Infrared finite coupling in Sudakov resummation

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Abstract: New arguments are presented to emphasize the interest of the infrared finite coupling approach to power corrections in the context of Sudakov resummation. The more regular infrared behavior of some peculiar combinations of Sudakov anomalous dimensions, free of Landau singularities at large $N_f$, is pointed out. A general conflict between the infrared finite coupling and infrared renormalon approaches to power corrections is explained, and a possible resolution is proposed, which makes use of the arbitrariness of the choice of exponentiated constant terms. A simple ansatz for a 'universal' non-perturbative Sudakov effective coupling at large $N_f$ follows naturally from these considerations. In this last version, a new result is presented: the striking emergence of an infrared finite perturbative effective coupling in the Drell-Yan process at large $N_f$ (at odds with the infrared renormalon argument) within the framework of Sudakov resummation for eikonal cross sections of Laenen, Sterman and Vogelsang. Some suggestions for phenomenology at finite $N_f$, alternative to the shape function approach, are given.
1. Introduction

The notion of an infrared (IR) finite coupling, and the related concept of universality, to parametrize power corrections in QCD has attracted much attention for a long time [1], [2]. In the present note, which is an extended version of a talk given at the FRIF workshop on non-perturbative effects in jets (Paris, January 10-14 2006), I display further evidence in favor of this assumption in the more specific framework of Sudakov resummation. A more detailed account of the topics covered here is postponed to a future publication. The paper is organized as follows. Section 2 gives a phenomenological incentive for an IR finite effective coupling to parametrize the tail of the Sudakov peak for the thrust distribution. Section 3 gives a theoretical incentive, where some remarkable IR properties of specific combinations of Sudakov anomalous dimensions are pointed out. Section 4 discusses a
general conflict between the renormalon and the IR finite coupling approaches to power corrections, and indicates a possible resolution based on the arbitrariness of the choice of constant terms exponentiated in Sudakov resummation along with the large logarithms. This issue is further discussed in sections 5-7, and a closed form solution is given at large $N_f$ in the case of deep inelastic scattering (DIS) in section 7.1. An application to the Drell-Yan process is given in section 7.2, where it is shown that the exponentiation of $O(N^0)$ terms advocated in [3] naturally leads to the emergence of the simplest ansatz for an IR finite perturbative coupling (albeit being at odds with the IR renormalon argument). Alternatively, an attempt to reconcile the IR renormalon and IR coupling approaches leads to another ansatz for a ‘universal’, but non-perturbative, Sudakov effective coupling at large $N_f$ proposed in section 8. Section 9 deals further with the issue of resummation of constant terms, and sketches a procedure for phenomenology at finite $N_f$. Section 10 contains the conclusions.

2. Incentive for IR finite coupling approach

Let us take as an example the case of thrust, and define

$$\frac{1}{\sigma_{\text{tot}}} \int_{\tau_{\text{max}}}^{0} d\tau \left[ \exp(-\nu \tau) \right] d\tau \equiv \exp \left[ E(Q^2, \nu) \right], \quad (2.1)$$

where $\tau \equiv 1 - T$. Consider now the case where $\nu$ is large, which corresponds to $\tau$ small, and assume the standard [4] exponentiation formula

$$E(Q^2, \nu) \equiv S(Q^2, \nu) + H(Q^2) + O(1/\nu), \quad (2.2)$$

where $H(Q^2)$ is a power series in $a_s(Q^2) \equiv \alpha_s(Q^2)/4\pi$ with coefficients independent of $\nu$, and the ‘Sudakov exponent’ $S(Q^2, \nu)$ is given by

$$S(Q^2, \nu) = \int_{0}^{1} \frac{dx}{x} \left[ \exp(-\nu x) - 1 \right] \left[ 4C_F \int_{xQ^2}^{Q^2} \frac{dk^2}{k^2} A_S(k^2) - 3C_F B_S(xQ^2) \right], \quad (2.3)$$

where $A_S(k^2)$ and $B_S(k^2)$, the ‘Sudakov effective couplings’, should be considered as two physical ‘effective charges’ [8] which can be expanded as power series in $a_s(k^2)$

$$A_S(k^2) = a_s(k^2) + A_1 a_s^2(k^2) + A_2 a_s^3(k^2) + ..., \quad (2.4)$$

and similarly for $B_S(k^2)$. It is convenient to interchange the $x$ and $k^2$ integrations, and write eq.(2.3) as a sum of two ‘renormalon integrals’ [8]

$$S(Q^2, \nu) = 4C_F \int_{0}^{Q^2} \frac{dk^2}{k^2} F_A(k^2/Q^2, \nu) A_S(k^2) - 3C_F \int_{0}^{Q^2} \frac{dk^2}{k^2} F_B(k^2/Q^2, \nu) B_S(k^2), \quad (2.5)$$

where $F_A(k^2/Q^2, \nu)$ and $F_B(k^2/Q^2, \nu)$, the ‘Sudakov distribution functions’, are given by
\[ F_A(k^2/Q^2, \nu) = \int_{k^2/Q^2}^{k/Q} \frac{dx}{x} \left[ \exp(-\nu x) - 1 \right], \quad (2.6) \]

and

\[ F_B(k^2/Q^2, \nu) = \exp(-\nu k^2/Q^2) - 1. \quad (2.7) \]

I note that

\[ F_A(k^2/Q^2, \nu) \to \ln(k/Q) \quad (2.8) \]

and

\[ F_B(k^2/Q^2, \nu) \to -1 \quad (2.9) \]

for \( \nu \to \infty \). Assume now that the effective couplings \( A_S(k^2) \) and \( B_S(k^2) \) are IR finite, and setting \( A_S(k^2) \equiv A_{IR} + \Delta A_S(k^2) \) and \( B_S(k^2) \equiv B_{IR} + \Delta B_S(k^2) \), assume further that \( \Delta A_S(k^2) \) and \( \Delta B_S(k^2) \) are \( O(k^2) \) for \( k^2 \to 0 \). Then, taking the \( \nu \to \infty \) limit under the integrals, one gets

\[
S(Q^2, \nu) \sim -4C_F K_1(\nu) A_{IR} - 3C_F K_0(\nu) B_{IR} \\
+ 4C_F \int_0^{Q^2} \frac{dk^2}{k^2} \ln(k/Q) \Delta A_S(k^2) + 3C_F \int_0^{Q^2} \frac{dk^2}{k^2} \Delta B_S(k^2), \quad (2.10)
\]

where the integrals over \( \Delta A_S(k^2) \) and \( \Delta B_S(k^2) \) are \( \nu \)-independent (and IR convergent), whereas

\[
K_1(\nu) = \int_0^1 \frac{dx}{x} \left[ \exp(-\nu x) - 1 \right] \ln x \sim \frac{1}{2} \ln^2 \nu \quad (2.11)
\]

and

\[
K_0(\nu) = \int_0^1 \frac{dx}{x} \left[ \exp(-\nu x) - 1 \right] \sim -\ln \nu \quad (2.12)
\]

for \( \nu \to \infty \). Thus we obtain at large \( \nu \)

\[ E(Q^2, \nu) \sim -4C_F K_1(\nu) A_{IR} - 3C_F K_0(\nu) B_{IR} + C(Q^2), \quad (2.13) \]

where

\[
C(Q^2) \equiv 4C_F \int_0^{Q^2} \frac{dk^2}{k^2} \ln(k/Q) \Delta A_S(k^2) + 3C_F \int_0^{Q^2} \frac{dk^2}{k^2} \Delta B_S(k^2) + H(Q^2) \quad (2.14)
\]

is \( \nu \)-independent. Hence for \( \nu \to \infty \)

\[
\frac{1}{\sigma_{tot}} \int_0^{\tau_{max}} d\tau \exp(-\nu \tau) \frac{d\sigma}{d\tau} \sim N(Q^2) \exp \left[ -2C_F A_{IR} \ln^2 \nu + O(\ln \nu) \right], \quad (2.15)
\]
where \( N(Q^2) \) is a non-perturbative \( Q \)-dependent normalization constant (its \( Q \)-dependence can eventually be estimated at large \( Q \)). Upon taking the inverse Laplace transform the present ansatz can thus potentially (if \( A_{IR} > 0 \)) reproduce the expected tail of the thrust distribution for \( \tau \to 0 \), which is parametrized essentially by the IR value \( A_{IR} \) of the effective coupling \( A_S(k^2) \) (\( B_{IR} \) gives a subdominant contribution) which could be fitted from the data. I note that no assumption that \( Q \) is large has been made, so this prediction goes beyond short distance physics. To fit the whole thrust distribution at all values of \( \tau \), one should make a more complete ansatz for the IR behavior of \( A_S(k^2) \) and \( B_S(k^2) \). This step requires a priori no more free parameters than the ‘shape function’ approach \[7\] and provides an alternative to it.

3. IR behavior of the Sudakov effective coupling in the \( N_f \to \infty \) limit

Let us take as a case study the example of the scaling violation for the non-singlet structure function in deep inelastic scattering (DIS). From the standard exponentiation formulas in Mellin \( N \)-space (I shall adopt in general the notations of \[8\]), one gets immediately the relation at large \( N \) for the ‘physical anomalous dimension’ \[5, 8\]

\[
\frac{d \ln F_2(Q^2, N)}{d \ln Q^2} = 4C_F H(Q^2) + 4C_F \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} A_S((1 - z)Q^2) + \mathcal{O}(1/N),
\]

where \( H(Q^2) \) is again given as a power series in \( a_s(Q^2) \) with \( N \)-independent coefficients and

\[
4C_F A_S(k^2) = A(a_s(k^2)) + \frac{dB(a_s(k^2))}{d \ln k^2},
\]

(3.2)

\( A \) (the universal ‘cusp’ anomalous dimension) and \( B \) are the standard Sudakov anomalous dimensions relevant to DIS, given as power series in \( a_s \): \( A(a_s) = A_0 a_s + A_1 a_s^2 + ... \) (with \( A_0 = 4C_F \)) and \( B(a_s) = B_0 a_s + B_1 a_s^2 + ... \) I assumed the conjecture of \[10\], relative to the vanishing to all orders of the third standard anomalous dimension \( D \) (the ‘soft function’), is correct. \( A_S = a_s + \mathcal{O}(a_s^2) \) shall be refered to as the ‘Sudakov effective coupling’, and we shall see (in the large \( N_f \) limit) that it has drastically different infrared properties then the individual Sudakov anomalous dimensions it is composed of.

Large \( N_f \) analysis: the Borel transform \( B[A_S](u) \) of the Sudakov effective coupling has been computed at large \( N_f \). It is defined by

\[
A_S(k^2) = \int_0^\infty du \ exp \left( -u \ln \frac{k^2}{\Lambda^2} \right) B[A_S](u),
\]

(3.3)

and is was found \[11, 10\] (this result is checked below using a different method) that at large \( N_f \), with the ‘naive non-abelization’ recipie \[12\]

\[
B[A_S](u) = \frac{1}{\beta_0} \exp(-du) \frac{\sin \pi u}{\pi u} \frac{1}{2} \left( \frac{1}{1-u} + \frac{1}{1-u/2} \right),
\]

(3.4)
where $\beta_0 = \frac{11C_2}{3} - \frac{2}{3}N_f$ is the one-loop coefficient of the beta function and $d$ is a scheme-dependent constant related to the renormalization of fermion loops: $d = -5/3$ in the $\overline{MS}$ scheme and $d = 0$ in the so-called ‘V-scheme’ (or ‘single dressed gluon scheme’). In the following I shall use the ‘V-scheme’ for simplicity. One should note that $B[A_S](u)$ is free of renormalons singularities, and in fact the corresponding standard perturbative series of $A_S$ in powers of $a_s$ has a finite convergence radius.

One first observes that for $k = \Lambda$ (the Landau pole of the one-loop V-scheme coupling) the Borel integral in eq.(3.3) converges and $A_S(k^2 = \Lambda^2)$ is well-behaved and finite. This behavior is in striking contrast with that of the cusp anomalous dimension, which has also been computed $[14], [15]$ at large $N_f$.

$$A_S(k^2) = \frac{4C_F}{\pi \beta_0} \sin \frac{\pi \beta_0 a_s}{\beta_0} \frac{\Gamma(4 + 2\beta_0 a_s)}{6\Gamma(2 + \beta_0 a_s)^2},$$ (3.5)

where $\beta_0 a_s = 1/\ln k^2/\Lambda^2$ at large $N_f$. It is easy to see, since $a_s \to \infty$ for $k \to \Lambda$, that the ratio of gamma functions in eq.(3.5) blows up in this limit, while the sin factor generates wild oscillations, resulting in a completely unphysical behavior around $k = \Lambda$. The much more tamed behavior of $A_S(k^2)$ suggests that it is well-behaved in the infrared region. It is actually possible to get an analytic expression for $A_S(k^2)$, valid at all $k^2$. One finds

$$A_S(k^2) = A_S^{\text{simple}}(k^2) + \frac{1}{2\pi \beta_0} \exp(-t) \frac{Ei(t + i\pi) - Ei(t - i\pi)}{2i}$$

$$- \frac{1}{2\pi \beta_0} \exp(-2t) \frac{Ei(2t + 2i\pi) - Ei(2t - 2i\pi)}{2i},$$ (3.6)

with

$$A_S^{\text{simple}}(k^2) = \frac{1}{\beta_0} \left[ \frac{1}{2} - \frac{1}{\pi} \arctan(t/\pi) \right],$$ (3.7)

where $t = \ln k^2/\Lambda^2$ and $Ei(x)$ is the exponential integral function. Eq.(3.6) can equivalently be written in term of the incomplete gamma function $\Gamma(0, x) = \int_x^\infty \frac{1}{z} \exp(-z)$ as

$$A_S(k^2) = A_S^{\text{simple}}(k^2) + \frac{1}{2\pi \beta_0} \exp(-t) \left[ \frac{\Gamma(0, -t + i\pi) - \Gamma(0, -t - i\pi)}{2i} \right] + \pi$$

$$- \frac{1}{2\pi \beta_0} \exp(-2t) \left[ \frac{\Gamma(0, -2t + 2i\pi) - \Gamma(0, -2t - 2i\pi)}{2i} \right] + \pi.$$ (3.8)

To investigate the infrared behavior, I now make use of the fact that $\Gamma(0, x) \simeq \frac{\exp(-x)}{x}(1 - 1/x + ...)$ for $|x| \to \infty$, which implies

$$\exp(-t) \left[ \frac{\Gamma(0, -t + i\pi) - \Gamma(0, -t - i\pi)}{2i} \right] \simeq \frac{\pi}{t^2}$$

$$\exp(-2t) \left[ \frac{\Gamma(0, -2t + 2i\pi) - \Gamma(0, -2t - 2i\pi)}{2i} \right] \simeq -\frac{2\pi}{4t^2}.$$ (3.9)
for $|t| \to \infty$. In the ultraviolet (UV) region ($t \to +\infty$) one thus obtain, as expected, a power series in $a_s(k^2) = 1/\beta_0$

$$A_S(k^2) \simeq a_s(k^2) + \frac{3}{4} \beta_0 a_s^2(k^2) + \left(\frac{5}{4} - \frac{\pi^2}{3}\right) \beta_0^2 a_s^3(k^2) + \ldots \quad (3.10)$$

I note that $A_S^{\text{simple}}(k^2)$ behaves as $a_s + \mathcal{O}(a_s^2)$ in this limit, while the last two terms on the right hand side of eq.(3.8) are $\mathcal{O}(a_s^2)$. $A_S^{\text{simple}}(k^2)$ turns out to coincide for real space-like $k^2 = \mu^2 > 0$ with the (integrated) time-like discontinuity at $k^2 = -\mu^2 < 0$ of the one-loop coupling. I stress however that here it should be considered as an analytic contribution ansatz for a $(\text{eventually finite})$ IR fixed point, yielding an IR convergent Sudakov integral. A simple strong IR fixed point of perturbative origin into a genuinely non-perturbative, but softer $\mathcal{O}(\Lambda)$ contribution axis. The trouble is in the too strongly divergent IR behavior, which gives a divergent for $k^2 \to -\infty$, the full $A_S(k^2)$ would have reached the same IR fixed point value of $1/\beta_0$, were it not for the two IR divergent contributions $\pi \exp(-t)$ and $\pi \exp(-2t)$. Thus in fact $A_S(k^2)$ approaches an infinite (and negative) IR fixed point for $k^2 \to 0$:

$$A_S(k^2) \simeq \frac{1}{2} - \frac{1}{\pi} \arctan(t/\pi) = \frac{1}{\pi} \ln(t + i\pi) - \ln(t - i\pi). \quad (3.11)$$

On the other hand, in the IR region ($t \to -\infty$) $A_S^{\text{simple}}(k^2)$ reaches a finite limit of $1/\beta_0$. Since the two incomplete gamma function contributions are $\mathcal{O}(1/t^2)$, and thus also vanish for $t \to -\infty$, the full $A_S(k^2)$ would have reached the same IR fixed point value of $1/\beta_0$, were it not for the two IR divergent contributions $\pi \exp(-t)$ and $\pi \exp(-2t)$. Thus in fact $A_S(k^2)$ approaches an infinite (and negative) IR fixed point for $k^2 \to 0$:

$$A_S(k^2) \simeq -\frac{1}{2\beta_0} \frac{\Lambda^4}{k^4} + \frac{1}{2\beta_0} \frac{\Lambda^2}{k^2} \quad (3.12)$$

This is not by itself an unphysical behavior, except for the negative sign in the infrared, which cannot reproduce a vanishing Sudakov tail. Moreover $A_S(k^2)$ is a causal function and has no unphysical Landau singularities on the whole first sheet of the $k^2$ plane: the only possible branch points occur for $t = \pm i\pi$, i.e. at $k^2/\Lambda^2 = -1$ on the time-like axis. The trouble is in the too strongly divergent IR behavior, which gives a divergent contribution $\frac{\Lambda^4}{k^2} \mathcal{O}(1/t^2)$ for $z \to 1$ to the integral on the right-hand side of eq.(3.1). This finding makes it more plausible the speculation that there exists a non-perturbative modification $\delta A_{S,NP}(k^2)$ of the coupling at very small momenta which might turn the too strong IR fixed point of perturbative origin into a genuinely non-perturbative, but softer (eventually finite) IR fixed point, yielding an IR convergent Sudakov integral. A simple ansatz for $\delta A_{S,NP}(k^2)$, localized in the IR region, and which thus does not induce large OPE-violating ultraviolet $\mathcal{O}(1/Q^2)$ power corrections, is given by

$$\delta A_{S,NP}(k^2) = \frac{1}{2\beta_0} \left[ -\frac{\Lambda^2}{k^2} \exp\left(-\frac{k^2}{\Lambda^2}\right) + \frac{\Lambda^4}{k^4} \exp\left(-\frac{k^2}{\Lambda^4}\right) \right] \quad (3.13)$$

Then

$$A_{S,NP}(k^2) = A_S(k^2) + \delta A_{S,NP}(k^2), \quad (3.14)$$

1The straightforward modification of the coupling obtained by just subtracting the IR divergent contribution eq.(3.12) yields, when viewed from the UV side, a too large short distance $\mathcal{O}(1/Q^2)$ correction.
which indeed yields for \( k^2 \to 0 \) a finite and positive IR fixed point: \( A_{S,NP}(k^2) \simeq 1/\beta_0 + O(k^2) \).

4. A clash between the infrared finite coupling and the IR renormalons approaches to power corrections

There is a potential clash between the infrared finite coupling and the IR renormalons approaches to power corrections (very closely connected to the well-known issue \([13], [14]\) of \(1/Q\) corrections in Drell-Yan) which can be summarized in general terms as follows. Consider a typical ‘renormalon integral’

\[
S(Q^2, N) = \int_0^{Q^2} \frac{dk^2}{k^2} F(k^2/Q^2, N) A_S(k^2),
\]

and introduce its (‘RS invariant’ \([16]\) Borel representation by

\[
S(Q^2, N) = \int_0^{\infty} du \exp \left( -u \ln \frac{Q^2}{\Lambda^2} \right) B[S](u, N).
\]

Using eq.(3.3) one obtains the Borel transform of \( S \) as

\[
B[S](u, N) = B[A_S](u) \tilde{F}(u, N),
\]

with

\[
\tilde{F}(u, N) = \int_0^1 \frac{dx}{x} F(x, N) \exp(-u \ln x).
\]

The factorized form \([10, 13]\) of the expression should be noted. Suppose now the Borel transform \( B[A_S](u) \) of the effective coupling \( A_S \) has a zero at some position \( u = u_0 \), which is not shared by \( B[S](u, N) \). Then necessarily \( \tilde{F}(u, N) \) in eq.(4.3) must have a pole\(^2\) at \( u = u_0 \). If \( u_0 > 0 \), this means an IR renormalon, and \( F(x, N) \) contains an \( O(x^{u_0}) \) contribution for \( x \to 0 \). Since this renormalon is not present in \( B[S](u, N) \), the standard IR renormalon philosophy would conclude that no corresponding power correction is present. On the other hand, this power correction is still expected in the IR finite coupling approach, where \( A_S(k^2) \) is assumed to have a finite IR fixed point and the low energy part of the integral in eq.(4.3) is well-defined. Then the distribution function \( F(k^2/Q^2, N) \) can be expanded in powers of \( k^2 \) for \( k^2 \to 0 \), yielding a non-vanishing power correction for each term in its low energy expansion, parametrized by a low energy moment of the effective coupling \( A_S(k^2) \). Even if \( B[S](u, N) \) is singular, rather than non-vanishing and finite, at \( u = u_0 \), there is still a clash, since the singularity of \( \tilde{F}(u, N) \) is necessarily stronger than that of \( B[S](u, N) \) in presence of a zero of \( B[A_S](u) \); thus the IR finite coupling approach will predict a more enhanced power correction than indicated by the renormalon argument.

\(^2\)This argument is usually presented \([14]\) as the existence of a zero in the Borel transform \( B[A_S] \) causing the vanishing of the residue of a would-be renormalon contained in \( \tilde{F}(u, N) \). Thus this remark is a particular case of the observation in \([16]\) that the renormalon residue is an all-order quantity, requiring the knowledge of all the perturbative coefficients of the effective coupling \( A_S \).
At this stage, there is no way to decide which of these two philosophies is correct without further information. I end this section by giving two examples of this situation at large $N_f$.

i) DIS: there eq.(3.4) implies that $B[A_S](u)$ vanishes for any integer $u$, $u \geq 3$.

ii) Drell-Yan: in this case the Sudakov effective coupling which occurs in the $Q^2$ derivative of the Drell-Yan cross section is given by

$$8C_F A_{S,DY}(k^2) = 2A(a_s(k^2)) + \frac{dD_{DY}(a_s(k^2))}{d\ln k^2},$$

and the corresponding Borel transform at $N_f = \infty$ is [14] (dropping an inessential $\exp(cu)$ factor which corresponds simply to a change of scale):

$$B[A_{S,DY}](u) = \frac{1}{\beta_0} \frac{1}{\Gamma(1+u)} \frac{\pi^{1/2}}{\Gamma(1/2-u)},$$

which vanishes for positive half-integer $u$.

In both cases, the zeroes are absent from the corresponding Sudakov exponent, and the IR finite coupling approach will lead to the prediction of extra power corrections in the exponent ($N^3/Q^6$, $N^4/Q^8$, etc...in DIS, and $N/Q$, $N^3/Q^3$, etc...in Drell-Yan).

5. Ambiguities in Sudakov resummation and exponentiation of constant terms

To make progress, we have to understand better the origin of zeroes in the Borel transform $B[A_S](u)$ of the Sudakov effective coupling. In this section, I shall show it is possible to make compatible the renormalon and IR finite coupling approaches. The main point which will be developed is the following: there is an ambiguity in the choice of the Sudakov distribution function (or, alternatively, in that of the Sudakov effective coupling), which corresponds to the freedom to select an arbitrary set of 'constant terms' to be included in the Sudakov exponent. One can eventually make use of this freedom to reconcile the two conflicting approaches.

To see this, let us consider the specific case of DIS scattering eq.(3.1). Putting

$$S(Q^2, N) = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} A_S[(1-z)Q^2],$$

it is again convenient to write eq.(5.1) as a renormalon integral (eq.(4.1)), where the Sudakov distribution function is now given by

$$F(k^2/Q^2, N) = (1 - k^2/Q^2)^{N-1} - 1.$$ 

I first note that order by order the perturbative series of $S(Q^2, N)$ contain both $O(N^0)$ 'constant terms' and terms which vanish as $N \to \infty$, e.g.

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3The natural occurrence of this combination of anomalous dimensions has been pointed out independently from a different point of view by G. Sterman [15].
\[ S(Q^2, N) = (\gamma_{01}L + \gamma_{00})a_s(Q^2) + (\gamma_{12}L^2 + \gamma_{11}L + \gamma_{10})a_s^2(Q^2) + \ldots + \mathcal{O}(1/N), \quad (5.3) \]

where \( L \equiv \ln N \).

A very useful simplification is achieved by making use of the following important scaling property: at large \( N \), \( F(k^2/Q^2, N) \) becomes a function of the variable \( \epsilon = \frac{Nk^2}{Q^2} \) only\(^4\), reflecting the relevance \(^{[18]}\) of the 'soft scale' \( Q^2/N \). Indeed defining

\[ F(k^2/Q^2, N) \equiv G(\epsilon, N) = (1 - \epsilon/N)^{N-1} - 1, \quad (5.4) \]

and taking the \( N \to \infty \) limit with \( \epsilon \) fixed one gets a finite result

\[ G(\epsilon, N) \to G(\epsilon, \infty) \equiv G(\epsilon), \quad (5.5) \]

with

\[ G(\epsilon) = \exp(-\epsilon) - 1. \quad (5.6) \]

Let us now redefine the Sudakov exponent by using \( G(\epsilon) \) as the new distribution function, i.e. eq.(4.1) is replaced by

\[ S_{\text{stan}}(Q^2, N) = \int_0^{Q^2} \frac{dk^2}{k^2} G(Nk^2/Q^2) A_S(k^2), \quad (5.7) \]

where the index ‘stan’ stands for ‘standard’. This step is legitimate since, order by order in perturbation theory, \( S_{\text{stan}}(Q^2, N) \) and \( S(Q^2, N) \) differ only by terms which vanish as \( N \to \infty \), and thus share the same \( \ln N \) and constant terms (eq.(5.3)). Indeed, eq.(5.7) can be written identically as

\[ S_{\text{stan}}(Q^2, N) = \int_0^1 dz \frac{\exp[-N(1-z)] - 1}{1-z} A_S[(1-z)Q^2], \quad (5.8) \]

and the equivalence between eq.(5.1) and eq.(5.8) up to \( \mathcal{O}(1/N) \) terms was proved in \(^{[19]}\).

I now assume an ansatz of the form of eq.(5.7)

\[ S_{\text{new}}(Q^2, N) = \int_0^{Q^2} \frac{dk^2}{k^2} G_{\text{new}}(Nk^2/Q^2) A_S^{\text{new}}(k^2), \quad (5.9) \]

and show that a unique solution for \( G_{\text{new}}(Nk^2/Q^2) \) and \( A_S^{\text{new}}(k^2) \) exists, under the condition to reproduce all divergent \( \ln N \) terms, together with an (a priori arbitrary) given set of constant terms, i.e.

\[ S_{\text{new}}(Q^2, N) = (\gamma_{01}L + \gamma_{00}^{\text{new}})a_s(Q^2) + (\gamma_{12}L^2 + \gamma_{11}L + \gamma_{10}^{\text{new}})a_s^2(Q^2) + \ldots + \mathcal{O}(1/N). \quad (5.10) \]

This statement can be checked order by order in perturbation theory. As is well-known, the \( \ln N \) structure in eq.(5.3) or (5.10) arises from performing the integrals in eq.(5.7) or (5.4)

\(^{4}\)For Drell-Yan, the corresponding scaling variable is \( \epsilon_{\text{DY}} = \frac{Nk}{Q} \).
over the renormalization group \( \ln(k^2/Q^2) \) logs which show up when \( A_S(k^2) \) is expanded in powers of \( a_s(Q^2) \):

\[
A_S(k^2) = a_s(Q^2) + (-\beta_0 \ln(k^2/Q^2) + A_1)a_s^2(Q^2) + \ldots
\]  
(5.11)

(where one should set \( k^2 = (1 - z)Q^2 \) if one uses instead eq. (5.1) or (5.8)). Using eq. (5.7) one thus gets

\[
S_{stan}(Q^2, N) = K_0(N)a_s(Q^2) + [-\beta_0 K_1(N) + A_1 K_0(N)] a_s^2(Q^2) + \ldots,
\]  
(5.12)

with

\[
K_p(N) = \int_0^N d\epsilon G(\epsilon) \ln(\epsilon/N).
\]  
(5.13)

Thus for large \( N \) one gets (see also eq. (2.12) and (2.11))

\[
K_0(N) = \int_0^N d\epsilon G(\epsilon) = c_{01} L + c_{00} + O(1/N),
\]  
(5.14)

and

\[
K_1(N) = \int_0^N d\epsilon G(\epsilon) \ln(\epsilon/N) = c_{12} L^2 + c_{11} L + c_{10} + O(1/N),
\]  
(5.15)

where the \( c_{ij} \)'s are known. Let us now try to modify \( G(\epsilon) \rightarrow G(\epsilon) + \Delta G(\epsilon) \equiv G_{\text{new}}(\epsilon) \), in such a way the coefficients of all positive powers of \( \ln N \) remain unchanged. I show that this cannot be achieved without changing simultaneously the Sudakov effective charge \( A_S \). At \( O(a_s) \) one gets \( c_{00} \rightarrow c_{00} + \Delta c_{00} = c_{00}^{\text{new}} \), and one should require that

\[
\Delta c_{00} = \int_0^\infty \frac{d\epsilon}{\epsilon} \Delta G(\epsilon) < \infty
\]  
(5.16)

in order to preserve the leading log coefficient \( c_{01} \). However at \( O(a_s^2) \) one gets \( c_{11} \rightarrow c_{11} + \Delta c_{11} = c_{11}^{\text{new}} \) with

\[
\Delta c_{11} = -\int_0^\infty \frac{d\epsilon}{\epsilon} \Delta G(\epsilon) = -\Delta c_{00}.
\]  
(5.17)

Thus the single log term in \( K_1(N) \) must be changed, and according to eq. (5.12), this implies a correlated change \( A_1 \rightarrow A_1^{\text{new}} \) if one wants to preserve the subleading single \( \ln N \) term at \( O(a_s^2) \) in \( S_{stan} \). Indeed, \( \Delta c_{00} \) is already fixed (given \( c_{00} \)) from the input \( c_{00}^{\text{new}} = c_{00}^{\text{new}} \) (compare e.g. eq. (5.3) and (6.12)). Then, the knowledge of \( \Delta c_{11} \) fixes \( \Delta A_1 \) requiring

\[
-\beta_0 \Delta c_{11} + \Delta A_1 = 0,
\]  
(5.18)
which fixes $A_1^{\text{new}}$, given $A_1$.

Furthermore one gets $c_{10} \rightarrow c_{10} + \Delta c_{10} = c_{10}^{\text{new}}$ with

$$\Delta c_{10} = \int_0^\infty \frac{d\epsilon}{\epsilon} \Delta G(\epsilon) \ln \epsilon.$$  \hfill (5.19)

Thus $\Delta c_{10}$ is fixed (given $c_{10}$ and $A_1^{\text{new}}$) from the input

$$-\beta_0 c_{10}^{\text{new}} + A_1^{\text{new}} c_{10}^{\text{new}} = \gamma_{10}^{\text{new}}$$

(compare again eq.(5.3) and (5.12)). One thus sees that in this process $\Delta G(\epsilon)$ is determined uniquely in principle by its input logarithmic moments $\Delta c_{i0} (i : 0 \rightarrow \infty)$ (see eq.(5.16) and (5.19)), as can be easily checked in the first few orders of perturbation theory. To determine $\Delta G(\epsilon)$, we need however input all the $\gamma_{i0}^{\text{new}}$'s and use all orders of perturbation theory. This procedure is clearly not tractable, so we next turn to large $N_f$ where the problem can be solved in closed form.

### 6. The asymptotic large $N$ Borel transform

Before doing this, it is useful to first introduce a simplification appropriate to the large $N$ limit, valid also at finite $N_f$ beyond the single dressed gluon approximation. I observe that the perturbative series eq.(5.3) of the Sudakov exponent eq.(5.7) still contains $O(1/N)$ terms at large $N$. It is useful to discard them, and find a Borel representation of the corresponding series $S_{\text{as}}(Q^2, N)$ which contains only $\ln N$ and $O(N^0)$ terms, with all the $O(1/N)$ terms in eq.(5.3) dropped. The (RS invariant) Borel transform of eq.(5.7) takes the factorized form (similarly to eq.(4.3) and (4.4))

$$B[S_{\text{stan}}](u, N) = B[A_S](u) \exp(u \ln N) \int_0^N \frac{d\epsilon}{\epsilon} G(\epsilon) \exp(-u \ln \epsilon).$$ \hfill (6.1)

Taking straightforwardly the large $N$ limit one gets a tentative ansatz for the looked for Borel transform

$$B[S_{\text{as}}](u, N) = B[A_S](u) \sin \frac{\pi u}{\pi} \exp(u \ln N) \int_0^\infty \frac{d\epsilon}{\epsilon} G(\epsilon) \exp(-u \ln \epsilon).$$ \hfill (6.2)

Since the large $N_f$ calculation is usually performed in the `dispersive approach’ \cite{20, 1} which makes use of a ‘Minkowskian’ coupling, it is convenient to use the equivalent ‘Minkowskian’ representation of $B[S_{\text{as}}](u, N)$, i.e. to introduce the Minkowskian counterpart of the Sudakov effective coupling, together with a ‘Sudakov characteristic function’ $\hat{G}(y)$ related to the Sudakov distribution function $G(\epsilon)$ by the dispersion relation

$$\hat{G}(y) = y \int_0^\infty \frac{d\epsilon}{(\epsilon + y)^2},$$ \hfill (6.3)

where $\hat{G} = -y d\hat{G}/dy$. It follows that$^5$

$$B[S_{\text{as}}](u, N) = B[A_S](u) \frac{\sin \frac{\pi u}{\pi}}{\pi} \exp(u \ln N) \int_0^\infty \frac{dy}{y} \hat{G}(y) \exp(-u \ln y).$$ \hfill (6.4)

$^5B[A_S](u) \frac{\sin \frac{\pi u}{\pi}}{\pi}$ is the Borel transform of the Minkowskian Sudakov effective coupling.
Comparing eq. (6.2) with eq. (6.4) yields the standard relation between ‘Euclidean’ and ‘Minkowskian’ Laplace transforms

\[
\int_0^\infty \frac{d\epsilon}{\epsilon} G(\epsilon) \exp(-u \ln \epsilon) = \sin \frac{\pi u}{\pi u} \int_0^\infty \frac{dy}{y} \hat{G}(y) \exp(-u \ln y).
\]

However, eq. (6.2) and (6.4) are not quite correct. For \( u = 0 \) the integrals on the right hand side are UV divergent, since \( \hat{G}(\infty) = G(\infty) = -1 \) (see eq. (5.6)). One actually needs to ‘renormalize’ these ‘bare’ Borel transforms, i.e. to add a subtraction term. One can show that the correct answer, in the case of the Euclidean representation, is

\[
B[S_{as}](u, N) = B[A_S](u) \left[ \exp(u \ln N) \int_0^\infty \frac{d\epsilon}{\epsilon} G(\epsilon) \exp(-u \ln \epsilon) - \frac{G(\infty)}{u} \right],
\]

whereas for the Minkowskian representation (as follows from eq. (6.3))

\[
B[S_{as}](u, N) = B[A_S](u) \sin \frac{\pi u}{\pi u} \left[ \exp(u \ln N) \int_0^\infty \frac{dy}{y} \hat{G}(y) \exp(-u \ln y) - \frac{\pi \hat{G}(\infty)}{\sin \pi u} \right].
\]

The corresponding ‘renormalon integral’ representation of \( S_{as}(Q^2, N) \) analogous to eq. (5.7) is

\[
S_{as}(Q^2, N) = \int_0^\infty \frac{dk^2}{k^2} G(Nk^2/Q^2) A_S(k^2) - G(\infty) \int_Q^\infty \frac{dk^2}{k^2} A_S(k^2)
\]

\[
\equiv \int_0^Q \frac{dk^2}{k^2} G(Nk^2/Q^2) A_S(k^2) + \int_Q^\infty \frac{dk^2}{k^2} [G(Nk^2/Q^2) - G(\infty)] A_S(k^2),
\]

where the second integral on the right hand side on the first line provides the necessary subtraction term (the first integral being UV divergent).

7. Exponentiation of constant terms at large \( N_f \): the single dressed gluon result

7.1 DIS case

Specializing now to large \( N_f \), the result for the Borel transform of \( d \ln F_2(Q^2, N)/d \ln Q^2 \) at finite \( N \) can be given in the ‘massive gluon’ or ‘single dressed gluon’ (SDG) dispersive Minkowskian formalism [20, 1] as (the V-scheme is assumed)

\[
B[d \ln F_2(Q^2, N)/d \ln Q^2]_{SDG}(u, N) = 4C_F \frac{1}{\beta_0} \frac{\sin \pi u}{\pi u} \int_0^\infty \frac{dx}{x} \hat{F}_{SDG}(x, N) \exp(-u \ln x),
\]

where the ‘characteristic function’ \( \hat{F}_{SDG}(x, N) \) has been computed in [1] (\( x = \lambda^2/Q^2 \) where \( \lambda \) is the ‘gluon mass’). I have checked that here too a similar scaling property holds at
large $N$, namely, putting $\mathcal{G}_{SDG}(y, N) \equiv \mathcal{F}_{SDG}(x, N)$ with $y \equiv Nx = N\lambda^2/Q^2$, one gets for $N \to \infty$ at fixed $y$

$$\tilde{\mathcal{G}}_{SDG}(y, N) \to \tilde{\mathcal{G}}_{SDG}(y, \infty) \equiv \check{\mathcal{G}}_{SDG}(y);$$

(7.2)

and one can derive from the result of [1] in the DIS case that

$$\check{\mathcal{G}}_{SDG}(y) = -1 + \exp(-y) - \frac{1}{2} y \exp(-y) - \frac{1}{2} y \Gamma(0, y) + \frac{1}{2} y^2 \Gamma(0, y).$$

(7.3)

Thus for $N \to \infty$

$$B[\ln F_2(Q^2, N)/\ln Q^2_{SDG}(u, N)] = 4C_F \beta_0 \frac{1}{\pi u} \exp(u \ln N) \int_0^\infty \frac{dy}{y} \check{\mathcal{G}}_{SDG}(y) \exp(-u \ln y),$$

(7.4)

which has also to be subtracted

$$B[\ln F_2(Q^2, N)/\ln Q^2_{SDG}(u, N)] = 4C_F \beta_0 \frac{1}{\pi u} \exp(u \ln N) \int_0^\infty \frac{dy}{y} \check{\mathcal{G}}_{SDG}(y) \exp(-u \ln y) + \Gamma_{SDG}(u),$$

(7.5)

where $\Gamma_{SDG}(0) = 1$ is finite. One can show that

$$\frac{\Gamma_{SDG}(u)}{u} = \int_0^\infty \frac{dy}{y} [\check{\nu}(y) + 1] \exp(-u \ln y),$$

(7.6)

where $\nu(y)$ is the (universal) virtual contribution [1] to the massive gluon characteristic function for space-like processes

$$\nu(y) = -\int_0^1 \frac{dz}{z-y} \ln \frac{z}{y}. $$

(7.7)

It follows that

$$\Gamma_{SDG}(u) = \left(\frac{\pi u}{\sin \pi u}\right)^2 \frac{1}{(1-u)(1-u/2)},$$

(7.8)

The above mentionned fact that $\nu(y)$ is universal suggests that $\Gamma_{SDG}(u)$ might also be universal for all space-like processes.

I note that the constant terms of the SDG result are obtained by setting $N = 1$ in eq. (7.5). Thus, if one wants to select as input an arbitrary subset of constant terms to be included into a new asymptotic Sudakov exponent $S_{\tilde{\alpha}_s}(Q^2, N)$, one should simply change the subtraction function $\Gamma_{SDG}(u)$ in eq. (7.5), namely define a new input $B[\ln F_2(Q^2, N)/\ln Q^2_{SDG}(u)]$ by

---

6 The normalization is half that of [1].
\[ B[\ln F_2(Q^2, N)/\ln Q^2_{SDG}(u, N)]_{SDG}(u, N) = 4C_F \frac{1}{\beta_0} \frac{\sin \pi u}{\pi u} \times \left[ \exp(u \ln N) \int_0^\infty \frac{dy}{y} \tilde{G}_{SDG}(y) \exp(-u \ln y) + \frac{\Gamma_{SDG}^{new}(u)}{u} \right], \]  

(7.9)

where \( \Gamma_{SDG}^{new}(u) \) (still with \( \Gamma_{SDG}^{new}(0) = 1 \)) takes into account the new selected set of constant terms to be included together with the \( \ln N \)'s into the new Sudakov exponent. The latter (output) should thus be given by (see eq.(6.7))

\[ B[S_{\tilde{a}a}](u, N) = B[A_{new}^S](u, N) \frac{\sin \pi u}{\pi u} \left[ \exp(u \ln N) \int_0^\infty \frac{dy}{y} \tilde{G}_{new}(y) \exp(-u \ln y) - \frac{\pi \tilde{G}(\infty)}{\sin \pi u} \right], \]  

(7.10)

where \( \tilde{G}(\infty) = -1 \) does not change (it determines the leading logs). Since there are no \( O(1/N) \) terms, one can now identify

\[ 4C_F B[S_{\tilde{a}a}](u, N) \equiv B[\ln F_2(Q^2, N)/\ln Q^2_{SDG}(u, N)], \]  

(7.11)

which yields the two master equations

\[ B[A_{new}^S](u) \int_0^\infty \frac{dy}{y} \tilde{G}_{new}(y) \exp(-u \ln y) = \frac{1}{\beta_0} \int_0^\infty \frac{dy}{y} \tilde{G}_{SDG}(y) \exp(-u \ln y), \]  

(7.12)

and

\[ B[A_{new}^S](u) = \frac{1}{\beta_0} \left( \frac{\sin \pi u}{\pi u} \right) \left( \frac{\Gamma_{SDG}^{new}(u)}{\tilde{G}(\infty)} \right) = \frac{1}{\beta_0} \left( \frac{\sin \pi u}{\pi u} \right) \Gamma_{SDG}^{new}(u). \]  

(7.13)

It also follows from eq.(7.3) that

\[ \int_0^\infty \frac{dy}{y} \tilde{G}_{SDG}(y) \exp(-u \ln y) = \Gamma(-u) \frac{1}{2} \left( \frac{1}{1-u} + \frac{1}{1-u/2} \right). \]  

(7.14)

Eq.(7.12) and (7.13) allow to determine both the Sudakov characteristic function \( G_{new}(y) \) and the Borel transform of the associated Sudakov effective coupling \( B[A_{new}^S](u) \) corresponding to a given input set of exponentiated constant terms which fix the subtraction function \( \Gamma_{SDG}^{new}(u) \). It is interesting that \( B[A_{new}^S](u) \) is given entirely by the subtraction term. Once \( G_{new}(y) \) is determined, one can derive the corresponding Sudakov distribution function \( G_{new}(\epsilon) \) using eq.(6.3) (\( G_{new}(\epsilon) \) is essentially the time-like discontinuity of \( G_{new}(y) \)), thus proving at large \( N_f \) the statement made below eq.(5.9).

In particular, one can easily obtain the result in the two extreme cases where all constant terms are included in the Sudakov exponent (the input is then given by eq.(7.5)), and that where none are. But these two cases are not interesting for the IR finite coupling...
approach, since one finds that $B[A_{S}^{new}](u)$ then contains renormalons. In particular, if one tries to include all constant terms one gets using eq.(7.8)

$$B[A_{S}^{glu}](u) = \frac{1}{\beta_0} \left( \frac{n u}{\sin \pi u} \right) \frac{1}{(1 - u)(1 - u/2)}.$$  \hspace{1cm} (7.15)

Despite the presence of IR renormalon poles, this result might be of interest since this coupling is expected to be universal for all space-like processes according to the remark made below eq.(7.8). However, the corresponding result for the Sudakov characteristic function

$$\int_{0}^{\infty} \frac{dy}{y} \hat{\varphi}_{all}(y) \exp(-u \ln y) = -\frac{1}{u} \frac{1}{\Gamma(1 + u)} \left( 1 - \frac{3u}{4} \right)$$ \hspace{1cm} (7.16)

shows that the latter actually does not exist, the inverse gamma function having no Mellin representation!

On the other hand, if no constant terms are included in the Sudakov exponent, one finds that the corresponding Sudakov distribution function $G_{logs}(\epsilon)$ is cut-off in the infrared

$$G_{logs}(\epsilon) = -\theta(\epsilon - 1),$$ \hspace{1cm} (7.17)

so that IR renormalons can show up only through the divergent expansion of the Sudakov effective coupling itself. Indeed one gets

$$B[A_{S}^{logs}](u) = \frac{1}{\beta_0} \frac{1}{\Gamma(1 + u)} \frac{1}{2} \left( \frac{1}{1 - u} + \frac{1}{1 - u/2} \right),$$ \hspace{1cm} (7.18)

which has IR renormalon poles at $u = 1, 2$.

More interesting is the simplest possible ansatz, namely $I_{SDG}^{simple}(u) \equiv 1$, which gives $B[A_{S}^{simple}](u) = \frac{1}{\beta_0} \frac{\sin \pi u}{\pi u}$ and corresponds to $A_{S}^{simple}(k^2)$ (eq.(3.7)), which is already IR finite at the perturbative level. However, the first two zeroes at $u = 1$ and $u = 2$ lead to two log-enhanced power corrections at large $N$ in the IR finite coupling framework: the right hand side of eq.(7.14) has double IR renormalon poles at $u = 1, 2$, resulting in $O(y \ln^2 y)$ and $O(y^2 \ln^2 y)$ terms at small $y$ in $\hat{G}_{simple}(y)$. In fact the Sudakov distribution function is now given by

$$G_{simple}(\epsilon) = \hat{G}_{SDG}(\epsilon),$$ \hspace{1cm} (7.19)

and eq.(7.3) indeed yields as $\epsilon \to 0$

$$\hat{G}_{SDG}(\epsilon) \approx \frac{1}{2} [\epsilon (\ln \epsilon + \gamma_E - 3) + \epsilon^2 (-\ln \epsilon - \gamma_E + 1) + \frac{5}{12} \epsilon^3 - \frac{1}{18} \epsilon^4 + ...].$$ \hspace{1cm} (7.20)

There is thus a discrepancy with the IR renormalon expectation, in agreement with the argument of section 4, and the question arises whether such an ansatz would violate the OPE at large $N$ already at the level of the two leading $(N/Q^2$ and $N^2/Q^4$) power corrections. Given the well-known \cite{21, 22} intricacies of the latter, the question is perhaps not yet settled.
Applying these results to the standard case, where $G(\epsilon)$ is known (eq.(5.6)) and fixes $\dot{G}(y)$ from eq.(6.3), and using eq.(7.12), one can also rederive the result eq.(8.4), where the first two zeroes are cancelled. Here the discrepancy with the IR renormalon prediction concerns only the higher order power corrections ($N^3/Q^6$, $N^4/Q^8$, ...), which are expected \[22, 10\] to be absent at large $N$ in the exponent.

7.2 Drell-Yan case: emergence of an IR finite perturbative effective coupling

The analogue of eq.(8.1 ) for the scaling violation of the short distance Drell-Yan cross-section is

$$\frac{d \ln \sigma_{DY}(Q^2, N)}{d \ln Q^2} = 4C_F \left( H_{DY}(Q^2) + S_{DY}(Q^2, N) \right) + O(1/N), \quad (7.21)$$

with

$$S_{DY}(Q^2, N) = \int_0^1 dz \frac{2z^{N-1} - 1}{1 - z} A_{S,DY}(1 - z)^2 Q^2]. \quad (7.22)$$

$S_{DY}(Q^2, N)$ can again be written as a renormalon integral

$$S_{DY}(Q^2, N) = \int_0^{Q^2} dk^2 \frac{k^2}{k^2} F_{DY}(k^2/Q^2, N) A_{S,DY}(k^2), \quad (7.23)$$

where $A_{S,DY}(k^2)$ is given in eq.(4.5), and the Sudakov distribution function

$$F_{DY}(k^2/Q^2, N) = (1 - k/Q)^{N-1} - 1, \quad (7.24)$$

involves both \[13\] even and odd powers of $k$ at small $k$. Taking the scaling limit $N \to \infty$ with $\epsilon_{DY} = Nk/Q$ fixed one thus get the analogue of eq.(8.4)

$$G_{DY}(\epsilon_{DY}) = \exp(-\epsilon_{DY}) - 1. \quad (7.25)$$

Now the work of \[3\] for eikonal cross sections suggests to exponentiate a new set of constant terms (for any $N_f$) using

$$S_{DY}^{new}(Q^2, N) = \int_0^{Q^2} \frac{dk^2}{k^2} G_{DY}^{new}(Nk/Q) A_{S,DY}^{new}(k^2), \quad (7.26)$$

with (we deal with the log-derivative of the Drell-Yan cross-section)

$$G_{DY}^{new}(\epsilon_{DY}) = 2 \frac{d}{d \ln Q^2} [K_0(2Nk/Q) + \ln(Nk/Q) + \gamma_E], \quad (7.27)$$

i.e.

$$G_{DY}^{new}(\epsilon_{DY}) = - \left[ 1 + x \frac{dK_0}{dx}(x = 2\epsilon_{DY}) \right], \quad (7.28)$$

where $K_0(x)$ is the modified Bessel function of the second kind.\footnote{But in the Drell-Yan case the analogue of eq.(12) is $G_{DY}^{new}(\epsilon_{DY}) \to -1$ for $\epsilon_{DY} \to \infty$, consistently with the large $\epsilon_{DY}$ limit of eq.(7.24).}
\[ B[S_{DY,as}](u, N) = B[S_{DY}](u) \exp(2u \ln N) \int_0^\infty \frac{d\epsilon}{\epsilon} G_{DY}(\epsilon) \exp(-2u \ln \epsilon), \quad (7.29) \]

and at large \( N_f \) we get, instead of eq. (7.12)

\[ B[A_{new S, DY}](u) \int_0^\infty \frac{d\epsilon}{\epsilon} G_{DY}^{\text{new}}(\epsilon) \exp(-2u \ln \epsilon) = \frac{1}{\beta_0} \frac{\sin \pi u}{\pi u} \int_0^\infty \frac{dy}{y} \hat{G}_{SDG, DY}(y) \exp(-u \ln y). \quad (7.30) \]

On the other hand from the large \( N_f \) calculation of [14] one gets the \( N \)-dependent part of the large \( N \) Borel transform

\[ \frac{\sin \pi u}{\pi u} \int_0^\infty \frac{dy}{y} \hat{G}_{SDG, DY}(y) \exp(-u \ln y) = -\frac{1}{u} \frac{\Gamma(1-u)}{\Gamma(1+u)}, \quad (7.31) \]

whereas eq. (7.28) yields

\[ \int_0^\infty \frac{d\epsilon}{\epsilon} G_{DY}^{\text{new}}(\epsilon) \exp(-2u \ln \epsilon) = -u \Gamma(-u)^2. \quad (7.32) \]

From eq. (7.30) one thus derive the large \( N_f \) result

\[ B[A_{new S, DY}](u) = \frac{1}{\beta_0} \frac{1}{\Gamma(1+u)\Gamma(1-u)} = \frac{1}{\beta_0} \frac{\sin \pi u}{\pi u}, \quad (7.33) \]

i.e. \( A_{new S, DY}^{\text{new}}(k^2) \) is nothing but the effective coupling \( A_{S, \text{simple}}^{\text{simple}}(k^2) \) of eq. (3.7). The latter, as we have seen, does have an IR finite fixed point, and no non-perturbative modification is a priori necessary in this case! As expected from the general discussion in section 4, the zeroes in \( B[A_{new S, DY}](u) \) lead to large \( N \) logarithmically enhanced power corrections in the IR finite coupling framework, at variance with the IR renormalon expectation. Indeed the new Sudakov distribution function has logarithmically enhanced contributions for \( \epsilon_{DY} \to 0 \)

\[ G_{DY}^{\text{new}}(\epsilon_{DY}) = \epsilon_{DY}^2 (2 \ln \epsilon_{DY} + 2\gamma_E - 1) + \epsilon_{DY}^3 (\ln \epsilon_{DY} + \gamma_E - \frac{5}{4}) + \mathcal{O}(\epsilon_{DY}^6 \ln \epsilon_{DY}). \quad (7.34) \]

The fact that this new Sudakov distribution function implies a new Sudakov effective coupling is of course one of the main point of the present paper.

8. Reconciling the IR renormalon and the IR finite coupling approaches:
   a large \( N_f \) ansatz for a ‘universal’ Sudakov effective coupling

I next turn to the question raised in the beginning of section 5 how to reconcile the IR renormalon and IR finite coupling approaches to power corrections. Actually, I should first stress it is not yet clear whether the two approaches should be necessarily reconciled. For instance, in the DIS case, it could be that the OPE at large \( N \) is consistent with the existence of two leading log-enhanced power corrections, as predicted by the \( A_{S, \text{simple}}^{\text{simple}}(k^2) \) ansatz

\[ \text{Recall also eq. (6.5).} \]
(eq. (3.7)), or with the existence of higher order power corrections \(N^3/Q^6, N^4/Q^8,\ldots\) at large \(N\) in the exponent (see section 7). If this turns out to be the case, the IR finite coupling approach would be consistent with the OPE (and at odds with the IR renormalon prediction) with the ‘simple’ ansatz of eq. (3.7), or eventually with the standard result of eq. (3.4). Similarly, the ‘simple’ ansatz of eq. (3.7) might be the correct one in the Drell-Yan case, arising from a ‘natural’ exponentiation of some \(O(N^0)\) terms, despite it also contradicts the standard IR renormalon expectation. It is interesting to note at this point that eq. (7.13) indicates that zeroes in \(B[A_{S}^{\text{new}}](u)\) arise from two distinct sources: either the ‘universal’ \(\sin \pi u / \pi u\) factor (simple zeroes at integer \(u\) can come only from there), or the ‘arbitrary’ \(\Gamma_{SDG}^{\text{new}}(u)\) subtraction term (zeroes at half-integer \(u\) can come only from there).

The previous discussion suggests that zeroes coming from the ‘universal’ \(\sin \pi u / \pi u\) factor need not be necessarily removed in the IR finite coupling approach, at the difference of the more ‘artificial’ zeroes (such as those occurring in the standard Drell-Yan case eq. (4.6)) coming from the subtraction term.

Notwithstanding the above remarks, I shall adopt in this section the attitude that the IR renormalon and IR finite coupling approaches to power corrections should always be made consistent with one another. For this purpose, one must remove all \(^9\) zeroes from \(B[A_{S}^{\text{new}}](u)\). The mathematically simplest \(^{10}\) ansatz suggested by eq. (7.13) is to choose

\[
\Gamma_{SDG}^{\text{new}}(u) = \Gamma(1 - u)
\]

which yields

\[
B[A_{S}^{\text{new}}](u) = \frac{1}{\beta_0} \left( \frac{\sin \pi u}{\pi u} \right) \Gamma(1 - u) = \frac{1}{\beta_0} \frac{1}{\Gamma(1 + u)}. \tag{8.2}
\]

It is interesting to compare this ansatz with the result obtained by eliminating the half-integer zeroes from the Drell-Yan standard result eq. (4.6). There the simplest ansatz is to define

\[
B[A_{S,\text{DY}}^{\text{new}}](u) = \frac{\Gamma(1/2 - u)}{\pi^{1/2}} B[A_{S,\text{DY}}](u), \tag{8.3}
\]

which yields again the ‘universal’ ansatz eq. (8.2).

Given an ansatz for \(B[A_{S}^{\text{new}}](u)\), eq. (7.12) then determines \(\hat{G}_{\text{new}}(y)\), hence the Sudakov distribution function \(G_{\text{new}}(\epsilon)\). In particular, in the DIS case one gets assuming the ansatz eq. (8.2)

\[
G_{\text{new}}(\epsilon) = -\frac{\epsilon(1 + \epsilon)}{2}, \quad 0 \leq \epsilon \leq 1
\]

\[
= -1, \quad \epsilon \geq 1, \tag{8.4}
\]

\(^9\) Except those eventually also present in \(\int_0^\infty \frac{dy}{y} \hat{G}_{SDG}(y) \exp(-u \ln y)\), such as the one at \(u = 4/3\) (see eq. (7.14)).

\(^{10}\) Apart from the obvious choice \(\Gamma_{SDG}^{\text{new}}(u) = \pi u / \sin \pi u\), where the Sudakov effective coupling is just the one-loop coupling, and is thus of little interest (having a Landau pole) for the IR finite coupling approach.
where $\epsilon = Nk^2/Q^2$. Thus all power corrections beyond the two leading ones are indeed absent from the Sudakov exponent, in agreement with the renormalon argument.

**Infrared behavior:** although I have not been able to find a closed form analytic expression for the Sudakov effective coupling corresponding to eq.(8.2)

$$\tilde{A}_{\text{new}}(k^2) = \frac{1}{\beta_0} \int_0^\infty du \, \exp \left( -u \ln \frac{k^2}{\Lambda^2} \right) \frac{1}{\Gamma(1+u)},$$

(8.5)

there is very strong numerical evidence that it blows up very fast (but remains positive) for $k^2 \to 0$:

$$\tilde{A}_{\text{new}}(k^2) \simeq \frac{1}{\beta_0} \exp \left( \frac{\Lambda^2}{k^2} \right).$$

(8.6)

Assuming that eq.(8.6) is correct, one can again speculate that there exists a non-perturbative modification $\delta A_{\text{S,NP}}(k^2)$ of the coupling at very small momenta which will eventually turn the too strong IR singularity into a non-perturbative, but finite IR fixed point. A simple ansatz for $\delta A_{\text{S,NP}}(k^2)$, localized in the IR region, can be constructed in analogy with eq.(3.13). Expanding the righthand side of eq.(8.6) for large $k^2$

$$\frac{1}{\beta_0} \exp \left( \frac{\Lambda^2}{k^2} \right) = \frac{1}{\beta_0} \left( 1 + \frac{\Lambda^2}{2k^2} + \frac{1}{2!} \frac{\Lambda^4}{k^4} + \frac{1}{3!} \frac{\Lambda^6}{k^6} + ... \right),$$

(8.7)

suggests to subtract term by term using

$$\delta A_{\text{S,NP}}^{\text{new}}(k^2) = -\frac{1}{\beta_0} \left[ \frac{\Lambda^2}{k^2} \exp \left( -\frac{k^2}{\Lambda^2} \right) + \frac{1}{2} \frac{\Lambda^4}{k^4} \exp \left( -\frac{k^4}{\Lambda^4} \right) + \frac{1}{3!} \frac{\Lambda^6}{k^6} \exp \left( -\frac{k^6}{\Lambda^6} \right) + ... \right].$$

(8.8)

Then for $k^2 \to 0$ one gets

$$\delta A_{\text{S,NP}}^{\text{new}}(k^2) \simeq -\frac{1}{\beta_0} \left( \frac{\Lambda^2}{k^2} + \frac{1}{2} \frac{\Lambda^4}{k^4} + \frac{1}{3!} \frac{\Lambda^6}{k^6} + ... \right) + \frac{1}{\beta_0} \left( 1 + \frac{1}{2} + \frac{1}{3!} + ... \right) + \mathcal{O}(k^2/\Lambda^2).$$

(8.9)

Thus defining

$$A_{\text{S,NP}}^{\text{new}}(k^2) = A_{\text{S}}^{\text{new}}(k^2) + \delta A_{\text{S,NP}}^{\text{new}}(k^2)$$

(8.10)

yields indeed for $k^2 \to 0$ a finite and positive IR fixed point

$$A_{\text{S,NP}}^{\text{new}}(k^2) \simeq \frac{1 + \epsilon}{\beta_0} + \mathcal{O}(k^2).$$

(8.11)

**9. Resummation of the ‘left-over’ constant terms at large $N_f$, and a strategy for phenomenology at finite $N_f$**

For a given choice of the exponentiated constant terms to be included in the Sudakov exponent $S_{\text{new}}(Q^2, N)$ (eq.(5.9)), there remains corresponding ‘left-over’ constant terms $H_{\text{new}}(Q^2)$ (see also eq.(3.1))
\[
\frac{d \ln F_2(Q^2, N)}{d \ln Q^2} = 4C_F (H_{\text{new}}(Q^2) + S_{\text{new}}(Q^2, N)) + \mathcal{O}(1/N),
\]
with
\[
H_{\text{new}}(Q^2) = \Delta \gamma_{i0}^{\text{new}} a_s(Q^2) + \Delta \gamma_{i0}^{\text{new}} a_s^2(Q^2) + \ldots,
\]
such that
\[
\gamma_i^{\text{new}} + \Delta \gamma_i^{\text{new}} = g_i,
\]
where the \(g_i\)'s are the full constant terms at large \(N\)

\[
\frac{d \ln F_2(Q^2, N)}{d \ln Q^2} = 4C_F \left[ (\gamma_{01} L + g_0) a_s(Q^2) + (\gamma_{12} L^2 + \gamma_{11} L + g_1) a_s^2(Q^2) + \ldots \right] + \mathcal{O}(1/N).
\]
From eq.\((7.5)\) and \((7.9)\) it follows that the Borel transform of \(H_{\text{new}}(Q^2)\) at large \(N_f\) is given by (in the \(V\)-scheme)
\[
B[H_{\text{new}}(Q^2)](u) = \frac{1}{\beta_0} \left( \frac{\sin \pi u}{\pi u} \right) \Gamma_{\text{SDG}}(u) - \Gamma_{\text{SDG}}^{\text{new}}(u) \frac{u}{u}.
\]
It is natural to try resumming the \(H_{\text{new}}(Q^2)\) series with a renormalon integral representation
\[
H_{\text{new}}(Q^2) = \int_0^\infty \frac{dk^2}{k^2} G_{0,\text{new}}(k^2/Q^2) A_S^{\text{new}}(k^2),
\]
using the same effective coupling \(A_S^{\text{new}}(k^2)\) as in the Sudakov exponent, which defines the ‘left-over constants’ distribution function \(G_{0,\text{new}}(k^2/Q^2)\). Eq.\((9.6)\) yields a representation for the Borel transform
\[
B[H_{\text{new}}(Q^2)](u) = B[A_S^{\text{new}}](u) \int_0^\infty \frac{dx}{x} G_{0,\text{new}}(x) \exp(-u \ln x).
\]
Comparing eq.\((9.5)\) with eq.\((9.7)\) allows to determine \(G_{0,\text{new}}(k^2/Q^2)\). For instance, in the case of the ‘new Sudakov effective coupling’ eq.\((8.2)\), one finds that \((x = k^2/Q^2)\)

\[
G_{0,\text{new}}(x) = 2x \left[ (2x - 1) \exp(1/x) \Gamma(0, 1/x) - 2x(\ln x - \gamma_E) - (\ln x - \gamma_E + 2) \right] - \theta(x - 1),
\]
and one can check that \(G_{0,\text{new}}(x) = \mathcal{O}(\ln x/x)\) for \(x \to \infty\), while for \(x \to 0\)
\[
G_{0,\text{new}}(x) \simeq -2x(\ln x - \gamma_E + 2).
\]
Eq.\((9.3)\) predicts in the IR finite coupling approach a log-enhanced \(\mathcal{O}(\ln Q^2/Q^2)\) \(N\)-independent power correction arising from the left-over constant terms. This is consistent with the IR renormalon expectation as can be checked by setting \(N = 1\) in eq.\((7.5)\).
Application to phenomenology at finite $N_f$: I suggest to use the same large $N_f$ Sudakov distribution function $G_{new}(\epsilon)$ and 'left-over constants' distribution function $G_{0,new}(x)$ at finite $N_f$, while adjusting the corresponding Sudakov effective coupling $A_{S,NP}^{new}(k^2)$ to the finite $N_f$ situation. For instance, assuming the ansatz eq.(8.3) is the correct one at large $N_f$, one would use the finite $N_f$ resummation formula

$$
\frac{d\ln F_{2}(Q^2, N_f)}{d\ln Q^2} = 4C_F \int_0^{Q^2} \frac{dk^2}{k^2} G_{new}(Nk^2/Q^2) A_{S}^{new}(k^2) + 4C_F \int_0^{\infty} \frac{dk^2}{k^2} G_{0,new}(k^2/Q^2) A_{S,C}^{new}(k^2) + 4C_F \Delta H_{new}(Q^2) + O(1/N),
$$

(9.10)

where the ($N_f$-independent) distribution functions $G_{new}(Nk^2/Q^2)$ and $G_{0,new}(k^2/Q^2)$ are the same as in eqs.(8.4) and (9.8), while $A_{S}^{new}(k^2)$ can be determined order by order in perturbation theory at finite $N_f$ in a standard way, matching eq.(9.10) with the perturbative expansion of the left hand side (only the $N$-dependent terms, i.e. the first line, are necessary for this purpose). I note that at finite $N_f$ there may still remain 'relic' constant terms contained in the function $\Delta H_{new}(Q^2)$, not accounted for by the two renormalon integrals on the right hand side of eq.(9.10), such that (see eq.(9.11))

$$
H_{new}(Q^2) = \int_0^{\infty} \frac{dk^2}{k^2} G_{0,new}(k^2/Q^2) A_{S}^{new}(k^2) + \Delta H_{new}(Q^2).
$$

(9.11)

The expansion of $\Delta H_{new}(Q^2)$ is expected to be better convergent than that of $H_{new}(Q^2)$, and can be dealt with in renormalization scheme invariant way by using the method of effective charges [1].

The right hand side of eq.(9.10) is entirely perturbative. To regularize the renormalons integrals in the infrared and deal with non-perturbative phenomena (such as power corrections), one can now assume a non-perturbative modification of the coupling at finite $N_f$, analogous to eq.(8.10). Its form should be determined phenomenologically. For instance, in the spirit of the method of effective charges, one could try a two-point Padé interpolation between the (known in perturbation theory) weak coupling UV form, and the strong coupling IR form of the non-perturbative beta function of the effective coupling $A_{S,NP}^{new}(k^2)$, assuming a non-perturbative IR fixed point (its value $A_{S,NP}^{new}(k^2 = 0)$ is then one of the free parameters to be fitted). $A_{S,NP}^{new}(k^2)$ could then be determined at all scales by integrating its own Gell-Mann-Low like renormalization group equation, and reported into eq.(9.10). This method (to be investigated in the near future) does not require the introduction of any IR cut-off, and represents an interesting alternative (especially if universality of the Sudakov effective coupling does hold), with a simple physical interpretation, to the shape function approach, as already remarked in section 2.

10. Conclusions

The IR finite coupling approach has many attractive features in the context of Sudakov resummation. On the phenomenological side, we have seen it is able to yield a very simple,
yet non-trivial prediction for the end tail of the Sudakov peak, which goes beyond short
distance physics. The parametrization of the low momentum piece of the Sudakov effective
coupling represents also an attractive alternative to the shape function approach. On the
more theoretical side, I have pointed out the rather unusual occurrence of a strongly IR
divergent, but nevertheless causal, effective coupling, in an all order, but still perturba-
tive, framework. The essential differences in the IR domain between the standard Sudakov
anomalous dimensions (such as cusp), which exhibit completely unphysical behavior at
low scales, and their specific combinations eq.(3.2) and (4.3) called here ‘Sudakov effective
couplings’, where these pathologies cancel out, has been stressed. Only the too strong IR
divergence localized at the origin requires the introduction of a non-perturbative modifica-
tion. I feel this case is basically different from the more familiar one of unphysical Landau
singularities at finite distances, and makes more plausible the (non-perturbative) IR finite
coupling hypothesis. The simplicity of the proposed non-perturbative large \(N_f\) ansatzes
represents, I believe, further encouragement to support this speculation. Moreover it was
found that the ‘natural’ resummation of a set of \(\mathcal{O}(N^0)\) terms within the procedure of [3]
for eikonal cross sections remarkably leads in the Drell-Yan case at large \(N_f\) to the simplest
difficulties within perturbation theory itself, with no a-priori need of a non-perturbative modification.

The IR renormalon and IR finite coupling approaches to power corrections are po-
tentially in conflict with each other. We have seen it is possible to make consistent these
two approaches, by appropriate use of the arbitrariness of exponentiated constant terms
in Sudakov resummation. This freedom in redefining the Sudakov exponent bears some
connection with alternative forms of Sudakov resummation previously discussed in the litera-
ture [14, 23]. The latter however involve an infrared cut-off\(^\text{11}\), so that IR renormalons
do not appear through integration over arbitrarily small momenta, but rather through
divergences of the redefined anomalous dimensions (or, as in [4], the divergence of some
initial condition sitting outside the exponent). This is in sharp contrast with the present
proposal, where integration at arbitrarily small momenta is essential, and makes sense
through the notion of IR finite effective coupling. The mathematically simplest solution
to resolve the above mentioned conflict leads to the proposal at large \(N_f\) of a ‘universal’
non-perturbative ansatz for the Sudakov effective coupling (eq.(8.10)).

Alternatively, it could be that the correct resolution of the conflict favors one approach
over the other, and that the answer provided by e.g. the ‘simple’ IR finite ansatz eq.(3.7)
turns out to be the correct one (as suggested by its natural occurrence in the Drell-Yan
process), despite being at odds with the IR renormalon prediction. These issues should
be resolved by a better understanding of OPE at large \(N\) in the DIS case. Another
related question to be settled is the application of the method of [3] to DIS, which should
determine the corresponding ‘natural’ Sudakov distribution function and effective coupling.
Even if the ‘simple’, purely perturbative IR finite effective coupling ansatz turns out to
represent an acceptable answer at the perturbative level at large \(N_f\), there may be extra
non-perturbative effects, similar to those which must occur in the case of the ‘universal’

\(^\text{11}\)The ‘purely logarithmic’ Sudakov exponent which excludes all constant terms gives an example at large
\(N_f\) of an alternative form of Sudakov resummation with a sharp IR cut-off, as we have seen in section 7.
coupling eq. (8.10), which could also take the form of a non-perturbative modification of the effective coupling at low momenta. Whatever the correct choice of the Sudakov distribution function turns out to be, it remains for future phenomenological work to determine the corresponding form of the Sudakov effective coupling at finite $N_f$ for each process, and test for eventual deviations from universality.

Acknowledgments

I thank Yu.L. Dokshitzer, J-P. Lansberg, G. Marchesini, G.P. Salam and G. Sterman for useful discussions. I am indebted to M. Beneke for reminding me the difficulties of the OPE at large $x$, and for pointing out the connection of the present findings with alternative forms of Sudakov resummation discussed in the literature.

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