Left-handed materials

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I. INTRODUCTION

Rapidly increasing interest in the left-handed materials (LHM) started after Pendry \textit{et. al.} predicted that certain man-made composite structure could possess, in a given frequency interval, a negative \textit{effective} magnetic permeability \(\mu_{\text{eff}}\). Combination of such a structure with negative effective permittivity medium - for instance the regular array of thin metallic wires\(^2,7\) - enabled the construction of meta-materials with both \textit{effective} permittivity and permittivity \textit{negative}. This was confirmed by experiments\(^8,9\).

Structures with negative permittivity and permeability were named “left-handed” by Veselago\(^10\) over 30 years ago to emphasize the fact that the intensity of the electric field \(\vec{E}\), the magnetic intensity \(\vec{H}\) and the wave vector \(\vec{k}\) are related by a left-handed rule.\(^11\) This can be easily seen by writing Maxwell’s equation for a plane monochromatic wave: \(\vec{k} \times \vec{E} = \frac{\omega}{c} \vec{H}\) and \(\vec{k} \times \vec{H} = -\frac{\omega}{c} \vec{E}\). Once \(\epsilon\) and \(\mu\) are both positive, then \(\vec{E}, \vec{H}\) and \(\vec{k}\) form a right set of vectors. In the case of negative \(\epsilon\) and \(\mu\), however, these three vectors form a left set of vectors.

In his pioneering work, Veselago described the physical properties of LH systems: Firstly, the direction of the energy flow, which is given by the Poynting vector

\[ \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} \]

does not depend on the sign of the permittivity and permeability of the medium. Then, the vectors \(\vec{S}\) and \(\vec{k}\) are parallel (anti-parallel) in the right-handed (left-handed) medium, respectively. Consequently, the phase and group velocity of an electromagnetic wave propagate in \textit{opposite} directions in the left-handed material. This gives rise to a number of novel physical phenomena, as were discussed already by Veselago. For instance, the Doppler effect and the Cherenkov effect are reversed in the LHM\(^10\).

If both \(\epsilon\) and \(\mu\) are negative, then also the refraction index \(n\) is negative\(^10,12\). This means the negative refraction of the electromagnetic wave passing through the boundary of two materials, one with positive and the second with negative \(n\) (negative Snell’s law). Observation of negative Snell’s law, reported experimentally\(^13\) and later in\(^14\), is today a subject of rather controversially debate\(^15-18\). Analytical arguments of the sign of the refraction index were presented in\(^12\). Numerically, negative phase velocity was observed in FDTD simulations\(^19\). Negative refraction index was calculated from the transmission and reflection data\(^20\). Finally, negative refraction on the wedge experiment was demonstrated also by FDTD simulations\(^21\).

Negative refraction allows the fabrication of flat lens\(^10\). Maybe the most challenging property of the left-handed medium is its ability to enhance the evanescent modes\(^22\). Therefore flat lens, constructed from left-handed material with \(\epsilon = \mu = -1\) could in principle work as perfect lens\(^22\) in the sense that it can reconstruct an object without any diffraction error.

The existence of the perfect lens seems to be in contradiction with fundamental physical laws, as was discussed in a series of papers\(^15,23,24\). Nevertheless, more detailed physical considerations\(^16,25,26,28,29\) not only showed that the construction of “almost perfect” lens is indeed possible, but brought some more insight into this phenomena\(^30-32,34,35,36\).

As Veselago also discussed in his pioneering paper, the permittivity and the permeability of the left-handed material must depend on the frequency of the EM field, otherwise the energy density\(^37\)

\[ U = \frac{1}{2\pi} \int d\omega \left[ \frac{\partial (\omega \epsilon')}{\partial \omega} |E|^2 + \frac{\partial (\omega \mu')}{\partial \omega} |H|^2 \right] \]  

would be negative for negative \(\epsilon'\) and \(\mu'\) (real part of the permittivity and permeability). Then, according to Kramers-Kronig relations, the imaginary part of the permittivity (\(\epsilon''\)) and of the permeability (\(\mu''\)) are non-zero in the LH materials. Transmission losses are therefore unavoidable in any LH structure. Theoretical estimation of losses is rather difficult problem, and led even to the conclusion that LH materials are not transparent\(^17\). Fortunately, recent experiments\(^14,38\) confirmed the more optimistic theoretical expectation\(^39\), that the losses in the LH structures might be as small as in conventional RH materials.

The number of papers about left-handed materials increased dramatically last year. The present paper is not the first review about left-handed materials. For recent reviews, see\(^40\) or the papers of Pendry\(^41\). Here we present typical structures of the left-handed materials (Sect. II), discuss a numerical method of simulation of the propagation of EM waves based on the transfer matrix (Sect. III), and present some recent results obtained by this method (Sects. IV, V). We discuss in Sect. V how the transmission depends on various structural and material parameters of LH structure. The method of calculation of the refractive index and of the effective permittivity and per-
meability is presented and applied to the LH structure. An unambiguous proof of the negative refraction index is given and the effective permittivity and permeability are calculated in Sects. VI, and VII. The obtained data for the permittivity are rather counter-intuitive and require some physical interpretation. Finally, in the Section VIII we discuss some new directions of the development of both theory and experiments.

FIG. 1. Left: Structure of the split ring resonator (SRR). The SRR consists of two splitted metallic “rings”. The SRR is characterized by the size \( w \), width of the rings \( d \), and two gaps: \( g \) and \( c \). The external magnetic field induces an electric current in both rings. The shape of the SRR (square or circular) is not crucial for the existence of the magnetic resonance. Right: Structure of the unit cell of the left-handed material. Each unit cell contains the split ring resonator located on the dielectric board, and one wire. Left-handed structure is created by regular lattices of unit cells. The EM wave propagates along the \( z \) direction. Periodic boundary conditions are considered in the \( x \) and the \( y \) direction, which assures the periodic distribution of the EM field.

II. STRUCTURE

LHM materials are by definition composites, whose properties are not determined by the fundamental physical properties of their constituents but by the shape and distribution of specific patterns included in them. The route of the construction of the of LH structure consists from three steps:

Firstly, the split ring resonators (SRR) (see fig. 1 for the structure of SRR) was predicted to exhibit the resonant magnetic response to the EM wave, polarized with \( \vec{H} \) parallel to the axis of the SRR. Then, the periodic array of SRR is characterized\(^1\) by the effective magnetic permeability

\[
\mu_{\text{eff}}(f) = 1 - \frac{F \nu^2}{\nu^2 - \nu_m^2 + i \nu \gamma}.
\]

(3)

In (3), \( \nu_m \) is the resonance frequency which depends on the structure of the SRR (fig. 1) as \((2 \pi \nu_m)^2 = 3 L_c c_{\text{light}}^2 / (\pi \ln(2 c / d) r^2)\). \( F \) is the filling factor of the SRR within one unit cell and \( \gamma \) is the damping factor \( 2 \pi \gamma = 2 L_c \rho / r \), where \( \rho \) is the resistivity of the metal.

Formula (3) assures that the real part of \( \mu_{\text{eff}} \) is negative at an interval \( \Delta \nu \) around the resonance frequency. If an array of SRR is combined with a medium with negative real part of the permittivity, the resulting structure would possess negative effective refraction index in the resonance frequency interval \( \Delta \nu \).

The best candidate for the negative permittivity medium is a regular lattice of thin metallic wires, which acts as a high pass filter for the EM wave polarized with \( \vec{E} \) parallel to the wires. Such an array exhibits negative effective permittivity

\[
\varepsilon_{\text{eff}}(\nu) = 1 - \frac{\nu_p^2}{\nu^2 + i \nu \gamma},
\]

(4)

\(^2, 4, 6\) with the plasma frequency \( \nu_p = c_{\text{light}}^2 / (2 \pi a^2 \ln(a / r))^2 \). Sarychev and Shalaev derived another expression for the plasma frequency, \( \nu_p = c_{\text{light}}^2 / (\pi a^2 L) \) with \( L = 2 \ln(a / \sqrt{2} r) + \pi / 2 - 3^6 \). Apart from tiny differences in both formulas, the two theories are equivalent\(^7\) and predict that effective permittivity is negative for \( \nu < \nu_p \).

By combining both the above structures, a left-handed structure can be created. This was done for the first time in the experiments of Smith et al.\(^8\). Left-handed material is a periodic structure. A typical unit cell of the left-handed structure is shown in fig. 1. Each unit cell contains a metallic wire and one split ring resonator (SRR), deposited on the dielectric board.

FIG. 2. Transmission of the EM wave, polarized with \( \vec{E} \parallel y \) and \( \vec{H} \parallel x \), through a periodic array of split ring resonators, wires, and of both SRR and wires.

Fig. 2 shows the transmission of the EM waves through the left-handed structure discussed above. The transmission through the array of the SRR is close to unity for all the frequencies outside the resonance interval (8.5-11 GHz in this particular case) and decreases to -120 dB in this interval, because \( \mu_{\text{eff}} \) is negative (Eq. 3). The transmission of the array of metallic wires is very small for all frequencies below the plasma frequency (which is \( \sim 20 \) GHz in this case), because \( \varepsilon_{\text{eff}} \) is negative (Eq. 4). The structure created by the combination of an array of SRR and wires exhibits high transmission \( T \sim 1 \) within the resonance interval, where both \( \varepsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \) are negative.
For frequencies outside the resonance interval, the product $\mu_{\text{eff}}\epsilon_{\text{eff}}$ is negative. The transmission decays with the system length, and is only $\sim -120 \text{ dB}$ in the example of fig. 2. Experimental analysis of the transmission of all the three structures was performed by Smith et al.\textsuperscript{8}.

We want to obtain a resonance frequency $\nu_\text{m} \approx 10 \text{ GHz}$. This requires the size of the unit cell to be 3-5 mm. The wavelength of the EM wave with frequency $\sim 10 \text{ GHz}$ is $\approx 4 \text{ cm}$, and exceeds by a factor of 10 the structural details of the left-handed materials. We can therefore consider the left-handed material as macroscopically homogeneous. This is the main difference between the left-handed structures and the “classical” photonic band gap (PBG) materials, in which the wave length is comparable with the lattice period.

It is important to note that the structure described in fig. 1 is strongly anisotropic. For frequencies inside the resonance interval, the effective $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ are negative only for EM field with $\vec{H} \parallel x$ and $\vec{E} \parallel y$. The left-handed properties appear only when a properly polarized EM wave propagates in the $z$ direction. The structure in fig. 1 is therefore effectively one-dimensional. Any EM waves, attempting to propagate either along the $x$ or along the $y$ direction would decay exponentially since the corresponding product $\epsilon_{\text{eff}}\mu_{\text{eff}}$ is negative. This structure is therefore not suitable for the realization of the perfect lens. To test the negative Snell’s law experimentally, a wedge type of experiment must be considered\textsuperscript{13} (fig. 3) in which the angle of refraction is measured outside, the left-handed medium (in air)\textsuperscript{13,14}.

Two dimensional structures have also been constructed. For instance, the anisotropy in the $x - z$ plane is removed if each unit cell contains two SRR located in two perpendicular planes\textsuperscript{9,13}. No three dimensional structures have been experimentally prepared yet.

![FIG. 3. Refraction experiment on the Left-handed material\textsuperscript{13,38}. The incident EM wave propagates from the left and hits perpendicularly the left boundary of the wedge. The angle of refraction is measured when the EM wave passes the right boundary of the inspected material and propagates for some time in the air. Two possible directions of the propagation of the refracted wave are shown: the right refraction for the conventional right-handed (RH) material, and left refraction for the left-handed material. This experimental design enables to use also strongly anisotropic one-dimensional LH samples, since the angle of refraction is measured outside the sample.](image)

![FIG. 4. Two positions of the metallic wire in the unit cell. Left: wire is deposited on the opposite side of the dielectric board\textsuperscript{9,14}. Right: wire is located along the split-ring resonator\textsuperscript{42}.](image)

### III. NUMERICAL SIMULATION

Various numerical algorithms were used to simulate the propagation of EM waves through the LH structure. We concentrate on the transfer matrix algorithm, developed in a series of papers by Pendry and co-workers\textsuperscript{45}. The transfer matrix algorithm enables us to calculate the transmission, reflection, and absorption as a function of frequency\textsuperscript{36,47}. Others use commercial software: either Microwave studio\textsuperscript{19,48,38} or MAFIA\textsuperscript{8,40}, to estimate the position of the resonance frequency interval. Time-dependent analysis, using various forms of FDTD algorithms are also used\textsuperscript{14,19,43,49,50}.

In the transfer matrix algorithm, we attach in the $z$ direction, along which EM wave propagates, two semi-infinite ideal leads with $\epsilon = 1$ and $\mu = 1$. The length of the system varies from 1 to 300 unit cells. Periodic boundary conditions along the $x$ and $y$ directions are used. This makes the system effectively infinite in the transverse directions, and enables us to restrict the simulated structure to only one unit cell in the transverse directions.

A typical size of the unit cell is 3.66 mm. Because of numerical problems, we are not able to treat very thin metallic structures. While in experiments the thickness of the SRRs is usually 17 $\mu$m, the thickness used in
the numerical calculations is determined by the minimal mesh discretization, which is usually \( \approx 0.33 \) mm. In spite of this constrain, the numerical data are in qualitative agreement with the experimental results. This indicates that the thickness of the SRR is not a crucial parameter, unless it decreases below or is comparable with the skin depth \( \delta \). As \( \delta \approx 0.7 \mu m \) at GHz frequencies of interest, we are far from this limitation. We will discuss the role of discretization in Section IV.

IV. TRANSMISSION

As discussed in Section II the polarization of the EM waves is crucial for the observation of the LH properties. The electric field \( E \) must be parallel to the wires, and the magnetic field \( H \) must be parallel to the axis of the SRRs. In the numerical simulations, we treat simultaneously both polarizations, \( \vec{E} \parallel x \) as well as \( \vec{E} \parallel y \). Due to the non-homogeneity of the structure, these polarizations are not separated: there is always non-zero transmission \( t_{xy} \) from the \( x \) to \( y \) polarized wave. As we will see later, this effect is responsible for some unexpected phenomena. At present, we keep in mind that they must be included into the formula for absorption

\[
A_x = 1 - |t_{xx}|^2 - |t_{xy}|^2 - |r_{xx}|^2 - |r_{xy}|^2
\]

(5)

and in the equivalent relation for \( A_y \).

Figure 5 shows typical data for the transmission in the resonance frequency region. A resonance frequency interval, in which the transmission increases by many orders of magnitude is clearly visible. Of course, high transmission does not guarantee negative refraction index. The sign of \( n \) must be obtained by other methods, which will be described in Sect. VI.

In contrast to the original experimental data, numerical data show very high transmission, indicating that LH structures could be as transparent as the "classical" right-handed ones. This is surprising, because due to the dispersion, high losses are expected. Fig. 6 shows the transmission as the function of the system length, for three different frequencies inside the resonance frequency interval.

\begin{align*}
\nu = 9.8 \text{ GHz} & \quad n = -3.26 + 0.0155 i \\
\nu = 10.5 \text{ GHz} & \quad n = -1.31 + 0.005 i \\
\nu = 11 \text{ GHz} & \quad n = -0.378 + 0.008 i
\end{align*}

FIG. 5. Transmission power \( T \) through the Left-handed meta material of various lengths. Left: frequency dependence, right: length dependence.

FIG. 6. System length dependence of the real part of the transmission through the left-handed structure for three various frequencies within the left-handed band. The system length is given as the number of unit cells in the propagation direction. Note the different scale on the \( y \) axis for the three cases.

Figure 5 shows also that the transmission never decreases below a certain limit. Due to the non-homogeneity of the structure there is a non-zero probability \( t_{yx} \) that the EM wave, polarized with \( \vec{E} \parallel y \), is converted into the polarization \( \vec{E} \parallel x \). The total transmission \( t_{yy} \) consists therefore not only from the "unperturbed" contribution \( t_{yy}^{(0)} \), but also from additional terms, which describe the conversion of the \( y \)-polarized wave into \( x \)-polarized and back to \( y \)-polarized:

\[
t_{yy}(0, L) = t_{yy}^{(0)}(0, L) + \sum_{z,z'} t_{xy}(0, z) t_{xx}(z, z') t_{yx}(z' L) + \ldots .
\]

(6)

\( t_{yy}^{(0)}(0, L) \) decreases exponentially with the system length.
L, while the second term, which represents the conversion of the y-polarized wave into x-polarized wave and back, remains system-length independent, because \( t_{xx}(z, z') \sim 1 \) for any distance \(|z - z'|\).

![Graph](image-url)

**FIG. 7.** The transmission peak for various sizes of the unit cell. In contrast to the structure shown in Fig. 1, the metallic wire is located on the opposite side of the dielectric board. This enables to compress the width of the unit cell as shown in the far left panel.

**V. STRUCTURE AND TRANSMISSION**

Numerical simulations confirm the existence of the resonance left-handed frequency interval. Before we proceed in the calculation of the effective system parameters, let us briefly discuss how the structure of the unit cell influences the position and the width of the resonance interval.

![Graph](image-url)

**FIG. 8.** Top panel: Effective permittivity \( \epsilon_{\text{eff}} \) for an array of metallic wires with radius 0.1 mm. The lattice constant (the distance between the wires) is 5 mm. The electric field is parallel to the wires. Note that the imaginary part of the effective permittivity is multiplied by a factor of 10. We present results of the analysis of the transfer matrix data for three different discretization of the unit cell. The lower panel shows the ratio \(|\epsilon''/\epsilon'|\) for an array of thin metallic wires \(100 \times 100 \mu m\) for two values of the metallic permittivity. The numerical results prove that electromagnetic losses decrease when the metallic conductance increases.

The resonance frequency depends on the structural parameters of the SRR and on the parameters of the unit cell\(^1\). Qualitative agreement between the theoretical formulas and the numerical results was obtained\(^{48,47}\). Another problem is the dependence of the resonance interval on the size and shape of the unit cell. As an example, we show in Fig. 7 how the transmission through the various LH structures depends on the width/high/length of the unit cell. Notice that the width of the LH frequency interval increases substantially as shown in the far left panel of fig. 7.

Another important parameter is the permittivity \( \epsilon_m \) of the metallic components. Within the first approximation, we can consider both the SRR and the wires made from a perfect metal. Then both the conductivity \( \sigma \) and the imaginary part of the metallic permittivity \( \epsilon''_m \) are infinite\(^{51}\). This option is often used in the simulations of the commercial software\(^{48}\). In the transfer matrix algorithm, however, \( \epsilon''_m \) is finite, of the order of \(10^5\). For copper, which is currently used in the experimentally fab-
ricate LH structures, $\epsilon''_m \approx 10^{7,51}$. Some test calculations with higher values of $\epsilon''_m$ gave us almost the same result, indicating that $\epsilon''_m \sim 10^{5}$ is already sufficient to simulate realistic materials.

As was mentioned in Sect. I, the left-handed structure must be dispersive. Dispersion requires a non-zero imaginary part for the permittivity and permeability. Therefore, transmission losses can not be avoided in LH systems. This seems to be in agreement with the first experimental data: the transmission measured in the experiments\textsuperscript{8,9}, was only of the order $10^{-3}$. Although recent experiments reported the transmission very close to unity, there are still serious doubts in the literature about the possibility to create highly transmitted LH structures. Results of numerical simulations, as shown in fig. 5, however give that the transmission through the LH structure could be very high, of order of unity.

The main argument against the expectation of high transmission in LH structures\textsuperscript{17} is that the effective permittivity of the array of thin metallic wires is mostly imaginary than real and negative\textsuperscript{49}. This is a serious objection, because the thickness of the wires in fig. 1 is 1 mm, which exceeds more than 10 times the realistic parameters of the experimentally analyzed structures. There is no chance to simulate very tiny metallic structures in the transfer matrix algorithm. Fortunately, Pendry et al. (formula (36) of\textsuperscript{6}) had shown that the problem of the thickness of the wires could be avoided by simultaneously re-scaling the metallic permittivity. An increase of the wire radius by factor $\alpha$ could be compensated by a decrease of the permittivity of the metal by factor of $\alpha^2$. As the metallic permittivity used in\textsuperscript{20} is $\approx 100$ times lower than the realistic permittivity of Cu, we assume that the size of the wires (1 mm) corresponds to Cu wires of thickness 0.1 mm, which is close to size used in experiments ($0.2 \times 0.017$ mm).

In fig. 8 we present results for the effective permittivity of an array of thin metallic wires. The left panel confirms that the transfer matrix algorithm provides us with realistic data for the transmission and reflection. Three different discretizations of the unit cells were used. For each of them the effective permittivity was calculated by the method explained in Sects. VI, VII. The right panel of figure 8 shows that transmission losses are small in realistic LH structures\textsuperscript{52}. Losses in the metallic components of the left-handed structure are therefore not responsible for the low transmission, measured in the experiments\textsuperscript{8,9,38}. The role of the material properties of the dielectric board on which SRR is deposited was also studied\textsuperscript{19}. As is shown in Fig. 9, very small imaginary part of $\epsilon_{\text{board}}$ causes a rapid decrease of the transmission. This surprising result seems to be in agreement with experiments\textsuperscript{38}. Much higher transmission was obtained in cases with extremely small losses in the dielectric board\textsuperscript{38}.

![Fig. 9. The dependence of the transmission peak on the imaginary part of the dielectric permittivity of the dielectric board. Note that standard dielectrics have $\epsilon'' \approx 10^{-7}$.](image)

![Fig. 10. To panel shows $n'kL$ as a function of the system length $L$ for the EM wave polarized with $E$ parallel to the wires. The slope is negative which confirms that $n'$ is negative. Bottom: $n'kL$ as a function of the system length $L$ for the EM wave polarized $E \parallel x$. The slope is positive which confirms that Re $n'$ is positive. A linear fit gives that $n' = 1.1$. There is almost no interaction of the LH structure with the EM wave.](image)

VI. EFFECTIVE INDEX OF REFRACTION

As was discussed above, the structural inhomogeneities of the LH materials are approximately ten times smaller
than the wavelength of the EM wave. It is therefore possible, within a first approximation, to consider the slab of the LH material as an homogeneous material. Then we can and use the textbook formulas for the transmission $t e^{ikL}$ and reflection $r$ for the homogeneous slab:

$$t^{-1} = \left[ \cos(nkL) - \frac{i}{2} \left( z + \frac{1}{z} \right) \sin(nkL) \right]$$ (7)

and

$$r = \frac{i}{2} \left( z - \frac{1}{z} \right) \sin(nkL)$$ (8)

Here, $z$ and $n$ are the effective impedance and the refraction index, respectively, $k$ is the wave vector of the incident EM wave in vacuum, and $L$ is the length of the LH slab. We only consider perpendicular incident waves, so that only the $z$ components of the effective parameters are important. To simplify the calculations, we neglect also the conversion of the polarized EM wave into another polarization, discussed in the Section IV. More accurate analysis should treat both $t$ and $r$ as $2 \times 2$ matrices. Here we assume that the off-diagonal elements of these matrices are negligible:

$$|t_{xy}| \ll |t_{yy}|$$ (9)

In the present analysis, we use the numerical data for the transmission and the reflection obtained by the transfer matrix simulation. The expressions for the transmission and the reflection can be inverted as

$$z = \pm \sqrt{\frac{(1 + r)^2 - t^2}{(1 - r)^2 - t^2}}$$ (10)

$$\cos(nkL) = X = \frac{1}{2t} \left( 1 - r^2 + t^2 \right)$$ (11)

The sign of $z$ is determined by the condition

$$z' > 0$$ (12)

which determines $z$ unambiguously. The obtained data for $z$ enables us also to check the assumption of the homogeneity of the system. We indeed found that $z$ is independent of the length of the system $L$.

The second relation, (11), is more difficult to invert since $\cos^{-1}$ is not an unambiguous function. One set of physically acceptable solutions is determined by the requirement

$$n'' > 0$$ (13)

which assures that the material is passive. The real part of the refraction index, $n'$, however, suffers from the ambiguity of $2\pi m/(kL)$ ($m$ is an integer). To avoid this ambiguity, data for various system length $L$ were used. As $n$ characterizes the material property of the system, it is $L$ independent. Using also the requirement that $n$ should be a continuous function of the frequency, the proper solution of (11) was found and the resonance frequency interval, in which $n'$ is negative was identified. Here, we use another method for the calculation of $n$ and $z$: Equation (11) can be written as

$$e^{-n''kL} [\cos(n'kL) + i \sin(n'kL)] = Y = X \pm \sqrt{1 - X^2}.$$ (14)

Relation (14) enables us to find unambiguously both the real and the imaginary part of the refraction index from the linear fit of $n''kL$ and $n'kL$ vs the system length $L$. The requirement (13) determines the sign in the r.h.s. of Eq. 14, because $|Y| < 1$. Then, the linear fit of $n'kL$ vs $L$ determines unambiguously the real part of $n'$.

Figure 10 shows the $L$ dependence of $n'kL$ for three frequencies inside the resonance interval (transmission data for these frequencies are presented in fig. 6). The numerical data proves that the real part of the refraction index, $n'$, is indeed negative in the resonance interval. For comparison, we present also $n'kL$ vs $L$ for the $x$ polarized wave outside the resonance interval. As expected, the slope is positive and gives that $n' = 1.13$, which is close to unity.

Besides the sign of the real part of the refraction index, the value of the imaginary part of $n$, $n''$ is important, since it determines the absorption of the EM waves inside the sample. Fortunately, $n''$ is very small, it is only of the order of $10^{-2}$ inside the resonance interval. As is shown in fig. 6, quite good transmission was numerically obtained also for samples with length of 300 unit cells (which corresponds to a length of the system $1.1$ m). This result is very encouraging and indicates that left-handed structures could be as transparent as right-handed materials.

There are two constrains in the present method. (i) While the above method works very well in the right side of the resonance interval, we had problems to estimate $n$ in the neighborhood of the left border of the resonance region, where $n'$ is very large and negative. This is, however, not surprising since in this frequency region the wavelength of the propagating wave becomes comparable with the size of the unit cell, so that the effective parameters have no physical meaning. (ii) Outside the resonance interval, we have serious problems to recover proper values of $n$. This is due to the conversion of the $x$ polarized wave into a $y$ polarized. As a result, we do not have enough numerical data for obtaining the $L$-dependence of the transmission $t$. As it is shown in Fig. 5, only data for 2-3 unit cells are representative for the $E \parallel y$ wave. The condition (9) is not any more fulfilled, and the present theory must be generalized as discussed above.
FIG. 11. Refraction index $n$, impedance $z$, permittivity $\varepsilon$ and permeability $\mu$ as a function of frequency in the resonance interval. Close to the resonance frequency $\nu = 9.8\,\text{GHz}$, there are problems to estimate $n$ and $z$, because the wave length of the propagated EM wave becomes comparable with the size of the unit cell of the left-handed structure. Dashed area shows the resonance interval.

VII. EFFECTIVE PERMITTIVITY AND PERMEABILITY

Once $n$ and $z$ are obtained

$$\varepsilon_{\text{eff}} = \frac{n}{z}, \quad \text{and} \quad \mu_{\text{eff}} = nz,$$

one can estimate unambiguously the effective permittivity and permeability of the investigated structure. Results are shown in fig. 11. Frequency dependence of the effective refraction index and $\varepsilon'_{\text{eff}}, \mu'_{\text{eff}}$ is also presented in\textsuperscript{20}. Our numerical results confirm the resonance behavior of the $\mu'_{\text{eff}}$, in agreement with theoretical predictions of Pendry et al.\textsuperscript{1}. In addition to these expected results, we also found rather strong electric response of the SRR, which manifests itself as the “anti-resonant” behavior of $\varepsilon'_{\text{eff}}$ in the resonance frequency interval and by the decrease of the plasma frequency (fig. 4 of ref.\textsuperscript{20}, see also\textsuperscript{53}).

The real parts of the permittivity and permeability, $\varepsilon'_{\text{eff}}$ and $\mu'_{\text{eff}}$ are negative in the resonance interval, as expected. Surprisingly, the imaginary part of the effective permittivity, $\varepsilon''_{\text{eff}}$ was also found negative. This seems to contradict our physical intuition, since, following\textsuperscript{37}, Sect. 80, the electro-magnetic losses are given by

$$Q = \frac{1}{2\pi} \int d\omega \omega \left[ |\varepsilon''| |E|^2 + |\mu''| |H|^2 \right]$$

In passive materials, we require $Q > 0$. This is trivially satisfied if we require that both $\varepsilon''$ and $\mu''$ are positive. Negative $\varepsilon''$ was obtained also in other structures\textsuperscript{53}.

More detailed analysis is also necessary for the energy of the EM wave inside the left-handed material. The formula for the total energy, given by Eq. 2, does not assure the positiveness of the energy because the first term in Eq. (2) is negative in the left part of the resonance interval. Indeed, as shown in fig. 11, both $\varepsilon'$ and $\partial \varepsilon'/\partial \omega$ are negative, so that

$$\frac{\partial (\omega \varepsilon')}{\partial \omega} < 0.$$  \hspace{1cm} (17)

Although formulas (2,16) are not the most general formulas for the energy of the EM field\textsuperscript{54}, we show that both $\varepsilon''_{\text{eff}} < 0$ and relation (17) are consistent with (16) and (2), respectively. We use the definition of the impedance\textsuperscript{37}, Sect. 83,

$$E^2 = \frac{\mu}{\varepsilon} H^2$$  \hspace{1cm} (18)
Then, with the help of relations (15), we re-write the relation (16) into the form

\[ Q = \frac{1}{2\pi} \int d\omega |H|^2 \times 2n''(\omega)z'(\omega) \]  

(19)

which assures that \( Q \) is positive thanks to the conditions that \( z' > 0 \) and \( n'' > 0 \) (12,13). There is therefore no physical constrain to the sign of the imaginary part of the permittivity and permeability.

With the help of the relation (18) Eq. 2 can be rewritten into the form

\[ U = \frac{1}{2\pi} \int d\omega |H|^2 \left[ |z|^2 \frac{\partial(\omega'\epsilon')}{\partial\omega} + \frac{\partial(\omega'\mu')}{\partial\omega} \right]. \]  

(20)

Using the numerical data for the impedance and for real part of the permittivity, we checked that indeed the expression in the bracket of the r.h.s. of (20) is always positive.

**VIII. FURTHER DEVELOPMENT, UNSOLVED PROBLEMS AND OPEN QUESTIONS**

We reviewed some recent experiments and numerical simulations on the transmission of the electromagnetic waves through left-handed structures. For completeness, we note that recently, Notomi\(^55\) has studied the light propagation in strongly modulated two dimensional photonic crystals (PC). In these PC structures the permittivity is periodically modulated in space and is positive. The permeability is equal to unity. Such PC behaves as a material having an effective refractive index controllable by the band structure. For a certain frequency range it was found by FDTD simulations\(^55\)\(^-\)\(^57\) that \( n_{\text{eff}} \) is negative. It is important to examine if left-handed behavior can be observed in photonic crystals at optical frequencies.

Negative refraction on the interface of a three dimensional PC structure has been observed experimentally by Kosaka et al\(^59\) and a negative refractive index associated to that was reported. Large beam steaming has been observed in\(^59\), that authors called “the superprism phenomena”. Similar unusual light propagation has been observed in one-dimensional and two dimensional refraction gratings. Finally, a theoretical work\(^60\) has predicted a negative refraction index in photonic crystals.

Studies of the left handed structures open a series of new challenging problems for theoreticians as well as for experimentalists. The complete understanding of the properties of left-handed structures requires the reevaluation of some “well known” facts of the electromagnetic theory. There is no formulas with negative permeability in classical textbooks of electromagnetism\(^37\)\(^,\)\(^51\). Application of the existing formulas to the analysis of left handed structures may lead to some strange results. The theory of EM field has to be reexamined assuming negative \( \mu \) and \( \epsilon \). We need to understand completely the relationship between the real and the imaginary parts of the permittivity and the permeability. Kramers-Kronig relations should be valid, but nobody have verified them yet in the case of the left-handed structures. The main problem is that we need \( \epsilon_{\text{eff}} \) and \( \mu_{\text{eff}} \) in the entire range of frequencies, which is difficult to obtain numerically. Then, due to the anisotropy of the structure as well as the nonzero transmission \( t_{xy} \) in Eq. (7), Kramers-Kronig relations should be generalized. We do not believe that today’s numerical data enables their verifications with sufficient accuracy.

Both in the photonic crystals and LHM literature there is a lot of confusion about what is the correct definitions of the phase and group refractive index and what is their relations to negative refraction. In additions, it is instructive to see how the LH behavior is related with the sign of the phase and group refractive indices for the PC system. The conditions of obtaining LH behavior in PC were recently examined in\(^96\). It was demonstrated that the existence of negative refraction is neither a prerequisite nor guarantees a negative effective refraction index and so LH behavior. Contrary, LH behavior can be seen only if phase refractive index \( n_{\text{phase}} \) is negative. Once \( n_{\text{phase}} \) is negative, the product \( \vec{S} \cdot \vec{k} \) is also negative, and the vectors \( \vec{k} \), \( \vec{E} \) and \( \vec{H} \) form a left handed set, as discussed in the Introduction.

Problems of causality arises also in connection with “well known” and accepted relations like \( \partial(n'(\omega)\omega)/\partial\omega > 0 \). It is evident, that this relation can not be valid in the vicinity of the left border of left-handed frequency interval, since both \( n' \) and \( \partial n'/\partial\omega \) are negative. The same problem arises for the real part of \( \epsilon_{\text{eff}} \). Also relation (17) requires more exact and complete treatment. We need more general relations for the energy of the EM field\(^74\) which incorporates all the allowed signs of the real and imaginary parts of the permittivity and permeability.

Following problems that are currently discussed in literature. The negative Snell’s law requires the understanding in more detail of the relationship between the Poynting vector, the group velocity, and the phase velocity. We believe, that there is no controversial in this phenomena\(^16\). Anisotropy of real left-handed materials inspires further development of super-focusing\(^32\).

We believe that further analysis, of what happens when EM wave crosses the boundary of the left-handed and right-handed systems, will bring more understanding of the negative refraction as well as perfect lensing. Numerical FDTD simulations of the transmission of the EM wave through the interface of the positive and negative refraction index\(^37\) showed that the wave is trapped temporarily at the interface and after a long time the wave front moves eventually in the direction of negative refraction. Computer simulations of the transmission through LH wedge\(^21\) also confirm that EM wave spends some time on the boundary before the formation of the left-handed wave front. Formation of surface waves\(^30\)\(^,\)\(^31\)\(^,\)\(^29\) can ex-
plain “perfect lensing” without violating causality. Recent development, both numerical and experimental indicate that perfect (although not absolutely perfect) lensing might be possible. We need also to understand some peculiar properties of the left handed structures due to its anisotropy, and bi-anisotropy.

Last but not least, let us mention an attempt to find new left-handed structures, both in the traditional metallic left-handed structures and in the photonic crystals.

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