WEIGHTED ESTIMATES OF SINGULAR INTEGRALS AND COMMUTATORS IN THE ZYGMUND DILATION SETTING

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ABSTRACT. The main purpose of this paper is to establish weighted estimates for singular integrals associated with Zygmund dilations via a discrete Littlewood–Paley theory, and then apply it to obtain the upper bound of the norm of commutators of such singular integrals with a function in the little bmo space associated with Zygmund dilations. Examples of such singular integrals associated with Zygmund dilations include a class of singular integrals studied by Ricci–Stein and Fefferman–Pipher, as well as a singular integral along a particular surface studied by Nagel–Wainger. We show that the lower bound of the norm of this commutator is not true for any singular integral in the class considered in Ricci–Stein and Fefferman–Pipher, but does in fact hold for the specific singular integral studied in Nagel–Wainger. In particular this implies that the family of singular integrals studied in these papers is not sufficiently general to contain the operator of Nagel–Wainger, which we show is of significance in this theory.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Modern harmonic analysis has had significant success in the last 50 years, notably the Calderón–Zygmund theory of singular integrals and its applications to partial differential equations and complex analysis. While harmonic analysis of one-parameter has been developed very successfully, less is known about the theory in the multi-parameter product setting, due to its complexity and technical difficulties. There has been great progress, yet there are still many open problems in this field. See for example the results on singular integrals, function spaces, covering lemmas, and commutators in [19, 6, 5, 4, 3, 10, 16, 14, 17, 31, 28, 29, 30, 34].

Classical Calderón–Zygmund theory deals with operators invariant under a group of dilations of one-parameter; the well-developed multi-parameter product theory is concerned with operators in \( \mathbb{R}^n \) that are invariant under the group of \( n \) independent dilation factors, such as tensor products of singular integral operators acting in each variable separately. However, in the context of symmetric spaces, the natural operators that arise are invariant under other multi-parameter dilation families. In a low dimensional setting, \( \mathbb{R}^3 \), one of the most natural and interesting examples of a group of dilations that lies in between the

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one-parameter and the full product setting is the so-called Zygmund dilation defined by 
\[ \rho_{s,t}(x_1, x_2, x_3) = (sx_1, tx_2, stx_3) \] for \( s, t > 0 \) (see for example [35, 15]). Corresponding to 
the Zygmund dilation, one has the maximal function (see for example in Cordoba [9])
\begin{equation}
\mathcal{M}_j f(x_1, x_2, x_3) = \sup_{R \ni (x_1, x_2, x_3)} \frac{1}{|R|} \int_R |f(u_1, u_2, u_3)| du_1 du_2 du_3,
\end{equation}
where the supremum above is taken over all rectangles in \( \mathbb{R}^3 \) with sides parallel to the axes 
and side lengths of the form \( s, t, \) and \( st \). See also the related Zygmund conjecture in [36].
The survey paper of R. Fefferman [13] has more information about research directions in 
this setting.

An explicit example of a singular integral which commutes with the Zygmund dilation 
\( \rho_{s,t}(x_1, x_2, x_3) \) is an operator studied by Nagel–Wainger [33] in late 1970s, which is the singular integral along certain surfaces in \( \mathbb{R}^3 \), defined as \( T f = f * \mathcal{K}, \) where
\begin{equation}
\mathcal{K}(x_1, x_2, x_3) = \operatorname{sgn}(x_1x_2) \left\{ \frac{1}{x_1^2x_2^2 + x_3^2} \right\}.
\end{equation}
It was shown in [33] that this \( T \) is bounded on \( L^2(\mathbb{R}^3) \).

Later, Ricci and Stein [35] introduced a class of singular integrals with more general 
dilations. They introduced the class of operators associated with Zygmund dilations of the form 
\( T_j f = f * \mathcal{K}, \) where
\begin{equation}
\mathcal{K}(x_1, x_2, x_3) = \sum_{k, j \in \mathbb{Z}} 2^{-2(k+j)} \phi \left( \frac{x_1}{2^j}, \frac{x_2}{2^k}, \frac{x_3}{2^{j+k}} \right)
\end{equation}
where \( \phi \) is supported in a unit cube in \( \mathbb{R}^3 \) (or Schwartz function on \( \mathbb{R}^3 \)) with a certain 
amount of uniform smoothness and satisfies cancellation conditions
\begin{equation}
\int_{\mathbb{R}^2} \phi(x_1, x_2, x_3) dx_1 dx_2 = \int_{\mathbb{R}^2} \phi(x_1, x_2, x_3) dx_2 dx_3 = \int_{\mathbb{R}^2} \phi(x_1, x_2, x_3) dx_3 dx_1 = 0.
\end{equation}
For more details on this class of operators, we refer to Definition 4.1. It was shown in [35] 
that operators in this class are bounded on \( L^p(\mathbb{R}^3) \) for all \( 1 < p < \infty \).

Later, in [15] it was observed that the cancellation conditions in (1.4) are also necessary 
for the boundedness of the above mentioned operators on \( L^2(\mathbb{R}^3) \). They investigated the 
boundedness of this class of operators on \( L^p_w(\mathbb{R}^3) \) for \( 1 < p < \infty \) when the weight \( w \) satisfies 
an analogous condition of Muckenhoupt with respect to rectangles whose dimensions are 
governed by the Zygmund dilations. We now recall this class of Muckenhoupt weights. For 
\( 1 < p < \infty \), a nonnegative measurable function \( w \) on \( \mathbb{R}^3 \) is called an \( A^p_k(\mathbb{R}^3) \)-Muckenhoupt 
weight if \( [w]_{A^p_k(\mathbb{R}^3)} < \infty \), where the quantity \( [w]_{A^p_k(\mathbb{R}^3)} \) is given by
\[ \sup_{R \in \mathcal{K}} \left( \frac{1}{|R|} \int_R w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right) \left( \frac{1}{|R|} \int_R w(x_1, x_2, x_3)^{-1/(p-1)} dx_1 dx_2 dx_3 \right)^{p-1}. \]
Here and throughout the paper, we use $\mathcal{R}_3$ to denote the rectangles in $\mathbb{R}^3$ such that $R = I \times J \times S$, with $\ell(S) = \ell(I) \times \ell(J)$.

The weights in $A^1_p(\mathbb{R}^3)$ are not as straightforward to deal with as the standard multi-parameter product weights. In the latter case, the condition that a weight belongs to the product $A_p$ Muckenhoupt class is equivalent to belonging to $A_p$ in each variable separately. This implies that a variety of weighted operator estimates can be obtained by iteration. For these Zygmund weights, $w \in A^1_p(\mathbb{R}^3)$ only implies that $w \in A_p(\mathbb{R}(x_j))$, where $j = 1, 2$. For example, the one-parameter maximal function in the variable $x_3$ fails to be bounded with respect to weights in the Zygmund class. In our present setting, a more careful iteration is needed to obtain weighted estimates for both continuous and discrete area functions, as in [15].

Motivated by these specific operators with convolution kernels as in (1.2) and (1.3), in [21] the authors introduced a more general class of singular integral operators of convolution type $Tf = K * f$ associated with the Zygmund dilation, characterized by suitable versions of regularity conditions and cancellation conditions for the convolution kernel $K$ (for the sake of simplicity, we provide the full notation and definition in Section 2, see Definition 2.1 below). In particular, the two operators studied in (1.2) and (1.3) (or more generally in Definition 4.1) are special examples in the class of operators in Definition 2.1.

The main purpose of this paper is to show that all singular integrals in this class are bounded on $L^p_w(\mathbb{R}^3)$ for $w \in A^1_p(\mathbb{R}^3)$, $1 < p < \infty$, and that the boundedness of their commutators with a multiplication operator by function $b$ can be characterized by the membership of $b$ in a suitable little BMO space associated with Zygmund dilation.

Our first result here is that the singular integral operators $T$ in Definition 2.1 is bounded on the weighted space $L^p_w(\mathbb{R}^3)$ for $1 < p < \infty$, where $w$ is in $A^1_p(\mathbb{R}^3)$.

**Theorem 1.1.** Suppose that $K$ is a function defined on $\mathbb{R}^3$ and satisfies the conditions (R) and (C.a) – (C.c) (or (R), (C’.a) – (C’.c)) in Definition 2.1, and in addition, $K^N_\epsilon$ (as defined in (2.1)) satisfies that the three integrals

$$
\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} \int_{|x_1| \leq 1} K^N_\epsilon(x_1, x_2, x_3) dx_1 dx_2 dx_3,
$$

$$
\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} K^N_\epsilon(x_1, x_2, x_3) dx_2 dx_3,
$$

$$
\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} K^N_\epsilon(x_1, x_2, x_3) dx_1 dx_3
$$

converge almost everywhere as $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \rightarrow (0, 0, 0)$ and $N = (N_1, N_2, N_3) \rightarrow (\infty, \infty, \infty)$. Then the operator $T$ as in Definition 2.1 extends to a bounded operator on $L^p_w(\mathbb{R}^3)$ for $w \in A^1_p(\mathbb{R}^3)$ with $1 < p < \infty$ and moreover, $\|K^N_\epsilon f\|_{L^p_w(\mathbb{R}^3)} \leq C_w \|f\|_{L^p_w(\mathbb{R}^3)}$ with the constant $C_w$ depending only on $[w]_{A^1_p(\mathbb{R}^3)}$ and the constant $C$ in Definition 2.1.
This generalizes the weighted estimate of $T_z f$ of [15] to a larger family of singular integrals associated with Zygmund dilations. For example, we note that from [21], the singular integral operator $T$ given in [33] with the convolution kernel as in (1.2) satisfies conditions (R) and (C’).a – (C’).c. Hence, from Theorem 1.1 we see that this operator $T$ is bounded on $L^p_w(\mathbb{R}^3)$ for $w \in A^p_z(\mathbb{R}^3)$ with $1 < p < \infty$.

To prove Theorem 1.1, we resort to the weighted Littlewood–Paley theory for square functions and almost orthogonality estimate associated with the Zygmund dilation. However, as pointed above, the weights in $A^p_z(\mathbb{R}^3)$ are not as straightforward to deal with as the standard multi-parameter product weights. Hence, it is not clear how to obtain the weighted Littlewood–Paley theory for square functions. Instead, we have to establish the weighted Littlewood–Paley theory for Lusin area functions, where the structure of the cone associated with Zygmund dilation plays an important role in dealing with the weight in $A^p_z(\mathbb{R}^3)$. And then we show that square function and area function are equivalent on $L^p_w(\mathbb{R}^3)$ for $w \in A^p_z(\mathbb{R}^3)$ with $1 < p < \infty$. The key step in proving this equivalence is to establish a discrete Calderón reproducing formula on suitable test function spaces, and then the weighted Littlewood–Paley theory for the discrete area function $S^d_3$ and square function $g^d_3$ associated with the Zygmund dilations. In the end we pass from the discrete version to the continuous version. In fact, the passage between the discrete and continuous square functions seems to be an essential ingredient, and the proof in this paper fills a gap in the argument of [15] \(^1\). Proving weighted $L^p$ estimates for square functions is a classical approach to obtaining the weighted theory of singular operators - and the vertical, or martingale, versions of these square functions are most useful (as in [16], for a relevant example). However, weighted estimates for the vertical square functions are not addressed in [21, 22] \(^2\). Therefore the estimates for the discrete square functions introduced here should be of independent interest. We refer the reader to Section 2.3 and to the equivalences in (2.44).

Next, to state the second main result of the paper, we introduce a function space measuring oscillation, the space $\text{bmo}_3(\mathbb{R}^3)$ associated with the Zygmund dilation as follows.

**Definition 1.2.** $\text{bmo}_3(\mathbb{R}^3) := \{ b \in L^1_{\text{loc}}(\mathbb{R}^3) : \| b \|_{\text{bmo}_3(\mathbb{R}^3)} < \infty \}$, where

$$\| b \|_{\text{bmo}_3(\mathbb{R}^3)} := \sup_{R \in \mathcal{R}_3} \frac{1}{| R |} \int_R \left| b(x_1, x_2, x_3) - b_R \right| dx_1 dx_2 dx_3,$$

with $b_R$ being the average of $b$ over a Zygmund rectangle $R$.

\(^1\)On p. 355 of [15], to complete the argument one would need to introduce the “vertical” square function as Fefferman and Stein did to obtain weighted estimates via square functions in the full product setting. In the Zygmund setting, this requires the new arguments in this paper which rely on the discrete square function built from the discrete Calderón reproducing formula.

\(^2\)On p. 25 of [22] they claimed that the weighted estimate of $g^d_3$ follows directly from [15, Theorem 2.9] (i.e. the weighted estimate of $S^d_3$). This claim is inaccurate.
We consider the commutator of a symbol function \( b \) and the singular integral operators of convolution type \( Tf = K * f \), associated with Zygmund dilations, defined as

\[
[b, T](f)(x_1, x_2, x_3) = b(x_1, x_2, x_3)Tf(x_1, x_2, x_3) - T(bf)(x_1, x_2, x_3).
\]

As a consequence of the weighted estimate in Theorem 1.1, we have the following estimates of the upper bound of commutators.

**Theorem 1.3.** Let \( T \) be a singular integral operator of convolution type \( Tf = K * f \) associated with Zygmund dilations as in Definition 2.1. Suppose \( b \in \text{bmo}_3(\mathbb{R}^3) \). Then for \( 1 < p < \infty \) and for every \( w \in A^p_\infty(\mathbb{R}^3) \), \( f \in L^p_w(\mathbb{R}^3) \), we have

\[
\| [b, T](f) \|_{L^p_w(\mathbb{R}^3)} \leq C\|b\|_{\text{bmo}_3(\mathbb{R}^3)} \|f\|_{L^p_w(\mathbb{R}^3)},
\]

where the constant \( C \) is independent of \( f \).

From the theorem above, one also immediately obtains the following analogous upper estimate for commutators of higher order.

**Corollary 1.4.** Let \( T \) be a singular integral operator of convolution type \( Tf = K * f \) associated with Zygmund dilations as in Definition 2.1. Suppose \( b_1, \ldots, b_k \in \text{bmo}_3(\mathbb{R}^3) \). Then for \( 1 < p < \infty \) and for every \( w \in A^p_\infty(\mathbb{R}^3) \), \( f \in L^p_w(\mathbb{R}^3) \), we have

\[
\| C^k_{b_1, \ldots, b_k}(T)(f) \|_{L^p_w(\mathbb{R}^3)} \leq C \left( \prod_{i=1}^{k} \|b_i\|_{\text{bmo}_3(\mathbb{R}^3)} \right) \|f\|_{L^p_w(\mathbb{R}^3)},
\]

where the \( k \)-th order commutator (for an integer \( k \geq 1 \)) is defined as

\[
C^k_{b_1, \ldots, b_k}(T)(f) := [b_k, \cdots, [b_2, [b_1, T]] \cdots](f)
\]

and the constant \( C \) is independent of \( f \).

We point out that there are several methods to obtain the upper bound of commutators in the classical setting by using, for example, the sharp maximal function with respect to the BMO space \([7, 2]\), the Cauchy integral trick \([7]\), suitable paraproduct decompositions \([26]\), sparse dominations \([32]\) and so on. However, in this specific multi-parameter setting with Zygmund dilations, most of them do not seem to apply. It is unclear whether the sharp maximal function \( f^\sharp \) with respect to the little bmo space associated with Zygmund dilations still has the properties parallel to the classical ones, such as \( \|f\|_p \leq C\|f^\sharp\|_p \). Moreover, the multiresolution analysis is not understood in this setting and it is unknown whether there is a suitable wavelet basis. Hence, it is not clear if the approach via paraproducts or sparse domination is applicable. We resort to the Cauchy integral trick in this paper for proving the upper bound, which relies totally on the a priori weighted estimates for singular integral operators associated with Zygmund dilations obtained in Theorem 1.1.
and a careful study of John-Nirenberg properties of the space \( \text{bmo}_1(\mathbb{R}^3) \), which is the main result in Section 3.

In addition, it is natural to ask whether the lower bound of the commutator \([b, T]\) holds for some specific operator \( T \) of the class given in Definition 2.1.

This is the best one can hope for, since we find that the commutator of the class of singular integrals studied by Ricci and Stein [35] (see Definition 4.1), which in particular includes \( Tf = K * f \) with \( K \) the kernel given by (1.3), does not have the desired lower bound. More precisely, we have the following result.

**Theorem 1.5.** Let \( T \) be a singular integral operator of convolution type: \( Tf = K * f \) with \( K \) the kernel as defined in Definition 4.1. Then there exists certain \( b_0 \in L^1_{\text{loc}}(\mathbb{R}^3) \) such that \( \| [b_0, T] \|_{L^p_{\text{loc}}(\mathbb{R}^3) \to L^q_{\text{loc}}(\mathbb{R}^3)} < \infty \) for some \( w \in A^1_2(\mathbb{R}^3) \). However, this \( b_0 \) is NOT in \( \text{bmo}_1(\mathbb{R}^3) \).

We then answer this question affirmatively by obtaining the lower bound of the commutator \([b, T]\) for the specific operator \( T \) studied by Nagel and Wainger [33] with kernel defined in (1.2). To be more precise, we have

**Theorem 1.6.** Let \( T \) be the singular integral operator of convolution type: \( Tf = K * f \) with \( K \) the kernel given by (1.2). Suppose \( 1 < p < \infty \) and \( w \in A^1_2(\mathbb{R}^3) \). Suppose that \( b \in L^1_{\text{loc}}(\mathbb{R}^3) \) and that \( \| [b, T] \|_{L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} < \infty \). Then we have that \( b \) is in \( \text{bmo}_1(\mathbb{R}^3) \) with \( \| b \|_{\text{bmo}_1(\mathbb{R}^3)} \leq C \| [b, T] \|_{L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \), where the constant \( C \) is independent of \( b \).

**Remark 1.7.** Theorems 1.5 and 1.6 above reveal that the class of operators studied in [15] and [35] is not sufficiently general to contain the particular operator given by (1.2) and studied in [33]. It is not immediately clear how to prove this conclusion directly just using the definitions (1.2), (1.3), or Definition 4.1. In general, lower bounds for commutators of singular integral operators only hold for specific canonical examples: the Riesz transforms in \( \mathbb{R}^n \) in the one-parameter setting ([7]), and the iterated Riesz transforms in the product setting ([31]). As we see, the operator of (1.2) is of significance for this theory, even though we have yet to identify the analogs of the Riesz transforms in this setting.

Via essentially the same argument of Theorem 1.6, one also obtains the following lower bound for higher order commutators:

**Corollary 1.8.** Let \( T \) be the same as in Theorem 1.6. Suppose \( b \in L^1_{\text{loc}}(\mathbb{R}^3) \) and suppose that \( \| C^k_{b, \ldots, b}(T) \|_{L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} < \infty \) for some \( k \geq 1 \), \( 1 < p < \infty \) and some \( w \in A^1_2(\mathbb{R}^3) \). Then we have that \( b \) is in \( \text{bmo}_1(\mathbb{R}^3) \) with \( \| b \|_{\text{bmo}_1(\mathbb{R}^3)} \leq C \| C^k_{b, \ldots, b}(T) \|_{L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \), where the constant \( C \) is independent of \( b \).
Remark 1.9. We point out that it is impossible to deduce the lower bound for general $C_{b_1,\ldots,b_k}(T)(f)$ with different $b_i$. That is, if $C_{b_1,\ldots,b_k}(T)$ is bounded on $L^p_w(\mathbb{R}^3)$, in general we can not expect that those $b_i$’s are in $\text{bmo}_1(\mathbb{R}^3)$ with a control of the $\text{bmo}$ norm by operator norm. In fact, consider $[b_2, [b_1, T]]$ with $T$ as in Theorem 1.6 and $b_1(x) = 1$, $b_2(x) = x_1$, the first component of $x$. Then it is clear that $b_1, b_2 \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $\|[b_2, [b_1, T]]\|_{L^p_w(\mathbb{R}^3)\rightarrow L^q_w(\mathbb{R}^3)} = 0 < \infty$ for arbitrary $w \in A^q_2(\mathbb{R}^3)$. However, $b_2$ is not in $\text{bmo}_3(\mathbb{R}^3)$.

Note that Theorems 1.3 and 1.6, combined together, say that the operator $T$ whose kernel as defined in (1.2) is a representative of the class of singular integrals defined in Definition 2.1 when it comes to boundedness of commutators. More precisely, if $T$ has a commutator with symbol $b$ that is bounded on some $L^p_w$ space, then the commutators $[b, T']$ for all $T'$ in the class of Definition 2.1 must be bounded on all $L^q_w$ spaces. Moreover, Theorem 1.3 and 1.6 also guarantee the following. If one perturbs the operator $T$, defined in (1.2), by any $T'$ from the class of singular integrals defined in Definition 2.1, as long as the norm (measured by constant $C$ in Definition 2.1) of $T'$ is sufficiently small, there always holds the lower bound for the perturbed commutator $[b, T + T']$.

This paper is organised as follows. In Section 2 we prove Theorem 1.1. In Section 3 we study the little bmo space associated with Zygmund dilations and use the result of Theorem 1.1 to prove Theorem 1.3, the upper bound of the commutator. In Section 4, we prove Theorem 1.5, showing that the lower bound of the commutator for the singular integrals $T$ studied in [35] is not true. In the last section, Section 5, we prove Theorem 1.6, the lower bound of the commutator for the particular singular integral $T$ studied in [33].

2. Weighted Estimates for Singular Integral Operators Associated with Zygmund Dilations: Proof of Theorem 1.1

The main result in this section establishes the weighted estimates for the class of singular integral operators associated with Zygmund dilations. To begin with, we recall the definition of the class of singular integral operators associated with Zygmund dilations.

Definition 2.1 ([21]). Suppose that $\mathcal{K}(x_1, x_2, x_3)$ is a function defined on $\mathbb{R}^3$ away from the union $\{0, x_2, x_3\} \cup \{x_1, 0, x_3\} \cup \{x_1, x_2, 0\}$. For integers $\alpha$, $\beta$ and $\gamma$ taking only values 0 or 1, we define

$$
\Delta_{x_1, h_1}^\alpha \mathcal{K}(x_1, x_2, x_3) = \alpha \mathcal{K}(x_1 + h_1, x_2, x_3) - \mathcal{K}(x_1, x_2, x_3), \quad \alpha = 0 \text{ or } 1;
$$

$$
\Delta_{x_2, h_2}^\beta \mathcal{K}(x_1, x_2, x_3) = \beta \mathcal{K}(x_1, x_2 + h_2, x_3) - \mathcal{K}(x_1, x_2, x_3), \quad \beta = 0 \text{ or } 1;
$$

and

$$
\Delta_{x_3, h_3}^\gamma \mathcal{K}(x_1, x_2, x_3) = \gamma \mathcal{K}(x_1, x_2, x_3 + h_3) - \mathcal{K}(x_1, x_2, x_3), \quad \gamma = 0 \text{ or } 1.
$$
For simplicity, we denote $\Delta_{x_1,h_1} = \Delta^1_{x_1,h_1}$, $\Delta_{x_2,h_2} = \Delta^1_{x_2,h_2}$ and $\Delta_{x_3,h_3} = \Delta^1_{x_3,h_3}$.

The regularity conditions of the kernels are characterized by

\[
\begin{align*}
(R) \quad |\Delta^\alpha_{x_1,h_1} \Delta^\beta_{x_2,h_2} \Delta^\gamma_{x_3,h_3} K(x_1, x_2, x_3)| &\leq C|h_1|^{\alpha \theta_1} |h_2|^{\beta \theta_1} |h_3|^{\gamma \theta_1} |x_1|^{\alpha \theta_1 + 1} |x_2|^{\beta \theta_1 + 1} |x_3|^{\gamma \theta_1 + 1} \left(\frac{1}{|x_1|^{\theta_1}} + \frac{|x_2|^{\theta_1}}{|x_1 x_2|^{\theta_2}} + \frac{|x_3|^{\theta_1}}{|x_1 x_3|^{\theta_2}}\right) \\
\quad &\leq C|h_2|^{\beta \theta_1} |h_3|^{\gamma \theta_1} \left(\frac{1}{|x_2|^{\theta_1}} + \frac{|x_3|^{\theta_1}}{|x_1 x_3|^{\theta_2}}\right) 
\end{align*}
\]

for all $0 \leq \alpha \leq 1$, $0 \leq \beta + \gamma \leq 1$ or $0 \leq \alpha + \gamma \leq 1$, $0 \leq \beta \leq 1$, and $|x_1| \geq 2|h_1| > 0$, $|x_2| \geq 2|h_2| > 0$, $|x_3| \geq 2|h_3| > 0$, $h_1, h_2, h_3 \in \mathbb{R}$ and some $0 < \theta_1 \leq 1 < \theta_2 < 2$.

We now recall the cancellation conditions given by

\[
\begin{align*}
(C.a) \quad &\int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} K(x_1, x_2, x_3)dx_1 dx_2 dx_3 \leq C \\
(C.b) \quad &\int_{\delta_1 \leq |x_1| \leq r_1} \Delta^\beta_{x_2,h_2} \Delta^\gamma_{x_3,h_3} K(x_1, x_2, x_3)dx_1 \\
\quad &\leq C|h_2|^{\beta \theta_1} |h_3|^{\gamma \theta_1} \left(\frac{1}{|x_2|^{\theta_1}} + \frac{|x_3|^{\theta_1}}{|x_1 x_3|^{\theta_2}}\right) \\
\quad &\quad \leq C|h_1|^{\alpha \theta_1} \quad \text{uniformly for all } \delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0;
\end{align*}
\]

for all $\delta_1, r_1 > 0$, $0 \leq \beta + \gamma \leq 1$, $|x_2| \geq 2|h_2| > 0$, $|z| \geq 2|h_3| > 0$;

\[
\begin{align*}
(C.c) \quad &\int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} \Delta^\alpha_{x_1,h_1} K(x_1, x_2, x_3)dx_2 dx_3 \\
\quad &\leq C|h_1|^{\alpha \theta_1} \quad \text{uniformly for all } \delta_2, \delta_3, r_2, r_3 > 0, |x_1| \geq 2|h_1| > 0 \text{ and } 0 \leq \alpha \leq 1.
\end{align*}
\]

Or

\[
\begin{align*}
(C'.a) \quad &\int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} K(x_1, x_2, x_3)dx_1 dx_2 dx_3 \leq C \\
(C'.b) \quad &\int_{\delta_2 \leq |x_2| \leq r_2} \Delta^\alpha_{x_1,h_1} \Delta^\gamma_{x_3,h_3} K(x_1, x_2, x_3)dx_2 \\
\quad &\leq C|h_1|^{\alpha \theta_1} |h_3|^{\gamma \theta_1} \left(\frac{1}{|x_2|^{\theta_1}} + \frac{|x_3|^{\theta_1}}{|x_1 x_3|^{\theta_2}}\right) \\
\quad &\quad \leq C|h_2|^{\beta \theta_1} \quad \text{uniformly for all } \delta_2, r_2 > 0, 0 \leq \alpha + \gamma \leq 1, |x_1| \geq 2|h_1| > 0 \text{ and } |x_3| \geq 2|h_3| > 0;
\end{align*}
\]

for all $\delta_2, r_2 > 0$, $0 \leq \alpha + \gamma \leq 1$, $|x_1| \geq 2|h_1| > 0$ and $|x_3| \geq 2|h_3| > 0$;

\[
\begin{align*}
(C'.c) \quad &\int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_1 \leq |x_1| \leq r_1} \Delta^\beta_{x_2,h_2} K(x_1, x_2, x_3)dx_1 dx_3 \leq C|h_2|^{\beta \theta_1} \quad \text{uniformly for all } \delta_1, \delta_3, r_1, r_3 > 0, |x_2| \geq 2|h_2| > 0 \text{ and } 0 \leq \beta \leq 1.
\end{align*}
\]

The operator $T$ is defined initially on $L^2(\mathbb{R}^3)$ as

\[
(2.1) \quad T(f) := K * f := \lim_{\epsilon \to 0^+} K_N^\epsilon * f,
\]

where $K_N^\epsilon(x_1, x_2, x_3) = K(x_1, x_2, x_3)$ if $\epsilon_1 \leq |x_1| \leq N_1$, $\epsilon_2 \leq |x_2| \leq N_2$ and $\epsilon_3 \leq |x_3| \leq N_3$ and $K_N^\epsilon(x_1, x_2, x_3) = 0$ otherwise, with $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ and $N = (N_1, N_2, N_3)$ for all $0 < \epsilon_1 \leq N_1 < \infty$, $0 < \epsilon_2 \leq N_2 < \infty$, and $0 < \epsilon_3 \leq N_3 < \infty$. 
2.1. Discrete Calderón Reproducing Formula. We note that in [22, Theorem 1.1], a special discrete Calderón reproducing formula was established.

Let \( \psi^{(1)} \in \mathcal{S}(\mathbb{R}) \) satisfy
\[
\text{supp } \hat{\psi}^{(1)}(\xi_1) \subset \{ \xi_1 : 1/2 < |\xi_1| \leq 2 \}
\]
and
\[
\sum_{j \in \mathbb{Z}} |\psi^{(1)}(2^j \xi_1)|^2 = 1 \quad \text{for all } \xi_1 \in \mathbb{R} \setminus \{0\},
\]
and let \( \psi^{(2)} \in \mathcal{S}(\mathbb{R}^2) \) satisfy
\[
\text{supp } \hat{\psi}^{(2)}(\xi_2, \xi_3) \subset \{ (\xi_2, \xi_3) : 1/2 < |(\xi_2, \xi_3)| \leq 2 \}
\]
and
\[
\sum_{k \in \mathbb{Z}} |\psi^{(2)}(2^k \xi_2, 2^k \xi_3)|^2 = 1 \quad \text{for all } (\xi_2, \xi_3) \in \mathbb{R}^2 \setminus \{0\}.
\]

Set
\[
\psi_{j,k}(x_1, x_2, x_3) := 2^{-2(j+k)} \psi^{(1)}(2^{-j} x_1) \psi^{(2)}(2^{-k} x_2, 2^{-(j+k)} x_3).
\]

Then
\[
f(x_1, x_2, x_3) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_3^{j,k}} |R| \psi_{j,k}(x_1 - x_I, x_2 - x_J, x_3 - x_S)(\psi_{j,k} * f)(x_I, x_J, x_S),
\]
where \( \mathcal{R}_3^{j,k} \) is the collection of Zygmund rectangles in \( \mathbb{R}^3 \) and \( R \in \mathcal{R}_3^{j,k} \) means that \( R = I \times J \times S \) with the side length \(|I| = 2^j, |J| = 2^k \) and \(|S| = 2^{j+k} \), \( \vec{x} \in \mathbb{R}^3 \), \( (x_I, x_J, x_S) \) denotes the “lower left corner” of \( R \) (i.e., the corner of \( R \) with the least value of each coordinate component), and the series converges in both \( \mathcal{S}_y(\mathbb{R}^3) \) (the Schwartz functions associated with the Zygmund dilations, i.e. \( f \in \mathcal{S}(\mathbb{R}^3) \) such that \( \int f(x_1, x_2, x_3)x_1^\alpha dx_1 = \int f(x_1, x_2, x_3)x_2^\beta x_3^\gamma dx_2 dx_3 = 0 \) for all nonnegative integers \( \alpha, \beta, \gamma \) and \( \mathcal{S}_y(\mathbb{R}^3) \) (the corresponding Schwartz distributions)).

However, we point out that in the proof of (2.7) in [22], \( (x_I, x_J, x_S) \) has to be chosen as the lower left corner of \( R \). This special discrete Calderón reproducing formula is not enough for us to build the weighted Littlewood–Paley theory for the vertical square functions. More precisely, in the proof of Theorem 2.7 in the next subsection, it is crucial that one can express \( f \) in a similar way as in (2.7) but with a free choice of \( (x_I, x_J, x_S) \). Therefore, we will first need to find a suitable test function space that is dense in \( L^2 \), on which we can build a more general discrete Calderón reproducing formula such that the points \( (x_I, x_J, x_S) \) can be chosen as an arbitrary point in the Zygmund rectangle \( R \). In particular, our test function space will be different from \( \mathcal{S}_y(\mathbb{R}^3) \).
We point out that the idea of such a discrete Calderón reproducing formula is not new, but it requires complicated technical estimates in order to deal with new difficulties brought by the Zygmund dilation. The main idea originates from Han and Sawyer \cite{20, 25} (see also \cite{11} for more details on space of homogeneous type, \cite{23} for this discrete Calderón’s reproducing formula in the tensor product setting, and see \cite{24} in the flag setting), which can be illustrated as follows:

(1) introduce a suitable test function space (non-convolution type) with an appropriate norm (semi norm);

(2) discretize the continuous Calderón reproducing formula (obtained by Fourier transform) and decompose it into an essential part and a remainder, i.e. via $Id = \mathcal{E} + \mathfrak{R}$ where $Id$ is the identity operator on the test function space, $\mathcal{E}$ denotes the operator with respect to the essential part and $\mathfrak{R}$ the remainder;

(3) show that the remainder operator $\mathfrak{R}$ maps the test function space into itself with norm strictly less than 1, hence $\mathcal{E}$ is invertible. Moreover, $\mathcal{E}^{-1}$ maps the test function space into itself as well;

(4) obtain the discrete version of Calderón reproducing formula by absorbing $\mathcal{E}^{-1}$ into the test function.

We now combine the idea outlined above with the techniques in the Zygmund dilation setting to establish the more general discrete Calderón reproducing formula that will work for our purposes. To begin with, we need to introduce a test function of non-convolution type associated with Zygmund dilations, which is modeled on the size, smoothness and cancellation conditions of the function $\psi_{r_1, r_2}$ with the dilations $r_1, r_2 \in (0, \infty)$:

\begin{equation}
\psi_{r_1, r_2}(x_1 - y_1, x_2 - y_2, x_3 - y_3) = r_1^{-2}r_2^{-2}\psi^{(1)}(\frac{x_1 - y_1}{r_1})\psi^{(2)}(\frac{x_2 - y_2}{r_2}, \frac{x_3 - y_3}{r_1r_2}),
\end{equation}

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R})$ satisfies (2.2) and (2.3), and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ satisfies (2.4) and (2.5).

**Definition 2.2.** A function $\psi(x_1, x_2, x_3)$ belongs to the test function space

$$G_{N_1,N_2}((y_1,y_2,y_3);r_1,r_2;\epsilon_1,\epsilon_2;M_1, M_2)$$

with respect to some fixed point $(y_1, y_2, y_3) \in \mathbb{R}^3$, some $\epsilon_1, \epsilon_2 \in (0,1]$, $r_1, r_2 \in (0, \infty)$, $M_1, M_2 \in (0, \infty)$ and $N_1, N_2 \in \mathbb{N}$ if $\psi(x_1, x_2, x_3)$ if it satisfies the following conditions for all $0 \leq \alpha \leq N_1$ and $0 \leq \beta + \gamma \leq N_2$:

\begin{equation}
|\frac{\partial^\alpha}{\partial x_1^\alpha} \frac{\partial^\beta}{\partial x_2^\beta} \frac{\partial^\gamma}{\partial x_3^\gamma} \psi(x_1, x_2, x_3)| \leq C \frac{r_1^{M_1+\alpha}}{(r_1 + |x_1 - y_1|)^{1+M_1+\alpha}} \cdot \frac{r_2^{M_2+\beta+\gamma}}{r_1(r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)^{2+M_2+\beta+\gamma}},
\end{equation}
\[
\left| \frac{\partial^\beta}{\partial x_2^\beta} \frac{\partial^\gamma}{\partial x_3^\gamma} \psi(x_1, x_2, x_3) - P_{N_1}^{(1)} \left( \frac{\partial^\beta}{\partial x_2^\beta} \frac{\partial^\gamma}{\partial x_3^\gamma} \psi(x_1', x_2, x_3) \right) (x_1) \right| \\
\leq C \left( \frac{|x_1 - x'_1|}{r_1 + |x_1 - y_1|} \right)^{N_1+\epsilon_1} \frac{r_1^{M_1}}{r_1^{M_2+\beta+\gamma}} \times \frac{r_1^{M_2+\beta+\gamma}}{r_1 (r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)^{1+M_1+\alpha}} \times r_1 (r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)^{2+M_2}
\]

for \(|x_1 - x'_1| \leq \frac{1}{2}(r_1 + |x_1 - y_1|)\), where \(P_{N_1}^{(1)} \left( \frac{\partial^\beta}{\partial x_2^\beta} \frac{\partial^\gamma}{\partial x_3^\gamma} \psi(x_1', x_2, x_3) \right) (x_1)\) denotes the Taylor polynomial of order \(N_1\) of \(\frac{\partial^\beta}{\partial x_2^\beta} \frac{\partial^\gamma}{\partial x_3^\gamma} \psi(x_1, x_2, x_3)\) (with \(x_2, x_3\) fixed) about the point \(x'_1\);

\[
\left| \frac{\partial^\alpha}{\partial x_1^\alpha} \psi(x_1, x_2, x_3) - \frac{\partial^\alpha}{\partial x_1^\alpha} \psi(x_1', x_2', x_3') \right| (x_2, x_3) \\
\leq C \left( \frac{|x_2 - x'_2| + r_1^{-1}|x_3 - x'_3|}{r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|} \right)^{N_2+\epsilon_2} \times \frac{r_1^{M_1+\alpha}}{r_1^{M_2}} \times \frac{r_2^{M_2}}{r_1 (r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)^{2+M_2}}
\]

for \(|x_2 - x'_2| + r_1^{-1}|x_3 - x'_3| \leq \frac{1}{2}(r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)\);

\[
\left[ \psi(x_1, x_2, x_3) - P_{N_1}^{(1)} (\psi(x_1', x_2, x_3)) (x_1) \right] \\
- P_{N_2}^{(2)} \left( \left[ \psi(x_1, x_2', x_3') - P_{N_1}^{(1)} (\psi(x_1', x_2', x_3')) (x_1) \right] \right) (x_2, x_3) \\
\leq C \left( \frac{|x_1 - x'_1|}{r_1 + |x_1 - y_1|} \right)^{N_1+\epsilon_1} \left( \frac{|x_2 - x'_2| + r_1^{-1}|x_3 - x'_3|}{r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|} \right)^{N_2+\epsilon_2} \times \frac{r_1^{M_1}}{r_1^{M_2}} \times \frac{r_2^{M_2}}{r_1 (r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)^{2+M_2}}
\]

for \(|x_1 - x'_1| \leq \frac{1}{2}(r_1 + |x_1 - y_1|)\) and \(|x_2 - x'_2| + r_1^{-1}|x_3 - x'_3| \leq \frac{1}{2}(r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)\);

\[
\left[ \psi(x_1, x_2, x_3) - P_{N_2}^{(2)} (\psi(x_1, x_2', x_3')) (x_2, x_3) \right] \left[ \psi(x_1, x_2', x_3') - P_{N_1}^{(1)} (\psi(x_1', x_2', x_3')) (x_1) \right] \\
- P_{N_1}^{(1)} \left( \left[ \psi(x_1', x_2, x_3) - P_{N_2}^{(2)} (\psi(x_1', x_2', x_3')) \right] \right) (x_1) \\
\leq C \left( \frac{|x_1 - x'_1|}{r_1 + |x_1 - y_1|} \right)^{N_1+\epsilon_1} \left( \frac{|x_2 - x'_2| + r_1^{-1}|x_3 - x'_3|}{r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|} \right)^{N_2+\epsilon_2} \times \frac{r_1^{M_1}}{r_1^{M_2}} \times \frac{r_2^{M_2}}{r_1 (r_2 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)^{2+M_2}}
\]
for $|x_1 - x'_1| \leq \frac{1}{2}(1 + |x_1 - y_1|)$ and $|x_2 - x'_2| + r_1^{-1}|x_3 - x'_3| \leq \frac{1}{2}(1 + |x_2 - y_2| + r_1^{-1}|x_3 - y_3|)$. Moreover, $\psi(x_1, x_2, x_3)$ satisfies the following cancellation conditions:

$$
\int \psi(x_1, x_2, x_3)x_1^{\alpha_1}dx_1 = \int \psi(x_1, x_2, x_3)x_2^{\beta_1}x_3^{\gamma_1}dx_2dx_3 = 0
$$

for all $0 \leq \alpha_1 \leq N_1$, $0 \leq \beta_1 + \gamma_1 \leq N_2$.

The norm of $\psi$ in $G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)$ is defined as

$$
\|\psi\|_{G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)} := \inf \{C : (2.9) - (2.13) \text{ hold}\}.
$$

We point out that the definition of the test function space above follows the idea of the definition of test functions on space of homogeneous type, see for example in [11], as well as the test function space in the tensor product setting, see for example in [23]. It is non-empty since it contains the following function $\psi(x_1 - y_1, x_2 - y_2, x_3 - y_3) = \psi^{(1)}(x_1 - y_1)\psi^{(2)}(x_2 - y_2, x_3 - y_3)$, with $\psi^{(1)} \in \mathcal{S}(\mathbb{R})$ satisfying (2.2) and (2.3), and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ satisfying (2.4) and (2.5).

Moreover, in $G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)$, $M_1$ and $M_2$ denote the polynomial decays; $N_1$ and $N_2$ denote the orders of cancellation; $\epsilon_1$ and $\epsilon_2$ denotes the Holder regularity, in the first variable and in the second and third variables, respectively. Also, $r_1$ and $r_2$ denotes the dilations in the first and second variable, respectively.

Also, the conditions (2.12) and (2.13) are the second order difference condition, which seems more natural and are easier to understand when $\psi(x_1, x_2, x_3)$ is of the form (2.8).

We observe that for any other fixed point $(y'_1, y'_2, y'_3) \in \mathbb{R}^3$ and $r'_1, r'_2 > 0$, the spaces $G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)$ and $G_{N_1, N_2}((y'_1, y'_2, y'_3); r'_1, r'_2; \epsilon_1, \epsilon_2; M_1, M_2)$ coincide and have equivalent norms. Furthermore, $G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)$ is a Banach space with respect to the norm $\|\psi\|_{G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)}$. Hence we will denote $G_{N_1, N_2}((y_1, y_2, y_3); r_1, r_2; \epsilon_1, \epsilon_2; M_1, M_2)$ by

$$
G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)
$$

for short.

For $\epsilon_1, \epsilon_2 \in (0, 1]$ and $\gamma > 0$, let $\overset{\circ}{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)$ be the completion of the space $G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)$ under the norm of $G_{N_1, N_2}(1, 1; M_1, M_2)$; of course when $\epsilon_1 = \epsilon_2 = 1$ we simply have $G_{N_1, N_2}(1, 1; M_1, M_2) = G_{N_1, N_2}(1, 1; M_1, M_2)$. We define the norm on $\overset{\circ}{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)$ by $\|\psi\|_{\overset{\circ}{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} := \|\psi\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)}$.

Then we have the following continuous Calderón’s reproducing formula on the space $\overset{\circ}{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)$. Note that $\overset{\circ}{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)$ is a subspace of $L^2$, hence the formula below is finer than the classical continuous Calderón reproducing formula on $L^2$.

**Theorem 2.3.** Let $\psi_{j,k}$ be defined as in (2.6). Suppose $\epsilon_1, \epsilon_2 \in (0, 1]$, $N_1, N_2 \in \mathbb{N}$, and $M_1, M_2 > 0$ with $N_1 + 1 < M_1$, $N_2 + 1 < M_2$ and $N_2 + 1 < N_1$. Then for
(2.14) \[ f(x_1, x_2, x_3) = \sum_{j, k} \psi_{j, k} * \psi_{j, k} * f(x_1, x_2, x_3) \]

where the series converges in \( \tilde{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \) for \( \epsilon'_i < \epsilon_i \) \( i = 1, 2 \) and \( M'_1, M'_2 \) with \( M'_1 + M'_2 < N_1 \) and \( M'_2 < N_2 - 1 \).

**Proof.** From the definition of \( \psi_{j, k} \), it follows immediately that (2.14) holds on \( L^2(\mathbb{R}^3) \). Note that \( f \) is in \( \tilde{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \). Without loss of generality, we assume that, with scaling \( r_1 = r_2 = 1 \), that \( f \in G_{N_1, N_2}((0, 0, 0); 1, 1; \epsilon_1, \epsilon_2; M_1, M_2) \) (see Definition 2.2).

We now show that the series converges in \( \tilde{G}_{N_1, N_2}(\epsilon'_1, \epsilon'_2; M'_1, M'_2) \). It suffices to prove

\[ \sum_{|j| > L_1, |k| \leq L_2} \psi_{j, k} * \psi_{j, k} * f(x_1, x_2, x_3), \quad \sum_{|j| \leq L_1, |k| > L_2} \psi_{j, k} * \psi_{j, k} * f(x_1, x_2, x_3) \]

and

\[ \sum_{|j| > L_1, |k| > L_2} \psi_{j, k} * \psi_{j, k} * f(x_1, x_2, x_3) \]

all tend to zero in \( \tilde{G}_{N_1, N_2}(\epsilon'_1, \epsilon'_2; M'_1, M'_2) \) as \( L_1 \) and \( L_2 \) tend to infinity. Note that \( \psi_{j, k} * \psi_{j, k} = (\psi * \psi)_{j, k} \), and \( \psi * \psi \) satisfies the same size, smoothness and cancellation conditions as \( \psi \) up to a multiplication of constants, so it suffices to show

\[ \sum_{|j| > L_1, |k| \leq L_2} \psi_{j, k} * f(x_1, x_2, x_3), \quad \sum_{|j| \leq L_1, |k| > L_2} \psi_{j, k} * f(x_1, x_2, x_3), \quad \sum_{|j| > L_1, |k| > L_2} \psi_{j, k} * f(x_1, x_2, x_3) \]

tends to zero in \( \tilde{G}_{N_1, N_2}(\epsilon'_1, \epsilon'_2; M'_1, M'_2) \) as \( L_1 \) and \( L_2 \) tend to infinity.

We just need to estimate \( \|\psi_{j, k} * f\|_{\tilde{G}_{N_1, N_2}(\epsilon'_1, \epsilon'_2; M'_1, M'_2)} \). Observe that:

\[ \psi_{j, k} * f(x_1, x_2, x_3) = \int_{\mathbb{R}^3} \psi_{j, k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) f(y_1, y_2, y_3) dy_1 dy_2 dy_3. \]

For each fixed \( (x_1, x_2, x_3) \in \mathbb{R}^3 \) and for each fixed \( j, k \), we consider \( \psi_{j, k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \) as a function of \( (y_1, y_2, y_3) \).

Similar to the proof of [21, Lemma 3.3] (also the proof of [11, Equation (3.50)]) for the one-parameter case, we split \( j \) and \( k \) into positive and negative cases. For the case \( j > 0 \) and \( k > 0 \), we use the cancellation condition on \( f \) for the variables \( y_1 \) and \( (y_2, y_3) \), and then use the size condition of \( f \) as in (2.9) as well as the smoothness of \( \psi_{j, k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \) for the variables \( y_1 \) and \( (y_2, y_3) \). and then use the size condition of \( \psi_{j, k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \) as well as the smoothness of \( f \) as in (2.12). For the case \( j > 0 \) and \( k < 0 \), we use the cancellation condition on \( f \) for the variables \( y_1 \) and the cancellation condition on \( \psi_{j, k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \) for the variables \( (y_2, y_3) \), and then use the smoothness of
\[ \psi_{j,k}(x_1-y_1, x_2-y_2, x_3-y_3) \] as well as the smoothness of \( f \) as in (2.11). Symmetrically, for the last case \( j < 0 \) and \( k > 0 \), we use the cancellation condition on \( \psi_{j,k}(x_1-y_1, x_2-y_2, x_3-y_3) \) for the variables \( y_1 \) and the cancellation condition on \( f \) for the variables \((y_2, y_3)\), and then use the smoothness of \( \psi_{j,k}(x_1-y_1, x_2-y_2, x_3-y_3) \) as well as the smoothness of \( f \) as in (2.10). Combing the estimates in all these cases, we obtain that

\[
|\psi_{j,k} \ast f(x_1, x_2, x_3)| 
\leq C\|f\|^2_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} 2^{-|\sigma_1|} 2^{-|\sigma_2|} 2^{-|y|} \frac{1}{(1 + |x_1|)^{1+M'_2}} \frac{1}{(1 + |x_2| + |x_3|)^{2+M'_2}},
\]

for \( \sigma_i > 0, i = 1, 2, \) and for \( M'_1, M'_2 \) with \( M'_1 + M'_2 < N_1 \) and \( M'_2 < N_2 + 1 \). Details of how this estimate is proved appear in the appendix, Section 6.

Note that \( \partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x,y,z) \) satisfies the same conditions as \( f \) for all indices \( \alpha, \beta, \) and \( \gamma \). The estimate above still holds when \( f \) is replaced by \( \partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x,y,z) \). Similarly, one can verify that (2.10)–(2.13) hold with \( r_1 = r_2 = 1 \), the extra factor \( 2^{-|\sigma_1|} 2^{-|\sigma_2|} \) in the coefficients for certain constants \( \sigma_1 \) and \( \sigma_2 \) depending on \( N_1, N_2, M_1 \) and \( M_2 \), and with \( M_i \) replaced by \( M'_i \) and \( \epsilon_i \) replaced by \( \epsilon'_i \). This shows that

\[
\sum_{|j| > L_1, |k| > L_2} \psi_{j,k} \ast f(x_1, x_2, x_3) \quad \text{and} \quad \sum_{|j| > L_1, |k| > L_2} \psi_{j,k} \ast f(x_1, x_2, x_3)
\]

tend to zero as \( L_1 \) and \( L_2 \) tend to infinity. The convergence of the other two terms follow similarly. The proof of Theorem 2.3 is complete.

We are now ready to establish the main result of this subsection, i.e. the following general discrete Calderón reproducing formula.

**Theorem 2.4.** Let \( \psi_{j,k} \) be the same as in (2.6). Suppose \( \epsilon_1, \epsilon_2 \in (0,1], N_1, N_2 \in \mathbb{N} \) and \( M_1, M_2 \) are large positive numbers with \( N_1 > M_1 + 1, N_2 > M_2 + 1 \). Then for \( f \in G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2) \), we have that

\[
f(x_1, x_2, x_3) \nonumber 
= \sum_{j,k \in \mathbb{Z}} \sum_{R = I \times J \times S \in \mathcal{R}_N(j,k)} |R| \psi_{j,k}(x_1, x_2, x_3, x_I, x_J, x_S) \psi_{j,k} \ast f(x_1, x_J, x_S),
\]

where the series converges in \( G_{N_1,N_2} (\epsilon'_1, \epsilon'_2; M'_1, M'_2) \) for \( \epsilon'_i < \epsilon, i = 1, 2, \) and \( M'_1, M'_2 \) with \( M'_1 + M'_2 < N_1 \) and \( M'_2 < N_2 + 1 \); for each fixed \( j, k \), \( \mathcal{R}_N(j,k) \) is the set of dyadic Zygmund rectangles which forms a partition of \( \mathbb{R}^3 \), that is, for each \( R = I \times J \times S \in \mathcal{R}_N(j,k) \), \( I, J, S \) are dyadic intervals in \( \mathbb{R} \) with \( |I| = 2^{j-N}, |J| = 2^{k-N} \), and \( |S| = 2^{j+k-2N} \). \( N \) is a large fixed positive integer; \( (x_I, x_J, x_S) \) is any fixed point in \( R \); for every \((x_1, x_J, x_S) \in R \) and every \( R \in \mathcal{R}_N(j,k) \), \( \tilde{\psi}_{j,k}(x_1, x_2, x_3, x_I, x_J, x_S) \) as a function of \((x_1, x_2, x_3) \) is in \( G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2) \) with respect to \((x_I, x_J, x_S) \).
Proof. For \( N_1, N_2 \in \mathbb{N} \) and \( M_1, M_2 \) are large positive numbers with \( N_1 > M_1 + 1, N_2 > M_2 + 1 \) and for \( f \in \mathcal{G}_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \), it is direct to see that \( f \) is in \( \mathcal{G}_{\tilde{N}_1, \tilde{N}_2}(\epsilon_1, \epsilon_2; \tilde{M}_1, \tilde{M}_2) \) such that \( \tilde{N}_i < N_i \) and \( \tilde{M}_i < M_i, i = 1, 2 \), and that \( \tilde{N}_1 + 1 < \tilde{M}_1, \tilde{N}_2 + 1 < \tilde{M}_2 \) and \( \tilde{N}_2 + 1 < \tilde{N}_1 \). Then from Theorem 2.4, we obtain that

\[
(2.17) \quad f(x_1, x_2, x_3) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x_1, x_2, x_3)
\]

\[
= \sum_{j,k} \int_{\mathbb{R}^3} \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \psi_{j,k} * f(y_1, y_2, y_3)dy_1dy_2dy_3
\]

\[
= \sum_{j,k} \sum_{R = I \times J \times S \in \mathcal{R}^N_{j,k}} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3)
\]

\[
\times \psi_{j,k} * f(y_1, y_2, y_3)dy_1dy_2dy_3
\]

\[
= \mathcal{E}(f)(x_1, x_2, x_3) + \mathcal{R}(f)(x_1, x_2, x_3)
\]

holds in \( \mathcal{G}_{N_1, N_2}(\epsilon'_1, \epsilon'_2; M'_1, M'_2) \) for \( \epsilon'_i < \epsilon_i, i = 1, 2 \), and \( M'_1, M'_2 \) with \( M'_1 + M'_2 < \tilde{N}_1 \) and \( M'_2 < \tilde{N}_2 + 1 \), where

\[
\mathcal{E}f(x_1, x_2, x_3) := \sum_{j,k} \sum_{R = I \times J \times S \in \mathcal{R}^N_{j,k}} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3)dy_1dy_2dy_3
\]

\[
\times \psi_{j,k} * f(x_1, x_J, x_S);
\]

\[
\mathcal{R}f(x_1, x_2, x_3) := \sum_{j,k} \sum_{R = I \times J \times S \in \mathcal{R}^N_{j,k}} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3)
\]

\[
\times (\psi_{j,k} * f(y_1, y_2, y_3) - \psi_{j,k} * f(x_1, x_J, x_S))dy_1dy_2dy_3.
\]

It is easy to see that the convolution operator \( \mathcal{R} \) has the kernel

\[
\mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3)
\]

\[
= \sum_{j,k} \sum_{R = I \times J \times S \in \mathcal{R}^N_{j,k}} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3)
\]

\[
\times (\psi_{j,k}(y_1 - v_1, y_2 - v_2, y_3 - v_3) - \psi_{j,k}(x_1 - v_1, x_J - v_2, x_S - v_3))dy_1dy_2dy_3.
\]

We further denote

\[
\mathcal{R}_{j,k}(x_1, x_2, x_3, v_1, v_2, v_3)
\]

\[
:= \sum_{R = I \times J \times S \in \mathcal{R}^N_{j,k}} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3)
\]

\[
\times (\psi_{j,k}(y_1 - v_1, y_2 - v_2, y_3 - v_3) - \psi_{j,k}(x_1 - v_1, x_J - v_2, x_S - v_3))dy_1dy_2dy_3,
\]

then it is direct that

\[
\mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3) = \sum_{j,k} \mathcal{R}_{j,k}(x_1, x_2, x_3, v_1, v_2, v_3).
\]
By looking carefully at the structure of the kernel $\mathcal{R}_{j,k}(x_1, x_2, x_3, v_1, v_2, v_3)$ as a function of $(x_1, x_2, x_3)$, we see that it satisfies the size, smoothness, and cancellation conditions as in Definition 2.2. Moreover, in the integrand, we see that $(y_1, y_2, y_3)$ and $(x_I, x_J, x_S)$ are in the same rectangle. Hence, this subtraction, by inserting an intermediate term, gives rise to the factor $2^{-N}$, which comes from the side-length of $R$ in terms of $I$ and of $J \times S$.

To be more precise, we have that

$$\left| \mathcal{R}_{j,k}(x_1, x_2, x_3, v_1, v_2, v_3) \right|$$

$$\leq \left| \sum_{R=I\times J \times S \in \mathcal{R}_N^J(j,k)} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \times 2^{-2(j+k)}(\psi^{(1)}(2^{-j}(y_1 - v_1)) - \psi^{(1)}(2^{-j}(x_I - v_1))) \times \psi^{(2)}(2^{-k}(y_2 - v_2), 2^{-(j+k)}(y_3 - v_3)) dy_1 dy_2 dy_3 \right|$$

$$+ \left| \sum_{R=I\times J \times S \in \mathcal{R}_N^J(j,k)} \int_R \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) 2^{-2(j+k)}\psi^{(1)}(2^{-j}(x_I - v_1)) \times (\psi^{(2)}(2^{-k}(y_2 - v_2), 2^{-(j+k)}(y_3 - v_3)) - \psi^{(2)}(2^{-k}(x_J - v_2), 2^{-(j+k)}(x_S - v_3))) dy_1 dy_2 dy_3 \right|$$

$$\leq \sum_{R=I\times J \times S \in \mathcal{R}_N^J(j,k)} \int_R \frac{2^{jM_1}}{2^{jM_1}} \frac{2^{-j2kM_2}}{2^{-j2kM_2}} \frac{|y_1 - x_I|}{2^j} \times \frac{2^{jM_1}}{2^{jM_1}} \frac{2^{-j2kM_2}}{2^{-j2kM_2}} \frac{|y_2 - x_J| + 2^{-j}|y_3 - x_S|}{(2^j + |x_I - y_1|)^{1+M_1}} \times \frac{2^{jM_1}}{2^{jM_1}} \frac{2^{-j2kM_2}}{2^{-j2kM_2}} \frac{|y_3 - x_3|}{(2^j + |x_J - y_2|)^{1+M_1}} \frac{2^{jM_1}}{2^{jM_1}} \frac{2^{-j2kM_2}}{2^{-j2kM_2}} \times dy_1 dy_2 dy_3$$

$$=: \text{Term}_1 + \text{Term}_2.$$
\[
\times \sum_{J \times S \in \mathbb{R}^2} \int_{J \times S} \frac{2^{-j2^{kM_2}}}{(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \cdot \frac{2^{-j2^{kM_2}}}{(2^k + |y_2 - v_2| + 2^{-j}|y_3 - v_3|)^{2+M_2}} dy_2 dy_3
\]
\[=: A \times B.\]

For term \(A\), by noting that \(|y_1 - x_I| \leq 2^{j-N}\) we get that
\[
A \leq \sum_{I \in \mathbb{R}^2} \int_{I} \frac{2^{jM_1}}{(2^j + |x_1 - y_1|)^{1+M_1}} \frac{2^{j-N}}{2^j + |y_1 - v_1|} \frac{2^{jM_1}}{(2^j + |y_1 - v_1|)^{1+M_1}} \frac{dy_1}{2^{jM_1}} \leq 2^{-N} \int_{\mathbb{R}} \frac{2^{jM_1}}{(2^j + |x_1 - y_1|)^{1+M_1}} \frac{2^{jM_1}}{(2^j + |y_1 - v_1|)^{1+M_1}} \frac{dy_1}{2^{jM_1}} \leq C2^{-N} \frac{2^{-j2^{kM_2}}}{(2^k + |x_2 - v_2| + 2^{-j}|x_3 - v_3|)^{2+M_2}}.
\]

As a consequence, we see that
\[
Term_1 \leq C2^{-N}2^{-2j-2k} \frac{1}{(1 + 2^{-j}|x_1 - v_1|)^{1+M_1}} \frac{1}{(1 + 2^{-k}|x_2 - v_2| + 2^{-j-k}|x_3 - v_3|)^{2+M_2}}.
\]

Similarly, we can get that
\[
Term_2 \leq C2^{-N}2^{-2j-2k} \frac{1}{(1 + 2^{-j}|x_1 - v_1|)^{1+M_1}} \frac{1}{(1 + 2^{-k}|x_2 - v_2| + 2^{-j-k}|x_3 - v_3|)^{2+M_2}}.
\]

which, together with the estimate for \(Term_1\), implies that
\[
|\mathfrak{R}_{j,k}(x_1, x_2, x_3, v_1, v_2, v_3)| \leq C2^{-N}2^{-2j-2k} \frac{1}{(1 + 2^{-j}|x_1 - v_1|)^{1+M_1}} \frac{1}{(1 + 2^{-k}|x_2 - v_2| + 2^{-j-k}|x_3 - v_3|)^{2+M_2}}.
\]

Similarly, since \(\psi^{(1)}\) and \(\psi^{(2)}\) are smooth, we can obtain that for all \(\alpha, \beta, \gamma \geq 0\)
\[
\left|\frac{\partial^\alpha}{\partial x_1^\alpha} \frac{\partial^\beta}{\partial x_2^\beta} \frac{\partial^\gamma}{\partial x_3^\gamma} \mathfrak{R}_{j,k}(x_1, x_2, x_3, v_1, v_2, v_3)\right| \leq C2^{-N}2^{-2j-2k} \frac{2^{-j(\alpha+\gamma)}}{(1 + 2^{-j}|x_1 - v_1|)^{1+M_1+\alpha+\gamma}} \frac{2^{-k(\beta+\gamma)}}{(1 + 2^{-k}|x_2 - v_2| + 2^{-j-k}|x_3 - v_3|)^{2+M_2+\beta+\gamma}}.
\]
Thus, by this observation and by following the proof of [21, Theorem 4.1], we see that $\mathcal{R}(x_1, x_2, v_1, v_2, v_3)$ satisfies:

\begin{equation}
|\partial_{x_1}^{\alpha} \partial_{x_2}^{\beta} \partial_{x_3}^{\gamma} \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3)| \leq \frac{2^{-N}C_{\alpha, \beta, \gamma, \theta_2}}{|x_1 - v_1|^{\alpha + 1}|x_2 - v_2|^{\beta + 1}|x_3 - v_3|^{\gamma + 1}} \times \left( \left| \frac{(x_1 - v_1)(x_2 - v_2)}{x_3 - v_3} \right|^{	heta_2} + \left| \frac{x_3 - v_3}{(x_1 - v_1)(x_2 - v_2)} \right|^{	heta_2} \right)
\end{equation}

for all $\alpha, \beta, \gamma \geq 0$ and for every $0 < \theta_2 < 1$;

\begin{equation}
\int_{\delta_1 \leq |x_1 - v_1| \leq r_1} \int_{\delta_2 \leq |x_2 - v_2| \leq r_2} \int_{\delta_3 \leq |x_3 - v_3| \leq r_3} |\mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3)| dx_1 dx_2 dx_3 \leq C2^{-N}
\end{equation}

uniformly for all $\delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0$;

\begin{equation}
\int_{\delta \leq |x_1 - v_1| \leq r} |\partial_{x_2}^{\beta} \partial_{x_3}^{\gamma} \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3)| dx_1 \leq \frac{2^{-N}C_{\beta, \gamma, \theta_2}}{|x_2 - v_2|^{\beta + 1}|x_3 - v_3|^{\gamma + 1}} \times \left( \left( \frac{1}{|r(x_2 - v_2)|} + \frac{|x_1 - v_1|}{r(x_2 - v_2)} \right)^{\theta_2} + \frac{1}{|r(x_2 - v_2)|} \right)
\end{equation}

for all $\delta, r, \beta, \gamma \geq 0$ and for every $0 < \theta_2 < 1$;

\begin{equation}
\int_{\delta_1 \leq |x_2 - v_2| \leq r_1} \int_{\delta_2 \leq |x_3 - v_3| \leq r_2} |\partial_{x_1}^{\alpha} \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3)| dx_2 dx_3 \leq \frac{2^{-N}C_{\alpha}}{|x_1 - v_1|^{\alpha + 1}}
\end{equation}

uniformly for all $\delta_1, \delta_2, r_1, r_2 > 0$ and $\alpha \geq 0$.

To sum up, $\mathcal{R}$ is a singular integral operator of Zygmund type with the associated kernel $\mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3)$ satisfying regularity condition (R) and the cancellation conditions (C.a)–(C.c) with $\theta_1 = 1$ and with an extra coefficient $2^{-N}$. In fact, we actually obtain that for all $\alpha$, $\beta$ and $\gamma$,

$$\int \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3) x_1^{\alpha} dx_1 = \int \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3) x_2^{\beta} x_3^{\gamma} dx_2 dx_3 = 0.$$
where λ_1 = N_1 - θ_2 and λ_2 = N_2 - θ_2 with θ_2 from (2.18)–(2.21) above which is an arbitrary constant in (0, 1).

Also note that the kernel \( \partial_{x_1}^\alpha \partial_{x_2}^\beta \partial_{x_3}^\gamma \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3) \) satisfies similar conditions as \( \mathcal{R}(x_1, x_2, x_3, v_1, v_2, v_3) \) for all indices \( \alpha, \beta, \) and \( \gamma \) and it possesses arbitrary order of cancellations for \( x_1 \) and for \( (x_2, x_3) \). Hence, by noting that by repeating the proof of [21, Lemma 3.2] we can verify all conditions (2.9)–(2.13) as in Definition 2.2 with an extra factor \( 2^{-N} \). Hence, we see that \( \mathcal{R}(f)(x_1, x_2, x_3) \) is in \( \tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \) and

\[
\| \mathcal{R}(f) \|_{\tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \leq C 2^{-N} \| f \|_{\tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)}.
\]

Now we obtain, by choosing a large fixed positive integer \( N \), that \( C 2^{-N} < 1 \), which further implies that

\[
\| \mathcal{R} \|_{\tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \to \tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} < 1.
\]

Note that from (2.17) we obtain that

\[ Id = \mathcal{E} + \mathcal{R} \]

in the sense of \( \tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \) and now we have that the operator norm of \( \mathcal{R} \) is strictly less than 1, which implies that \( \mathcal{E} \) is invertible with \( \mathcal{E}^{-1} = \sum_{n=0}^{\infty} \mathcal{R}^n \) and

\[
(2.22) \quad \| \mathcal{E}^{-1} \|_{\tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \to \tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \leq C < \infty.
\]

So now for \( f \in \tilde{G}_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2) \), from (2.17), we obtain that

\[
f(x_1, x_2, x_3) = \mathcal{E}^{-1} \circ \mathcal{E} f(x_1, x_2, x_3) = \mathcal{E}^{-1} \left( \sum_{j,k} \sum_{R=I \times J \times S \in \mathcal{R}^N_{I,j,k}} \int_R \psi_{j,k}(\cdot_1 - y_1, \cdot_2 - y_2, \cdot_3 - y_3) dy_1 dy_2 dy_3 \right) (x_1, x_2, x_3)
\]

\[
= \sum_{j,k} \sum_{R=I \times J \times S \in \mathcal{R}^N_{I,j,k}} \mathcal{E}^{-1} \left( \int_R \psi_{j,k}(\cdot_1 - y_1, \cdot_2 - y_2, \cdot_3 - y_3) dy_1 dy_2 dy_3 \right) (x_1, x_2, x_3)
\]

\[
\times \psi_{j,k} * f(x_1, x_J, x_S).
\]

Then we have

\[
f(x_1, x_2, x_3) = \sum_{j,k} \sum_{R=I \times J \times S \in \mathcal{R}^N_{I,j,k}} |R| \tilde{\psi}_{j,k}(x_1, x_2, x_3, x_I, x_J, x_S) \psi_{j,k} * f(x_1, x_J, x_S),
\]

where

\[
\tilde{\psi}_{j,k}(x_1, x_2, x_3, x_I, x_J, x_S) := \mathcal{E}^{-1} \left( \frac{1}{|R|} \int_R \psi_{j,k}(\cdot_1 - y_1, \cdot_2 - y_2, \cdot_3 - y_3) dy_1 dy_2 dy_3 \right) (x_1, x_2, x_3).
\]
Note that as a function of \((x_1, x_2, x_3)\),
\[
\frac{1}{|R|} \int_R \psi_{j,k}(\cdot - \cdot_1, \cdot - \cdot_2, \cdot - \cdot_3)dy_1dy_2dy_3
\]
is in \(G_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)\) since \(\psi_{j,k}\) as defined in (2.6) has arbitrary order of smoothness and cancellations. Next, from (2.22), we see that \(\tilde{\psi}_{j,k}(x_1, x_2, x_3, x_1, x_j, x_S)\) is also in \(G_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)\) with the same scaling \(r_1 = 2^{-j}\) and \(r_1 = 2^{-k}\) and associated with the point \((x_1, y_j, z_S)\), the center of \(R\), since \((y_1, y_2, y_3)\) is in \(R\).

The proof of Theorem 2.4 is complete. \(\square\)

2.2. Weighted Littlewood–Paley Theory: Discrete Version. To obtain the weighted estimates for the class of singular integral operators as in Definition 2.1, we now introduce the discrete Littlewood–Paley theory via the discrete area function and discrete (vertical) square function.

**Definition 2.5.** Let \(\psi_{j,k}\) be the same as in (2.6). For \(f \in L^p, 1 < p < \infty\), \(S^d_j(f)\), the discrete Littlewood–Paley area function of \(f\) associated with the Zygmund dilation is given by
\[
S^d_j(f)(x_1, x_2, x_3) = \left\{ \sum_{j,k \in \mathbb{Z}} \int_{|x_1-x_1|<2^{-j}} \int_{|x_2-x_2|<2^{-k}} \int_{|x_3-x_3|<2^{-j-k}} 2^{2j+2k} |\psi_{j,k} * f(y_1, y_2, y_3)|^2 dy_1dy_2dy_3 \right\}^{1/2}.
\]

We now prove that \(S^d_j\) satisfies the weighted estimate as \(S_j\) does. To be more precise, we have the following result.

**Theorem 2.6.** For \(w \in A^d_\infty(\mathbb{R}^3)\), there exist constants \(c, C > 0\), depending only on \([w]_{A^d_\infty(\mathbb{R}^3)}\), so that for all \(f \in L^p_w(\mathbb{R}^3)\),
\[
c\|f\|_{L^p_w(\mathbb{R}^3)} \leq \|S^d_j(f)\|_{L^p_w(\mathbb{R}^3)} \leq C\|f\|_{L^p_w(\mathbb{R}^3)}.
\]

**Proof.** We write \(\|S^d_j(f)\|^2_{L^p_w(\mathbb{R}^3)}\) as
\[
\int_{\mathbb{R}^3} S^d_j(f)(x_1, x_2, x_3)^2 w(x_1, x_2, x_3)dx_1dx_2dx_3
\]
\[
= \int_{\mathbb{R}^3} \sum_{j,k \in \mathbb{Z}} \int_{|x_1-x_1|<2^{-j}} \int_{|x_2-x_2|<2^{-k}} \int_{|x_3-x_3|<2^{-j-k}} 2^{2j+2k} |\psi_{j,k} * f(y_1, y_2, y_3)|^2 dy_1dy_2dy_3 w(x_1, x_2, x_3)dx_1dx_2dx_3
\]
\[
= \int_{\mathbb{R}^3} \sum_{j,k \in \mathbb{Z}} 2^{2j+2k} |\psi_{j,k} * f(y_1, y_2, y_3)|^2 w(R_{2^{-j},2^{-k}}(y_1, y_2, y_3))dy_1dy_2dy_3,
\]
where \(R_{2^{-j},2^{-k}}(y_1, y_2, y_3)\) denotes the rectangles with sides parallel to the axes, with center at \((y_1, y_2, y_3)\) and with side length \(2^{-j}\), \(2^{-k}\) and \(2^{-j-k}\).
Now fix $y_1 \in \mathbb{R}$ and $j \in \mathbb{Z}$. We consider the measure $w_{y_1,2^{-j}} = w_{y_1,2^{-j}}(y_2, y_3)dy_2dy_3$, where
\[ w_{y_1,2^{-j}}(y_2, y_3) = \int_{y_1 - 2^{-j} < \tilde{y}_1 < y_1 + 2^{-j}} w(\tilde{y}_1, y_2, y_3)d\tilde{y}_1. \]

From the definition of $A^2_\varphi(\mathbb{R}^3)$, we have that $w \in A^2_\varphi(\mathbb{R}^3)$ implies that $w_{y_1,2^{-j}}(y_2, y_3) \in A_2(\mathcal{R}_j)$, where $\mathcal{R}_j$ denotes the collection of rectangles in $\mathbb{R}^2$ with sides parallel to the axes and side length of the form $t$ and $2^{-j}t$, $t > 0$, and $A_2(\mathcal{R}_j)$ denotes the set of all weights in the plane satisfying the $A_2$ condition over all such rectangles in $\mathcal{R}_j$. Then we have
\[
\int_{\mathbb{R}^2} \sum_{k \in \mathbb{Z}} 2^{2k}|\psi_{j,k} * f(y_1, y_2, y_3)|^2 w(R_{2^{-j},2^{-k}}(y_1, y_2, y_3))dy_2dy_3
\]
\[
= \int_{\mathbb{R}^2} \sum_{k \in \mathbb{Z}} 2^{2k}|(f * 1/3_2 \psi_j^{(1)}(y_1, \cdot, \cdot)) * 2_3 \psi_j^{(2)}(y_2, y_3)|^2 w_{y_1,2^{-j}}(R_{2^{-k},2^{-j-k}}(y_2, y_3))dy_2dy_3.
\]

Here $* 1$ denotes the convolution only in the first variable $y_1$, and $* 2_3$ denotes the convolution only in the second and third variable $y_2, y_3$, $\psi_j^{(2)}(y_2, y_3) = \frac{1}{2^{-j} - 2^k} \psi_2(\frac{y_2}{2^{-j-k}}, \frac{y_3}{2^{-j-k}})$, and $R_{2^{-k},2^{-j-k}}(y_2, y_3)$ denotes the rectangle in $\mathbb{R}^2$ centered at $(y_2, y_3)$ whose side lengths are $2^{-k}$ and $2^{-j-k}$. Now set
\[
F_{y_1,2^{-j}}(y_1, y_2, y_3) = f(\cdot, y_2, y_3) * 1/3_2 \psi_j^{(1)}(y_1),
\]
then we observe that
\[
\int_{\mathbb{R}^2} \sum_{k \in \mathbb{Z}} 2^{2k}|(f(\cdot, \cdot, \cdot) * 1/3_2 \psi_j^{(1)}(y_1)) * 2_3 \psi_j^{(2)}(y_2, y_3)|^2 w_{y_1,2^{-j}}(R_{2^{-k},2^{-j-k}}(y_2, y_3))dy_2dy_3
\]
\[
= \int_{\mathbb{R}^2} |S_j(F_{y_1,2^{-j}})(y_2, y_3)|^2 w_{y_1,2^{-j}}(y_2, y_3)dy_2dy_3,
\]
where $S_j$ denotes the classical area function with respect to $\frac{1}{2^{-j} - 2^k} \psi_2(\frac{y_2}{2^{-j-k}}, \frac{y_3}{2^{-j-k}})$ with each fixed $j \in \mathbb{Z}$, that is
\[
S_j(h)(y_2, y_3) = \left( \sum_k \int_{|y_2 - \tilde{y}_2| < 2^{-j-k}} \int_{|y_3 - \tilde{y}_3| < 2^{-j-k}} \right) 2^{j+2k} |h * \frac{1}{2^{-j-2k}} \psi_2(\frac{\cdot}{2^{-j-k}}, \frac{\cdot}{2^{-j-k}})(\tilde{y}_2, \tilde{y}_3)|^2 d\tilde{y}_2d\tilde{y}_3
\]
for $h \in L^2(\mathbb{R}^2)$. By a change of variable (of $y_3$ and $\tilde{y}_3$) we obtain that
\[
(2.23) \quad \int_{\mathbb{R}^2} |S_j(F_{y_1,2^{-j}})(y_2, y_3)|^2 w_{y_1,2^{-j}}(y_2, y_3)dy_2dy_3
\]
\[
\approx \int_{\mathbb{R}^2} |F_{y_1,2^{-j}}(y_2, y_3)|^2 w_{y_1,2^{-j}}(y_2, y_3)dy_2dy_3,
\]
where the implicit constants depend only on the $A_2(\mathcal{R}_j)$ norm of $w_{y_1,2^{-j}}$ (which is uniformly bounded in $j$ by a quantity depending only on the $A^2_\varphi(\mathbb{R}^3)$ of $w$ in $\mathbb{R}^3$). This implies that, by taking the summation over $j$ with the factor $2^{2j}$ and taking the integration over $y_1$ on
both side of the equivalence \((2.23)\), we obtain that
\[
\int_{\mathbb{R}^3} |S^d_j(f)(x_1, x_2, x_3)|^2 w(x_1, x_2, x_3) dx_1 dx_2 dx_3
\]
\[
\approx \int \sum_j 2^{2j} \int_{\mathbb{R}^2} |F_{y_1, 2^{-j}}(y_2, y_3)|^2 w_{y_1, 2^{-j}}(y_2, y_3) dy_2 dy_3 dy_1
\]
\[
= \int_{\mathbb{R}^2} \sum_j 2^{2j} |f(x, y_2, y_3) * \psi_j(1)(y_1)|^2 \int_{y_1 - 2^{-j} < \tilde{y}_1 < y_1 + 2^{-j}} w(\tilde{y}_1, y_2, y_3) d\tilde{y}_1 dy_1 dy_2 dy_3.
\]
We now fix \(y_2\) and \(y_3\) to consider the term
\[
\int \sum_j 2^{2j} |f(x, y_2, y_3) * \psi_j(1)(y_1)|^2 \int_{y_1 - 2^{-j} < \tilde{y}_1 < y_1 + 2^{-j}} w(\tilde{y}_1, y_2, y_3) d\tilde{y}_1 dy_1.
\]
We rewrite it as
\[
\int \sum_j 2^{2j} |f(x, y_2, y_3) * \psi_j(1)(y_1)|^2 w_{y_2, y_3}(I_{y_1, j}) dy_1
\]
\[
= \int_{\mathbb{R}} S^2_{\psi(1)} f(x, y_2, y_3) (y_1) w_{y_2, y_3}(y_1) dy_1,
\]
where \(w_{y_2, y_3}(y_1) = w(y_1, y_2, y_3)\) and \(I_{y_1, j} = \{\tilde{y}_1 \in \mathbb{R} : |y_1 - \tilde{y}_1| < 2^{-j}\}\), and
\[
S^2_{\psi(1)}(h)(y_1) = \left( \sum_j \int_{|y_1 - \tilde{y}_1| < 2^{-j}} 2^j |h * \psi_j(1)(\tilde{y}_1)|^2 d\tilde{y}_1 \right)^{1/2}
\]
for \(h \in L^2(\mathbb{R})\), which is a classical (discrete) Littlewood–Paley area function on \(\mathbb{R}\). By the standard weighted estimate of the Littlewood–Paley area function, we obtain that
\[
\int_{\mathbb{R}} S^2_{\psi(1)} f(x, y_2, y_3) (y_1) w_{y_2, y_3}(y_1) dy_1 \approx \int_{\mathbb{R}} |f(y_1, y_2, y_3)|^2 w(y_1, y_2, y_3) dy_1
\]
and the implicit constants depend only on the \(A_2(\mathbb{R})\) constant of \(w_{y_2, y_3}(y_1)\) with \(y_2, y_3\) fixed, which depends only on the \(A^d_2(\mathbb{R}^3)\) of \(w\) in \(\mathbb{R}^3\) and is uniform in \((y_2, y_3)\).

As a consequence, we obtain that
\[
\int_{\mathbb{R}^3} |S^d_j(f)(x_1, x_2, x_3)|^2 w(x_1, x_2, x_3) dx_1 dx_2 dx_3
\]
\[
\approx \int_{\mathbb{R}^3} |f(y_1, y_2, y_3)|^2 w(y_1, y_2, y_3) dy_1 dy_2 dy_3.
\]
This finishes the proof of Theorem 2.6. \(\square\)

Next we also consider the Littlewood–Paley square function of \(f\) associated with the Zygmund dilation (defined in [21]), given by
\[
g^d_j(f)(x_1, x_2, x_3) := \left\{ \sum_{j,k \in \mathbb{Z}} |\psi_{j,k} * f(x_1, x_2, x_3)|^2 \right\}^{1/2}.
\]
The weighted estimate of \( g^d_3 \) can be established via \( S^d_3 \) according to Theorem 2.7 below. Here we point out that it is claimed in [22, page 25] that the weighted estimate of \( g^d_3 \) follows directly from [15, Theorem 2.9] (i.e. the weighted estimate of \( S_3 \)), which seems to be inaccurate. Our result fills in this gap.

**Theorem 2.7.** For \( w \in A^2_\infty(\mathbb{R}^3) \), we have that for all \( f \in L^2_0(\mathbb{R}^3) \),
\[
\|S^d_3(f)\|_{L^2_0(\mathbb{R}^3)} \approx \|g^d_3(f)\|_{L^2_0(\mathbb{R}^3)},
\]
where the implicit constants depend only on \( [w]_{A^2_\infty(\mathbb{R}^3)} \).

**Proof.** We now need the following almost orthogonality estimate for the test functions, whose proof follows from the approach and technique in the proof of [21, Lemma 3.3] (see also the proofs of [11, Lemmas 3.6 and 3.7] for the orthogonality estimate for the test function in the one-parameter setting).

Suppose that \( \tilde{\psi}_{j,k} \) and \( \tilde{\psi}_{j',k'} \) are test functions in \( \tilde{G}_{N_1,N_2}(\epsilon_1,\epsilon_2; M_1,M_2) \) with two fixed positive integers \( N_1,N_2 \) as in Definition 2.2 associated with the fixed points \((x_1,x_2,x_3)\) and \((y_1,y_2,y_3)\) in \( \mathbb{R}^3 \), respectively. Then
\[
\begin{align*}
\int_{\mathbb{R}^3} \tilde{\psi}_{j,k}(z_1,z_2,z_3)\tilde{\psi}_{j',k'}(z_1,z_2,z_3)dz_1dz_2dz_3 & \leq 2^{-|j-j'|(N_1+\epsilon_1)}2^{-|k-k'|(N_2+\epsilon_2)} \\
& \quad \times \frac{2^{M_1(j\vee j')}}{(2^{j\vee j'} + |x_1 - y_1|)^{1+M_1}2^{j'}(2^{k\vee k'} + |x_2 - y_2| + 2^{-j'}|x_3 - y_3|)^{2+M_2}},
\end{align*}
\]
where \( j^* = j \) if \( k \geq k' \) and \( j^* = j' \) if \( k < k' \).

Moreover, we recall the following technical result ([22, Lemma 3.3]), which is a generalisation of that of Frazier–Jawerth (in one-parameter in the Euclidean setting (See [18, pp. 147–148]) to the Zygmund dilation setting.

Given any nonnegative integer \( N \) and integers \( j,k,j',k' \). Let \( R \in \mathcal{R}^N_3(j,k) \) and \( R' = I' \times J' \times S' \in \mathcal{R}^N_3(j',k') \). Let \( \{a_{R'}\} \) be any given sequence and let \( x_{R'} = (x_{I'},x_{J'},x_{S'}) \) be any point in \( R' \). Then for any \( v = (v_1,v_2,v_3), v^* = (v_1^*,v_2^*,v_3^*) \in \mathbb{R}^3 \) we have
\[
\sum_{R' \in \mathcal{R}^N_3(j',k')} \frac{2^{(j\vee j')M_1}2^{(k\vee k')M_2}|R'|}{(2^{j\vee j'} + |v_1 - x_{I'}|)^{1+M_1}2^j(2^{k\vee k'} + |v_2 - x_{J'}| + 2^{-j'}|v_3 - x_{S'}|)^{2+M_2}} |a_{R'}| \\
\leq C 2^{[4N+2(j-j')_+ +2(k-k')_+]|\frac{1}{2}-1|2^{j-j'}| \left\{ \mathcal{M}_3\left( \sum_{R' \in \mathcal{R}^N_3(j',k')} |a_{R'}|^r \chi_{R'}(v_1^*,v_2^*,v_3^*) \right) \right\}^{\frac{1}{r}},
\]
where \( (j-j')_+ = \max(j-j',0) \), \( j^* = j \) if \( k < k' \) and \( j^* = j' \) if \( k \geq k' \), and moreover, \( \max\{\frac{1}{1+M_1},\frac{2}{2+M_2}\} < r \leq 1. \)
We now prove

\begin{equation}
||S^d_j(f)||_{L^2_\beta(\mathbb{R}^3)} \lesssim ||g^d_i(f)||_{L^2_\beta(\mathbb{R}^3)}.
\end{equation}

To see this, we begin with estimating \( \psi_{j',k'} \ast f(x_{I'}, x_{J'}, x_{S'}) \), where \((x_{I'}, x_{J'}, x_{S'})\) is any fixed point in a Zygmund rectangle \( R' = I' \times J' \times S' \) with \( \ell(I') = 2^{-j'}, \ell(J') = 2^{-k'} \) and \( \ell(S') = 2^{-j'-k'} \). Applying the discrete Calderón identity \((2.16)\) with \((x_I, x_J, x_S)\) being any point in \( R = I \times J \times S \), we get that

\[
\psi_{j',k'} \ast f(x_{I'}, x_{J'}, x_{S'}) = \sum_{j,k \in \mathbb{Z}} \sum_{R = I \times J \times S \in \mathcal{R}^N_{j,k}} |R| \psi_{j,k} \ast \left( \tilde{\psi}_{j,k}(1,2,3, x_I, x_J, x_S) \right) (x_{I'}, x_{J'}, x_{S'}) \psi_{j,k} \ast f(x_I, x_J, x_S).
\]

Note that \( \psi_{j',k'} \) is defined as in \((2.6)\), hence \( \psi_{j',k'} \) is in \( \mathcal{G}_{N_1, N_2} (\epsilon_1, \epsilon_2; M_1, M_2) \). Using the almost orthogonality estimate in \((2.25)\) for \( \psi_{j',k'} \ast \left( \tilde{\psi}_{j,k}(1,2,3, x_I, x_J, x_S) \right) (x_1, x_2, x_3) \) with choosing \( M \) such that \( \min\{\epsilon_1, \epsilon_2\} =: \epsilon > M \) and \( M < \min\{M_1, M_2\} \) and then applying the Frazier–Jawerth type estimate \((2.26)\) with \( M \), we have

\[
|\psi_{j',k'} \ast f(x_{I'}, x_{J'}, x_{S'})| 
\leq C \sum_{j,k \in \mathbb{Z}} 2^{-|j-j'|(N_1+\epsilon)-|k-k'|(N_2+\epsilon)} \sum_{R \in \mathcal{R}^N_{j,k}} |R| \frac{2^{(j\vee j')M}}{(2^{(2\vee j')} + |x_{I'} - x_I|)^{1+M}} |\psi_{j,k} \ast f(x_I, x_J, x_S)| \times 2^{-|j-j'|(N_1+\epsilon)-|k-k'|(N_2+\epsilon)} \frac{2^{4N+2(j-j')} + 2(k-k')_+ + 1}{2(j-j')} \psi_{j,k} \ast f(x_I, x_J, x_S)
\]

for any \((x_{I'}, x_{J'}, x_{S'}) \in 3R'\), where \( \frac{2}{2+M} \leq r < 1 \). Next, we note that for each fixed \( j, k \), we have that

\[
\sum_{R \in \mathcal{R}^N_{j,k}} |\psi_{j,k} \ast f(x_I, x_J, x_S)|^r \chi_R = \left( \sum_{R \in \mathcal{R}^N_{j,k}} \left| (\psi_{j,k} \ast f(x_I, x_J, x_S))^2 \chi_R \right| \right)^{\frac{r}{2}}
\]

since those \( R \in \mathcal{R}^N_{j,k} \) are pairwise disjoint. As a consequence we get that

\[
|\psi_{j',k'} \ast f(x_{I'}, x_{J'}, x_{S'})| 
\leq C \sum_{j,k \in \mathbb{Z}} 2^{-|j-j'|(N_1+\epsilon)-|k-k'|(N_2+\epsilon)} \frac{2^{4N+2(j-j')} + 2(k-k')_+ + 1}{2(j-j')} |\psi_{j,k} \ast f(x_I, x_J, x_S)| \times \left( \sum_{R \in \mathcal{R}^N_{j,k}} \left| (\psi_{j,k} \ast f(x_I, x_J, x_S))^2 \chi_R \right| \right)^{\frac{r}{2}} \chi_{3R'}(x_{I'}, x_{J'}, x_{S'})
\]
where $\frac{2}{2+M} < r < 1$. From the free choice of $(x_I, x_J, x_S)$ in the reproducing formula, we further have that for every $(x_P, x_{P'}, x_{S'}) \in 3R'$,

\begin{equation}
|\psi_{j', k'} * f(x_P, x_{P'}, x_{S'})| \\
\leq C \sum_{j, k \in \mathbb{Z}} 2^{-|j-j'|((N_1+\epsilon)-(k-k')2[4N+2(j-J')_+ + 2(k-k')_+]|(\frac{1}{2} - 1)2j-j'|} \\
\times \left\{ \mathcal{M}_3 \left( \left( \sum_{R \in R_3^N(j,k)} \inf_{(x_I, x_J, x_S) \in R} |\psi_{j,k} * f(x_I, x_J, x_S)|^2 \chi_R \right)^{\frac{1}{2}} \right) \right\}^2 \\
\times \chi_{3R'}(x_P^*, x_{P'}^*, x_{S'}^*) \right\}^{\frac{1}{2}}.
\end{equation}

We now consider $S_2^d(f)(x_1, x_2, x_3)$. First note that $\sum_{R' \in R_2(j', k')} \chi_{R'} \equiv 1$, where $R_2(j', k')$ is the set of dyadic Zygmund rectangles which forms a partition of $\mathbb{R}^3$, that is, for each $R' = I' \times J' \times S' \in R_2(j', k')$, $I', J', S'$ are dyadic intervals in $\mathbb{R}$ with $|I'| = 2^{-j'}$, $|J'| = 2^{-k'}$, and $|S'| = 2^{-j' - k'}$. Now from the definition of $S_2^d(f)$ and the partition above, we have that

\begin{align*}
S_2^d(f)(x_1, x_2, x_3) &= \left\{ \sum_{j, k \in \mathbb{Z}} \sum_{R' \in R_2(j', k')} \int \int \int_{(y_1, y_2, y_3) \in R'} |\psi_{j', k'} * f(y_1, y_2, y_3)|^2 dy_1 dy_2 dy_3 \right\}^{\frac{1}{2}} \\
&\leq \left\{ \sum_{j, k \in \mathbb{Z}} \sum_{R' \in R_2(j', k')} \int \int \int_{(y_1, y_2, y_3) \in R'} |\psi_{j', k'} * f(y_1, y_2, y_3)|^2 dy_1 dy_2 dy_3 \right\}^{\frac{1}{2}} \\
&\leq \left\{ \sum_{j, k \in \mathbb{Z}} \sum_{R' \in R_2(j', k')} \int \int \int_{(y_1, y_2, y_3) \in R'} |\psi_{j', k'} * f(y_1, y_2, y_3)|^2 \chi_{R'} dy_1 dy_2 dy_3 \right\}^{\frac{1}{2}} \\
&\leq \left\{ \sum_{j, k \in \mathbb{Z}} \sum_{R' \in R_2(j', k')} \left( \sum_{j, k \in \mathbb{Z}} \int \int \int_{(y_1, y_2, y_3) \in R'} |\psi_{j', k'} * f(y_1, y_2, y_3)|^2 \chi_{R'} dy_1 dy_2 dy_3 \right) \right\}^{\frac{1}{2}}.
\end{align*}
\[
\chi_{3R'}(x_1, x_2, x_3)^2 \right)^{\frac{1}{2}},
\]
where the first inequality follows from (2.28) with the fact that for each \(j', k', R' \in \mathcal{R}_j(j', k')\) and each \((y_1, y_2, y_3) \in R'\) with \(|x_1-y_1| < 2^{-j'}, |x_2-y_2| < 2^{-k'}, |x_3-y_3| < 2^{-j'-k'}, (x_1, x_2, x_3)\) is in \(3R'\).

Applying the Cauchy–Schwarz inequality and then summing over \(j', k'\) and \(R'\) yields (2.29)

\[
S_3^d(f)(x_1, x_2, x_3) \\
\leq C \left\{ \sum_{j, k \in \mathbb{Z}} \left\{ \mathcal{M}_3 \left( \sum_{R \in \mathcal{R}_j^N(j, k)} \inf_{(x_I, x_J, x_S) \in R} \left| \psi_{j, k} \ast f(x_I, x_J, x_S) \right|^2 \chi_R \right)^{\frac{1}{2}} \right\}^2 \right\}^{\frac{1}{2}}
\]

since

\[
\sum_{j, k \in \mathbb{Z}} 2^{-|j-j'|(|N_1+\epsilon|-|k-k'|(|N_2+\epsilon)|}|2^{4N+2|j-j'|+2(k-k'')}|\left( \frac{1}{r} - 1 \right) 2|j-j'| \leq C
\]

and

\[
\sum_{j', k' \in \mathbb{Z}} 2^{-|j-j'|(|N_1+\epsilon|-|k-k'|(|N_2+\epsilon)|)|2^{4N+2|j-j'|+2(k-k'')}|\left( \frac{1}{r} - 1 \right) 2|j-j'| \leq C,
\]

where the above two inequalities follows from the facts that \(\epsilon > M, \frac{2}{2+M} < r < 1\) and that \(N_1 \geq 1\).

Then, from (2.29), we have that

\[
\left\| S_3^d(f) \right\|^2_{L^d_2(\mathbb{R}^3)} \\
\leq \int_{\mathbb{R}^3} \sum_{j, k \in \mathbb{Z}} \left\{ \mathcal{M}_3 \left( \sum_{R \in \mathcal{R}_j^N(j, k)} \inf_{(x_I, x_J, x_S) \in R} \left| \psi_{j, k} \ast f(x_I, x_J, x_S) \right|^2 \chi_R \right)^{\frac{1}{2}} \right\}^2 \times w(x_1, x_2, x_3) dx_1 dx_2 dx_3
\]

\[
= \sum_{j, k \in \mathbb{Z}} \int_{\mathbb{R}^3} \left\{ \mathcal{M}_3 \left( \sum_{R \in \mathcal{R}_j^N(j, k)} \inf_{(x_I, x_J, x_S) \in R} \left| \psi_{j, k} \ast f(x_I, x_J, x_S) \right|^2 \chi_R \right)^{\frac{1}{2}} \right\}^2 \times w(x_1, x_2, x_3) dx_1 dx_2 dx_3
\]

\[
\leq C \sum_{j, k \in \mathbb{Z}} \int_{\mathbb{R}^3} \left\{ \left( \sum_{R \in \mathcal{R}_j^N(j, k)} \inf_{(x_I, x_J, x_S) \in R} \left| \psi_{j, k} \ast f(x_I, x_J, x_S) \right|^2 \chi_R(x_1, x_2, x_3) \right)^{\frac{1}{2}} \right\}^2 \times w(x_1, x_2, x_3) dx_1 dx_2 dx_3
\]

\[
= C \sum_{j, k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_j^N(j, k)} \inf_{(x_I, x_J, x_S) \in R} \left| \psi_{j, k} \ast f(x_I, x_J, x_S) \right|^2 \chi_R(x_1, x_2, x_3) \times w(x_1, x_2, x_3) dx_1 dx_2 dx_3,
\]
where the second inequality follows from the facts that \( w \) is in \( \mathcal{A}_2^{1}(\mathbb{R}^3) \) since \( w \in \mathcal{A}_2^{1}(\mathbb{R}^3) \) and \( r < 1 \), \( \mathcal{A}_2^{1}(\mathbb{R}^3) \subset \mathcal{A}_2^{r}(\mathbb{R}^3) \), that \( \mathcal{M}_j \) is bounded on \( L^2_w(\mathbb{R}^3) \), and that the constant \( C \) in the first inequality depends on \( [w]_{\mathcal{A}_2^{r}(\mathbb{R}^3)} \).

Next, by noting that

\[
\inf_{x_1, x_2, x_3 \in R} |\psi_{j,k} * f(x_1, x_2, x_3)|^2 \leq |\psi_{j,k} * f(x_1, x_2, x_3)|^2 \leq C \int_{\mathbb{R}^3} \sum_{j,k \in Z} \sum_{R \in R_N^j(j,k)} |\psi_{j,k} * f(x_1, x_2, x_3)|^2 \chi_{R}(x_1, x_2, x_3) \times w(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3,
\]

where the last inequality follows from the fundamental fact that for each fixed \( j \) and \( k \) in \( \mathbb{Z} \) and for each \( N \in \mathbb{N} \), \( \sum_{R \in R_N^j(j,k)} \chi_{R}(x_1, x_2, x_3) \leq 1 \).

As a consequence, we see that

\[
\|S_j^d(f)\|_{L^2_w(\mathbb{R}^3)} \leq C \left\{ \sum_{j,k \in Z} |\psi_{j,k} * f|^2 \right\}^{\frac{1}{2}} \lesssim \|g_j^d(f)\|_{L^2_w(\mathbb{R}^3)},
\]

which implies \((2.27)\).

We now prove

\[
(2.30) \quad \|g_j^d(f)\|_{L^2_w(\mathbb{R}^3)} \lesssim \|S_j^d(f)\|_{L^2_w(\mathbb{R}^3)}.
\]

To begin with, it is clear that

\[
g_j^d(f)(x_1, x_2, x_3) \leq C \left\{ \sum_{j', k' \in \mathbb{Z}} \sum_{R' \in R_N^{j',k'}(j',k',x')} \sup_{x' \in R'} |\psi_{j',k'} * f(x_1, x_2, x_3)|^2 \chi_{R'}(x_1, x_2, x_3) \right\}^{\frac{1}{2}}.
\]

Again, by using \((2.28)\), we obtain that

\[
g_j^d(f)(x_1, x_2, x_3) \leq C \left\{ \sum_{j', k' \in \mathbb{Z}} \sum_{R' \in R_N^{j',k'}(j',k',x')} \sup_{x' \in R'} |\psi_{j',k'} * f(x_1, x_2, x_3)|^2 \chi_{R'}(x_1, x_2, x_3) \right\}^{\frac{1}{2}}
\]

\[
\times \left\{ \mathcal{M}_j \left( \left\{ \sum_{R \in R_N^j(j,k)} \inf_{x_1, x_2, x_3 \in R} |\psi_{j,k} * f(x_1, x_2, x_3)|^2 \chi_{R}\right\}^\frac{1}{2} \right) \right\}^2
\]

\[
\chi_{3R'}(x_1, x_2, x_3) \right\}^{\frac{1}{2}}.
\]
\[ \leq C \left( \sum_{j,k \in \mathbb{Z}} \left\{ \mathcal{M}_j \left( \sum_{R \in \mathbb{R}^N_{(j,k)}} \left( \inf_{(x_I,x_J,x_S) \in R} |\psi_{j,k} \ast f(x_I,x_J,x_S)|^2 \chi_R(\cdot) \right)^{\frac{p}{2}} \right) \right\} \right)^{\frac{1}{2}}. \]

By using similar arguments as in the proof of (2.27), we obtain that
\[ \|g^d_j(f)\|_{L^p_w(\mathbb{R}^3)} \]
\[ \leq C \left\| \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathbb{R}^N_{(j,k)}} \left( \inf_{(x_I,x_J,x_S) \in R} |\psi_{j,k} \ast f(x_I,x_J,x_S)|^2 \chi_R(\cdot) \right)^{\frac{p}{2}} \right\} \right\|_{L^p_w(\mathbb{R}^3)} \]
\[ = C \left\| \left\{ \sum_{j,k \in \mathbb{Z}} \left( \int_{1-2^{-j} < |y_1| < 2^{-j}} \int_{2^{-k} < |y_1| < 2^{-k}} \int_{|y_3| < 2^{-j-k}} 2^{2j+2k} |\psi_{j,k} \ast f(y_1,y_2,y_3)|^2 dy_1 dy_2 dy_3 \right) \right\} \right\|_{L^p_w(\mathbb{R}^3)} \]
\[ \leq \|S^d_j(f)\|_{L^p_w(\mathbb{R}^3)}, \]
which implies (2.30).

As a consequence, we see that (2.24) holds. The proof of Theorem 2.7 is complete. \qed

Following the proof of Theorem 2.7 above, we can also obtain that

**Remark 2.8.** For \(1 < p < \infty\), \(w \in A^p_w(\mathbb{R}^3)\), for all \(f \in L^p_w(\mathbb{R}^3)\),
\[ (2.31) \]
\[ \|S^d_j(f)\|_{L^p_w(\mathbb{R}^3)} \approx \|g^d_j(f)\|_{L^p_w(\mathbb{R}^3)}, \]
where the implicit constants depend only on \([w]_{A^p_w(\mathbb{R}^3)}\).

Parallel to the test functions in the first case (as in (2.6)), we also consider the following test functions, grouping the variables in \(x_2\) and \((x_1, x_3)\). Let \(\psi^{(1)} \in \mathcal{S}(\mathbb{R})\) satisfy
\[ \text{supp} \widehat{\psi^{(1)}}(\xi_2) \subset \{ \xi_2 : 1/2 < |\xi_2| \leq 2 \} \]
and
\[ \sum_{j \in \mathbb{Z}} |\widehat{\psi^{(1)}}(2^j \xi_2)|^2 = 1 \quad \text{for all } \xi_2 \in \mathbb{R} \backslash \{0\}, \]
and let \(\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)\) satisfy
\[ \text{supp} \widehat{\psi^{(2)}}(\xi_1, \xi_3) \subset \{ (\xi_1, \xi_3) : 1/2 < |(\xi_1, \xi_3)| \leq 2 \} \]
and
\[ \sum_{k \in \mathbb{Z}} |\widehat{\psi^{(2)}}(2^k \xi_1, 2^k \xi_3)|^2 = 1 \quad \text{for all } (\xi_1, \xi_3) \in \mathbb{R}^2 \backslash \{0\}. \]
Set
\[
\psi_{j,k}(x_1, x_2, x_3) := 2^{-2(j+k)} \psi^{(1)}(2^{-j} x_2) \psi^{(2)}(2^{-k} x_1, 2^{-(j+k)} x_3).
\]

Then we can define a similar version of the discrete area function \( \tilde{S}_j^d \) and square function \( \tilde{g}_j^d \), and also have the following theorem.

**Theorem 2.9.** For \( 1 < p < \infty \), \( w \in A_p^1(\mathbb{R}^3) \), we have that for all \( f \in L^p_w(\mathbb{R}^3) \),
\[
\| \tilde{S}_j^d(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| \tilde{g}_j^d(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| f \|_{L^p_w(\mathbb{R}^3)},
\]
where the implicit constants depend only on \([w]_{A_p^1(\mathbb{R}^3)}\).

The proof of Theorem 2.9 follows from that of Theorem 2.7, by using the reproducing formula, almost orthogonality estimates and the Frazier–Jawerth type estimate with the minor change of swapping the roles of the variable \( x_1 \) and \( x_2 \). We omit the details here.

**2.3. Weighted Littlewood–Paley Theory: Continuous Version.** The continuous version of the Littlewood–Paley area function associated with Zygmund dilation was first introduced in [15]. We recall its precise definition below. The aim of this subsection is to introduce a continuous version of Littlewood–Paley (vertical) square function associated with Zygmund dilation, and then prove its weighted estimate. This result is also expected to be of independent interest.

We recall the definition by beginning with a technical decomposition as in [15, Lemma 2.5]. That is, suppose \( \tilde{\phi} \) is a function supported in a unit cube in \( \mathbb{R}^3 \) with a certain amount of uniform smoothness and satisfy cancellation conditions as in (1.4), then \( \tilde{\phi} \) can be decomposed as
\[
\tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2
\]
where both \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) satisfy a certain amount of uniform smoothness and moreover, the following cancellation conditions are satisfied:
\[
\int_{\mathbb{R}} \tilde{\phi}_1(x_1, x_2, x_3) dx_1 = 0 \quad \text{for each fixed } (x_2, x_3) \text{ in } \mathbb{R}^2;
\]
\[
\int_{\mathbb{R}^2} \tilde{\phi}_1(x_1, x_2, x_3) dx_2 dx_3 = 0 \quad \text{for each fixed } x_1 \text{ in } \mathbb{R}
\]
and
\[
\int_{\mathbb{R}} \tilde{\phi}_2(x_1, x_2, x_3) dx_2 = 0 \quad \text{for each fixed } (x_1, x_3) \text{ in } \mathbb{R}^2;
\]
\[
\int_{\mathbb{R}^2} \tilde{\phi}_2(x_1, x_2, x_3) dx_1 dx_3 = 0 \quad \text{for each fixed } x_2 \text{ in } \mathbb{R}.
\]

We recall the Littlewood–Paley area function in [15] with respect to the first group of cancellation conditions. Here we modify a bit on the Schwartz function class (by imposing a
stronger condition) that we use to define the area function. Let \( S(\mathbb{R}^i) \) denote the Schwartz class in \( \mathbb{R}^i, i = 1, 2, 3 \). We construct a function defined on \( \mathbb{R}^3 \) given by

\[
\phi(x_1, x_2, x_3) = \phi^{(1)}(x_1)\phi^{(2)}(x_2, x_3)
\]

where \( \phi^{(1)}(x_1) \) is an even function in \( C^\infty_c(\mathbb{R}) \) and \( \phi^{(2)}(x_2, x_3) \) is a radial function in \( C^\infty_c(\mathbb{R}^2) \) (both \( \phi_1 \) and \( \phi_2 \) are not identically 0), such that

\[
\supp \hat{\phi}^{(1)}(\xi_1) \subset \{ \xi_1 \in \mathbb{R} : 1/2 < |\xi_1| \leq 2 \}
\]

and

\[
\int_0^\infty |\hat{\phi}^{(1)}(s\xi_1)|^2 \frac{ds}{s} = 1 \quad \text{for all } \xi_1 \in \mathbb{R}\setminus\{0\},
\]

and let \( \phi^{(2)} \in S(\mathbb{R}^2) \) satisfy

\[
\supp \hat{\phi}^{(2)}(\xi_2, \xi_3) \subset \{ (\xi_2, \xi_3) \in \mathbb{R}^2 : 1/2 < |(\xi_2, \xi_3)| \leq 2 \}
\]

and

\[
\int_0^\infty |\hat{\phi}^{(2)}(t\xi_2, t\xi_3)|^2 \frac{dt}{t} = 1 \quad \text{for all } (\xi_2, \xi_3) \in \mathbb{R}^2\setminus\{0\}.
\]

For \( s, t > 0 \), we also set

\[
\phi_{s,t}(x_1, x_2, x_3) := \frac{1}{s^2t^2} \phi^{(1)}(\frac{x_1}{s})\phi^{(2)}(\frac{x_2}{t}, \frac{x_3}{st}).
\]

Then for \( f \in L^p(\mathbb{R}^3), 1 < p < \infty \), the Littlewood–Paley area function of \( f \) associated with the Zygmund dilation (defined in [15]) is given by

\[
S_3(f)(x_1, x_2, x_3) = \left\{ \iint_{\Gamma_3(x_1, x_2, x_3)} |\phi_{s,t} * f(y_1, y_2, y_3)|^2 \frac{dy_1 dy_2 dy_3 ds dt}{s^3t^3} \right\}^{1/2}
\]

where \( \Gamma_3(x_1, x_2, x_3) \) is the Zygmund-type cone with the vertex \( (x_1, x_2, x_3) \) defined as follows:

\[
\{(y_1, y_2, y_3, s, t) : |x_1 - y_1| < s, |x_2 - y_2| < s, |x_3 - y_3| < st, s > 0, t > 0\}.
\]

Then as pointed out in [15, Theorem 2.9], for \( w \in A^1_p(\mathbb{R}^3) \), there exist constants \( c, C > 0 \), depending only on \([w]_{A^1_p(\mathbb{R}^3)}\), so that for all \( f \in L^p_w(\mathbb{R}^3) \),

\[
c\|f\|_{L^p_w(\mathbb{R}^3)} \leq \|S_3(f)\|_{L^2_w(\mathbb{R}^3)} \leq C\|f\|_{L^p_w(\mathbb{R}^3)}.
\]

Then by the argument of Rubio de Francia’s extrapolation theorem as stated in [15], we obtain that (2.41) holds for \( L^p_w(\mathbb{R}^3) \) norm with \( 1 < p < \infty \) and \( w \in A^1_p(\mathbb{R}^3) \).

We now introduce a continuous version of the Littlewood–Paley square function as follows.
**Definition 2.10.** Let $\phi_{s,t}$ be the same as in (2.39). For $f \in L^p(\mathbb{R}^3)$, $1 < p < \infty$, $g_3(f)$, the continuous Littlewood–Paley square function of $f$ associated with the Zygmund dilation, is defined as

$$g_3(f)(x_1, x_2, x_3) = \left\{ \int_0^\infty \int_0^\infty |\phi_{s,t} \ast f(x_1, x_2, x_3)|^2 \frac{dsdt}{st} \right\}^{\frac{1}{2}}.$$ 

We point out that since there is no available atomic decomposition or harmonic function approach in the Zygmund dilation setting, it is not clear how to establish the weighted estimate for $g_3(f)$ directly or via the estimate of $S_3(f)$ as in (2.41).

However, it turns out that we can obtain the weighted estimate of $g_3(f)$ by using the result of its discrete version $g_3^d(f)$ obtained in Subsection 2.2.

**Theorem 2.11.** For $w \in A^1_{\infty}(\mathbb{R}^3)$, we have that for all $f \in L^2_w(\mathbb{R}^3)$,

$$(2.42) \quad \|g_3(f)\|_{L^2_w(\mathbb{R}^3)} \lesssim \|g_3^d(f)\|_{L^2_w(\mathbb{R}^3)},$$

where the implicit constants depend only on $[w]_{A^1_{\infty}(\mathbb{R}^3)}$.

**Proof.** In fact, by applying the discrete Calderón identity (2.16) for $f$ and then using the estimate (2.28), we see that

$$g_3(f)(x_1, x_2, x_3)^2$$

$$= \sum_{j'} \sum_{k'} \int_{2^{-j'}}^{2^{j'+1}} \int_{2^{-k'}}^{2^{k'+1}} |\phi_{s,t} \ast f(x_1, x_2, x_3)|^2 \frac{dsdt}{st}$$

$$\leq C \sum_{j'} \sum_{k'} \int_{2^{-j'}}^{2^{j'+1}} \int_{2^{-k'}}^{2^{k'+1}} \left( \sum_{j,k \in \mathbb{Z}} 2^{-|j-j'|(N_1+\epsilon)-|k-k'| (N_2+\epsilon)} 2^{4N+2(j-j')+2(k-k')} \right) \left( \sum_{j,k \in \mathbb{Z}} 2^{-|j-j'| (N_1+\epsilon)-|k-k'| (N_2+\epsilon)} 2^{4N+2(j-j')+2(k-k')} \right)$$

$$\times \left\{ \mathcal{M}_1 \left( \left( \inf_{R \in \mathcal{R}_N^N(j,k)} |\psi_{j,k} \ast f(x_1, x_2, x_3)|^2 \chi_R \right)^{\frac{1}{2}} \right)(x_1, x_2, x_3) \right\}^{\frac{1}{2}}$$

$$\times \chi_{3R'}(x_1, x_2, x_3)^2 \frac{dsdt}{st}$$

$$\leq C \sum_{j'} \sum_{k'} \left( \sum_{j,k \in \mathbb{Z}} 2^{-|j-j'| (N_1+\epsilon)-|k-k'| (N_2+\epsilon)} 2^{4N+2(j-j')+2(k-k')} \right) \left( \sum_{j,k \in \mathbb{Z}} 2^{-|j-j'| (N_1+\epsilon)-|k-k'| (N_2+\epsilon)} 2^{4N+2(j-j')+2(k-k')} \right)$$

$$\times \left\{ \mathcal{M}_1 \left( \left( \inf_{R \in \mathcal{R}_N^N(j,k)} |\psi_{j,k} \ast f(x_1, x_2, x_3)|^2 \chi_R \right)^{\frac{1}{2}} \right)(x_1, x_2, x_3) \right\}^{\frac{1}{2}}$$

$$\times \chi_{3R'}(x_1, x_2, x_3)^2.$$
Then similarly as in the proof of (2.27), we obtain that (2.42) holds by applying Hölder’s inequality and noting that the definition of \( g_t^d \) is independent of the choice of the bump function (up to a constant).

We note that rather than proving the reverse inequality in Theorem 2.11 via establishing another version of the reproducing formula and running the whole machinery as almost orthogonality and Frazier–Jawerth type inequality, we prove it indirectly by establishing the following equivalence between the \( L^p_w(\mathbb{R}^3) \) norm of \( g_1(f) \) and that of \( f \).

**Corollary 2.12.** Suppose \( 1 < p < \infty \). For \( w \in A^1_p(\mathbb{R}^3) \), we have that for all \( f \in L^p_w(\mathbb{R}^3) \),

\[
\| g_1(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| f \|_{L^p_w(\mathbb{R}^3)},
\]

where the implicit constants depend only on \([w]_{A^1_p(\mathbb{R}^3)}\).

**Proof.** For \( w \in A^1_p(\mathbb{R}^3) \), since \( \| g_1^d(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| f \|_{L^p_w(\mathbb{R}^3)} \) (from Theorems 2.6 and 2.7), by using Theorem 2.11 we obtain that \( \| g_1(f) \|_{L^p_w(\mathbb{R}^3)} \lesssim \| f \|_{L^p_w(\mathbb{R}^3)} \). From Rubio de Francia’s extrapolation theorem as stated in [15], we obtain that for \( 1 < p < \infty \) and \( w \in A^1_p(\mathbb{R}^3) \),

\[
\| g_1(f) \|_{L^p_w(\mathbb{R}^3)} \lesssim \| f \|_{L^p_w(\mathbb{R}^3)}.
\]

Now it suffices to prove the reverse. In fact, for \( 1 < p < \infty \) and \( w \in A^1_p(\mathbb{R}^3) \), for \( f \in L^p_w(\mathbb{R}^3) \cap S(\mathbb{R}^3) \) and \( h \in L^{p'}_{w'}(\mathbb{R}^3) \cap S(\mathbb{R}^3) \) with \( \| h \|_{L^{p'}_{w'}(\mathbb{R}^3)} = 1 \) where \( p' \) is the conjugate of \( p \) and \( w' \) is the conjugate of \( w \), we have that

\[
\langle f, h \rangle = \left| \int_0^\infty \int_0^\infty \phi_{s,t} * f \frac{dsdt}{st}, h \right| = \left| \int_{\mathbb{R}^3} \int_0^\infty \phi_{s,t} * h(y_1, y_2) \phi_{s,t} * f(y_1, y_2) \frac{dsdt}{st} d\mu \right| \leq \| g_1(f) \|_{L^p_w(\mathbb{R}^3)} \| g_1(h) \|_{L^{p'}_{w'}(\mathbb{R}^3)} \leq C \| g_1(f) \|_{L^p_w(\mathbb{R}^3)}.
\]

By duality, we see that

\[
\| f \|_{L^p_w(\mathbb{R}^3)} \lesssim \| g_1(f) \|_{L^p_w(\mathbb{R}^3)},
\]

which shows that (2.43) holds.

**Summary of the weighted Littlewood–Paley Theory in the Zygmund Setting:**

Suppose \( 1 < p < \infty \) and \( w \in A^1_p(\mathbb{R}^3) \). Then

\begin{itemize}
\item \( \| S_1(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| f \|_{L^p_w(\mathbb{R}^3)} \) ([15, Theorem 2.9]);
\item \( \| S_1^d(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| f \|_{L^p_w(\mathbb{R}^3)} \) (Theorem 2.6 + Rubio de Francia’s extrapolation);
\item \( \| g^2_1(f) \|_{L^p_w(\mathbb{R}^3)} \approx \| f \|_{L^p_w(\mathbb{R}^3)} \) (Theorem 2.7 and Remark 2.8);
\item \( \| g_1(f) \|_{L^{p'}_{w'}(\mathbb{R}^3)} \approx \| f \|_{L^{p'}_{w'}(\mathbb{R}^3)} \) (Corollary 2.12).
\end{itemize}
Hence, we have the following equivalence of area and square functions, both continuous and discrete:

$$
\|S_j(f)\|_{L^p_w(\mathbb{R}^3)} \approx \|S^d_j(f)\|_{L^p_w(\mathbb{R}^3)} \approx \|g^d_j(f)\|_{L^p_w(\mathbb{R}^3)} \approx \|g_j(f)\|_{L^p_w(\mathbb{R}^3)}.
$$

2.4. **Weighted Estimates for Singular Integrals.** To prove Theorem 1.1, we also need the almost orthogonality estimate for Zygmund singular integrals.

**Lemma 2.13** ([21, Proposition 3.1]). Suppose that $\psi_{j,k}$ is defined as in (2.6) and $\mathcal{K}$ is a function on $\mathbb{R}^3$ satisfying the conditions (R) and (C.a) – (C.c). Then, for $\lambda = \frac{1}{2} \min(\theta_1, \theta_2)$,

$$
|\psi_{j,k} * \mathcal{K} * \psi_{j',k'}(x_1, x_2, x_3)| \leq C_\lambda 2^{-|j-j'|} 2^{-|k-k'|} \frac{2^{-|\lambda j' k'|}}{(1 + 2^{-|\lambda j' k'|}|x_1|)^{1+\lambda}} \times \frac{2^{-|\lambda k' k'|}}{(1 + 2^{-|\lambda k' k'|}|x_2|)^{1+\lambda}} \frac{2^{-|\lambda j' j'|}}{(1 + 2^{-|\lambda j' j'|} (k k') |x_3|)^{1+\lambda}},
$$

where the constant $C_\lambda$ depends only on $\lambda$ and $a \lor b$ means $\max\{a, b\}$.

We now prove Theorem 1.1 by using the weighted estimate of the discrete Littlewood–Paley square function $g^d_j(f)$ and the almost orthogonality argument. We point out that we can also use the continuous square function $g_j(f)$.

**Proof of Theorem 1.1.** From Rubio de Francia’s extrapolation theorem as stated in [15], it suffices to prove the result in this Theorem with $p = 2$. So we let $w \in A^2_2(\mathbb{R}^3)$.

Suppose that $\mathcal{K}$ is a function defined on $\mathbb{R}^3$ and satisfies the conditions (R) and (C.a) – (C.c) and in addition the three integrals stated in Theorem 1.1 converge almost everywhere as $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ and $N_1, N_2, N_3 \to \infty$.

Suppose that $\psi_{j,k}$ is defined as in (2.6). From Lemma 2.13 above, we observe that

$$
|\psi_{j,k} * \mathcal{K} * \psi_{j',k'} * (\psi_{j',k'} * f)(x_1, x_2, x_3)| \leq C 2^{-|j-j'|} 2^{-|k-k'|} \mathcal{M}_3(\psi_{j',k'} * f)(x_1, x_2, x_3).
$$

Hence, by using the standard reproducing formula (which follows from the conditions of $\psi_{j,k}$ defined as in (2.6) and the Fourier transform)

$$
f = \sum_{j', k'} \psi_{j',k'} * \psi_{j',k'} * f
$$

we have that

$$
\|g^d_j(\mathcal{K} * f)\|_{L^p_w(\mathbb{R}^3)} = \left\| \left\{ \sum_{j,k} \left| \psi_{j,k} * \mathcal{K} * f \right|^2 \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^3)}
$$

\[= \left\| \left\{ \sum_{j,k} \sum_{j', k'} \psi_{j,k} * \mathcal{K} * \psi_{j',k'} * (\psi_{j',k'} * f) \right\}^2 \right\|_{L^p_w(\mathbb{R}^3)} \leq C \left\{ \sum_{j,k} \sum_{j', k'} 2^{-|j-j'|} 2^{-|k-k'|} \mathcal{M}_3(\psi_{j',k'} * f)^2 \right\}^{\frac{1}{2}}_{L^p_w(\mathbb{R}^3)}
$$
\[ \leq C \left\| \left\{ \sum_{j',k'} |\mathcal{M}_j(\psi_{j',k'} * f)|^2 \right\}^{\frac{1}{2}} \right\|_{L^2_w(\mathbb{R}^3)}, \]

where the last inequality follows from Hölder’s inequality and from the facts that
\[ \sum_{j',k'} 2^{-|j-j'|}2^{-|k-k'|} \leq C \quad \text{and} \quad \sum_{j,k} 2^{-|j-j'|}2^{-|k-k'|} \leq C. \]

By the fact that \( \mathcal{M}_j \) is bounded on \( L^2_w(\mathbb{R}^3) \), we get that
\[
\left\| \left\{ \sum_{j',k'} |\mathcal{M}_j(\psi_{j',k'} * f)|^2 \right\}^{\frac{1}{2}} \right\|_{L^2_w(\mathbb{R}^3)} \\
= \int_{\mathbb{R}^3} \sum_{j',k'} |\mathcal{M}_j(\psi_{j',k'} * f)(x_1, x_2, x_3)|^2 w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \\
= \sum_{j',k'} \int_{\mathbb{R}^3} |\mathcal{M}_j(\psi_{j',k'} * f)(x_1, x_2, x_3)|^2 w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \\
\leq C \sum_{j',k'} \int_{\mathbb{R}^3} |\psi_{j',k'} * f(x_1, x_2, x_3)|^2 w(x_1, x_2, x_3) dx_1 dx_2 dx_3,
\]

where the constant \( C \) depends only on \([w]_{A^2_w(\mathbb{R}^3)}\). This, together with (2.45), implies that
\[
\|g^d_k(\mathcal{K} * f)\|_{L^2_w(\mathbb{R}^3)} \leq C \left\| \left\{ \sum_{j',k'} |\psi_{j',k'} * f|^2 \right\}^{\frac{1}{2}} \right\|_{L^2_w(\mathbb{R}^3)} \\
= C \|g^d_k(f)\|_{L^2_w(\mathbb{R}^3)} \\
\leq C \|f\|_{L^2_w(\mathbb{R}^3)},
\]

where we use the Littlewood–Paley square function estimate as in Theorem 2.7 for \( L^2_w(\mathbb{R}^3) \) in the last inequality.

This gives
\[
\|\mathcal{K} * f\|_{L^2_w(\mathbb{R}^3)} \leq C \|g^d_k(\mathcal{K} * f)\|_{L^2_w(\mathbb{R}^3)} \leq C \|g^d_k(f)\|_{L^2_w(\mathbb{R}^3)} \leq C \|f\|_{L^2_w(\mathbb{R}^3)}.
\]

Next, suppose that \( \mathcal{K} \) is a function defined on \( \mathbb{R}^3 \) and satisfies the conditions (R) and \((C'\text{.a}) - (C'\text{.c})\) and in addition the three integrals stated in Theorem 1.1 converge almost everywhere as \( \epsilon_1, \epsilon_2, \epsilon_3 \to 0 \) and \( N_1, N_2, N_3 \to \infty \). Then for the corresponding test functions \( \psi_{j,k} \) and \( \psi_{j',k'} \) as defined in (2.32), we also have the fundamental fact that
\[
|\psi_{j,k} * \mathcal{K} * \psi_{j',k'} * (\psi_{j',k'} * f)(x_1, x_2, x_3)| \leq C 2^{-|j-j'|}2^{-|k-k'|}|\mathcal{M}_j(\psi_{j',k'} * f)(x_1, x_2, x_3)|.
\]

Then similarly to the estimate as in (2.45) above, by using the Littlewood–Paley square function estimate as in Theorem 2.9 for \( L^2_w(\mathbb{R}^3) \), we get that
\[
\|\mathcal{K} * f\|_{L^2_w(\mathbb{R}^3)} \leq C \|\tilde{g}^d_k(\mathcal{K} * f)\|_{L^2_w(\mathbb{R}^3)} \leq C \|\tilde{g}^d_k(f)\|_{L^2_w(\mathbb{R}^3)} \leq C \|f\|_{L^2_w(\mathbb{R}^3)}.
\]
Thus, combining the estimates of the operator $T$ associated with the two cases of kernels $K$, we obtain that Theorem 1.1 holds. \[\square\]

3. Little $\text{bmo}$ Space $\text{bmo}_3(\mathbb{R}^3)$ Associated with Zygmund Dilations and the Upper Bound of Commutators

In this section, we study the John-Nirenberg property of the little bmo space associated with Zygmund dilations, $\text{bmo}_3(\mathbb{R}^3)$, which implies in the end of the section the upper estimate of the commutator $[b, T]$ and its higher order analogue. In fact, we will work with a more general setting: little bmo with respect to general geometric bases satisfying certain uniformity and differentiability conditions, of which the Zygmund little bmo is a special case.

To start with, consider a measure space $(X, \Sigma, \mu)$ with $\sigma$-algebra $\Sigma$ and non-negative measure $\mu$. A basis $B \subset \Sigma$ is a collection of subsets of $X$ such that $B$ is $\mu$-measurable and $0 < \mu(B) < \infty$ for all $B \in B$. One can then define naturally the little bmo norm and $A_p$ characteristic associated to $B$ as follows:

$$\|b\|_{\text{bmo}_B} := \sup_{B \in B} \frac{1}{\mu(B)} \int_B |b(x) - b_B| \, d\mu,$$

where

$$b_B := \frac{1}{\mu(B)} \int_B b(x) \, d\mu,$$

and

$$[w]_{A_p,B} := \sup_{B \in B} \left( \frac{1}{\mu(B)} \int_B w \, d\mu \right) \left( \frac{1}{\mu(B)} \int_B w^{1-p'} \, d\mu \right)^{p-1}, \quad 1 < p < \infty.$$

We say a locally integrable function $b \in \text{bmo}_B$ if $\|b\|_{\text{bmo}_B} < \infty$, and a non-negative function $w$ satisfying $\int_B w \, d\mu < \infty$ for all $B \in B$ is in $A_{p,B}$, $1 < p < \infty$, if $[w]_{A_{p,B}} < \infty$. The following trivial properties hold: for all $1 < p \leq q < \infty$,

$$A_{p,B} \subset A_{q,B}, \quad w \in A_{p,B} \iff w^{1-p'} \in A_{p',B}.$$

In particular, this allows us to define $A_{\infty,B} = \cup_{p>1} A_{p,B}$.

It is demonstrated in [1] that if a linear operator $T$ is bounded on $L^p(w)$ for all $w \in A_{p,B}$, then the commutator $[b, T]$ satisfies the weighted upper bound

$$\|[b, T]f\|_{L^p(w)} \lesssim [w]_{A_{p,B}} \|b\|_{\text{BMO}_B} \|f\|_{L^p(w)}, \quad (3.1)$$

where in the above, the space $\text{BMO}_B$ is defined via the Orlicz norm

$$\|b\|_{\text{BMO}_B} := \sup_{B \in B} \|b - b_B\|_{\text{exp} L,B} = \sup_{B \in B} \inf \left\{ \lambda > 0 : \frac{1}{\mu(B)} \int_B e^{ \frac{|b(x) - b_B|}{\lambda} } \, d\mu \leq 2 \right\}.$$

In fact, a more general theorem is proved in [1, Theorem 3.17], where the operator $T$ is allowed to be just linearizable and the weight is allowed to satisfy a weaker condition.
Therefore, for our desired upper bound of the commutator \([b, T]\) to hold true, the key questions to ask are:

Q1. Whether the little \(BMO\) space \(bMO_B\) coincides with \(BMO_B\)?
Q2. Whether the linear operator \(T\) is bounded on weighted \(L^p\) with weight \(w \in A_{p,B}\)?

For the Zygmund setting, we have shown in the previous section that the class of singular integrals indeed has the desired weighted bound, that is, Theorem 1.1. Hence, Q2 is answered affirmatively.

In this section, we will focus on answering Q1.

The space \(BMO_B\) carries the John-Nirenberg inequality as part of the definition, which is crucial in the argument in [1] where a Cauchy integral trick that originates in [7] is used. It is shown in [1] that in the classical one-parameter Euclidean setting, \(BMO_B\) is the same as the John-Nirenberg \(BMO\), and in the bi-parameter Euclidean setting \(BMO_B\) coincides with the little \(bmo\) space. Both \(BMO\) and little \(bmo\) are special cases of our \(bmo_B\) as defined above.

It is unlikely, however, that \(BMO_B\) and \(bmo_B\) always coincide for all bases \(B\), but one can show that for a large class of bases this is indeed the case. A basis \(B\) is called uniform, if there exists a constant \(C > 0\) such that for all \(B \in B\), there exist a disjoint collection \(\{B_j\}_{B_j \in B}\) such that for all \(j, B_j \subset B\), \(\mu(B_j) \geq \frac{1}{C} \mu(B)\), and \(\mu(B \setminus \cup_j B_j) = 0\). Moreover, a basis \(B\) is said to be differentiable if at each point \(x\), there exists a sequence of elements \(\{B_n\}_{B_n \in B}\) such that \(x \in \cap_n B_n\) and \(\lim_{n \to \infty} \mu(B_n) = 0\).

**Theorem 3.1.** If a basis \(B\) is uniform and differentiable, then \(BMO_B = bmo_B\) with comparable norms.

As a consequence of Theorem 1.1 and (3.1), Theorem 3.1, once proved, will immediately imply that the desired upper bound estimate for the commutator \([b, T]\) with \(T\) being a Zygmund singular integral as in Definition 2.1. Similarly, it is straightforward to see that one can then iterate (3.1) to obtain Corollary 1.4. The rest of the section is devoted to the proof of Theorem 3.1.

**Lemma 3.2.** Let \(B\) be a uniform and differentiable basis. Given \(b \in bmo_B\), then there exist positive constants \(C_0\) and \(c_0\) such that for every \(B \in B\) and every \(t > 0\),

\[
\mu \left( \{ x \in B : |b(x) - b_B| > t \} \right) \leq C_0 e^{-\frac{c_0 t}{\|b\|_{bmo_B}} \mu(B)}.
\] (3.2)

**Proof.** The proof we give below is the standard proof of the Jonh-Nirenberg inequality, but with suitable modifications to work in the setting and generality at hand.

Noting that (3.2) is scale invariant, it suffices to assume that \(\|b\|_{bmo_B} = 1\). Now fix an element \(B \in B\) and fix \(\alpha > 1 = \|b\|_{bmo_B}\), and consider the following selection criterion for
S ⊂ B, S ∈ B:

\[
\frac{1}{\mu(S)} \int_S |b(x) - b_S| \, d\mu > \alpha.
\] (3.3)

Set \(B^{(0)} = B\) and subdivide \(B^{(0)}\) into \(\leq C\) sub-elements \(\{B_j\}_{B_j \in B}\), where \(C\) is the uniformity constant for \(B\). Select such an element \(B_j^{(1)}\) if it satisfies the selection criterion (3.3). Now further subdivide all the non-selected elements into \(\leq C\) elements, and then select among them those that satisfy (3.3). Continue this process indefinitely and we obtain a countable collection of elements \(\{B_j^{(1)}\}_j\) satisfying the following properties:

(A-1) The interior of every \(B_j^1\) is contained in \(B^{(0)}\);

(B-1) \(\alpha < \frac{1}{\mu(B_j^{(1)})} \int_{B_j^{(1)}} |b(x) - b_{B^{(0)}}| \, d\mu \leq C\alpha\);

(C-1) \(|b_{B_j^{(1)}} - b_{B^{(0)}}| \leq C\alpha\);

(D-1) \(\sum_j \mu(B_j^{(1)}) \leq \frac{1}{\alpha} \sum_j \int_{B_j^{(1)}} |b(x) - b_{B^{(0)}}| \, d\mu \leq \frac{1}{\alpha} \mu(B^{(0)})\);

(E-1) \(|b(x) - b_{B^{(0)}}| \leq \alpha\) for \(\mu\)-a.e. \(x \in B^{(0)} \setminus \bigcup_j B_j^{(1)}\).

In the above, the last property (E-1) follows from the fact that \(B\) is differentiable.

Next, fix \(B_j^{(1)}\) in the countable collection \(\{B_j^{(1)}\}_j\) selected above, and consider the following selection criterion for an element \(B_j^{(2)} \in B\) and \(B_j^{(1)} \subset B_j^{(2)}\):

\[
\frac{1}{\mu(B_j^{(2)})} \int_{B_j^{(2)}} |b(x) - b_{B_j^{(1)}}| \, d\mu > \alpha.
\] (3.4)

Repeat the process above for all \(B_j^{(1)}\) in the countable collection \(\{B_j^{(1)}\}_j\), and we end up with a countable collection of elements \(\{B_j^{(2)}\}_l\) of the second generation such that each \(B_j^{(2)}\) is contained in some \(B_j^{(1)}\) and that similar versions of (A-1)—(E-1) are satisfied with the superscript (2) replacing (1) and the superscript (1) replacing (0). We then iterate this procedure indefinitely to obtain a doubly indexed family of elements \(\{B_j^{(k)}\}_j\) satisfying the following properties:

(A-k) The interior of every \(B_j^k\) is contained in a unique \(B_i^{(k-1)}\);

(B-k) \(\alpha < \frac{1}{\mu(B_j^{(k)})} \int_{B_j^{(k)}} |b(x) - b_{B_i^{(k-1)}}| \, d\mu \leq C\alpha\);

(C-k) \(|b_{B_j^{(k)}} - b_{B_i^{(k-1)}}| \leq C\alpha\);

(D-k) \(\sum_j \mu(B_j^{(k)}) \leq \frac{1}{\alpha} \sum_l \mu(B_l^{(k-1)})\);

(E-k) \(|b(x) - b_{B_i^{(k-1)}}| \leq \alpha\) for \(\mu\)-a.e. \(x \in B_i^{(k-1)} \setminus \bigcup_j B_j^{(k)}\).

Applying (D-k) successively \(k - 1\) times, we obtain that

\[
\sum_j \mu(B_j^{(k)}) \leq \frac{1}{\alpha^k} \mu(B^{(0)}).
\] (3.5)
Moreover, by applying (E-k) and (C-k) successfully \( k - 1 \) times we obtain that

\[
|b(x) - b_{B(0)}| \leq Ck\alpha \quad \text{for } \mu - \text{a.e. } x \in B^{(0)} \setminus \bigcup_j B_j^{(k)},
\]

which implies that

\[
\{x \in B^{(0)} : |b(x) - b_{B(0)}| > Ck\alpha\} \subset \bigcup_j B_j^{(k)}.
\]

Now fix \( t > 0 \) and any constant \( \alpha > 1 \). Then if there exists \( k \geq 1 \) such that \( Ck\alpha < t \leq C(k + 1)\alpha \), we can conclude from (3.5) that

\[
\mu\left(\{x \in B : |b(x) - b_B| > t\}\right) \leq \mu\left(\{x \in B : |b(x) - b_B| > Ck\alpha\}\right)
\]

\[
\leq \sum_j \mu(B_j^{(k)}) \leq \frac{1}{\alpha^k} \mu(B^{(0)}) = \mu(B)e^{-k\log \alpha}
\]

\[
\leq \mu(B)e^{-\frac{t\log(\alpha)}{C\alpha}},
\]

since \(-k \leq 1 - \frac{t}{C\alpha}\).

On the other hand, if \( t \leq C\alpha \), then one has the trivial estimate

\[
\mu\left(\{x \in B : |b(x) - b_B| > t\}\right) \leq \mu(B) \leq \mu(B)e^{-\frac{t\log(\alpha)}{C\alpha}}.
\]

Combining the two cases above, we have proved (3.2) with constants \( C_0 = \alpha \) and \( c_0 = \frac{\log(\alpha)}{C\alpha} \), where \( \alpha \) is any fixed constant greater than 1.

Applying the John-Nirenberg inequality above for all \( t > 0 \), we have the following estimate for functions \( b \in \text{bmo}_B \).

**Corollary 3.3.** Given a basis \( \mathcal{B} \) that is uniform and differentiable. Then there exist constants \( C \geq 1 \) and 0 < \( \gamma \) < 1 such that for every \( b \in \text{bmo}_B \) and \( B \in \mathcal{B} \), we have that

\[
\frac{1}{\mu(B)} \int_B e^{\frac{|b(x) - b_B|}{\gamma \|b\|_{\text{bmo}_B}}} \, d\mu \leq C.
\]

Given any function \( b \in \text{bmo}_B \), in the corollary above, if \( C \leq 2 \), then one immediately has \( \|b\|_{\mathcal{B}\text{MO}_B} \leq \gamma^{-1}\|b\|_{\text{bmo}_B} \). If \( C < 2 \), let \( \lambda = \log_2 C \). Then by Hölder’s inequality, one has

\[
\frac{1}{\mu(B)} \int_B e^{\frac{|b(x) - b_B|}{\gamma \|b\|_{\text{bmo}_B}}} \, d\mu \leq \left( \frac{1}{\mu(B)} \int_B e^{\gamma \frac{|b(x) - b_B|}{\|b\|_{\text{bmo}_B}}} \, d\mu \right)^{\frac{1}{\lambda}} \leq C_2^{\frac{1}{\lambda}} = 2,
\]

hence \( \|b\|_{\mathcal{B}\text{MO}_B} \leq \log_2 C \cdot \gamma^{-1}\|b\|_{\text{bmo}_B} \). Therefore, \( \text{bmo}_B \subset \mathcal{B}\text{MO}_B \). On the other hand, it is straightforward to see that one always has \( \mathcal{B}\text{MO}_B \subset \text{bmo}_B \). Indeed, if \( b \in \mathcal{B}\text{MO}_B \), then by definition

\[
\frac{1}{\mu(B)} \int_B \frac{|b(x) - b_B|}{\|b - b_B\|_{\exp L_B}} \, d\mu \leq \frac{1}{\mu(B)} \int_B \left( e^{|b(x) - b_B|_{\exp L_B}} - 1 \right) \, d\mu \leq 1.
\]

Taking the supremum over \( B \in \mathcal{B} \) completes the proof in this direction and thus Theorem 3.1 is proved.
Before the end of this section, we introduce another property of the space \( \text{bmo}_\mathcal{B} \): the “exp-log” link between \( \text{bmo}_\mathcal{B} \) and \( A_{p,\mathcal{B}} \) weights. This is not relevant to Theorem 3.1 or the upper bound of the commutator, but is a natural consequence of the John-Nirenberg inequality and is expected to have some independent interest.

**Theorem 3.4.** Given a basis \( \mathcal{B} \) that is uniform and differentiable, and let \( A_{\infty,\mathcal{B}} \) be the weights as defined above. Then

(i) If \( w \in A_{\infty,\mathcal{B}} \), then \( \log w \in \text{bmo}_\mathcal{B} \);

(ii) If \( b \in \text{bmo}_\mathcal{B} \), then there exists some \( \delta > 0 \) such that \( e^{\delta b} \) is in \( A_{\infty,\mathcal{B}} \).

**Proof.** Suppose that \( w \in A_{\infty,\mathcal{B}} \), then there exists a \( 1 < p < \infty \) such that \( w \in A_{p,\mathcal{B}} \). Let \( \varphi = \log w \) and \( \sigma = \log \left( \frac{1}{w} \right) ^{\frac{1}{p-1}} = \frac{\varphi}{p-1} \). Then for any element \( B \in \mathcal{B} \), we have \( e^{\varphi_B} e^{(p-1)\sigma_B} = 1 \) and so the \( A_{p,\mathcal{B}} \) condition for \( w \) can be written as

\[
(3.7) \quad \left( \frac{1}{\mu(B)} \int_B e^{\varphi(x)-(\varphi)_B} \, d\mu \right) \left( \frac{1}{\mu(B)} \int_B e^{\sigma(x)-(\sigma)_B} \, d\mu \right)^{p-1} \leq [w]_{A_{p,\mathcal{B}}} < \infty.
\]

By Jensen’s inequality we have

\[
\frac{1}{\mu(B)} \int_B e^{\varphi(x)-(\varphi)_B} \, d\mu \geq e^{\frac{1}{\mu(B)} \int_B \varphi(x)-(\varphi)_B \, d\mu} = 1
\]

and similarly,

\[
\frac{1}{\mu(B)} \int_B e^{\sigma(x)-(\sigma)_B} \, d\mu \geq 1.
\]

Thus, we conclude that for any \( w \in A_{p,\mathcal{B}} \) there holds

\[
(3.8) \quad \left( \frac{1}{\mu(B)} \int_B e^{\varphi(x)-(\varphi)_B} \, d\mu \right) \leq \frac{[w]_{A_{p,\mathcal{B}}}}{(\mu(B))^{(\frac{1}{p})} \int_B e^{\sigma(x)-(\sigma)_B} \, d\mu} \leq [w]_{A_{p,\mathcal{B}}},
\]

and similarly,

\[
(3.9) \quad \left( \frac{1}{\mu(B)} \int_B e^{-(\varphi(x)-(\varphi)_B)/(p-1)} \, d\mu \right)^{p-1} \leq [w]_{A_{p,\mathcal{B}}}.
\]

Now for a given \( B \in \mathcal{B} \), let \( B_+ := \{ x \in B : \varphi - (\varphi)_B \geq 0 \} \) and \( B_- = B \setminus B_+ \). Then we have

\[
(3.10) \quad \frac{1}{\mu(B)} \int_B |\varphi(x) - (\varphi)_B| \, d\mu = \frac{1}{\mu(B)} \left( \int_{B_+} (\varphi(x) - (\varphi)_B) \, d\mu + \int_{B_-} -(\varphi(x) - (\varphi)_B) \, d\mu \right).
\]

For the first term in the right-hand side of the equality above, using the trivial estimate \( t \leq e^t \), we obtain that

\[
(3.11) \quad \frac{1}{\mu(B)} \int_{B_+} (\varphi(x) - (\varphi)_B) \, d\mu \leq \frac{1}{\mu(B)} \int_{B_+} e^{\varphi(x)-(\varphi)_B} \, d\mu
\]
\[ \leq \frac{1}{\mu(B)} \int_B e^{\varphi(x) - \langle \varphi \rangle_B} \, d\mu \]
\[ \leq [w]_{A_p,B}, \]

where the last inequality follows from (3.8).

For the second term, we first consider the case \( p - 1 \leq 1 \), where the estimate is slightly better. Using the trivial estimate \( t \leq e^t \) again we obtain
\[
\frac{1}{\mu(B)} \int_{B^-} - (\varphi(x) - \langle \varphi \rangle_B) \, d\mu \leq \frac{1}{\mu(B)} \int_{B^-} e^{-(\varphi(x) - \langle \varphi \rangle_B)} \, d\mu
\]
\[
= \frac{1}{\mu(B)} \int_{B^-} \left[ e^{-(\varphi(x) - \langle \varphi \rangle_B)/(p-1)} \right]^{p-1} \, d\mu
\]
\[
\leq \frac{1}{\mu(B)} \int_B \left[ e^{-(\varphi(x) - \langle \varphi \rangle_B)/(p-1)} \right]^{p-1} \, d\mu
\]
\[
\leq \left( \frac{1}{\mu(B)} \int_B e^{-(\varphi(x) - \langle \varphi \rangle_B)/(p-1)} \, d\mu \right)^{p-1}
\]
\[
\leq [w]_{A_p,B},
\]

where the third inequality follows from Hölder's inequality and the last inequality follows from (3.9).

We now consider the case \( p - 1 > 1 \). Again we have
\[
\frac{1}{\mu(B)} \int_{B^-} - (\varphi(x) - \langle \varphi \rangle_B) \, d\mu = \frac{p-1}{\mu(B)} \int_{B^-} - (\varphi(x) - \langle \varphi \rangle_B) \, d\mu
\]
\[
\leq \frac{p-1}{\mu(B)} \int_{B^-} e^{-(\varphi(x) - \langle \varphi \rangle_B)} \, d\mu
\]
\[
\leq (p-1)[w]_{A_p,B},
\]

where the last inequality follows from (3.9).

Now combining the estimates of the first and second terms on the right-hand side of (3.10), we obtain that
\[
\frac{1}{\mu(B)} \int_B |\varphi(x) - \langle \varphi \rangle_B| \, d\mu \leq [w]_{A_p,B} \max \left\{ [w]_{A_p,B}, (p-1)[w]_{A_p,B} \right\}
\]

Hence we obtain that \( \log w \in \text{bmo}_B \), which implies that (i) holds.

We now prove (ii). Suppose \( b \in \text{bmo}_B \). Then from Corollary 3.3, we obtain that there exist \( \gamma \in (0,1) \) and \( C \geq 1 \) such that
\[
\frac{1}{\mu(B)} \int_B e^{\frac{\gamma}{||b||_{\text{bmo}_B}}(b(x)-b_B)} \, d\mu \leq C
\]
\[
\frac{1}{\mu(B)} \int_B e^{\frac{\gamma}{||b||_{\text{bmo}_B}}(-b(x)+b_B)} \, d\mu \leq C.
\]

From the above two inequalities and by setting \( \delta = \frac{\gamma}{||b||_{\text{bmo}_B}} \), we see that
\[
\frac{1}{\mu(B)} \int_B e^{\delta b(x) - (\delta b)_B} \, d\mu \leq C^2,
\]
which gives
\[
\frac{1}{\mu(B)} \int_B e^{\delta b(x)} \, d\mu \leq C^2.
\]
This implies that \(e^{\delta b}\) is in \(A_{2, B}\). Similarly, by choosing a suitable \(\delta\) with respect to \(p\), we can also show that \(e^{\delta b}\) is in \(A_{p, B}\) for some \(p \in (1, \infty)\). The proof of Theorem 3.4 is complete. □

4. Lower Bound of Commutators of Singular Integrals associated with Zygmund Dilations: Proof of Theorem 1.5

We begin by recalling the detailed definition of the class of singular integral operators studied by Ricci and Stein in [35] which includes the kernel in (1.3), as stated in [15]. Recall that the dilations \(\{\rho_{s,t}\}_{s,t > 0}\) on \(\mathbb{R}^3\) are given by \(\rho_{s,t}(x, y, z) = (sx, ty, stz)\) for \((x, y, z) \in \mathbb{R}^3\).

**Definition 4.1.** We consider the singular integral operator introduced in [35] taking the form
\[
Tf = \mathcal{K} \ast f,
\]
where
\[
K(x, y, z) = \sum_{j, k \in \mathbb{Z}} 2^{-2j-2k} \psi_{j,k}(x, y, z) - \frac{y}{2^j}, \frac{z}{2^j+k} \]
with each \(\psi_{j,k} \in S_N\) and \(\|\psi_{j,k}\|_{S_N} \leq 1\) for a suitably large positive integer \(N\):
\[
S_N = \{\psi(x, y, z) \in C^\infty(\mathbb{R}^3) : \|\psi\|_{S_N} < \infty\},
\]
where
\[
\|\psi\|_{S_N} := \sup_{(x, y, z) \in \mathbb{R}^3} \left(1 + |(x, y, z)|^N\right) \sum_{\alpha, \beta, \gamma = 0}^N \left|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \psi(x, y, z)\right|.
\]
Moreover, each \(\psi_{j,k}\) satisfies the following cancellation conditions:
\[
\int_{\mathbb{R}^2} y^\sigma z^\kappa \psi_{j,k}(x, y, z) \, dydz = 0 \quad \text{for all fixed } x \in \mathbb{R} \text{ and all } \sigma, \kappa \leq N;
\]
\[
\int_{\mathbb{R}^2} x^\sigma z^\kappa \psi_{j,k}(x, y, z) \, dx dz = 0 \quad \text{for all fixed } y \in \mathbb{R} \text{ and all } \sigma, \kappa \leq N;
\]
\[
\int_{\mathbb{R}^2} x^\sigma y^\kappa \psi_{j,k}(x, y, z) \, dxdy = 0 \quad \text{for all fixed } z \in \mathbb{R} \text{ and all } \sigma, \kappa \leq N.
\]

In [15], to study the weighted estimates of the above operators \(T\) given in Definition 4.1, the authors introduced two multiplier classes, denoted by \(M_x^\delta\) and \(M_y^\delta\). These classes are natural analogues of the Hörmander and Marcinkiewicz classes, but with a certain asymmetry built into their definition. The authors then showed that any singular integral \(T\) as in Definition 4.1 can be decomposed as
\[
T = T_1 + T_2,
\]
where $T_1, T_2$ are Fourier multipliers with symbols in the class $M^x_\delta$ and $M^y_\delta$ respectively. More precisely,

**Definition 4.2 ([15]).** Consider the set

$$A^x = \left\{ (\xi, \eta, \zeta) \in \mathbb{R}^3 : \frac{1}{2} < |\xi| \leq 1, \frac{1}{2} < |(\eta, \zeta)| \leq 1 \right\},$$

whose dyadic lattices $\{\rho_{2^j, 2^k}(A^x)\}_{j,k \in \mathbb{Z}}$ form a partition of $\mathbb{R}^3$. Then by letting $N$ be some large positive integer, we define $M^x_\delta$ to be the set of all $m(\xi, \eta, \zeta)$ which are $C^N$ away from $\{\xi = 0\} \cup \{(\eta, \zeta) = (0, 0)\}$ and such that

$$\left| \partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma (m \circ \rho_{s,t})(\xi, \eta, \zeta) \right| \leq C$$

for all $0 \leq \alpha, \beta, \gamma \leq N$, for all $s, t > 0$, and for all $(\xi, \eta, \zeta) \in A^x$.

The class $M^y_\delta$ is defined analogously, by interchanging the roles of $x$ and $y$ in the definition of $M^x_\delta$.

In [15], the authors proved the following decomposition of the kernel in (4.1).

**Proposition 4.3 ([15]).** Any kernel $K$ of the type (4.1) can be split as

$$K = K_1 + K_2,$$

where $\hat{K}_1 \in M^x_\delta$ and $\hat{K}_2 \in M^y_\delta$.

We now turn to the proof of Theorem 1.5, to do so we first need the following result on the little bmo space $bmo_3(\mathbb{R}^3)$.

**Proposition 4.4.** The locally integrable function $b(x_1, x_2, x_3) := x_1$ is NOT in $bmo_3(\mathbb{R}^3)$.

**Proof.** We consider the Zygmund rectangles of the form

$$R = (a, 2a] \times (a, 2a] \times (a, a + a^2], \quad a > 0.$$ 

Then the average of $b$ over $R$ is

$$b_R := \frac{1}{|R|} \int_R b(x_1, x_2, x_3)dx_1dx_2dx_3 = \frac{1}{a^4} \int_a^{a+a^2} \int_a^{2a} \int_a^{2a} x_1 \, dx_1dx_2dx_3$$

$$= \frac{1}{a^2} \left. x_1^2 \right|_a^{2a} = \frac{3a}{2}.$$ 

Hence, we have

$$\frac{1}{|R|} \int_R |b(x_1, x_2, x_3) - b_R| \, dx_1dx_2dx_3$$

$$= \frac{1}{a^4} \int_a^{a+a^2} \int_a^{2a} \int_a^{2a} x_1 - \frac{3a}{2} \right| dx_1dx_2dx_3$$

$$= \frac{a}{4}.$$
It is clear that by letting $a \to \infty$,

$$\frac{1}{|R|} \int_{R} |b(x_1, x_2, x_3) - b_R| dx_1 dx_2 dx_3 \to \infty.$$ 

Hence, we see that $b(x_1, x_2, x_3) := x_1$ is NOT in $\text{bmo}_1(\mathbb{R}^3)$. \hfill \Box

**Remark 4.5.** Similarly, we see that $b(x_1, x_2, x_3) := x_2$ or $b(x_1, x_2, x_3) := x_3$ are NOT in $\text{bmo}_1(\mathbb{R}^3).

**Proof of Theorem 1.5.** Suppose $T$ is a singular integral operator as in Definition 4.1. We choose the weight $w \equiv 1$ (which is obviously an $A^1_2(\mathbb{R}^3)$ weight), and the function $b_0(x_1, x_2, x_3) := x_1$. From Proposition 4.4 we know that $b_0$ is NOT in $\text{bmo}_1(\mathbb{R}^3)$.

Now it suffices to prove that

$$(4.2) \quad \| [b_0, T] \|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} = C_0 < \infty.$$ 

To see this, for any $f \in L^2(\mathbb{R}^3)$, by using the decomposition given in Proposition 4.3, we have that

$$\| [b_0, T](f) \|_{L^2(\mathbb{R}^3)} = \| b_0 T(f) - T(b_0 f) \|_{L^2(\mathbb{R}^3)} \leq \| b_0 T_1(f) - T_1(b_0 f) \|_{L^2(\mathbb{R}^3)} + \| b_0 T_2(f) - T_2(b_0 f) \|_{L^2(\mathbb{R}^3)},$$ 

where $T_i f = \mathcal{K}_i * f$ for $i = 1, 2$.

Then by taking the Fourier transform and applying Plancherel’s theorem, we get that

$$\| [b_0, T](f) \|_{L^2(\mathbb{R}^3)} \leq \| \hat{b}_0 \hat{T}_1(f) - \hat{T}_1(\hat{b}_0 f) \|_{L^2(\mathbb{R}^3)} + \| \hat{b}_0 \hat{T}_2(f) - \hat{T}_2(\hat{b}_0 f) \|_{L^2(\mathbb{R}^3)}$$

$$= \| \partial_{x_1} (\hat{K}_1 \cdot \hat{f}) - \hat{K}_1 \cdot \partial_{x_1} \hat{f} \|_{L^2(\mathbb{R}^3)} + \| \partial_{x_1} (\hat{K}_2 \cdot \hat{f}) - \hat{K}_2 \cdot \partial_{x_1} \hat{f} \|_{L^2(\mathbb{R}^3)}$$

$$= \| \partial_{x_1} \hat{K}_1 \cdot \hat{f} \|_{L^2(\mathbb{R}^3)} + \| \partial_{x_2} \hat{K}_2 \cdot \hat{f} \|_{L^2(\mathbb{R}^3)}$$

$$\leq C \| \hat{f} \|_{L^2(\mathbb{R}^3)} + C \| \hat{f} \|_{L^2(\mathbb{R}^3)}$$

$$\leq C \| f \|_{L^2(\mathbb{R}^3)},$$ 

where the second inequality follows from Proposition 4.3, and the constant $C$ depends only on the constant in Definition 4.2. This implies that $(4.2)$ holds with the constant $C_0$ depending only on the constant in Definition 4.2.

The proof of Theorem 1.5 is complete. \hfill \Box

5. LOWER BOUND OF COMMUTATORS OF SINGULAR INTEGRALS ASSOCIATED WITH ZYGMUND DILATIONS: PROOF OF THEOREM 1.6

In this section we prove Theorem 1.6 and Corollary 1.8. To begin with, we first recall the definition of the median on a Zygmund rectangle $R = I \times J \times S \subset \mathbb{R}^3$. By a median
value of a real-valued measurable function $b$ over $R$ we mean a possibly non-unique, real number $m_R(b)$ such that
\[ |\{(x_1, x_2, x_3) \in R : b(x_1, x_2, x_3) > m_R(b)\}| \leq \frac{1}{2} |R| \quad \text{and} \quad |\{(x_1, x_2, x_3) \in R : b(x_1, x_2, x_3) < m_R(b)\}| \leq \frac{1}{2} |R|. \]

Recall that to prove the lower bound of commutator, there are a few results in one-parameter setting, by using Fourier transforms, weak factorisations, as well as a technical selection and decomposition of the balls in underlying space with respect to the BMO norm, see for example [7, 26, 38, 27, 32]. We now provide the proof based on the idea from [32].

**Proof of Theorem 1.6.** Let $T$ be the singular integral operator of convolution type studied in [33], i.e., $Tf = K*f$ with $K$ the kernel given by (1.2). Suppose $b \in L^1_{loc}(\mathbb{R}^3)$ and suppose that
\[ \|[b, T]\|_{L_p^\infty(\mathbb{R}^3) \to L_p^\infty(\mathbb{R}^3)} < \infty \]
for some $1 < p < \infty$ and $w \in A_p^\infty(\mathbb{R}^3)$. We aim to prove that for every Zygmund rectangle $R \subset \mathbb{R}^3$,

\[ \frac{1}{|R|} \int_R |b(x_1, x_2, x_3) - b_R| dx_1 dx_2 dx_3 \leq C \|[b, T]\|_{L_p^\infty(\mathbb{R}^3) \to L_p^\infty(\mathbb{R}^3)} [w]_{A_p^\infty(\mathbb{R}^3)}^{\frac{1}{p}}, \tag{5.1} \]

where the constant $C$ is independent of $R$ and $b$.

We note that in [12] they obtained the characterization of product little bmo space via commutator of product singular integrals (such as the double Riesz transforms). We follow the approach there to prove (5.1).

Now for any fixed Zygmund rectangle $R = I \times J \times S \subset \mathbb{R}^3$, we choose another Zygmund rectangle $\tilde{R} = \tilde{I} \times \tilde{J} \times \tilde{S} \subset \mathbb{R}^3$ such that $\tilde{I}$ is on the left-hand side of $I$, $\tilde{J}$ is on the left-hand side of $J$, $\tilde{S}$ is on the left-hand side of $S$, $\ell(I) = \ell(\tilde{I})$, $\ell(J) = \ell(\tilde{J})$, $\ell(S) = \ell(\tilde{S})$, and
\[ \text{dist}(I, \tilde{I}) = 5\ell(I), \text{dist}(J, \tilde{J}) = 5\ell(J), \text{dist}(S, \tilde{S}) = 47\ell(S). \]

Following the idea in [37, 32], we choose two measurable subsets $F_1, F_2$ of $\tilde{R}$ such that
\[ F_1 \subset \{(y_1, y_2, y_3) \in \tilde{R} : b(y_1, y_2, y_3) \leq m_{\tilde{R}}(b)\}, \]
\[ F_2 \subset \{(y_1, y_2, y_3) \in \tilde{R} : b(y_1, y_2, y_3) \geq m_{\tilde{R}}(b)\}, \]
where $m_{\tilde{R}}(b)$ is the median of $b$ on the Zygmund rectangle $\tilde{R}$ and that $\tilde{R} = F_1 \cup F_2$ with $|F_i| = \frac{1}{2} |\tilde{R}|$ for $i = 1, 2$. We also set
\[ E_1 = \{(x_1, x_2, x_3) \in R : b(x_1, x_2, x_3) \geq m_{\tilde{R}}(b)\}, \]
\[ E_2 = \{(x_1, x_2, x_3) \in R : b(x_1, x_2, x_3) < m_{\tilde{R}}(b)\}. \]
Then, for every \((x_1, x_2, x_3) \in E_1\) and every \((y_1, y_2, y_3) \in F_1\), it is obvious that
\[ b(x_1, x_2, x_3) - b(y_1, y_2, y_3) \geq 0 \]
and that
\[ b(x_1, x_2, x_3) - b(y_1, y_2, y_3) \geq b(x_1, x_2, x_3) - m_\tilde{R}(b). \]
Similarly, for every \((x_1, x_2, x_3) \in E_2\) and every \((y_1, y_2, y_3) \in F_2\), one has
\[ b(x_1, x_2, x_3) - b(y_1, y_2, y_3) \leq 0 \]
and
\[ b(x_1, x_2, x_3) - b(y_1, y_2, y_3) \leq b(x_1, x_2, x_3) - m_\tilde{R}(b). \]

Moreover, from the choice of \(\tilde{R}\), we see that
\[ \mathcal{K}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \]
\[ = \text{sgn}((x_1 - y_1) \cdot (x_2 - y_2)) \left\{ \frac{1}{|x_1 - y_1|^2 |x_2 - y_2|^2 + |x_3 - y_3|^2} \right\} \]
is always positive for every \((x_1, x_2, x_3) \in R\) and every \((y_1, y_2, y_3) \in \tilde{R}\). In addition,
\[ \mathcal{K}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \geq \frac{1}{|7\ell(I)|^2 |7\ell(J)|^2 + (49\ell(S))^2} \]
\[ = \frac{1}{2 \cdot 49^2 \ell(I)^2 \ell(J)^2} \]
\[ = \frac{1}{2 \cdot 49^2 |R|}, \]
where we used the fact that \(\ell(S) = \ell(I) \cdot \ell(J)\) since \(R = I \times J \times S\) is a Zygmund rectangle.

As a consequence, we obtain for \(f = \chi_{F_i}\) with \(i = 1, 2\), and \(w \in A_p^b(\mathbb{R}^3)\) as in the assumption that
\[
(5.2) \quad \frac{1}{|R|} \int_R |[b, T](f)(x_1, x_2, x_3)| dx_1 dx_2 dx_3 \\
\leq \left( \frac{1}{|R|} \int_R |[b, T](f)(x_1, x_2, x_3)|^p w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right)^{\frac{1}{p}} \left( \frac{1}{|R|} \int_R w^{-\mu'/p} \right)^{\frac{1}{p'}} \\
\leq \frac{1}{|R|^{\frac{1}{p}}} ||[b, T]||_{L_p^e(\mathbb{R}^3) \rightarrow L_p^e(\mathbb{R}^3)} ||f||_{L_p^e(\mathbb{R}^3)} \left( \frac{1}{|R|} \int_R w^{-\mu'/p} \right)^{\frac{1}{p'}} \\
= ||[b, T]||_{L_p^e(\mathbb{R}^3) \rightarrow L_p^e(\mathbb{R}^3)} \left( \frac{1}{|R|} \int_{F_i} w \right)^{\frac{1}{p}} \left( \frac{1}{|R|} \int_R w^{-\mu'/p} \right)^{\frac{1}{p'}} \\
\leq ||[b, T]||_{L_p^e(\mathbb{R}^3) \rightarrow L_p^e(\mathbb{R}^3)} \left( \frac{1}{|R|} \int_{F_i} w \right)^{\frac{1}{p}} \left( \frac{1}{|R|} \int_R w^{-\mu'/p} \right)^{\frac{1}{p'}} \\
\leq ||[b, T]||_{L_p^e(\mathbb{R}^3) \rightarrow L_p^e(\mathbb{R}^3)} [w]_{A_p^b(\mathbb{R}^3)}^{\frac{1}{p}}. 
\]
In the above, we have defined $\tilde{R}$ to be the smallest Zygmund rectangle containing both $R$, $\tilde{R}$. It is straightforward to verify that $|\tilde{R}| \sim |R|$ which explains the second to last inequality above. On the other hand,

\begin{equation}
\frac{1}{|R|} \int_R \| [b, T] (\chi_{F_1}) (x_1, x_2, x_3) \| dx_1 dx_2 dx_3 \\
\geq \frac{1}{|R|} \int_{E_1} \| [b, T] (\chi_{F_1}) (x_1, x_2, x_3) \| dx_1 dx_2 dx_3 \\
= \frac{1}{|R|} \int_{E_1} \int_{F_1} (b(x_1, x_2, x_3) - b(y_1, y_2, y_3)) \\
\quad \quad \times K(x_1 - y_1, x_2 - y_2, x_3 - y_3) dy_1 dy_2 dy_3 \, dx_1 dx_2 dx_3 \\
\geq \frac{1}{|R|} \int_{E_1} \int_{F_1} \left| b(x_1, x_2, x_3) - m_{\tilde{R}}(b) \right| \\
\quad \quad \times \left| K(x_1 - y_1, x_2 - y_2, x_3 - y_3) \right| dy_1 dy_2 dy_3 dx_1 dx_2 dx_3 \\
\geq \frac{1}{2} \cdot 49^2 \frac{1}{|R|} \int_{E_1} \left| b(x_1, x_2, x_3) - m_{\tilde{R}}(b) \right| \frac{|F_1|}{|R|} \, dx_1 dx_2 dx_3 \\
\geq \frac{1}{4} \cdot 49^2 \frac{1}{|R|} \int_{E_1} \left| b(x_1, x_2, x_3) - m_{\tilde{R}}(b) \right| \, dx_1 dx_2 dx_3,
\end{equation}

where the second inequality follows from the fact that both $b(x_1, x_2, x_3) - b(y_1, y_2, y_3)$ and $K(x_1 - y_1, x_2 - y_2, x_3 - y_3)$ do not change sign for every $(x_1, x_2, x_3) \in E_1$ and every $(y_1, y_2, y_3) \in F_1$ and the lower bound of $b(x_1, x_2, x_3) - b(y_1, y_2, y_3)$.

Similarly we get that

\begin{equation}
\frac{1}{|R|} \int_R \| [b, T] (\chi_{F_2}) (x_1, x_2, x_3) \| dx_1 dx_2 dx_3 \\
\geq \frac{1}{4} \cdot 49^2 \frac{1}{|R|} \int_{E_2} \left| b(x_1, x_2, x_3) - m_{\tilde{R}}(b) \right| \, dx_1 dx_2 dx_3.
\end{equation}

Combining estimates (5.2), (5.3) and (5.4), we obtain that

\[ 2 \| [b, T] \|_{L^p_{\infty} (\mathbb{R}^3) \rightarrow L^p_{\infty} (\mathbb{R}^3)} \| [u]_{A^p_{\infty} (\mathbb{R}^3)} \|_{p} \]
\[ \geq \frac{1}{|R|} \int_R \| [b, T] (\chi_{F_1}) (x_1, x_2, x_3) \| dx_1 dx_2 dx_3 + \frac{1}{|R|} \int_R \| [b, T] (\chi_{F_2}) (x_1, x_2, x_3) \| dx_1 dx_2 dx_3 \\
\geq \frac{1}{4} \cdot 49^2 \frac{1}{|R|} \int_{E_1 \cup E_2} \left| b(x_1, x_2, x_3) - m_{\tilde{R}}(b) \right| \, dx_1 dx_2 dx_3 \\
= \frac{1}{4} \cdot 49^2 \frac{1}{|R|} \int_R \left| b(x_1, x_2, x_3) - m_{\tilde{R}}(b) \right| \, dx_1 dx_2 dx_3, \]
which implies that (5.1) holds with a constant $C$ independent of $b$ and $R$ since it is direct that
\[
\frac{1}{|R|} \int_R |b(x_1, x_2, x_3) - b_R| dx_1 dx_2 dx_3 \leq \frac{2}{|R|} \int_R |b(x_1, x_2, x_3) - m_R(b)| dx_1 dx_2 dx_3.
\]
This finishes the proof of Theorem 1.6.

We end this section by briefly describing how the proof above can be modified to obtain Corollary 1.8, i.e. the lower bound for higher order commutator $C^k_{b,...,b}(T)$. Given a Zygmund rectangle $R$, one can construct $\tilde{R}$ and partition the rectangles $R, \tilde{R}$ in the analogous way as above. Estimate (5.2) still holds true with $C^k_{b,...,b}(T)$ replacing $[b, T]$. In estimate (5.3) and (5.4), the only difference is that instead of having $b(x_1, x_2, x_3) - b(y_1, y_2, y_3)$ in the kernel representation, one has $[b(x_1, x_2, x_3) - b(y_1, y_2, y_3)]^k$. Therefore, by the same deduction, one obtains in the end
\[
\frac{1}{|R|} \int_R |b(x_1, x_2, x_3) - m_R(b)|^k dx_1 dx_2 dx_3 \lesssim \|C^k_{b,...,b}(T)\|_{L^p_w(\mathbb{R}^3) \rightarrow L^p_w(\mathbb{R}^3)} [w]_{A^k_w(\mathbb{R}^3)}^{\frac{1}{p}},
\]
which by Hölder’s inequality immediately implies $b \in \text{bmo}_w(\mathbb{R}^3)$ and the desired norm estimate.

6. APPENDIX: PROOF OF (2.15)

Let $\psi_{j,k}$ be defined as in (2.6). Suppose $\epsilon_1, \epsilon_2 \in (0, 1], N_1, N_2 \in \mathbb{N}$ and $M_1 > 0, M_2 > 0$. Then for $f \in G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)$, without loss of generality, we assume that, with scaling $r_1 = r_2 = 1$, that $f \in G_{N_1, N_2}(0, 0, 0); 1, 1; \epsilon_1, \epsilon_2; M_1, M_2)$ (see Definition 2.2).

We now estimate $\psi_{j,k} * f(x_1, x_2, x_3)$ in the norm $\tilde{G}_{N_1, N_2}(\epsilon'_1, \epsilon'_2; M'_1, M'_2)$. We consider the following four cases according to the relationships between $j$ and $k$ being positive or negative.

Case 1: $j < 0, k < 0$.

In this case, by using the cancellation condition on $\psi_{j,k}$ for both $y_1$ and $(y_2, y_3)$, we get that

\[
\begin{align*}
\psi_{j,k} * f(x_1, x_2, x_3) &= \int_{\mathbb{R}^3} \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) f(y_1, y_2, y_3) dy_1 dy_2 dy_3 \\
&= \int_{\mathbb{R}^3} \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) \left[ f(y_1, y_2, y_3) - \mathcal{P}_{N_1}^{(1)}(f(x_1, y_2, y_3))(y_1) \right] \\
&\quad - \mathcal{P}_{N_2}^{(2)} \left[ f(y_1, x_2, x_3) - \mathcal{P}_{N_1}^{(1)}(f(x_1, x_2, x_3)) \right] (y_2, y_3) dy_1 dy_2 dy_3 \\
&= \text{Term}_{11} + \text{Term}_{12} + \text{Term}_{13} + \text{Term}_{14},
\end{align*}
\]
where each $\text{Term}_{1i}$ is the same as the right-hand side of the second equality above with $\int_{\mathbb{R}^3}$ replaced by $\int_{E_{1i}}$, for $i = 1, 2, 3, 4$, and

$$E_{11} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| < \frac{1}{2}(1 + |x_1|), |y_2 - x_2| + |y_3 - x_3| < \frac{1}{2}(1 + |x_2| + |x_3|) \right\};$$

$$E_{12} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| \geq \frac{1}{2}(1 + |x_1|), |y_2 - x_2| + |y_3 - x_3| < \frac{1}{2}(1 + |x_2| + |x_3|) \right\};$$

$$E_{13} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| < \frac{1}{2}(1 + |x_1|), |y_2 - x_2| + |y_3 - x_3| \geq \frac{1}{2}(1 + |x_2| + |x_3|) \right\};$$

$$E_{14} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| \geq \frac{1}{2}(1 + |x_1|), |y_2 - x_2| + |y_3 - x_3| \geq \frac{1}{2}(1 + |x_2| + |x_3|) \right\}.$$

Then, for $\text{Term}_{11}$, by using (2.12) for $f$ and using the size estimate for $\psi_{j,k}$, we have

$$\text{Term}_{11} \leq C\| f \|_{G_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{11}} \frac{2^j \tilde{M}_1}{(2^j + |x_1 - y_1|)^{1+M_1}} \cdot \frac{2^k \tilde{M}_2}{2^k(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3$$

$$\leq C\| f \|_{G_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} 2^{j(N_1+\epsilon_1-1)} \cdot \frac{1}{(1 + |x_1|)^{N_1+\epsilon_1}} \cdot \frac{2^k \tilde{M}_2}{2^k(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are chosen large enough.

Next, for $\text{Term}_{12}$, by using (2.11) for $f$ and using the size estimate for $\psi_{j,k}$, and noting that $|x_1 - y_1| > \frac{1}{2}(1 + |x_1|)$ we have

$$\text{Term}_{12} \leq C\| f \|_{G_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{12}} \frac{2^j \tilde{M}_1}{(2^j + |x_1 - y_1|)^{1+M_1}} \cdot \frac{2^k \tilde{M}_2}{2^k(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \cdot \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3$$

$$\leq C\| f \|_{G_{N_1,N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} 2^{j(N_1+\epsilon_1-1)} \cdot \frac{1}{(1 + |x_1|)^{N_1+\epsilon_1}} \cdot \frac{2^k \tilde{M}_2}{2^k(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}}.$$
\[
\begin{align*}
&\times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\quad \times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\quad \times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\quad \times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}}.
\end{align*}
\]

Next, for Term\(_{13}\), by using (2.10) for \(f\) and using the size estimate for \(\psi_{j,k}\), and noting that \(|y_2 - x_2| + |y_3 - x_3| \geq \frac{1}{2}(1 + |x_2| + |x_3|)\) we have

\[\text{Term}_{13}\]

\[
\begin{align*}
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\quad \times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} \\
&\quad \times \int_{\mathbb{R}^3} \frac{2^j(M_1 - N_1 - \epsilon_1)}{(2^j + |x_1 - y_1|)^{1+M_1 - N_1 - \epsilon_1}} \cdot \frac{2^k(M_2 - N_2 - \epsilon_2)}{(2^k + |x_2 - y_2| + |x_3 - y_3|)^{2+M_2 - N_2 - \epsilon_2}} dy_1 dy_2 dy_3 \\
&\leq C\|f\|_{g_{N_1,N_2(\epsilon_1,\epsilon_2;M_1,M_2)}} 2^{j(N_1+\epsilon_1-1)} 2^{k(N_2+\epsilon_2)} \left( \frac{1}{1 + |x_1|} \right)^{N_1+\epsilon_1} \left( \frac{1}{1 + |x_2| + |x_3|} \right)^{N_2+\epsilon_2} \\
&\quad \times \frac{1}{(1 + |x_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}}.
\end{align*}
\]
Case 2: \( j > 0, \ k > 0 \).

In this case, by using the cancellation condition on \( f \) for both \( y_1 \) and \((y_2, y_3)\), we get that

\[
\psi_{j,k} \ast f(x_1, x_2, x_3) = \int_{\mathbb{R}^3} \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) f(y_1, y_2, y_3) dy_1 dy_2 dy_3
= \int_{\mathbb{R}^3} \left[ \left( \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) - \mathcal{P}_{N_1}^{(1)} \psi_{j,k}(x_1 - 0, x_2 - y_2, x_3 - y_3) \right) - \mathcal{P}_{N_2}^{(2)} \left( \psi_{j,k}(x_1 - y_1, x_2 - 0, x_3 - 0) - \mathcal{P}_{N_1}^{(1)} \psi_{j,k}(x_1 - 0, x_2 - 0, x_3 - 0) \right) \right] f(y_1, y_2, y_3) dy_1 dy_2 dy_3
= \text{Term}_{21} + \text{Term}_{22} + \text{Term}_{23} + \text{Term}_{24},
\]

where each \( \text{Term}_{2i} \) is the same as the right-hand side of the second equality above with \( \int_{\mathbb{R}^3} \) replaced by \( \int_{E_{2i}} \), for \( i = 1, 2, 3, 4 \), and

\[
E_{21} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| < \frac{1}{2} (2^j + |x_1|), \ |y_2| + 2^{-j} |y_3| < \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\};
E_{22} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| \geq \frac{1}{2} (2^j + |x_1|), \ |y_2| + 2^{-j} |y_3| < \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\};
E_{23} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| < \frac{1}{2} (2^j + |x_1|), \ |y_2| + 2^{-j} |y_3| \geq \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\};
E_{24} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| \geq \frac{1}{2} (2^j + |x_1|), \ |y_2| + 2^{-j} |y_3| \geq \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\}.
\]

Then, for \( \text{Term}_{21} \), by using the smoothness conditions for \( \psi_{j,k} \) and using the size estimate for \( f \), we have that for arbitrary \( \tilde{M}_1, \tilde{M}_2 \),

\[
\text{Term}_{21} \leq C \| f \|_{G_{N_1, N_2}(e_1 e_2; M_1, M_2)} \int_{E_{21}} \left( \frac{|y_1|}{2^j + |x_1|} \right)^{N_1+1} \left( \frac{|y_2| + 2^{-j} |y_3|}{2^k + |x_2| + 2^{-j} |x_3|} \right)^{N_2+1} \times (2^j + |x_1|)^{1+M_1} \cdot (2^k + |x_2| + 2^{-j} |x_3|)^{2+M_2} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1, N_2}(e_1 e_2; M_1, M_2)} \int_{E_{21}} 2^{-j(N_1+2)} 2^{-k(N_2+1)} \times (2^j + |x_1|)^{1+M_1} \cdot (2^k + |x_2| + 2^{-j} |x_3|)^{2+M_2} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1, N_2}(e_1 e_2; M_1, M_2)} 2^{-j(N_1+2-M_1-2-M_2)} 2^{-k(N_2+1-M_2)}
\]

Therefore, we have

\[
\psi_{j,k} \ast f(x_1, x_2, x_3) \leq C \| f \|_{G_{N_1, N_2}(e_1 e_2; M_1, M_2)} 2^{-j(N_1+2-M_1-2-M_2)} 2^{-k(N_2+1-M_2)}
\]
where it is required that $\tilde{M}_2 < N_2 + 1$ and $\tilde{M}_1 + \tilde{M}_2 < N_1$, $N_1 + 1 < M_1$ and $N_2 + 1 < M_2$.

For Term $22$, by using the smoothness conditions for $\psi_{j,k}$ and using the size estimate for $f$, we have that for arbitrary $\tilde{M}_1, \tilde{M}_2$,

$$\text{Term}_{22} \leq C \|f\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2 ; M_1, M_2)} \int_{E_{22}} \left( \frac{|y_2| + 2^{-j}|y_3|}{2^k + |x_2| + 2^{-j}|x_3|} \right)^{N_2+1} \frac{1}{2^k M_2} \frac{1}{(1 + |y_1|)^{1 + \tilde{M}_1}} \frac{1}{(1 + |y_2| + |y_3|)^{2 + \tilde{M}_2}} dy_1 dy_2 dy_3$$

Next, for Term $23$, by using the smoothness conditions for $\psi_{j,k}$ and using the size estimate for $f$, we have that for arbitrary $\tilde{M}_1, \tilde{M}_2$,

$$\text{Term}_{23} \leq C \|f\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2 ; M_1, M_2)} \int_{E_{23}} \left( \frac{|y_1|}{2^j + |x_1|} \right)^{N_1+1} \frac{1}{2^k M_2} \frac{1}{(1 + |y_1|)^{1 + \tilde{M}_1}} \frac{1}{(1 + |y_2| + |y_3|)^{2 + \tilde{M}_2}} dy_1 dy_2 dy_3$$
Similarly, for $\text{Term}_{24}$, by using the size conditions on $\psi_{j,k}$ and using the size estimate for $f$, we have that for arbitrary $\tilde{M}_1, \tilde{M}_2$,

$$\text{Term}_{24} \leq C \|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{24}} \frac{2^{j\tilde{M}_1}}{(1 + |x_1|)^{1+M_1}} \cdot \frac{2^{k\tilde{M}_2}}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3$$

$$\leq C \|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{24}} 2^{-j(N_1+2)} 2^{-k(N_2+1)} \frac{2^{j\tilde{M}_1}}{(1 + |y_1|)^{1+M_1-(N_1+1)}} \cdot \frac{2^{k\tilde{M}_2}}{(1 + |y_2| + |y_3|)^{2+M_2-(N_2+1)}} dy_1 dy_2 dy_3$$

$$\leq C \|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} 2^{-j(N_1+2-M_1-2-M_2)} 2^{-k(N_2+1-M_2)}$$

To sum up, in this case we need $\tilde{M}_2 < N_2 + 1$ and $\tilde{M}_1 + \tilde{M}_2 < N_1$, $N_1 + 1 < M_1$ and $N_2 + 1 < M_2$.

Case 3: $j > 0$, $k < 0$. In this case, by using the cancellation condition on $\psi_{j,k}$ for $(y_2, y_3)$ and the cancellation condition on $f$ for $y_1$, we get that

$$\psi_{j,k} \ast f(x_1, x_2, x_3)$$

$$= \int_{\mathbb{R}^3} \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) f(y_1, y_2, y_3) dy_1 dy_2 dy_3$$

$$= \int_{\mathbb{R}^3} \left( \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) - \mathcal{P}_{N_1}^{(1)} \psi_{j,k}(x_1 - x_2, x_2 - y_2, x_3 - y_3) \right)$$

$$\times \left[ f(y_1, x_2, y_3) - \mathcal{P}_{N_2}^{(2)} f(y_1, x_2, x_3) (y_2, y_3) \right] dy_1 dy_2 dy_3$$

$$= \text{Term}_{31} + \text{Term}_{32} + \text{Term}_{33} + \text{Term}_{34},$$

where each $\text{Term}_{3i}$ is the same as the right-hand side of the second equality above with $\int_{\mathbb{R}^3}$ replaced by $\int_{E_{3i}}$, for $i = 1, 2, 3, 4$, and

$$E_{31} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| < \frac{1}{2} (2^j + |x_1|), |y_2 - x_2| + |y_3 - x_3| < \frac{1}{2} (1 + |x_2| + |x_3|) \right\};$$

$$E_{32} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| \geq \frac{1}{2} (2^j + |x_1|), |y_2 - x_2| + |y_3 - x_3| < \frac{1}{2} (1 + |x_2| + |x_3|) \right\};$$

$$E_{33} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| < \frac{1}{2} (2^j + |x_1|), |y_2 - x_2| + |y_3 - x_3| \geq \frac{1}{2} (1 + |x_2| + |x_3|) \right\};$$

$$E_{34} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| \geq \frac{1}{2} (2^j + |x_1|), |y_2 - x_2| + |y_3 - x_3| \geq \frac{1}{2} (1 + |x_2| + |x_3|) \right\};$$
\[ E_{34} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| \geq \frac{1}{2}(2^j + |x_1|), \ |y_2 - x_2| + |y_3 - x_3| \geq \frac{1}{2}(1 + |x_2| + |x_3|) \right\}. \]

Then, for Term 31, by using the smoothness conditions for \( \psi_{j,k} \) and using (2.11) for \( f \), we have that for arbitrary \( \tilde{M}_1, \tilde{M}_2 \),

\[
\text{Term} \ 31 \leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{31}} \left( \frac{|y_1|}{2^j + |x_1|} \right)^{N_1+1} 2^{j\tilde{M}_1} 2^{k\tilde{M}_2} \\
\times \left( \frac{2^j + |x_1|}{2^j + |x_1|} \right)^{1+M_1} \cdot \frac{2^j(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}}{2^j(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \cdot \frac{1}{(1 + |y_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{31}} 2^{-j(N_1+2-\tilde{M}_1-\tilde{M}_2-2)\epsilon_2} 2^{k(N_2+\epsilon_2)} 2^{k(M_2-2-\epsilon_2)} 2^{k(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \\
\times \left( \frac{2^j + |x_1|}{2^j + |x_1|} \right)^{1+M_1} \cdot \frac{1}{(1 + |y_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} 2^{-j(N_1+2-\tilde{M}_1-\tilde{M}_2-2)\epsilon_2} 2^{k(N_2+\epsilon_2)} 2^{k(M_2-2-\epsilon_2)} 2^{k(2^k + |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+M_2}} \\
\times \left( \frac{2^j + |x_1|}{2^j + |x_1|} \right)^{1+M_1} \cdot \frac{1}{(1 + |y_1|)^{1+M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+M_2}} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} 2^{-j(N_1+2-\tilde{M}_1-\tilde{M}_2-2)\epsilon_2} 2^{k(N_2+\epsilon_2)} 2^{k(M_2-2-\epsilon_2)}
\]
\[
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}}.
\]

For Term_{33}, by using the smoothness conditions for \(\psi_{j,k}\) and using (2.9) for \(f\), we have that for arbitrary \(\tilde{M}_1, \tilde{M}_2,\)

\[
\text{Term}_{33} \leq C\|f\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{33}} \left(\frac{|y_1|}{2^j + |x_1|}\right)^{N_1+1} 2^{j\tilde{M}_1} \cdot 2^{k\tilde{M}_2} \\
\times \frac{1}{(2^j + |x_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{2^j}(2^k |x_2 - y_2| + 2^{-j}|x_3 - y_3|)^{2+\tilde{M}_2} dy_1 dy_2 dy_3 \\
\leq C\|f\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{33}} 2^{-j(N_1+2-\tilde{M}_1-\tilde{M}_2-2)}2^{k(N_2+\epsilon_2)} \\
\times \frac{1}{(1 + |y_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3
\]

for Term_{34}, by using the size conditions for \(\psi_{j,k}\) and using (2.9) for \(f\), we have that for arbitrary \(\tilde{M}_1, \tilde{M}_2,\)

\[
\text{Term}_{34} \leq C\|f\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{34}} \left(\frac{2^j}{2^j + |x_1|}\right)^{1+\tilde{M}_1} \cdot \frac{2^{2\tilde{M}_2}}{2^{2\tilde{M}_2}} \\
\times \frac{1}{(1 + |y_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3 \\
\leq C\|f\|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{34}} 2^{-j(N_1+2-\tilde{M}_1-\tilde{M}_2-2)}2^{k(N_2+\epsilon_2)} \\
\times \frac{1}{(1 + |y_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3
\]

We see that in this case, it is required that \(\tilde{M}_2 > N_2 + \epsilon_2\) and \(\tilde{M}_1 + \tilde{M}_2 < N_1, N_1 + 1 < M_1.\)
Case 4: \( j < 0, k > 0 \). In this case, by using the cancellation condition on \( \psi_{j,k} \) for \( y_1 \) and the cancellation condition on \( f \) for \((y_2, y_3)\), we get that

\[
\psi_{j,k} \ast f(x_1, x_2, x_3) \\
= \int_{\mathbb{R}^3} \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) f(y_1, y_2, y_3) dy_1 dy_2 dy_3 \\
= \int_{\mathbb{R}^3} \left( \psi_{j,k}(x_1 - y_1, x_2 - y_2, x_3 - y_3) - P_{N_2}^{(2)} \psi_{j,k}(x_1 - y_1, x_2 - 0, x_3 - 0) \right) \\
\times \left[ f(y_1, y_2, y_3) - P_{N_1}^{(1)} (f(x_1, y_2, y_3))(y_1) \right] dy_1 dy_2 dy_3 \\
= \text{Term}_{41} + \text{Term}_{42} + \text{Term}_{43} + \text{Term}_{44},
\]

where each Term\(_i\) is the same as the right-hand side of the second equality above with \( \int_{\mathbb{R}^3} \) replaced by \( \int_{E_i} \), for \( i = 1, 2, 3, 4 \), and

\[
E_{41} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| < \frac{1}{2} (1 + |x_1|), |y_2| + 2^{-j} |y_3| < \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\}; \\
E_{42} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| \geq \frac{1}{2} (1 + |x_1|), |y_2| + 2^{-j} |y_3| < \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\}; \\
E_{43} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| < \frac{1}{2} (1 + |x_1|), |y_2| + 2^{-j} |y_3| \geq \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\}; \\
E_{44} = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1 - x_1| \geq \frac{1}{2} (1 + |x_1|), |y_2| + 2^{-j} |y_3| \geq \frac{1}{2} (2^k + |x_2| + 2^{-j} |x_3|) \right\}.
\]

Then, for Term\(_{41}\), by using the smoothness conditions for \( \psi_{j,k} \) and using (2.10) for \( f \), we have that for arbitrary \( \bar{M}_1, \bar{M}_2 \),

\[
\text{Term}_{41} \leq C \| f \|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{41}} \left( \frac{|x_1 - y_1|}{1 + |x_1|} \right)^{N_1 + \epsilon_1} \left( \frac{|y_2| + 2^{-j} |y_3|}{2^k + |x_2| + 2^{-j} |x_3|} \right)^{N_2 + 1} \\
\times \frac{2^j \bar{M}_1}{(2^j + |x_1 - y_1|)^{1 + \bar{M}_1}} \cdot \frac{2^k \bar{M}_2}{1 + |y_2| + |y_3|} dy_1 dy_2 dy_3 \\
\leq C \| f \|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \int_{E_{21}} 2^{j(N_1 + \epsilon_1 - (N_2 + \epsilon_1))} 2^{-k(N_2 + \epsilon_1 - \bar{M}_2)} \\
\times \frac{2^j (M_1 - N_1 - \epsilon_1)}{2^j (|x_1 - y_1|^{1 + M_1 - (N_1 + \epsilon_1)} \cdot (1 + |x_2| + |x_3|)^{2 + \bar{M}_2}} \\
\times \frac{1}{(1 + |x_1|)^{1 + M_1}} \cdot \frac{1}{(1 + |y_2| + |y_3|)^{2 + \bar{M}_2 - (N_2 + \epsilon_1)}} dy_1 dy_2 dy_3 \\
\leq C \| f \|_{G_{N_1, N_2}(\epsilon_1, \epsilon_2; M_1, M_2)} \frac{2^{j((N_1 + \epsilon_1) - (N_2 + \epsilon_1) - 1)}}{2^{-k(N_2 + \epsilon_1 - \bar{M}_2)}} \\
\times \frac{1}{(1 + |x_1|)^{1 + M_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2 + \bar{M}_2}}.
\]
where it is required that $N_2 < N_1$, $\tilde{M}_2 < N_2 + \epsilon_1$, $N_2 + \epsilon_1 < M_2$ and $N_1 + \epsilon_1 < \tilde{M}_1$.

For Term\textsubscript{42}, by using the smoothness conditions for $\psi_{j,k}$ and using (2.9) for $f$, we have that for arbitrary $\tilde{M}_1$, $\tilde{M}_2$,

\[
\text{Term}_{42} \leq C\|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{42}} \left( \frac{|y_2| + 2^{-j}|y_3|}{2^k + |x_2| + 2^{-j}|x_3|} \right)^{N_2+1} \frac{1}{2^k \tilde{M}_1} \left( 2^j + |x_1 - y_1| \right)^{1+\tilde{M}_1} \frac{1}{2^j \tilde{M}_2} \left( 2^{j-k} + |x_2| + 2^{-j}|x_3| \right)^{2+\tilde{M}_2} dy_1 dy_2 dy_3 
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3 
\leq C\|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{21}} 2^j((N_1+\epsilon_1)-(N_2+\epsilon_1)-1)2^{-k(N_2+\epsilon_1-\tilde{M}_2)} \frac{1}{2^j (\tilde{M}_1 - N_1 - \epsilon_1)} \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3 
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}}.
\]

For Term\textsubscript{43}, by using the size conditions for $\psi_{j,k}$ and using (2.10) for $f$, we have that for arbitrary $\tilde{M}_1$, $\tilde{M}_2$,

\[
\text{Term}_{43} \leq C\|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{43}} \left( \frac{|x_1 - y_1|}{1 + |x_1|} \right)^{N_1+\epsilon_1} \frac{1}{2^j \tilde{M}_1} \frac{1}{2^j \tilde{M}_2} \left( 2^j + |x_1 - y_1| \right)^{1+\tilde{M}_1} \left( 2^{j-k} + |x_2| + 2^{-j}|x_3| \right)^{2+\tilde{M}_2} dy_1 dy_2 dy_3 
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3 
\leq C\|f\|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{21}} 2^j((N_1+\epsilon_1)-(N_2+\epsilon_1)-1)2^{-k(N_2+\epsilon_1-\tilde{M}_2)} \frac{1}{2^j (\tilde{M}_1 - N_1 - \epsilon_1)} \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3 
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}}.
\]
For Term$_{44}$, by using the size conditions for $\psi_{j,k}$ and using (2.9) for $f$, we have that for arbitrary $\tilde{M}_1$, $\tilde{M}_2$,

$$
\text{Term}_{44} \leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{44}} \frac{2^j\tilde{M}_1}{(2^j + |x_1 - y_1|)^{1+\tilde{M}_1}} \cdot \frac{2^{k\tilde{M}_2}}{2^j(2^k + |x_2| + 2^{-j}|x_3|)^{2+\tilde{M}_2}}
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |y_2| + |y_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \int_{E_{21}} \frac{2^j(N_1+\epsilon_1)-(N_2+\epsilon_1)-1}{2-j(N_2+\epsilon_1-\tilde{M}_2)} 2^{k(N_2+\epsilon_1-\tilde{M}_2)}
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |y_2| + |y_3|)^{2+\tilde{M}_2}} dy_1 dy_2 dy_3
\leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \frac{1}{2^jN_1+\epsilon_1-\tilde{M}_2} 2^{k(N_2+\epsilon_1-\tilde{M}_2)}
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}}.
$$

And in this case it is required that $N_2 + 1 < N_1$, $\tilde{M}_2 < N_2 + \epsilon_1$, $N_2 + \epsilon_1 < M_2$ and $N_1 + \epsilon_1 < \tilde{M}_1$.

Combining all the estimates of the above four cases, we see that there exist $\sigma_1, \sigma_2 > 0$ such that

$$
|\psi_{j,k} * f(x_1,x_2,x_3)| \leq C \| f \|_{G_{N_1,N_2}(\epsilon_1,\epsilon_2;M_1,M_2)} \frac{2^{-|j|\sigma_1}2^{-|k|\sigma_2}}{(1 + |x_1|)^{1+M_1'} (1 + |x_2| + |x_3|)^{2+M_2'}}
\times \frac{1}{(1 + |x_1|)^{1+\tilde{M}_1}} \cdot \frac{1}{(1 + |x_2| + |x_3|)^{2+\tilde{M}_2}}
$$

where we can choose

$$
\sigma_1 = \min\{N_1 + \epsilon_1 - 1, N_1 - M_1', M_2', N_1 - N_2 - 1\},
$$

$$
\sigma_2 = \min\{N_2 + \epsilon_2, N_2 + 1 - M_2', N_2 + \epsilon_1 - M_2'\}
$$

and $M_1'$ and $M_2'$ satisfies $M_1' + M_2' < N_1$, $M_2' < N_2 - 1$.

Also, from the estimates above, we can see the restrictions of $N_1, N_2, M_1$ and $M_2$:

$$
N_1 + 1 < M_1, N_2 + 1 < M_2, N_2 + 1 < N_1.
$$

For example, we can choose $M_1$ large and $N_1 = M_1 - 2$, $N_2 = \lfloor N_1/3 \rfloor$, and $M_2 = N_2 + 2$.

Then $M_1' = M_2' = \lfloor N_1/4 \rfloor$.

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