Closed-form travelling wave solutions to the nonlinear space-time fractional coupled Burgers’ equation

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1. Introduction

Fractional calculus was first introduced by Leibniz in 1695 as a generalization of ordinary calculus (Diethelm, 2010). The behaviour of natural phenomena at sufficiently small scales can be better described by fractional order differential equation than the differential equation of integer order. As a result, fractional differential equations have gained considerable popularity and importance because of their realistic application in various fields of science and engineering, such as in signal processing, control theory, systems identification, solid state physics, plasma physics, optical fibres, chemical kinematics, electrical circuits, bio-genetics, fluid flow and other areas (Oldham and Spanier, 1974; Kilbas, Srivasta, & Trujillo, 2006). Researchers have recently paid great attention to constructing closed-form travelling wave solutions to the nonlinear evolution equations of fractional order to analyse natural phenomena. The closed-form travelling wave solutions of the fractional equations (Seadawy, 2017e; Baleanu, Diethelm, Scals, & Trujillo, 2012; Islam, Akbar, & Azad, 2018; Liu, Li, Zhang, & Liu, 2015) are very helpful for understanding the mechanisms of the phenomena, as well as their further application in practical life. Numerous influential methods have been proposed to investigate the exact travelling wave solutions to nonlinear partial differential equations (NPDEs) of fractional order as well as integer order, as for instance the symmetry group method (El-Shiekh, 2018), the \text{exp}(-\phi(\xi))-expansion method (Kaplan & Akbulut, 2018), the direct algebraic method (Seadawy, 2012), the fractional sub-equation method (Zhang & Zhang, 2011), the extended direct algebraic mapping method (Seadawy, 2016b), the Adomian decomposition method (El-Sayed, Behiry, & Raslan, 2010; Hu and He, 2016), the variational iteration method (Singh and Kumar, 2017; Tang, Fan, Zhao, & Wang, 2016), the modified extended direct algebraic mapping method (Seadawy, 2016c), the auxiliary equation mapping method and direct algebraic mapping method (Seadawy & Lu 2016), the \((G'/G)\)-expansion method and its various modifications (Alam & Akbar, 2014; Feng, Li, & Wan, 2011; Islam, Akbar, & Azad, 2017), the amplitude ansatz method (Seadawy & Lu 2017; Seadawy, 2017a), multiple scales methods (Seadawy, 2017b), the homotopy perturbation method (Cherif, Belghaba, & Ziane, 2016; He, 1999), the extended auxiliary equation method (Seadawy, 2017c), the mathematical methods (Seadawy, 2017d). The differential...
transformation method (Sepasgozar, Faraji, & Valipour, 2017), the extended modified mapping method (Seadawy, 2018), auxiliary equation method (Tariq and Seadawy, 2017), the finite element method (Gao, Sun, & Zhang, 2012; Huang, Huang, & Zhan, 2008), the extended tanh method to investigate exact analytic solutions of the equation mentioned above.

2. Explanation of the methods

In this section, we discuss the main steps of the suggested methods to examine closed-form travelling wave solutions to nonlinear evolution equations of fractional order. A fractional partial differential equation in the independent variables \( t, x_1, x_2, ..., x_n \) is supposed to be as follows:

\[
F(u_1, ..., u_k, \frac{\partial u_1}{\partial t}, \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_1}, ..., \frac{\partial u_1}{\partial x_n}, \frac{\partial u_1}{\partial x_n}, D_\xi^\alpha u_1, ..., D_\xi^\alpha u_k, D_\xi^\beta u_1, ..., D_\xi^\beta u_k) = 0,
\]

where \( u_i = u_i(t, x_1, x_2, ..., x_n) \), \( i = 1, ..., k \) are unknown functions, \( F \) is a polynomial in \( u_i \) and its various partial derivatives as well as the derivatives of fractional order.

The fractional composite transformation

\[
\xi = \zeta(t, x_1, x_2, ..., x_n), u_i = u_i(t, x_1, x_2, ..., x_n) = U_i(\xi),
\]

reduces Eq. (1) to the following ordinary differential equation of fractional order with respect to the variable \( \xi \):

\[
Q(u_1, ..., u_k, D_\xi^\alpha u_1, ..., D_\xi^\beta u_k, D_\xi^\alpha u_1, ..., D_\xi^\beta u_k) = 0
\]

We might integrate Eq. (3) term by term as many times possible, and the integral constant can be set to zero as soliton solutions are sought. Then the exact travelling wave solutions of Eq. (3) are constructed by the following three suggested methods.

2.1. The rational fractional \((D_\xi^\alpha G/G)-expansion method\)

In this sub-section, we discuss the main steps of the rational fractional \((D_\xi^\alpha G/G)-expansion method\) for finding exact analytic solutions of nonlinear partial differential equations of fractional order.

Step 1: According to the rational fractional \((D_\xi^\alpha G/G)-expansion method\), the wave solution is supposed to be expressed in the form

\[
u(\xi) = \sum_{i=0}^{n} a_i (D_\xi^\alpha G/G)^i \sum_{j=0}^{n} b_i (D_\xi^\beta G/G)^j,
\]

where \(a_i\)'s and \(b_i\)'s are unknown constants to be determined later and \(G = G(\xi)\) satisfies the following auxiliary nonlinear ordinary differential equation of fractional order:
\[ AGD_{\xi}^{2a}G - BGD_{\xi}^{2a}G - EG^2 - C(D_{\xi}^{2a}G)^2 = 0, \]  

where \( D_{\xi}^{2a}G(\xi) \) denotes the conformable fractional derivative of order \( a \) for \( G(\xi) \) with respect to \( \xi \); \( A, B, C \) and \( E \) are real parameters.

The nonlinear fractional complex transformation \( G(\xi) = H(\eta) \), \( \eta = \xi^\alpha / \Gamma(1 + \alpha) \) reduces Eq. (5) into the following second-order ordinary differential equation:

\[ AHH'' - BHH' - EH^2 - C(H')^2 = 0, \]

where solutions are well known. Since \( D_{\xi}^{2a}G(\xi) = D_{\xi}^{2a}H(\eta) = H'(\eta)D_{\xi}^{2a}H = H'(\eta) \), with the aid of the solutions of Eq. (6), we can obtain the solutions of Eq. (5) as follows:

Family 1: When \( B \neq 0 \) and \( \psi = A-C \) and \( \Omega = B^2 + 4E, \)

\[
(D_{\xi}^{2a}G/G) = \frac{B}{2\psi} + \frac{\sqrt{\Omega} C_1 \sinh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right) + C_2 \cosh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right)}{2\psi C_1 \cosh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right) + C_2 \sinh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right)}
\]

(7)

Family 2: When \( B \neq 0 \) and \( \psi = A-C \) and \( \Omega = B^2 + 4E, \)

\[
(D_{\xi}^{2a}G/G) = \frac{B}{2\psi} - \frac{\sqrt{\Omega} C_1 \sinh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right) + C_2 \cosh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right)}{2\psi C_1 \cosh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right) + C_2 \sinh \left( \frac{\sqrt{\Omega} \psi}{2A(\alpha + 1)} \right)}
\]

(8)

Family 3: When \( B \neq 0 \) and \( \psi = A-C \) and \( \Omega = B^2 + 4E, \)

\[
(D_{\xi}^{2a}G/G) = \frac{B}{2\psi} + \frac{\sqrt{\Omega} (\alpha + 1) C_2}{2\psi C_1 \Gamma(\alpha + 1) + C_2 \psi}
\]

(9)

Family 4: When \( B = 0 \) and \( \psi = A-C \) and \( \Delta = \psi E > 0, \)

\[
(D_{\xi}^{2a}G/G) = \sqrt{\Delta} \frac{C_1 \sinh \left( \frac{\sqrt{\Delta} \psi}{2A(\alpha + 1)} \right) + C_2 \cosh \left( \frac{\sqrt{\Delta} \psi}{2A(\alpha + 1)} \right)}{\psi C_1 \cosh \left( \frac{\sqrt{\Delta} \psi}{2A(\alpha + 1)} \right) + C_2 \sinh \left( \frac{\sqrt{\Delta} \psi}{2A(\alpha + 1)} \right)}
\]

(10)

Family 5: When \( B = 0 \) and \( \psi = A-C \) and \( \Delta = \psi E < 0, \)

\[
(D_{\xi}^{2a}G/G) = \frac{\sqrt{-\Delta} C_1 \sin \left( \frac{\sqrt{-\Delta} \psi}{2A(\alpha + 1)} \right) + C_2 \cos \left( \frac{\sqrt{-\Delta} \psi}{2A(\alpha + 1)} \right)}{\psi C_1 \cos \left( \frac{\sqrt{-\Delta} \psi}{2A(\alpha + 1)} \right) + C_2 \sin \left( \frac{\sqrt{-\Delta} \psi}{2A(\alpha + 1)} \right)}
\]

(11)

Step 2: Determine the positive constant \( n \) by taking homogeneous balance between the highest order linear and nonlinear terms appearing in Eq. (3).

Step 3: Utilize Eqs. (4) and (5) into Eq. (3) with the value of \( n \) obtained in step 2, we obtain polynomial in \( D_{\xi}^{2a}G/G \). Setting each coefficient of the resulted polynomial to zero gives a system of algebraic equations for \( a_i \)’s and \( b_i \)’s. Solve this system of equations for \( a_i \)’s and \( b_i \)’s by means of the symbolic computation software, such as Maple.

Step 4: Inserting the values of \( a_i \)’s and \( b_i \)’s into Eq. (4) along with Eqs. (7)–(11), the closed-form travelling wave solutions of the nonlinear evolution equation (1) are obtained.

### 2.2. The exp-function method

In this sub-section, the main steps of the Exp-function method are discussed for finding closed-form travelling wave solutions of nonlinear partial differential equations of fractional order.

Step 1: According to the Exp-function method, the wave solution is supposed to be expressed in the form

\[
u(\xi) = \sum_{p=0}^{d} \sum_{m=-p}^{q} a_n \exp(p \xi) + b_m \exp(m \xi),
\]

where \( p, q, c \) and \( d \) are positive integers which are known to be further determined, \( a_n \) and \( b_m \) are unknown constants.

Step 2: Balance the linear term of lowest order of Eq. (3) with the lowest order nonlinear term to determine the values of \( c \) and \( d \). Similarly, to determine the values of \( p \) and \( q \), balance the linear term of highest order of Eq. (3) with highest order nonlinear term.

Step 3: Substituting Eq. (12) into Eq. (3) with the values of \( c, d, p \) and \( q \) obtained in step 2, we obtain polynomials in \( \exp(\xi) \), for any integer \( r \). Equating like terms to zero gives a system of algebraic equations for \( a_i \) and \( b_i \). Solve this system for \( a_i \) and \( b_i \) by means of the symbolic computation software, such as Maple.

Step 4: Substituting the values that appeared in step 3 into Eq. (12), we obtain closed-form travelling wave solutions of the nonlinear evolution equation (1).

### 2.3. The extended tanh method

In this sub-section, the main steps of the extended tanh method are discussed for obtaining exact analytic wave solutions of nonlinear partial differential equations of fractional order.

Step 1: Suppose the wave solution is expressed as

\[
u(\xi) = \sum_{i=0}^{n} a_i Y^i + \sum_{i=1}^{n} b_i Y^{-i},
\]

for which

\[ Y = \tanh(\mu \xi), \]

where \( \mu \) is any arbitrary constant.

Step 2: Substituting Eq. (13) along with (14) into Eq. (3) with the value of \( n \) obtained at step 2 in 2.1., we obtain polynomial in \( Y \). Setting each coefficient of the resulted polynomial to zero gives a set of algebraic equations for \( a_i \)’s and \( b_i \)’s. Solve this set of equations for \( a_i \)’s and \( b_i \)’s by means of the symbolic computation software, such as Maple.

Step 3: Insert the values that appeared in step 3 into Eq. (13) along with Eq. (14) to construct exact travelling wave solutions to Eq. (1).
3. Formulation of the solutions

In this section, we employ the proposed rational fractional \( (D^2_tG/G) \)-expansion method, the Exp-function method and the extended tanh method to examine the travelling wave solutions to the following nonlinear space-time fractional coupled Burgers’ equations:

\[
\begin{align*}
D^\alpha_t u - D^{2\alpha} u + 2u D^\alpha u + ID^\alpha (uv) &= 0, \\
D^\beta v - D^{2\beta} v + 2v D^\beta v + mD^\beta (uv) &= 0,
\end{align*}
\]

where \( \alpha, \beta \) have appeared as model equation in mathematical physics (Esipov, 1995). It is very significant that the system is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity (Nee and Duan, 1998). The constants \( l \) and \( m \) depend on system parameters such as the Peclet number, the Stokes velocity of particles due to gravity, and the Brownian diffusivity.

Making use of the fractional compound transformation

\[
u(x,t) = u(\xi), \xi = k^{1/\alpha}x + w^{1/\beta}t,
\]

Eq. (15) is converted into the fractional order ODE,

\[
wd^\alpha_t u - k^2 D^{2\alpha} u + 2kuD^\alpha u + kID^\alpha (uv) = 0,
\]

\[
wd^\beta v - k^2 D^{2\beta} v + 2kvD^\beta v + kmD^\beta (uv) = 0,
\]

Now, construct the solutions as follows.

3.1. Solutions through rational fractional \( (D^2_tG/G) \)-expansion method

Considering the homogeneous balance between the highest order derivative and the highest order nonlinear term appearing in Eq. (17), the solution Eq. (4) takes the form

\[
\begin{align*}
\nu(\xi) &= a_0 + a_1 D^\alpha G/G, \\
\nu(\xi) &= b_0 + b_1 D^\beta G/G,
\end{align*}
\]

Substituting Eq. (18) into Eq. (17), the left-hand side is converted into a polynomial in \( D^2_t G/G \). Setting each coefficient of this polynomial to zero yields an over-determined set of algebraic equations for \( a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1, w \) and \( k \). Solving this set of equations with the aid of symbolic computation software, such as Maple, we obtain the following results:

Set 1 :

\[
\begin{align*}
a_0 &= \frac{k(l+1)\left(b_0^2\psi - b_1^2E + b_0b_1B\right) + a_1b_0A(lm-1)}{b_1A(lm-1)}, \\
c_0 &= \frac{d_0(m-1)\left(k(l+1)\left(b_0^2\psi - b_1^2E + b_0b_1B\right) + a_1b_0A(lm-1)\right)}{b_1A(lm-1)}, \\
c_1 &= \frac{a_1d_0(m-1)}{b_1(l-1)}, \\
d_1 &= \frac{b_1d_0}{b_0}.
\end{align*}
\]

where \( a_0, a_1, b_0, b_1, d_0 \) and \( k \) are all arbitrary constants.

Set 2 :

\[
\begin{align*}
2a_0A(lm-1) - kb_1B(l-1) &= 0, \\
c_0 &= \frac{a_0d_0(m-1)}{b_0(l-1)}, \\
c_1 &= \frac{-d_0k(m-1)\psi}{A(lm-1)}, \\
d_1 &= 0, \\
w &= \frac{-k\left(2a_0A(lm-1) - kb_1B(l-1)\right)}{b_0A(l-1)}.
\end{align*}
\]

where \( a_0, b_0, d_0 \) and \( k \) are all arbitrary constants.

Set 3 :

\[
\begin{align*}
a_1 &= \frac{c_1a_0}{c_0}, \\
b_0 &= 0, \\
b_1 &= \frac{d_1a_0(m-1)}{c_0(l-1)}, \\
d_0 &= 0, \\
k &= \frac{c_0A(lm-1)}{d_1E(m-1)}, \\
w &= \frac{-c_0(lm-1)\left(c_0\psi - 2c_1E\right)}{d_1^2E^2(m-1)^2},
\end{align*}
\]

where \( a_0, c_0, c_1 \) and \( d_1 \) are all arbitrary constants.

Inserting the values that appeared in Eq. (19) into solution Eq. (18) yields

\[
\begin{align*}
u(\xi) &= \frac{a_1}{b_1} + \frac{k(l+1)\left(b_0^2\psi - b_1^2E + b_0b_1B\right)}{b_1A(lm-1)(b_0 + b_1D^\alpha G/G)} \\
v(\xi) &= \frac{a_1(m-1)}{b_1(l-1)} + \frac{k(m-1)\left(b_0^2\psi - b_1^2E + b_0b_1B\right)}{b_1A(lm-1)(b_0 + b_1D^\alpha G/G)}
\end{align*}
\]

Eq. (22) along with Eq. (7) after simplification provides the following exact travelling wave solutions according as \( C_1 = 0 \) but \( C_2 \neq 0 \) and \( C_1 \neq 0 \) but \( C_2 = 0 \)
\begin{align*}
\frac{u_1(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\coth\left(\sqrt{\Omega}\xi^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_1(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\coth\left(\sqrt{\Omega}\xi^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{u_2(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tanh\left(\sqrt{\Omega}\xi^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_2(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tanh\left(\sqrt{\Omega}\xi^2/2\Delta\Gamma(1+x)\right)]},
\end{align*}

Using the value of \( \xi \), Eqs. (23) and (24) possess
\begin{align*}
\frac{u_1(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\coth\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_1(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\coth\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{u_2(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tanh\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_2(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tanh\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]},
\end{align*}

where \( w = \frac{k(k-1)[2b_0\psi + b_1B + 2a_1A(lm - 1)]}{b_1A(lm - 1)} \) and \( k \) is an arbitrary constant.

Eq. (22) with the aid of Eq. (8) after simplification gives the following closed-form travelling wave solutions under the conditions \( C_1 = 0 \) but \( C_2 \neq 0 \) and \( C_1 \neq 0 \) but \( C_2 = 0 \):
\begin{align*}
\frac{u_3(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\cot\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_3(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\cot\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{u_4(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tan\left(\sqrt{\Omega}\xi^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_4(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tan\left(\sqrt{\Omega}\xi^2/2\Delta\Gamma(1+x)\right)]},
\end{align*}

The substitution of the value for \( \xi \) in Eqs. (27) and (28) provides
\begin{align*}
\frac{u_3(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\cot\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_3(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\cot\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{u_4(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tan\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]}, \\
\frac{v_4(x,t)}{b_1} &= \frac{a_1}{b_1} + \frac{2k\psi(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)[2b_0\psi + b_1B + b_1\sqrt{\Omega}\tan\left(\sqrt{\Omega}(k^{1/2}x + w^{1/2}t)^2/2\Delta\Gamma(1+x)\right)]},
\end{align*}

where \( w = \frac{k(k-1)[2b_0\psi + b_1B + 2a_1A(lm - 1)]}{b_1A(lm - 1)} \) and \( k \) is an arbitrary constant.

Eq. (22) together with Eq. (9) possesses the wave solution
\begin{align*}
\frac{u_5(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{k(l-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)} \left( \frac{b_0 + B}{2\psi} + \left( \frac{C_1\Gamma(1+x)}{C_2\Gamma(1+x)} \right) \right), \\
\frac{v_5(\xi)}{b_1} &= \frac{a_1}{b_1} + \frac{k(m-1)(b_0^2\psi_b-b_1^2E + b_0b_1B)}{b_1A(lm - 1)} \left( \frac{b_0 + B}{2\psi} + \left( \frac{C_1\Gamma(1+x)}{C_2\Gamma(1+x)} \right) \right),
\end{align*}
Putting the value of $\zeta$ in Eq. (31), we obtain

$$u_s(x, t) = \frac{a_1}{b_1} + \frac{k(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left( b_0 + b_1 \left( \frac{B}{2 \psi} + \frac{C_1 \Gamma(1 + x)}{C_1 \Gamma(1 + x) + C_2 (k^{1/2} x + w^{1/2} t)^2} \right) \right)$$

$$v_s(x, t) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left( b_0 + b_1 \left( \frac{B}{2 \psi} + \frac{C_1 \Gamma(1 + x)}{C_1 \Gamma(1 + x) + C_2 (k^{1/2} x + w^{1/2} t)^2} \right) \right),$$

where $w = -\frac{k(k(1-l)(2b_0^2 \psi + b_1 E) + 2a_1 A(l-1))}{b_1 A(l-1)}$ and $k$ is arbitrary constant.

Eq. (22) with the help of Eq. (10) after simplification possesses the following exact travelling wave solutions as $C_1 = 0$ but $C_2 \neq 0$ and $C_1 \neq 0$ but $C_2 = 0$:

$$u_6(\zeta) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \cosh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_6(\zeta) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \cosh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right],$$

$$u_7(\zeta) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \tanh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_7(\zeta) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \tanh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right].$$

Utilizing the value of $\zeta$, Eqs. (33) and (34) become

$$u_6(x, t) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \cosh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_6(x, t) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \cosh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right],$$

$$u_7(x, t) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \tanh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_7(x, t) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{\Delta \tanh \left( \sqrt{\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right],$$

where $w = -\frac{k(k(1-l)(2b_0^2 \psi + b_1 E) + 2a_1 A(l-1))}{b_1 A(l-1)}$ and $k$ is arbitrary constant.

Eq. (22) along with Eq. (11) under the conditions $C_1 = 0$ but $C_2 \neq 0$ and $C_1 \neq 0$ but $C_2 = 0$ gives the travelling wave solutions

$$u_8(\zeta) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{-\Delta \cot \left( \sqrt{-\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_8(\zeta) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{-\Delta \cot \left( \sqrt{-\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right],$$

$$u_9(\zeta) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi - b_1 \sqrt{-\Delta \tan \left( \sqrt{-\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_9(\zeta) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi - b_1 \sqrt{-\Delta \tan \left( \sqrt{-\Delta \zeta^2 / A \Gamma(1 + x)} \right)} \right].$$

Making use of the value of $\zeta$, Eqs. (37) and (38) provide

$$u_6(x, t) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{-\Delta \cot \left( \sqrt{-\Delta (k^{1/2} x + \omega^{1/2} t)^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_6(x, t) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi + b_1 \sqrt{-\Delta \cot \left( \sqrt{-\Delta (k^{1/2} x + \omega^{1/2} t)^2 / A \Gamma(1 + x)} \right)} \right],$$

$$u_7(x, t) = \frac{a_1}{b_1} + \frac{k \psi(l-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi - b_1 \sqrt{-\Delta \tan \left( \sqrt{-\Delta (k^{1/2} x + \omega^{1/2} t)^2 / A \Gamma(1 + x)} \right)} \right]$$

$$v_7(x, t) = \frac{a_1 (m-1)}{b_1 (l-1)} + \frac{k \psi(m-1)(b_0^2 \psi - b_1^2 E + b_0 b_1 B)}{b_1 A(l-1)} \left[ b_0 \psi - b_1 \sqrt{-\Delta \tan \left( \sqrt{-\Delta (k^{1/2} x + \omega^{1/2} t)^2 / A \Gamma(1 + x)} \right)} \right].$$

where $w = -\frac{k(k(1-l)(2b_0^2 \psi + b_1 E) + 2a_1 A(l-1))}{b_1 A(l-1)}$ and $k$ is arbitrary constant.
In a similar procedure, Eqs. (20) and (21) with the aid of Eq. (18) along with the Eqs. (7)–(11) also provide exact travelling wave solutions to the suggested equation. For simplicity and for the convenience of the reader, all the solutions have not been recorded here.

3.2. Solutions by the exp-function method

Making use of the homogeneous balance the solution Eq. (12) takes the form

\[ u(\xi) = \frac{a_1 e^{-\xi} + a_0 + a_1 e^{\xi}}{b_{-1} e^{-\xi} + b_0 + b_{1} e^{\xi}} \]  \hspace{1cm} (41)

\[ v(\xi) = \frac{c_{-1} e^{-\xi} + c_0 + c_{1} e^{\xi}}{d_{-1} e^{-\xi} + d_0 + d_{1} e^{\xi}} \]  \hspace{1cm} (42)

Substituting Eqs. (41), (42) into Eq. (17), the left-hand side becomes a polynomial in \( e^{\eta} \), where \( n \) is any integer. Setting each coefficient of this polynomial to zero yields a set of algebraic equations (for simplicity, not shown here) for \( a_1, b_{-1}, c_0, d_1, k \) and \( w \). Solving this over-determined set of equations with the aid of computer algebra, like Maple, provides the following results:

Set 1: \( a_{-1} = \frac{b_{-1}(k^2-w)(l-1)}{2k(lm-1)}, a_0 = -\frac{b_0(k^2+w)(l-1)}{2k(lm-1)}, a_1 = 0, b_1 = 0, c_{-1} = 0 \)

\( c_0 = \frac{b_{-1}d_1(k^2-w)(m-1)}{2b_0k(lm-1)}, c_1 = -\frac{d_1(k^2+w)(m-1)}{2k(lm-1)}, d_{-1} = 0, d_0 = \frac{b_{-1}d_1}{b_0} \)  \hspace{1cm} (43)

where \( b_{-1}, b_0, d_1, k \) and \( w \) are all free parameters.

Set 2: \( a_1 = 0, b_{-1} = \pm \frac{a_{-1}\sqrt{-w}(lm-1)}{w(l-1)}, b_0 = \frac{a_1 b_1 w(l-1) - a_0^2 \sqrt{-w}(lm-1)}{a_0 w(l-1)}, c_{-1} = 0, c_0 = \frac{a_0 d_1 (m-1)}{b_1(l-1)}, c_1 = 0, d_{-1} = 0, d_0 = \pm \frac{a_0 d_1 \sqrt{-w}(lm-1)}{b_1 w(l-1)}, k = \mp \sqrt{-w} \)  \hspace{1cm} (44)

where \( a_{-1}, a_0, b_1, d_1 \) and \( w \) are all free parameters.

Set 3: \( a_{-1} = \frac{b_0 c_0 (l-1)}{d_1 (m-1)}, a_0 = 0, a_1 = 0, b_{-1} = \pm \frac{b_0 c_0 \sqrt{-w}(lm-1)}{d_1 w(m-1)}, b_1 = 0, c_{-1} = 0, c_1 = 0, d_{-1} = 0, d_0 = \pm \frac{c_0 \sqrt{-w}(lm-1)}{w(m-1)}, k = \mp \sqrt{-w} \)  \hspace{1cm} (45)

where \( b_0, c_0, d_1 \) and \( w \) are all free parameters.

Using the values that appeared in Eq. (43) into Eqs. (41), (42), we construct the following closed-form travelling wave solutions:

\[ u_1(\xi) = -\frac{l-1}{2k(lm-1)} \frac{b_{-1}(k^2-w)e^{-\xi} - b_0(k^2+w)}{b_{-1} e^{-\xi} + b_0} \]  \hspace{1cm} (46)

\[ v_1(\xi) = -\frac{m-1}{2k(lm-1)} \frac{b_{-1}(k^2-w) - b_0(k^2+w)e^{\xi}}{b_{-1} + b_0 e^{\xi}} \]  \hspace{1cm} (47)

If we assign the parameters as \( b_{-1} = b_0, k = \sqrt{2}, w = 1 \) in Eq. (46) and \( b_{-1} = b_0, k = \sqrt{2}, w = -1 \) in Eq. (47), then after simplification we obtain

\[ u_1(\xi) = -\frac{l-1}{2k(lm-1)} \left(1 + 2\tanh(\xi)/2 \right) \]  \hspace{1cm} (48)

\[ v_1(\xi) = -\frac{m-1}{2k(lm-1)} \left(1 - 2\tanh(\xi)/2 \right) \]  \hspace{1cm} (49)

The substitution of the value for \( \xi \) in Eqs. (48), (49) yields

\[ u_1(x, t) = -\frac{l-1}{2k(lm-1)} \left(1 + 2\tanh(k^{1/2}x + w^{1/2}t)/2 \right) \]  \hspace{1cm} (50)

\[ v_1(x, t) = -\frac{m-1}{2k(lm-1)} \left(1 - 2\tanh(k^{1/2}x + w^{1/2}t)/2 \right) \]  \hspace{1cm} (51)

Inserting the values appearing in Eq. (44) into Eqs. (41) and (42) provides the following exact travelling wave solutions:
where \( k = \mp \sqrt{-w} \) and \( w \) is arbitrary constant.

Substituting \( Eq. (44) \) into \( Eqs. (41), (42) \) gives the following wave solutions:

\[
\begin{align*}
  u_{2,3}(x, t) &= \frac{a_0 w(l-1) \{ \cosh(k^{1/2}x + w^{1/2}t) - \sinh(k^{1/2}x + w^{1/2}t) \} + a_1 b_1 w(l-1)}{\pm \sqrt{-w(l-1)} + w(l-1) \{ \cosh(k^{1/2}x + w^{1/2}t) + \sinh(k^{1/2}x + w^{1/2}t) \}}, \\
  v_{2,3}(x, t) &= \frac{w(m-1)}{\pm \sqrt{-w(l-1)} + w(l-1) \{ \cosh(k^{1/2}x + w^{1/2}t) + \sinh(k^{1/2}x + w^{1/2}t) \}}.
\end{align*}
\]

where \( k = \mp \sqrt{-w} \) and \( w \) is arbitrary constant.

The above obtained solutions containing many free parameters are new and more general than existing results.

### 3.3. Solutions constructed by the extended tanh method

The homogeneous balance reduces the solution \( Eq. (13) \) to the form

\[
\begin{align*}
  u(\zeta) &= a_0 + a_1 Y + b_1 Y^{-1}, \\
  v(\zeta) &= c_0 + c_1 Y + d_1 Y^{-1}.
\end{align*}
\]

Substituting \( Eq. (64) \) into \( Eq. (17) \) along with \( Eq. (14) \), the left-hand side becomes a polynomial in \( Y \). Setting each coefficient of this polynomial to zero, yields a set of algebraic equations (for simplicity, not shown here) for \( a_0, a_1, b_1, c_1, d_1, k \) and \( w \). Solving this over-determined set of equations with the aid of computer algebra, like Maple, provides the following results:

\[
\begin{align*}
  a_0 &= -\frac{w(l-1)}{2k(l-1)} , a_1 &= -\frac{k l (l-1)}{l m - 1}, b_1 &= -\frac{k l (l-1)}{l m - 1}, c_0 &= -\frac{w(m-1)}{2k(l-1)}, \\
  c_1 &= -\frac{k l (m-1)}{l m - 1}, d_1 &= -\frac{k l (m-1)}{l m - 1},
\end{align*}
\]

where \( k \) and \( w \) are arbitrary constants.
where $k$ and $w$ are arbitrary constants.

Using Eq. (65) into Eq. (64) together with Eq. (14), we have the following solitary wave solutions:

\[
\begin{align*}
  u_1(\zeta) &= -\frac{l-1}{2k(lm-1)} \left[ w + 2k^2 \mu \{ \tanh(\mu \zeta) + \coth(\mu \zeta) \} \right] \\
  v_1(\zeta) &= -\frac{m-1}{2k(lm-1)} \left[ w + 2k^2 \mu \{ \tanh(\mu \zeta) + \coth(\mu \zeta) \} \right]
\end{align*}
\]  

(68)

Using the value of $\zeta$, Eq. (68) becomes

\[
\begin{align*}
  u_1(x,t) &= -\frac{l-1}{2k(lm-1)} \left[ w + 2k^2 \mu \{ \tanh(\mu(k^{1/2}x + w^{1/2}t)) + \coth(\mu(k^{1/2}x + w^{1/2}t)) \} \right] \\
  v_1(x,t) &= -\frac{m-1}{2k(lm-1)} \left[ w + 2k^2 \mu \{ \tanh(\mu(k^{1/2}x + w^{1/2}t)) + \coth(\mu(k^{1/2}x + w^{1/2}t)) \} \right]
\end{align*}
\]  

(68)

where $k$ and $w$ are arbitrary constants.

Substituting Eq. (66) into Eq. (64) along with Eq. (14), provides the solutions

\[
\begin{align*}
  u_2(\zeta) &= -\frac{l-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \tanh(\mu \zeta) \right\} \\
  v_2(\zeta) &= -\frac{m-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \tanh(\mu \zeta) \right\}
\end{align*}
\]  

(70)

Putting the value of $\zeta$ in Eq. (70) yields

\[
\begin{align*}
  u_2(x,t) &= -\frac{l-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \tanh(\mu(k^{1/2}x + w^{1/2}t)) \right\} \\
  v_2(x,t) &= -\frac{m-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \tanh(\mu(k^{1/2}x + w^{1/2}t)) \right\}
\end{align*}
\]  

(71)

where $k$ and $w$ are arbitrary constants.

Inserting Eq. (67) into Eq. (64) together with Eq. (14) gives the following solitary wave solutions:

\[
\begin{align*}
  u_3(\zeta) &= -\frac{l-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \coth(\mu \zeta) \right\} \\
  v_3(\zeta) &= -\frac{m-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \coth(\mu \zeta) \right\}
\end{align*}
\]  

(72)

Use the value of $\zeta$ in Eq. (72) yields

\[
\begin{align*}
  u_3(x,t) &= -\frac{l-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \coth(\mu(k^{1/2}x + w^{1/2}t)) \right\} \\
  v_3(x,t) &= -\frac{m-1}{2k(lm-1)} \left\{ w + 2k^2 \mu \coth(\mu(k^{1/2}x + w^{1/2}t)) \right\}
\end{align*}
\]  

(73)

where $k$ and $w$ are arbitrary constants.

The above solutions obtained by the extended tanh method are new and more general. As far as we know, these results have not been recorded in the previous literature.
4. Conclusions

The solutions achieved throughout this article are new and more general than those obtained by others in the literature (Bekir and Guner, 2014). The mechanisms of complex nonlinear physical phenomena that occur in nature may effectively be elucidated by these solutions. Among the three methods implemented to study the space-time fractional coupled Burgers’ equation for closed-form travelling wave solutions, the proposed rational fractional \((D_\alpha^G/G)-\)expansion method is most useful, though the other two methods also show good performance. The new approach introduced in this article has shown efficiency and reliability, and may make a major contribution in investigating many other nonlinear evolution equations of fractional order emerging in various fields of science and engineering.

Disclosure statement

No potential conflict of interest was reported by the authors.

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