INDEPENDENT SETS, CLIQUES, AND COLORINGS IN GRAPHONS

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ABSTRACT. We study graphon counterparts of the chromatic and the clique number, the fractional chromatic number, the $b$-chromatic number, and the fractional clique number. We establish some basic properties of the independence set polytope in the graphon setting, and duality properties between the fractional chromatic number and the fractional clique number. We present a notion of perfect graphons and characterize them in terms of induced densities of odd cycles and its complements.

1. INTRODUCTION

The concepts of independent sets, cliques, and colorings are among the most studied in graph theory. Before stating its graphon counterparts, let us recall some fundamental concepts.

1.1. Review of basic properties for graphs. Suppose that $G = (V, E)$ is a graph. We say that a function $f : V \to [k]$ is a proper coloring of $G$ with $k$ colors if for all pairs $xy \in \binom{V}{2}$ we have $xy \notin E$ or $f(x) \neq f(y)$. The chromatic number $\chi(G)$ is defined as the minimal number of colors used in any proper coloring. Thus, in a proper coloring we paint the vertices avoiding that neighbors share the same color.

One can generalize the concept of proper coloring by adding multiple colors to each vertex of a graph in the following manner. Let $b \in \mathbb{N}$. A map $p : V \to \binom{[k]}{b}$ is a $b$-fold coloring of $G$ with $k$ colors if for all pairs $xy \in \binom{V}{2}$ we have $xy \notin E$ or $p(x) \cap p(y) = \emptyset$. The $b$-fold chromatic number $\chi_b(G)$ is the smallest number of colors necessary to construct a proper $b$-fold coloring of $G$.

A set $A \subseteq V$ is an independent set of $G$ if for all pairs $xy \in \binom{A}{2}$ we have $xy \notin E$. The size of the largest independent set of $G$ is the independence number $\alpha(G)$. Dual to the concept of an independent set is that of a clique of a graph $G$, a set $A \subseteq V$ where for all pairs $xy$ inside $A$ we have $xy \in E$. The clique number $\omega(G)$ is the size of the largest clique of $G$.

Now denote by $\mathcal{I}(G)$ the collection of independent sets in $G$. A function $c : \mathcal{I}(G) \to [0, 1]$ is a fractional coloring of $G$ if for every $v \in V(G)$ we have

$$\sum_{I \in \mathcal{I}(G), v \in I} c(I) \geq 1.$$ \hfill (1.1)

The fractional chromatic number is then defined as the infimum of $\sum_{I \in \mathcal{I}(G)} c(I)$ taken over all fractional colorings $c$. Fractional colorings are indeed a fractional relaxation of ordinary colorings. Indeed, when $c : \mathcal{I}(G) \to \{0, 1\}$, then (1.1) can be interpreted as the condition that in a coloring every vertex has to be covered by at least one independent set.

We say that a function $f : V \to [0, +\infty)$ is a fractional clique if for every $I \in \mathcal{I}(G)$ we have

$$\sum_{v \in I} f(v) \leq 1.$$ \hfill (1.2)

The fractional clique number of $G$ is defined as

$$\omega_{\text{frac}}(G) = \max \sum_{v \in V} f(v).$$
where the maximum is taken over all fractional cliques of $G$. In the same fashion as in fractional colorings, the fractional clique number is the relaxation of the problem of finding a maximum clique in a graph. Notice that when $f : V \to \{0, 1\}$, then (1.2) can be interpreted as the condition that every independent set has at most one vertex of any clique.

The study of these parameter, and in particular the study of the interplay of the integral and fractional versions of these parameters is central in graph theory.

1.2. Our contribution. We translate the basics of the theory regarding independent sets, cliques, and colorings to the setting of graphons (see Section 2 for basics). While turning a graph definition into a graphon one is typically straightforward (and in many cases had been done previously), counterparts to many natural facts from finite graphs turned out to be quite challenging. By optimizing over all objects of this type (independent sets, fractional cliques, . . .), we can define the corresponding numerical graphon parameter (independence number, fractional clique number, . . .). We study relations between these parameters. We also study continuity of these graphon parameters with respect to the cut-norm. It turns out that none of the quantities we introduce is continuous, and as is shown in all cases by the sequence of constant graphons $(Y_n \equiv \frac{1}{n})_n$ converging to $Y \equiv 0$. However, most of these parameters are semicontinuous with respect to the cut-norm. These results are summarized in Table 1.

|                          | (A) representation | (B) semicontinuity | (C) supremum over graphs |
|--------------------------|--------------------|--------------------|--------------------------|
| independence             | trivial            | Corollary 7        | meaningless              |
| chromatic                | Proposition 12     | Theorem 13 (a)     | Proposition 14           |
| fractional chromatic     | Proposition 16     | not true (Example 18) | not true (Example 18) |
| clique                   | trivial            | Proposition 24     | trivial                  |
| fractional clique        | Proposition 26     | Theorem 27         | Corollary 31             |

Table 1. Summary of results regarding graphon parameters we study. Column (A) refers to results that show equality of the graphon parameter to the corresponding graph parameter in case of graphon representation of a finite graph. Column (B) refers to results that show that these graphon parameters are lower semicontinuous (for chromatic number, fractional chromatic number, clique number, and fractional clique number) or upper semicontinuous (for independence number) in the cut-distance. In all cases the sequence $Y_n \equiv \frac{1}{n} \to Y \equiv 0$ shows that we do not have the complementary semicontinuity. Column (C) refers to results that show that for a general graphon, the value can be computed as the supremum over all finite graphs that appear in that graphon of the graph version of that parameter.

Further, in Section 3.1 we introduce a graphon counterpart to the independence set polytope, in Section 5 we treat the LP duality between fractional cliques and fractional colorings, and in Section 7 we introduce two notions of perfect graphons. Several fairly basic problems remain open.

Remark 1. All our notions only depend on the support of a graphon. That is, replacing a graphon for example by the indicator function of its support, the notions of independent sets, colorings, etc., the corresponding numerical parameters do no change. The only exception to this is a notion of perfect graphons introduced in Section 7.

The paper is organized as follows: in Section 2 we provide some notation and prove a preliminary lemma; In Section 3 we study independent sets; Section 4 is dedicated to develop different concepts of chromatic number; In Section 5 we introduce notions of clique numbers; In Section 6 we prove certain duality properties between chromatic and clique parameters; and finally, Section 7 is devoted to the study of perfect graphons.
2. Notation and preliminaries

Throughout, we fix an atomless Borel probability space \( \Omega \) equipped with a measure \( \nu \) (defined on an implicit sigma-algebra). For \( k \in \mathbb{N} \), we denote by \( \nu^{\otimes k} \) the product measure on \( \Omega^k \).

Graphons, introduced in [9, 1], are analytic objects that capture limit properties of dense graphs. We assume the reader’s familiarity with the basics of the theory. Our notation mostly follows Lovász’ treatise [3]. Our graphons will be mostly defined on \( \Omega^2 \). In particular, we shall work with the density and the induced density of a finite graph \( H \) in a graphon \( W \), defined by

\[
t(H, W) = \int_{\{x \in V(H) \}} \prod_{uv \in E(H)} W(x_u, x_v) \quad \text{and} \quad t_{\text{ind}}(H, W) = \int_{\{x \in V(H) \}} \prod_{uv \in E(H)} W(x_u, x_v) \cdot \prod_{uv \notin E(H)} (1 - W(x_u, x_v)),
\]

Also, we shall make use of inhomogeneous random graphs \( G(n, W) \) described for example in Section 10.1 of [8]. By a subgraphon of \( W \) obtained by restricting to a set \( A \subset \Omega \) of positive measure we mean a function \( W[A] : A^2 \to [0, 1] \) which is simply the restriction \( W|_{A \times A} \). When working with this notion, we need to turn \( A \) into a probability space. That is, we view \( W[A] \) as a graphon on the probability space \( A \) endowed with measure \( \nu_A(B) := \frac{\nu(B)}{\nu(A)} \) for every measurable set \( B \subset A \).

All subsets of \( \Omega \) or of \( \Omega^2 \) considered will be measurable; whenever a new set is constructed it follows immediately from the construction that the set is measurable. For sets \( A, B \subset \Omega, C, D \subset \Omega^2 \) we write \( A = B \mod 0, C = D \mod 0 \) for equality up to null sets, i.e., if \( \nu(A \Delta B) = 0 \) and \( \nu^{\otimes 2}(A \Delta B) = 0 \). Given a function \( f : X \to \mathbb{R} \), we denote by \( \text{supp}(f) \) the support of \( f \), \( \text{supp}(f) := \{ x \in X : f(x) \neq 0 \} \).

A graphon \( W : \Omega \times \Omega \to [0, 1] \) is a graphon representation of a finite graph \( G \) if there exists a partition \( \Omega = \bigsqcup_{v \in V(G)} \Omega_v \) of sets of measure \( \frac{1}{\|W(G)\|} \) such that \( W \) restricted to \( \Omega_u \times \Omega_v \) is either constant 0 or constant 1 (modulo a nullset), depending on whether \( uv \notin E(G) \) or \( uv \in E(G) \).

2.1. Structural properties of graphons with a given subgraph. In this section we state Lemma 2 which says that if \( t(H, W) > 0 \) for some finite graph \( H \) and some graphon \( W \) then we can find a subgraphon similar to the adjacency matrix of \( H \) in \( W \). The case \( H = C_{2\ell+1} \) of Lemma 2 appears in [3]. Our proof of Lemma 2 closely follows [4, Lemma 11].

**Lemma 2.** Suppose that \( W : \Omega \times \Omega \to [0, 1] \) is a graphon. Suppose that \( H \) is a graph on vertex set \( [k] \) with the property that \( t(H, W) > 0 \). For each \( \epsilon > 0 \) there exist \( \alpha > 0 \) and pairwise disjoint sets \( A_1, \ldots, A_k \subset \Omega \), each of measure \( \alpha \), such that for each \( ij \in E(H) \), we have that \( W \) is positive everywhere on \( A_i \times A_j \) except a set of measure at most \( \alpha^2 \).

**Proof.** Let \( n \in \mathbb{N} \) be such that

\[
(2.1) \quad n > \frac{k^2}{t(H, W)}.
\]

Take an arbitrary partition \( \Omega = \bigsqcup_{i=1}^n \Omega_i \) of \( \Omega \) into pairwise disjoint sets of measure \( \frac{1}{n} \). Set

\[
(2.2) \quad D := \left\{ x \in \Omega^k : \text{there are } i, j \in [k] \text{ and } \ell \in [n] \text{ such that } i \neq j \text{ and } x_i, x_j \in \Omega_\ell \right\}.
\]

Then we have

\[
(2.3) \quad \int_{x \in D} \prod_{1 \leq i, j \leq k, ij \in E(H)} W(x_i, x_j) \leq \nu^{\otimes k}(D) \leq \sum_{i,j=1,...,k} \frac{1}{n} \leq \frac{k^2}{n} \leq \frac{k^2}{t(H, W)} \leq \frac{k^2}{n^2} \leq t(H, W),
\]

From (2.3), we get

\[
\int_{x \in \Omega \setminus D} \prod_{1 \leq i, j \leq k, ij \in E(H)} W(x_i, x_j) > 0.
\]
Using the definition of $D$, we get that there are pairwise distinct integers $\ell_1, \ldots, \ell_k \in [n]$ such that
\[
\int_{\Omega_{t_1}} \int_{\Omega_{t_2}} \cdots \int_{\Omega_{t_k}} \prod_{1 \leq i < j \leq k, ij \in E(H)} W(x_i, x_j) > 0.
\]
We conclude that the set
\[
(2.4) \quad E := \left\{ x \in \Omega_{t_1} \times \Omega_{t_2} \times \cdots \times \Omega_{t_k} : \prod_{1 \leq i < j \leq k, ij \in E(H)} W(x_i, x_j) > 0 \right\}
\]
has positive measure.
Given $\epsilon > 0$, let $\delta > 0$ be such that
\[
(2.5) \quad \frac{\nu^\otimes k(E) - \delta}{\nu^\otimes k(E) + \delta} \geq 1 - \frac{\epsilon}{2}.
\]
Recall that the $\sigma$-algebra of all measurable subsets of $\Omega_{t_1} \times \cdots \times \Omega_{t_k}$ is generated by the algebra consisting of all finite unions of boxes. Thus there is a finite union $S = \bigcup_{i=1}^m R_i$ of boxes $R_1, \ldots, R_m$ in $\Omega_{t_1} \times \cdots \times \Omega_{t_k}$ such that $\nu^\otimes k(E \setminus S) + \nu^\otimes k(S \setminus E) \leq \delta$. Without loss of generality, we may assume that the boxes $R_1, \ldots, R_m$ are pairwise disjoint. Then we have
\[
(2.6) \quad \frac{\nu^\otimes k(E \cap S)}{\nu^\otimes k(S)} \geq \frac{\nu^\otimes k(E) - \nu^\otimes k(E \setminus S)}{\nu^\otimes k(E) + \nu^\otimes k(S \setminus E)} \geq \frac{\nu^\otimes k(E) - \delta}{\nu^\otimes k(E) + \delta} \geq 1 - \frac{\epsilon}{2}.
\]
The left-hand side of (2.6) can be expressed as
\[
\frac{\nu^\otimes k(E \cap S)}{\nu^\otimes k(S)} = \sum_{i=1}^m \frac{\nu^\otimes k(R_i)}{\nu^\otimes k(S)} \cdot \frac{\nu^\otimes k(E \cap R_i)}{\nu^\otimes k(R_i)},
\]
i.e. as a convex combination of $\frac{\nu^\otimes k(E \cap R_i)}{\nu^\otimes k(R_i)}$, $i = 1, \ldots, m$. Therefore, there is an index $i_0 \in [m]$ such that
\[
(2.7) \quad \frac{\nu^\otimes k(E \cap R_{i_0})}{\nu^\otimes k(R_{i_0})} \geq 1 - \frac{\epsilon}{2}.
\]
Let $R_{i_0}$ be of the form $R_{i_0} = B_1 \times \cdots \times B_k$. Find a natural number $p$ such that
\[
(2.8) \quad p \geq \frac{2k}{\epsilon \nu^\otimes k(R_{i_0})}.
\]
For every $i = 1, \ldots, k$, we consider a finite decomposition $B_i = B_i^0 \cup \bigcup_{j=1}^{q_i} B_i^j$ of $B_i$ into pairwise disjoint sets, such that $\nu(B_i^0) \leq \frac{1}{p}$ and $\nu(B_i^j) = \frac{1}{p}$ for $j = 1, \ldots, q_i$. Then we clearly have
\[
(2.9) \quad \nu^\otimes k \left( R_{i_0} \setminus \bigcup_{i=1}^k \bigcup_{j=1}^{q_i} B_i^j \right) \geq k \frac{p}{p} = \frac{k}{p},
\]
and so
\[
(2.10) \quad \frac{\nu^\otimes k \left( E \cap \bigcup_{i=1}^k \bigcup_{j=1}^{q_i} B_i^j \right)}{\nu^\otimes k \left( \bigcup_{i=1}^k \bigcup_{j=1}^{q_i} B_i^j \right)} \geq \frac{\nu^\otimes k \left( E \cap R_{i_0} \right) - \frac{k}{p}}{\nu^\otimes k \left( R_{i_0} \right)} \geq 1 - \frac{\epsilon}{2} - \frac{k}{p \nu^\otimes k \left( R_{i_0} \right)} \geq 1 - \epsilon.
\]
The left-hand side of (2.10) can be expressed as the following convex combination:

\[
\nu^\otimes k \left( \frac{E \cap \prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j}{\nu^\otimes k \left( \prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j \right)} \right)
= \sum_{j=1}^{q_1} \cdots \sum_{j=1}^{q_k} \nu^\otimes k \left( \prod_{i=1}^{q_i} B_i^j \right). \]

Therefore by (2.10), there are indices \(j_i \in [q_i], i = 1, \ldots, k\), such that

\[
\nu^\otimes k \left( \frac{\prod_{i=1}^{k} B_i^{j_i} \setminus E}{\nu^\otimes k \left( \prod_{i=1}^{k} B_i^{j_i} \right)} \right) \leq \epsilon.
\]

We set \(A_i = B_i^{j_i}\) for \(i = 1, \ldots, k\). Then \(A_1, \ldots, A_k\) are pairwise disjoint (as \(A_i \subseteq \Omega_{j_i}\) for every \(i\)), and each of these sets has the same measure \(\alpha = \frac{1}{p}\).

By (2.11) we have

\[
\left\{(x_1, x_2, \ldots, x_k) \in A_1 \times A_2 \times \cdots \times A_k : \prod_{1 \leq i < j \leq k, i,j \in E(H)} W(x_i, x_j) = 0 \right\} = \prod_{i=1}^{k} B_i^{j_i} \setminus E.
\]

By (2.11),

\[
\nu^\otimes k \left( \frac{\prod_{i=1}^{k} B_i^{j_i} \setminus E}{\nu^\otimes k \left( \prod_{i=1}^{k} B_i^{j_i} \right)} \right) \leq \epsilon \nu^\otimes k \left( \prod_{i=1}^{k} B_i^{j_i} \right) = \frac{\epsilon}{p^k}.
\]

Consider an arbitrary edge \(ij \in E(H)\). Observe that the set \(\prod_{i=1}^{k} B_i^{j_i} \setminus E\) contains all \(k\)-tuples \(x \in \prod_{i=1}^{k} A_i\) that have the property that \(W(x_i, x_j) = 0\). Therefore, we get from (2.12) that

\[
\nu^\otimes 2 \left\{ (x_i, x_j) \in A_i \times A_j : W(x_i, x_j) = 0 \right\} \leq \epsilon \alpha^2,
\]

as required. \(\square\)

3. INDEPENDENT SETS

While classically, the notion of independent sets and cliques are in one-to-one correspondence by taking complements, our definitions for graphons look at each of these concepts at a different scale. The independence number of a graphon is a number in \([0, 1]\) that should be interpreted as the fraction of vertices in a maximum independent set of a graph that corresponds to that graphon. That is, we scale down the independence number linearly. On the other hand, in Section 5 we define the clique number (and its fractional variant) which are all absolute with no additional rescaling introduced (see Section 5 for a discussion of subtleties). If a graphon contains a copy of \(K_{17}\) then its clique number will be at least 17. Of course, which of the two parameters is scaled and which one is not is just a matter of convention. Let us remark that in [3] a different scale of \(\log n\) is put on these parameters and studied in the context of inhomogeneous random graphs \(G(n, W)\).

The following is then the obvious graphon counterpart to independent sets.

Definition 3. Let \(W : \Omega \times \Omega \to [0, 1]\) be a graphon. A set \(A \subseteq \Omega\) is an independent set if \(W\) is zero almost everywhere on \(A \times A\). Denote by \(I(W)\) the set of independent sets of a graphon \(W\) and for each \(x \in \Omega\) denote by \(I_x(W) \subseteq I(W)\) the set of independent sets of \(W\) containing the point \(x\).

The definition of \(I_x(W)\) may look suspicious as it involves a measure-zero condition (a point belonging to a set). This will not cause a problem since we shall always work with collections \(I_x(W)\) for a set of \(x\)'s of positive measure.
We need to describe particular properties of independent sets in order to understand the behavior of the chromatic number and the clique number for graphons.

The next lemma which appears in [6, Lemma 20] asserts that the weak* limit of independent sets is again an independent set.

**Lemma 4.** Let $W$ be a graphon. Suppose $(A_n)_n$ is a sequence of sets in $\Omega$ with the property that

$$\lim_{n \to \infty} \int_{A_n \times A_n} W = 0.$$ 

Suppose that the indicator functions of the sets $A_n$ converge weak* to a function $f$. Then $\text{supp}(f)$ is an independent set in $W$.

It follows that for a convergent sequence of graphons, the weak* limit of independent sets in the sequence form an independent set in the limit graphon.

**Corollary 5.** Let $W_n : \Omega \times \Omega \to [0,1]$ be a sequence of graphons converging in the cut-norm to $W : \Omega \times \Omega \to [0,1]$. Let $I_n \subset \Omega$ be an independent set in $W_n$. Suppose that the indicator functions of the sets $I_n$ converge weak* to a function $f$. Then $\text{supp}(f)$ is an independent set in $W$.

**Proof.** Notice that

$$\int_{I_n \times I_n} W = \left| \int_{I_n \times I_n} W_n - W \right| \leq \|W_n - W\|_\Box.$$ 

Thus, $\lim_{n \to \infty} \int_{I_n \times I_n} W = 0$ and the claim follows from Lemma 4.

The defining property of weak* convergence gives us that in the setting of the corollary above, we have

$$\lim_n \nu(I_n) = \lim_n \int_\Omega 1_{I_n} = \int_\Omega f \leq \nu(\text{supp}(f)),$$ 

where the last inequality uses that $f$ is bounded above by 1, since it is a weak* limit of functions bounded above by 1. Thus, as a consequence of Corollary 5, we get that the supremum of the measures of independent sets in a graphon is attained. This leads us to the following definition.

**Definition 6.** The measure of the largest independent set of a graphon $W : \Omega \times \Omega \to [0,1]$ is called the independence number of $W$ and denoted by $\alpha(W)$.

Corollary 5 yields upper semicontinuity of the independence number. As mentioned in Section 1.2, the sequence $Y_n \equiv \frac{1}{n} \to Y \equiv 0$ shows that we do not have lower semicontinuity in general.

**Corollary 7.** Suppose that $(W_n)_n$ is a sequence of graphons that converges to $W$ in the cut-distance. Then $\alpha(W) \geq \limsup_n \alpha(W_n)$.

**Proof.** We may as well assume that $(W_n)_n$ converges to $W$ in the cut-norm, and that the limit $\lim_n \alpha(W_n)$ exists. Now, for each $n$, consider an independent set $I_n$ in $W_n$ of size $\alpha(W_n)$. Let $f$ be a weak* accumulation point of the sequence of indicator functions of the sets $(I_n)_n$, at least one such accumulation point exists by the sequential Banach–Alaoglu Theorem. By Corollary 5, $\text{supp}(f)$ is an independent set, and by the same calculation as in (3.1), the measure of $\text{supp}(f)$ is at least $\lim_n \alpha(W_n)$.

### 3.1. Structure of independent set

In this section, we make some observations about the structure of independent sets in a graphon. Often, rather than dealing with all independent sets, it is convenient to restrict attention just to maximal ones, from which all the remaining ones can be easily recovered. We denote the set of maximal independent sets (modulo nullsets) in a graphon $W$ by $\mathcal{I}_{\text{max}}(W) \subset \mathcal{I}(W)$. Even the set $\mathcal{I}_{\text{max}}(W)$ can be quite complicated, at least with respect to its cardinality. Indeed, let $W$ be a graphon representing a disjoint union of countably many complete bipartite graphs $(H_n = A_n \sqcup B_n)_{n=1}^\infty$, which occupy measure $2^{-n}$ each. Then the maximal independent sets in $W$ are all unions of maximal independent sets in all graphs $H_n$, (for example $A_1, A_2, A_3, B_4, B_5, \ldots$), which there are uncountably many.
Another perspective on the structure of $\mathcal{I}(W)$ comes from polyhedral combinatorics. To motivate this, let us first recall an approach common for finite graphs. Given a finite graph $G$, each independent set $I$ in $G$ can viewed as a vector in $\{0, 1\}^{V(G)} \subseteq \mathbb{R}^{V(G)}$. Taking a convex hull of all such vectors, one gets what is called an independent set (or stable set) polytope $\text{IND}(G) \subseteq \mathbb{R}^{V(G)}$. Among many classical results about $\text{IND}(G)$, let us mention perhaps the most basic one.

**Proposition 8** (e.g. [5 (9.1.3)]). For any graph $G$, each point $x \in \text{IND}(G)$ satisfies the following two families of conditions:

\begin{align}
(3.2) & \quad 0 \leq x_v \leq 1 \quad \text{for each } v \in V(G) \text{ and} \\
(3.3) & \quad x_u + x_v \leq 1 \quad \text{for each } uv \in E(G) .
\end{align}

Further, (3.2) and (3.3) characterize $\text{IND}(G)$ if and only if $G$ is bipartite.

Now, in the graphon setting we proceed as follows. We represent each set $I \in \mathcal{I}(W)$ of a graphon $W : \Omega \times \Omega \rightarrow [0, 1]$ by its characteristic function, which we view as an element in $\{0, 1\}^\Omega \subseteq \mathbb{R}^\Omega$. We can now take the closure (in the weak* topology) of the convex hull of such functions and get what we call independent set polyton $\text{IND}(W) \subseteq \mathbb{R}^\Omega$. Such an graphon approach to polyhedral combinatorics has been introduced [4], namely for the so-called matching polytope/polyton. To illustrate the potential of this area, let us prove a part of a counterpart to Proposition 8.

**Proposition 9.** For any graph $W : \Omega \times \Omega \rightarrow [0, 1]$, each point $x \in \text{IND}(W)$ satisfies the following two families of conditions:

\begin{align}
(3.4) & \quad 0 \leq x_v \leq 1 \quad \text{for almost all } v \in \Omega \text{ and} \\
(3.5) & \quad x_u + x_v \leq 1 \quad \text{for almost all } uv \in \text{supp}(W) .
\end{align}

Further, if $W$ is not bipartite, then (3.4) and (3.5) do not characterize $\text{IND}(W)$.

**Problem 10.** In analogy with Proposition 8 establish the other direction in Proposition 9.

**Proof of Proposition 9**. Obviously, every indicator function of an independent set $I \in \mathcal{I}(W)$ satisfies (3.4) and (3.5). These inequalities are then inherited to convex combinations and closure, thus proving the first part.

Suppose now that $W$ is not bipartite. We shall prove that the point $y \equiv \frac{1}{2}$ is not in $\text{IND}(W)$, even though it apparently satisfies (3.4) and (3.5). By [4, Proposition 21] (stated also as Proposition 14 below), we know that $t(C_{2\ell+1}, W) > 0$ for some $\ell \in \mathbb{N}$. By Lemma 2, we know that there exist disjoint sets $A_1, A_2, \ldots, A_{2\ell+1} \subseteq \Omega$ of the same measure $\alpha > 0$ such that that $W$ is positive everywhere on $A_i \times A_j$ except a set of measure at most $\frac{\alpha}{2^{2\ell}}$. Now, suppose for contradiction that $y$ is in the weak* closure of convex combinations of characteristic functions of independent sets. In particular, there exists a point $y^*$ such that we have

\begin{align}
\int_{\omega \in \bigcup_{i=1}^{2\ell+1} A_i} y^*(\omega) \geq \int_{\omega \in \bigcup_{i=1}^{2\ell+1} A_i} y(\omega) - \frac{\alpha}{10} = \frac{1}{2}(2\ell + 1)\alpha - \frac{\alpha}{10}
\end{align}

and such that $y^*$ is a convex combination of characteristic functions of independent sets, $y^* = \sum_{j=1}^{\ell} \alpha_j I_{j}$, where $\alpha_j \geq 0$ are the convex coefficients and $I_j \in \mathcal{I}(W)$. We get (3.6) must hold also for one of the terms appearing in the convex combination, i.e., there exists $j \in [\ell]$ such that

\begin{align}
\nu \left( I_j \cap \bigcup_{i=1}^{2\ell+1} A_i \right) \geq \frac{1}{2}(2\ell + 1)\alpha - \frac{\alpha}{10} .
\end{align}

Call an index $i \in [2\ell + 1]$ marked if $\nu (I_j \cap A_i) \geq \frac{\alpha}{10}$. Observe that we cannot have two consecutive marked indices $i$ and $i+1$ (with labeling considered is modulo $2\ell + 1$), since $W_{[A_i \times A_{i+1}}$ is positive on
most of the domain and \( I_j \) is an independent set. Hence, there are at most \( 2\ell \) marked indices. For a marked index \( i \) we have \( \nu(I_j \cap A_i) \leq \alpha \) and for an unmarked index \( i \) we have \( \nu(I_j \cap A_i) \leq \frac{\alpha}{5\ell} \). Hence,

\[
\nu \left( I_j \cap \bigcup_{i=1}^{2\ell+1} A_i \right) \leq \ell \cdot \alpha + (\ell + 1) \cdot \frac{\alpha}{5\ell}.
\]

This contradicts (5.7). \( \square \)

4. Coloring concepts

4.1. Ordinary colorings. First, we define a counterpart to the usual concept of coloring.

**Definition 11.** Let \( W : \Omega \times \Omega \to [0,1] \) be a graphon. We say that a measurable function \( f : \Omega \to [k] \) is a coloring of \( W \) with \( k \) colors if we have that for each \( i \in [k] \), the set \( f^{-1}(i) \) is an independent set in \( W \). The chromatic number \( \chi(W) \) is defined as the minimal number of colors used in any coloring. Here, as will be with other versions of the chromatic number, when no coloring of \( W \) exists, \( \chi(W) \) is taken to be infinity. In particular, this notion splits the space of graphons into graphons of finite and infinite chromatic number.

The next easy proposition shows that Definition 11 is consistent with the graph definition of the chromatic number.

**Proposition 12.** Suppose \( G \) is a finite graph and let \( W : \Omega \times \Omega \to [0,1] \) be its graphon representation. Then \( \chi(G) = \chi(W) \).

**Proof.** Let \( \Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_n \) be the partition for \( W \) corresponding to the vertices of \( G \). If \( U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k = V(G) \) is a \( k \)-coloring of \( G \), then the independent sets \( (\bigcup_{i \in [k]} \Omega_i) \) give a \( k \)-coloring of \( W \). Thus, \( \chi(W) \leq \chi(G) \). Conversely, suppose that \( I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k = \Omega \). Now, for every \( j \in [n] \), there exists at least one index, say \( \ell_j \in [k] \), such that \( \Omega_j \cap I_{\ell_j} \) has positive measure. Obviously, the map \( j \mapsto \ell_j \) is a proper coloring of \( G \). \( \square \)

The graphons \( Y_n \to Y \) from Section 1.2 show that the chromatic number is not upper continuous. Indeed, we have \( \chi(Y_n) = \infty \) and \( \chi(Y) = 1 \). On the other hand, the following theorem shows lower semicontinuity.

**Theorem 13.** Let \( W : \Omega \times \Omega \to [0,1] \) be a graphon.

\( a) \) For every sequence of graphs \( (H_n) \) that converges to \( W \) in the cut-distance we have \( \chi(W) \leq \liminf_{n \to \infty} \chi(H_n) \).

\( b) \) For the sequence \( (G_n : G(n,W)) \) we have \( \chi(W) = \lim_{n \to \infty} \chi(G_n) \), almost surely.

The next proposition, which can be found in [3] Proposition 21], is the key towards proving Theorem 13. We remark that the proof of this proposition given in [3] is short but non-trivial.

**Proposition 14.** Suppose \( k \in \mathbb{N} \) and \( W : \Omega \times \Omega \to [0,1] \) is a graphon. Then \( \chi(W) \leq k \) if and only if it has zero density of each finite graph \( G \) with \( \chi(G) \geq k + 1 \).

**Proof of Theorem 13(a).** Suppose that \( \chi(W) \geq k \). By (the more difficult part of) Proposition 14, there exists a graph \( F \) with chromatic number at least \( k \) that has positive density in \( W \). Since convergence in the cut-distance implies convergence of subgraph densities, we conclude that almost every graph \( H_n \) contains a copy of \( F \). Each such graph \( H_n \) has therefore chromatic number at least \( k \), as needed. \( \square \)

**Proof of Theorem 13(b).** Since the graphs \( G_n \) converge to \( W \) in the cut-distance almost surely, we have \( \chi(W) \leq \liminf_{n \to \infty} \chi(G_n) \) by (a). On the other hand, if \( \chi(W) \leq k \), then by (the easy part of) Proposition 14 each graph of chromatic number at least \( k + 1 \) has zero density. That is, the probability of \( W \)-sampling any such graph is zero. \( \square \)
4.2. Fractional colorings. The next definition is a straightforward counterpart to the graph definition of fractional colorings.

**Definition 15.** Suppose that \( W : \Omega \times \Omega \to [0, 1] \) is a graphon. We say that a function \( c : \mathcal{I}(W) \to [0, 1] \) is a fractional coloring of \( W \) if for almost every \( x \in \Omega \) it holds \( \sum_{I \in \mathcal{I}(W)} c(I) \geq 1 \). The fractional chromatic number of \( W \) is defined as

\[
\chi_{\text{frac}}(W) = \inf \sum_{I \in \mathcal{I}(W)} c(I),
\]

where the infimum is taken over the set of all fractional colorings of \( W \).

Notice that whenever \( \sum_{I \in \mathcal{I}(W)} c(I) \) is finite, then the support of the function \( c \) is at most countable. Thus it suffices to take the infimum over the set of all fractional colorings of \( W \) that are not zero at most in a countable subset of \( \mathcal{I}(W) \).

The next proposition shows that the definitions of fractional chromatic number for graphs and graphons are consistent.

**Proposition 16.** Suppose \( G \) is a finite graph and let \( W \) be its graphon representation. Then \( \chi_{\text{frac}}(G) = \chi_{\text{frac}}(W) \).

**Proof.** Denote by \( \mathcal{I}(W) \) and \( \mathcal{I}(G) \) the set of independent sets of \( W \) and \( G \), respectively. There is a map \( \pi : \mathcal{I}(G) \to \mathcal{I}(W) \) which maps independent sets in \( \mathcal{I}(G) \) to corresponding independent sets in \( \mathcal{I}(W) \). If \( c_G \) is a fractional coloring of \( G \) then we can define \( c_W : \mathcal{I}(G) \to [0, 1] \),

\[
c_W(I) := c_G(\pi(I)).
\]

Extending \( c_W \) by zeros to \( \mathcal{I}(W) \), we get a fractional coloring of \( W \) of the same size. This implies \( \chi_{\text{frac}}(G) \geq \chi_{\text{frac}}(W) \).

Now consider a fractional coloring \( c_W : \mathcal{I}(W) \to [0, 1] \) of \( W \). Let \( \Omega = \Omega_1 \cup \cdots \cup \Omega_n \) be the partition for \( W \) corresponding to the vertices of \( G \). For each \( v \in V(G) \), choose a point \( x_v \in \Omega_v \). We shall assume that \( \sum_{I \in \mathcal{I}(W)} c_W(I) \geq 1 \) for each \( v \in V(G) \); this is true for almost all choices of \( x_v \) in \( \Omega_v \). Now define \( c_G : \mathcal{I}(G) \to [0, 1] \) by

\[
c_G(J) := \sum_{I \in J} c_W(I),
\]

where \( I \) ranges over all sets in \( \mathcal{I}(G) \setminus \bigcup_{v \in V(G)} \mathcal{I}(x_v) \). The inclusion-exclusion formula implies that for each \( v \in V(G) \) we have \( \sum_{I \in \mathcal{I}(G)} c_G(J) = \sum_{I \in \mathcal{I}(W)} c_W(I) \geq 1 \), for each \( v \in V(G) \) we have \( \sum_{I \in \mathcal{I}(G)} c_G(J) \geq 1 \). Therefore, \( c_G \) is a valid fractional coloring of \( G \). Thus \( \chi_{\text{frac}}(G) \leq \chi_{\text{frac}}(W) \), which finishes the proof. \( \square \)

**Remark 17.** Observe that in the proof of Proposition 16 we did not use that each vertex of \( G \) is represented by a set of \( \Omega \) of measure \( \frac{1}{\chi(G)} \). That is, the same results holds true if vertices of \( G \) are represented by arbitrary sets \( \Omega_1 \cup \cdots \cup \Omega_n = \Omega \) of positive but not necessarily the same measure.

**Example 18.** In [7], Leader constructs a graph \( R \) with a countable vertex set, say \( \mathbb{N} \), fractional chromatic number of which is strictly greater than the supremum of the fractional chromatic numbers of its finite subgraphs. Considering an arbitrary partition \( \Omega = \bigsqcup_{n \in \mathbb{N}} \Omega_n \) into sets of positive measure, and taking \( W \) to be a graphon representation of \( R \) with respect to the partition \( \Omega = \bigsqcup_{n \in \mathbb{N}} \Omega_n \), we get by a version of Proposition 16 (see also Remark 17) that \( \chi_{\text{frac}}(W) > \sup \{ \chi_{\text{frac}}(G) : t(G, W) > 0 \} \). Note that this example also shows that \( \chi_{\text{frac}}(\cdot) \) is not lower semicontinuous. Indeed, for \( \ell = 1, 2, 3, \ldots \), we consider graphon \( W_\ell \) which is equal to \( W \) on \( \bigcup_{\ell=1}^{\ell} \Omega_\ell \) and 0 elsewhere. Obviously, the graphons \( (W_\ell)_\ell \) converge to \( W \) in the cut-norm (actually, even pointwise), but we have have \( \lim_\ell \chi_{\text{frac}}(W_\ell) = \sup \{ \chi_{\text{frac}}(G) : t(G, W) > 0 \} < \chi_{\text{frac}}(W) \).

We also note that in [7] Leader constructs another countable graph \( T \) in which \( \chi_{\text{frac}}(T) \) is not attained by any fractional coloring. Taking a graphon representation of \( T \), we see that the fractional chromatic number of a graphon need not be attained.

\(^1\)For countable graphs; the proof of such a version is mutatis mutandis.
4.3. $b$-fold colorings. Next, we define counterparts of $b$-fold colorings for graphons.

Definition 19. Suppose that $b \in \mathbb{N}$. For a graphon $W : \Omega \times \Omega \to [0,1]$, a measurable map $p : \Omega \to \{\frac{[k]}{b}\}$ is a $b$-fold coloring of $W$ with $k$ colors if for almost all pairs $(x,y) \in \Omega \times \Omega$ we have $W(x,y) = 0$ or $p(x) \cap p(y) = \emptyset$. The $b$-fold chromatic number $\chi_b(W)$ is the smallest number of colors necessary to construct a proper $b$-fold coloring of $W$.

As in the graph case, the relation between the fractional chromatic number, the $1$-fold chromatic number and the ordinary chromatic number is as follows: $\chi_{\text{frac}}(W) \leq \chi(W) = \chi_1(W)$. The next theorem shows that at least qualitatively, we can reverse the inequality.

Proposition 20. Let $W : \Omega \times \Omega \to [0,1]$ be a graphon. If $\chi_{\text{frac}}(W) < \infty$ then $\chi(W) < \infty$.

Proof. Suppose that $c$ is a fractional coloring of $W$ with $\sum_{I \in \mathcal{I}(W)} c(I) < \infty$. Thus, there is a finite number of independent sets $\{I_1, \ldots, I_k\}$ such that

\[
\sum_{i=1}^{k} c(I_i) > \sum_{I \in \mathcal{I}(W)} c(I) - 1.
\]

In particular, for each $x \in \Omega$ we have

\[
\sum_{I \in \mathcal{I}_x(W)} c(I) < \sum_{i=1}^{k} c(I_i) + 1.
\]

This implies that if $x \in \Omega \setminus \bigcup_{i=1}^{k} I_i$ then $\sum_{I \in \mathcal{I}_x(W)} c(I) < 0 + 1$. We conclude that $\Omega \setminus \bigcup_{i=1}^{k} I_i$ is a nullset. Therefore, $\{I_1, \ldots, I_k\}$ corresponds to a finite coloring of $W$. \hfill \Box

A sequence $\{a_n\}_{n=1}^\infty$ is said to be subadditive if for every $m$ and $n$ we have $a_{m+n} \leq a_m + a_n$. Fekete’s Lemma is a useful result concerning subadditive sequences.

Lemma 21 (Fekete’s Lemma). For every subadditive sequence $\{a_n\}_{n=1}^\infty$, the limit $\lim_{n \to \infty} \frac{a_n}{n}$ exists and is equal to $\inf \frac{a_n}{n}$.

If we have an $a$-fold coloring $p : \Omega \to \{\frac{[1, \ldots, k]}{a}\}$ of a graphon $W$ which uses a palette $\{1, \ldots, k\}$ and a $b$-fold coloring $q : \Omega \to \{\frac{[k+1, \ldots, k+\ell]}{b}\}$ which uses a palette $\{k+1, \ldots, k+\ell\}$, we observe that the map $x \mapsto p(x) \cup q(x)$ is an $(a+b)$-fold coloring of $W$ which uses a palette $\{1, \ldots, k+\ell\}$. Therefore, we have $\chi_{a+b}(W) \leq \chi_a(W) + \chi_b(W)$. Fekete’s lemma tells us that

\[
\lim_{b \to \infty} \frac{\chi_b(W)}{b} = \inf_{b \in \mathbb{N}} \frac{\chi_b(W)}{b}.
\]

The next theorem tells us that this limit equals to the fractional chromatic number of $W$.

Theorem 22. For a graphon $W$, it holds

\[
\chi_{\text{frac}}(W) = \lim_{b \to \infty} \frac{\chi_b(W)}{b} = \inf_{b \in \mathbb{N}} \frac{\chi_b(W)}{b}.
\]

Proof. First, we prove that $\chi_{\text{frac}}(W) \leq \inf_{b \to \infty} \frac{\chi_b(W)}{b}$. Given $b \in \mathbb{N}$, fix a proper $b$-fold coloring $c_b$ of $W$ for the $b$-chromatic number $\chi_b$, say using $\ell$ colors. We will construct a fractional coloring $c$ such that

\[
\sum_{I \in \mathcal{I}(W)} c(I) = \frac{\chi_b(W)}{b}.
\]

To this end, for each $j \in [\ell]$, consider the independent set $I_j := \{x \in \Omega : j \in c_b(x)\}$, and for each such set, define $c(I_j) = 1/b$. We have,

\[
\sum_{I \in \mathcal{I}(W)} c(I) = 1,
\]
for almost all \( x \in \Omega \), and
\[
\sum_{I \in \mathcal{I}(W)} c(I) = \frac{\chi_b}{b},
\]
as required.

It remains to prove that \( \chi_{\text{frac}}(W) \geq \inf_{b \to \infty} \frac{\chi_b(W)}{b} \). We can assume \( \chi_{\text{frac}}(W) \) is bounded. The next claim allows us to restrict ourselves to fractional colorings with weights that are a multiple of \( \frac{1}{b} \).

**Claim.** Suppose that \( \epsilon > 0 \) is fixed. For any \( b \) sufficiently large, there exists a fractional coloring \( c : \mathcal{I}(W) \to \{ \frac{j}{b} : j = 0, 1, \ldots, b \} \) with finite support such that \( \sum_{I \in \mathcal{I}(W)} c(I) \leq \chi_{\text{frac}}(W) + \epsilon \).

**Proof of Claim.** Take \( \delta > 0 \) such that
\[
\frac{\chi_{\text{frac}}(W) + \delta}{1 - \delta} \leq \frac{\chi_{\text{frac}}(W) + \epsilon}{2}.
\]
Take a fractional coloring \( c_0 \) of \( W \) such that \( \sum_{I \in \mathcal{I}(W)} c_0(I) \leq \chi_{\text{frac}}(W) + \delta \). Since \( \sum_{I \in \mathcal{I}(W)} c_0(I) \) is finite, we can find a finite subset \( \mathcal{I}_0 \subseteq \mathcal{I}(W) \) such that
\[
\sum_{I \in \mathcal{I}(W) \setminus \mathcal{I}_0} c_0(I) < \delta.
\]
Define a function \( c_1 \) by setting it equal to \( c_0 \) if \( I \subseteq \mathcal{I}_0 \) and 0 otherwise. The function \( c_1 \) is not necessarily a valid fractional coloring, but we can turn it into a fractional coloring \( c_2, c_2(I) := \frac{c_1(I)}{1 - \delta} \). The fact that \( c_2 \) is a valid fractional coloring follows from (4.3). We have
\[
\sum_{I \in \mathcal{I}(W)} c_2(I) = \frac{\sum_{I \in \mathcal{I}(W)} c_1(I)}{1 - \delta} \leq \frac{\chi_{\text{frac}}(W) + \delta}{1 - \delta} \leq \frac{\chi_{\text{frac}}(W) + \epsilon}{2},
\]
by our choice of \( \delta \).

Let \( b > \frac{2|\mathcal{I}_0|}{\epsilon} \). We can define \( c \) by rounding up \( c_2(I) \) to the closest multiple of \( 1/b \). This way, \( c \) is still a valid coloring, where we have increased the total sum of weight on independent sets by at most \( \frac{|\mathcal{I}_0|}{b} \). Thus, we obtain
\[
\sum_{I \in \mathcal{I}(W)} c(I) \leq \sum_{I \in \mathcal{I}(W)} c_2(I) + \frac{|\mathcal{I}_0|}{b} \leq \sum_{I \in \mathcal{I}(W)} c_2(I) + \frac{\epsilon}{2} \leq \chi_{\text{frac}}(W) + \epsilon.
\]
Now, \( c \) is a rational fractional coloring with common denominator \( b \), as required. That finishes the proof of the Claim.

Let us fix \( \epsilon > 0 \). Now, for each \( b \) sufficiently large, we now describe how to transform a fractional coloring \( c : \mathcal{I}(W) \to \{ \frac{j}{b} : j = 0, 1, \ldots, b \} \) from the Claim (for the given \( \epsilon \)) into a \( b \)-fold coloring \( p : \Omega \to (\frac{|K|}{b}) \), where \( K = b \sum_{I \in \mathcal{I}(W)} c(I) \) (recall that \( K \) is an integer). Actually, it is slightly easier to allow \( p \) to map to subsets of \( [K] \) of cardinality at least \( b \). Any such \( p \) can be obviously corrected eventually. Let us partition \( [K] \) into sets \( \bigcup_{I \in \mathcal{I}(W)} A_I \) so that the size of each set \( A_I \) is \( b \cdot c(I) \). Then for each \( x \in \Omega \) we define \( p(x) := \bigcup_{I \in A_x} A_I \). The map \( p \) is obviously a proper \( b \)-fold coloring with \( \sum_{I \in \mathcal{I}(W)} c(I) + \epsilon \geq \frac{\chi_b(W)}{b} \), as was needed.

### 5. Cliques and fractional cliques

Below, we shall define the clique number and the fractional clique number of a graphon.
5.1. Cliques. We first introduce the clique number of a graphon.

**Definition 23.** For a graphon \( W \), the **clique number** \( \omega(W) \) is defined as

\[
\omega(W) = \max \{ r : t(K_r, W) > 0 \}.
\]

Lower semicontinuity of the clique number follows immediately from the definition. (As mentioned in Section[1.2] the sequence \( Y_n \equiv \frac{1}{n} \to Y \equiv 0 \) shows that we do not have upper semicontinuity in general.)

**Proposition 24.** For every sequence of graphons \( (W_n) \) that converges to \( W \) in the cut-distance we have

\[
\omega(W) \leq \liminf_{n \to \infty} \omega(W_n).
\]

**Proof.** If \( \omega(W) \geq r \), then \( t(K_r, W) > 0 \). Since subgraph densities are continuous in the cut-distance, we have \( t(K_r, W_n) > 0 \) for almost all \( n \). In particular, \( \omega(W_n) \geq r \) for almost all \( n \). \( \square \)

5.2. Fractional cliques. In analogy with the finite counterpart, we make the following definition.

**Definition 25.** Suppose that \( W : \Omega \times \Omega \to [0, 1] \) is a graphon. We say that a measurable function \( f : \Omega \to [0, +\infty) \) is a **fractional clique** if for every \( I \in \mathcal{I}(W) \) we have \( \int_I f \leq 1 \). The size of a fractional clique \( f \) is \( ||f|| := \int f \). We define fractional clique number of \( W \) as

\[
\omega_{\text{frac}}(W) = \sup \int_{\Omega} f,
\]

where the supremum is taken over all fractional cliques in \( W \).

In analogy with Proposition[16], first we prove that the fractional clique number was introduced in a way that is consistent with the graph version.

**Proposition 26.** Suppose \( G \) is a finite graph and let \( W \) be its graphon representation. Then \( \omega_{\text{frac}}(G) = \omega_{\text{frac}}(W) \).

**Proof.** Let \( \Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_n \) be the partition for \( W \) corresponding to the vertices of \( G \). For any fractional clique \( x : V(G) \to [0, \infty) \) of the graph \( G \) define a function \( f : \Omega \to [0, \infty) \) with constant value \( x(v) \) in the interval \( \Omega_v \) for each \( v \in V(G) \). There is a map \( \pi : \mathcal{I}(G) \to \mathcal{I}(W) \) which maps independent sets in \( \mathcal{I}(G) \) to corresponding independent sets in \( \mathcal{I}(W) \). Let us verify that \( f \) is a fractional clique for \( W \). Let \( I \in \mathcal{I}(W) \) be arbitrary. Let \( J \subseteq V(G) \) consist of the vertices \( v \) for which \( I \cap \Omega_v \) has positive measure. Then \( J \) is an independent set in \( G \). We then have

\[
\int_I f \leq \sum_{v \in J} \int_{\Omega_v} f = \sum_{v \in J} x(v) \leq 1,
\]

where the last inequality follows uses that \( x \) in a fractional clique in \( G \). Obviously, for the size of \( f \) we have \( \int_I f = \sum_{v \in V(G)} x(v) \). Therefore, \( \omega_{\text{frac}}(G) \leq \omega_{\text{frac}}(W) \).

Now suppose \( f : \Omega \to [0, \infty) \) is any fractional clique of \( W \). Define a function \( x : V(G) \to [0, \infty) \), where \( x(v) = \int_{\Omega_v} f \). For each independent set of \( f \in \mathcal{I}(G) \), we have

\[
\sum_{v \in J} x(v) = \sum_{v \in J} \int_{\Omega_v} f = \int_{\pi(J)} f \leq 1.
\]

Thus, \( x \) is a valid fractional clique for \( G \) with size

\[
\sum_{v \in V(G)} x(v) = \sum_{v \in V(G)} \int_{\Omega_v} f = \int_{\Omega} f.
\]

That implies \( \omega_{\text{frac}}(G) \geq \omega_{\text{frac}}(W_G) \), and finishes the proof. \( \square \)

We now prove lower semicontinuity of the fractional clique number.

**Theorem 27.** Let \( (W_n) \) be a sequence of graphons converging to \( W \) in the cut-norm. Then

\[
\omega_{\text{frac}}(W) \leq \liminf_{n \to \infty} \omega_{\text{frac}}(W_n).
\]
### Proposition 30.

Let \( f : \Omega \to [0, \infty) \) be a fractional clique of \( W \). We will show that for each graphon \( W_n \) there is a fractional clique \( f_n \) such that \( \liminf_{n \to \infty} \int_{\Omega} f_n \geq \int_{\Omega} f \).

For each \( n \in \mathbb{N} \), define

\[
\omega_n = \sup_{I \in \mathcal{I}(W_n)} \int_I f.
\]

We claim that \( \limsup_{n \to \infty} \omega_n \leq 1 \). Suppose by contradiction there exists \( \delta > 0 \) such that \( \limsup_{n \to \infty} \omega_n > 1 + \delta \). Fix a subsequence \( (\omega_{n_i})_{i \in M} \) with \( \omega_{n_i} > 1 + \delta \) for each \( i \in M \). Thus, for each \( i \in M \) we can find \( I_i \in \mathcal{I}(W_{n_i}) \) such that \( \int_{I_i} f > 1 + \delta \). By the sequential Banach–Alaoglu Theorem, the sequence of indicator functions \( I_i \in L^\infty(\Omega) \) has a weak* accumulation point, say \( g \in L^\infty(\Omega) \). Note that \( g \) is bounded from above by 1. Define \( I = \text{supp} (g) \). By Corollary \( \Box \) \( I \) is an independent set of \( W \). We have

\[
1 + \delta \leq \liminf_{i \in M} \int_{I_i} f = \liminf_{i \in M} \int_{\Omega} f \cdot 1_{I_i} \leq \int_{\Omega} f \cdot g \leq \int_{\Omega} f,
\]

which contradicts the fact that \( f \) is a fractional clique and \( I \) is independent set for \( W \).

Since \( \limsup_{n \to \infty} \omega_n \leq 1 \), we can define for each \( W_n \) a valid fractional clique \( f_n : \Omega \to [0, \infty) \) for \( W_n \) by \( f_n = \frac{1}{\omega_n} \). Furthermore, \( (f_n)_n \) satisfy \( \liminf_{n \to \infty} \int_{\Omega} f_n \geq \int_{\Omega} f \), as required. \( \square \)

The next relations between independence number, fractional clique number, and chromatic number are expected from what we know in the graph world and this elementary proof is a straightforward adaptation from the discrete case.

### Theorem 28.

For any graphon \( W : \Omega \times \Omega \to [0, 1] \) we have \( 1 \leq \omega(W) \).

**Proof.** Define a function \( f : \Omega \to [0, \infty) \) by \( f(x) = 1/\alpha(W) \). Clearly, \( f \) is a valid fractional clique, since for every independent set \( I \in \mathcal{I}(W) \) we have \( \int_{I} f = \frac{\nu(I)}{\alpha(W)} \leq 1 \). Thus, \( \omega(W) = \int f = 1/\alpha(W) \) and the claim follows. \( \square \)

### Theorem 29.

For any graphon \( W : \Omega \times \Omega \to [0, 1] \) we have \( 1 \leq \chi(W) \).

**Proof.** Consider a decomposition of \( \Omega \) into independent sets \( I_1 \sqcup I_2 \sqcup \ldots \sqcup I_{\chi(W)} = \Omega \mod 0 \). Now, the inequality follows from

\[
1 = \sum_{j=1}^{\chi(W)} \int_{I_j} 1 \leq \sum_{j=1}^{\chi(W)} \alpha(W) = \chi(W) \alpha(W).
\]

\( \square \)

Alternatively, the proof of the previous result would follow from Theorem 28 if we would have the inequality \( \omega(W) \leq \chi(W) \). This inequality will only be obtained in Section 6 where we investigate duality properties between these parameters. That is why at this point we decided to get it from the same argument given for graphs.

However, certain properties can be more challenging to obtain. For instance, from what we know about fractional cliques in graphs it is not surprising that if a graph \( G \) satisfy \( t(G, W) > 0 \), then \( \omega(G) \leq \omega(W) \). In the discrete world, that is a direct consequence of the definition. However, for graphons this statement requires a technical proof.

### Proposition 30.

Suppose that \( G \) is a graph and \( W \) is a graphon with \( t(G, W) > 0 \). Then \( \omega(G) \leq \omega(W) \).

**Proof.** Let \( \epsilon > 0 \) and let \( g : V(G) \to [0, 1] \) be a fractional clique of \( G \). To prove the statement we shall find a fractional clique \( f : \Omega \to [0, \infty) \) for \( W \) such that

\[
\int_{\Omega} f \geq (1 - \epsilon) \sum_{v \in V(G)} g(v).
\]
Let $\delta = \left(\frac{\epsilon}{|V(G)|}\right)^2$. Since $t(G, W) > 0$, by Lemma 2 we can find $\beta > 0$ disjoint sets $A_v \subseteq \Omega, v \in V(G)$, with positive measure $\nu(A_v) = \beta$ such that for each $uv \in E(G)$,

\[
\mu^\otimes 2(A_v \times A_u \setminus \text{supp}(W)) < \delta \beta^2. 
\]

Define

\[
f(x) := \begin{cases} \frac{(1-\epsilon)}{\beta} g(v) & \text{for } x \in A_v \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, we obtain

\[
\int \Omega f = (1-\epsilon) \sum_{v \in V(G)} g(v).
\]

It remains to prove that for each $I \in \mathcal{I}(W)$ we have $\int_I f \leq 1$. It follows from (5.1) that for each $uv \in E(G)$ we have

\[
\mu(I \cap A_v) \leq \sqrt{\delta} \beta \text{ or } \mu(I \cap A_u) \leq \sqrt{\delta} \beta.
\]

Let $J \subset V(G)$ be the set of vertices for which $\mu(I \cap A_v) > \sqrt{\delta} \beta$. By (5.2) it follows that $J$ is an independent set. Therefore, $\sum_{v \in J} g(v) \leq 1$. We have

\[
\int_I f \leq \sum_{v \in J} \int_{A_v} f + \int_{I \setminus \cup_{v \in J} A_v} f.
\]

The first term we can bound by

\[
\sum_{v \in J} \int_{A_v} f \leq \sum_{v \in J} \beta \frac{(1-\epsilon)}{\beta} g(v) \leq (1-\epsilon) \sum_{v \in J} g(v) \leq 1 - \epsilon.
\]

For the second term, we have

\[
\int_{I \setminus \cup_{v \in J} A_v} f = \sum_{v \in V(G) \setminus J} \int_{I \cap A_v} f.
\]

For each $v \in V(G) \setminus J$, we have

\[
\int_{I \cap A_v} f \leq \sqrt{\delta} \beta \frac{(1-\epsilon)}{\beta} g(v)
\]

\[= \sqrt{\delta} (1-\epsilon) g(v)
\]

\[\leq \frac{\epsilon}{|V(G)|} (1-\epsilon) g(v)
\]

\[\leq \frac{\epsilon}{|V(G)|}.
\]

Therefore,

\[
\int_{I \setminus \cup_{v \in J} A_v} f \leq \sum_{v \in V(G) \setminus J} \frac{\epsilon}{|V(G)|} \leq \epsilon.
\]

By (5.3), (5.4), and (5.5), it follows that $\int_I f \leq 1$ and therefore $f$ is a valid fractional clique for $W$. That finishes the proof.

The results above allow us to express the fractional clique number of a graphon using the same parameter of finite graphs appearing in it.

**Corollary 31.** Suppose that $W$ is a graphon. Then $\omega_{\text{frac}}(W) = \sup_{G : t(G, W) > 0} \omega_{\text{frac}}(G)$. 

Theorem 34 \[ \text{since} \]

Proposition 32.

Proposition 30 gives \( \omega_{\text{frac}}(W) \geq \sup_{G: t(G,W) > 0} \omega_{\text{frac}}(G) \). On the other hand, taking \( G_n \sim G(n,W) \), we get a sequence \( (G_n) \) of graphs which all satisfy (almost surely) that \( t(G_n,W) > 0 \). Considering arbitrary graphon representations \( W_n \) of these graphs \( G_n \), we know that \( W_n \) converge to \( W \) in the cut-distance. Thus Theorem 33 tells us that \( \omega_{\text{frac}}(W) \leq \liminf_{n \to \infty} \omega_{\text{frac}}(W_n) \). As \( \omega_{\text{frac}}(W_n) = \omega_{\text{frac}}(G_n) \) by Proposition 26, we also prove the inequality \( \omega_{\text{frac}}(W) \leq \sup_{G: t(G,W) > 0} \omega_{\text{frac}}(G) \). \( \square \)

Last, we relate the fractional clique number and the integral clique number of a graphon.

Proposition 32. Suppose that \( W \) is a graphon. Then \( \omega_{\text{frac}}(W) \geq \omega(W) \).

Proof. By Corollary 31 we have \( \omega_{\text{frac}}(W) = \sup_{G: t(G,W) > 0} \omega_{\text{frac}}(G) \). Further, Definition 23 gives \( \omega(W) = \sup_{G: t(G,W) > 0} \omega(G) \). We thus get the statement combining these two relations together with the fact that \( \omega_{\text{frac}}(G) \geq \omega(G) \) for each finite graph. \( \square \)

6. Duality between fractional cliques and fractional coloring

The classical LP duality states that for a finite graph \( G \) we have that

\[
\omega_{\text{frac}}(G) = \chi_{\text{frac}}(G).
\]

Note that such a relation cannot hold for graphons in general. Indeed, taking the graphon \( W \) constructed in Example 18, we get that

\[
\omega_{\text{frac}}(G) = \sup_{G: t(G,W) > 0} \omega_{\text{frac}}(G) \quad \text{(6.1)}
\]

\[
\sup_{G: t(G,W) > 0} \chi_{\text{frac}}(G) < \chi_{\text{frac}}(W).
\]

So, in this section we establish the maximum that can be in this situation established: weak LP duality and complementary slackness.

Theorem 33 (Weak LP duality). For every graphon \( W: \Omega \times \Omega \to [0,1] \) we have

\[
\omega_{\text{frac}}(W) \leq \chi_{\text{frac}}(W).
\]

Proof. Suppose \( f: \Omega \to [0,1] \) is an arbitrary fractional clique of \( W \) and \( c: \mathcal{I}(W) \to [0,1] \) is an arbitrary fractional coloring of \( W \). Since \( \sum_{I \in \mathcal{I}(W)} c(I) \geq 1 \) for almost every \( x \in \Omega \), we can write

\[
\int_{x \in \Omega} f(x) \leq \int_{x \in \Omega} \left( f(x) \sum_{I \in \mathcal{I}(W)} c(I) \right) = \sum_{I \in \mathcal{I}(W)} \left( c(I) \int_{x \in \Omega} f(x) \right) \leq \sum_{I \in \mathcal{I}(W)} c(I),
\]

since \( \int_{I} f \leq 1 \) for each \( I \in \mathcal{I}(W) \). This finishes the proof. \( \square \)

Theorem 34 (Complementary slackness). Let \( W: \Omega \times \Omega \to [0,1] \) be a graphon with a fractional clique \( f: \Omega \to [0,1] \) and a fractional coloring \( c: \mathcal{I}(W) \to [0,1] \). We have

\[
\int_{\Omega} f = \sum_{I \in \mathcal{I}(W)} c(I)
\]

if and only if

\[
f(x) \left( \sum_{I \in \mathcal{I}(W)} c(I) - 1 \right) = 0
\]

for almost every \( x \in \Omega \) and

\[
c(I) \left( \int_{I} f - 1 \right) = 0
\]

for every \( I \in \mathcal{I}(W) \). In this case, for every \( A \subset \Omega \) with positive measure it holds \( f \mid_A = 0 \) almost everywhere if and only if

\[
\sum_{I \in \mathcal{I}(W)} c(I) > 1
\]
for almost every $x \in A$.

**Proof.** Notice that (6.3) holds if and only if both inequalities in (6.2) are at equality. Since $f$ and $c$ are nonnegative, (6.4) and (6.5) follow. This proves the first part. To get second part, it is enough to recall that $\sum_{I \in \mathcal{I}(W)} c(I) \geq 1$. \hfill $\square$

7. Perfect graphons

The notion of perfect graph is central in combinatorial optimization. Recall that a graph $G$ is perfect if for every induced subgraph $H$ of $G$, we have $\chi(H) = \omega(H)$. In the remarkable work of Chudnovsky, Robertson, Seymour, and Thomas [2], it is shown a forbidden subgraph characterization for perfect graphs. The result settles a problem that for four decades was known as the strong perfect graph conjecture. It can be stated as follows.

**Theorem 35** (Strong perfect graph theorem). A graph is perfect if and only if it contains no induced odd cycle of length more than 4 and no complement of an induced of cycle of length more than 4.

It is not clear what should be the right definition of perfect graphons. We offer two definitions. The first definition is in terms of subgraph densities.

**Definition 36.** We call a graphon $W : \Omega \times \Omega \to [0, 1]$ subgraph-perfect if for every finite graph $H$ with $t_{ind}(H, W) > 0$ it holds $\chi(H) = \omega(H)$. Further, we call $W$ properly subgraph-perfect if it is subgraph-perfect and $\chi(W) < \infty$.

The other definition of perfect graphons mimics more closely the graph definition. Note that the property $\chi(G) \geq \omega(G)$ which holds for finite graphs and was a motivation for the notion of perfect graphons holds for graphons, too.

**Definition 37.** We call a graphon $W : \Omega \times \Omega \to [0, 1]$ inheritance-perfect if for every set $A \subset \Omega$ of positive measure we have $\chi(W[A]) = \omega(W[A])$. Further, we call $W$ properly inheritance-perfect if it is induced-perfect and $\chi(W) < \infty$.

We can characterize subgraph-perfect graphons in the same fashion as perfect graphs by the density of induced odd cycles and its complements.

**Proposition 38.** A graphon $W$ is subgraph-perfect if and only if for every odd integer $\ell \geq 5$ we have $t_{ind}(C_{\ell}, W) = 0$ and $t_{ind}(\overline{C_{\ell}}, W) = 0$.

**Proof.** For every odd integer $\ell \geq 5$, we have $\omega(C_{\ell}) < \chi(C_{\ell})$ and $\omega(\overline{C_{\ell}}) < \chi(\overline{C_{\ell}})$. Thus, if $W$ is subgraph-perfect, then $t_{ind}(C_{\ell}, W) = 0$ and $t_{ind}(\overline{C_{\ell}}, W) = 0$.

Now, assume $W$ is not subgraph-perfect. Then, there is a finite imperfect graph $H$ with $t_{ind}(H, W) > 0$. By the Strong perfect graph theorem we have that $t_{ind}(C_{\ell}, H) > 0$ or $t_{ind}(\overline{C_{\ell}}, H) > 0$ for some odd integer $\ell \geq 5$. By Exercise 7.4 of [8] we get that $t_{ind}(C_{\ell}, W) > 0$ or $t_{ind}(\overline{C_{\ell}}, W) > 0$. This finishes the proof. \hfill $\square$

Recall that the complement of a graphon $W$ is defined as $W(x, y) = 1 - W(x, y)$ for every $x, y \in \Omega$. Naturally, we obtain the weak version of the last theorem.

**Corollary 39.** A graphon is subgraph-perfect if and only if its complement is subgraph-perfect.

We now relate subgraph-perfectness and induced-perfectness.

**Theorem 40.** Let $W : \Omega \times \Omega \to [0, 1]$ be a properly subgraph-perfect graphon. Then $W$ is inheritance-perfect.

**Proof.** We shall first prove that $\omega(W) = \chi(W)$. To this end, consider the sequence in the statement of Theorem 13(b). Let $G_n$ be the graphs in that sequence and let $W_n$ be their graphon representation. Now, we have

Exercise 7.6 in [8] is stated for the parameter $t(\cdot, \cdot)$ rather than $t_{ind}(\cdot, \cdot)$. The statement and the proof hold mutatis mutandis.
\[ (7.1) \quad \omega(W) = \lim_{n \to \infty} \omega(G_n). \]

Furthermore, by Theorem 13 we almost surely have that
\[ (7.2) \quad \chi(W) = \lim_{n \to \infty} \chi(G_n). \]

Since \( W \) is properly subgraph-perfect, the graphs \( G_n \) are all perfect (almost surely). Thus, \( \omega(G_n) = \chi(G_n) \), and \( \omega(W) = \chi(W) \) follows from (7.1) and (7.2).

Now, observe that induced-perfectness follows. Indeed, if \( W \) is subgraph-perfect and we are given a set \( A \subset \Omega \) of positive measure, we can run the above argument for the graphon \( W[A] \) (which is obviously subgraph-perfect) to conclude that \( \chi(W[A]) = \omega(W[A]). \) \[ \square \]

We cannot reverse Theorem 10 directly. For example, consider \( W \) to be a graphon representation of a triangle \( K_3 \). Now, take \( U := \frac{1}{2} W \), that is, \( U \) is a graphon representation of a triangle, where the edges are represented by value \( \frac{1}{2} \). Obviously, \( U \) is inheritance-perfect but we have \( t_{\text{ind}}(C_5, W) > 0 \). However, we do not know whether we can reverse Theorem 10 for \( \{0, 1\} \)-valued graphons. We pose this as an open problem.

**Problem 41.** Suppose that \( W \) is a \( \{0, 1\} \)-valued inheritance-perfect graphon. Is \( W \) also subgraph-perfect?

Our last open problem is the “weak perfect graph theorem” for inheritance-perfect graphons. Note that a positive answer to this problem would be automatically implied by a positive answer to Problem 41.

**Problem 42.** Suppose that \( W \) is a \( \{0, 1\} \)-valued inheritance-perfect graphon. Is its complement \( \overline{W} \) also inheritance-perfect?

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