Period Doubling in Small Multiply Connected Superconductors

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It is shown that for superconductors with circumference $2\pi R$ approaching BCS coherence length $\xi_0$ minimal period of the response of all thermodynamic quantities to external magnetic field is set by $\hbar c/e$ i.e. twice the corresponding value for the bulk case. This is explained by the dependence of internal energy of Cooper pairs on their center of mass motion which leads, in particular, to a transition offset between different current-carrying states. Explicit calculation of the transition offset is done for the case of $s$-wave superconducting cylinder with $R \gg \xi_0$ and turns out to be exponentially small. A possible enhancement of the effect for nodal superconductors is suggested. Similar conclusions should also apply to the response of charged or neutral superfluids to rotation.

It is well known that response of superconductors to electromagnetic field and inertial perturbations such as rotation involves such characteristics of Cooper pairs as their mass $2m_e$ and charge $2e$ with $m_e$ and $e$ being essentially, apart from tiny relativistic corrections, electron’s bare mass and charge. Although for all known superconductors the interparticle spacing is actually smaller (usually much smaller) than the size of the pairs, notions of pair’s mass and charge have good heuristic value in constructing phenomenological expressions for various thermodynamic and transport quantities on the basis of their counterparts in Bose superfluids by a simple replacement $m \rightarrow 2m$ and $e \rightarrow 2e$.

The implications of this prescription on a quantum of superfluid circulation have been experimentally confirmed on a variety of physical systems where internal degrees of freedom of Cooper pairs are either absent or irrelevant. As an example of the electromagnetic response one can quote classical flux quantization experiments in both conventional [1, 2] and $d$-wave superconductors [3]: inertial properties of Cooper pairs were tested, in particular, in observation of quantized circulation in $^3$He-B [4,5] as well as in high precision measurements of magnetic field induced in a superconductor by rotation [6]. While there are no known charged Bose superfluids so that the unit of flux quantization established in [1,2,3], namely $\hbar c/2e$, does not have its Bose counterpart, experiments with rotating superconductors [6] do have their analog—the Hess-Fairbank effect—in rotating $^3$He and seem to well establish the circulation unit of $h/m_e$ for $^4$He [7,8] and $h/2m_e$ for niobium in Ref. [6] thus confirming to the prescription indicated above.

On the theoretical side flux quantization in superconducting samples of annular geometry has been attributed to the condensation of pairs of particles into states with different momenta of the center of mass of the pairs [9,10]. In thermodynamic equilibrium, as the external field changes, discrete sequential transitions between these states lead to flux-periodic dependencies of all quantities characterizing the annulus, e.g. to a periodic dependence of induced magnetization. Even without detailed knowledge of the ground state and relying only on the gauge invariance principle one can conclude that the fundamental period of such dependencies will be $\hbar c/e$ [9,11]. However if the ground state possesses pair correlations of the type mentioned above with all or most pairs being in the same center of mass state then the response of the system will contain substantial $\hbar c/2e$ harmonic rendering the minimal flux period of the dependencies to the same value.

It has been noted a long time ago that there is no fundamental reason behind minimal flux periodicity being $\hbar c/2e$ [9,11]. Indeed calculation of the response of an $s$-wave ring under the assumption of independence of relative motion of electrons on the flux [12] as well as calculations of Little-Parks effect in $s$-wave rings [13,14] and magnetization of mesoscopic $d$-wave loops [15,16] have observed absence of $\hbar c/2e$ periods in the corresponding quantities. The simple argument presented here shows that this situation, namely doubling of the minimal period in the response of all thermodynamic quantities, is quite generic and happens irrespective of the particular form of pairing, interactions, temperature effects etc. when the circumference of the superconductor becomes small. For an $s$-wave superconductor where pair wave function (WF) decays exponentially, the effect becomes noticeable when the circumference approaches BCS coherence length $\xi_0$. However for a nodal superconductor, such as $d$-wave, there are directions in the real space where decay of the pair WF is algebraic which suggests that the effect may be noticeable at even larger than $\xi_0$ values of the circumference. Similar analysis goes through for rotating superconductors where the response is periodic in the rotation velocity with the minimal ‘unbroken’ period equal to $h/2m_e$.

Basic geometry considered in this work is that of a hollow cylinder with the wall thickness $d$ being smaller than the London penetration depth $\lambda_L$ and the radius of the cylinder $R \gg d$; same conclusions can be reached for an arbitrary geometry requiring only axial symmetry and $d \ll \lambda_L$ condition. Single particle states are specified by a set of three quantum numbers $(m, n)$ with $hm$ being a projection of angular momentum along the symmetry axis and $n \equiv (n_1, n_2)$ describing two other degrees of freedom responsible for the motion along and perpendicular to the symmetry axis.

In thermodynamic equilibrium periodicity in the response of $N$ paired fermions is attributed to the transitions between ground states of the following type

$$\Psi(m_0) = \frac{1}{N} \sum_{n_{(m0)}} (n, \mathbf{n}) a^\dagger_{n,m} a^\dagger_{\mathbf{n},-m+n_{m0}} |0\rangle$$

which describe condensation of $N/2$ spin-singlet pairs with the pair’s angular momentum along the symmetry axis equal
Pairing with respect to the other two quantum numbers $n$ is chosen in a standard way and connects state $n$ with its time reversal $\bar{n}$. Spin singlet symmetry of two-particle state requires that variational parameter $\chi$ satisfies $\chi_{-m+n}(\mathbf{n}, \mathbf{r}) = \chi_{m}(\mathbf{r}, \mathbf{n})$. The same condition is obeyed by the pair wave function in the momentum representation $F(m)$ defined as

$$ F(m)(\mathbf{r}, \mathbf{n}) = \langle a_{\mathbf{n},-m+n} | a_{\mathbf{n},m} \rangle $$

where the quantum mechanical average is taken between the states $|1\rangle$ with $N/2 - 1$ and $N/2$ pairs. Structure of the pair WF is more apparent in the coordinate representation where the quantum mechanical average is taken between the single pairs. As will be shown below this is indeed the case.

In the simplest case of $s$-wave superconductor the decay of the pair WF in real space is isotropic and exponential with the decay length equal to the BCS coherence length $\xi_0$. The internal energy difference for this case will be evaluated below under assumption $R \gg \xi_0$ and will turn out to be exponentially small. However for a superconductor with a nodal structure in the gap there are directions in the real space where pair WF decays algebraically which may lead to an enhancement of the effect.

Coming back to eqn. (3) and introducing superfluid velocity $\mathbf{v}_s$ and superfluid density $\rho_s$ for the pairs of particles which occupy volume $\Omega$ by usual relations $v_s \equiv \frac{\hbar}{2m_eR} (m_0 + \phi/\phi_0)$, $\rho_s \equiv m_eN/\Omega$ it becomes

$$ \Omega^{-1} E^{(m)}(\phi) = \frac{\hbar^2}{2m_eR^2} \rho_s v_s^{(0,1)} + \frac{1}{2} \rho_s v_s^2 $$

Being written in this form the expression for energy can, in fact, be generalized to include effects of non-zero temperature or other pair breaking perturbations by introducing corresponding changes in the superfluid density $\rho_s$ as well as other terms describing e.g. behavior of the excited component of the system. Such generalizations will not change qualitatively conclusions reached below and the following discussion will be limited to the simplest case specified by eqn. (3).
The discussion so far has led to the expressions (5) and (6) for the ground state energy as a function of pair’s angular momentum $h\kappa_0$ and total flux $\phi$. To find thermodynamically stable value of $m_0$ which is realized at a fixed external flux $\phi_e$, one needs to construct corresponding thermodynamic potential which is in this case Gibbs potential $G$. Although general expression for $G$ can be easily written down, for the purpose of this discussion it is enough to notice that, in thermodynamic equilibrium, the transition between $m_0 = 0$ and $m_0 = 1$ states occurs at the following value of the external flux:

$$\phi^{(0-1)} = \frac{1}{2} \frac{hc}{2e} \left[ 1 + \alpha (e^{(1)} - e^{(0)}) \right]$$

where $e^{(0,1)}$ is the dimensionless internal energy of the Cooper pair for the states $m_0 = 0, 1$ defined by eqn. (5). Appearance of parameter $\alpha \equiv 1 + 4Rd/L_\xi^2$ in (7) has purely geometrical reasons related to fact that, for the geometry considered here, field and condensate energies scale differently with $R$. As one can see from the result above, transition between different center of mass states is offset relative to the bulk case by an amount proportional to the internal energy difference.

Complete dependence of the induced flux obtained by minimization of the Gibbs potential corresponding to the energy (5) is illustrated on fig. 2. The effect of the non-vanishing internal energy difference is to shift the transitions between different current states by an amount proportional to $\pm (e^{(1)} - e^{(0)})$ thus breaking $hc/2e$ periodicity up to $hc/e$ (21). In an alternative interpretation one may refer to this effect as a change in Cooper pair’s charge. A very similar reasoning shows that the same change occurs for the Cooper pair mass in the rotating superconductors thus leaving gyromagnetic ratios like $2m_e c/e$ unchanged.

The preceding qualitative conclusions were reached without particular reference to a microscopic form of interaction. However for a quantitative analysis, which is most easily done in the momentum representation, a choice of specific microscopic model is needed. Using single-particle dispersion relation (1) and a contact interparticle interaction potential $V$ energy expectation value on a many-body ground state (1) is

$$E^{(\text{m})}(\phi) = \sum \varepsilon_{\alpha}(m, \mathbf{n}) \phi^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n})^2$$

$$- \sum V \mathcal{F}_{m-m_0/2}(\mathbf{n}, \mathbf{n}) \phi^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n})^2$$

where quantities $v$ and $\mathcal{F}$ are defined through earlier introduced $\chi$, eqn. (1), and $F$, eqn. (2), as $|\phi^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n})|^2 \equiv \lambda^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n}) \phi^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n})$ and $\mathcal{F}^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n}) \equiv F^{(\text{m})}_{m-m_0/2}(\mathbf{n}, \mathbf{n})$. Defined in such way $v$ and $\mathcal{F}$ are symmetric with respect to $m$, e.g. $v_{m}(\mathbf{n}, \mathbf{n}) = v_{-m}(\mathbf{n}, \mathbf{n})$. In what follows, to avoid complicated notation, $n$ dependence of all quantities will not be indicated explicitly unless mentioned otherwise.

Noticing that normalization condition for state (1) requires $2 \sum \phi^{(\text{m})}_{m-m_0/2} = N$, the flux dependence in (5) can be separated out thus establishing a connection with the expression for energy (5) used in the previous analysis. Identifying flux dependent part with the the center of mass contribution and everything else with the internal energy contribution one thus obtains for the latter

$$N e^{(\text{m})} = 2 \sum (\zeta + (m - m_0/2)^2) \phi^{(\text{m})}_{m-m_0/2}^2$$

$$- V \sum \mathcal{F}^{(\text{m})}_{m-m_0/2} \mathcal{F}^{(\text{m})}_{m-m_0/2}$$

and the interaction constant is now written in units of $\hbar^2/2m_e R^2$. It is also apparent that $e^{(\text{m})}$ defined by the equation above is the same for the same parity states, as was indicated above.

The actual forms of $v^{(\text{m})}$ and $\mathcal{F}^{(\text{m})}$ can be found by energy minimization in the same way as it is done in the standard BCS treatment which leads to $|\phi^{(\text{m})}_{m-m_0/2}|^2 = |\phi_0(\Delta^{(\text{m})})\mu^{(\text{m})})|^2$, $\mathcal{F}^{(\text{m})}_{m-m_0/2} = \mathcal{F}_\mu(\Delta^{(\text{m})})\mu^{(\text{m})})$ where, for a given pair state $m_0$, the gap parameter, $\Delta^{(\text{m})}$, and the chemical potential, $\mu^{(\text{m})}$, should be determined self-consistently from gap and normalization equations and are, in general, different for odd and even parity states. For fixed $\Delta^{(\text{m})}$ and $\mu^{(\text{m})}$ functions $v^{(\text{m})}$ and $\mathcal{F}^{(\text{m})}$ are given by standard BCS expressions.

In the momentum representation the reason for a non-vanishing value of the internal energy difference between odd and even parity states and, as a result, doubling of the flux periodicity, is less transparent than in the coordinate representation but from a mathematical point of view it can be traced down to two main factors. In the first place, the actual values of $m$ arguments which $v$ and $\mathcal{F}$ are being summed over in (5) are shifted by 1/2 for odd and even parity cases; secondly, there is a difference between odd and even values of the gaps and chemical potentials. It can be shown that the latter difference being itself caused by the former in the gap and normalization equations can be ignored in the leading approximation. The actual calculation of the energy difference can be found in the Appendix. Performing summation over $m$ with the help of Poisson summation formula and integrating afterwards with respect to two other quantum numbers $n$ it is, in fact, possible to get an analytic expression for $e^{(1)} - e^{(0)}$ in the limit $R \gg \xi_0$. With the relative accuracy $\xi_0/R$ one has

$$e^{(1)} - e^{(0)} = \frac{3}{R^2 \xi_0} \left[ \frac{R}{\xi_0} \right] e^{-2R/\xi_0} \cos(2k_F R)$$

where $k_F$ is the Fermi wave vector and $\xi_0$ is defined through the gap parameter $\Delta$ as $\xi_0 = h\kappa_F/\pi \Delta$. It should be emphasized that the accuracy of eqn. (10) does not allow one to distinguish between $\Delta^{(1)}$ and $\Delta^{(0)}$ (or $\mu^{(1)}$ and $\mu^{(0)}$) since the difference $\Delta^{(1)} - \Delta^{(0)}$ is itself exponentially small. There might be
however a substantial difference between either of $\Delta^{(0,1)}$ and the value of the gap for the corresponding bulk material.

One should observe that the sign of eqn. (10) is a rapidly oscillating function of the radius which is due to the oscillations of the pair WF in real space. Similar oscillatory effects have been found theoretically in Ref. [12] for the dependence of the gap on the thickness of a thin superconducting film. Strong dependence of the sign of the transition offset on the ring's size would make it more difficult to observe it on an ensemble of rings, such as the one used in [13] for Little-Parks measurement, because of possible variations in the rings' sizes.

Another limitation for experimental observation is the magnitude of the effect. According to eqn. (10) one may expect transition offset to be of the order of 0.1% for $R \approx 3\xi_0$; larger corrections for the same value of $R/\xi_0$ will most likely be achieved for nodal superconductors. Although reducing the radius will make the effect more pronounced, a little more care should be taken in extrapolating eqn. (10) to the ring sizes equal or lesser of the coherence length. This is due to the fact that when the diameter of the ring becomes equal to the coherence length the superfluid velocity needed to screen a quantum of flux reaches its critical value thus leading to the destruction of superconductivity around $\phi_c = \phi_0/2$ [19].

It is interesting to notice that, apart from the small sample size, there might be other ways to introduce correlations between the center of mass and internal energies of the Cooper pair. One of them could be a strong spin-orbit interaction where the small size limitation would not apply.

To conclude, it was suggested that in small superconductors, due to the dependence of internal energy of Cooper pairs on the center of mass state, the minimal flux periodicity is $hc/e$, twice the usually attributed value. The doubling of the periodicity is due to the offset of the transition between different current states. The magnitude of such offset was calculated for s-wave pairing. It was also suggested that other things being equal, the effect will be more pronounced for nodal superconductors.

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APPENDIX

In this appendix expression (10) for the internal energy difference between odd and even parity states of a large, $R \gg \xi_0$, s-wave superconducting cylinder or ring will be obtained. Unless explicitly otherwise stated all quantities with dimensions of energy are written in units of $\hbar^2/2m^2R^2$.

As it has been already remarked the minimization of the expectation value of energy given by eqn. (8) leads to the the following form of $v_m^{(m_0)}$ and $\mathcal{F}_m^{(m_0)}$:

\[
|v_m^{(m_0)}|^2 = |v_m(\Lambda^{(m_0)};\mu^{(m_0)})|^2
\]

\[
\mathcal{F}_m^{(m_0)} = \mathcal{F}_m(\Lambda^{(m_0)};\mu^{(m_0)})
\]

where, for given values of $\Delta$ and $\mu$, $v_m$ and $\mathcal{F}_m$ are specified by the standard BCS expressions:

\[
|v_m|^2 = \frac{1}{2} \left[ 1 - \frac{\epsilon_m}{\sqrt{\epsilon_m^2 + \Delta^2}} \right]
\]

\[
\mathcal{F}_m = \frac{\Delta}{2\sqrt{\epsilon_m^2 + \Delta^2}}
\]

with $\epsilon_m \equiv \zeta + m^2 - \mu$ being a single-particle energy counted from the chemical potential which corresponds to a given value of $m_0$.

Now, using eqn. (9), the internal energies for even and odd parity states are given by

\[
N\epsilon_0^{(0)} = 2 \sum (\zeta + m^2)|v_{m0}|^2 - V \sum \mathcal{F}_{m0} \mathcal{F}_{m'0}
\]

for even parity states, and

\[
N\epsilon_1^{(1)} = 2 \sum (\zeta + (m - 1/2)^2)|v_{m-1/2,1}|^2 - V \sum \mathcal{F}_{m-1/2,1} \mathcal{F}_{m+1/2,1}
\]

for odd parity states. The subscript 0 or 1 in all quantities indicates dependence on $\Delta^{(0)}$, $\mu^{(0)}$ or $\Delta^{(1)}$, $\mu^{(1)}$ respectively. The internal energy difference can then be written as

\[
\epsilon_1^{(1)} - \epsilon_0^{(0)} = \epsilon_1^{(1)} - \epsilon_0^{(0)} + \delta \epsilon^{(0)}
\]

where $\delta \epsilon^{(0)} \equiv \epsilon_1^{(0)} - \epsilon_0^{(0)}$ is the change of the ground state energy ($m_0 = 0$ state) as $\Delta$ and $\mu$ are varied from their $m_0 = 0$ to $m_0 = 1$ values. Since $\epsilon_0^{(0)}$ is the equilibrium i.e. minimal value of $\epsilon^{(0)}$, the last term on the r.h.s. of eqn. (A.7) is of the second order in $\Delta^{(1)} - \Delta^{(0)}$ and, as it will be seen below, can be ignored relative to the first two terms in the limit $R \gg \xi_0$. Thus to evaluate the internal energy difference in the leading order the energies themselves can be taken at the same values of $\Delta$ and $\mu$. At this accuracy any difference between $\Delta^{(0)}$ and $\Delta^{(1)}$ etc. is neglected; however, there might be a substantial difference between these values and the corresponding bulk parameters.

Using the gap equation $\Delta = V \sum \mathcal{F}_{m-1/2}$, in the leading approximation eqn. (A.7) becomes

\[
N(\epsilon^{(1)} - \epsilon^{(0)}) = 2 \sum (\zeta + m^2)|v_m|^2 - 2V \sum (\mathcal{F}_{m-1/2} - \mathcal{F}_m)
\]

where $v_m$ and $\mathcal{F}_m$ given by eqn.'s (A.3) and (A.4). The sums above run over all integers $m$ as well as over two other quantum numbers $n$ which represent other-than-azimuthal part of the dispersion $\zeta(n)$. With the help of the Poisson summation formula the sum over $m$ is converted to an integral; summation over $n$ is replaced by integration using the standard rule $\sum_n \rightarrow \int d\zeta g_2(\zeta)$ where $g_2(\zeta)$ is the (dimensionless) density of states for $\zeta(n)$ and can be considered to be a constant.

Keeping only leading exponential term in the Poisson summation series one obtains:

\[
N(\epsilon^{(1)} - \epsilon^{(0)}) = I + I_1
\]
where the following definitions are made
\[ I \equiv 4\mu \int d\zeta \int dx \frac{e(x, \zeta) e^{2\pi i x}}{\sqrt{e^2(x, \zeta) + \Delta^2}} \tag{A.10} \]
and
\[ I_1 \equiv 4\int d\zeta \int dx \left[ \sqrt{e^2(x, \zeta) + \Delta^2} - e(x, \zeta) \right] e^{2\pi i x} \tag{A.11} \]
with \( e(x, \zeta) \equiv \zeta + x^2 - \mu \). It is intuitively plausible and can, in fact, be shown that \( I_1 / I \sim \xi_0 / R \ll 1 \) so that the problem reduces to the calculation of the following integral:
\[ I \equiv 4\mu \int_{-\mu}^{\infty} \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dx \frac{(\zeta + x^2) e^{2\pi i x}}{\sqrt{(\zeta + x^2)^2 + \Delta^2}} \tag{A.12} \]
where and from now on \( \zeta \) will be counted relative to the chemical potential. The rest of the discussion is devoted to the calculation of the above integral under assumption \( R \gg \xi_0 \).

The integration over \( x \) in eqn. (A.12) is transformed by closing the integration contour in the upper half of the complex plane where the denominator \( \sqrt{(\zeta + x^2)^2 + \Delta^2} \) has two branch points. Choosing the branch cuts to point outwards away from the origin the integral over the real axis reduces to four integrals along both sides of the two branch cuts. The two integrals along one branch cut are complex conjugate of those along the other branch cut. Introducing new integration variable \( t \) along the first quarter branch cut by \( z = z_0(t + 1) \), where \( z_0 \) is the first quarter branch point, the oscillating exponent in (A.12) acquires a decaying part:
\[ I = 16\mu \Re \int_{-\mu}^{0} \int_{0}^{\infty} d\zeta \int_{0}^{\infty} dt \frac{(\zeta + z_0^2(t + 1))^2 e^{2\pi i (t + 1)}}{\sqrt{(\zeta + z_0^2(t + 1)^2)^2 + \Delta^2}} \tag{A.13} \]
where \( \Re \) denotes the real part of the corresponding expression and the decaying exponent is given by the imaginary part of \( z_0 \):
\[ \Re z_0 \equiv x_0 = \frac{1}{\sqrt{2}} \left( -\zeta + \sqrt{\zeta^2 + \Delta^2} \right)^{1/2} \tag{A.14} \]
\[ \Im z_0 \equiv y_0 = \frac{1}{\sqrt{2}} \left( \zeta + \sqrt{\zeta^2 + \Delta^2} \right)^{1/2} \tag{A.15} \]

The essential for the integration over \( r \) region in (A.13) is determined by the decaying exponent \( e^{-2\pi y_0 t} \) and extends from \( t_{\text{min}} = 0 \) to \( t_{\text{max}} = (2\pi y_0)^{-1} \). As a function of \( \zeta \), \( y_0 \) is monotonically increasing in the region \( (\mu - \infty) \) reaching its minimum at \( \zeta = -\mu \) with the value \( y_0(-\mu) \approx \Delta / 2 \sqrt{\mu} \); at the same time \( x_0 \) is monotonically decreasing with the maximum value \( x_0(-\mu) \approx \sqrt{\mu} \). Restoring dimensions i.e. supplying \( \hbar^2 / 2m_R R^2 \) denominators to \( \Delta \) and \( \mu \) gives
\[ y_0(-\mu) \approx R / \pi \xi_0, \quad x_0(-\mu) \approx k_F R \tag{A.16} \]
where \( \xi_0 \) is the BCS coherence length \( \xi_0 \equiv \hbar v_F / \pi \Delta \) and \( v_F \equiv 2\mu / m_0 \). The above analysis suggests that, under condition \( R \gg \xi_0 \), the integrand in (A.13) can be significantly simplified because in the important for the integration region \( t \ll 1 \) holds. After some algebraic manipulations followed by a Wick rotation of \( r \) variable to make the algebraic part of the integrand real for \( \zeta < 0 \), the expression for the integral takes the following form
\[ I = 8\mu \Re \int_{-\mu}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} dt i \frac{(-2t\zeta + \Delta) e^{2\pi i (t - 1)}}{\sqrt{(\Delta - \zeta t)}} \tag{A.17} \]
with further corrections being suppressed by a factor of order of \( \xi_0 / R \).

The integration over \( \zeta \) is considered separately for intervals of positive and negative values of \( \zeta \). After rescaling variable \( t \) by \( \Delta / |\zeta| \) in each interval the non-trivial dependence of the integrand on \( \Delta \) and \( \zeta \) is gathered in the exponent. For the integral over \( \zeta > 0 \) region the exponent has a sharp maximum and the integral can be evaluated by the saddle point method leading to its magnitude being of order of \( \exp(-2\pi \sqrt{\Delta}) \). As it will be seen below the integral over negative values of \( \zeta \) is of order of \( \exp(-2\pi \sqrt{\Delta / \mu}) \). Since the ratio of the two exponent is \( \sqrt{\Delta / \mu} \ll 1 \) the integration over \( \zeta \) in (A.17) can be limited to the interval \( \zeta < 0 \) leading to
\[ I = 8\mu \Delta \int_{-\mu}^{0} \int_{0}^{\infty} \int_{0}^{\infty} dt i \frac{(2t + 1) \sin Y e^{-X}}{\sqrt{1 + t}} \tag{A.18} \]
where
\[ X(\zeta, t) \equiv 2\pi y_0(\zeta) + x_0(\zeta) \frac{2\pi \Delta}{|\zeta|} \tag{A.19} \]
\[ Y(\zeta, t) \equiv -2\pi x_0(\zeta) + y_0(\zeta) \frac{2\pi \Delta}{|\zeta|} \tag{A.20} \]

with \( x_0 \) and \( y_0 \) given by eqn.’s (A.14) and (A.15).

Expressions for \( X(\zeta, t) \) and \( Y(\zeta, t) \) can be significantly simplified in the limit \( |\zeta| \gg \Delta \). In particular for \( \zeta = -\mu \)
\[ X(-\mu, t) \approx 4R (1/2 + t) \tag{A.21} \]
\[ Y(-\mu, t) \approx -2\pi \sqrt{\mu + \Delta / \mu} 2R t \tag{A.22} \]
where \( R \) is defined as
\[ R \equiv \pi \Delta / \sqrt{\mu} \tag{A.23} \]
and in dimensional units is equal to \( R / \xi_0 \), the ratio of the radius of the annulus to the coherence length. As it has been already mentioned it is assumed that \( R \gg 1 \).

The function in the exponent in eqn. (A.18), \( X(\zeta, t) \), is a positive monotonically growing function of \( \zeta \) for \( \zeta < 0 \) which reaches its minimum at the lower integration limit \( \zeta = -\mu \) and diverges at the upper limit \( \zeta = 0 \). To take the advantage of the simplified forms of \( X \) and \( Y \) at \( \zeta = -\mu \) one can perform repeated integration by parts in the integral over \( \zeta \) obtaining in that way following asymptotic expansion:
\[ I = 8g_2\mu A \int_0^{\infty} dt \frac{2t + 1}{\sqrt{R(t+1)}} \left( Z(\zeta, t) + \left[ \frac{Z(\zeta, t)}{X'(\zeta, t)} \right]' + \ldots \right) \frac{e^{-X(-\mu, t)}}{X'(-\mu, t)} \]  

(A.24)

where the prime sign denotes differentiation with respect to \( \zeta \) and function \( Z(\zeta, t) \) is defined as the \( \zeta \) dependent non-exponential part of the integrand in (A.18).

\[ Z(\zeta, t) \equiv \frac{1}{\sqrt{|\zeta|}} \sin Y(\zeta, t) \]  

(A.25)

Despite complicated at first sight \( t \) dependence of the integrand in (A.24) the integration over \( t \) can be carried out noticing that, because of the decaying exponent, the interval relevant to the integration is limited by \( \bar{R}^{-1} \ll 1 \). Dropping \( t \) dependence relative to constants of order 1 and integrating the expansion term by term gives with \( \bar{R}^{-1} \) relative accuracy

\[ I = 4g_2\mu^2 \Delta \pi^{1/2} S(\mu) \bar{R}^{-3/2} e^{-2R} \]  

(A.26)

where \( S(\mu) \) is defined as

\[ S(\mu) \equiv \left\{ Z(\zeta, 0) + \left[ \frac{Z(\zeta, 0)}{X'(\zeta, 0)} \right]' + \ldots \right\}_{\mu} \]  

(A.27)

The sum above cannot be limited to a finite number of terms because the \( n \)-th term in the sum is of order of \( \mu^{-1/2}(\mu/\Delta)^n \). However by rearranging summation one can notice that \( S(\mu) \) satisfies following differential equation

\[ S(\mu) = Z(-\mu, 0) - \frac{d}{d\mu} S(\mu) \]  

(A.28)

or, given explicit form of \( Z(-\mu, 0) \) and \( Z(-\mu, 0) \), eqn.’s (A.21) and (A.25), the differential equation for \( S(\mu) \) becomes

\[ S(\mu) = -\frac{1}{\mu} \cos 2\pi \sqrt{\mu} - \frac{d}{d\mu} \frac{3\mu^{3/2} S(\mu)}{\mu^{3/2}} \]  

(A.29)

It can be checked by a direct substitution that up to the terms of order \( \Delta/\mu \) this equation has the following solution:

\[ S(\mu) = \frac{\Delta}{\mu^{3/2}} \cos 2\pi \sqrt{\mu} \]  

(A.30)

so that the expression for the integral \( I \) reduces to

\[ I = 4g_2\pi^{1/2} \mu^{3/2} \bar{R}^{-1/2} e^{-2R} \cos 2\pi \sqrt{\mu} \]  

(A.31)

where \( g_2 \) is the density of states at the chemical potential for other-than-azimuthal part of the dispersion \( \xi(n) \). Using single-particle dispersion law for a thin walled cylinder and reverting to dimensional units one can see that \( g_2 \mu^{3/2} = \frac{3}{\pi} N \) so that

\[ I = \frac{3N}{\bar{R}^{1/2}} (R/\bar{R})^{1/2} e^{-2R/\bar{R}} 2\pi k_F R \]  

(A.32)

Comparing this equation with eqn. (A.9) gives the result (10) for the internal energy difference quoted in the main text.

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[20] There are two implicit assumptions required for the validity of this statement. In the first place, the decay length should not shrink as fast as the circumference itself. Secondly, the ring should be uniform enough to avoid a possibility of placing kinks in the pair WF in the regions where its magnitude is suppressed.
[21] It should be pointed out that this mechanism of breaking \( hc/2e \) periodicity is different from that proposed in Ref. [15] where it is the behavior of the excited component of the superfluid that breaks the periodicity. The position of the transitions between different center of mass states is una...