Abstract

Spectral geometric methods have brought revolutionary changes to the field of geometry processing – however, when the data to be processed exhibits severe partiality, such methods fail to generalize. As a result, there exists a big performance gap between methods dealing with complete shapes, and methods that address missing geometry. In this paper, we propose a possible way to fill this gap. We introduce the first method to compute compositions of non-rigidly deforming shapes, without requiring to solve first for a dense correspondence between the given partial shapes. We do so by operating in a purely spectral domain, where we define a union operation between short sequences of eigenvalues. Working with eigenvalues allows to deal with unknown correspondence, different sampling, and different discretization (point clouds and meshes alike), making this operation especially robust and general. Our approach is data-driven, and can generalize to isometric and non-isometric deformations of the surface, as long as these stay within the same semantic class (e.g., human bodies), as well as to partiality artifacts not seen at training time.

1. Introduction

Recent progress in spectral geometry processing has brought to significant qualitative leaps in a range of challenging applications such as deformable shape matching [34, 23], retrieval [36, 35], style [29] and pose transfer [19, 52] among other tasks. However, these methods typically operate in a controlled scenario, ignoring the fact that real-world data are ridden with partiality artifacts, non-isometric deformations, and that the data may be provided as different and often incompatible representations. In this paper, we consider precisely this challenging setting: a set of partial, deformable 3D shapes represent our data, and the task is to combine them into a complete 3D model. A typical pipeline to address this would solve for a set of partial correspondences, extract a set of (non-rigid) transformations from the correspondences, and merge the partial views...
into a consistent discretization. All these steps are hard to solve and error-prone, as testified by a wealthy literature on non-rigid shape matching and reconstruction. For example, the mere presence of inconsistent triangle tessellations can cause problems to most matching pipelines [30].

Here, we propose a different perspective. Currently, spectral geometric pipelines need explicit access to the extrinsic geometry of a target full shape in order to compute its underlying spectral quantities. We claim that this is not necessary in many cases, and propose to directly estimate the intrinsic properties of the sought full shape without having to materialize its surface geometry. This is done by translating the combination task from the spatial to a purely spectral domain. For each partial surface, our method takes as input the truncated sequence of its Laplacian eigenvalues, which act as a surrogate of the shape geometry, and yields as output the eigenvalue sequence of the 3D model obtained from the union of the two surfaces (or an isometric deformation thereof) – but not the 3D model itself. This prediction task is tough and ill-posed in general, but can be resolved by means of a data prior, namely by training a deep net on a few hundred examples. The advantages of this approach are numerous, and include compositionality, invariance to deformations and sampling, and generalization to different discrete representations for the input geometry. In a way, this recalls the notion of “homomorphic encryption” in secure computation [37], where the task is to perform calculations on encrypted data without decrypting it first.

**Contribution.** In this paper, we introduce a learning-based method to estimate the Laplacian spectrum of the union of partial non-rigid 3D shapes, without actually computing the 3D geometry of the union. Sidestepping the reconstruction means that we do not have to commit to one specific 3D embedding in the output (e.g., a specific pose for a human body), but leave this choice to task-specific blocks. Once a spectrum is predicted, it can be fed as-is to any existing spectral pipeline operating with eigenvalues. For example, if needed, we can reconstruct the full 3D geometry by using [29] as an output block. If the geometry is not needed, e.g., for tasks of shape retrieval [36] and region localization [35], we demonstrate that the same accuracy is obtained compared to the case where the full shape is given.

2. Related work

We discuss two lines of research that are most closely related to our spectral aggregation task: partial non-rigid aggregation of shapes in their extrinsic form (e.g., mesh or point cloud), and spectral analysis of partial shapes.

2.1. Nonrigid shape aggregation

Recovering deformable 3D shapes from partial scans has numerous applications in AR/VR, manufacturing, and robot manipulation. A common setting for this problem is non-rigid registration, where the scans are captured sequentially and exhibit mild inter-frame deformations, and significant overlap. In such cases, template-less methods have been shown to perform well by utilizing general deformation models such as thin-plate splines [6, 7] or as-rigid-as-possible energies [20]. Wand et al. [50, 49] used dynamic “surfels” to represent the input surfaces, and proposed a statistical model to recover a template model. Temporal coherence has been used in [46] to generate dense correspondences from robust landmarks, and in [31] to reconstruct a space-time surface embedded in 4D. Sharf et al. [42] incorporated a mass conservation prior to control the plausibility of the reconstructed surface. The “Dynamic Fusion” method of Newcomb et al. [33] and follow-up work [44], demonstrated real-time, template-free non-rigid reconstruction allowing both the object and the camera to move.

More related to our setting are cases where the input set is sparser, and the deformations between the scans can change significantly. In fact, we do not assume temporal coherence or an initial alignment. Similar to us, methods designed for these settings usually assume a strong prior on the shape category or even utilize a parametric model. In [53], a generic human template was used for building a personalized parametric human body model similar to SCAPE [1]. Chang and Zwicker [9, 10] assumed an articulated model and solved for joints and skinning weights. More recently, advances in geometric deep learning for processing point clouds and meshes were used to leverage data-driven priors (e.g., from [4]) for deformable shape completion and fusion [21, 16].

2.2. Eigenvalues and partiality

Spectral representations based on the Laplacian are widely used in the analysis of deformable shapes, mainly due to their isometric invariance. Much less attention has been given to the effect of partiality on the spectrum.

**Shape correspondence.** A first attempt at utilizing the Laplacian eigenfunctions to recover dense correspondences between a partial and a full shape was shown in [38], building upon the seminal functional maps framework [34]. This was further extended to matching shapes in the presence of clutter [12], and to a more efficient variant in [23]. In the context of partial shape aggregation, most relevant is an extension to the multi-part matching algorithm (a.k.a. “non-rigid puzzle”) proposed in [24]. Recently, deep learning techniques have also been utilizing Laplacian eigenfunctions for matching [22, 17, 15, 2, 41]. Replacing eigenfunctions with a basis learned from data was recently proven more robust and therefore applicable to challenging settings including point clouds and partiality [28].

**Reconstruction.** Aside from matching, several techniques
have investigated spectral methods for non-rigid completion and registration. In FARM [27] and its high-resolution variant [26], a functional maps representation is incorporated into a parametric model-based regression pipeline. A full recovery of the geometry from eigenvalues, also known as the problem of “hearing the shape of a drum” [18], was recently studied by Cosmo et al. [11] in practical rather than purely theoretical settings. Their procedure, dubbed “isospectralization”, was proven useful in multiple application scenarios, and extended in [29] by replacing the regularizers of [11] with a data-driven prior. Finally, [45] devised a framework to learn fuzzy representations that enable set operations on man-made objects. In our work, we learn to perform a union of partial deformable shapes directly in the spectral representation.

3. Proposed method

Let us be given two partial shapes X and Y, and let X ∪ Y denote their non-rigid alignment. We seek an answer to the following question: What can we say about X ∪ Y, without actually computing this union?

With no additional priors, the question is ill-posed; for example, there are infinitely many ways in which two sheets of paper can be glued together. This can be partly resolved by introducing a semantic prior, but the question remains of how much geometric information on X ∪ Y can be estimated from X and Y, without solving a correspondence or reconstruction problem in the process. In the sequel, we claim that coupling Laplacian eigenvalues with a data prior serves this purpose well.

Mathematical preliminaries. We model shapes as Riemannian manifolds M with boundary ∂M. Each manifold identifies an equivalence class of isometries, and thus has infinitely many embeddings in \( \mathbb{R}^3 \) (e.g. changes in pose). Let us be given two manifolds \( M_1 \) and \( M_2 \), together with a diffeomorphism \( \pi : R_1 \rightarrow R_2 \) between regions \( R_1 \subseteq M_1 \) and \( R_2 \subseteq M_2 \). A third manifold \( M_1 \cup M_2 \) can be obtained by attaching \( M_1 \) to \( M_2 \) over the common region via the map \( \pi \). We refer to \( M_1 \cup M_2 \) as the union shape.1

On each \( M \) we consider the Laplace-Beltrami operator \( \Delta \), extending the notion of Laplace operator from Euclidean geometry to surfaces. This operator admits a spectral decomposition:

\[
\Delta \phi_i(x) = \lambda_i \phi_i(x) \quad x \in \text{int}(M) \quad (1)
\]

\[
\phi_i(x) = 0 \quad x \in \partial M \quad (2)
\]

into eigenvalues \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) and associated eigenfunctions \( \phi_1, \phi_2, \phi_3, \ldots \); we adopt homogeneous Dirichlet boundary conditions (2). The set of eigenvalues forms a discrete spectrum, which we assume to be ordered non-decreasingly. In this paper, we consider truncated spectra of length \( k \), and introduce the vector-valued function:

\[
\lambda : \mathcal{M} \mapsto (\lambda_1, \ldots, \lambda_k). \quad (3)
\]

In particular, we completely discard the eigenfunctions \( \phi_1(x), \phi_2(x), \ldots \), which are point-based quantities and thus highly depend on shape discretization.

Remark. Since the Laplacian \( \Delta \) is invariant to isometries, so is its truncated spectrum encoded in \( \lambda \). This means that eigenvalues capture shape information up to pose, a fundamental property that is at the basis of our method.

Problem statement. In non-rigid alignment, one is given 3D embeddings (e.g. point clouds) for \( M_1 \) and \( M_2 \), and must recover a 3D embedding of their union \( M_1 \cup M_2 \). Since \( M_1 \) and \( M_2 \) may undergo wildly different deformations, there is no guarantee that they have the same 3D coordinates on the common region. Therefore, it is not clear how a 3D embedding for \( M_1 \cup M_2 \) should look like.

To resolve this, we propose to switch from a discrete representation of the 3D embedding of \( M_1 \cup M_2 \) to a discrete representation of the entire isometry class, given by \( \lambda(M_1 \cup M_2) \). Then, we translate the problem of recovering an alignment between 3D embeddings to the estimation of a parametric nonlinear operator \( U_{\phi_3} : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k \), such that:

\[
\lambda(M_1 \cup M_2) = U_{\phi_3}(\lambda(M_1), \lambda(M_2)). \quad (4)
\]

We call \( U_{\phi_3} \) the spectral union operator, and model it as a

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1We keep the mathematical description simple for the sake of clarity. Formally, this operation is called connected sum, denoted by \( M_1 \# M_2 \), and is part of the surgery theory of manifolds, see, e.g., [40].
deep net with learnable parameters $\Theta$. A specific definition for the architecture and the loss are given in Section 4.

**Remark.** In the general case, the spectrum of the union shape $M_1 \cup M_2$ is not simply the union of the spectra of $M_1$, $M_2$. This is only true if $M_1$ and $M_2$ are disconnected (see e.g. [38, Sec. 3.1]), while in this paper we consider the case in which $M_1$ and $M_2$ partially overlap. In our setting, the interactions between the two spectra are much more complex, which is why we choose to model the union via a nonlinear trainable operator.

**Difficulty settings and compositionality.** Estimating an operator $U_\Theta$ that makes Eq. (4) hold for many different pairs $(M_1, M_2)$ is an apparently simple problem, especially if one uses short sequences (in this paper we use $k = 20$). In fact, it is known that Laplacian spectra can vary wildly under partiality perturbations [13], and predicting these variations can be especially hard. At the same time, it has been empirically demonstrated that the types of partiality leading to high instability are rarely observed in practice [38].

Based on these observations, we consider two different scenarios with different characteristics:

1. $M_1 \cup M_2$ is a complete, watertight shape;
2. $M_1 \cup M_2$ is a partial shape itself.

Scenario 1 is simple enough to be easily solved with a feed-forward network, and generalizes well to unseen data, as shown in Figure 2. Scenario 2 is more difficult, since allowing partiality on the union shape introduces another dimension of variability, as well as more ambiguity on the possible output; see Figure 3 for examples.

Despite being more difficult to solve, the latter scenario lends itself to modeling more complex interactions. In particular, it allows us to model the composition of $m > 2$ partial shapes simply by aggregating pairwise unions:

$$\Lambda(M_1 \cup M_2 \cup \cdots \cup M_m) = U_\Theta(\cdots(U_\Theta(\Lambda(M_1), \Lambda(M_2)), \cdots), \Lambda(M_m))$$

(5)

See Figure 1 and the inset on the right for an illustration. Note that composing $m$ partial shapes resembles the ‘non-rigid puzzle’ setting seen in [24], although with a crucial difference: the method of [24] has access to the complete shape, which is instead unknown to us.

**4. Deep learning model**

Our learning model takes as input two sequences of $k$ eigenvalues, each associated to a partial shape, and outputs a sequence of $k$ eigenvalues, associated to the union shape.

**Eigenvalue embeddings.** The Laplacian eigenvalues of surfaces form a non-decreasing sequence that grows linearly with rate inversely proportional to surface area, a behavior described by Weyl’s asymptotic law [51]. This results in the input eigenvalues hugely varying depending on the area of the partial shape, resulting in an instability of the network. To mitigate this effect we encode the spectra via the offset representation:

$$\text{off} (\lambda_i) = \lambda_i - \lambda_{i-1},$$

with $\text{off} (\lambda_1) = \lambda_1$. This representation has the further advantage of imposing the increasing order constraint on the predicted eigenvalues, by just requiring the non-negativity of the predicted offset sequence.

In practice, the network sees each spectrum as a sequence of $k$ one-dimensional offset features $\Lambda = (\text{off} (\lambda_1), \ldots, \text{off} (\lambda_k)) \in \mathbb{R}^k$, one per eigenvalue, which are then embedded into a higher-dimensional representation of length $2\ell + 1$, constructed as follows:

$$\text{off} (\lambda_i) \mapsto (\vec{\theta}_1 | \text{off} (\lambda_i) \times \vec{\theta}_2 | \text{off} (\lambda_i)), $$

where $\vec{\theta}_1$ is a $\ell$-dimensional vector acting as a positional encoding for the $i$-th offset, and $\vec{\theta}_2$ is a linear mapping of the offset to a $\ell$-dimensional space. Both $\vec{\theta}_1$ and $\vec{\theta}_2$ are learned.

**Symmetric architecture.** Given the eigenvalue embeddings of $\Lambda_1$ and $\Lambda_2$ (associated to $M_1$ and $M_2$ respectively), our neural architecture learns how to perform their union without ever leaving the spectral domain. We require our model to be fully commutative, i.e., the result should not depend on which pair between $(\Lambda_1, \Lambda_2)$ or $(\Lambda_2, \Lambda_1)$ is given as input.
We gain this invariance by using a single transformer $T_A$, with encoder $E$ and decoder $D$. We first feed $\Lambda_1$ to the decoder $D$ and $\Lambda_2$ to the encoder $E$; this yields an embedding of $\Lambda_1$ that pays attention [48] to $\Lambda_2$. Then we do the symmetric operation, feeding $\Lambda_2$ to $D$, and $\Lambda_1$ to $E$. The two transformed embeddings are summed together to obtain a symmetry-invariant representation of the union.

The latter is fed to a second transformer $T_B$, whose task is to decode the union into a representation that can be easily reduced, via a simple linear layer $\rho$, from the high-dimensional embedding back to a sequence of eigenvalues. The whole architecture is illustrated in Figure 4.

**Training.** Our model is trained with a mean squared error loss $\ell_{10}$ between the predicted and ground truth spectra. Before entering the loss, the offset representation for the eigenvalues is decoded with a cumulative sum. The optimizer used is Adam with a learning rate of $2e-4$ and weight decay of $1e-5$. We use a learning rate scheduler to escape local minima and stabilize the training, in particular the cosine annealing with warm restarts scheduler [25], doubling at each restart the number of epochs between restarts.

**Data and evaluation.** In our experiments we use 3D data from the FAUST [3] and SURREAL [47] datasets of deformable human shapes with different identities. This provides us with a total of 50 different identities, each in 10 different poses. To produce partial data, we first extract surface patches of various sizes from the full shapes, and then combine the patches randomly to form two datasets:

- A dataset of $\sim$150 partial pairs, where each union covers the entire surface. We test in three different settings depending on the information given at training time: (i) known identity, unknown partiality; (ii) unknown identity, known partiality; (iii) both identity and partiality are unknown.
- A dataset of $\sim$100 partial pairs, whose union does not cover the entire surface. For training, the partial shapes are augmented by enlarging/shrinking the patches randomly. We define two test sets. In TEST A, the pose or the type of partiality have never been seen, but the predicted union may be seen in a different pose or identity at training time. TEST B is more challenging, since the union of the two parts has never been seen at training (neither in a different pose or identity, nor as a union of different partialities).

In Table 1 we report a quantitative analysis of the predictive power of our learning model, according to the mean squared error (mse) and mean absolute error (mae) metrics.

As shown in Figure 3, there are cases in which there is more than one valid solution to the union problem. Since we aim at learning a single-valued function, we remove this ambiguity in the training data by following a minimum union area principle and privileging “left-sided” symmetries.

We define two test sets. In TEST A, the pose or the type of partiality have never been seen, but the predicted union may be seen in a different pose or identity at training time. TEST B is more challenging, since the union of the two parts has never been seen at training (neither in a different pose or identity, nor as a union of different partialities).

In Table 1 we report a quantitative analysis of the predictive power of our learning model, according to the mean squared error (mse) and mean absolute error (mae) metrics.

| Data and evaluation | TEST A | TEST B |
|---------------------|--------|--------|
|                      |        |        |
|                      |        |        |

**5. Applications**

We can easily plug our method into existing pipelines that take as input Laplacian eigenvalues. Unique to our approach is that it addresses the scenario in which only partial views of the complete shape are available. We also refer to the Supplementary for further details and results.
Figure 5: Given two partial shapes as input, we compare the reconstruction obtained by running the method of [29] only on a partial input (the green shape), yielding the fourth shape, with the reconstruction obtained from our predicted full spectrum, yielding the last shape.

5.1. Geometry reconstruction

To recover the shape geometry from its predicted union eigenvalues, we use the data-driven method of [29], which takes eigenvalues as input and directly yields a 3D mesh embedding as output. An example is given in Figure 2, where we compare the geometry recovered from our estimated spectra with the one obtained from the ground truth eigenvalues. On the reconstructed surface we plot the Euclidean reconstruction error, where white color corresponds to zero error and dark red encodes a larger error. Our spectrum prediction is accurate enough to retain the core geometric information of the original partial shapes, as it can be seen in these examples. For these experiments we used the pre-trained network provided by the authors of [29], thus the network is not specifically trained to handle spectra predicted by our pipeline. Moreover, since the spectrum encodes just intrinsic properties (i.e. appearance) of the shape, all the reconstructions of [29] are in the T-pose.

To emphasize the importance of having an aggregated spectrum, as predicted by our model, in Figure 5 we show the reconstructions obtained with the method of [29] when using the spectrum of just one of the two partial shapes as input. The result in this case is quite different from what is expected, showing that existing state-of-the-art pipelines are not able to handle partial shapes correctly.

Table 2: Intersection over union (IoU) and accuracy in the region localization task, in different experimental settings. Model trained on a single identity, to show generalization.

|                      | IoU   | Acc.   |
|----------------------|-------|--------|
| known man            | 99.28%| 99.61% |
| unknown man          | 93.78%| 95.83% |
| unknown woman        | 94.19%| 96.32% |
| known man re-meshed  | 98.54%| 99.06% |
| unknown man re-meshed| 91.44%| 94.08% |
| unknown woman re-meshed| 93.47%| 95.56% |
| known man            | 97.96%| 98.55% |
| unknown man          | 87.58%| 92.52% |
| unknown woman        | 89.86%| 94.46% |
| known man re-meshed  | 83.64%| 91.33% |
| unknown man re-meshed| 80.54%| 90.98% |

Table 3: Performance when training on six different identities instead of a single identity (compare with Table 2).

|                      | IoU   | Acc.   |
|----------------------|-------|--------|
| known man            | 98.24%| 99.09% |
| unknown man          | 96.26%| 97.64% |
| unknown woman        | 96.17%| 98.04% |
| known man re-meshed  | 93.70%| 98.74% |
| unknown man re-meshed| 95.88%| 97.78% |
| unknown woman re-meshed| 96.04%| 97.66% |
| known man            | 97.43%| 99.14% |
| unknown man          | 93.31%| 98.23% |
| unknown woman        | 95.74%| 98.59% |
| known man re-meshed  | 97.61%| 99.11% |
| unknown man re-meshed| 90.85%| 97.63% |
| unknown woman re-meshed| 96.81%| 98.98% |

5.2. Region localization

This task, introduced in [35], consists in locating, on a fixed template, the region corresponding to a given partial shape. To solve this problem we devise a simple MLP network that takes as input the eigenvalues of the union, and outputs an indicator function over the vertices of the template. In the loss definition, one must take care of the potential ambiguities exemplified in Figure 3; we do so by implementing a symmetry-invariant loss, that does not penalize symmetric solutions. The MLP is trained using the train/test splits described in Section 4, with the difference
Given the eigenvalues of two partial shapes, we correctly predict an indicator function that represents the union of the two over a fixed template.

that we used just 6 different identities in the training phase.

To analyze the prediction quality on this task we adopt two metrics: intersection over union (IoU) of the predicted mask with the ground truth mask, and accuracy, i.e. the ratio of correctly predicted vertices over the full template. We show several qualitative results in Figure 6.

Robustness to remeshing. One key aspect of Laplacian eigenvalues is that they are robust to shape discretization and mesh connectivity. Our model inherits this robustness; see Figure 6, where we highlight the remeshed inputs by visualizing their surface triangulation. This is supported also by Tables 2 and 3, where the performance on the remeshed shapes is comparable with the original ones. In these experiments, we test our network with the eigenvalues computed from noisy, re-meshed partial shapes obtained by removing 30% of their vertices with an edge collapse algorithm [14].

Generalization to new identities. Our approach generalizes to identities unseen at training time as can be noted in Table 3. To further stress this aspect, we devised an experimental setup in which we used as training set just a single identity. The results of this setup are shown in Table 2.

Generalization to different datasets. In Figure 7 we use partial shapes from other datasets to localize regions on the fixed template. These shapes have different triangulation, vertex density and style, confirming generalization across datasets. More specifically: a shape from TOSCA [5] for humans (first row), one from SMAL [54] for the horses (second row) and a camel shape that has a different triangulation and comes from a different class (third row).

Generalization to point clouds. We obtain good results also on point clouds, as shown in Figure 8. For earphones, in the top row, we perform both training and testing on point clouds. In the bottom row, we show that our model trained on human meshes generalizes to point clouds. We compute the Laplacian for point clouds with the method of [43].

Compositionality. We can compute spectral unions of > 2 partial shapes hierarchically as described in Section 3. In Figure 9 we show qualitative results over three parts.

Interpolation. Finally, in Figure 10 we first interpolate the spectra of two partial shapes (in green), and then compute the union of the interpolated spectra with the spectrum of a fixed shape (in red). From each of these unions, we predict a mask on the given template (in yellow). We can see how
in the first example (top row) the mask changes smoothly. On the other hand, in the second example it is less obvious how to interpolate the completely missing leg, resulting in an abrupt discontinuity in the predicted mask.

5.3. Shape retrieval

This task consists in retrieving a query from a database of shapes that could undergo several deformations. A well-known spectral method to tackle this problem, ShapeDNA [36], adopts the Laplacian spectrum as a shape signature. In the space of these signatures, nearest-neighbor search yields the desired result. However, in order to work correctly, ShapeDNA needs the spectrum of a complete shape; extensions of this signature to the partial case have proven unsuccessful to date [39]. Our method applies directly to this case, since we can estimate the eigenvalues of the unknown complete shape whenever the input query is just a collection of its partial views.

We run our tests on a dataset of 440 complete shapes (44 identities in 10 poses each). For our method, we evaluate 4400 pairs of partial shapes; for each pair we predict its ShapeDNA signature, and use it to query the database. We compare with the accuracy obtained by standard ShapeDNA on each of the 440 complete shapes in the database. We measure the performance using top-k metrics, which count the number of times a shape with the correct identity is in the first k retrieved shapes; we use k = 1, 5, 10. The results are reported in Table 4, and show that our predicted eigenvalues are accurate enough to compete with, and even surpass, ShapeDNA for this task. The better performance is due to the robustness of our method to the noise induced by the pose change.

### Table 4: Comparisons on the shape retrieval task.

|       | top-1 | top-5 | top-10 |
|-------|-------|-------|--------|
| Ours  | 86.14%| 97.75%| 99.20% |
| ShapeDNA | 86.59%| 96.81%| 97.72% |

Figure 8: Region localization from partial point cloud spectra. The white mesh is just shown as a visual reference.

Figure 9: Example of compositionality. Note that all the human meshes involved have different connectivity.

Figure 10: Two examples of linear interpolation of eigenvalues (green shapes), and the resulting predicted masks (in yellow). Please refer to the main text for details.
6. Conclusion

We introduced a method to recover the aggregated Laplacian spectrum of a collection of partial deformable shapes, while avoiding the computational burden of computing correspondences or extrinsic alignments. Our method involves a deep net that, given two eigenvalue sequences as input, simply produces another eigenvalue sequence as output. In spite of its apparent simplicity, this method allows to address a number of applications that traditionally require solving for a correspondence, and retains a comparable number of empirical results. Enforcing this guarantee can be done by injecting a spectral term in the loss, at the cost of increased training time. We keep this possibility as an interesting direction of further research.

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A. Appendix A

We report additional results that due to lack of space were not included in the main manuscript. Further, we describe with more details the architectures involved in the proposed method.

A.1. Additional results

In this section, we collect additional results for the experiments and applications described in the main manuscript.

Additional results on geometry reconstruction In Fig. 11, we report additional examples of shape-from-spectrum recovery. These qualitative results confirm that we outperform [29] when applied to partial shapes.

Compositionality with more than 3 partial shapes In Fig. 12 we show an example of hierarchical spectral union, with four different partial shapes. Deeper hierarchical unions are more difficult since the prediction error in each step is amplified by the subsequent steps.

Additional results on the horse class We report additional results on the horse class. In these experiments, the spectral union operator is pre-trained on humans and fine-tuned to horses from the TOSCA dataset. Then, a region localization model is trained specifically for horses, slightly modified to account for the different number of vertices in
In Figure 13 we report qualitative results of region localization on horses; in Figure 14 we show composition examples; in Figure 15 we present qualitative results on horses with different triangulation, vertex density and style with respect to the horses used in the training phase; in Figure 16 we show that our method is able to generalize to non-isometric but similar enough deformations.

**Additional results on Point clouds** To show the flexibility of our approach we consider a further class of shapes composed by airplanes from [8] represented as point clouds. We report in Figure 17 some qualitative results on the region localization task with this class. We also report additional results on headphones [32] in Figure 18. These results show that our model generalizes to different source geometries, as long as the class shape does not change.

**A.2. Architecture**

In this section, we describe in detail the proposed neural architecture. Note that since surface area directly affects the magnitude of the eigenvalues, at test time the shapes are normalized to have the same area of the shapes seen at training time.
A.2.1 Spectral union model

In Fig. 19 we show the detailed architecture of the spectral union model.

Hyperparameters The dimensionality of each embedding is 32. $T_A$ has 8 heads, 6 layers, the dimensionality of the internal feed-forward layer is 64 and the dropout is 0.1. $T_B$ has 8 heads, 3 layers, the dimensionality of the feed-forward is 32 and the dropout is 0.1. Thus, $\rho$ reduces the embedding dimensionality from 32 to 1.

Training The model is trained until convergence. The training randomly augments online each input independently. The batch size is 32. The optimizer used is Adam with learning rate of $2e^{-4}$ and weight decay $1e^{-5}$. The learning rate changes according to cosine annealing with warm restarts scheduler and it restarts every 10 epochs, doubling the number of epochs between restarts at each restart.

A.2.2 Region Localization model

In Fig. 20 we show the detailed architecture of the region localization model for humans.

Hyperparameters The dense layers increase the dimensionality of the input sequence from 20 to 6890, for humans, i.e. the number of vertices in the fixed template. In particular, the layers apply the following transformations $20 \rightarrow 1300 \rightarrow 2600 \rightarrow 3900 \rightarrow 5200 \rightarrow 6890$. The dropout is always set to $p = 0.5$.

Training The model is trained to localize the region from both the predicted union eigenvalues and all the ground-truth eigenvalues, to which we add random noise. The model is early stopped, monitoring the IoU metric on a validation set. The batch size is 32. The optimizer used is Adam with learning rate of $5e^{-5}$ and weight decay $1e^{-6}$. The scheduler adopted is again the cosine annealing with warm restarts, with the same hyperparameters.

A.2.3 Data processing for point clouds

The aeroplanes from [8] and headphones from [32] are point clouds with semantic segmentation.

We performed some data processing to: (1) extract shapes with only given segments (e.g., discarding earphones or strange headphones), (2) extract random partialities from each shape and (3) find the segment-level matching between each shape and a fixed template for the region localization task. We did this by defining the graph of the segments for each shape, then searching for sub-graphs with determined properties for (2) and solving the graph isomorphism against the template for (3).

We obtained 75 headphones and 964 aeroplanes for the training set, we extract random pairs of partialities from each shape.

A.3. Code release

This supplementary material is accompanied by an interactive demo to reproduce the compositionality examples on humans. The demo contains the code for the region localization task and a pre-trained model, which must be downloaded from an external link due to its size.
Figure 17: Region localization on aeroplanes. The model is trained and tested on point clouds.
Figure 18: Region localization on headphones, trained and tested on point clouds. In the examples on the right column, despite significant changes in the geometry of the partialities, the model localizes the same correct region.
Figure 19: Detailed architecture of the spectral union operator.
Figure 20: Detailed architecture of the region localization MLP for humans.