Correspondence between conformal field theory and Calogero-Sutherland model

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Abstract

We use the Jack symmetric functions as a basis of the Fock space, and study the action of the Virasoro generators $L_n$. We calculate explicitly the matrix elements of $L_n$ with respect to the Jack-basis. A combinatorial procedure which produces these matrix elements is conjectured. As a limiting case of the formula, we obtain a Pieri-type formula which represents a product of a power sum and a Jack symmetric function as a sum of Jack symmetric functions. Also, a similar expansion was found for the case when we differentiate the Jack symmetric functions with respect to power sums. As an application of our Jack-basis representation, a new diagrammatic interpretation is presented, why the singular vectors of the Virasoro algebra are proportional to the Jack symmetric functions with rectangular diagrams. We also propose a natural normalization of the singular vectors in the Verma module, and determine the coefficients which appear after bosonization in front of the Jack symmetric functions.
1 Introduction and summary

In this paper, we revisit the relationship between two theories of quantum integrable systems, that is, the Virasoro algebra and the Calogero-Sutherland model.

The Virasoro algebra generated by $L_n$ ($n \in \mathbb{Z}$) and the center $c$ with the commutation relations

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{n(n^2 - 1)}{12} c \delta_{n+m,0},$$

was first introduced in the context of the string theory in 1970 \cite{1} and played a prominent role in the conformal field theory initiated by Belavin, Polyakov and Zamolodchikov in 1984 \cite{2}. If we bosonize the Virasoro algebra by the Feigin-Fuchs representation, we regard the Virasoro generators $L_n$’s as operators acting on the bosonic Fock space. This bosonization technique has been proved to be an extremely powerful tool to investigate the Virasoro algebra in many situations. This is mainly because we are able to construct many intertwining operators in terms of the primary fields and the screening charges. This is in some sense because we have a good control of the singular or co-singular vectors in the bosonic Fock space. However, we still do not fully understand the precise structures of the singular vectors of the Virasoro algebra constructed in the Verma module. Namely, explicit formulas for the singular vectors are not very well investigated before bosonization. Therefore, it is desired to construct a method which generates the Virasoro singular vectors in some systematic manner.

The Calogero-Sutherland model (CS model) also has its own background. In 1969, Calogero introduced a dynamical system with inverse square repulsive potential $U = 1/r^2$ and revealed that the model is exactly solvable \cite{3,4,5}. In 1972, Sutherland extended this model to a periodic potential case and obtained several exact results (\cite{6,7}, see also \cite{8}). In 1990’s, Ujino, Hikami and Wadati \cite{9,10} investigated the complete quantum integrability of these systems. The Calogero-Sutherland Hamiltonian is

$$H_{CS} = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \beta(\beta - 1) \sum_{i < j} \frac{(\pi L)^2}{\sin^2 \frac{\pi}{L} (q_i - q_j)},$$

where $q_i (0 \leq q_i \leq L)$ are the coordinates and $p_i = -\sqrt{-1} \frac{\partial}{\partial q_i}$ are the momenta. The Calogero-Sutherland model is a quantum integrable system in the following sense. First, the exact ground state of $H_{CS}$ has a simple factorized form

$$\psi_0 = \prod_{i < j} \sin \frac{\pi}{L} (q_i - q_j),$$

and all the excited states can be given by multiplying certain symmetric polynomials to $\psi_0$ as

$$\psi_\lambda = P_\lambda(x_1, x_2, \cdots, x_N; \beta) \psi_0,$$
where \( x_i = \exp(\frac{2\pi i \sqrt{-1}}{L} q_i) \). Here, \( P_\lambda(x_1, x_2, \cdots, x_N; \beta) \)'s are called the Jack symmetric polynomials \[11\], and are our central object in this paper \[12, 13, 14\]. The Jack symmetric functions have various interesting combinatorial properties, and of course play an essential role in studies of the Calogero-Sutherland model (see for example \[15, 16\]).

There is another aspect in the theory of the Jack symmetric polynomials. If one tries to find a good representation theory of the Virasoro algebra, the Jack polynomial automatically comes into the game by some mysterious reasons. The first example of such phenomena was found by Mimachi and Yamada \[17, 18\]. They found that the singular vectors of the Virasoro algebra represented in the bosonic Fock space are nothing but the Jack symmetric polynomials with rectangular diagrams. Our main aim in the present article is to introduce a full-use of the Jack symmetric functions for the Feigin-Fuchs representation of the Virasoro algebra, and have a much better understanding of these Virasoro-Jack correspondence.

We show our basic idea using some simple examples. Let us introduce a bosonic operators \( a_n \ (n \in \mathbb{Z}) \)

\[
[a_n, a_m] = n \delta_{n+m,0},
\]

and the Fock space with the Fock vacuum \( \langle A | 0 \rangle = A \langle A | \). Then, we can represent the generators \( L_n \)'s of the Virasoro algebra by these boson operators – the Feigin-Fuchs representation \[19\] (see Eq.(17)). Then, we can use the Jack symmetric functions as a basis of the Fock space. This mapping is simply given by replacing the power sum functions \( p_n \)'s by the bosonic modes \( a_n/\sqrt{2\beta} \)'s \[20, 21\]. See also \[14\]. Thus we have bosonization of the Jack symmetric functions, and we can study the action of \( L_n \)'s on the Jack symmetric functions.

Deferring the detailed definition to section 2, we give one impressive example.

\[
\langle A_{r+1,s+1} | \begin{array}{c}
\hline
\hline
\end{array} L_1
\begin{array}{c}
\hline
\hline
\end{array} \rangle = \langle A_{r+1,s+1} | \sqrt{2\beta} \left( \frac{\beta (1 + 2\beta)}{(1 + \beta)(3 + 2\beta)} A_{r-1,s-4} \right) \begin{array}{c}
\hline
\hline
\end{array}
\begin{array}{c}
\hline
\hline
\end{array} \\
+ \frac{\beta}{(1 + \beta)(2 + \beta)} A_{r-2,s-3} \begin{array}{c}
\hline
\hline
\end{array} \\
+ \frac{2 (3 + \beta)}{(2 + \beta)(3 + 2\beta)} A_{r-3,s-1} \begin{array}{c}
\hline
\hline
\end{array} \right).
\]

In this example, we immediately notice that the operation of the \( L_1 \) has the effect of adding one box to each possible place of the Young diagram, and factorized coefficients can be seen in front of each Jack symmetric functions. We performed careful calculations using Jack symmetric functions up to degree 10, and we found that the same structure do exist in general.

Firstly, for positive \( n \), the effect of \( L_n \) is represented by adding \( n \)-boxes in each possible ways to the original diagram. On the other hand, to apply the \( L_{-n} \) means to remove \( n \)
boxes in each possible ways. The crucial point for general \( n \) is that, for applying \( L_n \) (or \( L_{-n} \)), if we keep track of the ways to add (or subtract) the boxes from the original Young diagram, we have a good combinatorial rule to calculate the corresponding coefficients in a nice factorized form. Namely, the rational functions like \( \frac{\beta(1+2\beta)}{(1+\beta)(3+2\beta)} \) in the above example can be easily obtained by the combinatorial rule. In other words, if there are \( m \) possible ways to make one particular Young diagram (say \( \mu \)) from the original diagram (say \( \lambda \)), then \( m \)-terms with \( \langle A | J \mu \rangle \) will appear in \( \langle A | J \lambda L_n \rangle \). If we sum up these \( m \)-terms, we do not easily see any good combinatorial structure in the coefficient of \( \langle A | J \mu \rangle \). So, our decomposition with respect to the possible ways is crucial to our aim.

Second, as in the above example, we have a parameter \( A_{r,s} = \frac{1}{\sqrt{2}} (r\sqrt{\beta} - s \frac{1}{\sqrt{\beta}}) \) for the Fock vacuum. We can see that the action of \( L_n \) depends on this parameter in a systematic manner. As is easily seen from the example, we have a coefficient of the shape \( A_{r-s,s-s'} \) in each term, and this shift \((s,s')\) tells us the coordinate to where the box is added. For example, in the first term in R.H.S. of the above example, \( A_{r-1,s-4} \) appears, and added box in this term is at first row, fourth column of the Young diagram.

As an application of this Jack-basis, we are able to have another way to understand the Mimachi-Yamada theorem [18]. Actually, we can do a little better in the following sense. If we forget about the rational factors depending on \( \beta \) and keep the factors of the shape \( A_{r-s,s-s'} \) for simplicity, the action of the Virasoro generators looks like

\[
L_1 J \big| A_{r+1,s+1} \big> = A_{r-2,s-3} J \big| A_{r+1,s+1} \big> , \\
L_2 J \big| A_{r+1,s+1} \big> = A_{r-2,s-3} J \big| A_{r+1,s+1} \big> + A_{r-2,s-3} J \big| A_{r+1,s+1} \big> .
\]

Note that from the commutation relation (1), we can generate all the \( L_n, n > 0 \) only from \( L_1 \) and \( L_2 \). Note also that \( A_{0,0} = 0 \). Thus we can conclude that \( J \big| A_{r,s} \big> \) is a singular vector if \( r = 2, s = 3 \), i.e. when the highest weight is equal to \( A_{2+1,3+1} \). Usually we call this singular vector \( |\chi_{2,3} \rangle \) etc.

At present, we still do not have any good idea how we study the precise form of the Virasoro singular vectors in the Verma module. However, the following observation tells us that there may be a natural hidden structure in the Verma module which corresponds to the Jack-basis after bosonization. Let us try to normalize the singular vector \( |\chi_{r,s} \rangle \) at level \( n = rs \) as

\[
|\chi_{r,s} \rangle = (c_1 L_{-n} + c_2 L_{-n+1} L_{-1} + \cdots + 1 \times L_{n-1}^n) \big| A_{r+1,s+1} \big>, \tag{6}
\]

by letting the coefficient of \( L_{n-1}^n \) being unity. Then, in the lecture note [22], it was conjet-
tured that after bosonization of the normalized singular vector, we get

\[ |\chi_{r,s}\rangle = \prod_{i=1}^{r} \prod_{j=1}^{s} (i\beta - j) \cdot J_{(s')} |A_{r+1,s+1}\rangle, \quad (7) \]

\[ \langle \chi_{r,s} | = 0. \quad (8) \]

In section 4.2, we prove these equalities.

Let us consider a limiting case of the matrix elements for \( L_n \). If we take the limit \( A_{r,s} \to \infty \), then \( L_n \) becomes proportional to \( p_n \) or \( \partial / \partial p_n \). Thus in this limit, we obtain a formula for product of power sum \( p_n \) to the Jack symmetric functions, or for differentiating the Jack symmetric functions with respect to \( p_n \). These formulas can be easily obtained just by dropping all the factors of the shape \( A_{r-*,s-*} \) from the original formula. For example, in connection with the above example, we have

\[ J_{\begin{array}{c} \bullet \\ \bullet \end{array}} p_1 = \left( \begin{array}{c} \beta (1 + 2\beta) \\ (1 + \beta)(3 + 2\beta) \end{array} \right) J_{\begin{array}{c} \bullet \\ \bullet \end{array}} \\
+ \frac{\beta}{(1 + \beta)(2 + \beta)} J_{\begin{array}{c} \bullet \\ \bullet \end{array}} \\
+ \frac{2 (3 + \beta)}{(2 + \beta)(3 + 2\beta)} J_{\begin{array}{c} \bullet \\ \bullet \end{array}} \right). \]

Let us mention that in [23], the authors give the singular vectors for the \( c < 1 \) Fock modules over the Virasoro algebra in terms of Schur polynomials. This is done explicitly for the cases: \( (r, s) = (1, s), (2, s), (r, 1), (r, 2), (3, 3) \) and up to coefficients for the general case (see also the appendix for the action of the \( L_n \)).

In [24] a correspondence between Calogero–Sutherland model and correlators of vertex operators of a CFT with \( U(1) \) symmetry was also established.

The plan of our paper is as follows. In section 2, we briefly review how to bosonize the Virasoro algebra and Jack symmetric functions. In section 3, we give formulas for action of \( L_n \) operators on the Jack symmetric functions, and describe related formula of symmetric function. In section 4, we give the application of these formulas to the theory of singular vectors. A natural normalization of the Virasoro singular vectors in the Verma module is discussed, and whose bosonization is discussed.
2 Virasoro algebra and Jack symmetric functions

2.1 Fock space representation of the Virasoro algebra

In this section, we recall the Feigin-Fuchs representation of the Virasoro algebra. Introduce the Fock vacuum $|0\rangle$ by

$$a_n|0\rangle = 0 \quad (n \geq 0),$$

$$\langle 0|0\rangle = 1,$$  \hspace{1cm} (9)  \hspace{1cm} (10)

and the zero-mode operator $Q$ by the commutation relation

$$[a_n, Q] = \delta_{n,0}.$$ \hspace{1cm} (11)

Define the eigenstates of the zero-mode $a_0$ by

$$|A\rangle = e^{AQ}|0\rangle, \quad \langle A| = \langle 0|e^{-AQ},$$

satisfying

$$a_0|A\rangle = A|A\rangle, \quad \langle A|a_0 = \langle A|A. \quad (12)$$

The Fock space $\mathcal{F}_A$ is then defined by

$$\mathcal{F}_A = C[a_{-1}, a_{-2}, a_{-3}, \cdots]|A\rangle.$$ \hspace{1cm} (14)

We also define the dual space of $\mathcal{F}_A$ as

$$\mathcal{F}_A^* = \langle A|C[a_1, a_2, a_3, \cdots].$$ \hspace{1cm} (15)

If the central charge of the Virasoro algebra satisfies the condition

$$c = 1 - \frac{6(\beta - 1)^2}{\beta},$$

then we can represent $L_n$'s as (Feigin-Fuchs representation)

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} :a_{n-k}a_k: - A_{1,1}(n+1)a_n.$$ \hspace{1cm} (17)

Here, we used the usual normal ordering rule

$$:a_n a_m: = \begin{cases} a_n a_m & \text{(if } n \leq m), \\ a_m a_n & \text{(if } n > m), \end{cases}$$ \hspace{1cm} (18)

and the notation

$$A_{r,s} = \frac{1}{\sqrt{2}} \left( r \sqrt{\beta} - s \frac{1}{\sqrt{\beta}} \right).$$ \hspace{1cm} (19)

Note that the parameter $\beta$ is related to the central charge of the Virasoro algebra as $c = 1 - \frac{6(\beta - 1)^2}{\beta} = 1 - 12A_{1,1}^2$ and this $\beta$ also will be related to the coupling constant for the Calogero-Sutherland model, or in other words, the Jack symmetric polynomial.
2.2 Jack symmetric function

We now turn to the Calogero-Sutherland model and consider the Schrödinger equation defined by the Hamiltonian $H^{CS}$. Consider $N$ particles moving on a circle of length $L$. Let $q_i$’s be the coordinates of the particles. Then their chord distance is given by

$$\frac{L}{\pi} \sin \left( \frac{\pi}{L} (q_i - q_j) \right).$$

Thus the interaction term in $(2)$ expresses a strong repulsive interaction between the particle moving on the circle with the potential $1/r^2$. Set $x_i = \exp(2\pi\sqrt{-1}/L q_i)$. Then, using the exact ground state of $H^{CS}$ obtained by Sutherland, we can write all the eigenfunctions of the CS model as

$$\left( \prod_i x_i \right)^{l - \beta N} \prod_{i<j} (x_i - x_j)^\beta \cdot P_\lambda,$$

where $l \in \mathbb{Z}$ and $P_\lambda$ are some symmetric polynomials in $x_i$’s parameterized by partitions (or Young diagrams) $\lambda$. Then we notice that the $P_\lambda$’s are the eigenstates of the operator

$$H_\beta := \sum_{i=1}^N \left( x_i \frac{\partial}{\partial x_i} \right)^2 + \beta \sum_{i<j} \left( \frac{x_i + x_j}{x_i - x_j} \right) \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right).$$

Namely, the $P_\lambda$’s are completely characterized by the Schrödinger equation

$$H_\beta P_\lambda(x_1, x_2, \cdots x_N) = \epsilon_\lambda P_\lambda(x_1, x_2, \cdots x_N),$$

$$\epsilon_\lambda = \sum_{i=1}^N \left( \lambda_i^2 + \beta (N + 1 - 2i) \lambda_i \right),$$

and the expansion of the form

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu}(\beta) m_\mu.$$

Here, $m_\lambda$’s denote the monomial symmetric polynomials. These $P_\lambda$ are called Jack symmetric polynomials [7, 13].

For our purpose, we need to represent the Jack symmetric polynomials in terms of the power sums $p_n$. To this end, we need to let $N$ to be infinity. Such symmetric polynomials with infinitely many variables are called symmetric functions. So, in what follows, we consider Jack symmetric functions.

The Jack symmetric functions are uniquely characterized as a system of orthogonal functions with respect to the inner product

$$\langle p_\lambda, p_\mu \rangle := \delta_{\lambda,\mu} z_\lambda \beta^{-l(\lambda)},$$

where $z_\lambda = \prod_{i=1}^N x_i^{\lambda_i}$. This inner product is used to define the Jack symmetric functions.
if we work with expansion (25). Here, the power sum functions \( p_n \) are
\[
p_n := \sum_{i=1}^{\infty} x_i^n,
\]
and also write for a partition \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots) \) as
\[
p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i},
\]
where \( l(\lambda) \) is length of \( \lambda \), i.e. the largest \( i \) for which \( \lambda_i \neq 0 \). We also define the weight of \( \lambda \) as
\[
|\lambda| = \sum \lambda_i.
\]
We have set for a partition \( \lambda = (\cdots 3^{m_3} 2^{m_2} 1^{m_1}) \)
\[
z_\lambda := \prod_i i^{m_i} \cdot m_i!.
\]
Just from the inner product (26), we find a relationship between the Jack symmetric functions and the bosonic Fock space. In fact, we can identify these two by
\[
p_n \leftrightarrow \sqrt{2} \beta a_{-n}|A\rangle,
\]
\[
p_n \leftrightarrow \langle A|a_n \frac{1}{\sqrt{2\beta}}.
\]
More precisely we can show
\[
\left( \frac{\langle A|}{\prod_{i=1}^{l(\lambda)} a_{\lambda_i} \frac{1}{\sqrt{2\beta}} \prod_{j=1}^{l(\mu)} \frac{2}{\beta} a_{-\mu_j}|A\rangle}{}, \frac{2}{\beta} a_{-\mu_j}|A\rangle \right) = \delta_{\lambda,\mu} z_\lambda \beta^{-l(\lambda)}.
\]
Thus we can use these boson operators to represent the inner product (26), hence from the Macdonald’s existence theorem, we can represent the Jack symmetric functions by boson operators. Note that we will identify two \( \beta \)'s which appeared in \( H_{CS} \) and in the central charge of the Virasoro algebra. We will see under this identification, there is a good relationship between the Virasoro algebra and the Calogero-Sutherland model.

There are several ways to normalize the Jack symmetric functions. We use the so-called integral form \( J_\lambda \). To describe this normalization, we need some terminology (see section I.1 of [14]). Define the coordinate of Young diagram \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots) \) like as a matrix, i.e. if the box \( s \in \lambda \) is on the \( i \)-th row and \( j \)-th column, then the coordinate of \( s \) is \( (i, j) \). The conjugate of partition \( \lambda \) is a transpose of \( \lambda \) with respect to the main diagonal when expressed by diagram, and denoted by \( \lambda' \). The arm length \( a(s) \) and the leg length \( l(s) \) of \( s = (i, j) \in \lambda \) is defined by
\[
a(s) = \lambda_i - j, \quad l(s) = \lambda'_j - i.
\]

\(^1\)See page 379 of Macdonald’s book [14] as to the meaning of the expansion (25).
where $\lambda'_j$ means the $j$-th element of $\lambda'$. We linearly extend the definition of $a(s)$ and $l(s)$ when $s$ is outside of diagram.

In this section, we have been working with $P_{\lambda}$, whose norm is

$$
\langle P_{\lambda}, P_{\mu} \rangle = \delta_{\lambda,\mu} \prod_{s \in \lambda} \frac{a(s) + \beta l(s) + 1}{a(s) + \beta l(s) + \beta}.
$$

We give degree 3 of $P_{\lambda}$'s as examples.

$$
P_{\lambda_0} = \frac{2}{(1 + \beta)(2 + \beta)} p_3 + \frac{3\beta}{(1 + \beta)(2 + \beta)} p_2p_1 + \frac{\beta^2}{(1 + \beta)(2 + \beta)} p_1^3
$$

$$
P_{\lambda_1} = -\frac{1}{(1 + 2\beta)} p_3 + \frac{(1 - \beta)}{(1 + 2\beta)} p_2p_1 + \frac{\beta}{(1 + 2\beta)} p_1^3
$$

$$
P_{\lambda_2} = \frac{1}{3} p_3 - \frac{1}{2} p_2p_1 + \frac{1}{6} p_1^3
$$

However, it will become clear that integral form of the Jack symmetric functions

$$
J_{\lambda} = \prod_{s \in \lambda} \left( \frac{a(s)}{\beta} + l(s) + 1 \right) P_{\lambda},
$$

are more suitable for our purpose. Note that, when we use this integral form, the Jack symmetric functions become the polynomial of $\alpha = 1/\beta$ and $p_n$ where all the coefficients are integer. We also notice that in Jack symmetric function of degree $n$, the coefficient in front of $p_1^n$ is normalized as unity. We give a list of Jack symmetric functions up to degree 4.

$$
J_0 = p_1
$$

$$
J_1 = \alpha p_2 + p_1^2
$$

$$
J_2 = -p_2 + p_1^2
$$

$$
J_3 = 2\alpha^2 p_3 + 3\alpha p_2p_1 + p_1^3
$$

$$
J_4 = -\alpha p_3 + (\alpha - 1)p_2p_1 + p_1^3
$$

$$
J_5 = 2p_3 - 3p_2p_1 + p_1^3
$$

$$
J_6 = 6\alpha^3 p_4 + 8\alpha^2 p_3p_1 + 3\alpha^2 p_2^2 + 6\alpha p_2p_1^2 + p_1^4
$$

$$
J_7 = -2\alpha^2 p_4 + 2\alpha(\alpha - 1)p_3p_1 - \alpha p_2^2 + (3\alpha - 1)p_2p_1^2 + p_1^4
$$

$$
J_8 = -\alpha(\alpha - 1)p_4 - 4\alpha p_3p_1 + (1 + \alpha + \alpha^2)p_2^2 + 2(\alpha - 1)p_2p_1^2 + p_1^4
$$

$$
J_9 = 2\alpha p_4 - 2(\alpha - 1)p_3p_1 - \alpha p_2^2 + (\alpha - 3)p_2p_1^2 + p_1^4
$$

$$
J_10 = -6p_4 + 8p_3p_1 + 3p_2^2 - 6p_2p_1^2 + p_1^4
$$
The general form of the Jack symmetric functions $J_\lambda$ is

$$J_\lambda = \sum_{|\mu|=|\lambda|} c_{\lambda \mu} p_\mu$$  \hspace{1cm} (36)

with the normalization given by $c_{\lambda \mu} = 1$ for $\mu = (1^{\lambda})$.

3 Actions of Virasoro operators on Jack symmetric functions

In this section, we consider $\beta$, $r$ and $s$ as general real numbers.

3.1 Action of $L_n (n > 0)$ on Jack symmetric functions. Examples

The relationship between the Fock space and the Jack symmetric functions can be easily found as follows. Let us first compare the dimensions of both systems. The degree-$n$ subspace of the Fock space has dimension $p(n)$, where $p(n)$ stand for the partition number, i.e. the number of ways to express $n$ as a sum of all positive integers. On the other hand, number of the degree-$n$ Jack symmetric function is equal to the number of Young diagrams with $n$ boxes, i.e. $p(n)$. Then, from the orthogonality relation (34), the bosonized Jack symmetric functions form the orthogonal basis of the Fock space,

$$\mathcal{F}_A = \text{span}\{ J_\lambda | A \rangle | \text{any partition } \lambda \}.$$  \hspace{1cm} (37)

In the same way, dual of the Fock space is generated by the Jack symmetric functions as

$$\mathcal{F}_A^* = \text{span}\{ \langle A | J_\lambda \rangle | \text{any partition } \lambda \}.$$  \hspace{1cm} (38)

From now on, we consider the action of $L_n$ on the bosonized Jack symmetric functions, i.e. the matrix representation of $L_n$ with respect to the basis constructed by Jack symmetric functions. Before embarking on the general formula, we first consider some examples and clarify the combinatorial properties of the formula.

These formulas can be expressed as hooks on the Young diagrams. We prepare some legends of diagrammatic explanations we will use later. We express the hook within the Young diagram as broad line, and assign the term $(m + n\beta)$ to the hook with horizontal length $m$ and vertical length $n$. If there exist multiple hooks in one diagram, then we take product of each term. We now give some examples.

$$\begin{align*}
\text{\begin{figure}[h]
\begin{array}{c}
\end{array}
\end{figure}} &= (2 + \beta), \\
\text{\begin{figure}[h]
\begin{array}{c}
\end{array}
\end{figure}} &= (0 + \beta), \\
\text{\begin{figure}[h]
\begin{array}{c}
\end{array}
\end{figure}} &= \beta \cdot (2 + 2\beta).
\end{align*}$$
We need another terminology to make our discussion simple enough. A diagram being given, we define *outer-corners* (black dots) and *inner-corners* (white dots) as in the following picture:

First example is action of $L_1$ on $J_{\square}$.

$$
\langle A_{r+1,s+1} | J_{\square} L_1 = \langle A_{r+1,s+1} | \sqrt{2\beta} \\
	imes \left( \frac{\beta(2+2\beta)}{(2+\beta)(3+2\beta)} A_{r-1,s-4} J_{\square} \\
+ \frac{2\beta}{(2+\beta)(1+\beta)} A_{r-2,s-2} J_{\square} \\
+ \frac{1(3+\beta)}{(1+\beta)(3+2\beta)} A_{r-3,s-1} J_{\square} \right)
$$

In above equation, diagrams are inserted for explanation of rational functions of $\beta$. Two diagrams beside the horizontal line stand for numerators and denominators respectively. In the numerator, we join upper left corner of box “1” and all the outer-corners of $J_{\square}$ by hook. On the other hand, in the denominator, we join upper left corner of box “1” and all the inner-corners of $J_{\square}$. If added box – box “1” – has coordinate $(i, j)$, then the term $A_{r-i,s-j}$ appears.

As a second example, we show the action of $L_2$. 

11
\[ \langle A_{r+1,s+1} | J_{\mathbb{L}^2} L_2 = \langle A_{r+1,s+1} | \sqrt{2} \beta \]
\[ \times \left( \frac{\beta (2+2\beta)}{(2+\beta)(3+2\beta)} \times \frac{(3+2\beta)}{(3+\beta)(4+2\beta)} A_{r-1,s-5} \right) \]
\[ \times \left( \frac{\beta (2+2\beta)}{(2+\beta)(3+2\beta)} \times \frac{\beta}{(3+\beta)(1+\beta)} A_{r-2,s-2} \right) \]
\[ + \frac{2\beta}{(1+\beta)(2+\beta)} \times \frac{\beta}{(1+\beta)(3+2\beta)} A_{r-1,s-4} \]
\[ - \frac{\beta (2+2\beta)}{(2+\beta)(3+2\beta)} \times \frac{1}{(1+\beta)(4+2\beta)} A_{r-3,s-1} \]
\[ + \frac{3+\beta}{(1+\beta)(3+2\beta)} \times \frac{\beta}{(2+\beta)(3+3\beta)} A_{r-1,s-4} \]
\[ - \frac{2\beta}{(1+\beta)(2+\beta)} \times \frac{3+\beta}{(2+\beta)(3+2\beta)} A_{r-3,s-1} \]
\[ + \frac{3+\beta}{(1+\beta)(3+2\beta)} \times \frac{2}{(2+\beta)(1+2\beta)} A_{r-2,s-2} \]
\[ + \frac{2\beta}{(1+\beta)(2+\beta)} \times \frac{1}{(1+\beta)(2+\beta)} A_{r-2,s-3} \]
\[ - \frac{3+\beta}{(1+\beta)(3+2\beta)} \times \frac{3+2\beta}{(3+3\beta)(1+2\beta)} A_{r-4,s-1} \] \)

In this case, graphical meaning of the rational function of $\beta$ is almost the same as in the action of $L_1$. Note, however, we don’t take the corner created by the previously added box in the numerator. If the last added box –box “2”– has coordinate $(i, j)$, then factor $A_{r-i,s-j}$ appear. When row of box “1” is lesser than that of box “2”, then a minus sign appears.
3.2 Action of $L_n$: general formula

We take a general Young diagram $Y$ and $n > 0$, and consider $\langle A_{r+1,s+1}|J_Y L_n \rangle$. We parameterize $Y$ as

$$Y = Y^{(1)} = \left( s_1^{(1)} r_1^{(1)} - r_0^{(1)}, s_2^{(1)} r_2^{(1)} - r_1^{(1)}, \ldots, s_m^{(1)} r_m^{(1)} - r_{m-1}^{(1)} \right). \tag{39}$$

We take $r_0^{(1)} = 0$, $s_{j+1}^{(1)} = 0$. In this parameterization, the outer-corners of $Y^{(1)}$ are given by $(r_1^{(1)}, s_1^{(1)})$, $(r_2^{(1)}, s_2^{(1)})$, $\ldots$, $(r_{m^{(1)}}^{(1)}, s_{m^{(1)}}^{(1)})$, where symbol "(1)" indicate that this diagram is going to be added by the first box.

**General formula**

**Step 1:** Add one box, say to the place $(r_{i_1-1}^{(1)} + 1, s_{i_1}^{(1)} + 1)$, to $Y^{(1)}$, and denote it as $Y^{(1)} \cup (r_{i_1-1}^{(1)} + 1, s_{i_1}^{(1)} + 1)$. Associate a coefficient to $Y^{(1)}$, which is given by

$$\prod_{j=1}^{i_1-1} \frac{\left[ (a(r_j^{(1)}, s_j^{(1)} + 1) + l(r_j^{(1)}, s_j^{(1)} + 1) + 1) \beta \right]}{\left[ (a(r_{j-1}^{(1)} + 1, s_{i_1}^{(1)} + 1) + l(r_{j-1}^{(1)} + 1, s_{i_1}^{(1)} + 1) + 1) \beta \right]} \times \prod_{j=i_1}^{m^{(1)}} \frac{\left[ a(r_{j-1}^{(1)} + 1, s_j^{(1)}) + l(r_{j-1}^{(1)} + 1, s_j^{(1)}) + 1 \right]}{\left[ a(r_{j-1}^{(1)} + 1, s_{j+1}^{(1)} + 1) + l(r_{j-1}^{(1)} + 1, s_{j+1}^{(1)} + 1) + 1 \right] \beta}. \tag{40}$$

To make the meaning of the coordinates used in the above formula clearer, we give a schematic interpretation below. In this diagram, upper left diagram correspond to the numerator of the first term in above formula, and so on.
Step 2: Take a coordinate of $Y^{(2)} = Y^{(1)} \cup (r^{(1)}_{i_1-1} + 1, s^{(1)}_{i_1} + 1)$ as in $Y^{(1)}$, namely, set

$$Y^{(2)} = Y^{(1)} \cup (r^{(1)}_{i_1-1} + 1, s^{(1)}_{i_1} + 1) = \left( s^{(2)}_1 r^{(2)}_1 - r^{(2)}_0, s^{(2)}_2 r^{(2)}_2 - r^{(2)}_1, \ldots, s^{(2)}_{m(2)} r^{(2)}_{m(2)} - r^{(2)}_{m(2)-1} \right).$$

(41)

Add one more box to anywhere you want to this diagram if this addition gives us a Young diagram, and we obtain a similar factor as in Step 1. In this step, however, we need to work with the following “exception rule”. If the outer-corner $(r^{(2)}_k, s^{(2)}_k)$ is produced by the box just added in the last step, we should omit the factor corresponding to $(r^{(2)}_k, s^{(2)}_k)$ in the numerator.

Step 3: Denote the Young diagram after the second addition as $Y^{(3)} = Y^{(2)} \cup (r^{(2)}_{i_2-1} + 1, s^{(2)}_{i_2} + 1)$. Add third box to $Y^{(3)}$ and obtain similar factors as in Step 2. Note that we need to work with the exception rule. Repeat this manipulation recursively until $n$-th box is added.

Step 4: Multiply

$$\beta^{n-1}(-1)^{\#\{k | r^{(k)}_{i_k-1} < r^{(k+1)}_{k+1-1}\}} J_{Y \cup (r^{(1)}_{i_1-1} + 1, s^{(1)}_{i_1} + 1) \cup \ldots \cup (r^{(n)}_{i_n-1} + 1, s^{(n)}_{i_n} + 1)},$$

(42)

to the result of Step 3.
Step 5: Multiply
\[
\langle A_{r+1,s+1}|\sqrt{2\beta}A_{r-(r_{n-1}^{(n)})+1,s-(s_{n-1}^{(n)})+1}\rangle
\]
to the result of Step 4.

Step 6: Repeat Steps 1 to 5 for each way to add \(n\) boxes to \(Y\), and sum up all the terms.

Remark 3.1 Formula for \(p_n J_Y\) is obtained by doing steps 1 to 4 and 6.

3.3 Action of \(L_{-n}(n > 0)\) on Jack symmetric functions

Before giving a general formula, we show an example.

\[
\langle A_{r+1,s+1}|J_{[\beta]} L_{-1} = \langle A_{r+1,s+1}|\frac{1}{\sqrt{2\beta}} \cdot \frac{1}{\beta} \times \left( (2 + 2\beta)\beta \cdot \frac{1}{(2 + \beta)} A_{r+3-(2+2),s+3-(1+1)} \right)
\]

In the above diagram, the letter “1” stands for the first removed box. In the numerator, in contrast to the action of \(L_1\), we join lower right corner of box “1” with all inner corners of \([\beta]\). Similarly, in the denominator, we join lower right corner of box “1” with all outer corners of \([\beta]\). For other factors, see the general formula given below.

General formula

Introduce the parameterization on \(Y\). Then the formula for \(\langle A_{r+1,s+1}|J_Y L_{-n}\) is given as follows.

Step 1: Remove box \((r_{i_1}^{(1)}, s_{i_1}^{(1)})\) from \(Y\) and denote it as \(Y \setminus (r_{i_1}^{(1)}, s_{i_1}^{(1)})\). Associated coefficient is

\[
\prod_{j=0}^{i_1-1} \left[ a(r_j^{(1)}, s_{i_1}^{(1)}) + l(r_j^{(1)}, s_{i_1}^{(1)}) + 1 \right] \beta
\]

\[
\prod_{j=1}^{i_1-1} \left[ a(r_j^{(1)}, s_{i_1}^{(1)}) + l(r_j^{(1)}, s_{i_1}^{(1)}) + 1 \right] \beta
\]
\[
\prod_{j=i_1+1}^{m^{(1)}+1} \left[ \left( a(r_{i_1}^{(1)}, s_{j}^{(1)}) + 1 \right) + l(r_{i_1}^{(1)}, s_{j}^{(1)})\right] \times \prod_{j=i_1+1}^{m^{(1)}} \left[ a(r_{i_1}^{(1)}, s_{j}^{(1)}) + l(r_{i_1}^{(1)}, s_{j}^{(1)})\right].
\]

**Step 2:** Take a coordinate of \( Y \setminus (r_{i_1}^{(1)}, s_{i_1}^{(1)}) \) as in \( Y \), and remove the box \((r_{i_2}^{(2)}, s_{i_2}^{(2)})\). We obtain the same coefficient as in Step 1, except that in the numerator, we don’t incorporate term corresponding to the corner created by the last removal.

**Step 3:** Repeat Step 2 until \( n \)-th box is removed.

**Step 4:** Multiply
\[
\frac{1}{n!} \beta^n (-1)^\#\{k | r_{i_k}^{(k)} > r_{i_{k+1}}^{(k+1)} \} J_{(r_{i_1}^{(1)}, s_{i_1}^{(1)}) \cup \cdots \cup (r_{i_n}^{(n)}, s_{i_n}^{(n)})},
\]

(45) to the result of Step 3.

**Step 5:** Multiply
\[
\langle A_{r+1,s+1} \right| A \sum_{k=1}^{n} \left( r_{i}^{(n)} + r_{i}^{(k)} \right) \sum_{k=1}^{n} \left( s_{i}^{(n)} + s_{i}^{(k)} \right),
\]

(46) to the result of Step 4.

**Step 6:** Repeat Steps 1 to 5 for each way to remove \( n \) box from \( Y \), and sum up all the terms.

**Remark 3.2** Formula for \( \frac{\partial}{\partial p} J_{Y} \) is obtained by doing steps 1 to 4 and 6.

**Remark 3.3** So far, we have considered leftward actions of \( L_n \)’s only. To obtain formula \( L_n J_{Y} | A \), we use the identity
\[
\langle J_{Y'} | L_n \rangle = \langle J_{Y'} | (L_n | J_{Y}) \rangle,
\]

(47) and orthogonality of Jack symmetric functions. Especially, in \( L_n | J_{Y} \), the factor \( A_{r-i,s-j} \) appears when the first removed box has coordinate \((i, j)\). We use this property in the next section.

**Remark 3.4** Although we did not give rigorous mathematical proof of the formulas presented in this section (except for the simplest non-trivial case \( p_2 J_{Y} \)), numerous calculations involving Jack symmetric functions up to degree 10 strongly support these results.
4 Application to representation theory

4.1 Singular vectors of Virasoro algebra

We consider the Verma module defined by

\[ M(h) = C[L_{-1}, L_{-2}, L_{-3}, \cdots]|h\rangle, \]  

(48)

where \(|h\rangle\) is a highest weight vector defined as \(L_n|h\rangle = 0\) for \(n > 0\) and \(L_0|h\rangle = h|h\rangle\).

Then, from the Kac determinant formula \[25, 19, 26\], we know that there is a singular vector of degree \(rs\) if the highest weight is

\[ h_{r,s} = (r\beta - s)^2 - (\beta - 1)^2 \]  

\[ 4\beta. \]  

(49)

We denote this singular vector as \(|\chi_{r,s}\rangle\).

To compare the Verma module and the Fock space, using the bosonic representation \[17\], we have

\[ L_0|A_{r+1,s+1}\rangle = h_{r,s}|A_{r+1,s+1}\rangle, \]  

(50)

since \(h_{r,s} = \frac{1}{2}A_{r+1,s+1}A_{r-1,s-1}\). When we bosonize \(|\chi_{r,s}\rangle\), Mimachi-Yamada’s formula states that this is proportional to the bosonized Jack symmetric function with the partition \((s')\), i.e. the rectangle with horizontal length \(s\) and vertical length \(r\).

Our argument in the previous section gives us an intuitive reinterpretation of this fact. Consider the Jack symmetric function with rectangular partition of \(m\) rows and \(n\) columns and act by \(L_1\) and \(L_2\) respectively. Then, ignoring the factors independent of integers \(r, s\) in \(h_{r,s}\) for simplicity, we have

\[ L_1J_{\square}|A_{r+1,s+1}\rangle = A_{r-m,s-n}J_{\square}|A_{r+1,s+1}\rangle, \]  

(51)

\[ L_2J_{\square}|A_{r+1,s+1}\rangle = A_{r-m,s-n}(J_{\square} + J_{\square})|A_{r+1,s+1}\rangle. \]  

(52)

From the identity \(A_{0,0} = 0\), if we choose highest weight as \(h_{m,n}\), then above two actions vanish. Then from the commutation relation of the Virasoro algebra, we have \(L_pJ_{\square}|A_{m+1,n+1}\rangle = 0\) for general \(p > 0\). Thus we can conclude that \(J_{\{m\}}|A_{m+1,n+1}\rangle\) is the bosonized singular vector of highest weight \(h_{m,n}\).

4.2 Refinement of Mimachi-Yamada formula

In this section, we compute the singular vectors including the proportionality factor also. Let us normalize the singular vector \(|\chi_{r,s}\rangle\) in such a way that the coefficient of \(L_{-1}^s\) is
equal to 1 \[22\]. For example, we have

\[
|\chi_{2,2} \rangle = \left( -\frac{3(1-\beta)^2}{\beta} L_{-4} - \frac{2(1 - 3\beta + \beta^2)}{\beta} L_{-3} L_{-1} + \frac{(1 - \beta^2)^2}{\beta^2} L_{-2}^2 - \frac{2(1 + \beta^2)}{\beta} L_{-2} L_{-1}^2 + L_{-1}^3 \right) |h_{2,2} \rangle.
\]

We calculate the bosonization of $|\chi_{l,s} \rangle$. However, since we already know that this is proportional to the Jack symmetric function $J_{(s^r)}$, the only thing we have to do is to determine the proportionality factor. In this context, our normalization of Jack symmetric function is useful, because the coefficient of $p_l^n$ is equal to 1 for any Jack symmetric function of degree $n$. Thus, from identification \[30\], we need to consider only the coefficient of $a_{n-1}^n$ in the bosonized singular vector of degree $n$.

First note that in this problem, we have a nice property. We give two examples; We show Step 1: For this purpose, we define the subset $H \subset F_A$ as

\[
H = \text{span}\{a_{i_1}a_{i_2} \cdots a_{i_n}|A| |i_k \neq 0 \text{ for some } k > 1\}.
\] (55)

For example, $a_{-2}a_{-1}^2 \in H$ and $a_{-1}^4 \notin H$. Then we have

**Lemma** For some $g \in H$, we have

\[
L_{-n}^{i_n} \cdots L_{-2}^{i_2} L_{-1}^{i_1}|A\rangle = \begin{cases} 
\frac{A_{i_1}^{i_1}a_i^{i_1 + 2i_2}}{2} |A\rangle + g \ (i_3 = i_4 = \cdots = i_n = 0) \\
g \ (\text{otherwise})
\end{cases}
\] (56)

**Proof** is by induction.

**Step 1:** We show

\[
L_{i_1}^{i_1}|A\rangle = A^{i_1}a_{i_1}^{i_1}|A\rangle + g \ (g \in H).
\] (57)

From representation \[17\], $L_{-1}$ is expressed as a sum of terms $a_{-1}a_i$ ($i \geq 0$). From the condition $a_n|A\rangle = 0$ ($n > 0$), we have $L_{-1}|A\rangle = Aa_{-1}|A\rangle$. Let us assume for some $n \in \mathbb{Z}_{\geq 0}$, the relation $L_{-1}^n|A\rangle = A^n a_{-1}^n|A\rangle + g_1$ ($g_1 \in H$) holds. We can easily see that

\[
a_{-i_1}a_{i_1} \cdots a_{-m}^{l_m} a_{-(m-1)}^{l_{m-1}} \cdots a_{-1}^{l_1}|A\rangle = a_{-m}^{l_m} \cdots a_{-(i+1)}^{l_{i+1}} a_{-i}^{l_i} \cdots a_{-1}^{l_1}|A\rangle \in H,
\] (58)
if \( m \geq i \) (with a slight modification if \( m = i \)), otherwise act as 0. This means

\[
\sum_{i=1}^{\infty} a_{-i-1} a_i \cdot L_i^1 |A\rangle \in H.
\] (59)

We also notice that for \( g \in H, a_{-1}a_0 \cdot g \in H \). Therefore we have, for some \( g_2 \in H \)

\[
L_{-1} \cdot L_i^1 |A\rangle = a_{-1}a_0 (A^n a_{-1}^n |A\rangle) + g_2 = A^{n+1} a_{-1}^{n+1} |A\rangle + g_2.
\] (60)

From induction hypothesis, we conclude that (57) is valid for any \( i_1 \in \mathbb{Z}_{\geq 0} \).

**Step 2:** We consider the action of \( L_{-2} \), whose representation is given by

\[
L_{-2} = A_{1,1} a_{-2} + \frac{1}{2} a_{-1}^2 + a_{-2} a_0 + a_{-3} a_1 + a_{-4} a_2 + \cdots.
\] (61)

From the definition of \( H \), we notice that \( A_{1,1} a_{-2} \cdot \mathcal{F}_A \subset H \). On the other hand, terms like \( a_{-i-2} a_i (i > 0) \) act as

\[
a_{-i-2} a_i \cdot a_{-m}^l a_{-m}^{l-1} \cdots a_{-1}^i |A\rangle = a_{-m}^l \cdots a_{-i+2} a_{-i+1}^i a_{-i}^i \cdots a_{-1}^i |A\rangle \in H.
\] (62)

Thus, as in Step 1, we have

\[
L_{i_2}^2 L_{i_1}^1 |A\rangle = \frac{A_{i_1}^i}{2^i} a_{i_1}^{i+2i} |A\rangle + g \ (g \in H)
\] (63)

for general \( i_1, i_2 \in \mathbb{Z}_{\geq 0} \).

**Step 3:** We consider \( L_{i_2}^2 \cdots L_{i_1}^1 |A\rangle \) in case that for some \( n > 2, i_n \neq 0 \). By the representation (17), this \( L_{-n} \) is a sum of terms \( a_{-n} \) and \( a_{-i} a_{-j} \) \((i + j = n)\) with some coefficients. By the condition \( n > 2 \), we have \( a_{-n} \cdot \mathcal{F}_A \subset H \). We also have \( \max(i,j) \geq \frac{n}{2} > 1 \), in each \( a_{-i} a_{-j} \) \((i + j = n)\), there is at least one operator \( a_{-k}, k > 1 \). Therefore \( a_{-i} a_{-j} \cdot \mathcal{F}_A \subset H \), if \((i + j = n)\), and we have

\[
L_{-n}^i \cdots L_{i_2}^2 L_{i_1}^1 |A\rangle \in H \ (n > 2).
\] (64)

The proof of Lemma is now finished.

Thus, to calculate the proportionality factor we are interested in, we need only information about terms like \( L_{i_2}^2 L_{i_1}^1 |h\rangle \) in the singular vector, and this has already been calculated by Feigin-Fuchs [26] (with reference to [27]). To quote their result, we prepare some notations (see for example [28] for detail). For each positive integer \( k \), consider the subalgebra generated by \( L_{-k}, L_{-k-1}, L_{-k-2}, \cdots \), and denote its enveloping algebra by \( U_{-k} \). Let us write the singular vectors on the quotient algebra \( U_{-1}/U_{-3} \) as

\[
|\chi_{r,s}\rangle = \sigma_{j,j}^r (L_{-1}, L_{-2}) |h\rangle,
\] (65)
where \( r = 2j' + 1, \ s = 2j + 1 \). Then we have

\[
\sigma_{j'j}(L_{-1}, L_{-2})^2 = \prod_{-j \leq M \leq j} \left( L_{-1}^2 + 4(M\theta + M'\theta^{-1})^2L_{-2} \right),
\]

(66)

where \( \theta^2 = -1/\beta \). Note that we have \( [L_{-1}, L_{-2}] = 0 \) on \( U_{-1}/U_{-3} \).

Then, if we bosonize the factor \( (L_{-1}^2 + 4(M\theta + M'\theta^{-1})^2L_{-2}|_{h_{r,s}}) \) and use above Lemma and identification (30), we obtain the coefficient of \( p_1^s \) as

\[
\left( A_{r+1,s+1}^2 + 2\left( -\frac{M^2}{\beta} + 2MM' - M'^2\beta \right) \right) \frac{\beta}{2}
\]

= \((j' + 1 + M')\beta - (j + 1 + M)] \times [(j' + 1 - M')\beta - (j + 1 - M)]\.

(67)

Take product of above expression as in (66), and convert the result in language of symmetric functions. Then we notice that in the image of \( \sigma_{j'j}(L_{-1}, L_{-2})^2|h_{r,s} \), the coefficient of \( p_1^r \) is given by \( \prod_{i=1}^r \prod_{j=1}^s (i\beta - j) \). Since the coefficient of \( p_1^r \) in the integral form of the Jack symmetric function of degree \( rs \) is always 1, this factor itself is equal to the proportionality factor between the singular vectors in the Verma module and the Fock space. Thus, we have the following formula.

**Formula**

\[
|\chi_{r,s}\rangle = \prod_{i=1}^r \prod_{j=1}^s (i\beta - j)J_{(sr)}|A_{r,s}\rangle
\]

(68)

In the same way, we can also show that

\[
\langle \chi_{r,s} | = 0.
\]

(69)

To verify this, we first observe following two equations,

\[
L_{-n}|A_{r+1,s+1}\rangle = [(n-1)A_{1,1}a_{-n} + A_{r+1,s+1}a_{-n} + a_{-n+1}a_{-1} + \cdots]|A_{r+1,s+1}\rangle,
\]

(70)

\[
\langle A_{r+1,s+1}|L_n = \langle A_{r+1,s+1}|[-(n+1)A_{1,1}a_n + A_{r+1,s+1}a_n + a_{n-1}a_{-1} + \cdots].
\]

(71)

In the above equations, the expansion is in fact a finite sum. Comparing these two equations, we notice that to obtain \( \langle A_{r+1,s+1}|L_n \) from the result of \( L_{-n}|A_{r+1,s+1}\rangle \), we just replace \( (A_{r+1,s+1}, A_{1,1}, a_{-i}) \) in (70) by

\[
(A_{r+1,s+1}, A_{1,1}, a_{-i}) \rightarrow (A_{r-1,s-1}, A_{-1,-1}, a_{i}).
\]

(72)

The same relation holds if we consider a general vector

\[
L_{-n}^{i_1} \cdots L_{-2}^{i_2} L_{-1}^{i_1} |A_{r+1,s+1}\rangle
\]

(73)

and its dual

\[
\langle A_{r+1,s+1}|L_1^{i_1} L_2^{i_2} \cdots L_n^{i_n}
\]

(74)
since \([a_n, a_0] = 0 (n \in \mathbb{Z})\). Thus we can easily obtain dual of above Lemma.

Since \(|\chi_{r,s}\rangle\) is proportional to \(J_{(s^r)}\) in terms of \(p_n = \sqrt{2/\beta} a_{-n}\), we can use the replacement \(p_n \rightarrow 2p_n\) in stead of \(p_n\) (compare the two identifications (30) and (31)). Thus, in this case, we also have to consider the coefficient in front of \(p_{\tau s}\). To do this, we consider the dual of \((66)\). In particular, we have to calculate the term \((M, M') = (j, j')\), i.e.

\[
\left( L_1^2 + 4(j\theta + j'\theta^{-1})^2 L_2 \right). \tag{75}
\]

Then from the dual of Lemma and identification (31), we have

\[
\left( A_{-1,s-1}^2 + 2(j\theta + j'\theta^{-1})^2 \right) 2\beta = 0. \tag{76}
\]

Thus we obtain \(|\chi_{r,s}\rangle = 0\) as stated.

**Remark 4.1** Result in this section can be viewed as determination of proportionality factor between bosonized Jack symmetric functions and its analogue in the Verma module. Thus we can expect that similar coefficient occurs when general Jack symmetric function is embedded in Verma module.

**Remark 4.2** After completion of the manuscript, we noticed a paper of Adler and van Moerbeke [29]. They calculate the action of Virasoro operators on Schur symmetric functions. We also noticed a paper by B. Feigin and E. Feigin [30]. They consider the bosonized Jack symmetric functions as a basis of the Fock space and studied the representation of vertex operator algebra.

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