Cylindrical solutions in braneworld gravity

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Abstract

In this article we investigate exact cylindrically symmetric solutions to the modified Einstein field equations in the brane world gravity scenarios. It is shown that for the special choice of the equation of state $2U + P = 0$ for the dark energy and dark pressure, the solutions found could be considered formally as solutions of the Einstein-Maxwell equations in 4-D general relativity.

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I. INTRODUCTION

Since Einstein’s formulation of general theory of relativity, extra dimensions have always played a profound role in studies trying to unify gravity with other forces of nature. The very well known example is the Kaluza-Klein 5-dimensional theory whose main goal was the unification of gravitational and electromagnetic interactions via compactified fifth dimension. In recent years string inspired non-compactified extra dimensions have raised a lot of enthusiasm in the so called brane world gravity scenarios. In these scenarios, extra dimensions being large need not be compactified, instead we expect imprints of embedding and free gravitational field (in the 5-dimensional bulk spacetime of scenarios with one extra dimension) to show up in modified dynamical equations on the brane which are different from the usual four dimensional Einstein equations. Consequently, when we confine ourselves to the 3-brane, exact solutions to these modified field equations are expected to be either different from those in the usual four-dimensional general relativity or, if they are formally the same, to be reinterpreted in terms of some of the characteristics of the extra dimension. In 5-dimensional case, exact radial solutions with one and two parameters, are studied in [1] where it is shown that there are different solutions according to the different equations of state for dark radiation and dark pressure. Gravitational collapse of matter, producing black holes on a brane, is studied in [2]. There it is shown that a five dimensional black string solution intersects the brane in a Schwarzschild black hole. In the same scenario it is also shown that the two parameter Reissner-Nordstrom type solution could be interpreted in terms of mass and another parameter attributed to a characteristic of the bulk space [3]. In this paper, using the modified Einstein equations on the brane, we study exact solutions on the brane with cylindrical symmetry. Actually what we will show is another manifestation of the fact that any solution of the Einstein-Maxwell equations (with traceless energy-momentum tensor) in 4-dimensional general relativity could be interpreted as a vacuum solution of brane-world 5-dimensional gravity [3].
II. EINSTEIN FIELD EQUATIONS ON THE BRANE

A covariant generalization of Randall-Sundrum model \[4,5\] is given in \[6\] (the so called SMS braneworld \(^1\)) where the 4-dimensional world is described by a 3-brane \((M, q_{\mu\nu})\) as a fixed point of \(Z_2\) symmetry in a 5-dimensional space-time \((\mathcal{M}, g_{\mu\nu})\). Choosing the 5-th coordinate \(\chi\) such that the brane coincides with the hypersurface \(\chi = 0\), the 5-dimensional metric could be written as;

\[
ds^2 = (n_\mu n_\nu + q_{\mu\nu}) dx^\mu dx^\nu \quad (1)
\]

where \(n_\mu\) is the unit vector normal to the brane i.e \(n_\mu dx^\mu = d\chi\) and \(q_{\mu\nu}\) is the induced metric on the brane. Assuming that the matter is confined to the brane, the 5-dimensional Einstein equations read;

\[
(5) G_{\mu\nu} = \kappa_{(5)}^2 (5) T_{\mu\nu} \quad (5)
\]

\[
(5) T_{\mu\nu} = -\Lambda_{(5)} g_{\mu\nu} + \delta(\chi) [ -\lambda_b q_{\mu\nu} + (4) T_{\mu\nu}] \quad (2)
\]

where \(\Lambda_{(5)}\) is the bulk cosmological constant and \(\lambda_b\) is the vacuum energy on the brane or the brane tension in the bulk space. Since we assumed that the matter is confined to the brane, the 4-dimensional matter field Lagrangian determines the \(4) T_{\mu\nu}\) such that \(4) T_{\mu\nu} n^\mu = 0\). Using the Gauss-Codacci equations, along with the \(Z_2\) symmetry, one could project 5-dimensional curvature equations along the 4-dimensional hypersurface coincident with the brane. The effective Einstein field equations on the brane are found to be \[6\];

\[
(4) G_{\mu\nu} = -\Lambda q_{\mu\nu} + \kappa_{(4)}^2 (4) T_{\mu\nu} + \kappa_{(5)}^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu} \quad (3)
\]

where

\[
S_{\mu\nu} = \frac{1}{12} (4) T^{4\alpha\beta} T_{\alpha\beta}^\mu - \frac{1}{4} (4) T_{\mu\nu} (4) T^{\alpha\beta} (4) T_{\alpha\beta} - \frac{1}{12} (4) T^2 \quad (4)
\]

\(^1\)Note that there are reformulations of the SMS braneworld in the literature with respect to different aspects of that formulation \[7,8\].
\[ \kappa_{(4)}^2 = 8\pi G_N \]

\[ G_N = \frac{\kappa_{(5)}^4 \lambda_b}{48\pi} \]

\[ \Lambda = \frac{1}{2} \kappa_{(5)}^2 \left( \Lambda_{(5)} + \frac{1}{6} \kappa_{(5)}^2 \lambda_b^2 \right) \]

and

\[ \mathcal{E}_{\mu\nu} = (5)C_{\beta\gamma\delta}^{\alpha} n_\alpha n_\gamma q_\mu^\beta q_\nu^\delta \]

is the transmitted projection of the bulk Weyl tensor\(^2\). Note that it is symmetric and traceless. Equation (3) is the modified Einstein field equation on the brane due to the bulk effects. Compared with the usual 4-dimensional Einstein field equations, there are two main modifications: the first one is the presence of \( S_{\mu\nu} \) term which is quadratic in \(^4T_{\mu\nu}\). Comparison of the second and third terms in the right hand side of (3) shows that it is important in high energy limit and dominates when the energy density \( \rho \gg \lambda_b \) and is negligible for \( \rho \ll \lambda_b \). The second correction is due to \( \mathcal{E}_{\mu\nu} \), the projection of the bulk Weyl tensor on the brane. From the viewpoint of a brane-observer the former is local and the latter is non-local. One recovers the usual Einstein field equations, by taking the limit \( \kappa_{(5)} \to 0 \) while keeping \( G \) finite. Though there is a point to be made here, according to (5) it is impossible to define Newton’s gravitational constant \( G_N \) without an unambiguous definition of \( \lambda_b \) [9]. Using equation (3), the 4D contracted Bianchi identity \( D^\nu(4)G_{\mu\nu} = 0 \) along with the conservation of \(^4T_{\mu\nu}\), \( D^\nu(4)T_{\mu\nu} = 0 \) lead to the following constraint;

\[ D^\mu \mathcal{E}_{\mu\nu} = \kappa_{(5)}^4 D^\mu S_{\mu\nu}, \]

\(^2\)To remind the reader, the n-dimensional \((n \geq 3)\) Weyl tensor is defined by,

\[ (n)C_{\alpha\beta\gamma\delta} = (n)R_{\alpha\beta\gamma\delta} + \frac{1}{n-2} (g_{\alpha\delta}(n)R_{\beta\gamma} + g_{\beta\gamma}(n)R_{\delta\alpha} - g_{\alpha\gamma}(n)R_{\delta\beta} - g_{\beta\delta}(n)R_{\gamma\alpha}) \]

\[ + \frac{1}{(n-1)(n-2)} (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\gamma\beta})(n)R. \]
Where $\mathcal{D}^\mu$ is the covariant derivative with respect to $q_{\mu\nu}$. Being symmetric and traceless, $\mathcal{E}_{\mu\nu}$ could be decomposed irreducibly with respect to a chosen 4- velocity vector field $u^\mu$ as [10];

$$\mathcal{E}_{\mu\nu} = -\kappa^4 [U(u_\mu u_\nu + \frac{1}{3} h_{\mu\nu}) + P_{\mu\nu} + 2 Q_{(\mu} u_{\nu)}] \quad (8)$$

where $h_{\mu\nu} = q_{\mu\nu} + u_\mu u_\nu$ is the projection tensor orthogonal to $u^\mu$, $\kappa = \kappa_{(5)} / \kappa_{(4)}$ and

$$U = -\kappa^{-4} \mathcal{E}_{\mu\nu} u^\mu u^\nu$$
$$Q_{\mu} = \kappa^{-4} h_{\mu}^{\alpha} \mathcal{E}_{\alpha\beta} u^\beta$$

$$P_{\mu\nu} = -k^{-4} [h_{(\mu}^{\alpha} h_{\nu)}^{\beta} - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta}] \mathcal{E}_{\alpha\beta}$$

$U$ is an effective nonlocal energy density on the brane, arising from the bulk free gravitational field. Note that this nonlocal energy density need not be positive (in fact $U$ contributes to tidal acceleration on the brane in the off-brane direction [10]) and actually being negative is more consistent with the Newtonian picture in which gravitational field carries negative energy. Also $P_{\mu\nu}$ and $Q_{\mu}$ are respectively the effective nonlocal anisotropic stress and energy flux on the brane, both arising from the free gravitational field in the bulk.

III. STATIC CYLINDRICALLY SYMMETRIC SOLUTION

In this section we solve the 4-dimensional modified Einstein equation (3) in vacuum ($T_{\mu\nu} = 0$) for a static, cylindrically symmetric case. In the vacuum $T_{\mu\nu}$ and consequently $S_{\mu\nu}$ vanish and the constraint (7) becomes

$$\mathcal{D}^\mu \mathcal{E}_{\mu\nu} = 0 \quad (10)$$

One can choose $\Lambda = 0$ by taking $\Lambda_{(5)} = -1/6 \kappa_{(5)}^2 \lambda_b^2$ so that the effective Einstein equations on the brane reduce to

$$^{(4)}R_{\mu\nu} = -\mathcal{E}_{\mu\nu} \quad (11)$$
As pointed out in [3] the set of equations (10-11) form a closed system of equations for static solutions on the brane. In particular Einstein-Maxwell solutions (with traceless energy-momentum tensor) in 4-dimensional general relativity could be properly interpreted as \textit{vacuum} brane world solutions.

Representing nonlocal effects, one could specify $E_{\mu \nu}$ in an inertial frame at any point on the brane where $u^\mu = \delta^\mu_0$. In static spacetimes this corresponds to comoving frames along the timelike Killing vector field for which [3]

$$Q_\mu = 0 \quad , \quad h^{\nu}_{\mu} = \text{diag}(0, 1, 1, 1) \quad (12)$$

and the constraint for $E_{\mu \nu}$ takes the form

$$\frac{1}{3} \tilde{\nabla}_\mu U + \frac{4}{3} U A_\mu + \tilde{\nabla}^{\nu} P_{\mu \nu} + A^{\nu} P_{\mu \nu} = 0 \quad (13)$$

where $\tilde{\nabla}_\mu = h^{\nu}_{\mu} D_\nu$ is the projected covariant derivative orthogonal to $u^\mu$ and $A_\mu = u^{\nu} D_\nu u_\mu$ is the 4-acceleration. In the static cylindrically symmetric case we may choose

$$A_\mu = A(\rho) \rho_\mu \quad (14)$$

and

$$P_{\mu \nu} = P(\rho)(\rho_\mu \rho_\nu - \frac{1}{3} h_{\mu \nu}). \quad (15)$$

where $A(\rho)$ and $P(\rho)$ are scalar functions and $\rho_\mu$ is the radial unit vector. Now in the (inertial) comoving frame the projected bulk Weyl tensor takes the form;

$$E^{\mu}_{\nu} = -\kappa^4 \text{diag}(-U, \frac{1}{3}(U + 2P), \frac{1}{3}(U - P), \frac{1}{3}(U - P)) \quad (16)$$

Choosing the following general form for a static cylindrically symmetric line element on the brane,

$$ds^2 = -e^{2f(\rho)} dt^2 + e^{-2f(\rho)} e^{2K(\rho)} (d\rho^2 + dz^2) + e^{-2f(\phi)} W(\rho)^2 d\phi^2 \quad (17)$$

substituting it into the modified gravitational field equations (11) and using (16) we end up with,
\[-\frac{1}{W}(-2f''W - 2f'W' + f^2W + K''W + W'') = -\kappa^4 U e^{-2f} e^{2K} \quad (18)\]

\[-\frac{1}{W}(f'^2W - K'W') = -\frac{\kappa^4}{3} (U + 2P) e^{-2f} e^{2K} \quad (19)\]

\[\frac{1}{W}(f'^2W - K'W' + W'') = -\frac{\kappa^4}{3} (U - P) e^{-2f} e^{2K} \quad (20)\]

\[(f'^2 + K'') = -\frac{\kappa^4}{3} (U - P) e^{-2f} e^{2K} \quad (21)\]

The equation (10) could be written explicitly as

\[U' - (4f' - 2K' - \frac{2W'}{W}) U = 0 \quad (22)\]

From (19) and (20), we obtain

\[\frac{W''}{W'} = -\frac{\kappa^4}{3} (2U + P) e^{-2f} e^{2K}. \quad (23)\]

To solve the above equations we choose the following equation of state

\[2U + P = 0 \quad (24)\]

There is no physical restriction that implies this choice of equation of state, but it is not an easy task to solve the above equations in their general form, i.e for arbitrary \(U(\rho)\) and \(P(\rho)\), even in the spherical case [3]. On the other hand it is interesting to note that the above equation of state in terms of the components of the projected Weyl tensor reads;

\[\mathcal{E}_0^0 + \mathcal{E}_3^3 = 0 \quad (25)\]

This establishes, once more and now for cylindrical symmetry, the connection stated above between solutions of the Einstein-Maxwell equations in general relativity and those of the vacuum brane world. For, (25) is the equation satisfied by the energy momentum tensor of Maxwell fields leading to known cylindrically symmetric solutions to the Einstein-Maxwell equations [11]. We will discuss this in more detail in the next section.

By the above choice of the equation of state and from (23) we find \(W'' = 0\) or \(W = a\rho + b\), which by appropriate scaling of coordinates means either \(W = \rho\) or \(W = \text{const.}\) Choosing \(W = \rho\), (20) and (21) lead to the following solutions for \(e^{2K}\),
\[ e^{2K} = \rho^{2m^2} \] (26)

or

\[ e^{2K} = 1 \] (27)

Using the first solution and solving equations (18-22) we end up with

\[ e^{2f} = \frac{4m^2}{(c_1 \rho^{-m} + c_2 \rho^m)^2} \] (28)

\[ U(\rho) = \frac{16m^2 c_1 c_2}{\kappa^4} \frac{\rho^{-2-2m^2}}{(c_1 \rho^{-m} + c_2 \rho^m)^4} \] (29)

where \( m, c_1 \) and \( c_2 \) are constants and \( c_1 c_2 < 0 \). This last condition on the constants \( c_1 \) and \( c_2 \) could be inferred from the comparison of this solution with the corresponding Einstein-Maxwell solution (refer to the next section). For the latter choice of \( e^{2K} \) we arrive at

\[ e^{2f} = \frac{1}{(c_1 \ln \rho + c_2)^2} \] (30)

\[ U(\rho) = -\frac{c_1^2}{\kappa^4 \rho^2 (c_1 \ln \rho + c_2)^4} \] (31)

Now setting \( W = 1 \) and solving Einstein field equations along with the constraint equation (22) we find

\[ K = a \rho + b \]

\[ f = -\ln(\rho) \] (32)

\[ U = -\kappa^4 \frac{e^{-2(a \rho + b)}}{\rho^4} \]

**IV. RELATION TO EINSTEIN-MAXWELL SOLUTIONS**

As pointed out previously, the brane world solutions found in the last section could be treated as the solutions to the Einstein-Maxwell equations in general relativity. So one could use this analogy to find expressions for the corresponding electromagnetic fields in terms of the dark energy \( U \) (or dark pressure \( P \)). We note that the solutions (26-32) are
static cylindrically symmetric electrovacs discussed in the exact solutions literature (see section 22.2 of [11] and references therin). Assuming that the electromagnetic fields inherit the metric symmetry and are orthogonal to the orbits of the two dimensional orthogonally transitive group, the vector potential could be written as [11]

\[ A_\alpha dx^\alpha = \mathcal{P}(\rho) dt + \mathcal{Q}(\rho) d\phi. \]  

This vector potential corresponds to a magnetic field along the \( z \)-direction and an electric field along the \( \rho \)-direction. The non-zero components of \( F_{\mu\nu} \) are \( F_{01} = \mathcal{P}' \) and \( F_{13} = -\mathcal{Q}' \) so that the corresponding energy-momentum tensor components \( T^\mu_\nu \) are given by;

\[
T^0_0 = -\frac{1}{2} e^{-2K} \mathcal{P}'^2 - \frac{1}{2} e^{4f} e^{-2K} W^{-2} \mathcal{Q}'^2
\]  

\[
T^1_1 = -\frac{1}{2} e^{-2K} \mathcal{P}'^2 + \frac{1}{2} e^{4f} e^{-2K} W^{-2} \mathcal{Q}'^2
\]  

\[
T^2_2 = \frac{1}{2} e^{-2K} \mathcal{P}'^2 - \frac{1}{2} e^{4f} e^{-2K} W^{-2} \mathcal{Q}'^2
\]  

\[
T^3_3 = \frac{1}{2} e^{-2K} \mathcal{P}'^2 + \frac{1}{2} e^{4f} e^{-2K} W^{-2} \mathcal{Q}'^2
\]

It is clear from the above equations that

\[
T^0_0 + T^3_3 = 0
\]

Drawing analogies by comparing this equation with (25) we equate the right hand sides of equations (16) and (34-37) to arrive at

\[
\mathcal{Q} = \text{constant}
\]

obviously the equation of state (24) is also satisfied. Therefore dark energy and dark pressure in terms of the vector potential are given by

\[
U = -\frac{P}{2} = -\frac{1}{2} \kappa^{-4} e^{-2K} \mathcal{P}'^2
\]

or conversely the solutions (26-32) are solutions of the Einstein-Maxwell equations with the electromagnetic field given by the vector potential (33) with
\[ P = \kappa^2 \int e^K \sqrt{P(\rho)} \, d\rho \]  

(41)

and \( Q \) a constant which could be taken to be zero. It is noted from (40) that the dark energy, as expected from equations (29), (31) and (32), is a negative quantity. It is shown, by calculating the tidal acceleration on the brane in the off-brane direction, that negative dark energy enhances the localization of the gravitational field near the brane [10].

V. DISCUSSION AND SUMMARY

we have investigated exact static cylindrically symmetric solutions of the modified Einstein field equations for the induced metric on the brane. It is shown that, as in the case of the Reissner-Nordestrom type solutions [3], for the special choice of the equation of state \( 2U + P = 0 \) for the dark energy and dark pressure, the solutions found are that of the Einstein-Maxwell equations in the usual 4-dimensional general relativity. Of course there are no electromagnetic fields present, instead, using the analogy with the solutions to the Einstein-Maxwell solutions, the fields present could be interpreted as tidal fields (41) arising from the imprints, on the brane, of the bulk free gravitational field. The dark energy corresponding to the bulk free gravitational field is shown to be negative in all the solutions obtained, indicating that it acts in favour of confining the gravitational field near the brane. Finding an exact bulk solution that reduces to the exact induced metric on the brane is not an easy task and simple embeddings of the 4-d solution into the 5-d equations either results in equations which could not be solved or change the nature of the 4-d solution. As an example it could be seen that the solution given by equations (28) and (29) when embedded into the Randall-Sundrum [5] general form

\[ ds_\text{bulk}^2 = f(y)(dy^2 + ds_\text{brane}^2) \]

satisfies the bulk equations only when either \( c_1 = 0 \) or \( c_2 = 0 \). But in either case the 4-d solution reduces to the Levi-Civita’s Ricci flat solution [11]. This is already known to satisfy
the bulk equations since the Minkowski metric in Randall-Sundrum AdS solution could be replaced by *any* Ricci flat solution [2].

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