A COMBINATORIAL MODEL FOR THE TRANSITION MATRIX BETWEEN THE SPECHT AND WEB BASES

BYUNG-HAK HWANG, JIHYEUG JANG, AND JAESEONG OH

Abstract. We introduce a new class of permutations, called web permutations. Using these permutations, we provide a combinatorial interpretation for entries of the transition matrix between the Specht and web bases, which answers Rhoades’s question. Furthermore, we study enumerative properties of these permutations.

1. Introduction and the main result

In this article, we study the transition matrix between two famous bases, the Specht basis and the web basis, for the irreducible representation of the symmetric group $\mathfrak{S}_{2n}$ indexed by the partition $(n, n)$. Motivated by Rhoades’s work [Rho19], we give a combinatorial interpretation for entries of the transition matrix as a certain class of permutations, and present their interesting properties.

For an integer $n \geq 1$, let $\mathfrak{S}_{2n}$ be the symmetric group on the set $[2n] = \{1, \ldots, 2n\}$. It is well known that each irreducible representation of $\mathfrak{S}_{2n}$ can be indexed by a partition of $2n$. For a partition $\lambda$ of $2n$, we then denote by $\mathcal{S}^\lambda$ the irreducible representation indexed by $\lambda$, called the Specht module. In this article, we narrow our focus down to the Specht module indexed by the partition $(n, n)$, and two well-studied bases for $\mathcal{S}^{(n,n)}$.

A standard Young tableau of shape $(n, n)$ is an $2 \times n$ array of integers whose entries are $[2n]$, and each row and each column are increasing. See Figure 1 for example. The set of standard Young tableaux of shape $(n, n)$, denoted by $\text{SYT}(n, n)$, parametrizes the Specht basis

$$\{v_T \in \mathcal{S}^{(n,n)} : T \in \text{SYT}(n, n)\}$$

for $\mathcal{S}^{(n,n)}$. For more details on the Specht basis and related combinatorics, see [Ful97, Sag01].

A (perfect) matching on $[2n]$ is a set partition of $[2n]$ such that each block has size 2. We also depict a matching on $[2n]$ as a diagram consisting of $2n$ vertices and $n$ arcs where any pair of arcs has no common vertex. A crossing is a pair of arcs $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. A matching is called noncrossing if the matching has no crossing, and nonnesting if there is no pair of arcs $\{a, d\}$ and $\{b, c\}$ with $a < b < c < d$; see Figure 2.

\begin{center}
\begin{tabular}{ccc}
1 & 3 & 4 \\
2 & 5 & 7 & 8
\end{tabular}
\end{center}

\textbf{Figure 1.} A standard Young tableau of shape $(4, 4)$. 

1
For a matching $M$ and $\{i, j\} \in M$ with $i < j$, $i$ is called an opener and $j$ is called a closer. Let $\text{Mat}_{2n}$ ($\text{NC}_{2n}$ and $\text{NN}_{2n}$, respectively) stand for the set of (noncrossing and nonnesting, respectively) matchings on $[2n]$.

Note that there is a natural bijection between $\text{SYT}(n, n)$ and $\text{NN}_{2n}$. For $T \in \text{SYT}(n, n)$, connect two vertices lying on the same column of $T$ via an arc, then we obtain a nonnesting matching. For instance, the tableau in Figure 1 and the first matching in Figure 2 are under this correspondence. Using this correspondence, we index the Specht basis for $S(n,n)$ by nonnesting matchings of $[2n]$, instead of standard Young tableaux of shape $(n, n)$:

$$\{ v_M \in S(n,n) : M \in \text{NN}_{2n} \}.$$ 

We now consider the $2 \times 2n$ matrix

$$z = \begin{bmatrix} z_{1,1} & z_{1,2} & \cdots & z_{1,2n} \\ z_{2,1} & z_{2,2} & \cdots & z_{2,2n} \end{bmatrix},$$

where $z_{i,j}$’s are indeterminates. For $1 \leq i < j \leq 2n$, let $\Delta_{ij} := \Delta_{ij}(z)$ be the maximal minor of $z$ with respect to the $i$th and $j$th columns, i.e., $\Delta_{ij} = z_{1,i}z_{2,j} - z_{1,j}z_{2,i}$. For a matching $M \in \text{Mat}_{2n}$, let

$$\Delta_M := \Delta_M(z) = \prod_{\{i,j\} \in M} \Delta_{ij} \in \mathbb{C}[z_{1,1}, \ldots, z_{2,2n}].$$

It is important to note that the polynomials $\Delta_{ij}$ satisfy the following relation: For $1 \leq a < b < c < d \leq 2n$,

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}. \quad (1)$$

We define a vector space $W_n$ to be the $\mathbb{C}$-span of $\Delta_M$ for all $M \in \text{Mat}_{2n}$. In [KR84], it turns out that the set

$$\{ \Delta_M \in W_n : M \in \text{NC}_{2n} \}$$

forms a basis for $W_n$. We call this basis the web basis. (The web basis was developed in the SL$_2$-invariant theory due to Kuperburg [Kup96], and its original construction slightly differs from the one we describe above. But they are essentially the same; see [Rho19].)

In addition, there is a natural $\mathfrak{S}_{2n}$-action on $W_n$ as follows: Regarding a permutation $\sigma \in \mathfrak{S}_{2n}$ as a $2n \times 2n$ permutation matrix, define $\sigma \cdot \Delta_M(z) := \Delta_M(z\sigma^{-1})$. Then the space $W_n$ is closed under this action, and hence carries an $\mathfrak{S}_{2n}$-module structure. Furthermore, the $\mathfrak{S}_{2n}$-module $W_n$ is isomorphic to the Specht module $S^{(n,n)}$ [PPR09]. Therefore, due to Schur’s lemma, there is a unique (up to scalar) isomorphism between $W_n$ and $S^{(n,n)}$. 

---

Figure 2. Two matchings on $[8]$. The first one is nonnesting, while the second one is noncrossing.
We are now in a position to give the main purpose of this article. Let $M_0$ be the unique matching which is simultaneously noncrossing and nonnesting, i.e., $M_0 = \{\{1,2\}, \ldots, \{2n-1,2n\}\}$. Due to [RT19], the isomorphism maps $\Delta_{M_0}$ to $v_{M_0}$ up to scalar. Let $\varphi : \mathcal{W}_n \to \mathcal{S}^{(n,n)}$ be the unique isomorphism with $\varphi(\Delta_{M_0}) = v_{M_0}$. We also let $w_M := \varphi(\Delta_M)$ for each $M \in \text{NC}_2n$. Then the Specht basis can expand into (the image of) the web basis: For $M \in \text{NN}_2n$,

$$v_M = \sum_{M' \in \text{NC}_2n} a_{MM'} w_{M'}.$$ 

In [RT19], Russell and Tymoczko initiated the combinatorial study of the transition matrix $A = (a_{MM'})_{M \in \text{NN}_2n, M' \in \text{NC}_2n}$. They constructed directed graphs on the standard Young tableaux and noncrossing matchings, and using them, showed the unitriangularity of the matrix. They also gave some open problems related to their results. One of them is the positivity of the entries of $A$, which was proved by Rhoades soon after.

**Theorem 1.1 ([Rho19]).** The entries $a_{MM'}$ of the transition matrix $A$ are nonnegative integers.

Although Rhoades established the positivity phenomenon for entries of $A$, he did not find an explicit combinatorial interpretation of the nonnegative integer $a_{MM'}$, c.f. [Rho19, Problem 1.3]. Inspired by his work, we introduce a new family of permutations which are enumerated by the integers $a_{MM'}$, and study their enumerative properties.

Our strategy is based on Rhoades’s observation [Rho19]. He figured out that the entries $a_{MM'}$ are related to resolving crossings of matchings in the following sense: For a matching $M \in \text{Mat}_2n$, let $\{a,c\}$ and $\{b,d\}$ be a crossing pair in $M$ (if it exists) where $a < b < c < d$. Let $M'$ and $M''$ be the matchings identical to $M$ except that $\{a,b\}$ and $\{c,d\}$ in $M'$, and $\{a,d\}$ and $\{b,c\}$ in $M''$. Then, by the relation (1), we have

$$\Delta_M = \Delta_{M'} + \Delta_{M''}. \quad (3)$$

In addition, the number of crossing pairs in $M'$ (respectively, $M''$) is strictly less than the number of crossing pairs in $M$. Therefore, iterating the resolving procedure gives the expansion of $\Delta_M$ in terms of the basis (2). In other words, when we write

$$\Delta_M = \sum_{M' \in \text{NC}_2n} c_{MM'} \Delta_{M'}, \quad (4)$$

the coefficient $c_{MM'}$ is equal to the number of occurrences of the noncrossing matching $M'$ obtained by iteratively resolving crossings in $M$. Note that the order of the choice of crossing pairs does not affect the expansion of $\Delta_M$. Rhoades showed that for $M \in \text{NN}_2n$ and $M' \in \text{NC}_2n$, the entry $a_{MM'}$ of the transition matrix equals $c_{MM'}$. Hence, to give a combinatorial interpretation of $a_{MM'}$, we track the resolving process from a nonnesting matching to noncrossing matchings.

To state our main result, we need some preliminaries. First, we note that noncrossing matchings and nonnesting matchings are Catalan objects, that is, they are enumerated by
Catalan numbers. Another famous Catalan object is a Dyck path. A Dyck path of length \(2n\) is a lattice path from \((0,0)\) to \((n,n)\) consisting of \(n\) north steps \((1,0)\) and \(n\) east steps \((0,1)\) that does not pass below the line \(y = x\). We write \(N\) and \(E\) for the north step and the east step, respectively. We therefore regard a Dyck path as a sequence consisting of \(n\) \(N\)'s and \(n\) \(E\)'s. Let \(\text{Dyck}_{2n}\) be the set of Dyck paths of length \(2n\). Identifying a Dyck path with the region below the path, we give a natural partial order on \(\text{Dyck}_{2n}\) by inclusion, denoted by \(\subseteq\). For instance, the Dyck path \(N \cdots NE \cdots E\) where \(n\) \(N\)'s precede \(n\) \(E\)'s is the maximum path in \(\text{Dyck}_{2n}\) with respect to the partial order, while the path \(NENE \cdots NE\) is the minimum path. In Section 2 we define a map \(D : \text{Mat}_{2n} \rightarrow \text{Dyck}_{2n}\), and by abuse of notation, a map \(D : \mathfrak{S}_n \rightarrow \text{Dyck}_{2n}\). We also define a map \(M : \mathfrak{S}_n \rightarrow \text{NC}_{2n}\). Finally, we introduce a new family of permutations, called web permutations.

Theorem 1.2. For matchings \(M \in \text{NN}_{2n}\) and \(M' \in \text{NC}_{2n}\), the entry \(a_{MM'}\) is equal to the number of web permutations \(\sigma \in \mathfrak{S}_n\) such that \(D(\sigma) \subseteq D(M)\) and \(M(\sigma) = M'\).

The theorem follows almost immediately from the definition of the novel permutations. However, the definition does not directly tell us whether a given permutation is a web permutation or not. In Theorem 3.4 we thus explain how to characterize these permutations in terms of their cycle structures. Using this characterization, we deduce the results in [RT19, IZ21] concerning the unitriangularity of the transition matrix and a necessary and sufficient condition for additional vanishing entries.

The article is organized as follows. In Section 2 we give a new model, called a grid configuration, for representing matchings. Within this model, we resolve crossings in nonnesting matchings until there is no crossing. We then define web permutations from the noncrossing grid configurations, and prove the main theorem. In the next two sections, we study some properties of web permutations. In Section 3 we give a characterization of web permutations. We show that web permutations are closely related to André permutations. Section 4 provides some interesting enumerative properties of web permutations. One instance of them is that web permutations are enumerated by Euler numbers. We also give a conjecture for a relation between certain web permutations and the Seidel triangle. In Appendix A we give some computational data of the transition matrix and web permutations for small \(n\).

2. Grid configurations and web permutations

In this section, we define grid configurations which represent matchings in a ‘rigid’ setting. We describe the procedure of resolving crossings within this model. We then introduce a new class of permutations, called web permutations. This provides a combinatorial interpretation for the entries \(a_{MM'}\) of the transition matrix.

Consider an \(n\) by \(n\) (lattice) grid in the \(xy\)-plane with corners \((0,0)\), \((0,n)\), \((n,0)\) and \((n,n)\). We denote each cell by \((i,j)\) where \(i\) and \(j\) are the \(x\)- and \(y\)-coordinates of its upper-right corner. Let \(\sigma \in \mathfrak{S}_n\) be a permutation. For each \(1 \leq i \leq n\), mark the cell \((i,\sigma(i))\), and draw a horizontal line to the left and a vertical line to the top from the marked cell. We call this the empty grid configuration of \(\sigma\). A cell \((i,j)\) is a crossing if
there are both a vertical line and a horizontal line through the cell, that is, \( \sigma(i) < j \) and \( i < \sigma^{-1}(j) \). We denote by \( \text{Cr}(\sigma) \) the set of all crossings of \( \sigma \). For a subset \( E \subseteq \text{Cr}(\sigma) \), the grid configuration \( G(\sigma, E) \) of a pair \( (\sigma, E) \) is defined to be the empty grid configuration of \( \sigma \) where each crossing in \( E \) is replaced by an elbow as shown in Figure 3. In particular, the empty grid configuration of \( \sigma \) is \( G(\sigma, \emptyset) \).

For the \( n \) by \( n \) grid, we label leftmost vertical intervals from bottom to top with 1 through \( n \) and uppermost horizontal intervals from left to right with \( n + 1 \) through \( 2n \). With this label of boundary intervals, a grid configuration can be considered as a matching on \([2n]\) as follows: Each strand joining \( i \)-th and \( j \)-th boundary intervals represents an arc connecting \( i \) and \( j \); see Figure 4. We denote by \( M(\sigma, E) \) the matching associated to the grid configuration \( G(\sigma, E) \). For short, we write \( M(\sigma) = M(\sigma, \text{Cr}(\sigma)) \).

We define a partial order on cells of the \( n \) by \( n \) grid by \((x, y) \preceq (x', y')\) if \( x \leq x' \) and \( y \geq y' \). In other words, \((x, y) \preceq (x', y')\) if and only if the cell \((x, y)\) lies on the upper-left quadrant at \((x', y')\).

The relation (3) can be interpreted as a relation between grid configurations as follows. For a permutation \( \sigma \) and \( E \subseteq \text{Cr}(\sigma) \), let \( c = (i, j) \) be a maximal crossing in the grid configuration \( G(\sigma, E) \), i.e., there is no crossing on the upper-left quadrant at \( c \). One way of resolving \( c \) results a grid configuration \( G(\sigma, E \cup \{c\}) \). This procedure of resolving a crossing is called smoothing. The other way of resolving \( c \) results a grid configuration \( G(\sigma', E) \), where \( \sigma' \) is defined by

\[
\begin{aligned}
\sigma'(i) &= j, \\
\sigma'(\sigma^{-1}(j)) &= \sigma(i), \quad \text{and} \\
\sigma'(k) &= \sigma(k) \quad \text{for} \quad k \neq i, \sigma^{-1}(j).
\end{aligned}
\]

This procedure of resolving a crossing is called switching. Note that the crossing sets \( \text{Cr}(\sigma) \) and \( \text{Cr}(\sigma') \) are not the same. Nevertheless, by choosing \( c \) to be maximal, crossings not smaller than \( c \) (with respect to the partial order) are left unchanged under switching. In
particular, we have \( E \subseteq \text{Cr}(\sigma') \), so switching is well-defined. We often consider a grid configuration \( G \) as the vector \( \Delta_{M(G)} \). Therefore, we can write the relation (3) in terms of grid configurations as

\[
G(\sigma, E) = G(\sigma, E \cup \{c\}) + G(\sigma', E).
\]

For example, let \( \sigma = 1324 \in S_4 \) and \( E = \{(1, 3), (1, 4)\} \), and consider the grid configuration \( G(\sigma, E) \) which is shown in Figure 4. Resolving a maximal crossing \( c = (2, 4) \in \text{Cr}(\sigma) \setminus E \), we have

\[
\begin{align*}
\text{Here, the red dot indicates the crossing } c.
\end{align*}
\]

From the grid configuration \( G(id, \emptyset) \), we obtain two grid configurations by resolving a crossing by smoothing and switching, respectively. By resolving crossings until there is no crossing left, we get grid configurations of the form \( G(\sigma, \text{Cr}(\sigma)) \). For each remaining grid configuration \( G(\sigma, \text{Cr}(\sigma)) \), the permutation \( \sigma \) is called a web permutation of \([n]\) and we denote the set of web permutations of \([n]\) by \( \text{Web}_n \). In other words, we have

\[
G(id, \emptyset) = \sum G(\sigma, \text{Cr}(\sigma)),
\]

where the right hand side is the sum of all grid configurations obtained by resolving crossings from the grid configuration \( G(id, \emptyset) \) until there is no crossing left. This is reminiscent of (4). For example, starting from the grid configuration \( G(id, \emptyset) \) for \( n = 3 \), we have

\[
\begin{align*}
\text{Therefore we conclude that } \text{Web}_3 &= \{123, 213, 132, 231, 321\}. \text{ The following proposition justifies that web permutations are well-defined.}
\end{align*}
\]

**Proposition 2.1.** The expansion in (5) is unique. In other words, the grid configurations appearing in (5) does not depend on the order of resolving procedure (choice of maximal crossings). In addition, the permutations \( \sigma \) in (5) are all distinct.

**Proof.** Any total order extending the partial order \( \succeq \) on cells can be obtained from another total order by applying a sequence of changing the order of two incomparable cells.
Therefore it suffices to show that we can change the order of two maximal crossings. Let \( c \) and \( c' \) be two maximal crossings in a grid configuration \( G(\sigma, E) \) with the \( x \) coordinate of \( c \) is less than the \( x \)-coordinate of \( c' \). There are two cases: The \( y \)-coordinate of \( c \) and \( x \)-coordinate of \( c' \) are the same, or not. Two such cases are depicted in Figure 5, where the crossings \( c \) and \( c' \) are indicated by red dots.

For the first case, if we resolve both \( c \) and \( c' \) in the same way (both by smoothing or both by switching), the order of resolving \( c \) and \( c' \) is irrelevant. Therefore, it remains to show that if we resolve \( c \) in a way and \( c' \) in the other way results the same grid configuration when we resolve \( c' \) first and then \( c \), which can be checked directly. In addition, it is clear that the order of resolving crossings \( c \) and \( c' \) is irrelevant for the second case.

Let \( G(\sigma, E) \) be a grid configuration and \( c = (i, j) \) be a maximal crossing in \( G(\sigma, E) \). Suppose that we resolve \( c \) by smoothing and then resolve other crossings until there is no crossing to obtain a grid configuration of the form \( G(\tau, \text{Cr}(\tau)) \). Since there is an elbow at \( c \), we have \( \tau(i) < j \). On the other hand, suppose that we resolve \( c \) by switching and then resolve other crossings until there is no crossing to obtain a grid configuration of the form \( G(\rho, \text{Cr}(\rho)) \). Since there is a marking at \( c \), we have \( \rho(i) = j \). By this observation, we conclude that web permutations are all distinct.

For a matching \( M \), record \( N \) for openers and \( E \) for closers reading \( M \) from left to right. This gives the Dyck path \( D(M) \) in the \( n \) by \( n \) grid. It is known that the two restrictions of the map \( D : \text{Mat}_{2n} \to \text{Dyck}_{2n} \to \text{NC}_{2n} \) and \( \text{NN}_{2n} \) are bijections. To a permutation \( \sigma \), we associate the minimum Dyck path \( D(\sigma) \) where every cell \((i, \sigma(i))\) lies below the path; see Figure 6.

Given a nonnesting matching \( M \in \text{NN}_{2n} \), let \( E(M) \) be the set of cells in the \( n \) by \( n \) grid which are above the path \( D(M) \). It is easy to see that the matchings \( M \) and \( M(id, E(M)) \)
Figure 7. The grid configuration $G(id, E(M))$ and the Dyck path $D(M)$ where $M = \{(1, 2), (3, 5), (4, 7), (6, 8)\}$.

coincide. For example, let $M$ be the first matching in Figure 2. Then the corresponding path $D(M)$ is NENNENEE, and $E(M) = \{(1, 2), (1, 3), (1, 4), (2, 4)\}$. The grid configuration $G(id, E(M))$ is shown in Figure 7, and one can see $M(id, E(M)) = M$. Similarly to the definition of Web$_n$, we consider the equation 

$$G(id, E(M)) = \sum G(\sigma, Cr(\sigma)),$$

where the right hand side is the summation of grid configurations obtained by resolving crossings in $G(id, E(M))$ until there is no crossing. We then define Web$_M$ to be the set of permutations $\sigma$ appearing in the right hand side of the above equation. In particular, Web$_n = Web_M$ where $M = \{(1, n+1), (2, n+2), \ldots, (n, 2n)\}$.

Using the above notations, we prove one of our main results that tells us which web permutations contribute to the entry $a_{MM'}$.

**Proof of Theorem 1.2.** By the definition of web permutations, we have 

$$a_{MM'} = |\{\sigma \in Web_M : M(\sigma) = M'\}|.$$ 

Hence it is enough to show that 

$$Web_M = \{\sigma \in Web_n : D(\sigma) \subseteq D(M)\}.$$ 

(6)

We can obtain the grid configuration $G(id, E(M))$ from $G(id, \emptyset)$ by smoothing crossings in $E(M)$. Since Proposition 2.4 says that Web$_n$ does not depend on the order of resolving processes, we obtain Web$_M \subseteq Web_n$. From this, it is clear that 

$$Web_M = \{\sigma \in Web_n : E(M) \subseteq Cr(\sigma)\} = \{\sigma \in Web_n : (i, \sigma(i)) \notin E(M) \text{ for all } i\},$$

which proves the claim (6). 

3. Characterization of Web Permutations

In the previous section, we have introduced the new class of permutations which are obtained by tracking the resolving process. In fact, Theorem 1.2 is just a byproduct of the definition of web permutations. In this section, we provide a characterization of these permutations. This characterization depends only on their permutation structure. Using this characterization, we also prove the results in [RT19, IZ21].
We begin with recalling two ways to represent permutations. One way is the one-line notation which we have already used, that is, regarding a permutation as a word. More precisely, for a permutation \(\sigma: [n] \to [n]\), we write \(\sigma = \sigma_1\sigma_2 \ldots \sigma_n\) where \(\sigma_i = \sigma(i)\). Another way to write permutations is the cycle notation. Instead of the precise definition of this notation, we give an example; for the definition, see [Sta12]. Let \(\sigma = 564132 \in S_6\), then the cycle notation of \(\sigma\) is \((1, 5, 3, 4)(2, 6)\). We always use parentheses and commas for writing cycles.

To describe our characterization of web permutations, we review the notion of André permutations and define an analogue of them. André permutations were introduced by Foata and Schützenberger [FS73], and have been studied with several applications in the literature, see, e.g., [Sta94, FH16]. One of the interesting properties of them is that they are enumerated by Euler numbers; see Section 4.

We now think of permutations as words consisting of distinct positive integers. André permutations are defined recursively as follows. First, the empty word and each one-letter word are André permutations. For a permutation \(w \in S_n\), we only need to show that the permutation \(\max \{w_1, \ldots, w_n\}\) is an André cycle.

**Definition 3.1.** Let \(C = (a_1, \ldots, a_k)\) be a cycle with \(a_1 = \min \{a_1, \ldots, a_k\}\). We say that \(C\) is an André cycle if the permutation \(a_2 \cdots a_k\) is an André permutation.

For instance, a cycle \(C = (2, 3, 9, 1, 5, 4, 7)\) is an André cycle since \(C = (1, 5, 4, 7, 2, 3, 9)\) and the permutation 547239 is an André permutation.

For a cycle \(C = (a_1, \ldots, a_k)\), we write \(\min C = \min\{a_1, \ldots, a_k\}\) and \(\max C = \max\{a_1, \ldots, a_k\}\) for short. The following lemma is useful in the sequel.

**Lemma 3.2.** Let \(C = (a_1, \ldots, a_k)\) be an André cycle with \(a_1 = \min C\). Then \(a_k = \max C\).

**Proof.** By definition, the last letter of an André permutation is the largest element in the permutation. This fact directly gives the proof. \(\square\)

The following lemma gives how to obtain a new André cycle from old André cycles.

**Lemma 3.3.** Let \(C_1 = (a_1, \ldots, a_k)\) and \(C_2 = (b_1, \ldots, b_\ell)\) be André cycles with \(a_1 = \min C_1\) and \(b_1 = \min C_2\). If \(a_1 < b_1\) and \(a_k < b_\ell\), then the cycle \((a_1, \ldots, a_k, b_1, \ldots, b_\ell)\) is also an André cycle.

**Proof.** We induct on \(k\). First, consider the base case \(k = 1\). Since \(a_1 = \min\{a_1, b_1, \ldots, b_\ell\}\), we only need to show that the permutation \(b_1 \cdots b_k\) is an André permutation. This follows immediately from the definition of André permutations.

We now suppose \(k \geq 2\). Recall that the two permutations \(a_2 \cdots a_k\) and \(b_2 \cdots b_\ell\) are André permutations. In addition, by Lemma 3.2 and the assumption \(a_k < b_\ell\), we have \(\max\{a_2, \ldots, a_k\} < \max\{b_2, \ldots, b_\ell\}\). Thus, if \(b_1 < \min\{a_2, \ldots, a_k\}\), then the permutation
$a_2 \cdots a_kb_1 \cdots b_t$ is an André permutation. Otherwise, let $a_p = \min\{a_2, \ldots, a_k\}$ for some $p$, so that $a_p < b_1$ and both $a_2 \cdots a_{p-1}$ and $a_{p+1} \cdots a_k$ are André permutations. By the induction hypothesis, we have that the cycle $(a_p, \ldots, a_k, b_1, \ldots, b_t)$ is an André cycle. It is also clear that $a_{p-1} < a_k < b_t$. Again, by the induction hypothesis, we deduce that the cycle $(a_1, \ldots, a_k, b_1, \ldots, b_t)$ is an André cycle, which yields the desired result.

We now show another main result of the article, which gives a characterization of web permutations.

**Theorem 3.4.** A permutation $\sigma \in \mathfrak{S}_n$ is a web permutation if and only if each cycle of $\sigma$ is an André cycle.

**Proof.** Recall that the web permutations do not depend on the order of choices of maximal crossings. Hence we fix the following total order on the cells in the $n$ by $n$ grid, which completes the partial order, and we assume that our resolving process respects this total order: For two cells $(i, j)$ and $(i', j')$, we let $(i, j) > (i', j')$ if either $j > j'$, or $j = j'$ and $i < i'$. We first prove the “only if” part. Let $\sigma$ be a web permutation, and

$$(G^{(0)} = G(id, \emptyset), G^{(1)}, \ldots, G^{(r)} = G(\sigma, \text{Cr}(\sigma)))$$

be the sequence of grid configurations where $G^{(k)}$ is obtained from $G^{(k-1)}$ by resolving a single crossing for each $k$, with respect to the total order. We write $G^{(k)} = G(\sigma^{(k)}, E^{(k)})$. Also let $c^{(k)}$ be the crossing in $\text{Cr}(\sigma^{(k-1)}) \setminus E^{(k-1)}$ such that $G^{(k)}$ is obtained from $G^{(k-1)}$ by resolving $c^{(k)}$.

It is obvious that the identity permutation $\sigma^{(0)} = id$ consists of André cycles. We claim that each $\sigma^{(k)}$ also consists of André cycles for $1 \leq k \leq r$, in particular, so does $\sigma$. Fix an integer $1 \leq k \leq r$. We use an inductive argument, so suppose that each cycle of $\sigma^{(k-1)}$ is an André cycle. If $G^{(k)}$ is obtained by smoothing the crossing $c^{(k)}$ in $G^{(k-1)}$, then $\sigma^{(k-1)} = \sigma^{(k)}$ and thus there is nothing to prove. Therefore, we assume that $G^{(k)}$ is obtained from $G^{(k-1)}$ by switching the crossing $c^{(k)} = (i, j)$. Then

$$\sigma^{(k-1)}(i) < j \quad \text{and} \quad i < (\sigma^{(k-1)})^{-1}(j). \quad (7)$$

Let $C_1, \ldots, C_\ell$ be cycles of $\sigma^{(k-1)}$. We first observe that for each $1 \leq p \leq \ell$, all entries in $C_p$ except the minimum $\min C_p$ are greater than $j$. We justify this observation later. From this, we have that $i$ and $j$ are contained in different cycles of $\sigma^{(k-1)}$. Indeed, if $i$ and $j$ lie on the same cycle, then $\sigma^{(k-1)}(i)$ also lies on the cycle, but it is a contradiction to (7). Without loss of generality, let $C_1 = (a_1, \ldots, a_s)$ and $C_2 = (b_1, \ldots, b_t)$ contain $i$ and $j$ respectively with $a_1 = \min C_1$ and $b_1 = \min C_2$. By the first inequality of (7) and the observation, $a_1 = \sigma^{(k-1)}(i)$ and $b_1 = j$, so $a_s = i$ and $b_t = (\sigma^{(k-1)})^{-1}(j)$. By definition, resolving the crossing $(i, j)$ by switching merges two cycles $C_1$ and $C_2$ into the cycle $C = (a_1, \ldots, a_s, b_1, \ldots, b_t)$, and leaves other cycles unchanged. It therefore follows from Lemma 3.3 and (7) that the cycle $C$ is an André cycle. Note that $\min C = a_1 = \sigma^{(k-1)}(i) < j$, and there is no crossing on row $j$ in the grid configuration $G^{(k)}$. Hence the crossing $c^{(k+1)}$ lies below row $j$, which implies the observation inductively.
We now prove the “if” part. It suffices to show that we obtain any André cycle by iterating resolving processes to the identity permutation along the total order. We induct on the length of an André cycle where the base case being trivial. Suppose that $C = (a_1, \ldots, a_k)$ is an André cycle with $k \geq 2$ and $a_1 = \min C$. Then by definition, the permutation $a_2 \cdots a_k$ is an André permutation. Let $a_p = \min \{a_2, \ldots, a_k\}$ for some $p$, so $a_2 \cdots a_{p-1}$ and $a_{p+1} \cdots a_k$ are also André permutations. Thus, the cycles $(a_1, \ldots, a_{p-1})$ and $(a_p, \ldots, a_k)$ are André cycles. By the induction hypothesis, we can obtain the web permutation $\sigma = (a_1, \ldots, a_{p-1})(a_p, \ldots, a_k)$ by resolving processes. More precisely, we can obtain the grid configuration $G(\sigma, E)$ such that for $1 \leq i \leq n$ and $j \leq a_p$, $(i, j) \notin E$. Furthermore, one can easily check that the cell $(a_{p-1}, a_p)$ belongs to $\text{Cr}(\sigma)$, so $(a_{p-1}, a_p) \in \text{Cr}(\sigma) \setminus E$. We then obtain the desired cycle $C$ by switching the crossing $(a_{p-1}, a_p)$ in the grid configuration $G(\sigma, E)$, which completes the proof. 

As an application of the characterization, we show that the transition matrix $(a_{MM})$ is unitriangular with respect to a certain order on $\mathbb{N}^n$ and $\mathbb{N}C_2 n$, and determine which entries $a_{MM}$ vanish. These are already known due to Russell–Tymoczko [RT19] and Im–Zhu [IZ21].

Before we give the vanishing condition, we first show that the set $\text{Web}_n$ includes a well-studied class of permutations. For a permutation $\sigma = \sigma_1 \cdots \sigma_n$, we say that $\sigma$ contains a 312-pattern if there exist three indices $1 \leq i < j < k \leq n$ such that $\sigma_j < \sigma_k < \sigma_i$. A permutation is 312-avoiding if it does not contain a 312-pattern. Note that 312-avoiding permutations are a Catalan object. Furthermore, the restriction of $D : \mathfrak{S}_n \to \text{Dyck}_{2n}$ to the set of 312-avoiding permutations of $[n]$ is a bijection.

**Corollary 3.5.** A 312-avoiding permutation is a web permutation.

**Proof.** By Theorem 3.4, it suffices to show the following: For a permutation $\sigma$, if there is a cycle $C = (a_1, \ldots, a_\ell)$ which is not an André cycle in $\sigma$, then $\sigma$ contains a 312-pattern which consists of $a_i$’s.

We use induction on the length of $C$. Since any cycle of length less than 3 is an André cycle, the base case is when the length of $C$ is 3. The only case is of the form $(a_1, a_2, a_3)$, where $a_1 < a_3 < a_2$. Thus, $\sigma$ contains a 312-pattern $a_1 < a_3 < a_2$.

Now assume that the length of $C$ is larger than 3. Write $C = (a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_r)$, where $a_1$ and $b_1$ is the smallest and the second smallest elements of $C$, respectively. Then one of the following holds:

i) $C_1 = (a_1, a_2, \ldots, a_\ell)$ is not an André cycle.

ii) $C_2 = (b_1, b_2, \ldots, b_r)$ is not an André cycle.

iii) both $C_1$ and $C_2$ are André cycles and $\text{max } C_1 = a_\ell > b_r = \text{max } C_2$.

For the first case, by the induction hypothesis, there exist integers $0 \leq i, j, k \leq \ell$ such that $a_i < a_j < a_k$ and $a_{j+1} < a_{k+1} < a_{i+1}$ where the subscripts are interpreted modulo $\ell$. Note that $\sigma_{a_i} = a_{i+1}$ except for $\sigma_{a_\ell} = b_1$. Since $b_1$ is the second smallest element, replacing $a_1$ with $b_1$ does not change the pattern of $a_{i+1}a_{j+1}a_{k+1}$. Therefore, $\sigma$ contains a 312-pattern as we claimed. The second case can be proved similarly to the first case. For
Remark 3.6. In [RT19], Russell and Tymoczko defined a directed graph $\Gamma$ on $\text{NC}_{2n}$ and from $\text{NC}_{2n}$ to $\text{Dyck}_{2n}$. Then the maps $D$ induce a partial order on $\text{NN}_{2n}$ and $\text{NC}_{2n}$. Furthermore, when we choose a total order on $\text{Dyck}_{2n}$ that completes the partial order $\subseteq$, the maps $D$ give a total order on $\text{NN}_{2n}$ and $\text{NC}_{2n}$.

Corollary 3.7. In [RT19], Russell and Tymoczko defined a directed graph $\Gamma$ on $\text{NC}_{2n}$, and defined a partial order on $\text{NC}_{2n}$ using the digraph. The graph $\Gamma$ is an edge-labeled directed graph whose vertex set is the set of noncrossing matchings and its labeled edges are given as follows. For $M, M' \in \text{NC}_{2n}$, assign a labeled, directed edge $M \rightarrow M'$ if both of the following hold:

i) $M$ has arcs $\{j, k\}$ and $\{i, i + 1\}$ while $M'$ has arcs $\{j, i\}$ and $\{i + 1, k\}$ where $j < i < k$.

ii) Other arcs in $M$ and $M'$ are the same.

The graph $\Gamma$ defines a partial order on $\text{NC}_{2n}$ by letting $M \preceq M'$ if there is a directed path from $M'$ to $M$ in $\Gamma$. Russell and Tymoczko also defined a partial order on $\text{NN}_{2n}$ via a well-known bijection between $\text{NN}_{2n}$ and $\text{NC}_{2n}$. It is straightforward to see that their partial order on $\text{NC}_{2n}$ and $\text{NN}_{2n}$ coincides with ours.

We now take a total order on $\text{Dyck}_{2n}$ which completes the partial order $\subseteq$, and thus we have the induced total order on $\text{NN}_{2n}$ and $\text{NC}_{2n}$. We assume that orderings of rows and columns of the transition matrix $(a_{MM'})$ are the decreasing orders with respect to the total order on $\text{NN}_{2n}$ and $\text{NC}_{2n}$. Then the entry $a_{MM'}$ is on the diagonal if and only if $D(M) = D(M')$.

We are now ready to prove the unitriangularity of the transition matrix $(a_{MM'})$ and the conjecture of Russell and Tymoczko [RT19, Conjecture 5.8] concerning the condition of the vanishing entries, which is later proved by Im and Zhu [IZ21, Theorem 1.1].

Corollary 3.7. [RT19, IZ21] Let $M \in \text{NN}_{2n}$ and $M' \in \text{NC}_{2n}$. Then $a_{MM'} > 0$ if and only if $D(M') \subseteq D(M)$. In particular, the transition matrix $(a_{MM'})$ is upper-triangular. Moreover, there are ones along the diagonal of the transition matrix, and 312-avoiding permutations contribute to the ones.

Proof. Recall that by the argument in the proof of Theorem 1.2 we have

$$a_{MM'} = \{\sigma \in \text{Web}_M : M(\sigma) = M'\}.$$  

We first show that there are ones along the diagonal, i.e., $a_{MM'} = 1$ if $D(M) = D(M')$. Let $\sigma$ be a permutation in $\text{Web}_M$ satisfying $M(\sigma) = M'$. Denote the set of cells above the Dyck path $D(\sigma)$ by $E(\sigma)$. We claim that $E(\sigma) = \text{Cr}(\sigma)$. Since $E(\sigma) \subseteq \text{Cr}(\sigma)$ is obvious, suppose that we have $E(\sigma) \not\subseteq \text{Cr}(\sigma)$, and let $c$ be a maximal crossing in $\text{Cr}(\sigma) \setminus E(\sigma)$. Note that

$$D(M(\sigma, E(\sigma) \cup \{c\})) \subseteq D(M(\sigma, E(\sigma))) = D(M).$$
Thus, if we resolve all crossings as smoothing to obtain $G(\sigma, \text{Cr}(\sigma))$, the associated Dyck path $D(M') = D(M(\sigma))$ lies strictly below $D(M)$ which is a contradiction.

It is well known that 312-avoiding permutations are only permutations satisfying the condition $\text{Cr}(\sigma) = E(\sigma)$ and the map $D: \text{Web}_n \to \text{Dyck}_{2n}$ is a bijection when restricted to 312-avoiding permutations (see [Sta12 § 1.2]). Here, the restriction makes sense by Corollary 3.5. Combining these facts, it follows that each 312-avoiding permutation represents each one on the diagonal in the transition matrix.

To show the “only if” part of the first assertion, assume that $D(M') \not\subseteq D(M)$. Then there exists a cell below the Dyck path $D(M')$ and above the Dyck path $D(M)$, i.e., $E(M) \setminus E(M') \neq \emptyset$. Choose a maximal cell $c = (i, j)$ in $E(M) \setminus E(M') \neq \emptyset$. Let $\sigma$ be a web permutation. If $\sigma$ is in $\text{Web}_M$, then we have $c \in \text{Cr}(\sigma)$, thus $\sigma_i < j$. On the other hand, if $M(\sigma) = M'$, then we have $\sigma_i = j$, which is a contradiction. Therefore, we have $a_{MM'} = 0$.

For the “if” part, let $M''$ be the nonnesting matching such that $D(M'') = D(M')$. Then we have

$$a_{MM'} = |\{\sigma \in \text{Web}_n : D(\sigma) \subseteq D(M), M(\sigma) = M'\}|$$

$$\geq |\{\sigma \in \text{Web}_n : D(\sigma) \subseteq D(M') \setminus D(M''), M(\sigma) = M'\}|$$

$$= a_{M''M'} = 1.$$

This completes the proof. □

4. Enumeration of web permutations

In this section, we focus on the number of web permutations. More precisely, we give a relation between web permutations and André cycles (Theorem 4.1), and show that the numbers of web permutations equal Euler numbers. We also conjecture that the Seidel triangle can be recovered completely from the certain classes of web permutations.

We have characterized web permutations using André cycles (Theorem 3.4). We now present another relationship between web permutations and André cycles. Let us first review the Foata transformation $\hat{\cdot}: \mathfrak{S}_n \to \mathfrak{S}_n$. For a permutation $\sigma \in \mathfrak{S}_n$, the canonical cycle notation of $\sigma$ is a cycle notation of $\sigma$ such that its cycles are sorted based on the smallest elements of the cycles and the smallest element of each cycle is written in the last place of the cycle. We define $\hat{\sigma}$ to be the permutation obtained by dropping the parentheses in the canonical cycle notation of $\sigma$. A right-to-left minimum is an element $\sigma_i$ such that $\sigma_i < \sigma_j$ for all $j > i$. Using right-to-left minima of $\sigma$, one can easily construct the inverse of the Foata transformation. Note that the number of cycles of $\sigma$ equals the number of right-to-left minima of $\hat{\sigma}$.

We now introduce a map $\phi: \mathfrak{S}_n \to \mathfrak{S}_{n+2}$ as a slightly modification of the Foata transformation. For a permutation $\sigma \in \mathfrak{S}_n$, define the one-cycle permutation $\phi(\sigma) \in \mathfrak{S}_{n+2}$ by

$$\phi(\sigma) := (1, \hat{\sigma}_1 + 1, \ldots, \hat{\sigma}_n + 1, n + 2).$$

It follows immediately from the bijectivity of the Foata transformation that the map $\phi$ is injective, and its image $\phi(\mathfrak{S}_n)$ is the set of one-cycle permutations $\sigma \in \mathfrak{S}_{n+2}$ with
\(\sigma(n + 2) = 1\). For instance, let \(\sigma = 568479312 \in \mathcal{G}_9\). In the canonical cycle notation, 
\(\sigma = (5, 7, 3, 8, 1)(6, 9, 2)(4)\), so \(\tilde{\sigma} = 573816924\). Then we have 
\[\phi(\sigma) = (1, 6, 8, 4, 9, 2, 7, 10, 3, 5, 11) \in \mathcal{G}_{11}.
\]
The right-to-left minima of \(\tilde{\sigma}\) are 1, 2 and 4, which are the minima of cycles of \(\sigma\). Note that the permutation \(\sigma\) is a web permutation, and the cycle \(\phi(\sigma)\) is an André cycle. Surprisingly, this is not an accident.

**Theorem 4.1.** For \(n \geq 1\), let \(AC_{n+2} \subset \mathcal{G}_{n+2}\) be the set of André cycles consisting of \([n + 2]\). Then we have \(\phi(\text{Web}_n) = AC_{n+2}\). In particular, the number of web permutations of \([n]\) is equal to the number of André cycles consisting of \([n + 2]\).

**Proof.** Let \(\sigma\) be a web permutation of \([n]\). In the canonical cycle notation, we write 
\[\sigma = C_1C_2 \cdots C_k\]
where \(C_i = (c^{(i)}_1, \ldots, c^{(i)}_{n_i})\) is a cycle with \(\min C_i = c^{(i)}_{n_i}\) for each \(i = 1, \ldots, k\), and \(1 = c^{(1)}_{n_1} < \cdots < c^{(i)}_{n_i}\). Note that by Theorem 3.4 each cycle \(C_i\) is an André cycle, that is, each word \(c^{(1)}_1 \cdots c^{(i)}_{n_i-1}\) is an André permutation. We claim that the word 
\[c^{(1)}_1 \cdots c^{(1)}_{n_1} c^{(2)}_1 \cdots c^{(2)}_{n_2} \cdots c^{(k)}_{n_k} (n + 1)\]
oncluded by appending \(n + 1\) to the end of \(\tilde{\sigma}\) is an André permutation. Since \(c^{(1)}_{n_1}\) is the minimum in the word, and \(c^{(1)}_1 \cdots c^{(1)}_{n_1-1}\) is an André permutation, it suffices to show that the suffix \(c^{(2)}_1 \cdots c^{(2)}_{n_2} \cdots c^{(k)}_{n_k} (n + 1)\) is an André permutation. Then an appropriate inductive argument shows the claim. We now consider the cycle \(\phi(\sigma)\). Using the canonical cycle notation of \(\sigma\), we have 
\[\phi(\sigma) = (1, c^{(1)}_1 + 1, \ldots, c^{(1)}_{n_1} + 1, c^{(2)}_1 + 1, \ldots, c^{(2)}_{n_2} + 1, \ldots, c^{(k)}_1 + 1, \ldots, c^{(k)}_{n_k} + 1, n + 2)\]
Thus, by the claim, \(\phi(\sigma)\) is an André permutation of \([n + 2]\), as desired.

Conversely, let \(\tau\) be an André cycle of \([n + 2]\). One can directly check from the definition of André cycles that \(\tau\) forms \((1, a_1, \ldots, a_n, n + 2)\). Let \(a_{n_1}, \ldots, a_{n_k}\) be the right-to-left minima of the permutation \(a_1 \cdots a_n\) with \(n_1 < \cdots < n_k\) so that \(a_{n_1} < \cdots < a_{n_k}\). Then we only need to show that the permutation \((a_1, \ldots, a_{n_1})(a_{n_1+1}, \ldots, a_{n_2}) \cdots (a_{n_k-1+1}, a_{n_k})\) is a web permutation, or equivalently, due to Theorem 3.4 each cycle \((a_{n_k-1+1}, a_{n_k})\) is an André cycle. It is easily verified by a similar argument as in the previous claim and using the right-to-left minima. Hence we leave the details to the reader. \(\square\)

### 4.1. Euler and Entringer numbers

In this subsection, we give various enumerative properties of web permutations using Theorem 4.1.

We start with recalling Euler numbers. The **Euler numbers** \(E_n\) are defined via the exponential generation function 
\[E(z) := \sum_{n \geq 0} E_n \frac{z^n}{n!} = \sec z + \tan z.\]
The first few Euler numbers are 1, 1, 1, 2, 5, 16, 61; see [SI20] with ID number A000111. There are numerous combinatorial objects enumerated by Euler numbers $E_n$, e.g., alternating permutations, complete increasing binary trees, and etc. Especially, the Euler number $E_n$ counts André permutations of $[n]$. For details, we refer to [Sta10], which is a wonderful survey of Euler numbers and related topics. We provide another occurrence of Euler numbers.

**Corollary 4.2.** The Euler number $E_{n+1}$ enumerates the number of web permutations of $[n]$.

**Proof.** By definition, the number of André permutations of $[n]$ is equal to the number of André cycles of $[n + 1]$. Then Theorem 4.1 implies the desired result. □

**Remark 4.3.** One can prove the corollary without using the fact that the number of André permutations is equal to the Euler number. Indeed, let $w_n$ be the number of web permutations of $[n]$, $ac_n$ the number of André cycles of $[n]$, and

$$W(z) = \sum_{n \geq 0} w_n \frac{z^n}{n!}, \quad AC(z) = \sum_{n \geq 1} ac_n \frac{z^n}{n!},$$

where we set $w_0 = 1$. Then by a standard fact of generating functionology [Sta99, Corollary 5.1.6] and Theorem 3.1, we have

$$W(z) = \exp AC(z).$$

Meanwhile, Theorem 4.1 gives the ODE

$$W(z) = \frac{d^2}{dz^2} AC(z) = \frac{d^2}{dz^2} \log W(z)$$

whose unique solution is $W(z) = \sec z \tan z + \sec^2 z = E'(z)$, which implies $w_n = E_{n+1}$.

For a permutation $\sigma$, let $c(\sigma)$ be the number of cycles of $\sigma$, and $rlmin(\sigma)$ the number of right-to-left minima of $\sigma$. By convention, we set $c(\emptyset) = 0$ where $\emptyset$ is the empty permutation, and $Web_0 = \{\emptyset\}$. Since the Foata transformation gives the equidistribution of the two statistics $c(\sigma)$ and $rlmin(\sigma)$, we have the following corollary concerning the distribution of $c(\sigma)$ on $Web_n$.

**Corollary 4.4.** We have

$$\left(\frac{1}{1 - \sin z}\right)^t = \sum_{n \geq 0} \sum_{\sigma \in Web_n} t^{c(\sigma)} \frac{z^n}{n!}.$$  

**Proof.** In [Dis13, Proposition 1], the author showed that

$$\left(\frac{1}{1 - \sin z}\right)^t = \sum_{n \geq 1} \sum_{\sigma} t^{rlmin(\sigma)-1} \frac{z^{n-1}}{(n - 1)!}$$

where the inner sum is over all André permutations of $[n]$. Therefore the proof follows immediately from Theorem 4.1. □
We also recall Entringer numbers. The Entringer numbers are given by the generating function
\[
\frac{\cos x + \sin x}{\cos(x + y)} = \sum_{m,n \geq 0} E_{m+n,[m,n]} \frac{x^m y^n}{m! n!},
\]
where \([m,n]\) is \(m\) if \(m + n\) is odd, and \(n\) otherwise. These numbers refine Euler numbers in the following sense: For \(n \geq 1\),
\[
\sum_{k=1}^{n} E_{n,k} = E_{n+1}.
\]
We have a counterpart of this refinement.

**Corollary 4.5.** The Entringer number \(E_{n,k}\) is equal to the number of web permutations \(\sigma\) of \([n]\) with \(\sigma_1 = n + 1 - k\).

**Proof.** In [FH16 Theorem 1.1], the authors showed that \(E_{n,k}\) equals the number of André permutations \(\sigma\) of \([n+1]\) with \(\sigma_1 = n + 1 - k\). Combining this fact and Theorem 4.1 gives the proof. \(\square\)

### 4.2. Genocchi numbers and the Seidel triangle.

The Genocchi numbers are well-studied numbers with various combinatorial properties; see [Dun74, LW20]. The Genocchi numbers can be defined by the Seidel triangle as follows [Sei77]. Recall that the Seidel triangle is an array of integers \((s_{i,j})_{i,j \geq 1}\) such that \(s_{1,1} = s_{2,1} = 1\) and
\[
\begin{cases}
  s_{2i+1,j} = s_{2i+1,j-1} + s_{2i,j} & \text{for } j = 1, 2, \ldots, i + 1 \\
  s_{2i,j} = s_{2i+1,j} + s_{2i-1,j} & \text{for } j = i, i - 1, \ldots, 1,
\end{cases}
\]
where \(s_{i,j} = 0\) for \(j < 0\) or \(j > \lceil i/2 \rceil\). This Pascal type procedure is called the boustrophedon algorithm. The Genocchi numbers \(g_n\) are defined by
\[
g_{2n-1} = s_{2n-1,n} \quad \text{and} \quad g_{2n} = s_{2n,1}.
\]
In fact, the sequence \((g_n)\) is the interleaving of the Genocchi numbers of the first kind and the median Genocchi numbers. The first values of the Seidel triangle and Genocchi numbers (in red) are given in the following sequence.

| \(n\) \(\backslash\) \(k\) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 1 | 1 |
| 4 | 2 | 1 |
| 5 | 2 | 3 | 3 |
| 6 | 8 | 6 | 3 |
| 7 | 8 | 14 | 17 | 17 |
| 8 | 56 | 48 | 34 | 17 |
| 9 | 56 | 104 | 138 | 155 | 155 |
A COMBINATORIAL MODEL FOR THE TRANSITION MATRIX

Recall that we denote by $M_0$ for the unique matching which is simultaneously noncrossing and nonnesting, i.e., $M_0 = \{\{1,2\},\ldots,\{2n-1,2n\}\}$. To emphasize the size of the matchings, we denote this unique matching of $[2n]$ by $M_0^{(n)}$. Let $f(n)$ be the number of web permutations $\sigma$ of $[n]$ with $M(\sigma) = M_0^{(n)}$. In [Nak20], Nakamigawa showed the following theorem.

**Theorem 4.6 ([Nak20] Theorem 3.1).** For $n \geq 1$, we have $f(n) = g_n$.

Let $f(n,k)$ be the number of web permutations $\sigma$ of $[n]$ such that $M(\sigma) = M_0^{(n)}$ and $\sigma_1 = k$. Obviously, $f(n) = \sum_{1 \leq k \leq n} f(n,k)$. Some of these numbers vanish in the following cases.

**Proposition 4.7.** For $n \geq 1$ and $1 \leq k \leq \lfloor n/2 \rfloor$, we have $f(n,2k) = 0$.

**Proof.** Let $\sigma$ be a web permutation of $[n]$ with $\sigma_1 = 2k$. Then considering the grid configuration $G(\sigma, Cr(\sigma))$, the associated matching $M(\sigma)$ has an arc connecting $2k$ and some $j$ with $2k < j$. Since there is the arc connecting $2k - 1$ and $2k$ in $M_0^{(n)}$, we deduce $M(\sigma) \neq M_0^{(n)}$. □

**Proposition 4.8.** For $n > 1$, we have $f(n,n) = 0$.

**Proof.** Let $\sigma$ be a web permutation of $[n]$ with $\sigma_1 = n$. Since the elements 1 and $n$ are contained in the same cycle, we have $\sigma_n = 1$ by Lemma 3.2 and Theorem 3.4. Then there is a marking at $(n,1)$ in the grid configuration $G(\sigma, Cr(\sigma))$. Observe that the vertical line and horizontal line starting from the cell $(n,1)$ do not make a crossing. Hence we deduce $\{1,2n\} \in M(\sigma)$, which implies that $M(\sigma) \neq M_0^{(n)}$. □

By Propositions 4.7 and 4.8, we have

$$f(n) = \sum_{1 \leq k \leq \lfloor n/2 \rfloor} f(n,2k - 1).$$

We now propose a conjecture that the values appearing in the Seidel triangle are $f(n,k)$.

**Conjecture 4.9** (Verified up to $n = 6$). For $n \geq 1$, we have

$$\begin{cases} 
    f(2n - 1, 2k - 1) = s_{2n-2,k}, \\
    f(2n, 2k - 1) = s_{2n-1,n-k+1}.
\end{cases}$$

This conjecture includes Nakamigawa’s result. To elaborate, let $\sigma$ be a web permutation of $[n]$ such that $M(\sigma) = M_0^{(n)}$ and $\sigma_1 = 1$. Deleting the cycle (1) from $\sigma$ and decreasing each letter by 1, the resulting permutation is a web permutation of $[n-1]$ with $M(\sigma) = M_0^{(n-1)}$. In addition, this correspondence is bijective, so we deduce $f(n,1) = f(n-1)$. Thus the conjecture implies $f(n-1) = g_{n-1}$, which is Nakamigawa’s result.

**Acknowledgments**

The authors are grateful to Jang Soo Kim for several suggestions which improved the manuscript.
In this appendix, we give several computational results for some small $n$.

A.1. The transition matrices. All rows and columns are sorted with respect to the reverse lexicographic order on their corresponding Dyck paths. For example, let $n = 3$. The following is the list of 5 Dyck paths of length $2n = 6$ sorted in the reverse lexicographic order:

$$\text{Dyck}_6 = \{ \text{NNNEEE, NNENEE, NNEENE, NENNEE, NENENE} \}.$$ 

Then rows and columns of the transition matrix for $n = 3$ are indexed by $\text{NN}_6$ and $\text{NC}_6$ in order as follows:

$$\text{NN}_6 = \{ \text{NNNNNN}, \text{NNNNEN}, \text{NNNNEN}, \text{NNNNEE}, \text{NNNEEN} \}$$

and

$$\text{NC}_6 = \{ \text{NNNNNN}, \text{NNNNEN}, \text{NNNNEN}, \text{NNNNEE}, \text{NNNEEN} \}.$$ 

We omit zeros in the strictly lower-triangular part of $A$.

i) $n = 2$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

ii) $n = 3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

iii) $n = 4$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$
A.2. **Web permutations.** We present lists of all web permutations for \( n = 2, 3, 4, 5 \) with their corresponding Dyck paths and noncrossing matchings. Due to space limitation, the matchings are also represented as Dyck paths via the bijection \( D : \text{NC}_{2n} \rightarrow \text{Dyck}_{2n} \).

i) \( n = 2 \)

| Web permutations \( \sigma \) | \( D(\sigma) \) | \( M(\sigma) \) |
|-------------------------------|--------------|-------------|
| 12 = (1)(2)                  | NENE         | NENE        |
| 21 = (1,2)                   | NNEE         | NNEE        |

ii) \( n = 3 \)

| Web permutations \( \sigma \) | \( D(\sigma) \) | \( M(\sigma) \) |
|-------------------------------|--------------|-------------|
| 123 = (1)(2)(3)              | NENENENEN    | NENENENEN   |
| 132 = (1)(2,3)               | NENENENE     | NENENENE    |
| 213 = (1,2)(3)               | NNEENE       | NNEENE      |
| 231 = (1,2,3)                | NNEENE       | NNEENE      |
| 321 = (1,3)(2)               | NNEENE       | NNEENE      |

iii) \( n = 4 \)

| Web permutations \( \sigma \) | \( D(\sigma) \) | \( M(\sigma) \) |
|-------------------------------|--------------|-------------|
| 1234 = (1)(2)(3)(4)           | NENENENEN    | NENENENEN   |
| 1243 = (1)(2)(3,4)            | NENENENE     | NENENENE    |
| 1324 = (1)(2,3)(4)            | NENENENEN    | NENENENEN   |
| 1342 = (1)(2,3,4)             | NENNNEEEE    | NENNNEEEE   |
| 1432 = (1)(2,4)(3)            | NENNNEEEE    | NENNNEEEE   |
| 2134 = (1,2)(3)(4)            | NNEENE       | NNEENE      |
| 2143 = (1,2)(3,4)             | NNEENE       | NNEENE      |
| 2314 = (1,2,3)(4)             | NNEENE       | NNEENE      |
| 3214 = (1,3)(2)(4)            | NNEENE       | NNEENE      |
| 3412 = (1,3)(2,4)             | NNEENE       | NNEENE      |
| 2341 = (1,2,3,4)              | NNNNNNEEEE   | NNNNNNEEEE  |
| 2431 = (1,2,4)(3)             | NNNNNNEEEE   | NNNNNNEEEE  |
| 3241 = (1,3,4)(2)             | NNNNNNEEEE   | NNNNNNEEEE  |
| 4231 = (1,4)(2)(3)            | NNNNNNEEEE   | NNNNNNEEEE  |
| 3421 = (1,3,2,4)              | NNNNNNEEEE   | NNNNNNEEEE  |
| 4321 = (1,4)(2,3)             | NNNNNNEEEE   | NNNNNNEEEE  |
iv) \( n = 5 \)

| Web permutations \( \sigma \) | \( D(\sigma) \) | \( M(\sigma) \) |
|-----------------------------|-----------------|-----------------|
| 12345 = (1)(2)(3)(4)(5)   | NENENENENE      | NENENENENE      |
| 12354 = (1)(2)(3)(4,5)    | NENENENNEE      | NENENENNEE      |
| 12435 = (1)(2)(3,4)(5)    | NENENENEENE     | NENENENEENE     |
| 12453 = (1)(2)(3,4,5)     | NENENNEENE      | NENENNEENE      |
| 13245 = (1)(2,3)(4)(5)    | NENENENENE      | NENENENENE      |
| 13254 = (1)(2,3)(4,5)     | NENENNEENE      | NENENNEENE      |
| 13425 = (1)(2,3,4)(5)     | NENNNEENE      | NENNNEENE      |
| 14325 = (1)(2,4)(3)(5)    | NENNNEENE      | NENNNEENE      |
| 14523 = (1)(2,4)(3,5)     | NENNNNEE      | NENNNNEE      |
| 13452 = (1)(2,3,4,5)      | NENNNNEEE      | NENNNNEEE      |
| 13542 = (1)(2,3,5)(4)     | NENNNNEEEE     | NENNNNEEEE     |
| 14352 = (1)(2,4,5)(3)     | NENNNNEEE      | NENNNNEEE      |
| 15342 = (1)(2,5)(3)(4)    | NENNNNEEEE     | NENNNNEEEE     |
| 14532 = (1)(2,4,3,5)      | NENNNNEEEE     | NENNNNEEEE     |
| 15432 = (1)(2,4,3,5)      | NENNNNEEEE     | NENNNNEEEE     |
| 21345 = (1,2)(3)(4)(5)    | NNEENENENE      | NNEENENENE      |
| 21354 = (1,2)(3)(4,5)     | NNEENENENE      | NNEENENENE      |
| 21435 = (1,2)(3,4)(5)     | NNEENENENE      | NNEENENENE      |
| 21453 = (1,2)(3,4,5)      | NNEENENENE      | NNEENENENE      |
| 21543 = (1,2)(3,5)(4)     | NNEENNEENE      | NNEENNEENE      |
| 23145 = (1,2,3)(4)(5)     | NNEENENENE      | NNEENENENE      |
| 23154 = (1,2,3)(4,5)      | NNEENENENE      | NNEENENENE      |
| 23415 = (1,2,3,4)(5)      | NNEENENENE      | NNEENENENE      |
| 23514 = (1,2,3,4,5)       | NNEENENENE      | NNEENENENE      |
| 32145 = (1,3)(2)(4)(5)    | NNEENENENE      | NNEENENENE      |
| 32154 = (1,3)(2)(4,5)     | NNEENENENE      | NNEENENENE      |
| 34125 = (1,3)(2,4)(5)     | NNEENENENE      | NNEENENENE      |
| 34152 = (1,3)(2,4,5)      | NNEENENENE      | NNEENENENE      |
| 35142 = (1,3)(2,5)(4)     | NNEENENENE      | NNEENENENE      |
| 23415 = (1,2,3,4)(5)      | NNNNEENENE      | NNNNEENENE      |
| 24315 = (1,2,3,4)(5)      | NNNNEENENE      | NNNNEENENE      |
| 24513 = (1,2,3,4,5)       | NNNNEENENE      | NNNNEENENE      |
| 32415 = (1,3,4)(2)(5)     | NNNNEENENE      | NNNNEENENE      |
| 42315 = (1,4)(2)(3)(5)    | NNNNEENENE      | NNNNEENENE      |
| 42513 = (1,4)(2)(3,5)     | NNNNEENENE      | NNNNEENENE      |
| 43215 = (1,4)(2,3)(5)     | NNNNEENENE      | NNNNEENENE      |
| 35412 = (1,3,4)(2,5)      | NNNNEENENE      | NNNNEENENE      |
| 43512 = (1,4)(2,3,5)      | NNNNEENENE      | NNNNEENENE      |
| 45312 = (1,4,2,5)(3)      | NNNNEENENE      | NNNNEENENE      |
| Web permutations $\sigma$ | $D(\sigma)$ | $M(\sigma)$ |
|--------------------------|--------------|--------------|
| 23451 = (1,2,3,4,5)     | NNNNNNEEEE  | NNNENENENE   |
| 23541 = (1,2,3,5)(4)    | NNNNNNEEEE  | NNNENENENE   |
| 24351 = (1,2,4,5)(3)    | NNNNNNEEEE  | NNNENENENE   |
| 25341 = (1,2,5)(3)(4)   | NNNNNNEEEE  | NNNENENENE   |
| 24531 = (1,2,4,3,5)     | NNNNNNEEEE  | NNNENENENE   |
| 25431 = (1,2,5)(3,4)    | NNNNNNEEEE  | NNNENENENE   |
| 32451 = (1,3,4,5)(2)    | NNNNNNEEEE  | NNNENENENE   |
| 32541 = (1,3,5)(2)(4)   | NNNNNNEEEE  | NNNENENENE   |
| 42351 = (1,4,5)(2)(3)   | NNNNNNEEEE  | NNNENENENE   |
| 32531 = (1,4,3,5)(2)    | NNNNNNEEEE  | NNNENENENE   |
| 52431 = (1,5)(2)(3,4)   | NNNNNNEEEE  | NNNENENENE   |
| 34251 = (1,3,2,4,5)     | NNNNNNEEEE  | NNNENENENE   |
| 35241 = (1,3,2,5)(4)    | NNNNNNEEEE  | NNNENENENE   |
| 43251 = (1,4,5)(2,3)    | NNNNNNEEEE  | NNNENENENE   |
| 42531 = (1,4,3,5)(2)    | NNNNNNEEEE  | NNNENENENE   |
| 53241 = (1,5)(2,3)(4)   | NNNNNNEEEE  | NNNENENENE   |
| 34521 = (1,3,5)(2,4)    | NNNNNNEEEE  | NNNENENENE   |
| 35421 = (1,3,4,2,5)     | NNNNNNEEEE  | NNNENENENE   |
| 35231 = (1,5)(2,3,4)    | NNNNNNEEEE  | NNNENENENE   |
| 43521 = (1,4,2,3,5)     | NNNNNNEEEE  | NNNENENENE   |
| 53421 = (1,5)(2,3,4)    | NNNNNNEEEE  | NNNENENENE   |
| 45321 = (1,4,2,5)(3)    | NNNNNNEEEE  | NNNENENENE   |
| 54321 = (1,5)(2,4)(3)   | NNNNNNEEEE  | NNNENENENE   |

**References**

[Dis13] Filippo Disanto. André permutations, right-to-left and left-to-right minima. *Sém. Lothar. Combin.*, 70:Art. B70f, 13, 2013.

[Dum74] Dominique Dumont. Interprétations combinatoires des nombres de Genocchi. *Duke Math. J.*, 41:305–318, 1974.

[FH16] Dominique Foata and Guo-Niu Han. André permutation calculus: a twin Seidel matrix sequence. *Sém. Lothar. Combin.*, 73:Art. B73e, 54, [2014-2016].

[FS73] D. Foata and M.-P. Schützenberger. Nombres d’Euler et permutations alternantes. In *A survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)*, pages 173–187, 1973.

[Ful97] William Fulton. *Young Tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[IZ21] Mee Seong Im and Jieru Zhu. Transitioning Between Tableaux and Spider Bases for Specht Modules. *Algebras and Representation Theory*, 2021.

[KR84] Joseph P. S. Kung and Gian-Carlo Rota. The invariant theory of binary forms. *Bull. Amer. Math. Soc. (N.S.)*, 10(1):27–85, 1984.

[Kup96] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996.

[LW20] Alexander Lazar and Michelle L. Wachs. On the homogenized Linial arrangement: intersection lattice and Genocchi numbers. *Sém. Lothar. Combin.*, 82B:Art. 93, 12, 2020.
[Nak20] Tomoki Nakamigawa. The expansion of a chord diagram and the Genocchi numbers. *Ars Math. Contemp.*, 18(2):381–391, 2020.

[PPR09] T. Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades. Promotion and cyclic sieving via webs. *J. Algebraic Combin.*, 30(1):19–41, 2009.

[Rho19] Brendon Rhoades. The polytabloid basis expands positively into the web basis. *Forum Math. Sigma*, 7:Paper No. e26, 8, 2019.

[RT19] Heather M. Russell and Julianna S. Tymoczko. The transition matrix between the Specht and web bases is unipotent with additional vanishing entries. *Int. Math. Res. Not. IMRN*, (5):1479–1502, 2019.

[Sag01] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[Sei77] Ludwig Seidel. Über eine einfache Entstehungsweise der Bernoulli’schen Zahlen u. einiger verwandten Reihen. 1877.

[Slo20] Neil J. A. Sloane and The OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2020.

[Sta94] Richard P. Stanley. Flag $f$-vectors and the $cd$-index. *Math. Z.*, 216(3):483–499, 1994.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Sta10] Richard P. Stanley. A survey of alternating permutations. In *Combinatorics and graphs*, volume 531 of *Contemp. Math.*, pages 165–196. Amer. Math. Soc., Providence, RI, 2010.

[Sta12] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

Applied Algebra and Optimization Research Center, Sungkyunkwan University, Suwon, South Korea

*Email address*: byunghakhwang@gmail.com

Department of Mathematics, Sungkyunkwan University (SKKU), Suwon, Gyeonggi-do 16419, South Korea

*Email address*: 4242ab@gmail.com

Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, South Korea

*Email address*: jsoh@kias.re.kr