CONVERGENCE AND STABILITY OF GENERALIZED GRADIENT SYSTEMS BY LOJASIEWICZ INEQUALITY WITH APPLICATION IN CONTINUUM KURAMOTO MODEL

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Abstract. The existence and uniqueness/multiplicity of phase locked solution for continuum Kuramoto model was studied in [12, 29]. However, its asymptotic behavior is still unknown. In this paper we concern the asymptotic property of classic solutions to continuum Kuramoto model. In particular, we prove the convergence towards a phase locked state and its stability, provided suitable initial data and coupling strength. The main strategy is the quasi-gradient flow approach based on Łojasiewicz inequality. For this aim, we establish a Łojasiewicz type inequality in infinite dimensions for continuum Kuramoto model which is a nonlocal integro-differential equation. General theorems for convergence and stability of (generalized) quasi-gradient system in an abstract setting are also provided based on Łojasiewicz inequality.

1. Introduction.

1.1. Motivation. The Kuramoto model [17, 18], given by the following coupled oscillators

\[
\frac{d\theta_i}{dt} = \omega_i + \frac{\alpha}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i),
\]

has been a successful model for describing the synchronization process of large populations of weakly coupled oscillators which often appears in natural and engineering systems. Here, \( \theta_i \) is the phase of the \( i \)th oscillator, \( \omega_i \) is its natural frequency, and
the oscillators are all-to-all coupled through the sinusoidal function with strength $\alpha > 0$. This type of all-to-all coupling has played an important role in the analysis of (1). For example, for finite population of oscillators, there have been some studies on the complete synchronization of (1), where the analysis was built on the all-to-all coupling scheme [6, 11, 13]. In the limit as $N \to \infty$, there are two ways to recast the Kuramoto model. The first one is to use the one-oscillator probability density to derive a kinetic Kuramoto equation in the mean-field limit, see for example, [1, 2, 3, 8, 9, 26, 27, 28]. For the stability issue, in [4] Carrilo et al. studied the complete synchronization of kinetic Kuramoto model and derive some estimate for stability of measure-valued solutions in the Wasserstein distance. In [9], Dietert et al. discussed the stability of partially phase-locked states in kinetic Kuramoto model.

Another way to reformulate the Kuramoto model as $N \to \infty$ is to take a continuum limit in (1) so that we can derive a single integro-differential equation:

$$\frac{\partial \theta(x, t)}{\partial t} = \omega(x) + \alpha \int_0^1 \sin [\theta(x', t) - \theta(x, t)] \, dx', \quad t \geq 0, \ x \in [0, 1]. \quad (2)$$

This equation, referred as continuum Kuramoto model (CKM), can be formally constructed by interpreting the coupling term in (1) as a Riemann sum and sending $N \to \infty$, where $\theta(x, t)$ now describes a continuum of dynamical systems distributed along $I := [0, 1]$ and $\omega(x)$ describes the distributed natural frequencies. On the other hand, this model can be derived from (1) by applying Strong Law of Large Numbers with $N \to \infty$ when natural frequencies $\{\omega_i\}$ are subject to probability distributions that are independent and identically distributed [12]. A connection between the kinetic and continuum Kuramoto models is that the CKM can be regarded as a subclass of solutions of the kinetic Kuramoto model when the kinetic distribution $\delta_{\theta=\theta(t,x)} \, dx$ is taken. Moreover, this class is invariant under the evolution of the kinetic Kuramoto model.

In [12], Ermentrout derives a nonlinear equation and its equivalent formulation which gives a criterion for existence of phase-locked solutions of (2). Following this approach, Troy [29] investigated uniqueness or exact multiplicity of these solutions. He derived a criteria, depending on a parameter, to guarantee the uniqueness of phase-locked solution or coexistence of exactly two solutions. In [24, 25], Medvedev derived the continuum limit of Kuramoto model on networks with increasing number of nodes by using the ideas from graph limits [23], and discussed the relation between solutions of discrete model and the continuum limit. We should note that the dynamic evolution of (2) has not been touched. In [12, 29] they concerned only the static problems regarding its phase-locked solutions; in other words, the equilibrium of (2) was considered through the static equation

$$\omega(x) + \alpha \int_0^1 \sin [\theta(x') - \theta(x)] \, dx' = 0. \quad (3)$$

As pointed in [29], the convergence of system (2) and the stability of its equilibrium (3) are interesting problems. To the best of our knowledge, the only related work should be directed to the stability of the measure-valued solution of kinetic Kuramoto model in the Wasserstein distance [4], in the sense that CKM describes a subclass of solutions of the kinetic Kuramoto model. However, the dynamic properties of the system (2) in itself have not been studied. In this paper, we will study the convergence of solutions to system (2) in $C[0, 1]$ or $L^2[0, 1]$ (see Theorem 5.5 and Remark 4). We also consider the stability of its equilibrium (3) in the above
topologies. Motivated by this problem, we will establish a Lojasiewicz inequality in an infinite-dimensional Hilbert space for (2), and develop abstract theorems for the convergence and stability of generalized gradient systems in a Banach space which is continuously embedded into a Hilbert space (Theorems 4.1 and 4.4). In the following subsection, we briefly introduce our strategy.

1.2. Our strategy. The classic Lojasiewicz inequality was established in finite dimensions [22], which reveals a fundamental relation between a potential function $E : \mathbb{R}^N \to \mathbb{R}$ and its gradient, more precisely,

$$|E(x) - E(x_0)|^{1-\rho} \leq C\|\nabla E(x)\|, \quad x \in \mathcal{N}(x_0),$$

where $\mathcal{N}(x_0)$ is a neighborhood of $x_0$, $\rho \in (0, \frac{1}{2}]$ and $C > 0$ are constants independent of $x \in \mathcal{N}(x_0)$. The exponent $\rho$ that makes (4) hold is called Lojasiewicz exponent of $E$ at $x_0$. This inequality provides a powerful tool for proving the convergence of trajectories of gradient system $\dot{x} = -\nabla E(x)$ towards a single equilibrium. A celebrated result was given in [22] which claims that Lojasiewicz inequality holds for any real analytic function $E : \mathbb{R}^N \to \mathbb{R}$, although the computation of Lojasiewicz exponent $\rho$ can be a hard problem. It is worthy to mention that the computation of exponent is relevant to the convergence rate; precisely, $\rho = \frac{1}{2}$ implies exponential decay and $\rho < \frac{1}{2}$ implies algebraic decay. In [30] it was noticed that the discrete Kuramoto model (1) has a Lyapunov function so that it can be reformulated as a standard gradient system with an analytic potential. Based on this reformulation, in [10, 14, 20] the Lojasiewicz inequality in finite dimensions was used to prove the convergence towards a phase-locked state for system (1). Two exponents show up for (1), $\rho = \frac{1}{2}$ or $\frac{1}{3}$, depending on the configurations of phase-locked states [20]. For other studies on the second-order Kuramoto model with inertia by this inequality we refer to [7, 19, 21].

The Lojasiewicz inequality can be extended to abstract settings in infinite dimensions, for example, the so-called Lojasiewicz-Simon inequality [5]. However, unlike the case of finite dimensions, in infinite dimensions the Lojasiewicz type inequality may fail for a real analytic functional on a Hilbert space $\mathcal{H}$, see [15, Proposition 3.5]. Actually, this example also indicates that Lojasiewicz inequality can fail even when we confine the argument $x$ in a compact subset of the neighborhood $\mathcal{N}(x_0)$. Therefore, to verify a Lojasiewicz inequality for a real analytic functional in infinite dimensions is not a trivial work, even when we confine the argument in a compact subset of a neighborhood. In order to borrow the idea of Lojasiewicz inequality in finite dimensions to study the convergence and stability of CKM, we first need to prove a Lojasiewicz inequality for the specific system (2).

In this paper we will prove a Lojasiewicz gradient inequality in infinite dimensions in a suitable setting which enables us to address the convergence and stability of phase-locked solutions for CKM (2). We will consider classic solutions to (2) for given $\omega(\cdot) \in C^1[0, 1]$ and $\theta_0(\cdot) = \theta(\cdot, 0) \in C^1[0, 1]$. Let $F : C^1[0, 1] \to C^1[0, 1]$ be given by

$$F(\theta)(x) := \omega(x) + \alpha \int_0^1 \sin[\theta(x') - \theta(x)] \, dx',$$

so that (2) is reformulated as the following abstract differential equation in Banach space $C^1[0, 1]$: 

$$\dot{\theta} = F(\theta), \quad \theta(0) = \theta_0.$$
Define a functional $E : L^2[0, 1] \to \mathbb{R}$ as follows:

$$E(\theta) = -\int_0^1 \theta(x) \omega(x) dx - \frac{\alpha}{2} \int_0^1 \int_0^1 \cos [\theta(x') - \theta(x)] dx' dx.$$  \hspace{1cm} (6)

Then (2) can be regarded as a \textit{quasi-gradient} flow with potential $E$ (see Remark 3).

Two abstract theorems for the convergence and stability of quasi-gradient systems are established in a Banach space which is continuously embedded into a Hilbert space (Theorems 4.1 and 4.4). We prove a Lojasiewicz inequality for $E$ when we confine the argument $\theta$ in a relatively compact subset, i.e., $\mathcal{M}_\ell$ in Theorem 3.2. Using the Lojasiewicz inequality and abstract theorems we can obtain the desired results for the asymptotic behaviors of (2).

1.3. Contribution. The contribution of this paper is threefold. First, we prove the convergence and stability for CKM which was noticed as an open question in [29]. Second, we prove a Lojasiewicz type inequality with an exponent $\rho = \frac{1}{4}$ for infinite dimensional CKM which is a nonlocal integro-differential equation. As far as we know, this is the first result for this issue. Third, we present a novel setting for convergence and stability of generalized gradient systems in Banach spaces.

The rest of this paper is organized as follows. In Section 2, we show the existence and uniqueness of solution for (2). In Section 3, we prove a Lojasiewicz inequality for CKM. In Section 4, the abstract theorems are presented. Finally in Section 5 we give the main results in this paper for the convergence and stability of CKM. Then the conclusion is given in Section 6.

Notation. $C[0, 1] = \{ f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous} \}, \| f \|_{C[0, 1]} = \max_{x \in [0, 1]} |f(x)|$.

$C^1[0, 1] = \{ f : [0, 1] \to \mathbb{R} \mid f \text{ is continuously differentiable} \}, \| f \|_{C^1[0, 1]} = \| f \|_{C[0, 1]} + \| f' \|_{C[0, 1]}$.

$L^2[0, 1] = \{ f : [0, 1] \to \mathbb{R} \mid f \text{ is square-integrable} \}, \| f \|_{L^2[0, 1]} = \left( \int_0^1 f^2(x) dx \right)^{1/2}$.

2. Preliminaries.

2.1. On the model. We consider the following integro-differential equation

$$\frac{\partial \theta(x, t)}{\partial t} = \omega(x) + \alpha \int_0^1 \sin [\theta(x', t) - \theta(x, t)] dx', \hspace{1cm} (7)$$

$$\theta(x, 0) = \theta_0(x), \quad \theta_0(\cdot) \in C^1[0, 1], \quad \omega(\cdot) \in C^1[0, 1].$$

Let $\theta(x, t)$ be a solution of (7), we introduce natural frequency and phase fluctuations:

$$\hat{\omega}(x) = \omega(x) - \int_0^1 \omega(x) dx, \quad \hat{\theta}(x, t) = \theta(x, t) - t \int_0^1 \omega(x) dx,$$

then by (7) we have

$$\frac{\partial \hat{\theta}(x, t)}{\partial t} = \hat{\omega}(x) + \alpha \int_0^1 \sin \left[ \hat{\theta}(x', t) - \hat{\theta}(x, t) \right] dx'.$$

Note that the integral of right hand side goes to zero, so we have

$$\frac{\partial}{\partial t} \int_0^1 \hat{\theta}(x, t) dx = 0.$$
Without loss of any generality, let’s assume \( \int_0^1 \dot{\theta}(x,0) dx = 0 \), then we have \( \int_0^1 \dot{\theta}(x,t) dx = 0, \forall t \geq 0 \). We now drop the hat to simplify the notation and obtain the following system:

\[
\frac{\partial \theta(x,t)}{\partial t} = \omega(x) + \alpha \int_0^1 \sin[\theta(x',t) - \theta(x,t)] dx',
\]

\[
\theta(x,0) = \theta_0(x), \quad \theta_0(\cdot) \in C^1[0,1], \quad \omega(\cdot) \in C^1[0,1],
\]

\[
\int_0^1 \theta_0(x) dx = 0, \quad \int_0^1 \omega(x) dx = 0.
\]

Obviously, along the solution of (8) the mean phase is conserved, i.e.,

\[
\int_0^1 \theta(x,t) dx = 0, \quad \forall t \geq 0.
\]

Instead of (7), hereafter we will work on (8).

**Definition 2.1.** \( \theta^*(x) \in C[0,1] \) is an equilibrium of (8), if for all \( x \in [0,1] \),

\[
\omega(x) + \alpha \int_0^1 \sin[\theta^*(x') - \theta^*(x)] dx' = 0.
\]

In this paper, we will study the convergence of classic solutions of (8), or more precisely, whether there exists an equilibrium \( \theta^* \) of (8) such that

\[
\|\theta(\cdot,t) - \theta^*(\cdot)\|_{C[0,1]} \to 0, \quad \text{as } t \to \infty.
\]

In the remaining part of this section, we address the existence and uniqueness of solution to (8).

### 2.2. Existence and uniqueness.

**Proposition 2.2.** If \( \omega(\cdot) \in C^1[0,1] \) and \( \theta_0(\cdot) \in C^1[0,1] \), then the system (8) admits a unique solution \( \theta(\cdot,t) \in C^1[0,1] \).

**Proof.** Let \( X = C^1[0,1] \) and let

\[
F(\theta)(x) := \omega(x) + \alpha \int_0^1 \sin[\theta(x') - \theta(x)] dx',
\]

then \( X \) is a Banach space and \( F \) is a mapping from \( X \) to itself. Let \( \theta_1 \) and \( \theta_2 \) be arbitrarily chosen in \( X \), we have

\[
\|F(\theta_1) - F(\theta_2)\|_X
\]

\[
= \max_{x \in [0,1]} |F(\theta_1)(x) - F(\theta_2)(x)| + \max_{x \in [0,1]} \left| \frac{\partial F}{\partial x}(\theta_1)(x) - \frac{\partial F}{\partial x}(\theta_2)(x) \right|
\]

\[
= \max_{x \in [0,1]} \alpha \left| \int_0^1 \sin[\theta_1(x') - \theta_1(x)] dx' - \sin[\theta_2(x') - \theta_2(x)] dx' \right|
\]

\[
+ \max_{x \in [0,1]} \alpha \left| \frac{\partial \theta_1}{\partial x} \int_0^1 \cos[\theta_1(x') - \theta_1(x)] dx' - \frac{\partial \theta_2}{\partial x} \int_0^1 \cos[\theta_2(x') - \theta_2(x)] dx' \right|
\]

\[
\leq 2\alpha \|\theta_1 - \theta_2\|_{C[0,1]} + 2\alpha \left\| \frac{\partial \theta_1}{\partial x} - \frac{\partial \theta_2}{\partial x} \right\|_{C[0,1]} + \alpha \left\| \frac{\partial \theta_2}{\partial x} \right\|_{C[0,1]} \|\theta_1 - \theta_2\|_{C[0,1]}.
\]

Hence, the mapping \( F : X \to X \) is locally Lipschitz continuous. Consider the abstract differential equation in \( X \):

\[
\frac{d\theta}{dt} = F(\theta), \quad \theta(0) = \theta_0.
\]
Let

Lemma 2.3.

First we give the following Lemma.

By the standard Cauchy’s theory, we see that the system (9) admits a unique solution \( \theta \in C^1([0, T), \mathcal{X}) \). Next, we show that the local solution can be extended so that it is global in time. We note that \( \theta \) solution

\[\|F(\theta)\|_X = \max_{x \in [0, 1]} \left| \omega(x) + \alpha \int_0^1 \sin [\theta(x') - \theta(x)] \, dx' \right| \]
\[\quad + \max_{x \in [0, 1]} \left| \omega'(x) - \alpha \frac{\partial \theta}{\partial x} \int_0^1 \cos [\theta(x') - \theta(x)] \, dx' \right| \]
\[\leq M + \alpha \|\theta\|_X.\]

Let \( \theta(t) \) be the solution of (9) on \([0, T), T < +\infty\), then we have

\[\theta(t) = \theta_0 + \int_0^t F(\theta(s)) \, ds, \quad t \in [0, T),\]

and further

\[\|\theta(t)\|_X \leq \|\theta_0\|_X + \int_0^t (M + \alpha \|\theta(s)\|_X) \, ds.\]

This surely implies that \( \sup_{t \in [0, T]} \|\theta(t)\|_X < \infty \). By (10) we have

\[\sup_{t \in [0, T]} \|F(\theta(s))\|_X < \infty.\]

Therefore, \( \lim_{t \to T^-} \theta(t) \) exists in \( \mathcal{X} \). Then we can extend the solution to \([0, +\infty)\), and we conclude the existence and uniqueness of global solution for (9), and then (8).

Next, we discuss the time evolution of the \( x \)-derivative of classic solution to (8).

First we give the following Lemma.

**Lemma 2.3.** Let \( g \in C[0, 1] \). The following differential equation

\[
\frac{\partial a(x, t)}{\partial t} = g(x) - \alpha a(x, t) \int_0^1 \cos \left( \int_x^{x'} a(y, t) \, dy \right) \, dx',
\]

(11)

\[
a(\cdot, 0) = a_0 \in C[0, 1].
\]

admits a unique solution in \( C[0, 1] \).

**Proof.** For notational simplicity let’s denote \( \mathcal{V} = C[0, 1] \). We consider the differential equation in space \( \mathcal{V} \):

\[
\frac{da}{dt} = G(a) := g(x) - \alpha a(x, t) \int_0^1 \cos \left( \int_x^{x'} a(y, t) \, dy \right) \, dx'.
\]

Note that

\[
\|G(a) - G(\tilde{a})\|_\mathcal{V}
\]
\[
= \max_{x \in [0, 1]} \left| \alpha a(x) \int_0^1 \cos \left( \int_x^{x'} a(y) \, dy \right) \, dx' - \alpha \tilde{a}(x) \int_0^1 \cos \left( \int_x^{x'} \tilde{a}(y) \, dy \right) \, dx' \right|
\]
\[
\leq \alpha \|a - \tilde{a}\|_\mathcal{V} + \max_{x \in [0, 1]} \left| \alpha a(x) \left( \int_0^1 \cos \left( \int_x^{x'} a(y) \, dy \right) \, dx' - \int_0^1 \cos \left( \int_x^{x'} \tilde{a}(y) \, dy \right) \, dx' \right) \right|
\]
\[
\leq \alpha \|a - \tilde{a}\|_\mathcal{V} + \max_{x \in [0, 1]} 2\alpha a(x) \int_0^1 \sin \frac{1}{2} \int_x^{x'} a(y) - \tilde{a}(y) \, dy \sin \frac{1}{2} \int_x^{x'} a(y) + \tilde{a}(y) \, dy \, dx'
\]
\[
\leq \alpha \|a - \tilde{a}\|_\mathcal{V} + \alpha \|a\|_\mathcal{V} \|a - \tilde{a}\|_\mathcal{V}.
Thus, the mapping $G : \mathcal{V} \to \mathcal{V}$ is locally Lipschitz. Moreover, we have

$$\|G(a)\|_{\mathcal{V}} \leq M + \alpha \|a\|_{\mathcal{V}},$$

for some constant $M$. We use the similar argument as in Proposition 2.2 to see that the integro-differential equation (11) admits a unique solution on $[0, +\infty)$. 

Note that the solution of (8) satisfies

$$\theta(x, t) = \theta_0(x) + t \omega(x) + \alpha \int_0^t \int_0^1 \sin[\theta(x', s) - \theta(x, s)] \, dx' \, ds.$$  

Since $\theta(\cdot, t) \in \mathcal{C}^1[0, 1]$, we derive that

$$\frac{\partial}{\partial x} \theta(x, t) = \theta'_0(x) + t \omega'(x) - \alpha \int_0^t \frac{\partial \theta}{\partial x}(x, s) \int_0^1 \cos[\theta(x', s) - \theta(x, s)] \, dx' \, ds$$

$$= \theta'_0(x) + t \omega'(x) - \alpha \int_0^t \frac{\partial \theta}{\partial x}(x, s) \int_0^1 \cos \left[ \int_x^{x'} \frac{\partial \theta}{\partial x}(y, s) \, dy \right] \, dx' \, ds$$

We denote $a(x, t) = \frac{\partial}{\partial x} \theta(x, t)$, then the above equation implies

$$\frac{\partial}{\partial t} a(x, t) = \omega'(x) - \alpha a(x, t) \int_0^1 \cos \left[ \int_x^{x'} a(y, t) \, dy \right] \, dx' \quad (12)$$

By Lemma 2.3, there is a unique solution to the equation (12) with initial data $a(x, 0) = \theta'_0(x)$. Therefore, we see the following remark.

**Remark 1.** Let $\theta(x, t)$ be the unique solution to (8), which is guaranteed by Proposition 2.2, then its derivative $\frac{\partial}{\partial x} \theta(x, t)$ is the unique solution to (12).

### 3. Lojasiewicz inequality

In this section, we will establish a Lojasiewicz inequality for potential (6). For convenience, we drop the negative sign in (6) and consider the functional $\mathcal{E}_1 = -\mathcal{E}$, i.e.,

$$\mathcal{E}_1(\theta) = \int_0^1 \theta(x) \omega(x) \, dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \cos[\theta(x') - \theta(x)] \, dx' \, dx.$$  

**Lemma 3.1.** The gradient of $\mathcal{E}_1 : L^2[0, 1] \to \mathbb{R}$ at $\theta \in L^2[0, 1]$ is

$$\nabla \mathcal{E}_1(\theta)(x) = \omega(x) + \alpha \int_0^1 \sin[\theta(x') - \theta(x)] \, dx'.$$  

(13)  

Here, the gradient is defined through the standard inner product in Hilbert space $L^2[0, 1]$.

**Proof.** Let $\theta, v \in L^2[0, 1]$ and $h \in \mathbb{R}$. Note that

$$\mathcal{E}_1(\theta + hv) = \int_0^1 (\theta + hv) \omega dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \cos [\theta(x') - \theta(x) + h(v(x') - v(x))] \, dx' \, dx.$$  

Hence,

$$\frac{\mathcal{E}_1(\theta + hv) - \mathcal{E}_1(\theta)}{h}$$

$$= \int_0^1 \omega v \, dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \sin[\theta(x') - \theta(x) + h\xi(x')] \, dx' \, dx$$

$$= \int_0^1 \omega v \, dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \sin[\theta(x') - \theta(x) + \xi(x')] \, dx' \, dx$$
for some $\xi \in [0,h]$, which converges to

$$
\int_0^1 \omega v dx - \frac{\alpha}{2} \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] (v(x') - v(x)) dx' dx,
$$
as $h \to 0$. Since the function $\sin(\cdot)$ is odd, we find that

$$
- \frac{\alpha}{2} \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] (v(x') - v(x)) dx' dx
$$

$$
= - \frac{\alpha}{2} \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] v(x') dx' dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] v(x) dx' dx
$$

$$
= \frac{\alpha}{2} \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] v(x) dx' dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] v(x) dx' dx
$$

$$
= \alpha \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] v(x) dx' dx
$$

Hence,

$$
\frac{\mathcal{E}_1(\theta + h\nu) - \mathcal{E}_1(\theta)}{h} \to \int_0^1 \omega v dx + \alpha \int_0^1 \int_0^1 \sin [\theta(x') - \theta(x)] v(x) dx' dx, \text{ as } h \to 0.
$$

This implies (13). \qed

Throughout this paper, for $\theta \in \mathcal{C}^1[0,1]$ we will denote the phase diameter as

$$
D(\theta) := \max_{x,y \in [0,1]} |\theta(x) - \theta(y)|.
$$

For given constants $\ell > 0$ and $\bar{D} \in (0, \frac{\pi}{2})$, we let

$$
\mathcal{M}_\ell = \left\{ \theta(\cdot) \in \mathcal{C}^1[0,1] \bigg| \int_0^1 \theta(x) dx = 0, \ |\theta'(x)| \leq \ell, \ D(\theta) \leq \bar{D} \right\}.
$$

**Theorem 3.2.** Let $\ell$ and $\ell^*$ be positive constants, and let $\theta^* \in \mathcal{M}_\ell$ be an equilibrium of (8). Then there exist constants $\delta > 0$ and $\beta > 0$ such that

$$
|\mathcal{E}_1(\theta) - \mathcal{E}_1(\theta^*)|^{\frac{1}{p}} \leq \beta \|
abla \mathcal{E}_1(\theta)\|_{L^2([0,1])}
$$

for any $\theta \in \mathcal{M}_\ell$ with $\|\theta - \theta^*\|_{C^0[0,1]} < \delta$.

**Proof.** \bullet Step 1. Estimate of $\mathcal{E}_1(\theta) - \mathcal{E}_1(\theta^*)$.

Let’s set

$$
f(x) = \theta(x) - \theta^*(x), \quad \text{and} \quad \eta = \max_{x,x' \in [0,1]} |f(x) - f(x')| \geq 0.
$$

It is obvious that for $\theta \in \mathcal{M}_\ell$ with $\|\theta - \theta^*\|_{C^0[0,1]} < \delta$, we have $\eta \leq 2\delta$. Since $\theta^*$ is an equilibrium of (7), we have

$$
\omega(x) = -\alpha \int_0^1 \sin [\theta^*(x') - \theta^*(x)] dx'.
$$
Then we see that
\[ E_1(\theta) - E_1(\theta^*) \]
\[ = \int_0^1 \omega(x)[\theta(x) - \theta^*(x)]dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \cos[\theta(x') - \theta(x)] - \cos[\theta^*(x') - \theta^*(x)] \, dx' \, dx \]
\[ = -\alpha \int_0^1 \int_0^1 \sin[\theta(x') - \theta^*(x)] [\theta(x) - \theta^*(x)] \, dx' \, dx + \frac{\alpha}{2} \int_0^1 \int_0^1 \cos[\theta(x') - \theta(x)] - \cos[\theta^*(x') - \theta^*(x)] \, dx' \, dx \]
\[ =: I_1 + I_2. \]

We estimate the terms \( I_1 \) and \( I_2 \) as follows:
\[ I_1 = -\alpha \int_0^1 \int_0^1 \sin[\theta^*(x') - \theta^*(x)] [\theta(x) - \theta^*(x)] \, dx' \, dx \]
\[ = \frac{\alpha}{2} \int_0^1 \int_0^1 \sin[\theta^*(x') - \theta^*(x)] [\theta(x) - \theta^*(x) - \theta^*(x') + \theta^*(x)] \, dx' \, dx \]
\[ = \frac{\alpha}{2} \int_0^1 \int_0^1 \sin[\theta^*(x') - \theta^*(x)] [f(x') - f(x)] \, dx' \, dx, \]
and for \( I_2 \),
\[ I_2 = -\alpha \int_0^1 \int_0^1 2 \sin \left[ \frac{f(x') - f(x)}{2} \right] \sin \left[ \frac{\theta(x') - \theta(x) + \theta^*(x') - \theta^*(x)}{2} \right] \, dx' \, dx \]
\[ = \frac{\alpha}{2} \int_0^1 \int_0^1 [f(x') - f(x) + O(\eta^3)] \sin \left[ \frac{\theta(x') - \theta(x) + \theta^*(x') - \theta^*(x)}{2} \right] \, dx' \, dx. \]

Thus,
\[ I_1 + I_2 \]
\[ = \frac{\alpha}{2} \int_0^1 \int_0^1 \left( \sin[\theta^*(x') - \theta^*(x)] - \sin \left[ \frac{\theta(x') - \theta(x) + \theta^*(x') - \theta^*(x)}{2} \right] \right) \times \]
\[ \times (f(x') - f(x)) \, dx' \, dx + O(\eta^3) \]
\[ = \frac{\alpha}{2} \int_0^1 \int_0^1 2 \sin \left( \frac{f(x) - f(x')}{4} \right) \cos \left( \frac{\theta(x') - \theta(x) + 3\theta^*(x') - 3\theta^*(x)}{4} \right) \times \]
\[ \times (f(x') - f(x)) \, dx' \, dx + O(\eta^3) \]
\[ = \frac{\alpha}{2} \int_0^1 \int_0^1 -\frac{1}{2} (f(x') - f(x))^2 \cos \left( \frac{\theta(x') - \theta(x) + 3\theta^*(x') - 3\theta^*(x)}{4} \right) \, dx' \, dx + O(\eta^3) \]
Then we find
\[ |E_1(\theta) - E_1(\theta^*)| \leq \beta_1 \eta^2, \quad \text{for some } \beta_1 > 0. \quad (14) \]

• Step 2. Estimate of \( \nabla E_1(\theta) \).

Note that
\[ \|\nabla E_1(\theta)\|_{L^2[0,1]}^2 = \int_0^1 \left( \omega(x) + \alpha \int_0^1 \sin[\theta(x') - \theta(x)] \, dx' \right)^2 \, dx \]
\[ = \alpha^2 \int_0^1 \left( \int_0^1 \sin[\theta(x') - \theta(x)] - \sin[\theta^*(x') - \theta^*(x)] \, dx' \right)^2 \, dx. \quad (15) \]
We denote the integrand by \( q(x) \), that is,

\[
q(x) := \int_0^1 \sin [\theta(x') - \theta(x)] - \sin [\theta^*(x') - \theta^*(x)] \, dx' \\
= 2 \int_0^1 \sin \frac{\theta(x') - \theta(x) - \theta^*(x') + \theta^*(x)}{2} \cos \frac{\theta(x') - \theta(x) + \theta^*(x') - \theta^*(x)}{2} \, dx'.
\]

We notice that

\[
2\sin \frac{\theta(x') - \theta(x) - \theta^*(x') + \theta^*(x)}{2} = f(x') - f(x) - \frac{1}{24}(f(x') - f(x))^3 + o(\eta^3),
\]

and denote

\[
g(x, x') = \cos \frac{\theta(x') - \theta(x) + \theta^*(x') - \theta^*(x)}{2}.
\]

Then we find

\[
q(x) : = \int_0^1 \left( f(x') - f(x) - \frac{1}{24}(f(x') - f(x))^3 + o(\eta^3) \right) g(x, x') \, dx',
\]

where \( g(x, x') \geq \cos \bar{D} > 0, \forall x, \bar{x} \in [0, 1] \). We now set, for \( \theta \neq \theta^* \),

\[
f(\bar{x}) = \max_{x \in [0,1]} f(x) > 0 \quad \text{and} \quad f(\bar{x}) = \min_{x \in [0,1]} f(x) < 0.
\]

Then \( \eta = f(\bar{x}) - f(\bar{x}) \). Note that (8) implies \( \int_0^1 f(x') \, dx' = 0 \), so we can derive that

\[
q(\bar{x}) \geq \cos \bar{D} \int_0^1 [f(x') - f(\bar{x})] \, dx' - c_1 \eta^3 = -\cos \bar{D} f(\bar{x}) - c_1 \eta^3,
\]

and

\[
-q(\bar{x}) \geq \cos \bar{D} \int_0^1 [f(\bar{x}) - f(x')] \, dx' - c_2 \eta^3 = \cos \bar{D} f(\bar{x}) - c_2 \eta^3.
\]

Since

\[
\max \{q^2(\bar{x}), q^2(\bar{x})\} \geq \frac{1}{4} (q(\bar{x}) - q(\bar{x}))^2,
\]

there exists a constant \( \beta_2 > 0 \) such that

\[
\max \{q^2(\bar{x}), q^2(\bar{x})\} \geq \beta_2 \eta^2.
\]

Without loss of any generality, let’s say \( q^2(\bar{x}) \geq \beta_2 \eta^2 \). In order to find a lower bound for \( \|\nabla \mathcal{E}_1(\theta)\|_{L^2([0,1])}^2 \), we consider the \( x \)-derivative of \( q(x) \) and find that

\[
q'(x) = -\theta'(x) \int_0^1 \cos [\theta(x') - \theta(x)] \, dx' + \theta^*(x) \int_0^1 \cos [\theta^*(x') - \theta^*(x)] \, dx'.
\]

This implies that \( |q'(x)| \leq |\theta'(x)| + |\theta^*(x)| \leq \ell + \ell^* \), and

\[
||q(x) - q(\bar{x})|| \leq |q(x) - q(\bar{x})| = \int_{\bar{x}}^x q'(y) \, dy \leq (\ell + \ell^*) |x - \bar{x}|,
\]

which immediately implies

\[
|q(x)| \geq |q(\bar{x}) - (\ell + \ell^*) |x - \bar{x}|.
\]
If $|x - \bar{x}| \leq \frac{|q(x)|}{\sqrt{2(\ell + \ell^*)}}$, we have $|q(x)| \geq \frac{|q(x)|}{2}$ and $q^2(x) \geq \frac{q^2(x)}{4}$. Then we recall (15) to see that

$$
\|\nabla \mathcal{E}_1(\theta)\|_{L^2[0, 1]}^2 = \alpha^2 \int_0^1 q^2(x)dx \geq \alpha^2 \int_{[0, 1]\cap\{x:|x-\overline{x}|\leq \frac{|q(x)|}{\sqrt{2(\ell + \ell^*)}}\}} q^2(x)dx
$$

$$
\geq \frac{\alpha^2}{8(\ell + \ell^*)}|q(x)|^3 \geq \frac{\alpha^2 \beta_2 \sqrt{2\pi}}{8(\ell + \ell^*)} \eta^3.
$$

Finally, we combine Step 1 and Step 2, i.e., the relations (14) and (16), to obtain the desired inequality.

**Remark 2.** Unlike the usual Lojasiewicz inequality in either finite or infinite dimensions, for the Lojasiewicz inequality in Theorem 3.2, we need to confine the argument $\theta$ in a relatively compact subset, i.e., $\mathcal{M}_\ell$. The reason for this lies in the case that the equation (8) and potential (6) are nonlocal. The rationality for using such a subset lies in the fact that the trajectories of (2) do enter such a set and stay therein when suitable parameters are provided (see Section 5).

Next, we show that the equilibrium $\theta^*$ is an extremal point of the functional $\mathcal{E}_1$.

**Proposition 3.3.** In Theorem 3.2, the equilibrium $\theta^*$ is a local maximal point of $\mathcal{E}_1$ if $\delta > 0$ is chosen sufficiently small.

**Proof.** By the Step 1 in the proof of Theorem 3.2, there exists a constant $c > 0$ such that

$$
\mathcal{E}_1(\theta^*) - \mathcal{E}_1(\theta) \geq \frac{\alpha}{4} \int_0^1 \int_0^1 \frac{(f(x') - f(x))^2}{4} \cos \frac{\theta(x') - \theta(x) + 3\theta^*(x') - 3\theta^*(x)}{4} dx'dx
$$

$$
- \frac{\alpha}{4} c \int_0^1 \int_0^1 |f(x') - f(x)|^3 dx'dx
$$

$$
\geq \frac{\alpha}{4} \cos \bar{D} \int_0^1 \int_0^1 (f(x') - f(x))^2 dx'dx - \frac{\alpha}{4} c \eta \int_0^1 \int_0^1 (f(x') - f(x))^2 dx'dx.
$$

We then choose sufficiently small $\delta$ with $\delta < \frac{\cos \bar{D}}{2c}$ in Theorem 3.2 such that $\eta < \frac{\cos \bar{D}}{c}$, then

$$
\mathcal{E}_1(\theta^*) - \mathcal{E}_1(\theta) \geq \frac{\alpha}{4} \left( \cos \bar{D} - c\eta \right) \int_0^1 \int_0^1 (f(x') - f(x))^2 dx'dx \geq 0.
$$

Therefore, $\theta^*$ is a local maximal point for the functional $\mathcal{E}_1$.

**4. Convergence and stability of generalized gradient systems.** In this section, we present two results for the asymptotic property of an abstract differential equation based on Lojasiewicz inequality.

Let $\mathcal{V}$ be a Banach space and $\mathcal{H}$ be a Hilbert space such that $\mathcal{V}$ is continuously embedded into $\mathcal{H}$ (we write $\mathcal{V} \hookrightarrow \mathcal{H}$). Let $\mathcal{M}$ be a relatively compact subset in $\mathcal{V}$. We consider the following differential equation in $\mathcal{V}$:

$$
\dot{u} = -f(u), \quad u(0) = u_0, \quad (17)
$$

where $f : \mathcal{V} \rightarrow \mathcal{V}$ is locally Lipschitz continuous. Let $\mathcal{A} = \{u \in \mathcal{V} | f(u) = 0\}$ and we denote the $\omega$-limit set of $u_0$ by $\Omega(u_0)$, i.e., $\Omega(u_0)$ consists of all accumulation points in $\mathcal{V}$ of the trajectory $\{u(t)\}_{t \geq 0}$ initiated at $u_0$. 

4.1. Convergence.

**Theorem 4.1.** Assume that there exists a functional \( \mathcal{E} \in C^1(\mathcal{H}, \mathbb{R}) \) such that:

(i) \( \mathcal{A} = \{ u \in \mathcal{V} \mid \nabla \mathcal{E}(u) = 0 \} \);

(ii) there exists a constant \( C_1 > 0 \) such that

\[
(f(u), \nabla \mathcal{E}(u)) \geq C_1\|f(u)\|_\mathcal{H}\|\nabla \mathcal{E}(u)\|_\mathcal{H}, \quad \forall u \in \mathcal{V}.
\]

Let \( u : \mathbb{R}^+ \to \mathcal{V} \) be a solution to (17) with \( u(t) \in \mathcal{M} \) for all \( t \in \mathbb{R}^+ \). Assume that for any \( v \in \Omega(u_0) \), there exists a neighborhood \( N_v(v) \) of \( v \) in \( \mathcal{V} \), and two constants \( \rho \in (0, \frac{1}{2}] \) and \( C > 0 \), such that

\[
|\mathcal{E}(u) - \mathcal{E}(v)|^{1-\rho} \leq C\|\nabla \mathcal{E}(u)\|_\mathcal{H}, \quad \forall u \in N_v(v) \cap \mathcal{M}.
\]

Then, there exists \( u_0 \in \mathcal{A} \) such that

\[
\lim_{t \to \infty} \|u(t) - u_0\|_\mathcal{V} = 0.
\]

**Remark 3.** The abstract differential equation (17) which admits a functional \( \mathcal{E} \) satisfying (i) and (ii) is referred to as a (generalized) quasi-gradient system with potential \( \mathcal{E} \).

**Proof of Theorem 4.1.** The set \( \{ u(t) \mid t \in \mathbb{R}^+ \} \) is relatively compact in \( \mathcal{V} \), so there exists a sequence \( \{ t_n \} \) with \( t_n \to +\infty \) such that \( u(t_n) \to u_0 \) in \( \mathcal{V} \) for some \( u_0 \in \Omega(u_0) \).

\[
\frac{d}{dt} \mathcal{E}(u(t)) = (\nabla \mathcal{E}(u(t)), \dot{u}(t)) = -(\nabla \mathcal{E}(u(t)), f(u(t))) \leq -C_1\|\nabla \mathcal{E}(u(t))\|_\mathcal{H}\|f(u(t))\|_\mathcal{H}.
\]

(18)

So \( \mathcal{E}(u(t)) \) is monotonically decreasing. On the other hand, it is bounded, so we know that \( \mathcal{E}(u(t)) \) converges as \( t \to \infty \). Without loss of any generality, we assume \( \mathcal{E}(u(t)) \geq 0 \) and \( \lim_{t \to +\infty} \mathcal{E}(u(t)) = 0 \) (then \( \mathcal{E}(u_0) = 0 \)). If \( \mathcal{E}(u(t_1)) = 0 \) for some \( t_1 \), we must have \( \mathcal{E}(u(t)) = 0 \) for all \( t \geq t_1 \). Then (18) implies that \( u(t_1) \in \mathcal{A} \) and \( u(t) = u(t_1) \) for all \( t \geq t_1 \). This proves the convergence. In the following, we assume that \( \mathcal{E}(u(t)) > 0 \) for all \( t \in \mathbb{R}^+ \). We first recall that the gradient inequality holds at \( u_0 \), i.e.,

\[
|\mathcal{E}(u) - \mathcal{E}(u_0)|^{1-\rho} = |\mathcal{E}(u)|^{1-\rho} \leq C\|\nabla \mathcal{E}(u)\|_\mathcal{H}, \quad \forall u \in \mathcal{M} : \|u - u_0\|_\mathcal{V} < \sigma_*, \quad (19)
\]

for some constants \( \rho_0 \in (0, \frac{1}{2}] \) and \( \sigma_0 > 0 \). We can choose \( t_N \) such that \( \|u(t_N) - u_0\|_\mathcal{V} < \sigma_* \) and let

\[
\mathcal{T} = \{ t \in (t_N, +\infty) \mid \|u(t) - u_0\|_\mathcal{V} < \sigma_* \}.
\]

Then \( \mathcal{T} \neq \emptyset \) is open. We now claim that:

\[
\exists t_* > t_N \text{ such that: } \|u(t) - u_0\|_\mathcal{V} < \sigma_* \text{, } \forall t > t_*.
\]

(20)

**Proof of the Claim (20).** If the open set \( \mathcal{T} \) consists of finite number of open intervals, then as a result of \( t_n \in \mathcal{T} \) and \( t_n \to +\infty \), there must be an interval of the form \( (t_*, +\infty) \) such that \( (t_*, +\infty) \subset \mathcal{T} \). So, (20) is obviously true. In the following, we assume that the open set \( \mathcal{T} \) consists of infinitely many open intervals, and none of them has the form of \( (t_*, +\infty) \). In this case, we can find a sequence of open intervals, say \( (\alpha_k, \beta_k) \), and a subsequence \( \{ t_{n_k} \} \) of \( \{ t_n \} \) such that \( t_{n_k} \in (\alpha_k, \beta_k), \quad (k = 1, 2, \ldots) \). Let

\[
A = \{ u(\alpha_k) - u_0, u(\beta_k) - u_0 \mid k = 1, 2, \ldots \}.
\]
then

\[ \|a\|_V = \sigma_*, \quad \forall a \in A. \quad (21) \]

Since \( \{u(t) \mid t \in [0, +\infty)\} \subset M \) is relatively compact in \( V \), we see that \( A \) is relatively compact in \( V \) as well. Denote

\[ \sigma_1 = \inf_{a \in A} \|a\|_H. \quad (22) \]

Next we show that \( \sigma_1 > 0 \). Otherwise, there must exist \( \{a_i\} \subset A \) such that \( \|a_i\|_H \to 0 \). As \( A \) is relatively compact in \( V \), we can find a subsequence \( \{a_{i_n}\} \) of \( \{a_i\} \) and \( a_* \in V \) such that \( \|a_{i_n} - a_*\|_V \to 0 \). Note that \( \|a_{i_n}\|_V = \sigma_* \), so \( \|a_*\|_V = \sigma_* \). On the other hand, the relation \( \|a_{i_n} - a_*\|_H \to 0 \) implies \( \|a_*\|_H = 0 \), and hence \( a_* = 0 \). This contradicts \( (21) \). This proves that \( \sigma_1 > 0 \). We now choose a \( t_{n_{k_0}} \in (\alpha_{k_0}, \beta_{k_0}) \) such that

\[ \|u(t_{n_{k_0}}) - u_*\|_H < \frac{\sigma_1}{3}, \quad \|\mathcal{E}(u(t_{n_{k_0}}))\|^{\nu_*}_* < \frac{\rho_1 C_1}{3C} \sigma_1. \quad (23) \]

Obviously, we have \( \|u(t) - u_*\|_V < \sigma_* \) for all \( t \in (t_{n_{k_0}}, \beta_{k_0}) \). By assumption (ii) and relation \( (19) \) we can derive that

\[ \begin{align*}
- \frac{d}{dt} \mathcal{E}(u(t))^{\nu_*} &= \rho_* [\mathcal{E}(u(t))]^{\nu_*-1} \langle \nabla \mathcal{E}(u(t)), f(u(t)) \rangle \\
&\geq \rho_* C_1 [\mathcal{E}(u(t))]^{\nu_*-1} \|f(u(t))\|_H \|\nabla \mathcal{E}(u(t))\|_H \\
&\geq \frac{\rho_* C_1}{C} \|f(u(t))\|_H.
\end{align*} \quad (24) \]

So, we have

\[ [\mathcal{E}(u(t_{n_{k_0}}))]^{\nu_*} - [\mathcal{E}(u(\beta_{k_0}))]^{\nu_*} \geq \frac{\rho_* C_1}{C} \int_{t_{n_{k_0}}}^{\beta_{k_0}} \|f(u(t))\|_H dt, \]

which, together with \( (23) \), implies that

\[ \int_{t_{n_{k_0}}}^{\beta_{k_0}} \|f(u(t))\|_H dt < \frac{\sigma_1}{3}. \]

We combine this relation with \( (23) \) to derive

\[ \|u(\beta_{k_0}) - u_*\|_H \leq \|u(\beta_{k_0}) - u(t_{n_{k_0}})\|_H + \|u(t_{n_{k_0}}) - u_*\|_H \]

\[ \leq \int_{t_{n_{k_0}}}^{\beta_{k_0}} \|f(u(t))\|_H dt + \frac{\sigma_1}{3} < \frac{2\sigma_1}{3}. \]

However, the definition of \( \sigma_1 \) in \( (22) \) implies

\[ \|u(\beta_{k_0}) - u_*\|_H \geq \sigma_1, \]

which leads to a contradiction. The claim \( (20) \) is proved. \( \square \)

We recall the relation \( (24) \) to find that

\[ \int_{t_*}^{+\infty} \|f(u(t))\|_H dt < \frac{C}{\rho_* C_1} \mathcal{E}(u_*) < +\infty. \quad (25) \]

By the Cauchy principle for convergence, we have

\[ \lim_{t \to +\infty} \|u(t) - u_*\|_H = 0. \quad (26) \]

Next we show that \( \lim_{t \to +\infty} \|u(t) - u_*\|_V = 0 \). Suppose not, i.e.,

\[ \limsup_{t \to +\infty} \|u(t) - u_*\|_V > 0, \]
then there exists a sequence \( \{s_n\} \) with \( s_n \to +\infty \) such that
\[
\lim_{n \to +\infty} \|u(s_n) - u_*\|_V > 0.
\]
As \( \{u(s_n)\} \) is relatively compact, without loss of generality we can assume \( u(s_n) \to v_* \) in \( V \) for some \( v_* \). Then \( \|u_* - v_*\|_V > 0 \). On the other hand, \( u(s_n) \to v_* \) in \( H \), while (26) implies that \( u(s_n) \to u_* \) in \( H \), so we have \( u_* = v_* \). This is a contradiction. Therefore, we have
\[
\lim_{t \to +\infty} \|u(t) - u_*\|_V = 0.
\]

**Corollary 4.2.** Assume that \( f(u) = \nabla E(u) \) in \( H \) for \( u \in V \hookrightarrow H \).

(i) If \( \rho_* = \frac{1}{2} \) in (19), then \( u(t) \to u_* \) in \( H \) exponentially fast. In other words, there exist positive constants \( C, \lambda \) and \( T \) such that
\[
\|u(t) - u_*\|_H \leq Ce^{-\lambda t}, \quad t \geq T.
\]

(ii) If \( \rho_* < \frac{1}{2} \) in (19), then \( u(t) \to u_* \) in \( H \) algebraically slow. More precisely, there exist positive constants \( C \) and \( T \) such that
\[
\|u(t) - u_*\|_H \leq Ct^{-\frac{\rho_*}{1-\rho_*}}, \quad t \geq T.
\]

**Proof.** We recall (4.1) and take into account of Claim (20) to see that for all \( t > t_* \),
\[
\int_t^{t+\infty} \|f(u(s))\|_Hds < \frac{C}{\rho_* C_1} E^{\rho_*}(u(t)).
\]
Then, we invoke (19) and the assumption that \( f(u) = \nabla E(u) \) in \( H \), to see that
\[
\int_t^{t+\infty} \|\dot{u}(s)\|_Hds = \int_t^{t+\infty} \|f(u(s))\|_Hds < \frac{C}{\rho_* C_1} \|\nabla E(u(t))\|_H^{\frac{\rho_*}{1-\rho_*}} = \frac{C}{\rho_* C_1} \|\dot{u}(t)\|_H^{\frac{\rho_*}{1-\rho_*}}.
\]
Let
\[
\mu(t) = \int_t^{t+\infty} \|\dot{u}(s)\|_Hds.
\]
Then
\[
\|u(t) - u_*\|_H \leq \mu(t).
\]
Since \( \dot{\mu}(t) = -\|\dot{u}(t)\|_H \), we use (27) to find that
\[
\mu(t) < \frac{C}{\rho_* C_1} \left[ -\dot{\mu}(t) \right]^{\frac{1}{\rho_*}}.
\]
This implies that \( \mu(t) \) satisfies that following differential inequality:
\[
\dot{\mu}(t) < -L \mu(t)^{\frac{1}{\rho_*}}, \quad \forall t > t_*,
\]
for some constant \( L > 0 \). Consider the differential equation
\[
\dot{y}(t) = -Ly(t)^{\frac{1}{\rho_*}}, \quad t \geq t_*,
\]
\[
y(t_*) = \mu(t_*).
\]
By the principle of comparison, we have \( \mu(t) \leq y(t), \forall t > t_* \). Then the desired estimates in (i) and (ii) follows from (28) and solving the differential equation (29).
4.2. Stability. We assume that the differential equation (17) admits a unique global solution on \([0, +\infty)\). We will consider the stability of equilibrium of (17) in the following sense.

**Definition 4.3.** We say the equilibrium \(u_* \in \mathcal{A}\) is \(\mathcal{V} - \mathcal{H}\) stable with respect to \(\mathcal{M}\), if for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that for all \(u_0 \in \mathcal{M}\) with \(\|u_0 - u_*\|_\mathcal{V} < \delta\) we have

\[
\|u(t) - u_*\|_\mathcal{H} < \varepsilon, \quad \forall t \geq 0,
\]

where \(u(t)\) is the solution to (17) with initial value \(u_0\).

**Theorem 4.4.** Let \(u_* \in \mathcal{A}\). Assume that there exists a functional \(\mathcal{E} \in C^1(\mathcal{H}, \mathbb{R})\) such that:

(i) \(\nabla \mathcal{E}(u_*) = 0\);

(i) there exists a neighborhood \(\mathcal{N}_\mathcal{V}(u_*)\) of \(u_*\), and constants \(\rho \in (0, \frac{1}{2}]\) and \(C_1, C_2 > 0\), such that

\[
\langle f(u), \nabla \mathcal{E}(u) \rangle \geq C_1\|f(u)\|_\mathcal{H}\|\nabla \mathcal{E}(u)\|_\mathcal{H}, \quad \forall u \in \mathcal{V},
\]

\[
|\mathcal{E}(u) - \mathcal{E}(u_*)|^{1-\rho} \leq C_2\|\nabla \mathcal{E}(u)\|_\mathcal{H}, \quad \forall u \in \mathcal{N}_\mathcal{V}(u_*) \cap \mathcal{M}.
\]

(i) the relatively compact set \(\mathcal{M} \subset \mathcal{V}\) is positively invariant for (17), and \(u_*\) is a local minimal point of \(\mathcal{E}(\cdot)\) on \(\mathcal{N}_\mathcal{V}(u_*) \cap \mathcal{M}\).

Then, \(u_*\) is \(\mathcal{V} - \mathcal{H}\) stable with respect to \(\mathcal{M}\).

**Proof.** Without loss of any generality, we assume \(\mathcal{E}(u_*) = 0\). By the hypothesis (ii) and (iii), there exists a constant \(\sigma > 0\) such that for all \(u \in \mathcal{M}\) with \(\|u - u_*\|_\mathcal{V} < \sigma\) we have

\[
\mathcal{E}(u) \geq 0, \quad |\mathcal{E}(u)|^{1-\rho} \leq C_2\|\nabla \mathcal{E}(u)\|_\mathcal{H}.
\]

(31)

We may assume \(\|u\|_\mathcal{H} \leq \|u\|_\mathcal{V}\) for all \(u \in \mathcal{V}\). Let \(S_\sigma = \{u \in \mathcal{M} \mid \|u - u_*\|_\mathcal{V} = \sigma\}\), and let \(r_\sigma = \inf_{u \in S_\sigma} \|u - u_*\|_\mathcal{H}\). As \(S_\sigma\) is relatively compact, we have \(r_\sigma > 0\). By the continuity of \(\mathcal{E}(\cdot)\) and \(\mathcal{E}(u_*) = 0\), for any \(\varepsilon \in (0, r_\sigma)\), there exists \(\delta \in (0, \frac{\varepsilon}{2}) \cap (0, \sigma)\) such that

\[
\|u_0 - u_*\|_\mathcal{V} < \delta \implies |\mathcal{E}(u_0)|^{\rho} < \frac{\rho C_1}{2C_2} \varepsilon.
\]

For initial data \(u_0 \in \mathcal{M}\), we have \(u(t) \in \mathcal{M}\). Now, we let \(u_0 \in \mathcal{M}\) with \(\|u_0 - u_*\|_\mathcal{V} < \delta\), and let

\[
\mathcal{T} = \{t \mid \|u(s) - u_*\|_\mathcal{V} < \sigma, \quad \forall s \in [0, t)\}.
\]

By the continuity we see that \(\mathcal{T}\) is well-defined. Next we show that \(\sup \mathcal{T} = +\infty\). Assume the opposite, say \(\sup \mathcal{T} = t_1 < +\infty\), then we should have

\[
\|u(t_1) - u_*\|_\mathcal{V} = \sigma, \quad \text{and} \quad \|u(t) - u_*\|_\mathcal{V} < \sigma, \quad \forall t \in [0, t_1).
\]

(32)

By (17), (30) and (31), for \(t \in (0, t_1)\) we have

\[
-\frac{d}{dt}[\mathcal{E}(u(t))]^{\rho} = \rho[\mathcal{E}(u(t))]^{\rho-1}\langle \nabla \mathcal{E}(u(t)), f(u(t)) \rangle
\]

\[\geq \frac{\rho C_1}{C_2}\|f(u(t))\|_\mathcal{H}.
\]

We integrate this inequality to find

\[
|\mathcal{E}(u_0)|^{\rho} - |\mathcal{E}(u(t))|^{\rho} \geq \frac{\rho C_1}{C_2} \int_0^t \|f(u(s))\|_\mathcal{H} ds, \quad t \in (0, t_1),
\]
and hence,
\[
\int_0^t \|f(u(s))\|_H ds \leq \frac{C_2}{\rho C_1} \|\mathcal{E}(u_0)\|^p < \frac{\varepsilon}{2}, \quad t \in (0, t_1).
\]
Then we have
\[
\|u(t) - u_*\|_H \leq \|u_0 - u_*\|_H + \|u(t) - u_0\|_H
\]
\[
\leq \|u_0 - u_*\|_H + \int_0^t \|f(u(s))\|_H ds
\]
\[
\leq \|u_0 - u_*\|_H + \varepsilon, \quad t \in (0, t_1).
\]
Then we have \(\|u(t_1) - u_*\|_H \leq \varepsilon\). However, the relation (32) implies that \(\|u(t_1) - u_*\|_H \geq r_\sigma > \varepsilon\). This is a contradiction. So we have sup \(T = +\infty\), and (32) is true for \(t \in (0, +\infty)\). Now we can use the above argument on \((0, +\infty)\) to see that \(\|u(t) - u_*\|_H < \varepsilon, \forall t \geq 0\).

5. Convergence and stability of the Kuramoto model. In this section we will apply Theorems 4.1, 4.4 and 3.2 to prove the convergence and stability of CKM (8). In order to apply the above theorems for this model, we should find some framework to guarantee that the solution to (8) is confined in \(\mathcal{M}_{t_0}\) for some constant \(t_0 > 0\), i.e., \(\theta(\cdot, t) \in \mathcal{M}_{t_0}\). In order to do this we first present several lemmas as a prior estimates.

5.1. A prior estimates.

**Lemma 5.1.** Let \(f : [0, 1] \times [0, +\infty) \to \mathbb{R}\) be a function such that \(f(x, \cdot) \in C^1[0, +\infty)\) for each \(x \in [0, 1]\), and \(f(\cdot, t) \in C[0, 1]\) for each \(t \in [0, +\infty)\). Assume there exists a constant \(\ell > 0\) such that
\[
\left| \frac{\partial f(x, t)}{\partial t} \right| \leq \ell, \quad \forall (x, t) \in [0, 1] \times [0, +\infty).
\]

Let
\[
g_1(t) := \max_{x \in [0, 1]} f(x, t), \quad g_2(t) := \min_{x \in [0, 1]} f(x, t),
\]
then \(g_1(t), g_2(t)\) are locally absolutely continuous, and
\[
g_1'(t) = \max_{x \in M(t)} \frac{\partial f(x, t)}{\partial t}, \quad g_2'(t) = \min_{x \in m(t)} \frac{\partial f(x, t)}{\partial t}, \quad a.e.
\]

where \(M(t) = \{x \in [0, 1] \mid g_1(t) = f(x, t)\}\) and \(m(t) = \{x \in [0, 1] \mid g_2(t) = f(x, t)\}\).

**Proof.** We give a proof only for \(g_1(t)\), and it is similar for \(g_2(t)\).

(1) For any \(t \in [0, +\infty)\), let’s use \(\bar{x}_t\) to denote an element in \(M(t)\). Let \(a, b \in [0, +\infty)\). If \(g_1(b) \geq g_1(a)\), then we have
\[
g_1(b) - g_1(a) = f(\bar{x}_b, b) - f(\bar{x}_a, a) \leq f(\bar{x}_b, b) - f(\bar{x}_b, a) = \int_a^b \frac{\partial f}{\partial t}(\bar{x}_b, t) dt.
\]
If \(g_1(b) \leq g_1(a)\), similarly we have
\[
g_1(a) - g_1(b) = f(\bar{x}_a, a) - f(\bar{x}_b, b) \leq f(\bar{x}_a, a) - f(\bar{x}_a, b) = -\int_a^b \frac{\partial f}{\partial t}(\bar{x}_a, t) dt.
\]
In both cases, we could derive that \(\|g_1(a) - g_1(b)\| \leq \ell(b - a)\). Hence, \(g_1(t)\) is absolutely continuous.
(2) Let \( t \in [0, +\infty) \) and \( h > 0 \). We have
\[
\frac{g_1(t) - g_1(t - h)}{h} = \frac{f(\bar{x}, t) - f(\bar{x}, t - h)}{h} \leq \frac{f(\bar{x}, t) - f(x, t - h)}{h}.
\]
Taking the limit as \( h \to 0 \), we find that
\[
g'_1(t) \leq \frac{\partial f}{\partial t}(\bar{x}, t) \leq \max_{x \in M(t)} \frac{\partial f(x, t)}{\partial t}.
\]
On the other hand, for any \( \bar{x} \in M(t) \),
\[
\frac{g_1(t + h) - g_1(t)}{h} = \frac{f(\bar{x}, t + h) - f(\bar{x}, t)}{h} \geq \frac{f(\bar{x}, t) - f(\bar{x}, t)}{h}.
\]
Taking the limit as \( h \to 0 \), we find that
\[
g'_1(t) \geq \frac{\partial f}{\partial t}(\bar{x}, t), \quad \forall \bar{x} \in M(t).
\]
Due to the compactness of \( M(t) \), we have
\[
g'_1(t) = \max_{x \in M(t)} \frac{\partial f}{\partial t}(x, t).
\]

\[
D(\omega) := \max_{x,y \in [0,1]} |\omega(x) - \omega(y)|, \quad \lambda = \max_{x \in [0,1]} |\omega'(x)|.
\]  

The following two lemmas give estimates for the phase diameter \( D(\theta) \) and the derivative \( \frac{\partial f}{\partial t} \) along the solution. For \( \alpha > D(\omega) \), we denote by \( D_1^* \) and \( D_2^* \) the solutions of the equation
\[
\sin x = \frac{D(\omega)}{\alpha}
\]
with \( D_1^* \in \left(0, \frac{\pi}{2}\right) \) and \( D_2^* \in \left(\frac{\pi}{2}, \pi\right) \).

**Lemma 5.2.** Let \( \alpha > D(\omega) \). Assume that \( \theta(x, t) \) is the unique solution to (8) with initial data \( \theta(x, 0) = \theta_0(x) \) satisfying \( D(\theta_0) < D_2^* \). Then we have
(i) \( D(\theta(t)) < D_2^* \) for all \( t \geq 0 \), and \( D(\theta(t)) \) satisfies the following differential inequality:
\[
\frac{d}{dt} D(\theta(t)) \leq D(\omega) - \alpha \sin D(\theta(t)), \quad \text{a.e.} \ t \geq 0;
\]
(ii) For any \( \varepsilon > 0 \) with \( D_1^* + \varepsilon < \frac{\pi}{2} \), there exists a time \( T > 0 \) such that
\[
D(\theta(t)) < D_1^* + \varepsilon, \quad \forall t \geq T.
\]

**Proof.** (i) Let
\[
S = \left\{ s \in \mathbb{R}^+ \mid D(\theta(s)) < D_2^* \text{ for all } t \in (0, s) \right\}, \quad s_1 = \sup S.
\]
Then by continuity we see that \( S \) is nonempty and \( s_1 \) is well defined. We note that \( D(\theta(t)) = \max_{x \in [0,1]} \theta(x, t) - \min_{x \in [0,1]} \theta(x, t) \), and denote \( M_1(t) = \arg \max_{x \in [0,1]} \theta(x, t) \).
t) and \( m_1(t) = \arg\min_{x \in [0,1]} \theta(x, t) \). By Lemma 5.1, we see that \( D(\theta(t)) \) is continuous and locally absolutely continuous. Moreover, we have

\[
\frac{d}{dt} D(\theta(t)) = \max_{x \in M_1(t)} \frac{\partial \theta}{\partial t}(x, t) - \min_{x \in m_1(t)} \frac{\partial \theta}{\partial t}(x, t)
\]

\[
= \omega(x_M) + \alpha \int_0^1 \sin [\theta(x', t) - \theta(x_M, t)] \, dx' - \omega(x_m)
\]

\[
- \alpha \int_0^1 \sin [\theta(x', t) - \theta(x_m, t)] \, dx'
\]

\[
\leq D(\omega) + 2\alpha \int_0^1 \cos \left[ \left( \theta(x', t) - \frac{\theta(x_M, t) + \theta(x_m, t)}{2} \right) \right] \sin \left( \frac{\theta(x_m, t) - \theta(x_M, t)}{2} \right) \, dx'
\]

\[
\leq D(\omega) - 2\alpha \sin D(\theta(t)) \int_0^1 \cos \left( \frac{\theta(x', t)}{2} \right) \, dx'
\]

\[
\leq D(\omega) - \alpha \sin D(\theta(t)), \quad \text{a.e. } t \in [0, s_1],
\]

(37)

where \( x_M \in \arg\max_{x \in M_1(t)} \frac{\partial \theta}{\partial t}(x, t) \) and \( x_m \in \arg\min_{x \in m_1(t)} \frac{\partial \theta}{\partial t}(x, t) \). Note that \( D(\theta(t)) < D_2^* \) for \( t \in [0, s_1] \), and

\[
D(\omega) - \alpha \sin D(\theta(t)) < 0, \quad \text{for } D(\theta(t)) \in (D_1^*, D_2^*),
\]

we see that \( D(\theta(t)) \) will never exceed \( D_2^* \). So \( s_1 = +\infty \). Then, \( D(\theta(t)) < D_2^* \) for all \( t \geq 0 \) and the inequality (35) immediately follows from (37).

(ii) We consider two cases. If \( D(\theta_0) < \frac{\pi}{2} \), we consider the following differential equation

\[
\frac{dy}{dt} = D(\omega) - \alpha \sin y(t), \quad y(0) = D(\theta_0).
\]

(38)

By (35) and the principle of comparison [16], we see that \( D(\theta(t)) \leq y(t) \) for all \( t \geq 0 \). Note that the 1-dimensional flow (38) monotonically converges to the equilibrium \( y_c = D_1^* \); then we can derive the desired result.

If \( D(\theta_0) \geq \frac{\pi}{2} \), let us choose \( \varepsilon > 0 \) with \( D_1^* + \varepsilon < \frac{\pi}{2} \). Assume the opposite, that is, \( D(\theta(t)) \geq D_1^* + \varepsilon \), for all \( t \geq 0 \). Then by (35) we see that \( D(\theta(t)) \in [D_1^* + \varepsilon, D(\theta_0)] \) for all \( t \geq 0 \), which implies that

\[
\frac{d}{dt} D(\theta(t)) \leq D(\omega) - \alpha \min\{\sin (D_1^* + \varepsilon), \sin D(\theta_0)\} := L, \quad \text{a.e. } t \geq 0;
\]

where \( L \) is a constant with \( L < 0 \). This leads to \( D(\theta(t)) \to -\infty \), which contradicts the definition of \( D(\theta(t)) \).

\[
\square
\]

Lemma 5.3. Assume the conditions in Lemma 5.2 hold. Then for any \( \varepsilon > 0 \) with \( D_1^* + \varepsilon < \frac{\pi}{2} \) and any \( \varepsilon > 0 \), there exists \( T_1 > 0 \) such that

\[
\sup_{t \in [T_1, +\infty)} \max_{x \in [0,1]} \left| \frac{\partial \theta}{\partial x}(x, t) \right| \leq \frac{\lambda}{\alpha \cos (D_1^* + \varepsilon)} + \varepsilon,
\]

where \( \lambda \) is defined as in (33).

Proof. Denote \( a(x, t) = \frac{\partial \theta}{\partial x}(x, t) \), then \( a(x, t) \) satisfies (12), or equivalently,

\[
\frac{\partial a}{\partial t}(x, t) = \omega'(x) - \alpha a(x, t) \int_0^1 \cos [\theta(x', t) - \theta(x, t)] \, dx',
\]

\[
a(x, 0) = \theta_0'(x).
\]
Let $x_0 \in [0, 1]$, we obtain the following ODE:

$$
\frac{d}{dt}a(x_0, t) = \omega'(x_0) - \alpha a(x_0, t) \int_0^1 \cos[\theta(x', t) - \theta(x_0, t)] dx',
$$

$$
a(x_0, 0) = \theta_0'(x_0).
$$

Then we derive that

$$
\frac{d}{dt}(|a(x_0, t)|) = \text{sgn}(a(x_0, t)) \frac{d}{dt}a(x_0, t)
$$

\[= \omega'(x_0)\text{sgn}(a(x_0, t)) - \alpha a(x_0, t)\text{sgn}(a(x_0, t)) \int_0^1 \cos[\theta(x', t) - \theta(x_0, t)] dx'. \]

Now by Lemma 5.2, for any $\varepsilon > 0$ with $D_*^1 + \varepsilon < \frac{\pi}{2}$, there exists a time $T > 0$ such that

$$
D(\theta(t)) < D_*^1 + \varepsilon, \quad \forall t \geq T.
$$

So we derive from (39) that

$$
\frac{d}{dt}(|a(x_0, t)|) \leq \lambda - \alpha \cos(D_*^1 + \varepsilon) |a(x_0, t)|, \quad \forall t \geq T.
$$

Here, we used the fact $|\theta(x', t) - \theta(x_0, t)| \leq D(\theta(t))$. For comparison we consider

$$
\frac{dy}{dt} = \lambda - \alpha \cos(D_*^1 + \varepsilon) y(t), \quad t \geq T, \quad y(T) = |a(x_0, T)|.
$$

The solution of (40) converges to its equilibrium $y_e = \frac{\lambda}{\alpha \cos(D_*^1 + \varepsilon)}$. So for $\varepsilon > 0$, there exist a time $T_1 > T$ such that

$$
y(t) \in (y_e - \varepsilon, y_e + \varepsilon), \quad t \geq T_1.
$$

The principle of comparison immediately implies that

$$|a(x_0, t)| \leq y_e + \varepsilon = \frac{\lambda}{\alpha \cos(D_*^1 + \varepsilon)} + \varepsilon, \quad t \geq T_1.
$$

Then the desired estimate follows.

Next we consider the existence and regularity of the equilibrium of (8). For $\omega(\cdot) \in C^1[0, 1]$, $\omega \neq 0$ with $\int_0^1 \omega(x)dx = 0$, we denote

$$
\tilde{\omega} = \max_{x \in [0, 1]} |\omega(x)|, \quad \Delta(x) = \frac{\omega(x)}{\tilde{\omega}} \in [-1, 1], \quad r_* = \int_0^1 \sqrt{1 - \Delta^2(x)}dx > 0.
$$

(41)

**Lemma 5.4.** Assume that

$$
\alpha > \frac{\tilde{\omega}}{r_*}.
$$

(42)

Then the system (8) admits an equilibrium in $C^1[0, 1]$.

**Proof.** As an equilibrium of (8), $\theta(\cdot)$ should satisfy

$$
\omega(x) = \alpha \int_0^1 \sin[\theta(x) - \theta(x')]dx'.
$$
Then we have
\[ r \Delta(x) = \int_0^1 \sin [\theta(x) - \theta(x')] \, dx' \]
\[ = \int_0^1 \sin [(\theta(x) + \beta) - (\theta(x') + \beta)] \, dx' \]
\[ = \sin [\theta(x) + \beta] \int_0^1 \cos [\theta(x') + \beta] \, dx' - \cos [\theta(x) + \beta] \int_0^1 \sin [\theta(x') + \beta] \, dx', \]
where \( r = \frac{\omega}{\alpha} \). Note that the constant \( \beta \) can be chosen such that \( \int_0^1 \sin [\theta(x') + \beta] \, dx' = 0 \). So we can assume, without loss of generality, that \( \int_0^1 \sin \theta(x') \, dx' = 0 \). Then we see that the equilibrium \( \theta(x) \) satisfies
\[ r \Delta(x) = C \sin \theta(x). \quad (43) \]
Therefore, in order to find an equilibrium \( \theta(x) \) with \( \theta(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), it is equivalent to find the coefficient \( C \) which satisfies
\[ C = \int_0^1 \sqrt{1 - \frac{r^2}{\Delta^2(x)}} \, dx. \quad (44) \]
According to (42), we have \( 0 < r < r_* \). Let
\[ H(C, r) = C - \int_0^1 \sqrt{1 - \frac{r^2}{\Delta^2(x)}} \, dx. \]
Note that \( r_* = \int_0^1 \sqrt{1 - \Delta^2(x)} \, dx \), so we have
\[ H(r_*, r) = r_* - \int_0^1 \sqrt{1 - \frac{r^2}{r_*^2}} \, dx < 0, \quad H(1, r) = 1 - \int_0^1 \sqrt{1 - r^2 \Delta^2(x)} \, dx > 0. \]
Therefore, there exists \( C(r) \in (r_*, 1) \) such that \( H(C(r), r) = 0 \). This means that the equation \( (44) \), and further the equation \( (43) \), admits a solution corresponding to \( C(r) \).

We recall [29, Theorem 1.1] to see that \( (43) \) has a unique solution when \( r \in (0, r_*) \). Accordingly, this means that \( (8) \) has a unique solution (up to phase shift \( \theta \mapsto \theta + c \) with constant \( c \)). Let’s denote the unique solution by \( \phi(x) \), then by (43) we have
\[ \phi(x) = \arcsin^{-1} \frac{\omega(x)}{\alpha C(r)}. \]
According to (42), there exists \( D_c \in (0, \frac{\pi}{2}) \) such that
\[ \frac{|\omega(x)|}{\alpha C(r)} < \frac{|\omega(x)|}{\alpha r_*} < \sin D_c, \]
so we have
\[ \phi(x) \in [-D_c, D_c]. \]
As \( \omega(\cdot) \in C^1[0, 1] \), we have
\[ |\phi'(x)| = \left| \frac{1}{\cos \phi(x)} \frac{\omega'(x)}{\alpha C(r)} \right| \leq \frac{1}{\cos D_c} \cdot \frac{1}{\alpha C(r)} \max_{x \in [0, 1]} |\omega'(x)| \leq \frac{\lambda}{\alpha r_* \cos D_c}. \]
5.2. Convergence and stability. Now we present the main results of this paper. Let \( \mathcal{V} = C[0, 1] \) and \( \mathcal{H} = L^2[0, 1] \), then \( \mathcal{V} \hookrightarrow \mathcal{H} \).

**Theorem 5.5.** Let \( D_0 \in (0, \frac{\pi}{2}) \), and

\[
\alpha > \max \left\{ D(\omega), \frac{\bar{\omega}}{\bar{r}} \right\}.
\]

Assume that the initial data \( \theta_0(\cdot) \in C^1[0, 1] \) satisfies

\[
D(\theta_0) \leq D^*_2.
\]

Then the solution of (8) converges to some equilibrium \( \theta^* \):

\[
\|\theta(\cdot, t) - \theta^*(\cdot)\|_{\mathcal{V}} \to 0.
\]

Here, the terms \( D(\omega), D^*_2, \bar{\omega} \) and \( r_\ast \) are determined as in (33), (34) and (41).

**Proof.** Let \( \varepsilon \) be a fixed constant with \( D^*_1 + \varepsilon < \frac{\pi}{2} \). First of all, by Lemma 5.2 and 5.3, we see that there exists a time slice \( T > 0 \) such that the solution of (8) at \( T \) will enter the set

\[
\mathcal{M}_{\ell_0} = \left\{ \theta \in C^1[0, 1] \mid \int_0^1 \theta(x)dx = 0, |\theta'(x)| \leq \ell_0, D(\theta) \leq D^*_1 + \varepsilon \right\}
\]

with \( \ell_0 := \frac{\lambda}{\alpha \cos(D^*_1 + \varepsilon)} + \varepsilon \) and remains therein. We now regard the system (8) as a flow initialized at \( \theta(x, T) \); of course, \( \mathcal{M}_{\ell_0} \) is invariant for the flow (8). Note that \( \mathcal{M}_{\ell_0} \) is relatively compact in \( \mathcal{V} \). To apply Theorem 4.1, we let

\[
\mathcal{E}(\theta) = -\mathcal{E}_1(\theta) = -\int_0^1 \theta(x)\omega(x)dx - \frac{\alpha}{2} \int_0^1 \int_0^1 \cos[\theta(x') - \theta(x)] dx'dx.
\]

Then the system (8) is a generalized gradient system (17) with potential being \( \mathcal{E} \). We notice that

\[
\frac{d}{dt} \mathcal{E}(\theta(t)) = -\|\nabla \mathcal{E}(\theta(t))\|^2_{\mathcal{H}},
\]

which implies that \( \int_0^{+\infty} \|\nabla \mathcal{E}(\theta(t))\|^2_{\mathcal{H}} dt < +\infty \). Note that \( \theta : \mathbb{R}^+ \to \mathcal{V} \) is uniformly continuous with respect to \( t \in \mathbb{R}^+ \) and \( \nabla \mathcal{E}(\theta) \) is Lipschitz with respect to \( \theta \in \mathcal{V} \), so \( \|\nabla \mathcal{E}(\theta(t))\|^2_{\mathcal{H}} \) is uniformly continuous with respect to \( t \in \mathbb{R}^+ \). Then we have

\[
\lim_{t \to +\infty} \|\nabla \mathcal{E}(\theta(t))\|_{\mathcal{H}} = 0.
\]

We use \( \Omega(\theta_0) \subset \mathcal{V} \) to denote the \( \omega \)-limit set of \( \theta_0 \in \mathcal{V} \). As \( \mathcal{M}_{\ell_0} \) is positively invariant and relatively compact in \( \mathcal{V} \), we see that \( \Omega(\theta_0) \neq \emptyset \) and there exists sequence \( \{t_n\} \) with \( t_n \to +\infty \) such that \( \theta(t_n) \to \theta^* \) in \( \mathcal{V} \) for some \( \theta^* \in \Omega(\theta_0) \). Then we have \( \nabla \mathcal{E}(\theta^*) = 0 \) (in \( L^2[0, 1] \)). Now, Lemma 5.4 implies that \( \theta^* \in C^1[0, 1] \). Hence, \( \nabla \mathcal{E}(\theta^*) \) is continuous and \( \nabla \mathcal{E}(\theta^*) \equiv 0 \). As \( \theta^* \) is the limit of \( \theta(t_n) \) in \( \mathcal{V} \), we find that

\[
\int_0^1 \theta^*(x)dx = 0, \quad D(\theta^*) \leq \bar{D}.
\]

Therefore, \( \theta^* \in \mathcal{M}_{\ell_*} \) for some constant \( \ell_* > 0 \). By Theorem 3.2, the Lojasiewicz inequality holds at \( \theta^* \). We finally apply Theorem 4.1 to see that

\[
\lim_{t \to +\infty} \|\theta(t) - \theta^*\|_{\mathcal{V}} = 0.
\]

\( \square \)
Remark 4. For CKM (8), Lemma 3.1 says that the assumption in Corollary 4.2 holds true, i.e.,
\[ f(u) = \nabla E(u) \text{ in } H \] for \( u \in \mathcal{V} \rightarrow H \). So we can apply this corollary
and the result that \( \rho_* = \frac{1}{4} \) in Theorem 3.2 to see that

\[ \| \theta(t) - \theta^* \|_H \leq C t^{-\frac{1}{2}}, \quad t \geq T, \]

for some positive constants \( C \) and \( T \).

Remark 5. We acknowledge that in [4], the estimate of contractivity for the
measure-valued solutions to kinetic Kuramoto model was considered and an expo-
nential decay rate was obtained. This implies that the convergence is exponentially
fast. In Theorem 5.5 our estimate does not give the convergence rate in the uniform
topology \( C[0,1] \). However, as in Remark 4, we give an estimate for algebraically
slow convergene in \( L^2[0,1] \). According to Corollary 4.2, if we can refine the estimate
in Theorem 3.2 so that the Lojasiewicz exponent is \( \frac{1}{2} \), then we can lift the algebraic
rate to exponential rate. This leaves a problem in the future study.

Theorem 5.6. The equilibrium of (8) in Theorem 5.5 is \( \mathcal{V} - H \) stable with respect
to \( M_{\ell_0} \) in the sense of Definition 4.3.

Proof. We invoke Theorem 4.4 to obtain the desired result. Here we used Proposi-
tion 3.3, which claims that the assumption (iii) in Theorem 4.4 is valid.

6. Conclusion. In this paper, we considered the asymptotic property of classic
solutions of CKM. We proved that for suitable initial data and coupling strength,
the solution of CKM converges to a phase locked state, and the phase locked state
is stable. The analysis is based on the Lojasiewicz inequality approach. To show
the desired result, we proved the Lojasiewicz inequality for CKM, and established
two theorems on convergence and stability of generalized gradient system under a
suitable framework. As a unsolved problem, the relation between the discrete model
(1) and continuum model (2) should be addressed in the future study.

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