Newtonian limit of Einsteinian gravity and dynamics of Solar System

Arkady L. Kholodenko
375 H.L.Hunter Laboratories,
Clemson University, Clemson, SC 29634-0973, USA

Attempts to merge the Einsteinian gravity with Newtonian run into logical inconsistencies coming from the fact that in Newton’s gravity time is absolute and speed of propagation of gravity is assumed to be infinite. Such an assumption was in a focus of attention of many scientists in 19th century interested in finding out if, indeed, the speed of propagation of gravity is infinite. For this purpose, by analogy with electrodynamics, some retarded potentials replacing Newtonian were suggested. Using one of such potentials Gerber correctly calculated the perihelion shift for Mercury in 1902. However, subsequent attempts at calculation of the bending of light using Gerber-style calculations were not successful. Recently Gine’ (Chaos, Solitons and Fractals 42, 1893 (2009)) was able to reobtain both the perihelion shift and bending of light using retarded potential. His equations however are not those obtained by Einstein and the results coincide with those by Einstein only at the level of leading order terms of infinite series expansions. The obtained differential equations of motion are of delay-type. When applied to two-body dynamics such equations lead to orbital quantization. In this work, we use Einsteinian approach to reproduce this quantization. The developed formalism is tested by calculating the number of allowed stable orbits for planets and those for regular satellites of heavy planets resulting in good agreement with observational data. The paper also briefly discusses quantum mechanical nature of rings of heavy planets.

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I. Introduction

After almost hundred years since general relativity was formulated in its final form the Newtonian limit of Einsteinian relativity is still a hot topic of research e.g. see [1,2] and references therein. The problem stems from attempts to reconcile irreconcilable. For instance, in Einsteinian universe speed of
propagation of gravity is finite and is assumed to be that for the speed of light while in Newtonian it is infinite. Furthermore, as we demonstrated in [3-5], at least formally, in two dimensions the Einsteinian gravity is well defined/behaved while the Newtonian is not. These facts bring to life a whole array of recipes of either to modify Einsteinian or Newtonian gravity or both, e.g. [2,6]. The history of making such a recipes can be traced all the way back to 1804 when Soldner calculated bending of light using Newtonian mechanics [7] and got the value for the angle of deflection which is half of that obtained more than century later by Einstein. According to [8] calculation of (some or all) orbital precession for the Mercury had been "a fairly popular activity in 1890’s for physicists". Among these, the most notable is the calculation by Paul Gerber published in 1902 and reprinted in Annalen der Physik in 1917. Since Gerber’s paper came out long before Einstein’s with correct result for the perihelion shift for Mercury, Einstein was aware of his result. Republication of Gerber’s result in Annalen der Physik in 1917 was meant to demonstrate Einstein’s plagiarism. The situation in this case is very similar to that in atomic mechanics when Rutherford measured the differential cross section for scattering of electrons on hydrogen atom. The calculated by means of classical mechanics value of the cross section nicely matched experimental data. Everybody knows what happened next. In the case of relativity, Gerber was trying to use the velocity-dependent potential, whose origins remained obscure, to account for finite speed of propagation of gravity. Actually, his aim was not only to calculate the shift correctly but to use this calculation in order to figure out the speed of gravity. In 2010 this issue is still under active investigation [9]. Use of Gerber’s potential in calculations of bending of light not only produces result which is twice that known experimentally and correctly reproduced by Einstein [8] but, even more importantly, raises many questions regarding the way calculations were made. Specifically, in Einstein’s calculations the value for angular momentum for such calculation is infinite while in Newtonian-type calculations [7,8] it is obviously finite. It is possible to fiddle with Gerber’s equations and to use (artificially) the infinite value for the angular momentum. In which case one obtains the value for bending which is 1.5 bigger than that obtained by by Einstein. Using finite angular momentum produces already mentioned result which is twice bigger.

Photons are quantum objects and their treatment in general relativity is markedly different from that in Newton’s mechanics as we just mentioned. The bending of light obtained in both Newtonian and Einsteinian theories (perhaps with some non removable numerical discrepancies) is anticipated (recall that Soldner published his results for bending in 1804). Much more striking is not bending of light but existence of closed circular orbits for light predicted by general relativity. By rotating such an orbit around its symmetry axis one obtains what is known as "photon sphere" [10]. Even though such a sphere should exist around every black hole, apparently, it is difficult to probe it experimentally. This difficulty is eliminated recently when it became possible to model practi-

2E.g. see note on Paul Gerber in Wikipedia
http://en.wikipedia.org/wiki/Paul_Gerber
cally all known effects of general relativity in the laboratory [11-13]. Not only this is of intrinsic interest scientifically, but also this modelling has become a thriving area for designing of all kinds of photo and other devices in which the effects of general relativity are to be used commercially.

In this work we would like to discuss those aspects of general relativity which are related to celestial mechanics, not just to calculations of perihelion shift for Mercury. Incidentally, the results of such studies can also be tested both in the sky and in the lab. In particular, since the photon sphere can be reproduced in the lab [11], it can be analyzed both semiclassically (at the level of geometric optics) and quantum mechanically. In the last case one is confronted with the following problem: Can it be that only the photon sphere orbits could be treated quantum mechanically while the rest of massive orbits (geodesics) strictly classically? In this paper we argue in favor of treating both cases quantum mechanically.

Results and methods of this work differ substantially from that by Gine'. In a series of papers culminating in [2] he attempted to improve Gerber's calculations in order to reproduce Einstein's results for both the perihelion shift for Mercury and for bending of light. While formally succeeding in this task, he run across the delay differential equations replacing more familiar Newton's ordinary differential equations valid only in the absolute time. Methods of obtaining solutions of these delay equations are nontrivial and lead quite naturally to quantization. He checked his calculations using classical mechanics model of hydrogen atom accounting for delay(s) and got known quantum mechanical spectrum for hydrogen bound states. Next, he applied the same methods to the gravitational analog of hydrogen atom and got spectrum reproducing the Titius-Bode law of planetary distances. This law is also going to be discussed in this work in Section V. Gine' noticed, that in the case of gravitational quantization the Planck constant should be replaced by its celestial mechanics analog/equivalent. In this work we shall reach the same conclusion. Given this, his works are not without flaws. First, his equations of motion used for calculation of perihelion shift and for bending of light are not those used by Einstein for the same tasks. The obtained results agree with those by Einstein only in the leading terms. Second, the Titius-Bode law obtained in his work [14], and also reproduced by many authors cited in this paper is flawed. It is working well only for the planets closest to the Sun and is becoming increasingly inaccurate for more distant planets. More important is the fact that both Gine' and the rest of authors do not account for the fact that the number of planets around the Sun is finite. The same is true for the number of satellites of heavy planets. Unlike the case of Hydrogen atom, where the electroneutrality forbids more than one electron to be around proton, so that all excited states are either empty or can be occupied by the same electron excited from the ground state. In the sky there is no electroneutrality so that if one believes in the validity of the Titius-Bode law, one should anticipate a countable infinity of planets on a countable infinity of allowed orbits. This is not observed and is physically meaningless. Notice also, that classically the photon sphere can also accumulate unrestricted number of photons so that the observed mass of the black hole should grow in
observer’s time without bound. This is also physically meaningless. The photon devices made in the laboratory should provide an upper bound on the density of photons which photon sphere is capable of accommodating.

The ”quantization process” applied to celestial mechanics had been initiated not only in calculations by Gine’ and other authors (mentioned in his paper), it had also began in recent papers [15-18] discussing problems of space travel within Solar System. Mathematically, these problems are analogous to those encountered in quantum treatment of chemical kinetics of polyatomic molecules. The following quotation from the paper by Porter and Cvitanović [15] nicely illustrates the essence of these problems. "Almost perfect parallel between the governing equations of atomic physics and celestial mechanics implies that the transport mechanism for these two situations is virtually identical; on the celestial scale, transport takes a spacecraft from one Lagrange point to another until it reaches its desired destination. On the atomic scale, the same type of trajectory transports an electron initially trapped near the atom across the escape threshold (in chemical parlance, across a "transition state"), never to return. The orbits used to design space missions thus also determine the ionization rates of atoms and chemical reaction rates of molecules". This statement is nicely illustrated in the paper by Jaffe et al [18] in which it is reported that the transition state theory developed initially in chemistry (to describe the rates of chemical reactions) is working actually better in celestial mechanics where the discrepancy between the chemical theory and numerical simulations (done for celestial mechanics transport problems) is less than 1%. The current status of transition state theory at the quantum and classical levels in chemistry is nicely described in the recent book by Micha and Burghardt [19].

Before discussing the organization of the rest of this paper we would like to mention the following empirical facts. If $M_\odot$ is the mass of the Sun (or, respectively, heavy planet such as Jupiter, Saturn, etc.) and $m_i$ be the mass of an i-th planet (respectively, the i-th satellite of heavy planet). Make the ratio $r_i = \frac{m_i}{m_i + M_\odot}$. The analogous ratios can be constructed for respective heavy planets (Jupiter, Saturn, Uranus, Neptune) and for any of their satellites. The observational data indicate that with only two exceptions: Earth-Moon (for which $r \sim 10^{-2}$), and Pluto-Charon (for which $r \sim 10^{-1}$), all other ratios in the Solar System are of order $10^{-6} - 10^{-3}$[20]. Under such circumstances the center of mass of such a binary system practically coincides with that for $M_\odot$. And if this is so, then the respective trajectories can be treated as geodesics. Hence, not only motion of the Mercury can be treated in this way, as it was done by Einstein, but also motion of almost any satellite in the Solar System! These empirical facts, plus those discussed in Section V, are compelling enough to warrant calculations accounting for superiority of Einsteinian gravity over

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3That is point of equilibrium.

4E.g. see also paper by Convay al [17] for details.

5For planets $i = 1 \div 9$ while for satellites of heavy planets this number is different as explained below, in the text.

6Regrettably, not our Moon! The description of dynamics of Moon is similar to that for rings of heavy planets (to be discussed in Section V) and, as such, is also quantizable.
Newtonian already at the scales of our Solar System.

**Organization of the rest of this paper**

The paper is made of six sections and three appendices. In Section II we provide some historical discussion, beginning with works of Laplace and Poincaré on celestial mechanics. When these results are reinterpreted in modern terms and superimposed with the latest combinatorial formulations of quantum mechanics, they provide a foundation for thinking about dynamics of Solar System quantum mechanically. These general results are based on classical mechanics formalism. Such formalism is unable to treat photons (or neutrinos, etc.) and particles on the same footing. Only after discovery of general relativity this had become possible. Thus, in Section III classical results of Section II are reanalyzed using formalism of general relativity. In this section we provide detailed arguments in favor of the quantum nature of the motion of planets on geodesics. Such a conclusion is compatible with results of Section II. Section III can be looked upon as an analog of the mathematical proof of existence. To convert these abstract results into numbers requires development of this formalism. It is presented in Sections IV and V. In Section IV we argue that the formalism of quantum mechanics in its conventional form is not useful for development of quantum celestial mechanics. In the same section we extend this formalism in order to make it compatible with transformations normally used in general relativity. Based on formalism developed in Section IV, we present actual numerical calculations in Section V. These are done both for planets and for satellites of heavy planets culminating in Table 2 containing the main results of this paper. In Table 2 we compare our analytical calculations of available number of stable orbits for planets and of the number of stable orbits for regular satellites of heavy planets with the empirically available information. The obtained theoretical results are in surprisingly good agreement with empirically observed data. They are in accord with quantum mechanical rules for filling of stable orbits formulated in Section V. In the same section we provide some discussion on quantum nature of rings around heavy planets compatible with results of general relativity. Section VI is devoted to summary and discussion. In this section we stress that the quantization discussed in this paper is not exoteric phenomenon. This is so because most of known in Nature stochastic processes are of Poisson-Dirichlet-type. Quantization associated with these processes is more ubiquitous than is anticipated from reading of physics literature on stochastic processes. In the same section we also discuss some problems left for future work.

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7 The word “regular” is defined in Section V
8 Details on the Poisson-Dirichlet-type stochastic processes are given in Section VI.
A. From Laplace to Einstein and via Poincare' 

Even though classical Hamiltonians for Coulombic and Newtonian potentials look almost the same, they are far from being exactly the same. In the classical Hamiltonians for multielectron atoms all electron masses are the same, while for the Solar-like system the masses of all satellites are different. Using general relativity such difference can be made non existent. This fact does not come from the formalism of general relativity. It is just the empirical fact coming from a huge disparity in masses of the Sun and the planets (or the heavy planet and its satellites). In view of such a disparity one can proceed with formal quantization for both systems using the same formalism. To explain how this happens, we begin with two-body Kepler problem treated in representative physics textbooks [21]. Such treatments tend to ignore the equivalence principle-essential for the gravitational Kepler problem and nonexistent for the Coulomb-type problems. Specifically, the description of general relativity in Vol.2 of the world-famous Landau-Lifshitz course in theoretical physics [22] begins with the Lagrangian for the particle in gravitational field $\varphi$: $L = \frac{m v^2}{2} - m \varphi$. The Newton’s equation for such a Lagrangian reads:

$$\dot{v} = -\nabla \varphi.$$ (1)

Since the mass drops out of this equation, it is possible to think about such an equation as an equation for a geodesic in (pseudo)Riemannian space. This observation, indeed, had lead Einstein to full development of theory of general relativity, to his calculation of the Mercury’s perihelion shift, etc. The above example is misleading though. Indeed, let us discuss the 2-body Kepler problem for particles with masses $m_1$ and $m_2$ interacting gravitationally. The Lagrangian for this problem is given by

$$L = \frac{m_1}{2} \dot{r}_1^2 + \frac{m_2}{2} \dot{r}_2^2 + \gamma \frac{m_1 m_2}{|r_1 - r_2|}. \quad (2)$$

Introducing, as usual, the center of mass and relative coordinates via $m_1 r_1 + m_2 r_2 = 0$ and $r = r_1 - r_2$, the above Lagrangian acquires the following form:

$$L = \frac{\mu}{2} \dot{r}^2 + \gamma \frac{m_1 m_2}{|r|} \equiv \frac{m_1 m_2}{m_1 + m_2} \left( \frac{\dot{r}^2}{2} + \gamma \frac{(m_1 + m_2)}{|r|} \right), \quad (3)$$

where, as usual, we set $\mu = \frac{m_1 m_2}{m_1 + m_2}$. The constant $\frac{m_1 m_2}{m_1 + m_2}$ can be dropped and, after that, instead of the geodesic, Eq. (1), we obtain the equation for a fictitious point-like object of unit mass moving in the field of gravity produced by the point-like body of mass $m_1 + m_2$. Clearly, in general, one cannot talk about
geodesics in this case even though Infeld and Schild had attempted to do just this already in 1949 [23]. The case is far from being closed even in 2010 [24].

These efforts look to us mainly as academic (unless dynamics of binary stars is considered) for the following reasons. If, say, \( m_1 \gg m_2 \) as for the electron in Hydrogen atom or for the Mercury rotating around Sun one can (to a very good accuracy) discard mass \( m_2 \) thus obtaining the equation for a geodesic coinciding with Eq. (1). In the Introduction we defined the ratio \( r = \frac{m_2}{m_1 + m_2} \). If we do not consult general reality for guidance, the ratio \( r \) can have any nonnegative value. However, what is observed in the sky (and in atomic systems as well) leads us to the conclusion that (ignoring our Moon) all satellites of heavy planets as well as all planets of our Solar System move along geodesics described by Eq. (1), provided that we can ignore interaction between the planets/satellites. We shall call such an approximation the Einsteinian limit. It is analogous to the mean field Hartree-type approximation in atomic mechanics. If we believe Einstein, then such Hartree-type approximation does not require any corrections. This looks like "too good to be true". Indeed, the first who actually used Einstein’s limit (more then 100 years before Einstein!) in his calculations was Laplace [25], Vol.4. In his book [26], Vol.1, article 50, Poincare discusses Laplace’s work on dynamics of satellites of Jupiter. Quoting from Poincare:

"(Following Laplace) consider the central body of large mass (Jupiter) and three other small bodies (satellites Io, Europe and Ganymede), whose masses can be taken to be zero, rotating around a large body in accordance with Kepler’s law. Assume further that the eccentricities and inclinations of the orbits of these (zero mass) bodies are equal to zero, so that the motion is going to be circular. Assume further that the frequencies of their rotation \( \omega_1, \omega_2 \) and \( \omega_3 \) are such that there is a linear relationship

\[
\alpha \omega_1 + \beta \omega_2 + \gamma \omega_3 = 0
\]

with \( \alpha, \beta \) and \( \gamma \) being three mutually simple integers such that

\[
\alpha + \beta + \gamma = 0.
\]

Given this, it is possible to find another three integers \( \lambda, \lambda' \) and \( \lambda'' \) such that \( \alpha \lambda + \beta \lambda' + \gamma \lambda'' = 0 \) implying that \( \omega_1 = \lambda A + B, \omega_2 = \lambda' A + B, \omega_3 = \lambda'' A + B \) with \( A \) and \( B \) being some constants. After some time \( T \) it is useful to construct the angles \( T(\lambda A + B), T(\lambda' A + B) \) and \( T(\lambda'' A + B) \) describing current location of respective satellites (along their circular orbits) and, their differences: \((\lambda - \lambda')AT\) and \((\lambda - \lambda'')AT\). If now we choose \( T \) in such a way that \( AT \) is proportional to \( 2\pi \), then the angles made by the radius-vectors (from central body to the location of the planet) will coincide with those for \( T = 0 \). Naturally, such a motion (with zero satellite masses) is periodic with period \( T \).

The question remains: Will the motion remain periodic in the case if masses are small but not exactly zero? That is, if one allows the satellites to interact with each other?....

Laplace demonstrated that the orbits of these three satellites of Jupiter will differ only slightly from truly periodic. In fact, the locations of these satellites are oscillating around the zero mass trajectory"
Translation of this last paragraph into language of modern quantum mechanics reads: Laplace demonstrated that only the Einsteinian trajectories are subject to the Bohr-Sommerfel’d type quantization condition. That is at the scales of Solar System correctness of Einsteinian general relativity is assured by correctness of quantum mechanics (closure of the Laplace-Lagrange oscillating orbits [27]) and vice versa so that these two theories are inseparably linked together.

The attentive reader of this excerpt from Poincare could already realized that Laplace came to his conclusions based on Eq. (4) as starting point. Thus, the condition, Eq. (4), can be called quantization condition (since eventually it leads to the Bohr-Sommerfel’d condition). Interestingly enough, this condition was chosen by Heisenberg [28] as fundamental quantization condition from which all machinery of quantum mechanics can be deduced! This topic is discussed further in the next subsection. Before doing so, we notice that extension of work by Laplace to the full \( n + 1 \) body planar problem was made only in 20th century and can be found in the monograph by Charlier [29]. More rigorous mathematical proofs involving KAM theory have been obtained just recently by Fejoz [30] and Biasco et al [31]. The difficulty, of course, is caused by proper accounting of the effects of finite but nonzero masses of satellites and by showing that, when these masses are very small, the Einsteinian limit makes perfect sense and is stable. A sketch of these calculations for planar four-body problem (incidentally studied by de Sitter in 1909) can be found in a nicely written lecture notes by Moser and Zehnder [32].

B. From Laplace to Heisenberg and beyond

We begin with observation that the Schrödinger equation cannot be reduced to something else which is related to our macroscopic experience. It has to be postulated\(^9\) On the contrary, Heisenberg’s basic equation from which all quantum mechanics can be recovered is directly connected with experimental data and looks almost trivial. Indeed, following Bohr, Heisenberg looked at the famous equations for energy levels difference

\[
\omega(n, n - \alpha) = \frac{1}{\hbar}(E(n) - E(n - \alpha)),
\]

where both \( n \) and \( n - \alpha \) are some integers. He noticed [28] that this definition leads to the following fundamental composition law:

\[
\omega(n - \beta, n - \alpha - \beta) + \omega(n, n - \beta) = \omega(n, n - \alpha - \beta). \tag{7a}
\]

\(^9\) Usually used appeal to the DeBroigle wave-particle duality is of no help since the wave function in the Schrödinger’s equation plays an auxiliary role.
Since by design $\omega(k,n) = -\omega(n,k)$, the above equation can be rewritten in a symmetric form as

$$\omega(n,m) + \omega(m,k) + \omega(k,n) = 0.$$  \hspace{1cm} (7b)

In such a form it is known as the honeycomb equation (condition) in current mathematics literature [33-35] where it was rediscovered totally independently of Heisenberg’s key quantum mechanical paper and, apparently, with different purposes in mind. Connections between mathematical results of Knutson and Tao [33-35] and those of Heisenberg were noticed and discussed in recent papers by Kholodenko [36,37]. We would like to use some results from these works now.

We begin by noticing that Eq.(7b) due to its purely combinatorial origin does not contain the Planck’s constant $\hbar$. Such fact is of major importance for this work since the condition Eq. (4) can be equivalently rewritten in the form of Eq.(7b), where $\omega(n,m) = \omega_n - \omega_m$. It would be quite unnatural to think of the Planck’s constant in this case. Eq. (7b) looks almost trivial and yet, it is sufficient for restoration of all quantum mechanics. Indeed, in his paper of October 7th of 1925, Dirac [38], being aware of Heisenberg’s key paper, streamlined Heisenberg’s results and introduced notations which are in use up to this day. He noticed that the combinatorial law given by Eq.(7a) for frequencies, when used in the Fourier expansions for composition of observables, leads to the multiplication rule $a(mn)b(mk) = ab(mk)$ for the Fourier amplitudes for these observables. In general, in accord with Heisenberg’s assumptions, one expects that $ab(mk) \neq ba(mk)$. Such a multiplication rule is typical for matrices. In the traditional quantum mechanical language such matrix elements are written as $< n | \hat{O} | m > \exp(i\omega(n,m)t)$ so that Eq.(7b) is equivalent to the matrix statement

$$\sum_m < n | \hat{O}_1 | m > < m | \hat{O}_2 | k > \exp(i\omega(n,m)t) \exp(i\omega(m,k)t)$$

$$= < n | \hat{O}_1 \hat{O}_2 | k > \exp(i\omega(n,k)t).$$  \hspace{1cm} (8)

for some operator (observables) $\hat{O}_1$ and $\hat{O}_2$ evolving according to the rule: $\hat{O}_k(t) = U\hat{O}_kU^{-1}, k = 1,2$, provided that $U^{-1} = \exp(-i\hat{H}\hbar t)$. From here it follows that $U^{-1} | m >= \exp(-\hat{H}\hbar t) | m >$ if one identifies $\hat{H}$ with the Hamiltonian operator. Clearly, upon such an identification the Schrödinger equation can be obtained at once as is well known [39] and with it, the rest of quantum mechanics. In view of [33-37] it is possible to extend the traditional pathway: from classical to quantum mechanics and back. We begin discussion of this topic in the next section and shall continue doing so from other perspectives in the rest of this paper.

### III. Newtonian Limit of Einsteinian Gravity and Quantum Dynamics of

This paper was sent to Dirac by Heisenberg himself prior to its publication.
Solar System (Specifics)

After plausible and suggestive discussion of previous section it is instructive to arrive at the same conclusions via entirely different route. This task is accomplished in this section.

In developing his theory of general relativity Einstein was aware of previous efforts at calculation of bending of light and perihelion shift for Mercury based on Newton’s mechanics. In particular, bending of light based on Newtonian theory was calculated by Soldner in 1804 [7,8]. Calculation of the perihelion of Mercury, as well as other planets was done by Le Verrier [8] in 1851. His computations yielded 526.7 seconds of arc per century as compared with best modern theoretical value of 532. At the same time the observed precession is known to be 575. The difference between these two is 43 seconds of arc per century. It is famous Einstein’s result for perihelion shift for Mercury. In Einstein’s calculations no mention of Le Verrier’s results were made. This leaves us with the following problem. If we believe Einstein, then Mercury should move on a geodesic. But if this is so, then how to look at Le Verrier’s result obtained by explicit account of gravitational interactions between the Mercury and the rest of planets? This brings us back to results of previous section. This time, however, we would like to look at the same problem differently. For this purpose we would like to discuss briefly results by Le Verrier in a simplified form taken from [8].

A. Sketch of LeVerrier’s calculations

Taking into account planarity of motion in the gravitational field we introduce the polar coordinates \( r(t) \) and \( \varphi(t) \) of the particle of unit mass (e.g. see Eq. (3)) moving in the gravity field of massive body of mass \( M \). Newton’s equations are given by\(^{12}\)

\[
\ddot{r} - r\dot{\varphi}^2 = -\frac{M}{r^3} \quad \text{and} \quad r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0
\]

implying conservation of the angular momentum \( L = r^2\dot{\varphi} \) and allowing the first of Eqs (9) to be rewritten as follows

\[
\ddot{r} - \frac{L^2}{r^3} = -\frac{M}{r^2}.
\]

\(^1\)E.g. see Wikipedia
\[\text{http://en.wikipedia.org/wiki/Urbain_Le_Verrier}\]
\(^2\)In the system of units in which the gravitational constant \( \gamma \) was put equal to one.
Taking into account that \( \dot{\varphi} = \frac{L}{r^2} \), the following chain of transformations

\[
\dot{r} = \left( \frac{d\varphi}{dt} \frac{dt}{dr} \right) \frac{dr}{dt} = \frac{L^2}{r^2} \frac{dr}{d\varphi} \tag{11}
\]

and

\[
\ddot{r} = L \left( \frac{d\varphi}{dt} \frac{dt}{dr} \right) \frac{d}{dt} \left( r^{-2} \frac{dr}{d\varphi} \right) = \frac{L^2}{r^2} \frac{d}{d\varphi} \left( r^{-2} \frac{dr}{d\varphi} \right) \tag{12}
\]

is useful. Substituting this result in Eq.(10) produces

\[
\frac{d}{d\varphi} \left( r^{-2} \frac{dr}{d\varphi} \right) - \frac{1}{r} = -\frac{M}{L^2} \tag{13}
\]

Let now \( u = \frac{1}{r} \), then the above equation acquires especially simple form

\[
\frac{d^2 u}{dt^2} + u = \frac{M}{L^2} \tag{14}
\]

The constant term on the r.h.s can be easily eliminated so that we are left with equation of motion for the harmonic oscillator. Consider now the related, more general, equation

\[
\frac{1}{\Omega^2} \frac{d^2 u}{d\varphi^2} + u = \frac{1}{P} \tag{15}
\]

where both \( \Omega^2 \) and \( P \) are some constants. This equation admits solution

\[
u(\varphi) = \frac{(1 + k \cos(\Omega \varphi))}{P} \tag{16}
\]

in which \( k \) is a constant of integration. This result can be converted into polar equation for an ellipse. In case of Eq.(14) it is given by

\[
 r(\varphi) = \frac{L^2}{M} \frac{1}{1 + k \cos \varphi} \tag{17}
\]

We would like now to complicate situation as follows. Suppose that the obtained results, say, are for the Mercury. But then, in its current form they are unrealistic since Mercury is interacting only with the Sun. To account for interactions of Mercury with other planets requires much more work. This was done by Le Verier and takes about 150 pages\textsuperscript{13}. Fortunately, there is a much easier method known as the mass ring model described in [8] which produces results very close to those by Le Verier. The idea of the method lies in replacement of the planets other than Mercury by gravitating rings of masses \( m_i \) \((i = 1 - 8)\) centered at the Sun and located further away from the Sun at the radial distance \( R_i \). In such a case Mercury will be experiencing the inward force coming from the Sun and outward force coming from rings. The outward potential \( \psi(r) \) of a particular

\textsuperscript{13}E.g.see footnote 11.
ring at the distance \( r \) from the Sun and in the plane of a ring (which is the same for all planets) is given by

\[
\psi(r) = -\frac{m}{R^3} + \frac{1}{4} \left( \frac{r}{R} \right)^2 + \frac{9}{64} \left( \frac{r}{R} \right)^4 + \frac{25}{256} \left( \frac{r}{R} \right)^6 + \cdots
\]  

(18)

In view of this result, the Newton’s equation of motion, Eq.(10), is modified in the presence of a ring as follows

\[
\ddot{r} - \frac{L^2}{r^3} = -\frac{M}{r^2} + \alpha_1 r + \alpha_2 r^3 + \cdots
\]  

(19)

with \( \alpha_1 = \frac{m}{2R^2} \), \( \alpha_2 = \frac{m}{16R^4} \), and so on. Let \( r_1 \) and \( r_2 \) be the minimum and maximum radial distances from the Sun for Mercury. These numbers can be used for simplification of Eq.(19). Indeed, it can be brought to the form

\[
\ddot{r} - \frac{L^2}{r^3} = -\frac{A}{r^2} - \frac{B}{r^3},
\]  

(20)

provided that the constants \( A \) and \( B \) are determined from equations

\[
\frac{A}{r_1^2} + \frac{B}{r_1^3} = \frac{M}{r_1^2} - \alpha_1 r_1 - \alpha_2 r_1^3 - \cdots
\]  

(21a)

and

\[
\frac{A}{r_2^2} + \frac{B}{r_2^3} = \frac{M}{r_2^2} - \alpha_1 r_2 - \alpha_2 r_2^3 - \cdots.
\]  

(21b)

If \( r_1 \) differs not much from \( r_2 \) (small eccentricity) it is possible to introduce the mean orbital radius \( r_0 = \frac{1}{2}(r_1 + r_2) \) so that the constants \( A \) and \( B \) can be represented by the following power series expansions

\[
A = M - 4\alpha_1 r_0^3 - 6\alpha_2 r_0^5 - \cdots
\]  

(22a)

and

\[
B = 3\alpha_1 r_0^4 + 5\alpha_2 r_0^6 + \cdots.
\]  

(22b)

Using these results Eq.(20) can now be rewritten in the form of Eq.(14) (or Eq.(15)). This can be done as follows. By writing Eq.(20) as

\[
\ddot{r} - \left( \frac{L^2 - B}{r^3} \right) = -\frac{A}{r^2}
\]  

(23a)

and by assuming that the angular momentum \( L \) is conserved it is convenient to rewrite the above equation as follows

\[
\frac{1}{1 - \frac{L^3}{r^3}} \ddot{r} - \frac{L^2}{r^3} = -\frac{A}{r^2} \frac{1}{1 - \frac{L^3}{r^3}}.
\]  

(23b)

\footnote{We drop the subscript \( i \) for brevity.}
For this equation we can repeat the same steps as lead from Eq.(10) to Eq.(14) in order to obtain the equation analogous to Eq.(15) in which \( \Omega = \sqrt{1 - \frac{B}{L^2}} \) and \( P = \frac{(1 - \frac{B}{L^2})}{A} \). Elementary calculation, e.g. see [21], produces: \( L^2 = Mr_0 \).

Using this result we obtain,

\[
\Omega = \sqrt{1 - \frac{B}{L^2}} \simeq 1 - \frac{1}{2} \frac{B}{Mr_0}. \tag{24}
\]

It can be shown using this result for \( \Omega \) [8] that contribution of the particular ring to the Newtonian shift is given by

\[
\Delta \varphi = \frac{\pi B}{Mr_0}
= \pi \left( \frac{m}{M} \right) \frac{3}{2} \left( \frac{r_0}{R} \right)^3 + \frac{45}{16} \left( \frac{r_0}{R} \right)^5 + \cdots. \tag{25}
\]

To use this result for calculation of the perihelion shift for Mercury, we have to add up contributions, e.g. numerical values for masses and mean radius, for all rings/planets (excluding Pluto and including the asteroid belt). The obtained result is 549.7 seconds of arc per century compares well with Le Verrier’s 526.7. We shall analyse these results further in the next subsection.

B. Sketch of Einstein-type calculations

In [40] Le Verrier’s result for the perihelion shift for Mercury was characterized as “the first relativistic gravity effect observed”. Based on results of preceding subsection this statement is incorrect. Such a conclusion is in accord with commonly accepted point of view, e.g. read Chr.9, paragraph 5 of [41]. The situation however is more delicate as it appears. In this and the following subsection we would like to discuss why this is so.

Following Ref.[8], in Einstein’s case the equation of motion replacing Eq.(10) is given by \((c = 1)\)

\[
\ddot{r} - \frac{L^2}{r^3} = -\frac{M}{r^2} - \frac{3ML^2}{r^4}. \tag{25}
\]

Superficially, it differs from Eq.(10) only by the presence of last term. However, in Einstein’s case the evolution is taking place not in Newton’s time but in proper time [8,41]. Since this fact does not change the mathematical treatment of such an equation , we can go through the same steps as in previous subsection. That is, instead of Eq.(14), this time we obtain

\[
\frac{d^2u}{d\varphi^2} + u = \frac{M}{L^2} + 3Mu^2. \tag{26}
\]

Presence of the last term causes this equation to be nonlinear so that its exact solution, in principle, is much harder to find. To obtain Einstein’s result for the
perihelion shift of Mercury, it is sufficient to use just a perturbation theory. For this purpose, again, following [8], we rewrite Eq.(26) in the equivalent form

$$u = \frac{1}{6m}(1 - \sqrt{1 - 12\left(\frac{M}{L^2} - \dot{M}\right)}) \approx \frac{1}{6m}[\frac{1}{2}(12M\left(\frac{M}{L^2} - \dot{M}\right)) + \frac{1}{8}(12M\left(\frac{M}{L^2} - \ddot{M}\right))^2 + \cdots]. \quad (27)$$

Rearranging terms in this result brings us back to Eq.(15), in which \(\Omega^{-1} = \sqrt{1 + 6\left(\frac{M}{L^2}\right)^2}\), and \(P^{-1} = \frac{M}{L} + 3\frac{M^3}{L^4}\). In arriving at this result, in accord with [8], extra terms containing \(3M\ddot{u}^2\) were dropped. This is legitimate because of the following. From observational astronomy it is known that all three Kepler’s laws hold to a large degree of accuracy for planets of Solar System. This means that Eq.(15) is sufficiently adequate for description of planetary motion. But if this is correct, then perturbation coming from the term \(3M\ddot{u}^2\) should be negligible. Thus, we arrive at the following relativistic analog of Eq.(16)

$$r(\varphi) = \left[\frac{L^2/M}{1 + 3\left(\frac{M}{L}\right)^2}\right] \frac{1}{1 + k\cos(\Omega\varphi)}. \quad (28)$$

If \(\Omega\) would be equal to one, the above equation is converted into standard equation for an ellipse with origin at one focus and eccentricity \(k\). But since \(\Omega < 1\) the angle \(\varphi\) must go beyond \(2\pi\) in order for the radial distance to complete one cycle around the ellipse. This is being interpreted as precession. In the present case the precession angle per revolution is \(\Delta\varphi = 2\pi(\frac{1}{\Omega} - 1)\). Since

$$\Omega \approx 1 - \frac{3}{2}\left(\frac{M}{L}\right)^2 + \frac{27}{2}\left(\frac{M}{L}\right)^4 + \cdots$$

and expecting that \(\left(\frac{M}{L}\right) \ll 1\), we obtain celebrated Einstein’s result for perihelion shift of Mercury [42]

$$\Delta\varphi \approx 6\pi \left(\frac{M}{L}\right)^2. \quad (29)$$

C. From analysis to synthesis

In this subsection we connect the results obtained in this section thus far with those of Section II. Following Weinberg [41] (Ch.r 8, paragraph 6), we notice that among all tests of general relativity the measurement of the perihelion shift (say, for the Mercury) is the most important one. Such a measurement cannot be done directly though. In practice what is being measured is the total shift \(\Delta\varphi_{\text{obs}}\). Then, the Newtonian portion (e.g. calculated in subsection A) is subtracted from \(\Delta\varphi_{\text{obs}}\) (e.g. see Eq.(8.6.13) of Ref. [41]). The remainder is
the famous Einsteinian part of the shift theoretically calculated in subsection B. Although formally such a procedure makes sense, it is not in accord with results of Section II. In Section II the Einsteinian limit was defined in which planets/satellites are not interacting with each other in zeroth order approximation (the massless limit). Interaction effects are taken into account by considering stability of the Einsteinian orbits (geodesics) against small gravitational perturbations caused by interplanet/intersatellite interactions. These interactions caused Einsteinian orbits to become Laplace-Lagrange oscillating orbits. These are analogs of standing waves for electrons in Bohr’s model of atom. The above computational scheme uses Newtonian mechanics in which the Newtonian Eq.(1) is identified with equation for Einsteinian geodesic. Such an identification requires some care as explained in detail in [41] (Ch.r 8, paragraph 4). The difficulty stems from the fact that in Einsteinian theory gravitational interactions are propagating with finite speed while in Newtonian gravity the speed is infinite. This causes in Einsteinian theory to use the proper time. It is surely not the same as Newtonian time. To match these two theories in the limit of small velocities it is instructive to discuss the compatibility of Einsteinian dynamics with Keplerian laws. Fortunately, this issue is considered in detail in [8], (Ch.5.5). This allows us to reduce our discussion to the absolute minimum. The fact that Newton’s law of gravity was deduced from Kepler’s laws is well known. Much less is known that the Schwarzschild metric can also be restored using the 3rd Kepler’s law. To write this law explicitly requires few steps which would be unnecessary should popular textbooks contain needed information. In these textbooks the Lagrangian \( L \) for a particle of mass \( m_2 \) interacting with another particle of mass \( m_1 \) is given by Eq.(2) or, after usual reduction to center of mass coordinates, by

\[
L = \frac{\mu}{2} \dot{r}^2 + \frac{\gamma m_1 m_2}{|r|}. \tag{30}
\]

In such a form it is given, for example, in [21]. For our purposes it is more advantageous to use another form of the Lagrangian

\[
L = \frac{\dot{r}^2}{2} + \frac{\gamma M}{|r|}. \tag{31}
\]

also given in Eq.(3). Evidently, in Eq.(31) \( M \simeq m_1 \). If \( T \) is period of revolution of the fictitious particle of unit mass around the origin of coordinate system centered at \( M \), then the 3rd Kepler’s law reads (\( \gamma = 1 \) as before)

\[
T = 2\pi \sqrt{\frac{1}{M}} \text{ or } \omega^2 r_0^3 = M. \tag{32}
\]

In this formula we used \( r_0 = \frac{1}{2}(r_1 + r_2) \) as before and \( \omega = 2\pi / T \). But, because \( \omega = \dot{\varphi} = \frac{d}{d\tau} \varphi \), it is permissible to write as well \( \omega = \frac{d}{dt} \varphi = \frac{d}{d\tau} \varphi \left( \frac{d\tau}{dt} \right) \), where \( \tau \) is some function of Newton’s time \( t \). Now we would like to identify \( \tau \) with proper time of Einsteinian gravity. This is needed in view of the following. Recall that in polar coordinates Einsteinian’s equation of motion for fictitious
particle of unit mass is given by Eq.(25). It differs from the Newtonian’s Eq.(3) in two features. First, instead of Newtonian absolute time, in the present case the differentiation is taking place over the proper time. Second, there is an extra potential term absent in Newtonian mechanics. This difference between the Newtonian and Einsteinian formulations of mechanics is only apparent. It can be eliminated with help of the 3rd Kepler’s law, Eq.(32), as explained in [8]. Since this explanation is scattered all over this reference, we would like to describe it below in a concise form. Even without going into fine details, the results of previous subsections indicate that the Einsteinian result for perihelion shift of Mercury can be formally obtained simply by renormalizing the Newtonian potential(s). Incidentally, this was the original Le Verrier’s idea since he believed that between Mercury and Sun there should be yet another planet (Vulcan) which only remains to be discovered. Hence, such an homomorphism between the Newtonian and Einsteinian mechanics of point particles is assured from this (observational) point of view. Nevertheless, as is well known, absence of Vulcan had brought to life general relativity. It is instructive to supply the derivation of this homomorphism explicitly.

For this purpose we take into account that for the particle of unit mass
\[ L = \omega r^2. \]
Using this result Eq.(25) can be rewritten as
\[ \ddot{r} = -\frac{M}{r^2} + \omega^2 r (1 - \frac{3M}{r}). \]  
(33)
To analyze this equation further we need to recall that for a non-rotating body the Schwarzschild metric has the form
\[ (d\tau)^2 = [1 - \frac{2M}{r}] (dt)^2 - \frac{1}{1 - \frac{2M}{r}} (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2 \theta (d\phi)^2. \]  
(34)
Consider a special case of this metric for a motion in the equatorial plane \( \theta = \frac{\pi}{2} \) taking place on a circle of radius \( R \). In such a case the metric can be rewritten in polar coordinates as
\[ (d\tau)^2 = [1 - \frac{2M}{R}] (dt)^2 - R^2 (d\varphi)^2, \]  
(35)
where we have relabeled \( \phi \leftrightarrow \varphi \) for consistency with previous notations. If we treat \( R \) as \( r_0 \) (defined in Eq.(32)), then we can use the 3rd Kepler’s law in order to write
\[ R^2 = \frac{M}{(\omega^2 R)} = \left( \frac{dt}{d\varphi} \right)^2 \frac{M}{R}. \]  
(36)
Upon substitution of this result into Eq.(35) we obtain,
\[ \left( \frac{d\tau}{dt} \right)^2 = 1 - \frac{3M}{R}. \]  
(37)
This result can be used in Eq.(33) (evidently for closed orbits only with not too large eccentricity). Then, using Eq.(37) we obtain,
\[ \omega^2 (1 - \frac{3M}{r}) = \left[ \frac{d}{d\tau} \varphi \left( \frac{d\tau}{dt} \right) \right]^2 = \omega^2, \]  
(38)
where we relabeled $R \to r_0 \to r$ and used the fact that $\omega = \dot{\varphi} = d \varphi / d \tau$ is the time derivative of $\varphi$ with respect to the *Schwarzschild coordinate time*. This result allows us to rewrite the Einsteinian Eq.(33) in Newtonian form

$$\ddot{r} - \dot{\varphi}^2 r = -\frac{M}{r^2} \tag{39}$$

formally coinciding with Eq.(9). This result is only a homomorphism in view of the following. Recently [43], the reverse result was formally obtained. Beginning with Newtonian equations of motion, Einstein’s result, Eq.(25), was obtained after nontrivial (but otherwise ad hoc) changes in space and time coordinates. Naturally, since authors of [43] were using the absolute Newtonian time at the beginning, the derivation of Einstein’s result, Eq.(25), in their work is formal as the authors acknowledge. Nevertheless, Eq.(39) underscores the non triviality of time transformations in Newtonian mechanics. In particular, we notice that for $\dot{R} = 3M$ in Eq.(37) $\frac{d \tau}{dt} = 0$. This result is of physical significance. It describes the radius of the photon sphere. This result can be easily derived by noticing that in the case of light Eq.(26) acquires the form [8]

$$\frac{d^2 u}{d \varphi^2} + u = 3Mu^2 \tag{40}$$

formally implying that for photons $L \to \infty$. Both Eq.(26) and (40) possess a fixed point solution $u^* = \text{const}$ implying that either

$$u^* = \frac{M}{L^2} + 3Mu^{*2} \quad \text{or} \quad u^* = 3Mu^{*2}, \tag{41}$$

provided that

$$\frac{d}{d\tau} \varphi = Lu^{*2}. \tag{42}$$

In the "massive" case we have

$$L^2(u^* - 3Mu^{*2}) = M$$

so that

$$\left( \frac{d}{d\tau} \varphi \right)^2 = \frac{M}{r^{*2}(r^* - 3M)}. \tag{43}$$

The positivity of the l.h.s is implying that $r^* - 3M > 0$. That is for massive particles all circular orbits should have radius $r > 3M$. The radius $r^* = 3M$, e.g. see Eq.(41), is possible only for photons and describes the photon sphere.

Consider now the stability of the photon sphere. It can be easily calculated using the following representation of Eq.(40)

$$\frac{d u}{d \varphi} = p \tag{44a}$$

\[\text{That is, it is permissible to parametrize the proper time with help of Newtonian time but not the other way around.}\]
and
\[ \frac{dp}{d\phi} = u(3Mu - 1). \] (44b)

For \( u^* \) such that \( 3Mu^* = 1 \) standard stability analysis produces eigenvalues \( \varepsilon = \pm 1 \) for the stability matrix indicating that \textit{classically} the photon sphere orbits are unstable. Since photons are quantum objects such classical analysis may or may not be valid. Thus, in the case of photon sphere we are confronted with an unusual situation. While we began our analysis using classical mechanics (classical general relativity), we ended up with the necessity to discuss quantum mechanical effects in general relativity. Such situation is not totally unexpected in view of recent attempts to use the formalism of general relativity for designing of variety of photo devices in laboratory conditions [11-13]. Whether in real world or in laboratory, the progress in understanding quantum effects in general relativity begins with understanding the difference between Eqs(1) and (3) (or between Lagrangians given by Eqs (30) and (31)). This has been recognized by Infeld’ and Schild who attempted to analyze this difference already in 1949 [23]. The case is far from being closed even in 2010 [24]. The decisive attempt to describe the motion of extended objects in general relativity was made in seminal paper by Papapetrou [44] and continues up to the present day. From his papers it is known that, strictly speaking, the motion of extended bodies is not taking place on geodesics. And yet, for the Mercury such an approximation made by Einstein works extremely well as we just have demonstrated. Hence, the main issue is to understand the difference between the massless (photon) and the massive cases. Evidently, the Lagrangian, Eq.(31), remains the same as long as the mass \( m_2 \) in Eq.(3) or (31) is small but nonzero. In particular, this means that such a mass can be made arbitrarily small since relativity in its original form does not impose lower limit restrictions on the mass. Only the difference between the massive and massless cases matters. Indeed, to obtain results for perihelion shift or bending of light nowhere in our calculations we had used the mass \( m_2 \). There are treatments [22] involving such a mass. But, since it drops out at the end of calculations anyway, it is possible to perform all calculations without using this mass from the beginning [45]. In such a case the difference between different massive bodies disappears. Because of this, for the sake of argument, we can replace Mercury by, say, the atomic nucleus, or, just a single proton (or neutron), etc. In such a case not only photon sphere but the rest of massive orbits become formally quantum. Evidently, if we are able to apply this reasoning to Mercury, we can do the same for the rest of planets in view of smallness of their masses as compared to that of the Sun. In such a case, we can totally neglect their mutual attraction thus arriving at the situation considered already by Laplace and improved by Lagrange, Poincare’ and others. We discussed it in Section II. But we mentioned in the same section that to study the stability of Einsteinian orbits it is necessary to consider perturbations of the massless orbits caused by interaction between masses considered to be small. In the case of Mercury, the influence of other planets was discussed in subsection A resulting in Eqs(22) and (23). From these equations it follows that influence of other planets causes us to consider
the Newtonian-type equations of motion in which the mass $M$ and the square of angular momentum $L^2$ should be redefined without changing the form of Newton’s equations. Next, we demonstrated in this subsection that Einsteinian equations can be formally brought to the Newtonian form. As we discussed at the beginning of this subsection, the perihelion shift obtained by Einstein for Mercury is the difference between $\Delta \phi_{\text{obs}}$ and $\Delta \phi_{\text{Newton}}$. In view of previous remarks, this means that we can turn the above arguments around and to say that both contributions to perihelion shift can be obtained simultaneously from the Newton-looking Einstein-type equation for a geodesic in which parameters $M$ and $L^2$ are carefully chosen. Such a statement is formally compatible with results of Laplace discussed in Section II and those stated in [40] but is still not complete. It is not complete because we have not accounted yet for the relationship of the type given by Eq.(4). To account for relationship between angular frequencies is nontrivial. Indeed, now we know that if we begin with Newton’s Eq.(39), perhaps with properly redefined parameters $M$ and $\omega$, we can always use it in order to reobtain Einstein’s Eq.(33) for geodesics. But in Einsteinian universe geodesics should be parametrized only once while in the present case (without invoking quantization!) such parametrization is apparently different for each planet other than Mercury. This happens because LeVerier’s contributions are redefining geodesic parameters for different planets in apparently uncorrelated way. This apparent contradiction can be eliminated by assuming that parameters of the lowest possible stable geodesic orbit—for the Mercury—determine parameters for geodesics of other planets. This suggestion is compatible with the fact that we can place a proton instead of the Mercury on the same geodesic as we already mentioned. In such a case the problem about proton trajectory is no longer in the classical domain and, indeed, the parameters for higher orbits are determined by the parameters of the first allowed orbit which in the massive case should play the same role as photon sphere in the massless. Such a conclusion is compatible with results of Section II but requires explicit calculations for demonstration of its correctness. This is accomplished in the rest of this paper. We hope that the obtained below results could be tested not only against the empirical data in the sky (as it is done in this paper) but, subsequently, in laboratory [11-13].

IV. Space, Time and Space-Time in Classical and Quantum Mechanics

If one contemplates quantization of dynamics of celestial objects using traditional textbook prescriptions, one will immediately run into myriad of small and large problems. Unlike atomic systems in which all electrons repel each other, have the same masses and are indistinguishable, in the case of, say, Solar System all planets (and satellites) attract each other, have different masses and visibly distinguishable. Besides, in the case of atomic systems the Planck constant $\hbar$ plays prominent role while no such a role can be given to the Planck constant in the sky as it will be explained below. In two previous sections we demonstrated
that in the Einsteinian limit it is possible to remove almost all of the above objections so that, apparently, the only difference between the atomic and celestial quantum mechanics lies in replacement of the Planck constant by another constant and in attraction, instead of repulsion, between planets/satellites. Results of previous section indicate that this is not sufficient for quantization. Below we shall provide an explanation in several steps.

**A. Space and time in classical mechanics**

Although celestial mechanics based on Newton’s law of gravity is considered to be classical (i.e. non quantum), with such an assumption one easily runs into serious problems. Indeed, such an assumption implies that the speed of propagation of gravity is infinite and that time is absolute. Whether or not this is true or false can be decided only experimentally. As result, general relativity had emerged. Since at the spatial scales of Solar System one has to use radio signals to check correctness of Newton’s mechanics, all kinds of wave mechanical effects such as retardation, the Doppler effect, etc. become of use. Because of this, measurements are necessarily having some error margins. The error margins naturally will be larger for more distant objects. Accordingly, even at the level of classical Newtonian celestial mechanics we have to deal with inaccuracies of measurement similar in nature to those in atomic mechanics. These probabilistic effects are unavoidable but are not taken into account at the level of classical mechanics.

To make formalisms of both the atomic and celestial mechanics compatible we have to think carefully about space, time and space-time transformations at the level of classical mechanics first having in mind results of previous section. We begin with observation that in Hamiltonian mechanics the Hamiltonian equations by design remain invariant with respect to the canonical transformations. That is if sets \{q_i\} and \{p_i\} represent the "old" canonical coordinates and momenta while \(Q_i = Q_i((q_i), (p_i))\) and \(P_i = P_i((q_i), (p_i))\), \(i = 1 - N\), represent the "new", the Hamiltonian equations in the old and new canonical variables look the same.

We would like to complicate this familiar picture by investigating the "canonical" time changes in classical mechanics even though in relativity these transformations are not necessarily canonical. Fortunately, in the case of canonical time transformations the task is accomplished to a large extent in the monograph by Pars [46]. For the sake of space, we refer our readers to pages 535-540 of this monograph for details. In accord with Dirac [47], we believe that good quantization procedure should always begin with the Lagrangian formulation of mechanics. Hence, we begin with the Lagrangian \(L = L((q_i), (\dot{q}_i))\) whose equations of motion can be written in the form of Newton’s equations \(\dot{p_i} = F_i\), where the generalized momenta \(p_i\) are given by \(p_i = \delta L/\delta \dot{q}_i\) and the generalized forces \(F_i\) by \(F_i = -\delta L/\delta q_i\) as usual. In the case if the total energy \(E\) is conserved, it
is possible instead of "real" time \( t \) to introduce the fictitious time \( \theta \) via relation \( dt = u(\{q_i\})d\theta \) where the function \( u(\{q_i\}) \) is assumed to be nonnegative and is sufficiently differentiable with respect to its arguments. At this point we can enquire if Newton’s equations can be written in terms of new time variable so that they remain form-invariant. Notice, if we want to convert these results to relativistic form (e.g. see Section III) the form-invariance is going to be lost, while staying within Newton’s mechanics it is possible to preserve it. To do so, following Pars, we must: a) to replace \( L \) by \( uL \), b) to replace \( q_i \) by \( \frac{q'_i}{u} \), where \( q'_i = \frac{dq_i}{d\theta} \), c) to rewrite new Lagrangian in terms of such defined new time variables and, finally, d) to obtain Newton’s equations according to the described rules, provided that now we have to use \( p'_i \) instead of \( \dot{p}_i \). In the case if the total energy of the system is conserved, we shall obtain back the same form of Newton’s equations rewritten in terms of new variables. This means that by going from the Lagrangian to Hamiltonian formalism of classical mechanics we can write the Hamilton’s equations in which the dotted variables are replaced by primed. Furthermore, Hamilton’s equations will remain the same even if we replace the Hamiltonian \( H \) by some nonnegative function \( f(H) \) while changing time \( t \) to time \( \theta \) according to the rule \( d\theta/dt = df(H)/dH \mid_{H=E} \). Such a change while leaving classical mechanics form-invariant will affect quantum mechanics where the Schrödinger’s equation

\[
\frac{i\hbar}{\partial t}\Psi = \hat{H}\Psi \tag{45}
\]

is now replaced by

\[
\frac{i\hbar}{\partial \theta}\Psi = f(\hat{H})\Psi. \tag{46}
\]

Evidently, in view of results of Section III, the above results can be used for Einsteinian equations of motion written in Newtonian form so that everything in this subsection can be implemented for Einsteinian equations as well. With such information at our hands, we would like to discuss the extent to which symmetries of our (empty) space-time affect dynamics of particles "living" in it.

**B. Space and time in quantum mechanics**

Use of group-theoretic methods in quantum mechanics was initiated by Pauli in 1926. He obtained a complete quantum mechanical solution for the Hydrogen atom employing symmetry arguments. His efforts were not left without appreciation. Historically important references can be found in two comprehensive review papers by Bander and Itzykson [48]. In this subsection we pose and formally solve the following problem:

*Provided that the symmetry of (classical or quantum) system is known, will this information be sufficient for determination of this system uniquely?*
Below, we shall provide simple and concrete examples illustrating meaning of the word "determination". In the case of quantum mechanics this problem is known as the problem about hearing of the "shape of the drum" and is attributed to Mark Kac [49]. The problem can be formulated as follows. Suppose that the sound spectrum of the drum is known, will such an information determine the shape of the drum uniquely? The answer is "No" [50]. We would like to explain this non uniqueness using arguments much simpler than those used by Kac. For this purpose, we choose the most studied example of Hydrogen atom. The Keplerian motion of a particle (electron) in the centrally symmetric field is planar and is exactly solvable for both the scattering and bound states at the classical level [46]. The result of such a solution depends on two parameters: the energy and the angular momentum. The correspondence principle formulated by Bohr is expected to provide the bridge between the classical and quantum realities by requiring that in the limit of large quantum numbers the results of quantum and classical calculations for observables should coincide. However, this requirement may or may not be possible to implement. It is violated already for the Hydrogen atom! Indeed, according to the naive canonical quantization prescriptions, one should begin with the classical Hamiltonian in which one has to replace the momenta and coordinates by their operator analogs. Next, one uses such constructed quantum Hamiltonian in the Schrödinger's equation, etc. Such a procedure breaks down at once for the Hamiltonian of Hydrogen atom since the intrinsic planarity of the classical Kepler's problem is entirely ignored thus leaving the projection of the angular momentum without its classical analog. Accordingly, the scattering differential crosssection for the Hydrogen atom obtained quantum mechanically (within the 1st Born approximation) uses essentially 3-dimensional formalism and coincides with the result by Rutherford obtained for planar configurations using classical mechanics! Thus, even for the Hydrogen atom classical and quantum (or, better, pre quantum) Hamiltonians do not match thus formally violating the correspondence principle. Evidently, semiclassically we can only think of energy and the angular momentum thus leaving the angular momentum projection undetermined. Such a "sacrifice" is justified by the agreement between the observed and predicted Hydrogen atom spectra and by use of Hydrogen-like atomic orbitals for multielectron atoms, etc. Although, to our knowledge, such a mismatch is not mentioned in any textbooks on quantum mechanics, its existence is essential if we are interested in extension of these ideas to dynamics of Solar System. In view of such interest, we would like to reconsider traditional treatments of Hydrogen atom, this time being guided only by the symmetry considerations.

In April of 1940 Jauch and Hill [51] published a paper in which they studied the planar Kepler problem quantum mechanically. Their work was stimulated by earlier works by Fock of 1935 and by Bargmann of 1936 in which it was shown that the spectrum of bound states for the Hydrogen atom can be obtained by using representation theory of SO(4) group of rigid rotations of 4-dimensional Euclidean space while the spectrum of scattering states can be obtained by using the Lorentzian group SO(3,1). By adopting results of Fock and Bargmann to the planar configuration Jauch and Hill obtained the anticipated result: In the
planar case one should use $SO(3)$ group for the bound states and $SO(2,1)$ group for the scattering states. Although this result will be reconsidered almost entirely, we mention it now having several purposes in mind.

First, we would like to reverse arguments leading to the final results of Jauch and Hill in order to return to the problem posed at the beginning of this section. That is, we want to use the fact that the Kepler problem is planar (due to central symmetry of the force field) and the fact that the motion takes place in (locally) Lorentzian space-time in order to argue that the theory of group representations for Lorentzian $SO(2,1)$ symmetry group-intrinsic for this Kepler problem—correctly reproduces the Jauch-Hill spectrum. Nevertheless, the question remains: Is Kepler’s problem the only exactly solvable classical and quantum mechanical problem associated with the $SO(2,1)$ group? Below we argue that this is not the case! In anticipation of such negative result, we would like to develop our intuition by using some known results from quantum mechanics.

For the sake of space, we consider here only the most generic (for this work) example in some detail—the radial Schrödinger equation for the planar Kepler problem with the Coulombic potential. It is given by

$$
-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2}\right)\Psi(\rho) - \frac{Ze^2}{\rho} = E\Psi(\rho). \tag{47}
$$

Here $|m| = 0, 1, 2, \ldots$ is the angular momentum quantum number as required. For $E < 0$ it is convenient to introduce the dimensionless variable $x$ via $\rho = ax$ and to introduce the new wave function: $\psi(\rho) = \sqrt{\rho}\Psi(\rho)$. Next, by the appropriate choice of constant $a$ and by redefining $\psi(\rho)$ as $\psi(\rho) = \gamma x^{m/2} \exp(-y)\phi(y)$, where $y = \gamma x$, $-\gamma^2 = \frac{2Ze^2}{\hbar^2}\mu$, $a = \frac{\hbar^2}{\mu Ze^2}$, the following hypergeometric equation can be eventually obtained:

$$
\left\{ \frac{d^2}{dy^2} + 2\left[|m| + \frac{1}{2} - y \frac{d}{dy} + 2\left[\frac{1}{\gamma} - |m| - \frac{1}{2}\right]\right]\phi(y) = 0. \tag{48}
$$

Formal solution of such an equation can be written as $\phi(y) = F(-A(m), B(m), y)$, where $F$ is the confluent hypergeometric function. Physical requirements imposed on this function reduce it to a polynomial leading to the spectrum of the planar Kepler problem. Furthermore, by looking into standard textbooks on quantum mechanics, one can easily find that exactly the same type of hypergeometric equation is obtained for problems such as one-dimensional Schrödinger’s equation with the Morse-type potential, three dimensional radial Schrödinger equation for the harmonic oscillator and even three dimensional radial equation for the Hydrogen atom. Since the two-dimensional Kepler problem is solvable with help of representations of $SO(2,1)$ Lorentz group, the same should be.

---

16 The rationale for discussing the Coulombic potential instead of gravitational will be fully explained in the next section.
17 That is, $V(x) = A(\exp(-2ax) - 2\exp(-ax))$.
18 That is, $V(r) = \frac{A}{r^2} + B r^2$.
19 That is, $V(r) = \frac{A}{r^2} - \frac{B}{r}$.
true for all quantum problems just listed. That this is the case is demonstrated, for example, in the book by Wybourne [52]. A sketch of the proof is provided in Appendix A. This proof indicates that, actually, the *discrete spectrum* of all problems just listed is obtainable with help of representations for SO(2,1) group. The question still remains: If the method outlined in Appendix A provides the spectra of several quantum mechanical problems listed above, can we be sure that these are the only exactly solvable quantum mechanical problems associated with the SO(2,1) Lorentz group? Unfortunately, the answer is "No"!

As we are about to explain.

In Appendix A we provide a sketch of the so called spectrum-generating algebras (SGA) method. It is aimed at producing the exactly solvable one-variable quantum mechanical problems. In this subsection we would like to put these results in a broader perspective. In particular, in our recent works [36, 37] we demonstrated that *all exactly solvable quantum mechanical problem should involve hypergeometric functions of single or multiple arguments*. We argued that the difference between different problems can be understood topologically in view of the known relationship between the hypergeometric functions and braid groups. These results, even though quite rigorous, are not well adapted for immediate practical use. In this regard more useful would be to solve the following problem:

*For a given set of orthogonal polynomials find the corresponding many-body operator for which such set of orthogonal polynomials forms the complete set of eigenfunctions.*

At the level of orthogonal polynomials of one variable relevant for all exactly solvable two-body problems of quantum mechanics, one can think about the related problem of finding all potentials in one-dimensional radial Schrödinger’s equation, e.g. equation (A.1), leading to the hypergeometric-type solutions. Very fortunately, such a task was accomplished already by Natanzon [53]. Subsequently, his results were re-investigated by many authors with help of different methods, including SGA. To our knowledge, the most complete recent summary of the results, including potentials and spectra can be found in the paper by Levai [54]. Even this (very comprehensive) paper does not cover all aspects of the problem. For instance, it does not mention the fact that these results had been extended to relativistic equations such as Dirac and Klein-Gordon for which similar analysis was made by Cordero with collaborators [55]. In all cited cases (relativistic and non relativistic) the underlying symmetry group was SO(2,1). The results of Appendix A as well as of all other already listed references can be traced back to the classically written papers by Bargmann [56] and Barut and Fronsdal [57] on representations of SO(2,1) Lorentz group. Furthermore, the discovered connection of this problematics with supersymmetric quantum mechanics [58,59] can be traced back to the 19th century works by Gaston Darboux. The fact that representations of the *planar* SO(2,1) Lorentz group are sufficient to cover all known exactly solvable two-body problems (instead of the full SO(3,1) Lorentz group!) is quite remarkable and intuitively unexpected. It is also sufficient for purposes of this work but leaves open the question: Will use of the full Lorentz group produce exactly solvable quantum
mechanical problems not accounted by the SO(2,1) group symmetry? Since the answer to this problem is not affecting the results of the next section, we leave study of this problem outside the scope of this work. Instead, we proceed with our main task-quantization of Solar System dynamics- in the next section.

V. Some Classical and Quantum Aspects of Solar System Dynamics

While in previous sections we provided general background needed for development of quantum dynamics of Solar System, in this section we illustrate general principles by specific examples. Attentive reader probably noticed already that at the heart of quantization lies the honeycomb condition, Eq.(4) (or (7)), allowing to restore both Heisenberg’s and Schrodinger’s versions of quantum mechanics. In the previous section we demonstrated that the quantum-classical correspondence in general is not one-to-one. To reduce the number of options some additional physical arguments should be used. We shall provide these arguments in this section in a logical sequence needed for development of our arguments.

We begin by noticing that while Eq.(4) (or (7)) is only sufficient condition for quantization, the necessary condition in atomic and celestial mechanics lies in the non dissipativity. Recall that Bohr introduced his quantization prescription to avoid dissipation caused by the emission of radiation by electrons in orbits in general position. New quantum mechanics have not shed much light on absence of dissipation for stationary Bohr’s orbits. At the level of old Bohr theory absence of dissipation at the stationary Bohr orbit was explained by Boyer [60]. Subsequently, his result was refined by Puthoff [61]. In the case of Solar System absence of dissipation for motion on stable orbits (geodesics) was discussed by Goldreich [62] who conjectured that the dissipative (tidal) effects adjust the initial motion of planets/satellites in such a way that eventually the orbits become stable. Notice that dynamics of Solar System as considered by Poincare’ and by those who developed his ideas does not involve treatment of tidal effects. Treatment of tidal effects in general relativity is discussed in [10] and requires consideration of motion of extended objects, e.g. read Papapetrou, Ref. [44]. In view of results of Section III, following Einstein, we shall ignore finite sizes of planets and/or satellites of heavy planets thus removing the problem of tidal friction. This is justified by the numerical results to be obtained in this section.

Thus, following Bohr, we postulate that in the case of Solar System dynamics on stable orbits is non dissipative. This assumption then leads us to the following Table 1. In this table by accidental degeneracy we mean the condition given by Eq.(4).

Table 1
### A. Celestial spectroscopy and the Titius-Bode law of planetary distances

The atomic spectroscopy was inaugurated by Newton in the second half of 17th century. The celestial spectroscopy was inaugurated by Titius in the second half of 18th century and became more famous after it was advertised by Johann Bode, the Editor of the "Berlin Astronomical Year-book". The book by Nieto [53] provides extensive bibliography related to uses and interpretations of the Titius-Bode (T-B) law up to second half of 20th century. Unlike the atomic spectroscopy, where the observed atomic and molecular spectra were expressed using simple empirical formulas which were (to our knowledge) never elevated to the status of "law", in celestial mechanics the empirical T-B formula

\begin{equation}
    r_n = 0.4 + 0.3 \cdot 2^n, \quad n = -\infty, 0, 1, 2, 3, \ldots
\end{equation}

for the orbital radii (semimajor axes) of planets acquired the status of a law in the following sense. In the case of atomic spectroscopy the empirical formulas used for description of atomic/molecular spectra have not been used (to our knowledge) for making predictions. Their purpose was just to describe in mathematical terms what had been already observed. Since the T-B empirical formula for planetary distances was used as the law, it was used in search for planets not yet discovered. In such a way Ceres, Uranus, Neptune and Pluto were found [64]. However, the discrepancies for Neptune and Pluto were much larger than the error margins allowed by the T-B law. This fact divided the astronomical community. Without going into historical details, we would like to jump to the very end of the Titius-Bode story in order to use its latest version which we found in the paper by Neslušan [65] who, in turn, was motivated by...
the work of Lynch [66]. Instead of (49) these authors use another empirical power law dependence

\[ r_n = r_0 B^n, \quad n = 1, 2, 3, \ldots, 9. \]  

(50)

For planets (except Pluto and including the asteroid belt) Neslušan obtained \( r_0(\text{au}) = 0.203 \) and \( B = 1.773 \) with the rms deviation accuracy of 0.053. Analogous power law dependencies were obtained previously in the work by Dermott [67] for both planets and satellites of heavy planets such as Jupiter, Saturn and Uranus. It should be noted that because of noticed discrepancies the attempts were made to prove or disprove the Titius-Bode law by using statistical analysis, e.g. see papers by Lynch [66] and Hayes and Tremaine [68], with purpose of finding out to which extent the observed dependencies can be considered as non accidental. Following the logic of Bohr we would like to use the observed empirical radial dependencies as a guide to our calculations to be discussed below.

A. An attempt at quantization of Solar System dynamics

Being guided by the Table 1 we shall assume that planets do not interact since they move along geodesics as discussed in Section III. In the case of atomic mechanics it was clear from the beginning that such an approximation should sooner or later fail. The nonexisting electroneutrality in the sky provides strong hint that the T-B law must be of very limited use since the number of discrete levels for gravitating systems should be always finite. Otherwise, we would observe the countable infinity of satellites around Sun or of any of heavy planets. This is not observed and is physically incorrect. It is wrong because such a system would tend to capture all matter in the Universe. The same applies to the photon sphere discussed in Section III. In it we noticed that classically this orbit (sphere) is unstable. If such a sphere would be stable quantum mechanically, this would cause the black hole to grow in mass indefinitely since it would accumulate incoming photons with wavelengths lesser than \( 6\pi M \) if conditions for capture are right. Whether or not such accumulation is possible now can be investigated in laboratory conditions with potential practical applications as discussed already.

In the literature one can find many attempts at quantization of Solar System using standard prescriptions of quantum mechanics. Many of these papers are listed in [14]. Since this work is not a review, we do not provide references to papers whose results do not affect ours. Blind uses of standard rules of quantum mechanics for quantization of Solar System dynamics do not contain

\[ \text{In astronomical units (to be defined below).} \]

\[ \text{This result gives for the Earth in astronomical (au) units the result } r_3 \simeq 1.13. \text{ Much better result is obtained in case if we choose } B = 1.7. \text{ In this case we obtain: } r_3 \simeq 0.997339. \]

Lynch provides \( B = 1.706 \) and \( r_0 = 0.2139 \).
any provisions for finite number of energy levels/orbits for gravitating systems. To facilitate matters in the present case, we would like to make several additional observations.

First, we have to find an analog of the Planck constant. Second, we have to have some mechanical model in mind to make our search for physically correct answer successful. To accomplish the first task, we have to take into account the 3-rd Kepler’s law. In accord with Eq. (32), it can be written as $r_n^3/T_n^2 = \frac{4\pi^2}{\gamma(M_\odot + m)}$. In view of arguments presented in Section III, we can safely approximate the r.h.s. by $4\pi^2/\gamma M_\odot$, where $M_\odot$ is mass of the Sun. For the purposes of this work, it is convenient to restate this law as

$$3\ln r_n - 2\ln T_n = \ln 4\pi^2/\gamma M_\odot = \text{const} \quad (51)$$

Below, we choose the astronomical system of units in which $4\pi^2/\gamma M_\odot = 1$. By definition, in this system of units we have for the Earth: $r_3 = T_3 = 1$. Consider now the Bohr result, Eq. (6), and take into account that $E = \hbar \omega \equiv \hbar \frac{2\pi}{2\pi T}$. Therefore, Bohr’s result can be conveniently restated as $\omega(n, m) = \omega(n) - \omega(m)$. Taking into account Eqs. (6), (46), (50) and the third Kepler’s law, Eq. (51), we formally obtain:

$$\omega(n, m) = \frac{1}{c\ln \bar{A}}(nc\ln \bar{A} - mc\ln \bar{A}), \quad (52)$$

where the role of Planck’s constant is being played now by $c\ln \bar{A}$, $\bar{A} = B^2$ and $c$ is some constant which will be determined selfconsistently below\(^{23}\).

At first, one may think that what we obtained is just a simple harmonic oscillator spectrum. After all, this should come as not too big a surprise since both, the Newtonian Eq. (15) and the Einsteinian Eq. (26) (brought to the Newtonian-looking form) are classical equations for the harmonic oscillator. This result is also compatible with that of Appendix A. The harmonic oscillator option is physically undesirable though since the harmonic oscillator has countable infinity of energy levels. Evidently, such a spectrum is equivalent to the T-B law. But it is well known that this law is not working well for larger numbers. In fact, it would be very strange should it be working in this regime.

To make progress, we have to use the 3rd Kepler’s law once again. This time, we have to take into account that in the astronomical system of units $3\ln r_n = 2\ln T_n$. A quick look at equations (A.11), (A.12) suggests that the underlying mechanical system is likely to be associated with that for the Morse potential. This is so because the low lying states of such a system cannot be distinguished from those for the harmonic oscillator. However, this system does have only a finite number of energy levels which makes sense physically. The task remains to connect this system with planar Kepler’s problem. Although in view of results of Appendix A such a connection does exist, we want to demonstrate it explicitly at the level of classical mechanics first.

\(^{22}\)We have included the gravitation constant $\gamma$ in this expression.

\(^{23}\)Not to be confused with the speed of light!
Following Pars [46], the motion of a point of unit mass in the field of Newtonian gravity is described by the following equation

\[ \dot{r}^2 = \frac{(2Er^2 + 2\gamma Mr - \alpha^2)}{r^2}, \]  

(53)

where \( \alpha \) is the angular momentum [24] (e.g. see equation (5.2.55) of the book by Pars). Next, we replace \( r(t) \) by \( r(\theta) \) in such a way that \( dt = u(r(\theta))d\theta \). Let therefore \( r(\theta) = r_0 \exp(x(\theta)), -\infty < x < \infty \). Unless otherwise specified, we shall write \( r_0 = 1 \). In such (astronomical) system of units we obtain, \( \dot{r} = x'\frac{d\theta}{dt} \exp(x(\theta)) \). This result can be further simplified by choosing \( \frac{d\theta}{dt} = \exp(-x(\theta)) \). With this choice (53) acquires the following form:

\[ (x')^2 = 2E + 2\gamma M \exp(-x) - \alpha^2 \exp(-2x). \]  

(54)

Consider points of equilibria for the potential \( U(r) = -2\gamma M r^{-1} + \alpha^2 r^{-2} \). Using it, we obtain: \( r^* = \frac{\alpha^2}{\gamma M} \). According to [21] such defined \( r^* \) coincides with the major elliptic semiaxis. It can be also shown, e.g. Pars, equation (5.4.14), that for the Kepler problem the following relation holds: \( E = -\frac{\gamma M}{2r^*} \). Accordingly, \( r^* = -\frac{\gamma M}{2E} \), and, furthermore, using the condition \( \frac{dt}{dr} = 0 \) we obtain: \( \frac{\alpha^2}{\gamma M} = -\frac{\gamma M}{2E} \) or, \( \alpha^2 = -\frac{(\gamma M)^2}{2E} \). Since in the chosen system of units \( r(\theta) = \exp(x(\theta)) \), we obtain as well: \( \frac{\alpha^2}{\gamma M} = \exp(x^*(\theta)) \). It is convenient to choose \( x^*(\theta) = 0 \). This requirement sends the point \( x^*(\theta) = 0 \) to the origin of new coordinate system and implies that with respect to such chosen origin \( \alpha^2 = \gamma M \). In doing so some caution should be exercised since upon quantization equation \( r^* = \frac{\alpha^2}{\gamma M} \)

becomes \( r^*_n = \frac{\alpha^2_n}{\gamma M} \). By selecting the astronomical scale \( r^*_3 = 1 \) as the unit of length implies then that we can write the angular momentum \( \alpha^2_n \) as \( \frac{r^*_n}{r^*_3} \) and to define \( \alpha \) as \( \alpha^2_3 \equiv \alpha^2 \). Using this fact Eq.(54) can then be conveniently rewritten as

\[ \frac{1}{2}(x')^2 - \gamma M(\exp(-x) - \frac{1}{2} \exp(-2x)) = E \]  

(55a)

or, equivalently, as

\[ \frac{p^2}{2} + A(\exp(-2x) - 2 \exp(-x)) = E, \]  

(55b)

where \( A = \frac{\gamma M}{2} \). Since this result is exact classical analog of the quantum Morse potential problem, the transition to quantum mechanics now can be done

\[ \text{To comply with notations of the book by Pars we replaced } L \text{ by } \alpha. \]
straightforwardly. By doing so we have to replace the Planck’s constant \( \hbar \) by \( c \ln \tilde{A} \). After that, we can write the answer for the spectrum at once [69]:

\[
-\tilde{E}_n = \frac{\gamma M}{2} \left[ 1 - c \ln \tilde{A} \left( n + \frac{1}{2} \right) \right]^2.
\] (56)

This result contains an unknown parameter \( c \) to be determined now. To do so, it is sufficient to expand the potential in Eq.(55b) and to keep terms up to quadratic. Such a procedure produces the anticipated harmonic oscillator result

\[
\frac{p^2}{2} + Ax^2 = \tilde{E}
\] (57)

whose quantum spectrum is given by \( \tilde{E}_n = (n + \frac{1}{2})c^2 \pi \ln \tilde{A} \). In the astronomical system of units the spectrum reads: \( \tilde{E}_n = (n + \frac{1}{2})c^2 \pi \ln \tilde{A} \). This result is in agreement with Eq.(52). To proceed, we notice that in Eq.(52) the actual sign of the Planck-type constant is undetermined. Specifically, in our case (up to a constant) the energy \( \tilde{E}_n \) is determined by \( \ln \left( \frac{1}{T_n} \right) = -\ln \tilde{A} \) so that it makes sense to write \( -\tilde{E}_n \sim n \ln \tilde{A} \).

To relate the classical energy defined by the Kepler-type equation \( E = -\frac{\gamma M}{2r^*} \) to the energy we just have defined, we have to replace the Kepler-type equation by \( -\tilde{E}_n \equiv -\ln |E| = -2 \ln \sqrt{2\pi} + \ln r_n \). This is done in view of the 3rd Kepler’s law and the fact that the new coordinate \( x \) is related to the old coordinate \( r \) via \( r = e^x \). Using Eq.(50) (for \( n = 1 \)) in the previous equation and comparing it with the already obtained spectrum of the harmonic oscillator we obtain:

\[
-2 \ln \sqrt{2\pi} + \ln r_0 B = -c \pi \ln \tilde{A},
\] (58)

where in arriving at this result we had subtracted the nonphysical ground state energy. Thus, we obtain:

\[
c = \frac{1}{2 \pi \ln A} \frac{2\pi^2}{r_0 B}.
\] (59)

Substitution of this result back into Eq.(56) produces

\[
-\tilde{E}_n = 2\pi^2 \left[ 1 - \frac{(n + \frac{1}{2}) \ln \left( \frac{2\pi^2}{r_0 B} \right)}{4\pi^2} \right]^2 \simeq 2\pi^2 \left[ 1 - \frac{1}{9.87} (n + \frac{1}{2}) \right]^2
\]

\[
\simeq 2\pi^2 - 4(n + \frac{1}{2}) + 0.2(n + \frac{1}{2})^2.
\] (60)

To determine the number of bound states, we follow the same procedure as was developed long ago in chemistry for the Morse potential. For this purpose we introduce the energy difference \( \Delta \tilde{E}_n = \tilde{E}_{n+1} - \tilde{E}_n = 4 - 0.4(n + 1) \) first. Next,
the maximum number of bound states is determined by requiring $\Delta \tilde{E}_n = 0$. In our case, we obtain: $n_{\text{max}} = 9$. This number is in perfect accord with observable data for planets of our Solar System (with Pluto being excluded and the asteroid belt included). In spite of such a good accord, some caution must be still exercised while analyzing the obtained result. Should we not insist on physical grounds that the discrete spectrum must contain only finite number of levels, the obtained spectrum for the harmonic oscillator would be sufficient (that is to say, that the validity of the T-B law would be confirmed). Formally, such a choice also solves the quantization problem completely and even is in accord with numerical data [65]. The problem lies however in the fact that these data were fitted to the power law, Eq.(50), in accord with the original T-B empirical guess. Heisenberg’s honeycomb rule, Eq. (7b), does not rely on specific $n$–dependence. In fact, we have to consider the observed (the Titius-Bode-type) $n$–dependence only as a hint, especially because in this work we intentionally avoid use of any adjustable parameters. The developed procedure, when supplied with correctly interpreted numerical data, is sufficient for obtaining results without any adjustable parameters as we just demonstrated. In turn, this allows us to replace the T-B law in which the power $n$ is unrestricted by more accurate result working especially well for larger values of $n$. For instance, the constant $c$ was determined using the harmonic approximation for the Morse-type potential. This approximation is expected to fail very quickly as the following arguments indicate. Although $r'_n$'s can calculated using the T-B law given by Eq.(50), the arguments following this equation cause us to look also at the equation $-\tilde{E}_n \equiv - \ln |E| = -2 \ln \sqrt{2\pi} + \ln r_n$ for this purpose. This means that we have to use Eq.(60) (with ground state energy subtracted) in this equation in order to obtain the result for $r_n$. If we ignore the quadratic correction in Eq.(60) (which is equivalent of calculating the constant $c$ using harmonic oscillator approximation to the Morse potential) then, by construction, we recover the T-B result, Eq.(50). If, however, we do not resort to such an approximation, calculations will become much more elaborate. The final result will indeed replace the T-B law but the obtained analytical form is going to be too cumbersome for practitioners. Since corrections to the harmonic oscillator potential in the case of Morse potential are typically small, they do not change things qualitatively. Hence, we do not account for these complications in our paper. Nevertheless, accounting for these (anharmonic) corrections readily explains why the empirical T-B law works well for small $n$’s and becomes increasingly unreliable for larger $n$’s [63,64].

In support of our way of doing quantum calculations, we would like to discuss now similar calculations for satellite systems of Jupiter, Saturn, Uranus and Neptune. To do such calculations the astronomical system of units is not immediately useful since in the case of heavy planets one cannot use the relation $4\pi^2/\gamma M_\odot = 1$. This is so because we have to replace the mass of the Sun $M_\odot$ by the mass of respective heavy planet. For this purpose we write $4\pi^2 = \gamma M_\odot$, multiply both sides by $M_j$ (where $j$ stands for the $j$-th heavy planet) and divide both sides by $M_\odot$. Thus, we obtain: $4\pi^2 q_j = \gamma M_j$, where $q_j = M_j / M_\odot$. Since
the number $q_j$ is of order $10^{-3} - 10^{-5}$, it causes some inconveniences in actual calculations. To avoid this difficulty, we need to readjust Eq.(55a) by rescaling $x$ coordinate as $x = \delta \bar{x}$ and, by choosing $\delta^2 = q_j$. After transition to quantum mechanics such a rescaling results in replacing Eq.(56) for the spectrum by the following result:

$$-\tilde{E}_n = \frac{\gamma M}{2} \left[ 1 - \frac{c \delta \ln \tilde{A}}{\sqrt{\gamma M}} (n + \frac{1}{2}) \right]^2. \quad (61)$$

Since the constant $c$ is initially undetermined, we can replace it by $\tilde{c} = c \delta$. This replacement allows us to reobtain back equation almost identical to Eq.(60). That is

$$-\tilde{E}_n = 2\pi^2 \left[ 1 - \left( \frac{n + \frac{1}{2}}{4\pi^2} \ln \left( \frac{\gamma M_j}{(r_j)_1} \right) \right) \right]^2. \quad (62)$$

In this equation $\gamma M_j = 4\pi^2 q_j$ and $(r_j)_1$ is the semimajor axis of the satellite lying in the equatorial plane and closest to the $j$-th planet. Our calculations are summarized in the Table 2 below. Appendix B contains the input data used in calculations of $n_{\text{theory}}^*$. Observational data are taken from the web link: [http://nssdc.gsfc.nasa.gov/planetary/](http://nssdc.gsfc.nasa.gov/planetary/) Then, go to the respective planet and, then-to the "fact sheet" link for this planet.

| Satellite system | $n_{\text{max}}^*$ | $n_{\text{theory}}^*$ | $n_{\text{obs}}^*$ |
|------------------|---------------------|------------------------|---------------------|
| Solar system     | 9                   | 9                      | 9                   |
| Jupiter system   | 11-12               | 8                      |                     |
| Saturn system    | 20                  | 20                     |                     |
| Uranus system    | 40                  | 18                     |                     |
| Neptune system   | 33                  | 6-7                    |                     |

Since the discrepancies for Uranus and Neptune systems may be genuine or not we come up with the following general filling pattern to be considered in the next subsection.

**B. Filling patterns in Solar System: similarities and differences with atomic mechanics**

From atomic mechanics we know that the approximation of independent electrons used by Bohr fails rather quickly with increased number of electrons. For this reason alone to expect that the T-B law is going to hold for satellites of heavy planets is naive. At the same time, for planets rotating around the Sun such an approximation is seemingly good but also not without flaws. The $SO(2,1)$ symmetry explains why motion of all planets should be planar but it
does not explain why motion of all planets is taking place in the plane coinciding with the equatorial plane of the Sun or why all planets are moving in the same direction. The same is true for the regular satellites of all heavy planets as discussed by Dermott [67]. Such a configuration can be explained by a plausible hypothesis [70] that all planets of Solar System and regular satellites of heavy planets are originated from evolution of the pancake-like cloud. The assumption that all planets lie in the same (Sun’s equatorial) plane and move in the same direction coinciding with the direction of rotation of the Sun around its axis was used essentially by Poincare′ in his ”Les Methodes Nouvelles de la Mecanique Celeste” written between 1892 and 1899 [26]. These assumptions are compatible with the hypothesis [70] of how Solar System was formed. In 1898, while Poincare was still working on his ”Methodes” the shocking counter example to the Poincare′ theory was announced by Pickering who discovered the ninth moon of Saturn (eventually named Phoebe) rotating in the direction opposite to all other satellites of Saturn. Since that time the satellites rotating in the ”normal” direction are called ”regular” (or ”prograde”) while those rotating in the opposite direction called ”irregular”(or ”retrograde”). At the time of writing of this paper 103 irregular satellites were discovered (out of those, 93 were discovered after 1997 thanks to space exploration by rockets [27]. Furthermore, in the late 2009 Phoebe had brought yet another surprise to astronomers. Two articles in Nature [71,72] are describing the largest new ring of Saturn. This new ring lies in the same plane as Phoebe’s orbit and, in fact, the Phoebe’s trajectory is located inside the ring. The same arrangement is true for regular satellites and the associated with them rings.

To repair the existing theory of formation of Solar system one has to make an assumption that all irregular satellites are ”strangers”. That is that they were captured by the already existing and fully developed Solar System. Such an explanation would make perfect sense should the orbits of these strangers be arranged in a completely arbitrary fashion. But they are not! Without an exception it is known that: a) all retrograde satellite orbits are lying strictly outside of the orbits of prograde satellites, b) the inclinations of their orbits is noticeably different from those for prograde satellites, however, c) by analogy with prograde satellites they tend to group (with few exceptions) in orbits-all having the same inclination so that different groups of retrogrades are having differently inclined orbits in such a way that these orbits do not overlap if the retrograde plane of satellites with one inclination is superimposed with that for another inclination [28]. In addition, all objects lying outside the sphere made by the rotating plane in which all planets lie are arranged in a similar fashion [64]. Furthermore, the orbits of prograde satellites of all heavy planets lie in the respective equatorial planes- just like the Sun and the planets - thus forming miniature Solar-like systems. These equatorial planes are tilted with respect to the Solar equatorial plane since all axes of rotation of heavy planets are tilted [29].

27E.g. read ”Irregular moon” in Wikipedia
28As before, go to http://nssdc.gsfc.nasa.gov/planetary/ then, go to the respective planet, and then-to the ”fact sheet” link for this planet.
29That is the respective axes of rotation of heavy planets are not perpendicular to the Solar
with different angles for different Solar-like systems. These "orderly" facts make nebular origin of our Solar System questionable. To strengthen the doubt further we would like to mention that for the exoplanets\textsuperscript{30} it is not uncommon to observe planets rotating in the "wrong" direction around the respective stars\textsuperscript{31}. This trend goes even further to objects such as galaxies. In spiral galaxies the central bulge typically co-rotates with the disc. But for the galaxy NGC7331 the bulge and the disc are rotating in the opposite directions.

In the light of astronomical facts just described the Table 1 requires some extension. For instance, to account for the fact that all planets and regular satellites are moving in the respective equatorial planes requires to use the effects of spin-orbital interactions. Surprisingly, these effects exist in both Newtonian [20] and Einsteinian [73] gravities. At the classical (Newtonian) level the most famous example of spin-orbital resonance (but of a different kind) is exhibited by the motion of the Moon whose orbital period coincides with its rotational period so that it always keeps only one face towards the Earth. Most of the major natural satellites are locked in analogous 1:1 spin-orbit resonance with respect to the planets around which they rotate. Mercury represents an exception since it is locked into 3:2 resonance around the Sun (that is Mercury completes 3/2 rotation around its axis while making one full rotation along its orbit) [20]. Goldreich [62] explains such resonances as results of influence of dissipative (tidal) processes on evolutionary dynamics of Solar System. The resonance structures observed in the sky are stable equilibria in the appropriately chosen reference frames [20]. Clearly, the spin-orbital resonances just described are not explaining many things. For instance, while nicely explaining why our Moon is always facing us with the same side the same pattern is not observed for Earth rotating around the Sun with exception of Mercury. Mercury is treated as pointlike object in canonical general relativity. The spin-orbital interaction [73] causing planets and satellites of heavy planets to lie in the equatorial plane is different from that causing 1:1, etc. resonances. It is analogous to the NMR-type resonances in atoms and molecules where in the simplest case we are dealing, say, with the hydrogen atom. In it, the proton having spin 1/2 is affected by the magnetic field created by the orbital s-electron. In the atomic case due to symmetry of electron s-orbital this effect is negligible but nonzero! This effect is known in chemical literature as "chemical shift". In the celestial case situation is similar but the effect is expected to be much stronger since the orbit is planar (not spherical as for the s-electron in hydrogen atom). Hence, the equatorial location of planetary orbits and regular satellites is likely the result of such spin-orbital interactions. The equatorial plane in which planets (satellites) move can be considered as some kind of an orbital (in terminology of atomic physics). It is being filled in accordance with the equivalent of the Pauli principle: each orbit can be occupied by no more than one planet\textsuperscript{32}. Once the

\textsuperscript{30}E.g. see [http://exoplanets.org/]
\textsuperscript{31}E.g. read "Retrograde motion" in Wikipedia
\textsuperscript{32}The meteorite belt can be looked upon as some kind of a ring. We discussed such model rings in Section III and shall discuss the real rings below, in the next subsection.

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orbital is filled, other orbitals (planes) will begin to be filled out. Incidentally, such a requirement automatically excludes Pluto from the status of a planet. Indeed, on one hand, the T-B-type law, can easily accommodate Pluto, on another, not only this would contradict the data summarized in Table 1 and results of previous subsection but also, and more importantly, it would be in contradiction with the astronomical data for Pluto. According to these data the orbit inclination for Pluto is 17° as compared to the rest of planets whose inclination is within boundary margins of ±2° (except for Mercury for which it is 7°). Some of the orbitals can be empty and not all orbits belonging to the same orbital (a plane) must be filled (as it is also the case in atomic physics). This is indeed observed in the sky [64,67] and is consistent with results of Table 2. It should be said though that it appears (according to available data, that not all of the observed satellites are moving on stable orbits. It appears also as if and when the "inner shell"is completely filled, it acts as some kind of an s-type spherical orbital since the orbits of other (irregular) satellites lie strictly outside the sphere whose diameter is greater or equal to that corresponding to the last allowed energy level in the first shell. Such a situation resembles that for the photon sphere accommodating massless photons and more distant-massive-orbits discussed in Section III. The location of secondary planes appears to be quite arbitrary as well as the filling of their stable orbits. Furthermore, without account for spin-orbital interactions, one can say nothing about the direction of orbital rotation. Evidently, the "chemical shift" created by the motion of regular satellites lying in the s-shell is such that it should be more energetically advantageous to rotate in the opposite direction for irregulars. This proposition requires further study. In addition to planets and satellites on stable orbits there are many strangers in the Solar System: comets, meteorites, etc. These are moving not on stable orbits and, as result, should either leave the Solar System or eventually collide with those which move on "legitimate" orbits.

It is tempting to extend the picture just sketched beyond the scope of our Solar System. If for a moment we ignore relativistic effects, we can then find out that our Sun is moving along almost circular orbit around our galaxy center with the period \( T = 185 \cdot 10^6 \) years [74]. Our galaxy is also flat as our Solar System and the major mass is concentrated in the galaxy center. Hence, again, if we believe that stable stellar motion is taking place along geodesics around the galaxy center, then we have to accept that our galaxy is also a quantum object. It would be very interesting to estimate the number of allowed energy levels (stable orbits) for our galaxy and to check if the Pauli-like principle works for our and other galaxies.

C. Role of general relativity in the restricted 3-body problem and in dynamics of planetary rings
Although literature on the restricted 3-body problem is huge, we would like to discuss this problem from the point of view of its connection with general relativity and quantization of planetary orbits along the lines developed in this paper. We begin with several remarks. First, the existence of ring systems for all heavy planets is well documented [64]. Second, these ring systems are interspersed with satellites of these planets. Third, both rings and satellites lie in the respective equatorial planes (with exception of Phoebe’s ring) so that satellites move on stable orbits. From these observations it follows that:

a) While each of heavy planets is moving along the geodesics around the Sun, the respective satellites are moving along the geodesics around respective planets.

b) The motion of these satellites is almost circular (the condition which Laplace took into account while studying Jupiter’s regular satellites).

The restricted 3-body problem can be formulated now as follows. Given that the rings are made of some kind of small objects whose masses can be neglected as compared to masses of both satellite(s) and the respective heavy planet, we can ignore mutual gravitational interaction between these objects (as Laplace did). Under such conditions we end up with motion of a given piece of a ring (of zero mass) in the presence of two bodies of masses $m_1$ and $m_2$ respectively (the planet and one of the regular satellites). To simplify matters, it is usually being assumed that the motion of these two masses takes place on a circular orbit with respect to their center of mass. Complications associated with the eccentricity of such a motion are discussed in the book by Szebehely [75] and can be taken into account if needed. They will be ignored nevertheless in this discussion since we shall assume that satellites of heavy planets move on geodesics so that the center of mass coincides with the position of a heavy planet thus making our computational scheme compatible with methods of Einstein’s relativity. By assuming that ring pieces are massless we also are making their motion compatible with requirements of general relativity.

Thus far only motion of regular satellites in equatorial planes (of respective planets) was considered as stable (and, hence, quantizable). The motion of ring pieces was not accounted by these stable orbits. The task now lies in showing that satellites lying inside the respective rings of heavy planets are essential for stability of motion of these rings thus making it quantizable. For the sake of space, we would like only to provide a sketch of arguments leading to such a conclusion.

Our task is greatly simplified by the fact that very similar situation exists for 3-body system such as Moon, Earth and Sun. Dynamics of such a system was studied thoroughly by Hill whose work played pivotal role in Poincare’ studies of celestial mechanics [26]. Avron and Simon [76] adopted Hill’s ideas in order to develop formal quantum mechanical treatment of the Saturn rings. In this

\[33\] This approximation is known as Hill’s problem/approximation in the restricted 3-body problem [27, 46].
work we follow instead the original Hill’s ideas about dynamics of the Earth-
Moon-Sun system. When these ideas are looked upon from the point of view of
modern mathematics of exactly integrable systems, they enable us to describe
not only the Earth-Moon-Sun system but also the dynamics of rings of heavy
planets. These mathematical methods allow us to find a place for the Hill’s
theory within general quantization scheme discussed in previous sections.

To avoid repetitions, we refer our readers to the books of Pars [46] and
Chebotarev [74] for detailed and clear account of the restricted 3-body problem
and Hill’s contributions to Lunar theory. In a nutshell his method of studying
the Lunar problem can be considered as extremely sophisticated improvement of
previously mentioned Laplace method. Unlike Laplace, Hill realized that both
Sun and Earth are surrounded by the rings of influence\(^34\). The same goes for
all heavy planets. Each of these planets and each satellite of such a planet will
have its own domain of influence whose actual width is controlled by the Jacobi
integral of motion.

For the sake of argument, consider the Saturn as an example. It has Pan
as its the innermost satellite. Both the Saturn and Pan have their respective
domains of influence. Naturally, we have to look first at the domain of influence
for the Saturn. Within such a domain let us consider a hypothetical closed
Kepler-like trajectory. Stability of such a trajectory is described by the Hill
equation\(^35\). Since such an equation describes a wavy-type oscillations around
the presumably stable trajectory, the parameters describing such a trajectory
are used as an input (perhaps, with subsequent adjustment) in the Hill equation
given by
\[
\frac{d^2x}{dt^2} + (q_0 + 2q_1 \cos 2t + 2q_2 \cos 4t + \cdots) x = 0. \tag{63}
\]
If we would ignore all terms except \(q_0\) first, we would naively obtain:
\[x_0(t) = A_0 \cos(t/\sqrt{q_0} + \varepsilon).\]
This result describes oscillations around the equilibrium position along the trajectory with the constant \(q_0\) carrying information about this trajectory. The amplitude \(A\) is expected to be larger or equal to the average distance between the pieces of the ring. This naive picture gets very complicated at once should we use the obtained result as an input into Eq.(63). In this case the following equation is obtained
\[
\frac{d^2x}{dt^2} + q_0 x + A_0 q_1 \{\cos[t(\sqrt{q_0} + 2) + \varepsilon] + \cos[t(\sqrt{q_0} - 2) - \varepsilon]\} = 0 \tag{64}
\]
whose solution will enable us to determine \(q_1\) and \(A_1\) using the appropriate
boundary conditions. Unfortunately, since such a procedure should be repeated
infinitely many times, it is obviously impractical. Hill was able to design a
much better method. Before discussing Hill’s equation from perspective of
modern mathematics, it is useful to recall the very basic classical facts about

\(^{34}\)Related to the so called Roche limit [64, 46].

\(^{35}\)In fact, there will be the system of Hill’s equations in general [74]. This is so since
the disturbance of trajectory is normally decomposed into that which is perpendicular and
that which is parallel to the Kepler’s trajectory at a given point. We shall avoid these
complications in our work.
this equation summarized in the book by Ince [77]. For this purpose, we shall assume that solution of Eq.(63) can be presented in the form

$$x(t) = e^{\alpha t} \sum_{r=-\infty}^{\infty} b_r e^{irt}.$$  \hfill (65)

Substitution of this result into Eq.(63) leads to the following infinite system of linear equations

$$(\alpha + 2ri)^2 b_r + \sum_{k=-\infty}^{\infty} q_k b_{r-k} = 0, \ r \in \mathbb{Z}.$$  \hfill (66)

As in finite case, obtaining of the nontrivial solution requires the infinite determinant $\Delta(\alpha)$ to be equal to zero. This problem can be looked upon from two directions: either all constants $q_k$ are assigned and one is interested in the bounded type of solution for $x(t)$ for $t \to \infty$ or, one is interested in the relationship between constants made in such a way that $\alpha = 0$. In the last case it is important to know whether there is one or more than one of such solutions available. Although answers can be found in the book by Magnus and Winkler [78], we follow McKean and Moerbeke [79], Trubowitz [80] and Moser [81]. For this purpose, we need to bring our notations in accord with those used in these references. Thus, the Hill operator is defined now as $Q(q) = -\frac{d^2}{dt^2} + q(t)$ with periodic potential $q(t) = q(t+1)$. Eq.(63) can now be rewritten as

$$Q(q)x = \lambda x.$$  \hfill (67)

Since this is the second order differential equation, it has formally 2 solutions. These solutions depend upon the boundary conditions. For instance, for periodic solutions such that $x(t) = x(t+2)$ the "spectrum" of Eq.(67) is discrete and is given by

$$-\infty < \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots \uparrow +\infty.$$  

We wrote the word spectrum in quotation marks because of the following. Eq.(67) does have a normalizable solution only if $\lambda$ belongs to the (pre assigned) intervals $(\lambda_0, \lambda_1), (\lambda_2, \lambda_3), \ldots, (\lambda_{2i}, \lambda_{2i+1}), \ldots$ In such a case the eigenfunctions $x_i$ are normalizable in the usual sense of quantum mechanics and form the orthogonal set. Periodic solutions make sense only for vertical displacement from the reference trajectory. For the horizontal displacement the boundary condition should be chosen as $x(0) = x(1) = 0$. For such chosen boundary condition the discrete spectrum also exists but it lies exactly in the gaps between the intervals just described, i.e. $\lambda_1 \leq \mu_1 < \lambda_2 < \lambda_3 \leq \mu_2 < \lambda_4 < \cdots$. For such a spectrum there is also set of normalized mutually orthogonal eigenfunctions.

Thus in both cases quantum mechanical description is assured. One can do much more however. In particular, Trubowitz [80] designed an explicit procedure for recovering the potential $q(t)$ from the $\mu$-spectrum supplemented by information about normalization constants.

It is quite remarkable that the Hill’s equation can be interpreted in terms of the auxiliary dynamical (Neumann) problem. Such an interpretation is very
helpful for the purposes of this work since it allows to include the quantum mechanical treatment of Hill’s equation into general framework developed in this paper.

Before describing such connections, we would like to add few details to the results of previous subsection. First, as in the planetary case, the number of pre assigned intervals is always finite. This means that, beginning with some pre assigned \( i \), we would be left with \( \lambda_{2i} = \lambda_{2(i+1)} \forall i > i \). These double eigenvalues do not have independent physical significance since they can be determined by the set of single eigenvalues (for which \( \lambda_{2i} \neq \lambda_{2(i+1)} \)) as demonstrated by Hochstadt [82]. Because of this, potentials \( q(t) \) in the Hill’s equation are called the finite gap potentials. Hence, physically, it is sufficient to discuss only potentials which possess finite single spectrum. The auxiliary \( \mu \)-spectrum is then determined by the gaps of the single spectrum as explained above.

With this information in our hands, we are ready to discuss the exactly solvable Neumann dynamical problem. It is the problem about dynamics of a particle moving on \( n \)-dimensional sphere \( < \xi, \xi > = \xi_1^2 + \cdots + \xi_n^2 = 1 \) under the influence of a quadratic potential \( \phi(\xi) = < \xi, A\xi > \). Equations of motion describing the motion on \( n \)-sphere are given by

\[
\ddot{\xi} = -A\xi + u(\xi)\xi \quad \text{with} \quad u(\xi) = \phi(\xi) - < \dot{\xi}, \dot{\xi} > .
\]  

(68)

Without loss of generality, we assume that the matrix \( A \) is already in the diagonal form: \( A := \text{diag}(\alpha_1, \ldots, \alpha_n) \). With such an assumption we can equivalently rewrite Eq.(68) in the following suggestive form

\[
\left( -\frac{d^2}{dt^2} + u(\xi(t)) \right) \xi_k = \alpha_k \xi_k ; \quad k = 1, \ldots, n .
\]  

(69)

Thus, in the case if we can prove that \( u(\xi(t)) \) in Eq.(69) is the same as \( q(t) \) in Eq.(67), the connection between the Hill and Neumann’s problems will be established. The proof is presented in Appendix C. It is different from that given in the lectures by Moser [81] since it is more direct and much shorter.

This proof brought us the unexpected connection with hydrodynamics through the static version of the Korteweg-de Vries equation. Attempts to describe the Saturnian rings using equations of hydrodynamic are described in the recent monograph by Esposito [83]. This time, however, we can accomplish more using already obtained information.

Following Kirillov [84], we introduce the commutator for the fields (operators) \( \xi \) and \( \eta \) as follows: \( [\xi, \eta] = \xi \partial \eta - \eta \partial \xi \). Using the KdV equation (C.10), let us consider 3 of its independent solutions: \( \xi_0, \xi_{-1} \) and \( \xi_1 \). All these solutions can be obtained from general result: \( \xi_k = t^{k+1} + O(t^2) \), valid near zero. Consider now a commutator \( [\xi_0, \xi_1] \). Straightforwardly, we obtain, \( [\xi_0, \xi_1] = \xi_1 \). Analogously, we obtain, \( [\xi_0, \xi_{-1}] = -\xi_{-1} \) and, finally, \( [\xi_1, \xi_{-1}] = -2\xi_0 \). According to Kirillov, such a Lie algebra is isomorphic to that for the group \( SL(2, R) \) which

\[36\] Since there is only finite number of gaps \( \lambda_{1}, \lambda_{2}, \ldots, \lambda_{k} \), where the spectrum is forbidden.
is the center for the Virasoro algebra. Vilenkin [86] demonstrated that the group $SL(2, R)$ is isomorphic to $SU(1, 1)$. Indeed, by means of transformation:

$$w = \frac{z - i}{z + i},$$

it is possible to transform the upper half plane (on which $SL(2, R)$ acts) into the interior of unit circle on which $SU(1, 1)$ acts. Since, according to Appendix A, the group $SU(1, 1)$ is the connected component of $SO(2, 1)$, the anticipated connection with $SO(2, 1)$ group is established.

In Appendix C we noticed connections between the Picard-Fuchs, Hill and Neumann-type equations. In a recent paper by Veselov et al [87] such a connection was developed much further resulting in the Knizhnik-Zamolodchikov-type equations for the Neumann-type dynamical systems. We refer our readers to the original literature, especially to the well written lecture notes by Moser [81]. These notes as well and his notes in collaboration with Zehnder [32] provide an excellent background for study the whole circle of ideas discussed in this subsection.

VI. Concluding Remarks

Although Einstein was not happy with the existing formulation of quantum mechanics, results obtained in this work demonstrate harmonious coexistence of general relativity and quantum mechanics. It should be noted though that such harmony had been achieved at the expense of partial sacrificing of the correspondence principle. This principle is not fully working even for such well studied system as hydrogen atom as discussed in subsection IV.B. This fact should not be considered as too worrisome as it was to Einstein. Indeed, as Heisenberg correctly pointed out: all what we know about microscopic system is its spectrum (in the very best of cases). Results of our recent works [36,37] as well as those by mathematicians Knutson and Tao [33-35] indicate that there are numerous ways to develop quantum mechanics-all based on systematically analyzing combinatorics of the observed spectral data. Such an approach is not intrinsic to microscopic objects studied by traditionally developed quantum mechanics. In works by Knutson and Tao quantum mechanics was not discussed at all! Quantum mechanical significance of their work(s) is discussed in detail in our recent paper [36]. In addition, in [37] we developed brand new quantum mechanical formalism based on the theory of Poisson-Dirichlet-type processes. These stochastic processes are not necessarily microscopic. Mathematically rigorous detailed exposition of these processes can be found in [88]. The combinatorial formalism developed in [37] works equally well for quantum field and string theories. Not surprisingly such formalism can be successfully applied to objects as big as involved in the Solar System dynamics. Surely, the terminology "big" or "small" is by definition is relative. Given this, much more

\[37\text{Since connections between the KdV and the Virasoro algebra are well documented [85], it is possible in principle to reinterpret fine structure of the Saturn’s rings string-theoretically.}\]
surprising is the unifying role of gravity at the microscopic scales of hadron physics as discussed in our latest work on gravity assisted solution of the mass gap problem for Yang-Mills fields [89]. Role of gravity in solution of the mass gap problem for Yang-Mills fields is striking and unexpected. Although Einstein did not like quantum mechanics because, as he believed, it is incompatible with his general relativity, results of this work and those of reference [89] underscore the deep connections between gravity and quantum mechanics/quantum field theory. It is being hoped that our work will stimulate development of more detailed works on the same subjects in the future, especially those involving detailed study of spin-orbital interactions.

Appendix A: Some Quantum Mechanical Problems Associated With the Lie Algebra of SO(2,1) Group

Following Wybourne [52] consider the second order differential equation of the type

\[ \frac{d^2 Y}{dx^2} + V(x)Y(x) = 0 \]  

(A.1)

where \( V(x) = a/x^2 + bx^2 + c \). Consider as well the Lie algebra of the noncompact group SO(2,1) or, better, its connected component SU(1,1). It is given by the following commutation relations

\[ [X_1, X_2] = -iX_3; \quad [X_2, X_3] = iX_1; \quad [X_3, X_1] = iX_2 \]  

(A.2)

We shall seek the realization of this Lie algebra in terms of the following generators

\[ X_1 := \frac{d^2}{dx^2} + a_1(x); \quad X_2 := i[k(x)\frac{d}{dx} + a_2(x)]; \quad X_3 := \frac{d^2}{dx^2} + a_3(x). \]  

(A.3)

The unknown functions \( a_1(x), a_2(x), a_3(x) \) and \( k(x) \) are determined upon substitution of (A.3) into (A.2). After some calculations, the following result is obtained

\[ X_1 := \frac{d^2}{dx^2} + \frac{a}{x^2} + \frac{x^2}{16}; \quad X_2 := -i\left[\frac{d}{dx} + \frac{1}{2}\right]; \quad X_3 := \frac{d^2}{dx^2} + \frac{a}{x^2} - \frac{x^2}{16}. \]  

(A.4)

In view of this, (A.1) can be rewritten as follows

\[ [(\frac{1}{2} + 8b)X_1 + (\frac{1}{2} - 8b)X_3 + c]Y(x) = 0. \]  

(A.5)

This expression can be further simplified by the unitary transformation \( U X_1 U^{-1} = X_1 \cosh \theta + X_3 \sinh \theta; \quad U X_3 U^{-1} = X_1 \sinh \theta + X_3 \cosh \theta \) with \( U = exp(-i \theta X_2) \). By choosing \( \tanh \theta = -(1/2 + 8b)/(1/2 - 8b) \) (A.5) is reduced to

\[ X_3 \tilde{Y}(x) = \frac{c}{4\sqrt{-6}} \tilde{Y}(x), \]  

(A.6)
where the eigenfunction $\tilde{Y}(x) = UY(x)$ is an eigenfunction of both $X_3$ and the Casimir operator $X^2 = X_3^2 - X_2^2 - X_1^2$ so that by analogy with the Lie algebra of the angular momentum we obtain,

$$X^2\tilde{Y}_{jn}(x) = J(J + 1)\tilde{Y}_{jn}(x) \quad \text{and} \quad X_3\tilde{Y}_{jn}(x) = \frac{c}{4\sqrt{-b}}\tilde{Y}_{jn}(x) \equiv (-J + n)\tilde{Y}_{jn}(x); \quad n = 0, 1, 2, \ldots \quad (A.7a)$$

It can be shown that $J(J + 1) = -a/4 - 3/16$. From here we obtain: $J = -\frac{1}{2}(1 \pm \sqrt{\frac{1}{4} - a})$; $\frac{1}{4} - a \geq 0$. In the case of discrete spectrum one should choose the plus sign in the expression for $J$. Using this result in (A.7) we obtain the following result of major importance

$$4n + 2 + \sqrt{1 - 4a} = c \sqrt{-b}. \quad (A.8)$$

Indeed, consider the planar Kepler problem. In this case, in view of (3.5), the radial Schrödinger equation can be written in the following symbolic form

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{\nu}{r} + \frac{u}{r^2} + g \right] R(r) = 0 \quad (A.9)$$

By writing $r = x^2$ and $R(r) = x^{-\frac{1}{2}}R(x)$ this equation is reduced to the canonical form given by (A.1), e.g. to

$$\left( \frac{d^2}{dx^2} + \frac{4u + 1/4}{x^2} + 4g x^2 + 4\nu \right) R(x) = 0 \quad (A.10)$$

so that the rest of arguments go through. Analogously, in the case of Morse-type potential we have the following Schrödinger-type equation initially:

$$\left[ \frac{d^2}{dz^2} + pe^{2\alpha z} + qe^{\alpha z} + k \right] R(z) = 0 \quad (A.11)$$

By choosing $z = \ln x^2$ and $R(z) = x^{-\frac{1}{2}}R(x)$ (A11) is reduced to the canonical form

$$\left( \frac{d^2}{dx^2} + \frac{16k + \alpha^2}{4\alpha^2 x^2} + \frac{4p}{\alpha^2} x^2 + \frac{4q}{\alpha^2} \right) R(x) = 0. \quad (A.12)$$

By analogous manipulations one can reduce to the canonical form the radial equation for Hydrogen atom and for 3-dimensional harmonic oscillator.

Appendix B: Numerical Data Used for Calculations of $n_{\text{theory}}$ (Supplement to Table 2)

1 au = 149.598.10^6 km

Masses (in kg): Sun $1.988 \cdot 10^{30}$, Jupiter $1.8986 \cdot 10^{27}$, Saturn $5.6846 \cdot 10^{26}$,
Uranus $8.6832 \times 10^{-5}$, Neptune $10.243 \times 10^{-5}$.

$q_j$ : Jupiter $0.955 \times 10^{-3}$, Saturn $2.86 \times 10^{-4}$, Uranus $4.37 \times 10^{-5}$, Neptune $5.15 \times 10^{-5}$.

$(r_j)_1$ (km) : Jupiter $127.69 \times 10^3$, Saturn $133.58 \times 10^3$, Uranus $49.77 \times 10^3$, Neptune $48.23 \times 10^3$.

$\ln \left( \frac{\gamma M}{2r_1} \right)$ : Earth $4.0062$, Jupiter $3.095$, Saturn $1.844$, Uranus $0.9513$, Neptune $1.15$.

Appendix C: Connections Between the Hill and Neumann’s Dynamical Problems

Following our paper [90], let us consider the Fuchsian-type equation given by

$$y'' + \frac{1}{2} \phi y = 0,$$

where the potential $\phi$ is determined by the equation $\phi = [f]$ with $f = y_1/y_2$ and $y_1, y_2$ being two independent solutions of (C.1) normalized by the requirement $y'_1 y_2 - y'_2 y_1 = 1$. The symbol $[f]$ denotes the Schwarzian derivative of $f$. Such a derivative is defined as follows

$$[f] = \frac{f' f'''' - \frac{3}{2} (f'')^2}{(f')^2}.$$  

(C.2)

Consider (C.1) on the circle $S^1$ and consider some map of the circle given by $F(t + 1) = F(t) + 1$. Let $t = F(\xi)$ so that $y(t) = Y(\xi) \sqrt{F'(\xi)}$ leaves (C.1) form-invariant, i.e. in the form $Y'' + \frac{1}{2} \Phi Y = 0$ with potential $\Phi$ being defined now as $\Phi(\xi) = \phi(F(\xi)) [F''(\xi)]^2 + [F(\xi)]$. Consider next the infinitesimal transformation $F(\xi) = \xi + \delta \phi(\xi)$ with $\delta$ being some small parameter and $\phi(\xi)$ being some function to be determined. Then, $\Phi(\xi + \delta \phi(\xi)) = \phi(\xi) + \delta (\bar{T}_\phi)(\xi) + O(\delta^2)$. Here $(\bar{T}_\phi)(\xi) = \phi(\xi) \phi'(\xi) + \frac{1}{2} \phi''(\xi) + 2 \phi'(\xi) \phi(\xi)$. Next, we assume that the parameter $\delta$ plays the same role as time. Then, we obtain

$$\lim_{t \to 0} \frac{\Phi - \phi}{t} = \frac{\partial \phi}{\partial t} = \frac{1}{2} \phi'''(\xi) + \phi(\xi) \phi'(\xi) + 2 \phi'(\xi) \phi(\xi) $$  

(C.3)

Since thus far the perturbing function $\phi(\xi)$ was left undetermined, we can choose it now as $\phi(\xi) = \phi(\xi)$. Then, we obtain the Korteweg-de Vries (KdV) equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \phi'''(\xi) + 3 \phi(\xi) \phi'(\xi)$$  

(C.4)

determining the potential $\phi(\xi)$. For reasons which are explained in the text, it is sufficient to consider only the static case of KdV, i.e.

$$\phi'''(\xi) + 6 \phi(\xi) \phi'(\xi) = 0.$$  

(C.5)
We shall use this result as a reference for our main task of connecting the Hill and the Neumann’s problems. Using Eq.(68) we write

\[ u(\xi) = \phi(\xi) - \langle \dot{\xi}, \dot{\xi} \rangle . \]  

(C.6)

Consider an auxiliary functional \( \varphi(\xi) = \langle \xi, A^{-1} \xi \rangle . \) Suppose that \( \varphi(\xi) = u(\xi) . \)

Then,

\[ \frac{du}{dt} = 2 < \dot{\xi}, A\xi > - 2 < \ddot{\xi}, \dot{\xi} > . \]  

(C.7)

But \( < \ddot{\xi}, \dot{\xi} > = 0 \) because of the normalization constraint \( < \xi, \xi > = 1 . \) Hence, \( \frac{du}{dt} = 2 < \dot{\xi}, A\xi > . \) Consider as well \( \frac{d\varphi}{dt} . \) By using Eq.(68) it is straightforward to show that \( \frac{d\varphi}{dt} = 2 < \dot{\xi}, A^{-1} \xi > . \) Because by assumption \( \varphi(\xi) = u(\xi) , \) we have to demand that \( < \xi, A^{-1} \xi > = < \dot{\xi}, A\xi > \) as well. If this is the case, consider

\[ \frac{d^2 u}{dt^2} = 2 < \dot{\xi}, A^{-1} \xi > + 2 < \dot{\xi}, A^{-1} \dot{\xi} > . \]  

(C.8)

Using Eq.(68) once again we obtain,

\[ \frac{d^2 u}{dt^2} = - 2 + 2u \varphi + 2 < \dot{\xi}, A^{-1} \xi > . \]  

(C.9)

Finally, consider as well \( \frac{d^3 u}{dt^3} . \) Using (C.9) as well as Eq.(68) and (C.7) we obtain,

\[ \frac{d^3 u}{dt^3} = 2 \frac{du}{dt} \varphi + 4u \frac{du}{dt} = 6u \frac{du}{dt} . \]  

(C.10)

By noticing that in (C.5) we can always make a rescaling \( \phi(\xi) \rightarrow \lambda \phi(\xi) , \) we can always choose \( \lambda = -1 \) so that (C.5) and (C.10) coincide. This result establishes correspondence between the Neumann and Hill-type problems.

QED

[1] H.Blome, C.Chicone, F.Hehl, arXiv:1002.1425
[2] J.Gine, Chaos, Solitons and Fractals 42,1893 (2009).
[3] A.Kholodenko, J.Geom. &Physics 33, 23 (2000).
[4] A.Kholodenko, J. Geom.&Physics 33, 59 (2000).
[5] A.Kholodenko, J. Geom.&Physics 38, 81 (2001).
[6] A.Assis, Relational Mechanics (Aperion, Montreal, 1999).
[7] D.Soares, arXiv:physics/0508030.
[8] K.Brown, Reflections on Relativity,www.mathpages.com.
[9] A.Goldhaber and M.Nieto, Rev.Mod. Phys. 82, 939 (2010).
[10] A.Besse, Einstein Manifolds (Springer-Verlag, Berlin, 1987).
[11] D.Genov, S.Zhang and X. Zhang, NaturePhysics 5, 687 (2009).
[12] U.Leonhardt and Th.Philbin, New J.of Physics 8, 277 (2006).
[13] M.Novello, M.Visser and G.Volovik, Artificial Black Holes (World Scientific, Singapore, 2002).
[14] J.Gine, Chaos, Solitons and Fractals 32, 363 (2007).
[15] M.Porter and P. Cvitanovič, AMS Notices 52, 1020 (2005).
[16] J.Marsden and S. Ross, AMS Bulletin 43 43 (2006).
[17] B. Convay, C.Chilan and B.Wall, Cel. Mech. Dynam. Astron. 97, 73 (2007).
[18] C.Jaffe, S. Ross, M.Lo, J.Marsden, D. Farrelly and T. Uzer, PRL 89, 011101-1 (2002).
[19] D.Micha and I. Burghardt, Quantum Dynamics of Complex Molecular Systems (Springer-Verlag, Heidelberg, 2007).
[20] C. Murray and S. Dermott, Solar System Dynamics, (Cambridge University Press, Cambridge UK 1999).
[21] H.Goldstein, C. Poole and J. Safko, Classical Mechanics (Addison-Wesley, New York, 2002).
[22] L. Landau and E. Lifshitz, The Classical Theory of Fields (Pergamon, London, 1975).
[23] L. Infeld and A. Schield, Rev.Mod.Phys. 21, 408 (1949).
[24] A.Pound, Phys.Rev.D 81 024023 (2010).
[25] P.Laplace, Celestial Mechanics Vols 1-4 (Chelsea Publ. Co. Bronx, NY, 1966).
[26] H.Poincare’, Les Methodes Nouvelles de la Mechanique Celeste (Paris: Gauthier-Villars, Paris, 1892-1898).
[27] V. Arnol’d, V. Kozlov and A. Neishtadt, Mathematical Aspects of Classical and Celestial Mechanics (Springer, Heidelberg, 2006).
[28] W.Heisenberg, Z.Phys. 33, 879 (1925).
[29] C.Charlier, Die Mechanik des Himmels (Walter de Gruyter, Berlin, 1927).
[30] J.Fejoz, Erg.Th.&Dyn.Syst. 24, 1521 (2004).
[31] L.Biasco, L. Cherchia and E. Valdinoci, SIAM J.Math.Anal. 37, 1580 (2006).
[32] J.Moser and E. Zehnder, Notes on Dynamical Systems. (AMS Publishers, Providence, RI, 2005).
[33] A.Knutson and T. Tao, AMS Notices 48, 175 (2001).
[34] A. Knutson and T. Tao, AMS Journal 12,1055 (1999).
[35] A.Knutson, T. Tao and C.Woodward, 2004 AMS Journal 17, 19 (2004).
[36] A. Kholodenko, Math.Forum 4, 441(2009)
[37] A. Kholodenko, in Lev Davydovich Landau and His Impact on Contemporary Theoretical Physics, pp 37-75, (Nova Science Publishers, New York, NY, 2010).
[38] P.Dirac, Proc.Roy.Soc. A 109, 642 (1926)
[39] P. Dirac, Principles of Quantum Mechanics
[40] W-T Ni, Int.J.Mod.Phys.D 14, 901 (2005).
[41] S.Weiberg, *Gravitation and Cosmology* (J.Wiley&Sons, Inc. New York, NY, 1972).
[42] S.Carroll, *Spacetime and Geometry*, (Addison Wesley, New York, NY, 2004).
[43] K.Nandi,N.Migranov, J.Evans and M.Amedeke, Eur.J.Phys. 27,429 (2006).
[44] A. Papapetrou, Proc.Roy.Soc. A 209, 248 (1951).
[45] H.Stephani, *General Relativity* (Cambridge University Press, Cambridge, UK, 1990).
[46] L.Pars, *Analytical Dynamics* (Heinemann, London, 1968).
[47] P.Dirac, Canadian J.Math. 2, 129 (1950).
[48] M.Bander and C.Itzykson, Rev. Mod. Phys. 38, 330 (1966).
[49] M.Kac, Am.Math.Monthly 73, 1 (1966).
[50] A.Dhar, D.Rao, U. Shankar and S. Sridhar, Phys.Rev.E 68, 026208 (2003).
[51] J.Jauch and E. Hill, Phys.Rev. 57, 641 (1940).
[52] B.Wybourne, *Classical Groups for Physicists* (John Willey & Sons, New York, NY,1974).
[53] G.Natanzon, Theor.Math.Phys. 38, 219 (1979).
[54] G.Levai, J.Phys. A 27, 3809 (1994).
[55] P.Cordero, S.Holman, P.Furlan and G.Ghirardi, Il Nuovo Cim. 3A 807 (1971).
[56] V.Bargmann, Ann.Math. 48 568-640 (1947).
[57] A. Barut A and C. Fronsdal, Proc.Roy.Soc.London A 287, 532 (1965).
[58] F. Cooper, J. Ginoccio and A. Khare, Phys.Rev.D 36, 2458 (1987).
[59] G.Junker and P.Roy, Ann.Phys. 270. 155 (1998).
[60] T.Boyer, Phys.Rev.D 11, 790 (1975).
[61] H.Puthoff,Phys.Rev.D 35, 3266 (1987).
[62] P. Goldreich, Mon. Not.R. Astr.Soc. 130, 159 (1965).
[63] M.Nieto, *The Titius -Bode Law of Planetary Distances: Its History and Theory* (Pergamon Press, London,1972).
[64] A.Celletti and E. Perozzi,*Celestial Mechanics. The Waltz of the Planets* (Springer, Berlin, 2007).
[65] L.Neslušan, Mon. Not.R. Astr.Soc. 351, 133 (2004).
[66] P.Lynch, Mon.Not. R. Astr.Soc. 341, 1174 (2003).
[67] S.Dermott, Mon.Not.R.Astr.Soc. 141, 363 (1968).
[68] W.Hayes and S. Tremaine, Icarus 135, 549 (1998).
[69] L.Landau and E.Lifshitz, *Quantum Mechanics* (Pergamon, London, 1962).
[70] M. Woollson, *The Formation of the Solar System* (Singapore: World Scientific, Singapore, 2007).
[71] M. Tiscareno and M. Hedman, Nature 461,1064 (2009).
[72] A.Verbiscer, M. Skrutskie and D.Hamilton,
Nature 461, 1098 (2009).

[73] I. Khriplovich, *General Relativity* (Springer-Verlag, Berlin, 2005).

[74] G. Chebotarev, *Analytical and Numerical Methods of Celestial Mechanics* (Amsterdam: Elsevier, Amsterdam, 1968).

[75] V. Szebehely, *Theory of Orbits* (Academic Press, New York, NY, 1967).

[76] J. Avron and B. Simon, PRL 46, 1166 (1981).

[77] E. Ince, *Ordinary Differential Equations* (Dover Publishers, New York, NY, 1926).

[78] W. Magnus and S. Winkler, *Hill's Equation* (Interscience Publishers, New York, NY, 1966).

[79] H. McKean, and P. Moerbeke, Inv. Math. 30, 217 (1975).

[80] E. Trubowitz, Comm. Pure Appl. Math. 30, 321 (1977).

[81] J. Moser, in: *Dynamical Systems*, pp.233-290 (Boston: Birckhäuser, Boston, 1980).

[82] H. Hochstadt, Math. Zeit. 82, 237 (1963).

[83] L. Esposito, *Planetary Rings* (Cambridge University Press, Cambridge, UK, 2006).

[84] A. Kirillov, *Infinite dimensional Lie groups: their orbits, invariants and representations*, LNM 970, 101 (1982).

[85] V. Arnold and B. Khesin, *Topological Methods in Hydrodynamics* (Springer-Verlag, Berlin, 1998).

[86] N. Vilenkin, *Special Functions and Theory of Group Representations* (Nauka, Moscow, 1991).

[87] A. Veselov, H. Dullin, P. Richter and H. Waalkens, Physica D 155, 159 (2001).

[88] J. Bertoin, *Random Fragmentation and Coagulation Processes* (Cambridge University Press, Cambridge, UK, 2006).

[89] A. Kholodenko, arXiv:1001.0029.

[90] A. Kholodenko, J. Geom. Phys. 43, 45 (2002).