Clustering by Hill-Climbing: Consistency Results

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Abstract

We consider several hill-climbing approaches to clustering as formulated by Fukunaga and Hostetler [18] in the 1970’s. We study both continuous-space and discrete-space (i.e., medoid) variants and establish their consistency.

Keywords and phrases: clustering; gradient lines; gradient flow; dynamical systems; ordinary differential equations; Euler scheme; Morse theory; Mean Shift; Max Shift; Max Slope Shift; hill-climbing methods for clustering

1 Introduction

Clustering methods based on ‘climbing’ the density landscape date back to the 1970’s, in particular, to work by K. Fukunaga and his collaborators.1 Indeed, in 1975, Fukunaga and Hostetler [18] proposed to “assign each observation to the nearest mode along the direction of the gradient”. Formally, the gradient ascent line starting at a point \( x \) is the curve given by the image of \( \gamma_x \), the parameterized curve defined by the following ordinary differential equation (ODE)

\[
\gamma_x(0) = x; \quad \gamma_x(t) = \nabla f(\gamma_x(t)), \quad t \geq 0.
\] (1)

In this fashion, a point \( x \) is assigned to the (critical) point at the end of the gradient line above, meaning \( \gamma_x(\infty) \). Fukunaga and Hostetler [18] then added that “To accomplish this, one could move each observation a small step in the direction of the gradient and iteratively repeat the process on the transformed observations until tight clusters result near the modes.” This led them to propose what is known in numerical analysis as a forward Euler scheme (with step size being \( \rho > 0 \) here):

\[
x(0) = x; \quad x(k + 1) = x(k) + \rho \nabla f(x(k)), \quad k \geq 0.
\] (2)

Under some standard conditions, the scheme is consistent in that, as \( \rho \to 0 \), the sequence converges in an appropriate sense to the gradient line \( \gamma_x \). We will refer to this scheme as the Euler Shift.

In practice, the density needs to be estimated, and this is often done by kernel density estimation. This was already considered in [18], and the name ‘mean shift’ comes from the fact that when using a kernel density estimator, the gradient of that estimator is proportional to the shift in mean with

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1 Hill-climbing often refers to greedy approach to optimizing an objective function. Such strategies have been suggested in the context of clustering, including the algorithm of Kernighan and Lin [29], the \( k \)-means algorithms proposed by Lloyd [32] and Hartigan and Wong [23], and even the EM algorithm of Dempster et al. [17] when used to fit a mixture distribution. In the present paper, we reserve this term for approaches that climb the landscape defined by a density.
with respect to another kernel — what Cheng [13] called a ‘shadow’ of the kernel used to estimate the density. In the actual implementation proposed by Fukunaga and Hostetler [18] — which is nowadays known as Blurring Mean Shift — at each iteration all the sample points are moved and the density estimate is recomputed based on the new locations. Cheng [13] contrasted this with what he calls Mean Shift, where instead the density is estimated based on the original sample before the sample points are moved by mean shift as described above. We note that in both implementations the points are moved by successive mean shifts, which differs from applying the Euler scheme above.

The approach to clustering advocated by Fukunaga and Hostetler [18] has generated a good amount of enthusiasm over the past decades, leading to a number of methods. Carreira-Perpiñán [8] provides a fairly recent review of this work. Among them, arguably the simplest variant is what we call Max Shift, where at each step the location is changed to a point in the neighborhood with largest density value:

\[
x(0) = x_0; \quad x(k+1) \in \arg \max_{x \in B(x(k), \varepsilon)} f(x), \quad k \geq 0.
\]

(3)

Ties are broken in an arbitrary but deterministic way (for simplicity), and if \( f(x(k+1)) = f(x(k)) \), the process stops. The parameter \( \varepsilon > 0 \) defines the size of neighborhood around the present location where the maximization takes places to compute the next location in the sequence. This is effectively embedded in a method proposed by Chazal et al. [10], which they called ToMATo (for Topological Mode Analysis Tool) — although the overall approach is more sophisticated and also includes some merging of attraction basins based on (topological) persistence considerations.

In the present paper, we establish the consistency of Euler Shift, Mean Shift, Max Shift, and a few other variants (including a regularized version of the one proposed in [30]) in a concise and comprehensive manner. To be clear, consistency refers to the task of clustering in the sense of Fukunaga and Hostetler [18], where points are grouped according to the critical points where the density gradient ascent flow (1) leads them. By doing this, we contribute to the building of a mathematical foundation for this type of clustering methods, and adds to existing work in that area. For one, Euler Shift is known to be consistent [1, 5, 15], and there is a surrounding literature on the problem of estimating the gradient lines of a density [12] and even density ridges [19, 34]. Further, via the shadow kernel concept of Cheng [13], Mean Shift can be directly related to Euler Shift and thus proven to be also consistent.

The remainder of the paper is organized as follows. In Section 2, we introduce the framework and some concepts and notation. In Section 3.1, we introduce a prototypical hill-climbing algorithm and establish its consistency. We then specialize this to several variants. In Section 4, we establish the large-sample consistency of the corresponding methods. Throughout the paper, we distinguish between \textit{algorithm} — defined based on a given density function — and \textit{method} — an algorithm applied to an estimate of the density based on an iid sample from that density. In Section 5, we consider medoid variants of the previous algorithms where the sequence is constrained to be made of a given set of points (the sample points in practice). We discuss some open problems in Section 6 and gather some additional technical results in the Appendix.

2 Setting

We lay in this section the foundations, starting with our assumptions on the underlying density — which are very standard for this literature — and introducing some basic concepts regarding the gradient flow it defines. We also discuss the role of the point set \( \mathcal{Y} \) that appears in (36) and (37).
2.1 The density $f$

Throughout, we consider a density with respect to the Lebesgue measure on $\mathbb{R}^d$ — denoted $f$ everywhere — assumed to satisfy the following conditions:

- **Zero at infinity.** $f$ converges to zero at infinity, meaning, $f(x) \to 0$ as $\|x\| \to \infty$.
- **Twice differentiable.** $f$ is twice differentiable everywhere with bounded and uniformly continuous zeroth, first, and second derivatives.
- **Non-degenerate critical points.** The Hessian is non-singular at every critical point of $f$.

The first condition is equivalent to $f$ having bounded upper level sets, meaning that $U_s := \{ f \geq s \}$ is bounded, and therefore compact by continuity of $f$, for any $s > 0$. The second condition is simply a smoothness assumption on the density. (Note that the uniform continuity of the zeroth and first derivatives is implied by the fact that the second derivative exits everywhere and is bounded.)

And a function that satisfies the third condition is sometimes referred to as a Morse function [33]. Similar conditions are standard in the literature cited in the Introduction.

Define

$$
\kappa_0 = \sup_x f(x), \quad \kappa_1 = \sup_x \|\nabla f(x)\|, \quad \kappa_2 = \sup_x \|\nabla^2 f(x)\|.
$$

Then $f$ is $\kappa_1$-Lipschitz, meaning

$$
|f(y) - f(x)| \leq \kappa_1 \|y - x\|, \quad \forall x, y,
$$

and $\nabla f$ is $\kappa_2$-Lipschitz, meaning

$$
\|\nabla f(y) - \nabla f(x)\| \leq \kappa_2 \|y - x\|, \quad \forall x, y,
$$

and the following Taylor expansion holds

$$
|f(y) - f(x) - \nabla f(x)^\top (y - x)| \leq \frac{1}{2} \kappa_2 \|y - x\|^2, \quad \forall x, y.
$$

We will denote $N(x) := \nabla f(x)/\|\nabla f(x)\|$, which is well-defined whenever $\nabla f(x) \neq 0$, and in fact differentiable, with derivative equal to

$$
DN(x) = \frac{\nabla^2 f(x)}{\|\nabla f(x)\|} - \frac{\nabla f(x)\nabla f(x)^\top \nabla^2 f(x)}{\|\nabla f(x)\|^2}.
$$

In most of the paper, we will assume that the density is available. In practice, of course, it needs to be estimated, and this is most often done via kernel density estimation. Using some stability results implying roughly speaking that things do not change much if the estimation is accurate enough, we will port the consistency results established for the setting where the density is known to the setting where it is estimated. We differentiate between *algorithm*, which is applied with knowledge of the density; and *method*, which is applied without knowledge of the density except for an estimate of the density, typically derived from a sample. This terminology may not be standard, but we find it useful in the confined setting of the present work.

By *population* we mean the support of the density, which we will denote by $\text{supp} f$ on occasion. When we talk of a sample, we will assume it to be generated iid from $f$. 

2.2 Gradient lines and gradient flow

Under the above conditions on $f$, $\nabla f$ is Lipschitz, and this is enough for standard theory for ODEs [24, Sec 17.1] to justify the definition in (1) of the gradient ascent line $\gamma_x$ originating at any point $x$. This, and the fact that this is a gradient flow [24, Sec 9.3], gives the following.

Lemma 2.1. For any $x$, the function $\gamma_x$ defined in (1) is well-defined on $[0, \infty)$, with $\gamma_x(t)$ converging to a critical point of $f$ as $t \to \infty$.

The basin of attraction of a point $x_*$ is defined as $\{x : \gamma_x(\infty) = x_*\}$. Note that this set is empty unless $x_*$ is a critical point (i.e., $\nabla f(x_*) = 0$). In the gradient line view of clustering, we call a cluster any basin of attraction of a mode. It turns out that, if $f$ is a Morse function [33], then all these basins of attraction, sometimes called stable manifolds, provide a partition of the support up to a set of zero measure.

Lemma 2.2. Under the assumed regularity conditions, the basins of attraction of the local maxima, by themselves, cover the population, except for a set of zero measure.

Indeed, by Lemma 2.1, the basins of attraction partition the entire population. In addition, the set of critical points is discrete [4, Cor 3.3], the basin of attraction of each critical point that is not a local maximum is a (differentiable) submanifold of co-dimension at least one [4, Th 4.2], and therefore has zero Lebesgue measure. For more background on Morse functions and their use in statistics, see the recent articles of Chacón [9] and Chen et al. [11].

The following lemma gives the continuity of the gradient flow curve with respect to the Hausdorff metric when seen as a subset of $\mathbb{R}^d$ indexed by the starting point, which can be of independent interest — its proof is given in the Appendix. We note that this result is stronger than the well-known continuity of trajectories of gradient flows with respect to the starting points up to a fixed time point. See, for example, the main theorem in [24, Sec 17.3]. Let $d_H$ denote the Hausdorff metric.

Lemma 2.3. The gradient ascent flow, seen here as the function $x \mapsto \gamma_x([0, \infty))$ mapping $(\mathbb{R}^d, \|\cdot\|)$ to $(2^{\mathbb{R}^d}, d_H)$, is continuous in the basin of attraction of a mode.

For the sake of clarity, we will sometimes work with the gradient line parameterized by arc length. Equivalently, for a point $x$, this means considering the gradient flow of $N$, or more explicitly,

$$\zeta_x(0) = x; \quad \dot{\zeta}_x(t) = N(\zeta_x(t)). \quad (8)$$

Note that $\zeta_x$ is only defined over $[0, \ell_x]$, where $\ell_x$ is the length of the gradient ascent line originating at $x$ (meaning the length of $\gamma_x$). In view of our assumptions on $f$, assuming $x$ is not a critical point, $\zeta_x$ is twice continuously differentiable on $[0, \ell_x)$ with

$$\ddot{\zeta}_x(t) = DN(\zeta_x(t))\dot{\zeta}_x(t), \quad \text{with } DN \text{ given in (7)}. \quad (9)$$

2.3 Level sets

For a positive real number $s > 0$, the $s$-level set of $f$ is given by

$$\mathcal{L}_s := \{x : f(x) = s\}.$$
while the $s$-upper level set of $f$ is given by

$$U_s := \{ x : f(x) \geq s \}.$$ 

Throughout, whether specified or not, we will only consider levels that are in $(0, \kappa_0)$. Note that, because $f$ converges to zero at infinity and is continuous, its (upper) level sets are compact. We call any connected component of an upper level set a level cluster. This is in congruence with the level set definition of cluster offered by Hartigan [22]. Hartigan also defined what is now known as the cluster tree, which is the partial ordering between clusters that comes with the set inclusion operation: indeed, when two level clusters intersect, one of them must contain the other. We say that a level cluster $C$ is a leaf (level) cluster if the cluster tree does not branch out past $C$, or said differently, if all the descendants of $C$ have at most one child. Note that the last descendant of a leaf cluster is a singleton defined by a mode. For a point $x$ and $s \leq f(x)$, let $C_s(x)$ denote the $s$-level cluster that contains $x$, and $C(x)$ will be shorthand for $C_{f(x)}(x)$.

There is a large amount of literature on the estimation of level sets and on the estimation of the cluster tree, but in the present paper we will only use some basic results, including the following.

**Lemma 2.4.** Any level cluster contains at least one mode. Moreover, a mode coincides with the intersection of all the level clusters that contain that mode.

The following is a slightly different version of [2, Lem 5.9].

**Lemma 2.5.** Let $x_*$ be a mode of $f$ and let $s_* = f(x_*)$. Then there is a constant $\delta > 0$ such that $C \equiv C_\delta = \sup_{x \in \bar{B}(x_*, \delta)} \max\{ \sqrt{\lambda_{\max}(x)}, 2/\sqrt{\lambda_{\min}(x)} \}$ is finite, where $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ are the largest and smallest eigenvalues of $-\nabla^2 f(x)$, respectively; and

$$\bar{B}(x_*, 1/C \sqrt{s_* - s}) \subset C_\delta \subset B(x_*, C \sqrt{s_* - s}), \quad \text{for all } s \in (s_* - \delta^2/C^2, s_*).$$

## 3 Consistency: Algorithms

In this section we look at various hill-climbing clustering algorithms. As we indicated in the Introduction, an algorithm is defined based on an available density.

We adopt in this paper the definition of clustering proposed by Fukunaga and Hostetler [18], where we “assign each [point] to the nearest mode along the direction of the gradient”. That is, we assign a point $x$ to $\gamma_x(\infty)$, where $\gamma_x$ is the gradient ascent line originating from $x$, defined in (1), and $\gamma_x(\infty) := \lim_{t \to \infty} \gamma_x(t)$ is the endpoint where that line terminates. Consequently, the population — meaning the support of the density — is partitioned according to the basins of attraction of the density critical points. As just discussed in Section 2.2, this definition is justified, and even though not all the critical points are modes, it is true by Lemma 2.2 that the basins of attraction associated with modes are the ones that truly matter.

With the definition of clustering that we espouse here, we say that an algorithm is consistent if it moves almost any point $x$ in the support to $\gamma_x(\infty)$ when the neighborhood size is small enough. We make this more precise below.

### 3.1 Prototype

We start by discussing a prototypical hill-climbing algorithm that we prove to be consistent. We then show that a number of hill-climbing algorithms satisfy the same core properties, implying that these algorithms are consistent.
The prototypical algorithm that we consider, when initialized at some point in the support of the density, say \( x_0 \), produces a sequence, denoted \((x_k)\). The core properties we just alluded to are the following:

**Property 1 (The shifts are of comparable size).** For some positive function \( S \), for all \( k \), except perhaps for the last shift,

\begin{equation}
\varepsilon S(\| \nabla f(x_k) \|) \leq \|x_{k+1} - x_k\| \leq \varepsilon.
\end{equation}

The function \( S \) will be taken to be non-decreasing without loss of generality. The quantity \( \varepsilon \) can be made small by appropriately tuning the algorithm.

**Property 2 (The process converges to a mode when initialized in the vicinity of that mode).** Suppose that \( x_* \) is a mode. Then there is \( \delta > 0 \) such that, for \( \epsilon > 0 \) small enough, if \( x_0 \in B(x_*, \delta) \) then \((x_k)\) converges to \( x_* \).

**Property 3 (The shifts are close to the gradient at the corresponding location).** For all shifts, except perhaps for the last shift,

\begin{equation}
x_{k+1} - x_k = \|x_{k+1} - x_k\| N(x_k) \pm \|x_{k+1} - x_k\| R(\|x_{k+1} - x_k\|, \|\nabla f(x_k)\|),
\end{equation}

whenever \( \nabla f(x_k) \neq 0 \), where \((u, v) \mapsto R(u, v)\) is a continuous function that is increasing in \( u \) and decreasing in \( v \), and satisfies \( R(0, v) = 0 \) for all \( v > 0 \).

We note that the functions \( S \) and \( R \) may depend on the starting point \( x_0 \).

**Theorem 3.1.** Consider an algorithm that satisfies the above properties. Let \( x_* \) denote a mode. Then for any point \( x_0 \) in the basin of attraction of \( x_* \), when initialized at \( x_0 \) and with \( \epsilon \) made small enough, the algorithm produces a sequence that converges to \( x_* \).

**Proof.** Let \( \delta_* > 0 \) and \( \epsilon_* > 0 \) be as in Property 2, so that it suffices to show that the sequence that the algorithm with parameter \( \epsilon \) small enough that \( \epsilon \leq \epsilon_* \) constructs produces a sequence, denoted \((x_k)\) henceforth, that reaches \( B(x_*, \delta_*) \). Let \( \zeta \) be shorthand for \( \zeta_{x_0} \), defined in (8), and let \( \ell \) de shorthand for \( \ell_{x_0} \). We let \( Z_t := \zeta([0, t]) \), which is the gradient line up to time \( t \). (Note that ‘time’ represents length with the chosen parameterization of the gradient line.) Since \( Z_t \) joins \( x_0 \) and \( x_* \), the gradient line certainly enters that ball, and the idea is to show that \((x_k)\) remains close to that curve, at least until entering that ball.

Let \( t_\# := \inf\{t \geq 0 : \|\zeta(t) - x_*\| = \delta_*/3\} \). Then define \( \nu := \frac{1}{2} \min\{\|\nabla f(z)\| : z \in Z_{t_\#}\} \), and note that \( \nu > 0 \) by the fact that \( \|\nabla f\| \) is continuous and (strictly) positive on \( Z_{t_\#} \) because the gradient line \( Z_t \) does not contain a critical point other than \( x_* \) at its very end. By an application of (5), we have that \( \|\nabla f(y)\| \geq \nu \) for all \( y \) in the ‘tube’ \( T := B(Z_{t_\#}, \delta_{t_\#}) \), where \( \delta_{t_\#} := \nu / \kappa_2 \).

The sequence \((z_k)\). Define the sequence \( t_0 := 0 \) and \( z_0 := 0 \), and for \( k \geq 1 \), \( t_k := t_{k-1} + \epsilon_k \), where \( \epsilon_k := \|x_k - x_{k-1}\| \), and \( z_k := \zeta(t_k) \). Of course, as the discretization gets finer and finer, the sequence \((z_k)\) gets closer and closer to the gradient ascent line, and the basic idea is to compare the sequence \((x_k)\) to the sequence \((z_k)\). Let \( k_\# := \max\{k : t_k \leq t_\#\} \), and note that, since \( t_{k_\#+1} > t_\# \) and

\begin{equation}
t_{k_\#+1} = \sum_{k=1}^{k_\#+1} \epsilon_k \leq (k_\# + 1) \epsilon, \quad \text{we have } k_\# > t_\#/\epsilon - 1.
\end{equation}

Letting \( z_\# := \zeta(t_\#) \), we have that \( \|z_\# - x_*\| = \delta_*/3 \) by construction, and also

\begin{equation}
\|z_{k_\#} - z_{k_\#}\| = \|\zeta(t_\#) - \zeta(t_{k_\#})\| = t_\# - t_{k_\#} < t_{k_\#+1} - t_{k_\#} \leq \epsilon.
\end{equation}
Assuming \( \varepsilon \) is small enough that \( \varepsilon < \delta_\ast /3 \), we can guarantee that \( \| z_{k_\#} - x_* \| \leq 2\delta_\ast /3 \). Also, a Taylor expansion gives,

\[
z_k - z_{k-1} = \zeta(t_k) - \zeta(t_{k-1}) = (t_k - t_{k-1}) \zeta(t_{k-1}) + \frac{1}{2} \sup_{0 \leq t \leq t_k} \left\| \ddot{\zeta}(t) \right\| (t_k - t_{k-1})^2.
\]

While \( (t_k - t_{k-1}) \zeta(t_{k-1}) = \varepsilon_k N(z_{k-1}) \), based on (9) and (7), and the fact that \( \| \nabla f(z_{k-1}) \| \geq 2\nu \) for any \( k \leq k_\# \),

\[
\sup_{0 \leq t \leq t_k} \left\| \ddot{\zeta}(t) \right\| \leq \sup_{0 \leq t \leq t_k} \left\| D N(\zeta(t)) \right\| \leq \frac{\kappa_2}{2\nu} + \kappa_2.
\]

Hence,

\[
z_k - z_{k-1} = \varepsilon_k N(z_{k-1}) + C_1 \varepsilon^2, \quad \text{for all } 1 \leq k \leq k_\#.
\]

The sequence \((x_k)\). Define \( d_k := \| x_k - z_k \| \). We bound \( d_k \) by induction for \( 0 \leq k \leq k_\# \). Note that \( d_0 = 0 \) since \( z_0 = x_0 \). Recall the function \( R \) in (11), and define \( Q_1(\varepsilon) := R(\varepsilon, \nu) + C_1 \varepsilon \). Let \( C_2 := \kappa_2 / \nu + \kappa_2 \), and note that, due to (7), \( \| D N(x) \| \leq C_2 \) for all \( x \in T \). Recall the function \( S \) appearing in (10) and take \( \varepsilon \) small enough that

\[
Q_2(\varepsilon) := Q_1(\varepsilon) \frac{\exp[C_2 t_\# / S(\nu)] - 1}{C_2} \leq \delta_{\text{tube}},
\]

which is possible because \( Q_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). We are now ready to set the induction hypothesis: suppose that

\[
d_m \leq Q_1(\varepsilon) \frac{\exp[C_2 \mathrm{em}_1 - 1]}{C_2} \leq \delta_{\text{tube}}, \quad \forall m = 1, \ldots, k - 1.
\]

This is certainly true at \( k = 1 \), which primes our induction. Note that the two inequalities are part of the induction. We now bound \( d_k \), assuming that \( k \leq k_\# \). Since \( z_{k-1} = \zeta(t_{k-1}) \) with \( t_{k-1} \leq t_\# \), and \( \| x_{k-1} - z_{k-1} \| = d_{k-1} \leq \delta_{\text{tube}} \) by induction, we have \( \| \nabla f(x_{k-1}) \| \geq \nu \). With that, and Property 3 (specifically (11)), we derive

\[
x_k - x_{k-1} = \varepsilon_k N(x_{k-1}) \pm \varepsilon_k R(\varepsilon_k, \| \nabla f(x_{k-1}) \|)
\]

\[
= \varepsilon_k N(x_{k-1}) \pm \varepsilon R(\varepsilon, \nu).
\]

The bound (15) combined with (12) gives

\[
x_k - z_k = x_k - x_{k-1} + x_{k-1} - z_{k-1} + z_{k-1} - z_k
\]

\[
= \varepsilon_k N(x_{k-1}) + \varepsilon R(\varepsilon, \nu) + x_{k-1} - z_{k-1} - \varepsilon_k N(z_{k-1}) \pm C_1 \varepsilon^2,
\]

which after applying the triangle inequality, results in

\[
d_k \leq d_{k-1} + \varepsilon_k \| N(x_{k-1}) - N(z_{k-1}) \| + \varepsilon R(\varepsilon, \nu) + C_1 \varepsilon^2.
\]

Since the segment \([z_{k-1}, x_{k-1}]\) is inside the ball \( B(z_{k-1}, \delta_{\text{tube}}) \) (because, again, \( d_{k-1} \leq \delta_{\text{tube}} \)), and that ball is inside \( T \) (because, again, \( z_{k-1} = \zeta(t_{k-1}) \) with \( t_{k-1} \leq t_\# \), \( N \) is Lipschitz with constant \( C_2 \) inside \( B(z_{k-1}, \delta_{\text{tube}}) \), which then gives

\[
\| N(x_{k-1}) - N(z_{k-1}) \| \leq C_2 \| x_{k-1} - z_{k-1} \|.
\]
Hence, we have
\[
d_k \leq d_{k-1} + \varepsilon C_2 \|x_{k-1} - z_{k-1}\| + \varepsilon R(\varepsilon, \nu) + C_1 \varepsilon^2
\]
\[
= (1 + C_2 \varepsilon) d_{k-1} + \varepsilon Q_1(\varepsilon).
\]

Now, calling in the inequality (14) at \( m = k - 1 \), and simplifying using the fact that \( e^a - 1 - a \geq 0 \) for all \( a \), we deduce
\[
d_k \leq Q_1(\varepsilon) \frac{\exp[C_2 \varepsilon^k] - 1}{C_2}.
\]

This is only the first inequality that we needed to propagate. We now turn to the second one, which consists in bounding the right-hand side by \( \delta_{\text{tube}} \). Since we are considering \( k \leq k^\# \), we have \( t_k \leq t^\# \), and by using Property 1 (specifically (10)), we further get
\[
t^\# \geq t_k = \sum_{m=1}^{k} \varepsilon_m \geq \varepsilon \sum_{m=1}^{k} S(\|\nabla f(x_{m-1})\|).
\]

For \( m = 1, \ldots, k - 1 \), \( d_m \leq \delta_{\text{tube}} \) by induction, implying as we already saw above that \( x_m \in \mathcal{T} \), in turn implying that \( \|\nabla f(x_{m-1})\| \geq \nu \). Plugging this into (16), we get \( t^\# \geq k S(\nu) \varepsilon \), or \( k \varepsilon \leq t^\# / S(\nu) \). By monotonicity, we thus have
\[
Q_1(\varepsilon) \frac{\exp[C_2 \varepsilon^k] - 1}{C_2} \leq Q_2(\varepsilon) \leq \delta_{\text{tube}},
\]
the latter inequality being (13). Thus the induction proceeds. We have thus established that
\[
d_k = \|x_k - z_k\| \leq Q_2(\varepsilon), \quad \forall k = 0, \ldots, k^\#.
\]

**Conclusion.** In particular, we have \( d_{k^\#} = \|x_{k^\#} - z_{k^\#}\| \leq Q_2(\varepsilon) \), and taking \( \varepsilon \) small enough that \( Q_2(\varepsilon) < \delta_*/3 \), by the triangle inequality and the fact that \( \|z_{k^\#} - x_*\| \leq 2\delta_*/3 \), we can guarantee that \( \|x_{k^\#} - x_*\| < \delta_* \). This is what we needed to prove.

In this whole section we continue to use the same notation as in the proof above, except that we make the dependence on the starting point \( x \) explicit whenever needed as in, e.g., \( \nu(x) \) denoting \( \nu \) when associated with \( x \).

Next we provide a uniform version of Theorem 3.1, in the sense that with \( \varepsilon \) small enough, the result in Theorem 3.1 holds for almost all the starting points in \( A \), which is the union of the basins of attraction of all the modes. Its proof uses the continuity of \( \nu \), as given in Lemma A.1.

Recall functions \( S \) and \( R \) appearing in (10) and (11), respectively. We consider an algorithm satisfying the three properties with \( S \) and \( R \) that depend on \( x_0 \) only in a uniform way, as specified below. For any \( s \in (0, \kappa_0) \), let \( \mathcal{B}_s \) be the union of the basins of attraction for all the critical points in \( \mathcal{U}_s \) that are not modes. Note that if \( \mathcal{B}_s \) is not empty, it consists of finitely many \( k \)-dimensional submanifolds, where \( k = 0, \ldots, d - 1 \), and has zero Lebesgue measure, as indicated right below Lemma 2.2. For \( s, \delta > 0 \), define \( \Gamma_{\delta, s} = \mathcal{U}_s \cap B(\mathcal{B}_s, \delta)^C \). For any \( s, \delta > 0 \) such that \( \Gamma_{\delta, s} \) is not empty, we assume that there exist \( S = S_{\delta, s} \) and \( R \equiv R_{\delta, s} \) such that the properties hold for all \( x_0 \in \Gamma_{\delta, s} \). This is the case, for example, of Max Shift; see Lemma 3.1 and also Lemma 3.2 for example.

**Theorem 3.2.** Consider the prototypical algorithm satisfying the above properties and the assumption on the uniformity of \( R \) and \( S \). For every \( \eta > 0 \), there exists an \( \varepsilon_0 > 0 \) and a measurable set \( \Omega_\eta \) with probability measure at least \( 1 - \eta \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), the algorithm applied to any \( x_0 \in \Omega_\eta \) returns the associated mode, meaning, \( \lim_{t \to \infty} \gamma_{x_0}(t) \).
Proof. For an arbitrarily small but fixed $\eta > 0$, let $s_\eta$ be the largest $s > 0$ such that the probability measure of $U_\eta^C$ is not larger than $\eta/4$. Define $\Gamma_{\delta,s}^* := U_\delta \cap B(B_\eta, \delta)$. Note that as $\delta \to 0$, the Lebesgue measure of $\Gamma_{\delta,s_\eta}^*$ is of order $O(\delta) = o(1)$, and hence the probability measure of $\Gamma_{\delta,s_\eta}^*$ is also of order $o(1)$. Let $\delta_\eta$ be the largest $\delta > 0$ such that the probability measure of $\Gamma_{\delta,s_\eta}^*$ is not larger than $\eta/4$. Then the probability measure of $\Gamma_{\delta_\eta,s_\eta} = \Omega_\eta$ is at least $1 - \eta$. For simplicity, we denote $S = S_{\delta_\eta,s_\eta}$ and $R = R_{\delta_\eta,s_\eta}$.

Let $C$ be the union of balls $B(x_\ast, \delta_\ast/3)$, where $\delta_\ast$ is as in Property 2 and depends on the mode $x_\ast$. Let $\Gamma_\eta = \Omega_\eta \setminus C$. Define $\nu_\eta := \inf_{x \in \Gamma_\eta} \nu(x)$, which is positive, since $\Gamma_\eta$ is a compact set, and $\nu$ is continuous and positive on $A \supset \Gamma_\eta$ by Lemma A.1. Based on the proof of Theorem 3.1, especially (13), in order to guarantee that the prototypical algorithm returns the correct mode for all $x \in \Gamma_\eta$, we only need to choose $\varepsilon > 0$ small enough that

$$R(\varepsilon, \nu_\eta) + \kappa_2 \varepsilon \leq \frac{\kappa_2}{\exp[\kappa_0(\kappa_2/\nu_\eta + \kappa_2)/(\nu_\eta S(\nu_\eta))]} - 1. \quad (17)$$

Note that here for $t_\#$ appearing in (13), we have used $t_\#(x) \leq \kappa_0/\nu(x)$, which is implied by the following inequalities:

$$\kappa_0 \geq f(\zeta_x(t_\#(x))) - f(x) = \int_0^{t_\#(x)} \nabla f(\zeta_x(\tau)) \dot{\zeta}_x(\tau) d\tau = \int_0^{t_\#(x)} \| \nabla f(\zeta_x(\tau)) \| d\tau \geq \nu(x) t_\#(x).$$

Because $R(\varepsilon, \nu_\eta) + \kappa_2 \varepsilon \propto 0$ as $\varepsilon \propto 0$, we indeed have (17) for $\varepsilon$ small enough, and thus have shown that when $\varepsilon$ is small enough, for any starting point $x_0 \in \Omega_\eta$, we always have the correct clustering result using the prototype algorithm. \hfill $\Box$

### 3.2 Max Shift and Max Slope Shift

In this subsection, we show that Max Shift (3) and a related approach proposed early on by Koontz, Narendra, and Fukunaga [30] which we call Max Slope Shift (22), are consistent by showing that they both satisfy the three properties listed in Section 3.1.

#### 3.2.1 Max Shift

Max Shift — introduced in (3) — is arguably the simplest, and thus most prototypical, hill-climbing clustering algorithm. We show here that it satisfies the properties required of the prototypical algorithm of Section 3.1, establishing its consistency as the neighborhood size tends to zero ($\varepsilon \to 0$). We do so in a series of lemmas.

**Lemma 3.1.** Take any point $x_0$ in the population. If $\varepsilon$ is smaller than the minimum separation between $C(x_0)$ and any other level cluster at the same level, Max Shift initialized at $x_0$ converges (in a finite number of steps) to a mode belonging to $C(x_0)$.

It could be the case that $f(x_0) = 0$, but because $x_0 \in \text{supp} f$, it must be that for any $\varepsilon > 0$ there is $x \in B(x_0, \varepsilon)$ such that $f(x) > 0$. Therefore, the sequence does not stay at $x_0$, and even though $C(x_0)$ is, in this case, a connected component of $\text{supp} f$, the lemma does say that the sequence converges to some density mode.

**Proof.** If $x_0$ is a mode, then by choosing $\varepsilon$ small enough that it is a maximum inside $B(x_0, \varepsilon)$, the sequence immediately ends at $x_\infty = x_0$. Therefore, in the remaining of the proof, we consider a point $x_0$ which is not a mode.
Since \( \varepsilon \) is smaller than the minimum separation between \( C(x_0) \) and any other \( f(x_0) \)-level cluster, Max Shift initialized at \( x_0 \) outputs a sequence that must remain in \( C(x_0) \). Otherwise, there must be \( k \) such that \( x_k \equiv x(k) \in C(x_0) \) and \( x_{k+1} \notin C(x_0) \), and because \( \|x_{k+1} - x_k\| \leq \varepsilon \) and the separation between \( C(x_0) \) and any other connected component of \( U_f(x_0) \) exceeds \( \varepsilon \), it must be the case that \( x_{k+1} \notin U_f(x_0) \), triggering \( f(x_{k+1}) < f(x_0) \leq f(x_k) \), contradicting the rules governing the algorithm which makes it hill-climbing.

We now show that the sequence converges. We just saw that the sequence is inside \( C(x_0) \), which is compact. Therefore, the sequence has at least one accumulation point inside \( C(x_0) \). Let \( x_\infty \) be such a point. By the fact that the sequence of density values \( f(x_k) \) is increasing, it must be the case that \( f(x_\infty) \geq f(x_k) \) for all \( k \), and not just the \( k \)'s indexing the subsequence converging to \( x_\infty \).

Take \( k_1 \) such that \( x_{k_1} \) is within distance \( \varepsilon \) from \( x_\infty \). If \( x_{k_1+1} \neq x_\infty \), it must mean that \( f(x_{k_1+1}) \geq f(x_{k_1}) \), which then implies that \( f(x_{k_1+1}) \geq f(x_k) \) for all \( k \), itself implying that the sequence stops at, and thus converges to, \( x_{k_1+1} \).

We have thus established that the sequence converges to some point, say \( x_\infty \), inside \( C(x_0) \). And that the convergence happens in a finite number of steps: as soon as \( \|x_k - x_\infty\| \leq \varepsilon \), it must be that \( x_{k+1} = x_\infty \). This in turn implies that \( x_\infty \) is a mode within \( B(x_\infty, \varepsilon) \) — although perhaps not the only one. Indeed, let \( k_\infty \) denote the step at which the sequence stops at \( x_\infty \). By the rules governing the algorithm, it stops there because there are no points within that ball with a strictly higher density value.

\[ \square \]

**Lemma 3.2.** Let \( (x_k) \) denote the Max Shift sequence originating from some arbitrary point \( x_0 \) with \( t_0 := f(x_0) > 0 \). Suppose that \( \varepsilon \) is small enough that Lemma 3.1 applies, and small enough that any mode in \( C(x_0) \) is a maximum within a radius of \( \varepsilon \). Then, at each step \( k \), the shift \( x_{k+1} - x_k \) is of size exactly \( \varepsilon \), except possibly for the very last shift. In particular, Max Shift satisfies Property 1.

**Proof.** Suppose that, for some \( k \), \( \varepsilon_{k+1} := \|x_{k+1} - x_k\| < \varepsilon \). By how Max Shift constructs the sequence, this implies that \( x_{k+1} \) is a maximum in \( B(x_k, \varepsilon) \), and in particular a maximum in \( B(x_{k+1}, \varepsilon - \varepsilon_{k+1}) \), and therefore a mode. Because Lemma 3.1 applies, \( x_{k+1} \in C(x_0) \), and so the sequence must terminate at \( x_{k+1} \), because, by assumption, \( x_{k+1} \) is maximum in \( B(x_{k+1}, \varepsilon) \).

\[ \square \]

**Lemma 3.3.** Suppose that \( x_* \) is a mode. Then there is \( \delta > 0 \) such that, for \( \varepsilon > 0 \) small enough, Max Shift initialized at any point in \( B(x_*, \delta) \) converges to \( x_* \). In particular, Max Shift satisfies Property 2.

**Proof.** Our assumptions on the density imply that the critical points are isolated. Therefore, there is \( \delta_1 > 0 \) such that \( x_* \) is the only critical point in \( B(x_*, \delta_1) \). Let \( s_1 := \max \{ f(x) : x \in \partial B(x_*, \delta_1) \} \) by construction, \( s_1 < s_* := f(x_*) \) and \( C_{s_1}(x_*) \subset B(x_*, \delta_1) \). By Lemma 2.5, there is \( \delta \leq \delta_1 \) such that \( B(x_*, \delta) \subset C_{s_1}(x_*) \). Suppose \( \varepsilon > 0 \) is smaller than the separation between \( C_{s_1}(x_*) \) and any other \( s_1 \)-level cluster.

Now take a starting point \( x_0 \in B(x_*, \delta) \), and let \( s_0 := f(x_0) \). Since \( x_0 \in C_{s_1}(x_*) \), we have \( t_0 \geq s_1 \), and therefore we must have \( C(x_0) \subset C_{s_1}(x_*) \). Note that the separation between \( C(x_0) \) and any other \( s_0 \)-level cluster must exceed \( \varepsilon \). This is because any \( s_0 \)-level cluster must be inside a \( s_1 \)-level cluster, and \( C_{s_1}(x_*) \) cannot contain more than one \( s_0 \)-level cluster because of Lemma 2.4 and the fact that it only contains one mode. We are thus able to apply Lemma 3.1 to assert that the sequence converges to a local mode within \( C_{s_1}(x_*) \), which must be \( x_* \) since, again, \( C_{s_1}(x_*) \) does not contain any other mode by construction.

\[ \square \]

**Lemma 3.4.** Let \( (x_k) \) denote the Max Shift sequence originating from some arbitrary point \( x_0 \) in the population. Assume that \( \varepsilon \) is small enough that Lemma 3.2 applies. Then for each shift, except
perhaps for the last one,

\[ x_{k+1} - x_k = \varepsilon N(x_k) \pm \frac{C_0 \varepsilon^{3/2}}{\|\nabla f(x_k)\|^{1/2}}, \tag{18} \]

whenever \( \nabla f(x_k) \neq 0 \). In particular, Max Shift satisfies Property 3.

Remark 3.5. We note that if three times continuous differentiability of \( f \) is assumed, the \( \varepsilon^{3/2} \) in the remainder of (18) can be enhanced to \( \varepsilon^2 \) by performing higher order Taylor expansions in (6) and (19).

Proof. Let \( u_{k+1} := x_{k+1} - x_k \) and \( \varepsilon_{k+1} := \|u_{k+1}\| \). The result comes from comparing \( f(x_{k+1}) \), which by construction maximizes \( f(x) \) over \( x \in B(x_k, \varepsilon) \), with \( f(x'_{k+1}) \) where \( x'_{k+1} := x_k + \varepsilon_{k+1} N(x_k) \). Since by construction we must have \( \varepsilon_{k+1} \leq \varepsilon \), we have \( x'_{k+1} \in B(x_k, \varepsilon) \), triggering \( f(x_{k+1}) \geq f(x'_{k+1}) \).

In general, using (6), we have

\[ f(x + u) - f(x + \|u\|N(x)) = \nabla f(x)^\top u - \|u\|\|\nabla f(x)\| \pm \kappa_2 \|u\|^2, \tag{19} \]

so that

\[ f(x + u) - f(x + \|u\|N(x)) < 0 \quad \text{if} \quad \frac{\nabla f(x)^\top u}{\|\nabla f(x)\| \|u\|} < 1 - \frac{\kappa_2 \|u\|}{\|\nabla f(x)\|}. \]

Applying this to \( x_k, x_{k+1}, u_{k+1} \) as defined by the Max Shift, we obtain

\[ N(x_k)^\top u_{k+1} \geq (1 - \kappa_2 \varepsilon/\|\nabla f(x_k)\|) \varepsilon_{k+1}. \tag{20} \]

First, suppose that \( \varepsilon/\|\nabla f(x_k)\| > 1/\kappa_2 \). Then, taking \( C_0 \) large enough that \( C_0/\kappa_2^{1/2} \geq 2 \), we have

\[ \|x_{k+1} - x_k\| - \varepsilon N(x_k) = \pm 2 \varepsilon \quad \text{and} \quad C_0 \varepsilon^{3/2}/\|\nabla f(x_k)\|^{1/2} > \varepsilon C_0/\kappa_2^{1/2} \geq 2 \varepsilon, \]

so that (18) holds. Next, assume that \( \varepsilon/\|\nabla f(x_k)\| \leq 1/\kappa_2 \). Set \( u_{k+1} = aN(x_k) + v_{k+1} \) where \( a := N(x_k)^\top u_{k+1} \) and \( v_{k+1} \perp N(x_k) \). Using Lemma 3.2 and (20), we get that, except possibly for the last shift, \( a \geq (1 - \kappa_2 \varepsilon/\|\nabla f(x_k)\|) \varepsilon \geq 0 \). This and Pythagoras gives

\[ \|v_{k+1}\|^2 = \varepsilon_{k+1}^2 - a^2 \leq \varepsilon^2 - (1 - \kappa_2 \varepsilon/\|\nabla f(x_k)\|)^2 \varepsilon^2 \leq 2 \kappa_2 \varepsilon^3/\|\nabla f(x_k)\|, \tag{21} \]

so that

\[ u_{k+1} = \varepsilon N(x_k) \pm \kappa_2 \varepsilon^2/\|\nabla f(x_k)\| \pm (2 \kappa_2 \varepsilon^3/\|\nabla f(x_k)\|)^{1/2}, \]

from which we obtain (18) after some simplification. \( \square \)

Because it satisfies all three properties, by Theorem 3.1,

*Max Shift is consistent.*
3.2.2 Max Slope Shift

Soon after the original paper of Fukunaga and Hostetler [18], Koontz, Narendra, and Fukunaga [30] proposed a variant where, at each step, a point is moved to the point within a neighborhood window that results in the largest slope. This definition implies that a medoid algorithm which we give in (36). We consider a regularized version of this algorithm that disallows shifts shorter than some set fraction of the neighborhood size. This regularization enables us to consider a continuous (medoid-less) formulation that parallels Max Shift (3) quite closely, taking the following form

\[
x(0) = x_0; \quad x(k + 1) \in \begin{cases} 
\text{local maxima in } B(x(k), \varepsilon) \text{ if non-empty; otherwise} \\
\arg \max_{x \in A(x(k), x, \varepsilon)} \frac{f(x) - f(x(k))}{\|x - x(k)\|}, \quad k \geq 0,
\end{cases}
\]

where \( A(x, a, b) := \bar{B}(x, a) \setminus B(x, b) \). The constant \( 0 < c < 1 \) is arbitrary but fixed beforehand. We refer to this algorithm as Max Slope Shift.

Remark 3.6. It turns out that, without such a regularization, the algorithm fails. Indeed, consider the simplest setting of a unimodal density, for example, the standard normal distribution \( f(x) = \exp(-x^2/2)/\sqrt{2\pi} \). On the negative half-line, \( (-\infty, 0) \), the slope is maximum at the (unique) inflection point occurring at \( x = -1 \). It is not hard to see that, assuming that \( \varepsilon < 1 \), initialized at \( x_0 \leq -1 \), Max Slope Shift produces a sequence that converges (in a finite number of steps) to that inflection point.

We show that Max Slope Shift satisfies the three properties, and the arguments are almost identical to those just detailed for Max Shift.

For Property 1, it is satisfied by definition since all the shifts, except possibly for the very last one, are of size in between \( c \varepsilon \) and \( \varepsilon \).

For Property 2, we can see that Lemma 3.1 applies to Max Slope Shift, simply because in the algorithm: (i) the steps are of size at most \( \varepsilon \); (ii) it is hill-climbing; (iii) the process stops at the next step when there is a mode within distance \( \varepsilon \); (iv) if the process stops, it must stop at a mode. Having verified that Lemma 3.1 applies, Lemma 3.3 follows immediately, so that Max Slope Shift satisfies Property 2.

For Property 3, we retrace the arguments underlying Lemma 3.4. First, (20) holds in exactly the same way and for exactly the same reasons. As detailed below that, we may focus on the situation where \( \varepsilon/\|\nabla f(x_k)\| \leq 1/\kappa_2 \). We set \( u_{k+1} := aN(x_k) + v_{k+1} \) as before, although this time \( a \geq (1 - \kappa_2 \varepsilon/\|\nabla f(x_k)\|) \varepsilon_{k+1} \geq 0 \), which then gives

\[
\|v_{k+1}\|^2 = \varepsilon_{k+1}^2 - a^2 \leq \varepsilon_{k+1}^2 - (1 - \kappa_2 \varepsilon/\|\nabla f(x_k)\|)^2 \varepsilon_{k+1}^2 \leq 2\kappa_2 \varepsilon^3/\|\nabla f(x_k)\|.
\]

The last inequality being identical to the last inequality in (21), the remaining arguments apply verbatim, allowing us to conclude that Max Slope Shift satisfies Property 3.

Because it satisfies all three properties, by Theorem 3.1,

**Max Slope Shift is consistent.**

3.3 Euler Shift and Line Search Shift

We now consider Euler Shift (2) and its close variant, Line Search Shift introduced below in (27). We follow the blueprint that we detailed for the prototype algorithm of Section 3.1.
3.3.1 Euler Shift

Euler Shift has already been shown to be consistent. This was most definitely done in\(^3\) [1], following the same general proof architecture, which itself is well-known in the study of the Euler method, even at the level of textbooks as exemplified by the proof of [25, Th 1.1]. For completeness, we provide some details nonetheless.

For Property 1, we have

\[ \rho \| \nabla f(x_k) \| = \| x_{k+1} - x_k \| \leq \rho \kappa_1, \]

so that it is satisfied with \( \varepsilon := \rho \kappa_1. \)

**Lemma 3.7.** For \( \rho \) small enough, Euler Shift is hill-climbing, and when initialized inside a level cluster \( C \), Euler Shift converges to a critical point inside \( C \). In particular, Euler Shift satisfies Property 2.

**Proof.** Using (6), for any \( x \in \mathbb{R}^d \) and any \( s > 0 \), we get

\[
\begin{align*}
f(x + s \nabla f(x)) - f(x) &= \nabla f(x)^T (s \nabla f(x)) \pm \frac{1}{2} \kappa_2 \| s \nabla f(x) \| ^2 \\
&\geq (1 - \frac{\kappa_2}{2} s) \| \nabla f(x) \| ^2.
\end{align*}
\]

Therefore, if \( \rho > 0 \) in (2) is small enough that \( \rho < 2 / \kappa_2 \), Euler Shift is hill-climbing. Henceforth, we assume that \( 0 < \rho \leq 1 / \kappa_2 \), so that

\[
f(x_{k+1}) - f(x_k) \geq \frac{1}{2} \rho \| \nabla f(x_k) \| ^2.
\]

Because it is hill-climbing, any Euler Shift sequence \( (x_k) \) such that \( x_0 \in C \) must remain in \( C \). It must also have the property that \( (f(x_k)) \) converges — since it is increasing and \( f \) is assumed bounded — and in view of (2) this implies that \( \nabla f(x_k) \to 0 \). Hence, by continuity of the gradient, any accumulation point of \( (x_k) \) must be a critical point. Furthermore, the shift size converges to zero since \( \| x_{k+1} - x_k \| = \rho \| \nabla f(x_k) \| \), and because the critical points are assumed to be isolated, it must be the case (by elementary considerations) that \( (x_k) \) is convergent. And by what was said earlier, the limit must be a critical point inside \( C \).

That the algorithm satisfies Property 2 follows exactly as for Max Shift, as the same arguments given in the proof of Lemma 3.3 apply. \( \square \)

For Property 3, in view of (2), any Euler Shift sequence satisfies

\[
x_{k+1} - x_k = \| x_{k+1} - x_k \| N(x_k),
\]

simply because the shifts are exactly aligned with the gradients at the corresponding locations.

Because it satisfies all three properties, by Theorem 3.1,

*Euler Shift is consistent.*

---

\(^3\) An errata was issued shortly after the paper was published. Although the error was relatively minor, it was historically important as the same mistake had been made before in other works claiming to have established consistency, including [15]. This was pointed out by others working in the field [21, 31].
**Variants** Euler Shift refers to the forward Euler discretization of the gradient flow of \( f \). In their original proposal [18] advocated for using, instead, the gradient flow of \( \log f \), leading to the algorithm

\[
x(0) = x; \quad x(k + 1) = x(k) + \rho \frac{\nabla f(x(k))}{f(x(k))}, \quad k \geq 0.
\]

The same proof blueprint applies to this variant, and others like it. Specifically, consider

\[
x(0) = x; \quad x(k + 1) = x(k) + \rho \varphi(f(x(k))) \nabla f(x(k)), \quad k \geq 0,
\]

where \( \varphi \) is a non-increasing and positive function on \((0, \infty)\). Clearly, with \( \varphi(a) \propto 1/a \) we recover (24). Showing that the algorithm in (25) is consistent boils down to showing that it is hill-climbing when \( \rho \) is small enough. Using (6), as before, for any \( x \in \mathbb{R}^d \) and any \( s > 0 \), we get

\[
f(x + \rho \varphi(f(x)) \nabla f(x)) - f(x) = \rho \varphi(f(x)) \| \nabla f(x) \|^2 + \frac{1}{2} \kappa_2 \rho^2 \varphi(f(x))^2 \| \nabla f(x) \|^2 \\
\geq (1 - \frac{1}{2} \kappa_2 \rho \varphi(f(x))) \rho \varphi(f(x)) \| \nabla f(x) \|^2 \\
\geq 0, \quad \text{if } \frac{1}{2} \kappa_2 \rho \varphi(f(x)) \leq 1.
\]

Now, initialize the algorithm at some point \( x_0 \) with \( f(x_0) > 0 \), and let \( (x_k) \) denote the sequence that results. Take \( \rho > 0 \) small enough that \( \frac{1}{2} \kappa_2 \rho \varphi(f(x_0)) \leq 1 \). From (26), we have that \( f(x_1) \geq f(x_0) \). Suppose for induction that \( (f(x_m)) : m = 1, \ldots, k \) is non-decreasing. Because \( \varphi \) is non-increasing and the induction hypothesis implies that \( f(x_k) \geq f(x_0) \), we have \( \frac{1}{2} \kappa_2 \rho \varphi(f(x_k)) \leq \frac{1}{2} \kappa_2 \rho \varphi(f(x_0)) \leq 1 \), so that \( f(x_{k+1}) \geq f(x_k) \) by (26). Therefore, the induction proceeds, establishing that the algorithm is hill-climbing.

Other variants are possible. For example, in some previous work [2, 3] we found it useful to work with the following one

\[
x(0) = x; \quad x(k + 1) = x(k) + \rho \frac{\nabla f(x(k))}{\| \nabla f(x(k)) \|^2}, \quad k \geq 0.
\]

This is a discretization of the flow

\[
x(0) = x; \quad \dot{x}(t) = \frac{\nabla f(x(t))}{\| \nabla f(x(t)) \|^2},
\]

which has the interesting property that \( f(x(t)) = t \) for all (applicable) values of \( t \). (The square in the denominator is crucial, as without it we simply have the unit-speed flow given in (8).) Proving the consistency of such methods can be done by adapting the blueprint. Some adaptation is indeed necessary. Take this particular variant, for example. It is not a priori guaranteed that it is hill-climbing all the way to a mode. However, it is easy to see that it is hill-climbing until the gradient becomes too small. Indeed, using the usual route via (6), we have that \( f(x_{k+1}) \geq f(x_k) \) if \( \frac{1}{2} \kappa_2 \rho \leq \| \nabla f(x_k) \|^2 \). Although this is not enough to show that the algorithm satisfies Property 2, a quick look at the proof of Theorem 3.1 reveals that it is enough. Indeed, we care about what happens while the gradient remains \( \geq \nu \), and the process is hill-climbing in that region as long as \( \frac{1}{2} \kappa_2 \rho \leq \nu^2 \). Property 1 is also questionable, since \( \| x_{k+1} - x_k \| = \rho/\| \nabla f(x_k) \| \), but while the gradient is \( \geq \nu \), we do have \( \rho/\kappa_1 \leq \| x_{k+1} - x_k \| \leq \rho/\nu \), and this is enough.

### 3.3.2 Line Search Shift

In practice, an Euler scheme may not be monotone, and a way to force that is to perform a line search in the direction given by the gradient:

\[
x(0) = x; \quad x(k + 1) = x(k) + \rho_k \nabla f(x(k)), \quad \rho_k \in \text{arg} \max_{r \in [0, \rho]} f(x(k) + r \nabla f(x(k))), \quad k \geq 0.
\]

(27)
We call this algorithm Line Search Shift, and we prove below that it is consistent as well. The algorithm appears to be new, although the possibility of implementing a line search is briefly discussed in [8].

For Property 1, using (6), for any \( x \in \mathbb{R}^d \) and any \( r \in [0, \rho] \), we have

\[
    f(x + \rho \nabla f(x)) - f(x + r \nabla f(x)) = (\rho - r)\|\nabla f(x)\|^2 + \frac{1}{2} \kappa_2 (\rho^2 + r^2) \|\nabla f(x)\|^2 \\
    \geq (\rho - r - \kappa_2 \rho^2)\|\nabla f(x)\|^2.
\]

Therefore, if \( \rho > 0 \) in (2) is small enough that \( \kappa_2 \rho \leq 1/2 \), the last expression is strictly positive when \( r < \rho/2 \) and \( \nabla f(x) \neq 0 \). Hence, assuming \( \rho \) is that small, in the process of running the algorithm, until reaching a critical point (at which point the sequence has converged) \( \rho_k \) in (27) must satisfy \( \rho_k \in [\rho/2, \rho] \), which then implies that

\[
    \frac{1}{2} \rho \|\nabla f(x_k)\| \leq \rho_k \|\nabla f(x_k)\| = \|x_{k+1} - x_k\| \leq \rho \kappa_1.
\]

Hence, the algorithm satisfies Property 1.

Line Search Shift is hill-climbing by construction, and in view of (28), the arguments underlying Lemma 3.7 apply almost verbatim to show that the algorithm satisfies Property 2.

And the algorithm satisfies Property 3, in fact, (23) applies, for the same reason that the shifts are exactly aligned with the gradient directions.

Because it satisfies all three properties, by Theorem 3.1,

Line Search Shift is consistent.

### 3.4 Mean Shift

Given a kernel with bandwidth \( h > 0 \), denoted \( K_h(\cdot) := K(\frac{\cdot}{h}) \), define the mean shift at a point \( x \) as follows

\[
    \text{MS}_h(x) := \frac{\int K_h(y-x)(y-x)f(y)dy}{\int K_h(y-x)f(y)dy}.
\]

The Mean Shift algorithm is then defined as constructing the following sequence when initialized at some point \( x \):

\[
    x(0) = x; \quad x(k+1) = x(k) + \text{MS}_h(x(k)), \quad k \geq 0.
\]

We establish its consistency following, again, the blueprint detailed for the prototype algorithm of Section 3.1. Some elements of consistency for Mean Shift appear in the work of Cheng [13] and Comaniciu and Meer [14, 15], and a few others reviewed in [8]. As far as we are aware of, the consistency of Mean Shift per se is established here for the first time.

We start with the following result, which is a continuous version of [13, Th 1].

**Lemma 3.8.** Suppose that \( K(x) = K(\|x\|^2) \) for some nonnegative, nondecreasing, integrable function \( K \). Define \( L(u) := c \int_u^\infty K(v)dv \) and \( L(x) := L(\|x\|^2) \). Then

\[
    \text{MS}_h(x) = \frac{h^2 \nabla(L_h * f)(x)}{2c \ K_h * f(x)}.
\]

**Proof.** Assume without loss of generality that \( c = 1 \). Notice that

\[
    \nabla L_h(x) = \nabla \left[ h^{-d} \int_{|x/h|^2}^\infty K(v)dv \right] = -\frac{2}{h^2} h^{-d} K(\|x/h\|^2) x = -\frac{2}{h^2} K_h(x) x.
\]

\[
    \nabla L_h(x) = \nabla \left[ h^{-d} \int_{|x/h|^2}^\infty K(v)dv \right] = -\frac{2}{h^2} h^{-d} K(\|x/h\|^2) x = -\frac{2}{h^2} K_h(x) x.
\]

\[
    \nabla L_h(x) = \nabla \left[ h^{-d} \int_{|x/h|^2}^\infty K(v)dv \right] = -\frac{2}{h^2} h^{-d} K(\|x/h\|^2) x = -\frac{2}{h^2} K_h(x) x.
\]
Using this, we obtain
\[ \nabla (L_h \ast f)(x) = \nabla L_h \ast f(x) = \frac{2}{h^2} \int K_h(y-x)(y-x)f(y)dy. \]

Up to the scaling factor in front, we recognize the numerator in (29). And the denominator is simply \( K_h \ast f(x) \).

If \( K \) decays fast enough at infinity, \( L \) via \( L \) properly normalized is also a kernel, just like \( K \). We assume this is so henceforth. Cheng [13] calls \( L \) a ‘shadow kernel’ of \( K \). Assuming this is the case, we know from the extensive literature on kernel density estimation that, under some standard regularity conditions on \( f \), \( K_h \ast f(x) \rightarrow f(x) \) and \( \nabla (L_h \ast f)(x) = L_h \ast \nabla f(x) \rightarrow \nabla f(x) \) as \( h \rightarrow 0 \), implying that
\[ \frac{\nabla (L_h \ast f)(x)}{K_h \ast f(x)} \rightarrow \frac{\nabla f(x)}{f(x)}. \]

On the right-hand side we recognize the function driving the dynamic system (24), indicating that Mean Shift resembles this other algorithm with \( \rho = h^2/2c \). This is true, but to establish consistency, we take a different route which is arguably closer to the blueprint of Section 3.1, and overall more conceptual than calculatory.

In view of (30), Mean Shift can be described by the following Euler scheme
\[ x(0) = x; \quad x(k+1) = x(k) + \rho_h \frac{\nabla f_h^L(x(k))}{f_h^L(x(k))}, \quad k \geq 0, \]
where \( f_h^L := L_h \ast f \) and \( f_h^K := K_h \ast f \), and \( \rho_h := h^2/2c \). Note that this is a gradient flow of \( f_h^K \) — and not of \( f \) — with varying stepwise that is inversely proportional to \( f_h^K \). The idea is to work with that, showing via the blueprint that the algorithm converges to a mode of \( f_h^L \), and then argue that such a mode is close to a mode of \( f \) when the bandwidth \( h \) is small.

Before we start, we note two things make this different from (25) with \( f_h^K \) in place of \( f \). One is that the step size is not a function of \( f_h^L \) but of \( f_h^K \). The other is that \( f_h^L \) depends on the step size, which is \( \rho_h \). But it turns out that we can work with this.

By our smoothness assumption on the density \( f \), we have that \( f_h^K \) is twice differentiable with \( \nabla f_h^K = L_h \ast \nabla f \) and \( \nabla^2 f_h^K = L_h \ast \nabla^2 f \). In particular, we have \( \sup_x \| \nabla f_h^K(x) \| \leq \sup_x \| \nabla f(x) \| \) and \( \sup_x \| \nabla^2 f_h^K(x) \| \leq \sup_x \| \nabla^2 f(x) \| \), which then implies that (4) and (5), and thus also (6), apply to \( f_h^K \) with the meaning of \( \kappa_1 \) and \( \kappa_2 \) unchanged. In addition, we have the zeroth, first, and second derivatives of \( f_h^K \) converge uniformly to those of \( f \) as \( h \rightarrow 0 \). Let
\[ \eta_h^{L,0} := \sup_x |f_h^L(x) - f(x)|, \quad \eta_h^{L,1} := \sup_x \| \nabla f_h^L(x) - \nabla f(x) \|, \quad \eta_h^{L,2} := \sup_x \| \nabla^2 f_h^L(x) - \nabla^2 f(x) \|, \]
so that
\[ \eta_h^L := \max\{\eta_h^{L,0}, \eta_h^{L,1}, \eta_h^{L,2}\} \rightarrow 0, \quad \text{as } h \rightarrow 0. \]

The same is true of \( f_h^K \), meaning that
\[ \eta_h^K := \max\{\eta_h^{K,0}, \eta_h^{K,1}, \eta_h^{K,2}\} \rightarrow 0, \quad \text{as } h \rightarrow 0, \]
with analogous definitions, although we will only use the zeroth order convergence. In fact, more is true.
Lemma 3.9. For any non-critical level $t$ of $f$, as $h \to 0$, within $\mathcal{U}_t$, the modes of $f_h^t$ and their basins of attraction, as well as its $t$-upper level set, converge to those of $f$.

Proof. The result immediately follows from Lemma 4.1 and Lemma 4.2 given in Section 4. \qed

Lemma 3.10. Take a non-critical level $t > 0$. For $h$ small enough, the following happens. Let $C$ be a level cluster for $f_h^t$. Then, when initialized inside $C$, Mean Shift is hill-climbing for $f_h^t$ and converges to a critical point of $f_h^t$ inside $C$. In particular, Mean Shift satisfies Property 2 with respect to $f_h^t$.

Proof. Assume that $h$ is small enough that $\eta_h^t \leq t/2$, so that for any $x$ in any $t$-level cluster for $f_h^t$, $f(x) \geq f_h^t(x) - \eta_h^t \geq t - t/2 = t/2$. Take $h$ even smaller that $2\eta_h^t + \eta_h^t \leq t/2$ and that $\kappa_2 \rho_h = \kappa_2 h^2 / 2 \leq t/2$. And still smaller that all the $t$-level clusters are separated by more than $\rho_h(\kappa_1 + \eta_h^t)/(t/2)$. The latter is possible because as $h \to 0$ the $t$-level clusters of $f_h^t$ converge to those of $f$ by Lemma 3.9, and the latter are disjoint.

Now that $h$ is fixed, we proceed. Let $C$ denote a $t$-level cluster for $f_h^t$ and initialize the process at some $x_0 \in C$, and let $(x_k)$ denote the resulting sequence. Using a second-order Taylor development for $f_h^t$, we have

$$f_h^t(x_{k+1}) - f_h^t(x_k) = \rho_h \frac{\|\nabla f_h^t(x_k)\|^2}{f_h^t(x_k)} \pm \frac{\kappa_2}{2} \rho_h^2 \frac{\|\nabla f_h^t(x_k)\|^2}{f_h^t(x_k)^2} \geq \frac{1}{2} \rho_h \frac{\|\nabla f_h^t(x_k)\|^2}{f_h^t(x_k)} \text{ when } \kappa_2 \rho_h \leq f_h^t(x_k). \quad (31)$$

We prove by induction that $(f_h^t(x_k))$ is strictly increasing until convergence if it converges in a finite number of steps. Suppose that we have shown that $f_h^t(x_0) \leq \cdots \leq f_h^t(x_k)$, which is certainly true at $k = 0$. Because

$$\kappa_2 \rho_h \leq \frac{t/2}{f(x_0)/2} \leq f(x_0) - 2\eta_h^t - \eta_h^t \leq f_h^t(x_0) - \eta_h^t - \eta_h^t \leq f_h^t(x_k) - \eta_h^t - \eta_h^t \leq f(x_k) - \eta_h^t \leq f_h^t(x_k),$$

the condition in (31) is satisfied, and so the inequality holds, showing that the induction carries on. Therefore, the algorithm is hill-climbing for $f_h^t$.

In the process, we have shown that $f_h^t(x_k) \geq t/2$ for all $k$, implying the following upper bound on the shift size

$$\|x_{k+1} - x_k\| = \rho_h \frac{\|\nabla f_h^t(x_k)\|}{f_h^t(x_k)} \leq \rho_h \frac{\kappa_1 + \eta_h^t}{t/2}.$$

By assumption on $h$, the upper bound is strictly smaller than the separation between the $t$-level clusters of $f_h^t$, implying that the sequence remains in $(x_k)$ because it is hill-climbing for $f_h^t$.

The remaining arguments underlying Lemma 3.7 carry over verbatim to establish the entire statement, although here with reference to $f_h^t$ instead of $f$. \qed

Just like any other Euler scheme, Property 3 with respect to $f_h^t$ is trivially satisfied since

$$x_{k+1} - x_k = \|x_{k+1} - x_k\| \nabla f_h^t(x_k).$$
As for Property 1, it is also satisfied since, in the same context of Lemma 3.10, we have
\[ \rho_h \frac{\| \nabla f_k^h(x_k) \|_{\kappa_0}}{\kappa_0} \leq \| x_{k+1} - x_k \| = \rho_h \frac{\| \nabla f_k^h(x_k) \|}{f_k^h(x_k)} \leq \rho_h \frac{\| \nabla f_k^h(x_k) \|}{t/2}, \]
using the fact that \( \sup_x f_k^h(x) \leq \sup_x f(x) \) and \( f_k^h(x_k) \geq t/2 \) shown above in the induction.

Because it satisfies all three properties, by Theorem 3.1, we may conclude (with some care here) that, given a level \( s > 0 \), for \( h \) small enough, when initialized at any point in \( f_k^h(x_0) \geq s \) that is in the basin of attraction of some mode of \( f_k^h \), Mean Shift converges to that mode.

This is not quite what we want, as we are interested in the modes of \( f \) and their basins of attraction. Here we invoke Lemma 3.9. Take any point \( x_0 \) such that \( f(x_0) > 0 \) and take any level \( s > 0 \) such that \( f(x_0) \geq 2s \). Suppose in addition that \( x_0 \) is in the basin of attraction \( B \) of some mode \( x_* \) of \( f \). By the stability lemma, to each \( h \) small enough we may associate a mode \( x_{h,*}^k \) of \( f_k^h \) with basin of attraction \( B_{h,k}^l \) such that \( x_{h,*}^k \rightarrow x_* \) and \( B_{h,k}^l \rightarrow B \) as \( h \rightarrow 0 \). For \( h \) small enough, we have \( x_0 \in B_{h,k}^l \). For \( h \) even smaller, \( f_k^h(x_0) \geq s \). And for \( h \) still even smaller, the previous statement holds. When \( h \) is that small, Mean Shift initialized at \( x_0 \) converges to \( x_{h,*}^k \). And because \( x_{h,*}^k \rightarrow x_* \) as \( h \rightarrow 0 \), we may conclude that

Mean Shift is consistent.

4 Consistency: Methods

As announced in the Introduction, we call ‘method’ an algorithm applied to an estimate of the density computed based on a sample of points. As is usually the case, we assume that the sample was generated iid from the underlying density we are ultimately interested in. In this context, we establish the large-sample \((n \rightarrow \infty)\) consistency of the methods associated with the various algorithms considered in Section 3.

Continuing with same definition of clustering, we say that a method is consistent if it moves a fraction tending to one of data points to their associated mode when the sample size is sufficiently large — and the bandwidth defining the density estimate is appropriately chosen.

4.1 Some stability results

We first present some important results regarding the stability of the upper level sets, modes and their basins of attraction for density functions. Consider a density function \( \hat{f} \) on \( \mathbb{R}^d \), which is to be understood as a perturbed version of \( f \), and is really a placeholder for a kernel density estimator of \( f \) in practice. Denote \( \hat{U}_s := \{ x : \hat{f}(x) \geq s \} \), which is the upper s-level set of \( \hat{f} \). Let

\[ \eta_0 := \sup_x | \hat{f}(x) - f(x) |, \quad \eta_1 := \sup_x \| \nabla \hat{f}(x) - \nabla f(x) \|, \quad \eta_2 := \sup_x \| \nabla^2 \hat{f}(x) - \nabla^2 f(x) \|. \]

We use \( \mathcal{F}_\eta \) to denote the class of density functions \( \hat{f} \) satisfying \( \max \{ \eta_0, \eta_1, \eta_2 \} \leq \eta \), and the same second condition as \( f \) listed in Section 2.1. For an arbitrary quantity \( g \) that tends to zero with \( \eta \), we write \( g = \omega(\eta) \).

Lemma 4.1. In the present context, the following is true:
(a) For any \( s \in (0, \kappa_0) \) such that \( \mathcal{L}_s \) does not contain a critical point of \( f \):
   (a1) \( \sup \{ \mathcal{M}(\hat{U}_s, U_s) : \hat{f} \in \mathcal{F}_\eta \} = \omega(\eta) \);  
   (a2) for any \( \hat{f} \in \mathcal{F}_\eta \), all the critical points of \( \hat{f} \) in \( U_s \) are non-degenerate, when \( \eta \) is small enough.
(b) For any mode \( x_* \) of \( f \), there is \( \delta_* > 0 \) such that, when \( \eta \) is small enough, any function \( \hat{f} \in \mathcal{F}_\eta \)
has only one critical point (in fact, a mode) in \( B(x_*, \delta_*) \); moreover, if we denote the mode of \( \tilde{f} \) in \( B(x_*, \delta_*) \) by \( x_*[\tilde{f}] \), we have \( \sup\{\|x_* - x_*[\tilde{f}]\| : \tilde{f} \in \mathcal{F}_\eta\} = \omega(\eta) \).

(c) Let \( \mathcal{M} \) denote the set of all the modes of \( f \) and \( \mathcal{M}[\tilde{f}] \) denote the same for an arbitrary function \( \tilde{f} \). For any non-critical level \( s \) of \( f \), we have \( \sup\{d_H(\mathcal{U}_s \cap \mathcal{M}, \mathcal{U}_s \cap \mathcal{M}[\tilde{f}]) : \tilde{f} \in \mathcal{F}_\eta\} = \omega(\eta) \).

We note that similar ‘stability’ results can be found in the literature, e.g., [16, Th 1], [1, Lem 8], [20, Lem 3].

Proof. (a1) Notice that for any \( \tilde{f} \in \mathcal{F}_\eta \) and for \( \eta_0 \) small enough that \( s + 2\eta_0 < \kappa_0 \) and \( s - 2\eta_0 > 0 \),

\[ \mathcal{U}_{s+2\eta_0} \subset \tilde{\mathcal{U}}_{s+\eta_0} \subset \mathcal{U}_s \subset \tilde{\mathcal{U}}_{s-\eta_0} \subset \mathcal{U}_{s-2\eta_0}. \]

Therefore, using the fact that the upper level sets of any function are monotone for the inclusion,

\[
d_H(\mathcal{U}_s, \tilde{\mathcal{U}}_s) = \max \{d_H(\mathcal{U}_s \cap \tilde{\mathcal{U}}_s), d_H(\tilde{\mathcal{U}}_s \cap \mathcal{U}_s)\} \\
\leq \max \{d_H(\mathcal{U}_s \cap \mathcal{U}_{s+2\eta_0}), d_H(\mathcal{U}_{s-2\eta_0} \cap \mathcal{U}_s)\} \\
\leq d_H(\mathcal{U}_{s-2\eta_0}, \mathcal{U}_{s+2\eta_0}) \\
= d_H(\mathcal{L}_{s-2\eta_0}, \mathcal{L}_{s+2\eta_0}),
\]

where the last equality holds because of the monotonicity of the upper level sets as stated above and the fact that \( \mathcal{L}_v \) is the boundary of \( \mathcal{U}_v \) for all \( v \in (0, \kappa_0) \). Suppose that \( \eta_0 \) is small enough that there exists no critical points of \( f \) at any levels anywhere between \( s - 2\eta_0 \) and \( s + 2\eta_0 \), inclusive. This is possible because \( \mathcal{L}_s \) does not contain critical points, and the number of critical values of \( f \) above, say, \( s/2 \), is finite. The upper bound in (32) converges to zero as \( \eta_0 \to 0 \), as we have already shown in recent, related work [3, Th 4.2].

(a2) The result follows immediately from [4, Lem 5.32].

(b) Let \( -\lambda_* < 0 \) be the largest eigenvalue of \( \nabla^2 f(x_*) \). Due to the continuity of the second derivatives of \( f \), there exists \( \delta_* > 0 \) such that the largest eigenvalue of \( \nabla^2 f(x) \) is upper bounded by \( -\frac{1}{2}\lambda_* \) for all \( x \in B(x_*, \delta_*) \). A Taylor expansion gives, for all such \( x \),

\[
f(x_*) - f(x) \geq \frac{1}{4}\lambda_* \|x_* - x\|^2.
\]

Suppose that \( \eta_0 \) is small enough that \( \eta_0 > \frac{1}{8}\lambda_* \delta_*^2 \). For any \( x \in \partial B(x_*, \delta_*) \) and any \( \tilde{f} \in \mathcal{F}_\eta \),

\[
\tilde{f}(x_*) \geq f(x_*) - \eta_0 \geq f(x) + \frac{1}{4}\lambda_* \delta_*^2 - \eta_0 \geq \tilde{f}(x) + \frac{1}{4}\lambda_* \delta_*^2 - 2\eta_0 > \tilde{f}(x).
\]

Hence there must exist \( k \geq 1 \) modes of \( \tilde{f} \) in \( B(x_*, \delta_*) \). Below we show that \( k = 1 \) when \( \eta_2 \) is small enough. Because of the 1-Lipschitz continuity of the largest eigenvalue as a function on the space of symmetric matrices, the largest eigenvalue of \( \nabla^2 \tilde{f}(x) \) is upper bounded by \( -\frac{1}{4}\lambda_* \) for all \( x \in \bar{B}(x_*, \delta_*) \), when \( \eta_2 \leq \frac{1}{4}\lambda_* \). Note that this implies that any critical point of \( \tilde{f} \) in \( \bar{B}(x_*, \delta_*) \) must be one of its modes. Let \( \bar{x}_* \) be a mode of \( \tilde{f} \) in \( B(x_*, \delta_*) \). Then for any \( x \in B(x_*, \delta_*) \), it follows from a Taylor expansion that

\[
\tilde{f}(x) - \tilde{f}(\bar{x}_*) \leq \frac{1}{8}\lambda_* \|x - \bar{x}_*\|^2.
\]

In other words, \( \tilde{f}(x) < \tilde{f}(\bar{x}_*) \), for all \( x \in \bar{B}(x_*, \delta_*) \) different from \( \bar{x}_* \), which implies that \( \bar{x}_* \) is the only mode of \( \tilde{f} \) in \( B(x_*, \delta_*) \). Letting \( x = x_* \) in (33), we obtain

\[
\tilde{f}(x_*) - \tilde{f}(\bar{x}_*) \leq \frac{1}{8}\lambda_* \|x_* - \bar{x}_*\|^2.
\]
And using a similar Taylor expansion for $f$ derived above,

$$f(\tilde{x}_s) - f(x_s) \leq -\frac{1}{4} \lambda_s \|x_s - \tilde{x}_s\|^2.$$  

Combining the above two inequalities yields

$$\|x_s - \tilde{x}_s\|^2 \leq \frac{8}{3\lambda_s} \left(\|f(x_s) - \tilde{f}(x_s)\| + \|\tilde{f}(\tilde{x}_s) - f(\tilde{x}_s)\|\right) \leq \frac{16}{3\lambda_s} \eta_0 =: C_0 \eta_0. \quad (35)$$

Therefore $\sup_{f \in \mathcal{F}_\eta} \|x_s - \tilde{x}_s\| \to 0$ as $\eta \to 0$.

(c) Suppose there exists a critical point $x_\dagger$ of $f$ in $\mathcal{U}_s$, which is not a mode. Then the largest eigenvalue of $\nabla^2 f(x_\dagger)$ must be positive, denoted by $\lambda_\dagger$. Using the continuity of the second derivatives of $f$, there exists $\delta_\dagger > 0$ such that the largest eigenvalue of $\nabla^2 f(x)$ is lower bounded by $\frac{1}{2} \lambda_\dagger$ for all $x \in B(x_\dagger, \delta_\dagger)$. Suppose that $\eta_2 \leq \frac{1}{4} \lambda_\dagger$. Then the largest eigenvalue of $\nabla^2 \tilde{f}(x)$ is lower bounded by $\frac{1}{4} \lambda_\dagger$ for all $x \in \tilde{B}(x_\dagger, \delta_\dagger)$, implying that there is no mode of $\tilde{f}$ in $\tilde{B}(x_\dagger, \delta_\dagger)$.

Since $\mathcal{U}_s$ is a compact set, there are only finitely many critical points of $f$ in $\mathcal{U}_s$. Denote the union of all the balls near the critical points constructed above, including those in (b), by $\mathcal{B}$. Then $\mathcal{U}_s \setminus \mathcal{B}$ is a compact set, and there is no critical point of $f$ in $\mathcal{U}_s \setminus \mathcal{B}$ by construction. Using the continuity of $\|\nabla f\|$, there is a positive lower bound of $\|\nabla f\|$ on $\mathcal{U}_s \setminus \mathcal{B}$, denoted by $c$. Now suppose that $\eta_1 \leq \frac{1}{4c}$. Then $\|\nabla \tilde{f}(x)\| \geq \frac{1}{2} c$ for all $x \in \mathcal{U}_s \setminus \mathcal{B}$, and hence there is no critical point of $\tilde{f}$ in $\mathcal{U}_s \setminus \mathcal{B}$. Therefore, our construction guarantees that there is no mode of $\tilde{f}$ on $\mathcal{U}_s$, except that there is only one mode $\tilde{x}_s$ of $\tilde{f}$ in $B(x_s, \delta_s)$ for each mode $x_s$ of $f$ with its corresponding radius $\delta_s$ as defined above. Using (35) we conclude that $f$ and $\tilde{f}$ have the same number of modes on $\mathcal{U}_s$, and $\sup_{f \in \mathcal{F}_\eta} d_H(\mathcal{U}_s \cap \mathcal{M}, \mathcal{U}_s \cap \mathcal{M}[\tilde{f}]) \to 0$ as $\eta \to 0$. \qed

**Lemma 4.2.** In the present context, the following is true:

(a) Any point $x_0$ in the basin of attraction of $x_\ast$ relative to $f$ is also in the basin of attraction of $x_\ast[\tilde{f}]$ (as given in Lemma 4.1) relative to $\tilde{f}$, for any $\tilde{f} \in \mathcal{F}_\eta, x$ when $\eta$ is small enough.

(b) There is a measurable set $\Theta_\eta$ with probability at least $1 - \omega(\eta)$ such that $\sup\{\|\gamma_\ast(x) - \gamma[\tilde{f}](x)\| : x \in \Theta_\eta, \tilde{f} \in \mathcal{F}_\eta\} = \omega(\eta)$, where $\gamma_\ast$ and $\gamma[\tilde{f}]$ are the gradient flows of $f$ and $\tilde{f}$ starting at $x$, respectively.

**Proof.** (a) Denote $\tilde{N}(x) = \nabla \tilde{f}(x)/\|\nabla \tilde{f}(x)\|$. Consider the flow of $\tilde{N}$:

$$\tilde{\zeta}_x(0) = x; \quad \dot{\tilde{\zeta}}_x(t) = \tilde{N}(\tilde{\zeta}_x(t)).$$

We want to show that for any $x_\ast$ in the basin of attraction of $x_\ast$, $\tilde{\zeta} := \tilde{\zeta}_{x_\ast}$ ends at $\tilde{x}_\ast$, when $\eta$ is small enough. (Here and below, we use $\tilde{x}_\ast$ in place of $x_\ast[\tilde{f}]$ when it is clear what $\tilde{f}$ is.)

Using Lemma 2.5, there exists a constant $C > 0$ such that with $s_\ast := f(x_\ast)$ and $q_\ast := s_\ast - (\delta_\ast/C)^2 > 0$, $C_\ast : C_{q_\ast}(x_\ast)$ is a leaf cluster of $f$ contained in $B(x_\ast, \delta_\ast)$. Let $\tilde{C}_\ast$ be the cluster of $\tilde{f}$ at level $s$ containing $\tilde{x}_\ast$ whenever this is well defined. Denote $\eta_\ast := \frac{1}{4}(s_\ast - q_\ast)$, and define $s_\ast = q_\ast + \eta_\ast, s_\ast = q_\ast + 2\eta_\ast,$ and $s_\ast = q_\ast + 3\eta_\ast$. Correspondingly, denote $C_\ast = C_{s_\ast}(x_\ast), C_\ast = C_{s_\ast}(x_\ast),$ and $\tilde{C}_\ast = \tilde{C}_{s_\ast}(x_\ast)$. Suppose that $\eta_\ast$ is small enough that (34) holds and $\eta_0 < \eta_\ast$, so that

$$s_\ast := \tilde{f}(\tilde{x}_\ast) \geq \tilde{f}(x_\ast) \geq f(x_\ast) - \eta_0 > s_\ast > s_\ast,$$

which means $\tilde{C}_\ast := \tilde{C}_{s_\ast}(\tilde{x}_\ast)$ is well-defined. Using Lemma 2.5 again, $B(x_\ast, \delta_\ast) \subset C_\ast$, where $\delta_\ast := \sqrt{s_\ast - q_\ast}/C = \delta_\ast/C^2$. We require $\eta_0 \leq \delta_\ast/C_0$ so that $\tilde{x}_\ast \in B(x_\ast, \delta_\ast)$ by (35), implying in turn that $\tilde{x}_\ast \in C_\ast$. Then we must have $\tilde{C}_\ast \subset C_\ast$ for the following reason. Take $y \in \tilde{C}_\ast$. By the fact
that \( \tilde{C}_t \) is connected, there is a path \( p \subset \tilde{C}_t \) connecting \( \tilde{x}_* \) and \( y \). Any point \( x \in p \) satisfies \( f(x) \geq \tilde{f}(x) - \eta_0 \geq \eta_0 \geq \eta_* \), so that \( p \subset U_{\eta_*} \). By the fact that \( p \) intersects \( \mathcal{C}_s \), which is a connected component of \( U_{\eta_*} \), it must be that \( p \subset \mathcal{C}_s \), and in particular, \( y \in \mathcal{C}_s \). Hence \( \tilde{C}_t \subset B(x_*, \delta_s) \) by construction, which also means that \( \tilde{C}_t \) is a leaf cluster of \( \tilde{x}_* \), because \( \tilde{x}_* \) is the only critical point (in fact, mode) in \( B(x_*, \delta_s) \), that is, the gradient flow \( \tilde{\zeta} \) converges to \( \tilde{x}_* \) if initialized at any point \( x \in \tilde{C}_t \). By Lemma 2.5, when \( \eta_2 \) is small enough, we can find \( \delta \leq 2C \) such that \( B(\tilde{x}_*, \delta) \subset \mathcal{C}_t \), where

\[
\tilde{\delta} := \frac{1}{C} \sqrt{s_\mathcal{C} - s_t} > \frac{1}{C} \sqrt{s_\mathcal{D} - s_t} = \frac{1}{C} \sqrt{\frac{1}{2}(\delta_s/C)^2} \geq \frac{\delta_s}{4C^2}.
\]

We require \( \eta_0 \leq \tilde{\delta}^2/C \) so that \( x_* \in B(\tilde{x}_*, \delta) \subset \tilde{C}_t \), by (35). Furthermore, we must have \( \mathcal{C}_t \subset \tilde{C}_t \) for a similar reason as we have argued above for \( \tilde{C}_t \subset \mathcal{C}_s \), because here both \( \mathcal{C}_t \) and \( \tilde{C}_t \) are compact and connected, and for all \( x \in \mathcal{C}_t \), \( \tilde{f}(x) \geq f(x) - \eta_0 \geq s_t - \eta_0 \geq s_t \). Below we show that \( \tilde{\zeta} \) enters \( \mathcal{C}_t \) at some time point, and because \( \mathcal{C}_t \subset \tilde{C}_t \), it must be the case that it ends at \( \tilde{x}_* \). (As everywhere in this proof, this is understood to be true when \( \eta \) is small enough.)

Noticing that \( B(x_*, \delta_s) \subset \mathcal{C}_s \) and using Lemma 2.5, we have \( B(\tilde{x}_*, \delta_s/\sqrt{2}) \subset \mathcal{C}_t \). Let \( t_\# := \inf\{t \geq 0 : \|\zeta(t) - x_*\| \leq \delta_s/\sqrt{2}\} \), and \( t_0 := \inf\{t \geq 0 : \|\zeta(t) - x_*\| \leq \delta_s/(2\sqrt{2})\} \). We reuse the same notation as in the proofs of Theorem 3.1 and Lemma A.1: \( z_\#, \nu, T, \delta_{\text{tube}} \), \( S, \nu \), and \( \delta'_{\text{tube}} \). As shown in the proof of Lemma A.1, \( N \) is \( \kappa \)-Lipschitz on \( \nu \), where \( \kappa = \max\{\kappa_2/\nu + \kappa_2, 8/\delta'_{\text{tube}}\} \). Then the following is an immediate result of [24, Sec 17.5]:

\[
\|\zeta(t) - \tilde{\zeta}(t)\| \leq \frac{\eta_1}{\kappa} \exp(\kappa t) - 1 := \psi(t), \quad \forall t \in [0, t_\#].
\]

Here we require \( \eta_1 \) to be small enough that \( \psi(t_\#) \leq \frac{1}{2} \delta'_{\text{tube}} \), so that \( \tilde{Z}_{t_\#} := \{\tilde{\zeta}(t) : t \in [0, t_\#]\} \subset \nu \). Let \( \tilde{Z}_\# = \tilde{\zeta}(t_\#) \). Then \( \|z_\# - \tilde{z}_\#\| \leq \psi(t_\#) \). We further require \( \eta_1 \) to be small enough that \( \psi(t_\#) \leq \delta_\# := \eta_1/\kappa_1 \). For any \( y \in B(\mathcal{C}_\mathcal{D}, \delta_\#) \), there exists \( x \in \mathcal{C}_\mathcal{D} \) such that \( \|x - y\| \leq \delta_\# \), and hence by (4),

\[
f(y) \geq f(x) - \kappa_1 \delta_\# \geq s_\mathcal{D} - \eta_* = s_t,
\]

which implies \( B(\mathcal{C}_\mathcal{D}, \delta_\#) \subset \mathcal{C}_t \). Since \( z_\# \in \mathcal{C}_\mathcal{D} \) and \( \|z_\# - \tilde{z}_\#\| \leq \delta_\# \), we must have \( \tilde{z}_\# \in \tilde{C}_t \). We may thus conclude that \( \tilde{\zeta} \) ends at \( \tilde{x}_* \).

(b) For any \( \epsilon > 0 \), let \( s_\varepsilon \) be a non-critical level of \( f \) such that the probability measure of \( U_{\mathcal{F}_H}^{\mathcal{F}_H} \) is not larger than \( \epsilon \). By Lemma 4.1, with \( \eta \) small enough for any \( \tilde{f} \in \mathcal{F}_{\eta} \), all the critical points of \( \tilde{f} \) in \( \mathcal{U}_{\mathcal{F}_H} \) are non-degenerate, and \( \tilde{f} \) has the same number of modes as \( f \) in \( \mathcal{U}_{\mathcal{F}_H} \), whose locations match in such a way that their Hausdorff distance is \( \omega(\eta) \). Following the same arguments as in the proof of Theorem 3.2, we can extend the result in (a) to its uniform form, that is, \( \gamma_\mathcal{F}(\infty) \) and \( \tilde{\gamma}_\mathcal{F}(\infty) \) match in the above sense, for any starting point \( x_0 \) in \( \mathcal{U}_{\mathcal{F}_H} \), except for a small tube around the boundary of the basins of attraction of all the modes of \( f \) in \( \mathcal{U}_{\mathcal{F}_H} \), whose probability measure tends to zero as \( \eta \to 0 \). Since \( \epsilon \) is arbitrarily small, we arrive at the conclusion of this theorem. \( \square \)

### 4.2 Consistency of Hill-Climbing Methods

In this subsection, we show the consistency of various hill-climbing methods whose algorithms have been discussed in Section 3. When an i.i.d. sample \( \{x_1, \ldots, x_n\} \) from the density \( f \) is available, we estimate the density by the kernel density estimator (KDE)

\[
\hat{f}_{n,h}(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K_h(x - x_i).
\]
For simplicity, we take \( K(x) = \kappa(\|x\|^2) \) with a shadow kernel \( L(x) = L(\|x\|^2) \), and use \( \hat{f}_{n,h} \) to denote the KDE with the kernel \( L \). While the algorithms use the knowledge of the density and its derivatives, the methods rely on the KDE and its derivatives. The sequences \( \{\hat{x}(k) : k = 0, 1, \ldots, \} \) generated by these methods are defined by replacing \( f \) with \( \hat{f}_{n,h} \) (or \( \hat{f}_{n,h}^L \) for the Mean Shift) in the algorithms given in Section 3. Note that the update in the Mean Shift method can be equivalently written as

\[
\hat{x}(0) = x; \quad \hat{x}(k+1) = \frac{\sum_j K_h(x_j - \hat{x}(k))x_j}{\sum_j K_h(x_j - \hat{x}(k))}, \quad k \geq 0.
\]

Let

\[
\eta_{n,h}^{K,0} := \sup_x |\hat{f}_{n,h}(x) - f(x)|, \quad \eta_{n,h}^{K,1} := \sup_x \|\nabla \hat{f}_{n,h}(x) - \nabla f(x)\|, \quad \eta_{n,h}^{K,2} := \sup_x \|\nabla^2 \hat{f}_{n,h}(x) - \nabla^2 f(x)\|,
\]

and

\[
\eta_{n,h}^K := \max\{\eta_{n,h}^{K,0}, \eta_{n,h}^{K,1}, \eta_{n,h}^{K,2}\}.
\]

Define \( \eta_{n,h}^L \) analogously by replacing \( \kappa \) with \( L \) in the above notation. Denote \( \hat{U}_t = \{x : \hat{f}_{n,h}(x) \geq t\} \).

We first establish the consistency of all the above methods except for Mean Shift.

We assume that \( K : \mathbb{R}^d \to \mathbb{R} \) is nonnegative, twice continuously differentiable, and compactly supported. Note that \( L \) has the same property. The following is a standard uniform consistency result for KDE and its derivatives. See, for example, [1, Lem 2, Lem 3].

**Lemma 4.3.** Suppose that \( h = h_n \) is chosen such that \( h \to 0 \) and \( \log n/(nh_{d+4}) \to 0 \) as \( n \to \infty \). Then for any \( \eta > 0 \), there exists \( n_0 > 0 \) such that for all \( n \geq n_0 \), with probability at least \( 1 - \eta \),

\[
\max(\eta_{n,h}^L, \eta_{n,h}^K) \leq \eta.
\]

We are ready to give the consistency results of the methods considered in this paper. The proofs are straightforward using the results that have been established in this and previous sections, and hence are omitted.

We first consider all the hill-climbing methods except the Mean Shift. We use \( \theta \) to denote the neighborhood size parameter, meaning \( \varepsilon \) or \( \rho \) for the corresponding methods. Using Theorem 3.1, Lemma 4.1, Lemma 4.2, and Lemma 4.3, we immediately have the following result.

**Theorem 4.1.** Suppose the conditions of Lemma 4.3 are satisfied. Consider a mode \( x_* \) of \( f \). For any point \( x_0 \) in the basin of attraction of \( x_* \), the following is true with probability tending to one as \( n \to \infty \): when initialized at \( x_0 \), for \( \theta \) small enough, each of the above hill-climbing methods excluding the Mean Shift produces a sequence that converges to \( x_* \).

The Mean Shift method differs from all the other methods in that its sequence converges to a mode of \( \hat{f}_{n,h}^L \) rather than \( \hat{f}_{n,h} \), and its neighborhood size parameter \( h \) is the bandwidth used in KDE. The following theorem gives the consistency of the Mean Shift method. The proof follows the same arguments as in Section 3.4, where \( \hat{f}_{n,h} \) and \( \hat{f}_{n,h}^L \) play the role of \( f_h^L \) and \( f_h^L \).

**Theorem 4.2.** Suppose the conditions of Lemma 4.3 are satisfied. Consider a mode \( x_* \) of \( f \). For any point \( x_0 \) in the basin of attraction of \( x_* \), the following is true with probability tending to one as \( n \to \infty \): when initialized at \( x_0 \), the Mean Shift method produces a sequence that converges to \( x_* \).

We also have uniform consistency results for all the hill-climbing methods. Again, we first consider all the hill-climbing methods except the Mean Shift. The result is a consequence of Theorem 3.2.
Theorem 4.3. Suppose the conditions of Lemma 4.3 are satisfied. There is a measurable set $\Xi_{\theta,n,h}$ with probability measure tending to one as $\theta \to 0$ and $n \to \infty$, such that starting from any $x \in \Xi_{\theta,n,h}$, the sequence produced by each of the above hill-climbing methods excluding the Mean Shift converges to $\gamma_{x}(\infty)$.

Next, we establish the uniform consistency of the Mean Shift method.

Theorem 4.4. Suppose the conditions of Lemma 4.3 are satisfied. There is a measurable set $\Xi_{n,h}$ with probability measure tending to one as $n \to \infty$, such that starting from any $x \in \Xi_{n,h}$, the sequence produced by the Mean Shift method converges to $\gamma_{x}(\infty)$.

5 Medoid variants

While Euler Shift [18] and Mean Shift [13, 18] were defined as continuous-space algorithms, Max Slope Shift [30] was first introduced as a discrete-space algorithm: Assuming that a density $f$ is provided, and that a (locally finite) set of points $\mathcal{Y}$ is available, a point $x_0$ is moved as follows

$$x(0) = x_0; \quad x(k+1) \in \arg \max_{y \in \mathcal{Y} \cap B(x(k),\varepsilon)} \frac{f(y) - f(x(k))}{\|y - x(k)\|}, \quad k \geq 0.$$  

(36)

Ties are broken in some prescribed way, and if $f(x(k+1)) = f(x(k))$, the process stops. In [30], based on a sample, the density is estimated using a kernel, and the point set is the sample itself. Note that, defined as such, the algorithm is well-defined and, in particular, does not require regularization. Max Shift [10] — even though well-defined as a continuous-space algorithm as we did in (3) — was also first introduced as a discrete-space algorithm: In the same context as (36),

$$x(0) = x_0; \quad x(k+1) \in \arg \max_{y \in \mathcal{Y}} f(y), \quad k \geq 0,$$

(37)

where, again, the density is estimated in some way, and the point set $\mathcal{Y}$ is the sample itself.

We chose to work with the continuous-space versions of Max Shift and Max Slope Shift for simplicity. But it turns out there are good, practical reasons to work with discrete-space versions (beyond the fact that everything is necessarily discrete when implemented on a computer). For example, Sheikh, Khan, and Kanade [35] introduced a discrete-space variant of Mean Shift, which they named Medoid Shift, motivated by their view that “the relationship between the medoid shift algorithm and meanshift algorithm is similar to the relationship between the k-medoids and the k-means algorithms”: Based on a kernel $k$ and a sample $\mathcal{Y}$, a point $x_0$ is moved as follows

$$x(0) = x_0; \quad x(k+1) \in \arg \min_{y \in \mathcal{Y}} \sum_{y_s \in \mathcal{Y}} \|y_s - y\|^2 K(\|y_s - y\|^2/h^2), \quad k \geq 0.$$  

(38)

We adopt this terminology, and from now on refer to a discrete-size version of an algorithm as the medoid version, as in “medoid Max Shift”.

Note that for a medoid algorithm, consistency is also (necessarily) as the medoid set becomes dense enough.

5.1 Advantages

One of the advantages highlighted in [35] is that the algorithm can be easily adapted to the metric setting in which computing a mean may not even make sense — which is also one of the main advantages of k-Medoids over k-Means. Indeed, all it takes is replacing in (38) the Euclidean
metric with the available metric. The same is true of the medoid versions of Max Shift and Max Slope Shift.

There is another advantage that is not discussed in [35], but is really a basic principle in optimization: use a coarse discretization, effectively trading some accuracy for some computational complexity. Take Max Shift, for example. When implemented on a computer, the continuous-space maximization that happens at each stage in (3) needs to be discretized. This is often done by exhaustive search on a grid. (Some form of gradient ascent could be used, but the method would then be near identical to Euler Shift.) To avoid the use of a grid, which becomes quickly impractical in higher dimensions (very quickly in fact, as it is already impractical at $d \geq 5$), we can focus on the sample and simply use the medoid version of Max Shift. But if the sample is very large, some subsampling may help speedup the computations, perhaps substantially.

In the literature on the mean shift algorithm, a closely related idea appears in work of Jang and Jiang [26], would propose MeanShift++, a variant of Blurring Mean Shift where the points are binned using a regular partition of the space and means are taken over adjacent bins. This variant is shown in numerical experiments to be much faster. The authors state that MeanShift++ “runs in $O(n3^d)$ per iteration vs $O(n^2d)$ for [Blurring] MeanShift”, where $n$ denotes the sample size.

We claim that consistent clustering can be achieved with a computational cost of $O(nd)$ by using a relatively small subsample as medoid set. We focus on Max Shift, as it is the simplest algorithm among those studied in Section 3.

5.2 Medoid Max Shift

Consider therefore Max Shift in its medoid form (37). We assume for now that a density $f$, satisfying the usual assumptions listed in Section 2.1, is available. We assume that the medoid set $\mathcal{Y}$ is finite (or at least locally finite). Although the starting point can be any point, we start by looking at how a medoid point is moved by the algorithm. Note that the first point computed by the algorithm is necessarily a medoid point (assuming there is a medoid point within $\varepsilon$ of the starting point).

We adapt the arguments given to establish the consistency of Max Shift in Section 3.2.1, and our main tool to do so is simply the triangle inequality.

Define, for a level $s > 0$,

$$\alpha_s := \sup_{x \in \mathcal{U}_s} \inf_{y \in \mathcal{Y}} \|x - y\|.$$ 

Thus $\alpha_s$ quantifies how dense the medoid set is in the upper $s$-level set. Note that $s \mapsto \alpha_s$ is non-increasing.

**Lemma 5.1.** Take a level $s > 0$ and consider any 2s-level cluster $\mathcal{C}$. Then medoid Max Shift initialized at any point in $\mathcal{C}$ converges (in a finite number of steps) to a (medoid) point in $\mathcal{C}$ where the gradient has norm bounded by $C(\alpha_s/\varepsilon + \varepsilon)$.

**Proof.** Let $\alpha$ be short for $\alpha_s$. Note that $s$ is fixed, and when we assume that $\alpha$ is small enough, it is simply a condition on $\mathcal{Y}$ being dense enough in $\mathcal{U}_s$. Because the gradient is globally bounded, it is enough to show the result for $\alpha/\varepsilon$ and $\varepsilon$ small enough. We start by assuming that $\alpha \leq \varepsilon/2$ and that $\varepsilon$ is smaller than the minimum separation between $\mathcal{C}$ and any other $s$-level cluster. Then, initialized at an arbitrary point $x_0$ in $\mathcal{C}$, medoid Max Shift being hill-climbing, necessarily the sequence of medoids it computes must remain in $\mathcal{C}$. Let that sequence be denoted $(y_k)$.

It is also the case that the sequence of density values $(f(y_k))$ is strictly increasing until convergence, if it is the case that the sequence converges. But because there are finitely many medoids
in $\mathcal{C}$, the sequence must converge in finitely many steps, say $K$, and by construction, the endpoint $y_K$ must satisfy $f(y_K) \geq f(y)$ for all $y \in \mathcal{Y} \cap B(y_K, \varepsilon)$. We claim that
\[ \|\nabla f(y_K)\| \leq C_1(\alpha/\varepsilon + \varepsilon). \] (39)
Again, we only need to prove this for $\varepsilon$ small enough, and we assume that it satisfies $\varepsilon \leq s/\kappa_1$. Let $y$ be a medoid closest to $z := y_K + (\varepsilon - \alpha)N(y_K)$. By (4),
\[ f(z) \geq f(y_K) - \kappa_1\|z - y_K\| \geq 2s - \kappa_1\varepsilon \geq s, \] (40)
since $y_K \in \mathcal{C}$, $\|z - y_K\| \leq \varepsilon$, and our assumption on $\varepsilon$. Therefore, $\|y - z\| \leq \alpha$ by definition of $\alpha$. Then, by the triangle inequality, $\|y - y_K\| \leq \varepsilon - \alpha + \alpha \leq \varepsilon$, so that $f(y_K) \geq f(y)$. But using (5), (6), and the definition of $\kappa_1$, in that order, we derive
\[
0 \geq f(y) - f(y_K) \geq f(z) - f(y_K) - \kappa_1\|y - z\|
\geq \nabla f(y_K)^\top(z - y_K) - \frac{1}{2}\kappa_2\|z - y_K\|^2 - \kappa_1\alpha
\geq (\varepsilon - \alpha)\|\nabla f(y_K)\| - \frac{1}{2}\kappa_2\varepsilon^2 - \kappa_1\alpha
\geq \varepsilon\|\nabla f(y_K)\| - \kappa_1\alpha - \frac{1}{2}\kappa_2\varepsilon^2 - \kappa_1\alpha,
\]
from which we get that $\|\nabla f(y_K)\| \leq 2\kappa_1\alpha/\varepsilon + \frac{1}{2}\kappa_2\varepsilon$, confirming (39).

Below, we deviate from Property 1, but leave to the reader to either establish that property, or simply adapt the arguments in the proof of Theorem 3.1, which is easily done. Indeed, even a cursory look at the proof of that theorem reveals that all that is required is that a nontrivial fraction of the shifts are of size comparable to that of the largest shift, a fact which is then used to bound the total number of shifts from above — see (16) and its surroundings.

**Lemma 5.2.** In two consecutive shifts in any sequence produced by medoid Max Shift, at least one is size between $\varepsilon/2$ and $\varepsilon$, except possibly for the very last shift.

**Proof.** Suppose that, for some $k$, $\|y_{k+1} - y_k\| \leq \varepsilon/2$. If the sequence does not end at $y_{k+1}$ but continues, necessarily $y_{k+2}$ needs to be outside $B(y_k, \varepsilon)$ because $f(y_{k+2}) > f(y_{k+1})$ and $y_{k+1}$ is a maximum among medoids in $B(y_k, \varepsilon)$. And this forces $\|y_{k+2} - y_{k+1}\| > \varepsilon/2$, by the triangle inequality. \qed

**Lemma 5.3.** Suppose that $x_+$ is a mode and take any level $s$ such that $f(x_+) \geq 2s$. Then there is $\delta > 0$ such that medoid Max Shift initialized at any point in $B(x_+, \delta)$ converges to a (medoid) point within distance $C(\alpha_s/\varepsilon + \varepsilon)$ of $x_+$.

Thus, medoid Max Shift satisfies a weaker version of Property 2, where the convergence is not to a mode, but to a medoid point not too far from a mode.

**Proof.** By our assumptions on $f$ in Section 2.1, there exist $\lambda > 0$ and $\delta_1 > 0$ such that for all $x \in B(x_+, \delta_1)$, all the eigenvalues of $\nabla^2 f(x)$ are bounded from above by $-\lambda$. Using the fact that
\[
\nabla f(x) = \nabla f(x) - \nabla f(x_+)
= \int_0^1 \nabla^2 f(ux + (1 - u)x_+) (x - x_+) du,
\]
we then deduce, whenever $\|x - x_+\| \leq \delta_1$, that
\[ \|\nabla f(x)\| \geq \lambda\|x - x_+\|. \] (41)
Note that this implies that there are no other critical points, and therefore no other modes, inside $B(x_\ast, \delta_1)$.

Let $s_1 = \max \{ f(x) : x \in \partial B(x_\ast, \delta_1) \}$ and take $\delta_1$ even smaller if needed to have $s_1 \geq s$, which is possible since, by construction, $s_1 < s_\ast := f(x_\ast)$, and $s_1$ approaches $f(x_\ast)$ as $\delta_1$ approaches 0. Note that $C_{s_1}(x_\ast) \subset B(x_\ast, \delta_1)$; and by Lemma 2.5, there is $\delta_2 \leq \delta_1$ such that $B(x_\ast, \delta_2) \subset C_{s_1}(x_\ast)$.

Now take a starting point $x_0 \in B(x_\ast, \delta_2)$ and let $(y_k)$ denote the sequence produced by the algorithm. By Lemma 5.1, based on the fact that $x_0 \in C_{s_1}(x_\ast)$, we assert that $(y_k)$ converges to a medoid $y_\infty$ within $C_{s_1}(x_\ast)$ satisfying $\|\nabla f(y_\infty)\| \leq C_0(\alpha_\ast/\varepsilon + \varepsilon)$. Since $y_\infty \in B(x_\ast, \delta_1)$, (41) applies, so that we also have $\|\nabla f(y_\infty)\| \geq \lambda_\ast y_\infty - x_\ast$. Combining these two inequalities, and using the fact that $\alpha_\ast \leq \alpha_\ast$, we conclude.

**Lemma 5.4.** Let $(y_k)$ denote the medoid Max Shift sequence originating from some point $x_0$ in $U_{2s}$ for some level $s > 0$. At each step $k$, except perhaps for the last shift,

$$y_{k+1} - y_k = \|y_{k+1} - y_k\| N(y_k) + \frac{C_0(\alpha_s/\varepsilon + \varepsilon)^{3/2}}{\|\nabla f(y_k)\|^{1/2}}.$$

whenever $\nabla f(y_k) \neq 0$. In particular, medoid Max Shift satisfies Property 3.

**Proof.** Let $\alpha$ be short for $\alpha_\ast$. Let $u_{k+1} := y_{k+1} - y_k$. The result comes from comparing $f(y_{k+1})$, which by construction maximizes $f(y)$ over $y \in \mathcal{Y} \cap B(y_k, \varepsilon)$, with $f(y'_{k+1})$ where $y'_{k+1}$ is a closest point to $x'_{k+1} := y_k + (\varepsilon - \alpha)N(y_k)$. The arguments around (40) apply to give that $\|y_{k+1} - x'_{k+1}\| \leq \alpha$ if $\varepsilon$ is small enough. Hence, by the triangle inequality, $\|y_{k+1} - y_k\| \leq \varepsilon$. By construction of $y_{k+1}$, this implies that $f(y_{k+1}) \geq f(y'_{k+1})$.

Using (6), on the one hand we have

$$f(y_{k+1}) \leq f(y_k) + \nabla f(y_k)^T u_{k+1} + \frac{1}{2}\kappa_2 \varepsilon^2,$$

and on the other hand, with the help of (4) and the definition of $\kappa_1$, we have

$$f(y'_{k+1}) \geq f(x'_{k+1}) - \kappa_1 \alpha \
\geq f(y_k) + (\varepsilon - \alpha)\|\nabla f(y_k)\| - \frac{1}{2}\kappa_2 (\varepsilon - \alpha)^2 - \kappa_1 \alpha \
\geq f(y_k) + \varepsilon \|\nabla f(y_k)\| - C(\alpha + \varepsilon^2).$$

We thus have

$$\nabla f(y_k)^T u_{k+1} + \frac{1}{2}\kappa_2 \varepsilon^2 \geq \varepsilon \|\nabla f(y_k)\| - C_1(\alpha + \varepsilon^2),$$

implying

$$N(y_k)^T u_{k+1} \geq \varepsilon - C_2(\alpha + \varepsilon^2)/\|\nabla f(y_k)\| \
\geq (1 - C_2(\alpha/\varepsilon + \varepsilon)/\|\nabla f(y_k)\|)\varepsilon_{k+1},$$

with $\varepsilon_{k+1} := \|u_{k+1}\|$. (Note that we used the fact that $\varepsilon_{k+1} \leq \varepsilon$, by Lemma 5.2.) This is analogous to (20) in the proof of Lemma 3.4, and the remaining arguments are also analogous.

We have thus established that medoid Max Shift satisfies Properties 1 and 3, and a weaker version of Property 2. But this is enough, by the same arguments underlying Theorem 3.1, to show that, given a level $s > 0$, if $\alpha_s/\varepsilon$ and $\varepsilon$ are small enough, when initialized at any point in $U_{2s}$ that is in the basin of attraction of some mode, medoid Max Shift converges to a medoid point within distance $C(\alpha_s/\varepsilon + \varepsilon)$ of that mode. We may thus state that, as $\alpha_s \to 0$ and $\varepsilon \to 0$ in such a way that $\alpha_s/\varepsilon \to 0$, the algorithm converges to that mode. In that sense, we may conclude that
Medoid Max Shift is consistent.

This is for the algorithm. We now discuss the consistency of the method, which is, as usual, defined by applying the algorithm to a KDE \( \hat{f}_{n,h} \):

\[
\hat{x}(0) = x_0; \quad \hat{x}(k + 1) \in \arg\max_{y \in Y \cap B(\hat{x}(k), \varepsilon)} \hat{f}_{n,h}(y), \quad k \geq 0.
\]

The (uniform) consistency of this method can be established following the same line of arguments detailed in Section 4, most specifically, Section 4.2, from which we borrow some notation, and as we did there, we assume that \( h \to 0 \) and \( \log n/(nh^{d+4}) \to 0 \) as \( n \to \infty \). In addition, we also require that \( \alpha_{s_n}/\varepsilon \to 0 \) for a sequence of positive numbers \( \{s_n\} \) tending to zero as \( n \to \infty \). Then there is a measurable set \( \Xi_{\varepsilon,n,h} \) with probability measure tending to one as \( \varepsilon \to 0 \) and \( n \to \infty \), such that starting from any \( x \in \Xi_{\varepsilon,n,h} \), the sequence produced by the medoid Max Shift method converges to \( \gamma_x(\infty) \). Details of proof are omitted.

6 Discussion

Although we have covered a good amount of territory, there remain some interesting open questions in regards to the consistency of hill-climbing algorithms and methods.

**Blurring Mean Shift.** While we have established the consistency of Mean Shift, the behavior of Blurring Mean Shift is not as well understood at the moment, even though some results do exist \[6, 7, 13\]. Although the Blurring Mean Shift is often seen as a faster version of Mean Shift, we have reasons to speculate that these two approaches are in fact quite distinct.

**Quick Shift.** Vedaldi and Soatto \[36\] proposed the following hill-climbing algorithm. It is of medoid-type, with the sample being the default medoid set, as usual. Assuming that the density is available, starting at an arbitrary medoid point, the algorithm iteratively moves to the closest medoid point within a certain neighborhood radius whose density is strictly larger than the current value, or in formula,

\[
x(0) = x_0; \quad x(k + 1) \in \arg\min_{y \in Y \cap B(x(k), \varepsilon)} \|y - x(k)\| \text{ with } f(y) > f(x(k)), \quad k \geq 0.
\]

A population analog of quick shift is not straightforward to define as, by continuity of the density, there is no closest point in the population within distance \( h \) whose density value is strictly larger. Jiang \[27\] showed that Quick Shift can be used for a variety of tasks, including finding density modes, and in subsequent work, Jiang et al. \[28\] studied the consistency properties of this algorithm for the task of clustering. The latter is done for an initialization in some subset of the basin of attraction of a mode that, to quote the authors, “satisfy the property that any path leaving [this region] must sufficiently decrease in density at some point”. As is readily seen, in dimension \( d \geq 2 \), under the existence of a saddle point, this restricts the initialization to leaf clusters — incidentally, the same restriction as in the study of Max Shift in \[10\]. As far as we know, proving (or disproving) the consistency of Quick Shift over the entire basins of attraction remains an open problem.

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A Auxiliary results

Proof of Lemma 2.3

Proof. We use the same notation as used in the proof of Theorem 3.1, except that we make the dependence on the starting point x explicit whenever needed as in, e.g., \( t_\#(x) \) denoting \( t_\# \) when associated with \( x \). Let \( A_* \) be the basin of attraction associated with a mode \( x_* \), that is, \( A_* = \{ x \in \mathbb{R}^d : \lim_{t \to -\infty} \gamma_x(t) = x_* \} \). With the notation \( Z(x) := \zeta_x([0, \ell_x]) \), where \( \ell_x \) is the length of \( \gamma_x \), we only need to show that for every \( x \in A_* \),

\[
d_H(Z(x), Z(y)) \to 0, \quad \text{as } \|y - x\| \to 0.
\]

By Lemma 2.5, we can take \( \delta_* > 0 \) small enough that there exist \( \lambda > 0 \) and \( c_* \geq 1 \) such that \( B(x_*, c_* \delta_*) \subset A_* \), and all the eigenvalues of \( \nabla^2 f(x) \) for all \( x \in B(x_*, c_* \delta_*) \) are upper bounded by \( -\lambda \), and \( \inf_{x \in B(x_*, \delta_*/3)} f(x) > \sup_{x \in \partial B(x_*, c_* \delta_*)} f(x) \).

For any point \( x \in A_* \setminus B(x_*, \delta_*/3) \), define \( t_\varepsilon(x) := \inf \{ t \geq 0 : \| \gamma_x(t) - x_* \| = \delta_* / 6 \} \), and \( \mu(x) := \frac{1}{2} \min \{ \| \nabla f(z) \| : z \in Z_{t_\varepsilon(x)}(x) \} \). Notice that \( \| \nabla f(y) \| \geq \nu(x) \) for all \( y \in S(x) := B(Z_{t_\#}(x), \delta_{\text{tube}}') \).
where $\delta'_{\text{tube}} := \mu(x)/\kappa_2$ by (5). Similarly, by using (7),
\[
\sup_{y \in S(x)} \|DN(y)\| \leq \frac{\kappa_2}{\nu(x)} + \kappa_2.
\]
Define $\mathcal{V}(x) := B(\mathcal{Z}_{t_o}(x), \delta'_{\text{tube}}/2)$, and notice that $S(x) = B(\mathcal{V}(x), \delta'_{\text{tube}}/2)$. If $y, z \in \mathcal{V}(x)$ are such that $\|y - z\| \leq \delta'_{\text{tube}}/4$, then $z \in B(y, \delta'_{\text{tube}}/2) \subset S(x)$, and because that ball is convex, we have
\[
\|N(y) - N(z)\| \leq \left(\frac{\kappa_2}{\nu(x)} + \kappa_2\right)\|y - z\|.
\]
If, on the other hand, $\|y - z\| > \delta'_{\text{tube}}/4$, then we can simply write
\[
\|N(y) - N(z)\| \leq 2 = \frac{2}{\delta'_{\text{tube}}/4}\delta'_{\text{tube}}/4 \leq \frac{8}{\delta'_{\text{tube}}}\|y - z\|.
\]
Hence, $N$ is $\kappa$-Lipschitz on $\mathcal{V}(x)$, where $\kappa := \max\{\kappa_2/\nu(x) + \kappa_2, 8/\delta'_{\text{tube}}\}$.

Take a positive constant $\delta_o \leq \frac{1}{2}\exp\{-\kappa t_o(x)\}\delta'_{\text{tube}}$. Suppose that
\[
\mathcal{H}(x) := \{\zeta_y(\tau) : \tau \in [0, t_o(x)], y \in B(x, \delta_o)\} \subset \mathcal{V}(x).
\]
Then there exist $y \in B(x, \delta_o)$ and an escaping time $t_o \in (0, t_o(x))$ such that $\zeta_y(t_o) \in \partial \mathcal{V}(x)$ and $\{\zeta_y(\tau) : \tau \in [0, t_o(x)]\} \subset \mathcal{V}(x)$. This is impossible because applying a standard result on the dependence of the gradient flow on the initial condition, for example, the main theorem in [24, Sec 17.3], we have
\[
\|\zeta_x(t_o) - \zeta_y(t_o)\| \leq \|x - y\| \exp(\kappa t_o) < \frac{1}{2}\delta'_{\text{tube}},
\]
which would lead to $\zeta_y(t_o) \in \mathcal{V}(x)$, a contradiction against the definition of $t_o$ as $\mathcal{V}(x)$ is an open set. Therefore we must have $\mathcal{H}(x) \subset \mathcal{V}(x)$, and we can use the main theorem in [24, Sec 17.3] to obtain
\[
\|\zeta_x(t) - \zeta_y(t)\| \leq \|x - y\| \exp(\kappa t) \leq \frac{1}{2}\delta'_{\text{tube}}, \quad \forall t \in [0, t_o(x)], \quad \forall y \in B(x, \delta_o).
\]
We further require that

\[
\delta_o \leq \frac{1}{6}\delta_\star \exp(-\kappa t_o(x)).
\]
Then by the first inequality in (43), $\|\zeta_x(t_o(x)) - \zeta_y(t_o(x))\| \leq \frac{1}{6}\delta_\star$, and hence
\[
\|\zeta_y(t_o(x)) - x_\star\| \leq \|\zeta_x(t_o(x)) - \zeta_y(t_o(x))\| + \|\zeta_x(t_o(x)) - x_\star\| \leq \frac{1}{3}\delta_\star.
\]
This implies that $t_\#(y) \leq t_o(x)$, by definition of $t_\#(\cdot)$.

For $y \in B(x, \delta_o)$, without loss of generality, suppose that $t_\#(y) \geq t_\#(x)$. Notice that
\[
\mathcal{Z}_{t_\#(y)}(y) = \mathcal{Z}_{t_\#(x)}(y) \cup \{\zeta_y(t) : t \in [t_\#(x), t_\#(y))\}.
\]
With notation $\phi = \sup_{t \in [t_\#(x), t_\#(y))} \|\zeta_y(t) - \zeta_y(t_\#(x))\|$, we can write
\[
d_H(\mathcal{Z}_{t_\#(x)}(x), \mathcal{Z}_{t_\#(y)}(y)) \leq d_H(\mathcal{Z}_{t_\#(x)}(x), \mathcal{Z}_{t_\#(x)}(y)) + \phi \leq \sup_{t \in [0, t_\#(x)]} \|\zeta_x(t) - \zeta_y(t)\| + \phi \leq \|x - y\| \exp(\kappa t_\#(x)) + \phi.
\]

where the last equality is a result of (43).

For any \( z \in B(x_*, c_\delta_* ) \) with \( z \neq x_* \), using a Taylor expansion about \( z \), we obtain

\[
0 < f(x_*) - f(z) \leq (x_* - z)^\top \nabla f(z) - \frac{1}{2} \lambda \| x - x_* \|^2.
\]

Hence \( (x_* - z)^\top \nabla f(z) > \frac{1}{2} \lambda \| z - x_* \|^2 \). Denote \( \xi_z(t) = \| \zeta_z(t) - x_* \|^2 \). Then, for all \( t \in [t_\#, (x), \ell_x] \), where \( \ell_x \) is the length of \( \gamma_x \), we have \( \zeta_z(t) \in B(x_*, c_\delta_* ) \), and

\[
\dot{\xi}_z(t) = 2(\zeta_z(t) - x_*)^\top \dot{\zeta}_z(t) = 2 \frac{\langle \zeta_z(t) - x_* \rangle^\top \nabla f(\zeta_z(t)) \| \nabla f(\zeta_z(t)) \|}{\| \nabla f(\zeta_z(t)) \|^2} < -\lambda \frac{\| \zeta_z(t) - x_* \|^2}{\| \nabla f(\zeta_z(t)) \|^2} < 0.
\]

(46)

In other words, for all \( t \in [t_\#, (x), \ell_x] \), \( \zeta_z(t) \) stays in \( B(x_*, \delta_/3) \) and its distance to \( x_* \) strictly decreases as \( t \) increases. Since \( t_\#(y) \in [t_\#(x), t_\#(x)] \), we have

\[
\xi_z(t_\#(y)) = \xi_z(t_\#(x)) + \int_{t_\#(x)}^{t_\#(y)} \dot{\xi}_z(s) ds
\]

\[
< \frac{1}{9} \delta_* - \lambda \int_{t_\#(x)}^{t_\#(y)} \frac{\| \zeta_z(s) - x_* \|^2}{\| \nabla f(\zeta_z(s)) \|} ds
\]

\[
< \frac{1}{9} \delta_* - \frac{\lambda}{\kappa_{1}} (\delta_/3)^2 (t_\#(y) - t_\#(x))
\]

\[
=: \frac{1}{9} \delta_*^2 - C_1 (t_\#(y) - t_\#(x)).
\]

Noticing that by (43) and (44),

\[
\| \zeta_\gamma(t_\#(y)) - \zeta_\gamma(t_\#(x)) \| < \frac{1}{6} \delta_* < \frac{1}{3} \delta_* = \| \zeta_\gamma(t_\#(y)) - x_* \|
\]

and using the triangle inequality, we have

\[
\xi_z(t_\#(y)) = \| \zeta_z(t_\#(y)) - x_* \|^2
\]

\[
\geq \left( \| \zeta_\gamma(t_\#(y)) - x_* \| - \| \zeta_\gamma(t_\#(y)) - \zeta_\gamma(t_\#(y)) \| \right)^2
\]

\[
\geq \left( \| \zeta_\gamma(t_\#(y)) - x_* \| - \| \zeta_\gamma(t_\#(y)) - \zeta_\gamma(t_\#(y)) \| \right)^2
\]

\[
\geq \left( \delta_/3 - \| x - y \| \exp(\kappa_\#(y)) \right)^2
\]

\[
= \frac{1}{9} \delta_*^2 - \frac{2}{3} \delta_* \exp(\kappa_\#(y)) \| x - y \| + \exp(2\kappa_\#(y)) \| x - y \|^2
\]

\[
\geq \frac{1}{9} \delta_*^2 - \left\{ \frac{2}{3} \delta_* \exp(\kappa_\#(x)) - \| x - y \| \right\} \| x - y \|
\]

\[
=: \frac{1}{9} \delta_*^2 - C_2(x) \| x - y \|.
\]

Note that by (44),

\[
C_2(x) \geq 4 \delta_o \exp(2\kappa_o(x)) - \| x - y \| > 0.
\]

We thus obtain \( t_\#(y) - t_\#(x) \leq (C_2(x)/C_1) \| x - y \| \). Hence, using the fact that \( \zeta_\gamma \) is parameterized by arc length,

\[
\phi = \sup_{t \in [t_\#(x), t_\#(y)]} \left\| \int_{t_\#(x)}^{t} \dot{\zeta}_\gamma(t) dt \right\|
\]

\[
\leq t_\#(y) - t_\#(x)
\]

\[
\leq (C_2(x)/C_1) \| x - y \|.
\]
Combining this with (45), we obtain
\[ d_H(Z_t^\#(x), Z_t^\#(y)) \leq C_3(x) \|x - y\|, \tag{47} \]
where \( C_3(x) = \exp\{\kappa t^\#(x)\} + C_2(x)/C_1 \).

As shown in (46), once a gradient flow enters \( B(x_*, \delta_*/3) \), it never escapes from this ball. Using (47), we have \( d_H(Z(x), Z(y)) \leq \delta_* \) when \( \|x - y\| \) is small enough. Since \( \delta_* > 0 \) can be made arbitrarily small, we then obtain (42), which gives the conclusion of the lemma.

Lemma A.1. \( \nu \) is a continuous function on \( A \).

Proof. Noticing that \( A_* \) is an open set, and \( A \) is the union of all the \( A_* \)'s, we only need to show \( \nu \) is continuous \( A_* \). Without loss of generality, we again assume that \( \delta_* \) in Property 2 is small enough that the same assumption on \( \delta_* \) in the proof of Lemma 2.3 holds.

Note that \( \nu(x) = \frac{1}{2} \|\nabla f(x)\| \) for any \( x \in B(x_*, \delta_*/3) \). It is clear that \( \nu \) is continuous on \( \bar{B}(x_*, \delta_*/3) \), and thus we only need to show that \( \nu \) is continuous on \( A_* \setminus B(x_*, \delta_*/3) \).

By denoting \( d_\nu(A \mid B) = \sup_{v \in A} \inf_{w \in B} \|\nabla f(v)\| - \|\nabla f(w)\| \), it follows from (47) that
\[
\begin{align*}
|\nu(x) - \nu(y)| & \leq \max\{d_\nu(Z_t^\#(x) \mid Z_t^\#(x)) \mid d_\nu(Z_t^\#(y) \mid Z_t^\#(x))\} \\
& \leq \kappa_2 d_H(Z_t^\#(x), Z_t^\#(y)) \\
& \leq C_4(x) \|x - y\|,
\end{align*}
\]
where \( C_4(x) = \kappa_2 C_3(x) \). We have thus shown that \( \nu \) is a continuous function on \( A \).