Fast Track Communication

The $\epsilon$-expansion from conformal field theory

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Abstract
Conformal multiplets of $\varphi$ and $\varphi^3$ recombine at the Wilson–Fisher fixed point, as a consequence of the equations of motion. Using this fact and other constraints from conformal symmetry, we reproduce the lowest nontrivial order results for the anomalous dimensions of operators, without any input from perturbation theory.

Keywords: conformal field theory, Wilson–Fisher fixed point, epsilon-expansion

1. Introduction

The conformal bootstrap—using conformal symmetry to constrain the dynamics of various physically interesting critical theories—is an old idea [1–3] which in the last few years has been shown to be much more powerful than previously thought [4–11]. A flagship result of this line of research is the world’s most precise determination of the operator dimensions in the critical 3D Ising model [12]. Many other conformal theories, such as the $O(N)$ model [21, 22], models with supersymmetry in various dimensions [23–32], as well as other less familiar or conjectural conformal theories [33–39] are being studied. In the hectic excitement of exploring a new research field, much of the recent development has been numerical. It is a significant open problem to explain analytically the unreasonable effectiveness of the numerical bootstrap.

An analytic conformal bootstrap attack on the 3D Ising model looks difficult, since one has to deal with a totally isolated conformal field theory (CFT). The $d = 4 - \epsilon$ case might be

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5 See [13–17] for prior work and [18–20] for closely related results by another technique.
6 Nevertheless, there have been notable analytic results for the large $N$ [40–42], large $\Delta$ [43, 44], large $J$ [45–51], or SUSY [52, 53] cases. There is also a growing literature on the analytic expressions for the conformal correlation functions and conformal blocks, which is indispensable for the practical conformal bootstrap computations [54–73].
an easier target. Indeed, in the limit $\epsilon \to 0$ the CFT should become close to the free theory. One can then hope to find a solution of the bootstrap equations as a power series in $\epsilon$.

The critical interacting scalar theory in $4 - \epsilon$ dimensions is known as the Wilson–Fisher (WF) fixed point [74]. It is normally studied using the standard diagrammatic perturbation theory and the renormalization group. This method also yields an expansion of observables, e.g. the anomalous dimensions of local operators, as power series in $\epsilon$, called $\epsilon$-expansion series. An interesting open question is whether conformal bootstrap can reproduce these series and perhaps even extend them to a higher order$^7$.

The purpose of this note is to report some progress on the last question. Using CFT methods, we will be able to reproduce the leading $O(\epsilon)$ terms in the anomalous dimension of an infinite series of operators. For the lowest dimension scalar operator $\phi$ we will reproduce both $O(\epsilon)$ and $O(\epsilon^2)$ terms. This may seem not like much, given that the perturbative $\epsilon$-expansion series are known in some cases as far as $\epsilon^5$ [76] or even $\epsilon^6$ [77]. However, we believe that our discussion will lay the basis for any future extension to higher orders. Also, the modesty of the result is compensated by the relative simplicity of the derivation.

The paper is organized as follows. In section 2 we review the main structural properties of the WF fixed point. These properties are arrived at from the perturbative perspective, but we then formalize them in a series of axioms about CFT operators. The actual computation of operator dimensions, in section 3, proceeds from the axioms without recourse to the standard perturbation theory nor in fact to the Lagrangian. In section 4 we extend our computation is several directions, in particular generalizing to the $O(N)$ model. In section 6 we discuss open problems. Some technical details are relegated to the appendix.

2. Axioms

To set the notation, consider the massless $\phi^4$ theory in $d = 4 - \epsilon$ dimensions (see e.g. [78]):

$$S = \int d^d x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{4!} g_{\mu \nu} \phi^4 \right].$$ (2.1)

We will be interested in the IR fixed point of this theory, called the WF fixed point. It corresponds to the nontrivial zero of the $\beta$-function:

$$\beta(g) = -\epsilon g + \frac{3}{16\pi^2} g^2 + O(g^3).$$ (2.2)

which occurs for

$$g = g_\ast = \frac{16\pi^2}{3} \epsilon + O(\epsilon^2).$$ (2.3)

Using the standard Feynman diagram perturbation theory, correlation functions of the $\phi^4$ theory can be expanded in the coupling constant. Renormalization group can then be used to reach the WF fixed point in the IR. All properties of the WF fixed point, such as the anomalous dimensions of local operators, are thus computable as series in $\epsilon$.

Here, we will focus on the operators of the form $\phi^n$. As is well known, the anomalous dimension of $\phi$ arises first at two loops and is given by

$^7$ Numerically, conformal bootstrap equations in fractional dimensions were studied in [75], matching the $\epsilon$-expansion series.
\[ \gamma_\varphi = \frac{\epsilon^2}{108} + O(\epsilon^3), \]  
(2.4)

while the higher powers of \( \varphi \) acquire anomalous dimensions already at one loop\(^8\):

\[ \gamma_{\varphi^n} = \frac{1}{6(n - 1)}\epsilon + O(\epsilon^3), \quad n > 1. \]  
(2.5)

As a matter of fact, the anomalous dimensions of \( \varphi, \varphi^2 \) and \( \varphi^4 \) are known up to \( \epsilon^5 \) [76], but we will not need those expressions here.

Our main result will be to present an alternative method of computing the anomalous dimensions of \( \varphi^n \). Instead of the diagrammatic perturbation theory, our method will be based on conformal field theory. By this method, we will be able to reproduce equations (2.4) and (2.5) to the shown order.

We will formalize our method by stating a series of three axioms. The axioms themselves can be motivated and justified from perturbation theory. However, once the axioms are agreed upon, the computations will be done by CFT techniques, without further recourse to the Lagrangian.

**Axiom I. The WF fixed point is conformally invariant.**

This can be proven to all orders in perturbation theory by considering the Ward identities of the renormalized stress tensor operator\(^9\).

**Axiom II. Correlation functions of operators at the WF fixed point approach free theory correlators in the limit \( \epsilon \to 0 \).**

This is pretty self-evident, since the fixed point coupling \( g_\varphi = O(\epsilon) \). A more concrete formulation of the same axiom is as follows:

**Axiom II’. For every local operator \( \mathcal{O}_{\text{free}} \) in the free theory in \( d = 4 \), there exists a local operator at the WF fixed point, \( \mathcal{O}_{\text{WF}} \), which tends to \( \mathcal{O}_{\text{free}} \) in the limit \( \epsilon \to 0 \):**

\[ \lim_{\epsilon \to 0} \mathcal{O}_{\text{WF}} = \mathcal{O}_{\text{free}}. \]  
(2.6)

The limit is understood in the sense of correlators, i.e.

\[ \lim_{\epsilon \to 0} \langle \mathcal{O}_{\text{WF}}(x_1)\mathcal{O}_{\text{WF}}(x_2)\ldots \rangle_{d=4-\epsilon} = \langle \mathcal{O}_{\text{free}}(x_1)\mathcal{O}_{\text{free}}(x_2)\ldots \rangle_{d=4}. \]  
(2.7)

In particular, the scaling dimensions of WF operators must approach the free dimensions as \( \epsilon \to 0 \)\(^10\):

\[ \lim_{\epsilon \to 0} \left[ \mathcal{O}_{\text{WF}} \right] = \left[ \mathcal{O}_{\text{free}} \right]. \]  
(2.8)

In agreement with what perturbation theory suggests, we will assume that scaling dimensions have a power series expansion in integer powers in \( \epsilon \). In this paper we will consider these expansions as formal and will not be concerned with their convergence properties. It is in fact well known that the \( \epsilon \)-expansion series are not convergent but merely asymptotic [80]. Moreover, the range of \( \epsilon \) where the lowest-order result (equation (2.5)) can be trusted presumably becomes smaller and smaller for larger \( n \), due to the growth of the coefficient. This feature of the the \( \epsilon \)-expansion is also well known [81], and we have nothing to add here.

Axioms I, II by themselves would not be sufficient to constrain the WF fixed point. This is because correlators of the free scalar theory in \( d = 4 - \epsilon \) provide a trivial solution to both axioms. We need another axiom to distinguish the WF fixed point from the free theory. This

\(^8\) This result is usually given for \( n = 1, 2, 4 \), but generalization to arbitrary \( n \) is truly straightforward.

\(^9\) Conformal invariance follows from equation (4.16) in [79] by specializing to \( m^2 = 0 \) and going to the IR where the \( \beta \)-function vanishes.

\(^10\) Scaling dimensions of operators will be denoted by \( [\mathcal{O}] \) or \( \Delta_\mathcal{O} \).
Axiom can be inferred from the following reasoning. In the free theory all powers $q^n$ are independent local operators, with no derivative relations among them. From the CFT point of view, all these operators are what are called primaries. This is not so at the WF fixed point, where we have one nontrivial relation $\Box \equiv \partial^2$:

$$\Box q = \frac{1}{3!} g_{\mu \nu \rho} \partial^\mu q^3. \tag{2.9}$$

This is nothing but the classical equation of motion, and it survives in the quantum theory for renormalized operators. Consequently, at the WF fixed point, the $q^3$ is not an independent operator, but is a derivative of $q$. In CFT, such operators are called descendants, and their properties are completely fixed in terms of those of a primary. In particular, the total dimension (classical plus anomalous) of $q^3$ is predicted to be two plus that of $q$:

$$\Delta_{q^3} = \Delta_q + 2 \quad \text{(WF).} \tag{2.10}$$

This relation has to hold to all orders in $\epsilon$, and it can be verified to $O(\epsilon)$ using equations (2.4), (2.5).

Once $q^3$ is recognized as a descendant of $q$, its descendants also become descendants of $q$. A primary operator together with all its descendants constitute a conformal multiplet. We thus observe the phenomenon of multiplet recombination—two distinct multiplets in the free theory—those of $q$ and $q^3$—join and become a single conformal multiplet of $q$ at the WF fixed point:

$$\{q\} \text{WF} \approx \{q\} \text{free} + \{q^3\} \text{free}. \tag{2.11}$$

Here, $\{\ldots\}$ denotes a conformal multiplet, and $\approx$ means that the number of states at each level is conserved in the process of recombination, up to small anomalous dimensions.

Another way to see the need for multiplet recombination is as follows. The free scalar field satisfies the equation $\Box q_{\text{free}} = 0$, which acts as a shortening condition for the multiplet $\{q\} \text{free}$, setting many descendants to zero. There is no such shortening condition for the multiplet $\{q\} \text{WF}$, which thus contains more states than $\{q\} \text{free}$. These new states must come from some other multiplet. One can see that $\{q^3\} \text{free}$ contains the right number of states with the right dimensions to supply the difference\footnote{This can be also seen from the character formulas for the corresponding representations of the conformal group [82].}. Moreover, the free theory contains no other candidate multiplet able to fulfill this role. The recombination pattern in equation (2.11) is thus mandatory already from the point of view of counting states.

The just described phenomenon of multiplet recombination can be viewed as a sort of CFT analogue of the familiar Higgs mechanism from particle physics. Just as a massless vector boson can acquire mass only by ‘eating’ a scalar Goldstone particle, which provides a necessary extra state, so a short conformal multiplet can become long only by eating another multiplet.

Notice that $q$ and $q^3$ are the only free scalar primaries whose multiplets recombine\footnote{As will be discussed in section 6, recombination also happens for the multiplets of conserved currents of spins $l > 2$. In holographic theories, the Higgs mechanism analogy becomes literal for this recombination, as it is described via the usual Higgs mechanism in the bulk, see e.g. [83]. We are grateful to Leonardo Rastelli for emphasizing the last point to us.}. The multiplets of $q^2$ and of $q^n$ ($n \geq 4$) are already long in the free theory, and they remain long in the interacting theory.

From now on, we will adhere to a stricter notation for operators. The notation $q^n$ will be used to denote exclusively the free theory operators, while the WF operators $(q^n)_{\text{WF}}$ will be...
denoted by $V_n$. Operators $V_n$ and $q^n$ are related in the sense of axiom II:

$$\lim_{\epsilon \to 0} V_n = q^n. \quad (2.12)$$

Our last axiom summarizes the above discussion about the nature of the operators $V_n$. It will allow us to distinguish the WF fixed point from the free theory in $4 - \epsilon$ dimensions.

**Axiom III.** Operators $V_n$, $n \neq 3$, are primaries. Operator $V_3$ is not a primary but is proportional to $\Box V$:

$$\Box V_1 = \alpha V_3. \quad (2.13)$$

The proportionality coefficient $\alpha = \alpha(\epsilon)$ should be considered as an unknown at this stage. It would be contrary to our philosophy to fix it using the equation of motion equation (2.9). Our logic is that perturbation theory should be used to infer only robust, structural properties, like the fact that multiplets recombine. The actual computations should be done using CFT. Below we will be able to determine $\alpha$ by imposing equation (2.12).

### 3. Computation

As already said, we will be interested in the dimensions of the operators $V_n$, which can be written as

$$\Delta_n = n\delta + \gamma_n. \quad (3.1)$$

Here $\delta = 1 - \epsilon/2$ is the free scalar dimension in $4 - \epsilon$ dimensions, and $\gamma_n$ is the anomalous dimension, which can be expanded in $\epsilon$:

$$\gamma_n = y_{n,1} \epsilon + y_{n,2} \epsilon^2 + \ldots \quad (3.2)$$

As mentioned above, we will assume that only integer powers of $\epsilon$ occur in this expansion.

Perturbation theory results in equations (2.4), (2.5) are equivalently stated as:

$$y_{n,1} = \frac{1}{6} n(n - 1) \quad (n = 1, 2, \ldots), \quad y_{1,2} = 1/108. \quad (3.3)$$

In this section, we will show that these same results can be derived from the axioms using CFT reasoning.

The key idea is to consider correlators involving operators $V_n$ and $V_{n+1}$:

$$\left\langle V_n(x_1)V_{n+1}(x_2) \ldots \right\rangle, \quad (3.4)$$

where $\ldots$ stands for some other operator insertions. In the limit $\epsilon \to 0$ these correlators should approach

$$\left\langle q^n(x_1)q^{n+1}(x_2) \ldots \right\rangle. \quad (3.5)$$

These correlators will turn out to be useful because of their sensitivity to the multiplet recombination phenomenon. In the free theory the operator product expansion (OPE) $q^n \times q^{n+1}$ contains both operators $q$ and $q^3$, with coefficients independently computable by counting Wick contractions. On the other hand, in the interacting theory the OPE $V_n \times V_{n+1}$ contains only $V_1$ as a primary, while $V_3$ appears in the guise of $\Box V$, with a relative coefficient fixed by conformal symmetry. A nontrivial consistency condition will then arise from the requirement that the $\epsilon \to 0$ limit of the second OPE should agree with the first one.

Let us supply the details. It will be convenient to change the normalization of the free 4D scalar:
\[ \eta_{\text{new}} = 2\eta_{\text{old}}. \]  

(3.6)

In the new normalization the two-point function of \( \phi \) is unit-normalized:

\[ \langle \phi(x)\phi(0) \rangle = 1/|x|^2. \]  

(3.7)

Normalization of \( V \) can then be chosen so that

\[ \langle V(x)V(0) \rangle = 1/|x|^{2\Delta}. \]  

(3.8)

Axiom II' is satisfied.

Our first task is to find \( \alpha \) in axiom III. By repeated differentiation of equation (3.8), we get:

\[ \langle \Box V(x)\Box V(0) \rangle = h/|x|^{2\Delta+4}, \]  

(3.9)

\[ h = 16\delta_1 (\Delta_1 + 1)(\Delta_1 - \delta)(\Delta_1 + 1 - \delta) \approx 32\gamma_1 \quad (c \ll 1). \]  

(3.10)

On the other hand we must have

\[ \langle V_2(x)V_2(0) \rangle \rightarrow \langle \phi^3(x)\phi^3(0) \rangle = 6/|x|^6. \]  

(3.11)

Comparing the prefactors of the last two equations, we determine \( \alpha \) up to a sign:

\[ \alpha = 4\sigma(\gamma_1/3)^{1/2}, \quad \sigma = \pm 1. \]  

(3.12)

Later on, we will be able to fix the sign ambiguity and show that \( \sigma = 1 \).

Now, let us consider the correlators of the form of equations (3.4) and (3.5). It will be sufficient to analyze them in the region where the points \( x_1 \) and \( x_2 \) are much closer to each other than to the other operator insertions, so that we can use the OPE. We will not need the whole OPE, but only its leading part sensitive to the multiplet recombination. In the 4D free theory these are the \( \phi \) and \( \phi^3 \) terms:

\[ \phi^n(x) \times \phi^{n+1}(0) \supset f [|x|^{-2n}\left\{ \phi(0) + \phi |x|^2\phi^3(0) \right\}]. \]  

(3.13)

The OPE coefficients are found by counting the number of independent Wick contractions:

\[ f = (n+1)!, \quad q = n/2. \]  

(3.14)

The needed terms in the OPE at the WF fixed point are:

\[ V_n(x) \times V_{n+1}(0) \supset f \left[ |x|^{3\Delta_1 - \Delta_1 - \Delta_1} \left( 1 + q_1 x^\mu \partial_\mu + q_2 x^\mu x^\nu \partial_\mu \partial_\nu + q_3 x^2 \Box + \ldots \right) V_1(0) \right]. \]  

(3.15)

We will now consider the following two correlators in the free theory:

\[ \langle \phi^n(x)\phi^{n+1}(0)\phi(z) \rangle \quad \text{and} \quad \langle \phi^n(x)\phi^{n+1}(0)\phi_3(z) \rangle, \]  

(3.16)

and the WF correlators

\[ \langle V_n(x)V_{n+1}(0)V_1(z) \rangle \quad \text{and} \quad \langle V_n(x)V_{n+1}(0)V_3(z) \rangle, \]  

(3.17)

which have to tend to their counterparts in equation (3.16) for \( c \rightarrow 0 \). In the configuration \( |x| \ll |z| \) the leading behavior of the free correlators is found using the above OPE and is given by:

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13 All local operators in the free theory are assumed normal ordered.

14 This involves some combinatorics. One may wonder if the calculation of these and similar constants below can be simplified using the conformal symmetry (or the higher spin symmetry) of the free scalar theory, but we have not explored this here.
\[ \langle \phi^n(x) \phi^{n+1}(0) \phi(z) \rangle \approx f \left| x \right|^{-2n} \langle \phi(0) \phi(z) \rangle, \quad (3.18) \]
\[ \langle \phi^n(x) \phi^{n+1}(0) \phi^3(z) \rangle \approx f_0 \left| x \right|^{-2n+2} \langle \phi^3(0) \phi^3(z) \rangle. \quad (3.19) \]

Using the WF OPE, we also have the behavior of the WF correlators in the same configuration. For the first correlator we have

\[ \langle V_n(x) V_{n+1}(0) V_1(z) \rangle \approx \tilde{f} \left| x \right|^{2\delta_n - \delta_{n+1}} \langle V_1(0) V_1(z) \rangle. \quad (3.20) \]

This will match the free correlator if \( \tilde{f} \approx f \) for \( \epsilon \to 0 \), and consequently \( \tilde{f} \) will be non-vanishing. The second correlator is more interesting. We have:

\[ \langle V_n(x) V_{n+1}(0) V_3(z) \rangle \approx \tilde{f} \left| x \right|^{2\delta_n - \delta_{n+1}} \times \left( 1 + q_1 x^{\alpha} \partial_{\alpha} + q_2 x^{\alpha \beta} \partial_{\alpha} \partial_{\beta} + q_3 x^2 \Box + \ldots \right) \langle V_1(0) V_3(z) \rangle, \]

where the derivatives fall on the argument of \( V_1(0) \) inside the two-point function. The coefficients \( q_i \) are fixed by conformal symmetry in terms of the dimensions of the involved operators. Their expressions are computed in the appendix and will be discussed below.

We have to match equation (3.21) with the free correlator (equation (3.19)). First of all, notice that equation (3.19) predicts the subleading behavior \( O(x^2) \) with respect to equation (3.18). However, in equation (3.21) we see terms \( O(1) \) as well as \( x^\alpha \) and \( x^{\alpha \beta} \). For the matching to occur, all these offending terms have to vanish in the limit \( \epsilon \to 0 \). In fact, using equations (2.13) and (3.12) we obtain:

\[ \langle V_1(0) V_3(z) \rangle = \frac{4\Delta_1 (\Delta_1 - \delta) a^{-1}}{\left| x \right|^{2\Delta_2 + 2}} = \frac{\alpha \Delta_1 \sqrt{3}}{\left| x \right|^{2\Delta_2 + 2}}, \]

which goes to zero for \( \epsilon \to 0 \) as \( \sqrt{\epsilon} \). Therefore, the above-mentioned offending terms will go away as long as \( q_1, q_2 \) remain finite in the \( \epsilon \to 0 \) limit. We will see below that this condition is indeed satisfied.

On the other hand, to match the terms \( O(x^2) \), we must have that \( q_3 \) blows up in the \( \epsilon \to 0 \) limit at a rate which can be precisely determined. Namely, we must have\(^{15}\):

\[ q_3 \alpha \to q = n/2 \quad (\epsilon \to 0). \]

The rest of the argument will follow from equations (3.12), (3.23) and the explicit expressions for the coefficient \( q_3 \).

As already mentioned, \( q_i \) are fixed by conformal symmetry in terms of the dimensions of the involved operators. The full expressions will be given in the appendix. We have to consider separately two cases:

(i) \( n = 1 \) and \( n \geq 4 \),
(ii) \( n = 2 \) and \( n = 3 \). \( (3.24) \)

In case (i), both operators in the LHS of the OPE, \( V_n \) and \( V_{n+1} \), are primaries. In case (ii), one of them is the descendant \( V_3 \). Since descendants transform under the conformal group differently from primaries, their OPE has to be derived separately.

Let us begin by considering case (i). The coefficients \( q_i \) are then given by equations (A.3), where we have to put \( a = \Delta_n, b = \Delta_{n+1}, c = \Delta_1 \). It is easy to see that \( q_1, q_2 \) remain nonsingular as \( \epsilon \to 0 \), in agreement with the above discussion. On the other hand, \( q_3 \)

\(^{15}\) The coefficient of \( x^2 \) also gets a contribution from the \( q_3 \) term in the OPE. This extra contribution can be ignored in equation (3.23) because, as mentioned, \( q_2 \) will remain finite for \( \epsilon \to 0 \), while \( q_3 \) will blow up.
has a chance to become singular because of the \( c - \delta \) factor in the denominator. The asymptotic expression for \( q_3 \) in this limit is given by:

\[
q_3 \approx \frac{y_{n+1} - y_n + \eta_1}{16\gamma_1} \quad (\epsilon \ll 1).
\]  
(3.25)

The numerator in this formula is at least \( O(\epsilon) \). To have a singularity as \( \epsilon \to 0 \), the \( \gamma_1 \) in the denominator must be at least \( O(\epsilon^2) \), i.e.

\[
\gamma_{1,1} = 0.
\]  
(3.26)

Let us assume for the moment that \( \gamma_1 \) is precisely \( O(\epsilon^2) \), i.e.

\[
\gamma_1 \approx y_{1,2}\epsilon^2, \quad y_{1,2} \neq 0.
\]  
(3.27)

We will be able to justify this assumption later in this section. Substituting this asymptotics for \( \gamma_1 \) into equations (3.12), (3.25), we obtain:

\[
\alpha \approx 4\sigma \left(y_{1,2}/3\right)^{1/2} \epsilon, \quad (3.28)
\]

\[
q_3 \approx \frac{y_{n+1} - y_n}{\left(16y_{1,2}\epsilon\right)}. \quad (3.29)
\]

Using these expressions in equation (3.23) gives:

\[
y_{n+1} - y_n = 2\sigma \left(3y_{1,2}\right)^{1/2} \epsilon.
\]  
(3.30)

So far we have considered only case (i). We now turn to case (ii), which involves operator \( V_3 \) on the LHS of the OPE. Operator \( V_3 \) is a descendant of \( V_1 \) and its OPE can be worked out starting from the OPE of \( V_1 \). This computation is done in the appendix. The result for \( q_3 \) can be expressed in terms of \( \Delta_1 \) and the dimension of the third operator (\( V_2 \) or \( V_4 \)). Substituting \( \gamma_1 \approx y_{1,2}\epsilon^2 \), the result simplifies and we find (see equations (A.10), (A.12) where \( q_3 \) was denoted \( q_3' \)):

\[
q_3 \approx \begin{cases} 
(1 - y_{2,1})/\left(16y_{1,2}\epsilon\right) & (n = 2), \\
y_{4,1} - 1)/\left(16y_{1,2}\epsilon\right) & (n = 3).
\end{cases}
\]  
(3.31)

Now, notice that

\[
y_{3,1} = 1,
\]  
(3.32)

which follows from \( y_{1,1} = 0 \) and from the fact that \( \Delta_3 - \Delta_1 = 2 \) (see equation (2.10)). Using this fact, we can restate equation (3.31) as:

\[
q_3 \approx \frac{y_{n+1} - y_n}{\left(16y_{1,2}\epsilon\right)}. \quad (n = 2, 3).
\]  
(3.33)

Amazingly, this is the same equation as the one we found for \( n = 1 \) and \( n \geq 4 \) in equation (3.29). Thus we conclude that equation (3.29) is in fact true for all \( n \geq 1 \).

This allows us to complete the computation. Equation (3.30) was a consequence of equation (3.29) and we can now say that it’s true for all \( n \geq 1 \). Solving the recursion (3.30) with the initial condition \( y_{1,1} = 0 \), we find

\[
y_{n,1} = Kn(n - 1),
\]  
(3.34)
where

$$K = \sigma \left( 3y_{1,2} \right)^{1/2}. \quad (3.35)$$

The coefficient $K$ is determined by imposing that $y_{3,1} = 1$, which gives $K = 1/6$. Using the relation equation (3.35) we find $\sigma = +1$ and $y_{1,2} = 1/108$. Equation (3.3) has now been fully derived as promised.

As a side remark, it is reassuring that the value of the $\alpha$ coefficient implied by equation (3.28) can now be seen to agree with the proportionality coefficient $g_*/3!$ in the perturbative equation of motion equation (2.9), once the change in the field normalization equation (3.6) is taken into account.

It remains to justify the assumption we made above that $y_{1,2} \neq 0$. Consider equation (3.25) for $n = 1$:

$$q_1 \approx y_1/(16\gamma_1) \quad (n = 1). \quad (3.36)$$

Since by equation (3.23) this must go as $1/\alpha \sim 1/\sqrt{\eta}$ for $\epsilon \to 0$, we conclude that

$$y_1 \sim (y_1)^2. \quad (3.37)$$

Consider further the same coefficient for $n = 2$, whose asymptotics was given in the appendix, equation (A.9):

$$q_2 \approx (e - y_2)/(16\gamma_1) \quad (n = 2). \quad (3.38)$$

This also must go as $1/\alpha \sim 1/\sqrt{\eta}$, and hence

$$y_1 \sim (e - y_2)^2. \quad (3.39)$$

This equation can hold together with equation (3.37) only if $y_1 \sim e^2$, i.e. if $y_{1,2} \neq 0$. The derivation is now complete.

Looking at the above computation, one can’t help but wonder why equation (3.29), initially derived for the all primary case (i), eventually turned out to be also true when one of the involved operators is $V_3$, a descendant. An intuitive reason for why this happened is as follows. $V_3$ is not just any descendant, but a descendant which is on the verge of becoming null (it becomes null when $y_1 \to 0$). It is a fact familiar from 2D CFT that the leading null descendant in a conformal multiplet behaves like a primary.

One concrete relevant example of this phenomenon is as follows. Consider the correlator of three scalar primary operators $A, B, C$ of dimensions $a, b, c$, which is fixed by conformal symmetry to have the form [84]:

$$\langle A(x)B(y)C(z) \rangle = f |x - y|^{a-b} |y - z|^{b-c} |z - x|^{a-c}. \quad (3.40)$$

Now apply $\square_1$ to this equation, to compute the correlator

$$\langle \square_1 A(x)B(y)C(z) \rangle. \quad (3.41)$$

Even though this is also a correlator of three scalars, for general $a, b, c$ it will not be of the form equation (3.40). This is not surprising since $\square A$ is not a primary. However, let us now consider the limit $a \to 0$, which is precisely the limit when $\square A$ becomes null. It is easy to check that in this limit equation (3.41) takes the form of equation (3.40) with $a = \delta + 2$ and $f = (\delta + b - c)(\delta + c - b)f$. 

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16 An operator is called null if it has a zero two-point function with itself.
So, to summarize, the intuitive reason why equation (3.29) turns out to be valid also for $n = 2, 3$ is that $V_3$, though not a primary, is almost null and is behaving almost like a primary. It may be possible to derive equation (3.29) for $n = 2, 3$ by quantifying the ‘almost’ in the previous sentence. However, in our presentation we found it easier to do an explicit computation.

4. Extensions

We will now discuss a few simple extensions of the calculation from the previous section. The main idea is the same, so we will be brief.

4.1. Other primary scalars

Up to now, we focused on the scalar operators of the form $q^n$. The crucial idea was to consider the OPE $q^n \times q^{n+1}$, which contains the operator $q$. Here we will indicate how the same argument can be applied to other OPEs containing $q$.

So, let $A$ be a scalar primary operator in the free massless 4D scalar theory, and let $B$ be the normal-ordered product of $q$ and $A$:

$$B = qA.$$  \hfill (4.1)

Then $B$ is also a primary, and the OPE of $A$ and $B$ contains $q$. We denote by $\rho$ the relative coefficient of the operator $q^3$ in the same OPE:

$$A(x) \times B(0) \supset \int [x]^{-|A|-|B|+4} \left\{ q(0) + q x^2 q^3(0) \right\}$$  \hfill (free, $d = 4$).

This is the counterpart of equation (3.13). We assume that $f \neq 0$, while $q$ may or may not be zero.

We will denote by $A', B'$, the operators at the WF fixed point which tend to $A, B$ in the sense of axiom II'. The counterpart of equation (3.15) will be:

$$A'(x) \times B'(0) \supset \tilde{f} [x]^{-|A'|-|B'|+4} \left\{ V_1(0) + (q_3 \alpha)x^2 V_3(0) \right\}$$  \hfill (WF, $d = 4 - \epsilon$).

We kept only the terms in this OPE that will play a role in the subsequent discussion.

Repeating the argument of section 3, we can show that the coefficients of the subleading terms in equations (4.2) and (4.3) must match:

$$q_3 \alpha \approx q \quad (\epsilon \to 0).$$  \hfill (4.4)

On the other hand, we can determine $q_3$ using conformal symmetry. We will assume that operators $A', B'$ are primaries of the interacting CFT. The $q_3$ is then given by equation (A.3). Its asymptotic behavior of $q_3$ in the $\epsilon \to 0$ limit can then be written as:

---

17 Recall that the primary operators, by definition, satisfy the condition $[K_\mu, O(0)] = 0$, where $K_\mu$ is the special conformal transformation generator.

18 It can be shown generally that if the OPE of two scalar primaries contains $q$, then one of them is the normal-ordered product of $q$ with the other. We thank Balt van Rees for a discussion of this point.

19 The only case when this does not happen is if one of them is $V_3$. But then the other operator is necessarily $V_2$ or $V_4$, and these cases were already treated in section 3.
Here, we kept the dependence on $\epsilon$ only in the factors which have a chance to vanish for $\epsilon \to 0$. Comparing this equation with equation (4.4), we get a condition which relates the considered dimensions at $O(\epsilon)$:

$$\left[ B' \right] - \left[ A' \right] - \delta_{|_{\epsilon = 0}} \approx 4\sqrt{3}\gamma q = (2q/3) \epsilon + O\left( \epsilon^2 \right).$$

This equation generalizes the recursion relation for the dimensions of the $\phi^n$ operators from section 3.

The above discussion is meant to demonstrate that there is further potential for applying the recombination to constrain operator dimensions. We could continue by considering explicit examples of scalar primary operators containing derivatives. Unfortunately, even the simplest such operators are a bit awkward to work with (they have 4 $\phi$s and 4 derivatives). So we postpone full exploration of this topic to future work.

### 4.2. Generalization to the $O(N)$ model

We will now generalize to the $O(N)$ model. The fixed point for the $O(N)$ model is defined starting from the Lagrangian with $N$ massless scalar fields $\phi^n$ in $4 - \epsilon$ dimensions and turning on the interaction $(\phi^n \phi^o)^2 \equiv (\varphi^2)^2$, which preserves the global $O(N)$ symmetry. The operators will form multiplets of the global symmetry group. We will not attempt an exhaustive analysis, but will consider just two series of operators.

Our first series will consist of, on the free theory side, operators

$$\Phi_{2p+1}^n \equiv \phi^n (\varphi^2)^p$$

and

$$\Phi_{2p}^n \equiv (\varphi^2)^p.$$  \hspace{1cm} (4.7)

The fixed point operators tending to them in the sense of axiom II will be denoted as $W_{2p+1}^n$ and $W_{2p}$, with their respective anomalous dimensions $\gamma_{2p+1}$ and $\gamma_{2p}$.

The $W$ operators are primaries, except for $W_{2}^3$ which is a descendant of $W_{1}^1$:

$$\square W_{2}^3 = \alpha W_{1}^1.$$  \hspace{1cm} (4.8)

This relation is justified similarly to axiom III of the WF fixed point for $N = 1$. The discussion which led to equation (3.12) readily generalizes, and gives

$$\alpha = 4\sigma \left[ (2 + N)^{1/2} \right].$$  \hspace{1cm} (4.9)

Equation (3.13) splits into two equations, depending whether $n$ is even or odd:

$$\Phi_{2p}(x) \times \Phi_{2p+1}(0) \supset f_{2p} |x|^{-4p} \left\{ \Phi_{2p}(0) + \rho_{2p} x^2 \Phi_{2p+1}(0) \right\},$$

$$\Phi_{2p}^n(x) \times \Phi_{2p+2}(0) \supset f_{2p+1} |x|^{-4p-2} \left\{ \Phi_{2p}^n(0) + \rho_{2p+1} x^2 \Phi_{2p+2}^n(0) \right\}.$$  \hspace{1cm} (4.11)

We’ll normalize $\varphi$ so that the two-point function is $\langle \phi^n(x) \phi^o(0) \rangle = \delta^{n,o} / |x|^2$. In this normalization, the relative coefficients in the above OPEs are given by:

$$c_{2p} = \frac{3p}{2 + N}, \quad c_{2p+1} = \frac{3(2p + 1) + N - 1}{2(2 + N)}.$$  \hspace{1cm} (4.12)
The overall coefficients are not needed below, but we give them for completeness:

\[ f_n = \prod_{k=2}^{n+1} w_k, \quad w_{2p} = 2p, \quad w_{2p+1} = 2p + N. \]  

(4.14)

For \( N = 1 \) these formulas reduce to equation (3.14).

Now repeating the steps which led to equation (3.30), we obtain:

\[ y_{n+1,1} - y_{n,1} = 4\sigma \left[ (N + 2)y_{1,2} \right]^{1/2} \varrho_n. \]  

(4.15)

Just as in section 3, this equation is valid also for \( n = 2, 3 \) even though \( W^a_3 \) is not a primary. Summing the equations for \( n = 1 \) and \( n = 2 \), we get:

\[ y_{3,1} - y_{1,1} = 4\sigma \left[ (N + 2)y_{1,2} \right]^{1/2} \left( \varrho_1 + \varrho_2 \right). \]  

(4.16)

On the other hand, just as in section 3, we know that

\[ y_{1,1} = 0, \quad y_{3,1} = 1. \]  

(4.17)

It follows that \( \sigma = 1 \) and

\[ y_{1,2} = \frac{N + 2}{4(N + 8)^2}. \]  

(4.18)

Substituting this into equation (4.15), we obtain

\[ y_{n+1,1} - y_{n,1} = \frac{2(N + 2)}{N + 8} \varrho_n. \]  

(4.19)

From here and equation (4.17), all the unknown \( y_{n,1} \)'s can be found recursively, e.g. we get

\[ y_{2,1} = \frac{N + 2}{N + 8}, \quad y_{4,1} = 2. \]  

(4.20)

These results as well as equation (4.18) agree with diagrammatic methods (see e.g. [78]). The general solution of equation (4.19) is

\[ y_{2p+1,1} = \frac{p(N + 2 + 6p)}{N + 8}, \quad y_{2p,1} = \frac{p(N + 2 + 6(p - 1))}{N + 8}. \]  

(4.21)

in agreement with the diagrammatic result in equation (60) of [81].

The second series that we consider consists, on the free theory side, of the operators

\[ \Phi^{ab}_{2p} \equiv \left[ \varphi^a \varphi^b - N^{-1} \delta^{ab} \varphi \right] \left( \varphi^2 \right)^{p-1}, \]  

(4.22)

dumping as symmetric traceless 2-tensors. The relevant OPE sensitive to the multiplet recombination is

\[ \Phi_{2p-1}^c \times \Phi_{2p}^{ab} \supset [x]^{-2p+2} h_p \Gamma^{[\gamma][\rho][\lambda][\sigma]} \{ \Phi^e_{\gamma} + \kappa_p x^2 \Phi^e_{\gamma} \}, \]  

(4.23)

where \( \Gamma \) is the Clebsch–Gordan coefficient for the corresponding \( O(N) \) representations:

\[ \Gamma^{[\gamma][\rho][\lambda][\sigma]} = \delta^{\gamma\rho} \delta^{\lambda\sigma} + \delta^{\gamma\lambda} \delta^{\rho\sigma} - (2/N) \delta^{\gamma\rho} \delta^{\lambda\sigma}. \]  

(4.24)

The OPE coefficients are given by

\[ h_p = \frac{1}{2p(2 + N)} \prod_{k=1}^{p} 2k(2k + N), \quad \kappa_p = \frac{3p - 2}{2 + N}. \]  

(4.25)
Notice that to determine $\kappa_\mu$, the contribution of $\Phi^*_{\mu}$ has to be carefully disentangled from the contribution of the symmetric traceless 3-tensor operator
\[
q^a q^b q^c - \frac{1}{2 + N} \Phi^2 \Phi^{ab} + \Phi^{bc} + \Phi^{ca},
\] (4.26)
which has the same scaling dimension\(^{20}\). Applying the usual argument to equation (4.23) gives a relation between $O(\epsilon)$ anomalous dimensions of the involved operators at the WF fixed point:
\[
y_{ST}^{2p,1} - y_{2p-1,1} = \frac{2(N + 2)}{N + 8} \kappa_p.
\] (4.27)
In particular, for the anomalous dimension of $\Phi_{\mu}^{ab}$ we get
\[
y_{ST}^{2,1} = \frac{2}{N + 8},
\] (4.28)
which is a well-known result \([86]\), while for general $p$ we get
\[
y_{ST}^{2p,1} = \frac{N(p - 1) + 2p(3p - 2)}{N + 8}.
\] (4.29)

5. History and prior work

This project had a somewhat long gestation period, and several other people contributed to it at the initial stages. We would like to acknowledge their contributions here.

The problem of determining the $\epsilon$-expansion series by CFT techniques is rather old. It was discussed already by Polyakov in 1974 \([2]\); a comparison to his results will be given below.

This problem gained renewed attention at the Back to the Bootstrap II workshop (Perimeter Institute, June 2012), where the results of \([87]\) were reported. Among other things, that paper analyzed boundary conformal bootstrap equations in $4 - \epsilon$ dimensions, for the conformal two-point function $\langle \Phi \Phi \rangle$ in half-space with the Dirichlet or Neumann boundary conditions. In both cases, a one-parameter family of solutions of crossing symmetry constraints to $O(\epsilon)$ was found. All the solutions had $\gamma = O(\epsilon^2)$, and to pick one uniquely within the family, $\gamma_2$ had to be provided by some other means, e.g. from perturbation theory.

At the same workshop, one of us (S.R.) learned about the multiplet recombination phenomenon from Leonardo Rastelli and Balt van Rees. Shortly afterwards, Sheer El-Showk, Miguel Paulos, S.R. and David Simmons-Duffin found that multiplet recombination provides a constraint on the anomalous dimensions of $V_1$ and $V_2$. They looked at the conformal block of $V_1$ appearing in the decomposition of the four point function $\langle V_1(x_1) V_2(x_2) V_1(x_3) V_2(x_4) \rangle$, using the series representation from \([54]\), equation (2.32). Demanding that this block reduce for $\epsilon \to 0$ to the sum of the $\Phi$ and $\Phi^3$ blocks in the free theory, they arrived at a relation identical to the $n = 1$ case of equation (3.30). Then, borrowing $y_{1,1} = 0$, $y_{2,1} = 1/3$ from perturbation theory, they could determine $y_{1,2}$. This computation was later written up as a problem for the 2013 Mathematica School on Theoretical Physics \([88, 89]\).

Compared to this previous work, our paper added two main ideas. First, we showed that the multiplet recombination can be exploited already at the level of three point functions, which is rather easier than to analyze the four point function and its conformal blocks. Second, we realized that one should study all $\Phi^n \times \Phi^{n+1}$ OPEs together. In particular, this

\(^{20}\) Further details on this point and on other aspects of our paper can be found in \([85]\).
links $\phi$ to $\phi^3$ in two steps. This eventually allowed us to determine all the unknown quantities without any input from perturbation theory.

The pioneering computation of Polyakov [2], section 5, deserves a separate comment. He analyzed the $O(N)$ model four point function $\langle \phi^a \phi^b \phi^c \phi^d \rangle$ using CFT methods. The computation was done using not conformal blocks, but the ‘unitary blocks’ which he introduced and which satisfy crossing symmetry but violate OPE by logarithmic terms.21 His consistency condition was that these OPE-offending terms must cancel. He analyzed this condition to $O(\epsilon)$. Assuming that the anomalous dimension of $\phi^a$ arises at $O(\epsilon^2)$, he was able to determine the $O(\epsilon)$ anomalous dimensions of the operators $\Phi_2$ and $\Phi_2^{ab}$ appearing in the $\phi^a \times \phi^b$ OPE. We are not aware of any further work using Polyakov’s approach. It would be very interesting to understand it better and to explore its full potential.

6. Open problems

We will conclude by listing several related open problems, in the order of increasing difficulty.

1. Our results should admit a generalization to the fixed point of the $\phi^3$ theory in $6 - \epsilon$ dimensions.22

2. One should be able to extend our results to the theories with fermions, such as the UV fixed point of the Gross–Neveu model in $2 + \epsilon$ dimensions and the fixed point of the Yukawa model in $4 - \epsilon$ dimensions. These models are described by the Lagrangians

   \[
   \mathcal{L}_{\text{GN}} = \bar{\psi} \not{\partial} \psi + G (\bar{\psi} \psi)^2, \tag{6.1}
   \]

   \[
   \mathcal{L}_Y = \bar{\psi} \not{\partial} \psi + \frac{1}{2}(\partial \psi)^2 + m(\bar{\psi} \psi) + g \phi^4. \tag{6.2}
   \]

   The equations of motion imply that the multiplet of $\psi$, short in the free theory, should recombine with that of $\bar{\psi} (\bar{\psi} \psi)$ in Gross–Neveu, and with that of $\psi \phi$ in Yukawa. It appears likely that this recombination can be used to constrain the leading anomalous dimensions. One should consider OPEs which, in the free theory limit, contain the field $\psi$.

3. Let us come back to the WF fixed point, and discuss the recombinations for the higher spin currents, which we already mentioned in footnote 12. The story goes as follows [91]. The free scalar theory contains conserved currents of any even spin of the form

   \[
   J^{(l)} = \phi \overset{\rightarrow}{\partial}_{\mu_1} \cdots \overset{\rightarrow}{\partial}_{\mu_l} \phi - \text{traces}. \tag{6.3}
   \]

   All these operators are primaries in the free theory. The $l = 2$ case corresponds to the stress tensor, which remains conserved also at the WF fixed point, and so keeps its canonical dimension $d$. It is a well-known fact that all of the $l > 2$ currents acquire anomalous dimensions at the second order in $\epsilon$ [92];

21 These unitary blocks are closely related, if not identical, to the Witten diagrams of AdS/CFT [90].

22 After this paper was submitted, Yu Nakayama informed us about his unpublished results in this direction.

23 In the free $O(N)$ model for $N > 1$ there are also odd spin conserved currents, of the same form as equation (6.3) where the two $\phi$s carry antisymmetrized $O(N)$ indices.
\[
\gamma^{(l)} = \left(1 - \frac{6}{l(l+1)}\right) \frac{\epsilon^2}{54} + \ldots
\] (6.4)

As a consequence, they are not conserved at the WF fixed point\(^{24}\). Since current conservation is a shortening condition for its conformal multiplet, we conclude that the multiplets \([J^{(l)}]\), short in the free theory, should become long in the interacting theory. They can do this only by eating a multiplet of another primary, which must have an appropriate dimension and spin to match the quantum numbers of \(\partial \cdot J^{(l)}\). Thus, in analogy to equation (2.11), we must have

\[
\{ J^{(l)} \}_{\text{WF}} \approx \left\{ J^{(l)} \right\}_{\text{free}} + \left\{ j^{(l-1)} \right\}_{\text{free}},
\] (6.5)

where \(j^{(l-1)}\) must be a primary operator of spin \((l-1)\) and dimension

\[
\Delta_j = \Delta_j + 1 = l + 3.
\] (6.6)

Interestingly, one can show that for \(l = 2\), the free theory contains no primaries of spin 1 and dimension 5, and thus recombination equation (6.5) is impossible \([88, 89, 91]\). In other words, the stress tensor multiplet must remain short, i.e. conserved, also in the interacting theory. This provides a CFT proof why the WF fixed point must necessarily have an exactly conserved stress tensor operator. On the other hand, for higher spins, one does find candidates for the operator \(J\) with the right quantum numbers\(^{25}\).

To summarize, we know from perturbation theory and from the Coleman–Mandula-like theorem \([93, 94]\) that the recombination must occur for \(l > 2\). We also know that the free theory contains primaries with the right quantum numbers for this to happen. In analogy with section 3, one should be able to use this recombination to reproduce the \(O(\epsilon^3)\) anomalous dimensions of the spin \(l\) currents, equation (6.4). This is an open problem worth attacking. The challenge is to find OPEs sensitive to this recombination. For example, the OPE \(\varphi \varphi\) does not seem useful, as in the free theory it contains \(J l\) but not \(J\).

4. Finally, it would be extremely interesting and important to find a CFT way to compute higher order terms in the \(\epsilon\)-expansion. We do not have any concrete proposal for how this can be achieved. Analysis of three-point functions will certainly not suffice for this task, and one will have to consider the full-fledged bootstrap at the four-point function level.

Clearly, the OPE coefficients will also get corrections when one moves to \(4 - \epsilon\) dimensions. In this paper we were not sensitive to these corrections; e.g. in section 3 all we could say is that \(\bar{j} j \to 1\) as \(\epsilon \to 0\). However, they will come into play when higher-order corrections to the operator dimensions are considered. Notice that corrections to the OPE coefficients are hard to compute from perturbation theory, and almost nothing is known about them\(^{26}\).

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\(^{24}\) This is in agreement with the recent Coleman–Mandula-like theorem \([93, 94]\), which forbids the existence of conserved higher-spin currents in interacting higher-dimensional CFTs. The theorem has been proven in \(d = 3, 4\), but presumably holds for any dimensions \(d > 2\), integer or not.

\(^{25}\) This computation was carried out in \([88, 89, 91]\) but only for a few low spins, and it would be interesting to prove this for all \(l\).

\(^{26}\) The \(O(\epsilon^3)\) correction to the central charge was computed in \([95–98]\). See section IV of \([75]\) for a summary.
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Appendix. Conformal OPE

In this appendix we will study the structure of the conformal OPE (equation (3.15)). We will have to consider two cases (i) and (ii) defined in equation (3.24).

Consider first case (i) when all three operators $V_n, V_{n+1}, V_1$ are primaries. To simplify the notation, let us consider three scalar primary operators $\phi, \psi, \chi$ of dimensions $a, b, c$, so that we are dealing with the OPE:

$$\mathcal{A}^a(x) \times B^b(0) \supset \int dx^3 \sim^{-a-b} \left[ 1 + q_1 x^\mu \partial_\mu + q_2 x^\mu x^\nu \partial_\mu \partial_\nu + \ldots \right] \mathcal{C}(0)$$

$$\equiv \int dx^3 \sim^{-a-b} P^{a,b,c}(x, \partial_x) \mathcal{C}(0)$$

The differential operator $P$ incorporates abstractly all terms in the OPE. The overall OPE coefficient $f$ depends on the CFT, but the differential operator $P$ is universal. It depends only on the transformation properties of the operators under the conformal group.

An all-order integral representation for the operator $P$ was given in [99], equation (2.1). For our purposes, we need the expansion of $P$ up to the second order in $x$ shown in equation (A.1). This can be worked out simply as follows. Conformal symmetry implies the well-known functional form of the three-point function equation (3.40). On the other hand, the same correlation function can be computed using the OPE as:

$$f \mathcal{P}(x-y, \partial_x) \mathcal{C}(y) \mathcal{C}(z)$$

Expanding equation (3.40) in the limit $|x-y| \ll |x-z|$ and matching order by order to equation (A.2) we can find all coefficients in $P$. The three coefficients of interest to us are:

$$q_1 = (c + a + b)/(2c),$$

$$q_2 = (c + a - b)(c + a + b + 2)/[8c(c + 1)],$$

$$q_3 = -(c + a - b)(c + a + b)/[16c(c + 1)(c - \delta)], \quad \delta = d/2 - 1.$$

One interesting and crucial for us difference between these coefficients is that $q_1$ and $q_2$ remain finite in the limit $c \to \delta$, while $q_3$ has a first-order pole. Substituting $a = \Delta_n, b = \Delta_{n+1}, c = \Delta_1$ and taking the leading term in the $\varepsilon \to 0$ limit, the expression for $q_3$ reduces to equation (3.25).

We now turn to case (ii), when one of the two operators $V_n, V_{n+1}$ is the descendant $V_3$.

Continuing the abstract treatment, we will study the OPE:

$$\Box \mathcal{A}^a(x) \times B^b(0) = \int dx^3 \sim^{-a-b} \left[ 1 + \tilde{q}_1 x^\mu \partial_\mu + \tilde{q}_2 x^\mu x^\nu \partial_\mu \partial_\nu + \tilde{q}_3 x^2 \Box + \ldots \right] \mathcal{C}(0).$$

In practice we will have $\mathcal{A} = C = V_1$ but let us keep it general for the moment. The relative OPE coefficients $\tilde{q}_i$ can be obtained by simply applying $\Box$ to both sides of equation (A.1).

27 This pattern continues in higher orders: terms with at least one $\Box$ have a first-order pole for $c \to \delta$, while terms without $\Box$ remain finite. This follows from the mentioned all-order representation of $P$ [99].
We obtain:

\[
\begin{align*}
\tilde{q}_1 &= \frac{a + b - c - d}{a + b - c - d + 2}q_1, \\
\tilde{q}_2 &= \frac{a + b - c - d - 2}{a + b - c - d + 2}q_2, \\
\tilde{q}_3 &= \frac{(a + b - c - d)(a + b - c - d - 2)}{(a + b - c)(a + b - c - d + 2)}q_3 + 2q_2.
\end{align*}
\] (A.5) (A.6) (A.7)

We will now use these equations to analyze directly the two sub-cases of case (ii):

(a) \( n = 2 \). We have the OPE \( V_1 \times V_2 \supset V_i \). The order of the operators does not matter for the computation of the singular part of \( \tilde{q}_3 \) as long as \( \tilde{q}_1, \tilde{q}_2 \) remain nonsingular. Thus we can apply the above result for

\[
\square V_1 \times V_2 \supset V_i.
\] (A.8)

We substitute \( a = c = \Delta_1 = \delta + \gamma_1, \ b = \Delta_2 + \gamma_2, \ d = 4 - \epsilon \) into the above formulas. We will only need to consider the case when \( \gamma_1 \) is higher order in \( \epsilon \) than \( \gamma_2 \). In this case we find that \( \tilde{q}_1, \tilde{q}_2 \) are indeed nonsingular, while the leading asymptotics for \( \tilde{q}_3 \) takes the form:

\[
\tilde{q}_3 \approx (\epsilon - \gamma_2)/\left(16\gamma_1\right).
\] (A.9)

If we further specialize to \( \gamma_1 \approx \gamma_{2,1} \epsilon, \ y_1 \approx y_{1,2} \epsilon^2 \), we get

\[
\tilde{q}_3 \approx \left(\epsilon - \gamma_{2,1}\right)/\left(16y_{1,2}\epsilon\right).
\] (A.10)

(b) \( n = 3 \). We have the OPE \( V_1 \times V_4 \supset V_i \). We can apply directly the above result for

\[
\square V_1 \times V_4 \supset V_i.
\] (A.11)

We substitute \( a = c = \Delta_1 = \delta + \gamma_1, \ b = \Delta_4 + \gamma_4, \ y_4 \approx y_{4,1} \epsilon, \ y_1 \approx y_{1,2} \epsilon^2 \) into the above formulas. We find that \( \tilde{q}_1, \tilde{q}_2 \) are nonsingular, while

\[
\tilde{q}_3 \approx \left(y_{4,1} - 1\right)/\left(16y_{1,2}\epsilon\right).
\] (A.12)

References

[1] Ferrara S, Grillo A and Gatto R 1973 Tensor representations of conformal algebra and conformally covariant operator product expansion Ann. Phys. 76 161–88
[2] Polyakov A 1974 Nonhamiltonian approach to conformal quantum field theory Zh. Eksp. Teor. Fiz. 66 23–42
[3] Belavin A, Polyakov A M and Zamolodchikov A 1984 Infinite conformal symmetry in two-dimensional quantum field theory Nucl. Phys. B 241 333–80
[4] Rattazzi R, Rychkov V S, Tonni E and Vichi A 2008 Bounding scalar operator dimensions in 4D CFT J. High Energy Phys. JHEP12(2008)031
[5] Rychkov V S and Vichi A 2009 Universal constraints on conformal operator dimensions Phys. Rev. D 80 045006
[6] Caracciolo F and Rychkov V S 2010 Rigorous limits on the interaction strength in quantum field theory Phys. Rev. D 81 085037
[7] Poland D and Simmons-Duffin D 2011 Bounds on 4D conformal and superconformal field theories J. High Energy Phys. JHEP05(2011)017
[8] Rattazzi R, Rychkov S and Vichi A 2011 Central charge bounds in 4D conformal field theory Phys. Rev. D 83 046011
[9] Rattazzi R, Rychkov S and Vichi A 2011 Bounds in 4D conformal field theories with global symmetry J. Phys. A: Math. Theor. 44 035402
[10] Vichi A 2012 Improved bounds for CFT’s with global symmetries J. High Energy Phys. JHEP01 (2012)162
[11] Poland D, Simmons-Duffin D and Vichi A 2012 Carving out the space of 4D CFTs J. High Energy Phys. 2012 JHEP05(2012)110
[12] Simmons-Duffin D 2015 A Semidefinite program solver for the conformal bootstrap arXiv:1502.02033
[13] Rychkov S 2011 Conformal bootstrap in three dimensions? arXiv:1111.2115
[14] El-Showk S, Paulos M F, Poland D, Rychkov S, Simmons-Duffin D and Vichi A 2012 Solving the 3D ising model with the conformal bootstrap Phys. Rev. D 86 025022
[15] El-Showk S and Paulos M F 2013 Bootstrapping conformal field theories with the extremal functional method Phys. Rev. Lett. 111 241601
[16] El-Showk S, Paulos M F, Poland D, Rychkov S, Simmons-Duffin D and Vichi A 2014 Solving the 3D Ising model with the conformal bootstrap II. c-Minimization and precise critical exponents J. Stat. Phys. 157 869
[17] Kos F, Poland D and Simmons-Duffin D 2014 Bootstrapping mixed correlators in the 3D Ising model J. High Energy Phys. JHEP11(2014)109
[18] Gliozzi F 2013 More constraining conformal bootstrap Phys. Rev. Lett. 111 161602
[19] Gliozzi F and Rago A 2014 Critical exponents of the 3D Ising and related models from conformal bootstrap J. High Energy Phys. JHEP10(2014)042
[20] Gliozzi F, Liendo P, Meineri M and Rago A 2015 Boundary and interface CFTs from the conformal bootstrap J. High Energy Phys. JHEP05(2015)036
[21] Kos F, Poland D and Simmons-Duffin D 2014 Bootstrapping the O(N) vector models J. High Energy Phys. JHEP1406(2014)091
[22] Kos F, Poland D, Simmons-Duffin D and Vichi A 2015 Bootstrapping the O(N) archipelago arXiv:1504.07997
[23] Beem C, Rastelli L and van Rees B C 2013 The N = 4 superconformal bootstrap Phys. Rev. Lett. 111 071601
[24] Bashkirov D 2013 Bootstrapping the N = 1 SCFT in three dimensions arXiv:1310.8255
[25] Alday L F and Bissi A 2014 The superconformal bootstrap for structure constants J. High Energy Phys. JHEP09(2014)144
[26] Berkooz M, Yacoby R and Zait A 2014 Bounds on N = 1 superconformal theories with global symmetries J. High Energy Phys. JHEP08(2014)008
[27] Alday L F and Bissi A 2015 Generalized bootstrap equations for N = 4 SCFT J. High Energy Phys. JHEP02(2015)101
[28] Chester S M, Lee J, Pufu S S and Yacoby R 2014 The N = 8 superconformal bootstrap in three dimensions J. High Energy Phys. JHEP09(2014)143
[29] Chester S M, Lee J, Pufu S S and Yacoby R 2015 Exact correlators of BPS operators from the 3D superconformal bootstrap J. High Energy Phys. 2015 JHEP03(2015)130
[30] Beem C, Lemos M, Liendo P, Rastelli L and van Rees B C 2014 The N = 4 superconformal bootstrap arXiv:1412.7541
[31] Bobev N, El-Showk S, Mazac D and Paulos M F 2015 Bootstrapping the three-dimensional supersymmetric Ising model arXiv:1502.04124
[32] Bobev N, El-Showk S, Mazac D and Paulos M F 2015 Bootstrapping SCFTs with four supercharges arXiv:1503.02081
[33] Gaiotto D, Mazac D and Paulos M F 2014 Bootstrapping the 3D Ising twist defect J. High Energy Phys. JHEP03(2004)100
[34] Nakayama Y and Ohtsuki T 2014 Approaching the conformal window of O(n) x O(m) symmetric Landau-Ginzburg models using the conformal bootstrap Phys. Rev. D 89 126009
[35] Nakayama Y and Ohtsuki T 2014 Five dimensional O(N) -symmetric CFTs from conformal bootstrap Phys. Lett. B 734 193–7
[36] Nakayama Y and Ohtsuki T 2015 Bootstrapping phase transitions in QCD and frustrated spin systems Phys. Rev. D 91 021901
[37] Golden J and Paulos M F 2015 No unitary bootstrap for the fractal Ising model J. High Energy Phys. JHEP03(2015)167
[38] Bae J-B and Rey S-J 2014 Conformal bootstrap approach to O(N) fixed points in five dimensions arXiv:1412.6549
[39] Chester S M, Pufu S S and Yacoby R 2015 Bootstrapping O(N) vector models in 4 < d < 6 Phys. Rev. D 91 086014
[40] Heemskerk I, Penedones J, Polchinski J and Sully J 2009 Holography from conformal field theory J. High Energy Phys. 2009 JHEP10(2009)079
[41] Heemskerk I and Sully J 2010 More holography from conformal field theory J. High Energy Phys. JHEP09(2010)099
[42] Fitzpatrick A L and Kaplan J 2012 Unitarity and the holographic S-matrix J. High Energy Phys. JHEP12(2012)032
[43] Pappadopulo D, Rychkov S, Espin J and Rattazzi R 2012 OPE convergence in conformal field theory Phys. Rev. D 86 105043
[44] Goldberger W D, Khandker Z U and Prabhu S 2014 OPE convergence in non-relativistic conformal field theories arXiv:1412.8507
[45] Fitzpatrick A L, Kaplan J, Poland D and Simmons-Duffin D 2013 The analytic bootstrap and AdS superhorizon locality J. High Energy Phys. JHEP12(2013)004
[46] Komargodski Z and Zhiboedov A 2013 Convexity and liberation at large spin J. High Energy Phys. JHEP11(2013)140
[47] Fitzpatrick A L, Kaplan J and Walters M T 2014 Universality of long-distance AdS physics from the CFT bootstrap. J. High Energy Phys. JHEP08(2014)145
[48] Vos C 2014 Generalized additivity in unitary conformal field theories arXiv:1411.7941
[49] Kaviraj A, Sen K and Sinha A 2015 Analytic bootstrap at large spin arXiv:1502.01437
[50] Alday L F, Bissi A and Lukowski T 2015 Large spin systematics in CFT arXiv:1502.07707
[51] Kaviraj A, Sen K and Sinha A 2015 Universal anomalous dimensions at large spin and large twist arXiv:1504.00772
[52] Beem C, Lemos M, Liendo P, Peevaers E, Rastelli L and van Rees B 2015 Infinite chiral symmetry in four dimensions Commun. Math. Phys. 336 1359–433
[53] Beem C, Rastelli L and van Rees B C 2014 W Symmetry in six dimensions arXiv:1404.1079
[54] Dolan F and Osborn H 2001 Conformal four point functions and the operator product expansion Nucl. Phys. B 599 459–96
[55] Dolan F and Osborn H 2004 Conformal partial waves and the operator product expansion Nucl. Phys. B 678 491–507
[56] Fortin J-F, Intriligator K and Stergiou A 2011 Current OPEs in superconformal theories J. High Energy Phys. JHEP09(2011)071
[57] Costa M S, Penedones J, Poland D and Rychkov S 2011 Spinning conformal correlators J. High Energy Phys. JHEP11(2011)071
[58] Costa M S, Penedones J, Poland D and Rychkov S 2011 Spinning conformal blocks J. High Energy Phys. JHEP11(2011)154
[59] Dolan F and Osborn H 2011 Conformal partial waves: Further mathematical results arXiv:1108.6194
[60] Goldberger W D, Skiba W and Son M 2012 Superembedding methods for 4D N = 1 SCFTs Phys. Rev. D 86 025019
[61] Simmons-Duffin D 2014 Projectors, shadows, and conformal blocks J. High Energy Phys. JHEP04(2014)146
[62] Osborn H 2012 Conformal blocks for arbitrary spins in two dimensions Phys. Lett. B 718 169–72
[63] Goldberger W D, Khandker Z U, Li D and Skiba W 2013 Superembedding methods for current superfields Phys. Rev. D 88 125010
[64] Hogervorst M and Rychkov S 2013 Radial coordinates for conformal blocks Phys. Rev. D 87 106004
[65] Fitzpatrick A L, Kaplan J and Poland D 2013 Conformal blocks in the large D limit J. High Energy Phys. JHEP08(2013)107
[66] Hogervorst M, Osborn H and Rychkov S 2013 Diagonal limit for conformal blocks in d dimensions J. High Energy Phys. JHEP08(2013)014
[67] Dymarsky A 2013 On the four-point function of the stress-energy tensors in a CFT arXiv:1311.4546
[68] Fitzpatrick A L, Kaplan J, Khandker Z U, Li D, Poland D and Simmons-Duffin D 2014 Covariant approaches to superconformal blocks J. High Energy Phys. JHEP08(2014)129

19
[69] Behan C 2014 Conformal blocks for highly disparate scaling dimensions J. High Energy Phys. JHEP09(2014)005
[70] Khandker Z U, Li D, Poland D and Simmons-Duffin D 2014 N = 1 superconformal blocks for general scalar operators J. High Energy Phys. JHEP08(2014)049
[71] Li D and Stergiou A 2014 Two-point functions of conformal primary operators in N = 1 superconformal theories J. High Energy Phys. JHEP10(2014)137
[72] Costa M S and Hansen T 2015 Conformal correlators of mixed-symmetry tensors J. High Energy Phys. JHEP02(2015)151
[73] Elkhidir E, Karateev D and Serone M 2015 General three-point functions in 4D CFT J. High Energy Phys. JHEP01(2015)133
[74] Kleinert H and Schulte-Frohlinde V, Chetyrkin K and Larin S 1991 Five loop renormalization group functions of O(n) symmetric \(\phi^4\) theory and epsilon expansions of critical exponents up to \(\epsilon^3\) Phys. Lett. B272 39–44
[75] Kleinert H and Schulte-Frohlinde V 2001 Critical Properties of \(\phi^4\) Theories (Singapore: World Scientific)
[76] Brown L S 1980 Dimensional regularization of composite operators in scalar field theory Ann. Phys. 126 135
[77] Brézin E, le Guillou J and Zinn-Justin J 1977 Perturbation theory at large order: 1. The \(\phi^2N\) interaction Phys. Rev. D 15 1544–57
[78] Kehrein S, Wegner F and Pismak Y 1993 Conformal symmetry and the spectrum of anomalous dimensions in the N vector model in 4 - \(\epsilon\) dimensions Nucl. Phys. B402 669–92
[79] Barabanschikov A, Grant L, Huang L L and Raju S 2006 The spectrum of Yang Mills on a sphere J. High Energy Phys. JHEP01(2006)160
[80] Wilson K G 1975 Quantum field theory models in less than four-dimensions Phys. Rev. D7 2911–26
[81] Jack I and Osborn H 1984 Background field calculations in curved space-time. 1. General formalism and application to scalar fields Nucl. Phys. B352 616–70
[82] Maldacena J and Zhiboedov A 2013 Constraining conformal field theories with a higher spin symmetry J. Phys. A: Math. Theor. 46 214011
[83] Alba V and Diab K 2013 Constraining conformal field theories with a higher spin symmetry in \(d = 4\) arXiv:1307.8092
[84] Jack I and Osborn H 1984 Background field calculations in curved space-time. 1. General formalism and application to scalar fields Nucl. Phys. B352 616–70
[98] Petkou A 1996 Conserved currents, consistency relations, and operator product expansions in the conformally invariant $O(N)$ vector model *Ann. Phys.* **249** 180–221

[99] Ferrara S, Gatto R and Grillo A F 1975 Properties of partial wave amplitudes in conformal invariant field theories *Nuovo Cim.* A **26** 226