Cohomological Non-rigidity of Generalized Real Bott Manifolds of Height 2

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Abstract—We investigate the following problem: When do two generalized real Bott manifolds of height 2 have isomorphic cohomology rings with \( \mathbb{Z}/2 \) coefficients and also when are they diffeomorphic? It turns out that in general cohomology rings with \( \mathbb{Z}/2 \) coefficients do not distinguish those manifolds up to diffeomorphism. This gives a negative answer to the cohomological rigidity problem for real toric manifolds posed earlier by Y. Kamishima and the present author. We also prove that generalized real Bott manifolds of height 2 are diffeomorphic if they are homotopy equivalent.

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1. INTRODUCTION

A toric manifold is a compact smooth toric variety, and a real toric manifold is the set of real points of a toric manifold. In [7] we asked whether toric manifolds are diffeomorphic if their cohomology rings with \( \mathbb{Z} \) coefficients are isomorphic as graded rings, which is now called a cohomological rigidity problem for toric manifolds. No counterexample to the problem is known, and there are some partial affirmative solutions (see [3, 7]). If \( X \) is a toric manifold and \( X(\mathbb{R}) \) is the real toric manifold associated with \( X \), then \( H^*(X(\mathbb{R}); \mathbb{Z}/2) \) is isomorphic to \( H^2(\mathbb{X}; \mathbb{Z}) \otimes \mathbb{Z}/2 \) as a graded ring. Motivated by this, we posed in [5] the following analogous problem.

Cohomological rigidity problem for real toric manifolds. Are two real toric manifolds diffeomorphic if their cohomology rings with \( \mathbb{Z}/2 \) coefficients are isomorphic as graded rings?

We say that cohomological rigidity over \( \mathbb{Z}/2 \) holds for a family of closed smooth manifolds if the manifolds in the family are distinguished up to diffeomorphism by their cohomology rings with \( \mathbb{Z}/2 \) coefficients.

A real Bott manifold is the total space of an iterated \( \mathbb{R}P^1 \) bundle where each \( \mathbb{R}P^1 \) bundle is the projectivization of a Whitney sum of two real line bundles. A real Bott manifold is not only a real toric manifold but also a compact flat Riemannian manifold. We proved in [5] (and [6]) that cohomological rigidity over \( \mathbb{Z}/2 \) holds for the family of real Bott manifolds.

In this paper we consider real toric manifolds obtained as the total spaces of the projectivization of Whitney sums of real line bundles over a real projective space. We call such a real toric manifold a generalized real Bott manifold of height 2. In this paper we aim to find out when two such manifolds have isomorphic cohomology rings with \( \mathbb{Z}/2 \) coefficients and also when they are diffeomorphic. As a result, we will see that cohomological rigidity over \( \mathbb{Z}/2 \) fails to hold for some family of generalized real Bott manifolds of height 2, which gives a negative answer to the above cohomological rigidity problem for real toric manifolds. We also prove that generalized real Bott manifolds of height 2 are diffeomorphic if they are homotopy equivalent.

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2. COHOMOLOGICAL CONDITION

Let \( a \) and \( b \) be positive integers, and we fix them. Let \( \gamma \) be the tautological line bundle over \( \mathbb{RP}^{a-1} \), and let \( 1 \) denote a trivial real line bundle over an appropriate space. For a real vector bundle \( E \), we denote by \( P(E) \) the total space of the projectivization of \( E \). For an integer \( q \) such that \( 0 \leq q \leq b \), we set

\[
M(q) := P(q) \oplus (b - q)1.
\]

Note that

\[
M(q) \text{ is diffeomorphic to } M(b - q) \tag{2.1}
\]

because \( P(E \otimes L) \) and \( P(E) \) are diffeomorphic for any smooth vector bundle \( E \) and any line bundle \( L \) over a smooth manifold.

A simple computation shows that

\[
H^*(M(q); \mathbb{Z}/2) = \mathbb{Z}/2[x, y]/(x^a, (x + y)^qy^{b-q}), \tag{2.2}
\]

where \( x \) is the pullback of the first Stiefel–Whitney class of \( \gamma \) to \( M(q) \) and \( y \) is the first Stiefel–Whitney class of the tautological line bundle over \( M(q) \). One easily sees that the set \( \{x^iy^j \mid 0 \leq i < a, 0 \leq j < b \} \) is an additive basis of \( H^*(M(q); \mathbb{Z}/2) \).

**Lemma 2.1.** If \( 0 < q < b \), then both \( y^a \) and \( (x + y)^a \) are nonzero.

**Proof.** Suppose \( y^a = 0 \). Then it follows from (2.2) that there are constants \( c, d \in \mathbb{Z}/2 \) and a homogeneous polynomial \( f(x, y) \) in \( x, y \) over \( \mathbb{Z}/2 \) such that

\[
y^a = \begin{cases} cx^a & \text{if } a < b, \\ dx^a + f(x, y)(x + y)^qy^{b-q} & \text{if } a \geq b \end{cases}
\]

as polynomials in \( x, y \). Clearly the former does not occur and the latter does not occur either, because \( q > 0 \) by assumption. This is a contradiction, so \( y^a \neq 0 \).

If we set \( X = x \) and \( Y = x + y \), then \( x + y = Y \) and \( y = X + Y \), so that the roles of \( x \) and \( x + y \) will be interchanged. Since \( b - q > 0 \) by assumption, the above argument applied to \( Y \) instead of \( y \) proves that \( (x + y)^a \neq 0 \). \( \square \)

**Definition.** \( h(a) := \min\{n \in \mathbb{N} \cup \{0\} \mid 2^n \geq a\} \).

For example,

\[
h(1) = 0, \quad h(2) = 1, \quad h(3) = h(4) = 2, \quad h(5) = h(6) = h(7) = h(8) = 3,
\]

\[
h(9) = \ldots = h(16) = 4, \quad \ldots.
\]

**Lemma 2.2.** Let \( q \) and \( q' \) be nonnegative integers. Then \( \binom{q}{i} \equiv \binom{q'}{i} \pmod{2} \) for all \( i \) such that \( 0 \leq i < a \) if and only if \( q' \equiv q \pmod{2^{h(a)}} \), where \( \binom{n}{m} \) is understood to be 0 when \( n < m \), as usual.

**Proof.** When \( q' = q \), the lemma is trivial. We may assume that \( q' > q \) without loss of generality. We note that the former congruence relations in the lemma are equivalent to the following congruence relation of polynomials in \( t \) with \( \mathbb{Z}/2 \) coefficients:

\[
(1 + t)^{q'-q} \equiv 1 \pmod{t^a}. \tag{2.3}
\]