R²-IRRREDUCIBLE UNIVERSAL COVERING SPACES OF P²-IRRREDUCIBLE OPEN 3-MANIFOLDS

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Abstract. An irreducible open 3-manifold \( W \) is \( R² \)-irreducible if it contains no non-trivial planes, i.e. given any proper embedded plane \( \Pi \) in \( W \) some component of \( W - \Pi \) must have closure an embedded halfspace \( R² \times [0, \infty) \). In this paper it is shown that if \( M \) is a connected, \( P² \)-irreducible, open 3-manifold such that \( \pi_1(M) \) is finitely generated and the universal covering space \( \tilde{M} \) of \( M \) is \( R² \)-irreducible, then either \( \tilde{M} \) is homeomorphic to \( R³ \) or \( \pi_1(M) \) is a free product of infinite cyclic groups and fundamental groups of closed, connected surfaces other than \( S² \) or \( P² \). Given any finitely generated group \( G \) of this form, uncountably many \( P² \)-irreducible, open 3-manifolds \( M \) are constructed with \( \pi_1(M) \cong G \) such that the universal covering space \( \tilde{M} \) is \( R² \)-irreducible and not homeomorphic to \( R³ \); the \( \tilde{M} \) are pairwise non-homeomorphic. Relations are established between these results and the conjecture that the universal covering space of any irreducible, orientable, closed 3-manifold with infinite fundamental group must be homeomorphic to \( R³ \).

1. Introduction

Suppose \( M \) is a connected, \( P² \)-irreducible, open 3-manifold with \( \pi_1(M) \) finitely generated and non-trivial. It is easy to construct examples of such \( M \) for which the universal covering space \( \tilde{M} \) is not homeomorphic to \( R³ \). Start with any 3-manifold \( N \) satisfying the given conditions. Let \( U \) be a Whitehead manifold, i.e. an irreducible, contractible, open 3-manifold which is not homeomorphic to \( R³ \) (see e.g. [16], [1]). Choose end-proper embeddings of \([0, \infty)\) in each of \( N \) and \( U \). (A map between manifolds is end-proper if pre-images of compact sets are compact; it is \( \partial \)-proper if the pre-image of the boundary is the boundary; it is proper if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property.) Let \( X \) and \( Y \) be the exteriors of these rays. (The exterior of a submanifold is the closure of the complement of a regular neighborhood of it.) \( \partial X \) and \( \partial Y \) are each planes. We identify them to obtain a \( P² \)-irreducible open 3-manifold \( M \) with \( \pi_1(M) \cong \pi_1(N) \). Let \( p : \tilde{M} \to M \) be the universal covering map.

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Then \( \widetilde{M} \), \( p^{-1}(X) \), and \( p^{-1}(Y) \) are \( \mathbb{P}^2 \)-irreducible \([4]\). Each component \( \widetilde{Y} \) of \( p^{-1}(Y) \) has interior \( \widetilde{U} \) homeomorphic to \( U \) and so contains a compact, connected subset \( J \) which does not lie in a 3-ball in \( \widetilde{U} \). If \( M \) were homeomorphic to \( \mathbb{R}^3 \) then \( J \) would lie in a 3-ball \( B \) in \( M \). Standard general position and minimality arguments applied to \( \partial B \) and \( \partial \widetilde{Y} \) would then yield a 3-ball \( B' \) in \( \widetilde{U} \) containing \( J \), a contradiction. Alternatively, one could use the Tucker Compactification Theorem \([14]\) to obtain a compact polyhedron \( K \) in \( \widetilde{U} \) such that some component \( V \) of \( \widetilde{U} - K \) has non-finitely generated fundamental group. But this is impossible since the union of \( V \) and \( \widetilde{M} - \widetilde{U} \) is a component of \( \widetilde{M} - \widetilde{K} \) whose fundamental group is isomorphic to \( \pi_1(V) \).

In this example \( \partial \widetilde{Y} \) is a non-trivial plane in \( \widetilde{M} \), i.e. a proper plane \( \Pi \) such that no component of \( \widetilde{M} - \Pi \) has closure homeomorphic to \( \mathbb{R}^2 \times [0, \infty) \) with \( \Pi = \mathbb{R}^2 \times \{0\} \). This paper shows that it is harder to find examples if one rules out this behavior by requiring that \( M \) be \( \mathbb{P}^2 \)-irreducible in the sense that, in addition to being irreducible, it contains no non-trivial planes.

Define a closed surface group to be the fundamental group of a closed, connected 2-manifold.

**Theorem 1.** Let \( M \) be a connected, \( \mathbb{P}^2 \)-irreducible, open 3-manifold with \( \pi_1(M) \) finitely generated. If the universal covering space \( \widetilde{M} \) of \( M \) is \( \mathbb{R}^2 \)-irreducible, then either

1. \( \widetilde{M} \) is homeomorphic to \( \mathbb{R}^3 \) or
2. \( \pi_1(M) \) is a free product of infinite cyclic groups and infinite closed surface groups.

The second possibility can be disjoint from the first.

**Theorem 2.** Suppose \( G \) is a free product of finitely many infinite cyclic groups and infinite closed surface groups. Then there is a \( \mathbb{P}^2 \)-irreducible open 3-manifold \( M \) such that \( \pi_1(M) \cong G \) and \( \widetilde{M} \) is an \( \mathbb{R}^2 \)-irreducible Whitehead manifold. Moreover, for each given \( G \) there are uncountably many such \( M \) for which the \( \widetilde{M} \) are pairwise non-homeomorphic.

This generalizes an example of Scott and Tucker \([12]\) for which \( G \) is infinite cyclic. These results have a bearing on the following well-known problem.

**Conjecture 1** (Universal Covering Conjecture). Let \( X \) be a closed, connected, irreducible, orientable 3-manifold with \( \pi_1(X) \) infinite. Then the universal covering space \( \widetilde{X} \) of \( X \) is homeomorphic to \( \mathbb{R}^3 \).

Since there are only countably many homeomorphism types of closed 3-manifolds Theorem 2 implies that there must exist uncountably many \( \mathbb{R}^2 \)-irreducible Whitehead manifolds \( \widetilde{M} \) which cover open 3-manifolds \( M \) with \( \pi_1(M) \cong G \) but cannot cover a
closed 3-manifold. This generalizes a result of Tinsley and Wright [13] which shows that there must exist uncountably many non-$\mathbb{R}^2$-irreducible Whitehead manifolds $\tilde{M}$ which cover open 3-manifolds $M$ with $\pi_1(M)$ infinite cyclic but cannot cover a closed 3-manifold. Unfortunately this argument does not provide any specific such examples. Specific examples of non-$\mathbb{R}^2$-irreducible Whitehead manifolds $\tilde{M}$ which cover open 3-manifolds $M$ with $\pi_1(M)$ infinite cyclic or, more generally, a countable free group, but cannot cover a closed 3-manifold are given in [9] and [10], respectively. At the time of this writing the problem of providing specific examples of $\mathbb{R}^2$-irreducible Whitehead manifolds which non-trivially cover other open 3-manifolds but cannot cover a closed 3-manifold is still open.

One can make several conjectures related to Conjecture 1. We consider the selection below. In all of them $G$ is assumed to be a finitely generated group of covering translations acting on a Whitehead manifold $W$ with quotient a 3-manifold $M$.

Conjecture 2. $G$ is a free product of infinite cyclic groups and fundamental groups of $\partial$-irreducible Haken manifolds.

Conjecture 3. $G$ is a free group or contains an infinite closed surface group.

Conjecture 4. If $W$ is $\mathbb{R}^2$-irreducible, then $G$ is a free product of infinite cyclic groups and infinite closed surface groups.

A proper plane $\Pi$ in $W$ is equivariant if for each $g \in G$ either $g(\Pi) = \Pi$ or $\Pi \cap g(\Pi) = \emptyset$.

Conjecture 5 (Special Equivariant Plane Conjecture). If $G$ is not a free product of infinite cyclic groups and infinite closed surface groups, then $W$ contains a non-trivial equivariant plane.

Conjecture 6 (Equivariant Plane Conjecture). If $W$ contains a non-trivial plane, then it contains a non-trivial equivariant plane.

These conjectures are related as follows.

Theorem 3. $(4) \iff (1) \iff (2) \iff (3) \iff (5) \iff (4 + 6)$

Theorems 1 and 3 are proven in section 2. Theorem 2 is proven in sections 3–7. Section 3 presents a modified version of the criterion used by Scott and Tucker [12] for showing that a 3-manifold is $\mathbb{R}^2$-irreducible. Sections 4 and 5 treat, respectively, the special cases in which $G$ is an infinite cyclic group and an infinite closed surface group. The constructions and notation of these special cases are used in section 6, which treats the general case. Section 7 shows how to get uncountably many $M$ with non-homeomorphic $\tilde{M}$ for each group $G$. 
2. The proofs of Theorems 1 and 3

Lemma 2.1. Let $M$ be a connected, $\mathbb{P}^2$-irreducible, open 3-manifold. Let $Q$ be a compact, connected, 3-dimensional submanifold of $M$ such that $\partial Q$ is incompressible in $M$ and $\pi_1(Q)$ is not an infinite closed surface group. Let $p : \tilde{M} \to M$ be the universal covering map and $G$ the group of covering translations. Let $\tilde{Q}$ be a component of $p^{-1}(Q)$. Then

1. Each component of $p^{-1}(\partial Q)$ is a plane.
2. There is no component $\Pi$ of $\partial \tilde{Q}$ which is invariant under the subgroup $G_0$ of $G$ consisting of those covering translations which leave $\tilde{Q}$ invariant.
3. If each component of $\partial \tilde{Q}$ is a trivial plane, then $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$.

Proof. (1) follows from the incompressibility of $\partial Q$ in $M$.

Suppose $S$ is a component of $\partial Q$ and $\Pi$ is a component of $p^{-1}(S)$ which is invariant under $G_0$. Since the restriction of $p$ to $\tilde{Q}$ is the universal covering space of $Q$ and the restriction of $G_0$ to $\tilde{Q}$ is the group of covering translations we have that $\pi_1(S) \to \pi_1(Q)$ is an isomorphism, contradicting our assumption on $\pi_1(Q)$. This establishes (2).

We now prove (3). Suppose that each component $\Pi$ of $\partial \tilde{Q}$ bounds an end-proper halfspace $H_\Pi$ in $\tilde{M}$. Let $K_\Pi$ be the closure of the component of $\tilde{M} - \Pi$ which does not contain $\text{int}\ \tilde{Q}$.

Assume that for all such $\Pi$ we have $H_\Pi = K_\Pi$. Then $\tilde{M}$ is the union of $\tilde{Q}$ and an open collar attached to $\partial \tilde{Q}$, hence $\tilde{M}$ is homeomorphic to $\text{int}\ \tilde{Q}$. Since $Q$ is Haken, the Waldhausen Compactification Theorem [15] implies that $\tilde{Q}$ is homeomorphic to a closed 3-ball minus a closed subset of its boundary, hence $\text{int}\ \tilde{Q}$ is homeomorphic to $\mathbb{R}^3$, and we are done.

Thus we may assume that for some $\Pi$ we have $H_\Pi \neq K_\Pi$. Then $H_\Pi \cap K_\Pi = \Pi$ and $H_\Pi \cup K_\Pi = \tilde{M}$. Now $G_0$ has an element $g$ such that $g(\Pi) \neq \Pi$. Since $\tilde{Q} \subseteq H_\Pi$ and $g(\tilde{Q}) = \tilde{Q}$ we must have $g(K_\Pi) \subseteq H_\Pi$. Since $\mathbb{R}^2 \times [0, \infty)$ is $\mathbb{R}^2$-irreducible (see e.g. [8]) it follows that $K_\Pi$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$. Thus $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$. □

Proof of Theorem 1. By passing to a covering space of $M$, if necessary, we may assume that $\pi_1(M)$ is indecomposable with respect to free products and is neither an infinite cyclic group nor an infinite closed surface group. Let $C$ be the Scott compact core [14] of $M$, i.e. $C$ is a compact, connected, 3-dimensional submanifold of $M$ such that $\pi_1(C) \to \pi_1(M)$ is an isomorphism. The conditions on $\pi_1(M)$ imply that $\partial C$ is incompressible in $M$. We thus can apply Lemma 2.1 with $Q = C$ to finish the proof. □
Proof of Theorem 3. We first show that (1) ⇒ (2) ⇒ (3) ⇒ (1). If (1) is true, then M must be non-compact; this follows from the fact that if M were closed and non-orientable, then it would be Haken and so have universal covering space homeomorphic to \( \mathbb{R}^3 \). Let C be the Scott compact core for M. Since M is irreducible we may assume that no component of \( \partial C \) is a 2-sphere; it follows that C is irreducible. If C is \( \partial \)-irreducible, then we are done. If C is not \( \partial \)-irreducible, then there is a finite set of compressing disks for \( \partial C \) in C which express C as a \( \partial \)-connected sum of 3-balls and \( \partial \)-irreducible Haken manifolds, thus yielding (2). Clearly (2) ⇒ (3). Suppose (3) is true and M is closed. If \( \pi_1 \) is free, then M is by [2, Theorem 5.2] a connected sum of 2-sphere bundles over \( S^1 \), hence is not aspherical, hence W is not contractible. If \( \pi_1 \) contains an infinite closed surface group, then by a result of Hass, Rubinstein, and Scott [1] W is homeomorphic to \( \mathbb{R}^3 \).

Clearly Theorem 1 and the fact that M cannot be closed and non-orientable show that (1) ⇒ (4).

We now show that (1) ⇒ (5). Let C be the Scott compact core of M. Then the assumptions on \( \pi_1 \) imply that there is a set of compressing disks for \( \partial C \) in C such that some component Q of C split along this collection of disks satisfies the hypotheses of Lemma 2.1. Thus any component of the pre-image of \( \partial Q \) is an equivariant non-trivial plane.

We next show that (5) ⇒ (1). Assume M is closed. If \( \pi_1 \) is a free product of infinite cyclic groups and infinite closed surface groups, then we apply (3) to obtain (1). If \( \pi_1 \) is not such a group, then the existence of an equivariant plane, together with the compactness of M, implies that M is Haken, and so (1) follows by Waldhausen [15].

Finally we show that (4 + 6) ⇒ (1). If W is \( \mathbb{R}^2 \)-irreducible, then (4) implies the hypothesis of (2), hence implies (1). If W is not \( \mathbb{R}^2 \)-irreducible, then (6) implies as before that M is Haken, thus (1) holds.

3. Nice quasi-exhaustions and \( \mathbb{R}^2 \)-irreducibility

We shall reformulate a criterion due to Scott and Tucker [12] for a \( \mathbb{P}^2 \)-irreducible open 3-manifold to be \( \mathbb{R}^2 \)-irreducible. A proper plane \( \Pi \) in an open 3-manifold W is homotopically trivial if for any compact subset C of W the inclusion map of \( \Pi \) is end-properly homotopic to a map whose image is disjoint from C.

Lemma 3.1. Let W be an irreducible, open 3-manifold, and let \( \Pi \) be a proper plane in W. If \( \Pi \) is homotopically trivial, then \( \Pi \) is trivial.

Proof. This is Lemma 4.1 of [12].

Lemma 3.2. Let W be a connected, irreducible, open 3-manifold, and let \( \{ C_n \}_{n \geq 1} \), be a sequence of compact 3-dimensional submanifolds of W such that \( C_n \subseteq \text{int} \, C_{n+1} \) and
(1) each $C_n$ is irreducible,
(2) each $\partial C_n$ is incompressible in $W - \text{int} C_n$,
(3) if $D$ is a proper disk in $C_{n+1}$ which is in general position with respect to $\partial C_n$ such that $\partial D$ is not null-homotopic in $\partial C_{n+1}$, then $D \cap \partial C_n$ has at least two components which are not null-homotopic in $\partial C_n$ and bound disjoint disks in $D$.

Then any proper plane in $W$ can be end-properly homotoped off $C_n$ for any $n$.

Proof. This is Lemma 4.2 of [12].

The precise criterion we shall use is as follows.

Lemma 3.3. Let $W$ be a connected, irreducible, open 3-manifold. Suppose that for each compact subset $K$ of $W$ there is a sequence $\{C_n\}_{n \geq 1}$ of compact 3-dimensional submanifolds such that $C_n \subseteq \text{int} C_{n+1}$ and

1. each $C_n$ is irreducible,
2. each $\partial C_n$ is incompressible in $W - \text{int} C_n$ and has positive genus,
3. each $C_{n+1} - \text{int} C_n$ is irreducible, $\partial$-irreducible, and annular,
4. $K \subseteq C_1$.

Then $W$ is $\mathbb{R}^2$-irreducible.

Proof. Let $D$ be a disk as in part (iii) of Lemma 3.2. If every component of $D \cap \partial C_n$ is null-homotopic in $\partial C_n$, then one can isotop $D$ so that $D \cap C_n = \emptyset$ and hence $\partial C_{n+1}$ is compressible in $C_{n+1} - \text{int} C_n$. If only one component $\alpha$ of $D \cap \partial C_n$ is not null-homotopic in $\partial C_n$, then $\partial D \cup \alpha$ bounds an annulus $A$ which can be isotoped so that $A \cap \partial C_n = \alpha$, hence $C_{n+1} - \text{int} C_n$ is not annular. If no two of the components of $D \cap \partial C_n$ which are not null-homotopic in $\partial C_n$ bound disjoint disks in $D$, then these components must be nested on $D$. We can isotop $D$ to remove null-homotopic components and then intermediate annuli to again get an incompressible annulus joining $\partial C_{n+1}$ to $\partial C_n$. Now apply Lemma 3.2 and then Lemma 3.1.

Let $\{C_n\}$ be a sequence of compact, connected 3-dimensional submanifolds of an irreducible, open 3-manifold $W$ such that $C_n \subseteq \text{int} C_{n+1}$ such that $W - \text{int} C_n$ has no compact components. This will be called a quasi-exhaustion for $W$. A quasi-exhaustion for $W$ whose union is $W$ is an exhaustion for $W$. A quasi-exhaustion is nice if it satisfies conditions (1)–(3) of Lemma 3.3. Thus that lemma can be rephrased by saying that if every compact subset of $W$ is contained in the first term of a nice quasi-exhaustion, then $W$ is $\mathbb{R}^2$-irreducible.

We shall need some tools for constructing Whitehead manifolds with nice quasi-exhaustions. Define a compact, connected 3-manifold $Y$ to be nice it is is $\mathbb{P}^2$-irreducible, $\partial$-irreducible, and annular, it contains a two-sided proper incompressible surface, and it is not a 3-ball; define it to be excellent if, in addition, every connected, proper, incompressible surface of zero Euler characteristic in $Y$ is $\partial$-parallel.
So in particular an excellent 3-manifold is annular and atoroidal while a nice 3-manifold is annular but may contain a non-$\partial$-parallel incompressible torus.

A proper 1-manifold in a compact 3-manifold is excellent if its exterior is excellent; it is poly-excellent if the union of each non-empty subset of the set of its components is excellent.

**Lemma 3.4.** Every proper 1-manifold in a compact, connected 3-manifold whose boundary contains no 2-spheres or projective planes is homotopic rel $\partial$ to an excellent proper 1-manifold.

*Proof.* This is a special case of Theorem 1.1 of [6].

Define a $k$-tangle to be a disjoint union of $k$ proper arcs in a 3-ball.

**Lemma 3.5.** For all $k \geq 1$ poly-excellent $k$-tangles exist.

*Proof.* This is Theorem 6.3 of [7].

We shall also need the following criterion for gluing together excellent 3-manifolds to get an excellent 3-manifold.

**Lemma 3.6.** Let $Y$ be a compact, connected 3-manifold. Let $S$ be a compact, proper, two-sided surface in $Y$. Let $Y'$ be the 3-manifold obtained by splitting $Y$ along $S$. Let $S'$ and $S''$ be the two copies of $S$ which are identified to obtain $Y'$. If each component of $Y'$ is excellent, $S'$, $S''$, and $\partial Y' - \text{int} (S' \cup S'')$ are incompressible in $Y'$, and each component of $S$ has negative Euler characteristic, then $Y$ is excellent.

*Proof.* This is Lemma 2.1 of [6].

4. The infinite cyclic case

In [12] Scott and Tucker described an $\mathbb{R}^2$-irreducible Whitehead manifold which is an infinite cyclic covering space. This section gives a general procedure for constructing such examples. The construction introduced here will be incorporated into that for the general case in section 6.

Let $P_n = D_n \times [0, 1]$, where $D_n$ is the disk of radius $n$. We call $P_n$ a pillbox. Identify $D_n \times \{0\}$ with $D_n \times \{1\}$ to obtain a solid torus $Q_n$. Let $R_n$ be a solid torus and $H_n$ a 1-handle $D \times [0, 1]$ joining $\partial D_n \times (0, 1)$ to $\partial R_n$. Let $V_n = P_n \cup H_n \cup R_n$ and $M_n = Q_n \cup H_n \cup R_n$. We call $V_n$ an eyebolt. We embed $M_n$ in the interior of $M_{n+1}$ as follows.

We choose a collection of arcs $\theta_0, \theta_1, \alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon$ in $P_{n+1}$ which satisfy certain conditions described below. $\theta_0$, $\theta_1$, and $\alpha_0$ meet in a common end-point in $\text{int} P_{n+1}$ but are otherwise disjoint. The other endpoints of $\theta_0$ and $\alpha_0$ lie in $(\text{int} D_{n+1}) \times \{0\}$; that of $\theta_1$ lies in $(\text{int} D_{n+1}) \times \{1\}$. We let $\theta = \theta_0 \cup \theta_1$. All the other arcs are proper arcs in $P_{n+1}$ which are disjoint from each other and from $\theta \cup \alpha_0$. $\gamma_1$, $\beta_2$, and $\delta_2$ run from $(\text{int} D_{n+1}) \times \{0\}$ to itself. $\gamma_2$, $\beta_1$, and $\delta_1$ run from...
(int \; D_{n+1}) \times \{1\} to itself. \(\alpha_1\) runs from \((int \; D_{n+1}) \times \{0\}\) to \((int \; D_{n+1}) \times \{1\}\). \(\alpha_2\) runs from \((int \; D_{n+1}) \times \{1\}\) to \(int (P_{n+1} \cap H_{n+1})\). \(\varepsilon\) runs from \(int (P_{n+1} \cap H_{n+1})\) to itself. We denote the image in \(Q_{n+1}\) of an arc by the same symbol, relying on the context to distinguish an arc in \(P_{n+1}\) from its image in \(Q_{n+1}\). We require that \(\theta\) be a simple closed curve in \(Q_{n+1}\) and that \(\alpha_0 \cup \beta_1 \cup \gamma_1 \cup \delta_1 \cup \alpha_1 \cup \beta_2 \cup \gamma_2 \cup \delta_2\) is an arc consisting of subarcs which occur in the given order. We require that any union of these arcs which contains \(\alpha_0\) and at least one other arc has excellent exterior in \(P_{n+1}\), and that the same is true for any union of these arcs which contains neither \(\theta_0\), \(\theta_1\), nor \(\alpha_0\). This can be achieved as follows. Note that the exterior of \(\alpha_0\) in \(P_{n+1}\) is a 3-ball. Choose a poly-excellent 11-tangle in \(B\) and then slide its endpoints so that exactly two of the arcs meet a regular neighborhood of \(\alpha_0\). Extend them to meet \(\alpha_0\) in the desired configuration.

Next let \(\kappa_1\), \(\kappa_2\), and \(\kappa_3\) be product arcs in \(H_{n+1}\) joining \((int \; D) \times \{0\}\) to \((int \; D) \times \{1\}\). Let \(R_n \subseteq int R_{n+1}\) be any null-homotopic embedding. Let \(\lambda_1\) and \(\lambda_2\) be disjoint proper arcs in \(R_{n+1} \setminus int R_n\) with \(\lambda_1\) joining \(int (H_{n+1} \cap R_{n+1})\) to itself and \(\lambda_2\) joining \(int (H_{n+1} \cap R_{n+1})\) to \(\partial R_n\). We require \(\lambda_1 \cup \lambda_2\) to be excellent in \(R_{n+1} \setminus int R_n\). We also require that these arcs, together with \(\varepsilon\), fit into an arc whose subarcs form the sequence \(\kappa_1\), \(\lambda_1\), \(\kappa_2\), \(\varepsilon\), \(\kappa_3\), \(\lambda_2\) and that \(\kappa_1\) meets \(\alpha_2\) in a common endpoint.

Now we embed \(P_n\) in \(P_{n+1}\) as a regular neighborhood of the arc \(\theta\) so that the two disks of \(P_n \cap (D_{n+1} \times \{0, 1\})\) are identified to give an embedding of \(Q_n\) in \(Q_{n+1}\). Note that these embeddings are not consistent with the product structures. From the discussion above we have an arc \(\omega\) in \(M_{n+1} \setminus int (Q_n \cup R_n)\) running from \(\partial Q_n\) to \(\partial R_n\). We embed \(H_n\) as a regular neighborhood of \(\omega\). We change notation slightly by now letting \(\alpha_0\) be the old \(\alpha_0\) minus its intersection with the interior of \(Q_n\).

We let \(M\) be the direct limit of the \(M_n\) and let \(p : \tilde{M} \to M\) be the universal covering map. \(p^{-1}(Q_n) = p^{-1}(P_n)\) is the union of pillboxes \(P_{n,j} = D_n \times [j, j + 1]\) meeting along the \(D_n \times \{\{\}\}\) to form \(D_n \times R\). Note that this embedding is not the product embedding. \(p^{-1}(R_n)\) is a disjoint union of solid tori \(R_{n,j}\). \(p^{-1}(H_n)\) is a disjoint union of 1-handles \(H_{n,j}\) joining \(\partial D_n \times (j, j + 1)\) to \(\partial R_{n,j}\); these are regular neighborhoods of lifts \(\omega_j\) of \(\omega\). \(p^{-1}(\lambda_n) = p^{-1}(\epsilon_n)\) is the union of \(p^{-1}(P_{n,j})\), \(p^{-1}(H_{n,j})\), and \(p^{-1}(R_{n,j})\). It is the union of eyebolts \(V_{n,j} = P_{n,j} \cup H_{n,j} \cup R_{n,j}\) meeting along the \(D_n \times \{\{\}\}\). \(\tilde{M}\) is the union of \(p^{-1}(M_n)\).

Let \(\Sigma_n = \bigcup_{j=-m}^{m} V_{n,j}\) and \(\Lambda_n = P_{n,-(m+1)} \cup P_{n,m+1}\). Let \(\Phi_1^m = \emptyset\), and, for \(n \geq 2\), let \(\Phi_n^m = \bigcup_{j=m+2}^{m+n} (P_{n,-j} \cup P_{n,j})\). Note that \(\lambda_n^m\) and \(\Phi_n^m\) (for \(n \geq 2\)) are each disjoint unions of two 3-balls, \(\lambda_n^m \cap \Sigma_n^m\) is a pair of disjoint disks, and (for \(n \geq 2\)) so is \(\lambda_n^m \cap \Phi_n^m\). Define \(C_n^m = \Sigma_n^m \cup \Lambda_n^m \cup \Phi_n^m\).

**Lemma 4.1.** \(\{C_n^m\}\) is an exhaustion for \(\tilde{M}\). Each \(C_n^m\) is a nice quasi-exhaustion.
Proof. \( C^m_n \subseteq \text{int} C^m_{n+1} \), and \( C^m_n \subseteq C^{m+1}_n \). A given compact subset \( K \) of \( \tilde{M} \) lies in some \( p^{-1}(M_n) \) and thus in a finite union of \( V_{n,j} \) and hence in some \( \Sigma^m_n \subseteq C^m_n \subseteq C^q_\theta \), where \( q = \max\{m,n\} \). Thus \( \{C^m_n\} \) is an exhaustion for \( \tilde{M} \).

\( C^m_n \) is a cube with \( 2m + 1 \) handles. Let \( Y = C^m_{n+1} - \text{int} C^m_n \). We will show that \( Y \) is excellent by successive applications of Lemma 3.6.

Consider a \( P_{n+1,j} \) contained in \( C^m_{n+1} \). If \( |j| < m \), then it meets \( C^m_n \) in a regular neighborhood of the union of the \( j \)th copies of all the arcs in \( P_{n+1} \). Thus \( Y \cap P_{n+1,j} \) is excellent, and Lemma 3.6 implies that the union of these \( Y \cap P_{n+1,j} \) is excellent. For \( |j| \geq m \) some care must be taken so that one is always gluing excellent 3-manifolds with equal features in the appropriate type. Note that \( Y \cap (P_{n+1,m} \cup P_{n+1,m+1} \cup \cdots \cup P_{n+1,m+n-1} \cup P_{n+1,m+n}) \) is equal to the exterior of the \( m \)th copy of all the arcs but \( \beta_1 \) and \( \delta_2 \) in \( P_{n+1,n} \), together with the exterior of the \((m + 1)st\) copy of \( \beta_2, \delta_2 \), and \( \theta \) in \( P_{n+1,m+1} \), the exterior of the \( j \)th copy of \( \theta \) in \( P_{n+1,j} \) for \( m + 1 < j < m + n \), and the 3-ball \( P_{n+1,m+n} \). This space is homeomorphic to the exterior of the \( m \)th copy of all the arcs but \( \beta_1, \delta_1, \) and \( \theta_1 \) in \( P_{n+1,m+1} \) together with the exterior of the \((m + 1)st\) copy of \( \beta_2 \) and \( \delta_2 \) in \( P_{n+1,m+1} \), and the 3-ball consisting of the union of the \( P_{n+1,j} \) for which \( m + 1 < j \leq m + n \). This can be seen by taking the arc consisting of the \( m \)th copy of \( \theta \) and the \( j \)th copy of \( \theta \) for \( m < j < m + n \) and retracting it onto the endpoint in which it meets the rest of the graph. This space is then excellent by Lemma 3.6. Similar remarks apply for \( j \leq -m \), so these spaces can be added on to get that \( Y \cap \cup^m_{j=-m+n} P_{n+1,j} \) is excellent.

We fill in the remainder of \( Y \) by adding the exteriors of the \( j \)th copies of \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) in \( H_{n+1,j} \) and \( \lambda_1 \cup \lambda_2 \) in \( R_{n+1,j} - \text{int} R_{n,j} \) for \( |j| \leq m \). Since the first of these spaces is a product the union of the two spaces is homeomorphic to the second space, and Lemma 3.6 applies to complete the proof that \( Y \) is excellent.

It remains to show that each \( \partial C^m_n \) is incompressible in \( \tilde{M} - \text{int} C^m_n \). Since each \( C^m_{n+s+1} - \text{int} C^m_{m+q} \) is \( \partial \)-irreducible we have that \( \partial C^m_n \) is incompressible in \( C^m_{n+q} - \text{int} C^m_n \) for each \( q \geq 1 \). \( p^{-1}(M_{n+q}) \) is the union of \( C^m_{n+q} \) and the sets \( C^- = \cup_{j<0} V_{n+j, q} \) and \( C^+ = \cup_{j>0} V_{n+j, q} \). We have that \( C^- \cap C^+ \) and \( C^+ \cap C^m_{n+q} \) are disjoint disks, while \( C^- \cap C^+ = \emptyset \). It follows that \( \partial C^m_n \) is incompressible in \( p^{-1}(M_{n+q}) - \text{int} C^m_n \). Since \( \tilde{M} \) is the nested union of the \( p^{-1}(M_{n+q}) \) over all \( q \geq 1 \) we have the desired result. \( \square \)

5. The surface group case

Let \( F \) be a closed, connected surface other than \( S^2 \) or \( \mathbb{P}^2 \). Let \( n \geq 1 \). Regard \( F \) as being obtained from a \( 2k \)-gon \( E, k \geq 2 \), by identifying sides \( s_i \) and \( s'_i, 1 \leq i \leq k \). This induces an identification of the lateral sides \( S_i = s_i \times [-n,n] \) and \( S'_i = s_i \times [-n,n] \) of the prism \( P_n = E \times [-n,n] \) which yields \( Q_n = F \times [-n,n] \). Let \( R_n \) be a solid torus and \( H_n \) a 1-handle \( D \times [0,1] \). Let \( V_n = P_n \cup H_n \cup R_n \), where \( P_n \cap R_n = D \times \{1\} \) is a disk in \( \partial R_n \), and \( P_n \cap R_n = \emptyset \). We again call \( V_n \) an eyebolt. It is a solid torus.
whose image under the identification is \( M_n = Q_n \cup H_n \cup R_n \), a space homeomorphic to the \( \partial \)-connected sum of \( F \times [-n, n] \) and a solid torus.

We define an open 3-manifold \( M \) by specifying an embedding of \( M_n \) in the interior of \( M_{n+1} \) and letting \( M \) be the direct limit. The inclusion \( [-n, n] \subseteq [-n+1, n+1] \) induces \( P_n \subseteq P_{n+1} \) and hence \( Q_n \subseteq Q_{n+1} \). We let \( R_n \subseteq \text{int} R_{n+1} \) be any null-homotopic embedding. Again the interesting part of the embedding will be that of \( H_n \) in \( M_{n+1} \). It will be the regular neighborhood of a certain arc \( \omega \) in \( M_{n+1} - \text{int} (Q_n \cup R_n) \) joining \( \partial Q_n \) to \( \partial R_n \).

The arc \( \omega \) is the union of \( 4k + 7 \) arcs any two of which are either disjoint or have one common endpoint. The \( 4k + 2 \) arcs \( \alpha_0, \alpha_i, \beta_i, \gamma_i, \delta_i, 1 \leq i \leq k \), and \( \varepsilon \) lie in \( E \times [n, n+1] \) and are identified with their images in \( Q_{n+1} \); the three arcs \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) lie in \( H_{n+1} \), and the two arcs \( \lambda_1 \) and \( \lambda_2 \) lie in \( R_{n+1} \). These arcs will have special properties to be described later. We first describe their combinatorics. The arcs in \( P_{n+1} \) are all proper arcs in \( E \times [n, n+1] \). \( \alpha_0 \) runs from \( (\text{int} E) \times \{ n \} \) to \( \text{int} S_1 \). For \( 1 \leq i < k \), \( \alpha_i \) runs from \( \text{int} S_i \) to \( \text{int} S_{i+1} \). \( \alpha_k \) runs from \( \text{int} S_k' \) to \( \text{int} (P_{n+1} \cap H_{n+1}) \). For \( 1 \leq i \leq k \), \( \beta_i \) and \( \delta_i \) each run from \( \text{int} S_i' \) to itself, while \( \gamma_i \) runs from \( \text{int} S_i \) to itself. These arcs are chosen so that under the identification their endpoints match up in such a way as to give an arc which follows the sequence \( \alpha_0, \beta_1, \gamma_1, \delta_1, \alpha_1, \ldots, \beta_k, \gamma_k, \delta_k, \alpha_k \). We require \( \varepsilon \) to run from \( \text{int} (P_{n+1} \cap H_{n+1}) \) to itself. \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) are product arcs in \( H_{n+1} \) lying in \( (\text{int} D) \times [0, 1] \). \( \lambda_1 \) and \( \lambda_2 \) are proper arcs in \( R_{n+1} - \text{int} R_n \), with \( \lambda_1 \) running from \( \text{int} (H_{n+1} \cap R_{n+1}) \) to itself and \( \lambda_2 \) running from \( \text{int} (H_{n+1} \cap R_{n+1}) \) to \( \partial R_n \). These arcs are chosen so as to fit together into the sequence \( \kappa_1, \lambda_1, \kappa_2, \varepsilon, \kappa_3, \lambda_2 \) with the endpoint of \( \kappa_1 \) other than \( \kappa_1 \cap \lambda_1 \) being the same as the endpoint of \( \alpha_k \) other than \( \alpha_k \cap \delta_k \). This gives \( \omega \).

We now describe the special properties required of these arcs. We require that \( \alpha_0 \cup \beta_1 \cup \gamma_1 \cup \delta_1 \cup \alpha_1 \cup \ldots \cup \beta_k \cup \gamma_k \cup \delta_k \cup \varepsilon \) be a poly-excellent \((4k + 2)\)-tangle in \( E \times [n, n+1] \) and \( \lambda_1 \cup \lambda_2 \) to be an excellent 1-manifold in \( R_{n+1} - \text{int} R_n \).

We now consider the universal covering map \( p : \widetilde{M} \to M \). Our goal is to construct a sequence \( \{ C^m \} \) of nice quasi-exhaustions whose diagonal \( \{ C^m \} \) is an exhaustion for \( \widetilde{M} \).

The universal covering space \( \widetilde{F} \) of \( F \) is tessellated by copies \( E_j \) of \( E \). We fix one such copy \( E_1 \). We inductively define an exhaustion \( \{ F_n \} \) for \( \widetilde{F} \) as follows. \( F_1 = E_1 \). \( F_{m+1} \) is the union of \( F_m \) and all those \( E_j \) which meet it. Each \( F_m \) is a disk (which we call a star). The inner corona \( I_m \) of \( F_m \) is the annulus \( F_{m+1} - \text{int} F_m \). Each vertex on \( \partial F_m \) lies in either one or two of those \( E_j \) contained in \( F_m \). Each \( E_j \) in \( I_m \) meets \( F_m \) in either an edge or a vertex; in both cases it meets exactly two adjacent \( E_\ell \) of \( I_m \), and each of these intersections is an edge. For \( n \geq 2 \) we define the outer \( n \)-corona \( O_n^m \) to be the annulus \( F_{m+n} - \text{int} F_{m+1} \); we define \( O_n^m = \emptyset \). Let \( \sigma_2 \) be a proper arc in \( F_2 \) consisting of three edges of the polygons in \( F_2 \). Inductively define a proper arc \( \sigma_{m+1} \) in \( F_{m+1} \) by adjoining to \( \sigma_m \) two arcs spanning \( I_m \) which are edges.
of polygons in $I_m$. Thus each $\sigma_m$ is an edge path in $F_m$ splitting it into two unions of polygons $F'_m$ and $F''_m$.

We now consider the structure of $\tilde{M}$. For $n \geq 1$, $p^{-1}(Q_n) = p^{-1}(P_n)$ is the union of prisms $P_{n,j} = E_j \times [-n,n]$ meeting along their lateral sides to form $\tilde{F} \times [-n,n]$. $p^{-1}(R_n)$ is a disjoint union of solid tori $R_{n,j}$. $p^{-1}(H_n)$ is a disjoint union of 1-handles $H_{n,j}$ running from $E_j \times \{n\}$ to $\partial R_{n,j}$; these are regular neighborhoods of lifts $\omega_j$ of $\omega$. Now $p^{-1}(M_n) = p^{-1}(V_n)$ is the union of $p^{-1}(P_n)$, $p^{-1}(H_n)$, and $p^{-1}(R_n)$. It can be expressed as the union of the eyebolts $V_{n,j} = P_{n,j} \cup H_{n,j} \cup R_{n,j}$ meeting along the lateral sides of the $P_{n,j}$. Finally $\tilde{M}$ is the nested union of the $p^{-1}(M_n)$.

Let $\Sigma_n^m$ be the union of those $V_{n,j}$ such that $E_j$ is in the star $F_m$. Let $\Lambda_n^m$ be the union of those $P_{n,j}$ such that $E_j$ is in the inner corona $I_m$. Let $\Phi_n^m$ be the union of those $P_{n,j}$ such that $E_j$ is in the outer $n$-corona $O_n^m$. Note that $\Lambda_n^m$ and $\Phi_n^m$ (for $n \geq 2$) are solid tori, $\Lambda_n^m \cap \Sigma_n^m$ is an annulus which goes around $\Lambda_n^m$ once longitudinally and consists of those lateral sides of the prisms in $\Sigma_n^m$ which lie on $\partial \Sigma_n^m$, and (for $n \geq 2$) $\Lambda_n^m \cap \Phi_n^m$ is an annulus which goes around each of these solid tori once longitudinally. We now define $C_n^m = \Sigma_n^m \cup \Lambda_n^m \cup \Phi_n^m$.

**Lemma 5.1.** \{C_n^m\} is an exhaustion for $\tilde{M}$. Each $C_n^m$ is a nice quasi-exhaustion.

**Proof.** Note that $C_n^m \subseteq \text{int} \ C_{n+1}^m$, and $C_n^m \subseteq C_{n+1}^m$. Suppose $K$ is some compact subset of $\tilde{M}$. Then $K$ lies in some $p^{-1}(M_n)$ and thus in a finite union of $V_{n,j}$ and hence in some $\Sigma_n^m \subseteq C_n^m \subseteq C_q^q$, where $q = \max\{m, n\}$. Thus $\{C_n^m\}$ is an exhaustion for $\tilde{M}$.

Each $C_n^m$ is a cube with handles, so is irreducible. The number of handles is at least one, so $\partial C_n^m$ has positive genus. Let $Y = C_{n+1}^m - \text{int} \ C_n^m$. We will prove that $Y$ is excellent by successive applications of Lemma 3.6. Let $P_{n+1,j}^+$ and $P_{n+1,j}^-$ denote, respectively, $E_j \times [n, n + 1]$ and $E_j \times [-n + 1, -n]$.

Consider a $P_{n+1,j}$ contained in $\Sigma_{n+1}^m$. It meets $C_n^m$ in $P_{n+1,j}$ together with regular neighborhoods of certain arcs in $P_{n+1,j}^+$. These arcs consist at least of the $j^{th}$ copies of the $\alpha_i$, the $\gamma_i$, and $\varepsilon$ which are part of the lift $\omega_j$ of $\omega$. If another prism $P_{n+1,i}$ in $\Sigma_{n+1}^m$ meets $P_{n+1,j}$ in a common lateral side, then either $\omega_j$ or $\omega_i$ will meet this side; in the latter case this contributes a $\beta_i$ and $\delta_i$ to the subsystem of arcs in $P_{n+1,j}^+$. Since the full system of arcs was chosen to be poly-excellent this subsystem of arcs is excellent and so has excellent exterior $Y \cap P_{n+1,j}^+$. Let $U'$ be the union of those $Y \cap P_{n+1,j}^+$ such that $E_j \subseteq F'_m$. This space can be built up inductively by gluing on one $Y \cap P_{n+1,j}^+$ at a time, with the gluing being done along either a disk with two holes (when $P_{n+1,j}$ is glued along one lateral side) or a disk with four holes (when $P_{n+1,j}$ is glued along two adjacent lateral sides). No component of the complement of this surface in the boundary of either manifold is a disk, hence this surface is incompressible in each manifold. It follows that $U'$ is excellent. Similar remarks apply to the space $U''$ associated with $F''_m$. 

Next consider a $P_{n+1,j}^+$ contained in $\Lambda^m_{n+1}$. If $E_j \subseteq F_{m+1}$ and meets $F'_m$ in an edge of $E_\ell \subseteq F'_m$, then either $\omega_\ell$ misses $P_{n+1,j}^+$ or meets it in copies of $\beta_i$ and $\delta_i$. Thus enlarging $U'$ by adding $\partial C \cap P_{n+1,j}^+$ either adds a 3-ball along a disk in its boundary, giving a space homeomorphic to $U'$ or gives a new excellent 3-manifold. We adjoin all such $\partial C \cap P_{n+1,j}^+$ to $U'$. Then we consider those $E_j$ which meet $F'_m$ in a vertex. Then $P_{n+1,j}^+ = \partial C \cap P_{n+1,j}^+$; and one can successively adjoin these 3-balls along disks in their boundaries. We denote the enlargement of $U'$ from all these additions again by $U'$. Similar remarks apply to $U''$.

Now $(F_{m+1}^+ - \text{int } F^m_{m+1}) \times [n, n+1]$ is a 3-ball which meets $U'$ in a disk, so we adjoin it to $U'$ to get a new $U'$ homeomorphic to the old one. We then adjoin the 3-ball $(F_{m+1}^+ - \text{int } F^m_{m+1}) \times [-n, n]$ which meets this space along a disk to obtain our final $U'$. The same construction gives $U''$.

Now $U'$ and $U''$ are each excellent. $U' \cap U''$ is an annulus with a positive number of disks removed from its interior corresponding to its intersection with arcs passing from $F'_m \times [n, n+1]$ to $F'_m \times [n, n+1]$. No component of the complement of this surface in $\partial U'$ or in $\partial U''$ is a disk; this corresponds to the fact that $F'_m \times \{n\}$, $F'_m \times \{n\}$, $F'_m \times \{n+1\}$, and $F'_m \times \{n+1\}$ each meet some $\omega_j$. Thus this surface is incompressible in both $U'$ and $U''$, so $U' \cup U''$ is excellent.

Finally we add on the $Y \cap R_{n,j}$ for $E_j \subseteq F_m$ to $U' \cup U''$ to conclude that $Y$ is excellent.

It remains to show that each $\partial C^m_n$ is incompressible in $\widetilde{M} - \text{int } C^m_n$. First note that since each $C^m_{n+1} - \text{int } C^m_{n+1}$ is $\partial$-irreducible we must have that $\partial C^m_n$ is incompressible in $C^m_{n+1} - \text{int } C^m_n$ for each $q \geq 1$. Now consider the set

$$\widetilde{M}_{n+q} = p^{-1}(M_{n+q}) \cup (\widetilde{F} \times [-(n+q+1), -(n+q)]).$$

It can be obtained from $C^m_{n+q}$ as follows. First add the solid tori $R_{n+1, j} \cup H_{n+q, j}$ in $p^{-1}(M_{n+q})$ for which $E_j \subseteq F_{n+q+1}$; these meet $C^m_{n+q}$ in disks. Then add

$$(F_{m+q+n}^+ \times [-(n+q+1), -(n+q)]) \cup (\widetilde{F} - \text{int } F_{m+q+n}) \times [-(n+q+1), n+q]).$$

This is a space homeomorphic to $\mathbb{R}^2 \times [0, 1]$ which meets $C^m_{n+q}$ in the disk

$$(F_{m+q+n} \times \{-(n+q)\}) \cup ((\partial F_{m+q+n}) \times [-(n+q), n+q]).$$

Lastly add all the remaining solid tori $R_{n+q, j} \cup H_{n+q, j}$, where $E_j \subseteq \widetilde{F} - \text{int } F_{m+q+n}$; these do not meet $C^m_{n+q}$. This description shows that $C^m_{n+q} \cap (\widetilde{M}_{n+q} - \text{int } C^m_{n+q})$ consists of (finitely many) disjoint disks, and therefore $\partial C^m_{n+q}$ is incompressible in $\widetilde{M}_{n+q} - \text{int } C^m_{n+q}$. Finally since $\widetilde{M}$ is the nested union of the $\widetilde{M}_{n+q}$ over all $q \geq 1$ we have that $\partial C^m_n$ is incompressible in $\widetilde{M} - \text{int } C^m_n$. \[\square\]
6. The General Case

Suppose \( G_1, \ldots, G_k \) are infinite cyclic groups and infinite closed surface groups. For \( i = 1, \ldots, k \) let \( P_n^i \) be a pillbox or prism, as appropriate, with quotient \( Q_n^i \) a solid torus or product \( I \)-bundle over a closed surface, respectively. We let \( H_n^i \) be a 1-handle attached to \( P_n^i \) as before. We let \( R_n \) be a common solid torus to which we attach the other ends of all the \( H_n^i \). The union of the \( Q_n^i \) and \( H_n^i \) with \( R_n \) is called \( M_n \). As before we choose arcs in the \( P_n^i, H_n^i, \) and \( R_n \) and use them to define an embedding of \( M_n \) into the interior of \( M_{n+1} \).

The choice of arcs in \( R_{n+1} \setminus \text{int} R_n \), as well as the embedding \( R_n \subseteq \text{int} R_{n+1} \), requires some discussion, since we will want this family \( \lambda \) of arcs to be poly-excellent. Choose a poly-excellent \((2k+2)\)-tangle \( \lambda^+ \) in a 3-ball \( B \), with components \( \lambda^+_t, 1 \leq i \leq k+1, t = 1, 2 \). Construct a graph in \( B \) by sliding one endpoint of each \( \lambda^+_t \), \( 1 \leq i \leq k \), so that it lies on \( \text{int} \lambda^{k+1}_t \). Thus these \( \lambda^+_t \) now join \( \partial B \) to distinct points on \( \text{int} \lambda^{k+1}_t \); all the other \( \lambda^+_t \) still join \( \partial B \) to itself. Now choose disjoint disks \( E_1 \) and \( E_2 \) in \( \partial B \) such that \( E_t \) meets the graph in \( \partial \lambda^{k+1}_t \cap \text{int} E_t \). Glue \( E_1 \) to \( E_2 \) so that \( B \) becomes a solid torus \( R_{n+1} \) and \( \lambda^{k+1}_t \cup \lambda^{k+1}_t \) becomes a simple closed curve. The regular neighborhood of this simple closed curve is our embedding of \( R_n \) in the interior of \( R_{n+1} \). Clearly \( R_n \) is null-homotopic in \( R_{n+1} \). By Lemma 3.6 its exterior is excellent as is the exterior of the union of \( R_n \) with any of the \( \lambda^+_t, 1 \leq i \leq k, t = 1, 2 \).

Let \( p : \tilde{M} \to M \) be the universal covering map. Then \( p^{-1}(R_n) \) consists of disjoint solid tori whose union separates \( p^{-1}(M_n) \) into components with closures \( L_n^{i,\mu} \), where \( L_n^{i,\mu} \) is a component of \( p^{-1}(Q_n^i \cup H_n^i) \). Let \( Z_n^{i,\mu} \) be the union of \( L_n^{i,\mu} \) and all those components of \( p^{-1}(R_n) \) which meet it. Then \( Z_n^{i,\mu} = \cup_{n \geq 1} Z_n^{i,\mu} \) is an open subset of \( \tilde{M} \) which has a family \( \{C_n^{i,\mu,m}\} \) of quasi-exhaustions as previously described. We will develop from these families an appropriate family \( \{C_n^m\} \) of quasi-exhaustions of \( \tilde{M} \).

We start by choosing a component \( \tilde{R}_1 \) of \( p^{-1}(R_1) \). For each \( n \) there is then a unique component \( \tilde{R}_n \) of \( p^{-1}(R_n) \) which contains \( \tilde{R}_1 \). We define \( C_n^1 \) to be the union of \( \tilde{R}_n \) and the (finitely many) \( C_n^{i,\mu,1} \) which contain it; this space is a solid torus which meets each of these \( C_n^{i,\mu,1} \) in a 3-ball which is either a pillowbox and a 1-handle or a prism and a 1-handle. Suppose \( C_n^m \) has been defined and that it is the union of the \( C_n^{i,\mu,m} \) for which \( C_n^m \cap L_n^{i,\mu} \neq \emptyset \). We define \( C_n^{m+1} \) in two steps. We first take the union \( C' \) of all the \( C_n^{i,\mu,m} \) such that \( C_n^{i,\mu,m} \subseteq C_n^m \). This is just the union of the \( n \)-th elements of the \( (m+1) \)-st quasi-exhaustions for those \( Z_n^{i,\mu} \) such that \( \{i, \mu\} \) is in the current index set. The second step is to enlarge the index set by adding those \( \{i, \nu\} \) for which \( C' \cap L_n^{i,\nu} \neq \emptyset \) and then adjoin the \( C_n^{i,\nu,m+1} \) to \( C' \) in order to obtain \( C_n^{m+1} \). One can observe that the \( L_n^{i,\mu} \) and \( p^{-1}(R_n) \) give \( p^{-1}(M_n) \) a tree-like structure and that the passage from \( C_n^m \) to \( C_n^{m+1} \) goes out further along this tree.

Lemma 6.1. \( \{C_n^m\} \) is an exhaustion for \( \tilde{M} \). \( C_n^m \) is a nice quasi-exhaustion.
Proof. Again we have $C_n^m \subseteq \text{int} C_{n+1}^m$ and $C_n^m \subseteq C_n^{m+1}$ with the result that $\{C_m^m\}$ is an exhaustion for $\tilde{M}$.

As regards the excellence of $C_{n+1}^m - \text{int} C_m^m$ we note that the only thing new takes place in those components of $p^{-1}(R_{n+1})$ contained in $C_{n+1}^m$. Instead of two arcs $\lambda_1$ and $\lambda_2$ as before we have $\lambda_i^1$ and $\lambda_i^2$ as $i$ ranges over some non-empty subset of $\{1, \ldots, k\}$.

We then apply the poly-excellence of the full set of $\lambda_i^1$.

The incompressibility of $\partial C_n^m$ in $\tilde{M} - \text{int} C_n^m$ follows as before. We first note that $\partial C_n^m$ is incompressible in $C_{n+q}^m - \text{int} C_n^m$ for each $q \geq 1$. Now define $\tilde{M}_{n+q}$ to be the union of $p^{-1}(M_{n+q})$ and, for each of the surface group factors $G_i$ of $G$, the copy $\tilde{F}_{i,\mu} \times \left[ -(n+q+1), -(n+q) \right]$ of $\tilde{F}_i \times \left[ -(n+q+1), -(n+q) \right]$ contained in $Z_{i,\mu}^*$, where $\tilde{F}_i$ is the universal covering space of the surface $F_i$ with $\pi_1(F_i) \cong G_i$. Then the exterior of $C_{n+q}^m$ in $\tilde{M}_{n+q}$ meets it in a collection of disjoint disks, from which it follows that $\partial C_n^m$ is incompressible in $\tilde{M}_{n+q} - \text{int} C_n^m$, thus is incompressible in $\tilde{M} - \text{int} C_n^m$. $\square$

7. Uncountably many examples

We now describe how to get uncountably many examples for a given group $G$. We will use a trick introduced in [7]. Let $\{X_{n,s}\}$ be a family of exteriors of non-trivial knots in $S^3$ indexed by $n \geq 2$ and $s \in \{0, 1\}$; they are chosen to be anamnular, atoroidal, and pairwise non-homeomorphic. (One such family is that of non-trivial, non-trefoil twist knots.) One chooses a function $\varphi(n)$ with values in $\{0, 1\}$, i.e. a sequence of 0’s and 1’s indexed by $n$, and constructs a 3-manifold $\tilde{M}[\varphi]$ by embedding $X_{n,\varphi(n)}$ in $M_n - \text{int} M_{n-1}$ so that $\partial X_{n,\varphi(n)}$ in $M_n - \text{int} M_{n-1}$ (but is compressible in $M_n$). The idea is to do this in such a way that for “large” compact sets $C$ in $\tilde{M}[\varphi]$ one has components of $p^{-1}(X_{n,\varphi(n)})$ which lie in $\tilde{M} - C$ and have incompressible boundary in $\tilde{M} - C$ for “large” values of $n$; moreover, every knot exterior having these properties should be homeomorphic to some $X_{n,\varphi(n)}$. Thus if $\tilde{M}[\varphi]$ and $\tilde{M}[\psi]$ are homeomorphic one must have $\varphi(n) = \psi(n)$ for “large” $n$. One then notes that there are uncountably many functions which are pairwise inequivalent under this relation.

We proceed to the details. First assume $\varphi$ is fixed, so we can write $s = \varphi(n)$. The most innocuous place to embed $X_{n,s}$ is in $R_n - \text{int} R_{n-1}$ since this space is common to all our constructions. Recall that this space contains arcs $\lambda_1$, $\lambda_2$ or, if $G$ is a non-trivial free product, arcs $\lambda_i^1$, $\lambda_i^2$, $1 \leq i \leq k$; call this collection of arcs $\lambda$. We wish $X_{n,s}$ to lie in the complement of $\lambda$ in such a way that it is poly-excellent in $R_n - \text{int} (R_{n-1} \cup X_{n,s})$. We revise the construction of $\lambda$ from section 6 as follows. Let $B_0$ and $B_1$ be 3-balls. Choose disjoint disks $D_r$ and $D'_r$ in $\partial B_r$. Let $\zeta_r$ be a simple closed curve in $\partial B_r - (D_r \cup D'_r)$ which separates $D_r$ from $D'_r$. Let $A_r$ and $A'_r$ be the annuli into which $\zeta_r$ splits the annulus $\partial B_r - \text{int} (D_r \cup D'_r)$, with the notation chosen...
so that $A_r \cap D_r = \emptyset$. Let $\tau_r$ be a poly-excellent $(4k+4)$-tangle in $B_r$ which is the union of $(2k+2)$-tangles $\rho_r$ and $\rho'_r$ satisfying the following conditions. Each component of $\rho_0$ runs from $\text{int} D_0$ to $\text{int} A'_0$. Each component of $\rho_1$ runs from $\text{int} A'_1$ to $\text{int} D'_1$. Each component of $\rho'_0$ runs from $\text{int} D'_0$ to itself. Each component of $\rho'_1$ runs from $\text{int} D'_1$ to $\text{int} D_1$. We then glue $A'_0$ to $A'_1$ and $D'_0$ to $D'_1$ so as to obtain a space homeomorphic to a 3-ball minus the interior of an unknotted solid torus contained in the interior of the 3-ball. The 2-sphere boundary component is $D_0 \cup D_1$; the torus boundary component is $A_0 \cup A_1$. The gluing is done so that the endpoints of the arcs match up to give a system $\lambda^+$ of $2k+2$ arcs. Each arc in this system consists of an arc of $\rho_0$ followed by an arc of $\rho_1$ followed by an arc of $\rho'_0$ followed by an arc of $\rho'_1$. We then glue $X_{n,s}$ to this space along their torus boundaries so as to obtain a 3-ball $B$. We then apply the construction of section 6 to $\lambda^+$ to get a poly-excellent system $\lambda$ of arcs in $R_n - \text{int} R_{n-1}$. It is easily seen that this 3-manifold is nice and that $\partial X_{n,s}$ is, up to isotopy, the unique incompressible non-$\partial$-parallel torus in it; $\partial X_{n,s}$ is also, up to isotopy, the unique incompressible torus in the exterior $K_\sigma$ of any non-empty union $\sigma$ of components of $\lambda$.

**Lemma 7.1.** If $\tilde{M}[\varphi]$ and $\tilde{M}[\psi]$ are homeomorphic then there is an index $N$ such that $\varphi(n) = \psi(n)$ for all $n \geq N$.

**Proof.** Consider $\tilde{M}$. $Y = C^m_n - \text{int} C^m_{n-1}$ contains copies of $K_\sigma$ for various choices of $\sigma$. The closure of the complement in $Y$ of these copies consists of excellent 3-manifolds which meet the copies along incompressible planar surfaces. It follows that the various copies of $\partial X_{n,s}$ in $Y$ are, up to isotopy and for $n \geq 3$, the unique incompressible tori in $Y$. The incompressibility of $\partial C^m_n$ in $\tilde{M} - \text{int} C^m_n$ implies that these tori are also incompressible in $\tilde{M} - \text{int} C^m_{n-1}$.

Suppose $T$ is an incompressible torus in $\tilde{M} - \text{int} C^m_{n-1}$. Then $T$ lies in $\tilde{M}_{n+q}$ for some $q \geq 0$. The exterior of $C^m_{n+q}$ in $\tilde{M}_{n+q}$ consists of copies of $D \times \mathbb{R}$ and $\mathbb{R}^2 \times [0, 1]$ to which disjoint 1-handles have been attached. It meets $C^m_{n+q}$ in a set of disjoint disks. It follows that $T$ can be isotoped into $C^m_{n+q} - \text{int} C^m_{n-1}$. Since $\partial C^m_{n+u}$ for $1 \leq u < q$ is not a torus it is easily seen that $T$ can be isotoped into some $C^m_v - \text{int} C^m_{v-1}$ and thus is isotopic to some copy of $\partial X_{v,\varphi(v)}$. Thus any knot exterior $X$ incompressibly embedded in $\tilde{M} - \text{int} C^m_{n-1}$ is homeomorphic to some $X_{v,\varphi(v)}$.

Now consider two different functions $\varphi$ and $\psi$. We will show that if $\tilde{M}[\varphi]$ and $\tilde{M}[\psi]$ are homeomorphic then there is an $N$ such that $\varphi(n) = \psi(n)$ for all $n \geq N$. Suppose $h : \tilde{M}[\varphi] \to \tilde{M}[\psi]$ is a homeomorphism. Distinguish the various submanifolds arising in the construction of these two manifolds by appending $[\varphi]$ and $[\psi]$, respectively. For $n \geq 2$ there are incompressibly embedded copies $\tilde{X}_{n,\varphi(n)}$ of $X_{n,\varphi(n)}$ in $\tilde{M}[\varphi] - \text{int} C^1_n[\varphi]$. There is an index $\ell$ such that $h(C^1_n[\varphi]) \subseteq \text{int} C^1_\ell[\psi]$. By construction $\cup_{n \geq 2} \tilde{X}_{n,\varphi(n)}$ is end-proper in $\tilde{M}[\varphi]$, so there is an index $N$ such that for all $n \geq
we have \( h(\tilde{X}_{n,\varphi(n)}) \subseteq \tilde{M}[\psi] - \text{int} C^\ell_1[\psi] \). Since \( h(\partial \tilde{X}_{n,\varphi(n)}) \) is incompressible in \( \tilde{M}[\psi] - \text{int} C^\ell_1[\psi] \) it is incompressible in the smaller set \( \tilde{M}[\psi] - \text{int} C^\ell_1[\psi] \). Thus it is homeomorphic to \( X_{v,\psi(v)} \) for some \( v > \ell \). Since the knot exteriors are pairwise non-homeomorphic we must have \( n = v \) and \( \varphi(n) = \psi(v) = \psi(n) \).

\[ \square \]

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