time series (top) and probability distribution (bottom) of \(p(t)\) for different input noise intensities \(\alpha_2\). The other parameter values are \(m = 1.0, q = 8.0, \gamma = 1.0, c = 0.5, s = 1.0, D_2 = 0.1, \lambda_2 = 0.5,\) and \(\beta = 0.5\).](image)
low nutrient concentration. However, the increase in \( \alpha_2 \) causes the high-nutrient-concentration state (eutrophic state) to become populated, corresponding to an increase in the phosphorus concentration. This result indicates that regime shifts from the oligotrophic state to the eutrophic state can be induced by the fluctuation of the rate of phosphorus input (i.e., input noise) in the lake ecological system for a time delay in the recycling process. In contrast, it should be pointed out from Fig. 13 that the fluctuation of the rate of phosphorus loss (i.e., loss noise) can cause regime shifts from the eutrophic state to the oligotrophic state. It is shown in Fig. 12 (bottom panels) that at smaller \( \alpha_2 \) (see \( \alpha_2 = 0.01 \)), the probability distribution as a function of \( p \) shows a peak at a small \( p \) value, which corresponds to the oligotrophic state (low nutrient). As the value of \( \alpha_2 \) increases (see \( \alpha_2 = 0.25 \)), the position of the peak is shifted to a large value of \( p \). Fig. 13 shows what occurs when \( \alpha_2 \) is fixed and \( D_2 \) is changed. For small \( D_2 \) (see \( D_2 = 0.1 \)), the probability distribution as a function of \( p \) exhibits one peak. As the values of \( D_2 \) increase (see \( D_2 = 0.9 \)), the position of the peak is shifted to a small value of \( p \).

In Figs. 14 and 15, the time series of the phosphorus concentration \( p(t) \) and the probability distribution are plotted for different cross-correlation intensities \( \lambda_2 \) and nonlinear time delays \( \beta \),
respectively. For a negative $\lambda_2$ value (see $\lambda_2 = -0.9$ in Fig. 14), the phosphorus concentration $p(t)$ concentrates on the oligotrophic state. However, when the value of $\lambda_2$ is increased, the position of the peak is shifted to a large value of $p$. In Fig. 15, the probability distribution $Q(p)$ exhibits two peaks when $\beta = 0.0$. However, when the value of $\beta$ is increased, $Q(p)$ exhibits one single peak at a large value of $p$. As the value of $\beta$ continues to increase, the single peak becomes high rapidly. This result indicates that regime shifts from the oligotrophic state to the eutrophic state can be induced by cross-correlation between two noises and nonlinear time delay in the lake ecological system.

Finally, we compare the time delays in recycling with loss processes. For a time delay in the recycling process, the nonlinear time delay $\beta$ can induce a transition (Fig. 15), while the noise intensities $\alpha_2$ and $D_2$ and cross-correlation intensity $\lambda_2$ cannot induce it (Figs. 12–14). However, for a time delay in the loss process, the noise intensities $\alpha_1$ and $D_1$ and cross-correlation intensity $\lambda_1$ can induce a transition (Figs. 3–5), while a linear time delay $\tau$ cannot induce it (Fig. 6).

Global delay in both processes
Here, we consider a time delay $\theta$ in both processes (loss and recycling) in the lake ecosystem. The governing equation can be considered as
\[
\frac{dp}{dt} = c - sp_0 + \frac{\eta_3^2}{m^3 + p_0^3} - p_0 \eta_3(t) + \xi_3(t),
\]

where \( \eta_3(t) \) and \( \xi_3(t) \) are the Gaussian white noises with zero mean, and the correlation form between the two noises can satisfy the statistical property of Eq. 3, namely

\[
\langle \xi_3(t) \eta_3(t') \rangle = \langle \eta_3(t) \xi_3(t') \rangle = 2\lambda_3 \sqrt{D_3 \alpha_3} \delta(t - t'),
\]

where \( \alpha_3 \) and \( D_3 \) describe the intensity of white noise \( \xi_3(t) \) (input) and \( \eta_3(t) \) (loss), respectively, and \( \lambda_3 \) denotes the intensity of cross-correlation between \( \xi_3(t) \) and \( \eta_3(t) \). The variable \( p_0 \) is a global delay in the lake ecosystem.

This case combines the effects of the above two loss and recycling cases, and the impacts of the noise intensities \( \alpha_3 \) and \( D_3 \), the cross-correlation intensity \( \lambda_3 \), and global delay \( \theta \) on the time series and probability distribution of the nutrient concentration are shown in Figs. 16–19, respectively. The effects of \( \alpha_3 \), \( D_3 \), and \( \lambda_3 \) on the time series and probability distribution of the nutrient concentration (Figs. 16–18) play the same role as those of time delay in recycling process (Figs. 12–14). Compared with the time delay in the recycling process, the \( Q(p) \) is changed from two peaks to

Fig. 15. The time series (top) and probability distribution (bottom) of \( p(t) \) for different time delays of \( \beta \). The other parameter values are \( m = 1.0, q = 8.0, r = 1.0, c = 0.5, s = 1.0, \alpha_2 = 0.25, D_2 = 0.1, \) and \( \lambda_2 = 0.5. \)
one peak when the value of $\theta$ or $\beta$ is increased. The single peak rapidly becomes low and broad as the value of $\theta$ continues to increase (Fig. 19), but it becomes higher as the value of $\beta$ continues to increase (Fig. 15). Thus, the result of global delay $\theta$ indicates that the impacts of the different delays $\tau$ and $\beta$ on the time series and probability distribution are the same, and the impact of a nonlinear time delay $\beta$ on the time series and probability distribution of the nutrient concentration is stronger than that for a linear time delay $\tau$.

**The MFPT**

In this section, we are more interested in how regime shifts between two stable states are affected by the input and loss noises and the cross-correlation between two noises and time delays. In a certain range of values of the parameters, the model possesses two alternative stable states: One is the low nutrient state ($p_{s1}$), and the other is the high nutrient state ($p_{s2}$). Environmental perturbations and time delays present in the system can induce transitions between two stable states, and a variable of interest is the time from one state to the other state. This time is a random variable and is often referred to as the first passage time. A simple but general approach to the classical MFPT problem was offered by Fox (1986b). From this point of view, it is interesting to study the MFPT $T(p)$ of the transition between two stable states (Klosek-Dygas et al. 1988, Klosek-Dygas and Hagan 1998).
\[ A(p) \frac{\partial}{\partial p} T(p) + B(p) \frac{\partial^2}{\partial p^2} T(p) = -1, \quad (32) \]

with boundary conditions
\[ \left. \frac{dT(p)}{dp} \right|_{p=p_{s1}} = 0, \quad T(p) \bigg|_{p=p_{s2}} = 0, \quad (33) \]

which correspond to a reflecting boundary at \( p = p_{s1} \) and an absorbing boundary at \( p = p_{s2} \) and \( T(p) \) represents the MFPT of the transition between two stable states. Solving Eq. 32 with the boundary conditions in Eq. 33, the MFPT \( \tau_{\text{L}} \) of the process \( p(t) \) to reach the eutrophic state \( p_{s2} \) with the initial condition \( p(t = 0) = p_{s1} \) (the oligotrophic state) is given by (Hänggi et al. 1984)
\[ T(p_{s1} \rightarrow p_{s2}) = \int_{p_{s1}}^{p_{s2}} \frac{dp}{B(p)Q_{st}(p)} \int_{-\infty}^{0} Q_{st}(x)dx. \quad (34) \]

In order to characterize the regime shifts from the oligotrophic state to the eutrophic state, Fig. 20 shows the MFPT \( \tau_{\text{L}} \) as the function of input noise intensity \( \alpha_j \) \((j = 1, 2, \text{and } 3)\) of the rate of phosphorus input for different time delays \( \tau, \beta, \text{and } \theta \), respectively. The MFPT \( \tau_{\text{L}} \) is decreased as the value of \( \alpha_j \) increases. Therefore, the increase in \( \alpha_j \) leads to a decline in the MFPT \( \tau_{\text{L}} \) and enhances the likelihood of regime shifts to the eutrophic state. When \( \alpha_j \) is fixed, the MFPT \( \tau_{\text{L}} \) is decreased as the value of \( \tau \) or \( \theta \) increases, and it

Fig. 17. The time series (top) and probability distribution (bottom) of \( p(t) \) for different loss noise intensities \( D_3 \). The other parameter values are \( m = 1.0, q = 8.0, \gamma = 1.0, c = 0.5, s = 1.0, \alpha_3 = 0.25, \lambda_3 = 0.5 \), and \( \theta = 0.5 \).
slightly increases as $\beta$ increases. This implies that stability of the oligotrophic state can be weakened by the linear and global time delays and enhanced by the nonlinear time delay. That is to say, the mean lifetime of the oligotrophic state for the case of linear and global time delays is shorter than that without time delays, and that for the case of nonlinear time delay is slightly longer than it.

Fig. 21 shows the MFPTL as the function of the loss noise intensity $D_j$ ($j = 1, 2, \text{and} 3$) of the rate of phosphorus loss for different time delays $\tau$, $\beta$, and $\theta$, respectively. It is shown that the MFPTL vs. $D_j$ exhibits a maximum, which implies a long-lasting oligotrophic state and the existence of an appropriate noise intensity $D_j$ leading to a maximal MFPTL. This maximum for MFPTL as the function of the noise intensity $D_j$ identifies the NES. This NES was found numerically by Dayan et al. (1992), shown experimentally by Mantegna and Spagnolo (1996), and theoretically investigated by Mantegna and Spagnolo (1998), Spagnolo et al. (2004a), and Fiasconaro et al. (2010). In our work, this maximum for MFPTL implies that the stability of the oligotrophic state can be enhanced by the noise, and the mean lifetime of the oligotrophic state is longer than the deterministic decay time. Other work has emphasized the possibility that natural variations in the ecosystem can actually enhance the
resilience of an ecosystem (Holling and Meffe 1996), and the correlation between the noises may stabilize the transition state of a bistable system (Mondal and Bag 2015). But for linear or global time delays, our results show there is both a decrease in ecosystem resilience and an increase in the likelihood of regime shifts to the eutrophic state, and thus, linear or global time delays weaken the stability of the oligotrophic state. This finding has important ecological implications.

Fig. 22 depicts the MFPTL as the function of the cross-correlation intensity $\lambda_j$ ($j = 1, 2, \text{and} 3$) between two noises. Obviously, the MFPTL increases with increasing $\lambda_j$. From the physics point of view, the height of the potential well is related to the MFPTL of the transitions between two stable states. As shown in Fig. 9, a positive correlation between two noises shows that the phosphorus concentration $p(t)$ is more easily attracted by the left well and hard transitions to the right well. Hence, the positive cross-correlation between two noises enhances the stability of the right well. In an ecological sense, the positive cross-correlation between two noises can enhance the stability of the oligotrophic state; that is, the positive cross-correlation between two noises weakens the
regime shifts between two states. When \( k_j \) is fixed (Fig. 22), the MFPT\( _L \) is decreased as \( \tau \) or \( \theta \) increases, while slightly increased as \( \beta \) increases. This implies that the stability of the oligotrophic state can be weakened by the linear and global time delays, while it is enhanced by the nonlinear time delay. Global delay combines the effects of linear and nonlinear time delays. The competition between the linear and nonlinear time delay indicates that the impact of the different delays \( \tau \) and \( \beta \) on the MFPT\( _L \) can cancel each other out, but the impact of linear time delay \( \tau \) is stronger than that for nonlinear time delay \( \beta \).

Next, we discuss (another direction) regime shifts from the eutrophic state to the oligotrophic state. The MFPT\( _R \) of the process \( p(t) \) to reach the oligotrophic state \( p_{s1} \) with the initial condition \( p(t = 0) = p_{s2} \) (the eutrophic state) is given by (Hänggi et al. 1984)

\[
T(p_{s2} \rightarrow p_{s1}) = \int_{p_{s2}}^{p_{s1}} \frac{dp}{B(p)Q_{st}(p)} \int_{-\infty}^{p} Q_{st}(x)dx. \tag{35}
\]

In Figs. 23–25, the effects of \( \tau \), \( \beta \), and \( \theta \) on the MFPT\( _R \) are similar to those of \( \tau \), \( \beta \), and \( \theta \) on the MFPT\( _L \). By comparison with the MFPT\( _L \) from...
the oligotrophic state to the eutrophic state, it is found that the MFPT$_R$ vs. $\alpha_j$ exhibits a maximum, and the MFPT$_R$ vs. $D_j$ decreases monotonically. These are different from the results of the MFPT$_L$, where the MFPT$_L$ vs. $\alpha_j$ decreases monotonically and the MFPT$_L$ vs. $D_j$ exhibits a maximum.

**Conclusion**

In this study, we have developed a theoretical framework for the study of regime shifts between two states in lake eutrophication ecosystems driven by delayed processes, where environmental parameters are assumed to be disturbed by both input and loss external noises. The deterministic dynamics of Eq. 1 implies that bistability is affected by the maximum rate $r$ of recycling of phosphorus, at which the bistability expands with the increase in $r$. Noises and time delays present in lake eutrophication systems can induce transitions between two stable states. Primarily, we are interested in how regime shifts from one state to the other state are affected by input and loss external noises, cross-correlation between two noises, and time delays. Our main results are as follows. First, input noise, linear (or nonlinear)
time delay, and positive cross-correlation between two noises can induce regime shifts from the oligotrophic state to the eutrophic state, while the loss noise and the negative cross-correlation between two noises can induce regime shifts from the eutrophic state to the oligotrophic state. The mechanisms for fluctuation-induced regime shifts between the oligotrophic and eutrophic states can be explained from the point of view of the effective potential of the Fokker–Planck equation. Global delay can induce regime shifts from the eutrophic state to the oligotrophic state, which indicates that the impacts of the different time delays $\tau$ and $\beta$ on the regime shifts between two stable states are the same, while the impact of nonlinear time delay $\beta$ is stronger than that for linear time delay $\tau$. Theoretical analyses supported our numerical simulations. In addition, the input noise or loss noise (or cross-correlation between two noises) can induce a transition, while the linear time delay cannot induce it for a time delay in the loss process. However, for a time delay in the recycling process or a global delay in the loss and recycling processes, the nonlinear time delay or global delay can induce a transition, while the input noise or loss noise (or cross-correlation between two noises) cannot induce it. Therefore, it is shown that the fluctuation- and delay-induced regime shifts between

Fig. 22. Numerical simulation results for the MFPT$_L$ as a function of $\lambda$ for different time delays $\tau$, $\beta$, and $\theta$. The other parameter values are $\alpha_1 = \alpha_2 = \alpha_3 = 0.25$, $D_1 = D_2 = D_3 = 0.1$, $m = 1.0$, $q = 8.0$, $\gamma = 1.0$, $c = 0.5$, and $s = 1.0$. MFPT, mean first passage time.
oligotrophic and eutrophic states are physical examples of noise- and delay-induced transition. Finally, the linear time delay can lead to a decline in the MFPT_L (or MFPT_R) and enhance the likelihood of regime shifts to the eutrophic (or oligotrophic) state, while the nonlinear time delay can lead to an increase in the MFPT_L (or MFPT_R) and weaken the likelihood of regime shifts to the eutrophic (or oligotrophic) state. The MFPT_L vs. D_j (or MFPT_R vs. τ_j) exhibits a maximum, which implies a long-lasting oligotrophic (or eutrophic) state and the existence of an appropriate noise intensity D_j (or τ_j) leading to a maximal MFPT_L (or MFPT_R). This maximum for MFPT_L (or MFPT_R) implies that the stability of the oligotrophic (or eutrophic) state can be enhanced by the noises, and the mean lifetime of the oligotrophic (or eutrophic) state is longer than the deterministic decay time. Global delay combines the effects of the linear and nonlinear time delays. The competition between the linear and nonlinear time delay indicates that the impact of τ and β on the MFPT_L (or MFPT_R) can cancel each other out, but the impact of linear time delay τ is stronger than that for nonlinear time delay β. Interestingly, eutrophication of lakes has become a global problem that is likely to intensify in the

Fig. 23. Numerical simulation results for the MFPT_R as a function of α for different time delays τ, β, and θ. The other parameter values are D_1 = D_2 = D_3 = 0.1, λ_1 = λ_2 = λ_3 = 0.5, m = 1.0, q = 8.0, γ = 1.0, c = 0.5, and s = 1.0. MFPT, mean first passage time.
coming decades due to the increases in human population, demand for food, land conversion, fertilizer use, and nitrogen deposition. However, our results show that noises and time delays can induce (or accelerate) regime shifts from the eutrophic state to the oligotrophic state. Namely, noises and time delays can be used as control parameters of regime shifts between eutrophic and oligotrophic states. This finding has important ecological implications.

Ecological bistability is a topic of central importance in ecology. The nature of phosphorus loss or input dictates that stochastic delay is explicitly treated and understood in basic models. The noise- and delay-induced regime shifts discussed here are important properties of the stochastic delay in the lake eutrophication ecosystem. In addition, the stochastic-delayed dynamic approach can identify key ecological control parameters to which the behavior of specific lake eutrophication ecosystems is particularly sensitive. Such parameters might provide targets for environmental perturbation and time lag. In most empirical applications, it is possible to discern whether the system is driven by input or loss noise, and the magnitude of the dependency of the lake dynamics on the system’s present and past conditions.

Fig. 24. Numerical simulation results for the MFPT$_R$ as a function of $D$ for different time delays $\tau$, $\beta$, and $\theta$. The other parameter values are $a_1 = a_2 = a_3 = 0.25$, $\lambda_1 = \lambda_2 = \lambda_3 = 0.5$, $m = 1.0$, $q = 8.0$, $\gamma = 1.0$, $c = 0.5$, and $s = 1.0$. MFPT, mean first passage time.
can be evaluated. Therefore, our work not only offers a method to analyze some complex stochastic-delayed ecosystems, but also stimulates theoretical and experimental studies to illustrate the noise- and delay-induced regime shifts in a time-delayed lake eutrophication ecosystem.

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Fig. 25. Numerical simulation results for the MFPT as a function of λ for different time delays τ, β, and θ. The other parameter values are $a_1 = a_2 = a_3 = 0.25$, $D_1 = D_2 = D_3 = 0.1$, $m = 1.0$, $q = 8.0$, $γ = 1.0$, $c = 0.5$, and $s = 1.0$. MFPT, mean first passage time.
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