On a colored Turán problem of Diwan and Mubayi

Ander Lamaison ∗ Alp Muyesser† Michael Tait‡

Abstract

Suppose that $R$ (red) and $B$ (blue) are two graphs on the same vertex set of size $n$, and $H$ is some graph with a red-blue coloring of its edges. How large can $R$ and $B$ be if $R \cup B$ does not contain a copy of $H$? Call the largest such integer $\text{mex}(n, H)$. This problem was introduced by Diwan and Mubayi, who conjectured that (except for a few specific exceptions) when $H$ is a complete graph on $k + 1$ vertices with any coloring of its edges $\text{mex}(n, H) = \text{ex}(n, K_{k+1})$. This conjecture generalizes Turán’s theorem.

Diwan and Mubayi also asked for an analogue of Erdős-Stone-Simonovits theorem in this context. We prove the following asymptotic characterization of the extremal threshold in terms of the chromatic number $\chi(H)$ and the reduced maximum matching number $\mathcal{M}(H)$ of $H$.

$$\text{mex}(n, H) = \left(1 - \frac{1}{2(\chi(H) - 1)} - \Omega\left(\frac{\mathcal{M}(H)}{\chi(H)^2}\right)\right) \frac{n^2}{2}.$$

$\mathcal{M}(H)$ is, among the set of proper $\chi(H)$-colorings of $H$, the largest set of disjoint pairs of color classes where each pair is connected by edges of just a single color. The result is also proved for more than 2 colors and is tight up to the implied constant factor.

We also study $\text{mex}(n, H)$ when $H$ is a cycle with a red-blue coloring of its edges, and we show that $\text{mex}(n, H) \lesssim \frac{1}{2}\binom{n}{2}$, which is tight.

1 Introduction

Let $G_1, \ldots, G_r$ be not necessarily distinct graphs on the same vertex set, and let the edge set of $G_i$ be colored with color $i$. By $\bigcup_{i \in [r]} G_i$, we denote the $r$-edge colored multigraph formed by taking the union of the edge sets $E(G_i)$.

∗Freie Universität, Institut für Mathematik and Berlin Mathematical School, email: lamaison@zedat.fu-berlin.de.
†Freie Universität, Institut für Mathematik and Berlin Mathematical School, email: alp.muyesser@fu-berlin.de.
‡Villanova University Department of Mathematics and Statistics, email: michael.tait@villanova.edu. Research is partially supported by National Science Foundation grant DMS-2011553.
**Definition 1.1.** Let $H$ be an $r$-edge colored (multi-)graph. By $\text{mex}(r, n, H)$ we denote the maximum integer $T$ such that there exists graphs $G_1, \ldots, G_r$ on the same set of $n$ vertices such that $|E(G_i)| \geq T$ for all $i \in [r]$ and $\bigsqcup G_i$ does not contain a copy of $H$. We define $\text{mex}(r, H)$ to be $\lim_{n \to \infty} \frac{\text{mex}(r, n, H)}{(\binom{n}{2})}$.

When $r = 2$, Diwan and Mubayi [5] initiated the study of the above parameter. They were focused on the case when $H = K_{k+1}$ is a clique with an arbitrary coloring of its edges. They conjectured that when $k \geq 8$, regardless of the edge-coloring of $H$, the extremal threshold is the same as that in the colorless setting, namely, $1 - \frac{1}{k}$ (see Turán’s theorem [4]). When $k < 8$, they conjectured that the same result holds excluding some edge-colorings of $K_{k+1}$. For partial progress on this conjecture, we refer the reader to [5] and [13] (see Section 8). In this paper, we will be concerned with the question of determining $\text{mex}(r, H)$ when $H$ is not necessarily complete.

Recall the Erdős-Stone-Simonovits theorem, which states that for any graph $H$, $\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \left(\binom{n}{2}\right) + o(n^2)$. The theorem thus asymptotically characterizes the extremal threshold of any non-bipartite graph. Diwan and Mubayi asked for an analogue of the Erdős-Stone-Simonovits theorem in the colorful setting [5]. As we don’t even fully understand the behaviour of $\text{mex}(\cdot)$ on complete graphs, it is premature to hope for a single parameter that characterizes $\text{mex}(r, H)$ for any $r$-edge-colored $H$. Here, we aim to designate a parameter of edge-colored graphs that approximately determines their extremal threshold.

Note that any such bound on $\text{mex}(r, H)$ for general $H$ must take into account the coloring associated with the edges of $H$, and therefore depend on a parameter other than just the ordinary vertex-chromatic number of $H$. As an example, observe that for any monochromatic bipartite $H$, $\text{mex}(r, H) = 0$ whereas when $H$ is a 2-edge path colored red-blue, $\text{mex}(r, H) = 1/4$.

For a first general upper bound, consider when $r = 2$, and $H$ is a red/blue edge-colored $K_{k+1}$. It is easy to see that $\text{mex}(2, H) \leq 1 - \frac{1}{2k}$. Indeed, when $|R|, |B| > \left(1 - \frac{1}{2k}\right) \frac{n^2}{2}$, $|R \cap B| > \left(1 - \frac{1}{k}\right) \frac{n^2}{2}$ and by Turán’s theorem, $R \cap B$ contains a $K_{k+1}$. Now, regardless of the coloring of $H$, it will be possible to embed $H$ into $R \cup B$. Similarly, if $H$ was any $r$-edge-colored $(k+1)$-chromatic graph, if each color class has density more than $1 - \frac{1}{r(\chi(H)-1)}$, then the intersection of all of the colors has density more than $1 - \frac{1}{r(\chi(H)-1)}$. The Erdős-Stone theorem then tells us that a copy of a supergraph of $H$ where every edge has multiplicity $r$ can be found. Therefore we have $\text{mex}(r, H) \leq \left(1 - \frac{1}{r(\chi(H)-1)}\right)$ for any $H$. Perhaps surprisingly, we will see in Section 2 that for some $r$-edge-colored $(k+1)$-chromatic graphs, this bound is tight. Hence, any possible analogue of Erdős-Stone-Simonovits theorem in this context can only characterize how far away from the trivial upper bound $\text{mex}(r, H)$ is, taking into account the specific $r$-edge-coloring of $H$. 

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We will now introduce a parameter \( (\mathcal{M}(\cdot)) \) that is in some sense a measure of how monochromatic a bicolored \( H \) is. Using this parameter, we will provide a generalization of the celebrated Erdős-Stone-Simonovits theorem into the colorful setting. In short, \( \mathcal{M}(H) \) is, among the set of proper \( \chi(H)\)-colorings of \( H \), the largest set of disjoint pairs of color classes where each pair is connected by edges of just a single color. We give a more precise definition in what follows.

**Definition 1.2** (Reduced graph). Let \( G \) be an \( r \)-edge-colored graph. We say that an \( r \)-edge-colored multigraph \( P \) is a reduced graph of \( G \) if all of the following hold.

(a) there exists a function \( f : V(G) \to V(P) \) such that the existence of a \( c \)-colored edge \( \{x, y\} \) in \( G \) implies that there is a \( c \)-colored edge between \( f(x) \) and \( f(y) \) in \( P \)

(b) \( f \) induces a proper vertex-coloring of \( G \), i.e. each \( f^{-1}(\{v\}) \) is an independent set for any \( v \in V(P) \)

(c) No proper induced subgraph of \( P \) satisfies (a) and (b).

By \( \mathcal{R}(G) \), we denote the family of all reduced graphs of \( G \).

If \( P \) is a multigraph, we denote by \( M(P) \) the maximum size of a matching in the subgraph of \( P \) made up of the edges of multiplicity 1.

**Definition 1.3** (Reduced maximum matching). Let \( G \) be some \( r \)-edge-colored graph, and let

\[
\mathcal{M}(G) = \max_{P \in \mathcal{R}(G)} \frac{|P|}{\chi(G)}
\]

denote the “reduced maximum matching number” of \( G \).

We can now state our main theorem.

**Theorem 1.4** (Multigraph Erdős-Stone-Simonovits). Let \( G \) be some \( r \)-edge colored graph. Then,

\[
\max(r, G) \leq 1 - \frac{1}{r(\chi(G) - 1)} - \frac{\mathcal{M}(G)}{9r\chi(G)^2}.
\]

Further, there exist \( r \)-edge-colored graphs \( G \) so that the bound is best possible up to the multiplicative factor \( 1/9 \), whenever \( \mathcal{M}(G) \leq \chi(G)/10 \).

Although Theorem 1.4 is best possible in general up to the constant factors, it is not necessarily tight when \( \mathcal{M}(G) \) is near \( \chi(G)/2 \), for example when \( G \) is just a clique on \( \chi(G) \) vertices. There are other natural settings in which our Theorem 1.4 does not necessarily give a tight answer, and we discuss these problems more in Section 5.
We also study the parameter \( \text{mex}(G) := \text{mex}(2, G) \) for some specific classes of graphs. If \( G \) is bicolored and bipartite, observe that if \( R \cup B \) avoids \( G \), \( |R \cap B| = o(n^2) \). Indeed, otherwise \( R \cap B \) would contain a large complete bipartite graph by the Erdős-Stone-Simonovits theorem. Hence, here we may assume that \( R \cap B = \emptyset \) without changing densities. It follows that \( \text{mex}(G) \leq 1/2 \). Further, the hypothesis that \( R \cap B = \emptyset \) brings us to the setting of \([10]\). (Indeed, if \( R \) and \( B \) don’t meet, and they both have density at least \( \alpha \), their union contains a \((1/2)\)-balanced graph of density \( 2\alpha \).)

Using the Theorem 1.1 in \([10]\), we can then obtain a better upper bound on \( \text{mex}(G) \) when \( G \) is bipartite and “inevitable”\footnote{For definitions of (1/2)-balanced and “inevitable”, we refer the reader to \([10]\)}. In particular, in this case it will be that \( \text{mex}(G) \leq 1/2 - f(v(G)) \) where \( f \) is positive, but exponentially small in \( v(G) \).

From the previous discussion, it follows that for any even bicolored cycle, \( \text{mex}(G) \leq 1/2 \). We also show that odd bicolored cycles can also be found at this same threshold.

**Theorem 1.5.** Let \( C \) be any bicolored cycle. Then, \( \text{mex}(C) \leq 1/2 \).

We include the short proof of Theorem 1.5 in the Discussion section, along with some open problems. In the next section, we give a construction showing that Theorem 1.4 is tight up to the factor \( \frac{1}{9} \). Then in Sections 3 and 4, we prove the upper bound in Theorem 1.4.

### 2 Construction

Our goal in this section is to demonstrate the sharpness of Theorem 1.4. For even \( r \), we will construct an \( r \)-edge-colored graph \( H \), with chromatic number \( k \), such that \( \text{mex}(r, H) \geq 1 - \frac{1}{r(k-1)} \). First, assuming such an \( H = H(r, k) \) for every even \( r \) and \( k \), let us show that Theorem 1.4 is sharp. Given an even \( r \), and positive integers \( m \) and \( k \) such that \( 10m \leq k \) we will construct a graph \( H' \) for which \( \mathcal{M}(H') \geq m \), \( \chi(H') = k \) and

\[
\text{mex}(r, H') \geq 1 - \frac{1}{r(\chi(H') - 1)} - O \left( \frac{\mathcal{M}(H')}{r\chi(H')^2} \right).
\]

The construction of \( H' \) is as follows. Let \( H := H(r, k - 2m) \), and let \( H' \) be the union of \( H \) and a red \((2m)\)-clique, and we add all possible edges between the clique and \( H \) in red. It is clear that \( \mathcal{M}(H') \geq m \) and \( \chi(H') = k \). And as \( H' \) contains \( H \), it follows that

\[
\text{mex}(r, H') \geq 1 - \frac{1}{r(k - 2m - 1)} = 1 - \frac{1}{r(\chi(H') - 1)} - O \left( \frac{\mathcal{M}(H')}{r\chi(H')^2} \right)
\]

where in the last equality we used that \( 10m \leq k \).
2.1 The graph $H$

The construction of $H$ is as follows. $V(H)$ has size $tk$, where $t$ is a large constant that will be defined later. Denote the vertices by $v_{i,j}$, where $i \in [t]$ and $j \in [k]$. For every pair $j,j'$, between the vertex sets $\{v_{i,j}\}_{i=1}^{t}$ and $\{v_{i,j'}\}_{i=1}^{t}$ we have copy of a graph $F$, which is an $r$-colored complete bipartite graph $K_{t,t}$. We need the colors of the edges of $F$ to satisfy the following property: if the vertex set of $F$ has bipartition $X \cup Y$, then for any pair of sets $X' \subseteq X$, $Y' \subseteq Y$, each with size $\frac{t}{(rk)^{2}}$, the bipartite graph induced on $X', Y'$ has edges on every color in $[r]$.

Claim 2.1. If $r$ and $k$ are fixed, then for $t$ large enough there is a complete bipartite graph with $t$ vertices in each part and an $r$-coloring of the edges such if $X' \subseteq X$, $Y' \subseteq Y$, and $|X'|, |Y'| \geq \frac{t}{(rk)^{2}}$, the bipartite graph induced on $X', Y'$ has edges on every color in $[r]$.

Proof. Let $F$ be a complete bipartite graph with $t$ vertices in each part. Color the edges independently and uniformly at random from $[r]$. By the union bound, the expected number of pairs $X', Y'$ of size $\frac{t}{(rk)^{2}}$ which are missing some color is at most

$$r \left( \frac{t}{(rk)^{2}} \right)^{2} \left( \frac{r-1}{r} \right)^{2} \left( \frac{t}{(rk)^{2}} \right)^{2} \leq r(ek)^{\frac{2n}{(rk)^{2}}} e^{-\frac{t^2}{rk^2}},$$

which is less than 1 for $t$ large enough. \qed

Putting a copy of $F$ between each of the $k$ parts completes the construction of $H$. Since $H$ is a complete $k$-partite graph it has chromatic number $k$.

2.2 Constructing $G_1, \ldots, G_r$

Next we will construct graphs $G_1, \ldots, G_r$, each with $(1 - \frac{1}{r(k-1)} + o(1))\binom{2}{2}$ edges, such that $\bigcup G_i$ does not contain a colored copy of $H$. See Figure 1 for a diagram of the construction in the case when $r = 4$ and $k = 3$. Each $G_i$ will be on the same vertex set of size $n$ where $n = (k-1)rm$, for some integer $m$ (and $k$ and $r$ fixed as before). Denote the vertices in these graphs by $w_{x,y,z}$, for $x \in [k-1]$, $y \in [r]$ and $z \in [m]$.

It will be more convenient to define the complement $G^c_i$ of these graphs. The graph $G_r$ will be defined differently from $G_1, \ldots, G_{r-1}$. The edges in $G^c_r$ are precisely the edges of the form $w_{x,y,z}w_{x,y,z'}$ for all $1 \leq z, z' \leq m$. Another way to say this is that $G_r$ is a Turán graph on $(k-1)r$ parts.

To define the edge sets of $G_1, \ldots, G_{r-1}$, first let $U$ be a clique $K_r$, where the edges are properly colored with the colors in $[r-1]$ (such a coloring exists because $r$ is even).
Figure 1: An illustration of $G_1, \cdots, G_r$ when $r = 4$ and $k = 3$. The small circles represent subsets of $n/8$ vertices. $G_4$ is a Turán graph with these circles as the parts. $G_1$ can be obtained by starting with a complete graph, and then removing the edges in the bipartite graphs corresponding to the dotted lines. $G_2$ and $G_3$ is defined similarly, replacing dotted with solid and arrowed, respectively.

For each $i \in [r-1]$, the edges in $G^c_i$ are precisely those of the form $w_{x,y,z}w_{x,y',z'}$, where the edge $yy'$ receives color $i$ in $U$ and for all $1 \leq z, z' \leq m$.

Each of the $G^c_i$ have $(1 + o(1))\frac{1}{r(k-1)}\binom{n}{3}$ edges, so $e(G_i)/\binom{n}{3} \sim \left(1 - \frac{1}{r(k-1)}\right)$.

### 2.3 Showing that $\bigcup G_i$ is $H$-free

We will next prove that $\bigcup G_i$ indeed does not contain a copy of $H$. Suppose that there is such a copy of $H$. For each $i$, consider the vector $(x^i_1, y^i_1, x^i_2, y^i_2, \ldots, x^i_k, y^i_k)$, where $x^i_j$ and $y^i_j$ are the first and second coordinate of the image of $v_{i,j}$ in $\bigcup G_i$. By the pigeonhole principle, there is a subset $I \subseteq [t]$ of at least $\frac{t}{r(k-1)}$ values of $i$ for which the vector defined above is the same.

There are two values $j \neq j'$ such that $x^i_j = x^{i'}_{j'}$ for all $i \in I$. If we have $y^i_j = y^{i'}_{j'}$, then the images of all edges of the form $v_{i,j}v_{i',j'}$ for $i, i' \in I$ are in $G^c_r$. If on the other hand we have $y^i_j \neq y^{i'}_{j'}$, then the images of all edges of the form $v_{i,j}v_{i',j'}$ for $i, i' \in I$ are in $G^c_q$ for some $q \in [r-1]$. Regardless of the case, this contradicts the fact that the bipartite graph formed by these edges contains all colors.
3 Reduction to reduced graphs

Here, we show that the extremal threshold of any graph $G$ is at most as large as the threshold of any of its reduced graphs. We now state the main result of this section.

**Proposition 3.1.** Let $G$ be an $r$-edge-colored graph, and let $P \in R(G)$ be one of its reduced graphs. Then, $\text{mex}(r,G) \leq \text{mex}(r,P)$.

The proof will use the regularity lemma and is similar to the regularity based proof of the Erdős-Stone theorem. Before we begin, we must state the regularity lemma, starting with the necessary terminology. Let $G := (A,B)$ be a bipartite graph with $|A| = |B| = n$. For $X \subseteq A$ and $Y \subseteq B$ define $d(X,Y) := \frac{e(X,Y)}{|X||Y|}$. We call $G$ $\varepsilon$-regular if for all subsets $X \subseteq A$ and $Y \subseteq B$ with $|X|, |Y| \geq \varepsilon n$ we have $|d(X,Y) - d(A,B)| \leq \varepsilon$.

**Lemma 3.2** (Szemerédi [12]). For any $\varepsilon > 0$ there exists an $M := M(\varepsilon)$ such that any graph $G$ can be partitioned into $k$ (where $\frac{1}{2} \leq k \leq M$) equal sized parts $(V_i)_{i \in [k]}$ and a junk set $J$ with $|J| < \varepsilon n$ such that all but $\varepsilon$-fraction of the pairs $(V_i, V_j)$ are $\varepsilon$-regular.

We also record a multi-color version of the regularity lemma we will need, which can be proved by iterating the regularity lemma multiple times (c.f. [8] Theorem 1.18).

**Lemma 3.3.** For any $\varepsilon > 0$ there exists an $M := M(\varepsilon)$ such that any graph $\bigsqcup_{i \in [r]} G_i$ can be partitioned into $k$ (where $\frac{1}{2} \leq k \leq M$) equal sized parts $(V_i)_{i \in [k]}$ and a junk set $J$ with $|J| < \varepsilon n$ such that all but $\varepsilon$-fraction of the pairs $(V_i, V_j)$ are $\varepsilon$-regular in the color $i$ for all $i \in [r]$.

Given a multigraph $H$ with an $r$-coloring of its edges, we define a blow-up of $H$ with $s$ vertices in each part to be the graph where each vertex of $H$ is replaced by an independent set of size $s$ and each edge in $H$ of color $i$ is replaced by a complete bipartite graph in color $i$. We denote this blow-up by $H^{(s)}$.

In order to prove Proposition 3.1 it suffices to show that if $d(G_i) \geq \text{mex}(r,P) + \delta$ for all $i \in [r]$ and $n$ is sufficiently large, then $\bigsqcup G_i$ contains a blow-up of $P$ with $|G|$ vertices on each part. The fact that a sufficiently large blow-up of $P$ will contain a copy of $G$ follows from the definition of $P$ being a reduced graph of $G$.

The regularity lemma gives a streamlined method of finding such blow-ups in dense graphs. Given an $\varepsilon$-regular partition of an $r$-colored graph $G$ where the parts have size $\ell$ and a $d > 0$, we define the multicolor regularity multigraph with parameters $\varepsilon$, $\ell$, and $d$ to be the $r$-colored multigraph where vertices are indexed by the parts $V_i$ and there is an edge in color $c$ between $V_i$ and $V_j$ if the pair $(V_i, V_j)$ is $\varepsilon$-regular with density at least $d$ in color $c$. Applications of the regularity lemma often use
an embedding lemma which says that subgraphs of a blow-up of the regularity graph $R^{(s)}$ will also be found in the original graph. The following lemma is a colorful version of such an embedding lemma. The proof is a rewriting of the uncolored version (see Lemma 7.5.2 in Diestel [4]).

**Lemma 3.4.** For all $d \in (0, 1]$ and $\Delta, s \geq 1$, there exist an $\varepsilon_0$ and an $L$ such that if $G$ is an $r$-colored graph, $H$ is an $r$-colored multigraph with maximum degree $\Delta$ in each color, and $R$ is a multicolor regularity multigraph of $G$ with parameters $\varepsilon \leq \varepsilon_0$, $\ell \geq L$, and $d$, then

$$H \subset R^{(s)} \implies H \subset G$$

Using Lemma 3.4, it suffices to show that $P$ is a subgraph of the multicolor regularity multigraph $R$, for then $P^{(s)}$ will be a subgraph of $R^{(s)}$. Showing this will be a consequence of the regularity lemma. We would like to highlight that this proof is a standard application of the regularity method. We include the details for completeness.

**Proof of Proposition 3.1.** We let $n$ be sufficiently large, and fix $\bigcup_{i \in [r]} G_i$ to be some graphs on the same vertex set $[n]$ such that the density of each $G_i$ is at least $\operatorname{mex}(r, P) + \delta$ for some positive constant $\delta$. Choose $\varepsilon = \delta/16$ and $d = \delta/4$. Assume that $\varepsilon$ is also chosen small enough that it is less than the $\varepsilon_0$ from Lemma 3.4 with parameters $d$ and $\Delta(P)$ and so that $\operatorname{mex}(r, N, P) \leq \frac{N^2}{2} (\operatorname{mex}(r, P) + \delta/2)$ for all $N \geq \frac{1}{\varepsilon}$. Apply Lemma 3.3 to $\bigcup_{i \in [r]} G_i$ with regularity parameter $\varepsilon$. Assume that $\{J, V_1, \ldots, V_k\}$ are the parts of the partition and each $V_i$ has size $l$, and let $R$ be the multicolor regularity multigraph. For each color $c$ let $R_c$ be the subgraph of $R$ of the edges of color $c$. We now show that for each $c$, $e(R_c)$ is large. At most $\varepsilon l^2 \binom{k}{2}$ $c$-colored edges can be between pairs which are not $\varepsilon$-regular. At most $l^2 \binom{k}{2} d$ $c$-colored edges may be between pairs of density less than $d$. At most $k \binom{k}{2}$ $c$-colored edges may be within one of the parts $V_i$. At most $\varepsilon n^2$ $c$-colored edges may be incident with the junk set $J$. Finally, for each edge in $R_c$ there are at most $l^2$ edges in $G_c$ between the corresponding parts. In total, we have

$$e(G_c) \leq l^2 e(R_c) + \varepsilon l^2 \binom{k}{2} d + k \binom{l}{2} + \varepsilon n^2 \leq l^2 e(R_c) + \varepsilon \frac{l^2 k^2}{2} + d \frac{l^2 k^2}{2} + \varepsilon \frac{k^2}{2} + 2 \varepsilon \frac{l^2 k^2}{2},$$

where the last inequality uses $\varepsilon \geq \frac{1}{k}$ and $n = kl + |J| \leq kl + \varepsilon n$. Therefore

$$e(R_c) \geq \frac{k^2}{2} \left( \frac{e(G_c) - 4 \varepsilon - d}{l^2 k^2} \right) > \frac{k^2}{2} (\operatorname{mex}(r, P) + \delta/2),$$

by the choice of $\varepsilon$ and $d$. Since this inequality holds for all colors, and since $\varepsilon$ was chosen so that $\operatorname{mex}(r, N, P) \leq \frac{N^2}{2} (\operatorname{mex}(r, P) + \delta/2)$ for all $N \geq \frac{1}{\varepsilon}$, $P$ is a subgraph of
and therefore $P(s)$ is a subgraph of $R(s)$. If $n$ (and thus $l$) is large enough, applying Lemma 3.4 shows that $P(s)$ is a subgraph of $\bigcup_{i \in [r]} G_i$ and hence $G$ is a subgraph of $\bigcup_{i \in [r]} G_i$.

\[\square\]

4 The upper bound

In this section, we prove the upper bound in Theorem 1.4. Our main tool will be the following stability result of Füredi.

**Theorem 4.1** (Füredi, [7]). Let $G$ be a graph on $n$ vertices without a $K_{k+1}$, and let $t := ex(n, K_{k+1}) - e(G)$. Then, $G$ can be made $k$-partite by deleting at most $t$ edges.

Let $G$ be an $r$-edge colored multigraph with $E_1, \ldots, E_r$ denoting the edge sets in colors $1, \ldots, r$ respectively. Let $\chi(G) = k+1$ and let $U_1, \ldots, U_{k+1}$ be the color classes of a proper $\chi$-coloring of $G$. Define an $r$-edge colored multigraph $P$ on vertex set $[k+1]$ where $xy$ is an edge of $P$ in color $i$ if and only if $U_x \cup U_y$ contains an edge in $E_i$. Then $P$ is a reduced graph of $G$.

Let $m := M(P)$ be the maximum size of a matching in the subgraph made of the edges of multiplicity 1 in $P$. Let $e_1, \ldots, e_m$ be a maximum matching of multiplicity 1 edges in $P$. Define an $r$-edge colored multigraph $P'$ on vertex set $[k+1]$ that has a matching of size $m$ of single edges with the same colors as $e_1, \ldots, e_m$ and the remaining edges have multiplicity $r$ with 1 edge of every color. Then $P$ is a subgraph of $P'$ and so by Proposition 3.1, it suffices to show that

$$\text{mex}(r, P) \leq \text{mex}(r, P') \leq 1 - \frac{1}{rk} - \frac{m}{9rk^2},$$

To show this, assume that $H$ is an $n$-vertex $r$-edge colored multigraph and for $1 \leq i \leq r$ let $H_i$ be the simple graph of the color $i$ edges of $H$. Assume that

$$e(H_i) > \left(1 - \frac{1}{rk} - \frac{m}{9rk^2}\right) \binom{n}{2},$$

for all $i$. We will show that $H$ contains $P'$ as a subgraph.

Let $R$ be the subgraph of $H$ of edges of multiplicity $r$. That is, $E(R) = \bigcap H_i$. Then

$$e(R) \geq \left(1 - \frac{1}{k} - \frac{m}{9k^2}\right) \binom{n}{2}.$$

If $R$ contains $K_{k+1}$ as a subgraph, then $H$ contains $rK_{k+1}$ and hence $P'$, so we may assume that $R$ is $K_{k+1}$-free. Hence, by Turán’s theorem

$$e(R) \leq \left(1 - \frac{1}{k}\right) \binom{n}{2}.$$
By Theorem 4.1, \( R \) has a \( k \)-partite subgraph \( R' \) satisfying
\[
e(R') \geq \left(1 - \frac{1}{k} - \frac{2m}{9k^2}\right) \binom{n}{2}.
\]
Assume that \( V_1, \ldots, V_k \) are the partite sets of \( R' \). Let \( H'_i \) be the subgraph of \( H_i \) consisting of all edges that have both endpoints in one of the partite sets. That is
\[
H'_i = \bigcup_{j=1}^{k} H_i[V_j].
\]
Then we have
\[
e(H'_i) \geq \left(1 - \frac{1}{rk} - \frac{m}{9rk^2}\right) \binom{n}{2} - \left(1 - \frac{1}{k}\right) \binom{n}{2} = \left(\frac{r - 1}{r} - \frac{m}{9rk^2}\right) \binom{n}{2},
\]
for each \( i \).

Assume that \( e_1, \ldots, e_m \) have colors \( c_1, \ldots, c_m \) respectively. Choose \( \pi \in S_k \) uniformly at random. For \( 1 \leq j \leq m \), let \( F'_i \) be the graph in color \( c_i \) induced by \( V_{\pi(i)} \). That is, \( F'_i = H_{c_i}[V_{\pi(i)}] \). Let \( F'_i \) be the subgraph of \( F_i \) given by a maximum cut of \( F_i \). Finally, consider the uncolored simple graph
\[
H' = F'_1 \cup \cdots \cup F'_m \cup R'.
\]
We claim that if \( H' \) contains \( K_{k+1} \), then \( P' \) is a subgraph of \( H \). To see this, assume that \( K_{k+1} \) is a subgraph of \( H' \). We argue that the vertices corresponding to those of \( K_{k+1} \) in \( H' \) induce a \( P' \) in \( H \). Note that all edges of the \( K_{k+1} \) that are not contained in a single part \( V_i \) (i.e. edges that come from \( R' \) have multiplicity \( r \) in \( H \). Further, the \( K_{k+1} \) can draw at most two vertices from each \( V_i \), as each \( F'_i \) is a bipartite graph. Thus, the edges of \( H' \) that come from \( F'_i \) form a matching. Hence, \( H' \) corresponds to a multigraph in \( H \) all of whose edges have multiplicity \( r \) with 1 edge of every color, except for a matching of size at most \( m \) with single edges whose colors are a submultiset of \( \{c_1, \ldots, c_m\} \). This structure contains a \( P' \).

To complete the proof, we show that there is a choice of \( \pi \) so that \( H' \) contains \( K_{k+1} \) by showing that \( \mathbb{E}[e(H')] \geq \left(1 - \frac{1}{k}\right) \binom{n}{2} \) and applying Turán’s theorem.

Note that
\[
\mathbb{E}[e(H')] = e(R') + \sum_{j=1}^{m} \mathbb{E}[e(F'_j)] \geq \left(1 - \frac{1}{k} - \frac{2m}{9k^2}\right) \binom{n}{2} + \frac{1}{2} \sum_{j=1}^{m} \mathbb{E}[e(F_i)].
\]
Now, for each \( j \), each edge in \( H'_{c_j} \) has a \( \frac{1}{k} \) chance of being in \( F_j \). Therefore,
\[
\mathbb{E}[e(F_j)] \geq \frac{1}{k} \left(\frac{r - 1}{r} - \frac{m}{9rk^2}\right) \binom{n}{2},
\]
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for $1 \leq j \leq m$. Using $r \geq 2$ and $m < k$ gives
\[
\mathbb{E}[H'] > \left(1 - \frac{1}{k} - \frac{2m}{9k^2}\right) \binom{n}{2} + \frac{m}{2k} \left(\frac{1}{2} - \frac{1}{18k}\right) \binom{n}{2} = \left(1 - \frac{1}{k}\right) \binom{n}{2}.
\]

5 Discussion

As we remarked in the introduction, it makes sense to study the $\mex(\cdot)$ parameter for specific families of graphs. When $r = 2$, and when $G$ is bipartite, we showed that this problem reduces to a setting investigated in [10]. At any rate, the bound $\mex(G) := \mex(2,G) \leq 1/2$ is true, regardless of the coloring of $G$. If one wishes to see at which threshold a bicolored cycle will emerge in $R \cup B$ regardless of the coloring, the previous bound only leaves open the case of odd cycles. As stated in Theorem 1.5, it turns out that the same upper bound of $1/2$ also holds for odd cycles.

Proof of Theorem 1.5. Let $C$ be a bicolored odd cycle. If $C$ is monochromatic, then the result follows from the (uncolored) Erdős-Stone-Simonovits theorem, and so we may assume that there is a path on two edges in $C$ with colors RB. Since $C$ has an odd number of edges, it also contains a path on two edges colored either RR or BB. Assume that there is a RR path (if there is only a BB path, switch the colors in the remainder of the proof). Let $T_1$ be the bicolored triangle with one double edge, and the other two edges colored red. And let $T_2$ be the bicolored triangle with one double edge, one red edge, and one blue edge. It follows that $C$ can be embedded in a sufficiently large blow-up of either $T_1$ or $T_2$.

So the bound $\mex(\{T_1, T_2\}) \leq 1/2$ would suffice to deduce the theorem, by Proposition 3.1. We will actually show something stronger.

Claim 5.1. Let $G$ be a bicolored graph avoiding both $T_1$ and $T_2$. Then $e(G) \leq \frac{n^2}{2}$

Therefore, in any $T_1/T_2$ avoiding $R \cup B$, one color class has at most $n^2/4$ edges, which implies that $\mex(\{T_1, T_2\}) \leq 1/2$.

To prove the claim, hence the theorem, we will proceed by induction. The base cases of $n = 1$ and $n = 2$ are clear. Let us fix an edge $\{x, y\}$ that is colored red (possibly also blue). If there wasn’t one, there are at most $\binom{n}{2}$ edges (all blue) in the graph, and we are done.

By induction, $e(G \setminus \{x, y\}) \leq \frac{(n-2)^2}{2}$. Further, for any $v \in G \setminus \{x, y\}$, $d(\{v\}, \{x, y\}) \leq 2$. Indeed, otherwise $v$ must send at least 3 edges to $\{x, y\}$ and so they must include at least 1 double edge. Then $\{v, x, y\}$ is either a $T_1$ or a $T_2$. In total,
\[
e(G) \leq e(G \setminus \{x, y\}) + e(G \setminus \{x, y\}, \{x, y\}) + 2 \leq \frac{(n-2)^2}{2} + 2(n-2) + 2 = \frac{n^2}{2}
\]
\[\square\]
Note that our general Erdős-Stone-Simonovits type theorem (Theorem 1.4) cannot give the above optimal bounds in the above problem as \( \chi(G) = 3 \) and \( M(G) \leq 1 \) when \( G \) is an odd cycle. Hence, we had to take a more direct approach. Although in general Theorem 1.4 is tight, the example we gave for tightness in Section 2 featured a graph that is as dense as possible while having a particular chromatic number. Indeed, it was critical for our construction that the graph \( H \) had clique number equal to its chromatic number, in fact, \( H \) contained a large blow up of a clique of order \( \chi(H) \). Outside this domain, we do not know if Theorem 1.4 is tight. It could be interesting to investigate \( \text{mex}(\cdot) \) in other sparser settings in an effort to determine if a more specific Erdős-Stone-Simonovits theorem could be established in this setting, which would give a more complete answer to the question asked by Diwan and Mubayi.

One natural direction is to study the average case. Say \( G := G(n, 1/2) \), the uniformly random graph, equipped with a uniformly random red-blue coloring of its edges. We know that \( \chi(G) = (1/2 + o(1))n/\log_2 n \) with high probability. It would be interesting to give bounds on \( \text{mex}(G) \) with high probability. It seems likely that the upper bound from our Theorem 1.4 would not be tight here.

Another intriguing open problem is the one Diwan and Mubayi originally studied, namely determining \( \text{mex}(G) \) when \( G := K_n \) and \( G \) is equipped with an arbitrary coloring of its edges. This problem seems quite hard, and restricting attention to almost all colorings of \( K_n \) while aiming to determine \( \text{mex}(K_n) \) already seems to be a challenging question.

Finally, there are many natural variants or particular cases of this problem that have been studied and interesting open questions are pervasive. We conclude with a selection of these: forbidding rainbow triangles was considered in [1] and [11]; forbidding nonmonochromatic triangles was considered in [3]; a survey of a more general problem with weights was given in [9]; an inverted version of the problem was asked in [2]; finding multiple rainbow cliques was studied in [6].

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