Locally distinguishable bipartite orthogonal quantum states in a $d \otimes n$ system

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In this article, we mainly study the locally distinguishability of bipartite orthogonal quantum states (OQS) in $d \otimes n$ ($2 < d \leq n$) by local operations and classical communication (LOCC). We first show the sufficient and necessary condition (SNC) for some bipartite distinguishable pure OQS in $d \otimes n$ by LOCC. Then the local indistinguishability of OQS in $d \otimes d$ can be proved using the SNC. Furthermore, we generalize this result into $d \otimes n$. Similarly, we show the SNC for some bipartite distinguishable mixed OQS in $d \otimes n$ by LOCC. Our study further reveals quantum nonlocality in $d \otimes n$ quantum system.

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I. INTRODUCTION

Quantum entanglement is an important manifestation of quantum nonlocality. Quantum entangled states, especially the maximally entangled states, play an important role in quantum information theory. On the one hand, quantum nonlocality of the maximally entangled states has attracted considerable attention in recent years, since it can assist us in managing quantum information and in understanding the fundamental principles of quantum mechanics. Quantum nonlocality of maximally entangled states has been studied, which includes the distinguishability and the indistinguishability. Bandyopadhyay et al. revealed an important conclusion: unilaterally transformable quantum states are distinguishable by one-way local operations and classical communication (OW-LOCC). With this conclusion, they proved that maximally entangled states are exactly indistinguishable by OW-LOCC in $d \otimes d$ when $d = 4, 5, 6$. Zhang et al. showed two classes of indistinguishable maximally entangled states by OW-LOCC for $d - 1$ and $d - 2$ ($d = 7, 8, 9, 10$). On the other hand, quantum nonlocality of orthogonal product states (OPS) was widely studied, which included the local distinguishability and the local indistinguishability. Duan et al. exhibited the distinguishable OPS by separable operations in $3 \otimes 3$ and $2 \otimes 2 \otimes 2$ dimensional quantum systems, respectively. However, this result is not always true in LOCC protocols because LOCC operations are weaker than separable operations. These OPS in Refs. are all indistinguishable by LOCC, which exhibits quantum nonlocality without entanglement.

The distinguishability of complete OPS can be proved by using the sufficient and necessary conditions (SNC) in Refs. Walgate and Hardy also gave a SNC for distinguishing bipartite OQS in $2 \otimes n$ by LOCC. The SNC was used to prove the indistinguishability of a class OPS. Zhang et al. extended and proved the indistinguishability of OPS in $d \otimes d$ ($d$ is odd) and Duan et al. proved the distinguishability of mixed states in $2 \otimes n$ using the theorem in Ref. In this paper, we focus on the study of locally distinguishable bipartite orthogonal quantum states in $d \otimes n$ ($2 < d \leq n$) shared by Alice and Bob. We firstly discuss and prove the SNC for some locally distinguishable bipartite pure OQS in $3 \otimes n$. Then the SNC can be used to prove the local indistinguishability of OPS in $d \otimes d$ ($d \geq 3$). When $d \geq 3$ is odd, we show the tiling structure of OQS and the indistinguishability of OQS in $3 \otimes 3$ can be directly proved using the SNC in $3 \otimes n$. When $d \geq 4$ is even, the tiling structure is similar to the case that $d$ is odd and the set of indistinguishable OQS in $4 \otimes 4$ is given. The local indistinguishability of generalize Bell states in $3 \otimes 3$ can also be proved by the SNC. Second, we show the SNC for some locally distinguishable bipartite pure OQS in $d \otimes n$. Moreover, we analyze the number of OPS in $d \otimes n$. Finally, the sufficient and necessary condition is given for some locally distinguishable bipartite mixed OQS in $d \otimes n$.

The rest of this paper is organized as follows. Sec. presents bipartite distinguishable pure orthogonal quantum states in $d \otimes n$. In Sec. we present bipartite distinguishable mixed orthogonal quantum states in $d \otimes n$. Finally, the conclusions are shown in Sec.

II. BIPARTITE DISTINGUISHABLE PURE ORTHOGONAL QUANTUM STATES IN $d \otimes n$

Now, we first introduce a lemma in Ref. in which $2 \otimes n$ orthogonal quantum states can be distinguished with Alice going first.

**Lemma 1.** Alice and Bob share a $2 \otimes n$ dimensional quantum system: Alice has a qubit, and Bob has a $n$ dimensional system that may be entangled with that qubit. If Alice goes first, a set of $l$ orthogonal states $\{ |\psi_i \rangle \}$ is
exactly locally distinguishable if and only if there is a basis \(\{|0\rangle, |1\rangle\}_A\) such that in that basis
\[
|\psi_i⟩ = |0⟩_A|ν_{0i}⟩_B + |1⟩_A|ν_{1i}⟩_B,
\]
where \(\langle ν_{0i} | ν_{0j}⟩ = \langle ν_{1i} | ν_{1j}⟩ = 0\) if \(i \neq j\).

In the following proof of theorems, the element \(M_A^† M_m\) of positive operator-value measure (POVM) is a diagonal matrix when the basis space \(d\) is larger than or equal to 2. Now, we give the reasons. In [29], Nielsen and Chuang pointed out that a POVM operator \(M_A^† M_m\) is a semi-positive operator, which is necessarily normal operator. There is a remarkable theorem called spectral decomposition in which an operator is a normal operator if and only if it is diagonalizable.

Next, we show the bipartite distinguishable pure orthogonal quantum states by OW-LOCC in \(3 \otimes n\), where the proof techniques refer to Ref. [28].

**Theorem 1.** A three-dimensional quantum system that belongs to Alice may be entangled with a \(n\)-dimensional quantum system that belongs to Bob. They share this \(3 \otimes n\) dimensional quantum system: there is a basis \(\{|0⟩, |1⟩, |2⟩\}_A\) such that in that basis
\[
|\psi⟩ = |0⟩_A|ν_{0}⟩_B + |1⟩_A|ν_{1}⟩_B + |2⟩_A|ν_{2}⟩_B,
\]
where \(\langle ν_{0} | ν_{0}⟩ = \langle ν_{1} | ν_{1}⟩ = \langle ν_{2} | ν_{2}⟩ = 0\) (\(i \neq j\)),

then if Alice goes first, a set of orthogonal states is exactly locally distinguishable if and only if \(\langle ν_{1} | ν_{1}⟩ = \langle ν_{2} | ν_{2}⟩ = 0\) (\(i \neq j\)).

**Proof:** The sufficiency is obvious. Suppose that there exists a set \(\{|ψ_i⟩\}_{i=1}^k\) each of \(\{|ψ_i⟩\}_{i=1}^k\) has the form in Eq.(2). Alice sends the result to Bob by measuring in the basis \(\{|0⟩, |1⟩, |2⟩\}_A\), and then Bob can successfully distinguish these states by measuring with the corresponding orthogonal basis \(\{|ν_{0}⟩, |ν_{1}⟩\}_B\) or \(\{|ν_{2}⟩\}_B\).

The necessity is shown as follows. Suppose that Alice goes first and the \(k\) states \(\{|ψ_i⟩\}_{i=1}^k\) must be reliably distinguished. After Alice’s measuring, all Bob’s reserved states must be orthogonal. Therefore, for all pairs of states \(\{|ψ_i⟩, |ψ_j⟩\}\) and for all measurement results \(m\), either that pair remains orthogonal postmeasurement i.e. \(\langle ψ_i | M^†_m M_m | ψ_j⟩ = 0\) or else one of that pair of states has been eliminated i.e. \(\langle ψ_i | M^†_m M_m | ψ_i⟩ = 0\) or \(\langle ψ_j | M^†_m M_m | ψ_j⟩ = 0\).

One of the POVM elements must be nontrivial. So in the basis space \(\{|0⟩, |1⟩, |2⟩\}_A\), the POVM elements \(M^†_m M_m\) can be expressed as
\[
M^†_m M_m = α|0⟩⟨0| + β|1⟩⟨1| + γ|2⟩⟨2|,
\]
where \(β > α > γ \geq 0\) or \(β > α = γ \geq 0\). Two orthogonal states \(|ψ_i⟩\) and \(|ψ_j⟩\) must remain orthogonal if Alice eliminates neither pair from the running. Combined with \(\langle ν_{0} | ν_{0}⟩ = 0\), we get
\[
\langle ψ_i | M^†_m M_m | ψ_j⟩ = β\langle ν_{1} | ν_{1}⟩ + γ\langle ν_{2} | ν_{2}⟩ = 0,
\]
\[
\langle ψ_i | ψ_j⟩ = \langle ν_{1} | ν_{1}⟩ + \langle ν_{2} | ν_{2}⟩ = 0.
\]

Thus \((β - γ)\langle ν_{1} | ν_{1}⟩ = 0\). Since \(β > γ \geq 0\), we get \(β - γ > 0\). Therefore, \(\langle ν_{1} | ν_{1}⟩ = 0\) holds. \(\langle ν_{2} | ν_{2}⟩ = 0\) also holds. Hence, in the case that all states are not eliminated, this pair of states must be in the form given in the theorem.

Next, consider the special case, i.e. Alice achieves a negative identification herself that is \(\langle ψ_i | M^†_m M_m | ψ_i⟩ = 0\) or \(\langle ψ_j | M^†_m M_m | ψ_j⟩ = 0\). Suppose she has eliminated \(|ψ_i⟩\), so we consider \(\langle ψ_i | M^†_m M_m | ψ_i⟩ = 0\) and \(\langle ψ_i | ψ_i⟩ = 1\). We get two equations
\[
α\langle ν_{0} | ν_{0}⟩ + β\langle ν_{1} | ν_{1}⟩ + γ\langle ν_{2} | ν_{2}⟩ = 0,
\]
\[
\langle ν_{0} | ν_{0}⟩ + \langle ν_{1} | ν_{1}⟩ + \langle ν_{2} | ν_{2}⟩ = 1.
\]

Thus \((α + β - γ)\langle ν_{0} | ν_{0}⟩ + (β - γ)\langle ν_{1} | ν_{1}⟩ = -γ \geq 0\).

When \(β = max(α, γ) \geq 0\), the results \(α - γ > 0, β - γ > 0, γ \leq 0\) and \(0 \leq \langle ν_{1} | ν_{1}⟩, \langle ν_{2} | ν_{2}⟩ \leq 1\). So we obtain \(γ = 0\) and \(\langle ν_{0} | ν_{0}⟩ = \langle ν_{1} | ν_{1}⟩ = 0\). It implies that \(\langle ν_{2} | ν_{2}⟩ = 1\), i.e. \(|ψ_i⟩ = |2⟩_A|ν_{2}⟩_B\). In this case, the other state must take the form
\[
|ψ_j⟩ = |0⟩_A|ν_{0}⟩_B + |1⟩_A|ν_{1}⟩_B + |2⟩_A|ν_{2}⟩_B.
\]

As we can see, in all cases, any pair of states must be in the form of Theorem 1. Therefore, all states in the basis \(\{|0⟩, |1⟩, |2⟩\}_A\) are written as
\[
|ψ_i⟩ = |0⟩_A|ν_{0}⟩_B + |1⟩_A|ν_{1}⟩_B + |2⟩_A|ν_{2}⟩_B.
\]

where \(\langle ν_{0} | ν_{0}⟩ = \langle ν_{1} | ν_{1}⟩ = \langle ν_{2} | ν_{2}⟩ = 0\) (\(i \neq j\)).

In Fig.1, we give the general structure of indistinguishable \(d^2 + d + \frac{d}{2}\) (\(d\) is odd) orthogonal quantum states. Then Theorem 1 can be used to prove the indistinguishability.

**Corollary 1.** In \(d \otimes d\) dimensional quantum system, there are \(\frac{d^2}{4} + d + \frac{d}{2}\) (\(d\) is odd) indistinguishable orthogonal product states \(|ϕ_i⟩\) (in Eq.(5)) by LOCC whoever goes first.

\[
|ϕ_1⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_2⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_3⟩ = |1⟩_A|3⟩_B + |2⟩_A|2⟩_B + |3⟩_A|1⟩_B,
\]
\[
|ϕ_4⟩ = |1⟩_A|3⟩_B + |2⟩_A|2⟩_B - |3⟩_A|1⟩_B,
\]
\[
|ϕ_3⟩ = |1⟩_A|3⟩_B + |2⟩_A|2⟩_B - |3⟩_A|1⟩_B,
\]
\[
|ϕ_2⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_3⟩ = |1⟩_A|3⟩_B + |2⟩_A|2⟩_B + |3⟩_A|1⟩_B,
\]
\[
|ϕ_4⟩ = |1⟩_A|3⟩_B + |2⟩_A|2⟩_B - |3⟩_A|1⟩_B,
\]
\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_4⟩ = |1⟩_A|3⟩_B + |2⟩_A|2⟩_B + |3⟩_A|1⟩_B,
\]
\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
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\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
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\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
\]
\[
|ϕ_5⟩ = |0⟩_A|2⟩_B + |1⟩_A|1⟩_B + |2⟩_A|0⟩_B,
draw from the subspace \{1\}, \{2\}, \{3\}\}_A, we make the same argument. Then we obtain results \(a_{11} = a_{22} = a_{33} = a\) and \(a_{12} = a_{13} = a_{21} = a_{31} = a_{23} = a_{32} = 0\). In the same way, for the subspace \{2\}, \{3\}, \{4\}\}_A, \{3\}, \{4\}, \{5\}\}_A, \ldots and \{(d - 3), \{d - 2\}, \{d - 1\}\}_A, we obtain the results \(a_{22} = a_{33} = \cdots = a_{d - 1, d - 1} = a\) and \(a_{23} = a_{24} = \cdots = a_{d - 3, d - 2} = a_{d - 3, d - 1} = \cdots = a_{d - 2, d - 1} = 0\).

Because the POVM elements \(M^\dagger_mM_m\) are Hermitian, the equation \((M^\dagger_mM_m)^\dagger = M^\dagger_mM_m\) holds. Then we obtain \(a^* = a\), \(a_{30} = a_{03}\), \(a_{40} = a_{04}\), \ldots , \(a_{d - 1, d - 4} = a_{d - 4, d - 1}\). Now the \(M^\dagger_mM_m\) can be rewritten as

\[
M^\dagger_mM_m = \begin{pmatrix}
0 & 0 & a_{03} & a_{04} & \cdots & a_{0, d - 1} \\
0 & a & 0 & 0 & a_{14} & \cdots & a_{1, d - 1} \\
0 & 0 & a & 0 & 0 & \cdots & a_{2, d - 1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{d - 1, 0} & a_{d - 1, 1} & a_{d - 1, 2} & a_{d - 1, 3} & \cdots & a_{d - 1, d - 1}
\end{pmatrix},
\]

where \(a\) is a real number. We now consider the states

\[
|\phi_3\rangle = |1\rangle_A|3\rangle_B + |2\rangle_A|2 + 3\rangle_B + |3\rangle_A|1\rangle_B,
\]

and the subspace \{0\}, \{3\}\}_A. After Alice’s measuring, the result is either remaining orthogonal post-measurement or distinguishing them outright. If the result is the states orthogonal post-measurement, we demand that \(\langle \phi_2 | M^\dagger_mM_m | \phi_3 \rangle = a_{03} = 0\). So, we obtain \(a_{03} = a_{00} = 0\). For the subspace \{0\}, \{4\}\}_A, \{0\}, \{5\}\}_A, \ldots, \{(d - 4), \{d - 1\}\}_A, we can obtain \(a_{30} = a_{03} = a_{40} = a_{04} = \cdots = a_{d - 1, d - 4} = a_{d - 4, d - 1} = 0\). Now the \(M^\dagger_mM_m\) is proportional to the identity. However, if Alice distinguishes the state \(|\phi_2\rangle\) outright, we get the result \(\langle \phi_2 | M^\dagger_mM_m | \phi_2 \rangle = 0\). We can also obtain the result \(\langle \phi_2 | M^\dagger_mM_m | \phi_2 \rangle = a\). Therefore \(a = 0\). Since POVM elements must be positive, \(M^\dagger_mM_m\) is the null matrix. □

FIG. 2. Schematic diagram of OPS in \(3 \otimes 3\).

In \(3 \otimes 3\) quantum system (See Fig. 2), Theorem 1 can be directly used to prove the indistinguishability of \(3 \otimes 3\) dimensional OPS.

\[
|\phi_1\rangle = |0 + 1\rangle_A|0\rangle_B, \quad |\phi_2\rangle = |0\rangle_A|1\rangle_B, \quad |\phi_3\rangle = |2\rangle_A|1 + 2\rangle_B, \quad |\phi_4\rangle = |0\rangle_A|2\rangle_B + |1\rangle_A|1 + 2\rangle_B - 2\rangle_A|0\rangle_B,
\]

(6)
When \( d \) is even, the construction in \( d \otimes d \) \((d \geq 4)\) is similar to Fig. 1, where the number of states is \( \frac{1}{2}d^2 + d - 2 \). We introduce an indistinguishable orthogonal produce states in \( 4 \otimes 4 \) (Fig. 3). Theorem 1 can also be used to prove its indistinguishability.

\[
|\varphi_1\rangle = |0 + 1\rangle A|0\rangle B, \quad |\varphi_2\rangle = |0\rangle A|1\rangle B, \quad |\varphi_3\rangle = |0\rangle A|3\rangle B, \\
|\varphi_4\rangle = |3\rangle A|2 + 3\rangle B, \quad |\varphi_5\rangle = |0\rangle A|1 + 2\rangle B \mp |2\rangle A|0\rangle B, \\
|\varphi_6\rangle = |0\rangle A|2\rangle B \mp |1\rangle A|1 - 2\rangle B \mp |2\rangle A|0\rangle B, \\
|\varphi_7\rangle = |1\rangle A|3\rangle B \mp |2\rangle A|2 + 3\rangle B + |3\rangle A|1\rangle B, \\
|\varphi_8\rangle = |1\rangle A|3\rangle B \mp |2\rangle A|2 - 3\rangle B \mp |3\rangle A|1\rangle B. 
\]

In the following, a simple example given in Fig. 3 can be directly distinguishable by Theorem 1.

\[
|\varphi_1\rangle = |00\rangle_{AB} + \omega|11\rangle_{AB} + \omega^2|22\rangle_{AB}, \\
|\varphi_2\rangle = |00\rangle_{AB} + \omega^2|11\rangle_{AB} + \omega|22\rangle_{AB}, \\
|\varphi_3\rangle = |01\rangle_{AB} + |12\rangle_{AB} + |20\rangle_{AB}, 
\]

where \( \omega = e^{\frac{2\pi i}{3}} \). It can be rewritten as

\[
|\varphi_1\rangle = |\xi_1\rangle_A|\xi_2\rangle_B + |\xi_2\rangle_A|\xi_1\rangle_B + |\xi_3\rangle_A|\xi_3\rangle_B, \\
|\varphi_2\rangle = |\xi_1\rangle_A|\xi_3\rangle_B + |\xi_2\rangle_A|\xi_1\rangle_B + |\xi_3\rangle_A|\xi_2\rangle_B, \\
|\varphi_3\rangle = |\xi_1\rangle_A|\xi_1\rangle_B + |\xi_2\rangle_A|\xi_2\rangle_B + |\xi_3\rangle_A|\xi_3\rangle_B. 
\]

where \( |\xi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), |\xi_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle) \), and \( |\xi_3\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle) \).

The proof of the necessity is similar to Theorem 1. Suppose the quantum states \( \{|\psi_i\rangle\}_{i=1}^k \) can be expressed as the Eq.(8). Alice measures with the basis \( \{|0\rangle, |1\rangle, \ldots, (d - 2), |(d - 1)\rangle\}_A \) and sends the measurement outcomes to Bob. Bob successfully distinguishes these states by measuring with the orthogonal base \( \{|\eta_0^i\rangle\}_B \) or \( \{|\eta_1^i\rangle\}_B \) or \( \{|\eta_2^i\rangle\}_B \) or \( \{|\eta_{d-1}^i\rangle\}_B \) or \( \{|\eta_{d-2}^i\rangle\}_B \).

The proof of the necessity is similar to Theorem 1. Suppose that Alice goes first and the \( k \) states \( \{|\psi_i\rangle\}_{i=1}^k \) can be distinguished. Therefore, for all pairs of states \( \{|\psi_i\rangle, |\psi_j\rangle\} \) and relevant measurement results \( m \), we have \( \langle\psi_1|M_{m}^iM_{m}\rangle \psi_i = 0 \) or \( \langle\psi_1|M_{m}^iM_{m}\rangle \psi_j = 0 \) or \( \langle\psi_1|M_{m}^iM_{m}\rangle \psi_j = 0 \).

One of the POVM elements must be nontrivial. So under the basis space \( \{|0\rangle, |1\rangle, |2\rangle, \ldots, |(d - 1)\rangle\}_A \), the POVM elements \( M_{m}^iM_{m} \) can be expressed as

\[
M_{m}^iM_{m} + m = a_0|0\rangle|0\rangle + a_1|1\rangle|1\rangle + \cdots + a_{d-1}|(d - 1)\rangle|(d - 1)\rangle,
\]

where \( a_{d-2} \geq a_0 \geq a_1 \geq \cdots \geq a_{d-3} > a_{d-1} \geq 0 \) or \( a_{d-2} > a_0 \geq a_1 \geq \cdots \geq a_{d-3} \geq a_{d-1} \geq 0 \).

Identically, those Bob’s orthogonal states must remain orthogonal. Combining with \( \langle\eta_0^i|\eta_0^j\rangle = \langle\eta_1^i|\eta_1^j\rangle = \cdots = \langle\eta_{d-3}|\eta_{d-3}\rangle = 0 \), we get

\[
\langle\psi_1|M_{m}^iM_{m}\rangle \psi_j = \alpha_{d-2}\langle\eta_{d-2}^i|\eta_{d-2}^j\rangle + \alpha_{d-1}\langle\eta_{d-1}^i|\eta_{d-1}^j\rangle = 0.
\]

Thus, we get

\[
(\alpha_{d-2} - 2 \alpha_{d-3})\langle\eta_{d-2}^i|\eta_{d-2}^j\rangle = 0.
\]

Thus, we get

\[
(\alpha_0 - \alpha_{d-1})\langle\eta_{d-1}^i|\eta_{d-1}^j\rangle = 0.
\]

When \( \alpha_{d-2} \geq \alpha_0 \geq \alpha_1 \geq \alpha_{d-1} \geq \alpha_{d-2} \geq \alpha_{d-3} \geq \cdots \geq \alpha_{d-1} \geq 0 \) or \( \alpha_{d-2} > \alpha_0 \geq \alpha_1 \geq \alpha_{d-1} \geq \alpha_{d-2} \geq \alpha_{d-3} \geq \cdots \geq \alpha_{d-1} \geq 0 \). When \( \alpha_{d-2} > \alpha_0 \geq \alpha_1 \geq \alpha_{d-1} \geq \alpha_{d-2} \geq \alpha_{d-3} \geq \cdots \geq \alpha_{d-1} \geq 0 \) or \( \alpha_{d-2} > \alpha_0 \geq \alpha_1 \geq \alpha_{d-1} \geq \alpha_{d-2} \geq \alpha_{d-3} \geq \cdots \geq \alpha_{d-1} \geq 0 \).
\[ \langle \eta^0 | \eta^0 \rangle + \cdots + \langle \eta^2-3 | \eta^2-3 \rangle + \langle \eta^2-1 | \eta^2-1 \rangle = 1. \] So we get \[ |\psi\rangle = |0\rangle_A |\eta^0\rangle_B + |1\rangle_A |\eta^1\rangle_B + \cdots + |(d-3)\rangle_A |\eta^2-3\rangle_B + |(d-1)\rangle_A |\eta^2-1\rangle_B. \] In this case, the other state must be expressed as the form
\[ |\psi_j\rangle = |0\rangle_A |\eta^0\rangle_B + \cdots + |(d-3)\rangle_A |\eta^2-3\rangle_B + |(d-2)\rangle_A |\eta^2-2\rangle_B + |(d-1)\rangle_A |\eta^2-1\rangle_B. \]

As we can see, the state is the standard form in Theorem 2. Then, we get the conclusion: in the basis \{(0), (1), \cdots, (d-1)\}_A, all states are represented as
\[ |\psi_j\rangle = |0\rangle_A |\eta^0\rangle_B + |1\rangle_A |\eta^1\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-2\rangle_B + |(d-1)\rangle_A |\eta^2-1\rangle_B. \]

where \[ \langle \eta^0 | \eta^0 \rangle = \langle \eta^1 | \eta^1 \rangle = \cdots = \langle \eta^2-1 | \eta^2-1 \rangle = 0 \text{ if } i \neq j. \]

According to the above results, we analyze the number of product states in \(d \otimes d\) and easily obtain Theorem 3.

**Theorem 3.** In \(d \otimes n\), \(3n\) orthogonal states (in Eq.(11)) expressed as the Eq.(8) can be exactly locally distinguished if and only if at most \(2n\) of those states are product states.

\[ 3n \text{ d } \otimes \text{ n orthogonal quantum states can be written as } \]
\[ |\psi_1\rangle = |1\rangle_A |\eta^1\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-1\rangle_B + |(d-1)\rangle_A |\eta^2-1\rangle_B, \]
\[ |\psi_2\rangle = |1\rangle_A |\eta^2\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-2\rangle_B + |(d-1)\rangle_A |\eta^2-2\rangle_B, \]
\[ |\psi_3\rangle = |1\rangle_A |\eta^3\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-3\rangle_B + |(d-1)\rangle_A |\eta^2-3\rangle_B, \]
\[ \vdots \]
\[ |\psi_n\rangle = |1\rangle_A |\eta_n\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-n\rangle_B + |(d-1)\rangle_A |\eta^2-n\rangle_B, \]
\[ |\psi_{n+1}\rangle = |1\rangle_A |\eta_{n+1}\rangle_B + |(d-2)\rangle_A |\eta^2-n-1\rangle_B + |(d-1)\rangle_A |\eta^2-n-1\rangle_B, \]
\[ |\psi_{n+2}\rangle = |1\rangle_A |\eta_{n+2}\rangle_B + |(d-2)\rangle_A |\eta^2-n-2\rangle_B + |(d-1)\rangle_A |\eta^2-n-2\rangle_B, \]
\[ \vdots \]
\[ |\psi_{3n}\rangle = |1\rangle_A |\eta_{3n}\rangle_B + |(d-2)\rangle_A |\eta^2-3n\rangle_B + |(d-1)\rangle_A |\eta^2-3n\rangle_B. \]

**Proof:** Suppose the set in Eq.(10) is locally distinguishable: Alice firstly measures her particles A with the basis \{(1), (2), \cdots, (d-1), (d)\}_A such that Bob’s quantum states collapse. Thus, we get \(\langle \eta^q | \eta^s \rangle = 0 \) (\(q \neq s\)), where \(q = 1, 2, \cdots, d\) and \(r, s = 1, 2, \cdots, 3n\). But it does not exist \(3n\) mutually orthogonal states in Bob’s \(n\)-dimensional Hilbert space. Therefore, in each of these \(2n\) states must be product states. These quantum states in Eq.(11) can be simplified into
\[ |\psi_1\rangle = |1\rangle_A |\eta^1\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-1\rangle_B, \]
\[ |\psi_2\rangle = |1\rangle_A |\eta^2\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-2\rangle_B, \]
\[ |\psi_3\rangle = |1\rangle_A |\eta^3\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-3\rangle_B, \]
\[ \vdots \]
\[ |\psi_d\rangle = |1\rangle_A |\eta_n\rangle_B + \cdots + |(d-2)\rangle_A |\eta^2-n\rangle_B, \]
\[ |\psi_{d+1}\rangle = |d-1\rangle_A |\eta_{d-1}\rangle_B, \]
\[ |\psi_{d+2}\rangle = |d-1\rangle_A |\eta_{d-2}\rangle_B, \]
\[ |\psi_{d+3}\rangle = |d-1\rangle_A |\eta_{d-3}\rangle_B, \]
\[ |\psi_{2d}\rangle = |d-1\rangle_A |\eta_{d-1}\rangle_B, \]
\[ |\psi_{3d}\rangle = |d-1\rangle_A |\eta_{d}\rangle_B. \]

3n d \otimes n orthogonal states can be exactly locally distinguished if and only if at most 2n of those states are product states.

**III. BIPARTITE DISTINGUISHABLE MIXED ORTHOGONAL QUANTUM STATES IN d \otimes n**

Similar to Theorem 2, the \(2 \otimes n\) dimensional mixed quantum system in Ref. [20] can be extended to \(d \otimes n\) (\(d > 2\)) as follows. \(\{\rho_1, \rho_2, \ldots, \rho_\lambda\}\) is a set of orthogonal states and the support of a positive operator \(\rho\) is expressed as \(Supp(\rho)\) presented in Definition 2 and Lemma 1 of Ref. [20].

The \(d \otimes n\) dimensional mixed quantum states \(\rho_i = \sum_{m,n=1}^{d} |m\rangle\langle n| \otimes \rho_{nn}\) (\(i = 1, 2, \ldots, \lambda\)) can be distinguished with the help of Theorem 5. Now we will present Theorem 5 and its proof technique is similar to Ref. [20].

**Theorem 5.** A d dimensional quantum system that belongs to Alice may be entangled with a n dimensional quantum system that belongs to Bob. They share this orthogonal \(d \otimes n\) (\(2 < d \leq n\)) dimensional quantum system in which orthogonal quantum states \(\rho_1, \rho_2, \ldots, \rho_\lambda\) are randomly chosen: there is a basis \{(1), (2), (3), \cdots, d\}_A such that in that basis
\[ |\psi_i\rangle = |1\rangle_A |\psi^i_1\rangle_B + |2\rangle_A |\psi^i_2\rangle_B + \cdots + |(d-1)\rangle_A |\psi^i_{d-1}\rangle_B + |d\rangle_A |\psi^i_d\rangle_B, \]

where \(\langle \psi^i_1 | \psi^i_1 \rangle = \cdots = \langle \psi^i_{d-2} | \psi^i_{d-2} \rangle = 0 (i \neq j). \]

then if Alice goes first, a set of quantum states is exactly locally distinguishable if and only if \(\langle \psi_{d-1}^i | \psi_{d-1}^j \rangle = \langle \psi^i_1 | \psi^j_1 \rangle = 0 (i \neq j). \]

**Proof:** Density operators are used to prove the Theorem 5. Firstly, the sufficiency is obvious. When the Eq.(12) holds, \(\rho_i\) should be of the following form:
\[ \rho_i = \sum_{m,n=1}^{d} |m\rangle\langle n| \otimes \rho_{nn}, \]

where \(\rho_{nn}\) is a set of orthogonal states and \(m = 1, 2, 3, \cdots, d\). Alice sends the output m measured in such a basis \{(1), (2), (3), \cdots, d\}_A to Bob. Since these quantum states are orthogonal, Alice can perfectly discriminate them by a projective measurement. If \(m = 1\), then Bob’s state is one of \(\rho_{11}\). The cases \(m = 2, \cdots, d\) can be analyzed similarly.

Secondly, the necessity is analysed. Suppose that \(\rho_1, \rho_2, \ldots, \rho_\lambda\) can be distinguished with Alice going first. Suppose there exists a non-trivial operator \(M_m\) that maintains orthogonality. Then for any \(|\psi_i\rangle \in Supp(\rho_i)\) and \(|\psi_j\rangle \in Supp(\rho_j)\), we have \(\langle \psi_i | M_m^\dagger M_m \otimes I | \psi_j \rangle = 0. \)
Since $\rho_i$ and $\rho_j$ are orthogonal, $\langle \rho_i | \rho_j \rangle = 0 \implies \langle \psi_i | \psi_j \rangle = 0$ holds. Let the spectral decomposition is

$$M^i_m M_m = \alpha_1 |1\rangle \langle 1| + \alpha_2 |2\rangle \langle 2| + \cdots + \alpha_d |d\rangle \langle d|,$$

where $\alpha_{d-1} \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_d \geq 0$ or $\alpha_{d-1} > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_d \geq 0$ by the nontriviality. Combining with $\langle \psi_i | \psi_i \rangle = \cdots = \langle \psi_{d-2} | \psi_{d-2} \rangle = 0$, we have $\alpha_{d-1} \langle \psi_{d-1}^i | \psi_{d-1}^j \rangle + \alpha_d \langle \psi_d^i | \psi_d^j \rangle = 0$ and $\langle \psi_{d-2}^i | \psi_{d-2}^j \rangle + \langle \psi_d^i | \psi_d^j \rangle = 0$. By solving two equations, we get $\langle \psi_{d-1}^i - \psi_d^i | \psi_{d-1}^j \rangle = 0$. Since $\alpha_{d-1} > \alpha_d \geq 0$ holds, we get $\langle \psi_{d-1}^i | \psi_{d-1}^j \rangle = \langle \psi_d^i | \psi_d^j \rangle = 0$. \(\Box\)

Thus, the $d \otimes n$ dimensional mixed quantum states

$$\rho_i = \sum_{m,n=1}^d |m\rangle \langle n| \otimes \rho_{mn},$$

can be distinguished with the help of Theorem 5.

IV. CONCLUSION

We have provided sufficient and necessary conditions respectively for distinguishing bipartite orthogonal pure quantum states and mixed quantum states in $d \otimes n$ ($2 < d \leq n$) by LOCC. We construct a set of OQS in $d \otimes d$ ($d \geq 3$) and prove the indistinguishability by the SNC in Theorem 1. The distinguishability of generalized Bell states in $3 \otimes 3$ can also be proved by Theorem 1. Moreover, we analyze the number of OPS in $d \otimes d$ to get the conclusion: $3n$ distinguishable OQS in the form of Eq.(8) include at most $2n$ OPS. These results exhibit the phenomenon of quantum nonlocality.

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