A-INFINITY STRUCTURE ON EXT-ALGEBRAS

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Abstract. Let A be a connected graded algebra and let E denote its Ext-algebra $\bigoplus_i \text{Ext}^i_A(k_A, k_A)$. There is a natural $A_\infty$-structure on E, and we prove that this structure is mainly determined by the relations of A. In particular, the coefficients of the $A_\infty$-products $m_n$ restricted to the tensor powers of $\text{Ext}^1_A(k_A, k_A)$ give the coefficients of the relations of A. We also relate the $m_n$'s to Massey products.

Introduction

The notions of $A_\infty$-algebra and $A_\infty$-space were introduced by Stasheff in the 1960s [St1]. Since then, more and more theories involving $A_\infty$-structures (and its cousins, $E_\infty$, $L_\infty$, and $B_\infty$) have been discovered in several areas of mathematics and physics. Kontsevich's talk [Ko] at the ICM 1994 on categorical mirror symmetry has had an influence in developing this subject. The use of $A_\infty$-algebras in non-commutative algebra and the representation theory of algebras was introduced by Keller [Ke1, Ke2, Ke3]. Recently the authors of this paper used the $A_\infty$-structure on the Ext-algebra $\text{Ext}^*_A(k_A, k_A)$ to study the non-Koszul Artin-Schelter regular algebras A of global dimension four [LP3]. The information about the higher multiplications on $\text{Ext}^*_A(k_A, k_A)$ is essential and very effective for the work [LP3].

Throughout let $k$ be a commutative base field. The definition of an $A_\infty$-algebra will be given in Section 1. Roughly speaking, an $A_\infty$-algebra is a graded vector space $E$ equipped with a sequence of "multiplications" $(m_1, m_2, m_3, \cdots)$: $m_1$ is a differential, $m_2$ is the usual product, and the higher $m_n$'s are homotopies which measure the degree of associativity of $m_2$. An associative algebra $E$ (concentrated in degree 0) is an $A_\infty$-algebra with multiplications $m_n = 0$ for all $n \neq 2$, so sometimes we write an associative algebra as $(E, m_2)$. A differential graded (DG) algebra $(E, d)$ has multiplication $m_2$ and derivation $m_1 = d$; this makes it into an $A_\infty$-algebra with $m_n = 0$ for $n \geq 3$ and so it could be written as $(E, m_1, m_2)$.

Let $A$ be a connected graded algebra, and let $k_A$ be the right trivial $A$-module $A/A_{\geq 1}$. The Ext-algebra $\bigoplus_{i \geq 0} \text{Ext}^i_A(k_A, k_A)$ of $A$ is the homology of a DG algebra, and hence by a theorem of Kadeishvili, it is equipped with an $A_\infty$-algebra structure. We use $\text{Ext}^i_A(k_A, k_A)$ to denote both the usual associative Ext-algebra and the Ext-algebra with its $A_\infty$-structure. By [LP3, Ex. 13.4] there is a graded algebra $A$ such that the associative algebra $\text{Ext}^*_A(k_A, k_A)$ does not contain enough information to recover the original algebra $A$; on the other hand, the information from the $A_\infty$-algebra $\text{Ext}^*_A(k_A, k_A)$ is sufficient to recover $A$. This is the point of Theorem A below, and this process of recovering the algebra from its Ext-algebra is one of the main tools used in [LP3].

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We need some notation in order to state the theorem. Let \( m = \bigoplus_{i \geq 1} A_i \) be the augmentation ideal of \( A \). We say that a graded vector space \( V = \bigoplus V_i \) is \textit{locally finite} if each \( V_i \) is finite-dimensional. We write the graded \( k \)-linear dual of \( V \) as \( V^\# \). As our notation has so far indicated, we use subscripts to indicate the grading on \( A \) and related vector spaces. For example,

\[
(m^\otimes m)_i = \bigoplus_{i_1 + \cdots + i_m = i} A_{i_1} \otimes \cdots \otimes A_{i_m}.
\]

Also, the grading on \( A \) induces a bigrading on \( \text{Ext} \). We write the usual, homological, grading with superscripts, and the second, induced, grading with subscripts.

Let \( Q = m/m^2 \) be the graded vector space of generators of \( A \). The relations in \( A \) naturally sit inside the tensor algebra on \( Q \). In Section 4 we choose a vector space embedding of each graded piece \( A_s \) into the tensor algebra on \( Q \): a map

\[
A_s \hookrightarrow (\bigoplus_{m \geq 1} Q^\otimes m)_s,
\]

which splits the multiplication map, and this choice affects how we choose the minimal generating set of relations. See Lemma 5.2 and the surrounding discussion for more details.

**Theorem A.** Let \( A \) be a connected graded locally finite algebra, and let \( E \) be the \( A_\infty \)-algebra \( \text{Ext}^*_A(k_A, k_A) \). Let \( Q = m/m^2 \) be the graded vector space of generators of \( A \). Let \( R = \bigoplus_{s \geq 2} R_s \) be a minimal graded space of relations of \( A \), with \( R_s \) chosen so that \( R_s \subset \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \subset \bigoplus_{m \geq 2} Q^\otimes m \). For each \( n \geq 2 \) and \( s \geq 2 \), let \( i_s : R_s \to (\bigoplus_{m \geq 2} Q^\otimes m)_s \) be the inclusion map and let \( i^\#_s \) be the composite

\[
R_s \xrightarrow{i_s} (\bigoplus_{m \geq 2} Q^\otimes m)_s \to (Q^\otimes n)_s.
\]

Then in any degree \( -s \), the multiplication \( m_n \) of \( E \) restricted to \( (E^1)^\otimes n \) is equal to the map

\[
(i^\#_s)^* : ((E^1)^\otimes n)_{-s} = ((Q^\otimes n)_s)^\# \to R^\#_s \subset E^2_{-s}.
\]

In plain English, the multiplication maps \( m_n \) on classes in \( \text{Ext}^1_A(k_A, k_A) \) are determined by the relations in the algebra \( A \).

Note that the space \( Q \) of generators need not be finite-dimensional – it only has to be finite-dimensional in each grading. Thus it applies to infinitely generated algebras like the Steenrod algebra.

The authors originally announced the result in the following special case; this was used heavily in [LP3].

**Corollary B** (Keller’s higher-multiplication theorem in the connected graded case). Let \( A \) be a graded algebra, finitely generated in degree 1, and let \( E \) be the \( A_\infty \)-algebra \( \text{Ext}^*_A(k_A, k_A) \). Let \( R = \bigoplus_{n \geq 2} R_n \) be a minimal graded space of relations of \( A \), chosen so that \( R_n \subset A_1 \otimes A_{n-1} \subset A_1^\otimes n \). For each \( n \geq 2 \), let \( i_n : R_n \to A_1^\otimes n \) be the inclusion map and let \( i_n^\# \) be its \( k \)-linear dual. Then the multiplication \( m_n \) of \( E \) restricted to \( (E^1)^\otimes n \) is equal to the map

\[
i^\#_n : (E^1)^\otimes n = (A_1^\#)^\otimes n \to R^\#_n \subset E^2.
\]

Keller has the same result for a different class of algebras; indeed, his result was the inspiration for Theorem A. His result applies to algebras the form \( k\Delta/I \) where \( \Delta \) is a finite quiver and \( I \) is an admissible ideal of \( k\Delta \); this was stated in [Ke1, Proposition 2] without proof. This class of algebras includes those in Corollary B but since the algebra \( A \) in Theorem A need not be finitely generated, that theorem
is not a special case of Keller’s result. A version of Corollary B was also proved in a recent paper by He and Lu [HL] for \( N \)-graded algebras \( A = A_0 \oplus A_1 \oplus \cdots \) with \( A_0 = k^{\oplus n} \) for some \( n \geq 1 \), and which are finitely generated by \( A_0 \oplus A_1 \). Their proof was based on the one here (see [HL, page 356]).

Here is an outline of the paper. We review the definitions of \( A_\infty \)-algebras and Adams grading in Section 1. In Section 2 we discuss Kadeishvili’s and Merkulov’s results about the \( A_\infty \)-structure on the homology of a DG algebra. In Section 3 we use Merkulov’s construction to show that the \( A_\infty \)-multiplication maps \( m_n \) compute Massey products, up to a sign – see Theorem 3.1 for details. In Section 4 the bar construction is described: this is a DG algebra whose homology is Ext, and so leads to an \( A_\infty \)-structure on Ext algebras. Then we give a proof of Theorem A in Section 5, and in Section 6 we give a few examples.

This paper began as an appendix in [LP3].

1. Definitions

In this section we review the definition of an \( A_\infty \)-algebra and discuss grading systems. Other basic material about \( A_\infty \)-algebras can be found in Keller’s paper [Ke3]. Some examples of \( A_\infty \)-algebras related to ring theory were given in [LP1]. Here is Stasheff’s definition.

**Definition 1.1.** [SI] An \( A_\infty \)-algebra over a base field \( k \) is a \( \mathbb{Z} \)-graded vector space

\[ A = \bigoplus_{p \in \mathbb{Z}} A^p \]

endowed with a family of graded \( k \)-linear maps

\[ m_n : A^\otimes n \to A, \quad n \geq 1, \]

of degree \( 2 - n \) satisfying the following Stasheff identities:

**SI(n)**

\[ \sum (-1)^{r+s+t} m_u(id^\otimes r \otimes m_s \otimes id^\otimes t) = 0 \]

for all \( n \geq 1 \), where the sum runs over all decompositions \( n = r + s + t \) (\( r, t \geq 0 \) and \( s \geq 1 \)), and where \( u = r + 1 + t \). Here, \( id \) denotes the identity map of \( A \). Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule. Some authors also use the terminology strongly homotopy associative algebra (or sha algebra) for \( A_\infty \)-algebra. The degree of \( m_1 \) is 1 and the identity **SI(1)** is \( m_1 m_1 = 0 \). This says that \( m_1 \) is a differential of \( A \). The identity **SI(2)** is

\[ m_1 m_2 = m_2(m_1 \otimes id + id \otimes m_1) \]

as maps \( A^{\otimes 2} \to A \). So the differential \( m_1 \) is a graded derivation with respect to \( m_2 \). Note that \( m_2 \) plays the role of multiplication although it may not be associative. The degree of \( m_2 \) is zero. The identity **SI(3)** is

\[ m_2(id \otimes m_2 - m_2 \otimes id) = m_1 m_3 + m_3(m_1 \otimes id \otimes id + id \otimes m_1 \otimes id + id \otimes id \otimes m_1) \]

as maps \( A^{\otimes 3} \to A \). If either \( m_1 \) or \( m_3 \) is zero, then \( m_2 \) is associative. In general, \( m_2 \) is associative up to a chain homotopy given by \( m_3 \).

When \( n \geq 3 \), the map \( m_n \) is called a higher multiplication. We write an \( A_\infty \)-algebra \( A \) as \( (A, m_1, m_2, m_3, \cdots) \) to indicate the multiplications \( m_i \). We also assume that every \( A_\infty \)-algebra in this paper contains an identity element \( 1 \) with respect to the multiplication \( m_2 \) that satisfies the following strictly unital condition:
If \( n \neq 2 \) and \( a_i = 1 \) for some \( i \), then \( m_n(a_1, \ldots, a_n) = 0 \).

In this case, 1 is called the strict unit or identity of \( A \).

We are mainly interested in graded algebras and their Ext-algebras. The grading appearing in a graded algebra may be different from the grading appearing in the definition of the \( A_\infty \)-algebra. We introduce the Adams grading for an \( A_\infty \)-algebra, as follows. Let \( G \) be an abelian group. (In this paper, \( G \) will always be free abelian of finite rank.) Consider a bigraded vector space

\[
A = \bigoplus_{p \in \mathbb{Z}} A^p_i
\]

where the upper index \( p \) is the grading appearing in Definition 1.1 and the lower index \( i \) is an extra grading, called the \( G \)-Adams grading, or Adams grading if \( G \) is understood. We also write

\[
A^p = \bigoplus_{i \in G} A^p_i \quad \text{and} \quad A_i = \bigoplus_{p \in \mathbb{Z}} A^p_i.
\]

The degree of a nonzero element \( p \) in the \( A^p_i \) is \((p, i)\), and the second degree is called the Adams degree. For an \( A_\infty \)-algebra \( A \) to have an Adams grading, the map \( m_n \) in Definition 1.1 must be of degree \((2 - n, 0)\): each \( m_n \) must preserve the Adams grading. When \( A \) is an associative \( G \)-graded algebra \( A = \bigoplus_{i \in G} A_i \), we view \( A \) as an \( A_\infty \)-algebra (or a DG algebra) concentrated in degree 0, viewing the given grading on \( A \) as the Adams grading. The Ext-algebra of a graded algebra is bigraded; the grading inherited from the graded algebra is the Adams grading, and we keep using the lower index to denote the Adams degree.

Assume now that \( G = \mathbb{Z} \), since we are mainly interested in this case. Write

\[
A_{\geq n} = \bigoplus_{p \geq n} A^p \quad \text{and} \quad A_{\leq n} = \bigoplus_{i \geq n} A_i,
\]

and similarly for \( A_{\leq n} \) and \( A_{\leq n} \). An \( A_\infty \)-algebra \( A \) with a \( \mathbb{Z} \)-Adams grading is called Adams connected if (a) \( A_0 = k \), (b) \( A = A_{\geq 0} \) or \( A = A_{\leq 0} \), and (c) \( A_i \) is finite-dimensional for all \( i \). When \( G = \mathbb{Z} \times G_0 \), we define Adams connected in the same way after omitting the \( G_0 \)-grading. If \( A \) is a connected graded algebra which is finite-dimensional in each degree, then it is Adams connected when viewed as an \( A_\infty \)-algebra concentrated in degree 0.

The following result is a consequence of Theorem A and it will be proved at the end. There might be several quasi-isomorphic \( A_\infty \)-structures on \( E := \text{Ext}_A^*(k_A, k_A) \): we call these different structures models for the quasi-isomorphism class of \( E \).

**Proposition 1.2.** Let \( A \) be a \( \mathbb{Z} \oplus G \)-Adams graded algebra, such that with respect to the \( \mathbb{Z} \)-grading, \( A \) is locally finitely generated. Then there is an \( A_\infty \)-model for \( E \) such that the multiplications \( m_n \) in Theorem A preserve the \( \mathbb{Z} \oplus G \) grading.

2. Kadeishvili’s theorem and Merkulov’s construction

Let \( A \) and \( B \) be two \( A_\infty \)-algebras. A morphism of \( A_\infty \)-algebras \( f : A \to B \) is a family of \( k \)-linear graded maps

\[
f_n : A^\otimes n \to B
\]

of degree \( 1 - n \) satisfying the following Stasheff morphism identities:

\[
\text{MI}(n) \sum (-1)^{r + st} f_n(id^\otimes r \otimes m_s \otimes id^\otimes t) = \sum (-1)^w m_n(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n})
\]
for all $n \geq 1$, where the first sum runs over all decompositions $n = r + s + t$ with $s \geq 1$ and $r, t \geq 0$, where $u = r + 1 + t$, and the second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + \cdots + i_q$ with all $i_s \geq 1$. The sign on the right-hand side is given by

$$w = (q - 1)(i_1 - 1) + (q - 2)(i_2 - 1) + \cdots + 2(i_{q-2} - 1) + (i_{q-1} - 1).$$

When $A$ and $B$ have a strict unit (as we always assume), an $A_\infty$-morphism is also required to satisfy the following extra unital morphism conditions:

$$f_1(1_A) = 1_B$$

where $1_A$ and $1_B$ are the strict units of $A$ and $B$ respectively, and

$$f_n(a_1 \otimes \cdots \otimes a_n) = 0$$

if $n \geq 2$ and $a_i = 1_A$ for some $i$.

If $A$ and $B$ have Adams gradings indexed by the same group, then the maps $f_i$ are required to preserve the Adams degree.

A morphism $f$ is called a quasi-isomorphism if $f_1$ is a quasi-isomorphism. A morphism is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism is the strict morphism $f$ such that $f_1$ is the identity of $A$. When $f$ is a strict morphism from $A$ to $B$, then the identity $\text{MI}(n)$ becomes

$$f_1 m_n = m_n(f_1 \otimes \cdots \otimes f_1).$$

A morphism $f = (f_i)$ is called a strict isomorphism if it is strict with $f_1$ a vector space isomorphism.

Let $A$ be an $A_\infty$-algebra. Its cohomology ring is defined to be

$$HA := \ker m_1 / \text{im} m_1.$$
Note that if $A$ has an Adams grading, then the decompositions above will be chosen to respect the Adams grading, and all maps constructed below will preserve the Adams grading.

Let $p = Pr_H : A \to A$ be a projection to $H := \bigoplus_n H^n$, and let $G : A \to A$ be a homotopy from $id_A$ to $p$. Hence we have $id_A - p = \partial G + G\partial$. The map $G$ is not unique, and we want to choose $G$ carefully, so we define it as follows: for every $n$, $G^n : A^\otimes n \to A^{n-1}$ is the map which satisfies

- $G^n = 0$ when restricted to $L^n$ and $H^n$, and
- $G^n = (\partial^{n-1}|_{L^{n-1}})^{-1}$ when restricted to $B^n$.

So the image of $G^n$ is $L^{n-1}$. It follows that $G^{n+1}\partial^n = Pr_{L^n}$ and $\partial^{n-1}G^n = Pr_{B^n}$.

Define a sequence of linear maps $\lambda_n : A^{\otimes n} \to A$ of degree $2 - n$ as follows. There is no map $\lambda_1$, but we formally define the “composite” $G\lambda_1$ by $G\lambda_1 = -id_A$. $\lambda_2$ is the multiplication of $A$, namely, $\lambda_2(a_1 \otimes a_2) = a_1 \cdot a_2$. For $n \geq 3$, $\lambda_n$ is defined by the recursive formula

$$\lambda_n = \sum_{s+t=n, s \geq 1} (-1)^{s+1} \lambda_2[G\lambda_s \otimes G\lambda_t].$$

We abuse notation slightly, and use $p$ to denote both the map $A \to A$ and also (since the image of $p$ is $HA$) the map $A \to HA$; we also use $\lambda_i$ both for the map $A^{\otimes i} \to A$ and for its restriction $(HA)^{\otimes i} \to A$ to $HA^{\otimes i}$.

Merkulov reproved Kadeishvili’s result in [Mc].

**Theorem 2.2.** [Mc] Let $m_i = p\lambda_i$. Then $(HA, m_2, m_3, \cdots)$ is an $A_{\infty}$-algebra.

We can also display the quasi-isomorphism between $HA$ and $A$ directly.

**Proposition 2.3.** Let $\{\lambda_n\}$ be defined as above. For $i \geq 1$ let $f_i = -G\lambda_i : (HA)^{\otimes i} \to A$, and for $i \geq 2$ let $m_i = p\lambda_i : (HA)^{\otimes i} \to HA$. Then $(HA, m_2, m_3, \cdots)$ is an $A_{\infty}$-algebra and $f := \{f_i\}$ is a quasi-isomorphism of $A_{\infty}$-algebras.

**Proof.** This construction of $\{m_i\}$ and $\{f_i\}$ is a special case of Kadeishvili’s construction. \hfill \Box

Any $A_{\infty}$-algebra constructed as in Theorem 2.2 and Proposition 2.3 is called a Merkulov model of $A$, denoted by $H_{Mer}A$. The particular model depends on the decomposition (2.1.1), but all Merkulov models of $A$ are quasi-isomorphic to each other. If $A$ has an Adams grading, then by construction all maps $m_i$ and $f_i$ preserve the Adams degree.

Next we consider the unital condition.

**Lemma 2.4.** Suppose $H^0$ is chosen to contain the unit element of $A$. Then $H_{Mer}A$ satisfies the strictly unital condition, and the morphism $f = \{f_i\}$ satisfies the unital morphism conditions.

**Proof.** First of all, $1 \in H^0$ is a unit with respect to $m_2$. We use induction on $n$ to show the following, for $n \geq 3$:

(a)$_n$: $f_{n-1}(a_1 \otimes \cdots \otimes a_{n-1}) = 0$ if $a_i = 1$ for some $i$.

(b)$_n$: $\lambda_n(a_1 \otimes \cdots \otimes a_n) \in L := \bigoplus_n L^n$ if $a_i = 1$ for some $i$.

(c)$_n$: $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some $i$.

The strictly unital condition is (c)$_n$. The unital morphism condition is (a)$_n$.

We first prove (a)$_3$. For $a \in H$,

$$f_2(1 \otimes a) = -G\lambda_2(1 \otimes a) = -G(a) = 0,$$
since \( G|_H = 0 \). Similarly, \( f_2(a \otimes 1) = 0 \). This proves \((a)_3\). Now suppose for some \( n \geq 3 \) that \((a)_i\) holds for all \( 3 \leq i \leq n \). By definition,

\[
\lambda_n = \sum_{s=1}^{n-1} (-1)^{s+1} \lambda_2(f_s \otimes f_{n-s}).
\]

If \( a_1 = 1 \), \((a)_n\) implies that

\[
\lambda_n(a_1 \otimes \cdots \otimes a_n) = f_{n-1}(a_2 \otimes \cdots \otimes a_n) \in L.
\]

Similarly, if \( a_n = 1 \), we have \( \lambda_n(a_1 \otimes \cdots \otimes a_n) \in L \). If \( a_i = 1 \) for \( 1 < i < n \), then \( \lambda_n(a_1 \otimes \cdots \otimes a_n) = 0 \). Therefore \((a)_i\) for \( i \leq n \) implies \((b)_n\). Since \( p(L) = 0 \), \((c)_n\) follows from \((b)_n\). Since \( G(L) = 0 \), \((a)_{n+1}\) follows from \((b)_n\). Induction completes the proof.

**Lemma 2.5.** Let \((A, \partial)\) be a DG algebra and let \( e \in A^0 \) be an idempotent such that \( \partial(e) = 0 \). Let \( D = eAe \) and \( C = (1 - e)A + A(1 - e) \).

(a) If \( HC = 0 \), then we can choose Merkulov models so that \( H_{Mer}A \) is strictly isomorphic to \( H_{Mer}D \). As a consequence \( A \) and \( D \) are quasi-isomorphic as \( A_\infty\)-algebras.

(b) If moreover \( HA \) is Adams connected, then \( H^0_{Mer}A \) and \( H^0_{Mer}D \) in part (a) can be chosen to contain the unit element.

**Proof.** First of all, \( D \) is a sub-DG algebra of \( A \) with identity \( e \). Since \( A = D \oplus C \) as chain complexes, the group of coboundaries \( B^n \) decomposes as \( B^n = B^n_D \oplus B^n_C \), where \( B^n_D = B^n \cap D \) and \( B^n_C = B^n \cap C \). Since \( HC = 0 \), we can choose \( H \) and \( L \) so that they decompose similarly (with \( HC = 0 \)), giving the following direct sum decompositions:

\[
A^n = D^n \oplus C^n = (B^n_D \oplus H^n_D \oplus L^n_D) \oplus (B^n_C \oplus L^n_C),
\]

\[
A^n = B^n \oplus H^n \oplus L^n = (B^n_D \oplus B^n_C) \oplus H^n_D \oplus (L^n_D \oplus L^n_C).
\]

It follows from the construction before Theorem 2.2 that \( H_{Mer}A = H_{Mer}D \). We choose \( H^n_D \) to contain \( e \). By Lemma 2.4, \( e \) is the strict unit of \( H_{Mer}D \); hence \( e \) is the strict unit of \( H_{Mer}A \), but note that the unit \( 1 \) of \( A \) may not be in \( HA \).

Now suppose \( HA \) is Adams connected with unit \( u \). Let \( H^0 = ku \oplus H^0_{\geq 1} \) (or \( H^0 = ku \oplus H^0_{\leq -1} \) if negatively connected graded). Replace \( H^0 \) by \( k1 \oplus H^0_{\leq 1} \) and keep the other subspaces \( B^n, H^n, \) and \( L^n \) the same. Let \( \overline{H_{Mer}A} \) denote the new Merkulov model with the new choice of \( H^0 \). Then by Lemma 2.4 \( 1 \) is the strict unit of \( \overline{H_{Mer}A} \). By construction, we have \((\overline{H_{Mer}A})_{\geq 1} = (H_{Mer}A)_{\geq 1} \) as \( A_\infty \)-algebras without unit. By the unital condition, we see that \( \overline{H_{Mer}A} \) is strictly isomorphic to \( H_{Mer}A \).

**3. Massey products**

It is common to view \( A_\infty \)-algebras as algebras which are strongly homotopy associative: not associative on the nose, but associative up to all higher homotopies, as given by the \( m_n \)'s. Any \( A_\infty \)-algebra in which the differential \( m_1 \) is zero, such as the cohomology of a DG algebra, is strictly associative, though, and in such a case, it is natural to wonder about the role of the higher multiplications. On the other hand, the cohomology of a DG algebra is the natural setting for Massey products. With Merkulov’s construction in hand, we give a proof of a folk theorem which connects the higher multiplication maps with Massey products: we prove that they
are essentially the same, up to a sign. We start by reviewing Massey products. We use the sign conventions from May [Ma]; see also Ravenel [Ra A1.4].

If $a$ is an element of a DG algebra $A$, we write $\pi$ for $(-1)^{1+\deg a}a$. (This notation helps to keep some formulas simple.)

The length two Massey product $\langle \alpha_1, \alpha_2 \rangle$ is the ordinary product: $\langle \alpha_1, \alpha_2 \rangle = \alpha_1 \circ \alpha_2$. (For consistency with the higher products, one could also define $\langle \alpha_1, \alpha_2 \rangle$ as being the set $\{ \alpha_1, \alpha_2 \}$, but we do not take this point of view.)

The Massey triple product is defined as follows: suppose given classes $\alpha_1, \alpha_2, \alpha_3 \in HA$ which are represented by cocycles $a_{01}, a_{12}, a_{23} \in A$, respectively, and suppose that $\alpha_1 \circ \alpha_2 = 0 = \alpha_2 \circ \alpha_3$. Then there are cochains $a_{02}$ and $a_{13}$ so that $\partial(a_{02}) = \pi_{01}a_{12}$ and $\partial(a_{13}) = \pi_{12}a_{23}$. Then

$$\pi_{02}a_{23} + \pi_{01}a_{13}$$

is a cocycle, and so represents a cohomology class. One can choose different cochains for $a_{02}$ and $a_{13}$: one can replace $a_{02}$ with $a_{02} + z$ for any cocycle $z$, for instance, and this can produce a different cohomology class. The length 3 Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is the set of cohomology classes which arise from all such choices of $a_{02}$ and $a_{13}$.

More generally, for any $n \geq 3$, the length $n$ Massey product is defined as follows. Suppose that we have cohomology classes $\alpha_i$ for $1 \leq i \leq n$. Suppose that whenever $i < j$ and $j-i < n-1$, each length $j-i+1$ Massey product $\langle \alpha_i, \ldots, \alpha_j \rangle$ is defined and contains zero. Then the length $n$ Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ exists and is defined as follows: using induction on $j-i+1$, one defines cochains $a_{ij}$ as follows: $a_{i-1,i}$ is a cocycle representing the cohomology class $\alpha_i$. Given $a_{km}$ for all $k < m$ with $m-k+1 < j-i+1$, choose $a_{ij}$ so that

$$\partial(a_{ij}) = \sum_{i<k<j} \pi_{ik}a_{kj}.$$  

Then $\langle \alpha_1, \ldots, \alpha_n \rangle$ is the set of cohomology classes represented by cocycles of the form

$$\sum_{0<i<n} \pi_{0i}a_{in}.$$  

(It is tedious but straightforward to check that each such sum is a cocycle.) One can see that $\langle \alpha_1, \ldots, \alpha_n \rangle \subset H^{s-(n-2)}A$, where $s$ is the sum of the degrees of the $\alpha_i$’s, which means that $\langle \alpha_1, \ldots, \alpha_n \rangle$ is in the same degree as $m_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$. This is not a coincidence.

**Theorem 3.1.** Let $A$ be a DG algebra. Up to a sign, the higher $A_\infty$-multiplications on $HA$ give Massey products. More precisely, if $HA$ is given an $A_\infty$-algebra structure by Merkulov’s construction, then for any $n \geq 3$, if $\alpha_1, \ldots, \alpha_n \in HA$ are elements such that the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined, then

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \ldots, \alpha_n \rangle,$$

where

$$b = 1 + \deg \alpha_{n-1} + \deg \alpha_{n-3} + \deg \alpha_{n-5} + \cdots.$$  

The authors have been unable to find an account of this theorem in its full generality, but for some related results, see [Ka p. 233], [St2 Chapter 12], and [JL 6.3–6.4].

Now, there are choices made in Merkulov’s construction – the choices of the splittings (2.1.1) – and different choices can (depending on $A$) lead to different elements.
in the Massey products, as well as different (but quasi-isomorphic) $A_\infty$-algebra structures. In any case, any choice of $A_\infty$-structure via Merkulov’s construction gives a “coherent” set of choices for an element of each Massey product. Of course, the $A_\infty$-multiplications are also universally defined, not just when certain products are zero.

**Proof.** The proof is by induction on $n$.

The theorem discusses the situation when $n \geq 3$, but we will also use the formula when $n = 2$ in the induction: when $n = 2$, we have

$$m_2(\alpha_1 \otimes \alpha_2) = \alpha_1 \alpha_2 = (\alpha_1, \alpha_2) = (-1)^{1+\deg 01} a_1 \alpha_2.$$

Now let $n = 3$. We use Merkulov’s construction for the $A_\infty$-algebra structure on $HA$, so we choose splittings as in (2.1.11), and we define the multiplication maps $m_n$ as in Theorem (2.2). We use a little care when choosing the elements $a_{ij} \in A$: we define $a_{02}$ by $G(a_{01}a_{12}) = a_{02}$, so $\delta(a_{02}) = \alpha_{01}a_{12}$ and $G\lambda_2(\alpha_1 \otimes \alpha_2) = (-1)^{1+\deg 01} a_{02}$. We define $a_{13}$ similarly. Then we have

$$m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = p\lambda_2(G\lambda_1 \otimes G\lambda_2 - G\lambda_2 \otimes G\lambda_1)(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

$$= p \left((-1)^{1+\deg 01+1+\deg 02} a_{01}a_{13} + (-1)^{1+\deg 01+\deg 02} a_{02}a_{23}\right)$$

$$= p \left((-1)^{1+\deg 02} a_{01}a_{13} + (-1)^{1+\deg 02} a_{02}a_{23}\right)$$

$$= (-1)^{1+\deg 02} p(\bar{a}_{01}a_{13} + \bar{a}_{02}a_{23}).$$

(Some signs here are due to the Koszul sign convention; for example, the map $G\lambda_2$ has degree 1, so $(G\lambda_1 \otimes G\lambda_2)(a_{01} \otimes a_{12} \otimes a_{23}) = (-1)^{\deg a_1} G\lambda_1(a_{01}) \otimes G\lambda_2(a_{12} \otimes a_{23}).$)

The map $p$ is the projection map from $A$ to its summand $H$. Loosely, for any coycle $z$, $p(z)$ is the cohomology class represented by $z$; more precisely, $p(z)$ is the unique class in $H \subset A$ which is cohomologous to $z$. In the situation here, the term in parentheses is a cocycle whose cohomology class is in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, so we get the desired result.

Assume that the result is true for $m_i$ with $i < n$. Therefore for all $i < j$ with $j - i < n - 1$, we may choose elements $a_{i-1,j}$ by the formula

$$G\lambda_{j-i+1}(\alpha_i \otimes \cdots \otimes \alpha_j) = (-1)^{1+\deg a_{j-1}+\deg a_{j-3}+\cdots} a_{i-1,j}.$$

We write $b_{ij}$ for the exponent of $-1$ here:

$$b_{ij} = 1 + \deg a_{j-1} + \deg a_{j-3} + \cdots.$$

The last term in this sum is $\deg a_{j-(2k+1)}$, where $k$ is the maximum such that $j - (2k+1) \geq i$. Note also for use with the Koszul sign convention that $G\lambda_i$ has degree $1 - i$. Then
\[ m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) = p\lambda_2 \left( \sum_{s=1}^{n-1} (-1)^{s+1} G\lambda_s \otimes G\lambda_{n-s} \right) (a_0 \otimes \cdots a_{n-1,n}) \]

\[ = p \left( \sum_{s=1}^{n-1} (-1)^{s+1+(1-n+s)(\deg \alpha_1 + \cdots + \deg \alpha_s)} G\lambda_s (\alpha_1 \otimes \cdots \otimes \alpha_s) G\lambda_{n-s} (\alpha_{s+1} \otimes \cdots \otimes \alpha_n) \right) \]

\[ = p \left( \sum_{s=1}^{n-1} (-1)^{s+1+(1-n+s)(\deg \alpha_1 + \cdots + \deg \alpha_s) + b_{1,n} + b_{s+1,n}} a_0 a_{sn} \right) \]

\[ = p \left( \sum_{s=1}^{n-1} (-1)^{1-s+1+\deg \alpha_1 + \cdots + \deg \alpha_s + b_{1,n} + b_{s+1,n}} \bar{a_0 a_{sn}} \right) \]

\[ = p \left( \sum_{s=1}^{n-1} (-1)^{b_{1,n}} \bar{a_0 a_{sn}} \right), \]

where \( b_{1,n} = b \) is the sign as in the theorem: if \( n - s \) is even, then the sign is \((-1)^{1+b_{1,n}+b_{s+1,n}}\), and with \( n - s \) even, we have \( b_{1,n} = 1 + b_{1,s} + b_{s+1,n} \). If \( n - s \) is odd, then the sign is \((-1)^{1+b_{s+1,n}+1+\deg \alpha_s+\deg \alpha_{s-2}+\deg \alpha_{s-4}} = (-1)^{b_{1,n}}, \)

as claimed. As with the \( n = 3 \) case, since the sum \( \sum \bar{a_0 a_{sn}} \) is a cocycle, \( p \) sends it to the cohomology class that it represents, which is an element of the Massey product \( \langle \alpha_1, \ldots, \alpha_n \rangle \). This finishes the proof. \( \square \)

See Section 6 for some examples.

4. The bar construction and Ext

The bar/cobar construction is one of the basic tools in homological algebra. Everything in this section is well-known – see [FHT], for example – but we need the details for the proof in the next section.

Let \( A \) be a connected graded algebra and let \( k \) be the trivial \( A \)-module. Of course, the \( i \)-th Ext-group \( \text{Ext}_A^i(k_A, k_A) \) can be computed by the \( i \)-th cohomology of the complex \( \text{Hom}_A(P_A, k_A) \) where \( P_A \) is any projective (or free) resolution of \( k_A \). Since \( P_A \) is projective, \( \text{Hom}_A(P_A, k_A) \) is quasi-isomorphic to \( \text{Hom}_A(P_A, k_A) = \text{End}_A(P_A) \); hence \( \text{Ext}_A^i(k_A, k_A) \cong H^i(\text{End}_A(P_A)) \). Since \( \text{End}_A(P_A) \) is a DG algebra, the graded vector space \( \text{Ext}_A^i(k_A, k_A) := \bigoplus_{i \in \mathbb{Z}} \text{Ext}_A^i(k_A, k_A) \) has a natural algebra structure, and it also has an \( A_\infty \)-structure by Kadeishvili’s result Theorem 2.1. By [Ad Chap.2], the Ext-algebra of a graded algebra \( A \) can also be computed by using the bar construction on \( A \), which will be explained below.
First we review the shift functor. Let \((M, \partial)\) be a complex with differential \(\partial\) of degree 1, and let \(n\) be an integer. The \(n\)th shift of \(M\), denoted by \(S^n(M)\), is defined by
\[
S^n(M)^i = M^{i+n}
\]
and the differential of \(S^n(M)\) is
\[
\partial_{S^n(M)}(m) = (-1)^n \partial(m)
\]
for all \(m \in M\). If \(f : M \to N\) is a homomorphism of degree \(p\), then \(S^n(f) : S^n(M) \to S^n(N)\) is defined by the formula
\[
S^n(f)(m) = (-1)^{pn} f(m)
\]
for all \(m \in S^n(M)\). The functor \(S^n\) is an automorphism of the category of complexes.

The following definition is essentially standard, although sign conventions may vary; we use the conventions from [FHT, Sect.19]. Let \(A\) be an augmented DG algebra with augmentation \(\epsilon : A \to k\), viewing \(k\) as a trivial DG algebra. Let \(I\) be the kernel of \(\epsilon\) and \(SI\) the shift of \(I\). The tensor coalgebra on \(SI\) is
\[
T(SI) = k \oplus SI \oplus (SI)^{\otimes 2} \oplus (SI)^{\otimes 3} \oplus \cdots ,
\]
where an element \(Sa_1 \otimes Sa_2 \otimes \cdots \otimes Sa_n\) in \((SI)^{\otimes n}\) is written as
\[
[a_1|a_2|\cdots|a_n]
\]
for \(a_i \in I\), together with the comultiplication
\[
\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].
\]
The degree of \([a_1|\cdots|a_n]\) is \(\sum_{i=1}^{n} (\deg a_i - 1)\).

**Definition 4.1.** Let \((A, \partial_A)\) be an augmented DG algebra and let \(I\) denote the augmentation ideal \(\ker(A \to k)\). The bar construction on \(A\) is the coaugmented differential graded coalgebra (DG coalgebra, for short) \(BA\) defined as follows:
- As a coaugmented graded coalgebra, \(BA\) is the tensor coalgebra \(T(SI)\).
- The differential in \(BA\) is the sum \(d = d_0 + d_1\) of the coderivations given by
\[
d_0([a_1|\cdots|a_m]) = -\sum_{i=0}^{m} (-1)^{n_i}[a_1|\cdots|\partial_A(a_i)|\cdots|a_m]
\]
and
\[
d_1([a_1]) = 0
\]
\[
d_1([a_1|\cdots|a_m]) = \sum_{i=2}^{m} (-1)^{n_i}[a_1|\cdots|a_{i-1}a_i|\cdots|a_m]
\]
where \(n_i = \sum_{j<i} (-1 + \deg a_j) = \sum_{j<i} \deg [a_j]\).

The cobar construction \(\Omega C\) on a coaugmented DG coalgebra \(C\) is defined dually [FHT, Sect.19]. We omit the definition since it is used only in two places, one of which is between Lemma 5.3 and Lemma 5.4, and the other is in Lemma 5.5.

In the rest of this section we assume that \(A\) is an augmented associative algebra. In this case \(SI\) is concentrated in degree \(-1\); hence the degree of \([a_1|\cdots|a_m]\) is \(-m\). This means that the bar construction \(BA\) is graded by the *negative* of tensor length. The degree of the differential \(d\) is 1. We may think of \(BA\) as a complex
with \((-i)\)th term equal to \(I^{\otimes i}\), the differential \(d\) mapping \(I^{\otimes i}\) to \(I^{\otimes i-1}\). If \(A\) has an Adams grading, denoted \(\text{adeg}\), then \(BA\) has a bigradation that is defined by
\[
\text{deg} [a_1] \cdots [a_m] = (-m) \sum_i \text{adeg} a_i.
\]
The second component is the Adams degree of \([a_1] \cdots [a_m]\).

The bar construction on the left \(A\)-module \(A\), denoted by \(B(A, A)\), is constructed as follows. As a complex \(B(A, A) = BA \otimes A\) with \((-i)\)th term equal to \(I^{\otimes i} \otimes A\). We use
\[
[a_1] \cdots [a_m] x
\]
to denote an element in \(I^{\otimes i} \otimes A\) where \(x \in A\) and \(a_i \in I\). The degree of \([a_1] \cdots [a_m] x\) is \(-m\). The differential on \(B(A, A)\) is defined by
\[
d(x) = 0 \quad (m = 0 \text{ case}),
\]
and
\[
d([a_1] \cdots [a_m] x) = \sum_{i=2}^m (-1)^{i-1} [a_1] \cdots [a_{i-1} a_i] \cdots [a_m] x + (-1)^m [a_1] \cdots [a_{m-1}] a_m x.
\]
Then \(B(A, A)\) is a complex of free right \(A\)-modules. One basic property is that the augmentations of \(BA\) and \(A\) make it into a free resolution of \(k_A\),
\[
(4.1.1) \quad B(A, A) \to k_A \to 0
\]
(see [FHT] 19.2 and [AM] Chap.2).

**Remark 4.2.** In the next section we use the tensor \(\otimes\) notation instead of the bar \(|\cdot\rangle\) notation, which seems more natural when we concentrate on each term of the bar construction.

We now assume that with respect to the Adams grading, \(A\) is connected graded and finite-dimensional in each degree. Then \(B(A, A)\) is bigraded with Adams grading on the second component, and the differential of \(B(A, A)\) preserves the Adams grading. Let \(B^\# A\) be the graded \(k\)-linear dual of the coalgebra \(BA\). Since \(BA\) is locally finite, \(B^\# A\) is a locally finite bigraded algebra. With respect to the Adams grading, \(B^\# A\) is negatively connected graded. The DG algebra \(\text{End}_A(\mathcal{B}(A, A)_A)\) is bigraded too, but not Adams connected. Since \(B(A, A)\) is a left differential graded comodule over \(BA\), it has a left differential graded module structure over \(B^\# A\), which is compatible with the right \(A\)-module structure. By an idea similar to [FHT] Ex. 4, p. 272 (also see [LP2]) one can show that the natural map \(B^\# A \to \text{End}_A(B(A, A)_A)\) is a quasi-isomorphism of DG algebras.

Define the *Koszul dual* of a connected graded ring \(A\) to be the DG algebra \(\text{End}_A(P_A)_A\), where \(P_A\) is any free resolution of \(k_A\). By the following lemma, this definition makes sense up to quasi-isomorphism in the category of \(A_\infty\)-algebras.

**Lemma 4.3.** Let \(A\) be a connected graded algebra which is finite-dimensional in each degree, and let \(P_A\) and \(Q_A\) be two free resolutions of \(k_A\).

(a) \(\text{End}_A(P_A)\) is quasi-isomorphic to \(\text{End}_A(Q_A)\) as \(A_\infty\)-algebras.

(b) \(\text{End}_A(P_A)\) is quasi-isomorphic to \(B^\# A\) as \(A_\infty\)-algebras.

**Proof.** (a) We may assume that \(Q_A\) is a minimal free resolution of \(k_A\). Then \(P_A = Q_A \oplus I_A\) where \(I_A\) is another complex of free modules such that \(HI_A = 0\) [APH] 10.1.3 and 10.3.4]. In this case \(D := \text{End}_A(Q_A)\) is a sub-DG algebra of
$E := \text{End}_A(P_A)$ such that $D = eEe$ where $e$ is the projection onto $Q_A$. Let $C = (1-e)E + E(1-e)$. Then

$$C = \text{Hom}_A(I_A, Q_A) + \text{Hom}_A(Q_A, I_A) + \text{Hom}_A(I_A, I_A),$$

and $HC = 0$. By Lemma 2.8, $D$ and $E$ are quasi-isomorphic.

(b) Since $B(A, A)$ is a free resolution of $k_A$, then part (a) says that $\text{End}_A(P_A)$ is quasi-isomorphic to $\text{End}_A(B(A, A))$. The assertion follows from the fact that $\text{End}_A(B(A, A))$ is quasi-isomorphic to $B^\#A$ [PH1] Ex. 4, p. 272]. □

So we may think of the bigraded DG algebra $B^\#A$ as the Koszul dual of $A$. This viewpoint of Koszul duality is also taken by Keller in [Ke1]. By results in [LP2], we can define the Koszul dual of any connected graded (or augmented) $A_\infty$-algebra, and the double Koszul dual is quasi-isomorphic to the original $A_\infty$-algebra.

The classical Ext-algebra $\text{Ext}^*_A(k_A, k_A)$ is the cohomology ring of $\text{End}_A(P_A)$, where $P_A$ is any free resolution of $k_A$. The above lemma demonstrates the familiar fact that this is independent of the choice of $P_A$. Since $E := \text{End}_A(P_A)$ is a DG algebra, by Proposition 2.3, $\text{Ext}^*_A(k_A, k_A) = HE$ has a natural $A_A$-structure, which is called an $A_\infty$-Ext-algebra of $A$. By abuse of notation we use $\text{Ext}^*_A(k_A, k_A)$ to denote an $A_\infty$-Ext-algebra.

5. $A_\infty$-structure on Ext-algebras

In this section we consider the multiplications on an $A_\infty$-Ext-algebra of a connected graded algebra, and finally give proofs of Theorem A and Proposition 1.2.

Consider a connected graded algebra

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots,$$

which is viewed as an $A_\infty$-algebra concentrated in degree 0, with the grading on $A$ being the Adams grading. Let $Q \subset A$ be a minimal graded vector space which generates $A$. Then $Q \cong m/m^2$ where $m := A_{\geq 1}$ is the unique maximal graded ideal of $A$. Following Milnor and Moore [MM 3.7], we call the elements of $m/m^2$ the indecomposables of $A$, and by abuse of notation, we also call the elements of $Q$ indecomposables. Let $R \subset T(Q)$ be a minimal graded vector space which generates the relations of $A$ ($R$ is not unique). Then $A \cong T(Q)/(R)$ where $(R)$ is the ideal generated by $R$, and the start of a minimal graded free resolution of the trivial right $A$-module $k_A$ is

$$\cdots \to R \otimes A \to Q \otimes A \to A \to k \to 0. \tag{5.0.1}$$

Lemma 5.1. Let $A$ be a connected graded algebra. Then there are natural isomorphisms of graded vector spaces

$$\text{Ext}^1_A(k_A, k_A) \cong Q^\# = \bigoplus Q^\#_s \quad \text{and} \quad \text{Ext}^2_A(k_A, k_A) \cong R^\# = \bigoplus R^\#_s.$$  

Proof: This follows from the minimal free resolution (5.0.1). □

In the rest of the section, we assume that $A$ is Adams locally finite: each $A_i$ is finite-dimensional. Let $E$ be the $A_\infty$-Ext-algebra $\text{Ext}^*_A(k_A, k_A)$. We would like to describe the $A_\infty$-structure on $E$ by using Merkulov’s construction.

We first fix some notation. For each Adams degree $s$, we choose a vector space splitting $A_s = Q_s \oplus D_s$: the elements in $Q_s$ are indecomposable, while those in $D_s$ are “decomposable” in terms of the indecomposables of degree less than $s$. More precisely, we define $Q_s$ and $D_s$ inductively: we start by setting $Q_1 = A_1$ and
\(D_1 = 0.\) Now assume that \(Q_i\) and \(D_i\) have been defined for \(i < s\); then we set \(D_s\) to be the image in \(A_s\) of the multiplication map
\[
\mu_s : \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \rightarrow A_s.
\]
We choose \(Q_s\) to be a vector space complement of \(D_s.\)

Now for each \(s \geq 2,\) the multiplication map
\[
\mu_s : \bigoplus_{1 \leq i \leq s} Q_i \otimes A_{s-i} \rightarrow A_s
\]
is onto. Define the \(k\)-linear map \(\xi_s : A_s \rightarrow \bigoplus_{1 \leq i \leq s} Q_i \otimes A_{s-i}\) so that the composition
\[
(5.1.1) \quad A_s \xrightarrow{\xi_s} \bigoplus_{1 \leq i \leq s} Q_i \otimes A_{s-i} \xrightarrow{- \mu_s} A_s
\]
is the identity map of \(A_s.\) Further, we choose \(\xi_s\) so that with respect to the direct sum decomposition \(A_s = Q_s \oplus D_s,\) we have
\[
(5.1.2) \quad \text{im}(\xi_s|_{Q_s}) = Q_s \otimes A_0, \quad \text{im}(\xi_s|_{D_s}) \subset \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i}.
\]
(The second of these holds for any choice of \(\xi_s;\) the first need not.) Define \(\xi_1 = \theta_1 = \text{id}_{A_1},\) and inductively set \(\theta_s = \sum_{i+j=s} (\text{id}_{Q_i} \otimes \theta_j) \circ \xi_s;\) that is, \(\theta_s\) is the composition
\[
A_s \xrightarrow{\xi_s} \bigoplus_{i+j=s} Q_i \otimes A_j \xrightarrow{\sum \text{id}_{Q_i} \otimes \theta_j} \bigoplus_{i+k+l=s} Q_i \otimes Q_k \otimes A_l \xrightarrow{\sum \text{id}_{Q_i} \otimes \text{id}_{Q_k} \otimes \theta_l} \cdots \xrightarrow{\bigoplus_{n \geq 1} \bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}}.
\]
Here, the subscripts on the \(Q_i\)'s are positive, while those on the \(A_i\)'s are non-negative.

Let \(R = \bigoplus_{s \geq 2} R_s \subset T(Q)\) be a minimal graded vector space of the relations of \(A.\) Note that with respect to tensor length, the elements of \(R\) need not be homogeneous, but they are homogeneous with respect to the Adams grading – the grading induced by that on \(Q.\)

Let \(T(Q)_s\) denote the part of \(T(Q)\) in Adams degree \(s;\) thus
\[
T(Q)_s = \bigoplus_{n \geq 1} \bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.
\]
We write \(\mu\) for the map \(\mu : T(Q) \rightarrow A \cong T(Q)/(R).

**Lemma 5.2.** For each \(s,\) \(R_s\) may be chosen so that
\[
R_s \subset \bigoplus_{1 \leq i \leq s-1} Q_i \otimes \theta_{s-i}(A_{s-i}) \subset T(Q)_s.
\]
Hence \(R_s\) may also be viewed as a subspace of \(\bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i},\) via the composite
\[
R_s \hookrightarrow \bigoplus_{1 \leq i \leq s-1} Q_i \otimes \theta_{s-i}(A_{s-i}) \xrightarrow{\sum \text{id}_{Q_i} \otimes \theta_{s-i}(A_{s-i})} \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i}.
\]

**Proof.** Let \((R)_{s-i}\) be the degree \(s - i\) part of the ideal \((R).\) Then it is generated by the relations of degree at most \(s - i,\) and we have a decomposition
\[
T(Q)_{s-i} = \theta_{s-i}(A_{s-i}) \oplus \ker \mu = \theta_{s-i}(A_{s-i}) \oplus (R)_{s-i},
\]
where \(\mu : T(Q)_{s-i} \rightarrow A_{s-i}\) is multiplication. Hence we have
\[
T(Q)_s = Q_s \oplus \bigoplus_{1 \leq i \leq s-1} [(Q_i \otimes \theta_{s-i}(A_{s-i})) \oplus (Q_i \otimes (R)_{s-i})].
\]
Any relation \(r \in R_s\) has no summands in \(Q_s,\) and hence is a sum of \(r' \in \bigoplus Q_i \otimes \theta_{s-i}(A_{s-i})\) and \(r'' \in \bigoplus Q_i \otimes (R)_{s-i}.\) Modulo the relations of degree less than \(n,\) we may assume \(r'' = 0.\) Hence the first part of the lemma is proved.

The map \(\theta_{s-i} : A_{s-i} \rightarrow T(Q)_{s-i}\) is an inclusion, and up to a sign its left inverse is the multiplication map \(\mu : T(Q)_{s-i} \rightarrow A_{s-i}.\) Once \(R_s\) has been chosen
to be a subspace of $\bigoplus Q_i \otimes \theta_{s-i}(A_{s-i})$, composing with $\mu$ takes it injectively to $\bigoplus Q_i \otimes A_{s-i}$.

The minimal resolution $[\ref{sec:resolution}]$ is a direct summand of any other resolution, and in particular it is a summand of the bar resolution $[\ref{sec:bar-resolution}]$

$$\cdots \to \mathfrak{m} \otimes 2 A \to \mathfrak{m} \otimes A \to A \to k \to 0.$$  

We have made several choices up to this point: choosing the splittings $A_s = Q_s \oplus D_s$, and now choosing $R_s$ as in the lemma, so that $R_s \subset \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \subset \mathfrak{m} \otimes \mathfrak{m}$. These choices give a choice for this splitting of resolutions, at least in low degrees.

Since $A$ is concentrated in degree 0, the grading on the differential graded coalgebra $T(\mathrm{Sm})$ is by the negative of the wordlength, namely, $\langle T(\mathrm{Sm}) \rangle^{-i} = \mathfrak{m}^i$. The differential $d = (d^i)$ of the bar construction $T(\mathrm{Sm})$ is induced by the multiplication $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ in $A$. For example,

$$d^{-1}(\{a_1\}) = 0 \quad \text{and} \quad d^{-2}(\{a_1|a_2\}) = (-1)^{-1}[a_1a_2]$$

for all $a_1, a_2 \in \mathfrak{m}$. There is a natural decomposition of $\mathfrak{m}$ with respect to the Adams grading,

$$\mathfrak{m} = A_1 \oplus A_2 \oplus A_3 \oplus \cdots,$$

which gives rise to a decomposition of $\mathfrak{m} \otimes \mathfrak{m}$ with respect to the Adams grading:

$$\mathfrak{m} \otimes \mathfrak{m} = (A_1 \otimes A_1) \oplus (A_1 \otimes A_2 \oplus A_2 \otimes A_1) \oplus \cdots.$$  

As mentioned above, we are viewing $R_s$ as being a subspace of $\bigoplus Q_i \otimes A_{s-i}$.

**Lemma 5.3.** Let $W_s$ be the Adams degree $s$ part of $\mathfrak{m} \otimes \mathfrak{m}$; that is, let

$$W_s = \bigoplus_{1 \leq i \leq s-1} A_i \otimes A_{s-i}.$$  

Then there are decompositions of vector spaces

$$W_s = \im(d_{s-3}^s) \oplus R_s \oplus \xi_s(D_s),$$

$$\ker(d_{s-2}^s) = \im(d_{s-3}^s) \oplus R_s,$$

where $R_s$ and $\xi_s(D_s)$ are subspaces of

$$\bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \subset \bigoplus_{1 \leq i \leq s-1} A_i \otimes A_{s-i} = W_s.$$  

**Proof.** It is clear that the injection

$$D_s \xrightarrow{\xi} \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \to W_s$$

defines a projection from $W_s$ to $D_s$. Since $d_{s-2}^s : W_s \to D_s$ is a surjection, we have a decomposition

$$W_s = \ker(d_{s-2}^s) \oplus \xi_s(D_s).$$

Since $R^\# \cong \text{Ext}^2_A(k_A, k_A) = H^2((T(\text{Sm}))^\#)$ by Lemma $[\ref{sec:ext}]$, there is a decomposition $\ker(d_{s-2}^s) = \im(d_{s-3}^s) \oplus R_s$. Hence the assertion follows. □

Since $A$ is Adams locally finite, $(\mathfrak{m}^\#)^n \cong (\mathfrak{m}^\#)^{\otimes n}$ for all $n$. Let $\Omega A^\#$ be the cobar construction on the DG coalgebra $A^\#$. Via the isomorphisms

$$B^\# A = (T(\text{Sm}))^\# \cong T((\text{Sm})^\#) \cong T(S^{-1} \mathfrak{m}^\#) = \Omega A^\#,$$

we identify $B^\# A = (T(\text{Sm}))^\#$ with $\Omega A^\# = T(S^{-1} \mathfrak{m}^\#)$. The differential $\partial$ on $B^\# A$ is defined by

$$\partial(f) = -(-1)^{\deg f} f \circ d$$

for all $f \in T(S^{-1} \mathfrak{m}^\#)$. 
We now study the first two nonzero differential maps of $\Omega A^\#$, 
$$\partial^1 : m^\# \to (m^\#)^{\otimes 2} \quad \text{and} \quad \partial^2 : (m^\#)^{\otimes 2} \to (m^\#)^{\otimes 3}.$$ 

For all $s$ and $n$, let 
$$T^n = (m^\#)^{\otimes n}$$
and 
$$T^n_s = \bigoplus_{i_1 + \cdots + i_n = s} A_{i_1}^\# \otimes \cdots \otimes A_{i_n}^\#.$$ 

Since we are working with $m^\#$, all subscripts here and in what follows are positive. Fix Adams degree $-s$, and consider 
$$\partial^1_{-s} : A_{i}^\# \to \bigoplus_{i+j=s} A_i^\# \otimes A_j^\#,$$ 
$$\partial^2_{-s} : \bigoplus_{i+j=s} A_i^\# \otimes A_j^\# \to \bigoplus_{i_1+i_2+i_3=s} A_{i_1}^\# \otimes A_{i_2}^\# \otimes A_{i_3}^\#.$$ 

The decomposition \(2.14\) for $T^1_{-s}$ is 
$$B^1_{-s} = 0, \quad H^1_{-s} = Q^\#, \quad L^1_{-s} = D^\#_{-s}.$$ 

for all $s \geq 1$. The decomposition \(2.14\) for $T^2_{-s}$ is given in the following lemma.

**Lemma 5.4.** Fix $s \geq 2$. With notation as above, we have the following.

(a) Define the duals of subspaces by using the decompositions given in Lemma \(\ref{lem:decomp} \). Then $\text{im} \partial^1_{-s} = (\xi_s(A_s))^\#$ and $\text{ker} \partial^2_{-s} = (\xi_s(A_s))^\# \oplus R^\#_{-s}$.

(b) The decomposition \(2.14\) for $T^2_{-s}$ can be chosen to be

$$T^2_{-s} = B^2_{-s} \oplus H^2_{-s} \oplus L^2_{-s} = (\xi_s(A_s))^\# \oplus R^\#_{-s} \oplus (\text{im} d^3)^\#.$$ 

The projections onto $R_{-s}^\#$ and $(\xi_s(A_s))^\#$ kill $\bigoplus_{2 \leq i \leq s-1} D^i_{-s} \oplus A_{s-i-1}^\#$.

(c) Let $G$ be the homotopy defined in Merkulov’s construction for the DG algebra $T(m^\#)$. Then we may choose $G^2_{-s}$ to be equal to $-(\xi_s)^\#$, restricted to $T^2_{-s}$.

**Proof.** (a) This follows from Lemma \(\ref{lem:decomp} \) and a linear algebra argument.

(b) This follows from Lemma \(\ref{lem:decomp} \) part (a), and the fact that $R_s$ and $\xi_s(A_s)$ are subspaces of $\bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i}$.

(c) Let $\xi_s$ also denote the map $D_s \to \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-1} \to W_s$. Since $d^2_s = -\mu_s$, the composite $d^2_s \circ \xi_s : D_s \to W_s \to D_s$ is the identity map (see \(\ref{eq:ident} \)). Since $(d^2_s \circ \xi_s)^\# = (\xi_s)^\# \circ (d^2_s)^\#$,

$$(\xi_s)^\# \circ (d^2_s)^\# : D^\#_{-s} \to T^2_{-s} \to D^\#_{-s}$$

is the identity map of $D^\#$, which is the summand $L^1_{-s}$ of $T^1$. Since $(d^2_s)^\# = -\partial^1_{-s}$, we may choose the homotopy $G$ to be $-(\xi_s)^\#$ when restricted to $T^2_{-s}$. \(\square\)

Now we start to construct the higher $A_\infty$-multiplication maps on $\text{Ext}^*_A(k_A, k_A)$, using Merkulov’s construction from Section 2. Lemma \(\ref{lem:merkulov} \) tells us what the homotopy $G$ is. The maps $\lambda_1 : (T(S^{-1}m^\#))^n \to T(S^{-1}m^\#)$ are defined as in Section 2; in particular, recall that we formally set $G\lambda_1 = -id_T$, and $\lambda_2$ is the multiplication of $T(S^{-1}m^\#)$.

Recall that $T(S^{-1}m^\#)$ is a free (or tensor) DG algebra generated by $S^{-1}m^\#$. To distinguish among the various tensor products occurring here, we use $\otimes$ when tensoring factors of $T(S^{-1}m^\#)$ together; in particular, we write $\lambda_2$ as

$$\lambda_2 : (S^{-1}m^\#)^{\otimes n} \otimes (S^{-1}m^\#)^{\otimes m} \to (S^{-1}m^\#)^{\otimes (n+m)}.$$
Then for $a_1 \otimes \cdots \otimes a_n \in (S^{-1}m^\#)^\otimes n$ and $b_1 \otimes \cdots \otimes b_m \in (S^{-1}m^\#)^\otimes m$ we have
\begin{equation}
(5.4.1) \quad \lambda_2((a_1 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes \cdots \otimes b_m)) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m.
\end{equation}
By the above formula, we see that $\lambda_2$ changes $\otimes$ to $\otimes$, so it is like the identity map.

**Lemma 5.5.** Let $E^1 = \text{Ext}_A^1(k_A, k_A) = Q^\#$. Fix $n \geq 2$ and $s \geq 2$.

(a) When restricted to $(E^1)^\otimes n$ in Adams degree $s$, the map $\lambda_n$ has image in
\[ T^2_s = \bigoplus_{q+r=s} A^\#_q \otimes A^\#_r. \]
Hence the image of $G\lambda_n$ is in $D^\#_s$.

(b) When restricted to $(E^1)^\otimes n$ in Adams degree $s$, the map $-G\lambda_n$ is the $k$-linear dual of the composite
\[ A_s \xrightarrow{\theta} \bigoplus_{m \geq 1} \bigoplus_{i_1 + \cdots + i_m = s} Q_{i_1} \otimes \cdots \otimes Q_{i_m} \rightarrow \bigoplus_{i_1 + \cdots + i_m = s} Q_{i_1} \otimes \cdots \otimes Q_{i_m}. \]

(c) When restricted to $(E^1)^\otimes n$ in Adams degree $s$, the map $m_n = P r_H \lambda_n$ is the $k$-linear dual of the canonical map
\[ R_s \rightarrow \bigoplus_{1 \leq i \leq n-1} Q_i \otimes A_{s-i} \xrightarrow{\sum id \otimes \theta_{i-1}} \bigoplus_{i_1 + \cdots + i_m = s} Q_{i_1} \otimes \cdots \otimes Q_{i_m}. \]

**Proof.** We use induction on $n$.

(a) By definition,
\[ \lambda_n = \lambda_2 \sum_{i+j=n} (-1)^{i+1} G\lambda_i \otimes G\lambda_j. \]
For $n = 2$, the claim follows from (5.4.1). Now assume $n > 2$ and consider $\lambda_n$ applied to $b_1 \otimes \cdots \otimes b_n$, with $b_m \in Q^\#_{i_m}$ for each $m$. By Lemma 5.4(c), when restricted to $T^2_s$, for any $r$, $-G$ is dual to the map $\xi_r : A_r \rightarrow \bigoplus Q_i \otimes A_{s-i} \subset \bigoplus A_i \otimes A_{s-i}$, so by induction, for each $m < n$,
\[ G\lambda_m(b_1 \otimes \cdots \otimes b_m) \in G(\bigoplus_{i_1 + \cdots + i_m = s} A^\#_i \otimes A^\#_j) \subset A^\#_{i_1 + \cdots + i_m}, \]
and similarly for $G\lambda_{n-m}(b_{m+1} \otimes \cdots \otimes b_n)$. Thus the first statement follows. The second statement follows from the assumption $[5.1.2]$ about the map $\xi_s$.

(b) When $n = 2$, $\theta_2 = \xi_2$, and the claim follows from Lemma 5.4(c). Now we assume $n > 2$. When restricted to $(E^1)^\otimes n$ in Adams degree $s$, part (a) says that if $i > 1$, then the image of $G\lambda_i \otimes G\lambda_j$ is in $\bigoplus_{q+r=s} D^\#_q \otimes D^\#_r \subset \bigoplus_{q+r=s} D^\#_q \otimes A^\#_r$. By Lemma 5.4(b,c),
\[ G\lambda_2(D^\#_q \otimes A^\#_r) = G(D^\#_q \otimes A^\#_r) = (-\xi_s)^\#(D^\#_q \otimes A^\#_r) = 0. \]
Therefore, when restricted to $((E^1)^\otimes n)_s$, we have
\[ G\lambda_n = G\lambda_2(-1)^2(-id) \otimes G\lambda_{n-1} = -G\lambda_2(id \otimes G\lambda_{n-1}). \]
By induction on $n$, in any Adams degree $r$, $G\lambda_{n-1}$ is $-(\theta_r)^\#$ composed with projection to a summand, and by Lemma 5.4(c) we see that in Adams degree $q$, we have $G = -(\xi_q)^\#$. Hence when applied to $b_1 \otimes b_2 \otimes \cdots \otimes b_n$ with $b_1$ in Adams degree $i$ and $b_2 \otimes \cdots \otimes b_n$ in Adams degree $s - i$, we have
\[ G\lambda_n = (\xi_s)^\# \lambda_2(id_{E^1} \otimes -\theta_{s-i})^\# = -((id_{Q_i} \otimes \theta_{s-i}) \circ \xi_s)^\#. \]
and thus $G\lambda_n$ is exactly $-\theta_s^\#$ followed by projection onto the tensor length $n$ summand

$$\bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.$$ 

(c) Since we assume that $R_s$ is a subspace of $\bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i}$, the dual of the inclusion

$$R_s \to \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i}$$

is $Pr_H$ restricted to $\left( \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \right)^\#$. Hence the dual of

$$R_s \to \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i} \to (Q \otimes Q^\otimes(n-1))_s$$

is equal to $Pr_H \circ \left( \sum_{q+r=s} id \otimes \theta_{s-i} \right)^\#$.

By Lemma 5.4(b), $Pr_H$ is zero when applied to $D_q^\# \otimes A_r^\#$ for all $q$. By (b),

$$Pr_H \lambda_n = Pr_H \lambda_2 \left( - \sum_{q+r=s} id_{Q_q^\#} \otimes G\lambda_r \right) = Pr_H \left( \sum_{q+r=s} id_{E_1^q} \otimes (\theta_r)^\# \right),$$

which is the desired map. \qed

Proof of Theorem A. First of all by Lemma 5.2, we may assume that

$$R_s \subset \bigoplus_{1 \leq i \leq s-1} Q_i \otimes A_{s-i}.$$ 

Then we appeal to Lemmas 5.3, 5.4 and 5.5. The canonical map in Lemma 5.5(c) is just the inclusion, and so the assertion holds. \qed

Proof of Corollary B. Note that under the assumptions in the corollary, the vector space of indecomposables is canonically isomorphic to $A^\#_1$. Thus various parts of Theorem A simplify; for example, $E^1$ is isomorphic to $A^\#_1$, and hence is concentrated in Adams degree $-1$. \qed

The following corollary is immediate.

Corollary 5.6. Let $A$ and $E$ be as in Theorem A.

(a) The algebra $A$ is determined by the maps $m_n$ restricted to $(E^1)^\otimes n$ for all $n$.

(b) The $A_\infty$-structure of $E$ is determined up to quasi-isomorphism by the maps $m_n$ restricted to $(E^1)^\otimes n$ for all $n$.

Proof. (a) By Theorem A, the map $R \to T(Q)$ can be recovered from $m_n$ restricted to $(E^1)^\otimes n$. Hence the structure of $A$ is determined.

(b) After $A$ is recovered, the $A_\infty$-structure of $E$ is determined by $A$. Therefore the structure of $E$ is determined by the restriction of $m_n$ on $(E^1)^\otimes n$, up to quasi-isomorphism. \qed

Proof of Proposition 1.2. By the construction given above, it is clear that if the grading group for the Adams grading is $\mathbb{Z} \otimes G$ for some abelian group $G$, then all of the maps including $m_n$ preserve the $G$-grading. The assertion follows. \qed
6. Examples

Several examples of $A_\infty$-algebras $E$ are given in [LP1] Exs. 3.5, 3.7, 13.4 and 13.5]. We conclude this paper with a few more examples.

**Example 6.1.** Fix a field $k$ and consider the free algebra $B = k\langle x_1, x_2 \rangle$ on two generators, each in Adams degree $1$. Fix an integer $q \geq 2$, and let $f(x_1, x_2)$ be an element in Adams degree $q$, and let $A = B/(f)$. Then the minimal resolution (5.0.1) for $k_A$ has the form

$$\cdots \to Ar \xrightarrow{i} A e_1 \oplus A e_2 \to A \to k_A \to 0,$$

where the generator $e_i$ corresponds to $x_i$, and $r$ corresponds to $f(x_1, x_2)$. Order the monomials in $B$ left-lexicographically, setting $x_1 < x_2$, and assume that with respect to this ordering, $f(x_1, x_2)$ has leading term $x_1^r x_2^q$ with $0 < i < q$. Then one can show that the map $i : Ar \to A e_1 \oplus A e_2$ is injective, so

$$\text{Ext}^s_A(k_A, k_A) = \begin{cases} 
  k & s = 0, \\
  k(-1) \oplus k(-1) & s = 1, \\
  k(-q) & s = 2, \\
  0 & \text{else}
\end{cases}$$

$\text{Ext}^1$ is dual to $A_1$, and we choose $(y_1, y_2)$ to be the dual basis to $(x_1, x_2)$. We write $z$ for the generator of $\text{Ext}^2$ dual to $f$. For degree reasons, the $A_\infty$-algebra structure on $\text{Ext}$ has the property that $m_n = 0$ unless $n = q$. By Theorem $A$, the map $m_q$ is “dual to the relations”:

$$m_q(y_{i_1} \otimes \cdots \otimes y_{i_q}) = \alpha z \quad \text{if } \alpha x_{i_1} \cdots x_{i_q} \text{ is a summand in } f(x_1, x_2).$$

So for example, if $q > 2$, then as an associative algebra, $\text{Ext}^s_A(k_A, k_A)$ has trivial multiplication no matter what $f$ is, so one cannot recover $A$ from the ordinary algebra structure. One can recover $A$ from the $A_\infty$-algebra structure, though.

**Example 6.2.** Let $A = k[x_2, x_3]/(x_2^3 - x_3^2)$, graded by giving each $x_i$ Adams degree $i$. The graded vector space $Q$ has two nonzero graded pieces: $Q_i$ is spanned by $x_i$ when $i = 2, 3$. Let $(b_2, b_3)$ be the graded basis for $Q^\#$ which is dual to $(x_2, x_3)$. The space $R$ of relations has two graded pieces also: there is the degree 5 relation $r_5 = x_2 x_3 - x_3 x_2$, and the degree 6 relation $r_6 = x_2^3 - x_3^2$. Let $(s_5, s_6)$ be the graded basis for $R^\#$ which is dual to the basis $(r_5, r_6)$. Thus in low dimensions, the Ext algebra is given by

$$\text{Ext}^n_A(k_A, k_A) = \begin{cases} 
  k & n = 0, \\
  k(-2) \oplus k(-3) & n = 1, \\
  k(-5) \oplus k(-6) & n = 2.
\end{cases}$$

Indeed, by viewing $A$ as a subalgebra of $k[y]$ (with $A \hookrightarrow k[y]$ defined by $x_i \mapsto y^i$), one can construct a minimal resolution for $k_A$ to find that if $n > 0$,

$$\text{Ext}^n_A(k_A, k_A) = k(3n + 1) \oplus k(-3n),$$
with vector space basis \((b_2b_3^{n-1}, b_3^n)\). Theorem \(\square\) gives us the following formulas in Ext:

\[
\begin{align*}
m_2(b_2 \otimes b_3) &= s_5, \\
m_2(k_1 \otimes b_2) &= -s_5, \\
m_2(b_3 \otimes b_3) &= s_6, \\
m_3(b_2 \otimes b_2 \otimes b_2) &= -s_6.
\end{align*}
\]

All other instances of \(m_2\) and \(m_3\) on classes from \(E^1\) are zero, for degree reasons.

**Example 6.3.** Fix \(p > 2\) and let \(A = k[x]/(x^p)\) with \(x\) in Adams degree \(2d\) (so that \(A\) is graded commutative). Its Ext algebra is

\[
\text{Ext}^*_{A}(k_A, k_A) \cong \Lambda(y_1) \otimes k[y_2],
\]

with \(y_i\) in Ext\(i\), with \(y_1\) in Adams degree \(-2d\) and \(y_2\) in Adams degree \(-2dp\). Then Theorem \(\square\) tells us that we may choose \(y_1\) and \(y_2\) so that \(m_p(y_1 \otimes \cdots \otimes y_1) = y_2\). It is a standard Massey product computation that the \(p\)-fold Massey product \((y_1, \ldots, y_1)\) equals a generator of Ext\(2\) (with no indeterminacy, for degree reasons), and Theorem \(\square\) tells us that with our choice of \(y_1\) and \(y_2\), we have \(\langle y_1, \ldots, y_1 \rangle = \{-1\}^{(p+1)/2}/y_2\}^\perp_{\exists i\in\mathbb{Z}}\).

**Example 6.4.** Let \(k\) be a field of characteristic 2, and define the \(k\)-algebra \(A\) by

\[
A = k\langle x_1, x_2 \rangle/(x_1^2, x_1x_2x_1 + x_2^2),
\]

graded by putting \(x_i\) in Adams degree \(i\). This is the sub-Hopf algebra \(A(1)\) of the mod 2 Steenrod algebra, and its cohomology can be computed using spectral sequences — see Wilkerson [Wi 2.4] or Ravenel [Ra 3.1.25], for instance. In low degrees, it has

\[
\text{Ext}^n_{A}(k_A, k_A) = \begin{cases} 
k, & n = 0, \\
k(-1) \oplus k(-2), & n = 1, \\
k(-2) \oplus k(-4), & n = 2.
\end{cases}
\]

Following Wilkerson, we write \((h_0, h_1)\) for the basis of Ext\(1\), with \(h_i\) in Adams degree \(-2^i\). Then Theorem \(\square\) tells us that \(m_2(h_0 \otimes h_0) \neq 0\) and \(m_2(h_1 \otimes h_1) \neq 0\), so \((h_0^2, h_1^2)\) is a basis for Ext\(2\). Theorem \(\square\) also gives the formula

\[
m_3(h_0 \otimes h_1 \otimes h_0) = h_1^2.
\]

This reflects the Massey product computation \(\langle h_0, h_1, h_0 \rangle = \{h_1^2\}\).

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