Well-posedness of Non-autonomous Evolutionary Inclusions.

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Abstract. A class of non-autonomous differential inclusions in a Hilbert space setting is considered. The well-posedness for this class is shown by establishing the mappings involved as maximal monotone relations. Moreover, the causality of the so established solution operator is addressed. The results are exemplified by the equations of thermoplasticity with time dependent coefficients and by a non-autonomous version of the equations of viscoplasticity with internal variables.

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1 Introduction

As it was pointed out in [15], the classical equations of mathematical physics share a common form, namely

$$\partial_0 v + Au = f,$$

where\( \partial_0 \) denotes differentiation with respect to time and \( A : D(A) \subseteq H \to H \) is a suitable linear operator on a Hilbert space \( H \). The equation needs to be completed by a constitutive relation linking the unknowns \( u \) and \( v \). We consider a certain class of such constitutive relations, which actually occurs frequently in mathematical physics. It can be written in the form

$$\partial_0 v = \partial_0 M_0 u + M_1 u,$$

where \( M_0, M_1 \in L(H) \) with \( M_0 \) selfadjoint and strictly positive definite on its range and \( \Re M_1 = \frac{1}{2}(M_1 + M_1^*) \) is strictly positive definite on the kernel of \( M_0 \) (in [15] this case is called the (P)-degenerate case, since it typically occurs for parabolic-type problem). Thus, we end up with an equation of the form

$$\partial_0 (M_0 + M_1 + A) u = f,$$  \( (1) \)

whose well-posedness was proved in [15] in the case of a skew-selfadjoint operator \( A \). Later on the well-posedness of problems of the form (1) was shown in the case of \( A \) being a maximal monotone operator in [22, 23] (for the topic of maximal monotone operators we refer to the monographs [4, 8, 20]). In [18] a non-autonomous version of (1) was considered in the sense that the operators \( M_0 \) and \( M_1 \) were replaced by operator-valued functions \( M_0, M_1 : \mathbb{R} \to L(H) \) and the well-posedness of the corresponding evolutionary problem

$$(\partial_0 M_0(\cdot) + M_1(\cdot) + A) u = f$$

was shown in the case of a skew-selfadjoint operator \( A \). The aim of this article is to generalize this well-posedness result to the case of \( A \) being a maximal monotone relation, i.e. providing a solution theory for differential inclusions of the form

$$(u, f) \in \partial_0 M_0(\cdot) + M_1(\cdot) + A.$$  \( (2) \)

In the literature we find several approaches to the well-posedness of non-autonomous differential equations and inclusions and, depending on the techniques involved, several notions of solutions. For example, one classical approach, established in a general Banach spaces setting, is the theory of evolution families introduced by Kato in [10] in the case of evolution equations and generalized by Crandall and Pazy in [6] to evolution inclusions. This strategy, which carries over the idea of semigroup theory to the case of non-autonomous problems, requires that the differential inclusion is given as a Cauchy-problem, i.e. an inclusion of the form

$$(u(t), f(t)) \in \partial_0 A(t),$$  \( (3) \)

which corresponds to the case of an invertible mapping \( M_0 \) in our setting. However, in the approach presented here, \( M_0 \) is allowed to have a non-trivial kernel, which makes the inclusion (2) to be a differential-algebraic problem, which in general may not be accessible by the theory of semigroups or evolution families in a straightforward way. Another approach to problems
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of the form (3) is to approximate the differential inclusion by difference inclusions, i.e. one replaces the derivative with respect to time by suitable difference quotients. The corresponding solutions of the difference inclusion then uniformly converge to a so-called “weak” solution of (3) (see e.g. [7, 11]). Another notion of solution of differential inclusions of the form (3) are so-called “integral solutions”, introduced by Bénilan [3] for autonomous inclusions and generalized in [11] to non-autonomous problems, which satisfy a certain integral inequality. Under suitable assumptions on $A(t)$ one can show that the notion of “weak” solutions, i.e. the limit of solutions of the difference inclusions, coincides with the notion of “integral solutions”.

We emphasize that in all classical approaches, the operator $A$ is time-dependent. However, looking at concrete examples, in many cases the coefficients depending on time while the spatial differential operator is indeed time-independent. Since the coefficients can usually be incorporated in the operators $M_0$ and $M_1$, we are led to assume that these operators depend on time while $A$ is time-independent. This point of view has the advantage that the time dependent operators are bounded and thus, we avoid the technicalities arising when dealing with time-dependent unbounded operators, whose domain may also depend on time.

In our approach we consider the operator $\partial_0 M_0(\cdot) + M_1(\cdot) + A$ on the right hand side of (2) as an object in time and space. More precisely, the operators involved are defined on an exponentially weighted $L_2$-space of Hilbert-space valued functions, and we are seeking for solutions $u$ of (2) in the sense that

$$(u, f) \in \partial_0 M_0(\cdot) + M_1(\cdot) + A,$$

where the closure is taken with respect to the topology on this $L_2$-space of Hilbert space valued functions. This can be seen as an $L_2$-analogue to the notion of weak solutions due to Brezis (see [4, Definition 3.1]). Thus, the well-posedness of (2) relies on the invertibility of $\partial_0 M_0(\cdot) + M_1(\cdot) + A$, which will be shown by proving that $\partial_0 M_0(\cdot) + M_1(\cdot) + A - c$ defines a maximal monotone operator in time and space for some $c > 0$. For doing so, we establish the time derivative $\partial_0$ in an exponentially weighted $L_2$-space in order to obtain a normal, boundedly invertible operator (cf. [19, 16]). The operator $\partial_0 M_0(\cdot) + M_1(\cdot)$ turns out to be strictly maximal monotone as an operator in time and space and hence, the well-posedness of (2) can be shown by applying well-known perturbation results for maximal monotone relations.

In addition to the well-posedness, we address the question of causality (see e.g. [12] or [25] for an alternative definition), which is a characteristic property for processes evolving in time. Roughly speaking, causality means that the behavior of the solution $u$ of (2) should not depend on the future behavior of the given right hand side (for the exact definition in our framework see Definition 3.12).

The article is structured as follows. In Section 2 we recall the definition of the time derivative $\partial_0$ and some basic facts on maximal monotone operators in Hilbert spaces. Section 3 is devoted to the proof of our main theorem (Theorem 3.4), stating the well-posedness result for problems of the form (2) and the causality of the corresponding solution operator. In the concluding section we apply our results to two examples from the theory of plasticity. The first one deals with a non-autonomous version of the equations of thermoplasticity, where the inelastic part of the strain and the stress are coupled by an differential inclusion. In the second one we consider the non-autonomous equations of viscoplasticity, where the inelastic strain is given in terms of an internal variable (for constitutive equations with internal variables we refer to the monograph [1]).
In the following let \( H \) be a complex Hilbert space with inner product \( \langle \cdot | \cdot \rangle \), assumed to be linear in the second and conjugate linear in the first argument and we denote by \(| \cdot | \) the induced norm.

## 2 Preliminaries

### 2.1 The time derivative

Following the strategy in [10], we introduce the derivative as a normal, boundedly invertible operator in an exponentially weighted \( L_2 \)-space. For the proofs of the forthcoming statements we refer to [9, 10]. For \( \varrho \in \mathbb{R} \) we define the space \( H_{\varrho,0}(\mathbb{R}; H) \) as the completion of \( C_c^\infty(\mathbb{R}; H) \) – the space of arbitrarily often differentiable functions with compact support in \( \mathbb{R} \) taking values in \( H \) – with respect to the norm induced by the inner product

\[
\langle \phi | \psi \rangle_{H_{\varrho,0}(\mathbb{R}; H)} := \int_{\mathbb{R}} \langle \phi(t) | \psi(t) \rangle e^{-2\varrho t} \, dt \quad (\phi, \psi \in C_c^\infty(\mathbb{R}; H)).
\]

Note that in the case \( \varrho = 0 \), this is just the usual \( L_2 \)-space of (equivalence classes of) square integrable functions with values in \( H \), i.e. \( H_{0,0}(\mathbb{R}; H) = L_2(\mathbb{R}; H) \). On the Hilbert space \( H_{\varrho,0}(\mathbb{R}; H) \) we define the derivative \( \partial_{\varrho,0} \) as the closure of the linear operator

\[
C_c^\infty(\mathbb{R}; H) \subseteq H_{\varrho,0}(\mathbb{R}; H) \rightarrow H_{\varrho,0}(\mathbb{R}; H)
\]

\[
\phi \mapsto \phi'.
\]

Then \( \partial_{\varrho,0} \) is a normal operator with \( \partial_{\varrho,0}^* = -\partial_{\varrho,0} + 2\varrho \) and consequently \( \Re \partial_{\varrho,0} = \varrho \). In the case \( \varrho = 0 \) this operator coincides with the usual weak derivative on \( L_2(\mathbb{R}; H) \) with domain \( H^1(\mathbb{R}; H) = W_2^1(\mathbb{R}; H) \). For \( \varrho \neq 0 \) the operator \( \partial_{\varrho,0} \) has a bounded inverse with \( \| \partial_{\varrho,0}^{-1} \|_{L(H_{\varrho,0}(\mathbb{R}; H))} = \frac{1}{|\varrho|} \) (see [9, Corollary 2.5]). More precisely, the inverse is given by

\[
(\partial_{\varrho,0}^{-1} u)(t) = \begin{cases} 
\int_{-\infty}^t u(s) \, ds & \text{if } \varrho > 0, \\
\int_{t}^{\infty} u(s) \, ds & \text{if } \varrho < 0 
\end{cases} \quad (u \in H_{\varrho,0}(\mathbb{R}; H), \ t \in \mathbb{R} \ a.e.).
\]

(4)

Since we are interested in the forward causal case (see Definition 3.12 below), throughout we may assume that \( \varrho > 0 \). Next, we state an approximation result for elements in the domain of \( \partial_{\varrho,0} \). For this we denote by \( \tau_h \) for \( h \in \mathbb{R} \) the translation operator on \( H_{\varrho,0}(\mathbb{R}; H) \) given by \( (\tau_h u)(t) := u(t + h) \) for \( u \in H_{\varrho,0}(\mathbb{R}; H) \) and almost every \( t \in \mathbb{R} \). Moreover we define the Hilbert space \( H_{\varrho,1}(\mathbb{R}; H) \) as the domain of \( \partial_{\varrho,0} \) equipped with the inner product

\[
\langle u | v \rangle_{H_{\varrho,1}(\mathbb{R}; H)} = \langle \partial_{\varrho,0}u | \partial_{\varrho,0}v \rangle_{H_{\varrho,0}(\mathbb{R}; H)} \quad (u, v \in D(\partial_{\varrho,0})).
\]

**Proposition 2.1.** Let \( u \in H_{\varrho,0}(\mathbb{R}; H) \). Then \( u \in H_{\varrho,1}(\mathbb{R}; H) \) if and only if the set of difference quotients \( \{ \frac{1}{h} (\tau_h u - u) | \ h \in ]0,t[ \} \) is bounded in \( H_{\varrho,0}(\mathbb{R}; H) \) for some \( t > 0 \). Moreover, for \( u \in H_{\varrho,1}(\mathbb{R}; H) \) we have

\[
\frac{1}{h} (\tau_h u - u) \rightarrow \partial_{\varrho,0}u \text{ in } H_{\varrho,0}(\mathbb{R}; H) \text{ as } h \rightarrow 0 +.
\]
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Proof. For $h > 0$ we define the operator

$$D_h : H_{p,1}(\mathbb{R}; H) \to H_{p,0}(\mathbb{R}; H)$$

$$u \mapsto \frac{1}{h} (\tau_h u - u).$$

Obviously, this operator is linear and we estimate

$$|D_h u|_{H_{p,0}(\mathbb{R}; H)}^2 = \int_\mathbb{R} \left| \frac{1}{h} (u(t+h) - u(t)) \right|^2 e^{-2ht} \, dt$$

$$= \int_\mathbb{R} \frac{1}{h^2} \left( \int_0^h \partial_0 \phi (t+s) \, ds \right)^2 e^{-2ht} \, dt$$

$$\leq \int_\mathbb{R} \frac{1}{h} \left( \int_0^h |\partial_0 \phi (t+s)|^2 \, ds \right) e^{-2ht} \, dt$$

$$= \frac{1}{h} \int_0^h \int_\mathbb{R} |\partial_0 \phi (t+s)|^2 e^{-2ht} \, dt \, ds$$

$$\leq e^{2ht} |u|_{H_{p,1}(\mathbb{R}; H)}^2$$

for each $u \in H_{p,1}(\mathbb{R}; H)$. Thus, the family $(D_h)_{h \in [0,t]}$ is uniformly bounded in the space

$L(H_{p,1}(\mathbb{R}; H), H_{p,0}(\mathbb{R}; H))$ for every $t > 0$. Moreover, for $\phi \in C_c^\infty(\mathbb{R}; H)$ we have

$$|D_h \phi - \partial_0 \phi|_{H_{p,0}(\mathbb{R}; H)} \to 0 \quad (h \to 0+)$$

by the dominated convergence theorem. Since $C_c^\infty(\mathbb{R}; H)$ is dense in $H_{p,1}(\mathbb{R}; H)$ we derive that

$$D_h u \to \partial_0 \phi$$

in $H_{p,0}(\mathbb{R}; H)$ as $h \to 0+$, if $u \in H_{p,1}(\mathbb{R}; H)$. Assume now that $\{ \frac{1}{h} (\tau_h u - u) \mid h \in [0,t] \}$ is bounded in $H_{p,0}(\mathbb{R}; H)$ for some $t > 0$. Then we can choose a sequence $(h_n)_{n \in \mathbb{N}}$ in $[0,t]$ such that $h_n \to 0$ as $n \to \infty$ and

$$(\frac{1}{h_n} (\tau_n u - u))_{n \in \mathbb{N}}$$

is weakly convergent. We denote its weak limit by $w \in H_{p,0}(\mathbb{R}; H)$. Then we compute for $\phi \in C_c^\infty(\mathbb{R}; H)$

$$\langle w | \phi \rangle_{H_{p,0}(\mathbb{R}; H)} = \lim_{n \to \infty} \int_\mathbb{R} \frac{1}{h_n} (u(t+h_n) - u(t)) \phi(t) e^{-2ht} \, dt$$

$$= \lim_{n \to \infty} \int_\mathbb{R} \langle u(t) | \frac{1}{h_n} (\phi(t-h_n) e^{2ht_n} - \phi(t)) \rangle e^{-2ht} \, dt$$

$$= \int_\mathbb{R} \langle u(t) | - \phi'(t) + 2\phi(t) \rangle e^{-2ht} \, dt.$$
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by the dominated convergence theorem. Thus, we have for all \( \phi \in C^\infty_c(\mathbb{R}; H) \)

\[
(w|\phi)_{H_{e,0}(\mathbb{R};H)} = (u|\partial_{0,e}^\phi)_{H_{e,0}(\mathbb{R};H)}.
\]

Since \( C^\infty_c(\mathbb{R}; H) \) is a core for \( \partial_{0,e}^\phi \) we obtain \( u \in H_{e,1}(\mathbb{R}; H) \).

\[
\square
\]

2.2 Maximal monotone relations

In this section we recall some basic results on maximal monotone operators. Instead of con-
sidering the operators as set-valued mappings, we prefer to use the notion of binary relations.
The proofs of the results can be found for instance in the mono graphs \([4, 8]\).

**Definition 2.2.** A (binary) relation \( A \subseteq H \oplus H \) is called monotone, if for all pairs \( (u,v), (x,y) \in A \) the inequality

\[
\Re \langle u - x | v - y \rangle \geq 0
\]

holds. Moreover, \( A \) is called maximal monotone, if \( A \) is monotone and there exists no proper monotone extension of \( A \), i.e. for every monotone \( B \subseteq H \oplus H \) with \( A \subseteq B \) it follows that \( A = B \).

In order to deal with relations we fix some notation, which will be used in the forthcoming sections.

**Definition 2.3.** For two relations \( A, B \subseteq H \oplus H \) and \( \lambda \in \mathbb{C} \) we define the relation \( \lambda A + B \) by

\[
\lambda A + B := \{ (x, \lambda y + z) | (x, y) \in A, (x, z) \in B \}.
\]

The inverse relation \( A^{-1} \) is given by

\[
A^{-1} := \{ (y, x) | (x, y) \in A \}.
\]

Furthermore, for a subset \( M \subseteq H \) we define the pre-set of \( M \) under \( A \) by

\[
[M]A := \{ x \in H | \exists y \in M : (x, y) \in A \}
\]

and the post-set\(^1\) of \( M \) under \( A \) by

\[
A[M] := \{ y \in H | \exists x \in M : (x, y) \in A \}.
\]

Moreover, \( A \) is called bounded, if for every bounded set \( M \subseteq H \) the post-set \( A[M] \) is bounded.

In 1962 G. Minty proved the following characterization of maximal monotonicity.

**Theorem 2.4** (G. Minty, \([13]\)). Let \( A \subseteq H \oplus H \) be a monotone relation. Then the following statements are equivalent:

(i) \( A \) is maximal monotone,

\(^1\)These notions generalize the well-known concepts of pre-image and image in the case of mappings. However, since it seems to be inappropriate to speak of images in case of a relation, we choose the notions pre- and post-set.
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(iii) there exists \( \lambda > 0 \) such that \( (1 + \lambda A)[H] = H \),

(iii) for every \( \lambda > 0 \) we have \( (1 + \lambda A)[H] = H \).

In order to formulate inclusions of the form (2) we need to extend a relation \( A \subseteq H \oplus H \) to a relation on \( H_{q,0}(\mathbb{R}; H) \) for \( q > 0 \). This is done by setting

\[
A_q := \{ (u,v) \in H_{q,0}(\mathbb{R}; H) \oplus H_{q,0}(\mathbb{R}; H) \mid (u(t),v(t)) \in A \text{ for almost every } t \in \mathbb{R} \}.
\]

The relation \( A_q \) then interchanges with the translation operator \( \tau_h \) for \( h \in \mathbb{R} \) in the sense that

\[
(u,v) \in A_q \iff \forall h \in \mathbb{R} : (\tau_h u, \tau_h v) \in A_q.
\]

We recall the following result from [4] on extensions of maximal monotone relations.

Lemma 2.5 ([4, Example 2.3.3]). Let \( A \subseteq H \oplus H \) be maximal monotone and \( q \geq 0 \). If \( (0,0) \in A \), then \( A_q \) is maximal monotone, too.

If \( A \subseteq H \oplus H \) is a monotone relation it follows that \( (1 + \lambda A)^{-1} \) is a Lipschitz-continuous mapping with Lipschitz-constant less than or equal to 1 for each \( \lambda > 0 \). Furthermore, if \( A \) is maximal monotone, the mapping \( (1 + \lambda A)^{-1} \) is defined on the whole space \( H \) by Theorem 2.4.

Definition 2.6. Let \( A \subseteq H \oplus H \) be maximal monotone and \( \lambda > 0 \). The Yosida approximation \( A_\lambda \) of \( A \) is defined by

\[
A_\lambda := \lambda^{-1} \left( 1 - (1 + \lambda A)^{-1} \right).
\]

The mapping \( A_\lambda \) is monotone and Lipschitz-continuous with a Lipschitz-constant less than or equal to \( \lambda^{-1} \) (see [4, Proposition 2.6]).

We close this subsection by stating some perturbation results for maximal monotone relations, which provide the key argument for our proof of well-posedness of evolutionary inclusions of the form (2).

Proposition 2.7. Let \( A \subseteq H \oplus H \) be maximal monotone and \( B : H \to H \) be Lipschitz-continuous. Furthermore, let \( A + B \) be monotone. Then \( A + B \) is maximal monotone.

Proof. If \( B \) is constant the assertion holds trivially. Assume now \( B \) is not constant. By Theorem 2.4 it suffices to check that there exists \( \lambda > 0 \) such that \( (1 + \lambda(A + B))[H] = H \). Let \( 0 < \lambda < |B|_{\text{Lip}}^{-1} \) and \( y \in H \). Then by the contraction mapping theorem there exists a fixed point \( x \in H \) of the mapping

\[
H \ni u \mapsto (1 + \lambda A)^{-1}(y - \lambda B(u)).
\]

This fixed point satisfies

\[
(x,y) \in 1 + \lambda(A + B).
\]
**Corollary 2.8.** Let \( B : H \to H \) be monotone and Lipschitz-continuous. Then \( B \) is maximal monotone.

**Proof.** This follows from Proposition 2.7 with \( A = 0 \). \(\square\)

**Corollary 2.9 ([4 Lemme 2.4]).** Let \( A \subseteq H \oplus H \) be maximal monotone and \( B : H \to H \) be a monotone, Lipschitz-continuous mapping. Then \( A + B \) is maximal monotone.

**Proof.** The statement follows from Proposition 2.7 since the sum of two monotone relations is again monotone. \(\square\)

Let \( A, B \subseteq H \oplus H \) be maximal monotone. Then, by Corollary 2.9, the relation \( A + B_\lambda \) is maximal monotone for each \( \lambda > 0 \) and thus, for \( y \in H \) there exists a unique \( x_\lambda \in H \) such that

\[
(x_\lambda, y) \in 1 + A + B_\lambda,
\]

according to Minty’s theorem (Theorem 2.4). Using this observation, one can show the following perturbation result.

**Proposition 2.10 ([8 Proposition 3.1]).** Let \( A, B \subseteq H \oplus H \) be maximal monotone with \([H]A \cap [H]B \neq \emptyset\) and \( y \in H \). Moreover, for \( \lambda > 0 \) let \( x_\lambda \in H \) such that \((x_\lambda, y) \in 1 + A + B_\lambda\). Then, there exists \( x \in H \) with \((x, y) \in 1 + A + B\) if and only if the family \((B_\lambda(x_\lambda))_{\lambda \in [0, \infty]}\) is bounded.

**Corollary 2.11 ([21 Proposition 1.22]).** Let \( A, B \subseteq H \oplus H \) be maximal monotone with \([H]A \cap [H]B \neq \emptyset\). Moreover, assume that \( B \) is bounded. Then \( A + B \) is maximal monotone.

### 3 Solution theory

In this section we provide a solution theory for differential inclusions of the form (2). More precisely we show that the problem

\[
(u, f) \in \partial_{0, \varrho}M_0(m) + M_1(m) + A_\varrho
\]

is well-posed in \( H_{\varrho,0}(\mathbb{R}; H) \) for sufficiently large \( \varrho > 0 \) and that the corresponding solution operator

\[
\left( \partial_{0, \varrho}M_0(m) + M_1(m) + A_\varrho \right)^{-1} : H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)
\]

is causal. Throughout let \( A \subseteq H \oplus H \) be maximal monotone with \((0, 0) \in A\) and \( M_0, M_1 : \mathbb{R} \to L(H) \) be strongly measurable and bounded functions. Then we denote the associated multiplication operators on \( H_{\varrho,0}(\mathbb{R}; H) \) by \( M_0(m) \) and \( M_1(m) \), respectively, where \( m \) serves as a reminder for the “multiplication by the argument”, i.e.

\[
M_0(m) : H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)
\]

with \((M_0(m)u)(t) := M_0(t)u(t)\) for every \( u \in H_{\varrho,0}(\mathbb{R}; H) \) and almost every \( t \in \mathbb{R} \) and analogously for \( M_1(m) \). Following [13] we require that the pair \((M_0, M_1)\) satisfies the following conditions.

\[^{4}\text{Note that due to the boundedness of } M_0 \text{ and } M_1 \text{ the operators } M_0(m) \text{ and } M_1(m) \text{ are bounded on } H_{\varrho,0}(\mathbb{R}; H) \text{ for each } \varrho > 0 \text{ with } \|M_0(m)\|_{L(H)} \leq |M_0|_{\infty} \text{ and } \|M_1(m)\|_{L(H)} \leq |M_1|_{\infty}.\]

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Conditions.

(a) $M_0$ is Lipschitz-continuous and for every $t \in \mathbb{R}$ the operator $M_0(t)$ is selfadjoint.

(b) There exists a set $N \subseteq \mathbb{R}$ of Lebesgue-measure 0, such that for every $x \in H$ the mapping

$$\mathbb{R} \setminus N \ni t \mapsto M_0(t)x$$

is differentiable.\(^5\)

(c) The kernel of $M_0(t)$ is independent of $t \in \mathbb{R}$, i.e. $N(M_0) := \{[0]\}M_0(0) = \{[0]\}M_0(t)$ for all $t \in \mathbb{R}$.

We denote by $\iota_0 : N(M_0) \to H$ the canonical embedding of $N(M_0)$ into $H$. Then an easy computation shows that $\iota_0^* \iota_1 : H \to H$ is the orthogonal projector onto the closed subspace $N(M_0)$ (see e.g. [17, Lemma 3.2]). In the same way we denote by $\iota_1 : N(M_0)^\perp \to H$ the canonical embedding of $N(M_0)^\perp = M_0(t)[H]$ into $H$.

Finally, we require that

(d) There exist $c_0, c_1 > 0$ such that for all $t \in \mathbb{R}$ the operators $\iota_1^* M_0(t) \iota_1 - c_0$ and $\iota_0^* \Re M_1(t) \iota_0 - c_1$ are monotone in $N(M_0)^\perp$ and $N(M_0)$, respectively.

Note that in [18] condition (c) is not required and (d) is replaced by a more general monotonicity constraint. However, in order to apply perturbation results, which will be a key tool for proving the well-posedness of (7), we need to impose the constraints (c) and (d) above (compare [18, Theorem 2.19]).

Remark 3.1. Note that under the conditions above, the operators $\iota_1^* M_0(m) \iota_1$ and $\iota_0^* \Re M_1(m) \iota_0$ are continuously invertible as operators in $L(H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp))$ and $L(H_{\varrho,0}(\mathbb{R}; N(M_0)))$, respectively.

We recall the following result from [18].

Lemma 3.2 ([18, Lemma 2.1]). For $t \in \mathbb{R}$ the mapping

$$M_0'(t) : H \to H$$

$$x \mapsto \begin{cases} (M_0(\cdot)x)'(t) & \text{if } t \in \mathbb{R} \setminus N, \\ 0 & \text{otherwise} \end{cases}$$

is a bounded linear operator with $\|M_0'(t)\|_{L(H)} \leq |M_0|_{\text{Lip}}$ and thus, gives rise to a bounded multiplication operator $M_0'(m) \in L(H_{\varrho,0}(\mathbb{R}; H))$ for each $\varrho \in ]0, \infty[$. Furthermore, $M_0'(m)$ is selfadjoint. Moreover, for $u \in H_{\varrho,1}(\mathbb{R}; H)$ the chain rule

$$\partial_{0,\varrho} M_0(m)u = M_0(m) \partial_{0,\varrho} u + M_0'(m)u$$

(8)

holds.

\(^5\)If $H$ is separable, this assumptions already follows by the Lipschitz-continuity of $M_0$. 

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3.1 Well-posedness

Remark 3.3. From \( \square \) we derive

\[
\partial_{0,\varrho} (\iota^*_1 M_0(m) t_1) u = \iota^*_1 \partial_{0,\varrho} M_0(m) t_1 u \\
= \iota^*_1 M_0(m) \partial_{0,\varrho} t_1 u + \iota^*_1 M_0'(m) t_1 u \\
= (\iota^*_1 M_0(m) t_1) \partial_{0,\varrho} u + (\iota^*_1 M_0'(m) t_1) u \tag{9}
\]

for \( u \in H_{0,1}(\mathbb{R}; N(M_0)^\perp) \).

In the following two subsections we prove our main theorem.

Theorem 3.4 (Solution Theory). Let \((M_0, M_1)\) be a pair of \(L(H)\)-valued strongly measurable functions satisfying (a)-(d). Moreover, let \( A \subseteq H \oplus H \) be a maximal monotone relation with \((0,0) \in A\). Then there exists \( \varrho_0 > 0 \) such that for every \( \varrho \geq \varrho_0 \)

\[
(\partial_{0,\varrho} M_0(m) + M_1(m) + A_\varrho)^{-1} : H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)
\]

is a Lipschitz-continuous, causal mapping. Moreover, the mapping is independent of \( \varrho \) in the sense that, for \( \nu, \varrho \geq \varrho_0 \) and \( f \in H_{\varrho,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H) \)

\[
(\partial_{0,\nu} M_0(m) + M_1(m) + A_\nu)^{-1} (f) = (\partial_{0,\varrho} M_0(m) + M_1(m) + A_\varrho)^{-1} (f)
\]

as functions in \( L_{2,\text{loc}}(\mathbb{R}; H) \).

3.1 Well-posedness

We begin with characterizing the elements belonging to the domain of \( \partial_{0,\varrho} M_0(m) \).

Lemma 3.5. Let \( \varrho > 0 \). Then an element \( u \in H_{\varrho,0}(\mathbb{R}; H) \) belongs to \( D(\partial_{0,\varrho} M_0(m)) \) if and only if \( \iota^*_1 u \in H_{0,1}(\mathbb{R}; N(M_0)^\perp) \).

Proof. Let \( u \in H_{\varrho,0}(\mathbb{R}; H) \). First we assume that \( u \in D(\partial_{0,\varrho} M_0(m)) \). Then for \( \phi \in C_c^\infty(\mathbb{R}; N(M_0)^\perp) \) we compute, using the continuous invertibility of \( \iota^*_1 M_0(m) t_1 \) and the chain rule \( \square \)

\[
\langle \iota^*_1 u | \partial_{0,\varrho} \phi \rangle_{H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp)} = \langle (\iota^*_1 M_0(m) t_1)^{-1} \iota^*_1 M_0(m) t_1 \iota^*_1 u | \partial_{0,\varrho} \phi \rangle_{H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp)} \\
= \langle \iota^*_1 M_0(m) u | (\iota^*_1 M_0(m) t_1)^{-1} \partial_{0,\varrho} \phi \rangle_{H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp)} \\
= \langle \iota^*_1 M_0(m) u | \partial_{0,\varrho} (\iota^*_1 M_0(m) t_1)^{-1} \phi \rangle_{H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp)} \\
+ \langle \iota^*_1 M_0(m) u | (\iota^*_1 M_0(m) t_1)^{-1} (\iota^*_1 M_0'(m) t_1) (\iota^*_1 M_0(m) t_1)^{-1} \phi \rangle_{H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp)} \\
= \langle (\iota^*_1 M_0(m) t_1)^{-1} (\partial_{0,\varrho} \iota^*_1 M_0(m) u + (\iota^*_1 M_0'(m) t_1) \iota^*_1 u) | \phi \rangle_{H_{\varrho,0}(\mathbb{R}; N(M_0)^\perp)}.
\]

This proves that \( \iota^*_1 u \in D(\partial_{0,\varrho}^* \phi) = D(\partial_{0,\varrho} \phi) \), since \( C_c^\infty(\mathbb{R}; N(M_0)^\perp) \) is dense in \( H_{0,1}(\mathbb{R}; N(M_0)^\perp) \). On the other hand if \( \iota^*_1 u \in D(\partial_{0,\varrho} \phi) \) the assertion follows by the chain rule \( \square \).
3 Solution theory

As it was done in [22] in the autonomous case we prove the strict maximal monotonicity of
the operator \( \partial_{0,e}M_0(m) + M_1(m) \) for sufficiently large \( q \).

**Lemma 3.6.** Let \( q > 0 \) and \( a \in \mathbb{R} \). Then for each \( u \in D(\partial_{0,e}M_0(m)) \) and \( \varepsilon > 0 \) the estimate
\[
\Re \int_{-\infty}^{a} \langle (\partial_{0,e}M_0(m) + M_1(m))u(t), u(t) \rangle e^{-2qt} \, dt \\
\geq (q_0 - \frac{1}{2}|M_0|_{\text{Lip}} - |M_1|_{\infty} - \frac{1}{\varepsilon}|M_1|^2_{\infty}) \int_{-\infty}^{a} |\partial_t^* u(t)|^2 e^{-2qt} \, dt + (c_1 - \varepsilon) \int_{-\infty}^{a} |\partial_t^* u(t)|^2 e^{-2qt} \, dt \\
\] holds. In particular, by letting a tend to infinity, we have
\[
\Re(\partial_{0,e}M_0(m) + M_1(m))u, u \rangle_{H_{q,0}(\mathbb{R}; H)} \\
\geq (q_0 - \frac{1}{2}|M_0|_{\text{Lip}} - |M_1|_{\infty} - \frac{1}{\varepsilon}|M_1|^2_{\infty}) |\partial_t^* u(t)|^2_{H_{q,0}(\mathbb{R}; N(M_0))} + (c_1 - \varepsilon)|\partial_t^* u(t)|^2_{H_{q,0}(\mathbb{R}; N(M_0))}. \\
\] (10)
Moreover, for each \( 0 < \tilde{c} < c_1 \) there exists \( q_0 > 0 \) such that for all \( q \geq q_0 \) the operator \( \partial_{0,e}M_0(m) + M_1(m) - \tilde{c} \) is maximal monotone.

**Proof.** Let \( \phi \in C_c^{\infty}(\mathbb{R}; N(M_0)) \). Then we compute
\[
\Re \int_{-\infty}^{a} \langle (\partial_{0,e}M_0(m)\phi(t))\phi(t) \rangle e^{-2qt} \, dt \\
= \frac{1}{2} \int_{-\infty}^{a} \langle (\partial_{0,e}M_0(m)\phi(t))\phi(t) + \phi(t)(\partial_{0,e}M_0(m)\phi(t)) \rangle e^{-2qt} \, dt \\
= \frac{1}{2} \int_{-\infty}^{a} \langle (\partial_{0,e}M_0(m)\phi(t))\phi(t) + \phi(t)|\partial_t^* M_0(t)\phi(t)|^2_{H_{q,0}(\mathbb{R}; N(M_0))} \rangle e^{-2qt} \, dt \\
= \frac{1}{2} \int_{-\infty}^{a} \langle |\partial_t^* M_0(t)|^2_{H_{q,0}(\mathbb{R}; N(M_0))} |\phi(t)|^2 e^{-2qt} \, dt + \frac{1}{2} \int_{-\infty}^{a} \langle |\phi(t)|^2 e^{-2qt} \, dt \\
\geq \frac{1}{2} \langle |\partial_t^* M_0(t)|^2_{H_{q,0}(\mathbb{R}; N(M_0))} |\phi(t)|^2 e^{-2qt} \, dt \int_{-\infty}^{a} \langle |\phi(t)|^2 e^{-2qt} \, dt \\
\geq (q_0 - \frac{1}{2}|M_0|_{\text{Lip}}) \int_{-\infty}^{a} \langle |\phi(t)|^2 e^{-2qt} \, dt.
\]
Since \( C_c^{\infty}(\mathbb{R}; N(M_0)) \) is dense in \( H_{q,1}(\mathbb{R}; N(M_0)) \) and \( \chi_{[-\infty,0]}(m) \) is continuous in \( H_{q,0}(\mathbb{R}; H) \)

\(^{\text{For a function } g \in L_\infty(\mathbb{R}) \text{ we denote by } g(m) \text{ the corresponding multiplication operator on } H_{q,0}(\mathbb{R}; H), \text{ i.e. } (g(m)u)(t) := g(t)u(t) \text{ for } u \in H_{q,0}(\mathbb{R}; H) \text{ and almost every } t \in \mathbb{R}.}\)
and by means of Lemma 3.5, we obtain
\[
\Re \int_{-\infty}^{a} \langle (\partial_{0,\nu} M_0)(m) \phi \rangle_{H_{\nu,0}(\Re; H)} \, e^{-\nu t} \, dt \geq (g_0 - \frac{1}{2}|M_0|_{\text{Lip}}) \int_{-\infty}^{a} |\phi(t)|^2 e^{-\nu t} \, dt
\]
for all \( u \in D(\partial_{0,\nu} M_0(m)) \). Moreover, we compute
\[
\Re \int_{-\infty}^{a} \langle (\partial_{0,\nu} M_0 + M_1(m)) u(t) \rangle_{H_{\nu,0}(\Re; H)} \, e^{-\nu t} \, dt
\]
\[
= \Re \int_{-\infty}^{a} \langle (\partial_{0,\nu} M_0)(m) \phi \rangle_{H_{\nu,0}(\Re; H)} \, e^{-\nu t} \, dt + \Re \int_{-\infty}^{a} \langle \phi \rangle_{H_{\nu,0}(\Re; H)} \, e^{-\nu t} \, dt
\]
\[
+ \Re \int_{-\infty}^{a} \langle M_1(t) u(t) \rangle_{H_{\nu,0}(\Re; H)} \, e^{-\nu t} \, dt + \Re \int_{-\infty}^{a} \langle (\partial_{0,\nu} M_0)(m) \phi \rangle_{H_{\nu,0}(\Re; H)} \, e^{-\nu t} \, dt
\]
\[
\geq (g_0 - \frac{1}{2}|M_0|_{\text{Lip}} - |M_1|_{\infty}) \int_{-\infty}^{a} |\phi(t)|^2 e^{-\nu t} \, dt + c_1 \int_{-\infty}^{a} |\phi(t)(m)|^2 e^{-\nu t} \, dt
\]
\[
- 2|M_1|_{\infty} \int_{-\infty}^{a} |\phi(t)|^2 e^{-\nu t} \, dt + (g_0 - \frac{1}{2}|M_0|_{\text{Lip}} - |M_1|_{\infty} - \frac{1}{c_1 - c}|M_1|_{\text{Lip}}^2) \int_{-\infty}^{a} |\phi(t)|^2 e^{-\nu t} \, dt
\]
for all \( u \in D(\partial_{0,\nu} M_0(m)) \). Let now \( 0 < \tilde{c} < c_1 \) and set \( g_0 := \frac{1}{c_1} \left( \tilde{c} + \frac{1}{2}|M_0|_{\text{Lip}} + |M_1|_{\infty} + \frac{1}{c_1 - c}|M_1|_{\text{Lip}}^2 \right) \). Then by (10), the operator \( \partial_{0,\nu} M_0(m) + M_1(m) - \tilde{c} \) is monotone for each \( g \geq g_0 \). To show the maximal monotonicity of \( \partial_{0,\nu} M_0(m) + M_1(m) - \tilde{c} \) we need to determine the domain of \( (\partial_{0,\nu} M_0(m))^* \). Let \( v \in D(10) \). Then for each \( \phi \in C_c^\infty(\Re; H) \) we compute, using (3)
\[
\langle \phi \rangle_{H_{\nu,0}(\Re; H)} = (\partial_{0,\nu} M_0(m) \phi \rangle_{H_{\nu,0}(\Re; H)}
\]
\[
= (M_0(m) \partial_{0,\nu} \phi \rangle_{H_{\nu,0}(\Re; H)} + (M_0(m) \phi \rangle_{H_{\nu,0}(\Re; H)} + (\partial_{0,\nu} \phi \rangle_{H_{\nu,0}(\Re; H)} + (\phi \rangle_{H_{\nu,0}(\Re; H)}
\]
yielding that \( M_0(m) v \in D(\partial_{0,\nu} M_0(m)) \) is dense in \( H_{\nu,0}(\Re; H) \). Thus,
\[
D((\partial_{0,\nu} M_0(m))^*) \subseteq D(\partial_{0,\nu} M_0(m))
\]
which yields by (10) that \( (\partial_{0,\nu} M_0(m) + M_1(m))^* \) is injective and hence \( \partial_{0,\nu} M_0(m) + M_1(m) \) is onto. Thus, Theorem 3.4 implies that \( \partial_{0,\nu} M_0(m) + M_1(m) - \tilde{c} \) is maximal monotone.
3 Solution theory

With the latter lemma, the uniqueness of a solution of $(7)$ and its continuous dependence on the given right hand side for $g$ sufficiently large can easily be proved.

**Proposition 3.7.** Let $0 < \tilde{c} < c_1$ and $\rho > 0$ sufficiently large such that $\partial_{0,\rho} M_0(m) - M_1(m) - \tilde{c}$ is maximal monotone. Moreover, let $B \subseteq H_{\rho,0}(\mathbb{R}; H) \oplus H_{\rho,0}(\mathbb{R}; H)$ be monotone. Then for $(u, f), (v, g) \in \partial_{0,\rho} M_0(m) + M_1(m) + B$ the estimate
\[
|u - v|_{H_{\rho,0}(\mathbb{R}; H)} \leq \frac{1}{\tilde{c}} |f - g|_{H_{\rho,0}(\mathbb{R}; H)}
\]
holds, or, in other words, the inverse relation $(\partial_{0,\rho} M_0(m) + M_1(m) + B)^{-1}$ is a Lipschitz-continuous mapping.

**Proof.** Let $(u, f), (v, g) \in \partial_{0,\rho} M_0(m) + M_1(m) + B$. Then
\[
\begin{align*}
\Re \langle f - g, u - v \rangle_{H_{\rho,0}(\mathbb{R}; H)} &= \Re \langle f - (\partial_{0,\rho} M_0(m) + M_1(m)) u - (g - (\partial_{0,\rho} M_0(m) + M_1(m)) v, u - v \rangle_{H_{\rho,0}(\mathbb{R}; H)} \\
&= \Re \langle (\partial_{0,\rho} M_0(m) + M_1(m)) (u - v), u - v \rangle_{H_{\rho,0}(\mathbb{R}; H)} \\
&\geq \tilde{c} |u - v|^2_{H_{\rho,0}(\mathbb{R}; H)},
\end{align*}
\]
where we have used the monotonicity of $\partial_{0,\rho} M_0(m) + M_1(m) - \tilde{c}$ and of $B$. The assertion now follows from the Cauchy-Schwarz-inequality. \(\square\)

It is left to show that $(7)$ possesses a solution $u \in H_{\rho,0}(\mathbb{R}; H)$ for every right hand side $f \in H_{\rho,0}(\mathbb{R}; H)$. Instead of considering the problem $(7)$ we study an inclusion of the form
\[
(u, f) \in \partial_{0,\rho} M_0(m) - M_0'(m) + \delta + A_{\rho}
\]
where $\delta > 0$. This means we specify $M_1(m)$ to be of the form $\delta - M_0'(m)$. It is easy to see that the pair $(M_0, \delta - M_0')$ satisfies the conditions (a)-(d).\footnote{Note that $M_0'(m) = 0$.} We show that for $f \in H_{\rho,1}(\mathbb{R}; H)$ there exists $u \in H_{\rho,0}(\mathbb{R}; H)$ satisfying (11). For doing so, we employ Proposition 2.10 and thus, we have to consider the approximate problem
\[
(\partial_{0,\rho} M_0(m) - M_0'(m) + \delta) u_\lambda + A_{\rho,\lambda}(u_\lambda) = f,
\]
for each $\lambda > 0$, where we denote by $A_{\rho,\lambda}$ the Yosida approximation of $A_{\rho,\lambda}$.\footnote{One can show that $A_{\rho,\lambda}$ equals the extension of $A_{\lambda}$ given by (4).}

**Lemma 3.8.** Let $0 < \tilde{c} < \delta$ and $\rho > 0$ such that $\partial_{0,\rho} M_0(m) - M_0'(m) + \delta - \tilde{c}$ is maximal monotone. Moreover, let $f \in H_{\rho,1}(\mathbb{R}; H)$ and $u_\lambda \in D(\partial_{0,\rho} M_0(m))$ satisfying (12) for $\lambda > 0$. Then $u_\lambda \in H_{\rho,1}(\mathbb{R}; H)$.

**Proof.** We decompose $u_\lambda$ into the orthogonal parts $\iota_u^* u_\lambda$ and $\iota_v^* u_\lambda$ lying in $H_{\rho,0}(\mathbb{R}; N(M_0)\perp)$ and $H_{\rho,0}(\mathbb{R}; N(M_0))$, respectively. Since $u_\lambda \in D(\partial_{0,\rho} M_0(m))$, Lemma 3.5 yields that $\iota_u^* u_\lambda \in H_{\rho,0}(\mathbb{R}; H)$.
Thus, by Corollary 2.8, \( D(\partial_{0,0}) \). Thus, it suffices to prove that also \( \psi_{0}^{*} u_{\lambda} \in D(\partial_{0,0}) \), which will be shown by using Proposition 2.1. We apply \( \psi_{0}^{*} \) on (12) and obtain the equality

\[
\delta \psi_{0}^{*} u_{\lambda} + \psi_{0}^{*} A_{\psi,\lambda}(\psi_{0}^{*} u_{\lambda} + \psi_{1}^{*} u_{\lambda}) = \psi_{0}^{*} f,
\]

since \( \psi_{0}^{*} M_{0}(m) = 0 = \psi_{0}^{*} M_{1}(m) \). We define the mapping \( B_{\lambda} \) on \( H_{0,0}(\mathbb{R}; N(M_{0})) \) given by

\[
B_{\lambda}(v) = \psi_{0}^{*} A_{\psi,\lambda}(v_{0} + \psi_{1}^{*} u_{\lambda}) \quad (v \in H_{0,0}(\mathbb{R}; N(M_{0}))).
\]

Then (13) can be written as

\[
\delta \psi_{0}^{*} u_{\lambda} + B_{\lambda}(\psi_{0}^{*} u_{\lambda}) = \psi_{0}^{*} f. \tag{14}
\]

\( B_{\lambda} \) is monotone, since for \( v, w \in H_{0,0}(\mathbb{R}; N(M_{0})) \) we estimate

\[
\Re(\langle B_{\lambda}(v) - B_{\lambda}(w), v - w \rangle)_{H_{0,0}(\mathbb{R}; N(M_{0}))} = \Re(\langle \psi_{0}^{*} A_{\psi,\lambda}(v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(w_{0} + \psi_{1}^{*} u_{\lambda}), v - w \rangle)_{H_{0,0}(\mathbb{R}; N(M_{0}))}
\]

\[
= \Re(\langle A_{\psi,\lambda}(v_{0} + \psi_{1}^{*} u_{\lambda}) - A_{\psi,\lambda}(w_{0} + \psi_{1}^{*} u_{\lambda}), v - w \rangle)_{H_{0,0}(\mathbb{R}; H)}
\]

\[
= \Re(\langle A_{\psi,\lambda}(v_{0} + \psi_{1}^{*} u_{\lambda}) - A_{\psi,\lambda}(w_{0} + \psi_{1}^{*} u_{\lambda}), (v_{0} + \psi_{1}^{*} u_{\lambda}) - (w_{0} + \psi_{1}^{*} u_{\lambda}) \rangle)_{H_{0,0}(\mathbb{R}; H)}
\]

\[
\leq 0,
\]

where we have used the monotonicity of \( A_{\psi,\lambda} \). Moreover, \( B_{\lambda} \) is Lipschitz-continuous. Indeed, for \( v, w \in H_{0,0}(\mathbb{R}; N(M_{0})) \) we have that

\[
|B_{\lambda}(v) - B_{\lambda}(w)|_{H_{0,0}(\mathbb{R}; N(M_{0}))} = |\psi_{0}^{*} A_{\psi,\lambda}(v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(w_{0} + \psi_{1}^{*} u_{\lambda})|_{H_{0,0}(\mathbb{R}; N(M_{0}))}
\]

\[
\leq \lambda^{-1} |v_{0} - w|_{H_{0,0}(\mathbb{R}; H)}
\]

\[
= \lambda^{-1} |v - w|_{H_{0,0}(\mathbb{R}; H)}.
\]

Thus, by Corollary 2.8, \( B_{\lambda} \) is maximal monotone. Hence, we find a unique solution \( v \in H_{0,0}(\mathbb{R}; N(M_{0})) \) of

\[
\delta v + B_{\lambda}(v) = \psi_{0}^{*} f,
\]

which thus coincides with \( \psi_{0}^{*} u_{\lambda} \) by (14). Furthermore, we compute for \( h > 0 \)

\[
\delta \tau_{h} v + B_{\lambda}(\tau_{h} v)
\]

\[
= \delta \tau_{h} v + \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda})
\]

\[
= \delta \tau_{h} v + \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}) + \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda})
\]

\[
= \tau_{h} (\delta v + B_{\lambda}(v)) + \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda})
\]

\[
= \tau_{h} \delta v + \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}),
\]

where we have used that \( A_{\psi,\lambda} \circ \tau_{h} = \tau_{h} \circ A_{\psi,\lambda} \), which follows from (6). Thus, we estimate

\[
\Re(\langle \tau_{h} \delta v + \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}), v - \psi_{0}^{*} f |_{H_{0,0}(\mathbb{R}; N(M_{0}))} = \Re(\langle \delta v + B_{\lambda}(v), v - \psi_{0}^{*} f |_{H_{0,0}(\mathbb{R}; N(M_{0}))}
\]

\[
\geq 0,
\]

where we have used that \( A_{\psi,\lambda} \circ \tau_{h} = \tau_{h} \circ A_{\psi,\lambda} \), which follows from (6). Thus, we estimate

\[
\Re(\langle \tau_{h} \delta v + v_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}) - \psi_{0}^{*} A_{\psi,\lambda}(\tau_{h} v_{0} + \psi_{1}^{*} u_{\lambda}), v - \psi_{0}^{*} f |_{H_{0,0}(\mathbb{R}; N(M_{0}))} = \Re(\langle \delta v + B_{\lambda}(v), v - \psi_{0}^{*} f |_{H_{0,0}(\mathbb{R}; N(M_{0}))}
\]

\[
\geq 0.
\]
3 Solution theory

and hence, by the Cauchy-Schwarz-inequality and the Lipschitz-continuity of \( A_{\theta, \lambda} \)

\[
|\tau_h v - v|_{H_{\theta,0}(\mathbb{R}; N(M))} \leq \frac{1}{\delta} \left( |\tau_0^*(\tau_h f - f)|_{H_{\theta,0}(\mathbb{R}; N(M))} + \frac{1}{\lambda} |(\tau_h - 1) \tau_0^* \lambda|_{H_{\theta,0}(\mathbb{R}; N(M))} \right).
\]

Dividing the latter inequality by \( h \) and using that \( f \) and \( \tau_0^* \lambda \) are in \( D(\partial_{\theta,0}) \), we derive that \( (h^{-1}(\tau_h - 1)v)_{h \in [0,1]} \) is bounded, which yields that \( \tau_0^* \lambda = v \in D(\partial_{\theta,0}) \) by Proposition 2.1.

This completes the proof.

\[ \square \]

We are now able to state the existence result for (11).

**Proposition 3.9.** Let \( f \in C_\infty^\infty(\mathbb{R}; H) \). Then there exists \( \rho_0 > 0 \) such that for every \( \rho \geq \rho_0 \) we find \( u \in H_{\theta,0}(\mathbb{R}; H) \) such that

\[
(u, f) /in D_{\theta,0} M_0(m) - M_0'(m) + \delta + A_{\theta}.
\]

**Proof.** Let \( 0 < \tilde{c} < \delta \). We choose \( \rho_0 > 0 \) such that \( \partial_{\theta,0} M_0(m) - M_0'(m) + \delta - \tilde{c} \) and \( \partial_{\theta,0} M_0(m) + \delta - \tilde{c} \) are maximal monotone for all \( \rho \geq \rho_0 \). Let \( \rho \geq \rho_0 \). For \( \lambda > 0 \) let \( u_\lambda \in H_{\theta,0}(\mathbb{R}; H) \) such that (12) is satisfied. Then, by Lemma 3.8 \( u_\lambda \in H_{\theta,1}(\mathbb{R}; H) \). In order to show the assertion we have to prove that the family \( (A_{\theta, \lambda}(u_\lambda))_{\lambda > 0} \) is bounded (see Proposition 2.10). For doing so, we define

\[
B_\lambda : D(B_\lambda) /subseteq H_{\theta,0}(\mathbb{R}; H) \rightarrow H_{\theta,0}(\mathbb{R}; H)
\]

\[
v \mapsto \partial_{\theta,0} A_{\theta, \lambda}(\partial_{\theta,0}^{-1} v)
\]

for \( \lambda > 0 \) with maximal domain \( D(B_\lambda) := \{ v \in H_{\theta,0}(\mathbb{R}; H) | A_{\theta, \lambda}(\partial_{\theta,0}^{-1} v) \in H_{\theta,1}(\mathbb{R}; H) \} \). Then for \( v \in D(B_\lambda) \) we estimate, using Proposition 2.1,

\[
\text{Re}(B_\lambda(v) | v)_{H_{\theta,0}(\mathbb{R}; H)} = \text{Re}(\partial_{\theta,0} A_{\theta, \lambda}(\partial_{\theta,0}^{-1} v) | v)_{H_{\theta,0}(\mathbb{R}; H)}
\]

\[
= \lim_{h \rightarrow 0^+} \frac{1}{h^2} \text{Re}(\tau_h A_{\theta, \lambda}(\partial_{\theta,0}^{-1} v) - A_{\theta, \lambda}(\partial_{\theta,0}^{-1} v) \tau_h \partial_{\theta,0}^{-1} v - \partial_{\theta,0}^{-1} v)_{H_{\theta,0}(\mathbb{R}; H)}
\]

\[
= \lim_{h \rightarrow 0^+} \frac{1}{h^2} \text{Re}(A_{\theta, \lambda}(\tau_h \partial_{\theta,0}^{-1} v) - A_{\theta, \lambda}(\partial_{\theta,0}^{-1} v) \tau_h \partial_{\theta,0}^{-1} v - \partial_{\theta,0}^{-1} v)_{H_{\theta,0}(\mathbb{R}; H)}
\]

\[
\geq 0,
\]

(15)

due to the monotonicity of \( A_{\theta, \lambda} \). Since \( u_\lambda \in H_{\theta,1}(\mathbb{R}; H) \) for each \( \lambda > 0 \) we obtain, using Proposition 2.1,

\[
\sup_{h \in [0,1]} \frac{1}{h} |\tau_h A_{\theta, \lambda}(u_\lambda) - A_{\theta, \lambda}(u_\lambda)|_{H_{\theta,0}(\mathbb{R}; H)} = \sup_{h \in [0,1]} \frac{1}{h} |A_{\theta, \lambda}(\tau_h u_\lambda) - A_{\theta, \lambda}(u_\lambda)|_{H_{\theta,0}(\mathbb{R}; H)}
\]

\[
\leq \frac{1}{\lambda} \sup_{h \in [0,1]} \frac{1}{h} |\tau_h u_\lambda - u_\lambda|_{H_{\theta,0}(\mathbb{R}; H)} < \infty
\]

and thus, again by Proposition 2.1 \( A_{\theta, \lambda}(u_\lambda) \in H_{\theta,1}(\mathbb{R}; H) \) for each \( \lambda > 0 \). In other words,

\[ \text{Note that also the pair } (M_0, \delta) \text{ satisfies the conditions (a)-(d) and thus, Lemma 3.8 is applicable.} \]

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this means that $\partial_{0,\varrho}u_\lambda \in D(B_\lambda)$ for each $\lambda > 0$. Moreover, using (8) we obtain

$$B_\lambda(\partial_{0,\varrho}u_\lambda) = \partial_{0,\varrho}A_{\varrho,\lambda}(u_\lambda) = \partial_{0,\varrho}(f - (\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta)u_\lambda) = \partial_{0,\varrho}f - \partial_{0,\varrho}M_0(m)\partial_{0,\varrho}u_\lambda - \delta\partial_{0,\varrho}u_\lambda,$$

which means that $\partial_{0,\varrho}u_\lambda$ solves the differential equation

$$(\partial_{0,\varrho}M_0(m) + \delta + B_\lambda) (\partial_{0,\varrho}u_\lambda) = \partial_{0,\varrho}f$$

for each $\lambda > 0$. Then we estimate, using (11) and the monotonicity of $\partial_{0,\varrho}M_0(m) + \delta - \tilde{c}$

$$\Re(\partial_{0,\varrho}f|\partial_{0,\varrho}u_\lambda|_{H_{0,0}(\mathbb{R};H)}) = \Re\langle(\partial_{0,\varrho}M_0(m) + \delta)\partial_{0,\varrho}u_\lambda|\partial_{0,\varrho}u_\lambda\rangle_{H_{0,0}(\mathbb{R};H)} + \Re(B_\lambda(\partial_{0,\varrho}u_\lambda)|\partial_{0,\varrho}u_\lambda\rangle_{H_{0,0}(\mathbb{R};H)}$$

and hence,

$$|\partial_{0,\varrho}u_\lambda|_{H_{0,0}(\mathbb{R};H)} \leq \frac{1}{\tilde{c}}|\partial_{0,\varrho}f|_{H_{0,0}(\mathbb{R};H)}.$$

Thus, since $u_\lambda$ solves (12), we get that

$$|A_{\varrho,\lambda}(u_\lambda)|_{H_{0,0}(\mathbb{R};H)} = |f - (\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta)u_\lambda|_{H_{0,0}(\mathbb{R};H)} \leq |f|_{H_{0,0}(\mathbb{R};H)} + |M_0(m)\partial_{0,\varrho}u_\lambda|_{H_{0,0}(\mathbb{R};H)} + |\delta|u_\lambda|_{H_{0,0}(\mathbb{R};H)} \leq |f|_{H_{0,0}(\mathbb{R};H)} + \frac{1}{\tilde{c}}|M_0|\infty|\partial_{0,\varrho}f|_{H_{0,0}(\mathbb{R};H)} + |\delta|u_\lambda|_{H_{0,0}(\mathbb{R};H)}$$

for each $\lambda > 0$. Now using that $(0,0)$ also satisfies (12), we can estimate, using Proposition 3.7

$$|u_\lambda|_{H_{0,0}(\mathbb{R};H)} \leq \frac{1}{\tilde{c}}|f|_{H_{0,0}(\mathbb{R};H)}$$

for each $\lambda > 0$. Summarizing we get that

$$\sup_{\lambda>0}|A_{\varrho,\lambda}(u_\lambda)|_{H_{0,0}(\mathbb{R};H)} \leq \left(1 + \frac{\delta}{\tilde{c}}\right)|f|_{H_{0,0}(\mathbb{R};H)} + \frac{1}{\tilde{c}}|M_0|\infty|\partial_{0,\varrho}f|_{H_{0,0}(\mathbb{R};H)}$$

for each $\lambda > 0$, which completes the proof.

We summarize our so far findings.

**Theorem 3.10.** For each $\delta > 0$ there exists $\varrho_0 > 0$ such that for all $\varrho \geq \varrho_0$ the problem (17) is well-posed, i.e. the inverse relation $\left(\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta + A_\varrho\right)^{-1}$ is a Lipschitz-continuous mapping defined on the whole space $H_{0,0}(\mathbb{R};H)$. More precisely, for each $0 < \tilde{c} < \delta$ there exists $\varrho_0 > 0$ such that for each $\varrho \geq \varrho_0$ the relation $\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta + A_\varrho - \tilde{c} \subseteq H_{0,0}(\mathbb{R};H) \cup H_{0,0}(\mathbb{R};H)$ is maximal monotone.

**Proof.** Let $0 < \tilde{c} < \delta$. By Lemma 3.6 there exists $\varrho_0 > 0$ such that for every $\varrho \geq \varrho_0$ the relation $\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta + A_\varrho - \tilde{c}$ is monotone, which then also holds for its closure.
3 Solution theory

Moreover, Proposition 3.3 shows that \((\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta + A_\varrho) [H_{\varrho,0}(\mathbb{R}; H)]\) contains the test functions (if we choose \(\varrho\) large enough) and therefore it is dense in \(H_{\varrho,0}(\mathbb{R}; H)\). The latter yields that 
\[
(\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta + A_\varrho) [H_{\varrho,0}(\mathbb{R}; H)] = H_{\varrho,0}(\mathbb{R}; H),
\]
which in turn implies the maximal monotonicity of \(\partial_{0,\varrho}M_0(m) - M_0'(m) + \delta + A_\varrho - \tilde{c}\) by Theorem 2.4. \(\square\)

Now we come back to our original problem (7). It turns out that the well-posedness for this inclusion just relies on the perturbation result stated in Proposition 2.7.

**Theorem 3.11** (Well-posedness). For every \(0 < \tilde{c} < c_1\) there exists \(\varrho_0 > 0\) such that for every \(\varrho \geq \varrho_0\) the relation 
\[
(\partial_{0,\varrho}M_0(m) + M_1(m) + A_\varrho - \tilde{c})^{-1} : H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)
\]
is a Lipschitz-continuous mapping. In other words, the problem (7) is well-posed, i.e. for each right hand side \(f \in H_{\varrho,0}(\mathbb{R}; H)\) there exists a unique \(u \in H_{\varrho,0}(\mathbb{R}; H)\) satisfying (7) and depending continuously on \(f\).

**Proof.** Let \(0 < \tilde{c} < c_1\). According to Theorem 3.10 there exists \(\varrho_0 > 0\) such that for all \(\varrho \geq \varrho_0\) the relation 
\[
\partial_{0,\varrho}M_0(m) - M_0'(m) + 2\tilde{c} + A_\varrho - \tilde{c}
\]
is maximal monotone. Furthermore, by Lemma 3.6 there exists \(\varrho_1 > 0\) such that for all \(\varrho \geq \varrho_1\) 
\[
\partial_{0,\varrho}M_0(m) + M_1(m) + A_\varrho - \tilde{c}
\]
is monotone. Thus, for \(\varrho \geq \max\{\varrho_0, \varrho_1\}\) the relation 
\[
\partial_{0,\varrho}M_0(m) + M_1(m) + A_\varrho - \tilde{c} = \partial_{0,\varrho}M_0(m) - M_0'(m) + 2\tilde{c} + A_\varrho - \tilde{c} + (M_1(m) + M_0'(m) - 2\tilde{c})
\]
is maximal monotone by Proposition 2.7. \(\square\)

3.2 Causality

In this section we show that our solution operator \((\partial_{0,\varrho}M_0(m) + M_1(m) + A_\varrho)^{-1}\) corresponding to the differential inclusion (7) is causal in \(H_{\varrho,0}(\mathbb{R}; H)\) and independent of the parameter \(\varrho\) in the sense that for \(f \in H_{\varrho,0}(\mathbb{R}; H) \cap H_{\nu,0}(\mathbb{R}; H)\) we have 
\[
(\partial_{0,\varrho}M_0(m) + M_1(m) + A_\varrho)^{-1} (f) = (\partial_{0,\nu}M_0(m) + M_1(m) + A_\nu)^{-1} (f)
\]
as functions in \(L_{2,loc}(\mathbb{R}; H)\). The definition of causality in our framework is the following:

**Definition 3.12.** Let \(F : H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H)\). \(F\) is called forward causal (or simply causal), if for each \(a \in \mathbb{R}\) and \(u, v \in H_{\varrho,0}(\mathbb{R}; H)\) the implication 
\[
\chi_{]-\infty,a]}(m)(u - v) = 0 \Rightarrow \chi_{]-\infty,a]}(m)(F(u) - F(v)) = 0
\]
is satisfied.
which yields \( \chi_{|a,\infty|}(m)(u - v) = 0 \Rightarrow \chi_{|a,\infty|}(m)(F(u) - F(v)) = 0 \) holds.

**Proposition 3.13.** There exists \( \varrho_0 > 0 \) such that for every \( \varrho \geq \varrho_0 \) the solution operator 
\[
\left( \partial_0 \varrho M_0(m) + M_1(m) + A_\varrho \right)^{-1}
\]
holds.

**Proof.** We choose \( \varrho_0 > 0 \) according to Theorem 3.11. Let \( a \in \mathbb{R}, \varrho \geq \varrho_0 \) and \( f, g \in (\partial_0 \varrho M_0(m) + M_1(m) + A_\varrho)[H_{\varrho,0}(\mathbb{R}; H)] \). Then there exist two pairs \((u, v), (x, y) \in A_\varrho\) such that \( u, x \in D(\partial_0 \varrho M_0(m))\) and 
\[
\partial_0 \varrho M_0(m)u + M_1(m)u + v = f \quad \text{and} \quad \partial_0 \varrho M_0(m)x + M_1(m)x + y = g.
\]

By using Lemma 3.6 and the monotonicity of \( A_\varrho \) we estimate 
\[
\Re \int_{-\infty}^{a} \langle f(t) - g(t)\rangle u(t) - x(t) e^{-2\varrho t} \, dt 
\]
\[
= \Re \int_{-\infty}^{a} (\partial_0 \varrho M_0(m) + M_1(m))(u - x)(t) + v(t) - y(t)|u(t) - x(t)|e^{-2\varrho t} \, dt 
\]
\[
\geq \tilde{c} \int_{-\infty}^{a} |u(t) - x(t)|^2 e^{-2\varrho t} \, dt. \tag{16}
\]

Now, let \( f, g \in H_{\varrho,0}(\mathbb{R}; H) \) with \( \chi_{|a,\infty|}(m)(f - g) = 0 \). Then there exist two sequences \((f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \) in \((\partial_0 \varrho M_0(m) + M_1(m) + A_\varrho)[H_{\varrho,0}(\mathbb{R}; H)]\) such that \( f_n \to f \) and \( g_n \to g \) in \( H_{\varrho,0}(\mathbb{R}; H) \) as \( n \to \infty \). We define 
\[
u := \left( \partial_0 \varrho M_0(m) + M_1(m) + A_\varrho \right)^{-1} f = \lim_{n \to \infty} \left( \partial_0 \varrho M_0(m) + M_1(m) + A_\varrho \right)^{-1} f_n
\]
as well as 
\[
x := \left( \partial_0 \varrho M_0(m) + M_1(m) + A_\varrho \right)^{-1} g = \lim_{n \to \infty} \left( \partial_0 \varrho M_0(m) + M_1(m) + A_\varrho \right)^{-1} g_n.
\]

Using inequality (16) and the continuity of the cut-off operator \( \chi_{|a,\infty|}(m) \) we obtain 
\[
0 = \Re \int_{-\infty}^{a} \langle f(t) - g(t)\rangle u(t) - x(t) e^{-2\varrho t} \, dt 
\]
\[
\geq \tilde{c} \int_{-\infty}^{a} |u(t) - x(t)|^2 e^{-2\varrho t} \, dt,
\]
which yields \( \chi_{|a,\infty|}(m)(u - x) = 0 \). \( \square \)
3 Solution theory

The last part of this subsection is devoted to the independence of the solution operator of problem \(^{7}\) on the parameter \(g > 0\). For doing so, we need the following auxiliary results.

**Proposition 3.14.** Let \(X_0, X_1\) be two Banach spaces and \(V\) a vector space with \(X_0, X_1 \subseteq V\). Moreover, let \(F_0, G_0 : X_0 \to X_0\) and \(F_1, G_1 : X_1 \to X_1\) be Lipschitz-continuous mappings with \(|F_0|_{\text{Lip}} \cdot |G_0|_{\text{Lip}} < 1\) and \(|F_1|_{\text{Lip}} \cdot |G_1|_{\text{Lip}} < 1\). Then for \(i \in \{0, 1\}\) the relation \((F_i^{-1} + G_i)^{-1}\) is a Lipschitz-continuous mapping with domain equal to \(X_i\). Furthermore, if \(F_0|_{X_0 \cap X_1} = F_1|_{X_0 \cap X_1}\) and \(G_0|_{X_0 \cap X_1} = G_1|_{X_0 \cap X_1}\), then

\[
(F_0^{-1} + G_0)^{-1}|_{X_0 \cap X_1} = (F_1^{-1} + G_1)^{-1}|_{X_0 \cap X_1}.
\]

**Proof.** Let \(i \in \{0, 1\}\). For \(x, y \in X_i\) we observe that

\[
(x, y) \in (F_i^{-1} + G_i)^{-1} \iff (y, x) \in F_i^{-1} + G_i
\]

\[
\iff (y, x - G_i(y)) \in F_i^{-1} \iff y = F_i(x - G_i(y)).
\]

(17)

Since for each \(x \in X_i\) the mapping \(y \mapsto F_i(x - G_i(y))\) has a unique fixed point, according to the contraction mapping theorem, we obtain that \((F_i^{-1} + G_i)^{-1}\) is a mapping defined on the whole space \(X_i\). Moreover, using (17), we estimate for \((x_0, y_0), (x_1, y_1) \in (F_i^{-1} + G_i)^{-1}\)

\[
|y_0 - y_1|_{X_i} = |F_i(x_0 - G_i(y_0)) - F_i(x_1 - G_i(y_1))|_{X_i}
\]

\[
\leq |F_i|_{\text{Lip}}(|x_0 - x_1|_{X_i} + |G_i(y_0) - G_i(y_1)|_{X_i})
\]

\[
\leq |F_i|_{\text{Lip}}|x_0 - x_1|_{X_i} + |F_i|_{\text{Lip}}|G_i|_{\text{Lip}}|y_0 - y_1|_{X_i},
\]

and thus

\[
|y_0 - y_1|_{X_i} \leq \frac{|F_i|_{\text{Lip}}}{1 - |F_i|_{\text{Lip}}|G_i|_{\text{Lip}}}|x_0 - x_1|_{X_i},
\]

which proves the Lipschitz-continuity of \((F_i^{-1} + G_i)^{-1}\). Now assume that \(F_0|_{X_0 \cap X_1} = F_1|_{X_0 \cap X_1}\) and \(G_0|_{X_0 \cap X_1} = G_1|_{X_0 \cap X_1}\) and let \(x \in X_0 \cap X_1\). Set \(y_0 := 0 \in X_0 \cap X_1\). For \(n \in \mathbb{N}\) we define

\[
y_{n+1} := F_0(x - G_0(y_n)).
\]

Noting that for \(z \in X_0 \cap X_1\) we have \(G_0(z) = G_1(z) \in X_0 \cap X_1\) and \(F_0(x - G_0(z)) = F_1(x - G_1(z)) \in X_0 \cap X_1\), we can show inductively that \(y_n \in X_0 \cap X_1\) for every \(n \in \mathbb{N}\). Moreover, by the contraction mapping theorem and (17) we get that

\[
(F_0^{-1} + G_0)^{-1}(x) = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} F_0(x - G_0(y_n)) = \lim_{n \to \infty} F_1(x - G_1(y_n)) = (F_1^{-1} + G_1)^{-1}(x).
\]

**Lemma 3.15.** Let \(\nu \geq g > 0\), \(a \in \mathbb{R}\) and \(u \in H_{\nu,0}(\mathbb{R}; H)\) such that the support \(\text{spt } u \subseteq [a, \infty[\). Then \(u \in H_{\nu,0}(\mathbb{R}; H)\). Moreover, if \(u \in H_{\nu,1}(\mathbb{R}; H)\) then \(u \in H_{\nu,1}(\mathbb{R}; H)\) with

\[
\partial_{0,\nu} u = \partial_{0,\nu} u.
\]
3.2 Causality

Proof. We estimate

$$\int_{\mathbb{R}} |u(t)|^2 e^{-2\nu t} \, dt = \int_{a}^{\infty} |u(t)|^2 e^{-2\nu t} \, dt \leq |u|_{H_{\nu,0}(\mathbb{R};H)}^2 e^{2(\nu-\nu)a},$$

(18)

which yields the first assertion. Assume now that \( u \in H_{\nu,1}(\mathbb{R};H) \). Let \( \phi \in C_c^\infty(\mathbb{R};H) \) and define \( \psi(t) := e^{2(\nu-\nu)t} \phi(t) \) for \( t \in \mathbb{R} \). Then clearly \( \psi \in C_c^\infty(\mathbb{R};H) \) and we compute

$$\langle u|\partial_{0,\nu}^* \phi \rangle_{H_{\nu,0}(\mathbb{R};H)} = \int_{a}^{\infty} \langle u(t)| - \phi'(t) + 2\nu \phi(t) \rangle e^{-2\nu t} \, dt$$

$$= \int_{a}^{\infty} \langle u(t)| - \psi'(t) + 2\nu \psi(t) \rangle e^{-2\nu t} \, dt$$

$$= \langle u|\partial_{0,\nu}^* \psi \rangle_{H_{\nu,0}(\mathbb{R};H)}$$

$$= \langle \partial_{0,\nu} u|\psi \rangle_{H_{\nu,0}(\mathbb{R};H)}$$

$$= \int_{\mathbb{R}} \langle (\partial_{0,\nu} u)(t)| \phi(t) \rangle e^{-2\nu t} \, dt.$$

If \( \phi \) is supported on \([-\infty, a]\), the latter yields that

$$\int_{\mathbb{R}} \langle (\partial_{0,\nu} u)(t)| \phi(t) \rangle e^{-2\nu t} \, dt = 0.$$

Thus, the fundamental lemma of variational calculus implies \( \text{spt} \partial_{0,\nu} u \subseteq [a, \infty[ \) and thus, \( \partial_{0,\nu} u \in H_{\nu,0}(\mathbb{R};H) \) by (18). Hence, by the computation above

$$\langle u|\partial_{0,\nu}^* \phi \rangle_{H_{\nu,0}(\mathbb{R};H)} = \langle \partial_{0,\nu} u|\phi \rangle_{H_{\nu,0}(\mathbb{R};H)}$$

for each \( \phi \in C_c^\infty(\mathbb{R};H) \), which yields the assertion, since \( C_c^\infty(\mathbb{R};H) \) is a core for \( \partial_{0,\nu}^* \). \( \square \)

We are now able to prove the independence of the parameter \( \nu \) of the solution operator associated with (11).

**Lemma 3.16.** Let \( \delta > 0 \) and choose \( \nu_0 > \delta \) according to Theorem 3.10. Let \( \nu \geq \delta \geq \nu_0 \) and \( f \in H_{\nu,0}(\mathbb{R};H) \cap H_{\nu,0}(\mathbb{R};H) \). Then

$$(\partial_{0,\nu} M_0(m) - M_0'(m) + \delta + A_\phi)^{-1}(f) = (\partial_{0,\nu} M_0(m) - M_0'(m) + \delta + A_\nu)^{-1}(f).$$

**Proof.** We begin to show the assertion for \( f \in C_c^\infty(\mathbb{R};H) \subseteq H_{\nu,0}(\mathbb{R};H) \cap H_{\nu,0}(\mathbb{R};H) \). Then there exists \( a \in \mathbb{R} \) such that \( \chi_{[a,\infty[}(m)f = f \). Due to Proposition 3.9 there exist \( u_\nu \in D(\partial_{0,\nu} M_0(m)) \) and \( u_\phi \in D(\partial_{0,\phi} M_0(m)) \) such that

$$(u_\phi, f) \in \partial_{0,\phi} M_0(m) - M_0'(m) + \delta + A_\phi \text{ and } (u_\nu, f) \in \partial_{0,\nu} M_0(m) - M_0'(m) + \delta + A_\nu.$$

Furthermore by using Proposition 3.13 and \((0,0) \in A \) we get that \( \text{spt} u_\phi \subseteq [a, \infty[ \) and hence
3 Solution theory

\[ u_\varphi \in H_{\nu,0}(\mathbb{R};H) \]  
and

\[ \partial_{0,\varphi} M_0(m) u_\varphi = \partial_{0,\nu} M_0(m) u_\varphi \]

by Lemma 3.15. Thus, we get that

\[ (u_\varphi, f) \in \partial_{0,\nu} M_0(m) - M'_0(m) + \delta + A_\nu \]

and by the uniqueness of the solution, this yields \( u_\varphi = u_\nu \). Now let \( f \in H_{\varphi,0}(\mathbb{R};H) \cap H_{\nu,0}(\mathbb{R};H) \). Then there exists a sequence \( (f_n)_{n \in \mathbb{N}} \in C_c^\infty(\mathbb{R};H)^N \) which converges to \( f \) in both spaces \( H_{\varphi,0}(\mathbb{R};H) \) and \( H_{\nu,0}(\mathbb{R};H) \) (cf. [24, Lemma 3.5]). Due to the continuity of the solution operators and what we have shown above, we get that

\[
\begin{align*}
\left( \partial_{0,\nu} M_0(m) - M'_0(m) + \delta + A_\nu \right)^{-1} (f) &= \lim_{n \to \infty} \left( \partial_{0,\nu} M_0(m) - M'_0(m) + \delta + A_\nu \right)^{-1} (f_n) \\
&= \lim_{n \to \infty} \left( \partial_{0,\varphi} M_0(m) - M'_0(m) + \delta + A_\varphi \right)^{-1} (f_n) \\
&= \left( \partial_{0,\varphi} M_0(m) - M'_0(m) + \delta + A_\varphi \right)^{-1} (f).
\end{align*}
\]

Finally, we prove the independence of the parameter \( \varphi \) of the solution operator corresponding to our original problem [7].

**Proposition 3.17.** There exists \( \varphi_0 > 0 \) such that for every \( \nu \geq \varphi \geq \varphi_0 \) and \( f \in H_{\varphi,0}(\mathbb{R};H) \cap H_{\nu,0}(\mathbb{R};H) \) we have that

\[
\left( \partial_{0,\varphi} M_0(m) + M_1(m) + A_\varphi \right)^{-1} (f) = \left( \partial_{0,\nu} M_0(m) + M_1(m) + A_\nu \right)^{-1} (f).
\]

**Proof.** Let \( \delta > 2(|M_1|_{\infty} + |M_0|_{\text{Lip}}) \). The proof will be done in two steps.

(a) According to Theorem 3.10 there exists \( \varphi_1 > 0 \) such that for every \( \varphi \geq \varphi_1 \) the inverse relation \( \left( \partial_{0,\varphi} M_0(m) - M'_0(m) + \delta + A_\varphi \right)^{-1} \) is a Lipschitz-continuous mapping with

\[
\left| \left( \partial_{0,\varphi} M_0(m) - M'_0(m) + \delta + A_\varphi \right)^{-1} \right|_{\text{Lip}} \leq \frac{2}{\delta}.
\]

For \( \nu \geq \varphi \geq \varphi_1 \) we set \( X_0 := H_{\varphi,0}(\mathbb{R};H) \) and \( X_1 := H_{\nu,0}(\mathbb{R};H) \). Moreover, we define \( F_0 := \left( \partial_{0,\varphi} M_0(m) - M'_0(m) + \delta + A_\varphi \right)^{-1} \) and \( F_1 := \left( \partial_{0,\nu} M_0(m) - M'_0(m) + \delta + A_\nu \right)^{-1} \). Then by Lemma 3.16 we have that

\[
F_0|_{X_0 \cap X_1} = F_1|_{X_0 \cap X_1}.
\]

Moreover, we set \( G_0 \) and \( G_1 \) as \( M_1(m) + M'_0(m) \) interpreted as bounded linear operators in \( X_0 \) and \( X_1 \), respectively. Then by definition

\[
G_0|_{X_0 \cap X_1} = G_1|_{X_0 \cap X_1}.
\]
Furthermore, $|F_i|_{\text{Lip}} \cdot |G_i|_{\text{Lip}} \leq \frac{1}{2}$ for $i \in \{0, 1\}$ and thus, by Proposition 3.14

$$
\left( \partial_{\theta,e} M_0(m) + M_1(m) + \delta + A_e \right)^{-1} |_{X_0 \cap X_1} = (F_0^{-1} + G_0)^{-1} |_{X_0 \cap X_1}
= (F_1^{-1} + G_1)^{-1} |_{X_0 \cap X_1}
= \left( \partial_{\theta,\nu} M_0(m) + M_1(m) + \delta + A_\nu \right)^{-1} |_{X_0 \cap X_1}.
$$

(b) Let $\varrho_1$ be as in (a). By Theorem 3.11 there exists $\varrho_0 \geq \varrho_1$ such that for every $\varrho \geq \varrho_0$ the inverse relation $\left( \partial_{\theta,e} M_0(m) + M_1(m) + \delta + A_e \right)^{-1}$ is a Lipschitz-continuous mapping with

$$
\left| \left( \partial_{\theta,e} M_0(m) + M_1(m) + \delta + A_e \right)^{-1} \right|_{\text{Lip}} < \frac{1}{\delta}.
$$

Let $\nu \geq \varrho \geq \varrho_0$ and set $X_0 := H_{\varrho,0}(\mathbb{R}; H)$ and $X_1 := H_{\nu,0}(\mathbb{R}; H)$. Moreover, we define $F_0 := \left( \partial_{\theta,e} M_0(m) + M_1(m) + \delta + A_e \right)^{-1}$ and $F_1 := \left( \partial_{\theta,\nu} M_0(m) + M_1(m) + \delta + A_\nu \right)^{-1}$ and by (a) we have

$$
F_0|_{X_0 \cap X_1} = F_1|_{X_0 \cap X_1}.
$$

Defining $G_0$ and $G_1$ as $-\delta$, interpreted as bounded linear operators in $X_0$ and $X_1$, respectively, we derive the assertion by using Proposition 3.14.

\[ \square \]

4 Examples

In this section we apply our solution theory to two concrete systems out of the theory of plasticity. The first one, dealing with thermoplasticity, couples the heat equation with the equations of plasticity, where the stress tensor and the inelastic part of the strain tensor are linked via a maximal monotone relation. The second example deals with the equations of viscoplasticity, where the inelastic strain tensor is given in terms of an internal variable, satisfying a differential inclusion. Before we can state the equations, we have to define the differential operators involved. Throughout, let $\Omega \subseteq \mathbb{R}^3$ be an arbitrary domain.

**Definition 4.1.** We define the operator $\text{grad}_c$ (the gradient with Dirichlet-type boundary conditions) as the closure of

$$
\text{grad} |_{C_c^\infty(\Omega)} : C_c^\infty(\Omega) \subseteq L_2(\Omega) \rightarrow L_2(\Omega)^3,
$$

$$
\phi \mapsto (\partial_i \phi)_{i \in \{1,2,3\}}
$$

and the operator $\text{div}_c$ (the divergence with Neumann-type boundary conditions) as the closure of

$$
\text{div} |_{C_c^\infty(\Omega)^3} : C_c^\infty(\Omega)^3 \subseteq L_2(\Omega)^3 \rightarrow L_2(\Omega)
$$

$$
(\psi_i)_{i \in \{1,2,3\}} \mapsto \sum_{i=1}^3 \partial_i \psi_i.
$$
4 Examples

Using integration by parts one easily sees that
\[
\begin{align*}
\text{grad} c & \subseteq (- \text{div} c)^*, \\
\text{div} c & \subseteq (- \text{grad} c)^*.
\end{align*}
\]

We set \( \text{grad} := (- \text{div} c)^* \) (the distributional gradient with maximal domain in \( L_2(\Omega) \)) and \( \text{div} := (- \text{grad} c)^* \) (the distributional divergence with maximal domain in \( L_2(\Omega)^3 \)). Moreover, we define the matrix-valued symmetrized gradient and the vector-valued divergence, by setting \( \text{Grad}_c \) to be the closure of \( \text{Grad}|_{C_c^\infty(\Omega)^3} : C_c^\infty(\Omega)^3 \subseteq L_2(\Omega)^3 \rightarrow L_{2,\text{sym}}(\Omega)^{3 \times 3} \)
\[
(\psi_i)_{i \in \{1,2,3\}} \mapsto \frac{1}{2} (\partial_i \psi_j + \partial_j \psi_i)_{i,j \in \{1,2,3\}}
\]
and by defining \( \text{Div}_c \) as the closure of 
\[
\begin{align*}
\text{Div}|_{C_c^\infty(\Omega)^{3 \times 3}} : C_c^\infty(\Omega)^{3 \times 3} \subseteq L_{2,\text{sym}}(\Omega)^{3 \times 3} \rightarrow L_2(\Omega)^3 \\
(T_{ij})_{i,j \in \{1,2,3\}} \mapsto \left( \sum_{j=1}^3 \partial_j T_{ij} \right)_{i \in \{1,2,3\}}
\end{align*}
\]

Here \( L_{2,\text{sym}}(\Omega)^{3 \times 3} \) denotes the space of \( L_2 \) functions attaining values in the space of symmetric \( 3 \times 3 \) matrices, equipped with the Frobenius inner product
\[
\langle T | S \rangle_{L_{2,\text{sym}}(\Omega)^{3 \times 3}} := \int_\Omega \text{trace} \left( (T(x))^* S(x) \right) \, dx \quad (T, S \in L_{2,\text{sym}}(\Omega)^{3 \times 3})
\]
and \( C_c^\infty(\Omega)^{3 \times 3} \) denotes the space of test functions with values in the space of symmetric \( 3 \times 3 \) matrices. Like in the scalar-valued case we obtain
\[
\begin{align*}
\text{Grad}_c & \subseteq (- \text{Div}_c)^* =: \text{Grad}, \\
\text{Div}_c & \subseteq (- \text{Grad}_c)^* =: \text{Div}.
\end{align*}
\]

4.1 Thermoplasticity

We denote by \( u : \mathbb{R} \rightarrow L_2(\Omega)^3 \) the displacement field, by \( \theta : \mathbb{R} \rightarrow L_2(\Omega) \) the temperature density and by \( T : \mathbb{R} \rightarrow L_{2,\text{sym}}(\Omega)^{3 \times 3} \) the stress tensor of a body \( \Omega \subseteq \mathbb{R}^3 \). The equations of thermoplasticity read as
\[
\begin{align*}
\partial_{t,\varrho} M \partial_{t,\varrho} u - \text{Div} T &= f, \quad (19) \\
\partial_{t,\varrho} w \theta - \text{div} \kappa \text{grad} \theta + \tau_0 \text{trace} \text{Grad} \partial_{t,\varrho} u &= g. \quad (20)
\end{align*}
\]

Here \( w : \mathbb{R} \rightarrow L_\infty(\Omega) \), modelling the time-dependent mass density, is assumed to be bounded and Lipschitz-continuous, \( w(t) \) is real-valued for each \( t \in \mathbb{R} \) and \( w \) is uniformly positive.\(^\text{10}\)

Likewise \( M : \mathbb{R} \rightarrow L(L_2(\Omega)^3) \) is a bounded, Lipschitz-continuous function, \( M(t) \) is selfadjoint

\(^{10}\)This means that there exists a positive constant \( c > 0 \) such that \( w(t) \geq c \) for each \( t \in \mathbb{R} \).
4.1 Thermoplasticity

for each $t \in \mathbb{R}$ and $M$ is uniformly strictly positive definite and $\tau_0 > 0$ is a numerical parameter. The function $\kappa : \mathbb{R} \rightarrow L(L_2(\Omega)^3)$ describes the (time-dependent) heat conductivity of $\Omega$ and is assumed to be measurable, bounded and $\Re \kappa(t) = \frac{1}{2} (\kappa(t) + \kappa(t)^*) \geq \bar{c} > 0$ for every $t \in \mathbb{R}$. The right hand sides $f : \mathbb{R} \rightarrow L_2(\Omega)^3$ and $g : \mathbb{R} \rightarrow L_2(\Omega)$ are given source terms. The operator trace is defined by

$$\text{trace} : L_2(\Omega)^{3 \times 3} \rightarrow L_2(\Omega)$$

$$(S_{ij})_{i,j \in \{1,2,3\}} \mapsto \sum_{i=1}^{3} S_{ii}.$$  

The stress tensor $T$ is coupled with the strain tensor $\text{Grad} u$ via the constitutive relation\(^{11}\)

$$T = C (\text{Grad} u - \varepsilon_p) - c \text{trace}^* \theta,$$  

where $C : \mathbb{R} \rightarrow L(L_2(\Omega)^{3 \times 3})$ is bounded, Lipschitz-continuous and uniformly strictly positive definite, $C(t)$ is selfadjoint for every $t \in \mathbb{R}$ and $c > 0$ is a coupling parameter. The function $\varepsilon_p : \mathbb{R} \rightarrow L_2(\Omega)^{3 \times 3}$ denotes the inelastic part of the strain tensor and is linked to $T$ by

$$(T, \partial_0, \varepsilon_p) \in \mathbb{I},$$

where $\mathbb{I} \subseteq L_2(\Omega)^{3 \times 3} \oplus L_2(\Omega)^{3 \times 3}$ is a bounded maximal monotone relation with $(0,0) \in \mathbb{I}$ and for every element $S \in \mathbb{I} \left( L_2(\Omega)^{3 \times 3} \right)$ we have that trace $S = 0$. This system was considered for the autonomous case in \(^{22}\) and earlier by \(^{5}\) for the autonomous, quasistatic case. Following \(^{22}\), we rewrite the system in the following way: Define $v := \partial_0 u$ and $q := c\tau_0^{-1} \kappa \text{grad} \theta$. Then \(^{19}\) and \(^{20}\) can be written as

$$\partial_0 M v - \text{div} T = f,$$

$$\partial_0 c\tau_0^{-1} w \theta - \text{div} q + \text{trace} c \text{Grad} v = c\tau_0^{-1} g.$$  

Moreover, applying $\partial_0 C^{-1}$ to \(^{21}\) yield\(^{15}\)

$$\partial_0 C^{-1} T + \partial_0 \varepsilon_p + \partial_0 C^{-1} c \text{trace}^* \theta = \text{Grad} v,$$  

which gives

$$\partial_0 c \text{trace} c C^{-1} T + \partial_0 \text{trace} c C^{-1} c \text{trace}^* \theta = \text{trace} c \text{Grad} v,$$

where we have used that $\partial_0 \varepsilon_p \in \mathbb{I} \left( L_2(\Omega)^{3 \times 3} \right) \subseteq \{0\}$ trace by \(^{22}\). Using this representation of trace $c \text{Grad} v$, \(^{23}\) reads as

$$\partial_0 \left( c\tau_0^{-1} w + \text{trace} c C^{-1} c \text{trace}^* \right) \theta + \partial_0 c \text{trace} c C^{-1} T - \text{div} q = c\tau_0^{-1} g.$$  

\(^{11}\)The adjoint of the operator trace is given by

$$\text{trace}^* f = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix} (f \in L_2(\Omega)).$$  

\(^{12}\)Note that $C(t)$ is invertible for each $t \in \mathbb{R}$ and $C^{-1} : t \mapsto C(t)^{-1}$ is bounded, Lipschitz-continuous and uniformly strictly positive definite.
4 Examples

Using these equations, the system (19)-(22) can be written as

\[
\begin{pmatrix}
  v \\
  T \\
  \theta \\
  q
\end{pmatrix}
\in \partial_{0,2}
\begin{pmatrix}
  M & 0 & 0 & 0 \\
  0 & C^{-1} & C^{-1} c \text{trace}^* & 0 \\
  0 & \text{trace } c C^{-1} C_0^{-1} w + \text{trace } c C^{-1} c \text{trace}^* & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

Indeed, this system fits into our general framework with

\[
M_0(t) := \begin{pmatrix}
  M(t) & 0 & 0 & 0 \\
  0 & C(t)^{-1} & C(t)^{-1} c \text{trace}^* & 0 \\
  0 & \text{trace } c C(t)^{-1} C_0^{-1} w(t) + \text{trace } c C(t)^{-1} c \text{trace}^* & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
M_1(t) := \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & \kappa(t)^{-1} c^{-1} C_0
\end{pmatrix}
\]

and

\[
A := \begin{pmatrix}
  0 & -\text{Div} & 0 & 0 \\
  -\text{Grad} & I & 0 & 0 \\
  0 & 0 & 0 & -\text{div} \\
  0 & 0 & -\text{grad} & 0
\end{pmatrix}
\]

The assumptions on the coefficient ensure that our solvability conditions (a)-(d) are satisfied. Moreover, by imposing suitable boundary conditions, e.g. Dirichlet boundary conditions for \( v \) and \( \theta \), we obtain that

\[
\begin{pmatrix}
  0 & -\text{Div} & 0 & 0 \\
  -\text{Grad}_c & 0 & 0 & 0 \\
  0 & 0 & 0 & -\text{div} \\
  0 & 0 & -\text{grad}_c & 0
\end{pmatrix}
\]

is skew-selfadjoint and hence, maximal monotone. Since \( I \) is bounded and maximal monotone, Corollary 2.11 yields the maximal monotonicity of \( A \). Moreover, \( (0, 0) \in A \) by assumption.

4.2 Viscoplasticity with internal variables

In this section we consider the equations of viscoplasticity with internal variables. The problem belongs to the class of constitutive equations which are studied in [1].

As in the thermoplastic case we denote by \( u: \mathbb{R} \to L_2(\Omega)^3 \) the displacement field of a medium \( \Omega \subseteq \mathbb{R}^3 \) and by \( T: \mathbb{R} \to L_{2,\text{sym}}(\Omega)^{3 \times 3} \) the stress tensor. Moreover, \( z: \mathbb{R} \to L_2(\Omega)^N \), where \( N \in \mathbb{N} \), is the vector of internal variables. The model equations of viscoplasticity with internal
variables are
\[ \partial_{0,e}M \partial_{0,e}u - \text{Div} T = f, \]
\[ T = D(\text{Grad} u - Bz), \]
\[ (B^*T - Lz, \partial_{0,e}z) \in g, \]
where \( M : \mathbb{R} \to L(L_2(\Omega)^3), \) the elasticity tensor \( D : \mathbb{R} \to L(L_{2,\text{sym}}(\Omega)^{3 \times 3}) \) and \( L : \mathbb{R} \to L(L_2(\Omega)^N) \) are bounded, Lipschitz-continuous, uniformly strictly positive definite functions and \( M(t), D(t), L(t) \) are selfadjoint for each \( t \in \mathbb{R} \). Furthermore, \( g \subseteq L_2(\Omega)^N \otimes L_2(\Omega)^N \) is a bounded, maximal monotone relation with \( (0, 0) \in g \) and \( B : L_2(\Omega)^N \to L_{2,\text{sym}}(\Omega)^{3 \times 3} \) is linear and continuous. The mapping \( B \) is linked to the inelastic part of the strain tensor \( \varepsilon = \text{Grad} u \) by \( \varepsilon_p = Bz \). The volume force \( f : \mathbb{R} \to L_2(\Omega)^3 \) is given. This system was studied in \( [2] \) in the autonomous, quasi-static case, where the focus was on the regularity of solutions.

To apply our solution theory to these equations we have to rewrite the system. For doing so, we define \( v := \partial_{0,e}u \) and \( w := B^*T - Lz \). Thus, we obtain \( z = L^{-1}(B^*T - w) \) and we can reformulate \( (25) \) and \( (27) \) by
\[ \partial_{0,e}Mv - \text{Div} T = f, \]
\[ (w, \partial_{0,e}L^{-1}(B^*T - w)) \in g. \]
Moreover, applying \( \partial_{0,e}D^{-1} \) to equation \( (26) \) yields
\[ \partial_{0,e}D^{-1}T = \partial_{0,e}(\text{Grad} u - BL^{-1}(B^*T - w)) = \text{Grad} v - \partial_{0,e}BL^{-1}B^*T + \partial_{0,e}BL^{-1}w. \]
Hence, the system \( (25)-27 \) can be written as
\[
\begin{pmatrix}
  v \\
  w \\
  T
\end{pmatrix}
= \begin{pmatrix}
  f \\
  0 \\
  0
\end{pmatrix}
\in \partial_{0,e}
\begin{pmatrix}
  M & 0 & 0 \\
  0 & L^{-1} & -L^{-1}B^* \\
  0 & -BL^{-1} & D^{-1} + BL^{-1}B^*
\end{pmatrix}
+ \begin{pmatrix}
  0 & 0 & -\text{Div} \\
  0 & g & 0 \\
  -\text{Grad} & 0 & 0
\end{pmatrix},
\]
which fits into our general framework. The operator
\[
M_0(t) := \begin{pmatrix}
  M(t) & 0 & 0 \\
  0 & L^{-1}(t) & -L(t)^{-1}B^* \\
  0 & -BL(t)^{-1} & D(t)^{-1} + BL(t)^{-1}B^*
\end{pmatrix}
\]
satisfies the solvability conditions (a) and (b). Concerning (c) and (d), we observe that by means of a symmetric Gauß step
\[
\begin{pmatrix}
  L^{-1}(t) & -L(t)^{-1}B^* \\
  -BL(t)^{-1} & D(t)^{-1} + BL(t)^{-1}B^*
\end{pmatrix}
\]
is strictly positive definite if and only if
\[
\begin{pmatrix}
  L^{-1}(t) & 0 \\
  0 & D(t)^{-1}
\end{pmatrix}
\]
References

is strictly positive definite, which holds by our assumptions on $L$ and $D$. The maximal monotonicity of

$$A := \begin{pmatrix} 0 & 0 & -\text{Div} \\ 0 & g & 0 \\ -\text{Grad} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\text{Div} \\ 0 & 0 & 0 \\ -\text{Grad} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

can be obtained by imposing suitable boundary conditions on $v$ and $T$ in order to make

$$\begin{pmatrix} 0 & 0 & -\text{Div} \\ 0 & 0 & 0 \\ -\text{Grad} & 0 & 0 \end{pmatrix}$$

skew-selfadjoint and using Corollary 2.11 (compare Example 4.1).

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