Interrelation of nonclassicality conditions through stabiliser group homomorphism

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Abstract

In this paper, we show that coherence witness for a single qubit itself yields conditions for nonlocality and entanglement inequalities for multiqubit systems. It also yields a condition for quantum discord in two-qubit systems. It is shown by employing homomorphism among the stabiliser group of a single qubit and those of multi-qubit states. Interestingly, globally commuting homomorphic images of single-qubit stabilisers do not allow for consistent assignments of outcomes of local observables. We employ these observables for the construction of conditions for nonlocality, entanglement, and quantum discord. As an application, we show that CHSH inequality can be straightforwardly generalised to nonlocality inequalities for multiqubit GHZ states. It also reconfirms the fact that quantumness prevails even in the large $N$-limit if coherence is sustained. The mapping provides a way to construct many nonlocality inequalities, given a seed inequality. This study gives us a motivation to gain better control over multiple degrees of freedom and multi-party systems. It is because, in multi-party systems, the same nonclassical feature, viz, coherence may appear in many avatars.

1. Introduction

Quantum mechanics has a number of distinctive features, e.g., quantum coherence [1], nonlocality [2], steering [3], quantum entanglement [4], and, quantum discord [5], etc. They act as resources in quantum-computing [6–8], quantum-search [9, 10], and, quantum communication algorithms [11–13]. Owing to their resource-theoretic importance, recent times have witnessed an unprecedented surge of interest in the detection and quantification of these features. Despite being closely related, there are subtle differences in the definitions of different nonclassical features. For example, the definition of entanglement (separability) a priori assumes quantum mechanics, whereas the assumption of locality is independent of quantum mechanics. Quantum coherence underlies all the nonclassical correlations, be it entanglement or nonlocality or quantum discord. In fact, the interrelations between coherence, entanglement and quantum discord have already been studied from different viewpoints [14–17]. For example, in [14], the measures for coherence of a state have been obtained by the maximum entanglement that they can produce under incoherent operations and incoherence auxiliary quantum systems (ancilla). In [15–17], negative probability has been shown to underlie quantum coherence, nonlocality and entanglement in two-qubit and multi-qubit systems. In [18], using orbital angular momentum entangled photons in different angular positions, the interferometric visibility of two-photon interference (related to quantum coherence) has been related to an entanglement monotone, viz, concurrence. In [19], a relation between entanglement concurrence and coherence concurrence has been derived.

Nonlocality, in particular, plays a pivotal role in various quantum communication tasks, such as device-independent quantum key distribution [20] and randomness generation [21], etc. Thanks to its crucial applications, several approaches have been employed to derive nonlocality inequalities for numerous families of states such as graph states, cluster states, etc (see, for example [2, 22, 23], and references therein). Many nonlocality inequalities have been derived by employing stabiliser formalism, logical qubits [24] and...
by employing the action of abelian and non-abelian groups [25, 26]. Additionally, a procedure for lifting Bell inequalities has also been proposed [27]. A natural question in this program is how new nonlocality inequalities can be obtained and how the new ones are related to the already existing ones. Stabiliser formalism has also been studied in other contexts, most extensively in quantum error-correcting codes [28], and, in finding entanglement witnesses [29].

In this work, we study the interrelation of nonclassicality features in Hilbert spaces of different dimensions. We first show that there exists a homomorphic mapping between stabiliser groups of a single qubit state and an N-qubit GHZ state. A key feature that we employ is the existence of several stabiliser operators of an N-qubit system, homomorphic to a single stabiliser of an M-qubit system (M < N). We harness it to establish a relation between conditions for coherence in a single qubit, and, those for nonclassical correlations (e.g., nonlocality, entanglement, and quantum discord [5]) in multiqubit systems. Different stabilisers of an N-qubit system are products of locally noncommuting operators. So, these operators might play a crucial role in systematic constructions of nonlocality inequalities. Locally noncommuting observables are required to detect entanglement as well as nonlocality. That is why we choose sets of those lower dimensional stabilisers, whose image(s) involves locally noncommuting observables [30]. For the emergence of a nonlocality inequality, a natural requirement is that the inequality gets satisfied purely by classical probability rules, without taking recourse to quantum mechanics [31]. Additionally, it should also get violated by (at least) one quantum mechanical state. In particular, while deriving nonlocality inequalities, a key aspect is the identification of observables leading to violation of a constraint put by local-hidden-ariable (LHV) models. It is facilitated via homomorphism between the stabiliser groups. This feature provides us with a prescription for constructing a series of nonlocality inequalities, given a seed nonlocality inequality. In fact, while constructing a nonlocality inequality, the main task is identification of appropriate observables. Stabiliser group homomorphism serves two purposes at the same time: it shows the interrelation between different nonclassical features. Secondly, it helps us to identify the observables for construction of nonlocality or entanglement inequalities. Out of all the homomorphic images, we choose those images, which are globally commuting and locally noncommuting, for construction of entanglement and nonlocality inequalities. The underlying reason is that two or more parties of entangled/nonlocal states share correlation in two or more bases which are mutually noncommuting locally [30].

We harness this feature of entangled states to choose appropriate observables for construction of entanglement inequalities. In this way, this work shows how a single nonlocality condition (e.g., for coherence in a single logical qubit system) may give rise to a family of nonclassicality conditions (e.g., for entanglement and nonlocality in the constituent multiqubit physical systems) by employing suitable homomorphic mappings.

The central results of the paper are as follows. (I) We start by showing how various nonlocality inequalities involving dichotomic observables emerge from CHSH inequality [32]. (II) Entanglement inequalities for the simplest systems, i.e., two-qubit systems, yield several nonlocality inequalities for multiqubit systems, thanks to the homomorphism between stabiliser groups of a two-qubit system and that of a multi-qubit system. (III) This approach naturally unravels interconnection among various two-qubit entanglement inequalities and multi-qubit nonlocality inequalities. (IV) Coherence leads to various nonclassical correlations depending on which homomorphic image of the stabiliser of a single qubit is employed. It turns out that a less stringent coherence witness and a more stringent coherence witness are required to obtain conditions for entanglement and nonlocality respectively.

The importance of these results is twofold. The first one is from a foundational perspective and the other one is from the viewpoint of applications. From a foundational perspective, coherence underlies all the nonclassical correlations (see, for example [1], and references therein) and conditions for coherence for a single qubit are relatively easier to obtain. The results in this paper clearly show that a single nonclassicality condition gives rise to conditions for different forms of nonclassicalities in different physical systems. This work shows that they give rise to many different nonclassicality conditions for different nonclassical correlations in multiqubit systems by employing suitable homomorphic mappings. The observables can be chosen by identifying those elements from the homomorphic maps which conform to the conditions laid down in sections 3.3 and 3.4.

1 An example is two-qubit Bell state $|\Psi^+\rangle = (|01\rangle + |10\rangle) / \sqrt{2}$, which is SU(2) × SU(2) invariant state. Here $|0\rangle, |1\rangle$ represent the eigenstates of the operator Z. This state shares correlations in the eigenbases of the three operators, X, Y, Z, which are mutually noncommuting locally. It may be seen in the following way. If we rewrite the same state in the eigenbasis of X, i.e., $\{\pm\} = \{0\pm|1\rangle\}$, it acquires the form, $\sqrt{2} \frac{1}{\sqrt{2}} (|+\rangle + |\mp\rangle)$ (up to an overall phase). Similarly, the same state, if written in the eigenbasis of Y, which are represented by $|R\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)$ and $|L\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle)$, will acquire the form $\frac{1}{\sqrt{2}} (|RL\rangle - |LR\rangle)$ (again, up to an overall phase)
In a separate work, we have also shown that it admits a straightforward generalisation to higher dimensional systems in [33]. In fact, studies of interrelations of various nonclassical features have been made by employing isomorphism of Hilbert spaces of identical dimensions [34, 35]. The study of interrelations of nonclassical features by employing homomorphism between different stabiliser groups, however, has been relatively less explored. We undertake that study in this paper.

Isomorphism of Hilbert spaces endows us with conditions for contextuality in a single system of a suitable dimension, given a condition for nonlocality or entanglement in a multiqubit systems. Similarly, by employing homomorphic mapping, we show how conditions for different nonclassical correlations emerge purely from the one for coherence. Thus, this work shows how nonclassicality conditions derived for Hilbert spaces of nonidentical dimensions are related to one another. In fact, various nonlocality and entanglement inequalities, which have earlier been obtained through diverse considerations [36–38], straightaway emerge by employing this approach. Additionally, we have also been able to construct new nonlocality inequalities by employing this approach. In fact, by employing the prescriptions laid down in sections 3.3 and 3.4, many more such inequalities can be constructed.

From the viewpoint of applications, these nonlocality and entanglement inequalities may be employed as security checks in various quantum key distribution protocols and other secure quantum communication protocols.

The paper is organised as follows: in section 2, we set up the notation to be used in the paper. In section 3, we set up the framework. Section 4 is central to the paper, in which the emergence of nonlocality inequalities has been shown. In section 5, we discuss the relation of this work with the previous works. Section 6 concludes the paper with closing remarks.

2. Notation

In this section, we set up compact notations to be used henceforth in the paper for expressing the results.

(a) For an \( N \)-party system, the observables \( A_i, B_i \) refer to the \( i \)th subsystem. Observables belonging to the same subsystem are distinguished by primes in the superscript, such as \( A_i', A_i'', \ldots \).

(b) The symbols \( X_i, Y_i, Z_i \) are reserved for Pauli matrices acting over the space of the \( i \)th qubit.

3. Framework

In this section, we elaborate on the framework to be used throughout the paper.

3.1. Interrelation between coherence in a logical qubit and entanglement/nonlocality in its constituent qubits

We first illustrate how, under different choices of logical qubits, coherence in a logical qubit maps to entanglement/nonlocality in multiqubit systems. The basic idea is that pure logical qubits have their supports only in the logical subspace (which is relatively smaller and whose dimension is given by the Schmidt rank of the state), whereas they belong to a higher dimensional Hilbert space, thanks to the presence of many physical constituent qubits. That is why if we employ the observables that have supports only in the logical subspace, they detect coherence in the logical qubit. On the other hand, if we employ those observables that have support in the full Hilbert space of physical qubits and conform to the condition of local noncommutativity, they may detect entanglement or other forms of nonclassical correlations in the logical qubit. We take a few examples to illustrate it.

(a) Suppose that there is a single qubit,

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).
\]  

(1)

Its stabiliser group is \( \{ 1_x, X \} \), where \( X = |0\rangle\langle 1| + |1\rangle\langle 0| \). Instead, if we assume that both \( |0\rangle \) and \( |1\rangle \) are logical qubits, i.e.,

\[
|\psi_L\rangle = \frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle),
\]  

(2)

and \( |0_L\rangle \equiv |00\rangle \) and \( |1_L\rangle \equiv |00\rangle \). Then, the state \( |\psi_L\rangle \) acquires the following form,

\[
|\psi_L\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
\]  

(3)
The stabiliser group of $|\psi_L\rangle$ will be $\{1_L, X_L\}$, where $X_L \equiv |0_L\rangle\langle 1_L| + |1_L\rangle\langle 0_L|$. Note that the operators $1_L$ and $X_L$ have their supports only the two-dimensional subspace spanned by $\{|0_L\rangle, |1_L\rangle\}$.

The operator $X_L$ detects coherence in the subspace spanned by $|0_L\rangle$ and $|1_L\rangle$. The crux of the matter is that the actions of both the operators-$1_L$ and $X_L$-on the state $|\psi_L\rangle$ are identical to the tensor products of the following local operators,

$$1_L \rightarrow 1_4, Z_1Z_2; \quad X_L \rightarrow X_1X_2, -Y_1Y_2. \quad (4)$$

Here $X_i, Y_i, Z_i$ are Pauli operators acting over the space of $i$th qubit ($i = 1, 2$). All the operators $1_4, X_1X_2, -Y_1Y_2, Z_1Z_2$ have their supports over the full four-dimensional Hilbert space.

Of special interest to us are the following properties of these images:

- The elements $X_1X_2$ and $Y_1Y_2$ are globally commuting but locally noncommuting, hence they may be used for differentiating separable and entangled states.
- Only the operator $X_1X_2$ (or $Y_1Y_2$) cannot discriminate between separable and entangled states because they do not conform to the condition of local noncommutativity.

(b) Under the choice $|0_L\rangle \equiv |000\rangle, |1_L\rangle \equiv |111\rangle$, the coherent logical qubit, $\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$, acquires the form of a three-qubit GHZ state,

$$|\text{GHZ}_3\rangle \equiv \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \quad (5)$$

Again, the stabiliser group of the state $\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$ will be $\{1_1, X_1\}$, where $X_1$ and $1_1$ are Pauli $X$ operator and identity operator respectively acting over the two-dimensional logical space spanned by $\{|0_L\rangle, |1_L\rangle\}$. In this case as well, the operator $X_1$ detects coherence in the logical space spanned by $|0_L\rangle$ and $|1_L\rangle$. The crucial point is that the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ belongs to an eight-dimensional Hilbert space. That is why the action of both the operators-$1_L$ and $X_L$-on the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is identical to those of the following operators,

$$1_L \mapsto 1_8, Z_1Z_2, Z_1Z_3, Z_2Z_3, \quad X_L \mapsto X_1X_2X_3, -X_1X_2Y_3, -Y_1X_2Y_3, -Y_1Y_2X_3. \quad (6)$$

We wish to stress that all the eight operators, viz, $1_8, Z_1Z_2, Z_1Z_3, Z_2Z_3, X_1X_2X_3, -X_1X_2Y_3, -Y_1X_2Y_3, -Y_1Y_2X_3$, have their supports in the entire eight-dimensional Hilbert space. At this juncture, it is pertinent to look at the properties of the elements $\{1_8, Z_1Z_2, Z_1Z_3, Z_2Z_3, X_1X_2X_3, -X_1X_2Y_3, -Y_1X_2Y_3, -Y_1Y_2X_3\}$. The elements in the first set are all mutually commuting, locally as well as globally. On the other hand, the elements in the set $\{X_1X_2X_3, -X_1X_2Y_3, -Y_1X_2Y_3, -Y_1Y_2X_3\}$ are globally commuting, whereas they are locally noncommuting. In fact, these are the observables which give rise to Mermin’s nonlocality in a three-qubit GHZ state [37].

(c) Under the choice of logical states $|0_L\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle(|00\rangle + |11\rangle)$ and $|1_L\rangle \equiv \frac{1}{\sqrt{2}}(|11\rangle(|00\rangle - |11\rangle)$, the single logical qubit coherent state, $\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$ assumes the form $\frac{1}{2} (|00\rangle(|00\rangle + |11\rangle + |11\rangle(|00\rangle - |11\rangle))$.

As before, the images of $1_L$ and $X_L$ can be identified.

These observations can be succinctly formulated in terms of homomorphism among stabiliser groups, which forms the basis of the framework used throughout the paper. The framework essentially hinges on homomorphism between the stabiliser groups of $M$ and $N$-qubit systems ($M < N$). The homomorphism entails sets of many ‘locally noncommuting’ operators, which have identical actions on an entangled state. We plan to employ homomorphism of stabiliser groups in the reverse direction for the case of Pauli observables throughout this paper. This is because the procedure for obtaining conditions for entanglement and nonlocality is more involved. So, we believe that by employing this prescription, one can identify the observables and the bounds can be set by using the rationale given in the subsequent sections 3.3 and 3.4. We illustrate it with the simplest example of a single qubit and a two-qubit state.

### 3.2. Homomorphic mapping between the stabiliser groups

Consider a single-qubit state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad (7)$$

whose stabiliser group is $G \equiv \{1, X\}$. Here, the symbol $1_2$ represents a $2 \times 2$ identity matrix and $X$ represents Pauli $X$ operator. This set forms a group under matrix multiplication.
This group bears a homomorphism with the stabiliser group, \( \{ 1_4, X_1X_2, -Y_1Y_2, Z_1Z_2 \} \), of the two-qubit state, \( \frac{1}{\sqrt{2}} (|00⟩ + |11⟩) \). The homomorphic mapping is given by,
\[
\begin{align*}
\{ X_1X_2, -Y_1Y_2 \} &\mapsto X, \\
\{ 1_4, Z_1Z_2 \} &\mapsto I_2.
\end{align*}
\]

Moving on to the Hilbert space of three qubits (i.e., \( \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \)), the group \( G \) bears a homomorphism with the group,
\[
\{ 1_8, X_1X_2X_3, -X_1Y_2Y_3, -Y_1X_2X_3, -Y_1Y_2Y_3, Z_1Z_2Z_3, Z_1Z_2, Z_2Z_3, Z_3Z_1 \}.
\]

The mapping is given by,
\[
\begin{align*}
\{ 1_8, Z_1Z_2, Z_1Z_3, Z_2Z_3 \} &\mapsto I_2, \\
\{ X_1X_2X_3, -X_1Y_2Y_3, -Y_1X_2X_3, -Y_1Y_2Y_3 \} &\mapsto X.
\end{align*}
\]

Another homomorphic mapping of the group,
\[
\{ 1_8, Y_1X_2X_3, -Y_1Y_2Y_3, X_1X_2Y_3, X_1X_2Y_3, Z_1Z_3, Z_2Z_3, Z_3Z_1 \},
\]

to the group \( \{ I_2, Y \} \) is as follows:
\[
\begin{align*}
\{ 1_8, Z_1Z_2, Z_1Z_3, Z_2Z_3 \} &\mapsto I_2, \\
\{ Y_1X_2X_3, -Y_1Y_2Y_3, X_1X_2Y_3, X_1X_2Y_3 \} &\mapsto Y.
\end{align*}
\]

Its extension to the stabiliser group of an \( N \)-qubit system is straightforward. An important point is that homomorphic images of single qubit stabilisers are locally noncommuting. So, they can be naturally employed for constructing nonlocality inequalities. In this way, this mapping plays a crucial role in, (i) identifying the forms of observables while constructing a nonlocality inequality, and, (ii) showing the interrelation between coherence witness for a single qubit, entanglement witness for a two-qubit system, and nonlocality inequalities for \( N \)-qubit system (\( N \geq 3 \)). The observables are provided by this mapping and the bound can be set by considering the notion of classicality (e.g., LHV models, separability, etc).

3.3. Rationale underlying choice of stabiliser group elements yielding nonlocality inequalities

Since we plan to obtain nonlocality inequalities starting from group homomorphism, it is worthwhile to discuss beforehand the rationale underlying the choices of observables for deriving such inequalities, which is given below:

(a) Nonlocality in a multi-party system refers to correlations between outcomes of one set of locally non-commuting observables on the first party with another set of locally non-commuting observables on the second party. Guided by this, we choose those elements of the stabiliser group, which ensure that the ensuing inequality involves the correlations, such as \( \langle A_1A_2 \rangle \) and \( \langle A'_1A'_2 \rangle \) or their higher powers (e.g., \( \langle A_1A_2 \rangle^2 \) and \( \langle A'_1A'_2 \rangle^2 \)), with the proviso that the commutators \( [A_1, A'_1] \neq 0; [A_2, A'_2] \neq 0 \) (please note that the observables \( A_1, A'_1 \) and \( A_2, A'_2 \) act over the spaces of the first and the second party respectively).

(b) A further requirement is, of course, that, the local-hidden-variable (LHV) bound should be put such that all the LHV models satisfy the inequality.

We next give the rationale for the choice of observables for constructing entanglement inequalities.

3.4. Rationale underlying choice of stabiliser group elements yielding entanglement inequalities

We choose the observables through stabiliser group homomorphism, keeping the following considerations in mind:

(a) We choose those elements of a stabiliser group, which ensure that the ensuing inequality involves the correlations, such as \( \langle A_1A_2 \rangle \) and \( \langle A'_1A'_2 \rangle \) or their higher powers (e.g., \( \langle A_1A_2 \rangle^2 \) and \( \langle A'_1A'_2 \rangle^2 \)), with the proviso that the commutators \( [A_1, A'_1] \neq 0; [A_2, A'_2] \neq 0 \) (please note that the observables \( A_1, A'_1 \) and \( A_2, A'_2 \) act over the spaces of the first and the second party respectively).

(b) A further requirement is, of course, that, the bound on the inequality should be put such that all the separable states satisfy the inequality.

In what follows, we shall consider some choices of coherent logic qubits for the purpose of illustration. The framework, however, is amenable to all the choices of coherent logical qubits. It is because every multiqubit entangled state can be written as a logical state of the form,
\[ |\psi_2\rangle \equiv \alpha|0_2\rangle + \beta|1_2\rangle, \alpha \neq 0, \beta \neq 0, |\alpha|^2 + |\beta|^2 = 1, \tag{12} \]

for different choices of \(|0_2\rangle\) and \(|1_2\rangle\). The Hilbert space to which the state \(|\psi_2\rangle\) belongs depends on the choices of \(|0_2\rangle\) and \(|1_2\rangle\). Consider an operator,

\[ O_L \equiv \langle \psi_L | (\psi_L^L) \langle \psi_L^x |, \text{ where } |\psi_L^L\rangle \equiv -\beta^*|0_L\rangle + \alpha^*|1_L\rangle. \tag{13} \]

The stabiliser group of that multiqubit physical entangled state \(|\psi_L\rangle\) would obviously bear a homomorphic map with the stabiliser group \{1, O_L\} of the logical state \(|\psi_L^L\rangle\). The operators, \(1_L, O_L\), have their supports in two-dimensional Hilbert space spanned by \{|\psi_L\rangle, |\psi_L^L\rangle\}.

The operator \(O_L\) detects coherence in the basis, \(|\phi_L^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\psi_L\rangle \pm |\psi_L^L\rangle\rangle\)}, which is mutually unbiased with respect to the two-dimensional basis \{|\psi_L\rangle, |\psi_L^L\rangle\}. It is because \(\langle \phi_L^+ | O_L | \phi_L^-\rangle = \langle \phi_L^- | O_L | \phi_L^+\rangle = 0\) and its expectation value is maximum for the state \(|\psi_L\rangle\), which is the maximally coherent state in the basis \{|\phi_L^\pm\rangle\}. Many stabilising transformations having support in full Hilbert space will map to the same transformation having support in the logical subspace (as has been shown through various examples in sections 3.1 and 3.2). Out of all the elements of the stabiliser group of the multiqubit entangled state \(|\psi_L\rangle\), those elements which are locally noncommuting, can be employed for constructing entanglement inequalities. The bounds on the inequalities can be set by following the steps given in sections 3.3 and 3.4.

4. Application: coherence witness \(\mapsto\) entanglement witnesses \(\mapsto\) nonlocality inequalities

This section contains the central results of the paper. In this section, we start by showing how a coherence witness for a single qubit system gets mapped to an entanglement witness for a two-qubit system and to a nonlocality inequality for a three-qubit (more generally, a tripartite) system.

4.1. Coherence witness \(\mapsto\) entanglement witness I

Consider the coherence witness,

\[ 0 \leq \langle X \rangle \leq \frac{1}{2}. \tag{14} \]

We look at the possible homomorphisms to the group of stabilisers of two-qubit Bell state and show that the entanglement inequalities and conditions for quantum discord follow. The coherence witness, \(\langle X \rangle \leq \frac{1}{2}\), and a coherent state, \(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle\rangle\)}, get mapped to an entanglement witness, \(\langle X_1X_2 - Y_1Y_2 \rangle \leq 1\), and to an entangled state, \(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle\rangle\rangle\)} respectively, under the homomorphism given in equation (8), i.e.,

\[ 2\langle X \rangle \leq 1 \implies 2\langle X_1X_2 - Y_1Y_2 \rangle \leq 1, \tag{15} \]

\[ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle\rangle\rangle\rangle \implies \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle\rangle\rangle\langle X_1X_2 - Y_1Y_2 \rangle \leq 1. \]

The images \(X_1X_2\) and \(-Y_1Y_2\) have been chosen by employing the prescription given in section 3.4. These two operators are globally commuting, i.e., \([X_1X_2, -Y_1Y_2] = 0\) and locally noncommuting, i.e., \([X_1, Y_1] \neq 0\) and \([X_2, Y_2] \neq 0\). In fact, if we had chosen only \(X_1X_2\) (or, only \(Y_1Y_2\)), that would have given rise to condition for quantum coherence in two-qubit systems, and not the condition for entanglement because it does not involve locally noncommuting observables. The bound of 1 in the inequality \(\langle X_1X_2 - Y_1Y_2 \rangle \leq 1\) ensures that all the separable states satisfy this inequality. Note that we have employed the mapping, \(|0\rangle \mapsto |00\rangle\) and \(|1\rangle \mapsto |11\rangle\) in the last line of equation (15).

4.2. Coherence witness \(\mapsto\) entanglement inequality II

Secondly, the coherence witness, \(\langle X \rangle < 0\), maps to the entanglement witness \(\langle \mathbb{1} + X_1X_2 + Y_1Y_2 + Z_1Z_2 \rangle < 0\) as follows:

\[ \langle X \rangle < 0 \implies \langle (1 - 1 + X + X) \rangle < 0. \tag{16} \]

Employing the mapping, \(\{X_1X_2, Y_1Y_2\} \mapsto X, \{1_4, Z_1Z_2\} \mapsto 1_2\), equation (16) assumes the following form:

\[ \langle \mathbb{1} + X_1X_2 + Y_1Y_2 + Z_1Z_2 \rangle < 0, \tag{17} \]

which is the optimal linear entanglement witness for two-qubit systems [38]. We have chosen these observables according to the rationale given in section 3.4. That is why these observables are globally
commuting but locally noncommuting. In this manner, we observe that choices of observables appearing in two-qubit entanglement witnesses are governed by the observables appearing in coherence witnesses for a single-qubit system. We show, in the subsequent sections, that a similar conclusion holds for nonlocality as well.

### 4.3. Coherence witness ↔ Mermin’s nonlocality inequality

We next move on to show that the same coherence witness, \(\langle X \rangle \leq \frac{1}{2}\), maps to Mermin’s nonlocality inequality, under the following choice of logical qubits,

\[
\begin{align*}
|000\rangle & \equiv |0_2\rangle, \\
|111\rangle & \equiv |1_2\rangle.
\end{align*}
\]

We employ the homomorphism given in equation (9). So, the coherence witness, \(\langle X \rangle \leq \frac{1}{2}\), maps to the following inequality,

\[
\langle X_1(X_2X_3 - Y_2Y_3) - Y_1(X_2Y_3 + Y_2X_3) \rangle \leq 2,
\]

which is Mermin’s inequality [37]. We have chosen these observables according to the rationale given in section 3.3. This procedure admits a straightforward generalisation to a higher number of parties, provided we choose the observables as per the prescription laid down in section 3.3.

At this juncture, it is pertinent to enquire into the special choice of \(\frac{1}{2}\) on the right-hand side of coherence witness in equation (14). This value has been chosen because, the ensuing inequality for a two-qubit turns out to be an entanglement inequality and the one for \(N\)-qubit system turns out to be Mermin’s nonlocality inequality. We next show that any value less than \(\frac{1}{2}\) may be used to construct a witness for quantum discord.

Secondly, it is also worth noticing that we have taken all possible homomorphic maps of \(X\) in the group \(\{12, X\}\), i.e., we have taken all the four elements \(X_1X_2X_3, -X_1Y_2Y_3, -Y_1X_2Y_3, -Y_1Y_2X_3\). It is because, in order to detect nonlocality/entanglement, it is required to have locally noncommuting operators in the expression. In the next section, we show how the condition for quantum coherence for a single qubit and quantum discord for a two-qubit system are related to each other.

### 4.4. Coherence witness ↔ condition for quantum discord

Consider a coherence witness \(\|X\| \leq \epsilon\) for a single qubit, with \(0 \leq \epsilon \leq \frac{1}{2}\). Under the homomorphic mapping \(\{X_1X_2, -Y_1Y_2\} \mapsto X\), it gets mapped to two conditions, \(\|X_1X_2\| \leq \epsilon\) and \(\|(-Y_1Y_2)\| \leq \epsilon\), which are conditions for quantum discord for a two-qubit system. The proof is by explicit construction.

By definition, a state whose discord, \(D^{1\rightarrow 2}\) vanishes has the structure,

\[
\rho_{12} = \sum_k p_k |\phi_{1k}\rangle \langle \phi_{1k}| \otimes \rho_k,
\]

where, \(\sum_k p_k |\phi_{1k}\rangle \langle \phi_{1k}|\) is the resolution of \(\rho_1\) in its eigenbasis.

We choose two orthogonal observables \(X_1, Y_1\) for the first qubit with the stipulation that one of them say, \(X_1\) shares its eigenbasis with \(\rho_1\) (which is the reduced density matrix of \(\rho_{12}\)). The eigen-basis of \(Y_1\) is unbiased with respect to that of \(X_1\). Thus, a partial tomography is warranted. For this reason, the condition for discord does not qualify as a witness. Note that these observables \(X_1\) and \(Y_1\) do not commute locally. That is why, they give rise to condition for quantum discord, which is a nonclassical correlation.

### 4.5. Coherence witness ↔ CHSH inequality

We now show how CHSH inequality emerges as a descendant of a more stringent coherence witness \(\langle X \rangle \leq \frac{1}{\sqrt{2}}\). Under the homomorphic map given in equation (8), this inequality gets mapped to,

\[
\langle X_1X_2 - Y_1Y_2 \rangle \leq \sqrt{2},
\]

which can be reexpressed as,

\[
\left( \frac{X_1 - Y_1}{\sqrt{2}} + \frac{X_1 + Y_1}{\sqrt{2}} \right) X_2 + \left( \frac{X_1 - Y_1}{\sqrt{2}} - \frac{X_1 + Y_1}{\sqrt{2}} \right) Y_2 \right) \leq 2.
\]

At this juncture, we wish to stress that the choice of observables has been made by following the rationale given in section 3.3. That is to say, these observables are mutually noncommuting locally.

We may replace the observables acting over a two-qubit system by generic dichotomic observables \(A_{12}, A'_{12}, A'_{12}', A'_{12}\), which renders the seminal CHSH inequality [32],

\[
\langle X_1X_2 - Y_1Y_2 \rangle \leq \sqrt{2},
\]
We are now in a position to start straightaway from two-qubit entanglement witnesses and show the three single qubit Pauli operators assumed that the second qubit is a logical one and itself composed of two qubits. Employing group homomorphism, we may write the logical operators as a tensor product of local operators, i.e.,

\[ X \equiv X_1 \otimes X_2, \quad Y \equiv Y_1 \otimes Y_2, \quad Z \equiv Z_1 \otimes Z_2. \]

Now, thanks to this mapping, the inequality (23) will assume the following form:

\[ \langle A_1(A_2 + A_2') + A_1'(A_2 - A_2') \rangle \leq 2. \]

The observables acting over the second qubit may be chosen as, \( A_2 \equiv X_2L, A_2' \equiv Y_2L \). It is because for the following choice of observables, \( A_1 \equiv \frac{X_1 + Y_1}{\sqrt{2}}, A_1' \equiv \frac{X_1 - Y_1}{\sqrt{2}} \) and \( A_2 \equiv X_2L, A_2' \equiv Y_2L \), the inequality assumes the form, \( \langle X_1X_2L - Y_1Y_2L \rangle \leq \sqrt{2} \), which is maximally violated by the Bell state, \( \frac{1}{\sqrt{2}}(|000_L\rangle + |111_L\rangle) \). We have assumed that the second qubit is a logical one and itself composed of two qubits. Employing group homomorphism between stabiliser group of the Bell state, which is \( \{1, X_1X_2L, -Y_1Y_2L, Z_1Z_2L\} \) and that of the three-qubit GHZ state \( \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \), the following mapping is straightforward,

\[
X_1X_2X_3; -X_1Y_2Y_3 \mapsto X_1X_2L \\
X_1X_2Y_3; X_1Y_2X_3 \mapsto -Y_1Y_2L.
\]

Now, thanks to this mapping, the inequality (23) will assume the following form:

\[ \langle A_1(A_2A_3 + A_2'A_3') + A_1'(A_2A_3' - A_2'A_3) \rangle \leq 2, \]

which is Mermin’s inequality [37]. Note that while choosing the observables, we have ensured that they are not locally commuting observables (as per the prescription laid down in section 3.3). That is to say, the operators \( X_2X_3, -Y_2Y_3 \) do not commute, i.e., \([X_2, Y_2] \neq 0, [X_3, Y_3] \neq 0\). It can be further extended for finding nonlocality inequalities for an \( N \)-party systems similarly.

### 4.6. CHSH inequality \( \mapsto \) Mermin’s inequality

So far we have shown how different nonlocality inequalities emerge from conditions for quantum coherence. We now study how different nonlocality inequalities are themselves related to each other. We start with finding the descendants of CHSH inequality under different homomorphic maps. Consider, once again, the CHSH inequality [32],

\[ \langle A_1(A_2 + A_2') - A_1'(A_2 - A_2') \rangle \leq 2. \]

In this section, we show how CHSH inequality for a different choice of observables, coupled with homomorphic map, yields Das–Datta–Agrawal inequality [39]. The seminal CHSH inequality [32] can be written as,

\[ \langle (A_1 + A_1')A_2 + (A_1 - A_1')A_2' \rangle \leq 2. \]

It gets violated by the state \( \frac{1}{\sqrt{2}}(|0\rangle\langle 1|_L - |1\rangle\langle 0|_L) \) for the following choice of observables,

\[ A_1 = \frac{X_1 + Z_1}{\sqrt{2}}, \quad A_1' = \frac{X_1 - Z_1}{\sqrt{2}}, \quad A_2L = X_2L, \quad A_2' = Z_2L. \]

We may write the logical operators as a tensor product of local operators, i.e., \( X_L \mapsto X_2X_3 \) and \( Z_L \mapsto Z_2 \). In terms of generic observables, it will lead to the following inequality,

\[ \langle (A_1 + A_1')A_2A_3 + (A_1 - A_1')A_2A_3' \rangle \leq 2, \]

which is the same as Das–Datta–Agrawal inequality for a tripartite system. In this language, various inequalities, which have been derived through diverse considerations may be regarded as descendants of a smaller set of nonlocality inequalities under different homomorphic maps. At this juncture, we wish to stress that the choice of observables has been made according to the rationale given in section 3.3. That is to say, for violation of the inequality (27), the observables \( A_2 \) and \( A_2' \) have to be noncommuting (\( A_2 \equiv X_2 \) and \( A_2' \equiv Z_2 \)).

Up to this point, we have considered only the homomorphic mappings to the group \( \{I_2, X\} \) or \( \{I_2, Y\} \) or \( \{I_2, Z\} \) separately or at most, to two of them at a time. We now consider the homomorphic maps of all the three single qubit Pauli operators \( X, Y \) and \( Z \) simultaneously in the sections to follow.

### 4.7. CHSH inequality \( \mapsto \) Das–Datta–Agrawal inequality

In this section, we show how CHSH inequality for a different choice of observables, coupled with homomorphic map, yields Das–Datta–Agrawal inequality [39]. The seminal CHSH inequality [32] can be written as,

\[ \langle (A_1 + A_1')A_2 + (A_1 - A_1')A_2' \rangle \leq 2. \]

It gets violated by the state \( \frac{1}{\sqrt{2}}(|0\rangle\langle 1|_L - |1\rangle\langle 0|_L) \) for the following choice of observables,

\[ A_1 = \frac{X_1 + Z_1}{\sqrt{2}}, \quad A_1' = \frac{X_1 - Z_1}{\sqrt{2}}, \quad A_2L = X_2L, \quad A_2' = Z_2L. \]

We may write the logical operators as a tensor product of local operators, i.e., \( X_L \mapsto X_2X_3 \) and \( Z_L \mapsto Z_2 \). In terms of generic observables, it will lead to the following inequality,

\[ \langle (A_1 + A_1')A_2A_3 + (A_1 - A_1')A_2A_3' \rangle \leq 2, \]

which is the same as Das–Datta–Agrawal inequality for a tripartite system. In this language, various inequalities, which have been derived through diverse considerations may be regarded as descendants of a smaller set of nonlocality inequalities under different homomorphic maps. At this juncture, we wish to stress that the choice of observables has been made according to the rationale given in section 3.3. That is to say, for violation of the inequality (27), the observables \( A_2 \) and \( A_2' \) have to be noncommuting (\( A_2 \equiv X_2 \) and \( A_2' \equiv Z_2 \)).

Up to this point, we have considered only the homomorphic mappings to the group \( \{I_2, X\} \) or \( \{I_2, Y\} \) or \( \{I_2, Z\} \) separately or at most, to two of them at a time. We now consider the homomorphic maps of all the three single qubit Pauli operators \( X, Y \) and \( Z \) simultaneously in the sections to follow.

### 4.8. Multi-party nonlocality inequalities from two-qubit entanglement inequalities

We are now in a position to start straightaway from two-qubit entanglement witnesses and show the emergent hierarchical structure of inequalities. We can simply assume that one of the two qubits is a logical one. The logical qubit itself is composed of many qubits. The crux of the matter is that the operators
appearing in the entanglement inequalities corresponding to the local qubit become logical operators. These operators may, in turn, be written as direct products of local operators in more ways than one.

To illustrate it, in this section, we map the entanglement inequality [40],

$$\langle X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 \rangle \leq 1,$$  \hspace{1cm} (28)

to various nonlocality inequalities for three and four party systems. The symbols $X_{2L}, Y_{2L}, Z_{2L}$ represent that the second qubit may be a logical one. The inequality (28) gets maximally violated by the state \(\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\). If we assume the second qubit to be a logical one, the state will become \(\frac{1}{\sqrt{2}}(|011\rangle - |100\rangle)\), i.e., it maps to a three-qubit GHZ state.

### 4.8.1. First inequality: tripartite state

We assume that the second logical qubit consists of two qubits. We observe the homomorphic map, $X_2 X_3 \mapsto X_{2L}, X_2 Y_3 \mapsto Y_{2L}$, and $Z_2 Z_3 \mapsto Z_{2L}$. Employing this homomorphic map, the left-hand side of equation (28) maps to,

$$\langle X_1 (X_2 X_3 - Y_2 Y_3) + Y_1 (X_2 Y_3 + Y_2 X_3) + Z_1 (Z_2 + Z_3) \rangle \leq 4.$$  \hspace{1cm} (29)

The bound of 4 has been set by considering all the LHV models. The inequality (29) is a nonlocality inequality as it gets satisfied by all possible LHVs. Note that unlike in the previous section, the bound has been set by considering all the LHV models. We may easily make a reverse substitution in inequality (14) to obtain a corresponding condition for entanglement in the effective two-qubit system. It gets maximally violated by the three-qubit GHZ state $\frac{1}{\sqrt{2}}(|011\rangle - |100\rangle)$, the modulus of whose expectation value is 6 for the operator given in (29). The inequality (29) can be generically written as,

$$\langle A_1 (A_2 A_3 - A'_2 A'_3) + A'_1 (A_2 A'_3 + A'_2 A_3) + A''_1 (A''_2 + A''_3) \rangle \leq 4,$$ \hspace{1cm} (30)

where $A_i, A'_i, A''_i$ are dichotomic observables, with outcomes ±1. The observables have been chosen according to the rationale given in section 3.3. The detailed derivation of the LHV bound has been given in appendix A.

At this stage, we wish to point out a difference between homomorphic images of $X, Y$ and $Z$. Though the homomorphic images of $X$ and $Y$ do not allow for consistent assignments of joint outcomes of local observables, those of $Z$ do. We believe that this feature has its genesis in the fact that the Pauli operators $X$ and $Y$ detect coherence in a single qubit state, whereas the operator $Z$ does not (in the computational basis).

### 4.8.2. Second inequality: four-party state

The second logical qubit in equation (28) may be assumed to be composed of three qubits. Under this assumption, in this section, we replace $X_{2L}, Y_{2L}$ and $Z_{2L}$ in equation (28) by their homomorphic images acting over the tensor product space of three-qubits, i.e., $\mathcal{H}_i \otimes \mathcal{H}_i \otimes \mathcal{H}_i$. That is, we make the following replacement, $X_{2L} \mapsto X_2 X_3 X_4, Y_{2L} \mapsto -X_2 Y_3 Y_4, Z_{2L} \mapsto Z_2 Z_3 Z_4,$ and, $Y_{2L} \mapsto X_2 Y_3 Y_4, Y_{2L} \mapsto Y_2 X_3 X_4, -Y_2 Y_3 Y_4$ and $Z_{2L} \mapsto Z_2 Z_3 Z_4$. This substitution yields the following inequality,

$$\langle X_1 (X_2 X_3 X_4 - X_2 Y_3 X_4 - Y_2 X_3 Y_4 + Y_2 Y_3 X_4 + X_2 Y_3 X_4 + X_2 X_3 Y_4 - Y_2 Y_3 Y_4) + Z_1 (Z_2 + Z_3 + Z_4 + Z_2 Z_3 Z_4) \rangle \leq 8.$$  \hspace{1cm} (31)

While choosing the observables, we have followed the rationale given in section 3.3. The detailed derivation of the bound is given in appendix B. The bound on the right-hand side has been so fixed that the inequality gets satisfied by all the LHVs. It gets violated by the four-qubit GHZ state, for which the expectation value is equal to the algebraic bound, i.e., 12.

This procedure admits a straightforward generalisation to $N$-qubits provided we choose the observables as per the prescription laid down in section 3.3. At this juncture, it is worth-mentioning that if we use these stabilisers, different variants of Mermin’s inequality [37] and Das–Datta–Agrawal inequality [39] emerge naturally.

We now show how a nonlocality inequality, derived originally for a cluster state, may be looked upon as a descendant of a three-qubit entanglement witness.

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2 All the statements have been made in the computational basis \{00, 11\}, which consists of the eigenstates of $Z$. 

4.9. Rederivation of Bell’s inequality for cluster state and its descendants

We have, so far, mapped single qubit logical states $|0\rangle_L, |1\rangle_L$ to $|0\ldots0\rangle, |1\ldots1\rangle$, i.e., to states having concatenated zeros and ones. A natural question is what happens if we replace single qubit logical states with arbitrary superpositions of multi-qubit states. This is not without interest as we observe that in five-qubit error correction code, the logical qubits are superpositions of multi-qubit states [41]. We give one such example in this section. A nonlocality inequality, getting maximally violated by the following five-qubit error correction code, the logical qubits are superpositions of multi-qubit states [41]. We give one with arbitrary superpositions of multi-qubit states. This is not without interest as we observe that in

\[
|\psi\rangle = \frac{1}{2}(|00\rangle(|00\rangle + |11\rangle) + |11\rangle(|00\rangle - |11\rangle)),
\]

has been derived in [42]. We show how it can be looked upon as a descendant of a three-qubit entanglement witness. Assume that, $|0\rangle_L \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |1\rangle_L \mapsto \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. We now attempt to find the operators having the same effect as $Y_X Y_X Y_Y Y_X Y_Y$ on single qubit states $|0\rangle_L$ and $|1\rangle_L$. The requisite mapping is as follows:

\[
Y_3Y_4, X_3Y_4 \mapsto Y_X X_4 - Y_3Y_4 \mapsto -Z_L.
\]

At this juncture, we wish to stress that the observables have been chosen according to the rationale given in section 3.3. Under this mapping, the entanglement inequality,

\[
\langle X_1X_2X_3 + Z_2Z_3 \rangle \leq 1,
\]

for a three-qubit system gives rise to the following nonlocality inequality,

\[
\langle X_1X_2(X_3Y_4 + Y_3X_4) + Z_2Z_2(X_3Y_4 - Y_3X_4) \rangle \leq 2.
\]

Note that in this section, we have assumed that single logical qubits, $|0\rangle_L, |1\rangle_L$ to map to a coherent superposition of two states, viz, $|00\rangle$ and $|11\rangle$.

4.10. Multiparty nonlocality inequalities from nonlinear entanglement inequalities

We now turn our attention to derivation of multiparty nonlinear nonlocality inequalities starting from a two-qubit nonlinear entanglement inequality. A nonlinear entanglement inequality is given as [43],

\[
\langle X_1X_2 + Y_1Y_2 + Z_1Z_2 \rangle - \frac{1}{2} (\langle X_1 + X_2 \rangle^2 + \langle Y_1 + Y_2 \rangle^2) \leq 1.
\]

In this case, we assume that both the first and the second qubits are logical and composed of three qubits each. We employ the homomorphic mappings of single qubit operators $X, Y, Z$ to three-qubit operators (given in equations (9) and (11)) to arrive at the following nonlocality inequality,

\[
\begin{align*}
\langle (X_1X_3 - X_1Y_3 - Y_1X_3 + Y_1Y_3)(X_4X_6 - X_4Y_6 - Y_4X_6 + Y_4Y_6) \\
+ (Y_1X_1 + X_1Y_1 + X_1X_1 - Y_1Y_1)(X_4X_6 + X_4Y_6 + X_4X_6 - Y_4Y_6) \\
+ (Z_1 + Z_2 + Z_3 + Z_2Z_3)(Z_4 + Z_5 + Z_6 + Z_4Z_5Z_6) \rangle - \frac{1}{2} \left[ \langle X_1X_3 - X_1Y_3 - Y_1X_3 - Y_1Y_3 \rangle^2 \right. \\
+ \left. \langle X_4X_6 - X_4Y_6 - Y_4X_6 - Y_4Y_6 \rangle^2 \right] \leq 32.
\end{align*}
\]

The LHV bound is 32, whereas quantum mechanical maximum value is 48, which is achieved by six-qubit GHZ state. The LHV bound has been proved in appendix C. While constructing this inequality, we have chosen those images which are locally noncommuting as per the rationale given in section 3.3.

4.11. Descendants of Mermin’s inequality

We know that the tri-partite Mermin’s inequality is given as [37],

\[
\langle (A_1A_2 + A'_1A'_2)A_3 + (A_1A'_2 - A'_1A_2)A'_3 \rangle \leq 2.
\]

The above inequality gets maximally violated by the three-qubit GHZ state, $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, for the following choices of observables,

\[
A_1 \equiv X_1; \quad A'_1 \equiv Y_1; \quad A_2 \equiv X_2; \quad A'_2 \equiv Y_2; \quad A_3 \equiv X_3; \quad A'_3 \equiv -Y_3.
\]

If we assume that the third party is a logical qubit and composed of $N$-qubits, the corresponding operators will have the following forms,
\[ A_{3L} \equiv X_{3L}; \quad A'_{3L} \equiv -Y_{3L}. \]  \hfill (40)

The forms of the logical operators are given below:
\[ X_{3L} \equiv |0_1\rangle\langle 1_1| + |1_1\rangle\langle 0_1|; \quad Y_{3L} \equiv -i(|0_1\rangle\langle 1_1| - |1_1\rangle\langle 0_1|). \]  \hfill (41)

We next find out the products of local operators corresponding to \( X_{3L} \) and \( Y_{3L} \). We consider the two cases of \( |0_1\rangle \) and \( |1_1\rangle \) comprising of two and three qubits separately.

### 4.11.1. Logical qubits comprising of two qubits

In this case, we employ the following mapping,
\[ \{X_3X_4, -Y_3Y_4\} \mapsto X_{3L}; \quad \{X_3Y_4, Y_3X_4\} \mapsto Y_{3L}, \]  \hfill (42)

to find the descendant inequality of Mermin’s inequality. The descendant inequality is given as,
\[ \langle (A_1A_2 + A'_1A'_2)(A_3A_4 - A'_3A'_4) + (A_1A'_2 - A'_1A_2)(A'_3A'_4 + A_3A_4) \rangle \leq 4. \]  \hfill (43)

Here, \( A_i, A'_i \) represent dichotomic observables with outcomes \( \pm 1 \). For example, if we assume \( A_i = X_i \) and \( A'_i = Y_i \). These observables have been chosen according to the rationale laid down in section 3.4, that is, they are locally noncommuting so that they can bring out the difference between local and nonlocal states. The derivation of LHV bound of 4 has been shown in appendix D.

### 4.11.2. Logical qubits comprising of three qubits

In this case, we employ the following mapping,
\[ \{X_3X_4X_5, -X_3Y_4Y_5, -Y_3X_4X_5, -Y_3Y_4Y_5\} \mapsto X_{3L}, \]  \hfill (44)
\[ \{-Y_3Y_4Y_5, X_3X_4X_5, X_3Y_4X_5, X_3Y_4Y_5\} \mapsto Y_{3L}, \]

to find the descendant inequality of Mermin’s inequality. The descendant of Mermin’s inequality is given as,
\[ \langle (A_1A_2 + A'_1A'_2)(A_4A_5 - A'_4A'_5) - A'_3A_4A'_5 - A'_3A'_4A_5 \rangle \]  \hfill (44)
\[ + \langle A_1A'_2 - A'_1A_2 \rangle (-A'_3A'_4A'_5 + A'_3A_4A_5 + A_3A'_4A_5 + A_3A_4A'_5) \rangle \leq 4. \]  \hfill (45)

Here \( A_i, A'_i \) represent dichotomic observables with outcomes \( \pm 1 \). In fact, if we assume \( A_i = X_i \) and \( A'_i = Y_i \), the above inequality gets maximally violated by the GHZ state of five qubits. The derivation of LHV bound is given in appendix E. In a similar manner, it can be generalised to an arbitrary \( N \)-qubit system provided the observables are chosen in such a way so as to conform to the condition of local noncommutativity, which has been laid down in section 3.4.

### 4.12. Descendants of Svetlichny inequality

In this same way, we can find the descendants of Svetlichny inequality. The tripartite Svetlichny inequality is given as [36],
\[ \langle (A_1A_2 + A'_1A'_2)A_3 + (A'_1A_2 - A_1A'_2)A'_3 \rangle \leq 4, \]  \hfill (46)
where \( A_1 = B_1 + B'_1 \) and \( A'_1 = B_1 - B'_1 \). It gets maximally violated by the three-qubit GHZ state \( \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \). As before, if we assume that the third qubit is a logical qubit and itself consists of three qubits, we may find descendant of Svetlichny inequality for a five party system. The descendant inequality is given by,
\[ \langle (A_1A_2 + A'_1A'_2)A_3A_4A_5 + (A'_1A_2 - A_1A'_2)A'_3A'_4A'_5 \rangle \leq 4, \]  \hfill (47)
which is obtained by employing the mapping \( A_3 \mapsto A_3A_4A_5 \) and \( A'_3 \mapsto A'_3A'_4A'_5 \). Note that the observables \( A_i \) and \( A'_i \) may be chosen to be locally noncommuting observables, hence we have chosen this mapping. For example, if we choose \( A_1 = \sqrt{2}X_1, A'_1 = \sqrt{2}Y_1, A_2 = X_2, A'_2 = -Y_2, A_3 = X_3, A'_3 = Y_3, A_4 = X_4, A'_4 = Y_4, A_5 = X_5, A'_5 = Y_5 \). Since, \( A_1 = B_1 + B'_1 \) and \( A_2 = B_1 - B'_1 \), the observables \( B_1 \) and \( B'_1 \) are of the form \( B_1 = \frac{1}{\sqrt{2}}(X_1 + Y_1) \) and \( B'_1 = \frac{1}{\sqrt{2}}(X_1 - Y_1) \). For this choice of observables, the above inequality assumes the following form,
\[ \langle (X_1X_2 - Y_1Y_2)X_3X_4X_5 + (Y_1X_2 + X_1Y_2)Y_3Y_4Y_5 \rangle \leq 2\sqrt{2}. \]  \hfill (48)

This inequality is maximally violated by the five qubit GHZ state \( \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \). The LHV bound of 4 has been derived in appendix F.

This concludes the applications of the proposed approach. In this way, we can employ this approach to find descendants of any given inequality.
5. Relation with previous work

Stabilisers have been employed to construct nonlocality and entanglement inequalities [24, 29]. Nevertheless, homomorphism among stabiliser groups has not been hitherto studied, to the best of our knowledge, to derive a number of nonlocality inequalities given a seed nonlocality inequality. Furthermore, this approach also shows an interrelation between coherence witnesses for a single qubit and entanglement and nonlocality inequalities for two and multi-qubit systems.

6. Conclusion

In summary, we have developed a procedure, based on homomorphism of stabiliser groups, that shows interconnection among coherence witnesses of a single qubit system, entanglement witnesses, and nonlocality inequalities of multiparty systems, and condition for quantum discord. We have applied the procedure to a number of inequalities to find their descendants. In particular, we have shown (i) how nonlocality and entanglement inequalities can be obtained from conditions for quantum coherence, (ii) how a nonlocality inequality for, say, two-qubit system can be generalised to three and four-qubit systems. (iii) How linear and nonlinear entanglement inequalities for two-qubit systems yields conditions for nonlocality in a multiqubit system. In general, this approach can be employed to obtain a series of (multiparty) nonlocality inequalities, contingent on the knowledge of one seed inequality for a bipartite system. The framework is quite generic. We have considered but some examples. Its generalisation to qudits forms an interesting study that will be taken up separately. It also shows that more and more substructures of quantumness emerge as we probe more particles that are in coherent superposition. Since multipartite nonlocality may also be regarded as a special case of sequential contextuality in single-party systems [44], the proposed procedure may also be employed to render a number of contextuality inequalities.

This work also leaves open a lot of questions: (i) whether an arbitrary entanglement witness can be converted into a nonlocality inequality by suitably extending the number of parties?, (ii) whether a very weak (less stringent) coherence witness for a single qubit system be converted to condition for nonclassical correlations in multiparty systems? If yes, how many subsystems are required as a function of bound of coherence witnesses? The answers to these questions will hopefully improve the understanding of interrelation among nonclassicality criteria and also serve to deliver a number of conditions for different nonclassical correlations. Furthermore, since nonlocality and entanglement inequalities are, by construction, contextuality inequalities, the inequalities derived via this approach are also contextuality inequalities.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Derivation of LHV bound in inequality (30)

This derivation follows the approach given in [45]. The inequality (30) is given as,

\[ \langle A_1(A_2 A_3 - A'_2 A'_3) + A'_1( A_2 A'_2 + A'_3 A_3) + A''_1(A''_2 + A''_3) \rangle \leq 4. \]  

(A.1)

Let the outcomes of $A_1, A_2, A'_1, A'_2, A''_1, A''_2$ be represented by $a_1, a_2, a'_1, a'_2, a''_1, a''_2$. Obviously, due to the dichotomic nature of these observables, $a_1, a_2, a'_1, a'_2, a''_1, a''_2$ take values ±1. Then, under the assumption of a
Appendix B. Derivation of LHV bound in inequality (31)

The inequality (31) is given as follows:

\[
\langle X_1(X_1X_1X_1 - X_2Y_1Y_1 - Y_2X_1Y_1 - Y_2Y_1X_1) + Y_1(Y_2X_1X_1 + X_2Y_1X_1 + X_2X_1Y_1 - Y_2Y_1X_1) \\
+ Z_1(Z_2 + Z_3 + Z_4 + Z_2Z_3Z_4) \rangle \leq 8. \tag{B.1}
\]

The LHV bound of 8 can be set in the following manner. Let the outcomes of \(X_i, Y_i, Z_i\) be represented by \(x_i, y_i, z_i\), where \(i \in \{1, 2, 3, 4\}\). where \(x_1, y_1, z_1\) can take values \(\pm 1\). Under the assumption of a LHV model, the lhs of equation (31) assumes the following form:

\[
\langle x_1(x_2x_3x_4 - x_2y_3y_4 - y_2x_3y_4 - y_2y_3x_4) + y_1(y_2x_3x_4 + x_2y_3x_4 + x_2x_3y_4 - y_2y_3y_4) \\
+ z_1(z_2 + z_3 + z_4 + z_2z_3z_4) \rangle.
\]

We fix the LHV bounds of the I, II and III combinations as follows:

(a) The first term can take a maximum value of 2. It is because the product of all the four terms \(x_1x_2x_3x_4\), \(-x_1x_2y_3y_4\), \(-x_1y_2x_3y_4\) and \(-x_1y_2y_3x_4\) is equal to \(-1\), hence one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the first combination is 2.

(b) The second term can also take a maximum value of 2. It is because the product of all the four terms \(y_1y_2x_3x_4, y_1y_2x_3y_4, y_1y_2x_3y_4\) and \(-y_1y_2y_3x_4\) is equal to \(-1\), hence one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the second combination is 2.

(c) The third term can take a maximum value of 4. It is because the product of all the four terms \(z_1z_2, z_1z_3, z_1z_4\) and \(z_1z_2z_3z_4\) can simultaneously take values +1 under the assumption of a LHV model.

If we allow for an arbitrary distribution of hidden variables of all the observables \(x_i, y_i, z_i\), we arrive at the inequality (31). That is why the LHV bound on the inequality (31) is 8.

Appendix C. Derivation of LHV bound in inequality (37)

The inequality (37) is given as follows:
The LHV bound of 32 can be set in the following manner. Let the outcomes of $x_i, y_i, z_i$ be represented by $x_i, y_i, z_i$ where $i \in \{1, \ldots, 6\}$, where $x_i, y_i, z_i$ can take values $\pm 1$. Under the assumption of an LHV model, the lhs of equation (37) assumes the following form:

$$\langle (x_1x_2x_3 - x_1y_2y_3 - Y_1x_2Y_3 - Y_1y_2Y_3)(x_4x_5x_6 - x_4y_5y_6 - Y_4x_5Y_6 - Y_4y_5Y_6) \rangle$$

$$+ \langle (y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3 - Y_1x_2Y_3)(y_4x_5x_6 + x_4y_5x_6 + x_4x_5y_6 - Y_4x_5Y_6) \rangle$$

$$+ \langle (z_1 + z_2 + z_3 + Z_1Z_2Z_3)(Z_4 + Z_5 + Z_6 + Z_4Z_5Z_6) \rangle$$

$$- \frac{1}{2} \langle x_1x_2x_3 - x_1y_2y_3 - Y_1x_2Y_3 - Y_1y_2Y_3 \rangle \langle x_4x_5x_6 - x_4y_5y_6 - Y_4x_5Y_6 - Y_4y_5Y_6 \rangle \rangle \leq 32.$$

(C.1)

We fix the LHV bounds of the I, II, III, IV and V combinations as follows:

(a) The first combination can take a maximum value of 4. It is because the product of all the four terms $x_1x_2x_3, -x_1y_2y_3, -y_1x_2y_3$ and $-y_1y_2y_3$ is equal to $-1$, hence one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination $(x_1x_2x_3 - x_1y_2y_3 - Y_1x_2Y_3 - Y_1y_2Y_3)$ is 2. Similarly, the product of all the four terms $x_4x_5x_6, -x_4y_5y_6, -y_4x_5y_6$ and $-y_4y_5y_6$ is equal to $-1$, hence one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination $(x_4x_5x_6 - x_4y_5y_6 - Y_4x_5Y_6 - Y_4y_5Y_6)$ is 2. Hence the maximum value taken by the first combination is 4 (and the minimum value is $-4$).

(b) For the same reason as above, the second combination can also take a maximum value of 4, under the assumption of a LHV model. It is because the product of all the four terms $y_1x_2x_3, x_1y_2y_3, x_1x_2y_3$ and $-y_1y_2y_3$ is equal to $-1$, hence one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination $(y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3 - y_1y_2y_3)$ is 2. Similarly, the product of all the four terms $y_4x_5x_6, y_4y_5x_6, y_4x_5y_6$ and $y_4y_5y_6$ is equal to $-1$. That is why, one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination $(y_4x_5x_6 + y_4y_5x_6 + x_4y_5x_6 - y_4y_5y_6)$ is equal to 2. That is why, the maximum value taken by the second combination is 4 (and the minimum value is $-4$).

(c) The third combination can take a maximum value of 16 (and a minimum value of $-8$). It is because all the eight terms $z_1, z_2, z_3, z_1z_2z_3, z_4, z_5, z_6, z_4z_5z_6$ can simultaneously take values $\pm 1$ under the assumption of a LHV model.

(d) The fourth and fifth combinations can each take a value of 2 under the assumption of a LHV model for the same reason as has been discussed above.

If we allow for an arbitrary distribution of hidden variables of all the observables $x_i, y_i, z_i$, we arrive at the inequality (37). Hence the LHV bound on the inequality (37) is given by $4 + 4 + 16 + \frac{1}{2}(2 + 2)^2 = 24 + 8 = 32$. 

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Appendix D. Derivation of LHV bound in inequality (43)

The inequality (43) is given as follows:

\[ \langle (A_1A_2 + A'_1A'_2) (A_3A_4 - A'_3A'_4) + (A_1A'_2 - A'_1A_2) (A'_3A_4 + A_3A'_4) \rangle \leq 4, \]  

which can be rewritten as,

\[ \langle A_1 \left( A_2 (A_3A_4 - A'_3A'_4) + A'_1 (A'_3A_4 + A_3A'_4) \right) \rangle + \langle A'_1 \left( A'_2 (A'_3A_4 - A'_3A'_4) - A_2 (A'_3A_4 + A_3A'_4) \right) \rangle \leq 4. \]  

(D.2)

The LHV bound of 4 can be set in the following manner. Let the outcomes of \( A_i, A'_i, i \in \{1, \ldots, 4\} \) be represented by \( a_i, a'_i \), where \( a_i, a'_i \) can take values \( \pm 1 \). Under the assumption of a LHV model, the lhs of equation (43) assumes the following form:

\[ a_1 \{a_2(a_3a_4 - a'_3a'_4) + a'_2(a'_3a_4 + a_3a'_4)\} + a'_1 \{a'_2(a'_3a_4 - a'_3a'_4) - a_2(a'_3a_4 + a_3a'_4)\}. \]  

(D.3)

We fix the LHV bounds for the I and II combinations as follows:

(a) The first combination can take the maximum value +2 under the assumption of an underlying LHV model. It is because the product of all the four terms, viz, \( a_1a_2a_3a_4, -a_1a_2a'_3a'_4, a_1a'_2a'_3a_4 \) and \( a_1a'_2a_3a'_4 \) is equal to \( -1 \). Hence, one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination \( a_1 \{a_2(a_3a_4 - a'_3a'_4) + a'_2(a'_3a_4 + a_3a'_4)\} \) is 2.

(b) The second combination can take the maximum value +2 under the assumption of an underlying LHV model for the same reason as above. The product of all the four terms, viz, \( a'_1a'_2a_3a_4, a_1a'_2a'_3a_4, -a'_1a_2a'_3a'_4 \) and \( -a_1a_2a'_3a'_4 \) is equal to \( -1 \). Hence, one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination \( a'_1 \{a'_2(a'_3a_4 - a'_3a'_4) - a_2(a'_3a_4 + a_3a'_4)\} \) is 2.

If we allow for an arbitrary distribution of hidden variables of all the observables \( a_i, a'_i \), we arrive at the inequality (43). Hence the LHV bound of the inequality (43) is given by 4.

Appendix E. Derivation of LHV bound in inequality (45)

The inequality (45) is given as follows:

\[ \langle (A_1A_2 + A'_1A'_2)(A_3A_4 - A'_3A'_4) + (A_1A'_2 - A'_1A_2)(A'_3A_4 + A_3A'_4) \rangle \leq 4. \]  

(E.1)

The LHV bound of 8 can be set in the following manner. Let the outcomes of \( A_i, A'_i, i \in \{1, \ldots, 5\} \) be represented by \( a_i, a'_i \), where \( a_i, a'_i \) can take values \( \pm 1 \). Under the assumption of a LHV model, the lhs of equation (45) assumes the following form:

\[ a_1a_2a_3a_4a_5 + \{a_1a_2 + a'_1a'_2\} \{a_3a_4a_5 - a_3a_4a'_5 - a'_3a'_4a_5 - a'_3a'_4a'_5\} \]  

\[ + \{a_1a'_2 - a'_1a_2\} \{a'_3a_4a_5 + a_3a'_4a_5 + a_3a'_4a'_5\}. \]  

(E.2)

We fix the LHV bounds for the I and II combinations as follows:

(a) The first combination can take the maximum value +4 under the assumption of an underlying LHV model. It is because the product of all the four terms, viz, \( a_1a_2a_3a_5, -a_1a_2a'_3a'_5, -a'_1a_2a'_3a_5 \) and \( -a'_1a_2a'_3a'_5 \) is equal to \( -1 \). Hence, one of the four terms has to have an opposite sign with respect to the other three terms. So, under the assumption of a LHV model, the maximum value taken by the combination \( a_1a_2a_3a_5 - a_3a_4a'_5 - a'_3a'_4a_5 - a'_3a'_4a'_5 \) is 2. The value taken by the combination \( a_1a_2 + a'_1a'_2 \) is 2, since both the terms can take value +1 simultaneously. Hence, the value taken the I combination is 4 under the assumption of a LHV model.

(b) In this case, if the first combination takes a value 4, the second combination takes a value 0. It is because of the presence of the first term, \( (a_1a'_2 - a'_1a_2) \). If \( a_1 = a_2 = a'_1 = a'_2 = 1 \), then the term \( (a_1a'_2 - a'_1a_2) \) takes a value 0 and the term \( (a_1a_2 + a'_1a'_2) \) takes a value 2.
If we allow for an arbitrary distribution of hidden variables of all the observables $a_i, a'_i$, we arrive at the inequality (45). Hence the LHV bound of the inequality (45) is given by 4.

**Appendix F. Derivation of LHV bound in inequality (47)**

We follow the procedure given in [39] for deriving the LHV bound. The inequality (47) is given as follows:

$$\langle (A_iA_2 + A'_iA'_2)A_3A_4A_5 + (A'_iA_2 - A_iA'_2)A'_3A'_4A'_5 \rangle \leq 4.$$  \hspace{1cm} (F.1)

Employing $B_1 \equiv A_1 + A'_1$ and $B'_1 \equiv A_1 - A'_1$, the above inequality can be reexpressed as,

$$\langle \{(B_1 + B'_1)A_2 + (B_1 - B'_1)A'_2\}A_3A_4A_5 + \{(B_1 - B'_1)A_2 - (B_1 + B'_1)A'_2\}A'_3A'_4A'_5 \rangle \leq 4. $$

The LHV bound of 4 can be set in the following manner. Under the assumption of a factorisable LHV model, the above expectation value gets factorised. Thereafter, we can apply triangle inequality,

$$\langle \{(B_1 + B'_1)A_2 + (B_1 - B'_1)A'_2\}A_3A_4A_5 + \{(B_1 - B'_1)A_2 - (B_1 + B'_1)A'_2\}A'_3A'_4A'_5 \rangle \leq \langle \{(B_1 + B'_1)A_2 \}A_3A_4A_5 \rangle + \langle \{(B_1 - B'_1)A_2 - (B_1 + B'_1)A'_2\}A'_3A'_4A'_5 \rangle. $$  \hspace{1cm} (F.2)

Since the outcomes of the observables $A_1, A_4, A_5$ and $A'_1, A'_4, A'_5$ are ±1, their expectation values are bounded by 1 in magnitude. So, we can apply Cauchy–Schwartz inequality,

$$\langle \{(B_1 + B'_1)A_2 \}A_3A_4A_5 \rangle + \langle \{(B_1 - B'_1)A_2 - (B_1 + B'_1)A'_2\}A'_3A'_4A'_5 \rangle \leq \langle \{(B_1 + B'_1)A_2 \}A_3A_4A_5 \rangle + \langle \{(B_1 - B'_1)A_2 - (B_1 + B'_1)A'_2\}A'_3A'_4A'_5 \rangle \leq 4.$$  \hspace{1cm} (F.3)

The last step follows because the combination represents Svetlichny polynomial for a tripartite system, whose LHV bound is 4 [36].

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