ITERATED EXTENSIONS IN MODULE CATEGORIES

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Abstract. Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $\mathcal{F} = \{M_\alpha : \alpha \in I\}$ be a family of orthogonal points in $\text{Mod}(R)$ such that $\text{End}_R(M_\alpha) \cong k$ for all $\alpha \in I$. Then $\text{Mod}(\mathcal{F})$, the minimal full subcategory of $\text{Mod}(R)$ which contains $\mathcal{F}$ and is closed under extensions, is a full exact Abelian sub-category of $\text{Mod}(R)$ and a length category in the sense of Gabriel [8].

In this paper, we use iterated extensions to relate the length category $\text{Mod}(\mathcal{F})$ to noncommutative deformations of modules, and use some new methods to study $\text{Mod}(\mathcal{F})$ via iterated extensions. In particular, we give a new proof of the characterization of uniserial length categories, which is constructive. As an application, we give an explicit description of some categories of holonomic and regular holonomic $D$-modules on curves which are uniserial length categories.

INTRODUCTION

Let $C = \text{Mod}(R)$ be the category of left modules over an associate ring $R$. We shall assume that $\mathcal{F} = \{M_\alpha : \alpha \in I\}$ is a family of non-zero, pairwise non-isomorphic objects in $C$, and consider the minimal full sub-category $\text{Mod}(\mathcal{F})$ of $C$ which contains $\mathcal{F}$ and is closed under extensions.

An alternative and explicit description of $\text{Mod}(\mathcal{F})$ is useful: An object $M$ of $C$ is in $\text{Mod}(\mathcal{F})$ if and only if there is a cofiltration

$$M = C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 = 0$$

in $C$ such that $f_i : C_i \to C_{i-1}$ is surjective with kernel $K_i \cong M_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$. Equivalently, $M$ is in $\text{Mod}(\mathcal{F})$ if and only if there is a filtration

$$0 = F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = M$$

in $C$ such that $K_i = F_{i-1}/F_i \cong M_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$.

In general, $\text{Mod}(\mathcal{F})$ is not an exact Abelian sub-category of $C$: It does not necessarily contain its kernels and cokernels. Ringel [15] has shown that $\text{Mod}(\mathcal{F})$ is a full, extension closed and exact Abelian sub-category of $\text{Mod}(R)$ with $\mathcal{F}$ as its simple objects if and only if $\mathcal{F}$ is a family of orthogonal points in $C$. In this paper, we shall assume that this is the case. So by definition, $\text{End}_R(M_\alpha)$ is a division ring and $\text{Hom}_R(M_\alpha, M_\beta) = 0$ for all $\alpha, \beta \in I$. This means that $\text{Mod}(\mathcal{F})$ is a length category in the sense of Gabriel [8].

Let us assume that the family $\mathcal{F}$ of orthogonal points in $C$ is given. We shall consider the following problem: Classify the indecomposable objects in $\text{Mod}(\mathcal{F})$, up to isomorphism. This problem is fundamental, but there is nothing original about it - we cite Gabriel [8]:

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The main and perhaps hopeless purpose of representation theory is to find an efficient general method for constructing the indecomposable objects by means of the simple objects, which are supposed to be given.

It is not plausible to expect a full solution to this problem. It is well-known that the problem is wild in many cases, for instance when \( \mathcal{F} \) is a complete family of simple left modules over the first Weyl algebra \( A_1(k) \) over any algebraically closed field \( k \) of characteristic 0.

It is maybe better to consider the following problem: Give necessary and sufficient conditions for the category \( \text{Mod}(\mathcal{F}) \) to be tame. We would of course also like to classify the indecomposable modules in \( \text{Mod}(\mathcal{F}) \) in these cases. We are not able to give a complete solution to this problem at present. However, we shall introduce some new methods which we believe are useful for attacking the problem, and we shall give the solution to the problem in some special cases.

Before we go on to study the above problem in more detail, we remark that so far, we have only treated the case when \( C = \text{Mod}(R) \) is the category of left modules over an associative ring \( R \). It makes sense to consider the problem above when \( C \) is any Abelian category. Most of the results will remain true with mild restrictions (and in many cases, none) on the Abelian category \( C \). We shall only treat the case \( C = \text{Mod}(R) \) in this paper, since this will lead to a much more readable exposition, and leave it to the reader to figure out how to generalize the results to more general Abelian categories \( C \). The only exception is one of our applications, where we assume that \( C \) is the category of graded modules over a graded ring \( R \), in which case all results of this paper remain valid.

Let us consider the category \( \text{Ext}(\mathcal{F}) \) of iterated extensions of the family \( \mathcal{F} \): The objects of \( \text{Ext}(\mathcal{F}) \) are couples \((M,C)\), where \( M \) is an object of \( C \) and \( C \) is a cofiltration of \( M \) of the type considered above, see section 1 for details. Clearly, there is a forgetful functor \( \text{Ext}(\mathcal{F}) \to \text{Mod}(R) \), and image of this functor is exactly the category \( \text{Mod}(\mathcal{F}) \). The reason why we would like to consider the category \( \text{Ext}(\mathcal{F}) \), is that many useful invariants are naturally defined there.

Let \((M,C)\) be an object of \( \text{Ext}(\mathcal{F}) \). We define the length \( n \) to be the length of the cofiltration \( C \), and the order vector \( \alpha \in I^n \) to be the vector defined by \( K_i \cong M_{\alpha(i)} \) for \( 1 \leq i \leq n \). All objects in an isomorphism class of \( \text{Ext}(\mathcal{F}) \) has the same length and order vector. We define the extension type of the object \((M,C)\) to be the ordered quiver \( \Gamma \) with vertices \( \{\alpha(i) \in I : 1 \leq i \leq n\} \), and arrows \( \{a_{i-1,i} : 2 \leq i \leq n\} \), where \( a_{12} < a_{23} < \cdots < a_{n-1,n} \) is the total ordering of the arrows, and \( a_{i-1,i} \) is an arrow from vertex \( \alpha(i-1) \) to vertex \( \alpha(i) \). The extension type \( \Gamma \) is a convenient way of representing the invariants given by the length \( n \) and the order vector \( \alpha \).

Let us assume that \( R \) is an algebra over an algebraically closed (commutative) field \( k \). Under some finiteness conditions, we shall show that the category \( \text{Ext}(\mathcal{F}) \), and therefore also the category \( \text{Mod}(\mathcal{F}) \), is determined by the noncommutative deformations of the family \( \mathcal{F} \).

Let \((M,C)\) be an object of \( \text{Ext}(\mathcal{F}) \), and let \( \Gamma \) be the extension type of \((M,C)\). We associate with \( \Gamma \) the \( k \)-algebra \( k[\Gamma] \), see section 2 for details. This is a \( p \)-pointed \( k \)-algebra, where \( p \) is the number of vertices in \( \Gamma \), so there are natural maps \( k^p \to k[\Gamma] \to k^p \). Moreover, the radical \( I \) of \( k[\Gamma] \) satisfies \( I^n = 0 \), where \( n \) is the length of \((M,C)\). So \( k[\Gamma] \) is a complete Artinian ring in the \( I \)-adic topology, and therefore \( k[\Gamma] \) is an object in the category \( \text{Art}(n) \) of complete Artinian \( p \)-pointed algebras.

There is a theory of noncommutative deformations of modules, due to Laudal, see Laudal [11], [12], [13]. We refer to the preprint Eriksen [7] for a convenient form
of the results we need in this paper. Let us recall the main theorem: For any finite family \( F = \{ M_\alpha : \alpha \in I \} \) of left \( R \)-modules, there is a noncommutative deformation functor \( \text{Def}_F : \mathfrak{a}_p \to \text{Sets} \). If \( \text{Ext}(M_\alpha, M_\beta) \) is a finite dimensional vector space over \( k \) for \( i = 1, 2 \) and for all \( \alpha, \beta \in I \), then \( \text{Def}_F \) has a pro-representing hull \( H \), which is unique up to (non-canonical) isomorphism in \( \mathfrak{a}_p \). Both the hull \( H \) and the corresponding versal family \( M_H \in \text{Def}_F(H) \) are in principle constructible.

Let \( \Gamma \) be a fixed extension type, and consider iterated extensions of the family \( F \) with extension type \( \Gamma \). Clearly, these are all iterated extensions of the finite sub-family of \( F \) given by the vertices of \( \Gamma \). Let us denote by \( E(F, \Gamma) \) the set of isomorphism classes of iterated extensions in \( \text{Ext}(F) \) with extension type \( \Gamma \). We give a proof of the following result, due to Laudal:

**Theorem 1** (Laudal). Let \( F \) be a finite family of \( p \) orthogonal points in \( \text{Mod}(R) \), and let \( \Gamma \) be the extension type of some object \((M, C)\) of \( \text{Ext}(F) \) with \( p \) vertices. Then there is a bijective correspondence between \( \text{Def}_F(k[\Gamma]) \) and \( E(F, \Gamma) \).

We show that \( X(F, \Gamma) = \text{Mor}(H, k[\Gamma]) \) has a natural structure of an affine scheme over \( k \), and by definition of the pro-representing hull \( H \), there is a surjection \( X(F, \Gamma) \to E(F, \Gamma) \). The forgetful functor \( \text{Ext}(F) \to \text{Mod}(F) \) maps the isomorphism classes of \( E(F, \Gamma) \) to a subset of isomorphism classes of modules in \( \text{Mod}(F) \), and we shall denote this set of isomorphism classes by \( M(F, \Gamma) \). So there are natural surjections \( X(F, \Gamma) \to E(F, \Gamma) \to M(F, \Gamma) \). In particular, the sets \( E(F, \Gamma) \) and \( M(F, \Gamma) \) are quotients of the affine variety \( X(F, \Gamma) \).

The *species* of \( \text{Mod}(F) \) is given by \( (K_\alpha, E_{\alpha, \beta}) \), where \( K_\alpha = \text{End}_R(M_\alpha) \) is a division ring and \( E_{\alpha, \beta} = \text{Ext}^1_R(M_\alpha, M_\beta) \) is a \( K_\beta \)-\( K_\alpha \) bimodule for all \( \alpha, \beta \in I \). From now on, we shall assume that \( R \) is an algebra over an algebraically closed field \( k \), that \( F \) is a family of orthogonal points in \( \text{Mod}(R) \), and that \( \text{End}_R(M_\alpha) \cong k \) for all \( \alpha \in I \). In this case, the species of \( \text{Mod}(F) \) is called a \( k \)-quiver, because it is completely determined by the Gabriel quiver, defined by the set of vertices \( I \) and \( \text{dim}_k E_{\alpha, \beta} \) arrows from \( \alpha \) to \( \beta \) for each pair of vertices \( \alpha, \beta \in I \).

It is known that the species, and therefore the Gabriel quiver, contains a lot of information about the length category \( \text{Mod}(F) \). In fact, we see from Laudals theorem above that the only information which is not present in the Gabriel quiver is the obstruction theory of the family \( F \). Under the same finiteness conditions as in Laudals theorem, it is shown in Deng, Xiao [6] that if \( \text{Mod}(F) \) is a hereditary category, then it is equivalent to the category of small representations of the Gabriel quiver of \( F \). A representation is called small if it is nilpotent (and finite dimensional).

The hereditary case mentioned above is an example of an unobstructed case, where \( \text{Mod}(F) \) is completely determined by its species. If the Gabriel quiver is without loops, the category of small representations of the quiver is just the usual category of finite dimensional representations of the quiver. So in this case, it is well-known when the category \( \text{Mod}(F) \) is wild, tame and finite. If the Gabriel quiver has loops, we ask when the category of its small representations is wild, tame and finite. We do not know the complete answer to this question.

We say that a module \( M \) in \( \text{Mod}(F) \) is uniserial if the lattice of submodules of \( M \) is a chain. Moreover, we say that the length category \( \text{Mod}(F) \) is uniserial if all indecomposable modules in \( \text{Mod}(F) \) are uniserial. It is maybe not so obvious, but the uniserial case is also unobstructed in the sense that the obstructions for deforming the family \( F \) do not survive in the category \( \text{Mod}(F) \). In other words, the category \( \text{Mod}(F) \) is completely determined by its species in the uniserial case.

**Theorem 2.** Let \( F \) be a family of orthogonal points such that \( \text{End}_R(M_\alpha) \cong k \) for all \( \alpha \in I \). Then the category \( \text{Mod}(F) \) is uniserial if and only if each connected
component of the Gabriel quiver of $\mathcal{F}$ is either a cycle or a linear quiver. Moreover, if this is the case, then $\text{IM}(\mathcal{F}, \Gamma)$ has a single element if $\Gamma$ is admissible, and otherwise it is empty.

By definition, an extension type $\Gamma$ is admissible if the corresponding path appears in the Gabriel quiver of $\text{Mod}(\mathcal{F})$. The set $\text{IM}(\mathcal{F}, \Gamma)$ denotes the subset of $\text{M}(\mathcal{F}, \Gamma)$ consisting of indecomposable isomorphism classes.

The first part of this theorem is known, see Gabriel [8], Amdal and Ringdal [1]. As far as we know, our proof is new. It has the good property of being constructive, which is manifested in the second part of the theorem. In fact, we can construct the indecomposable modules corresponding to admissible extension types explicitly.

We take advantage of this fact when we apply the theorem to some categories of regular holonomic $D$-modules over curves of characteristic 0. In particular, we show that the category of graded holonomic $D$-modules over the first Weyl algebra is uniserial when $k$ has characteristic 0. We also describe the graded holonomic modules explicitly in this case.

Clearly, the length category $\text{Mod}(\mathcal{F})$ is tame (or even finite) when the condition of the theorem holds (that is, when $\text{Mod}(\mathcal{F})$ is uniserial). On the other hand, a sufficient condition for the category $\text{Mod}(\mathcal{F})$ to be wild in a strong sense is known. To be more precise, let $W = k < x, y >$ be the free associative $k$-algebra on two generators, and let $\text{fdMod}(W)$ be the category of left $W$-modules which are finite dimensional as vector spaces over $k$. We shall say that the length category $\text{Mod}(\mathcal{F})$ is wild if there is a full exact embedding of $\text{fdMod}(W)$ into $\text{Mod}(\mathcal{F})$.

We recall a well-known argument to show how hopeless it is to classify the indecomposable modules when $\text{Mod}(\mathcal{F})$ is wild: Let $R$ be any $k$-algebra which has finite dimension as vector space over $k$. Then there is a full exact embedding of the category of left $R$-modules which have finite dimension over as vector spaces over $k$ into $\text{fdMod}(W)$, and therefore into $\text{Mod}(\mathcal{F})$. Since full exact embeddings preserve indecomposable modules, a classification of indecomposable modules in $\text{Mod}(\mathcal{F})$ would contain a classification of all finite dimensional indecomposable modules over all finite dimensional $k$-algebras.

The following theorem, essentially due to Klingler, Levy [10], gives a sufficient condition for the length category $\text{Mod}(\mathcal{F})$ to be wild in the above sense:

**Theorem 3** (Klingler-Levy). Let $\mathcal{F}$ be a family of orthogonal points such that $\text{End}_R(M_\alpha) \cong k$ for all $\alpha \in I$. If the Gabriel quiver of $\text{Mod}(\mathcal{F})$ contains the quiver $Q_5$ given by

```
 1  2  3  4  5
\downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow
0
```

then there is a full exact embedding of $\text{fdMod}(W)$ into $\text{Mod}(\mathcal{F})$, and all modules in the image of this embedding has socle-height 2. In particular, $\text{Mod}(\mathcal{F})$ is wild in this case.

We have described two extreme cases: The uniserial case, where we have found a very nice description of the length category $\text{Mod}(\mathcal{F})$, and a wild case, where it would be hopeless to describe $\text{Mod}(\mathcal{F})$. It would be interesting to know what happens with $\text{Mod}(\mathcal{F})$ in all the intermediate cases. We would expect that an understanding of the obstructions of the deformations of $\mathcal{F}$ is necessary to give the full answer to this question. But as we have noted above, even in the hereditary case this question is open.
1. Categories of iterated extensions

Let $R$ be an associative ring, and let $\mathcal{F} = \{M_\alpha : \alpha \in I\}$ be a fixed family of non-zero, pairwise non-isomorphic left $R$-modules. We shall define the category $\text{Ext}(\mathcal{F})$ of \textit{iterated extensions} of the family $\mathcal{F}$.

Let us first consider the category $\text{CoFilt}(\mathcal{F})$ of \textit{modules with cofiltration} over the family $\mathcal{F}$, defined in the following way: An object of $\text{CoFilt}(\mathcal{F})$ is a couple $(M, C)$, where $M$ is a left $R$-module and $C$ is a cofiltration of $M$ of the form

$$M = C_n \to C_{n-1} \to \cdots \to C_2 \to C_1 \to C_0 = 0,$$

such that $f_i : C_i \to C_{i-1}$ is surjective, and $K_i = \ker(f_i) \cong M_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$. The integer $n \geq 0$ is called the \textit{length} of the cofiltration, and the modules $K_1, \ldots, K_n$ are called the \textit{factors} of the cofiltration. Let $(M, C)$ and $(M', C')$ be objects of $\text{CoFilt}(\mathcal{F})$ of lengths $n, n' \geq 0$, and let $N = \max\{n, n'\}$. A morphism $\phi : (M, C) \to (M', C')$ is a collection $\{\phi_i \in \text{Hom}_R(C_i, C'_i) : 0 \leq i \leq N\}$ of $R$-linear homomorphisms such that $\phi_{i-1}f_i = f'_i\phi_i$ for $1 \leq i \leq N$. By convention, $C_i = M$ if $i > n$ and $C'_i = M'$ if $i > n'$.

Similarly, we consider the category $\text{Filt}(\mathcal{F})$ of \textit{modules with filtration} over the family $\mathcal{F}$, defined in the following way: An object of $\text{Filt}(\mathcal{F})$ is a couple $(M, F)$, where $M$ is a left $R$-module and $F$ is a filtration of $M$ of the form

$$0 = F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_0 = M,$$

such that $K_i = F_{i-1}/F_i \cong M_{\alpha(i)}$ with $\alpha(i) \in I$ for $1 \leq i \leq n$. The integer $n \geq 0$ is called the \textit{length} of the filtration, and the modules $K_1, \ldots, K_n$ are called the \textit{factors} of the filtration. Let $(M, F)$ and $(M', F')$ be objects in $\text{Filt}(\mathcal{F})$ with lengths $n, n' \geq 0$, and let $N = \max\{n, n'\}$. A morphism $\phi : (M, F) \to (M', F')$ is a homomorphism $\phi \in \text{Hom}_R(M, M')$ such that $\phi(F_i) \subseteq F'_i$ for $1 \leq i \leq N$. By convention, $F_i = 0$ if $i > n$ and $F'_i = 0$ if $i > n'$.

Clearly, the categories $\text{CoFilt}(\mathcal{F})$ and $\text{Filt}(\mathcal{F})$ are equivalent, since filtrations and cofiltrations are dual notions: If a cofiltration $C$ of $M$ of length $n \geq 0$ is given, let $F$ be the filtration defined by $F_i = \ker(M \to C_i)$ for $0 \leq i \leq n$. Then the assignment $(M, C) \mapsto (M, F)$ defines a functor $\text{CoFilt}(\mathcal{F}) \to \text{Filt}(\mathcal{F})$. Conversely, if a filtration $F$ of $M$ of length $n \geq 0$ is given, let $C$ be the cofiltration defined by $C_i = M/F_i$ for $0 \leq i \leq n$, with the natural surjections $f_i : C_i \to C_{i-1}$. Then the assignment $(M, F) \mapsto (M, C)$ defines a functor $\text{Filt}(\mathcal{F}) \to \text{CoFilt}(\mathcal{F})$. We see that these functors are inverses of each other, and therefore define an equivalence of categories between $\text{CoFilt}(\mathcal{F})$ and $\text{Filt}(\mathcal{F})$. Moreover, this equivalence preserves the length $n$ and the factors $K_1, \ldots, K_n$.

We say that an object $(M, C)$ in $\text{CoFilt}(\mathcal{F})$ is an \textit{iterated extension} of the family $\mathcal{F}$, and we define the category $\text{Ext}(\mathcal{F})$ of \textit{iterated extensions} of the family $\mathcal{F}$ to equal the category $\text{CoFilt}(\mathcal{F})$. Moreover, we say that the length of an iterated extension $(M, C)$ is the length $n$ of the cofiltration $C$, and the factors of $(M, C)$ are the factors $K_1, \ldots, K_n$ of the cofiltration $C$.

Clearly, an iterated extension $(M, C)$ of the family $\mathcal{F}$ of length $n \leq 1$ is given in the following way: If $n = 0$ then $M = 0$, with the trivial filtration $C_0 = 0$. If $n = 1$, then $M \in \mathcal{F}$, and the filtration $C$ is given by $C_1 = M$, $C_0 = 0$.

As the name suggests, iterated extensions of the family $\mathcal{F}$ of length $n \geq 2$ can be characterized in terms of extensions. Recall that given a pair $M', M''$ of left $R$-modules, $M$ is said to be an \textit{extension} of $M''$ by $M'$ if there exists an exact sequence $0 \to M' \to M \to M'' \to 0$ of left $R$-modules.

\textbf{Lemma 4.} Let $M$ be a left $R$-module and let $n \geq 2$ be an integer. Then the following conditions are equivalent:
(1) There exists a cofiltration $C$ of $M$ such that $(M, C)$ is an object of $\text{Ext}(\mathcal{F})$ of length $n$.

(2) $M$ is an extension of $M''$ by $M'$, where $(M', C'), (M'', C'')$ are objects of $\text{Ext}(\mathcal{F})$ of lengths $n', n''$, with $n' + n'' = n$ and $n', n'' < n$.

Proof. If $(M, C)$ is an iterated extension of the family $\mathcal{F}$ of length $n$, then $M$ is an extension of $C_{n-1}$ by $K_n$. But $K_n \in \mathcal{F}$ and $C_{n-1}$ is clearly an iterated extension of the family $\mathcal{F}$ of length $n - 1$. For the other implication, assume that $(M', C')$ and $(M'', C'')$ are iterated extensions of the family $\mathcal{F}$ of lengths $n', n''$. We construct a cofiltration $C$ of $M$ of length $n = n' + n''$ in the following way: Let $f : M' \to M$ and $g : M \to M''$ be the maps given by the extension $0 \to M' \to M \to M'' \to 0$, let $F'$ be the filtration of $M'$ corresponding to the cofiltration $C'$, and let $F''$ be the filtration of $M''$ corresponding to $C''$. We define $F_i = g^{-1}(F''_i)$ for $0 \leq i \leq n''$, and $F_i = f(F'_i \cap M)$ for $n'' < i < n$. Then $F$ is a filtration of $M$, and we have $F_{n''-1}/F_i \cong \ker(M \to C'_{n''-1})/\ker(M \to C'_i) \cong K_i''$ for $0 \leq i \leq n''$, $F_{n''-1}/F_i \cong K_{n''-i}$ for $n'' < i < n$. Let $C$ be the cofiltration of $M$ corresponding to the filtration $F$. Then $(M, C)$ is an iterated extension of the family $\mathcal{F}$ of length $n = n' + n''$. \hfill \square

Let $(M, C)$ be an iterated extension of the family $\mathcal{F}$ of length $n$. For $1 \leq i \leq n$, we have $K_i = \ker(C_i \to C_{i-1}) = M_{\alpha(i)}$ for a unique $\alpha(i) \in I$. The resulting vector $\alpha = (\alpha(1), \ldots, \alpha(n)) \in I^n$ is called the order vector of $(M, C)$. Clearly, it is uniquely defined by the cofiltration $C$ since the family $\mathcal{F}$ consists of pairwise non-isomorphic modules.

Moreover, for $2 \leq i \leq n$, the filtration $C$ induces the following commutative diagram of left $R$-modules

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_i & \longrightarrow & C_i & \longrightarrow & C_{i-1} & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \uparrow & & \uparrow & \\
0 & \longrightarrow & K_i & \longrightarrow & \ker(f_{i-1} \circ f_i) & \longrightarrow & K_{i-1} & \longrightarrow & 0,
\end{array}
\]

where the rows are exact. We denote the extensions corresponding to the upper and lower row by $\xi_i \in \text{Ext}^1_R(C_{i-1}, K_i)$ and $\tau_i \in \text{Ext}^1_R(K_{i-1}, K_i)$ respectively. The commutativity of the above diagram means that $\xi_i \mapsto \tau_i$ under the map

\[
\text{Ext}^1_R(K_{i-1} \to C_{i-1}, K_i) \to \text{Ext}^1_R(C_{i-1}, K_i) \to \text{Ext}^1_R(K_{i-1}, K_i)
\]

induced by the inclusion $K_{i-1} \to C_{i-1}$.

Let $\phi : (M, C) \to (M', C')$ be an isomorphism in $\text{Ext}(\mathcal{F})$. Then the homomorphisms $\phi_i : C_i \to C'_i$ are all isomorphisms. This proves the following useful result:

**Lemma 5.** Let $\phi : (M, C) \to (M', C')$ be an isomorphism in $\text{Ext}(\mathcal{F})$. Then $(M, C)$ and $(M', C')$ have the same length and order vector. Moreover, the isomorphism $\phi$ induces isomorphisms of Abelian groups

\[
\begin{align*}
\text{Ext}^1_R(C_{i-1}, K_i) & \to \text{Ext}^1_R(C'_i, K'_i) \\
\text{Ext}^1_R(K_{i-1}, K_i) & \to \text{Ext}^1_R(K'_i, K'_i)
\end{align*}
\]

for $2 \leq i \leq n$, where $n$ is the common length of $(M, C)$ and $(M', C')$. Under these isomorphisms, $\xi_i \mapsto \xi'_i$ and $\tau_i \mapsto \tau'_i$.

There is a forgetful functor $\text{Ext}(\mathcal{F}) \to \text{Mod}(R)$ given by $(M, C) \mapsto M$, where $\text{Mod}(R)$ denotes the category of left $R$-modules. The image of this functor defines a full subcategory of $\text{Mod}(R)$, which we denote by $\text{Mod}(\mathcal{F})$. Clearly, a left $R$-module $M$ is an object of $\text{Mod}(\mathcal{F})$ if and only if there exists a cofiltration $C$ of
\( M \) such that \((M, C)\) is an object of \( \text{Ext}(F) \). Lemma 5 translates to the following characterization of the category \( \text{Mod}(F) \):

**Proposition 6.** Let \( F \) be a family of non-zero, pairwise non-isomorphic left \( R \)-modules. Then the category \( \text{Mod}(F) \) is the minimal full sub-category of \( \text{Mod}(R) \) which contains \( F \) and is closed under extensions.

It follows that \( \text{Mod}(F) \subseteq \text{Mod}(R) \) is a full, exact subcategory which is closed under extensions. But in general, \( \text{Mod}(F) \) is not an exact Abelian sub-category: It does not necessarily contain the kernels, images and cokernels of its morphisms. However, we have the following result, due to Ringel:

**Proposition 7.** Let \( F \) be a family of non-zero, pairwise non-isomorphic left \( R \)-modules. Then \( \text{Mod}(F) \subseteq \text{Mod}(R) \) is a full, extension closed, exact Abelian subcategory and \( F \) is the set of simple objects in \( \text{Mod}(F) \), up to isomorphism, if and only if the following conditions hold:

\( 1 \) End\(_R\)(\( M_\alpha \)) is a division ring for all \( \alpha \in I \),

\( 2 \) Hom\(_R\)(\( M_\alpha, M_\beta \)) = 0 for all \( \alpha, \beta \in I \) with \( \alpha \neq \beta \).

**Proof.** This follows from Ringel [18], theorem 1.2 and the comments preceding the theorem. \( \square \)

Following Ringel, we shall say that an object \( M_\alpha \) in \( F \) is a point if \( \text{End}_R(M_\alpha) \) is a division ring, and that a pair \((M_\alpha, M_\beta)\) of objects in \( F \) are orthogonal if \( \text{Hom}_R(M_\alpha, M_\beta) = \text{Hom}_R(M_\beta, M_\alpha) = 0 \).

If \( F \) is a family of orthogonal points, we see that \( \text{Mod}(F) \) is a length category in the sense of Gabriel [8]. In the rest of this paper, we shall assume that \( F \) is a family of orthogonal points, unless otherwise specified.

To simplify notation, we shall sometimes say that a left \( R \)-module \( M \) is an iterated extension of \( F \) when \( M \) is an object of \( \text{Mod}(F) \). Since \( \text{Mod}(F) \) is a length category, there is a Jordan-Hölder theorem for \( \text{Mod}(F) \). Consequently, every cofiltration \( C \) of \( M \) has the same length \( n \) and the same (composition) factors \( K_1, \ldots, K_n \), up to a permutation.

We say that an object \( M \) in \( \text{Mod}(F) \) is uniserial if the lattice of sub-modules of \( M \) is a chain (which is the unique composition series of \( M \)). If \( M \) is uniserial, then clearly \( M \) is indecomposable. But in general, the class of indecomposable modules in \( \text{Mod}(F) \) is larger than the class of uniserial modules. We shall later give a characterization of when these classes coincide.

2. Ordered quivers and extension types

Let \( R \) be an associative ring, and let \( F = \{ M_\alpha : \alpha \in I \} \) be a family of orthogonal points. We shall define the extension type \( \Gamma \) of an iterated extension of the family \( F \). To give the extension type is equivalent to giving the length \( n \) and the order vector \( \alpha \), and we may therefore consider the extension type as a (discrete) invariant.

A quiver or directed graph is a graph \( \Gamma \), given by a set \( N \) of vertices, a set \( E \) of arrows, and maps \( s, e : E \to N \). The maps \( s, e \) define the starting node \( s(a) \) and the ending node \( e(a) \) of each arrow \( a \in E \), and we picture \( a \) as an arrow from node \( s(a) \) to node \( e(a) \). A quiver is said to be finite if \( N \) and \( E \) are finite sets, and connected if the underlying graph is connected. We shall only consider quivers which are finite and connected.

An ordered quiver is a quiver \( \Gamma \) together with a total order on the set \( E \) of edges of \( \Gamma \), such that \( e(a) = s(b) \) whenever \( a, b \in E \) and \( b \) is the successor of \( a \). Recall that \( b \) is a successor of \( a \) if \( a < b \) and the set \( \{ c \in E : a < c < b \} \) is empty. To fix notation, we shall sometimes write \( N = \{ 1, 2, \ldots, p \} \) and \( E = \{ a_{12}, a_{23}, \ldots, a_{n-1,n} \} \), where \( p \) is the number of nodes and \( n - 1 \) is the number of edges in \( \Gamma \). The underlying total
order of $E$ is given by $a_{12} < a_{23} < \cdots < a_{n-1,n}$, and the definition of an ordered quiver dictates that $1 \leq p \leq n$ and that $e(a_{i-1,i}) = s(a_{i,i+1})$ for $2 \leq i \leq n-1$.

Let $(M,C)$ be an object in $\text{Ext}(\mathcal{F})$ of length $n$ and with order vector $\alpha \in I^n$. We let $I(M,C) = \{ \alpha(i) : 1 \leq i \leq n \}$ be the minimal subset $I(M,C) \subseteq I$ such that $(M,C)$ is an iterated extension of the family $\mathcal{F}' = \{ M_\alpha : \alpha \in I(M,C) \}$.

We define the extension type $\Gamma$ of the iterated extension $(M,C)$ to be the ordered quiver given by $N = I(M,C)$, $E = \{ a_{i-1,i} : 2 \leq i \leq n \}$, and $s(a_{i-1,i}) = \alpha(i-1)$, $e(a_{i-1,i}) = \alpha(i)$.

We remark that the extension type $\Gamma$ only depends upon the length $n$ and the order vector $\alpha$ of $(M,C)$, so isomorphic extensions of extensions of $\mathcal{F}$ have the same extension type.

As an example, let us draw all the different extension types of extensions of length $n = 3$, the first interesting case. When $n = 3$, we must have $1 \leq p \leq 3$, and there are 5 isomorphism classes of ordered directed graphs:

\begin{align*}
(p=3) & \quad 1 \xrightarrow{a_{12}} 2 \xrightarrow{a_{23}} 3 \\
(p=2) & \quad 1 \xrightarrow{a_{12}} \xrightarrow{a_{23}} 2 \\
(p=1) & \quad 1 \xrightarrow{a_{12}} \xrightarrow{a_{23}} 2
\end{align*}

Let $k$ be an algebraically closed field, and let $\Gamma$ be an ordered quiver with $p$ vertices and $n-1$ arrows. There is a $k$-algebra $k[\Gamma]$ associated with the ordered quiver $\Gamma$. This algebra is in a natural way an object of the category $\mathbf{a}_p$, the category of complete Artinian $p$-pointed algebras. We shall briefly recall the definition of $\mathbf{a}_p$.

The category $\mathbf{A}_p$ is the category of $p$-pointed $k$-algebras: An object of $\mathbf{A}_p$ is an associative ring $S$ with structural ring homomorphisms

\[
\begin{array}{ccc}
S & \xrightarrow{f} & S' \\
\downarrow & & \downarrow \\
S & \xrightarrow{g} & k^p
\end{array}
\]

such that that composition $g \circ f = \text{id}$, and the morphisms in $\mathbf{A}_p$ are ring homomorphisms such that the natural diagrams commute (that is, ring homomorphisms $\phi : S \to S'$ such that $\phi \circ f = f'$ and $g' \circ \phi = g$). For each object $S$ in $\mathbf{A}_p$, we denote by $I = I(S) = \ker(g)$ the radical ideal of $S$. The category $\mathbf{a}_p$ is the full sub-category of $\mathbf{A}_p$ consisting of objects $S$ such that $S$ is Artinian and complete in the $I$-adic topology.

We recall some basic facts about $\mathbf{a}_p$. If $S$ is an object of $\mathbf{A}_p$, then $S$ is in $\mathbf{a}_p$ if and only if $S$ has finite dimension as vector space over $k$ and the radical $I$ is nilpotent. In this case, $I$ is the Jacobson radical of $S$. We denote by $\mathbf{a}_p(n)$ the full sub-category of $\mathbf{a}_p$ consisting of objects $S$ such that $I(S)^n = 0$. Furthermore, any object $S$ in $\mathbf{A}_p$ is a matrix ring in the following sense: Denote by $e_1, \ldots, e_p$ the idempotents $e_i = (0, \ldots, 1, \ldots, 0) \in k^p$ for $1 \leq i \leq p$, and let $S_{ij} = e_i S e_j$ for all $1 \leq i,j \leq p$. Then $S = \oplus S_{ij}$, and $S_{ij} S_{kl} \subseteq \delta_{jk} S_{il}$. We write $S = (S_{ij})$.

Let $\Gamma$ be an ordered quiver with $p$ vertices, and let the edges of $\Gamma$ be denoted $\{a_{12}, \ldots, a_{n-1,n}\}$ as usual. We define $k[\Gamma]$ to be the object in $\mathbf{a}_p(n)$ given by generators $x_{i-1,i} \in k[\Gamma]_{(i-1),i}$, with the relations

\[x_{j-1,j} x_{i-1,i} = 0 \text{ unless } i < j\]
for $2 \leq i,j \leq n$. It follows that $k[\Gamma]$ is a finite dimensional vector space over $k$: It has a natural basis consisting of the non-zero monomials in $\{x_{12}, \ldots, x_{n-1,n}\}$ of length at most $n-1$ (including $e_1, \ldots, e_p$, which are considered as monomials of length 0). Furthermore $I^0 = 0$, where $I$ is the radical of $k[\Gamma]$. So by construction, $k[\Gamma]$ is an object of $a_p(n)$ when $\Gamma$ is an ordered quiver with $p$ vertices and $n-1$ arrows.

Let us continue the example of iterated extensions of length $n = 3$. In this case, the algebras $k[\Gamma]$ associated to the ordered quivers $\Gamma$ shown above are the following:

$$(p=3) \begin{pmatrix} ke_1 & 0 & 0 \\ ke_{12} & ke_2 & 0 \\ ke_{23} & ke_3 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}$$

$$(p=2) \begin{pmatrix} ke_1 + ke_{12} & 0 \\ ke_{23} + ke_{23}ke_{12} & ke_2 \end{pmatrix} = \begin{pmatrix} k[\epsilon] & 0 \\ k[\epsilon] & k \end{pmatrix}$$

$$(p=1) \begin{pmatrix} ke_1 + ke_{12} + ke_{23} + ke_{23}ke_{12} \end{pmatrix} = k\{ke_{12}, ke_{23}\}/(ke_{12}ke_{12}ke_{23})$$

Notice that for each $\Gamma$, we have given two different descriptions of the algebra $k[\Gamma]$ in $a_p$: To the left, we indicate the natural $k$-linear basis of $k[\Gamma]$, and to the right, we give the multiplicative structure of $k[\Gamma]$ (recall that $\epsilon^2 = 0$). The multiplicative structure can be worked out from the natural $k$-linear basis and equation (1).

Finally, let us mention that an ordered quiver can be considered as a quiver with relations: Indeed, let $\Gamma$ be the underlying quiver of an ordered quiver, and consider the relations $a_{j-1,i}a_{i-1,i} = 0$ for all $i,j$ with $2 \leq j \leq i \leq n$.

### 3. Noncommutative deformations of modules

Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $F = \{M_1, \ldots, M_p\}$ be a finite family of left $R$-modules. There is a deformation functor

$${\text{Def}}_F : a_p \to \text{Sets},$$

describing the simultaneous deformations of the family $F$ of left $R$-modules, and a theory of noncommutative deformations of modules related to this functor: This theory is due to Laudal, and it is described in several preprints, see Laudal [11, 12, 13]. However, we find it more convenient to give references to Eriksen [7], which is a version of the theory adapted to the study of left modules.

Let us briefly recall the definition of the deformation functor $\text{Def}_F$: Let $S$ be an object in $a_p$. A lifting of the family $F$ to $S$ is a left $R \otimes_k S^{\text{op}}$-module $M_S$ together with isomorphisms $\eta_i : M_S \otimes_S k_i \to M_i$ of left $R$-modules for $1 \leq i \leq p$, such that $M_S \cong (M_i \otimes_k S_{ij})$ considered as right $S$-modules. Recall that $k_i$ is the image of the $i$'th projection of $k^p$, which is an $S$-module via $\gamma : S \to k^p$. We let $\text{Def}_F(S)$ denote the set of all equivalence classes of liftings of the family $F$ to $S$, where $M_S$ and $M'_S$ are equivalent liftings if there is an isomorphism $\tau : M_S \to M'_S$ of left $R \otimes_k S^{\text{op}}$-modules such that the natural diagrams commute (that is, such that $\eta'_i \circ (\tau \otimes_S k_i) = \eta_i$ for $1 \leq i \leq p$).

For the rest of this sections, we shall assume that $\text{dim}_k \text{Ext}_R^m(M_i, M_j)$ is finite for $1 \leq i,j \leq p, m = 1,2$. In this case, there exists a pro-representable hull...
$H = H(\mathcal{F})$ for the deformation functor $\text{Def}_\mathcal{F}$ (which is unique up to non-canonical isomorphism):

**Theorem 8** (Laudal). Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $\mathcal{F} = \{M_1, \ldots, M_p\}$ be a finite family of left $R$-modules such that $\dim_k \text{Ext}^m_R(M_i, M_j)$ is finite for $1 \leq i, j \leq p$, $m = 1, 2$. Then there exists a pro-representable hull $H$ for the deformation functor $\text{Def}_\mathcal{F} : a_p \to \text{Sets}$.

**Proof.** The hull $H$ can be constructed along well-known lines, via the obstruction morphism $O : T^2 \to T^1$ (see Eriksen [7], theorem 4.2), or via (non-symmetric) matrix Massey products (outlined in Laudal [13]).

Recall that a pro-representable hull $H$ for the functor $\text{Def}_\mathcal{F}$ is an object of the pro-category $\hat{a}_p$ such that there exists a smooth morphism of functors on $a_p$

$$\text{Mor}(H, -) \to \text{Def}_\mathcal{F},$$

which is an isomorphism when restricted to a morphism of functors on $a_p(2)$. The pro-category $\hat{a}_p$ is the full sub-category of $A_p$ consisting of objects $S$ which are complete in the $I$-adic topology and such that $S_n = S/I^n$ is an object in $a_p(n)$ for all $n \geq 1$.

**Proposition 9.** Let $H$ be the pro-representable hull of the deformation functor $\text{Def}_\mathcal{F} : a_p \to \text{Sets}$, and let $S$ be any object in $a_p$. Then $\text{Mor}(H, S)$ has a natural structure as an affine scheme over $k$. In particular, $\text{Mor}(H, k[\Gamma])$ is an affine scheme over $k$ for any ordered quiver $\Gamma$ with $p$ nodes.

**Proof.** Let $V_{ij}^m = \text{Ext}^m_R(M_j, M_i)^*$ for $1 \leq i, j \leq p$, $m = 1, 2$, and choose $k$-linear bases $\{s_{ij}(\alpha) : 1 \leq \alpha \leq d_{ij}\}$ for $V_{ij}^1$ and $\{t_{ij}(\beta) : 1 \leq \beta \leq r_{ij}\}$ for $V_{ij}^2$. Then $T^1$ is the (free) formal matrix ring with generators $\{s_{ij}(\alpha)\}$, $T^2$ is the (free) formal matrix ring with generators $\{t_{ij}(\beta)\}$, and the obstruction morphism is a morphism $O : T^2 \to T^1$ in $\hat{a}_p$. Let $f_{ij}(\beta) = o(t_{ij}(\beta)) \in T_{ij}^1$ for $1 \leq i, j \leq p$, $1 \leq \beta \leq r_{ij}$. There is a natural surjection $T^1 \to H$, and its kernel is generated by the relations $\{f_{ij}(\beta) : 1 \leq i, j \leq p, 1 \leq \beta \leq r_{ij}\}$. Clearly this surjection induces an injective map of sets $\text{Mor}(H, S) \to \text{Mor}(T^1, S)$. Since $I(S)^n = 0$ for some $n \geq 1$, we have that

$$\text{Mor}(T^1, S) = \text{Mor}(T^1, S) = \prod_{i, j} \text{Hom}_k(V_{ij}^1, W_{ij}),$$

where $W_{ij} = I(S)_{ij}$ with basis $\{w_{ij}(\gamma) : 1 \leq \gamma \leq v_{ij}\}$. So $\text{Mor}(T^1, S) \cong A^N$, where $N = \sum d_{ij}v_{ij}$. We obtain a set of coordinates $\{z_{ij}(\alpha, \gamma)\}$ for $A^N$, where the coordinate $z_{ij}(\alpha, \gamma)$ corresponds to a morphism $\phi_{ij}(\alpha, \gamma) \in \text{Mor}(T^1, S)$ given by

$$\phi_{ij}(\alpha, \gamma)(s_{ij}(\alpha')) = \delta_{\alpha, \alpha'}w_{ij}(\gamma)$$

for $1 \leq i, j \leq p$, $1 \leq \gamma \leq v_{ij}$, $1 \leq \alpha, \alpha' \leq d_{ij}$. Let $\phi = (z_{ij}(\alpha, \gamma)) \in \text{Mor}(T^1, S)$, then $\phi \in \text{Mor}(H, S)$ if and only if $\phi(f_{ij}(\beta)) = 0$ for all $1 \leq i, j \leq p$, $1 \leq \beta \leq r_{ij}$. But we have

$$\phi(f_{ij}(\beta)) = \sum_\gamma f_{ij}(\beta)(\{a_{ij}(\alpha, \gamma) : 1 \leq \alpha \leq d_{ij}\}) w_{ij}(\gamma),$$

so $\phi(f_{ij}(\beta)) = 0$ if and only if we have the equations

$$f_{ij}(\beta)(\{a_{ij}(\alpha, \gamma) : 1 \leq \alpha \leq d_{ij}\}) = 0$$

for $1 \leq i, j \leq p$, $1 \leq \beta \leq r_{ij}$, $1 \leq \gamma \leq v_{ij}$. Notice that $\phi(I(T^1)^n) = 0$, so the above equations corresponds to polynomial relations $R_{ij}(\beta, \gamma) \in k[[z_{ij}(\alpha, \gamma)]]$, and therefore $\text{Mor}(H, S)$ is the affine sub-scheme of $\text{Mor}(T^1, S) \cong A^N$ defined by those relations. \qed
We denote by $X(\mathcal{F}, \Gamma)$ the affine scheme $\text{Mor}(H, k[\Gamma])$. Moreover, we denote by $M_H$ be the versal family $M_H \in \text{Def}_F(H)$ corresponding to the smooth morphism $\text{Mor}(H, -) \to \text{Def}_F$ via Yoneda lemma. For each point $\phi \in X(\mathcal{F}, \Gamma)$, there is a deformation $M_\phi = \text{Def}_F(\phi)(M_H) \in \text{Def}_F(k[\Gamma])$. By construction, the morphism $\text{Mor}(H, -) \to \text{Def}_F$ is smooth, so the map of sets

$$X(\mathcal{F}, \Gamma) \to \text{Def}_F(k[\Gamma]),$$

given by $\phi \mapsto M_\phi$, is surjective.

We shall explain how to calculate the surjection above in concrete terms: Let $\phi \in X(\mathcal{F}, \Gamma)$, so $\phi : H \to k[\Gamma]$ is a morphism in $\mathcal{F}_p$. Let $M_H \in \text{Def}_F(H)$ be the versal family defined over $H$. Then the deformation $M_\phi \in \text{Def}_F(k[\Gamma])$ is given by

$$M_\phi = (M_i \otimes_k k[\Gamma]_{ij}),$$

considered as a right $k[\Gamma]$-module, and the left $R$-module structure of $M_\phi$ is determined by

$$r(m_i \otimes e_i) = (\text{id} \otimes \phi)(r(m_i \otimes e_i))$$

for all $r \in R$, $m_i \in M_i$. The expression $r(m_i \otimes e_i)$ on the right hand side of equation (2) is the left multiplication of $r \in R$ with $m_i \otimes e_i$ considered as an element of $M_H$. This makes it possible to compute $M_\phi \in \text{Def}_F(k[\Gamma])$ when $\phi \in X(\mathcal{F}, \Gamma)$ is given, assuming that the versal family $M_H$ can be computed.

For the rest of this section, assume that $\mathcal{F} = \{M_1, \ldots, M_p\}$ is a finite family of non-zero, pairwise non-isomorphic left $R$-modules, and consider the category $\text{Ext}(\mathcal{F})$ of iterated extensions of the family $\mathcal{F}$. For any ordered quiver $\Gamma$ with vertices $N = \{1, 2, \ldots, p\}$, we denote by $\text{E}(\mathcal{F}, \Gamma)$ the set of isomorphism classes of extensions of extensions of the family $\mathcal{F}$ with extension type $\Gamma$. It is clear that the forgetful functor $\text{Ext}(\mathcal{F}) \to \text{Mod}(\mathcal{F})$ maps $\text{E}(\mathcal{F}, \Gamma)$ to a set of isomorphism classes of left $R$-modules, which we shall denote by $\text{M}(\mathcal{F}, \Gamma)$. Moreover, the above construction defines a natural surjective map $\text{E}(\mathcal{F}, \Gamma) \to \text{M}(\mathcal{F}, \Gamma)$.

**Theorem 10** (Laudal). Let $\mathcal{F} = \{M_1, \ldots, M_p\}$ be a finite family of non-zero, pairwise non-isomorphic left $R$-modules such that $\dim_k \text{Ext}^n_R(M_i, M_j)$ is finite for $1 \leq i, j \leq p$, $m = 1, 2$, and let $\Gamma$ be an ordered quiver with vertices $N = \{1, 2, \ldots, p\}$. Then there is a natural bijection between $\text{Def}_F(k[\Gamma])$ and $\text{E}(\mathcal{F}, \Gamma)$.

**Proof.** Let the $R$-$k[\Gamma]$ bimodule $(M_i \otimes_k k[\Gamma]_{ij})$ be a lifting of the family $\mathcal{F}$ to $k[\Gamma]$. We show how to construct an iterated extension $(M, C)$ of the family $\mathcal{F}$ with extension type $\Gamma$: We let $M' = (M_i \otimes_k k[\Gamma]_{i, \alpha(\Gamma)}) \subseteq (M_i \otimes_k k[\Gamma]_{ij})$, this is by construction invariant under left multiplication by $R$. Consider sequences $j = (j_1, \ldots, j_r)$ with $1 \leq j_1 < j_2 < \cdots < j_r \leq n - 1$ and $\alpha(j_h + 1) = \alpha(j_{h+1})$ for $1 \leq h \leq r - 1$. We say that $j$ is broken if $j_h + 1 \neq j_{h+1}$ for some $h$, otherwise $j$ is unbroken. Let us denote by $M'(j)$ the $k$-linear space $M_{j_r+1} \otimes x_{j_r, j_r+1} \cdots x_{j_1, j_1+1}$, by $M'(B)$ the sum of all $k$-linear spaces $M'(j)$ with $j$ broken, and $M(U)$ the sum of all $k$-linear spaces $M'(j)$ with $j$ unbroken. Notice that $M'(B)$ is invariant under left multiplication with $R$. We define $M = M' / M'(B)$, which has a natural left $R$-module structure, and clearly $M \cong M(U)$ considered as a $k$-linear space. For $0 \leq i \leq n$, let $F_i$ be the sum of all $k$-linear spaces $M'(j)$ where $j$ is unbroken of length at least $i$. Then $F_i \subseteq M$ are also invariant under left multiplication with $R$, so $C$ is a co-filtration of $M$ when $C_i = M / F_i$ for $0 \leq i \leq n$. Clearly, this co-filtration satisfies $K_i = \ker(C_i \to C_{i-1}) \cong M_{\alpha(i)}$ for $1 \leq i \leq n$. It follows that $(M, C)$ is an extension of extensions of the family $\mathcal{F}$ of extension type $\Gamma$. It is also straightforward to check that equivalent liftings of $\mathcal{F}$ to $k[\Gamma]$ gives isomorphic iterated extensions: Any isomorphism between liftings will map $M'$ to $M'$, $M'(B)$ to $M'(B)$, and $F_i$ to $F_i$. So we have constructed a well-defined map from $\text{Def}_F(k[\Gamma])$ to $\text{E}(\mathcal{F}, \Gamma)$. 


It only remains to see that this map is a bijection: But given an extension of extensions \((M,C)\) of the family \(\mathcal{F}\), consider \((M_i \otimes_k k[\Gamma]_{ij})\) as a right \(k[\Gamma]\)-module. It is easy to see that the left \(R\)-module structure of the co-filtration \(C\) on \(M\) will generate a left \(R\)-module structure on \((M_i \otimes_k k[\Gamma]_{ij})\) compatible with the right \(k[\Gamma]\)-module structure. Furthermore, isomorphic iterated extensions give equivalent liftings of \(\mathcal{F}\) to \(k[\Gamma]\). So we have constructed an inverse to the map of sets described above.

\(\square\)

**Corollary 11.** There are natural surjections \(X(\mathcal{F}, \Gamma) \to E(\mathcal{F}, \Gamma) \to M(\mathcal{F}, \Gamma)\). In particular, the sets \(E(\mathcal{F}, \Gamma)\) and \(M(\mathcal{F}, \Gamma)\) are quotients of the affine scheme \(X(\mathcal{F}, \Gamma)\).

We remark that these quotients are computable, in principle: In proposition \(\text{[9]}\) we have shown how to construct the affine scheme \(X(\mathcal{F}, \Gamma)\) when the hull \(H\) of \(\text{Def}_F\) is known. Moreover, we have shown above how to calculate the surjection \(X(\mathcal{F}, \Gamma) \to \text{Def}_F(k[\Gamma])\) when the versal family \(M_{H} \in \text{Def}_F(H)\) is known. The identification \(\text{Def}_F(k[\Gamma]) \cong E(\mathcal{F}, \Gamma)\) is explicitly given in the proof of theorem \(\text{[10]}\) and the surjection \(E(\mathcal{F}, \Gamma) \to M(\mathcal{F}, \Gamma)\) is natural, induced by the forgetful functor \((M, C) \mapsto M\).

4. Species

We say that \(S = (K_{\alpha}, E_{\alpha,\beta})\) is a species indexed by the set \(I\) if \(K_{\alpha}\) is a division ring and \(E_{\alpha,\beta}\) is a \(K_{\beta} - K_{\alpha}\) bimodule for all \(\alpha, \beta \in I\). Let \(k\) be a fixed commutative field. We say that \(S\) is a \(k\)-species if \(k\) is contained in \(K_{\alpha}\) for all \(\alpha \in I\) in such a way that \(c\xi = \xi c\) for all \(c \in k, \xi \in E_{\alpha,\beta}\) and \(\dim_k K_{\alpha}\) is finite. Moreover, \(S\) is called a \(k\)-quiver if in addition \(K_{\alpha} = k\) for all \(\alpha \in I\).

If \(S\) is a \(k\)-quiver, it is completely determined by the Gabriel quiver of \(S\), formed in the following way: The set of vertices of the Gabriel quiver is \(I\), and for each pair of vertices \(\alpha, \beta \in I\), there are \(\dim_k E_{\alpha,\beta}\) arrows from \(\alpha\) to \(\beta\).

Let \(\mathcal{F}\) be a family of orthogonal points in \(\text{Mod}(R)\) indexed by \(I\), and consider the corresponding length category \(\text{Mod}(\mathcal{F})\). We define the species of \(\text{Mod}(\mathcal{F})\) to be the species indexed by \(I\) given by \(K_{\alpha} = \text{End}_R(M_{\alpha})\) and \(E_{\alpha,\beta} = \text{Ext}^1_R(M_{\alpha}, M_{\beta})\) for all \(\alpha, \beta \in I\).

In what follows, we shall be particularly interested in the case when \(k\) is an algebraically closed commutative field, \(R\) is a \(k\)-algebra, and \(\mathcal{F}\) is a family of orthogonal points in \(\text{Mod}(R)\) such that \(\text{End}_R(M_{\alpha}) = k\) for all \(\alpha \in I\). In this case, the species of \(\text{Mod}(\mathcal{F})\) is clearly a \(k\)-quiver.

It is well-known that the species of the length category \(\text{Mod}(\mathcal{F})\) contains a lot of information about the category, see for instance Gabriel \(\text{[3]}\). This fact can be explained by a general principle: The length category \(\text{Mod}(\mathcal{F})\) is completely determined by its species \(S\) and the obstruction theory of the family \(\mathcal{F}\) of simple objects. In theorem \(\text{[10]}\) we have proved this principle under some finiteness conditions, using non-commutative deformations of modules.

5. The hereditary case

In the previous section, we have seen that the length category \(\text{Mod}(\mathcal{F})\) is determined by its species \(S\) and the obstruction theory of \(\mathcal{F}\). We shall consider the unobstructed cases, which are the easiest ones. That is, we shall consider the cases in which \(\text{Mod}(\mathcal{F})\) is completely determined by its species \(S\).

We say that \(\text{Mod}(\mathcal{F})\) is hereditary if \(\text{Ext}^2_R(M_{\alpha}, M_{\beta}) = 0\) for all \(\alpha, \beta \in I\). This is clearly a sufficient (but not necessary) condition for the length category \(\text{Mod}(\mathcal{F})\) to be unobstructed. To understand how \(\text{Mod}(\mathcal{F})\) is related to the species \(S\) in this case, we shall use some results of Deng, Xiao \(\text{[9]}\):
In the rest of this section, let $k$ be an algebraically closed commutative field, let $R$ be a $k$-algebra, and let $\mathcal{F}$ be a family of orthogonal points in $\text{Mod}(R)$ such that $\text{Mod}(\mathcal{F})$ is a hereditary category and $\text{End}_R(M_\alpha) = k$ for all $\alpha \in I$. We shall also assume that the following finiteness conditions hold:

- $I$ is a finite set,
- $\dim \text{Ext}^1_R(M_\alpha, M_\beta)$ is finite for all $\alpha, \beta \in I$.

The species $S$ of the hereditary length category $\text{Mod}(\mathcal{F})$ is clearly a $k$-quiver, so it is completely determined by the corresponding Gabriel quiver $Q$. The finiteness conditions above means that the Gabriel quiver is finite.

We consider the finite representations $V$ of $Q$: These consist of a finite dimensional vector space $V_\alpha$ over $k$ for all $\alpha \in I$, and a $k$-linear map $V_\alpha : V_\alpha \rightarrow V_\beta$ for all arrows $\alpha : \alpha \rightarrow \beta$. We say that a finite representation $V$ is small or nilpotent if there is a positive integer $n \geq 1$ such that $V_p = 0$ for all paths $p$ of length $n$ in $Q$.

**Theorem 12** (Deng-Xiao). Let $\mathcal{F}$ be a family of left $R$-modules satisfying the conditions above. Then there is a natural exact equivalence of categories between $\text{Mod}(\mathcal{F})$ and the category of small representations of Gabriel quiver $Q$.

**Proof.** See Deng, Xiao [6, theorem 1.5].

In other words, the category $\text{Mod}(\mathcal{F})$ is completely determined by the Gabriel quiver $Q$, and therefore also by the species of $\text{Mod}(\mathcal{F})$. Notice that if $Q$ is loop-free, then all finite representations are small. In this case, we can tell when the category $\text{Mod}(\mathcal{F})$ is finite, tame or wild by considering the well-known classification of quivers into these classes.

An interesting question is the following: Let $Q$ be a finite quiver with loops. When is the category of small representations of $Q$ finite, tame and wild? We do not know if the answer to this question is known.

## 6. Indecomposable and Uniserial Objects

Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $\mathcal{F} = \{M_\alpha : \alpha \in I\}$ be a family of orthogonal points. We are interested in the indecomposable modules in the length category $\text{Mod}(\mathcal{F})$.

Since $\text{Mod}(\mathcal{F})$ is a length category, we have a Krull-Remak-Schmidt-Azumaya theorem. So every object in $\text{Mod}(\mathcal{F})$ has a finite indecomposable decomposition, unique up to a permutation, and $\text{End}_R(M)$ is a local ring for all indecomposable objects $M \in \text{Mod}(\mathcal{F})$. We say that $\text{Mod}(\mathcal{F})$ is a uniserial category if every indecomposable object in $\text{Mod}(\mathcal{F})$ is uniserial.

**Lemma 13.** Let $M$ be an object of $\text{Mod}(\mathcal{F})$, and consider the following conditions:

1. $M$ is uniserial,
2. $M$ has a unique minimal submodule,
3. $M$ is indecomposable.

Then we have $1) \Rightarrow 2) \Rightarrow 3)$. In particular, all conditions are equivalent if and only if $\text{Mod}(\mathcal{F})$ is a uniserial category.

**Proof.** The implication $1) \Rightarrow 2)$ is obvious. If $M = N_1 \oplus N_2$ is a direct decomposition with $N_i \neq 0$ for $i = 1, 2$, there are minimal submodules $K_i \subseteq N_i$ for $i = 1, 2$. This shows that $2) \Rightarrow 3)$. The last part is clear.

The implication $3) \Rightarrow 1)$ in the above lemma clearly holds if $M$ has length $n = 2$. But already in the case $n = 3$, it is very easy to come up with counterexamples.

**Lemma 14.** Let $\mathcal{F}$ be a family of orthogonal points. If $\mathcal{F}$ contains modules $S, T$ such that $\text{End}_R(K) = k$ for $K = S, T$ and $\dim_k \text{Ext}^1_R(S, T) \geq 2$, then there exists
an object $M$ in $\text{Mod}(F)$ of length $n = 3$ such that $M$ is a non-uniserial $R$-module with a unique simple submodule.

Proof. From McConnel, Robson \[14\], proposition 3.3 it follows that there exists non-split extensions $U,V$ of $S$ by $T$ such that $U$ and $V$ are not isomorphic as $R$-modules. Let $M$ be the cokernel of the diagonal map $T \to U \oplus V$, then $M$ is a non-uniserial module with the unique simple submodule $T$ by McConnell, Robson \[14\], proposition 6.1.

Lemma 15. Let $F$ be a family of orthogonal points. If $F$ contains modules $S,T,U$ such that $\text{End}_R(K) = k$ for $K = S,T,U$, $\text{Ext}^1_R(U,V)$, $\text{Ext}^1_R(U,T)$, $\text{Ext}^1_R(U,S)$, $\text{Ext}^1_R(U,U) \neq 0$, and $S,T$ are non-isomorphic, then there exists an object $M$ in $\text{Mod}(F)$ of length $n = 3$ such that $M$ is indecomposable and such that $S$ and $T$ are simple submodules of $M$.

Proof. Let $\xi_1, \xi_2 \neq 0$ be extensions of $U$ by $S$ and $U$ by $T$, and let $\psi_i$ be representatives of $\xi_i$ in Hochschild cohomology for $i = 1,2$. Let furthermore $M$ be the extension of $U$ by $S \oplus T$ given by $\langle \xi_1, \xi_2 \rangle$. Then $M \cong S \oplus T \oplus U$ as a vector space over $k$, and the $R$-module structure of $M$ is defined by the representatives $\psi_1, \psi_2$. A calculation shows that $\text{End}_R(M) = k$ if $U$ is not isomorphic to any of $S,T$, and that $\text{End}_R(M) = k[x]/(x^2)$ if $U$ is isomorphic to one of the modules $S,T$. In either case, $\text{End}_R(M)$ is a local ring, and therefore $M$ is indecomposable. Since $S \oplus T \subseteq M$, it is clear that $M$ has simple submodules $S,T$. \[\square\]

Lemma 16. Let $F$ be a family of orthogonal points. If $F$ contains modules $S,T,U$ such that $\text{End}_R(K) = k$ for $K = S,T,U$, $\text{Ext}^1_R(S,U)$, $\text{Ext}^1_R(T,U)$, $\text{Ext}^1_R(U,S)$, $\text{Ext}^1_R(U,U) \neq 0$, and $S,T$ are non-isomorphic, then there exists an object $M$ in $\text{Mod}(F)$ of length $n = 3$ such that $M$ is indecomposable but not uniserial.

Proof. The proof is similar to the proof of the lemma \[15\]. We consider $M = U \oplus S \oplus T$ considered as a vector space over $k$, and let the $R$-module structure of $M$ be given by non-split extension $\xi_S \in \text{Ext}^1_R(S,U)$ and $\xi_T \in \text{Ext}^1_R(T,U)$ via Hochschild cohomology. A calculation shows that $\text{End}_R(M) = k$ if $U$ is not isomorphic to $S$ or $T$, and $\text{End}_R(M) \cong k[x]/(x^2)$ otherwise, so $M$ is indecomposable in both cases. On the other hand, $M$ has to submodules of length 2, so $M$ is not uniserial. \[\square\]

Corollary 17. Let $F$ be a family of orthogonal points such that the species of $F$ is a $k$-quiver. If $\text{Mod}(F)$ is a uniserial category, then we have

\begin{align*}
1. \sum_{\beta \in I} \dim_k (\text{Ext}^1_R(M_\alpha, M_\beta)) &\leq 1 \text{ for all } \alpha \in I, \\
2. \sum_{\alpha \in I} \dim_k (\text{Ext}^1_R(M_\alpha, M_\beta)) &\leq 1 \text{ for all } \beta \in I.
\end{align*}

Proof. This follows from lemma \[14\], \[15\] and \[16\] \[\square\]

We shall later see that these conditions are also sufficient for $\text{Mod}(F)$ to be a uniserial category. We remark that this criterion has been known since the 60’s, see Amdal, Ringdal \[1\] and Gabriel \[8\], section 8.3.

7. The uniserial case

Let $k$ be an algebraically closed commutative field, let $R$ be an associative $k$-algebra, and let $F$ be a family of orthogonal points in $\text{Mod}(R)$ such that the species of $F$ is a $k$-quiver. We shall determine when $\text{Mod}(F)$ is a uniserial category. In the process, we shall also show that if $\text{Mod}(F)$ is uniserial then it is unobstructed, regardless if it is hereditary or not.

We recall that the species of $F$ is a $k$-quiver if and only if $\text{End}_R(M_\alpha) \cong k$ for all $\alpha \in I$, which is equivalent to the condition that for any $\alpha \in I$ and any endomorphism $\phi \in \text{End}_R(M_\alpha)$, $\phi$ is algebraic over $k$. By Quillen’s lemma, this is the case when $M$ is any simple module over a ring $R = \text{Diff}(A)$ of $k$-linear
differential operators on $A$, such that $\text{gr Diff}(A)$ is a finitely generated $k$-algebra, see Quillen [17]. In particular, any family $\mathcal{F}$ of simple modules over the first Weyl algebra $R = A_1(k)$ satisfy these conditions.

In the first part of this section, we shall assume that the family $\mathcal{F}$ satisfy the following additional conditions:

\[(*) \sum_{\beta \in I} \dim_k(\text{Ext}^1_R(M_\alpha, M_\beta)) \leq 1 \text{ for all } \alpha \in I, \]
\[\sum_{\alpha \in I} \dim_k(\text{Ext}^1_R(M_\alpha, M_\beta)) \leq 1 \text{ for all } \beta \in I.\]

When $\mathcal{F}$ satisfy $(*)$, we shall classify all iterated extensions $(M, C)$ in $\text{Ext}(\mathcal{F})$ such that $M$ is indecomposable, up to isomorphism in $\text{Ext}(\mathcal{F})$, and all indecomposable modules $M \in \text{Mod}(\mathcal{F})$ up to isomorphism. However, it is useful to start looking at iterated extensions $(M, C)$ such that $\xi_i \neq 0$ for $2 \leq i \leq n$:

**Lemma 18.** Let $\mathcal{F}$ be a family satisfying $(*)$, and let $(M, C)$ be an iterated extension of the family $\mathcal{F}$ of length $n \geq 2$ such that $\xi_i \neq 0$ for $2 \leq i \leq n$. For any module $K \in \mathcal{F}$, the map $\text{Ext}^1_R(C_{i-1}, K) \to \text{Ext}^1_R(K_{i-1}, K)$ induced by the inclusion $K_{i-1} \subseteq C_{i-1}$ is an isomorphism for $2 \leq i \leq n$.

**Proof.** We show the result by induction on $n$. Since $C_1 = K_1$ by definition, the result is clearly true for $n = 2$. So let $n \geq 3$, and assume that the result holds for all integers less than $n$. In particular, this implies that

\[\text{Ext}^1_R(C_{n-2}, K_{n-1}) \to \text{Ext}^1_R(K_{n-2}, K_{n-1})\]

is an isomorphism, and consequently $\text{Ext}^1_R(K_{n-2}, K_{n-1}) \neq 0$. Consider the long exact sequence of the functor $\text{Hom}_R(\cdot, K)$ applied to the extension $\xi_{n-1}$. Since $\xi_{n-1} \neq 0$ and $\text{End}_R(K_{n-1}) \cong k$, it follows that $\text{Hom}_R(K_{n-1}, K) \to \text{Ext}^1_R(C_{n-2}, K)$ is injective, and therefore the sequence

\[0 \to \text{Hom}_R(K_{n-1}, K) \to \text{Ext}^1_R(C_{n-2}, K) \to \text{Ext}^1_R(C_{n-1}, K) \to \text{Ext}^1_R(K_{n-2}, K)\]

is exact. But $\text{Hom}_R(K_{n-1}, K) \cong \text{Ext}^1_R(K_{n-2}, K)$, since $\text{Ext}^1_R(K_{n-2}, K) \neq 0$ if and only if $K \cong K_{n-1}$. So $\text{Hom}_R(K_{n-1}, K) \to \text{Ext}^1_R(C_{n-2}, K)$ is an isomorphism by the induction hypothesis. So the map $\text{Ext}^1_R(C_{n-1}, K) \to \text{Ext}^1_R(K_{n-1}, K)$ is injective. If $K = K_n$, then this map is also surjective, since it maps $\xi_n$ to $\tau_n$, and $\xi_n \neq 0$. If $K \neq K_n$, then $\text{Ext}^1_R(K_{n-1}, K) = 0$ and the map is an isomorphism as well. \(\square\)

**Corollary 19.** Let $\mathcal{F}$ be a family satisfying $(*)$, and let $(M, C)$ be an iterated extension of the family $\mathcal{F}$ such that $\xi_i \neq 0$ for $2 \leq i \leq n$. Then $\tau_i \neq 0$ for $2 \leq i \leq n$. In particular, the extension type of $(M, C)$ is uniquely determined by $(\alpha(1), n) \in I \times \mathbb{N}$.

It is also useful to notice that if $(M, C)$ is an iterated extension of the type considered above, then $M$ is an indecomposable left $R$-module:

**Lemma 20.** Let $\mathcal{F}$ be a family satisfying $(*)$, and let $(M, C)$ be an iterated extension of $\mathcal{F}$ of length $n$ such that $\xi_i \neq 0$ for $2 \leq i \leq n$. Then $M$ is indecomposable.

**Proof.** Because of lemma 18, it is enough to show that $K_n \subseteq M$ is the unique minimal submodule of $M$: Assume that $K \in \mathcal{F}$ and that $\phi : K \to M$ is injective. Because of lemma 18, we may assume that $M \cong K_n \oplus \cdots \oplus K_1$ as a vector space over $k$. Moreover, we may assume that the left $R$-module structure of $M$ is given by

\[r(k_n, \ldots, k_1) = (rk_n + \psi^n_n(k_{n-1}), rk_{n-1} + \psi^{n-1}_r(k_{n-2}), \ldots, rk_2 + \psi^2_r(k_1), rk_1)\]

for all $r \in R$, $(k_n, \ldots, k_1) \in M$, where $\psi^i \in \text{Der}_k(R, \text{Hom}_k(K_{i-1}, K_i))$ represents $\tau_i \in \text{Ext}^1_R(K_{i-1}, K_i)$ in Hochschild cohomology. Clearly, there are $k$-linear maps
φ₁ : K → Kᵢ such that φ(k) = (φ₁(k), ..., φₙ(k)), and φₙ : K → Kₙ is R-linear as well. But a simple calculation, using the fact that φ is R-linear and Endᵦ(Kₙ) ≅ k, shows that φₙ ≠ 0. So φₙ(K) = Kₙ ⊆ M, and therefore Kₙ ⊆ M is the unique minimal submodule of M.

Let (α, n) ∈ I × N. We consider the set of order vectors ξ ∈ Iⁿ such that α(1) = α and Ext₁ᵦ(Mₗₐₓ₁, Mₗₐₓ₂) ≠ 0 for 2 ≤ i ≤ n. We say that the couple (α, n) ∈ I × N is admissible if such an order vector exists. If this is the case, we know from (*) that the order vector ξ is uniquely determined by (α, n), and we shall say that ξ the order vector associated with the admissible couple (α, n).

Let (α, n) ∈ I × N be an admissible couple, and let ξ be the associated order vector. We denote by θ(α, n) the ordered quiver defined by the order vector ξ. If ℱ is a family satisfying (*) and (M, C) is an iterated extension of the family ℱ such that ξᵢ ≠ 0 for 2 ≤ i ≤ n, then the extension type of (M, C) is θ(α, n) with α = α(1) by corollary 19.

Let (M, C) be an iterated extension of the family ℱ of extension type θ. To simplify notation, we shall write (M, C) for the isomorphism class of (M, C) in E(ℱ, θ), and M for the isomorphism class of M in M(ℱ, θ).

We shall denote by IE(ℱ, θ) ⊆ E(ℱ, θ) the subset of isomorphism classes (M, C) in E(ℱ, θ) such that M is indecomposable, and by *E(ℱ, θ) ⊆ E(ℱ, θ) the subset of isomorphism classes (M, C) such that ξᵢ ≠ 0 for 2 ≤ i ≤ n. We denote by IM(ℱ, θ) and *M(ℱ, θ) the images of IE(ℱ, θ) and *E(ℱ, θ) under the natural surjection E(ℱ, θ) → M(ℱ, θ).

Let (α, n) be an admissible couple, and consider the ordered quiver θ(α, n). We shall find the classification spaces *E(ℱ, θ) and *M(ℱ, θ) when ℱ is family satisfying (*) and θ = θ(α, n). For this, we only need the following simple lemma:

Lemma 21. Let M, N be non-zero left R-modules, let E, E' be extensions of M by N, and let ξ, ξ' ∈ Ext₁ᵦ(M, N) be the corresponding classes. If ξ' = ψξφ for automorphisms φ ∈ Autᵦ(M), ψ ∈ Autᵦ(N), then E ≅ E' considered as left R-modules. In particular, E ≅ E' as R-modules when ξ' = αξ with α ∈ k*.

Proof. Clearly, we have k ≅ k idᵦ(M) ⊆ Endᵦ(M) and k* ≅ Autᵦ(M) when M is a non-zero R-module. The rest follows from McConnel, Robson [14], proposition 3.3 and the preceding paragraph. □

Proposition 22. Let ℱ be a family satisfying (*), let (α, n) ∈ I × N be an admissible couple, and let θ = θ(α, n). Then we have *E(ℱ, θ) ≅ (k*)ⁿ−₁ and *M(ℱ, θ) = {∗}.

Proof. Let θ = θ(α, n), let Kᵢ = Mₗₐₓᵢ and choose a basis for Ext₁ᵦ(Kᵢ₋₁, Kᵢ) for 2 ≤ i ≤ n. There is a map *E(ℱ, θ) → (k*)ⁿ−₁ given by the composition (M, C) → (ξ₂, ..., ξₙ) → (τ₂, ..., τₙ) → (k*)ⁿ−₁. This map is clearly injective, since Ext₁ᵦ(Cᵢ₋₁, Kᵢ) → Ext₁ᵦ(Kᵢ₋₁, Kᵢ) is an isomorphism and the extensions ξᵢ determine (M, C). Furthermore, the map is surjective by lemma 20. Finally, lemma 21 shows if (M, C) and (M', C') are any isomorphism classes in *E(ℱ, θ), then M ≅ M' considered as left R-modules. □

Let (M, C) be an iterated extension of the family ℱ with extension type θ, and we have by now obtained a complete classification of all extensions of extensions (M, C) of the family ℱ such that ξᵢ ≠ 0 for 2 ≤ i ≤ n. Notice that all iterated extensions (M, C) in this classification is such that M is indecomposable. In fact, the condition ξᵢ ≠ 0 for 2 ≤ i ≤ n is equivalent with the condition that that M is indecomposable when ℱ satisfy (*):
Proposition 23. Let $F$ be a family satisfying (*), and let $(M, C)$ be an iterated extension of the family $F$ of length $n$ such that $M$ is indecomposable. Then $\xi_i \neq 0$ for $2 \leq i \leq n$.

Proof. The result is obviously true if $n \leq 2$, so let us proceed by induction on $n$: We assume that $n \geq 3$, and let $(M, C)$ be an extension of extensions of $F$ of length $n$ with $M$ indecomposable. Clearly, $C_{n-1}$ has finite length and therefore an indecomposable decomposition

$$C_{n-1} = N_1 \oplus \cdots \oplus N_q.$$ 

Suppose $q > 1$. Since $N_j$ is a left $R$-module of finite length for $1 \leq j \leq q$, $N_j$ has a co-filtration of length $n_j < n$

$$N_j = C_{j,n_j} \to C_{j,n_j-1} \to \cdots \to C_{j,1} \to C_{j,0} = 0,$$

with $K_{ji} = \ker(C_{j,i} \to C_{j,i-1}) \cong M_{i(j,i)}$ for $1 \leq i \leq n_j$, with $\alpha(j,i) \in I$. Since $\xi_n \in \text{Ext}^1_R(C_{n-1}, K_n) \cong \oplus \text{Ext}^1_R(N_j, K_n)$, we can write $\xi_n = (\xi_{n,1}, \ldots, \xi_{n,q})$ with $\xi_{n,j} \in \text{Ext}^1_R(N_j, K_n)$. We claim that $\xi_{n,j} \neq 0$ for all $j$: Assume that $\xi_{n,j} = 0$ for some $j$, then we may assume $j = 1$ with no loss of generality. For each $j$, we choose a representative $\psi(j) \in \text{Der}_k(R, \text{Hom}_k(N_j, K_n))$ of $\xi_{n,j}$, and let $\phi \in \text{Hom}_k(N_1, K_n)$ satisfy $r \phi - \phi r = \psi(1)$, for all $r \in R$. We may assume that $M \cong K_n \oplus (N_1 \oplus \cdots \oplus N_q)$ considered as a vector space over $k$, and that the left $R$-module structure of $M$ is given by

$$r(k_{n_1}, \ldots, k_{n_q}) = (rk + \sum_{j=1}^q \psi(j)_r(n_j), r_{n_1}, \ldots, r_{n_q})$$

for all $r \in R$, $(k_{n_1}, \ldots, k_{n_q}) \in M$. Let $M' \cong M$ considered as a vector space over $k$, and let $M'$ have a left $R$-module structure given by

$$r(k_{n_1}, \ldots, k_{n_q}) = (rk + \sum_{j=2}^q \psi(j)_r(n_j), r_{n_1}, \ldots, r_{n_q})$$

for all $r \in R$, $(k_{n_1}, \ldots, k_{n_q}) \in M'$. Then the homomorphism $\pi : M \to M'$ given by $(k_{n_1}, \ldots, k_{n_q}) \to (k + \phi(n_1), n_1, \ldots, n_q)$ defines an isomorphism of left $R$-modules. But $M'$ has $N_1$ as a direct summand, and $M'$ is indecomposable since $M$ is, so this implies $q = 1$. Since we have assumed that $q > 1$, we must have $\xi_{n,j} \neq 0$ for all $j$.

Clearly, $N_j$ is indecomposable of length $n_j < n$ for $1 \leq j \leq q$, so by the induction hypothesis, the extensions $\xi_{ji} \in \text{Ext}^1_R(C_{j,i-1}, K_{i-1}) \neq 0$ for $2 \leq i \leq n_j$. So $(N_j, C_j)$ is an extension of families of the family $F$ which is part of the classification in proposition 22. Let $K_n = M_{\alpha(n)}$ with $\alpha(n) \in I$, and let $\alpha(n-1) \in I$ be the unique index such that $\alpha(n) = \sigma(\alpha(n-1))$. From the proof of lemma 18, we see that $\text{Ext}^1_R(N_j, K_n) \cong \text{Ext}^1_R(K_{n,j}, K_n)$ and $K_{j,n} \cong M_{\alpha(n-1)}$ for $1 \leq j \leq q$. We may therefore assume that

$$M \cong K_n \oplus (N_1 \oplus \cdots \oplus N_q) \cong K_n \oplus (\oplus_{j=1}^q (\oplus_{i=1}^{n_j} \text{K}_{j,i})),$$

considered as a vector space over $k$, and the $R$-module structure on $M$ maps $K_{j,i}$ into $K_{j,i+1}$ when $i < n_j$, and $K_{j,n_j}$ into $K_n$.

With no loss of generality, we may assume that $n_1 \leq n_2$. Let us choose representatives $\psi(j) \in \text{Der}_k(R, \text{Hom}_k(K_{j,n_j}, K_n))$ of $\tau_{n,j} \in \text{Ext}^1_R(K_{j,n_j}, K_n) \cong k^*$ for $1 \leq j \leq q$. Then we can find $c \in k^*$ such that $\psi(1) = c \psi(2)$. Let us also choose representatives $\psi(j, i) \in \text{Der}_k(R, \text{Hom}_k(K_{j,n_j-i}, K_{j,n_j-i+1}))$ corresponding to the extensions $\tau_{n,j-i+1}(N_j, C_j) \in \text{Ext}^1_R(K_{j,n_j-i}, K_{n_j-i+1})$ for $1 \leq j \leq 2$, $1 \leq i \leq n_j - 1$. Since $K_{1,n_1} \cong K_{2,n_2}$, we also have $K_{1,n_1-i} \cong K_{2,n_2-i}$ for $1 \leq i \leq n_1 - 1$, so we can
find $c_i \in k^*$ such that $\psi(1, i) = c_i \psi(2, i)$ for $1 \leq i \leq n_1 - 1$. Let $M' \cong M$ considered as a vector space over $k$, and let $M'$ have the left $R$-module structure given by

$$r(k, n_1, \ldots, n_q) = (rk + \sum_{j=2}^{q} \psi(j)r(k_j, n_j), rn_1, \ldots, rn_q)$$

for all $r \in R$, $(k, n_1, \ldots, n_q) \in M'$, where we write $n_j = (k_j, n_j, \ldots, k_j, 1)$. Let us also write $C(i) = c_1 c_2 \cdots c_i$ for $1 \leq i \leq n_1 - 1$. Then the map $\pi : M \to M'$ given by

$$(k, n_1, \ldots, n_q) \mapsto (k, n_1, n_2, \ldots, n_q)$$

$$+ (0, 0, c(k_{1, n_1}, C(1)k_{1, n_1-1}, \ldots, C(n_1 - 1)k_{1, 0}, 0, \ldots, 0), 0, \ldots, 0)$$

defines an $R$-linear isomorphism of left $R$-modules. But $N_1$ is a direct summand of $M'$, and $M'$ is indecomposable since $M$ is, so this implies $q = 1$. We must therefore conclude that $C_{n-1}$ is indecomposable. But this implies that $\xi_{n-1} \neq 0$. By induction, it follows that $\xi_i \neq 0$ for $2 \leq i \leq n$. □

**Theorem 24.** Let $\mathcal{F}$ be a family satisfying (*), and let $\Gamma$ be an ordered quiver. There exists an iterated extension of the family $\mathcal{F}$ with extension type $\Gamma$ such that $M$ is indecomposable if and only if $\Gamma = \Gamma(\alpha, n)$ for some admissible couple $(\alpha, n)$ in $I \times N$. Moreover, $\text{IE}(\mathcal{F}, \Gamma) \cong (k^*)^{n-1}$ and $\text{IM}(\mathcal{F}, \Gamma) = \{\ast\}$ in this case.

**Proof.** Proposition [23] shows that $\text{IE}(\mathcal{F}, \Gamma) = \ast \mathcal{E}(\mathcal{F}, \Gamma)$ when $\mathcal{F}$ satisfy (*). The rest is clear. □

Let us denote by $M(\alpha, n)$ the $R$-module representing the isomorphism class $\ast$ in the above classification for each admissible couple $(\alpha, n) \in I \times N$. We remark that the proofs given in this section are constructible, and therefore we may in principle construct $M(\alpha, n)$. In fact, the simple modules $\mathcal{F}$ and their extensions is enough to construct the left $R$-module $M(\alpha, n)$.

**Theorem 25.** Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $\mathcal{F} = \{M_\alpha : \alpha \in I\}$ be a family of orthogonal points in $\text{Mod}(R)$ such that the species of $\mathcal{F}$ is a $k$-quiver. Then the category $\text{Mod}(\mathcal{F})$ is uniserial if and only if $\mathcal{F}$ satisfies the condition (*). If this is the case, there is a complete classification of all indecomposable modules in $\text{Mod}(\mathcal{F})$.

**Proof.** The modules $M(\alpha, n)$ are uniserial when $(\alpha, n) \in I \times N$ is an admissible couple and $\mathcal{F}$ satisfy (*). So if $\mathcal{F}$ satisfy (*), all indecomposable modules in $\text{Mod}(\mathcal{F})$ are uniserial. Conversely, if all indecomposable modules in $\text{Mod}(\mathcal{F})$ are uniserial, then $\mathcal{F}$ satisfy (*) by lemma 13 and 16. □

In the uniserial case, the list of indecomposable modules in $\text{Mod}(\mathcal{F})$ is of course given by

$$\{M(\alpha, n) : (\alpha, n) \in I \times N \text{ is an admissible couple }\}.$$ 

Moreover, we have seen that the left $R$-modules $M(\alpha, n)$ are constructible. That is, given the simple modules in $\mathcal{F}$ and the extensions $t_i \in \text{Ext}^1_R(K_{i-1}, K_i)$ expressed in Hochschild cohomology, we can construct the modules $M(\alpha, n)$ in concrete terms (for instance in terms of generators and relations).

8. **Applications: Regular holonomic D-modules in dimension 1**

We find many examples of uniserial length categories among the categories of regular holonomic D-modules on curves. We shall prove this in a number of examples, and at the same time give the corresponding classification of all indecomposable objects.
In most cases, this gives new proofs of known results. But the classification of graded holonomic \( D \)-modules when \( D \) is the first Weyl algebra \( D = A_1(k) \) or the ring of differential operators \( D = \text{Diff}(A) \) over an affine monomial curve \( A \) is new.

8.1. The local analytic case. Let \( k = \mathbb{C} \) be the complex numbers, and consider the local ring \( A = k\{t\} \) of convergent power series with coefficients in \( k \). Let furthermore \( D = \text{Diff}(A) = A < \partial > \) be the ring of \( k \)-linear differential operators on \( A \), with \( \partial = d/dt \). Explicitly, \( D \) is the skew polynomial ring

\[
D = \{ \sum_{i=0}^{d} p_i \partial^i : d \geq 0, \ p_i \in A \text{ for } 0 \leq i \leq d \}
\]

with the relation \( \partial t = t\partial + 1 \).

Let us consider the holonomic \( D \)-modules with regular singularities. These were completely classified by Boutet de Monvel [9]. The same classification result was later obtained by Briançon, Maisonobe [4,5], using division algorithms and perverse sheaves. For definitions of the terms holonomic \( D \)-modules with regular singularities, we refer the reader to van den Essen [20]. However, for the ring \( D \) considered in this section, it is useful to notice the following facts: A \( D \)-module is holonomic if and only if it has finite length. Moreover, if \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of holonomic \( D \)-modules, then \( M \) has regular singularities if and only if \( M' \) and \( M'' \) has regular singularities.

Let \( I = \{ \alpha \in k : 0 < \Re(\alpha) < 1 \} \cup \{0, 1\} \), and let \( \mathcal{F} = \{ M_\alpha : \alpha \in I \} \) be the family of left \( R \)-modules given by \( M_0 = D/D\partial \), \( M_1 = D/Dt \), and \( M_\alpha = D/D(E - \alpha) \) with \( E = t\partial \) for all \( \alpha \in I \setminus \{0, 1\} \). It is well-known that \( \mathcal{F} \) is the family of all simple left \( D \)-modules with regular singularities, up to isomorphism. From the comments above, we see that the category \( \text{Mod}(\mathcal{F}) \) is the category of regular holonomic \( D \)-modules.

Since \( \mathcal{F} \) consists of simple, non-isomorphic \( D \)-modules, it is clear that \( \mathcal{F} \) is a family of orthogonal points in \( \text{Mod}(D) \). Moreover, it is easy to check that the species of \( \mathcal{F} \) is a \( k \)-quiver satisfying the condition (*) of section 7. It follows that the category \( \text{Mod}(\mathcal{F}) \) of regular holonomic \( D \)-modules is a uniserial category. Moreover, each indecomposable object \( M \) in \( \text{Mod}(\mathcal{F}) \) can be constructed explicitly with the methods from section 7. With our methods, we therefore reprove the classification of regular holonomic \( D \)-modules:

**Theorem 26.** Let \( k = \mathbb{C} \) and let \( D \) be the ring of differential operators on the \( k \)-algebra \( A = k\{t\} \) of convergent power series with coefficients in \( k \). Then the category of regular holonomic \( D \)-modules is uniserial. Moreover, any regular holonomic \( D \)-module is a finite direct sum of the indecomposable ones, given by

\[
\{ M(\alpha, n) : \alpha \in I, n \geq 1 \},
\]

where \( M(\alpha, n) = D/Dw(\alpha, n) \) for \( \alpha = 0, 1 \), and \( M(\alpha, n) = D/D(E - \alpha)^n \) for \( \alpha \in I \setminus \{0, 1\} \).

For \( \alpha = 0, 1 \), we use the notation \( w(\alpha, n) \) for the alternating word in the letters \( t, \partial \) of length \( n \), ending in \( \partial \) if \( \alpha = 0 \) and ending in \( t \) if \( \alpha = 1 \). Notice that any couple \( (\alpha, n) \in I \times \mathbb{N} \) is admissible.

Let \( A' \subseteq A \) be a sub-algebra of \( A = k\{t\} \) such that \( \dim_k A/A' \) is finite. Then \( D' = \text{Diff}(A') \) is Morita equivalent to \( D = \text{Diff}(A) \) by Smith, Stafford [18], proposition 3.3, and this Morita equivalence preserves regular holonomic modules by van den Essen [21], theorem 3.1. It follows that the category of regular holonomic \( D' \)-modules is uniserial, and there is a classification of regular holonomic \( D' \)-modules similar to theorem 26.
Let \( A_n = k\{t_1, \ldots, t_n\} \) be the ring of convergent power series in \( n \) variables with coefficients in \( k \), and let \( D_n = \text{Diff}(A_n) \) be the ring of differential operators on \( A_n \).

The category of regular holonomic \( D_n \)-modules supported by an irreducible analytic curve is equivalent to the category of regular holonomic \( D \)-modules (with \( D = D_1 \) as above), by van Doorn, van den Essen \[23\]. It follows that the category of regular holonomic \( D_n \)-modules supported by an irreducible analytic curve is uniserial, and there is a classification of regular holonomic \( D \)-modules supported by an irreducible analytic curve similar to theorem \[26\].

8.2. The formal case. Let \( k \) be an algebraically closed field of characteristic 0, and let \( A = k[[t]] \) be the local ring of formal power series in \( t \). Let furthermore \( D = \text{Diff}(A) = A < \partial > = B_1(k) \) be the ring of \( k \)-linear differential operators on \( A \), with \( \partial = d/dt \). Explicitly, \( D \) is the skew polynomial ring

\[
D = \{ \sum_{i=0}^{d} p_i \partial^i : d \geq 0, p_i \in A \text{ for } 0 \leq i \leq d \}
\]

with the relation \( \partial t = t \partial + 1 \).

Let us consider the holonomic \( D \)-modules. These fall into 2 classes: The regular holonomic \( D \)-modules and the irregular holonomic \( D \)-modules. The classification of the regular ones is completely parallel to the classification of regular holonomic \( D \)-modules in the local analytic case. Again, we refer the reader to van den Essen \[20\] for definitions of the terms holonomic and regular holonomic \( D \)-modules. For the ring \( D \) we consider in this section, it remains valid that a module is holonomic if and only if it has finite length, and for a short exact sequence of holonomic \( D \)-modules \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \), we have that \( M \) is regular if and only if \( M' \) and \( M'' \) are regular.

The classification of the irregular holonomic \( D \)-modules were first obtained by Puninski \[16\]. It is based upon the classification of the simple irregular \( D \)-modules of van den Essen, Levelt \[22\]. We remark that this classification is very similar to the classification of simple modules over the first Weyl algebra in Block \[2\]: The simple \( D \)-modules which are torsion free \( D \)-modules are parametrized by the simple \( \text{Diff}(K) \)-modules, where \( K = k((t)) \) is the quotient field of \( A \).

It is clear that any simple \( \text{Diff}(K) \)-module is of the form \( N = \text{Diff}(K)/\text{Diff}(K)P \), where \( P \in \text{Diff}(K) \) is irreducible, since \( \text{Diff}(K) \) is a left and right principal ideal domain. Moreover, the simple \( A \)-torsion free module \( M \) corresponding to \( N \) is given by \( \text{soc}_D N \). But if \( N \cong K \), then \( \text{soc}_D N = A \), and otherwise \( \text{soc}_D N = N \). We denote by \( I' \) the set of equivalence classes of irreducible elements in \( \text{Diff}(K) \), and by \( M_\alpha \) the simple, \( A \)-torsion free \( D \)-module corresponding to \( \alpha \in I' \).

Let \( I = I' \cup \{t\} \), and let \( M_t = D/Dt \) be the simple \( D \)-module which has \( A \)-torsion. Then \( \mathcal{F} = \{M_\alpha : \alpha \in I\} \) is the set of simple \( D \)-modules (up to isomorphism) by van den Essen, Levelt \[22\]. This is clearly a family of orthogonal points in \( \text{Mod}(D) \), since it consists of simple, pairwise non-isomorphic \( D \)-modules. Moreover, it follows from Puninski \[16\], fact 2.3 that the species of \( \mathcal{F} \) is a \( k \)-quiver, and from Puninski \[16\], proposition 3.4 that this species satisfy condition (*) of section 7.

If follows that the category of holonomic \( D \)-modules is uniserial, and we can construct the holonomic \( D \)-modules explicitly with the methods of section 7. This gives a new proof of Puninskis classification of holonomic \( D \)-modules:

**Theorem 27** (Puninski). Let \( k \) be a algebraically closed field of characteristic 0, and let \( D \) be the ring of differential operators on the ring \( A = k[[t]] \) of formal power series with coefficients in \( k \). Then the category of holonomic \( D \)-modules is uniserial.
Moreover, any holonomic $D$-module is a finite direct sum of the indecomposable ones, given by
\[ \{ M(\alpha, n) : \alpha \in I, n \geq 0 \}, \]
where $M(\alpha, n) = D/Dw(\alpha, n)$ for $\alpha = t, \partial$ and $M(\alpha, n) = M_{n-1}(\alpha)$ in Puninskis notation for $\alpha \in I \setminus \{ t, \partial \}$.

For $\alpha = t, \partial$, we use the notations $w(\alpha, n)$ for the alternating word in the letter $t, \partial$ of length $n$ ending in $\alpha$. Notice that all couples $(\alpha, n) \in I \times \mathbb{N}$ are admissible. We also remark that it does not seem to be possible to use the same methods to obtain a complete classification of holonomic $D$-module in the local analytic case.

Let $D_n = \text{Diff}(k[[t_1, \ldots, t_n]])$, and consider the category of holonomic $D_n$-modules supported by an irreducible curve. By the equivalence of categories between holonomic $D_n$-modules supported on an irreducible curve and holonomic $D_1$-modules, we conclude that the category of holonomic $D_n$-modules supported by an irreducible curve is uniserial, and a classification result similar to theorem 27 holds for this category.

8.3. The graded algebraic case. Let $k$ be an algebraically closed field of characteristic 0, and let $A = k[t]$ be the polynomial ring in one variable over $k$. Let furthermore $D = \text{Diff}(A) = A < \partial >= A_1(k)$ be the first Weyl algebra, with $\partial = d/dt$. Explicitly, $D$ is the skew polynomial ring
\[ D = \{ \sum_{i=0}^d p_i \partial^i : d \geq 0, p_i \in A \text{ for } 0 \leq i \leq d \}, \]
with the relation $\partial t = t \partial + 1$.

The ring $A = k[t]$ is a $\mathbb{Z}$-graded $k$-algebra in a natural way, such that $t^i$ is homogeneous of degree $i$ for all integers $i \geq 0$. This induces a natural $\mathbb{Z}$-grading of the Weyl algebra $D$, such that $D$ is a $\mathbb{Z}$-graded $k$-algebra: We say that a differential operator $P \in D$ is homogeneous of weight $w \in \mathbb{Z}$ if $P * A_i \subseteq A_{i+w}$ for all integers $i \in \mathbb{Z}$. Explicitly, $D$ is the $\mathbb{Z}$-graded $k$-algebra generated by the homogeneous monomials $t^i \partial^j$ of weight $i - j$ for all $i, j \geq 0$.

Let $C = \operatorname{grMod}(D)$ be the category of graded $D$-modules. The objects of $\operatorname{grMod}(D)$ is the $\mathbb{Z}$-graded, left $D$-modules, and the morphisms are the homogeneous homomorphisms (of any degree) between graded $D$-modules. In this section, we shall study the full sub-category $\operatorname{grHol}(D)$ of $\operatorname{grMod}(D)$, consisting of all objects in $\operatorname{grMod}(D)$ which have finite length. Even though the notations we have introduced and the results we have obtained in this paper are stated for length categories in $C = \operatorname{Mod}(R)$, we shall feel free to use them in the case $C = \operatorname{grMod}(D)$ as well, see the note in the introduction.

Let $M$ be a graded $D$-module. We shall denote by $\underline{M}$ the module $M$ considered as a $D$-module, forgetting the graded structure. First, notice that $M$ is simple in the category $\operatorname{grMod}(D)$ if and only if $\underline{M}$ is simple in $\operatorname{Mod}(D)$: One implication is obvious, the other follows from Năstăsescu, van Oystaeyen [15], theorem II.7.5 and the fact that if $M$ is a simple object of $\operatorname{grMod}(D)$, then $\underline{M}$ is a $D$-module of finite length.

It follows that a graded $D$-module $M$ is of finite length if and only if $\underline{M}$ is a $D$-module of finite length. On the other hand, it is well known that for the first Weyl algebra $D = A_1(k)$, a $D$-module is holonomic if and only if it has finite length. So a graded $D$-module has finite length if and only if $\underline{M}$ is holonomic. It is therefore natural to define the category of graded holonomic $D$-modules to be the category $\operatorname{grHol}(D)$. We notice that if $M$ is an object of $\operatorname{grHol}(D)$, then $\underline{M}$ is a regular holonomic $D$-module in the sense of van den Essen [20].
Let us classify the simple objects in \( \text{grHol}(D) \): We know that these are exactly the simple graded \( D \)-modules. On the other hand, all simple \( D \)-modules have been classified by Block \[2\]. We shall adapt this classification to obtain a classification of all simple objects in \( \text{grHol}(D) \), up to graded isomorphism.

Let \( M \) be a simple graded \( D \)-module. Then \( M \) is either torsion or torsion free considered as a module over the sub-ring \( A \subseteq D \). If \( M \) is a torsion module over \( A \), then \( M \cong D/D(t - \alpha) \) for some \( \alpha \in k \) by Block \[2\], proposition 4.1 and corollary 4.1. So \( M \cong D/Dt \) in \( \text{grHol}(D) \), where \( D/Dt \) has the natural graded structure inherited from \( D \).

There is a bijective correspondence, given by localization, between simple graded \( D \)-modules which are torsion free over \( A \) and simple graded \( \text{Diff}(T) \)-modules, where \( T = k[t, t^{-1}] \). This follows from Block \[2\], lemma 2.2.1 and corollary 2.2, slightly adapted to the graded situation. But any simple graded \( \text{Diff}(T) \)-module is of the form \( N_\alpha = \text{Diff}(T)/\text{Diff}(T)(E - \alpha) \) for some \( \alpha \in k \), where \( E = i\partial \). Moreover, \( N_\alpha \cong N_\beta \) as graded modules if and only if \( \alpha - \beta \in \mathbb{Z} \).

Let \( I' \subseteq k \) be a subset of \( k \) containing 0 such that the natural composition \( I' \rightarrow k \rightarrow k/\mathbb{Z} \) is a bijection. For each non-zero \( \alpha \in I' \), the simple graded \( D \)-module \( M_\alpha = D/D(E - \alpha) \) corresponds to \( N_\alpha \) in the correspondence above. Moreover, the simple graded \( D \)-module \( M_0 = D/D\partial \) corresponds to \( N_0 \).

Let \( I = I' \cup \{1\} \), and let \( M_I = D/Dt \). It follows that \( \mathcal{F} = \{ M_\alpha : \alpha \in I \} \) is a family of simple graded \( D \)-modules with the following property: Any simple graded \( D \)-module \( M \) is isomorphic to \( M_\alpha \) (as a graded \( D \)-module) for a unique \( \alpha \in I \). In this sense, \( \mathcal{F} \) is the family of simple objects in \( \text{grHol}(D) \), up to graded isomorphism.

Clearly, \( \mathcal{F} \) is a family of orthogonal points in \( \text{grMod}(D) \), since it is a family of simple, non-isomorphic objects. It is also easy to check that the species of \( \mathcal{F} \) is a \( \mathbb{k} \)-quiver which satisfy the condition (*) of section \[4\]. So the sub-category \( \text{Mod}(\mathcal{F}) \) of \( \text{grMod}(D) \) is a uniserial category. In fact, \( \text{Mod}(\mathcal{F}) \) is the length category \( \text{grHol}(D) \) of graded holonomic \( D \)-modules. So we conclude that the category \( \text{grHol}(D) \) of graded holonomic \( D \)-modules is a uniserial category. Moreover, each indecomposable object in \( \text{grHol}(D) \) can be constructed explicitly with the methods of section \[4\]. We obtain the following classification result:

**Theorem 28.** Let \( k \) be an algebraically closed field of characteristic 0, and let \( D = A_1(k) \) be the first Weyl algebra over \( k \). Then the category of graded holonomic \( D \)-modules is uniserial. Moreover, any graded holonomic \( D \)-module is a finite direct sum of the indecomposable ones, given by

\[
\{ M(\alpha, n) : \alpha \in I, n \geq 1 \},
\]

where \( M(\alpha, n) = D/Dw(\alpha, n) \) for \( \alpha = 0, 1 \) and \( M(\alpha, n) = D/D(E - \alpha)^n \) for \( \alpha \in I \setminus \{0, 1\} \).

For \( \alpha = 0, 1 \), we use the notation \( w(\alpha, n) \) for the alternating word in the letters \( t, \partial \) of length \( n \), ending in \( \partial \) if \( \alpha = 0 \) and ending in \( t \) in \( \alpha = 1 \). Notice that any couple \( (\alpha, n) \in I \times \mathbb{N} \) is admissible.

Let \( A' \) be an affine monomial curve over \( k \), and let \( D' = \text{Diff}(A') \) be the corresponding ring of differential operators. By Smith, Stafford \[13\], the category of holonomic \( D \)-modules and the category of holonomic \( D' \)-modules are equivalent, and clearly graded structures are conserved under this equivalence. It follows that the category of graded holonomic \( D' \)-modules is uniserial, and there is a classification of graded holonomic \( D' \)-modules similar to theorem \[28\] for any affine monomial curve \( A' \).
9. The wild case

Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $\mathcal{F}$ be a family of orthogonal points in $\text{Mod}(R)$ such that $\text{End}_R(M_\alpha) = k$ for all $\alpha \in I$. In this section, we shall mention a result which gives a sufficient condition for the length category $\text{Mod}(\mathcal{F})$ to be wild in a strong sense.

Let $W = k < x, y >$ be the free associative $k$-algebra on two generators, and consider the category $\text{fdMod}(W)$ of left $W$-modules which are finitely dimensional as vector spaces over $k$. We say that the category $\text{Mod}(\mathcal{F})$ is wild if there is a full exact embedding of $\text{fdMod}(W)$ into $\text{Mod}(\mathcal{F})$, following Klingler, Levy [10]. It is well-known that if $\text{Mod}(\mathcal{F})$ is wild in this sense, a classification of the indecomposable modules in $\text{Mod}(\mathcal{F})$ would contain a classification of all indecomposable modules of finite dimension over $k$ over any finite dimensional $k$-algebra.

In Klingler, Levy [10], it was shown that the category of holonomic modules over the first Weyl algebra is wild in the above sense if $k$ has characteristic 0. In fact, a full exact embedding can be chosen such that every module in its image has socle-height 2.

**Theorem 29** (Klingler-Levy). Let $k$ be an algebraically closed field, let $R$ be an associative $k$-algebra, and let $\mathcal{F}$ be a family of orthogonal points in $\text{Mod}(R)$ such that $\text{End}_R(M_\alpha) = k$ for all $\alpha \in I$. If the Gabriel quiver of $\text{Mod}(\mathcal{F})$ contains the quiver $Q_5$ given by

```
 1 ----> 2 ----> 3 ----> 4 ----> 5
   |        |        |        |        |
   |        |        |        |        |
   |        |        |        |        |
   |        |        |        |        |
   |        |        |        |        |
   |        |        |        |        |
   |        |        |        |        |
   |        |        |        |        |
```

then there is a full exact embedding of $\text{fdMod}(W)$ into $\text{Mod}(\mathcal{F})$, and all modules in the image of this embedding has socle-height 2. In particular, $\text{Mod}(\mathcal{F})$ is wild in this case.

**Proof.** This follows from Klingler, Levy [10], theorem 2.12 with some minor changes. $\square$

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