Wave function for $GL(n, \mathbb{R})$ hyperbolic Sutherland model

S. Kharchev*, S. Khoroshkin**,

*Institute for Theoretical and Experimental Physics, B. Cheremushkinskaya, 25, Moscow 117259, Russia;
**Institute for Information Transmission Problems RAS (Kharkevich Institute), Bolshoy Karetny per. 19, Moscow, 127994, Russia;
***National Research University Higher School of Economics, Moscow, Russia.

Abstract

We obtain certain Mellin-Barnes integrals which present wave functions for $GL(n, \mathbb{R})$ hyperbolic Sutherland model with arbitrary positive coupling constant.

1 Introduction

In the paper [9] A. Gerasimov, S. Kharchev and D. Lebedev applied the famous technique of Gelfand–Zetlin basis [8] for the derivation of integral presentation for $GL(n, \mathbb{R})$ Whittaker functions, equivalently, for wave functions of the open Toda chain. They used formulas for the action of Lie algebra generators on Gelfand–Zetlin patterns to construct certain infinite-dimensional representation of the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ in the functional space of meromorphic functions, where the Lie algebra acts by difference operators with rational coefficients. In this representation the Whittaker vectors and nondegenerate pairing where found, so that the pairing of two dual Whittaker vectors gives Mellin-Barnes presentation for the Whittaker functions. The mentioned Whittaker vectors are given by products of Euler Gamma functions, and the pairing is the integration on the imaginary plane in $\mathbb{C}^{n(n-1)/2}$ with the Sklyanin measure also factorized into a product of Gamma functions.

Besides, in [9] the Mellin-Barnes presentation for zonal spherical functions of the symmetric space $GL(n, \mathbb{R})/O(n)$ was obtained, which are, in turn, the wave functions of the hyperbolic Sutherland model for a special value of the coupling constant. Precisely, in the above infinite dimensional representation of $\mathfrak{gl}(n)$ the spherical vector was found. It is given by another products of Euler Gamma functions, so that the corresponding matrix elements is an eigenvector of Sutherland operator

$$H_2 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2 \sum_{i<j} \frac{g(g-1)}{\sinh^2(x_i - x_j)}, \quad (1.1)$$

again presented by the integral of Mellin-Barnes type for the particular case $g = 1/2$. 
Thus in the framework of Representation Theory, the wave function for Sutherland model with the coupling constant \( g = 1/2 \) admits the integral presentation

\[
\Psi^{(1/2)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) = \prod_{i<j} \sinh^{1/2}|x_i - x_j| \times \int_{\mathbb{R}^{n(n-1)/2}} \prod_{i=1}^{n-1} \prod_{j=1}^{i+1} \prod_{k=1}^{i+j-1} \Gamma \left( \frac{\gamma_{ij} - \gamma_{i+1,k} + g}{2} \right) \Gamma \left( \frac{\gamma_{i+1,k} - \gamma_{ij} + g}{2} \right) e^{\sum_{i,j=1}^{n} (\gamma_{ij} - \gamma_{i-1,j})x_i} \prod_{i=1}^{n-1} d\gamma_{ij},
\]

where \( \lambda_i = \gamma_{n,i}, \lambda_i \in i\mathbb{R} \) and \( \gamma_{i,j} = 0 \) if \( i < j \).

On the other hand, in their research on generalized hypergeometric functions associated to root systems, G. Heckman and E. Opdam studied in particular the properties of the wave functions of Sutherland Hamiltonian for general coupling constant \( g \). G. Heckman showed in [7] the existence of the wave function \( \Psi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) \) such that the function

\[
\Phi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) = \prod_{i<j} \sinh^{-g}|x_i - x_j| \Psi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n)
\]

(now called Heckman-Opdam hypergeometric function) is real analytical and invariant with respect to the permutations of the coordinates \( x_k \). See also [12] for their further analytical properties. These results were obtained by studying the recurrence relations on the coefficients of Taylor expansions of the solutions to corresponding differential equation.

Our paper is devoted to precise construction of the Sutherland wave function. This results to analytical version of Heckman-Opdam hypergeometric series in the case of \( GL(n, \mathbb{R}) \). Set

\[
\Psi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) = \prod_{j<k} \sinh^g|x_j - x_k| \times \int_{\mathbb{R}^{n(n-1)/2}} \prod_{i=1}^{n-1} \prod_{j=1}^{i+1} \prod_{k=1}^{i+j-1} \Gamma \left( \frac{\gamma_{ij} - \gamma_{i+1,k} - g}{2} \right) \Gamma \left( \frac{\gamma_{i+1,k} - \gamma_{ij} + g}{2} \right) e^{\sum_{i,j=1}^{n} (\gamma_{ij} - \gamma_{i-1,j})x_i} \prod_{i=1}^{n-1} d\gamma_{ij},
\]

where \( \lambda_i = \gamma_{n,i}, \lambda_i \in i\mathbb{R}, g > 0, \) and \( \gamma_{i,j} = 0 \) if \( i < j \). Our main result is

**Theorem 1** The function \( \Psi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) \) is the eigenfunction of the Hamiltonian (1.1),

\[
H_2 \Psi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n) = -\lambda^2 \Psi^{(g)}_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n).
\]

where \( \lambda^2 = \lambda_1^2 + \ldots + \lambda_n^2 \).

Due to the strong convergency of the integral, see (3.4) the hypergeometric function (1.3) is analytical in \( (x_1, \ldots, x_n) \) in a small strip around real hyperplane (see (3.5) for more detail).

The crucial new point in the formula (1.4) is the denominator of the integral kernel, which can be regarded as a deformation of Sklyanin measure, or as a degeneration of the
weight function used in the scalar product of Macdonald polynomials [11]. It may reveals a new type of Barnes integrals associated to integrable systems related to DAHA [2] and quantum toroidal algebras [3].

Our proof of Theorem 1 is essentially simple, but uses unexpected arguments from the representation theory. Usually, the technique of matrix elements in the group theory works for special values of parameters, related to real, complex of quaternionic symmetric spaces. The formula (1.2) is a typical example of this approach, yielding the wave function only for \( g = 1/2 \). However, we use Laplace operator in Gelfand–Zetlin representation of Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \) constructed in [9] as a Hamiltonian in the space of rational functions and derive, using its properties, the second order differential equation on the integral in the right hand site of (1.4) for arbitrary positive coupling constant. The coupling constant \( g \) appears as a parameter \( i\hbar/2 \) in the representation, constructed in [9].

Note also that we use Gelfand–Zetlin formulas in a full range, for all generators of Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \).

The plan of the paper is as follows. In Section 2, following [9], we collect necessary information about the so called Gelfand–Zetlin representation. We are interested in the second order Laplace operator and in rational functions \( a^{kj}_i(\gamma) \) and \( b^{kj}_i(\gamma) \), which constitute constant terms of second order difference operators \( e_{i,j} e_{j,i} \). Here \( e_{k,l} \) are standard generators of the Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \). For this purpose we present (well known to specialists) precise expressions for the action of all generators \( e_{k,l} \) and formulate basic identities on the mentioned rational functions responsible for the validness of \( \mathfrak{gl}(n, \mathbb{R}) \) commutation relations. Here the shifts of the arguments by \( 2g \) are essential.

In Section 3 we derive the differential equation on the wave function given by integral (1.4). To do this, we imitate standard tricks with Laplace operator, where the rational coefficients \( a^{kj}_i(\gamma) \) and \( b^{kj}_i(\gamma) \) are used now for a number of proper deformations of the integration contour. Take note here on Lemma 3.1, which establishes the difference relations on these coefficients and the integration kernel with the shift by step 2. It indicates the use of some two–periodic properties of Gelfand–Zetlin coefficients.

Finally, in Section 4 we give \( n = 2 \) example, which can be regarded as Barnes integral presentation of the Legendre function.

After this work was completed, we found that M. Hallnäs and S. Ruijsenaars wrote a series of papers [4, 5, 6], where they suggest a general construction of eigenfunctions for Ruijsenaars systems based on a precise kernel function found in [13]. In particular, degeneration of their construction to hyperbolic Sutherland system [5] yields another presentation of the wave functions given by iterated beta integrals over space variables.

2 Gelfand–Zetlin representation

2.1 Laplace operator

In the paper [9] A. Gerasimov, S. Kharchev and D. Lebedev used famous Gelfand–Zetlin basis for the construction of infinite–dimensional representation of the Lie algebra \( \mathfrak{gl}(n) \), which we also name as Gelfand–Zetlin representation. More precisely, they interpreted Gelfand–Zetlin formulas [8] for the action of simple root generators in finite-dimensional
representations of \( \mathfrak{gl}(n, \mathbb{R}) \) as difference operators, presenting the action of \( \mathfrak{gl}(n, \mathbb{R}) \) in the space of meromorphic functions of \( n(n - 1)/2 \) variables.

Rewrite formulas \([9, (2.1)]\) replacing \( i\hbar \) factor by parameter \( 2g \):

\[
\begin{align*}
e_{i,i} &= \frac{1}{2g} \left( \sum_{k=1}^{i} \gamma_{i,k} - \sum_{k=1}^{i-1} \gamma_{i-1,k} \right), \\
e_{i,i+1} &= -\frac{1}{2g} \sum_{k=1}^{i} \left( \prod_{r=1}^{i-1} (\gamma_{i,k} - \gamma_{i+1,r} - g) \prod_{s \neq k} (\gamma_{i,k} - \gamma_{i,s}) \right) T_{\gamma_{i,k}}^{-g}, \\
e_{i+1,i} &= \frac{1}{2g} \sum_{k=1}^{i} \left( \prod_{r=1}^{i-1} (\gamma_{i,k} - \gamma_{i-1,r} + g) \prod_{s \neq k} (\gamma_{i,k} - \gamma_{i,s}) \right) T_{\gamma_{i,k}}^{g}.
\end{align*}
\] (2.1a, 2.1b, 2.1c)

Here \( T_{\gamma} f(\gamma) = f(\gamma + 2g) \). If \( i \) ranges from 1 to \( n \) and \( \gamma_{n,i} \) specialize to constants \( \gamma_{n,i} = \lambda_i \), \( i = 1, \ldots, n \), the relations (2.1) define a representation of \( U(\mathfrak{gl}(n, \mathbb{R})) \) in the space of meromorphic functions on \( \gamma_{i,j} \), \( 1 \leq j \leq i < n - 1 \), realized by difference operators with the step \( 2g \) and some rational coefficients. It was proved in [9] that the center \( Z(\mathfrak{gl}(n, \mathbb{R})) \) of the algebra \( U(\mathfrak{gl}(n, \mathbb{R})) \) acts by scalar operators.

We need here precise evaluation of the first two Laplace operators,

\[
L_1 = \sum_{i=1}^{n} e_{i,i}, \quad L_2 = \sum_{i,j=1}^{n} e_{i,j} e_{j,i}.
\]

**Lemma 2.1** Laplace operators \( L_1 \) and \( L_2 \) are realized in Gelfand-Zetlin representation by the following scalar operators

\[
L_1 = \frac{1}{2g} \left( \sum_{j} \lambda_j \right) \text{Id}, \quad L_2 = \frac{1}{4g^2} \left( \lambda^2 - 4g^2 \rho^2 \right) \text{Id},
\] (2.2)

where \( \rho \) is the Weyl vector with components \( \rho_i = \frac{1}{2}(n - 2i + 1) \), \( i = 1, \ldots, n \) such that \( \rho^2 = \frac{n(n^2 - 1)}{12} \).

**Proof.** The relations (2.2) can be extracted from the relation [9, (2.19)] where the generating function of central elements in Gelfand-Zetlin representation was computed in a form of Capelli determinant. \( \square \)

### 2.2 Root vectors and related identities

We need further formulas for the action of all root vectors \( e_{i,j} \) in Gelfand–Zetlin representation. Using the recurrent relations

\[
\begin{align*}
e_{i,i+p} &= [e_{i,i+p-1}, e_{i+p-1,i+p}], \\
e_{i+p,i} &= [e_{i+p-1,i}, e_{i+p,i+p-1}].
\end{align*}
\] (2.3a, 2.3b)
for any \( p = 1, \ldots, j - i, \ j > i \), one gets the following formulas:

\[
e_{i,j} = -\frac{1}{2g} \sum_{k_i, \ldots, k_{j-1}} \frac{1}{\prod_{m=i}^{m+1} (\gamma_{m,k_m} - \gamma_{m+1,r_{m+1}} - g)} \cdot \prod_{s_m \neq k_m} \gamma_{m,k_m} \cdot \prod_{m=i}^{j-1} T_{\gamma_{m,k_m}}^{-g}, \tag{2.4a}
\]

\[
e_{j,i} = \frac{1}{2g} \sum_{k_i, \ldots, k_{j-1}} \frac{1}{\prod_{m=i}^{m+1} (\gamma_{m,k_m} - \gamma_{m-1,r_{m-1}} + g)} \cdot \prod_{s_m \neq k_m} \gamma_{m,k_m} \cdot \prod_{m=i}^{j-1} T_{\gamma_{m,k_m}}^{-g}, \tag{2.4b}
\]

were sum is performed over integers \( k_i, \ldots, k_{j-1} \), such that \( 1 \leq k_m \leq m \) for all \( m = i, \ldots, j-1 \).

Our next goal is to use commutation relations in Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \) and the results of the previous subsection to establish certain identities on the coefficients of difference operators (2.4a) and (2.4b).

For any pair \((i, j)\), \( i < j \) denote by \( S_{i,j} \) the set of all \((j - i)\)-tuple of integers \( k = (k_i, \ldots, k_{j-1}) \), such that \( 1 \leq k_m \leq m \) for all \( m = i, \ldots, j-1 \), and for each \( k = (k_i, \ldots, k_{j-1}) \in S_{i,j} \) set

\[
c_{i,j}^k(\gamma) = \prod_{m=i}^{m+1} (\gamma_{m,k_m} - \gamma_{m+1,r_{m+1}} - g) \cdot \prod_{r_m \neq k_m} (\gamma_{m,k_m} - \gamma_{m,r_m}), \tag{2.5a}
\]

\[
c_{j,i}^k(\gamma) = \prod_{m=i}^{m+1} (\gamma_{m,k_m} - \gamma_{m-1,r_{m-1}} + g) \cdot \prod_{r_m \neq k_m} (\gamma_{m,k_m} - \gamma_{m,r_m}). \tag{2.5b}
\]

The rational functions \( c_{i,j}^k, c_{j,i}^k \) are coefficients of difference operators presenting \( e_{i,j} \) and \( e_{j,i} \) (2.4).

Next, introduce

\[
a_{i,j}^k(\gamma) = c_{i,j}^k(\gamma) T_{\gamma_{i,k_i}}^{-g} c_{j,i}^k(\gamma) T_{\gamma_{j,k_j}}^{-g}, \tag{2.6}
\]

where \( T_{\gamma_{i,k_i}}^g \) is the shift operator

\[
T_{\gamma_{i,k_i}}^g = T_{\gamma_{i,k_i}}^g \cdots T_{\gamma_{j-1,k_{j-1}}}^g.
\]
Due to (2.5) and (2.6) these function can be presented by the following products

\[ a^k_{i,j}(\gamma) = \prod_{m=i}^{m+1} \frac{\prod_{r_{m+1}=1, r_{m+1} \neq k_{m+1}}^{m+1} (\gamma_{m,k_m} - \gamma_{m+1,r_{m+1}} - g)}{\prod_{r_{m+1}=1, r_{m+1} \neq k_{m+1}}^{m+1} (\gamma_{m,k_m} - \gamma_{m+1,r_{m+1}} - g)} \]

(2.7a)

\[ b^k_{i,j}(\gamma) = \prod_{m=i}^{m+1} \frac{\prod_{r_{m+1}=1, r_{m+1} \neq k_{m+1}}^{m+1} (\gamma_{m,k_m} - \gamma_{m+1,r_{m+1}} + g)}{\prod_{r_{m+1}=1, r_{m+1} \neq k_{m+1}}^{m+1} (\gamma_{m,k_m} - \gamma_{m+1,r_{m+1}} + g)} \]

(2.7b)

Finally, denote by \( h_i(\gamma) \) the linear functions

\[ h_i(\gamma) = \sum_{j=1}^{i} \gamma_{i,j} - \sum_{j=1}^{i-1} \gamma_{i-1,j} \]

(2.8)

so that the operator \( e_{i,i} \) is the operator of multiplication by \( h_i(\gamma)/2g \).

**Lemma 2.2** For any pair \((i, j)\), \(1 \leq i < j \leq n\) one has the identity

\[ \sum_{k \in S_{i,j}} (b^k_{i,j}(\gamma) - a^k_{i,j}(\gamma)) = 2g(h_i(\gamma) - h_j(\gamma)). \]

(2.9)

**Proof.** This is direct corollary of the relation

\[ e_{i,j}e_{j,i} - e_{j,i}e_{i,j} = e_{i,i} - e_{j,j} \]

in GZ representation of \( U(\mathfrak{gl}(n, \mathbb{R})) \). Indeed, the LHS of (2.10) can be written as

\[ e_{i,j}e_{j,i} - e_{j,i}e_{i,j} = \frac{1}{4g^2} \sum_{k, k' \in S_{i,j}} (c^k_{j,i}(\gamma)T^g_k c^k_{i,j}(\gamma)(T^g_k)^{-1} - c^k_{i,j}(\gamma)(T^g_k)^{-1} c^k_{j,i}(\gamma)T^g_k, \]

\[ = \frac{1}{4g^2} \sum_{k \in S_{i,j}} (c^k_{j,i}(\gamma)T^g_k c^k_{i,j}(\gamma)(T^g_k)^{-1} - c^k_{i,j}(\gamma)(T^g_k)^{-1} c^k_{j,i}(\gamma)T^g_k) \]

(2.11)

and then (2.9) follows from (2.1a). The key point here is the fact that in (2.11) the sum of the terms with \( k \neq k' \in S_{i,j} \) vanishes and thus the final result does not contain the shift operators. \( \square \)

The next lemma is a consequence of properties of Laplace operators.

**Lemma 2.3** We have the following identity of rational functions

\[ \sum_{i<j} \sum_{k \in S_{i,j}} (a^k_{i,j}(\gamma) + b^k_{i,j}(\gamma)) = \sum_{i=1}^{n} h_i^2(\gamma) - \lambda^2 + 4g^2 \rho^2. \]

(2.12)

**Proof.** Analogous direct calculations based on the formulas (2.4) and (2.8). \( \square \)
3 Wave function

For a set of variables $\gamma_{i,j}$, $1 \leq j \leq i \leq n$ and real positive $g$ define a kernel

$$K^{(g)}(\gamma) = \prod_{i=1}^{n-1} \frac{\prod_{j=1}^{i+1} \Gamma \left( \frac{\gamma_{i,j} - \gamma_{i+1,k} + g}{2} \right) \Gamma \left( \frac{\gamma_{i+1,k} - \gamma_{i,j} + g}{2} \right)}{\prod_{1 \leq r \neq s \leq i} \Gamma \left( \frac{\gamma_{i,r} - \gamma_{i,s} + 2g}{2} \right) \Gamma \left( \frac{\gamma_{i,r} - \gamma_{i,s} + 2g}{2} \right) \Gamma \left( \gamma_{i,r} - \gamma_{i,s} + 2g \right)}. \quad (3.1)$$

Consider the wave function

$$\Phi_{\lambda}^{(g)}(x) = \int_C K^{(g)}(\gamma)e^{\sum_{i=1}^{n} h_i(\gamma) x_i} d\gamma \quad (3.2)$$

assuming that

$$\gamma_{n,i} := \lambda_i \in i\mathbb{R}, \quad i = 1, \ldots, n \quad (3.3)$$

are fixed parameters and $h_i(\gamma)$ are defined in (2.8). The integration contour $C$ is an imaginary plane $i\mathbb{R}^{\frac{n(n-1)}{2}}$,

$$C : \text{Re}\gamma_{i,j} = 0, \quad 1 \leq j \leq i \leq n - 1$$

The integral (3.2) absolutely converges. The proof is identical to that of [9, 10]. For instance, we can use an elegant estimate [10, (30)] by N. Iorgov and V. Shadura, which states that for fixed $\gamma_{n,i} = \lambda_i \in i\mathbb{R}, \quad i = 1, \ldots, n$,

$$|K^{(g)}(\gamma)| < P(\gamma) \exp \left(-\frac{\pi}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} |\gamma_{i,j}| \right) \quad (3.4)$$

where $P(\gamma)$ is locally integrable function of not more than polynomial growth.

Moreover the estimate (3.4) shows that the function (3.2) is analytical on $x$ in a strip

$$|\text{Im}x_k| < \frac{\pi}{n}, \quad k = 1, \ldots, n. \quad (3.5)$$

Fix a pair $(i, j)$, $1 \leq i < j \leq n$ and a tuple $\mathbf{k} = (k_i, \ldots, k_{j-1}) \in S_{i,j}$. The following statement establishes a set of difference equations on the kernel $K^{(g)}(\gamma)$ with coefficients $a_{i,j}^{\mathbf{k}}(\gamma)$ and $b_{i,j}^{\mathbf{k}}(\gamma)$.

**Lemma 3.1** We have the relation

$$T_{\mathbf{k}} \left(a_{i,j}^{\mathbf{k}}(\gamma) K^{(g)}(\gamma)\right) = b_{i,j}^{\mathbf{k}}(\gamma) K^{(g)}(\gamma). \quad (3.6)$$

Here $T_{\mathbf{k}}$ is the shift operator with the step 2:

$$T_{\mathbf{k}} = T_{\gamma_{i,k_i}} \cdots T_{\gamma_{j-1,k_{j-1}}}, \quad \text{where} \quad T_{\gamma}f(\gamma) = f(\gamma + 2). \quad (3.7)$$

**Proof.** This is a direct consequence of the fundamental functional relation on the Euler $\Gamma$ function, $\Gamma(x + 1) = x\Gamma(x)$. Indeed, in the product $K^{(g)}(\gamma)a_{i,j}^{\mathbf{k}}(\gamma)$ we may incorporate all the factors

$$\frac{1}{2} (\gamma_{n,r} - \gamma_{m,k_m} + lg), \quad l = 0, 1, 2, \quad r \neq k_n$$
from \( a_{i,j}^k(\gamma) \) to the shifts of the corresponding \( \Gamma \) functions,

\[
\Gamma \left( \frac{\gamma_{m,s} - \gamma_{m,km} + lg}{2} \right) \mapsto \Gamma \left( \frac{\gamma_{m,s} - \gamma_{m,km} + lg}{2} + 1 \right)
\]  

(3.8)

The application of the shift \( T_k \) returns \( \Gamma \) functions (3.8) back to the initial position but makes a shift in other \( \Gamma \) functions,

\[
\Gamma \left( \frac{\gamma_{m,km} - \gamma_{m,r} + lg}{2} \right) \mapsto \Gamma \left( \frac{\gamma_{m,km} - \gamma_{m,r} + lg}{2} + 1 \right)
\]

which in its turn can be achieved by the multiplication of the factor

\[
\frac{1}{2} (\gamma_{m,km} - \gamma_{m,r} + lg) \quad l = 0, 1, 2, \quad r \neq k_n
\]

of \( b_{i,j}^k(\gamma) \). The number of 1/2 factors is the same in both sides, so that we can cancel them.

Since the relation (3.6) plays the crucial role in our arguments, we repeat the proof by means of exact calculations. Using (2.7a) one has, certainly

\[
T_k (a_{i,j}^k(\gamma)) = \prod_{j=1}^{m+1} \prod_{r_{m+1}=1, r_{m+1} \neq k_{m+1}}^{m} \left( \gamma_{m,km} - \gamma_{m,r_{m+1}} + 2 - g \right)
\]

\[
\prod_{m=i} \prod_{s \neq k_m} \left( \gamma_{m,km} - \gamma_{m,s} + 2 \right) \left( \gamma_{m,km} - \gamma_{m,s} + 2 - 2g \right)
\]

(3.9)

One the other hand, the explicit calculation results to

\[
T_k (K'(\gamma)) = K'(\gamma).
\]

\[
\prod_{m=i} \prod_{r_{m+1}=1, r_{m+1} \neq k_{m+1}}^{m} \left( \gamma_{m,km} - \gamma_{m,r_{m+1}} + g \right)
\]

\[
\prod_{m=i} \prod_{s \neq k_m} \left( \gamma_{m,km} - \gamma_{m,s} + 2 - g \right)
\]

(3.10)

One can see that the product of right hand sides of (3.9) and (3.10) is precisely \( b_{i,j}^k(\gamma) \cdot K'(\gamma)(\gamma) \) in accordance with (2.7b).

\[\blacksquare\]

Remark. Note that the rational functions \( a_{i,j}^k(\gamma) \) and \( b_{i,j}^k(\gamma) \) themselves satisfy the difference relation with the step 2g:

\[
T^g_k(a_{i,j}^k(\gamma)) = b_{i,j}^k(\gamma).
\]  

(3.11)

Denote by \( A_{i,j}^k(x) \) and \( B_{i,j}^k(x) \) the following functions of variables \( x_1, \ldots, x_N \):

\[
A_{i,j}^k(x) = \int_C a_{i,j}^k(\gamma) K'(\gamma)(\gamma)e^{\sum_{i=1}^n h_r(\gamma)x_r} d\gamma,
\]  

(3.12a)

\[
B_{i,j}^k(x) = \int_C b_{i,j}^k(\gamma) K'(\gamma)(\gamma)e^{\sum_{i=1}^n h_r(\gamma)x_r} d\gamma.
\]  

(3.12b)
Under our assumptions on the integration contour we have the following direct corollary of Lemma 3.1:

**Proposition 3.1** For any pair \((i, j)\), \(1 \leq i < j \leq n\) and a tuple \(k = (k_i, \ldots, k_{j-1}) \in S_{i,j}\) the following relations hold:

\[
A^k_{i,j}(x) = e^{2(x_i-x_j)} B^k_{i,j}(x) \tag{3.13}
\]

**Proof.** In the integral (3.12a) perform the change of variables \(\gamma_{a,b} \to T^k \gamma_{a,b}, \ k \in S_{i,j}\). Then, by Lemma 3.1 and the relation

\[
T^k h_r(\gamma) = h_r(\gamma) + 2(\delta_{i,r} - \delta_{j,r}),
\]

we get the equality

\[
A^k_{i,j}(x) = e^{2(x_i-x_j)} \int_{T^k C} h_{i,j}(\gamma) K^{(g)}(\gamma) e^{\sum_{r=1}^n h_r(\gamma) x_r} d\gamma.
\]

However, we can move the contour \(T^k C\) back to initial position of imaginary plane since the nominator of \(b^k_{i,j}(\gamma)\) kills all the poles of \(K^{(g)}(\gamma)\) which could prevent this deformation of the contour, while the poles of \(b^k_{i,j}(\gamma)\) are located in zeroes of \(K^{(g)}(\gamma)\). Thus one arrives to (3.13).

□

Now we are ready to derive the differential equation on \(\Phi^{(g)}(x)\), gathering the statements of Lemmas 2.1, 2.2, 2.3, 3.1 and Proposition 3.1.

Set

\[
A_{i,j}(x) = \sum_{k \in S_{i,j}} A^k_{i,j}(x), \tag{3.14a}
\]

\[
B_{i,j}(x) = \sum_{k \in S_{i,j}} B^k_{i,j}(x). \tag{3.14b}
\]

Since

\[
\frac{\partial}{\partial x_i} \Phi^{(g)}(x) = \int_{C} h_{i}(\gamma) K^{(g)}(\gamma) e^{\sum_{r=1}^n h_r(\gamma) x_r} d\gamma, \tag{3.15}
\]

Lemma 2.2 implies the equality

\[
B_{i,j}(x) - A_{i,j}(x) = 2g \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Phi^{(g)}(x) \tag{3.16}
\]

This relation together with (3.13) gives the system of two linear equations on \(A_{i,j}(x)\) and \(B_{i,j}(x)\) which solution is

\[
A_{i,j}(x) = 2g \frac{e^{2(x_i-x_j)}}{1 - e^{2(x_i-x_j)}} (\partial_{x_i} - \partial_{x_j}) \Phi^{(g)}(x), \tag{3.17a}
\]

\[
B_{i,j}(x) = 2g \frac{1}{1 - e^{2(x_i-x_j)}} (\partial_{x_i} - \partial_{x_j}) \Phi^{(g)}(x), \tag{3.17b}
\]
Summing up, we get the relation
\[ A_{i,j}(x) + B_{i,j}(x) = -2g \cosh(x_i - x_j)(\partial_{x_i} - \partial_{x_j})\Phi_{\lambda}(x). \] (3.18)

On the other hand, Lemma 2.3 together with (3.15) results in the relation
\[ \sum_{i<j}(A_{i,j}(x) + B_{i,j}(x)) = \left( \sum_{i=1}^{n} \partial_{x_i}^2 - \lambda^2 + 4g^2\rho^2 \right)\Phi_{\lambda}(x). \] (3.19)

Hence, comparison of (3.19) with the sum of (3.18) results in the following Proposition 3.2

**Proposition 3.2** The function \( \Phi_{\lambda}(x) \) satisfies the equation
\[ \left( \sum_{i=1}^{n} \partial_{x_i}^2 + 2g \sum_{i<j} \cosh(x_i - x_j)(\partial_{x_i} - \partial_{x_j}) \right)\Phi_{\lambda}(x) = \left( \lambda^2 - 4g^2\rho^2 \right)\Phi_{\lambda}(x). \] (3.20)

It is well known that differential equation (3.20) is related to the original Sutherland equation by means of the conjugation by the function \( \prod_{p<q}|\sinh(x_p - x_q)|^g \). Set
\[ \Psi_{\lambda}(x) = \prod_{p<q}|\sinh(x_p - x_q)|^g\Phi_{\lambda}(x) = \prod_{p<q}|\sinh(x_p - x_q)|^g \int_{C} K(\gamma)e^{\sum_{k=1}^{N} h_k x_k d\gamma} \] (3.21)

Proposition 3.2 implies

**Theorem 1** The function \( \Psi_{\lambda}(x) \), is the wave function of the Sutherland system,
\[ H_1 \Psi_{\lambda}(x) = \left( \sum_{i} \lambda_i \right)\Psi_{\lambda}(x), \]
\[ H_2 \Psi_{\lambda}(x) = -\left( \sum_{i} \lambda_i^2 \right)\Psi_{\lambda}(x), \] (3.22)

where
\[ H_1 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}, \quad H_2 = -\sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{i<j} \frac{g(g-1)}{\sinh^2(x_i - x_j)}. \]

### 4 Example

Consider the case \( n = 2 \) where \( h_1(\gamma) = \gamma, h_2(\gamma) = \lambda_1 + \lambda_2 - \gamma \) and two basis rational functions (2.7) are
\[ a(\gamma) = (\gamma - \lambda_1 - g)(\gamma - \lambda_2 - g), \quad b(\gamma) = (\gamma - \lambda_1 + g)(\gamma - \lambda_2 + g). \] (4.1)

According to Lemmas 2.2 and 2.3, they satisfy the relations
\[ a(\gamma) - b(\gamma) = h_1(\gamma) - h_2(\gamma) = 2g(2\gamma - \lambda_1 - \lambda_2), \]
\[ a(\gamma) + b(\gamma) = h_1^2(\gamma) + h_2^2(\gamma) - \lambda_1^2 - \lambda_2^2 + 2g^2. \] (4.2)
Consider the wave function:

\[
\Phi_{\lambda_1, \lambda_2}^{(g)}(x_1, x_2) = \int \exp\left\{ (\lambda_1 + \lambda_2 - \gamma) x_2 + \gamma x \right\} \times \Gamma\left(\frac{\gamma - \lambda_1 + g}{2}\right) \Gamma\left(\frac{\lambda_1 - \gamma + g}{2}\right) \Gamma\left(\frac{\lambda_2 - \gamma + g}{2}\right) d\gamma,
\]

where \(\lambda_1, \lambda_2 \in i\mathbb{R}\) and \(g > 0\). Since

\[
T_\gamma a(\gamma) = (\gamma - \lambda_1 + 2 - g)(\gamma - \lambda_2 + 2 - g),
\]

\[
T_\gamma K^{(g)}(\gamma) = K^{(g)}(\gamma) \frac{(\gamma - \lambda_1 + g)(\gamma - \lambda_2 + g)}{(\gamma - \lambda_1 + 2 - g)(\gamma - \lambda_2 + 2 - g)},
\]

then

\[
T_\gamma (a(\gamma) K^{(g)}(\gamma)) = b(\gamma) K^{(g)}(\gamma)
\]

in accordance with Lemma 3.1.

One can calculate integral (4.3) in explicit terms. Performing the shift \(\gamma = s + \lambda_1\) of the integration variable, we rewrite the \(\mathfrak{sl}(2, \mathbb{R})\)-part

\[
\Phi_{\lambda_1 - \lambda_2}^{(g)}(x_1 - x_2) = \exp\left\{ -\frac{1}{2}(\lambda_1 + \lambda_2)(x_1 + x_2) \right\} \Phi_{\lambda_1, \lambda_2}^{(g)}(x_1, x_2)
\]

of the wave function in a form

\[
\phi_{\lambda}^{(g)}(x) = e^{\frac{\lambda x}{2}} \int_{\mathbb{R}} \Gamma\left(\frac{g + s}{2}\right) \Gamma\left(\frac{g - s}{2}\right) \Gamma\left(\frac{g + \lambda + s}{2}\right) \Gamma\left(\frac{g - \lambda - s}{2}\right) \exp\{sx\} ds,
\]

with \(\lambda = \lambda_1 - \lambda_2, x = x_1 - x_2\). The poles of the kernel are:

\[
\left\{ \begin{array}{l}
s_k = -g - 2k, \\
s_k = -\lambda - g - 2k, \\
s_k = -\lambda + g + 2k,
\end{array} \right. 
\]

where \(k \in \mathbb{Z}_{\geq 0}\). Assuming that \(x > 0\) and enclosing the contour of integration in the left half plane, one has

\[
\frac{1}{4\pi i} \phi_{\lambda}^{(g)}(x) = e^{(\lambda/2-g)x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(k + g) \Gamma\left(k + g - \frac{\lambda}{2}\right) \Gamma\left(\frac{g - \lambda}{2}\right) e^{-2kx} +
\]

\[
+ e^{(-\lambda/2-g)x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(k + g) \Gamma\left(k + g + \frac{\lambda}{2}\right) \Gamma\left(\frac{g + \lambda}{2}\right) e^{-2kx}.
\]

This can be written in terms of hypergeometric functions:

\[
b^{-\frac{1}{2}} \phi_{\lambda}^{(g)}(x) = e^{(\lambda/2-g)x} \frac{\Gamma\left(\frac{g - \lambda}{2}\right)}{\Gamma\left(\frac{1 - \lambda}{2}\right)} F\left(g, g - \frac{\lambda}{2}, 1 - \frac{\lambda}{2}; e^{-2x}\right) -
\]

\[
e^{-\frac{1}{2}} e^{(-\lambda/2-g)x} \frac{\Gamma\left(\frac{g + \lambda}{2}\right)}{\Gamma\left(1 + \frac{\lambda}{2}\right)} F\left(g, g + \frac{\lambda}{2}, 1 + \frac{\lambda}{2}; e^{-2x}\right).
\]
where \( \beta = \frac{4\pi^2 \Gamma(g)}{\sin \left( \frac{2\pi g}{2} \right)} \). These particular hypergeometric functions are related to Legendre functions of the second kind (see [1, 3.2(45)]):

\[
e^{-\pi i \mu} Q^\mu_v (\cosh x) \frac{e^{-\nu(x+1)}}{\sqrt{\pi} 2^\mu \sinh^\nu x} = e^{-\nu(x+1)} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} F(\nu + \frac{1}{2}, \nu + \mu + 1, \nu + \frac{3}{2}; e^{-2x}), \quad (4.11)
\]

and therefore

\[
\phi^{(g)}_\lambda(x) = \beta e^{\pi i \frac{1}{2} - g} \sinh^{\frac{1}{2} - g} x \left( Q^{g - \frac{1}{2} - \frac{1}{2}}_{\frac{1}{2} - \frac{1}{2}} (\cosh x) - Q^{g - \frac{1}{2} - \frac{1}{2}}_{\frac{1}{2} - \frac{1}{2}} (\cosh x) \right). \quad (4.12)
\]

Using the formula [1, 3.3.1 (9)] which connects Legendre functions of the first and the second kinds:

\[
e^{-\pi i \mu} \left\{ Q^\mu_{\nu+1}(z) - Q^\mu_{\nu}(z) \right\} = \Gamma(\mu + \nu + 1) \Gamma(\mu - \nu) P^\mu_{\nu}(z). \quad (4.13)
\]

we arrive to the following expression

\[
\phi^{(g)}_\lambda(x) = 4i \pi^2 2^{\frac{3}{2} - g} \Gamma(g) \Gamma(g - \frac{1}{2}) \Gamma(g + \frac{1}{2}) \cdot \sinh^{\frac{1}{2} - g} x \frac{1}{2} \phi^{(g)}_\lambda(x). \quad (4.14)
\]

The function (4.14) satisfies the equation

\[
(\partial^2_x + 2g \coth x \partial_x) \phi^{(g)}_\lambda(x) = \left( \frac{1}{4} \lambda^2 - g^2 \right) \phi^{(g)}_\lambda(x). \quad (4.15)
\]

### Acknowledgements

The work of the first author was supported in part by RFBR and NSFB according to the research project number 19-51-18006. The second author appreciates the support of Russian Science Foundation, projects No. 20-41-09009, used for the proof of the statements of Sections 1 and 3. Besides, section 2 was prepared within the framework of the HSE University Basic Research Program.

### References

[1] Bateman manuscript project, ed A.Erdélyi, *Higher transcendental functions*, vol 1. McGraw-Hill, 1953.

[2] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge (2005).

[3] B. Feigin, M. Jimbo, E. Mukhin, *Integrals of motion from quantum toroidal algebras* Journal of Physics A Mathematical and Theoretical 50 (46) (2017) 464001.

[4] M. Hallnäs, S. Ruijsenaars, *Joint Eigenfunctions for the Relativistic Calogero–Moser Hamiltonians of Hyperbolic Type: I. First Steps*, International Mathematics Research Notices 2014 (16) (2014) 4400-4456.
[5] M. Hallnäs, S. Ruijsenaars, *A recursive construction of joint eigenfunctions for the hyperbolic nonrelativistic Calogero-Moser Hamiltonians*, International Mathematics Research Notices 2015 (20) (2015) 10278-10313.

[6] M. Hallnäs, S. Ruijsenaars, *Joint Eigenfunctions for the Relativistic Calogero–Moser Hamiltonians of Hyperbolic Type II. The Two-and Three-Variable Cases*, International Mathematics Research Notices 2018 (14) (2018) 4404-4449.

[7] G.J. Heckman, *Root systems and hypergeometric functions. II*, Compositio Mathematica 64 (3) (1987) 353-373.

[8] I.M. Gelfand, M.L. Tsetlin, *Finite-dimensional representations of the group of unimodular matrices*, Doklady Akademii Nauk SSSR, 71 (1950) 825–828.

[9] A. Gerasimov, S. Kharchev, D. Lebedev, *Representation theory and quantum inverse scattering method: the open Toda chain and the hyperbolic Sutherland model*, International Mathematics Research Notices, 2004. 17 (2004) 823-854.

[10] N. Iorgov, V. Shadura, *Wave functions of the Toda chain with boundary interaction* Theoretical and mathematical physics 142 (2) (2005) 289-305.

[11] I.G. Macdonald, *Symmetric functions and Hall polynomials* Second edition, Oxford University Press (1998).

[12] Opdam, E. M. *Root systems and hypergeometric functions IV*, Compositio Mathematica 67 (2) (1988) 191-209.

[13] S. Ruijsenaars, *Zero-eigenvalue eigenfunctions for differences of elliptic relativistic Calogero-Moser Hamiltonians*, Theoretical and Mathematical Physics 146 (1) (2006) 25-33.

[14] B. Sutherland, *Exact results for a quantum many-body problem in one dimension. I & II*, Physical Review A4(5) (1971) 2019-2021 (1971), & A5 (3) (1972) 1372-1376.