A SHORT NOTE ON A THEOREM BY ELIAHOU AND FROMENTIN

MATHEUS BERNARDINI AND PATRICK MELO

Abstract. In this note, we give an alternative proof for a theorem by Eliahou and Fromentin, which exhibit a remarkable property of the sequence \( n'_g \), where \( n'_g \) denotes the number of gapsets with genus \( g \) and depth at most 3.

1. Introduction

A numerical semigroup \( S \) is a submonoid of the set of non-negative integers, \( \mathbb{N}_0 \), equipped with the usual addition, such that \( G(S) := \mathbb{N}_0 \setminus S \), the set of gaps of \( S \), is finite. In [2], the concept of a gapset was formally introduced in the following way. A gapset is a finite set \( G \subset \mathbb{N} \) satisfying the following property: let \( z \in G \) and write \( z = x + y \), with \( x \) and \( y \in \mathbb{N} \); then \( x \in G \) or \( y \in G \). There is a bijective map between the set of numerical semigroups and the set of gapsets given by \( S \mapsto \mathbb{N}_0 \setminus S \). Thus, a gapset is the set of gaps of some numerical semigroup and one can define some invariants of a gapset by using the invariants of its complement in \( \mathbb{N}_0 \). For instance, the genus, the multiplicity, the conductor and the depth of a gapset \( G \) are \( g(G) := |G| \), \( m(G) := \min\{s \in \mathbb{N} : s \notin G\} \), \( c(G) := \min\{s \in \mathbb{N} : s + n \notin G, \forall n \in \mathbb{N}_0\} \) and \( q(G) := \left\lceil \frac{c(G)}{m(G)} \right\rceil \), respectively, where \( \mathbb{N} \) denotes the set of positive integers.

A central problem in numerical semigroup theory is Bras-Amorós’ conjecture, which was originally stated with three items (see [1]). It consists in understanding the behaviour of the sequence \( (n_g) \) where \( n_g \) denotes the number of gapsets (or numerical semigroups) with a fixed genus \( g \). Two of the items of the conjecture are about asymptotic behaviour for \( n_g \) ((1) \( \lim_{g \to \infty} \frac{n_g}{n_{g-1}} = \phi \), the golden ratio, and (2) \( \lim_{g \to \infty} \frac{n_{g+1} + n_{g-2}}{n_g} = 1 \); observe that (1) \( \Rightarrow \) (2)) and were proved by Zhai [5]. The key ingredient of the proof was observing that “almost all” gapsets with a fixed genus have depth at most 3. The only item of Bras-Amorós’ conjecture that remains as an open problem is “is it true that \( n_g + n_{g+1} \leq n_{g+2} \), for all \( g \)?”. Also, a weaker version of this conjecture, namely “is \( (n_g) \) a non-decreasing sequence?”, is still an open problem. See [3] for more details.

Eliahou and Fromentin [2] studied a related problem to this one. They proved a remarkable property about the behaviour of the sequence \( (n'_g) \), where \( n'_g \) denotes the number of gapsets with a fixed genus \( g \) and depth at most 3. It is stated as follows.

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\(^1\)This sequence is registered as A007323 at OEIS.
Theorem 1 ([2]). Let \((n'_{g})\) be the sequence of the number of gapsets with genus \(g\) and depth at most 3. Then \(n'_{g-1} + n'_{g-2} \leq n'_{g} \leq n'_{g-1} + n'_{g-2} + n'_{g-3}\).

In that paper, the authors prove Theorem 1 in two parts. The first inequality is a consequence of the construction of two injective maps: one of them has domain given by the set of gapsets with genus \(g - 1\) and depth at most 3 and the other one has domain given by the set of gapsets with genus \(g - 2\) and depth at most 3; both of them have counter-domain given by the set of gapsets with genus \(g\) and depth at most 3 and the images of those maps are disjoint sets. For the second inequality, the authors split the set of gapsets with genus \(g\) and depth at most 3 into three disjoint parts, \(\Gamma'_{1}(g) \cup \Gamma'_{2}(g) \cup \Gamma'_{3}(g)\), and they construct three injective maps, where \(\Gamma'_{k}(g)\) is mapped into the set of gapsets with genus \(g - k\) and depth at most 3, for \(k \in \{1, 2, 3\}\).

In this note, we give an alternative proof for Theorem [1]. In section 2, we introduce the Kunz coordinates for gapsets and we prove that the depth of a gapset is its largest coordinate. In section 3, we prove Theorem [1] by using the Kunz coordinates of a gapset and functions which have been considered in [2]. We prefer the gapset language to write this note, but it could also be done using numerical semigroup theory.

2. The Kunz coordinates of a gapset

In this section, we introduce the Apéry set and the Kunz coordinates of a gapset \(G\). This definition arises in a natural way when we consider the numerical semigroup \(\mathbb{N}_{0} \setminus G\).

Let \(S\) be a numerical semigroup with multiplicity \(m\). The Apéry set of \(S\) (on \(m\)) is defined as \(\text{Ap}(S) = \{w_{0}, w_{1}, \ldots, w_{m-1}\}\), where \(w_{i} = \min\{s \in S : s \equiv i \pmod{m}\}\), for \(i \in [0, m - 1]\). Notice that \(w_{0} = 0\) and there is a \(k_{i} \in \mathbb{N}\) such that \(w_{i} = mk_{i} + i\). The Kunz coordinates of \(S\) (on \(m\)) is \(\text{Kunz}(S) = (k_{1}, k_{2}, \ldots, k_{m-1})\), where \(k_{i} = (w_{i} - i)/m\).

Now, we transfer this terminology to a gapset \(G\) with multiplicity \(m\); the Apéry set of \(G\) (on \(m\)) and the Kunz coordinates of \(G\) (on \(m\)) are given by \(\text{Ap}(G) := \text{Ap}(\mathbb{N}_{0} \setminus G)\) and \(\text{Kunz}(G) := \text{Kunz}(\mathbb{N}_{0} \setminus G)\), respectively.

Proposition 2. Let \(G\) be a gapset with multiplicity \(m\), such that \(\text{Ap}(G) = \{w_{0}, w_{1}, \ldots, w_{m-1}\}\). Then \(w_{i} = m + \max\{z \in G : z \equiv i \pmod{m}\}\) if \(i \neq 0\) and \(w_{0} = 0\).

Proof. First, observe that \(w_{0} = 0\). Also, by the definition of \(w_{i}\), we conclude that \(G \cap \{n \in \mathbb{N} : n \equiv i \pmod{m}\} = \{i, i + m, \ldots, i + (k_{i} - 1)m\}\), since \(mx + i \notin G\) for all \(x \geq k_{i}\). Hence, \(w_{i} = i + (k_{i} - 1)m + m\) and the result follows. \(\square\)

As a consequence, we obtain:

Corollary 3. Let \(G\) be a gapset with multiplicity \(m\), such that \(\text{Kunz}(G) = (k_{1}, k_{2}, \ldots, k_{m-1})\). Then \(k_{i} = \#\{z \in G : z \equiv i \pmod{m}\}\) for all \(i \in [1, m - 1]\).
Proof. It follows from the fact that \( G \cap \{ n \in \mathbb{N} : n \equiv i \text{ (mod } m) \} = \{ i, i + m, \ldots, i + (k_i - 1)m \} \) has \( k_i \) elements. \( \square \)

Example 4. The gapset \( G = \{1, 2, 4, 5, 8, 11\} \) has multiplicity 3. In this case, \( \text{Ap}(G) = \{0, 7, 14\} \) and \( \text{Kunz}(G) = (2, 4) \).

One can characterize numerical semigroups with multiplicity \( m \) in terms of its Kunz coordinates, namely \( \text{Kunz}(S) = (k_1, k_2, \ldots, k_{m-1}) \). As a matter of fact, a tuple in \( \mathbb{N}^{m-1} \) is the Kunz coordinates of some numerical semigroup with multiplicity \( m \) if, and only if, it satisfies the following system of inequalities (cf. [4]):

\[
\begin{aligned}
X_i &\in \mathbb{N} \\
X_i + X_j &\geq X_{i+j}, \quad \text{for } 1 \leq i \leq j \leq m-1; i+j < m; \\
X_i + X_j + 1 &\geq X_{i+j-m}, \quad \text{for } 1 \leq i \leq j \leq m-1; i+j > m
\end{aligned}
\]

(1)

In particular, we conclude that the Kunz coordinates of a gapset with multiplicity \( m \) also must satisfy the system (1).

Example 5. The tuple \((2, 3, 3, 1)\) is the Kunz coordinates of the gapset \( \{1, 2, 3, 4, 6, 7, 8, 12, 13\} \). However the tuple \((1, 3, 3, 2)\) is not the Kunz coordinates of any gapset, since \( k_1 + k_1 = 2 < 3 = k_2 \).

The canonical partition of a gapset \( G \) was introduced in [2] as

\[
G = G_0 \cup G_1 \cup \ldots \cup G_{q-1},
\]

where \( G_0 = [1, m-1] \cap \mathbb{Z} \) and \( G_{i+1} \subseteq G_i + m \). Basically, it is a clipping of the set \( G \) into \( q \) parts where each part lies in an interval of integers of the type \([am+1, (a+1)m-1] \cap \mathbb{Z}\), for some \( a \in \mathbb{N} \).

Proposition 6. Let \( G \) be a gapset, with \( \text{Kunz}(G) = (k_1, k_2, \ldots, k_{m-1}) \), genus \( g \) and depth \( q \). Then \( g = \sum_{i=1}^{m-1} k_i \) and \( q = \max\{k_i\} \).

Proof. By Corollary 3, \( G \cap \{ n \in \mathbb{N} : n \equiv i \text{ (mod } m) \} \) has \( k_i \) elements, if \( i \neq 0 \) and \( G \cap m\mathbb{N} = \emptyset \). Thus the formula for the the genus can be obtained by summing up the coordinates of \( \text{Kunz}(G) \). Consider the canonical partition of \( G \), \( G_0 \cup G_1 \cup \ldots \cup G_{q-1} \) and let \( x \in G_{q-1} \) (that exists, since the depth of \( G \) is \( q \)). In particular, \( x - \ell m \in G \) for all \( \ell \in \mathbb{N}_0 \) such that \( x - \ell m > 0 \) and there is exactly one element in each \( G_i \) that is congruent to \( x \) modulo \( m \). Using Corollary 3 again, we conclude that the depth of \( G \) (which is also the quantity of parts of the canonical partition of \( G \)) coincides with the quantity of elements that are congruent to \( x \) modulo \( m \). Hence, \( q = k_x \text{ (mod } m) \) and we are done. \( \square \)
3. An alternative proof for Theorem $\Box$

In this section, we give an alternative proof for Theorem $\Box$. Here, we deal with the Kunz coordinates of a gapset. It is important to observe that the functions that we consider are the same as the ones considered in [2]. However, we look at how the Kunz coordinates are changed under those maps.

Proof. Let $\Gamma'(g)$ be the set of gapsets with genus $g$ and depth at most 3. First, we identify the gapsets of $\Gamma'(g)$ with tuples such that its coordinates are 1, 2 or 3, the sum of its coordinates is $g$ and that satisfies the system $\Box$.

Now we prove the first inequality by considering the following functions: $f_1 : \Gamma'(g - 1) \to \Gamma'(g)$ and $f_2 : \Gamma'(g - 2) \to \Gamma'(g)$, which are described by the Kunz coordinates. If $G$ has multiplicity $m$ and $\text{Kunz}(G) = (k_1, k_2, \ldots, k_{m-1})$, then we define $\text{Kunz}(f_1(G)) = (k_1, k_2, \ldots, k_{m-1}, 1)$ and $\text{Kunz}(f_2(G)) = (k_1, k_2, \ldots, k_{m-1}, 2)$. Notice that both functions are injective and their images are disjoint sets since the last coordinate of $\text{Kunz}(f_1(G))$ is always 1 and the last coordinate of $\text{Kunz}(f_2(G))$ is always 2. Also, if $G \in \Gamma'(g - 1)$ and $\text{Kunz}(G) = (k_1, \ldots, k_{m-1})$, then the genus of $G$ is $\sum_{i=1}^{m-1} k_i = g - 1$, $k_i \in \{1, 2, 3\}$, for all $i$, the genus of $f_1(G)$ is $g$ and its depth is at most 3; if $G \in \Gamma'(g - 2)$ and $\text{Kunz}(G) = (k_1, \ldots, k_{m-1})$, then the genus of $G$ is $\sum_{i=1}^{m-1} k_i = g - 2$, $k_i \in \{1, 2, 3\}$, for all $i$, the genus of $f_2(G)$ is $g$ and its depth is at most 3. It remains to show that $(k_1, \ldots, k_{m-1}, k_m)$, where $k_m \in \{1, 2\}$ satisfy the system $\Box$. If $1 \leq i \leq j \leq m - 1$ and $i + j < m$, then $k_i + k_j \geq k_{i+j}$ (by hypothesis). If $1 \leq i \leq j \leq m - 1$ and $i + j = m$, then $k_i + k_j \geq 2$ and $k_m \leq 2$; thus $k_i + k_j \geq k_{i+j} = k_m$. Finally, if $1 \leq i \leq j \leq m$ and $i + j > m + 1$, then $k_i + k_j + 1 \geq 3$ and $k_{i+j-m} \leq 3$; thus $k_i + k_j + 1 \geq k_{i+j-m}$.

Now we prove the second inequality by considering three functions that have as domains subsets of $\Gamma'(g)$. The first one has as domain the set of gapsets such that its Kunz coordinates has last coordinate equals one, namely $\Gamma'_1(g)$, the second one has as domain the set of gapsets such that its Kunz coordinates has last coordinate equals two, namely $\Gamma'_2(g)$, and the third one has as domain the set of gapsets such that its Kunz coordinates has last coordinate equals three, namely $\Gamma'_3(g)$. Notice that it splits the set $\Gamma'(g)$ into three (disjoint) parts. Consider the following functions $h_1 : \Gamma'_1(g) \to \Gamma'(g - 1), h_2 : \Gamma'_2(g) \to \Gamma'(g - 2) \text{ and } h_3 : \Gamma'_3(g) \to \Gamma'(g - 3)$, which are described by the Kunz coordinates. If $\text{Kunz}(G) = (k_1, k_2, \ldots, k_{m-1}, k_m)$, with $k_m \in \{1, 2, 3\}$, then we define $\text{Kunz}(h_1(G)) = \text{Kunz}(h_2(G)) = \text{Kunz}(h_3(G)) = (k_1, k_2, \ldots, k_{m-1})$. Notice that those functions are injective and if $G \in \Gamma'(g)$, then the genus of $h_1(G), h_2(G)$ and $h_3(G)$ are $g - 1, g - 2$ and $g - 3$, respectively. It remains to show that $(k_1, \ldots, k_{m-1})$ satisfy the system $\Box$. If $1 \leq i \leq j \leq m - 1$ and $i + j < m$, then $k_i + k_j \geq k_{i+j}$ (by hypothesis). Finally, if $1 \leq i \leq j \leq m - 1$ and $i + j > m$, then $k_i + k_j + 1 \geq 3$ and $k_{i+j-m} \leq 3$; thus $k_i + k_j + 1 \geq k_{i+j-m}$ and we are done. $\square$
We observe that, in general, all the tuples with coordinates 1, 2 or 3 satisfy the condition $X_i + X_j + 1 \geq X_{i+j}$ of the system (\Pi). For larger values of one (or more) coordinate, it does not hold true and it can be a reason for the difficulty to construct (injective) maps that preserve the gapset property, at least for gapsets with depth greater than 3. In particular, the functions considered in this note cannot be extended to gapsets with fixed genus and larger depths. For instance, if Kunz($G$) = (4, 2, 1), then the tuples obtained by adding 1 and 2 as the last coordinate are (4, 2, 1, 1) and (4, 2, 1, 2), respectively, but they are not the Kunz coordinates of any gapset (in both cases, $2k_3 + 1 = 3 < 4 = k_1$). Also, if Kunz($G$) = (4, 1, 4), then the tuple obtained removing the last coordinate is (4, 1), but it is not the Kunz coordinates of any gapset ($2k_2 + 1 = 3 < 4 = k_1$).

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Universidade de Brasília, Faculdade do Gama, Brasília, DF, Brazil

*Email address: matheusbernardini@unb.br*

Universidade de Brasília, Faculdade do Gama, Brasília, DF, Brazil

*Email address: patrickmelo13@gmail.com*