“INFORMATION PARADOX”
AND SCHWARZSCHILDIAN GEODESICS

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Abstract. We show that the “Information Paradox” follows from inappropriate considerations on the geodesics of a Schwarzschildian manifold created by a gravitating point-mass. In particular, we demonstrate that the geometric differential equation which gives the radial coordinate as a function of the angular coordinate of the geodesics does not represent fully all the consequences following from the metric tensor. We remark that: \(i\) it does not yield the conditions characterizing the circular orbits; (this fact has been ignored in the previous literature); \(ii\) it “neglects” the space region in which the radial coordinate is minor or equal to twice the mass of the gravitating point (in suitable units of measure).

Summary – 1. Introduction and résumé of the main theses. – 1bis. On Kundt’s physical explanations of the observational data about the believed BHs. – 2, 3, 3bis. A precise treatment à la Hilbert of the geodesics of a Schwarzschild’s manifold. – 4. Physical meaning of the \(t\)-parametrization of the mentioned geodesics. – 5. Independently of any specific instance, the formal structure of GR excludes the existence of any “Information Paradox”. – 6. Legenda regarding De Jans’ diagrams of geodesics at the end of the present paper. – Appendix A: Computative verifications of the inadequacy of geometric eq. \(7\) in the treatment of the circular orbits. – Appendix B: On binaries composed of two mass-points according approximate calculations of Numerical Relativity founded on a \((3+1)\)-decomposition of Einstein field equations. – Diagrams of Schwarzschildian geodesics.

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1. – A glance over the recent literature about the Einsteinian gravitating point-mass shows that the subject is still interesting \([1]\). The diffuse interpretation of Schwarzschild’s solution which has given origin to the notion of black hole (BH) is unfounded, see \([2]\). A unpleasant consequence of this notion is the belief in the so-called “Information Paradox”, according to which GR would be in contradiction with the time reversibility. Indeed, it has been affirmed that in the instance of Schwarzschild’s manifold created by a gravitating mass-point the test-particles and the light-rays go beyond the space surface \(R(r) = 2m\) – where \(R(r)\) is the radial coordinate \([3]\) and \(m\) is the mass of the gravitating point \((c = G = 1)\) – and disappear from the “external” world for ever, with an irreversible process. We shall show in this
Note that if one takes into account all the assumptions which characterize the deduction of the geometric differential equation of the geodesic trajectories \[ R(\varphi) = \text{a function of } \varphi, (0 \leq \varphi \leq 2\pi) \] of test-particles and light-rays, one obtains a confirmation of the dynamical results by Droste [4] and De Jans [5]: the geodesics that arrive on the surface \( R(r) = 2m \) find here their end: the “Information Paradox” does not exist.

Of course, this conclusion is implicitly contained in [2], but we think useful to give an explicit and detailed proof of the erroneousness of a widespread belief.

1bis. – Astrophysics is an observational and experimental science. All phenomena that the current “δδξα” ascribes to BHs can be actually explained in quite physical ways – see, e.g., Kundt [6]. According to this Author: i) the believed stellar-mass BHs are neutron stars inside accretion disks; ii) the central engine of an Active Galactic Nucleus (AGN) is a nuclear-burning disk.

2. – Schwarzschild’s manifold of a gravitating point-mass \( m \) is characterized by the following \( ds^2 \):

\[
(1) \quad ds^2 = \frac{R(r)}{R(r) - \alpha} [dR(r)]^2 + [R(r)]^2 (d\varphi^2 + \sin^2 \varphi d\varphi^2) - \frac{R(r) - \alpha}{R(r)} \ dt^2; \quad (c = G = 1),
\]

where \( \alpha \equiv 2m \), and \( R(r) \) is a regular function of \( r \) such that the \( ds^2 \) becomes Minkowskian if \( r \to \infty \). (In the standard solution \( R(r) \equiv r \), in the original Schwarzschild’s solution \( R(r) = (r^3 + \alpha^3)^{1/3} \), in Brillouin’s solution \( R(r) \equiv r + \alpha \), etc.). The geodesics of test-particles and light-rays are plane trajectories and obey the following equations – see [7], eqs. (41)÷(44):

\[
(2) \quad \frac{R}{R - \alpha} \left( \frac{dR}{dp} \right)^2 + R^2 \left( \frac{d\varphi}{dp} \right)^2 - \frac{R - \alpha}{R} \left( \frac{dt}{dp} \right)^2 = A (\leq 0); \\
(3) \quad R^2 \left( \frac{d\varphi}{dp} \right) = B; \\
(4) \quad \frac{R - \alpha}{R} \frac{dt}{dp} = C; \\
(5) \quad \frac{d}{dp} \left( \frac{2R - \alpha}{R - \alpha} \frac{dR}{dp} \right) + \frac{\alpha}{(R - \alpha)^2} \left( \frac{dR}{dp} \right)^2 - 2R \left( \frac{d\varphi}{dp} \right)^2 + \frac{\alpha}{R^2} \left( \frac{dt}{dp} \right)^2 = 0.
\]

\( A, B, C \) are integration constants (which respect to the affine parameter \( p \)): \( A \) is zero for the light-rays, negative for the test particles; we can put
C = 1, by virtue of the arbitrariness of \( p \). The Lagrangean eq. (5) for \( R \) is connected with eqs. (2), (3), (4); indeed, we have the identity:

\[
\frac{d[2]}{dp} - 2 \frac{d\varphi}{dp} \frac{d[3]}{dp} + 2 \frac{dt}{dp} \frac{d[4]}{dp} = \frac{d[R]}{dp} \quad (5) ,
\]

where the brackets denote the left sides of eqs. (2), (3), (4), (5). The elimination of \( dp \) and \( dt \) from (2), (3), (4) gives the geometric differential equation of the geodesics:

\[
\left( \frac{d\varphi}{d\varphi} \right)^2 = 1 + \frac{A}{B^2} - \frac{A\alpha}{B^2} q - q^2 + \alpha q^3 ,
\]

where \( q := 1/R \). Since for the circular orbits \( dR/dp = 0 \), in this case identity (6) is not a consequence of (2), (3), (4). Consequently, as it is easy to verify, eq. (7) does not give the correct restrictions on the above orbits (see Appendix A).

3. – Of course, eq. (2) implies \( R > \alpha \), but eq. (7) “neglects” this condition. The substitutions \((R-\alpha) \leftrightarrow -t\) give for \( R \leq \alpha \) a non-static metric for which the temporal and the radial coordinates interchange their roles; in particular, eq. (7) becomes:

\[
\left( \frac{d[1/(\alpha-t)]}{d\varphi} \right)^2 = \frac{1 + A}{B^2} - \frac{A\alpha}{B^2} \frac{1}{(\alpha-t)} - \frac{1}{(\alpha-t)^2} + \frac{\alpha}{(\alpha-t)^3} .
\]

This is not, however, a significant result, because the geodesic parametrization with \( p \) – or with the proper time \( s \) – and the parametrization with \( \varphi \) of eq. (7) – would give a geodesic surpassing of \( R = \alpha \) with the original coordinates \( R \) and \( t \).

For the circular orbits eq. (5) gives:

\[
-2R \left( \frac{d\varphi}{dp} \right)^2 + \frac{\alpha}{R^2} \left( \frac{dt}{dp} \right)^2 = 0 ;
\]

from which the circular velocity \( v \):

\[
v^2 = \left( R \frac{d\varphi}{dp} \right)^2 = \frac{\alpha}{2R} .
\]

For the test-particle geodesics, we have from eq. (2) – with \( A < 0 \) – and eq. (5') that

\[
R > \frac{3}{2} \alpha ,
\]

\[
v < \frac{1}{\sqrt{3}} .
\]
And for the light-rays ($A = 0$):

\begin{equation}
R = \frac{3}{2} \alpha \quad ;
\end{equation}

\begin{equation}
v = \frac{1}{\sqrt{3}} \quad ,
\end{equation}

The restrictions (8), (8') and (9), (9') are not deducible from the geometric equation (7).

3bis. – The metric generated in the space domain $R(r) \leq \alpha$ by the substitutions $R(r) - \alpha \leftrightarrow -t$ is a non-static metric for a static problem. From the standpoint of the differential geometry, this fact does not represent a difficulty. But from the physical point of view, things stand otherwise, because the non-static character implies clearly the existence of transport forces, that are extraneous to our problem. This means that the above metric is only a formal trick, which cannot give a physical significance to space domain $R(r) \leq \alpha$, that in reality does not belong to Schwarzschild’s manifold. Remark that for the radial coordinates by Schwarzschild and by Brillouin (cf. sect. 2) this space domain is reduced to a singular point.

We emphasize finally that also the well-known metric of Kruskal and Szekeres is non-static, and therefore introduces transport forces in a static problem.

4. – As it was emphasized by von Laue [8], in Schwarzschild’s manifold of a gravitating material point the Systemzeit $t$ has a clear physical meaning, as it is specially attested by the red-shift of the spectral lines.

Now, with the $t$-parametrization of the dynamical evolution – which is privileged by Droste [4] –, we have that the velocities and the accelerations of the geodesics at $R = \alpha$ are equal to zero: Hilbertian repulsion by the event horizon $R(r) = \alpha$. [7].

5. – Back to the “Information Paradox”. We could affirm a priori, i.e. without the detailed examination of the geodesics in a Schwarzschildian manifold of a gravitating point-mass, that it cannot have a real existence in a theory as the GR, that has been devised in a manner which is independent of the directions of the spacetime coordinates, and in particular independent of the direction of the temporal coordinate.

Any contradiction to this fact must be ascribed to an erroneous interpretation of a given aspect of the formalism.
6. – At the end of paper [5b]), De Jans emphasizes that the solutions of eq. (7) can be divided into four categories, that he illustrates with some diagrams. For each figure he gives the values of \( \sigma \equiv -\alpha^2 A/B^2 \) and \( \tau \equiv \alpha^2/\beta^2 \), where \( A, B \) are the constants of our eqs. (2) and (3). For the radial and circular geodesics we have no figure. It is remarkable that for no geodesic there is the surpassing of the space surface \( R = \alpha \).

**Categorie A.** – Orbits with pericentre: periodic orbits (Figs. 1a and 1b); limiting orbits; open orbits (Figs. 2a and 2b).

**Categorie B.** – Finite orbits, with apocentre and without pericentre (Figs. 3a and 3b).

**Categorie C.** – Finite orbits, with apocentre and without pericentre (Fig. 4); infinite orbits without apsides, without asymptote; infinite orbits without apsides, with an asymptote (Figs. 5a and 5b); radial orbits.

**Categorie D.** – Transition orbits between categories B and C (cf. Figs. 3); orbits with apocentre and internal asymptotic circle (Fig. 6); orbits without apocentre, with internal asymptotic circle, without asymptote; orbits without apocentre, with internal asymptotic circle, with an asymptote (Fig. 7); orbits without pericentre and with internal asymptotic circle (Fig. 8).

The subdivision into the above categories depends on the discriminant \( \Delta \) of Weierstraß’ elliptic function \( P \) which gives the general solution of eq. (7):

\[
\frac{\alpha}{4R} = P (\varphi + K) + \frac{1}{12},
\]

where \( K \) is a constant of integration; \( \Delta \) is given by the following equality:

\[
\Delta := g_2^3 - g_3^2 ,
\]

where:

\[
g_2 := \frac{1}{12} + \frac{\alpha^2}{4B^2} ,
\]

\[
g_3 := \frac{1}{216} \left( 1 - 9 \alpha^2 A B^2 - \frac{27 \alpha^2}{2B^2} \right).
\]

We have: \( \Delta > 0 \) for **Categorie A**, **Categorie B**; \( \Delta < 0 \) for **Categorie C**; \( \Delta = 0 \) for **Categorie D** and for the circular orbits. –

See the diagrams of [5b)] at the end of the present paper; they are referred to the standard radial coordinate \( R(r) \equiv r \).

**APPENDIX A**

To show the inadequacy of eq. (7) in the treatment of the circular geodesics, it is sufficient to consider the instance of the light-rays, for which \( A = 0 \). Eq. (7) becomes:
For a circular orbit we must have:

\[
(A2) \quad \frac{1}{B^2} = \frac{1}{R^2} - \frac{\alpha}{R^3}.
\]

Let us put:

\[
(A3) \quad R = k \frac{\alpha}{2}, \quad \text{with} \quad k \geq 3;
\]

then:

\[
(A4) \quad \frac{1}{B^2} = \frac{1}{k^3 \alpha^2} (4k - 8) > 0;
\]

in particular, for \(k = 3\):

\[
(A5) \quad \frac{1}{B^2} = \frac{1}{27 \alpha^2},
\]

which is the unique value prescribed by the dynamical solution. We see, however, that the geometric eq. (7) allows all the trajectories for which \(k > 3\).

If \(k = (3 - \eta)\), with \(\eta > 0 \text{ and } \leq 3\), the right-hand side of (A4) becomes \(\frac{4}{(3-\eta)^3 \alpha^2} (1 - \eta)\). We see that for \(\eta \geq 1\) no circular orbit is possible.

**APPENDIX B**

The Numerical Relativity investigates, in particular, existence and behaviour of binaries composed of two mass-points with approximate calculations that hardly can have an exact counterpart [9]. A fortiori, this consideration holds for the three believed supermassive BHs residing in a quasar triplet [10]. Clearly, also for these instances no “Information Paradox” exists.

The numerical computations make use of \((3 + 1)\)-decompositions of Einstein field equations. Now, a \((3 + 1)\)-decompositions is not fully equivalent to Einstein gravitational theory, which considers also reference frames corresponding to metrics not belonging to the class of metrics characterized by any \((3 + 1)\)-decomposition. The \((3 + 1)\)-decompositions are an obvious generalization of the Gaussian frames, which were devised by Hilbert in 1916 [7]. In general, the metrics of GR must only satisfy the well-known conditions that \(g_{44}\) be negative and the quadratic form with the coefficients \(g_{\alpha \beta}, (\alpha, \beta = 1, 2, 3)\), be positive-definite.

Finally, it is contrary to the spirit of GR to ascribe a conceptual importance to any spacetime “foliation”. From a purely geometric standpoint, a given “foliation” has the same value as a given Gaussian frame.
Deux exemples d'orbites périodiques
\((\sigma > \tau)\)

Deux exemples d'orbites ouvertes
\((\sigma < \tau)\)

N.B. Les orbites limites \((\sigma = \tau)\) ont la même allure générale que les orbites ouvertes, mais sont dépourvues d'asymptotes.
Les orbites pour lesquelles $\sigma = \tau$ ont la même allure générale que celles des figures 5, mais sont dépourvues d'asymptotes.

La figure 4. correspond au cas de $R_a > 3\alpha$. 

N.B.
Par suite d’une erreur, le nombre 6 a été omis dans la numérotation des figures [5b], p.91.–

References

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[2] See, e.g., sects. 3b and 3c in A. Loinger and T. Marsico, arXiv:1205.3158 [physics.gen-ph] 13 May 2012.

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