GROMOV-WITTEN THEORY OF TAME DELIGNE-MUMFORD STACKS
IN MIXED CHARACTERISTIC

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ABSTRACT. We define Gromov-Witten classes and invariants of smooth proper tame Deligne-Mumford stacks of finite presentation over a Dedekind domain. We prove that they are deformation invariants and verify the fundamental axioms. For a smooth proper tame Deligne-Mumford stack over a Dedekind domain, we prove that the invariants of fibers in different characteristics are the same. We show that genus zero Gromov-Witten invariants define a potential which satisfies the WDVV equation and we deduce from this a reconstruction theorem for genus zero Gromov-Witten invariants in arbitrary characteristic.

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1. Introduction

Gromov-Witten theory of orbifolds was introduced in the symplectic setting in [7] and in the algebraic setting in [1] and [2], where Abramovich, Graber and Vistoli developed the Gromov-Witten theory of Deligne-Mumford stacks in characteristic zero, using the moduli stack of twisted stable maps into \( X \), denoted by \( K_{g,n}(X, \beta) \). This stack was constructed in [4] and it is the necessary analogue of Kontsevich’s moduli stack of stable maps for smooth projective varieties when replacing the variety with a Deligne-Mumford stack.

In this paper we define Gromov-Witten classes and invariants associated to smooth proper tame Deligne-Mumford stacks of finite presentation over a Dedekind domain. The main motivation for us is to compare the invariants in different characteristics for stacks defined in mixed characteristic. We hope that this approach could give a useful insight into the Gromov-Witten theory in characteristic zero, providing a new technique for computing Gromov-Witten invariants.

We consider a modified version, which we denote by \( K_{g,n}(X/s, \beta) \), of Abramovich, Graber and Vistoli’ stack of twisted stable maps. The stack \( K_{g,n}(X/s, \beta) \) parametrizes twisted stable maps to \( X \), but we take \( \beta \) to be a cycle class over the generic fiber \( X \) of \( X \) rather than over \( X \) itself (section 2). This stack turns out to be more convenient when we want to compare the Gromov-Witten invariants in mixed characteristic.

The fundamental ingredient for the construction of Gromov-Witten invariants is the virtual fundamental class \( [K_{g,n}(X, \beta)]^\virt \in A_*(K_{g,n}(X, \beta)) \). In the language of [6], a virtual fundamental class \( [\mathcal{M}]^\virt \in A_*(\mathcal{M}) \) is defined in the Chow group with rational coefficients, for a Deligne-Mumford stack \( \mathcal{M} \) endowed with a perfect obstruction theory. The main problem in developing Gromov-Witten theory in positive or mixed characteristic is that in general the stack

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\( \mathcal{K}_{g,n}(X, \beta) \) is not Deligne-Mumford. For instance this happens for \( \mathcal{K}_{0,0}(\mathbb{P}^1_k, p) \), when \( k \) is a field of characteristic \( p > 0 \), because the map \( f: \mathbb{P}^1_k \to \mathbb{P}^1_k \) such that \( f(x_0, x_1) = (x_0^p, x_1^p) \) is stable but has stabilizer

\[
\mu_p = \text{Spec } k[x]/(x^p - 1) = \text{Spec } k[x]/(x - 1)^p,
\]

which is not reduced. When the base is a field of characteristic \( p > 0 \), then \( \mathcal{K}_{g,n}(X, \beta) \) is still Deligne-Mumford for certain values of the fixed discrete parameters \( g, n, \beta \) which are big with respect to \( p \) ([1]). However, this is not satisfactory from the point of view of Gromov-Witten theory, because most of the properties of Gromov-Witten invariants (e.g. WDVV equation, Getzler relations) involve all the invariants at the same time.

The definition of virtual fundamental class for Artin stacks was not feasible at the beginning because of the lack for Artin stacks of two useful technical devices: Chow groups and the cotangent complex. We now have these devices at our disposal. Chow groups and intersection theory for Artin stacks over a field are defined in [10]. A working theory for the cotangent complex of a morphism of Artin stacks is provided by [11], [18], [12]. Nonetheless the presence of these tools is not enough to overcome all the difficulties in the absolute case. However, for the purpose of this work, it is enough to define a relative version of the virtual fundamental class of an Artin stack. The crucial point is to observe that both Kresch’s intersection theory and the construction of the virtual fundamental class in [6] 7 generalize to Artin stacks over a Dedekind domain. In section 3 we apply this to the natural forgetful functor \( \theta: \mathcal{K}_{g,n}(X, \beta) \to \mathcal{M}_{g,n}^{\text{tw}} \) into the stack of twisted curves \( \mathcal{M}_{g,n}^{\text{tw}} \) constructed in [4], after we exhibited a perfect relative obstruction theory for \( \theta \), and we construct a virtual fundamental class \( [\mathcal{K}_{g,n}(X, \beta)]^{\text{virt}} \in A_*\left( \mathcal{K}_{g,n}(X, \beta) \right) \).

A Dedekind domain \( D \) can be thought of as a space whose points corresponds to fields of different characteristics; a Deligne-Mumford stack \( \mathcal{Y} \) over \( D \) is a family of Deligne-Mumford stacks - the fibers - each of which is defined over a point of \( D \). We prove the following result, providing a comparison between invariants in different characteristics (section 4).

1. **Theorem.** Let \( \mathcal{Y} \) be a smooth proper tame Deligne-Mumford stack of finite presentation over a Dedekind domain \( D \). Then the Gromov-Witten theories of the geometric fibers of \( \mathcal{Y} \) are equivalent (i.e., the Gromov-Witten invariants of the fibers are the same).

When the base is an algebraically closed field \( k \), we prove that Gromov-Witten invariants define an associative and supercommutative product on the quantum cohomology ring

\[
H^*_\text{et}(X) = \sum_r H^r(\mathcal{I}_\mu(X), \mathbb{Q}_l(\tau)),
\]

where the right hand side is the \( l \)-adic étale cohomology, for a prime \( l \) different from the characteristic of \( k \), of the rigidified cyclotomic inertia stack \( \mathcal{I}_\mu(X) \) (section 5).

**Future work.** A natural generalization would be to develop a Gromov-Witten theory for tame Artin stacks, using the moduli stack of twisted stable maps constructed in [3]. The main problem is that the natural forgetful functor \( \theta: \mathcal{K}_{g,n}(X, \beta) \to \mathcal{M}_{g,n}^{\text{tw}} \) is not Deligne-Mumford type in general, and therefore the relative cotangent complex of \( \theta \) has three terms, so that one cannot use the construction described in [3].

In another direction, it would be interesting to prove a degeneration formula in the mixed characteristic setting. This would give a useful tool to compute Gromov-Witten invariants of Deligne-Mumford stacks in characteristic zero out of simpler invariants of tame Deligne-Mumford stacks in positive characteristic. For instance, this would apply to the fake projective plane constructed by Mumford in [14] using \( p \)-adic uniformization. We imagine this is far from easy, but we hope to return to these points in a future paper.

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**Notations.** We write \((\text{Sch}/S)\) for the category of schemes over a base scheme \(S\). For a scheme \(X \in (\text{Sch}/S)\), we denote by \(A_*(X/S)\) the group of numerical equivalence classes of cycles. All stacks are Artin stacks in the sense of [5], [11] and are of finite type over a base scheme. Unless otherwise specified, the words \"stack of twisted stable maps\" refer to \(\mathcal{K}_{g,n}(X/S, \beta_\eta)\) in Definition 2.4. We recall that a Deligne-Mumford stack \(X\) is tame if for every morphism \(\pi: \text{Spec} \, k \to X\), with \(k\) algebraically closed, the stabilizer group of \(\pi\) in \(X\) has order invertible in \(k\).

### 2. The stack of twisted stable maps

Let \(D\) be a Dedekind domain and set \(S = \text{Spec} \, D\). Let \(X\) be a proper tame Deligne-Mumford stack of finite presentation over \(S\), admitting a projective coarse moduli scheme \(\bar{X}\). We fix an ample invertible sheaf \(\mathcal{O}(1)\) on \(X\). We fix integers \(g \geq 0\), \(n \geq 0\). Let \(\eta\) be the generic point of \(S\) and set \(X_\eta = X \times_S \eta\). Fix \(\beta_\eta \in A_1(X_{\eta}/\eta)\).

#### 2.1. Twisted curves and twisted stable maps

For any closed point \(s \in S\), we denote by \(X_s\) the fiber over \(s\). Let \(m_s \subset D\) be the maximal ideal corresponding to \(s\) and consider the localization \(R = D_{m_s}\) of \(D\) at \(m_s\). Let us set \(X_s = X \times_S \text{Spec} \, R\) and let \(X_s \to X\) and \(X_\eta \to X\) be the natural inclusions. Notice that \(R\) is a discrete valuation ring and, by [10] 20.3, there exists a specialization homomorphism

\[
\sigma_\eta: A_*(X_{\eta}/\eta) \to A_*(X_\eta/S),
\]

sending a cycle \(\alpha\) to \(\sigma_\eta^* \alpha\), for some \(\alpha \in A_*(X_\eta/R)\) such that \(j^* \sigma_\eta^* \alpha = \alpha\). By [10] 20.3, there exists an induced specialization homomorphism

\[
\sigma_\eta: A_*(X_{\eta}/\eta) \to A_*(X_{\eta}/\bar{\eta}),
\]

where \(\bar{\eta}\) and \(\bar{\bar{\eta}}\) are geometric points over \(\eta\) and \(S\). We denote by \(\bar{\beta_\eta} \in A_1(X_{\bar{\eta}}/\bar{\bar{\eta}})\) the cycle class induced by \(\beta_\eta\) and we notice that \(\sigma_\eta(\bar{\beta_\eta}) = \sigma_\eta(\beta_\eta)\).

#### 2.1. Definition

Let \(T\) be a scheme over \(S\). A stable \(n\)-pointed map of genus \(g\) and class \(\beta_\eta\) into \(X\) is the data \((C \xrightarrow{\pi} T, t_1, \ldots, t_n, f)\), where

1. the morphism \(\pi\) is a projective flat family of curves;
2. the geometric fibers of \(\pi\) are reduced with at most nodes as singularities;
3. the sheaf \(\pi_* \omega_{C/T}\) is locally free of rank \(g\) (where \(\omega_{C/T}\) is the relative dualizing sheaf);
4. the morphisms \(t_1, \ldots, t_n\) are sections of \(\pi\) which are disjoint and land in the smooth locus of \(\pi\);
5. \(f: C \to X\) is a morphism of \(\text{S-schemes}\);
6. the group scheme \(\text{Aut}(C, f, \pi, t_1)\) of automorphisms of \(C\), which commute with \(f\), \(\pi\) and \(t_1\), is finite over \(T\);
7. for every geometric point \(\bar{t} \to T\), we consider the following induced morphisms

\[
C_\bar{t} = C \times_T \bar{t} \xrightarrow{j_{\bar{t}}} X_\bar{t} = X \times_S \bar{t} \xrightarrow{\pi} X_\bar{\eta} = X \times_S \bar{\eta} \to X = X \times_S \bar{\eta} \to X,
\]

where \(s = \text{Spec} \, k \in S\) is the image of \(\bar{t}\) and \(\bar{\eta} = \text{Spec} \, \bar{k}\), with \(\bar{k}\) a separable closure of \(k\), then we have \(j_{\bar{t}} \sigma_\eta(C_\bar{t}) = \tau^* \sigma_\eta(\bar{\beta_\eta})\).

#### 2.2. Remark

Notice that a stable map of class \(\beta_\eta\) is a stable map of class \(\beta\) (in the sense of [4] 4.3.1) for some \(\beta \in A_1(X/S)\) such that \(j^* \beta = \beta_\eta\).
2.3. Definition. Let $T$ be a scheme over $S$. A twisted stable $n$-pointed map of genus $g$ and class $\beta$ into $X$ over $T$ is the data $(C \to T, \{\Sigma^C_i\}_{i=1}^n, f : C \to X)$ where

1. the following natural diagram is commutative

\[
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & X
\end{array}
\]

2. $(C \to T, \{\Sigma^C_i\}_{i=1}^n)$ is a twisted nodal $n$-pointed curve of genus $g$ over $T$;
3. the morphism $C \to X$ is representable;
4. let $\Sigma^C_i$ be the image of $\Sigma^C$ in $C$, then $(C \to T, \{\Sigma^C_i\}_{i=1}^n, f : C \to X)$ is a stable $n$-pointed map of class $\beta$.

2.4. Definition. We denote by $K_{g,n}(\mathcal{X}/S, \beta)$ the category fibered in groupoids over $(\text{Sch}/S)$ of twisted stable $n$-pointed maps of genus $g$ and class $\beta$ into $X$.

2.5. Theorem. The category $K_{g,n}(\mathcal{X}/S, \beta)$ is a proper Artin stack over $S$, admitting a projective coarse moduli scheme $K_{g,n}(\mathcal{X}/S, \beta) \to S$.

Proof. Let $d = \deg \beta$. It is enough to show that $K_{g,n}(\mathcal{X}/S, \beta)$ is an open and closed substack of $K_{g,n}(\mathcal{X}/S, d)$ and then apply [4] 1.4.1. Notice that $K_{g,n}(\mathcal{X}/S, \beta)$ is an open substack of $K_{g,n}(\mathcal{X}/S, d)$, because it is a union of open substacks. On the other hand $K_{g,n}(\mathcal{X}/S, \beta) = \bigcup K_{g,n}(\mathcal{X}/S, \beta)$ is open, where the union is over $\beta \in A_1(\mathcal{X}/S)$ such that $\deg \beta = d$. Let us denote $\beta \mapsto \beta$. It follows that $K_{g,n}(\mathcal{X}/S, \beta)$ is a closed substack of $K_{g,n}(\mathcal{X}/S, d)$.

2.6. We denote by $M_{g,n/S}^{\text{tw}}$ the stack of twisted $n$-pointed curves of genus $g$ as defined in [4] 4.1.2. Recall that $M_{g,n/S}^{\text{tw}}$ is a smooth Artin stack, locally of finite type over $S$. Moreover, the stack $M_{g,n/S}^{\text{tw},N}$ classifying twisted curves $(C, \{\Sigma^C_i\})$ such that the order of the stabilizer group at every point is at most $N$ and the coarse space $C$ of $C$ has dual graph $\Gamma$, is a smooth Artin stack of finite type over $S$ (17 1.9–1.12).

2.7. Definition. Let $C \to M_{g,n/S}^{\text{tw}}$ be the universal twisted nodal curve. We define the algebraic stack $\underline{\text{Hom}}_{M_{g,n}^{\text{tw}}}(C, \mathcal{X})$ over $M_{g,n/S}^{\text{tw}}$ as follows

1. for every $S$-scheme $T$, an object of $\text{Hom}_{M_{g,n}^{\text{tw}}}(C, \mathcal{X})(T)$ is a twisted pointed curve $(C_T \to T, \{\Sigma^C_T\}_{i=1}^n)$ over $T$ together with a representable morphism of $S$-stacks $f : C_T \to \mathcal{X}$;
2. a morphism from $(C_T \to T, \{\Sigma^C_T\}, f)$ to $(C_{T'} \to T', \{\Sigma^C_{T'}\}, f')$ consists of data $(F, \alpha)$, where $F : C_T \to C_{T'}$ is a morphism of twisted curves and $\alpha : f \to f' \circ F$ is an isomorphism.

2.8. Remark. There is a canonical functor $\theta : \underline{\text{Hom}}_{M_{g,n}^{\text{tw}}}(C, \mathcal{X}) \to M_{g,n/S}^{\text{tw}}$ which forgets the map into $\mathcal{X}$. Moreover, since stability is an open condition, the stack $K_{g,n}(\mathcal{X}/S, \beta)$ is an open substack of $\underline{\text{Hom}}_{M_{g,n}^{\text{tw}}}(C, \mathcal{X})$.

2.9. Proposition. The natural forgetful functor

$$\theta : K_{g,n}(\mathcal{X}/S, \beta) \to M_{g,n/S}^{\text{tw}}$$

which forgets the morphism into $\mathcal{X}$ is of Deligne-Mumford type.

Proof. Let $U \to M_{g,n/S}^{\text{tw}}$ be a morphism from a scheme $U$ over $S$ and let us denote $C_U = C \times_{M_{g,n/S}^{\text{tw}}} U$ the corresponding twisted pointed curve over $U$. Form the fiber diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\text{Hom}_{M_{g,n}^{\text{tw}}}(C, \mathcal{X})} & \theta \\
\downarrow & & \downarrow \\
U & \longrightarrow & M_{g,n/S}^{\text{tw}}
\end{array}
$$
and notice that $V = \text{Hom}_{\mathfrak{p}}(\mathcal{C}_U, \mathcal{X})$. Since $\mathcal{C}_U$ and $\mathcal{X}$ are Deligne-Mumford stacks it follows, by [16] 1.1, that $V$ is a Deligne-Mumford stack and hence $\overline{\theta}$ is of Deligne-Mumford type. The statement follows from the fact that $K_{g,n}(X/S, \beta_1)$ is an open substack of $\text{Hom}_{\text{DM}_g}^w(C, \mathcal{X})$. □

2.10. REMARK. For every $S$-scheme $T$, a morphism $T \to K_{g,n}(X/S, \beta_1)$ corresponds to a stable map $(C_T \arrow T, t_i, f_T)$ over $T$, then, by descent theory, the identity of $K_{g,n}(X/S, \beta_1)$ corresponds to a universal stable map $(\mathcal{G} \arrow T, K_{g,n}(X/S, \beta_1), \sigma_t, \psi)$.

2.2. Evaluation maps. Let $\mathcal{F}_\mu(X)$ be stack of cyclotomic gerbes of $X$ as described in [2] 3.3. Recall that $\mathcal{F}_\mu(X)$ is proper, since $X$ is proper; moreover, if $X$ is smooth then $\mathcal{F}_\mu(X)$ is smooth (2) 3.4).

2.11. REMARK ([2] 3.5). There is an involution $\iota: \mathcal{F}_\mu(X) \rightarrow \mathcal{F}_\mu(X)$ defined over each $\mathcal{F}_\mu(X)$ as follows. Consider the inversion automorphism $\tau: \mu_r \rightarrow \mu_r$ sending $\xi$ to $\xi^{-1}$. For every object $(\mathcal{G}, \psi)$ of $\mathcal{F}_\mu(X)$, we can change the banding of the gerbe $\mathcal{G} \rightarrow T$ through $\tau$ and get another object $\tau \mathcal{G} \rightarrow X$ of $\mathcal{F}_\mu(X)$.

2.12. Notation. We denote $\Delta: \mathcal{F}_\mu(X) \rightarrow \mathcal{F}_\mu(X)^2$ the morphism, which we will call diagonal, induced by the identity and the involution $\iota$.

2.13. Definition ([2] 4.4.1). The $i$-th evaluation map $e_i: K_{g,n}(X/S, \beta_1) \rightarrow \mathcal{F}_\mu(X)$ is the morphism that associates to every twisted stable map $(C \rightarrow T, \{\Sigma_\tau^i\}_i, \mathcal{I}: C \rightarrow X)$ the diagram

$$\begin{array}{ccc}
\Sigma_i^i & \xrightarrow{f} & X \\
\downarrow & & \\
T & & \\
\end{array}$$

The $i$-th twisted evaluation map $e_i: K_{g,n}(X/S, \beta_1) \rightarrow \mathcal{F}_\mu(X)$ is the composition $\iota \circ e_i$, where $\iota$ is the involution described in Remark 2.11.

2.14. REMARK. Let us notice that the evaluation map $e_i$ is the composition

$$K_{g,n}(X/S, \beta_1) \xrightarrow{\Gamma_{e_i}} K_{g,n}(X/S, \beta_1) \times_S \mathcal{F}_\mu(X) \xrightarrow{\pi} \mathcal{F}_\mu(X),$$

where $\Gamma_{e_i}$ is the graph of $e_i$ and $\pi$ is the projection. By the following cartesian diagram

$$\begin{array}{ccc}
K_{g,n}(X/S, \beta_1) & \xrightarrow{e_i} & \mathcal{F}_\mu(X) \\
\downarrow & & \downarrow \Delta \\
K_{g,n}(X/S, \beta_1) \times_S \mathcal{F}_\mu(X) & \xrightarrow{e_i \times \text{id}} & \mathcal{F}_\mu(X) \times_S \mathcal{F}_\mu(X)
\end{array}$$

it follows that $\Gamma_{e_i}$ is a regular local immersion, hence, by [16] 6.1, there exists a Gysin map $\Gamma_{e_i}^\times$. Moreover $\text{Hom}_{\text{DM}_g}^w(C, \mathcal{X})$ is flat over $\mathcal{M}_g^w$, therefore, since $\mathcal{M}_g^w$ is smooth over $S$ and $K_{g,n}(X/S, \beta_1)$ is an open substack of $\text{Hom}_{\text{DM}_g}^w(C, \mathcal{X})$, we get that $\pi$ is flat. Then we can define the pull-back $e_i^\times = \Gamma_{e_i}^\times \circ \pi^\times$.

2.15. Notation. We write $e^\times(\gamma) = \bigcap_{i=1}^n e_i^\times(\gamma_i)$ for every $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_n$.

3. A VIRTUAL FUNDAMENTAL CLASS

Let $D$ be a Dedekind domain and set $S = \text{Spec } D$. Let $X$ be a smooth proper tame Deligne-Mumford stack of finite presentation over $S$, admitting a projective coarse moduli scheme $\mathcal{X}$. We want to define a virtual fundamental class for $K_{g,n}(X/S, \beta_1)$ relative to the forgetful morphism

$$\theta: K_{g,n}(X/S, \beta_1) \rightarrow \mathcal{M}_g^{w, \text{vir}},$$

following the construction of [14] 7. For this, we need a perfect relative obstruction theory for $\theta$. 

5
3.1. The stack of morphisms. With notations as in Definition 3.1 notice that \( \overline{\mathcal{C}} = C \times_{\text{Grw}_{g,n}} \text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X}) \) is a universal family for \( \text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X}) \) and we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\psi} & \mathcal{X} \\
\downarrow{\pi} & & \downarrow{\pi} \\
K_{g,n}(X/S, \beta) & \rightarrow & \text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X})
\end{array}
\]

3.1. Lemma. We have \( F^\bullet = R\pi_*(\psi^! \Omega_{X/S} \otimes \omega_{\mathcal{F}})[-1] \in D_{\text{coh}}^{(-1,0)}(\text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X})) \) and \( h^j/\mu(F^\bullet) \) is a vector bundle stack.

Proof. Since \( \mathcal{X} \) is smooth over \( S \), the sheaf \( \Omega_{X/S} \) is a vector bundle over \( \mathcal{X} \). The dualizing sheaf \( \omega_{\mathcal{F}} \) is a line bundle over \( \overline{\mathcal{C}} \), because \( \mathcal{F} \) is a family of curves with at most nodal singularities (which are Gorenstein). Hence \( \psi^! \Omega_{X/S} \otimes \omega_{\mathcal{F}} \) is a vector bundle on \( \overline{\mathcal{C}} \). Recall that the cohomology of the total pushforward is the higher pushforward sheaf. Moreover, \( \mathcal{T} \) is a flat projective morphism of relative dimension 1, so the \( i \)-pushforward vanishes for \( i > 1 \) by the cohomology and base-change theorem ([S Corollary 8.3.4]), therefore

\[
R\pi_*(\psi^! \Omega_{X/S} \otimes \omega_{\mathcal{T}}) \in D_{\text{coh}}^{(0,1)}(\text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X})).
\]

Set \( \mathcal{F} = \psi^\ast \Omega_{X/S} \otimes \omega_{\mathcal{T}} \). Let \( \mathcal{L} \) be a \( \mathcal{T} \)-ample line bundle then, for \( n \) big enough, \( \mathcal{F} \otimes \mathcal{L}^n \) is generated by global sections and \( R^0\pi_*(\mathcal{F} \otimes \mathcal{L}^{-n}) = 0 \). Thus we have a surjection

\[
\mathcal{J} = \pi^\ast \pi_*(\mathcal{F} \otimes \mathcal{L}^{-n}) \otimes \mathcal{L}^{-n} \rightarrow \mathcal{F},
\]

and we denote by \( \mathcal{K} \) the kernel. Notice that \( \mathcal{K} \) is a vector bundle because it is the kernel of a surjection of vector bundles. Hence we get the following exact sequence

\[
0 \rightarrow R^0\pi_* \mathcal{K} \rightarrow R^0\pi_* \mathcal{J} \rightarrow R^0\pi_* \mathcal{F} \rightarrow R^1\pi_* \mathcal{K} \rightarrow R^1\pi_* \mathcal{J} \rightarrow R^1\pi_* \mathcal{F} \rightarrow 0.
\]

Since \( R^0\pi_* \mathcal{J} = 0 \) and \( R^0\pi_* \mathcal{K} = 0 \), as vector bundles and \( F^\bullet \) is quasi-isomorphic to \([R^1\pi_* \mathcal{K} \rightarrow R^1\pi_* \mathcal{J}] \). 

3.2. We define a morphism \( \overline{\mathcal{F}}: F^\bullet \rightarrow \mathcal{T}^{-1}L^\bullet_{g,n} \) as follows. By adjunction, this is equivalent to define a morphism

\[
(\psi^! \Omega_{X/S} \otimes \omega_{\mathcal{T}})[-1] \rightarrow L\overline{\mathcal{F}}(L^\bullet_{g,n}).
\]

Recall that if \( \pi \) is a flat proper Gorenstein morphism of relative dimension \( N \), then \( L\pi^\bullet = \pi^\bullet \otimes \omega_{\mathcal{X}}[-N] \). This applies in our case with \( N = 1 \) and we get \( L\pi^\bullet = \pi^\bullet \otimes \omega_{\mathcal{T}}[-1] \). Hence to give the morphism \( \overline{\mathcal{F}} \) is equivalent to giving a morphism \( \psi^! \Omega_{X/S} \rightarrow \mathcal{T} L^\bullet_{g,n} \). Notice that \( \Omega_{X/S} = L^\bullet_{X/S} \), since \( \mathcal{X} \) is smooth over \( S \). We define the morphism \( \overline{\mathcal{F}}: L^\bullet_{X/S} \rightarrow \pi^! L^\bullet_{g,n} \) as the composition

\[
\overline{\mathcal{F}}: L^\bullet_{X/S} \rightarrow L^\bullet_{g,n} \rightarrow \mathcal{L}^\bullet_{g,n} \cong \pi^! L^\bullet_{g,n},
\]

where \( \mathcal{C} \) is the universal curve of \( \text{Grw}_{g,n/S} \), the isomorphism \( L^\bullet_{g,n} \cong \pi^! L^\bullet_{g,n} \) follows from the fact that \( \overline{\mathcal{C}} = C \times_{\text{Grw}_{g,n/S}} \text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X}) \) and the morphism \( C \rightarrow \text{Grw}_{g,n/S} \) is flat ([S 8.1]).

3.3. Proposition. The pair \((F^\bullet, \overline{\mathcal{F}})\) defined above is a perfect relative obstruction theory for \( \overline{\mathcal{C}} \).

Proof. Let \( \text{Spec} \overline{K} \xrightarrow{\overline{\mathcal{F}}} \text{Hom}_{\text{Grw}_{g,n}}(C, \mathcal{X}) \) be a geometric point. Then \( \overline{\mathcal{F}} \) corresponds to a twisted pointed curve \( \overline{C}_\mathcal{F} \) over \( \overline{K} \) together with a representable morphism \( \overline{\mathcal{F}}: \overline{C}_\mathcal{F} \rightarrow \mathcal{X} \), obtained by pulling back \((\overline{\mathcal{C}}, \psi)\) along \( \overline{\mathcal{F}} \). Using Serre duality and cohomology and base change theorem ([S Corollary 8.3.4]), we have

\[
H^i(\overline{C}_\mathcal{F}, \overline{\mathcal{F}}^! T_{X/S}) = H^{1-i}(\overline{C}_\mathcal{F}, \overline{\mathcal{F}}^! (\psi^! \Omega_{X/S} \otimes \omega_{\mathcal{T}}))^\vee = h^{i-1}((F^\bullet[-1])^\vee) = h^i((L\pi^\bullet F^\bullet)^\vee).
\]
Now, let $A' \to A = A'/i$ be a small extension in $(\mathbb{A}^r/\partial S_w)$ and consider a commutative diagram

$$
\begin{array}{ccc}
\text{Spec } A & \xrightarrow{g} & \text{Hom}_{\text{tw}}^{\text{tw}}(C, \mathcal{X}) \\
\downarrow & & \downarrow \circ \\
\text{Spec } A' & \xrightarrow{h'} & \mathcal{M}_{g,n/S}
\end{array}
$$

In particular $h'$ corresponds to a family of twisted curves $C_{A'}$ over $A'$, obtained by pulling back $C \to \mathcal{M}_{g,n/S}$ along $h'$, and $g$ corresponds to a family of twisted curves $C_A$ over $A$ together with a representable morphism $\psi_A: C_A \to \mathcal{X}$, obtained by pulling back $(\mathcal{E}, \psi)$ along $g$. Thus $g$ extends to $g'$: Spec $A' \to \text{Hom}_{\text{tw}}^{\text{tw}}(C, \mathcal{X})$ if and only if $\psi_A$ extends to $\psi_{A'}: C_{A'} \to \mathcal{X}$ if and only if, by deformation theory, $h^1(\mathcal{F}^*/(\text{ob}(\mathcal{F}(g, h')))$ is zero in $H^1(C_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}T_{X/S}}) \otimes I$. Moreover, the extensions, if they exist, form a torsor under $H^0(C_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}T_{X/S}}) \otimes I$. By [6] 4.5, $(\mathcal{F}^*, \mathcal{F})$ is a relative obstruction theory for $\mathcal{F}$ and, by Lemma 3.1, $\mathcal{F}^*$ is perfect.

3.2. A perfect obstruction theory for $K_{g,n}(X/S, \beta_n)$.  

3.4. Corollary. Let $E^* = R\pi_+(\psi^*\Omega_{X/S} \otimes \omega_n)[-1]$ and let $\varphi: E^* \to \tau_{\geq -1}L^*_{g}$ be the morphism induced by $\mathcal{F}$. Then $(E^*, \varphi)$ is a perfect relative obstruction theory for $\mathcal{F}$.  

Proof. Since the natural inclusion $j: K_{g,n}(X/S, \beta_n) \to \text{Hom}_{\text{tw}}^{\text{tw}}(C, \mathcal{X})$ is an open immersion, it follows that $Lj^*L^* = L^*g$, $Lj^*F^* = E^*$, and $\varphi = j^*\mathcal{F}$. Hence, by Lemma 3.1 we have $E^* \in D_{\text{coh}}^{(-1,0)}(K_{g,n}(X/S, \beta_n))$. By Proposition 3.3 we know that $(\mathcal{F}^*, \mathcal{F})$ is a perfect obstruction theory for $\mathcal{F}$, hence $h^0(\mathcal{F})$ is an isomorphism and $h^1(\mathcal{F})$ is surjective. Since the pullback $j^*$ is an exact functor, we have that $h^0(\varphi)$ is an isomorphism and $h^1(\varphi)$ is surjective, which implies the statement.

3.5. Definition. We define the virtual fundamental class of $K_{g,n}(X/S, \beta_n)$ to be

$$[K_{g,n}(X/S, \beta_n)]^{\text{virt}} = [K_{g,n}(X/S, \beta_n), E^*]^{\text{virt}} \in A_*(K_{g,n}(X/S, \beta_n)/S).$$

3.6. Remark. Consider the vector bundle stack $\mu: \mathcal{E}_\mathcal{F} = h^1(\mathcal{F}^*) \to K_{g,n}(X/S, \beta_n)$. Then, for a geometric point $\mathcal{T}$ of a component $\mathcal{K}$ of $K_{g,n}(X/S, \beta_n)$, by Riemann-Roch theorem (2.7.2.1),

$$\text{rk } \mathcal{F}^* \mathcal{E}_\mathcal{F} = \dim h^{-1}(LX^* \mathcal{F}^*) - \dim h^0(LX^* \mathcal{F}^*) = \dim H^1(\mathcal{E}_\mathcal{F}, \psi^*T_{X/S}) - \dim H^0(\mathcal{E}_\mathcal{F}, \psi^*T_{X/S}) = (g - 1) \text{rk } (\psi^*T_{X/S}) - c_1(\psi^*T_{X/S}) \cdot [\mathcal{E}_\mathcal{F}] + \sum_{i=1}^{n} \text{age}(\Sigma_i) = (g - 1) \text{dim } \mathcal{X} - c_1(T_{X/S}) \cdot \psi_{X/S}[\mathcal{E}_\mathcal{F}] + \sum_{i=1}^{n} \text{age}(\Sigma_i),$$

where $\text{age}(\Sigma_i) = \text{age}((\psi^*T_{X/S})|_{\Sigma_i})$ denotes the age of a locally free sheaf as defined in [2] 7.1 (recall that the age is constant on connected components of $\mathcal{F}(\mathcal{X})$). Thus the dimension of $[\mathcal{K}]^{\text{virt}}$ is

$$\text{dim } \mathcal{M}_{g,n/S} - \text{rk } \mathcal{E}_\mathcal{F} = \text{dim } \mathcal{X} - (1 - g) + c_1(T_{X/S}) \cdot \psi_{X/S}[\mathcal{E}_\mathcal{F}] - \sum_{i=1}^{n} \text{age}(\Sigma_i) + n.$$

3.3. Properties.  

3.7 (2.5.1). Let $D_{\text{tw}}(g_1, A|g_2, B)$ be the category fibered in groupoids over $(\text{Sch}/S)$ which parametrizes nodal twisted curves with a distinguished node separating the curve in two components, one of genus $g_1$ containing the markings in a subset $A \subset \{1, \ldots, n\}$, the other of genus $g_2$ containing the markings in the complementary set $B$. The category $D_{\text{tw}}(g_1, A|g_2, B)$ is a
smooth algebraic stack, locally of finite presentation over \( S \). Let \( g = g_1 + g_2 \), there is a natural representable morphism
\[
gl : \mathcal{D}^{tw}(g_1, A|g_2, B) \rightarrow \mathfrak{M}^{tw}_{g,n}
\]
induced by gluing the two families of curves into a family of reducible curves with a distinguished node.

3.8. Proposition. (1) Consider the evaluation morphisms \( \tilde{e}_*: K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \rightarrow \mathcal{I}_\mu(\mathcal{X}) \) and \( e_*: K_{g_2, B|\bullet}(\mathcal{X}, \beta_2) \rightarrow \mathcal{I}_\mu(\mathcal{X}) \). There exists a natural representable morphism
\[
K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \times \mathcal{I}_\mu(\mathcal{X}) \rightarrow K_{g_1+g_2, A|\bullet}(\mathcal{X}, \beta_1 + \beta_2).
\]

(2) Consider the evaluation morphisms \( \tilde{e}_* \times e_*: K_{g-1, A|\bullet \bullet}(\mathcal{X}, \beta_\eta) \rightarrow \mathcal{I}_\mu(\mathcal{X})^2 \) and the diagonal \( \Delta: \mathcal{I}_\mu(\mathcal{X}) \rightarrow \mathcal{I}_\mu(\mathcal{X})^2 \) [2.12]. There exists a natural representable morphism
\[
K_{g-1, A|\bullet \bullet}(\mathcal{X}, \beta_\eta) \rightarrow K_{g-1, A|\bullet \bullet}(\mathcal{X}, \beta_\eta).
\]

(3) We have a cartesian diagram
\[
\begin{aligned}
\begin{array}{c}
K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \times \mathcal{I}_\mu(\mathcal{X}) \\
\beta_1 + \beta_2 = \beta_n
\end{array}
\end{aligned}
\]
\[
\begin{aligned}
\mathcal{D}^{tw}(g_1, A|g_2, B) & \xrightarrow{\text{gl}} \mathfrak{M}^{tw}_{g_1+g_2, A|\bullet B} \\
\end{aligned}
\]

Proof. Follows in the same way as in [2] 5.2. \( \square \)

3.9. By [2] 6.2.4, the morphism gl is finite and unramified, therefore, by [10] 4.1, it induces a pull-back homomorphism on Chow groups
\[
gl^*: A_*(K_{g,n}(\mathcal{X}/s, \beta_\eta)) \rightarrow \bigoplus_{\beta_1 + \beta_2 = \beta_n} A_*(K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \times \mathcal{I}_\mu(\mathcal{X}) K_{g_2, B|\bullet}(\mathcal{X}, \beta_2)).
\]

3.10. Proposition. Consider the diagonal \( \Delta: \mathcal{I}_\mu(\mathcal{X}) \rightarrow \mathcal{I}_\mu(\mathcal{X})^2 \) [2.12]. We have

(1) \( \text{gl}[K_{g,A|\bullet B}(\mathcal{X}, \beta_\eta)]^{\text{virt}} = \sum_{\beta_1 + \beta_2 = \beta_n} \Delta^*(\text{[}K_{g_1, A|\bullet}(\mathcal{X}, \beta_1)]^{\text{virt}} \times [K_{g_2, B|\bullet}(\mathcal{X}, \beta_2)]^{\text{virt}}); \)

(2) \( \text{gl}[K_{g,A}(\mathcal{X}, \beta_\eta)]^{\text{virt}} = \Delta^*[K_{g-1, A|\bullet \bullet}(\mathcal{X}, \beta_\eta)]^{\text{virt}}. \)

Proof. For the first part, by Proposition 3.8 and [6] 7.2,
\[
\text{gl}[K_{g,A|\bullet B}(\mathcal{X}, \beta_\eta)]^{\text{virt}} = \sum_{\beta_1 + \beta_2 = \beta_n} [K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \times \mathcal{I}_\mu(\mathcal{X}) K_{g_2, B|\bullet}(\mathcal{X}, \beta_2)]^{\text{virt}}.
\]

Let us denote for simplicity \( K^{(1)} = K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \) and \( K^{(2)} = K_{g_2, B|\bullet}(\mathcal{X}, \beta_2) \). Let \( E_j^* \) be the perfect obstruction theory of \( K^{(j)} \) as constructed in section 3.2, then \( E_1^* \oplus E_2^* \) is the perfect obstruction theory of \( K^{(1)} \times_k K^{(2)} \). Let \( E_{1,2}^* \) be the perfect obstruction theory of \( K^{(1)} \times \mathcal{I}_\mu(\mathcal{X}) K^{(2)} \).

By considering the normalization sequence for a family of nodal curves with a distinguished node \( \Sigma \) over \( K^{(1)} \times \mathcal{I}_\mu(\mathcal{X}) K^{(2)} \), we get the following distinguished triangle, as in [2] 5.3.1,
\[
E_{1,2}^* \rightarrow E_1^* \oplus E_2^* \rightarrow E_{\Sigma}^*.
\]

where \( E_{\Sigma}^* \) can be identified with the cotangent complex of the map \( \Delta \) in the same way as in [2] 3.6.1. Then, by [6] 7.5, we get
\[
\Delta^*[K_{g_1, A|\bullet}(\mathcal{X}, \beta_1)]^{\text{virt}} \times [K_{g_2, B|\bullet}(\mathcal{X}, \beta_2)]^{\text{virt}} = [K_{g_1, A|\bullet}(\mathcal{X}, \beta_1) \times \mathcal{I}_\mu(\mathcal{X}) K_{g_2, B|\bullet}(\mathcal{X}, \beta_2)]^{\text{virt}}.
\]

For the second part of the statement, we observe that, since \( \Delta \) is a regular embedding,
\[
\Delta^*[K_{g-1, A|\bullet \bullet}(\mathcal{X}, \beta_\eta)]^{\text{virt}} = [K_{g-1, A|\bullet \bullet}(\mathcal{X}, \beta_\eta) \times \mathcal{I}_\mu(\mathcal{X})^2 \mathcal{I}_\mu(\mathcal{X})]^{\text{virt}},
\]
and, by [6] 7.2, the right-hand side is equal to \( \text{gl}[K_{g,A}(\mathcal{X}, \beta_\eta)]^{\text{virt}}. \) \( \square \)
4. Gromov-Witten classes and invariants

4.1. Gromov-Witten classes. Let $D$ be a Dedekind domain, set $S = \text{Spec} \, D$ and denote by $\eta$ the generic point of $S$. Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over $S$, admitting a projective coarse moduli scheme $X$. Set $X_\eta = X \times_S \eta$. Fix $\beta_\eta \in A_1(X_\eta/\eta)$ and $g, n \geq 0$ with $2g + n \geq 3$.

4.1. Remark. If $S = \text{Spec} \, k$ with $k$ an algebraically closed field and if $l$ is a prime different from the characteristic of $k$, we can define the $l$-adic étale cohomology as

$$H^r(\overline{T}_\mu(\mathcal{X}), \mathbb{Z}_{l^n}) = \lim_{\to} H^r_{\text{ét}}(\overline{T}_\mu(\mathcal{X}), \mathbb{Z}/l^n).$$

Moreover $H^r(\overline{T}_\mu(\mathcal{X}), \mathbb{Q}_l) = H^r(\overline{T}_\mu(\mathcal{X}), \mathbb{Z}_l \otimes \mathbb{Q}_l)$ and we have the cycle map

$$\text{cl}: A^r(\overline{T}_\mu(\mathcal{X})/k) \to H^{2r}(\overline{T}_\mu(\mathcal{X}), \mathbb{Q}_l(r))$$

as described in [13] VI.9. We set $H^*(\overline{T}_\mu(\mathcal{X})) = \sum_r H^r(\overline{T}_\mu(\mathcal{X}), \mathbb{Q}_l(r))$, where $r$ is the integral part of $r/2$.

4.2. Definition (Gromov-Witten classes). We define the linear operator

$$I_{g,n,\beta_\eta}^X: A^*(\overline{T}_\mu(\mathcal{X})/S)_{\mathbb{Q}}^{\otimes n} \to A^*(\mathcal{M}_{g,n/\eta}/S)_{\mathbb{Q}}$$

such that, given $\gamma \in A^*(\overline{T}_\mu(\mathcal{X})/S)_{\mathbb{Q}}^{\otimes n}$,

$$I_{g,n,\beta_\eta}^X(\gamma_1 \otimes \cdots \otimes \gamma_n) = q_*(e^*(\gamma) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\text{virt}}),$$

where $q: K_{g,n}(\mathcal{X}/S, \beta_\eta) \to \mathcal{M}_{g,n/\eta}$ forgets the map to $\mathcal{X}$, passes to the coarse curve and stabilizes. If moreover $S = \text{Spec} \, k$ with $k$ an algebraically closed field, we can define

$$I_{g,n,\beta_\eta}^X: H^*(\overline{T}_\mu(\mathcal{X}))^{\otimes n} \to H^*(\mathcal{M}_{g,n/\eta})$$

as above, where, abusing the notation, we write $[K_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\text{virt}}$ instead of the corresponding homology class $\text{cl} \left( [K_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\text{virt}} \right)$.

4.3. Definition (Gromov-Witten invariants). We define

$$\langle I_{g,n,\beta_\eta}^X(\gamma) \rangle = \int_{K_{g,n}(\mathcal{X}/S, \beta_\eta)} \left( e^*(\gamma) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\text{virt}} \right),$$

for $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_n \in A^*(\overline{T}_\mu(\mathcal{X})/S)_{\mathbb{Q}}^{\otimes n}$. If $S = \text{Spec} \, k$ with $k$ an algebraically closed field then $\langle I_{g,n,\beta_\eta}^X(\gamma) \rangle$ is defined for every $\gamma \in H^*(\overline{T}_\mu(\mathcal{X}))^{\otimes n}$.

4.4. Notation. When $S = \text{Spec} \, k$, we have $X_\eta = X$ and hence we will simply write $\beta$ instead of $\beta_\eta$.

4.5. Remark. We have that

$$\int_{\overline{M}_{g,n/\eta}} I_{g,n,\beta_\eta}^X(\gamma) = \int_{\overline{M}_{g,n/\eta}} q_*(e^*(\gamma) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\text{virt}}) = \int_{K_{g,n}(\mathcal{X}/S, \beta_\eta)} \left( e^*(\gamma) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\text{virt}} \right) = \langle I_{g,n,\beta_\eta}^X(\gamma) \rangle.$$

4.6. Definition ([2] 6.1.1). We define a locally constant function $r: T_\mu(\mathcal{X}) \to \mathbb{Z}$ by evaluating on geometric points, $r(\mathfrak{P}, \mathcal{G}) = r$, where $\mathcal{G}$ is a gerbe banded by $\mu_r$. We can view $r$ as an element of $A^0(\overline{T}_\mu(\mathcal{X}))$. 
4.7 (Alternative definition). Following the formalism of \[2\], we could define $I_{g,n,\beta}^\chi$ as a linear operator $A^*(\mathcal{T}_\mu(\mathcal{X}/S)_Q^\otimes) \to A^*(\mathcal{T}_\mu(\mathcal{X}/S)_Q)$ such that

$$I_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = r \cdot \hat{e}_{n+1}^\chi_{\text{virt}} \left( \left( \prod_{i=1}^n e_i^\chi(\gamma_i) \right) \cap [K_{g,n+1}(\mathcal{X}/S, \beta_\eta)]_{\text{virt}} \right).$$

With this definition,

$$\int_{\mathcal{T}_\mu(\mathcal{X})} \frac{1}{r} I_{g,n-1,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1}) \cap \epsilon^n(\gamma_n)$$

$$= \int_{\mathcal{T}_\mu(\mathcal{X})} \hat{e}_{n\ast} \left( \left( \prod_{i=1}^{n-1} e_i^\chi(\gamma_i) \right) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]_{\text{virt}} \cap \epsilon_n^\chi(\gamma_n) \right)$$

$$= \int_{\mathcal{T}_\mu(\mathcal{X})} \hat{e}_{n\ast} \left( \left( \prod_{i=1}^{n-1} e_i^\chi(\gamma_i) \right) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]_{\text{virt}} \cap \epsilon_n^\chi(\gamma_n) \right)$$

$$= \int_{K_{g,n}(\mathcal{X}/S, \beta_\eta)} \left( \left( \prod_{i=1}^{n-1} e_i^\chi(\gamma_i) \right) \cap [K_{g,n}(\mathcal{X}/S, \beta_\eta)]_{\text{virt}} \cap \epsilon_n^\chi(\gamma_n) \right)$$

$$= (\bar{I}_{g,n,\beta}(\gamma)).$$

4.8. Remark. Let $\mathcal{M}$ be a proper Artin stack over a field $k$. Let $L$ be a finite algebraic extension of $k$, then $\mathcal{M}_L = \mathcal{M} \times_k L$ is smooth and finite of degree $[L : k]$. By \[9\] 1.7.4, $\rho_L \circ \rho_L^* = [L : k]$, therefore $\rho_L^*$ gives an isomorphism $A_*(\mathcal{M}/k)_Q \cong A_*(\mathcal{M}_L/L)_Q$. Let $\overline{L}$ be an algebraic closure of $k$ and set $\overline{\mathcal{M}} = \mathcal{M} \times_k \overline{k}$, then $A_*(\overline{\mathcal{M}}) = \lim_{\to} A_*(\mathcal{M}_L/L)$, where the limit is over all finite algebraic extensions $L$ of $k$ such that $L \subset \overline{L}$. There is an induced homomorphism $\rho: A_*(\mathcal{M}/k) \to A_*(\overline{\mathcal{M}})$ which gives an isomorphism $A_*(\mathcal{M}/k)_Q \cong A_*(\overline{\mathcal{M}})_Q$; for all $\beta \in A_*(\mathcal{M}/k)$ we set $\overline{\beta} = \rho(\beta)$. The same holds for bivariant Chow groups $A^*(\bullet)_Q$.

4.9. Proposition. Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over a field $k$, admitting a projective coarse moduli scheme $\mathcal{X}$, and set $\overline{\mathcal{X}} = \mathcal{X} \times_k \overline{k}$. Then, for all $\gamma \in A^*(\mathcal{T}_\mu(\mathcal{X}/k)/S)_Q^\otimes$,

$$\bar{I}_{g,n,\beta}(\gamma) = \bar{I}_{g,n,\beta}(\gamma).$$

Proof. Let $L$ be a finite algebraic extension of $k$ and set $\mathcal{X}_L = \mathcal{X} \times_k L$. Let $\beta_L = \rho_L^* \beta$. Notice that $K_{g,n}(\mathcal{X}_L/\beta_L) = K_{g,n}(\mathcal{X}/\beta) \times_k L$ and thus, by \[10\] 7.2,

$$[K_{g,n}(\mathcal{X}/\beta)]_{\text{virt}} = [K_{g,n}(\mathcal{X}_L/\beta)]_{\text{virt}} \subset A_*(\mathcal{K}_{g,n}(\mathcal{X}/\beta)_L/L)_Q \cong A_*(\mathcal{K}_{g,n}(\mathcal{X}/\beta)_L/L)_Q.$$

Then for all $\gamma \in A^*(\mathcal{T}_\mu(\mathcal{X}/k)/S)_Q^\otimes$, we have $I_{g,n,\beta}(\gamma) = I_{g,n,\beta}(\gamma)$ and therefore, passing to the limit, we get $\bar{I}_{g,n,\beta}(\gamma) = \bar{I}_{g,n,\beta}(\gamma).$ \qed

4.2. Comparison of invariants in mixed characteristic. Let $D$ be a Dedekind domain, set $B = \text{Spec} D$. We denote by $\eta = \text{Spec} K$ the generic point of $B$ and let $b_0, b_1 \in B$ be closed points of $B$. Let $\pi: \mathcal{Y} \to B$ be a smooth proper tame Deligne-Mumford stack of finite presentation over $B$, admitting a projective coarse moduli scheme $\overline{\mathcal{Y}}$ and set $\mathcal{Y}_h = \mathcal{Y} \times_B \mathcal{Y}$ for $h = 0, 1$. By \[11\] 20.3, there are specialization morphisms $\sigma_h: A_*(\mathcal{Y}_h/\eta) \to A_*(\mathcal{Y}_h/b_h)$ for $h = 0, 1$. Let $b_h = \text{Spec} k_h$ and let $\overline{k}_h$ be an algebraic closure of $k_h$ for $h = 0, 1$. We set $\overline{b}_h = \text{Spec} \overline{k}_h$. Recall that the cosepecialization map gives an isomorphism $H^*(\mathcal{T}_\mu(\mathcal{Y}_0)) \cong H^*(\mathcal{T}_\mu(\mathcal{Y}_1))$, where $\mathcal{Y}_h = \mathcal{Y}_h \times_{b_h} \overline{k}_h$ for $h = 0, 1$ (\[12\] VI.4.1).

4.10. Theorem. Let $\beta \in A_1(\mathcal{Y}_h/\eta)$ and set $\beta_h = \sigma_h(\beta)$ for $h = 0, 1$. Then

$$\bar{I}_{g,n,\beta}\overline{(\gamma)} = \bar{I}_{g,n,\beta}\overline{(\gamma)},$$

for every $\gamma \in H^*(\mathcal{T}_\mu(\mathcal{Y}_0))$.
Proof. Let \( R_h \) be the localization of \( D \) at \( b_h \) for \( h = 0, 1 \), then \( R_h \) is a discrete valuation ring with generic point \( \eta \) and closed point \( b_h \). Let \( \hat{R}_h \) be the completion of \( R_h \), then \( \hat{R}_h \) is a complete discrete valuation ring with closed point \( b_h \) and generic point \( \eta \times \hat{R}_h \). Moreover \( R_0 \otimes_D R_1 = K \) and hence \( \eta \times R_0 \hat{R}_0 = \eta \times R_1 \hat{R}_1 \). We denote by \( \hat{\eta} = \text{Spec} \hat{K} \) the generic point of \( \hat{R}_h \). Set \( \hat{Y}_h = Y \times_D \hat{R}_h \) and \( \hat{Y}_\eta = Y \times_D \hat{\eta} \). Let \( i_h : Y_h \to \hat{Y}_h \) and \( j_h : \hat{Y}_\eta \to \hat{Y}_h \) be the natural inclusions. Let \( \hat{\beta} \in A_1(\hat{Y}_\eta) \) be the pullback of \( \beta \). We have the following cartesian diagram

\[
\begin{array}{ccc}
\mathcal{K}_{g,n}(Y_\hat{\eta}/\hat{\beta}) & \overset{j}{\to} & \mathcal{K}_{g,n}(Y_h/R_h, \hat{\beta}) \\
\mathcal{M}_{g,n,\eta} & \overset{j}{\to} & \mathcal{M}_{g,n,R_h} \\
\mathcal{M}_{g,n,h} & \overset{i}{\to} & \mathcal{M}_{g,n,h} \\
\end{array}
\]

Let \( \hat{K} \) be an algebraic closure of \( \hat{K} \). We set \( \hat{\beta} = \rho(\hat{\beta}) \in A_1(\hat{Y}_\eta) \), where \( \eta = \text{Spec} \hat{K} \) and \( \hat{Y}_\eta = Y \times_D \eta \). By [13] VI.4.1, there are isomorphisms

\[
\hat{\sigma}_h : A_k(\mathcal{K}_{g,n}(\hat{Y}_\eta), \hat{\beta})_Q \to A_k(\mathcal{K}_{g,n}(\hat{Y}_\eta), \hat{\beta})_Q,
\]

and, by the functoriality of the virtual fundamental class ([14] 7.2),

\[
\hat{\sigma}_h([\mathcal{K}_{g,n}(\hat{Y}_\eta)(\hat{\beta})]^\text{virt}) = ([\mathcal{K}_{g,n}(\hat{Y}_\eta)(\hat{\beta})]^\text{virt})^\text{virt}.
\]

By [13] VI.4.1, there are isomorphisms \( H^*(\hat{I}_\mu(\hat{Y}_\eta)) \cong H^*(\hat{I}_\mu(\hat{Y}_h)) \) for \( h = 0, 1 \), compatible with evaluation maps. It follows that, for \( h = 0, 1 \), \( \hat{I}_{g,n,h}(\hat{\gamma}) = \hat{I}_{g,n,h}(\hat{\gamma}) \) for \( \gamma \in H^*(\hat{I}_\mu(\hat{Y}_h))^{\otimes_n} \). \( \square \)

4.11. Corollary. Let \( X \) be a smooth proper tame Deligne-Mumford stack of finite presentation over a field \( k \), admitting a projective coarse moduli scheme \( X \). Then the Gromov-Witten invariants \( (P^X_{g,n,\beta}) \) are invariant under deformations of \( X \).

4.3. Axioms. Let \( \mathcal{X} \) be a smooth proper tame Deligne-Mumford stack of finite presentation over an algebraically closed field \( k \), admitting a projective coarse moduli scheme \( X \).

4.3.1. Effectivity. Let \( A_1(X/k)_+ \) be the set of \( \beta \in A_1(X/k) \) such that \( \beta \cdot c_1(L) \geq 0 \) for every ample line bundle \( L \). Then \( I^X_{g,n,\beta} = 0 \), for all \( \beta \notin A_1(X/k)_+ \).

Proof. If \( \mathcal{K}_{g,n}(X/k)(\beta) \neq 0 \) then \( \beta = f_*(C) \) for some stable map \( (C, x_i, f) \), hence \( \beta \in A_1(X/k)_+ \). It follows that \( \mathcal{K}_{g,n}(X/k)(\beta) = 0 \) for every \( \beta \notin A_1(X/k)_+ \), and thus \( [\mathcal{K}_{g,n}(X/k, \beta)]^{\text{virt}} = 0 \). \( \square \)

4.3.2. \( S_n \)-covariance. For all \( \gamma_j \in H^{m_j}(\hat{I}_\mu(\mathcal{X})) \), we have

\[
I_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n) = (-1)^{m_i m_{i+1}} I_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n).
\]

Proof. The statement follows from the following ([13] VI.8)

\[
\gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n = (-1)^{m_i m_{i+1}} \gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n \in H^*(\hat{I}_\mu(\mathcal{X}))^\otimes_n. \quad \square
\]

4.3.3. Grading. Let us set \( H^*_n(\mathcal{X}) = H^*(\hat{I}_\mu(\mathcal{X})) \). We consider \( H^*_n(\mathcal{X}) \) as a graded group with the following grading \( H^*_n(\mathcal{X}) = \bigoplus \Omega H^{m-2\text{age}(\Omega)}(\Omega) \), where the sum is taken over all connected components \( \Omega \) of \( \hat{I}_\mu(\mathcal{X}) \). We have

\[
I^X_{g,n,\beta} : \bigotimes_{i=1}^n H^*_m(\mathcal{X}) \to H^\sum_{m_i+2(g-1)\text{dim}_k \mathcal{X} - c_1(T_{X/k})} \mathcal{X}_{\text{log}}(\mathcal{K}_{g,n,\beta}).
\]

Proof. Let \( \overline{\mathcal{X}} = (\mathcal{C} \to \mathcal{X}, \Sigma_1, \ldots, \Sigma_n) \) be a geometric point of a component \( \mathcal{K} \) of \( \mathcal{K}_{g,n}(X, \beta) \) then, for \( i = 1, \ldots, n \), we have evaluation maps \( e_i : \mathcal{K} \to \Omega_i \) for connected components \( \Omega_j \) of
\( \mathcal{I}_\mu(\mathcal{X}) \). Since the age only depends on the connected component, we have \( \text{age}(\Sigma_i) = \text{age}(\Omega_i) \). The virtual fundamental class \( [K]^{\text{virt}} \) is a cycle class of dimension

\[
(\dim S \mathcal{X} - 3)(1 - g) + c_1(T_{\mathcal{X}/k}) \cdot \beta - \sum_{i=1}^{n} \text{age}(\Sigma_i) + n.
\]

Notice that \( \gamma_i \in H_{\text{st}}^{m_i}(\mathcal{X}) = H^{m_i-2\text{age}(\Omega_i)}(\Omega_i) \). It follows that \( I_{g,n,\beta,\Omega_{n+1}}^\mathcal{X}(\gamma) \) has degree

\[
2(3g-3+n)-2 \left( (\dim_k \mathcal{X} - 3)(1-g) + n + c_1(T_{\mathcal{X}/k}) \cdot \beta - \sum_{i=1}^{n} \text{age}(\Sigma_i) \right) + \\
+ \sum_{i=1}^{n} (m_i - 2\text{age}(\Omega_i)) = \sum_{i=1}^{n} m_i + 2((g-1) \dim_k \mathcal{X} - c_1(T_{\mathcal{X}/k}) \cdot \beta). \quad \square
\]

4.3.4. **Fundamental class.** Let \( \varphi_n: \mathcal{M}_{0, n+1/k} \to \mathcal{M}_{0, n/k} \) be the natural functor that forgets the last marked point and stabilizes. We have

\[
I_{g,n,\beta,\Omega_{n+1}}^\mathcal{X}(\gamma_1 \otimes \gamma_2 \otimes 1) = \begin{cases} 
\int \mathcal{I}_{g,n,\beta}^\mathcal{X}(\mathcal{X}) & \text{if } \beta = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Let us form the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{j} & \mathcal{N} \\
\downarrow{\theta} & & \downarrow{\bar{\theta}} \\
\mathcal{M}_{g,n+1/k} & \xrightarrow{\varphi_n} & \mathcal{M}_{g,n/k} \\
\end{array}
\]

and notice that \( \mathcal{M} \) is the algebraic stack of twisted stable maps of genus \( g \) and class \( \beta \) with \( n+1 \) gerbes which remain stable if we forget the last gerbe. In particular there is a regular embedding \( i: \mathcal{M} \to K_{g,n+1}(\mathcal{X}/k, \beta) \) which commute with \( \theta_{n+1} \) and \( \theta \). If we define a virtual fundamental class \( [\mathcal{M}]^{\text{virt}} \) relative to \( \bar{\theta} \) as described in section 3.2 then

\[
i^*[K_{g,n+1}(\mathcal{X}/k, \beta)]^{\text{virt}} = [\mathcal{M}]^{\text{virt}}.
\]

If we define a virtual fundamental class \( [\mathcal{N}]^{\text{virt}} \) relative to \( \bar{\theta} \) then, by \[6\] 7.2,

\[
j^*[\bar{\phi}^*[K_{g,n}(\mathcal{X}/k, \beta)]^{\text{virt}} = j^*[\mathcal{N}]^{\text{virt}} = [\mathcal{M}]^{\text{virt}}.
\]

Let \( \bar{\varphi}: \mathcal{N} \to \mathcal{M}_{g,n+1/k} \) and let \( \pi: \mathcal{I}_\mu(\mathcal{X})^{n+1} \to \mathcal{I}_\mu(\mathcal{X})^n \) be the projection on the first \( n \) components. Moreover we denote \( \bar{e} = e_{(n)} \circ \bar{\varphi}, \hat{e} = e_{(n+1)} \circ \bar{i} \) and observe that \( q_{n+1} \circ i = q \circ j \). We have
that

\[ I_{g,n+1,\beta}^X(\gamma \otimes \text{id}) = q_{n+1,*} \left( e_{(n+1)}^* (\gamma \otimes \text{id}) \cap [K_{g,n+1}(X/k,\beta)]^{\text{virt}} \right) \]
\[ = q_{n+1,*} \left( \hat{e}_n^* (\gamma \otimes \text{id}) \cap i_!^* [K_{g,n+1}(X/k,\beta)]^{\text{virt}} \right) \]
\[ = \tilde{q}_n \left( j_* \hat{e}_n^* (\gamma \otimes \text{id}) \cap \tilde{\varphi}_* [K_{g,n}(X/k,\beta)]^{\text{virt}} \right) \]
\[ = \tilde{q}_n \left( e_{(n)}^* (\gamma) \cap [K_{g,n}(X/k,\beta)]^{\text{virt}} \right) \]
\[ = \varphi_n^* q_{(n)} \left( e_{(n)}^* (\gamma) \cap [K_{g,n}(X/k,\beta)]^{\text{virt}} \right) \]
\[ = \varphi_n^* I_{g,n,\beta}^X(\gamma). \]

The remaining part of the proof follows from the same arguments of [2] 8.2.1. \( \square \)

4.3.5. **Divisor.** We have, for all \( \gamma \in H^2(X) \),

\[ \varphi_* I_{g,n+1,\beta}^X (\bullet \otimes \gamma) = (\beta \cdot \gamma) I_{g,n,\beta}^X (\bullet). \]

**Proof.** Consider the functor

\[ \overline{\varphi} : K_{g,n+1}(X/k,\beta) \to K_{g,n}(X/k,\beta) \]

which forgets the last gerbe and stabilizes, and let

\[ \tilde{\varphi} = \varphi \times e_{n+1} : K_{g,n+1}(X/k,\beta) \to K_{g,n}(X/k,\beta) \times_k \mathcal{I}_\mu(X) \]

By the Künneth formula ([13] VI.8), we can write

\[ \tilde{\varphi}_*[K_{g,n+1}(X/k,\beta)]^{\text{virt}} = [K_{g,n}(X/k,\beta)]^{\text{virt}} \otimes \beta' + \alpha, \]

where \( \beta' \in H^n(\mathcal{I}_\mu(X)) \) and \( \alpha \in H^m(\mathcal{K}_{g,n}(X/k,\beta)) \otimes H^1(\mathcal{I}_\mu(X)) \), with \( m \) less than the degree of \([K_{g,n}(X/k,\beta)]^{\text{virt}}\). The class \( \beta' \) can be calculated by restricting to what happens over a generic point of \( K_{g,n}(X/k,\beta) \). Representing such a point by \( \xi = (C, \Sigma_1, \ldots, \Sigma_n, f) \), we have the cartesian diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & \xi \times_k \mathcal{I}_\mu(X) \\
\downarrow & & \downarrow i \\
K_{g,n+1}(X/k,\beta) & \xrightarrow{\tilde{\varphi}} & K_{g,n}(X/k,\beta) \times_k \mathcal{I}_\mu(X) \\
\end{array}
\]

where, for \( \xi \) generic, the map \( i \) is a regular embedding, hence

\[ i_!^* \tilde{\varphi}_*[K_{g,n+1}(X/k,\beta)]^{\text{virt}} = f_* i_!^* [K_{g,n+1}(X/k,\beta)]^{\text{virt}} = f_* [C] = \beta, \]

on the other hand

\[ i_!^* \tilde{\varphi}_*[K_{g,n+1}(X/k,\beta)]^{\text{virt}} = i_! \left( [K_{g,n}(X/k,\beta)]^{\text{virt}} \otimes \beta' + \alpha \right) = \beta'. \]
It follows that \( \beta' = \beta \). Then

\[
\varphi^* \mathcal{I}_{g,n+1,\beta}(\gamma \otimes \gamma) = \varphi^* q_{n+1*} \left( e^*_n(\gamma \otimes \gamma) \cap [\mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)]^{\text{virt}} \right) \\
= q_{n+1*} \pi_* (\mathcal{I}_{g,n+1,\beta}(\gamma \otimes \gamma) \cap [\mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)]^{\text{virt}}) \\
= q_{n+1*} \left( (e_n \otimes \text{id})^* (\gamma \otimes \gamma) \cap [\mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)]^{\text{virt}} \right) \\
= q_{n+1*} \left( (e_n \otimes \text{id})^* (\gamma \otimes \gamma) \cap \left( [\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\text{virt}} \times \beta + \alpha \right) \right) \\
= q_{n+1*} \left( e_n^*(\gamma) \cap [\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\text{virt}} (\beta \cdot \gamma) \right) \\
= (\beta \cdot \gamma)_{g,n,\beta}(\gamma).
\]

\[\square\]

4.3.6. Splitting. Let \( g_1, g_2, n_1, n_2 \geq 0 \) be integers with \( 2g_j + n_j + 1 \geq 3 \), and set \( g = g_1 + g_2 \), \( n = n_1 + n_2 \). Let

\[\text{gl}: \mathcal{M}_{g_1, n_1 + 1/k} \times_k \mathcal{M}_{g_2, n_2 + 1/k} \to \mathcal{M}_{g, n/k},\]

be the natural functor that identifies the last marked points. Let \( \gamma = \gamma_1 \otimes \cdots \otimes \gamma_n \), then

\[\text{gl}^! \mathcal{I}_{g,n,\beta}(\gamma) = \sum_{\beta_1 + \beta_2 = \beta} \mathcal{I}_{g_1, n_1 + 1, \beta_1} \otimes \mathcal{I}_{g_2, n_2 + 1, \beta_2}(\gamma \otimes [\Delta]).\]

where \( \Delta \) is the diagonal in \( \mathcal{T}_\mu(\mathcal{X})^2 \).

Proof. Let us notice that \( A_1(\mathcal{X}/k)_+ \) is a commutative semigroup then, by effectivity, the sum is finite. Denote for simplicity \( \mathcal{K}(\beta_j) = \mathcal{K}_{g_j, n_j+2}(\mathcal{X}/k, \beta_j) \) for \( j = 1, 2 \). Let us consider the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{K}(\beta_1) \times_k \mathcal{K}(\beta_2) & \xrightarrow{\Delta} & \mathcal{K}(\beta_1) \times_\mathcal{T}_\mu(\mathcal{X}) \mathcal{K}(\beta_2) \\
\downarrow \scriptstyle{q_{1,2}} & & \downarrow \scriptstyle{q} \\
\mathcal{M}_{g_1, n_1 + 1/k} \times_k \mathcal{M}_{g_2, n_2 + 1/k} & \xrightarrow{\text{gl}} & \mathcal{M}_{g, n/k}
\end{array}
\]

where the square is cartesian by Proposition 3.8. Moreover, we have the following cartesian diagram

\[
\begin{array}{ccc}
\mathcal{K}(\beta_1) \times_\mathcal{T}_\mu(\mathcal{X}) \mathcal{K}(\beta_2) & \xrightarrow{\Delta} & \mathcal{K}(\beta_1) \times_k \mathcal{K}(\beta_2) \\
\downarrow \scriptstyle{e_{1,2}} & & \downarrow \scriptstyle{e_{1,2}} \\
\mathcal{T}_\mu(\mathcal{X})^{n+1} & \xrightarrow{\text{id} \times \Delta} & \mathcal{T}_\mu(\mathcal{X})^{n+2}
\end{array}
\]

By Proposition 3.10

\[\text{gl}^! [\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\text{virt}} = \sum_{\beta_1 + \beta_2 = \beta} \Delta^! ([\mathcal{K}_{g_1, n_1 + 1}(\mathcal{X}, \beta_1)]^{\text{virt}} \times [\mathcal{K}_{g_2, n_2 + 1}(\mathcal{X}, \beta_2)]^{\text{virt}}).\]
Then we have

\[
\text{gl}^! I_{g,n,\beta} (\gamma) = \text{gl}^! q_* \left( e^* (\gamma) \cap [K_{g,n}(X/k, \beta)]^{\text{virt}} \right) \\
= q_* \text{gl}^! \left( e^* (\gamma) \cap [K_{g,n}(X/k, \beta)]^{\text{virt}} \right) \\
= q_* \left( \tilde{e}^* \pi^* (\gamma) \cap \text{gl}^![K_{g,n}(X/k, \beta)]^{\text{virt}} \right) \\
= \sum_{\beta_1 + \beta_2 = \beta} q_{1,2*} \tilde{D}_* \left( \tilde{e}^* (\gamma \otimes \text{id}) \cap \Delta \left( [K_{g_1, n_1+1}(X, \beta_1)]^{\text{virt}} \times [K_{g_2, n_2+1}(X, \beta_2)]^{\text{virt}} \right) \right) \\
= \sum_{\beta_1 + \beta_2 = \beta} q_{1,2*} \left( \tilde{e}_1^* (\gamma \otimes \Delta) \cap \left( [K_{g_1, n_1+1}(X, \beta_1)]^{\text{virt}} \times [K_{g_2, n_2+1}(X, \beta_2)]^{\text{virt}} \right) \right) \\
= \sum_{\beta_1 + \beta_2 = \beta} \Delta^* \left( \tilde{e}^* (\gamma \otimes \Delta) \cap \left( [K_{g_1, n_1+1}(X, \beta_1)]^{\text{virt}} \times [K_{g_2, n_2+1}(X, \beta_2)]^{\text{virt}} \right) \right) \\
\sum_{\beta_1 + \beta_2 = \beta} I_{g_1, n_1+1, \beta_1} \otimes I_{g_2, n_2+1, \beta_2} (\gamma \otimes \Delta). \\
\]

4.3.7. **Genus reduction.** Let \( \text{gl} : \overline{M}_{g-1, n+2/k} \to \overline{M}_{g, n/k} \) be the natural functor that identifies the last gerbes. We have

\[
\text{gl}^! I_{g,n,\beta} (\bullet) = I_{g-1, n+2,\beta} (\bullet \otimes [\Delta]),
\]

where \( \Delta \) is the diagonal in \( \mathcal{I}_\mu(X)^2 \).

**Proof.** Let us consider the following commutative diagram

\[
\begin{array}{ccc}
K_{g-1, n+2}(X/k, \beta) & \xrightarrow{\Delta} & K_{g-1, n+2}(X/k, \beta) \times \mathcal{I}_\mu(X)^2 \mathcal{I}_\mu(X) \to K_{g,n}(X/k, \beta) \\
q_{n+2} \downarrow & & \downarrow q_n \\
\overline{M}_{g-1, n+2/k} & \xrightarrow{\text{gl}} & \overline{M}_{g, n/k}
\end{array}
\]

where the square is cartesian. Moreover, we have the following cartesian diagram

\[
\begin{array}{ccc}
K_{g-1, n+2}(X/k, \beta) \times \mathcal{I}_\mu(X)^2 \mathcal{I}_\mu(X) & \xrightarrow{\Delta} & K_{g-1, n+2}(X/k, \beta) \\
\tilde{e} \downarrow & & \downarrow e_{n+2} \\
\mathcal{I}_\mu(X)^{n+1} & \xrightarrow{\text{id} \times \Delta} & \mathcal{I}_\mu(X)^{n+2}
\end{array}
\]

By Proposition [3.10]

\[
\text{gl}^! [K_{g,n}(X/k, \beta)]^{\text{virt}} = \Delta^! ([K_{g-1,n+2}(X, \beta)]^{\text{virt}}).
\]
Then we have
\[ gl^* I_{g,n,\beta}^{X} = gl^* q_{n*} \left( e_{(n)}^* (\gamma) \cap [K_{g,n}(X/k)]^{\text{virt}} \right) = q_{*} gl^* \left( e_{(n)}^* (\gamma) \cap [K_{g,n}(X/k)]^{\text{virt}} \right) = \tilde{q}_* \tilde{e}^* \left( e^* (\gamma) \cap [K_{g,n}(X/k)]^{\text{virt}} \right) = q_{n+2*} \Delta_* \left( e^* (\gamma \otimes \text{id}) \cap \Delta_1 [K_{g-1,n+2}(X,\beta)]^{\text{virt}} \right) = q_{n+2*} \left( \Delta_* \tilde{e}^* (\gamma \otimes \text{id}) \cap [K_{g-1,n+2}(X,\beta)]^{\text{virt}} \right) = q_{n+2*} \left( e_{(n+2)}^* (\gamma \otimes [\Delta]) \cap [K_{g-1,n+2}(X,\beta)]^{\text{virt}} \right) = I_{g-1,n+2,\beta}^{X} (\gamma \otimes [\Delta]). \]

5. Genus zero invariants in positive characteristic

5.1. Gromov-Witten potential. Let \( X \) be a smooth proper tame Deligne-Mumford stack of finite presentation over an algebraically closed field \( k \) (of arbitrary characteristic), admitting a projective coarse moduli scheme \( X \). Fix \( \beta \in A_1(X/k) \) and \( n \geq 0 \). Let \( l \) be a prime different from the characteristic of \( k \).

5.1. Remark. Recall that we defined on the group \( H^m_{st}(X) = H^r(T_{\mu}(X)) \) the following grading \( H^m_{st}(X) = \bigoplus_{\Omega} H^{m-2\text{deg}(\Omega)}(\Omega) \), where the sum is taken over all connected components \( \Omega \) of \( T_{\mu}(X) \). By [13] V.1.11, \( H^m_{st}(X) = \sum_r H^r(T_{\mu}(X), \mathbb{Q}(\mathcal{P})) \) is finitely generated over \( \mathbb{Q} \). Let \( T_0 = 1, T_1, \ldots, T_m \) be generators for \( H^1_{st}(X) \). For each \( i = 1, \ldots, m \), we introduce a variable \( t_i \) of the same degree of \( T_i \), such that the \( t_i \) supercommute, which means

\[ t_i t_j = (-1)^{\text{deg}(t_i) \cdot \text{deg}(t_j)} t_j t_i, \]

and \( t_i^2 = 0 \) if \( t_i \) has odd degree.

5.2. Remark. If \( \gamma_i \in H^m_{st}(X) \) then \( \langle I_{0,n,\beta}^{X}, \gamma_i \rangle \) is zero unless

\[ \sum_{i=1}^n m_i = 2(\dim_k X + c_1(T_{X/k}) \cdot \beta). \]

5.3. Notation. We denote the vector \( (a_0, \ldots, a_m) \) as \( \vec{a} \); we set \( |\vec{a}| = a_0 + \cdots + a_m \) and \( \vec{a}! = a_0! \cdots a_m! \). Moreover we set \( \langle I_{0,n,\beta}^{X}, \gamma \rangle = 0 \) for \( n < 3 \).

5.4. Definition. Let \( \gamma = \sum_{i=0}^m t_i T_i \) (regarding \( T_i \) and \( t_i \) as supercommuting variables). We define the genus 0 Gromov-Witten potential as the formal series

\[ \Phi(\gamma) = \sum_{n \geq 0} \sum_{\beta \in A_1(X/k)} \frac{1}{n!} \langle I_{0,n,\beta}^{X}, \gamma^n \rangle q^\beta, \]

where \( q^\beta \) is a free variable of degree \( \beta \cdot c_1(T_{X/k}) \) and

\[ \frac{1}{n!} \langle I_{0,n,\beta}^{X}, \gamma^n \rangle = \sum_{|\vec{a}| = n} \epsilon(\vec{a}) \langle I_{0,n,\beta}^{X}, (T_{\vec{a}}) \rangle \frac{t_{\vec{a}}}{\vec{a}!}, \]

with \( \epsilon(\vec{a}) = \pm 1 \) determined by

\[ (t_0 T_0)^{a_0} \cdots (t_m T_m)^{a_m} = \epsilon(\vec{a}) T_0^{a_0} \cdots T_m^{a_m} t_0^{a_0} \cdots t_m^{a_m}. \]

5.5. Remark. By effectivity axiom, the Gromov-Witten potential is a formal series in \( \mathcal{R} = R[[t_0, \ldots, t_m]], \) with \( R = \mathbb{Q}[q^\beta; \beta \in A_1(X/k)_+]. \)
5.2. Quantum product. By VI.8, \( H^*(\mathcal{T}_\mu(X) \times_k \mathcal{T}_\mu(X)) = H^*(\mathcal{T}_\mu(X)) \otimes H^*(\mathcal{T}_\mu(X)) \). Let \( \Delta \) be the diagonal in \( \mathcal{T}_\mu(X)^2 \), then
\[
[\Delta] = \sum_{e,f} g^e T_e \otimes T_f.
\]

5.6. Definition. We define
\[
T_i * T_j = \sum_{e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^e T_f,
\]
Extending this linearly gives the (big) quantum product on \( H^*_a(X, \mathcal{R}) \).

5.7. Remark. Notice that the Gromov-Witten invariants with \( n < 3 \) do not affect the quantum product.

5.8. Lemma. For all \( i, j, h, \) we have
\[
\frac{\partial^3 \Phi(\gamma)}{\partial t_i \partial t_j \partial t_h} = \sum_{n, \beta} \left( \frac{1}{n!} \left( \begin{array}{c} n \\ \beta \end{array} \right) (T^a_i \otimes T^a_j \otimes T^a_h \otimes \gamma^n) q^\beta \right).
\]

Proof. For simplicity, we will assume that \( H^*_a(X, \mathcal{R}) \) has only even cohomology so that we don’t have to worry about signs. We have
\[
\frac{\partial^3 \Phi(\gamma)}{\partial t_i \partial t_j \partial t_h} = \frac{\partial^3}{\partial t_i \partial t_j \partial t_h} \sum_{n, \beta} \left( \begin{array}{c} n \\ \beta \end{array} \right) (T^a_i \otimes T^a_j \otimes T^a_h \otimes \gamma^n) q^\beta = \sum_{n, \beta} \left( \begin{array}{c} n \\ \beta \end{array} \right) (I^n_{0, n+3, \beta}) (T^a_i \otimes T^a_j \otimes T^a_h \otimes \gamma^n) q^\beta,
\]
where \( \text{deg} a = a_i - e_j - e_h \) and \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 in the \( i \)-th position. Moreover
\[
\sum_{n, \beta} \left( \begin{array}{c} n \\ \beta \end{array} \right) (I^n_{0, n+3, \beta}) (T^a_i \otimes T^a_j \otimes T^a_h \otimes \gamma^n) q^\beta = \sum_{n, \beta} \left( \begin{array}{c} n \\ \beta \end{array} \right) (I^n_{0, n+3, \beta}) (T^{a_i+e_i+e_h} \otimes \gamma^n) q^\beta = \sum_{n, \beta} (I^n_{0, n+3, \beta}) (T^{a_i+e_i+e_h} \otimes \gamma^n) q^\beta.
\]

5.9. Theorem (WDVV equation). The Gromov-Witten potential satisfies the equation
\[
\sum_{e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^e f \frac{\partial^3 \Phi}{\partial t_j \partial t_k \partial t_l} = (-1)^{\text{deg} t_i \text{deg} t_j + \text{deg} t_k} \sum_{e,f} \frac{\partial^3 \Phi}{\partial t_j \partial t_k \partial t_e} g^e f \frac{\partial^3 \Phi}{\partial t_j \partial t_l \partial t_e},
\]
for all \( i, j, h, l \).

Proof. For simplicity, we will assume that \( H^*_a(X, \mathcal{R}) \) has only even cohomology so that we don’t have to worry about signs. If we set
\[
F(ij|hl) = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^e f \frac{\partial^3 \Phi}{\partial t_j \partial t_h \partial t_l},
\]
then we want to show that \( F(ij|hl) = F(jh|il) \). Consider the following cartesian diagram
\[
\begin{array}{ccc}
D(ij|hl) & \xrightarrow{\rho} & \overline{\mathcal{M}}_{0, n+4/jk} \\
\Spec k - \overline{\mathcal{M}}_{0, \{i,j\} \cup \{k\}} \times_k \overline{\mathcal{M}}_{0, \{i,k\}} \cup \{g\} & \xrightarrow{\gamma} & \overline{\mathcal{M}}_{0, 4/jk}
\end{array}
\]
where the image of \( gl \) is a boundary point of \( \overline{\mathcal{M}}_{0, 4/jk} \). Since the boundary points are linearly equivalent, the same is true for the fibers of \( \rho \) over these points, hence \( D(ij|hl) \) and \( D(jh|il) \)
are linearly equivalent divisors in $\overline{\mathcal{M}}_{0,n+4:k}$. Let $A \cup B$ be a partition of $\{1, \ldots, n+4\}$ such that $i, j \in A$ and $h, l \in B$. Let us set $\overline{\mathcal{M}}_{A,B} = \overline{\mathcal{M}}_{0,A \cup \ast:k} \times_k \overline{\mathcal{M}}_{0,B \cup \ast:k}$ and form the fiber square

$$
\begin{array}{ccc}
D(A|B) & \rightarrow & D(ij|hl) \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_{A,B} & \rightarrow & \overline{\mathcal{M}}_{0,n+4:k}
\end{array}
$$

then $D(ij|hl) = \bigsqcup_{A \cup B = \{1, \ldots, n+4\}} D(A|B)$. We set

$$
\overline{K}^{(\beta_1, \beta_2)} = K_{0,A \cup \ast}(\chi/k, \beta_1) \times_k K_{0,B \cup \ast}(\chi/k, \beta_2).
$$

Let us set for simplicity $\gamma_{n_1} = T_i \otimes T_j \otimes \gamma^n$ and $\gamma_{n_2} = T_h \otimes T_l \otimes \gamma^n$. Then, by Lemma 5.8 and splitting axiom,

$$
F(ij|hl) = \sum_{\beta_1, \beta_2, n_1, n_2, e, f} \frac{1}{n_1! n_2!} (T^X_{0,n_1+3, \beta_1})(T_e \otimes \gamma_{n_1}) g^{ef} (T^X_{0,n_2+3, \beta_2})(T_f \otimes \gamma_{n_2}) q^{\beta_1 + \beta_2}
$$

$$
= \sum_{\beta, n} \sum_{\beta_1 + \beta_2 = \beta} \frac{1}{n_1! n_2!} \int_{[K(\beta_1, \beta_2)]^\text{virt}} g^{ef} e_{(1,2)}^* (T_e \otimes \gamma_{n_1} \otimes T_f \otimes \gamma_{n_2}) q^\beta
$$

$$
= \sum_{\beta, n} \sum_{A \cup B = \{1, \ldots, n+4\}} \frac{1}{n_1! n_2!} \int_{\overline{\mathcal{M}}_{A,B}} g^! T^X_{0,n+4, \beta}(T_i \otimes T_j \otimes T_h \otimes T_l \otimes \gamma^n) q^\beta
$$

$$
= \sum_{\beta, n} \frac{1}{n_1!} \int_{D(ij|hl)} T^X_{0,n+4, \beta}(T_i \otimes T_j \otimes T_h \otimes T_l \otimes \gamma^n) q^\beta.
$$

Since $D(ij|hl)$ and $D(jh|il)$ are linearly equivalent, it follows that $F(ij|hl) = F(jh|il)$.

5.10. **Proposition.** The quantum product is supercommutative with identity $T_0$ and associative.

**Proof.** By Lemma 5.8 and $S_n$-covariance axiom,

$$
T_i \ast T_j = \sum_{\beta, n, e, f} \frac{1}{n!} (T^X_{0,n+3, \beta})(T_i \otimes T_j \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta
$$

$$
= \sum_{\beta, n, e, f} \frac{1}{n!} (-1)^{\deg T_e \deg T_f} (T^X_{0,n+3, \beta})(T_j \otimes T_i \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta
$$

$$
= (-1)^{\deg T_i \deg T_j} T_j \ast T_i.
$$
Let $\Delta: \mathcal{T}_\mu(X) \to \mathcal{T}_\mu(X)^2$ be the diagonal (2.12) and let $p_i: \mathcal{T}_\mu(X)^2 \to \mathcal{T}_\mu(X)$ be the natural projections for $i = 1, 2$. By the fundamental class axiom,

$$T_i = p_{2*} \Delta_* (\Delta^1 p_i^* (T_i)) = p_{2*} (p_i^* (T_i) \cup [\Delta]) = \sum_{e,f} g^{ef} p_{2*}((\iota^* (T_i) \otimes T_0) \cup (T_e \otimes T_f)) = \sum_{e,f} g^{ef} p_{2*}((\iota^* (T_i) \cup T_e) \otimes T_f) = \sum_{e,f} (I_{0,3,0}^X)(T_0 \otimes T_i \otimes T_e) g^{ef} T_f.$$

Moreover, we have $\langle I_{0,n+3,\beta}^X(\bullet \otimes T_0) \rangle = 0$ unless $\beta = 0$ and $n = 3$. Therefore

$$T_0 * T_i = \sum_{\beta, n, e, f} \frac{1}{n!} (I_{0,n+3,\beta}^X)(T_0 \otimes T_i \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta = \sum_{e, f} (I_{0,3,0}^X)(T_0 \otimes T_i \otimes T_e) g^{ef} T_f = T_i.$$

Finally, we prove that the quantum product is associative. For simplicity, we will assume that $H_{st}^*(X, \mathbb{R})$ has only even cohomology so that we don’t have to worry about signs. We have

$$(T_i * T_j) * T_h = \sum_{e, f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} T_e * T_h = \sum_{e, f, d, e, f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} \frac{\partial^3 \Phi}{\partial t_f \partial t_h \partial t_e} g^{cd} T_d$$

and

$$T_i * (T_j * T_h) = (-1)^{\deg T_i (\deg T_j + \deg T_h)} (T_j * T_h) * T_i,$$

since the quantum product is supercommutative. Therefore, associativity follows from Theorem 5.9.

5.3. **Reconstruction for genus zero Gromov-Witten invariants.**

5.11. **Theorem.** If $H_{st}^*(X)$ is generated by $H_{st}^2(X)$ then every genus zero Gromov-Witten invariant can be uniquely reconstructed starting with the following system of Gromov-Witten invariants

$$\left\{ I_{0,3,\beta}^X(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \big| \beta \cdot c_1(X/k) \leq \dim_k X + 1, \deg \gamma_3 = 2 \right\}.$$

**Proof.** Apply the WDVV equation (Theorem 5.9) to $\gamma_1 \otimes \cdots \otimes \gamma_{n+1}$ with indices $\{i, j, h, l\} = \{1, 2, n, n + 1\}$. Let us define a partial order on pairs $(\beta, n)$, with $n \geq 3$ and $\beta \in A_1(X/k)_+$, by setting $(\beta, n) > (\beta', n')$ if and only if either $\beta = \beta' + \beta''$ or $\beta = \beta'$ and $n > n'$. Then there are four terms of higher order in the WDVV equation each of the form

$$I_{a,b} = \sum_{e,f} (I_{0,3,0}^X)(\gamma_a \otimes \gamma_b \otimes T_e) g^{ef} (I_{0,n-1,\beta}^X)(T_f \otimes (\otimes_{s \neq a, b} \gamma_s)),$$

with $(a, b) \in \{(1, 2), (n, n + 1), (2, n), (1, n + 1)\}$. As shown in the proof of Proposition 5.10 we have

$$\gamma_a \cup \gamma_b = \sum_{e, f} (I_{0,3,0}^X)(\gamma_a \otimes \gamma_b \otimes T_e) g^{ef} T_f,$$

hence $I_{a,b} = (I_{0,3,0}^X)(\gamma_a \cup \gamma_b \otimes (\otimes_{s \neq a, b} \gamma_s))$. Let us consider $(I_{0,n,\beta}^X)(\gamma_1 \otimes \cdots \otimes \gamma_n)$. If $\deg \gamma_n = 2$, then we can apply divisor axiom to reduce $n$. Otherwise, since $H_{st}^*(X)$ is generated by $H_{st}^2(X)$, we can write $\gamma_n = \sum \delta_i \cup \delta_i$, with $\deg \delta_i = 2$. By linearity, we can assume $\gamma_n = \delta' \cup \delta$, with
deg δ = 2. Apply the construction above with γ_n = δ' and γ_{n+1} = δ. Then, by WDVV equation, we get
\[ \pm (\Omega_{0,n-1,\beta}^X) (\gamma_1 \cup \gamma_2 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta' \cup \delta) \pm (\Omega_{n,n-1,\beta}^X) (\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta' \cup \delta) \]
\[ \pm (\Omega_{0,n-1,\beta}^X) (\gamma_1 \cup \delta \otimes \gamma_2 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta') \pm (\Omega_{n,n-1,\beta}^X) (\gamma_1 \otimes \gamma_2 \otimes \delta' \otimes \gamma_3 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta) = \]
= a combination of higher order terms.

By divisor axiom, the first and the fourth summands are lifted from \( \overline{\mathcal{M}}_{0,n-1/k} \). Moreover in the third summand we have deg δ' < deg γ_n. If deg δ' = 2 then, by divisor axiom, we can reduce n, otherwise we repeat this trick and in a finite number of iterations we will reduce n. Finally, we can apply the procedure described above to \( (\Omega_{0,3,\beta}^X) (\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \) and decrease deg γ_3 ≥ 2. □

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