Investigating the Numerical Range
of Non Square Matrices

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Abstract. A presentation of numerical range for rectangular matrices is undertaken in this paper, introducing two different definitions and elaborating basic properties. Then we are extended to the treatment of rank-k numerical range.

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1 Introduction

Let $\mathcal{M}_{m,n}(\mathbb{C})$ be the set of matrices $A = [a_{ij}]_{i,j=1}^{m,n}$ with entries $a_{ij} \in \mathbb{C}$. For $m = n$, the set

$$F(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\|_2 = 1\}$$

is the well known numerical range or field of values of $A$, for which basic properties can be found in [11], [8] and [9, chapter 22]. Equivalently, we say that $F(A) = f(S_n)$, where $S_n$ is the unit sphere of $\mathbb{C}^n$ and the function $f$ on $S_n$ is defined by the bilinear mapping $g : S_n \times S_n \to \mathbb{C}$, such that $f(x) = g(x, x) = x^*Ax$. It is remarkable that $F(A)$ is closed and convex set and contains the set of eigenvalues of $A$.

For $m \neq n$, the motivation herein is to investigate "how the numerical range $w(A)$ can be defined for a rectangular matrix $A$" based on the inner product and to develop some basic and fundamental properties. As we may see, the results vary and the approach is undertaken in two ways, firstly we consider a natural extension of (1.1) and on the other hand, introducing the idea of restriction or extension of dimensions of $A$, we are led to the relationship of $w(A)$ with the numerical range of square matrices via projection matrices. Hence, generalizing the notion of definition (1.1), we consider the

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bilinear mapping $g : \mathcal{S}_n \times \mathcal{S}_m \to \mathbb{C}$, $g(x, y) = y^*Ax$, which gives rise to the numerical range of $m \times n$ matrix $A$, as the set

$$w(A) = \{ y^*Ax : x \in \mathbb{C}^n, y \in \mathbb{C}^m, \|x\|_2 = \|y\|_2 = 1 \}$$

which is equal to $g(S_n \times S_m)$. Note that for $m > n$ we have

$$F([A \ 0_{m \times (m-n)}]) = \left\{ y^*Ax : y = \begin{bmatrix} x \\ \omega \end{bmatrix} \in \mathbb{C}^m, x \in \mathbb{C}^n, \|y\|_2 = 1 \right\}$$

$$\subseteq w(A).$$

Proceeding, it is proved that $w(A)$ is identified with the circular disc $\{ z \in \mathbb{C} : |z| \leq \|A\|_2 \}$, since the unit vectors $x$ and $y$ belong to different dimensional spaces. An approximation of $w(A)$ from within, following, is shown, assuming that the vectors $x, y$ in (1.2) belong to subspaces $\mathcal{F} \subset \mathbb{C}^n$ and $\mathcal{G} \subset \mathbb{C}^m$, respectively. Recently, has been proposed [7] as numerical range of $A \in \mathcal{M}_{m,n}$ with respect to matrix $B \in \mathcal{M}_{m,n}$ the compact and convex set

$$w_{\|\cdot\|}(A, B) = \bigcap_{z_0 \in \mathbb{C}} \{ z \in \mathbb{C} : |z - z_0| \leq \|A - z_0B\| \}.\tag{1.3}$$

The (1.3) is an extension of definition of $F(A)$ for square matrices in [1] and clearly the numerical range, as in [1], [2], is based on the notion of matrix norm. In [7] has been proved that $w_{\|\cdot\|}(A, B)$ coincides with the disc

$$\left\{ z \in \mathbb{C} : |z - \frac{\langle A, B \rangle}{\|B\|^2}| \leq \|A - \frac{\langle A, B \rangle}{\|B\|^2}B\| \sqrt{1 - \|B\|^2} \right\}\tag{1.4}$$

when $\|B\| \geq 1$ and the matrix norm $\|\cdot\|$ is induced by the inner product $\langle \cdot, \cdot \rangle$. The complicated formulation of numerical range of $A$ and the necessity of independence of $w_{\|\cdot\|}(A, B)$ by the matrix $B$ in (1.3) and (1.4), are signified in section 2.

Another proposal for the definition of numerical range for rectangular matrices, which will be further exploited in section 3, is the projection onto the lower or the higher dimensional subspace. Let $m > n$ and the vectors $v_1, \ldots, v_n$ of $\mathbb{C}^m$ be orthonormal basis of $\mathbb{C}^n$. Clearly, the matrix $P = HH^*$, where $H = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$, is an orthogonal projector of $\mathbb{C}^m \to \mathbb{C}^n$. In this case, for $A \in \mathcal{M}_{m,n}$, we define with respect to $H$:

$$w_l(A) = F(H^*A)\tag{1.5}$$

where obviously $H^*A$ is $n \times n$ matrix. Moreover, the vector $y = Hx \in \mathbb{C}^m$ is projected onto $\mathbb{C}^n$ along $\mathcal{K}$, where $\mathcal{K}$ is any direct complement of $\mathbb{C}^n$, i.e.
\[ C^m = C^n \oplus K. \] Since, \( \|y\| = (x^*H^*Hx)^{1/2} = \|x\| \), instead of (1.5), it can also be provided a treatment of the numerical range \( w_h(A) \) of higher dimensional \( m \times m \) matrix \( AH^* \), namely,

\[
(1.6) \quad w_h(A) = F(AH^*).
\]

Similarly, if \( m < n \), then \( x = Hy \) and consequently

\[
(1.7) \quad w_l(A) = F(AH), \quad w_h(A) = F(HA).
\]

Apparently, by (1.5)-(1.7), the numerical range of \( A \in M_{m,n} \) via the projection of unit vectors onto \( C^n \) or \( C^m \) is referred to the numerical range of square matrix, indicating obviously the convexity of \( w_l(A) \) and \( w_h(A) \).

Accepted the definition (1.2), an equivalent representation of \( w(A) \) in (1.2) is

\[
(1.8) \quad w(A) = \{z \in \mathbb{C} : PAQ = zS, \text{ where } P = yy^*, Q = xx^*, S = yy^* \text{ and } x \in C^n, y \in C^m, \|x\|_2 = \|y\|_2 = 1\}
\]

In (1.8) the matrices \( P, Q \) are rank-1 orthogonal projections of \( C^m \) and \( C^n \) and \( S \) satisfies the equation \( PXQ = X \). In this way, in section 4 we are led to the generalization of rank-\( k \) numerical range for square matrices

\[
(1.9) \quad \Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank } - k \text{ orthogonal projection } P\}
\]

which has been presented and extensively studied by Choi et al in [3], [4], [5], [6] and later by other researchers in [10], [14], [13] and [12]. In this paper, for the \( m \times n \) matrix \( A \) and a positive integer \( k \geq 1 \), the rank-\( k \) numerical range of \( A \) is defined by the set

\[
(1.10) \quad \phi_k(A) = \{z \in \mathbb{C} : PAQ = zS, \text{ for some rank } - k \text{ orthogonal projections } P \text{ and } Q \text{ and } S = PSQ\}
\]

For \( m = n \) and \( P = Q \), \( \phi_k(A) = \Lambda_k(A) \).

These sets satisfy, analogously to \( \Lambda_k(A) \), the inclusion relationship

\[ w(A) = \phi_1(A) \supseteq \phi_2(A) \supseteq \ldots \supseteq \phi_\tau(A) \]

where \( \tau = \min \{m, n\} \) and the proof is established in the final section 4. Then it is proved that \( \phi_k(A) \), under constraints for the index \( k \), is a circular ring or a disc, presenting the non-emptiness and the convexity of the set for special cases.
2 Properties of $w(A)$

Recalling the definition of $w(A)$ in [1,2], we readily recognize the property

$$w(kA) = kw(A).$$

The convexity of $w(A)$ is confirmed indirectly by the next statement.

**Proposition 1.** For each $m \times n$ matrix $A$, $w(A) = \{z \in \mathbb{C} : |z| \leq \|A\|_2 \}$. 

**Proof.** Let $m > n$. Since the rows $\tilde{a}_1, \ldots, \tilde{a}_n$ of $A$ are linear dependent, we consider the unit vector $y_0$ such that $y_0^* A = 0$. Then, for a unit vector $x$, we have $y_0^* A x = 0$, i.e., $0 \in w(A)$. Also, due to Cauchy-Schwarz inequality, we obtain

$$|y^* A x| = |\langle Ax, y \rangle| \leq \|Ax\|_2 \|y\|_2 = \|Ax\|_2 \leq \max_{\|x\|_2=1} \|Ax\|_2 = \|A\|_2 = \sigma_{\text{max}}(A).$$

If $z = re^{i\theta} \in \{z : |z| \leq \|A\|_2 \}$, obviously $0 < r \leq \|A\|_2$. Evenly, there exists a unit vector $\hat{x}$ such that $\|A \hat{x}\|_2 = r$, since the function $f(x) = \|A x\|_2 : \mathcal{S}_n \rightarrow (0, \|A\|_2]$ is continuous, where $\mathcal{S}_n$ is the compact unit sphere of $\mathbb{C}^n$. Thus, for $\hat{y} = A \hat{x}/(\|A \hat{x}\|_2 e^{i\theta})$, clearly $\|\hat{y}\|_2 = 1$ and

$$\hat{y}^* A \hat{x} = \frac{\hat{x}^* A^* A \hat{x}}{\|A \hat{x}\|_2} e^{-i\theta} = \|A \hat{x}\|_2 e^{i\theta} = re^{i\theta} = z. $$

Moreover, the boundary $\partial w(A) = \{z : |z| = \|A\|_2 \}$ is attained, since, by the unit eigenvectors $A^* A x = \sigma_{\text{max}}^2 x$ and $A A^* y = \sigma_{\text{max}}^2 y$, we receive the point $|y^* A x| = |y^*(\sigma_{\text{max}} x)| = \sigma_{\text{max}}$. Due to the fact that the singular values of $A$ and $e^{i\theta} A$ are identical, the points $y^*(e^{i\theta} A) x$ are also boundary points of the circular disc $\{z : |z| \leq \|A\|_2 \}$.

We remark that the proof is not specially simplified if we consider the singular value decomposition of $A$ and the invariant under unitary equivalence.

**Corollary 2.** Let $A \in M_{m,n}$ and $z = y^* A x \in w(A)$. Then we have

I. $z \in F\left(0_{n \times m} 2A \right)$ and corresponds to unit vector $\omega = \frac{1}{\sqrt{2}} \begin{bmatrix} y \\ x \end{bmatrix}$,

II. $w(A) = \bigcap_{0 \leq \theta \leq 2\pi} \{\text{half plane} : e^{-i\theta} \{z : \text{Re} \ z \leq \sigma_{\text{max}}(A) \}\}$,

III. if $A = a \in \mathbb{C}^n$, $w(a) = \mathcal{D}(0, \|a\|_2)$.

**Proof.** I. By Proposition 1, we have $\text{Re} \ z \in [-\sigma_{\text{max}}(A), \sigma_{\text{max}}(A)]$ and after some algebraic manipulations, we obtain $\text{Re} \ z = \omega^* \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \omega$, where $\omega =$

\[\frac{1}{\sqrt{2}} \begin{bmatrix} y \\ x \end{bmatrix}\]
\[
\frac{1}{\sqrt{2}} \begin{bmatrix} y \\ x \end{bmatrix}. \text{ Similarly, } \text{Im } z = \omega^* \begin{bmatrix} 0 & -iA \\ iA^* & 0 \end{bmatrix} \omega, \text{ i.e. } \text{Im } z \in [-i\sigma_{\max}(A), i\sigma_{\max}(A)]
\]
and consequently \( z = \omega^* \begin{bmatrix} 0 & 2A \\ 0 & 0 \end{bmatrix} \omega. \)

II. The graph of \( \partial w(A) \) is constructed only by the values \( e^{i\theta} \sigma_{\max}(A) \).

III. For \( A = a \in \mathbb{C}^n \), the unique singular value of \( a \) is \( \sigma = \|a\|_2 \). \( \square \)

**Corollary 3.** Let \( A, B \in \mathcal{M}_{m,n} \), then holds:

I. \( w(A) = w(A^*) \),

II. \( w(\hat{A}) \subseteq w(A) \), for any \( p \times q \) submatrix \( \hat{A} \) of \( A \),

III. \( w(diag(A, B)) = \max\{w(A), w(B)\} \), where \( A \in \mathcal{M}_{m,n} \), \( B \in \mathcal{M}_{n,m} \),

IV. \( w(A + B) \subseteq w(A) + w(B) \),

V. \( w(U^*AV) = w(A) \), where \( U \in \mathcal{M}_m \), \( V \in \mathcal{M}_n \) are unitary matrices.

**Proof.** Statement (I) is an immediate consequence of Proposition 1 and (II) is implied using the inequality \( \| \hat{A} \|_2 \leq \| A \|_2 \) (II, Cor. 3.1.3, p.149), where \( \hat{A} \) is \( p \times q \) submatrix of \( A \). Following, assertion (III) can be deduced from the condition \( \| diag(A, B) \|_2 = \max\{\| A \|_2, \| B \|_2\} \) and for (IV), (V) the triangle inequality and the unitarily invariant property of \( \| \cdot \|_2 \) are applied, respectively. \( \square \)

The computation of \( w(A) \) from inside is presented by the next proposition.

**Proposition 4.** Let \( l, k \) be positive integers less than \( m, n \), respectively. Then

\[
(2.1) \quad w(A) = D \left( 0, \max_{\xi_1, \ldots, \xi_l \in \mathbb{C}^m} \max_{\eta_1, \ldots, \eta_k \in \mathbb{C}^n} \left\| \left[ \xi_i^* A \eta_j \right]_{i,j=1}^{l,k} \right\|_2 \right)
\]

where \( \{\xi_1, \ldots, \xi_l\} \) and \( \{\eta_1, \ldots, \eta_k\} \) are orthonormal vectors of \( \mathbb{C}^m \) and \( \mathbb{C}^n \), respectively.

**Proof.** Any vectors \( x \in \mathbb{C}^n \) and \( y \in \mathbb{C}^m \) belong to subspaces \( \mathcal{F} \subseteq \mathbb{C}^n \) and \( \mathcal{G} \subseteq \mathbb{C}^m \). If \( \{\eta_1, \ldots, \eta_k\} \) and \( \{\xi_1, \ldots, \xi_l\} \) are orthonormal bases of \( \mathcal{F} \) and \( \mathcal{G} \), respectively, then

\[
x = \begin{bmatrix} \eta_1 & \ldots & \eta_k \end{bmatrix} u, \quad y = \begin{bmatrix} \xi_1 & \ldots & \xi_l \end{bmatrix} v
\]

where \( u \in \mathbb{C}^k \) and \( v \in \mathbb{C}^l \). Since \( \|x\| = \|y\| = 1 \), \( u \) and \( v \) are also unit vectors and we have

\[
y^* A x = v^* \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_l \end{bmatrix} A \begin{bmatrix} \eta_1 & \ldots & \eta_k \end{bmatrix} u = v^* \left[ \xi_i^* A \eta_j \right]_{i,j=1}^{l,k} u.
\]
Thus, we verify \( \left\| [\xi^* A \eta]^{l,k}_{i,j=1} \right\|_2 \leq \|A\|_2 \) and then the equation (2.1). \( \square \)

Note that in (2.1) for \( k = n \) and \([\eta_1 \ldots \eta_n] = I_n \) we have

\[
\nu(A) = D(0, \max_{\Xi \in M, l} \|\Xi^* A\|_2),
\]

where \( \Xi = [\xi_1 \ldots \xi_l] \).

For a pair of matrices \( A, B \in M_{m,n} \) the numerical \( \nu(A, B) \) as it has been presented in (1.3) and (1.4), imposes the question "how \( \nu(A, B) \) in (1.4) is independent of \( B \)?" An answer is given in the next proposition.

**Proposition 5.** Let \( A, B \in M_{m,n} \) such that \( \|B\|_F \geq 1 \). Then

I. \( \bigcup_{\|B\|_F \geq 1} \nu_{\|B\|_F}(A, B) = D(0, \|A\|_F). \)

II. If \( \text{rank} B = k \) and \( \|\sigma\|_F \geq \sqrt{k} \), where the vector \( \sigma = (\sigma_1, \ldots, \sigma_k) \) corresponds to the singular values of \( B \), then the centers of the discs in (1.4), \( (\nu_{\|B\|_F})_{\|B\|_F} \in D(0, \|A\|_2) \).

**Proof.**

I. Let \( z \in \bigcup_{\|B\|_F \geq 1} \nu_{\|B\|_F}(A, B) \), then there exists a matrix \( B_0 \in M_{m,n} \) with \( \|B_0\|_F \geq 1 \), such that \( |z - \nu_{\|B_0\|_F}(A, B_0)| \leq \|A - \nu_{\|B_0\|_F}(A, B_0)\|_F \sqrt{1 - \|B_0\|_{F}^{-2}} \).

Hence,

\[
|z| \leq \left| \frac{\langle A, B_0 \rangle}{\|B_0\|_F^2} \right| + \|A - \frac{\langle A, B_0 \rangle}{\|B_0\|_F^2} B_0\|_F \sqrt{1 - \|B_0\|_{F}^{-2}}
\]

and it suffices to show that the right part of (2.2) is less than \( \|A\|_F \). In fact, the relationship \( (\|A\|_F - \left| \langle A, B_0 \rangle \right|)^2 \geq 0 \) is equivalent to

\[
\left( \|A\|_F^2 - \frac{|\langle A, B_0 \rangle|^2}{\|B_0\|_F^2} \right) (1 - \|B_0\|_{F}^{-2}) \leq \left( \|A\|_F^2 - \left| \langle A, B_0 \rangle \right| \right)^2.
\]

Since, \( \|A\|_F^2 - \frac{|\langle A, B_0 \rangle|^2}{\|B_0\|_F^2} = \|A - \frac{\langle A, B_0 \rangle}{\|B_0\|_F^2} B_0\|_F \), we have

\[
\frac{|\langle A, B_0 \rangle|}{\|B_0\|_F^2} + \|A - \frac{\langle A, B_0 \rangle}{\|B_0\|_F^2} B_0\|_F \sqrt{1 - \|B_0\|_{F}^{-2}} \leq \|A\|_F.
\]

Moreover, if \( B_0 = Ae^{-i\theta}/\|A\|_F, \) \( \theta \in [0, 2\pi) \), then \( \|B_0\|_F = 1 \) and

\[
\frac{\langle A, B_0 \rangle}{\|B_0\|_F^2} = \|A\|_F e^{i\theta}, \quad \|A - \frac{\langle A, B_0 \rangle}{\|B_0\|_F} B_0\|_F = 0.
\]

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Thus, by (1.4) we have
\[ |z - \|A\|_F e^{i\theta}| \leq 0 \Rightarrow z = \|A\|_F e^{i\theta} \Rightarrow |z| = \|A\|_F \]
thereby, the boundary of \( D(0, \|A\|_F) \) is attained.

II. Denoting by \( \lambda(\cdot) \) and \( \sigma(\cdot) \) the eigenvalues and singular values of matrices and making use of known inequalities [11, p.176,177] it follows that
\[
\frac{|\langle A, B \rangle|}{\|B\|_F^2} = \frac{|\text{tr}(B^*A)|}{\|B\|_F^2} = \frac{\sum |\lambda(B^*A)|}{\|B\|_F^2} \leq \frac{\sum |\lambda(B^*A)|}{\|B\|_F^2} \leq \frac{\sum \sigma(B^*A)}{\|B\|_F^2}.
\]
(2.3)
\[
\leq \sum \frac{\sigma(B^*)\sigma(A)}{\|B\|_F^2} \leq \sigma_{\max}(A) \sum \frac{\sigma(B)}{\sum \sigma^2(B)}.
\]
Since \( \|\sigma\|_F \geq \sqrt{k} \), then \( \sum \sigma^2(B) = \|\sigma\|_F^2 \geq \sqrt{k} \|\|\sigma\|_F \geq \langle 1, \sigma \rangle = \sum \sigma(B) \) and consequently by (2.3),
\[
\frac{|\langle A, B \rangle|}{\|B\|_F^2} \leq \sigma_{\max}(A) = \|A\|_2.
\]
\[\square\]

The conclusions of proposition 5 strengthen the definition \( w(A) \) in (1.2) since the independence of \( w_{\|\|_F}(A, B) \) by the matrix \( B \) leads to a circular disc.

**Proposition 6.** Let \( A \in M_{m,n} \), then
\[
w(A) = \{ (A, B) : B \in M_{m,n}, \rank B = 1, \|B\|_F = 1 \}.
\]

**Proof.** Let \( z \in w(A) \), then there exist unit vectors \( x \in \mathbb{C}^n, y \in \mathbb{C}^m \) such that
\[
z = y^*Ax = \text{tr}(y^*Ax) = \text{tr}(Ayx^*) = \langle A, yx^* \rangle
\]
Denoting by \( B = yx^* \), obviously \( \rank B = 1 \) and
\[
\|B\|_F^2 = \text{tr}(B^*B) = \text{tr}(yx^*yx^*) = \text{tr}(xx^*) = \text{tr}(x^*x) = 1.
\]
Conversely, if \( \rank B = 1 \) then \( B = yx^* \) and evenly \( \langle A, yx^* \rangle = \text{tr}(yx^*A) = y^*Ax \). Since, \( 1 = \|B\|_F^2 = \text{tr}(yx^*yx^*) = \|x\|_2^2 \|y\|_2^2 \), to the case where \( x, y \) are not unit vectors, let \( \|y\| \geq 1 \), then \( \|x\| \leq 1 \) and we verify that the point \( y^*Ax = \frac{y^*}{\|y\|} A \frac{x}{\|x\|} \) belongs to \( w(A) \). \[\square\]

**Example.** If \( A = \begin{bmatrix} 6 + i & 0 & 1/2 \\ -4 & -3 - 6i & 0 \end{bmatrix} \), Propositions 5 and 6 are illustrated in the next figure, where the drawing discs \( w_{\|\|_F}(A, B) \) in (1.4), for six different matrices \( B \) with \( \|B\|_F \geq 1 \), approximate the disc \( D(0, \|A\|_F) \). The dashed circle is \( w(A) \) in (1.2).
3 Properties of \( w_l(A) \) and \( w_h(A) \)

In the introduction we have been referred to the numerical ranges \( w_l(A) \) and \( w_h(A) \) for rectangular matrices with respect to unitary \( m \times n \) matrix \( H \). Let \( A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \), (or \( A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \)), with \( A_1 \) to be square. Then for \( H = \begin{bmatrix} I \\ 0 \end{bmatrix} \), by (1.5)-(1.6) we have

\[
w_l(A) = F(A_1) \quad \text{and} \quad w_h(A) = F\left(\begin{bmatrix} A & 0_{m \times (m-n)} \end{bmatrix}\right), \quad \text{when} \quad m > n
\]

and by (1.7) we have

\[
w_l(A) = F(A_1) \quad \text{and} \quad w_h(A) = F\left(\begin{bmatrix} A & 0_{(n-m) \times n} \end{bmatrix}\right), \quad \text{when} \quad m < n.
\]

**Proposition 7.** Let the vector \( a = \begin{bmatrix} a_1 & a_2 & \ldots & a_m \end{bmatrix}^T \in \mathbb{C}^m \), then \( w_h(a) \) with respect to \( H = \begin{bmatrix} I_1 \\ 0 \end{bmatrix} \) is the elliptical disc with focal points 0 and \( a_1 \), the major axis has length \( \|a\|_2 \) and the minor axis has length \( \|b\|_2 \), where \( b = \begin{bmatrix} a_2 & \ldots & a_m \end{bmatrix}^T \).

**Proof.** By the definition \( w_h(a) = F\left(\begin{bmatrix} a & 0_{m \times (m-1)} \end{bmatrix}\right) \). If \( b \) is not collinear of \( \varepsilon_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T \in \mathbb{C}^{m-1} \), we consider the Householder matrix \( H = I_{m-1} - 2\frac{uu^*}{\|u\|^2} \), with \( u = b - \frac{\|b\|_2 a_2}{|a_2|} \varepsilon_1 \). Then

\[
\begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & H^* \end{bmatrix} = \text{diag} \left( \begin{bmatrix} \frac{a_1}{\|b\|_2 a_2} & 0 \\ 0 & 0 \end{bmatrix}, 0_{m-2} \right).
\]
Proposition 8. Let $m > n$ and $A \in M_{m,n}$. If $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, where $A_1$ is the principal $n \times n$ submatrix of $A$, then

I. $w_l(A) \subseteq w_h(A)$ for every unitary $H \in M_{m,n}$.

II. $w(A) = \bigcup_H w_l(A) = \bigcup_H w_h(A)$.

III. $\text{Re } w_h(A) = F\left( \begin{bmatrix} \mathcal{H}(A_1) & A_2^*/2 \\ A_2/2 & 0_{m-n} \end{bmatrix} \right)$, $\text{Im } w_h(A) = F\left( \begin{bmatrix} S(A_1) & -A_2^*/2 \\ A_2/2 & 0_{m-n} \end{bmatrix} \right)$

with respect to unitary $H = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, where $\mathcal{H}(\cdot)$ and $S(\cdot)$ denote the hermitian and skew-hermitian part of matrix, respectively.

IV. $\sigma(A_1) \subseteq w_h(A) \subseteq w(A)$ with $H = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$.

Proof. I. Let the unitary matrix $U = \begin{bmatrix} H & R \end{bmatrix} \in M_{m,m}$, where $H \in M_{m,n}$. Then

$$w_h(A) = F(AH^*) = F(U^*AH^*U) = F\left( \begin{bmatrix} H^*A & 0 \\ R^*A & 0 \end{bmatrix} \right)$$

whereupon $w_l(A) = F(H^*A) \subseteq w_h(A)$.

II. Suppose $z \in \bigcup w_l(A) = \bigcup_H F(H^*A)$, then for a $m \times n$ unitary matrix $H$

$$|z| \leq r(H^*A) \leq \|H^*A\|_2 \leq \|H^*\|_2 \|A\|_2 = \|A\|_2$$

where $r(\cdot)$ denotes the numerical radius of matrix. Thereby, $\bigcup w_l(A) = \bigcup_H F(H^*A) \subseteq w(A)$. On the other side, if $z = y^*Ax \in w(A)$, then there exists a $m \times n$ unitary matrix $H$ such that $y = Hx$ and $z = x^*(H^*A)x \in F(H^*A)$. The assertion $\bigcup w_h(A) = w(A)$ is established similarly.

III. It is enough to confirm that for the $m \times n$ unitary matrix $H = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$

$$\text{Re } w_h(A) = \text{Re } F\left( \begin{bmatrix} A & 0 \end{bmatrix} \right) = F(\mathcal{H}\left( \begin{bmatrix} A & 0 \end{bmatrix} \right)),$$

where $\mathcal{H}(\cdot)$ denotes the hermitian part of matrix. Similarly, for $\text{Im } w_h(A)$.

IV. We need merely to apply cases (I) and (II) for the $m \times n$ unitary matrix $H = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$. 

\[ \square \]
By the definitions (1.5), (1.6) or (1.7) it is clear that the concept of sharp point \([11]\) p.50 of \(F(AH^*)\) or \(F(H^*A)\) is transferred to the sharp point of \(w_h(A)\) or \(w_l(A)\), respectively. Especially, we note:

**Proposition 9.** Let \(A \in \mathcal{M}_{m,n}\), \(m > n\) and \(\lambda_0(\neq 0)\) be sharp point of \(w_h(A) = F(AH^*)\) for \(H \in \mathcal{M}_{m,n}\), \(H^*H = I_n\). Then \(\lambda_0 \in \sigma(H^*A)\) and is also sharp point of \(w_l(A) = F(H^*A)\).

**Proof.** For the sharp point \(\lambda_0 \in \partial w_h(A) = \partial F(AH^*)\) with \(H^*H = I_n\) apparently, \(\lambda_0 \in \sigma(AH^*) = \sigma(U^*AH^*U) = \sigma(H^*A) \cup \{0\}\), for the unitary matrix \(U = \begin{bmatrix} H & R \end{bmatrix} \in \mathcal{M}_{m,m}\), i.e. \(\lambda_0 \in \sigma(H^*A) \subseteq F(H^*A) = w_l(A)\).

Moreover, for \(\lambda_0\), according to the definition of sharp point, there exist \(\theta_1, \theta_2 \in [0, 2\pi)\), \(\theta_1 < \theta_2\) such that

\[
\Re(e^{i\theta} \lambda_0) = \max \left\{ \Re a : a \in e^{i\theta} w_h(A) \right\}
\]

for all \(\theta \in (\theta_1, \theta_2)\). Since \(w_h(A) \supseteq w_l(A)\) we have

\[
\Re(e^{i\theta} \lambda_0) = \max_{a \in e^{i\theta} w_h(A)} \Re a \geq \max_{b \in e^{i\theta} w_l(A)} \Re b
\]

for all \(\theta \in (\theta_1, \theta_2)\).

Furthermore, for every \(\theta \in (\theta_1, \theta_2)\)

\[
\Re(e^{i\theta} \lambda_0) \in \Re(e^{i\theta} F(H^*A)) \leq \max \left\{ \Re b : b \in e^{i\theta} F(H^*A) \right\}
\]

and thus \(\Re(e^{i\theta} \lambda_0) = \max \{ \Re b : b \in e^{i\theta} F(H^*A) \} \) for all \(\theta \in (\theta_1, \theta_2)\), concluding that \(\lambda_0(\neq 0)\) is sharp point of \(F(H^*A) = w_l(A)\). \(\Box\)

For \(m \times n\) unitary matrix \(H = \begin{bmatrix} I_n \\ 0 \end{bmatrix}\) we may obviously see the following corollary.

**Corollary 10.** Let \(A_1 \in \mathcal{M}_{n,n}\) be the principal submatrix of \(A \in \mathcal{M}_{m,n}\) and \(\lambda_0(\neq 0)\) be sharp point of \(w_h(A) = F([A \quad 0])\). Then \(\lambda_0 \in \sigma(A_1)\) and is also sharp point of \(w_l(A) = F(A_1)\).

It is noticed here that the converse of Proposition 9 does not hold as it is illustrated in the next figure. If \(A = \begin{bmatrix} 1 + i & -7 & 0 \\ 5i & 0.02 & 0 \\ 0 & 0 & 6 - i \end{bmatrix}\) and \(H = \begin{bmatrix} 0 \\ I_3 \end{bmatrix}\), \(\lambda_0 = 5i\) is sharp point of \(w_l(A)\) but not of \(w_h(A)\). Note that by \(\ast\) are denoted the eigenvalues 0 and \(5i\) of \(AH^*\).
4 The rank-\(k\) numerical range

In this section, initially, we note the easily confirmed properties of \(\phi_k(A)\) in (1.10):

\[
\phi_k(cA) = c\phi_k(A), \quad c \in \mathbb{C} \quad \text{and} \quad \phi_k(A^*) = \overline{\phi_k(A)}.
\]

Also, in the next proposition we generalize some necessary and sufficient conditions [3] for \(\Lambda_k(A)\) in (1.9), which are extended to \(\phi_k(A)\).

Proposition 11. Let \(A \in \mathcal{M}_{m,n}\). The next expressions are equivalent.

\begin{enumerate}
  \item \(z \in \phi_k(A)\).
  \item There exist subspaces \(J \subseteq \mathbb{C}^m\) and \(K \subseteq \mathbb{C}^n\) such that \(\dim J = \dim K = k\) and \((A - zS)K \perp J\).
  \item There exist orthonormal matrices \(M \in \mathcal{M}_{m,k}\) and \(N \in \mathcal{M}_{n,k}\) such that \(M^*AN = zI_k\).
  \item \(\langle Av, u \rangle = z \langle \tilde{v}, \tilde{u} \rangle\), where \(v = N\tilde{v}\), \(u = M\tilde{u}\) and \(M, N\) are the matrices in (III).
  \item There exist subspaces \(\mathcal{L} \subseteq \mathbb{C}^m\) and \(\mathcal{G} \subseteq \mathbb{C}^n\) of dimension \(k\), where \(\langle Av, u \rangle = z \|v\|\|u\|\), for every \(u \in \mathcal{L}\) and \(v \in \mathcal{G}\).
\end{enumerate}

Proof. We prove that (I) is equivalent to (II), (III), (IV) and (V).

\(\text{II.}\) For \(z \in \phi_k(A)\), clearly by (1.10) \(P(A - zS)Q = 0\). If \(J = \text{Im}(P) \subseteq \mathbb{C}^m\) and \(K = \text{Im}(Q) \subseteq \mathbb{C}^n\), then \(\dim J = \dim K = k\) and for every \(x \in K\), \(y \in J\), we have

\[
\langle (A - zS)x, y \rangle = \langle (A - zS)Qx', Py' \rangle = \langle P^*(A - zS)Qx', y' \rangle = \langle P(A - zS)Qx', y' \rangle = 0
\]
whereupon \((A - zS)\mathcal{K} \perp \mathcal{J}\). Conversely, by orthogonality we have:

\[
\langle (A - zS)x, y \rangle = 0 \quad \forall x \in \mathcal{K}, y \in \mathcal{J} \Rightarrow \langle (A - zS)x', y' \rangle = 0 \quad \forall x', y' \Rightarrow P(A - zS)Q = 0 \Rightarrow z \in \phi_k(A).
\]

**III.** Let the matrices \(M = [ u_1 \ldots u_k ]\) and \(N = [ v_1 \ldots v_k ]\), where their columns \(u_j, v_i\) constitute orthonormal bases of \(\mathcal{J}\) and \(\mathcal{K}\) in (II), respectively. Then, by statement (II):

\[
0 = \langle (A - zS)v_i, u_j \rangle = \langle Av_i, u_j \rangle - z \langle Sv_i, u_j \rangle.
\]

Denoting by \(S = MN^* = \sum_{l=1}^k u_l v_l^*\), we obtain

\[
\langle Av_i, u_j \rangle = z \sum_{l=1}^k u_l v_l^* v_i, u_j = z \sum_{l=1}^k u_l^* u_l v_l^* v_i = z
\]

for \(l = i = j\), and thereby \(M^*AN = zI_k\). For the converse, by the equation \(M^*AN = zI_k\) with \(M^*M = N^*N = I_k\) we have \(\langle Av_i, u_j \rangle = \delta_{ij}z\), for \(i, j = 1, \ldots, k\), where \(\delta_{ij}\) is the Kronecker symbol. Hence, \(PAQ = zS\), where \(P = MM^*, Q = NN^*\) and \(S = MN^*, i.e. z \in \phi_k(A)\).

**IV.** If \(u = \lambda_1 u_1 + \ldots + \lambda_k u_k\) and \(v = \mu_1 v_1 + \ldots + \mu_k v_k\), then by (III):

\[
\langle Av, u \rangle = u^*Av = [ \bar{\lambda}_1 \ldots \bar{\lambda}_k ] M^*AN \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}
\]

\[
= z [ \bar{\lambda}_1 \ldots \bar{\lambda}_k ] \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix} = z \langle \bar{v}, \bar{u} \rangle.
\]

Conversely, by the equation \(\langle Av, u \rangle = z \langle \bar{v}, \bar{u} \rangle\), for \(v = v_i = Ne_i\) and \(u = u_j = Me_j\), where \(e_i, e_j\) are vectors of standard basis of \(\mathbb{C}^k\), we have

\[
\langle Av_i, u_j \rangle = z \langle e_i, e_j \rangle \Rightarrow u_j^*Av_i = e_j^*M^*AN e_i = \delta_{ij}z ; i, j = 1, \ldots, k
\]

or equivalently \(M^*AN = zI_k\), i.e. \(z \in \phi_k(A)\).

**V.** Let \(z \in \phi_k(A)\) and \(u \in \text{span} \{ u_2, \ldots, u_k \}^\perp\), \(v \in \text{span} \{ v_2, \ldots, v_k \}^\perp\), where \(u_j \in \mathbb{C}^m, v_i \in \mathbb{C}^n\) are orthonormal vectors. Denoting by

\[
M = \begin{bmatrix} u_1 & u_2 & \ldots & u_k \end{bmatrix}, N = \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix}
\]

clearly \(M^*M = N^*N = I_k\). By statement (II) and for \(P = MM^*, Q = NN^*\) and \(S = MN^*\) we have \((A - zS)\mathcal{G} \perp \mathcal{L}\), where \(\mathcal{G} = \text{Im}(Q), \mathcal{L} = \text{Im}(P)\). Thus, we obtain

\[
\langle (A - zS)v, u \rangle = 0 \Rightarrow \langle Av, u \rangle = z \langle Sv, u \rangle = z \langle MN^*v, u \rangle = z \langle N^*v, M^*u \rangle
\]
\[ z v^* u^* \|v\| \|u\| = z \|v\| \|u\|. \]

The converse is received trivially, completing the proof. \[ \square \]

**Proposition 12.** The rank-k numerical range \( \phi_k(A) \) for a rectangular matrix \( A \in M_{m,n} \) satisfies the relationship

\[ w(A) = \phi_1(A) \supseteq \phi_2(A) \supseteq \ldots \supseteq \phi_r(A) \]

where \( \tau = \min \{ m, n \} \).

**Proof.** Let \( z \in \phi_k(A) \) and \( u, v \) are unit vectors of \( \mathbb{C}^m \) and \( \mathbb{C}^n \). Then, by proposition 11(V), we derive \( \langle Av, u \rangle = z \), i.e. \( z \in \phi_1(A) = w(A) \). Hence, \( \phi_k(A) \subseteq \phi_1(A) \) for every \( k \). Besides, if \( z \in \phi_k(A) \), by Proposition 11(III) we have \( M^* A N = z I_k \), where \( M = [ u_1 \ldots u_k ] = [ M_1 \ u_k ] \) and \( N = [ v_1 \ldots v_k ] = [ N_1 \ v_k ] \) are orthonormal matrices. Then \( M^*_1 A N_1 = z I_{k-1} \), i.e. \( z \in \phi_{k-1}(A) \), concluding that \( \phi_k(A) \subseteq \phi_{k-1}(A) \) for \( k = 2, \ldots, \tau \), where \( \tau = \min \{ m, n \} \).

Following, we present some additional properties:

**Proposition 13.** Let \( A \in M_{m,n} \), then for \( \phi_k(A) \) in \([1,10]\), holds:

I. \( \phi_k(U^* A V) = \phi_k(A) \), where \( U \in M_{m,m} \) and \( V \in M_{n,n} \) are unitary matrices.

II. \( \phi_k(A) = \phi_k(e^{i\theta} A) \) for every \( \theta \in [0, 2\pi) \).

III. If \( z \in \phi_k(A) \), then \( \text{Re} \ z \in \Lambda_k \left[ \begin{array}{c} 0 & A \\ A^* & 0 \end{array} \right] = [-\sigma_k, \sigma_k] \) and \( \text{Im} \ z \in \Lambda_k \left[ \begin{array}{cc} 0 & -iA \\ iA^* & 0 \end{array} \right] = [-i\sigma_k, i\sigma_k] \), where \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_q \geq 0 \) denote the decreasingly ordered singular values of \( A \), counting multiplicities.

**Proof.** I. Let \( z \in \phi_k(U^* AV) \), then for suitable unitary matrices \( M \in M_{m,k} \) and \( N \in M_{n,k} \) we have \( M^* U^* AV N = z I_k \Rightarrow (UM)^* (A V N) = z I_k \) i.e. \( z \in \phi_k(A) \). Thus \( \phi_k(U^* AV) \subseteq \phi_k(A) \).

Conversely, if \( z \in \phi_k(A) \), then \( R^* T = z I_k \), where \( R \in M_{m,k} \) and \( T \in M_{n,k} \) are unitary. Clearly we can write \( R = UM \) and \( T = VN \), where \( U \) and \( V \) are defined by orthonormal bases of \( \mathbb{C}^m \) and \( \mathbb{C}^n \), respectively. Therefore, \( M^* (U^* AV) N = z I_k \), i.e. \( z \in \phi_k(U^* AV) \).

II. Assume \( M \in M_{m,k} \) and \( N \in M_{n,k} \) such that \( M^* M = N^* N = I_k \), then

\[ \phi_k(A) = \{ z \in \mathbb{C} : M^* A N = z I_k \} \]
\[ = \{ z \in \mathbb{C} : (M^* e^{-i\theta}) e^{i\theta} A N = z I_k \} \]
\[ = \{ z \in \mathbb{C} : e^{i\theta} M^* (e^{i\theta} A) N = z I_k \} \]
\[ = \{ z \in \mathbb{C} : M_1^* (e^{i\theta} A) N = z I_k \} = \phi_k(e^{i\theta} A) \]
for every \( \theta \in [0,2\pi) \), since \( M_1^*M_1 = M^*M = I_k \). That is, the set \( \phi_k(A) \) is circular.

III. By \( \text{[11,10]} \), let \( PAQ = zS \), where \( P = MM^* \), \( Q = NN^* \) and \( S = MN^* \). Then, \( M^*AN = zI_k \) and consequently

\[
(\text{Re } z)I_k = \frac{1}{2} [M^* N^*] \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}
\]

Denoting by \( T = \frac{1}{\sqrt{2}} \begin{bmatrix} M \\ N \end{bmatrix} \in M_{(m+n),k} \), then \( T^*T = \frac{1}{2}(M^*M + N^*N) = I_k \) and the \((m+n) \times (m+n)\) matrix \( G = TT^* = \frac{1}{2} \begin{bmatrix} P & S \\ S^* & Q \end{bmatrix} \) is rank-\( k \) orthogonal projector, because \( \text{rank } T = k \) and

\[
G^2 = \frac{1}{4} \begin{bmatrix} P^2 + SS^* & PS + SQ \\ S^*P + QS^* & S^*S + Q^2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} P + P & 2S \\ 2S^* & Q + Q \end{bmatrix} = G.
\]

Thus, by \( \text{[4.1]} \) we obtain \( (\text{Re } z)G = G \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \), i.e. \( \text{Re } z \in \Lambda_k(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}) \).

Similarly, we derive \( \text{Im } z \in \Lambda_k(\begin{bmatrix} 0 & -iA \\ iA^* & 0 \end{bmatrix}) \). Moreover, due to the \((m+n) \times \)

\((m+n)\) hermitian matrix \( \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \) having eigenvalues \( -\sigma_1 \leq -\sigma_2 \leq \ldots \leq -\sigma_q < 0 < \sigma_q \leq \ldots \leq \sigma_2 \leq \sigma_1 \), \( \text{[10]} \) and the multiplicity of \( \lambda = 0 \) being equal to \( m + n - 2q \), we verify \( \text{[9, Th. 2.4]} \) that \( \Lambda_k(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}) = [-\sigma_k, \sigma_k] \) and \( \Lambda_k(\begin{bmatrix} 0 & -iA \\ iA^* & 0 \end{bmatrix}) = [-i\sigma_k, i\sigma_k] \). \( \square \)

A more precise description of \( \phi_k(A) \) is given in the next proposition.

**Proposition 14.** Let \( A \in M_{m,n} \) and \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min\{m,n\}} \) be its singular values.

I. If for the index \( k \), \( \max \{ m, n \} < k \leq \frac{m+n+1}{3} \), then \( \phi_k(A) \) is equal to the ring \( \mathcal{R}(0; \sigma_{m+n-2k+1}, \sigma_k) \).

II. If \( k > \frac{m+n+1}{3} \), then \( \phi_k(A) \) is the empty set.

III. If \( k \leq \min \{ m, n \} \), then \( \phi_k(A) \) is identified with the circular disc \( \mathcal{D}(0, \sigma_k) \).

**Proof.** I. Consider that \( A = U\Sigma V^* \) is the singular value decomposition of \( A \), then by Proposition 13(I), \( \phi_k(A) = \phi_k(\Sigma) \). If \( z \in \phi_k(\Sigma) \), then for suitable \( m \times k \) and \( n \times k \) unitary matrices \( M \) and \( N \) we have \( zI_k = M^*\Sigma N \).
Denoting by $\tilde{U} = [M \quad M_1]$ and $\tilde{V} = [N \quad N_1]$ the augmented unitary square matrices, then the singular values of matrix

$$(4.2) \quad \tilde{U}^*\Sigma\tilde{V} = \begin{bmatrix} M^*\Sigma N & M^*\Sigma N_1 \\ M_1^*\Sigma N & M_1^*\Sigma N_1 \end{bmatrix}$$

are also $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min\{m,n\}}$ and the singular values of submatrix $M^*\Sigma N = zI_k$ are equal to $\beta_1 = \beta_2 = \ldots = \beta_k = |z|$. Thus, by Th.1 in [15], we have

$$(4.3) \quad \begin{align*}
\sigma_i &\geq \beta_i = |z|, \quad \text{for } i = 1, \ldots, k, \\
\beta_k &\geq \sigma_{i+m+n-2k}, \quad \text{for } i = 1, \ldots, \min\{2k-m, 2k-n\}.
\end{align*}$$

Since $k \leq \frac{m+n+1}{2}$, then clearly $\sigma_{m+n-2k+1} \leq \sigma_k$. The validity of all inequalities \((4.3)\) confirms $\sigma_{m+n-2k+1} \leq |z| \leq \sigma_k$ and by the circular property of $\phi_k(\Sigma)$ in Prop. 13(II) we have that $z$ belongs to the ring $\mathcal{R}(0; \sigma_{m+n-2k+1}, \sigma_k)$. Consequently, if $z \in \mathcal{R}(0; \sigma_{m+n-2k+1}, \sigma_k)$, then

$$\sigma_{\min\{m,n\}} \leq \ldots \leq \sigma_{m+n-2k+1} \leq |z| \leq \sigma_k \leq \sigma_{k-1} \leq \ldots \leq \sigma_1$$

and by Th.2 in [15], we have that there exist $m \times k$ and $n \times k$ unitary $M \in \mathcal{M}_{m,k}$ and $N \in \mathcal{M}_{n,k}$ such that $\beta_1 = \ldots = \beta_k = |z|$ are the singular values of the submatrix $M^*\Sigma N$ in \((4.2)\). Due to the singular values of $zI_k$ and $M^*\Sigma N$ being identified, the matrices are related by the equation

$$W_1(zI_k)W_2^* = M^*\Sigma N$$

where $W_1, W_2$ are $k \times k$ unitary matrices. Hence, we have $(MW_1)^*\Sigma(NW_2) = zI_k$, yielding that $z \in \phi_k(\Sigma)$.

Note that, for $k = \frac{m+n+1}{2} \Rightarrow \sigma_k = \sigma_{m+n-2k+1}$, i.e. the ring is degenerated to the circle $\{z : |z| = \sigma_k\}$.

\textbf{II.} If $k > \frac{m+n+1}{3}$, then $\sigma_k < \sigma_{m+n-2k+1}$ and should be $z \in \{z : |z| \leq \sigma_k\} \cap \{z : |z| \geq \sigma_{m+n-2k+1}\} = \emptyset$. Therefore, $\phi_k(A) = \emptyset$.

\textbf{III.} To the case $k \leq \max\{\frac{m}{2}, \frac{n}{2}\}$, obviously $\min\{2k-m, 2k-n\} \leq 0$ and then only inequalities $\sigma_i \geq \beta_i = |z|$ for $i = 1, \ldots, k$ are valid, establishing $\phi_k(A) = \mathcal{D}(0, \sigma_k)$. \hfill \Box

\textbf{Corollary 15.} Let $A \in \mathcal{M}_{m,n}$ and $\sigma_1 \geq \ldots \geq \sigma_{\min\{m,n\}}$ be its singular values. If $\max\{\frac{m}{2}, \frac{n}{2}\} < k \leq \frac{m+n+1}{3}$ and $\sigma_r = 0$, for $k \leq r \leq m+n-2k+1$, then $\phi_k(A)$ coincides with the circular disc $\mathcal{D}(0, \sigma_k)$.

\textbf{Proof.} Apparently, by Proposition 14(II), $\phi_k(A) = \mathcal{R}(0; \sigma_{m+n-2k+1}, \sigma_k)$. Since index $r$ satisfies $k < r \leq m+n-2k+1$, we have $\sigma_k \geq 0 \geq \sigma_{m+n-2k+1}$. To the case $k = r$, $\sigma_k = \sigma_r = 0$ and $\phi_k(A)$ is degenerated to the origin. \hfill \Box
We remark here that if \( \|A\|_2 = \sigma_1 \) with multiplicity \( k \), as it is stated in Corollary 15, and \( \sigma_l = 0 \) for \( k < l \leq m + n - 2k + 1 \), then \( \phi_k(A) = \mathcal{D}(0, \sigma_1) \). The boundary points of this disc are reached, using the eigenvectors of \( A^*A \) corresponding to \( \sigma_1^2 \).

**Proposition 16.** Let the matrix \( A \in \mathcal{M}_{m,n} \). If \( \mathcal{L} \) is \( (m-k+1) \)-dimensional subspace of \( \mathbb{C}^m \) and \( \mathcal{G} \) is \( (n-k+1) \)-dimensional subspace of \( \mathbb{C}^n \), then for any positive integer \( k \geq 1 \)
\[
\phi_k(A) \subseteq \bigcap_{\mathcal{L}} w(P_L A) \quad \text{and} \quad \phi_k(A) \subseteq \bigcap_{\mathcal{G}} w(AQ_G)
\]
where the numerical range \( w(\cdot) \) has been defined in \([1,2]\) and \( P_L, Q_G \) are orthogonal projectors onto \( \mathcal{L} \) and \( \mathcal{G} \), respectively.

**Proof.** Assume \( z \in \phi_k(A) \). By Proposition 11(V) there exist subspaces \( \mathcal{L}' \) and \( \mathcal{G}' \) of \( \mathbb{C}^m \) and \( \mathbb{C}^n \), respectively, with \( \dim \mathcal{L}' = \dim \mathcal{G}' = k \), such that \( z = \langle Av, u \rangle \) for unit vectors \( v \in \mathcal{G}', u \in \mathcal{L}' \). Then, following the arguments in \([3]\), for a unit vector \( \tilde{u} \in \mathcal{L} \cap \mathcal{L}' \) we readily see that \( z = \langle Av, \tilde{u} \rangle = \langle Av, P_L \tilde{u} \rangle = \langle P_L^* Av, \tilde{u} \rangle \in w(P_L A) \), where \( P_L \) is orthogonal projector of \( \mathbb{C}^m \) onto \( \mathcal{L} \). Hence, \( \phi_k(A) \subseteq \bigcap_{\mathcal{L}} \{ w(P_L A) : P_L \text{ orthogonal projector onto } \mathcal{L} \} \).

Similarly, considering the subspace \( \mathcal{G} \) of dimension \( n-k+1 \) we conclude the second inclusion. \( \square \)

**Remark.** It is worth noticing, finally, the containment
\[
\mathcal{D}(0, \sigma_k(A)) \subseteq \bigcap_{\dim \mathcal{G} = n-k+1} w(AQ_G) = \mathcal{D}(0, \min_{\mathcal{G}} \|AQ_G\|_2),
\]
since \([11]\) p.148
\[
\min_{\mathcal{G} \subseteq \mathbb{C}^n} \|AQ_G\|_2 = \min_{\mathcal{G}} \max \{ \|AQ_G x\|_2 : x \in \mathbb{C}^n, \|x\|_2 = 1 \} \\
\geq \min_{\mathcal{G}} \max \{ \|AQ_G x\|_2 : x \in \mathcal{G}, \|x\|_2 = 1 \} = \sigma_k(A).
\]

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