Dynamical algebra and Dirac quantum modes in Taub-NUT background

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Abstract

The $SO(4,1)$ gauge-invariant theory of the Dirac fermions in the external field of the Kaluza-Klein monopole is investigated. It is shown that the discrete quantum modes are governed by reducible representations of the $o(4)$ dynamical algebra generated by the components of the angular momentum operator and those of the Runge-Lenz operator of the Dirac theory in Taub-NUT background. The consequence is that there exist central and axial discrete modes whose spinors have no separated variables.

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1 Introduction

In the relativistic quantum mechanics in the context of general relativity an important problem is how could be found the conserved operators which

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should be involved in the sets of commuting observables defining quantum modes. From this point of view examples of geometries giving rise to large sets of conserved quantities are useful for understanding the connection among symmetries and conservation laws in general relativity.

One of the most interesting geometries is that of the Euclidean Taub-NUT space since it has not only the usual isometries but also admits a hidden symmetry of the Kepler type if a cyclic variable is gotten rid of. Moreover in the Taub-NUT geometry there are four Killing-Yano tensors among them the first three are covariantly constant and realize the quaternionic algebra related to the hyper-Kähler structure of the Taub-NUT manifold. The last one, that exists by virtue of the metric being of type $D$, has a non-vanishing field strength. These Killing-Yano tensors represent a certain square root of three Stäckel-Killing tensors connected with the components of an analogue to the Runge-Lenz vector of the Kepler type problem since they give rise to constants of the geodesic motion (i.e. prime integrals) quadratic in 4-velocities.

In other respects, the Taub-NUT metric is involved in many modern studies in physics. For example the Kaluza-Klein monopole was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. On the other hand, in the long-distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space. The Dirac equation in the Kaluza-Klein monopole field was studied in the mid eighties. An attempt to take into account the Runge-Lenz vector of this problem was done in. We have continued this study showing that the Dirac equation is analytically solvable and determining the energy eigenspinors of the central modes. Moreover we derived all the conserved observables of this theory, including those associated with the hidden symmetries of the Taub-NUT geometry. Thus we obtained the Runge-Lenz vector-operator of the Dirac theory, pointing out its specific properties.

In the present paper we investigate the consequences of the existence of the Runge-Lenz operator in the theory of the Dirac field in Taub-NUT background. We show that the dynamical algebras corresponding to different spectral domains are the same as in the scalar case but the representations of these algebras are different. Thus for each discrete energy level we obtain two irreducible representations of the $o(4)$ algebra determining different quantum modes. This is a new phenomenon due to the spin terms of the Runge-Lenz operator since in the scalar case for each energy level we have only one
irreducible representation of the $o(4)$ algebra \[ \mathfrak{o}(4) \]. This conjecture offers us the opportunity to define new Dirac modes, called natural central or axial modes, the energy eigenspinors of which can be easily written using only algebraic properties.

In Sec.2 we briefly present the Killing vectors of the isometries of the Taub-NUT background and the Killing tensors corresponding to the hidden symmetries of this geometry, pointing out that these are related to the four specific Killing-Yano tensors. The general form of the conserved operators of the Dirac theory is discussed in the next section while in Sec.4 we write down the conserved operators arising from isometries or hidden symmetries, presenting their algebraic properties. Sec.5 is devoted to the study of the representations of the dynamical algebra of the discrete energy levels. These can be completely investigated since the action of the Dirac operators can be calculated at the level of two-component Pauli spinors where the structure of the corresponding operators is simpler. This method helps us to define the natural discrete Dirac modes in the next section where, in addition, we completely determine the form of the energy eigenspinors of these modes in terms of the spinors of the simple modes defined in Appendix. We work in natural units with $\hbar = c = 1$.

## 2 The Taub-NUT geometry

The background of the gauge-invariant five-dimensional theory of the Dirac fermions in the external field of the Kaluza-Klein monopole \[ \mathbb{1} \] is the space of the Taub-NUT gravitational instanton with the time coordinate trivially added. Herein it is convenient to consider the static chart of Cartesian coordinates $x^\mu$, $(\mu, \nu, ... = 0, 1, 2, 3, 5)$, with the line element

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = dt^2 - \frac{1}{V} dl^2 - V (dx^5 + A_i dx^i)^2,$$

(1)

where $dl^2 = (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is the usual Euclidean three-dimensional line element. This involves the Cartesian physical space coordinates $x^i$ ($i, j, ... = 1, 2, 3$) which cover the domain $D$. The other coordinates are the time, $x^0 = t$, and the Cartesian Kaluza-Klein extra-coordinate, $x^5 \in D_5$. Another chart suitable for applications is that of spherical coordinates, $x^i = (t, r, \theta, \phi, \chi)$, among them the first four are the time and the common spherical coordinates associated with $x^i$, ($i = 1, 2, 3$), while
\[ \chi + \phi = -\mu x^5. \] The real number \( \mu \) is the main parameter of the theory which enters in the expression of the function \( 1/V(r) = 1 + \mu/r. \) The unique non-vanishing component of the vector potential in spherical coordinates is \( A_\phi = \mu (1 - \cos \theta). \) In Cartesian coordinates we have
\[
\text{div} \vec{A} = 0, \quad \vec{B} = \text{rot} \vec{A} = \mu \vec{x}/r^3.
\] (2)

We note that the space domain \( D \) is defined such that \( 1/V > 0. \)

The spacetime defined by (1) has the symmetry given by the isometry group \( G_s = SO(3) \otimes U_5(1) \otimes T_t(1) \) of rotations of the Cartesian space coordinates and \( x^5 \) and \( t \) translations. The \( U_5(1) \) symmetry is important since this eliminates the so called NUT singularity if \( x^5 \) has the period \( 4\pi \mu. \) On the other hand, the \( SO(3) \) isometry transformations involve all of the space coordinates, \( x^i \) and \( x^5, t, \) since these are given by a non-linear representation of the \( SO(3) \) group. For this reason the corresponding Killing vectors, \( k_{(i)} \) \( (i = 1, 2, 3), \) have the components \( k_{(i)}^j = -\varepsilon_{ijk} x^k \) and \( k_{(i)}^5 = -\varepsilon_{ijk} A_k - \mu x^i/r \) while the other two Killing vectors of the translation groups have usual constant components, \( k_{(0)}^\mu = \delta_0^\mu \) and \( k_{(5)}^\mu = \delta_5^\mu. \)

The Taub-NUT geometry possesses four Killing-Yano tensors, \( f_{(i)} \) \( (i = 1, 2, 3) \) and \( f^Y, \) of valence 2 (which satisfy \( f_{\mu\nu} = -f_{\nu\mu}, \) and \( f_{\mu\nu;\sigma} + f_{\mu\sigma;\nu} = 0). \) The first three,
\[
f_{(i)} = f_{(i)}^{\alpha\beta} \hat{e}^\alpha \wedge \hat{e}^\beta = 2 \hat{e}^5 \wedge \hat{e}^i + \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k, \tag{3}
\]
are rather special since they are covariantly constant (with vanishing field strength). The fourth Killing-Yano tensor of the Taub-NUT space,
\[
f^Y = -\frac{x^i}{r} f_{(i)} + \frac{2x^i}{\mu V} \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k, \tag{4}
\]
is not covariantly constant having the non-vanishing components of its field strength
\[
f^Y_{r\theta;\phi} = \frac{2r^2}{\mu V} \sin \theta, \tag{5}
\]
which are completely antisymmetric in \( r, \theta \) and \( \phi. \)

The hidden symmetries of the Taub-NUT geometry are encapsulated in the non-trivial Stäckel-Killing tensors \( k_{(i)\mu\nu} \) \( (i = 1, 2, 3) \). They can be expressed as symmetrized products of Killing-Yano tensors (3) and (4):\[ k_{(i)\mu\nu} = -\mu (f^Y_{\mu\lambda} f_{(i)\lambda}^{\nu} + f^Y_{\nu\lambda} f_{(i)\lambda}^{\mu}) + \frac{1}{2\mu} (k_{(5)\mu} k_{(i)\nu} + k_{(5)\nu} k_{(i)\mu}). \tag{6} \]
In fact only the product of Killing-Yano tensors \( f^{(i)} \) and \( f^Y \) leads to non-trivial Stäckel-Killing tensors, the last term in the r.h.s. of (3) being a simple product of Killing vectors. This term is usually added to write the Runge-Lenz vector in the standard form of the scalar (classical, Schrödinger or Klein-Gordon) theory [2, 4, 12].

3 The operators of the Dirac theory

The theory of the Dirac field in Cartesian charts of the Taub-NUT geometry takes the simplest form if one considers the local frames given by pentad fields, \( e(x) \) and \( ˆe(x) \), as defined in [13]. Their components, have the usual orthonormalization properties and give the components of the metric tensor, \( g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}^\mu_{\hat{\alpha}} \hat{e}^\nu_{\hat{\beta}} \) and \( g^{\mu\nu} = \eta^{\hat{\alpha}\hat{\beta}} \hat{e}_\mu^{\hat{\alpha}} \hat{e}_\nu^{\hat{\beta}} \). In our notation [9], \( \eta = \text{diag}(1, -1, -1, -1, -1) \) is the flat metric which raises or lowers the hated indices (ranging from 0 to 5). Its gauge group, \( G(\eta) = \text{SO}(4, 1) \), has as universal covering group, \( \tilde{G}(\eta) \), a subgroup of the group \( \text{SU}(2, 2) \) carried by the space of four-dimensional Dirac spinors. Therefore, the five matrices \( \tilde{\gamma}^{\hat{\alpha}} \), that must satisfy \( \{ \tilde{\gamma}^{\hat{\alpha}}, \tilde{\gamma}^{\hat{\beta}} \} = 2\eta^{\hat{\alpha}\hat{\beta}} \), can be defined in terms of the standard Dirac matrices [14, 15] as \( \tilde{\gamma}^0 = \gamma^0 \), \( \tilde{\gamma}^i = \gamma^i \), \( (i = 1, 2, 3) \) and \( \tilde{\gamma}^5 = i\gamma^5 \). These are self-adjoint with respect to the usual Dirac conjugation, i.e. \( \tilde{\gamma}^{\hat{\alpha}} = \gamma^0 (\tilde{\gamma}^{\hat{\alpha}})^{\dagger} \gamma^0 = \tilde{\gamma}^{\hat{\alpha}} \), and give the covariant basis generators of the group \( \tilde{G}(\eta) \), denoted by \( S^{\hat{\alpha}\hat{\beta}} = i[\tilde{\gamma}^{\hat{\alpha}}, \tilde{\gamma}^{\hat{\beta}}]/4 \).

The Dirac field \( \psi \) of mass \( M \), defined on the space domain \( D \times D_5 \), has the gauge-invariant action [14, 7],

\[
S[\psi] = \int d^5 x \sqrt{g} \left\{ \frac{i}{2} \left[ \bar{\psi} \tilde{\gamma}^{\hat{\mu}} \tilde{\nabla}_{\hat{\mu}} \psi - (\bar{\psi} \tilde{\nabla}_{\hat{\mu}} \psi) \tilde{\gamma}^{\hat{\mu}} \psi \right] - M \bar{\psi} \psi \right\} \tag{7}
\]

where \( \tilde{\nabla}_{\hat{\mu}} \) are the components of the spin covariant derivatives with local indices [9],

\[
\tilde{\nabla}_i = i\sqrt{V} P_i + \frac{i}{2} V \sqrt{\varepsilon_{ijk}} \Sigma^*_j B_k, \quad \tilde{\nabla}_5 = \frac{i}{\sqrt{V}} P_5 - \frac{i}{2} V \sqrt{\bar{\Sigma}^*} \cdot \bar{B}. \tag{8}
\]

These depend on the momentum operators \( P_i = -i(\partial_i - A_i \partial_5) \) and \( P_5 = -i\partial_5 \), which in Taub-NUT geometry obey the commutation rules \([P_i, P_j] = i\varepsilon_{ijk} B_k P_5 \) and \([P_1, P_5] = 0 \). The spin matrices which give the spin connection are defined by

\[
\Sigma^*_i = S_i + \frac{1}{2} \gamma^5 \gamma^i, \quad S_i = \frac{1}{2} \varepsilon_{ijk} S^{jk}. \tag{9}
\]
The action (7) leads to the Dirac equation, \( \mathcal{D}\psi = M\psi \), given by the Dirac operator \([7, 9]\)

\[
\mathcal{D} = i\gamma^0 \partial_t - \sqrt{V} \gamma^5 \vec{P} - \frac{i}{\sqrt{V}} \gamma^5 \vec{P}_5 - \frac{i}{2} V \sqrt{V} \gamma^5 \vec{\Sigma}^* \cdot \vec{B},
\]

having the usual time-dependent term \([15]\) and a static part, \( \mathcal{D}_s \).

In the standard representation of the Dirac matrices (with diagonal \( \gamma^0 \) \([15]\) ) the Hamiltonian operator is \([9]\) \( \tilde{H} = H + \gamma^0 M \). The massless Hamiltonian,

\[
H = \gamma^0 \mathcal{D}_s = \begin{pmatrix} 0 & V \pi^* \frac{1}{\sqrt{V}} \\ \sqrt{V} \pi & 0 \end{pmatrix},
\]

is expressed in terms of the operators \( \pi = \sigma P - i V^{-1} P_5 \) and \( \pi^* = \sigma P + i V^{-1} P_5 \) where \( \sigma = \vec{\sigma} \cdot \vec{L} \) involves the Pauli matrices, \( \sigma_i \). These Pauli operators give the space part of the massless Klein-Gordon operator as,

\[
\Delta = V \pi^* \pi = V \vec{P}^2 + \frac{1}{V} P_5^2.
\]

In what follows, we are interested especially by the form and the action of the conserved operators of the Dirac theory which, by definition, are the operators that commute with the Hamiltonian operator \( \tilde{H} \). Since the mass term is simple and does not rise difficulties to keep the study of the operators as simple as possible we leave it aside. We shall consider the massless case \( (M = 0) \) and the Hamiltonian operator reduces to (11) without affecting the symmetries and conserved observables of the Dirac theory. The conserved operators depend on Pauli operators obeying several conditions requested by the above definition. Two types of Dirac operators are important for our further developments. The first one are the \( Q \)-operators introduced in \([9]\) as

\[
Q(X) = \left\{ H, \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \right\} = i \left[ Q_0, \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \right],
\]

where \( Q_0 = i\mathcal{D}_s = i\gamma^0 H \). The linear mapping \( Q \) associates Dirac operators to the Pauli operators, \( \pi, \pi^*, \sigma_p, \sigma_L = \vec{\sigma} \cdot \vec{L}, \sigma_r = \vec{\sigma} \cdot \vec{x}/r \), etc. \([16]\). The remarkable property of the \( Q \)-operators is that if \( [X, \Delta] = 0 \) then we can write

\[
Q(X)Q(Y) = H Q(XY), \quad Q(Y)Q(X) = Q(YX)H,
\]
for any other Pauli operator $Y$. Hereby it results that $Q(X)$ commutes with $H$ since $H = Q(1)$. Moreover, one can verify that if both the operators, $X$ and $Y$, commute with $\Delta$ and commute or anticommute between themselves then $Q(X)$ and $Q(Y)$ commute or anticommute each other. An important property is that all of the $Q$-operators anticommute with $\gamma^0$. Another type of conserved operators are of the diagonal form $T = \text{diag}(T^{(1)}, T^{(2)})$. Then the condition $[T, H] = 0$ requires the Pauli operators $T^{(1)}$ and $T^{(2)}$ to satisfy

$$T^{(2)}\sqrt{V}\pi = \sqrt{V}\pi T^{(1)}, \quad V^{\pi*} \frac{1}{\sqrt{V}} T^{(2)} = T^{(1)} V^{\pi*} \frac{1}{\sqrt{V}}. \quad (15)$$

Hereby it results $[T^{(1)}, \Delta] = 0$ and the useful relations

$$TQ(X) = Q(T^{(1)}X), \quad Q(X)T = Q(XT^{(1)}), \quad (16)$$

that can be used for deriving new algebraic properties.

## 4 Symmetries and conserved observables

A class of conserved operators are the generators of the operator-valued representations of the isometry group carried by the spaces of physical fields. In the scalar case, these are orbital operators defined up to the factor $-i\hbar$ as the Killing vector fields associated to the isometries \[17\]. For the fields with spin the conserved generators get, in addition, specific spin terms determined by the form of the Killing vectors and the gauge fixing. Thus, in the case of the Dirac field (in four or five dimensions), for each Killing vector $k$ one can write the operator \[17, 18\]

$$X_k = -i\kappa^\mu \varepsilon^\mu_{\hat{\alpha} \hat{\beta}} \nabla_{\hat{\alpha}} + \frac{1}{2} k_{\mu \nu} \varepsilon^\mu_{\hat{\alpha}} \varepsilon^\nu_{\hat{\beta}} S^\hat{\alpha}^\hat{\beta} \quad (17)$$

which commutes with the Dirac operator $D$.

In Cartesian coordinates and our gauge fixing, the $U_t(1)$ generator is $-i\partial_t$, while the $U_5(1)$ one is $P_5$. In spherical coordinates it is convenient to replace $P_5$ with $Q = -\mu P_5 = -i\partial_\chi$. The other three Killing vectors give the $SO(3)$ generators which are the components of the whole angular momentum $\vec{J} = \vec{L} + \vec{S}$ as in the flat spacetimes. The difference is that here the orbital angular momentum,

$$\vec{L} = \vec{x} \times \vec{P} - \mu \frac{\vec{x}}{r} P_5, \quad (18)$$
depends on $P_5$ since the $SO(3)$ isometries are non-linear transformations. The consequence is that the irreducible representations of the $o(3)$ algebra generated by $L_i$ are similar with the linear ones but with a supplementary restriction upon the angular quantum numbers [13]. On the other hand, the operators $J_i = \text{diag}(J_i, J_i)$ have diagonal form where $J_i = L_i + \sigma_i/2$ are just the angular momentum operators of the Pauli theory. For this reason $J_i$ commute with $\mathcal{D}$ and $\gamma^0$ and, therefore, they are conserved, commuting with $H$. Moreover, they satisfy the canonical commutation rules among themselves and with the components of all the other vector operators (e.g. coordinates, momenta, spin, etc.).

However, there are other types of conserved operators directly related to the specific geometric objects of the Taub-NUT geometry, as Stäckel-Killing and Killing-Yano tensors. The Killing-Yano tensors give rise to conserved observables defined as Dirac-type operators of the form

$$Q_f = f_{\dot{\alpha}\dot{\beta}} \bar{\partial}^{\dot{\alpha}} \bar{\nabla}^{\dot{\beta}} - \frac{1}{6} \gamma^\mu(x)\gamma^\nu(x)\gamma^\lambda(x)f_{\mu\nu\lambda}(x)$$

(19)

where $f_{\dot{\alpha}\dot{\beta}} = f_{\mu\nu}e^\mu_{\dot{\alpha}}e^\nu_{\dot{\beta}}$ and $\gamma^\mu(x) = e^\mu_{\dot{\alpha}}\bar{\gamma}^{\dot{\alpha}}$. According to an important result of Ref.[18], these operators anticommute with the Dirac operator $\mathcal{D}$.

Starting with the first three Killing-Yano tensors, from Eq.(19), after some algebra, we obtain the Dirac-type operators

$$Q_i = f^i_{\dot{\alpha}\dot{\beta}} \bar{\partial}^{\dot{\alpha}} \bar{\nabla}^{\dot{\beta}} = Q(\sigma_i),$$

(20)

which anticommute with $Q_0$ and $\gamma^0$, commute with $H$ and obey

$$Q_iQ_j = \delta_{ij}H^2 + i\varepsilon_{ijk}Q_kH, \quad [J_i, Q_j] = i\varepsilon_{ijk}Q_k.$$  

(21)

Hereby we find the $N = 4$ superalgebra [9]

$$\{Q_A, Q_B\} = 2\delta_{AB}H^2, \quad A, B, \ldots = 0, 1, 2, 3.$$  

(22)

The corresponding Dirac-type operator of the fourth Killing-Yano tensor, $f^Y$, calculated with the general rule (19), is

$$Q^Y = -Q(\sigma_r) + \frac{2i}{\mu\sqrt{V}} \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

(23)

where $\lambda = \tilde{\sigma} \cdot (\tilde{x} \times \tilde{P}) + 1 = \sigma_L + 1 + \mu\sigma_rP_5$ is the operator studied in [13]. Other equivalent forms are given in [10]. Again one can verify that $Q^Y$ commutes with $H$ and anticommutes with $Q_0$ and $\gamma^0$. 
Of a special interest is the Runge-Lenz operator of the Dirac theory associated to the Killing tensor $\vec{k}_{\mu\nu}$. This can be constructed with the help of the conserved Dirac-type operators generated by the Killing-Yano tensors. One defines first the vector operator $\vec{N}$ of components

$$N_i = \frac{\mu}{4} \{Q^Y, Q_i\} - J_i P_5$$

(24)

which commutes with $H$ since the operators $Q^Y$ and $Q_i$ are commuting with $H$. Consequently, $\vec{N}$ has diagonal form, its diagonal blocks, $\vec{N}^{(1)}$ and $\vec{N}^{(2)}$, obeying the general conditions (15). One finds that the first block which commutes with $\Delta$,

$$\vec{N}^{(1)} = \vec{K} + \frac{\vec{s}}{2} P_5,$$

(25)

contains not only the orbital Runge-Lenz operator,

$$\vec{K} = \frac{1}{2}(\vec{P} \times \vec{L} - \vec{L} \times \vec{P}) - \frac{\mu}{2} \frac{\vec{x}}{r} \Delta + \frac{\mu}{2} \frac{\vec{x}}{r} P_5^2,$$

(26)

but a spin term too. Furthermore, by taking into account that the components of $\vec{K}$ commute with $\Delta$ and satisfy

$$[L_i, K_j] = i \varepsilon_{ijk} K_k,$$

$$[K_i, K_j] = i \varepsilon_{ijk} L_k F^2, \quad F^2 = P_5^2 - \Delta,$$

(27)

(28)

one can calculate the algebraic properties given in (10) among them the commutation relations

$$[N_i, N_j] = i \varepsilon_{ijk} J_k F^2 + \frac{i}{2} \varepsilon_{ijk} Q_k H, \quad F^2 = P_5^2 - H^2,$$

(29)

suggested us to define the components of the Runge-Lenz operator of the Dirac theory, $\vec{K}$, as follows

$$K_i = N_i + \frac{1}{2} H^{-1}(\mathcal{F} - P_5)Q_i.$$

(30)

Then one obtains the commutation relations

$$[\mathcal{K}_i, H] = 0, \quad [\mathcal{K}_i, J_j] = i \varepsilon_{ijk} \mathcal{K}_k,$$

$$[\mathcal{K}_i, P_5] = 0, \quad [\mathcal{K}_i, Q_j] = i \varepsilon_{ijk} Q_k \mathcal{F},$$

(31)

(32)
and
\[ [\mathcal{K}_i, \mathcal{K}_j] = i\varepsilon_{ijk}J_k \mathcal{F}^2. \]  
Since there are no zero modes \[9\] the operator \( H \) is invertible such that our definition of the Runge-Lenz operator is correct. It is worthy to note that \( \mathcal{K}_i \) are diagonal commuting with \( \gamma^0 \) which means that they are also conserved in the massive case \([\mathcal{K}_i, \hat{H}] = 0\).

5 Dynamical algebra

The large collection of conserved observables we have obtained will help us to select many different complete sets of commuting observable which should define static quantum modes, with a given energy \( E > 0 \). In other respects, since \( P_5 \) commutes with all conserved observables, its eigenvalue, \( \hat{q} \), will play the role of a general parameter. When one uses the operator \( Q \) instead of \( P_5 \) then we have to consider as parameter its eigenvalue \( q = -\mu \hat{q} \).

In these conditions we can re-scale the Runge-Lenz operator in order to recover the dynamical algebras \( \mathfrak{o}(4) \), \( \mathfrak{o}(3, 1) \) or \( \mathfrak{e}(3) \), corresponding to different spectral domains of the Kepler-type problems \[2\]. If we define

\[ \mathcal{R}_i = \begin{cases} \mathcal{F}^{-1}\mathcal{K}_i & \text{for } \mu < 0 \text{ and } E < |\hat{q}| \\ \mathcal{K}_i & \text{for any } \mu \text{ and } E = |\hat{q}| \\ i\mathcal{F}^{-1}\mathcal{K}_i & \text{for any } \mu \text{ and } E > |\hat{q}| \end{cases} \]

then the operators \( \mathcal{J}_i \) and \( \mathcal{R}_i \) \((i = 1, 2, 3)\) will generate either a representation of the \( \mathfrak{o}(4) \) algebra for the discrete energy spectrum in the domain \( E < |\hat{q}| \) or a representation of the \( \mathfrak{o}(3, 1) \) algebra for continuous spectrum in the domain \( E > |\hat{q}| \). The dynamical algebra \( \mathfrak{e}(3) \) corresponds only to the ground energy of the continuous spectrum, \( E = |\hat{q}| \).

The manipulation of the dynamical algebras seems to be difficult in view of the complicate structure of their generators but in fact the calculations can be done at the level of Pauli operators because of the special properties of the common eigenspinors of \( H \) and \( P_5 \). These have the form \( U_{E,\hat{q}} = (u_{E,\hat{q}}^{(1)}, u_{E,\hat{q}}^{(2)})^T \) depending on two-component Pauli spinors which satisfy the equations \[\]

\[ \Delta u_{E,\hat{q}}^{(1)} = E^2 u_{E,\hat{q}}^{(1)}, \]  
\[ u_{E,\hat{q}}^{(2)} = E^{-1} \sqrt{V} \pi u_{E,\hat{q}}^{(1)}, \]
equivalent with the eigenvalue problem \((H - E)U_{E,\hat{q}} = 0\). This means that \(u_{E,\hat{q}}^{(1)}\) may be any solution of the scalar equation (35) which is just the static Klein-Gordon equation in Taub-NUT geometry. Moreover, it is clear that \(u_{E,\hat{q}}^{(2)}\) is completely determined by Eq.(36) if the form of \(u_{E,\hat{q}}^{(1)}\) is known. Since the Klein-Gordon equation is scalar, the form of the spinor \(u_{E,\hat{q}}^{(1)}\) and, therefore, that of \(U_{E,\hat{q}}\) is strongly dependent on the choice of the other observables included in the complete set of commuting operators which defines the quantum modes. Technically speaking, we have to start with the first Pauli spinor of the form \(u_{E,\hat{q}}^{(1)} \sim \sum f_\sigma \hat{\xi}_\sigma\) where \(\hat{\xi}_\sigma\) are the usual Pauli eigenspinors of polarizations \(\sigma = \pm 1/2\) while \(f_\sigma\) are scalar solutions of Eq.(35). The next step is to precise this linear combination with the help of the other conserved operators of the complete set of commuting operators and to write \(u_{E,\hat{q}}^{(2)}\) according to (36). The action of these operators can be easily calculated using the mentioned general properties (15) of the diagonal operators or the structure of the off-diagonal ones [9]. For example, it is not difficult to show that the action of the Runge-Lenz operator upon the energy eigenspinors is

\[
\vec{K} U_{E,\hat{q}} = \left( E^{-1} \sqrt{\pi} \vec{K}' u_{E,\hat{q}}^{(1)} \right), \quad \vec{K}' = \vec{K} + F \frac{\vec{\sigma}}{2},
\]

which means that this reduces to the action of the new operator \(\vec{K}'\) upon the first Pauli spinor. We say that \(\vec{K}'\) is the Pauli operator associated with \(\vec{K}\). In fact, because of the central role played here by \(u_{E,\hat{q}}^{(1)}\), this property is general, each conserved observables having its own associated Pauli operator. Thus any problem of the Dirac theory in Taub-NUT background can be rewritten in terms of Pauli operators acting upon the first two-component spinors of the spinors \(U_{E,\hat{q}}\).

In order to illustrate how works this mechanism, let us study the representations of the \(o(4)\) dynamical algebra of the discrete quantum modes of the Dirac field. As mentioned, these arise in the domain \(E < |\hat{q}|\), only when \(\mu < 0\), and have the same energy levels as the scalar Klein-Gordon (or Schrödinger) [9] equation,

\[
E_n^2 = \frac{2}{\mu^2} \left[ n \sqrt{n^2 - q^2} - (n^2 - q^2) \right].
\]

These are labeled only by the principal quantum number \(n\) which takes all the integer values larger than \(|q|\) [9]. The quantization rule of the scalar
modes is related to the irreducible unitary finite-dimensional representations of the \( o(4) \) dynamical algebra generated by \( L_i \) and the components of the rescaled Runge-Lenz vector, \( R_i = F^{-1}K_i \). Since the second Casimir operator of this algebra is \( C_2 = \vec{L} \cdot \vec{R} = 0 \), the first one, \( C_1 = \vec{L}^2 + \vec{R}^2 \), can take only the eigenvalues \( c_1 = n^2 - 1 \) (with \( n > |q| \)) which give the energy levels (38).

In terms of \( su(2) \) weights these representations of \( o(4) \sim su(2) \times su(2) \) are denoted by \( (n - 1/2, n - 1/2) \).

In the Dirac case the dynamical algebra is the same but its representations are generated by \( J_i \) and \( R_i' = F^{-1}K_i' \) acting upon the first Pauli spinor. However, since \( J_i = L_i + \sigma_i/2 \) and \( R_i' = R_i + \sigma_i/2 \), we draw the conclusion that the Dirac discrete modes are governed by the reducible representation

\[
\left( \frac{n-1}{2}, \frac{n-1}{2} \right) \otimes \left( \frac{1}{2}, 0 \right) = \left( \frac{n}{2}, \frac{n-1}{2} \right) \oplus \left( \frac{n-1}{2}, \frac{n-1}{2} \right).
\]

The Casimir operators, \( C'_1 = \vec{J}^2 + \vec{R}'^2 \) and \( C'_2 = \vec{J} \cdot \vec{R}' \) take now the eigenvalues, \( c'_1 \) and \( c'_2 \), such that \( c'_1 - 2c'_2 = n^2 - 1 \) for both irreducible representations while \( c'_2 = (2n + 1)/4 \) for the representation \( (n - \frac{1}{2}, \frac{n-1}{2}) \) and \( c'_2 = -(2n - 1)/4 \) for the representation \( (\frac{n}{2} - 1, \frac{n-1}{2}) \). This suggests us to use the new Pauli operator

\[
C = 2C'_2 - 1/2 = \sigma_R + \sigma_L + 1, \quad \sigma_R = \vec{\sigma} \cdot \vec{R}
\]

in order to distinguish between the irreducible representations resulted from the decomposition (39). The advantage is that this has the simplest eigenvalues, \( c = \pm n \).

6 Discrete quantum modes

Since the energy levels \( E_n \) are degenerated, we need to use complete sets of commuting operators for determining quantum modes. Fortunately the set of conserved observables is large enough to offer us many possibilities of choice. An appropriate option is to consider the natural modes involving only one irreducible representation of (39). The complete sets of commuting observables of these modes must include the operators \( H \), \( P_5 \) and the new operator \( C = 2\vec{J} \cdot \vec{R} - 1/2 \) associated with the Pauli operator \( C \) defined above. Then only two more operators we need for defining these Dirac quantum modes.
We say that the set \{H, P_5, C, \vec{J}^2, J_3\} defines the natural central modes. Its common eigenspinors, \(U_{n,\hat{q},c,j,m_j}\), correspond to the eigenvalues \(E_n, \hat{q}, c, j(j+1)\) and \(m_j\). Another possibility is to take the set \{\(H, P_5, C, R_3, J_3\)\} of the natural axial modes the common eigenspinors of which, \(U_{n,\hat{q},c,m_r,m_j}\), correspond to the eigenvalues \(E_n, \hat{q}, c, m_r\) and \(m_j\). On the other hand, our previous results \[12,9\] lead to the conclusion that there are simple modes having eigenspinors with separated variables. These are central and axial simple modes defined by the sets \{\(H, P_5, \vec{J}^2, J_3, Q(\sigma_L + 1)\)\} and \{\(H, P_5, R_3, J_3, Q_3\)\} respectively, as it is shown in Appendix. Since neither \(Q(\sigma_L + 1)\) nor \(Q_3\) do not commute with \(C\), it results that the natural modes do not have eigenspinors with separated variables. Therefore these must be linear combinations of the spinors of simple modes. Then it is interesting to try to write down these linear combinations using only the algebraic method based on the relation between Dirac and Pauli operators.

According to this method, we observe the first Pauli spinors, \(u_{n,\hat{q},c,j,m}^{(1)}\), of the eigenspinors \(U_{n,\hat{q},c,j,m}\) of the natural central modes must be the eigenspinors of the set \{\(\Delta, P_5, C, \vec{J}^2, J_3\)\} corresponding to the eigenvalues \(E_n^\Delta, \hat{q}, c, j(j+1)\) and \(m_j\). On the other hand, the superalgebra
\[
\{\sigma_R, \sigma_L + 1\} = 0, \quad (\sigma_R)^2 = C_1 - \vec{L}^2 - \sigma_L,
\]
allows us to demonstrate that the first Pauli spinors of the eigenspinors of the simple central modes (A.1) have the remarkable property
\[
\sigma_R u_{n,\hat{q},j,m}^{(1)\pm} = \left[n^2 - \left(j + \frac{1}{2}\right)^2\right]^{1/2} u_{n,\hat{q},j,m}^{(1)\mp}.
\]
Hereby it results that the Dirac eigenspinors of the natural central modes can be expressed as
\[
U_{n,\hat{q},c=\pm n,j,m_j} = \frac{1}{\sqrt{2n}} \left[\pm \sqrt{n \pm \left(j + \frac{1}{2}\right)U_{n,\hat{q},j,m}^{\pm}} + \sqrt{n \mp \left(j + \frac{1}{2}\right)U_{n,\hat{q},j,m}^{\mp}}\right].
\]

In the same way the eigenspinors of the natural axial modes can be expressed as linear combinations of the spinors of the simple axial modes presented in Appendix. As in previous case the calculations reduce to the eigenspinors of the set of Pauli operators \{\(\Delta, P_5, C, R_3', J_3\)\} associated to
that of the Dirac operators of the natural axial modes. Using the identity 
\( \{ C, \sigma_3 \} = 2(R'_3 + J_3) \) it is not difficult to show that the eigenspinors of the 
natural axial modes are

\[
U_{n, \hat{q},c=\pm n,m_r,m_j} = \frac{1}{\sqrt{2n}} \left[ \pm \sqrt{n \pm |m_r + m_j|} \right] \left[ \pm \sqrt{n \pm |m_r + m_j|} U_{n, \hat{q},m_r,m_j}^{\pm} \right].
\]

(44)

Thus we see that in the Dirac theory the algebraic method offers us 
the mechanisms of constructing new quantum modes having no separated 
variables. We can say that this method and that of separation of variables 
complete each other, helping us to find many types of different quantum 
modes related among themselves.

7 Concluding remarks

The aim of this paper was to complete the previous studies of the Dirac 
equation in Taub-NUT space [9, 10]. In the case of the Dirac equation the 
existence of the Killing-Yano tensors in the Taub-NUT geometry allowed us 
to construct some Dirac-type tensors the anticommutators of which generate 
operators that commute with the standard Dirac one. We are attempting 
to find non-trivial operators of the Dirac theory connected with the hidden 
symmetry of the Taub-NUT space. These operators are constructed in Sec.4 
and they represent the quantum analogue of the classical Runge-Lenz vector 
from the Kepler problem. Thus we get an example of gauge-invariant theory 
of Dirac fermions in a geometry with high manifest or hidden symmetries 
for which we can write all the corresponding conserved operators. The main 
consequence is that we can explicitly use the dynamical algebra for defining 
new quantum modes.

In the literature there is a detailed description of the classical geodesic motion for scalar particles and a quantum treatment through the Schrödinger 
equation in Taub-NUT space [1, 2, 12]. On the other hand, the pseudo-classical limit of the Dirac theory of a spin\(-\frac{1}{2}\) fermion in curved spacetime is described by supersymmetric extension of the ordinary relativistic point particle [20]. In the pseudo-classical models the spin degrees of freedom are characterized in terms of anticommuting Grassmann variables. The constants of motion related to the symmetries of the manifold generally contain
spin-dependent parts [21]. It is interesting that these specific spin contributions are similar to the spin terms of the conserved operators of the quantum theory. A notable example is the spin contribution to the Runge-Lenz vector which re-confirms the result from pseudo-classical approach [4]. Moreover, since both the pseudo-classical and the quantum theory are completely solvable, we have the opportunity to compare their physical meaning in all details.

Appendix A: The simple discrete modes

The energy eigenspinors of the simple central modes have the form [9]

\[ U_{n,\hat{q},\hat{j},m_j}^{\pm} = \left( \begin{array}{c} u_{n,\hat{q},\hat{j},m_j}^{(1)\pm} \\ u_{n,\hat{q},\hat{j},m_j}^{(2)\pm} \end{array} \right) \sim \left( \begin{array}{c} h_{n,q,j}^{\pm}\Psi_{q,j,m_j}^{\pm} + f_{n,q,j}^{\pm}\Psi_{q,j,m_j}^{\mp} \\ g_{n,q,j}^{\pm}\Psi_{q,j,m_j}^{\pm} \end{array} \right), \quad (A.1) \]

where the radial functions \( f^{\pm}, g^{\pm} \) and \( h^{\pm} \) depend only on \( r \) while \( \Psi^{\pm} \) are the two-component spherical spinors that solve the common eigenvalue problems of the Pauli operators \( Q = -\mu P_5, \vec{J}^2, J_3, \sigma_L + 1 \) for the eigenvalues \( q = -\mu \hat{q}, j(j + 1), m_j, \) and \( \pm(j + \frac{1}{2}) \), respectively [9]. We observe that only the first Pauli spinor has separated variables. Its radial function, \( f \), is a solution of the radial Klein-Gordon equation while the other two radial functions arise from (36). The quantum numbers of these modes must satisfy the selection rule \( |q| < j + \frac{1}{2} < n \). Thus we find that \( u_{n,\hat{q},\hat{j},m_j}^{(1)\pm} \) is the common eigenspinor of the set \( \{ \Delta, P_5, \vec{J}^2, J_3, \sigma_L + 1 \} \) corresponding to the eigenvalues \( E_n^2, \hat{q}, j(j + 1), m_j, \) and \( \pm(j + \frac{1}{2}) \) respectively. This means that (A.1) is the common eigenspinor of the operators \( \{ H, P_5, \vec{J}^2, J_3, Q(\sigma_L + 1) \} \) whose eigenvalues are \( E_n, \hat{q}, j(j + 1), m_j, \) and \( \pm E_n(j + \frac{1}{2}) \).

The simple axial modes can be constructed in a similar way. We start with scalar common eigenfunctions \( f_{n,\hat{q},\hat{m},m} \) of the commuting operators \( \Delta, P_5, R_3 \) and \( L_3 \), corresponding to the eigenvalues \( E_n^2, \hat{q}, \hat{m} \) and \( m \). These eigenfunction can be easily calculated in parabolic coordinates as the axial solutions of the Schrödinger equation [12], with the unique difference that the quantity \( 2E \) from the Schrödinger case must be replaced here by \( E^2 \). Furthermore, we construct the eigenspinors of the simple axial modes, \( U_{n,\hat{q},\hat{m},r,m_j}^{\pm} \), by choosing their first Pauli spinors as

\[ u_{n,\hat{q},\hat{m},r,m_j}^{(1)\pm} \sim f_{n,\hat{q},\hat{m},m}^{\pm}\zeta_{\sigma=\pm1/2} \quad (A.2) \]
and calculating from (36) the second Pauli spinors that have no more separated variables. It is clear that (A.2) is the common eigenspinor of the set of Pauli operators \{\Delta, P_5, R_3, J_3, \sigma_3\} associated with the Dirac ones from the set defining the simple axial modes. The eigenvalues of these operators are 

\[ E_n, \hat{q}, m_r = \tilde{m} + \sigma, m_j = m + \sigma \]

and \( \sigma \). Consequently, \( U_{n,\hat{q},m_r,m_j}^\pm \) are the common eigenspinors of the set \( \{H, P_3, R_3, J_3, Q_3\} \) corresponding to the eigenvalues \( E_n, \hat{q}, m_r, m_j \) and \( E_n \sigma \).

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