On the derivation of guiding center dynamics without coordinate dependence

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The fundamental of the classical guiding center theory is gyro-phase averaging, which cannot be well defined over a non-trivial magnetic field topology. The local gyro-phase coordinate frame hides the geometric nature of gyro-symmetry. A coordinate-free geometric representation should be a more appropriate alternative for a deeper understanding of the guiding center dynamics. In this paper, the motion of a charged particle is described by a Lagrangian one-form on a seven-dimensional phase space. The Lagrangian one-form is geometrically decomposed by constructing a coordinate-free gyro-averaging method. As a result, we obtain the coordinate-free expression of the non-relativistic guiding-center dynamics in the time-dependent slow-varying electromagnetic field.

I. INTRODUCTION

The guiding center dynamics has been a subject of interest to researchers for decades [1]. The purpose of guiding center theory is to decompose particle motion into vertical gyro-motion and horizontal drift motion. The classical decomposition method is based on the averaging of the gyro-phase. The gyro-phase is defined on a predefined local orthogonal coordinate frame, which may not exist globally over a non-trivial magnetic field. This issue was raised and discussed by Sugiyama [2, 3] and Krommes [4] in 2009. Soon after, Burby and Qin [5] recognized that the magnetic field inhomogeneity obstructs the existence of the global gyro-phase. The global gyro-phase is not a necessary condition for gyro-symmetry. The gyro-symmetry depends only on the homogeneity of the electromagnetic field within the range of gyro-motion. The local gyro-phase coordinate frame hides the geometric nature of gyro-symmetry and prevents us from identifying this issue. A coordinate-free geometric representation should be a more appropriate alternative for a deeper understanding of the guiding center dynamics. A series of recent works by Burby on slow manifolds of near-periodic Hamiltonian systems has shown the importance of geometric tools for understanding gyro-symmetry [6–8].

The base of guiding-center theory is the symmetry of trajectories of the charged particles in electromagnetic fields, called gyro-symmetry. Kruskal [9] pointed out that the set of charged particles with a common guiding center constitutes a topological ring in phase space, called Kruskal’s ring [10]. In a slowly varying electromagnetic field, the charged particles on the same Kruskal’s ring have similar phase space trajectories. As long as the particles do not resonate with the field, Kruskal’s ring will not break, but will only be slightly deformed. In other words, the guiding center is the center of Kruskal’s ring. The decomposition of gyro-motion is to decompose the Kruskal’s ring from the phase space of particle trajectories.

This paper aims to construct coordinate-free guiding-center dynamics, or rather, Kruskal’s ring dynamics. The description of Kruskal’s ring relies on two vector fields, the roto-rate vector and the gyro-radius. The roto-rate vector is the generator of gyro-symmetry, named by Kruskal [9]. Omohundro [11] showed the coordinate-free expression of the roto-rate vector. The gyro-radius is the vector from the guiding center to the ringmates, representing the mapping between Kruskal’s ring and the guiding center. In an inhomogeneous electromagnetic field, gyro-symmetry is not absolute. The gyro-decomposition is an asymptotic approximation to the exact particle motion. The form of gyro-radius is not unique—different forms of gyro-radius correspond to different decompositions [12]. We will show that a proper definition of the roto-rate vector and the gyro-radius may yield a concise expression of the guiding center dynamics.

In this paper, we use the Lagrangian one-form (or called Poincaré-Cartan one-form) to represent the motion of charge particle in a time-dependent slow-varying electromagnetic field. The extended phase space of particle is seven-dimensional contact manifold. The Lagrangian formalism provides simple and explicit expressions for the variational principle and Noether’s theorem [13]. From the geometric point of view, the Lagrangian one-form is the dual counterpart of the trajectory. The decomposition of the trajectory can be achieved by decomposing the Lagrangian one-form. We will show that the non-existence of global gyro-phase is a natural conclusion of the Lagrangian one-form decomposition. As the result go geometric descomposition, we obtain the coordinate-free expression of the non-relativistic guiding-center dynamics in the time-dependent slow-varying electromagnetic field.

The derivation in this paper uses knowledge of elementary differential geometry and Lie groups. For interested readers, Marsden and Ratiu’s book [14] would be a good reference.

This paper is organized as follows. In Section II, we recall the general geometric setting of the Lagrangian formalism for a time-dependent system and discuss the relation between Poincaré-Cartan integral invariant and Noether’s
II. GEOMETRIC SETTING

Considering a time-dependent Hamiltonian system with the extended phase space \( P \), the action integral is an line integral along phase space trajectory \( \lambda \),

\[
\mathcal{A}[\lambda] = \int_\lambda \eta ,
\]

where one-form \( \eta = pdq - Hdt \) is called the Lagrangian one-form. From Hamiltonian principle, the variation of trajectory \( \lambda \) gives the Hamiltonian equations

\[
\iota_\tau d\eta = 0 ,
\]

where \( \iota_\tau \) means interior product with vector field \( \tau \), and \( d\eta \) is the exterior derivative of one-form \( \eta \). The Hamiltonian flow is an one-parameter group of \( t \) whose infinitesimal generator is \( \tau \),

\[
\Psi_H^t = \exp (t\tau).
\]

If we add a closed one-form to the Lagrangian one-form \( \eta' = \eta + \alpha \) and \( d\alpha = 0 \), the result of Hamiltonian equations Eq.(2) does not change. The extended phase space \( P \) is a 2n+1 dimensional manifold endowed with a one-form \( \eta \) that satisfies the nonintegrable condition \( \eta \wedge (d\eta)^n \neq 0 \). This structure \( (P, \eta) \) is a contact structure, and \( \eta \) is also called contact form. The Lagrangian one-form (contact one-form) \( \eta \) plays an important role in time-dependent mechanics, which provides simple and explicit expressions for the variational principles and Noether’s theorem [13, 14].

A. Poincaré-Cartan integral invariant

Consider a curve \( O \) encircles a tube of phase trajectories in extended phase space \( P \), the action integral on \( O \) is denoted as \( \mu_P[O] \equiv \oint_O \eta \). Let \( O \) move along the same tube of phase trajectories, the action integral on the image of Hamiltonian flow \( \Psi_H^t \) looks like

\[
\mu_P[\Psi_H^t \circ O] = \oint_{\Psi_H^t \circ O} \eta = \oint_O \Psi_H^t \ast \eta
\]

\[
= \oint_O (\eta + tL_\tau \eta + \cdots)
\]

\[
= \mu_P[O] + t \oint_O (\iota_\tau d\eta + di_\tau \eta) + \cdots.
\]

Because \( \iota_\tau d\eta = 0 \) and the circle integral of closed form \( di_\tau \eta \) is zero, the second and higher order terms of Eq.(3) will vanish. Then, we can say that the action integral on a closed phase space curve, \( \mu_P[O] \), is a constant of motion. The one-form \( \eta \) is also called the of Poincaré’s relative integral invariant or Poincaré-Cartan one-form [15]. The action integral \( \mu_P[O] \) is independent with the shape of \( O \), which captures the topological property of the bundle of trajectories. The Hamiltonian flow preserves the action integral over arbitrarily closed loop in phase space. However, without additional constraints, the Hamiltonian flow would not preserve the compactness of the loop \( O \).

B. Noether’s theorem

Consider an one-parameter Lie group \( \Phi_{\theta_s} = \exp (s\partial_\theta), s \in \mathbb{R} \), which is generated by a vector field \( \partial_\theta \). The action integral of the infinitesimally transformed trajectory is

\[
\mathcal{A}[\Phi_{\theta_s} \circ \lambda] = \int_{\Phi_{\theta_s} \circ \lambda} \eta = \int_\lambda \Phi_{\theta_s}^* (\eta) = \int_\lambda \exp (sL_{\partial_\theta}) \eta = \mathcal{A}[\lambda] + s \int_\lambda L_{\partial_\theta} \eta + O(s^2) ,
\]
where $\mathcal{L}_{\partial_0}$ is the Lie derivative along the vector $\partial_0$. If $\mathcal{L}_{\partial_0}\eta = 0$, the higher order term of $s$ will vanish and action integral $\mathcal{A}[\lambda]$ is preserved under transformation of the group $\Phi_\theta$. We shall say $\Phi_\theta$ is a Noether symmetry on the Hamiltonian system $(P, \eta)$. Using the Cartan’s Magic Formula

$$0 = \mathcal{L}_{\partial_0} \eta = \iota_{\partial_0} d\eta + d\iota_{\partial_0} \eta,$$  \hfill (5)

we get

$$\iota_{\partial_0} d\eta = -d\iota_{\partial_0} \eta = -d\mu,$$ \hfill (6)

where

$$\mu \equiv \iota_{\partial_0} \eta,$$ \hfill (7)

is the moment map induced by $\partial_0$ \cite{14}. Putting $\iota_\tau$ on Eq. (5), yields

$$0 = \iota_{\tau} \iota_{\partial_0} d\eta = -\iota_{\tau} d\iota_{\partial_0} \eta = -\iota_{\tau} d\mu \hfill (8)$$

It is easy to verify that $\mu$ is also an invariant under the action of $\Phi_\theta$

$$\mathcal{L}_{\partial_0} \mu = 0, \quad \Phi_\theta^\ast (\mu) = \mu.$$ \hfill (9)

Putting $\mathcal{L}_\tau$ on Eq. (5), yields

$$\mathcal{L}_\tau \mathcal{L}_{\partial_0} \eta = \mathcal{L}_\tau (\iota_{\partial_0} d\eta + d\iota_{\partial_0} \eta) = \iota_{[\tau, \partial_0]} d\eta - i_{\partial_0} \mathcal{L}_\tau d\eta + d\mathcal{L}_\tau (\iota_{\partial_0} \eta) = 0,$$ \hfill (10)

where $[\cdot, \cdot]$ denotes the commutator of vector fields. If $d\eta$ is not degenerate, the symmetry vector $\partial_0$ should commutes with Hamiltonian vector $[\tau, \partial_0] = 0$.

Noether’s theorem requires the Lie derivative of Poincaré-Cartan one-form $\eta$ along the symmetry vector $\partial_0$ vanish, which is only a local constraint on the one-form $\eta$. The Noether’s theorem Eq.(8) can not tell us the global topology of the symmetry. For the same Lie algebra $\partial_0$, the orbit $O$ of Lie group $\Phi_\theta$ may be isomorphic to $S^1$ or $\mathbb{R}$. The global topology of the Lie group $\Phi_\theta$ depends on the nature of the extended phase space $(P, \eta)$. If the symmetry group $\Phi_\theta$, is a compact one-parameter group, it is isomorphic to the group $S^1 \cong U(1)$, whose orbit $O$ is a closed curve \cite{10}. The action integral on the closed orbit $O$ is

$$\mu_P [O] = \oint_O \eta = \int_0^1 \Phi_\theta^\ast (\iota_{\partial_0} \eta) \, ds = \mu.$$ \hfill (11)

which is consistent with the moment map $\mu$.

Noether’s theorem determines the invariance of the moment map $\mu$. In many cases, there is no guarantee that the exact symmetry always exists everywhere in the phase space, and $\mu$ will not be a globally valid exact invariant. If the inhomogeneity of the phase space is a small quantity in the range of the closed loop $O$, then the variation of $\mu$ along the particle trajectory is bounded, and $\mu$ is called adiabatic invariant \cite{17}. Liouville’s theorem determines the invariance of action integral $\mu_P$ on the closed loop $O$, which is an absolute invariant. Kruskal pointed out that points in the phase space of a near-periodic dynamical system form closed loops that drift along phase space trajectories, preserving their topology with only slight deformations \cite{9}. The phase space inhomogeneity leads to the deformation of $O$, and the deviation between $\mu$ and $\mu_P$. Since the deviation is bounded, we use $\mu$ as the asymptotic approximation to $\mu_P$, which can preserve the invariance to arbitrary orders \cite{7, 3}. We call these loops Kruskal’s ring or invariant tori in Arnold’s book \cite{17}. The existence of Kruskal’s ring implies that the divergent Hamiltonian flow is constrained by a local compact group $\Phi_\theta$. Our aim is to decompose Kruskal’s ring $O$ from the phase space $P$ to obtain a quotient manifold $P/O$.

III. KRUSKAL’S RING AND GUIDING CENTER

A. Kruskal’s ring

Consider the phase space $(P, \eta)$ and a local compact Lie group $\Phi_\theta$ called gyro-transformation or gyro-symmetry. If the Lagrangian one-form $\eta$ is invariant to the action of $\Phi_\theta$ throughout the phase space $P$, we say the phase space $P$
is uniform to the gyro-transformation $\Phi_\theta$. The orbit of $\Phi_\theta$ is called Kruskal’s ring $O$, and the points on the same orbit are called ringmates, $O_z = \{ \Phi_\theta(z) \}$. The gyro-symmetry $\Phi_\theta$ induce a moment map $\mu : P \to \mathbb{R}$, which projects Kruskal’s ring to a constant of motion along the particle trajectory. If the gyro-transformation has a fixed point $\Phi_\theta(Z) = Z$, we call it the guiding center and use it as the representative of Kruskal’s ring $O$.

The trajectories of Kruskal’s ring constitute the quotient manifold $P/O$. The guiding center trajectory is isomorphic to the Kruskal’s ring trajectory $\bar{P} \simeq P/O$. Then, a natural projection arises

$$
\Pi_{-\rho} : P \to P/O \times O \to \bar{P} \times \bar{O},
$$

where the minus sign indicates the transformation is in the opposite direction of the gyro-radius, and the superscript 'bar' indicates that it is defined at the guiding center. Let inverse projection $\Pi_\rho$ be a one-parameter transformation generated by gyro-radius $\Pi_{r,\rho} = \exp(r \partial_\rho)$, the pullback from ringmate to guiding center is a formal power series of the Lie derivative $L_{\partial_\rho}$

$$
\Pi_\rho^* \alpha = \sum_{n \geq 0} \frac{r^n}{n!} L^n_{\partial_\rho} \alpha|_Z ,
$$

where $\alpha|_Z$ is the quantity defined at the guiding center.

Pushing forward $\partial_\theta$ to the guiding center, yields the rote-rate vector

$$
\partial_\theta \equiv \Pi_{-\rho} \partial_\theta = \Pi_\rho^* \partial_\theta ,
$$

which is named by Kruskal [9]. The orbit of rote-rate vector $\partial_\theta$ is a circle in the velocity space

$$
\Phi_{\bar{\theta}} = \exp(2\pi s \partial_\theta) \leftrightarrow \bar{O} \subset T_Z M ,
$$

called limiting ring. Therefore, we say the guiding center is a particle with "spin", whose magnetic moment $\mu$ is equal to the action integral over the limiting ring $\bar{O}$.

### B. Decomposition

To decompose the motion of Kruskal’s ring, we split the Hamiltonian vector $\tau$ into two parts, the horizontal part $\tau_\parallel$ and the vertical part $\partial_\theta$,

$$
\tau = \tau_\parallel + \partial_\theta .
$$

Substituting $\tau$ to Eq.(2), yields the equation of horizontal motion

$$
\iota_{\tau_\parallel} d\eta = -\iota_{\partial_\theta} d\eta = d\mu .
$$

Pushing forward $\tau_\parallel$ to the guiding center, we get the Hamiltonian vector field of guiding center

$$
\bar{\tau}_\parallel = \Pi_{-\rho}^* \tau_\parallel = \Pi_\rho^* \tau_\parallel .
$$

It is straightforward to verify that the horizontal motion is commute with vertical motions

$$
[\partial_\theta, \tau] = [\partial_\theta, \tau_\parallel] = 0 ,
$$

and $\mu$ is a constant of motion in both directions

$$
\iota_{\tau_\parallel} d\mu = \iota_{\partial_\theta} d\mu = 0 .
$$

We also split the Lagrangian one-form into two parts

$$
\eta = \eta_\parallel + \eta_\perp ,
$$

where the horizontal part is orthogonal to $\partial_\theta$

$$
\iota_{\partial_\theta} \eta_\parallel = 0 ,
$$

$$
\iota_{\partial_\theta} \eta_\perp = 0 .
$$
and the vertical part gives the moment map $\mu$

$$\iota_\theta \eta_\perp = \mu.$$  \hfill (23)

From Eq. (17), the horizontal vector field is given by

$$\star \tau_\parallel = \frac{1}{2! \vol_\theta} d\mu \wedge \eta_\perp \wedge d\eta \wedge d\eta,$$  \hfill (24)

where

$$\star \vol_\theta = \frac{1}{3!} \eta_\perp \wedge d\eta \wedge d\eta \wedge d\eta,$$  \hfill (25)

is the phase space volume form introduced by $\eta_\perp$. Here, the Hodge operator $\star$ maps the $p$-form to $(n-p)$-form, and the superscript $\flat$ means lowering of indices [18]. Substituting Eq. (24) into Eq. (17), one can verify that

$$\iota_\tau_\parallel d\eta = \frac{\vol_\theta}{\vol_\mu} \eta_\perp,$$  \hfill (26)

where

$$\vol_\mu = \star \left( \frac{1}{3!} d\mu \wedge d\eta \wedge d\eta \wedge d\eta \right) = \iota_\tau d\mu = 0,$$  \hfill (27)

is the degenerate phase space volume form introduced by $d\mu$.

### C. Perturbation

The guiding center has practical significance only if the trajectories of the ringmates are similar. In other words, the deformation of Kruskal's ring should be limited. In the uniform phase space, Kruskal's ring $\mathcal{O}$ is a circle. The deformation of Kruskal's ring came from the inhomogeneity of phase space. Let a small quantity $\varepsilon \sim L \partial \bar{\rho}$ denote the phase space inhomogeneity with respect to the gyro-radius $\partial \bar{\rho}$. In a slowly varying system, $\varepsilon \ll 1$, the deformation is a near-identity transformation generated by a perturbation vector field $G$,

$$\Psi_{\varepsilon G} : \mathcal{O} \to \mathcal{O}_{\varepsilon}, \quad \Psi_{\varepsilon G} = \exp (\varepsilon G).$$  \hfill (28)

Further, we also have

$$\mathcal{O}_{\varepsilon} = \Psi_{\varepsilon G} \mathcal{O} = \Psi_{\varepsilon G} \circ \Pi_\rho \circ \mathcal{O} = \Pi_{\rho_{\varepsilon}} \circ \mathcal{O},$$  \hfill (29)

where $\Pi_{\rho_{\varepsilon}} \equiv \Psi_{\varepsilon G} \circ \Pi_\rho$ is the perturbed guiding-center projection. The perturbed gyro-transformation $\Phi_{\varepsilon \theta}$ is given by

$$\Phi_{\varepsilon \theta} \equiv \Pi_{\varepsilon \rho} \circ \Phi_\theta \circ \Pi_{-\varepsilon \rho} = \Psi_{\varepsilon G} \circ \Pi_\rho \circ \Phi_\theta \circ \Pi_{-\rho} \circ \Psi_{-\varepsilon G} = \Ad_{\varepsilon G} \circ \Ad_\rho \circ \Phi_\theta.$$  \hfill (30)

Then, one can verify that the perturbed gyro-transformation $\Phi_{\varepsilon \theta}$ preserve the Noether’s theorem

$$\mathcal{A} [\Phi_{\varepsilon \theta} \circ \lambda_\varepsilon] = \int_{\Phi_{\varepsilon \theta} \circ \lambda_\varepsilon} \eta_\varepsilon = \int_{\lambda_\varepsilon} \Phi_{\varepsilon \theta}^* (\eta_\varepsilon)$$

$$= \int_{\lambda_\varepsilon} \Pi_{-\varepsilon \rho_{\varepsilon}}^* \circ \Phi_{\theta}^* \circ \Pi_{\rho_{\varepsilon}}^* (\eta_\varepsilon)$$

$$= \int_{\lambda_\varepsilon} \Pi_{-\varepsilon \rho_{\varepsilon}} \circ \exp (s L_{\partial_\theta}) \circ \Pi_{\rho_{\varepsilon}}^* (\eta_\varepsilon)$$

$$= \mathcal{A}[\lambda_\varepsilon] + s \int_{\Pi_{-\varepsilon \rho_{\varepsilon}} \circ \lambda_\varepsilon} L_{\partial_\theta} \Pi_{\rho_{\varepsilon}}^* (\eta_\varepsilon) + \cdots$$

$$= \mathcal{A}[\lambda_\varepsilon] + s \int L_{\partial_\theta} \eta + \cdots,$$  \hfill (31)
where the second and higher order terms will vanish at $L_{\partial \psi} \bar{\eta} = 0$. Pullback $\bar{\mu}$ to the perturbed trajectory, the perturbed moment map $\mu_\epsilon = \iota_{\partial \psi} \eta_\epsilon$ is a constant of motion along the perturbed trajectory

$$0 = \Pi^*_{\mu_\epsilon} \left( \iota_T d\bar{\mu} \right) = \iota\Pi^*_{\mu_\epsilon} d\Pi^*_{\mu_\epsilon} \bar{\mu} = \iota_T d\mu_\epsilon ,$$

(32)

Following the Lie perturbation method developed by Littlejohn[19, 20] and Cary [21, 22]. The perturbed moment map $\mu_\epsilon$ may be calculated to arbitrary order.

The stable Kruskal’s ring implies that ringmates’ trajectories have symmetry $\Phi_\theta \leftrightarrow \mathcal{O}$. The deformed Kruskal’s ring correspond to the perturbed gyro-symmetry $\Phi_{\theta \epsilon} \leftrightarrow \mathcal{O}_\epsilon$. The action integral along the Kruskal’s ring $\mu_P [\mathcal{O}_\epsilon]$ is an exact invariant. As long as the symmetry group $\Phi_\theta$ is compact, the exact invariant $\mu_P$ always exist. Therefore, we say the perturbed moment map $\mu_\epsilon$ is an asymptotical approximation of $\mu_P$.

### D. Gyro-averaging

The inhomogeneities in phase space break the gyro-symmetry. The purpose of gyro-averaging is to eliminate the perturbation and to obtain the gyro-invariant unperturbed Lagrangian. The gyro-averaging is an integral along the Kruskal’s ring

$$\langle \alpha \rangle = \int_0^1 \Phi^*_\bar{\theta}_s (\alpha) \, ds .$$

(33)

Pulling it back to the limiting ring, yields

$$\langle \alpha \rangle = \int_0^1 \Phi^*_\bar{\theta}_s \circ \Pi^*_\mu (\alpha) \, ds = \int_0^1 \sum_{n \geq 0} \frac{\varepsilon^n}{n!} L^\mu_{\Phi^*_\bar{\theta}_s, \partial \theta} \Phi^*_\bar{\theta}_s (\alpha) \, ds .$$

(34)

If $\alpha|_\mathcal{O}$ is gyro-independent, the gyro-average $\langle \alpha \rangle$ only dependent on the rotation of gyro-radius $\Phi^*_\bar{\theta}_s \partial \theta$, which is a formal power series of $L_{\partial \theta}$

$$\Phi^*_\bar{\theta}_s \partial \theta = \exp \left( 2\pi s L_{\partial \theta} \right) (\partial \theta) = \sum_{n \geq 0} \frac{(2\pi s)^n}{n!} L^\mu_{\partial \theta} (\partial \theta) .$$

(35)

Note that the form of gyro-radius $\partial \theta$ and rote-rate vector $\partial \theta$ are not unique, as long as the mapping from the limiting ring $\mathcal{O}$ to the Kruskal’s ring $\mathcal{O}$ holds,

$$\Pi_{\theta} \mathcal{O} \leftrightarrow \mathcal{O} \leftrightarrow \partial \theta = \Pi_{\theta^*} \partial \theta .$$

(36)

To simplify the gyro-averaging, we let gyro-radius $\partial \theta$ be ‘complex-like’ under the action of rote-rate vector $\partial \theta$

$$L^2_{\partial \theta} \partial \theta = - \partial \theta .$$

(37)

Then, the rotation of gyro-radius $\Phi^*_\bar{\theta}_s \partial \theta$ is a simple trigonometric polynomial

$$\Phi^*_\bar{\theta}_s \partial \theta = \partial \theta + (2\pi s) \partial \theta - \frac{1}{2} (2\pi s)^2 \partial \theta - \frac{1}{3!} (2\pi s)^3 \partial \theta + \cdots$$

$$= \sum_{n \geq 0} \frac{(-1)^n (2\pi s)^{2n}}{2n!} \partial \theta + \sum_{n \geq 0} \frac{(-1)^n (2\pi s)^{2n+1}}{(2n+1)!} \partial \theta$$

$$= \cos (2\pi s) \partial \theta + \sin (2\pi s) \partial \theta \quad (38)$$

where $\partial \theta \equiv L_{\partial \theta} \partial \theta$ is the dual vector orthogonal to $\partial \theta$. And, the gyro-average $\langle \alpha \rangle$ is a polynomials in $L_{\partial \theta}$ and $L_{\partial \theta}$

$$\langle \alpha \rangle = \sum_{n \geq 0} \frac{1}{n!} \int_0^1 ds \left( \cos (2\pi s) L_{\partial \theta} + \sin (2\pi s) L_{\partial \theta} \right)^n \alpha|_\mathcal{O} .$$

(39)
IV. CHARGED PARTICLE MOTION IN A SLOW-VARYING ELECTROMAGNETIC FIELD

A. Poincaré-Cartan-Einstein one-form

The motion of charged particle was considered in the four-dimensional Minkowski space $M = E^{1,3}$ with global Cartesian coordinates $x^\alpha \equiv (x^0, x^i) = (ct, \mathbf{x})$ and the metric tensors takes the form $g = (-1, 1, 1, 1)$. The phase space $P$ is the cotangent bundle $T^* M$ of $M$ with constraint condition,

$$P \equiv \{ (x, p) \mid x \in M, p \in T^*_x M, g^{-1}(p, p) = -m^2 c^2 \},$$

where $p \equiv (p^0, \mathbf{p})$ is the four-momentum. For a charged particle in an electromagnetic field the Lagrangian one-form (also known as the Poincaré-Cartan-Einstein one-form [23]) is given by

$$\eta = A + \mathbf{p} = \mathbf{A} + \mathbf{p} \cdot d\mathbf{x} - (\phi + p^0) \, dt,$$

where $A^\alpha \equiv (\phi, \mathbf{A})$ is the four-potential. The natural unit system is adopted, let $m = c = e = 1$, and only consider the non-relativistic case,

$$p^0 = \frac{v^2}{2}, \quad \mathbf{p} = v \cdot \mathbf{u}.$$  \hfill (42)

The Hamiltonian vector $\tau$ is solved from Hamiltonian equation Eq.(2),

$$\begin{align*}
\frac{\partial \tau}{\partial x^i} &= \frac{\partial}{\partial \mathbf{p}} = v, \\
\frac{\partial \tau}{\partial t} &= \mathbf{v} \times \mathbf{B} + \mathbf{E},
\end{align*}$$

where the electromagnetic fields are expressed in terms of the potentials as $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla \phi - \partial_t \mathbf{A}$.

Define two auxiliary vector fields, one is the unit vector along the direction of the magnetic field

$$\mathbf{b} \equiv \frac{\mathbf{B}}{|\mathbf{B}|},$$

and the other is the $\mathbf{E} \times \mathbf{B}$ drift velocity

$$\mathbf{D} \equiv \frac{\mathbf{E} \times \mathbf{B}}{B^2}.$$  \hfill (45)

The velocity $\mathbf{v}$ is decomposed into horizontal part $\mathbf{v}_\parallel$ and vertical part $\mathbf{v}_\perp$

$$\begin{align*}
\mathbf{v}_\parallel &\equiv \mathbf{v} \cdot \mathbf{b} \mathbf{b} + \mathbf{D}, \\
\mathbf{v}_\perp &\equiv \mathbf{v} - \mathbf{v} \cdot \mathbf{b} \mathbf{b} - \mathbf{D}.
\end{align*}$$

If $\mathbf{D} \neq 0$, $\mathbf{v}_\perp$ and $\mathbf{v}_\parallel$ are not orthogonal

$$\mathbf{v}_\parallel^2 + \mathbf{v}_\perp^2 = \mathbf{v}^2 - 2\mathbf{v}_\perp \cdot \mathbf{D}.$$  \hfill (48)

The decomposition of the Hamiltonian vector $\tau = \tau_\parallel + \tau_\perp$ looks like

$$\begin{align*}
\tau_\parallel &= \mathbf{v}_\parallel \cdot \partial_x + \mathbf{E} \cdot \mathbf{b} \mathbf{b} \cdot \partial_v + \partial_t, \\
\tau_\perp &= \mathbf{v}_\perp \cdot \partial_x + \mathbf{v}_\perp \times \mathbf{B} \cdot \partial_v.
\end{align*}$$

The gyro-radius in configuration space is a three-dimensional vector field

$$\rho \equiv \frac{\mathbf{b} \times \mathbf{v}_\perp}{|\mathbf{B}|}.$$  \hfill (51)

Using $\rho$, we decompose the the four-momentum $p$ into vertical part

$$\begin{align*}
p_\perp &\equiv -i \rho_\partial_x dA = \rho \times \mathbf{B} \cdot d\mathbf{x} - \mathbf{E} \cdot d\mathbf{t} \\
&= (\mathbf{v} - \mathbf{v} \cdot \mathbf{b} \mathbf{b} - \mathbf{D}) \cdot d\mathbf{x} - (\mathbf{v} - \mathbf{D}) \cdot d\mathbf{t} \\
&= \mathbf{v}_\perp \cdot (d\mathbf{x} - D dt),
\end{align*}$$

\hfill (52)
and horizontal part
\[ p_{\parallel} = p - p_{\perp} \]
\[ = (v \cdot \mathbf{b} + D) \cdot dx - \frac{(v - D)^2 + D^2}{2} dt \]
\[ = v_{\parallel} \cdot dX - \frac{v_{\parallel}^2 + v_{\perp}^2}{2} dt \]
\[ = v_{\parallel} \cdot dX - \frac{v_{\parallel}^2}{2} dt - \mu |B| dt . \]

Appending a closed form \(-d (\rho \cdot \mathbf{A})\) to the Poincaré-Cartan-Einstein one-form Eq. (41), yields
\[ \eta = A + p_{\parallel} = \mathcal{L}_{\rho} \partial_x A , \] (54)
which will simplify our subsequent derivation.

**B. Gyro-transformation**

First consider the case that the electromagnetic field is homogeneous, \( \varepsilon = 0 \). The gyro-transformation is a rotation of the gyro-radius \( \rho \)
\[ \Phi_{\theta}(z) = (t, x + \rho (\cos \theta - 1) + \rho \times \mathbf{b} \sin \theta, v + |B| \rho \times \mathbf{b} (\cos \theta - 1) - |B| \rho \sin \theta) , \] (55)
whose generator is obtained from vertical Hamiltonian vector field \( \tau_{\perp} \) (Eq. (50))
\[ \partial_\theta = \frac{d}{ds} \Phi_{\theta}|_{s=0} = \rho \times \mathbf{b} \cdot \partial_x - |B| \rho \cdot \partial_x = \tau_{\perp} / |B| . \] (56)
The action integral along the Kruskal’s ring \( O \) is
\[ \mu_{P}[O] = \frac{1}{2\pi} \int_{O} (A + p) \]
\[ = -\frac{1}{2\pi} \int_{D_{O}} \mathbf{B} \cdot dS + \frac{1}{2\pi} \int_{0}^{2\pi} \Phi_{\theta}^* (\partial_x p) d\theta \]
\[ = -\frac{|B| \rho^2}{2} + \frac{v_{\perp}^2}{|B|} = \frac{|B| \rho^2}{2} = \frac{v_{\perp}^2}{2 |B|} , \] (57)
where \( D_{O} \) is the area enclosed by \( O \), and the Stokes’ theorem is applied. The Eq. (57) is the original definition of the magnetic moment [24].

The calculation of the moment map \( \mu = \iota_{\partial_\theta} \eta \) is tricky because the potential \( A(t, x) \) of uniform electromagnetic field are not constant. The traditional solution is to expand \( \eta \) around the guiding center \( Z = (t, X, V) \) with respect the gyro-radius \( \rho \) [9]. Expanding Eq. (54) with respect \( \rho \cdot \partial_x \), yields
\[ \bar{\eta} = \eta|_X = p_{\parallel} + A + \mathcal{L}_{\rho} \partial_x A + \frac{1}{2} \mathcal{L}^2_{\rho} \partial_x A \]
\[ - \mathcal{L}_{\rho} \partial_x A - \varepsilon \mathcal{L}^2_{\rho} \partial_x A + O (\varepsilon^2) \]
\[ = A + p_{\parallel} - \frac{1}{2} \rho \cdot dV + O (\varepsilon) . \] (58)

Pushing \( \partial_\theta \) forward to the guiding center, yields the limiting rote-rate vector \( \partial_{\bar{\theta}} \) as follows
\[ \partial_{\bar{\theta}} = \Pi_{(-\rho \partial_x)} \partial_\theta = -|B| \rho \cdot \partial V + O (\varepsilon) . \] (59)

It is straightforward to verify that the Lagrangian one-form \( \bar{\eta} \) (Eq. 58) is gyro-invariant in a uniform electromagnetic field
\[ \mathcal{L}_{\partial_{\bar{\theta}}} \bar{\eta} = 0 , \]
\[ \iota_{\partial_{\bar{\theta}}} d\bar{\eta} = -d \partial_{\bar{\theta}} \bar{\eta} = -d \bar{\mu} . \] (60)
and the magnetic moment $\mu$ is given by

$$\bar{\mu} = \imath_{\partial_\theta} \bar{\eta} = \frac{|B| \rho^2}{2},$$

which is equal to the action integral along the Kruskal’s ring $\mu_P[O]$.

### C. Gyro-average

For the inhomogeneous case $\varepsilon \neq 0$, we need pull the Lagrangian one-form $\eta$ back to the guiding center. Substituting Eq. (54) to Eq. (13), yields

$$\eta_\varepsilon = \sum_{n \geq 0} \varepsilon^n \frac{n!}{n!} (\varepsilon A + p_\parallel - \mathcal{L}_{\partial_\rho} A) = A + p_\parallel + \sum_{n > 0} \frac{\varepsilon^n}{n!} \eta_n,$$

where

$$\eta_n \equiv \mathcal{L}_{\partial_\rho} p_\parallel - \frac{n}{n+1} \mathcal{L}_{\partial_\rho}^{n+1} A.$$  

(63)

The head order does not depend on the gyro-radius $\partial_\rho$

$$\eta_\parallel = A + p_\parallel,$$

which is the horizontal Lagrangian one-form. The other orders are polynomials of Lie derivative $\mathcal{L}_{\partial_\rho}$

$$\eta_\perp = \sum_{n > 0} \frac{\varepsilon^n}{n!} \eta_n,$$

(66)

which form the vertical Lagrangian one-form.

Since $p_\parallel$ and $A(t, X)$ are gyro-independent, the gyro-transformation of $\eta_\varepsilon$ only depend on the rotation of gyro-radius $\partial_\rho$

$$\Phi_{\partial_\rho}^{\varepsilon} \eta_n = \mathcal{L}_{\Phi_{\partial_\rho}^{\varepsilon} \partial_\rho} p_\parallel - \frac{n}{n+1} \mathcal{L}_{\Phi_{\partial_\rho}^{\varepsilon} \partial_\rho}^{n+1} A.$$

(67)

Let the gyro-radius $\partial_\rho$ satisfy the complex-like condition Eq. (37), then the gyro-averaged Lagrangian one-form $\langle \eta_n \rangle$ are polynomials of the Lie derivative $\mathcal{L}_{\partial_\rho}$ (see Eq. (69)),

$$\langle \eta_n \rangle = \int_0^1 ds (\cos (2\pi s) \mathcal{L}_{\partial_\rho} + \sin (2\pi s) \mathcal{L}_{\partial_\rho})^n p_\parallel
- \frac{n}{n+1} \int_0^1 ds (\cos (2\pi s) \mathcal{L}_{\partial_\rho} + \sin (2\pi s) \mathcal{L}_{\partial_\rho})^{n+1} A.$$

(68)

Let the rote-rate vector be

$$\partial_\rho = - |B| \rho \cdot \partial_V.$$

(69)

The gyro-radius is obtaineded from the complex-like condition Eq. (37)

$$\partial_\rho \equiv \rho \cdot \hat{\Omega}_{\omega} + \varepsilon (-\rho \cdot \nabla b \times b \times (V-D) + \rho \cdot \nabla D) \cdot \partial_V,$$

(70)

whose second term come from the perturbation caused by the inhomogeneity. There are different equivalent forms of $\partial_\rho$ and $\partial_\theta$ that correspond to different decomposition of gyro-motion [12]. How to find a suitable decomposition for the gyro-motion is a problem worthy of in-depth discussion.
D. The guiding-center dynamic

In order to obtain the dynamics of the guiding center, we first perform the gyro-average on the Lagrangian one-form $\bar{\eta} = \langle \eta \rangle$. The vertical part is given by Eq. (68),

$$\bar{\eta}_\perp = \langle \eta_\perp \rangle = -\frac{1}{4} \varepsilon \left( \mathcal{L}_{\bar{\theta}_p} A + \mathcal{L}_{\bar{\theta}_c} A \right) + O (\varepsilon^2).$$  \hfill (71)

Substituting Eq. (70), yields

$$\bar{\eta}_\perp = -\frac{1}{2} \rho \cdot dV_\perp + \tilde{\mu} \left( b \cdot \nabla \times b \right) b \cdot dX + \tilde{\mu} \left( b \cdot \nabla \times D \right) dt,$$  \hfill (72)

where

$$\tilde{\mu} \equiv \iota_\partial \bar{\eta}_\perp = \iota_\partial \bar{\eta}_\perp = \frac{V^2}{2 |B|},$$  \hfill (73)

is the magnetic moment.

Extracting the factor $\mu$ from $\bar{\eta}_\perp$, yields a dimensionless one-form

$$\frac{\eta_\perp}{\mu} = \sigma \equiv -dc \cdot a + \frac{1}{2} \left( b \cdot \nabla \times b \right) b \cdot dX - \frac{1}{2} b \cdot \nabla \times D dt,$$  \hfill (74)

where

$$c \equiv \frac{V_\perp}{|V_\perp|} = a \times b,$$  \hfill (75)

$$a \equiv -\mathcal{L}_{\bar{\theta}_c} c = b \times c = \frac{b \times V_\perp}{|V_\perp|},$$  \hfill (76)

are two unit vector fields perpendicular to the direction of magnetic field $b$. The dimensionless one-form $\sigma$ only depends on the spatial-temporal inhomogeneity of the electromagnetic field. The interior product of $\partial_\bar{\theta}$ and $\sigma$ is unit one, $\iota_\partial \sigma = 1$. A natural question arises. Is $\sigma$ the covector of gyrophase $d\theta$? Or, can $\sigma$ define the global gyrophase $\theta$?

To answer this question, we check the exterior derivative of $\sigma$

$$d\sigma = \frac{1}{2} \nabla \times R \times dX \wedge dX - (\nabla R + \partial_t R) \cdot dX \wedge dt,$$  \hfill (77)

where

$$R \equiv \nabla c \cdot a - \frac{1}{2} \left( b \cdot \nabla \times b \right) b,$$  \hfill (78)

$$R \equiv -\partial_t c \cdot a - \frac{1}{2} b \cdot \nabla \times D,$$  \hfill (79)

is a four-vector $(R, R)$ in the configuration space. Since $d\sigma \neq 0$, the dimensionless one-form $\sigma$ is not an exact form $\sigma \neq d\theta$, and we cannot define the global gyrophase $\theta$ from $\sigma$.

The traditional guiding center theory defines the gyrophase on a predefined orthogonal coordinate frame, which may not exist globally over a non-trivial field topology. This issue was raised and discussed by Sugiyama \cite{2,3} and Krommes \cite{4} in 2009. Soon after, Burdy and Qin \cite{5} identified the obstruction to the global existence of gyrophase is the vector field $\nabla \times (\nabla c \cdot a)$. Bohosian \cite{25} gave similar results in his earlier work. However, the global gyrophase is not a necessary condition for gyro-symmetry. The gyro-symmetry depends only on the homogeneity of the phase space within the range of Kruskal’s ring. The classical approach obscures the geometric meaning of the gyro-symmetry. A coordinate-free geometric representation is a more suitable alternative. The Eq. (77) shows that the existence of gyrophase depends on the integrability of $\sigma$, or requires two-form $d\sigma = 0$ to vanish everywhere. This is a straightforward conclusion of the geometric method.

Continuing the derivation of guiding center dynamics, the horizontal Lagrangian one-form is gyro-independent

$$\bar{\eta}_\parallel = \langle \eta \rangle = A + p_\parallel = (A + V_\parallel) \cdot dX - \left( \phi + \frac{V_\parallel^2}{2} - \mu |B| \right) dt,$$  \hfill (80)
which no additional calculations are required. Using Eq. (77), the Lagrangian two-form is rewritten as

\[ d\bar{\eta} = d\bar{\eta}_\parallel + d\bar{\eta}_\perp = \Omega^\parallel + d\mu \wedge \sigma, \quad (81) \]

where

\[ \Omega^\parallel \equiv d\eta_\parallel + \mu d\sigma = \frac{1}{2} B^\parallel \times dX \wedge dX + E^\parallel \cdot dX \wedge dt + b \cdot dV \wedge b \cdot dX - (V - D) \cdot dV \wedge dt, \quad (82) \]

is the gyro-independent part of Lagrangian two-form, and

\[ B^\parallel \equiv B + \nabla \times V_\parallel + \mu \nabla \times R, \quad (83) \]

\[ E^\parallel \equiv E - \partial_t V_\parallel - \nabla (V_\parallel^2 + \mu |B|) - \mu (\nabla R + \partial_t R), \quad (84) \]

are effective electromagnetic fields. Then, the Eq. (17) is simplified as

\[ * \tau_\parallel = \frac{3! d\mu \wedge \sigma \wedge \Omega^\parallel \wedge \Omega^\parallel}{2! \sigma \wedge \Omega^\parallel \wedge \Omega^\parallel \wedge \Omega^\parallel} = \frac{d\mu \wedge \sigma \wedge \Omega^\parallel \wedge \Omega^\parallel}{2B^\parallel \cdot b}. \quad (85) \]

Substituting Eq. (82), yields the Hamiltonian vector field of guiding center

\[ \tau_\parallel = \frac{B^\parallel \cdot b}{|B|}, \quad (86) \]

\[ \tau_\parallel X = \frac{V \cdot bB^\parallel + E^\parallel \times b}{|B|} - b \times \nabla \mu, \quad (87) \]

\[ \tau_\parallel V = \left( \frac{B^\parallel \cdot E^\parallel}{|B|} + B^\parallel \cdot \nabla \mu \right) b + |V_\perp| (\tau_\parallel X \cdot R - \tau_\parallel R) a - \frac{1}{2} |V_\perp| (\tau_\parallel X \cdot \nabla \mu + \tau_\parallel \partial_t \ln \mu) c. \quad (88) \]

where \( \tau_\parallel \) is the drift motion of the guide center, and \( \tau_\parallel \) is the acceleration of Kruskal’s ring. In Eq. (88), the first term is the acceleration of the guiding center in the direction of the magnetic field, the second term represents the expansion of Kruskal’s ring, and the third term represents the rotational acceleration of Kruskal’s ring. One can simply verify that \( \tau_\parallel \) is gyro-independent and orthogonal to the gyro-motion \( \mathcal{L}_{\partial_\theta} \tau_\parallel = i_{\tau_\parallel} \eta_\parallel = 0 \). The magnetic moment \( \mu \) is a constant for guiding center motion \( i_{\tau_\parallel} d\mu = 0 \).

V. SUMMARY AND DISCUSSION

To summarize, we discuss the dynamics of a charged particle in a time-dependent, slowly varying electromagnetic field, whose Lagrangian is a one-form \( \eta \) on the seven-dimensional contact manifold \( P \). The orbit of gyro-symmetry \( \Phi_\theta \) is a closed ring in the phase space, called Kruskal’s ring \( O \). The guiding center is the center of Kruskal’s ring, the fixed point of gyro-symmetry \( \Phi_\theta \). By properly defining the rate-rate vector \( \partial_\parallel \) (Eq. (69)) and the gyro-radius \( \partial_\parallel \) (Eq. (70)), we give a general expression for the gyro-averaging (Eq. (83)). Further, the geometric decomposition of the gyro-motion is given and verified (Eq. (72) and (80)). As a result, we obtain the coordinate-free expression of the non-relativistic guiding-center dynamics in the time-dependent slow-varying electromagnetic field. (Eq. (87) and (88)).

We understand the gyro-symmetry as the similarity of the trajectory of ringmates on the same Kruskal’s ring, which depends only on the local homogeneity of the electromagnetic field. The gyro-averaging is integral along the Kruskal’s ring. The guiding center is a particle with “spin”, whose magnetic moment \( \mu \) is equal to the action integral over the Kruskal’s ring. The guiding center dynamics is the dynamics of Kruskal’s ring. The purpose of guiding center
theory is to decompose particle motion into vertical gyro-motion and horizontal drift motion. In the classical guiding center theory, the vertical part of Lagrangian is expressed as $md\theta$. We now recognize that $d\theta$ may not be globally defined on the non-trivial magnetic field. Even if we ignore the global validity and consider only the local domain, $d\theta$ is still an ambiguous expression. Applying the local gyro-phase coordinate frame to the dimensionless one-form $\theta$, yields $\theta = d\theta + d\mathbf{X} \cdot \nabla \cdot \mathbf{a} + dt \cdot \partial_{\theta} \mathbf{c} \cdot \mathbf{a} + O(\varepsilon)$, which means $d\theta$ only make scenes when the electromagnetic field is homogeneous. The geometric decomposition method can avoid the confusion caused by the local gyro-phase coordinates frame. The expansion (Eq. (E5)) and averaging method (Eq. (E6)) ensure that each order of the expansion is gyro-independent.

In the subsequent work, the geometric methods established in this paper will be applied to Lie perturbations, gyro-kinetics theory, and others. Thus, the rich geometric of gyro-dynamics should be further revealed.

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