Chern insulator with large Chern numbers. Chiral Majorana fermion liquid

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Abstract

In the framework of Hofstadter’s approach we provide a detailed analysis of a realization of exotic topological states such as the Chern insulator (CI) with large Chern numbers. In a transverse homogeneous magnetic field a one-particle spectrum of fermions transforms to an intricate spectrum with a fine topological structure of the subbands. In a weak magnetic field \( H \) for a rational magnetic flux, a topological phase with a large Chern number is realized near the half filling. There is an abnormal behavior of the Hall conductance \( \sigma_{xy} \simeq \frac{e^2}{2H} \). At half-filling, the number of chiral Majorana fermion edge states increases sharply forming a new state, called the chiral Majorana fermion liquid.

1. Introduction

In the case of a rational magnetic flux through the unit cell \( \phi = p/q \) (\( p \) and \( q \) are coprime integers), a new physics of the 2D model systems subjected to a perpendicular magnetic field arises from its two magnetic and lattice competing scales [1]. A strong periodic potential leads to an intricate spectrum of the topological system, the problem brings to play the commensurability of these two scales [2]. For the particular case, when the magnetic flux assumes either the continued fraction approximations towards the golden mean or the golden mean itself, the multifractal properties of Harper’s equation, the winding numbers were discussed in [3]. The problem of realization of the Hofstadter Hamiltonian with ultracold atoms in optical lattices is addressed, for example, in [4, 5].

Wiegmann and Zabrodin pointed out [6] that Hofstadter’s model responsible is closely related to the quantum group \( U_q(sl_2) \), the model Hamiltonian is determined in terms of the generators of the quantum group (see also the approach of Faddeev and Kashaev [7]). Using the exact solution of the Hofstadter model [6, 7] for the semi-classical limit at \( p = 1 \) and \( q \to \infty \), the behavior of the wavefunction was calculated at zero energy (for the center of the spectrum) \(|\psi|^{2} = G_{j} = \frac{2}{\text{det}(\gamma - 2i/\beta q)} \) (\( j \) is the site of the lattice) [8]. Near the edge \( j = 0 \), the finite size correction is given by \(|\psi_{2j+1}|^{2} = Q(j)G_{2j+1} \), with \( Q(0) = \frac{5}{4} \), \( Q(1) = \frac{50}{16} \), \( Q(2) = \frac{81\pi}{256} \). The authors of [8] noted that power-low behavior of \(|\psi|^{2} \) is critical and un-normalizable. For the golden mean flux, the wavefunction is multifractal and critical, has a clear self-similar branching structure. Harper et al [9] shown that the Hofstadter model converges to continuum Landau levels in the limit of small flux per plaquette (in the \( q \to \infty \) limit).

Filled \( r \) bands with the Chern numbers \( C_{\gamma} \) (1 \( \leq \gamma \leq r \) yield a Hall conductance

\[
\sigma_{xy} = \frac{e^2}{4}|\sigma_r|, \quad \sigma_r = \sum_{\gamma=1}^{r} C_{\gamma}.
\]

The relationship between the Hall conductance \( \sigma_{xy} \) and energy spectrum, the Chern numbers of isolated subbands are discussed in [10]. Dana et al [11] proposed that magnetic translational symmetry yields the diophantine equation \( p|\sigma_{1}| + 2pq = 2r \) (two deuces in the equation take into account spin degeneracy). The solution of the equation can be uniquely determined by imposing the condition \( |\sigma_{1}| \leq q \), that is realized in the model.

The dynamics of a charged particle in a magnetic field applied perpendicular to the plane of the lattice and to the electric field in the plane of the lattice is considered in [12]. A non-zero magnetic field crucially modifies this
dynamics of fermions along the edges. Exact results [6, 7, 13] and numerical calculations [14, 15] show that the behavior of 2D fermion systems in a transverse magnetic field does not depend on the symmetry of the lattice.

However strange as it may seem at first glance, the behavior of 2D fermions in a transverse magnetic field largely depends on the boundary conditions. We show that in the case of a rational magnetic flux the Hall conductivity has an anomalous behavior in a small magnetic field near the half-filling due to large number of chiral gapless modes localized in wide regions near boundaries. In the case of an irrational magnetic flux, the magnetic scale tends to $\infty$ (in reality, $q > N$, where $N$ is the size of the system), therefore it is necessary to take into account the boundary conditions. We show that the number of chiral modes is anomalously large, they form the chiral Majorana fermionic liquid state.

Non-trivial topological order in a band is indicated by non-zero Chern number. The topological insulating phase state is determined by the Chern number of all fermion bands below the Fermi energy. The Chern number is a topological invariant which can be easily defined for the $\gamma$-band isolated from all other bands by the formula

$$C_{\gamma} = \frac{1}{2\pi} \int_{\text{BZ}} \mathcal{B}_\gamma(k) \, d^2k$$

(1)

integrating the Berry curvature $\mathcal{B}_\gamma(k) = \nabla_k \times \mathcal{A}_\gamma(k)$ over the Brillouin zone (BZ). The Berry potential

$$\mathcal{A}_\gamma(k) = -i \langle u^\gamma(k) | \nabla_k | u^\gamma(k) \rangle$$

is defined in terms of the Bloch states $u^\gamma(k)$. If bands in the set are isolated, $\mathcal{G}_\gamma = \sum_{\gamma \in \Gamma} C_{\gamma}$.

2. The model

We will analyze a model of the 2D CI determined in the framework of Hofstadter’s approach [11]. In the presence of a transverse homogeneous magnetic field $H_{e}$, the model Hamiltonian is determined according to [11]

$$\mathcal{H} = \sum_{n,m} \left[ t^x(n,m) a_{n,m}^\dagger a_{n+1,m} + i t^y(n,m) a_{n,m}^\dagger a_{n,m+1} + H_{c} \right]$$

(2)

where $a_{n,m}^\dagger$ and $a_{n,m}$ are the spinless fermion operators on a site $(n, m)$ with the usual anticommutation relations. The Hamiltonian describes the nearest-neighbor hoppings along the $x$-direction with the magnitude $t$ ($t^x(n, m) = t$) and along the $y$-direction with the hopping integral $t^y(n, m) = \exp[2i\pi(n - 1)\phi]$, where $\phi = \Phi/\Phi_0$ is a magnetic flux $\Phi = H$ through the unit cell measured in quantum flux unit $\Phi_0 = h/e$, a homogeneous field $H$ is represented by its vector potential $A = H e_x$. The case $t = 1$ corresponds to the original Hofstadter model [11]. We focus on the 2D system in the form of a hollow cylinder, with periodic boundary conditions along the $y$-direction and size $N$ along the $x$-direction.

3. Topological structure of the spectrum

3.1. The flux $\phi = \frac{1}{2}$

We consider an evolution of the one-particle spectrum of the Hamiltonian (2) in a transverse homogeneous magnetic field with rational magnetic flux $\phi = \frac{2}{3}$. In the case $q > 2$ the magnetic field breaks a time reversal symmetry [16, 17], leads to topological states of fermions [18]. Our starting point is $q = 3$, when the one-particle spectrum consists of three topologically non-trivial subbands with the Chern numbers $\{1, -2, 1\}$ (see in figures 1(a), (b)). A detailed calculation of the Chern numbers for a rational flux is given in [19, 20]. In an external field the spectrum of fermions is intricate, the band is split into the topological subbands $\gamma$ with non-trivial topological index $C_{\gamma}$. The structure and number of subbands in the fine structure of the spectrum depend on the value of the magnetic flux. The excitation spectrum of the sample in stripe geometry consists of the chiral edge modes, which are indicated in red color (see in figures 1(a), (b)).

They are localized near the boundaries of the sample, the amplitudes of the wave functions decrease exponentially with receding from the boundaries. The energies of the edge modes are determined by the wave vector component that is parallel to the boundary, they intersect each other at the Dirac point, merge with the bulk states. The edge modes with different chirality are localized at the different boundaries, they are associated with Majorana operators that belong to the boundaries. The chiral gapless edge modes do exist in the gap if the Chern number of isolated bands located below the gap is non-zero. The phase state of fermions is characterized by a chiral current along the boundary of the 2D system, the direction of the current is determined by the sign of the Chern number.

The structure of the spectrum does not depend on the value of $t$, the gaps close at $t = 0$ and open for $t > 0$. The values of gaps are equal to $\Delta = \frac{1}{2}\lvert 1 + t - \sqrt{1 + t^2 - 2t/3} \rvert$. The value of $\Delta$ is shown in figure 1(c). The topology of the spectrum is not changed at the point $t = 0$. The gaps close at $t = 0$, but the spectrum of the
excitations has not the Dirac-type in the $k_x$-direction ($k$ is a wave vector). Let us consider the spectrum of the excitations in the case of a weak coupling along the $x$-direction at small values of $t$ (see in figure 1(b)). In the $t \to 0$ limit the fermions form the non-interacting chains along the $y$-direction (see in figure 1(d)). The energies of fermion excitations in the chains intersect at the energies equal to $\pm 1$ for $q = 3$ (see in figure 1(b)). Due to the hoppings of fermions between the nearest chains the gaps open at the energies equal to $\pm 1$. The tunneling of fermions between chains dominates in the regions of the crossings of energies of the nearest chains, since the conservation of the energy and the momentum of fermions are realized automatically.

The Hamiltonian, which takes into account the low energy excitations of Majorana fermions, is determined as follows

$$\mathcal{H}_\text{eff}^l = it \sum_{n=1}^{N-1} g(n, k_{n,n+1}) f(n + 1, k_{n,n+1}),$$

where $k_{1,2} = \frac{2\pi}{l}$ or $\frac{2\pi}{3}$, $k_{3,4} = 0$ or $\pi$, $k_{5,1} = \frac{4\pi}{3}$ or $\frac{2\pi}{3}$, $1$, $2$, $3$ numerate the chains for $q = 3$ (the sets values of $k$ for the energies $-1$ or $1$, see in figure 1(b)). Majorana operators $g(n, k)$, $f(n, k)$ describe the chiral modes (with opposite velocities) of the chain $n$, Majorana state operators $\gamma(j)$ defined by the algebra $[\gamma(j), \gamma(i)] = 2\delta_{j,i}$ and $\gamma(j)^\dagger = \gamma(j)^\dagger$.

Chiral Majorana fermions from different nearest chains are paired together in the Hamiltonian (3), Majorana fermions form the chain of non-interacting dimers along the $x$-direction as noted in figure 2. According to Kitaev [21] the Hamiltonian (3) is determined as $\mathcal{H}_\text{eff} = t \sum_{n=1}^{N-1} (2c_{n,k}^\dagger \epsilon_{n,k} - 1)$, where $c_{n,k}^\dagger$, $\epsilon_{n,k}$ are the Fermi operators constructed from the two Majorana bound states located at the nearest $y$-chains chains (see in figure 2). The Kitaev chain Hamiltonian [21] describes topological superconductivity in a chain of spinless fermions.

The operators $f\left(1, \frac{2\pi}{3}\right)$ and $g\left(N, \frac{2\pi}{3}\right)$ are free Majorana fermions at the energy equal to $1$, they remain unpaired and form the edge states. The gapless edge modes existing in the energy range of about $-1$ are associated with the Majorana operators $g\left(1, \frac{4\pi}{3}\right)$ and $f\left(N, \frac{4\pi}{3}\right)$ (see in figure 1(b)). They are localized near the

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Figure 1. Energy levels calculated on a cylinder with open boundary conditions along $y$-direction for $t = 1$ (a), $t = \frac{1}{10}$ (b) at $\phi \approx \frac{1}{3}$. The value of gaps that separated the band on the isolated topological subbands, as a functions of $t$ (red line) and its asymptotic at small $t$ $\Delta \approx |2t(1 - \frac{1+1+1}{\pi})|$ (dotted line) (c). The hopping integrals are determined for $q = 3$, the fermion chains are indicated in red color (d).
The chiral gapless edge modes do exist in the gaps, connecting the lower and upper fermion subbands (see in figures 1(a), (b)).

3.2. The flux $\phi = 1_5$, rational fluxes

The spectrum of the excitations contains five subbands and four gaps, which are determined by two values of $D_1$ and $D_2$ (see in figure 3(a)), with different structure of edge modes in the regions of the gaps. At the energies equal to $\pm 2 \cos \frac{\phi}{2}$ the edge states with two chiral modes $f \left( \frac{1}{2}, \frac{2}{2} \right)$ and $g \left( N, \frac{2}{2} \right)$ into the gap $D_1$ are described by the Hamiltonian (3).

Figure 2. Pairing of Majorana fermions in the Hamiltonian (3) and forming chiral edge states $f \left( 1, k \right)$ and $g \left( N, k \right)$.

The Hamiltonian (4) describes the low energy spectrum $\mathcal{H}_{\text{eff}}^{\parallel} = \sum_{n=1}^{N-2} g(n, k_{n,n+2}) f(n + 2, k_{n,n+2})$, where in a weak coupling $\tau \simeq t^2$ or according to numerical calculations $\tau = \left( 1 + \frac{1}{t^2} \right)^2$, $k_{n,n+2} = \frac{3\pi}{5}$ or $\frac{8\pi}{5}$, $k_{2,4} = \frac{3\pi}{5}$ or $\frac{8\pi}{5}$, $k_{3,5} = \frac{9\pi}{5}$ or $\frac{4\pi}{5}$, $k_{4,6} = \frac{7\pi}{5}$ or $\frac{2\pi}{5}$, $k_{5,2} = \pi$ or $0$ for the energies $\pm 2 \cos \frac{\phi}{2}$. The Hamiltonian (4)
describes the dimers that contain Majorana fermions located at the next-nearest neighbor sites in the $x$-chains. Majorana fermions $f \{1, \frac{2}{3}, \frac{5}{3}\}$ and $g \{2, \frac{5}{3}\}$ and $g(N, 0)$ and $f(N - 1, 0)$ are free Majorana fermions, they remain unpaired and form edge states into the gap $\Delta_3$ at the energy $2 \cos \frac{2\pi}{N}$ (see in figure 3(c)).

When the flux is rational, the Hamiltonian (2) admits a periodic representation on a magnetic unit cell comprising $q$ cells along the $x$-direction. We use Bloch’s theorem and calculate the Chern numbers of the isolated subbands. The topological picture of model's spectrum is defined by the Chern numbers of each subband $C_{\nu}$. The Chern numbers \{1, 1, -4, 1, 1, 1\} for $p = 1$ and \{-2, 1, 2, 1, -2\} for $p = 2$ determine the topological structure of the spectrum at $q = 5$. The following relations follow from symmetry and numerical calculations: $\sum_{\nu} C_{\nu} = 0$ and $C_{\nu}(\phi) = -C_{\nu}(-\phi)$. The Chern numbers are defined for isolated subband for a rational flux $\phi = \frac{p}{q}$, $C_{\nu}\left(\frac{q}{q}\right) = -C_{\nu}\left(\frac{q - p}{q}\right)$. $p$ and $q$ are integers, $q$ enlarges a magnetic unit cell and thus defines the number of subbands $\gamma = 1, \ldots, q$. From numerical calculations and exact solution [6], it follows that for $p = 1$ and odd $q = 2s + 1$ all subbands are isolated by the gaps, therefore $C_{\nu}$ is defined for arbitrary $\gamma$. Useful relations are obtained for sets values of rational fluxes: $|C_{\nu + 1}\left(\frac{1}{2\nu + 1}\right)| = 2s, |C_{\nu + 1}\left(\frac{2}{4\nu + 1}\right)| = 2s$ and $|C_{\nu + 1}\left(\frac{s}{2\nu + 1}\right)| = 2.$ The Chern number of the middle subband for given $q = 2s + 1$ is varied from the maximal value $q - 1$ at $p = 1$ to the minimal value $2$ at $p = (q - 1)/2, 2 \leq |C_{\nu}| \leq q - 1$. The Chern number $|\sigma|$ corresponds to the total number of edges modes with respect to their chirality, even for groups of intersecting bands, when the Fermi energy lies in the rth gap from the bottom. The calculating the total numbers of the edge modes, intersecting the gaps, we obtain the diophantine equation $p|\sigma| = 2r + s$ (where $s$ is an integer).

The phase state of CI with a large Chern number and corresponding Hall conductivity $\sigma_{xy} = (q - 1)^2/2N \approx \frac{q}{4N}$ is realized in a weak magnetic field in a narrow region near the half filling when filling is $\frac{1}{2} \left(1 \pm \frac{1}{N}\right)$Gapless chiral edge modes are localized in a wide region near the boundaries, from $1$ to $\frac{N}{2}$. $\left(N - \frac{q - s}{2}\right)$ lattice cites. There are two different separated regions of the phase states of fermions: itinerant fermions in the bulk at $\frac{q - 1}{2} < n < -\frac{q - s}{2}$ and localized chiral gapless edge Majorana state at $1 \leq n \leq \frac{q - 1}{2}$ and $N - \frac{q - s}{2} \leq n \leq N$. In the limit of a weak magnetic field, the region of the existence of the localized chiral gapless edge Majorana state has a large size $\sim \frac{q}{N}$.

Figure 4, which shows the calculations of a hyperfine structure of the middle subband, illustrates what was said above. In figure 4(a) the center of the spectrum is calculated using periodic boundary conditions for flux $\phi = \frac{1}{20N}$ at $t = 1$, when the fermion states are determined by Bloch's function. Well-defined subbands separated by the gaps, here the wave vector directed along the $y$-direction and $k_y$ is perpendicular to the plane of the figure. The similar calculation of spectrum for the sample with $N = 1005$ and open boundary conditions is shown in figure 4(b). Chiral edge states of Majorana fermions and band states of itinerant fermions form the spectrum of the system. In quasi-irrational limit at $N = 177$, when $q > N$, the spectrum contains predominantly chiral edge modes figure 4(c).

3.3. Chiral Majorana fermion liquid, irrational flux $\phi = \frac{1}{\sqrt{q}}$

Numerical calculations of the excitation spectrum for a rational flux in Hofstadter strips with open boundary conditions were obtained for samples of sizes $N > q$, more precisely $\frac{N}{q}$ is integer. When calculating the spectrum for an irrational flux which is approximated by rational numbers $\phi \approx \frac{p}{q}$, we assume that $N < q$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{A low energy spectrum (a hyperfine structure of middle subband), calculated for flux $\phi = \frac{1}{20N}$ at $t = 1$: with periodic boundary conditions (a), with open boundary conditions along the $y$-direction at $N = 1003$ (b) and at $N = 177$ (c), $k$ is the wave vector along the $y$-direction.}
\end{figure}
Consider the excitation spectrum, calculated at $t = 1$, which can be approximated by the set
\[
\left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{23}{45}, \frac{29}{82}, \frac{35}{99}, \frac{134}{379}, \frac{169}{478}, \frac{204}{577}, \ldots \right\}.
\]
analyzing the results of calculations presented in figure 5. Three bands separated by wide gaps, connected by the chiral gapless edge modes, are topological ones with the Chern numbers $\{-1, 2, 1\}$, as in the case $q = 3$ (see in figure 1(a)). The number of gapless edge modes, indicated in red lines, determines the values of the Chern numbers of the isolated subbands. The number of these gapless edge modes is conserved which is confirmed by numerical calculations of the spectrum for a set of rational fluxes that correspond to an irrational flux. For an irrational flux the structure of the fermion spectrum differs from that in the case of a rational flux. The gaps in the spectrum are determined by crossings of the energies of non-interacting chains.

The tunneling of fermions between chain opens the gaps in the spectrum near the crossing points, in the gaps chiral localized modes are formed. The corresponding low energy effective Hamiltonian has the following form.

Consider the excitation spectrum, calculated at $t = 1$, which can be approximated by the set
\[
\left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{23}{45}, \frac{29}{82}, \frac{35}{99}, \frac{134}{379}, \frac{169}{478}, \frac{204}{577}, \ldots \right\}.
\]

Figure 5. Energy levels calculated for the flux $\phi = \frac{169}{478}$ at $t = 1$ and $N = 450$: a total spectrum (a), a fine structure of the middle subband of the spectrum (b), a hyperfine structure of a fine structure of the middle subband (c) and (d).

Figure 6. The center of the spectrum, calculated for the flux $\phi = \frac{134}{379}$, $N = 100$ at $t = 0.01$ (a) and $t = 1$ (b), the dotted blue lines correspond to the crossing points of the energies of non-interacting chains.
where the constant of effective tunneling between chains $\tau(\delta(n))$ depends on the crossing point or distance between these chains $\delta(n)$ (in (5) is summed over the crossing points).

An irrational flux breaks a translation invariance in the system, Majorana fermions form isolated dimers (not chains of dimers, as is the case for rational flux) and free gapless Majorana fermions. We can not use a perturbation theory for calculation of the value $\tau(\delta(n))$ because the effective hopping of fermions in the chains decreases with $N$. According to numerical calculations (see illustration in figure 6(a)) $\tau(\delta(n))$ weakly depends on $n$ in the case of weak interaction between chains. A hyperfine structure of the middle subband with different scaling is shown in figures 5(c), (d). Each subband has a fine structure, a middle subband is shown in figures 5(b)–(d). The state of itinerant fermions are connected by the chiral edge modes. The density of the edge states increases when the energy approaches zero, which corresponds to half filling. At the half filling two component fermion liquid contains both itinerant and chiral Majorana fermion states. An anomalously large Hall conductivity is due to chiral Majorana localized states.

4. Conclusions

We studied the behavior of 2D spinless fermions in the framework of Hofstadter’s model, focusing on the half filling. A topological structure of the Hofstadter model is calculated for the CI state with a rational magnetic flux through the unit cell. In the limit of weak flux per plaquette we obtained abnormal behavior of a Hall conductivity, when the magnetic field is weak, $\sigma_{yx} \approx \frac{e}{2\pi}$, that is a result of the large Chern numbers $|C| \approx \frac{1}{\phi}$ of topological subband at zero energy. Such behavior of the system is determined by two different liquid states of spinless fermions at half filling, namely, the liquid of itinerant fermions and chiral Majorana fermion liquid. The chiral Majorana fermion liquid determines the anomalous behavior of the Hall conductivity in a weak magnetic field.

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