Numerical homogenization of elastoplastic deformations of composite material with small proportion of inclusions

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Abstract. Numerical simulation of the stress-strain state of a composite material may be difficult due to large computational complexity associated with a grid resolution of a large number of inclusions. To overcome the problem one may use the homogenization method. But for material with plastic properties, proper modeling of yield stress and hardening may be overcomplicated. In this work we use some simplification associated with a small proportion of inclusions and restriction of stress values by matrix material strength. As model problem, we use deformation of concrete deep beam reinforced with steel or basalt fiber inclusions. For the numerical solution, the finite element method was applied using the FEniCS computing platform.

1. Introduction
In the concrete industry one of the most innovative reinforcements are inclusions in the form of fibers, which can significantly enhance the strength characteristics of concrete [1]. In recent decades, interest in the study of composite reinforcement by basalt plastic for the subsequent replacement of metal rods in load-bearing reinforced concrete structures in construction has been increased [2]. One of the innovative trends in the development of the North of Russia are technologies related to the production of basalt continuous fiber and composite materials based on it. The presence of huge stocks of raw materials and the relatively low cost of fibers produced from it makes it possible to consider their prospects in economic terms.

When modeling the problem of calculating the stress-strain state of a composite material with inclusions in the form of fibers, there is a problem associated with a large quantity of the cells needed for the grid resolution of each fiber. One way to solve this problem is to use a multiscale method or a method of numerical homogenization. However, in real applications, the number of fibers can be so large that the solution to the problem of constructing local bases for the classical multiscale method will also be quite complex. Of course, one can use the method of numerical homogenization on the representative element and solve the problem at the macro level using effective coefficients [3, 4]. But for the material with plastic properties, proper modeling of yield stress and hardening may be overcomplicated. In this work, we use
Material | Young modulus $E, GPa$ | Poisson ratio $\nu$  
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Concrete | 40 | 0.15  
Basalt | 89 | 0.26  
Steel | 200 | 0.3  

Table 1: Elastic parameters of concrete matrix, basalt and steel fibers

some simplification associated with a small proportion of inclusions. As model problem, we use deformation of concrete deep beam reinforced with steel or basalt fiber inclusions [5].

For a numerical solution, we have used the implementation of the finite element method using the FEniCS computing platform [6–8]. Results of comparison of deformations between a solution with a full grid resolution of fibers and a solution with effective coefficients are presented. Two different variants of elastic properties of inclusions in concrete were used: ones for basalt and steel material.

2. Problem Statement
Let us consider mathematical model describing stress-strain state of composite concrete with basalt or steel fiber inclusions

$$\text{div } \sigma = 0, \quad x \in \Omega = \Omega_1 \cup \Omega_2,$$

where $\Omega_1$ is subdomain of concrete, $\Omega_2$ is subdomain of fibers, $\sigma$ is stress tensor.

For plasticity model [9,10] strain tensor consists of elastic and plastic parts:

$$\varepsilon = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \end{pmatrix} = \varepsilon^e + \varepsilon^p.$$  

(2)

For convenience, we use Voight notation. There is also a linear relation between stress and elastic strain tensors.

$$\sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = C \varepsilon^e,$$

$$C = \begin{pmatrix} C_{1111} & C_{1122} & C_{1112} \\ C_{2211} & C_{2222} & C_{2212} \\ C_{1211} & C_{1222} & C_{1212} \end{pmatrix},$$

here $C$ is elastic tensor, which is represented as follows

In the case of isotropic body, elastic tensor can be uniquely determined by Young modulus $E$ and Poisson coefficient $\nu$. Elastic parameters of materials under the investigation are presented in Table 1 [11, 12]. Hardening parameter and yielding stress equal to 21.5 GPa and 30.0 MPa, respectively

Concrete is modeled as plastic material with isotropic hardening, which can be represented as figure 1.

In this article, we limit the maximum value of stress that occurs in concrete around the destruction values. So we can model inclusion materials as pure elastic.

Equation of solid body is supplemented with boundary conditions related to the fixation on the left side and tangent stress on the right side, which is illustrated in figure 2. These conditions can be defined with following Dirichlet and Neumann boundary conditions:

$$u_Y = 0, \quad x \in \Gamma_D, \quad \sigma_n = (0, -P), \quad x \in \Gamma_N.$$  

(3)
Pressure stress grows from 0 to 10 MPa. At the final moment of time, pressure value is high enough to produce stresses more than tensile strength of the material, which is roughly equal to 100 MPa.

To approximate the space, the finite element method is used. The original differential problem is reduced to a discrete variational problem. The numerical implementation is performed using the FEniCS computing platform.

3. Homogenization Method

For homogenized plasticity, we use homogenized elastic tangent calculated using the asymptotic homogenization method for elastic coefficients. Plastic parameters, such as yield stress and hardening coefficient, remain the same as matrix material, i.e. concrete. We assume that such approximation can be suitable for stress-strain computation of composites with a low volumetric concentration of inclusions limited by stress values equal to matrix material strength. Without such restrictions, one must use multiple yielding points with tensor values for the homogenization parameter, which could describe the anisotropic nature of composite material.

The usage of the asymptotic homogenization method is associated with the computation of an effective elastic tensor. Let denote an average value of function $\psi$ by the following notation

$$\langle \psi \rangle = \frac{\int_\omega \psi \, dx}{\int_\omega \, dx},$$

where $\omega$ is a periodic domain. Thus, an average stress tensor can be expressed by

$$\langle \sigma \rangle = \langle C \varepsilon \rangle = C^* \langle \varepsilon \rangle,$$
where $C^*$ is the effective elastic tensor. To compute components of the effective elastic tensor, we consider three problems in the periodic domain $\omega$ with the following force sources:

(i) $f = -\nabla \cdot C \varepsilon((x_1, 0))$,
(ii) $f = -\nabla \cdot C \varepsilon((0, x_2))$,
(iii) $f = -\nabla \cdot C \varepsilon((x_2/2, x_1/2))$.

To ensure the uniqueness of the solution we fix displacement to $(0,0)$ in the middle point of the domain $\omega$.

4. Comparative Analysis

In figures 4, 3, comparison of the maximum displacement magnitude between plastic, elastic, homogenized plastic and homogenized elastic solutions over time steps are presented for concrete composite with steel and basalt fiber inclusions. According to these results, our approximation provides good accuracy for the case of basalt fiber inclusions. Our approximation for composite with steel inclusions provides results with bigger error, but still tolerable. It is confirmed by a comparison of the displacement results. Its error for steel fiber inclusions is presented in figures 5–8.

![Figure 3: Comparison of the maximum displacement magnitude between plastic, elastic, homogenized plastic and homogenized elastic solutions for concrete composite with steel fiber inclusions over time steps.](image)

5. Conclusion

We have made a solution for the plasticity problem for a composite material with fiber inclusion using the homogenization method. Results demonstrate that the homogenized plasticity model with homogenized elasticity tangent, yield stress and hardening values equal to ones of the matrix can be applied to a model stress-strain state of the composite material in the whole stress range until it reach tensile strength, in other words, the values are high enough to produce cracks and fractures.

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Figure 4: Comparison of the maximum displacement magnitude between plastic, elastic, homogenized plastic and homogenized elastic solutions for concrete composite with basalt fiber inclusions over time steps.

Figure 5: Horizontal displacement value at the last time step of plasticity solution for concrete composite with steel fiber inclusions over time steps.

Figure 6: Vertical displacement value at the last time step of plasticity solution for concrete composite with steel fiber inclusions over time steps.

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Figure 7: Horizontal displacement error at the last time step of homogenized plasticity solution for concrete composite with steel fiber inclusions over time steps.

Figure 8: Vertical displacement error at the last time step of homogenized plasticity solution for concrete composite with steel fiber inclusions over time steps.

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