THE SIGNIFICANCE OF THE CONTRIBUTIONS OF CONGRUENCES TO THE THEORY OF CONNECTEDNESSES AND DISCONNECTEDNESSES FOR TOPOLOGICAL SPACES AND GRAPHS.

STEFAN VELDSMAN

Nelson Mandela University, Port Elizabeth, South Africa, and
La Trobe University, Melbourne, Australia.

E-Mail veldsman@outlook.com

Dedicated to the memory of Izak Broere

Abstract. This is a survey of some of the consequences of the recently introduced congruences on the theory of connectednesses (radical classes) and disconnectednesses (semisimple classes) of graphs and topological spaces. In particular, it is shown that the connectednesses and disconnectednesses can be obtained as Hoehnke radicals and a connectedness has a characterization in terms of congruences resembling the classical characterization of its algebraic counterpart using ideals for a radical class. But this approach has also shown that there are some unexpected differences and surprises: an ideal-hereditary Hoehnke radical of topological spaces or graphs need not be a Kurosh-Amitsur radical and in the category of graphs with no loops, non-trivial connectednesses and disconnectednesses exist, but all Hoehnke radicals degenerate.

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1. Introduction. In this survey, we report on the work providing the last piece of the puzzle to fully establish the correspondence between the radical theory of the algebraic structures like rings, nearrings and groups on the one hand, and the non-algebraic categories of topological spaces and graphs on the other. This link has been made possible by the recent definition and development of a theory of congruences for graphs and for topological spaces. It is then also possible to define the radical classes for topological spaces and graphs (in these categories, such classes are called connectednesses) as Hoehnke radicals. This opened the door to explore other similarities and differences between the algebraic and non-algebraic theories and we will report on some of these unexpected and surprising differences.

The origins of radical theory goes back to the early twentieth century with the work of Wedderburn on finite dimensional algebras. This was extended to
ring theory with Köthe's nilradical, the Jacobson radical and subsequently many other radicals. These, together with some developments in group theory, led to the axiomization of the radical concept by Kurosh and independently Amitsur for rings, groups and omega-groups in the early fifties. In the sixties, Hoehnke used congruences to define a radical for universal algebras; now known as a Hoehnke radical. In an environment where a congruence is completely determined by one of its congruence classes, as for example in Ω-groups, it was then shown under which conditions a Hoehnke radical will be a Kurosh-Amitsur radical. A second main stream that contributed to the development and existence of general radical theory has its origins in the torsion theory of modules. Torsion theories of modules were defined in terms of equivalence classes of injective modules, but when Dickson generalized torsion theories to abelian categories, this approach was abandoned. It turns out that the torsion and torsion-free classes correspond to the radical and semisimple classes of general radical theory respectively. It is then interesting and rather pleasing that the hereditary torsion theories of topological spaces have a connection with injective topological spaces (= indiscrete spaces) as will be seen below. The third contribution to the general radical theory is to be found in the category of topological spaces. Connectednesses and disconnectednesses of topological spaces were defined by Preuß as classes of spaces on which certain maps are constant. Then Arhangel'skii and Wiegandt showed that these classes corresponds to the Kurosh-Amitsur radical and semisimple classes of rings and groups by replacing the algebraic notions involved in one of their characterizations with their categorical suitable topologial versions. This was followed by a theory of connectednesses and disconnectednesses for graphs and more generally for abstract relational structures. The graphs that we refer to in the preceding lines are graphs for which loops are allowed. When no loops are permitted, the radical theory has many different and unusual characteristics.

With such similar radical theories in so many divergent branches of mathematics, the need arose for a common language to describe them all. Category theory proved to be a suitable tool; initially only catering for the radical theories from an algebraic environment, for example by Sul'geifer [27], Suliński [28] and Holcombe and Walker [19]. Such categories exclude the connectednesses and disconnectednesses of topological spaces and graphs. A unified treatment for these two cases were given by Fried and Wiegandt [15, 16] by considering graphs and topological spaces as abstract relational structures. But this approach excluded the algebraic cases. In [5] and [29] less stringent conditions were imposed on a general category to make it suitable to describe the radical theory of the classical algebraic structures as well as those of topological spaces and graphs. Subsequently, the most comprehensive theory covering most known radical theories was given by Márki, Mlitz and Wiegandt [20] in a general categorical setting but with an universal algebraic flavor.

Quite recently it has been shown that the connectednesses and disconnectednesses of both topological spaces and graphs (i.e the Kurosh-Amitsur radical and semisimple classes) can be obtained from Hoehnke radicals as has been done for universal algebras using congruences. In [4] and [30] it was shown how to define congruences on graphs and topological spaces respectively. These congruences
then lead in a natural way to appropriate versions of the algebraic isomorphism theorems and subdirect products. As in universal algebra, one can then define a Hoehnke radical for graphs and topological spaces. Necessary and sufficient conditions to ensure that the Hoehnke radical becomes a Kurosh-Amitsur radical have been determined for algebraic structures (Mlitz [21]), and they can also be adopted for the non-algebraic structures. Furthermore, it is then shown that congruences can be used to give a characterization of the connectednesses of topological spaces and graphs which resembles the well-known and classical characterizations of the algebraic radical classes ([33, 35].

The fact that connectednesses and disconnectednesses of topological spaces and graphs can be defined as Hoehnke radicals, has opened up a number of new questions to explore; sometimes with interesting consequences. In the classical torsion theory, torsionfree classes (= semisimple classes) are always hereditary and a hereditary torsion theory means that the associated torsion class (= radical class) is hereditary. For associative rings, the semisimple classes are also always hereditary, so also here a hereditary radical would mean the associated radical class is hereditary. But for more general classes, e.g. not necessarily associative rings or near-rings, semisimple classes need not be hereditary. Thus, for a radical in general, it is now customary to call it ideal-hereditary if both its semisimple class and its radical class are hereditary. The relationships between Hoehnke radicals, Kurosh-Amitsur radicals, connectednesses and disconnectednesses and torsion theories with or without additional properties (like hereditariness) have been investigated and clarified for most concrete categories. For associative rings and similar types of algebraic categories, in fact for Ω-groups in general, it is well-known that any ideal-hereditary Hoehnke radical is a Kurosh-Amitsur radical. In terms of torsion theories, this statement says that any hereditary torsion theory is Kurush-Amitsur. It is thus interesting to know if an ideal-hereditary Hoehnke radical of topological spaces or graphs is a Kurosh-Amitsur radical (i.e., whether every hereditary torsion theory of topological spaces or graphs gives rise to a corresponding pair of connectedness and disconnectedness). Wiegandt [38] has shown that for S-acts this is not the case: a hereditary torsion theory need not determine a Kurosh-Amitsur radical. It will be seen below that this is also the case for topological spaces and graphs (see [31, 34]). In this vein of the unexpected, we will find the radical theory of graphs that do not admit loops. Also here there is a theory of congruences and Hoehnke radicals can be defined. But they all degenerate, and this in spite of the fact that there are non-trivial connectednesses and disconnectednesses which, by the way, all come as complementary pairs [36].

Our objective here is to give an overview of the value added to the general radical theory by this approach to the connectednesses and disconnectednesses via congruences. To set the scene and to compare and appreciate the correspondences and differences between the theories, the appropriate definitions and main results on the general radical theory of algebraic structures, actually mainly for associative rings, will be recalled in the next section. In Sections 3, 4 and 5, one each for the category of topological spaces, the category of graphs for which loops are allowed and the category of graphs for which loops are not allowed respectively, we present a brief summary of the congruence theory and the radical theory for
each, concluding with some interesting features of this theory for that category. In this survey, the earlier works of the author and his co-authors on the congruences of topological spaces and graphs and their radical theory are freely used and often quoted verbatim without reference. In particular, we refer to [4, 30, 31, 33, 34, 35, 36].

2. Rings. Here we give a quick overview of Kusorh-Amitsur radicals, Hoehnke radicals, torsion theories and the relations between them. The terminology here is largely motivated by that in use for associative rings. Moreover, the results recalled below are mostly for associative rings and may not necessarily be valid for other classes or rings or algebras. Gardner and Wiegandt [17] can be consulted for a thorough overview of the radical theory of associative rings. To appreciate the similarities between the algebraic and the non-algebraic with the use of congruences in the radical theory presented in the next sections, [22] can be consulted.

A property \( R \) that a ring may possess, is called a radical property provided the following three conditions are fulfilled:

\( \text{(R1)} \) Any homomorphic image of a ring with property \( R \) has property \( R \).

\( \text{(R2)} \) Any ring has a largest ideal, called the \( R \)-radical of the ring, which, as a ring, has property \( R \), and contains all other ideals of the ring which has property \( R \).

\( \text{(R3)} \) The quotient of any ring by its \( R \)-radical contains no non-trivial ideals with property \( R \) (such rings are called \( R \)-semisimple).

The class of all rings which has property \( R \) is usually also denoted by \( R \) and no distinction is made between a ring in the class \( R \) and a ring with the property \( R \). If \( R \) is a radical property, then \( R \) is called a radical class, the \( R \)-radical of a ring \( A \) is written as \( R(A) \) and the class \( S := \{ \text{rings} \ A \mid R(A) = 0 \} \) is called the semisimple class of \( R \). In general, a class of rings is called a semisimple class if it is the semisimple class of some radical class. In recognition of their contributions in establishing this abstract approach to radicals by Kurosh and independently Amitsur, these radicals are called KA-radicals.

For a class of rings \( \mathcal{M} \) and a ring \( A \), we let \( \mathcal{M}(A) := \sum \{I \triangleleft A \mid I \in \mathcal{M}\} \) and \( (A)\mathcal{M} := \cap \{I \triangleleft A | A/I \in \mathcal{M}\} \). The class \( \mathcal{M} \) is said to be: hereditary if for all rings \( A, I \triangleleft A \in \mathcal{M} \) implies \( I \in \mathcal{M} \); homomorphically closed if for any surjective homomorphism \( \theta : A \rightarrow B \) with \( A \in \mathcal{M} \) implies \( B \in \mathcal{M} \); inductive if \( I_1 \subseteq I_2 \subseteq \ldots \) is an ascending chain of ideals of the ring \( A \) with \( I_n \in \mathcal{M} \) for all \( n = 1, 2, 3, \ldots \), then \( \bigcup_{n=1}^{\infty} I_n \in \mathcal{M} \); closed under extensions if \( I \triangleleft A \) with both \( I \) and \( A/I \) in \( \mathcal{M} \), then also \( A \in \mathcal{M} \); regular if every non-zero ideal of the ring \( A \) has a non-zero homomorphic image which is in \( \mathcal{M} \), then \( A \in \mathcal{M} \); and closed under subdirect sums whenever rings \( A_j \in \mathcal{M} \) for all \( j \in J \), then so is their subdirect sum.

**Theorem 2.1.** For a class of rings \( \mathcal{R} \), the four conditions (1), (2), (3) and (4) below are equivalent:

1. \( \mathcal{R} \) is a KA-radical class.
2. \( \mathcal{R} \) fulfills the following three conditions:
   a. \( \mathcal{R} \) is homomorphically closed.
(b) \( \mathcal{R}(A) \in \mathcal{R} \) for all rings \( A \).
(c) \( \mathcal{R}(A/\mathcal{R}(A)) = 0 \) for all rings \( A \).
(3) \( \mathcal{R} \) fulfills the following condition:

For any ring \( A, A \in \mathcal{R} \iff \) every non-zero homomorphic image of \( A \) has a non-zero ideal which is in \( \mathcal{R} \).
(4) \( \mathcal{R} \) fulfills the following three conditions:
(a) \( \mathcal{R} \) is homomorphically closed.
(b) \( \mathcal{R} \) is inductive.
(c) \( \mathcal{R} \) is closed under extensions.

Semisimple classes can be characterized in their own right:

**Theorem 2.2.** For a class of rings \( \mathcal{S} \), the four conditions (1), (2), (3) and (4) below are equivalent:

(1) \( \mathcal{S} \) is a KA-semisimple class.
(2) \( \mathcal{S} \) fulfills the following four conditions:
   (a) \( \mathcal{S} \) is regular.
   (b) \( \mathcal{S} \) is closed under subdirect sums.
   (c) \( \mathcal{S} \) is closed under extensions.
   (d) \( ((A)\mathcal{S})S \triangleright A \) for all rings \( A \).
(3) \( \mathcal{S} \) fulfills the following condition:

For any ring \( A, A \in \mathcal{S} \iff \) every non-zero ideal of \( A \) has a non-zero homomorphic image which is in \( \mathcal{S} \).
(4) \( \mathcal{S} \) fulfills the following three conditions:
   (a) \( \mathcal{S} \) is hereditary.
   (b) \( \mathcal{S} \) is closed under subdirect sums.
   (c) \( \mathcal{S} \) is closed under extensions.

Condition (3) in each of the previous two theorems has become the defining condition of a radical class and a semisimple class respectively in a wide-ranging and diverse range of categories. Hoehnke’s approach to radicals [18] has prompted the following: an ideal-mapping \( \rho \) is a mapping which assigns to each ring \( A \) an ideal \( \rho(A) \), usually written as \( \rho_A \). The Hoehnke radical is defined as follows:

An \( H \)-radical is an ideal-mapping \( \rho \) which fulfills the following two conditions:

\((H1)\) For any surjective homomorphism \( \theta : A \to B, \theta(\rho(A)) \subseteq \rho(B) \).
\((H2)\) For all rings \( A, \rho(A/\rho(A)) = 0 \).

For the \( H \)-radical \( \rho, \mathcal{R}_\rho := \{\text{rings } A \mid \rho(A) = A\} \) is the associated radical class and \( \mathcal{S}_\rho := \{\text{rings } A \mid \rho(A) = 0\} \) is the associated semisimple class.

**Theorem 2.3.** Let \( \mathcal{M} \) be a class of rings which contains the zero ring and is closed under isomorphic copies. Then \( \rho \) defined by \( \rho(A) = \cap\{I \triangleright A \mid A/I \in \mathcal{M}\} \) for all rings \( A \) is an \( H \)-radical with \( \mathcal{S}_\rho \) the subdirect closure of the class \( \mathcal{M} \). Conversely, if \( \rho \) is an \( H \)-radical, there is a class \( \mathcal{M} \) of rings which contains the zero ring, is closed under isomorphic copies, has \( \rho(A) = \cap\{I \triangleright A \mid A/I \in \mathcal{M}\} \) for all rings \( A \) and \( \mathcal{S}_\rho \) is the subdirect closure of \( \mathcal{M} \).
To clarify the connection between KA-radicals and H-radicals, we need two more definitions. An ideal mapping $\rho$ is: complete if for all rings $A$, $\rho(I) = I \triangleleft A$ implies $I \subseteq \rho(A)$; and idempotent if $\rho(\rho(A)) = \rho(A)$ for all rings $A$. In view of the next result, a complete and idempotent H-radical is called a KA-radical.

**Theorem 2.4.** For an ideal-mapping $\rho$, the following three conditions are equivalent:

1. $\rho$ is an H-radical which is complete and idempotent.
2. $\rho$ is an H-radical for which the corresponding radical class $R_\rho$ is a KA-radical class with associated semisimple class $S_\rho$.
3. $\rho$ is an H-radical for which the corresponding semisimple class $S_\rho$ is a KA-semisimple class with associated radical class $R_\rho$.

Conversely, if $R$ is a KA-radical class, then the ideal mapping $\rho$ defined by $\rho(A) = \sum \{ I \triangleleft A \mid I \in R \}$ for all rings $A$ is a complete, idempotent H-radical with $R_\rho = R$; or equivalently, if $S$ is a KA-semisimple class, then the mapping $\rho$ defined by $\rho(A) = \cap \{ I \triangleright A \mid A/I \in S \}$ for all rings $A$ is a complete, idempotent H-radical with $S_\rho = S$.

An ordered pair $(R, S)$ of classes of rings $R$ and $S$ is called a torsion theory if:

1. $R \cap S = 0$.
2. $R$ is homomorphically closed.
3. $S$ is hereditary.
4. Every ring $A$ has an ideal $B$ with $B \in R$ and $A/B \in S$.

The class $R$ is called a torsion class and $S$ is a torsionfree class. A hereditary torsion theory is a torsion theory $(R, S)$ with hereditary torsion class $R$. This abstract approach to torsion theories was introduced by Dickson [12] and has its origins in abelian categories.

**Theorem 2.5.** The following are equivalent:

1. $(R, S)$ is a torsion theory.
2. $R$ is a KA-radical class with corresponding semisimple class $S$ and $R(I) \subseteq R(A)$ for all rings $A$ and $I \triangleright A$.
3. $S$ is a hereditary KA-semisimple class with corresponding radical class $R$.

An ideal mapping $\rho$ is called ideal-hereditary if $\rho(I) = \rho(A) \triangleleft I$ for all rings $A$ and $I \triangleright A$. Recall, $\rho$ is a KA-radical if it is an idempotent and complete H-radical. For such a radical $\rho$, let $R$ and $S$ be the corresponding radical and semisimple class respectively. Then:

1. $R$ is hereditary iff $\rho(A) \triangleleft I \subseteq \rho(I)$ for all rings $A$ and $I \triangleright A$.
2. $S$ is hereditary iff $\rho(I) \subseteq \rho(A) \triangleleft I$ for all rings $A$ and $I \triangleright A$.

If $\rho$ is only a Hoehnke radical, then:

3. $\rho(A) \triangleleft I \subseteq \rho(I)$ for all rings $A$ and $I \triangleright A$ implies $\rho$ is idempotent and $R_\rho$ is hereditary.
4. $\rho(I) \subseteq \rho(A) \triangleleft I$ for all rings $A$ and $I \triangleright A$ implies $\rho$ is complete and $S_\rho$ is hereditary.
We know that an idempotent and complete H-radical of rings is a KA-radical; hence an ideal-hereditary H-radical of associative rings coincides with an ideal-hereditary KA-radical (and both the radical and semisimple classes are hereditary) which in turn is the same as a hereditary torsion theory.

3. Topology. In the early seventies, Preuß [23, 24, 25] defined and developed a general theory of connectednesses and disconnectednesses for topological spaces. This showed and clarified, amongst others, the relationship between the separation axioms and non-connectedness on the one hand and connected spaces on the other, establishing a Galois connection between them. Subsequently many papers appeared on connectednesses and disconnectednesses of topological spaces, see for example Castellini [6, 7, 8], Castellini and Holgate [9], Clementino [10] and Clementino and Tholen [11] and their references. Of particular interest here are the results of Arhangel’skii and Wiegandt [1]. They showed that the theory of connectednesses and disconnectednesses of topological spaces is, from a categorical perspective, the same as the KA-radical and -semisimple theory of rings and related algebraic structures and also to the torsion theory of abelian categories. In algebra, these radicals can be obtained as H-radicals being the intersection of congruences (ideals) for which the quotients are semisimple. In [30] it was shown that this can also be done for topological spaces using the recently defined congruences for topological spaces: every disconnectedness of topological spaces (and so also every connectedness) can be obtained as an H-radical of the intersection of all congruences for which the corresponding weak quotient is in the disconnectedness. Moreover, the connectednesses of topological spaces can be characterized in terms of congruences in precisely the same way that the radical classes of associative rings are characterized in terms of ideals. But not all the salient features of the radical theory of rings hold for their topological counterparts. We shall see that a hereditary torsion theory (= ideal-hereditary H-radical) of topological spaces need not be a KA-radical. In fact, we give all the ideal-hereditary H-radicals; in particular also showing which ones are KA-radicals. Details of the work presented here can be found in [30, 31, 33].

We start with an overview of the congruence theory for topological spaces. A topological space will usually be denoted by \((X, \mathcal{T})\) and when there is no need to specify the topology, just by \(X\). The one-element space will be denoted by \(T\) and we identify all one-element spaces with \(T\): A trivial space is either \(T\) or the empty space \(\emptyset\): A subset of a topological space will always be regarded as a topological space with respect to the relative topology, unless explicitly mentioned otherwise. Homeomorphic topological spaces may be denoted by \(\cong\).

3.1. Topological congruences.

**Definition 3.1.** Let \((X, \mathcal{T})\) be a topological space. A congruence \(\rho\) on \(X\) is a pair \(\rho = (\sim, \mathcal{T})\) where:

\(\text{(C1)} \sim\) is an equivalence relation on \(X\).

\(\text{(C2)} \mathcal{T}\) is a topology on \(X\) with \(\mathcal{T} \subseteq \mathcal{T}\). \(\mathcal{T}\) is called the congruence topology.
(C3) For all \( x \in X \), \( x \in U \in \mathbb{T} \) implies \([x] \subseteq U\).

A congruence \( \rho = (\sim, \mathbb{T}) \) is a strong congruence on \( X \) if \( \mathbb{T} = \{ U \subseteq X | U \) is open in \( X \) and \( x \in U \) implies \([x] \subseteq U\)\).

Here \([x]\) denotes the equivalence class of \( x \in X \) with respect to the equivalence relation \( \sim \) on \( X \). By (C3), if \( U \in \mathbb{T} \), then \( U = \bigcup_{a \in U} [a] \) and if the congruence is strong, then \( \mathbb{T} \) consists precisely of all open sets \( U \) with this property. As examples of congruences on \((X, T)\) we mention the identity congruence on \( X \), \( \iota_X = (\sim, \mathbb{T}) \) where \(\sim = \emptyset\) if and only if \(a = b\), the universal congruence on \( X \), \( \nu_X = (\sim_\sim, \mathbb{T}_X) \) where \( \sim_\sim \) is the universal equivalence relation on \( X \) with \( a \sim_\sim b \) for all \(a, b \in X\) and \( \mathbb{T}_X \) is the indiscrete topology on \( X \) and the kernel of a continuous map \( f : X \to Y \), denoted by \( \ker f = (\sim_f, \mathbb{T}_f) \), with \( a \sim_f b \iff f(a) = f(b) \) for \( a, b \in X \) and \( \mathbb{T}_f = \{ f^{-1}(V) \mid V \subseteq Y \) open\}. For this congruence \([x] = f^{-1}(f(x))\) for all \( x \in X \) and for any \( U \in \mathbb{T}_f \), \( f^{-1}(f(U)) = U \). The strong kernel of \( f \), denoted by \( \sker f = (\sim_f, \mathbb{T}_{sf}) \), is the congruence on \( X \) with \( \sim_f \) as defined above and \( \mathbb{T}_{sf} = \{ U \subseteq X \mid U \) is open in \( X \) and \( U = f^{-1}(f(U)) \}\). Note that for a topological space \((X, T)\) and any topology \( T \) on \( X \) with \( T \subseteq T \), \((\sim, \mathbb{T})\) is a congruence on \( X \) and is called a trivial congruence on \( X \). A trivial congruence on \( X \) need not be the identity congruence \( \iota_X \), but a strong congruence is trivial if and only if it coincides with the identity congruence \( \iota_X \).

Every congruence \( \rho = (\sim, \mathbb{T}) \) on \( X \) determines a topological space \((X/\rho, T/\rho)\) with \( X/\rho = \{ [x] \mid x \in X \} \) and topology \( T/\rho = \{ \pi_\rho(U) \mid U \in \mathbb{T}\} \) where \( \pi_\rho : X \to X/\rho \) defined by \( \pi_\rho(x) = [x] \) is a surjective continuous map with \( \ker \pi_\rho = \rho \). This space \( X/\rho \) is called the weak quotient space determined by \( \rho \) and \( \pi_\rho \) is the weak quotient map (or just called the canonical map). In general \( X/\rho \) need not be a quotient space with \( \pi_\rho \), a quotient map. But if \( \rho \) is a strong congruence, \( X/\rho \) is a quotient space and \( \pi_\rho \) is a quotient map. Using condition (C3) and the surjectivity of the map \( \pi_\rho \), it can be shown that \( T/\rho = \{ W \subseteq X/\rho \mid \pi_\rho^{-1}(W) \in \mathbb{T}\} \). Two expected quotients are \((X/\iota_X, T/\iota_X) \cong (X, T)\) and for \( X \neq \emptyset \), \((X/\nu_X, T/\nu_X) \cong T\). In fact, it can be shown that \( X/\rho \cong X \iff \rho = \iota_X \); for \( X \neq \emptyset \), \( X/\rho \cong T \iff \rho = \nu_X \); and for any \( X \), \( \iota_X = \nu_X \iff X \cong T \) or \( X = \emptyset \).

**Ordering of congruences.** For two congruences \( \rho = (\sim_\rho, \mathbb{T}_\rho) \) and \( \gamma = (\sim_\gamma, \mathbb{T}_\gamma) \) on \( X \), \( \rho \) is contained in \( \gamma \), written as \( \rho \subseteq \gamma \), provided \( \sim_\rho \subseteq \sim_\gamma \) and \( \mathbb{T}_\rho \subseteq \mathbb{T}_\gamma \). For any congruence \( \rho \) on \( X \), \( \iota_X \subseteq \rho \subseteq \nu_X \) and \( \rho \subseteq \rho \). This ordering \( \subseteq \) is a partial order on the class \( \operatorname{Con}(X) := \{ \theta \mid \theta \) is a congruence on \( X\} \). In fact, \( \operatorname{Con}(X) \) is a bounded complete lattice. The meet of the congruences \( \theta_i = (\sim_i, \mathbb{T}_i) \in \operatorname{Con}(X) \), \( i \in I \), is given by the intersection of congruences \( \bigcap_{i \in I} \theta_i = (\sim_{\cap}, \mathbb{T}_{\cap}) \) defined by:

For \( a, b \in X \), \( a \sim_{\cap} b \iff a \sim_i b \) for all \( i \in I \) and the congruence topology \( \mathbb{T}_{\cap} \) is given by the topology on \( X \) with basis \( \mathcal{B} := \{ B \subseteq X \mid B \) is a finite intersection \( B = \bigcap_{j=1}^n U_j \) where \( U_j \in \mathbb{T}_{i_j} \) for some \( i_j \in I, j = 1, 2, 3, ..., n, n \geq 1 \}\). The join of the congruences \( \theta_i = (\sim_i, \mathbb{T}_i) \) is given by the sum \( \sum_{i \in I} \theta_i = (\sim_{\Sigma}, \mathbb{T}_{\Sigma}) \) defined by:

\[
\mathbb{T}_{\Sigma} := \bigcap_{i \in I} \mathbb{T}_i \text{ and for } a, b \in X,
\]

\( a \sim_{\Sigma} b \iff \) there are \( i_1, i_2, ..., i_n \in I \) and \( a_{i_1}, a_{i_2}, ..., a_{i_n} \in X, n \geq 2 \), such that \( a = a_{i_1} \sim_{i_1} a_{i_2} \sim_{i_2} a_{i_3} \sim_{i_3} ... \sim_{i_{n-2}} a_{i_{n-1}} \sim_{i_{n-1}} a_{i_n} = b \).
Note that if $\circ$ denotes the usual composition of two binary relations, then
\[ a \sim_X b \iff a(\sim_{i_1} \circ \sim_{i_2} \circ \sim_{i_3} \circ \ldots \circ \sim_{i_{n-1}} \circ \sim_{i_n})b \]
for some $n \geq 1, i_1, i_2, \ldots, i_n \in I$.

If $\theta_i$ is a strong congruence for all $i$, then it can easily be shown that the sum
\[ \sum_{i \in I} \theta_i \]
is also a strong congruence. The sum of two congruences $\alpha$ and $\beta$ on a space $X$ is written as $\alpha + \beta$. The usual relationships are valid: $\alpha + \iota_X = \alpha = \alpha \cap \iota_X$, $\alpha \cap \iota_X = \iota_X$, $\alpha \cup \iota_X = \iota_X$ and $\alpha \subseteq \beta \iff \alpha = \alpha \cap \beta \iff \beta = \alpha + \beta$. Since $x \sim x$ for any equivalence $\sim$, there is no loss of generality in writing $a \sim_{\alpha + \beta} b$ as $a = a_{i_1} \sim_{\alpha} a_{i_2} \sim_{\beta} a_{i_3} \sim_{\alpha} \ldots \sim_{\alpha} a_{i_{n-1}} \sim_{\beta} a_{i_n} = b$ for some $n \geq 1$ and $a_{i_1}, a_{i_2}, \ldots, a_{i_n} \in X$.

**Homeomorphism Theorems.**

**Theorem 3.2.** (First Homeomorphism Theorem) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{F})$ be a surjective continuous map with $\alpha = \ker f$. Then $(X/\alpha, \mathcal{T}/\alpha) \cong (Y, \mathcal{F})$.

Let $(X, \mathcal{T})$ be a topological space with $\rho = (\sim, \mathbb{T})$ a congruence on $X$ and $\pi_\rho : X \to X/\rho$ the canonical map $\pi_\rho(a) = [a]$. Let $S$ be a non-empty subset of $X$. Then $\rho$ induces a congruence on the subspace $S$, denoted by $S \cap \rho = (\sim_S, \mathbb{T}_S)$, by restricting $\rho$ to $S$ in a natural way: For all $a, b \in S$, $a \sim_S b \iff a \sim b$ and $\mathbb{T}_S = \{ U \cap S \mid U \in \mathbb{T} \}$. This congruence $S \cap \rho$ is called the restriction of the congruence $\rho$ to $S$. The subspace of $X/\rho$ determined by $\pi_\rho(S)$ will be denoted by $(S + \rho)/\rho$.

**Theorem 3.3.** (Second Homeomorphism Theorem) Let $(X, \mathcal{T})$ be a topological space, $S$ a subspace of $X$ and $\rho = (\sim, \mathbb{T})$ a congruence on $X$. Then the weak quotient space of $S$ determined by the congruence $S \cap \rho$ on $S$ is homeomorphic to the subspace $\{ [a] \mid a \in S \}$ of $(X/\rho, \mathcal{T}/\rho)$; i.e., $S/\rho \cong S + \rho/\rho$.

For the next homeomorphism theorem, we need the quotient of two congruences. Let $(X, \mathcal{T})$ be a topological space; let $\alpha = (\sim_\alpha, \mathbb{T}_\alpha)$ and $\beta = (\sim_\beta, \mathbb{T}_\beta)$ be two congruences on $X$ with $\alpha \subseteq \beta$. The quotient of $\beta$ by $\alpha$, written as $\beta/\alpha = (\sim_{\beta/\alpha}, \mathbb{T}_{\beta/\alpha})$, is the congruence on the weak quotient space $(X/\alpha, \mathcal{T}/\alpha)$ defined as follows: for $[a]_\alpha, [b]_\alpha \in X/\alpha$, $[a]_\alpha \sim_{\beta/\alpha} [b]_\alpha \iff a \sim_{\beta} b \cap \mathbb{T}_{\beta/\alpha} = \{ W \subseteq X/\alpha \mid W = \pi_\alpha(U) \text{ for some } U \in \mathbb{T}_\beta \}$ where $\pi_\alpha : X \to X/\alpha$ is the canonical map. In fact we have:

**Lemma 3.4.** Let $(X, \mathcal{T})$ be a topological space with $\alpha = (\sim_\alpha, \mathbb{T}_\alpha)$ a congruence on $X$. Then $\gamma = (\sim, \mathbb{T})$ is a congruence on the weak quotient space $(X/\alpha, \mathcal{T}/\alpha)$ if and only if $\gamma = \beta/\alpha$ for some congruence $\beta = (\sim_\beta, \mathbb{T}_\beta)$ on $X$ with $\alpha \subseteq \beta$.

**Theorem 3.5.** (Third Homeomorphism Theorem) Let $(X, \mathcal{T})$ be a topological space and let $\alpha = (\sim_\alpha, \mathbb{T}_\alpha)$ and $\beta = (\sim_\beta, \mathbb{T}_\beta)$ be two congruences on $X$ with $\alpha \subseteq \beta$. Then $\beta/\alpha$ is a congruence on $X/\alpha$ and $(X/\alpha)/(\beta/\alpha)$ is homeomorphic to $X/\beta$.

**Corollary 3.6.** Let $\theta$ be a congruence on $X$. Then there is a one-to-one correspondence between the set of all congruences $\alpha$ on $X$ for which $\theta \subseteq \alpha$ and the set of all congruences on $X/\theta$ given by $\alpha \mapsto \alpha/\theta$. This correspondence preserves containment, joins and intersections.
Image of a congruence. Let \( f : X \to Y \) be a surjective continuous map with \( \rho = (\sim_\rho, \mathbb{T}_\rho) \) a congruence on \( X \). Then \( f(\rho) = (\sim_{f(\rho)}, \mathbb{T}_{f(\rho)}) \) is the congruence on \( Y = f(X) \) defined by \( f(a) \sim_{f(\rho)} f(b) \Leftrightarrow \) there are \( a_1, a_2, \ldots, a_n \) in \( X \) with \( a \sim_\rho a_1, f(a_1) = f(a_2), a_2 \sim_\rho a_3, f(a_3) = f(a_4), a_4 \sim_\rho a_5, \ldots, a_{n-1} \sim_\rho a_n, f(a_n) = f(b) \) and \( \mathbb{T}_{f(\rho)} = \{ V \subseteq Y \text{ open} | f^{-1}(V) \in \mathbb{T}_\rho \} \). It is clear that \( f(a) \sim_{f(\rho)} f(b) \Leftrightarrow a \sim_{\rho + \ker f} b \Leftrightarrow a' \sim_{\rho + \ker f} b' \) for \( f(a') = f(a) \) and \( f(b') = f(b) \). We note that if \( \rho \) is strong, so is \( f(\rho) \) and in view of the First Homeomorphism Theorem, if \( \alpha = \ker f \), then \( f(\rho) = (\rho + \alpha)/\alpha \) where \( Y \cong X/\alpha \). As is to be expected, \( f(\iota_X) = \iota_Y \) and \( f(\iota_Y) = \iota_Y \).

Subdirect products. Let \( \prod_{i \in I} X_i \) denote the product of the topological spaces \( X_i, i \in I \) with \( p_j : \prod_{i \in I} X_i \to X_j \) the \( j \)-th projection. A subspace \( Y \) of \( \prod_{i \in I} X_i \) is said to be a subdirect product of the spaces \( X_i, i \in I \), if \( p_i(Y) = X_i \) for all \( i \in I \). As is the case for algebra, subdirect products can be characterized in terms of congruences:

**Theorem 3.7.** A topological space \( Y \) is a subdirect product of spaces \( X_i, i \in I \), if and only if for every \( i \in I \) there is a congruence \( \theta_i \) on \( Y \) such that \( Y/\theta_i \cong X_i \) and \( \bigcap_{i \in I} \theta_i = \iota_Y \).

### 3.2. Radical theory.

As is usual in radical theory, all considerations will be in a universal class \( \mathcal{W} \) of topological spaces. This means \( \mathcal{W} \) is a non-empty class of spaces which is hereditary (if \( Y \) is a subspace of \( X \in \mathcal{W} \), then \( Y \in \mathcal{W} \)) and closed under continuous images (if \( f : X \to Y \) is a surjective continuous map with \( X \in \mathcal{W} \), then also \( Y \in \mathcal{W} \)). Clearly then, \( \mathcal{W} \) contains the trivial spaces (empty space and one-point spaces). By assumption, any subclass of \( \mathcal{W} \) under discussion, will be an abstract class; i.e., it contains the trivial spaces and all homeomorphic copies of spaces from the class (this assumption will mostly not even be mentioned explicitly).

From [23], we recall: a class \( \mathcal{C} \) of topological spaces in \( \mathcal{W} \) is called a connectedness if there is a class of spaces \( \mathcal{P} \) in \( \mathcal{W} \) such that \( \mathcal{C} = \{ X \in \mathcal{W} | \text{every continuous mapping } X \to Y \in \mathcal{P} \text{ is constant} \} \) and a class of spaces \( \mathcal{D} \) in \( \mathcal{W} \) is called a disconnectedness if there is a class of spaces \( \mathcal{Q} \) in \( \mathcal{W} \) such that \( \mathcal{D} = \{ X \in \mathcal{W} | \text{every continuous mapping } Y \to X \text{ with } Y \in \mathcal{Q} \text{ is constant} \} \).

Let \( \mathcal{C} \) and \( \mathcal{D} \) be be subclasses of \( \mathcal{W} \). A subspace \( Y \) of a space \( X \) is called a \( C \)-subspace of \( X \) if \( Y \in \mathcal{C} \). A continuous image \( Y \) of a space \( X \) is called a non-trivial continuous image of \( X \) provided \( Y \) is not a trivial space. Likewise, a subspace \( S \) of \( X \) is a non-trivial subspace of \( X \) if it is not a trivial space. Motivated by the terminology from ring theory, a class \( \mathcal{C} \) of spaces is called a KA-radical class if it satisfies: \( X \in \mathcal{C} \Leftrightarrow \text{every non-trivial continuous image of } X \text{ has a non-trivial } \mathcal{C} \)-subspace. \( \mathcal{D} \) is a KA-semisimple class if it satisfies: \( X \in \mathcal{D} \Leftrightarrow \text{every non-trivial subspace of } X \text{ has a non-trivial continuous image which is in } \mathcal{D} \). Arhangel’skii and Wiegandt [1] have shown that a class of spaces \( \mathcal{C} \) is a connectedness if and only if it is a KA-radical class and a class of spaces \( \mathcal{D} \) is a disconnectedness if and only if it is a KA-semisimple class. We explicitly recall these two statements for later reference.
Proposition 3.8. ([1]) Let \( C \) and \( D \) be abstract classes of topological spaces in \( \mathcal{W} \).

(1) \( C \) is a connectedness if and only if \( C \) satisfies the condition: \( X \in C \) if and only if every non-trivial continuous image of \( X \) has a non-trivial \( C \)-subspace.

(2) \( D \) is a disconnectedness if and only if \( D \) satisfies the condition: \( X \in D \) if and only if every non-trivial subspace of \( X \) has a non-trivial continuous image in \( D \).

Two operators \( \mathcal{U} \) and \( S \) on a class \( M \subseteq \mathcal{W} \), called the upper radical operator and semisimple operator respectively, are defined by:

\[
\mathcal{U}M = \{ X \in \mathcal{W} \mid X \text{ has no non-trivial continuous image in } M \} \quad \text{and} \quad SM = \{ X \in \mathcal{W} \mid X \text{ has no non-trivial subspace in } M \}.
\]

Note that \( M \cap \mathcal{U}M = \{ T, \emptyset \} = M \cap SM \). The class \( \mathcal{U}M \) is always closed under continuous images and \( SM \) is always hereditary. If \( M \) is hereditary, then \( \mathcal{U}M \) is a connectedness and if \( M \) is closed under continuous images, then \( SM \) is a disconnectedness. It can be shown that any connectedness \( C \) is closed under continuous images; hence \( SC \) is a disconnectedness corresponding to \( C \). Likewise, any disconnectedness \( D \) is hereditary, hence \( UD \) is a connectedness called the connectedness corresponding to \( D \). Moreover, \( C \subseteq \mathcal{W} \) is a connectedness if and only if \( C = \mathcal{U}SC \) and \( D \subseteq \mathcal{W} \) is a disconnectedness if and only if \( D = SUD \).

A mapping \( \sigma \) which assigns to each \( X \in \mathcal{W} \) a congruence \( \sigma(X) = (\sim_{\sigma_X}, T_{\sigma_X}) \) on \( X \), is called a H-radical on \( \mathcal{W} \) if it satisfies the following two conditions:

(H1) For any surjective continuous map \( f : X \to Y \), \( f(\sigma_X) \subseteq \sigma_Y \).

(H2) For all \( X \in \mathcal{W} \), \( \sigma(X/\sigma_X) = \iota_{X/\sigma_X} \), the identity congruence on \( X/\sigma_X \).

The class \( S_\sigma = \{ X \in \mathcal{W} \mid \sigma_X = \iota_X \} \) is called the associated semisimple class and \( R_\sigma = \{ X \in \mathcal{W} \mid \sigma_X = \nu_X \} \) the associated radical class. Note that \( S_\sigma \cap R_\sigma = \{ T, \emptyset \} \), \( R_\sigma \) is always closed under continuous images and \( R_\sigma = \mathcal{U}S_\sigma \).

Hoehnke radicals are very general as is shown in the next result which also gives most of the salient features of these radicals:

Theorem 3.9. (1) Let \( \sigma \) be an H-radical on \( \mathcal{W} \). Then, for every \( X \in \mathcal{W} \), \( \sigma(X) = \cap \{ \theta \mid \theta \text{ is a congruence on } X \text{ for which } X/\theta \in S_\sigma \} \) and \( S_\sigma \) is closed under subdirect products.

(2) Conversely, let \( M \subseteq \mathcal{W} \) be any abstract class. Then \( \sigma \) defined by \( \sigma(X) = \cap \{ \theta \mid \theta \text{ is a congruence on } X \text{ for which } X/\theta \in M \} \) for all \( X \in \mathcal{W} \) is an H-radical on \( \mathcal{W} \) and \( S_\sigma = \overline{M} \), the subdirect closure of \( M \).

In [30] it was shown that any disconnectedness (and hence also every connectedness) can be obtained from an H-radical provided it fulfills two additional requirements. An H-radical \( \sigma \) on \( \mathcal{W} \) is:

- complete if \( X \in \mathcal{W} \) and \( \theta \) is a strong congruence on \( X \) with \([a]_\theta \in R_\sigma \) for all \( a \in X \), then \( \theta \subseteq \sigma_X \); and

- idempotent if for all \( X \in \mathcal{W} \) and \( a \in X \), we have \([a]_{\sigma_X} \in R_\sigma \).

Then:
Theorem 3.10. (1) Let $\sigma$ be an $H$-radical on $W$. Suppose $\sigma$ is complete, idempotent and for all $X \in W$, the congruence $\sigma_X$ is strong. Then $S_\sigma$ is a disconnectedness and $R_\sigma = US_\sigma$ is a connectedness.

(2) Let $D \subseteq W$ be a disconnectedness. Then there is an $H$-radical $\sigma$ on $W$ which is complete, idempotent, for all $X \in W$ the congruence $\sigma_X$ is strong, $S_\sigma = D$ and $R_\sigma = UD$. The congruence $\sigma_X$ is given by $\sigma_X = \cap \{ \theta \mid \theta$ is a congruence on $X$ for which $X/\theta \in D \}$. 

In view of this result, a Hoehnke radical $\sigma$ on $W$ which is complete, idempotent and for which $\sigma_X$ is a strong congruence for all $X$ is called a $KA$-radical. Next we give a characterization of a connectedness in terms of congruences. We start by recalling two fundamental theorems for connectedness and disconnectednesses from [1]. For this we firstly fix some terminology. Let $C$ be a class of topological spaces. Then $C$ is second additive if, whenever a space $X$ is covered by a family $F$ of $C$-subspaces with $\cap F \neq \emptyset$, then $X \in C$; and $C$ is $q$-reversible if $f : X \to Y$ is a quotient map with $Y$ and $f^{-1}(y)$ in $C$ for all $y \in Y$, then also $X \in C$. Recall, a subspace $Y$ of a space $X$ is a $C$-subspace if $Y \in C$.

Theorem 3.11. ([1]) Let $C$ be an abstract class of topological spaces in $W$. Then statements (1) and (2) below are equivalent:

1. $C$ is a connectedness.
2. $C$ satisfies the following three conditions:
   a. $C$ is closed under continuous images.
   b. $C$ is second additive.
   c. $C$ is $q$-reversible.

Theorem 3.12. ([1]) Let $D \subseteq W$ be a disconnectedness with associated connectedness $C = UD$. For any $X \in W$:

1. There is a quotient map $q_X : X \to X_s$ with $X_s \in D$.
2. For every surjective continuous map $g : X \to Y \in D$, there is a continuous map $h : X_s \to Y$ such that $h \circ q_X = g$.
3. For every $t \in X_s$, $q_X^{-1}(t) \in C$.
4. If $Y$ is a subspace of $X$ with $Y \in C$, then $Y \subseteq q_X^{-1}(t)$ for some $t \in X_s$.

For an abstract class $C$ of topological spaces, a congruence $\rho$ on $X$ is a $C$-congruence if it is a strong congruence and for each $x \in X$, $[x] \in C$ holds. It will be useful to rephrase Theorem 3.12 above in terms of congruences.

Theorem 3.13. ([1]) Let $D \subseteq W$ be a disconnectedness with associated connectedness $C = UD$. For any $X \in W$:

1. The congruence $\sigma_X = \cap \{ \theta \mid \theta$ is a congruence on $X$ for which $X/\theta \in D \}$ is a strong congruence and the weak quotient map $\pi_X : X \to X/\sigma_X$ is a quotient map with $X/\sigma_X \in D$.
2. For every surjective continuous map $g : X \to Y \in D$, there is a continuous map $h : X/\sigma_X \to Y$ such that $h \circ \pi_X = g$ (or equivalently, $\sigma_X \subseteq \ker g$).
3. $\sigma_X$ is a $C$-congruence on $X$.
4. If $Y$ is a $C$-subspace of $X$, then $Y \subseteq [x]_{\sigma_X}$ for some $x \in X$. 

For our characterization of connectednesses, we start with:

**Proposition 3.14.** Let \( C \) be an abstract class of topological spaces in \( \mathcal{W} \). Then \( C \) is a connectedness if and only if \( C \) satisfies the condition: \( X \in C \) if and only if every non-trivial continuous image of \( X \) has a non-trivial \( C \)-congruence.

Let \( C \) be an abstract class of topological spaces. For any space \( X \), let \( \rho(X) \) be the congruence on \( X \) defined by \( \rho(X) = \sum \{ \alpha \mid \alpha \text{ is a } C \text{-congruence on } X \} \). Mostly we will write \( \rho(X) \) as \( \rho_X \) but there are occasions when the former will be better to use. A sum of strong congruences is a strong congruence, hence \( \rho_X \) is a \( C \)-congruence on \( X \) if and only if \( X \in C \). When \( C \) is a connectedness, then more can be said as will be seen below.

**Proposition 3.15.** Let \( C \) be a connectedness with associated disconnectedness \( D = SC \) in \( \mathcal{W} \). Then \( \rho_X \) is a \( C \)-congruence and \( \rho_X = \sigma_X \) for all \( X \) where \( \rho_X = \sum \{ \alpha \mid \alpha \text{ is a } C \text{-congruence on } X \} \) and \( \sigma_X = \cap \{ \theta \mid \theta \text{ is a congruence on } X \text{ with } X/\theta \in D \} \).

A class \( C \) of topological spaces is closed under extensions if it satisfies: whenever \( \alpha \) and \( \beta \) are congruences on \( X \) with \( \alpha \subseteq \beta \), \( \alpha \) a \( C \)-congruence on \( X \) and \( \beta/\alpha \) a \( C \)-congruence on \( X/\alpha \), then \( \beta \) is a \( C \)-congruence on \( X \). This brings us to the main result of this section which shows that a connectedness of topological spaces can be characterized in terms of the congruence \( \rho_X \) and also in terms of \( C \)-congruences (for associative rings, one would say in terms of the radical and in terms of the radical ideals respectively). In particular, this shows that the characterization of connectednesses of topological spaces is in complete harmony with the characterization of the radical classes of associative rings (compare the next theorem with Theorem 2.1 above). A last concept we need is: A class \( C \) of topological spaces has the inductive property if it satisfies the following condition: whenever \( \alpha_1 \subseteq \alpha_2 \subseteq \alpha_3 \subseteq \ldots \) is a chain of \( C \)-congruences on a topological space \( X \in \mathcal{W} \), then \( \sum \{ \alpha_i \mid i = 1, 2, 3, \ldots \} \) is a \( C \)-congruence on \( X \).

**Theorem 3.16.** Let \( C \) be an abstract class of spaces in \( \mathcal{W} \). Then statements (1), (2) and (3) below are equivalent:

1. \( C \) is a connectedness.
2. \( C \) satisfies the following three conditions:
   a. For every surjective continuous map \( f : X \rightarrow Y \), \( f(\rho_X) \subseteq \rho_Y \).
   b. For every space \( X \), \( \rho_X \) is a \( C \)-congruence on \( X \) and it contains all \( C \)-congruences on \( X \).
   c. For every space \( X \), \( \rho(X/\rho_X) = \nu_X/\rho_X \).
3. \( C \) satisfies the following three conditions:
   a. If \( \alpha \) and \( \beta \) are congruences on \( X \) with \( \alpha \subseteq \beta \) and \( \beta \) a \( C \)-congruence, then \( \beta/\alpha \) is a \( C \)-congruence on \( X/\alpha \).
   b. \( C \) has the inductive property.
   c. \( C \) is closed under extensions.
3.3. Hereditary torsion theories. For this section, the universal class \( \mathcal{W} \) is the class of all topological spaces. For any set \( X \), we will use \( \mathcal{I}_X \) and \( \mathcal{D}_X \) to denote the indiscrete and discrete topologies on \( X \) respectively. We use \( I_2 \) to denote the two-element indiscrete space, \( S_2 \) for the Sierpiński space (with topology \( S_2 = \{\emptyset, \{0\}, \{0, 1\}\} \)) and \( D_2 \) for the two-element discrete space. On \( I_2 \) there are only two congruences \( I_{I_2} \) and \( I_{D_2} \) and on \( S_2 \) there are three namely \( S_{I_2}, (\emptyset, I_2) \) and \( S_{D_2} \). The space \( D_2 \) has five congruences \( D_{I_2}, (\emptyset, S_2), (\emptyset, \{0, 1\}, \{0, 1\}), (\emptyset, I_2) \) and \( D_{D_2} \). Recall, an object \( Q \) in a category is called injective if for any given morphism \( g : C \to Q \) and monomorphism \( f : C \to B \) there exists a morphism \( h : B \to Q \) such that \( h \circ f = g \). It is known (in any case easy to prove) that a topological space is injective in the category of all topological spaces precisely when it is an indiscrete space.

Definition 3.17. Let \( \rho \) be a Hoehnke radical of topological spaces. Then \( \rho \) is called:

1. \( r \)-hereditary if for every space \( X \) and subspace \( Y \), \( \rho_X \cap Y \subseteq \rho_Y \).
2. \( s \)-hereditary if for every space \( X \) and subspace \( Y \), \( \rho_Y \subseteq \rho_X \cap Y \).
3. Ideal-hereditary if it is both \( r \)-hereditary and \( s \)-hereditary.
4. A hereditary torsion theory if it is an ideal-hereditary Hoehnke radical.

Then:

Proposition 3.18. Let \( \rho \) be an ideal-hereditary Hoehnke radical (= hereditary torsion theory) in a universal class of topological spaces. Then \( \rho \) is idempotent, complete and both the radical class \( \mathcal{R}_\rho \) and the semisimple class \( \mathcal{S}_\rho \) are hereditary.

For all the well-known classes of algebras, any \( \rho \) as above (ideal-hereditary Hoehnke radical) will be a Kurosh-Amitsur radical. For topological spaces, this need not be the case. To conclude this section, we give all the ideal-hereditary Hoehnke radicals \( \rho \) of topological spaces. There are exactly five such radicals of which three are Kurosh-Amitsur radicals, i.e. for such \( \rho \), the classes \( \mathcal{R}_\rho \) and \( \mathcal{S}_\rho \) form a corresponding pair of connectednesses and disconnectednesses. This sounds better than it actually is, of these three, two are trivial with the connectedness and disconnectedness coinciding with the class of all spaces respectively. More details about the radical-theoretic properties of the classes of topological spaces in the next result can be found in [31].

Theorem 3.19. Let \( \rho \) be an ideal-hereditary Hoehnke radical of topological spaces. Then \( \rho \) is one of the following five radicals:

(a) \( \rho_X = \{\} \) for all \( X \). This is a KA-radical with \( \mathcal{R}_\rho \) the class of all spaces and \( \mathcal{S}_\rho = \{T\} \).
(b) \( \rho_X = (\sim, T_X) \) is the congruence with \( x \sim y \) iff for any \( U \subseteq X \) open, \( x \in U \Leftrightarrow y \in U \) and \( T_X = T \). This is a KA-radical with \( \mathcal{R}_\rho = \{X \mid X \text{ is an indiscrete space}\} \) and \( \mathcal{S}_\rho = \{X \mid X \text{ is a } T_0\text{-space}\} \).
(c) \( \rho_X = \mathcal{I}_X \) for all \( X \). This is a KA-radical with \( \mathcal{R}_\rho = \{T\} \) and \( \mathcal{S}_\rho \) is the class of all spaces.
(d) \( \rho_X = (\emptyset, \mathcal{I}_X) \) for all \( X \) which is not a KA-radical. Here \( \mathcal{R}_\rho = \{T\} \) and \( \mathcal{S}_\rho \).
In the next section we shall see that the radical-theoretic properties presented here for topological spaces are also valid for graphs that can have loops.

4. Graphs that admit loops. As for topological spaces in the previous section, here we present the congruence approach to the connectednesses and disconnectednesses of graphs that allow loops. Connectednesses and disconnectednesses for such graphs have been defined and developed by Fried and Wiegandt [14] showing that they are just the KA-radical and -semisimple classes respectively in this category. Here it will be shown that they can be obtained as H-radicals and then they will be characterized using congruences with conditions resembling the classical algebraic conditions for characterizing radical classes of rings. We have seen in the previous section that there are hereditary torsion theories of topological spaces that can be added to those of S-acts as examples of ideal-hereditary H-radicals which are not KA-radicals. Graphs provide a third example. In fact, there are exactly eight ideal-hereditary H-radicals for graphs and of these, only three are KA-radicals. Details of the work presented here can be found in [4, 34, 35].

4.1. Congruences. We start with some graph theoretic preliminaries. A graph $G$ with vertex set $V$ and edge set $E$ will typically be denoted by $G = (V_G, E_G)$, often without the subscripts. When we write $a \in G$, it actually means $a$ is a vertex of $G$, i.e., $a \in V_G$. By a graph, we mean a non-empty vertex set, edges are not directed, no multiple edges are allowed but loops are. For $a, b \in G$, an edge between $a$ and $b$ is written as $ab$ and $aa$ is the loop at $a$. The set of all possible edges on a graph $G$ is denoted by $C_G := \{ab \mid a, b \in V_G\}$. A (graph) homomorphism is an edge preserving mapping from the vertex set of a graph into the vertex set of a graph. A strong homomorphism is a homomorphism that sends “no edges” to “no edges” and if it is also a bijection, it is called an isomorphism. Isomorphic graphs $G$ and $H$ will be denoted by $G \cong H$. For a graph $G = (V_G, E_G)$, a subgraph $H = (V_H, E_H)$ of $G$ is a graph with $V_H \subseteq V_G$ and $E_H \subseteq \{ab \mid a, b \in V_H \text{ and } ab \in E_G\}$. When $E_H = \{ab \mid a, b \in V_H \text{ and } ab \in E_G\}$, then $H$ is called an induced subgraph of $G$. For a homomorphism $f : G \to H$, the image graph $f(G)$ will always be the induced subgraph of $H$ on the vertex set $f(V_G)$. In general, unless mentioned otherwise, if a subset $V_H$ of $V_G$ is regarded as a graph, it will be the subgraph induced by $G$ on $V_H$. There are two (non-isomorphic) one-vertex graphs, called the trivial graphs; the one with a loop $T_0$ and the one without a loop $T$. For an equivalence relation $\sim$ on a vertex set $V$, we use $[a]$ to denote the equivalence class of $a \in V$. For $A, B \subseteq V_G$, $AB$ is the set $AB = \{ab \mid a \in A, b \in B\} \subseteq C_G$. In particular, $[a][b] = \{st \mid s \in [a], t \in [b]\}$.

**Definition 4.1.** Let $G = (V_G, E_G)$ be a graph. A congruence on $G$ is a pair $\theta = (\sim, \mathcal{E})$ which fulfills the following three conditions:
(i) $\sim$ is an equivalence relation on $V_G$.
(ii) $E$ is a set of unordered pairs of elements from $V_G$, called the congruence edge-set, with $E_G \subseteq E \subseteq C_G$.
(iii) (Substitution Property of $E$ with respect to $\sim$) for $x, y \in V_G$, if $xy \in E$, then $[x][y] \subseteq E$.

A strong congruence on $G$ is a pair $\theta = (\sim, E(\sim))$ where $\sim$ is an equivalence relation on $V_G$ and $E(\sim) = \{xy \mid x, y \in V_G \text{ and } [x][y] \cap E_G \neq \emptyset\}$.

Congruences are partially ordered by the relation “contained in”: for two congruences $\alpha = (\sim_\alpha, E_\alpha)$ and $\beta = (\sim_\beta, E_\beta)$ on $G$, $\alpha$ is contained in $\beta$, written as $\alpha \subseteq \beta$, if $\sim_\alpha \subseteq \sim_\beta$ and $E_\alpha \subseteq E_\beta$. Let $\equiv$ denote the identity relation on $V_G$ (also called the diagonal; i.e., $x \equiv y$ if and only if $x = y$). The congruence $\iota_G := (\sim, E_G)$ is called the identity congruence on $G$ and is the smallest congruence on $G$. The universal congruence on $G$ is the pair $\upsilon_G = (\sim_0, E)$ where $\sim_0$ is the universal relation (i.e., $a \sim_0 b$ for all $a, b \in V_G$) and $E = \{ab \mid a, b \in V_G\}$. Any congruence on $G$ contains $\iota_G$ and is contained in $\upsilon_G$. Given any graph homomorphism $f : G \to H$, the kernel of $f$, written as $\ker f = (\sim_f, E_f)$, is the congruence with $\sim_f = \{((x, y) \mid x, y \in V_G, f(x) = f(y))\}$ and $E_f = \{uv \mid u, v \in V_G, f(u)f(v) \in E_H\}$. With $f$ is also associated the strong kernel of $f$, defined by $\sker f = (\sim_f, E(\sim_f))$. This is a strong congruence on $G$ and $\sker f \subseteq \ker f$; in fact, if $\theta = (\sim_f, E)$ is any congruence on $G$ for some $E$, then $\sker f \subseteq \theta$. If $f$ is a strong homomorphism, then $\ker f = \sker f$. Given any congruence $\theta = (\sim, E)$ on a graph $G = (V_G, E_G)$, we define a new graph, denoted by $G/\theta = (V_{G/\theta}, E_{G/\theta})$ and called the quotient of $G$ modulo $\theta$, by taking $V_{G/\theta} = \{[x] \mid x \in V_G\}$ and $E_{G/\theta} = \{[xy] \mid xy \in E\}$. The natural or canonical mapping $p_{\theta} : G \to G/\theta$ given by $p_{\theta}(x) = [x]$ is a surjective homomorphism with $\ker p_{\theta} = \theta$. It is a strong congruence, then $p_{\theta}$ is a strong homomorphism with $\sker p_{\theta} = \theta$. In general, if we take $\theta = (\sim, E)$ for some suitable $E$ to make $\theta$ a congruence on $G$, then $G/\theta$ is the graph with vertex set $V_{G/\theta} = V_G$ and edge set $E_{G/\theta} = E$ (here we identify $[x] = \{x\}$ with $x$). If $\upsilon_G$ is the universal congruence on $G$, then $G/\upsilon_G$ is isomorphic to the trivial graph $T_0$ with a loop.

**Ordering of congruences.** For a given graph $G$, we denote the set of all congruences on $G$ by $\text{Con}(G)$ which is a partially ordered set with respect to containment $\subseteq$. But we can say more. $\text{Con}(G)$ is a bounded complete lattice with the meet and join of $\theta_i \in \text{Con}(G), i \in I$, given by the intersection $\bigcap_{i \in I} \theta_i$ and sum $\sum_{i \in I} \theta_i$ respectively. These are defined as follows: for any collection of congruences $\{\theta_i = (\sim_i, E_i) \mid i \in I\} \subseteq \text{Con}(G)$ the greatest lower bound in $\text{Con}(G)$ is given by $\bigcap_{i \in I} \theta_i = (\sim_\cap, E_\cap)$ where $a \sim_\cap b \iff a \sim_i b$ for all $i \in I$ and $ab \in E_\cap \iff ab \in E_i$ for all $i \in I$. The smallest upper bound is $\sum_{i \in I} \theta_i = (\sim_\Sigma, E_\Sigma)$ where $a \sim_\Sigma b \iff \exists c_1, c_2, \ldots, c_n \in G$ with $a = c_1 \sim_{i_1} c_2 \sim_{i_2} c_3 \sim_{i_3} \ldots \sim_{i_{n-1}} c_n = b$ for some $i_1, i_2, \ldots, i_{n-1} \in I, n \geq 2$ and $E_\Sigma = \{ab \mid ab \in [x]_\Sigma[y]_\Sigma \text{ for some } xy \in \bigcup_{i \in I} E_i\}$. If $\theta_i$ is strong for all $i \in I$, then so is $\sum_{i \in I} \theta_i$. For two congruences $\alpha$ and $\beta$, we write the
Image of a congruence. Let \( f : G \to H \) be a surjective homomorphism with \( \theta = (\sim_\theta, \mathcal{E}_\theta) \) a congruence on \( G \) and \( \alpha = \ker f \). The image of \( \theta \) under \( f \) is the congruence \( f(\theta) = (\sim_{f(\theta)}, \mathcal{E}_{f(\theta)}) \) of \( H \) where \( f(a) \sim_{f(\theta)} f(b) \Leftrightarrow a \sim_{\alpha + \theta} b \). Recall that \( x \sim_{\ker f} y \Leftrightarrow f(x) = f(y) \); hence \( f(a) \sim_{f(\theta)} f(b) \Leftrightarrow \exists c_1, c_2, \ldots, c_n \in G \) with \( f(a) = f(c_1), c_1 \sim \theta c_2, f(c_2) = f(c_3), c_3 \sim \theta c_4, \ldots, f(c_{n-1}) = f(c_n), c_n \sim \theta b \). Let \( \mathcal{E}_{f(\theta)} = \{cd | cd \in [f(x)f(y)] \mathcal{E}_{f(\theta)} \mathcal{E}_{f(\theta)} \} \) for some \( x, y \in G \) with \( xy \in \mathcal{E}_\theta \) or \( f(x)f(y) \in E_H \). It can be checked that \( \sim_{f(\theta)} \) is well-defined and in the chain of equalities and equivalences, it does not matter whether one starts or ends with an equality or equivalence. Then \( f(\theta) \) is a congruence which is strong if \( \theta \) is strong.

Isomorphism theorems for congruences.

**Theorem 4.2.** (First Isomorphism Theorem) Let \( f : G \to H \) be a surjective homomorphism. Then \( G/\ker f \) is isomorphic to \( H \).

Let \( G \) be a graph with induced subgraph \( H \). Then a congruence \( \theta = (\sim, \mathcal{E}) \) on \( G \) induces a congruence \( H \cap \theta = (\sim_H, \mathcal{E}_H) \) on \( H \) with \( \sim_H = (V_H \times V_H) \cap \sim = \{(a, b) | a, b \in V_H \text{ and } a \sim b \} \) and \( \mathcal{E}_H = \{ab | a, b \in V_H \} \cap \mathcal{E} = \{ab | a, b \in V_H \text{ with } ab \in \mathcal{E} \} \). The mapping \( f : H \to G/\theta \) defined by \( f(a) = [a] \) for all \( a \in V_H \) is a homomorphism with \( \ker f = H \cap \theta \). Now \( f(V_H) \) is a set of vertices of \( G/\theta \) on which we form the induced subgraph of \( G/\theta \), denoted by \( (H + \theta)/\theta \). Then, by the First Isomorphism Theorem, we have:

**Theorem 4.3.** (Second Isomorphism Theorem) Let \( H \) be an induced subgraph of a graph \( G \). Let \( \theta \) be a congruence on \( G \). Then \( H \cap \theta \) as defined above is a congruence on \( H \) and \( H/H \cap \theta \cong (H + \theta)/\theta \) where \( (H + \theta)/\theta \) is the induced subgraph of \( G/\theta \) on the vertex set \( \{[a] | a \in V_H \} \).

**Theorem 4.4.** (Third Isomorphism Theorem) Let \( G \) be a graph with two congruences \( \theta_1 = (\sim_1, \mathcal{E}_1) \) and \( \theta_2 = (\sim_2, \mathcal{E}_2) \) on \( G \) for which \( \theta_1 \subseteq \theta_2 \). Then \( \theta_2/\theta_1 := (\sim, \mathcal{E}) \) is a congruence on \( G/\theta_1 \) where \( [a]_1 \sim_2 [b]_1 \Leftrightarrow a \sim_2 b \) and \( [a]_1[b]_1 \in \mathcal{E} \Leftrightarrow ab \in \mathcal{E}_2 \). Moreover, \( (G/\theta_1)/(\theta_2/\theta_1) \) is isomorphic to \( G/\theta_2 \).

**Corollary 4.5.** Let \( G \) be a graph with \( \theta \) a fixed congruence on \( G \). Any congruence \( \xi \) on the graph \( G/\theta \) is of the form \( \alpha/\theta \) for some congruence \( \alpha \) on \( G \) with \( \theta \subseteq \alpha \). Moreover, there is a one-to-one correspondence between \( \{\alpha | \alpha \text{ is a congruence on } G \text{ with } \theta \subseteq \alpha \} \) and \( \text{Con}(G/\theta) \) which preserves inclusions, intersections and unions of congruences.

**Subdirect product of graphs.** For an index set \( I \), let \( G_i = (V_i, E_i) \) be a graph for all \( i \in I \). The product \( \prod_{i \in I} G_i \) of the graphs \( G_i \) is the graph \( \prod_{i \in I} G_i := (\prod_{i \in I} V_i, E) \) where \( \prod_{i \in I} V_i \) is just the usual Cartesian product of the sets \( V_i \) and \( E = \{fg | f, g \in \prod_{i \in I} V_i \text{ with } f(i)g(i) \in E_i \text{ for all } i \in I \} \). For every \( j \in I \), the \( j \)-th
projection \( \pi_j : \prod_{i \in I} G_i \rightarrow G_j \) defined by \( \pi_j(f) = f(j) \) for all \( f \in \prod_{i \in I} V_i \) is a surjective homomorphism. An induced subgraph \( H \) of \( \prod_{i \in I} G_i \) is called a subdirect product of the graphs \( G_i, i \in I \), provided the restriction of each projection \( \pi_j \) to \( H \) is a surjective mapping onto \( G_j \). As in universal algebra, subdirect products can be characterized in terms of congruences and quotients:

**Theorem 4.6.** A graph \( G \) is a subdirect product of graphs \( G_i, i \in I \), if and only if for every \( i \in I \) there are congruences \( \theta_i \) on \( G \) with \( G_i \) isomorphic to \( G/\theta_i \) and \( \bigcap_{i \in I} \theta_i = \iota_G \).

### 4.2. Radical theory.

All radical theoretic considerations are in a universal class of graphs \( \mathcal{W} \). This means \( \mathcal{W} \) is non-empty, closed under homomorphic images and closed under the taking of subgraphs (= strongly hereditary). We do not distinguish between isomorphic graphs. From the definition, it follows that \( \mathcal{W} \) contains a one-vertex graph, and consequently also the class \( \mathcal{T} = \{ T_0, T \} \) of all trivial graphs is contained in \( \mathcal{W} \). Note that there is a unique congruence on \( T_0 \) since here \( \iota_{T_0} = \iota_{T_0} \), but on \( T \) these two congruences are distinct. A class of graphs \( \mathcal{M} \) in \( \mathcal{W} \) is an abstract class provided it is closed under isomorphic copies and it contains the trivial graph \( T_0 \). All subclasses of \( \mathcal{W} \) under consideration will be assumed to be abstract, even though it may not always be explicitly stated. Since \( G/\iota_G \cong T_0 \) for any graph \( G \), there is always at least one congruence \( \theta \) on a graph \( G \) for which \( G/\theta \) is in any abstract class. For a class \( \mathcal{M} \) in \( \mathcal{W} \), we use \( \overline{\mathcal{M}} \) to denote the subdirect closure of \( \mathcal{M} \), i.e., the class of all graphs that are subdirect products of graphs from \( \mathcal{M} \). Clearly \( \mathcal{M} \subseteq \overline{\mathcal{M}} \) and we say \( \mathcal{M} \) is subdirectly closed if \( \mathcal{M} = \overline{\mathcal{M}} \). We start with the definition of the Hoehnke radical for which we will need the image of a congruence under a homomorphism as defined in the previous section.

**Definition 4.7.** An H-radical on \( \mathcal{W} \) is a function \( \varrho \) that assigns to every graph \( G \) in \( \mathcal{W} \) a congruence \( \varrho(G) = \varrho_G \) on \( G \) such that:

(H1) For every surjective homomorphism \( f : G \rightarrow H \), \( f(\varrho(G)) \subseteq \varrho(H) \).

(H2) For any graph \( G \), \( \varrho(G/\varrho_G) = \iota_G/\varrho_G \), the identity congruence on \( G/\varrho_G \).

For an H-radical \( \varrho \), if \( \varrho(G) = \iota_G \), then \( G \) is called semisimple (actually, \( \varrho \)-semisimple), the class \( \mathcal{S}_\varrho = \{ G \in \mathcal{W} \mid \varrho(G) = \iota_G \} \) is called the associated semisimple class and \( \mathcal{R}_\varrho = \{ G \in \mathcal{W} \mid G/\varrho_G \) is a trivial graph} \) is the associated radical class. For an H-radical \( \varrho \) on \( \mathcal{W} \), note that \( \{ T_0 \} \subseteq \mathcal{S}_\varrho \cap \mathcal{R}_\varrho \subseteq \mathcal{T} \subseteq \mathcal{R}_\varrho \) and the radical class \( \mathcal{R}_\varrho \) is always homomorphically closed. If \( G/\varrho_G \) is trivial, then \( \varrho_G = (\sim, \mathcal{C}_G) \) for \( E_G \neq \emptyset \) and when \( E_G = \emptyset \), then \( \varrho_G \) can be \( (\sim, \mathcal{C}_G) \) or \( (\sim, \emptyset) \). An H-radical is very general and is always of a prescribed form as the next result shows.

**Theorem 4.8.** Let \( \varrho \) be a mapping that assigns to any graph \( G \) in \( \mathcal{W} \) a congruence \( \varrho(G) = \varrho_G \) on \( G \). Then \( \varrho \) is an H-radical on \( \mathcal{W} \) if and only if there is an abstract class of graphs \( \mathcal{M} \) in \( \mathcal{W} \) such that for all \( G \in \mathcal{W} \), \( \varrho(G) = \cap \{ \theta \mid \theta \) is a congruence on \( G \) for which \( G/\theta \in \mathcal{M} \} \). Furthermore, \( \mathcal{S}_\varrho = \overline{\mathcal{M}} \).

The salient properties of an H-radical are contained in the next corollary.
Corollary 4.9. (1) The semisimple class of any \( H \)-radical is subdirectly closed.

(2) For an \( H \)-radical \( \varrho \), \( \varrho(G) \) is the smallest congruence on \( G \) for which \( G/\varrho(G) \) is semisimple (i.e., if \( \theta \) is a congruence on \( G \) with \( G/\theta \in \mathcal{S}_\varrho \), then \( \varrho(G) \subseteq \theta \)). Or, equivalently, \( G/\varrho(G) \) is the largest semisimple image of \( G \) (in the following sense: if \( g : G \to H \) is a surjective homomorphism with \( H \in \mathcal{S}_\varrho \), then there is a homomorphism \( h : G/\varrho(G) \to H \) such that \( h \circ p = g \) where \( p : G \to G/\varrho(G) \) is the canonical quotient map).

(3) For any abstract class of graphs \( \mathcal{M} \) in \( \mathcal{W} \) and \( G \in \mathcal{W} \), define \( \varrho(G) := \bigcap \{ \theta \mid \theta \in \text{Con}(G) \text{ with } G/\theta \in \mathcal{M} \} \). Then \( \varrho \) is an \( H \)-radical with \( \mathcal{S}_\varrho = \overline{\mathcal{M}} \), i.e., every semisimple graph is a subdirect product of graphs from \( \mathcal{M} \).

A class \( \mathcal{M} \) of graphs is said to be hereditary (respectively strongly hereditary) if \( G \in \mathcal{M} \) implies all the induced subgraphs of \( G \) (respectively all the subgraphs of \( G \)) are in \( \mathcal{M} \). Hereditariness is retained under subdirect closure. Next we recall the definitions of Kurosh-Amitsur radical and semisimple classes of graphs. As for topological spaces, these classes of graphs are called connectednesses and disconnectednesses respectively. The terminology used is obvious when looking at the many examples of these classes ([14]). A class \( \mathcal{C} \subseteq \mathcal{W} \) is a connectedness (= KA-radical class) if it satisfies the following condition: A graph \( G \in \mathcal{W} \) is in \( \mathcal{C} \) if and only if every non-trivial homomorphic image of \( G \) has a non-trivial induced subgraph which is in \( \mathcal{C} \). A class \( \mathcal{D} \subseteq \mathcal{W} \) is a disconnectedness (= KA-semisimple class) if it satisfies the following condition: A graph \( G \in \mathcal{W} \) is in \( \mathcal{D} \) if and only if every non-trivial induced subgraph of \( G \) has a non-trivial homomorphic image which is in \( \mathcal{D} \). For examples and many additional statements and properties of these classes of graphs, see Fried and Wiegandt [14]. The class of trivial graphs \( \mathcal{T} \) is always contained in any connectedness and also in any disconnectedness. It is easy to find examples of connectednesses and disconnectednesses: If \( \mathcal{M} \subseteq \mathcal{W} \) is a hereditary class, then \( UM := \{ G \in \mathcal{W} \mid G \text{ has no non-trivial homomorphic image in } \mathcal{M} \} \) is a connectedness and if \( \mathcal{H} \subseteq \mathcal{W} \) is a homomorphically closed class, then \( SH := \{ G \in \mathcal{W} \mid G \text{ has no non-trivial induced subgraph in } \mathcal{H} \} \) is a disconnectedness. From the preceding, we thus have: If \( \mathcal{C} \) is a connectedness, then \( SC \) is a disconnectedness and if \( \mathcal{D} \) is a disconnectedness, then \( UD \) is a connectedness. Moreover, it can be shown that a class \( \mathcal{C} \subseteq \mathcal{W} \) is a connectedness if and only if \( \mathcal{C} = USC \) and a class \( \mathcal{D} \subseteq \mathcal{W} \) is a disconnectedness if and only if \( \mathcal{D} = SU D \). If \( \varrho \) is a Hoehnke radical, then \( R_\varrho = US_\varrho \) and if \( S_\varrho \) is hereditary, then \( S_\varrho \subseteq SR_\varrho \). Two properties that a Hoehnke radical \( \varrho \) on \( \mathcal{W} \) may satisfy are:

**Complete:** If \( \theta \) is a strong congruence on \( G \in \mathcal{W} \) with \( [a]_\theta \in R_\varrho \) for all \( a \in V_G \), then \( \theta \subseteq \varrho_G \); and

**Idempotent:** For \( G \in \mathcal{W} \) and all \( a \in V_G \), \( [a]_{\varrho_G} \in R_\varrho \).

Then we have:

Theorem 4.10. Let \( \varrho \) be an \( H \)-radical on \( \mathcal{W} \) which is complete, idempotent and such that for all \( G \in \mathcal{W} \), \( \varrho_G \) is a strong congruence on \( G \). Then \( S_\varrho \) is a disconnectedness and \( R_\varrho = US_\varrho \) is a connectedness. Conversely, suppose \( \mathcal{D} \) is a disconnectedness in \( \mathcal{W} \) with corresponding connectedness \( \mathcal{C} \). Then there is an \( H \)-radical \( \varrho \) on
$W$ which is complete, idempotent and for all $G \in W$, $\varrho_G$ is a strong congruence on $G$. Moreover, $S_\varrho = \mathcal{D}$ and $R_\varrho = UD = C$.

This result motivates the following terminology: An H-radical $\varrho$ which is complete, idempotent and for which $\varrho_G$ is a strong congruence for all $G$ is called a $KA$-radical. Next we show that a connectedness of graphs can be characterized with conditions using congruences which correspond to the conditions characterizing a radical class of associative rings using ideals. In particular, it is shown that any graph has a largest radical congruence which contains all other radical congruences on the graph and the quotient of a graph by this largest radical congruence has no non-trivial radical congruences. A significant property of $KA$-semisimple classes in general radical theory is that any object in the universal class has a maximal semisimple image. This is also the case for disconnectednesses of graphs as Fried and Wiegandt have shown [14]. This result is recalled below, together with their characterization of connectednesses. Remember, whenever a subset of vertices of a graph $G$ is considered as a graph and nothing else is mentioned, it is the subgraph induced by the graph $G$.

**Theorem 4.11.** ([14]) Let $C$ be a connectedness with corresponding disconnectness $\mathcal{D} = SC$. Then:

1. For every $G \in W$, there is a strong homomorphism $q : G \to G_D$ with $G_D \in \mathcal{D}$ and if $f : G \to H$ is any surjective homomorphism with $H \in \mathcal{D}$, then there is a homomorphism $g : G_D \to H$ such that $g \circ q = f$. (In categorical terms, this means $\mathcal{D}$ is an epi-reflective subcategory of $W$.) $G_D$ is called the maximal $D$-image of $G$.

2. For every $a \in G_D$, $q^{-1}(a) \in C$ and it is maximal in the sense that it is not properly contained in any other induced subgraph of $G$ which is in $C$.

3. If $H$ is an induced subgraph of $G$ with $H \in C$, then there is an $a \in G_D$ such that $H \subseteq q^{-1}(a)$.

Let $C$ be an abstract subclass of $W$. $C$ closed under forming strings means if a graph $G$ is the union of induced subgraphs $G_i$, $i \in I$ with each $G_i \in C$ and for every $j, k \in I$ there are finitely many indices $j = i_1, i_2, ..., i_n = k$ in $I$ with $G_{i_{t-1}} \cap G_{i_t} \neq \emptyset$ for $t = 2, 3, ..., n$, then $G \in C$; and $C$ weakly extensive means whenever $f : G \to H$ is a strong homomorphism with $H$ and $f^{-1}(b)$ in $C$ for all $b \in H$, then $G \in C$. The intersection $G_{i_{t-1}} \cap G_{i_t}$ refers to the vertex sets of the graphs.

**Theorem 4.12.** ([14]) Let $C$ be an abstract subclass of $W$. Then $C$ is a connectedness if and only if it satisfies the following three conditions:

1. $C$ is homomorphically closed.
2. $C$ is closed under forming strings.
3. $C$ is weakly extensive.

For an abstract subclass $C$ of $W$, a subgraph $H$ of $G$ is a $C$-subgraph of $G$ provided $H \in C$; likewise an induced $C$-subgraph is defined. A congruence $\alpha$ on a graph $G$ is a $C$-congruence on $G$ if $\alpha$ is a strong congruence and $[a]$ is an induced $C$-subgraph of $G$ for all $a \in G$. A congruence on a graph $G$ is called trivial if
the following three conditions:

(1) Theorem 4.15. $[a] = \{a\}$ for all $a \in G$. Note that for any $\mathcal{E}$ with $E_G \subseteq \mathcal{E} \subseteq C_G$, $\alpha = (\sim, \mathcal{E})$ is a trivial congruence on $G$, but it need not coincide with the identity congruence $\iota_G := (\sim, E_G)$ on $G$. If however, $\alpha$ is a trivial strong congruence, then $\alpha = \iota_G$ since in this case, $\mathcal{E}(\sim) = E_G$. For our characterization of connectednesses, we start with:

**Proposition 4.13.** Let $\mathcal{C}$ be an abstract subclass of $\mathcal{W}$. Then $\mathcal{C}$ is a connectedness if and only if it satisfies the following condition: A graph $G \in \mathcal{W}$ is in $\mathcal{C}$ if and only if every non-trivial homomorphic image of $G$ has a non-trivial $\mathcal{C}$-congruence.

Let $\mathcal{C}$ be an abstract class of graphs in $\mathcal{W}$. For any graph $G$, define a congruence $\varrho(G)$ on $G$ by $\varrho(G) = \sum \{\alpha \mid \alpha \text{ is a } \mathcal{C}\text{-congruence on } G\}$. This sum is not void, since $\iota_G$ is always a $\mathcal{C}$-congruence on $G$. Moreover, $\varrho(G)$ is a strong congruence on $G$ and it contains all $\mathcal{C}$-congruences on $G$. Usually we write $\varrho(G)$ as $\varrho_G$, but there are instances when the former will be better to use. Note that $\varrho_G$ is a $\mathcal{C}$-congruence on $G$ if and only if $G \in \mathcal{C}$.

**Proposition 4.14.** Let $\mathcal{C}$ be a connectedness in $\mathcal{W}$ with associated disconnectedness $\mathcal{D} = \mathcal{S}\mathcal{C}$. For any graph $G$ in $\mathcal{W}$, $\varrho_G = \sum \{\alpha \mid \alpha \text{ is a } \mathcal{C}\text{-congruence on } G\}$ is a $\mathcal{C}$-congruence on $G$ which contains all $\mathcal{C}$-congruences on $G$ and $\varrho_G = \sigma_G$ where $\sigma_G = \cap \{\theta \mid \theta \in \text{Con}(G) \text{ with } G/\theta \in \mathcal{D}\}$.

In terms of congruences and the terminology introduced in this section, the condition weakly extensive can be rephrased as: whenever $\alpha$ is a $\mathcal{C}$-congruence of $G$ and $G/\alpha \in \mathcal{C}$, then $G \in \mathcal{C}$. Furthermore, this condition can be formulated using only congruences: The class $\mathcal{C}$ is weakly extensive if and only if it satisfies: whenever $\alpha$ and $\beta$ are congruences on $G$ with $\alpha \subseteq \beta$, $\alpha$ a $\mathcal{C}$-congruence on $G$ and $\beta/\alpha$ a $\mathcal{C}$-congruence on $G/\alpha$, then $\beta$ is a $\mathcal{C}$-congruence on $G$. One last definition: $\mathcal{C}$ is *inductive* if it satisfies: whenever $\alpha_1 \subseteq \alpha_2 \subseteq \alpha_3 \subseteq \ldots$ is a chain of $\mathcal{C}$-congruences on the graph $G$, then $\sum_{i=1}^{\infty} \alpha_i$ is a $\mathcal{C}$-congruence on $G$. We now give the main result which shows that a connectedness of graphs can be characterized using congruences in exactly the same way that a radical class of associative rings can be characterized using ideals (again, compare this with Theorem 2.1).

**Theorem 4.15.** Let $\mathcal{C}$ be an abstract class of graphs in $\mathcal{W}$. Then statements (1), (2) and (3) below are equivalent:

1. $\mathcal{C}$ is a connectedness.
2. If for all $G \in \mathcal{W}$, $\varrho_G := \sum \{\alpha \mid \alpha \text{ is a } \mathcal{C}\text{-congruence on } G\}$, then $\mathcal{C}$ satisfies the following three conditions:
   (a) For every surjective homomorphism $f : G \rightarrow H$, $f(\varrho_G) \subseteq \varrho_H$.
   (b) For any graph $G$ in $\mathcal{W}$, $\varrho_G$ is a $\mathcal{C}$-congruence on $G$ which contains all $\mathcal{C}$-congruences on $G$.
   (c) $\varrho(G/\varrho_G) = \iota_G/\varrho_G$ for all graphs $G$.
3. $\mathcal{C}$ satisfies the following three conditions:
   (a) If $\alpha$ is a congruence on $G$ and $G \in \mathcal{C}$, then $G/\alpha \in \mathcal{C}$.
   (b) $\mathcal{C}$ is inductive.
   (c) $\mathcal{C}$ is weakly extensive.
4.3. Hereditary torsion theories. In this section, we take the universal class \( \mathcal{W} \) to be the class of all graphs.

**Definition 4.16.** Let \( \varrho \) be a Hoehnke radical of graphs. Then \( \varrho \) is called:

1. \( r \)-hereditary if for every graph \( G \) and induced subgraph \( H \) of \( G \), \( \varrho_G \cap H \subseteq \varrho_H \).
2. \( s \)-hereditary if for every graph \( G \) and induced subgraph \( H \) of \( G \), \( \varrho_H \subseteq \varrho_G \cap H \).
3. Ideal-hereditary if it is both \( r \)-hereditary and \( s \)-hereditary.
4. A hereditary torsion theory if it is an ideal-hereditary Hoehnke radical.

We will need:

**Proposition 4.17.** Let \( \varrho \) be a Hoehnke radical.

1. If \( \varrho \) is \( r \)-hereditary, then \( \varrho \) is idempotent and \( \mathcal{R}_\varrho \) is hereditary.
2. \( \varrho \) is \( s \)-hereditary if and only if \( \mathcal{S}_\varrho \) is hereditary. Any one of these two conditions, implies that \( \varrho \) is complete.

**Corollary 4.18.** Let \( \varrho \) be an ideal-hereditary Hoehnke radical (= hereditary torsion theory). Then \( \varrho \) is idempotent, complete and both the associated radical class \( \mathcal{R}_\varrho \) and the associated semisimple class \( \mathcal{S}_\varrho \) are hereditary.

For all the well-known classes of algebras, any \( \varrho \) as above (ideal-hereditary \( H \)-radical) will be a KA-radical meaning the associated radical class and semisimple class is a KA-radical class and a KA-semisimple class respectively. For graphs, as is the case for topological spaces (previous section) and \( S \)-acts (see Wiegandt [38]), this need not be the case. Below it is shown that there are exactly eight ideal-hereditary \( H \)-radicals of which three are KA-radicals. As will be seen, they are determined by the six non-isomorphic two-vertex graphs which will be denoted by \( B_i, i = 1, 2, 3, ..., 6 \) where \( V_{B_i} = \{0, 1\} \) and \( E_{B_i} = \emptyset \) (empty set), \( E_{B_2} = \{01\}, E_{B_3} = \{00\}, E_{B_4} = \{01, 11\}, E_{B_5} = \{01, 11\} \) and \( E_{B_6} = \{00, 01, 11\} \). Let \( \mathcal{B} \) denote the set consisting of the six two-vertex graphs and for a graph \( G \), \( L_G \) will be the set of all vertices from \( G \) which has a loop, i.e., \( L_G = \{t \in G \mid tt \in E_G\} \).

Recall, for an equivalence \( \sim \) on the vertex set \( V_G \), \((\sim, \mathcal{E}(\sim))\) denotes the strong congruence on \( G \) determined by \( \sim \). As was the case for topological spaces, of the three ideal-hereditary \( H \)-radicals which are KA-radical, only one is non-trivial.

**Theorem 4.19.** Let \( \varrho \) be an ideal-hereditary Hoehnke radical. Then \( \varrho \) is one of the following eight radicals:

(a) \( \varrho_G = (\sim_G, \mathcal{E}(\sim_G)) \) for all \( G \). This is a Kurosh-Amitsur radical with \( \mathcal{R}_\varrho \) the class of all graphs, \( \mathcal{S}_\varrho = \mathcal{T} \) and \( \mathcal{S}_\varrho \cap \mathcal{B} = \emptyset \).

(b) \( \varrho_G = \nu_G \) for all \( G \). This is not a Kurosh-Amitsur radical, \( \mathcal{R}_\varrho \) is the class of all graphs, \( \mathcal{S}_\varrho = \{T_0\} \) and \( \mathcal{S}_\varrho \cap \mathcal{B} = \emptyset \).

(c) For all \( G, \varrho_G = (\sim_G, \mathcal{E}_G) \) where \( \sim_G \) is the equivalence relation with equivalence classes \( \{a \mid a \in V_G - L_G\} \cup \{L_G\} \) and \( \mathcal{E}_G = \mathcal{E}(\sim_G) \). This is a Kurosh-Amitsur radical with \( \mathcal{R}_\varrho = \{G \mid G \) is non-trivial, then every vertex of \( G \) has a loop\}, \( \mathcal{S}_\varrho = \{G \mid G \) has at most one loop\} and \( \mathcal{S}_\varrho \cap \mathcal{B} = \{B_1, B_2, B_3, B_5\} \).

(d) For all \( G, \varrho_G = (\sim_G, \mathcal{E}_G) \) where \( \sim_G \) is the equivalence relation with equivalence classes \( \{a \mid a \in L_G\} \cup \{V_G - L_G\} \) and \( \mathcal{E}_G = \mathcal{E}_G \cup \{tt \mid t \in V_G - L_G\} \).
This is not a Kurosh-Amitsur radical, \( R_G = T, S_G = \{ G \mid \text{every vertex of } G \text{ has a loop} \} \) and \( S_G \cap B = \{ B_4, B_6 \} \).

(e) \( \varrho_G = (\varnothing, \mathcal{C}_G) \) for all \( G \). This is not a Kurosh-Amitsur radical, \( R_G = T, S_G \) is the class of all complete graphs with a loop at every vertex and \( S_G \cap B = \{ B_6 \} \).

(f) \( \varrho_G = \iota_G \) for all \( G \). This is a Kurosh-Amitsur radical with \( R_G = T, S_G \) the class of all graphs and \( B S_G \).

(g) For all \( G, \varrho_G = (\varnothing, \mathcal{E}_G) \) where \( \mathcal{E}_G = E_G \cup \{ ab \mid a, b \in L_G \} \). This is not a Kurosh-Amitsur radical, \( R_G = T, S_G = \{ G \mid \text{if } G \text{ is non-trivial, then } ab \in E_G \text{ for all } a, b \in L_G \} \) and \( S_G \cap B = \{ B_1, B_2, B_3, B_5, B_6 \} \).

(h) For all \( G, \varrho_G = (\varnothing, \mathcal{E}_G) \) where \( \mathcal{E}_G = E_G \cup L_G V_G, R_G = T, S_G = \{ G \mid \text{if } a \in L_G, \text{ then } ab \in E_G \text{ for all } b \in V_G \} \) and \( S_G \cap B = \{ B_1, B_2, B_5, B_6 \} \). This is not a Kurosh-Amitsur radical.

In the next section, we continue on our well-trodden path of presenting the congruence theory and then the radical theory, in this case for graphs which do not admit loops. But we will be taken to new destinations.

5. Graphs with no loops. A congruence theory for graphs that do not allow loops was introduced by Broere, Heidema and Pretorius [3] and precedes the congruence theories for graphs that admit loops and topological spaces in [4] and [30] respectively. They showed that, as for universal algebra, congruences give rise to isomorphism theorems as well as the characterization of subdirect products in terms of the intersection of congruences. When loops are not allowed, there is a significant restriction on the possible number of homomorphisms on a graph. A radical theory for such graphs was only developed recently, and we shall see that this restriction has a degenerative impact on certain aspects of the radical theory for these graphs. In particular, the semisimple class of a Hoehnke radical coincides with the class of all graphs in this case. But in spite of this, there are non-trivial connectednesses and disconnectednesses for such graphs. It should be mentioned that an earlier use of a congruence for graphs with no loops has already appeared in Sabidussi [26], quoting and using results (unpublished) from his PhD student Fawcett [13]. They used a congruence in a limited sense as the kernel of a strong graph homomorphism and no general theory of congruences was mentioned or discussed. In particular, they showed that subdirect products of graphs can be described in terms of congruences and they applied these to formulate and prove a version of Birkhoff’s Theorem for graphs: every graph is a subdirect product of subdirectly irreducible graphs. In their set-up, the subdirectly irreducible graphs are the complete graphs and the almost complete graphs.

In this section, all graphs considered are undirected and without multiple edges. Graphs have non-empty vertex sets, edge sets may be empty and no loops are allowed. For a graph \( G = (V_G, E_G) \), the set of all possible edges on \( G \) will be denoted by \( K_G := \{ ab \mid a, b \in V_G, a \neq b \} \). Here, for a homomorphism \( f : G \rightarrow H \) and \( a, b \in V_G \), if \( ab \in E_G \), then \( f(a) \neq f(b) \).
5.1. Congruences. A congruence on a graph $G = (V_G, E_G)$ [3] is a pair $\theta = (\sim, \mathcal{E})$ such that
(i) $\sim$ is an equivalence relation on $V_G$;
(ii) $\mathcal{E}$ is a set of unordered pairs of different elements from $V_G$, called the congruence edge set, with $E_G \subseteq \mathcal{E} \subseteq K_G$;
(iii) when $x \sim y$, then $xy \notin \mathcal{E}$; and
(iv) (Substitution Property of $\mathcal{E}$ with respect to $\sim$) when $x, y, x', y' \in V_G, x \sim x', y \sim y'$ and $x'y' \in \mathcal{E}$.

A strong congruence on $G$ is a pair $\theta = (\sim, \mathcal{E})$ where $\sim$ is an equivalence relation on $V_G$, $\mathcal{E} = \{xy \mid x, y \in V_G \text{ and there are } x', y' \in V_G \text{ with } x \sim x', y \sim y' \text{ and } x'y' \in E_G\}$ and condition (iii) is fulfilled.

Requirement (iii) ensures that the equivalence classes $[x] := \{y \in V \mid x \sim y\}$ are independent sets of vertices with respect to $\mathcal{E}$, i.e., if $a, b \in [x]$, then $ab \notin E_G$. It can easily be verified that a strong congruence is also a congruence. Congruences are partially ordered by the relation “contained in”: for two congruences $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ and $\beta = (\sim_\beta, \mathcal{E}_\beta)$ on $G$, $\alpha$ is contained in $\beta$, written as $\alpha \subseteq \beta$, if $\sim_\alpha \subseteq \sim_\beta$ and $\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta$. We will always use $\sim$ to denote the identity relation (diagonal) on a set. The congruence $\iota_G := (\sim, E_G)$ on $G$, called the identity congruence on $G$, is the smallest congruence on $G$. It is a strong congruence on $G$. If $G$ is a complete graph, i.e., $E_G = K_G$, then $G$ can have only one congruence namely the identity congruence $\iota_G$.

The kernel of a homomorphism. Given any graph homomorphism $f : G \rightarrow H$, a congruence on $G$, called the kernel of $f$ and written as $\ker f = (\sim_f, \mathcal{E}_f)$, is defined by $\sim_f = \{(x, y) \mid x, y \in V_G, f(x) = f(y)\}$ and $\mathcal{E}_f = \{uv \mid u, v \in V_G, f(u)f(v) \in E_H\}$. It is immediately clear that $\ker f$ is a congruence on $G$. With $f$ is also associated the strong kernel of $f$, written as $\text{sker } f = (\sim_{sf}, \mathcal{E}_{sf})$ with the same equivalence relation but $\mathcal{E}_{sf} = \{xy \mid x, y \in V_G \text{ and there are } x', y' \in V_G \text{ with } x \sim_f x', y \sim_f y' \text{ and } x'y' \in E_G\}$. This is a strong congruence on $G$ and $\text{sker } f \subseteq \ker f$; in fact, if $\theta = (\sim_f, \mathcal{E})$ is any congruence on $G$ for some $\mathcal{E}$, then $\text{sker } f \subseteq \theta$. If $f$ is a strong homomorphism, then $\ker f = \text{sker } f$. Note that a homomorphism $f$ is injective if and only if $\sim_f = \sim$. Moreover, if $f$ is a surjective strong homomorphism, then $f$ is an isomorphism if and only if $\ker f = \iota_G$.

Quotients. Given any congruence $\theta = (\sim, \mathcal{E})$ on a graph $G = (V_G, E_G)$, a new graph, denoted by $G/\theta = (V_{G/\theta}, E_{G/\theta})$ and called the quotient of $G$ by $\theta$, is defined by taking $V_{G/\theta} := \{[x] \mid x \in V_G\}$ and $E_{G/\theta} := \{[x][y] \mid xy \in \mathcal{E}\}$. The natural (canonical) mapping $p_\theta : G \rightarrow G/\theta$ given by $p_\theta(x) = [x]$ is a surjective homomorphism with $\ker p_\theta = \theta$. In particular, for $\theta = \iota_G$ we have $G/\iota_G$ is isomorphic to $G$. If $\theta$ is a strong congruence, then $p_\theta$ is a strong homomorphism with $\ker p_\theta = \theta = \ker \theta$. In general, if we take $\theta = (\sim, \mathcal{E})$ for some suitable $\mathcal{E}$ to make $\theta$ a congruence on $G$, then $G/\theta$ is the graph with vertex set $V_{G/\theta} = V_G$ and edge set $E_{G/\theta} = \mathcal{E}$ (here we identify $[x] = \{x\}$ with $x$).

The semilattice of congruences. For a given graph $G$, we denote the set of all congruences on $G$ by $\text{Con}(G)$. We already know that $\text{Con}(G)$ is a partially
ordered set with respect to containment $\subseteq$. But we can say more. Any collection of congruences $\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\} \subseteq \text{Con}(G)$ has a greatest lower bound in $\text{Con}(G)$ given by $\bigcap_{i \in I} \theta_i = (\sim, \mathcal{E})$ where $a \sim b \iff a \sim_i b$ for all $i \in I$ and $ab \in \mathcal{E} \iff ab \in \mathcal{E}_i$ for all $i \in I$. This ensures that $\text{Con}(G)$ is a complete meet-semilattice with the meet given by the intersection as defined above.

**Image of a congruence.** Let $f : G \to H$ be a homomorphism and $\theta = (\sim, \mathcal{E})$ a congruence on $G$. Then $f(\theta)$ means the pair $(f(\sim), f(\mathcal{E}))$ with $f(\sim) := \{(f(a), f(b)) \mid a, b \in V_G, a \sim b\} \subseteq V_H \times V_H$ and $f(\mathcal{E}) := \{f(a)f(b) \mid ab \in \mathcal{E}\} \subseteq \{xy \mid x, y \in V_H\}$. Note that $f(\theta)$ need not be a congruence on the graph $H$. Despite this, for a congruence $\beta = (\sim_\beta, \mathcal{E}_\beta)$ on $H$, we will compare $f(\theta)$ with $\beta$ in the usual sense: $f(\theta) \subseteq \beta$ if $f(\sim) \subseteq \sim_\beta$ and $f(\mathcal{E}) \subseteq \mathcal{E}_\beta$.

**Isomorphism theorems for congruences.** When dealing with radicals, the basic tools are the appropriate versions of the algebraic isomorphism theorems for graph congruences. These theorems are discussed in detail in Broere, Heidema and Pretorius [3], but summarized below for ease of reference.

1. **(First Isomorphism Theorem)** [3] Let $f : G \to H$ be a surjective homomorphism. Then $G/\ker f$ is isomorphic to $H$.

   The Second Isomorphism has also been given in [3], but we will present it here using a different (and more suggestive) notation. Let $G$ be a graph with induced subgraph $H$. Then a congruence $\theta = (\sim, \mathcal{E})$ on $G$ induces a congruence $H \cap \theta = (\sim_H, \mathcal{E}_H)$ on $H$ with $\sim_H = \{(a, b) \mid a, b \in V_H \text{ and } a \sim b\}$ and $\mathcal{E}_H = \{ab \mid a, b \in V_H \text{ with } ab \in \mathcal{E}\}$. The mapping $f : H \to G/\theta$ defined by $f(a) = [a]$ for all $a \in H$ is a homomorphism with $\ker f = H \cap \theta$. Now $f(V_H)$ is a set of vertices of $G/\theta$ on which we form the induced subgraph of $G/\theta$, denoted by $(H + \theta)/\theta$. Then, by the First Isomorphism Theorem, we have:

2. **(Second Isomorphism Theorem)** [3] Let $H$ be an induced subgraph of a graph $G$. Let $\theta$ be a congruence on $G$. Then $H/H \cap \theta \cong (H + \theta)/\theta$ where the latter graph is the induced subgraph of $G/\theta$ on the vertex set $\{[a] \mid a \in V_H\}$.

3. **(Third Isomorphism Theorem)** [3] Let $G$ be a graph with $\theta_1 = (\sim_1, \mathcal{E}_1)$ and $\theta_2 = (\sim_2, \mathcal{E}_2)$ two congruences on $G$ for which $\theta_1 \subseteq \theta_2$. Then $\theta_2/\theta_1 := (\sim, \mathcal{E})$ is a congruence on $G/\theta_1$ where $[a]_1 \sim [b]_1 \iff a \sim_2 b$ and $[a]_1[b]_1 \in \mathcal{E} \iff ab \in \mathcal{E}_2$. Moreover, $(G/\theta_1)/(\theta_2/\theta_1)$ is isomorphic to $G/\theta_2$.

A related result often used, also from [3], is:

4. Let $G$ be a graph with $\theta$ a fixed congruence on $G$. Any congruence $\xi$ of the graph $G/\theta$ is of the form $\alpha/\theta$ for some congruence $\alpha$ on $G$ with $\theta \subseteq \alpha$. Moreover, there is a one-to-one correspondence between $\{\alpha \mid \alpha \text{ is a congruence on } G \text{ with } \theta \subseteq \alpha\}$ and $\text{Con}(G/\theta)$ which preserves inclusions and intersections.

**Products, subdirect products and Birkhoff’s Theorem.** For an index set $I$, let $G_i = (V_i, E_i)$ be a graph for all $i \in I$. The product $\prod_{i \in I} G_i$ of the graphs $G_i$ is the
graph \( \prod_{i \in I} G_i = (\prod_{i \in I} V_i, E) \) where \( \prod_{i \in I} V_i \) is just the usual Cartesian product of the sets \( V_i \) and \( E = \{ fg \mid f, g \in \prod_{i \in I} V_i \text{ with } f(i)g(i) \in E_i \text{ for all } i \in I \} \). For every \( j \in I \), the \( j \)-th projection \( \pi_j : \prod_{i \in I} G_i \to G_j \) defined by \( \pi_j(f) = f(j) \) for all \( f \in \prod_{i \in I} V_i \) is a surjective homomorphism. An induced subgraph \( H \) of \( \prod_{i \in I} G_i \) is called a subdirect product of the graphs \( G_i, i \in I \), provided the restriction of each projection \( \pi_j \) to \( H \) is surjective. As in universal algebra, subdirect products can be expressed in terms of congruences and quotients:

**Theorem 5.1.** ([3]) A graph \( G \) is a subdirect product of graphs \( G_i, i \in I \), if and only if for every \( i \in I \) there are congruences \( \theta_i \) on \( G \) with \( G_i \) isomorphic to \( G/\theta_i \) and \( \bigcap_{i \in I} \theta_i = \iota_G \).

A graph \( G \) which has the property that whenever it is a subdirect product of graphs \( G_i, i \in I \), then at least one of the \( G_i \)'s must be isomorphic to \( G \) is called subdirectly irreducible. In view of the theorem above, a graph \( G \) is subdirectly irreducible if and only if any intersection of congruences on \( G \) which is the identity congruence, must already include one congruence which is the identity. A graph which is not subdirectly irreducible, is called subdirectly reducible. The corresponding notions for algebra are important, especially in the context of the well-known theorem of Birkhoff [2]: every non-trivial algebra is a subdirect product of subdirectly irreducible algebras. And, of course, knowing exactly what the subdirectly irreducible algebras are is the essence of this result. This theorem has meaning and validity for many other mathematical structures as well. For example, every non-trivial topological spaces is a subdirect product of copies of the Sierpiński space and the two-element indiscrete space (see, for example, Proposition 2.4 in [31]), noting that both these two two-element spaces are subdirectly irreducible topological spaces. Birkhoff’s Theorem is also valid in the category of graphs that admit loops [32], where it says that any non-trivial graph is a subdirect product of subdirectly irreducible graphs. Here the subdirectly irreducible graphs are \( B_4, B_5, B_6 \) (see Section 4.3 for the definitions of the \( B_i \)'s) and \( A_3 \) where \( A_3 \) is the three-vertex graph \( V_{A_3} = \{0, 1, 2\} \) and \( E_{A_3} = \{00, 11, 22, 01, 21\} \). Sabidussi and Fawcett’s version of Birkhoff’s theorem (cf. [26] and [13]) is not valid in our more general setting. In our case, a graph is subdirectly irreducible if and only if it is a complete graph and then:

**Theorem 5.2.** ([36]) Every graph is a subdirect product of complete graphs.

### 5.2. Radical theory.

The prohibition on loops imposes a significant restriction on the admissible maps between graphs and will consequently have an impact on the radical theory for these graphs. In fact, it will be shown that the radical theory in the category of graphs that do not admit loops has significant differences with all existing radical theories. In particular, all Hoehnke radicals are degenerative, but there are non-trivial connectednesses (radical classes) and disconnectednesses (semisimple classes). Contrary to radical theories in other categories, this means
that objects need not have maximal semisimple images. We start by defining a Hoehnke radical and report on their salient features. But this is really just a wild goose chase, since they all are trivial. Then we define the connectednesses and disconnectednesses, give non-trivial examples and show that they always come as complementary pairs. More detail on what is presented here can be found in [36].

We will work in a universal class \( W \) of graphs (non-empty, closed under homomorphic images and closed under the taking of subgraphs (= strongly hereditary)). From the definition, it follows that \( W \) contains a one-vertex graph, and consequently all one-vertex graphs. We identify all the one-vertex-graphs, denote them by \( T \) and call them the trivial graphs. For any graph \( G \), the completion of \( G \) is the graph \( G^c \) with the same vertex set as \( G \) and edge set \( K_G \). Since \( G^c \) is a homomorphic image of \( G \), \( W \) contains the completion of all graphs \( G \in W \). We will assume \( W \) is non-trivial, i.e., it has at least one graph with two or more vertices. Since \( W \) is strongly hereditary, it thus contains the two-vertex graph with no edges \( B_1 \) and its completion \( B_1^c = B_2 \). All considerations relating to the radicals of graphs will be inside the class \( W \).

**Definition 5.3.** An \( H \)-radical on \( W \) is a function \( \rho \) that assigns to every graph \( G \) in \( W \) a congruence \( \rho(G) = \rho_G \) on \( G \) such that:

1. (H1) if \( f : G \to H \) is a surjective homomorphism in \( W \), then \( f(\rho_G) \subseteq \rho_H \); and
2. (H2) for all graphs \( G \in W \), \( \rho(G/\rho_G) = \iota_{G/\rho_G} \), the identity congruence on \( G/\rho_G \).

For an \( H \)-radical \( \rho \), the class \( S_\rho = \{ G \in W \mid \rho(G) = \iota_G \} \) is called the associated semisimple class and \( R_\rho = \{ G \in W \mid G/\rho_G \text{ is the one-element graph} \} \) is the associated radical class. \( R_\rho \) is always homomorphically closed, i.e., if \( G \in R_\rho \) and \( f : G \to H \) is a surjective homomorphism, then \( H \in R_\rho \) and \( S_\rho \cap R_\rho = \{ T \} \). The essence of this radical is given in the next result. A class of graphs \( M \) in \( W \) is an abstract class provided it contains all the one element graphs in \( W \) and it is closed under isomorphic copies. All subclasses of \( W \) under consideration will be assumed to be abstract, even though it may not always be explicitly stated. For a class \( M \) in \( W \), we use \( \overline{M} \) to denote the subdirect closure of \( M \), i.e., the class of all graphs that are subdirect products of graphs from \( M \). Clearly \( M \subseteq \overline{M} \) and we say \( M \) is subdirectly closed if \( M = \overline{M} \).

**Theorem 5.4.** (1) Let \( \rho \) be a mapping that assigns to any graph \( G \) in \( W \) a congruence \( \rho(G) = \rho_G \) on \( G \). If \( \rho \) is an \( H \)-radical on \( W \), then there is an abstract class of graphs \( M \) in \( W \) such that for all \( G \) in \( W \), \( \rho(G) = \cap \{ \theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in M \} \). Furthermore, \( S_\rho = \overline{M} \) and \( S_\rho \) is closed under subdirect products.

2. For any abstract class of graphs \( M \) in \( W \) for which every \( G \) in \( W \) has a congruence \( \theta \) on \( G \) with \( G/\theta \in M \), define a mapping \( \rho \) by \( \rho(G) = \cap \{ \theta \mid \theta \in \text{Con}(G) \} \) with \( G/\theta \in M \). Then \( \rho \) is an \( H \)-radical and \( S_\rho = \overline{M} \).

In the context of the theorem above, we say that the \( H \)-radical \( \rho \) is determined by the class \( M \) if \( M \subseteq S_\rho \) and \( \overline{M} = S_\rho \); the class \( M \) is not necessarily unique. In
categories where congruences can be identified by a distinguished subobject (e.g., for rings, a congruence is completely determined by an ideal which is the congruence class of the additive identity), any class \( M \) of objects will fulfill the requirement of (2) above, but in general it need not be the case. Fortunately we can say exactly when it will be in our case. Let \( \mathcal{K}_W \) be the set of all complete graphs in \( W \).

**Lemma 5.5.** Let \( M \) be an abstract class of graphs in \( W \). Then for every \( G \) in \( W \) there is a congruence \( \theta \) on \( G \) with \( G/\theta \in M \) if and only if \( \mathcal{K}_W \subseteq M \).

This result is not good for the well-being of Hoehnke radicals in this universal class, for we have:

**Theorem 5.6.** Let \( \rho \) be an \( H \)-radical on \( W \). Then \( \rho = W \), i.e., \( \rho(G) = \iota_G \) for all \( G \in W \), and \( \mathcal{R}_\rho = \{T\} \).

Any \( H \)-radical on a universal class of graphs that do not admit loops is thus degenerative in the sense that \( \rho(G) = \iota_G \) for all \( G \). On the other hand, in general radical theory, the semisimple objects are usually regarded as well-behaved and sought after objects. One interpretation of the result \( S \rho = W \) is that all the graphs that do not admit loops are good graphs! Next we define the connectednesses and disconnectednesses in the universal class \( W \). A class \( \mathcal{C} \subseteq W \) is a connectedness (\( KA \)-radical class) if it satisfies the following condition: A graph \( G \) is in \( \mathcal{C} \) if and only if every non-trivial homomorphic image of \( G \) has a non-trivial induced subgraph which is in \( \mathcal{C} \). A class \( \mathcal{D} \subseteq W \) is a disconnectedness (\( KA \)-semisimple class) if it satisfies the following condition: A graph \( G \) is in \( \mathcal{D} \) if and only if every non-trivial induced subgraph of \( G \) has a non-trivial homomorphic image which is in \( \mathcal{D} \). The trivial graph \( T \) is always in any connectedness and also in any disconnectedness. If \( M \subseteq W \) is a hereditary class, then \( U M := \{ G \in W \mid G \) has no non-trivial homomorphic image in \( M \} \) is a connectedness and if \( H \subseteq W \) is a homomorphically closed class, then \( S H := \{ G \in W \mid G \) has no non-trivial induced subgraph in \( H \} \) is a disconnectedness.

**Proposition 5.7.** Any connectedness is homomorphically closed and any disconnectedness is strongly hereditary and closed under subdirect products.

From the preceding, we thus have: If \( \mathcal{C} \) is a connectedness, then \( \mathcal{D} := SC \) is a disconnectedness, called the disconnectedness corresponding to \( \mathcal{C} \), and if \( \mathcal{D} \) is a disconnectedness, then \( \mathcal{C} := UD \) is a connectedness, called the connectedness corresponding to \( \mathcal{D} \). Moreover, it can be shown that a class \( \mathcal{C} \subseteq W \) is a connectedness if and only if \( \mathcal{C} = USC \) and a class \( \mathcal{D} \subseteq W \) is a disconnectedness if and only if \( \mathcal{D} = SU D \). By Theorem 5.6 we immediately have:

**Proposition 5.8.** Let \( \mathcal{D} \) be a disconnectedness. Suppose there is an \( H \)-radical \( \rho \) such that its semisimple class \( S \rho = \mathcal{D} \). Then \( \mathcal{D} = W \) and the corresponding connectedness \( \mathcal{C} \) is \( \mathcal{C} = \{T\} \).

The significance of this result is in stark contrast to the following feature of all known radical theories, which, for example we have seen in all three the previous sections. For a radical theory in a given universal class \( W \) of objects and
a semisimple class \( D \) in \( W \), any object in \( W \) has a maximal semisimple image in the following sense. For any \( A \in W \), there is a surjective morphism \( g : A \to A_D \) with \( A_D \in D \) such that for any surjective morphism \( f : A \to D \) with \( D \in D \), there is a morphism \( g : A_D \to D \) for which \( g \circ q = f \). If there is a congruence theory available in \( W \), then this means for every \( A \in W \), there is a congruence \( \delta_A \) on \( A \) with \( A/\delta_A \cong A_D \) and for every congruence \( \theta \) on \( A \) for which \( A/\theta \in D \), we have \( \theta \subseteq \delta_A \). Thus we have an \( H \)-radical \( \delta \) on \( W \) with \( S_\delta = D \) and \( R_\delta = UD \). In view of Proposition 5.8 above, this raises the question whether there are any non-trivial connectednesses and disconnectednesses in \( W \). By example we will show that there are, and we will give a very special example (which turns out to be not so special after all). For a connectedness \( C \) with corresponding disconnectedness \( D \) in \( W \), we know \( C \cap D = \{ T \} \) and the pair \(( C, D )\) is called complementary if \( C \cup D = W \). Of course, there are two trivial such pairs, namely \(( W, \{ T \} \) and \(( \{ T \}, W \)\). The existence of non-trivial complementary pairs was first observed in the category of graphs which allow loops by Fried and Wiegandt [14]. Such pairs are also to be found in the radical theory for \( S \)-acts, see Wiegandt [38], with a general condition for their existence given in [37] (which, however, is not applicable here).

**Example 5.9.** Suppose \( K \), the class of all complete graphs, is contained in \( W \). The class \( K \) is homomorphically closed, hence \( D := SK \) is a disconnectedness and the corresponding connectedness is \( C = USK \). It follows that \( D = \{ G \in W \mid E_G = \emptyset \} \) and \( C = \{ G \in W \mid G = T \ or \ E_G \neq \emptyset \} \) which is clearly a complementary pair. It is a non-trivial pair since \( W \) contains all the complete graphs and \( W \) is strongly hereditary (so \( D \neq \{ T \} \)). Moreover, no graph \( G \) in \( W \) with \( E_G \neq \emptyset \) has a homomorphic image in \( D \) and thus certainly not a maximal one. \( D \) is not a connectedness (not homomorphically closed) and \( C \) is not a disconnectedness (not strongly hereditary). Any graph \( G \in W \) has a maximal homomorphic image in \( C \) in the following sense: the completion \( G^c \) of \( G \) is a homomorphic image of \( G \), \( G^c \) is in \( C \) and if \( H \) is any other complete homomorphic image of \( G \), then \( H \subseteq G^c \).

This example shows a unique feature of the general radical theory in this universal class. Here is a very natural universal class with a radical theory in which all the Hoehnke radicals degenerate and it contains non-trivial KA-radicals. But there are more surprises. Recall, \( B_1 \) is the graph with two vertices and no edges. For any connectedness \( C \) with corresponding disconnectedness \( D \), we must have \( B_1 \in C \) or \( B_1 \in D \). By \( K_n \) we denote the complete graph on \( n \) vertices, \( n \geq 1 \). Then \( K_2 = B_1^c \) and \( K_1 = T \). The next result shows that there is really nothing special about the example above in this universal class.

**Proposition 5.10.** Let \( C \subseteq W \) be a connectedness with corresponding disconnectedness \( D = SC \). Then \(( C, D )\) is a complementary pair. If \( B_1 \in C \), then the pair \( ( C, D ) = ( W, \{ T \} \) is trivial, and if \( B_1 \in D \), then the pair \( ( C, D ) \) need not be trivial.

This is another special feature of the radical theory in this universal class \( W \): any connectedness (respt. disconnectedness) comes as a connectedness (respt. disconnectedness) in a complementary pair. We conclude with a general example, which has Example 5.9 as a special case.
Example 5.11. Suppose $K_n \in \mathcal{W}$ for $n = 1, 2, 3, \ldots$. For each $n \geq 1$, let $C_n := \{G \in \mathcal{W} \mid G = T \text{ or if } G \neq T, \text{ then } K_n \text{ is an induced subgraph of } G\}$ and let $D_n := \{G \in \mathcal{W} \mid G = T \text{ or if } G \neq T, \text{ then } K_n \text{ is not an induced subgraph of } G\}$. Then $(C_n, D_n)$ forms a complementary pair of connectednesses and disconnectednesses. Clearly $C_1 = \mathcal{W}$ and $C_2 = \{G \in \mathcal{W} \mid G = T \text{ or } E_G \neq \emptyset\}$. Let $C$ be any connectedness in $\mathcal{W}$. If $C \neq \mathcal{W}$, then $C \subseteq C_2$. This means $\mathcal{W}$ has a largest proper connectedness $C_2$ and thus a smallest non-trivial disconnectedness $D_2 = \{G \in \mathcal{W} \mid E_G = \emptyset\}$. If $C$ contains a graph with $n$ vertices, $n \geq 2$, then $K_n \in C$ in which case $C_n \subseteq C$. If the universal class $\mathcal{W}$ contains at least one graph with an infinite number of vertices, then $C := \{G \in \mathcal{W} \mid G = T \text{ or } E_G \text{ is infinite}\}$ and $D := \{G \in \mathcal{W} \mid E_G \text{ is finite}\}$ is a corresponding pair of non-trivial connectedness and disconnectedness respectively and $C_n \not\subseteq C$ for all $n \geq 1$.

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