Toric $\mathcal{Q}$-Gorenstein Singularities

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Abstract

For an affine, toric $\mathcal{Q}$-Gorenstein variety $Y$ (given by a lattice polytope $Q$) the vector space $T^1$ of infinitesimal deformations is related to the complexified vector spaces of rational Minkowski summands of faces of $Q$. Moreover, assuming $Y$ to be an isolated, at least 3-dimensional singularity, $Y$ will be rigid unless it is even Gorenstein and $\dim Y = 3$ ($\dim Q = 2$). For this particular case, so-called toric deformations of $Y$ correspond to Minkowski decompositions of $Q$ into a sum of lattice polygons. Their Kodaira-Spencer-map can be interpreted in a very natural way.

We regard the projective variety $\mathbb{P}(Y)$ defined by the lattice polygon $Q$. Data concerning the deformation theory of $Y$ can be interpreted as data concerning the Picard group of $\mathbb{P}(Y)$.

Finally, we provide some examples (the cones over the toric Del Pezzo surfaces). There is one such variety yielding $\text{Spec } \mathcal{O}[[\varepsilon]]/\varepsilon^2$ as the base space of the semi-universal deformation.

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1 Introduction

(1.1) In [Al 1] and [Al 2] we investigated the deformation theory of affine toric varieties $Y = \text{Spec } \mathcal{O}[\sigma \cap M]$ (cf. [Al 1] for an explanation of the notations):
It is always the first step to look at the vector space $T^1_Y$ of infinitesimal deformations - it equals (if $Y$ admits isolated singularities only) the tangent space on the base space $S$ of the semi-universal deformation of $Y$. For toric $Y$ the space $T^1_Y$ is $M$-graded, and the homogeneous pieces were computed in [Al 1] (cf. (2.2) of the present paper).

In [Al 2] we were interested in describing toric deformations of $Y$. They are defined as those deformations (i.e. flat maps $f : X \to S$ endowed with an isomorphism $f^{-1}(0 \in S) \cong Y$) such that the total space $X$ together with the embedding of the special fiber $Y \hookrightarrow X$ are toric.

Toric deformations are “really existing” deformations - in the sense that they admit reduced (even smooth) base spaces. Moreover, we conjecture that (in case of isolated singularities $Y$) the semi-universal deformation of $Y$ is toric over each irreducible component of the reduced base space. (This is true for $\dim Y = 2$ and also for examples of higher dimension.)

Toric deformations always arise as relative deformations of $Y$ inside a greater affine toric variety $X$ containing $Y$ as a relatively complete intersection. More strictly speaking, $Y \subseteq X$ is defined by a so-called toric regular sequence $x^{r_0^1} - x^{r_1^1}, \ldots, x^{r_m^0} - x^{r_m^m} \in \Gamma(X, \mathcal{O}_X)$.

On the other hand, each toric regular sequence can be regarded as a flat map $X \to \mathcal{A}^m$ by itself. This $m$-parameter deformation of $Y$ is called the standard toric deformation induced by the given sequence.

It is possible to compute the Kodaira-Spencer-map $\theta : \mathcal{A}^m \to T^1_Y$ corresponding to these standard toric deformations. Then, the first observation is that $\theta$ maps the $i$-th canonical basis vector $e^i \in \mathcal{A}^m$ into the homogeneous piece $T^1_Y(-\bar{r}^i)$, and $\bar{r}^i \in M$ is defined as the exponent of the common image of $x^{r_0^i}$ and $x^{r_1^i}$ via the surjection $\Gamma(X, \mathcal{O}_X) \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y)$.

**Definition:** A toric regular sequence is called strongly homogeneous if only $m + 1$ different elements occur in the set $\{r_0^1, r_1^1, \ldots, r_m^0, r_m^m\}$. Then, the corresponding images $\bar{r}_1^1, \ldots, \bar{r}_m^m$ coincide, and this element will be denoted by $\bar{r} \in \sigma \cap M$. It equals the negative degree of the Kodaira-Spencer-map.

(Toric regular sequences of length one are always strongly homogeneous.)

The main result of [Al 2] is a complete combinatorial description of those standard toric deformations that are induced by strongly homogeneous toric regular sequences. They arise from certain Minkowski decompositions of affine slices (induced by $\bar{r} \in M$) of the cone $\sigma$ (cf. (3.1) of the present paper).

(1.2) The aim of this paper is to apply the previous results to the special case
of toric $\mathcal{Q}$-Gorenstein singularities. On the one hand, this notion is the next one if we are looking for a wider class than that of complete intersections (which yields no interesting deformation theory). On the other hand, the property “$\mathcal{Q}$-Gorenstein” admits a very clear description in the language of toric varieties and convex cones:

In general, the dualizing sheaf $\omega$ on a Cohen-Macaulay variety is defined as

(i) $\omega_P := \Omega_{P}^{\dim P}$ (sheaf of the highest differential forms) if $P$ is smooth, and

(ii) $\pi_*\omega_Y := \text{Hom}_{\mathcal{O}_P}(\pi_*\mathcal{O}_Y, \omega_P)$ for flat and finite maps $\pi : Y \to P$.

**Definition:** A variety $Y$ is called ($\mathcal{Q}$-) Gorenstein if (the reflexive hull of some tensor power of) $\omega_Y$ is an invertible sheaf on $Y$.

Since toric varieties are normal, the dualizing sheaf can be obtained as the push forward of the canonical sheaf on its smooth part. Hence, in our special situation, $\omega_Y$ equals the $T($orus)-invariant complete fractional ideal that is given by the order function mapping each fundamental generator onto $1 \in \mathbb{Z}$ (cf. Theorem I/9 in [Ke]).

In particular, we obtain the following

**Fact:** Let $Y = \text{Spec} \mathcal{O}[\mathcal{\tilde{\sigma}} \cap M]$ be an affine toric variety given by a cone $\mathcal{\tilde{\sigma}} = \langle a^1, \ldots , a^n \rangle$. (The fundamental generators $a^i$ are assumed to be primitive elements of the lattice that is dual to $M$.)

Then, $Y$ is $\mathcal{Q}$-Gorenstein, if and only if there is a primitive element $R^* \in M$ and a natural number $g \in \mathbb{N}$ such that

$$\langle a^i, R^* \rangle = g$$

for each $i = 1, \ldots , N$.

$Y$ is Gorenstein if and only if $g = 1$, in addition.

(1.3) Affine toric varieties of dimension two are always $\mathcal{Q}$-Gorenstein. The deformation theory of these varieties (the two-dimensional cyclic quotient singularities) is well known. For instance, $T^1_Y$ and the number and dimension of the components of the reduced base space of the semi-universal deformation have been computed (cf. [H], [Ar], [Ch], [St]). Therefore, our investigation concerns the case of $Y$ being smooth in codimension 2.

Assume that $Y = \text{Spec} \mathcal{O}[\mathcal{\tilde{\sigma}} \cap M]$ is a $\mathcal{Q}$-Gorenstein variety (i.e. $\langle a^i, R^* \rangle = g$ for each $i = 1, \ldots , N$), which is smooth in codimension 2. Denote by $Q$ the lattice polyhedron $Q := \text{Conv}(a^1, \ldots , a^N)$. Then, we obtain the following results:

(1) The graded pieces of $T^1_Y$ equal the vector spaces of Minkowski summands of certain faces of $Q$ (cf. Theorem (2.7)).
(2) For the special case of an isolated singularity $Y$ this implies:

- If $g \geq 2$, then $Y$ will be rigid, i.e. $T_Y^1 = 0$.
- If $Y$ is Gorenstein (i.e. $g = 1$) and at least 4-dimensional, then $Y$ will be rigid.
- Let $Y$ be a 3-dimensional Gorenstein singularity given by a plane, convex $N$-gon $Q$. Then, $T_Y^1$ is concentrated in the single degree $-R^*$, and it equals to the $(N - 3)$-dimensional vector space of Minkowski summands of $Q$.

(Cf. (2.8) and (2.9).)

(3) Keep assuming that $Y$ is a 3-dimensional, isolated, toric, Gorenstein singularity. Then, toric $m$-parameter deformations of $Y$ correspond to Minkowski decompositions of $Q$ into a sum of $m + 1$ lattice polygons. Using the previous description of $T_Y^1$, the Kodaira-Spencer-map is the natural one.

(1.4) Let us assume, for a moment, that the semi-universal deformation is, indeed, toric over each component of the reduced base space $S_{\text{red}}$. Then, the absence of proper lattice summands of $Q$ would mean that $S$ is only 0-dimensional.

On the other hand, this is possible even for $Y$ admitting a non-trivial $T_Y^1$ (which equals the tangent space of $S$). In (1.4.3) we will give a (three-dimensional) example of this phenomenon: The base space $S$ is a point with non-reduced structure, i.e. each deformation is obstructed.

(1.5) A toric Gorenstein variety $Y = \text{Spec} \mathcal{O}[\sigma \cap M]$ is given by a lattice polytope $Q$ ($\sigma = \text{Cone}(Q)$). On the other hand, each lattice polytope $Q$ induces a projective toric variety (defined by the inner normal fan) endowed with an ample line bundle $\mathcal{O}(1)$. We call this the polar variety of $Y$ - denoted by $\mathbb{P}(Y)$.

Then, the cone $P(Y)$ over the embedded projective variety $\mathbb{P}(Y)$ equals the affine toric variety which is given by the cone dual to that of $Y$.

In §4 we interprete data of the deformation theory of 3-dimensional $Y$ as data concerning divisors on the varieties $\mathbb{P}(Y)$ or $P(Y) \setminus \{0\}$. Assuming $Y$ having only an isolated singularity we obtain the following relations:

(4) $T_Y^1 = \text{Pic}(P(Y) \setminus \{0\}) \otimes \mathbb{Z} \mathcal{O} = \text{Pic} \mathbb{P}(Y) / \mathcal{O}(1) \otimes \mathbb{Z} \mathcal{O}$.

(5) Toric $m$-parameter deformations of $Y$ correspond to splittings of $\mathcal{O}_{\mathbb{P}(Y)}(1)$ into a tensor product of $m + 1$ invertible sheaves that are nef.
(1.6) **Acknowledgement:** I am grateful to Duco van Straten for computing several semi-universal deformations on the computer (using Macaulay). Moreover, I want to thank Bernd Sturmfels for his special lecture concerning the cone of Minkowski summands and the Gale transform of a given polytope.

## 2 The \( T^1 \) of toric \( \mathcal{Q} \)-Gorenstein singularities

### (2.1)

Let us start with introducing some basic notations and recalling the \( T^1 \)-formulas of [Al 1]:

Let \( M, N \) be free \( \mathbb{Z} \)-modules of finite rank - endowed with a perfect pairing \( \langle ., . \rangle : N \times M \rightarrow \mathbb{Z} \). Denote by \( M_{\mathbb{R}} \) and \( N_{\mathbb{R}} \) the corresponding vector spaces (dual to each other) obtained via base change with \( \mathbb{R} \).

Let \( \sigma = \langle a^1, \ldots, a^N \rangle \subseteq N_{\mathbb{R}} \) be a top-dimensional, rational, polyhedral cone with apex. The fundamental generators \( a^i \in N \) are assumed to be primitive elements of the lattice \( N \).

The dual cone \( \check{\sigma} \subseteq M_{\mathbb{R}} \) of \( \sigma \) is defined as \( \check{\sigma} := \{ r \in M_{\mathbb{R}} | \langle a, r \rangle \geq 0 \text{ for each } a \in \sigma \} \). Denote by \( E \subseteq \check{\sigma} \cap M \) the minimal (finite) set that generates the semigroup \( \check{\sigma} \cap M \).

In particular, \( Y := \text{Spec} \mathcal{O}[\check{\sigma} \cap M] \subseteq \mathcal{O}^E \).

### (2.2) Theorem: (cf. (2.3) and (4.4) of [Al 1])

1. The vector space \( T_Y^1 \) of infinitesimal deformations of \( Y \) is \( M \)-graded. For a fixed element \( R \in M \) the homogeneous piece of degree \( -R \) can be computed as

\[
T_Y^1(-R) = \left( L(E') / \sum_{i=1}^N L(E_i) \right)^* \otimes_{\mathbb{R}} \mathcal{O}
\]

\( (E_i := \{ s \in E | (0 \leq) \langle a^i, s \rangle < \langle a^i, R \rangle \} \quad (i = 1, \ldots, N); \quad E' := \bigcup_{i=1}^N E_i; \) \( L(\text{set}) := \mathbb{R}\)-vector space of all linear dependences between its elements).

2. If \( Y \) is smooth in codimension 2 (i.e. if all 2-dimensional faces \( \langle a^i, a^j \rangle < \sigma \) are spanned by a part of a \( \mathbb{Z} \)-basis of the lattice \( N \)), then with

\[
V_i := \text{span}_{\mathbb{R}}(E_i) = \begin{cases} 0 & \text{for } \langle a^i, R \rangle \leq 0 \\ [a^i = 0] \subseteq M_{\mathbb{R}} & \text{for } \langle a^i, R \rangle = 1 \\ M_{\mathbb{R}} & \text{for } \langle a^i, R \rangle \geq 2 \end{cases}
\]
we obtain the second formula

\[ T_Y^1(-R) = \text{Ker} \left[ \frac{V_1 \oplus \ldots \oplus V_N}{\sum_{\langle a^i, a^j \rangle < \sigma} V_i \cap V_j \rightarrow V_1 + \ldots + V_N} \right]^*. \]

(2.3) Lemma: Let \( Y \) be a \( \Phi \)-Gorenstein variety, which is smooth in codimension 2. If \( R \in M \) is a degree such that \( \langle a^i, R \rangle \geq 2 \) for some \( i \in \{1, \ldots, N\} \), then \( T_Y^1(-R) = 0 \).

Proof: Let \( R^* \in M \) be as in (1.2), i.e. \( \langle a^i, R^* \rangle = g \) for \( i = 1, \ldots, N \). Then,

\[ H := \{ a \in N_R | \langle a, g R - R^* \rangle = 0 \} \]

is a hyperplane in \( N_R \) that subdivides the set of fundamental generators of \( \sigma \). \( H^-, H, \) and \( H^+ \) contain the elements \( a^i \) meeting the properties \( \langle a^i, R \rangle \leq 0 \), \( \langle a^i, R \rangle = 1 \), and \( \langle a^i, R \rangle \geq 2 \), respectively. Let us assume that the latter class of generators is not empty.

To use the second \( T^1 \)-formula of the previous theorem, we fix a map

\[ \varphi : \{ i | \langle a^i, R \rangle = 1 \} \rightarrow \{1, \ldots, N\} \]

such that for each \( a^i \in H \) the element \( a^{\varphi(i)} \) is contained in \( H^+ \) and adjacent to \( a^i \) (i.e. \( \{a^i, a^{\varphi(i)}\} \subseteq \sigma \) spans a 2-dimensional face of \( \sigma \)).

Now, assume that we are given an element \( v = (v_1, \ldots, v_N) \in V_1 \oplus \ldots \oplus V_N \) such that \( v_1 + \ldots + v_N = 0 \). Adding the terms \( [-v_i \cdot e^i + v_i \cdot e^{\varphi(i)}] \) (for \( \langle a^i, R \rangle = 1 \)) does not change the equivalence class of \( v \) in \( V_1 \oplus \ldots \oplus V_N \). However, non-trivial components survive for \( \langle a^i, R \rangle \geq 2 \) (corresponding to \( V_i = M_R \)) only.

The set of these special generators \( a^i \) is connected by 2-dimensional faces of \( \sigma \). Moreover, by slightly disturbing \( R \) inside \( M_R \), we can find a unique \( a^* \) among these edges on which \( R \) is maximal. Then, each \( a^i \in H^+ \) is connected with \( a^* \) by an \( R \)-monotone path (consisting of 2-faces of \( \sigma \)) inside \( H^+ \).

Now, we can use the previous method of cleaning the components of \( v \) once more - the steps from \( a^i \) to \( a^{\varphi(i)} \) are replaced by the steps on the path from \( a^i \) to \( a^* \). It remains an \( N \)-tuple \( v \) which is non-trivial at most at the \( a^* \)-place. On the other hand, the components of \( v \) sum up to 0, but this yields \( v = 0 \).

(2.4) If \( \langle a^i, R \rangle \leq 1 \) for every \( i \in \{1, \ldots, N\} \), then equality holds on some face \( \tau < \sigma \). Now, \( \tau \) is a top-dimensional cone in the linear subspace \( \tau - \tau \subseteq N_R \), and
it defines a variety \( Y_\tau = \text{Spec} \mathcal{A}[\tau^\vee \cap M/_{\tau^\perp} \cap M] \), which is even Gorenstein. The corresponding element \( R^*_\tau \in M/_{\tau^\perp} \cap M \) can be obtained as the image of \( R \) as well as of \( \frac{1}{2} R^* \) using the canonical projection \( M_{\mathbb{R}} \to M_{\mathbb{R}}/_{\tau^\perp} \).

**Lemma:** In general (even the \( Q \)-Gorenstein assumption can be dropped) let \( \tau < \sigma \) be a face such that \( \langle a^i, R \rangle \geq 1 \) for \( a^i \in \tau \) and \( \langle a^i, R \rangle \leq 0 \) otherwise. Then, \( T^1_{Y_\tau}(-R) = T^1_{Y_\tau}(-R^*_\tau) \).

**Proof:** The formula of Theorem (2.2)(1) remains true if \( E \) is replaced by an arbitrary (not necessarily minimal) generating subset of \( \sigma^\vee \cap M \) - even a multiset could be allowed. Hence, for computing \( T^1_{Y_\tau}(-R^*_\tau) \) we can use the image \( \bar{E} \) of \( E \) under the projection \( \sigma^\vee \cap M \to (\sigma^\vee + \tau^\perp) \cap M/_{\tau^\perp} \cap M \).

For \( a^i \in \tau \), the corresponding sets \( \bar{E}_i \subseteq \bar{E} \) coincide with the images of the subsets \( E_i \). For \( a^i \notin \tau \), the notion \( \bar{E}_i \) does not make sense, and the \( E_i \) are empty, anyway. It remains to show that the canonical map

\[
L(E')/\sum_{a^i \in \tau} L(E_i) \to L(\bar{E}')/\sum_{a^i \in \tau} L(\bar{E}_i)
\]

is an isomorphism.

The vector space \( \tau^\perp \) is generated by \( \tau^\perp \cap (\bigcap_{a^i \in \tau} E_i) \). Hence, by choosing a basis among these elements, we can embed \( \tau^\perp \) into \( \mathbb{R}^{\tau^\perp \cap (\bigcap_{a^i \in \tau} E_i)} \) to obtain a section of

\[
\mathbb{R}^{\tau^\perp \cap (\bigcap_{a^i \in \tau} E_i)} \subseteq \bigcap_{a^i \in \tau} L(\bar{E}_i) \subseteq L(\bar{E}') \to \tau^\perp \quad (\ldots, \lambda_r, \ldots)_{r \in E'} \mapsto \sum_{r \in E'} \lambda_r \cdot r \in \tau^\perp \subseteq M_{\mathbb{R}}.
\]

In particular, we obtain

\[
L(\bar{E}_i) = L(E_i) \oplus \tau^\perp \quad (a^i \in \tau) \quad \text{and} \quad L(\bar{E}') = L(E') \oplus \tau^\perp.
\]

(2.5) Let \( Q = \text{Conv}(a^1, \ldots, a^N) \) be a \( K \)-polytope contained in some \( K \)-vector space \( \mathcal{A} \) (\( K = \mathbb{Q} \) or \( \mathbb{R} \)). It can be described by inequalities

\[
\langle \bullet, -c^v \rangle \leq \eta^v \quad (c^v \in \mathcal{A}^*, \eta^v \in K)
\]

corresponding to the facets of \( Q \). (The \( c^v \) are the inner normal vectors of \( Q \).)
Let us denote by $\sigma \subset A \times K$ the cone over $Q$ (embedded as $Q \times \{1\}$). Then, the pairs $[c^v, \eta^v] \in A^* \times K$ are exactly the fundamental generators of the dual cone $\hat{\sigma}$. For $i = 1, \ldots, N$ we define

$$F_i := \{[c^v, \eta^v] \in A^* \times K \mid \langle a^i, -c^v \rangle = \eta^v\} = \{\text{fundamental generators of the face } (a^i)^\perp \cap \hat{\sigma} < \hat{\sigma}\}.$$ 

Finally, denoting the set of all fundamental generators of $\hat{\sigma}$ by $F' := \bigcup_{i=1}^N F_i$, we can construct the $K$-vector space

$$\tilde{T}^1(Q) := \left( L(F') \bigg/ \sum_{i=1}^N L(F_i) \right)^*.$$

Now, each Minkowski summand $Q'$ of a scalar multiple of $Q$ is given by inequalities $\langle \bullet, -c^v \rangle \leq \eta^v$. We can define its class $\varrho(Q') \in \tilde{T}^1(Q)$ as

$$\varrho(Q') (q \in L(F')) := \sum_v q_v \eta^v \in K.$$ 

This definition is correct and depends on the translation class of $Q'$ only. Moreover, scalar multiples of $Q$ yield the zero class.

On the other hand, the translation classes of Minkowski summands of scalar multiples of $Q$ form a convex polyhedral cone which contains “$Q$” as an interior point. Dividing by the relation “$Q = 0$” yields a $K$-vector space which we will call the vector space of Minkowski summands of scalar multiples of $Q$. It has one dimension less than the cone of Minkowski summands.

**Theorem:** (cf. [Sm]) The map $\varrho$ induces an isomorphism between the vector space of Minkowski summands of scalar multiples of $Q$ and the vector space $\tilde{T}^1(Q)$.

**Remark:** The constructions made in (2.5) do not depend on the linear, but on the affine structure of $A$.

**Lemma:** Let $Y$ be an affine toric Gorenstein variety induced from a lattice polytope $Q$. Then, the vector space $T_Y^1(-R^*)$ equals the complexified vector space $\tilde{T}^1(Q) \otimes \mathcal{O}$ of (rational or real) Minkowski summands of scalar multiples of $Q$ (modulo translations and scalar multiples of $Q$ itself).

**Proof:** We will use the first $T^1$-formula of Theorem (2.2). For the special degree $-R^*$ the sets $E_i$ equal

$$E_i = \{s \in E \mid \langle a^i, s \rangle = 0\} = E \cap (a^i)^\perp.$$
In particular, they contain the sets \( F_i \) constructed above. We obtain a natural linear map
\[
\theta : T^1_Y(-R^*) = \left( \frac{L(E')}{\sum_{i=1}^N L(E_i)} \right)^* \otimes \mathcal{C} \longrightarrow \left( \frac{L(F')}{\sum_{i=1}^N L(F_i)} \right)^* \otimes \mathcal{C} = \tilde{T}^1(Q) \otimes \mathcal{C},
\]
and it remains to prove that \( \theta \) is an isomorphism.

Let \( s \in E' \subseteq \partial \sigma^\vee \) be an element that is not a fundamental generator of \( \sigma^\vee \). Then, there is a minimal face \( \alpha < \sigma^\vee \) containing \( s \) (as a relatively interior point), and we can choose some fundamental generators \( s_1, \ldots, s_k \in \alpha \) such that \( s = \sum_{j=1}^k \lambda_j s^j \) (\( \lambda_j \in Q \geq 0 \)).

Now, for each \( E_i \) containing \( s \) (equivalent to \( \alpha < (a^i)^\perp \cap \sigma^\vee \)) we have a decomposition
\[
L(E_i) = L(E_i \setminus \{s\}) \oplus \mathbb{R} \cdot \text{relation } s = \sum_{j=1}^k \lambda_j s^j.
\]

In particular, the second summand can be reduced in the expression for \( T^1_Y \). Since the map \( \theta \) consists of such steps only, we are done. \( \square \)

**Theorem:**

1. Let \( R \in M \), then
\[
T^1_Y(-R) = \begin{cases} 
\tilde{T}^1(Q \cap [\frac{1}{g}R^* - R]^{\perp}) = \tilde{T}^1(\text{Conv}\{a^i \mid \langle a^i, R \rangle = 1\}) & \text{for } \frac{1}{g}R^* \geq R \text{ on } \sigma \text{ (i.e. } \langle a^i, R \rangle \leq 1 \forall i) \text{ otherwise.} \\
0
\end{cases}
\]

2. Let \( \tau < \sigma \) be a face of \( \sigma \), then \( T^1_Y \left( [\frac{-1}{g}R^* + \text{int}(\sigma^\vee \cap \tau^{\perp})] \cap M \right) = \tilde{T}^1(Q \cap \tau) \). \( T^1_Y \) vanishes in the remaining degrees.

**(2.7)** We collect the results of (2.3) - (2.6): Let \( Y \) be an affine toric \( Q \)-Gorenstein variety given by a lattice polytope \( Q = \text{Conv}(a^1, \ldots, a^N) \) contained in an affine hyperplane \( \langle \bullet, R^* \rangle = g \subseteq N_{R^1} \) of lattice-distance \( g \) from \( 0 \in N_{R^1} \). Moreover, assume that \( Y \) is smooth in codimension 2.

Then, the graded pieces of \( T^1_Y \) are related to the vector spaces of Minkowski summands of faces of \( Q \). Using the notations of (2.5) (and \( \tilde{T}^1(\emptyset) := 0 \)) we obtain the following two equivalent descriptions of \( T^1_Y \):

**Theorem:**

1. Let \( R \in M \), then
\[
T^1_Y(-R) = \begin{cases} 
\tilde{T}^1(Q \cap [\frac{1}{g}R^* - R]^{\perp}) = \tilde{T}^1(\text{Conv}\{a^i \mid \langle a^i, R \rangle = 1\}) & \text{for } \frac{1}{g}R^* \geq R \text{ on } \sigma \text{ (i.e. } \langle a^i, R \rangle \leq 1 \forall i) \text{ otherwise.} \\
0
\end{cases}
\]

2. Let \( \tau < \sigma \) be a face of \( \sigma \), then \( T^1_Y \left( [\frac{-1}{g}R^* + \text{int}(\sigma^\vee \cap \tau^{\perp})] \cap M \right) = \tilde{T}^1(Q \cap \tau) \). \( T^1_Y \) vanishes in the remaining degrees.

**(2.8)** With the same assumptions as in (2.7) we obtain the following applications of the previous theorem:
Corollary:

(1) If every 2-face of $Q$ is a triangle (for instance, if $Y$ is smooth in codimension 3), then $Y$ is rigid, i.e. $T^1_Y = 0$.

(2) If $Y$ is Gorenstein ($g = 1$) of dimension at least 4 ($\dim Q \geq 3$), then the existence of a 2-face of $Q$ that is not a triangle implies $\dim T^1_Y = \infty$.

(3) Let $Y$ be not Gorenstein, i.e. $g \geq 2$. Then, $\dim T^1_Y < \infty$ implies $T^1_Y = 0$.

Proof: (1) If $Q$ is shaped that every 2-face is a triangle, then every (at least 2-dimensional) face of $Q$ will have this property, too. On the other hand, Smilanski has shown that polytopes with only triangular 2-faces admit at most trivial Minkowski decompositions (cf. [Sm], Corollary (5.2)), i.e. $\tilde{T}^1 = 0$.
Moreover, faces of dimension smaller or equal than 1 of $Q$ cannot be non-trivially decomposed, anyway.

(2) Two-dimensional polygons with at least 4 vertices have a non-trivial $\tilde{T}^1$. Hence, a non-triangular 2-face of $Q$ yields a proper face $\tau < \sigma$ with $\tilde{T}^1(\cap \tau) \neq 0$.
On the other hand, “proper” means that $[-R^* + \text{int}(\cap \tau^\perp)] \cap M$ contains infinitely many elements, and $T^1_Y$ is non-trivial in all those degrees.

(3) Assume that $T^1_Y \neq 0$, then there must be a face $\tau < \sigma$ and an element $-R \in [-\frac{1}{g}R^* + \text{int}(\sigma^\perp \cap \tau)] \cap M$ such that $T^1_Y(-R) = \tilde{T}^1(\cap \tau) \neq 0$.
For $\tau = \sigma$ we would obtain $[-\frac{1}{g}R^* + \text{int}(\sigma^\perp \cap \tau^\perp)] \cap M = \{-\frac{1}{g}R^*\} \cap M = \emptyset$. Hence, $\tau < \sigma$ must be a proper face, and we can argue as in (2). \(\square\)

(2.9) Finally, we want to mention the case $\dim Y = 3$. Then, $Y$ is an isolated singularity, and it is given by a lattice $N$-gon $Q$.

Case1: $Y$ is not Gorenstein (i.e. $g \geq 2$).
Then, $Y$ is rigid. (This follows from (3) of the corollary in (2.8).)

Case 2: $Y$ is Gorenstein (i.e. $g = 1$).
Then, $T^1_Y$ is concentrated in degree $-R^*$, and $T^1_Y(-R^*) = \tilde{T}^1(Y)$ has dimension $N - 3$.
(The proper faces of $Q$ are Minkowski indecomposable. Hence, to produce a non-trivial contribution to $T^1_Y$, the face $\tau$ in Theorem (2.7)(2) has to equal $\sigma$. That means, $T^1_Y$ is concentrated in degree $-R^*$ only.
On the other hand, for computing the dimension of $T^1_Y(-R^*) = \tilde{T}^1(Y)$ in our special case, use the second formula of Theorem (2.2).)
3 Really existing deformations of toric Gorenstein singularities

(3.1) As in the previous chapter, we start with recalling the general result concerning arbitrary affine toric varieties. We use the notations of (2.1).

Let \( \bar{r} \in \tilde{\sigma} \cap M \) be a primitive element. Then, each (strongly homogeneous) toric regular sequence of degree \(-\bar{r}\) and its corresponding standard deformation of \( Y \) arise in the following way:

(i) Define \((A, \mathbb{L})\) as the affine space (with lattice) induced by \( \bar{r} \in M \)

\[
(A, \mathbb{L}) := (N, N) \cap \{ a \in N | \langle a, \bar{r} \rangle = 1 \}.
\]

By choosing an arbitrary base point \( 0 \in \mathbb{L} \) the pair \((A, \mathbb{L})\) can be regarded as a vector space with lattice. Moreover, we obtain an isomorphism of lattices \( \mathbb{L} \times \mathbb{Z}^m \rightarrow N \) via \((a, g) \mapsto (a - 0) + g \cdot 0 \).

(ii) \( Q := \sigma \cap A \) is a (not necessarily compact) rational polyhedron in \( A \). Fix a Minkowski decomposition \( Q = R_0 + \ldots + R_m \) such that for each vertex of \( Q \) at least \( m \) of its \( m + 1 \) \( R_i \)-summands (which are uniquely determined vertices of the polyhedra \( R_i \)) are contained in the lattice \( \mathbb{L} \).

(iii) Define \( P \subseteq A \times \mathbb{R}^{m+1} \) as the convex polyhedron

\[
P := \text{conv} \left( \bigcup_{i=0}^{m} R_i \times \{ e^i \} \right)
\]

and \( \tilde{\sigma} := \overline{R_{\geq 0} \cdot P} \) as the closure of its cone in \( A \times \mathbb{R}^{m+1} \).

Moreover, if \( r^i \) denotes the projection of \( \mathbb{L} \times \mathbb{Z}^{m+1} \) onto the \( i \)-th component of \( \mathbb{Z}^{m+1} \), we have found elements \( r^0, \ldots, r^m \in \tilde{\sigma}^\vee \cap (\mathbb{L} \times \mathbb{Z}^{m+1})^* \).

(iv) \( \sigma \subseteq N_{R_1} \) is the cone over \( Q \subseteq A \). Hence, the affine embedding \( A \hookrightarrow A \times \mathbb{R}^{m+1} \) \((a \mapsto (a; 1, \ldots, 1))\) induces an embedding of lattices \( N \hookrightarrow \mathbb{L} \times \mathbb{Z}^{m+1} \) such that \( N = (\mathbb{L} \times \mathbb{Z}^{m+1}) \cap \bigcap_{i=1}^{m} (r^i - r^0) \perp \) and \( \sigma = \tilde{\sigma} \cap N_{R_1} \).

(v) \( \{ x^1 - x^0, \ldots, x^m - x^0 \} \) is a toric regular sequence in \( X := \text{Spec} \mathcal{O}[\tilde{\sigma}^\vee \cap (\mathbb{L} \times \mathbb{Z}^{m+1})^*] \), and \( Y \) is equal to the special fiber of the corresponding (flat) map \( X \rightarrow \mathcal{O}^m \).

(The proof can be found in §4 of [Al 2].)

Remark: (1) The assumption that the degree \(-\bar{r}\) has to be a primitive element of the lattice \( M \) is not essential. However, the description of the corresponding toric regular sequences becomes slightly more complicated in the general case (cf. §3 of [Al 2]).
and we do not need it in the present paper.

(2) The previous method yields deformations of degrees contained in \(-(\sigma^\vee \cap M)\) only. Nevertheless, \(T^1\) can be non-trivial in other degrees, too.

(3.2) As a direct consequence we obtain

**Theorem:** Let \(Y\) be an affine toric Gorenstein variety induced from a lattice polytope \(Q\). Then, toric \(m\)-parameter deformations of degree \(-R^*\) correspond to Minkowski decompositions of \(Q\) into a sum \(Q = R_0 + \ldots + R_m\) of \(m + 1\) lattice polytopes.

The Kodaira-Spencer-map maps the parameter space \(Q^m\) onto the linear subspace \(\text{span}(g(R_0), \ldots, g(R_m)) \subseteq T^1(Y) = T^1_Y(-R^*) \subseteq T^1_Y\).

**Proof:** \((A, II)\) defined in (i) of the previous theorem is exactly that affine space containing our polytope \(Q\). Moreover, \(Q\) coincides with the polyhedron \(Q := \sigma \cap A\) defined in (ii). Since \(Q\) is a lattice polytope, the conditions for the summands \(R_i\) (cf. (ii) of the previous theorem) are equivalent to the property of being lattice polytopes, too.

Finally, the claim concerning the Kodaira-Spencer-map follows from the definitions of the maps \(g\) and \(\theta\) in (2.5) and (2.6), respectively, and from Theorem (5.2) of [Al 2].

(3.3) Let us focus on the special case of \(\dim Y = 3\). Let \(Y\) be given by a 2-dimensional lattice polygon \(Q = \text{Conv}(a^1, \ldots, a^N)\) with primitive edges \(\vec{e}_i := a^{i+1} - a^i (i \in \mathbb{Z}/N\mathbb{Z})\), i.e. \(Y\) has an isolated singularity in \(0 \in Y\). Then, we obtain

**Theorem:** Non-trivial toric \(m\)-parameter deformations of \(Y\) correspond to non-trivial Minkowski decompositions of \(Q\) into a sum of \(m + 1\) lattice polygons, i.e. to decompositions of the set of edges of \(Q\) into a disjoint union of \(m + 1\) subsets each suming up to 0.

**Proof:** This is an immediate consequence of Theorem (3.2) and the fact that \(T^1_Y\) is concentrated in degree \(-R^*\) (cf. (2.7)). Nevertheless, beeing a little more carefully, for this conclusion the following fact has to be used: Toric, regular sequences inducing a trivial Kodaira-Spencer-map always yield trivial (standard) deformations. This is proved in §6 of [Al 2].

4 Polarity and Examples

(4.1) We start with recalling some general facts concerning the relation be-
tween lattice polytopes and projective toric varieties (cf. Chapter 2 of [Od]).

Let $Q \subseteq (\mathbb{A}, \mathbb{L})$ be a lattice polytope. Then, the inner normal fan $\Sigma$ induces a projective toric variety $\mathbb{P}(Q)$, and $Q$ itself corresponds to an ample line bundle on it.

Equivariant Weil divisors on $\mathbb{P}(Q)$ are described by maps $\Sigma^{(1)} \xrightarrow{h} \mathbb{Z}$. Modulo principal divisors, they generate the whole divisor class group $\text{Div}(\mathbb{P}(Q))$. Let $D_h$ be a Weil divisor on $\mathbb{P}(Q)$.

(i) $D_h$ is Cartier if and only if, on each top dimensional cone $\alpha = \langle c^1, \ldots, c^k \rangle \in \Sigma$, the map $h$ can be represented as

$$h(c^j) = \langle a_\alpha, c^j \rangle \quad (j = 1, \ldots, k) \quad \text{with} \quad a_\alpha \in \mathbb{L}.$$

(ii) A Cartier divisor $D_h$ is nef if and only if, moreover, $h \leq \langle a_\alpha, \bullet \rangle$ holds (for each top dimensional $\alpha \in \Sigma$) on the whole 1-skeleton $\Sigma^{(1)}$. Then, the elements $a_\alpha$ form the vertex set of a polytope with inner normal cones containing the corresponding cones of $\Sigma$.

On the other hand, if $Q'$ is a lattice polytope such that $\Sigma$ is a subdivision of its inner normal fan (i.e. $Q'$ is a Minkowski summand of a scalar multiple of $Q$), then we can use its vertices to define a map $h(Q') : \Sigma^{(1)} \to \mathbb{Z}$ via (i). We obtain a nef Cartier divisor $D_{Q'}$ on $\mathbb{P}(Q)$ again.

The divisor $D_{Q'}$ is even ample if and only if $Q'$ and $Q$ induce the same inner normal fan $\Sigma$. Equivalently, the elements $a_\alpha$ yield different lattice points for different cones $\alpha$.

Finally, we remark that variations by principal divisors correspond to translations of the polytopes by lattice vectors only.

(4.2) Let $Y$ be a 3-dimensional affine toric Gorenstein variety induced by a lattice polygon $Q$, let $Y$ having an isolated singularity in $0 \in Y$. Then, we call $\mathbb{P}(Q)$ the polar variety of $Y$, it will be denoted by $\mathbb{P}(Y)$.

(I) $\text{Pic} \mathbb{P}(Y)$ equals the group generated by the lattice Minkowski summands of scalar multiples of $Q$. If we denote by $\mathcal{O}_{\mathbb{P}(Y)}(1)$ the ample line bundle corresponding to $Q$ itself, then (2.9) tells us that

$$T^1_Y = \text{Pic} \mathbb{P}(Y) \left/ \mathcal{O}(1) \otimes \mathbb{Z} \mathcal{C} \right.$$  

Let $P(Y)$ be the cone over $(\mathbb{P}(Y), \mathcal{O}(1))$. The pull back of $\mathcal{O}(1)$ is a principal divisor on $P(Y) \setminus \{0\}$. Hence,

$$T^1_Y = \text{Pic}(P(Y) \setminus \{0\}) \otimes \mathbb{Z} \mathcal{C}.$$
Theorem (3.3) deals with Minkowski decompositions of \(Q\) into a sum of lattice polygons. In the language of the polar variety we obtain that non-trivial toric \(m\)-parameter deformations of \(Y\) correspond to non-trivial decompositions of \(\mathcal{O}_{\mathbb{P}(Y)}(1)\) into a tensor product
\[
\mathcal{O}_{\mathbb{P}(Y)}(1) = L_0 \otimes \ldots \otimes L_m
\]
of \(m+1\) nef invertible sheaves on \(\mathbb{P}(Y)\). The tangent plane to this deformation inside the semi-universal base space \(S\) is spanned by the classes \([L_0], \ldots, [L_m] \in \text{Pic}\ \mathbb{P}(Y) / \mathcal{O}(1) = \text{Pic}(\mathbb{P}(Y) \setminus \{0\}) \subseteq T^1_Y\).

**Remark:** The condition “\(0 \in Y\) is an isolated singularity” can be translated into the \(\mathbb{P}(Y)\)-language, too: Each closed equivariant subvariety of \(\mathbb{P}(Y)\) equals a linearly embedded projective space.

**Conjecture (4.3):** Take an arbitrary Minkowski decomposition of \(Q\) into a sum of lattice polytopes (equivalently: a decomposition of \(\mathcal{O}_{\mathbb{P}(Y)}(1)\) into a tensor product of nef line bundles), project the summands into \(T^1_Y = \tilde{T}^1(Q) \otimes \mathbb{R} \mathcal{C} = \text{Pic}(P(Y) \setminus \{0\}) \otimes \mathbb{Z} \mathcal{C}\), and form their linear hull. Then, the union of all linear subspaces obtained in this way equals the reduced base space of the semi-universal deformation of \(Y\).

The *cone* of Minkowski summands (i.e. the nef sheaves in \(\text{Pic}\ \mathbb{P}(Y)\)) contains much more information than the so-called *space* of Minkowski summands of \(Q\) (i.e. \(\text{Pic}(P(Y) \setminus \{0\}) \otimes \mathbb{Z} \mathcal{R}\)). Apart from the toric context, is there any such cone in deformation theory? Projecting a certain interior point onto 0 has to yield \(T^1_Y\), then.

Does the cone of Minkowski summands contain any information about the non-reduced structure of the base space?

Finally, we want to present a special class of examples. We are looking for those three-dimensional \(Y\) that are, additionally to the usual assumptions, cones over projective toric varieties. (Do not mistake this property for \(P(Y)\) being the cone over \(\mathbb{P}(Y)\).)

In §4 of [Ba] it is shown that these \(Y\) can be characterized as the cones over two-dimensional toric Fano varieties with Gorenstein singularities. (Then, \(P(Y)\) admits the same property.)

The corresponding \(Q\) are exactly those lattice polygons containing one and only one interior lattice point ("reflexive polygons"). Choosing this point as the origin, the polar polygon \(Q^\vee := \{r \in \mathbb{A}^1 | \langle Q, r \rangle \leq 1\}\) is a lattice polygon, too. Then, \(\tilde{\sigma} = \text{Cone}(Q^\vee)\), and \(Y\) is the cone over the projective variety corresponding to \(Q^\vee\).
Reflexive polygons were classified in (4.2) of [Ko]. Our additional assumption of $Y$ having only an isolated singularity causes that only five polygons $Q$ survive from the original list (containing 16 items). Including the polar polygons $Q^\vee$, we will see nine ones, however.

$Y_1$ is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(2, 2)$.

$Q_1$ is a quadrangle, hence $T^1$ is one-dimensional. Moreover, $Q_1$ is the Minkowski sum of two line segments, i.e. there is a really existing toric 1-parameter deformation. The total space is an isolated 4-dimensional cyclic quotient singularity.

In particular, the base space $S_1$ of the semi-universal deformation of $Y_1$ equals $\mathcal{O}^1$.

$Y_2$ is the cone over $\mathbb{P}^2$ embedded by $\mathcal{O}(3)$. Since $Q_2$ is a triangle, $Y_2$ is rigid.

$Y_3$ is the cone over the Del Pezzo surface of degree 8 (the blowing up of $(\mathbb{P}^2, \mathcal{O}(3))$ in one point).

The vector space $T^1$ is one-dimensional. The two-dimensional cone of the rational Minkowski summands of scalar multiples of $Q_3$ is generated by two triangles.

However, there are no lattice polygons that are non-trivial Minkowski summands of $Q_3$. That means, $Y_3$ does not admit any toric deformation at all. Indeed, as Duco van Straten has computed with Macaulay, the semi-universal base
space $S_3$ of $Y_3$ equals $\text{Spec} \mathcal{A}^/[\varepsilon]/\varepsilon^2$.

(4.4.4) 

Polygons $Q_4$ and $Q_4^\vee$ 

$Y_4$ is the cone over the Del Pezzo surface of degree 7 (obtained from $\mathbb{P}^2, \mathcal{O}(3)$) by blowing up two points, or from $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 2))$ by blowing up one point).

$T^1$ is two-dimensional, but $Q_4$ admits one decomposition into a Minkowski sum of two lattice polygons only. $Q_4$ equals the sum of a line segment and a triangle - this yields a 1-parameter deformation of $Y_4$, the total space is the cone over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$.

The semi-universal base space $S_4$ is a complex line with one embedded component (computed by Duco van Straten using Macaulay).

(4.4.5) 

Polygon $Q_5 = Q_5^\vee$ 

$Y_5$ is the cone over the Del Pezzo surface of degree 6 (obtained by blowing up the projective variety of (4.4.4) in one more point).

$T^1$ is three-dimensional, and $Q_5$ admits two different extremal Minkowski decompositions:

(i) $Q_5$ equals the sum of two triangles, the corresponding 1-parameter family admits the cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as its total space.

(ii) $Q_5$ also equals the sum of three line segments. This corresponds to a two-parameter family with the cone over $\mathbb{P}^2 \times \mathbb{P}^2$ as its total space.

Again, Duco van Straten has computed the semi-universal base space - it is reduced and equals the transversal union of a complex plane with a complex line. These components correspond to the toric deformations we have already seen.

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