On the Reidemeister spectrum and the $R_\infty$ property for some free nilpotent groups

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Abstract

We describe the Reidemeister spectrum $\text{Spec}_\varphi G$ for $G = N_{rc}$, the free nilpotent group of rank $r$ and class $c$, in the cases: $r \in \mathbb{N}$ and $c = 1$; $r = 2, 3$ and $c = 2$; $r = 2$ and $c = 3$, and prove that any group $N_{2c}$ for $c \geq 4$ satisfies to the $R_\infty$ property. As a consequence we obtain that every free solvable group $S_{2t}$ of rank 2 and class $t \geq 2$ (in particular the free metabelian group $M_2 = S_{22}$ of rank 2) satisfies to the $R_\infty$ property. Moreover, we prove that any free solvable group $S_{rt}$, of rank $r \geq 2$ and class $t$ big enough also satisfies to the $R_\infty$ property.

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1 Introduction

Let $G$ be a group, and $\varphi : G \rightarrow G$ be an automorphism of $G$. One says that the elements $g, f \in G$ are $\varphi$-twisted conjugated, denoted by $g \sim \varphi f$, if and only if

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there exists \( x \in G \) such that \( (x\varphi)g = fx \). A class of equivalence \([g]_\varphi\) is called the **Reidemeister class** (or the \( \varphi \)-conjugacy class of \( \varphi \)). The number \( R(\varphi) \) of Reidemeister classes is called the **Reidemeister number** of \( \varphi \). We define the **Reidemeister spectrum** of \( G \) by

\[
\text{Spec}_R(G) = \{ R(\varphi) \mid \varphi \in \text{Aut}(G) \}.
\]

One says that a group \( G \) has the **\( R_\infty \) property** for automorphisms, denoted by \( G \in R_\infty \), if for every automorphism \( \varphi : G \to G \) one has \( R(\varphi) = \infty \). The class of groups with \( R_\infty \) property is very interesting in view of various of applications in Nielsen-Reidemeister fixed point theory, representation theory, dynamic systems, algebraic geometry and so on. See for instances \( [3], [4], [5], [6], [8], [9], [10], [15], [18], [21], [20], [19] \).

In the paper by D. Gonçalves and P. Wong \( [9] \) mainly devoted to finitely generated nilpotent groups it was shown that any free nilpotent group \( N_{2c} \) of rank 2 and class \( c \geq 8 \) satisfies to \( R_\infty \) property. On the other hand the authors of \( [9] \) noted that they do not know how to extend their techniques to \( N_{rc} \) for \( r \geq 3 \), and \( c \geq 3 \). In the paper by V. Roman’kov \( [16] \) it was proved that any free nilpotent group \( N_{rc} \) of rank \( r = 2 \) or \( r = 3 \) and class \( c \geq 4r \), or rank \( r \geq 4 \) and class \( c \geq 2r \), satisfies to the \( R_\infty \) property. This result provides a purely algebraic proof of the fact that any absolutely free group \( F_r \), \( r \geq 2 \), has the \( R_\infty \) property. Note that in \( [16] \) this statement was derived by group-geometrical techniques.

In this paper we mainly consider the free nilpotent groups of small class. We obtain the Reidemeister spectrum of a group \( N_{rc} \) for \( r \in \mathbb{N} \) and \( c = 1 \) (free abelian case), for \( r = 2, 3 \) and \( c = 2 \) (2-nilpotent case), for \( r = 2 \) and \( c = 3 \) (3-nilpotent case). As a main statement we prove that any group \( N_{2c} \) for \( c \geq 4 \) satisfies to the \( R_\infty \) property. This result completes the case of rank 2. As a consequence we obtain that every free solvable group \( S_{2t} \) of rank 2 and class \( t \geq 2 \) (in particular the free metabelian group \( M_2 = S_{22} \) of rank 2) satisfies to the \( R_\infty \) property. Moreover every free solvable group \( S_{rt} \) of rank \( r \geq 3 \) and class \( t \) big enough also satisfies to the \( R_\infty \) property.

### 2 Preliminaries

The results of this section in fact are known by paper \( [9] \). Nevertheless we present them for completeness of the paper.

Let \( G \) be a finitely generated group, and let \( C \) be a central subgroup of \( G \). For any automorphism \( \varphi : G \to G \) define a central subgroup

\[
L(C, \varphi) = \{ c \in C \mid \exists \, x \in G : x\varphi = cx \}.
\]

It was shown in \( [16] \) that any pair of elements \( c_1, c_2 \in C \) are \( \varphi \)-conjugated in \( G \) if and only if
$c_1^{-1}c_2 \in L(C, \varphi)$.  

(3)

Thus in the case of a finitely generated abelian group $A$ the set of all $\varphi$–conjugacy classes coincides with the set of all cosets $A$ w.r.t. $L(A, \varphi)$. Note that

$L(A, \varphi) = Im(\varphi - id)$.  

(4)

Hence

$R(\varphi) = [A : L(A, \varphi)]$.  

(5)

Let $A(r) = \mathbb{Z}^r$ be a free abelian group of rank $r \in \mathbb{N}$, and $\varphi : A(r) \to A(r)$ be any automorphism.

Easy to see that

$Spec(\mathbb{Z}) = \{2\} \cup \{\infty\}$.  

(6)

We claim that for $r \geq 2$ the spectrum is full, i.e.

$SpecA(r) = \mathbb{N} \cup \{\infty\}$.  

(7)

To prove it enough to find for every number $k \in \mathbb{N}$ a matrix $A(r, k) \in GL_r(\mathbb{Z})$ such that $rank(A(r, k) - E) = k$.

If $r = 2$ or $r = 3$ one can take

$A(2, k) = \begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix}$,  

(8)

and

$A(3, k) = \begin{pmatrix} 1 & k & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,  

(9)

respectively.

In general case for odd $r = 2t + 1$ we put

$A(r, k) = diag(A(3, k), A(2, 1), \ldots, A(2, 1))$,  

(10)

and for even $r = 2t$

$A(r, k) = diag(A(2, k), A(2, 1), \ldots, A(2, 1))$,  

(11)

where $A(2, 1)$ repeats $t - 1$ times.

It remains to note that $R(id) = \infty$ for any group $A(r)$.

Let $N$ be a finitely generated torsion free nilpotent group of class $k$. Let

$\zeta_0 N = 1 < \zeta_1 N < \ldots < \zeta_{k-1} N < \zeta_{k} N = N$  

(12)

be the upper central series in $N$. It is well known (see [11], [12]) that all quotients

$N_i = N/\zeta_i N, \ A_i = \zeta_{i+1} N/\zeta_i N, \ i = 0, 1, \ldots, k - 1$,  

(13)
are finitely generated torsion free groups. In particular, every $A_i$, $i = 0, 1, ..., k−1$, is a free abelian group of a finite rank.

Let $\varphi : N \to N$ be any automorphism. Then there are the induced automorphisms

$$\varphi_i : N_i \to N_i, \bar{\varphi_i} : A_i \to A_i, i = 0, 1, ..., k−1.$$ (14)

If $R(\varphi_i) = \infty$ for some $i = 0, 1, ..., k−1$, then $R(\varphi_j) = \infty$ for every $j < i$, in particular, $R(\varphi) = \infty$ in $N$. Moreover, if for some $i = 0, 1, ..., k−1$, there is a non trivial element $\bar{a} \in A_i$ such that $\bar{a} \bar{\varphi_i} = \bar{a}$, then $R(\varphi_j) = \infty$ for every $j < i$, and again $R(\varphi) = \infty$. To explain the last assertion we assume that $i$ is maximal with the property $\bar{a} \bar{\varphi_i} = \bar{a} \neq 1$. Then the group $N_{i+1}$ does not admit a non trivial element $x$ such that $x \varphi_i = x$. Hence we have $L(N_i, \varphi_i) = L(A_i, \varphi_i)$. Obviously $[A_i : L(A_i, \varphi_i)] = \infty$. Then by Lemma 2.1 from [16] we derive that $R(\varphi_i) = \infty$ in $N_i$, and so $R(\varphi_i) = \infty$ for any $j < i$, in particular $R(\varphi) = \infty$.

Suppose that $R(\varphi) < \infty$. Then we have that the subgroup of $\varphi_i$–invariant elements $Fix_{\varphi_i}(N_i) = 1$ for all $i = 0, 1, ..., k−1$. We have $[A_i : L(N_i, \varphi_i)] = [A_i : L(A_i, \varphi_i)] = q_i, \ i = 0, 1, ..., k−1$. Then we have

**Lemma 2.1.** Let $N$ be a finitely generated torsion free nilpotent group of class $k$, and $\varphi : N \to N$ be any automorphism. Suppose that $R(\varphi) < \infty$, and in notions as above $[A_i : L(N_i, \varphi_i)] = [A_i : L(A_i, \varphi_i)] = q_i, \ i = 0, 1, ..., k−1$. Then

$$R(\varphi) = \prod_{i=0}^{k−1} q_i.$$ (15)

**Proof.** The formula (15) is based on Lemma 2.4 from [16]. In this lemma a group $G$ is considered with a central $\varphi$–admissible subgroup $C$ for an automorphism $\varphi : G \to G$. It states that the pre image of any $\varphi$–conjugacy class $[g]_\varphi$ of the induced automorphism $\varphi : G/C \to G/C$ is a disjoint union of $s = [C : L(C, \varphi_g)]$ $\varphi$–conjugacy classes. Here $\varphi_g = \varphi \circ \sigma_g$, where $\sigma_g \in InnG, \sigma_g : h \mapsto g^{-1}hg$ for all $h \in G$.

We apply this statement consequently to groups $G = N_i$ and central subgroups $C = A_i, i = 0, 1, ..., k−1$. By our assumption $Fix_{\varphi_i}(N_{i+1}) = 1$, so if $x \varphi_i = cx, c \in A_i$, then $x \in A_i$. We see that $L(A_i, \varphi_i) = L(A_i, (\varphi_i)_g)$ for every $g \in N_i$. Hence every pre image of $[g]_{\varphi_{i+1}}$ in $N_i$ is a disjoint union of exactly (independent of $g$) $q_i = [A_i : L(A_i, \varphi_i)]$ $\varphi_i$–conjugacy classes. It follows that

$$R(\varphi_i) = R(\varphi_{i+1}) \cdot q_i, \ i = 0, 1, ..., k−1.$$ (16)

Repeating such process we derive (15).
3 The Reidemeister spectrum of $N_{rc}$ for $r = 2$ and $c = 2, 3$; and $r = 3$ and $c = 2$

Let $N = N_{22}$ be the free nilpotent group of rank 2 and class 2 (also known as the discrete Heisenberg group). Let $x, y$ be a free basis of $N$. Then the center $\zeta_1 N$ (which coincides with the derived subgroup $N'$) is generated by a single basic commutator $(x, y)$.

Let an automorphism $\varphi : N \to N$ induces the automorphism of abelianization $\bar{\varphi} : N/N' \to N/N'$, with matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (17)$$

matching to the basis $x, y$. We assume that $\det(A - E) = \tr A = k \neq 0$, because in other case $R(\bar{\varphi}) = R(\varphi) = \infty$. Moreover, since $(x, y)\varphi = (x, y)^{\det(A)}$ we assume that $\det(A) = -1$, because $\det(A) = 1$ implies $(x, y)\varphi = (x, y)$ and so $R(\varphi) = \infty$ again. Now we have $(x, y)\varphi = (x, y)^{-1}$, and so $[\zeta_1 N : L(N, \varphi)] = [\zeta_1 N : L(\zeta_1 N, \varphi)] = 2$. By Lemma 2.1 we obtain that $R(\varphi) = 2|\tr A| \in 2\mathbb{N}$. It remains to take the matrices $A(2, k)$ from the previous section to conclude that

$$\text{Spec}_{R}(N_{22}) = 2\mathbb{N} \cup \{\infty\}. \quad (18)$$

In paper [12] it was proved that each even number belongs to $\text{Spec}_{R}(N_{22})$, but nothing was said about odd numbers.

Let now $N = N_{23}$ be the free nilpotent group of rank 2 and class 3. Let $x, y$ be a basis of $N$.

Then the center $\zeta_1 N$ has a basis consisting from the basic commutators of weight 3: $g_1 = (x, y, x), g_2 = (x, y, y)$. Here and so far we suppose that the brackets in any long commutator stand from left to right, in particular $(h_1, h_2, h_3) = ((h_1, h_2), h_3)$, and so on.

It is well known (see [11]) that every automorphism of a free abelian quotient $N/N'$ of any free nilpotent group $N$ is induced by some automorphism of $N$ itself. Moreover any endomorphism $\eta : N \to N$, invertible $\mod (N')$ (inducing an automorphism in the abelianization $N/N'$), is automorphism. We will use this fact later many times without mention.

By direct calculation we derive that an automorphism $\varphi : N \to N$, with matrix on the abelianization $N/N'$ as in (17) (we assume again that $\det(A) = -1$ and $\det(A - E) = \tr A \neq 0$) induces the automorphism $\bar{\varphi}_0 : \zeta_1 N \to \zeta_1 N$ with matrix in the basis $g_1, g_2$

$$A_3 = \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}. \quad (19)$$

We see that

$$\det(A_3) = -1, \det(A_3 - E) = \alpha + \delta = \tr A. \quad (20)$$
Since every number $N$ is obviously realized as $k = trA_{[k]}$ for some matrix $A_{[k]} \in GL_2(\mathbb{Z})$ we complete our consideration as above to conclude that 

$$Spec_R(N_{23}) = \{2k^2 | k \in \mathbb{N}\} \cup \{\infty\}. \quad (21)$$

At last, let $N = N_{32}$ be the free nilpotent group of rank 3 and class 2. Let $x, y, z$ be any basis of $N$. Then the center $\zeta_1 N$ (which coincides with the derived subgroup $N'$) has a basis $h_1 = (x, y), h_2 = (x, z)$, and $h_3 = (y, z)$.

Let an automorphism $\varphi : N \to N$ induces the automorphism $\bar{\varphi} : N/N' \to N/N'$ with a matrix

$$A = (a_{ij}), \ i, j = 1, 2, 3, \quad (22)$$

in a basis of the abelianization $N/N'$ matching to $x, y, z$. Then the matrix of the induced automorphism $\bar{\varphi}_0 : N' \to N'$ is

$$B = \begin{pmatrix} M_{33} & M_{32} & M_{31} \\ M_{23} & M_{22} & M_{21} \\ M_{13} & M_{12} & M_{11} \end{pmatrix}, \quad (23)$$

where $M_{ij}$ means the minor deriving by deleting $i$-row and $j$-column of $B$.

We can assume that $det(A - E) \neq 0$, in other case $R(\varphi) = \infty$. By direct calculation we get $det(B) = 1$. Also by direct calculation we obtain

$$det(A - E) = detA - (M_{11} + M_{22} + M_{33}) + a_{11} + a_{22} + a_{33} - 1, \quad (24)$$

and

$$det(B - E) = det(B) - det(A)(a_{11} + a_{22} + a_{33}) + M_{11} + M_{22} + M_{33} - 1. \quad (25)$$

We see that these numbers in view $det(A) = \pm 1$ and $det(B) = 1$ have the same parity. If they both are even then their product is divided by 4, if both are odd this product is odd too. It follows that numbers $4l + 2$ can not appear.

On the other hand a matrix

$$D_{[n]} = \begin{pmatrix} n & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (26)$$

gives an example of automorphism $\varphi(n)$ which induces the automorphism of abelianization $N/N'$ with matrix $D_{[n]}$ gives an example of $R(\varphi_n) = 2n - 1$.

A matrix

$$F_{[n]} = \begin{pmatrix} n + 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (27)$$

in the same way presents an example of an automorphism $\psi(n)$ for which $R(\psi(n)) = 4n$. 


It follows that

\[ \text{Spec}_{R} N_{23} = \{2n - 1|n \in \mathbb{N}\} \cup \{4n|n \in \mathbb{N}\} \cup \{\infty\}. \] (28)

4 Every free nilpotent group $N_{2c}$ for $c \geq 4$ satisfies to the $R_{\infty}$ property

The main result of this section will follow from the next principal statement.

**Theorem 1.** Let $N = N_{24}$ be the free nilpotent group of rank 2 and class 4. Then every automorphism $\varphi : N \to N$ induces the automorphism $\bar{\varphi}_0 : \zeta_1 N \to \zeta_1 N$ such that $\det(\bar{\varphi}_0 - E) = 0$.

**Proof.** Let $x, y$ be a basis of $N$. Then the basic commutators of the weight 4

\[ f_1 = (x, y, x, x), f_2 = (x, y, y, x), \text{ and } f_3 = (x, y, y, y) \]

present a basis of $C = \zeta_1 N$ (see [11]). Since $N$ is metabelian one has identity

\[ (g, f, h_1, h_2) = (g, f, h_2, h_1). \] (29)

Now suppose that any automorphism $\varphi : N \to N$ induces the automorphism of the abelianization $\bar{\varphi} : N/N' \to N/N'$ with a matrix $A$ in the basis matching to $x, y$. So, $\det(A) = \pm 1$. We assume that $\det(A) = -1$, in other case $(x, y)\bar{\varphi} = (x, y)mod\zeta_2 N$, and so $R(\varphi) = \infty$. Moreover, we can assume that $tr(A) \neq 0$, by a similar reason (see previous section). Let $\bar{\varphi}_0 : \zeta_1 N \to \zeta_1 N$ be the automorphism induced by $\varphi$.

By direct calculation we define the matrix of $\bar{\varphi}_0$ in the basis $f_1, f_2, \text{ and } f_3$

\[ B = \begin{pmatrix} -\alpha^2 & -2\alpha\beta^2 & -2\beta^2 \\ -\alpha\gamma & -\alpha\delta + \beta\gamma & -\beta\delta \\ -\gamma^2 & -2\gamma\delta & -\delta^2 \end{pmatrix}. \] (30)

By direct calculation we derive that $\det(B) = -1$, and $\det(B - E) = 0$.

Since by our assumptions $\text{Fix}_{\varphi_1}(N_1) = 1$, where as above $N_1 = N/\zeta_1 N$ is the free nilpotent group $N_{23}$ of rank 2 and class 3 (see previous section), we conclude that $L(N, \varphi) = L(\zeta_1 N, \varphi_0)$ has infinite index in $\zeta_1 N$. Hence by Lemma 2.4 from [10] we obtain that $R(\varphi) = \infty$.

Theorem is proved.

**Corollary 4.1.** Every group $N_{2c}$ for $c \geq 4$ satisfies to the $R_{\infty}$ property.

Since $N_{24}$ is metabelian and in view of Theorem 2 in [10] we immediately obtain

Since $N_{24}$ is metabelian and in view of Theorem 2 in [10] we immediately obtain
Theorem 2. 1) Every free solvable group $S_{2t}$ of rank 2 and class $t \geq 2$ (in particular, the free metabelian group $M_2 = S_{22}$ of rank 2) satisfies to the $R_\infty$ property.

2) Every free solvable group $S_{3t}$ of rank 3 and class $t \geq 4$ satisfies to the $R_\infty$ property.

3) Every free solvable group $S_{rt}$ of rank $r \geq 4$ and class $t \geq \log_2(2r+1)$ satisfies to the $R_\infty$ property.

Proof. We have for every $t \geq 2$

$$N_{24} = S_{2t}/\gamma_5 S_{2t},$$

(31)

where $\gamma_5 S_{2t}$ is an automorphic admissible subgroup of $S_{2t}$. Any automorphism $\tilde{\varphi} : S_{2t} \to S_{2t}$ induces the automorphism $\varphi : N \to N$ for which $R(\varphi) = \infty$. Hence $R(\tilde{\varphi}) = \infty$ too.

2) By Theorem 2 in [16] one has $N_{3,12} \in R_\infty$. Since for every $t \geq 4$

$$N_{3,12} = S_{3t}/\gamma_{13} S_{3t},$$

(32)

by the similar argument we conclude that $S_{3t}$ satisfies to the $R_\infty$ property.

3) By Theorem 2 in [16] one has $N_{r,2r+1} \in R_\infty$. Since for every $t \geq \log_2(2r+1)$

$$N_{r,2r} = S_{rt}/\gamma_{2r+1} S_{rt},$$

(33)

we again derive $S_{rt} \in R_\infty$.

Theorem is proved.

Remark. The similar results can be proved for any varieties of groups (not just the varieties $A^t$ of all solvable groups of given class $t$) which admit a natural homomorphisms onto free nilpotent groups of class big enough.

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