ON THE COMBINATORICS OF $B \times B$-ORBITS ON GROUP COMPACTIFICATIONS

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ABSTRACT. It is shown that there is an order isomorphism $\varphi'$ from the poset $V$ of $B \times B$-orbits on the wonderful compactification of a semi-simple adjoint group $G$ with Weyl group $W$ to an interval in reverse Chevalley-Bruhat order on a non-canonically associated Coxeter group $\tilde{W}$ (in general neither finite nor affine). Moreover, $\varphi'$ preserves the corresponding Kazhdan-Lusztig polynomials. Springer’s (partly conjectural) construction of Kazhdan-Lusztig polynomials for the analogues of $V$ for general Coxeter groups $W$ is completed by reducing it by a similar order isomorphism to known results involving a “twisted” Chevalley-Bruhat order on $W$.

INTRODUCTION

Let $G$ be a connected semi-simple adjoint algebraic group over an algebraically closed field. Denote by $B$ a Borel subgroup of $G$ and by $T$ a maximal torus in $B$. $T$ and $B$ determine the Weyl group $W$ of $G$ and the set $S$ of its simple reflections. For any $I \subseteq S$, we write $W_I$ for the standard parabolic subgroup of $W$ generated by $I$ and $W^I$ for the set of minimal left coset representatives of $W_I$ in $W$.

According to [11, §1], $G$ has a “wonderful” compactification $X$, which is a smooth irreducible projective $G \times G$-variety and contains $G$ as an open subvariety. The $B \times B$-orbits on $X$ can be parametrized as $O_v$ for $v \in V$, where $V = \{ [I,a,b] \mid I \subseteq S, a \in W^I, b \in W \}$. $V$ is a poset endowed with the partial order $w \leq v$ if $O_w$ is contained in the closure of $O_v$ in $X$. By [11] Proposition 2.4, $[I,a_1,b_1] \leq [J,a_2,b_2]$ in $V$ if and only if $I \subseteq J$ and there exist $c \in W_I$, $d \in W_J \cap W^I$ with $a_2 d c^{-1} \leq_0 a_1$, $b_1 c \leq_0 b_2 d$, and $l_0(b_2 d) = l_0(b_2) + l_0(d)$, where $\leq_0$ and $l_0$ are the Chevalley-Bruhat order and the length function on $W$.

Let $R = \mathbb{Z}[u,u^{-1}]$ be the integral Laurent polynomial ring in the indeterminate $u$, $M$ be the free left $R$-module with a basis $\{ \tilde{m}_v \mid v \in V \}$, and $H$ be the Iwahori-Hecke algebra of $W$. By [11, Lemma 3.2], $M$ admits an $(H,H)$-bimodule structure. Springer constructs a semi-linear $(H,H)$-bimodule map $\Delta : M \to M$ in [11, 3.3] with respect to a certain involution on $H \otimes RH$. Write $\Delta(\tilde{m}_v) = \sum_{w \in V} \tilde{b}_{w,v} \tilde{m}_w$ for some $\tilde{b}_{w,v} \in R$ ($v,w \in V$). $\Delta$ is an involution, that is, $\Delta^2 = 1$ (see [11, §3]) and the $\tilde{b}_{w,v}$ may be regarded as (appropriately normalized) analogues of the Kazhdan-Lusztig $R$-polynomials $R_{x,y}$. Using the $\tilde{b}_{w,v}$ in place of the $R_{x,y}$, Springer defines analogues $c_{w,v}$ of the Kazhdan-Lusztig polynomials $P_{x,y}$ and shows that they compute the Poincaré series at a point in $O_w$ of the local intersection cohomology of the closure of $O_v$.

Now let $(\tilde{W}, \tilde{S})$ be any Coxeter system containing $(W,S)$ as a standard parabolic subgroup and such that there is a bijection $\theta : S \to R := \tilde{S} \setminus S$ such that for $r,s \in S$, $r$ and $\theta(s)$ are joined by an edge in the Coxeter graph of $(\tilde{W}, \tilde{S})$ (i.e. do not commute in $\tilde{W}$) iff $r = s$. Let $c_R$ be a Coxeter element of
the standard parabolic subgroup $\hat{W}_R$ and let $w_S$ be the longest element of $W = \hat{W}_S$. Let $\Omega$ denote the interval $[w_{SCR \emptyset} W, 1]$ in $\hat{W}$ in the order induced by the reverse Chevalley-Bruhat order. The main result of this paper implies that there is a poset isomorphism $\varphi' : V \to \Omega$ satisfying $c_{w,v} = Q_{\varphi'(w), \varphi'(v)}$, where the $Q_{x,y}$ denote the inverse Kazhdan-Lusztig polynomials \cite[(2.1.6)]{10} for $\hat{W}$.

More generally, let $(W, S)$ be a Coxeter system with finitely many simple reflections. Springer extends in \cite[§6]{11} the constructions of the poset $V$, $(\mathcal{H}, \mathcal{H})$-bimodule $\mathcal{M}$, and map $\Delta$ to $(W, S)$; the only needed modification in this more general setting is thinking of $\Delta$ as a semi-linear map from $\mathcal{M}$ to its completion $\hat{\mathcal{M}}$. Springer conjectures in \cite[6.10]{11} that $\Delta$ is still an involution in the general setting, which would imply that there exist analogues of the Kazhdan-Lusztig polynomials $c_{w,v}$ and inverse Kazhdan-Lusztig polynomials $c_{w,v}^{\text{inv}}$ for $V$ (see \cite[Formulas (7) and (8)]{12}).

On the other hand, in \cite{13}, there are attached to any suitable subset $A$ (initial section of a reflection order) of the reflections $\hat{T}$ of an arbitrary Coxeter system $(\hat{W}, \hat{S})$ the analogue $\leq_A$ of Chevalley-Bruhat order, analogues $R^A_{x,y}$ of the $R$-polynomials $R_{x,y}$, analogues $P_A(x, y)$ of the Kazhdan-Lusztig polynomials $P_{x,y}$ etc; to define $\leq_A$, $R^A_{x,y}$, $P_A(x, y)$ etc, one just regards the standard length function on which suitable standard definitions (of Chevalley-Bruhat order, $R_{x,y}$, $P_{x,y}$ etc) implicitly depend as a parameter and replaces it by a more general length function $l_A$ depending on $A$ (the fact one obtains well-defined notions for general $l_A$ is not obvious).

In this paper, we prove Springer’s conjecture by showing that for $(\hat{W}, \hat{S})$ associated to $(W, S)$ exactly as for finite Weyl groups and $A := \hat{T} \setminus W$, Springer’s $V$, $\mathcal{M}$, $\Delta$, $\hat{b}_{w,v}$, $c_{w,v}$ etc may be described directly in terms of corresponding objects attached to $\leq_A$ on $\hat{W}$. In the case of a finite Coxeter group $W$, $\hat{W}$ in $\leq_A$ is order isomorphic to $\hat{W}$ in reverse Chevalley-Bruhat order, and we recover the above-mentioned interpretation of $c_{w,v}$ as an inverse Kazhdan-Lusztig polynomial.

The arrangement of this paper is as follows. Section 2 briefly recalls some details of Springer’s combinatorial constructions. Section 3 describes our results and lists some of their immediate consequences. Section 4 gives results we need concerning an arbitrary Coxeter system $(\hat{W}, \hat{S})$ in the specific order $\leq_A$, where $A = \hat{T} \setminus W$ and $(W, S)$ is a standard parabolic subsystem of $(\hat{W}, \hat{S})$. The main result there is an identity (Lemma 1.3) expressing $R^A_{a \hat{b}, 1}$ for $a, b \in W$ and $z \in \hat{W}_S \setminus S$ in terms of the classical $R$-polynomials of $(\hat{W}, \hat{S})$. In Section 5, we finally prove the results of Section 4 by showing that for $(\hat{W}, \hat{S})$ associated to $(W, S)$ as described previously, this identity specializes to the initial condition \cite[(b)]{14} for Springer’s recurrence formula for $\hat{b}_{w,v}$.

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1. Springer’s Combinatorial Constructions

Let $(W, S)$ be a Coxeter system with finitely many simple reflections. Denote the length function on $W$ by $l_0$, the Chevalley-Bruhat order on $W$ by $\leq_0$, and the identity of $W$ by 1. If $I \subseteq S$, then write $W_I$ for the standard parabolic subgroup of $W$ generated by $I$. $W^I = \{ x \in W \mid x \leq_0 xx \text{ for all } s \in I \}$ denotes the set of minimal left coset representatives of $W_I$ in $W$. It is well-known that for a fixed $I \subseteq S$,
\( l_q(xy) = l_q(x) + l_q(y) \) for \( x \in W_I \) and \( y \in W_J \), and that every element of \( W \) is uniquely expressible in the form \( xy \) for some such \( x \) and \( y \).

1.1. The poset \((V, \leq)\). In his paper \[11, 6.4\], Springer introduces a poset

\[ V = \{ [I, a, b] : I \subseteq S, a \in W_I, b \in W \}. \]

Let \([I, a_1, b_1], [J, a_2, b_2] \in V\). Define \([I, a_1, b_1] \leq [J, a_2, b_2]\) if \( I \subseteq J, a_2 \leq a_1, b_1 \leq b_2\), and define \([I, a_1, b_1] \leq [J, a_2, b_2]\) if \( I \subseteq J, b_1 = b_2c \) for some \( c \in W_J \) with \( l_q(b_1) = l_q(b_2) + l_q(c) \), \( a_2c \leq a_1\).

The partial order \( \leq \) on \( V \) is generated by \( \leq_1 \) and \( \leq_2 \). That is, \( v \leq w \) if there is a sequence \( v = v_0, v_1, \ldots, v_{n-1}, v_n = w \) in \( V \) such that either \( v_{i-1} \leq v_i \) or \( v_{i-1} \leq v_i \) for \( i = 1, 2, \ldots, n \).

Given \([I, a, b] \in V\), define

\[ d([I, a, b]) = -l_q(a) + l_q(b) + |I|, \]

where \( |I| \) is the cardinality of \( I \). Springer calls \( d \) the dimension function on \( V \) due to its relation to the dimensions of \( B \times B \)-orbits on \( X \) in the case \( W \) is a finite Weyl group (see \[13, Lemma 1.3\]).

1.2. The Iwahori-Hecke algebra of \((W, S)\). Set \( W = W \times W \) and \( S = (S \times \{1\}) \cup (\{1\} \times S) \). \((W, S)\) is a Coxeter system. By abuse of notation, we write \( \leq_0 \) and \( l_0 \) for the Chevalley-Bruhat order and length function on \( W \), respectively.

Let \( R = \mathbb{Z}[u, u^{-1}] \) be the integral Laurent polynomial ring in the indeterminate \( u \). \( R \) admits a ring involution \( \alpha \mapsto \bar{\alpha} \) satisfying \( \bar{u} = u^{-1} \). To simplify our notation, we define \( \alpha = u - u^{-1} \). \( \alpha \) satisfies the equation \( \bar{\alpha} = -\alpha \). Note that \( u \) corresponds to the notation \( q^{1/2} \) used in \[13, \S 1\].

The Iwahori-Hecke algebra \( H \) of \((W, S)\) is the free left \( R \)-module with a basis \( \{ \tilde{T}_x : x \in W \} \) and subject to the multiplication law

\[ \tilde{T}_s \tilde{T}_x = \begin{cases} 
\tilde{T}_{sx} & \text{if } x \leq_0 sx \\
\tilde{T}_{sx} + \alpha \tilde{T}_x & \text{if } sx \leq_0 x,
\end{cases} \]

where \( s \in S \) and \( x \in W \). \( \tilde{T}_1 \) is the multiplicative identity of \( H \) and every \( \tilde{T}_x \) is invertible in \( H \).

1.3. The \( H \)-module \( M \). If \( a \in W_I \) and \( s \in S \), then there are exactly three possibilities for \( sa \):

(a) \( a \leq_0 sa \) and \( sa \in W_I \);
(b) \( a \leq_0 sa \) and \( sa = at \) for some \( t \in I \);
(c) \( sa \leq_0 a \) in which case \( sa \in W_I \).

By \[11, 6.4 (b)\] there is a left action of \( W \) on \( V \) satisfying

\[ (s, 1).[I, a, b] = \begin{cases} 
[I, sa, b] & \text{if } s, a \text{ satisfy (a) or (c)} \\
[I, a, bt] & \text{if } s, a \text{ satisfy (b)},
\end{cases} \]

\[ (1, s).[I, a, b] = [I, a, sb] \]

for \( s \in S \) and \( [I, a, b] \in V \).
Let $\mathcal{M}$ be the free left $\mathcal{R}$-module with a basis $\{\tilde{m}_v \mid v \in V\}$. By [11] 6.3 there is an $H$-module structure on $\mathcal{M}$ defined by

$$\tilde{T}_s \tilde{m}_v = \begin{cases} \tilde{m}_v & \text{if } d(v) < d(s.v) \\ \tilde{m}_v + \alpha \tilde{m}_v & \text{if } d(s.v) < d(v) \end{cases}$$

for $s \in S$. Note that $\tilde{m}_v = u^{-d(v)}m_v$ for the $m_v$ appearing in [11] 3.1).

1.4. The map $\Delta$. Let $\tilde{\mathcal{M}}$ be the completion of $\mathcal{M}$ consisting of formal $\mathcal{R}$-linear combinations of elements $\tilde{m}_v$ for $v \in V$. Springer defines a semi-linear map $\Delta: \mathcal{M} \to \tilde{\mathcal{M}}$ with respect to the Kazhdan-Lusztig involution on $\mathcal{H} \otimes_\mathcal{R} \mathcal{H}$ in [11] 6.8. Write

$$\Delta(\tilde{m}_v) = \sum_{w \in V} \tilde{b}_{w,v} \tilde{m}_w$$

for some $\tilde{b}_{w,v} \in \mathcal{R}$ ($w,v \in V$). By [11] 6.9(i), 3.8, 6.6, $\Delta$ is uniquely determined by the conditions

(a) $\Delta(\tilde{T}_s \tilde{m}) = \tilde{T}_s^{-1} \Delta(m)$ for $s \in S$ and $m \in \mathcal{M}$, and

(b) if $\tilde{R}_{a,b}(u) = (-u)^{-l(b)+l(a)}R_{a,b}(u^2)$ for the polynomials $R_{a,b}$ described in [4] §2 or [4] 3.1 – 3.7, then

$$\tilde{b}_{IJ,a_1,b_1},IJ,a_2,1 = \begin{cases} 0 & \text{if } I \not\subseteq J \text{ or } b_1 \not\in W_J \\ (-\alpha)^{|J|-|I|}\tilde{R}_{a_2b_1,a_1} & \text{otherwise.} \end{cases}$$

2. Statement of Results

Maintain the notations and assumptions of Section 1.

2.1. Fix a Coxeter system $(\hat{W}, \hat{S})$ with the following properties:

(a) $W$ is the standard parabolic subgroup $W = \hat{W}_S$ of $\hat{W}$ generated by $S \subseteq \hat{S}$ and

(b) there is a bijection $\theta: S \to \hat{S} \setminus S$ such that for $r,s \in S$, $r$ and $\theta(s)$ are joined by an edge of the Coxeter graph of $(\hat{W}, \hat{S})$ (i.e. do not commute in $\hat{W}$) iff $r = s$.

Note that the order of $r\theta(r)$ for $r \in S$ is assumed to be three or greater but is not specified more precisely, and that there is no assumption made about the orders of the products $\theta(r)\theta(s)$ for distinct $r, s \in S$. In particular, if $W$ is non-trivial, there are infinitely many possible choices of $(\hat{W}, \hat{S})$ up to isomorphism.

For example, if $(W, S)$ is of type $A_1$, then $(\hat{W}, \hat{S})$ could be taken to be any irreducible dihedral Coxeter system, for instance of type $A_2$. If $(W, S)$ is of type $A_2$ (resp., $A_3, B_2$) then $(\hat{W}, \hat{S})$ could be chosen to be of type $A_4$ (resp., $E_6, F_4$) with the Coxeter graph of $(W, S)$ embedded as the full subgraph of that of $(\hat{W}, \hat{S})$ on the vertex set obtained by deleting all of the terminal vertices of the latter graph. If $(W, S)$ is irreducible but not of type $A_1, A_2, A_3$ or $B_2$, then $(\hat{W}, \hat{S})$ cannot be of finite or affine type.

2.2. Fix once and for all an enumeration of $S$ as $S = \{r_1, \ldots, r_n\}$ with distinct $r_i$. For $I \subseteq S$, define $z_I \in \hat{W}$ as $z_I = \theta(r_{i_1}) \cdots \theta(r_{i_m})$, where $S \setminus I = \{i_1, \ldots, i_m\}$ with $i_1 < \cdots < i_m$ (note $z_I$ may depend on the fixed enumeration of $S$).

Let $\hat{T}$ denote the set of reflections of $(\hat{W}, \hat{S})$ and set $A = \hat{T} \setminus W$. Then by [3] 2.11, $A$ is an initial section of a reflection order of $\hat{T}$. Correspondingly, there are a length function $l_A: \hat{W} \to \mathbb{Z}$, a partial
order \( \leq_A \) on \( \hat{W} \) and elements \( R^A_{x,y} \in \mathcal{R} \), \( P_A(x, y) \in \mathbb{Z}[q] \) defined for \( x, y \in \hat{W} \) (some of their definitions and basic properties are summarized in Section 3). The following is the main result of this paper.

2.3. Theorem. Let \( \Omega = \bigcup_{I \subseteq S} W z_I W \subseteq \hat{W} \), regarded as a poset in the order induced by \( \leq_A \).

(i) \( \Omega \) is locally closed in \((\hat{W}, \leq_A)\), i.e. if \( x \leq_A y \leq_A z \) with \( x, z \in \Omega \) and \( y \in \hat{W} \), then \( y \in \Omega \).

(ii) The map \( \varphi : V \to \Omega \) given by \( \varphi([I, a, b]) = azb^{-1} \) is a poset isomorphism.

(iii) For \( v \in V \), \( l_A(\varphi(v)) = d(v) - |S| \).

(iv) \( \varphi(a, b, v) = a\varphi(v)b^{-1} \) for any \( v \in V \) and \((a, b) \in W\).

(v) For \( v, w \in V \), \( \hat{b}_{w,v} = R^A_{\varphi(w),\varphi(v)} \).

The theorem allows one to directly transfer many known properties of the twisted Chevalley-Bruhat order \( \leq_A \) to the poset \( V \). For example, one obtains the following.

2.4. Corollary. Let \( v, w \in V \) and \( n = d(v) - d(w) \).

(i) \( \sum_{w \leq z \leq v} \hat{b}_{w,z}(u)\hat{b}_{z,v}(w^{-1}) = \delta_{w,v} \), where \( \delta_{w,v} \) is the Kronecker delta.

(ii) if \( w \leq v \), the closed interval \([w, v]\) and the opposite poset \([w, v]^\text{op}\) are pure EL-shellable posets (in which every maximal chain contains \( n + 1 \) elements) in the sense of \footnote{[4]}. Definition 2.1.

(iii) if \( w \leq v \) and \( n \geq 2 \), then the order complex (the abstract simplicial complex with totally ordered subsets as simplexes) of the open interval \((w, v)\) is a combinatorial \((n-2)\)-sphere.

Corollary 2.4 (i) asserts that \( \Delta \) (or more precisely its natural extension to a map \( \hat{M} \to \hat{M} \)) is an involution; this was proved geometrically by Springer in the case \( W \) is a finite Weyl group, and conjectured by him to hold in general. Once it is known, it becomes possible to define analogues \( c_{w,v} \) (resp., \( c_{w,v}^{\text{inv}} \)) of the Kazhdan-Lusztig polynomials \( P_{x,y} \) (resp., inverse Kazhdan-Lusztig polynomials \( Q_{x,y} \)) as in \footnote{[4]}. Regarding these, we record

2.5. Corollary. For \( w, v \in V \), we have \( c_{w,v} = P_A(\varphi(v), \varphi(w)) \) and \( c_{w,v}^{\text{inv}} = P_{\hat{F} \setminus A}(\varphi(w), \varphi(v)) \), where the \( P_{\hat{F} \setminus A} \) and \( P_A \) are the polynomials to be described in 3.9.

2.6. Introduce a graph, also denoted \( V \), with vertex set \( V \) and an edge from \( w \) to \( v \) if \( \hat{b}'_{w,v}(1) \neq 0 \), where \( p'(1) \) denotes the value at 1 of the derivative with respect to \( u \) of a Laurent polynomial \( p \in \mathcal{R} \). By \footnote{[4] 3.10} (extended to general \((W, S)\)) and Theorem 2.3, one has an edge from \( w \) to \( v \) iff either \( d(w) < d(v) \) and there is a reflection \( r \in W \) such that \( w = rv \), or \( w = [I, a, b] \), \( v = [J, c, d] \), \( I \subset J \), \( |J \setminus I| = 1 \) and there exists \( f \in W_J \) with \( a = cf \) and \( b = df \).

2.7. Corollary.

(i) If \( v, w \in V \), then \( w \leq v \) iff there is a directed path in the graph \( V \) from \( w \) to \( v \).

(ii) (Deodhar’s inequality) For \( w \leq z \leq v \) in \( V \), the graph \( V \) has at least \( d(v) - d(w) \) edges with \( z \) as one vertex and both vertices in \([w, v]\).

2.8. Other consequences of the theorem include results on the structure of “quotients” of \( V \) in \footnote{[4]} (such as posets of shortest orbit representatives in \( V \) for orbits of parabolic subgroups of \( W \) for left or right multiplication), the analogue for \( \hat{b}_{w,v} \) in terms of the above graph for the generating function formula \footnote{[4]}
§3 for $R^A_{x,y}$, recurrence formulas [4, §4] for the $c_{w,v}$ similar to the classical ones [4] for $P_{x,y}$, existence of three-parameter versions $c^B_{w,v}$ of the $c_{w,v}$ with conjecturally non-negative coefficients (involving as an additional parameter $B$ an arbitrary initial section of a reflection order of $\hat{W}$), and existence of highest weight representation categories for which the $c_{w,v}$ conjecturally describe Verma module multiplicities etc. We refrain from describing any of these in detail, but record the following.

2.9. Corollary. Suppose that $W$ is finite with the longest element $w_S$. Define a map $\varphi' : V \to \hat{W}$ by $\varphi'(I, a, b) = azrb^{-1}w_S$.

(i) The map $\varphi'$ restricts to an order isomorphism from $V$ to the interval $[w_Sz_0w_S, 1]$ in reverse Chevalley-Bruhat order.

(ii) $l_0(\varphi'(v)) = -d(v) + |S| + l_0(w_S)$ for $v \in V$.

(iii) $\varphi'((a, b), v) = a\varphi'(v)w_Sb^{-1}w_S$ for $v \in V$ and $(a, b) \in W$.

(iv) $\tilde{b}_{w,v} = \tilde{R}_{\varphi'(v),\varphi'(w)}$.

(v) $c_{w,v} = Q_{\varphi'(v),\varphi'(w)}$ and $c^\text{inv}_{w,v} = P_{\varphi'(v),\varphi'(w)}$.

2.10. Remark. The above corollary can be proved independently of the theory of the orders $\leq_A$; the main step, analogous to [4, 3] and proved similarly, is to show that for any Coxeter system $(\hat{W}, \hat{S})$ with finite standard parabolic subgroup $(W, S)$, one has

\[
\tilde{R}_{z_1z_2z_3w_S, a_2z_3b_2w_S} = \begin{cases} 
\tilde{R}_{z_1z_2} \tilde{R}_{a_1b_2^{-1}} & \text{if } b_2 \in W_1, \\
0 & \text{otherwise},
\end{cases}
\]

where $z_i \in \hat{W}_{\hat{S} \setminus S}$, $I_i = S \cap z_iS z_i^{-1}$, $a_i \in W^{I_i}$, and $b_2 \in W$.

3. Twisted Chevalley-Bruhat Orders

Let $(W, S)$ be an arbitrary Coxeter system and $T = \bigcup_{x \in W} xSx^{-1}$ be the set of reflections in $W$.

3.1. Reflection orders on $T$. By [4, Definition 2.1], a total order $\leq$ on $T$ is called a reflection order if for any dihedral reflection subgroup $W'$ of $W$ with its canonical simple reflections $S' = \{r, s\}$ with respect to $S$, either $r \prec rsr \prec \cdots \prec srs \prec s$ or $s \prec srs \prec \cdots \prec rsr \prec r$.

$A \subseteq T$ is called an initial section of reflection orders on $T$ if there is a reflection order $\leq$ on $T$ such that $x \prec y$ for all $x \in A$ and $y \in T \setminus A$.

3.2. Twisted Chevalley-Bruhat order. We give a brief description of twisted Chevalley-Bruhat orders here. The reader can refer to [4, §1] for the details.

Regard the power set $\mathcal{P}(T)$ of $T$ as an additive abelian group under the symmetric difference $A + B = (A \cup B) \setminus (A \cap B)$. Define a map $N : W \to \mathcal{P}(T)$ by $N(x) = \{ t \in T \mid tx \leq_{0} x \}$ for $x \in W$. We have $l_0(x) = |N(x)|$. $N(x)$ may be characterized by $N(xy) = N(x) + xN(y)x^{-1}$ for $x, y \in W$ and $N(s) = \{ s \}$ for $s \in S$. There is an action of $W$ on $\mathcal{P}(T)$ given by $x.A = N(x) + xA x^{-1}$ for $x \in W$ and $A \in \mathcal{P}(T)$.

Fix an initial section $A$ of a reflection order on $T$. Define the twisted length function $l_A$ on $W$ by

\[
l_A(x) = l_0(x) - 2|N(x^{-1}) \cap A| \quad \text{for } x \in W.
\]
The twisted Chevalley-Bruhat order \( \leq_A \) on \( W \) is generated by the relation \( x \leq_A tx \) for \( x \in W \) and \( t \in T \) with \( l_A(x) < l_A(tx) \). By \cite[Proposition 1.2]{4}, we have \( x \leq_A tx \) iff \( t \not\in x.A \).

3.3. The Coxeter system \((W', S')\). Set \( S' = \{ t \in T \mid t.A = \{ t \} + A \} \). Let \( W' \) be the subgroup of \( W \) generated by \( S' \) and set \( T' = \bigcup_{x \in W'} xS'x^{-1} \). Note that the map \( N': W' \to \mathcal{P}(T') \) given by \( N'(x) = x.A + A \) satisfies \( N'(xy) = N'(x) + xN'(y)x^{-1} \) for \( x, y \in W' \) and \( N'(s) = \{ s \} \) for \( s \in S' \). By \cite[§2]{4}, \((W', S')\) is a Coxeter system.

For \( x \in W \), define \( L_A(x) = \{ s \in S \mid sx \leq_A x \} \) and \( R_A(x) = \{ t \in S' \mid xt \leq_A x \} \). By \cite[1.8]{4}, we see that if \( s \in L_A(x) \), then \( l_A(sx) = l_A(x) - 1 \) and that if \( t \in R_A(x) \), then \( l_A(xt) = l_A(x) - 1 \). By \cite[Proposition 1.9]{4}, the twisted Chevalley-Bruhat order \( \leq_A \) satisfies the following properties.

3.4. Proposition. Let \( x, y \in W \), \( s \in S \), and \( t \in S' \):

(i) If \( sx \leq_A x \) and \( sy \leq_A y \), then \( sx \leq_A sy \) iff \( sx \leq_A y \) iff \( x \leq_A y \).

(ii) If \( xt \leq_A x \) and \( yt \leq_A y \), then \( xt \leq_A yt \) iff \( xt \leq_A y \) iff \( x \leq_A y \).

We call (i) (resp., (ii)) in Proposition 3.4 the left (resp., right) \( Z \)-property of \( \leq_A \).

3.5. Lemma. Regard the power set of \( W \) as a monoid under the product \( XY = \{ xy \mid x \in X, y \in Y \} \). For \( w \in W \), let \([1, w] = \{ z \in W \mid 1 \leq_0 z \leq_0 w \} \). Suppose that \( B \) is a subset of \( W \) with a maximum (resp., minimum) element in the order \( \leq_A \). Then \([1, w]B \) has a maximum (resp., minimum) element in \( \leq_A \).

Proof. We treat only the case \( B \) has a maximum element \( m \). It is well-known that if \( r_1 \cdots r_n \) is a reduced expression for \( w \), then \([1, w] = [1, r_1] \cdots [1, r_n] \). Hence we may assume \( w = r \in S \). Let \( m' \) be the maximum of \( m \) and \( rm \) in \( \leq_A \). The left \( Z \)-property implies that \( m' \) is a maximum element of \([1, r]B \). \( \square \)

3.6. A relation on \( W \times W \). For \((x, y), (x', y') \in W \times W \), write \((x, y) \rightarrow (x', y') \) (see \cite[§2]{4}) if there exists \( s \in S \) satisfying one of the following conditions:

(a) \( s \in L_A(x), s \in L_A(y), x' = sx, y' = sy \),
(b) \( s \notin L_A(x), s \notin L_A(y), x' = sx, y' = sy \),
(c) \( s \notin L_A(x), s \in L_A(y), x' = x, y' = sy \),
(d) \( s \notin L_A(x), s \in L_A(y), x' = sx, y' = y \).

Define the relation \( \rightarrow \) on \( W \times W \) to be the (reflexive) transitive closure of the relation \( \rightarrow' \). It is known by \cite[Lemma 2.2 and Proposition 2.5]{4} that if \((v, w) \rightarrow (1, 1) \) then \( v \leq_A w \) and the closed interval \([v, w] \) in \( \leq_A \) is finite.

3.7. Lemma. Suppose \( v, w \in W \) with \((v, w) \rightarrow (1, 1) \). If \( v \leq_A z \leq_A w, J \subseteq S \) and \( wv^{-1} \in W_J \), then \( zv^{-1}, wz^{-1} \in W_J \).

Proof. This holds by \cite[Lemma 2.12]{4}. \( \square \)
3.8. $R^A$-polynomials. Recall that $R = \mathbb{Z}[u, u^{-1}]$ and $\alpha = u - u^{-1}$. By [4, Corollary 3.6 (1)], there is a unique family of polynomials $R^A_{x,y} \in \mathbb{Z}[\alpha]$ defined for $x, y \in W$ with $(x, y) \rightarrow (1, 1)$ such that

(a) $R^A_{1,1} = 1$;
(b) if $x, y \in W$ satisfy $(x, y) \rightarrow (1, 1)$ and $s \in \mathcal{L}(y)$, then

$$R^A_{x,y} = \begin{cases} R^A_{s,sy} & \text{if } s \in \mathcal{L}(x) \\ R^A_{s,ys} + \alpha R^A_{s,y} & \text{if } s \notin \mathcal{L}(x), \end{cases}$$

where it is understood that in case $s \notin \mathcal{L}(A(x))$ and $sx \not\leq_A sy$, the term $R^A_{s,ys}$ is zero.

3.9. Generating function formula. The proof that the $R^A_{x,y}$ are well-defined is accomplished in conjunction with the proof of a generating function formula

$$R^A_{x,y} = \sum_{(t_1,\ldots,t_n)} \alpha^n,$$

for $(x, y) \rightarrow (1, 1)$, where the sum is over $n \in \mathbb{N}$ and $n$-tuples $(t_1,\ldots,t_n)$ of elements of $T$ satisfying $x \leq_A t_1x \leq_A t_2t_1x \leq_A \cdots \leq_A t_n \cdots t_2t_1x = y$ and $t_1 \leq t_2 \leq \cdots \leq t_n$ with $\leq$ a fixed but arbitrary reflection order on $T$.

By [4, Corollaries 3.6 and 3.7], if $(x, y) \rightarrow (1, 1)$, then $\sum_{x \leq_A z \leq_A y} R^A_{x,z}(u)R^A_{z,y}(u^{-1}) = \delta_{xy}$ and $R^A_{x,y}$ is a monic polynomial in $\alpha$ of degree $l_A(y) - l_A(x)$. We also note that if $A = \emptyset$, then $R^A_{x,y} = \tilde{R}_{x,y}$ is exactly the polynomial described in [4].

3.10. From [4, §4], there are unique $p_A(v, w) \in \mathbb{Z}[u, u^{-1}]$ defined for $(v, w) \rightarrow (1, 1)$, such that

(a) $p_A(v, w) = 1$;
(b) $p_A(v, w) \in u^{-1}\mathbb{Z}[u^{-1}]$ if $v \neq w$;
(c) $p_A(v, w) = \sum_{v \leq_A z \leq_A w} R_A(v, z)p_A(z, w)$

We define the “Kazhdan-Lusztig” polynomials $P_A(v, w) = P_A(v, w)(q) \in \mathbb{Z}[q]$ for $\leq_A$ by the formula

$$(u^{l_A(w)} - l_A(v))p_A(v, w) = P_A(v, w)(u^2),$$

if $(v, w) \rightarrow (1, 1)$.

The following lemma summarizes some simple relations we shall require between objects associated to varying initial sections $A$.

3.11. Lemma. Let $A$ be an initial section of reflection orders on $T$ and $v, w \in W$. Then

(i) $T + A$ and $x.A$ for $x \in W$ are initial sections of reflection orders on $T$.
(ii) We have $l_{T+A}(w) = -l_A(w)$ and $(W, \leq_{T+A}) = (W, \leq_A)^{op}$, $R^A_{w,v} = R^A_{v,w}$.
(iii) For a fixed $x \in W$, the map $v \mapsto vx$ defines a poset isomorphism $\phi: (W, \leq_A) \rightarrow (W, \leq_A)$ satisfying $l_A(\phi(w)) = l_{x.A}(w) + l_A(x)$, $R^A_{\phi(v),\phi(w)} = R^A_{v,w}$, and $P_A(\phi(v),\phi(w)) = P_{x.A(V),v}$.

(Here, equality of $R^A$-polynomials or $P_A$-polynomials is interpreted in the sense that if one side is not defined then neither is the other.)

Proof. Part (i) is in [4, Lemma 2.7]. For (ii) and (iii), it is enough by the definitions of $\leq_A$, $R^A$, $P_A$ to prove the assertions on the length functions; for (ii), this is immediate from the definition, and for (iii), it is established in the proof of [4, Proposition 1.1].
3.12. Modules over the Iwahori-Hecke algebra. Let $\mathcal{H}$ and $\mathcal{H}'$ be the Iwahori-Hecke algebras of $(W, S)$ and $(W', S')$, respectively. Denote the standard basis of $\mathcal{H}'$ over $\mathcal{R}$ by $\{ \tilde{T}'_x \mid x \in W \}$. Let $\mathcal{H}_A$ be the set of formal $\mathcal{R}$-linear combinations $\sum_{x \in W} a_x \tilde{T}_x$ such that there exists $y \in W$ so that $a_x = 0$ unless $x \leq_A y$. $\mathcal{H}_A$ admits an $(\mathcal{H}, \mathcal{H}')$-bimodule structure (see [4, §4]). If $s \in S$, then $\tilde{T}'_s(\sum_{x \in W} a_x \tilde{T}_x) = \sum_{x \in W} b_x \tilde{T}_x$, where

$$b_x = \begin{cases} a_{sx} & \text{if } s \notin \mathcal{L}_A(x) \\ a_{sx} + \alpha a_x & \text{if } s \in \mathcal{L}_A(x). \end{cases}$$

If $t \in S'$, then $\left( \sum_{x \in W} a_x \tilde{T}_x \right) \tilde{T}'_t = \sum_{x \in W} c_x \tilde{T}_x$, where

$$c_x = \begin{cases} a_{xt} & \text{if } t \notin \mathcal{R}_A(x) \\ a_{xt} + \alpha a_x & \text{if } t \in \mathcal{R}_A(x). \end{cases}$$

In the rest of this section, we assume for simplicity that $x \leq_A y$ in $W$ iff $(x, y) \rightarrow (1, 1)$.

3.13. An Involution on $\mathcal{H}_A$. By [4, §1], there are semi-linear ring involutions on $\mathcal{H}$ and $\mathcal{H}'$ given by

$$\sum_{x \in W} a_x \tilde{T}_x = \sum_{x \in W} \tilde{a}_x (\tilde{T}'_{x^{-1}}), \quad \sum_{x \in W} a_x \tilde{T}'_x = \sum_{x \in W} \tilde{a}_x (\tilde{T}'_{x^{-1}})^{-1}. $$

(Here, semi-linearity is with respect to the involution $a \mapsto \tilde{a}$ of $\mathcal{R}$.) By [4, Proposition 4.2 (1)], the $\mathcal{R}$-module $\mathcal{H}_A$ admits a semi-linear involution defined by

$$\sum_{x \in W} a_x \tilde{T}_x = \sum_{x, y \in W, x \leq_A y} \tilde{a}_y R^A_{x,y}(u) \tilde{T}_x.$$

3.14. Lemma. These involutions are compatible; namely, $h \cdot m \cdot h' = \tilde{h} \cdot \tilde{m} \cdot \tilde{h}'$ for $m \in \mathcal{H}_A$, $h \in \mathcal{H}$, and $h' \in \mathcal{H}'$.

Proof. See [4, Proposition 4.2 (2) and Corollary 4.16].

The following is the right analogue of [4, §5] (b).

3.15. Lemma. If $x, y \in W$ and $t \in S'$ satisfy $(x, y) \rightarrow (1, 1)$ and $t \in \mathcal{R}_A(y)$, then

$$R^A_{x,y} = \begin{cases} R^A_{x,t,y} & \text{if } t \in \mathcal{R}_A(x) \\ R^A_{x,t,y} + \alpha R^A_{x,y} & \text{if } t \notin \mathcal{R}_A(x). \end{cases}$$

Proof. It is easy to compute

$$\overline{(\tilde{T}'_x)} = \left\{ \sum_{z \in W} (R^A_{x,z} + \alpha R^A_{x,z}) \tilde{T}'_z \right\}_{t \in \mathcal{R}_A(x)}$$

$$\overline{(\tilde{T}'_x)} = \left\{ \sum_{z \in W} R^A_{x,z,t} \tilde{T}'_z \right\}_{t \notin \mathcal{R}_A(x)}$$

$$\overline{(\tilde{T}'_x)} = \left\{ \sum_{z \in W} R^A_{x,z,t} \tilde{T}'_z + \sum_{z \in W} (R^A_{x,z,t} + \alpha R^A_{x,z}) \tilde{T}'_z. \right\}_{t \notin \mathcal{R}_A(z)}$$

Since $\overline{(\tilde{T}'_x)} = \overline{(\tilde{T}'_y)}$, comparing the coefficients of $\tilde{T}'_t$ gives the desired recurrence formula. \(\square\)
4. Twisted Chevalley-Bruhat Order Associated to Standard Parabolic Subgroups

In this section, \((\hat{W}, \hat{S})\) denotes an arbitrary Coxeter system. Fix \(S \subseteq \hat{S}\) and set \(W = \hat{W}_{S}\). By \([4\text{, Proposition 2.11}]\), \(A := \hat{T} \setminus W\) is an initial section of a reflection order of \(\hat{W}\) and the corresponding partial order \(\leq A\) on \(W\) satisfies \(v \leq A w\) iff \((v, w) \rightarrow (1, 1)\).

4.1. Lemma. Let either \(B = A\) or \(B = 0\).

(i) \(S \subseteq \{ t \in \hat{T} \mid t \cdot B = B + \{ t \} \}\).

(ii) For \(w \in \hat{W}^S\) and \(x \in W\), \(l_A(wx) = l_0(x) - l_0(w)\) and \(l_0(wx) = l_0(x) + l_0(w)\).

(iii) Every element \(z\) of \(\hat{W}^S\) is the minimum element of its coset \(zW\) in the order \(\leq_B\).

(iv) The map \(\pi: W \to \hat{W}^S\) such that \(\pi(w) \in \hat{W}^S \cap wW\) is order-preserving for \(\leq_B\), i.e. \(x \leq_B y\) implies \(\pi(x) \leq_B \pi(y)\).

(v) The restrictions of \(\leq A\) and reverse Chevalley-Bruhat order to partial orders on \(\hat{W}^S\) coincide.

Proof. Part (i) follows by simple calculations which we omit. For (ii) and (v), see \([4\text{, Lemma 5.3 (1) and}\ (3)]\). For (iii), note that if \(x \in W\) has a reduced expression \(s_1 \cdots s_m\), then for \(z \in \hat{W}^S\), \(z \leq_B zs_1 \leq_B z_1 \cdots z_m \leq_B z\) by (ii) and the definition of \(\leq_B\). Finally, for (iv), suppose in the proof of (iii) that \(w \leq_B z\). Repeated application of the right \(Z\)-property for \(\leq_B\) gives some \(v \in W\) with \(wv \leq_B z\). Then by (iii), \(\pi(w) = \pi(wv) \leq_B wv \leq_B z = \pi(zx)\). \(\square\)

4.2. Lemma. Let \(z \in \hat{W}_{S_{\setminus S}}\) and \(I_z = S \cap zSz^{-1}\).

(i) \(I_z = \{ s \in S \mid sr = rs \text{ for all } r \in \text{Supp}(z) \}\), where for any \(x \in \hat{W}\), \(\text{Supp}(x)\) stands for the set of all simple reflections occurring in some (equivalently, all) reduced expression of \(x\).

(ii) \(z\) is the unique element of minimal length in the double coset \(WzW\).

(iii) For \(a \in \hat{W}^{I_z}\) and \(b \in W\) we have \(l_0(azb) = l_0(a) + l_0(z) + l_0(b)\); in particular, \(az \in \hat{W}^S\).

(iv) Every element of \(WzW\) is uniquely expressible in the form \(azb\), where \(a \in \hat{W}^{I_z}\) and \(b \in W\).

(v) \(W \cap zWz^{-1} = W_{I_z}\).

(vi) For \(a \in \hat{W}^{I_z}\) and \(b \in W\), \(l_A(azb) = -l_0(a) - l_0(z) + l_0(b)\).

(vii) \(\hat{W}_{S_{\setminus S}} \subseteq \hat{W}^S \cap (\hat{W}^S)^{-1}\).

Proof. Clearly, \(l_0(zr) = l_0(rz) = l_0(z) + 1\) for all \(r \in S\), which implies (vii). It is well-known that this implies (ii)–(v) (see \([3\text{, Proposition 2.7.5}]\)). Then (vi) follows from (iii) and \([3\text{, (ii)}\). (i)

(i) Clearly the right hand side is contained in the left hand side. For the reverse inclusion, it will suffice to show that if \(z, z' \in \hat{W}_{S_{\setminus S}}\), \(s, t \in S\) and \(sz = z't\) then \(z = z'\), \(s = t\) and \(s\) commutes with all \(r \in \text{Supp}(z)\). To see this, note that since \(\text{Supp}(sz) = \text{Supp}(z) \cup \{ s \}\) and \(\text{Supp}(zt) = \text{Supp}(z) \cup \{ t \}\) are disjoint unions, we must have \(s = t\). The remainder of the proof is by induction on \(l_0(z)\). Write \(z = s_1 \cdots s_m\) (reduced) and let \(m \geq 1\). By the exchange property \(ss_1 \cdots s_{m-1}\) equals either \(z'\) or \(z''\) with \(z'' \leq_B z'\). The first case is impossible, because \(s \notin \text{Supp}(z')\). By induction \(z'' = s_1 \cdots s_{m-1}\), and \(s\) commutes with \(s_1, \ldots, s_{m-1}\). Then

\[
ss_1 \cdots s_m = s_1 \cdots s_{m-1} \cdot s m = s_1 \cdots s_m s,
\]
and one sees that $s$ also commutes with $s_m$. \qed

4.3. Lemma. Assume that $z_1, z_2 \in \hat{W}_{\hat{S} \setminus S}$, $a_1 \in W^{l_{z_1}}$, $a_2 \in W^{l_{z_2}}$, and $b_1, b_2 \in W$ satisfy $a_1 z_1 b_1 \leq_A a_2 z_2 b_2$.

(i) If $x \in \hat{W}$ satisfies $a_1 z_1 b_1 \leq_A x \leq_A a_2 z_2 b_2$, then there exist $z_3 \in \hat{W}_{\hat{S} \setminus S}$, $a_3 \in W^{l_{z_3}}$, and $b_3 \in W$ such that $x = a_3 z_3 b_3$.

(ii) $z_2 \leq_\emptyset z_1$.

Proof. We use the map $\pi$ from Lemma 4.1. Note $\pi(a_i z_i b_i) = a_i z_i$ and $\pi(z_i^{-1} a_i^{-1}) = z_i^{-1}$ by Lemma 4.2. We get by Lemma 4.1 (iv) that $a_1 z_1 \leq_A \pi(x) \leq_A a_2 z_2$; so by Lemma 4.1 (v), $a_2 z_2 \leq_\emptyset \pi(x) \leq_\emptyset a_1 z_1$. Taking inverses throughout, applying $\pi$ then taking inverses again gives $z_2 \leq_\emptyset v := \left(\pi(\pi(x)^{-1})\right)^{-1} \leq_\emptyset z_1$. Since $v$ is a subword of $z_1$ (or by Lemma 3.7 we have $v \in \hat{W}_{\hat{S} \setminus S}$; by definition, $v \in W x W$). Applying Lemma 4.2 gives (i), and we get (ii) by taking $x = a_1 z_1 b_1$ above. \qed

4.4. Lemma. Let $z_1, z_2 \in \hat{W}_{\hat{S} \setminus S}$. Then $R^A_{z_1, z_2} = \hat{R}_{z_2, z_1}$.

Proof. Observe that for any $z \in \hat{W}_{\hat{S} \setminus S}$ and $s \in \hat{S} \setminus S$, we have $z \leq_A sz$ iff $sz \leq_\emptyset z$. We shall proceed by induction on $l_\emptyset(z_1)$.

If $l_\emptyset(z_1) = 0$, then $z_1 = 1$. If either $R^A_{z_1, z_2}$ or $\hat{R}_{z_2, z_1}$ is non-zero, then by Lemma 4.1 (v) and 3.9 it follows that $z_2 = 1$ and that $R^A_{z_1, z_2} = 1 = \hat{R}_{z_2, z_1}$.

Suppose that $R^A_{z_1, z_2} = \hat{R}_{z_2, z_1}'$ for any $z_1', z_2' \in \hat{W}_{\hat{S} \setminus S}$ with $l_\emptyset(z_1') < l_\emptyset(z_1)$. Pick an $s \in \hat{S}$ such that $s z_1 \leq_\emptyset z_1$. If $s z_2 \leq_\emptyset z_2$, then by the induction hypothesis and the recurrence formula in 3.8 we obtain $R^A_{z_1, z_2} = R^A_{sz_1, sz_2} = \hat{R}_{sz_2, sz_1} = \hat{R}_{z_2, z_1}$. If $z_2 \leq_\emptyset sz_2$, then $R^A_{z_1, z_2} = R^A_{z_1, sz_2} + \bar{a} R^A_{z_1, sz_2} = R^A_{z_1, sz_2} + \bar{a} R^A_{z_1, sz_2} = \hat{R}_{z_2, z_1}$. \qed

4.5. Lemma. Assume that $z_1, z_2 \in \hat{W}_{\hat{S} \setminus S}$, $a_1 \in W^{l_{z_1}}$, $a_2 \in W^{l_{z_2}}$, and $b_1 \in W$. Then

$$R^A_{a_1 z_1 b_1, a_2 z_2} = \hat{R}_{z_2, z_1} \hat{R}_{a_2 b_1^{-1}, a_1} \chi(I_{z_2, b_1}),$$

where $\chi(I, b) := \begin{cases} 1 & \text{if } b \in \hat{W}^I \\ 0 & \text{otherwise.} \end{cases}$

Proof. We shall proceed by induction on $l_\emptyset(a_1)$.

If $l_\emptyset(a_1) = 0$, then $a_1 = 1$. If the LHS is non-zero, then $z_1 \leq_A z_1 b_1 \leq_A a_2 z_2 \leq_A z_2$, so $b_1 = a_2 = 1$ by Lemma 3.7. If the RHS is non-zero, then $a_2 b_1^{-1} \leq_\emptyset 1$ implies $a_2 = b_1 = 1$ since $l_\emptyset(a_2 b_1^{-1}) = l_\emptyset(a_2) + l_\emptyset(b_1)$. Hence if either side is non-zero, then $a_2 = b_1 = 1$ and the assertion reduces to Lemma 4.4. On the other hand, if both sides are zero, they are equal.

Suppose that $R^A_{a_1 z_1 b_1', a_2 z_2'} = \hat{R}_{z_2, z_1} \hat{R}_{a_2 b_1'^{-1}, a_1} \chi(I_{z_2, b_1'})$ whenever $z_1', z_2' \in \hat{W}_{\hat{S} \setminus S}$, $a_1' \in W^{l_{z_1'}}$, $a_2' \in W^{l_{z_2'}}$, and $b_1' \in W$ subject to $l_\emptyset(a_1') < l_\emptyset(a_1)$. Pick an $s \in S$ with $sa_1 \leq_\emptyset a_1$. By Lemma 4.2 (vi), we have $a_1 z_1 b_1 \leq_A sa_1 z_1 b_1$ (we omit further reference to our use of Lemma 4.2 (vi) below in view of their frequency). We need to deal with three different situations.
(1) If \( sa_2 \leq a_2 \), then \( a_2z_2 \leq_A sa_2z_2 \). We obtain by \( \ref{lem:induction} \) and induction that
\[
R^A_{a_1z_1b_1,a_2z_2} = R^A_{sa_1z_1b_1,sa_2z_2} = \tilde{R}_{z_2z_1} \tilde{R}_{sa_2b_1^{-1},sa_1} \chi(I_{z_2},b_1) = \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},a_1} \chi(I_{z_2},b_1).
\]
(2) If \( a_2 \leq sa_2 \) and \( sa_2 \in W^{I_{z_2}} \), then \( sa_2z_2 \leq_A a_2z_2 \). We obtain
\[
R^A_{a_1z_1b_1,a_2z_2} = R^A_{sa_1z_1b_1,a_2z_2} + \alpha R^A_{a_1z_1b_1,a_2z_2} = (\tilde{R}_{z_2z_1} \tilde{R}_{sa_2b_1^{-1},sa_1} + \alpha \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},sa_1}) \chi(I_{z_2},b_1) = \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},a_1} \chi(I_{z_2},b_1).
\]
(3) If \( a_2 \leq sa_2 \) and \( sa_2 = a_2t \) for some \( t \in W_{I_{z_2}} \) then \( a_2z_2 \leq_A a_2z_2t = sa_2z_2t \). Note \( \chi(I_{z_2},b_1t) = \chi(I_{z_2},b_1) \). We obtain \( R^A_{a_1z_1b_1,a_2z_2} = R^A_{sa_1z_1b_1,sa_2z_2} = R^A_{sa_1z_1b_1,a_2z_2t} \). If \( b_1t \leq b_1 \), then Lemma \( \ref{lem:induction} \) gives
\[
R^A_{a_1z_1b_1,a_2z_2} = R^A_{sa_1z_1b_1t,a_2z_2} = \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},a_1} \chi(I_{z_2},b_1) = \tilde{R}_{z_2z_1} \tilde{R}_{sa_2b_1^{-1},sa_1} \chi(I_{z_2},b_1) = \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},a_1} \chi(I_{z_2},b_1).
\]
If \( b_1 \leq b_1t \), then using Lemma \( \ref{lem:induction} \) again,
\[
R^A_{a_1z_1b_1,a_2z_2} = R^A_{sa_1z_1b_1,a_2z_2} + \alpha R^A_{sa_1z_1b_1,t,a_2z_2} = (\tilde{R}_{z_2z_1} \tilde{R}_{sa_2b_1^{-1},sa_1} + \alpha \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},sa_1}) \chi(I_{z_2},b_1) = \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},sa_1} \chi(I_{z_2},b_1) = \tilde{R}_{z_2z_1} \tilde{R}_{a_2b_1^{-1},a_1} \chi(I_{z_2},b_1).
\]

5. **Proof of Results**

In this section, we fix a Coxeter system \((W, S)\) with finitely many simple reflections and let \((\tilde{W}, \tilde{S})\), \(z_I \in \tilde{W}_{\tilde{S}, S} \) for \( I \subseteq S, \Omega, \varphi \) be as in Section \( \ref{sec:intro} \). We let \( A, I_z \) for \( z \in \tilde{W}_{\tilde{S}, S}, \leq_A, R^A_{z_I}, H \) etc be associated to \((\tilde{W}, \tilde{S})\) and \((W, S)\) as in Section \( \ref{sec:modular} \).

5.1. **Proof of Theorem \( \ref{thm:main1} \)**. Note that \( z_0 \) is a Coxeter element of \( \tilde{W}_{\tilde{S}, S} \) and for \( I, J \subseteq S, z_I \leq_A z_J \) iff \( z_J \leq z_I \) if \( I \subseteq J \), using \( \ref{lem:order} \) and the definition of the \( z_I \). From Lemma \( \ref{lem:order} \) (i), one sees that \( I_{z_J} = J \) for \( J \subseteq S \). Now Theorem \( \ref{thm:main1} \) (i) holds by Lemma \( \ref{lem:order} \). By Lemma \( \ref{lem:order} \) (iv) and (vi), \( \varphi \) is a bijection and Theorem \( \ref{thm:main1} \) (iii) holds. The definitions give Theorem \( \ref{thm:main1} \) (iv) first for \((a,b) = (r,1)\) or \((a,b) = (1,r)\) with \( r \in S \) and then it follows immediately in general. Let \( \mathcal{H}'_A \) denote the \( R \)-submodule of \( \mathcal{H}_A \) with a basis \( \tilde{l}_x \) for \( x \in \Omega \). It becomes an \( H \)-module under the \( R \)-linear action given by \( \tilde{T}_{(a,b)m} = \tilde{T}_a \tilde{T}_{b^{-1}} \) and...
the map $\beta : \mathcal{M} \to \mathcal{H}'_A$ given by $\tilde{m}_v \mapsto \tilde{t}_{\varphi(v)}$ is immediately seen to be an $H$-module isomorphism from Theorem 2.3 (iii) and (iv), since $\varphi$ is bijective. This isomorphism extends to an $H$-module isomorphism $\hat{\beta} : \hat{\mathcal{M}} \to \hat{\mathcal{H}}'_A$ of their completions (consisting of formal $\mathcal{R}$-linear combinations of their standard basis elements).

Define $\Delta' : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ as $\Delta'(m) = \hat{\beta}^{-1}(\hat{\beta}m)$, where $\sum_{x \in \Omega} a_x \hat{t}_x = \sum_{x, y \in \Omega} u_x R^A_{y, x} \hat{t}_y$. That is, $\Delta'(\tilde{m}_v) = \sum_{w \in V} R^A_{\varphi(w), \varphi(v)} \tilde{m}_w$ for $v \in V$. It is easy to see from 3.8 with $A = \emptyset$ that for $I, J \subseteq S$, $\hat{R}_{I, J}$ is equal to $(\hat{\alpha}) |J| - |I|$ if $I \subseteq J$ and is zero otherwise. By Lemma 3.3 and an argument similar to the proof of Lemma 3.15 it easily follows that $\Delta'$ has the properties 1.3 (a) and (b), so $\Delta' = \Delta$, establishing Theorem 2.3 (v).

Finally, the fact that $\varphi$ is a poset isomorphism follows from (v), since $v \leq w$ in $V$ iff $b_{w, v} \neq 0$ and $x \leq_A y$ in $\Omega$ iff $R^A_{x, y} \neq 0$ from 3.9.

5.2. Proofs of the corollaries. Corollary 2.4 (i) follows immediately from Theorem 2.3 and 3.9. The proofs of Corollary 2.4 (ii) and (iii) are similar using [1] Proposition 3.9 and Corollary 3.10.

Corollary 2.5 follows immediately from the definitions and Theorem 2.3.

For Corollary 2.3 (i), note $b'_{w, v}(1) \neq 0$ is equivalent to $(R^A_{\varphi(w), \varphi(v)})'(1) \neq 0$ which is equivalent in turn to the requirement that as a polynomial in $\hat{\alpha}$, the coefficient of $\hat{\alpha}$ in $R^A_{\varphi(w), \varphi(v)}$ is non-zero. From this is in turn equivalent to the condition that $\varphi(w) = t \varphi(v) \leq_A \varphi(v)$ for some $t \in \hat{T}$. Now Corollary 2.7 (i) and (ii) follow from the definition of $\leq_A$ and [7] respectively.

For the proof of Corollary 2.6, assume $W$ is finite. We have

$$w_S.A = N(w_S) + w_S A w_S = T + w_S (\hat{T} \setminus W) w_S = T + (\hat{T} \setminus T) = \hat{T},$$

so $\leq_{w_S, A}$ is the reverse Chevalley-Bruhat order. It now follows from Theorem 2.3 and Lemma 3.11 that $\varphi'$ defines an order isomorphism of $V$ with a locally closed subset $\Omega$ of $W$ in reverse Chevalley-Bruhat order, satisfying all listed properties except perhaps $\Omega = \{w_S z \varnothing \}$. But $\Omega = \{1, w_S\} \sqcup \{1, z \varnothing\} \sqcup \{w_S\}$ has maximum and minimum elements in reverse Chevalley-Bruhat order by Lemma 3.3, so $\Omega$ is an interval since it is locally closed. We have $l_\varnothing(1) = 0$ (resp., $l_\varnothing(w_S z \varnothing) = 2l_\varnothing(w_S) + |S|$) which is obviously minimal (resp., maximal) among the values $l_\varnothing(c)$ for $c \in \Omega$, so $\Omega = \{w_S z \varnothing\}$ as required.

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