Extension of Quantum Mechanics to Individual Systems

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Abstract

The Copenhagen interpretation describes individual systems using the same Hilbert space formalism as does the statistical ensemble interpretation (SQM). This leads to the well-known paradoxes surrounding the Measurement Problem. We extend this common mathematical structure to encompass certain natural bundles with Hamiltonian-dependent connections over the Hilbert sphere $S$. This permits a consistent extension of the statistical interpretation to interacting individual systems.

Suppose $V$ is a physical system in interaction with another system $W$. The standard state vector $\Gamma(t)$ of the two interacting systems has a set of polar decompositions $\Gamma = \sum_k q_k \phi_k \otimes \psi_k$, with the $q_k$ complex. These are parameterized by the right toroid $T$ of amplitudes $q = (q_k)_k$ and comprise a singular bundle over $S$, the enlarged state space of $U = V + W$. The evolution of $q$ is determined via the connection on this bundle. We prove that each fiber $T$ has a unique natural convex partition $\{p_1, p_2, \ldots\}$ yielding the correct SQM probabilities, since the circle of unit vectors which generate the ray corresponding to the SQM state of $\Gamma$ intersects $p_j$ in an arc of length $|q_j|^2$. In the extended theory, $V$ is in the state $\phi_j$ (and synchronously $W$ is in the state $\psi_j$) precisely when $q \in p_j$. This refines the assertion of SQM which assigns to $V$ only the mixed state $\sum_k |q_k|^2 |\phi_k\rangle\langle\phi_k|$.

In the new interpretation, rays in Hilbert space correspond to ensembles, while unit vectors in a ray correspond to individual members of such an ensemble. The apparent indeterminism of SQM is thus attributable to the effectively random distribution of initial phases.

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1 Introduction

There is geometry in the humming of the strings. *Pythagoras*

The purpose of this paper is to extend the formalism of quantum mechanics to model the dynamics of individuals interacting with one another. The extension will first require a modest reinterpretation and clarification of the existing formalism. In the new interpretation, rays in Hilbert space $\mathcal{H}$, i.e. the elements of the projective space $P := P(\mathcal{H})$ will correspond to ensembles, while unit vectors in a ray will correspond to individual members of such an ensemble. Second, we need to augment the unit sphere $S := S(\mathcal{H})$ in $\mathcal{H}$ by equipping it with a toroidal bundle $P$ over $S$ together with a Hamiltonian-dependent connection on $P \times \mathbb{R}$. We call $P$ the polar bundle. A key aspect of this extension of the mathematical structure consists of a novel but natural way of partitioning each of the toroidal fibers of $P$.

1.1 IQM State = SQM State + Phases

We denote by IQM the proposed new theory of individual physical systems, while we use SQM to denote the standard statistical assertions of quantum mechanics, which IQM extends.

For much of this paper we will adopt a “top-down” perspective: we start from a total system $S$ and consider a subsystem $S_1$ together with its complement $S_2$, symmetrically. The IQM state of a composite system $S = S_1 + S_2$ consists of an element of the fiber $P_\Gamma$ above a vector $\Gamma$ in the unit sphere $S$ of $\mathcal{H}$. This fiber can be thought of as the set of all the possible complex polar decompositions of $\Gamma$:

$$\Gamma = \sum_k q_k \Gamma_k,$$

where the $q_k \in \mathbb{C}$, and the $\Gamma_k$ are bi-orthonormal. Thus each $\Gamma_k$ is of the form $\phi_k \otimes \psi_k$ where the $\phi_k$ and the $\psi_k$ are orthonormal. We are using here the conventional practice of taking our (pure) vector states for $S_1 + S_2$ from the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The requisite properties of the polar decompositions of $\Gamma$ are elaborated in Appendix A. We note that their totality forms a right toroid $T(r, \Gamma)$ isometric to $\prod_k S^1(r_k)$, the product of circles with radii $r_k := |q_k|$. (We reserve the term torus for the equal radii case.) Thus the additional information they carry about the state of $S$ beyond $\Gamma$ is a compounding of the extra phase data utilized in the first step of refining the ray to a unit vector $\Gamma$ in the ray. In other words, the polar bundle $P$ can be regarded as the natural amalgamation of the polar decomposition with the Hopf bundle $S \to P$.

We stress that the state represented by an element of $P$ is a state of the composition $S = S_1 + S_2$; it depends on all three of $S, S_1, S_2$, although any two of them determine the other. To emphasize this contextuality, we refer to these states as polar states of $S = S_1 + S_2$. Another name that might be appropriate is joint state, although this does not signal the special way it is formed.

Throughout this paper, whenever $\alpha$ is a non-zero vector in a Hilbert space, $[\alpha]$ will denote the ray $\mathbb{C}\alpha$ it generates. In the polar decomposition, only $[\phi_k]$ and $[\psi_k]$ are determined, i.e. $\phi_k$ and $\psi_k$ are only determined up to phase factors but once they are fixed then the $q_k \in \mathbb{C}$ are uniquely
determined. Let $\Gamma = \Gamma(0)$ be specified at $t = 0$. Then the $r_k = |q_k|$ are determined. There is precisely a toroid $T(r)$ of possibilities for the $q := (q_1, \ldots)$. Suppose $q = q(0)$ is fixed at $t = 0$. Then the $\Gamma_k(0) := \Gamma_k := \phi_k \otimes \psi_k$ are also determined, although $\phi_k$ can still be multiplied by an arbitrary phase factor as long as $\psi_k$ is multiplied by its reciprocal.

In Section 2, we define the **dynamical connection** on the bundle $S \times \mathbb{R} \to P \times \mathbb{R}$ as the unique connection with the property that for any unitary evolution and any $\Gamma \in S$, the paths $t \to (\Gamma(t), t)$ are horizontal. This connection defines a unique evolution of the bi-orthonormal frame $\Gamma_k(0), t \mapsto \Gamma_k(t)$, compatible with the Schrödinger evolution $t \to \Gamma(t)$ of $\Gamma(0)$. For the $[\Gamma_k(t)]$ are uniquely determined by $\Gamma(t)$, and so once the $\Gamma_k(0)$ are specified, the connection determines the $\Gamma_k(t)$. This, in turn, defines an evolution of the amplitudes $q_k = q_k(0)$, viz. $t \to q(t) := (q_1(t), \ldots)$. Now $q(t)$ lies in the toroid $T(r, \Gamma(t))$ above $\Gamma(t)$.

We prove in Section 3 a precise theorem whose rough content is as follows:

*There is a unique natural way of partitioning any $n$-dimensional right toroid with basepoint 1 into $n$ convex subsets so that the $k$-th member $p_k$ of the partition $\Psi$ meets every translate of the diagonal circle in an arc of length $2\pi r_k^2$.*

Each $p_k$ is, essentially, a convex neighborhood of the circle $C_k \equiv S^1(r_k) \Gamma_k + \sum_{j \neq k} r_j \Gamma_j$ in $T(\Gamma(t))$. It is a property of $\Psi$ that, regarding the initial phase of $\Gamma(0)$ as randomly uniformly distributed on the circle $S^1(1)\Gamma$, the probability that $S_1$ is in the state $\phi_k$ is $|q_k|^2$. This implies that IQM is consistent with SQM. Thus, the probabilistic nature of SQM derives from the indeterminacy of the initial data. Therefore, we may, without contradicting SQM, go beyond it, and hypothesize that $S_1$ is in the projective state $[\phi_k]$ and simultaneously $S_2$ is in the projective state $[\psi_k]$ whenever $q(t)$ is in the interior of $p_k$.

We discuss the relation of this hypothesis to the current interpretation of SQM and its new implications in greater detail in Section 4.

Thus a joint or polar state of $S = S_1 + S_2$ assigns a **conditional spectral** state $[\phi_k]$ to $S_1$. We call it **spectral** since $[\phi_k]$ corresponds bi-uniquely to the one dimensional spectral projection $P_{\phi_k}$ of the mixed state $\sum_k |q_k|^2 |\phi_k\rangle \langle \phi_k|$, which is all that SQM assigns to $S_1$.

Even if it is granted that $S_1$ has a pure state, why should it be one of the eigenprojections of the density operator? After all, as first stressed by Fano in [17], the mixed state can be realized by many convex combinations of various pure states, some of which can be made to naturally arise in experiments. The reason for choosing a **spectral** projection is that it is the only natural choice. This somewhat vague statement is intuitively obvious, and its precise formulation and proof are given in Appendix 4. We shall see in Section 4 that the experiment described by Fano can be handled using spectral projections.

The extension IQM of SQM to individual systems is complete, natural, thus symmetry-preserving, and essentially unique. Our main purpose is to rationalize the foundations of QM (hopefully, only in the good sense of that word), but it is this uniqueness property which may be of interest in “real” physics.

We believe that what is canonically constructed in the mathematical formalism of a physical model has a physical meaning in that model. The history of physics, from Maxwell’s equations to Einstein’s gravitational theory and the Dirac equation, bears out this principle. It is this idea that was formulated by Dirac in [13, p.60].
The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical formalism in terms of physical entities.

Our main approach has been to follow the path indicated by Dirac; we will refer back to it after taking some steps along these lines.

1.2 SQM=Standard or Statistical Quantum Mechanics.

Our point of departure is the standard Hilbert space formulation of quantum mechanics, which we conservatively take to apply to statistical ensembles. Ensembles are idealized objects which can be realized, to a good approximation, by sufficiently weak beams.

The irreducible ensembles or pure states $\mathcal{S}$ of $\mathcal{S}$ correspond to the set $\mathbf{P}(\mathcal{H})$ of rays in $\mathcal{H}$, with the transition probability given by $|\langle \alpha, \beta \rangle|^2$ for unit vectors $\alpha, \beta$ representing pure states. The model is extended to encompass mixed states, observables, dynamics, symmetry, etc.

The mixed states are modeled by the positive trace 1 operators on $\mathcal{H}$, together with the transition probability $\text{Trace}(XY')$. This extends the previous pure version where the rays in $\mathcal{H}$ correspond to the 1-dimensional orthogonal projections on them.

To a composite system $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, SQM associates a tensor product of their respective Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, provided the states of $\mathcal{S}_1$ and of $\mathcal{S}_2$ are distinguishable. The case where some states of each may be indistinguishable is also very important, especially for the theory of identical particles. These require special treatment which we do not carry out in this paper except, briefly in Appendix [3], for the key case of two identical systems.

As long as the theory is applied to the computation of probabilities which are physically realized as statistical relative frequencies, no interpretational problems arise. Of course, the question of the nature and behavior of the individuals comprising the ensembles is not touched in this formulation.

Even measurement in this setting is unproblematical. The usual formulation goes as follows. SQM models the measurement of some quantity such as the spin of the atoms in some pure beam with state $[\Phi] \in \mathbf{P}$, by considering an ensemble of systems $\mathcal{S}$ composed of the atom $\mathcal{S}_1$ and, say, a Stern-Gerlach apparatus $\mathcal{S}_2$. The ensemble of atoms is sufficiently well represented by the beam (although finer experiments would undoubtedly reveal unwanted correlations unless the beam was very sparse.) A single macroscopic apparatus at a sequence of slightly different times can be interpreted as the second components $\mathcal{S}_2$ of the ensemble.

It is assumed initially that $\phi = \sum_k g_k \phi_k$ is a superposition of the spin states $\phi_k$. The mathematical formulation of the measurement process then considers a unitary evolution of $\mathcal{S}$ with initial state $[\phi \otimes \psi]$ at time $t = 0$ so that

$$\Gamma(0) := \phi \otimes \psi \text{ evolves to } \Gamma(1) = \sum_k q_k \phi_k \otimes \psi_k.$$ 

In the last equation, the $\phi_k$ and the $\psi_k$ are orthonormal. There is a well-defined mixed state attributed to $\mathcal{S}_1$ after the measurement, namely the reduced trace, $\text{Tr}_{\mathcal{S}_1}(P_{\Gamma})$, where $P_{\Gamma} = P_{\mathcal{S}_1 \Gamma}$.
denotes projection on \( C \Gamma \). In this situation the density operator is \( \sum_k |q_k|^2 P_{\phi_k} \). The mixed state \( \text{Tred}_1(P_{\Gamma}) \) is interpreted to mean that with probability \( |q_k|^2 \), \( S_1 \) is in state \( [\phi_k] \).

The paper of Kochen [25] was the first to define a pure state of an interacting system as a spectral ray \( [\phi_k] \) of the polar decomposition, thereby extending the interpretation of SQM for the explicit purpose of resolving the measurement problem. There was however no proposal for the dynamics leading to the particular \( [\phi_k] \). The present extension of the mathematical formalism of SQM to IQM supplies the dynamics and toroidal partitions, leading to a natural choice of \( k \) for each time.

### 1.3 Individual systems.

Our main goal is to find a consistent extension IQM of SQM which models individual systems such as single atoms. Of course, it is generally taken that the mathematical formalism outlined in the previous section does double-duty in modeling individuals as well as statistical ensembles.

The interpretational difficulties of this ambiguous usage begin with the notion that \( |q_k|^2 \) is the probability that an individual is in the state \( [\phi_k] \). In applications, it is interpreted, as usual, in terms of the relative frequency of an ensemble of systems all in state \( [\phi] \). This subjective interpretation of probability is awkward but, by itself, consistent. However, this is only the beginning of the difficulties: they end with the need to somehow identify the pure state \( \sum_k q_k \phi_k \otimes \psi_k \) with the mixed state \( \sum_k |q_k|^2 P_{\phi_k \otimes \psi_k} \). This identification is usually called “collapse of the wave function”.

From this brief description, it can be seen that the heart of the problem is to consistently attribute a pure state to an individual system, including one such as \( S_1 \) which is a subsystem of \( S \) interacting with \( S_2 \), and then to mathematically model these pure states. The possibility of such a model is formally equivalent to the consistency of IQM, but the acceptability of the model will depend upon its naturalness, uniqueness, and intrinsic interest.

The first step in constructing the new model is to represent states of individuals, in part, by unit vectors instead of rays. In all but the most foundational studies of QM, states are already so represented. Usually, there is an early, one-time warning that vectors differing by a phase factor give the same pure state. \( \sum_k q_k \phi_k \otimes \psi_k \) with the mixed state \( \sum_k |q_k|^2 P_{\phi_k \otimes \psi_k} \). This identification is usually called “collapse of the wave function”.

Relative phases occur in the literature, along with some controversy as to what observable or self-adjoint operator should be used to represent them. The absolute phases we are invoking could be avoided but they clearly foreshadow the additional “moduli” necessary for a complete individual state description: a torus of phases.

### 2 Enlarging the State Space

In the introduction, we already indicated that IQM extends the SQM state space of rays to unit vectors representing individuals. The mathematical relation of the unit vector \( \phi \) being in the ray \( [\phi] \) mirrors the physical relation of the individual system in state \( \phi \) being in the ensemble with state \( [\phi] \). We can think of an ensemble with pure state \( [\phi] \) as being composed of individuals having state vectors of the form \( \zeta \phi \), where the \( \zeta \) are uniformly distributed random phase factors. Perhaps
a reason this minor extension of the model for SQM has not explicitly appeared in the literature is that it requires acceptance of the non-applicability of SQM to individuals.

In this section, we must compound this extension and simultaneously include all the phases arising in polar decompositions of vectors in a tensor product:

\[ \Gamma = \sum q_k \Gamma_k = \sum q_k \phi_k \otimes \psi_k \]

2.1 An *ad hoc* construction of the polar bundle \( \mathcal{P} \).

We consider the set of all possible pairs \((\Gamma, q) = (\Gamma(t), q(t))\) which can arise; we form the (right) toroids \( T(r, \Gamma) \) consisting of any \( q \) arising from a possible polar decomposition \( \sum_k q_k \Gamma_k \). We thus get a family of toroids parameterized by the \( \Gamma \in S \) for which they occur as the polar decomposition coefficients.

Let \( T \) def = \( \{ (\Gamma, q) \mid (\exists \Gamma_k) \Gamma = \sum_k q_k \Gamma_k \text{ is a polar decomposition} \} = \prod_{r \in S} T(r, \Gamma) \).

Then \( T \) is the total space of a generalized bundle \( \mathcal{P} \) def = \( [T \xrightarrow{\Pi} S] \), where \( \Pi(\Gamma, q) = \Gamma \). We are talking then of a *bundle* in its most general form, a map between spaces. We call \( \mathcal{P} \) a toroidal bundle since each fiber is a toroid. We may consider \( \mathcal{P} \) as a singular torus bundle, with torus group \( T^n(1) \) def = \( \prod_k S^1(1) \). Indeed, just as the unit circle \( S^1(1) \) acts on any circle \( S^1(r) \subset \mathbb{C} \) by \( S^1(r) \ni re^{i\theta} \mapsto re^{i(r+\theta)} \in S^1(r) \), (even if \( r = 0 \)) so does \( T^n(1) \) act on \( T^n(r) \), provided a bouquet of generating circles \( S^1(r_k) \) is distinguished. This distinguishing is automatic when the \( r_k \) are all distinct. Thus, if we restrict the bundle to the set \( S^\text{reg} \) of regular \( \Gamma \), i.e. those with positive distinct \( r_k \), then \( \mathcal{P}^\text{reg} \) def = \( \left[ \Pi^{-1}(S^\text{reg}) \xrightarrow{\Pi} S^\text{reg} \right] \) is a (standard) smooth principal \( T^n \)-bundle on which we have put a metric on each fiber.

The coefficient \( q_k(t) \) might be thought of as “the amplitude that S is in the state \( \Gamma_k(t) \)” using typical QM textbook language.

We will take as our IQM state space of \( S = S_1 + S_2 \) the total space \( T = T(\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2) \) of the bundle \( \mathcal{P} \). This bundle is not a locally trivial bundle, since the base space is connected, while the fibers do not have constant dimension. Less trivially, if we restrict to the stable points \( S^\text{stab} \) of \( S \), i.e. those for which no \( r_k = 0 \), then the fibers do have constant dimension, but the bundle is not locally a product at those points where the \( r_k \) are not all distinct. If we restrict the bundle to the regular points \( S^\text{reg} \) of \( S \), then it is indeed a trivial or product bundle. Nevertheless, even restricted to \( S^\text{reg} \), the connection \( A^H \) we will employ is not trivial; it has curvature and, when the Hamiltonian \( H \neq 0 \), even torsion.

2.2 A natural construction of \( \mathcal{P} \) using moment maps.

In the introduction, we stressed the importance of canonical constructions in extending physical models. In this section we show that the polar bundle can be realized canonically. The idea behind this construction makes use of the moment (or momentum) map from symplectic geometry and general bundle operations.
\[ H \] has a natural symplectic structure \( \omega_0 \), which, in terms of coordinates \( z_k \) with respect to an orthonormal basis, can be written (as in [32, p.130])
\[
\omega_0 = \frac{i}{2} \sum_k dz_k \wedge \overline{dz}_k.
\]
The action of the unitary group \( \mathbb{U}(H) \) on \( H \) is then Hamiltonian so that, denoting the Lie algebra \( \mathfrak{L}(\mathbb{U}(H)) \) by \( u \), there exists a moment map [32, p.162]
\[
\mu : H \rightarrow u^* \cong u, \quad \mu(z) = \frac{i}{2} zz^* = \frac{i}{2} \|z\|^2 P_C z.
\]
We used here the natural identification of the Lie algebra \( u \) with its dual \( u^* \), deriving from the Hilbert-Schmidt inner product.

If \( G \) is a subgroup of \( \mathbb{U}(H) \) then its moment map \( \mu_G \) is given by composing \( \mu \) with projection on the subalgebra \( \mathfrak{L}(G) \), i.e. \( \mu_G = \mathfrak{P}(G) \circ \mu \). We are going to apply this together with the fact (see Appendix A.2) that the reduced trace is itself, essentially, a projection in the space of operators with respect to the Hilbert-Schmidt Hilbert space inner product.

Now suppose we have a Hilbert space factorization:
\[ H = H_1 \otimes H_2. \]
Let \( G \) be the compact group generated by \( \mathbb{U}(H_1) \otimes \mathbb{I}_2, \mathbb{I}_1 \otimes \mathbb{U}(H_2) \). \( G \) is almost isomorphic to the direct product \( \mathbb{U}(H_1) \times \mathbb{U}(H_2) \), i.e. the bottom row of the following diagram of Lie algebras is exact:
\[
\begin{array}{ccc}
0 & \xrightarrow{\delta} & u(H_1) \times u(H_2) & \xrightarrow{} & g & \xrightarrow{\sigma} & 0 \\
& \nearrow^\tau & \searrow^{\mu_G} & & & \searrow^g & & \xrightarrow{\sigma} & \xrightarrow{\mathfrak{P}} & 0
\end{array}
\]
where \( u(H_i) = \mathcal{L}(\mathbb{U}(H_i)), g = \mathcal{L}(G), \delta(aI) = (aI_1, aI_2) \) and \( \sigma(u_1, u_2) = u_1 \otimes I_2 - I_1 \otimes u_2 \). Moreover, defining
\[
\tau(z) = \frac{i}{2} (\text{Tred}_1(P_{Cz}), \text{Tred}_2(P_{Cz}))
\]
the triangle commutes. If we pass to the corresponding projective spaces we get a moment map \( \tilde{\tau} : \mathbb{P}(H) \rightarrow \tilde{g} \cong \mathfrak{su}(H_1) \times \mathfrak{su}(H_2) =: \mathcal{L}_{12} \) given in finite dimensions by the formula
\[
\tilde{\tau}([z]) = \frac{i}{2} \left( \text{Tred}_1(P_{Cz}) - \frac{1}{n_1} I_1, \text{Tred}_2(P_{Cz}) - \frac{1}{n_2} I_2 \right)
\]
with \( z \in \mathbb{S} \).

The pair of reduced traces gives a map to the equi-spectral density operators. We have shown that this map is essentially the moment map for the group \( G \) of automorphisms of \( \mathbb{P}(H) \) preserving the tensor product decomposition: \( G \) is \( \mathbb{SU}(H_1) \times \mathbb{SU}(H_2) \) modulo its center.

If \( \Gamma = \sum_k q_k \phi_k \otimes \psi_k \) is a polar decomposition of \( \Gamma \in \mathbb{S} \), then
\[
\tilde{\tau}([\Gamma]) = \frac{i}{2} \left( \sum_k |q_k|^2 P_{C\phi_k} - \frac{1}{n_1} I_1, \sum_k |q_k|^2 P_{C\psi_k} - \frac{1}{n_2} I_2 \right).
\]
In any case, the fibers of $\tilde{\tau}$ are the toroids $T(\Gamma) \overset{\text{def}}{=} \{ \sum_k q_k' \phi_k \otimes \psi_k \mid |q_k'| = |q_k| \}$. The (generalized) bundle $\tilde{\tau} : P \to \mathfrak{L}_{12}$ induces a bundle, with the same fibers, over any space $M$ mapping to $\mathfrak{L}_{12}$. So $M \overset{\eta}{\to} \mathfrak{L}_{12}$ yields a commutative diagram

$\begin{array}{ccc}
\eta^* P & \xrightarrow{\eta^*} & P \\
\downarrow & & \downarrow \tilde{\tau} \\
M & \xrightarrow{\eta} & \mathfrak{L}_{12}
\end{array}$

We can, in particular, apply this construction to the case where $M = P, \eta = \tilde{\tau}$. In general, this bundle induced by the projection map ($\tilde{\tau}$ in this case) is called the square of the original bundle [39, p.49]. It always leads to a bundle with a cross-section which is as smooth as is locally possible. In the present case, it yields a toroidal bundle $B$ over $P$. We have the Hopf map $S \to P$. We can use the Hopf map to induce from $[B \to P]$, a bundle $[T \to S]$, which is our polar bundle $\mathcal{P}$. Using this squaring operation, it is not hard to justify the statements made at the end of the previous section about product bundles.

**Review of the natural construction of $\mathcal{P}$.** We start with the essentially classical notion of the moment map from $P \to \mathfrak{L}_{12}$, the Lie algebra of the natural group acting on the tensor product. We then take the square of this bundle, using traditional terminology. Thinking of the bundle as the map $\tilde{\tau}$ this makes the space of the new bundle, our polar state space, a kind of square-root of $P$. The final step is the familiar Hopf bundle construction, $S \to P$. The last two steps could be carried out in the opposite order. We have not explicitly carried out the natural construction in the infinite dimensional case, although the naturality implies this is possible by a limiting procedure. At any rate, the final bundle is well-defined in infinite, as well as finite, dimensions.

The idea that quantum mechanics arises as a kind of square-root of standard structures is an intuition which is here made precise. A precise representation-theoretic version of this square-root operation was established in [1, Ax].

We note that if $\mathcal{H}_1 = \mathcal{H}, \mathcal{H}_2 = \mathbb{C}$, then the bundle $B \to P$ is equivalent to $S \to P$. When squared, this becomes diffeomorphic to the product bundle $S \times S^1 \to S$.

### 2.3 The Geometric and Dynamical Connections

In the original papers of Berry and Simon, their successors (and predecessor Pancharatnam), collected in [38], the physical and mathematical significance of the canonical unique lifting of smooth curves in $P$ to $S$ was explicated. This lifting is by means of the canonical connection. A good reference for connections with a physics orientation is Frankel’s book [20]. We recall that one way of specifying a connection on a principal bundle is by giving a 1-form $A$ on the total space which is compatible with its principal bundle structure. In the present case, this 1-form $A$ is on the total space $S$ of the principal $S^1$-bundle $S \overset{\pi}{\to} P$. It is given by the following formula.

$$A := A^0 \overset{\text{def}}{=} \langle z, dz \rangle = \sum \bar{z}_k d\bar{z}_k, \text{ for } ||z|| = 1. \quad (2.1)$$
A curve \( t \to z(t) \in S \) is horizontal (over \( P \) with respect to the canonical connection) if, and only if, the induced 1-form on \( \mathbb{R} \) vanishes, i.e.

\[
\forall t \; \langle z(t), \dot{z}(t) \rangle = 0.
\]

The curve \( t \to z(t) \in S \) is the unique horizontal lift of the curve \( t \to [z(t)] \in P \), which begins at \( z(t) \) for \( t = 0 \).

This 1-form \( A \) takes values in \( \mathfrak{i}\mathbb{R} \) which is naturally identified with the Lie algebra \( \mathcal{L}S^1 \) of \( S^1 \). It is easy to see that

(i) \( A \) is invariant under the natural action of \( S^1 \) on \( S \).

Moreover, for any fixed \( w \in S \) and variable \( \zeta = e^{i\theta} \in S^1 \), we have

(ii) \( \langle \zeta w, d(\zeta w) \rangle = \zeta d\zeta = i\theta \).

Since, we are going to be dealing with some slight generalizations of \( A^0 \), we indicate the two compatibility conditions required for a 1-form \( A \) on a general principal \( G \)-bundle \( P \overset{\pi}{\rightarrow} M \) (where \( G \) is a Lie group and \( M \) is a manifold.) \( A \) is now required to be \( \mathfrak{g} := \mathcal{L}G \)-valued. Thus \( G \) acts naturally on \( A \) by combining its action on \( P \) with its Ad-action on \( \mathfrak{g} \). These two conditions are:

1. \( A \) is invariant under the action of \( G \);
2. For all points \( p \in P \), the pull-back \( \gamma^*(A) \) to a 1-form on \( 4G \) is the right-invariant Maurer-Cartan form on \( G \) (which is \( i\theta \) when \( G = S^1 \).)

For the canonical connection, these conditions are satisfied, in view of (i) and (ii).

The horizontal lift of a closed loop in \( P(\mathcal{H}) \) need not be closed in \( S(\mathcal{H}) \), but will differ at its endpoints by a phase factor. The argument of this phase factor is Berry’s phase. The original papers, and so far as we know, all subsequent papers on the subject separated the total phase change of various kinds of liftings of curves in \( P \) to \( S \) as being the sum of a geometrical phase and a dynamical phase. The first is Berry’s phase, determined by the canonical connection, the second is the remaining phase angle needed to comply with the Schrödinger evolution of a vector in \( \mathcal{H} \).

We are going to reverse this procedure because we prefer an equivalent but more unified treatment of these phase factors by getting the total phase from a new connection, which we call the dynamical connection. This will be useful in the sequel, where we study certain horizontal liftings of non-evolutionary curves in \( P \). But it is clear from the consideration of an energy eigenvector that no such connection can exist on \( P \)! Namely, if \( H\Gamma(0) = E\Gamma(0) \), then \( \Gamma(t) = e^{-iEt}\Gamma(0) \). Of course, no phase determined by a connection can be involved here, since \( [\Gamma(t)] \) is the constant curve.

For this reason we will work with the corresponding contact manifold \( P \times \mathbb{R} \). This is also a convenient space for the consideration of time-dependent Hamiltonians \( H(t) \). Its analog appears in classical mechanics, where it is sometimes called the “extended phase space”, as in [2, p.236].

### 2.3.1 The dynamical connection on time extended phase space.

**Lemma 2.1** We suppose given a (possibly time-dependent) bounded Hamiltonian \( H(t) \) on \( \mathcal{H} \). The evolutions \( t \rightarrow \Gamma(t) \) in \( P(\mathcal{H}) \) correspond bijectively to the maps \( t \rightarrow (\Gamma(t), t) \) in \( P \times \mathbb{R} \).

There exists a connection on \( S \times \mathbb{R} \overset{\pi \times 1}{\rightarrow} P \times \mathbb{R} \) so that every curve \( t \rightarrow (\Gamma(t), t) \) corresponding to a Schrödinger evolution \( t \rightarrow \Gamma(t) \) is horizontal. It is given by the 1-form:

\[
A^H \overset{\text{def}}{=} \langle z, dz \rangle + i\langle z, H(t)z \rangle dt =: A^0 + i\mathcal{E}^H(z)dt
\]

(2.3)
We can characterize $A^H$ as the unique connection compatible with the Schrödinger evolution which agrees with $A^0$ on constant time slices.

The condition for a curve $t \to (\Gamma(t), t) \in S \times \mathbb{R}$ to be horizontal with respect to $A^H$ is the following variation of Eq. (2.2), with which it coincides when $H(t) \equiv 0$.

$$\forall t \ (\Gamma(t), \dot{\Gamma}(t)) = -i(\Gamma(t), H(t)\Gamma(t)).$$  \hspace{1cm} (2.4)

**Proof.** Eq. (2.3) defines a 1-form $A^H$ on $S \times \mathbb{R}$ which clearly satisfies compatibility condition 1).

To be more precise, in this equation we should actually replace the 1-form $A^0$ by its pullback to $S \times \mathbb{R}$. This form is independent of $t$ and it has the same expression $\langle z, dz \rangle$ as does $A^0$, so we neglect this nicety. To see that 2) is also satisfied, we note that $A^H$ has the same pullback to $S^1$ as does $A^0$. Thus $A^H$ does define a connection.

To obtain Eq. (2.4) we take the tangent vector $(\dot{\Gamma}(t), \partial_t)$ to the time-extended phase space and apply $A^H$, as defined by Eq. (2.3).

Now suppose $t \to (\Gamma'(t), t) \in S \times \mathbb{R}$ is a horizontal lift of $t \to [\Gamma(t)] \times \mathbb{R}$, where $\Gamma(t)$ satisfies the Schrödinger equation. Then from Eq. (2.4), we have $(\Gamma', \partial_t \Gamma') = -i(\Gamma', H(t)\Gamma') = -i(\Gamma, H(t)\Gamma)$. Since $\Gamma'(t)$ differs from $\Gamma(t)$ only by a phase factor $\zeta(t)$, it follows that $\langle \zeta, \partial_t (\zeta \Gamma) \rangle = -i(\Gamma, H(t)\Gamma) \Rightarrow \langle \Gamma, \partial_t \Gamma \rangle + \bar{\zeta} \zeta = -i(\Gamma, H(t)\Gamma)$. It follows that $\dot{\zeta} = 0$ and hence $\Gamma'(t)$ and $\Gamma(t)$ differ only by a constant phase factor. Thus $t \to \Gamma(t)$ is horizontal.

The uniqueness of $A^H$ follows from the fact that the tangent vectors at a point $(z, t)$ are generated by $\partial_t$ and the tangent space of the constant time slice.

A technique for finding $A^H$—horizontal versions $(\Omega(t), t) \in S \times \mathbb{R}$ of a $A^0$-horizontal curve $\Omega^0(t) \in S$ follows from the argumentation of the proof. Namely, write $\Omega(t) = \zeta(t)\Omega^0(t)$ and take inner products:

$$\langle \Omega, \partial_t \Omega \rangle = \langle \zeta \Omega^0, \partial_t (\zeta \Omega^0) \rangle = \langle \Omega^0, \partial_t \Omega^0 \rangle + \bar{\zeta} \zeta = \bar{\zeta} \zeta.$$

Now we want Eq. (2.4) to hold, so we require

$$\bar{\zeta} \zeta = -i(\Omega, H\Omega) = -i(\Omega^0, H\Omega^0) \Rightarrow \partial_t \ln \zeta(t) = -i(\Omega^0(t), H(t)\Omega^0(t)).$$  \hspace{1cm} (2.5)

Thus we can express the $A^H$-horizontalizing phase factor $\zeta$ in terms of the given $\Omega^0$:

$$\zeta(t) = e^{-i \int_0^t \langle \Omega^0(s), H(s)\Omega^0(s) \rangle ds}. \hspace{1cm} (2.6)$$

We express this result in a lemma for reference.

**Lemma 2.2** If $t \to \Omega^0(t) \in S$ is a curve horizontal with respect to the canonical connection and if

$$\Omega^H(t) = \zeta(t)\Omega^0(t) \hspace{1cm} (2.7)$$

then $\Omega^H(0) = \Omega^0(0)$ and $(\Omega^H(t), t)$ is horizontal with respect to $A^H$ if and only if

$$\zeta(t) = e^{-i \int_0^t \langle \Omega^0(s), H(s)\Omega^0(s) \rangle ds}. \hspace{1cm} (2.8)$$

■
If we make the substitution \( \zeta(t) := e^{i\alpha(t)} \),

\[
\alpha(t) = -\int_0^t \langle \Omega^0(s), H(s)\Omega^0(s) \rangle ds.
\] (2.9)

We can compare this equation with Equation (3) in the paper of Aharanov-Anandan, contained in the previously referenced collection [38]. There we see the total phase \( \phi \) represented as the geometric phase \( \beta \) plus \( \alpha \). While that paper and others are at pains to separate out \( \beta \) and obtain it from the canonical connection, we reverse this procedure so as to get the total phase from a connection.

The canonical connection is the dynamical connection formed with a trivial Hamiltonian. The following lemma is therefore a generalization of the previous one and is proved similarly.

**Lemma 2.3** If \( t \to \Omega'(t) \in S \) is a curve horizontal with respect to the dynamical connection formed with respect to the Hamiltonian \( H' \), and if \( \Omega''(0) = \Omega'(0) \), then (dropping the second component) \( t \to \Omega''(t) = \zeta(t)\Omega'(t) \) is horizontal with respect to the dynamical connection formed with respect to the Hamiltonian \( H = H' + H'' \) if, and only if,

\[
\zeta(t) = e^{-i\int_0^t \langle \Omega'(s), H''(s)\Omega'(s) \rangle ds}.
\] (2.10)

\[\blacksquare\]

### 2.3.2 The dynamical connection as a Lagrangian.

Suppose \( C := C_\Gamma := [0, 1] \ni t \to (\Gamma(t), t) \in S \times \mathbb{R} \) is a critical curve for the curve-functional

\[
S(C) := \int_0^1 C^*(A^H) = \int_0^1 (\langle \Gamma(t), \dot{\Gamma}(t) \rangle + i\langle \Gamma(t), H(t)\Gamma(t) \rangle)dt =: \int_0^1 L(\Gamma, \dot{\Gamma})dt.
\]

Here \( C^* \) is the pullback map.

By the Euler–Lagrange equation \( \frac{\partial L}{\partial \Gamma} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\Gamma}} \right) \), we get the Schrödinger equation:

\[
\dot{\Gamma}(t) = iH(t)\Gamma(t) \Rightarrow \dot{\Gamma}(t) = -iH(t)\Gamma(t).
\] (2.11)

We thus see that the connection form \( A^H \) is a Lagrangian form. The defining property of the 1-form \( A^H \) is that Schrödinger evolutions are horizontal. It then turns out that Schrödinger evolutions are critical values of the \( A^H \) action integral.

This is part of a very general situation, in which horizontality with respect to a connection on a bundle over a manifold yields critical values of a related Lagrangian, e.g. the projections of horizontal curves in the tangent bundle of a Riemannian manifold \( M \) are the geodesics of \( M \) as in [24, Vol.1, Prop. 6.3].
2.3.3 The relativistic dynamical connection.

Extending the SQM state space $\mathbf{P}$ to $\mathbf{P} \times \mathbb{R}$ is a convenient way to exhibit the phase of the constant SQM states corresponding to energy eigenstates. But it has the immediate effect of ruining the symmetry of the extensions of SQM to bundles over $\mathbf{P}$. For suppose a group $G$ acts on $\mathbf{P}$. Then we need a natural extension of this action to $\mathbf{P} \times \mathbb{R}$ which, in general, will be affine on the second “time” factor $\mathbb{R}$. This is fine for Galilean relativity, but it won’t do for Poincaré covariance.

This suggests that to make the theory relativistic we begin by extending the SQM phase space $\mathbf{P}$ to $\mathbf{P} \times \mathfrak{m}$, where $\mathfrak{m} := M^4$ is Minkowski space with its usual action by the simply connected cover $G$ of the Poincaré group. We also assume that $G$ acts unitarily on $\mathcal{H}$ and thence on $\mathbf{P} = \mathbf{P}(\mathcal{H})$.

There is as yet no generally accepted rigorous version of the quantum field theory or even QED which is required to model the non-trivial dynamics of interacting relativistic particles, despite the best efforts of constructive quantum field theorists. The analytic considerations of QFT are beyond the scope of this paper, so we proceed merely formally. We assume that $G$ acts compatibly on the attendant additional structures, such as the distribution-valued field operators, which we formally treat as ordinary unbounded operators. In particular the (unbounded) energy operator $H$ may be combined with the canonical connection, as before, to produce the dynamical connection with respect to a given inertial frame with coordinates $\mathbf{x} = (x_0 = t, x_1, x_2, x_3)$. For a curve of vectors $\Gamma(t)$ analytic for $H$, we can consider the connection 1-form in Eq. (2.3).

$$A^H \overset{\text{def}}{=} \langle \mathbf{z}, d\mathbf{z} \rangle + i \langle \mathbf{z}, H\mathbf{z} \rangle dt = A^0 + i \langle \mathbf{z}, H\mathbf{z} \rangle dt. \quad (2.12)$$

We want to formally exhibit a manifestly covariant version of this form on $\mathbf{P} \times M$ which along a time-like line in an inertia frame agrees with this expression, interpreting $t$ as the proper time. Let

$$\Lambda(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$$

be the usual Lorentzian inner product. Let $\mathbf{p} = (p_0 = H, p_1, p_2, p_3)$ denote the 4-vector of energy momentum operators, i.e. the generators of the translation group action on $\mathbf{P}$. The connection form $A^0$ is already invariant under $G$, so we need only modify the imaginary part of $A^H$. The new, manifestly covariant form is:

$$\langle \mathbf{z}, d\mathbf{z} \rangle + i \Lambda(\langle \mathbf{z}, \mathbf{p}\mathbf{z} \rangle, \mathbf{x}). \quad (2.13)$$

It now follows, at least formally, that $G$ acts naturally on the $A^H$—horizontal lifts in $\mathcal{P}$ of curves in $\mathbf{P}$. It follows that our theory faces no new insuperable obstacles from special relativity. Those already present in SQM are, of course, quite sufficient.

2.4 Evolution in the polar bundle.

Let us examine the possible polar decompositions appearing in

$$\Gamma = \sum_k q_k \phi_k \otimes \psi_k = \sum_k q_k \Gamma_k \quad (2.14)$$

especially in the non-degenerate case.
The expression on the right hand side of Eq. (2.14) is not unique for it is possible to multiply the \( \Gamma_k \) by any phase factor, and simultaneously multiply the \( q_k \) by the inverse or conjugate phase factor. It is tempting to try and pick some unique representation, for example by requiring the \( q_k > 0 \). This choice is affirmed by recognizing that the \( |q_k| \) comprise the eigenvalues of the positive part \( P = +\sqrt{\Gamma^\star \Gamma} \) of the polar decomposition of \( \Gamma = +\sqrt{\Gamma^\star \Gamma} \), where \( U \) is an isometric linear map \( \mathcal{H}_2 \to \mathcal{H}_1 \) satisfying \( U(\psi_k) \to \phi_k \). See Appendix A. In the completely non-degenerate case, where the \( \phi_k \) comprise an orthonormal basis for \( \mathcal{H}_1 \) and the \( \psi_k \) comprise an orthonormal basis for \( \mathcal{H}_2 \), \( U \) is a well-defined unitary operator: \( U = P^{-1}\Gamma \), since the \( |q_k| \) are assumed positive and distinct.

The trouble with this apparently natural choice, is that it is not sustainable under the natural evolution of the \( \Gamma_k \). This evolution is the lift by the connection \( A^H \) of the curve \( t \to [\Gamma_k(t)] \in \mathcal{P} \), where the \( \Gamma_k(t) \) are bi-orthonormal polar components of \( \Gamma(t) \). To see what is involved here, let us suppose we have some Hamiltonian \( H \) on \( \mathcal{H} \) generating a unitary evolution \( t \to U(t) = e^{-itH} \), taking units for which \( \hbar = 1 \).

Let
\[
\Gamma(0) = \sum_k q_k(0) \Gamma_k(0). \tag{2.15}
\]
be an initial fixed polar decomposition. Set \( \Gamma(t) = U(t) \Gamma(0) \). Let
\[
\Gamma(t) = \sum_k q_k(t) \Gamma_k(t). \tag{2.16}
\]
be any polar decomposition of \( \Gamma(t) \), with some choice of smooth \( q_k(t) \) and \( \Gamma_k(t) \).

The curve \( t \to \gamma_k(t) := [\Gamma_k(t)] \in \mathcal{P} \) is independent of any such choice. Let \( \Gamma_k^H(t) \) be the unique curve horizontally lifting \( \gamma_k(t) \) with respect to the connection form \( A^H \). We loosely use this expression when we really mean: \( (\Gamma_k^H(t),t) \) is the horizontal lift of \( (\gamma_k(t),t) \) to \( S \times \mathbb{R} \) which begins at \( (\Gamma_k^H(0),0) \). By Eq. (2.4), this means \( \langle \Gamma_k^H(t), \dot{\Gamma}_k^H(t) \rangle = -i \langle \Gamma_k^H(t), H(t) \Gamma_k^H(t) \rangle \). Of course, this condition alone is far from characterizing the \( \Gamma_k^H(t) \). This requires the additional condition that \( [\Gamma_k^H(t)] = \gamma_k(t) \), i.e. we also require \( [\Gamma_k^H(t)] = [\phi_k(t) \otimes \psi_k(t)] \).

Now the \( q_k^H(t) \) are uniquely determined by finally requiring
\[
\Gamma(t) = \sum_k q_k^H(t) \Gamma_k^H(t). \tag{2.17}
\]
In the next section, we exhibit the equations determining \( \Gamma_k^H(t) \) and \( q_k^H(t) \); but we can easily see that positivity of the \( q_k^H(t) \) is ruled out in general by Eq. (2.8) because the horizontalizing factor needed to go from \( A^0 \)–horizontal to \( A^H \)–horizontal is given by
\[
\zeta(t) = e^{-i \int_0^t \langle \Gamma^0(s),H(s) \Gamma^0(s) \rangle ds}, \tag{2.18}
\]
which can be any phase factor for some \( t \) and some \( H \).

### 2.4.1 Evolutionary equations for the amplitudes \( q_k^H \).

If you are out to describe the truth, leave elegance to the tailor. Einstein
Let $\Gamma(t) := \Gamma^H(t)$ evolve according to Schrödinger’s equation with Hamiltonian $H$. We want to determine the behavior of the $q_k^H(t)$ when $\Gamma(t) = \sum_k q_k^H(t)\Gamma_k^H(t)$ is a polar decomposition in which the $(\Gamma_k^H(t), t)$ are horizontal with respect to $A^H$. Using Lemma A.2 we can get this from the easier situation where $\Gamma(t) = \sum_k q_k^0(t)\Gamma_k^0(t)$ is a polar decomposition in which the $\Gamma_k^0(t)$ evolve horizontally with respect to $A^0$.

So we now examine the polar decompositions with reference to $A^0$. A horizontal lift of $[\Gamma_k(t)]$ can be taken of the form $\phi_k(t)\otimes \psi_k(t)$ where $\phi_k(t)$ and $\psi_k(t)$ are horizontal with respect to their own (lower-dimensional) canonical connections on $S(H_1)$ and $S(H_2)$. Indeed,

$$\langle \phi_k, \partial_t \phi_k \rangle = 0 \& \langle \psi_k, \partial_t \psi_k \rangle = 0 \Rightarrow \langle \phi_k \otimes \psi_k, \partial_t (\phi_k \otimes \psi_k) \rangle = 0.$$  \hspace{1cm} (2.19)

Let $A(t) := \widetilde{\Gamma_0(t)}\Gamma^0(t)$, $B(t) := \Gamma^0(t) \cdot \Gamma_0(t)$, as in Appendix A and let the $r_j^2 = |q_j|^2$ be the common eigenvalues of $A(t)$ and $B(t)$. Because these $\phi_k(t)$ and $\psi_k(t)$ are assumed non-degenerate eigenvectors of $A(t)$ and $B(t)$, we can apply first-order perturbation theory (see Appendix A.) Then

$$\dot{\phi}_j = \sum_{k \neq j} \langle \phi_k, \hat{A} \phi_j \rangle \frac{1}{r_j^2 - r_k^2} \phi_k, \quad \dot{\psi}_j = \sum_{k \neq j} \langle \psi_k, \hat{B} \psi_j \rangle \frac{1}{r_j^2 - r_k^2} \psi_k.$$  \hspace{1cm} (2.20)

Moreover we can also make $\hat{A}(t)$ and $\hat{B}(t)$ more explicit by taking reduced traces, as in [Blum, p.72]. Let $P_1$ be the 1-dimensional orthogonal projection on $\Gamma \in H_1 \otimes H_2$. Then

$$\dot{\hat{A}}(t) = \frac{1}{i} \text{Tred}_1[H, P_1(t)], \quad \dot{\hat{B}}(t) = \frac{1}{i} \text{Tred}_2[H, P_1(t)].$$  \hspace{1cm} (2.21)

We make use of the abbreviations:

$$H_{jk, mn} := \langle H(\phi_j \otimes \psi_k), \phi_m \otimes \psi_n \rangle.$$  \hspace{1cm} (2.22)

$$\beta_{ab} := -i(\langle q_b H_{ab, kk} - q_a \sum_k q_k H_{kk, ba} \rangle)$$  \hspace{1cm} (2.23)

$$\beta'_{ab} := -i(\langle q_b H_{ba, kk} - q_a \sum_k q_k H_{kk, ab} \rangle).$$  \hspace{1cm} (2.24)

Combining these abbreviations with Eq. (2.20), Eq. (2.19), and Lemma A.3, we are going to prove (using Appendix B) for the canonically horizontal evolutions that the following system of highly coupled, highly non-linear (usually infinite) autonomous system of first order ODE’s holds:

$$\dot{q}_a = -i \sum_k H_{aa, kk} q_k$$  \hspace{1cm} (2.25)

$$\dot{\phi}_a = \sum_{k \neq a} \frac{\beta_{ka}}{|q_a|^2 - |q_k|^2} \phi_k$$  \hspace{1cm} (2.26)

$$\dot{\psi}_a = \sum_{k \neq a} \frac{\beta'_{ka}}{|q_a|^2 - |q_k|^2} \psi_k.$$  \hspace{1cm} (2.27)
Theorem 2.4 Let $\phi_k(t), \psi_k(t)$ be horizontal with respect to their own (lower-dimensional) canonical connections on $S(H_1)$ and $S(H_2)$. Let $\Gamma^\phi_k(t) = \phi_k(t) \otimes \psi_k(t)$. The $\Gamma^\phi_k(t)$ are horizontal with respect to $A^0$. Let $\Gamma(t) = \sum_k q_k(t) \Gamma^\phi_k(t)$ be a polar decomposition. Then the $q_a, \phi_a, \psi_a$ satisfy the above system of autonomous ODE's.

Proof. Taking reduced traces, as in [9, p.72],
\[
\dot{\rho}_{S_1}(t) = \frac{1}{i} \text{Tred}_1[H, P\Gamma(t)]
\] (2.28)
Now we can express this in terms of polar bases:
\[
\langle \phi_a, \text{Tred}_1 H P\Gamma(\phi_b) \rangle = \sum_k \langle \phi_a \otimes \psi_k, H P\Gamma(\phi_b \otimes \psi_k) \rangle
\] (2.29)
\[
= \sum_k \langle \phi_a \otimes \psi_k, \langle \Gamma, \phi_b \otimes \psi_k \rangle H \Gamma \rangle = \sum_k \langle \Gamma, \phi_b \otimes \psi_k \rangle \langle \phi_a \otimes \psi_k, H \Gamma \rangle
\] (2.30)
\[
= \sum_k \delta_{kb} \overline{q_b} \langle \phi_a \otimes \psi_k, H \Gamma \rangle = \overline{q_b} \langle \phi_a \otimes \psi_b, H \Gamma \rangle
\] (2.31)
Similarly, for the oppositely ordered product in the commutator,
\[
\langle \phi_a, \text{Tred}_1 P\Gamma H(\phi_b) \rangle = \sum_k \langle \phi_a \otimes \psi_k, P\Gamma H(\phi_b \otimes \psi_k) \rangle
\] (2.32)
\[
= \sum_k \langle P\Gamma(\phi_a \otimes \psi_k), H(\phi_b \otimes \psi_k) \rangle
\] (2.33)
\[
= \sum_k \langle \langle \Gamma, \phi_a \otimes \psi_k \rangle \Gamma, H(\phi_b \otimes \psi_k) \rangle = \sum_k \langle \Gamma, \phi_a \otimes \psi_k \rangle \langle H\Gamma, \phi_b \otimes \psi_k \rangle
\] (2.34)
\[
= q_b \langle H\Gamma, \phi_b \otimes \psi_a \rangle
\] (2.35)
Thus we can write Eq. (2.28) as
\[
\langle \phi_a, \dot{\rho}_{S_1}(\phi_b) \rangle = \frac{1}{i} (\overline{q_b} \langle \phi_b \otimes \psi_a, H\Gamma \rangle - q_a \langle H\Gamma, \phi_a \otimes \psi_b \rangle)
\] (2.36)
We also have,
\[
H\Gamma = \sum_k q_k H \Gamma_k = \sum_k q_k H(\phi_k \otimes \psi_k)
\] (2.37)
Thus
\[
\langle H\Gamma, \phi_b \otimes \psi_a \rangle = \sum_k \overline{q_k} \langle H(\phi_k \otimes \psi_k), \phi_b \otimes \psi_a \rangle = \sum_k \overline{q_k} H_{kk,ba}
\]
\[
\langle \phi_a \otimes \psi_b, H\Gamma \rangle = \sum_k q_k \langle \phi_a \otimes \psi_b, H(\phi_k \otimes \psi_k) \rangle = \sum_k q_k H_{ab,kk}
\]
\[
\langle \phi_a, \dot{\rho}_{S_1}(\phi_b) \rangle = \frac{1}{i} (\overline{q_b} \sum_k q_k \overline{H_{kk,ab}} - q_b \sum_k q_k H_{kk,ba})
\] (2.38)
This, combined with Eq. (B.18) yields Eq. (2.26).
For $S_2$, we get
\[
\langle \psi_a, \dot{S}_2(\psi_b) \rangle = \frac{1}{i} \left( q_b \sum_k q_k H_{kk,ba} - q_a \sum_k q_k H_{kk,ab} \right)
\]
(2.39)

This yields Eq. (2.27). The proof of Eq. (2.25) is similar, but since we want to establish a more general version that will be needed later, the proof is given in Lemma A.3.

Theorem 2.5 Let $\Gamma(t) = \sum_k q^H_k(t)\Gamma^H_k(t)$ be a polar decomposition in which the $\Gamma^H_k(t)$ evolve horizontally with respect to $A^H$. Then, up to a constant phase factor, the $q^H_k(t)$ can be expressed in terms of the $q^k(t)$ in the last theorem as
\[
q^H_k(t) = q^k(t) e^{\int_0^t i H^k_{kk,kk}(s) ds}
\]
(2.40)

Proof. By Lemma 2.2, $\Gamma^H_k(t) = \sum_k q^k(t) e^{-i \int_0^t H^k_{kk,kk}(s) ds} = \sum_k q^k(t) e^{-i \int_0^t H^k_{kk,kk}(s) ds}$.

3 Natural Partitions of Toroids

I don’t believe it; you’ve actually found a practical use for geometry!

B. Simpson

The polar state space of $S = S_1 + S_2$ replaces each point of the SQM state space $P$ by a toroid of phase factors. It turns out this toroid has a canonical partition into convex subsets, one for each circular factor. This is exactly what is needed to associate to the pair $S_1$ and $\Gamma = \sum_k q^k \phi_k \otimes \psi_k$ precisely one of $[\phi_k]$. The reason for considering the Pythagorean type of partition is discussed in Section 3.

In this section, we carry out this purely mathematical analysis of the partitioning of the right toroids which have arisen as the fibers of the polar bundle. We state and discuss the precise theorems in the next section, and the proofs are in the following sections.

3.1 Right toroids and their mappings.

By a right toroid $T(r)$ is meant the direct product of circles. Let $I$ be an index set which is either all positive integers $\mathbb{Z}_{>0}$ or just those in $[1,n]$ where $n \in \mathbb{Z}_{>0}$. Occasionally we abuse notation to write $\mathbb{Z}_{>0} = [1,n]$ with $n = \infty$. Recall, that if $r = (r_k), r_1 \geq r_2 \geq \cdots \geq 0$, then
\[
T(r) = \prod_{k \in I} S^I(r_k) = \prod_{k \in I} (\mathbb{R}/2\pi r_k \mathbb{Z})..
\]

We have a canonical surjection $\varpi : \mathbb{R}^n \to T(r)$, which is a local isometry. We denote the set of all these right toroids by $\mathcal{T}$. An allowable map (or morphism)
\[
\iota : T(r) \to T(r')
\]
is of the form $(\zeta_j)_{j<n} \rightarrow (\zeta'_k)_{k<n'}$. 

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where for every $k < n'$ there exists $j < n$ so that $\zeta_k = \zeta_j$ or $\zeta'_k = 1$. We also require that $\iota$ be injective, so that $\iota(T(r))$ is just a sub-product of $T(r')$. In other words, the morphisms split. $T(r) \hookrightarrow T(r')$ is just an inclusion map, which can usually be omitted. It is easy to check that if $T(r) \hookrightarrow T(r')$ is a morphism, then so is the induced map $T(r) \cap T(r') \hookrightarrow T(r)$.

For each $k \in I$, $C_k := C_k(r) \overset{\text{def}}{=} \{(\zeta_j)_{j \in I}\}$, where $\zeta_j = 1$ if $j \neq k$, are the circles bijectively corresponding to the imbedded image of $S^1(r_k)$. For example, $C_1 = (S^1(r_1), 1 \cdots)$, is a member of the bouquet $\{C_j \mid j \in I\}$. These right toroids arise as the amplitudes in polar decompositions, i.e. the fibers of the polar bundle discussed in Section 3. The main diagonal circle (or subgroup) $\bar{D} := \bar{D}(r) \overset{\text{def}}{=} \{(r_j e^{i\theta}) \mid \theta \in \mathbb{R}\}$ plays a central role. For any $t \in T(r)$, we call a translate $t \cdot \bar{D} \subset T(r)$ of $\bar{D}$ a diagonal.

**Pythagorean Partitions.**

We are interested in the natural way(s) of partitioning right toroids. So we form the set $\Psi \Sigma$ of partitioned right toroids $(T(r), \Psi(r))$. Here the partitions $\Psi$ are of the Pythagorean type $\Psi = \{p_j\}_{j \in I}$; this means $(T(r), \Psi(r))$ has the following three properties:

**Partition Property**: $T(r) = \cup_{j \in I} p_j$, the interiors $\bar{p}_j$ are disjoint and each is equal to the closure of its interior: $\bar{p}_j = \overline{p_j}$.

**Convexity Property**: For all $j \in I$ there exists a compact convex set $A_j$ in $\mathbb{R}^n$ with $\varpi(A_j) = p_j$ and $\varpi|A_j$ is injective.

**Diagonal Property**: Every diagonal $\varpi(\mathbb{R}^2)$ of $T(r)$ intersects $p_j$ in an arc of length $2\pi r_j^2$.

We want to show there is a uniform way of partitioning any right toroid. In other words, we want to find a natural procedure $P$ which, when applied to $T(r)$, yields a Pythagorean partition $\Psi(r)$ of $T(r)$, i.e.

$$\Sigma \ni T(r) \overset{P}{\rightarrow} (T(r), \Psi(r)) \in \Psi \Sigma.$$  \hspace{1cm} (3.1)

We mean natural in the general categorical sense reviewed in Appendix [3], which in this specific situation amounts to the following

**Naturality Property**: If $T(r) \hookrightarrow T(r')$ then for all $k \in I'$, there exists a $j \in I$ so that $T(r) \cap \bar{p}_k' \subset p_j(r)$. (Here we are denoting $p_k(r')$ by $p_k'$.)

All we really need are the cases of the Naturality Property when $T(r)$ has dimension $n < \infty$, and $T(r) \cap p_k'$ also has dimension $n$.  

**Example** $n = 2$. Refer to the rectangle OABC in Figure 3-1 in which the sides have lengths $2\pi r_1, 2\pi r_2$ in the golden ratio. We have $T(r_1, r_2) = OABC, C_1 = OC, C_2 = OA, D = OB$. Let $AE, CF \perp OB$. We can take $OE'CF$ for $A_1$, and $A_2 = OF'AE$. Set $A = A_1 \cup A_2$ so that $T(r_1, r_2) = \bar{A}$. The Naturality Property above entails that $T(r_1) = p_1 \subset p'_1(r_1, r_2) \cap T(r_1, 1) = OC$. It follows that $C_1 \subset p'_1(r_1, r_2) \subset C_1$, i.e. $C_1 = p_1(r_1, r_2) \cap T(r_1, 1)$. In particular, $C_1 \subset p_1(r_1, r_2)$. This always happens: For all $k$, $C_k \subset p_k$, whenever they exist (Lemma 3.4). A similar argument applies to $C_2 = OA$.

We want to prove the following result.

**Theorem 3.1** There exists exactly one natural procedure $P$ satisfying the relation (3.4), or in the language of Appendix [3]: there is a manifest functor $P$ from $\Sigma$ to $\Psi \Sigma$.  

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Figure 3-1: Shown are $A_1 = OFCE'$, $A_2 = OF'AE$. Also shown are $s_1^1(1, 1) = OFC$, $s_1^1(1, \sigma) = OE'C$, $s_1^1(2, 1) = OF'A$, $s_1^1(2, \sigma) = OEA$, $B_1 = OA'CB$, $B_2 = OC'AB$, which are defined below.

The existence part of the proof begins with a description of the partition for dimension $n$. We then graphically illustrate it for $n = 3$, as we have already done for $n = 2$. This gives the geometric idea motivating the proof for general $n$. Then we give an existence proof for any $n$. By the naturality and taking limits, this even holds for $n = \infty$. We conclude with a proof of the more difficult uniqueness assertion.

### 3.2 The construction of the Pythagorean partition.

The idea behind the existence proof is the following: We start with an $n$-dimensional box (rectangular parallelepiped) which covers the toroid almost isometrically. The box has a natural partition into $n!$ simplices obtained by slicing it with $n$ hyperplanes through the main diagonal which are perpendicular to the faces. Then each simplex can be partitioned into $n$ convex subsets (slabs) by slicing it with hyperplanes perpendicular to the main diagonal, which acts as the hypotenuse. This hypotenuse corresponds to a Hopf circle whose phase determines which state of the subsystem $S_1$ obtains. The slices perpendicular to the main diagonal are made at each vertex in the given simplex. Now comes the surprising part (even for $n = 3$): these slabs can be translated to the generating edges of the box in only one way and when this is done, we are left with $n$ convex neighborhoods of these $n$ edges. The interiors of these parts map isometrically to the desired members of the partition of the toroid. The desired partition is determined by the following definitions, where the $e_j$ denotes the standard unit vectors.

$s_j = 2\pi r_j, f_j = s_j e_j, s = \sum_j f_j, g_j = r_j^2 s - f_j.$

$A_k := A_k^n \overset{\text{def}}{=} \text{parallelotope generated by } \{g_j, g_j + r_j^2 s \mid 1 \leq j \neq k \leq n\}.$

$L^n(s_1, \cdots, s_n) := L^n(s) \overset{\text{def}}{=} \bigoplus_j \mathbb{Z} f_j,$ the lattice generated by the $f_j$. 

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\( \varpi : \mathbb{R}^n \mapsto T^n(r_1, \ldots, r_n) \overset{\text{def}}{=} \mathbb{R}^n / L^n(s) \) is the canonical local isometric surjection mentioned above. 
\( \bar{x} \overset{\text{def}}{=} \varpi(x) \) for any \( x \in \mathbb{R}^n \) (so as to reduce the number of \( \varpi \)'s and parentheses.) 
\( p_k \overset{\text{def}}{=} \bar{A}_k \). This defines the partition \( \mathcal{P} = \{ p_1, p_2, \ldots, p_n \} \). The reader can check that Figure 3-1 comports with it, for \( n = 2 \). Figure 3-2 and Figure 3-3 show the component parts, before translation, of \( A_3 \) for a box with edge-length ratios 5:4:3.

Figure 3-2: \( st(3, 312), st(3, 123), st(3, 231) \)

Figure 3-3: \( st(3, 132), st(3, 321), st(3, 213) \)

In Figure 3-1, the parts are lattice-translated to form the parallelepiped \( A_3 \).
In Figure 3-2, we illustrate the \( A_k \) for the 3-cube.
In Figure 3-4, we illustrate the \( A_k \) for a 3-dimensional box.
In Figure 3-5, we illustrate a portion of the corresponding tiling.
In Figure 3-6, we illustrate the same portion where each tile has been linearly shrunk towards its centroid by a factor .8 so as to better reveal how successive layers appear twisted, although they are obtained by lattice translations.
3.3 The existence of a Pythagorean partition

We will need some more definitions which we collect here. These are required for the proofs of existence and uniqueness.

\[ \Sigma_n \overset{\text{def}}{=} \text{symmetric group on } \{1, \ldots, n\}, \quad 1 := \text{identity element of } \Sigma_n. \]

For all \( U \subset \mathbb{R}^n \), \( [U] \overset{\text{def}}{=} \text{convex hull of } U \).

\( U + v \overset{\text{def}}{=} \{ u + v \mid u \in U \subset \mathbb{R}^n \} \).

\( D \overset{\text{def}}{=} [0, s], \quad D^\perp \overset{\text{def}}{=} \{ x \in \mathbb{R}^n \mid s \cdot x = 0 \} \).

\( \text{Stab}(v, w, U) \overset{\text{def}}{=} [D^\perp + v, D^\perp + w] \cap U \quad \forall U \subset \mathbb{R}^n \).

\( (\forall \sigma \in \Sigma_n) \quad S_\sigma \overset{\text{def}}{=} [0, f_{\sigma(1)} + f_{\sigma(2)} + \cdots + \sum_{\nu=1}^n f_{\sigma(\nu)} = s] \).

In particular \( S_1 = [0, f_1, f_1 + f_2, \ldots , s] \).

\( v_{k\sigma} \overset{\text{def}}{=} \sum_{\nu<\sigma^{-1}(k)} f_{\sigma(\nu)} \).

\( \text{Sl}(k, \sigma) \overset{\text{def}}{=} \text{Stab}(v_{k\sigma}, v_{k\sigma} + f_k, S_\sigma) \).

\( \text{Sl}^*(k, \sigma) \overset{\text{def}}{=} \text{Sl}(k, \sigma) - v_{k\sigma} \).

We say two sets in \( \mathbb{R}^n \) of dimension \( n \) are \textit{quasi-disjoint} if their intersection is of smaller dimension.

Figure 3-4: \( A_3 = \bigcup_{\sigma \in \Sigma_3} \text{Sl}^*(3, \sigma) \)
Figure 3-5: The partition for a cube.

Figure 3-6: The partition for a box with edge ratios 5:4:3.
Figure 3-7: The tiling for a cube.

Figure 3-8: The shrunken tiling for a cube.
Lemma 3.2 \( \{S_\sigma \mid \sigma \in \Sigma_n\} \) is a partition of the box \( B \) generated by \( f_1, \ldots, f_n \).

Proof. The \( S_\sigma \) are quasi-disjoint. Each \( S_\sigma \subset B \), so it suffices to show \( B \subset \bigcup_{\sigma \in \Sigma_n} S_\sigma \). Let \( v \in B \). Then \( v = \sum_{\nu=1}^n a_\nu f_\nu \), \( a_\nu = \langle v, f_\nu \rangle \). Also \( a_\nu \in [0,1] \) and there exists \( \sigma \in \Sigma_n \) so that

\[ a_\sigma(1) \geq a_\sigma(2) \geq \cdots \geq a_\sigma(n) \geq a_\sigma(n+1) := 0. \]

\[ \therefore v = \sum_{\nu=1}^n (a_\sigma(\nu) - a_\sigma(n+1))(f_\sigma(1) + \cdots + f_\sigma(\nu)) \in S_\sigma. \]

\[ \blacksquare \]

Proposition 3.3 \( \forall k \leq n \{st^\dagger(k, \sigma) \mid \sigma \in \Sigma_n\} \) is a partition of \( A_k \).

Proof. \[ st(k, S_\sigma) = [D^\perp + v_{k_\sigma}, D^\perp + v_{k_\sigma} + f_k] \cap S_\sigma \]

\[ \therefore st^\dagger(k, \sigma) = [D^\perp, D^\perp + f_k] \cap (S_\sigma - v_{k_\sigma}) \]

\[ S_\sigma - v_{k_\sigma} = \left[ \begin{array}{c}
\sigma^{-1}(k) \\
- \sum_{\nu=1}^{\sigma^{-1}(k)} f_\sigma(\nu), \sum_{\nu=2}^{\sigma^{-1}(k)} f_\sigma(\nu) + \cdots + \sum_{\nu=\sigma^{-1}(k)+1}^{n} f_\sigma(\nu) \\
\end{array} \right] \]

\[ B_k := \bigcup_\sigma (S_\sigma - v_{k_\sigma}), E\{n, k\} := \{1, \ldots, n\} - \{k\} \]

\[ \therefore B_k = \left[ - \sum_{j \in M} f_j, f_k + \sum_{j \in M} f_j \mid M \subset E\{n, k\} \right] \]

Set \( v_k = \sum_{j \neq k} f_j \). Replacing \( M \) by its complement in \( E\{n, k\} \) we get

\[ B_k = \left[ \sum_{j \in M} f_j - v_k, f_k + \sum_{j \in M} f_j \mid M \subset E\{n, k\} \right]. \]

Set \( F_k = \left[ \sum_{j \in M} f_j \mid M \subset E\{n, k\} \right] \), one of the 2 facets of the box \( B \) which are perpendicular to \( f_k \).

\[ \therefore B_k = [F_k - v_k, F_k + f_k]. \]

Now \( F_k + f_k \) is the other facet of \( B \) parallel to \( F_k \) and \( F_k - v_k \) is a parallel facet of the box \( B - v_k \) (which also contains \( 0 \) as a vertex.)

\[ v_M := \sum_{j \in M} f_j - v_k \in \text{Vertices}(F_k - v_k), w_M := \sum_{j \in M} f_j + f_k \in \text{Vertices}(F_k + f_k) \]

Thus \( B_k \) is a parallelotope which is the convex hull of its edges \[ [v_M, w_M], M \subset E\{n, k\}. \]

We have \( w_M - v_M = v_k + f_k = s \) which implies the \( [v_M, w_M] \) are parallel translates of \( D = [0,s] \), i.e.

\[ [v_M, w_M] = D + v_M. \]
The last three relations imply
\[ \cup_{\sigma \in \Sigma_n} s_t^\dagger(k, \sigma) = \left[D^\perp, D^\perp + f_k \right] \cap \cup_{\sigma \in \Sigma_n} (S_{\sigma} - v_{k\sigma}) = \left[D^\perp, D^\perp + f_k \right] \cap B_k \]
\[ D^\perp \cap [v_M, w_M] \neq \emptyset \implies (v_M - f_k) \cdot s < 0, (w_M - f_k) \cdot s \geq 0. \]

The last three relations imply
\[ \cup_{\sigma \in \Sigma_n} s_t^\dagger(k, \sigma) = \left[D^\perp \cap B_k, (D^\perp + f_k) \cap B_k \right]. \]
\[ \therefore \quad \cup_{\sigma \in \Sigma_n} s_t^\dagger(k, \sigma) = \left[D^\perp \cap (v_M, w_M), (D^\perp + f_k) \cap (v_M, w_M) \right] \left| M \subset E\{n, k\} \right] = \left[D^\perp \cap (D + v_M), (D^\perp + f_k) \cap (D + v_M) \right] \left| M \subset E\{n, k\} \right] \]
\[ = \left[D^\perp \cap (D - \sum_{j \in E\{n, k\} - M} f_j), (D^\perp + f_k) \cap (D - \sum_{j \in E\{n, k\} - M} f_j) \right] \left| M \subset E\{n, k\} \right] = \left[D^\perp \cap (D - \sum_{j \in M} f_j), (D^\perp + f_k) \cap (D - \sum_{j \in M} f_j) \right] \left| M \subset E\{n, k\} \right] \]
\[ = \left[(D^\perp + \sum_{j \in M} f_j) \cap D - \sum_{j \in M} f_j, (D^\perp + f_k + \sum_{j \in M} f_j) \cap D - \sum_{j \in M} f_j \right] \left| M \subset E\{n, k\} \right] \]
\[ = \left[\sum_{j \in M} r_j^2 s - \sum_{j \in M} f_j, \sum_{j \in M} r_j^2 s + r_k^2 s - \sum_{j \in M} f_j \right] \left| M \subset E\{n, k\} \right] = A_k, \text{ i.e.} \]
\[ A_k = \cup_{\sigma \in \Sigma_n} s_t^\dagger(k, \sigma) \]

and the \( s_t^\dagger(k, \sigma) \) are quasi-disjoint.

\[ \Leftarrow \]

**Theorem 3.4** \( \varpi | \hat{A}_k \) is injective (\( \hat{X} \) denotes the interior of \( X \).) \( p \triangleq \{ p_k = \varpi(A_k) \mid k = 1, \ldots, n \} \) is a partition of \( T(r) \).

**Proof.** Both assertions follow from the fact that the \( A_k \) are unions of translates, \( s_t(j, S_\sigma) - v_{k\sigma} \), by elements of the lattice \( L(s) \) of quasi-disjoint subsets \( s_t(j, S_\sigma) \) of the box \( B \), which is a fundamental domain for \( T(r) \).

\[ \Delta \triangleq \varpi(D), \text{ the closed, diagonal subgroup of } T(r). \]

**Lemma 3.5** \( \forall k \) every coset \( C \) of \( \Delta \) in \( T^n(r) \) intersects \( p_k \).
Proof. By the theorem, there exists a line \( L \subset \mathbb{R}^n \), parallel to \( D \), such that \( \varpi(L) = C \), which intersects some \( A_j \). It follows that \( L \) intersects the \((n - 1)\)-dimensional parallelogram generated by the \( \{g_h | h \neq j\} \). Thus \( L + g_j - g_k \) intersects \( A_k \) and

\[
L + g_j - g_k = L + (r^2 s - f_j) - (r^2 s - f_k) = L + f_k - f_j \equiv L \pmod{\mathcal{L}(s)}.
\]

Remark. \( C \) is a geodesic circle contained in \( T(r) \) which is “parallel” to \( \Delta \).

Lemma 3.6 For all \( k \) every coset \( C \) of \( \Delta \) in \( p_k \cap T(r) \) is a geodesic segment of length at least \( 2\pi r_k^2 \).

Proof. In the proof of the previous lemma, \((L + g_j - g_k) \cap \hat{A}_k \) is an interval of length \( 2\pi r_k^2 \) whose interior, by Theorem 3.4, is mapped injectively by \( \varpi \) into \( T(r) \).

\[ \hat{p}_k \equiv \varpi(\hat{A}_k). \] Since \( \varpi \) is a covering map, We have \( \hat{p}_k \) is open and \( \text{Dim}(p_k - \hat{p}_k) < n \). Since \( \sum_j 2\pi r_j^2 = 2\pi = \text{Length}(D) = \text{Length}(\Delta) \), and by the lemma, \( \text{Length}(\hat{p}_k \cap C) \geq 2\pi r_k^2 \), we obtain

Lemma 3.7 For all \( k \) every coset \( C \) of \( \Delta \) in \( T(r) \), \( C \cap \hat{p}_k \) is either empty or a geodesic segment of length exactly \( 2\pi r_k^2 \).

Each \( p_k \) is toroidally convex in the sense that it differs by a closed lower dimensional set (namely \( \varpi(\partial A_k) \)) from a subset \( \hat{p}_k \), which is isometric via \( \varpi^{-1} \) with an open convex subset \( \hat{A}_k \) in Euclidean space. We will just use the term convex for this notion in the sequel.

We have an exact sequence

\[ 0 \to \Delta \to T(r) \to T(r)/\Delta \to 0. \]

Now

\[
T(r)/\Delta \cong (\mathbb{R}^n/\mathcal{L}(s))/((\mathbb{R}s + \mathcal{L}(s))/\mathcal{L}(s)) \cong \mathbb{R}^n/(\mathbb{R}s + \mathcal{L}(s))
\]

Let \( p_{D^\perp} \) denote the orthogonal projection on \( D^\perp \).

Then \( g_j = -p_{D^\perp}(f_j) \). Thus \( T(r)/\Delta \cong D^\perp/\mathcal{G} \), where \( \mathcal{G} \) is the lattice in \( D^\perp \) generated by any \( n - 1 \) of the \( g_j \). (Note \( \sum_j g_j = 0 \).) Thus the cosets of \( \Delta \) in \( T(r) \) are given by the points of the toroid \( G^{n-1} \equiv D^\perp/\mathcal{G} \). We can now state

Lemma 3.8 For all \( k \) and for all but a lower dimensional set of cosets \( C \in G^{n-1} \), \( C \cap \hat{p}_k \) is a geodesic segment of length exactly \( 2\pi r_k^2 \).

In our present notation, we have \( C_k = \varpi([0,f_k]) \) and \( 1 = \varpi(0) \).

Lemma 3.9 \( \forall k C_k \subset p_k, \forall k C_k - \varpi(0) \subset \hat{p}_k \).
This follows from the above construction of the $A_k$.

With notation as above, we can collect our results in the following statement.

**Theorem 3.10** The above partition $\mathcal{P} \stackrel{\text{def}}{=} \{ p_k \mid k = 1, \ldots, n \}$ of $T(r)$ satisfies:

(I) $\forall k \; C_k \subset p_k \, \forall k \, C_k - \varpi(0) \subset \hat{p}_k$.

(II) For all $k$ and for all but a lower dimensional set of cosets $C \in G^{n-1}$, $C \cap \hat{p}_k$ is a geodesic segment of length exactly $2\pi r_k^2$.

(III) $p_k$ is convex.

(IV) There exist quasi-disjoint paralleloctopes $A_k = A_k \subset \mathbb{R}^n$ so that

- (o) $p_k = \varpi(A_k)$;
- (i) $[0, f_k] \subset A_k$;
- (ii) $\cup_k A_k$ is a fundamental domain for $\mathbb{R}^n / L(s)$;
- (iii) For every line $L$ in $\mathbb{R}^n$ which is parallel to $D$, $L \cap \hat{A}_k = \emptyset$ or an interval of length $2\pi r_k^2$.

We note that we have arrived at our partition of the toroid by means of a new tiling of the covering space $\mathbb{R}^n$ by the translates of sets $\cup_{k=1}^n A_k$, which is non-convex (unless the box is a cube). In any case, this tiling is not face-to-face and projects onto $D^\perp$ to yield a tiling of $\mathbb{R}^{n-1}$ by zonotopes.

For $n = 3$, this is a tiling of $\mathbb{R}^2$ by hexagons as shown in Figures 3-5 and 3-6. For $n = 4$, it projects to a tiling of $\mathbb{R}^3$ by rhombic dodecahedra.

### 3.4 Uniqueness of Natural Partitions of Right Toroids.

**Theorem 3.11** The functor $P : \mathcal{S} \rightsquigarrow \mathcal{P} \mathcal{S}$, defined by the procedure of Section 3.3, is inverse to the forgetful functor and so is unique.

**Proof.** We have to show for each toroid $T = T(r)$ with distinct $r_k$, in a Hilbert space $\mathcal{H}$, there is only one (functorial) way to endow it with a Pythagorean partition. We use the functoriality to argue inductively on the dimension $d$ of $T$. But first, we note that the partition of $T$ induces a tiling of $R^n$ by convex compact sets, which must then be polytopes.

The rough idea of the proof is to start with the faces of the box, where the induction yields the desired induced partition. Then we must extend the unique determination of the polytopic parts into the interior of the box, using some simple connectivity properties.

Now if $d = 1$, we must take $A_1 = [0, f_1] = 2\pi e_1$ and the uniqueness is trivially true. If $d > 1$ then there exists a morphism $S^1(r_j) \hookrightarrow T$. Then the partition of $T$ restricted to $S^1(r_j)$ must be the partition associated to $S^1(r_j)$, i.e. the trivial partition. This means that $S^1(r_j) \subset p_i$, for some $i$. Since the morphism splits we must have $r_i = r_j$ and we can assume $i = j$. Thus $S^1(r_j) = C_j \subset p_j$.

We have shown:

A natural partition of $T(r) \in \mathcal{S}$ must preserve its bouquet.

Let us now take $d = 2$. Then $\hat{p}_j$ is isometric, via $\varpi$ to $A_j \subset \mathbb{R}^2$, where the $A_j$ are convex in $\mathbb{R}^2$.

Moreover, we can assume $[0, f_j] \subset A_j$, $(0, f_j) \subset \hat{A}_j, \varpi(0) = 1$. We also know every diagonal line $L$ (a line parallel to $D$) in $\mathbb{R}^2$ must intersect $A_j$ either trivially or in a line segment of length $2\pi r_j^2$. 

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It is this latter possibility which must hold for every diagonal through a \( p \in (0, f_j) \). It follows that 
\( A_j \) is the union of open line segments of length \( 2\pi r_j^2 \), since the total area (2-dimensional Lebesgue measure) these comprise is \( 2\pi r_j^2 \times |g_j| = 2\pi r_1 r_2 \) and these sum to the area \( 4\pi^2 r_1 r_2 \) of \( T \).

We claim that \( 0 \) is a vertex of the convex polygon \( A_j \). It is a point of \( \partial A_j \), since it lies in \( \bigcap_k A_k \). So we must eliminate the possibility that \( 0 \) is an interior point of an edge \( E \) of \( A_j \). To accomplish this, we resort again to the functoriality of the tiling. Each \( T \) possesses an isometric involution, namely its geodesic symmetry about 1, or, more simply, the inverse operation. This isometry preserves the special circles \( C_j \) and hence must preserve the \( p_j \). Thus \( \tilde{A}_j = A_j = p_j \). If now \( 0 \) were interior to \( E \), then \( 1 = \tilde{0} \) would be interior to \( \varpi((–A_j) \cup A_j) = p_j \), a contradiction.

We know that every diagonal \( L \) which intersects \( A_j \) must intersect it in a closed interval of length \( 2\pi r_j^2 \). For the main diagonal \( \mathbb{R}D \), this segment is the edge \( E \) which has 0 as a vertex. There are thus two possibilities: \( E = r_j^2 D \) or \( E = –r_j^2 D \). The latter possibility can be eliminated using a variant of the “inversion argument” we used before. Specifically, if \( E = –r_j^2 D \), then the angle between \( E \) and \( f_j \) would be obtuse. It follows that \( –A_j \) would intersect \( (0, f_j) \) non-trivially for \( j' \neq j \), a contradiction.

We now know that \( A_j \) contains the triangle \( t_j \) with vertices \( 0, r_j^2 s, f_j \). Similar arguments show that \( f_j \) must also be a vertex of \( A_j \) and that the triangle \( t_j' \) with vertices \( f_j', f_j – r_j^2 s, 0 \) is contained in \( A_j \). Since the total area of \( t_j \cup t_j' \) is \( 4\pi^2 r_1 r_2 r_j^2 \), the area of \( A_j \), therefore \( A_j = t_j \cup t_j' \) and we have shown that \( A_j \) is the one previously constructed, and so, unique.

We take \( n > 2 \) and inductively assume we have proven the uniqueness of our tilings for all toroids of dimension less than \( n \). For any \( n \)-dimensional right toroid \( T = T^n \), we set \( T_{−k} := T_{−k}^{n−1} := \) the \((n−1)\)-dimensional toroid generated by \( C_j, j \neq k \). Suppose \( \{p_k \mid k = 1, \ldots, n\} \) is a Pythagorean tiling of \( T \). Then, by functoriality, \( p_{−j} := p_{n−1} = \{p_k \cap T_{−j} \mid k = 1, \ldots, n, k \neq j\} \) is a tiling of \( T_{−j} \). We tentatively drop the assumption that \( r_1 > \ldots > r_n \) while still requiring that the \( r_k \) be positive distinct reals. We do this so that we may consider that \( T_{−n} \) is an arbitrary \( T_{−j} \), for the sake of simplifying notation.

**The three remaining steps of the proof.**

We know from the existence proof, that

\[
\forall k \leq n \quad A_k = \bigcup_{\sigma \in \Sigma_n} s^\dagger(k, \sigma).
\]

The rest of the proof consists of three steps, showing:

1) the \( s^\dagger(k, \sigma) \) are the (convex) hulls of their intersections with \( \partial B \) and \( D \);
2) \( A_k \cap \partial B' = A_k \cap \partial B' \), for any lattice translate \( B' \) of \( B \);
3) \( A_k \cap D' = A_k \cap D' \), for any lattice translate \( D' \) of \( D \).

Since the \( A_k \) are convex the theorem will then follow. These three steps will be established in the following lemmas. In accordance with the notation introduced in Section 3.3, \( S_1 \) is the simplex corresponding to the identity permutation \( 1 \). We sometimes abbreviate:

\[
R_{k\sigma} := \left( \sum_{j \leq \sigma^{-1}(k)} r_j^2 \right) s, \quad R_{k\sigma} := \left( \sum_{j > \sigma^{-1}(k)} r_j^2 \right) s, \quad v_k := v_{k1}, \quad D_{k\sigma} = [R_{k\sigma}, R_{k\sigma}],
\]
Lemma 3.12

\[ si(k, 1) = \left( \partial B \cap si(k, 1), D_{k1} \right) \]  \hspace{1cm} (3.2)

**Proof.** \( si(k, 1) \) is convex since it is the intersection of a convex slab and a simplex. Thus, it is the convex closure of its vertices which are the intersections of the edges of \( S_1 \) with the hyperplanes \( D^\perp + v_{k1} \) and \( D^\perp + v_{k1} + f_k \). Now all the edges of \( S_1 \) are contained in \( \partial B \) except for \( D \). To see this, note that \( j < k \Rightarrow [v_j, v_k] \subset [v_j, v_{j+1}, \ldots, v_k] \) which is a \((k - j)\)-dimensional face of \( S_1 \) and so contained in \( \partial B \) unless \( j = 0 \) and \( k = n \). The result follows since \( (D^\perp + v_{k1}) \cap D = R_{k1}^\perp \) and \( (D^\perp + v_{k1} + f_k) \cap D = R_{k1} \). \hfill \blacksquare

Lemma 3.13

\[ si(k, \sigma) = \left( \partial B \cap si(k, \sigma), D_{k\sigma} \right) \]  \hspace{1cm} (3.3)

**Proof.** We have \( R_{k\sigma} = \left( \sum_{j < \sigma - 1} r_{j_{\sigma}(j)}^2 \right) s \) and \( R_{k\sigma}^\perp = \left( \sum_{j \leq \sigma - 1} r_{\sigma(j)}^2 \right) s \). So the result follows by permuting the \( f_k \) with \( \sigma \).

This completes the first step.

Lemma 3.14 *For any Pythagorean partition \( \{p_1, \ldots, p_n\} \) with corresponding \( \{A_1, \ldots, A_n\} \) and for any lattice translate \( B' \) of \( B \)

\[ \forall k \ A_k \cap \partial B' = A_k \cap \partial B'. \]  \hspace{1cm} (3.4)

**Proof.** Let \( E \) be a facet of \( B' = B + v \), where \( v \) is a lattice vector.

\[ \widetilde{A_k} \cap E \subset \widetilde{A_k} \cap \tilde{E} = p_k \cap T_{-j} \]

for some \( j \), namely that \( j \) such that \( f_j \perp E \). By renumbering, we can take \( j = n \). The \( A_k \cap E \) are convex. \( \cup_k (p_k \cap T_{-n}) = T_{-n} \). We can apply the functor \( F \) to \( T_{-n} \) yielding \( F(T_{-n}) \hookrightarrow F(T) \). This implies \( p_k^{n-1} = p_i^n \cap T_{-n} \) provided \( i < n \). Inductively, \( p_i^{n-1} = A_i^{n-1} \). Thus \( A_i \cap E \) and \( A_i \cap E \) are both lifts of \( p_i^{n-1} \). The interior of \( A_i \cap E \), which equals \( \tilde{A_i} \cap E \), is a lift of an open subset \( U \subset p_i^{n-1} \) and \( U \) is isometric to the convex set \( \tilde{A_i} \cap E \). Thus, there is an open subset \( \tilde{U} \subset A_i \cap E \) which is also a lift of the simply connected set \( U \). Both \( \tilde{U} \) and \( \tilde{A_i} \cap E \) contain \((0, f_i] \). Since lifts, with a common starting point (say \( \frac{1}{2} f_i \)), of simply connected sets with respect to the covering map \( \pi \) are unique, we get \( \tilde{U} = \tilde{A_i} \cap E \). Taking closures, we get \( A_i \cap E = \tilde{A_i} \cap E \). The lemma follows by applying this argument to an arbitrary facet of \( B' \).

This completes the second step. The third step will be more involved as we have to investigate the properties of the \( A_k \) in the interior of the \( B' \).

**Proposition 3.15** \( \partial B \cap si(k, 1) \) has as vertices \( V_k \cup V_{k+1} \), where the \( V_k \) are defined below.
Proof. $S_1 \cap \partial B$ is a union of some of the facets of $S_1$, its external facets. Every facet of $S_1$ is obtained by taking a subset $T \subset \{0 = v_1, \ldots, v_{n+1} = s\}$ of size $n$ and forming $[T]$; the external facets being those which do not contain $[0, s]$, i.e. $T = \{0 = v_1, \ldots, v_n\}$ or $T = \{v_2, \ldots, v_{n+1} = s\}$.

\[ \therefore S_1 = [0 = v_1, \ldots, v_{n+1} = s]. \]  

(3.5) \[ S_1 \cap \partial B = [0 = v_1, \ldots, v_n] \cup [v_2, \ldots, v_{n+1} = s]. \]  

(3.6) As above, $s l (k, 1) = S l a b (v_k 1, v_k + f_k, S_1)$.

Note:

\[ (\forall U \subset \mathbb{R}^n) (\forall u_1, u_2 \in \mathbb{R}^n) [D^\perp + u_1, D^\perp + u_2] \cap U = \{ u \in U \mid s \cdot u_1 \leq s \cdot u \leq s \cdot u_2 \}. \]

\[ S_1 \cap (D^\perp + v_k) = \left\{ u \in S_1 \mid u.s = 2\pi \sum_{j<k} r_j^2 \right\}. \]

Also

\[ \partial B \cap S_1 \cap (D^\perp + v_k) = \bigcup_{i < h} [v_i, v_h] \cap (D^\perp + v_k), \]

where the prime over the union indicates that the inner edge $[0, s]$ is not included.

\[ \therefore V_k \overset{\text{def}}{=} \partial B \cap S_1 \cap (D^\perp + v_k) = \{ a_{ihk}v_i + (1 - a_{ihk})v_h \mid i \leq k \leq h, (i, h) \neq (1, n + 1) \}, \]

where $a = a_{ihk}$ satisfies $a \sum_{j<i} r_j^2 + (1 - a) \sum_{j<h} r_j^2 = \sum_{j<k} r_j^2$.

\[ \therefore i < h \Rightarrow a = \frac{\sum_{k<j<h} r_j^2}{\sum_{i<j<h} r_j^2}. \]

$V_k$ are vertices of $\partial B \cap s l (k, 1)$ which are on the face of $s l (k, 1)$ perpendicular to $s$ closest to $0$, i.e. the face contained in $D^\perp + v_{k-1}$. The others are $V_{k+1}$.

Proposition 3.16 Let $k < m \leq n, 0 < b \leq c < \sum_{i \leq n} r_i^2$. Then $[V_m, bs] \cap [V_k, cs] \neq \emptyset$.

Proof. We can assume $b < c$ The hyperplane $D^\perp + v_{m1}$ disconnects the simplex $S_1$:

\[ S_1 - (D^\perp + v_{m1}) = \left\{ u \in S_1 \mid u \cdot s < 2\pi \sum_{i \leq m} r_i^2 \right\} \text{ disjoint } \bigcup \left\{ u \in S_1 \mid u \cdot s > 2\pi \sum_{i \leq m} r_i^2 \right\} . \]

Now $S_1 \cap (D^\perp + v_{m1}) = [V_m, as]$, $a = \sum_{i \leq m} r_i^2$. There exists (cf. below) a piecewise linear homeomorphism $\phi : S_1 \rightarrow S_1$ fixing $S_1 \cap \partial B$ such that $\phi(bs) = as$ and such that $\phi([V_m, as]) = [V_m, bs]$. Thus $[V_m, bs]$ also disconnects $S_1$ and $V_k$ lies in one component (the connected component of $0$) and $bs$ lies in the other (the connected component of $s$). The result follows.

The construction of the map $\phi$. By Eq. (3.6) $V_k \subset \partial B \cap S_1 = [0 = v_1, \ldots, v_n] \cup [v_2, \ldots, v_{n+1} = s].$

Set $V_k^- = V_k \cap [0 = v_1, \ldots, v_n]$. Then $[V_k^-]$ is a $g$–simplex, $g \leq n - 2$, with vertices $V_k^-$.
\[ V_k \cap [v_2, \ldots, v_{n+1} = s], [V_k^+] \text{ is a } g' \text{-simplex, } g' \leq n - 2, \text{ with vertices } V_k^+. \] If \( 1 < k \leq n \), \( e.g. \) when \( k = m \), then \( \max(g, g' = n - 2) \). Then \( (D^+ + v_k) \cap \partial B = [V_k^-] \cup [V_k^+] \), the union of two simplices. Thus \( [V_m, as] = [V_m^-, as] \cup [V_m^+, as] \) is the union of two simplices. Likewise, \( [V_m, bs] = [V_m^-, bs] \cup [V_m^+, bs] \). Hence we can define \( \phi \) piecewise by requiring \( \phi|V_m^+ = I_{V_m^+} \), \( \phi(0) = 0, \phi(as) = bs \). By linearity, \( \phi \) extends uniquely to the \( n \)-simplex \( [0, as, V_m^+] \), mapping it homeomorphically to \( [0, bs, V_m^+] \). Similarly, \( \phi \) extends uniquely to the \( n \)-simplex \( [0, as, V_m^-] \), mapping it homeomorphically to \( [0, bs, V_m^-] \). Moreover, these extensions agree on the intersections of their domains, since they agree on the vertices \( V_m^- \cap V_m^+ \), namely the two extensions are the identity on this set. We complete the definition of \( \phi : S_1 \to S_1 \) by similarly defining \( \phi|[s, as, V_m^-] \to [s, bs, V_m^-] \) and \( \phi|[s, as, V_m^+] \to [s, bs, V_m^+] \).

**Lemma 3.17** \( \forall k \ A_k \cap D = [0, r_k^2s] \).

**Proof.** \( 0 \in A_k \cap D \Rightarrow A_k \cap D \) is an interval of length \( 2\pi r_k^2 \). It therefore suffices to prove that \( ts \in A_k \Rightarrow t \geq 0 \). Assume \( t < 0 \), Then \( A_k \supset [0, ts, f_k] \). We can take \( k = n \). Then \( [0, g_1] \subset A_n \).

**Lemma 3.18** \( (\forall k) \ \epsilon \in \mathbb{R}, \epsilon s \in A_k \Rightarrow \epsilon \geq 0 \).

**Proof.** We can take \( k = n \). We know inductively that \( \delta(s - f_1) \in A_1 \) where

\[
\delta = \frac{r_n^2}{\sum_{\mu=2}^{n} r_\mu^2} > 0.
\]

So \( s \in A_k, t \in [0, 1] \Rightarrow \Xi \overset{\text{def}}{=} te + (1 - t)\delta(s - f_1) \in A_k \). If \( \epsilon < 0 \) we can take \( t = \frac{\delta}{\delta - \epsilon} \), then \( A_k \supset \Xi = \frac{\delta}{\delta - \epsilon} \epsilon f_1 \overset{\text{def}}{=} \xi f_1 \). Since \( \xi \in (0, 1) \) this contradicts the interior disjointness of \( A_1, A_n \).

**Lemma 3.19** \( \forall k A_k \cap \mathbb{R}s = [0, r_k^2s] \)

**Proof.** We know \( 0 \in A_k \), so the result follows from the preceding lemma.

**Lemma 3.20** \( \forall k A_k \cap B = A_k \cap B \)

**Proof.** We know from Lemma 3.13, \( si(k, 1) = \left[ \partial B \cap si(k, 1), [R^-_{k1}, R^+_{k1}] \right] \). In particular, \( si(1, 1) = \left[ \partial B \cap si(1, 1), [R^-_{11}, R^+_{11}] \right] = \left[ \partial B \cap si(1, 1), [0, r_1^2s] \right] \). Thus \( A_1 \cap B \supset si(1, 1) \). Similarly, for any permutation \( \sigma \) such that \( \sigma(1) = 1 \) we get \( A_1 \cap B \supset si(1, \sigma) \). Thus \( A_1 \cap B \supset A_1 \cap B \). We can similarly show \( A_k \cap B \supset A_k \cap B \). It now follows that none of these inclusions can be proper: \( \forall k A_k \cap B = A_k \cap B \).

**Lemma 3.21** \( \forall j, k A_k \cap si(j, 1) = A_k \cap si(j, 1) \).
Proof. The translates \( A'_k \) of \( A_k \) exhaust \( S_1 \) and hence the \( s^l(j, 1) \). By Lemma 3.14, we know \((A_k + v) \cap s(j, 1) \cap \partial B = (A_k + v) \cap s(j, 1) \cap \partial B \) for any lattice vector \( v \). Therefore, by Lemma 3.13, it suffices to show \((A_k + v) \cap D = (A_k + v) \cap D \). For this, it suffices to take \( v = v_{j1} \) for some \( j = 1, \ldots, n \), since \( s_l(j, 1) = s^l(j, 1) + v_{j1} \). (For \( j = 1 \), this follows from the previous lemma.) If \((A_k + v_{j1}) \cap D \neq \emptyset \), then we know it is a subinterval of length \( 2\pi r_k^2 \). Since these intersections must exhaust \( D \), it must be that \((A_k + v_{j1}) \cap D = D_{k\sigma} \) for some permutation \( \sigma \). By Proposition 3.16, the disjointness of the (interiors of the) \( A_k + v_{j1} \) implies \( \sigma = 1 \). Thus \((A_k + v_{j1}) \cap D = D_{k1} = (A_k + v_{j1}) \cap D \). The result now follows.

Lemma 3.22 \( \forall j, k, \sigma \) \( A_k \cap s^l(j, \sigma) = A_k \cap s^l(j, \sigma) \).

Proof. We have only to reorder the \( f_j \) in the previous lemma.

Theorem 3.23 For all \( k \) we have \( A_k = A_k \).

Proof. This follows from \( A_k = \bigcup_\sigma s^l(k, \sigma) \).

We have thus established the uniqueness of the functor \( P \); there is only one natural way to partition right toroids satisfying the diagonal property.

4 Perspective States

...each quality or property of a thing is, in reality, nothing else but its capability of exercising certain effects upon other things... it can never depend upon the nature of one agent alone, but exists only in relation to, and dependent on, the nature of some second object, which is acted upon.

Helmholtz

The root change we are making in going from SQM to IQM is in the concept of state. Classical mechanics and SQM share the concept of state as adhering to a system simpliciter, without reference to other systems. In our view, the difficulties in the application of QM to individuals mandate relativizing the notion of state. In fact we deny the existence of an absolute state of an individual system. In order to be precise in a confusing area, we need to make some formal definitions.

Definition. The polar states of \( S = S_1 + S_2 \) are the elements of the polar bundle \( P \).

An element of \( P \) can be written \( p = (\Gamma, q) \in S \times T(r, \Gamma) \). Here, we take \( q \) from a polar decomposition

\[
\Gamma = \sum_l q_k \Gamma_k = \sum_l q_k \phi_k \otimes \psi_k.
\]

The notation \( p = (\Gamma, q) \) is slightly redundant since the absolute values \( r_k = |q_k| \) are already determined by \( \Gamma \). We could replace \( q \) by \( \hat{\theta} \) or \( \zeta \) where \( q_k = e^{i\theta_k} r_k = \zeta_k r_k \). However, the notation \( (\Gamma, q) \) is more direct and seems to cause no problem.
If a regular \( \Gamma \) is given, then \( q \) and \( (\Gamma_k)_k \) determine one another. Another name we sometimes use for the polar state is **joint state** to emphasize the analogy of the wave function with a probability density.

The \( (\Gamma, q) \) parameterize the new phase space of the composite system. They give extra phase angle data compared with SQM, the arguments \( \theta_k \) of the \( q_k \). These phases, combined with the Pythagorean partition of the toroidal fibers, give a classical way of specifying a particular SQM state \([\phi_k]_S\) of \( S_1 \), the conditional state of \( S_1 \). Namely, the map \( P_{\phi_k} \mapsto p_k \in \mathfrak{P} \) extends to a faithful representation of the Boolean algebra generated by the projections \( P_{\phi_k} \) onto a field of subsets of the toroidal fiber. This comprises a bridge between quantum and classical logic, but it is contextually restricted to the Boolean algebra of subsets of \( \{[\phi_k] \mid k = 1, 2, \ldots\} \). The same applies to \( S_2 \) and \( \{[\psi_k] \mid k = 1, 2, \ldots\} \).

To get a dynamical description, we need the further specification of a (possibly time-dependent) Hamiltonian \( H \) for \( S \). Once \( (\Gamma, q) \) is given at time \( t \), it is determined for all \( t \) by the Hamiltonian evolution on \( S \) induced by \( H \) and the dynamical connection \( A^H \).

This completes the description of the polar state of an interacting pair of systems.

**Definition.** The **conditional spectral state** of \( S \) with respect to the polar state \( (\Gamma, q, H) \) of \( S \) is \([\phi_{k(t)}]_S\). Symbolically, we write this as \([\phi_{k(t)}] = [(\Gamma, q)]_S \), defining the value of the function \( t \mapsto k(t) \) as the index \( k \) for which \( q(t) \in p_k(t) \).

A crucial element of the extension of SQM we are describing here consists of the existence and determination of the function \( t \mapsto k(t) \). The ray \([\phi_{k(t)}] \) can be identified with a **spectral** projection \( P_{\phi_{k(t)}} \) of the mixed state \( \text{Tred}_1(P_{\text{CT}}) \) assigned by SQM to \( S_1 \). Now \( \text{Tred}_1 \), which is an orthogonal projection of the (Jordan) algebra of observables of \( S \) to that of \( S_1 \), is the non-commutative analogue of the projection of algebras of random variables used in the theory of conditional probabilities. This non-commutative analogue can also be expressed in the terminology of **partial** Boolean algebras, as in [26, (iv)]. The conditional spectral state \([(\Gamma, q)]_S \) is the IQM extension of this notion.

We now make some technical remarks concerning the considerations required to handle the lower dimensional situations where the polar decomposition

\[
\Gamma = \sum_k q_k \phi_k \otimes \psi_k
\]

has either the special property 1) the \( |q_k| \) are not distinct, or 2) \( q \) lies on the boundary of two \( p_k \).

In either case, the function \( k(t) \) becomes undefined; in fact, in the first case the \( [\phi_k] \) are not well-defined.

For 1) we need to use the dynamical behavior and assume that \( \Gamma(t) \) is an analytic function of \( t \), i.e. \( \Gamma(0) \) is an analytic vector for the Hamiltonian. From perturbation theory [23, Chap.II, Th.6.1] it follows that the spectral states (i.e. the spectral projections of the reduced density operator \( \rho_1 \)) can be analytically continued across any **isolated degeneracy** where the eigenvalues of \( \rho_1 \) are not all distinct. This restores the definition of \( k(t) \). It is possible to encounter a **permanent degeneracy** as in the case of identical particles mentioned in Section E. We do not elaborate here on the special considerations required for this case.

In 2) we have the ambiguity of \( k(t) \) when \( q \) lies on a lower dimensional boundary. We can invoke the analogy with classical physical theories which sometimes regard the functions and subsets of
phase spaces to be just representative of the corresponding \(\sigma\)-Boolean algebra entities formed by factoring out sets of measure zero, including the lower dimensional boundaries of the \(p_k\).

### 4.1 Special case: \(S_2\) is empty = SQM.

The ordinary QM of \(S\) is the special case when either \(S_1\) or \(S_2\) is empty. Taking \(S = S_1\), we can take \(H_1 = H\) and \(H_2 = \mathbb{C}\). We assume all the Hamiltonians are time-independent. Then the polar state of \(S_1\) is just \((\Gamma, q)\) with \(q \in \mathbb{C}\) and the conditional spectral state of \(S_1\) is \([\Gamma, q = 1] = [\Gamma]\). This is ordinary quantum theory, \(i.e.\) the quantum mechanics of an isolated system. In this case, we can speak about the state of \(S_1 = S\), as is customary.

### 4.2 Compounding of perspectives.

If we want to consider interactions of subsystems of \(S_1\) or more general multiple interactions, then we iteratively need more data at each new level, specifying the vector states \(\phi_k\), not just the \([\phi_k]\). A grandiose example would be the situation, where \(S, S_1\), and \(S_2\) are respectively, the entire universe, the subsystem of bosons and the subsystem of fermions. Then one might want a representation of some subsystem of \(S_1\), \(e.g.\) the microwave background. The details are left to the reader.

An iteratively complete description of the state of \(S = S_1 + S_2\) requires expanding the data above. It is also necessary to specify Hamiltonians \(H_1\), and \(H_2\) for \(S_1\), and \(S_2\) respectively. These will usually be time-dependent. Then \(H = H_0 + H_1 \otimes I_2 + I_1 \otimes H_2\) is the total Hamiltonian of \(S\), where the interaction Hamiltonian \(H_0\) is thereby defined.

It is now also necessary to be given conditional vector states, \(i.e.\) particular \(\phi_k(t) \in [(\Gamma, q)]S_1\) or, equivalently in the presence of \(\Gamma_k(t)\), \(\psi_k(t) \in [(\Gamma, q)]S_2\). The simplest way these can be determined is via the Hamiltonians \(H_i\) and the connections \(A^H_i\), starting with initial values \(\phi_k(0)\) for all \(k\). Now we have the necessary data to treat a decomposition \(S_1 = S_{11} + S_{12}\). This process can then be iterated, but always requiring additional information at each new stage.

Simple behavior for conditional states is ruled out by the fact that a spectral projection of the reduced trace of a spectral projection of a density operator \(\rho\) is not in general a spectral projection of the reduced trace of \(\rho\). We will elaborate on this cruel fact of life when we deal with EPR in Section 7. It seems that the closer one gets to the truth, the more relational is the required formulation.

### 4.3 Evolution of conditional vectors states.

The collective evolution of the \([\phi_k]\) is equivalent to the evolution of the density matrix \(\text{Tred}_1(\Gamma)\); it is a standard part of SQM. The particular one of these which obtains at a given time \(t\), \([\phi_k(t)]\) is given by the conditional spectral projective state \([(\Gamma, q)]S_1\). It is completely determined by the evolution of this joint state. However, if it is desired to iterate this procedure it is then necessary to obtain a conditional spectral vector state, \(i.e.\) a lift of \([\Gamma, q]\) to \(S(\mathcal{H}_1)\). Another reason to study this situation is to facilitate the proof that no interaction implies no jumping (\(i.e.\) no change in the spectral states), without which IQM measurement theory would be of questionable meaning. In this section we investigate the possibility of determining the evolution of conditional vector states.
First we show that essentially this requires being given separate or “free” Hamiltonians $H_i$ for each subsystem, together with an interaction Hamiltonian $H_0$.

We start with the usual composite system $S = S_1 + S_2$ together with a Hamiltonian $H$. Let $t \mapsto D_1(t) =: D_1$ be the curve of density operators on $H_1$ given by $\text{Tred}_1(P_1(t))$. Using $\Im$ to denote the imaginary part, we have by Eq. (B.17)

$$
\dot{\phi}_j = \sum_{k \neq j} \frac{\langle \phi_k, D_1 \phi_j \rangle}{x_j - x_k} \phi_k + \langle \phi_j, \dot{\phi}_j \rangle \phi_j = \sum_{k \neq j} \frac{\langle \phi_k, D_1 \phi_j \rangle}{x_j - x_k} \phi_k + i\Im \langle \phi_j, \dot{\phi}_j \rangle \phi_j.
$$

We can write this as

$$
\dot{\phi}_j = \sum_{k \neq j} \frac{\beta_{kj}}{x_j - x_k} \phi_k + i\Im \langle \phi_j, \dot{\phi}_j \rangle \phi_j,
$$

where the $\beta_{kj}$ have been defined in Eq. (2.23).

If we take the $\phi_j$ and $\psi_j$ to be canonically horizontal over $P(H_1)$ and $P(H_2)$, then we must have the $\langle \phi_j, \dot{\phi}_j \rangle = \langle \psi_j, \dot{\psi}_j \rangle = 0$. This choice would lead to the $\Gamma_j = \phi_j \otimes \psi_j$ being canonically horizontal over $P$. But we have required that the $\Gamma_j$ be $A^H$-horizontal. The most general way to satisfy this condition, while lifting the $\phi_k$ and the $\psi_k$ by connections on $P(H_1)$ and $P(H_2)$, is to require

$$
\dot{\phi}_a = \sum_{k \neq a} \frac{\beta_{ka}}{|q_a|^2 - |q_k|^2} \phi_k + i f_a \phi_a
$$

$$
\dot{\psi}_a = \sum_{k \neq a} \frac{\beta'_{ka}}{|q_a|^2 - |q_k|^2} \psi_k + i g_a \psi_a
$$

where $f_j$ and $g_j$ are a real-valued smooth functions of $t$, depending on the $\phi_j$, and where

$$
\langle \Gamma_j, H \Gamma_j \rangle = \mathcal{E}(\Gamma_j(t)) = f_j(t) + g_j(t).
$$

To do this plausibly would require partitioning the expected energy $\mathcal{E}(\Gamma_j(t))$ of $\Gamma_j(t)$ between $\phi_j(t)$ and $\psi_j(t)$. The moral here is that $f_j$ and $g_j$ cannot be separately determined by any data we have so far considered, at least without some new principle. That is why, when we need the separate evolutions of the $\phi_k$ and $\psi_k$, we specify individual Hamiltonians $H_i$ on $H_i$, for $i = 1, 2$.

### 4.4 The case of non-interacting subsystems.

The important special case where $H_0 = 0$ is now treated. As we have said, we prove there is no jumping in this case.

Here we assume that the states of $S_1$ and $S_2$ evolve separately. This means that the total Hamiltonian $H$ for $H = H_1 \otimes H_2$ is of the form

$$
H = H_1 \otimes I_2 + I_1 \otimes H_2.
$$

Then it is natural to lift the $[\phi_k]$ by the dynamical connection $A^{H_1}$ defined in a fashion analogous to Eq. (2.23)

$$
A^{H_1} \overset{\text{def}}{=} \langle z, dz \rangle + i \langle z, H_1 z \rangle dt =: A^0 + i \mathcal{E}^{H_1}(z) dt
$$

(4.10)
and similarly for $[\psi_k]$.

We now show that $\Gamma_k(t) \overset{\text{def}}{=} H_k^1(t) \otimes H_k^2(t)$ is $A^H$-horizontal.

$$A^H(\hat{\Gamma}_k, \partial t) = (\Gamma_k, \hat{\Gamma}_k) + i(\Gamma_k, H\Gamma_k) =$$

(4.11)

$$\langle \phi_k^H \otimes \psi_k^H, \hat{\Gamma}_k \rangle + i\langle \phi_k^H \otimes \psi_k^H, H_1 \phi_k^H \otimes \psi_k^H \rangle + i\langle \phi_k^H \otimes \psi_k^H, \phi_k^H \otimes H_2 \psi_k^H \rangle =$$

(4.13)
almost all $\Gamma$ have a polar decomposition $\sum k q_k \phi_k \otimes \psi_k$ with constant $q_k$. It follows that $H$ has the non-interactive form $H = H_1 \otimes I_2 + I_1 \otimes H_2$.

**Remark.** It can sometimes happen, that even with a non-trivial interaction, the $r_k(t)$ are constant. Then there is jumping but no net or statistical jumping. This happens for certain values of the parameters in the hyperfine example of Appendix \footref{hyp}, namely when $k = Cl$.

### 4.5 Interaction Hamiltonians

We now consider the general case. We can obtain evolutions of the conditional vector states when $H_0 \neq 0$, provided we have the proper sort of decomposition of the total Hamiltonian. Using the bi-orthonormality property of polar decompositions we see

**Lemma 4.2** Let $H = H_0 + H_1 \otimes I + I \otimes H_2$.

\[ j \neq k \Rightarrow H_{jj, kk} = (H_0)_{jj, kk}. \]  

**Proof.** Let the $\phi_k$ be an $A^{H_1}$-horizontal lift of the spectral projection $[\phi_k]$ associated to $\Gamma(t)$ and likewise for $\psi_k$. Set $\Gamma^{12}_k := \phi_k(t) \otimes \psi_k(t)$ and $H_{12} := H_1 \otimes I + I \otimes H_2$. By Section \ref{sec:4.4}, the $\Gamma^{12}_k$ are $A^{H_{12}}$-horizontal. Using their bi-orthonormality and $\mathbf{r} \cdot \mathbf{r} = 0$, we get

\[ \Gamma^{12}(t) = \sum_k r_k(t) \Gamma^{12}_k(t) \text{ is } A^{H_{12}}-\text{horizontal}. \]  

Using Lemma \ref{lem:2.3}

\[ \therefore \Gamma := e^{-i \int_0^t (\Gamma^{12}, H_0 \Gamma^{12}) ds} \Gamma^{12}(t) \text{ is } A^H-\text{horizontal}. \]  

Since $\Gamma$ and $\Gamma'$ have the same reduced traces and are both $A^H$-horizontal, they differ by a constant phase factor which can be absorbed into the $\Gamma_k$. Thus we can take $\Gamma = \Gamma'$. Again using Lemma \ref{lem:2.3}, we get

\[ \Gamma_k := e^{-i \int_0^t (\Gamma^{12}_k, H_0 \Gamma^{12}_k) ds} \Gamma^{12}_k(t) \text{ is } A^H-\text{horizontal}. \]
Define the $q_k(t)$ implicitly by
\[ \Gamma(t) = \sum_k q_k(t) \Gamma_k(t). \]  
(4.20)

Combining the last three equations, we see that Eq. (4.15) holds with
\[ \Upsilon_k(s) = \langle \Gamma^{12}, H_0 \Gamma^{12} \rangle - \langle \Gamma_k^{12}, H_0 \Gamma_k^{12} \rangle = \langle \Gamma, H_0 \Gamma \rangle - \langle \Gamma_k, H_0 \Gamma_k \rangle. \]

Finally, from Eq. (A.14)
\[ \dot{r}_j = \sum_k \Im (H_{jj,kk}) r_k = \sum_k \Im ((H_0)_{jj,kk}) r_k + \sum_k \Im ((H_{12})_{jj,kk}) r_k. \]  
(4.21)

Now, in the non-interacting case, we know the coefficients $q_{j12}$ are constant and so are their absolute values, $|q_{j12}|$. But we also know from Lemma [A.3], that
\[ \partial_t |q_{j12}| = \sum_k \Im ((H_{12})_{jj,kk}) |q_{k12}| \]
which suggests all the $\Im ((H_{12})_{jj,kk}) = 0$ in Eq. (4.21). This fact follows directly from the definitions and establishes the last assertion of the theorem.

**Conclusion:** All the characteristics (such as the frequency) of the jumping between states are determined by the interactive part $H_0$ of the Hamiltonian $H = H_0 + H_1 \otimes I + I \otimes H_2$. This generalizes the no-jumping result of Section 4.4.

**Remark:** Up to a scalar (multiple of the identity operator) and ignoring all analytic difficulties, $H$ alone determines natural choices for $H_0, H_1,$ and $H_2$. Namely, let $H_1 \otimes I_2$ be the orthogonal projection with respect to the Hilbert-Schmidt inner product of $H$ into the space of Hermitian operators on $\mathcal{H}$ of the form $h_1 \otimes I_2$ where $h_1$ is Hermitian. Similarly, let $I_1 \otimes H_2$ be the orthogonal projection with respect to the Hilbert-Schmidt inner product of $\mathcal{H}$ into the space of Hermitian operators on $\mathcal{H}$ of the form $I_1 \otimes h_2$, where $h_2$ is Hermitian and traceless. Of course, these definitions require that $H$ be Hilbert-Schmidt, and $h_2$ be trace-class, but we proceed formally. Finally, we set $H_0 = H - H_1 \otimes I_2 - I_1 \otimes H_2$. This gives a “minimal” interaction Hamiltonian.

The above mentioned orthogonal projections are strongly related to reduced traces. For example, in finite dimensions it is not hard to see that $H_1 = \frac{1}{n_2} \text{Tr}_{\Gamma} (H)$, where $n_2$ is the dimension of $\mathcal{H}_2$; this fact follows from Proposition [A.8].

5 The Interpretation of IQM

In this section, we summarize the main features of the new model of quantum mechanics. We then compare IQM with the standard treatments and discuss some of the ramifications. We conclude by applying it to the hyperfine splitting example in Section 5.2.

With reference to Dirac’s dictum quoted in the introduction, we have extended “the mathematical formalism” of SQM to include the right toroids $T(r)$ and $q(t) \in T(r) = \mathcal{T}_r$ for $\Gamma \in \mathcal{S}$. The naturality of the extension leads to the existence of a natural evolution in the enlarged state space.
consisting of the polar bundle $P$. We can say this adds a “success” in our “attempts to perfect and generalize the existing mathematical formalism”. As another step, we have shown there is a natural way of partitioning the toroids. The corresponding tilings of Euclidean spaces appear to be mathematically new. Our approach to “try to interpret the new mathematical formalism in terms of physical entities” has already been explicated, starting with our introductory remarks in Section I.1. The main points will be discussed below.

5.1 Summary of the extension of the mathematical formalism.

The extension of the mathematical formalism of SQM to IQM has four main constituents, upon which we elaborate in the ensuing discussion.

I) Interacting systems have pure states, associated to a particular one of the eigenprojections of the density operator.

II) Unit vectors within the rays assigned to the system by SQM are incorporated into the representation of pure states.

III) These (eigen)vectors evolve by lifting the SQM evolution of the eigenprojections by the dynamical connection.

IV) The choice of the particular eigenprojection is via the Pythagorean partition.

5.1.1 I) The pure state of an interacting system.

The use of the polar decomposition to define a pure spectral projection for a system in interaction with another was first proposed by Kochen in [25]. The main objection raised to it was the lack of dynamics, which this paper is largely devoted to remedying. The dynamics are summarized in III) and IV). On the other hand, its acceptance resolves the main paradox of SQM in its attempted application to individual systems, the measurement problem. The resolution is short enough to warrant repeating here.

A measurement of a system $S_1$ by an apparatus $S_2$ is viewed simply as an interaction between two quantum systems. Neither the size of $S_2$, nor the observer plays any part in our analysis. We assume that throughout the interaction, the system $S = S_1 + S_2$ is isolated from its environment.

This means the SQM state of $S$ is of the form $[Q(t)]$, which is undergoing Schrödinger evolution. The more detailed IQM description requires the actual vector $\Gamma$. It also requires a particular polar decomposition of $\Gamma$, which we write as $\Gamma = \sum_k q_k \phi_k \otimes \psi_k$. In [25], Kochen proposed that $S_1$ has the pure state $[\phi_k]$, the measured state of $S_1$ and synchronously $S_2$ has the pure state $[\psi_k]$, the corresponding state of the apparatus. No mechanism for determining which $k$ obtains was proposed except (implicitly) that the choice should satisfy the probabilistic requirements of SQM. More recently in [8, Bacciagaluppi and the references therein] various stochastic evolutions (essentially of the $k(t)$) have been considered.

In the present theory, $q$ lies in a particular part $p_k$ of the partition $P$. This $k$ then determines the conditional spectral state $[\phi_k] = [(\Gamma, q)]S_1$. There is no collapse of $\Gamma$, i.e. no mysterious transition from the pure state, $\Gamma$, to a mixed state, which is usually taken to $\sum_k |q_k|^2 P_{\phi_k \otimes \psi_k}$. In fact, the only remnant of such a transition is in our description of the passage from SQM to IQM and this
transition goes in the opposite direction: we replace the density operator \( \sum_k |q_k|^2 P_{\phi_k} \) (which is a mixed state of \( S_1 \) and hence an inappropriate description of an individual) by \( |\phi_k\rangle \).

### 5.1.2 II) Using \( S \) instead of \( P \).

The use of unit vectors in place of rays in representing pure states of isolated systems is harmless and inessential, but very useful. We employ it because it allows for a more uniform treatment of the toroidal phases, which are essential to our treatment.

Actually, the use of unit vectors to represent pure states is a convenience of which all physicists avail themselves, while occasionally paying lip service to \( P \) by noting that \( \langle \phi, A\phi \rangle \) and \( |\langle \phi, \psi \rangle|^2 \) are all that is measurable and these are independent of the “phases” of \( \phi \) and \( \psi \). On the other hand, the existence of relative phases has long been noted theoretically, e.g. the geometric phase of Berry and its generalizations, see [38], and has been measured in interference experiments. It is true, however, that there is no generally accepted way of associating a Hermitian operator to this measurable quantity (see [3, Barnett-Pegg and its many references].) This is a gap in the formalism of SQM applied to individuals (somewhat patched up by the formalism of POVs as in [12, Davies], and [14 and the references therein]), since measurable quantities should be observables and observables should be represented by Hermitian operators, according to the formalism of SQM. It is not coincidental that our approach to unraveling the mystery of the quantum theory of individuals starts with this loose thread of phases in the cloak of the individual interpretation of SQM.

The transition from a relative to an absolute phase merely entails the choice of a basepoint, i.e. a reference phase. The phase, relative to this basepoint, is an “absolute” phase. We can interpret the experiments demonstrating interference between different laser sources, e.g. [30] and [35], as providing instances of such measurement.

Even if absolute phase is generally not directly measurable, that is not an absolute argument against its existence or appropriateness. Many physical constructs have been introduced without an expectation of being subject to direct measurement. We quote Feynman, Vol I, 38-8 in [19],

> It is not true that we can pursue science by using only those concepts which are directly subject to measurement.

In quantum mechanics itself there is a wave function amplitude, there is a potential, and there are many constructs that we cannot measure directly.

The use of the phase as a parameter to distinguish different individual systems belonging to an ensemble with the same pure state appears rather natural; for the ensemble of systems has a statistical state represented by an ensemble of unit vectors in a ray. However simply assigning a unit vector to an isolated system does not suffice to treat interacting systems. In fact, the no-go theorems of [3, Bell] and [27, Kochen-Specker] show that such non-contextual assignments of state run afoul of the predictions of SQM. To get a “go” theorem, we need to combine the phases of the various polar components of the vector \( \Gamma \) associated to a composite system \( S = S_1 + S_2 \).

The ubiquity and utility of amplitudes suggest that possibly every individual physical system does indeed have a “phase”, i.e. we can represent their states by unit vectors \( \Gamma \) in \( \mathcal{H} \), not just equivalence classes \( [\Gamma] \) in \( \mathbb{P}(\mathcal{H}) \); more precisely, that it is consistent to so model individuals, which
is what we end up doing. Indeed, the mathematical structures we employ are a uniquely determined extension of the present mathematical structure of SQM.

5.1.3 III) The evolution of the eigenvectors.

The evolution of the reduced density operator of an interacting system is a standard part of SQM (see [3]). The evolution of its eigenprojections in $P$ has been increasingly used in, e.g. [7, 34, 16, 21] in connection with the Jaynes-Cummings model. In IQM, the individual state vectors lie in the polar bundle $P$, and the evolution of the eigenprojection in $P$ lifts to $P$ by means of the dynamical connection $A_H$. When the Hamiltonian $H$ is 0, the connection reduces to the canonical connection used in obtaining the geometric phase of Berry. When the Hamiltonian describes separate evolution for the subsystems, i.e. when $H$ has the form $H_1 \otimes I_2 + I_1 \otimes H_2$, it turns out that there is no jumping between the states vectors, as reviewed in the next section.

5.1.4 IV) the Pythagorean partition $\mathcal{P}$ of the phases.

Assuming the extensions in (I-III) have been accepted, we can argue for the role of $\mathcal{P}$ as follows. We know from our assumption that $q$ lies in the toroidal fiber $\mathcal{P}_\Gamma$ of $\mathcal{P}$ above $\Gamma$. This toroid with base point is isometric to $T(r) = \prod_k S^1(r_k)$. Hence the toroid must be partitioned in subsets $p_k^*$ physically by (I-III). Indeed, $q \in p_k^*$ if, and only if, $\phi_k$ is, in fact, the state of $S_1$. We should note, in this connection, that after a measurement, even a Copenhagen adherent would allow that $S_1$ is in a pure state. Returning to the physically defined partition $\mathcal{P}^* = \{p_1^*, \ldots\}$, the diagonal property of Section 3 follows from the probability requirements of SQM. The naturality property of the partition says, in effect, that a partition of a (right) toroid is consistent with the partitions of the subtoroids it contains. Thus, the partitions are defined uniformly for all finite dimensions. This implies the results hold even for $\infty$-dimensions mutatis mutandis without dealing with it explicitly. Physically, and in $n$ dimensions, the naturality property is a continuity property of the partitions: the partition of $T(r_1, \ldots, r_n)$ approaches the partition of $T(r_1, \ldots, r_{n-1})$ as $r_n \to 0$.

If we assume that $\mathcal{P}^*$ is a convex partition, then Theorem 3.1 stated in Section 3 and proved in Section 3.3, shows $\mathcal{P}^* = \mathcal{P}$. What is our rationale for assuming that we are dealing with convex partitions? The convexity for the corresponding tiling of $\mathbb{R}^n$ is a simple and natural property that is usually assumed in mathematical discussions of tilings. We are however concerned with physical reasons for convexity. Consider the special case in which all the polar vectors are energy eigenstates, i.e. when for all $k$, there exist constants $r_k, E_k$ so that $q_k = r_k e^{-iE_k t}$. Then the frequency of jumping of the $k(t)$ specifying the state $\phi_k(t)$ is minimized under the convexity hypothesis. Conversely, this “minimum jumping property” implies the convexity property of our partitions. This follows from the characterization of convex subsets of a Euclidean space as those which intersect every line in an interval.

It is possible that a deeper understanding of the physical processes that lead to the partition would, in general, allow the convexity to be derived from a physically natural variational problem. Mathematically, it is likely that these partitions minimize the co-dimension-one volume of the boundary set, as is true when $n = 2$. More speculatively, if the parts are analogous to different
thermodynamical phases there may be some physically plausible partition function whose critical values define the boundary. Conceivably, such a function could lead to a non-convex partition of the toroidal fiber above $\Gamma$, but one which approaches $\mathfrak{P}$ as $(\Gamma, H\Gamma) \to 0$.

Here is one way we may intuitively think about the individual states of interacting systems. First, an isolated system has a phase that changes uniformly with time at a rate proportional to its energy. This phase can be regarded as an internal clock or pulse of the system (cf. Feynman [18, QED].) If the system interacts with another, the pulse quickens (if all energies are positive) and becomes a complicated but still smooth function of time. For instance, in the spin-spin interaction of the hydrogen atom, a discussion of which follows in the next section, it is shown that the phase of the electron is essentially an elliptic function of time. The partition boundaries may be considered as thresholds between different quantum states. As such a threshold is crossed, a different quantum state is assumed. As an example, an excited state of an atom plus a weak exterior electromagnetic field changes smoothly with time, but the condition of the field changes abruptly when the atom decays and emits a photon.

5.2 How IQM works in an example.

The classification of the constituents of a chaos, nothing less is here essayed.

from Moby Dick by Herman Melville

We now apply the general IQM theory to a specific case which is, essentially, the simplest possible non-trivial example. We take the case of the spin of an electron in a hydrogen atom, using the simplified model discussed in [19, Feynman] and worked out in detail in Appendix C.

It is modeled by two spin $\frac{1}{2}$ systems, the “electron” and the “proton”. Thus $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \approx \mathbb{C}^2 \otimes \mathbb{C}^2 \approx \mathbb{C}^4$. The Hamiltonian $H$ is a pure interaction Hamiltonian with $H = \mu \vec{\sigma}_1 \otimes \vec{\sigma}_2$. For notation and details not given here, see Appendix C and in particular, the glossary given in C.3.

We assume at $t = 0$, the data $\Gamma(0)$, and $\theta_{\pm} := \arg q_{\pm}$ are given. These are enough to determine an initial polar decomposition $\Gamma = q_+ \phi_+ \otimes \psi_+ + q_- \phi_- \otimes \psi_-$, up to reciprocal phase factors for the $\phi_{\pm}, \psi_{\pm}$. These would be important if we were interested in further details of the component systems, which are ruled-out by the simplicity of our model, i.e. no subsystems of the component $S_i$ are possible here because $\dim \mathcal{H}_i = 2$. However, if we were to treat, say the proton as a composite system, then the actual phase of the $\psi_{\pm}$ would come into play. For this simple example, we are free to choose these original phases so as to make the $\phi_{\pm}, \psi_{\pm}$ real. This is accomplished in conjunction with assuming the $q_{\pm}$ are also real at $t = 0$, as in the appendix. We have also arranged the axes in $\mathbb{R}^3$ so that initially the $z$–axis bisects the spin vectors. We show in the appendix that the polarization vector representing the mixed state of the electron (and the proton) has an ellipse in a plane perpendicular to the $z$–axis as trajectory. We further require that the $x, y$–axes are initially aligned along the major and minor axes of this ellipse, which implies that $q_{\pm} \in \mathbb{R}$.

Then we can determine the polar decomposition for all $t$, using the dynamical connection of IQM:

$$\Gamma(t) = q_+ \phi_+ \otimes \psi_+ + q_- \phi_- \otimes \psi_-, \quad (5.1)$$
where all the components are explicitly determined functions of $t$. In particular, we find

$$q_\pm = \sqrt{\frac{1 \pm \sqrt{\Delta}}{2}} \exp\left(\nu + \sigma_\pm \mp \frac{C^2 t}{k} \Pi(e^2; \omega t | S^2 e^2) - \frac{\omega t}{2}\right)$$  \hfill (5.2)

where we here briefly recall the definitions of the functions involved in this formula.

$$\nu = \frac{C^2}{\sqrt{1 - S^2 e^2}} \arctan(\sqrt{1 - S^2 e^2} \tan \omega t),$$

and where the arctan is taken so that the resulting function is a continuous function of $t$, vanishing at 0, as is possible.

$$\sigma_\pm = \arctan\left(\frac{C^2 t^2 \pm \sqrt{\Delta}}{k^2 \sqrt{\Delta}} \tan \omega t\right).$$

Also

$$\Pi(n; \varphi|m) = \int_0^{\varphi} \frac{d\rho}{(1 - n \sin^2 \rho) \sqrt{1 - m \sin^2 \rho}}$$

is Legendre’s elliptic integral of the third kind.

$$\phi_\pm = \pm e^{i\tau_\pm}\left(\frac{\alpha}{\beta_\pm}\right), \quad \psi_\pm = e^{i\tau_\pm}\left(\frac{\alpha}{-\beta_\pm}\right),$$

where

$$\tau_\pm = \frac{l}{2} \arctan\left(\frac{Cl}{k} \tan \omega t\right) \pm \frac{Cl^2}{2k} \Pi(e^2; \omega t | S^2 e^2).$$

The spectral states $[\phi_\pm(t)]$ and $[\psi_\pm(t)]$ of the electron and proton give the axes of spin of the particles, and these two axes are antipodal on the ellipse, which is the trajectory of the SQM mixed states. These spectral states do not tell when the spin is up or down for each axis (although they are synchronous). To find out when $[\phi_k(t)]$ is $[\phi_+(t)]$ or $[\phi_-(t)]$, we must consider the complex vector $\textbf{q}(t) = (q_+(t), q_-(t)) = (r_+ e^{i\theta_+}, r_- e^{i\theta_-})$ and determine, for each $t$, whether $(S_+, S_-) := (r_+ \theta_+, r_- \theta_-)$ lies in $\varpi^{-1}(p_+)$ or $\varpi^{-1}(p_-)$. Here the $p_\pm = p_\pm(t)$ comprise the Pythagorean partition $\mathcal{P} = \mathcal{P}(t)$ constructed in Section 3.

In the terminology of that section, we have $p_\pm = \varpi(A_\pm)$. It will be convenient to replace the $A_\pm$ by the union $\mathcal{A}_\pm$ of the appropriate basic building blocks $s(k, \sigma)$ from which they are constructed. This is justifiable since $\varpi(A_\pm) = \varpi(A_\pm)$. In this case, each $\mathcal{A}_\pm$ is just the union of 2 triangles as in Figure 3-1. We can take $\mathcal{A}_+ = \text{OCF+ABE}$ and $\mathcal{A}_- = \text{OAE+BCF}$. Thus, denoting the box OABC by $B$,

$$\mathcal{A}_+ = \left\{ (S_+, S_-) \in B \mid S_- < \frac{r_- S_+}{r_+}, S_- r_+ + S_+ r_- < 2\pi r_+^2 \right\} \cup \left\{ (S_+, S_-) \in B \mid S_+ > \frac{r_+ S_-}{r_-}, S_- r_+ + S_+ r_- > 2\pi r_-^2 \right\},$$ \hfill (5.3)

$$\mathcal{A}_- = \left\{ (S_+, S_-) \in B \mid S_- > \frac{r_- S_+}{r_+}, S_- r_+ + S_+ r_- < 2\pi r_+^2 \right\} \cup \left\{ (S_+, S_-) \in B \mid S_- < \frac{r_- S_+}{r_+}, S_- r_+ + S_+ r_- > 2\pi r_-^2 \right\}. \hfill (5.4)$$

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Of course, $S_\pm, r_\pm$, and even $B$ are all time-dependent. Let $\overline{S}_\pm(t)$ denote the least non-negative residue of $S_\pm(t)$ modulo $2\pi r_\pm(t)$. Then the condition that $k(t) = +$ is equivalent to $\overline{S}_\pm(t), r_\pm(t)$ satisfying the inequalities coming from Eq. (5.3).

We set
\[
c_1(t) := \frac{r_-, \overline{S}_+}{r_+}, \quad c_2(t) := 2\pi r_+^2 - \frac{\overline{S}_+ r_+}{r_-}, \quad c_3(t) := 2\pi r_- - \frac{\overline{S}_- r_-}{r_-}.
\]

Then $q$ lies in $\mathbf{p}_+$ if, and only if, either $\overline{S}_- < c_1, c_2$ or $\overline{S}_- > c_1, c_3$. For numerical values, say $\theta = \frac{3\pi}{\tau}, q_+(0) = .94$, we can compute these functions as in Figure 5-9. For example, restoring here $\hbar$, we can see that $4\frac{\hbar}{\tau} t \in [3\pi, 10\pi] \Rightarrow q \in \mathbf{p}_+, i.e. \text{the conditional spectral state of the electron is } +$.

6 Observables

We have avoided the use of observables; we now consider their place in IQM.

6.1 States Versus Observables.

The question of which to consider first, states or observables, has been likened to the chicken and egg problem. But there is really no doubt about priority, at least for IQM. Our theory takes states as primary. In fact, we have completed its basic structure without even mentioning within
it the observables which are usually taken as basic. Lattice theory, favored by foundationalists, and $C^*$–algebras predominating in QFT, are the main observable-first approaches. In fact, the mathematical notion of state is, essentially, a positive linear functional on a $C^*$–algebra. But, of course, this mathematical concept, which the algebra of observables must precede, derived from the standard models of quantum mechanics. These approaches have their roots in Born and Jordan’s version of Heisenberg’s matrix mechanics.

But physically, it is difficult to find a referent for an observable without something to observe, *e.g.* a weak beam of atoms. From this point of view, state-first approaches appear to us to be more direct. Of course, the original state-first approach was Schrödinger’s.

In such a formulation of SQM, the pure states are identified with equivalence classes of irreducible ensembles. These are the idealizations of the weak beams. The set of these has a metric $m(\alpha, \beta) = 1 - p(\alpha, \beta)$, where $p(\alpha, \beta)$ is the transition probability. This set turns out to be metrically identifiable with $P = P(H)$ with $p(\alpha, \beta) = |\langle a, b \rangle|^2$, where $a$ and $b$ are unit vectors representing $\alpha$ and $\beta$ Then $\theta(\alpha, \beta) = 2 \arcsin(1 - p(a, b))$ is also a metric, the Fubini-Study metric on $P$. Alternatively starting from the $\theta$-metric, we have that $\sin^2 \frac{\theta}{2}$ being differentiable, subadditive for $\theta \in [0, \pi]$, and taking 0 at 0 transforms $\theta$ into a smoothly equivalent metric $1 - p$ on $P$ with the same invariance group.

Thus the basic projective Hilbert space model of SQM can be expressed in terms only involving the geometry of the physical states of ensembles. An early attempt at deriving the mathematical structure of SQM through use of the geometry and symmetry was postulated by Landé in [28, 29]. His conjectures were settled, mostly affirmatively, by Ax in [4]. More recent attempts at a geometric founding of SQM can be found in [13 and 22].

### 6.2 Measurability Versus Observability.

Even though we take a state-first approach, we should have, by the end, some correspondent to the observables of SQM. In this subsection, we show how to get the truly measurable observables. These are, essentially, the strongly repeatable instruments of Davies in [12, Chap.4].

The standard way, since von Neumann, of representing the measurement process is to consider a composite system $S = S_1 + S_2$, where $S_1$ is the system to be measured and $S_2$ is the apparatus. Given the observable $A$, an experimental arrangement of $S$, including a Hamiltonian $H$, is posited so as to model measurement.

We reverse this procedure, by starting with essentially arbitrary states of $S = S_1 + S_2$, and see what can be considered as being measured. We assume, therefore, that we have an evolving polar decomposition for $t \geq 0$.

$$\Gamma = \Gamma(t) = \sum_k q_k \phi_k \otimes \psi_k.$$ 

We require, for simplicity, that it be regular at the time of measurement, say $t = 1$. If the procedure is to have the usual repeatability requirement, then we must require that the evolutions be free for $t \geq 1$. Thus, we take the total Hamiltonian $H$ to be, for $t \geq 1$, effectively of the non-interacting form $H = H_1 \otimes I_2 + I_1 \otimes H_2$. By the result of Section [13], we have no jumping, *i.e.* $r_k(t)$ is constant for $t \geq 1$. This guarantees the repeatability. This procedure can be regarded as a method of measuring
the observable corresponding to the Hermitian Hilbert-Schmidt operator \( A = \sum_k r_k P_{\phi_k} \), \( r_k := |q_k| \).

Since we have assumed the \( r_k \) distinct, this is equivalent to giving a finite-dimensional-projection-valued measure on \( Z \). These operators are dense in the strongly repeatable instruments, which by [12, Th. 3.1] can be identified with the projection-valued measures on \( Z \).

In this way, we recover the truly measurable observables. They have the property that we can sensibly assign a state to the result of an \( A \)-measurement of a state. Indeed we have the formula that the result \( \rho_1 \) of measuring a state (represented by the density operator) \( \rho_0 \) by the instrument \( A \) is the state

\[
\rho_1 = \sum_k P_{\phi_k} \rho_0 P_{\phi_k},
\]

as well as

\[
\rho_1 = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{iA} \rho_0 e^{-iA} ds = Q(\rho_0),
\]

Here \( Q \) denotes the projection (orthogonal with respect to the Hilbert-Schmidt norm) on the set of density operators intersected with the commutant of \( A \).

As pointed out by Davies, in general there is no comparable formula to Eq. (6.5) for arbitrary bounded Hermitian operators. The same applies to Eq. (6.6). For example, suppose the position operator (multiplication by \( f(x) = x \)) on \( L^2(\mathbb{R}) \) commutes with some density operator \( \rho \). Let \( P_I \) denote the spectral projection of the position operator corresponding to an interval \( I \). Then \( P_I \) is multiplication by the characteristic function of \( I \). Let \( v \) be an eigenfunction of \( \rho \) with eigenvalue \( r > 0 \). Then all the \( P_I(v) \) are eigenfunctions of \( \rho \) with eigenvalue \( r \). There is an interval \( I \) on which the essential infimum of \( \pm v(x) \) is positive. Then using a subinterval \( J \) of \( I \), it follows that there must be an infinite dimensional space of eigenvectors of \( \rho \) with eigenvalue \( r \). This contradiction to \( \rho \) being trace class, shows that the position operator is not measurable within the framework of SQM.

In other words, only sufficiently “small” Hermitian operators correspond to truly measurable quantities. Others have expectation values and can be approximated by instruments, but they cannot sensibly give an after-measurement state.

### 6.3 The values of SQM observables in IQM.

Let \( B \) be an Hermitian operator on the Hilbert space \( \mathcal{H}_1 \). For simplicity, take \( B \) to be positive with distinct eigenvalues \( b_k \) and eigenvectors \( \phi_k \). A basic assumption of SQM is that the expected value of \( B \) in any SQM state \( |\phi\rangle \) is \( \langle B\phi, \phi \rangle \).

We want to compare the SQM description of an observable with operator \( B \) with its IQM version. For this, we need
1) another system \( S_2 \) with Hilbert space \( \mathcal{H}_2 \) and \( \psi \in \mathcal{S}(\mathcal{H}_2) \)
2) a Hamiltonian \( H \) on \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) and an orthonormal set \( \{ \psi_k \} \) in \( \mathcal{H}_2 \) such that

\[
U(\phi \otimes \psi) = \phi_k \otimes \psi_k, \text{ where } U_t = e^{iHt} \text{ and } U = U_1.
\]

These are the same requirements as are invoked in the theory of measurement in SQM. Discussions of this situation with some proposals as to the formation of \( H \) are contained in [11], [12, Chap.4], and [11].
Now let \( \phi \in S(\mathcal{H}_1) \). Then we have

\[
\phi = \sum_k q_k \phi_k \Rightarrow U(\phi \otimes \psi) = \sum_k q_k \phi_k \otimes \psi_k.
\]

In this situation, SQM says that the probability that \( B \) assumes the value \( b_j \) in state \([\phi_j]\) is \(|\langle \phi, \phi_j \rangle|^2 = |q_j|^2\). In IQM more information is available because a particular \( j \) is picked out by virtue of the fact that \( S_1 \) is actually in one of the states \([\phi_j]\). Thus a definite value for \( B \) is determined, namely \( b_j \). Moreover, for \( e^{i\theta} \in S^1 \), the probability that \( e^{i\theta} \phi \) would yield the value \( b_j \) is \(|q_j|^2\), by the diagonal property of Pythagorean partitions. This probabilistic fact is independent of the choices made in 1) and 2). If we make different choices of \( H \) and \( \psi \) or the particular \( \phi \in [\phi] \), we would, in general, get a different \( j \) and so a different value of \( B \). We note that in an important case, the value of \( B \) is independent of these choices. This happens when \([\phi] \) is one of the \([\phi_k] \), say \([\phi_j]\). Then by Eq. (6.7), \( \phi \otimes \psi \) evolves to \( \phi_j \otimes \psi_j \) and so \( B \) takes on the value \( b_j \), as it should.

Thus IQM can predict the actual value an observable attains in a state of \( S_1 \) provided that the complete experimental setup is specified, including the Hamiltonian \( H \). This added contextuality makes the set of dynamical variables on \( S_1 \) a much more complicated object than the algebra of observables assigned to it by SQM.

Thus for IQM, the state-first approach appears natural. At any rate, the appropriate notion of observable for individual systems would require some reworking as did the notion of state.

7 EPR

The EPR paper [15] raised the important issue of entanglement and the spectre of spooky non-local effects which haunts physics to this day. EPR seems to present a truly puzzling aspect of the physical world, a “Z-mystery” in the terminology of [33, Penrose]. We agree that it reveals a deep truth, but contend that this is the inescapable relativity of states. We follow Penrose in contrasting EPR with the Measurement Problem, a paradox exemplified by Schrödinger’s cat. We claim that the reduction of the state vector in the latter is not a real physical effect. It is, according to IQM, really a misapplication of a statistical theory to individuals. One should not be surprised at inconsistencies arising from confusing states of ensembles with states used in representing individuals.

We show how our interpretation leads to a description of the EPR experiment without any paradoxical features. We shall study it in a general setting to emphasize that it is a pervasive effect of any multiple interactions, and not, as in [15] and [4], partially the result of degeneracies of state.

We first describe the EPR situation using SQM absolute projective states, in order to examine its puzzling aspect.

7.1 The orthodox account of EPR.

Two systems \( S_1 \) and \( S_2 \) interact and then are spatially isolated from each other. We use parenthesized ligatures as variables for states of composite systems to keep track of which systems are
involved at each stage. We denote the state after interaction, at \( t = 0 \), of \( S_1 + S_2 \) by \((\alpha\beta)_0\) and write this fact symbolically:

\[
(S_1 + S_2)(0) = (\alpha\beta)_0.
\]

A measurement is going to be performed on \( S_2 \) by an apparatus system \( S_3 \), which is initially in a state \( S_3(0) = \gamma_0 \). The state of \((S_1 + S_2) + S_3\) is assumed to be, initially,

\[
((S_1 + S_2) + S_3)(0) = (\alpha\beta)_0 \otimes \gamma_0
\]

At the time \( t = 1 \) of measurement and after, \( S_2 \) and \( S_3 \) are both spatially separated from \( S_1 \), and so we can consider their compound state as if they comprised an isolated system. In polar form, we assume this state to be given by

\[
(S_2 + S_3)(1) = \sum_i z_i \beta_i \otimes \gamma_i
\]  

In other words, \( S_3 \) is measuring which \([\beta_i]\) is the state of \( S_2 \). We also have a polar decomposition

\[
(\alpha\beta)_i := ((S_1 + S_2) + S_3))(1) = \sum_i y_i (\alpha\beta)_i \otimes \gamma_i.
\]  

The parenthetical association of systems within \( S = S_1 + S_2 + S_3 \) in the above equation serves only as an additional mnemonic device to indicate which polar decomposition we are considering. Indeed, we use below that \(((S_1 + S_2) + S_3))(1) = (S_1 + (S_2 + S_3))(1)\). While systems can be freely associated; the situation for their relative states is more complicated. Eq. (7.2) says that \( S_3 \) is (also) measuring which \([\alpha\beta]_i\) is the state of \( S_1 + S_2 \). Thus an outcome or reading \([\gamma_i]\) for \( S_3 \) is synchronous with both \( S_2 \) being in state \([\beta_i]\) and \( S_1 + S_2 \) being in state \([\alpha\beta]_i\). It follows that \([\alpha\beta]_i\) has the form \([\alpha_i \otimes \beta_i]\) for some \( \alpha_i \in S(\mathcal{H}_1) \). For suppose \([\alpha\beta]_i\) has a general polar decomposition

\[
(\alpha\beta)_i = \sum_h q'_h \alpha'_i \otimes \beta'_h.
\]

with non-zero \( q'_h \). If there is more than one non-zero \( q'_h \), say for \( h = 1, 2 \), then at least one of \([\beta'_1], [\beta'_2]\), say \( \beta'_1 \) is unequal to \([\beta_i]\). Thus there would be a positive probability that both \([\gamma_i]\) and \([\beta'_i] \neq [\beta_i]\) would simultaneously occur, contradicting Eq. (7.1).

**According to the orthodox interpretation:** After a the measurement of \( S_2 \) by \( S_3 \), with \( S_3 \) “reading” \([\gamma_i]\) for some \( i \), the state of \( S_2 \) has collapsed to \([\beta_i]\), according to Eq. (7.3).

Thus \( S_1 \) must be in state \([\alpha_i]\).

The puzzling aspects are:

**Z1-** that whereas \( S_1 \) was in no definite state after the interaction of \( S_1 \) and \( S_2 \) (since no observation was made), \( S_1 \) suddenly enters the state \([\alpha_i]\) as a result of the possibly distant interaction of \( S_2 \) and \( S_3 \);

**Z2-** that moreover, if \( S_3 \) were to be arranged differently, perhaps just rotated, then an entirely different \( \alpha'_j \) would result, where \([\alpha'_j]\) need not even be among the \([\alpha_i]\).

This is indeed mysterious as long as one uses absolute states.
7.2 The IQM resolution.

Using the polar and conditional spectral states of IQM: Now \((S_1 + S_2 + S_3, S_1)\) has a polar state \(((\alpha \beta \gamma)(t), q(t))\) at each time \(t\) corresponding to a polar decomposition

\[
(\alpha \beta \gamma) = \sum_k q_k \alpha_k^1 \otimes (\beta \gamma)_k. \tag{7.3}
\]

from which the spectral projection \([\alpha^1]\) assigned to \(S_1\) is derivable. Indeed, \([\alpha^1]\) is the conditional state \([\alpha^1_{k(1)}]\) := \(((\alpha \beta \gamma)(1), q(1))|S_1\).

Projection on \([\alpha^1_{k(1)}]\) is an eigenprojection of \(A(t) := \text{Tred}_1 \left( P_C(\alpha \beta \gamma)(t) |t\right)\), at \(t = 1\). Because of the assumed isolation of \(S_1\) from \(S_2 + S_3\) for \(t > 0\), we can take the total Hamiltonian for \((S_1 + (S_2 + S_3))\) to be of the form \(H_1 \otimes I_2 + I_1 \otimes H_{23}\). Thus we have

\[
A(t) = e^{-itH_1} A(0) e^{itH_1} = \sum_k |q_k|^2 e^{-itH_1} P_{\alpha_k^1} e^{itH_1}.
\]

It follows that for \(t > 0\) there exists \(k(t)\) so that the conditional spectral state of \(S_1\) relative to \(S_1 + S_2 + S_3\) at time \(t\) is \([\alpha^1_{k(t)}(t)] = [e^{-itH_1} \alpha^1_{k(t)}(0)]\).

We now combine this essentially standard analysis with the underlying hidden phases, which according to IQM, determine the particular value \(k(1)\). Actually \(k(t)\) is determined by IQM by \(q_k \in p_{k(t)}\). It follows from Section 4.4 that there is no jumping since there is no interaction Hamiltonian. Thus \(k(t)\) is a constant for \(t > 0\). In particular it does not depend on the (later, distant) interaction between \(S_2\) and \(S_3\) and certainly not on the particular outcome \([\beta_i]\) previously found for the spectral state of \(S_2\), or even whether any such measurement is made.

Nevertheless, the state \([\alpha_i]\), obtained as in (⋆) above, represents “an element of reality” since we know empirically that if we test \(S_1\) for the property \(P_{\alpha_i}\) (after having obtained the “reading” \([\gamma_i]\) before), we will definitely obtain an affirmative answer.

The IQM account is that the conditional spectral state of \(S_1 + S_2\) relative to \((S_1 + S_2) + S_3\) at \(t = 1\) is given, for some \(q'\) by

\[
[((\alpha \beta \gamma), q')|S_1 + S_2] = [((\alpha \beta)_i]|S_1 + S_2]
\]

from Eq. (7.2) and the ensuing discussion. The index \(i\) is determined by the toroidal part \(p_i\) which contains \(q'\) at \(t = 1\). Then the conditional spectral state of \(S_1\) relative to \(S_1 + S_2\) is derivable from a polar decomposition of \(\alpha_i \otimes \beta_i\), which is already in polar form. Hence it is just \(P_{\alpha_i}\), i.e. this iterated conditional spectral state is \([\alpha_i]\). So this “element of reality” is faithfully represented in IQM as the iterated conditional spectral state,

\[
[\alpha_i] = [((\alpha \beta)_i, q'')|S_1], \text{ where } [((\alpha \beta)_i)|S_1 + S_2] \tag{7.4}
\]

for some \(q''\). This is an example of the compounding of perspectives discussed on Section 4.2. At the same time, the conditional spectral state of \(S_1\) relative to \(S_1 + S_2 + S_3\) is \([\alpha^1_{k(1)}(1)]\).

Returning to the puzzling aspects mentioned above, we see that:

Z1 is obviated in IQM by \(S_1\) always having a conditional state with respect to any supersystem;
Z2 is clarified in IQM by the fact that these conditional states of $S_1$ with respect to $S$, actually do depend upon $S$, which may have distant parts.

This analysis of EPR enables us to answer a possible objection to our interpretation. It has been argued in the literature that the spectral resolution of the density operator has no privileged role among the different decompositions of the operator into convex combinations of one-dimensional projections. Fano [17] in particular argued that no such particular convex combination is “intrinsically relevant apart from analytic convenience . . .”.

In Fano’s striking example, a beam of atoms filtered to have total angular momentum $J = 1$ and $z$-component $J_z = 0$ emits photons, after which the $z$-component of angular momentum $J_z$ of the atom is measured by a Stern-Gerlach apparatus. In the example, the eigenstates of the measured atoms are confined to a frame of eigenprojections of $J_z$ in a three-dimensional Hilbert space. The corresponding eigenvalues are 0, $\pm 1$. According to which member $[\beta_i]$ of the frame occurs in the measurement, the polarization of the photon will be in a state $[\alpha_i]$, after (approximately) confining the photons to a given direction. Then the $2 \times 2$ density matrix of the polarization is, as Fano argues, most naturally represented as a convex combination of the three eigenprojections $P_{\alpha_i}$. This is not the spectral decomposition (which has at most two summands), which would be relevant in an ordinary direct measurement of the polarization of the photon.

Now Fano’s example is clearly an experiment of the EPR type which we analyzed above, if we take $S_1$ to be the photon, $S_2$ the atom, and $S_3$ the Stern-Gerlach apparatus. The mixed state of $S_1$ is given in our notation by the density matrix $\text{Tred}_1(\alpha\beta)$, which has the non-spectral decomposition $\sum_{i=1}^3 |y_i|^2 P_{\alpha_i}$. In our analysis, however, the rays $[\alpha_i]$ of the photon do arise as (iterated) spectral states: because the photon state is measured indirectly, via the spin of the atom with which it has previously interacted, the ray $[\alpha_i]$ occurs as an iterated spectral state. It is actually, in the terminology introduced in Section 4, a conditional spectral state of $S_1$ with respect to a polar state of $S_1 + S_2$ which itself represents a conditional spectral state of $S_1 + S_2$ with respect to a polar state of $S_1 + S_2 + S_3$, as is made explicit in Eq. (7.4).

This is a case of a quite general situation. It is easy to prove that every convex decomposition of a density operator into one-dimensional projections arises as a two-step iterated spectral decomposition, as above.

It requires a shift in thinking to become reconciled to the necessary perspectivity of states. An analogy from special relativity may be helpful. Consider two particles $S_1$ and $S_2$ colliding and the subsequent distant collision of $S_2$ with a particle $S_3$. The center-of-mass inertial frame $F$ of $S_2 + S_3$ does not change as a result of the collision of $S_2$ and $S_3$; hence, any characteristic property of $S_1$ (corresponding to the state $[\alpha_k^{(1)}]$) such as mass is unchanged by this collision from the point of view of an observer with frame $F$. On the other hand, the inertial frame of $S_2$ is changed by this collision. Thus the mass of $S_1$ from the perspective of $S_2$ suddenly changes (corresponding to the state $[\alpha_i]$).

It is difficult to drop the idea of the absoluteness of some quantities, such as mass, time and the state of a system. The situation in QM is actually more serious than in special relativity where we have the possibility of passing from a description in one inertial frame to that of another by means of the Poincaré group. The intransitivity of spectral projections prohibits so neat an extrication from the net of compounded perspectives. This is also the reason that SQM is more tractable than
IQM: reduced traces are transitive. But if one is interested in modeling individual systems, then this extra complication seems necessary.

Everything should be made as simple as possible, but not simpler. *Einstein*


APPENDICES

A  Polar Decompositions, Reduced Traces and Moment Maps

In this section we recall some basic facts about the polar decompositions including the relation with
the reduced trace. Then we show, that the reduced trace is really a moment map. The main fact
we need is that the natural moment map associated with the action of \( \mathbb{U}(n_1) \times \mathbb{U}(n_2) \) on \( \mathbb{P}(\mathbb{C}^{n_1 \times n_2}) \)
is given by a pair of reduced traces.

A.1 Polar decompositions and reduced traces

**Lemma A.1** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces. Let \( \Gamma \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). Then \( \Gamma \) has a polar
decomposition, i.e. there exist scalars \( q_k \) and orthonormal \( \phi_k \in \mathcal{H}_1 \) and orthonormal \( \psi_k \in \mathcal{H}_2 \) so
that

\[
\Gamma = \sum_k q_k \phi_k \otimes \psi_k \tag{A.1}
\]

The \( r_k := |q_k| > 0 \) are unique. If they are distinct then the \( q_k, \phi_k, \psi_k \) are all unique up to phase
factors. We sometimes also write \( \Gamma_k = \phi_k \otimes \psi_k \). Then

\[
\Gamma = \sum_k q_k \Gamma_k. \tag{A.2}
\]

Again, if the \( r_k \) are distinct, then the \( \Gamma_k \) are unique up to phase factors. If either the set of \( q_k \) or
the set of \( \Gamma_k \) is specified, then the other set is uniquely determined. We sometimes refer to the \( \Gamma_k \)
as bi-orthonormal.

The main content of this lemma is

**Lemma A.2** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces. Let \( \Gamma \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). Then there exist non-negative
scalars \( q_k \in \mathbb{R} \) and orthonormal \( \phi_k \in \mathcal{H}_1 \) and orthonormal \( \psi_k \in \mathcal{H}_2 \) so that

\[
\Gamma = \sum_k q_k \phi_k \otimes \psi_k. \tag{A.3}
\]

**Proof.** This is well-known and can be found, e.g., in \([25]\), but we include a proof which introduces
some maps used later. The map

\[
\mathcal{H}_1 \times \mathcal{H}_2 \to \overline{\text{Hom}}(\mathcal{H}_2, \mathcal{H}_1), \text{ given by } (\phi, \psi) \mapsto [\psi' \mapsto \langle \psi', \psi \rangle \phi] \tag{A.4}
\]
is \( \mathbb{C} \)-bilinear to the conjugate-linear maps. It therefore induces a map

\[
\tilde{\cdot} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \overline{\text{Hom}}(\mathcal{H}_2, \mathcal{H}_1), \phi \otimes \psi \mapsto [\psi' \mapsto \langle \psi', \psi \rangle \phi] \tag{A.5}
\]

In particular, we have the conjugate-linear operator \( \tilde{\Gamma} : \mathcal{H}_2 \to \mathcal{H}_1 \) associated to any \( \Gamma \in \mathcal{H}_1 \otimes \mathcal{H}_2 \).
We also have a conjugate-linear bijective isometry \( \dagger : \mathcal{H}_2 \to \mathcal{H}_2 \), whose inverse is explicitly induced
by \( \psi \mapsto [\psi' \mapsto \langle \psi', \psi \rangle] \). Any bounded linear operator has a polar decomposition [36, Theorem VI.10, p.197]. We can apply this fact to \( \tilde{\Gamma} \circ \dagger : \mathcal{H}_2 \to \mathcal{H}_1 \). It follows that there exists a unique positive self-adjoint operator \( V : \mathcal{H}_2 \to \mathcal{H}_2 \) and a partial conjugate-linear isometry \( U : \mathcal{H}_2 \to \mathcal{H}_1 \) with domain the image of \( V \) so that \( \tilde{\Gamma} = UV \).

\[
\tilde{\Gamma} = UV, \quad V = \sqrt{(\tilde{\Gamma}^* \tilde{\Gamma})}, \quad \ker U = \ker V = \ker \tilde{\Gamma}, \quad \text{ran} \ U = \text{ran} \tilde{\Gamma} \tag{A.6}
\]

\( V \) is a positive self-adjoint Hilbert-Schmidt operator. It has an explicit spectral decomposition of the form

\[
V = \sum_j q_j P_j, \quad q_j > 0, \quad \sum_j q_j^2 = 1, \tag{A.7}
\]

where the \( P_j \) are finite-dimensional mutually orthogonal projections. We are mostly interested in the case where \( \dim P_j = 1 \) for all \( j \). Then there exist \( \psi_j \in \mathcal{H}_2, |\psi_j| = 1 \) so that

\[
V(\psi) = \sum_j q_j \langle \psi_j, \psi \rangle \psi_j, \quad q_j > 0, \quad \sum_j q_j^2 = 1 \tag{A.8}
\]

The 1-dimensional projections \( P_j \) in Eq. (A.7) determine the \( \psi_j \) only up to phase factors which however do not change the value of \( V \) in the last formula. From it and Eq. (A.6)

\[
\tilde{\Gamma}(\psi_j) = UV(\psi_j) = U(q_j \psi_j) = q_j \phi_j, \quad \text{where} \quad \phi_j \overset{\text{def}}{=} U(\psi_j) \tag{A.9}
\]

\[
\therefore \quad \tilde{\Gamma}(\psi) = \sum_j q_j \langle \psi, \psi_j \rangle \phi_j \tag{A.10}
\]

since the operator defined by the RHS of this formula is conjugate linear and agrees with \( \tilde{\Gamma} \) on the orthogonal complement of the kernel of \( \tilde{\Gamma} \). It follows now from the definition of \( \sim \) in Eq. (A.5) and its injectivity

\[
\Gamma = \sum_j q_j \phi_j \otimes \psi_j \tag{A.11}
\]

which is Eq. (A.3). \textbf{This is the polar decomposition of \( \Gamma \).}

Moreover, if we change

\[
\psi_j \rightarrow \tau_j \psi_j, \quad \text{then} \quad \phi_j = U(\psi_j) \rightarrow U(\tau_j \psi_j) = \tau_j U(\psi_j) = \tau_j \phi_j, \tag{A.12}
\]

so that the only remaining indeterminacies are reciprocal phase factors in each pair \( \phi_j, \psi_j \).

\[\blacksquare\]

**Lemma A.3** Let \( \Gamma(t) \) evolve according to the Schrödinger equation (with \( \hbar = 1 \)) with Hamiltonian \( H \). Let there be given a smooth curve of polar decompositions:

\[
\Gamma(t) = \sum_j q_j(t) \phi_j(t) \otimes \psi_j(t) = \sum_j q_j(t) \Gamma_j(t) \tag{A.13}
\]

Set \( H_{jj, kk} = \langle \Gamma_j, H \Gamma_k \rangle \) and \( r_j = |q_j| \). If each \( q_j \) never vanishes, then

\[
\dot{r}_j = \sum_k \Im(H_{jj, kk}) r_k. \tag{A.14}
\]
If the $q_j$ are all real then
\[ \dot{q}_j = \sum_k \Im(H_{jj,kk})q_k. \] (A.15)

If the $\phi_k, \psi_k$ are horizontal with respect to the canonical connection, then
\[ \dot{q}_j = -i \sum_k H_{jj,kk}q_k. \] (A.16)

**Proof.** The first assertion follows from the second, since if the $r_j = |q_j|$ never vanish, we can replace the $q_j$ by the $r_j$ and smoothly compensate with a phase modification of the $\phi_j$. Differentiating Eq. (A.13), we get
\[ -iH\Gamma = \dot{\Gamma} = \sum_j (\dot{q}_j \phi_j \otimes \psi_j + q_j \dot{\phi}_j \otimes \psi_j + q_j \phi_j \otimes \dot{\psi}_j), \] (A.17)

Using inner products which are conjugate linear in the first variable, and making use of the biorthonormality of the $\phi_j \otimes \psi_j$, we find by taking inner products of both sides with $\phi_j \otimes \psi_j$
\[ \langle \phi_j \otimes \psi_j, \dot{\Gamma} \rangle = \langle \phi_j \otimes \psi_j, -iH\Gamma \rangle = -i \langle \phi_j \otimes \psi_j, H\Gamma \rangle = -i \sum_k H_{jj,kk}q_k \] (A.18)

\[ = \dot{q}_j + q_j \langle \phi_j \otimes \psi_j, \dot{\phi}_j \otimes \psi_j \rangle + q_j \langle \phi_j \otimes \psi_j, \phi_j \otimes \dot{\psi}_j \rangle \] (A.19)

Since $||\phi_j|| = ||\psi_j|| = 1$, $\langle \phi_j, \dot{\phi}_j \rangle, \langle \psi_j, \dot{\psi}_j \rangle$ are purely imaginary and thus so are the last two terms of Eq. (A.19). If the $q_j$ are real, we deduce Eq. (A.15). If the $\phi_k, \psi_k$ are horizontal with respect to the canonical connection, the last two terms of Eq. (A.19) vanish and we deduce Eq. (A.16).

\[ \blacksquare \]

**A.1.1 Quaternionic version**

Quaternions came from Hamilton after his really good work had been done; and though beautifully ingenious, have been an unmixed evil to those who have touched them in any way.

*Lord Kelvin*

Suppose that the $\mathcal{H}_1$ and $\mathcal{H}_2$ are left Hilbert spaces over the quaternions $\mathbb{Q}$. Then for $q \in \mathbb{Q}$ and $v \in \mathcal{H}_j$, $q \cdot v$ is defined and satisfies the standard rules, as in [40, Vol.I]. We can also regard them as right $\mathbb{Q}$–Hilbert spaces by using the definition $v \cdot q = \overline{q} \cdot v$. Of course, this works from right to left as well. This permits the formation of the tensor product $\mathcal{H}_1 \otimes_{\mathbb{Q}} \mathcal{H}_2$ by using the rule $vq \otimes qw = v \otimes qw$. A $\mathbb{Q}$-Hilbert space $\mathcal{H}$ can be regarded as a $\mathbb{C}$-Hilbert space together with a $\mathbb{Q}$-**structure map** $j_{\mathcal{H}}$: this means $j_{\mathcal{H}}$ is $\mathbb{R}$–isometric, conjugate-linear, and satisfies $j_{\mathcal{H}}^2 = -1$. Here conjugation means for all $a, b, c, d \in \mathbb{R}$, $a + bi + cj + dk = a - bi - cj - dk$. Then $j_{\mathcal{H}}$ is just left multiplication by $j$. Also $j := j_{\mathcal{H}}$ is an isometry: $\langle jv, jw \rangle = \langle v, w \rangle$

**Lemma A.4** Lemma [A.3] holds for $\mathbb{Q}$-Hilbert spaces.
Proofs. Making the necessary changes of \(\mathbb{C}-(\text{bi})\text{linear to left(\text{+ right bi})linear, etc., including the quaternionic version of the polar decomposition, whose proof also follows the complex version, the proof of Lemma A.2 works.}

Another proof uses the following fact:

We can identify \(\mathcal{H}_1 \otimes \mathcal{Q} \mathcal{H}_2\) with the \((-1)\)-eigenspace of the operator \(j_{\mathcal{H}} = j_{\mathcal{H}_1} \otimes \mathbb{C} j_{\mathcal{H}_2}\) acting on \(\mathcal{H}_1 \otimes \mathbb{C} \mathcal{H}_2\). This follows from [1, p.30].

Now let \(\Gamma \in \mathcal{H}_1 \otimes \mathcal{Q} \mathcal{H}_2 \subset \mathcal{H}_1 \otimes \mathbb{C} \mathcal{H}_2\). Then \(\Gamma\) has a positive polar decomposition: \(\Gamma = \sum_k r_k \Gamma_k = \sum_k r_k \phi_k \otimes \mathbb{C} \psi_k\). We first assume this polar decomposition has distinct positive \(r_k\). Applying \(j_{\mathcal{H}} = j_{\mathcal{H}_1} \otimes \mathbb{C} j_{\mathcal{H}_2}\), we get \(-\Gamma = \sum_k r_k (j_{\mathcal{H}_1} \otimes \mathbb{C} j_{\mathcal{H}_2})(\Gamma_k)\). Since the \(j_{\mathcal{H}_1}\) are isometric, the \(j_{\mathcal{H}_1}(\phi_k)\) are orthonormal, as are the \(j_{\mathcal{H}_2}(\psi_k)\). Thus the \(\Gamma'_k := (j_{\mathcal{H}_1} \otimes \mathbb{C} j_{\mathcal{H}_2})\Gamma_k\) are bi-orthonormal. By the uniqueness of positive polar decompositions with distinct positive \(r_k\), we must have \(\Gamma'_k = -\Gamma_k\). This means \(\Gamma_k \in \mathcal{H}_1 \otimes \mathcal{Q} \mathcal{H}_2\). This completes the proof in the case of distinct positive \(r_k\). The general case follows by taking limits.

Lemma A.5 Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be \(\mathbb{Q}\)-Hilbert spaces. Let \(\Gamma \in \mathcal{H}_1 \otimes \mathcal{H}_2\). Then \(\Gamma\) has a polar decomposition, i.e., there exist scalars \(q_k \in \mathbb{Q}\) and orthonormal \(\phi_k \in \mathcal{H}_1\) and orthonormal \(\psi_k \in \mathcal{H}_2\) so that

\[
\Gamma = \sum_k q_k \phi_k \otimes \psi_k
\]  

(A.20)

The \(r_k := |q_k| > 0\) are unique. If they are distinct then the \(q_k, \phi_k, \psi_k\) are all unique up to \(\mathbb{Q}\)-phase factors, i.e., unit quaternions. We sometimes also write \(\Gamma_k = \phi_k \otimes \psi_k\). Then

\[
\Gamma = \sum_k q_k \Gamma_k.
\]  

(A.21)

Again, if the \(r_k\) are distinct, then the \(\Gamma_k\) are unique up to \(\mathbb{Q}\)-phase factors. If either the set of \(q_k\) or the set of \(\Gamma_k\) is specified, then the other set is uniquely determined.

Remark. This allows the extension of our work to \(\mathbb{Q}\)-Hilbert spaces. The \(\mathbb{Q}\)-version of the polar bundles will thereby have \(\mathbb{Q}\)-toroidal fibers which are isometric to \(\prod_k \mathbb{SU}(2)(r_k)\), where \(\mathbb{SU}(2)(r_k)\) is \(\mathbb{SU}(2)\) with its invariant Riemannian metric normalized to give its maximal tori (which look like \(S^1\)) total arclength \(2\pi r_k\).

For now, we follow the advice implicit in Lord Kelvin’s opinion, and avoid quaternions elsewhere in this paper. We remark for later work, however, that the crucial partitioning theorems extend to \(\prod_k \mathbb{SU}(2)(r_k)\), because of functoriality and the conjugacy of maximal tori.

A.2 Reduced traces.

A good reference for reduced traces is [12].

Definition. The reduced, or partial, trace, \(\rho_1 \overset{\text{def}}{=} \text{Tred}_{\mathcal{H}_1}(\rho) := \text{Tred}_1(\rho)\) of a trace class operator \(\rho\)
in \(\text{Hom}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_1 \otimes \mathcal{H}_2)\) is defined implicitly via the sesquilinear form
\[
\langle \text{Tred}_1(\rho)(\phi), \phi' \rangle \overset{\text{def}}{=} \sum_k (\rho(\phi \otimes \psi'_k), \phi' \otimes \psi'_k),
\] (A.22)
where the \(\psi'_k\) comprise any orthonormal basis for \(\mathcal{H}_2\).

If we apply the partial trace to \(\rho := \rho \Gamma := P \Gamma = \text{orthogonal projection on } C \Gamma\) and use the polar decomposition of \(\Gamma\), Eq. (A.1), we get
\[
\sum_k \langle P(\phi \otimes \psi'_k), \phi' \otimes \psi'_k \rangle = \sum_k |q_k|^2 |\langle \phi_k, \phi \rangle|^2.
\] (A.23)
Thus
\[
\langle \text{Tred}_1(\rho)(\phi), \phi' \rangle = \sum_k |q_k|^2 |\langle \phi_k, \phi \rangle|^2.
\] (A.24)
Set
\[
\rho_1 \overset{\text{def}}{=} \sum_j |q_j|^2 P_{\phi_j},
\] (A.25)
Then we see that
\[
\rho_1 = \text{Tred}_1(\rho).
\] (A.26)
So, explicitly, we have the following relation between polar decompositions and reduced traces.

**Proposition A.6** If the \(\phi_j \otimes \psi_j\) are bi-orthonormal then
\[
\text{Tred}_1(\sum_j q_j \phi_j \otimes \psi_j) = \sum_j |q_j|^2 P_{\phi_j}.
\] (A.27)

**Examples.** 1) \(\text{Tred}_1(P_{\phi_1 \otimes \phi_2}) = P_{\phi_1}\).
2) If \(\mathcal{H}_1 = \mathcal{H}_2\) and \(\phi_j\) is ON then
\[
\text{Tred}_1(P_{\frac{1}{\sqrt{2}}(\phi_1 \otimes \phi_2 \pm \phi_2 \otimes \phi_1)}) = \frac{1}{2} P_{\phi_1} + \frac{1}{2} P_{\phi_2}.
\] (A.28)

Let \(\text{Trace}^j\) denote the ordinary trace for \(\mathcal{H}_j\), and \(\text{Trace}\) denote the ordinary trace for \(\mathcal{H}_1 \otimes \mathcal{H}_2\).

**Proposition A.7** If the \(A \in \text{Hom}(\mathcal{H}_1, \mathcal{H}_1)\) is bounded, then
\[
\rho_1 = \text{Tred}_1(\rho) \Rightarrow \text{Trace}^1(\rho_1 A) = \text{Trace}^1(\text{Tred}_1(\rho) A) = \text{Trace}(\rho \cdot (A \otimes I_2)).
\] (A.29)
In other words, the reduced trace map is adjoint to \(A \rightarrow A \otimes I_2\).

**Proof.** Let \(\phi_j, \psi_k\) be bases for \(\mathcal{H}_1, \mathcal{H}_2\).
\[
\text{Trace}^1(\rho_1 A) = \sum_j \langle \phi_j, \rho_1 A \phi_j \rangle = \sum_j \sum_k \langle \phi_j \otimes \psi_k, (A \phi_j) \otimes \psi_k \rangle = \sum_j \sum_k \langle \phi_j \otimes \psi_k, (\rho \cdot (A \otimes I_2)) \phi_j \otimes \psi_k \rangle = \text{Trace}(\rho (A \otimes I_2)).
\] (A.30)
For any Hilbert space $\mathcal{H}$, we let $\mathcal{L}(\mathcal{H})$ denote the $C^*$-algebra of bounded linear operators on $\mathcal{H}$. If $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, then we regard $\mathcal{L}(\mathcal{H}_1)$ as a subalgebra of $\mathcal{L}(\mathcal{H})$ via the injection

$$\mathcal{L}(\mathcal{H}_1) \hookrightarrow \mathcal{L}(\mathcal{H}) \text{ given by } L_1 \mapsto L_1 \otimes I_2.$$ 

More generally, if we are just given two Hilbert spaces and a unital $C^*$-algebra homomorphism $\mathcal{L}(\mathcal{H}_1) \hookrightarrow \mathcal{L}(\mathcal{H})$, then there exists a Hilbert space $\mathcal{H}_2$ so that the situation is as above. This follows from [14, Th.5.40]. This is a familiar fact for finite matrix algebras to which we now confine our attention.

Thus we have $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim \mathcal{H}_i = n_i < \infty$. We will also make use of the Hilbert-Schmidt inner product on $\mathcal{L}(\mathcal{H})$ given by

$$\langle L, L' \rangle \text{ def } = \text{Trace}(LL').$$

Proposition A.8 $P := \frac{1}{n_2^2} \text{Tred}_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_1)$ is orthogonal projection on $\mathcal{L}(\mathcal{H}_1)$.

Proof. From Eq. (A.22), it follows that $P$ is idempotent. To see that it is self-adjoint with respect to the Hilbert-Schmidt inner product, we observe from Proposition A.7

$$\text{Trace}(P(L) \cdot L') = \frac{1}{n_2^2} \text{Trace}(\text{Tred}_1(L)L') = \frac{1}{n_2^2} \text{Trace}^1(\text{Tred}_1(L)\text{Tred}_1(L'))$$

$$= \frac{1}{n_2^2} \text{Trace}^1(\text{Tred}_1(L')\text{Tred}_1(L)) = \text{Trace}(P(L)L) = \text{Trace}(L \cdot P(L')).$$

$\blacksquare$

A.3 Reduced traces and moment maps

Let $\mathcal{H}$ be a Hilbert space. It has a natural symplectic structure given by

$$\omega(X, Y) = \Im \langle X, Y \rangle,$$

where $X, Y \in \mathcal{H}$. Here $\mathcal{H}$ is identified, as a real vector space, with its own tangent space.

Let $G$ be a subgroup of $\text{U}(\mathcal{H})$, and $\mathfrak{g}$ its Lie algebra which consists of those skew-Hermitian operators $\xi$ for which $\exp \xi \in G$. We assume $\mathfrak{g}$ is a norm-closed Lie subalgebra of the skew-Hermitian operators $\mathfrak{u}(\mathcal{H}) \subset \text{Hom}(\mathcal{H}, \mathcal{H})$. A moment(um) map $\mu : \mathcal{H} \to \mathfrak{g}^*$ is characterized by

$$\forall \xi \in \mathfrak{g}, \mu(X)(\xi) = -\frac{i}{2} \langle \xi(X), X \rangle. \quad (A.32)$$

Now we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the pairing on $(\mathfrak{g} \times \mathfrak{g})$

$$(\xi, \eta) \to \text{Trace}(\xi^* \eta)$$

provided $\mathfrak{g}$ is contained in the Hilbert-Schmidt operators. This suggests that the “right” group of unitaries are those which are logarithmically Hilbert-Schmidt. We have from [32, p.163], in our notation, the moment map is given by the formula:

$$\mathcal{H} \ni X \to \mu(X) = \frac{i}{2} \langle X, \cdot \rangle X \in \mathfrak{g} \cong \mathfrak{g}^*.$$ \quad (A.33)
The projective version of Eq. (A.32) is
\[
\forall \xi \in \mathfrak{g}, \quad \mu([X])(\xi) = -\frac{i}{2} \frac{\langle \xi(X), X \rangle}{\|X\|^2}. \tag{A.34}
\]
which is in agreement with [31, page 335].

Suppose now that \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) and
\[
G = \cup(\mathcal{H}_1) \times \mathcal{H}_2 \Rightarrow \mathfrak{g} = u(\mathcal{H}_1) \times u(\mathcal{H}_2). \tag{A.35}
\]
We can take the pairing on \( \mathfrak{g} \) to be
\[
\langle (\chi_1, \chi_2), (\xi_1, \xi_2) \rangle = \text{Trace}^1 (\chi_1^* \xi_1) + \text{Trace}^2 (\chi_2^* \xi_2) \tag{A.36}
\]
where the last two traces are the ordinary traces for trace class operators on \( \mathcal{H}_1, \mathcal{H}_2 \).

Applying Eq. (A.32) to this case, we get
\[
\mu(X)(\xi_1, \xi_2) = -\frac{i}{2} \langle (\xi_1, \xi_2)(X), X \rangle = -\frac{i}{2} \langle (\xi_1 \otimes I_2 + I_1 \otimes \xi_2)(X), X \rangle \tag{A.37}
\]
\[
= -\frac{i}{2} \{ \langle (\xi_1 \otimes I_2)(X), X \rangle + \langle (I_1 \otimes \xi_2)(X), X \rangle \}
\]
Let \( \text{Trace} \) denote the usual trace on the trace class operators on \( \mathcal{H} \).
\[
\therefore \mu(X)(\xi_1, \xi_2) = -\frac{i}{2} \left\{ \text{Trace} \left( (X, \cdot)X(\xi_1 \otimes I_2) \right) + \text{Trace} \left( (X, \cdot)X(I_1 \otimes \xi_2) \right) \right\} \tag{A.38}
\]
Then, from Proposition A.7
\[
\mu(X)(\xi_1, \xi_2) = -\frac{i}{2} \left\{ \text{Trace}^1 \left( \text{Tred}_1((X, \cdot)X)\xi_1 \right) + \text{Trace}^2 \left( \text{Tred}_2((X, \cdot)X)\xi_2 \right) \right\} \tag{A.39}
\]
\[
\therefore \mu(X)(\xi_1, \xi_2) = -\frac{i}{2} \left\{ \langle \text{Tred}_1((X, \cdot)X), \text{Tred}_2((X, \cdot)X)\xi_1, \xi_2 \rangle \right\} \tag{A.40}
\]
This establishes

**Proposition A.9** \( \mu(X) = -\frac{i}{2}(\text{Tred}_1((X, \cdot)X), \text{Tred}_2((X, \cdot)X)) \).

**Corollary A.10** The range of \( \mu \) consists of those pairs \( (-\frac{i}{2}B_1, -\frac{i}{2}B_2) \) of trace class skew-Hermitian matrices with \( B_1, B_2 \) positive operators with the same non-zero spectral components.

**Corollary A.11** Using the notation of the last corollary, assume the non-zero eigenvalues \( r_1^2, r_2, \cdots \) of \( B_1 \) are distinct. These are also the non-zero eigenvalues of \( B_2 \). Let \( \phi_1, \cdots, \phi_n \) (respectively \( \psi_1, \cdots, \psi_n \)) be corresponding unit eigenvectors for \( B_1 \) (respectively \( B_2 \)). Let \( \mathcal{L}_{2\pi r} \) denote the lattice generated by the \( r_k e_k \). There is an isometry \( \iota \) to the right toroid with radii \( r_k \), given by
\[
\iota\left( \sum_{\nu} q_{\nu} \phi_{\nu} \otimes \psi_{\nu} \right) = (r_{\nu}, \arg(q_{\nu}))_{\nu} \in \mathbb{R}^n / \mathcal{L}_{2\pi r} = \mathbb{T}_{2\pi r}.
\]
Then \( \mu^{-1} \left( \left( -\frac{i}{2}B_1, -\frac{i}{2}B_2 \right) \right) = \left\{ \sum_{\nu} q_{\nu} \phi_{\nu} \otimes \psi_{\nu} \mid |q_{\nu}| = r_{\nu} \right\} \cong \mathbb{T}_{2\pi r} \).
**Proposition A.12** The fiber \( C(\Gamma) \) of the moment map above \( \mu(\Gamma) \) is isometric to a right toroid of dimension \( \text{rank}(\Gamma) \).

If \( \Gamma = \sum_{\nu=1}^{n} q_{\nu} \phi_{\nu} \otimes \psi_{\nu} \) with \( r_{\nu} = |q_{\nu}| \) distinct for \( \nu = 1, \ldots, n \leq \infty \) then \( C(\Gamma) \) is an \( n \)--dimensional toroid with \( n \) uniquely-defined (1-dimensional) foliations by circles. Namely, the \( k \)--th circle \( C_{k}(\Gamma) \subset C(\Gamma) \) through \( \Gamma' = \sum_{\nu=1}^{n} q'_{\nu} \phi_{\nu} \otimes \psi_{\nu} \) consists of the elements of the form \( \Gamma'' = \sum_{\nu=1}^{n} q''_{\nu} \phi_{\nu} \otimes \psi_{\nu} \), where \( q''_{\nu} = q'_{\nu} \) if \( \nu \neq k \) and \( q''_{k} = e^{i\tau} q'_{k}, \tau \in \mathbb{R} \). These circular foliations are well-defined in the case where the \( r_{\nu} \) are distinct because then the \( \mathbb{C}\phi_{\nu}, \mathbb{C}\psi_{\nu} \) and hence the \( \mathbb{C}(\phi_{\nu} \otimes \psi_{\nu}) \) are unique.

**Proposition A.13** For all \( \Gamma \) with \( r_{\nu} = |q_{\nu}| \) distinct and each \( X \in C(\Gamma) \), the \( \text{rank}(\Gamma) \)--toroid \( C(\Gamma) \) contains a uniquely defined bouquet \( \{ C_{k}(X) | k = 1, \ldots, n \} \) of circles through \( X \).

### B Perturbation Theory and Connections on the Hopf Bundles

Let \( \rho = \rho(t) \) be a smoothly varying curve of compact self-adjoint operators, with a non-degenerate smoothly varying normalized eigensystem consisting of the eigenvalues \( x_{j} = x_{j}(t) \) and the eigen-projections \( P_{j} = P_{\phi_{j}(t)} \) near \( t = 0 \). We use below that \( \| \dot{\rho}(t) \| \) is bounded. The application we have in mind, is where \( \rho(t) \) is a curve of reduced density operators. In this section we use Dirac’s suggestive bra-ket notation.

**Theorem B.1** For \( t \) sufficiently close to \( 0 \),

\[
P_{j}(t) = \sum_{k \neq j} \left( \frac{\langle \phi_{k} | \dot{\rho} | \phi_{j} \rangle}{x_{j} - x_{k}} \phi_{j} \phi_{k} + \frac{\langle \phi_{j} | \dot{\rho} | \phi_{k} \rangle}{x_{j} - x_{k}} \phi_{k} \phi_{j} \right). \tag{B.1}
\]

**Proof.** Near \( 0 \), we can write, with \( \rho' \) bounded,

\[
P_{j}(t) = \frac{-1}{2\pi i} \oint_{C_{j}} (\rho - \zeta)^{-1} d\zeta = \frac{-1}{2\pi i} \oint_{C_{j}} (\rho(0) + t\rho' - \zeta)^{-1} d\zeta, \tag{B.2}
\]

where \( C_{j} \) is a sufficiently small circle in \( \mathbb{C} \) centered at \( x_{j}(0) \), and \( \rho' \rightarrow \dot{\rho}(0) \) as \( t \rightarrow 0 \).

\[
\therefore P_{j}(t) = \frac{-1}{2\pi i} \oint_{C_{j}} (\rho(0) - \zeta)^{-1} \left( 1 + t\rho'(\rho(0) - \zeta)^{-1} \right)^{-1} d\zeta = \tag{B.3}
\]
\[ -\frac{1}{2\pi i} \oint_{C_j} (\rho(0) - \zeta)^{-1} \left( 1 + \sum_{k>0} (-t)^k (\rho'(0) - \zeta)^{-1} k \right) d\zeta; \]  

the series converges by the boundedness of \( \rho' \).

\[ \therefore P_j(t) = \frac{t}{2\pi i} \oint_{C_j} (\rho(0) - \zeta)^{-1} \rho'(0)(\rho(0) - \zeta)^{-1} d\zeta \text{ mod } t^2 \]

Letting \( t \to 0 \), we get

\[ \dot{P}_j(0) = \frac{1}{2\pi i} \oint_{C_j} (\rho(0) - \zeta)^{-1} \rho(0)(\rho(0) - \zeta)^{-1} d\zeta \]

In order to evaluate the integral, we now deface the pristine beauty of this operator equation, by evaluating matrix elements with respect to unit eigenvectors.

\[ \therefore \langle \phi_k(0) | \dot{P}_j(0) | \phi_h(0) \rangle = \frac{1}{2\pi i} \oint_{C_j} \langle \phi_k(0) | (\rho(0) - \zeta)^{-1} \rho(0)(\rho(0) - \zeta)^{-1} | \phi_h(0) \rangle d\zeta = \]  

\[ \frac{1}{2\pi i} \oint_{C_j} \langle \rho(0) - \zeta)^{-1} \phi_k(0) | \rho(0)(\rho(0) - \zeta)^{-1} \phi_j(0) \rangle d\zeta = \]  

\[ \frac{1}{2\pi i} \oint_{C_j} \langle (x_k(0) - \zeta)^{-1} \phi_k(0) | \rho(0)(x_h(0) - \zeta)^{-1} \phi_h(0) \rangle d\zeta = \]  

\[ \frac{1}{2\pi i} \oint_{C_j} \langle (x_k(0) - \zeta) (x_h(0) - \zeta)^{-1} \phi_k(0) | \rho(0)(x_h(0) - \zeta)^{-1} \phi_h(0) \rangle d\zeta = \quad \text{by a constant phase factor, leaves the equation unchanged,} \]

\[ \therefore \langle \phi_k(0) | \dot{P}_j(0) | \phi_h(0) \rangle = \frac{\langle \phi_k(0) | \rho(0)(\phi_h(0)) \rangle}{x_h(0) - x_k(0)} \]

if exactly one of \( h, k \) equals \( j \). In the remaining equations of this proof, we omit the argument 0, which can actually be any \( t \). We get

\[ \langle \phi_k | \dot{P}_j | \phi_h \rangle = 0 \text{ if } k, h \neq j \text{ or if } k = h = j. \]

\[ P_j = | \phi_j \rangle \langle \phi_j | \Rightarrow \dot{P}_j = | \dot{\phi}_j \rangle \langle \phi_j | + | \phi_j \rangle \langle \dot{\phi}_j |. \]

\[ \therefore h \neq j \Rightarrow \langle \phi_h | \dot{P}_j | \phi_j \rangle = \langle \phi_h | \dot{\phi}_j \rangle \& \langle \phi_j | \dot{P}_j | \phi_h \rangle = \langle \dot{\phi}_j | \phi_h \rangle. \]

\[ \therefore h \neq j \Rightarrow \langle \phi_h | \dot{\phi}_j \rangle = \frac{\langle \phi_h | \dot{P}_j | \phi_j \rangle}{x_j - x_h}. \]

What about \( \langle \phi_j | \dot{\phi}_j \rangle ? \) From Eq. (B.13),Eq. (B.12), we get only

\[ 0 = \langle \phi_j | \dot{\phi}_j \rangle \langle \phi_j | \phi_j \rangle + \langle \phi_j | \phi_j \rangle \langle \phi_j | \dot{\phi}_j \rangle + \langle \dot{\phi}_j | \phi_j \rangle + \langle \dot{\phi}_j | \phi_j \rangle = 2 \text{ Re} \langle \phi_j | \dot{\phi}_j \rangle \]

\[ \therefore \dot{\phi}_j = \sum_{k \neq j} \frac{\langle \phi_k | \dot{\phi}_j \rangle}{x_j - x_k} \phi_k + \langle \phi_j | \dot{\phi}_j \rangle \phi_j = \sum_{k \neq j} \frac{\langle \phi_k | \rho(0)(\phi_j) \rangle}{x_j - x_k} \phi_k + i3 \langle \phi_j | \dot{\phi}_j \rangle \phi_j \]

Notice that changing the \( \phi_k, (k \neq j), \) by a constant phase factor, leaves the equation unchanged, while if we so change \( \dot{\phi}_j \) all terms on both sides change by this same phase.
If the $\phi_j$ are horizontal with respect to the canonical connection $A^0$, then the last term can be omitted and we get

$$\dot{\phi}_j = \sum_{k \neq j} \frac{\langle \phi_k | \dot{\rho} | \phi_j \rangle}{x_j - x_k} \phi_k$$

(B.18)

From Eq. (B.17),

$$\dot{P}_j = |\dot{\phi}_j\rangle \langle \phi_j| + |\phi_j\rangle \langle \dot{\phi}_j| = \sum_{k \neq j} \frac{\langle \phi_k | \dot{\rho} | \phi_j \rangle}{x_j - x_k} |\phi_j\rangle + i \Im \langle \phi_j | \dot{\phi}_j \rangle |\phi_j\rangle + \text{HC},$$

(B.19)

(B.20)

where the last summand is the Hermitian Conjugate of the preceding summand. The middle purely imaginary number times the projection $P_j = |\phi_j\rangle \langle \phi_j|$ disappears after being added to its HC and so we get:

$$\dot{P}_j = \sum_{k \neq j} \frac{\langle \phi_k | \dot{\rho} | \phi_j \rangle}{x_j - x_k} |\phi_j\rangle + \text{HC}.$$  

(B.21)

Each summand in this equation is independent of the phase of the $\phi_k$. This equation does NOT depend on the horizontality of the $\phi_k$.

If we want a simple equation for the eigenvectors, rather than the projections, we have

**Corollary B.2** The $\phi_j(t)$ are $A^0$–horizontal if, and only if,

$$\dot{\phi}_j = \sum_{k \neq j} \frac{\langle \phi_k | \dot{\rho} | \phi_j \rangle}{x_j - x_k} \phi_k.$$  

(B.22)

This shows the relation between first order perturbation theory and the canonical connection, which does not seem to be explicitly mentioned in the literature despite (or perhaps because of) its simplicity. We now extend this relation to the dynamical connection defined in subsection 2.3.1.

**Corollary B.3** Let the $\phi_j(t)$ be smoothly evolving eigenvectors of $\rho(t)$ which is itself evolving by means of the Hamiltonian $H_1 = H_1(t)$. Then the $\phi_j(t)$ are $A^{H_1}$–horizontal if, and only if, they satisfy

$$\dot{\phi}_j = \sum_{k \neq j} \frac{\langle \phi_k | \dot{\rho} | \phi_j \rangle}{x_j - x_k} \phi_k - i \langle \phi_j | H_1 | \phi_j \rangle \phi_j.$$  

(B.23)

**Proof.** By Lemma 2.1, the $\phi_j(t)$ are $A^{H_1}$–horizontal if, and only if,

$$\langle \phi_j | \dot{\phi}_j \rangle = -i \langle \phi_j | H_1 | \phi_j \rangle.$$  

(B.24)

Thus the result follows from Eq. (B.17).
C  Example: Two Spin 1/2 Systems

We now give the details of the hyperfine splitting example. A nice treatment of the basics appears in [19, Feynman, Vol. III, Chap. 12]. The polar decomposition in this example was worked out in [25]. We shall use the results and notations of this last treatment, with some minor modifications, to investigate the dynamical behavior of the two subsystems. The definitions of most of the many new variables introduced are collected in a glossary in Section C.3.

The Hamiltonian is given by

$$H = \mu \vec{\sigma} \otimes \vec{\sigma}' \overset{\text{def}}{=} \mu \sigma_x \otimes \sigma'_x + \mu \sigma_y \otimes \sigma'_y + \mu \sigma_z \otimes \sigma'_z$$

(C.1)

In terms of the standard notation of, e.g., Feynman

$$H = \mu \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(C.2)

We find the eigenvalues $E_1 = -3\mu, E_2 = \mu$ with respective eigenspaces with the indicated eigenvectors

$$< |+\rangle - |-\rangle > \leftrightarrow < \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} >,$$

$$< |+\rangle, |+\rangle + |-\rangle, |+\rangle > \leftrightarrow < \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} >,$$

$$< |+\rangle, |+\rangle + |+\rangle > \leftrightarrow < \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} >$$

The Schrödinger equation (with $\hbar = 1$) is

$$i\partial_t \Gamma = H\Gamma$$

(C.4)

and thus has for its general solution:

$$\Gamma(t) = \begin{pmatrix}
d_1 e^{-i\mu t} \\
d_2 e^{-i\mu t} + d_3 e^{3i\mu t} \\
d_2 e^{-i\mu t} - d_3 e^{3i\mu t} \\
d_4 e^{-i\mu t}
\end{pmatrix}$$

(C.5)

We want to choose axes in physical space $\mathbb{R}^3$ so as to make $\Gamma(0)$ simple. Since the Hamiltonian is rotation invariant, we can keep its matrix Eq. (C.2) while rotating the $z$-axis to bisect the initial spin vectors (in $\mathbb{R}^3$) of the two spin systems. Let the angle between them be $2\theta$. Thus we have

$$\Gamma(0) = q_{+}(0) \begin{pmatrix} e^{i\varphi} \cos \frac{\theta}{2} \\ e^{-i\varphi} \cos \frac{\theta}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\varphi} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \end{pmatrix} + q_{-}(0) \begin{pmatrix} -e^{i\varphi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\varphi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} =: \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}$$

(C.6)
for some real $\varphi$. To simplify calculations, we restrict to the case where $\varphi = 0$. Then

$$
\begin{pmatrix}
a_0 \\
b_0 \\
c_0 \\
d_0
\end{pmatrix} = \begin{pmatrix}
q_+(0) \cos^2 \frac{q}{2} - q_-(0) \sin^2 \frac{q}{2} \\
-(q_+(0) + q_-(0)) \cos \frac{q}{2} \sin \frac{q}{2} \\
(q_+(0) + q_-(0)) \cos \frac{q}{2} \sin \frac{q}{2} \\
-q_+(0) \sin^2 \frac{q}{2} + q_-(0) \cos^2 \frac{q}{2}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}(Cl + k) \\
-\frac{1}{2}Sl \\
\frac{1}{2}Sl \\
\frac{1}{2}(Cl - k)
\end{pmatrix},
$$

where (C.7)

$$
C := \cos \theta, S := \cos \theta, k := q_+(0) - q_-(0), l := q_+(0) + q_-(0).
$$

Note that Eq. (C.7) is a polar decomposition of $\Gamma(0)$. We are still free to rotate about the $z-$axis. Later, we use this to make the $q_{\pm}(0)$ real.

From Eq. (C.6), (C.7), we have

$$
\Gamma(t) = \begin{pmatrix}
a_0 e^{-i\mu t} \\
b_0 e^{3i\mu t} \\
-b_0 e^{3i\mu t} \\
d_0 e^{-i\mu t}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}(Cl + k)e^{-i\mu t} \\
-\frac{1}{2}Se^{3i\mu t} \\
\frac{1}{2}Se^{3i\mu t} \\
\frac{1}{2}(Cl - k)e^{-i\mu t}
\end{pmatrix} = \begin{pmatrix}
a \\
b \\
-b \\
d
\end{pmatrix},
$$

We now calculate the polar decomposition of $\Gamma(t)$ via the prescription in Section A. We use the associated matrix operator $\Gamma(t)$.

$$
\tilde{\Gamma}(t) := \begin{pmatrix}
a \\
b \\
-b \\
d
\end{pmatrix}
$$

(C.10)

$\tilde{\Gamma}(t)$ is the matrix operator (with respect to the standard bases) of the $\mathbb{C}$-linear operator associated with $\Gamma(t)$ via the isomorphism $\mathbb{H}_1 \otimes \mathbb{H}_2 \cong \text{Hom}(\mathbb{H}_2^*, \mathbb{H}_2)$ defined so that

$$(\forall h^* \in \mathbb{H}_2^*) \ h_1 \otimes h_2(h^*) = h^*(h_2)h_1.$$  Here $\mathbb{H}_2^*$, the dual space of $\mathbb{H}_2$, can be $\mathbb{C}$-linearly identified with the set of $h$ for $h \in \mathbb{H}_2$ by means of $\tilde{h}(h') = \langle h, h' \rangle$ for all $h' \in \mathbb{H}_2$. Then

$$
\tilde{\Gamma}(t) \Gamma(t) = \begin{pmatrix}
a & -b \\
b & d
\end{pmatrix} = \begin{pmatrix}
A & B \\
B^* & D
\end{pmatrix},
$$

(C.11)

where

$$
A = \bar{a}a + bb, D = bb + dd \in \mathbb{R}, \ A + D = 1, \ B = \bar{a}b - \bar{b}d
$$

(C.12)

We have $\text{Det}(\tilde{\Gamma}^* \tilde{\Gamma}) = AD - BB = |\text{Det}(\tilde{\Gamma})|^2 = |ad + b|^2$ and $\text{Trace}(\tilde{\Gamma}^* \tilde{\Gamma}) = A + D = 1$. The eigenvalues $x_\pm$ satisfy $x_\pm^2 - x_\pm = (AD - BB) = 0$. Thus they are

$$
x_\pm = \frac{1 \pm \sqrt{\Delta}}{2}, \ \Delta = 1 - 4(AD - B^*B) = 1 - 4|ad + b|^2
$$

(C.13)

As (unnormalized) eigenvectors for $\tilde{\Gamma}^* \tilde{\Gamma}$ we can take

$$
\tilde{\psi}_{\pm} = -2 \begin{pmatrix}
B & 2B \\
-A + x_\pm
\end{pmatrix} = -\begin{pmatrix}
2(\bar{a}b - \bar{b}d) \\
D - A \pm \sqrt{\Delta}
\end{pmatrix} = \begin{pmatrix}
2(\bar{a}b - \bar{b}d) \\
|d|^2 - |a|^2 \pm \sqrt{\Delta}
\end{pmatrix} \defeq \begin{pmatrix}
\tilde{\alpha} \\
\tilde{\beta}_{\pm}
\end{pmatrix}.
$$

(C.14)

The $\phi_j$ are defined analogously to the $\psi_j$ using the matrix operator $\bar{\Gamma}^* \Gamma$. We can take

$$
\phi_{\pm} = \pm \begin{pmatrix}
2(b\bar{d} - \bar{a}b) \\
|d|^2 - |a|^2 \pm \sqrt{\Delta}
\end{pmatrix} \defeq \begin{pmatrix}
\alpha \\
\beta_{\pm}
\end{pmatrix}.
$$

(C.15)
Then a polar decomposition of $\Gamma(t)$ is given by

$$
\Gamma = \sum_{k=\pm} q_k \frac{\phi_k}{||\phi_k||} \otimes \frac{\psi_k}{||\psi_k||}.
$$

(C.16)

Note that this reduces to the polar decomposition Eq. (C.6) at $t = 0$. We have

$$
||\psi_{\pm}|| = ||\bar{\psi}_{\pm}|| = ||\phi_{\pm}|| = \sqrt{2\sqrt{\Delta} (|d|^2 - |a|^2)}.
$$

Using the equality of these norms and applying equation Eq. (C.16) to $\psi_{\pm}$, we get

$$
q_{\pm} \phi_{\pm} = \bar{\Gamma}(\bar{\psi}_{\pm}) = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \begin{pmatrix} \alpha_{\pm} \\ -\beta_{\pm} \end{pmatrix} = \begin{pmatrix} a\alpha_{\pm} - b\beta_{\pm} \\ -b\alpha_{\pm} - d\beta_{\pm} \end{pmatrix}.
$$

(C.17)

Thus

$$
q_{\pm} = \frac{\langle e_j, \bar{\Gamma}(\bar{\psi}_{\pm}) \rangle}{\langle e_j, \phi_{\pm} \rangle}, j = 1, 2.
$$

(C.18)

Looking at the 2nd components in equation Eq. (C.18), we see

$$
q_{\pm} = \pm \left( b\alpha_{\pm} + d \right).
$$

(C.19)

### C.1 The Trajectories of the Spinors

The spinors $\phi_{\pm}, \psi_{\pm}$ represent spins in various directions. In this subsection we calculate these directions.

The operator representing (in SQM) the observable of 2×spin in the direction $\mathbf{n}$ is

$$
\sigma_\mathbf{n} = \mathbf{n} \cdot \vec{\sigma} = x\sigma_x + y\sigma_y + z\sigma_z, \ x^2 + y^2 + z^2 = 1.
$$

It has unit column eigenvectors with eigenvalues $\pm 1$:

$$
\frac{1}{\sqrt{2(1+z)}} \begin{pmatrix} x - iy \\ \pm 1 - z \end{pmatrix}
$$

The spin up or (+) state is represented by

$$
\frac{1}{\sqrt{2(1-z)}} \begin{pmatrix} x - iy \\ 1 - z \end{pmatrix}.
$$

The unit vectors $\phi_{\pm}$ are of this form if we take

$$
x = 3 \frac{\alpha}{\sqrt{\Delta}}, \ y = -3 \frac{\alpha}{\sqrt{\Delta}}, \ z = \frac{|a|^2 - |d|^2}{\sqrt{\Delta}}.
$$

We continue to use the abbreviations: $C \overset{\text{def}}{=} \cos \theta$ and $S \overset{\text{def}}{=} \cos \theta$. From Eq. (C.9),

$$
|a|^2 - |d|^2 = |a_0|^2 - |d_0|^2 = C \left( |q_+(0)|^2 - |q_-(0)|^2 \right).
$$
Let $\mathfrak{Z}$ denote the plane in $\mathbb{R}^3$ defined by $z = |a_0|^2 - |d_0|^2$. The spin axis trajectory of $\phi_{\pm}(t)$ intersects $\mathfrak{Z}$ in a curve $\mathcal{C}$ given by:

$$\mathcal{C} : x = \Re \alpha, \quad y = -\Im \alpha$$

We next find an equation in $x$ and $y$ for this curve in the plane $\mathfrak{Z}$. From Eq. (C.15), we have

$$\alpha = 2(bd - ab) = E \cos \omega t + F \sin \omega t - iG \sin \omega t,$$

where

$$E = S \left( |q_+ (0)|^2 - |q_- (0)|^2 \right), \quad F = 2S \Im (q_+ (0)q_- (0)),$$

and

$$G = SC (1 + 2 \Re (q_+ (0)q_- (0))) = SC |q_+ (0) + q_- (0)|^2.$$

Hence the curve $\mathcal{C}$ is given by:

$$x = E \cos \omega t + F \sin \omega t, \quad y = G \sin \omega t, \quad z = |a_0|^2 - |d_0|^2.$$

Eliminating the parameter $t$, gives

$$G^2 x^2 - 2FGxy + (E^2 + F^2)y^2 = E^2 G^2.$$

The discriminant of this conic is $-4G^2E^2 \leq 0$, so if $GE \neq 0$ the curve $\mathcal{C}$ is an ellipse. The $x, y$–axes coincide with the elliptic axes if and only if $FG = 0$. Now $G = 0$ means $\theta = 0$ or $\theta = \frac{\pi}{2}$ when the curve is a single point, or $q_+ (0) = -q_- (0)$, which is a case with a degenerate polar decomposition. Note that $GE = 0$ also leads to similar degenerate cases. Thus, aside from degenerate cases, alignment of the $x, y$–axes with the elliptic axes is equivalent to $F = 0$ or equivalently, $\Im (q_+ (0)q_- (0)) = 0$, i.e. $q_+ (0)$ and $q_- (0)$ have the same phase modulo $\pi$. Hence, multiplying $\Gamma(0)$ by a phase factor, if necessary, we may assume that $q_+ (0)$ and $q_- (0)$ are real, $q_+ (0) \geq 0$, and that the $x$–axis is the major axis and the $y$–axis is the minor axis. This choice entails that $k^2 \geq C^2 l^2$.

We can relate the present situation to the standard representation of mixed states in $\mathbb{C}^2$ by means of polarization vectors in the ball of radius $\frac{1}{2}$ in $\mathbb{R}^3$, as in [Blum, p.9]. To do this we will reconcile the differing conventions by contracting the unit sphere we have been using to the boundary of the polarization ball, i.e. multiply by $\frac{1}{2}$.

The rays $[\phi_{\pm}] \in \mathbf{P}(\mathbb{C}^2) \approx S^2$ give the pure projective states of the electron which correspond to a pair of antipodal points on the sphere of radius $\frac{1}{2}$. The line joining these antipodal points is called the spin axis of the electron. The spin axis is determined by the unique point $e(t)$ in which it intersects the ellipse $\frac{1}{2}\mathcal{C}$. This point, regarded as a vector from $0$, is the polarization vector, i.e. the mixed state of the electron, which is all that SQM accords to the subsystem $S_1$. The same applies to the proton states, whose spin axis passes through a point $p(t)$ on the ellipse $\frac{1}{2}\mathcal{C}$ antipodal (with respect to the center of the ellipse) to $e(t)$.

We show in Figure [C-11], the polarization ball containing the ellipse which is the trajectory of the density operators represented by the blue (proton) and red (electron) balls.
C.2 Horizontalizing the Spin States

We begin with two lemmas.

**Lemma C.1** If $t \to \chi = \chi(t)$ is any curve of non-zero vectors in a Hilbert space, then $e^{i\tau \chi}$ is horizontal with respect to the canonical connection if, and only if,

$$\dot{\tau} = i \frac{\langle \chi, \dot{\chi} \rangle}{\| \chi \|^2}. \quad (C.20)$$

**Proof.** The condition for horizontality is:

$$0 = \langle e^{i\tau \chi}, \partial_t (e^{i\tau \chi}) \rangle = \langle e^{i\tau \chi}, i\dot{\tau} e^{i\tau \chi} + e^{i\tau \dot{\chi}} \rangle = i\dot{\tau} \| \chi \|^2 + \langle \chi, \dot{\chi} \rangle. \quad (C.21)$$

**Remark.** $\dot{\tau}$ is real if, and only if, $\langle \chi, \dot{\chi} \rangle$ is purely imaginary if, and only if, $\| \chi \| = \text{constant}$.

**Lemma C.2** If $t \to \chi = \chi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$ is a curve of non-zero vectors in the Hilbert space $\mathbb{C}^2$, with $\beta$ real, then $e^{i\tau \frac{\chi}{\| \chi \|}}$ is horizontal with respect to the canonical connection if, and only if,

$$\dot{\tau} = -3 \frac{\alpha \dot{\alpha}}{\| \chi \|^2}. \quad (C.22)$$
Proof. Apply the preceding lemma to $\chi_1 \overset{\text{def}}{=} \frac{\chi}{\|\chi\|}$. Let $L = \|\chi\|$.

\[ i\dot{\tau} = -\langle \chi_1, \dot{\chi}_1 \rangle = -\frac{\hat{\alpha}\hat{\alpha}}{L^2} - \frac{|\alpha|^2\hat{L}}{L^3} + \frac{\beta\hat{L} - \hat{\beta}\hat{L}}{L^2} = -\frac{\hat{\alpha}\hat{\alpha}}{L^2} + \text{real.} \quad (C.23) \]

Using the previous remark, we find that

\[ \dot{\tau} = -\Im \left( \frac{\hat{\alpha}\hat{\alpha}}{L^2} \right) \quad (C.24) \]

We now want to horizontalize $\phi_\pm, \psi_\pm$ using Lemma C.2. It follows from the lemma that the same factor $e^{i\tau_\pm}$ can be used for $\phi_+; \psi_+$ and similarly for $\phi_-, \psi_-$, where

\[ \tau_\pm = -\Im \left( \frac{\hat{\alpha}\hat{\alpha}}{L_\pm} \right), \quad L_\pm = \|\phi_\pm\|^2 = \|\psi_\pm\|^2 \quad (C.25) \]

\[ L_\pm = |\alpha|^2 + |\beta_\pm|^2 = 2\sqrt{\Delta}((d_0^2 - |a_0|^2)) = 2\sqrt{\Delta}((\sqrt{\Delta} + Ck)l). \quad (C.27) \]

\[ \therefore \tau_\pm = -\frac{CS\beta_k\omega}{\sqrt{\Delta}((\sqrt{\Delta} + Ck)l)}. \quad (C.28) \]

\[ \Gamma = q_0^\pm \phi_+ \otimes \psi_0^\pm + q_0^\pm \phi_- \otimes \psi_0^- \quad (C.30) \]

We now calculate the corresponding canonically horizontalized $q_0^\pm$. We have

\[ q_0^\pm = q_0 e^{-2i\tau_\pm} \]

since

\[ \Gamma = q_0^\pm \phi_+ \otimes \psi_0^\pm + q_0^\pm \phi_- \otimes \psi_0^- \quad (C.30) \]

Note that $\arctan(Cl/k \tan\omega t) = -\arg \alpha$. Thus, using Eq. (C.30) and Eq. (C.19),

\[ q_0^\pm = \sqrt{\frac{\beta_\pm}{|\alpha|}} (b\alpha + d) e^{\pm\frac{C^2 l}{2k}\Pi(\omega t; S^2e^2)} = \quad (C.31) \]
\[
\pm \frac{-ie^{-iut}}{2k\sqrt{1-e^2\sin^2\omega t}} \left((k^2 \pm \sqrt{\Delta}) \cos \omega t + i(C^2l^2 \pm \sqrt{\Delta}) \sin \omega t\right) e^{\mp i\frac{C^2l}{k}\Pi(e^2;\omega t|S^2e^2)}. \]
\[
\therefore \arg q_0^\pm = \sigma_\pm \mp \frac{C^2l}{k}\Pi(e^2;\omega t|S^2e^2) - \frac{\omega t}{4},
\]
where
\[
\sigma_\pm = \arctan \left(\frac{(k^2 \pm \sqrt{\Delta}) \tan \omega t}{(C^2l^2 \pm \sqrt{\Delta}) \tan \omega t}\right).
\]

Then for the dynamically horizontalized \(q_H^\pm\) we have
\[
\arg q_H^\pm = \nu + \sigma_\pm \mp \frac{C^2l}{k}\Pi(e^2;\omega t|S^2e^2) - \frac{\omega t}{2},
\]
where
\[
\nu = \frac{C^2}{\sqrt{1-S^2e^2}} \arctan(\sqrt{1-S^2e^2 tan \omega t}).
\]

In terms of the angle \(\eta\) between the electron and the spin axis (cf. below), we have
\[
\nu = \frac{C^2}{\sqrt{1-S^2e^2}} \eta.
\]

We have given the details in the case where \(\varphi = 0\). The calculations in the general case are more complicated, but lead to similar results. The polarizing vectors of the electron and the proton again move antipodally on elliptic trajectories. The phases of the \(q_H^\pm\) in the general case are also given by elliptic integrals. The main difference is that the argument \(\omega t\) of the elliptic function is replaced by \(\omega(t-t_0)\) for some \(t_0\).

### C.2.1 Hamiltonian in the polar basis.

In Section 2.4 we described the evolution of a composite system in the polar bundle. This led us to an autonomous system of ODE’s satisfied by the canonically horizontal spectral states \(\phi_k(t), \psi_k(t)\) together with \(q(t)\). These equations required us to give the Hamiltonian as a matrix \((H_{jk,mn})\) in the polar basis, as well as the related matrices \((\beta_{ab})\). In this section and the next we calculate these two matrices for the two spin-\(\frac{1}{2}\) systems, allowing one to write the system Eqs. 2.25, 2.26, 2.27 explicitly.

We first compute the change of basis matrix \(Q\) from the standard basis to the basis
\[
\left(\frac{\phi_0^i \otimes \psi_0^j}{\|\phi_0^i \otimes \psi_0^j\|}\right)_{i,j=\pm}.
\]
We have
\[
\phi_0^0 = \pm e^{ir^\pm} \left(\begin{array}{c}
\alpha \\
\beta_\pm
\end{array}\right), \quad \psi_0^0 = e^{ir^\pm} \left(\begin{array}{c}
-\alpha \\
-\beta_\pm
\end{array}\right)
\]
\[
\|\phi_0^0\|^2 = \|\psi_0^0\|^2 = |\alpha|^2 + \beta_\pm^2 = \beta_\pm(\beta_+ - \beta_-).
\]
\[ \frac{\phi_+^0 \otimes \psi_+^0}{\| \phi_+^0 \otimes \psi_+^0 \|} = \frac{e^{2i\tau_+}}{\beta_+ (\beta_+ - \beta_-)} \left( \frac{\alpha}{\beta_+} \right) \otimes \left( \frac{\alpha}{-\beta_-} \right) = \frac{e^{2i\tau_+}}{\beta_+ - \beta_-} \left( \frac{\alpha^2}{\beta_+} \right) \frac{\beta_+ - \beta_-}{\alpha} \right). \] (C.33)

\[ \frac{\phi_+^0 \otimes \psi_-^0}{\| \phi_+^0 \otimes \psi_-^0 \|} = \frac{e^{i(\tau_+ - \tau_-)}}{\alpha (\beta_+ - \beta_-)} \left( \frac{\alpha}{\beta_+} \right) \otimes \left( \frac{\alpha}{-\beta_-} \right) = \frac{e^{i(\tau_+ - \tau_-)}}{\beta_+ - \beta_-} \left( \frac{\alpha^2}{\beta_+} \right) \frac{\beta_+ - \beta_-}{\alpha} \right). \] (C.34)

\[ \frac{\phi_-^0 \otimes \psi_+^0}{\| \phi_-^0 \otimes \psi_+^0 \|} = \frac{e^{i(\tau_+ + \tau_-)}}{\beta_+ - \beta_-} \left( \frac{\alpha^2}{\beta_+} \right) \frac{\beta_+ - \beta_-}{\alpha} \right), \quad \frac{\phi_-^0 \otimes \psi_-^0}{\| \phi_-^0 \otimes \psi_-^0 \|} = \frac{e^{2i\tau_-}}{\beta_+ - \beta_-} \left( \frac{\alpha^2}{\beta_+} \right) \frac{\beta_+ - \beta_-}{\alpha} \right). \] (C.35)

We find:

\[ Q = \frac{1}{\beta_+ - \beta_-} \begin{pmatrix}
\frac{\alpha^2}{\beta_+} e^{2i\tau_+} & -\frac{\alpha^2}{\beta_+} e^{i(\tau_+ + \tau_-)} & -\frac{\alpha^2}{\beta_+} e^{i(\tau_+ + \tau_-)} & \frac{\alpha^2}{\beta_+} e^{2i\tau_-} \\
-\alpha e^{2i\tau_+} & -\alpha e^{i(\tau_+ + \tau_-)} & \alpha e^{i(\tau_+ + \tau_-)} & -\alpha e^{2i\tau_-} \\
\alpha e^{2i\tau_+} & \alpha e^{i(\tau_+ + \tau_-)} & -\alpha e^{i(\tau_+ + \tau_-)} & \alpha e^{2i\tau_-} \\
-\beta_+ e^{2i\tau_+} & -\beta_+ e^{i(\tau_+ + \tau_-)} & \beta_+ e^{i(\tau_+ + \tau_-)} & -\beta_+ e^{2i\tau_-}
\end{pmatrix}. \] (C.36)

Set \( B = -\frac{\beta_+ + \beta_-}{2|\alpha|} \) and \( \tau = \tau_+ - \tau_- = \frac{c q}{\hbar} \Pi(e^2; \omega t |S|^2 e^2) \).

We now transform

\[ H = \mu \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]

from the standard basis to the new basis Eq. (C.32) (see Eq. (2.22)):

\[ (H_{jk,nn}) = Q^* H Q = \frac{\mu}{B^2 + 1} \begin{pmatrix}
B^2 - 1 & 2B e^{-2i\tau} & -2B e^{-2i\tau} & -2e^{-4i\tau} \\
2B e^{2i\tau} & 1 - B^2 & -2B^2 & 2B e^{-2i\tau} \\
-2B e^{2i\tau} & -2B^2 & 1 - B^2 & -2B e^{-2i\tau} \\
-2e^{4i\tau} & 2B e^{2i\tau} & -2B e^{2i\tau} & B^2 - 1
\end{pmatrix}. \] (C.37)

We have that \( B = \tan \eta \) where \( \eta \) is the angle between the spin axis of the electron and the \( z \)-axis, so that \( 2\eta \) is the angle between the spin axes of the electron and the proton. We may then write:

\[ Q^* H Q = \mu \begin{pmatrix}
\cos 2\eta & \sin 2\eta e^{-2i\tau} & -\sin 2\eta e^{-2i\tau} & -\sin 2\eta e^{-2i\tau} \left( -1 + \cos 2\eta \right) e^{-4i\tau} \\
-\sin 2\eta e^{2i\tau} & -\cos 2\eta & 1 + \cos 2\eta & \sin 2\eta e^{-2i\tau} \\
1 + \cos 2\eta & -\cos 2\eta & -\sin 2\eta & -\sin 2\eta e^{-2i\tau} \\
(-1 + \cos 2\eta) e^{4i\tau} & \sin 2\eta e^{2i\tau} & -\sin 2\eta e^{2i\tau} & \cos 2\eta
\end{pmatrix}. \] (C.38)
C.2.2 Computation of the matrix \((\beta_{ab})\).

We use the equation \(\dot{\rho} = (\beta_{ab})\) in the basis \(\frac{\phi^0_j}{\|\phi^0_j\|}, j = \pm\).

Let \(R\) be the change of basis matrix from the standard basis to

\[ R = \frac{1}{\sqrt{\beta_+ - \beta_-}} \left( \frac{\alpha}{\|\phi^0_+\|} \sqrt{-\beta_- e^{i\tau_+}} \frac{-\alpha}{\|\phi^0_-\|} \sqrt{\beta_+ e^{i\tau_-}} \right) \]  

(C.39)

In the new basis

\[ \dot{\rho} = R^* \partial_t \left( |\rho_+|^2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} R^* \right) R = (\beta_{ab}). \]  

(C.40)

\[ (\beta_{ab}) = \frac{1}{2} R^* \begin{pmatrix} 0 & \dot{\alpha} \\ \alpha & 0 \end{pmatrix} R. \]  

(C.41)

\[ (\beta_{ab}) = \frac{\omega S^2 \|q\|^2}{\sqrt{\Delta}} \begin{pmatrix} -k^2 e^2 \cos \omega t \sin \omega t & -e^{2i\tau} Ckl(k^2 e^2 \cos \omega t \sin \omega t + i\sqrt{\Delta}) \\ -e^{-2i\tau} Ckl(k^2 e^2 \cos \omega t \sin \omega t - i\sqrt{\Delta}) & k^2 e^2 \cos \omega t \sin \omega t \end{pmatrix}. \]  

(C.42)

C.3 Glossary

\( C = \cos \theta \)
\( S = \sin \theta \)
\( k = q_+(0) - q_-(0) \)
\( l = q_+(0) + q_-(0) \)
\( a = \frac{1}{2}(Cl + k)e^{-i\mu t} \)
\( d = \frac{1}{2}(Cl - k)e^{-i\mu t} \)
\( b = -\frac{1}{2}Sle^{3i\mu t} \)
\( \Delta = l^2 (k^2 + S^2(C^2 l^2 - k^2) \sin^2 \omega t) \)
\( r^2 = |\phi_\pm|^2 = \frac{1 + \sqrt{\Delta}}{2} \)
\( \|\phi_\pm\|^2 = \|\psi_\pm\|^2 = 2\sqrt{\Delta}(\sqrt{\Delta} \mp Ckl) \)
\( \alpha = Sl(k \cos \omega t - iCl \sin \omega t) \)
\( \beta_\pm = -Ckl \pm \sqrt{\Delta} \)
\( e = \frac{\sqrt{k^2 - C^2 l^2}}{k} = \text{eccentricity of ellipse} \)
\( \omega = 4\mu \)
\( |\alpha|^2 = -\beta_+ \beta_- \)
D  Categorical Naturality

In this appendix, we briefly consider a few of the basic concepts of category theory. Our main use of them will be to make the notion of naturality precise.

D.1  Some concepts and examples from category theory.

The basic concept of category theory can be axiomatically described.

Definition. A category $C$ is a collection of primitive entities, called “arrows”, with a partially defined associative multiplication.

This means that when the arrows $ab$ and $bc$ are defined so are $(ab)c$ and $a(bc)$ and they are equal. It is also required that for every arrow $a$ there are right and left identities: $e_L,a = ae_R = a$. We can then think of these identities as being or representing objects.

For example, we could take $C$ to be the collection of continuous maps between topological spaces; this category might be denoted $TOP$. With the same “objects”, but using arrows corresponding to homotopy classes of continuous maps we get, say $HTOP$. Here, the arrows are not merely maps. Other examples are homomorphisms between groups, defining $GROUP$ and isometric maps between metric spaces.

Definition. A functor $F$ from $C_1$ to $C_2$ is a homomorphic function $F : C_1 \hookrightarrow C_2$ in the sense that $F(ab) = F(a)F(b)$, when $ab$ is defined.

The maps which take topological spaces to their (singular) homology groups define functors from $TOP \hookrightarrow GROUP$ and even $HTOP \hookrightarrow GROUP$.

A forgetful functor is one for which some of the structure and attendant restrictions on morphisms are omitted. An example which makes the idea clear is the functor $FDP : DIFF \hookrightarrow TOP$ which assigns to differentiable manifolds the underlying topological space or in terms of the arrow representatives, regards each differentiable map as just a continuous one. Usually, such functors have no inverse. For instance, $FDP$ does not: there is no natural way to put a differentiable structure on a topological space.

We are interested in some cases where the forgetful functor has an inverse, i.e. the additional structure is canonically definable.

Definition. A manifest functor is a functor $E : C_1 \hookrightarrow C_2$ which has a forgetful functor $F : C_2 \hookrightarrow C_1$ as an inverse.

In other words, $E$ adds structure in a natural way, so that every arrow in $C_1$ preserves the additional structure. $E$ makes manifest the hidden structure already possessed by the objects of $C_1$.

A typical example of a manifest functor can be obtained from the fact (see, e.g. [37, III.7]) that every non-singular cubic surface $S$ in $\mathbb{P}(\mathbb{C}^4)$ contains exactly 27 (complex) lines $L$. Let $CS$ denote the category of such $S$ and $CSL$ the category of pairs $(S,L)$ where $L$ is the set of lines $L \subset S$. The morphisms in each category are those induced by automorphisms of $\mathbb{P}(\mathbb{C}^4)$. Then the forgetful functor from $CSL$ to $CS$ has an inverse, a manifest functor.
D.2 The naturality of certain constructions.

The raison d’être of our terminology is to make the statement that the functor $P$ which assigns to a right toroid, a Pythagorean partition is manifest. This is a precise way of saying that such partitions exist, are natural, and unique.

As another usage of this terminology, we now justify our assertion, made at the end of Section 1.1 about naturally assigning projective states to density operators.

Let $MIX$ denote the category of convex spaces arising as (mixed) state spaces of algebras of the form $B(\mathcal{H})$, bounded operators on Hilbert spaces of some specified range of dimensions. For example, we could allow all Hilbert spaces or just those of dimension $n$. The morphisms are those affine maps induced by unital homomorphisms of algebras. Let $MIXR$ denote the category of triples $(S,e,P)$ where $S \in MIX$, $P$ is the set of pure states in $S$, i.e. the extreme points of $S$, and $e : S^{\text{reg}} \to P$ is any map of the regular mixed states (those with distinct positive eigenvalues) to the pure states.

**Remark.** There exist manifest functors $E : MIX \rightsquigarrow MIXR$.

For instance, $E(S) := (S,e,P)$, where $e(\rho)$ is the eigenprojection corresponding to the smallest eigenvalue of $M$ is such a functor.

**Theorem D.1** For any manifest functor $E : MIX \rightsquigarrow MIXR$, and for any $\rho \in S \in MIX$, we have for the map $e$ of the triple $(S,e,P) = E(S)$, that $e(\rho)$ is an eigenprojection of $\rho$.

**Proof.** The isomorphisms of $MIX$ correspond in the usual way to conjugation by unitary and conjugate-unitary operators. Suppose now that the functor $E$ assigns to $S$ the triple $(S,e,P)$ and that $e(\rho) = \gamma$ for some regular state $\rho \in S$. Let $U$ be any unitary commuting with $\rho$. Then the isomorphism $\iota : MIX \to MIX$ defined by $\iota(\sigma) = U^*\sigma U$ fixes $\rho$. Thus $E(\iota)$ must fix $\gamma := e(\rho)$. But $E(\iota)((S,e,P)) = (U^*SU,U^*eU,U^*PU)$. It follows that $U$ commutes with $\gamma$. Since this holds for every unitary commuting with $\rho$, it follows that $\gamma$ is in the double commutant of the algebra $X$ generated by $\rho$. By the spectral theorem, $X$ is generated by the eigenprojections $P_j$ of the self-adjoint operator $\rho$. Since $\gamma$ is a one-dimensional projection, it must be a sub-projection of one of the $P_j$. Thus $\gamma$ is an eigenprojection of $\rho$. ■

E Two Identical Systems

There are situations where there is (something like) a subsystem $S_1$ of $S$, but no natural complementary subsystem. The most prominent of such situations is that of identical particles. In order to show that even such systems pose no insuperable obstacle to IQM, we adumbrate the simplest of such cases: two identical particles (systems) where the one particle state has space $H_1$. The state space of the two particle system can then be identified with either the subspace of $\mathcal{H} = H_1 \otimes H_1$ consisting of the symmetric tensors $\mathcal{H}_+ := H_1 \otimes^+ H_1$ or the anti-symmetric tensors $\mathcal{H}_- := H_1 \otimes^- H_1$. Here

$$\mathcal{H}_\pm = \{ \Gamma \in \mathcal{H} \mid \sigma(\Gamma) = \pm \Gamma \}, \quad (E.1)$$

where $\sigma$ is the $\mathbb{C}$–linear map determined by $\sigma(\phi \otimes \psi) = \psi \otimes \phi$ for all $\phi, \psi \in H_1$. 73
Now we have, as usual, some polar decomposition
\[ \Gamma = \sum_k q_k \phi_k \otimes \psi_k. \] (E.2)

Applying the linear map \( \sigma \), we find that \( \Gamma \in H_{\pm} \) if, and only if, \( \sigma(\Gamma) = \pm \Gamma \). This is equivalent, for regular \( \Gamma \), to having for all \( k \) that there exists a \( j \) and a \( \zeta_j \in S^1 \) so that \( \phi_k = \zeta_j \psi_j \). In the regular boson case (distinct positive \( |q_k| \)), using the uniqueness property of polar decompositions, it follows that there exists a polar decomposition of the form
\[ \Gamma = \sum_k q_k \phi_k \otimes \phi_k. \] (E.3)

Thus we can proceed as before in what amounts to a specialized polar decomposition. We can speak of an \( S^1 \)-spectral state \( \phi_k \) of \( \mathcal{S} \) but not of the state of the first particle.

In the fermion case, using Eq. (E.1), the pairing \( \phi_k, \phi_j \), yields a partition of the indices. By re-indexing using positive integers we can write
\[ \Gamma = \sum_k q_k (\phi_{2k} \otimes \phi_{2k-1} - \phi_{2k-1} \otimes \phi_{2k}) =: \sum_k q_k \Gamma_k. \] (E.4)

We are in a situation of permanent degeneracy. No such vector has a regular polar decomposition. The notion of regularity must be redefined to cover this case.

We proceed no further here. We have shown enough to define the \( q_k \) and hence the appropriate polar bundle \( \mathcal{P} \). Moreover, the natural connection \( A^H \) can be used to define the evolution within \( \mathcal{P} \) when the Hamiltonian has the symmetry properties mandated by SQM, taking into account the assumed indistinguishability of the systems. Now we can no longer speak of a spectral vector state of \( \mathcal{S}_1 \). We can speak of an \( S^1 \)-spectral state \( \{\phi_{2k}, \phi_{2k-1}\} \) of \( \mathcal{S} \) meaning that one of the two particles is in spectral state \( \phi_{2k} \), and the other is in \( \phi_{2k-1} \). This assumes the polar decompositions are regular, which means that the \( |q_k| \) are distinct. In the anti-symmetric case, we must take extra care not to think we are asserting that the state of \( \mathcal{S} \) is (collapsed to) \( \Gamma_k \).

We have thus shown that although some circumlocution is required, especially in the anti-symmetric case, IQM can incorporate identical particles.

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