Electrodynamics of moving magnetoelectric media: variational approach

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Abstract

Recently, Feigel has predicted a new effect in magnetoelectric media. The theoretical evaluation of this effect requires a careful analysis of a dynamics of the moving magnetoelectric medium and, in particular, the derivation of the energy-momentum of the electromagnetic field in such a medium. Then, one can proceed with the study of the wave propagation in this medium and derive the mechanical quantities such as the energy, the momentum, and their fluxes and the corresponding forces. In this paper, we develop a consistent general-relativistic variational approach to the moving dielectric and magnetic medium with and without magnetoelectric properties. The old experiments in which the light pressure was measured in fluids are reanalysed in our new framework.

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I. INTRODUCTION

The discussion of the electrodynamics of moving media has a long history. At present, the general structure of classical electrodynamics appears to be well established. In particular, in the generally covariant pre-metric approach to electrodynamics [1, 2, 3, 4, 5, 6], the electric charge and the magnetic flux conservation laws manifest themselves in the Maxwell equations for the excitation $H = (\mathcal{D}, \mathcal{H})$ and the field strength $F = (E, B)$, namely $dH = J, \ dF = 0$. These equations should be supplemented by a constitutive law $H = H(F)$. The latter relation contains the crucial information about the underlying physical continuum (i.e., spacetime and/or material medium), in particular, about the spacetime metric. Mathematically, this constitutive law arises either from a suitable phenomenological theory of a medium or from the electromagnetic field Lagrangian. It can be a nonlinear or even nonlocal relation between the electromagnetic excitation and the field strength. The constitutive law is called a spacetime relation if it applies to spacetime (“the vacuum”) itself.

Among many physical applications of classical electrodynamics, the problem of the interaction of the electromagnetic field with matter occupies a central position. The fundamental question, which arises in this context, is about the definition of the energy and momentum in the possibly moving medium. The discussion of the energy-momentum tensor in macroscopic electrodynamics is quite old. The beginning of this dispute dates back to Minkowski [7], Einstein and Laub [8], and Abraham [9]. Nevertheless, up to now the question was not settled and there is an on-going exchange of conflicting opinions concerning the validity of the Minkowski versus the Abraham energy-momentum tensor, see, e.g., the review [10]. Even experiments were not quite able to make a definite and decisive choice of electromagnetic energy and momentum in material media. A consistent solution of this problem has been proposed in [4, 11] (cf. also the earlier work [12]) in the context of a new axiomatic approach to electrodynamics.

Recently Feigel [13] has studied, theoretically, the dynamics of a dielectric magnetoelectric medium in an external electromagnetic field. He predicted that the contributions of the quantum vacuum waves (or “virtual photons”) could transfer a nontrivial momentum to matter. This prediction was made with the help of the non-relativistic formalism. In our opinion, a proper relativistic analysis is needed for a better understanding of the physics and of the viability of this phenomenon. Here we begin to reconsider this problem in a
covariant framework as developed earlier in [4, 11]. As a first step, we develop a variational approach to the description of the dynamics of a moving magnetoelectric medium. The corresponding energy-momentum of matter plus electromagnetic field that arises can be derived straightforwardly in this formalism from the variation of the total action with respect to the spacetime metric.

II. PRELIMINARIES: THE ESSENCE OF THE FEIGEL EFFECT

The Feigel effect [13] can be described in simple terms as follows: Let us consider an isotropic homogeneous medium with the electric and magnetic constants $\varepsilon, \mu$. Electromagnetic waves are propagating in such a medium absolutely symmetrically, with the Fresnel equation describing the unique light cone. This is easily derived from the constitutive relations $D = \varepsilon\varepsilon_0 E$ and $H = (\mu\mu_0)^{-1}B$.

However, if a medium is placed in crossed constant external electric and magnetic fields, then it acquires magnetoelectric properties. As a result, we have the anisotropic magnetoelectric medium with $\varepsilon, \mu$, plus the magnetoelectric matrix $\beta$ (determined by the external fields) which modifies the constitutive relations to $D = \varepsilon\varepsilon_0 E + \beta \cdot B$ and $H = (\mu\mu_0)^{-1}B - \beta^T \cdot E$; here $^T$ denotes the transposed matrix.

Accordingly, the wave propagation in such a medium also becomes anisotropic and birefringent, with the wave covectors now belonging to two light cones. Applying this to vacuum waves (or, perhaps, better to say to the “vacuum fluctuations” or “virtual photons”) propagating in the magnetoelectric body, Feigel [13] computed the total momentum carried by these waves and concluded that it is nontrivial. In accordance with this derivation, a body should move with a small but non-negligible velocity. Earlier the Feigel process was discussed in [14, 15, 16, 17, 18].

In order to evaluate the possible Feigel effect, it is necessary to substitute the “vacuum waves” into the energy-momentum tensor. This paper is devoted to the derivation of the latter in the framework of a variational approach.
III. CONSTITUTIVE RELATION

Within the axiomatics of the premetric generally covariant framework [4], the projection technique is used to define the electric and magnetic phenomena in an arbitrarily moving medium. As in [4], we assume that the spacetime is foliated into spatial slices with time $\sigma$ and transverse vector field $n$.

When applying the projection technique to the 2-forms of the electromagnetic excitation $H$ and the electromagnetic field strength $F$, we obtain the three-dimensional objects: the magnetic and electric excitations $\mathcal{H}$ and $\mathcal{D}$ as longitudinal and transversal parts of $H$ and, similarly, electric and magnetic fields $E$ and $B$ as longitudinal and transversal parts of $F$, respectively, namely

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D} \quad \text{and} \quad F = E \wedge d\sigma + B. \quad (3.1)$$

This foliation is called the laboratory foliation, with the coordinate time variable $\sigma$ labeling the slices of this foliation.

The spacetime metric $g$ introduces the scalar product in the tangent space and defines the line element. With respect to the laboratory foliation coframe it reads $(a, b, \ldots = 1, 2, 3)$

$$ds^2 = N^2 d\sigma^2 + g_{ab} dx^a dx^b = N^2 d\sigma^2 - (3) g_{ab} dx^a dx^b. \quad (3.2)$$

Here $N^2 = g(n, n)$ is the length square of the foliation vector field $n$, and $dx^a = dx^a - n^a d\sigma$ is the transversal 3-covector basis, in accordance with the definitions above. The 3-metric $(3) g_{ab}$ is the positive definite Riemannian metric on the spatial 3-dimensional slices corresponding to fixed values of the time $\sigma$. This metric defines the 3-dimensional Hodge duality operator $\star$.

The constitutive relation which links the electromagnetic field strength to the electromagnetic excitation, $H = H(F)$, can be nonlocal and nonlinear, in general. Here we will confine our attention to the local and linear constitutive relation.

Then, if we write the the excitation 2-form in terms of its components in a local coordinate system $\{x^i\}$, $(\mathcal{H}, \mathcal{D}) = H = H_{ij} dx^i \wedge dx^j / 2$ (with $i, j, \cdots = 0, 1, 2, 3$), the local and linear constitutive relation means that the components of the excitation are local linear functions of the components of the field strength $(E, B) = F = F_{ij} dx^i \wedge dx^j / 2$:

$$H = \kappa(F), \quad H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} F_{kl}. \quad (3.3)$$
Along with the original constitutive $\kappa$-tensor, it is convenient to introduce an alternative representation of the constitutive tensor:

$$\chi_{ijkl} := \frac{1}{2} \epsilon_{ijmn} \kappa_{mn}^{\phantom{mn}kl}. \quad (3.4)$$

Performing a $(1 + 3)$-decomposition of covariant electrodynamics, as described above, we can write $H$ and $F$ as column 6-vectors with the components built from the magnetic and electric excitation 3-vectors $\mathcal{H}_a, \mathcal{D}^a$ and the electric and magnetic field strengths $E_a, B^a$, respectively. Then the linear spacetime relation (3.3) reads:

$$\begin{pmatrix} \mathcal{H}_a \\ \mathcal{D}^a \end{pmatrix} = \begin{pmatrix} \mathcal{C}_a^b & \mathcal{B}_{ba} \\ \mathcal{A}^{ba} & \mathcal{D}_b^a \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix}. \quad (3.5)$$

Here the constitutive tensor is conveniently represented by the $6 \times 6$-matrix

$$\kappa_I^K = \begin{pmatrix} \mathcal{C}_a^b & \mathcal{B}_{ba} \\ \mathcal{A}^{ba} & \mathcal{D}_b^a \end{pmatrix}, \quad \chi^{I}_K = \begin{pmatrix} \mathcal{B}_{ab} & \mathcal{D}_b^a \\ \mathcal{C}_b^{\phantom{b}a} & \mathcal{A}_b^a \end{pmatrix}. \quad (3.6)$$

Assuming that the skewon and the axion pieces are absent, we find that the constitutive matrices satisfy $\mathcal{A}^{ab} = \mathcal{A}_b^a, \mathcal{B}_{ab} = \mathcal{B}_{ba},$ and $\mathcal{D}_b^a = \mathcal{C}_b^a, \text{ with } \mathcal{C}_a^a = 0.$

The dynamics of a material medium is encoded in the structure of another foliation $(\tau, u)$ which is determined by the four-vector field of the velocity $u$ of matter and the proper time coordinate $\tau$. Accordingly, we have to formulate the constitutive law with respect to this, so called material foliation. As a first step, we observe that the relation between the two coframe bases, namely those of the laboratory foliation $(d\sigma, dx^a)$ and of the material foliation $(d\tau, dx^a)$ is as follows [4]:

$$\begin{pmatrix} d\sigma \\ dx^a \end{pmatrix} = \begin{pmatrix} \gamma c/N & v_b/(cN) \\ \gamma v^a & \delta^a_b \end{pmatrix} \begin{pmatrix} d\tau \\ dx^b \end{pmatrix}. \quad (3.7)$$

Here, for the relative velocity 3-vector, we introduced the notation

$$v^a := \frac{c}{N} \left( \frac{u^a}{u(\sigma)} - n^a \right), \text{ with } \gamma := \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (3.8)$$

Substituting (3.7) into (3.2), we find for the line element in terms of the new variables

$$ds^2 = c^2 d\tau^2 - \tilde{g}_{ab} dx^a dx^b, \text{ where } \tilde{g}_{ab} = (3)_{ab} - \frac{1}{c^2} v_a v_b. \quad (3.9)$$
The metric $\hat{g}_{ab}$ of the material foliation has the inverse

$$\hat{g}^{ab} = (3) g^{ab} + \frac{\gamma^2}{c^2} v^a v^b, \quad (3.10)$$

where $(3) g^{ab}$ is the inverse of $(3) g_{ab}$. For the determinant one finds $(\det \hat{g}_{ab}) = (\det (3) g_{ab}) \gamma^{-2}$.

We will denote the 3-dimensional Hodge star defined by the metric $\hat{g}_{ab}$ as $\hat{\star}$.

Given the transformation (3.7) between the 1-form bases of the two foliations (laboratory and material), it is straightforward to calculate the components of the constitutive tensor with respect to the moving reference frame from the constitutive tensor which describes the matter at rest. The corresponding explicit general transformation formulas for the 3-dimensional matrices $A, B, C$ can be found in [4]. A nontrivial outcome of this formalism is that the isotropic moving medium is described by the constitutive relation based on the so-called optical metric (first introduced by Gordon [19]). We will use this fact in the subsequent derivations.

IV. RELATIVISTIC FLUID

The earlier work includes that for the ideal fluids [20, 21, 22] and the case of the fluids in electromagnetic fields was discussed in [23, 24, 25, 26]. Recently, a new study was reported in [27, 28].

An ideal fluid which consists of structure-less elements (particles) is characterized in the Eulerian approach by the fluid 4-velocity $u^i$, the internal energy density $\rho$, the particle density $\nu$, the entropy density $s$, and the identity (Lin) coordinate $X$. Normally, it is assumed that the motion of a fluid is such that the number of particles is constant and that the entropy and the identity of the elements is conserved. In other words,

$$\nabla_i (\nu u^i) = 0, \quad (4.1)$$

$$u^i \partial_i s = 0, \quad (4.2)$$

$$u^i \partial_i X = 0. \quad (4.3)$$

By the conservation of the entropy only the reversible processes are allowed. In a variational approach to continuous media, these assumptions are considered as constraints imposed on the dynamics of the fluid by means of Lagrange multipliers. Then the fluid Lagrangian
reads $V_{\text{mat}} = L_{\text{mat}} \sqrt{-g} d\sigma \wedge \hat{\epsilon}$ with $(\hat{\epsilon} = \text{spatial volume 3-form})$

$$L_{\text{mat}} = -\frac{\rho(\nu, s)}{c} + \Lambda_0(u^i u_i - c^2) + \Lambda_1 \nabla_i(\nu u^i) + \Lambda_2 u^i \partial_i s + \Lambda_3 u^i \partial_i X.$$ \hspace{1cm} (4.4)

The Lagrange multipliers $\Lambda_1, \Lambda_2, \Lambda_3$ impose the constraints [4.1]-[4.3] on the dynamics of the fluid, whereas $\Lambda_0$ provides the standard normalization condition for the 4-velocity

$$g_{ij} u^i u^j = c^2.$$ \hspace{1cm} (4.5)

For the description of the thermodynamical properties of the fluid, the usual thermodynamical law ("Gibbs relation") is used,

$$T \, ds = d(\rho/\nu) + p\, d(1/\nu),$$ \hspace{1cm} (4.6)

where $T$ is the temperature and $p$ the pressure.

When the medium possesses dielectric and magnetic properties, then we have to add the Lagrangian of the electromagnetic field

$$V_{\text{em}} = -\frac{1}{2} H \wedge F = -\frac{1}{2} d\sigma \wedge (F^a B_a - D \wedge E)$$

$$= -\frac{1}{2} d\sigma \wedge \hat{\epsilon} \left( A^{ab} E_a E_b + B_{ab} B^a B^b - 2C^a b E_a B^b \right).$$ \hspace{1cm} (4.7)

At first, let us consider the moving isotropic dielectric fluid without magnetoelectric properties, that is, $C^a b = 0$. Then the electromagnetic field Lagrangian reads $V_{\text{em}} = L_{\text{em}} \sqrt{-g} d\sigma \wedge \hat{\epsilon}$ with

$$L_{\text{em}} = -\frac{\lambda_0}{4\mu} g_{ij}^{\text{opt}} g_{kl}^{\text{opt}} F_{ik} F_{jl}.$$ \hspace{1cm} (4.8)

Here $\lambda_0$ is the vacuum admittance, and

$$g_{ij}^{\text{opt}} = g_{ij} - \frac{1 - n^2}{c^2} u^i u^j$$ \hspace{1cm} (4.9)

is the optical metric of Gordon [19].

\textbf{A. Equations of motion without a magnetoelectric effect}

At first, we consider the case when the fluid does not have magnetoelectric properties: $C^a b = 0$. We also assume that there are no free charges and currents in the fluid.

The dynamics of the system (fluid+field) is thus described by the total Lagrangian $V = V_{\text{mat}} + V_{\text{em}}$. The corresponding equations of motion are then derived from the variational
principle for $V$ with the independent variables $u^i, \nu, s, X, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3,$ and $A_i$ (the covector of the electromagnetic potential).

Variation with respect to the Lagrange multipliers $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3$ yields the constraints (4.1)-(4.3) and (4.5), whereas variation of $V$ with respect to $\nu, s, X$ yields, respectively,

\[
\nu u^i \partial_i \Lambda_1 + \frac{\rho + p}{c} \mu = 0, \tag{4.10}
\]

\[
\nabla_i (\Lambda_2 u^i) + \frac{T \nu}{c} = 0, \tag{4.11}
\]

\[
\nabla_i (\Lambda_3 u^i) = 0. \tag{4.12}
\]

In order to derive these results, we used the thermodynamical law (4.6), from which one has $\partial \rho/\partial s = \nu T$ and $\partial \rho/\partial \nu = (\rho + p)/\nu$.

Finally, the variation of $V$ with respect to the 4-velocity yields

\[
2 \Lambda_0 u_i - \nu \partial_i \Lambda_1 + \Lambda_2 \partial_i s + \Lambda_3 \partial_i X + \frac{\lambda_0}{\mu c^2} (1 - n^2) \mu c^2 F_{ik} F^{jk} u_j = 0. \tag{4.13}
\]

The last term emerges from $V_{\text{em}}$. Contracting the last equation with $u^i$ and making use of (4.1)-(4.3), (4.5), and (4.10), we find

\[
2 \Lambda_0 = - \frac{1}{c^2} \left( \frac{\rho + p}{c} + \frac{\lambda_0}{\mu c^2} (1 - n^2) \mu c^2 F_{ik} u^k F^{il} u_l \right). \tag{4.14}
\]

The variation with respect to $A_i$ gives the Maxwell equations for the moving fluid

\[
\nabla_j \left( g_{\text{opt}}^{jk} g_{\text{opt}}^{il} F_{kl} \right) = 0. \tag{4.15}
\]

When the fluid has free currents and charges, we have to add the interaction term $A_i J^i$ to the electromagnetic field Lagrangian $L_{\text{em}}$, and then the right-hand side of (4.15) picks up a nontrivial current $J^i$.

### B. Energy-momentum tensor

The above consideration is not only relativistically covariant, but actually generally covariant. The metric $g_{ij}$ above describes an arbitrary curved spacetime. The invariance of the total Lagrangian $L$ under general coordinate transformations (using the standard Noether machinery) yields the conservation of the energy-momentum of the system (fluid+field). We obtain the energy-momentum tensor as usual from the variation of the total Lagrangian with
respect to the metric, i.e.,

\[ T_{ij} := 2c \frac{\delta \left[ \sqrt{-g} (L_{\text{mat}} + L_{\text{em}}) \right]}{\sqrt{-g} \delta g^{ij}}. \]  

(4.16)

The computation is straightforward, and it yields for (4.4) and (4.8)

\[
T_{ij} = -\delta_{ij} p + \frac{p + \rho}{c^2} u_i u^j + \frac{1 - n^2}{\mu_0 \mu c^4} u_i u^j \mathcal{F}_{mk} u^k \mathcal{F}_{ml} u_l \\
+ \frac{1}{\mu_0 \mu} \left[ - \mathcal{F}_{ik} \mathcal{F}^{jk} + \frac{1}{4} \delta_i^j \mathcal{F}_{kl} \mathcal{F}^{kl} \\
+ \frac{1 - n^2}{c^2} \left( \mathcal{F}_{ik} u^k \mathcal{F}^{jl} u_l - \frac{1}{2} \mathcal{F}_{mk} u^k \mathcal{F}^{ml} u_l \delta_i^j \right) \right].
\]  

(4.17)

In the course of its derivation, we made use of the results of the previous subsection, in particular of the equations (4.10) and (4.14). The first line in (4.17) originates from \( V_{\text{mat}} \), whereas the rest comes from \( V_{\text{em}} \). It is interesting though that it is possible to rearrange the contributions of the fluid and of the field in such a way that the total energy-momentum is recast into the nice form

\[
T_{ij} = -\delta_{ij} p_{\text{eff}} + \frac{p_{\text{eff}} + \rho_{\text{eff}}}{c^2} u_i u^j \\
+ \frac{1}{\mu_0 \mu} \left[ - \mathcal{F}_{ik} \mathcal{F}^{jk} + \frac{1}{4} \delta_i^j \mathcal{F}_{kl} \mathcal{F}^{kl} + \frac{1 - n^2}{c^2} \mathcal{F}_{ik} u^k \mathcal{F}^{jl} u_l \delta_i^j \right].
\]  

(4.18)

Here we introduced the effective pressure and the effective energy density by means of

\[
p_{\text{eff}} := p + \frac{1 - n^2}{2 \mu_0 c^2} \mathcal{F}_{ij} u^j \mathcal{F}^{ik} u_k, \tag{4.19}
\]

\[
\rho_{\text{eff}} := \rho + \frac{1 - n^2}{2 \mu_0 c^2} \mathcal{F}_{ij} u^j \mathcal{F}^{ik} u_k. \tag{4.20}
\]

General covariance underlies the conservation law of the total energy-momentum tensor

\[ \nabla_j T_{ij} = 0, \]  

(4.21)

which yields the equations of motion of the medium.

C. Extension to the magnetoelectric case

The above results all refer to the isotropic dielectric and magnetic medium in motion without magnetoelectric crossterms. The extension to the magnetoelectric case, see O’Dell
is straightforward. The electromagnetic field Lagrangian (4.8) is replaced by the more general one
\[ L_{\text{em}} = -\frac{\lambda_0}{4\mu} g_{ij} g_{kl} F_{ik} F_{jl} - \frac{1}{2c^2} \beta_{ij} \beta^{ij} F_{i}^{\prime} u^{i} \eta^{ijkl} F_{lm} u_{k}. \] (4.22)
Here the tensor $\beta_{ij}$ gives the relativistic generalization of the magnetoelectric matrix $\beta$. The totally antisymmetric Levi-Civita tensor is defined as usual by $\eta_{ijkl} = \varepsilon_{ijkl} / \sqrt{-g}$, with the permutation symbol $\varepsilon_{ijkl}$ that have the only nontrivial component, $\varepsilon_{0123} = 1$. The dimension $[\beta_{ij}] = [\lambda_0]$.

The tensor $\beta_{ij}$ is assumed to be traceless. If a nontrivial trace is included, this will describe an axion contribution. Indeed, the trace $\beta_{ij} = \delta_{ij} \tilde{\alpha}$, when substituted into (4.22), yields the term $\sim \tilde{\alpha} F_{i}^{\prime} u^{i} \eta^{ijkl} F_{lm} u_{k}$. Using the identity (which states that a totally antisymmetric tensor of the 5-th rank is zero in four dimensions) $2u^{[i} \eta^{jklm]} = u^{k} \eta^{jlm} + u^{l} \eta^{jkm} + u^{m} \eta^{jkl}$, we find $\tilde{\alpha} F_{i}^{\prime} u^{i} \eta^{ijkl} F_{lm} u_{k} = \frac{c^2}{4} \tilde{\alpha} \eta^{ijkl} F_{ij} F_{kl}$. Thus, indeed the trace $\tilde{\alpha}$ adds an axion contribution to the electromagnetic Lagrangian. We assume that the axion is absent in this paper.

Furthermore, it is easy to see that only a projection of the magnetoelectric matrix on the rest frame of the 4-velocity $u$ enters the Lagrangian (4.22), namely
\[ -\frac{1}{2c^2} \beta_{ij} \beta^{ij} F_{i}^{\prime} u^{i} \eta^{ijkl} F_{lm} u_{k} = -\frac{1}{2c^2} \tilde{\beta}_{ij} \beta_{ij}^{\prime} F_{i}^{\prime} u^{i} \eta^{ijkl} F_{lm} u_{k}, \] (4.23)
where
\[ \tilde{\beta}_{ij} = P_{i}^{k} P_{j}^{l} \beta_{kl}, \] (4.24)
with the projector defined by $P_{i}^{k} := \delta_{ik} - u^{i} u_{k} / c^2$. By construction, $u^{i} \tilde{\beta}_{ij} = 0$ and $u_{i} \tilde{\beta}_{ij} = 0$, which means that only the 3-dimensional transversal part of the magnetoelectric tensor contributes.

Computation of the electromagnetic excitation $H = -\partial V_{\text{em}} / \partial F$ is straightforward. In the (1+3)-decomposed form, this yields the constitutive relation (3.5), where the 3-dimensional matrices read explicitly:
\[ A^{ab} = -\varepsilon^{ab} - m_{e} \varepsilon_{ab}, \] (4.25)
\[ B_{ab} = (\mu^{-1})_{ab} + (\mu^{-1})_{ab}, \] (4.26)
\[ C^{a}_{b} = \gamma^{a}_{b} + m_{e} \gamma^{a}_{b}. \] (4.27)
Here the first terms on the right-hand sides describe the contributions of the isotropic moving
medium
\[\varepsilon^{ab} = \frac{\lambda \sqrt{(3)g}}{c(1 - v^2/c^2)} \left[ (3)g^{ab} \left( n - \frac{v^2}{nc^2} \right) + \frac{v^a v^b}{c^2} \left( \frac{1}{n} - n \right) \right], \tag{4.28}\]
\[\left( \mu^{-1} \right)^{ab} = \frac{\lambda c}{\sqrt{(3)g(1 - v^2/c^2)}} \left[ (3)g_{ab} \left( \frac{1}{n} - \frac{nv^2}{c^2} \right) + \frac{v^a v^b}{c^2} \left( n - \frac{1}{n} \right) \right], \tag{4.29}\]
\[\gamma^{ab} = \frac{\lambda}{1 - v^2/c^2} \left( n - \frac{1}{n} \right) (3)\eta^{c} \varepsilon_{eb} v^{c}. \tag{4.30}\]

Here \(\lambda \ := \sqrt{\varepsilon\varepsilon_0/\mu\mu_0}\) and \(n \ := \sqrt{\varepsilon\mu}\) is the refraction index of the medium. The 3-dimensional indices are raised and lowered with the help of the 3-dimensional spacetime metric \((3)g_{ab}\), see the line element (3.9).

The magnetoelectric contributions to the constitutive matrices read
\[\varepsilon^{ab} = -\frac{2}{1 - v^2/c^2} \beta^{(a}_c \varepsilon^{b)cd} v_d / c^2, \tag{4.31}\]
\[\left( \mu^{-1} \right)^{ab} = -\frac{2}{1 - v^2/c^2} \beta^{(a}_c \varepsilon^{b)cd} v^d, \tag{4.32}\]
\[\gamma^{ab} = \frac{1}{1 - v^2/c^2} \tilde{\beta}^{ab}. \tag{4.33}\]

Here we introduced the notation
\[\tilde{\beta}^{ab} := \beta^{ab} - \frac{1}{c^2} \varepsilon^{acd} \varepsilon_{bd} \varepsilon^{c} \varepsilon^{d} v^c v^d, \tag{4.34}\]
with the “doubly projected” magnetoelectric matrix
\[\beta^{a}_b := \left( \delta^{a}_c - \frac{1}{c^2} v^a v_c \right) \left( \delta^{b}_d - \frac{1}{c^2} v^b v^d \right) \tilde{\beta}^{cd}. \tag{4.35}\]

For the medium at rest (with \(v^a = 0\)), we straightforwardly verify that the above formulas reduce to the constitutive law
\[D^a = \varepsilon\varepsilon_0 E^a + \beta^a_b B^b, \tag{4.36}\]
\[H_a = -\beta^b_a E^b + \frac{1}{\mu\mu_0} B_a. \tag{4.37}\]

From these, the constitutive relations of the moving magnetoelectric medium (4.25)-(4.33) can be alternatively derived with the help of the transformation (3.7) that links the laboratory to the material foliations.

The new magnetoelectric term in the Lagrangian (4.22) will modify the equations of motion of the medium and the total energy-momentum of the system. Direct computation
yields:

\[ T^j_i = -\delta^j_i \rho_{\text{eff}} + \frac{p_{\text{eff}} + \rho_{\text{eff}}}{c^2} u_iu^j + \frac{2}{c} q_i(u^j) \]

\[ + \frac{1}{\mu_0 \mu} \left[ - F_{ik}F^{jk} + \frac{1}{4} \delta^j_i F_{kl}F^{kl} + \frac{1-n^2}{c^2} F_{ik}u^k F^{jl}u_l \right]. \tag{4.38} \]

Here the effective energy density and the effective pressure remain the same (4.19), (4.20), whereas the vector \( q_i \) is defined by

\[ q_i = \frac{1}{2} \beta^i_{i'} P^j_{i'} \eta^{i'}_{j} F_{lm}F_{i'}^{j} u^k. \tag{4.39} \]

Clearly, it is orthogonal to the velocity of the medium, \( u^i q_i = 0 \). The terms with the structure \( 2u_i(q_j)/c \) are well known in the models of relativistic fluids; they usually describe fluids with fluxes of energy. It is interesting that the magnetoelectric parameters introduce such an additional energy flux term in the total energy-momentum tensor.

At the beginning, we assumed that the axion is absent. However, it is instructive to check how it could contribute to the energy-momentum. For this, we relax for a moment the tracefree condition for the magnetoelectric tensor and allow for a nontrivial trace \( \beta^i_j = \delta^i_j \tilde{\alpha} \). This yields a contribution \( \sim \frac{1}{2} \tilde{\alpha} P^j_i \eta^{i'}_{j} F_{lm}F_{i'}^{j} u^k \) in (4.39). With the help of the identity 

\[ 2u_i[q_j]/c \equiv u_j \eta^{i'}_{j} F_{lm}F_{i'}^{j} u^k, \]

this is transformed into \( \frac{1}{2} \tilde{\alpha} P^j_i u_j \eta^{i'}_{j} F_{lm}F_{i'}^{j} u^k \). It vanishes because of the identity \( P^j_i u_j \equiv 0 \). In other words, this demonstrates that the axion does not contribute to the energy-momentum, in accordance with its general theory [4].

D. Explicit components of the energy-momentum

All the results of the previous two subsections are generally covariant, i.e., they are valid in an arbitrary curved spacetime with any metric \( g_{ij} \). Now, returning to a more narrow case actually discussed by Feigel, we will specialize to the Minkowski spacetime with \( g_{ij} = \text{diag}(c^2, -1, -1, -1) \).

In order to write down the separate components of the energy-momentum tensor, we first notice that \( u^0 = \gamma, u^a = \gamma v^a \), with the 3-velocity \( v^a \) and the usual relativistic (Lorentz) factor \( \gamma = 1/\sqrt{1 - v^2/c^2} \). Then, denoting \( X_i := F_{ij}u^j \), we find

\[ X_0 = -\gamma (vE), \quad X_a = \gamma (E + v \times B)_a. \tag{4.40} \]

Accordingly, we have the invariant \( \text{that enters (4.19), (4.20)} \)

\[ X_iX^i = -\gamma^2 \left[ E^2 - (vE)^2/c^2 + v^2B^2 - (vB)^2 + 2(E \cdot [v \times B]) \right]. \tag{4.41} \]
Here we denoted the scalar product \((vE) = (v \cdot E)\) and similarly \((vB)\).

After these preliminaries, we find, componentwise, the energy density

\[
T_0^0 = u = \gamma^2 \left[ \rho_{\text{eff}} + \frac{v^2}{c^2} p_{\text{eff}} + \frac{\varepsilon_0 (1 - n^2)}{\mu c^2} (vE)^2 \right] \\
+ \frac{1}{2\mu} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{2\gamma v^a}{c} q_a, \tag{4.42}
\]

the energy flux density (or Poynting vector)

\[
T_0^a = s^a = \gamma^2 \left[ (\rho_{\text{eff}} + p_{\text{eff}}) v + \frac{\varepsilon_0 (1 - n^2)}{\mu} (vE)(E + v \times B) \right]^a \\
+ \frac{1}{\mu\mu_0} [E \times B]^a - \gamma c \left( \delta^a_b - \frac{1}{c^2} v^a v_b \right) q^b, \tag{4.43}
\]

the momentum density

\[
T_a^0 = -p_a = -\gamma^2 \left[ \rho_{\text{eff}} + \frac{\varepsilon_0 (1 - n^2)}{\mu} (vE)(E + v \times B) \right]_a \\
+ \frac{\varepsilon_0}{\mu} [B \times E]_a + \gamma c \left( \delta^a_b - \frac{1}{c^2} v^a v_b \right) q^b, \tag{4.44}
\]

and the stress tensor

\[
T_a^b = S_a^b = -p_{\text{eff}} \left( \delta^b_a + \frac{\gamma^2}{c^2} v_a v^b \right) - \gamma^2 \rho_{\text{eff}} \frac{v_a v^b}{c^2} \\
+ \frac{\varepsilon_0}{\mu} \left( E_a E^b - \frac{1}{2} \delta^b_a E^2 \right) + \frac{1}{\mu\mu_0} \left( B_a B^b - \frac{1}{2} \delta^b_a B^2 \right) \\
- \frac{\varepsilon_0 (1 - n^2)}{\mu} \gamma^2 (E + v \times B)_a (E + v \times B)^b - 2\gamma q_a v^b. \tag{4.45}
\]

As we see, the vector \(q_a\) indeed induces an additional energy flux in (4.43) and adds a corresponding magnetoelectric term in the electromagnetic momentum density (4.44).

For completeness, let us also give the explicit expressions for the effective energy density and the effective pressure of the medium:

\[
\rho_{\text{eff}} = \rho - \frac{\varepsilon_0 (1 - n^2)}{2\mu} \gamma^2 \left[ E^2 - (vE)^2/c^2 + v^2 B^2 - (vB)^2 + 2 (E \cdot [v \times B]) \right], \tag{4.46}
\]

\[
p_{\text{eff}} = p - \frac{\varepsilon_0 (1 - n^2)}{2\mu} \gamma^2 \left[ E^2 - (vE)^2/c^2 + v^2 B^2 - (vB)^2 + 2 (E \cdot [v \times B]) \right]. \tag{4.47}
\]

The energy-momentum tensor obtained here is different from both the Minkowski and the Abraham energy-momentum tensors. It is thus necessary to check whether the new expression is consistent with the main experiments in phenomenological electrodynamics. As a first step, we consider usual matter without magnetoelectric properties.
V. EXPERIMENTS WITH LIGHT PRESSURE

In order to test how the formalism works, we analyse in this section the famous experiments of Richards and Jones [30, 31]. Their measurements have shown that the radiation pressure on a metallic plate immersed into a dielectric fluid is proportional to the refraction index \( n \) of the medium.

The problem of the propagation of a plane wave in a dielectric medium is solved exactly. Suppose a wave travels along the \( x \) axis, with the metal plate surface located at \( x = 0 \). For \( x < 0 \), we have dielectric fluid with the permeability and permittivity \( \varepsilon_1, \mu_1 \), and for \( x > 0 \) we have the metal characterized by \( \varepsilon_2, \mu_2 \) and by the electric conductivity \( \sigma \). Then the electric and magnetic fields that describe the plane waves normally incident on and reflected from the metal surface are (\( x \leq 0 \))

\[
E_y = a e^{-i\omega t} \left( e^{i\omega n_1 c x} + R e^{-i\omega n_1 c x} \right), \quad (5.1) \\
B_z = \frac{n_1}{c} a e^{-i\omega t} \left( e^{i\omega n_1 c x} - R e^{-i\omega n_1 c x} \right). \quad (5.2)
\]

The fields in the metal (\( x \geq 0 \)) read

\[
E_y = T a e^{-i\omega t} e^{i\omega n_2 c K x}, \quad (5.3) \\
B_z = \frac{n_2}{c} T K a e^{-i\omega t} e^{i\omega n_2 c K x}. \quad (5.4)
\]

Without losing generality, we assume that the electric field is directed along the \( y \) axis. We use the complex representation to simplify the formulas. Here the complex variable \( a \) describes the amplitude of the incident wave, whereas the complex reflection and transmission coefficients are denoted \( R \) and \( T \), respectively. As usual, \( n_1 = \sqrt{\varepsilon_1 \mu_1} \) and \( n_2 = \sqrt{\varepsilon_2 \mu_2} \).

Solving the Maxwell equations, we find \( K = k + i\delta \) with

\[
k = \left[ \frac{1}{2} \left( \sqrt{1 + \left( \frac{\sigma}{\varepsilon_0 \omega} \right)^2} + 1 \right) \right]^{1/2}, \quad \delta = \left[ \frac{1}{2} \left( \sqrt{1 + \left( \frac{\sigma}{\varepsilon_0 \omega} \right)^2} - 1 \right) \right]^{1/2}. \quad (5.5)
\]

From the matching conditions at the boundary \( x = 0 \), we obtain explicitly

\[
R = \frac{\sqrt{\varepsilon_1 \mu_1} - K \sqrt{\varepsilon_2 \mu_2}}{\sqrt{\varepsilon_1 \mu_1} + K \sqrt{\varepsilon_2 \mu_2}}, \quad T = \frac{2 \sqrt{\varepsilon_1 \mu_1}}{\sqrt{\varepsilon_1 \mu_1} + K \sqrt{\varepsilon_2 \mu_2}}. \quad (5.6)
\]

All the formulas are simplified in the case of the strongly conducting matter, when \( \sigma \gg \varepsilon \varepsilon_0 \omega \) (see [10], e.g.), but this condition is actually not necessary in our analysis.
It is now straightforward to compute the force per area of the metal surface, which is most conveniently given in terms of the stress tensor components, \( f_x = -S_{xx} \). We put \( v^a = 0 \) since the dielectric fluid is at rest. Then (4.45) yields

\[
f_x = p + \frac{\varepsilon_1\varepsilon_0}{2} |E|^2 + \frac{1}{2\mu_1\mu_0} |B|^2 = p + \frac{n_1}{c} (1 + |\mathcal{R}|^2) s^x_{(i)}.
\]

Here the right-hand side is evaluated at \( x = 0 \), and we used (4.43) to substitute the energy flux (Poynting vector) of the incident wave \( s^x_{(i)} \) along the \( x \) axis. As usual, the averaging over a period was performed in these calculations. The hydrodynamical pressure term \( p \) is obviously cancelled by the same fluid pressure force acting on the metal plate from the opposite side. So, the net force per area on the is described only by the last term in (5.7), in complete agreement with the experimental results (cf. also with the formula (4.22) in [10]).

Another application of our formalism to the analysis of the important experiments of Ashkin and Dziedzic [32, 33] will be considered elsewhere.

VI. DISCUSSION AND CONCLUSION

In this paper we have developed a model for a moving electrodynamical medium within the framework of a variational approach. The application to the observations of Richards and Jones [30, 31] demonstrates the viability of the model. Our next aim is to use this model for the analysis of the possibility of the Feigel effect [13] predicted recently.

We have derived the total energy-momentum of moving matter and the electromagnetic field. In particular, we found that the magnetoelectric properties contribute with the vector \( q_i \) to an additional energy flux and momentum density.

We are now in a position to compare the quantities derived here with those used by Feigel [13]. A direct inspection shows that the expression of Feigel is not obtained from the formulas above in the non-relativistic limit, when we assume that \( v^2/c^2 \ll 1 \). This fact casts certain doubts on the validity of the original conclusions of Feigel.

In the meantime, an interesting new paper [34] appeared where the Feigel effect was analysed with the help of a Green-function techniques. A similar discussion, which is based on the quantum-mechanical formalism and using the above variational formalism, will be
published elsewhere.

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