Assessment of Rate Effects in Piezocone Tests from Poroelastic Cavity Expansion Analysis

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Abstract. Cavity expansion solutions are often used in geomechanics modelling to investigate problems such as the bearing capacity of piles or interpretation of cone and pressuremeter tests. Conceived as a simplified approach to capturing the transient flow effects in the soil around an expanding cylinder, a non-linear poroelastic model is formulated in this paper based on the concept of proportional poroplasticity. The latter is used to assess rate effects and associated drainage conditions during piezocone penetration. In this context, cone tests performed in gold tailings at penetration rates ranging from 0.1 mm/s to 57 mm/s are used to validate the proposed approach. The model predictions are directly compared with in situ testing results in the space of normalized velocity $V_h$ to cone resistance $Q$ and to degree of drainage $U$, demonstrating the model capability of capturing the transitions from drained to partially drained and undrained soil regimes.

Keywords: cylinder expansion, finite element analysis, in situ tests, non-linear poroelasticity, transient flow.

1. Introduction

With the growth of the mining industry, large-scale mining operations have increased the challenge to the design of safe and economical Tailings Storage Facilities (TSF). The exploitation of minerals, such as gold, zinc, bauxite, produces large amounts of mine tailings which are often in slurry form with high water content and compressibility. Recent failures (Tonglvshan Mine, China 2017; Mariana’s Dam, Brazil 2015; Mount Polley mine, Canada 2014, to cite a few) reinforce that although there has been significant improvement in the state of practice (e.g. Vick, 1990; Martin & McRoberts, 1999; Davies & Martin, 2000; Fahey et al., 2002), available engineering technology for the design, construction, operation and closure of tailing deposits still causes important environmental impacts.

Central in the design of TSF, the geo-characterization of tailings faces difficulties coming from the heterogeneous nature of waste products, the hydraulic depositional processes and the change in the constitutive parameters during the lifetime of deposits. The experience gathered in the last 20 years (e.g. Schnaid et al., 2013; Jamilolkowski et al., 2003) shows that the deposition process often produces tailings in the so-called intermediate permeability range of $10^{-3}$ to $10^{-8}$ m/s (silty materials). In intermediate soils, including natural clayey and sandy-silts, tailings and other geomaterials, partially drained behavior can occur when in situ tests as cone penetration are performed at the standardized velocity (20 mm/s), introducing errors in interpretation.

For in situ tests, partially drained effects in silty soils (transient materials) can be and are usually evaluated considering a non-dimensional velocity $V_h$ [e.g. Randolph & Hope, 2004]:

$$V_h = \frac{v \cdot d}{c_h}$$

where $d$ is the probe diameter, $v$ the loading rate and $c_h$ the coefficient of horizontal consolidation.

The use of expression (1) is essentially based on experimental observation, in which normalized testing data are interpreted in a space that correlates the degree of drainage $U$ to a non-dimensional velocity parameter $V_h$ (House et al., 2001; Schnaid et al., 2004; Randolph & Hope, 2004; Chung et al., 2006; DeJong et al., 2013). At the same time, attempts have been made to develop a structured theoretical framework to anticipate the drained conditions in intermediate permeability soils. Most of the current developments are based on cavity expansion solutions which are recognized as useful in interpreting the response of piles (e.g. Soderberg, 1962; Vesci, 1972; Randolph & Wroth, 1979; Carter et al., 1979; Osman & Randolph, 2012; Yu, 2000) and in in situ tests (e.g. Gibson & Anderson, 1961; Baligh, 1985; Teh & Houlshy, 1991; Burns & Mayne, 2002; Yu & Mitchell, 1998) with a variety of soil stress-strain models.

Following a literature review, analytical solutions for both spherical and cylindrical cavity expansion assuming a Mohr-Coulomb criterion and considering the volume change in a plastic region can be found in the work of Vesci
(1972). In Randolph & Wroth (1979) a cavity expansion approach is used to study the pore pressure dissipation after pile installation. Semi-analytical solutions based on critical state models are reported by Yu & Houlsby (1991), Collins et al. (1992), Collins & Stimpson (1994), Cao et al. (2001), Chang et al. (2001) for the drained and undrained cylindrical expansion.

Finite element solutions can be used to analyze the consolidation fields in a saturated porous medium under cylindrical and spherical symmetries. One may see the studies of Jang et al. (2003), Silva et al. (2006), Zhao et al. (2007), LeBlanc & Randolph (2008), Xu & Lehane (2008), Wang (2000), Jaeger et al. (2010), Jaeger (2012), Suryasentana & Lehane (2014), modelling the soil as Tresca, Mohr Coulomb or Modified Cam-Clay under different expansion rates, capturing undrained (very fast rate) and fully drained behaviour (very slow rate).

Aiming to analyze the consolidation and rate effects induced by the expansion of a rigid cylinder embedded within an isotropic elastic medium of infinite extent, a constitutive model has been formulated in Dienstmann et al. (2017). The model is based on the local equivalence between the response of a perfectly plastic behavior to monotonic loading and an appropriate fictitious non-linear poroelastic behavior. Closed-form expressions for pore pressure distribution were derived according to a simplified framework, while stresses and displacements are computed numerically. The theoretical features of the model are briefly described and applied herein to the interpretation of drainage conditions during piezocone tests executed in Fazenda Brasileiro gold TSF.

2. Problem Idealization

The analytical cylindrical cavity expansion solution proposed and described in Dienstmann et al. (2017) is structured as a consolidation analysis of a rigid cylinder deeply embedded within an isotropic fully saturated poroelastic medium of infinite extent. The problem, idealized in Fig. 1, defines that from an initial stress and pore pressure state \((\sigma_0, p_0)\), a prescribed radial displacement of magnitude \(\alpha R\) is applied at the cylinder wall \(r = R\). The influence zone of the applied displacement is defined from \(R \leq r \leq a\) to characterize the extent of the region whose poromechanical state is no longer affected by cylinder installation and subsequent expansion (e.g. Blight, 1968; Randolph and Wroth, 1979; Osman and Randolph, 2012). This means in particular that displacement and excess pore pressure vanish at \(r = a\).

The soil surrounding the cylinder is modeled as a fully saturated, isothermal poroelastic material undergoing infinitesimal strains. In all that follows, the sign convention of positive stress values for tension and negative stress values for compression will be adopted.

2.1. Constitutive formulation

The theoretical features of the model are based on the local equivalence between a perfectly plastic behavior response to monotonic loading and an appropriate fictitious non-linear poroelastic behavior. The non-linear behavior was considered through a secant shear modulus \(G\) which evolves with the level of strains. The model is briefly described in the sequel considering a null initial stress and pore pressure \(i.e.\), \(\sigma_i = p_i = 0\). The extension to non-zero initial state and details of the model can be found in the original work, Dienstmann et al. (2017).

To capture some of the non-linear features of the soil behavior, the shear modulus \(G\) was defined by a dependence law of the form \(G = G(\varepsilon_d, \varepsilon_v, \sigma')\) where \(\varepsilon_d = \sqrt{(\varepsilon - \frac{1}{3}\text{tr}\sigma):(\varepsilon - \frac{1}{3}\text{tr}\sigma)}\) and \(\varepsilon_v = \text{tr}\varepsilon\) refer respectively to equivalent deviatoric strain and volumetric strain, and \(\sigma\) is the pore pressure.

Assuming a Drucker-Prager yield condition, the plasticity of the soil is described by

\[
F(\sigma') = \sigma_d + T(\sigma'_m - h) \leq 0
\]

where \(\sigma' = \sigma + p\) is the Terzaghi effective stress, \(\sigma_d = \sqrt{(\sigma - \frac{1}{3}\text{tr}\sigma): (\sigma - \frac{1}{3}\text{tr}\sigma)}\) is the equivalent deviatoric stress and \(\sigma'_m = \frac{1}{3}\text{tr}\sigma' = \sigma_m + p\) is the mean Terzaghi effective stress. The previous relations are defined considering a sign convention of positive stress values for tension and negative stress values for compression. Additionally, according to the classical poromechanics formulation, the pore pressure (or the fluid pressure) is defined as: \(p = u_h + u\), where \(u_h\) refers to the hydrostatic pore pressure and \(u\) to the excess pore pressure. Parameters \(h\) and \(T\) respectively characterize the tensile strength and the friction coefficient of the Drucker-Prager yield condition, and can be derived from Mohr Coulomb parameters, \(c\) - cohesion and \(\phi\) - friction angle.

Considering that the stress associated with the secant behavior

\[
\sigma = \sigma_0 + (K - \frac{1}{3}G)\text{tr}\varepsilon + 2G\varepsilon - b\Delta p
\]

Figure 1 - Idealized geometry and loading conditions for consolidation around an infinite expanding cylinder.
meets asymptotically the above yield condition (2), that is
\[
\lim_{\frac{G}{\Delta p} \to \infty} F(\sigma' = \sigma + p) = 0
\]
where \( \varepsilon_{\text{ref}} \ll 1 \) is a reference strain that physically represents the order of magnitude of the shear strain mobilized at yielding.

In Eq. 3 \( b \) is the Biot coefficient, \( \Delta p \) defines the pore pressure variation, \( G \) is the shear modulus and \( K \) is the bulk modulus.

Adopting a constant value for the bulk modulus \( K \), and considering an approach similar to that developed in Lemarchand et al. (2002) and Maghouss et al. (2009), a simple way to meet the above condition consists in considering the following law [Maghouss et al. (2009)].

\[
G(\varepsilon_d, \varepsilon_v, p) = \frac{1}{2} \left[ T(h - K \varepsilon_v) - (1 - b) p \right] \times \frac{1}{1 + \frac{\varepsilon_v}{\varepsilon_{\text{ref}}}}
\]

(5)

Figure 2 provides a non-linear poroelastic representation of the Drucker-Prager behavior as defined by Eq. 5.

Coming back to the expansion problem, the strains and pore pressures mobilized must satisfy the momentum balance equation \((\text{div} \Delta \sigma = 0)\), which should be complemented by the mechanical boundary conditions related to the displacement \( \xi \):

\[
\begin{cases}
 \xi = a R \varepsilon_v & \text{at } r = R \\
 \xi = 0 & \text{at } r = a > R \quad \forall t > 0
\end{cases}
\]

(6)

The first condition in Eq. 6 implies that a radial displacement \( aR \) is imposed at the cylinder wall, while the second condition indicates that the displacement induced by the expansion of the rigid cylinder is null at a distance \( a > R \).

Due to problem symmetry, the displacement distribution can be sought in the form

\[
\frac{\xi}{a} = f(r) \varepsilon_v
\]

(7)

The corresponding strain tensor reads

\[
\varepsilon = \varepsilon_{\text{ref}} \varepsilon_v \otimes \varepsilon_v + \varepsilon_{00} \varepsilon_0 \otimes \varepsilon_v
\]

(8)

with \( \varepsilon_{\text{ref}} = f'(r) \); and \( \varepsilon_{00} = \frac{f}{r} \)

and the stress increment associated with poroelastic law is:

\[
\Delta \sigma = \frac{\lambda}{G} \left( f'(r) + \frac{f(r)}{r} \right) - 2G \left( f'(r) \varepsilon_v \otimes \varepsilon_v + \frac{f(r)}{r} \varepsilon_0 \otimes \varepsilon_0 \right) \Delta \rho
\]

(9)

The displacement function \( f(r) \) is obtained by the integration of the local equilibrium equation in projection following the radial direction, observing that

\[
\begin{align*}
\Delta \sigma_{rr} &= K \left( f'(r) + \frac{f(r)}{r} \right) + \frac{2}{3} G \left( 2 f'(r) + \frac{f(r)}{r} \right) - b \Delta \rho \\
\Delta \sigma_{00} &= K \left( f'(r) + \frac{f(r)}{r} \right) + \frac{2}{3} G \left( 2 f'(r) - \frac{f(r)}{r} \right) - b \Delta \rho
\end{align*}
\]

(10)

Equation 9 can be conveniently written as

\[
\frac{d}{dr} \left( K f'(r) + \frac{f(r)}{r} \right) + \frac{2}{3} G \left( 2 f'(r) - \frac{f(r)}{r} \right) - b \Delta \rho = 0
\]

(11)

with \( G = G(\varepsilon_d, \varepsilon_v, p) = G(f, p) \) since

\[
\varepsilon_d = \sqrt{\frac{2}{3} f'(r)^2 + \frac{f(r)^2}{r^2} - f'(r) \frac{f(r)}{r}}
\]

(12)

Equation 11 is a differential equation that couples the skeleton displacements (function \( f(r) \)) and pore pressure \( \Delta \rho \). The second equation for consolidation around the expanding cylinder is deduced from the solution to the fluid flow problem.

2.2. Pore fluid problem

To obtain the pore flow component of the problem, it is used the second poroelastic state equation

\[
\Delta \Phi = b \Delta \sigma + \frac{1}{M} \Delta \rho
\]

(13)

which relates the Lagrangian porosity change \( \Delta \Phi = \Phi - \Phi_0 \) to the skeleton volumetric strain and pore pressure change. Eq. 13 presents the Biot parameter \( b \) and the Biot Modulus \( M \), and must be combined with the fluid mass balance and a flow law (Darcy law).

Neglecting the variations of fluid density, the fluid mass balance in infinitesimal skeleton strains is written as:

\[
\frac{\partial \Phi}{\partial t} + \text{div} q = 0
\]

(14)

where \( q \) is the filtration vector. The latter is connected to the excess pore pressure \( u \) through Darcy’s law.


\[ q = -k \cdot \nabla u \]  \hspace{1cm} (15)

where \( k \) denotes the permeability tensor. Considering an isotropic medium, \( k = kI \) where \( k \) is the permeability and \( I \) is the identity matrix.

Reporting both the second poroelastic state Eq. 13 and Darcy’s law (Eq. 15) into the fluid mass balance (Eq. 14) yields:

\[ \frac{\partial \text{tr} e}{\partial t} + M \frac{\partial \Delta p}{\partial t} = k \nabla^2 u \]  \hspace{1cm} (16)

where \( \nabla^2 \) stands for the Laplacian operator.

From the classical reasoning for a non-linear poroelastic medium, the following generalized Navier equation can be derived (assuming irrotational displacement field)

\[ \left[ K + \frac{4}{3} G_e \right] \nabla \text{tr} e + 2 \nabla G \cdot \left( \frac{\varepsilon_1}{3} - \frac{\text{tr} e}{3} \right) = b \nabla (\Delta p) \]  \hspace{1cm} (17)

Equation 17, emphasizes the strong coupling between skeleton strains and pore pressure. In classical formulations, the pore pressure is only affected by the volumetric part \( \varepsilon_1 \) of skeleton strains, considerably simplifying the pore fluid flow solution. However, in the adopted solution the pore pressure is also explicitly related to the deviatoric part \( \varepsilon_r \) of skeleton strains through the shear modulus \( G(e_r, \varepsilon_r, p) \).

In this case, determination of pore pressure distribution and displacement field (function \( r \rightarrow f(r) \)) requires solving the coupled system defined by the set of partial differential Eqs. 11, 16 and 17.

A time incremental procedure has been implemented in Dienstmann et al. (2017) to provide semi-analytical solutions to this problem. The basic idea consists in approximating, during the time interval \( t_n \leq t \leq t_{n+1} \), the shear modulus \( G \) of the medium by an equivalent mean value \( G \approx G_0 \) that is constant in time and space. Accordingly, Eq. 17 reduces to

\[ \left[ K + \frac{4}{3} G_0 \right] \nabla \text{tr} e = b \nabla (\Delta p) \]  \hspace{1cm} (18)

which can be integrated to relate \( \text{tr} e \) to \( \Delta p \).

Observing that when the hydrostatic pore pressure \( u_0 \) turns to be constant \( \frac{\partial \Delta p}{\partial t} = \frac{\partial u_0}{\partial t} \) and following Eq. 2 the uncoupled pressure diffusion equation can be obtained:

\[ \frac{\partial u}{\partial t} = c_r \nabla^2 u \quad \text{with} \quad c_r = kM \frac{K + \frac{4}{3} G_0}{M + \frac{4}{3} G_0} \]  \hspace{1cm} (19)

where \( c_r \) is the fluid diffusivity coefficient. An average shear modulus is therefore introduced for each time interval \([t_n, t_{n+1}]\) as:

\[ G_{eq} = \frac{1}{a} \int_R^a G(e_r(t_n), \varepsilon_r(t_n), p(t_n))dr \]  \hspace{1cm} (20)

The approximation \( G \approx G_0 \) is introduced for the diffusion equation only, while the non-linear form of the shear modulus is kept within the equilibrium equation.

The diffusion Eq. 19 together with the boundary and initial conditions (Eqs. 21-23) defines the solution of the hydraulic problem, which is characterized as a distribution of excess pore pressure over the time interval \([t_n, t_{n+1}]\).

\[ \frac{\partial u}{\partial t} = 0 \quad \text{at} \quad r = R \forall t \in [t_n, t_{n+1}] \]  \hspace{1cm} (21)

\[ u = 0 \quad \text{at} \quad r = a \forall t \in [t_n, t_{n+1}] \]  \hspace{1cm} (22)

\[ u = u(r, t_n) \quad \text{at} \quad t = t_n \forall r \in [R, a] \]  \hspace{1cm} (23)

Equations 21 to 23 refer to the impermeable condition of the expanding cylinder, the negligible influence of pore pressure condition at the radius of influence, and the initial pore pressure state at time \( t_n \), respectively. An initial pore pressure \( u_0 = u_0(r) \) at \( t = 0 \) is intended to account for the excess pore pressure produced by the rigid cylinder insertion. This point will be discussed in the next section.

The general solution to the boundary value problem mentioned above is:

\[ u = \sum_{i=1}^{\infty} C_i^* \left[ f \omega_0(a, r) - J_0(a, r) \right] e^{-\omega_i(t-t_n)} \]  \hspace{1cm} (24)

where

\[ C_i^* = \frac{\int_0^a u(r, t_n) [\omega_i Y_0(\alpha R) - J_0(\alpha R)]dr}{\int_0^a [\omega_i Y_0(\alpha R) - J_0(\alpha R)]^2 dr} \]  \hspace{1cm} (25)

Functions \( J_0 \) and \( Y_0 \) are zero-order Bessel functions of the first and second kind, respectively. Scalar \( \alpha_i \) is the \( i \)-th root of the following algebraic equation with respect to variable \( \alpha \)

\[ Y_1(x\alpha)J_0(\alpha a) - J_1(x\alpha)Y_0(\alpha a) = 0 \]  \hspace{1cm} (26)

where \( J_1 \) and \( Y_1 \) are first-order Bessel functions referring respectively to the first and second kind. Scalar \( \omega_i \) is computed from \( \alpha_i \) as

\[ \omega_i = -\frac{Y_1(\alpha_i R)}{J_1(\alpha_i R)} \]  \hspace{1cm} (27)

From Eqs. 24 to 27 the excess pore pressure \( u \) is computed, letting the pore pressure increment be determined as \( \Delta p = u - u_0 \). The final expression of pore pressure distribution \( \Delta p \) is substituted into Eq. 11, which simplifies to be an ordinary differential equation governing the displacement function \( f(r) \) along time interval \([t_n, t_{n+1}]\). The latter function and the corresponding strain and stress distribution are de-
2.3. Initial excess pore pressure distribution

The solution developed in Dienstmann et al. (2017) is generalized to consider initial states of stress and pore pressure different from zero, as would be expected for problems such as pile installation and in situ tests. Expressions for initial excess of pore pressure \( u_0 \) (or equivalently \( p_0 \)) generated by insertion of a rigid cylinder within the medium can be found in the literature (e.g. Vesic, 2000; Randolph & Wroth, 1979; Poulos & Davis, 1980) and have been based on laboratory or field data. Most of collected data have shown that the pore pressure gradients are essentially radial, and that the maximum excess pore pressure \( u_{0,max} \) generated by the installation process is observed close to the cylinder (i.e., at \( r = R \)). The magnitude of the excess pore pressure appears to decrease with the radius \( r \) measured from the cylinder axis, and becomes negligibly small at a certain distance \( r = a \), referred to as the initial radius of influence. Combining hyperbolic and logarithmic distributions, a new expression has been proposed in Dienstmann et al. (2017). It extends classical expressions to account for the flow restriction at the cylinder wall:

\[
 u_0(r) = u_{0,max} \frac{F(r)}{F(R)} \quad \text{with} \quad F(r) = 1 - \frac{a}{r} + \frac{\ln a}{R} \quad (28)
\]

for \( R \leq r \leq a \)

where \( u_{0,max} \) refers to the maximum value of pore pressure generated by cylinder insertion.

Regarding \( u_{0,max} \) correlating a pile insertion into the soil with a cavity expansion problem, typical solutions derived in this context (e.g. Vesic, 1972; Randolph & Wroth, 1979) suggest that \( u_{0,max} \) can be evaluated from the stress paths developed in undrained triaxial tests carried out until the ultimate state (undrained strength) is attained. The excess pore pressure is thus deduced from the variation in mean and shear stresses, leading to the following estimate for normally consolidated soils (Vesic, 1972):

\[
 u_{0,max} = \frac{p_{c0}}{2} (1 + M_{cs}) \quad (29)
\]

where \( p_{c0} \) denotes a reference initial consolidation pressure, and \( M_{cs} \) is the slope of the critical state line. It should be emphasized that expression (29) implicitly assumes that cylinder installation is fast, the elapsed time between installation and expansion is sufficiently short for pore pressures to not dissipate, thus justifying the assumption of undrained hydromechanical evolution of soil.

2.4. Cylindrical cavity expansion to cone resistance estimation

The cylindrical cavity expansion solution provides expressions for stress \( (\sigma) \), strain \( (\varepsilon) \) and pore pressure \( (p) \) fields. Seeking a direct comparison to in situ test as cone penetration some considerations must be drawn. In this context, the calculated cavity expansion limit pressures \( (\sigma) \) at the cylinder wall have been converted to cone resistance \( (q) \) using an approach similar to that proposed by Rohani & Baladi (1981), and used by Silva (2005) and LeBlanc & Randolph (2008). In the approach the cone tip resistance is estimated from the vertical projections of all forces acting on the cone. The expression of the vertical resistive force \( F_z \) can be calculated from the integration of stresses over the cone tip length \( L \) (see Fig. 3):

\[
 F_z = \int_0^L [(\sigma' + u) \sin \beta + \tau \cos \beta] 2\pi r dl \quad (30)
\]

Expression 30 can directly be used to determine the cone tip resistance

\[
 q_e = \frac{F_z}{S_e} \quad (31)
\]

where \( S_e = \pi L \tan \beta \) denotes the piezocone cross-section area. In the context of cylindrical cavity expansion, it is more convenient to express \( q_e \) in terms of the radial stresses using \( \tau = \sigma'_r \tan \delta \tan \beta \), together with the equations expressing locally the horizontal and vertical force equilibrium at any point along the cone tip:

\[
 q_e = \sigma'_r \left( \frac{1 + \tan \beta}{\tan \delta} \right) + u \quad (32)
\]

where \( \beta \) is half of the angle of the cone tip (usual value 60°/2 = 30°), \( \delta \) is the interface friction taken as the soil friction angle \( \phi' \), \( \sigma'_r \) and \( u \) are respectively the radial stress and pore pressure at the expanding cylinder radius.

Figure 3 describes the idealization of stresses acting on a small portion \( (dl) \) of the cone tip length \( (L) \).

3. Gold Tailings

The present paper describes the use of a cylindrical cavity solution to assist the drainage behavior interpreta-
tion of experimental data from a gold TSF located in northeast Brazil (Fig. 4). The Fazenda Brasileiro Mine includes a producing gold mine and approximately 197,000 hectares of adjacent exploration properties. The site is being a subject of study for the past two decades (Bedin et al., 2012, Schnaid et al., 2013, Schnaid et al., 2016, Dienstmann et al., 2018), with a research project including site evaluation, field and laboratory tests.

A general view of Fazenda Brasileiro Mine is presented in Fig. 4, in which locations for the site investigations conducted in 2006, 2013, and 2016 are displayed. A total of eleven (11) different islands of investigation were defined, with sample collecting, and in situ tests as piezocone (conventional CPTu, and with seismic measurements SCPTu), seismic dilatometer (SDMT) and vane tests. Previous studies have shown that the material disposed is predominantly silty sand (Fig. 5) with an average in situ solids content (ratio of weights obtained before and after a drying process) of about 30%, in situ water content of 35% (ratio between water and solid weights), low to non-plastic and with high specific gravity ($2.89 < G_s < 3.2 \text{ g/cm}^3$). In that respect, data referring to the gold tailings are reported in Table 1. Triaxial tests described in Bedin et al. (2012) and Schnaid et al. (2013) allowed the critical state line CSL to be established for tests carried out in compression and extension and sheared under monotonic loading. A highly-nonlinear shape of the CSL under undrained loading was observed indicating that the tailings exhibit severe strainsoftening with high compressibility that at low stresses may lead to flow liquefaction.

A typical piezocone profile for tests carried out at the standard 20 mm/s penetration rate is presented in Fig. 6 and includes cone tip resistance $q_c$, penetration pore pressure $u_2$, pore pressure ratio $B_q = (u_2 - u_h)/(q_c - \sigma_v)$ - where $u_h$ is the hydrostatic pore pressure, $q_c$ is the total cone tip resistance and $\sigma_v$ is the total vertical stress - and soil behavior index $I_{ch}$, according to Robertson and Wride (1998). The soil profile identified by SCPTu and CPTu data, reveals a drained layer near the surface underlain by silty soils (from 3 m to 10 m) with $I_{ch}$ (soil behavior type) values typically in the 2.0 to 3.5 range, indicating clay and silt layers where excess pore pressures are relatively high, yielding $B_q$ (pore pressure ratio) in the 0.3 to 0.6 range. Some thin drained layers were detected at different penetration depths. SCPTu data indicate thin drained layers at 5 m, 6 m and 7 m.

To evaluate the effect of drainage conditions on the piezocone (CPTu) measurements, a series of soundings were performed at the Fazenda Brasileiro site with the penetration rates from 0.1 mm/s to 57 mm/s. Results of adjacent (2 m spacing) soundings performed at different penetration rates are presented in Fig. 7, in which values of $q_c$, $u_2$ and $B_q$ are plotted against depth. Steady penetration velocity profiles of 20 mm/s and 57 mm/s are compared to one profile carried out at variable penetration rates (labeled “variable”). Considerable excess pore pressure $\Delta u$ was gen-

### Table 1 - Gold tailings characterization.

| Study Region | Region | Physical characterization | Consolidation | Critical state | Friction angle |
|--------------|--------|---------------------------|---------------|----------------|---------------|
| Bedin (2010) | PZC01 to PZC05 | $w$ (w%), $G_s$ (g/cm$^3$) | $q_c$ (kN/m$^3$) | $n$ | $\phi$ |
|              |        | 40.1                      | 13.2          | 1.23           | 18.6          | 0.048         |
|              |        | 30                        | 3.15          | 1.1            | 20.5          | 0.058         |
|              |        | 38.7                      | 3.3           | 1.25-1.3       | 10-19.3       | 0.045         |
| Klahold (2013) | Cluster 01 and Cluster 02 | PZC06 to PZC08 | 38.7-37.8 | 2.85-2.86 | 0.02-1.08 | 18.94-19.63 |
erated over the entire depth interval in all tests (even for the slowest penetration rate of 0.1 mm/s). The pore pressure is shown to increase with increasing penetration rates, inducing maximum $Bq$ values in the 0.6 to 0.7 range. There is no appreciable difference in measured values of $q_c$ and $u^2$ for tests carried out at 20 mm/s and 57 mm/s, which appears to indicate that tests are essentially undrained for these penetration rates.

For a better comparison with modeling, results of the tailings database were reinterpreted in terms of a normalized resistance $Q = q_{nc}/\sigma^*_v$, and normalized pore pressure $U = \Delta u/\sigma^*_v$, plotted against a normalized velocity $V_h$ (Eq. 1). In this analysis $q_{nc} = (q_c - \sigma^*_v)$ and $\Delta u = (u^2 - u^h)$. Results are displayed in Fig. 8, from which it is possible to identify a region characterized by normalized velocities $V_h$ in the range of about 0.01 to 10 where partial drainage appears to occur during cone penetration. The lowest (undrained) penetration resistance ($Q_{UD}$) is of the order of 2.0, while the drained penetration resistance ($Q_{D}$) is 42.0, yielding a drained to undrained ratio $Q_D/Q_{UD}$ of about 21, which is consistent with previously reported data from Jaeger et al. (2010) and Lehane et al. (2009). When comparing results in terms of $Q$ vs. $V_h$ and $U$ vs. $V_h$, a slightly larger scatter is observed from data on the $U$ vs. $V_h$ space, although it is possible to identify that at fully undrained conditions the normalized pore pressure $\Delta u/\sigma^*_v$ ratio is of the order of 2.2, and reduces to zero for drained penetration.

Adjusting curves in Fig. 8 are represented by empirical upper and lower limits, obtained considering a hyperbolic cosine function as suggested by Schnaid (2005):

$$Q = Q_{mn} + \left( a + (1-a) \frac{1}{\cosh(b V_h)} \right) \times (Q_{max} - Q_{mn})$$

(33)

where $a$, $b$ and are fitting parameters, and are shown in Fig. 8. From a practical point of view, parameter $a$ embodies the difference between the undrained and the drained effective penetration, whereas $b$ and $c$ control the rate of change from drained to undrained. These parameters are representative of measured data in the $Q$ vs. $V$ space and, as an overall trend, they also cover results plotted in the $\Delta u$ vs. $V$ space.

### 3.1. Modelling the gold tailings

To help understand rate effects on cone measurements, properties of gold tailings were analytically mod-
Figure 6 - Typical piezocone profile.

Figure 7 - Values of $q_c$, $u_c$, and $B_q$ measured at different penetration rates.
eled using the approach developed by Dienstmann et al. (2017). Finite Element (FE) results from a cavity expansion analysis in ABAQUS are additionally added to reinforce the applicability of the analytical model. The cavity expansion simulated in Abaqus was defined according to an axisymmetric model, with unit height and infinite extent (classical approach, see Fig. 9). The same initial and boundary conditions used in the analytical model were adopted in the finite element approach. For the constitutive model in the FE analysis, the Drucker Prager combined with linear elasticity was considered. Elements used are 8-node axisymmetric quadrilateral, biquadratic displacement, bilinear pore pressure and reduced integration (Abaqus finite element type: CAX8RP).

Parameters used in the analysis are summarized in Table 2, and were defined by the average values obtained from in situ and laboratory tests. The friction angle $\phi^*$ is $32^\circ$ and the $h$ and $T$ parameters from the Drucker Prager criterion are defined according to Mohr Coulomb yield surface condition. The initial field of excess pore pressure $u(e)\theta$ applied in both models is calculated from Eq. 28. The initial stress distribution is an isotropic and uniform (in the plane of analysis) stress field; $\sigma_0 = \sigma_0 \mathbf{1}$ is considered along with the subsequent simulation.

To evaluate the influence of the extension zone defined by $a$, the radius that defines the extension of the region affected by cylinder installation, two approaches were considered for analysis:

- $a$ as a function of the soil Rigidity Index ($I_r$): $a = \sqrt{I_r R}$ with $I_r = 874$ (defined from triaxial results, Bedin et al., 2012), corresponding to a radius of influence of about 30 times the cylinder radius ($a = 30R$).
- and $a$ defined arbitrarily as $a = 10R$ to comply with previous studies (Vesic, 1972; Randolph & Wroth, 1979; Osman 2010, Osman & Randolph, 2012).

Applied displacements are limited to produce maximum local strains of 10% to comply with the model small strain assumptions. Although, from a practical engineering perspective, strain levels as high as 10% are often admitted for geotechnical testing interpretation. It is therefore implicitly assumed that results characterized from both the simplified model and the numerical approach are reasonable approximations for cylinder expansion, which would be more appropriately formulated by large strains.

Results in sequence are directly displayed in the normalized velocity space combined to normalized cone tip resistance $Q/Q_{\text{ref}}$ and normalized pore pressure $\Delta u/\Delta u_{\text{ref}}$, where $Q_{\text{ref}}$ is the maximum value of $Q$ ($Q_{\text{ref}} = Q_{\text{max}}$) corres-

![Figure 8](image-url) - Rate effects in the (a) $Q$ vs. $V_h$ space and (b) $U$ vs. $V_h$ space.

![Figure 9](image-url) - Finite-element mesh detail (not in to scale).
ponding to the drained penetration, and $\Delta u_{ref}$ is the maximum value of mobilized pore pressure ($\Delta u_{ref} = \Delta u_{max}$) corresponding to undrained penetration. Field measurements are directly compared to analytical and numerical predictions in Figs. 10 and 11. It is important to highlight that the goal of comparisons is not to data-fit the experimental results or to find the parameter values that most closely match the data. The analysis is performed by predicting results using the set of parameters defined from independent tests, with the aim of developing a proper understanding of the parameters controlling rate effects.

From Fig. 10 it is possible to observe that the predictions are generally able to capture the experimental trends in the $Q$ vs. $V_h$ and $U$ vs. $V_h$ spaces, with some discrepancies. Both models, analytical and numerical, capture the transition from drained to partially drained to undrained regimes, but underestimate the $Q/DQ$ measured ratio of 21. Numerical and analytical prediction in the space $U$ vs. $V_h$ are shifted to the left with respect to pore pressure measurements, producing lower $V_h$ for the onset of drained conditions. Typical values for the transition from undrained to partially drained behavior are in the range of 1 to 10, while predictions are underestimated by at least one log-cycle.

To assess the influence of the size of the deformable zone, in Fig. 11 results from predictions using $a = 10R$ are displayed. It can be observed that a reduction of the influence zone has a significant effect on the variation of pore pressure with time, producing acceptable comparisons of measured and predicted rate effects.

4. Conclusions

Accurate prediction and assessment of drainage conditions during geotechnical testing is a challenging issue for relevant interpretation of experimental data. For this purpose, theoretical poromechanical formulations may reveal useful in the understanding of different geotechnical problems. Cavity-expansion solutions have notably proven successful in providing valuable support for the evaluation of bearing capacity of piles or the interpretation of cone and pressuremeter tests. In this context, analytical solutions have been derived for the problem of cylinder expansion within a non-linear poroelastic medium. The fully coupled approach has been then applied to the investigation of rate effects that arise during piezocone tests performed in a gold tailing deposit. Field data are directly compared to analytical predictions as well as to finite element solutions in terms of normalized velocity vs. normalized cone tip re-

**Figure 10** - Comparisons between analytical and numerical predictions to field data in tailings for $a/R = 30$.

**Figure 11** - Comparisons between analytical and numerical predictions to field data in tailings for $a/R = 10$. 
Assessment of Rate Effects in Piezocone Tests from Poroelastic Cavity Expansion Analysis

Table 2 - Gold Tailings constitutive parameters.

| Constitutive parameters | Value     |
|-------------------------|-----------|
| \( p_0 \)               | 100 kPa   |
| \( \sigma_0' \)          | -50 kPa   |
| \( \phi; \theta''; T''' \) | 32°; 0.98; 0.98 |
| \( u_{max} = p_0(1 + M_\gamma) / 2 \) | 100 kPa |
| \( \sigma_u = \sigma_0' - p \) | -150 kPa |
| \( \gamma_c \)           | 10 kN/m²  |
| \( k \)                 | 1.00E-08 m/s |
| \( K \)                 | 5814 kPa  |
| \( b_{ref} \)           | 0.01      |
| \( b \)                 | 0.999     |
| \( M \)                 | 2.3 GPa   |
| \( R \)                 | 2.5 cm    |
| \( a \)                 | 10R 30R cm |

*Reference consolidation pressure used to define the maximum value of excess pore pressure \( u_{max} \).
**M_\gamma \) is the slope of the critical state line obtained directly from \( \phi \).
***\( T \) is the frictional coefficient of the Drucker Prager model obtained directly from \( \phi \).

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List of Symbols

\( a \): radius of the zone of influence
\( b \): Biot coefficient
\( B \): pore pressure ratio
\( c \): fitting parameter for the hyperbolic cosine function
\( c_h \): horizontal coefficient of consolidation
\( c_f \): fluid diffusivity coefficient
CPTU: piezocone test
\( d \): probe diameter size
\( dL \): small portion of the cone tip length
\( e_0 \): initial void ratio
\( F \): vertical resistive force
\( G \): specific gravity
\( G_e \): shear modulus
\( G_{eq} \): equivalent shear modulus
\( h \): Drucker Prager tensile strength
\( I_{RBI} \): soil behaviour index according Robertson & Wride (1998)
\( I_c \): Rigidity Index
\( I_0 \): zero-order Bessel functions of the first kind
\( I_1 \): first-order Bessel functions of the first kind
\( K \): Bulk Modulus
\( k \): permeability
\( k_f \): permeability tensor
\( L \): cone tip length
\( M \): Biot modulus
\( M_d \): slope of the critical state line
\( p \): pore pressure - hydrostatic plus excess pore pressure
\( p_i \): initial pore pressure - hydrostatic plus excess pore pressure
\( p_{0i} \): reference initial consolidation pressure
\( Q \): normalized resistance
\( Q_{on} \): drained normalized resistance
\( Q_{on}^{ref} \): reference normalized resistance
\( Q_{max} \): maximum normalized resistance
\( Q_{min} \): minimum normalized resistance
\( Q_{ud} \): undrained normalized resistance
\( q \): filtration vector
\( q_{ct} \): cone tip resistance
\( q_{net} \): cone tip net resistance
\( q_t \): total cone tip resistance
\( r \): radial distance
\( R \): cylinder radius
\( SCPTu \): seismic piezocone test
\( s \): piezocone cross-section area
TSF: Tailings Storage Facilities
\( T \): Drucker Prager friction coefficient
\( t \): time of analysis
\( t_i \): time of analysis
\( t_{m} \): time of analysis
\( U \): degree of drainage
\( u \): excess pore pressure
\( u_i \): initial excess pore pressure
\( u_{0max} \): maximum initial excess pore pressure
\( u_{pen} \): penetration pore pressure (position 2)
\( u_{on} \): reference penetration pore pressure
\( u_{max} \): maximum penetration pore pressure
\( u_{min} \): minimum penetration pore pressure
\( u_h \): hydrostatic pore pressure
\( w \): water content
\( V_h \): horizontal normalized penetration velocity
\( v \): loading rate
\( Y_0 \): zero-order Bessel functions of the second kind
\( Y_1 \): first-order Bessel functions of the second kind
\( \alpha \): radial displacement ratio
\( \alpha_{1} \): \( 1^{st} \) root of an algebraic function
\( \beta \): half of the angle of the cone tip
\( \Gamma \): specific volume of CSL (Critical State Line) at \( p' = 1 \text{kPa} \)
\( \gamma \): specific weight
\( \Delta u \): pore pressure decay
\( \Delta u_{max} \): maximum pore pressure decay
\( \Delta u_{min} \): minimum pore pressure decay
\( \delta \): interface friction
\( \xi \): displacement vector
\( \xi \): radial displacement

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\( \varepsilon_v \): equivalent deviatoric strain
\( \varepsilon_r \): volumetric strain
\( \varepsilon_{\text{ref}} \): reference strain
\( \varepsilon_{\text{eq}} \): strain matrix
\( \varepsilon_r \): radial strain
\( \varepsilon_{\text{eq}} \): orthogonal strain
\( \hat{e}_r \): unit vector in the radial direction
\( \hat{e}_v \): unit vector in the orthogonal direction
\( \sigma \): total stress matrix
\( \sigma' \): effective stress matrix
\( \sigma'_{\text{eq}} \): equivalent deviatoric stress
\( \sigma_m \): mean total stress
\( \sigma_m' \): mean effective stress
\( \sigma_n' \): effective normal pressure

\( \sigma_r \): radial pressure
\( \sigma_r' \): effective radial pressure
\( \sigma_v' \): total vertical pressure
\( \sigma_v' \): effective vertical pressure
\( \sigma_0 \): initial pressure

\( N \): specific volume of NCL (Normal Consolidation Line) at \( p' = 1 \) kPa
\( \lambda \): slope of the NCL (Normal Consolidation Line)
\( \Phi \): Lagrangian porosity
\( \phi \): friction angle
\( X \): variable
\( \sigma \): scalar computed from \( \alpha \)
\( \psi \): state parameter
\( I \): identity matrix