A Dyson realization and a Holstein-Primakoff realization for the quantum superalgebra $U_q[gl(n/m)]$

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Abstract. The Holstein-Primakoff and the Dyson realizations of the Lie superalgebra $gl(n/m)$ are generalized to the class of the quantum superalgebras $U_q[gl(n/m)]$ for any $n$ and $m$. It is shown how the elements of $U_q[gl(n/m)]$ can be expressed via $n - 1$ pairs of Bose creation and annihilation operators and $m$ pairs of Fermi creation and annihilation operators.

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Recently an analogue of the Dyson (D) and of the Holstein-Primakoff (HP) realization for the superalgebras $gl(n/m)$ of any rank was given [1]. In the present paper the results are extended to the quantum superalgebras $U_q[gl(n/m)]$. The elements of $U_q[gl(n/m)]$ are expressed via $n-1$ pairs of Bose creation and annihilation operators (CAOs) and of $m$ pairs of Fermi CAOs. In the case $m=0$ the results reduce to those announced in [2], namely to D and HP realizations of $U_q[gl(n)]$ in terms of $n-1$ pairs of only Bose operators.

Initially the D and the HP realizations were given for $sl(2)$ [3, 4]. The generalization for $gl(n)$ is due to Okubo [5]. The “quantum case” was worked out also first for $U_q[sl(2)]$ [6] and $U_q[sl(3)]$ [7]. Very recently it was extended to $U_q[sl(n)]$ [2]. To the best of our knowledge analogues of D and of HP realizations for quantum superalgebras have not been published in the literature so far. The available realizations are of Jordan-Schwinger type, requiring $n$ pairs of Bose CAOs and $m$ pairs of Fermi CAOs in case of $U_q[gl(n/m)]$ [8].

The motivation in the present work stems from the various applications of the Holstein-Primakoff and of the Dyson realizations in theoretical physics. Beginning with [2] and [3] the HP and D realizations were constantly used in condensed matter physics. Some early applications can be found in the book of Kittel [9] (more recent results are contained in [10]). For applications in nuclear physics see [11, 12] and the references therein, but there are, certainly, several other publications. Once the $q$-analogues of D and of HP realizations for $U_q[sl(2)]$ and $U_q[sl(3)]$ were established, they found also immediate applications [13-18]. One can expect therefore that the generalization of the results to an arbitrary rank $U_q[gl(n/m)]$ superalgebra may prove useful too.

To begin with we recall the definition of $U_q[gl(n/m)]$ in the sense of Drinfeld [19], keeping close to the notation in [20]. Let $\mathbb{C}[[h]]$ be the complex algebra of formal power series in the indeterminate $h$, $q = e^{h/2} \in \mathbb{C}[[h]]$. Then $U_q[gl(n/m)]$ is a Hopf algebra, which is a topologically free $\mathbb{C}[[h]]$ module (complete in the $h$–adic topology), with generators $h_j$, $(j = 1, 2, \ldots, r \equiv n + m)$ and $e_i, f_i$ $(i = 1, 2, \ldots, r-1)$ subject to the following relations (unless stated otherwise, the indices below run over all possible values):

The Cartan-Kac relations:

\[
[h_i, e_j] = (\delta_{ij} - \delta_{i,j+1})e_j;
\]
\[
[h_i, f_j] = (-\delta_{ij} + \delta_{i,j+1})f_j;
\]
\[
e_i f_j - f_j e_i = 0, \quad \text{if} \quad i \neq j;
\]
\[
e_i f_i - f_i e_i = \frac{q^{h_i - h_{i+1}} - q^{h_i + h_{i+1}}}{q - q^{-1}}, \quad \text{if} \quad i \neq n;
\]
\[
e_n f_n + f_n e_n = \frac{q^{h_n + h_{n+1}} - q^{h_n - h_{n+1}}}{q - q^{-1}};
\]
The Serre relations for the $e_i$ ($e$-Serre relations):

$$e_i e_j = e_j e_i, \quad \text{if} \quad |i - j| \neq 1; \quad e_i^2 = 0;$$

$$e_i^2 e_{i+1} - (q + q^{-1})e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0,$$

for $i \in \{1, \ldots, n - 1\} \cup \{n + 1, \ldots, n + m - 2\};$ (6)

$$e_{i+1}^2 e_i - (q + q^{-1})e_{i+1} e_i e_{i+1} + e_{i+2}^2 e_i = 0,$$

for $i \in \{1, \ldots, n - 2\} \cup \{n, \ldots, n + m - 2\};$ (7)

$$e_n e_{n-1} e_n e_{n+1} + e_{n-1} e_n e_{n+1} e_n + e_n e_{n+1} e_n e_{n-1}
+ e_{n+1} e_{n-1} e_n - (q + q^{-1})e_n e_{n-1} e_n e_n = 0;$$

for $i \in \{1, \ldots, n - 1\} \cup \{n + 1, \ldots, n + m - 2\};$ (8)

The relations obtained from (6)-(9) by replacing every $e_i$ by $f_i$ ($f$-Serre relations).

Let

$$\theta_i = \begin{cases} 0, & \text{if } i < n; \\ 1, & \text{if } i \geq n. \end{cases}$$

Then

$$\deg(h_i) = 0, \quad \deg(e_j) = \deg(f_j) = \theta_j - 1 + \theta_j,$$

i.e. the generators $e_n$ and $f_n$ are odd and all other generators are even.

We do not write the other Hopf algebra maps ($\Delta, \varepsilon, S$) (see [20]), since we will not use them. They are certainly also a part of the definition.

The Dyson and the Holstein-Primakoff realizations are different embeddings of $U_q[gl(n/m)]$ into the algebra $W(n-1/m)$ of $n-1$ pairs of Bose CAOs and $m$ pairs of Fermi CAOs. The precise definition of $W(n-1/m)$ is the following. Let $A_1^\pm, \ldots, A_{n+m-1}^\pm$ be $\mathbb{Z}_2$-graded indeterminates:

$$\deg(A_i^\pm) = \theta_i.$$ (12)

Then $W(n-1/m)$ is a topologically free $C[[h]]$ module and an associative unital superalgebra with generators $A_1^\pm, \ldots, A_{n+m-1}^\pm$ subject to the relations

$$[A_i^-, A_j^+] = \delta_{ij}, \quad [A_i^+, A_j^+] = [A_i^-, A_j^-] = 0.$$ (13)

Here and throughout

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)}yx, \quad [x, y]_q = xy - (-1)^{\deg(x)\deg(y)}qyx$$ (14)

for any two homogeneous elements $x$ and $y$. With respect to the supercommutator $[x, y] W(n-1/m)$ is also a Lie superalgebra.
From (13) one concludes that $A^+_i, \ldots, A^+_{n-1}$ are Bose CAOs, which are even variables; $A^-_n, \ldots, A^-_{n+m-1}$ are Fermi CAOs, which are odd. The Bose operators commute with the Fermi operators.

In the physical applications it is often more convenient to consider $h$ and $q$ as complex numbers, $h, q \in \mathbb{C}$. Then all our considerations remain true provided $q$ is not a root of 1. The replacement of $q \in \mathbb{C}[[h]]$ with a number corresponds to a factorization of $U_q[gl(n/m)]$ and $W(n-1/m)$ with respect to the ideals generated by the relation $q = \text{number}$. The factor-algebras $U_q[gl(n/m)]$ and $W(n-1/m)$ are complex associative algebras. However the completion in the $h$-adic topology leaves a relevant trace: after the factorization the elements of $U_q[gl(n/m)]$ and of $W(n-1/m)$ are not simply polynomials of their generators. In particular the functions of the CAOs, which appear in the D and in the HP realizations (see below) are well defined as elements from $W(n-1/m)$.

Now we are ready to state our main results. Let

$$\tilde{q} = q^{-1}, \quad [x] = \frac{q^x - \bar{q}^x}{q - \bar{q}}, \quad N_i = A^+_i A^-_i, \quad N = N_1 + \ldots + N_{n+m-1}. \quad (15)$$

**Proposition 1 (Dyson realization).** The linear map $\varphi : U_q[gl(n/m)] \to W(n-1/m)$, defined on the generators as

$$\varphi(h_1) = p - N, \quad \varphi(h_i) = N_{i-1}, \quad i = 2, \ldots, n + m \equiv r,$$

$$\varphi(e_1) = \frac{[N_1 + 1]}{N_1 + 1} (p - N) A^+_1, \quad \varphi(e_i) = \frac{[N_1 + 1]}{N_i + 1} A^+_i A^-_{i-1}, \quad i = 2, \ldots, n - 1,$$

$$\varphi(e_i) = A^+_i A^-_{i-1}, \quad i = n, \ldots, n + m - 1, \quad (16)$$

is a homomorphism of $U_q[gl(n/m)]$ into $W(n-1/m)$ for any $p \in \mathbb{C}$.

The proof is straightforward. One has to verify that Eqs. (1)-(9) with $\varphi(h_i), \varphi(e_i), \varphi(f_i)$ substituted for $h_i, e_i$ and $f_i$, respectively, hold. In the intermediate computations the relations

$$[N_i, A^+_i] = \pm A^+_i, \quad [N, A^+_i A^-_j] = 0. \quad (17)$$

$$F(N_i) A^+_i = A^+_i F(N_i + 1), \quad F(N_i) A^-_i = A^-_i F(N_i - 1), \quad (18)$$

are repeatedly used. The verification of Eqs. (4), (5) and (7)-(9) is based also on the identity

$$[x + 1] - (q + \bar{q})[x] + [x - 1] = 0. \quad (19)$$

are repeatedly used. The verification of Eqs. (4), (5) and (7)-(9) is based also on the identity

$$q^{N_n} = 1 - N_n + q N_n, \quad \bar{q}^{N_n} = 1 - N_n + \bar{q} N_n. \quad (20)$$

Similar as for $gl(n-1/m)$ [1], the Dyson realization defines an infinite-dimensional representation of $U_q[gl(n/m)]$ (for $n > 0$) in the Fock space $\mathcal{F}(n-1/m)$ with orthonormed basis

$$|l\rangle \equiv |l_1, \ldots, l_{r-1}\rangle = \frac{(A^+_1)^{l_1} \ldots (A^+_r)^{l_{r-1}}}{\sqrt{l_1! \ldots l_{r-1}!}} |0\rangle, \quad (20)$$

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where \( l_1, \ldots, l_{n-1} \in \mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}; \ l_n, \ldots, l_{r-1} \in \mathbb{Z}_2 \equiv \{0, 1\} \).

If \( p \) is a positive integer, \( p \in \mathbb{N} \), the representation is indecomposable: the subspace
\[
\mathcal{F}_1(p; n-1, m) = \text{lin.env.}\{ |l_1, \ldots, l_{r-1}| \mid l_1 + \ldots + l_{r-1} > p \}
\]  
(21)
is an invariant subspace, whereas its orthogonal compliment
\[
\mathcal{F}_0(p; n-1, m) = \text{lin.env.}\{ |l_1, \ldots, l_{r-1}| \mid l_1 + \ldots + l_{r-1} \leq p \}
\]  
(22)
is not an invariant subspace. If \( p \notin \mathbb{N} \), the representation is irreducible. In all cases however, and this is the disadvantage of the D realization, the representation of \( U_q[gl(n/m)] \) in \( \mathcal{F}(n-1, m) \) is not unitarizable with respect to the antilinear anti-involution \( \omega : U_q[gl(n/m)] \to U_q[gl(n/m)] \), defined on the generators as
\[
\omega(h_i) = h_i, \quad \omega(e_i) = f_i.
\]  
(23)

In order to turn \( \mathcal{F}_0(p; n-1, m) \) into an unitarizable \( U_q[gl(n/m)] \) module we pass to introduce the HP realization. To this end set
\[
\langle N_i + c \rangle = \left( \frac{[N_i + c]}{N_i + c} \right)^{\frac{1}{2}}.
\]  
(24)

**Proposition 2 (Holstein-Primakoff realization).** The linear map \( \pi : U_q[gl(n/m)] \to W(n-1/m) \), defined on the generators as:
\[
\pi(h_1) = p - N, \quad \pi(h_i) = N_{i-1}, \quad i = 2, \ldots, n + m \equiv r,
\]
\[
\pi(e_1) = \sqrt{[p - N]} \langle N_i + 1 \rangle A_i^-, \quad \pi(f_1) = \sqrt{[p - N + 1]} \langle N_i \rangle A_i^+,
\]
\[
\pi(e_i) = \langle N_{i-1} \rangle \langle N_i + 1 \rangle A_{i-1}^- A_i^-, \quad i = 2, \ldots, r - 1,
\]
\[
\pi(f_i) = \langle N_{i-1} + 1 \rangle \langle N_i \rangle A_i^+ A_{i-1}^-, \quad i = 2, \ldots, r - 1
\]  
(25)
is a homomorphism of \( U_q[gl(n/m)] \) into \( W(n-1/m) \). If \( p \in \mathbb{N} \), then \( \mathcal{F}_0(p; n-1, m) \) and \( \mathcal{F}_1(p; n-1, m) \) are invariant subspaces; \( \mathcal{F}_0(p; n) \) carries a finite-dimensional irreducible representation; it is unitarizable with respect to the anti-involution (23) and the metric defined with the orthonormed basis (20), provided \( h \in \mathbb{R} \).

The representations, corresponding to different \( p \in \mathbb{N} \) are inequivalent.

We scip the proof. The circumstance that \( \mathcal{F}(n-1, m) \) is a direct sum of its invariant subspaces \( \mathcal{F}_0(p; n-1, m) \) and \( \mathcal{F}_1(p; n-1, m) \) is due to the the factor \( \sqrt{[p - N]} \) in \( \pi(e_1) \) and \( \sqrt{[p - N + 1]} \) in \( \pi(f_1) \). If \( h \in \mathbb{R} \), then \( (, ) \) denotes the scalar product
\[
(\pi(h_i)|l), (\pi(h_i)|l') = (|l|, \pi(h_i)|l') \quad (\pi(e_i)|l), (\pi(e_i)|l') = (|l|, \pi(f_i)|l')
\]
for all \( |l|, |l'| \in \mathcal{F}_0(p; n-1, m) \); \( i = 1, \ldots, r - 1 \). Hence the representation of \( U_q[gl(n/m)] \) in \( \mathcal{F}_0(p; n-1, m) \) is unitarizable.
Let us say a few words about the place of the Fock representations among all known representations. Any highest weight finite-dimensional irreps \( \psi \) of \( gl(n/m) \) or \( U_q[gl(n/m)] \) is labeled by its signature \( \{m\} \equiv \{m_1, m_2, \ldots, m_r\} \), where each \( m_i \) is determined from \( \psi(h_i)x_0 = m_ix_0 \). Here \( x_0 \) is the highest weight vector. So far explicit expressions for all (finite-dimensional) irreps are available only for \( gl(n/m) \) with \( m = 1 \) [21, 22]. Each such representation can be deformed also to an irreps of \( U_q[gl(n/1)] \) [23].

In case of \( gl(n/m) \) or \( U_q[gl(n/m)] \) with \( m \neq 1 \) explicit constructions were carried out for the so called essentially typical representations [24, 20]. A representation is essentially typical, if

\[
\{l_1, l_2, \ldots, l_n\} \cap \{l_{n+1}, l_{n+1} + 1, l_{n+1} + 2, \ldots, l_r\} = \emptyset,
\]

where \( l_i = m_i - i + n + 1 \) for \( 1 \leq i \leq n \) and \( l_j = -m_j + j - n \) for \( n + 1 \leq j \leq r \).

In case of \( \mathcal{F}_0(p; n - 1, m) \) the highest weight vector is the vacuum. Then \( m_i = p\delta_{1i}, \; i = 1, \ldots, r \) and therefore (26) is not fulfilled. Hence the Fock space representations of \( U_q[gl(n/m)] \) in \( \mathcal{F}_0(p; n - 1, m), \; p \in \mathbb{N} \) describe finite-dimensional irreps in addition to those studied in [20]. As mentioned already, if \( q \) is taken to be a number, it should not be a root of 1.

In conclusion we note that the operators [25]

\[
\hat{A}_i^- = (N_i + 1)A_i^- , \quad \hat{A}_i^+ = (N_i)A_i^+ , \quad \hat{N}_i = N_i , \quad i = 1, \ldots, n + m - 1
\]

satisfy the relations

\[
[\hat{A}_i^-, \hat{A}_j^+] = \delta_{ij}q^{-\hat{N}_i} , \quad [\hat{N}_i, \hat{A}_j^\pm] = \pm \delta_{ij} \hat{A}_j^\pm , \quad [\hat{A}_i^\pm, \hat{A}_k^\pm] = [\hat{N}_i, \hat{N}_k] = 0. \quad i \neq k.
\]

Therefore \( \hat{A}_i^\pm, \; i = 1, \ldots, n - 1 \), give a representation of the algebra of the deformed Bose operators [26-28], whereas \( \hat{A}_i^\pm, \; i = n, \ldots, n + m - 1 \), yield a representation of the deformed Fermi operators [8]. In terms of the deformed operators Eqs. (25) read:

\[
\pi(h_1) = p - \hat{N}, \quad \pi(e_1) = \sqrt{|p - \hat{N}|} \hat{A}_1^- , \quad \pi(f_1) = \sqrt{|p - \hat{N} + 1|} \hat{A}_1^+, \quad (29a)
\]

\[
\pi(h_i) = \hat{N}_{i-1}, \quad \pi(e_i) = \hat{A}_{i-1}^- \hat{A}_i^- , \quad \pi(f_i) = \hat{A}_i^+ \hat{A}_{i-1}^- , \quad i \neq 1. \quad (29b)
\]

The equations (29) could be called a \( q \)-deformed analogue of the Holstein-Primakoff realization for \( gl(n/m) \) [1], whereas only Eqs. (29b) correspond to a \( q \)-deformed Jordan-Schwinger realization of \( U_q[gl(n - 1/m)] \) [.

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