CONGRUENCES OF 5-SECANT CONICS AND THE RATIONALITY OF SOME ADMISSIBLE CUBIC FOURFOLDS

FRANCESCO RUSSO* AND GIOVANNI STAGLIANÒ

Abstract. The works of Hassett and Kuznetsov identify countably many divisors $C_d$ in the open subset of $\mathbb{P}^5 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3)))$ parametrizing all cubic 4-folds and conjecture that the cubics corresponding to these divisors are precisely the rational ones. Rationality has been known classically for the first family $C_{14}$. We use congruences of 5-secant conics to prove rationality for the first three of the families $C_d$, corresponding to $d = 14, 26, 38$ in Hassett’s notation.

Introduction

One of the most challenging open problems in classical and modern algebraic geometry is the rationality of smooth cubic hypersurfaces $X \subset \mathbb{P}^5$. The very general cubic fourfold is expected to be irrational although no such example is known. Recent work of Hassett via Hodge Theory in [Has99, Has00] (see also [Has16]) and of Kuznetsov via derived categories in [Kuz10, Kuz16] lead to the definition of infinitely many irreducible divisors $C_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds corresponding to the admissible values $d \in \mathbb{N}$ and to the conjecture that the cubics belonging to the admissible $C_d$ should be precisely the rational ones (Kuznetsov Conjecture), see [AT14, Kuz10, Kuz16, Has16]. The admissible values are the even integers $d > 6$ not divisible by 4, by 9 and nor by any odd prime of the form $2 + 3m$ (see for example [Has99, Has00]), so that the first admissible values are $d = 14, 26, 38$. Fano showed the rationality of a general $[X] \in C_{14}$, see [Fan43, BRS15, Mor40] and Section 2 for a different proof. Our main result is the following:

Theorem. Every cubic fourfold in the irreducible divisors $C_{26}$ and $C_{38}$ is rational.

The Hodge theoretic definition of the divisors $C_d$ reduces to the existence of a rank two lattice $\langle h^2, S \rangle \subseteq H^{2,2}(X, \mathbb{Z})$ of discriminant $d$, where $h$ is the class of a hyperplane section of $X$ and $S$ is the class of an algebraic 2-cycle on $X$. The elements of $C_d$ are called special cubic fourfolds and $C_d \neq \emptyset$ if and only if $d > 6$ and $d \equiv 0, 2 \pmod{6}$, see [Has99, Has00]. The geometrical definition of $C_d$ for small values of $d$ is by means of a particular surface $S_d \subset X$, which is obviously not unique. For example, $C_8$ is usually described as the locus of smooth cubic fourfolds containing a plane, while $C_{14}$ can be described either as the closure of the locus of cubic fourfolds containing a smooth quartic rational normal scroll or as the closure of the locus of those containing a smooth quintic del Pezzo surface, the so called Pfaffian locus. In [Nue15] there are similar descriptions for the values $12 \leq d \leq 44$, $d \neq 42$, while for $d = 42$ one can consult [Lai16].

*Partially supported by the PRIN Geometria delle varietà algebriche and by the FIR2014 Aspetti geometrici e algebrici della Weak e Strong Lefschetz Property of the University of Catania; the author is a member of the G.N.S.A.G.A. of INDAM.
Every \( [X] \in C_{14} \) has been proved to be rational by showing that it contains either a smooth surface with one apparent double point or one of its small degenerations, see [BRS15] and also Theorem 2 here for a different proof. The extension of this geometrical approach to rationality for other (admissible) values \( d \) appeared to be impossible due to the paucity of surfaces with one apparent double point, which essentially can be used only for \( d = 14 \).

Our discovery is the existence of irreducible surfaces \( S_d \subset \mathbb{P}^5 \) contained in a general element of \( C_d \), for \( d = 14, 26 \) or \( 38 \), admitting a four-dimensional irreducible family of 5-secant conics such that through a general point of \( \mathbb{P}^5 \) there passes a unique conic of the family. We dubbed these families \textit{congruences of 5-secant conics to} \( S_d \). Once established the existence of such a congruence we deduce that a general cubic in \( |H^0(I_{S_d}(3))| \) is a rational section of the universal family of the congruence of 5-secant conics. In particular, such a general cubic is birational to the parameter space of the congruence. Then the rationality of the parameter space of the congruence (or the existence of a particular, sufficiently general, singular rational cubic hypersurface through \( S_d \)) shows that every smooth cubic in \( |H^0(I_{S_d}(3))| \) is rational (and hence that a general \( X \in C_d \) is rational); see Theorem 1 for a precise and general formulation of this principle.

For \( d = 26 \), which was the first open case, we take \( S_{26} \subset \mathbb{P}^5 \) to be the surface with one node obtained as the projection of a smooth del Pezzo surface \( S \subset \mathbb{P}^7 \) of degree seven from a line intersecting the secant variety of \( S \) transversally. For \( d = 14 \), we use smooth projections of rational octic surfaces in \( \mathbb{P}^6 \) of sectional genus 3, which are the birational images of \( \mathbb{P}^2 \) via the linear system of plane quartic curves with eight general base points. For \( d = 38 \), we consider smooth surfaces of degree 10 in \( \mathbb{P}^5 \) of sectional genus 6, which are the birational images of \( \mathbb{P}^2 \) via the linear system of plane curves of degree 10 having ten general base points of multiplicity three and which were also used by Nuer in [Nue15] to describe some birational properties of the divisor \( C_{38} \).

As far as we know, the existence of congruences of 5-secant conics to surfaces in \( \mathbb{P}^5 \) has never been considered before, even in the classical literature. The surfaces \( S_d \subset \mathbb{P}^5 \) with \( d = 14, 26, 38 \) we choose can be analyzed better nowadays via computational systems revealing remarkable algebraic properties of their defining equations.

The classification of irreducible surfaces in \( \mathbb{P}^5 \) admitting a congruence of 5-secant conics is an important and pressing open question we shall consider elsewhere. An obvious necessary condition is that the surface is not contained in a quadric hypersurface. This restriction plays a key role for the existence of these surfaces and it explains some geometrical aspects of the paucity of examples.

Our fundamental tool for determining the existence of congruences of 5-secant conics has been the Hilbert scheme of lines passing through a general point of a projective variety and contained in the variety; see also [Rus16] for other applications. The study of these lines for suitable birational images of \( \mathbb{P}^5 \) via the cubic equations defining the right \( S_d \subset X \) revealed the existence of the congruences of 5-secant conics to \( S_d \), which a posteriori have also natural geometrical constructions. To analyze the geometry of the images of \( \mathbb{P}^5 \) via the linear system of cubics defining \( S_d \subset \mathbb{P}^5 \) we used Macaulay2 [GS17]. The necessary ancillary files containing the scripts to verify some geometrical properties claimed throughout the paper are posted on arXiv; see also Section 5 and also [Sch13] for the philosophy behind these verifications for random surfaces or hypersurfaces (although we work over \( \mathbb{Q} \) and not over finite fields).
So far \(C_{14}, C_{26}\) and \(C_{38}\) are the only loci of codimension one in \(C\) whose elements are known to be rational. There exist infinitely many other irreducible loci of codimension at least two in \(C\) (some of them contained also in non admissible \(C_{d}\)'s), whose general element is rational, see [Has99] and [AHTVA16]. The next admissible value \(d = 42\) seems to deserve some special attention and it might be a watershed in order to confirm the validity of the Kuznetsov Conjecture also for the countably many admissible values \(d > 38\).

**Acknowledgements.** We wish to thank Ciro Ciliberto for useful and stimulating discussions and the referee’s for some suggestions about the presentation of the results.

1. **Rationality via congruences of \(((r e - 1))\)-secant curves of degree \(e\)**

Let \(S \subset \mathbb{P}^5\) be an irreducible surface and let \(\mathcal{H}\) be an irreducible proper family of (rational or of fixed arithmetic genus) curves of degree \(e\) in \(\mathbb{P}^5\) whose general element is irreducible. Let

\[\pi : \mathcal{D} \to \mathcal{H}\]

be the universal family over \(\mathcal{H}\) and let

\[\psi : \mathcal{D} \to \mathbb{P}^5\]

be the tautological morphism. Suppose moreover that \(\psi\) is birational and that a general member \([C] \in \mathcal{H}\) is \(((r e - 1))\)-secant to \(S\), that is \(C \cap S\) is a length \(r e - 1\) scheme, \(r \in \mathbb{N}\). We shall call such a family a **congruence of \(((r e - 1))\)-secant curves of degree \(e\) to \(S\)**. Let us remark that necessarily \(\dim(\mathcal{H}) = 4\).

The classification of irreducible surfaces in \(\mathbb{P}^5\) admitting a congruence of \(((r e - 1))\)-secant rational curves of degree \(e \geq 2\) (and with \(r \geq 3\)) is an open problem, never considered before. In the sequel we shall construct some surfaces admitting a congruence of 5-secant conics. We also know some irreducible surfaces in \(\mathbb{P}^5\) admitting a congruence of 8-secant twisted cubics and others admitting a congruence of 14-secant quintic rational normal curves.

Let \(S \subset \mathbb{P}^5\) be a surface admitting a congruence of \(((r e - 1))\)-secant curves of degree \(e\) parametrized by \(\mathcal{H}\) and let notation be as above. The birationality of \(\psi : \mathcal{D} \to \mathbb{P}^5\) and Zariski Main Theorem yield that the locus of points \(q \in \mathbb{P}^5\) through which there pass at least two \(((r e - 1))\)-secant curves of degree \(e\) in the family \(\mathcal{H}\) has codimension at least two in \(\mathbb{P}^5\). Let \(X \in |H^0(\mathcal{I}_S(r))|\) be fixed and irreducible and let \(p \in X\) be a general point. Then through \(p\) there passes a unique curve \(C_p\) of the family \(\mathcal{H}\). An irreducible hypersurface \(X \in |H^0(\mathcal{I}_S(r))|\) is said to be **transversal to the congruence** \(\mathcal{H}\) if the unique curve of the congruence passing through a general point \(p \in X\) is not contained in \(X\). The next result will be crucial for our analysis.

**Theorem 1.** Let \(S \subset \mathbb{P}^5\) be a surface admitting a congruence of \(((r e - 1))\)-secant curves of degree \(e\) parametrized by \(\mathcal{H}\). If \(X \in |H^0(\mathcal{I}_S(r))|\) is an irreducible hypersurface transversal to \(\mathcal{H}\), then \(X\) is birational to \(\mathcal{H}\).

If the map \(\Phi = \Phi_{|H^0(\mathcal{I}_S(r))|} : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(\mathcal{I}_S(r)))\) is birational onto its image, then a general hypersurface \(X \in |H^0(\mathcal{I}_S(r))|\) is said to be **transversal to the congruence** \(\mathcal{H}\) if the unique curve of the congruence passing through a general point \(p \in X\) is not contained in \(X\). The next result will be crucial for our analysis.

Moreover, under the previous hypothesis on \(\Phi\), if a general element in \(|H^0(\mathcal{I}_S(r))|\) is smooth, then every \(X \in |H^0(\mathcal{I}_S(r))|\) with at worst rational singularities is birational to \(\mathcal{H}\).
Proof. Let notation be as above and suppose that the irreducible hypersurface $X \in |H^0(I_S(r))|$ is transversal to $\mathcal{H}$. Then, for $p \in X$ general, the curve $C_p$ is not contained in $X$ and it cuts $X$ outside $S$ only in the point $p$ by Bézout Theorem. Thus $X$ naturally becomes a rational section of the family $\pi : \mathcal{D} \rightarrow \mathcal{H}$. Indeed, we get a rational map $\eta : X \dashrightarrow \mathcal{D} \overset{\pi}{\rightarrow} \mathcal{H}$ by associating to a general point $p \in X$ the same point $p \in C_p$ on the unique curve $C_p = \mathcal{D}_{\pi(p)}$ passing through $p$ and belonging to the family $\mathcal{H}$. The rational map $\varphi : \mathcal{H} \dashrightarrow X$ is defined by taking $[C] \in \mathcal{H}$ and setting $\varphi([C]) = C \cap (X \setminus S)$ to be the unique point in $X \cap C$ outside $S$. For $p \in X$ general the map $\varphi$ is defined at $\eta(p)$ by the transversality hypothesis, $\varphi(\eta(p)) = p$ and $\eta$ is birational, as claimed.

Suppose that the map $\Phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(I_S(r)))$ is birational onto its image. Then, for $p \in \mathbb{P}^5$ general, the curve $\Phi(C_p)$ is a line $L_p$ through $\Phi(p)$ and contained in $\Phi(\mathbb{P}^5)$. A general hypersurface $X \in |H^0(I_S(r))|$ passing through $p$ is sent into a general hyperplane section $H$ of the image passing through $\Phi(p)$. Thus $H$ does not contain $L_p$ and a general $X \in |H^0(I_S(r))|$ which is irreducible, is transversal to $\mathcal{H}$ and hence birational to $\mathcal{H}$ by the first part. Assume also that a general member of $|H^0(I_S(r))|$ is smooth. Then every $X \in |H^0(I_S(r))|$ with at worst rational singularities is birational to $\mathcal{H}$ by [KT17, Theorem 1 and Theorem 16].

Until now the above construction has been applied only for $e = 1$ and $r = 3$ to show that a smooth cubic hypersurface $X \subset \mathbb{P}^5$ containing a surface $S \subset \mathbb{P}^5$ which admits a congruence of secant lines, the so called surfaces with one apparent double point, is rational. Indeed, such irreducible surfaces $S \subset \mathbb{P}^5$ are known to be rational, $\mathcal{H}$ is birational to the rational fourfold $S^{(2)}$ (see [Mat68]) and cubics through such a $S$ satisfy the hypothesis in the second part of Theorem 1.

In the sequel we shall apply Theorem 1 for $e = 2$ and $r = 3$ to a general element of $\mathcal{C}_d$ with $d = 14, 26, 38$, showing in different ways the rationality of $\mathcal{H}$.

2. Rationality of cubics in $\mathcal{C}_{14}$ via congruences of 5-secant conics

Let $S_{14} \subset \mathbb{P}^5$ be a smooth isomorphic projection of an octic smooth surface $S \subset \mathbb{P}^6$ of sectional genus 3 in $\mathbb{P}^6$, obtained as the image of $\mathbb{P}^2$ via the linear system of quartic curves with 8 general base points.

Theorem 2. A surface $S_{14} \subset \mathbb{P}^5$ as above admits a congruence of 5-secant conics parametrized by a rational variety birational to $S_{14}^{(2)}$. Moreover, every $[X] \in \mathcal{C}_{14}$ is rational.

Proof. Let $S \subset \mathbb{P}^6$ be an octic surface of sectional genus 3 as described above. The surface $S$ has ideal generated by seven quadratic forms defining a Cremona transformation:

$$\psi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^6,$$

whose base locus scheme is exactly $S$ and whose inverse is defined by forms of degree four, see [ST70, HKS92]. The secant variety to $S$, Sec($S$) $\subset \mathbb{P}^6$, is an irreducible hypersurface of degree 7 by the double point formula (see [Ful98, Theorem 9.3]) and it is also the exceptional locus of $\psi$, that is the closure of the locus of points where $\psi$ is not an isomorphism. Thus Sec($S$) is contracted to a four dimensional irreducible variety $W \subset \mathbb{P}^6$, which has degree 8 and which is the base locus scheme of $\psi^{-1}$ (see [ST70, HKS92]). The variety $W$ birationally parametrizes the secant lines to $S$ and it is thus birational to the rational variety $S^{(2)}$. 

Let $q \in \mathbb{P}^6 \setminus \text{Sec}(S)$ be general, let $q' = \psi(q)$, let $\pi_q : \mathbb{P}^6 \to \mathbb{P}^5$ be the projection from $q$ onto a hyperplane and let $S_{14} = \pi_q(S) \subset \mathbb{P}^5$. The map $\psi^{-1}$ is given by forms of degree four and the strict transform via $\psi^{-1}$ of a line $L$ through $q'$ intersecting $W$ transversally in a point $r'$ is a rational curve $C = \psi^{-1}(L)$ of degree three, passing through $q$. The curve $C$ is 5-secant to $S$ because $\psi(C) = L$ and $\psi$ is given by quadratic forms. Hence the linear span of $C$ cannot be a plane and $C$ is a twisted cubic. The twisted cubic $C$ projects from $q$ into a 5-secant conic to $S_{14}$. Thus we have produced a four dimensional family of 5-secant conics to $S_{14}$, whose parameter space $\mathcal{H}$ is birational to the rational variety $\mathbb{W}$.

We claim that through a general point $p \in \mathbb{P}^5$ there passes a unique 5-secant conic to $S_{14}$ belonging to the family $\mathcal{H}$. If $L_p = < q, p > \subset \mathbb{P}^6$, then $L_p \cap \text{Sec}(S) = \{q_1, \ldots, q_7\}$ and $D_p = \psi(L_p)$ is a conic because $\psi$ is given by forms of degree two and $L_p \cap S = \emptyset$. Let $\mathbb{P}_p^2 = \langle D_p \rangle \subset \mathbb{P}^6$, remark that $D_p \cap W = \{\psi(q_1), \ldots, \psi(q_7)\}$ ($\psi^{-1}$ is given by forms of degree four and $\psi^{-1}(D_p) = L_p$), that $q' = \psi(q) \in D_p$ and that the scheme $\mathbb{P}_p^2 \cap W$ is zero dimensional because through $p$ there pass at most finitely many 5-secant conics in the family $\mathcal{H}$ and no trisecant line to $S$. Let $\mathbb{P}_q^8 = \mathbb{P}_p^2 \cap W \setminus \{\psi(q_1), \ldots, \psi(q_7)\}$ and let $L'_q = \langle q', q'_i \rangle$. The line $L'_q$ cuts $D_p$ in another point $s' \neq \psi(q_i)$, $i = 1, \ldots, 7$. Indeed, if $s' = \psi(q_i)$ for some $i = 1, \ldots, 7$, then $\psi^{-1}(L'_q)$ would be a 3-secant conic to $S$ passing through $q$ and $q_i$. Hence $\pi_q(\psi^{-1}(L'_q))$ would be a 3-secant line to $S_{14}$ passing through $p$, which is impossible by the generality of $p$. Therefore $\psi^{-1}(L'_q)$ is a twisted cubic, which passes through $q$ and cuts $\langle q, p \rangle$ in the point $s = \psi^{-1}(s')$, and the conic $\pi_q(\psi^{-1}(L'_q))$ passes through $p$.

Any other 5-secant conic to $S_{14}$ passing through $p$ and belonging to $\mathcal{H}$, which is necessarily irreducible by the generality of $p$, would determine another point of intersection of $D_p$ (and a fortiori of $\langle D_p \rangle$) with $W$, which is impossible because $\deg(W) = 8$. Thus $S_{14}$ admits a congruence of 5-secant conics parametrized by a rational variety $\mathcal{H}$ birational to $S^{(2)}$.

The irreducible component $S_{14}$ of the Hilbert scheme parametrizing the surfaces $S_{14} \subset \mathbb{P}^5$ has dimension 49 and it is generically smooth. Indeed, one can verify that $h^1(N_{S/\mathbb{P}^5}) = 0$ and that $h^0(N_{S/\mathbb{P}^5}) = 49$ for a general $[S] \in S_{14}$. Let $h$ be the class of a hyperplane section of $X$, let $h^2$ be the class of 2-cycles $h \cdot h$ and remark that $h^2 \cdot h^2 = h^4 = 3$ and $h^2 \cdot S_{14} = 8$. The double point formula for $S_{14} \subset X$ (see [Ful98, Theorem 9.3]) yields $S_{26}^2 = 26$ and the restriction of the intersection form to $h^2$, $S_{14}$ has discriminant $3 \cdot 26 - 64 = 14$. Let $\mathcal{V} \subset |H^0(\mathcal{O}_{\mathbb{P}^5}(3))| = \mathbb{P}^{55}$ be the open set corresponding to smooth cubic hypersurfaces. We have $h^0(\mathcal{I}_{S_{14}}(3)) = 13$ so that the locus

$$\mathcal{C}_{14} = \{([S], [X]) : S \subset X\} \subset S_{14} \times \mathcal{V},$$

has dimension $49 + 12 = 61$. The image of $\pi_2 : \mathcal{C}_{14} \to \mathcal{V}$ has dimension at most 54 because the general cubic does not contain any $S$ belonging to $S_{14}$. For every $[X] \in \pi_2(\mathcal{C}_{14})$ we have

$$\dim(\pi^{-1}_2([X])) \geq \dim(\mathcal{C}_{14}) - \dim(\pi_2(\mathcal{C}_{14})) = 61 - \dim(\pi_2(\mathcal{C}_{14})) \geq 61 - 54 = 7.$$

Since $h^0(N_{S/X}) \geq \dim[\pi_2^{-1}([X])]$ for every $[S] \in \pi_2^{-1}([X])$, to show that a general $X \in \mathcal{C}_{14}$ contains a surface $S_{14}$ it is sufficient to verify that $h^0(N_{S_{14}/X}) = 7$ for a general $S_{14}$ and for a smooth $X \in |H^0(\mathcal{I}_{S_{14}}(3))|$, see also [Nue15, pp. 284–285] for a similar argument. We verified this via Macaulay2 and we can conclude that a general $[X] \in \mathcal{C}_{14}$ contains a surface $S_{14}$ as above.
Let $\Phi : \mathbb{P}^5 \dasharrow Z \subset \mathbb{P}^{12}$ be given by the linear system $|H^0(I_{S_{14}}(3))|$. The map $\Phi$ is birational onto its image $Z$ (one can look at the resolution of the homogeneous ideal of $S_{14}$, which is generated by cubic forms, and then apply the argument in the proof [Ver01, Proposition 2.8]; or by a direct computation). By Theorem 1 a general $X \in C_{14}$ is rational being birational to $\mathcal{H}$. From [KT17, Theorem 1] we deduce that every $[X] \in C_{14}$ is rational. \qed

Although it is well known that a general $[X] \in C_{14}$ is rational, the previous result has been included in order to point out a uniform approach via congruences of 5-secant conics to the rationality of general smooth cubics of discriminant equal to the first three admissible values 14, 26 and 38.

**Remark 3.** The existence of the congruence of 5-secant conics was firstly discovered by studying the map $\Phi$ defined above. The image $Z = \Phi(\mathbb{P}^5) \subset \mathbb{P}^{12}$ is an irreducible projective variety of degree 28 cut out by 16 quadrics and such that through a general point $z = \Phi(p) \in Z$, $p \in \mathbb{P}^5$ general, there pass 8 lines contained in $Z$. Since $S_{14} \subset \mathbb{P}^5$ has seven secant lines through the point $p$ by the double point formula, we deduce the existence of a congruence of $(3e - 1)$-secant rational curves of degree $e$ to $S_{14}$. Indeed, letting $L_1, \ldots, L_7$ be the seven secant lines to $S_{14}$ passing through the general point $p$, they are mapped by $\Phi$ to seven lines $L_1', \ldots, L_7'$ contained in $Z$ and passing through the general point $z = \Phi(p)$. Let $L_p'$ be the unique line passing through $z = \Phi(p)$ and different from the lines $L_1', \ldots, L_7'$, which all belong to an irreducible family of lines (a birational image of the family of secant lines to $S_{14}$). Let $C_p \subset \mathbb{P}^5$ be the strict transform of $L_p'$ in $\mathbb{P}^5$ via $\Phi^{-1}$. Then $C_p$ is a rational curve of degree $e \geq 2$ passing through $p$ and cutting $S_{14}$ along a zero-dimensional scheme of degree $3e - 1$ since $\Phi(C_p) = L_p'$. One then verifies that $e = 2$, see Section 5, although this value is irrelevant for the rationality of the cubic fourfolds through the previous $S_{14}$.

### 3. Rationality of Cubics in $C_{26}$ via Congruences of 5-Secant Conics

Let $S \subset \mathbb{P}^6$ be a septimic surface with a node, which is the projection of a smooth del Pezzo surface of degree seven in $\mathbb{P}^7$ from a general point on its secant variety. Let $S_{26} \subset \mathbb{P}^5$ be the projection of $S$ from a general point outside the secant variety $\text{Sec}(S) \subset \mathbb{P}^6$.

**Theorem 4.** A surface $S_{26} \subset \mathbb{P}^5$ as above admits a congruence of 5-secant conics parametrized by a variety birational to a rational singular cubic hypersurface in $\mathbb{P}^5$. Moreover, every $[X] \in C_{26}$ is rational.

**Proof.** A septimic surface with a node $S \subset \mathbb{P}^6$ as above has ideal generated by seven quadratic forms and one cubic form. The linear system $|H^0(I_S(2))|$ defines a Cremona transformation:

$$\psi : \mathbb{P}^6 \dasharrow \mathbb{P}^6,$$

whose base locus scheme is $S \cup P$, with $P \subset \mathbb{P}^6$ a plane cutting $S$ along a cubic curve. The surface $S \cup P$ has degree 8 and arithmetic sectional genus three, and it is a projective degeneration of the base locus scheme of the Cremona transformation considered in the proof of Theorem 2.

The exceptional locus $E \subset \mathbb{P}^6$ of $\psi$ is thus a hypersurface of degree seven, whose defining polynomial is the g.c.d. of the seven degree eight polynomials defining $\psi^{-1} \circ \psi$. The secant variety to $S$, $\text{Sec}(S) \subset \mathbb{P}^6$, is an irreducible hypersurface of degree five by the double point formula and it is contracted by $\psi$. Therefore, there exists a quadric hypersurface $Q \subset \mathbb{P}^6$.
such that $E = \text{Sec}(S) \cup Q$. Since $\psi$ is given by forms of degree two vanishing on $S \cup P$, the projective join

$$J(P, S) = \bigcup_{x \neq y, x \in P, y \in S} < x, y > \subset \mathbb{P}^6$$

is also contracted by $\psi$, so that $Q = J(P, S)$ is a rank four quadric hypersurface containing the base locus of $\psi$. Since $\psi^{-1}$ is given by forms of degree four, the hypersurface $E$ has points of multiplicity four along $S \cup P$ (or better, a general point of any irreducible component is of multiplicity four for $E$). From $E = Q \cup \text{Sec}(S)$ and from the fact that $Q$ is non singular at a general point of $S$ and has double points along $P$, we deduce that $\text{Sec}(S)$ has points of multiplicity three along $S$ and double points along $P$ (facts which can be also verified directly). Since $\psi$ is defined by quadratic forms, the irreducible variety $\psi(Q) = W_2$ is contained in a hyperplane $H \subset \mathbb{P}^6$. The secant variety to $S$ is contracted to an irreducible variety $W_1 \subset \mathbb{P}^6$. Clearly, $W_1 \cup W_2$ is the base locus of $\psi^{-1}$ and we claim that $\dim(W_i) = 4$ for $i = 1, 2$, that $\deg(W_1) = 5$ and that $\deg(W_2) = 3$. Indeed, a general plane $\Pi \subset \mathbb{P}^6$ is mapped by $\psi^{-1}$ to a degree eight surface $\Sigma \subset \mathbb{P}^6$, which is the residual intersection of four general quadric hypersurfaces through $S \cup P$. Thus $\Sigma$ has sectional genus three and the restriction of $\psi^{-1}$ to $\Pi$ is given by a linear system of quartics with eight base points. Moreover, $\Pi \cap H = L$ is a general line in $H$ and the strict transform $\psi^*^{-1}(L) \subset \Sigma \cap \Pi$ has degree one or two because $\Sigma \subset \mathbb{P}^8$ has ideal generated by seven quadratic forms (see the proof of Theorem 2). Hence $W_2 \subset H$ is a hypersurface of degree equal to $4 - \deg(\psi^*^{-1}(L))$, and $W_2$ is birational to $S \times P$ since the general fibre over $W_2$ is a line joining a point of $S$ with a point of $P$. The curve $C$ in $\Sigma \cap (S \cup P)$ has degree twelve and arithmetic genus seven. Moreover, it contains $\psi^*^{-1}(L)$ and is represented on $\Pi$ as a curve of degree seven with double points at the eight base points. Let $D = C - \psi^*^{-1}(L)$ be the corresponding divisor on $\Sigma$, let $G = S \cap P$ and recall that $\deg(G) = 3$. Then, taking intersections in $\Sigma$ and in $P$, we deduce $G \cdot \psi^*^{-1}(L) = D \cdot \psi^*^{-1}(L) \in \{3, 4\}$, yielding $\deg(\psi^*^{-1}(L)) = 1$ and $\deg(W_2) = 3$. Since $8 - \deg(\Sigma) = \deg(\Pi \cap (W_1 \cup W_2))$ and since $\deg(W_2) = 3$, there are five points of intersection between $\Pi$ and $W_1$, proving $\dim(W_1) = 4$ and $\deg(W_1) = 5$ by the generality of $\Pi$. The rest of the proof is similar to that of Theorem 2.

Let $q \in \mathbb{P}^6 \setminus \text{Sec}(S)$ be a general point, let $q' = \psi(q)$, let $\pi_q : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5$ be the projection from $q$ and let $S_{26} = \pi_q(S) \subset \mathbb{P}^5$. The strict transform of a line $L$ through $q'$ intersecting $W_2$ transversally in a general point $r'$ is a degree three irreducible curve $C = \psi^{-1}(L)$ passing through $q$ and 5-secant to $S \cup P$. We claim that $C$ is 5-secant to $S$. Indeed, let $a \geq 0$ be the number of points in $C \cap S$ and let $b \geq 0$ be the number of points in $C \cap P$. Since $L \cap W_1 = \emptyset$ and since $L \cap W_2 = r'$, we deduce from Bézout Theorem and from the description of the multiplicities of $Q$ and $\text{Sec}(S)$ along $S$ and along $P$, $\deg(C) \cdot \deg(Q) - 1 = 5 = a + 2b$ and $\deg(C) \cdot \deg(\text{Sec}(S)) = 15 = 3a + 2b$. The unique solution of this system is $(a, b) = (5, 0)$. Hence the linear span of $C$ cannot be a plane and $C$ is a twisted cubic. This twisted cubic projects from $q$ into a 5-secant conic to $S_{26}$.

So far we have constructed a four dimensional family $\mathcal{H}$ of 5-secant conics to $S_{26}$, whose parameter space $\mathcal{H}$ is birational to the four dimensional singular rational cubic hypersurface $W_2$. We claim that through a general point $p \in \mathbb{P}^5$ there passes a unique 5-secant conic to

\[ 1 \text{Similarly, the lines through } q' \text{ and a general point of } W_1 \text{ determine twisted cubics which are 4-secant to } S \text{ and cut } P \text{ in one point because } (4, 1) \text{ is the unique solution to } 6 = a + 2b \text{ and } 14 = 3a + 2b. \]
$S_{26}$ belonging to $\mathcal{H}$. If $L_p = \langle q,p \rangle > \mathbb{P}^5$, then $L_p \cap \text{Sec}(S) = \{q_1, \ldots, q_5\}$, $L_p \cap Q = \{r_1, r_2\}$ and $D_p = \psi(L_p)$ is a conic. Let $\mathbb{P}^2 = \langle D_p \rangle \subset \mathbb{P}^5$, remark that $D_p \cap W_1 = \{\psi(q_1), \ldots, \psi(q_5)\}$, that $D_p \cap W_2 = \{\psi(r_1), \psi(r_2)\}$, that $q' \in D_p$ and that the scheme $\mathbb{P}^5 \cap W$ is zero dimensional because through $p$ there pass at most finitely many 5-secant conics in the family $\mathcal{H}$ and no trisecant line to $S$. Let $r_3' = (\mathbb{P}^2 \cap W_2) \setminus \{\psi(r_1), \psi(r_2)\}$ and let $L_3' = \langle q', r_3' \rangle$. We claim that the line $L_3'$ cuts $D_p$ in a point $s' \in D_p \setminus W_2$. Indeed, since $q' \not\in H$, $L_3' \cap H = r_3' = L_3' \cap W_2$. Then $\psi^{-1}(L_3')$ is a twisted cubic, which cuts $\langle q,p \rangle$ in the point $s = \psi^{-1}(s')$ and the conic $\pi_q(\psi^{-1}(L_3'))$ passes through $p$.

Any other 5-secant conic to $S_{26}$ passing through $p$ and belonging to $\mathcal{H}$, necessarily irreducible by the generality of $p$, would determine another point of intersection of $D_p$ (and a fortiori of $\langle D_p \rangle$) with $W_2$, which is impossible because $\deg(W_2) = 3$. In conclusion, $S_{26}$ admits a congruence of 5-secant conics birationally parametrized by $W_2$.

We claim that a general $[X] \in \mathcal{C}_{26}$ contains a $S_{26}$ as above. The irreducible component of the Hilbert scheme parametrizing these surfaces $S_{26}$ has dimension 42 and it is generically smooth. Using that $S_{26}^2 = 25$ (which follows from the double point formula for surfaces with nodes, see [Ful98, Theorem 9.3]) and that $h^0(I_{S_{26}}(3)) = 14$, by reasoning exactly as in the last part of the proof of Theorem 2 to prove the claim it suffices to produce a general $S_{26} \subset \mathbb{P}^5$ and a smooth $X \in |H^0(I_{S_{26}}(3))|$ such that $h^0(N_{S_{26}/X}) = 1 = 42 + 13 - 54$. An explicit computational verification, see for example the end of Section 5, shows that this holds, proving that a general $[X] \in \mathcal{C}_{26}$ contains a surface $S_{26}$.

The homogeneous ideal of $S_{26} \subset \mathbb{P}^5$ is generated by 14 cubic forms defining a map $\Phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^{13}$, which is birational onto its image $Z = \varphi(\mathbb{P}^5) \subset \mathbb{P}^{13}$. By Theorem 1 a general $X \in \mathcal{C}_{26}$ is rational being birational to $\mathcal{H}$ and hence to $W_2$. From [KT17, Theorem 1] we deduce that every $[X] \in \mathcal{C}_{26}$ is rational.

**Remark 5.** We firstly discovered the congruence of 5-secant conics to the surfaces $S_{26}$ described above via the map $\Phi$ defined above. The variety $Z \subset \mathbb{P}^{13}$ has degree 34, it cut out by 20 quadrics, and through a general point $z \in Z$ there pass 6 lines contained in $Z$, see Section 5. By the double point formula, we deduce that through a general point $p \in \mathbb{P}^5$ there pass five secant lines to $S_{26}$. The strict transforms via $\Phi^{-1}$ of the lines through a general point, not coming from secant lines to $S_{26}$, define a congruence of $(3e-1)$-secant curves to $S_{26}$ of degree $e \geq 2$. By a standard computation, one sees that $e = 2$ although the value of $e$ is insignificant for the rationality conclusions.

**Remark 6.** The rationality of a general $[X] \in \mathcal{C}_{26}$ can also be proved by taking as $S_{26} \subset \mathbb{P}^5$ the rational septimic scroll with three nodes recently considered by Farkas and Verra in [FV18]. Also these surfaces admit a congruence of 5-secant conics, whose parameter space is rational. Moreover, the associated map $\Phi$ is birational onto its image.

4. RATIONALITY OF CUBICS IN $\mathcal{C}_{38}$ VIA CONGRUENCES OF 5-SECANT CONICS

Let $S_{38} \subset \mathbb{P}^5$ be a general degree 10 smooth surface of sectional genus 6 obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having 10 fixed triple points, which were studied classically by Coble.

**Theorem 7.** A general surface $S_{38} \subset \mathbb{P}^5$ admits a congruence of 5-secant conics. Moreover, every $[X] \in \mathcal{C}_{38}$ is rational.
Proof. As shown by Nuer in [Nue15], these surfaces are contained in a general \([X] \in C_{38}\). Moreover \(h^0(I_{S_{38}}(2)) = 0 = h^1(I_{S_{38}}(3))\) and the homogeneous ideal of \(S_{38}\) is generated by 10 cubic forms, whose first syzygies are generated by the linear ones. Via the argument in the proof of [Ver01, Proposition 2.8], we deduce that the linear system \([H^0(I_{S_{38}}(3))]\) defines a birational map onto the image \(\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9\). An explicit calculation with Macaulay2, see Section 5, shows that the image \(Z = \Phi(\mathbb{P}^5) \subset \mathbb{P}^9\) is an irreducible variety of degree 20 cut out by 16 cubics and such that through a general point \(z = \varphi(p) \in Z\) there pass 8 lines contained in \(Z\). Since \(S_{38} \subset \mathbb{P}^5\) has seven secant lines passing through a general point \(p \in \mathbb{P}^5\), we deduce the existence of a congruence of \((3e - 1)\)-secant rational curves of degree \(e\) to \(S_{38}\).

We have, once again, \(e = 2\) although this is irrelevant for the rationality conclusions.

There exist singular rational cubic hypersurfaces through a general \(S_{38}\) with a unique ordinary double point, as one can directly verify by using Macaulay2. Theorem 1 yields the rationality of the parameter space \(H\) of the congruence of 5-secant conics to \(S_{38}\). Then every cubic through \(S_{38}\) (with at most rational singularities) is rational by Theorem 1 and a general cubic \([X] \in C_{38}\) is then rational. From [KT17, Theorem 1] we deduce that every \([X] \in C_{38}\) is rational.

Remark 8. The irreducible boundary cubics of \(C_{38}\) corresponding to singular elements in \([H^0(I_{C_{38}}(3))]\) are all rational by Theorem 1. These rational cubic hypersurfaces with a rational double point also lie on the boundary of \(C_{44}\), see also [Nue15, p. 285].

Let \(S_d \subset \mathbb{P}^5\) with \(d = 44 - 6j\) and with \(j = 0, \ldots, 6\) be the smooth surfaces constructed by Nuer to describe \(C_d\) and which are flat projective deformations of \(S_{38} \subset \mathbb{P}^5\) (although belonging to different irreducible components of the Hilbert scheme), see [Nue15, p. 286]. In Table 1 below, we give the number of 5-secant conics to \(S_d\) passing through a general point of \(\mathbb{P}^5\).

5. Congruences of 5-secant conics from a computational point of view

In this section we illustrate how one can detect the 5-secant conic congruence property in specific examples using the computer algebra system Macaulay2 [GS17].

We start by loading the file Cubics.m2, which is available as an ancillary file to our arXiv submission. This file contains equations for three explicit examples related to the three divisors \(C_d, d = 14, 26, 38\), but here, for brevity, we will consider only the case of \(C_{26}\). After loading the file, the tools for working with rational maps provided by the Cremona package (included with Macaulay2) will be also available.

Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "Cubics";

The next line of code produces two rational maps, \(f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5\) and \(\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^{13}\), which are birational parameterizations of a del Pezzo surface of degree 7 with a node \(S = S_{26} \subset \mathbb{P}^5\) and of a subvariety \(Z \subset \mathbb{P}^{13}\), respectively; here \(\mathbb{Q}\) specifies that we want the ground field to be \(\mathbb{Q}\). The base locus of \(\varphi\) is exactly the (closure of the) image of \(f\), and the information on the projective degrees says that \(\varphi\) is birational onto a degree 34 variety.

i2 : time (f,phi) = example(26,QQ);
   -- used 0.630698 seconds

i3 : f;
Table 1. Smooth surfaces $S \subset \mathbb{P}^5$ of degree 10 and sectional genus 6 cut out by 10 cubics and contained in the generic cubic fourfold of $C_d$. The 3rd and 4th columns contain, respectively, the number of 2-secant lines and the number of 5-secant conics to $S$ passing through a general point of $\mathbb{P}^5$. The 5th column contains the multidegree of the map defined by $|H^0(\mathcal{I}_S(3))|$. \\

| $d$ | Surface $S \subset \mathbb{P}^5$ | 2-secant lines | 5-secant conics | Multidegree |
|-----|---------------------------------|----------------|-----------------|-------------|
| 8   | Image of the plane via the linear system of quintic curves with 15 general base points | 12 | 6 | 1, 3, 9, 17, 21, 15 |
| 14  | Image of the plane via the linear system of sextic curves with 10 general simple base points and 4 general double points | 11 | 5 | 1, 3, 9, 17, 21, 16 |
| 20  | Image of the plane via the linear system of septic curves with 6 general simple base points, 6 general double points and one general triple point | 10 | 4 | 1, 3, 9, 17, 21, 17 |
| 26  | Image of the plane via the linear system of septic curves with 3 general simple base points and 9 general double points | 9 | 3 | 1, 3, 9, 17, 21, 18 |
| 32  | Image of the plane via the linear system of curves of degree 9 with one general simple base point, 4 general double points and 6 general triple points | 8 | 2 | 1, 3, 9, 17, 21, 19 |
| 38  | Image of the plane via the linear system of curves of degree 10 with 10 general triple points | 7 | 1 | 1, 3, 9, 17, 21, 20 |
| 44  | Fano embedded Enriques surface | 6 | 0 | 1, 3, 9, 17, 21, 21 |

Note that the code above does not make any computation as the data are stored internally, but it is not difficult to perform this kind of computations (for instance $Z$ can be quickly obtained with $\text{image}(2, \phi)$, see [Sta18]). Now we choose a random point $^2 p \in \mathbb{P}^5$ and compute the locus $E \subset \mathbb{P}^5$ consisting of the union of all secant lines to $S$ passing through $p$, and the locus $V \subset Z \subset \mathbb{P}^{12}$ consisting of the union of all lines contained in $Z$ and passing through $\varphi(p) \in Z$. The former is obtained using standard elimination techniques, while for the latter we use the procedure given in [Rus16, p. 57].
Five of the six lines in $V$ come from lines in $E$. So, the extra line $L = V \setminus \phi(E)$ can be determined with a saturation computation.

The 5-secant conic to $S$ passing through $p$ is the inverse image of the extra line $L$.

Finally, in the following code, we take a smooth cubic hypersurface $X \subset \mathbb{P}^5$ containing the surface $S$ and compute $h^0(N_{S/X})$, where $N_{S/X}$ is the normal sheaf of $S$ in $X$.
[Has16] ———, *Cubic fourfolds, $K3$ surfaces, and rationality questions*, Rationality Problems in Algebraic Geometry: Levoic Terme, Italy 2015 (R. Pardini and G. P. Pirola, eds.), Springer International Publishing, Cham, 2016, pp. 29–66.

[HKS92] K. Hulek, S. Katz, F. O. Schreyer, *Cremona transformations and syzygies*, Math. Z. 209 (1992), 419–443.

[KT17] M. Kontsevich, Y. Tschinkel, *Specialization of birational types*, preprint [https://arxiv.org/abs/1708.05699](https://arxiv.org/abs/1708.05699), 2017.

[Kuz10] A. Kuznetsov, *Derived categories of cubic fourfolds*, in *Cohomological and Geometric Approaches to Rationality Problems*, Progress in Mathematics, vol. 282, Birkhäuser Boston, 2010, pp. 219–243.

[Kuz16] ———, *Derived categories view on rationality problems*, Rationality Problems in Algebraic Geometry: Levoic Terme, Italy 2015 (R. Pardini and G. P. Pirola, eds.), Springer International Publishing, Cham, 2016, pp. 67–104.

[Lai16] K. Lai, *New cubic fourfolds with odd degree unirational parametrizations*, Algebra & Number Theory 11 (2017), 1597–1626.

[Mat68] A. Mattuck, *The field of multisymmetric functions*, Proc. Amer. Math. Soc. 19 (1968), 764–765.

[Mor40] U. Morin, *Sulla razionalità dell’ipersuperficie cubica dello spazio lineare $S_5$*, Rend. Semin. Mat. Univ. Padova 11 (1940), 108–112.

[Nue15] H. Nuer, *Unirationality of moduli spaces of special cubic fourfolds and $K3$ surfaces*, Algebraic Geom. 4 (2017), no. 3, 281–289.

[Rus16] F. Russo, *On the Geometry of some Special Projective Varieties*, Lect. Notes Unione Mat. Ital., vol. 18, Springer International Publishing, 2016.

[Sch13] F. O. Schreyer, *Computer aided unirationality proofs of moduli spaces*, in *Handbook of Moduli, Vol. III*, Adv. Lect. Math. (ALM), vol. 26 (G. Farkas and I. Morrison, eds.) Int. Press, Somerville, MA, 2013, pp. 257–280.

[ST70] J. G. Semple, J. A. Tyrrell, *The $T_{2,4}$ of $S_6$ defined by a rational surface $^3F^4_8$*, Proc. Lond. Math. Soc. 20 (1970), 205–221.

[Sta18] G. Staglianò, *A Macaulay2 package for computations with rational maps*, J. Softw. Alg. Geom. 8 (2018), 61–70.

[Ver01] P. Vermeire, *Some results on secant varieties leading to a geometric flip construction*, Comp. Math. 125 (2001), no. 3, 263–282.

Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria 5, 95125 Catania, Italy

E-mail address: frusso@dmi.unict.it, giovannistagliano@gmail.com