A simple remark on infinite series is presented. This applies to a particular recursion scenario, which in turn has applications related to a classical theorem on Euler’s phi-function and to recent work by Ron Brown on natural density of square-free numbers.

1 A Basic Fact about Infinite Series

In a recent paper [1], Ron Brown has computed the natural density of the set of square-free numbers divisible by $a$ but relatively prime to $b$, where $a$ and $b$ are relatively prime square-free integers. Here we note a simple remark on infinite series, one of whose consequences generalizes a key argument in that work. We then derive a consequence of a well-known result on the Euler $\phi$-function. The \(m = p\) case of that consequence follows from En-Naoui [2] who anticipates some of our arguments.

Remark 1 Let \(\sum_{i=1}^{\infty} a_i\) be an absolutely convergent series of complex numbers, and for \(i \geq 1\), \(f_i : \mathbb{N} \cup \{0\} \to \mathbb{C}\) with \(\lim_{N \to \infty} f_i(N) = D\) (independent of \(i\)) and the \(f_i\) uniformly bounded. Then \(\lim_{N \to \infty} \sum_{i=1}^{\infty} a_i f_i(N) = D \sum_{i=1}^{\infty} a_i\).

Proof. This is a special case of the Lebesgue Dominated Convergence Theorem (using the counting measure and applied to the sequence \(\{a_i f_i(n)\}_{n=1}^{\infty}\)). To preserve the elementary character of the arguments here, we give an "Introductory Analysis" proof.

Let \(\varepsilon > 0\) be given. By uniform boundedness, there is a constant \(B\) for which \(|f_i(N) - D| < B\) for all \(i\) and \(N\). Choose \(k \in \mathbb{N}\) with \(\sum_{i=k+1}^{\infty} |a_i| < \frac{\varepsilon}{2B}\), and choose \(M\) such that for all \(N \geq M\) and \(1 \leq i \leq k\), \(|f_i(N) - D| < \varepsilon / (1 + 2 \sum_{j=1}^{k} |a_j|)\).
Then we have for \( N \geq M \)
\[
\left| \sum_{i=1}^{\infty} a_i f_i(N) - D \sum_{i=1}^{\infty} a_i \right| = \left| \sum_{i=1}^{k} a_i (f_i(N) - D) \right|
\leq \sum_{i=1}^{k} |a_i| \left| (f_i(N) - D) \right| + \sum_{i=k+1}^{\infty} |a_i| \left| (f_i(N) - D) \right|
\leq \sum_{i=1}^{k} |a_i| \cdot \varepsilon / (1 + 2 \sum_{i=1}^{k} |a_i|) + \frac{\varepsilon}{2B} \cdot B < \varepsilon.
\]

\[\blacksquare\]

2 A Consequence and Some Applications

For all applications of the remark above, we first derive the following consequence involving a "linear division-based" recursion.

**Lemma 2** Let, \( F, G : \mathbb{N} \cup \{0\} \to \mathbb{C} \), \( 1 < m \in \mathbb{N} \), \( \alpha, \beta, D \in \mathbb{C} \) satisfy the conditions (1) \( \lim_{N \to \infty} F(N)/N = D \), \( \beta \) \( \beta \) is even, \( \beta \) is odd, \( G(N) = \alpha F(\lfloor N/m \rfloor) + \beta G(\lfloor N/m \rfloor) \), and (4) \( F(0) = G(0) = 0 \). Then \( \lim_{N \to \infty} G(N)/N = \frac{\alpha D}{m - \beta} \).

**Proof.** Recursively expand (using condition (3) and \( \lfloor a/b \rfloor/c = \lfloor a/(bc) \rfloor \) for positive integers \( a, b, c \)) we have for \( N > 0 \)
\[
G(N)/N = \frac{\alpha}{m} \frac{F(\lfloor N/m \rfloor)}{N/m} + \frac{\alpha \beta}{m^2} \frac{F(\lfloor N/m^2 \rfloor)}{N/m^2} + \cdots + \frac{\alpha \beta^{j-1}}{m^j} \frac{F(\lfloor N/m^j \rfloor)}{N/m^j} + \frac{\alpha \beta^{j-1}}{m^j} \frac{G(\lfloor N/m^j \rfloor)}{N/m^j}
\]

By properties (1), (2) and (4), this implies we have
\[
G(N)/N = \sum_{i=1}^{\infty} \frac{\alpha \beta^{i-1} F(\lfloor N/m^i \rfloor)}{m^i N/m^i}
\]

After all, for any fixed \( N \) this is actually a finite sum by (4) and the final term in display (*) above is 0 for large \( j \). Now by Lemma 1, taking \( a_i = \frac{\alpha \beta^{i-1}}{m^i} \) and \( f_i(N) = \frac{F(\lfloor N/m^i \rfloor)}{N/m^i} \), it follows that \( \lim_{N \to \infty} G(N)/N = D \sum_{i=1}^{\infty} \frac{\alpha \beta^{i-1}}{m^i} = \frac{D\alpha}{m - \beta} \).

\[\blacksquare\]

We can derive some simple applications.

**Application 1.** Let \( m \) be an integer greater than 1. Call an integer \( n \) oddly divisible by \( m \) if the largest nonnegative integer \( i \) with \( m^i | n \) is odd. Similarly define evenly divisible. (Note that by this definition, a number not divisible by \( m \) is evenly divisible by \( m \).) Set \( F(n) = n \) and \( G(n) = |\{ i \in \mathbb{N} : 1 \leq i \leq n, i \text{ oddly divisible by } m \}| \). Since there is a 1-1 correspondence between \( \{ i \in \mathbb{N} : 1 \leq i \leq n, i \text{ oddly divisible by } m \} \)
and \( \{i \in \mathbb{N} : 1 \leq i \leq \lfloor n/m \rfloor \text{ and } i \text{ is evenly divisible by } m\} \), we quickly see that \( G(n) = F(\lfloor n/m \rfloor) - G(\lfloor n/m \rfloor) \). Now apply the Lemma with \( D = \alpha = -\beta = 1 \) to get \( \lim_{N \to \infty} G(N)/N = \frac{1}{m+1} \). So the natural density of numbers oddly divisible by \( m \) is \( \frac{1}{m+1} \). (This is also easily arrived at by an inclusion-exclusion argument.)

**Application 2.** In Brown\[1\] the natural density of the set of square-free numbers divisible by primes \( p_1, \cdots, p_k \) is shown to be \( \frac{6}{\pi^2} \prod_{j=1}^{k} \frac{1}{p_j} \). (In fact, he more generally computes the density of the set of such numbers also not divisible by a further set of primes and reduces that problem to this one.) Using that the natural density of the set of square-free numbers is \( \frac{6}{\pi^2} \), the cited result follows directly from \[?\] Lemma 3, which states that, for a square-free integer \( t \) and a prime \( p \) not dividing \( t \), if the natural density of the set of square-free numbers divisible by \( t \) is \( D \), then the natural density of the set of square-free numbers divisible by \( tp \) is \( D/(p+1) \). To do this (converting to our notation), letting \( C \) be the set of square-free numbers, \( F(x) = |\{r \in C : t|r, r \leq x\}| \) and \( G(x) = |\{r \in C : pt|r, r \leq x\}| \) Brown quickly establishes that \( F(x/p) = G(x/p) + G(x) \). Noting that we can replace arguments here with their greatest integers, and that all hypotheses are in place, we can apply Lemma 2 with \( \alpha = 1, \beta = -1, m = p \) to arrive at \( \lim_{N \to \infty} G(N)/N = \frac{D}{p+1} \).

### 3 Application to a Classical Theorem on Euler’s \( \varphi \)-function

It is well-known that \( \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{\varphi(n)}{n} \right)/N = \frac{6}{\pi^2} \). (See for example \[3\].)

From this we can derive the following proposition, where we sum only over multiples of an integer \( m \):

**Proposition 3** Let \( m \) be a positive integer, and let \( p_1, \cdots, p_k \) the distinct prime divisors of \( m \). Then

\[
\lim_{N \to \infty} \left( \sum_{m|n \leq N} \frac{\varphi(n)}{n} \right)/N = \frac{6}{\pi^2 m} \prod_{j=1}^{k} \frac{p_j}{1+p_j}
\]

Some numerical evidence:

\( N = 1000, m = 5 \). Here \( \sum_{m|n \leq 1000} \frac{\varphi(n)}{n} \approx .1016 \) while \( \frac{6}{5 \pi^2} \approx .1013 \).

\( N = 100000, m = 200 \). Here \( \sum_{m|n \leq 100000} \frac{\varphi(n)}{n} \approx .001691 \), while \( \frac{6}{200 \pi^2} \cdot \frac{2}{3} \cdot \frac{5}{6} \approx .001689 \).
\[ N = 1000000, \ m = 12348. \] Here \( \frac{\sum_{n \leq 1000000} \varphi(n)}{1000000} \approx .00002153, \) while \( \frac{6 \cdot 3 \cdot 7}{12348} \approx .00002154. \)

**Proof.** The result will follow inductively from the following Claim: Let \( p \) be a prime, \( k \) a positive integer and \( t \) an positive integer not divisible by \( p \). Then if \( \lim_{N \to \infty} \left( \sum_{n \leq N} \frac{\varphi(n)}{n} \right) / N = L \), it follows that

\[
\lim_{N \to \infty} \left( \sum_{tp^j \mid n \leq N} \frac{\varphi(n)}{n} \right) / N = \frac{L}{p^{j+1}}.
\]

To establish the claim, we first handle the case \( j = 1 \). We set \( F(N) = \sum_{t \mid n \leq N} \frac{\varphi(n)}{n} \), \( G(N) = \sum_{pt \mid n \leq N} \frac{\varphi(n)}{n} \). We can bijectively correspond the set \( A \) of integers divisible by \( t \) and less than or equal to \( N/p \) with the set \( B \) of multiples of \( pt \) less than or equal to \( N \) by multiplication by \( p \). We write \( A = A_1 \cup A_2 \), with multiples of \( p \) in \( A_1 \) and nonmultiples of \( p \) in \( A_2 \), and note that (from the usual computation of \( \varphi \) in terms of prime factorization) for \( n \in A_1, \varphi(n)/n = \varphi(pn)/(pn) \), while for \( n \in A_2, \varphi(n)/n = \frac{p-1}{p} \varphi(pn)/(pn) \). So

\[
G(N) = \sum_{pt \mid n \leq N} \frac{\varphi(n)}{n} = \sum_{n \in A_1} \frac{\varphi(np)}{np} + \sum_{n \in A_2} \frac{\varphi(np)}{np} = \sum_{n \in A_1} \frac{\varphi(n)}{n} + \frac{p-1}{p} \sum_{n \in A_2} \frac{\varphi(n)}{n} = \frac{p-1}{p} F([N/p]) + \frac{1}{p} G([N/p])
\]

Applying our lemma with \( m = p, \alpha = \frac{p-1}{p}, \beta = \frac{1}{p}, D = L \) we get

\[
\lim_{N \to \infty} G(N)/N = \frac{D\alpha}{m - \beta} = \frac{L}{p + 1}.
\]

Now we can proceed to the general case of the claim. We now bijectively correspond the set \( A \) of integers divisible by \( t \) and less than or equal to \( N/p^j \) with the set \( B \) of multiples of \( p^jt \) less than or equal to \( N \) by multiplication by \( p^k \), and similarly \( j = 1 \) case write \( A = A_1 \cup A_2 \), with multiples of \( p \) in \( A_1 \) and
nonmultiples of $p$ in $A_2$. Then
\[
\sum_{p^ti|n \leq N} \frac{\varphi(n)}{n} = \sum_{t|n \leq N/p^j} \frac{\varphi(p^j i)}{p^i}.
\]

Dividing through by $N$ we get
\[
\sum_{p^t|n \leq N^j} \frac{\varphi(n)}{n} / N = \frac{p-1}{p^{j+1}} \sum_{t|n \leq N/p^j} \frac{\varphi(i)}{i} + \frac{1}{p} \sum_{pt|n \leq N/p^j} \frac{\varphi(i)}{i}.
\]

where the first limit of the first term is given by the hypothesis $\lim_{N \to \infty} \left( \sum_{t|n \leq N} \frac{\varphi(n)}{n} \right) / N = L$ and the limit of the second term follows from the $j = 1$ case above. That concludes the proof of the claim, and hence the proposition. \qed

References

[1] Brown R., What Proportion of Square-Free Numbers are Divisible by 2? Or by 30, but not by 7?, Private Communication 1/2021

[2] En-Naoui E., Some Remarks on Sum of Euler’s Totient Function, arXiv:2101.02040v1

[3] P. Erdos and H. N. Shapiro, Canad. J. Math. 3 (1951), 375-385.