Dynamics of compact homogeneous universes

MASAYUKI TANIMOTO *, TATSUKI KOIKE **, AND AKIO HOSOYA ***

Department of Physics, Tokyo Institute of Technology, Oh-Okayama 2-12-1, Meguro-ku, Tokyo 152, Japan

ABSTRACT

A complete description of dynamics of compact locally homogeneous universes is given, which, in particular, includes explicit calculations of Teichmüller deformations and careful counting of dynamical degrees of freedom. We regard each of the universes as a simply connected four dimensional spacetime with identifications by the action of a discrete subgroup of the isometry group. We then reduce the identifications defined by the spacetime isometries to ones in a homogeneous section, and find a condition that such spatial identifications must satisfy. This is essential for explicit construction of compact homogeneous universes. Some examples are demonstrated for Bianchi II, VI, VII, and I universal covers.
1 Introduction

In relativistic and observational cosmology, we often use a simplified spacetime model having restricted dynamical degrees of freedom. In particular, the well-known homogeneous and isotropic (FRW) models \cite{1}, in which the spatial sections are assumed to be homogeneous and isotropic, have been successful. On the other hand, a wider class of models, known as the Bianchi homogeneous models \cite{2, 3, 4}, in which the spatial sections are assumed to be homogeneous but not isotropic, have been largely used in relativity and quantum cosmology \cite{3, 5, 6}. In the models except type IX, each spatial section has been regarded as open. The open topology, however, is not a sole possibility. For example, the “open” model in the homogeneous and isotropic models, which has constant negative spatial curvature, and belongs to Bianchi type V, can be regarded to be spatially compact if spatial points are appropriately identified with some discrete subgroup of the isometry group. The purpose of this paper is, in fact, to investigate a class of the models in which each spatial section has compact topology. The compactness of space is physically reasonable due to its finite spatial volume.

The crucial point of the arguments in this paper is that the compactness of locally homogeneous space, in general, brings about new degrees of freedom of deformations, known in mathematics as Teichmüller deformations. They preserve the local geometry but change the global one. This can be easily understood if we regard the Teichmüller deformations of a compact locally homogeneous space as a homogeneous space (i.e., a covering space) with varying identifications. A space spanned by independent Teichmüller deformations, and its coordinates are referred to as the Teichmüller space and the Teichmüller parameters, respectively. We shall shed light on these degrees of freedom of the Teichmüller deformations, which in fact have been often disregarded so far. The field of (2+1)-gravity \cite{7} was exceptional, but we shall take a somewhat different approach. The Teichmüller deformations are of great interest, because they would carry part of the dynamical degrees of freedom. We shall give a complete framework to do a concrete analysis concerning the Teichmüller deformations. We also carefully count the total dynamical degrees of freedom.
In Ref. [8], referred to as I hereafter, we presented a treatment of three-dimensional compact homogeneous Riemannian manifolds, which will be a basis of our arguments in the present work. We, there, (1) gave the possible eight types (a\sim h) of homogeneous universal covers, which are closely related to Thurston’s eight geometries [9, 10], (2) classified compact quotients (a1/1, b/1, etc.), and (3) gave Teichmüller spaces by explicitly finding embeddings of covering groups in the isometry groups of the universal covers, which enable us to perform explicit calculations. To investigate the dynamics of compact homogeneous universes, we must first show how we can adapt such knowledge of compact homogeneous three-manifolds in the context of relativity in four dimensions.

Our strategy to this will be as follows. We begin with considering a four-dimensional universal cover \( (4)\tilde{M}, \tilde{g}_{ab} \), which is a simply connected Lorentzian manifold, and then take identifications in \((4)\tilde{M}, \tilde{g}_{ab}\) so as to make each 3-surface \((\tilde{M}_t, \tilde{h}_{ab})\) compact. To utilize our knowledge about compact homogeneous 3-manifolds, we translate the identifications in \((4)\tilde{M}, \tilde{g}_{ab}\) into those in \((\tilde{M}_t, \tilde{h}_{ab})\). We find that the identifications must be extendible isometries of \((\tilde{M}_t, \tilde{h}_{ab})\), which have natural extension in \((4)\tilde{M}, \tilde{g}_{ab}\). We make \((\tilde{M}_t, \tilde{h}_{ab})\) compact by the action of a discrete subgroup \( \Gamma \) of the group \( \text{Esom}\tilde{M}_t \subset \text{Isom}\tilde{M}_t \) of extendible isometries, where \( \text{Isom}\tilde{M}_t \) is the isometry group of \((\tilde{M}_t, \tilde{h}_{ab})\). Once given \( \Gamma \subset \text{Esom}\tilde{M}_t \) on \((\tilde{M}_t, \tilde{h}_{ab})\), all necessary identifications in \((4)\tilde{M}, \tilde{g}_{ab}\) are automatically determined by the natural extension of \( \Gamma \). The quotient \((4)M, g_{ab}\) is a solution of a local equation, e.g., Einstein’s equation, if and only if \((4)\tilde{M}, \tilde{g}_{ab}\) is a solution of the same equation.

The organization of this paper is as follows. In section 2, we first give the definition of Teichmüller deformations, and then briefly review the classification of compact homogeneous 3-manifolds given in I. In section 3, we establish the prescription for identifications, and discuss possible four-dimensional universal covers. We also discuss how we can eliminate the “gauge” degrees of freedom, and thereby we give how to find the dynamical degrees of freedom. In section 4, we apply the framework of the previous section to concrete models. There, we give the time-development of the Teichmüller parameters, establish the dynamical variables and give the number of the dynamical degrees of freedom for each case. The final section is devoted to conclusions. We employ the abstract
index notation (See e.g. [3]) throughout this paper.

2 Preliminaries

We give definitions of Teichmüller deformations in the first subsection. In the next subsection, we briefly sketch the classification scheme, given in I, of compact homogeneous 3-manifolds, though we will not duplicate the results of the classification. This will be helpful for the subsequent discussions. For the explicit results, see tables 1 and 2 in I, and section V of I. We consider only complete Riemannian manifolds, and shall drop the word “complete” hereafter.

2.1 Teichmüller deformations

We define Teichmüller deformations of a Riemannian manifold \((M, h_{ab})\) as follows.

**Definition 1 (Teichmüller deformations)** Let \((M, h_{ab})\) be a Riemannian manifold. Then, smooth and non-isometric deformations of \(h_{ab}\) are called Teichmüller deformations if they leave the universal cover \((\tilde{M}, \tilde{h}_{ab})\) globally conformally isometric.

In this definition, a globally conformal isometry means a conformal isometry with constant conformal factor. For definition of coverings, see, e.g., Ref.[12].

In other words, Teichmüller deformations are deformations induced by variations of a covering group \(\Gamma\) of the universal cover \((\tilde{M}, \tilde{h}_{ab})\). Here, a covering group \(\Gamma\) is a representation (or an embedding) of the fundamental group \(\pi_1 M\) into \(\text{Isom}\tilde{M}\), the isometry group of \((\tilde{M}, \tilde{h}_{ab})\), where \(M\) can be realised as \(\tilde{M}/\Gamma\). We denote the space of all covering groups as \(\text{Rep}(\pi_1 M, \text{Isom}\tilde{M})\). Note that not all variations of \(\Gamma\) correspond to independent Teichmüller deformations. In fact, two Riemannian manifolds \(\tilde{M}/\Gamma'\) and \(\tilde{M}/\Gamma\) are isometric if

\[
\Gamma' = a \circ \Gamma \circ a^{-1}
\]  

(1)

holds for an isometry \(a \in \text{Isom}\tilde{M}\). \(\Gamma'\) is called the conjugation of \(\Gamma\) by \(a\). If we denote the equivalence relation by conjugation as \(\sim\), the Teichmüller space, \(\text{Teich}(M, \tilde{h}_{ab})\), for
\( M \) and \( \tilde{h}_{ab} \), is defined as

\[
\text{Teich}(M, \tilde{h}_{ab}) = \frac{\text{Rep}(\pi_1 M, \text{Isom}\tilde{M})}{\sim}.
\]  

(2)

Although in this paper we are interested only in locally homogeneous metrics, it is worth noting that the definition of Teichmüller deformations here concerns wider classes of metrics. For example, even though the universal cover admits only one Killing vector, the quotients can smoothly deform if we smoothly vary the identifications along the Killing orbit.

### 2.2 The classification of compact homogeneous 3-manifolds

We briefly sketch the classification scheme, given in I, of compact homogeneous 3-manifolds.

Consider a pair \((M, G)\) of a manifold \(M\) and a group \(G\) acting transitively on \(M\) with compact isotropy subgroup. Note that we can construct a homogeneous manifold \((M, h_{ab})\) by first giving an arbitrary metric at a point \(p\) in \(M\), averaging it by the isotropy subgroup, then finally sending it by the actions of \(G\). Since \(G\) is transitive, the resulting metric \(h_{ab}\) is guaranteed to be homogeneous. The isometry group of \((M, h_{ab})\) would be isomorphic to \(G\), or contain \(G\) as a subgroup of the isometry group. Note that if we give various metrics at \(p\), then we obtain many homogeneous metrics on \(M\). Hence the pair \((M, G)\) can be considered as an equivalence class of homogeneous manifolds whose isometry groups are isomorphic to \(G\), or contain \(G\) as a subgroup of the isometry groups. Such a pair \((M, G)\) is called a *geometry*. If two geometries \((M, G)\) and \((M, G')\) have an inclusion relation \(G \subset G'\), then \((M, G)\) is called a *subgeometry* of \((M, G')\). If geometry \((M, G)\) is not a subgeometry of any geometry, then \((M, G)\) is called a *maximal* geometry, and if geometry \((M, G)\) does not have any subgeometry, then \((M, G)\) is called a *minimal* geometry.

Our starting point of classification of compact homogeneous manifolds is following Thurston’s theorem \[9\].

**Theorem 1** Any maximal, simply connected three-dimensional geometry which admits a compact quotient is equivalent to the geometry \((M, \text{Isom}M)\) where \(M\) is one of \(E^3\), \(H^3\), \(S^3\), \(S^2 \times \mathbb{R}\), \(H^2 \times \mathbb{R}\), \(\widetilde{SL}(2, \mathbb{R})\), Nil, and Sol.
A brief proof and accounts of the eight geometries in the theorem are found in Ref. [10].

Note that this theorem concerns only maximal geometries. This seems, however, too restricted for our purpose. For example, while we are interested in the closed FRW universe, of which a spatial section corresponds to maximal geometry \((S^3, \text{SO}(4))\), we are also, and maybe more, interested in the Bianchi IX universe, of which a spatial section is subgeometry \((S^3, \text{SU}(2))\) of geometry \((S^3, \text{SO}(4))\). Hence we should concern all non-maximal geometries which admit compact quotients, too. The above theorem, however, is still of great use, because of the following fact. That is, the “maximized” geometry of any non-maximal geometry admitting a compact quotient must admit a compact quotient, because the group of the maximal geometry must contain the covering group which makes the non-maximal geometry compact, and it must make the maximal geometry compact. This simple fact and Thurston’s theorem lead us to the investigation of all possible subgeometries of Thurston’s eight maximal geometries. Of course, we must note that not all subgeometries of the eight geometries admit compact quotients. We need to check explicitly that the subgeometry certainly admits a compact quotient.

Although, as we have seen, the use of “geometry” is convenient to carry out classification of homogeneous manifolds, it is useful to switch to the conventional “Riemannian manifold” representation for physical applications. As we have noted, a geometry is an equivalence class of homogeneous Riemannian manifolds. Let \(R\) be the set of all homogeneous manifolds belonging to a geometry, and let \(\tilde{R}\) be the quotient set of \(R\) by all possible diffeomorphisms and globally conformal transformations. Our classification for all possible universal covers admitting compact quotients is described in terms of \(\tilde{R}\)'s. In I, we labeled those \(\tilde{R}\)'s as “a1, a2, b, c, ..., h”. For each element of quotient \(\tilde{R}\), we choose a representative element \((M, h_{ab})\), and call metric \(h_{ab}\) the representative metric or the standard metric, whose explicit form for each type is also given in I.

In getting the standard metrics, the Bianchi-Kantowski-Sachs-Nariai (BKSN) classification [13, 14, 15] is useful, which concerns all minimal 3-geometries;

**Theorem 2** Any minimal, simply connected three-dimensional geometry is equivalent to \((M, G)\) where \(M = \mathbb{R}^3\); \(G = \text{one of Bianchi I to Bianchi VIII groups}; M = S^3\); \(G = \text{Bianchi IX group}; \) or \(M = S^2 \times \mathbb{R}, G = \text{SO}(3) \times \mathbb{R}\).
This is useful in that the invariant metrics for these groups are well known. These geometries are called the $BKSN$ minimal geometries. We take this opportunity to show the correspondence of Thurston’s eight geometries and the $BKSN$ minimal geometries, which is shown in table 1. (Such a correspondence was first pointed out by Fagundes [16] without referring to subgeometry.)

| Thurston’s maximal geometries | $BKSN$ minimal geometries |
|------------------------------|----------------------------|
| $(E^3, \text{Isom} E^3)$     | $(\mathbb{R}^3, \text{BI}), (\mathbb{R}^3, \text{BVI}_{0})$ |
| $(H^3, \text{Isom} H^3)$     | $(\mathbb{R}^3, \text{BV}), (\mathbb{R}^3, \text{BVI}_{a})$ |
| $(S^3, \text{Isom} S^3)$     | $(S^3, \text{BIX})$ |
| $(S^2 \times \mathbb{R}, \text{Isom} S^2 \times \mathbb{R})$ | $(S^2 \times \mathbb{R}, \text{KSN})$ |
| $(H^2 \times \mathbb{R}, \text{Isom} H^2 \times \mathbb{R})$ | $(\mathbb{R}^3, \text{BIII})$ |
| $(\widetilde{\text{SL}}(2, \mathbb{R}), \text{Isom} \widetilde{\text{SL}}(2, \mathbb{R}))$ | $(\mathbb{R}^3, \text{BVIII}), (\mathbb{R}^3, \text{BIII})$ |
| $(\text{Nil}, \text{Isom} \text{Nil})$ | $(\mathbb{R}^3, \text{BII})$ |
| $(\text{Sol}, \text{Isom} \text{Sol})$ | $(\mathbb{R}^3, \text{BVI}_0)$ |

Table 1: The correspondence between Thurston’s maximal geometries and $BKSN$ minimal geometries. BI is an abbreviation for Bianchi I group, and similar for BII, BIII, etc. KSN is equivalent to $\text{Isom} S^2 \times \text{Isom} \mathbb{R}$. The missing Bianchi types IV and VI$_a$ do not admit compact quotients.

It should be noted that an $\bar{R}$ does not always correspond to a single geometry, though the converse is true. For example, consider Bianchi I minimal geometry $(\mathbb{R}^3, \mathbb{R}^3)$, where the left $\mathbb{R}^3$ stands for the manifold homeomorphic to $\mathbb{R}^3$, while the right $\mathbb{R}^3$ stands for the 3-dimensional translation group acting on the manifold $\mathbb{R}^3$. Then the corresponding $R$ is the set of Riemannian manifolds $(\mathbb{R}^3, h_{ab})$’s, where $h_{ab} = h_{\mu \nu} (dx^\mu)_a (dx^\nu)_b$ with $h_{\mu \nu}$ being positive definite symmetric $3 \times 3$ matrix. All such $(\mathbb{R}^3, h_{ab})$’s, however, are isometric to $(\mathbb{R}^3, \eta_{ab})$, where $\eta_{ab}$ is the standard Euclid metric $\eta_{ab} = \delta_{\mu \nu} (dx^\mu)_a (dx^\nu)_b$. We thus have $\bar{R} = \{(\mathbb{R}^3, \eta_{ab})\}$. On the other hand, it is manifest that maximal geometry $(\mathbb{R}^3, \text{IO}(3))$ also gives rise to the same $\bar{R}$, since $\text{IO}(3)$ is the isometries of $\eta_{ab}$. (The $\bar{R}$ of this example is classified to type a2 in our classification. Similarly, Bianchi types II, VII$_0$, and VI$_0$ give
rise, respectively, to universal cover types b, a1, and f, on which compact models will be discussed specifically in subsequent sections.)

Let us summarize the actual procedure for our classification. First, list up Thurston's eight geometries and all their subgeometries. Enumerate all possible compact quotients of Thurston's eight geometries, and check whether the subgeometries admit them. Find out $\tilde{R}$'s for the subgeometries which admit a compact quotient. Note that this includes the explicit determinations of the standard metrics. This completes the classification of the universal covers which admit a compact quotient. The classification of the compact quotients that those universal covers admit can be carried out by using the results in Ref.[17, 18, 19]. The Teichmüller parameters are usually defined as components of the identification generators acting on the standard metric. Our classification, as a result, consists of the classification of universal covers, the classification of compact quotients, and explicit parametrizations of the Teichmüller spaces.

3 The framework of construction

In the first subsection, we show a method of construction of compact homogeneous universes, and then in the second subsection, we discuss the dynamical degrees of freedom of a system of compact homogeneous universes. Note, however, that the second subsection is not independent of the first one. Our construction of a system of compact homogeneous universes is completed by the discussions there.

3.1 Universal covers and identifications

We mean by a compact homogeneous universe a smooth Lorentzian 4-manifold $({}^4M, g_{ab})$ which admits a foliation by compact homogeneous spatial leaves (sections), and denote the universal cover of $({}^4M, g_{ab})$ as $({}^4\tilde{M}, \tilde{g}_{ab})$. It is important that $({}^4M, g_{ab})$ inherits all the local properties from $({}^4\tilde{M}, \tilde{g}_{ab})$ by a covering map. Hence, we can think that the local and the global geometries are carried by the universal cover $({}^4\tilde{M}, \tilde{g}_{ab})$ and the covering map, respectively, and thus we can look into them separately. First, we shall consider how we can take the covering map, or "identifications", when a universal cover $({}^4\tilde{M}, \tilde{g}_{ab})$ is
given. After that, we shall consider what universal covers are appropriate for our purpose.

The identifications in \((\tilde{M}, \tilde{g}_{ab})\) act on each homogeneous 3-section \((\tilde{M}_t, \tilde{h}_{ab})\) of \(t = \text{const.}\), where \(t\) parametrizes the homogeneous sections of \((\tilde{M}, \tilde{g}_{ab})\). For simplicity, we omit the argument \(t\) of the metric \(\tilde{h}_{ab}\) as far as no confusions occur. Let \(\text{Isom}\tilde{M}_t\) be the isometry group of \((\tilde{M}_t, \tilde{h}_{ab})\). It is very important to note that we cannot make, in general, the homogeneous 3-manifold \((\tilde{M}_t, \tilde{h}_{ab})\) compact by the action of a discrete subgroup of \(\text{Isom}\tilde{M}_t\), since for \((M, g_{ab})\) to be a smooth Lorentzian manifold, the covering group, \(\Gamma\), of the section needs to preserve the extrinsic curvature, as well as the spatial metric. To give a suitable prescription of compactification, we convert this requirement of the smoothness of \((M, g_{ab})\) into the following statement; Since \((M, g_{ab})\) is obtained by taking identifications in \((\tilde{M}, \tilde{g}_{ab})\), for any two points which are identified, there should exist an isometry of \((\tilde{M}, \tilde{g}_{ab})\) (not of \((\tilde{M}_t, \tilde{h}_{ab})\)) which maps one to the other. Hence, if we define the extendible isometry group of \((\tilde{M}_t, \tilde{h}_{ab})\), \(\text{Esom}\tilde{M}_t \subset \text{Isom}\tilde{M}_t\), as below, then we obtain a complete prescription for construction of a compact homogeneous universe, as shown subsequently;

**Definition 2 (Extendible isometry group)** Let \((\tilde{M}_t, \tilde{h}_{ab})\) be a spatial section of \((\tilde{M}, \tilde{g}_{ab})\). An extendible isometry is the restriction on \(\tilde{M}_t\) of an isometry of \((\tilde{M}, \tilde{g}_{ab})\) which preserves \(\tilde{M}_t\). They form a subgroup of \(\text{Isom}\tilde{M}_t\). We call it the extendible isometry group, and denote it as \(\text{Esom}(\tilde{M}_t, (\tilde{M}), \text{Esom}\tilde{M}_t\), or simply \(\text{Esom}\tilde{M}_t\). Obviously, an extendible isometry \(a \in \text{Esom}\tilde{M}_t\) has the natural extension on \((\tilde{M}, \tilde{g}_{ab})\) which is an element of \(\text{Isom}(\tilde{M})\) and preserves \(\tilde{M}_t\). We call such the natural extension on \((\tilde{M}, \tilde{g}_{ab})\) the extended isometry of \(a\), or simply the extension of \(a\).

**Proposition 1** The identifications on an initial surface \((\tilde{M}_t, \tilde{h}_{ab})\) must be implemented in \(\text{Esom}(\tilde{M}_t, (\tilde{M}))\),

\[
\Gamma \subset \text{Esom}(\tilde{M}_t, (\tilde{M})),
\]

(3)

to get a compact homogeneous universe out of a given four-dimensional universal cover \((\tilde{M}, \tilde{g}_{ab})\). Moreover, the identifications acting on whole \((\tilde{M}, \tilde{g}_{ab})\) are determined by the action of the extension of \(\Gamma\) on \((\tilde{M})\).
For example, Kasner type metric,
\[ ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \]  
where \( p_1 \sim p_3 \) are constants, has Euclid spaces as its spatial part. Suppose \((\tilde{M}_t, \tilde{h}_{ab})\) is such a Euclid space, where the metric is given by
\[ dl^2 = t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2. \]
As long as a generic case is concerned where \( p_\alpha \)'s are all different, the (continuous) isometries of the 4-metric (4) are only the translations generated by \( \partial/\partial x, \partial/\partial y, \) and \( \partial/\partial z, \) which form group \( \mathbb{R}^3. \) Since they all preserve the spatial sections, we find \( \text{Esom}_0 \tilde{M}_t \cong \mathbb{R}^3. \) (\( \text{Esom}_0 \tilde{M}_t \) is the identity component of \( \text{Esom} \tilde{M}_t, \) and similar for \( \text{Isom}_0 \tilde{M}_t. ) \) On the other hand, rotations generated by vector
\[ k_3 \equiv -t^{-(p_1-p_2)} y \frac{\partial}{\partial x} + t^{p_1-p_2} x \frac{\partial}{\partial y}, \]
and the similar vectors obtained by permutations of indices also form intrinsic isometry group, \( \text{SO}(3), \) of \((\tilde{M}_t, \tilde{h}_{ab}). \) This shows \( \text{Esom}_0 \tilde{M}_t \neq \text{Isom}_0 \tilde{M}_t \cong \text{ISO}(3). \) On a homogeneous section \((\tilde{M}_t, \tilde{h}_{ab}),\) if we choose a covering group \( \Gamma \) in \( \text{Esom} \tilde{M}_t \) such as to make \((\tilde{M}_t, \tilde{h}_{ab})\) compact, we obtain a compact homogeneous universe as a four-dimensional manifold through proposition II.

We now discuss what universal covers are appropriate for our purpose. If the topology of \((4)M\) is fixed, then manifold \((4)\tilde{M}\) is uniquely determined. Hence we only need to consider possible metrics, \( \tilde{g}_{ab}'s. \) Let us consider the case where each homogeneous spatial section \((\tilde{M}_t, \tilde{h}_{ab})\) corresponds to a Bianchi minimal geometry. It is also straightforward to adapt the following description for the KSN minimal geometry. By the definition of our compact homogeneous universes, the metric should be of the form
\[ ds^2 = -N^2(t,x) dt^2 + h_{\alpha\beta}(t)(N^\alpha(t,x) dt + \sigma^\alpha)(N^\beta(t,x) dt + \sigma^\beta), \]
where \((t,x)\) are local coordinates, \( \sigma^\alpha \) the invariant 1-forms, and \( \alpha, \beta, \ldots \) run from 1 to 3. The spatial metric \( h_{\alpha\beta}(t)\sigma^\alpha\sigma^\beta \) is, indeed, homogeneous on each section \( t = \text{const}. \)

For a metric with generic lapse and shift functions, the extendible isometry group \( \text{Esom} \tilde{M}_t \) contains only the identity element so that we cannot use the prescription for
the compactification. One might argue that there would remain possibility to find a
discrete group $\Gamma'$ for the compactification if the lapse and shift functions have some
periodicity. However, such a discrete group does not contain continuous parameters.
Since we are interested in the case of maximal number of continuous parameters in the
initial identifications, we demand that $E_{\text{som}} \tilde{M}_t$ is transitive and therefore the lapse and
shift functions are independent of the spatial coordinates $x$;

$$ds^2 = -N^2(t)dt^2 + h_{\alpha\beta}(t)(N^\alpha(t)dt + \sigma^\alpha)(N^\beta(t)dt + \sigma^\beta). \quad (8)$$

This metric becomes the following form

$$ds^2 = -dt^2 + h_{\alpha\beta}(t)\sigma^\alpha\sigma^\beta, \quad (9)$$

by the induced map of a diffeomorphism which preserves each homogeneous section of
t = const.. We shall mainly focus on this type of metrics hereafter.

We here comment on an intuitive prescription for identifications, which is also useful
particularly for the metric (9). Note that the normal geodesics emerging from a section
$\tilde{M}_t$ are uniquely defined, provided that they are parametrized by proper time $\tau$. We refer
to the exponential map $\exp(\tau n^a(t))$ which is defined with respect to the normal vector field
$n^a(t)$ on $\tilde{M}_t$ as the normal map. (The image of $\tilde{M}_t$ by the normal map $\exp(\tau n^a(t))$ is not
generally $\tilde{M}_{t'}$ for some $t'$ when considering metric (7). For metric (9), we of course have
$\exp(\tau n^a(t)) : \tilde{M}_t \rightarrow \tilde{M}_{t+\tau}$.) Obviously, if two points, $a$ and $b$, on $\tilde{M}_t$ are identified, any two
points mapped by the normal map should continue to be identified,

$$\forall \tau \in \mathbb{R}; \ (\exp(\tau n^a(t)))(a) \sim (\exp(\tau n^a(t)))(b). \quad (10)$$

Hence, we can determine how the identifications evolve in time, in terms of geodesics
in a given four-dimensional universal cover. For the metric (9), since the hypersurface-
orthogonal geodesics coincide with the t-axes, we immediately obtain the following useful
proposition.

**Proposition 2** In terms of the coordinates $(t, x)$ of metric (9), if at the initial surface
t = $t_0$ an identification is specified as $(t_0, x) \sim (t_0, a x)$, where $a$ is a free action on the
coordinate space, then at any time $t$ we must have $(t, x) \sim (t, a x)$. 

11
That is, if we take identifications in $\text{Esom} \tilde{M}_t$ on an initial surface of the metric (9), and describe them in terms of the spatial coordinates $x$, then the description of the identifications holds for any time $t$.

By proposition 4, one might conclude that no interesting global, i.e. Teichmüller, deformations occur for the metric (9), since the identifications on each homogeneous section in terms of the spatial coordinates do not vary with time. However, this is not the case. Remember that the Teichmüller deformations are defined with respect to the intrinsic geometries of the three dimensional sections $(M_t, h_{ab})$. Variation of metric with time does cause Teichmüller deformations with time, and if there exists difference between $\text{Esom} \tilde{M}_t$ and $\text{Isom} \tilde{M}_t$, as in the example below proposition 1, the situation becomes much richer. We will comment on this point again at the end of the next subsection.

### 3.2 Dynamical degrees of freedom

Let us consider a universe characterized by an initial data set $(h_{ab}, K_{ab})$, where $h_{ab}$ and $K_{ab}$ are the spatial metric and the extrinsic curvature of the initial spatial section $M$. To give a possible $(h_{ab}, K_{ab})$ in the case that $M$ has nontrivial topology and $(h_{ab}, K_{ab})$ is locally homogeneous, we first need to cover $M$ with some open patches, define coordinates for each patch, and give a transformation function for each overlap of two patches. That is, we need to set an atlas. After that, if we give an $(h_{\mu\nu}, K_{\mu\nu})$ on a point with respect to the coordinates defined in a patch, we can send $(h_{\mu\nu}, K_{\mu\nu})$ to all points in the patch by the transitive group action, and finally assign the values of the data set to all patches by the transformation functions and the group actions. Hence the information of the data $(h_{ab}, K_{ab})$ is equivalent to the information of the value of $(h_{\mu\nu}, K_{\mu\nu})$ on a point and the way of taking transformation functions if patches are fixed. However, it can be easily imagined that it is very difficult to count the number of possible $(h_{\mu\nu}, K_{\mu\nu})$’s and the number of possible ways of taking transformation functions up to diffeomorphism. We can accomplish this counting, using coverings, as follows.

As stated in the previous subsection, we think that a set, $U$, of universal covers, $(\tilde{M}, \tilde{g}_{ab})$’s, carries the degrees of freedom of local geometry like local curvatures, and the covering maps do the degrees of freedom of global geometry like Teichmüller param-
eters. In this approach, it is evident that we need to eliminate the degrees of freedom of all possible diffeomorphisms in $U$. We introduce the equivalence relation in $U$ by diffeomorphisms, and denote the resulting set of universal covers as $\bar{U}$. If we fix a homogeneous surface $(\bar{M}_t, \bar{h}_{ab})$ in a $u \equiv (^{(4)}\bar{M}, \bar{g}_{ab}) \in \bar{U}$ [21], and suppose $\Gamma \subset \text{Esom} \bar{M}_t$ makes $(\bar{M}_t, \bar{h}_{ab})$ compact, then we can identify a pair $(u, \Gamma)$ with a compact homogeneous universe through proposition 1. We denote the set of $(u, \Gamma)$’s of all possible $\Gamma$’s for a fixed $u$ as $C_u$. If $\Gamma' = \phi \circ \Gamma \circ \phi^{-1}$ holds for a $\phi \in \text{Esom} \bar{M}_t$, then the resulting compact homogeneous universe, $(u, \Gamma')$, is isometric to $(u, \Gamma)$. In this sense, the freedom of taking conjugations of $\Gamma$ by $\text{Esom} \bar{M}_t$ also corresponds to “gauge” freedom (cf. Sec. 2.1). Introducing the equivalence relation in $C_u$ by the conjugations, we get the quotient set $\bar{C}_u$.

Now, our dynamical system, $\bar{C}$, of compact homogeneous universes is equivalent to the set $\bar{C} \equiv \{c | c \in \bar{C}_u, \, u \in \bar{U}\}$.

If we choose a homogeneous section arbitrarily for each element in $\bar{C}$, we will have a set, $\bar{I}$, of initial data sets, $(h_{ab}, K_{ab})$’s, on a fixed compact 3-manifold. In $\bar{I}$, there are no elements which are isometric to each other, since for any different elements in $\bar{C}$ are non-isometric, and the development of an initial data set is unique. The set $\bar{I}$ is therefore equivalent to the set we considered at the beginning of the subsection.

The following proposition is now trivial.

**Proposition 3** The number, $\dim \bar{C}$, of degrees of freedom of a system of compact homogeneous universes is the sum of the number, $\dim \bar{U}$, of the degrees of freedom of the four-dimensional universal covers up to isometry, and the number, $\dim \bar{C}_u$, of degrees of freedom of initial identifications, i.e. covering groups on an initial section, up to conjugations taken by the extendible isometry group.

Hereafter, by a construction of compact homogeneous universes, we mean a construction with explicit determination of representatives of the universes in the above sense, so that the number of arbitrary parameters in a universal cover should be $\dim \bar{U}$, and the number of arbitrary parameters in the identification generators should be $\dim \bar{C}_u$. Note that for vacuum solutions of Einstein’s equation for Bianchi class A [22] (i.e. types I, II, VI$_0$, VII$_0$, VIII, and IX) and type V, the metric components $h_{\alpha\beta}(t)$ in Eq. (9) is “diagonalizable”,

13
i.e. becomes diagonal by diffeomorphisms. Hence we will begin with the diagonal form metric to construct compact homogeneous universes on Bianchi class A or type V.

Note that we take conjugations for \((\tilde{M}_t, \tilde{h}_{ab})\) only by Esom\(\tilde{M}_t\) to obtain the initial identification parameters as stated in proposition \[1\], while the Teichmüller parameters are defined with respect to conjugations by full Isom\(\tilde{M}_t\) (cf. Sec. 2.1). Roughly speaking, the difference between the freedom of Esom\(\tilde{M}_t\) and that of Isom\(\tilde{M}_t\) corresponds to the freedom of giving initial ‘velocities’ of Teichmüller parameters, as we will see more explicitly through the examples in the next section.

4 Four compact homogeneous universe models

In this section, we construct four explicit models of compact homogeneous universe, the b/1, f1/1, a1/1, and a2/1 models. For each model, we count the number of dynamical degrees of freedom and give the time-development of the Teichmüller parameters.

To get the Teichmüller parameters of a compact section \((M_t, h_{ab})\), we need to compare two mathematical representations, i.e., \((\tilde{M}_t, \tilde{h}_{ab})\) with the covering group \(\Gamma\), and the standard universal cover \((\tilde{M}, \tilde{h}_{ab}^{\text{std}})\) with the covering group, \(A\), parametrized by the Teichmüller parameters. (In I, the standard metrics were called the representative metrics.) \(\Gamma\) and \(A\) are generated by the same number, \(n\), of generators, \(\{g_i\}\) and \(\{a_i\}\) \((i = 1, \cdots, n)\), respectively. \(\{g_i\}\) and \(\{a_i\}\) satisfy the same multiplication rule of an extendible isometry group. We can get the Teichmüller parameters by finding the automorphism of Esom\(\tilde{M}\) which relates the two sets of generators. We shall do this first for the b/1 model, where we will see the most typical calculation to get Teichmüller parameters. Then the f1/1 and a1/1 models follow. Finally, for the a2/1 model, we present a different method in getting the time-development of the Teichmüller parameters.

Our universal cover metrics are synchronous (Eq.(9)) and diagonal (See Sec. 3.2).
4.1 The b/1 model: a compact model on Bianchi II geometry

We start with the multiplication rule of Nil (=Bianchi II group);
\[
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
h_1 \\
h_2 \\
h_3 \\
\end{pmatrix} =
\begin{pmatrix}
g_1 + h_1 \\
g_2 + h_2 \\
g_3 + h_3 + g_1 h_2 \\
\end{pmatrix},
\]
(11)
where \(g, h \in \text{Nil}\), and we shall use superscripts to denote the components of a group element. We use the same components \((x^1, x^2, x^3) \equiv (x, y, z)\) as coordinates of \(\tilde{M}_t\). The action of Nil on \(\tilde{M}_t\) is defined by the left action on \((x, y, z) \in \text{Nil}\). A Nil-invariant (diagonal) metric is given by
\[
dl^2 = h_{11} dx^2 + h_{22} dy^2 + h_{33} (dz - xdy)^2,
\]
(12)
where \(h_{\alpha\alpha} (\alpha = 1 \sim 3)\) are constants, i.e. independent of \((x, y, z)\). The four dimensional universal cover metric of our concern is of the form
\[
ds^2 = -dt^2 + dl^2
\]
(13)
with \(h_{\alpha\alpha}\) being functions of \(t\). The vacuum solution is, of course, known, but we proceed with calculation, leaving \(h_{\alpha\alpha}\) free, since they are complicated functions in the synchronous gauge and, moreover, it enable us to apply the result also to models other than the vacuum model.

We consider manifold “b/1 \((n = 1)\)”, classified in I, which is probably the most stereotypical compact manifold modeled on Bianchi II geometry. The fundamental group \(\pi_1\) is given by (See Eq.(118) in I)
\[
\pi_1 = \langle g_1, g_2, g_3; [g_1, g_2]g_3^{-1}, [g_1, g_3], [g_2, g_3] \rangle.
\]
(14)
The topology of b/1 is illustrated in Fig.1.

To represent the generators of \(\pi_1\), \(g_i\)’s, in \(\text{Esom}\tilde{M}_t = \text{Nil}\), we put them as
\[
g_i = \begin{pmatrix}
g_i^1 \\
g_i^2 \\
g_i^3 \\
\end{pmatrix}, \ (i = 1 \sim 3),
\]
(15)
Fig.1. The topology of b/1. Letters A and F show how the front and back sides are identified. The right and left sides, and the top and bottom sides are identified normally. The five arrows shows actions of \( g_i \)'s, illustrating the first relation in Eq.(12), \( g_3g_2g_1 = g_1g_2 \).

and substitute these in the relations of \( \pi_1 \) (Eq.(14)). We then get the following:

\[
\begin{align*}
g_1 &= \begin{pmatrix} g_1^1 \\ g_1^2 \\ g_1^3 \end{pmatrix}, \\
g_2 &= \begin{pmatrix} g_2^1 \\ g_2^2 \\ g_2^3 \end{pmatrix}, \\
g_3 &= \begin{pmatrix} 0 \\ 0 \\ \bar{g}_3^3 \end{pmatrix},
\end{align*}
\]

where \( \bar{g}_3^3 \equiv g_1^1g_2^2 - g_1^2g_2^1 \neq 0 \).

We then consider the possible conjugations by \( \text{Esom}\bar{M}_t = \text{Nil} \). For the conjugation of \( g_i \)'s by \( h = (h^1, h^2, h^3) \in \text{Nil} \) (For typographical convenience, we sometimes write components of group horizontally), we have

\[
\begin{align*}
h g_1 h^{-1} &= \begin{pmatrix} g_1^1 \\ g_1^2 \\ g_1^3 + h^1g_1^2 - h^2g_1^1 \end{pmatrix}, \\
h g_2 h^{-1} &= \begin{pmatrix} g_2^1 \\ g_2^2 \\ g_2^3 + h^1g_2^2 - h^2g_2^1 \end{pmatrix}, \\
h g_3 h^{-1} &= g_3.
\end{align*}
\]

We can make the third components of \( g_1 \) and \( g_2 \) zero if we take \( h \) as \( h^1 = (g_1^3g_2^1 - g_1^1g_2^3)/\bar{g}_3^3, h^2 = (g_1^3g_2^2 - g_1^2g_2^3)/\bar{g}_3^3 \). After all, our representation of \( \pi_1 \) in Nil reduces
to
\[ g_1 = \begin{pmatrix} g_1^1 \\ g_1^2 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} g_2^1 \\ g_2^2 \\ 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 \\ 0 \\ \tilde{g}_3^3 \end{pmatrix}. \] (18)

The nonvanishing four independent components in these \(g_i\)’s determine the initial identifications in the universal cover with metric \([13]\).

To proceed further calculations, we here cite the definition given in I of the Teichmüller parameters for \(b/1\) and some related properties. We denote the standard universal cover as \((\mathbb{R}^3, \tilde{h}_{ab}^{\text{std}})\), where the standard metric \(\tilde{h}_{ab}^{\text{std}}\) is given by (Eq.(75) in I)
\[ dl^2 = dx^2 + dy^2 + (dz - xdy)^2. \] (19)

Any compact homogeneous 3-manifold classified in \(b/1\) is globally conformally isometric to manifold \((\mathbb{R}^3, \tilde{h}_{ab}^{\text{std}})/A\), where \(A\) is a covering group whose generators are given by
\[ a_1 = \begin{pmatrix} a_1^1 \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_2^1 \\ a_2^2 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 \\ 0 \\ a_1^{-1}a_2^{-2} \end{pmatrix}. \] (20)

Then, the Teichmüller parameters are \(\tau = (a_1^1, a_2^1, a_2^2)\) (Eq.(129) in I). We can see that the map
\[ s_\theta : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} R_\theta \begin{pmatrix} x \\ y \end{pmatrix} \\ z + \zeta_\theta(x, y) \end{pmatrix} \] (21)
is a 1-parameter isometry for \((\mathbb{R}^3, \tilde{h}_{ab}^{\text{std}})\), where \(R_\theta\) is the rotation matrix by angle \(\theta\), and \(\zeta_\theta\) is defined by
\[ \zeta_\theta(x, y) \equiv \frac{1}{2}(x^2 - y^2)\cos \theta - 2xy\sin \theta)\sin \theta. \] (22)

We here remark that \(s_\theta\) is not an element of \(\text{Esom}\tilde{M}_t\) but of \(\text{Isom}\tilde{M}_t\), and therefore \(\text{Esom}\tilde{M}_t \neq \text{Isom}\tilde{M}_t\) in the \(b/1\) model. For an element \(h \in \text{Nil}\), conjugation by \(s_\theta\) is given by
\[ s_\theta \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} s_\theta^{-1} = \begin{pmatrix} R_\theta \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \\ h^3 + \zeta_\theta(h^1, h^2) \end{pmatrix}. \] (23)
Note that metric (12) is rewritten as
\[ dl^2 = \frac{h_{11}h_{22}}{h_{33}}(dx'^2 + dy'^2 + (dz' - x'dy')^2) \] (24)
with
\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} = \begin{pmatrix}
  \sqrt{\frac{h_{33}}{h_{22}}} x \\
  \sqrt{\frac{h_{33}}{h_{11}}} y \\
  \frac{h_{33}}{\sqrt{h_{11}h_{22}}} z
\end{pmatrix},
\] (25)
where \( h_{\alpha\alpha} \) are regarded as constants. If we view this coordinate transformation as a diffeomorphism and drop the constant conformal factor of metric (24), the resulting metric coincides with the standard metric (19). This diffeomorphism is obviously an element of the HPDs [23, 8], from the form of metric (24), so that the transformation \((x, y, z) \rightarrow (x', y', z')\) is an (outer-) automorphism of Nil. The image of \( g_i \)'s, which acts on metric (24) (or metric (19)), is
\[
\begin{pmatrix}
  g_1^{1'} \\
  g_1^{2'} \\
  0
\end{pmatrix}, \quad
\begin{pmatrix}
  g_2^{1'} \\
  g_2^{2'} \\
  0
\end{pmatrix}.
\] (26)
Here,
\[
g_1^{1'} = \sqrt{\frac{h_{33}}{h_{22}}} g_1^1, \quad g_1^{2'} = \sqrt{\frac{h_{33}}{h_{11}}} g_1^2, \quad g_2^{1'} = \sqrt{\frac{h_{33}}{h_{22}}} g_2^1, \quad g_2^{2'} = \sqrt{\frac{h_{33}}{h_{11}}} g_2^2.
\] (27)
Generator \( g_3 \) is automatically determined by \( g_1 \) and \( g_2 \) (see Eq.(18)), so we will concentrate on \( g_1, g_2 \) and the images of them by automorphisms. Since Eq.(26) is not of the form of Eq.(20), it does not yet give the Teichmüller parameters. To get them, we take a conjugation of Eq.(26) by the (full) isometry of Nil, which is given by Nil itself with \( s_\theta \). We can “rotate” the two-dimensional vectors \((g_i^{1'}, g_i^{2'}) (i = 1, 2)\) by conjugations by \( s_\theta \) (Eq.(23)), leaving the third components zero by a conjugation by Nil like the way we obtained Eq.(18). So, we arrive at
\[
a_1 = h s_\theta g_i s_\theta^{-1} h^{-1} = \begin{pmatrix}
  \sqrt{(g_1^{1'})^2 + (g_2^{2'})^2} \\
  0 \\
  0
\end{pmatrix}
\] (28)
and
\[
a_2 = \begin{pmatrix} g_2^{1r} \cos \theta_1 - g_2^{2r} \sin \theta_1 \\ g_2^{1r} \sin \theta_1 + g_2^{2r} \cos \theta_1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\left(g_1^{1r}\right)^2 + \left(g_1^{2r}\right)^2}} \begin{pmatrix} g_1^{1r}g_2^{1r} + g_1^{2r}g_2^{2r} \\ g_1^{1r}g_2^{2r} - g_1^{2r}g_2^{1r} \\ 0 \end{pmatrix},
\]
(29)
where
\[
\cos \theta_1 = \frac{g_1^{1r}}{\sqrt{\left(g_1^{1r}\right)^2 + \left(g_1^{2r}\right)^2}}, \quad \sin \theta_1 = \frac{-g_1^{2r}}{\sqrt{\left(g_1^{1r}\right)^2 + \left(g_1^{2r}\right)^2}},
\]
(30)
and \( h \) is an element of Nil. Using Eq.(27), we obtain the final form of the Teichmüller parameters;
\[
a_1^1 = \frac{\sqrt{h_{33}}}{h_{22}}(g_1^1)^2 + \frac{h_{33}}{h_{11}}(g_1^2)^2, \\
a_2^1 = \frac{1}{a_1^1} \left( \frac{h_{33}}{h_{22}}g_1^1g_2^1 + \frac{h_{33}}{h_{11}}g_1^2g_2^2 \right), \\
a_2^2 = \frac{\bar{g}_3^3}{a_1^1 \sqrt{h_{11}h_{22}}}.
\]
(31)

In Eq.(31), parameters \( g_1^1, g_1^2, g_2^1, g_2^2 \) and hence \( \bar{g}_3^3 \) are constants, and \( h_{\alpha\alpha} \)'s are functions of \( t \). The metric components \( h_{11}, h_{22}, \) and \( h_{33} \) are determined by substituting Eqs.(13) and (12) into Einstein’s equation, and moreover we must exclude the degrees of freedom of HPDs from them (See Sec. 3.2). Hence, the number of free parameters that the metric components can have coincides with the known number of degrees of freedom of the conventional (open) Bianchi models [24]. For the vacuum Bianchi II, the number of free parameters in the metric functions is two, i.e., \( \dim \bar{U} = 2 \). With the four parameters specifying the initial identifications, \( \dim \bar{C}_u = 4 \), the total number of dynamical degrees of freedom of the present vacuum b/1 model is six. (cf. proposition 3) The dynamical variables are the Teichmüller parameters \( a_1^1, a_2^1, \) and \( a_2^2 \), and the total volume
\[
v = (\bar{g}_3^3)^2 \sqrt{h_{11}h_{22}h_{33}}.
\]
(32)

Remember that the Teichmüller parameters are defined with respect to the standard universal cover which is isometric to the universal cover \((\tilde{M}_t, \tilde{h}_{ab})\) up to a global conformal factor. In fact, it is clear that, if we know the values of them, we can completely construct the original compact 3-manifold.
Additional remarks:

We end this subsection with some common remarks to the subsequent subsections, which will be helpful for understanding the rather unique concept employed throughout this section. Since as in Sec.2 any compact locally homogeneous 3-manifold can always be smoothly specified by some Teichmüller parameters, curvature parameters, and the volume, we are regarding them, rather than $h_{aa}$’s, as the dynamical variables of the compact homogeneous universe. In the b/1 case, they are the four parameters $(a_1^1, a_2^1, a_2^2, v)$, as pointed out. (There are no curvature parameters in this case). One remark we want to emphasize here is that we are on the standpoint that we do not ask whether or not such the dynamical variables directly fulfill some dynamical differential equations, since we do not need them to obtain the time-development of the “dynamical variables”. (We will however discuss this problem in a separate work. See also Sec.3.) Of course, this is not to say we do not need Einstein’s equation. The role of Einstein’s equation in our calculation is to fix the universal cover, i.e. to fix $h_{aa}$’s.

One of the main conclusions in this subsection was that once the universal cover is set fixed by Einstein’s equation, we automatically get the time-development of the dynamical variables through Eqs.(31) and (32). In this sense, Eqs.(31) and (32) are the kinematical relation between the universal cover and the dynamical variables. Another remark is therefore the fact that even if some matter fields are included and the form of $h_{aa}$’s accordingly vary, Eqs.(31) and (32) are invariant. Correspondingly, while the degrees of freedom, dim $\bar{U}$, of the universal cover may vary, those, dim $\bar{C}_u$, of the initial identifications are invariant.

4.2 The f1/1(n) model: the compact Bianchi VI$_0$ model

The multiplication rule of Sol (=Bianchi VI$_0$ group) is given by:

$$\begin{pmatrix} g^1 \\ g^2 \\ g^3 \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = \begin{pmatrix} g^1 + e^{-g^3}h^1 \\ g^2 + e^{g^3}h^2 \\ g^3 + h^3 \end{pmatrix}, \quad (33)$$
where \( g, h \in \text{Sol} \). We can easily check that the 1-forms
\[
\sigma^1 = \frac{1}{\sqrt{2}}(e^zd\bar{x} + e^{-z}d\bar{y}), \quad \sigma^2 = \frac{1}{\sqrt{2}}(-e^zd\bar{x} + e^{-z}d\bar{y}), \quad \sigma^3 = dz
\]
are invariant under the left action of \( \text{Sol} \) if \((x, y, z)\) is identified with an element of \( \text{Sol} \). We therefore have the following invariant metric
\[
dl^2 = h_{11}(\sigma^1)^2 + h_{22}(\sigma^2)^2 + h_{33}(\sigma^3)^2. \quad (35)
\]
As in the case of Bianchi II, the four dimensional universal cover metric is Eq.(13) with the above \( \dl^2 \). In contrast to the Bianchi II case, the isometries and the extendible isometries coincide; \( \text{Isom} \tilde{M}_t = \text{Esom} \tilde{M}_t = (\text{Sol plus three discrete elements}) \). For future use, we present one of the three discrete elements here. It is
\[
h : (x, y, z) \rightarrow (-x, -y, z). \quad (36)
\]

The fundamental group \( \pi_1 \) of a compact manifold modeled on Bianchi VI\(_0\) geometry is given by (See Eq.(145) in I)
\[
\pi_1 = \langle g_1, g_2, g_3; [g_1, g_2], g_3g_1g_3^{-1}g_2^{-1}, g_3g_2g_3^{-1}g_1g_2^{-n} \rangle, \quad (37)
\]
where \(|n| > 2\). Because of the coincidence of the two isometry groups, we need not do new calculations to find embeddings of \( \pi_1 \) in \( \text{Esom} \tilde{M}_t \) other than those shown in I. We simply show the results.

Let
\[
c_3 \equiv \ln \left| \frac{n + \sqrt{n^2 - 4}}{2} \right|. \quad (38)
\]
If \( n > 2 \), then \( e^{-c_3} \) and \( e^{c_3} \) are the eigenvalues of matrix
\[
\begin{pmatrix}
0 & 1 \\
-1 & n
\end{pmatrix},
\]
and so are \( -e^{-c_3} \) and \( -e^{c_3} \), if \( n < -2 \). Let \((u_1, v_1)\) and \((u_2, v_2)\) be the normalized eigenvectors corresponding to the two eigenvalues, i.e.,
\[
\begin{pmatrix}
u_1 \\
v_1
\end{pmatrix} = \frac{1}{\sqrt{|n|}} \begin{pmatrix}\sqrt{\frac{|n + \sqrt{n^2 - 4}|}{2}} \\
\sqrt{\frac{|n - \sqrt{n^2 - 4}|}{2}}\end{pmatrix}, \quad \begin{pmatrix}u_2 \\
v_2
\end{pmatrix} = \frac{1}{\sqrt{|n|}} \begin{pmatrix}\sqrt{\frac{|n - \sqrt{n^2 - 4}|}{2}} \\
\sqrt{\frac{|n + \sqrt{n^2 - 4}|}{2}}\end{pmatrix}. \quad (39)
\]
Then we can embed the generators of $\pi_1$ in $\text{Esom}\tilde{M}_t$ as (See Eqs.(156) and (157) in I)

\[
g_1 = \begin{pmatrix} \alpha_0 u_1 \\ \alpha_0 u_2 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \alpha_0 v_1 \\ \alpha_0 v_2 \\ 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix}
\]

(40)

for $n > 2$, or

\[
g_1 = \begin{pmatrix} \alpha_0 u_1 \\ \alpha_0 u_2 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \alpha_0 v_1 \\ \alpha_0 v_2 \\ 0 \end{pmatrix}, \quad g_3 = h \circ \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix}
\]

(41)

for $n < -2$, where $h$ is defined in Eq.(36). We thus find that the parameter for the initial identifications is only $\alpha_0$ in Eq.(40) or Eq.(41).

Before giving the time-development of the Teichmüller parameters, we take this opportunity to present a pictorial account to the topology of a compact Sol: $f/1(n)$. Manifold $f/1(n)$ is a torus-bundle over $S^1$. The relation (37) implies that $g_1$ and $g_2$ generate the fiber torus. We can understand the topology of $f/1(n)$ by observing the gluing map generated by $g_3$ which maps generators of a torus to another generators of a torus. From the relation (37), we observe that

\[
\tilde{g}_1 \equiv g_3 g_1 g_3^{-1} = g_2, \quad \tilde{g}_2 \equiv g_3 g_2 g_3^{-1} = g_2^n g_1^{-1}.
\]

(42)

This means that $g_3$ maps a parallelogram spanned by $g_1$ and $g_2$ on an $x$-$y$ plane to another parallelogram spanned by $\tilde{g}_1$ and $\tilde{g}_2$ on another $x$-$y$ plane (Fig.2 (A) and (B)). If we “cut” and translate the second parallelogram by the actions of $g_1$ and $g_2$, then we can take a fundamental region as a parallelopiped of which the bottom and top surfaces are spanned by $g_1$ and $g_2$ (Fig.2 (B) and (C)). When identifying the bottom surface to the top, the surface is stretched by $n$ times, and then folded. (A geodesic congruence along $z$-axis in $f/1(n)$ will therefore behave extremely chaotically after a journey over some periods.)

Let us return to the operation to identify the Teichmüller parameter. Note that we can transform the spatial metric (35) into

\[
dl^2 = h_{33} \left[ \frac{1}{2} \sqrt{\frac{h_{11}}{h_{22}} (e^{\epsilon'} dx' + e^{-\epsilon'} dy')^2} + \frac{1}{2} \sqrt{\frac{h_{22}}{h_{11}} (-e^{\epsilon'} dx' + e^{-\epsilon'} dy')^2} + dz'^2 \right],
\]

(43)
Fig. 2: The topology of f/1(n = 4). $g_1$ and $g_2$ span a parallelogram in a $x$-$y$ plane (Fig.(A)). The stretched parallelogram, shown in Fig.(B), spanned by $\tilde{g}_1 = g_2$ and $\tilde{g}_2 = g_2^4 g_1^{-1}$ in $z = z_0 + c_3$ plane can be identified by the actions of $g_1, g_1g_2^{-1}, ..., g_1g_2^{-4}$ with the parallelogram spanned by $g_1$ and $g_2$ in a way that letters A~E in Fig.(B) label the corresponding regions. The corresponding regions in Fig.(A) are labeled by the same letters. So, the up and down sides of the parallelopiped in Fig.(C), which shows a fundamental region of f1/1(4), are identified as indicated by circles and triangles. The front and back sides, and the right and left sides are identified in the trivial way.
where
\[
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
(h_{11}h_{22})^{1/4}x \\
(h_{11}h_{22})^{1/4}y \\
z
\end{pmatrix}.
\tag{44}
\]
The transformation \((x, y, z) \rightarrow (x', y', z')\) defined by Eq.\(44\) is an automorphism of Sol.

So, the appropriate action of identifications on \((x', y', z')\) is given by
\[
g'_1 = \left(\frac{(h_{11}h_{22})^{1/4}}{\sqrt{h_{33}}} \begin{array}{c}
\alpha_0 u_1 \\
\alpha_0 u_2 \\
0
\end{array} \right),
g'_2 = \left(\frac{(h_{11}h_{22})^{1/4}}{\sqrt{h_{33}}} \begin{array}{c}
\alpha_0 v_1 \\
\alpha_0 v_2 \\
0
\end{array} \right),
g'_3 = \begin{pmatrix}
0 \\
0 \\
c_3
\end{pmatrix}
\tag{45}
\]
for \(n > 2\), or \(g'_3 = h \circ (0, 0, c_3)\) for \(n < -2\). This already coincides with the parametrization of the Teichmüller space, i.e. the only Teichmüller parameter is
\[
\alpha(t) = \frac{(h_{11}h_{22})^{1/4}}{\sqrt{h_{33}}} \alpha_0.
\tag{46}
\]

It is worth noting that we can observe from Eq.\(45\) that the Teichmüller deformations of \(f_{1/1(n)}\) are the variations of the ratio of the area of the fiber torus to the length of the base \(S^1\).

The dynamical variables (in configuration space) are the Teichmüller parameter \(\alpha\), the 3-volume \(v = h_{11}h_{22}h_{33}(\alpha_0)^4\) up to a function depending on \(n\), and the curvature control parameter \(\lambda = \ln(h_{11}/h_{22})\). The number of dynamical degrees of freedom is four; one is for \(\alpha_0\), and three is for the parameters contained in the four-dimensional universal cover.

### 4.3 The \(a_{1/1}\) model: a compact model on Bianchi \(VII_0\)

The multiplication rule of Bianchi \(VII_0\) group is given by;
\[
\begin{pmatrix}
g^1 \\
g^2 \\
g^3
\end{pmatrix} \begin{pmatrix}
h^1 \\
h^2 \\
h^3
\end{pmatrix} = \begin{pmatrix}
g^1 \\
g^2 \\
g^3 + h^3
\end{pmatrix} + R_{g^3} \begin{pmatrix}
h^1 \\
h^2
\end{pmatrix},
\tag{47}
\]
where \(g, h \in \text{Bianchi } VII_0\) group, and \(R_{g^3}\) is the rotation matrix by angle \(g^3\). The 1-forms
\[
\sigma^1 = \coszd + \sinzd, \quad \sigma^2 = -\sinzd + \coszd, \quad \sigma^3 = dz
\tag{48}
\]
are invariant under the left action of Bianchi VII$_0$. Hence the invariant metric is
\[ dl^2 = h_{11}(\sigma^1)^2 + h_{22}(\sigma^2)^2 + h_{33}(\sigma^3)^2. \] (49)

As usual, the four dimensional universal cover metric is Eq.(13) with the above \( dl^2 \). Since, as in the Bianchi VI$_0$ case in the previous subsection, the isometries and the extendible isometries of \((\tilde{M}_t, \tilde{h}_{ab})\) coincide, Isom\(\tilde{M}_t = Esom\tilde{M}_t = (\text{Bianchi VII}_0 \text{ group plus three discrete elements})\), our calculations to do will be similar to those in the Bianchi VI$_0$ case.

The compact 3-manifold we consider here is a1/1, which is homeomorphic to the 3-torus \(T^3\). The three generators, \(g_1, g_2\) and \(g_3\), of the fundamental group of a1/1 are hence all commutative.

The embedding of the fundamental group in Esom\(\tilde{M}_t\) up to conjugacies by Esom\(\tilde{M}_t (=\text{Isom}\tilde{M}_t)\) is already given in I, which reads
\[
\begin{align*}
g_1 &= \begin{pmatrix} g_{11}^1 \\ 0 \\ 2l\pi \end{pmatrix}, \\
g_2 &= \begin{pmatrix} g_{21}^2 \\ g_{22}^2 \\ 2m\pi \end{pmatrix}, \\
g_3 &= \begin{pmatrix} g_{31}^3 \\ g_{32}^3 \\ 2n\pi \end{pmatrix},
\end{align*}
\] (50)

where \(l, m, n\) are integers.

To obtain the time-development of the Teichmüller parameters, we, as usual, first note that we can transform the spatial metric (49) into
\[
dl^2 = h_{33} \left( \sqrt{h_{11}} \, (\cos z' dx' + \sin z' dy')^2 + \sqrt{h_{22}} \, (-\sin z' dx' + \cos z' dy')^2 + dz'^2 \right), \quad (51)
\]

where
\[
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} (h_{11} h_{22})^{1/4} x \\ (h_{11} h_{22})^{1/4} y \\ z \end{pmatrix}. \quad (52)
\]

Here, the metric (51) coincides with the standard metric, given in I, of Bianchi VII$_0$ up to global conformal factor.

Since the transformation \((x, y, z) \rightarrow (x', y', z')\) is an automorphism of the Bianchi VII$_0$ group, we can easily obtain the actions on the standard metric (51) of the generators (50).
We immediately get
\[
\begin{pmatrix}
a_1^1 \\
0 \\
2l\pi
\end{pmatrix},
\begin{pmatrix}
a_2^1 \\
a_2^2 \\
2m\pi
\end{pmatrix},
\begin{pmatrix}
a_3^1 \\
a_3^2 \\
2n\pi
\end{pmatrix},
\]
where
\[
a_i^j = \frac{(h_{11} h_{22})^{1/4}}{\sqrt{h_{33}}} g_i^j, (i, j) = (1, 1), (2, 1), (2, 2), (3, 1), \text{and} (3, 2).
\]
Here, \(a_i\)'s are the images of \(g_i\)'s by the automorphism, and \(a_1^1 \sim a_3^2\) are the Teichmüller parameters.

To summarize, the five constants \(g_1^1 \sim g_3^2\) determine the initial identifications on an initial surface in \((\tilde{M}, \tilde{g}_{ab})\). The universal cover \((\tilde{M}, \tilde{g}_{ab})\) have three arbitrary parameters in vacuum, and hence the number of the dynamical degrees of freedom is 8 (= 5 + 3). We have seven dynamical variables; the five Teichmüller parameters, the curvature control parameter \(\lambda = \ln(h_{11}/h_{22})\), and the 3-volume \(v = \sqrt{h_{11} h_{22} h_{33}} \det(g_1, g_2, g_3)\).

4.4 The a2/1 model: a compact model on Bianchi I

In this subsection, we give the time-development of the Teichmüller parameters of the b2/1 model \((\simeq T^3)\) on vacuum Bianchi I, by a method other than the one finding an automorphism explicitly. This is done by calculating invariants under the automorphisms, like lengths of minimal loops and angles between the loops of the compact homogeneous manifold. To this end, we introduce a matrix representing such invariants as follows.

Let \(\Gamma\) be a covering group acting on \((\tilde{M}, \tilde{h}_{ab})\). For \(a \in \Gamma, p \in \tilde{M}\), let the map \(\gamma_{a,p} : \mathbb{R} \to \tilde{M}\) be the geodesic satisfying
\[
\gamma_{a,p}(0) = p,
\gamma_{a,p}(1) = a(p).
\]
Then, we define the map \(v_p : \Gamma \to V_p\) by relating \(\Gamma\) to the geodesic generator at \(p\);
\[
v_p(a) = \frac{d\gamma_{a,p}(\lambda)}{d\lambda} \bigg|_{\lambda=0}.
\]
Finally, let all the independent generators of $\Gamma$ be $a_i$ ($i = 1, \cdots, n$). Then the loop matrix defined by

$$H_{ij}(p) = \tilde{h}(v_p(a_i), v_p(a_j))$$

(57)

will contain all the information concerning the global geometry of $M = \tilde{M}/\Gamma$. Here, we have dropped the abstract indices of the metric and the vectors in the r.h.s.. It is worth noting that the $p$-dependence of $H_{ij}(p)$ decides whether $M$ is locally homogeneous or globally homogeneous. That is, if $H_{ij}(p)$ is independent of $p$, then $M$ is globally homogeneous, and if not so, then homogeneity of $M$ is local.

We are in a position to calculate the time-development of the $a2/1$ model in vacuum. Our four dimensional universal cover is the Kasner solution = $(R^4, \tilde{g}_{ab})$, where with the usual coordinates $(t, x, y, z)$, $\tilde{g}_{ab}$ is given by Eq.(4) with $\sigma \equiv p_1 + p_2 + p_3 = 1 = (p_1)^2 + (p_2)^2 + (p_3)^2$. Each homogeneous spatial section is given by $(R^3, \tilde{h}_{ab})$ with $\tilde{h}_{ab}$ being Eq.(5). The covering group may be generated by three commuting generators, for which we write as

$$g_i = \begin{pmatrix} g_i^1 \\ g_i^2 \\ g_i^3 \end{pmatrix}, \quad (i = 1 \sim 3).$$

(58)

Here, the action of $g_i$ on $p = (x_0, y_0, z_0)$ on $(R^3, \tilde{h}_{ab})$ is given by

$$g_i \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} g_i^1 + x_0 \\ g_i^2 + y_0 \\ g_i^3 + z_0 \end{pmatrix}.$$  

(59)

Since the extendible isometry group of each slice is isomorphic to the commutative group $R^3$, we see that no nontrivial conjugation occurs. This implies that we cannot simplify the components of $g_i$’s more than the original form of Eq.(58).

We can at this point count the dynamical degrees of freedom of the present model. Immediately can we see that the Kasner parameter carries the part of dynamical degrees of freedom in the universal cover, dim $\bar{U} = 1$, while $g_i^\alpha$’s in Eq.(58) carry the part in the initial identifications, dim $\bar{C}_u = 9$. The total dynamical degrees of freedom is therefore 10(= dim $\bar{U}$ + dim $\bar{C}_u$). (cf. proposition 3)

27
Now, return back to the procedure to get the Teichmüller parameters. We can easily find the geodesics satisfying Eq. (55), and then get the generator at \( p \) as follows.

\[
\gamma_{g_i,p}(\lambda) = \begin{pmatrix} g_i^t + x_0 \\ g_i^2 + y_0 \\ g_i^3 + z_0 \end{pmatrix}, \quad v_p(g_i) = \begin{pmatrix} g_i^1 \\ g_i^2 \\ g_i^3 \end{pmatrix}.
\] (60)

From this, we have

\[
H_{ij}(p) = \tilde{h}(v_p(g_i), v_p(g_j)) = \sum_{\alpha=1}^3 t^{2p_\alpha} g_\alpha g_\alpha.
\] (61)

On the other hand, any flat \( T^3 \) (a2/1) can be implemented in the standard Euclid metric

\[
d\ell^2 = dx^2 + dy^2 + dz^2
\] (62)

with six Teichmüller parameters \([25]\) in three generators

\[
a_1 = \begin{pmatrix} a_1^1 \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_2^1 \\ a_2^2 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} a_3^1 \\ a_3^2 \\ a_3^3 \end{pmatrix}.
\] (63)

Components of \( v_p(a_i) \) is the same as Eq. (61) with \( g_i^t \) replaced by \( a_i^t \) with \( a_1^2 = a_1^3 = a_2^3 = 0 \). Using Eq. (62) as \( \tilde{h} \), we have

\[
H_{ij} = a_i^1 a_j^1 + a_i^2 a_j^2 + a_i^3 a_j^3
\]

\[
= \begin{pmatrix} (a_1^1)^2 & a_1^1 a_2^1 & a_1^1 a_3^1 \\ (a_2^1)^2 + (a_2^2)^2 & a_2^1 a_3^1 + a_2^2 a_3^2 \\ (\text{sym.}) & (a_3^1)^2 + (a_3^2)^2 + (a_3^3)^2 \end{pmatrix}
\] (64)

We set equal the two expressions Eqs. (74) and (51) to get \( a_i^t \) as time functions with initial parameters \( g_i^\alpha \). Elementary calculations lead to the following results;

\[
(a_1^1)^2 = \sum_\alpha t^{2\alpha} g_1^\alpha, \quad a_2^1 = \frac{(\sum_\alpha t^{2\alpha} g_1^\alpha g_2^\alpha)}{a_1^1}, \quad a_3^1 = \frac{(\sum_\alpha t^{2\alpha} g_1^\alpha g_3^\alpha)}{a_1^1},
\]

\[
(a_2^2)^2 = \frac{\Delta^2}{(a_1^1)^2}, \quad a_3^2 = -a_2^2 \frac{(\sum_\alpha t^{2(\sigma-\alpha)} g_3^\alpha g_2^\alpha)}{\Delta^2}, \quad a_3^3 = (\det g)^2 t^{2\sigma}/\Delta^2.
\] (65)
where \( \Delta^2 \equiv \sum_\alpha t^{2(\sigma - p_\alpha)}(\Delta_3^\alpha)^2 \), \( \sigma \equiv \sum_\alpha p_\alpha = 1 \), and \( \bar{g}_i^\alpha \) is the \((i, \alpha)\)th cofactor of the matrix \( (g_i^\alpha) \). It would be useful, especially in getting \( a_3^3 \), to note that the determinant of \( H_{ij} \) is given by

\[
\det H = (\det g)^2 t^{2\sigma} = (a_1^1)^2(a_2^2)^2(a_3^3)^2.
\]

(66)

5 Conclusions

We have given a general method of construction of compact homogeneous universes. This is accomplished by taking identifications in a universal cover \( (\tilde{M}, \tilde{g}_{ab}) \). The universal cover must satisfy Einstein’s equation, and the degrees of freedom of all the possible diffeomorphisms must be subtracted. The identifications in the universal cover are implemented by a discrete subgroup of the extendible isometries, \( \text{Esom}\tilde{M}_t \). At this stage, one takes the conjugations by \( \text{Esom}\tilde{M}_t \), and finally we can obtain a system of compact homogeneous universes which is free from any diffeomorphisms, i.e., the free parameters in the metric and in the identifications are the dynamical degrees of freedom of the system. This method of construction is evidently applicable to any system of compact homogeneous universes (i.e., compact models on the Bianchi class A, class B, and the Kantowski-Sachs-Nariai models).

We have considered the dynamical variables of the system to be parameters specifying spatial sections completely. In this sense, the Teichmüller parameters are dynamical variables, as well as the 3-volume and the possible curvature parameters. It is important to note that the number, \( f \), of dynamical degrees of freedom is less than double the number, \( d \), of dynamical variables. As we have seen in the explicit examples, not all of the initial velocities can be arbitrarily chosen. In some cases, \( f \) is less than \( 2d \) by 2. This could be explained by the Hamiltonian constraint. In the other cases, however, \( f \) is less than \( 2d - 2 \). These could be well understood if we study whether the dynamical system admits a canonical structure. This is also needed in canonical quantization of compact homogeneous universes. This problem will be discussed in a separate work.

Although we focused on the time-developments of the Teichmüller parameters and the dynamical degrees of freedom, our framework of compact homogeneous universes should
be useful in wide variety of problems in astrophysics, observational cosmology, fundamental problems of relativity, quantum cosmology, and quantum gravity. For example, the behavior of geodesics in a compact universe can become chaotic (cf. Sec.4.2), which fact may provide an interesting cosmological model. The problem of strong cosmic censorship for compact homogeneous universes is also of great interest, which is being investigated by some groups (See, e.g., [26]).

Acknowledgments

M. T. thanks Soryushi Shogakukai for financial support. T. K. acknowledges financial support from the Japan Society for the Promotion of Science and the Ministry of Education, Science and Culture. This work is partially supported by the Grant-in-Aid for Scientific Research of the Ministry of Education, Science, and Culture of Japan (No.02640232)(A.H.).

References

[*] Present address: Department of Physics, Kyoto University, Kyoto 606-01, Japan. E-mail: prince@tap.scphys.kyoto-u.ac.jp

[**] Present address: Department of Physics, Keio University, Hiyoshi, Yokohama, Kanagawa 230, Japan. E-mail: koike@rk.phys.keio.ac.jp

[***] E-mail address: ahosoya@th.phys.titech.ac.jp

[1] See, e.g., S. Weinberg, Gravitation and cosmology. (Wiley, New York, 1972)

[2] L. D. Landau, E. M. Lifshitz, Classical theory of Fields (MIT, Reading, 1971).

[3] M. P. Ryan, L. C. Shepley, Homogeneous Relativistic Cosmologies. Princeton Series in Physics (Princeton University Press, Princeton, 1975)

[4] R. M. Wald, General Relativity (University of Chicago, Chicago, 1984).

[5] V. A. Belinski, I. M. Khalatnikov and E. M. Lifshitz, Adv. in Phys. 19, 525 (1970)
[6] See, e.g., J. J. Halliwell, in ed. S. Coleman et al., Quantum cosmology and baby universes, (World Scientific, Singapore, 1991), pp.159-243, and references therein.

[7] A. Hosoya and K. Nakao, Class. Quantum Gravit. 7, 163 (1990). For recent status of (2+1)-gravity and more references, see e.g. S. Carlip, Lectures on (2+1)-Dimensional Gravity (gr-qc/9503024, 1995).

[8] T. Koike, M. Tanimoto and A. Hosoya, J. Math. Phys. 35, 4855 (1994).

[9] W. P. Thurston, The Geometry and Topology of 3-manifolds (To be published by Princeton University Press).

[10] P. Scott, Bull. London Math. Soc. 15, 401 (1983).

[11] For type d (, which correspond to Bianchi VIII), the representative metric (or standard metric, see Sec.2.2) is incorrect, due to incorrectness of the appendix. For detail, see the erratum, which will appear in the journal.

[12] I. M. Singer and J. A. Thorpe, Lecture notes on elementary topology and geometry. (Scott, Foresman and Company, Glenview, 1967).

[13] L. Bianchi, Mem. della Soc. Italiana delle Scienze Ser. 3a, 11, 267 (1897).

[14] H. Nariai, Sci. Rep. Tohoku Univ., I, 34, 160 (1950); ibid., 35, 62 (1951).

[15] R. Kantowski and R. K. Sachs, J. Math. Phys. 7, 443 (1967).

[16] H. V. Fagundes, Phys. Rev. Lett. 54, 1200 (1985).

[17] J. A. Wolf, Spaces of constant curvature, Fifth Edition, (Wilmington, Publish or Perish 1984).

[18] K. Ohshika, Topology and its Application 27, 75 (1987).

[19] R. Kulkarni, K. B. Lee, and F. Raymond, Geometry and Topology, Lecture Notes Math. 1167, 180 (1985).
[20] Here, Eq. (9) is understood. The extrinsic curvature therefore has the same homogeneity as the spatial metric.

[21] For simplicity, we identify the set $\tilde{U}$ with a set of representatives. Similar identifications are understood also for the sets of equivalence classes defined subsequently.

[22] G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. 12, 108 (1969)

[23] A. Ashtekar and J. Samuel, Class. Quantum Gravit. 8, 2191 (1991).

[24] See, e.g., D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, Exact Solutions of Einstein’s Field Equations. (Cambridge Univ. Press, Cambridge, 1980)

[25] For convenience, we here do not distinguish the 3-volume from the Teichmüller parameters. If do so, we need to regard, e.g., $a_3^3$ as a function of other Teichmüller parameters and the 3-volume.

[26] P. T. Chruściel and A. D. Rendall, Ann. Phys. 242, 349 (1995).