Anomalies, counterterms and the $\mathcal{N} = 0$ Polchinski-Strassler solutions

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Abstract

The singularity structure of many IIB supergravity solutions asymptotic to $AdS_5 \times S^5$ becomes clearer when one considers the full ten dimensional solution rather than the dimensionally reduced solution of gauged supergravity. It has been shown that all divergences in the gravitational action of the dimensionally reduced spacetime can be removed by the addition of local counterterms on the boundary. Here we attempt to formulate the counterterm action directly in ten dimensions for a particular class of solutions, the $\mathcal{N} = 0$ Polchinski-Strassler solutions, which are dual to an $\mathcal{N} = 4$ SYM theory perturbed by mass terms for all scalars and spinors. This involves constructing the solution perturbatively near the boundary. There is a contribution to the Weyl anomaly from the mass terms (which break the classical conformal invariance of the action). The coefficient of this anomaly is reproduced by a free field calculation indicating a non-renormalisation theorem inherited from the $\mathcal{N} = 4$ theory. We comment on the structure of the full solutions and their construction from uplifting particular $\mathcal{N} = 0$ flows in five dimensions.

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I. INTRODUCTION

The general form of the Maldacena conjecture [1], [2], [3], [4] asserts that there is an equivalence between a gravitational theory in a $(d+1)$-dimensional anti-de Sitter spacetime and a field theory in a $d$-dimensional spacetime. An interesting consequence of the Maldacena conjecture is the natural definition of the gravitational action for asymptotically anti-de Sitter spacetimes without reference to a background [5], [6].

The calculation of the gravitational action has a long history, particularly in the context of black hole thermodynamics [7]. One difficulty that has always plagued this approach is that the gravity action diverges. The traditional approach to this problem is to use a background subtraction whereby one compares the action of a spacetime with that of a reference background, whose asymptotic geometry matches that of the solution in some well-defined sense. However, this approach breaks down when there is no appropriate or obvious background.

The AdS/CFT correspondence tells us that if, as we expect, the dual conformal field theory has a finite partition function, then we must be able to remove the divergences of the gravitational action without background subtraction. The framework for achieving this is by defining local counterterms on the boundary [3], [8], [9], [5], [10]. For example consider the Einstein action in $(d+1)$ dimensions

$$S = -\frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{g} (\mathcal{R} + d(d-1)l^2) - \frac{1}{8\pi G_{d+1}} \int_N \sqrt{\gamma} K$$

(1.1)

where $G_{d+1}$ is the Newton constant and $\mathcal{R}$ is the Ricci scalar. As usual a boundary term must be included for the equations of motion to be well-defined [7], with $K$ the trace of the extrinsic curvature of the $d$-dimensional boundary $N$ embedded into the $(d+1)$-dimensional manifold $M$. Then provided that the metric near the conformal boundary can be expanded in the asymptotically anti-de Sitter form

$$ds^2 = \frac{dr^2}{l^2r^2} + \frac{1}{r^2} \gamma_{ij} dx^i dx^j$$

(1.2)

where in the limit $r \to 0$ the metric $\gamma$ is non-degenerate, we may remove divergent terms in the action by the addition of a counterterm action dependent only on $\gamma$ and its covariant derivatives of the form [4], [11]

$$S_{ct} = \frac{1}{8\pi G_{d+1}} \int_N d^d x \sqrt{\gamma} \left((d-1)l + \frac{1}{2(d-2)}l R(\gamma) + ..\right)$$

(1.3)

$R(\gamma)$ and $R_{ij}(\gamma)$ are the Ricci scalar and the Ricci tensor for the boundary metric respectively. Combined these counterterms are sufficient to cancel divergences for $d \leq 4$, with a number of exceptions. Firstly, in even dimensions $d = 2n$ one has logarithmic divergences in the partition function which can be related to the Weyl anomalies in the dual conformal field theory [5]. Secondly, if the boundary metric becomes degenerate one can no longer remove divergences by counterterm regularisation [11]; this is a manifestation of the fact that the dual conformal field theory does not have a finite partition function in the degenerate limit. Thirdly, if there are gauge and scalar fields on $M$ additional counterterms may be needed to regulate the action [12].
Work on anomalies and counterterms has so far been in the context of gauged supergravity theories; that is, we have worked with the spherical reduction of a ten or eleven dimensional supergravity solution rather than in the higher dimension. Given the interest in extensions of the Maldacena conjecture to non-conformal theories [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], particularly to those which exhibit confinement and condensates, it would be nice to understand counterterm regularisation of the action for the dual supergravity backgrounds. For such bulk solutions, one expects to have counterterms and anomalies related to, for example, the masses which one has switched on in the gauge theory, in addition to (1.3).

Such solutions are better interpreted in ten (or eleven) dimensions rather than in the context of some $d$-dimensional gauged supergravity. This was particularly manifest in the discussions of [35], [36], [37] where the true IR structure of the GPPZ $\mathcal{N} = 1$ flow only became manifest when one uplifted to ten dimensions. Another class of supergravity solutions for which the ten-dimensional interpretation is essential are the duals of fractional branes on conifolds [28], [29], [30], [31], [32], [33], [34]. So we would like to work with the higher-dimensional solution and formulate the corresponding counterterm actions and anomalous divergences in terms of the higher-dimensional fields. This will shed light on the nature of the warped compactification and the angular dependences of the ten-dimensional fields.

In this paper we will consider the perturbative expansion of one class of supergravity solutions which are dual to non-conformal gauge theories, and determine the corresponding counterterms and anomalies. Starting from the duality between $\mathcal{N} = 4$ super Yang-Mills (SYM) and IIB string theory on $AdS_5 \times S^5$, we perturb by the addition of mass terms in the gauge theory which preserve either less or no supersymmetry. At first sight such a perturbation appears to produce a spacetime with a naked singularity in which even basic quantities are incalculable.

However, Polchinski and Strassler [27] suggested that the naked singularity is replaced by an expanded brane source. This is because the mass perturbation in the field theory corresponds to a 3-form perturbation in the bulk. The basic idea is that the field strength couples to the D3-brane and so the D3-branes polarise into D5-branes with worldvolume $R^4 \times S^2$ by the Myers dielectric effect [38]. The background metric for large distances is dominated by the D3-brane charges but as one moves towards the IR one starts probing near the D5 brane and so the metric should be dominated by the D5 branes. The work of [35], [36], [37] showed that the interpretation is rather more subtle than this: by partially uplifting the GPPZ flow [26] to ten dimensions, which gives the equal mass Polchinski-Strassler solution, they found that if one approaches the IR from a generic direction in the sphere one sees a 7-brane. It is only when one approaches from a direction which is consistent with having an IR vacuum in the field theory that one sees five branes.

Explicit construction of the Polchinski-Strassler supergravity solutions in ten dimensions is difficult for a number of reasons. If one starts with a solution of $\mathcal{N} = 2$ gauged supergravity in five dimensions and tries to uplift the solution using [39], [40], [41], [42] then the metric on the sphere is so warped that explicit analytic construction of all ten-dimensional fields is very complex and has not so far been achieved. However, if one starts from the ten-dimensional perspective and works with the IIB equations directly, all bosonic fields are switched on which makes solving even the first order Killing spinor equations very complex.
Switching on a mass perturbation breaks down the $R$ invariance of the field theory and hence the isometry group of the supergravity solution; all fields acquire an angular dependence.

We can simplify the problem a little by trying to preserve the largest possible isometry group when we make the perturbation. To understand this, let us consider the $\mathcal{N} = 4$ theory in the language of $\mathcal{N} = 1$ supersymmetry; it consists of a vector multiplet $V$ and three chiral multiplets $\Phi_a$ in the adjoint of the gauge group. We can partially break supersymmetry by adding mass terms of the form

$$m_{ab} \text{Tr}(\Phi_a \Phi_b)$$

(1.4)

to the superpotential. Such terms break the $R$-symmetry; this is manifest since they transform in the $\bar{6}$ of the $SU(3)$ flavour symmetry subgroup of the full $SO(6)$. However, if the matrix is diagonal with three equal eigenvalues, then the the resulting $\mathcal{N} = 1^*$ theory retains an $SO(3)$ invariance. This was exploited in [35]: writing the metric in a manifestly $SO(3)$ symmetric form was helpful in displaying the IR behaviour.

However, even determination of the linearised ten-dimensional solution in this more symmetric case where we switch on equal mass perturbations required a great deal of calculation [14]. To calculate the counterterms and anomalies we will need to go beyond the linearised solution: in effect we need the solution to second order near the boundary at infinity. We find it therefore calculationally easier to make a further simplification: we consider bulk solutions in which a residual $[SO(3)]^2$ isometry group is preserved. To preserve this symmetry there is a price: we have to switch on a mass for the gluino and break the remaining supersymmetry.

We do not expect that this breaking of supersymmetry will manifest itself in the asymptotic solution: after all, the mass perturbation is relevant and so is a subleading effect in the near boundary solution. However, there are a number of issues we will need to address in the absence of supersymmetry, particularly whether vacua of the type considered in [27] exist even at the classical level and whether they are stable.

The plan of this paper is as follows. In §II we set up the framework for constructing $\mathcal{N} = 0$ Polchinski-Strassler solutions in type IIB supergravity. In §III we perturbatively solve the equations of motion about the boundary to adequate order to determine all (UV) divergences in the action. This is computationally quite intensive and the reader may wish to skip over the details to §IV where we summarise the resulting anomalous and divergent terms in the action and discuss how to define an appropriate counterterm action.

In §V we consider the action of $SL(2, \mathbb{R})$ on the solutions we have constructed. This enables us to construct more general solutions in which the corresponding UV complex coupling and masses in the field theory take general values.

In §VI we consider the dual field theory interpretation of the solutions we have constructed. This involves a discussion of the appropriate form for the scalar potential in the $\mathcal{N} = 0$ theory and the corresponding vacua. In §VII we discuss five-brane probes moving in the bulk solution we have constructed. This enables us to fix the normalisation between the masses in the field theory and the perturbations we have switched on in the bulk.

In §VIII we show that the anomalous terms in the action resulting from switching on mass terms are exactly reproduced by a weak coupling (free field) calculation. This is indicative
of a non-renormalisation theorem inherited from the $\mathcal{N} = 4$ theory to which it flows in the UV.

In §IX we give some general comments about the full Polchinski-Strassler solutions. We discuss how one might use the asymptotic form of the $\mathcal{N} = 0$ solution discussed here to construct the full solution. The most tractable method is probably to uplift the corresponding five-dimensional solution which, though not supersymmetric, should be straightforward to construct. We comment on the unequal mass solutions and provide further evidence for the mass anomaly being reproduced by a free field calculation. We also discuss the running of the coupling in the dual theory.

II. FIELD EQUATIONS AND ASYMPTOTIC SOLUTIONS

The IIB equations can be derived from the Einstein frame action \[ S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} R - \frac{1}{4\kappa^2} \int \left( d\Phi \wedge *d\Phi + e^{2\Phi} dC \wedge *dC + \right. \]
\[ g e^{-\Phi} H_3 \wedge *H_3 + g e^\Phi \bar{F}_3 \wedge *F_3 + \frac{1}{2} g^2 \bar{F}_5 \wedge *\bar{F}_5 + g^2 C_4 \wedge H_3 \wedge F_3 \right), \]

supplemented by the self-duality condition
\[ *\bar{F}_5 = \bar{F}_5. \] (2.2)

The other fields are defined as
\[ \bar{F}_3 = F_3 - C H_3, \quad F_3 = dC_2, \]
\[ F_5 = F_5 - C_2 \wedge H_3, \quad F_5 = dC_4. \] (2.3)

Note that $2\kappa^2 = (2\pi)^7 \alpha'^4 g^2$. The field equations are \[ \nabla^2 \Phi = e^{2\Phi} \partial_M C \partial^M C - \frac{g}{12} e^{-\Phi} H_{MNP} H^{MNP} + \frac{g}{12} e^\Phi \bar{F}_{MNP} \bar{F}^{MNP}, \]
\[ \nabla^M (e^{2\Phi} \partial_M C) = -\frac{g}{6} e^\Phi H_{MNP} \bar{F}^{MNP}, \]
\[ d * (e^\Phi \bar{F}_3) = g F_5 \wedge H_3, \]
\[ d * (e^{-\Phi} H_3 - C e^\Phi \bar{F}_3) = -g F_5 \wedge F_3, \]
\[ d * \bar{F}_5 = -F_3 \wedge H_3, \] (2.4)

whilst the Einstein equations are
\[ R_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{1}{2} e^{2\Phi} \partial_M C \partial_N C + \frac{g^2}{96} F_{MPQRS} F^{MPQRS} \]
\[ + \frac{g}{4} (e^{-\Phi} H_{MPQ} H_{NPQ} + e^\Phi \bar{F}_{MPQ} \bar{F}_{NPQ}) \]
\[ - \frac{g}{48} (e^{-\Phi} H_{PQR} H_{PQR} + e^\Phi \bar{F}_{PQR} \bar{F}_{PQR}) G_{MN}. \] (2.5)

We use indices $\{ M, N \}$ in ten dimensions. The Bianchi identities are
\( dF_3 = -dC \wedge H_3, \)  
\( d\bar{F}_5 = -F_3 \wedge H_3. \)  
(2.6)

As is by now well-known these equations of motion admit a class of solutions

\[
 ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} dx^m dx^n, \\
 e^\Phi = g, \quad C = \frac{\theta}{2\pi}, \\
 \bar{F}_5 = d\chi_4 + *d\chi_4, \quad \chi_4 = \frac{1}{gH} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,
\]

with both \( g \) and \( \theta \) constant and the three-form fields vanishing. Here the indices \( \{\mu, \nu\} \) run between \( 0, .., 3 \) whilst \( \{m, n\} = 4, .., 9 \). \( H \) is an arbitrary harmonic function. For \( AdS_5 \times S^5 \),

\[ H = \frac{R^4}{x^4}, \quad R^4 = 4\pi g N \alpha'^2. \]  
(2.8)

We will now look for a solution near the boundary \( x \to \infty \) in which we switch on three-form fields. For ease of notation we will set \( g = R = 1 \) during the calculations and reinstate these factors at the end. We choose the following coordinate system for the leading order metric:

\[
 ds^2 = \frac{\gamma_0}{r^2} dx^\mu dx^\nu + \frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta (d\phi^2 + \sin^2 \phi d\psi_1^2) + \cos^2 \theta (d\chi^2 + \sin^2 \chi d\psi_2^2), \]  
(2.9)

where the range of \( \theta \) is \( \pi/2 \) and the other angular variables take their usual range. Note that \( r = 1/x \). \( \gamma^0 \) is the background for the dual theory and is taken to be arbitrary.

The reason for the unconventional choice for the metric on the 5-sphere will soon become apparent: we find it convenient to use a metric in which an \( SO(3)^2 \) symmetry group is manifest since our ansatz will preserve this symmetry group. This \( SO(3)^2 \) symmetry group is related to the invariance of the dual \( \mathcal{N} = 0 \) theory when all four fermionic mass perturbations are equal.

To obtain the series expansion of the Polchinski-Strassler solution about the boundary \( r = 0 \) we need to switch on three-form perturbations. From the approximate solutions given in [27] and in [44], we know that we should take the leading order perturbation of the fields to be

\[ \Phi = O(r^2), \quad \bar{F}_3, H_3 = O(r). \]  
(2.10)

The power of the latter ensures that the solution corresponds to switching on a non-normalisable relevant perturbation.

We now have to use an appropriate ansatz for the angular dependence of the three-form fields. In [27], the leading order form form the three-form perturbations was given for the general case in terms of antisymmetric imaginary-self-dual tensors on \( R^6 \) satisfying

\[ *_6 T_{mnp} = -iT_{mnp}, \]  
(2.11)

where \( x^m \) are coordinates on \( R^6 \); it is convenient for what follows to set \( m = 4, 9 \). Given that we want to write the fluxes explicitly in terms of coordinates on the sphere it is more convenient to proceed as follows. We are going to present the explicit details of the calculation for the special case for which the RR scalar \( C \) vanishes identically and hence
\begin{align}
\tilde{F}_3 & \equiv F_3, \quad F^{MNP} H_{MNP} \equiv 0. 
\end{align}

We will discuss the more general case in a later section; it is more complicated in details, though not in principle.

Motivated by the form of the angular metric in (2.9) there is now a natural choice of ansatz for the leading order three-form perturbations:

\begin{align}
H_3 &= d(b(x^\mu, \theta) r \sin \phi d\phi \wedge d\psi_1) \equiv d(b(x^\mu, \theta) r d\Omega_1), \\
F_3 &= d(c(x^\mu, \theta) r \sin \chi d\chi \wedge d\psi_2) \equiv d(c(x^\mu, \theta) r d\Omega_2),
\end{align}

which manifestly satisfies the condition (2.12). This ansatz preserves an \([SO(3)]^2\) subgroup of the original \(SO(6)\) symmetry group of the sphere and, in addition, evidently satisfies (2.12). We will demonstrate explicitly in \(\S VI\) that this perturbation corresponds to switching on masses for all of the fermions in the dual theory.

Switching on such a perturbation will necessarily induce corrections in the metric and other fields. We are going to look for series solutions expanded around the boundary; this will involve expanding the three-form perturbations in powers of \(br\) and \(cr\). Given such an ansatz for the three-form fields, the natural choice of ansatz for the metric expansion is the following:

\begin{align}
\gamma_{\mu\nu} = \gamma^{0}_{\mu\nu}(x^\rho) + \gamma^{2}_{\mu\nu}(x^\rho, \theta) r^2 + \gamma^{4}_{\mu\nu}(x^\rho, \theta) r^4 + h^{4}_{\mu\nu}(x^\rho, \theta) r^4 \ln r + \ldots
\end{align}

where \(e^i\) is the sphere vielbein which follows from (2.9). The ellipses denote “cross-terams” in the metric such as \(G_{r\theta}\), which will not be needed to first order; we will defer discussion of these to later sections.

Following the approach of [9], [48] we will expand \(\gamma\) as

\begin{align}
\gamma_{\mu\nu} = \gamma^{0}_{\mu\nu}(x^\rho) + \gamma^{2}_{\mu\nu}(x^\rho, \theta) r^2 + \gamma^{4}_{\mu\nu}(x^\rho, \theta) r^4 + h^{4}_{\mu\nu}(x^\rho, \theta) r^4 \ln r + \ldots
\end{align}

This ansatz for the metric is the natural generalisation of that given in [49], [48]. We expect that the series breaks down at higher order but the terms in (2.13) should be adequate to determine all divergences in the action.

For any solution for which the metric is a warped product asymptotic to \(AdS_5 \times S^5\) the solution should really be interpreted in ten dimensions to understand the IR behaviour. This means that we should work out the series expansion and counterterms to the action in ten dimensions. The example considered here is the first of many where one would have to do this. Other examples include continuous distributions of D3-branes [14], [16], [17], [18], [50]; uplifts of flows such as [26], [33], [37], [40], [51], [52]; branes on conifolds [28], [29], [30], [31], [32], [33], [34].

Actually we should be a little careful about making an expansion of the form (2.13). According to [43], [48], such an expansion always exists if a \((d+1)\)-dimensional Einstein manifold \(M\) with negative cosmological constant has a regular conformal boundary \(N\) in the sense of Penrose [53]. This is manifestly not going to be the case here. The full ten-dimensional metric is not Einstein of negative curvature and furthermore will have a degenerate conformal boundary. However we could still dimensionally reduce our ten-dimensional solution to a solution of gauged supergravity in five dimensions using a warped ansatz to take account
of the non-trivial angular dependence. Hence we expect that a well defined expansion will exist and that demanding that this expansion is well defined as \( r \to 0 \) will fix the angular dependence of and impose other conditions on the matter fields induced on \( N \).

An equivalent way of looking at this is to say that we are going to have to derive the boundary conditions for a metric to be asymptotically \( AdS_5 \times S^5 \). This leads to natural extensions of the work of Henneaux and Teitelboim [54] on asymptotically AdS spacetimes and of Hawking [53] on boundary conditions for gauged supergravity theories.

We still have to decide how we are going to expand the functions \( G_i \) in (2.14). Given the form of the expansion of the four-dimensional part of the metric (2.15) it is reasonable to take the expansion of \( G_i \) to be

\[
G_i(r, \theta, x^\mu) = 1 + h^i(\theta, x^\mu)r^2 + j^i(\theta, x^\mu)r^4 + k^i(\theta, x^\mu)r^4 \ln r + ..., \tag{2.16}
\]

where by the symmetry of our ansatz \((h^i, j^i, k^i)\) satisfy \( h^\phi = k^\psi_1, \ h^\chi = k^\psi_2 \) and so on. We expect that the field equations will explicitly determine the angular dependence of each of these terms; the \( x^\mu \) dependence will also be determined in terms of derivatives of the functions \( b \) and \( c \) and the curvature of the metric \( \gamma^0 \). Finally, we should expand the dilaton (and later the RR scalar also) as

\[
\Phi = \phi^{(2)}(x^\rho, \theta)r^2 + \phi^{(4)}(x^\rho, \theta)r^4 + \phi^{(4)}(x^\rho, \theta)r^4 \ln r + ..., \tag{2.17}
\]

where the first order term has vanished since we have taken \( g = 1 \).

**III. PERTURBATIVE SOLUTION OF FIELD EQUATIONS**

**A. First order solution**

Following the approach of [5] we are going to solve first for the leading order corrections to the metric and other fields around the boundary. This means that we should keep the leading order terms in the three-form, proportional to the perturbations \( b \) and \( c \), and we should keep the corrections of order \( r^2 \) in all the other fields.

Given the ansätze for the fields, by matching powers of \( r \) we can work out in which order we need to solve the field equations. It turns out that we should first solve the coupled equations for the three-form perturbations. Once we have solved these equations we can substitute into the dilaton (and RR scalar) equations of motion and determine their perturbations. Finally, we should solve the coupled Einstein equations and five-form equations of motion for the \( r^2 \) perturbations of the metric and five-form.

So let us first solve the equations for the three-form fields

\[
d \ast (e^\Phi F_3) = F_5 \wedge H_3, \tag{3.1}
\]

\[
d \ast (e^{-\Phi} H_3) = -F_5 \wedge F_3, \tag{3.1}
\]

which to leading order are

\[
3 \frac{\cos^2 \theta}{\sin^2 \theta} b(x^\mu, \theta) - \partial_\theta \left( \frac{\cos^2 \theta}{\sin^2 \theta} \partial_\theta b(x^\mu, \theta) \right) = -4 \partial_\theta c(x^\mu, \theta), \tag{3.2}
\]

\[
3 \frac{\sin^2 \theta}{\cos^2 \theta} c(x^\mu, \theta) - \partial_\theta \left( \frac{\sin^2 \theta}{\cos^2 \theta} \partial_\theta c(x^\mu, \theta) \right) = 4 \partial_\theta b(x^\mu, \theta).
\]
The solution to these equations
\begin{align}
  b(x^\mu, \theta) &= b \sin^3 \theta, \\
  c(x^\mu, \theta) &= b \cos^3 \theta,
\end{align}
(3.3)
where the \( x^\mu \) dependence of \( b \) is implicit and suppressed in most of what follows, and \( b \) is for the moment arbitrary. We should note here that we can also find a leading order solution to (3.1) which is
\begin{align}
  H_3 &= d(a \cos^3 \theta d\Omega_2) ; \\
  F_3 &= -d(a \sin^3 \theta d\Omega_1),
\end{align}
(3.4)
where again \( a \) is an arbitrary function of \( x^\mu \). However, it is evident from the RR scalar field equation that we will not be able to switch on both \( a \) and \( b \) simultaneously without switching on a perturbation of \( C \). Put another way, \( ab \) acts as a source for the RR scalar. We will find it convenient to discuss first the case in which \( a = 0 \) and then extend our solutions to the more general case in which \( a \) and the RR scalar are non-zero.

In [27] and [44], it was found convenient to combine the R-R and NS-NS three-forms into a complex three-form \( G_3 \) by defining
\begin{equation}
  G_3 = F_3 - \hat{\tau} H_3,
\end{equation}
(3.5)
where as usual
\begin{equation}
  \tau = C + ie^{-\Phi},
\end{equation}
(3.6)
and the hat denotes unperturbed fields. This combination is natural when one considers just the leading order solution and its first order correction (directly equivalent to the linearised solutions in [27], [44]) because of the symmetry between \( F_3 \) and \( H_3 \) which is manifest above. However at higher orders in the radial expansion there is an asymmetry between the perturbations of \( F_3 \) and \( H_3 \) caused by the metric and scalar perturbations and we will find it more convenient to work mostly with these fields rather than \( G_3 \).

The dilaton equation of motion depends only on the three-form perturbations we have just found and on the leading order metric and five-form:
\begin{equation}
  -4\Phi(x^\mu, \theta) + \frac{1}{\sin^2 \theta \cos^2 \theta} \partial_\theta (\sin^2 \theta \cos^2 \theta \partial_\theta \Phi(x^\mu, \theta)) = 4b^2(\sin^2 \theta - \cos^2 \theta).
\end{equation}
(3.7)
The first correction to \( \Phi \) is hence
\begin{equation}
  \Phi(x^\mu, \theta) = \frac{1}{4} b^2(\cos^2 \theta - \sin^2 \theta)r^2.
\end{equation}
(3.8)
We now have all the ingredients needed to solve the coupled equations for the corrections to the metric and the five-form. Let us write the four-form potential as
\begin{equation}
  C_4 = C(r, \theta, x^\mu)d\eta_4^0 + \bar{C}(r, \theta, x^\mu)d\Omega_4,
\end{equation}
(3.9)
where we will use the natural shorthand notation
\[ dx_4^0 = \sqrt{\gamma^0} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \]
\[ d\Omega_4 = d\Omega_1 \wedge d\Omega_2. \]  
(3.10)

Then to get the first corrections to the five form we should expand the functions in (3.9) as

\[ C = -\frac{1}{r^4} - \frac{\alpha^{(2)}}{2r^2}, \]  
(3.11)
\[ \bar{C} = 4 \int \sin^2 \theta \cos^2 \theta d\theta + \beta^{(2)} r^2. \]

It is convenient here to make a further assumption, namely that

\[ \beta^{(2)} = \frac{1}{2} b^2 \sin^3 \theta \cos^3 \theta, \]  
(3.12)

which implies that \( \bar{F}_{r\Omega_4} = \bar{F}_{\mu\Omega_4} = 0 \) and

\[ \bar{F}_{\theta\Omega_4} = \sin^2 \theta \cos^2 \theta (4 - \frac{3}{2} b^2 r^2). \]  
(3.13)

Using one of the constraints arising from the self-duality of the five-form we find that the vanishing of \( \bar{F}_{r\Omega_4} \) implies the vanishing of \( \bar{F}_{\theta\Omega_4} \) and hence that \( \alpha^{(2)} \) does not depend on \( \theta \). Although at this stage there is no justification for this assumption if we keep \( \beta^{(2)} \) arbitrary during the calculation we find that the \( [SO(3)]^2 \) symmetry will later force it to take the value (3.12). Fixing \( \beta \) is effectively a coordinate choice not a gauge choice; we have already fixed the gauge in the choice of the ansatz (3.9). Reasons for this choice of \( \beta \) will be discussed at the end of this section. Note the simplicity of the ansatz for the five-form for this symmetric case compared to that given for the linearised solution in [44] which corresponds to massless gluinos.

Given the ansatz for the five-form and the metric, as well as the solutions we already have for the three-form perturbations, we can now work out the explicit expansions of the Einstein equations. The angular Einstein equations are:

\[ R_{\theta\theta} = 2h^\theta - \frac{1}{2} \text{Tr}((\gamma^0)^{-1} \gamma^2)_{,\theta\theta} - \frac{1}{2} \sum_{i \neq \theta} h^i_{,\theta\theta} - (2h^\phi - h^\theta)_{,\theta} \frac{\cos \theta}{\sin \theta} + (2h^\chi - h^\theta)_{,\theta} \frac{\sin \theta}{\cos \theta}; \]
\[ = \frac{b^2}{4} - 4 \sum_{i \neq \theta} h^i. \]
\[ 
\]

\[ R_{\phi\phi} = -\frac{\cos \theta}{2 \sin \theta} \text{Tr}((\gamma^0)^{-1} \gamma^2)_{,\phi\phi} + h^\phi(6 - \frac{1}{\sin^2 \theta}) + h^\theta(\frac{1}{\sin^2 \theta} - 4) - \frac{1}{2} h^\phi_{,\theta\theta} + h^\phi(\frac{\sin \theta}{\cos \theta} - \frac{2 \cos \theta}{\sin \theta} + \frac{1}{2} (h^\phi_{,\theta} - h^\chi_{,\phi}) \frac{\cos \theta}{\sin \theta}; \]
\[ = b^2(4 \cos^2 \theta - \frac{15}{4}) - 4 \sum_{i \neq \phi} h^i. \]
\[ 
\]

\[ R_{\chi\chi} = \frac{\sin \theta}{2 \cos \theta} \text{Tr}((\gamma^0)^{-1} \gamma^2)_{,\chi\chi} + h^\chi(6 - \frac{1}{\cos^2 \theta}) + h^\theta(\frac{1}{\cos^2 \theta} - 4) - \frac{1}{2} h^\phi_{,\theta\theta} + h^\chi(\frac{2 \sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta}) - \frac{1}{2} (h^\phi_{,\theta} - h^\chi_{,\phi}) \frac{\sin \theta}{\cos \theta}; \]
\[ = \frac{\sin \theta}{2 \cos \theta} \text{Tr}((\gamma^0)^{-1} \gamma^2)_{,\chi\chi} + h^\chi(6 - \frac{1}{\cos^2 \theta}) + h^\theta(\frac{1}{\cos^2 \theta} - 4) - \frac{1}{2} h^\phi_{,\theta\theta} + h^\chi(\frac{2 \sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta}) - \frac{1}{2} (h^\phi_{,\theta} - h^\chi_{,\phi}) \frac{\sin \theta}{\cos \theta}; \]
\[ R_{r\theta} = 0; \]
\[ = -\text{Tr}((\gamma^0)^{-1}\gamma^2)_{,\theta} - \sum_{i \neq \chi} h^i_{,\theta} + 2(h^\theta - h^\phi) \frac{\cos \theta}{\sin \theta} - 2(h^\theta - h^\chi) \frac{\sin \theta}{\cos \theta}. \] (3.14)

\[ R_{\mu\theta} = 0; \]
\[ = -\frac{1}{2} \sum_{i \neq \theta} h^i_{,\mu} + \frac{1}{2}(\gamma^0)^{\nu\rho}(\nabla^0_{\mu}\gamma^2_{\nu\rho} - \nabla^0_{\rho}\gamma^2_{\mu\nu})_{,\theta} + (h^\theta_{,\mu} - h^\phi_{,\mu}) \frac{\cos \theta}{\sin \theta} - (h^\theta_{,\mu} - h^\chi_{,\mu}) \frac{\sin \theta}{\cos \theta}. \] (3.15)

In this list, we give first the curvature calculated using the metric ansatz and second the trace adjusted stress energy tensor calculated using the fields already found and the ansätze for the other fields. Here \( \nabla^0 \) is the covariant derivative associated with \( \gamma^0 \). The \( SO(3)^2 \) symmetry of the ansatz implies that the \( R_{\phi\psi} \) equations are equivalent to the \( R_{\chi\chi} \) equations. Furthermore, the symmetry of the ansatz for the three-form and five-form is going to force a symmetry between the \( h^\phi \) and \( h^\chi \) perturbations.

The other Einstein equations are:

\[ R_{rr} = -2 \sum_i h^i = 4\text{Tr}((\gamma^0)^{-1}\gamma^2) - 2\alpha^{(2)} - \frac{3}{4} b^2. \]

\[ R_{\mu\nu} = \frac{1}{2} bb_{,\mu} = -\gamma^0_{,\rho}(\nabla^0_{\mu}\gamma^2_{\rho\nu} - \nabla^0_{\nu}\gamma^2_{\rho\mu}) - \frac{3}{2} \sum_i h^i_{,\mu}. \] (3.16)

where again we give the explicitly calculated Ricci tensor followed by the stress energy tensor. We have already mentioned one constraint arising from the self-duality of the five-form; the second constraint relates \( \bar{F}_{r\eta i} \) to \( \bar{F}_{\theta\Omega i} \) and gives the final equation of motion:

\[ \alpha^{(2)} = 2\text{Tr}((\gamma^0)^{-1}\gamma^2) - 2 \sum_i h^i - \frac{3}{2} b^2. \] (3.16)
The final constraint induced by the field equations is that the torsion associated with the metric $\gamma^0$ vanishes, the same constraint as was found in \cite{9}.

Let us mention here that we could also make a perturbation such that

$$h^\theta = \frac{1}{6} \beta (\cos^2 \theta - \sin^2 \theta);$$
$$h^\phi = \frac{1}{6} \beta \cos^2 \theta; \quad h^\chi = -\frac{1}{6} \beta \sin^2 \theta;$$
$$G_{\mu\theta} = \frac{1}{12} \beta r^2 \cos \theta \sin \theta; \quad G_{r\theta} = \frac{1}{6} \beta r \cos \theta \sin \theta; \quad F_{\theta\Omega_4} = \sin^2 \theta \cos^2 \theta \left(4 + \beta r^2 (\cos^2 \theta - \sin^2 \theta)\right);$$
$$F_{r\Omega_4} = \frac{2}{3} \beta r \cos^3 \theta \sin^3 \theta; \quad \bar{F}_{r\eta} = \frac{4}{r^5};$$

with the three-form fields vanishing. This is precisely the effect of rescaling

$$\theta \rightarrow \theta + \frac{1}{12} \beta \cos \theta \sin \theta r^2$$

leaving all other coordinates fixed, and hence we can always eliminate $G_{r\theta}$, $G_{\mu\theta}$ terms to first order with a suitable choice of $\beta$. We expect, however, that we will need to include such cross terms in the metric ansatz at higher order in order to find a solution. Coordinate transformations cannot remove cross terms to all orders.

A final comment is that the solution we have found here is closely related to the (extremal limit of the) linearised solution presented in \cite{44}. The difference is that our solution corresponds to switching on four equal fermion masses rather than three. We will expand on this in a later section when we discuss the form for the more general Polchinski-Strassler solutions.

**B. Logarithmic terms**

To determine all divergent terms in the action we need to carry on expanding the fields about the boundary. It turns out that we need to solve next for the logarithmic corrections to the fields, that is, metric corrections of order $r^4 \ln r$ and corresponding terms in the other fields, rather than for second order perturbations to the fields, meaning metric corrections of order $r^4$. The reason for this is that radial derivatives of the logarithmic corrections will contribute to the $r^4$ terms.

The ansatz for the logarithmic terms in the metric is contained in (2.13) and (2.14). We also need to allow for cross-terms in the metric; although the field redefinition discussed
above can remove the leading order cross-terms we should expect that the following terms in the metric could be present:

\[ G_{r\theta} = r^3 k^\theta(\theta, x^\mu) \ln r; \]
\[ G_{r\mu} = r^3 k_\mu(\theta, x^\mu) \ln r; \]
\[ G_{\mu\theta} = r^4 k_\theta(\theta, x^\mu) \ln r. \] (3.21)

These are the only terms allowed by the symmetry of our ansatz. Notice that we fix the power of \( r \) in these terms by demanding that the contributions to the Ricci tensor are of the appropriate order. For simplicity we will anticipate our results and set \( k r^\theta = k_\mu = 0 \) from the start.

As in the previous section, we need to solve the equations of motion in the order three-form equations, scalar field equations and Einstein/five-form equations. So let us solve first the three-form field equations. Taking the ansatz to be

\[ H_3 = d(br \sin^3 \theta + \lambda(\theta, x^\mu) r^3 \ln r) d\Omega_1; \]
\[ F_3 = d(br \cos^3 \theta + \bar{\lambda}(\theta, x^\mu) r^3 \ln r) d\Omega_2, \]

we find a similar pair of equations to (3.2) which admit the solutions

\[ \lambda(\theta, x^\mu) = \lambda \sin^3 \theta; \]
\[ \bar{\lambda}(\theta, x^\mu) = \lambda \cos^3 \theta. \] (3.23)

Given these solutions we can solve the dilaton equation to get the following logarithmic contribution

\[ \Phi = .... + \frac{1}{2} \lambda b(\cos^2 \theta - \sin^2 \theta) r^4 \ln r. \] (3.24)

From the solutions for the three-form we can substitute these into the coupled five-form and Einstein equations to determine metric and five-form perturbations in terms of \( \lambda \) and \( b \). The ansatz we will use for the five-form is the following:

\[ \bar{F}_{\theta\Omega_4} = 4 \sin^2 \theta \cos^2 \theta (1 + ... - \frac{3}{4} \lambda br^4 \ln r); \]
\[ \bar{F}_{r\eta} = \left( \frac{4}{r^5} + ... + \alpha_{(4)}^{(4)} \frac{\ln r}{r} \right), \]

where the ellipses denote the first order perturbations found in the previous section which are not relevant here and all other logarithmic terms vanish. Again we are using hindsight to write down this ansatz; there is no reason a priori why all the other components should vanish. Note that the absence of a \( (\ln(r))^2 \) term in \( \bar{F}_{\theta\eta} \) will require that \( \alpha_{(4)}^{(4)} \) is independent of \( \theta \), a condition which we will find is indeed satisfied by our solution.

Using this ansatz and the solutions for the three-form perturbations, we can write the angular Einstein equations as

\[ R_{\theta\theta} = -\frac{1}{2} \text{Tr}((\gamma^0)^{-1}k^4),_{\theta\theta} - \frac{1}{2} \sum_{i \neq \theta} k^i,_{\theta\theta} + (k^\theta - 2 k^\phi),_\theta \cos \theta \sin \theta - (k^\theta - 2 k^\chi),_\theta \sin \theta \cos \theta; \]
\[
\begin{align*}
R_{\phi \phi} &= \frac{-\cos \theta}{\sin^2 \theta} \text{Tr}((\gamma^0)^{-1}h^4),_\theta + (k^\phi - k^\theta)(4 - \frac{1}{\sin^2 \theta}) - \frac{1}{2} k^\phi,_{\theta \theta} \\
&\quad + (k^\theta - 4 k^\phi - 2 k^\chi)_\theta \frac{\cos \theta}{\sin \theta} + k^\phi \frac{\sin \theta}{\cos \theta}; \\
&= -6 \lambda b \sin^2 \theta - 4 \sum_{i \neq \phi} k^i. \\
R_{\chi \chi} &= \frac{\sin \theta}{\cos^2 \theta} \text{Tr}((\gamma^0)^{-1}h^4),_\theta + (k^\chi - k^\theta)(4 - \frac{1}{\cos^2 \theta}) - \frac{1}{2} k^\chi,_{\theta \theta} \\
&\quad - (k^\theta - 4 k^\chi - 2 k^\phi)_\theta \frac{\sin \theta}{\cos \theta} - k^\chi \frac{\cos \theta}{\sin \theta}; \\
&= -6 \lambda b \cos^2 \theta - 4 \sum_{i \neq \chi} k^i. \\
R_{r \theta} &= -2 \sum_{i \neq \theta} k^i + 4(k^\theta - k^\phi) \frac{\cos \theta}{\sin \theta} - 4(k^\theta - k^\chi) \frac{\sin \theta}{\cos \theta} - \frac{3}{2} \text{Tr}((\gamma^0)^{-1}h^4),_\theta; \\
&= 0. \\
R_{\mu \theta} &= (\gamma^0)^{\nu \rho}((\nabla^0\eta^4)_{\mu \nu} - \nabla^0_{\mu} h^4),_\theta - \frac{5}{4} \sum_{i \neq \theta} k^i; \\
&= 0,
\end{align*}
\]

where as before we have written first the curvature terms and secondly the trace adjusted stress energy tensor. The other Einstein equations are

\[
\begin{align*}
R_{rr} &= -8 \sum_i k^i - 4 \text{Tr}((\gamma^0)^{-1}h^4); \\
&= -2 \alpha_\log^{(4)} + 4 \text{Tr}((\gamma^0)^{-1}h^4), \\
R_{\mu \nu} &= 2(\gamma^0)^{\nu \rho}((\nabla^0\eta^4)_{\mu \nu} - \nabla^0_{\mu} h^4) - \frac{5}{2} \sum_{i \neq \theta} k^i - 4k^r; \\
&= \frac{1}{2}(b\lambda_{r \mu} + 3 \lambda b_{\mu}); \\
R_{\mu \nu} &= -4h^4_{\mu \nu} - \frac{1}{2} h^4_{\mu \nu, \theta \theta} - h^4_{\mu \nu, \theta} \left(\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}\right) + 2 \sum_i k^i (\gamma^0),_{\mu \nu} + 2 \text{Tr}((\gamma^0)^{-1}h^4)_{\gamma^0 \mu \nu}; \\
&= (-3 \lambda b - 2 \alpha_\log^{(4)} + 2 \text{Tr}((\gamma^0)^{-1}h^4))_{\gamma^0 \mu \nu}.
\end{align*}
\]

Here \(R^0\) is the curvature associated with the metric \(\gamma^0\). The final equation is given by the self-duality condition on the five-form

\[
\alpha_\log^{(4)} = -2 \sum_i k^i + 2 \text{Tr}((\gamma^0)^{-1}h^4) - 3 \lambda b.
\]

These equations may again be solved by assuming that the unknowns depend only on the first harmonics in \(\theta\) which reduces most of the differential equations to algebraic equations. Solving everything but the \(R_{\mu \nu}\) equation we find the following solutions for the metric and five-form perturbations
\[ \alpha^{(4)}_{\text{log}} = 0; \]
\[ \text{Tr}((\gamma^0)^{-1}h^4) = \frac{3}{4}\lambda b; \]
\[ k^\theta = \frac{1}{4}\lambda b; \quad (3.29) \]
\[ k^\phi = k^{\psi_1} = \frac{1}{4}\lambda b(1 - 4 \sin^2 \theta); \]
\[ k^\lambda = k^{\psi_2} = \frac{1}{4}\lambda b(1 - 4 \cos^2 \theta); \]

The \( R_{\mu r} \) equation then determines the logarithmic contribution to the \( G_{\mu r} \) component to be

\[ k^r = \frac{1}{16}b\lambda_{\mu} - \frac{3}{16}\lambda b_{\mu}. \quad (3.30) \]

Only the trace of \( h^4 \) is fixed by these equations; the rest of \( h^4 \) will be determined by the equations of motion for the non-logarithmic terms at the same radial order. Note that the trace of \( h^4 \) does not vanish. At first sight, this seems a little worrying since it implies we might end up getting divergences of the form \((\ln \epsilon)^2\) in the action, where \( \epsilon \) is the cut-off length scale; such divergences do not have a natural interpretation. However, it will turn out that there are other contributions to these divergences and the net contribution cancels in a non-trivial way.

C. Second order terms

Given the solutions for the first order and logarithmic perturbations, we can now solve for the \( r^4 \) corrections to the metric and corresponding terms. Once again, we should start with the coupled equations for the three forms. Taking the ansatz to be

\[ H_3 = d(br^3 \cos^2 \theta + \lambda \sin^3 \theta r^3 \ln r + B(\theta, x^\mu)r^3)d\Omega_1; \]
\[ F_3 = d(br^3 \cos^2 \theta + \lambda \cos^3 \theta r^3 \ln r + C(\theta, x^\mu)r^3)d\Omega_2, \]

and substituting into the field equations, using the explicit values of the first order terms, we find the following coupled equations:

\[ -\partial_\theta(\frac{\cos^2 \theta}{\sin^2 \theta}B,\theta) + 3B \frac{\cos^2 \theta}{\sin^2 \theta} = -4C,\theta + 24\lambda^3 \sin^4 \theta \cos \theta + K \sin^2 \theta \cos \theta; \quad (3.32) \]
\[ -\partial_\theta(\frac{\sin^2 \theta}{\cos^2 \theta}C,\theta) + 3C \frac{\sin^2 \theta}{\cos^2 \theta} = -4B,\theta + 24\lambda^3 \cos^4 \theta \sin \theta + K \cos^2 \theta \sin \theta, \]

where

\[ K = (\Box^0 - \frac{1}{6}R^0)b + 2\lambda - \frac{53}{6}b^3. \quad (3.33) \]

Note that there is a contribution here from the radial derivatives of the logarithmic terms in \( F_3 \) and \( H_3 \). This is why we had to work out the logarithmic corrections to the fields before solving for the \( r^4 \) corrections. Solving (3.32) turns out to be quite subtle. Let us first substitute in

15
\[ B(\theta, x^\mu) = \hat{B}(\theta, x^\mu) - \frac{1}{2} b^3 \sin^5 \theta, \quad C(\theta, x^\mu) = \hat{C}(\theta, x^\mu) - \frac{1}{2} b^3 \cos^5 \theta, \quad (3.34) \]

so that

\[ -\partial_\theta \left( \frac{\cos \theta}{\sin^2 \theta} \hat{B}_{,\theta} \right) + 3 \hat{B} \frac{\cos^2 \theta}{\sin^2 \theta} = -4 \hat{C}_{,\theta} + \hat{K} \sin^2 \theta \cos \theta; \quad (3.35) \]


\[ -\partial_\theta \left( \frac{\sin^2 \theta}{\cos^2 \theta} \hat{C}_{,\theta} \right) + 3 \hat{C} \frac{\sin^2 \theta}{\cos^2 \theta} = -4 \hat{B}_{,\theta} + \hat{K} \cos^2 \theta \sin \theta, \]

where now

\[ \hat{K} = (\Box^0 - \frac{1}{6} R^0) b + 2 \lambda + \frac{1}{6} b^3. \quad (3.36) \]

We now want to argue that a sensible (non-divergent) solution to (3.35) exists only when \( \hat{K} = 0 \). When \( \hat{K} \neq 0 \) there is effectively an inhomogeneous perturbation which coincides with the solution to the homogeneous equations. This will mean that the resulting solution will be singular at some value of \( \theta \) and since the range of \( \theta \) is finite this singularity will result in a real physical singularity of the metric and other fields. This effect is directly analogous to resonance when an oscillatory system is forced at its natural frequency.

Unfortunately, explicit solutions to (3.35) when \( \hat{K} \neq 0 \) in terms of elementary functions do not seem to be accessible. The best way to demonstrate that the solutions are divergent is to expand \( \hat{B} \) and \( \hat{C} \) in harmonics on the \( S^5 \): the resulting series show divergences at \( \theta = 0 \) or \( \theta = \pi/2 \).

When \( \hat{K} = 0 \), solution of (3.35) proceeds as in the previous sections so that the three-form fields are

\[ H_3 = d(br \sin^3 \theta + \lambda \sin^3 \theta r^3 \ln r + (B - \frac{1}{2} b^3 \sin^2 \theta)r^3 \sin^3 \theta) d\Omega_1; \quad (3.37) \]

\[ F_3 = d(br \cos^3 \theta + \lambda \cos^3 \theta r^3 \ln r + (B - \frac{1}{2} b^3 \cos^2 \theta)r^3 \cos^3 \theta) d\Omega_2. \]

\( B(x^\mu) \) is a new field, deriving from the homogeneous part of the solution to (3.35). It gives rise to a normalisable perturbation and corresponds to switching on a vacuum expectation value for fermion bilinears. We can now substitute directly into the dilaton equation of motion to get the following solution

\[ \Phi = \left( \frac{1}{4} b^2 r^2 + \frac{1}{2} Bb - \frac{1}{144} R^0 b + \frac{71}{576} b^4 \right)r^4 + \frac{1}{2} \lambda b r^4 \ln r \right) (\cos^2 \theta - \sin^2 \theta). \quad (3.38) \]

We need to give an ansatz for the metric and for the five-form. As well as the terms given in (2.13) and (2.16), we should also include the following cross-terms

\[ G_{r\theta} = f_{r\theta} r^3; \]

\[ G_{r\mu} = f_{r\mu} r^3; \]

\[ G_{\mu\theta} = f_{\mu\theta} r^3. \quad (3.39) \]

We have derived the powers of \( r \) by demanding that the contribution to the Einstein tensor is of appropriate order. Thus as anticipated we are being forced to derive the falloff of cross
terms in the metric, or in another words the conditions for a spacetime to be asymptotically $AdS_5 \times S^5$.

The appropriate ansatz for the five-form field is then

\[
\begin{align*}
\bar{F}_{\theta\alpha_4} &= 4 \sin^2 \theta \cos^2 \theta \{1 - \frac{3b^2 r^2}{8} - \frac{3Br^4}{4} + \frac{3fr^4}{16} (\cos^2 \theta - \sin^2 \theta) \\
& \quad + f_+ r^4 (\frac{3}{16} - \sin^2 \theta \cos^2 \theta) + b'^4 r^4 (\frac{9}{32} - \frac{1}{4} \sin^2 \theta \cos^2 \theta)\}, \\
\bar{F}_{r\alpha_4} &= \{f_+ - \lambda b + f_+(\cos^2 \theta - \sin^2 \theta)\} r^3 \cos^3 \theta \sin^3 \theta, \\
\bar{F}_{\mu\alpha_4} &= \frac{1}{4} \{f_+ + f_- (\cos^2 \theta - \sin^2 \theta)\} \mu r^3 \cos^3 \theta \sin^3 \theta, \\
\bar{F}_{\theta\eta_4} &= \frac{4}{r^5} \{f_+ + f_- (\cos^2 \theta - \sin^2 \theta)\} \theta r^3 \cos^3 \theta \sin^3 \theta, \\
\bar{F}_{r\eta_4} &= \frac{4}{r^5} \{b^2 \frac{5}{12r^3} - R^0_0 \frac{5}{72} (R^0)^2\} + \frac{3}{r^5} \cos \theta \sin \theta. 
\end{align*}
\]

There are various comments to make about this ansatz. Firstly, we need to introduce two new parameters \{f_-, f_+\} to obtain a general enough form; when we solve the equations of motion these parameters will however be fixed in terms of \{b, B\}.

Secondly, in this ansatz we have already partially implemented self-duality; that is, we have exploited the duality between \(\bar{F}_{r\alpha_4}\) and \(\bar{F}_{\theta\alpha_4}\). Note that the term \(\bar{F}_{\mu\alpha_4}\) must correspond to a dual term \(\bar{F}_{r\theta\rho\sigma}\) which we have not included in our ansatz. However, keeping track of powers of \(r\) we can see that the latter is higher order, does not contribute to the field equations and can be ignored in all that follows.

Thirdly, we have already implemented the \(\bar{F}_5\) field equation in this ansatz, namely that

\[
d(\ast \bar{F}_5) = - F_3 \wedge H_3. \tag{3.41}
\]

This condition determines the relationship between \(\bar{F}_{r\alpha_4}\), \(\bar{F}_{\theta\alpha_4}\) and \(\bar{F}_{\mu\alpha_4}\), using the already determined expressions for \(F_3\) and \(H_3\).

The condition that \(F_5\) is exact imposes the further constraint that

\[
\partial_\theta \alpha^{(4)} = 0, \tag{3.42}
\]

since we have found in the previous section that there is no logarithmic term in \(\bar{F}_{\theta\alpha_4}\). The final constraint from self-duality requires that

\[
\alpha^{(4)} = 2\text{Tr}((\gamma^0)^{-1} \gamma^4) - 2 \sum_i j^i - 3Bb + \frac{3}{4} f_- (\cos^2 \theta - \sin^2 \theta) \\
+ f_+ (\frac{3}{4} - 4 \sin^2 \theta \cos^2 \theta) + b'^4 (\frac{1301}{1382} - 2 \sin^2 \theta \cos^2 \theta) \tag{3.43}
\]

Using the fields already determined as well as the ansätze for the metric and five-form perturbations, we can work out the angular Einstein equations which are

\[
R_{\theta\theta} = - \frac{1}{8} (b (\Box^0 - \frac{1}{6} R^0) b + (\partial b)^2) - \frac{1}{2} \text{Tr}((\gamma^0)^{-1} \gamma^4)_{\theta\theta} - \frac{1}{2} \sum_{i \neq \theta} j^i_{\theta\theta} - \frac{1}{2} \lambda b,
\]
\begin{align*}
  \text{the trace adjusted stress energy tensor.} \\
  \text{Once again, we have given first the calculated curvature using the metric ansatz and second} \\
  \text{derivative } \nabla. \text{ Note that the d'Alambertian appears in its conformal form.} \\
  \text{The only other Einstein equations which we will need are} \\
  R_{rr} &= -4 \text{Tr}((\gamma^0)^{-1} \gamma^4) - 8 \sum_i j^i - \frac{3}{4} \lambda b + \frac{37}{36} b^4 - 3b^4 \sin^2 \cos^2 \theta \\
  &= 4 \text{Tr}((\gamma^0)^{-1} \gamma^4) - 2 \alpha^{(4)} + \frac{3}{4} \lambda b + b^4 (-\frac{83}{288} + \sin^2 \cos^2) - \frac{1}{8} (\partial b)^2 \\
  &= -\frac{1}{2} (R^0)_{\mu \nu} R^0_{\mu \nu} + \frac{5}{36} (R^0)^2 + \frac{1}{18} b^2 R^0. \\
  \end{align*}

(3.45)
\[(\gamma^0)^{\mu\nu} R_{\nu\mu} = 8\text{Tr}((\gamma^0)^{-1}\gamma^4) - \frac{1}{8}\square^0 (b^2) + \frac{5}{72} b^2 R^0 + \frac{1}{9} (R^0)^2 + \frac{1}{2} (R^0)^{\mu\nu} R^0_{\mu\nu} - \frac{1}{8} \text{Tr}((\gamma^0)^{-1}\gamma^4)_{,\theta} - \text{Tr}((\gamma^0)^{-1}\gamma^4)_{,\rho} \frac{(\cos \theta - \sin \theta \cos \theta)}{\sin \theta} + 8 \sum_i j^i - \frac{3}{2} \lambda b + b^4 (4 \cos^2 \theta \sin^2 \theta - \frac{403}{288}) - 4(j^\theta + 2 j^\phi (\cos \theta - \sin \theta \cos \theta)) \]

\[= 16\text{Tr}((\gamma^0)^{-1}\gamma^4) - 8\alpha^{(4)} - 12Bb - \lambda b + \frac{353}{144} b^4 - 2(R^0)^{\mu\nu} R^0_{\mu\nu} + \frac{5}{9} (R^0)^2 + \frac{13}{72} b^2 R^0.\]

Given the complexity of equations (3.40), (3.44) and (3.45) even in the SO(3)² symmetric case we can now justify why we considered this simpler case. Even for the \(N = 1^*\) case in which all three mass perturbations are equal, we would have four times as many components in (3.44) and (3.40)!

As before, we solve these equations by expanding each unknown in spherical harmonics and reducing most of the differential equations to algebraic equations. After considerable algebra we find the following solution for metric and five-form perturbations

\[j^\theta = -\frac{1}{4} \lambda b \cos \theta \sin \theta;\]
\[f_- = -\frac{1}{4} \lambda b;\]
\[f_+ = 12Bb - \frac{65}{18} b^4;\]
\[\alpha^{(4)} = -\frac{1}{4} (R^0)^{\mu\nu} R^0_{\mu\nu} + \frac{1}{24} (R^0)^2 + \frac{1}{96} b^2 R^0 + \frac{1}{16} (\partial b)^2 + \frac{1}{96} b^4;\]
\[\text{Tr}((\gamma^0)^{-1}\gamma^4) = \frac{1}{16} (R^0)^{\mu\nu} R^0_{\mu\nu} - \frac{1}{72} (R^0)^2 - \frac{5}{576} b^2 R^0 + \frac{1}{32} (\partial b)^2 - \frac{3}{2} Bb + \frac{277}{576} b^4 - (12Bb + \frac{121}{36} b^4) \sin^2 \theta \cos^2 \theta.\]

Note that as anticipated the solutions depend only on \(B(x^\mu), b(x^\mu)\) and the latter’s derivatives. The remaining parts of the metric perturbations are such that if expand out the angular perturbations as

\[j^\theta = (\alpha \cos^4 \theta + \beta \cos^2 \theta \sin^2 \theta + \gamma \sin^4 \theta);\]
\[j^\phi = (\alpha \cos^4 \theta + \eta \cos^2 \theta \sin^2 \theta + \delta \sin^4 \theta);\]
\[j^\chi = (\zeta \cos^4 \theta + \xi \cos^2 \theta \sin^2 \theta + \gamma \sin^4 \theta),\]

then

\[\alpha = -\frac{1}{8} \lambda b + \frac{1}{2} Bb - \frac{501}{2304} b^4;\]
\[\beta = Bb - \frac{501}{1152} b^2;\]
\[\gamma = \frac{1}{8} \lambda b + \frac{1}{2} Bb - \frac{501}{2304} b^4;\]
\[ \delta = \zeta = -\frac{195}{768} b^4; \quad (3.48) \]
\[ \eta = -\frac{5}{2} Bb + \frac{527}{384} b^4 - \frac{1}{8} \lambda b; \]
\[ \xi = -\frac{5}{2} Bb + \frac{527}{384} b^4 + \frac{1}{8} \lambda b. \]

We now have the field perturbations to adequate order to determine all divergences in the action. The full solution hence depends on only two independent fields, \( b(x^\mu) \) and \( B(x^\mu) \), which have a natural dual interpretation in terms of normalisable and non-normalisable perturbations which we will discuss below in more detail. To reinstate dimensional factors we note that \( b \) has dimensions of length and \( B \) has dimensions of (length)\(^3\).

The remaining Einstein equations \( R_{\mu \theta}, R_{\mu r} \) and \( R_{\mu \nu} \) will fix the off-diagonal elements in the metric \( G_{\mu r} \) and \( G_{\mu \theta} \) as well as \( h^4_{\mu \nu} \). We do not need these to calculate the action divergences and the explicit expressions are not particularly illuminating. The only contribution to the trace of the \( h^4 \) is that given in (3.29).

Note that the Einstein equations do not fix \( \gamma^4_{\mu \nu} \), completely. Only the trace of \( \gamma^4 \) and its covariant divergence are determined. Extra data from the field theory is needed to fix the rest: the undetermined part is specified by the expectation value of the dual stress energy tensor \[56\]. However the trace is sufficient to determine the divergences in the action.

Before proceeding it is useful for reference in later sections to summarise (parts of) the solution when \( b \) is constant and \( \gamma^0 \) is flat

\[
\begin{align*}
ds^2 &= \frac{R^2}{r^2} (1 + b^2 r^2 R^4) \eta_{\mu \nu} dx^\mu dx^\nu + \frac{R^2}{r^2} dr^2 + \frac{R^2}{r^2} d\theta^2 + \frac{b^2 r^2}{8 R^4} d\theta^2 + \frac{b^2 r^2}{8 R^4} (4 \sin^2 \theta) d\Omega_2^2 \\
\Phi &= \ln g + \frac{b^2 r^2}{4 R^4} (\cos^2 \theta - \sin^2 \theta); \\
C_2 &= \frac{1}{g^2} (br + \frac{Br^3}{R^2}) \cos^3 \theta d\Omega_2 \\
B_2 &= (br + \frac{Br^3}{R^2}) \sin^3 \theta d\Omega_1; \\
C_4 &= -\frac{1}{4 g^4} (br^4 + \frac{b^2 r^2}{24 g^2}) d\eta_4 + \frac{4 R^4}{g} \int \sin^2 \theta \cos^2 \theta \theta d \theta + \frac{1}{2 g} b^2 r^2 \sin^3 \theta \cos^3 \theta d\Omega_4,
\end{align*}
\]

where we have also reinstated factors of \( R \) and \( g \) explicitly. For the purposes of calculating the counterterms and anomalies, we have been considering solutions dual to field theories in general backgrounds with position dependent masses. The latter is natural from the supergravity perspective but is somewhat unconventional from the field theory point of view. When we discuss the field theory in \( \S \text{VI} \) and onwards we will restrict to the most interesting case of flat backgrounds where the mass is constant.

### IV. THE ACTION AND THE CONFORMAL ANOMALY

Having determined the series expansion of this Polchinski-Strassler solution, we can now determine the divergent terms in the action, the corresponding counter-terms needed to
render the action finite and the mass contributions to the conformal anomaly. The on-shell contribution to the volume term in the action (2.1) is

\[ S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left( \frac{1}{12}(e^{-\Phi} H_3^2 + e^{\Phi} F_3^2) - \frac{1}{48} F_5^2 \right) - \frac{3}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3. \] (4.1)

Now to the order that we need

\[ \frac{1}{12}(e^{-\Phi} H_3^2 + e^{\Phi} F_3^2) = 5b^2r^2 + 12\lambda br^4 \ln(r) \] (4.2)

\[ + r^4(12Bb + \lambda b + \frac{1}{2}(\partial b)^2 - \frac{61}{16}b^4), \]

whilst

\[ - \frac{F_5^2}{48} = -8 + 3b^2r^2 + 8r^4 \sum_i j^i + 3\lambda br^4(\cos^2 \theta - \sin^2 \theta) \] (4.3)

\[ + Bbr^4(-12 + 96 \cos^2 \theta \sin^2 \theta) + b^4r^4(\frac{67}{12} - \frac{448}{9} \cos^2 \theta \sin^2 \theta) + 6\lambda br^4 \ln(r). \]

Expanding out the metric determinant we get

\[ \sqrt{-G} = \sqrt{\frac{\gamma}{r^5}} \sin^2 \theta \cos^2 \theta \sin \phi \sin \chi \left( 1 + r^2\left(\frac{b^2}{48} - \frac{R^0}{12}\right) + \frac{1}{2}r^4 \sum_i j^i + \frac{1}{2}r^4 \text{Tr}((\gamma^0)^{-1} \gamma^4) \right. \]

\[ - \frac{1}{16} (R^0)^{\mu\nu} R^0_{\mu\nu} + \frac{5}{288} (R^0)^2 r^4 + \frac{1}{144} b^2 R^0 r^4 + b^4 r^4 (\frac{1}{4} \sin^2 \theta \cos^2 \theta - \frac{427}{4608}) \left. \right). \] (4.4)

Note that the logarithmic terms have cancelled in this expression, since \( \sum_i k^i + \text{Tr}((\gamma^0)^{-1} h^4) = 0 \). Furthermore,

\[ - \frac{3}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3 = \int d^{10}x \sqrt{-\gamma^0} r^3 \sin^2 \theta \cos^2 \theta \sin \phi \sin \chi \left( \frac{9}{2} b^2 - 18Bb^2 - \frac{9}{2} \lambda br^2 \right. \]

\[ - 18\lambda br^2 \ln(r) + \frac{3}{4} R^0 b^2 r^2 + \frac{105}{16} b^4 r^2 - 6b^4 \sin^2 \theta \cos^2 \theta \left. \right). \] (4.5)

Putting all of this together, the volume term in the action is

\[ S = \frac{\pi^3}{2\kappa^2 r^5} \int dr d^4 x \sqrt{-\gamma^0} \left( -8 + \left(\frac{2R^0}{3} + \frac{10b^2}{3}\right)r^2 + r^4 \left(\frac{1}{4}(R^0)^{\mu\nu} R^0_{\mu\nu} - \frac{1}{12}(R^0)^2 \right) \right. \]

\[ + \frac{3}{8} b(\Box^0 b - \frac{1}{6} R^0 b)r^4 + \frac{1}{24} b^4 r^4 \right), \] (4.6)

where we have carried out the angular integrations. Note that terms proportional to \( B(x^\mu) \) have cancelled as have terms proportional to \( \ln(r)/r \). The cancellation of the latter is necessary to ensure that there is no \( \ln^2(\epsilon) \) divergence in the action. It is convenient to reinstate factors of \( R \) and \( g \) at this stage. Introducing a cut-off \( \epsilon \) and performing the \( r \) integration we find that the (UV) divergent terms are

\[ S = -\frac{N^2}{8\pi^2} \int d^4 x \sqrt{-\gamma^0} \left( \frac{2}{\epsilon^4} \left(\frac{R^0}{3} + \frac{5b^2}{3}\right) - \frac{1}{\epsilon^2} + A \ln \epsilon \right). \] (4.7)
We have switched to the natural dual parameters, using \(2\kappa^2 = (2\pi)^7\alpha'^4 g^2\) and \(R^4 = 4\pi g N \alpha'^2 = g^2_{YM} \alpha'^2\), where \(g_{YM}\) is the Yang-Mills coupling. We have also introduced what will turn out to be the more natural gauge theory parameter \(\tilde{b}\) which is defined as

\[
\tilde{b} = \frac{b}{\alpha' \sqrt{g^2_{YM} N}}.
\] (4.8)

\(\mathcal{A}\) is the conformal anomaly which is

\[
\mathcal{A} = \frac{1}{4} (R^0)^{\mu\nu} R^0_{\mu\nu} - \frac{1}{12} (R^0)^2 + \frac{3}{8} \tilde{b} (\Box \tilde{b} - \frac{1}{6} R^0 \tilde{b}) + \frac{1}{24} \tilde{b}^4.
\] (4.9)

The curvature terms are in agreement with those found in [9] whilst the other contributions to the anomaly depend on the field \(b(x^\mu)\). Since \(B(x^\mu)\) corresponds to a vacuum expectation value in the dual theory it was expected that it should not appear in the anomaly from the field theory perspective; cancellation of the \(B\) terms from the supergravity perspective was very non-trivial.

There are additional divergent terms in the action arising from the Gibbons/Hawking boundary term

\[
S = \frac{1}{\kappa^2} \int d^9 x \sqrt{-H} K = -\frac{\pi^3}{\kappa^2} \int d^4 x \sqrt{-\gamma^0} \left( -\frac{4}{\epsilon^4} + \left( \frac{R^0}{6} - \frac{b^2}{24 R^4} \right) \frac{1}{\epsilon^2} + \ldots \right),
\] (4.10)

where \(K\) is the trace of the second fundamental form on the induced boundary which has induced metric \(H\). These divergent terms from the volume and surface terms should be cancelled by introducing counterterms. However it is not immediately apparent how we can do this here in a coordinate invariant way. Since the induced metric on the boundary is nine-dimensional we cannot write a counterterm action in terms of a four-dimensional restriction of this such as \(\gamma\) in (2.14) since this is not well-defined under coordinate transformations. Worse than this, \(\gamma\) does depend on the angular coordinates explicitly.

The best we can do in ten dimensions is the following. Suppose we have a metric \(g\) which we know lies in the class of solutions considered here. Then we should write \(g\) in a coordinate system in which [i] it is a direct product of a five-dimensional metric and a sphere as one approaches the boundary at infinity and [ii] cross-terms between the two metrics fall off faster than \(\Omega^2\) where \(\Omega\) is the conformal factor at the boundary of the five-dimensional metric in the sense of Penrose [23]. This is equivalent to saying that we should bring the metric into the coordinate system of (2.14) up to coordinate transformations which affect only \((r, x^\mu)\) or the angular coordinates separately. We want the metric to be in a form where it is manifestly asymptotic to \(AdS_5 \times S^5\). We then identify \(H\) with the metric induced on a nine-dimensional hypersurface close to the boundary and write the counterterm action as

\[
S_{ct} = -\frac{R^3}{\kappa^2} \int d^6 x \sqrt{-H} \left( 3 + \frac{1}{4}(R(H) + \frac{b^2}{R^4}) \right),
\] (4.11)

where \(R(H)\) is the curvature. The first two contributions to this action are equivalent to those found in [3], [3] and [10]. This whole procedure is not very satisfactory in that it is very dependent on fixing the coordinate choice; in this sense, the counterterm action is defined much more naturally by considering the solution in the context of five-dimensional
gauged supergravity instead. Then one has already taken account of integrating out the angular dependence in a natural way by implementing the dimensional reduction ansatz; of course one still has to single out the spherical directions.

The counterterm action (4.11) is also very restrictive in that it applies only to solutions which are of this specific Polchinski-Strassler form. To work out a generally applicable counterterm action involving matter fields would be computationally extremely difficult since one would have to have a much more general ansatz. One can understand this simply from the five-dimensional perspective. Suppose we are in a consistent subsector of the $\mathcal{N} = 8$ supergravity in which only the graviton and some number $k$ of scalar fields are switched on. Then to calculate the divergences in the action we will have to perturbatively solve the coupled Einstein equations and $k$ equations for the scalars, to find (typically) $(k + 1)$ terms in both counterterm action and anomaly. As $k$ increases this process becomes more and more computationally intensive even if $\gamma^0$ is flat. The simplification we have made here corresponds to switching on only a small number of scalars in five dimensions but choosing a particularly interesting combination from the dual perspective.

V. GENERAL ANSATZ AND SL(2,R) INVARINCE

So far we have discussed the special case of the Polchinski-Strassler solution with residual $SO(3)^2$ invariance and the RR scalar vanishing. It turns out to be quite simple to generalise to the case in which the leading order value of $C$ is non-zero and in which we switch on perturbations of the RR scalar.

We already have a clue as to how to construct the most general $SO(3)^2$ invariant solution: as well as switching on a three-form perturbation of the type (3.3) we should also switch on a perturbation of the type (3.4). To consistently solve the field equations we then have to allow for non-zero perturbations of the RR scalar.

Of course another way of constructing the general solution would be to use the action of the $SL(2,R)$ of IIB supergravity. For example, if one uses the S transformation $\tau \rightarrow -1/\tau$ we go from the solution with only $b$ non-zero to the corresponding solution with only $a$ non-zero and interchange the spheres on which $C_2$ and $B_2$ are non-zero. However it turns out to be more illuminating to explicitly construct the more general solution and then interpret the results in terms of the $SL(2,R)$.

In particular, the scalar field perturbations are

$$\Phi = \ln g + \left(\frac{1}{4}(b^2 - a^2)r^2 + \frac{1}{2}(\lambda_b - \lambda_a)a r^4 \ln r\right) \left(\cos^2 \theta - \sin^2 \theta\right) + ...$$

$$C = C^0 - \left(\frac{ab}{2g}r^2 + \frac{1}{2g}(\lambda_a b + \lambda_b a)r^4 \ln r\right) \left(\cos^2 \theta - \sin^2 \theta\right) + ... \quad (5.1)$$

where ellipses denote terms in $r^4$ which do not contribute to the action (or field equations). $C^0$ is the background value for the RR scalar; we have also reinstated factors of $g$ explicitly, though we are still suppressing factors of $R$ for notational simplicity. The resulting solution for the three-form fluxes is the following

$$\bar{F}_3 = -\frac{1}{g} d \left(a + Ar^2 + F_0 \sin^2 \theta r^2 + \lambda_a r^2 \ln r\right) r \sin^3 \theta d\Omega_1$$
The five-form is given by

\[ + \frac{1}{g} d \left( b + Br^2 + F_b \cos^2 \theta r^2 + \lambda_b r^2 \ln r \right) r \cos^3 \theta d\Omega_2 \]

\[ + \frac{ab}{2g} r^2 (\cos^2 \theta - \sin^2 \theta) d \left( b \sin^3 \theta r d\Omega_1 + a \cos^3 \theta r d\Omega_2 \right). \]  (5.2)

\[ H_3 = d \left( b + Br^2 + F_b \sin^2 \theta r^2 + \lambda_b r^2 \ln r \right) r \sin^3 \theta d\Omega_1 \]

\[ + d \left( a + Ar^2 + F_a \cos^2 \theta r^2 + \lambda_a r^2 \ln r \right) r \cos^3 \theta d\Omega_2. \]

In these equations,

\[ F_a = -\frac{1}{2} a (a^2 + b^2); \]

\[ F_b = -\frac{1}{2} b (a^2 + b^2), \]

with the following relations being satisfied

\[ (\square^0 - \frac{1}{6} R^0) b = -2\lambda_b - \frac{1}{6} b (a^2 + b^2); \]  (5.4)

\[ (\square^0 - \frac{1}{6} R^0) a = -2\lambda_a - \frac{1}{6} a (a^2 + b^2). \]

There are hence now four independent fields, \( \{a, b, A, B\} \), rather than two. Note that, since the perturbation to \( C \) is non-zero, \( F_3 \) is no longer exact and hence one needs the non-exact term in (5.2).

Let us consider the first two terms in the \( SL(2, R) \) combination \( \tau = C + i e^{-\Phi} \). Combining \( \beta = b - ia \) then we can write

\[ \tau = (C^0 + \frac{i}{g}) - \frac{\beta^2 r^2}{4g} (\sin^2 \theta - \cos^2 \theta) + .. \]  (5.5)

Thus the \( a \neq 0, C^0 \neq 0 \) solution is related to the special solution we have already constructed by an \( SL(2, R) \) transformation chosen such that

\[ \frac{i}{g} \rightarrow (C^0 + \frac{i}{g}); \]

\[ \frac{b^2}{g} \rightarrow \frac{\beta^2}{g}. \]  (5.6)

The five-form is given by

\[ F_{\Phi \Phi} = \frac{4}{g} \sin^2 \theta \cos^2 \theta \{ 1 - \frac{3(b^2 + a^2)r^2}{8} - \frac{3(Bb + Aa)r^4}{4} + \frac{3f_- r^4}{16} (\cos^2 \theta - \sin^2 \theta) \]

\[ + f_+ r^4 \left( \frac{3}{16} - \sin^2 \theta \cos^2 \theta \right) + (b^2 + a^2)^2 r^4 \left( \frac{9}{32} - \frac{1}{4} \sin^2 \theta \cos^2 \theta \right) \}. \]

\[ \bar{F}_{\tau \tau} = \frac{1}{g} \{ f_- - \lambda_b b + \lambda_a a + f_+ (\cos^2 \theta - \sin^2 \theta) - \lambda_b b C^0 - \lambda_a a C^0 \} r^3 \cos^3 \theta \sin^3 \theta. \]

\[ \bar{F}_{\mu \Omega_4} = \frac{1}{4g} \{ f_- + f_+ (\cos^2 \theta - \sin^2 \theta) \} \mu r^3 \cos^3 \theta \sin^3 \theta. \]  (5.7)

\[ \bar{F}_{\theta \Omega_4} = \frac{1}{g} \{ -f_- + \lambda_b b - \lambda_a a - \lambda_b b C^0 - \lambda_a a C^0 - f_+ (\cos^2 \theta - \sin^2 \theta) + 4f^a \} \cos \theta \sin \theta. \]

\[ \bar{F}_{r \Omega_4} = \frac{1}{g} \left( \frac{4}{r^2} + \frac{(a^2 + b^2)}{12r^2} - \frac{R^0}{3r^3} + \frac{\alpha^{(4)}}{r} \right). \]
Here \( f_\pm \) and \( f_\mp \) are parameters which are now fixed by the field equations in terms of the four independent variables \( \{a, b, A, B\} \).

The key point is that the only place in all of these field perturbations where the background constant \( C^0 \) appears is in the \( F_\mu\Omega_4 \) component and its dual. What this means is that the solutions for the metric and five-form perturbations with \( C^0 \neq 0 \) and \( a, b \neq 0 \) are identical to those already presented, with the replacements in (3.46), (3.47) and (3.48) of

\[
\begin{align*}
\lambda b &\rightarrow \lambda b + \lambda_a a; \\
Bb &\rightarrow (Bb + Aa); \\
b^2 &\rightarrow (b^2 + a^2).
\end{align*}
\] (5.8)

The first place in the metric where the value of \( C^0 \) appears explicitly is in the perturbation \( G_{r\theta} \); this follows from the form of \( \bar{F}_\mu\Omega_4 \). However, neither of these two perturbations will contribute to the divergent terms in the action. So we come to the following conclusion. For the more general three-form perturbation given in (5.2), about a background in which \( C^0 \neq 0 \), the anomaly in the action (4.17) is given by

\[
\begin{align*}
\frac{1}{4}(R_0^{\mu
u}R_0^{\rho\sigma} - \frac{1}{12}(R_0^2)^2) + \frac{3}{8}b(\Box b - \frac{1}{6}R_0^2b) + \frac{1}{24}(b^2 + a^2)^2 + \frac{3}{8}(\Box a - \frac{1}{6}R_0^2a),
\end{align*}
\] (5.9)

whilst the counterterm action should now be taken as

\[
S_{ct} = -\frac{R^3}{\kappa^2} \int d^3x \sqrt{-H} \left( 3 + \frac{1}{4}(R(\gamma) + \frac{(a^2 + b^2)}{R^4}) \right).
\] (5.10)

Note that neither \( C^0 \) nor \( g \) appear explicitly in the anomaly when written in natural dual variables: it is invariant under \( SL(2, R) \) transformations of the UV parameter \( \tau \). As we have seen this more general case corresponds to switching on a complex dual mass term \( m \sim b - ia \) rather than a real mass \( m \sim b \). This will become clearer now as we consider the field theory interpretation. Note that the anomaly depends not just on the magnitude of the mass but also non-trivially on its phase through the derivative terms.

**VI. DUAL FIELD THEORY INTERPRETATION**

Switching on a three-form perturbation corresponds to perturbing the dual \( \mathcal{N} = 4 \) SYM theory with mass terms. In the language of four-dimensional \( \mathcal{N} = 1 \) supersymmetry, the \( \mathcal{N} = 4 \) theory consists of a vector multiplet \( V \) and three chiral multiplets \( \Phi_a \) in the adjoint representation of the gauge group. Normalising the Kähler potential as

\[
K = \frac{2}{\sqrt{2} g_{YM}} \sum_a \text{Tr}(\bar{\Phi}_a \Phi_a),
\] (6.1)

then the theory has interactions summarised in the superpotential

\[
W = \frac{2\sqrt{2}}{\sqrt{2} g_{YM}} \text{Tr}([\Phi_1, \Phi_2]|\Phi_3).
\] (6.2)

Supersymmetry can be partially broken by adding mass terms to the superpotential.
Choosing a basis in which the mass matrix is diagonal with eigenvalues $m_a$, then if two masses are equal and the third is zero the theory has $\mathcal{N} = 2$ supersymmetry; otherwise it is the $\mathcal{N} = 1^*$, in the notation of \cite{27}. When the mass matrix vanishes, there is an $SO(6)$ R-symmetry containing an $SU(3)$ flavour symmetry under which the $\Phi_a$ are triplets. Mass terms generically totally break this symmetry: $m_{ab}$ transforms in the $\bar{6}$ of $SU(3)$.

When the mass matrix is diagonal the F-term equations for a supersymmetric vacuum read

$$[\Phi_a, \Phi_b] = -\frac{m_c}{\sqrt{2}} \epsilon_{abc} \Phi_c.$$  \hfill (6.4)

The classical vacua of this theory were described in \cite{57} whilst the quantum theory was discussed in \cite{58}. Since this was also discussed in detail in \cite{27} we give only a brief description here focusing on issues relevant here. For the purpose of describing the vacua, we can rescale the fields so that all three masses are equal. Then the solutions to these equations are given by $N$-dimensional representations of $SU(2)$, both reducible and irreducible \cite{57}. In the so-called Higgs vacuum the representation is irreducible; there are also Coulomb vacua in which the representation is the product of two or more irreducible representations and there is an unbroken gauge group of at least $U(1)$.

The dual representation of these vacua discussed in \cite{27} is as follows. Let us write the lowest component of $\Phi_a$ as

$$\phi_a = (A_a + i\bar{A}_a),$$  \hfill (6.5)

where $A_a, \bar{A}_a$ are real. Consider a vacuum in which

$$\phi_a = -(\frac{1}{4}\epsilon_{abc}m_b m_c)^{\frac{1}{2}} L_a,$$  \hfill (6.6)

where $L_a$ is an $N$-dimensional representation of $SU(2)$ satisfying the usual commutator relation

$$[L_a, L_b] = \epsilon_{abc} L_c.$$  \hfill (6.7)

One now interprets the scalars $\phi_a$ as the collective coordinates of the bulk D3-branes, normalised as $z^a = 2\pi\alpha'\phi_a$, where $z^a = x^{a+3} + ix^{a+6}$ written in terms of coordinates $x^m$ on $R^6$. Suppose that the $m_a$ are real. Then, for the Higgs phase, it was proposed that the branes should all lie on an ellipsoid in the $(x^4, x^5, x^6)$ directions with the radii of the ellipsoid $r_m$ determined by

$$r_m^2 \equiv x^m x^m \approx \frac{1}{2} \epsilon_{abc} m_b m_c N^2.$$  \hfill (6.8)

When the $m_a$ are complex, the ellipsoid is effectively phase rotated into the $(x^7, x^8, x^9)$ directions. The idea of is that this system should be equivalent to a single D5-brane with worldvolume of $M^4$ times an ellipsoid, with $N$ units of magnetic flux through this ellipsoid.
The more general Coulomb vacua are then represented by a set of D5-branes wrapped around ellipsoids of different radii determined by the dimensions of the irreducible representations contained in $L_a$. When all three mass eigenvalues are equal, the branes should all lie on spheres rather than ellipses. Thus the supergravity solution will have an $SO(3)$ symmetry related to the $SO(3)$ invariance of rotations between chiral superfields.

The supergravity story seems to be more subtle than this, though, since it was found in [35] that the uplift of the GPPZ flow [26] gave rise to a ten-dimensional metric which had an IR limit corresponding either to seven-branes or to five-branes on a ring, depending on the direction in the $S^5$. One only sees five-branes when one approaches along a direction consistent with an IR vacuum.

Now let us consider the perturbation in terms of masses for the four Weyl fermions $\lambda_\alpha$

$$\frac{1}{g_5^2 M} m_{\alpha\beta} \lambda^\alpha \lambda^\beta + \text{h.c.} \quad (6.9)$$

As discussed in [27], a diagonal mass term transforms the same way as an imaginary-self-dual antisymmetric 3-tensor (2.11) whose components are given in terms of the complex coordinates $z^a$ as

$$T = m_1 dz^1 \wedge d\bar{z}^3 + m_2 d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + m_3 d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + m_4 d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3. \quad (6.10)$$

The complex coordinates should be related to the coordinate system of (2.9) as

$$z^1 = x^4 + ix^7 = \frac{1}{r} (\cos \theta \cos \chi + i \sin \theta \cos \phi);$$
$$z^2 = x^5 + ix^8 = \frac{1}{r} (\cos \theta \sin \chi \cos \psi_2 + i \sin \theta \sin \phi \cos \psi_1);$$
$$z^3 = x^6 + ix^9 = \frac{1}{r} (\cos \theta \sin \chi \sin \psi_2 + i \sin \theta \sin \phi \sin \psi_1). \quad (6.11)$$

Then we define the form

$$S_2 = \frac{1}{2} T_{mnp} x^m dx^n dx^p, \quad (6.12)$$

using the components of $T$. The leading order behaviour of the perturbations of the bulk three-form $G_3$ defined in (3.5) lie in the $\text{Tr}$ of $SO(6)$. The non-normalisable modes are

$$G_3 \sim d(\frac{R^4 r^4}{g} S_2), \quad (6.13)$$

which is of order $r$ whilst the normalisable modes are

$$G_3 \sim d(\frac{R^6 r^6}{g} S_2), \quad (6.14)$$

which is of order $r^3$. Thus terms in $b$ in the bulk solution correspond to mass perturbations in the field theory whilst the terms in $B$ correspond to inducing vacuum expectation values of the fermion bilinears $\bar{\lambda}\lambda$.  

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When all four masses are equal and non-vanishing, the $SO(4)$ symmetry of the fermionic Lagrangian is manifest in the perturbation (6.9); the Lagrangian is invariant under rotations of the $\lambda^\alpha$ which preserve $\lambda^\alpha \lambda^\alpha$. It is convenient to rewrite (6.10) in terms of the real coordinates for which

$$T = 4m(dx^4 \wedge dx^5 \wedge dx^6 - idx^7 \wedge dx^8 \wedge dx^9), \quad (6.15)$$

which is manifestly $[SO(3)]^2$ symmetric. Furthermore, we can bring $T$ into the form

$$T = 4md\left(\frac{1}{r^3} \cos^3 \theta d\Omega_2 - i \frac{1}{r^3} \sin^3 \theta d\Omega_1\right), \quad (6.16)$$

which from (6.13) and (6.14) gives rise to the form for $G_3$ which we have been using. To fix the normalisation between $T$ and the three-form perturbations in the bulk we can consider probes moving in the perturbed background, which we will do in the following section. First however we should consider a little more carefully what the Lagrangian and the vacua of this theory are.

When we switch on a mass for the gluino, there is no longer a supersymmetric completion of the Lagrangian (beyond terms linear in the mass [27]). So we need to add “by hand” quadratic mass terms for the scalars. We want to do this in such a way as to obtain maximal symmetry of the Lagrangian when $m_4 = m$ with the linear terms fixed by the fermionic mass perturbation. This effectively fixes the quadratic mass terms for the scalars: the scalar part of the potential is of the form

$$V \propto \frac{1}{g^2_{YM}} \text{Tr} \left( \sum_{a<b} |\phi_a, \phi_b| + \frac{m}{\sqrt{2}} \epsilon_{abc} \phi_c |^2 + \frac{2}{3\sqrt{2}} \epsilon_{abc} \text{Re}(m_4 \phi_a \phi_b \phi_c) + \sum_c \frac{|m_4|^2}{6} |\phi_a|^2 \right), \quad (6.17)$$

where $m_4$ is the gluino mass. Let us again assume that the masses are real; then when the gluino mass is equal to the other mass scale we can conveniently write the potential in the form

$$V \propto \frac{1}{g^2_{YM}} \text{Tr} \left( \sum_{a<b} ([A_a, A_b]^2 + [A_a, \bar{A}_b]^2 + [\bar{A}_a, A_b]^2 + [\bar{A}_a, \bar{A}_b]^2) \right) + \frac{8m}{3\sqrt{2}} \epsilon_{abc} A_a A_b A_c + \sum_a \frac{2m^2}{3} (\bar{A}_a^2 + A_a^2) \right). \quad (6.18)$$

This manifestly displays an $SO(3)^2$ symmetry corresponding to flavour rotations of the $(A_a, \bar{A}_a)$. The classical potential (6.18) is extremised when the following equations are satisfied:

$$2[A_b, [\bar{A}_a, A_b]] + 2[\bar{A}_b, [\bar{A}_a, \bar{A}_b]] + \frac{4m^2}{3} \bar{A}_a = 0;$$

$$2[A_b, [A_a, A_b]] + 2[\bar{A}_b, [A_a, \bar{A}_b]] + 4\sqrt{2}m \epsilon_{abc} A_a A_c + \frac{4m^2}{3} A_a = 0, \quad (6.19)$$

where summation over repeated indices is implicit. In the trivial vacuum in which $A_a = \bar{A}_a = 0$ the potential vanishes. We can also find solutions for which $A_a = 0$ and for which
\[ A_a = -\lambda L_a, \]  
with \( L_a \) defined in (6.7). Then (6.19) and (6.18) fix
\[ \lambda_\pm = \left( \frac{m}{\sqrt{2}} \pm \frac{\sqrt{2}m}{3} \right); \]  
\[ V_\pm = \pm \frac{2m^2}{9} \text{Tr}(L_a L_a). \]

Now for a generic \( N \)-dimensional representation which is reducible into \( k \) irreducible representations of dimension \( n_k \) such that \( N = \sum_k n_k \)
\[ \text{Tr}(L_a L_a) = \sum_k \frac{1}{4} n_k (n_k^2 - 1). \]  
This is manifestly maximised when there is a single partition, that is, the \( N \)-dimensional representation is irreducible. So classically the favoured configuration seems to be a Higgs phase, in which all \( N \) of the branes lie on a sphere of radius \( \pi \alpha' \lambda_+ N \) where we have again identified the scalars \( A_a \) with the locations of the branes \( x^m \).

From the form of the potential (6.17) we can show that classical vacua in which
\[ [\phi_a, \phi_b] = -\lambda \epsilon_{abc} \phi_c, \]  
persist in the sense that the potential is minimised at finite \( \lambda \) provided that \( |m_4| \) is less than about \( 3m \).

VII. PROBE POTENTIAL

In this section we will consider five-brane probes moving in the background which we have constructed perturbatively. If one retains only first order corrections to the fields, this is equivalent to considering probes in the linearised solution as in [14]. Let us consider probe D5-branes with worldvolume \( M^4 \times S^2 \) and D3-brane charge \( n \ll N \) with \( n \gg \sqrt{gN} \). For simplicity we will consider only D5-brane probes within the background of \( \gamma^0 \) flat, \( C = 0 \), \( a = 0 \) and \( b \) a (real) constant. The relevant terms in the action for the D5-branes are
\[ S = -\frac{\mu_5}{g} \int d^6 \xi \left[ -\det(G_{M^4}) \det(g^{-\frac{1}{2}} e^\frac{\Phi}{2} G_{S^2} + 2\pi \alpha' F) \right]^\frac{1}{2} \]  
\[ +\mu_5 \int (C_6 + 2\pi \alpha' F_2 \wedge C_4), \]  
where
\[ 2\pi \alpha' F_2 = 2\pi \alpha' F_2 - B_2. \]  
The dilaton factors follow from the fact that we are in the Einstein as opposed to the string frame metric. \( G_{M^4} \) is the induced metric in the \( M^4 \) directions of the worldvolume and \( G_{S^2} \) is the induced metrics on the two sphere. We will choose to identify the worldvolume coordinates \( \xi^\mu \) with the bulk coordinates \( x^\mu \) with \( \mu = (0, 3) \).
The bulk potential $C_6$ is defined by the equation

$$dC_6 - H_3 \wedge C_4 = -g^{-1} e^\Phi \ast F_3.$$  \hspace{1cm} (7.3)

This definition follows from the field equations (2.4) as well as from the D-brane action (7.1): we require that both the action and the field strength $F_3$ are invariant under gauge transformations of the type $\delta C_4 = d\chi$ with $\delta C_6 = -H_3 \wedge \chi$ and that the field equations (2.4) are satisfied. In our solution the potential $C_6$ is hence given by

$$C_6 = \left[-\frac{2bR^4}{3gr^3} - \frac{11b^3}{24rR^2} + \frac{b^3}{2rR^2} \sin^2 \theta + \ldots\right] \sin^3 \theta d\eta_4 \wedge d\Omega_1,$$  \hspace{1cm} (7.4)

where the ellipses denote terms which are finite as $r \to 0$. Note that the second parameter in the solution, $B$, does not appear in this potential.

We have fixed four of the collective coordinates of the D5-brane; let us denote the remaining six by the (complex) fields $z^a$, with $a = (1,3)$. Then we should take the brane configuration to be

$$z^a = z e^a, \quad e^a e^a = 1,$$  \hspace{1cm} (7.5)

where $e^a$ is real and parametrises the $S^2$; let us suppose that worldvolume $S^2$ coordinates are $(\vartheta, \phi)$. We should then identify $z^a$ with the bulk coordinates on the $R^6$ as in (5.11). Note that with this identification we have $|z| = R^2 r^{-1}$. As mentioned above, we want the D5-brane to have a D3-brane charge of magnitude $n$. Hence we need to take a worldvolume flux of the form

$$F_2 = -\frac{1}{2} n \sin \vartheta d\vartheta \wedge d\phi.$$  \hspace{1cm} (7.6)

The minus sign here is necessary for a D3-brane probe to remain static since with our choice of $F_5$ the bulk solution has negative 3-brane charge. The solutions for the bulk fields summarised in (3.49) along with the definition of the embedding of the brane into the bulk give us all the ingredients necessary to expand the action (7.1) as

$$S = -\frac{\mu_5}{g} \int d^4x \left( \frac{2R^8}{n\alpha' r^4} + \frac{8\pi bR^4}{3r^3} \sin^3 \theta + \frac{2\pi^2 n b^2 \alpha'}{3r^2} \right.$$  \hspace{1cm} (7.7)

$$+ \frac{\pi b^3}{r} \sin^3 \theta \left( \frac{5}{3} - 2 \sin^2 \theta \right) + \frac{\pi^2 b^4 n \alpha'}{12 R^4} \ln(r/R) + \ldots \right),$$

where we have integrated out the worldvolume angular dependence and the ellipses denote terms which are finite or tend to zero as $r \to 0$. We have used the conditions that $n \ll N$ and $n\alpha' \gg R^2$ to retain only the leading order coefficient at each power of $r$. At leading order there is a large cancellation between the Born-Infeld and Chern-Simons terms, since a D3-brane probe feels no force from D3-branes. The perturbations of $B_2$ and $\phi$ contribute at a subleading level to the action and only affect terms of order $1/r$ or smaller. Notice that the second parameter in the bulk solution, $B$, does not appear in this action at all.

We would now like to match this action to the potentials discussed in the previous section. To do this, we should first introduce a new radial coordinate $\rho$ such that
\begin{align*}
\rho &= \frac{1}{r} \left[ 1 + \frac{\tilde{b}^2 r^2}{24} (5 - 6 \sin^2 \theta) + \frac{\pi^2 \tilde{b}^4 r^4 \ln(r)}{96 g_{YM}^2 N} + \ldots \right], \tag{7.8}
\end{align*}

where we have switched to the natural gauge theory parameters, \( \tilde{b} \), introduced previously, and \( g_{YM} \); ellipses denote terms which are finite or zero as \( r \to 0 \). This rescaling absorbs the last two divergent terms in (7.7). Both (7.8) and (7.7) are applicable only when \( r \tilde{b} < 1 \) so that our perturbative construction of the bulk solution is valid.

Now we should relate the gauge theory scalar \( \phi \) to the D-brane collective coordinates as

\begin{equation}
\phi = \sqrt{g_{YM}^2 N / 2 \pi} \rho e^{i \theta}. \tag{7.9}
\end{equation}

With this identification, the probe action becomes

\begin{equation}
S = -\frac{16}{n g_{YM}^2} \int d^4 x \left[ |\phi|^4 - \frac{\tilde{b} n}{6 \sqrt{2}} (3 |\phi|^2 \text{Im}(\phi) + \text{Im}(\phi)^3) + \frac{\tilde{b}^2 n^2}{24} |\phi|^2 \right], \tag{7.10}
\end{equation}

where we have used \( \mu_5^{-1} = (2\pi)^5 \alpha'^3 \). To make a comparison with the field theory, we should return to the potential (6.17). Let us make the matrix scalar fields to be of the form

\begin{equation}
\phi_a = -\frac{2i}{n} \phi L_a, \tag{7.11}
\end{equation}

where \( \phi \) is a scalar and \( L_a \) is the \( n \)-dimensional irreducible representation of \( SU(2) \). Substituting into (6.17) and making use of (6.22) the scalar potential is

\begin{equation}
\frac{4}{n g_{YM}^2} \left( |\phi|^4 + \frac{n}{3 \sqrt{2}} (3 |\phi|^2 \text{Im}(m\phi) + \text{Im}(m\phi^3)) + \frac{n^2}{6} |m|^2 |\phi|^2 \right), \tag{7.12}
\end{equation}

which agrees with the form (7.10) when \( m \) is real. It was mentioned in [27] that there is an ambiguity in the quadratic term of this potential: we can add another term corresponding to traceless combination of masses for the scalar bilinears. We chose this to vanish in (6.17) but even if we had not we could have absorbed it in the matching between gauge theory scalar and collective coordinate. Comparison with (7.10) then gives the identification

\begin{equation}
m \equiv -\frac{\tilde{b}}{2}, \tag{7.13}
\end{equation}

which means that the anomaly from the mass perturbation is

\begin{equation}
S = -\frac{N^2}{8 \pi^2} \int d^4 x \sqrt{-\gamma} \left( \frac{3}{2} m (\Box m - \frac{1}{6} R^0 m) + \frac{2}{3} m^4 \right). \tag{7.14}
\end{equation}

We will interpret this in the next section. Now returning to the probe action (7.10) we see that in addition to the extremum at \( \phi = 0 \) there will also be an extremum when

\begin{equation}
\phi = \phi_e = \frac{in\tilde{b}}{2\sqrt{24}} (\sqrt{3} + 1), \tag{7.15}
\end{equation}

which is real.
which is at an absolute minimum of the potential. Thus it seems that is energetically
favourable for the D5-brane to sit at finite radius in the bulk. To check this we need to match
the gauge theory scalar back to the collective D-brane coordinate and consider carefully the
range of validity of our bulk solution. From (7.8), (7.15) and (7.9) the coordinate position
of the minima is at
\[ \tilde{br}_e \approx \sqrt{\frac{gN}{n}} \ll 1, \]  
(7.16)
at which point our solution is still valid. More generally we could consider \((c, d)\) probes and
find their minima: for example, we expect that an NS5-brane should sit at \(\theta = 0\) at a radius
approximately \(g\) smaller than the D5-brane minima.

VIII. CONFORMAL ANOMALIES IN THE FIELD THEORY

The addition of mass terms breaks conformal invariance and so the stress energy tensor
is not traceless even classically. For example if we consider free massive scalar fields in a flat
background:
\[ S = \int d^4x \sqrt{\eta} ((\partial \phi)^2 + m^2 \phi^2), \]  
(8.1)
then the action is manifestly not invariant under conformal transformations of the metric
\(\eta_{\mu\nu} \to \Omega^2 \eta_{\mu\nu}\). On dimensional grounds the anomalous scaling of the action under conformal
transformations must be
\[ S \sim m^4 \ln \epsilon \int d^4x, \]  
(8.2)
where \(\epsilon\) has dimensions of length. Now let us consider the coefficient in the mass perturbed
YM theory. As usual calculation of the coefficient in the regime of large t'Hooft coupling
where the supergravity calculations are valid is inaccessible. However, it is straightforward to
calculate the coefficients in the weak coupling (free field) regime. Naively there is no reason
why the weak and strong coupling coefficients should agree. However, like the curvature
part of the conformal anomaly \([1]\), the mass anomaly term depends only on \(N\) and the UV
mass parameter \(m\) and does not involve the coupling explicitly. This is indicative that the
coefficient may be reproduced by a weak coupling calculation and is not renormalised.

So let us now determine the coefficient for massive multiplets in the adjoint representation
of \(SU(N)\) in the zero coupling limit in a flat background. To do this, we can use zeta function
regularisation to calculate the action for free massive scalars and Weyl spinors. Both scalar
and spinor results are well-known and can be determined very easily. For example, for
the scalars, the eigenvalues of the differential operator associated with (8.1) (in Euclidean
spacetime) are
\[ \lambda_k = k^2 - m^2. \]  
(8.3)
Then if we define the generalised zeta function as \(\zeta(s) = \sum_k (\lambda_k)^{-s}\) the anomalous scaling
of the associated Euclidean action is given by
\[ S_{\text{anomalous}} = -\ln(\epsilon)\zeta(0), \quad (8.4) \]

where \( \epsilon \) is the scale. Here we have

\[ \zeta(s) = \frac{V}{(2\pi)^s} \int d^4k (k^2 - m^2)^{-s} = \frac{V m^4}{(2\pi)^s} \int d^4u (u^2 - 1^2)^{-s}, \quad (8.5) \]

where \( V \) is the volume of the spacetime, given by the regularisation of \( \int d^4x \sqrt{\eta} \), and in the second inequality we explicitly demonstrate the dimensions. Carrying out the integral above then gives the anomalous part of the scalar action to be

\[ S_{\text{scalar}} = -\frac{m^4}{16\pi^2} \ln \epsilon \int d^4x, \quad (8.6) \]

for a free (complex) scalar. This is in agreement with the result obtained in [60] and [61]. Both the scalar result and the anomalous part of the action for Weyl spinors

\[ S_{\text{spinor}} = -\frac{m^4}{16\pi^2} \ln \epsilon \int d^4x, \quad (8.7) \]

can be found in the review article [62].

For the \( \mathcal{N} = 0 \) theory with all four fermionic masses equal to \( m \), the three complex scalars acquire masses of \( \sqrt{4m^2/3} \): this follows from the potential (6.17). Since the fields transform in the adjoint representation of \( SU(N) \), in the free field limit the anomalous part of the action is given by

\[ S_{\text{anomalous}} = (N^2 - 1)/(S_{\text{scalar}} + S_{\text{spinor}}); \]

\[ = (N^2 - 1)(-\frac{m^4}{3\pi^2} + \frac{m^4}{4\pi^2}) \ln \epsilon \int d^4x = -\frac{m(N^2 - 1)}{12\pi^2} \ln \epsilon \int d^4x, \quad (8.8) \]

which is in agreement with (7.14) in the large \( N \) limit! Given this agreement one would expect that a calculation along the lines of [63], [64], [65] and [66] would also reproduce the mixed parts of the anomaly in (7.14) for a position dependent mass term.

In the \( \mathcal{N} = 4 \) theory, coefficient of the gravitational conformal anomaly is not renormalised; agreement between strong and weak coupling results is an indication of the existence of the non-renormalisation theorems discussed in [67], [68], [69], [70]. Here however the theory is no longer supersymmetric or conformal even classically and so we would not expect such theorems to go through. Since conformal anomalies are intimately connected with UV renormalisation and our theory flows to the \( \mathcal{N} = 4 \) theory in the UV, the non-renormalisation of the mass anomaly must be inherited from the conformal theory. We leave as an open question the issue of making this statement more precise.

**IX. GENERAL COMMENTS ON POLCHINISKI-STRASSLER SOLUTIONS**

In this section we will make a number of comments about the full form for this Polchinski-Strassler solution and about the general solutions with unequal fermion masses.

The range of validity of the solution which we have constructed is \( br \ll 1 \); as we go further into the interior the perturbative series breaks down. To probe the IR physics in
the field theory, we would have to construct the full bulk solution or at least match our asymptotic solution to some interior near-brane solution. Obviously to find the IR structure of our solution would be very interesting. Since the bulk solution has $SO(3)^2$ symmetry, all fields depend only on two coordinates and it may be tractable to construct the full solution in ten dimensions, using the asymptotic solution for guidance. For example, the two-forms must be of the form

$$B_2 = B(r, \theta) d\Omega_1; \quad C_2 = C(r, \theta) d\Omega_2,$$

and the asymptotic solution will guide us as to appropriate ansätze for the other fields. However, since no supersymmetry is preserved we would have to solve the full set of second-order equations which would be challenging.

Another way to obtain insight about the full solution would be to try a matching to a near shell metric along the lines of that in [27]. Consider the near shell solution obtained by simply replacing the harmonic function in (2.7) with that for 3-branes distributed on a shell of radius $w$

$$H = \frac{R^4}{(x^2 + 2xw \cos \tilde{\theta} + w^2)(x^2 - 2xw \cos \tilde{\theta} + w^2)},$$

where $x$ is the AdS radius and $\tilde{\theta}$ is an angle such that the branes are located on the sphere at $\tilde{\theta} = 0$. This metric cannot be matched even to first order to the asymptotic solution we have constructed here, by any matching between $(x, \tilde{\theta})$ and $(r, \theta)$, and is hence not the appropriate near-shell solution.

This is understandable from the supergravity point of view in that as we flow into the interior the 3-form perturbations become significant and so we do not expect the leading order metric to be the same as in the $G_3 = 0$ case even if the correct interpretation is in terms of 3-branes distributed on a sphere. In fact given the work of [35] we should not expect that the IR solution looks like five-branes wrapped on a sphere. It is likely that the IR physics is far more subtle: the supergravity approximation probably fails at some scale short of the infra-red. It is also quite possible that the flow runs into a “brick wall” as in [35].

Since there is a high degree of symmetry in our solution, it might be profitable to try and construct the full solution in five dimensions and then uplift it. It is somewhat ironic, given the motivation of this paper, that in the example specifically considered here it is probably easier to work with the five-dimensional fields. Unfortunately since there is no supersymmetry we will have to use the Einstein equations rather than the by now familiar first order equations for flows involving the superpotential [51]. However, this is probably still the most tractable method of obtaining the full solution, as we will only need to switch on a small subset of the scalars and the graviton in five dimensions.

This follows from restricting to the $[SO(3)]^2$ invariant sector of the scalar manifold of gauged $\mathcal{N} = 8$ supergravity in five dimensions: the resulting manifold will be only four-dimensional. The four scalars are dual to the gauge coupling, the theta angle and a single complex fermion bilinear

$$\mathcal{O} = \sum_{\alpha=1}^{4} \text{Tr}(\lambda_{\alpha} \lambda_{\alpha}).$$
Following [26] and [40] it is probably consistent to keep only the latter. Then if $\sigma$ is the corresponding supergravity scalar the scalar potential will be of the form

$$V = -\frac{2}{R^2} \left( 4 + e^{-\sigma} + e^{\sigma} \right), \quad (9.4)$$

where $\sigma$ is canonically normalised. This form for the potential follows from restricting the $SO(3)$ invariant potentials of [40], [71] further. The only critical point of this potential is at $\sigma = 0$, corresponding to the UV conformal fixed point: we expect this since there is no critical point preserving $SO(3)^2$ invariance [71]. Thus the $\mathcal{N} = 0$ flow is a flow to Hades in the sense of [71] and both the five and the ten dimensional solutions are going to be singular in the IR. With a one-dimensional potential solving for five-dimensional flows may be tractable. Even more subtle, however, will be uplifting the solution to ten dimensions and correctly matching higher and lower dimensional fields. We hope to report on this elsewhere.

Our second group of comments relate to switching on general fermion masses, discussed in the context of five-dimensional flows in [37]. An interesting insight into the structure of the unequal mass solutions comes from considering the linearised dilaton perturbations induced by switching on the three-forms. These are particularly easy to construct since they are independent of the linearised metric perturbations. Let us assume that the masses are real so as to prevent source terms for the RR scalar field. Then the dilaton equation of motion is just

$$D^M \partial_M \Phi = \frac{g^2}{24} \left[ G_{mnp} G^{mnp} + c.c. \right]. \quad (9.5)$$

Using the explicit forms for $G$ from (6.10), (6.12) and (6.13), we find the following simple result

$$D^M \partial_M \Phi = 8r^2(m_1 m_4 + m_2 m_3)(\sin^2 \theta - \cos^2 \theta), \quad (9.6)$$

which leads to the linearised solution

$$\Phi = \ln g + \frac{1}{2}(m_1 m_4 + m_2 m_3)r^2(\cos^2 \theta - \sin^2 \theta). \quad (9.7)$$

There are various observations to make about this. Firstly, the masses do not appear symmetrically as one might have naively expected. The masses will also not appear symmetrically in either the counterterms or the anomaly. The anomaly in fact vanishes except when $m_4 \neq 0$. We can prove this from both the field theory and the supergravity perspective. From the field theory point of view, if the anomaly is given by that in the free field limit, then it must vanish unless $m_4 \neq 0$ since the contribution from each massive chiral superfield is zero from (8.6) and (8.7).

However, the supergravity result can also be derived from the work of [12] and [26]. A result given in [12] is that there is no anomaly if the scalar potential for a canonically normalised scalar $\varphi$ satisfies the following conditions at the critical point $\varphi = 0$

$$V = -\frac{12}{R^2}; \quad \frac{\partial^2 V}{\partial^2 \varphi} = -\frac{8}{R^2}, \quad (9.8)$$

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The GPPZ potential which corresponds to equal masses for three fermions (and zero gluino condensate) indeed satisfies this condition and so the anomaly vanishes. (Note that the potential (9.4) does not satisfy this condition permitting a non-zero anomaly.) Hence we have further evidence for non-renormalisation of the mass anomaly.

Secondly, although in general the symmetry group is totally broken down, when the masses are real the first correction to the dilaton depends only on the angle $\theta$. When the masses are complex, the dilaton depends on more angular coordinates. Also second order corrections which become significant as one flows further into the interior probably depend on more of the $S^5$ coordinates, though we would have to go beyond the linearised level to show this.

Thirdly, if we switch off the mass perturbations but retain expectation values for the fermion bilinears the leading order corrections to the dilaton vanish. This is apparent from the form of (6.14) and in particular using (3.49). From the supergravity point of view this is not obvious a priori but in the field theory we do not expect the coupling to run unless masses are switched on [40], [52]. The vacuum expectation value is not expected to affect the UV physics, as we have found here, but to crucially determine the IR behaviour.

Finally, we come to the most significant point: the coordinate dependence of the dilaton (or more generally the complex coupling $\tau$). The flow of the dilaton should be interpreted in terms of the scale dependence of the field theory gauge coupling. However, this interpretation is not easy to make precise. Non-trivial operator mixings may cause the dilaton to become some other coupling of the effective action as one flows towards the IR. Also the dilaton does not depend just on the UV mass scales and the radius, to be interpreted as the energy scale in the dual theory; it also depends upon the bulk angular coordinates. This was commented on in [44], [35] and was also discussed in the context of $\mathcal{N} = 2$ flows in [40]. The flow knows about which directions are getting massive; however, to make sense of this we would need to understand how this direction is encapsulated in the effective action and how it is seen from the brane perspective.

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REFERENCES

[1] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.
[2] S. Gubser, I. Klebanov and A. Polyakov, Phys. Lett. B428 (1998) 105, hep-th/9802109.
[3] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.
[4] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. 323 (2000) 183, hep-th/9905111.
[5] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208 (1999) 413, hep-th/9902123.
[6] P. Kraus, F. Larsen and R. Siebelink, Nucl. Phys. B563 (1999) 259, hep-th/9901017.
[7] G. Gibbons and S. Hawking, Phys. Rev. D15 (1977) 2752.
[8] H. Liu and A. Tseytlin, Nucl. Phys. B533 (1998) 88, hep-th/9804083.
[9] M. Henningson and K. Skenderis, JHEP 9807 (1998) 023, hep-th/9806087, hep-th/9812032.
[10] R. Emparan, C. Johnson and R. Myers, Phys. Rev. D60 (1999) 104001, hep-th/9903238.
[11] M. Taylor-Robinson, hep-th/0001177.
[12] M. Taylor-Robinson, hep-th/0002125.
[13] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, hep-th/9803131.
[14] J. Russo, Nucl. Phys. B543 (1999) 183, hep-th/9808117.
[15] C. Csaki, Y. Oz, J. Russo and J. Terning, Phys. Rev. D59 (1999) 065012, hep-th/9810186.
[16] J. G. Russo and K. Sfetsos, Adv. Theor. Math. Phys. 3 (1999) 131, hep-th/9901050.
[17] M. Cvetic and S. S. Gubser, JHEP 9904 (1999) 024, hep-th/9902195.
[18] M. Cvetic and S. S. Gubser, JHEP 9907 (1999) 010, hep-th/9903132.
[19] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B546 (1999) 155, hep-th/9811035.
[20] J. A. Minahan, JHEP 9901 (1999) 020, hep-th/9811156.
[21] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B547 (1999) 143, hep-th/9812089.
[22] J. A. Minahan, JHEP 9904 (1999) 007, hep-th/9902074.
[23] A. Kehagias and K. Sfetsos, Phys. Lett. B454 (1999) 270, hep-th/990225.
[24] S. S. Gubser, hep-th/9902153.
[25] A. Kehagias and K. Sfetsos, Phys. Lett. B456 (1999) 22, hep-th/9903109.
[26] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, Nucl. Phys. B 569 (2000) 451, hep-th/9909047.
[27] J. Polchinski and M. Strassler, hep-th/0003136.
[28] I. Klebanov and E. Witten, Nucl. Phys. B 536 (1998) 199, hep-th/9807058.
[29] I. Klebanov and N. Nekrasov, Nucl. Phys. B574 (2000) 263, hep-th/9911096.
[30] I. Klebanov and A. Tseytlin, Nucl. Phys. B578 (2000) 123, hep-th/0002159.
[31] I. Klebanov and M. Strassler, JHEP 0008 (2000) 052, hep-th/0007191.
[32] L. Pando Zayas and A. Tseytlin, JHEP 0011 (2000) 028, hep-th/0010088.
[33] L. Pando Zayas and A. Tseytlin, hep-th/0101043.
[34] A. Buchel, C. Herzog, I. Klebanov, L. Pando Zayas and A. Tseytlin, hep-th/0102103.
[35] K. Pilch and N. Warner, hep-th/0006060.
[36] N. Warner, hep-th/0011207.
[37] A. Khavaev and N. Warner, Phys. Lett. B495 (2000) 215, hep-th/0009159.
[38] R. Myers, JHEP 9912 (1999) 022, hep-th/9910053.
[39] H. Nastase and D. Vaman, Nucl. Phys. B 583 (2000) 211, hep-th/0002028.
[40] K. Pilch and N. Warner, Nucl. Phys. B594 (2001) 209, hep-th/0004063.
[41] K. Pilch and N. Warner, Phys. Lett. B487B (2000) 22, hep-th/0002192.
[42] M. Cvetic, H. Lu, C. Pope, A. Sadrzadeh and T. Tran, Phys. Rev. D62 (2000) 064028, hep-th/0003103.
[43] M. Grana and J. Polchinski, Phys. Rev. D63 (2001) 026001, hep-th/0009211.
[44] D. Freedman and J. Minahan, JHEP 0101 (2001) 036, hep-th/0007250.
[45] E. Bergshoeff, H. Boonstra and T. Ortin, Phys. Rev. D53 (1996) 7206, hep-th/9508091.
[46] J. Schwarz, Nucl. Phys. B226 (1983) 269.
[47] P. Howe and P. West, Nucl. Phys. B238 (1984) 181.
[48] C. Graham and J. Lee, Adv. Math. 87 (1991) 186.
[49] C. Fefferman and C. Graham, “Conformal Invariants”, in Elie Cartan et les Mathématiques d ’aujourd’hui (Astérisque, 1985) 95.
[50] D. Freedman, S. Gubser, K. Pilch and N. Warner, JHEP 0007 (2000) 38, hep-th/9906194.
[51] D. Freedman, S. Gubser, K. Pilch and N. Warner, hep-th/9904017; JHEP 0007 (2000) 023, hep-th/9906194.
[52] M. Petrini and A. Zaffaroni, hep-th/0002172.
[53] R. Penrose and W. Rindler, Spinors and Spacetime, volume 2, chapter 9 (Cambridge University Press, Cambridge, 1986).
[54] M. Henneaux and C. Teitelboim, Phys. Lett. B142 (1984) 355.
[55] S. Hawking, Phys. Lett. B126 (1983) 175.
[56] S. de Haro, K. Skenderis and S. Solodukhin, hep-th/0002230.
[57] C. Vafa and E. Witten, Nucl. Phys. B431 (1994) 3, hep-th/9408074.
[58] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299, hep-th/9510101.
[59] D. Kabat and W. Taylor, Phys. Lett. B426 (1998) 297, hep-th/9712183.
[60] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888.
[61] J. Iliopoulos, C. Itzykson and A. Martin, Rev. Mod. Phys. 47 (1975) 165.
[62] A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Rept. 266 (1996) 1, hep-th/9505061.
[63] S. Coleman and R. Jackiw, Ann. Phys. 67 (1971) 552.
[64] M. Chanowitz and J. Ellis, Phys. Rev. D7 (1973) 2490.
[65] S. Deser, M. Duff and C. Isham, Nucl. Phys. B111 (1976) 45.
[66] J. Collins, M. Duncan and S. Joglekar, Phys. Rev. D16 (1977) 438.
[67] D. Anselmi, D. Freedman, M. Grisaru and A. Johansen, Nucl. Phys. B 526 (1998) 543, hep-th/9708042; D. Anselmi, J. Erlich, D. Freedman and A. Johansen, Phys. Rev. D57 (1998) 7570, hep-th/9711033.
[68] S. Gubser and I. Klebanov, Phys. Lett. B413 (1997) 41, hep-th/9708005.
[69] P. Howe, E. Sokatchev and P. West, Phys. Lett. B444 (1998) 341, hep-th/9808162.
[70] A. Petkou and K. Skenderis, Nucl. Phys. B561 (1999) 100, hep-th/9906030.
[71] A. Khavaev, K. Pilch and N. Warner, Phys. Lett. B487 (2000) 14, hep-th/9812035.