Totally irregular total labeling of some caterpillar graphs

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Abstract

Assume that $G(V, E)$ is a graph with $V$ and $E$ as its vertex and edge sets, respectively. We have $G$ is simple, connected, and undirected. Given a function $\lambda$ from a union of $V$ and $E$ into a set of $k$-integers from 1 until $k$. We call the function $\lambda$ as a totally irregular total $k$-labeling if the set of weights of vertices and edges consists of different numbers. For any $u \in V$, we have a weight $wt(u) = \lambda(u) + \sum_{uy \in E} \lambda(uy)$. Also, it is defined a weight $wt(e) = \lambda(u) + \lambda(uv) + \lambda(v)$ for each $e = uv \in E$. A minimum $k$ used in $k$-total labeling $\lambda$ is named as a total irregularity strength of $G$, symbolized by $ts(G)$. We discuss results on $ts$ of some caterpillar graphs in this paper. The results are $ts(S_{p,2,2,q}) = \lceil \frac{p+q-1}{2} \rceil$ for $p, q$ greater than or equal to 3, while $ts(S_{p,2,2,2,p}) = \lceil \frac{2p-1}{2} \rceil$, $p \geq 4$.

Keywords: totally irregular total $k$-labeling, total irregularity strength, weight, caterpillar graph.

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1. Introduction

Graph theory is one of branch of mathematics. In this field, many real life problems can be solved, especially on optimization problem [8]. Given a graph $G(V, E)$ which is assumed as connected, simple, and undirected graph. A function that assigns a set of elements (vertex/edge) of $G$ into a set of integers is mentioned as labeling (Wallis [12]). The labeling is said to be a total labeling if the domain is a union of vertex and edge sets.

A function $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ is named a vertex irregular total $k$-labeling if $wt_f(u) \neq wt_f(v)$ for each $u \neq v \in V(G)$, where $wt(u) = f(u) + \sum_{uz \in E} f(uz)$ [1]. A minimum $k$ in which there exists a vertex irregular total $k$-labeling of $G$ is named as a total vertex irregularity strength (tvs) of $G$. Indriati et al. [4] obtained tvs of generalized helm. Recently, the tvs of comb product of two cycles and two stars has been found in [10]. Meanwhile, Nurdin et al. [11] proved tvs of tree $T$ which does not have vertex of degree two and has $n$ pendant nodes, i.e.

$$tvs(T) = \left\lceil \frac{n + 1}{2} \right\rceil.$$ (1)

Further, a total $k$-labeling $g$ that assigns a union of $V$ and $E$ into $\{1, 2, \ldots, k\}$ is called an edge irregular when the requirement $wt(xy) \neq wt(x'y')$ is satisfied for each pair $xy \neq x'y'$ in $E(G)$ with $wt(xy) = g(x) + g(xy) + g(y)$. Bača et al. [1] mentioned the minimum $k$ required in labeling $g$ as a total edge irregularity strength (tes) of $G$. The exact value of tes of generalized web graphs was given in [2]. Recent research has found tes of some $n$-uniform cactus chain graphs and related chain graphs [6]. In addition, tes of any tree has been given in [7], i.e. $tes(T)$ is equal to

$$\max \left\{ \left\lceil \frac{|E(T)| + 2}{3} \right\rceil, \left\lceil \frac{(\Delta(T) + 1)}{2} \right\rceil \right\}.$$ (2)

Furthermore, the total $k$-labeling $g$ becomes a totally irregular total $k$-labeling if the set of all weights of vertices and edges contains distinct numbers [9]. A minimum $k$ needed in the labeling $g$ is named as total irregularity strength (ts) of $G$. Marzuki, et al. observed

$$ts(G) \geq \max \{tes(G), tvs(G)\}.$$ (3)

Different with tes and tvs, the value of ts of tree has not been obtained. In order to find ts of tree, we have started the investigation for double stars $S_{p,q}$ and related graphs $S_{p,2,q}$ ([3], [5]). In this research, we verify ts of caterpillar graphs $S_{p,2,2,q}$ and $S_{p,2,2,2,p}$.

We use the notion of caterpillar $S_{p,2,2,q}$. It is a graph which is formed from double-star $S_{p,q}$ by putting two vertices on the path which are connected to the two centers of stars in $S_{p,q}$. The value of tes of graph $S_{p,2,2,q}$ can be found by (2), that is

$$tes(S_{p,2,2,q}) = \max \left\{ \left\lceil \frac{\max\{p, q\} + 1}{2} \right\rceil, \left\lceil \frac{p + q + 3}{3} \right\rceil \right\} = \left\lceil \frac{p + q + 3}{3} \right\rceil.$$ (4)

This graph has two vertices of degree two. Therefore, (1) cannot be used for determining tvs of this graph. The next theorem gives this parameter.
Theorem 1.1. Let $S_{p,2,2,q}$ be a caterpillar with $p,q$ greater than or equal to 3. The graph $S_{p,2,2,q}$ has

$$tvs(S_{p,2,2,q}) = \left\lceil \frac{p+q-1}{2} \right\rceil.$$  

Proof. Without loss of generality, we can assume that $p \leq q$. We know that $S_{p,2,2,q}$ contains $p+q-2$ pendants, two vertices with degree two, one vertex with degree $p$, and one vertex of degree $q$. The smallest weight of each vertex is at least two. Each pendant vertex has the smallest weight which is not less than $p+q-1$, i.e. the weight is a sum of two labels. Then, the largest number to label pendant vertices is not less than $\left\lceil \frac{p+q-1}{2} \right\rceil$.

The graph $S_{p,2,2,q}$ consists of $V(S_{p,2,2,q}) = \{v_r^1 : 1 \leq r \leq q-1\} \cup \{v_r^4 : 1 \leq r \leq p-1\} \cup \{v_s^s : s = 1,2,3,4\}$ and $E(S_{p,2,2,q}) = \{v_r^1v_r^1 : 1 \leq r \leq q-1\} \cup \{v_r^4v_r^4 : 1 \leq r \leq p-1\} \cup \{v_s^sv_s^{s+1} : s = 1,2,3\}$.

Next we will distinguish the following three cases, i.e. $p = q = 3$, $p = q \geq 4$ and $3 \leq p < q$. Assume $k = \left\lceil \frac{p+q-1}{2} \right\rceil$ for all cases, and define a total $k$-labeling $\lambda$ on each element $x \in V(S_{p,2,2,q}) \cup E(S_{p,2,2,q})$ as follows.

| $x$   | $\lambda(x)$ | Case for $p,q$ |
|-------|---------------|----------------|
| $v_r^1$ | $1$, $1 \leq r \leq p-1$; $s = 1$ | $p = q \geq 3$ |
|        | $1$, $1 \leq r \leq p-1$; $s = 4$ | $p = q \geq 3$ |
|        | $1$, $1 \leq r \leq k$; $s = 1$ | $3 \leq p < q$ |
|        | $r - k + 1$, $k + 1 \leq r \leq q - 1$; $s = 1$ | $3 \leq p < q$ |
|        | $q - k + r$, $1 \leq r \leq p - 1$; $s = 4$ | $3 \leq p < q$ |
| $v_s^s$ | $1$, $s = 1, 2$ | $p = q = 3$ |
|        | $k$, $s = 3, 4$ | $p = q = 3$ |
|        | $1$, $s = 1, 3$ | $4 \leq p = q$ |
|        | $2$, $s = 2$ | $4 \leq p = q$ |
|        | $4$, $s = 4$ | $4 \leq p = q$ |
|        | $1$, $s = 1$ | $3 \leq p < q$ |
|        | $\left\lceil \frac{p-q+5}{2} \right\rceil$, $s = 2$ | $3 \leq p < q$ |
|        | $\left\lceil \frac{p-q}{2} \right\rceil$, $s = 3$ | $3 \leq p < q$ |
|        | $4$, $s = 4$ | $3 \leq p < q$ |
We observe that each vertex and each edge has been labeled with a number which is at most $k = \left\lceil \frac{p+q-1}{2} \right\rceil$. Further, each vertex $x \in V(S_{p,2,2,q})$ has a weight as follows.

| $x$       | $\lambda(x)$ | Case for $p, q$ |
|-----------|--------------|-----------------|
| $v^s v^r_*$ | $r$, $1 \leq r \leq p-1; s = 1$ | $p = q \geq 3$ |
|           | $k$, $1 \leq r \leq p-1; s = 4$ | $p = q = 3$ |
|           | $i$, $1 \leq r \leq k; s = 1$ | $3 \leq p < q$ |
|           | $k$, $k+1 \leq r \leq q-1; s = 1$ | $p, q \geq 4$ |
|           | $k$, $1 \leq r \leq p-1; s = 4$ | $p < q; p, q \geq 3$ |

| $v^s v^{s+1}$ | $k$, $s = 1, 3$ | $p = 3$ |
|               | $k - 1$, $s = 2$ | $p = q = 3$ |
|               | $p - 1$, $s = 1, 3$ | $p, q \geq 4$ |
|               | $p$, $s = 2$ | $3 \leq p < q$ |
|               | $\left\lceil \frac{p+q-4}{2} \right\rceil$, $s = 1$ | |
|               | $p$, $s = 2$, | |
|               | $k$, $s = 3$ | |

It is shown above, each vertex has a distinct weight under total labeling $f$. Therefore, $tvs(S_{p,2,2,q}) = k = \left\lceil \frac{p+q-1}{2} \right\rceil$.

Furthermore, an exact value of $ts$ of $S_{p,2,2,q}$ is proved in the next theorem.

**Theorem 1.2.** Given a caterpillar $S_{p,2,2,q}$ with $p, q$ greater than or equal to 3. We get

$$ts(S_{p,2,2,q}) = \left\lceil \frac{p + q - 1}{2} \right\rceil.$$
Proof. According to (3), by using Equality (4) and Theorem 1.1, the lower bound is as follows:

\[ ts(S_{p,2,2,q}) \geq \max \left\{ \left\lceil \frac{p + q + 3}{3} \right\rceil, \left\lceil \frac{p + q - 1}{2} \right\rceil \right\} = \left\lceil \frac{p + q - 1}{2} \right\rceil. \]  

(5)

Furthermore, we use total k-labeling \( \lambda \) constructed in Theorem 1.1 to get a totally irregular total k-labeling. Under labeling \( \lambda \), we obtain the edge-weights below.

**Case 1:** For \( p = q = 3 \).

\[
wt(v^s_v^s) = \begin{cases} 
  r + 2, & 1 \leq r \leq p - 1, \ s = 1, \\
  2k + r, & 1 \leq r \leq p - 1, \ s = 4.
\end{cases}
\]

\[
wt(v^s_v^{s+1}) = \begin{cases} 
  k + 2, & s = 1, \\
  2k, & s = 2, \\
  3k, & s = 3.
\end{cases}
\]

**Case 2:** For \( p = q \geq 4 \) and \( 3 \leq p < q \).

\[
wt(v^s_v^s) = \begin{cases} 
  r + 2, & 1 \leq r \leq q - 1, \ s = 1, \\
  q + 4 + r, & 1 \leq r \leq p - 1, \ s = 4.
\end{cases}
\]

\[
wt(v^s_v^{s+1}) = \begin{cases} 
  q + 2, & s = 1, \\
  q + 3, & s = 2, \\
  q + 4, & s = 3.
\end{cases}
\]

It can be seen that each edge has a different weight. This concludes that \( \lambda \) is totally irregular total k-labeling. Thus, \( ts(S_{p,2,2,q}) = k = \left\lceil \frac{p + q - 1}{2} \right\rceil \). \( \square \)

2. A graph \( S_{p,2,2,2,p} \)

A graph that is formed from the double-star \( S_{p,p} \) by inserting three vertices on the path connecting two centers of the two stars in \( S_{p,p} \) is called as a caterpillar \( S_{p,2,2,2,p} \). Hence, \( S_{p,2,2,2,p} \) is a kind of tree with \(|E(S_{p,2,2,2,p})| = 2p + 2\) and it has maximal degree \( \Delta = p \). Based on (2), \( tes \) of \( S_{p,2,2,2,p} \) is

\[
tes(S_{p,2,2,2,p}) = \max \left\{ \left\lceil \frac{p + 1}{2} \right\rceil, \left\lceil \frac{2p + 4}{3} \right\rceil \right\} = \left\lceil \frac{2p + 4}{3} \right\rceil.
\]

(6)

Meanwhile, \( tvs \) of \( S_{p,2,2,2,p} \) is given in Theorem 2.1.

**Theorem 2.1.** If \( S_{p,2,2,2,p} \), \( p \geq 4 \) is a caterpillar with \( p \geq 4 \), then

\[ tvs(S_{p,2,2,2,p}) = p. \]
Proof. The graph $S_{p,2,2,2,p}$ is a tree that consists of $2p - 2$ pendant vertices, three vertices of degree two, and it has two vertices with degree $p \geq 4$. By the similar reason as in Theorem 1.1 we get $tvs(S_{p,2,2,2,p}) \geq p$. Let $V(S_{p,2,2,2,p}) = \{v_r^s : 1 \leq r \leq p - 1, s = 1, 5\} \cup \{v^s : s = 1, 2, 3, 4, 5\}$ and $E(S_{p,2,2,2,p}) = \{v^sv_r^s : 1 \leq r \leq p - 1, s = 1, 5\} \cup \{v^sv^{s+1} : s = 1, 2, 3, 4\}$. To find $tvs$ of $S_{p,2,2,2,p}$, we create a total labeling $f$ of an element $x$, $x \in V(S_{p,2,2,2,p}) \cup E(S_{p,2,2,2,p})$ as follows.

| $x$ | $f(x)$ | Case for $p$ |
|-----|--------|-------------|
| $v_r^s$ | $1$, $1 \leq r \leq p - 1; s = 1$, $r$, $1 \leq r \leq p - 1; s = 5$ | $p \geq 4$ |
| $v^sv_r^s$ | $r$, $1 \leq r \leq p - 1; s = 1$, $\left\lceil \frac{2p - 1}{2} \right\rceil$, $1 \leq r \leq p - 1; s = 5$ | $p \geq 4$ |
| $v^s$ | $1$, $s = 1, 2$, $2$, $s = 3$, $4$, $s = 4, 5$ | $p = 4$ |

| $x$ | $f(x)$ | Case for $p$ |
|-----|--------|-------------|
| $1$, $s = 1, 2$, $2$, $s = 3, 4$, $5$, $s = 5$ | $p \geq 5$ |
| $v^sv^{s+1}$ | $p$, $s = 1, 2, 4$, $p - 2$, $s = 3$, $p$, $s = 1, 2, 3$, $p - 2$, $s = 4$ | $p = 4$, $p \geq 5$ |
Under labeling $f$, we can see that each vertex has label at most $\lceil \frac{2p-1}{2} \rceil$.

| $x$ | $wt(x)$ | Case for $p$ |
|-----|---------|-------------|
| $v^r_r$ | $r + 1$, $1 \leq r \leq p - 1; s = 1$ | $p \geq 4$ |
|      | $p + r$, $1 \leq r \leq p - 1; s = 5$ |             |
| $v^s_s$ | $1/2(p^2 + p) + 1$, $s = 1$ | $p \geq 4$ |
|      | $2p + 1$, $s = 2$ | $p \geq 4$ |
|      | $2p$, $s = 3$ |             |
|      | $2p + 2$, $s = 4$ | $p = 4$ |
|      | $5p$, $s = 5$ |             |
|      | $2p + 2$, $s = 3$ | $p \geq 5$ |
|      | $2p$, $s = 4$ |             |
|      | $p^2 + 3$, $s = 5$ |             |

Moreover, the weight for each $x \in V(S_{p,2,2,2,p})$ is shown above. We can see that each vertex has a distinct weight. Therefore, $tvs(S_{p,2,2,2,p}) = k = \lceil \frac{2p-1}{2} \rceil$.

The exact value of $ts$ of $S_{p,2,2,2,p}$ is discussed in the next theorem.

**Theorem 2.2.** If $S_{p,2,2,2,p}$ is a caterpillar with $p \geq 4$, then

$$ts(S_{p,2,2,2,p}) = p.$$  

**Proof.** Based on (3), by using Theorem 2.1 and Equality (6) we get the lower bound of $ts$ of $S_{p,2,2,2,p}$ as follows:

$$ts(S_{p,2,2,2,p}) \geq \max\left\{ \left\lceil \frac{2p + 4}{3} \right\rceil, p \right\} = p.$$  

To construct totally irregular total $k$-labeling, we use the vertex irregular total $k$-labeling $f$ defined in Theorem 2.1. Under labeling $f$, we get the edge-weights as follows.

| $xy$ | $wt(xy)$ | Case for $p$ |
|-----|---------|-------------|
| $v^s_s v^r_r$ | $r + 2$, $1 \leq r \leq p - 1; s = 1$ | $p \geq 4$ |
|      | $2p + r$, $1 \leq r \leq p - 1; s = 5$ | $p = 4$ |
|      | $p + 5 + r$, $1 \leq r \leq p - 1; s = 5$ | $p \geq 5$ |
| $v^s_s v^{s+1}$ | $p + 2$, $s = 1$ | $p \geq 4$ |
|      | $p + 3$, $s = 2$ |             |
|      | $p + 4$, $s = 3$ |             |
|      | $3p$, $s = 4$ | $p = 4$ |
|      | $p + 5$, $s = 4$ | $p \geq 5$ |

It is obvious that each edge has a different weight. Hence, the labeling $f$ is desired a totally irregular total $k$-labeling with $ts(S_{p,2,2,2,p}) = k = p$; $p \geq 4$.

**Conjecture 1.** For $p, q \geq 4$: $ts$ of $S_{p,2,2,2,q}$ is $\lceil \frac{p+q-1}{2} \rceil$.  

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