A New Class of Elliptic Finite-Gap Densities of the Polar Operator and Stationary Solutions of the Harry Dym equation

L. A. Dmitrieva, D. A. Pyatkin
Department of Mathematical 
& Computational Physics, St.-Petersburg State University, 
198504 St.-Petersburg, Russia
e-mail: mila@JK1454.spb.edu
danila@DP2627.spb.edu

Abstract

A new family of one- and two-gap elliptic densities of the polar operator has been constructed by modifying the so-called ”higher time approach” to constructing finite-gap solutions of the Harry Dym equation. Auto-Bäcklund transformations of the obtained stationary solutions have been constructed and their properties have been studied.

1 Introduction

The present work is devoted to constructing elliptic one- and two-gap densities of the polar operator

\[ L = -r^2(x) \frac{d^2}{dx^2}, \]

with periodic function \( r(x): r(x + T) = r(x) \) on the real axis \( x \in \mathbb{R} \). Function \( r(x) \) is referred to as density of the polar operator. It is easy to see that the operator \( L \) is intimately connected with the string equation

\[ \psi''_{xx} + \lambda \rho(x) \psi = 0 \]  

provided that \( \rho(x) = r^{-2}(x) \) where \( \rho(x) \) is the density of the string and \( \psi \) the amplitude of its oscillation. We construct and study functional properties of some finite-gap densities of the operator \( L \), namely 1-gap and 2-gap elliptic densities of the operator \( L \).

Despite classical character of the spectral problem \( L \) for the operator \( L \) and a great number of papers devoted to the string equation (see works \( \text{[1]}, \text{[2]}, \text{[3]}, \text{[4]}, \text{[5]} \) and reference therein to name a just few), hitherto there weren’t any explicit analytical formulae for the finite-gap densities \( r(x) \) with the exception of expression for smooth 1-gap density constructed explicitly in \( \text{[6]} \). Here by explicit representation we assume representation of the form

\[ r(x) = R(y), \]

where \( y = y(x) \) is solution of the functional equation

\[ x = F(y), \]

1) we call a density elliptic if it can be expressed in terms of elliptic functions.
and the functions \( R \) and \( F \) are given explicitly.

The absence of analytical formulae for the finite-gap densities doesn’t seem strange despite the relation between the string equation \([12]\) and the one-dimensional Schrödinger equation

\[-\Omega''_{yy} + u(y) \Omega = \lambda \Omega\]  

(13)

for which the class of finite-gap potentials is well known \([8]\). This relation has the form

\[u(y) = \frac{1}{2} r^2_x(x) - \frac{1}{4} r(x) r_{xx}(x) = \left( R^{-1/2} \right)_{yy} R^{1/2}\]

\[y = \int \frac{dx'}{r(x')}\]

\[\psi(x, \lambda) = \Omega(y, \lambda) r^{1/2}(x)\]

So if the density \( r(x) \) is known then the potential \( u(y) \) can be reconstructed easily. However the above relations can not be used for effective constructing the densities of the polar operator directly.

At the same time the fact of finite-gap densities’ existence was proved by M. G. Krein more than 40 years ago. In the paper \([1]\) a theorem was stated (without proof) asserting that there exist \( 2^{N-1} \) \( N \)-gap densities of the string equation provided that the left edge of the operator’s \( L \) zonal spectrum is situated at the point \( \lambda = 0 \). The last condition in the Krein’s theorem is essential \([2]\), since in case of it’s violation generally speaking the number of \( N \)-gap densities is greater than \( 2^{N-1} \). In the present paper we illustrate this effect by studying arrangement of the operator’s spectrum in regard to the point \( \lambda = 0 \) and in case of 1- and 2-gap elliptic densities show that the set of all 1- and 2-gap densities is considerably wider and contains also some singular densities which were out of consideration in \([1]\).

In order to construct smooth and singular 1- and 2-gap densities of the operator \( L \) we make use of the relation of linear spectral problem for the polar operator \( L \) with solutions of nonlinear evolutionary integrable Harry Dym equation \([10]\):

\[r_t + r^3 r_{xxx} = 0, \quad r = r(x, t),\]  

(14)

and it’s higher analogues \([3]\). This relation is based on the fact that \([6]\) holding the time variable \( t \) in the solution of the higher Harry Dym equation the resulting function turns out to be finite-gap density of the string equation. Because of this and the purpose of our paper (as we interested mostly in densities of operator \( L \)) we will not distinguish between densities of the string equation and solutions of Harry Dym equation.

For the first time the finite-gap solutions of the Harry Dym hierarchy were constructed in \([6]\) using higher times method developed in \([7]\). In \([6]\) the finite-gap solutions were constructed in terms of multi-dimensional Riemann \( \Theta \)-functions and needed considerable effectivization. In the present paper we modify the scheme for construction finite-gap solutions of the Harry Dym equation and reduce the \( \Theta \)-functions form in case of 2-gap densities to elliptic functions. This reduction becomes possible after the Riemann surface of the polar operator’s Bloch functions is chosen to possess symmetry of the special sort.

Along with higher times technique we use auto-Bäcklund transformation \([11]\) for higher equations of Harry Dym hierarchy. We show that auto-Bäcklund transformation allows one to construct new 1-gap densities of the polar operator. We also construct auto-Bäcklund transformations in 2-gap case and study their properties leaving the property of spectrum preservation as hypothesis.

---

2) note that location of the Schrödinger zonal spectrum’s edge in the point \( \lambda = 0 \) is not essential

3) hierarchy of the Harry Dym equations is described later
2 Modified scheme for construction of finite-gap solutions of Harry Dym equation.

The scheme for construction of \( N \)-gap \((N < \infty)\) solutions of the Harry Dym equation proposed in [6] is the following. The \( N \)-gap solution of the eq. (14) \( r(x, t_1) \) is assumed to depend on \( N - 1 \) additional parameters — higher times \( t_2, \ldots, t_N \) in such a way that solution \( r(x, t) \equiv r(x, t_1, \ldots, t_N) \) with respect to \( t_m, 1 \leq m \leq N \), obeys the \( m \)-th higher Harry Dym equation:

\[
rt_m + r^3 (\partial_x^2 r Ir)^m r^{-3} r_x = 0, \quad m = 1, \ldots, N.
\] (25)

Here the operator \( I \) is defined by equations

\[
I \partial_x r - 1 = r - 1 - 1 \quad \text{and} \quad I \partial_y \varphi(x, t) = \varphi(x, t),
\]

if \( \varphi \) is any differential polynomial of \( r \).

The \( N \)-gap solutions of the system (25) were obtained [6] in the form:

\[
r(x, t) = R(y, t) = 1 + \langle \alpha, f(y, t) \rangle
\] (26)

where \( y = x + \varepsilon(x, t) \). The phase function \( \varepsilon(x, t) \) is related with solution \( r(x, t) \) by equation \( \varepsilon_x = 1/r - 1 \) and finally should be determined by solving functional equation

\[
\varepsilon(x, t) = E(y, t) = \int_0^y (1 - R(y', t)) dy'.
\]

In (26) by \( < \cdot, \cdot > \) we denote the inner product in \( \mathbb{R}^N \). The constant vector \( \alpha \) and the function \( f(y, t) \) have been described explicitly in [6] in terms of \( N \)-dimensional Riemann \( \Theta \)-function and the periods of holomorphic differentials on the hyperelliptic curve

\[
\Lambda^2 = \prod_{i=0}^{2N} (\lambda - E_i), \quad E_i \neq E_j
\] (27)

with real numbers \( E_i \) coinciding with edges of spectral gaps of the polar operator (11) associated with Harry Dym equation.

Remind that explicit expressions for \( f_m(y, t) \) were obtained in [6] basing on the relation

\[
f_m = \varepsilon t_m(x, t),
\]

and on the link [11] between Harry Dym and Korteweg-de Vries (KdV) hierarchies. Namely let \( u(y, t_1, \ldots, t_N) \) be a solution of the first \( N \) equations of the KdV hierarchy, then

\[
u_{t_m} = \frac{1}{2} \partial_y \varepsilon_{t_{m+1}}, \quad m = 1, \ldots, N.
\] (28)

In particular \( u = -1/2 \varepsilon t_1 \).

In the present paper we modify the scheme of construction of finite-gap solutions of the Harry Dym equation using Hamiltonian structure of the KdV hierarchy. Namely the system of the first \( N \) equations of the hierarchy

\[
u_{t_m} = -L^m u_y, \quad m = 1, \ldots, N,
\]

where

\[
L = \partial_y^2 - 4u - 2u_y \int_0^y dy.
\]

can be expressed in Hamiltonian form:

\[
u_{t_m} = (-1)^m \partial_y \left( \frac{1}{2} \delta C_{m+1}[u] \right).
\] (29)
Here the functionals $C_m[u]$ represent the countable set of conservation laws of KdV equation:

$$C_m = (-1)^m(2m + 1)^{-1} \int_{-\infty}^{\infty} dy L^m\{y u_{yy} + 2u_y(y, t)\},$$  \hspace{1cm} (210)$$

and the symbol $\delta/\delta u$ in (29) denotes variational derivative:

$$\frac{\delta}{\delta u} = \sum_{j=0}^{\infty} \left( -\frac{d}{dx} \right)^j \frac{\partial}{\partial u^{(j)}}. \quad u^{(j)} = \frac{\partial^j u}{\partial y^j}$$

In case of periodic function $u$ one should regard integrals in (210) as integrals over period of $u$. But for us the most important fact is that from (24), (210) it is readily seen that variational derivatives of conservation laws $C_m[u]$ appear to be differential polynomials of $u(y, t)$. Comparing (28) and (29) we obtain

$$\varepsilon_{\ell_m} = (-1)^m \frac{\delta C_m[u]}{\delta u} + \beta_m,$$  \hspace{1cm} (211)$$

where $\beta_m$ are constants which in periodic case can’t be deduced from relations (28) and (29). Computation of $\beta_m$ can be carried out in each particular case (1-gap, 2-gap, etc.).

Thus the meaning of the scheme of solutions construction modification proposed in the present paper is in representation of $f_m(y, t) = \varepsilon_{\ell_m}$ in the form of differential polynomial of $u(y, t)$ by virtue of (211). Then, knowing the solution of KdV equation possessing desired spectral properties we can compute all the necessary $f_m(y, t)$ and putting them in (26) obtain solution of the Harry Dym equation possessing the same spectrum.

In particular from (211) and (210) it follows that

$$f_1(y, t) \equiv \varepsilon_{t_1} = -2u + \beta_1$$
$$f_2(y, t) \equiv \varepsilon_{t_2} = 6u^2 - 2uu_y + \beta_2$$
$$f_3(y, t) \equiv \varepsilon_{t_3} = -2[u_y^3 - 10uu_{yy}^2 - 5(u_y)^2 + 10u^3] + \beta_3$$

3 Construction of one-gap densities of polar operator

In order to construct a family of one-gap densities of the polar operator we start with smooth one-gap solution of the KdV equation of the form

$$u(y, t_1) = -2\wp(i(y + 4C t_1) + \omega) + \frac{2}{3} C.$$  \hspace{1cm} (314)$$

Here $\wp(z)$ is elliptic Weierstrass function defined in a standard manner by real parameters $e_j$, $j = 1, 2, 3$ such that $e_3 < e_2 < e_1$, $\sum_j e_j = 0$; $\omega$, $\omega'$ are correspondingly real and purely imaginary semi-periods of the function $\wp(z)$: $\wp(\omega_1|\omega, \omega') = e_\alpha$, $\alpha = 1, 2, 3$, where

$$\omega_1 = \omega, \quad \omega_2 = \omega + \omega', \quad \omega_3 = \omega',$$

and $C \in \mathbb{R}$ is arbitrary constant.

Since in what follows we are interested in the densities of the polar operator rather than solutions of the Harry-Dym equation, everywhere below we omit the dynamics simply putting $t = 0$. However it should be noted that eqs. (24) have been derived from HDE hierarchy and completely are based on dynamics.

The spectrum of the Schrödinger equation (13) (with potential $u(y) = u(y, 0)$) has only one gap $(E_1, E_2)$ and the spectrum boundary points have form $E_0 = e_3 + \frac{2}{3} C$, $E_1 = e_2 + \frac{2}{3} C$, $E_2 = e_1 + \frac{2}{3} C$. 

4
To obtain all 1-gap densities of the polar operator generated by the potential (314) one has to insert (314) and (212) into (26). This yields

\[ R(y) = \frac{C}{\varphi(iy + \omega) + \frac{2}{3}C + \frac{2i}{T}}. \]

Now one has to check the validity of relation \( u = (R^{-1/2})_{yy} R^{1/2} \) where \( u \) is given by (314). It turns out that it holds not for any values of \( C \) and \( \beta_1 \). Namely we come to the following

**Lemma 1.** Potential \( u \) from (314) gives rise to density of the polar operator only if \( \hat{C} \equiv 2/3 \) \( C \) is solution of the equation:

\[ 4\hat{C}^3 - g_2\hat{C} + g_3 = 0 \]

and \( \beta_1 = 0 \).

Since the solution of the latter equation reads \( \hat{C} = -e_\alpha \), \( \alpha = 1, 2, 3 \) one sees that there exist 1-gap densities only with following arrangement of spectrum boundary points: \( E_0 = e_3 - e_\alpha, E_1 = e_2 - e_\alpha, E_2 = e_1 - e_\alpha \). So by all means one of the boundary points coincides with 0.

Expressions for finite-gap densities (emerging if we put \( t_1 = 0 \) in solution of the Harry Dym equation) read:

\[
\begin{align*}
r_\alpha(x) &= A_\alpha[e_\alpha - \varphi(iy + \omega_1 + \omega_\alpha)], \\
x &= e_\alpha A_\alpha y - iA_\alpha[\zeta(iy + \omega + \omega_\alpha) - \eta - \eta_\alpha], \quad \alpha = 1, 2, 3.
\end{align*}
\]

Here

\[
A_\alpha = \frac{3/2 e_\alpha}{H_\alpha^2}, \quad H_\alpha^2 = (e_\alpha - e_\beta)(e_\alpha - e_\gamma), \quad \alpha = 1, 2, 3,
\]

\( \zeta \) is Weierstrass \( \zeta \)-function: \(-\zeta'(z) = \varphi(z) \) and \( \eta_\alpha = \zeta(\omega_\alpha) \).

Let us briefly overview functional properties of constructed densities.

The density \( r_\alpha(x) \), \( \alpha = 1, 2, 3 \) is periodic function with real period

\[
T_\alpha = e_\alpha A_\alpha 2|\omega'| - 2iA_\alpha \eta_3,
\]

it’s spectrum consists of spectral bands \([E_0, E_1]\) and \([E_2, +\infty]\).

In case \( \alpha = 3 \) one has: \( E_0 = 0, E_1 = e_2 - e_3, E_2 = e_1 - e_3 \). Function \( r_3(x) \) is even (i. e. \( r_3(-x) = r_3(x) \)), periodic, smooth and doesn’t turn into 0. Therefore the string density arising in eq. (12) \( \rho_3 = 1/r_3^2 \), possesses the same properties. It is density which M. G. Krein wrote about in his paper [1] and therefore we refer to function \( r_3(x) \) as the Krein density. According to Krein’s theorem there exists just one periodic even density of the string equation \( \rho = 1/r^2 \) having one-gap spectrum which starts at point 0. It should be noted that the density \( r_3(x) \) was first constructed in [1]. It’s period \( T_3 \) first had been found in [14] in terms of elliptic Jacobi functions. It is this density which has been constructed here explicitly.

In case \( \alpha = 1 \) \( r_1(x) \) is a discontinuous function. Therefore instead of smooth generating solution of the KdV equation (314) we take singular one, namely

\[
u(y,t_1) = -2\varphi(i(y + 4Ct_1)) + 2/3 C.
\]

Using the same technique we arrive to the density

\[
\hat{r}_1(x) = A_1[e_1 - \varphi(iy + \omega)],
\]

\[
\hat{x} = e_1 A_1 y - iA_1[\zeta(iy + \omega) - \eta],
\]

\[
\hat{C} = \frac{2}{3} A_1 C + \frac{2i}{T_1},
\]

where \( T_1 = e_1 A_1 2|\omega'| - 2iA_1 \eta_1 \).
As formula (316) for periods of densities $r_{\alpha}(x)$ remains to be valid in this case we conclude that density $\hat{r}_1(\hat{x})$ is a periodic even function with period $T_1 = 2e_1 A_1 |\omega' - 2iA_1\eta|$ having singularities of the "cusp" type:

$$\hat{r}_1(\hat{x}) \sim \text{Const} \hat{x}^2/3$$

in points $\hat{x}_n = nT_1$, $n \in \mathbb{Z}$. Therefore we refer to this density as cusp-periodic density. The boundary spectrum points are $E_0 = e_3 - e_1$, $E_1 = e_2 - e_1$, $E_2 = 0$.

This density is remarkable for it possesses degeneration property, namely if the first spectral zone shrinks into single point ($E_0 = E_1 = -3\gamma$) and continuous spectrum fills $\mathbb{R}^+$ ($E_2 = 0$) then $\hat{r}_1(\hat{x})$ turns into well-known soliton density of the Harry Dym equation:

$$\hat{R}_1(y) = \text{th}^2(\sqrt{3\gamma} y),$$

$$\hat{E}_1(y) = \frac{1}{\sqrt{3\gamma}} \text{th} (\sqrt{3\gamma} y).$$

In case $\alpha = 2$ we obtain singular density similar to $r_1$. Again changing the initial generating solution of the KdV equation by

$$u(y, t_1) = -2\wp(i(y + 4Ct_1) + \omega') + 2/3 C$$

we arrive to the density

$$\hat{r}_2(\hat{x}) = A_2[e_2 - \wp(iy + \omega)],$$

$$\hat{x} = e_2 A_2 y - iA_2[\zeta(iy + \omega) - \eta].$$

Here the properties are quite similar to the properties of the $\hat{r}_1$. Namely it has "cusps" at the points $\hat{x} = \frac{T_2}{2} + nT_2$, where $T_2$ is it’s period with respect to $\hat{x}$. Here the boundary spectrum points are $E_0 = e_3 - e_2$, $E_1 = 0$, $E_2 = e_1 - e_2$.

Note that when trying to degenerate $\hat{r}_2$ while the first spectral zone shrinks into single point ($E_0 = E_1 = -3\gamma$) we get $A_2 \to \infty$ and thus $\hat{r}_2$ doesn’t possess degeneracy property analogous to that of $\hat{r}_1$.

### 4 Two-gap densities of the polar operator

In this section we apply the technique described in sect. 3 to construct elliptic two-gap densities of the polar operator. As far as we know these densities weren’t present in literature before.

Consider the case when the Riemann surface (27) possesses a special sort of symmetry, namely let

$$E_0 = -\sqrt{3g_2} + 3a, \quad E_\alpha = -3e_\alpha + 3a, \quad \alpha = 1, 2, 3, \quad E_4 = \sqrt{3g_2} + 3a, \quad (417)$$

where $a$ — arbitrary real parameter. The points of additional spectrum $\{\mu_k\}_{k=1}^2$ necessary to define 2-gap potential of the Schrödinger equation unambiguously we choose (following [12]) to be the following:

$$\mu_1 = E_2 = -3e_2 + 3a, \quad \mu_2 = E_3 = -3e_3 + 3a,$$

i.e. we put additional spectrum at the ends of the "main" spectrum. Then according to [12] the general formula for potential in terms of 2-dimensional $\Theta$-functions is reduced to to form:

$$u(y) = -6\wp(iy + \omega|\omega, \omega') + 3a, \quad (418)$$

i.e. it is two-gap Lame potential.
In 2-gap case insertion of (418), (212) and (213) into (26) yields
\[ R(y) = K \frac{\varphi^2(iy + \omega) + a\varphi(iy + \omega) + b'}{\varphi'}, \tag{419} \]
where
\[ b = a^2 - \frac{g_2}{4} + \frac{5}{24}a\beta_1 + \beta_2, \quad K = \frac{15}{18}a^2 - \frac{7}{24}g_2. \]

Now as in 1-gap case one has to check the validity of the relation
\[ u(y) = \left( R^{1/2} \right)_{yy} R^{1/2} \]
with \( u \) being the generated 2-gap potential (418). The following statement is valid:

**Lemma 2.** Potential \( u \) from (418) gives rise to density of the polar operator only if \( a \) is solution of the equation:
\[ \left( a^2 - \frac{g_2}{3} \right)(4a^3 - g_2a - g_3) = 0, \]
and the constants \( \beta_1 \) and \( \beta_2 \) are related as follows:
\[ \frac{5}{24}a\beta_1 + \beta_2 = 0. \]

One sees that parameter \( a \) may take only 5 values:
\[ a = \pm \sqrt{\frac{g_2}{3}} \ e_1, e_2, e_3. \]
This means that we get 5 two-gap densities of the form (419). Due to eqs. (417) one of the spectrum boundary points by all means coincides with zero.

Each density \( r_\alpha(x) \), \( \alpha = \pm, 1, 2, 3 \) as in 1-gap cases is defined by pair of equations
\[ r_\alpha(x) = R_\alpha(y), \quad x = y - \int_0^y (1 - R_\alpha(y') dy'. \]

In the case \( a^2 = \pm \sqrt{\frac{g_2}{3}} \) (\( \alpha = \pm \)) we obtain two densities
\[ r_\pm(x) = \frac{g_2/3}{\varphi(iy + \omega) \pm \frac{1}{2} \sqrt{g_2/3}}^2, \tag{420} \]
\[ x = y \frac{a^3}{a^3 - g_3} + \frac{ia^2}{a^3 - g_3} \{ \zeta(iy + \omega - \gamma_\pm) + \zeta(iy + \omega + \gamma_\pm) - 2\eta \}. \tag{421} \]
where \( \gamma_\pm \) is a solution of the equation \( \pm \frac{1}{2} \sqrt{g_2/3} = -\varphi(\gamma_\pm) \).

The periods of functions \( r_\pm(x) \) implicitly described by (420) and (421) are
\[ T_\pm = \frac{2a^2}{a^3 - g_3} (a|\omega'| + 2i\eta'), \quad a = \pm \sqrt{\frac{g_2}{3}}. \]

In the case \( a = e_\alpha \) (\( \alpha = 1, 2, 3 \)) we obtain densities
\[ r_\alpha(x) = \frac{15}{8}e_\alpha - \frac{7}{24}g_2 \frac{\varphi(iy + \omega) - e_\beta}{(\varphi(iy + \omega) - e_\beta)(\varphi(iy + \omega) - e_\gamma)}, \tag{422} \]
\[ x = -y \cdot \left( A \left( \frac{e_\beta}{H_\beta} - \frac{e_\gamma}{H_\gamma} \right) \right) - \]
\[ -iA \left( \frac{1}{H_\gamma} \zeta(iy + \omega + \omega_\gamma) - \frac{1}{H_\beta} \zeta(iy + \omega + \omega_\beta) \right) + iA \left( \frac{\eta_\gamma}{H_\gamma} - \frac{\eta_\beta}{H_\beta} \right), \]
\[ A = \frac{K}{e_\beta - e_\gamma}. \]
Here it is assumed that symbols \( \{\alpha, \beta, \gamma\} \) represent some transposition of numbers \( \{1, 2, 3\} \).

Each constructed density \( r_\alpha(x), \alpha = 1, 2, 3 \) is a periodic function with period:

\[
T_\alpha = 2A|\omega'| \left( \frac{e_\gamma}{H_\gamma^2} - \frac{e_\beta}{H_\beta^2} \right) - 2A \eta' \left( \frac{1}{H_\gamma^2} - \frac{1}{H_\beta^2} \right), \quad \alpha = 1, 2, 3
\]

Among these five densities the most interesting is \( r_+ (x) \). It is smooth, even, nonvanishing function. All of it’s spectral bands \( [E_0, E_1], [E_2, E_3], [E_4, +\infty] \) lie on \( \mathbb{R}^+ \) and \( E_0 = 0 \). Thus the density \( r_+ (x) \) is two-gap density of Krein’s type \([1]\).

Another four densities are discontinuous functions. Their analysis is out of scope of this paper.

## 5 Auto-Bäcklund transformations of finite-gap densities

In this section we analyze the densities which can be obtained by auto-Bäcklund transformation of the one- and two-gap densities constructed above. Let’s return for a while to a KdV and HD equations. Remind the following statements \([11], [3]\). The pair of KdV solutions \( u(y, t) \) and \( \tilde{u}(y, t) \) related by the auto-Bäcklund transformation can be written in terms of the HDE solution \( r(x,t) = R(y,t) \) as

\[
\tilde{u} = u + \partial_y^2 \ln R,
\]

where \( R \) corresponds to \( u \) under the relation \( u = (R^{-1/2})_{yy} R^{1/2} \).

Let by \( \tilde{r}(\tilde{x}, t) = \tilde{R}(y, t) \) denote the HDE solution related to \( \tilde{u} \) in the same manner, namely:

\[
\tilde{u} = \left( \tilde{R}^{-1/2} \right)_{yy} \tilde{R}^{1/2}.
\]

Then as follows from \([11], [3]\)

\[
\tilde{R} = \frac{b}{R},
\]

\[
\tilde{x} = y - \int_0^y (1 - \tilde{R}(y', t)) dy'.
\]

In case of decreasing density \( (r \to 1 \text{ when } |x| \to \infty) \) the constant \( b = 1 \) \([11]\). In case of periodic density this constant can not be obtained from the general relations and some additional considerations are needed. It turns out that constant \( b \) values are different for individual densities. Below we construct explicitly auto-Bäcklund transformation for 1- and 2-gap densities presented above and compute the value of constant \( b \) in each case.

### 1-gap case

Here we compute auto-Bäcklund transformation for smooth 1-gap Schrödinger potentials, \( u_\alpha(y) = -2\phi(iy + \omega) - e_\alpha \) which has been used in section \([3]\) for generating the densities \( R_\alpha(y) = r_\alpha(x) \) given by \([315]\).

From \((523), (525)\) it follows that

\[
\tilde{u}_\alpha(y) = \left( R^{1/2}_\alpha(y) \right)_{yy} R^{-1/2}_\alpha(y).
\]

The result of calculations is following. Potentials \( \tilde{u}_\alpha(y) \) are:

\[
\tilde{u}_\alpha(y) = -2\phi(iy + \omega + \omega_\alpha) - e_\alpha.
\]
When $\alpha = 3$ we obtain smooth potential which differs from its auto-Bäcklund transformation $u_3(y)$ by shift in $y$. When $\alpha = 1, 2$ we obtain singular potentials. Thus in 1-gap case the auto-Bäcklund transformation for the KdV equation doesn’t lead out the class of known 1-gap potentials.

In order to calculate auto-Bäcklund transformation $\hat{r}_\alpha(x) = \hat{R}_\alpha(y)$ of 1-gap densities $r_\alpha(x) = R_\alpha(y)$ the whole procedure of densities $R_\alpha$ construction should be repeated starting from potentials $\hat{u}_\alpha$ rather than $u_\alpha$. The properties of densities $\hat{r}_1(x)$ and $\hat{r}_2(x)$ have been already considered in section 3. The density $\hat{r}_3(x)$ differs from the smooth density $r_3(x)$ which possesses no roots only by shift in $x$ by $T_3/2$.

Thus we have got 5 distinct 1-gap densities of the polar operator. Densities $\hat{r}_1(x)$ and $\hat{r}_2(x)$ are continuous cusp-periodic densities. Their auto-Bäcklund transformations $r_1(x)$ and $r_2(x)$ correspondingly are singular functions. Densities $r_3(x)$ and $\hat{r}_3(x)$ differ only in shift by half-period $T_3/2$ in $x$ and represent unique smooth 1-gap Krein density whose spectrum starts in 0 and spectral zones lie at $R^+$.

Concluding analysis of auto-Bäcklund transformation in 1-gap case we put expressions for the constant $b$ from (523). Expressions are:

$$ b \equiv b_\alpha = \frac{(3/2 e_\alpha)^2}{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)}, \quad \alpha = 1, 2, 3. $$

2-gap case

The scheme of calculating auto-Bäcklund transformation of two-gap densities $r_\alpha(x) = R_\alpha(y)$, $\alpha = \pm, 1, 2, 3$ (see eqs. (420), (422)) is the following: one computes $\hat{u}_\alpha(y)$ by means of relation (523) and then use this function as generating Schrödinger potential in the scheme of constructing polar operator densities presented in section 3. The result reads:

$$ \hat{r}_\pm(x) = K_\pm \left( \varphi(iy + \omega) \pm \frac{1}{2} \sqrt{g_2/3} \right)^2 $$

$$ \hat{x} = \frac{K_\pm g_2}{6} y - \frac{iK_\pm}{6} \varphi'(iy + \omega) \mp i aK_\pm [\zeta(iy + \omega) - \eta] $$

where

$$ K_\pm = \frac{75a^2 - 7g_2}{12a(a^2 - g_3)a^2}, \quad a = \pm \sqrt{\frac{g_2}{3}}. $$

Another three densities have the form:

$$ \hat{r}_\alpha(x) = K_\alpha (\varphi(iy + \omega) - e_\beta)(\varphi(iy + \omega) - e_\gamma), \quad \alpha = 1, 2, 3. $$

$$ \hat{x} = K_\pm \left( \frac{g_2}{12} + e_\beta e_\gamma \right), \quad y - \frac{iK_\pm}{6} \varphi'(iy + \omega) - ie_\alpha K_\pm [\zeta(iy + \omega) - \eta] $$

where

$$ K_\alpha = \frac{(75a^2 - 7g_2)}{2(3a^2 - g_2)(12a^2 - g_2)} \left( \frac{15}{8} e_\alpha - \frac{7}{24} g_2 \right), \quad a = e_\alpha, \alpha = 1, 2, 3. $$

Before describing properties of the densities obtained let us note the following. Since we don’t prove the fact that auto-Bäcklund transformation in periodic case preserves the structure of spectrum of the corresponding linear problem for the polar operator we can not state that spectra of the densities $\hat{r}_\pm(x)$, $\hat{r}_\alpha(x)$ consist of only two bands. Nevertheless properties of these densities outlined below suggest the validity of the said hypothesis.

Now we briefly describe properties of the obtained densities. The density $\hat{r}_+(x)$ is the smooth periodic symmetric nonvanishing density of the Krein type. The density $\hat{r}_-(x)$ has cusp-type singularities

$$ \hat{r}_-(x) \sim Const (\hat{x} - \hat{x}_0)^{4/5}. $$
The periodic densities $\hat{r}_\alpha(\hat{x})$ also have cusp singularities although of another type:

$$\hat{r}_\alpha(\hat{x}) \xrightarrow{\hat{x} \to \hat{x}_\alpha} \text{Const} (\hat{x} - \hat{x}_\alpha)^{2/3}.$$ 

Let us note that to implement the whole scheme which led to the above expressions the constants $\beta_1$ and $\beta_2$ in (212) and (213) should satisfy condition

$$(30a\beta_1 + \beta_2) = 180a^2.$$ 

Finally calculations for the constants $b$ in (525) for each pair of densities linked by the auto-Bäcklund transformation give:

$$b_\pm = \frac{75a^2 - 7g_2}{12a(a^3 - g_3)a^2}, \quad a = \pm \sqrt{\frac{g_2}{3}}$$

$$b_\alpha = \frac{(75a^2 - 7g_2)}{2(3a^2 - g_2)(12a^2 - g_2)\left(\frac{15}{8}e_\alpha - \frac{7}{24}g_2\right)}, \quad a = e_\alpha, \alpha = 1, 2, 3.$$ 

6 Acknowledgements

The authors acknowledge RFBR Grant # 99-01-00696 for support of this work.

References

[1] Krein M.G. Dokl. Akad. Nauk SSSR (in Russian) 93 (5), (1953) pp. 767–790
[2] Krein M.G., Cats I. S., 1968, On spectral functions of the string, in: Atkinson F. V., Discrete and continuous boundary value problems, Moscow, 748 p. (in Russian)
[3] Atkinson F. V., Discrete and continuous boundary value problems, Academic Press, N.-Y.-London, 1964
[4] Kuperin Yu. A., in: Problems of Mathematical Physics 9, LGU publishing house, Leningrad, 1977, pp. 54–63
[5] Dmitrieva L. A., Khlabystova M. A., Lett. Math. Phys. 39 (1997), 355–366
[6] Dmitrieva L. A., Phys. Lett. A 182 (1993), 65–70
[7] Dmitrieva L. A., J. Phys. A; Math. Gen. 26 (1993), 6005–6020
[8] Its A.R., Matveev V. B., in: Problems of Mathematical Physics 8, LGU publishing house, Leningrad, 1976, pp. 70–92
[9] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, Theory of solitons (Consultants Bureau, New York, 1984)
[10] Calogero F., Degasperis A., Spectral Transform and Solitons I, (North-Holland, Amsterdam, 1982)
[11] Rogers C., Nucci M. C., Phys. Sci. 33 (1986), 289–292
[12] Enolsky V.Z., Preprint ITPE-83-112P, Kiev, 1983
[13] Abramowitz M., Stigun I., Handbook of Mathematical Functions National Bureau of Standards, Applied Mathematics Series - 55, 1964.
[14] Bordag L. A., Properties of the Multi Phase Solutions of the HD Equation, Poster presentation book, Int. Conf. KdV'95, Amsterdam, 21–23, April, 1995, p. 5