FINITELY STAR REGULAR DOMAINS (*)

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Abstract. Let R be an integral domain, Star(R) the set of all star operations on R and StarFC(R) the set of all star operations of finite type on R. Then R is said to be star regular if |Star(T)| ≤ |Star(R)| for every overring T of R. In this paper we introduce the notion of finitely star regular domain as an integral domain R such that |StarFC(T)| ≤ |StarFC(R)| for each overring T of R. First, we show that the notions of star regular and finitely star regular domains are completely different and do not imply each other. Next, we extend/generalize well-known results on star regularity in Noetherian and Prüfer contexts to finitely star regularity. Also we handle the finite star regular domains issued from classical pullback constructions to construct finitely star regular domains that are not star regular and enriches the literature with a such class of domains.

1. Introduction

Let R be an integral domain with quotient field L, F(R) the set of nonzero fractional ideals of R and f(R) the set of nonzero finitely generated fractional ideals of R. A mapping * : F(R) → F(R), E → E’, is called a star operation on R if the following conditions hold for all a ∈ L \ {0} and E, F ∈ F(R):

(I) (aE)’ = aE’;
(II) E ⊆ E’; if E ⊆ F, then E’ ⊆ F’; and
(III) (E’)’ = E’.

The simplest star operations are the d-operation defined by E’ = E for every E ∈ F(R), and the v-operation defined by E’ = (E’v)−1 (where E’v = (R : E) = {x ∈ L |xE ⊆ R}) for every E ∈ F(R). A star operation * is said to be of finite type (or of finite character) if for each nonzero (fractional) ideal E of R, E’ = ⋃F’ ∈ F where the union is taken over all nonzero finitely generated subideals F of E. Also a star operation is stable if (E ∩ F)’ = E’ ∩ F’ for each E, F ∈ F(R). To any star operation * on R, we associate a star operation of finite type *f, and a stable star operation of finite type *w by setting respectively E’f = ⋃{F | F ∈ f(R), F ⊆ E} and E’w = ⋃{(E : F) | F ∈ f(R), F’ = R}. Notice that v_f = t and t_w = w. For star operations * and *’ on R, * ≤ *’ provided that E’ ⊆ E’’ for every E ∈ F(R). Clearly d ≤ w ≤ t ≤ v and for every star operation * on R, d ≤ * ≤ v and d ≤ *’ ≤ t. We denote by Star(R) the set of all star operations on R and StarFC(R) the set of all star operations of finite type on R.

Recently, motivated by well-known characterizations of integrally closed and Noetherian divisorial domains \[22,38\], the author of this paper, together with E. Houston and M. H. Park, started a long and deep study of some ring-theoretic properties of integral domains having only finitely many star operations in different contexts of integral domains. Namely, complete characterizations are given in the cases of

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local Noetherian domains with infinite residue field and integrally closed domains see [25, 26, 27, 28, 29] (see also [11, 30, 24, 44], and [46, 47]). In [28], the authors studied, for a Noetherian domain $R$, how $|\text{Star}(R)|$ affects $|\text{Star}(T)|$ for each proper overring $T$ of $R$ with the emphasis on the case where $\text{Star}(R)$ is finite. They introduced the notion of a star regular domain as a domain $R$ such that $|\text{Star}(T)| \leq |\text{Star}(R)|$ for each overring $T$ of $R$. Notice that a Noetherian domain $R$ (which is not a field) with finitely many star operations has Krull dimension one. The authors constructed a Noetherian domain $R$ with $|\text{Star}(R)| = 1$ (equivalently, $R$ is a divisorial domain), but having an overring $T$ with $|\text{Star}(T)| = \infty$. Next, they showed that for a one-dimensional Noetherian domain $R$, if $R$ is locally star regular, then it is star regular, and the converse holds if $\text{Star}(R)$ is finite. They conjectured that “if $R$ is a local Noetherian domain with $1 < |\text{Star}(R)| < \infty$, then $R$ is star regular”, and proved that this conjecture holds if $R$ has infinite residue field. They also considered the question of whether finiteness of $\text{Star}(T)$ for each proper overring of a Noetherian domain $R$ implies finiteness of $\text{Star}(R)$, and showed that this occurred when $R$ is non-local.

In [29], the authors investigated star regular domains in the context of Prüfer domains. They showed that star regularity for Prüfer domains with only finitely many star operations reduces to star regularity of Prüfer domains $R$ possessing a nonzero prime ideal $P$ contained in the Jacobson radical of $R$ such that $\text{Spec}(R/P)$ is finite (Theorem 3.1)). More precisely, they proved that if $R$ is a semi-local Prüfer domain with more than one maximal ideal such that $R/P$ is strongly discrete (where $P$ is the largest prime ideal contained in the Jacobson radical of $R$), then $R$ is star regular (Theorem 3.11)).

In [34], the authors investigated some ring-theoretic properties of certain classes of integral domains with only finitely many star operations of finite type. Several generalizations/analogues of well-known results on integral domains with finitely many star operations were extended to integral domains with finitely many star operations of finite type. Namely in Noetherian-like settings such as Mori domains, pullback constructions and more.

The purpose of this paper is to introduce and study the notion of finitely star regular domains, that is, integral domains $R$ such that $|\text{StarFC}(T)| \leq |\text{StarFC}(R)|$ for each overring $T$ of $R$. The class of finitely star regular domains includes the class of Prüfer domains (since $|\text{StarFC}(R)| = 1$ for every Prüfer domain $R$), and coincides with the class of star domains in the Noetherian context. Motivated by the fact that Prüfer domains are finitely star regular domains, but not always star regular, in section 2, we start by showing that the two notions of star and finitely star regular domains are completely different and do not imply each other, (see Example 2.3 and Example 2.4). Next, we extend [26] Theorem 2.3 to a one-dimensional domain of finite character (i.e., every nonzero non-unit element is contained in a finitely many maximal ideals). That is, for a one-dimensional domain $R$ of finite character such that $\text{StarFC}(R)$ is finite, $|\text{StarFC}(R)| = \prod_{M \in \text{Max}(R)} |\text{StarFC}(R_M)|$ (Theorem 2.10). The second main theorem asserts that if $R$ is a one-dimensional quasi-Prüfer domain (i.e. its integral closure is a Prüfer domain) such that $\text{StarFC}(R)$ is finite, then $R$ is finitely star regular if and only if $R_M$ is finitely star regular for every maximal ideal $M$ of $R$ (Theorem 2.11). The third section deals with certain pullbacks that are
finely star regular. The main result asserts that a large class of integral domains related to valuation domains, namely the class of pseudo-valuation domains (PVD for short), are always finely star regular, but not star regular in general. Next, we deal with the classical pullbacks issued from valuation domains in order to enriches the literature with such a class of integral domains.

Throughout, $R$ denotes an integral domain (which is not a field), $R'$ its integral closure and $\overline{R}$ its complete integral closure. The set of all maximal ideals of $R$ is denoted by $\text{Max}(R)$, and if $S$ is an overring of $R$, $[R,S]$ denotes the set of all intermediate rings between $R$ and $S$ (i.e. rings $T$ such that $R \subseteq T \subseteq S$).

2. General Results

Recall that an integral domain $R$ is said to be a star regular domain if $|\text{Star}(T)| \leq |\text{Star}(R)|$ for every overring of $R$. Next, we introduce the notion of finely star regular domains.

**Definition 2.1.** Let $R$ be an integral domain. We say that $R$ is finely star regular if $|\text{StarFC}(T)| \leq |\text{StarFC}(R)|$ for each overring $T$ of $R$.

The next proposition deals with the class of integrally closed domains and shows that it is an important class of finely star regular domains that are not necessarily star regular.

**Proposition 2.2.** Let $R$ be an integrally closed domain. Then $R$ is finely star regular.

**Proof.** Assume that $R$ is integrally closed. If $|\text{StarFC}(R)| = \infty$, we are done. Assume that $\text{StarFC}(R)$ is finite. Then by [27, Theorem 3.1], $R$ is a Prüfer domain. Thus every overring $T$ of $R$ is Prüfer and so $|\text{StarFC}(T)| = |\text{StarFC}(R)| = 1$, as desired. □

Our next three examples show that the notions of star regular domain and finely star regular domain do not implies each other. The first one is an example of a finely star regular domain $R$ with $\text{Star}(T)$ finite for every overring $T$ of $R$, but which is not star regular (as it has an overring $T$ with $|\text{Star}(R)| < |\text{Star}(T)|$). The second one is an example of a star regular domain which is not finely star regular with $\text{StarFC}(R)$ finite but $\text{Star}(R)$ is infinite. In the third example, $R$ is finely star regular but not star regular with $\text{Star}(R)$ finite and having an overring $T$ with $|\text{Star}(T)| = \infty$. Notice that if $V$ is a valuation domain, then $|\text{Star}(V)| \leq 2$.

**Example 2.3.** Let $V$ be a valuation domain with a principal maximal ideal $M$ and suppose that $V$ has a nonzero non-maximal ideal $P$ such that $P = P^2$. Clearly for every overring $T$ of $V$, $|\text{StarFC}(T)| = |\text{StarFC}(V)| = 1$. Thus $V$ is a finely star regular domain. However, $|\text{Star}(V_p)| = 2$ while $|\text{Star}(V)| = 1$. Hence $V$ is not star regular.

**Example 2.4.** The following is an example of a star regular domain which is not finely star regular. Let $k$ be an infinite field, $X$ and $Y$ indeterminates over $k$ and set $D = k[X^3, X^7]$ and $D_1 = k[X^3, X^7, X^8]$. By [41, Theorem 2.2], $D$ is a divisorial Noetherian domain and so $|\text{Star}(D)| = |\text{StarFC}(D)| = 1$, while $D_1$ is a Noetherian overring of $D$ with $|\text{Star}(D_1)| = |\text{StarFC}(D_1)| = \infty$. Indeed, let $M_1 = (X^3, X^7, X^8)$. Then $M_1$ is a maximal ideal of $D_1$ with $(D_1 : M_1) = k[X^3, X^4, X^5]$ and so $(D_1)_M$ is a Noetherian local domain satisfying the conditions of [26, Theorem 3.9]. Thus $|\text{StarFC}(D_1)_M| = |\text{Star}(D_1)_M| = \infty$. By [26, Theorem 2.3], $|\text{StarFC}(D_1)| = |\text{Star}(D_1)| = \infty$. Now set $V = k(X)[[Y]] = k(X) + Yk(X)[[Y]]$, $R = D + Yk(X)[[Y]] = D + M$ and $T = D_1 + M$. By
Theorem 4.4, |StarFC(R)| = |StarFC(D)| = 1, but |StarFC(T)| = |StarFC(D)| = ∞.
Thus R is not finitely star regular. However, |Star(R)| = ∞ and so R is star regular.
Indeed, for every maximal ideal P of k[X], set S_P = k[X]_P + M. Then S_P is a fractional
overring of R, and (R : S_P) = M, so that (S_P)_P = (R : M) = V. Now, by Proposition
2.7, *P = δ(d_S_P, v) defined by E^P = ES_P ∩ E_0 is a star operation on R. Moreover, for
P ≠ Q maximal ideals of k[X], (S_P)^Q = S_P ∩ V = S_Q and (S_P)^Q = S_Q S_P ∩ V = V ∩ V = V.
Hence *P ≠ *Q. Since k[X] has infinitely many maximal ideals that induce different
star operations on R, R has infinitely many star operations.

The following is an example of a non star regular Prüfer domain R with only
finitely many star operations having an overring T with |Star(T)| = ∞. The Example
is given in [29 Example 2.4], and for the convenience of the reader, we include it
here.

Example 2.5. The following is an example of a finitely star regular which is not
star regular. Let A be the direct sum of countably infinitely many copies of G,
where G is the totally ordered group R ⊕ Z with lexicographic order (i.e., for
(a, b), (a', b') ∈ R ⊕ Z, (a, b) ≥ (a', b') if a > a' or a = a' and b ≥ b'). For (a_i)_∞^i=1 ∈ A,
define (a_i) ≥ 0 if a_i ≥ 0 for all i ≥ 1. Then A is a lattice ordered group. According to
Jaffard and Ohm [33, 43], every lattice ordered group is a group of divisibility of a
Bézout domain. Thus there is a Bézout domain R with K'/U(R) ≃ A, where U(R) is
the set of units of R. Since A is the weak direct product of G's, R has finite character.
It follows easily from Theorem 3.2 and Lemma 3.4 that R is an h-local
Prüfer domain with dim R = 2. Thus each overring T of R is a Prüfer domain
and so |StarFC(T)| = |StarFC(R)| = 1. Hence R is finitely star regular. Since for each
maximal ideal M of R, R_M has value group G, MD_M is principal. Since R has finite
character, each M is invertible. Hence R is a divisorial domain [22 Theorem 5.1].
Let M be a maximal ideal of R, and let P be the nonzero prime contained in M.
Since the value group of R_P is equal to R, PR_P is not principal. Hence P = P^2, and
so R_M is not strongly discrete. By Theorem 2.3], there is an overring T of R
with |Star(T)| = ∞ (and we may take T = ∩ R_P, where the intersection is taken over
the height-one primes of R).
Notice that, for each positive integer n, if in the definition of A, we replace n of the
G by R ⊕ R, then the resulting R satisfies |Star(R)| = 2^n by Theorem 3.1, and
R has an overring T with |Star(T)| = ∞ (29 Remark 2.5)).

The next two lemmas are crucial. The first one is Proposition 2.7] and we shall use it whenever we consider a proper overring T of the base ring R and a star
operation on T. The second one is a direct combination of Proposition 2.4 and
Proposition 4.6] and we shall use it whenever we consider the particular overring
T = R_M for some maximal ideal M.

Lemma 2.6. (25 Proposition 2.7] Let R be an integral domain, T an overring of R, * a
star operation on R and *' a star operation on T. Then the map δ(*', *) : F(R) → F(R),
E → E^δ(*', *') := (ET)^* ∩ E', defines a star operation on R. Moreover, if * and *' are of finite
type, then so is δ(*', *).

Lemma 2.7. Let R be an integral domain, M a maximal ideal of R and * ∈ StarFC(R).
Then *'(M) defined on R_M by (AR_M)^*(M) = A' R_M for every A ∈ f(R) is a star operation on
R_M. Moreover, if R is v-coherent, then every star operation on R_M is of this form, that is,
if *' ∈ StarFC(R_M), then *' = *(M) for some * ∈ StarFC(R).
Theorem 2.8. Let $R$ be an integral domain such that $(R : \overline{R}) = M$ is a maximal ideal of $R$ and suppose that $\overline{R}$ is a PID. Then for every $T \in [R, \overline{R}]$, $|\Star(T)| \leq |\Star(R)|$ and $|\StarFC(T)| \leq |\StarFC(R)|$.

Proof. First notice that $M^{-1} = (M : M) = \overline{R}$ and so $M$ is a divisorial ideal of $R$. Let $T \in [R, \overline{R}]$. Since $R$ is a PID, $|\StarFC(R)| = 1 \leq |\StarFC(R)|$, so without loss of generality we may assume that $R \subseteq T \subseteq \overline{R}$. Thus $(R : T) = M$ and so $M$ is an ideal of $T$ and $T = M^{-1} = \overline{R}$. Notice that if $S$ is a fractional overring of $T$, then $S \subseteq \overline{S} = T = M^{-1}$. Since $(M^{-1})^* = M^{-1}$ is a fractional overring of $T$, then $(M^{-1})^* = M^{-1}$ for every $* \in \Star(T)$. Thus for every nonzero fractional ideal $I$ of $T$ with $IM^{-1} = xM^{-1}$, $I^* \subseteq (IM^{-1})^* = (xM^{-1})^* = xM^{-1} = IM^{-1}$. Now let $*_1 \neq *_2$ be star operations (resp. star operations of finite type) on $T$ and let $A$ be a fractional ideal (resp. a finitely generated fractional ideal) of $T$ such that $A^*_1 \neq A^*_2$. Without loss of generality, we may assume that $A \subseteq M$ (for if $0 \neq d \in T$ is such that $dA \subseteq T$ and $0 \neq m \in M$, $mdA \subseteq mT \subseteq M$ and $(mdA)^*_1 = m(A^*_1) = (mdA)^*_2 = m(A^*_2)$). Notice that $A$ is not a divisorial ideal of $T$ and so $A$ is not an invertible ideal of $T$. Thus $A(R : A) \neq R$. Now set $AM^{-1} = aM^{-1}$. Then $AM = AMM^{-1} = aM$ and $M : A = (M : A) = (M : aM^{-1}) = (R : AM^{-1}) = (R : AM^{-1}) = (R : aM^{-1}) = a^{-1}M$. Thus $A(M : A) = a^{-1}AM = M$. Hence $M = A(M : A) \subseteq A(R : A) \subseteq R$ and by maximality of $M, M = A(M : A) = A(R : A)$. Thus $A(R : A) = (M : A) = a^{-1}M$ and so $A_0 = aM^{-1} = AM^{-1}$. Now, by Lemma 2.6 $A^{(*, v)} = A^*_1 \cap A_0 = A^*_1 \cap AM^{-1} = A^*_1$ and $A^{(*, 2)} = A^*_2 \cap A_0 = A^*_2 \cap AM^{-1} = A^*_2$. Thus $\delta(*_1, v) \neq \delta(*_2, v)$.

For star operations of finite type $*_1 \neq *_2$ and $A$ a finitely generated ideal of $T (A \subseteq M)$ with $A^*_1 \neq A^*_2$, let $B$ be a finitely generated ideal of $T$ such that $BT = A$ (for $A = \sum_{i=1}^{n} a_i T$, just take $B = \sum_{i=1}^{n} a_i T$). If since $B(R : B) \subseteq A(T : A) \subseteq T, B(R : B) \subseteq R$. Moreover, $BM^{-1} = AM^{-1} = aM^{-1}$. Thus $(M : B) = (M_0 : B) = ((R : BM^{-1}) : B) = (R : BM^{-1}) = (R : aM^{-1}) = a^{-1}M_0 = a^{-1}M$. Hence $B(M : B) = a^{-1}BM = M$ and so $M = B(M : B) \subseteq R$. Thus $M = B(M : M) = M(R : M)$ and so $(R : B) = (M : B) = a^{-1}M$. Hence $B_1 = B_0 = aM^{-1} = BM^{-1}$. Finally, $B^{(*, v)} = (BT)^*_1 \cap B_1 = A^*_1 \cap AM^{-1} = A^*_1$ and $B^{(*, 2)} = (BT)^*_2 \cap B_1 = A^*_2 \cap AM^{-1} = A^*_2$. It follows that $\delta(*_1, v) \neq \delta(*_2, v)$. In both cases, the map $\delta(\cdot, v) : \Star(T) \to \Star(R), * \mapsto \delta(\cdot, v)$ (resp. $\delta(\cdot, t) : \StarFC(T) \to \StarFC(R), * \mapsto \delta(\cdot, t)$) is one-to-one. Hence $|\Star(T)| \leq |\Star(R)|$ (resp. $|\StarFC(T)| \leq |\StarFC(R)|$), as desired.

Recall that an integral domain $R$ is said to be conducive if the conductor $(R : T) \neq 0$ for every overring $T$ of $R$ with $T \nsubseteq qf(R)$, equivalently $(R : V) \neq 0$ for some valuation overring $V$ of $R$. The next corollary generalizes [28, Proposition 1.10].

Corollary 2.9. ([28, Proposition 1.10]) Let $(R, M)$ be a local Noetherian domain such that $M^{-1}$ is a valuation domain. Then $R$ is star regular.

Proof. Since $M^{-1}$ is a valuation domain, $R$ is a conducive domain and clearly $M^{-1} = (M : M) = \overline{R}$. Thus $[R, L] = [R, \overline{R}] \cup \{L\}$. By Theorem 2.8, $R$ is star regular. □

It is well-known that if $R$ is a Noetherian domain (which is not a field) with $\Star(R) = \StarFC(R)$ finite, then $R$ has Krull dimension $\dim(R) = 1$ ([26, Theorem 2.1]). Therefore $R$ is of finite character, that is, each nonzero nonunit element is contained in only finitely many maximal ideals. In [26, Theorem 2.3], it was proved
Theorem 2.10. Let \( R \) be a one-dimensional domain of finite character such that \( \text{StarFC}(R) \) is finite. Then \( |\text{StarFC}(R)| = \prod_{M \in \text{Max}(R)} |\text{StarFC}(R_M)| \).

Proof. First set \( M_1 := \{ M \in \text{Max}(R) : |\text{StarFC}(R_M)| = 1 \} \) and \( M_{\geq 2} := \{ M \in \text{Max}(R) : |\text{StarFC}(R_M)| \geq 2 \} \). Clearly \( \text{Max}(R) \) is the disjoint union of \( M_1 \) and \( M_{\geq 2} \). Now if \( M_{\geq 2} = \emptyset \), then \( |\text{StarFC}(R_M)| = 1 \) and so \( d_M = t(M) \) for every \( M \in \text{Max}(R) \). Let \( \ast \in \text{StarFC}(R) \) and let \( I \) be a finitely generated ideal of \( R \). Then for every \( M \in \text{Max}(R) \), \( I_R = (I_R)^{t(M)} = (I_R)^{d(M)} = I_R \). Hence \( I = I \) and so \( t = d \). Thus \( \text{StarFC}(R) = |d| \) and therefore \( |\text{StarFC}(R)| = 1 = \prod_{M \in \text{Max}(R)} |\text{StarFC}(R_M)| \). Assume that \( M_{\geq 2} \neq \emptyset \).

Claim: \( M_{\geq 2} \) is finite. By way of contradiction suppose that \( M_{\geq 2} \) is infinite. Then for every positive integer \( n \) consider a subset \( \{ M_1, \ldots, M_n \} \subseteq M_{\geq 2} \), and consider the map \( \phi : \prod_{i=1}^{n} \text{StarFC}(R_{M_i}) \rightarrow \text{StarFC}(R) \), \( \ast = (\ast_i)_{i=1}^{n} \mapsto \phi(\ast) = \ast \phi \), where \( \ast \phi \) is defined by \( A^{\ast \phi} = \bigcap_{i=1}^{n} (AR_{M_i})^{\ast_i} \cap \bigcap_{M \in \text{Max}(R), M \neq M_j} (AR_M) \). By Th. 2, \( \ast \phi \in \text{StarFC}(R) \) and so \( \phi \) is well-defined. Now, let \( \ast \neq (\ast_i)_{i=1}^{n} \neq (\ast_i')_{i=1}^{n} = \ast' \). Then \( \ast_j \neq \ast_j' \) for some \( j \in \{1, \ldots, n\} \). Let \( E \) be a finitely generated integral ideal of \( R_{M_j} \) such that \( E^{\ast_i} \neq E^{\ast_i'} \) and let \( A = E \cap R \). Since \( \dim(R_{M_j}) = 1 \), \( E \) is \( M_{\ast_j} \)-primary and so \( A \) is \( M_{\ast_j} \)-primary. Hence \( \text{Max}(R, A) = \{ M_j \} \) and so \( AR_{M_j} = R_{M_j} \) for every \( M \in \text{Max}(R) \) \( \setminus \{ M_j \} \). So \( A^{\ast \phi} = (AR_{M_j})^{\ast_j} \cap \bigcap_{M \in \text{Max}(R), M \neq M_j} R_M = E^{\ast_j} \cap \bigcap_{M \in \text{Max}(R), M \neq M_j} R_M \). Thus \( A^{\ast \phi} \cap R_{M_j} = E^{\ast_j} \cap R_{M_j} \). Similarly \( A^{\ast \phi'} = E^{\ast'} \cap R_{M_j} \). So if \( A^{\ast \phi} = A^{\ast \phi'} \), then \( E^{\ast_j} \cap R_{M_j} = A^{\ast \phi} \cap R_{M_j} = A^{\ast \phi'} \cap R_{M_j} = E^{\ast'} \cap R_{M_j} \). Hence \( E^{\ast_j} = (E^{\ast_j} \cap R_{M_j}) = (E^{\ast'} \cap R_{M_j}) \) \( \geq |\text{StarFC}(R_{M_j})| \), which is absurd. Hence \( A^{\ast \phi} \neq A^{\ast \phi'} \) and so \( \phi(\ast) \neq \phi(\ast') \). Thus \( \phi \) is a one-to-one and therefore \( \prod_{i=1}^{n} |\text{StarFC}(R_{M_i})| \leq |\text{StarFC}(R)| \), for every positive integer \( n \). So \( |\text{StarFC}(R)| = \infty \), which is a contradiction, completing the proof of the claim.

Now assume that \( M_{\geq 2} = \{ M_1, \ldots, M_r \} \). Let \( \varphi : \text{StarFC}(R) \rightarrow \prod_{M \in \text{Max}(R)} \text{StarFC}(R_M) \) defined by \( \varphi(\ast) = (\varphi(M))_{M \in \text{Max}(R)} \). Clearly \( \varphi \) is one-to-one and so

\[
|\text{StarFC}(R)| \leq \prod_{M \in \text{Max}(R)} |\text{StarFC}(R_M)| = \prod_{i=1}^{r} |\text{StarFC}(R_{M_i})| \prod_{M \not\in M_1} |\text{StarFC}(R_M)| = \prod_{i=1}^{r} |\text{StarFC}(R_{M_i})| \leq |\text{StarFC}(R)| \quad \text{(since } \prod_{M \not\in M_1} |\text{StarFC}(R_M)| = 1 \text{ and } \prod_{i=1}^{r} |\text{StarFC}(R_{M_i})| \leq |\text{StarFC}(R)| \text{ by the proof of the claim)}. \]

It follows that \( |\text{StarFC}(R)| = \prod_{M \in \text{Max}(R)} |\text{StarFC}(R_M)| \).

\( \square \)
In [28, Theorem 1.5] it was proved that for a Noetherian domain $R$ with $\text{Star}(R)$ finite, $R$ is star regular if and only if $R_M$ is star regular for every maximal ideal $M$ of $R$. Our next theorem generalizes this result to one-dimensional quasi-Prüfer domain such that $\text{StarFC}(R)$ is finite. Notice that, for a Noetherian domain with $\text{Star}(R)$ finite, $\dim(R) = 1$ and so $R'$ is a Dedekind domain and so $R$ is a quasi-Prüfer domain.

**Theorem 2.11.** Let $R$ be a one-dimensional quasi-Prüfer domain such that $\text{StarFC}(R)$ is finite. Then $R$ is finitely star regular if and only if $R_M$ is finitely star regular for every maximal ideal $M$ of $R$.

**Proof.** Notice that each overring $T \subsetneq qf(R)$ of $R$ is a one-dimensional quasi-Prüfer domain.

$\Longleftrightarrow$) Assume that $R_M$ is finitely star regular for every $M \in \text{Max}(R)$. Let $T$ be a proper overring of $R$. By Theorem 2.10, $|\text{StarFC}(T)| = \prod_{N \in \text{Max}(R)} |\text{StarFC}(R_N)| \leq |\text{StarFC}(T)| = |\text{StarFC}(R)|$. Thus $R$ is finitely star regular.

$\Rightarrow$) We mimic the proof of [28, Theorem 1.5]. Assume that $R$ is finitely star regular and let $M$ be a maximal ideal of $R$ and suppose that there is an overring $T$ of $R_M$ such that $|\text{StarFC}(T)| > |\text{StarFC}(R_M)|$. Let $B := \bigcap_{Q \in \text{Max}(R) \setminus \{M\}} Q$ and $S = T \cap B$. We first note that for every maximal ideal $N$ of $T$, $N \cap R_M = MR_M$ since $R_M$ is one-dimensional local domain. So $N \cap R = M$. But since $T$ is one-dimensional quasi-Prüfer domain, $T$ is $h$-local and so $T$ has only finitely many maximal ideals, say $N_1, \ldots, N_r$. Since $R$ is a one-dimensional quasi-Prüfer domain, it is $h$-local, and hence $B_{R\setminus M} = K$ by [39, Theorem 22]. This yields $S_{R\setminus M} = T_{R\setminus M} \cap B_{R\setminus M} = T \cap K = T$. It follows that $N_i \cap S \neq N_j \cap S$ for $i \neq j$. It is clear that if $Q$ is a maximal ideal of $S$ different from the $N_i \cap S$, then $Q \cap R = P$ for some maximal ideal $P$ of $R$ distinct from $M$ and hence $P = PR \cap S$ and $S_Q = R_P$. We then have $\text{Max}(S) = \{N_i \cap S \mid i = 1, \ldots, n\} \cup \{PR \cap S \mid P \neq M\}$. Therefore,

$$
|\text{StarFC}(S)| = \prod_{i=1}^{n} |\text{StarFC}(S_{N_i \cap S})| \cdot \prod_{M \neq N} |\text{StarFC}(R_M)|
$$

$$
= \prod_{i=1}^{n} |\text{StarFC}(T_{N_i \cap S})| \cdot \prod_{M \neq N} |\text{StarFC}(R_M)|
$$

$$
= |\text{StarFC}(T)| \cdot \prod_{M \neq N} |\text{StarFC}(R_M)|
$$

$$
> |\text{StarFC}(R_N)| \cdot \prod_{M \neq N} |\text{StarFC}(R_M)|
$$

$$
= |\text{StarFC}(R)|.
$$

Therefore, $R$ is not star regular. \qed

**Corollary 2.12.** Let $R$ be a domain with $\text{StarFC}(R)$ is finite and which satisfies one of the following conditions:

1. Each proper overring of $R$ is Archimedean;
(2) Each proper valuation overring of $R$ satisfies the accp; 
(3) Each proper overring of $R$ is a Mori domain.

Then $R$ is finitely star regular if and only if $R_M$ is finitely star regular for every maximal ideal $M$ of $R$.

Proof. By [6 Proposition 3.1(b), Lemma 3.2 and Corollary 3.9], either $R$ is a valuation domain or $\dim(R') = 1$ and $R'$ is Prüfer. If $R = V$ is a valuation domain, then $R$ is finitely star regular. Assume that $\dim(R') = 1$ and $R'$ is Prüfer. Then $R$ has the finite character and for each overring $T$ of $R$, either $T$ is a valuation or $\dim(T) = 1$ and $T'$ is Prüfer. If $T$ is valuation, $|\text{StarFC}(T)| = 1 \leq |\text{StarFC}(R)|$. Assume that $\dim(T) = 1$ and $T'$ is Prüfer. By Theorem 2.10, $|\text{StarFC}(T)| = \prod_{N \in \text{Max}(T)} |\text{StarFC}(T_N)| \leq |\text{StarFC}(R)|$ as desired. 

Corollary 2.13. ([28 Theorem 1.5]) Let $R$ be a Noetherian domain with Star$(R)$ finite. Then $R$ is star regular if and only if $R_M$ is star regular for every maximal ideal $M$ of $R$.

Proof. Notice that Star$(R) = \text{StarFC}(R)$ and $\dim R = 1$. Since $R' = \overline{R}$ is a Krull domain, $R'$ is a Dedekind domain. The conclusion follows now from Theorem 2.11.

3. Pullback Constructions

Let $T$ be a domain, $M$ a maximal ideal of $T$, $K$ its residue field, $\phi : T \to K$ the canonical surjection, $D$ a proper subring of $K$, and $k := qf(D)$. Let $R$ be the pullback issued from the following diagram of canonical homomorphisms:

$$
\begin{array}{ccc}
R := \phi^{-1}(D) & \to & D \\
\downarrow & & \downarrow \\
T & \overset{\phi}{\to} & K = T/M.
\end{array}
$$

Clearly, $M = (R : T)$ and $D \cong R/M$. For ample details on the ideal structure of $R$ and its ring-theoretic properties, we refer the reader to [23 9 10 13 14 19]. The case where $T = V$ is a valuation domain is crucial and we will refer to this case as a classical diagram of type $(\square)$. Notice that for the classical diagram, if $I$ is a (fractional) ideal of $R$, then either $I = \phi^{-1}(E)$ for some $E \in F(D)$ if $M \subseteq I$, or $I$ is an ideal of $V$ or $I = a\phi^{-1}(E)$ for some $0 \neq a \in M$ and $E$ a $D$-submodule of $K$ with $D \subseteq E \subseteq K$ if $I$ is not an ideal of $V$ (the proof similar to that of [9 Theorem 2.1]). For more on star operations on pullbacks, see [15 16].

Theorem 3.1. For the classical diagram of type$(\square)$, assume that $qf(D) = K$. Then $R$ is finitely star regular if and only if $D$ is finitely star regular.

Proof. Assume that $D$ is finitely star regular and let $T$ be an overring of $R$. If $V \subseteq T$, then $T$ is a valuation domain and so $|\text{StarFC}(T)| = 1 \leq |\text{StarFC}(R)|$. Assume that $R \subsetneq T \subseteq V$. Then $T = \phi^{-1}(D_1)$ where $D_1$ is an overring of $D$. Now by [24 Theorem 4.4], $|\text{StarFC}(T)| = |\text{StarFC}(D_1)| \leq |\text{StarFC}(D)| = |\text{StarFC}(R)|$, and therefore $R$ is finitely star regular.

Conversely, assume that $R$ is finitely star regular and let $D_1$ be an overring of $D$. Set $T = \phi^{-1}(D_1)$. Then $T$ is an overring of $R$ and again by [24 Theorem 4.4], $|\text{StarFC}(D_1)| = |\text{StarFC}(T)| \leq |\text{StarFC}(R)|$, and therefore $D$ is finitely star regular. 

□
Example 3.2. Let $k$ be a field, $X$ an indeterminate over $k$ and set $D = k[[X^3, X^4, X^5]]$, $V = k((X))[Y] = k((X)) + M$ and $R = D + M$. Clearly $R$ is neither integrally closed nor Noetherian domain. By \cite[Theorem 3.8]{[26]}, $|\text{StarFC}(D)| = |\text{Star}(D)| = 3$. Since the only overrings of $D$ are $D_1 = k[[X^2, X^3]]$, $D_2 = k[[X]]$ and $af(D) = k((X))$ and since $D_1$ and $D_2$ are Noetherian divisorial domains, $D$ is finitely star regular. Hence $R$ is finitely star regular by Theorem \cite[Theorem 3.8]{[26]}. In fact the only proper overrings of $R$ are $T_1 = D_1 + M$, $T_2 = D_2 + M$ and $V$. By \cite[Theorem 3.4]{[24]}, $|\text{StarFC}(T_1)| = |\text{StarFC}(T_2)| = |\text{StarFC}(V)| = 1$ while $|\text{StarFC}(R)| = 3$.

Our next theorem deals with an important class of finitely star regular domains that are not in general star regular. It shows that any PVD is a finitely star regular domain. Recall from Hedstrom and Houston \cite{[21]} that a domain $R$ is pseudo-valuation domain if it is quasilocal and shares its maximal ideal with a valuation domain which necessarily must contain $R$ and be unique. In terms of pullbacks, according to \cite[Proposition 2.6]{[3]}, $R$ is a pseudo-valuation domain if and only if there is a valuation domain $V$ with maximal ideal $M$ and a subfield $k$ of $V/M = K$ such that $R$ is the pullback in the following diagram

$$
\begin{array}{c}
R \longrightarrow k \\
\downarrow \hspace{1cm} \downarrow \\
V \overset{\phi}{\longrightarrow} K = V/M
\end{array}
$$

Notice that a PVD which is not a valuation domain is a $TV$-domain, that is, the $t$- and $v$-operations are the same \cite[Proposition 4.3]{[31]}). We start with the following useful lemma.

Lemma 3.3. Let $R$ be a PVD (which is not a valuation domain), $V$ its associated valuation overring, $M$ its maximal ideal, $k = R/M$ and $K = V/M$. If $\text{StarFC}(R)$ is finite, then $K$ is algebraic over $k$.

Proof. Assume that $\text{StarFC}(R)$ is finite and suppose that $K$ is transcendental over $k$. Let $\lambda \in K$ transcendental over $k$ and set $T = \phi^{-1}(k[\lambda])$. Since $(k : k[\lambda]) = (0)$, $T^{-1} = (R : T) = \phi^{-1}(k : k[\lambda]) = \phi^{-1}(0) = M$, and so $T = T_v = V$. Now, for every nonzero prime ideals $p \neq q$ of $k[\lambda]$, set $T_p = \phi^{-1}(k[\lambda]_p)$ and $T_q = \phi^{-1}(k[\lambda]_q)$. By Lemma \cite[2.6]{[24]}, $\delta(d_{T_p, t})$ and $\delta(d_{T_q, t})$ are star operations on $R$ of finite type and $T^{\delta(d_{T_p, t})} = T_T \cap T = T_p \cap V = T_p$ and $T^{\delta(d_{T_q, t})} = T_T \cap T = T_q \cap V = T_q$. Thus $\delta(d_{T_p, t}) \neq \delta(d_{T_q, t})$. As $\text{Spec}(k[\lambda])$ is infinite, $\text{starFC}(R)$ would be infinite, which is absurd. It follows that $K$ is algebraic over $k$. \hfill $\square$

Theorem 3.4. Any PVD is finitely star regular.

Proof. First, if $R$ is a valuation domain, then for every overring $T$ of $R$, $|\text{StarFC}(T)| = |\text{StarFC}(R)| = 1$. So, without loss of generality, we may assume that $R$ is not a valuation domain and $\text{StarFC}(R)$ is finite. Let $V$ be the associated valuation overring, $M$ its maximal ideal, $k = R/M$ and $K = V/M$. By Lemma \cite[3.3]{[3]}, $K$ is algebraic over $k$. Now, let $T$ be a proper overring of $R$. If $V \subset T$, then $T$ is a valuation domain and so $|\text{StarFC}(T)| = 1 \leq |\text{StarFC}(R)|$, as desired. Assume that $R \subset T \subset V$. Then $T = \phi^{-1}(F)$ where $F$ is a subfield of $K$ with $k \not\subseteq F \not\subseteq K$. We claim that $\delta(-, t)$: $\text{StarFC}(T) \rightarrow \text{StarFC}(R), * \mapsto \delta(*, t)$ is a one-to-one map. Indeed, let $* \neq *' \in \text{StarFC}(T)$ and let $A$ be a finitely generated integral ideal of $T$ such that $A* \neq A*$. Necessarily $A \subseteq M$ and $A$ is not an ideal of $V$. Then $A = a\phi^{-1}(W)$ where $F \subset W \not\subseteq K$ is a finite
dimensional $k$-subspace of $K$. Set $W = \sum_{i=1}^{r} F \lambda_i$ and let $H = \sum_{i=1}^{r} k \lambda_i$ and $B = a \phi^{-1}(H)$.

Then $B$ is a finitely generated ideal of $R$, $BT = A$ and $B_1 = B_2 = A_{V_T} = aV$. Thus $B^{\delta(\cdot, \cdot)} = (BT)^* \cap B_1 = A^* \cap A_{V_T} = A^*$. Similarly, $B^{\delta(\cdot, \cdot)} = (BT)^* \cap B_1 = A^* \cap A_{V_T} = A^*$. Hence $B^{\delta(\cdot, \cdot)} \neq B^{\delta(\cdot, \cdot)}$ and therefore $\delta(\cdot, \cdot)$ is one-to-one. It follows that $|\text{StarFC}(T)| \leq |\text{StarFC}(R)|$ and therefore $R$ is finitely star regular.

Notice that a PVD is not necessarily a star regular domain as shown by the following example.

**Example 3.5.** Let $V$ be a valuation domain with a principal maximal ideal $M$ and a non-maximal prime ideal $P$ such that $P = P^2$. Suppose that $K = V/M$ is a quadratic extension of a field $k$ and let $R$ be the PVD arising from the diagram:

$$
R := \phi^{-1}(k) \longrightarrow k
$$

$$
\downarrow \quad \downarrow
$$

$$
V \quad \phi \quad K = V/M.
$$

Since $[K : k] = 2$, $R$ is a divisorial domain and so $|\text{Star}(R)| = 1$. However, $V_P$ is an overring of $R$ and $|\text{Star}(V_P)| = 2$. Hence $R$ is not star regular.

**Example 3.6.** Let $k = \mathbb{Z}_2$ and $K$ an extension of $k$ with $[K : k] = 4$ (for instance, let $x$ be a root of the irreducible polynomial $f(Y) = Y^4 + Y^3 + 1 \in k[Y]$ and $K = k(x)$). Let $X$ be an indeterminate over $k$ and set $V = K[[X]] = K + M$ and $R = k + M$. Let $T$ be a proper overring of $R$. If $V \subseteq T$, $T$ is a valuation domain and so $|\text{StarFC}(T)| = 1 \leq |\text{StarFC}(R)|$. If $R \ngeq T \nsubseteq V$, then $T = \phi^{-1}(F)$ where $k \nsubseteq F \nsubseteq K$ is a subfield of $K$. Necessarily $[K : F] = 2$ and so $T$ is a divisorial PVD. Hence $|\text{StarFC}(T)| = |\text{Star}(T)| = 1 \leq |\text{Star}(R)| = |\text{StarFC}(R)| = 9$ by [44].

Recall that for the general pullback of type $(\square)$, every star operation $*$ on $R$ induces a star operation $*_{\phi}$ on $D$ defined by $f^* = \phi((f^*)^*)$ for every $f \in D$ (\cite[Proposition 2.7 and Proposition 2.6]{15}). In this context, it is easy to check that if $*$ is of finite type on $R$, then $*_{\phi}$ on $D$ is of finite type on $D$.

**Theorem 3.7.** For the classical pullback diagram of type $(\diamondsuit)$, let $T = \phi^{-1}(D_1)$ be an overring of $R$. Then $|\text{StarFC}(T)| \leq |\text{StarFC}(D_1)||\text{StarFC}(R)|$. In particular, if $|\text{StarFC}(D_1)| = 1$ for every $D_1 \in [D, K]$, then $R$ is finitely star regular.

**Proof.** Set $T = \phi^{-1}(D_1)$ and consider the map $\delta : \text{StarFC}(T) \longrightarrow \text{StarFC}(D_1) \times \text{StarFC}(R)$, $* \mapsto (\delta_1, \delta_2, \delta_3, \delta_4)$. We claim that $\delta$ is one-to-one. Indeed, let $* \neq *' \in \text{StarFC}(T)$ and let $A$ be a finitely generated integral ideal of $T$ such that $A* \neq A*'$. Necessarily $A$ is not an ideal of $V$. If $M \nsubseteq A$, then $A = \phi^{-1}(J)$ for some finitely ideal $J$ of $D_1$. In this case $A* = \phi^{-1}(J^*)$ and $A*' = \phi^{-1}(J'^*)$. Thus $J^* \neq J'^*$ and so $* \neq *'$. Assume that $A \subseteq M$ and set $A = a\phi^{-1}(W)$ where $D_1 \subseteq W \subseteq K$ is a finitely generated $D_1$-module. If $(D_1 : W) \neq 0$, then $W \subseteq qf(D_1)$ and so $W$ would be a finitely generated fractional ideal of $D_1$. Thus $A* = a\phi^{-1}(W^*)$ and $A*' = a\phi^{-1}(W'^*)$. Thus $W^* \neq W'^*$ and so $* \neq *'$.

Assume that $(D_1 : W) = 0$. Set $W = \sum_{i=1}^{r} D_1 \lambda_i$ and let $H = \sum_{i=1}^{r} D_1 \lambda_i$ and $B = a\phi^{-1}(H)$.
Then $B$ is a finitely generated ideal of $R$, $BT = A$ and $B_1 = B_2 = A_{T^2} = aV$. Thus $B^{δ(c,t)} = (BT)^* \cap B_1 = A^* \cap A_{T^2} = A^*$. Similarly, $B^{δ(c,t)} = (BT)^* \cap B_2 = A^* \cap A_{T^2} = A^*$. Hence $B^{δ(c,t)} \neq B^{δ(c,t)}$. Thus $δ(\star) \neq δ(\star)$ and hence $δ$ is one-to-one. It follows that $|\text{StarFC}(T)| \leq |\text{StarFC}(D_1)||\text{StarFC}(R)|$.

Now assume that $|\text{StarFC}(D_1)| = 1$ for every $D_1 \in [D,K]$ and let $T$ be an overring of $R$. If $V \subseteq T$, then $|\text{StarFC}(T)| = 1 \leq |\text{StarFC}(R)|$. Let $T$ be a proper overring of $R$. Assume that $R \subseteq T \subseteq V$ and $T = \phi^{-1}(D_1)$ where $D_1 \subseteq D \subseteq K$. Then $|\text{StarFC}(T)| \leq |\text{StarFC}(D_1)||\text{StarFC}(R)| = |\text{StarFC}(R)|$ and therefore $R$ is finitely star regular.

\[\square\]

**Example 3.8.** Let $Q$ be the field of rational numbers, and $X$ and $Y$ indeterminates over $Q$. Set $D = Q[[X^2, X^3]]$, $V = Q(\sqrt{2}, (X))[[Y]] = Q(\sqrt{2}, (X)) + M$ and $R = D + M$. Clearly $[D,K] = [D,Q[[X]], Q((X)), Q(\sqrt{2})[[X^2, X^3]], Q(\sqrt{2})[[X]], K]$ and every $D_1 \in [D,K]$ is divisorial. Thus $R$ is finitely star regular.

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