COHEN-MACAUŁAY CHORDAL GRAPHS

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ABSTRACT. We classify all Cohen-Macaulay chordal graphs. In particular, it is shown that a chordal graph is Cohen-Macaulay if and only if it is unmixed.

INTRODUCTION

To each finite graph \( G \) with vertex set \([n] = \{1, \ldots, n\}\) and edge set \( E(G) \) one associates the edge ideal \( I(G) \subset K[x_1, \ldots, x_n] \) which is generated by all monomials \( x_i x_j \) such that \( \{i, j\} \in E(G) \). Here \( K \) is an arbitrary field. The graph \( G \) is called Cohen-Macaulay over \( K \), if \( K[x_1, \ldots, x_n]/I(G) \) is a Cohen-Macaulay ring, and is called Cohen-Macaulay if it is Cohen-Macaulay over any field.

Given a field \( K \). The general problem is to classify the graphs which are Cohen-Macaulay over \( K \). In this generality the problem is as hard as to classify all Cohen-Macaulay simplicial complexes, because given a simplicial complex \( \Delta \), one can naturally construct a finite graph \( G \) such that \( G \) is Cohen-Macaulay if and only if \( \Delta \) is Cohen-Macaulay. In fact, if \( P \) is the face poset of \( \Delta \) (the poset consisting of all faces of \( \Delta \), ordered by inclusion), then \( \Delta \) is Cohen-Macaulay if and only if the order complex \( \Delta(P) \) of \( P \) is Cohen-Macaulay. Since the order complex \( \Delta(P) \) is flag, i.e., every minimal non-face is a 2-element subset, it follows that there is a finite graph \( G \) such that \( I(G) \) coincides with the Stanley–Reisner ideal of \( \Delta(P) \).

Thus one cannot expect a general classification theorem. On the other hand, the first positive result was given by Villarreal [4] who determined all Cohen-Macaulay trees. This result has been recently widely generalized in [2] where all bipartite Cohen-Macaulay graphs have been described. It turned out that the Cohen-Macaulay property of a bipartite graph does not depend on the field \( K \).

In this note we classify all Cohen-Macaulay chordal graphs. Again it turns out that for chordal graphs the Cohen-Macaulay property is independent of the field \( K \). Indeed we show that \( G \) is Cohen-Macaulay if and only if the edge ideal \( I(G) \) is height unmixed. One of our tools is Dirac’s theorem [1] in a version as presented in [3].

1. Preliminaries

Let \( G \) be a finite graph on \([n] \) without loops, multiple edges and isolated vertices, and \( E(G) \) its edge set. The graph \( G \) is called chordal if all cycles of length \( \geq 3 \) has a chord.

A stable subset or clique of \( G \) is a subset \( F \) of \([n] \) such that \( \{i, j\} \in E(G) \) for all \( i, j \in F \) with \( i \neq j \). We write \( \Delta(G) \) for the simplicial complex on \([n] \) whose faces are the stable
Let $R$ be a Noetherian ring, $S$.

Lemma 1.2. For the proof of our main theorem we need the following algebraic fact:

Lemma 1.1. Let $G$ be a chordal graph. Then $\Delta(G)$ is a quasi-forest.

We recall the definition of a quasi-forest introduced in [5]: let $\Delta$ be a simplicial complex, and $\mathcal{F}(\Delta)$ the set of its facets. A facet $F \in \mathcal{F}(\Delta)$ is called a leaf, if there exists a facet $G$ (called a branch of $F$) with $G \neq F$ and such that $H \cap F \subset G \cap F$ for all $H \in \mathcal{F}(\Delta)$ with $H \neq F$. We say that $\Delta$ is a quasi-forest, if there exists an order $F_1, \ldots, F_r$ of the facets of $\Delta$ such that for each $i = 1, \ldots, r$, $F_i$ is a leaf of the simplicial complex $\langle F_1, \ldots, F_i \rangle$ (whose facets are $F_1, \ldots, F_i$).

Let $K$ be a field. A graph $G$ is called Cohen-Macaulay over $K$ if the edge ideal $I(G) = \langle \{x_i x_j : \{i, j\} \in E(G)\} \rangle$ of $G$ is a Cohen-Macaulay ideal in $S = K[x_1, \ldots, x_n]$, in other words, if $S/I(G)$ is Cohen-Macaulay.

Suppose $G$ is Cohen-Macaulay over $K$. Then we say $G$ is of type $r$ over $K$, if $r$ is the Cohen-Macaulay type of $S/I(G)$, that is, if $r$ is the minimal number of generators of the canonical module of $S/I(G)$. The Cohen-Macaulay type of a Cohen-Macaulay ring $R$ can also be computed as the socle dimension of the residue class ring of $R$ modulo a maximal regular sequence. The ring $R$ is Gorenstein, if and only if the Cohen-Macaulay type of $R$ is 1. We say that $G$ is Gorenstein over $K$, if $S/I(G)$ is Gorenstein over $K$.

Finally we say that $G$ is Cohen-Macaulay, of type $r$, or Gorenstein, if $G$ has the corresponding property over any field.

The minimal prime ideals of $I(G)$ correspond to the minimal vertex covers of $G$. Recall that a vertex cover of $G$ is a subset $C \subset [n]$ such that $C \cap \{i, j\} \neq \emptyset$ for all $\{i, j\} \in E(G)$. It is called minimal if no proper subset of $C$ is a vertex cover of $G$. If we denote by $\mathcal{C}(G)$ the set of minimal vertex covers, then the set of ideals $\mathcal{I} = \{\langle \{i : i \in C\} : C \in \mathcal{C}(G)\}$ is precisely the set of minimal prime ideals of $I(G)$.

Suppose again that $G$ is Cohen-Macaulay over $K$. Then the ideal $I(G)$ is height unmixed. Thus all minimal vertex covers of $G$ have the same cardinality.

For the proof of our main theorem we need the following algebraic fact:

Lemma 1.2. Let $R$ be a Noetherian ring, $S = R[x_1, \ldots, x_n]$ the polynomial ring over $R$, $k$ an integer with $0 \leq k < n$, and $J$ the ideal $I_1 x_1, \ldots, I_k x_k, \{x_i x_j\}_{1 \leq i < j \leq n} \subset S$, where $I_1, \ldots, I_k$ are ideals in $R$. Then the element $x = \sum_{i=1}^{n} x_i$ is a non-zerodivisor on $S/J$.

Proof. For a subset $T \subset [n]$ we let $L_T$ be the ideal generated by all monomials $x_i x_j$ with $i, j \in T$ and $i < j$, and we set $I_T = \sum_{j \in T} I_j$ and $X_T = \langle \{x_j : j \in T\} \rangle$.

It is easy to see that

$$L_T = \bigcap_{\ell \in T} X_T \setminus \{\ell\}.$$
Hence we get
\[
J = (I_1x_1, \ldots, I_kx_k, L_n) = \bigcap_{T \subset [k]} (I_T, X_{[k] \setminus T}, L_n) \\
= \bigcap_{T \subset [k]} (I_T, X_{[k] \setminus T}, L_n \setminus ((k) \setminus T)) = \bigcap_{T \subset [k], \ell \in [n] \setminus (k) \setminus T} (I_T, X_{[k] \setminus \ell}).
\]

Thus in order to prove that \(x\) is a non-zerodivisor modulo \(J\) it suffices to show that \(x\) is a non-zerodivisor modulo each of the ideals \((I_T, X_{[n] \setminus \ell})\). To see this we first pass to the residue class ring modulo \(I_T\), and hence if we replace \(R\) by \(R/I_T\) it remains to be shown that \(x\) is a non-zerodivisor on \(R[x_1, \ldots, x_n]/(x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n)\). But this is obviously the case.

\[\square\]

2. The classification

**Theorem 2.1.** Let \(K\) be a field, and let \(G\) be a chordal graph on the vertex set \([n]\). Let \(F_1, \ldots, F_m\) be the facets of \(\Delta(G)\) which admit a free vertex. Then the following conditions are equivalent:

(a) \(G\) is Cohen-Macaulay;
(b) \(G\) is Cohen-Macaulay over \(K\);
(c) \(G\) is unmixed;
(d) \([n]\) is the disjoint union of \(F_1, \ldots, F_m\).

**Proof.** (a) \(\Rightarrow\) (b) is trivial.

(b) \(\Rightarrow\) (c): Since any Cohen-Macaulay ring is height unmixed it follows that \(G\) is unmixed.

(c) \(\Rightarrow\) (d): Let \(G\) be an unmixed chordal graph on \([n]\) and \(E(G)\) the set of edges of \(G\). Let \(F_1, \ldots, F_m\) denote the facets of \(\Delta(G)\) with free vertices. Fix a free vertex \(v_i\) of \(F_i\) and set \(W = \{v_1, \ldots, v_m\}\). Suppose that \(B = [n] \setminus (\bigcup_{i=1}^m F_i) \neq \emptyset\) and write \(G|B\) for the induced subgraph of \(G\) on \(B\). Since \(\{v_i, b\} \notin E(G)\) for all \(1 \leq i \leq m\) and for all \(b \in B\), if \(X \subset B\) is a minimal vertex cover of \(G|B\), then \(X \cup (\bigcup_{i=1}^m F_i)\) is a minimal vertex cover of \(G\). In particular \(G|B\) is unmixed. Since the induced subgraph \(G|B\) is again chordal, by working with induction on the number of vertices, it follows that if \(H_1, \ldots, H_s\) are the facets of \(\Delta(G|B)\) with free vertices, then \(B\) is the disjoint union \(B = \bigcup_{j=1}^s H_j\). Let \(v'_j\) be a free vertex of \(H_j\) and set \(W' = \{v'_1, \ldots, v'_s\}\). Since \((\bigcup_{i=1}^m F_i) \setminus W) \cup (B \setminus W')\) is a minimal vertex cover of \(G\) and since \(G\) is unmixed, every minimal vertex cover of \(G\) consists of \(n-(m+s)\) vertices.

We claim that \(F_i \cap F_j = \emptyset\) if \(i \neq j\). In fact, if, say, \(F_1 \cap F_2 \neq \emptyset\) and if \(w \in [n]\) satisfies \(w \in F_i\) for all \(1 \leq i \leq \ell\), where \(\ell \geq 2\), and \(w \notin F_i\) for all \(\ell < i \leq m\), then \(Z = (\bigcup_{i=1}^m F_i) \setminus \{w, v_{\ell+1}, \ldots, v_m\}\) is a minimal vertex cover of the induced subgraph \(G' = G|_{[n] \setminus B}\) on \([n] \setminus B\). Let \(Y\) be a minimal vertex cover of \(G\) with \(Z \subset Y\). Since \(Y \cap B\) is a vertex cover of \(G|B\), one has \(|Y \cap B| \geq |B| - s\). Moreover, \(|Y \cap ([n] \setminus B)| \geq n - |B| - (m - \ell + 1) > n - |B| - m\).

Hence \(|Y| > n-(m+s)\), a contradiction.
Consequently, a subset $Y$ of $[n]$ is a minimal vertex cover of $G$ if and only if $|Y \cap F_i| = |F_i| - 1$ for all $1 \leq i \leq m$ and $|Y \cap H_j| = |H_j| - 1$ for all $1 \leq j \leq s$.

Now, since $\Delta(G|_B)$ is a quasi-forest, one of the facets $H_1, \ldots, H_s$ must be a leaf of $\Delta(G|_B)$. Let, say, $H_1$ be a leaf of $\Delta(G|_B)$. Let $\delta$ and $\delta'$, where $\delta \neq \delta'$, be free vertices of $H_1$ with $\{\delta, a\} \in E(G)$ and $\{\delta', a'\} \in E(G)$, where $a$ and $a'$ belong to $[n] \setminus B$. If $a \neq a'$ and if $\{a, a'\} \in E(G)$, then one has either $\{\delta, a'\} \in E(G)$ or $\{\delta', a\} \in E(G)$, because $G$ is chordal and $\{\delta, \delta'\} \in E(G)$. Hence there exists a subset $A \subset [n] \setminus B$ such that

(i) $\{a, b\} \not\in E(G)$ for all $a, b \in A$ with $a \neq b$,

(ii) for each free vertex $\delta$ of $H_1$, one has $\{\delta, a\} \in E(G)$ for some $a \in A$, and

(iii) for each $a \in A$, one has $\{\delta, a\} \in E(G)$ for some free vertex $\delta$ of $H_1$.

In fact, it is obvious that a subset $A \subset [n] \setminus B$ satisfying (ii) and (iii) exists. If $\{a, a'\} \in E(G)$, $\{\delta, a\} \in E(G)$ and $\{\delta', a'\} \not\in E(G)$ for some $a, a' \in A$ with $a \neq a'$ and for a free vertex $\delta$ of $H_1$, then every free vertex $\delta'$ of $H_1$ with $\{\delta', a'\} \in E(G)$ must satisfy $\{\delta', a\} \in E(G)$. Hence $A \setminus \{a'\}$ satisfies (ii) and (iii). Repeating such the technique yields a subset $A \subset [n] \setminus B$ satisfying (i), (ii) and (iii), as required.

If $s > 1$, then $H_1$ has a branch. Let $w_0 \not\in H_1$ be a vertex belonging to a branch of the leaf $H_1$ of $\Delta(G|_B)$. Thus $\{\xi, w_0\} \in E(G)$ for all nonfree vertices $\xi$ of $H_1$. We claim that either $\{a, w_0\} \not\in E(G)$ for all $a \in A$, or one has $a \in A$ with $\{a, \xi\} \in E(G)$ for every nonfree vertices $\xi$ of $H_1$. To see why this is true, if $\{a, w_0\} \in E(G)$ and $\{\delta, a\} \in E(G)$ for some $a \in A$ and for some free vertex $\delta$ of $H_1$, then one has a cycle $(a, \delta, \xi, w_0)$ of length four for every nonfree vertex $\xi$ of $H_1$. Since $\{\delta, w_0\} \not\in E(G)$, one has $\{a, \xi\} \in E(G)$.

Let $X$ be a minimal vertex cover of $G$ such that $X \subset [n] \setminus (A \cup \{w_0\})$ (resp. $X \subset [n] \setminus A$) if $\{a, w_0\} \not\in E(G)$ for all $a \in A$ (resp. if one has $a \in A$ with $\{a, \xi\} \in E(G)$ for every nonfree vertices $\xi$ of $H_1$). Then, for each vertex $\gamma$ of $H_1$, there is $w \not\in X$ with $\{\gamma, w\} \in E(G)$. Hence $H_1 \subset X$, in contrast to our considerations before. This contradiction guarantees that $B = \emptyset$. Hence $[n]$ is the disjoint union $[n] = \bigcup_{i=1}^{m} F_i$, as required.

Finally suppose that $s = 1$. Then $H_1$ is the only facet of $\Delta(G|_B)$. Then $X = \bigcup_{i=1}^{m} (F_i \setminus v_i)$ is a minimal free vertex cover $G$ with $H_1 \subset X$, a contradiction.

(d) $\Rightarrow$ (c): Let $F_1, \ldots, F_m$ denote the facets of $\Delta(G)$ with free vertices and, for each $1 \leq i \leq m$, write $F_i$ for the set of vertices of $F_i$. Given a minimal vertex cover $X \subset [n]$ of $G$, one has $|X \cap F_i| \geq |F_i| - 1$ for all $i$ since $F_i$ is a clique of $G$. If, however, for some $i$, one has $|X \cap F_i| = |F_i|$, i.e., $F_i \subset X$, then $X \setminus \{v_i\}$ is a vertex cover of $G$ for any free vertex $v_i$ of $F_i$. This contradicts the fact that $X$ is a minimal vertex cover of $G$. Thus $|X \cap F_i| = |F_i| - 1$ for all $i$. Since $[n]$ is the disjoint union $[n] = \bigcup_{i=1}^{m} F_i$, it follows that $|X| = n - m$ and $G$ is unmixed, as desired.

(c) and (d) $\Rightarrow$ (a): We know that $G$ is unmixed. Moreover, if $v_i \in F_i$ is a free vertex, then $[n] \setminus \{v_1, \ldots, v_m\}$ is a minimal vertex cover of $G$. In particular it follows that $\dim S/I(G) = m$.

For $i = 1, \ldots, m$, we set $y_i = \sum_{j \in F_i} x_j$. We will show that $y_1, \ldots, y_m$ is a regular sequence on $S/I(G)$. This then yields that $G$ is Cohen-Macaulay.

Let $F_i = \{i_1, \ldots, i_k\}$, and assume that $i_{t+1}, \ldots, i_k$ are the free vertices of $F_i$. Let $G' \subset G$ be the induced subgraph of $G$ on the vertex set $[n] \setminus \{i_1, \ldots, i_k\}$. Then $I(G) = (I(G'), J_1x_{i_1}, J_2x_{i_2}, \ldots, J_{\ell}x_{i_{\ell}})$, where $J_j = (\{x_r : \{r, i_j\} \in E(G)\})$ for $j = 1, \ldots, \ell$, and where $J = (\{x_{i_r}x_{i_s} : 1 \leq r < s \leq k\})$. 

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Since \( [n] \) is the disjoint union of \( F_1, \ldots, F_m \) it follows that all generators of the ideal \( (I(G'), y_1, \ldots, y_{i-1}) \) belong to \( K[\{x_i\}_{i \in [n] \setminus F}] \). Thus if we set
\[
R = K[\{x_i\}_{i \in [n] \setminus F}]/(I(G'), y_1, \ldots, y_{i-1}),
\]
then
\[
(S/I(G))/(y_1, \ldots, y_{i-1})(S/I(G)) \cong R[x_{i_1}, \ldots, x_{i_k}]/(I_{i_1}x_{i_1}, \ldots, I_{i_k}x_{i_k}, \{x_{i_r}x_{i_s} : 1 \leq r < s \leq k\}),
\]
where for each \( j \), the ideal \( I_j \) is the image of \( J_j \) under the residue class map onto \( R \). Thus Lemma \([1,2]\) implies that \( y_j \) is regular on \( (S/I(G))/(y_1, \ldots, y_{i-1})(S/I(G)) \). \(\square\)

Let \( G \) be an arbitrary graph on the vertex set \([n]\). An independent set of \( G \) is a set \( S \subset [n] \) such that \( \{i, j\} \notin E(G) \) for all \( i, j \in S \). With this notion we can describe the type of a Cohen-Macaulay chordal graph.

**Corollary 2.2.** Let \( G \) be a chordal graph, and let \( F_1, \ldots, F_m \) be the facets of \( \Delta(G) \) which have a free vertex. Let \( i_j \) be a free vertex of \( F_j \) for \( j = 1, \ldots, m \), and let \( G' \) be the induced subgraph of \( G \) on the vertex set \([n] \setminus \{i_1, \ldots, i_m\} \). Then

(a) the type of \( G \) is the number of maximal independent subsets of \( G' \);

(b) \( G \) is Gorenstein, if and only if \( G \) is a disjoint union of edges.

**Proof.** (a) Let \( F \subset [n] \) and \( S = K[x_1, \ldots, x_n] \). We note that if \( J \) is the ideal generated by the set of monomials \( \{x_ix_j : i, j \in F \text{ and } i < j\} \), and \( x = \sum_{i \in F} x_i \), then for any \( i \in F \) one has that
\[
(S/J)/x(S/J) \cong S_i/((\{x_j : j \in F, j \neq i\})^2,
\]
where \( S_i = K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \).

Thus if we factor by a maximal regular sequence as in the proof of Theorem \(2.1\) we obtain a 0-dimensional ring of the form
\[
A = T/(P_1^2, \ldots, P_m^2, I(G''))).
\]
Here \( P_j = (\{x_k : k \in F_j, k \neq i_j\}) \), \( G'' \) is the subgraph of \( G \) consisting of all edges which do not belong to any \( F_j \), and \( T \) is the polynomial ring over \( K \) in the set of variables \( X = \{x_k : k \in [n], k \neq i_j \text{ for all } j = 1, \ldots, m\} \). It is obvious that \( A \) is obtained from the polynomial ring \( T \) by factoring out the squares of all variables of \( T \) and all \( x_i, x_j \) with \( \{i, j\} \in E(G') \). Therefore \( A \) has a \( K \)-basis of squarefree monomials corresponding to the independent subsets of \( G' \), and the socle of \( A \) is generated as a \( K \)-vector space by the monomials corresponding to the maximal independent subsets of \( G' \).

(b) If \( G \) is a disjoint union of edges, then \( I(G) \) is a complete intersection, and hence Gorenstein.

Conversely, suppose that \( G \) is Gorenstein. Then \( A \) is Gorenstein. Since \( A \) a 0-dimensional ring with monomial relations, \( A \) is Gorenstein if and only if \( A \) is a complete intersection. This is the case only if \( E(G') = \emptyset \), in which case \( G \) is a disjoint union of edges. \(\square\)
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