GEOMETRY OF KOTTWITZ-VIEHMANN VARIETIES

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Abstract. We study basic geometric properties of Kottwitz-Viehmann varieties, which are certain generalizations of affine Springer fibers that encode orbital integrals of spherical Hecke functions. Based on previous work of A. Bouthier and the author, we show that these varieties are equidimensional and give a precise formula for their dimension. Also we give a conjectural description of their number of irreducible components in terms of certain weight multiplicities of the Langlands dual group and we prove the conjecture in the case of unramified conjugacy class.

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1. Introduction

1.1. Background and motivation. In this article we study certain analogue of affine Springer fibres that we call Kottwitz-Viehmann varieties whose underlying set is defined as

\[ X^\gamma_\lambda = \{ g \in G(F)/G(O) | g^{-1}g \in G(O) \varpi^\lambda G(O) \} \]

where

- \( G \) is a connected reductive algebraic group over a field \( k \);
- \( F = k((\varpi)) \) is the field of Laurent series with coefficients in \( k \) and \( O = k[[\varpi]] \) is the ring of power series;
- \( \gamma \in G(F) \) is a regular semisimple element;
- \( \lambda : G_m \to T \) is a cocharacter of a maximal torus \( T \) of \( G \) and \( \varpi^\lambda := \lambda(\varpi) \in G(F) \).

Also we will consider a closely related set \( X^{\leq \lambda}_\gamma \) defined by Replacing the double coset \( G(O) \varpi^\lambda G(O) \) in the definition of \( X^\gamma_\lambda \) by the union

\[ G(O) \varpi^\lambda G(O) = \cup_{\mu \leq \lambda} G(O) \varpi^\mu G(O) \]

These sets were first studied by Kottwitz and Viehmann in [KV12]. More general versions of them (replacing \( G(O) \) by parahoric subgroups of \( G(F) \)) have also been studied by Lusztig in [Lus15]. When \( k \) is a finite field, they arise naturally in the study of orbital integrals of functions in the spherical Hecke algebra \( \mathcal{H}(G(F), G(O)) \) consisting of \( G(O) \)-biinvariant locally constant functions with compact support on \( G(F) \).

It turns out that \( X^\lambda_\gamma \) can be realized as the set of \( k \)-rational points of some algebraic variety over \( k \). We view them as group analogue of affine Springer fibers for Lie algebras studied by Kazhdan and Lusztig in [KL88]:

\[ X_\gamma = \{ g \in G(F)/G(O) | \text{ad}(g)^{-1} \gamma \in \mathfrak{g}(O) \} \].

Here \( \mathfrak{g} \) is the Lie algebra of \( G \), \( \gamma \in \mathfrak{g}(F) \) is a regular semisimple element and “\( \text{ad} \)” denotes the adjoint action of \( G \) on \( \mathfrak{g} \).

Basic geometric properties of these affine Springer fibers \( X_\gamma \) have been well understood through the works of Kazhdan and Lusztig [KL88], Bezrukavnikov [Bez96], Ngô [Ngô10]. A key ingredient in their approach is the symmetry on \( X_\gamma \) arising from the centralizer \( G_\gamma(F) \). More precisely, the group \( G_\gamma(F) \) has a dense open orbit \( X^\gamma_{\lambda,\text{reg}} \) (the “regular locus”) and geometric properties of \( X^\lambda_\gamma \) are reduced to the commutative algebraic group \( G_\gamma(F) \) (more precisely certain finite dimensional quotient \( P_\gamma \) of the infinite dimensional loop group \( G_\gamma(F) \)).

We would like to generalize these methods to study the Kottwitz-Viehmann varieties \( X^\lambda_\gamma \). Similar to Lie algebra case, the (connected) centralizer \( G^0_\gamma(F) \) acts naturally on \( X^\lambda_\gamma \) and we consider the open orbits \( X^\lambda_{\gamma,\text{reg}} \) (the “regular locus”). However, there are the following notable differences from the Lie algebra situation:

- In general the action of \( G^0_\gamma(F) \) on \( X^\lambda_{\gamma,\text{reg}} \) is not transitive.
- A more serious problem is that in general the “regular locus” \( X^\lambda_{\gamma,\text{reg}} \) is not dense in \( X_\gamma \) and there might be irreducible components disjoint from \( X^\lambda_{\gamma,\text{reg}} \).

Thus \( X^\lambda_\gamma \) may have more irreducible components than \( X^\lambda_{\gamma,\text{reg}} \). This makes it more difficult to reduce geometric properties of \( X^\lambda_\gamma \) to the commutative group \( G^0_\gamma(F) \).

1.2. Main results. Our first goal is to prove a dimension formula of \( X^\lambda_\gamma \).

**Theorem 1.2.1.** \( X^\lambda_\gamma \) and \( X^{\leq \lambda}_\gamma \) are \( k \)-schemes locally of finite type, equidimensional with dimension

\[ \dim X^\lambda_\gamma = \dim X^{\leq \lambda}_\gamma = \langle \rho, \lambda \rangle + \frac{1}{2}(d(\gamma) - c(\gamma)) \]

where

- \( \rho \) is half sum of the positive roots for \( G \);
- \( d(\gamma) \) is the discriminant valuation of \( \gamma \) (cf. Definition 3.1.2);
- \( c(\gamma) = \text{rank}(G) - \text{rank}_F(G_\gamma) \), the difference between the dimension of the maximal torus of \( G \) and the dimension of the maximal \( F \)-split subtorus of the centralizer \( G_\gamma \).
In [Bou15a] and [BC17], this theorem is proved when $G$ is semisimple and simply-connected. In this article we prove it for any split connected reductive group.

As in the Lie algebra case, there are two major steps. First we prove the dimension formula for the regular open subset, this step generalize the method of Kazhdan-Lusztig in [KL88]. The second step is to show that

$$\dim X^\lambda_{\gamma,\text{reg}} = \dim X^\lambda_{\gamma}$$

For this the argument of Kazhdan-Lusztig in [KL88] does not generalize, since otherwise it would imply that the complement of the regular open subset has strictly smaller dimension (see [Ngô10, Proposition 3.7.1]), which in our situation may not be true due to the possible existence of irregular components. In general, actually most components of $X^\lambda_{\gamma}$ will be irregular, see Remark 3.9.14. Instead, we bypass this difficulty by studying the global analogue of Kottwitz-Viehmann varieties, the Hitchin-Frenkel-Ngô fibration. Similar ideas occured previously in [BC17].

This major difference from Lie algebra case lead us naturally to the question of determining the number of irreducible components of $X^\lambda_{\gamma}$, which is our second goal. We will formulate a conjecture on the number of irreducible components of $X^\lambda_{\gamma}$ and prove the conjecture in the case where $\gamma$ is an unramified (or split) conjugacy class. One formulation of the conjecture involves the Newton point $\nu_\gamma \in (X^*_T \otimes \mathbb{Q})^+$ of $\gamma$, which is an element in the dominant rational coweight cone. By the discussion in §3.9, if $X^\lambda_{\gamma}$ is nonempty, there exists a unique smallest dominant integral coweight $\mu$ such that $\nu_\gamma \leq_\mathbb{Q} \mu$ and $\mu \leq \lambda$.

**Conjecture** (Conjecture 3.9.8). Let $\mu$ be as above. The number of $G^0_k(F)$-orbits on the set of irreducible components of $X^\lambda_{\gamma}$ equals to $m_{\mu,\gamma}$, which is the dimension of $\mu$-weight space in the irreducible representation $V_\lambda$ of the Langlands dual group $\hat{G}$ with highest weight $\lambda$.

We remark that there is a similar conjecture made by Miaofen Chen and Xinwen Zhu on the irreducible components of affine Deligne-Lusztig varieties, see [HV17] and [XZ17] for statements.

In fact we will also give a conceptually better formulation of this Conjecture using the extended Steinberg base of Vinberg monoid. See Conjecture 3.9.8 for more details.

**Theorem 1.2.2.** The Conjecture is true if $\gamma \in G(F)^{rs}$ is split.

This is proved in Corollary 3.5.3.

**Remark 1.2.3.** Although we restrict to equal characteristic local field, we expect that most results involving only local arguments in this paper could also be generalized to mixed characteristic Kottwitz-Viehmann varieties, which could be defined based on the work of X.Zhu [Zhu17]. However, the dimension formula in full generality involves global argument and currently it’s not clear how to generalize this to mixed characteristic case. It would be interesting to see if there is a purely local argument to prove dimension formula.

### 1.3. Organization of the article

In §2, we review certain facts needed from the theory of reductive monoids. In §3, we prove dimension formula and the conjecture on irreducible components in the unramified case. In §4, we review basic facts of Hitchin-Frenkel-Ngô fibration. The main result we establish in this chapter is properness of the fibration over anisotropic open subset. In §5, we relate Kottwitz-Viehmann varieties and Hitchin-Frenkel-Ngô fibrations and finish the prove of dimension formula for $X^\lambda_{\gamma}$.

### 1.4. Notations and conventions.

#### 1.4.1. Group theoretic notations.

We assume throughout the article that $k$ is an algebraically closed field. $F = k((\varpi))$ and $\mathcal{O} = k[[\varpi]]$. We let $G$ be a (split) connected reductive group over $k$. Assume that either $\text{char}(k) = 0$ or $\text{char}(k) > 0$ does not divide the order of Weyl group of $G$.

Denote by $G_{\text{der}}$ the derived group of $G$, a semisimple group of rank $r$. Let $G^{\text{sc}}$ be the simply-connected cover of $G_{\text{der}}$ and $G_{\text{ad}}$ the adjoint group of $G$.

Fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ containing $G$. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots determined by $T \subset B$. Let $\Lambda := X^*_T(T)$ (resp. $\Lambda := X_*(T)$) be the weight (resp. coweight) lattice. Let $\Lambda^+$ (resp. $\Lambda^+$) be the set of dominant weights (resp. dominant coweights). Let $W$ be the Weyl group of $G$ and $S \subset W$ the set of simple reflections associated to the simple roots $\Delta$. There is a unique longest element $w_0$ of $W$ under the Bruhat order determined by $S$. Then $w_0$ is a reflection and $-w_0$ defines a bijection on the sets $\Delta$, $\Lambda^+$ and $\Lambda^+$. 
Let $\hat{G}$ be the Langlands dual group of $G$, viewed as a complex reductive group. For each $\lambda \in \Lambda^+$, viewed as a dominant weight for $\hat{G}$, let $V(\lambda)$ be the irreducible representation of $\hat{G}$ with highest weight $\lambda$. For any $\mu \in \Lambda^+$ with $\mu \leq \lambda$, let $m_{\lambda \mu}$ be the dimension of $\mu$ weight space in $V(\lambda)$.

1.4.2. Scheme theoretic notations. For a scheme $X$ over Spec $F$, let $LX$ be the loop space of $X$. More precisely, $LX$ is the $k$-space that associates to any $k$-algebra $R$ the set $LX(R) = X(R((t)))$.

For a scheme $X$ over Spec $O$, let $L^+_n X$ be its $n$-th jet space. In other words, $L^+_n X$ is the $k$-space whose set of $R$ points is $L^+_n X(R) = X(R[t]/t^n)$ for any $k$ algebra $R$. Let $L^+_X := \varprojlim L^+_n X$ be the arc space of $X$.

If $X$ is a $k$-scheme, then we denote $LX := L(X \otimes_k F)$, $L^+_n X := L^+_n (X \otimes_k O)$ and $L^+_X := L^+ (X \otimes_k O)$.

For any scheme $X$, we denote by $\text{Irr}(X)$ the set of its irreducible components.

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2. Review on reductive monoids

In this section we summarize some results on reductive monoids needed later. We will roughly follow the exposition in [Bou15a], with several modifications and improvements. We refer the reader to [Vin95], [Rit98], [Rit01] for more backgrounds on this subject.

2.1. Construction of Vinberg monoid. In this section, we assume that $G$ is semisimple simply connected.

The Vinberg monoid for $G$ is an algebraic monoid $\text{Vin}_G$ such that the derived group of its unit group is isomorphic to $G$, and it is characterized by certain nice universal properties. For our purpose, we construct it in an explicit manner as follows.

Let $\omega_1, \ldots, \omega_r \in X_*(T)_+$ be the fundamental weights. For each $1 \leq i \leq r$, let $\rho_{\omega_i} : G \to \text{GL}(V_{\omega_i})$ be the irreducible representation with highest weight $\omega_i$.

We introduce the extended group $G_+ := (T \times G)/Z$ where $Z$, the center of $G$, embeds anti-diagonally in $T \times G$. Then $G_+$ is a reductive group with center $Z_+ = (T \times Z)/Z \cong T$ and derived group $G$. Let $T_+ = (T \times T)/Z$ be a maximal torus of $G^+$. We extend the representations $\rho_{\omega_i}$ representations of $G_+$:

$$\rho_i^+ : G_+ \rightarrow \text{GL}(V_{\omega_i})$$

$$(t, g) \mapsto t \rho_{\omega_i}(g)$$

For each $1 \leq i \leq r$, we also extend the simple roots $\alpha_i$ to $\alpha_i^+ : G^+ \to G_m$ by $t \alpha_i^+(t, g) = \alpha_i(t)$. Altogether, we get the following homomorphism

$$(\alpha^+, \rho^+) : G^+ \rightarrow G_m^r \times \prod_{i=1}^r \text{GL}(V_{\omega_i})$$

**Definition 2.1.1.** The Vinberg monoid of $G$, denoted by $\text{Vin}_G$, is the normalization of the closure of $G_+$ in the product

$$\mathbb{A}^r \times \prod_{i=1}^r \text{End}(V_{\omega_i}).$$

Then $\text{Vin}_G$ is an algebraic monoid with unit group $G_+$. It has a smooth dense open subvariety $\text{Vin}_G^0$ defined as the inverse image of the following product in $\text{Vin}_G$

$$\mathbb{A}^r \times \prod_{i=1}^r \left( \text{End}(V_{\omega_i}) \setminus \{0\} \right)$$

**Definition 2.1.2.** The abelianization of the monoid $\text{Vin}_G$ is the invariant quotient

$$A_G := \text{Vin}_G/(\mathbb{G} \times G)$$

Let $\alpha : \text{Vin}_G \rightarrow A_G$ be the quotient map.
Using the maps $\alpha^+$ we get a canonical isomorphism $\mathbb{A}^r$. The adjoint torus $T_{ad}$ embeds via the simple roots as the open subset where all the $\tau$-coordinates are nonzero.

Note that the fibers of $\alpha$ over points in $T_{ad}$ are isomorphic to $G$. One can construct a canonical section of the abelianization map $\alpha$ as follows.

Let $T_{\text{diag}}$ be the image of the diagonal embedding $T \to T_\tau$. Then there is a canonical isomorphism $T_{\text{diag}} \cong T_{ad}$ which extends to an isomorphism $T_{\text{diag}} \cong A_G$ between the closure of $T_{\text{diag}}$ in $\text{Vin}_G$ and $A_G$. The inverse of this isomorphism defines a section of the abelianization map $\alpha$, which we denote by

$$s : A_G \to \text{Vin}_G$$

The group $G_+ \times G_+$ acts by left and right multiplication on $\text{Vin}_G$. More precisely, for all $(x, y) \in G_+ \times G_+$ and $\gamma \in \text{Vin}_G$, the action is given by $(x, y) \cdot \gamma = x\gamma y^{-1}$. The $G_+ \times G_+$-orbits on $\text{Vin}_G$ correspond bijectively to pairs $(I, J)$ of subsets of $\Delta$ such that no connected component (in the sense of Dynkin diagram) of the complement of $J$ is entirely contained in $I$. Each orbit $O_{I,J}$ contains an idempotent $e_{I,J} \in \text{Vin}_G$, defined up to conjugation. We can choose $e_{I,J} \in T_{I,J}$, the closure of $T_{I,J}$ in $\text{Vin}_G$. Then it is well-defined up to $W$-conjugation.

Fix such a pair $(I, J)$. Let $J^\circ$ be the complement of $J$ in $\Delta$ and $J^0$ be the interior of $J$, i.e. the elements in $J$ that is not connected to any element of $J^\circ$ in the Dynkin diagram. Let $M := I \cap J^\circ \cup J^\circ$. Let $P_+(M)$ be the corresponding standard parabolic subgroup of $G_+ \times G_+$, and $\text{L}(M)$ their common Levi subgroup. Denote by $\delta : P_+(M) \to L_+(M)$ and $\delta_- : P_+(M) \to L_-(M)$ the canonical projections. The following lemma is [Rit97, Thm 21.1]:

**Lemma 2.1.3.** The stabilizer of $e_{I,J}$ under $G_+ \times G_+$ is the subgroup of $P_+(M) \times P_+(M)_-$ consisting of pairs $(g, g_-)$ such that

$$\delta(g) \equiv \delta(g_-) \mod L_+(J^\circ),$$

where $T_{I,J}$ is certain torus of $T_\tau$.

**2.4.** The adjoint action of $G$ on the Vinberg monoid $\text{Vin}_G$ is the restriction of left and right multiplication by $G \times G$ along the diagonal. In other words, for any $g \in G$ and $\gamma \in \text{Vin}_G$, the adjoint action is given by $\text{Ad}(g)(\gamma) := g\gamma g^{-1}$. Note that this action factors through the adjoint group $G_{ad}$.

For any $\gamma \in \text{Vin}_G$, we let $G_\gamma$ be the centralizer of $\gamma$ in $G$, i.e. the stabilizer of $\gamma$ under the adjoint action of $G$. If $\gamma \in G_+$ belongs to the unit group of $\text{Vin}_G$, we know that $\dim G_\gamma \geq \dim T = r$. By upper-semicontinuity of stabilizer dimension (cf. [ABD+65, VI B.4, Prop. 4.1]), we see that $\dim G_\gamma \geq r$ for all $\gamma \in \text{Vin}_G$.

**Definition 2.1.5.** An element $\gamma \in \text{Vin}_G$ is **regular** if $\dim G_\gamma = r$ (i.e. smallest possible). Let $\text{Vin}^{reg}_G \subset \text{Vin}_G$ be the open subset consisting of regular elements.

**Definition 2.1.6.** The **extended Steinberg base** is defined to be the invariant quotient $\mathcal{C}_+ := \text{Vin}_G//\text{Ad}(G)$. We denote the canonical quotient map by $\chi_+ : \text{Vin}_G \to \mathcal{C}_+$.

The functions $\alpha^+_i$ define a canonical map $\beta : \mathcal{C}_+ \to A_G$ so that $\alpha = \beta \circ \chi_+$. The following result is [Bou15a, Proposition 1.7]:

**Theorem 2.1.7.** The closed embedding $\mathcal{C}_+ \subset \text{Vin}_G$ induces an isomorphism $\mathcal{C}_+//W \cong \mathcal{C}_+$. Moreover, the functions $\alpha_+$ and $\text{Tr}(\rho^+_i)$ define isomorphism

$$\mathcal{C}_+ \cong A_G \times \mathbb{A}^r \cong \mathbb{A}^{2r}.$$
In general, an element \( w \in W \) is called a Coxeter element if it is conjugate to an \( S \)-Coxeter element in \( W \).

Let \( N := \chi_+^{-1}(0) \) be the nilpotent cone in the Vinberg monoid \( \text{Vin}_G \). Let \( N^0 := N \cap \text{Vin}_G^\reg \) and \( N^\reg := N \cap \text{Vin}_G^\reg \) be the corresponding open subsets.

2.2.2. Our approach in this part follows a suggestion of Xinwen Zhu. For any subset \( J \subset \Delta \), denote \( J^c := \Delta \setminus J \), then we have

\[
O_{\emptyset, J} \cong (G/G_{J^c} \cdot U_{J^c} \times G/G_{J^c} \cdot U_{J^c})/Z(L_{J^c}).
\]

where \( Z(L_{J^c}) \), the center of the Levi \( L_{J^c} \) acts diagonally on the product. There is a canonical map \( \pi_{\emptyset, J} : O_{\emptyset, J} \to G/P_J \times G/P_{J^c} \).

The diagonal \( G \)-orbits on the product \( G/P_J \times G/P_{J^c} \) corresponds bijectively to \( J^c W_{J^c} \). The element \( w \in J^c W_{J^c} \) corresponds to the \( G \)-orbit of \((\dot{w}, 1)\) for any representative \( \dot{w} \) of \( w \) in \( G \). We denote this \( G \)-orbit by \( Y_{\emptyset, J, \dot{w}} \) and let \( X_{\emptyset, J, \dot{w}} \) be its inverse image under \( \pi_{\emptyset, J} \). Then we have

\[
(2.2) \quad X_{\emptyset, J, \dot{w}} = \text{Ad}(G)(Z(L_{J^c})\dot{w} e_{\emptyset, J}).
\]

The \( G \)-orbit \( Y_{\emptyset, J, \dot{w}} \) has codimension \( l(w) \) in \( G/P_J \times G/P_{J^c} \). Hence we have

\[
\dim X_{\emptyset, J, \dot{w}} = 2\dim(G/P_{J^c}) - l(w) + \dim Z(L_{J^c})
\]

\[
= \dim G - \dim L_{J^c} - l(w) + |J|.
\]

**Lemma 2.2.3.** \( X_{\emptyset, J, \dot{w}} \subset N \) if and only if \( J \subset \text{Supp}(w) \).

**Proof.** First suppose \( X_{\emptyset, J, \dot{w}} \subset N \). Then in particular \( \dot{w} e_{\emptyset, J} \in N \). Recall that the idempotent \( e_{\emptyset, J} \) acts as projector to highest weight space in the representation \( V_{\omega_i} \) if \( i \in J \) and acts by 0 if \( i \notin J \). If there exists \( j \in J \) but \( j \notin \text{Supp}(w) \), then \( \rho_{\omega_j}(\dot{w}) \) preserves the highest weight space in \( V_{\omega_j} \) and hence \( \text{Tr}(\rho_{\omega_j}(\dot{w} e_{\emptyset, J})) \neq 0 \), contradiction the assumption that \( \dot{w} e_{\emptyset, J} \in N \).

Conversely suppose that \( J \subset \text{Supp}(w) \). Let \( x = t\dot{w} e_{\emptyset, J} \) where \( t \in Z(L_{J^c}) \subset T \). Then \( \rho_{\omega_i}(x) = 0 \) if \( i \notin J \). If \( i \in J \), so \( i \in \text{Supp}(w) \), then by a standard result in root system we have \( w(\omega_i) \neq \omega_i \) (see, for example [HT06, Lemma 3.5]). Thus we have \( \text{Tr}(\rho_{\omega_i}) = 0 \) as \( t \in T \) preserve the weight spaces and \( \dot{w} \) maps the highest weight space into the weight space with weight \( w(\omega_i) \). Thus \( x \in N \).

**Corollary 2.2.4.** (a) There is a stratification of \( N \) into \( \text{Ad}(G) \)-stable pieces

\[
N = \bigsqcup_{J \subset \Delta} \bigsqcup_{\dot{w} \in J^c W_{J^c}} \bigsqcup_{J \subset \text{Supp}(w) \supset J} X_{\emptyset, J, \dot{w}}.
\]

(b) \( N^0 = \bigsqcup_{\dot{w} \in W} X_{\emptyset, \Delta, \dot{w}}. \)

(c) For each \( w \in \text{Cox}(W, S) \)(cf. Definition 2.2.1), \( X_{\emptyset, \Delta, w} \) is a single \( \text{Ad}(G) \)-orbit and \( N^\reg = \bigsqcup_{w \in \text{Cox}(W, S)} X_{\emptyset, \Delta, w}. \)

In particular \( N^\reg \subset N^0 \).

(d) \( \dim N = \dim N^\reg = \dim G - r \) and the dimension of the complement \( N \setminus N^\reg \) is strictly less than \( \dim N \).

**Proof.** Part (a) and (b) are immediate from Lemma 2.2.3. For each strata \( X_{\emptyset, J, \dot{w}} \subset N \) as in Lemma 2.2.3, we have \( l(w) \geq |J| \) since \( J \subset \text{Supp}(w) \). From (2.3) we see that

\[
\dim X_{\emptyset, J, \dot{w}} \geq \dim G - \dim L_{J^c} \geq \dim G - r
\]

and equality is reached precisely when \( J = \Delta \) and \( l(w) = r \). This condition means that \( w \in \text{Cox}(W, S) \). Hence part (d) follows from part (c).

It remains to show that for each \( w \in \text{Cox}(W, S) \), \( X_{\emptyset, \Delta, w} \) is a single \( \text{Ad}(G) \)-orbit. By (2.2), we have

\[
X_{\emptyset, \Delta, w} = \text{Ad}(G)(T \dot{w} e_{\emptyset, \Delta}).
\]

So it suffices to show that for each \( t \in T \), the elements \( T \dot{w} e_{\emptyset, \Delta} \) and \( \dot{w} e_{\emptyset, \Delta} \) are conjugate. Since \( w \) is a Coxeter element, by [Ste65, Lemma 7.6] there exists \( s \in T \) such that \( t = s^{-1} \dot{w} s \dot{w}^{-1} \). This implies that \( s^{-1} \dot{w} e_{\emptyset, \Delta} s = t \dot{w} e_{\emptyset, \Delta} \) since \( s, t \in T \) and hence commute with \( e_{\emptyset, \Delta} \).
Remark 2.2.5. Another way to show that \( X_{\varnothing, \Delta, w} \) consists of a single \( \text{Ad}(G) \)-orbit is to show that the centralizer of \( \varnothing \varnothing \Delta \) in \( G \) has dimension \( r \), i.e. \( \varnothing \varnothing \Delta \in \mathcal{N}^{\text{reg}} \). For then the \( \text{Ad}(G) \)-orbit of \( \varnothing \varnothing \Delta \) is contained in the irreducible set \( X_{\varnothing, \Delta, w} \) and has the same dimension, thus equals to \( X_{\varnothing, \Delta, w} \).

Corollary 2.2.6. The morphism \( \chi_+ : \text{Vin}_G \to \mathfrak{c}_+ \) is flat.

Proof. There exists a nonempty open subset \( U \subset \mathfrak{c}_+ \) such that the fibres of \( \chi_+ \) over \( U \) have dimension \( \dim \text{Vin}_G - \dim \mathfrak{c}_+ = \dim G - r \). Since \( \chi_+ \) is \( Z_+ \) equivariant, \( U \) is \( Z_+ \)-stable. By Corollary 2.2.4(d) we know that \( 0 \in U \) and hence we have \( U = \mathfrak{c}_+ \). By [BK05, 6.2.9], \( \text{Vin}_G \) is Cohen-Macaulay. Moreover, \( \mathfrak{c}_+ \cong \mathbb{A}^r \) is regular and hence \( \chi_+ \) is flat.

Corollary 2.2.7. \( \text{Vin}_G^{\text{reg}} \subset \text{Vin}_G^0 \).

Proof. Let \( F := \text{Vin}_G^{\text{reg}} \setminus \text{Vin}_G^0 \). By Corollary 2.2.4(c), we have \( \mathcal{N}^{\text{reg}} \subset \mathcal{N}^0 \) and hence \( F \cap \mathcal{N} = \emptyset \). On the other hand, \( F \) is a \( Z_+ \)-stable closed subset of \( \text{Vin}_G^{\text{reg}} \), so we must have \( F = \emptyset \).

Proposition 2.2.8. The nilpotent cone \( \mathcal{N} \) is connected and equidimensional. Moreover, there exist bijections

\[
\text{Cox}(W, S) \xrightarrow{\sim} \text{Irr}(\mathcal{N}^{\text{reg}}) \xrightarrow{\sim} \text{Irr}(\mathcal{N}^0) \xrightarrow{\sim} \text{Irr}(\mathcal{N})
\]

which send \( w \in \text{Cox}(W, S) \) to the irreducible component containing \( \varnothing \varnothing \Delta \).

Proof. Since \( \chi_+ \) is flat, its fibre \( \mathcal{N} = \chi_+^{-1}(0) \) is equidimensional. Since \( \chi_+ \) is the invariant quotient under a reductive group, there is a unique closed orbit in \( \mathcal{N} \), namely \( 0 \in \mathcal{N} \). In particular, \( \mathcal{N} \) is connected.

From Corollary 2.2.4, we see that \( \mathcal{N}^{\text{reg}} \) is dense in \( \mathcal{N}^0 \) and \( \text{Irr}(\mathcal{N}^{\text{reg}}) \) is in bijection with \( \text{Cox}(W, S) \). Hence \( \text{Irr}(\mathcal{N}^0) \) and \( \text{Irr}(\mathcal{N}) \) are also in bijection with \( \text{Cox}(W, S) \).

Remark 2.2.9. It is worthwhile to compare with the Lie algebra case. The fibers of the Chevalley map \( \mathfrak{g} \to \mathfrak{c} \) are irreducible and \( \mathfrak{g}^{\text{reg}} \) is the union of the unique open \( G \)-orbit in each fiber.

2.2.10. Discriminant divisor. Recall that on \( T \) we have the discriminant function

\[
\text{Disc}(t) := \prod_{\alpha \in \Phi}(1 - \alpha(t))
\]

which is \( W \)-equivariant and descends to a regular function on the Steinberg base \( \mathfrak{c} := T//W \). We extend the function \( \text{Disc} \) to a function \( \text{Disc}_+ \) on \( T_+ = (T \times T)/Z_G \) by

\[
\text{Disc}_+(t_1, t_2) := 2\rho(t_1)\text{Disc}(t_2).
\]

Then \( \text{Disc}_+ \) extends to a regular function on \( T_+ \), which further descends to a regular function on \( \mathfrak{c}_+ \). The vanishin loci of \( \text{Disc}_+ \) is a principal divisor on \( \mathfrak{c}_+ \) which we call extended discriminant divisor and denote by \( \mathfrak{D}_+ \).

From the definition, we see that \( \text{Disc}_+ \) is an eigenfunction for the \( Z_+ \)-action on \( T_+ \) and \( \mathfrak{c}_+ \), with eigen-value \( 2\rho \). Hence the subschemes \( \mathfrak{D}_+ \) is \( Z_+ \)-invariant.

For \( t_+ = (t, t^{-1}) \in T_+ \subset T_+ \), we have

\[
\text{Disc}_+(t_+) = 2\rho(t) \prod_{\alpha \in \Phi_+}(1 - \alpha(t))(1 - \alpha(t^{-1})) = (-1)^{||\Phi_+||} \prod_{\alpha \in \Phi_+}(1 - \alpha(t))^2
\]

(2.4)

For each \( \alpha \in \Phi_+ \), \( \text{Disc}_+(t_+) = (1 - \alpha(t))^2 \) extends to a polynomial function on \( T_+ \) such that \( \mathfrak{D}_+ \cong \mathbb{A}^r \).

2.2.11. Adjoint orbits in extended Steinberg fibre. An element \( \gamma \in \text{Vin}_G \) is called semisimple if it is \( G \)-conjugate to an element in \( T_+ \). Let \( \text{Vin}_G^{\text{reg}} \) be the subset of \( \text{Vin}_G \) consisting of elements that are both regular and semisimple.

Lemma 2.2.12. The centralizer of any semisimple element \( \gamma \in \text{Vin}_G \) in \( G \) is a Levi subgroup of \( G \).

Proof. We may assume that \( \gamma \in T_+ \) so that \( \gamma = te_{1,J} \) for some \( t \in T_+ \) and idempotent \( e_{1,J} \).

For any \( g \in G_+ \), we have \( g\gamma g^{-1} = \gamma \) if and only if

\[
t^{-1}g e_{1,J} g^{-1} = e_{1,J}.
\]
By the description of the stabilizer of $e_{I,J}$ under the action of $G_+ \times G_+$, we see that $g \in (G_+)_\gamma$ if and only if the following 2 conditions are satisfied:

- $(t^{-1}gt, g) \in P_M \times P_{M_i}$;
- $\delta(t^{-1}gt)\delta(-g)^{-1} \in (L_{J^c})_{\text{det}}T_{I,J}$.

Here $M := I \cap J^0 \cup J^c$. Since $t \in L_M$, the first condition implies that $g \in L_M$. Since the roots in $I \cap J^0$ and $J^c$ are orthogonal to each other, the second condition implies that $(G_+)_\gamma$ is the subgroup of $L_M$ generated by $T_+, L_{J^c}$ and the centralizer of $t$ in $L_{I \cap J^0}$. This shows that $(G_+)_\gamma$ is a Levi subgroup of $G_+$ and hence $G_\gamma$ is a Levi subgroup of $G$.

\textbf{Lemma 2.2.13.} For any closed point $c \in \mathcal{C}_+$, the fibre $\chi_+^{-1}(c)$ is connected and equidimensional of dimension $\dim G - r$. The open $\text{Ad}(G)$-orbits in $\chi_+^{-1}(c)$ are precisely the regular conjugacy classes in $\chi_+^{-1}(c)$. On the other hand, there is a unique closed $\text{Ad}(G)$-orbit in $\chi_+^{-1}(c)$ which is also the unique semisimple conjugacy class in $\chi_+^{-1}(c)$.

\textbf{Proof.} By Corollary 2.2.6, $\chi_+$ is flat. Hence $\chi_+^{-1}(c)$ is equidimensional of dimension $\dim G - r$. Since $\chi_+$ is the invariant quotient by the reductive group $G$, there is a unique closed orbit in $\chi_+^{-1}(c)$. This closed orbit is connected since $G$ is connected. Consequently $\chi_+^{-1}(c)$ is also connected.

The regular conjugacy classes in $\chi_+^{-1}(c)$ are locally closed subsets of the same dimension as $\chi_+^{-1}(c)$. Hence they are precisely the open $\text{Ad}(G)$-orbits in $\chi_+^{-1}(c)$.

Finally by [Ren88], closed $\text{Ad}(G)$-orbits are precisely the semisimple conjugacy classes. \hfill $\square$

Unlike the group case, there might be more than one regular conjugacy class in an extended Steinberg fibre $\chi_+^{-1}(c)$, as we see in Proposition 2.2.8 for the nilpotent cone $\mathcal{N} = \chi_+^{-1}(0)$. On the other hand, regular semisimple conjugacy classes are the only $\text{Ad}(G)$ orbit in the extended Steinberg fibre they live in. We give another characterization of regular semisimple conjugacy classes using the discriminant function $\text{Disc}_+$. The following is a generalization of [Bou15a, 2.19]

\textbf{Proposition 2.2.14.} Denote $T_{+}^{\text{reg}} := T_+ \cap \text{Vin}_G^{\text{reg}}$. For any $\gamma \in T_+$, the following are equivalent:

1. $\gamma \in T_+^{\text{reg}}$;
2. $\text{Disc}_+(\gamma) \neq 0$;
3. The map $q : T_+ \to \mathcal{C}_+$ is étale at $\gamma$;
4. $G_\gamma = T$.

\textbf{Proof.} (1)⇒(2): Suppose $\gamma \in T_+^{\text{reg}}$. By Corollary 2.2.7, we have $\gamma \in \text{Vin}_G^0 \cap T_+$. After conjugation and multiplying by the center $Z_+$, we may assume that $\gamma \in T_\text{diag}$. If $\text{Disc}_+(\gamma) = 0$, then there exists $\alpha \in \Phi_+$ such that $D_\alpha(\gamma) = 0$. This implies that $\gamma$ lies in the closure of the diagonal embedded $\ker(\alpha)$. Since the centralizers of elements in $\ker(\alpha)$ have dimension at least $r+1$, the same is true for $G_\gamma$ by upper semicontinuity of centralizer dimension. This contradicts the assumption that $\gamma$ is regular and we must have $\text{Disc}_+(\gamma) \neq 0$.

(1)⇔(3)⇒(4): Since $\mathcal{C}_+ = T_+//W$, the finite cover $q : T_+ \to \mathcal{C}_+$ is étale at $\gamma$ if and only if the stabilizer of $\gamma$ in $W$ is trivial, which is equivalent to the fact $G_\gamma = T$ since $G_\gamma$ is a standard Levi subgroup of $G$ by the proof of Lemma 2.2.12.

(2)⇒(1): Let $V \subset T_+$ be the open subset where $\text{Disc}_+$ is nonzero and we need to show that $V = T_+^{\text{reg}}$. In the implication “(1)⇒(2)” we proved that $T_+^{\text{reg}} \subset V$.

Consider the stratification of $T_+$ induced by the $T_\text{ad}$-orbits on $A_G = A^r$. The open strata is $T_+$, the unit group of $T_+$. The codimension 1 strata are described as follows: for each $1 \leq i \leq r$, let $O_i$ be the codimension 1 strata consisting of $x \in T_+$ such that the $i$-th coordinate of $\alpha(x)$ vanishes and the other coordinates are nonzero. Consider the complement $F := V \setminus T_+^{\text{reg}}$, which is a closed subset of $V$. It is a classical fact that $F \cap T_+ = \emptyset$. Also, we have $e_{0,1} \in T_+^{\text{reg}}$ by direct calculation of its centralizer. Hence $e_{0,1}$ lies in the closure $\overline{O}_1$ for all $1 \leq i \leq r$. This shows that the generic point of $O_i$ lies in $T_+^{\text{reg}}$ for all $i$, which implies that $F$ has codimension at least 2 in $T_+$. By the equivalence “(1)⇔(3)” we just proved and purity of branch locus (see, for example [Sta17, Tag 0BMB]), the complement $T_+ \setminus T_+^{\text{reg}}$ is pure of codimension 1 in $T_+$. This forces $F$, an open subset of $T_+ \setminus T_+^{\text{reg}}$ to be empty and hence $V = T_+^{\text{reg}}$. \hfill $\square$

\textbf{Corollary 2.2.15.} $\text{Vin}_G^{\text{reg}} = \chi_+^{-1}(\mathcal{C}_+ \setminus D_+)$. Moreover, $G$ acts transitively on each fibre of $\chi_+$ over $\mathcal{C}_+ \setminus D_+$. 
Proof. By Proposition 2.2.14, we have $\text{Vin}^G_C \subset \chi^{-1}_+(\mathcal{C}_+ \setminus \mathcal{D}_+)$. Let $c \in \mathcal{C}_+ \setminus \mathcal{D}_+$. By Lemma 2.2.13 and Proposition 2.2.14, the unique closed orbit in $\chi^{-1}_+(c)$ is also open. Hence $\chi^{-1}_+(c)$ is a single $\text{Ad}(G)$-orbit consisting of elements that are both regular and semisimple. This proves the inverse inclusion. □

For this reason, we denote $\mathcal{C}_+^c := \mathcal{C}_+ \setminus \mathcal{D}_+$ and call it the regular semisimple open subset of $\mathcal{C}_+$.

2.2.16. Extended Steinberg section. For each $S$-Coxeter element $w \in \text{Cox}(W, S)$ (cf. Definition 2.2.1), each choice of representatives $\tilde{s}_i \in N_G(T)$ of the simple roots $s_i$, Steinberg defines a section $\varepsilon^w : \mathcal{C}_G \rightarrow G$ of the adjoint quotient map $\chi_G : G \rightarrow \mathcal{C}_G$. Moreover, it is shown that the equivalence class of $\varepsilon^w$ depends neither on $w$ nor the choices $\tilde{s}_i$, see [Ste65, 7.5 and 7.8]. Here we say that two sections $\varepsilon, \varepsilon'$ are equivalent if for all $a \in \mathcal{C}_G$, $\varepsilon(a)$ and $\varepsilon'(a)$ are conjugate under $G$.

Following [Bou15a], we extend the Steinberg sections $\varepsilon^w$ to the Vinberg monoid $\text{Vin}_G$ as follows. For each $(b, a) \in \mathcal{C}_+ \cong \mathbb{A}^{2r}$ where $b \in A_G \cong \mathbb{A}^r$, define a map

$$
\varepsilon^w_+ : \mathcal{C}_+ \rightarrow \text{Vin}_G
$$

by $\varepsilon^w_+(b, a) := \varepsilon^w(a) \sigma(b)$ where $\sigma : A_G \rightarrow \text{Vin}_G$ is the section of the abelianization map $\alpha$ defined in § 2.1.

Proposition 2.2.17. The map $\varepsilon^w_+$ is a section of the adjoint quotient $\chi_+ : \text{Vin}_G \rightarrow \mathcal{C}_+$. Moreover, the image of $\varepsilon^w_+$ is contained in $\text{Vin}_G^{\text{reg}}$.

Proof. The first statement is [Bou15a, Proposition 1.10]. The second statement is Proposition 1.16 in loc. cit. □

Remark 2.2.18. For each $w \in \text{Cox}(W, S)$, the equivalence class of the extended section $\varepsilon^w_+$ is independent of the choice of representatives $\tilde{s}_i$ of the simple reflections. However, for two different $w, w' \in \text{Cox}(W, S)$, the sections $\varepsilon^w_+$ and $\varepsilon^{w'}_+$ are not equivalent since, as we will see, $\varepsilon^w_+(0)$ and $\varepsilon^{w'}_+(0)$ are not conjugate.

Next we examine the interaction of the extended Steinberg section $\varepsilon^w_+$ with the action of the central torus $Z_+^c$.

To this end, we drop the semisimple simply connected assumption and allow $G$ to be any connected reductive group. Then the adjoint action of $G_{\text{ad}}$ on $\text{Vin}_{G^{\text{reg}}}$ induces an action of $G$ on $\text{Vin}_{G^{\text{reg}}}$ which we also denote by “$\text{Ad}$”. Let $\mathcal{C}_+ = \text{Vin}_{G^{\text{reg}}}/\text{Ad}(G) = \text{Vin}_{G^{\text{reg}}}/\text{Ad}(G^{\text{reg}})$ be the extended Steinberg base for $\text{Vin}_{G^{\text{reg}}}$. The central torus $Z_+^{\text{reg}} = T^{\text{reg}}$ acts naturally on $\text{Vin}_{G^{\text{reg}}}$ and $\mathcal{C}_+$ such that the morphism $\chi_+ : \text{Vin}_{G^{\text{reg}}} \rightarrow \mathcal{C}_+$ is $T^{\text{reg}}$-equivariant. Hence $\chi_+$ induces a morphism between stacks

$$
[\chi_+] : [\text{Vin}_{G^{\text{reg}}}/(\text{Ad}(G) \times T^{\text{reg}})] \rightarrow [\mathcal{C}_+/T^{\text{reg}}]
$$

We would like to see if $\varepsilon^w_+$ induces a section $[\chi_+]$. It turns out that this is not true in general. To remedy it we consider the homomorphism $\psi : T^{\text{reg}} \rightarrow G_{\text{ad}}$ defined as the following composition

$$
\psi : T^{\text{reg}} \xrightarrow{\omega_*} \mathbb{G}_m^r \xrightarrow{\sigma} G_+^{\text{reg}} \rightarrow G_{\text{ad}}
$$

where the first arrow is $\omega_* := (\omega_1, \ldots, \omega_r)$, the second arrow is induced by the canonical section of the abelianization $\alpha$ (cf. Equation 2.1) and the third arrow is the canonical quotient morphism.

Consider the action of $T^{\text{reg}} \times T^{\text{reg}}$ on $\text{Vin}_{G^{\text{reg}}}$ where the first copy of $T^{\text{reg}}$ acts by composing $\psi$ with the adjoint action of $G_{\text{ad}}$ and the second copy of $T^{\text{reg}}$ acts as central torus. In [Bou15a, Proposition 1.11], by examining the action on weight vectors of fundamental representations, it is shown that for all $a_+ \in \mathcal{C}_+$ and $z \in T^{\text{reg}}$ we have

$$
\varepsilon^w_+(z \cdot a_+) = z \cdot \sigma(\omega_*(z)) \varepsilon^w_+(a_+) \sigma(\omega_*(z))^{-1}
$$

This shows that $\varepsilon^w_+$ is equivariant with respect to the diagonal embedding $T^{\text{reg}} \rightarrow T^{\text{reg}} \times T^{\text{reg}}$ and hence induces a morphism

$$
[\mathcal{C}_+/T^{\text{reg}}] \rightarrow [\text{Vin}_{G^{\text{reg}}}/\psi(T^{\text{reg}}) \times T^{\text{reg}}]
$$

If $G = G_{\text{ad}}$, then this leads to a section $[\varepsilon^w_+]$ of $[\chi_+]$. In general, let $c = |Z(G_{\text{der}})|$ be the order of the center of the derived group $G_{\text{der}}$. Then by extracting $c$-th roots, we would get a lifting $\psi[c] : T^{\text{reg}} \rightarrow G_{\text{der}} \subset G$ of $\psi$. 

More precisely, $\psi|_c$ is defined by the following commutative diagram

$$
\begin{array}{ccc}
T^{\text{sc}} & \xrightarrow{\psi|_c} & G \\
\downarrow{c} & & \downarrow{\chi|_c} \\
T^{\text{sc}} & \xrightarrow{\psi} & G_{\text{ad}}
\end{array}
$$

where the left vertical map is raising to $c$-th power.

The $c$-th power map $T^{\text{sc}} \to T^{\text{sc}}$ induces a morphism between classifying stacks $\mathbb{B}T^{\text{sc}} \to \mathbb{B}T^{\text{sc}}$. Base changing $[\chi_+]$ along this map, we obtain a Cartesian diagram

$$
\begin{array}{ccc}
[Vin_{G^{\text{sc}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}]_[c] & \xrightarrow{[\chi_+]} & [Vin_{G^{\text{reg}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}]_[c] \\
\downarrow{[\chi_+]_c} & & \downarrow{[\chi_+]_c} \\
[\mathcal{X}_+/T^{\text{sc}}]_[c] & \xrightarrow{[\chi_+]_c} & [\mathcal{X}_+/T^{\text{sc}}]
\end{array}
$$

where on the left, the $T^{\text{sc}}$ action is the composition of the $c$-th power map and the usual action.

**Proposition 2.2.19.** The map $\epsilon^w_n$ induces a section $\epsilon^w_n, [\chi_+]: [\mathcal{X}_+/T^{\text{sc}}]_[c] \to [Vin_{G^{\text{sc}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}]_[c]$ of $[\chi_+]_c$ whose image lies in the open substack

$$[Vin_{G^{\text{reg}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}]_[c].$$

**Proof.** By what we have discussed, $\epsilon^w_n$ induces a morphism

$$[\mathcal{X}_+/T^{\text{sc}}]_[c] \to [Vin_{G^{\text{sc}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}] [c]$$

where on the right, the second copy of $T^{\text{sc}}$ acts by composing the $c$-th power map and the usual action. Since $\psi|_c(T^{\text{sc}}) \subset G$, there is a canonical morphism

$$[Vin_{G^{\text{sc}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}] [c] \to [Vin_{G^{\text{reg}}/\mathcal{Y}(\text{Ad}(G) \times T^{\text{sc}})}] [c].$$

Composing the two morphisms above we obtain the morphism $\epsilon^w_n, [\chi_+|$ with the desired property. \qed

### 2.3. Regular centralizer for the group.

In this section we let $(G, G')$ be a pair of connected reductive groups equipped with an isomorphism of their derived groups $G_{\text{ad}} \cong G'_{\text{ad}}$. Assume moreover that the derived group of $G$ is simply connected. Then there is a natural adjoint action of $G'$ on $G$ and the action factors through $G'_{\text{ad}} \cong G_{\text{ad}}$. Let $\mathcal{E}_G := G/\mathcal{Y}(\text{Ad}(G'))$ be the invariant quotient. Then there is a canonical isomorphism $\mathcal{E}_G \cong T/W$. The natural map $T \to \mathcal{E}_G$ is finite flat and its restriction to $\mathcal{E}_G^{\text{reg}}$ is a Galois étale cover with Galois group $W$.

Consider the centralizer group scheme $I_G$ over $G$ defined by

$$I_G := \{(g, x) \in G \times G | \text{Ad}(g)x = x\}.$$  

In other words, the fiber of $I_G$ over $x \in G$ is the centralizer $G_x$ of $x$ in $G'$. Since the derived group of $G$ is simply connected, $G_x$ is connected for semisimple $x \in G$. If moreover $x \in G^{rs}$ is regular semisimple, then $G'_x$ is a maximal torus in $G'$. More generally, the restriction $I_{G^{\text{reg}}}|_{G^{\text{reg}}}$ to the regular open subscheme $G^{\text{reg}}$ is a smooth commutative group scheme of relative dimension $\dim(T)$. The following lemma is the group version of [Ngô10, Lemme 2.1.1]

**Lemma 2.3.1.** There exists a unique smooth commutative group scheme $J_{G'}$ over $\mathcal{E}_G$ such that we have a $G'$-equivariant isomorphism

$$(\chi^*J_{G'})_{G^{\text{reg}}} \cong I_{G'}|_{G^{\text{reg}}}.$$  

Moreover, this isomorphism extends uniquely to a homomorphism $\chi^*J_{G'} \to I_{G'}$.

**Proof.** The proof of [Ngô10, Lemme 2.1.1] generalize verbatim to our situation. For the last statement, we use the fact that the complement of $G^{\text{reg}}$ in $G$ has codimension at least 2, c.f. [Ste65]. \qed

Fix a maximal torus $T' \subset G'$. Consider the Weil restriction of the torus $T' \times T$ on $T$ to $\mathcal{E}_G$:

$$\Pi_G := \Pi_{T/\mathcal{E}_G}(T' \times T).$$  

In other words, for any \( \mathcal{C} \)-scheme \( S \), we have
\[
\Pi_G(S) = \text{Hom}_T(S \times_G T, T' \times T)
\]
The diagonal action of \( W \) on \( T' \times T \) induces an action of \( W \) on \( \Pi_G \). The fixed point subscheme of \( \Pi_G^W \) is a closed smooth subscheme of \( \Pi_G \) since the characteristic of the base field does not divide the order of \( W \).

**Proposition 2.3.2.** There exists a canonical open embedding \( J_G \to \Pi_G^W \).

**Proof.** We follow the argument for the Lie algebra case in [Ngô10, §2.4]. First we define a morphism \( J \to \Pi_G^W \).

By adjunction, this is the same as giving a morphism \( q^*J \to T \times T \) where \( q : T \to \mathcal{C} \) and we view \( T \times T \) as a constant group scheme over \( T \). One construct this morphism by descent along the smooth morphism \( \tilde{\chi} : G \to T \) which sits in the Cartesian diagram

\[
\begin{array}{ccc}
\tilde{\chi} & \longrightarrow & G \\
\downarrow & & \downarrow \\
q & \longrightarrow & \mathcal{C}_G
\end{array}
\]

Hence it suffices to construct a \( G \)-equivariant morphism \( (\tilde{\chi}^{\text{reg}})^* q^* J_G \to T \times G^{\text{reg}} \). The upshot is that for all \( x \in G \) and Borel subgroup \( x \in B \subset G \), we have \( I_r \subset B \) by the argument of [Ngô10, Lemme 2.4.3]. Hence when composed with the quotient \( B \to T \), we obtain a map \( I_r \to T \) depending on the choice of Borel \( B \) containing \( x \). Thus we get the desired morphism \( (\tilde{\chi}^{\text{reg}})^* q^* J_G \cong q^* I_{\text{reg}} \to T \times G^{\text{reg}} \) which is \( G \)-equivariant by construction.

To show that the morphism \( J_G \to \Pi_G^W \) constructed above is an isomorphism, it suffices to show the isomorphism over an open subset of \( \mathcal{C} \) whose complement has codimension at least 2.

For each simple root \( \alpha \in \Phi^+ \), let \( T_\alpha \) be the kernel of \( \alpha \), which is a subscheme of codimension 1 in \( T \). Then the discriminant divisor \( \mathcal{D} \subset \mathcal{C} \) is the union of \( q(T_\alpha) \) for all simple root \( \alpha \). Let \( T_\alpha^c \subset T _\alpha \) be the open subscheme consisting of points that does not lie in \( T_\beta \) for any \( \beta \neq \alpha \). Then

\[
\mathcal{C}^c \cup \left( \bigcup_{\alpha \in \Phi^+} q(T_\alpha^c) \right)
\]

is an open subset of \( \mathcal{C} \) whose complement has codimension 2. It follows from construction that it is an isomorphism over \( \mathcal{C}^c \). Hence it remains to show that \( J_G \to \Pi_G^W \) is an isomorphism when restricted to \( q(T_\alpha^c) \) for each positive root \( \alpha \).

Let \( t \in T_\alpha^c \) and we will show that \( J \to \Pi_G^W \) is an isomorphism in an étale neighbourhood of \( t \). Let \( G_\alpha \) be the centralizer of \( T_\alpha \) in \( G \) and \( \mathcal{C}_{G_\alpha} \) its adjoint quotient. Then the natural morphism \( \pi_\alpha : \mathcal{C}_{G_\alpha} \to \mathcal{C} \) is étale in a neighbourhood of \( q_\alpha(t) \) where \( q_\alpha : T \to \mathcal{C}_{G_\alpha} \) is the natural map. This implies that in an étale neighbourhood of \( q_\alpha(t) \) the group schemes \( \Pi_G^W \times_{\mathcal{C}} \mathcal{C}_{G_\alpha} \) and \( \Pi_{G_\alpha}^W \) are isomorphic.

There is a natural open embedding \( G^{\text{reg}} \cap G_\alpha \subset G_{\alpha}^{\text{reg}} \). Consider the open subset

\[
\mathcal{C}_{G_\alpha}^{G^{\text{reg}}} := \chi_{G^{\text{reg}}} \cap G_\alpha
\]

As \( t \in T_\alpha^c \), one can choose a unipotent element \( u \in G_\alpha \) such that \( tu \in G^{\text{reg}} \cap G_\alpha \). In particular, \( q_\alpha(t) \in \mathcal{C}_{G_\alpha}^{G^{\text{reg}}} \).

It is clear that

\[
I_{G_\alpha}|_{G^{\text{reg}} \cap G_\alpha} \cong I_G|_{G^{\text{reg}} \cap G_\alpha}
\]

This implies that \( (\pi_\alpha J_G)|_{\mathcal{C}_{G_\alpha}^{G^{\text{reg}}}} \cong (J_{G_\alpha})|_{\mathcal{C}_{G_\alpha}^{G^{\text{reg}}}} \).

In summary, the base change of \( J_G \) and \( \Pi_G^W \) to an étale neighbourhood of \( q(t) \) are isomorphic to the corresponding groups defined for the group \( G_\alpha \). Note that by assumption, \( G_\alpha \) is of rank 1 and has semisimple derived group, thus isomorphic to the product of a torus with either \( GL_2 \) or \( SL_2 \). So we are finally reduced to the case of \( GL_2 \) and \( SL_2 \), on which the isomorphism follows by direct calculation.

\( \square \)

2.4. **Regular centralizer for Vinberg monoid.** In this section we let \( G \) be an arbitrary connected reductive group over \( k \). Let \( G^{\text{sc}} \) be the simply-connected cover of its derived group. Then there is a natural adjoint action of \( G \) on \( \text{Vin}_{G^{\text{sc}}} \) and the action factors through \( G_{\text{ad}} \).

Consider the centralizer group scheme \( I \) over \( \text{Vin}_{G^{\text{sc}}} \) defined by

\[
I = \{(g, \gamma) \in G \times \text{Vin}_{G^{\text{sc}}} | \text{Ad}(g) \gamma = \gamma \}
\]
Then $\mathcal{I}|_{\text{Vin}_{G_{sc}}^{\text{reg}}}$ is smooth of relative dimension $r$. By [Ren88], the fibres of $\mathcal{I}$ over $\text{Vin}_{G_{sc}}^{\text{reg}}$ are maximal tori in $G$. In particular, $\mathcal{I}|_{\text{Vin}_{G_{sc}}^{\text{reg}}}$ is commutative. Hence $\mathcal{I}|_{\text{Vin}_{G_{sc}}^{\text{reg}}}$ is also commutative.

2.4.1. **Open cover of regular locus.** For each $w \in \text{Cox}(W, S)$, define $\mathcal{J}^w := (\chi^w)^{-1}\mathcal{I}$. Then $\mathcal{J}^w$ is a smooth commutative group scheme over $\mathcal{C}_+$. The morphism

$$
c_w : G \times \mathcal{C}_+ \longrightarrow \text{Vin}_{G_{sc}}^{\text{reg}}$$

$$
(g, a) \longmapsto gc_w(a)g^{-1}
$$

factors through $(G \times \mathcal{C}_+)/\mathcal{J}^w$ and induces a quasi-finite morphism

$$
c_w : (G \times \mathcal{C}_+)/\mathcal{J}^w \rightarrow \text{Vin}_{G_{sc}}^{\text{reg}}
$$

Since $c_w$ is an isomorphism over $G_{sc, \text{reg}}^{\text{sc}}$, it is birational. Since $\text{Vin}_{G_{sc}}^{\text{reg}}$ is normal, $c_w$ is an open embedding by Zariski Main Theorem.

Denote by $\text{Vin}_{G_{sc}}^{w}$ the image of $c_w$, which is an open subscheme of $\text{Vin}_{G_{sc}}^{\text{reg}}$. The union $U := \bigcup_{w \in \text{Cox}(W, S)} \text{Vin}_{G_{sc}}^{w}$ is a $\mathbb{Z}_{\text{sc}}$-stable open subset of $\text{Vin}_{G_{sc}}^{\text{reg}}$. By Proposition 2.2.8, it coincides with $\text{Vin}_{G_{sc}}^{\text{reg}}$ over $0 \in \mathcal{C}_+$. Hence it equals to $\text{Vin}_{G_{sc}}^{\text{reg}}$. In other words, the sets $\text{Vin}_{G_{sc}}^{w}$ form an open cover of $\text{Vin}_{G_{sc}}^{\text{reg}}$:

$$(2.7) \quad \text{Vin}_{G_{sc}}^{\text{reg}} = \bigcup_{w \in \text{Cox}(W, S)} \text{Vin}_{G_{sc}}^{w}$$

We generalize Lemma 2.3.1 to $\text{Vin}_{G_{sc}}^{w}$:

**Lemma 2.4.2.** There is a unique smooth commutative group scheme $\mathcal{J}$ over $\mathcal{C}_+$ such that we have a $G$-equivariant isomorphism $(\chi^w_{G_{sc}})^{-1} \mathcal{J} \cong \mathcal{I}|_{\text{Vin}_{G_{sc}}^{\text{reg}}}$. Moreover, this isomorphism extends uniquely to a homomorphism $\chi_+^w \mathcal{J} \rightarrow \mathcal{I}$.

**Proof.** By the same argument as Lemma 2.3.1, for each $w \in \text{Cox}(W, S)$, $\mathcal{J}^w$ is the unique commutative smooth group scheme over $\mathcal{C}_+$ such that

$$(\chi^w_{G_{sc}})^{-1} \mathcal{J} \cong \mathcal{I}|_{\text{Vin}_{G_{sc}}^{w}}$$

Next we show that for any $w, w' \in \text{Cox}(W, S)$, the group schemes $\mathcal{J}^w$ and $\mathcal{J}^{w'}$ are canonically isomorphic. It suffices to show that they are canonically isomorphic over certain open subset whose complement has codimension at least 2. From Lemma 2.3.1, we have the isomorphism over the open subset $\mathcal{C}_+^w$. Over $\mathcal{C}_+^w$, each fiber of $\chi_+$ consists of a single $\text{Ad}(G)$ orbit by Lemma 2.2.13. In other words, $G$ acts transitively on each fiber of $\chi_+$ over $\mathcal{C}_+^w$. Hence $\text{Vin}_{G_{sc}}^{w} \subset \text{Vin}_{G_{sc}}^{\text{reg}}$ for all $w \in \text{Cox}(W, S)$. Thus by uniqueness of $\mathcal{J}^w$ we see that $\mathcal{J}^w$ and $\mathcal{J}^{w'}$ are isomorphic over $\mathcal{C}_+^w$.

The complement of $\mathcal{C}_+^w$ is the union of the closure of codimension 1 strata in $\mathcal{C}_+$. Since the idempotent $e_G$ is regular semisimple and belongs each of the strata closure we see that on each strata, the regular semisimple locus is nonempty open. Hence the complement of $\mathcal{C}_+^w \subset \mathcal{C}_+^w \subset \mathcal{C}_+$ has codimension at least 2.

Consequently there is a unique commutative smooth group scheme $\mathcal{J}$ over $\mathcal{C}_+$ which comes with a unique isomorphism $(\chi^w_{G_{sc}})^{-1} \mathcal{J} \cong \mathcal{I}|_{\text{Vin}_{G_{sc}}^{\text{reg}}}$. We know from Lemma 2.3.1 that this isomorphism extends uniquely to a homomorphism between $\chi_+^w \mathcal{J}$ and $\mathcal{I}$ over the open subset $G_{sc}^w \subset \text{Vin}_{G_{sc}}^{\text{reg}}$ whose complement has codimension at least 2. Hence it extends further to the whole space $\text{Vin}_{G_{sc}}^{\text{reg}}$. □

**Proposition 2.4.3.** The classifying stack $B\mathcal{J}$ acts naturally on $[\text{Vin}_{G_{sc}}^{\text{reg}}/\text{Ad}(G)]$. The action preserves the open substacks $[\text{Vin}_{G_{sc}}^{0}/\text{Ad}(G)]$, $[\text{Vin}_{G_{sc}}^{\text{reg}}/\text{Ad}(G)]$ and $[\text{Vin}_{G_{sc}}^{w}/\text{Ad}(G)]$ for each $w \in \text{Cox}(W, S)$. Moreover, the morphism

$$[\chi^w_{G_{sc}}] : [\text{Vin}_{G_{sc}}^{w}/\text{Ad}(G)] \rightarrow \mathcal{C}_+
$$

induced by $\chi_+$ is a $B\mathcal{J}$ gerbe, neutralized by the extended Steinberg section $e_+^w$.

The proof is the same as [Ngô10, Proposition 2.2.1].

**Proposition 2.4.4.** The number of irreducible components of the fibers of the map

$$\chi^w_{G_{sc}} : \text{Vin}_{G_{sc}}^{\text{reg}} \rightarrow \mathcal{C}_+
$$

is bounded above by $|\text{Cox}(W, S)|$ and equality is achieved at $N^{\text{reg}} = (\chi^w_{G_{sc}})^{-1}(0)$. 
Proof. The first statement follows from (2.7). The second statement is in Proposition 2.2.8. \qed

Remark 2.4.5. Consequently, unless all simple factors of $G^{sc}$ are SL$_2$, the action of $\mathcal{B}J$ on $[\text{Vin}^{\text{reg}}_{G^{sc}}/\text{Ad}(G)]$ is not transitive. In other words, $[\text{Vin}^{\text{reg}}_{G^{sc}}/G]$ is not a $\mathcal{B}J$-gerbe, but rather a finite union of $\mathcal{B}J$-gerbes as in Proposition 2.4.3. This is different from Lie algebra situation, cf [Ngô10, Proposition 2.2.1].

2.4.6. Galois description of universal centralizer. Let $\prod_{T_+/\mathcal{E}+} (T \times T_+)$ be the restriction of scalar which associates to any $\mathcal{E}+$-scheme $S$ the set

$$\prod_{T_+/\mathcal{E}+} (T \times T_+)(S) = \text{Hom}_{T_+/\mathcal{E}+}(S \times \mathcal{E}+, T \times T_+)$$

Then $W$ acts diagonally on $\prod_{T_+/\mathcal{E}+} (T' \times T_+)$ and we consider its fixed point subscheme

$$J^1 := (\prod_{T_+/\mathcal{E}+} T \times T_+)^W.$$ 

The following is proved in [Bou17, Proposition 11].

Proposition 2.4.7. $J^1$ is a smooth commutative group scheme over $\mathcal{E}+$. Moreover, there exists an open embedding $J \rightarrow J^1$ whose restriction to $\mathcal{E}+_{pr}$ is an isomorphism.

2.5. Arc space of Vinberg monoid. In this section, we assume that $G$ is semisimple and simply connected.

For each $\lambda \in X_*(\text{Ad})_+$, define the affine scheme $\text{Vin}^\lambda_G$ over Spec $\mathcal{O}$ by the following Cartesian diagram

$$\begin{array}{ccc}
\text{Vin}^\lambda_G & \longrightarrow & \text{Vin}_G \times \text{Ad} \\
\downarrow & & \downarrow \\
\text{Spec} \mathcal{O} & \xrightarrow{\mathcal{O}^{-w_0(\lambda)}} & A_G
\end{array}$$

where the right vertical arrow is the product of the abelianization map $\alpha_G$ and the natural embedding $\text{Ad} \hookrightarrow A_G$, the lower horizontal arrow corresponds to the point $\mathcal{O}^{-w_0(\lambda)} \in A_G(\mathcal{O})$. Replacing $\text{Vin}_G$ by its open subscheme $\text{Vin}^\lambda_G$ in the above diagram, we define an open subscheme $\text{Vin}^{\lambda,0}_G \subset \text{Vin}^\lambda_G$.

There is a stratification of the space of nondegenerate arcs of $A_G \supset \text{Ad}$ by $\text{Ad}((\mathcal{O})$ orbits:

$$A_G(\mathcal{O}) \cap \text{Ad} = \bigsqcup_{\lambda \in X_*(\text{Ad})_+} \text{Ad}(\mathcal{O}) \mathcal{O}^{-w_0(\lambda)}$$

The inverse image of $\text{Ad}(\mathcal{O}) \mathcal{O}^{-w_0(\lambda)}$ under the abelianization map is precisely $L^+\text{Vin}^\lambda_G(k)$. In other words we get a stratification of the space of nondegenerate arcs of $\text{Vin}_G \supset G_+$ into $G_+(\mathcal{O})$-stable pieces:

$$\text{Vin}_G(\mathcal{O}) \cap G_+(F) = \bigsqcup_{\lambda \in X_*(\text{Ad})_+} L^+\text{Vin}^\lambda_G(k)$$

Also we note that

$$L^+\text{Vin}^{\lambda,0}_G(k) = L^+\text{Vin}^\lambda_G(k) \cap \text{Vin}^0_G(\mathcal{O}).$$

Lemma 2.5.1. For any $g_+ \in G_+(F)$, we have $g_+ \in L^+\text{Vin}^\lambda_G$ if and only if $\alpha(g_+) \in \mathcal{O}^{-w_0(\lambda)}\text{Ad}(\mathcal{O})$ and the image of $g_+$ in $\text{Ad}(F)$ belongs to $G_+(\mathcal{O}) \mathcal{O}^{-w_0(\lambda)}G_+(\mathcal{O}) = \bigsqcup_{\mu \in X_*(\text{Ad})_+} G_+(\mathcal{O}) \mathcal{O}^{-\mu}G_+(\mathcal{O})$.

Moreover, $g_+ \in L^+\text{Vin}^{\lambda,0}_G$ if and only if $\alpha(g_+) \in \mathcal{O}^{-w_0(\lambda)}\text{Ad}(\mathcal{O})$ and the image of $g_+$ in $\text{Ad}(F)$ belongs to the double coset $G_+(\mathcal{O}) \mathcal{O}^{-\lambda}G_+(\mathcal{O})$.

Proof. The coweight lattice for $T_+$ can be expressed as

$$X_+(T_+) = \{ (\lambda_1, \lambda_2) \in X_*(\text{Ad}) \times X_*(\text{Ad}) | \lambda_1 + \lambda_2 \in X_*(T) \}$$

For $(\lambda_1, \lambda_2) \in X_+(T_+)$, we have $\mathcal{O}^{(\lambda_1, \lambda_2)} \in L^+\text{Vin}^\lambda_G$ if and only if

...
\begin{itemize}
\item $\alpha(\varpi^{(\lambda_1, \lambda_2)}) \in \varpi^{-w_0(\lambda)} T_{\text{ad}}(O)$
\item The matrix $\rho_{\lambda_i}^+(\varpi^{(\lambda_1, \lambda_2)}) \in \text{End}(V_{\omega_i})$ has entries in $O$ for all $1 \leq i \leq r$.
\end{itemize}
Since $\alpha(\varpi^{(\lambda_1, \lambda_2)}) = \varpi^{\lambda_1}$, the first condition means that $\lambda_1 = -w_0(\lambda)$. Then the second condition means that
\[
\langle (-w_0(\lambda), \lambda_2), \chi_+ \rangle \geq 0
\]
for all $1 \leq i \leq r$ and all weights $\chi_+$ in the $G_+$-representation $\rho_\lambda^+$. Since the weights of the representation $\rho_\lambda^+$ lie in the convex hull of the $W$-orbit of the highest weight $(\omega_i, \omega_i)$ where $W$ acts on the second factor, the above inequality is equivalent to
\[
\langle (-w_0(\lambda), \omega_i), (\lambda_2, w(\omega_i)) \rangle = \langle (-w_0(\lambda), \lambda_2), (\omega_i, w(\omega_i)) \rangle \geq 0
\]
for all $w \in W$ and $1 \leq i \leq r$. This can be further reformulated as
\[
\langle \lambda - w(\lambda_2), \omega_i \rangle \geq 0
\]
for all $w \in W$ and $1 \leq i \leq r$.

By the discussion so far, we have
\[
L^+ \text{Vin}_G^\lambda(k) \cap T_+ (F) = \bigcup_{\mu \in X_+ (T_{\text{ad}})} \varpi^{-w_0(\lambda)} \mu T_+ (O)
\]
where $\mu_{\text{dom}}$ denotes the unique dominant coweight in the $W$-orbit of $\mu$. As $L^+ \text{Vin}_G^\lambda$ is stable under the action of $G_+(O) \times G_+(O)$, it is a union of $G_+(O)$ double cosets in $G_+(F)$. Thus by Cartan decomposition we get
\[
L^+ \text{Vin}_G^\lambda(k) = \bigcup_{\mu \in X_+ (T_{\text{ad}})^+} G_+(O) \varpi^{-w_0(\lambda)} \mu G_+(O)
\]
Similarly we can get a description of $L^+ \text{Vin}_G^0$. The difference is that we require furthermore that $\rho_\lambda^+(\varpi^{(-w_0(\lambda), \lambda_2)})$ have nonzero reduction mod $\varpi$ for all $1 \leq i \leq r$. Hence besides the inequality $\langle \lambda - w(\lambda_2), \omega_i \rangle \geq 0$ for all $w \in W$ and $1 \leq i \leq r$, we require furthermore that for each $i$, there exists $w \in W$ such that $\langle \lambda - w(\lambda_2), \omega_i \rangle = 0$. This condition means that implies that $\lambda_2$ is in the $W$-orbit of $\lambda$ and hence
\[
L^+ \text{Vin}_G^\lambda_0(k) = G_+(O) \varpi^{-w_0(\lambda)} \lambda G_+(O).
\]
From these description the lemma follows. \hfill \square

**Lemma 2.5.2.** Suppose $n \geq b(\lambda) := \max_{1 \leq i \leq r} \langle \lambda, \omega_i - w_0(\omega_i) \rangle$. Then for all $\gamma, \gamma' \in L^+ \text{Vin}_G^\lambda(k)$ having the same image in $\text{Vin}_G^\lambda(O/\varpi^nO)$, there exists $g \in G_+(O)$ such that $\gamma' = \gamma g$.

**Proof.** The following argument is due to Zhiwei Yun. Let $i \mapsto i^*$ be the involution on the set $\{1, \ldots, r\}$ such that $\omega_{i^*} = -w_0(\omega_i)$. For each $1 \leq i \leq r$, there is a natural pairing between $V_i$ and $V_{i^*}$ such that for all $x \in G_+$, $v \in V_i$ and $v^* \in V_{i^*}$, we have
\[
\langle \rho_i^+(x) v, \rho_{i^*}^+(x) v^* \rangle = \langle \omega_i + \omega_{i^*} \rangle (\alpha(x)) \langle v, v^* \rangle.
\]
Thus for each $x \in G_+(F)$, under the natural pairing above, the lattice $\rho_i^+(x) V_i(O)$ in $V_i(F)$ is dual to the lattice
\[
\langle \omega_i + \omega_{i^*} \rangle (\alpha(x)^{-1}) \rho_{i^*}^+(g) V_{i^*}(O) \subset V_{i^*}(F).
\]
For $\gamma \in L^+ \text{Vin}_G^\lambda \subset \text{Vin}_G^\lambda(O)$, we have $\rho_i^+(\gamma) V_i(O) \subset V_i(O)$ for all $1 \leq i \leq r$. Taking duals, we get
\[
V_{i^*}(O) \subset \varpi^{-(\lambda, \omega_i + \omega_{i^*})} \rho_{i^*}^+(\gamma) V_{i^*}(O).
\]
In other words, we have shown that for all $1 \leq i \leq r$,
\[
\varpi^{\langle \lambda, \omega_i + \omega_{i^*} \rangle} V_i(O) \subset \rho_{i^*}^+(\gamma) V_i(O) \subset V_i(O).
\]
Thus if $\gamma$ and $\gamma'$ have the same image in $\text{Vin}_G^\lambda(O/\varpi^nO)$ for $n \geq b(\lambda)$, the lattices $\rho_i^+(\gamma) V_i(O)$ and $\rho_i(\gamma') V_i(O)$ are the same and hence $\gamma' = \gamma g$ for some $g \in G_+(O)$. \hfill \square

2.6. **Local lifting property of extended Steinberg map.** We review certain infinitesimal lifting property of the extended Steinberg morphism $\chi_+$ needed in § 4.4. This is based on some result of Gabber-Ramero in [GR03]. Our exposition below follow the treatment in [Bou17] and [Yun15b].
2.6.1. We start by recalling certain results in [GR03, §5.4]. Let $A$ be a ring and $B$ an $A$-algebra of finite presentation. Let $f : \text{Spec } B \to \text{Spec } A$ be the natural morphism. Choose a presentation $B \cong P/J$ where $P := A[X_1, \ldots, X_N]$ and $J \subset P$ is a finitely generated ideal. Then the map $f$ is factored as the composition of a closed embedding $i : \text{Spec } B \hookrightarrow \mathbb{A}^N_A$ and the natural projection $p : \mathbb{A}^N_A \to \text{Spec } A$. Define the ideal $H_A(P, J) := \text{Ann}_P \text{Ext}^1_B(\mathbb{L}_{B/A}, J/J^2) \subset P$

where $\mathbb{L}_{B/A}$ is the cotangent complex of the morphism $\chi$. Notice that $J \subset H_A(P, J)$.

Consider the ideal in $B$ defined by $H_{B/A} := H_A(P, J)B = H_A(P, J)/J$. Let $\Sigma_f := \text{Spec } B/H_{B/A}$ be the closed subscheme of $\text{Spec } B$ defined by $H_{B/A}$. We remark that $H_{B/A}$ depends on the choice of presentation $B \cong P/J$. The following is [GR03, Lemma 5.4.2]:

**Lemma 2.6.2.**

1. For all $B$-module $N$, $H_{B/A}$ annihilates $\text{Ext}^1_B(\mathbb{L}_{B/A}, N)$.
2. The complement of $\Sigma_f = \text{Spec } B/H_{B/A}$ in $\text{Spec } B$ is the smooth locus of the morphism $\chi : \text{Spec } B \to \text{Spec } A$.
3. For any $A$-algebra $A'$, let $B' = B \otimes_A A'$ and $f' : \text{Spec } B' \to \text{Spec } A'$ be the induced morphism. Define the ideal $H_{B'/A'} \subset B'$ in the same way as above, using the presentation of $B'$ induced from $B \cong P/J$. Then we have $H_{B/A}B' \subset H_{B'/A'}$, or in other words $\Sigma_f \subset \Sigma_f \times_Z \mathbb{Z}'$.

2.6.3. When $A = k[[\omega]]$, we define the conductor of $f$ to be the smallest integer $h$ such that $\omega^h \in H_{B/A}$. Note that the conductor depends on the presentation of $B$.

The next lemma is the key step in establishing the local lifting result. To state it we let $\sigma$ be the discriminant valuation of $\mathfrak{m}/\mathfrak{m}^2$. By Corollary 2.2.15, $\sigma$ is a principal divisor, there exists a positive integer $\sigma(\mathfrak{m})$ such that $\mathfrak{m}^{\sigma(\mathfrak{m})}$ is contained in $H_{B/A}$.

**Lemma 2.6.4.** Suppose $n \geq h$ and $\sigma : \text{Spec } A/\omega^n I \to \text{Spec } A$ is a morphism such that the composition $f \circ \sigma$ is the natural embedding $\text{Spec } A/\omega^n I \hookrightarrow \text{Spec } A$. Then there exists a section $\sigma : \text{Spec } A \to \text{Spec } B$ such that the restriction of $\sigma$ to $\text{Spec } A/\omega^{n-h} I$ coincides with $\sigma$.

**Proof.** The obstruction of extending $\sigma$ to $\text{Spec } A$ is an element $\omega \in \text{Ext}^1_B(\mathbb{L}_{B/A}, \omega^n I)$ where we view $\omega^n I$ as a $B$-module via the map $\sigma^* : B \to A/\omega^n I$. By definition of conductor $h$, we have $\omega^h \in H_{B/A} + \mathfrak{m}B$. By Lemma 2.6.2(1) and the assumption that $\mathfrak{m} \cdot I = 0$ we see that $\omega^h \omega = 0$. This implies that the image of $\omega$ in $\text{Ext}^1_B(\mathbb{L}_{B/A}, \omega^{n-h} I)$ vanishes by noticing that the multiplication map $\omega^n I \to \omega^n I$ can be factored as the composition of the natural embedding $\omega^n I \hookrightarrow \omega^{n-h} I$ and an isomorphism $\omega^{n-h} I \cong \omega^n I$. Hence we get the desired lifting of the restriction of $\sigma$ to $\text{Spec } A/\omega^{n-h} I$.

2.6.5. We apply the general discussion above to the situation where $\text{Spec } A = \mathcal{C}_+$, $\text{Spec } B = \text{Vin}_G$ and $f = \chi_+$. Choose a presentation $B = P/J$ where $P = A[X_1, \ldots, X_N]$ and $J \subset P$ is a finitely generated ideal as above. Recall that we have the discriminant divisor $\mathfrak{D}_+ \subset \mathcal{C}_+$ defined by the extended discriminant function $\text{Disc}_+$. By Corollary 2.2.15, $\chi_+$ is smooth over $\mathcal{C}_+ - \mathfrak{D}_+$. Hence $\chi_+(\Sigma_{\chi_+})$ is contained in $\mathfrak{D}_+$ set-theoretically by Lemma 2.6.2(2).

Since $\mathfrak{D}_+$ is a principal divisor, there exists a positive integer $m_0$ (depending on the presentation $B \cong P/J$) such that $\chi_+(\Sigma_{\chi_+}) \subset m_0 \mathfrak{D}_+$ scheme-theoretically. To state the main result in this section, we consider an artin local $k$-algebra $R$ with maximal ideal $\mathfrak{m}$ and let $I \subset R$ be an ideal such that $I \cdot \mathfrak{m} = 0$.

**Proposition 2.6.6.** Let $\delta \in \text{Vin}_G(R[[\omega]])$ and $a_0 \in \mathcal{C}_+(k[[\omega]])$ be the reduction mod $\mathfrak{m}$ of $\chi_+(\delta)$. Let $d := \nu(a_0^R \mathfrak{D}_+)$ be the discriminant valuation of $a_0^R$ (suppose that $d$ is a finite number). Then for any integer $N \geq m_0d$ and any $a \in \mathcal{C}_+(R[[\omega]])$ such that $a \equiv \chi_+(\delta) \mod \omega^N I$, there exists $\gamma \in \text{Vin}_G(R[[\omega]])$ such that $\chi_+(\gamma) = a$ and $\gamma \equiv \delta \mod \omega^{N-m_0d} I$.

**Proof.** Consider the following diagram in which the right square is Cartesian

\[
\begin{array}{ccc}
V_0 & \xrightarrow{\chi_+} & \text{Vin}_G \\
\downarrow{\chi_0} & & \\
\text{Spec }R[[\omega]]/\omega^N I & \xrightarrow{a} & \mathcal{C}_+
\end{array}
\]

Also, let $\chi_{a_0} : V_{a_0} \to \text{Spec } k[[\omega]]$ be the reduction mod $\mathfrak{m}$ of $\chi_0$. Let $h$ be the conductor of $\chi_{a_0}$. By Lemma 2.6.2(3), we have $\Sigma_{\chi_{a_0}} \subset V_{a_0} \cap \Sigma_{\chi_+}$. Since $\chi_+(\Sigma_{\chi_+}) \subset m_0 \mathfrak{D}_+$, we have $h \leq m_0d$. By Lemma 2.6.4,
there exists a section \( \gamma \) of \( \chi_\delta \) such that the restriction of \( \gamma \) to \( \text{Spec} R[[\varpi]]/\varpi^N-m_{\text{od}} \) coincides with \( \delta \). Thus the element in \( V_{16}(R[[\varpi]]) \) determined by \( \gamma \) is the lifting we want. \[\square\]

3. Kottwitz-Viehmann varieties

We fix a connected reductive group \( G \). Let \( T \subset G \) be a maximal torus and \( \lambda \in X_*(T)_+ \) a dominant coweight. Let \( \gamma \in G^{rs}(F) \) be a regular semisimple element.

We study the following sets associated to the pair \((\gamma, \lambda)\), which we both refer to as Kottwitz-Viehmann varieties:

\[
X^\lambda_\gamma = \{ g \in G(F)/G(O) | \text{Ad}(g)^{-1}(\gamma) \in G(O)\varpi^\lambda \}
\]

\[
X^{\leq \lambda}_\gamma = \{ g \in G(F)/G(O) | \text{Ad}(g)^{-1}(\gamma) \in G(O)\varpi^{\leq \lambda} G(O) \}
\]

3.1. Nonemptiness. The first immediate question is when the sets \( X^\lambda_\gamma, X^{\leq \lambda}_\gamma \) are nonempty. To answer this we need to recall the notion of Newton points and Kottwitz map.

3.1.1. Newton Points. Following [KV12, §4], for each \( \gamma \in G(F)^{rs} \), one associate a rational dominant coweight \( \nu_\gamma \in X_*(T)_+^\lambda \), called the Newton point of \( \gamma \).

**Definition 3.1.2.** The discriminant valuation for \( \gamma \in G(F)^{rs} \) is defined by

\[
d(\gamma) := \text{val} \det(\text{Id} - \text{ad}_\gamma : g(F)/g_\gamma(F) \rightarrow g(F)/g_\gamma(F))
\]

where \( g \) is the Lie algebra of \( G \) and \( \gamma_\gamma \) is the centralizer of \( \gamma \), i.e. the fixed locus of the adjoint action \( \text{ad}_\gamma \).

**Lemma 3.1.3.** Let \( \gamma \in G(F)^{rs} \) and \( \nu_\gamma \in \Lambda^\lambda_\gamma \) its Newton point. Let \( \bar{\gamma} \in T(\bar{F})^{rs} \) be a \( G(\bar{F}) \)-conjugate of \( \gamma \) such that \( \text{val}(\alpha(\bar{\gamma})) \geq 0 \) for all positive root \( \alpha \). Then we have

\[
d(\bar{\gamma}) = 2 \sum_{\alpha \in \Phi^+} \text{val}(\alpha(\bar{\gamma}) - 1) - \langle 2\rho, \nu_\gamma \rangle
\]

where we have extended the valuation on \( F \) to its separable closure \( \bar{F} \).

**Proof.** From the definition we see that

\[
d(\gamma) = \sum_{\alpha \in \Phi} \text{val}(\alpha(\gamma) - 1).
\]

Separate the sum over \( \Phi \) according to whether \( \langle \alpha, \nu_\gamma \rangle = 0 \) or not, then we get

\[
d(\gamma) = \sum_{\langle \alpha, \nu_\gamma \rangle = 0} \text{val}(\alpha(\gamma) - 1) + \sum_{\langle \alpha, \nu_\gamma \rangle < 0} \langle \alpha, \nu_\gamma \rangle.
\]

By our assumption that \( \text{val}(\alpha(\gamma)) \geq 0 \) for \( \alpha \in \Phi^+ \), the first term in (3.1) equals to

\[
2 \sum_{\langle \alpha, \nu_\gamma \rangle = 0} \text{val}(\alpha(\gamma) - 1) = 2 \sum_{\alpha \in \Phi^+} \text{val}(\alpha(\gamma) - 1)
\]

while the second term of (3.1) equals to

\[
\sum_{\alpha \in \Phi^-} \langle \alpha, \nu_\gamma \rangle = - \sum_{\alpha \in \Phi^+} \langle \alpha, \nu_\gamma \rangle = -(2\rho, \nu_\gamma).
\]

Hence the lemma follows. \[\square\]
3.1.4. Kottwitz map. Let \( \pi_1(G) := X_\ast(T)/X_\ast(T^\infty) \) be the quotient of the coweight lattice by the coroot lattice and \( p_G : X_\ast(T) \to \pi_1(G) \) be the canonical projection. Following [KV12], one defines a group homomorphism
\[
\kappa_G : G(F) \to \pi_1(G)
\]
which we refer to as Kottwitz homomorphism. Note that in loc. cit., this map is denoted by \( \omega_G \).

**Lemma 3.1.5.** Suppose that \( \kappa_G(\gamma) = p_G(\lambda) \). Then there exists an element \( \gamma_\lambda \in G_+^e(F) \) such that
- the image of \( \gamma_\lambda \) in \( G_+^e(F) \) coincides with the image of \( \gamma \) in \( G_+^e(F) \);
- \( \alpha(\gamma_\lambda) = \omega^{-\nu_\lambda} \in T_\ast(F) \cap A^\ast_{G_+^e}(O) \) where \( \lambda \in X_\ast(T)_+ \) is the image of \( \lambda \in X_\ast(T)_+ \).

Moreover, \( \gamma_\lambda \) is uniquely determined up to multiplication by an element in \( Z_{G_+^e}(F) \).

**Proof.** Let \( \gamma_\lambda \in G_+^e(F) \) be the image of \( \gamma \). Choose any \( \tilde{\gamma} \in G_+^e(F) \) that maps to \( \gamma_\lambda \). Suppose \( \alpha(\tilde{\gamma}) \in \omega T_\ast(O) \) for \( \mu \in X_\ast(T)_+ \). By the assumption \( \kappa_G(\gamma) = p_G(\lambda) \), we have \( \lambda_\lambda = \mu \in X_\ast(T) \). Let \( \gamma_\lambda := \omega^{\lambda_\lambda - \mu} \tilde{\gamma} \) where we view \( \omega^{\lambda_\lambda - \mu} \in T(F) = Z_\ast(F) \) as a central element in \( G_+^e(F) \). Then we have \( \alpha(\gamma_\lambda) = \omega^{-\nu_\lambda} \) and the image of \( \gamma_\lambda \) in \( G_+^e(F) \) equals to \( \gamma_\lambda \).

Suppose \( \gamma_\lambda, \gamma_\lambda' \in G_+^e(F) \) both satisfy the requirement of the Lemma. Then \( \gamma_\lambda' \gamma_\lambda^{-1} \in G_+^e(F) \) and its image in \( G_+^e(F) \) is the identity. Hence \( \gamma_\lambda' \gamma_\lambda^{-1} \in Z_{G_+^e}(F) \). \( \square \)

Now we can state the non-emptiness criterions.

**Proposition 3.1.6.** The following are equivalent:
1. \( X_\lambda^\ast \) is nonempty;
2. \( X_{\leq \lambda}^\ast \) is nonempty;
3. \( \kappa_G(\gamma) = p_G(\lambda) \) and \( \nu_\gamma \leq \lambda \), i.e. \( \lambda - \nu_\gamma \) is a \( \mathbb{Q} \)-linear combination of simple coroots with non-negative coefficients;
4. \( \kappa_G(\gamma) = p_G(\lambda) \) and \( \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(O) \), where \( \gamma_\lambda \in G_+^e(F) \) is defined in Lemma 3.1.5.

**Proof.** The implication "(1)\( \Rightarrow \) (2)" is tautological. The implication "(1)\( \Rightarrow \) (3)" is done in [KV12, Corollary 3.6].

(3)\( \Rightarrow \) (4): Let \( F'/F \) be a finite extension of degree \( e \) so that \( \gamma \) (and hence \( \gamma_\lambda \)) is split in \( G(F'/F) \). Let \( \omega' = \omega^h \) be a uniformizer of \( F' \) and \( O' = k[[\omega']] \subset F' \) be the ring of integers. Then \( e \cdot \nu_\gamma \in X_\ast(T)_+ \) and \( \gamma \) is \( G(F') \)-conjugate to an element in \( (\omega')^e \nu_\lambda T(O') \). From (3) we deduce that \( \gamma_\lambda \in G_+^e(F') \)-conjugate to an element in \( \text{Vin}_{G_+^e}(O') \). Therefore
\[
\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(O') \subset \mathfrak{C}_+(F) = \mathfrak{C}_+(O).
\]

(2)\( \Rightarrow \) (4): Let \( g \in X_{\leq \lambda}^\ast \). Then \( \text{Ad}(g)^{-1}(\gamma) \in G(O) \omega^\mu G(O) \) for some \( \mu \in X_\ast(T)_+ \) with \( \mu \leq \lambda \). Then we have \( \omega \gamma(\gamma) = p_G(\mu) = p_G(\lambda) \). In particular we can define the element \( \gamma_\lambda \in G_+^e(F) \) as in Lemma 3.1.5. Then by Lemma 2.5.1 we have \( \text{Ad}(g)^{-1}(\gamma_\lambda) \in \lambda^\ast \text{Vin}_{G_+^e} \subset \text{Vin}_{G_+^e}(O) \). Thus \( \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(O) \).

(4)\( \Rightarrow \) (1): Let \( a_\lambda := \chi_+(\gamma_\lambda) \). So \( a_\lambda \in \mathfrak{C}_+(O) \) by condition (4). Then for any Coxeter element \( w \in \text{Cox}(W,S) \) (cf. Definition 2.2.1), we have \( \omega^w(a_\lambda) \in \text{Vin}_{G_+^e}(O) \). It remains to show that there exists \( h \in G(F) \) such that \( \text{Ad}(h)^{-1}(\gamma_\lambda) = e^w(a_\lambda) \), for then \( h \in X_\lambda^\ast \). To see this, notice that the transporter from \( \gamma \) to \( e^w(a_\lambda) \) in \( G \) is a torsor under the torus \( G_{\lambda} \) over \( F \). Any such torsor is trivial since \( H^1(F,G_{\gamma_\lambda}) \) by a theorem of Steinberg (using the fact that the residue field \( k \) is algebraic closed). Thus the transporter has an \( F \)-point \( h \in G(F) \). \( \square \)

3.2. Ind-scheme structure.

3.2.1. First approach. We will equip the sets \( X^\ast_\lambda \) and \( X_{\leq \lambda}^\ast \) with an ind-scheme structure. We present two approaches, one based on the original definition, the other using Vinberg monoid.

Let \( \text{Gr}_G := LG/L^+G \) be the affine Grassmannian for \( G \), which are known to be ind-projective ind-scheme over \( k \). The positive loop group \( L^+G \) acts by left multiplication on \( \text{Gr}_G \). Let \( (LG)_\lambda := L^+G \omega^\lambda L^+_G \) (resp. \( (LG)_{\leq \lambda} \)) be the k-scheme whose set of k-points is \( G(O) \omega^\lambda G(O) \) (resp. \( G(O) \omega^\lambda G(O) \)).

**Definition 3.2.2.** Let \( X_{\gamma}^\ast \) be the k-functor which associates to any k-algebra \( R \) the set
\[
X_{\gamma}^\ast(R) = \{ g \in \text{Gr}_G(R) \ | g^{-1} \gamma g \in (LG)_\lambda(R) \}.
\]

Also, we define the k-functor \( X_{\leq \lambda}^\ast \) by replacing \( (LG)_\lambda \) with \( (LG)_{\leq \lambda} \) in the above definition

By definition, \( X_{\lambda}^\ast \) is a closed sub-indscheme of \( \text{Gr}_G \) and \( X_{\gamma}^\ast \) is an open sub-indscheme of \( X_{\leq \lambda}^\ast \). Let \( X_{\gamma}^\ast \) (resp. \( X_{\leq \lambda}^\ast \)) be the reduced structure of \( X_{\gamma}^\ast \) (resp. \( X_{\leq \lambda}^\ast \)).
3.2.3. Second approach. Now we use Vinberg monoids to define certain analogue of affine Springer fibers, which turns out to be isomorphic to Kottwitz-Viehmann varieties.

Let $\mathfrak{C}_+$ be the extended Steinberg base for the monoid $V_{G^0}^\circ$. Let $a \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_+(\mathcal{O})^\tau$ and suppose that

$$\beta(a) \in \varpi^{-w_0(\lambda_{ad})} T_{ad}(\mathcal{O}) \subset A_{G^0}(\mathcal{O}) \cap T_{ad}(F)$$

where $\lambda_{ad} \in X_+(T_{ad})$ is the image of $\lambda \in X_+(T)$. Moreover, let $\gamma_+ \in G_+^0(F)$ be an element such that $\gamma_+ = a$.

**Definition 3.2.4.** The generalized affine Springer fibre $Sp_{G,\gamma_+}$ associates to any $k$-algebra $R$ the set of isomorphism classes of pairs $(h, \iota)$ where $h$ is the horizontal arrow in the following commutative diagram

$$\text{Spec } R[[\varpi]] \xrightarrow{h} \text{Spec } R[\text{Vin}_{G^0}]$$

and $\iota$ is an isomorphism between the restriction of $h$ to $\text{Spec } R((\varpi))$ and the composition

$$\text{Spec } R((\varpi)) \xrightarrow{\gamma_+} \text{Vin}_{G^0} \rightarrow [\text{Vin}_{G^0}/\text{Ad}(G)].$$

Also, we define $k$-functors $Sp_{G,\gamma_+}^0$ (resp. $Sp_{G,\gamma_+}^{reg}$) by replacing $Vin_{G^0}$ with $G_{G^0}$ (resp. $Vin_{G^0}^{reg}$).

By definition $Sp_{G,\gamma_+}$ is a closed sub-indscheme of $G_G$ and $Sp_{G,\gamma_+}^{reg} \subset Sp_{G,\gamma_+}$ are its open sub-indschemes.

We let $Sp_{G,\gamma_+}$ (resp. $Sp_{G,\gamma_+}^0$, $Sp_{G,\gamma_+}^{reg}$) be the reduced structures of $Sp_{G,\gamma_+}$ (resp. $Sp_{G,\gamma_+}^0$, $Sp_{G,\gamma_+}^{reg}$).

The isomorphism class of $Sp_{G,\gamma_+}$ and $Sp_{G,\gamma_+}^0$ only depends on $a = \chi_+(\gamma_+)$, so we will also denote them by $Sp_{G,a}$ and $Sp_{G,a}^0$. We will simplify notation as $Sp_{G,a}^0$, $Sp_{G,a}$ etc. if the group $G$ is clear from the context.

Next we relate the two definitions given above. Let $(\gamma, \lambda)$ be as in the beginning of this chapter. Suppose that the ind-scheme $X^\lambda_\gamma$ is nonempty. Then by Proposition 3.1.6 we have $\kappa_G(\gamma) = p_G(\lambda)$ and $a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$ where $\gamma_\lambda \in G_+^0(F)$ is defined in Lemma 3.1.5. It is not hard to see that

$$X^\lambda_\gamma \cong Sp_{a}^0 \quad \text{and} \quad X^{\leq \lambda}_\gamma \cong Sp_{a}.$$  

Conversely, let $a \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_+(\mathcal{O})^\tau$ and suppose that

$$\beta(a) \in \varpi^{-w_0(\lambda)} T_{ad}(\mathcal{O}) \subset A_{G^0}(\mathcal{O}) \cap T_{ad}(F)$$

for some $\lambda \in X_+(T_{ad})_+$. Let $\gamma_\lambda^w \in G_{ad}(F)$ be the image of $c_\lambda^w(a) \in G_+(F) \cap \text{Vin}_{G^0}(\mathcal{O})$ under the natural quotient $G_+(F) \rightarrow G_{ad}(F)$. Then we have

$$Sp_{a} \cong X^\lambda_{\gamma_\lambda^w} \quad \text{and} \quad Sp_{a}^0 \cong X^{\lambda}_{\gamma_\lambda^w}.$$  

Note that the isomorphism class of $X^\lambda_{\gamma_\lambda^w}$ and $X^{\leq \lambda}_{\gamma_\lambda^w}$ does not depend on the choice of $w \in \text{Cox}(W,S)$.

3.3. Symmetries. Assume $X^\lambda_\gamma$ is nonempty. Then by Proposition 3.1.6 we have $\kappa_G(\gamma) = p_G(\lambda)$ and $a = \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$.

Let $J_a$ be the commutative group scheme over $\text{Spec } \mathcal{O}$ obtained by pulling back $\mathcal{J}$ along $a : \text{Spec } \mathcal{O} \rightarrow \mathfrak{C}_+$. Since $a$ is generically regular semisimple, there is a canonical isomorphism $LJ_a \cong LG_\gamma$ which allows us to identify the positive loop group $L^+J_a$ as a subgroup of $LG_\gamma^0$. Consider the quotient group

$$P_a := L/J_a/L^+J_a \cong LG_\gamma^0/L^+J_a.$$  

In other words, $P_a$ is the affine Grassmanian of $J_a$ classifying isomorphism classes of $J_a$-torsors on $\text{Spec } \mathcal{O}$ with a trivialization of its restriction to $\text{Spec } \mathcal{O}$.

The loop group $LG_\gamma^0$ acts naturally on $X^\lambda_\gamma$ and this action factors through $P_a$. Using the isomorphism $X^\lambda_\gamma \cong Sp_{a}$, the $P_a$ action is induced by the $\mathbb{B}J$ action on $[V_{G^0}/\text{Ad}(G)]$ in Proposition 2.4.3. Moreover, $P_a$ preserve the open subspaces $Sp_{a}^{reg}$ and $Sp_{a}^{w}$ for each $w \in \text{Cox}(W,S)$.

**Proposition 3.3.1.** For each $w \in \text{Cox}(W,S)$, $Sp_{a}^{w}$ is a torsor under $P_a$.

**Proof.** This is a consequence of 2.4.3. \(\square\)
Lemma 3.3.3. There is a canonical isomorphism

\[
\text{Lie}(\gamma) = \text{Lie}(\gamma^\circ)
\]

and by 3.3.3,

In this section we closely follow \[GHKR06, \S 5\].

Remark 3.3.2. Unlike the Lie algebra case, \(\text{Sp}_a^{\text{reg}}\) may not be a \(P_a\)-torsor in general. See the discussion in \(\S\ 3.9.10\).

Let \(R_a\) be the finite free \(O\)-algebra defined by the Cartesian diagram

\[
\begin{array}{c}
\Xi_a := \text{Spec } R_a \\
\text{Spec } O \\
\end{array}
\]

\[\begin{array}{c}
\downarrow \\
\cong \\
\downarrow \\
\end{array}
\]

\[
T_+ \\
\mathcal{X}_+
\]

Let \(R^\circ_a\) be the normalization of \(R_a\) and \(\Xi^\circ_a := \text{Spec } R^\circ_a\). Then \(W\) acts naturally on the \(O\)-algebras \(R_a\) and \(R^\circ_a\).

Let \(J^\circ_a\) be the finite type Neron model of \(J_a\). Hence \(J^\circ_a\) is a smooth commutative group scheme over \(O\) such that \(J_a^\circ(F) = J_a(F) = G^\circ_a(F)\) and \(J^\circ_a(O)\) is the maximal bounded subgroup of \(G^\circ_a(F)\).

Lemma 3.3.3. There is a canonical isomorphism

\[
J^\circ_a \cong (\prod_{R^\circ_a/O} T \times \Xi^\circ_a)^W
\]

Proof. The proof is the same as [Ngô10, Proposition 3.8.2].

Corollary 3.3.4. \(\text{Lie}(P_a) = (t \otimes_k (R^\circ_a/R_a))^W\)

Proof. The quotient \(L^+J_a^\circ/L^+J_a\) is an open subgroup of \(P_a\). Hence we have isomorphism of \(O\) modules

\[
\text{Lie}P_a \cong \text{Lie}(L^+J_a^\circ)/\text{Lie}(L^+J_a).
\]

On the other hand, by 2.4.7, we have

\[
\text{Lie}L^+J_a = (t \otimes_k R_a)^W
\]

and by 3.3.3,

\[
\text{Lie}L^+J^\circ_a = (t \otimes_k R^\circ_a)^W.
\]

Hence the Corollary follows.

3.4. Admissible subsets of loop spaces. In this section we closely follow [GHKR06, \S 5].

Let \(M\) be a standard Levi subgroup of \(G\) and \(P = MN\) the standard parabolic subgroup where \(N\) is the unipotent radical of \(P\). Let \(Z(M)^0\) be the central component of the center of \(M\). Then \(Z(M)^0\) is a subtorus of \(T\). Let \(\Phi_N\) be the set of roots of \(Z(M)^0\) acting on \(N\) and \(\Phi^N_X\) the corresponding set of coroots. For each \(\alpha \in \Phi_N\), let \(N_\alpha\) be the corresponding root subgroup. Then each \(N_\alpha\) is isomorphic to a product of several copies of \(G_a\) and is preserved by the adjoint action of \(M\). Denote \(\delta_N\) half sum of elements in \(\Delta_N\).

For each \(\alpha \in \Delta_N\), denote \(ht_N(\alpha) := (\delta_N, \alpha)\). Let \(l = \max_{\alpha \in \Phi_N} ht_N(\alpha)\). For each \(1 \leq i \leq l\), let \(N[i]\) be the subgroup of \(N\) generated by root groups \(N_\alpha\) with \(ht_N(\alpha) \geq i\). Also we denote \(N[l+1] = 1\). Then \(N[1] = N\) and for each \(1 \leq i \leq s + 1\), \(N[i]\) is a normal subgroup of \(N\) and the successive quotients \(N[i] := N[i]/N[i+1]\) are commutative groups isomorphic to products of some copies of \(G_a\). Let \(L_N\) and \(L^+N\) be the loop space and arc space of \(N\). For each integer \(n \geq 0\), let \(N_n := \ker(L^+N \to L^+N)\). Then \(\{N_n\}_{n \geq 0}\) form a decreasing sequence of compact open subgroups of \(L_N\).

For each \(\gamma \in M(F) \cap G(F)^{\text{reg}}\), consider the map

\[
f_\gamma : LN \to LN
\]

\[
u \mapsto u^{-1} \gamma u \gamma^{-1}
\]

Then \(f_\gamma\) preserves the root subgroups \(N_\alpha\) and hence each normal subgroup \(N[i]\). In particular, \(f_\gamma\) induces morphism \(f_\gamma[i] : LN[i] \to LN[i]\) and \(f_\gamma(i) : LN(i) \to LN(i)\).

For each \(1 \leq i \leq l\), denote \(r_i := \text{val det}(f_\gamma(i))\). Note that there is a \(M\)-equivariant isomorphism \(N[i] \cong \text{Lie}N(i)\) from which we see that

\[
r_i = \text{val det}(\text{ad} \gamma : \text{Lie}N(i)(F) \to \text{Lie}N(i)(F)).
\]

Consider the following invariant of \(\gamma\):

\[
r_N(\gamma) := \text{val det}(\text{ad} \gamma : \text{Lie}N(F) \to \text{Lie}N(F))
\]
Then we also have $r_N(\gamma) = \sum_{i=1}^l r_i$.

Now assume that $\gamma \in M(F)_+$, we have $f_\gamma(U_n) \subset U_n$ for all $n \geq 0$.

Let $f_0 : L^+ \to L^+ N$ be the restriction of $f_\gamma$ to the arc space $L^+ N$.

**Lemma 3.4.1.** For any $1 \leq i \leq l + 1$ and any positive integer $n$ such that $n \geq \sum_{j=1}^{l+1} r_j$ we have $N[i]_n \subset f_\gamma(L^+ N[i])$.

**Proof.** We prove by descending induction on $i$. The case $i = l + 1$ is trivial since $N[l + 1] = 1$. Assume the statement is true for $i + 1$. Let $x \in N[i]_n$. To show that $x \in f_\gamma(L^+ N[i])$ it suffices to find $u \in N[i](O)$ with $x * u = 1$, for then $f_\gamma(x) = x$.

Let $x_i \in N(i)_n$ be the image of $x$. Since $\det(f_\gamma(i)) = r_i$, we have $\varpi^{r_i} N(i)(O) \subset f_\gamma(i)(N(i)(O))$. Hence there exists $u_i \in N[i]_{n-r_i}$ such that $x_i * u_i = 1$ in $N(i)(O)$ and hence $x * u_i \in N[i + 1]_{n-r_i}$. By induction hypothesis, there exists $v \in N[i + 1](O)$ such that $(x * u_i) * v = 1$. Then $u = u_i v$ satisfies $x * u = 1$. □

A subset of $L^+ N$ is *admissible* if it is the pre-image of a locally closed subset of $L^+_n N$ for some $n$. A subset $Z$ of $L^+ N$ is *admissible* of it is conjugate under $G(F)$ to an admissible subset of $L^+ N$.

**Lemma 3.4.2.** Let $V$ be an admissible subset of $L^+ N$. Let $n \geq r_N(\gamma)$ be a positive integer such that $V$ is right invariant under $N_n$. Suppose moreover that $V \subset f_0(L^+ N)$. Then the set $f_0^{-1}(V)$ is admissible and right invariant under $N_n$. Moreover, $f_0$ induces a smooth surjective map

$$f_0^{-1}(V)/N_n \to V/N_n$$

whose fibers are isomorphic to $A^{r_N(\gamma)}$.

**Proof.** Let $f_0 : L^+_n N \to L^+_n N$ be the map induced by $f_0$. Since $V$ is right invariant under $N_n$, a straightforward calculation shows that $f_0^{-1}(V)$ is also right invariant under $N_n$. Denote $\overline{V} := V/U_n$. Then we have $f_0^{-1}(V)/U_n = f_0^{-1}(\overline{V})$, a locally closed subset of $L^+_n N$. In particular, $f_0^{-1}(V)$ is admissible. Since $V \subset f_0(L^+ N)$, the induced map $f_0^{-1}(\overline{V}) \to \overline{V}$ is surjective and it remains to show that it is smooth with fibers isomorphic to $A^{r_N(\gamma)}$.

Denote $H := L^+_n N$, $H[i] := L^+_n (N[i])$ and $H(i) := L^+_n (N(i))$. Then for each $1 \leq i \leq l + 1$, $H[i]$ is a normal subgroup of $H$ and $H[i]/H[i + 1] \cong H(i)$. For each $1 \leq j \leq n$, we define a normal subgroup $H_j := \ker(H \to L^+_j N)$ of $H$; and similarly we define normal subgroups $H[i]_j$ (resp. $H(i)_j = \varpi^j H(i)$) of $H[i]$ (resp. $H(i)$).

Consider the right action of $H$ on itself defined by $v * u := u^{-1} v \gamma \gamma^{-1} u$ for $u, v \in H(k) = N(O/\varpi^n O)$. Then $f_0(u) = 1 * u$ and hence $f_0$ is the orbit map at 1 of the $H$-action. In particular, all fibres of $f_0$ are isomorphic to the stabilizer $S := f_0^{-1}(1)$.

Now we take a closer look at the structure of the stabilizer $S$. First note that the action $*$ induces actions of $H[i]$ and $H(i)$ on $S[i]$ and $S(i)$ respectively. Let $S[i]$ (resp. $S(i)$) be the stabilizer of 1 under the $H[i]$ (resp. $H(i)$) action.

We claim that for all $i$, the canonical homomorphism $S[i] \to S(i)$ is surjective. Let $s \in S(i)$ and choose a representative $h \in H[i]$ of $s$. Since

$$S(i) = \ker(f_0(i)) \subset \varpi^n - r_i H(i)$$

we have $h \in H[i]_{n-r_i}$ and $1 * h \in H[i]_{n-r_i} \cap H[i + 1] = H[i + 1]_{n-r_i}$. By assumption $n - r_i \geq \sum_{j=1}^{l+1} r_j$, then we can apply Lemma 3.4.1 to obtain an element $h' \in H[i + 1]$ such that $1 * (hh') = 1$. Thus $hh' \in S[i]$ maps to $s \in S(i)$ and the claim follows.

The kernel of the surjective homomorphism $S[i] \to S(i)$ is $S[i] \cap H[i + 1] = S[i + 1]$. Moreover, we have

$$S(i) \cong (f_0(i))^{-1}(\varpi^n N(i))/\varpi^n N(i) \cong A^{r_N(\gamma)}.$$ 

From this we see that $S \cong A^{r_N(\gamma)}$ as a scheme. □

The proof of the following lemma is inspired by [KV12, Lemma 3.8].

**Lemma 3.4.3.** For any $n \geq r_N(\gamma)$, we have $f_\gamma^{-1}(N_n) \subset N_{n-r_N(\gamma)}$.

**Proof.** Let $u \in N(F)$ with $f_\gamma(u) \in N_n$. We will show by induction that $u \in N[i](F) \cdot N_{n-\sum_{j<i} r_j}$.
The case $i = 1$ says $u \in N[1](F) = N(F)$ which is clear and the case $i = s + 1$ gives the lemma since $\sum_{i=1}^{s} r_i = r_N(\gamma)$ and $N[s + 1] = 1$.

It remains to finish the induction step. By induction hypothesis we have $u = u_i v$ with $u_i \in N[i](F)$ and $v \in N_{n-\sum_{j < i} r_j}$. By assumption,

$$f_{\gamma}(u) = f_{\gamma}(u_i v) = v^{-1} \cdot u_i^{-1} \gamma u_i \gamma^{-1} \cdot \gamma v \gamma^{-1} \in N_n$$

from which it follows that

$$u_i^{-1} \gamma u_i \gamma^{-1} \in N[i](F) \cap v \cdot N_n \cdot (\gamma v^{-1} \gamma^{-1}) \subset N[i]_{n-\sum_{j < i} r_j}$$

Let $\bar{u}_i \in N(i)$ be the image of $u_i$. Then we have

$$f_{\gamma}(i)(\bar{u}_i) \in N(i)_{n-\sum_{j < i} r_j}$$

Since $\text{val det}(f_{\gamma}(i)) = r_i$, we get that $\bar{u}_i \in N(i)_{n-\sum_{j < i+1} r_j}$ and hence

$$u = u_i v \in N[i+1](F) \cdot N_{n-\sum_{j < i+1} r_j}$$

This finishes the induction step. \hfill \Box

**Proposition 3.4.4.** Let $Z$ be an admissible subset of the loop space $LN$. Then $f_{\gamma}^{-1}(Z)$ is admissible and there exists a positive integer $m$ such that for all $n \geq m$, $f_{\gamma}^{-1}(Z)$ and $Z$ are right invariant under the group $N_n$ and the map

$$f_{\gamma}^{-1}(Z)/N_n \rightarrow Z/N_n$$

induced by $f_{\gamma}$ is smooth surjective whose geometric fibers are irreducible of dimension $r_N(\gamma)$. 

**Proof.** Let $n_0 \geq r(\gamma)$ be a positive integer. Choose a coweight $\mu_0 \in X_*(Z(M)^0)$ such that

$$Z^{\mu_0} := \text{Ad}(\varpi^{\mu_0})(Z) \subset N_{n_0}.$$ 

Then by Lemma 3.4.3 we have

$$f_{\gamma}^{-1}(Z^{\mu_0}) \subset f_{\gamma}^{-1}(N_{n_0}) \subset N_{n_0-r(\gamma)} \subset L^+ N$$

Hence in particular

$$\text{Ad}(\varpi^{\mu_0})(f_{\gamma}^{-1}(Z)) = f_{\gamma}^{-1}(Z^{\mu_0}) = f_0^{-1}(Z^{\mu_0})$$

Moreover, since $Z^{\mu_0}$ is an admissible subset of $L^+ N$, $f_0^{-1}(Z^{\mu_0})$ is an admissible subset of $L^+ N$ by Lemma 3.4.2. This shows that $f_{\gamma}^{-1}(Z)$ is admissible.

Let $n_0 > n_0$ be a positive integer such that $Z^{\mu_0}$ and $f_{\gamma}^{-1}(Z^{\mu_0})$ are invariant under right multiplication by $N_{n_1}$. For all $n \geq n_1$, since the map $f_{\gamma}$ commutes with conjugation by $\varpi^{\mu_0}$, $Z$ and $f_{\gamma}^{-1}(Z)$ are right invariant under the group $N_{n}^{\mu_0} := \varpi^{-\mu_0} N_n \varpi^{\mu_0}$. Then we get the following commutative diagram

$$
\begin{array}{ccc}
Z_{n_1}^{-\mu_0} & \cong & Z_{n_1}^{-\mu_0} \\
\downarrow \cong & & \downarrow \cong \\
Z_{n_1}^{\mu_0}/N_n & \cong & Z_{n_1}^{\mu_0}/N_n \\
\end{array}
$$

where the horizontal arrows are induced by $f_{\gamma}$ and the vertical arrows are isomorphisms induced by $\text{Ad}(\varpi^{\mu_0})$.

By Lemma 3.4.1, $Z^{\mu_0} \subset N_{n_0} \subset f_{\gamma}(L^+ N)$. Therefore we can apply Lemma 3.4.2 to conclude that the lower horizontal map is surjective smooth whose fibers are isomorphic to $A^{r_N(\gamma)}$. Hence the same is true for the upper horizontal map.

Let $m$ be a positive integer such that for all $n \geq m$, $N_n \supset N_{n'}^{\mu_0}$ for some $n' \geq n_1$. Consider the following diagram

$$
\begin{array}{ccc}
f_{\gamma}^{-1}(Z)/N_{n'}^{-\mu_0} & \rightarrow & Z/N_{n'}^{-\mu_0} \\
\downarrow & & \downarrow \\
\rightarrow & & \\
\downarrow & & \\
f_{\gamma}^{-1}(Z)/N_n & \rightarrow & Z/N_n \\
\end{array}
$$

This finishes the proof. \hfill \Box
The two vertical maps are smooth surjective with fibers isomorphic to the irreducible scheme $U_n/U_n^{-\mu}$ and the upper horizontal map is smooth surjective with fibers isomorphic to $\mathcal{A}_{\nu}^{r_N(\gamma)}$ as we have just seen. Hence the lower horizontal map is smooth surjective with irreducible fibers of dimension $r_N(\gamma)$. □

3.5. The case of unramified conjugacy class. In this section we assume that $\gamma \in G(F)^{rs}$ is an unramified regular semisimple element. Since the residue field $k$ is algebraically closed, after conjugation we may assume that $\gamma \in \varpi^{\mu}T(O) \cap G^{rs}(F)$, where $\mu = \nu_{r} \in X_*(T)_+^r$ is the Newton points of $\gamma$. In this case, we have $G_{\gamma}^0 = T$.

By Lemma 3.1.3 the discriminant valuation for $\gamma$ is

$$d(\gamma) = 2 \sum_{\alpha \in \Phi^+} \text{val}(\alpha(\gamma) - 1) - 2\rho, \mu.\tag{3.6}$$

We will apply the results in previous section to the case $N = U$ is a maximal unipotent subgroup. In this case, the corresponding invariant for $\gamma$ is

$$r(\gamma) := r_U(\gamma) = \sum_{\alpha \in \Phi^+} \text{val}(\alpha(\gamma) - 1) = \frac{1}{2} d(\gamma) + \langle \rho, \mu \rangle.\tag{3.5}$$

Fix a dominant coweight $\lambda \in \Lambda^+$ such that $\mu \leq \lambda$. By Proposition 3.1.6, this implies that $X^\lambda_{\gamma}$ is nonempty.

3.5.1. Relation with MV-cycles. Let $Y^\lambda_{\gamma}$ be the locally closed sub-indscheme of $X^\lambda_{\gamma}$ whose set of $k$-points is

$$Y^\lambda_{\gamma}(k) = \{ u \in U(F)/U(O) | Ad(u)^{-1} \gamma \in G(O) \varpi^{\lambda} G(O) \}$$

To understand the structure of $Y^\lambda_{\gamma}$, we use the map $f_{\gamma} : LU \to LU$ (cf. (3.3)). In the following, we denote $K := L^+G$. Then we have

$$Y^\lambda_{\gamma} = (f_{\gamma}^{-1}(K \varpi^{\lambda} K \varpi^{-\mu} \cap LU))/L^+U\varpi^\mu$$

Recall the Mirkovic-Vilonen cycles in the affine Grassmannian:

$$S_{\mu} \cap \text{Gr}_{\lambda} = (LU \varpi^{\mu} K \cap K \varpi^{\lambda} K)/K$$

From this description we get an isomorphism

$$(LU \cap K \varpi^{\lambda} K \varpi^{-\mu})/\varpi^{\mu} L^+U \varpi^{-\mu} \xrightarrow{u} S_{\mu} \cap \text{Gr}_{\lambda}\tag{3.6}$$

In summary, we have the following diagram

$$\xymatrix{ f_{\gamma}^{-1}(K \varpi^{\lambda} K \varpi^{-\mu} \cap LU) \ar[d] \ar[r]^{f_{\gamma}} & K \varpi^{\lambda} K \varpi^{-\mu} \cap LU \ar[d] \\
Y^\lambda_{\gamma} \ar[d] & S_{\mu} \cap \text{Gr}_{\lambda} \ar[d] \\
Y^\lambda_{\gamma} & S_{\mu} \cap \text{Gr}_{\lambda}}$$

where the left vertical arrow is an $L^+U$-torsor and the right vertical arrow is a torsor under the group $\varpi^{\mu} L^+U \varpi^{-\mu}$.

**Theorem 3.5.2.** $Y^\lambda_{\gamma}$ is an equi-dimensional quasi-projective variety of dimension $\langle \rho, \lambda \rangle + \frac{1}{2} d(\gamma)$, where $d(\gamma)$ is the discriminant valuation, cf. Definition 3.1.2. Moreover, the number of irreducible components of $Y^\lambda_{\gamma}$ equals to $m_{\lambda \mu}$, the dimension of $\mu$-weight space in the irreducible representation $V_{\lambda}$ of $\tilde{G}$ with highest weight $\lambda$.

**Proof.** Apply Proposition 3.4.4 to the admissible subset $Z = K \varpi^{\lambda} K \varpi^{-\mu} \cap LU$ of $LU$, we see that there exists a large enough positive integer $n$ such that in the following diagram

$$\xymatrix{ f_{\gamma}^{-1}(K \varpi^{\lambda} K \varpi^{-\mu} \cap LU)/U_n \ar[d] \ar[r]^{f_{\gamma}} & (K \varpi^{\lambda} K \varpi^{-\mu} \cap LU)/U_n \ar[d] \\
Y^\lambda_{\gamma} \ar[d] & S_{\mu} \cap \text{Gr}_{\lambda} \ar[d] \\
Y^\lambda_{\gamma} & S_{\mu} \cap \text{Gr}_{\lambda}}$$

(1) All schemes are of finite type;
(2) The map \( \tilde{f}_\gamma \) induced by \( f_\gamma \) is smooth surjective whose geometric fibers are irreducible of dimension \( r(\gamma) \), where we recall that \( r(\gamma) \) is defined in (3.5);

(3) \( U_n \) is contained in \( \varpi^\mu L^+U/\varpi^{-\mu} \), hence also \( L^+U \);

(4) The left vertical map is smooth surjective with fibers isomorphic to the irreducible scheme \( L^+U/U_n \);

(5) The right vertical map is smooth with fibers isomorphic to the irreducible scheme \( \varpi^\mu L^+U/\varpi^{-\mu}/U_n \).

Since \( Y_\gamma \) is of finite type, it is a locally closed subscheme of a closed Schubert variety. In particular, \( Y_\gamma \) is quasi-projective since closed Schubert varieties are projective.

Recall that the MV-cycle \( S_\mu \cap Gr_\lambda \) is equidimensional of dimension \( (\rho, \lambda + \mu) \). Hence by (2)-(5) we see that \( Y_\gamma \) is equidimensional of dimension

\[
\dim Y_\gamma = \dim(S_\mu \cap Gr_\lambda) + \dim \varpi^\mu U(O)\varpi^{-\mu}/U_n^{\lambda_0} + r(\gamma) - \dim U(O)/U_n^{\lambda_0} \tag{3.7}
\]

Moreover, \( \det(\gamma) \) is a scheme locally of finite type and the assertions about equidimensionality and dimension formula follows from the corresponding statements for \( Y_\gamma \).

Moreover, by [Sta17, Tag 037A] the 3 maps in the diagram above induces a canonical bijections between set of irreducible components

\[
\operatorname{Irr}(Y_\gamma) \xrightarrow{\sim} \operatorname{Irr}(S_\mu \cap Gr_\lambda).
\]

Hence the number of irreducible components of \( Y_\gamma \) equals to the number of irreducible components of the MV-cycle \( S_\mu \cap Gr_\lambda \), which is known to be \( m_{\lambda_\mu} \).

\begin{corollary}
Suppose \( \gamma \in G(F)^{rs} \) is unramified (i.e. split) and \( \nu_\gamma = \mu \in X_*(T)_+ \), then \( X_\gamma \) is a scheme locally of finite type, equidimensional of dimension

\[
\dim X_\gamma = (\rho, \lambda) + \frac{1}{2} d(\gamma).
\]

Moreover, the number of \( G_\gamma^0(F) \)-orbits on its set of irreducible component \( \operatorname{Irr}(X_\gamma) \) equals to \( m_{\lambda_\mu} \).

\begin{proof}
There is a natural morphism

\[
Y_\gamma \times X_*(T) \longrightarrow X_\gamma
\]

\[
(u, \nu) \longmapsto u\varpi^\nu
\]

which induces bijection on \( k \)-points and a stratification of \( X_\gamma \) such that each strata is isomorphic to \( Y_\gamma \). Thus \( X_\gamma \) is a scheme locally of finite type and the assertions about equidimensionality and dimension formula follows from the corresponding statements for \( Y_\gamma \).

The \( \Lambda^0_\gamma \) action on the set \( \operatorname{Irr}(X_\gamma) \) factors through \( \pi_0(\Lambda^0_\gamma) = X_*(T) \) and hence \( \Lambda_\gamma \)-orbits on \( \operatorname{Irr}(X_\gamma) \) corresponds bijectively to the set \( \operatorname{Irr}(Y_\gamma) \). Thus the number of orbits equals to the weight multiplicity \( m_{\lambda_\mu} \).
\end{proof}

\end{corollary}

3.6. Finiteness of Kottwitz-Viehmann varieties. In this section we let \( \gamma \in G(F)^{rs} \) be any regular semisimple element and \( \lambda \in \Lambda^+ \). Assume without loss of generality that \( X_\gamma \) is nonempty and \( \det(\gamma) = \det(\varpi^\lambda) \). Then we get an element \( \gamma \in V_{G^0}(F) \) as in Lemma 3.1.5. Moreover, the Newton point of \( \gamma \) satisfies \( \nu_{\gamma} \leq \underline{\lambda} \) and \( \chi(\gamma) \in \mathfrak{C}_{\leq \lambda} \) by Proposition 3.1.6.

We show in this section that \( X_\gamma \), a priori an ind-scheme, is actually a scheme locally of finite type. This has already been proved for unramified conjugacy classes in Corollary 3.5.3. It remains to reduce the general case to the unramified case. This reduction step is completely analogous to the Lie algebra case. For the reader’s convenience, we include the details, following the exposition in [Yun15a, §2.5]. See also [Bou15a].

Let \( F'/F \) be a finite extension of degree \( e \) so that \( \gamma \) splits over \( F' \). Let \( \varpi' = \varpi^{1/e} \in F' \) be a uniformizer and \( O' = k[[\varpi']] \) the ring of integers in \( F' \). Let \( \sigma \) be a generator of the cyclic group \( \operatorname{Gal}(F'/F) \)

Choose \( h \in G(F') \) such that \( \operatorname{Ad}(h)G_\gamma^{0} = T \). Then \( h\sigma(h)^{-1} \in N_{G}(T)(F') \) and we let \( w \in W \) be its image.

Consider the embedding

\[
\iota_\gamma : \Lambda := X_*(T) \longrightarrow G_\gamma(F')
\]

\[
\mu \longmapsto \operatorname{Ad}(h)^{-1}\varpi^\mu
\]

Let \( \Lambda_\gamma := \iota_\gamma^{-1}(G_\gamma(F')) \). It follows immediately that \( \Lambda_\gamma \subseteq \Lambda^w \) where \( \Lambda^w \) is the fixed point set of \( w \) on \( \Lambda \). Moreover, \( \Lambda_\gamma \) can be identified with the coweight lattice of the maximal \( F \)-split subtorus of \( G_\gamma \). In particular, \( (\Lambda_\gamma)_Q = (\Lambda^w)_Q \) so that \( \Lambda_\gamma \subseteq \Lambda^w \) is a subgroup of finite index.
Proposition 3.6.1. There exists a closed subscheme $Z \subset X_\gamma^\lambda$ which is projective over $k$ such that $X_\gamma^\lambda = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot Z$. Here $\ell \in \Lambda_\gamma$ acts on $X_\gamma^\lambda$ via the embedding $\iota_\gamma$.

Proof. We rephrase the argument in [Yun15a, §2.5.7]. Let $\tilde{X}_\gamma^\lambda$ be the generalized affine Springer fiber of coweight $e\lambda$ for $\gamma$ in $Gr_{G,F'}$. Then $\sigma$ acts naturally on $\tilde{X}_\gamma^\lambda$ and the fixed points sub-ind scheme $(\tilde{X}_\gamma^\lambda)^\sigma$ contains $X_\gamma^\lambda$ (but they are not equal in general). Let $\gamma' = h\gamma h^{-1} \in T(F')$ and $\tilde{X}_{\gamma'}^\lambda$, the corresponding generalized affine Springer fiber in $Gr_{G,F'}$. Then

$$\tilde{X}_{\gamma'}^\lambda = h \cdot \tilde{X}_\gamma^\lambda$$

By Theorem 3.5.2, there is a locally closed subscheme $\tilde{Y}_\gamma^\lambda$ of $\tilde{X}_\gamma^\lambda$ such that

$$\tilde{X}_\gamma^\lambda = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot \tilde{Y}_\gamma^\lambda.$$

Let $\tilde{Z}$ be the closure of $h^{-1}\tilde{Y}_\gamma^\lambda$ in $\tilde{X}_\gamma^\lambda$. Then $\tilde{Z}$ is projective over $k$ and $\tilde{X}_\gamma^\lambda = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot \tilde{Z}$.

Recall that $w \in W$ is represented by $h\sigma(h)^{-1}$. One can check that $\sigma(\tilde{Z}) = \tilde{Z}$ and more generally $\sigma(\ell \cdot \tilde{Z}) = w(\ell) \cdot \tilde{Z}$ for all $\ell \in \Lambda$. Consequently,

$$(\tilde{X}_\gamma^\lambda)^\sigma = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot \tilde{Z} = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot (C \cdot \tilde{Z})$$

where $C \subset \Lambda$ is a finite set of representatives of the quotient $\Lambda/\Lambda_\gamma$. Hence, $C \cdot \tilde{Z}$ is a finite type scheme.

Finally let $Z := (C \cdot \tilde{Z}) \cap X_\gamma^\lambda$. Then $Z$ is a finite type subscheme of $X_\gamma^\lambda$. Hence $Z$ is projective over $k$ and $X_\gamma^\lambda = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot Z$. \hfill \Box

As a consequence, we immediately get:

Theorem 3.6.2. The ind-scheme $X_\gamma^\lambda$ is a finite dimensional $k$-scheme, locally of finite type. Moreover, the lattice $\Lambda_\gamma$ acts freely on $X_\gamma^\lambda$ and the quotient $X_\gamma^\lambda/\Lambda_\gamma$ is representable by a proper algebraic space over $k$.

3.7. Dimension of the regular locus. Recall that the regular locus $X_\gamma^\lambda,\text{reg}$ is an open subscheme of $X_\gamma^\lambda$ on which the action of $P_a = L\gamma G_0 / L^+ J_a$ is free (but not necessarily transitive).

Theorem 3.7.1.

$$\dim P_a = \dim X_\gamma^\lambda,\text{reg} = \langle \rho, \lambda \rangle + \frac{d(\gamma) - c(\gamma)}{2}$$

where

- $d(\gamma) := \text{val}(\det(\text{Id} - \text{ad}(\gamma)) : g(F)/g_\gamma(F) \to g(F)/g_\gamma(F))$.
- $c(\gamma) := \text{rank}(G) - \text{rank}_{F} G_\gamma$, where $\text{rank}_F G_\gamma$ is the dimension of the maximal $F$-split torus of $G_\gamma$.

Moreover, $X_\gamma^\lambda,\text{reg}$ is equidimensional.

Proof. The first equality follows from the fact that the $P_a$-orbits in $X_\gamma^\lambda,\text{reg}$ are open and the action is free.

When $\gamma$ is unramified (hence split as $k$ is algebraically closed), the second equality follows from Corollary 3.5.3. It remains to reduce to this case. The argument is similar to that of Bezrukavnikov’s in Lie algebra case, cf. [BZ96], which we reformulate using the Galois description of universal centralizer.

Let $A$ be the finite free $\mathcal{O}$-algebra defined by the Cartesian diagram (3.2) and $A^\circ$ the normalization of $A$. Then $W$ acts naturally on the $\mathcal{O}$-algebras $A$ and $A^\circ$ by and 3.3.4, we get

$$\dim P_a = \dim_k(t \otimes_k (A^\circ/A))^W.$$

Let $\tilde{F}/F$ be a ramified extension of degree $e$, with ring of integers $\hat{O} = k[[\pi^\pm]]$, such that $\gamma$ is split over $\tilde{F}$. Let $\sigma$ be a generator of the cyclic group $\Gamma := \text{Gal}(\tilde{F}/F)$. Let $A := A \otimes_{\hat{O}} \hat{O}$ and $\tilde{A}$ its normalization. We remark that $\tilde{A}$ is not the same as $A^\circ \otimes_{\hat{O}} \hat{O}$ in general. Let $\tilde{P}_a = L\gamma G_{\tilde{F}} / L^+ J_{\tilde{a},\tilde{F}}$. Then by the dimension formula in split case, we have

$$\dim_k(t \otimes_k \tilde{A}^\circ/\tilde{A})^W = \dim \tilde{P}_a = \langle \rho, e\lambda \rangle + \frac{1}{2} e \cdot d(\gamma).$$

As $\gamma$ split over $\hat{O}$, we have

$$\tilde{A}^\circ \cong \hat{O}[W] := \hat{O} \otimes_k k[W]$$
as \( W \)-module. Here \( W \) acts on \( \hat{O}[W] \) via right regular representation. Moreover, there exists an element \( w_\gamma \in W \) of order \( e \) such that under the above isomorphism, the natural action of \( \sigma \in \Gamma \) on \( \hat{A}^\gamma \) becomes \( \sigma \otimes l_{w_\gamma} \) where \( l_{w_\gamma} \) denotes the left regular action of \( w_\gamma \) on \( k[W] \). In particular, the action of \( W \) and \( \Gamma \) commutes with each other. With these considerations, we obtain an isomorphism

\[
(t \otimes_k \hat{A}^\gamma)^W \cong t \otimes_k \hat{O}
\]

which intertwines the action of \( \sigma \in \Gamma \) on the left hand side with the action of \( w \otimes \sigma \) on the right hand side.

Moreover, we have an equality

\[
(t \otimes_k \hat{A}^\gamma) = t \otimes_k A^\delta
\]

which remains true after taking \( W \)-invariants since the \( \Gamma \) action commutes with \( W \) action. In particular, we have

\[
M := (t \otimes_k \hat{O}) = (t \otimes_k A)^W
\]

Moreover, it is clear that from the definition of \( W \) action that

\[
(t \otimes_k \hat{A})^W = (t \otimes_k A)^W \otimes \hat{O}.
\]

Thus we get

\[
\dim \mathcal{P}_a = \dim_k (t \otimes_k A^\gamma/A)^W = \frac{1}{e} \dim_k (t \otimes_k (A^\gamma/A) \otimes \hat{O})^W = \frac{1}{e} \dim_k \left( \frac{M \otimes \hat{O}}{(t \otimes_k A)^W} \right)
\]

\[
= \langle \rho, \lambda \rangle + \frac{1}{e} d(\gamma) - \frac{1}{e} \dim_k \left( \frac{t \otimes_k \hat{O}}{M \otimes \hat{O}} \right)
\]

Since the element \( w_\gamma \in W \) has order \( e \), its eigenvalues are \( e \)-th roots of unit. Let \( \zeta \) be a primitive \( e \)-th root of unit and \( t(i) \) the subspace of \( t \) on which \( w_\gamma \) acts via the scalar \( \zeta^i \). In particular, \( t(0) = t^{w_\gamma} \) is the \( w_\gamma \) invariant subspace. Then we have

\[
M := (t \otimes_k \hat{O}) = \bigoplus_{i=0}^{e-1} t(i) \otimes_k \zeta^{-i}
\]

The existence of a \( W \)-invariant nondegenerate symmetric bilinear form on \( t \) guarantees that \( \dim_k t(i) = \dim_k t(e - i) \), from this we obtain that

\[
\dim_k \left( \frac{t \otimes_k \hat{O}}{M \otimes \hat{O}} \right) = e (\dim_k t - \dim_k t^{w_\gamma}) = e \cdot c(\gamma)
\]

Combined with (3.8), we obtain

\[
\dim \mathcal{P}_a = \langle \rho, \lambda \rangle + \frac{1}{2} (d(\gamma) - c(\gamma)).
\]

Finally, \( X_\gamma^{\lambda, \text{reg}} \) is equidimensional since it is a finite union of \( \mathcal{P}_a \)-torsors.

### 3.7.2. Some \( \theta \)-dimensional generalized affine Springer fibers

Suppose \( X_\lambda^{\gamma} \) is nonempty. Then there exists \( \gamma_\lambda \in G_+^{sc} \) satisfying the conclusion of Lemma 3.1.5. Let \( a := \chi_+((\gamma_\lambda)) \in \mathcal{C}_+(\hat{O}) \cap \mathcal{C}_{G_+^{sc}}(F) \). Recall the extended discriminant divisor \( \mathcal{D}_+ \subset \mathcal{C}_+ \) defined in § 2.2.10. We define the extended discriminant valuation to be

\[
d_+(a) := \text{val}(a^* \mathcal{D}_+) \in \mathbb{Z}
\]

From equation (3.1) we get

\[
d_+(a) = 2 \cdot \text{val}(\rho(\alpha(\gamma_\lambda))) + d(\gamma)
\]

\[
= \langle 2\rho, \lambda \rangle + d(\gamma)
\]

\[
\sum_{\alpha \in \Phi : \langle \alpha, \rho \rangle > 0} \text{val}(\alpha(\gamma) - 1) + \langle 2\rho, \lambda - \nu_\gamma \rangle
\]

**Proposition 3.7.3.** Suppose \( d_+(a) = 0 \). Then \( \gamma \) is split and \( \dim X_\gamma^{\lambda} = 0 \). Moreover, \( X_\gamma^{\lambda} = X_\gamma^{\lambda, \text{reg}} \) and it is a torsor under \( P_a \).
Proof. The assumption $d_+(a) = 0$ implies that $a \in \mathfrak{c}_+^\ast(\mathcal{O})$. Let $\bar{X}_a = \text{Spec} \, R_a$ be defined by the Cartesian diagram

$$
\begin{array}{ccc}
\bar{X}_a & \xrightarrow{} & \mathcal{T}_+ \\
\downarrow & & \downarrow \\
\text{Spec} \, \mathcal{O} & \xrightarrow{} & \mathfrak{c}_+
\end{array}
$$

By Proposition 2.2.14, $\bar{X}_a$ is an étale cover $\text{Spec} \, \mathcal{O}$ which must be trivial since the residue field $k$ is algebraically closed. Then we see that $\gamma_\lambda \in T_+(F)$ is split and hence $\gamma \in T(F)$ is split.

Since $X_\lambda$ is nonempty, we have $\nu_\gamma \leq 0 \lambda$ and hence the terms on the right hand side of (3.9) are non-negative. In particular, $d_+(a) = 0$ implies that $\nu_\gamma = \lambda$. Thus the proposition follows from Corollary 3.5.3. □

3.8. The case of central coweight. In this section we deal with the case where $\lambda \in X_\ast(T)_+$ is a central coweight, i.e. $(\lambda, \alpha) = 0$ for all roots $\alpha$. Then $\lambda \in X_\ast(Z^0)$ where $Z^0$ is the maximal torus in the center of $G$. Consequently we have $X_\lambda \cong X_{0_{\lambda}}$. Hence the essential case is when $\lambda = 0$ and the corresponding Kottwitz-Viehmann variety becomes

$$X_\gamma := \{ g \in G(F)/G(\mathcal{O}) | \text{Ad}(g)^{-1} \gamma \in G(\mathcal{O}) \}.$$ 

We first do some routine reductions. Let $P = MN$ be a standard parabolic subgroup with standard Levi $M$ and unipotent radical $N$. For $\gamma \in M(\mathcal{O}) \cap G^\circ(F) \subset M(\mathcal{O}) \cap M^{rs}(F)$, we consider the Kottwitz-Viehmann variety $X_\gamma$ (resp. $X^M_\gamma$) defined for the groups $G$ (resp. $M$). We have the discriminant valuation $d(\gamma)$ (resp. $d_M(\gamma)$) defined for $G$ (resp. $M$). The two discriminant valuations are related by

$$(3.10) \quad d(\gamma) = d_M(\gamma) + 2r_N(\gamma)$$

where $r_N(\gamma)$ is defined in (3.4).

Proposition 3.8.1. With notation as above, we have

$$\dim X_\gamma = \dim X^M_\gamma + \frac{d_G(\gamma) - d_M(\gamma)}{2}$$

Proof. Let $P = MN$ be the standard parabolic subgroup with Levi factor is $M$ and unipotent radical $N$. The connected components of $\text{Gr}_M$ and $\text{Gr}_P$ both corresponds bijectively to $\pi_1(M)$, the quotient of $X_\ast(T)$ by the coroot lattice of $M$. The canonical map $\text{Gr}_P \rightarrow \text{Gr}_G$ induces bijection on $k$-points by generalized Iwasawa decomposition. For each $\lambda \in \pi_1(M)$, let $X_{\gamma, \lambda}$ be the intersection of $X_\gamma$ and the connected component of $\text{Gr}_P$ corresponding to $\lambda$. Similarly, let $X^M_{\gamma, \lambda}$ be the intersection of $X^M_\gamma$ with the connected component of $\text{Gr}_M$ corresponding to $\lambda$. Then there is a canonical morphism

$$p^M_{\gamma} : X_{\gamma, \lambda} \rightarrow X^M_{\gamma, \lambda}$$

It suffices to show that the fibres of this map have dimension $r_N(\gamma)$.

Let $h \in X^M_{\gamma, \lambda}$. Then $\gamma_h := h^{-1} \gamma h \in M(\mathcal{O})$ and we consider the fibre $Y_h := (p^M_{\gamma})^{-1}(h)$. Its set of $k$-points is

$$Y_h(k) = \{ u \in N(F)/N(\mathcal{O}) | u^{-1} \gamma_h u \in G(\mathcal{O}) \}$$

In other words, we have

$$Y_h = f^{-1}_h(N(\mathcal{O}))/N(\mathcal{O})$$

where $f_{\gamma_h} : N(F) \rightarrow N(F)$ is defined by $f_{\gamma_h}(u) = u^{-1} \gamma_h u \gamma_h^{-1}$. Apply Proposition 3.4.4 to the admissible set $Z = N(\mathcal{O})$ we see that $Y_h$ is an irreducible affine space of dimension

$$\dim Y_h = r_N(\gamma_h) = r_N(\gamma)$$

and hence we conclude by (3.10). □

Corollary 3.8.2. Let $\lambda \in X_\ast(T)$ be a central coweight and $\gamma \in G(F)^{\ast}$. Then

$$\dim X^\lambda_\gamma = \frac{1}{2}(d_\gamma - c_\gamma).$$

Moreover, $X^\lambda_\gamma^{\text{reg}}$ is a torsor under $P_a$ and the dimension of the complement of $P_a = X^{\lambda, \text{reg}}_\gamma$ in $X^\lambda_\gamma$ is strictly smaller than the dimension of $X^\lambda_\gamma$. 

Proof. We first assume that γ ∈ G(𝒪) is topologically unipotent mod center. In other words, the reduction mod ω of γ is unipotent mod center. After multiplying by an element in Z(𝒪), we may assume that γ ∈ G^{sc}(𝒪) is topologically unipotent. Then when G = G^{sc} the argument of [KL88, §4] and [Ngô10, Proposition 3.7.1] generalize verbatim to our situation and proves dim X^\text{reg}_γ = dim X_γ and the complement of X^\text{reg}_γ has strictly smaller dimension. In particular the dimension formula follows in this situation. More generally, we argue as in [Tsa16, Lemma 4.1] to reduce to the case G = G^{sc}.

It remains to reduce to the case where γ is topologically unipotent mod center. After multiplying γ ∈ G(𝒪) by an element in Z(𝒪) we may assume that γ ∈ G^{sc}(𝒪). Then G_γ is a maximal torus in G and γ ∈ G_γ(F) ∩ G(𝒪). Let S be the maximal split subtorus in the centralizer G_γ. After conjugation we may assume that S ⊂ T. Let M = C_G(S) be the centralizer of S in G. Then M is a standard Levi subgroup of G and γ ∈ M(𝒪). Let a_M := χ_M(γ) ∈ C_M(𝒪). Then the pullback of T along a_M : Spec 𝒪 → C_M is a totally ramified cover of Spec 𝒪 and we deduce that γ is topologically unipotent mod center in M(𝒪). Thus the result follows from the case already proved and Proposition 3.8.1. □

We highlight the following special case:

**Corollary 3.8.3.** Let λ ∈ X_∗(T) be a central coweight and γ ∈ G(F)^{ss}. If d_γ ≤ 1, then X^λ_γ = X^\text{reg}_λ = P_a and they are 0-dimensional.

**Proof.** By Corollary 3.8.2, if d_γ ≤ 1, we must have dim X^λ_γ = 0. Moreover, the complement of X^\text{reg}_λ in X^λ_γ has strictly smaller dimension, hence must be empty. □

### 3.9. Irreducible components.

#### 3.9.1. Stratification on dominant coweight cone.

Let Λ := X_∗(T) and Λ_Q := Λ ⊗_ℤ Q. Let D ⊂ Λ_Q be the positive coroot cone. In other words, D consists of Q-linear combinations of simple coroots with non-negative coefficients.

For λ ∈ Λ^+, we define the **dominant coweight polytope** to be:

\[ P_λ := Λ_Q^+ ∩ \text{Conv}(W ⋅ λ) = Λ_Q^+ ∩ (λ − D). \]

where Conv(W ⋅ λ) denotes the convex hull of the W-orbit of λ.

**Lemma 3.9.2.** For each λ_1, λ_2 ∈ Λ^+ with κ_G(λ_1) = κ_G(λ_2), there exists μ ∈ Λ^+ such that μ ≤ λ_1, μ ≤ λ_2 and

\[ (λ_1 − D) ∩ (λ_2 − D) = μ − D. \]

In particular, we have P_λ_1 ∩ P_λ_2 = P_μ.

**Proof.** Since κ_G(λ_1) = κ_G(λ_2), the difference λ_1 − λ_2 lies in the coroot lattice. There exists a partition of the set of simple coroots \( Δ^< = Δ^<_1 ∪ Δ^<_2 \) such that

\[ λ_1 − λ_2 = β_1 − β_2 \]

where β_i is a non-negative integral linear combinations of simple coroots in Δ^<_i for \( i ∈ \{1, 2\} \). Let Δ = Δ_1 ∪ Δ_2 be the corresponding partition of the set of simple roots. Consider the coweight μ := λ_1 − β_1 = λ_2 − β_2. Then clearly μ ≤ λ_1 and μ ≤ λ_2.

We claim that μ ∈ Λ^+. Take any simple root α ∈ Δ_1. Since β_1 is positive linear combination of coroots in Δ_2, we have \( ⟨α, β_2⟩ ≤ 0 \) and hence \( ⟨μ, α⟩ = ⟨λ_2 − β_2, α⟩ ≥ 0 \). Similarly, using μ = λ_1 − β_1, we see that for all α ∈ Δ_2, \( ⟨μ, α⟩ ≥ 0 \). Thus we conclude that μ ∈ Λ^+.

It is clear that μ − D ⊂ (λ_1 − D) ∩ (λ_2 − D). Now we prove the reverse inclusion. Let ν ∈ (λ_1 − D) ∩ (λ_2 − D). Then for \( i ∈ \{1, 2\} \), λ_1 − ν ∈ D is a non-negative Q-linear combination of simple coroots and we need to show that μ − ν ∈ D. For any fundamental weight ω, there exists \( i ∈ \{1, 2\} \) so that ω is orthogonal to all coroots in Δ^<_i. Without loss of generality assume \( i = 1 \), then we have

\[ ⟨μ − ν, ω⟩ = ⟨λ_1 − β_1 − ν, ω⟩ = ⟨λ_1 − ν, ω⟩ ≥ 0. \]

This means that ν ≤_Q μ, or ν ∈ (μ − D). Therefore we have shown that μ − D = (λ_1 − D) ∩ (λ_2 − D).

Finally, taking intersection with Λ_Q^+, we get P_λ_1 ∩ P_λ_2 = P_μ. □
For each $\lambda \in \Lambda^+$, define
\begin{equation}
P^\circ_\lambda := P_\lambda - \bigcup_{\mu \in \Lambda^+, \mu < \lambda} P_\mu.
\end{equation}

**Corollary 3.9.3.** For any $\lambda_1, \lambda_2 \in \Lambda_+$ with $\lambda_1 \neq \lambda_2$, we have $P^\circ_{\lambda_1} \cap P^\circ_{\lambda_2} = \emptyset$. In particular, we get a well-defined stratification
\[
\{ \nu \in \Lambda^+_G \mid p_{G, \mathcal{Q}}(\nu) \in X_*(G_{ab}) \subset \pi_1(G)_{\mathcal{Q}} \} = \bigcup_{\lambda \in \Lambda^+} P^\circ_{\lambda}.
\]

**Proof.** If $\det(\omega^{\lambda_1}) \neq \det(\omega^{\lambda_2})$, it is clear that $P_{\lambda_1}$ and $P_{\lambda_2}$ are disjoint. Suppose $\det(\omega^{\lambda_1}) = \det(\omega^{\lambda_2})$. Then by Lemma 3.9.2, there exists $\mu \in \Lambda^+$ such that $\mu \leq \lambda_1, \mu \leq \lambda_2$ and $P^\circ_{\lambda_1} \cap P^\circ_{\lambda_2} \subset P_{\lambda_1} \cap P_{\lambda_2} = P_{\mu}$.

But by (3.11), we have $P_\mu \cap P^\circ_{\lambda_i} = \emptyset$ since $\mu \leq \lambda_i$ for $i \in \{1, 2\}$. Therefore $P^\circ_{\lambda_1} \cap P^\circ_{\lambda_2} = \emptyset$. \qed

**3.9.4. Stratification on extended Steinberg base.** To get a conceptually simpler formulation of the conjecture on irreducible components, we introduce a stratification on $\mathcal{C}_+(O) \cap \mathcal{C}_{Gc}^e$.

Recall that $\mathcal{C}_+ \cong A_{Gc} \times A^r$. Consider the strata
\[
\mathcal{C}^\lambda_+ := \omega^{-w_0(\lambda_{ad})} T_{ad}(O) \times O^r \subset \mathcal{C}_+(O)
\]
where $\lambda_{ad} \in X_*(T_{ad})_+ \text{ is the image of } \lambda$.

For each $\mu \in \Lambda^+$ such that $\mu \leq \lambda$, we have an embedding
\[
i_{\mu \lambda} : \mathcal{C}^\mu_+ \hookrightarrow \mathcal{C}^\lambda_+
\]
defined by the formula
\begin{equation}
i_{\mu \lambda}(a_1, \ldots, a_r, b_1, \ldots, b_r) = (\omega(-w_0(\lambda - \mu), a_1), \ldots, \omega(-w_0(\lambda - \mu), a_r), \omega(-w_0(\lambda - \mu), b_1), \ldots, \omega(-w_0(\lambda - \mu), b_r))
\end{equation}

Note that we need to choose a uniformiser to define the embedding $i_{\mu \lambda}$ but its image does not depend on this choice.

**Proposition 3.9.5.** For any $\lambda, \mu_1, \mu_2 \in \Lambda^+$ with $\mu_1 \leq \lambda$ and $\mu_2 \leq \lambda$, there exists $\mu_3 \in \Lambda^+$ such that $\mu_3 \leq \mu_1, \mu_3 \leq \mu_2$ and
\[
i_{\mu_1 \lambda}(\mathcal{C}^\mu_1) \cap i_{\mu_2 \lambda}(\mathcal{C}^\mu_2) = i_{\mu_3 \lambda}(\mathcal{C}^\mu_3).
\]

**Proof.** By Lemma 3.9.2, there exists $\mu_3 \in \Lambda^+$ such that
\begin{equation}(\mu_1 - D) \cap (\mu_2 - D) = \mu_3 - D
\end{equation}
To prove the proposition, it suffices to show that
\[
i_{\mu_1 \lambda}(\mathcal{C}^\mu_1) \cap i_{\mu_2 \lambda}(\mathcal{C}^\mu_2) \subset i_{\mu_3 \lambda}(\mathcal{C}^\mu_3)
\]
Let $\iota$ be the involution on the set $\{1, \ldots, r\}$ such that $\omega_{i}(\iota) = -w_0(\omega_i)$ for all $1 \leq i \leq r$. For each $c = (c_1, \ldots, c_r) \in O^r$, let $a_i := \text{val}(c_{i(\iota)})$.

Suppose that $\omega(-w_0(\lambda_{ad}), c) \in i_{\mu_1 \lambda}(\mathcal{C}^\mu_1) \cap i_{\mu_2 \lambda}(\mathcal{C}^\mu_2)$, then we get
\begin{equation}a_i \geq \langle \lambda - \mu_1, \omega_i \rangle \text{ and } a_i \geq \langle \lambda - \mu_2, \omega_i \rangle \text{ for all } 1 \leq i \leq r
\end{equation}
and we need to show that $a_i \geq \langle \lambda - \mu_3, \omega_i \rangle$ for all $1 \leq i \leq r$.

Let $\mu'_1 := \sum_{i=1}^r (\mu_1, \omega_i) a_i$ and define $\mu'_2, \mu'_3$. Then we have $\mu'_1, \mu'_2, \mu'_3 \in \Lambda^+_0$. Consider the coweight $\nu := \sum_{i=1}^r (\langle \lambda, \omega_i \rangle - a_i) a_i \in A_0$. By (3.13) and (3.14) we have
\[
\nu \in (\mu'_1 - D) \cap (\mu'_2 - D) = \mu'_3 - D.
\]
This implies that
\[
\langle \lambda, \omega_i \rangle - a_i = \langle \nu, \omega_i \rangle \leq \langle \mu'_3, \omega_i \rangle = \langle \mu_3, \omega_i \rangle
\]
which is what we want. \qed
For any $\lambda, \mu \in \Lambda^+$ with $\mu \leq \lambda$, define
\begin{equation}
\mathcal{E}^\lambda_+ := i_{\mu \lambda}(\mathcal{E}^\mu_+) - \bigcup_{\nu \in \Lambda_+ \atop \nu \leq \mu} i_{\nu \lambda}(\mathcal{E}^\nu_+). 
\end{equation}

**Corollary 3.9.6.** For any $\lambda, \mu_1, \mu_2 \in \Lambda^+$ with $\mu_1 \neq \lambda$ and $\mu_2 \leq \lambda$, we have $\mathcal{E}^\mu_1 \cap \mathcal{E}^\mu_2 = \emptyset$. In particular, we get well-defined stratifications
\[
\mathcal{E}^\lambda_+ = \bigsqcup_{\mu \in \Lambda_+ \atop \mu \leq \lambda} \mathcal{E}^\mu_+, \quad \mathcal{E}^\nu_{G}(F) \cap \mathcal{E}^\lambda_+ = \bigsqcup_{\lambda, \mu \in \Lambda_+ \atop \mu \leq \lambda} \mathcal{E}^\lambda_+
\]

**Proof.** The argument is similar to the proof of Corollary 3.9.3, using Proposition 3.9.5 instead of Lemma 3.9.2. \qed

The following lemma relates the strata (3.15) the strata (3.11).

**Lemma 3.9.7.** For any $\lambda \in \Lambda^+$ and $\gamma \in G(F)^s$ with $\nu_\gamma \leq Q \lambda$, there exists a unique dominant integral coweight $\mu \in \Lambda_+$ with $\mu \leq \lambda$ that satisfies any (hence all) of the following equivalent conditions:

1. $\mu \in \Lambda_+$ is a minimal dominant integral coweight such that $\nu_\gamma \leq Q \mu$;
2. $\nu_\gamma \in P^\mu_\gamma$, cf. (3.11);
3. $\chi_+^\gamma(\gamma) \in \mathcal{E}^\lambda_+$, cf. (3.15).

**Proof.** The equivalence between (1) and (2) follows from the definition of $P_\mu$. The equivalence of (1) and (3) follows from Proposition 3.1.6.

Finally, the uniqueness of $\mu$ follows from Lemma 3.9.2 or Proposition 3.9.5. \qed

Now we state our conjecture on irreducible components of $X_\gamma^\lambda$:

**Conjecture 3.9.8.** Let $\lambda \in \Lambda^+$ and $\gamma \in G(F)^s$ with $\nu_\gamma \leq Q \lambda$. Let $\mu \in \Lambda_+$ be the “best integral approximation” of $\nu_\gamma$, i.e. the unique dominant coweight that satisfies the equivalent conditions in Lemma 3.9.7. Then the number of $G^0_\gamma(F)$-orbits on $\text{Irr}(X_\gamma^\lambda)$ equals to the weight multiplicity $m_{\lambda \mu}$.

By Conjecture 3.5.3, this conjecture is true when $\gamma$ is an unramified conjugacy class.

**Remark 3.9.9.** For irreducible components of affine Deligne-Lusztig varieties, there is a similar conjecture made by Chen-Zhu, see the discussion in [HV17] and [XZ17]. In their setting, they also approximate Newton points of twisted conjugacy classes by integral coweight. However, the “best integral approximation” as defined in [HV17] is the largest integral coweight dominated by the Newton point. Whereas in the formulation of Conjecture 3.9.8, we use the smallest integral coweight dominating the Newton point. Simple examples suggest that these two integral approximations are very likely in the same Weyl group orbit, so we expect the two weight multiplicities to be the same.

3.9.10. **Components of the regular locus.** The $G^0_\gamma(F)$-orbits on $\text{Irr}(X_\gamma^\lambda, \text{reg})$ corresponds bijectively to $G^0_\gamma(F)$ orbits on $X_\gamma^\lambda, \text{reg}$, which are precisely the $P_\gamma$-orbits of maximal dimension on $\text{Sp}^0_a \cong X_\gamma^\lambda$. We know from Proposition 3.3.1 that these are the varieties $X_\gamma^\lambda w = \text{Sp}^w_\gamma$ for $w \in \text{Cox}(W, S)$.

However, for two different $w, w' \in \text{Cox}(W, S)$, $X_\gamma^\lambda w$ and $X_\gamma^\lambda w'$ might coincide. For example, in the case $\lambda = 0$ and $\gamma \in G(O)$, all $X_\gamma^\lambda w$ coincide (hence equal to $X_\gamma^\lambda, \text{reg}$). So in this particular case $X_\gamma^\lambda, \text{reg}$ is the unique $P_\gamma$-orbit of maximal dimension. In general, we know from (2.7) that the number of $G^0_\gamma(F)$ orbits in $X_\gamma^\lambda, \text{reg}$ is bounded above by the Cardinality of $\text{Cox}(W, S)$. We will see that in many situations, this upper bound can be achieved (in other words $X_\gamma^\lambda w$ are mutually disjoint).

**Theorem 3.9.11.** Let $\lambda \in \Lambda^+$ and $\gamma \in G(F)^s$ with $\nu_\gamma \leq Q \lambda$. Let $\mu \in \Lambda^+$ be the “best integral approximation” of the Newton point $\nu_\gamma$, as in Lemma 3.9.7. Then we have an inequality
\[
|\{G^0_\gamma(F) \text{ orbits on } X_\gamma^\lambda, \text{reg}\}| \leq |\text{Cox}(W, S)|
\]
where $\text{Cox}(W, S)$ is the set of $S$-Coxeter elements defined in Definition 2.2.1. Moreover, when $\lambda$ lies in the interior of the Weyl chamber and $\lambda - \mu$ lies in the interior of the positive coroot cone, the equality is achieved.
Proof. It remains to show the last statement. Suppose λ lies in the interior of the Weyl chamber and λ − µ lies in the interior of the dominant coroot cone. Consider the following Cartesian diagram

\[
\begin{array}{ccc}
\chi_+^{-1}(a) & \longrightarrow & \text{Vin}_{G^c} \\
\downarrow & & \downarrow \\
\text{Spec } \mathcal{O} & \overset{a}{\longrightarrow} & \mathcal{C}_+ \\
\end{array}
\]

For \( g \in G(F) \) such that \( gG(O) \in X^\gamma_{\text{reg}} \), let \( \text{Ad}(g)^{-1} \gamma \) be the reduction mod \( \mathcal{C} \) of \( \text{Ad}(g)^{-1} \gamma \in \text{Vin}_{G^c}^\text{reg}(O) \). The condition that \( \lambda \) lies in the interior of the Weyl chamber means that \( \langle \lambda, \alpha_i \rangle > 0 \) for all simple roots \( \alpha_i \). Hence the special fiber of \( \chi_+^{-1}(a) \) lies in the asymptotic semigroup \( \text{As}(G^c) := \alpha^{-1}(0) \) and in particular \( \text{Ad}(g)^{-1} \gamma \in \text{As}(G^c) \cap \text{Vin}_{G^c}^\text{reg} \).

Furthermore, the assumption that \( \lambda - \mu \) lies in the interior of the positive coroot cone implies that \( \langle \lambda - \mu, \omega_i \rangle > 0 \) for all fundamental weight \( \omega_i \). Therefore the reduction mod \( \mathcal{C} \) of \( a \) equals to 0 and the special fiber of \( \chi_+^{-1}(a) \) is the nilpotent cone \( \mathcal{N} \). In particular, we get \( \text{Ad}(g)^{-1} \gamma \in \mathcal{N}^\text{reg} \).

Consequently there is a bijection between \( G^c_{\gamma}(F) \) orbits on \( X^\lambda_{\text{reg}} \) and \( G \) orbits on \( \mathcal{N}^\text{reg} \), the latter of which corresponds bijectively to \( \text{Cox}(W, S) \) by Proposition 2.2.8. □

As an immediate consequence, we mention the following purely combinatorial result, which might be of independant interest:

**Corollary 3.9.12.** Let \( \lambda \geq \mu \) be dominant weights of a complex reductive group \( G \). Suppose that \( \lambda \) lies in the interior of the Weyl chamber and \( \lambda - \mu \) lies in the interior of the positive root cone (the “wide cone”). Then we have the following lower bound for the weight multiplicity

\[
m_{\lambda, \mu} \geq |\text{Cox}(W, S)|
\]

where the set \( \text{Cox}(W, S) \) is defined in §2.2.1.

**Proof.** We consider the dual group \( G' \) of \( G \) over \( k \). Then \( \lambda \geq \mu \) are dominant coweights for \( G' \). Let \( T' \subset G' \) be a maximal torus and \( \gamma \in \mathfrak{a}_{T'(O)} \cap G'(F)^{rs} \). Then the generalized affine Springer fibre \( X^\lambda_{\gamma} \) is nonempty and by Corollary 3.5.3, the number of \( G^c_{\gamma, 0}(F) \)-orbits on \( \text{Irr}(X^\lambda_{\gamma}) \) equals to \( m_{\lambda, \mu} \). On the other hand, by Theorem 3.9.11, the number of \( G^c_{\gamma, 0}(F) \)-orbits on \( \text{Irr}(X^\lambda_{\gamma}) \) equals to \( |\text{Cox}(W, S)| \), hence the inequality. □

**Remark 3.9.13.** If \( G_{\text{ad}} \) is simple of rank \( r \), then \( |\text{Cox}(W, S)| = 2^{r-1} \). In general, if the simple factors of \( G_{\text{ad}} \) has rank \( r_1, \ldots, r_m \), then

\[
|\text{Cox}(W, S)| = \prod_{i=1}^{m} 2^{r_i-1}.
\]

We expect that there should be a more straightforward proof of Corollary 3.9.12.

**Remark 3.9.14.** In general, the weight multiplicity \( m_{\lambda, \mu} \) will increase with \( \lambda \) while the right hand side in Corollary 3.9.12 is a fixed constant independant of \( \lambda, \mu \). Thus in general there will be much more irreducible components in \( X^\lambda_{\gamma} \) than the regular open subvariety \( X^\lambda_{\gamma} \text{reg} \).

4. THE HITCHIN-FRENKEL-Ngô FIBRATION

In this section we study global analogue of Kottwitz-Viehmann varieties, the Hitchin-Frenkel-Ngô fibration. These are certain group analogue of Hitchin fibrations, first introduced in [FN11] and later studied in more detail in [Bou17] and [Bou15b].

Throughout this section we let \( X \) be a projective smooth curve of genus \( g \) over \( k \) and \( G \) a connected reductive group over \( k \).

4.1. First definitions. Let \( \mathcal{L} \) be a \( \mathbb{Z}^c_{\text{sc}} = T^c \) torsor on \( X \). Then we can twist the schemes \( \text{Vin}_{G^c} \) (resp. \( \mathcal{C}_+, A_{G^c} \)) by \( \mathcal{L} \) to form corresponding affine spaces \( \text{Vin}_{G^c}^\mathcal{L} \) (resp. \( \mathcal{C}_+^\mathcal{L}, A_{G^c}^\mathcal{L} \)) over \( X \).

**Definition 4.1.1.** The Hitchin-Frenkel-Ngô moduli stack associated to the \( T^c \)-torsor \( \mathcal{L} \) is the mapping stack

\[
\mathcal{M}_\mathcal{L} := \text{Hom}(X, [\text{Vin}_{G^c}^\mathcal{L}, \text{Ad}(G)])
\]
In other words, $M_L$ classifies pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a $G$-torsor on $X$ and $\varphi$ is a section of $\mathcal{E} \otimes^G \text{Vin}_{G_{sc}}^L$ where $G$ acts on $\text{Vin}_{G_{sc}}^L$ by adjoint action, and the action factors through $G_{ad}$. We refer to such pairs $(\mathcal{E}, \varphi)$ as Higgs-Vinberg pairs.

Replacing $\text{Vin}_{G_{sc}}$ by $\text{Vin}_{G_{sc}}^\text{reg}$ (resp. $\text{Vin}_G^\text{reg}$) in the definition of $M_L$, we define open substacks $M_{L}^0$ (resp. $M_L^\text{reg}$ $\subset M_L$). Also we define

$$A_L := \text{Hom}_X(X, \mathcal{E}_L), \quad B_L := \text{Hom}_X(X, \mathcal{E}_{G_{sc}}^L)$$

as the space of sections of the affine space $\mathcal{E}_L$ (resp. $\mathcal{E}_{G_{sc}}^L$) over $X$. More concretely, we can describe $A_L$ and $B_L$ as follows.

For each $\omega \in X^*(T)$, let $\omega(L)$ be the invertible sheaf on $X$ defined by pushing $L$ along the morphism $\omega : T \to G_m$. Then we have

$$B_L = H^0(X, A_{G_{sc}}^L) = \bigoplus_{i=1}^r H^0(X, \alpha_i(L))$$

and

$$A_L = B_L \oplus \bigoplus_{i=1}^r H^0(X, \omega_i(L)).$$

**Definition 4.1.2.** The Hitchin-Frenkel-Ngô fibration is the morphism

$$h_L : M_L \to A_L$$

induced by $\chi_+ : \text{Vin}_{G_{sc}} \to \mathcal{E}_+.$

Let $\beta_L : A_L \to B_L$ be the natural projection and $\alpha_L := \beta_L \circ h_L : M_L \to B_L$ be the map induced by $\alpha : \text{Vin}_{G_{sc}} \to A_{G_{sc}}$. We call the fibres of $\alpha_L$ restricted Hitchin-Frenkel-Ngô moduli stack.

4.1.3. Each point $b \in B_L$ can be written as $b = (b_1, \ldots, b_r)$ where $b_i \in H^0(X, \alpha_i(L))$. Let $B_L^\text{sc} \subset B_L$ be the open subset consisting of those $b$ such that $b_i$ is nonzero for all $i$. To each point $b \in B_L^\text{sc}$, we can associate an $X_+(T_{ad})_+$-valued divisor $\lambda_b$ on $X$ defined by

$$\lambda_b := \sum_{i=1}^r \tilde{\omega}_i D(b_i)$$

where $D(b_i)$ is the effective divisor on $X$ associated to $b_i$ and $\tilde{\omega}_i$ is the $i$-th fundamental coweight. For any $a \in A_L$ with $\beta_L(a)_+ \in B_L^\text{sc}$, we denote $\lambda_a := \lambda_{\beta_L(a)}$.

**Definition 4.1.4.** The generically regular semisimple locus $A_L^\text{reg}$ is the open subset of $A_L$ consisting of sections $a : X \to \mathcal{E}_L^+$ such that $\beta_L(a) \in B_L^\text{sc}$ and $a(X)$ generically lies in the open subset $\mathcal{E}_+^{\text{reg}, L} = \mathcal{E}_L^+ - \mathcal{D}_L^+$.

4.1.5. Global Steinberg section. Let $c = |Z(G_{\text{der}})|$ be the order of the center of the derived group of $G$. Suppose there exists a $T^{\text{reg}}$-torsor $\mathcal{L}'$ such that $\mathcal{L} \cong (\mathcal{L}')^{\otimes c}$. By definition, there is a canonical map $[ev]_L : A_L \times X \to [\mathcal{E}_+/T^{\text{reg}}]$ making the following diagram commutative:

$$\begin{array}{ccc}
A_L \times X & \xrightarrow{[ev]_L} & [\mathcal{E}_+/T^{\text{reg}}] \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mathcal{L}} & T^{\text{reg}}
\end{array}$$

Here the left arrow is projection to $X$ and the bottom arrow corresponds to the $T^{\text{reg}}$-torsor $\mathcal{L}$.

The choice of $c$-th root $\mathcal{L}'$ of $\mathcal{L}$ defines a morphism $[ev]_{L'} : A_L \times X \to [\mathcal{E}_+/T^{\text{reg}}]$ lifting $[ev]_L$. Then for each $w \in \text{Cox}(W, S)$ (cf. Definition 2.2.1), the composition of $[ev]_{L'}$ and the section $e_L^{w, [c]}$ of $[\mathcal{E}_+/T^{\text{reg}}]$ (cf. Proposition 2.2.19) induces a section of $h_L$: $e_L^{w} : A_L \to M_{L}^\text{reg} \subset M_L$.

We refer to $e_L^{w}$ as the global Steinberg section.
4.2. Symmetries of Hitchin-Frenkel-\Ngo fibration.

**Definition 4.2.1.** Let $\mathcal{P}_\mathcal{L}$ be the Picard stack over $\mathcal{A}_\mathcal{L}$ that associates to any $S$-point $a \in \mathcal{A}_\mathcal{L}(S)$ the Picard groupoid $\mathcal{P}_a$ of $\mathcal{J}_a$ torsors on $X \times S$. Here $\mathcal{J}_a$ is the pull back of the universal centralizer $\mathcal{J}_\mathcal{L}$ on $\mathcal{C}_+^\mathcal{L}$ along the map $a : X \times S \to \mathcal{C}_+^\mathcal{L}$.

**Proposition 4.2.2.** $\mathcal{P}_\mathcal{L}$ is a smooth Picard stack over $\mathcal{A}_\mathcal{L}$.

*Proof.* The argument of [\Ngo10, Proposition 4.3.5] generalize verbatim to our situation. The point is that $\mathcal{J}_a$ is a smooth group scheme and the obstruction to deforming a $\mathcal{J}_a$-torsor lives in $H^2(X, \text{Lie}(\mathcal{J}_a))$, which vanishes since $X$ is a curve.

The action of $\mathcal{B}\mathcal{J}$ on $[\text{Vin}_{\mathcal{G}_0}/\text{Ad}(G)]$ (resp. $[\text{Vin}^\text{reg}_{\mathcal{G}_0}/\text{Ad}(G)]$) induces action of $\mathcal{P}_\mathcal{L}$ on $\mathcal{M}_\mathcal{L}$ (resp. $\mathcal{M}^\text{reg}_\mathcal{L}$).

To understand the connected components of the fibres of $\mathcal{P}_\mathcal{L}$, we utilize camera covers.

**Definition 4.2.3.** The camera cover associated to each $a \in \mathcal{A}_\mathcal{L}(k)$ is the finite flat cover $\pi_a : \tilde{X}_a \to X$ defined by the following Cartesian diagram

\[
\begin{array}{c}
\tilde{X}_a \\
\downarrow \pi_a \\
X \\
\downarrow a \quad \downarrow \mathcal{C}_+^\mathcal{L}
\end{array}
\]

For any closed point $a \in \mathcal{A}_\mathcal{L}^\circ$, we define the discriminant divisor for $a$ to be the effective divisor

\[\Delta_a := a^{-1}(\mathcal{D}_+^\mathcal{L})\]

Over the nonempty open subset $U_a := X - \Delta_a$, the camera cover $\pi_a$ is Galois étale with Galois group $W$. Choosing a point $\tilde{u} \in \tilde{X}_a$ with $u := \pi_a(\tilde{u}) \in U_a$, we get a homomorphism

\[\rho_a : \pi_1(U_a, u) \to W\]

whose image is a subgroup $W_a \subset W$. Note that the conjugacy class of $W_a$ in $W$ is independent of the choice of base point $\tilde{u}$.

Let $\mathcal{J}_a^0 \subset \mathcal{J}_a$ be the fibrewise neutral component and consider the Picard stack $\mathcal{P}_a' := \text{Bun}_{\mathcal{J}_a^0}$ of $\mathcal{J}_a^0$-torsors on $X$. Then there is a natural homomorphism of Picard stacks $\mathcal{P}_a' \to \mathcal{P}_a$. The following Lemma is parallel to [\Ngo10, Lemme 4.10.2] with exactly the same proof.

**Lemma 4.2.4.** The homomorphism $\mathcal{P}_a' \to \mathcal{P}_a$ is surjective with finite kernel. Same is true for the induced homomorphism $\pi_0(\mathcal{P}_a') \to \pi_0(\mathcal{P}_a)$.

**Corollary 4.2.5.** $\pi_0(\mathcal{P}_a)$ is finite if and only if $T^{W_a}$ is finite.

*Proof.* By previous lemma, $\pi_0(\mathcal{P}_a)$ is finite if and only if $\pi_0(\mathcal{P}_a')$ is finite. By [\Ngo06, Corollaire 6.7], $\pi_0(\mathcal{P}_a') = \tilde{T}^{W_a}$. Since the finiteness of $T^{W_a}$ is equivalent to the finiteness of $\tilde{T}^{W_a}$, the result follows. \[\square\]

**Definition 4.2.6.** The anisotropic locus is the subset $\mathcal{A}_\mathcal{L}^\text{ani} \subset \mathcal{A}_\mathcal{L}^\circ$ consisting of $a \in \mathcal{A}_\mathcal{L}^\circ$ such that the component group $\pi_0(\mathcal{P}_a)$ is finite.

For each subset $I \subset \Delta$, we consider the invariant quotient $T^{W_I}_+$. Then the natural morphism $T^{W_I}_+ : \mathcal{C}_+^\mathcal{L}$ is finite and $Z^\mathcal{L} = T^{\mathcal{L}}$ equivariant. Denote $T^{W_I,\mathcal{L}}_+ := T^{W_I}_+ \times Z^\mathcal{L}$. Let $\mathcal{A}_\mathcal{L}^{W_I} := H^0(X, T^{W_I,\mathcal{L}}_+)$ be the space of sections of the affine scheme $T^{W_I,\mathcal{L}}_+$ over $X$. Consider the map

\[\nu_I : \mathcal{A}_\mathcal{L}^{W_I} \to \mathcal{A}_\mathcal{L}\]

induced by the finite morphism $T^{W_I,\mathcal{L}}_+ \to \mathcal{C}_+^\mathcal{L}$. Let $\mathcal{A}_\mathcal{L}^{W_I,\circ} := \nu_I^{-1}(\mathcal{A}_\mathcal{L}^\circ)$.

**Proposition 4.2.7.** Suppose $G$ is semisimple. Then the complement of $\mathcal{A}_\mathcal{L}^\text{ani}$ in $\mathcal{A}_\mathcal{L}^\circ$ is a finite union

\[\mathcal{A}_\mathcal{L}^\circ \setminus \mathcal{A}_\mathcal{L}^\text{ani} = \bigcup_{I \subset \Delta} \nu_I(\mathcal{A}_\mathcal{L}^{W_I,\circ}).\]
Proof. Let \( a \in \mathcal{A}_L^\varnothing - \mathcal{A}_L^{\text{ani}} \). Then by Corollary 4.2.5, \( T^W_a \) contains a nontrivial torus \( S \). Since \( G \) is semisimple, the centralizer of \( S \) is a proper Levi subgroup of \( G \) whose simple roots form a proper subset \( I \not\subseteq W \). Then we have \( W_a \subset W_I \).

Consider the following diagram in which both squares are Cartesian:

\[
\begin{array}{ccc}
\tilde{X}_a & \xrightarrow{\pi_a} & X \\
\downarrow & & \downarrow \\
T^\varnothing_+ & \xrightarrow{\pi_I} & \mathcal{E}_+ \\
\end{array}
\]

Let \( \tilde{Y}_a \subset \tilde{X}_a \) be the union of all irreducible components that contain a point in the \( W_I \)-orbit of \( \tilde{u} \). Then the image of \( \tilde{Y}_a \) in \( Y_a \) is isomorphic to \( X \) and hence gives a section of the morphism \( \pi_I \). In other words, there is a section \( a_I : X \to T^\varnothing_{W,\mathcal{L}} \) such that \( \nu_I(a_I) = a \). This proves that

\[
\mathcal{A}_L^\varnothing \setminus \mathcal{A}_L^{\text{ani}} \subset \bigcup_{I \not\subseteq \Delta} \nu_I(\mathcal{A}_L^{W_I,\varnothing}).
\]

Conversely, for any \( I \not\subseteq \Delta \) and \( a_I \in \mathcal{A}_L^{W_I,\varnothing} \) with \( \nu_I(a_I) = a \), the morphism \( \pi_I \) in the diagram above has a section given by \( a_I \). This implies that \( W_a \subset W_I \) so that \( T^W_a \) is not finite. By Corollary 4.2.5 again we see that \( a \in \mathcal{A}_L^{\text{ani}} \).

\textbf{Corollary 4.2.8.} Suppose \( G \) is semisimple. Then \( \mathcal{A}_L^{\text{ani}} \) is an open subset of \( \mathcal{A}_L^\varnothing \). Moreover, for any \( b \in \mathcal{B}_L^\varnothing \) and any integer \( N \) with \( N > \max\{2g - 2, r g\} \), if \( \deg \omega_i(\mathcal{L}) > N \) for all \( 1 \leq i \leq r \), then the complement of \( \mathcal{A}_L^{\text{ani}} \) in \( \mathcal{A}_{L,b}^\varnothing \) has codimension at least \( N - r g \).

\textbf{Proof.} By valuative criterion and [Ngô06, Lemme 7.3] we see that \( \nu_I \) is proper. So the images \( \nu_I(\mathcal{A}_{W_I,\varnothing}) \) are closed subsets of \( \mathcal{A}_L^\varnothing \) and their complement \( \mathcal{A}_L^{\text{ani}} \) is open. It remains to calculate the dimension of \( \mathcal{A}_L^{W_I} \).

Let \( I \not\subseteq \Delta \) and \( L_I \) a corresponding Levi subgroup of \( G^\text{sc} \). We label the fundamental weights \( \omega_1, \ldots, \omega_s \) of \( G^\text{sc} \) so that \( \omega_1, \ldots, \omega_s \) are fundamental weights for \( L_I \) where \( s = |I| < r \). There is a natural morphism

\[
q_I : T^\varnothing_{W_I} \to G^\text{sc} \times \mathbb{A}^s
\]

given by the \( W_I \)-invariant functions \( (\alpha_i, 0) \) for \( 1 \leq i \leq r \) and \( (\omega_i, \chi^I_{\omega_i}) \) for \( 1 \leq i \leq s \), where \( \chi^I_{\omega_i} \) is the character of the irreducible representation of \( L_I \) with highest weight \( \omega_i \). The map \( q_I \) induces a map

\[
q_X^I : \mathcal{A}_{L,\varnothing} \to \mathcal{B}_L^\varnothing \oplus \bigoplus_{i=1}^s H^0(X, \omega_i(\mathcal{L}))
\]

The fibres of \( q_I \) over the open subset \( T_{\text{ad}} \times \mathbb{A}^r \subset G^\text{sc} \times \mathbb{A}^s \) are isomorphic to \( \mathbb{G}_{r-s}^\text{m} \). This implies that the nonempty fibres of \( q_X^I \) are \((k^*)^{r-s}\). Hence

\[
\dim \mathcal{A}_{L,\varnothing} \leq \dim \mathcal{B}_L + \sum_{i=1}^s (\deg(\omega_i(\mathcal{L}))) + 1 - g + r - s.
\]

Therefore, the codimension of \( \mathcal{A}_{L,b}^\varnothing - \mathcal{A}_{L,b}^{\text{ani}} \) is bounded below by

\[
\sum_{i=1}^r (\deg(\omega_i(\mathcal{L}))) + 1 - g - \sum_{i=1}^s (\deg(\omega_i(\mathcal{L}))) + 1 - g + r - s \geq N - r g.
\]

Denote \( \mathcal{M}_L^{\text{ani}} := h^{-1}_L(\mathcal{A}_L^{\text{ani}}) \) the anisotropic open substack. This is nonempty when \( G \) is semisimple. Also, let \( \mathcal{P}_L^{\text{ani}} \) be the restriction of \( \mathcal{P}_L \) to \( \mathcal{A}_L^{\text{ani}} \).

\textbf{Proposition 4.2.9.} \( \mathcal{M}_L^{\text{ani}} \) and \( \mathcal{P}_L^{\text{ani}} \) are Deligne-Mumford stacks.
Proof. Let $(\mathcal{E}, \varphi) \in \mathcal{M}_{\text{ani}}^{\text{reg}}(k)$ and $a = h_{\mathcal{E}}(\mathcal{E}, \varphi)$. Then the $k$-group $\text{Aut}(\mathcal{E}, \varphi)$ classifies sections of the group scheme $\text{Aut}_{\mathcal{G}}(\mathcal{E})_{\mathcal{E}}$ over $X$, which is the closed subscheme of centralizer of $\varphi$ in the group scheme $\text{Aut}_{\mathcal{G}}(\mathcal{E})$.

Choose a geometric point $\bar{\eta}$ over the generic point $\eta$ of $X$. Restricting the fiber to $a$, we obtain a homomorphism $\rho_{\bar{\eta}}^a : \text{Aut}(\bar{\mathcal{E}}) \to W$. Let $W_{\eta}$ be the image of $\rho_{\bar{\eta}}^a$. Furthermore, choose a trivialization of $\mathcal{E}$ over the generic point $\eta$ under which $\varphi$ maps to a regular semisimple element in $T_+(k(X))$. With these choices, we get a closed embeddings $\text{Aut}(\mathcal{E}, \varphi) \subset T_{W_{\eta}}$ and $H^0(X, J_{a}) \subset T_{W_{\eta}}$.

Since $a \in \mathcal{A}_{\text{ani}}$, $T_{W_{\eta}}$ is finite. Since $\text{char}(k)$ is coprime to the order of $W$, $T_{W_{\eta}}$ is finite and $k$-group. This shows that $\mathcal{M}_{\text{ani}}^{\text{reg}}$ and $\mathcal{P}_{\text{ani}}^\text{reg}$ are Deligne-Mumford stacks. □

**Theorem 4.2.10.** Assume that the $T_{\text{sc}}$-torsor $\mathcal{L}$ admits a $c$-th root $\mathcal{L}'$. Then for any $a \in \mathcal{A}_{\text{ani}}$, there is a homeomorphism of quotient stacks

$$[\mathcal{M}_a/\mathcal{P}_a] \cong \prod_{x \in X - U_a} [\text{Sp}_{a_x}/\mathcal{P}_{a_x}]$$

In particular, we have

$$\dim \mathcal{M}_a - \dim \mathcal{P}_a = \sum_{x \in \text{Supp}(\Delta_a)} (\dim \text{Sp}_{a_x} - \mathcal{P}_{a_x}).$$

**Proof.** Choose a Coxeter element $w \in \text{Cox}(W, S)$. The $c$-th root $\mathcal{L}'$ of $\mathcal{L}$ induces a global Steinberg section $\epsilon_{\mathcal{L}_w}'$, in particular a base point $\epsilon_{\mathcal{L}_w}(a) \in \mathcal{M}_a^{\text{reg}}$. Using Corollary 2.2.15, we argue as in the proof of [Ngô06, Théorème 4.6] to show that there is a morphism as (4.1) inducing equivalence of groupoids on $k$-points. Then the argument of [Ngô10] shows that the map (4.1) is a homeomorphism. □

4.3. **Properness over the anisotropic locus.** Throughout this section, we assume $G$ is semisimple so that $\mathcal{A}_{\text{ani}}$ is nonempty. Our goal is to show that the morphism $h_{\mathcal{L}}^{\text{ani}} : \mathcal{M}_{\text{ani}}^{\text{reg}} \to \mathcal{A}_{\text{ani}}$ is proper.

4.3.1. **Finiteness properties.** We first show that the Hitchin-Frenkel- Ngô fibration is of finite type over the anisotropic locus.

We start with a more general situation. Let $\rho : G \to \text{GL}(V)$ a finite dimensional representation such that $\text{ker}(\rho)$ is contained in the center of $G$. Fix a torus $T$ and a Borel subgroup $B$ containing $T$. Let $V^{(1)}, \ldots, V^{(m)}$ be the irreducible constituents of $V$ (counted with multiplicity) and $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be the corresponding highest weights.

For each $V^{(j)}$, we choose a basis $\{e_{(j)}^{(i)}, 1 \leq i \leq d_j\}$ (where $d_j = \dim V^{(j)}$) as follows. Each $e_{(j)}^{(i)}$ is a weight vector with weight $\lambda_{(j)}^{(i)} \in X^*(T)$. Then we can express $\lambda_{(j)}^{(i)} - \lambda_{(j)}^{(j)}$ as a linear combination of simple roots with non-negative integer coefficients and we call the sum of coefficients the height of $e_{(j)}^{(i)}$. The basis elements $e_{(j)}^{(i)}$ are indexed so that the height is non-decreasing with respect to $i$. In particular, $e_{(j)}^{(1)}$ is a highest weight vector and $e_{(j)}^{(d_j)}$ is a lowest weight vector in $V^{(j)}$.

Then under the basis $\{e_{(j)}^{(i)}, 1 \leq i \leq d_j, 1 \leq j \leq m\}$, $\rho(B)$ consists of upper triangular matrices in $\prod_j \text{End}(V_j)$, which are the stabilizers of the standard flags $0 = L^{(j)}_{0} \subset \cdots \subset L^{(j)}_{d_j} = V^{(j)}$ where $L^{(j)}_{i} = \text{Span}(e_{(1)}^{(i)}, \ldots, e_{(j)}^{(i)})$ for $1 \leq i \leq d_j$.

Let $I \subset \Delta$ be a subset of simple roots and $P_I \subset G$ the standard parabolic subgroup whose Levi factor has simple roots in $I$. Then there exists standard parabolic subalgebras $p_{(j)}^{(I)} \subset \text{End}(V^{(j)})$ such that

$$\rho(P_I) = \rho(G) \cap (\bigoplus_{j=1}^{m} p_{(j)}^{(I)}).$$

More precisely, $p_{(j)}^{(I)}$ is the stabilizer of the partial flag in $V^{(j)}$ obtained from the standard flag by replacing $L^{(j)}_{i}$ with the span of $L^{(j)}_{i}$ and all basis vectors whose corresponding weight differs from the weight of $e_{(j)}^{(i)}$ by a linear combination of simple roots in $I$.

Fix a divisor $D$ on a smooth projective curve $X$. Consider the following stack

$$\mathcal{M}_{X} := \text{Hom}(X, \left(\prod_{j=1}^{m} \text{End}(V^{(j)})(D)/G\right))$$
where the action of $G$ on $\prod_{i=1}^{m} \text{End}(V(i))$ is induced by $\rho$. More concretely, the moduli stack $\mathcal{M}_V$ classifies tuples $(E, \varphi_j, 1 \leq j \leq m)$ where $E$ is a $G$-torsor and $\varphi_j : \rho_j E \to \rho_j E(D)$ is a meromorphic endomorphism of the vector bundle $\rho_j E := E \wedge (G, \rho) V(j)$.

From the definition, we have

$$\mathcal{M}_V = \mathcal{M}_1 \times_{\text{Bun}_G} \mathcal{M}_2 \times_{\text{Bun}_G} \cdots \times_{\text{Bun}_G} \mathcal{M}_m$$

where for each $1 \leq j \leq m$, we define

$$\mathcal{M}_j = \text{Hom}(X, [(\text{End}(V(j))(D))/G]).$$

By ??? we know that there exists a constant $C > 0$ such that for any $G$-torsor $E$ on $X$ there exists a Borel reduction $E_B$ of $E$ so that $\deg(E_B)$ belongs to

$$C := \{ H \in \Lambda_Q, \alpha(H) \geq -c \forall \alpha \in \Delta \}.$$

Let $N$ be a positive integer which is larger than the sum of coefficients of $\lambda^{(j)} - \lambda^{(j)}_i$ under the basis $\Delta$ for all $i, j$. Let $d$ be an integer such that

\begin{equation}
\tag{4.2}
d > \deg(D) + 2Nc
\end{equation}

For each subset $I \subset \Delta$, consider the following cone

$$C_I := \{ H \in \Lambda_Q, \alpha(H) \leq d \forall \alpha \in I \text{ and } \alpha(H) \geq d \forall \alpha \in \Delta - I \}.$$  

**Lemma 4.3.2.** Let $(E, \varphi_j) \in \mathcal{M}_V$ and $E_B$ a $B$-reduction of $E$. Suppose that $\deg E_B \in C \cap C_I$, then we have

$$\varphi \in \mathfrak{p}(D) \wedge^B E_B$$

where $\mathfrak{p} = \bigoplus_{j=1}^{m} \mathfrak{p}_I^{(j)}$

**Proof.** We can treat each factor $\mathcal{M}_I$ separately and assume that $V$ is irreducible. It suffices to prove that under the adjoint action of $\varphi$, $E_B \wedge^B b$ is sent into $E_B \wedge^B \mathfrak{p}(D)$. Consider a filtration of $\text{End}(V(j))$:

$$(0) = b_0 \subset b_1 \subset \cdots \subset b_s = b \subset \mathfrak{p} = \mathfrak{p}_s \subset \mathfrak{p}_{s-1} \subset \cdots \subset \mathfrak{p}_0 = \text{End}(V(j)).$$

stable under adjoint action of $B$, with one-dimensional successive quotients.

Suppose the image of $E_B \wedge^B b$ under $\text{ad}(\varphi)$ is not contained in $E_B \wedge^B \mathfrak{p}(D)$. Then there exists $0 < i \leq r$ and $0 \leq j < s$ such that $\text{ad}(\varphi)$ induces a non-zero homomorphism of line bundles

$$E_B \wedge^B (b_i/b_{i-1}) \to E_B \wedge^B (\mathfrak{p}_j/\mathfrak{p}_{j+1})(D).$$

In particular, the degree of the source is not larger than the degree of the target. More precisely, let $\gamma$ be the weight of $B$ on $b_i/b_{i+1}$ and $\delta$ the weight of $B$ on $\mathfrak{p}_j/\mathfrak{p}_{j+1}$. Then we have the inequality

$$\langle \deg E_B, \gamma - \delta \rangle \leq \deg D.$$

Note that $\gamma$ is the difference between the highest weight $\lambda^{(j)}$ and certain weight of the $G$-representation $V(j)$, hence a non-negative linear combination of simple roots with the sum of coefficients bounded by $N$. Since $\deg E_B \in C$, we then have

$$\langle \deg E_B, \gamma \rangle \geq -Nc.$$

On the other hand, by definition of $\mathfrak{p} = \mathfrak{p}_I^{(j)}$, we see that $-\delta$ is a non-negative linear combination of simple roots such that the sum of coefficients is bounded by $N$ and the coefficient of some root in $\Delta - I$ is positive. Hence because $\deg E_B \in C \cap C_I$, we have

$$\langle \deg E_B, -\delta \rangle \geq d - Nc.$$

Combining the above two inequalities, we get $d - 2Nc \leq \deg D$ which contradicts (4.2) and thus the lemma follows.

**Proposition 4.3.3.** The stack $\mathcal{M}^{\text{ani}}_E$ is of finite type.
Proof. The natural morphism $\mathcal{M}^\text{ani} \to \text{Bun}_G$ is of finite type. For each $\nu \in X^*(T)$, the moduli stack $\text{Bun}_B^\nu$ of $B$-bundles on $X$ with degree $\nu$ is of finite type. It suffices to show that there is a finite subset $S \subset X^*(T)$ such that the image of $\mathcal{M}^\text{ani}$ in $\text{Bun}_G$ is contained in the image of $\cup_{\nu \in S} \text{Bun}_B^\nu$ in $\text{Bun}_G$.

Let $m = (\mathcal{E}, \phi) \in \mathcal{M}_E^\text{ani}(k)$ and $\mathcal{E}_B$ a $B$-reduction of $\mathcal{E}$ such that $\text{deg}(\mathcal{E}_B) \in C$. Let $a = h_C(m) \in \mathcal{A}_E^\text{ani}$. Suppose that $\text{deg}(\mathcal{E}_B) \in C_I$ for some proper subset $I \subset \Delta$. Then $\varphi$ maps the generic point of the curve into the proper parabolic subgroup $P_{I,+}$ of $G^\nu$. This implies that $W_\nu$ is contained in the Weyl group of the Levi $L_I$ and hence $T^{W_\nu}$ is not finite, contradicting the fact that $a \in \mathcal{A}_E^\text{ani}(k)$. Consequently, we have $\det \mathcal{E}_B$ lies in the intersection of $C$ and the complement of $C_I$ for any proper subgroup $I \subset \Delta$. This intersection is a bounded subset of $X^*(T)\mathbb{R}$ and hence the set of weights $S \subset X^*(T)$ lying in the intersection is a finite set. \hfill $\square$

4.3.4. Valuative criterion. First we have the existence part of the valuative criterion, which is true over the larger open subset $\mathcal{A}_E^{\nu}$.

**Proposition 4.3.5.** Let $R$ be a complete discrete valuation ring with algebraically closed residue field containing $k$. Let $K$ be the fraction field of $R$. Then for all $a \in \mathcal{A}_E^{\nu}(R)$ and $m_K \in \mathcal{M}_E^\text{ani}(K)$ such that $h_C(m_K) = a$, there exists a finite extension $K'$ of $K$ and $m \in \mathcal{M}_E^\text{ani}(R')$, where $R'$ is the integral closure of $R$ in $K'$, such that

1. The image of $m$ in $\mathcal{M}_E^\text{ani}(K')$ is isomorphic to that of $m_K$;
2. $h_C(m) = a$.

Proof. The argument is the same as [CL10, §8.4]. The key points are: 1. Any $G$-torsor extends uniquely over a codimension 2 subset; 2. the universal twisted monoid $\mathcal{V}_G$ over $\mathcal{A}_G \times X$ is affine, so that Higgs fields extend over any codimension 2 subset. \hfill $\square$

**Proposition 4.3.6.** Suppose $G$ is semisimple. Let $R$ be a complete discrete valuation ring with algebraically closed residue field $\kappa$ containing $k$. Let $m, m' \in \mathcal{M}^\text{ani}(R)$ be two elements and $m_K, m'_K \in \mathcal{M}^\text{ani}(K)$ their base change. Suppose that the following two conditions are satisfied:

1. $h(m) = h(m')$;
2. there exists an isomorphism $\iota_K : m_K \to m'_K$.

Then there exists a unique isomorphism $\iota : m \to m'$ extending $\iota_K$.

Proof. We follow the argument in [CL10, §9]. Let $m = (\mathcal{E}, \phi)$ and $m' = (\mathcal{E}', \phi')$. Consider the local ring $B$ of the generic point of the special fiber of $X_R$. Then $B$ is a discrete valuation ring whose residue field is the function field $\kappa(X)$ of $X_k$ and whose fraction field is the function field $F$ of $X_R$.

By §9.2 of loc. cit., it suffices to extend $\iota_K$ to an isomorphism of $G$-torsors $\iota : \mathcal{E} \to \mathcal{E}'$ over Spec $B$. As in §9.3 of loc. cit., it suffices to show that for some finite extension $K'$, the base change $\iota_K'$ of $\iota_K$ extends to an isomorphism between $\mathcal{E}, \mathcal{E}'$ over Spec $B'$. Here $B'$ is the integral closure of $B$ in the function field $F'$ of $X_R'$ where $R'$ is the integral closure of $R$ in $K'$.

To achieve this, after taking a finite extension $K'/K$ one can assume that $\mathcal{E}, \mathcal{E}'$ are trivial over Spec $B$ (since by [DS95, Theorem 2], they will be trivial in a Zariski open neighbourhood of the generic point of the special fibre of $X_R$ after a finite extension of $K$). Moreover, as in [CL10, Lemma 9.3.1], one can choose trivialization of $\mathcal{E}$ and $\mathcal{E}'$ over Spec $B$ such that they map the “Higgs fields” $\phi$ and $\phi'$ to some element $\gamma \in \text{Vin}^n_G(B)$. Under these trivializations, the isomorphism $\iota_K$ is identified with an element $g \in G(F)$ such that $g^{-1}\gamma g = \gamma$. In other words, $g \in G_\gamma(F)$. Since $m, m'$ lies in the anisotropic open substack and $\gamma \in \text{Vin}^n_G(B)$, $G_\gamma$ is an anisotropic torus over Spec $B$ and hence $G_\gamma(B) = G_\gamma(F)$. Thus in particular, $g \in G(B)$ and the isomorphism $\iota_K$ extends. \hfill $\square$

**Theorem 4.3.7.** The morphism $h_C : \mathcal{M}_E^\text{ani} \to \mathcal{A}_E^\text{ani}$ is proper.

Proof. This follows from what have been proved in this section and the valuative criterion of properness for algebraic stacks. \hfill $\square$

4.4. **Singularities of restricted Hitchin-Frenkel-Ngô moduli stack.** Later when proving equi-dimensionality of Kottwitz-Viehmann varieties, we will need the transversality theorem of Bouthier in [Bou17], where it was shown that the singularities of certain open substack of restricted Hitchin-Frenkel-Ngô moduli stack are the same as some closed Schubert varieties in the affine Grassmanian. The method of Bouthier was later simplified by Yun in [Yun15b]. In [Bou17] and [Yun15b] it is assumed that the group is simply-connected but
the argument works without this assumption. For the reader’s convenience we review this result following [Yun15b].

4.4.1. Fix a \(X_1(T_\text{ad}) \)-valued divisor \(\lambda = \sum_{i=1}^m \lambda_i x_i\) on the curve \(X\). Then \(\lambda\) defines a \(T_\text{ad}\)-torsor \(L_\lambda\). We assume that \(L_\lambda\) can be lifted to a \(T^e\)-torsor \(L\). Then \(\lambda\) can be identified with a closed point of \(B_L = H^0(X, A^E_\text{ad})\). Let \(\mathcal{M}_{\leq \lambda} := a^{-1}_L(\lambda)\) be the corresponding restricted Hitchin-Frenkel-Ng\^o moduli stack. Let \(A_{\leq \lambda} := \beta^{-1}(\lambda)\) and \(h_{\leq \lambda} : \mathcal{M}_{\leq \lambda} \to A_{\leq \lambda}\) be the restricted Hitchin-Frenkel-Ng\^o fibration. Let \(\mathcal{M}_\lambda := \mathcal{M}_{\leq \lambda} \cap M_\lambda^c\) be the open substack where the Higgs-Vinberg field lands in \([\text{Vin}_{G_{\text{adm}}}/T^\text{sc} \times \text{Ad}(G)]\).

Assume moreover that \(L\) admits a \(c\)-th root \(L'\) where \(c = |Z(\text{G}_{\text{der}})|\). Then by the discussion in § 4.1.5, there exists global Steinberg section \(c_{\text{ES}}^E : A_{\leq \lambda} \to M_{\leq \lambda}^\text{reg}\) for each choice of Coxeter element \(w \in \text{Cox}(W, S)\).

4.4.2. For each \(a \in A_{\leq \lambda}^C\), we write the associated discriminant divisor as

\[\Delta(a) = \Delta(a)_{\text{sing}} + \Delta(a)_{\text{triv}}\]

where \(\Delta(a)_{\text{triv}}\) is multiplicity free and the multiplicity of \(\Delta(a)_{\text{sing}}\) at each point is at least 2.

**Definition 4.4.3.** Let \(S \subset \text{Supp}(\lambda)\) be a nonempty subset. The transversal subset \(A_{\leq \lambda}^S \subset A_{\leq \lambda}\) consists of \(a \in A_{\leq \lambda}^C\) satisfying the following two conditions

- \(\text{Supp}(\Delta(a)) \cap \text{Supp}(\lambda) \subset S\)
- For each \(1 \leq i \leq r\)

\[2g - 2 + m_0(\deg \Delta(a)_{\text{sing}} + |S|) + \sum_{s \in S} b(\lambda_s) < \deg \omega_i(\mathcal{L})\]

where \(m_0\) is the positive integer defined in the paragraph before Proposition 2.6.6 and \(b(\lambda_s)\) is the non-negative integer in Lemma 2.5.2.

We call \(M_{\leq \lambda}^S := h^{-1}_{\leq \lambda}(A_{\leq \lambda}^S)\) the transversal open substack and denote \(M_\lambda^S := M_{\leq \lambda}^S \cap M_\lambda\).

4.4.4. **Local evaluation map.** For each \(s \in S\), the arc space \(L^+ G\) acts by left multiplication on \(\text{Gr}_{\leq \lambda_s} := \text{Gr}_{\leq \lambda_s}^G\) and the action factors through \(L^+ G_{\text{adm}}\). We let \(N\) be a positive integer such that for all \(s \in S\), the action of \(L^+ G\) on \(\text{Gr}_{\leq \lambda_s}\) factors through the \(N\)-th jet space \(L_N^+ G\). Then the product group \(L^+_N G := \prod_{s \in S} L_N^+ G\) acts naturally on \(\prod_{s \in S} \text{Gr}_{\leq \lambda_s}\) and we define the local evaluation map

\[\text{ev}_{NS} : M_{\leq \lambda} \to [L^+_N G \backslash \prod_{s \in S} \text{Gr}_{\leq \lambda_s}]\]

by choosing trivialisations of \(G\)-torsors on the \(N\)-th infinitesimal neighbourhood of points \(s \in S\). Let \(\text{ev}_{NS}^b\) be the restriction of \(\text{ev}_{NS}\) to \(M_\lambda^b\). From the first condition in Definition 4.4.3, we see that for any \((\mathcal{E}, \varphi) \in M_{\leq \lambda}^b\) the restriction of the Higgs-Vinberg field \(\varphi\) to points in \(\text{Supp}(\lambda) \setminus S\) lands in the open substack \([\text{Vin}_{G_{\text{adm}}}/T^\text{sc} \times \text{Ad}(G)]\), which is contained in \([\text{Vin}_{G_{\text{adm}}}/T^\text{sc} \times \text{Ad}(G)]\) by Corollary 2.2.7. Hence the inverse image of the open strata \([L^+_N G \backslash \prod_{s \in S} \text{Gr}_{\lambda_s}]\) under \(\text{ev}_{NS}^b\) is precisely \(M_\lambda^b = M_{\leq \lambda} \cap M_\lambda\).

**Theorem 4.4.5 ([Bou17],[Yun15b]).** The morphism

\[\text{ev}_{NS}^b : M_\lambda^b \to [L^+_N G \backslash \prod_{s \in S} \text{Gr}_{\leq \lambda_s}]\]

is smooth.

The proof proceeds in several steps which occupy the rest of this section.

4.4.6. Let \(M_{\leq \lambda, NS}\) be the stack classifying triples \((\mathcal{E}, \varphi, \tau_{NS})\) where \((\mathcal{E}, \varphi)\) is a point in \(M_\lambda^b\) and \(\tau_{NS}\) is a trivialisation of \(\mathcal{E}\) on the \(N\)-th infinitesimal neighbourhoods of \(s\) for all \(s \in S\). Then \(M_{\leq \lambda, NS}\) is a \(L^+_N G\)-torsor over \(M_{\leq \lambda}\) and to prove the smoothness of \(\text{ev}_{NS}^b\), it suffices to prove the smoothness of its base change

\[\tilde{\text{ev}}_{NS}^b : M_{\leq \lambda, NS} \to \prod_{s \in S} \text{Gr}_{\leq \lambda_s}\]

Notice that the source and target of \(\tilde{\text{ev}}_{NS}^b\) are locally of finite type. Hence it suffices to show that \(\tilde{\text{ev}}_{NS}^b\) is formally smooth. In other words, we need to check the infinitesimal lifting property.
4.4.7. Let $R$ be an artin local $k$-algebra with maximal ideal $m$ and let $I \subset R$ be an ideal with $I \cdot m = 0$. Denote $\tilde{R} := R/I$. Consider a triple $(\tilde{\mathcal{E}}, \tilde{\varphi}, \tilde{\tau}_{NS}) \in \mathcal{M}_{\leq \lambda}(\tilde{R})$ whose image under the map $\tilde{v}_{NS}^b$ is $((\tilde{\gamma}_s))_{s \in S} \in \prod_{s \in S} \text{Gr}_{\leq \lambda_s}(\tilde{R})$. Let $(\gamma_s)_{s \in S} \in \prod_{s \in S} \text{Gr}_{\leq \lambda_s}(R)$ be a lifting of $((\tilde{\gamma}_s))_{s \in S}$. We need to find a lifting of $(\tilde{\mathcal{E}}, \tilde{\varphi}, \tilde{\tau}_{NS})$ to a point in $\mathcal{M}_{\leq \lambda}(R)$ whose image under $\tilde{v}_{NS}^b$ coincides with $((\gamma_s))_{s \in S}$.

Extend $\tau_{NS}$ to a trivialisation $\tau_{\infty S}$ of $\tilde{\mathcal{E}}$ on $\prod_{s \in S} D_t \otimes \tilde{R}$ lifting $\tau_{\infty S}$.

4.4.8. Let $\tilde{a}_{\lambda} := h_{\leq \lambda}(\tilde{\mathcal{E}}, \tilde{\varphi}) \in \mathcal{A}_{\leq \lambda}(\tilde{R})$ and $a_{\lambda} \in \mathcal{A}_{\leq \lambda}(k)$ its reduction mod $m$. We have the discriminant divisor $\Delta(a_0)$ associated to $a_0$. For each $v \in X$, let $d_v$ be the multiplicity of $\Delta(a_0)$ at $v$. Let $S' := S' \cup \text{Supp}(\Delta(a_0)_{\text{sing}})$ and $T := S' \setminus S$ so that $S' = S \sqcup T$. Note that the first condition in Definition 4.4.3 implies that $T \cap \text{Supp}(\lambda) = \emptyset$.

For each $s \in S$, under the trivialisation $\tau_{\infty S}$, the Taylor expansion of $\tilde{\varphi}$ at $s$ corresponds to an element $\tilde{\gamma}_s \in L^+ \text{Vin}_{\lambda}^\lambda(\tilde{R})$ whose image in $\text{Gr}_{\leq \lambda_s}(\tilde{R})$ is $[\tilde{\gamma}_s]$. Since the morphism $L^+ \text{Vin}_{\lambda}^\lambda \rightarrow \text{Gr}_{\leq \lambda_s}$ is formally smooth, there exists a lifting $\gamma_s \in L^+ \text{Vin}_{\lambda}^\lambda(\tilde{R})$ of $\tilde{\gamma}_s$ whose image in $\text{Gr}_{\leq \lambda_s}$ equals to the $[\gamma_s]$ given above. Let $a_{s} := \chi_+(\gamma_s) \in \mathcal{C}_+(\tilde{O}_{t} \otimes \tilde{R})$. Then $\tilde{a}_s = a \in \mathcal{C}_+(\tilde{O}_{t} \otimes \tilde{R})$.

For each $t \in T = S' \setminus S$, choose a trivialisation $\tau_{\infty t}$ of $\tilde{\mathcal{E}}$ on $D_t \otimes \tilde{R}$, under which the Taylor expansion of $\varphi$ at $t$ corresponds to an element $\gamma_t \in L^+ G_{\tilde{t}}^\lambda(\tilde{R})$. We lift $\gamma_t$ arbitrarily to an element $\gamma_t \in G_{\tilde{t}}^\lambda(\tilde{O}_{t} \otimes \tilde{R})$ and let $a_t := \chi_+(\gamma_t) \in \mathcal{C}_+(\tilde{O}_{t} \otimes \tilde{R})$. Then in particular $\tilde{a}_t = a \in \mathcal{C}_+(\tilde{O}_{t} \otimes \tilde{R})$.

4.4.9. Consider the local evaluation map of the base space

$$\mathcal{A}_{\leq \lambda} \rightarrow \bigoplus_{s \in S} \mathcal{C}_s^+(\mathcal{E}_s/\mathcal{E}_s^{mod_t} \otimes \mathcal{E}_t) \times \bigoplus_{t \in T} \mathcal{C}(\mathcal{O}_t/\mathcal{E}_t^{mod_t}).$$

By the inequality in Definition 4.4.3, this is a surjective linear map between $k$-vector spaces, hence it is smooth when viewed as morphisms between affine $k$-schemes. So there exists $a \in \mathcal{A}_{\leq \lambda}(\tilde{R})$ lifting $a \in \mathcal{A}_{\leq \lambda}(\tilde{R})$ such that $a \equiv a_0 \mod \mathcal{E}_s^{\infty}$ for all $v \in S'$.

Then for each $s \in S$, we have $a \equiv \chi_+(\gamma_s) \mod \mathcal{E}_s^{\infty} \otimes \mathcal{E}_s^{mod_t} \otimes \mathcal{E}_t$. By Proposition 2.6.6 there exists $\theta_s \in \text{Vin}_G(R[[\mathcal{E}_s]])$ such that $\chi_+(\theta_s) = a$ and $\theta_s \equiv \gamma_s \mod \mathcal{E}_s^{\infty} \otimes \mathcal{E}_s^{mod_t} \otimes \mathcal{E}_t$. By Lemma 2.5.2, this implies that the image of $\theta_s$ in $\text{Gr}_{\leq \lambda_s}(R)$ coincides with $[\gamma_s]$.

For each $t \in T$, by Proposition 2.6.6 again, there exists $\theta_t \in G_{\tilde{t}}^\lambda(\tilde{O}_{t} \otimes \tilde{R})$ such that $\chi_+(\theta_t) = a$.

4.4.10. For each $v \in \text{Supp}(\lambda) \setminus S$, the restriction of the Higgs-Vinberg field $\varphi$ to $v$ lands in $[\text{Vin}_G^R/T \times \text{Ad}(G)]$ by the first condition in Definition 4.4.3. For each point $v$ in the complement of $\text{Supp}(\lambda) \cup S' = \text{Supp}(\lambda) \cup T$, the restriction of $\varphi$ to $v$ lands in $[\text{Vin}_G^R/T \times \text{Ad}(G)]$ by Corollary 3.8.3.

Therefore the restriction of $(\tilde{\mathcal{E}}, \tilde{\varphi})$ to $(X - S') \otimes_k R$ lands in the stack $[\text{Vin}_G^R/T \times \text{Ad}(G)]$ which is a $\mathbb{B}J$ gerbe neutralized by a global Steinberg section $e_v^{\infty}$. By the same reasoning as in Proposition 4.2.2, there exists a Higgs-Vinberg pairs $(\mathcal{E}', \varphi')$ over $(X - S') \otimes_k R$ together with trivialisations $\tau_{\infty v}'$ of $\mathcal{E}'$ over the formal punctured disc at each $v \in S'$ that lifts $(\tilde{\mathcal{E}}, \tilde{\varphi})|_{(X - S') \otimes_k R}$ and the restrictions of the trivialisations $\tau_{\infty v}$ to the punctured disc and moreover $\chi_+(\mathcal{E}', \varphi') = a \in \text{Hom}(X - S', \mathcal{E}_t^+)$. 

4.4.11. Finally we construct the desired lifting $(\tilde{\mathcal{E}}, \tilde{\varphi}, \tau_{\infty S})$ by Beauville-Laszlo gluening lemma. For each $v \in S'$, restricting $\varphi'$ to the formal punctured disc $D_{\mathcal{E}_t}^\bullet \otimes_k R$ and using the trivialisation $\tau_{\infty v}'$, we obtain an element $\theta'_v \in G^\lambda_{\tilde{t}}(R((\mathcal{E}_t^+)))$ with $\chi_+(\theta'_v) = a$. Recall that we have constructed elements $\theta_v \in L^+ \text{Vin}_{\lambda}^\lambda(R) \subset G^\lambda_{\tilde{t}}(R((\mathcal{E}_t^+)))$ with $\chi_+(\theta_v) = a$. Since $a \in \mathcal{C}_s^+(\mathcal{E}_s^+ R((\mathcal{E}_t^+) ))$, the transporter $\text{Isom}(\theta_v, \theta'_v)$ from $\theta_v$ to $\theta'_v$ is a torsor under the smooth centralizer $J_{\tilde{t}}^d \otimes \tilde{R}$. After reduction mod $I$, we know that $\theta_v$ and $\theta'_v$ comes from a globally defined Higgs-Vinberg pair $(\mathcal{E}', \varphi')$. In other words, $\text{Isom}(\theta_v, \theta'_v)$ has an $R$-point. By smoothness, this $R$-point lifts to an $R$-point of $\text{Hom}(\theta_v, \theta'_v)$. Consequently by Beauville-Laszlo lemma, the Higgs-Vinberg pairs $(\mathcal{E}', \varphi')$ over $(X - S') \otimes_k R$ with the trivialisations $\tau_{\infty v}'$ over the formal punctured discs $D_{\mathcal{E}_t}^\bullet \otimes_k R$ can be glued with the Higgs-Vinberg pair $(\mathcal{E}_0, \theta_v)$ (where $\mathcal{E}_0$ is the trivial $G$-torsor) on the formal discs $D_{\mathcal{E}_t} R \otimes_k R$ to get a Higgs-Vinberg pair $(\mathcal{E}, \varphi) \in \mathcal{M}_{\leq \lambda}^\bullet(R)$. By construction it comes with tautological trivialisations $\tau_{\infty S}$ on
Moreover, the local Picard $\text{Pic}_{\lambda}^{\mathcal{C}}$ is Cohen-Macaulay and its open substack $\mathcal{M}_{\lambda}$ is smooth.

**Proof.** This follows from Theorem 4.4.5 and the fact that $\text{Gr}_{\leq \lambda}$ is Cohen-Macaulay and $\text{Gr}_{\lambda}$ is smooth for all $s \in S$. □

5. From global to local

In this section we finish the proof of Theorem 1.2.1.

Let $\lambda \in \mathcal{X}(T_{+})$ and $\gamma \in G(F)^{rs}$. Suppose that $\kappa_{\gamma}(\gamma) = p_{G}(\lambda)$ and $\nu_{\gamma} \leq_{Q} \lambda$ so that the generalized affine Springer fibres $X_{\gamma}^{\lambda}$ and $X_{\gamma}^{\leq \lambda}$ are nonempty. Let $a := \chi_{+}(\gamma) \in \mathcal{E}_{+}(O) \cap \mathcal{C}_{G_{sc}^{rs}}^{\mathbf{F}}(F)$ where $\gamma_{\lambda} \in G_{sc}^{rs}(F)$ is defined in Lemma 3.1.5. Then we have isomorphisms

$$X_{\gamma}^{\leq \lambda} \cong \text{Sp}_{a_{\lambda}}, \quad X_{\gamma}^{\lambda} \cong \text{Sp}_{a}^{0}.$$ 

Moreover, the local Picard $\text{Pic}_{\lambda}^{\mathcal{C}}$ acts on $\text{Sp}_{a_{\lambda}}$ and $\text{Sp}_{a}^{rs}$ is the union of open orbits.

5.1. Local constancy of Kottwitz-Viehmann varieties. This subsection is devoted to the proof of the following:

**Theorem 5.1.1.** There exists an integer $N$ such that for all $a \in \mathcal{C}_{+}(O_{\mathbb{C}}) \cap \mathcal{C}_{G_{sc}^{rs}}^{\mathbf{F}}(F)$ with $a^{'\mathbb{C}} \equiv a$ mod $\varpi^{N}$, $\text{Sp}_{a^{\mathbb{C}}}$ equipped with the action of $P_{a^{\mathbb{C}}}$ is isomorphic to $\text{Sp}_{a}$ equipped with the action of $P_{a}$.

First we make some standard reductions. Notice that for any $a \in \mathcal{C}_{+}(O_{\mathbb{C}}) \cap \mathcal{C}_{G_{sc}^{rs}}^{\mathbf{F}}(F)$, $\text{Sp}_{P_{a^{\mathbb{C}}}}$ is a union of certain connected components of $\text{Sp}_{G_{w}}^{w\mathbb{C}}$, the latter of which is isomorphic to the quotient of $\text{Sp}_{G_{w}}^{w\mathbb{C}}$ by the coweight lattice $X_{w}(T)^{w}$ of the central torus of $G_{w}$. Hence we may assume that $G = G_{w}$ and for simplicity omit $G$ in the notation.

Fix a Coxeter element $w \in \text{Cox}(W, S)$, cf. 2.2.1. Let $\gamma_{0} := \epsilon_{w}^{w_{f}}(a)$ (resp. $\gamma_{0}^{'} := \epsilon_{w}^{w_{f}}(a^{'})$) be the extended Steinberg sections for $a$ (resp. $a^{'}$). Then we have canonical isomorphism between groups schemes over $\text{Spec} O$:

$$J_{\alpha} \cong I_{\gamma_{0}}, \quad J_{\alpha^{'}} \cong I_{\gamma_{0}^{'}}.$$

**Lemma 5.1.2.** For any $g \in G(F)$, we have $\text{Ad}(g)^{-1}(\gamma_{0}) \in \text{Vin}_{G^{w}}(O)$ if and only if

$$\text{Ad}(g)^{-1}(\gamma_{0}I_{\gamma_{0}}(O)) \subset \text{Vin}_{G_{sc}^{w}}(O).$$

**Proof.** Since $\gamma_{0} \in \gamma_{0}I_{\gamma_{0}}(O)$, the condition is sufficient. Now assume that $\gamma := \text{Ad}(g)^{-1}(\gamma_{0}) \in \text{Vin}_{G^{w}}(O)$. Then the centralizer $I_{\gamma}$ is a group scheme over $\text{Spec} O$. By Lemma 2.4.2, the isomorphism of $F$ groups

$$\text{Ad}(g)^{-1} : J_{\alpha,F} = I_{\gamma_{0},F} \rightarrow I_{\gamma,F}$$

extends to $\text{Spec} O$. Thus we have

$$\text{Ad}(g)^{-1}(I_{\gamma_{0}}(O)) \subset I_{\gamma}(O) \subset G(O)$$

from which we obtain

$$\text{Ad}(g)^{-1}(\gamma_{0}I_{\gamma_{0}}(O)) = \gamma_{0}\text{Ad}(g)^{-1}(I_{\gamma_{0}}(O)) \subset \text{Vin}_{G^{w}}(O).$$

□

**Lemma 5.1.3.** Let $a, a^{'} \in \mathcal{E}_{+}(O) \cap \mathcal{C}_{G_{sc}^{rs}}^{\mathbf{F}}(F)$ with $a \equiv a^{'}$ mod $\varpi^{N}$. Suppose that there exists a $W$-equivariant isomorphism between the cameral covers $\tilde{X}_{a}$ and $\tilde{X}_{a^{'}}$ lifting the identity modulo $\varpi^{N}$. Let $\gamma_{0} := \epsilon_{w}^{w_{f}}(a)$ and $\gamma_{0}^{'} := \epsilon_{w}^{w_{f}}(a^{'})$. Then there exists $g \in G(O)$ such that

$$\text{Ad}(g)^{-1}(\gamma_{0}I_{\gamma_{0}}(O)) = \gamma_{0}^{'}I_{\gamma_{0}^{'}}(O).$$

**Proof.** We follow the argument of [Ng10, Lemme 3.5.4]. Let $\tilde{X}_{a} = \text{Spec} R_{a}$ and $\tilde{X}_{a^{'}} = \text{Spec} R_{a^{'}}$ where $R_{a}$, $R_{a^{'}}$ are finite flat $O$-algebras. Let $F_{a} := R_{a} \otimes_{O} F$ (resp. $F_{a^{'}} := R_{a^{'}} \otimes_{O} F$) and $R_{a}^{0}$ (resp. $R_{a^{'}}^{0}$) be the normalization of $R_{a}$ (resp. $R_{a^{'}}$) in $F_{a}$ (resp. $F_{a^{'}}$).

By assumption, we have $R_{a}/\varpi^{N} = R_{a^{'}}/\varpi^{N}$ and there exists a $W$-equivariant $O$-isomorphism

$$
\iota : R_{a} \xrightarrow{\sim} R_{a^{'}}
$$

that lifts the identity modulo $\varpi^{N}$.
By Proposition 2.4.7, the isomorphism $\iota: R_a \cong R_{a'}$ induces an isomorphism $\iota_I: I_{\gamma_0} \to I_{\gamma_0}'$ between group schemes over $\text{Spec} \, \mathcal{O}$. Since $\gamma_0 \in I_{\gamma_0}(F)$, we have $\iota_I(\gamma_0) \in I_{\gamma_0}'(F)$. We can choose $h \in G(R_a)$ and $h' \in G(R_{a'}')$ such that on $F$-points, the map $\iota_I$ is given by the following composition

$$I_{\gamma_0}(F) \xrightarrow{\iota} T(F_a)^{W} \xrightarrow{\iota} T(F_{a'})^{W} \xrightarrow{h'} I_{\gamma_0}'(F),$$

where the first map is $\text{Ad}(h)$ the and third map is $(h')^{-1}$. In other words, $\iota_I = \text{Ad}(h^{-1}I(h))$ on $F$-points. In particular, we have

$$\chi_+ (\iota_I(\gamma_0)) = \chi_+ (\gamma_0) = a.$$

The assumption that $\iota$ is identity modulo $\varpi^N$ implies that $\text{Ad}(h^{-1}I(h)) \equiv \text{Id} \pmod{\varpi^N}$. Thus we get

$$\iota(I_{\gamma_0}(F) \cap \text{Vin}_{G_{w}}(O)) \subset I_{\gamma_0}'(F) \cap \text{Vin}_{G_{w}}(O).$$

In particular, we have $\iota(\gamma_0) \in I_{\gamma_0}' \cap \text{Vin}_{G_{w}}(O)$ and moreover

$$\iota_I(\gamma_0) = \gamma_0 = \gamma_0' \text{ in } \text{Vin}_{G_{w}}^{w}(O/\varpi^{N}).$$

Since the map

$$G \times \text{Vin}_{G_{w}}^{w} \to \text{Vin}_{G}^{w} \times_{\eta} \text{Vin}_{G_{w}}^{w},$$

is smooth and surjective, there exists $g \in G(O)$ with $g \equiv 1 \pmod{\varpi^{N}}$ such that $\text{Ad}(g)^{-1}(\gamma_0) = \iota_I(\gamma_0)$. Therefore

$$\text{Ad}(g)^{-1}(I_{\gamma_0}) = I_{\gamma_0(g)} = I_{\gamma_0}' = I_{\gamma_0}'.$$  

Finally by Lemma 2.5.2, we have $(\gamma_0')^{-1} \iota_I(\gamma_0) \in G(O) \cap I_{\gamma_0}'(F) = I_{\gamma_0}(O)$ which implies that $\iota_I(\gamma_0) \in \gamma_0' I_{\gamma_0}'(O)$ and hence we are done. \(\square\)

### 5.2. Dimension of Kottwitz-Viehmann varieties

By Theorem 3.7.1, the dimension formula for $X_{\gamma}^\wedge \cong \text{Sp}_a$ is reduced to the following statement which we prove in this subsection:

**Theorem 5.2.1.** $\text{dim } \text{Sp}_a = \text{dim } P_a$.

If $C \subset G$ is the maximal torus in the center of $G$, then $\text{Sp}_{G/C,a} \cong \text{Sp}_{G,a}/X_{\ast}(C)$ and similar isomorphism holds for the local Picard $P_a$. Thus we may assume that $G$ is semisimple.

Let $X$ be a projective smooth curve over $k$ and $x \in X$ a closed point. Let $\mathcal{O}_x$ be the completed local ring at $x$ and $F_x$ its fraction field. Choose a uniformiser $\varpi_x$ at $x$ so that we have $\mathcal{O}_x = k[[\varpi_x]]$ and $F_x = k((\varpi_x))$. Also we let $X' = X - \{x\}$ be the open curve.

We view $a \in \mathcal{C}_+(\mathcal{O}_x)$ as a power series in $\varpi_x$ with coefficients in $\mathcal{C}_+$. Form the Cartesian diagram

$$\begin{array}{ccc}
\pi_x & \longrightarrow & \pi \\
\downarrow & & \downarrow \\
\text{Spec} \, \mathcal{O} & \xrightarrow{a} & \mathcal{C}_+ \\
\end{array}$$

where $X_a = \text{Spec} \, R_a$ for a finite flat $\mathcal{O}$ algebra $R_a$. Moreover, $F_a = R_a \otimes_{\mathcal{O}} F$ is a product of finite tamely ramified extension of $F$ of degree $e$ by our assumption that char($k$) is coprime to the order of Weyl group. Then $a(\varpi_x^e) \in \mathcal{C}_+(\mathcal{O}_x) \cap \mathcal{C}_{G_{w}}^{w}(F_x)$ will be a split conjugacy class.

For each $s \in k$ we define

$$a_s := a(s \varpi_x + (1-s) \varpi_x^e) \in \mathcal{C}_+(\mathcal{O}) \cap \mathcal{C}_{G_{w}}^{w}(F)^{w}.$$

Then $a_1 = a$ and $a_0 = a(\varpi_x^e)$. For each $s \neq 0$, $\text{Sp}_{a_s}$ is isomorphic to $\text{Sp}_a$ since $a_s$ is obtained from $a = a_1$ by changing uniformizer.
5.2.2. Let $N > 0$ be a positive integer such that both $\text{Sp}_a$ and $\text{Sp}_{a_0}$ only depends on $a$ (resp. $a_0$) modulo $\mathbb{Z}_x^N$. Then for all $s \leq k$, $\text{Sp}_a$ only depends on $a_\beta$ modulo $\mathbb{Z}_x^N$.

Now we choose a $T^{\text{sc}}$-torsor $L$ on $X$ trivialized on the formal neighbourhood of $x$ such that

1. There exists a $T^{\text{sc}}$-torsor $L'$ and an isomorphism $(L')^{\otimes c} \cong L$;
2. For all $y \in X' = X - x$, choosing a trivialisation of $L$ on a formal neighbourhood of $y$, the local evaluation map

\[
A_L = H^0(X, \mathcal{E}_x^\ast) \to \mathcal{E}_+(\mathcal{O}_x / \mathbb{Z}_x^N) \times \mathcal{E}_+(\mathcal{O}_y / \mathbb{Z}_y^N)
\]

is surjective.

By Riemann-Roch, condition 2 is satisfied if for all $1 \leq i \leq r$ we have $\deg(\alpha_i(L)) \geq 2g + N$ and $\deg(\omega_i(L)) > 2g + N$.

Recall that for each $a_+ \in A_L^\Sigma$, we associate an $X_+(T_{\text{ad}})_+$-valued divisor $\lambda_{a_+}$ on $X$ as in § 4.1.3.

**Lemma 5.2.3.** Let $\Sigma \subset X$ be a finite subset. The subset $A_L^\Sigma \subset A_L^\circ$ consisting of $a_+ \in A_L^\circ$ such that

\[
\text{Supp}(\lambda_{a_+}) \cap \text{Supp}(\Delta_{a_+}) \subset \Sigma
\]

is constructible.

**Proof.** For each $1 \leq i \leq r$, consider the closed subscheme $D_i \subset A_L^\circ \times X$ whose fibre over $a_+ \in A_L^\circ$ is the effective divisor $D(b_i)$ where $b_i$ is the $i$-th coordinate of $\beta_L(a_+)$ as above. Similarly, we have the closed subscheme $\Delta \subset \mathcal{A}_x^\ast \times X$ whose fibre over $a_+$ is the discriminant divisor $\Delta_{a_+}$. Let $D_{i \Sigma} = D_i \cap (A_L^\circ \times (X - \Sigma))$ and $\Delta_{i \Sigma} := \Delta \cap (A_L^\circ \times (X - \Sigma))$. Then $D_{i \Sigma} \cap \Delta_{i \Sigma}$ is a locally closed subset of $A_L^\circ \times X$. By construction $A_L^\circ$ is the image of $\bigcup_{1 \leq i \leq r} (D_{i \Sigma} \cap \Delta_{i \Sigma})$ in $A_L^\circ$, hence constructible. \hfill $\square$

5.2.4. The one-parameter family (5.2) defines a curve $C$ in $\mathcal{E}_+(\mathcal{O}_x / \mathbb{Z}_x^N)$. Let $L_C \subset A_L$ be the closed subset defined as the inverse image of $C$ under the map (5.3). For all $s \leq k$, let $L_{a_+} \subset A_L$ be the inverse image of $a_+$ under the map (5.3). Since $a_+ \in \mathcal{E}_{G^s}(F)$ for all $s \leq k$, we have $L_C \subset A_L^\circ$.

**Definition 5.2.5.** Let $Z_C \subset L_C$ be the subset consisting of $a_+ \in L_C$ with $b = \beta_L(a_+)$ such that

- $a_+ \in A_{L_i}^{\text{ani}}$;
- $\text{Supp}(\lambda_{a_+}) \cap \text{Supp}(\Delta_{a_+}) \subset \{x\}$;
- $a_+(X')$ intersects the discriminant divisor $\mathcal{D}_x^\ast$ transversally, where $X' = X - \{x\}$.

**Lemma 5.2.6.** $Z_C$ is a constructible subset of $L_C$ that is fibrewise dense with respect to the projection $L_C \to C$. In particular, there exists a fibrewise dense open subset $U_C$ of $L_C$ such that $U_C \subset Z_C$.

**Proof.** First we show that $Z_C$ is constructible. The first condition in Definition 5.2.5 defines an open subset of $L_C$. By Lemma 5.2.3, the set $L_{C_+}^\circ := L_C \cap A_L^\circ$ determined by the second condition in Definition 5.2.5 is a constructible subset of $L_C$.

Let $U \subset X' \times L_C$ be the open subset whose fibre over $a_+ \in L_C$ is the open curve $X' - \text{Supp}(\lambda_{a_+})$. The local evaluation maps define a morphism

\[
U \to T\mathcal{E}_x^\ast
\]

where $T\mathcal{E}_x^\ast$ is the relative tangent bundle of $\mathcal{E}_x^\ast$ over $X$. Let $U_1$ be the inverse image of

\[
T\mathcal{E}_x^{\ast, \text{sm}} \cup (T\mathcal{E}_x^{\ast, \text{sing}} \times \mathcal{E}_x \mathcal{D}_x^{L, \text{sing}}).
\]

Then the image of $U_1$ in $L_C$ is a constructible subset that satisfies the third condition in Definition 5.2.5. Hence $Z_C$ is a constructible subset of $L_C$.

Next we show that $Z_C$ is fibrewise dense with respect to the map $L_C \to C$. We fix a point $a_+ \in C$.

For any closed point $y \in X'$, the map

\[
L_{a_+} \to T\mathcal{E}_x^{\ast, y} = \mathcal{E}_x^\ast \otimes_{\mathcal{O}_y} \mathcal{O}_y / m_y^2
\]

is surjective by our choice of $L$.

Let $X'' := X' \setminus \text{Supp}(\lambda)$. By the same argument as in [Ngô10, Lemme 4.7.2], we know that the subset $Z \subset L_{a_+}$ consisting of $a_+ \in L_{a_+}$ such that $a_+(X'')$ intersects $\mathcal{D}_x^\ast$ transversally is dense in $L_{a_+}$.

For each $y \in \text{Supp}(\lambda) - \{x\}$, since the map $\text{ev}_y : L_{a_+} \to \mathcal{E}_x^{\ast, y}$ is surjective, the subset $\Sigma_y := \text{ev}_y^{-1}(\mathcal{D}_x^\ast) \subset L_{a_+}$ has codimension 1.
Finally, since $L_{a_s}$ has codimension $2rN$ in $\mathcal{A}_L^\vee$ and the complement of $\mathcal{A}_L^\text{uni}$ in $\mathcal{A}_L^\vee$ has codimension strictly larger than $2rN$, we see that

$$Z_{a_s} = (Z - \bigcup_{y \in \text{Supp}(\lambda_y)} \Sigma_y) \cap \mathcal{A}_L^\text{uni}$$

is dense in $L_{a_s}$. 

5.2.7. Thus we can choose a section $\sigma$ of the surjective linear map (5.3) such that $C' := \sigma(C) \cap U_C$ is nonempty and contains the point $\sigma(a_0)$.

By the product formula 4.2.10, we have

$$\dim M_{\sigma(a_0)} - P_{\sigma(a_0)} = \sum_{v \in \text{Supp}(\Delta_v) \setminus \{x\}} (\dim \text{Sp}_{\sigma(a_0)_v} - \dim P_{\sigma(a_0)_v})$$

where $\sigma(a_0)_v$ denotes the image of $\sigma(a_0)$ in $\mathcal{C}_+(\mathcal{O}_v)$.

For summands with $v \neq x$, since $\sigma(a_0) \in Z_C$ we have in particular $\lambda_{\sigma(a_0)_v} = 0$ and hence by Corollary 3.8.2 $\dim \text{Sp}_{\sigma(a_0)_v} = \dim P_{\sigma(a_0)_v}$. On the other hand, for the term $v = x$, we know that $\sigma(a_0)_x = a_0$ is split and hence by Corollary 3.5.3 $\dim \text{Sp}_{a_0} = \dim P_{a_0}$. Thus the above equality simplifies to

$$\dim M_{\sigma(a_0)} - P_{\sigma(a_0)} = 0.$$

Since $C' \subset \mathcal{A}_L^\text{uni}$, the restriction of the Hitchin-Frenkel-Ngô fibration to $C'$ is proper. Hence by upper semicontinuity of fibre dimension we have for

$$\dim M_{\sigma(a_s)} \leq \dim M_{\sigma(a_0)} = \dim P_{\sigma(a_0)}$$

for all $\sigma(a_s) \in C'$ with $s \neq 0$. Since $P$ is smooth over $\mathcal{A}_L$ by Proposition 4.2.2, we have $\dim P_{\sigma(a_s)} = P_{\sigma(a_0)}$, which forces

$$\dim M_{\sigma(a_s)} = \dim P_{\sigma(a_s)}.$$

Apply product formula 4.2.10 again we get

$$0 = \dim M_{\sigma(a_s)} - \dim P_{\sigma(a_s)} = \sum_{v \in \text{Supp}(\Delta_v) \setminus \{x\}} (\dim \text{Sp}_{\sigma(a_s)_v} - \dim P_{\sigma(a_s)_v})$$

By similar reasoning as above, all terms in the right hand side where $v \neq x$ are zero; at $v = x$ notice that $\sigma(a_s)_x = a_s$ and then we get

$$\dim \text{Sp}_{a_s} - P_{a_s} = 0.$$

Since $s \neq 0$, we have $\text{Sp}_{a_s} \cong \text{Sp}_a$ and hence

$$\dim \text{Sp}_a = \dim P_a.$$

This finishes the proof of Theorem 5.2.1 and hence the dimension formula in Theorem 1.2.1.

5.3. Equidimensionality. To finish the proof of Theorem 1.2.1 it remains to show the equi-dimensionality statement. Again our argument is of global nature, this time using a restricted Hitchin-Frenkel-Ngô moduli stack instead of the whole moduli stack. As in the previous subsection, we may assume that $G$ is semisimple.

5.3.1. Recall that by Theorem 5.1.1 there exists a positive integer $N > 0$ such that the isomorphism class of $\text{Sp}_a$ equipped with the action of $P_a$ only depends on $a$ modulo $\mathbb{Z}^N$. Let $X$ be the projective smooth curve as in the previous section. Fix two distinct closed points $x, x_0 \in X$. We consider an $X_+(T_{ad})^\text{-valued divisor on } X$ of the form $\lambda[x] + \lambda_0[x_0]$, where $\lambda_0 \in X_+(T_{ad})^+$ is chosen such that the following properties are satisfied:

- The $T_{ad}$-torsor associated to the divisor $\lambda[x] + \lambda_0[x_0]$ lifts to a $T^\text{sc}$-torsor $\mathcal{L}$ and there exists a $T^\text{sc}$-torsor $\mathcal{L}'$ together with an isomorphism $(\mathcal{L}')^\text{sc} \cong \mathcal{L}$.
- For each $1 \leq i \leq r$, the following three inequalities are satisfied:

$$\langle \omega_i, \lambda + \lambda_0 \rangle > 2g - 2 + (N + 3)r$$

$$\langle \omega_i, \lambda + \lambda_0 \rangle > 2g - 2 + m_0(d_a + 1) + b(\lambda)$$

$$\langle \omega_i, \lambda + \lambda_0 \rangle > \max\{nr, 2g - 2, rg\} + 1 + rg$$

where $d_a = d_\gamma + \langle \rho, \lambda \rangle$ is the valuation of extended discriminant divisor of $a = \chi(\gamma_\lambda) \in \mathcal{C}_+(\mathcal{O})$ and the numbers $m_0, b(\lambda)$ are as in Definition 4.4.3.
5.3.2. Let \( h_{\leq \lambda} : M_{\leq \lambda} \to A_{\leq \lambda} \) be the restricted Hitchin-Frenkel-Ngô moduli stack associated to the divisor \( \lambda[x] + \lambda_0[x_0] \). For simplicity we have omitted \( \lambda_0 \) from the notation.

We apply the result of 4.4 to our current situation. The set \( S \) in Definition 4.4.3 is taken to be \( \{ x \} \) in the current situation. Then we get open subset \( A_{\leq \lambda}^p \) and open substack \( M_{\leq \lambda}^{b, an} \). For some positive integer \( n > 0 \) large enough, there is a local evaluation map

\[
ev_{n x}^b : M_{\leq \lambda}^p \to [L_n^+ G \backslash G_{\leq \lambda}]
\]

which is smooth by Theorem 4.4.5. Moreover, the inverse image of \([L_n^+ G \backslash G_{\leq \lambda}]\) under \( ev_{n x}^b \) is the open substack \( M_{\leq \lambda}^1 \).

**Corollary 5.3.3.** Consider the restriction of the Hitchin-Frenkel-Ngô to the transversal anisotropic open substack \( h_{\leq \lambda}^{b, an} : M_{\leq \lambda}^{b, an} \to A_{\leq \lambda}^{b, an} \) is flat.

**Proof.** By product formula (Theorem 4.2.10) and Theorem 5.2.1 we have dim \( M_a = \dim \mathcal{P}_a \) for each \( a \in A_{\leq \lambda}^{b, an} \). In particular, the fibre dimension of \( h_{\leq \lambda}^{b, an} \) is constant since \( \mathcal{P} \) is a smooth Deligne-Mumford stack over \( A_{\leq \lambda}^{b, an} \). By Corollary 4.4.12, the source \( M_{\leq \lambda}^{b, an} \) is Cohen-Macaulay and hence we conclude that the morphism is flat. \( \square \)

**Lemma 5.3.4.** There exists a point \( a_+ \in A_{\leq \lambda}^{b, an} \) such that

- \( x_0 \notin \text{Supp}(\Delta(a_+)) \),
- \( \text{Supp}(\Delta(a_+)_{\text{sing}}) \subset \{ x \} \), in other words, \( a_+(X - x) \) is transversal to the discriminant divisor,
- \( a_+ \equiv a \mod \omega_x^N \).

**Proof.** The proof is similar to Lemma 5.2.6. Let \( L \subset A_{\leq \lambda} \) be the linear subspace consisting of \( a_+ \in A_{\leq \lambda} \) such that \( a_+ \equiv a \mod \omega_x^N \). Since \( a \in \mathcal{E}_\lambda^1(\mathcal{O}) \) is generically regular semisimple, we have \( L \subset A_{\leq \lambda}^{b, an} \).

The first inequality in \( \S \) 5.3.1 implies that for any point \( y \in X - \{ x, x_0 \} \), the local evaluation map

\[
A_{\leq \lambda} \to \mathcal{E}_\lambda^1(\mathcal{O}_x/\omega_x^N) \times \mathcal{E}(\mathcal{O}_y/\omega_y^2) \times \mathcal{E}_\lambda^1(\mathcal{O}_{x_0}/\omega_{x_0})
\]

is surjective. By similar argument as in the proof of Lemma 5.2.6, this implies that there is dense subset \( Z \subset L \) consisting of points \( a_+ \in A_{\leq \lambda}^{b, an} \) such that \( a_+(X - x) \) intersects the discriminant divisor transversally and \( a_+(x_0) \) does not intersect with the discriminant divisor.

Then for each \( a_+ \in Z \), we have \( \Delta(a_+)_\text{sing} = d_a[x] \) and hence the second inequality in \( \S \) 5.3.1 ensures that \( Z \subset A_{\leq \lambda}^{b, an} \).

Finally since \( Z \subset A_{\leq \lambda} \) is a subset of codimension \( Nr \), the third inequality in \( \S \) 5.3.1 ensures that \( Z \) has nonempty intersection with \( A_{\leq \lambda}^{b, an} \) by Corollary 4.2.8. Then any point \( a_+ \in Z \cap A_{\leq \lambda}^{b, an} \) satisfies the condition we want. \( \square \)

Choose \( a_+ \in A_{\leq \lambda}^{b, an} \) as in the Lemma above. Then by Theorem 4.2.10 and Corollary 3.8.3, there is a homeomorphism of stacks

\[
[M_{\leq \lambda, a_+}/\mathcal{P}_{a_+}] \cong [\text{Sp}_a/\mathcal{P}_a].
\]

Corollary 5.3.3 implies that \( M_{\leq \lambda, a_+} \) is equi-dimensional. Therefore \( \text{Sp}_a \) and its open substack \( \text{Sp}_a^0 \) are also equidimensional. This finishes the proof of Theorem 1.2.1.

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