Universality near zero virtuality

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Abstract

In this paper we study a random matrix model with the chiral and flavor structure of the QCD Dirac operator and a temperature dependence given by the lowest Matsubara frequency. Using the supersymmetric method for random matrix theory, we obtain an exact, analytic expression for the average spectral density. In the large-$n$ limit, the spectral density can be obtained from the solution to a cubic equation. This spectral density is non-zero in the vicinity of eigenvalue zero only for temperatures below the critical temperature of this model. Our main result is the demonstration that the microscopic limit of the spectral density is independent of temperature up to the critical temperature. This is due to a number of ‘miraculous’ cancellations. This result provides strong support for the conjecture that the microscopic spectral density is universal. In our derivation, we emphasize the symmetries of the partition function and show that this universal behavior is closely related to the existence of an invariant saddle-point manifold.

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1 Introduction

Since its initial application to level correlations in nuclear spectra [1], random matrix theory has been applied to a variety of physical phenomena ranging from resonant cavities [2] to lattice gauge theory [3]. In particular, the spectra of classically chaotic systems have been analyzed in great detail within this framework [4, 5, 6]. Recently, a great deal of progress has been made in the realm of mesoscopic physics. For example, universal conductance fluctuations have been understood thoroughly within the framework of random matrix theory [7, 8, 9, 10].

In initial applications, random matrix theories were introduced in the hope of representing the complicated strong interactions of nuclear physics. It was soon realized that the average spectral density cannot be described by means of random matrix theories. (For a large class of invariant random matrix theories, the spectral density is given by a semicircle. In contrast, the spectral density increases with energy for typical physical systems.) However, random matrix theories proved themselves capable of providing a remarkably accurate description of the correlations between eigenvalues on the scale of the average level spacing. Apparently, some of the properties of random matrices are universal while others are not. Such behavior is familiar from the theory of critical phenomena where, for example, critical exponents are universal, but the critical temperature is not. There, universal phenomena are usually associated with the soft modes which arise due to the spontaneous breaking of a symmetry; non-universal properties are determined by all modes. A similar separation of scales takes place in random matrix theories. Consider, for example, the average spectral density (with variations only over many level spacings) and level correlations on the scale of the average level spacing.

In the supersymmetric formulation of random matrix theory, universal properties, e.g., the level correlations, are associated with the existence of a saddle-point manifold which is intimately related to the symmetries of the theory. Non-universal properties, such as the average spectral density, can be calculated by a saddle-point approximation.

In this paper we study a model which was introduced in [11, 12]. This model is a random matrix model which possesses the chiral and flavor structure of the QCD Dirac operator and a schematic temperature dependence corresponding to the lowest Matsubara frequency. Otherwise, all matrix elements of the Dirac operator are completely random.
The temperature dependence is such that this model has a second-order phase transition with mean field critical exponents. Below the critical temperature, chiral symmetry is broken spontaneously; above the critical temperature, it is restored. According to the Banks-Casher formula \([13]\), the order parameter is the spectral density at eigenvalue zero. Physical motivation for this model comes from two rather different directions. First, at zero temperature, it satisfies all Leutwyler-Smilga sum-rules \([15]\), which are identities for chiral QCD in a finite volume. Second, Kocić and Kogut have recently suggested \([16]\) that the chiral phase transition in fermionic systems is driven towards a mean field description because of the fact that the lowest Matsubara frequency non-zero.

The chiral structure of the Dirac operator forces all eigenvalues to appear in pairs \(\pm \lambda\). The spectrum is symmetric about zero. As we shall soon see, it is useful to introduce the microscopic limit of the spectral density which probes the spectrum around zero on a scale set by the distance between adjacent eigenvalues:

\[
\rho_S(u) = \lim_{N \to \infty} \frac{1}{N^2} \rho\left(\frac{u}{N^2}\right) .
\]  

(1)

Here, \(\Xi\) is the temperature-dependent chiral condensate which, according to the Banks-Casher formula \([13]\), is given by

\[
\Xi = \frac{\pi \rho(0)}{N} ,
\]  

(2)

and \(N\) is the total number of eigenvalues.

The zero-temperature version of this model has been studied extensively in the literature \([17, 18, 19, 20, 23, 21]\). It is known as the Laguerre ensemble or the chiral Gaussian Unitary Ensemble (chGUE). Two types of universal behaviour are known to exist in chiral random matrix theories: Spectral correlations in the bulk of the spectrum are universal; the microscopic limit of the spectral density, just introduced, is universal. It has been shown \([22, 23]\) that the chiral structure of the random matrix ensemble does not affect eigenvalue correlations in the bulk of the spectrum. Such level correlations have been observed both experimentally and numerically in many systems \([1, 2, 3, 4]\). Further, analytic arguments have been presented \([4, 6]\) in favor of the universality of correlations in the bulk of the spectrum of classically chaotic systems.

In \([25]\) we conjectured that the microscopic limit of the spectral density is universal as well. The first argument in support of this conjecture came from instanton liquid
calculations [26], where we were able to generate ensembles large enough to permit the calculation of the spectral density in the microscopic limit. A slightly less direct argument came from lattice QCD calculations of the dependence of the chiral condensate on the valence quark mass [27, 28]. Another hint came from the work of the MIT group [29], who studied the chGUE using the supersymmetric method. They found that the microscopic limit of the spectral density is determined by a saddle-point manifold associated with the spontaneous breaking of a symmetry. The first convincing analytical arguments in favor of this conjecture came from recent work by Brézin, Hikami and Zee [21]. They considered families of random matrix models all possessing the chiral structure of the Dirac operator. They discovered the same microscopic limit in all the models they investigated.

In this paper, we offer further evidence in support of the universality of the microscopic limit of the spectral density. Specifically, we investigate the effect of temperature in the chiral random matrix model introduced in [11, 12]. This model differs structurally from the models in [21]. In the random matrix models considered in [21], the unitary symmetry of the probability distribution leaves the spectrum of each element of the ensemble invariant. This invariance is not realized for temperatures $T \neq 0$, and analytic proofs are consequently somewhat more difficult. Using the supersymmetric method of random matrix theory, we obtain an exact expression for the spectral density which is valid for any dimension, $n$, of the matrices. This enables us to take the microscopic limit. This limit also requires the large-$n$ limit of the spectral density (see (2)), which can be evaluated conveniently by means of a saddle-point approximation. In [11], this spectral density was evaluated numerically. It was found to have the well-known semi-circular shape at zero temperature. At high temperature, the shape is given by two disjoint semi-circles with centers located at $\pm \pi T$. In that paper, we also announced the analytic result for the shape of the average spectral density. The result merely requires the solution of a cubic equation. This result has also been obtained by Stephanov [30], who applied an extension of this model to the problem of the relation between the $Z_N$ phase of the theory and the restoration of chiral symmetry.

The organization of this paper is as follows. In section 2 we give a definition of the random matrix model and the supersymmetric partition function. In section 3 the partition function is reduced to a finite-dimensional integral. Symmetries and convergence
questions of the partition function are analyzed in section 4. The exact two-dimensional integral for the resolvent is obtained in section 5. In section 6 we derive the large-\(n\) limit of the spectral density and discuss its properties. The microscopic limit of the partition function is evaluated in section 8, and concluding remarks are made in section 9. Our notation and conventions are explained in Appendix A, and a perturbative calculation of the large-\(n\) limit of the spectral density is presented in Appendix B.

2 Definition of the random matrix model

In this paper we study the spectrum of the ensemble of matrices

\[
H = \begin{pmatrix} 0 & W + \pi T \\ W^\dagger + \pi T & 0 \end{pmatrix}
\]

(3)

Here, \(T\) is the temperature dependence as given by the lowest Matsubara frequency, and \(W\) is a complex \(n \times n\) matrix distributed according to

\[
\exp[-n\Sigma^2 \text{Tr} WW^\dagger].
\]

(4)

The average spectral density can be expressed as

\[
\rho(\lambda) = -\lim_{\epsilon \to 0} \frac{2n}{\pi} \text{Im} \ G(\lambda + i\epsilon).
\]

(5)

where the average resolvent \(G(z)\),

\[
G(z) = \frac{1}{2n} \text{Tr} \frac{1}{z + i0 - H} = -\frac{1}{2n} \frac{\partial \log Z(J)}{\partial J} \bigg|_{J=0},
\]

(6)

can be obtained from the partition function

\[
Z(J) = \int \mathcal{D}W \frac{\det(z-H)}{\det(z+J-H)} \exp[-n\Sigma^2 \text{Tr} WW^\dagger].
\]

(7)

The integration measure, \(\mathcal{D}W\), is the Haar measure normalized so that \(Z(0) = 1\). For a Hermitean matrix, \(H\), the resolvent is analytic in \(z\) in the upper complex half-plane. This allows us to calculate the resolvent for purely imaginary \(z\) and to perform the analytic
continuation to real $z$ at the end of the calculation. As will be seen below, this improves the convergence properties of the integrals in the partition function.

Some properties of this model all already known. At $T = 0$, this model reduces to the well-known Laguerre ensemble. The joint probability distribution of the eigenvalues is known explicitly, as are all correlation functions. In particular, the average spectral density is a semicircle:

$$\rho(\lambda) = \frac{n\Sigma^2}{\pi} \sqrt{\frac{4}{\Sigma^2} - \lambda^2}. \quad (8)$$

We wish to stress that the largest eigenvalue is larger than a typical matrix element by a factor on the order of $\sqrt{n}$.

The temperature dependence of this model was analyzed in [11]. It was shown that, in the thermodynamic limit, this model shows a chiral phase transition at a critical temperature of

$$T_c = \frac{1}{\pi \Sigma}. \quad (9)$$

The order parameter is the chiral condensate $\Xi$ with $\Xi \neq 0$ below $T_c$ and $\Xi = 0$ above $T_c$. This chiral symmetry is broken spontaneously. For each finite value of $n$, $\Xi = 0$. A non-zero value of $\Xi$ is obtained only in the thermodynamic limit. Below $T_c$ it was found that in this limit

$$\Xi = \Sigma \left(1 - \pi^2 T^2 \Sigma^2\right)^{1/2}. \quad (10)$$

The Banks-Casher formula (2) allows us to convert this value of $\Xi$ into the spectral density $\rho(0)$. In [11], the complete spectral density of this model was determined numerically. At $T = 0$ the result (8) was reproduced; at $T = T_c$ we found that $\rho(\lambda) \sim \lambda^{1/3}$. For $T \gg T_c$, the spectral density reduced to two semi-circles centered at $\pm \pi T$ with a radius independent of $T$. We will present an analytic derivation of these results.
3 Ensemble average of the partition function

In order to perform the Gaussian integrals, we write the determinant as an integral over the fermionic variables $\chi$ and $\chi^*$:

$$
\det(z - H) = (2\pi)^n \int \prod_{i=1}^n d[\chi_i^*]d[\chi_i] \prod_{i=1}^n d[\chi_{2i}^*]d[\chi_{2i}]
\times \exp \left( \frac{\chi_1^*}{\chi_2^*} \right) \left( \begin{array}{cccc}
{z} & -W - \pi T \\
-W^\dagger - \pi T & {z}
\end{array} \right) \left( \begin{array}{c}
\chi_1 \\
\chi_2
\end{array} \right). \quad (11)
$$

Similarly, the inverse determinant can be written as an integral over the bosonic variables $\phi$ and $\phi^*$:

$$
\det^{-1}(z - H) = \frac{1}{(2\pi)^n} \int \prod_{i=1}^n d[\phi_{1i}]d[\phi_{1i}^*] \prod_{i=1}^n d[\phi_{2i}]d[\phi_{2i}^*]
\times \exp \left( \frac{\phi_1^*}{\phi_2^*} \right) \left( \begin{array}{cccc}
{z + J} & -W - \pi T \\
-W^\dagger - \pi T & {z + J}
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right). \quad (12)
$$

The conventions for the Gaussian integrals are defined in Appendix A. The factor $i$ in the exponent in $\text{(12)}$ is chosen so that the integral is convergent for $z$ in the upper complex half-plane. This choice is consistent with the $i\epsilon$ prescription in $\text{(6)}$. The integral in $\text{(11)}$ converges independent of the overall phase of the exponent. The present choice of phase ensures that the product of the fermionic and bosonic integrals is one.

The Gaussian integral over $W$ can be performed by completing the squares. The result is a term of fourth order in the integration variables,

$$
\exp \left( -\frac{1}{n\Sigma^2} (\chi_{2j}^* \chi_{1i} + \phi_{2j}^* \phi_{1i}) (\chi_{1i}^* \chi_{2j} + \phi_{1i}^* \phi_{2j}) \right). \quad (13)
$$

We apply the Hubbard-Stratonovitch transformation to each of the terms of fourth order in the bosonic and fermionic variables. Two of the four factors can be decoupled with the help of real integration variables:

$$
\exp \left( -\frac{1}{n\Sigma^2} \phi_1^* \cdot \phi_1 \phi_2^* \cdot \phi_2 \right) = \int \frac{d\sigma_1 d\sigma_2}{I_b} \exp \left[ -n\Sigma^2(\sigma_1^2 + \sigma_2^2) - (\sigma_1 + i\sigma_2)\phi_1^* \cdot \phi_1 + (\sigma_1 - i\sigma_2)\phi_2^* \cdot \phi_2 \right],
$$

$$
\exp \left( \frac{1}{n\Sigma^2} \chi_1^* \chi_1 \chi_2^* \chi_2 \right) = \int \frac{d\rho_1 d\rho_2}{I_b} \exp \left[ -n\Sigma^2(\rho_1^2 + \rho_2^2) - (\rho_1 - i\rho_2)\chi_1^* \cdot \chi_1 - (\rho_1 + i\rho_2)\chi_2^* \cdot \chi_2 \right]. \quad (14)
$$
The terms which involve mixed bilinears can be decoupled with the help of Grassmann integrations. We do not encounter convergence problems in the process.

\[
\exp \left( -\frac{1}{n} \sum \chi_1 \cdot \phi_1 \chi_2 \cdot \phi_2 \right) = I_f \int \frac{d\alpha d\beta}{(i/2)} \exp \left[ -n \sum \alpha^* \beta + \alpha^* \chi_1 \cdot \phi_1 - \beta \phi_2 \cdot \chi_2 \right],
\]

\[
\exp \left( \frac{1}{n} \sum \chi_1 \cdot \phi_1^* \chi_2 \cdot \phi_2^* \right) = I_f \int \frac{d\beta d\alpha}{(i/2)} \exp \left[ -n \sum \beta^* \alpha - \alpha \chi_1 \cdot \phi_1^* + \beta \phi_2 \cdot \chi_2^* \right].
\] (15)

The constants \( I_b \) and \( I_f \) are defined such that \( I_f/I_b = 1 \). (See Appendix A.)

Thus, we obtain the partition function as

\[
Z(J) = \int \prod_i^n d[\phi_i] \ d[\chi_i] \ d[\sigma] \exp \left[ -n \sum (\sigma_1^2 + \sigma_2^2 + \rho_1^2 + \rho_2^2 + \alpha^* \beta + \beta^* \alpha) \right]
\]

\[
\times \exp i \left( \begin{array}{cccc}
\phi_1^* \\
\phi_2^* \\
\chi_1^* \\
\chi_2^*
\end{array} \right) \left( \begin{array}{cccc}
z + J + i\sigma_1 - \sigma_2 & -\pi T & i\alpha & 0 \\
-\pi T & z + J - i\sigma_1 - \sigma_2 & 0 & i\beta \\
i\alpha^* & 0 & z + i\rho_1 + \rho_2 & -\pi T \\
0 & i\beta^* & -\pi T & z + i\rho_1 - \rho_2
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\chi_1 \\
\chi_2
\end{array} \right),
\] (16)

where

\[
d[\sigma] = d\sigma_1 \ d\sigma_2 \ d\rho_1 \ d\rho_2 \ \frac{d\alpha \ d\alpha^* \ d\beta \ d\beta^*}{(i/2)^2}.
\] (17)

If \( i\sigma_1, \sigma_2, z, \) and \( J \) are all real, the matrix \( A \) appearing in the exponent of (16) is a graded Hermitean matrix. Then the Gaussian integrals can be performed according to (73). This results in

\[
Z(J) = \int d[\sigma] \exp \left[ -n \sum (\sigma_1^2 + \sigma_2^2 + \rho_1^2 + \rho_2^2 + \alpha^* \beta + \beta^* \alpha) \right] \ \text{det} g^{-n} A,
\] (18)

where \( \text{det} g A \) is the graded determinant of \( A \). For a matrix with Grassmann blocks \( \rho \) and \( \sigma \) and commuting blocks \( a \) and \( b \), it can be shown that

\[
\text{det} \left( \begin{array}{cc}
a & \sigma \\
b & \rho
\end{array} \right) = \text{det}^{-1} b \ \text{det}(a - \sigma b^{-1} \rho).
\] (19)

In our case, all blocks \( a \) and \( b \) are \( 2 \times 2 \) matrices, which permits us to evaluate all expressions directly.

We note, however, that the result (18) was obtained by interchanging the \( \phi_i \) and \( \sigma_i \) integrations in (16). This is allowed only if the \( \phi \) integral is uniformly convergent in \( \sigma \). Unfortunately, this is not the case when the \( \sigma_1 \) and \( \sigma_2 \) integration paths are along the
real axis. This problem can be circumvented by a suitable deformation of the integration
paths. Previous studies of random matrix theories within the framework of the sigma
model formulation of the Anderson model [14] have stressed the importance of deforming
the integration contours in a manner which is consistent with the symmetries of the
problem. The same is true for the present problem. In order to interchange the $\phi$ and
$\sigma$ integrals in (16), we must deform the integration contour so that the $\phi$ integration
is uniformly convergent in $\sigma$. In order to motivate our choice of contour, we must first
consider the symmetries of the partition function.

4 Symmetries

We wish to study the partition function in the microscopic limit, i.e., the limit $n \to \infty$
with $zn$ held constant. For $z = 0$ (and $J = 0$), the partition function has an additional
symmetry. For all temperatures, the bosonic part of the partition function is invariant
under the non-compact symmetry operation

$$
\phi_1 \to e^{+t}\phi_1, \quad \phi_1^* \to e^{+t}\phi_1^*,
$$

$$
\phi_2 \to e^{-t}\phi_2, \quad \phi_2^* \to e^{-t}\phi_2^*.
$$

(20)

This induces a hyperbolic rotation of the variables $\sigma_1$ and $\sigma_2$ of the preceding section,

$$
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix} \to
\begin{pmatrix}
cosh t & i\sinh t \\
-i\sinh t & \cosh t
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix},
$$

(21)

which clearly reveals the $O(1,1)$ nature of the transformation.

The fermionic part of the partition function is invariant under

$$
\chi_1 \to e^{+iu}\chi_1, \quad \chi_1^* \to e^{+iu}\chi_1^*,
$$

$$
\chi_2 \to e^{-iu}\chi_2, \quad \chi_2^* \to e^{-iu}\chi_2^*.
$$

(22)

where, a priori, $u$ can be either real or complex. In terms of the $\rho$ variables of [14], this
induces the transformation

$$
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix} \to
\begin{pmatrix}
\cos u & \sin u \\
-sin u & \cos u
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix}.
$$

(23)
Because the integration over the Grassmann variables is finite, the volume of the symmetry group must be finite as well. Therefore, \( u \) must be real with \( u \in [0, 2\pi] \). In other words, the symmetry group is \( O(2) \).

The terms of a mixed fermionic-bosonic nature are also affected by the transformations (20) and (23). This induces a transformation of Grassmann variables introduced through the Hubbard-Stratonovitch transformation.

As is known from studies of random matrix theories, the parameterization of the variables \((\sigma_1, \sigma_2)\) and \((\rho_1, \rho_2)\) is dictated by the above symmetries. It is natural and convenient to choose integration variables which lie along and perpendicular to the invariant manifold. In the microscopic limit, integrations along this manifold must be performed exactly, whereas the perpendicular integrations can be performed by saddle-point methods in the limit \( n \to \infty \). For the \( \sigma \) variables, we thus choose the parametrization

\[
\begin{align*}
\sigma_1 &= -i(\sigma - i\epsilon) \sinh s/\Sigma , \\
\sigma_2 &= (z - i\epsilon) + J + (\sigma - i\epsilon) \cosh s/\Sigma ,
\end{align*}
\]  

(24)

where \( \sigma \in [-\infty, +\infty] \) and \( s \in [-\infty, +\infty] \). After the \( \phi_i \) integration, the \( i\epsilon \) appears only in the combination \( \sigma - i\epsilon \). Below, we will not write the \( i\epsilon \) term explicitly, but it is always understood that it is included in the variable \( \sigma \). This parametrization renders the \( \phi_1 \) and \( \phi_2 \) integrations uniformly convergent in \( \sigma_1 \) without jeopardizing the convergence of the \( \sigma \) and \( s \) integrations. This allows us to interchange the \( \phi_i \) and \( \sigma_i \) integrations leading to the final result (18) of the last section. The term \( \sigma_1^2 + \sigma_2^2 \) appearing in the first exponent in (18) becomes \( \sigma^2 + (z + J)^2 + 2(z + J)\sigma \cosh s \) in the parametrization (24). It is clear that the integral over \( s \) can be convergent only when \( z + J \) is purely imaginary. (Recall that \( \sigma \) contains the term \(-i\epsilon\).) The transformation (20) for \( z + J = 0 \) reduces to a translation of \( s \) and leaves \( \sigma \) invariant.

The \( \rho \) variables are also parametrized along and perpendicular to the saddle-point manifold according to

\[
\begin{align*}
\rho_1 &= iz + \rho \cos \varphi/\Sigma , \\
\rho_2 &= \rho \sin \varphi/\Sigma .
\end{align*}
\]  

(25)

The rotation (23) leads to a translation of the angle \( \varphi \) and leaves \( \rho \) invariant.
Our strategy in dealing with (18) is to perform the Grassmann integrations first, i.e., to collect the coefficient of $\alpha \alpha^* \beta \beta^*$. This leaves us with a four-dimensional integral which is the exact analytical result for the partition function for any finite $n$. In the thermodynamic limit and for $z \sim \mathcal{O}(1)$, the remaining integrations can be performed with a saddle-point approximation. This result is obtained in section 6. In the microscopic limit, $z$ will be $\mathcal{O}(1/n)$, and the integration over the invariant manifold must be performed exactly. The radial integrals can be approximated to leading order in $1/n$. (See section 5.)

The $T = 0$ problem has been investigated previously using the supersymmetric method [29, 21]. In [29], the saddle-point manifold was constructed in a manner similar to that used for the problem of invariant random matrix ensembles. (In this regard, see [31, 32].) In [21], the convergence difficulties were circumvented in an elegant fashion by the use of spherical coordinates for the variables $\phi_1$ and $\phi_2$. Unfortunately, a direct generalization of this approach is not possible for the present case of non-zero temperatures. When $T \neq 0$, the angles between the complex vectors $\phi_1$ and $\phi_2$ also enter in the integration variables.

5 Exact result for the spectral density at finite $n$

Because the fermionic blocks of the matrix $A$ are nilpotent, the right side of (13) can be expanded in a finite number of terms. The $n$-th power of the inverse of the graded determinant can be written as

$$
\det^{-n} \left( \begin{array}{cc} a & \sigma \\ \rho & b \end{array} \right) = \left( \frac{\det b}{\det a} \right)^n \left( 1 + n \text{Tr} a^{-1} \sigma b^{-1} \rho + \frac{n}{2} \text{Tr} (a^{-1} \sigma b^{-1} \rho)^2 + \frac{n^2}{2} \text{Tr}^2 a^{-1} \sigma b^{-1} \rho \right).$

(26)

Terms in the partition function which are of fourth order in the Grassmann variables can be obtained by supplementing the above terms with factors $\alpha^* \beta$ and $\beta^* \alpha$ from the exponent in (18). The result is

$$
Z(J) = \frac{n^2 \Sigma^4}{\pi^2} \int d\sigma_1 d\sigma_2 d\rho_1 d\rho_2 \left[ (1 - \frac{\pi^2 T^2}{D \Delta \Sigma^2})^2 - \frac{(D + \pi^2 T^2)(\Delta + \pi^2 T^2) + \pi^2 T^2(\Delta - D)/n}{D^2 \Delta^2 \Sigma^4} \right]$

$$
\times \left( \frac{\Delta}{D} \right)^n \exp\left[ -n \Sigma^2 (\sigma_1^2 + \sigma_2^2 + \rho_1^2 + \rho_2^2) \right],$

(27)
where $D$ is the determinant of the boson-boson block,

$$D = (z + J + i\sigma_1 - \sigma_2)(z + J - i\sigma_1 - \sigma_2) - \pi^2 T^2,$$

and $\Delta$ is the determinant of the fermion-fermion block,

$$\Delta = (z + i\rho_1 + \rho_2)(z + i\rho_1 - \rho_2) - \pi^2 T^2.$$

In (27) the variables $\sigma_i$ and $\rho_i$ are parametrized according to (24) and (25). 

In (27) the variables $\sigma_i$ and $\rho_i$ are parametrized according to (24) and (25). temperature by

$$t = \pi T \Sigma$$

the resulting form of $Z(J)$ simplifies to

$$Z(J) = \frac{-in^2}{\pi^2} \int_\infty^{-\infty} \sigma d\sigma \int_\infty^{-\infty} ds \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi \int_0^{-\infty} d\rho \int_\infty^{-\infty} d\sigma \Delta^2 \left[ (1 - \frac{t^2}{(t^2 - \sigma^2)(\rho^2 + t^2)})^2 + \frac{\sigma^2 \rho^2 + t^2(\sigma^2 + \rho^2)/n}{(\sigma^2 - t^2)^2(\rho^2 + t^2)^2} \right]$$

$$\times \left( \frac{\rho^2 + t^2}{t^2 - \sigma^2} \right)^n e^{-n(\sigma^2 + \rho^2 + 2(z + J)\Sigma \sigma \cosh s + 2iz\Sigma \rho \cos \varphi + \Sigma^2 ((z + J)^2 - z^2))}.$$

The integrations over $s$ and $\varphi$ can be expressed in terms of Bessel functions. Differentiation of the partition function with respect to $J$ at $J = 0$ gives us the resolvent which we desire. Thus, the final result of this section is

$$G(z) = \frac{2in}{\pi} \int \sigma d\sigma \rho d\rho \Delta^2 \left[ (1 - \frac{t^2}{(t^2 - \sigma^2)(\rho^2 + t^2)})^2 + \frac{\sigma^2 \rho^2 + t^2(\sigma^2 + \rho^2)/n}{(\sigma^2 - t^2)^2(\rho^2 + t^2)^2} \right]$$

$$\times \left( \frac{\rho^2 + t^2}{t^2 - \sigma^2} \right)^n (2z\Sigma^2 K_0(2n\Sigma z\sigma) - 2n\Sigma \sigma K_1(2n\Sigma z\sigma))J_0(2n\Sigma z\rho) \exp[-n(\sigma^2 + \rho^2)].$$

Again, we remind the reader that $\sigma$ contains a term $-i\epsilon$. The integral over $\sigma$ can thus be performed by successive partial integrations. Details regarding the calculation of this kind of integral can be found in [33]. This result has been obtained for $z$ purely imaginary. Since the modified Bessel functions have a cut for $z\sigma < 0$, we can analytically continue this expression anywhere in the upper half-plane.
6 The large-$n$ limit of the average spectral density

For $n \rightarrow \infty$ and $z \sim O(1)$, all integrals in (31) can be performed by a saddle-point approximation. Because we started with a supersymmetric partition function, the Gaussian fluctuations about the saddle point give an overall constant of unity, i.e., $Z(0) = 1$ in (31). Using the relation (3) to determine the resolvent from the partition function (31), we find that

$$G(z) = \Sigma(\Sigma z + \bar{\sigma} \cosh \bar{s}) ,$$

(33)

where $\bar{\sigma}$ and $\bar{s}$ are the saddle-point values of these variables. The saddle-point equation for $s$ is trivial with solution $\bar{s} = 0$. The equation for $\bar{\sigma}$ is more complicated

$$\frac{\bar{\sigma}}{\sigma^2 - t^2} - (\bar{\sigma} + \Sigma z) = 0 .$$

(34)

This equation can be rewritten as an equation for the ensemble averaged resolvent

$$G^3/\Sigma^4 - 2zG^2/\Sigma^2 + G(z^2 - \pi^2T^2 + 1/\Sigma^2) - z = 0 .$$

(35)

At $T = 0$, this equation reduces to

$$(G - z)(G^2/\Sigma^2 - zG + 1) = 0$$

(36)

with a non-trivial solution

$$G(z) = \Sigma^2 \frac{z \pm i(4/\Sigma^2 - z^2)^{1/2}}{2} .$$

(37)

As indicated in (4) above, the associated spectral density is simply the imaginary part of the branch of $G(z)$ with the negative sign,

$$\rho(\lambda) = \frac{n\Sigma^2}{\pi} (4/\Sigma^2 - \lambda^2)^{1/2} ,$$

(38)

which is the familiar semicircle normalized to the total number of eigenvalues.

For $z = 0$, the saddle-point equation (33) simplifies to

$$G^3 + \Sigma^4G(1/\Sigma^2 - \pi^2T^2) = 0$$

(39)

with the solution

$$G(0) = -i\Sigma\sqrt{1 - \pi^2T^2/\Sigma^2} .$$

(40)
This corresponds to the spectral density

\[ \rho(0) = \frac{2n\Sigma}{\pi} \sqrt{1 - \pi^2T^2\Sigma^2}. \]  

(41)

Using the Banks-Casher formula (2), we immediately obtain the chiral condensate (10) in agreement with [11].

In order to determine the high-temperature limit of the spectral density, it is most convenient to return to the saddle-point equation (34). It is clear that, for \( z \approx \pi T \), this equation can only be satisfied for \( \bar{\sigma} \approx -t \). In the partition function (31) we can approximate the logarithmic term

\[ \log(t^2 - \sigma^2) \approx \log(2t) + \log(t + \sigma). \]  

(42)

This leads us to the high-temperature limit of the saddle-point equation

\[ -\frac{1}{\bar{\sigma} + t} - (\bar{\sigma} + \Sigma z) = 0. \]  

(43)

The solution for the resolvent is

\[ G(z) = \frac{\Sigma}{2}(\Sigma z - t - i\sqrt{2 - (\Sigma z - t)^2}) . \]  

(44)

This results in a semicircular spectral density of radius \( \sqrt{2} \) located at \( z = \pi T \). An identical argument leads to another semicircular contribution to the spectral density of radius \( \sqrt{2} \) centered at \( z = -\pi T \). Of course, we can arrive at the same conclusion working directly from (33). For \( z \approx \pi T \), the resolvent \( G(z) \sim \mathcal{O}(1) \), and the first term in (33) will be sub-leading in the high-temperature limit. This leads immediately to (14).

Finally, we consider the case at the critical temperature, \( T = \Sigma/\pi \), with \( z \) in the neighborhood of 0. Then, the saddle-point equation for \( G \) reduces to

\[ G^3 = z \]  

(45)

with solutions \( (z\Sigma^4)^{1/3} \), \( (z\Sigma^4)^{1/3} \exp(\pi i/3) \) and \( (z\Sigma^4)^{1/3} \exp(2\pi i/3) \). Only the last of these gives rise to a positive definite spectral density with

\[ \rho(\lambda) = \frac{n\Sigma\sqrt{3}}{\pi}(\lambda\Sigma)^{1/3} \]  

(46)

in agreement with the mean field critical exponent of \( \delta = 3 \) for this model.
The equation for the resolvent (35) also enables us to obtain a simple recursion relation for the moments of the spectral density. Expanding $G(z)$ in terms of these moments,

$$G(z) = \sum_n \frac{M_{2n}}{z^{2n+1}},$$

we obtain

$$M_{2n+2} = (T^2 - \frac{1}{\Sigma^2})M_{2n} + \frac{2}{\Sigma^2} \sum_{k+l=n} M_{2k}M_{2l} - \frac{1}{\Sigma^4} \sum_{k+l+m=n-1} M_{2k}M_{2l}M_{2m}.$$ (48)

The evident initial condition, $M_0 = 1$, immediately leads us to et cetera. Without too much effort, it is possible to use standard combinatoric methods to find the general result for the $(2n)$-th moment,

$$M_{2n} = \sum_{k=0}^n y^{2k}T(2n-k) \frac{1}{k+1} \binom{n}{k} \binom{2n}{k}.$$ (49)

7 The microscopic limit of the partition function

The ‘microscopic limit’ denotes the investigation of the spectral density in the vicinity of $z = 0$ on a scale set by the average level spacing. More precisely, we take the limit $n \to \infty$ while keeping $nz$ fixed, as indicated in (1). We start from the expression (18) for the partition function. In the thermodynamic limit, the $\sigma$ and $\rho$ integrations can be performed by a saddle-point method. The saddle-point equations read

$$\frac{\rho}{\rho^2 + t^2} - \rho = 0,$$

$$\frac{\sigma}{t^2 - \sigma^2} - \sigma = 0,$$ (50)

with solutions

$$\bar{\rho}^2 = 1 - t^2,$$

$$\bar{\sigma}^2 = t^2 - 1.$$ (51)

For temperatures less than the critical temperature, $\bar{\rho}$ is real and $\bar{\sigma}$ is purely imaginary. The integration range of $\rho$ is the positive real axis. Therefore, the sign of $\bar{\rho}$ is positive.
The $\sigma$ integration ranges from $-\infty$ to $+\infty$. In order to reach the $\sigma$ saddle point, we must deform the integration contour. Because of the modified Bessel functions which appear in our expression (32) for the resolvent, there is a cut in the complex $\sigma$-plane for $\sigma z$ on the negative real axis. The cut of the modified Bessel function is then $i\epsilon$ above the positive real axis for negative $z$ and $i\epsilon$ above the negative real axis for positive $z$. Therefore, independent of the sign of $z$, only the saddle point with a negative imaginary part can be reached by a deformation of the contour. Thus,

$$\bar{\sigma} = -i \left(1 - t^2\right)^{1/2},$$

for $t < 1$.

At the saddle point, the pre-exponential factor vanishes:

$$\left(1 - \frac{t^2}{(t^2 - \sigma^2)(\bar{\rho}^2 + t^2)}\right)^2 + \frac{\sigma^2 \bar{\rho}^2 + t^2 (\bar{\sigma}^2 + \bar{\rho}^2)/n}{(t^2 - \sigma^2)^2 (\bar{\rho}^2 + t^2)^2} = 0.$$  

(53)

Given (51), it is trivial that this equation is satisfied when $t = 0$. However, the vanishing of this pre-exponential factor for arbitrary $t$ is remarkable and unexpected. This fact is responsible for the ‘universal’ behaviour of the microscopic limit of the spectral density. As a consequence, the $O(1/n)$ term in this factor does not contribute to the resolvent to leading order in $1/n$. The $zK_0$ term in the pre-exponent is also of subleading order ($z \sim O(1/n)$). This leads to the following result for the resolvent in the microscopic limit:

$$G(z) = -\frac{4n^2i\Sigma}{\pi} \int \sigma d\sigma \rho d\rho$$

$$\times \left[ \left(1 - \frac{t^2}{(t^2 - \sigma^2)(\bar{\rho}^2 + t^2)}\right)^2 + \frac{\sigma^2 \bar{\rho}^2}{(\sigma^2 - t^2)^2 (\rho^2 + t^2)^2} \right]$$

$$\times \sigma K_1(2n\Sigma z\sigma) J_0(2n\Sigma z\rho) \exp[-n(\sigma^2 + \rho^2 + \log(t^2 - \sigma^2) - \log(\rho^2 + t^2))] .$$

(54)

In order to proceed, we make the substitution

$$\sigma = \bar{\sigma} + \delta \sigma ,$$

$$\rho = \bar{\rho} + \delta \rho$$

in (54) and keep only those terms which contribute to leading order in $1/n$, i.e., terms through second order in $\delta \rho$ and $\delta \sigma$. The exponent in (54) then becomes

$$\exp(2n\sigma^2 \delta \sigma^2 - 2n\bar{\rho}^2 \delta \rho^2) .$$

(56)
The product of the terms in square brackets in (54) and \( \sigma^2 \rho \) can be expanded as
\[
\bar{\sigma}^2 \bar{\rho} [2\bar{\rho}^3 \delta \rho + 2\bar{\sigma}^3 \delta \sigma + \bar{\rho}^2 (-1 + 8t^2) \delta \rho^2 + \bar{\sigma}^2 (+1 + 8t^2) \delta \sigma^2 - \bar{\sigma} \bar{\rho} (2 + 8t^2) \delta \rho \delta \sigma] .
\] (57)

It is clear already at this point that all temperature dependence enters through the scale factors \( \bar{\sigma} \) and \( \bar{\rho} \). The terms of \( O(\delta \rho \delta \sigma) \) vanish upon integration with the exponential factor (56). Since \( \langle \delta \rho^2 \rangle = \langle \delta \sigma^2 \rangle \) and \( \bar{\sigma}^2 = -\bar{\rho}^2 \), the other terms involving \( 8t^2 \) cancel as well. To complete the calculation, we need only expand the Bessel functions to first order
\[
K_1(2n \Sigma z \bar{\sigma}) J_0(2n \Sigma z \bar{\rho}) = K_1 J_0 + 2n \Sigma z \delta \sigma K_1' J_0 + 2n \Sigma z \delta \rho K_1 J_0' .
\] (58)

where the Bessel functions \( K_1 \) and \( J_0 \) and their derivatives appearing on the right of this equation are to be evaluated at their saddle points which are \( 2n \Sigma z \bar{\sigma} \) and \( 2n \Sigma z \bar{\rho} \), respectively. In order to arrive at the final result, we form the product of this expression and (57), collect the coefficients of \( \delta \rho^2 \) and \( \delta \sigma^2 \), and perform the Gaussian integrations over \( \delta \rho \) and \( \delta \sigma \) according to
\[
\langle \delta \rho^2 \rangle = \frac{1}{2} \frac{\sqrt{\pi}}{(2n \bar{\rho}^2)^{3/2}} \frac{\sqrt{\pi}}{(-2n \bar{\sigma}^2)^{1/2}} ,
\]
\[
\langle \delta \sigma^2 \rangle = \frac{1}{2} \frac{\sqrt{\pi}}{(2n \bar{\rho}^2)^{1/2}} \frac{\sqrt{\pi}}{(-2n \bar{\sigma}^2)^{3/2}} .
\] (59)

The result is
\[
G(z) = \frac{i \Sigma \bar{\sigma}^2}{2 \bar{\rho}^2} \left[ K_1 J_0(\bar{\sigma}^2 - \bar{\rho}^2) + 4nz \bar{\rho}^3 \Sigma K_1 J_0' + 4nz \bar{\sigma}^3 \Sigma K_1' J_0 \right] .
\] (60)

If we make use of the identities
\[
J_0' = -J_1 ,
\]
\[
K_1'(z) = -K_0(z) - \frac{1}{z} K_1(z) ,
\] (61)
we discover that the terms proportional to \( K_1 J_0 \) cancel. This leaves us with
\[
G(z) = i2nz \Sigma^2 (1 - t^2)(K_1 J_1 + iK_0 J_0) .
\] (62)

Finally, we can explicitly separate the resolvent into its real and imaginary parts by using two more elementary identities:
\[
K_1(-iz) = -\frac{\pi}{2} [J_1(z) + iN_1(z)] ,
\]
\[
K_0(-iz) = \frac{\pi}{2} i [J_0(z) + iN_0(z)] .
\] (63)
The final result for the microscopic spectral density is thus

$$
\rho(\lambda) = 2n^2\lambda\Sigma^2(1 - t^2)(J_1^2(2n\lambda\Sigma\sqrt{1 - t^2}) + J_0^2(2n\lambda\Sigma\sqrt{1 - t^2})).
$$

(64)

As noted above, the microscopic limit of this model has previously been considered for the special case \( t = 0 \) [25, 26, 29, 21]. Our result is in agreement with this earlier work. Now, however, we can also consider the microscopic limit for general \( t \neq 0 \). At finite temperature, the temperature enters only through the temperature-dependent modification of the chiral condensate which was obtained in [11]. As defined in (1) with \( \rho(0) \) given by (41), the microscopic limit is strictly independent of the temperature:

$$
\rho_S(u) = \frac{u}{2}[J_0^2(u) + J_1^2(u)].
$$

(65)

### 8 Conclusions

In this paper, we have studied a random matrix model with the chiral structure of the QCD Dirac operator and a temperature dependence characteristic of the lowest Matsubara frequency. This model possesses the global color and flavor symmetries of QCD. It undergoes a chiral phase transition with critical exponents given by mean field theory.

Using the supersymmetric method for random matrix theories, we have found an exact, analytic expression for the average spectral density of this model. The result has the form of a two-dimensional integral which is valid for matrices of any dimension. In the large-\( n \) limit, these integrals can be performed using a saddle-point approximation. The spectral density then follows from the solution of an elementary cubic equation and nicely confirms our earlier numerical work [11].

Our primary result is that the spectral density in the microscopic limit is strictly independent of the temperature below the critical temperature of this model. This result supports the recent work of Brézin, Hikami and Zee, who investigated several families of random matrix models and found the same microscopic limit of the spectral density in all cases. As noted in the introduction, our model differs from the models considered by these authors in an essential way. In each of their models, the spectrum of each element in the
ensemble is strictly invariant under the unitary symmetry of the probability distribution. In the present model this symmetry is violated for $T \neq 0$. Thus, agreement between the microscopic limit of the spectral density for our model and the models of Brézin, Hikami and Zee increases our confidence in the universality of this quantity.

In lattice QCD simulations the microscopic limit of the spectral density enters in the valence quark mass dependence of the chiral condensate. This quantity has been calculated for a variety of temperatures [27], and it has been shown that the results below the critical temperature and not too large valence quark masses fall on a universal curve that can be obtained from the microscopic limit of the spectral density [28]. The present work provides a proper theoretical foundation of this analysis.

In our derivations, the symmetries of the partition function played a crucial role. The universal behavior was closely related to the existence of an invariant saddle-point manifold generated by these symmetries. This suggests that the ‘miraculous’ cancellation of the temperature dependence of the microscopic spectral density found here is not a coincidence. It would be very interesting to obtain this result using more general arguments. Recent work by Guhr [34] on the superposition of two matrix ensembles appears to offer a promising method towards this goal. Work in this direction is in progress.

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Appendix A: Notations and conventions

In this appendix we summarize our notations and conventions. For a more detailed discussion regarding the motivation for these conventions, we refer to [31] [32].

The integration measure for complex Gaussian integrals is defined such that

$$\int \frac{d\phi^* \, d\phi}{2\pi} \exp +i\phi^*\phi = 1.$$  \hspace{1cm} (66)

For Grassmann integrals, the measure is defined so that

$$2\pi \int d\chi^* \, d\chi \exp +i\chi^*\chi = 1.$$  \hspace{1cm} (67)
A graded vector or supervector is defined by
\[ \Phi = \left( \begin{array}{c} \phi \\ \chi \end{array} \right), \] (68)
with \( \phi \) a commuting vector of length \( n \) and \( \chi \) an anti-commuting vector of dimension \( m \).

The corresponding supermatrix which acts on this vector has the structure
\[ A = \left( \begin{array}{cc} a & \sigma \\ \rho & b \end{array} \right), \] (69)
where \( a \) and \( b \) are complex matrices of dimension \( n \times n \) and \( m \times m \), respectively. The entries in the \( n \times m \) dimensional matrix \( \sigma \) and the \( m \times n \) dimensional matrix \( \rho \) are Grassmann variables. The graded trace of the matrix \( A \) is defined as
\[ \text{Tr}_{g} A = \text{Tr}a - \text{Tr}b. \] (70)

The Hermitean conjugate of \( A \) is defined as
\[ A^\dagger = \left( \begin{array}{cc} a^\dagger & \rho^\dagger \\ -\sigma^\dagger & b^\dagger \end{array} \right), \] (71)
where the \( \dagger \) denotes transposition and complex conjugation. A graded matrix is called Hermitean if \( A^\dagger = A \). We use complex conjugation of the second kind for Grassmann variables, \( i.e., \chi^{**} = -\chi \). The graded determinant is defined as
\[ \det_{g} A = \exp(\text{Tr}_{g} \log A). \] (72)

With this definition, we obtain the following natural result for a Hermitean, graded matrix:
\[ \int \prod_{i=1}^{n} d[\phi^*_{i}] \ d[\phi_{i}] \ d[\chi^*_{i}] \ d[\chi_{i}] \ \exp + i \Phi^* A \Phi = \frac{1}{\det_{g} A}. \] (73)

**Appendix B: Perturbative evaluation of the average spectral density**

In this appendix, we derive the large-\( n \) limit of the resolvent without employing the supersymmetric method. Because the operator \( \mathcal{B} \) has only a finite support, it is possible
to expand the resolvent in a geometric series in $1/(z - K)$ for $z$ sufficiently large. Here, $K$ is the matrix

$$K = \begin{pmatrix} 0 & \pi T \\ \pi T & 0 \end{pmatrix}. \quad (74)$$

One finds by inspection that $G(z)$ satisfies

$$G(z) = \Tr \frac{1}{z - K} + \Tr \frac{1}{z - K} \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \mathcal{G} \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \mathcal{G} \quad (75)$$

where $\mathcal{G}$ is the matrix

$$\mathcal{G} = \frac{1}{z - \mathcal{H}}, \quad (76)$$

and the bar denotes averaging over the probability distribution $\Pi$. It should be clear that $\mathcal{G}$ is block diagonal with the block structure

$$\mathcal{G} = \begin{pmatrix} g_{1n} & h_{1n} \\ h_{1n} & g_{1n} \end{pmatrix}, \quad (77)$$

where $1_n$ is the $n \times n$ identity matrix. Therefore, we find that $G(z) = g$. The average over $W$ can be carried out immediately to give

$$\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \mathcal{G} \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} = \frac{1}{n\Sigma^2} \begin{pmatrix} g_{1n} & 0 \\ 0 & g_{1n} \end{pmatrix}. \quad (78)$$

This yields the following matrix equation for $g$ and $h$:

$$\begin{pmatrix} z & -\pi T \\ -\pi T & z \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix} = 1 + \frac{1}{\Sigma^2} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix}, \quad (79)$$

which leads to the two independent equations

$$zg - \pi Th = 1 + \frac{1}{\Sigma^2} g^2, \quad (80)$$

$$zh - \pi Tg = \frac{1}{\Sigma^2} gh. \quad (80)$$

Elimination of $h$ yields the equation

$$zg - \frac{\pi^2 T^2 g}{z - g/\Sigma^2} = 1 + \frac{1}{\Sigma^2} g^2, \quad (81)$$

which agrees with (34). Evidently, it can be rewritten as a cubic equation for $g$. 

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