SHORT INJECTIVE PROOFS OF THE ERDŐS-KO-RADO AND HILTON-MILNER THEOREM:  
A CANONICAL PARTITION OF SHIFTED INTERSECTING SET SYSTEMS  

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\textbf{Abstract.} We give a canonical partition of shifted intersecting set systems, from which one can obtain unified and elementary proofs of the Erdős-Ko-Rado and Hilton-Milner Theorem, as well as a characterization of maximal shifted $k$-uniform intersecting set systems over $[n]$.

1. Introduction

A $k$-uniform intersecting set system on $[n] = \{1, 2, \ldots, n\}$ is a collection of $k$-element subsets of $[n]$ such that any two subsets in the collection have nonempty intersection. One of the classical and fundamental result in extremal set theory is the Erdős-Ko-Rado Theorem [1] on the maximum size of a $k$-uniform intersecting set system. In particular, it shows that the maximum size of a $k$-uniform intersecting set system is attained by the collection of all $k$-element subsets of $[n]$ containing a fixed element.

\textbf{Theorem 1.1.} Let $F$ be a $k$-uniform intersecting set system on $[n]$. If $n \geq 2k$, then $|F| \leq \binom{n-1}{k-1}$. Furthermore, if $n > 2k$, equality occurs if and only there exists $x$ such that $F$ consists of all subsets of size-$k$ of $[n]$ containing $x$.

A strengthening of the Erdős-Ko-Rado Theorem was shown by Hilton and Milner [6], which gives a tight upper bound on the size of an intersecting $k$-uniform set system in which no element is contained in all sets in the system.

\textbf{Theorem 1.2.} Let $F$ be a $k$-uniform intersecting set system on $[n]$. Assume that $n > 2k$, and no element is contained in all sets of $F$. Then $|F| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Furthermore, when $k \geq 4$, equality is attained if and only if $F$ is isomorphic to the family $\{A \in \binom{[n]}{k} : 1 \in A\} \cup \{\{2, 3, \ldots, k+1\}\}$, and when $k = 3$, equality is attained if and only if $F$ is isomorphic to either the family $\{A \in \binom{[n]}{3} : 1 \in A\} \cup \{\{2, 3, 4\}\}$, or the family $\{A \in \binom{[n]}{3} : |A \cap \{1, 2, 3\}| = 2\}$.

Since the original papers, there have been several alternative proofs of the Erdős-Ko-Rado and Hilton-Milner Theorem, [4, 3, 5, 7, 8]. In this paper, we provide short elementary injective proofs of the Erdős-Ko-Rado and Hilton-Milner Theorem on intersecting set systems (with equality cases characterized). Our approach is based on a canonical partition of the set system, and allows to prove the Hilton-Milner Theorem with a unifying and essentially identical argument as for the Erdős-Ko-Rado Theorem.

The canonical partition of the set system also allows to give a characterization of maximal shifted $k$-uniform intersecting set systems over $[n]$ via appropriate intersecting set systems over $[2k - 1]$ for any $n$. As an immediate corollary, we can show that the number of maximal shifted $k$-uniform intersecting set systems over $[n]$ is bounded by a constant depending only on $k$.

We expect that the same argument would be useful for studying intersecting set systems with additional constraints.
2. Main results

2.1. Compression. First, we introduce the standard compression (shifting) technique together with its basic properties.

Definition 2.1. A set system $\mathcal{F}$ on the universe $[n] = \{1, 2, \ldots, n\}$ is said to be shifted if for all $i < j$ and all $S \in \mathcal{F}$ with $j \in S$ and $i \notin S$, the set $S' = S - \{j\} + \{i\} \in \mathcal{F}$.

Any set system can be transformed into a shifted set system by using the following operations: Define an $(i, j)$-shift of $\mathcal{F}$ to be the set system $\mathcal{F}'$ obtained by replacing $S \in \mathcal{F}$ with $j \in S$ and $i \notin S$ with $S' = S - \{j\} + \{i\}$ if $S' \notin \mathcal{F}$. It is easy to verify that upon applying an $(i, j)$-shift, an intersecting set system $\mathcal{F}$ remains intersecting. Furthermore, upon finitely many $(i, j)$-shifts, we obtain a shifted set system.

For our later application to the Hilton-Milner Theorem, we will need the following result of Frankl [3].

Lemma 2.2. Let $n \geq 2k \geq 4$. Suppose that $\mathcal{F}$ is a $k$-uniform intersecting set system of $[n]$ with no element contained in all sets of $\mathcal{F}$. Then there exists an intersecting set system $\mathcal{F}'$ with no element contained in all sets of $\mathcal{F}'$, $|\mathcal{F}'| = |\mathcal{F}|$, and $\mathcal{F}'$ is shifted.

In particular, for the proof of Theorem 1.1 and Theorem 1.2, we can assume without loss of generality that the set system $\mathcal{F}$ is shifted.

2.2. Partitioning the set system. In this subsection, we introduce the key idea in our elementary proofs of Theorems 1.1 and 1.2, based on a canonical way to partition a shifted intersecting set system. Throughout this subsection and the remaining part of the paper, we assume that the intersecting set system $\mathcal{F}$ is intersecting and shifted.

Given two sets $A, B$ of integers of size $k$, we write $A \geq B$ (or $B \leq A$) if the elements of $A$ are $a_1 < a_2 < \ldots < a_k$ and the elements of $B$ are $b_1 < b_2 < \ldots < b_k$ and $a_i \geq b_i$ for all $i = 1, \ldots, k$.

Claim 2.3. If $\mathcal{F}$ is shifted and $A \in \mathcal{F}$ then $B \in \mathcal{F}$ for any $B \leq A$.

Proof. Assume that $B \leq A$. Since $b_1 \leq a_1$ and $\mathcal{F}$ is shifted, we have $\{b_1, a_2, \ldots, a_k\} \in \mathcal{F}$ (by considering the $(b_1, a_1)$-shift of $\mathcal{F}$ if $b_1 \neq a_1$). Inductively we can guarantee that $\{b_1, \ldots, b_i, a_{i+1}, \ldots, a_k\} \in \mathcal{F}$ for any $i \leq k$, so $B \in \mathcal{F}$. \hfill \Box

Lemma 2.4. If $\mathcal{F}$ is intersecting and shifted and $S \in \mathcal{F}$ has size $k$, then there exists $i \in [0, k - 1]$ with $|S \cap [2k - i - 1]| \geq k - i$.

Proof. Assume that $|S \cap [2k - i - 1]| < k - i$ for all $i \in [0, k - 1]$. Define $T$ to be the set of the first $k$ integers not contained in $S$. We claim that $T \leq S$. Indeed, let the elements of $T$ be $t_1 < t_2 < \ldots < t_k$. If $t_j > s_j$ for some $j \in [1, k]$, then $|S \cap [t_j - 1]| \geq j$. Furthermore, $t_j \leq k + j$ since any integer less than $t_j$ is either of the form $t_i, i < j$ or an element of $S$. Thus, $|S \cap [2k - (k - j) - 1]| \geq |S \cap [t_j - 1]| \geq j$, contradicting our assumption on $S$. Hence, there exists a set $T \leq S$ with $T$ disjoint from $S$. Since $\mathcal{F}$ is shifted, $T \in \mathcal{F}$, contradicting our assumption that $\mathcal{F}$ is intersecting. \hfill \Box

We next introduce our key partition of the set system $\mathcal{F}$. For each $S \in \mathcal{F}$, by Lemma 2.4, there exists $i \in [0, k - 1]$ such that $|S \cap [2k - i - 1]| \geq k - i$. Let $i_S$ be the smallest such $i$. We then define the subcollection of sets $\mathcal{F}_i$ to be the sets $S$ for which $i_S = i$. We say that $S \in \mathcal{F}_i$ has type $i$.

Lemma 2.5. We have $\mathcal{F} = \bigcup_{i=0}^{k-1} \mathcal{F}_i$, and for any $S \in \mathcal{F}_i$, $|S \cap [2k - i - 1]| = k - i$ and $2k - i \notin S$. 

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Proof. For each \( S \in F \), let \( i = i_S \) be the smallest integer in \( [0, k-1] \) such that \( |S \cap [2k-i-1]| \geq k - i \) (so \( S \in F_i \)). If \( i = 0 \), then \( |S \cap [2k-1]| \geq k \) so \( |S \cap [2k-1]| = k \) and \( S \setminus [2k-1] = \emptyset \), so \( S \in F_0 \). Next, assume that \( i \geq 1 \). Since \( i \) is smallest, \( |S \cap [2k-i]| < k - i + 1 \) and \( |S \cap [2k-i-1]| \geq k - i \). In particular, \( |S \cap [2k-i-1]| = |S \cap [2k-i]| = k - i \), which implies \( 2k - i \notin S \), so \( S \in F_i \).

For each set \( S \in \mathcal{F}_i \), define \( \pi_i(S) = S \cap [2k-i-1] \) and \( \psi_i(S) = S \setminus [2k-i-1] \subseteq [2k-i+1,n] \).

Lemma 2.6. The collection of sets \( \pi_i(F_i) := \{ \pi_i(S), S \in F_i \} \) is intersecting.

Proof. Assume that there exists \( S, S' \in F_i \) with \( \pi_i(S) \cap \pi_i(S') = \emptyset \). Then \( |\pi_i(S) \cup \pi_i(S')| = 2k - i \) and \( |[2k-i] \setminus (\pi_i(S) \cup \pi_i(S'))| = 1 \). Let \( T = [2k-i] \setminus (\pi_i(S) \cup \pi_i(S')) \). It is trivial that \( T \not\subseteq \psi_i(S') \), and thus, using shifts we can obtain the set \( \pi_i(S') \cup T \) from \( S' \). In particular, \( \pi_i(S') \cup T \in \mathcal{F} \), but this contradicts the assumption \( \mathcal{F} \) is intersecting as \( S \cap (\pi_i(S') \cup T) = \emptyset \).

2.3. Proof of the Erdős-Ko-Rado Theorem.

Proof of Theorem 1.1. We prove the theorem by induction on \( k \) and on \( n \). When \( k = 1 \), the claim is trivial for any \( n \geq 1 \).

Next, consider \( k \geq 2 \). For \( n = 2k \), the conclusion directly follows as subsets of \([2k]\) of size \( k \) can be partitioned into \( \frac{1}{2} \binom{2k}{k} = \binom{2k-1}{k-1} \) pairs of sets, such that the sets in each pair are disjoint. Thus, each intersecting system of \( k \)-sets in \([2k]\) has size at most \( \binom{2k-1}{k-1} \).

Next, consider \( n > 2k \). Observe that

\[
|\mathcal{F}| \leq \sum_{i=0}^{k-1} |\pi_i(F_i)|.
\]

For \( i > 0 \), we have that \( \pi_i(F_i) \) is a \((k-i)\)-uniform intersecting set system on the universe \([2k-i-1]\), and by the inductive hypothesis, \( |\pi_i(F_i)| \leq \binom{2k-i-2}{k-i-1} \). Since each set in \( F_i \) can be chosen by picking a set in \( \pi_i(F_i) \), and picking the remaining \( i \) elements in \([n] \setminus [2k-i] \) in \( \binom{n-2k+i}{i} \) ways,

\[
|\mathcal{F}| \leq |\pi_i(F_i)| \binom{n-2k+i}{i} \leq \binom{2k-i-2}{k-i-1} \binom{n-2k+i}{i}.
\]

Furthermore, consider the set system \( \mathcal{F}' \) as follows. For each set \( S \in \mathcal{F}_i \), define a set \( S' = \pi_i(S) \cup \{2k-i+1, \ldots, 2k\} \) and include it in \( \mathcal{F}' \). Then \( |\mathcal{F}'| = \sum_{i=1}^{k-1} |\pi_i(F_i)| \). Furthermore, \( \mathcal{F}' \) is intersecting: since \( \mathcal{F} \) is shifted, \( S \in \mathcal{F}_i \) implies that \( S' \in \mathcal{F}_i \). Hence, \( \mathcal{F}' \) is an intersecting system of \( k \)-sets of \([2k]\), which satisfies \( |\mathcal{F}'| \leq \binom{2k-1}{k-1} \) by the base case above. Thus, \( \sum_{i=0}^{k-1} |\pi_i(F_i)| \leq \binom{2k-1}{k-1} \).

Hence, we have

\[
|\mathcal{F}| \leq \sum_{i=1}^{k-1} |\pi_i(F_i)| \binom{n-2k+i}{i} + |\mathcal{F}_0|
\]

\[
\leq \binom{2k-1}{k-1} + \sum_{i=1}^{k-1} |\pi_i(F_i)| \left( \binom{n-2k+i}{i} - 1 \right)
\]

\[
\leq \binom{2k-1}{k-1} + \sum_{i=1}^{k-1} \binom{2k-i-2}{k-i-1} \left( \binom{n-2k+i}{i} - 1 \right)
\]

\[
= \sum_{i=0}^{k-1} \binom{2k-i-2}{k-i-1} \binom{n-2k+i}{i} + \binom{2k-1}{k-1} - \binom{2k-1}{k-1} - \sum_{i=1}^{k-1} \binom{2k-i-2}{k-1}
\]

\[
= \binom{n-1}{k-1}.
\]
where we have used that
\[
\binom{2k-2}{k-1} + \sum_{i=1}^{k-1} \binom{2k-i-2}{k-1} = \sum_{j=0}^{k-1} \binom{k-1+j}{k-1} = \sum_{j=0}^{k-1} \left( \binom{k+j}{k} - \binom{k+j-1}{k} \right) = \binom{2k-1}{k-1},
\]
(1)
and
\[
\sum_{i=0}^{k-1} \binom{2k-i-2}{k-i-1} \binom{n-2k+i}{i} = \binom{n-1}{k-1}.
\]
(2)
This latter equality can be justified as follows: for each subset of \( [n-1] \) of size \( k-1 \), there is a unique \( i \in [0, k-1] \) so that the set contains exactly \( i \) elements in \([2k-i, n-1]\) and \( k-i-1 \) elements in \([2k-i-2]\) (i.e., can be uniquely written as the union of a subset of \([2k-i, n-1]\) of size \( i \) and a subset of \([2k-i-2]\) of size \( k-i-1 \) (i.e., \( i \) is the largest integer so that the set contains at least \( i \) elements in \([2k-i, n-1]\)).

This shows the desired inequality. Furthermore, to attain equality, when \( n > 2k \), it must be the case that \( |\pi_i(F_i)| = \binom{2k-i-2}{k-i-1} \) for each \( i \geq 1 \) and \( |F_0| = \binom{2k-2}{k-1} \). By the inductive hypothesis, for \( i \geq 1 \), \( \pi_i(F_i) \) must only consist of sets that contain 1 (recall that \( \pi_i(F_i) \) is shifted). Then \( F_0 \) cannot contain any set without 1 as well, since the complement of that set in \([2k]\) must be a set of type \( i \geq 1 \) that contains 1. In particular, \( F \) can only consist of sets containing 1.

\[\square\]

2.4. Proof of the Hilton-Milner Theorem.

**Proof of Theorem 1.2.** We follow a similar scheme to our proof of Theorem 1.1. Consider \( n > 2k \) and the same decomposition of \( F \) into the families \( F_i \). We again have that
\[
|F| \leq \sum_{i=0}^{k-1} |F_i|,
\]
and as before, for \( i > 0 \), \( |\pi_i(F_i)| \leq \binom{2k-i-2}{k-i-1} \) and
\[
|F_i| \leq |\pi_i(F_i)| \binom{n-2k+i}{i} \leq \binom{2k-i-2}{k-i-1} \binom{n-2k+i}{i}.
\]
We also have
\[
\sum_{i=0}^{k-1} |\pi_i(F_i)| \leq \binom{2k-1}{k-1}.
\]
Since \( F \) contains at least one set that does not contain 1 and \( F \) is shifted, it must be the case that \( F \) contains \( \{2, 3, \ldots, k+1\} \). Then, \( F \) cannot contain the set \( \{1, k+2, \ldots, 2k\} \), and in particular, \( F \) contains no set of type \( k-1 \), i.e. \( |F_{k-1}| = |\pi_{k-1}(F_{k-1})| = 0 \).

Hence, we have that
\[
|F| \leq \sum_{i=0}^{k-1} |F_i| \leq \binom{2k-1}{k-1} + \sum_{i=1}^{k-2} |\pi_i(F_i)| \left( \binom{n-2k+i}{i} - 1 \right)
\]
\[
\leq \binom{2k-1}{k-1} + \sum_{i=1}^{k-2} \binom{2k-i-2}{k-i-1} \left( \binom{n-2k+i}{i} - 1 \right) - \left( \binom{n-k+1}{k-1} - 1 \right)
\]
\[
= \binom{2k-1}{k-1} + \sum_{i=1}^{k-2} \binom{2k-i-2}{k-i-1} \left( \binom{n-2k+i}{i} - 1 \right) - \left( \binom{n-k+1}{k-1} - 1 \right)
\]
\[
= \binom{n-1}{k-1} - \left( \binom{n-k+1}{k-1} - 1 \right),
\]
as desired (here we have used (1) and (2)).
Equality occurs only if \(|\pi_i(F_i)| = \binom{2k-i-2}{k-i-1}\) for each \(i \in [1, k-2]\), and \(|F_0| = \binom{2k-2}{k-1} + 1\). We then have that each \(F_i\), with \(2 \leq i \leq k-2\) is the collection of all sets of type \(i\) which contains 1.

If \(k \geq 4\), since one can check that the set \(\{1, k+1, k+3, \ldots, 2k\}\) has type \(k-2\), \(\mathcal{F}_{k-2}\) contains this set. In particular, the set \(\{2, 3, \ldots, k, k+2\}\) is not contained in \(\mathcal{F}\). Since \(\mathcal{F}\) is shifted, this implies that there can be no set in \(\mathcal{F}\) not containing 1 that is different from the set \(\{2, 3, \ldots, k+1\}\). Hence, we conclude that if equality occurs, then \(\mathcal{F}\) must be the family consisting of sets containing 1 that intersect \(\{2, 3, \ldots, k+1\}\), together with the set \(\{2, 3, \ldots, k+1\}\).

If \(k = 3\), then \(\pi_1(F_1)\) is a shifted intersecting family of size 3 consisting of subsets of size 2 in \(\{1, 2, 3, 4\}\). One can then check that \(\pi_1(F_1)\) is either the family of sets containing the element 1, or the family \(\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}\). From this, one can obtain that \(\mathcal{F}\) is either the family of sets containing the element 1 together with \(\{2, 3, 4\}\), or the family of sets intersecting \(\{1, 2, 3\}\) in a subset of size exactly 2. \(\square\)

2.5. A characterization of maximal shifted intersecting families. Here we record a characterization of maximal shifted intersecting families based on the partition of the set system in Section 2.2. Let \(\mathcal{F}\) be a maximal intersecting family which is shifted. Given a subset \(A'\) of \([2k-i]\) of size \(k-i\), we denote by \(S_i(A') = \{B \subseteq [n] : \pi_i(B) = A', B \setminus \pi_i(B) \subseteq [2k-i+1]\}\). For each \(A\) of type \(i\), define \(S(A) = S_i(\pi_i(A))\).

Lemma 2.7. Let \(A \in \mathcal{F}\) be set of type \(i\). Then \(\mathcal{F}\) contains \(S(A)\).

Proof. We show that any set \(B \in S(A)\) intersects all sets in \(\mathcal{F}\). Indeed, assume that \(C \in \mathcal{F}\) is disjoint from \(B\). Then \(C \cap \pi_i(A) = \emptyset\). Since \(\mathcal{F}\) is shifted, by shifting \(C\), we obtain that \([2k-i] \setminus \pi_i(A) \in \mathcal{F}\) (note that \(|[2k-i] \setminus \pi_i(A)| = k\), which is a contradiction as \(A \cap ([2k-i] \setminus \pi_i(A)) = \emptyset\).

Since \(\mathcal{F}\) is maximal, we then have that \(S(A) \subseteq \mathcal{F}\). \(\square\)

Corollary 2.8. There is a bijection between maximal shifted intersecting families on \([n]\) and shifted intersecting set systems \(\mathcal{G} = \bigcup_{i=0}^{k-1} \mathcal{G}_i\) over \([2k-1]\) where \(\mathcal{G}_i\) consists of sets of size \(k-i\) contained in \([2k-1]\).

Proof. To each maximal shifted intersecting family \(\mathcal{F}\) on \([n]\), we can associate a shifted intersecting set system \(\mathcal{G} = \bigcup_{i=0}^{k-1} \mathcal{G}_i\) over \([2k-1]\) where \(\mathcal{G}_i\) consists of sets of size \(k-i\) contained in \([2k-1]\), by defining \(\mathcal{G}_i\) to be the collection of \(\pi_i(A)\) for \(A \in \mathcal{F}\) of type \(i\). \(\mathcal{G}\) is an intersecting set system by Lemma 2.6. Conversely, for each shifted intersecting set system \(\mathcal{G} = \bigcup_{i=0}^{k-1} \mathcal{G}_i\) over \([2k-1]\), by Lemma 2.7, we can recover the maximal shifted intersecting set system \(\mathcal{F}\) on \([n]\) as the union of \(S_i(A_i)\) for \(A_i \in \mathcal{G}\) of size \(i\). One can easily check that this gives a bijection between maximal shifted intersecting families on \([n]\) and shifted intersecting set systems \(\mathcal{G} = \bigcup_{i=0}^{k-1} \mathcal{G}_i\) over \([2k-1]\). \(\square\)

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