QUIVERS AND EQUATIONS A LA PLÜCKER FOR THE HILBERT SCHEME

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Abstract:

Several moduli spaces parametrising linear subspaces of the projective space admit a natural projective embedding in which they are cut out by linear and quadratic equations (Grassmannians, flag varieties, and Schubert varieties). The aim of this paper is to prove that a similar statement holds when one replaces linear subspaces with algebraic subschemes of the projective space. We exhibit equations of degree 1 and 2 that define schematically the Hilbert schemes $\text{Hilb}_p \mathbb{P}^n$ for all (possibly non-constant) Hilbert polynomials $p$. The equations are reminiscent of the Plücker relations on the Grassmannians: they are built formally with wedge products and permutations on indexes on the Plücker coordinates. Our method relies on a new description of the Hilbert scheme as a quotient of a scheme of quiver representations.

1. Introduction

The Plücker coordinates on a Grassmannian satisfy the well known Plücker relations. Similarly, the flag varieties are defined by quadratic equations and Schubert varieties are defined by quadratic and linear equations [23, 8]. The Grassmannians, flag varieties and Schubert varieties parametrize linear subspaces in a projective space. The goal of this paper is to prove that analog results hold in a non-linear context. We consider the Hilbert schemes parametrising the algebraic subschemes of a projective space and we prove that they are defined by simple explicit linear and quadratic equations in their natural embedding.

The Hilbert schemes carry in general a natural non-reduced structure inherited from their functorial construction. Our equations take into account the non-reduced structure and define the Hilbert schemes schematically.

More specifically, let $\text{Hilb}_p \mathbb{P}^n$ be the Hilbert scheme parametrising closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p$ over a field $k$. There is a Grassmannian embedding $\text{Hilb}_p \mathbb{P}^n \subset \mathbb{G}^{p(R)}_{S_R}$, where $R$ is any integer larger or equal to the Castelnuovo-Mumford-Gotzmann number $r$ of $p$ and $S_R = H^0 \mathcal{O}_{\mathbb{P}^n}(R)$. Composing with the Plücker embedding $\mathbb{G}^{p(R)}_{S_R} \subset \mathbb{P}^{N(R)-1}$, $N(R) := \binom{\dim S_R}{p(R)}$, we consider the problem of finding equations for the Hilbert scheme in $\mathbb{P}^{N(R)-1}$.

The question of finding equations for the Hilbert scheme as a subscheme of a Grassmannian has been addressed many times after its introduction by Grothendieck. The equations that arise depend much on the way the Hilbert scheme is described. The initial construction of the Hilbert scheme involved flattening stratifications [15, Lemme 3.4]. Techniques were developed to compute local equations for the flat stratum corresponding to the Hilbert scheme [9][12, Proposition 0.5]. The work by Gotzmann [11] leads to a
description of the Hilbert scheme as a determinantal locus. This determinantal approach was also used by Bayer in his PhD thesis [3] to build up a set of equations defining set theoretically the Hilbert scheme. It was proved by Iarrobino and Kleiman [19, Appendix +C] exploiting an argument of Grothendieck that the Bayer equations hold scheme theoretically. Haiman and Sturmfels obtained the Bayer equations schematically as a special case of their own construction of the multigraded Hilbert scheme [17]. In [4] and [20], Brachat, Lella, Mourrain and Roggero define the Hilbert scheme using a functor which involves the action of $GL_n$ to use the symmetry of the Hilbert scheme. See also [21] for techniques using Border bases.

The various approaches lead to equations of different degrees: for instance degree $n+1$, only depending on the “ambient” space $\mathbb{P}^n$, for those by Bayer, Iarrobino-Kleiman and Haiman-Sturmfels, degree $\text{deg}(p)+2$, only depending on the Hilbert polynomial, for those by Brachat-Lella-Mourrain-Roggero.

We will see that it is possible to find equations of degree 1 and 2 that cut out the Hilbert scheme when $p$ is non-constant, and equations of degree 2 when $p$ is constant. These are obviously the smallest possible degrees since in general the Hilbert scheme is not a linear space, not even a linear section of a Grassmannian [4, Section 7.2].

It was remarked by Haiman and Sturmfels [17] that their quite theoretical construction of the Hilbert scheme provides access to equations hardly accessible by direct computation. In cryptography, systems built with rich structures are possibly fragile because attackers may extract information from the structure. The above list of examples suggest that a similar principle could hold in our context: a new description of the Hilbert scheme could reveal a structure providing access to some new equations.

Starting from these remarks, our approach is to produce a new description for the Hilbert scheme and to extract equations of small degree from the construction.

We consider the description by Nakajima of $\text{Hilb}^p_{\mathbb{A}^2}$, when $p$ is a constant polynomial. It is related to the framed moduli space of torsion free sheaves on $\mathbb{P}^2$, monads and adhm-structures, quivers or commuting matrices [22]. We seek a description in the same vein for $\text{Hilb}^p_{\mathbb{P}^n}$, i.e. we want to replace the constant $p$ by any polynomial $p$ and the affine plane $\mathbb{A}^2$ by a projective space $\mathbb{P}^n$ of any dimension.

An extension of Nakajima’s description has been realized by Bartocci, Bruzzo, Lanza and Rava in [2]. They replaced the affine plane $\mathbb{A}^2$ with the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$. They use a description of the moduli space parametrising isomorphism classes of framed sheaves on the Hirzebruch surface $\Sigma_n$. The computations of the paper show that it is not possible to extend the initial description by Nakajima directly. In the sheaf context, the trivialization at infinity of the sheaf is responsible for the loss of projectivity. Replacing the surface by a higher dimensional variety or considering a non-constant Hilbert polynomial weakens the link between sheaves and Hilbert schemes.

We may reformulate the above difficulties in matrix terms. Recall that a zero-dimensional subscheme $Z \subset \mathbb{A}^2$ is represented by a pair of commuting matrices $X,Y$ corresponding to the multiplication by the variables $x,y$ on the vector space $O_Z \simeq k^{\text{length}(Z)}$, together with a cyclic vector $v \in k^{\text{length}(Z)}$ for the pair $(X,Y)$. The matrices are determined up to the choice of the base of $O_Z$, and the cyclic vector is the algebraic counterpart of the constant function $1 \in O_Z$ generating $O_Z$ as a $k[x,y]$-module. In a nutshell, the Hilbert scheme is constructed as a $\text{GIT}$-quotient of an open set of a commuting variety parametrising pairs $(X,Y)$ of commuting matrices.
Considering now any subscheme $Z \subset \mathbb{P}^n$ with Hilbert polynomial $p$, we try to characterize $Z$ using matrices corresponding to multiplication by the variables, up to the choice of the base. The multiplication by the variable $x_i$ yields a morphism $M_i : H^0(O_Z(j)) \to H^0(O_Z(j+1))$, where $j$ is chosen fixed and larger than or equal to the Castelnuovo-Mumford regularity of $Z$. However, the source space and the target space are different and the commutativity $M_iM_j = M_jM_i$ does not make sense. When $p$ is non-constant, the underlying matrices $M_i$ are not square matrices and their size are incompatible. When $p$ is constant, the matrix sizes are compatible but we miss a trivialization at infinity to identify $H^0(O_Z(j))$ with $H^0(O_Z(j+1))$. Finally, in the affine case, the constant function 1 generates $O_Z$ as a $k[z_0, \ldots, z_n]$-module. In the projective case, there is no privileged element in $H^0(O_Z(j))$ and no natural cyclic vector notion.

The above analysis shows that for a description of $\text{Hilb}_{\mathbb{P}^n}^p$ based on the multiplicative action of the variables, we require a framework where we can formulate substitute conditions for the commutativity and cyclic conditions. In the first part of the paper, we introduce a quiver and we formulate these substitutes as technical conditions on the representations of the quivers that we consider. We proceed as follows.

We choose any integer $R$ larger than or equal to the Gotzmann number $r$ of $p$ and we consider the quiver $Q_p$ with 4 vertices, $2n + 3$ arrows, dimension vector $((R-R^{-1}), (R^{-1}), p(R), p(R+1))$ and corresponding vector spaces $S_{R-1}, S_R, k^p(R), k^{p(R+1)}$, where $S := k[x_0, \ldots, x_n]$.

Then we consider the representations $\mu_0, \ldots, \mu_n, \rho, M_0, \ldots, M_n$ of the quiver such that:

- The map $\mu_i$ is the multiplication by the variable $x_i$.
- The map $\rho$ is surjective
- The images of the $M_i$ satisfy the condition $\text{Im}(M_0) + \cdots + \text{Im}(M_n) = k^{p(R+1)}$.
- $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$ for every $i, j \in \{0, \ldots, n\}$.

There is a natural functor $\mathcal{H}_C$ associated to the above representations, which is represented by a scheme $C^p$. There is an action of $GL_{p(R)} \times GL_{p(R+1)}$ on $C^p$ corresponding to the base changes on the last two vertices of the quiver. Our description of the Hilbert scheme is summarized in the following theorem.

**Theorem 1.1.** $C^p$ is a $GL_{p(R)} \times GL_{p(R+1)}$ principal bundle over the Hilbert scheme $\text{Hilb}_{\mathbb{P}^n}^p$.

The theorem provides a new universal property for the Hilbert scheme: it is possible to describe locally a family of subschemes of $\mathbb{P}^n$ using families of matrices from the quiver description, up to action of the group. Describing schemes in terms of linear algebra up to action may be more convenient than the usual description in terms of polynomial ideals (see [5, Prop. 3.14] for an explicit example).

Recall that Grassmannians are quotients of Stiefel varieties, and that Plücker coordinates are computable from Stiefel coordinates [10]. In our context, the “Stiefel” coordinates are on $C^p$, they are the entries of the matrices $\rho, M_0, \ldots, M_n$. The following proposition describes similarly the Plücker coordinates of the Hilbert schemes in terms of the “Stiefel coordinates of $C^p$”.
Proposition 1.2. The Plücker coordinates in $\mathbb{G}^{p(R)}_{S_R}$ are the maximal minors of $\rho$. The Plücker coordinates in $\mathbb{G}^{p(R+1)}_{S_{R+1}}$ are the maximal minors of $\sum_{i=0}^{n} (M_i \circ \rho) : S_R^{n+1} \to k^{p(R+1)}$.

The notations to formulate our equations are as follows. If $l = x_j$ is a variable, if $m_i \in S_R, n_i \in S_{R+1}$ are monomials, $lm_1 \wedge \cdots \wedge lm_{p(R)+1} \wedge n_1 \wedge \cdots \wedge n_{p(R+1)-p(R)-1}$ is a product of $p(R+1)$ monomials in $S_{R+1}$ and it is a Plücker coordinate on the Grassmannian $\mathbb{G}^{p(R+1)}_{S_{R+1}}$. If $l = L = a_0 x_0 + \cdots + a_n x_n$ is the generic linear form with indeterminate coefficients $a_i$, the multilinear expansion of $Lm_1 \wedge \cdots \wedge Lm_{p(R)+1} \wedge n_1 \wedge \cdots \wedge n_{p(R+1)-p(R)-1}$ is a linear combination of Plücker coordinates. This expansion is a polynomial in the variables $a_i$ and we denote by $E(m, n, x)$ the coefficients of this polynomial. Similarly, we denote by symbols $F(m, n, x)$ the coefficients of the expansion of $Lm_1 \wedge \cdots \wedge Lm_{p(R)} \wedge n_1 \wedge \cdots \wedge n_{p(R+1)-p(R)}$. Both $E(m, n, x)$ and $F(m, n, x)$ are linear combinations of Plücker coordinates on $\mathbb{G}^{p(R+1)}_{S_{R+1}}$.

Theorem 1.3. Suppose that $p$ is a non-constant Hilbert polynomial, $r$ its Gotzmann number, $R \geq r$, and consider the composed embedding $\text{Hilb}^p_{\mathbb{P}^n} \hookrightarrow \mathbb{G}^{p(R+1)}_{S_{R+1}} \hookrightarrow \mathbb{P}^{N(R+1)}-1$. Let $I$ be the ideal generated by:

- the quadratic Plücker relations of the Grassmannian,
- the linear forms $E(m, n, x)$
- the quadrics $F(m_1, n_1, x_1)F(m_2, n_2, x_2) - F(m_2, n_1, x_1)F(m_1, n_2, x_2)$

Then $\text{Hilb}^p_{\mathbb{P}^n} \subset \mathbb{P}^{N(R+1)}-1$ is the subscheme defined by the ideal $I$.

When $p$ is constant, we have the same result as above, except that the set of linear forms $E(m, n, x)$ is empty. Thus $I$ is generated by the quadrics of the first and third items in the list.

There is always an ambiguity for the signs of the Plücker coordinates. Our convention in these equations is to consider Plücker coordinates of the quotient.

Overview of the proof of Theorem 1.3. In the first part of the proof, the equations $E(m, n, x) = 0$ are obtained as a direct algebraic consequence of our quiver description. Recall that the Plücker coordinates in degree $R+1$ appear as determinants of $(M_0 \circ \rho, \ldots, M_n \circ \rho)$ by Proposition 1.2. When $p$ is non-constant, the composition $M_i \circ \rho$ is not surjective for obvious dimensional reasons and we get the vanishing of the corresponding determinant/Plücker coordinate. After a few algebraic manipulations to get the maximum from this idea, we get the linear equations $E(m, n, x) = 0$.

In the second part of the proof, we investigate the geometrical meaning of these algebraic vanishings. We characterize the locus $H$ defined by the equations $E(m, n, x) = 0$ in terms of locally free sheaves (Proposition 6.7).

More specifically, let $H \subset \mathbb{G}^{p(R+1)}_{S_{R+1}}$ be the locus in the Grassmannian cut out by all the linear equations $E(m, n, x) = 0$. A closed point $P \in \mathbb{G}^{p(R+1)}_{S_{R+1}}$ parametrises a vector space $I_{R+1} \subset S_{R+1}$ and we consider $(I_{R+1} : l) \subset S_R$ for any linear form $l$. For any $P$ and a general $l$, $\text{codim} (I_{R+1} : l) \geq p(R)$. If $P \in H$, we have the equality $\text{codim} (I_{R+1} : l) = p(R)$. The set of linear forms $l$ such that the inequality holds depend on $P$. To work functorially with families, a linear form suitable for all $I_{R+1}$ simultaneously is necessary. To bypass this difficulty, we follow Grothendieck and we use non-closed points: the generic linear form $L = a_0 x_0 + \cdots + a_n x_n$ with indeterminate coefficients may be used uniformly for all $I_{R+1}$. Technically, we work over the residual field $k(L) = k(a_0, \ldots, a_n)$ and we show
that $\mathcal{O}_{H \times k(L)} \otimes S_R/(I_{R+1} : L)$ is a locally free sheaf of rank $p(R)$ on some nice open set $U_H \subset H \times_k k(L)$ (6.7).

The next step is to compare this geometrical interpretation in terms of locally free sheaves to the Gotzmann-Iarrobino-Kleiman description of the Hilbert scheme. The analysis of the difference leads to the missing quadratic equations $F(m_1, n_1, x_1)F(m_2, n_2, x_2) = F(m_2, n_1, x_1)F(m_1, n_2, x_2)$, as follows.

Let $I_{R+1} \in \mathcal{G}^p_{S_{R+1}}(R)$. By the above, $I_{R+1} \in H$ iff $\text{codim} (I_{R+1} : l) = p(R)$ for $l$ general while Gotzmann’s description says that $I_{R+1} \in \text{Hilb}^p_{\mathbb{P}^n}$ iff $\text{codim} (I_{R+1} : S_1) = p(R)$.

Heuristically, if $L$ is generic and $l$ is general, we have the following sequence of inclusion:

$$(I_{R+1} : L) \subset (I_{R+1} : S_1) \subset (I_{R+1} : l).$$

Indeed, if $t \in S_R$ satisfies $tL \in I_{R+1}$ for the generic $L$ then $tl' \in I_{R+1}$ for every linear form $l'$ specialization of $L$. The left inclusion follows and the right inclusion is obvious. When $I_{R+1} \in H$, $(I_{R+1} : L)$ and $(I_{R+1} : l)$ have the same codimension $p(R)$, we conclude that $(I_{R+1} : L) = (I_{R+1} : l) = (I_{R+1} : S_1)$ and by Gotzmann’s description, $I_{R+1} \in \text{Hilb}^p_{\mathbb{P}^n}$.

This heuristic is not correct since it is careless about the residual fields. In general, $(I_{R+1} : L)$ is a $k(L)$-point (i.e. the computation yields a formula depending on the coefficients $a_i$ of the generic form $L = a_0 x_0 + \cdots + a_n x_n$) whereas $(I_{R+1} : l)$ is a $k$-point. However, if $(I_{R+1} : L)$ is a $k$-point, the above reasoning makes sense and this yields an equivalence: if $I_{R+1} \in H$, then $I_{R+1} \in \text{Hilb}^p_{\mathbb{P}^n}$ if and only if $(I_{R+1} : L)$ is a $k$-point. (Proposition 7.1).

It remains to prove that this condition on the base field of $(I_{R+1} : L)$ corresponds to the quadratic equations $F(m_1, n_1, x_1)F(m_2, n_2, x_2) = F(m_2, n_1, x_1)F(m_1, n_2, x_2)$ (Proposition 7.7). To settle this, we compute the (superabundant) Plücker coordinates of $(I_{R+1} : L)$ which are elements in $k(L)$ (Proposition 7.5). The formula obtained and the simple cross product remark 7.6 show that $(I_{R+1} : L)$ is a $k$-point exactly when the quadratic equations hold.

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2. Embeddings of the Hilbert scheme

Until section 8, we work over a field $k$ algebraically closed of arbitrary characteristic. We will prove in section 8 that our equations are valid on any field.

In this section, we recall some of the classical material used to embed Hilbert schemes into Grassmannians.

**Notation 2.1.** Let $n \in \mathbb{N}$, $S = k[x_0, \ldots, x_n]$ and $S_A = A[x_0, \ldots, x_n]$ for any $k$-algebra $A$. We denote by $S_d \subset S$ and $S_{A,d} \subset S_A$ the free submodules of homogeneous polynomials of degree $d$. We denote by the same letter $\mu_i : S \rightarrow S$ and $\mu_i : S_A \rightarrow S_A$ the multiplication by the variable $x_i$.

Recall [13, p.80] that if $p$ is the Hilbert polynomial of a subscheme $Z \subset \mathbb{P}^n$, then

$$p(d) = \binom{d+a_1}{a_1} + \binom{d+a_2-1}{a_2} + \cdots + \binom{d+a_r-(r-1)}{a_r}, \quad a_1 \geq a_2 \geq a_r \geq 0.$$
where the number $r$ of binomials is called the Gotzmann number of $p$. It depends on $p$, but not on $n$. For every $Z \subset \mathbb{P}^n$ and every $d \geq r$, the Hilbert function $H_Z$ and the Hilbert polynomial $p$ satisfy $H_Z(d) = p(d)$ [19, Corollary C.15]. The Gotzmann number coincides with the Castelnuovo-Mumford regularity of $p$, i.e. the smallest integer $m$ such that every $Z \subset \mathbb{P}^n$ with Hilbert polynomial $p$ is $m$-regular [19, Proposition C.24].

From now on, $p$ will denote a Hilbert polynomial for subschemes of $\mathbb{P}^n$, $r$ its Gotzmann number and $R$ any number $\geq r$.

We recall the following result which realizes $Hilb^p_{\mathbb{P}^n}$ as a closed subscheme of the product of Grassmannians $\mathbb{G}^{p(r)}_{S_R} \times \mathbb{G}^{p(r+1)}_{S_{R+1}}$ [11, Bemerkung 3.2],[19, Proposition C.28, Theorem C.29], [17].

**Theorem 2.2.** The Hilbert scheme $Hilb^p_{\mathbb{P}^n}$ represents a functor $\mathcal{H}_{Hilb^p_{\mathbb{P}^n}}$ from $k$-algebras to sets where $\mathcal{H}_{Hilb^p_{\mathbb{P}^n}}(A) = \{(I_{A,R}, I_{A,R+1}) \mid I_{A,R} \subset S_{A,R}, I_{A,R+1} \subset S_{A,R+1}, S_{A,R}/I_{A,R}$ and $S_{A,R+1}/I_{A,R+1}$ locally free $A$-submodules of rank $p(R)$ and $p(R+1)$ respectively, and for each variable $x_i$, $x_i I_{A,R} \subset I_{A,R+1}\}$. In particular, $Hilb^p_{\mathbb{P}^n}$ is a closed subscheme of the product of Grassmannians $\mathbb{G}^{p(r)}_{S_R} \times \mathbb{G}^{p(r+1)}_{S_{R+1}}$.

Moreover, the first (resp. second) projection gives an embedding $Hilb^p_{\mathbb{P}^n} \xrightarrow{i_R} \mathbb{G}^{p(R)}_{S_R}$ (resp. $Hilb^p_{\mathbb{P}^n} \xrightarrow{i_{R+1}} \mathbb{G}^{p(R+1)}_{S_{R+1}}$).

Let us fix any positive integer $d$. The binomial expansion of a positive integer $c$ in base $d$, also called the $d$-th Macaulay representation of $c$, is the unique expression

$$c = \left( \begin{array}{c} k_d \\ d \\ \end{array} \right) + \left( \begin{array}{c} k_{d-1} \\ d-1 \\ \end{array} \right) + \cdots + \left( \begin{array}{c} k_1 \\ 1 \\ \end{array} \right)$$

with $k_d > k_{d-1} \cdots > k_1 \geq 0$. Let

$$c^{<d>} = \left( \begin{array}{c} k_d+1 \\ d+1 \\ \end{array} \right) + \left( \begin{array}{c} k_{d-1}+1 \\ d \\ \end{array} \right) + \cdots + \left( \begin{array}{c} k_1+1 \\ 2 \\ \end{array} \right)$$

and

$$c_{<d>} = \left( \begin{array}{c} k_d-1 \\ d \\ \end{array} \right) + \left( \begin{array}{c} k_{d-1}-1 \\ d-1 \\ \end{array} \right) + \cdots + \left( \begin{array}{c} k_1-1 \\ 1 \\ \end{array} \right).$$

**Theorem 2.3.** (Macaulay)[13, p.79]. Let $W \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ with codimension $c$, $W_1$ the image of the multiplication $W \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(d+1))$, and $c_1$ the codimension of $W_1$. Then $c_1 \leq c^{<d>}$. 

**Theorem 2.4.** (Green)[13, p.77]. Let $W \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ with codimension $c$, $W_H \subset H^0(\mathcal{O}_H(d))$ the restriction of $W$ to a general hyperplane $H$, and $c_H$ the codimension of $c_H$. Then $c_H \leq c_{<d>}$. 

**Theorem 2.5.** Let $Z \subset \mathbb{P}^n$ be a subscheme with Hilbert polynomial $p$ and consider a degree $R \geq r$. Then $\text{codim}(I_{R+1}(Z), S_{R+1}) = \text{codim}(I_R(Z), S_R)^{<R>}$, i.e. $p(R + 1) = p(R)^{<R>}$. 

**Proof.** In degree $R \geq r$, the codimension is computed by the Hilbert polynomial, whose value is $p(R) = \left( \begin{array}{c} R+a_1 \\ R \\ \end{array} \right) + \left( \begin{array}{c} R+a_2-1 \\ R-1 \\ \end{array} \right) + \cdots + \left( \begin{array}{c} R+a_s-(s-1) \\ R-(s-1) \\ \end{array} \right)$ with $a_1 \geq a_2 \cdots \geq a_s \geq 0$ according to the Gotzmann regularity Theorem [13, p.80]. The relation between $p(R + 1)$ and $p(R)$ follows immediately. \qed
3. Description of the Hilbert scheme

In this section, we give the description of the Hilbert scheme in terms of representation of quivers.

**Notation 3.1.** If $\varphi_j : E \rightarrow F$, for $j = 0, \ldots, i$, are morphisms of $A$-modules and $A \rightarrow B$ is a morphism of $k$-algebras, we will use the following notations

- $\varphi_j \otimes_A B : E \otimes_A B \rightarrow F \otimes_A B$ is the morphism of modules with $(\varphi_j \otimes_A B)(e \otimes b) = \varphi_j(e) \otimes b$, 
- $\varphi$ is the list $(\varphi_0, \ldots, \varphi_i)$,
- $\oplus \varphi$ is the morphism $E \oplus \cdots \oplus E \rightarrow F \oplus \cdots \oplus F$ given by $\oplus \varphi(e_0, \ldots, e_i) = (\varphi_0(e_0), \ldots, \varphi_i(e_i))$,
- $\Sigma \varphi$ is the morphism $E \oplus \cdots \oplus E \rightarrow F$ given by $\Sigma \varphi(e_0, \ldots, e_i) = \varphi_0(e_0) + \cdots + \varphi_i(e_i)$.

We recall the quiver $Q_p$ from the introduction. To build the variety $C^p$ above the Hilbert scheme, a subset of representations of the quiver $Q_p$ is considered. The following definition introduces these representations in a functorial way.

**Definition 3.2.** Let $A$ be a $k$-algebra. Let $\mathcal{H}_{C^p}(A) = \{((\mu, \rho, M))\}$ where:

- $\mu = (\mu_0, \ldots, \mu_n)$ and $\mu_i : S_{A,R} \rightarrow S_{A,R}$ is the multiplication by the variable $x_i$,
- $M = (M_0, \ldots, M_n)$ and $M_i : A^p(R) \rightarrow A^p(R+1)$ is a morphism of $A$-modules,
- $\Sigma M : (A^p(R))^{n+1} \rightarrow A^p(R+1)$ is surjective,
- $\rho : S_{A,R} \rightarrow A^p(R)$ is a surjective morphism of $A$-modules,
- for every pair $(i, j) \in \{0, \ldots, n\}$, $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$.

**Remark 3.3.** The set $\mathcal{H}_{C^p}(A)$ and the map $\mu$ depend on $R$, but for brevity $R$ is not included in our notation. Similarly, we will use the notation $\mathcal{H}_{C^p}(A) = \{(\rho, M)\}$ as a shortcut for $\mathcal{H}_{C^p}(A) = \{((\mu, \rho, M))\}$ since there is only one possible choice for $\mu$.

Since the tensorisation preserves the surjectivity, for any map of $k$-algebras $A \rightarrow B$, we have a morphism $\mathcal{H}_{C^p}(A) \rightarrow \mathcal{H}_{C^p}(B)$ which sends $(\rho, M)$ to $(\rho \otimes_A B, M \otimes_A B)$. This makes $\mathcal{H}_{C^p}$ a functor from the category of $k$-algebras to the category of sets.

**Proposition 3.4.** There exists a scheme $C^p$ such that:

- $\mathcal{H}_{C^p}(A) = \text{Hom}(\text{Spec}(A), C^p)$
- the $k$-points of $C^p$ are representations of the quiver $Q_p$.

**Proof.** The non-trivial fact is the first item. It follows immediately that the $k$-points are representations of $Q_p$.

Let $\widetilde{\mathcal{H}}_{C^p}$ be the extension of $\mathcal{H}_{C^p}$ to the category of $k$-schemes, i.e. $\widetilde{\mathcal{H}}_{C^p}(Z) = \{((\mu, \rho, M))\}$ where:

- $\mu = (\mu_0, \ldots, \mu_n)$ and $\mu_i : S_{R-1} \otimes \mathcal{O}_Z \rightarrow S_R \otimes \mathcal{O}_Z$ is the multiplication by the variable $x_i$,
- $M = (M_0, \ldots, M_n)$ and $M_i : \mathcal{O}_Z^{p(R)} \rightarrow \mathcal{O}_Z^{p(R+1)}$ is a morphism of $\mathcal{O}_Z$-modules,
- $\Sigma M : (\mathcal{O}_Z^{p(R)})^{n+1} \rightarrow \mathcal{O}_Z^{p(R+1)}$ is surjective.
\[ \rho : S_R \otimes \mathcal{O}_Z \to \mathcal{O}_Z^{(R)} \] is a surjective morphism of \( \mathcal{O}_Z \)-modules,

- for every pair \((i, j) \in \{0, \ldots, n\}\), \(M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i\).

It suffices to prove that \( \mathcal{H}_{C_\rho} \) is representable to obtain the first item of the proposition.

Consider the functor \( \mathcal{G} \) defined as follows. If \( Z \) is a \( k \)-scheme, an element of \( \mathcal{G}(Z) \) is a couple \((M, \rho)\) where:

- \( \Sigma M : (\mathcal{O}_Z^{(R)})^{n+1} \to \mathcal{O}_Z^{(R+1)} \) is a morphism of \( \mathcal{O}_Z \)-modules,

- \( \rho : S_R \otimes \mathcal{O}_Z \to \mathcal{O}_Z^{(R)} \) is a (possibly not surjective) morphism of \( \mathcal{O}_Z \)-modules.

For any map of \( k \)-schemes \( \varphi : Z_2 \to Z_1 \), we have a morphism \( \mathcal{G}(Z_1) \to \mathcal{G}(Z_2) \) which sends \((\rho, M)\) to \((\varphi^* \rho, \varphi^* M)\).

For any finite dimensional \( k \)-vector space \( V \), let us denote by \( \mathcal{H}_V \) the functor defined by \( \mathcal{H}_V(Z) = H^0(\mathcal{O}_Z) \otimes_k V \) and, for any map of \( k \)-schemes \( Z_2 \to Z_1 \), the map \( \mathcal{H}_V(Z_1) \to \mathcal{H}_V(Z_2) \) sends \( H^0(\mathcal{O}_Z) \otimes_k V \) to \( H^0(\mathcal{O}_Z) \otimes_k V \) by pullback. It is well known that \( \mathcal{H}_V \) is represented by \( t(V) = \text{Spec}(\text{Sym}(V^*)) \), the scheme associated to \( V \). In particular, \( \mathcal{G} \) is represented by \( t(V) \) with \( V = \text{Hom}( (k^{(R)})^{n+1}, k^{(R+1)} ) \oplus \text{Hom}(S_R, k^{(R)}) \).

We recall the notion of relative representability from [14]. Let \( F,G \) be functors from the category of \( k \)-schemes to sets. Suppose that \( F \) is a subfunctor of \( G \). The inclusion \( F \subset G \) is relative representable if, for every \( k \)-scheme \( Z \) with functor \( h_Z \), and every morphism of functors \( h_Z \to G \), the cartesian product \( h_Z \times_G F \) is representable. Grothendieck, [14, Lemme 3.6] proves that if \( G \) is representable and if \( F \subset G \) is relatively representable, then \( F \) is representable.

In our case, \( \mathcal{G} \) is representable and its subfunctor \( \mathcal{H}_{C_\rho} \) is defined by the surjectivity of \( \Sigma M \) and \( \rho \), and by the equality \( M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i \). Thus it suffices to prove that a subfunctor defined by the surjectivity of a morphism of locally free sheaves is relatively representable, and that a subfunctor defined by the equality of morphisms of locally free sheaves is relatively representable.

The locus in \( \text{Spec}(A) \) where two matrices \( M, N \in \text{Hom}(\text{Spec}(A), k^{pq}) \) of size \( p \times q \) with coefficients \( m_{ij}, n_{ij} \) in \( A \) coincide is closed. Indeed, if \( \text{Spec}(B) \to \text{Spec}(A) \) is a morphism, then the pullback matrices \( M_B, N_B \in \text{Hom}(\text{Spec}(B), k^{pq}) \) satisfy \( M_B = N_B \) if and only if the morphism \( \text{Spec}(B) \to \text{Spec}(A) \) factorizes through the closed subscheme \( Z = \text{Spec}(A/J) \) where the ideal \( J \) is generated by the elements \((m_{ij} - n_{ij})\). It follows that if \( F, G \) are locally free sheaves on \( Z \), and if \( f, g \in \text{Hom}_{\mathcal{O}_Z}(F, G) \) are two morphisms of sheaves, there exists a closed subscheme \( i_W : W \to Z \) such that for all \( \varphi : Y \to Z \), \( \varphi^* f = \varphi^* g \) iff \( \varphi \) factorizes through \( W \). Let \( G \) be a functor such that \( G(Z) = \{(f, g, \ldots)\} \), i.e. \( G(Z) \) is a tuple, and two components \( f, g \) of this tuple correspond to a morphism of locally free sheaves \( F_Z \to G_Z \) naturally associated to \( Z \). Let \( F \) be the subfunctor of \( G \) defined by the condition \( f = g \). By Yoneda, a morphism \( h_Z \to G \) is defined by an element \( G(Z) \). By the above, \( h_Z(Y) \times_G h_Z(Y) F(Y) \) can be identified with the set of morphisms \( h_W(Y) \). Thus \( h_Z \times_G F \simeq h_W \) and \( F \subset G \) is a relatively representable functor. It follows that the condition \( M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i \) defines a relatively representable (closed) subfunctor of \( \mathcal{G} \).

The fact that the surjectivity condition on a morphism of sheaves defines an open subfunctor is a classical argument used in the construction of the Grassmannians [16, Lemme 9.7.4.6].

Thus \( \mathcal{H}_{C_\rho} \) is representable as it is a locally closed subfunctor of the representable functor \( \mathcal{G} \).
We denote by $GL_i(A)$ the group of invertible matrices with coefficients in $A$ and we use the abbreviation $GL_i = GL_i(k)$.

**Proposition 3.5.** There is an action of the group $GL_{p(R)} \times GL_{p(R+1)}$ on $C^p$.

**Proof.** At the functorial level, if $(M, \rho) \in \mathcal{H}_{C^p}(A)$ is an $A$-point of $C^p$ and $g, h$ are two matrices with $(g, h) \in GL_{p(R)}(A) \times GL_{p(R+1)}(A)$, the action is defined by $(g, h)(M, \rho) = (M', \rho')$ with $\rho' = g \circ \rho$ and $M'_i = h \circ M_i \circ g^{-1}$. \hfill $\square$

Our goal is to prove that the Hilbert scheme $Hilb^n_{p, \Sigma}$ is a geometric quotient of $C^p$ by the above natural action. We start with the construction of a morphism.

**Proposition 3.6.** There exists a morphism $\pi : C^p \to Hilb^n_{p, \Sigma}$. More specifically, if $(\rho, M) \in \mathcal{H}_{C^p}(A)$ is an $A$-point of $C^p$, then $\pi((\rho, M))$ is the $A$-point of the Hilbert functor defined by the ideal $I_A \subset S_A$ generated by $I_{A,R} := \text{Ker}(\rho)$.

**Proof.** We specify the notation, all appearing in the diagram 3.1. Since the argument will involve several degrees, we let $\mu_R := (\mu_0, \ldots, \mu_n)$ instead of $\mu = (\mu_0, \ldots, \mu_n)$ for the multiplications $S_R \overset{\mu_i^R}{\to} S_{R+1}$. Let $P = (\rho, \rho, \ldots, \rho)$ ($n + 1$ copies).

We consider the maps $P \oplus M$ and $\Sigma M$ obtained from $P, \mu, M$ following the conventions introduced in Notation 3.1. For readability, we let $P \oplus \rho := \oplus P$. Moreover, we denote by $\Sigma_{\mu,R} \mu_R$ the restriction of $\Sigma M_R$ to $(I_{A,R})^{n+1}$, and we let $I_{A,R+1} := \Sigma M_R(I_{A,R})^{n+1}$.

**Claim:** We can define a morphism $\beta : S_{A,R+1} \to A^{p(R+1)}$ such that the following diagram is commutative with exact rows.

\[
\begin{array}{ccc}
(S_{A,R-1})^{n+1} & \to & (A^{p(R)})^{n+1} \\
\oplus \mu_{R-1} & \downarrow & \downarrow \Sigma M \\
\Sigma_{\mu,R} \mu_R & \to & A^{p(R+1)} \\
\end{array}
\]

We observe that:

- $\oplus \rho, \Sigma_{\mu,R} \mu_R$ and $\Sigma M$ are surjective by hypotheses and/or by construction,
- by construction the first row is exact and the square on the left commutes.

We use all these properties in order to define the dash arrow $\beta$ so that also the last line is exact and all the diagram commute.

We define $\beta$ by diagram chasing in the following way: by the surjectivity of $\Sigma M_R$ every element of $S_{A,R+1}$ can be written (not uniquely) as $\Sigma x_i f_i$ where $f := (f_0, \ldots, f_n) \in (S_{A,R})^{n+1}$; then we set $\beta(\Sigma x_i f_i) = \Sigma M(\oplus \rho(f))$.

To verify that $\beta$ is well defined, we prove that when $\Sigma x_i f_i = 0$ we have $\Sigma M(\oplus \rho(f)) = 0$. This is obvious if $f = (0, \ldots, 0, f_n)$, since $\Sigma x_i f_i = 0$ implies $f_n = 0$. Then, we prove the assertion for $f = (0, \ldots, 0, f_{j-1}, \ldots, f_n)$ assuming it holds for elements of the form $(0, \ldots, 0, f_j, \ldots, f_n)$.

For every $i = j, \ldots, n$ we set $f_i = x_{j-1} f_i' + f_i''$ with $f_i' \in S_{A,R-1}$ and $x_{j-1}$ not appearing in $f_i''$. The equality $\sum_{i=0}^n x_i f_i = 0$ implies $f_{j-1} + \sum_{i=j}^n x_i f_i' = 0$ and $\sum_{i=j}^n x_i f_i'' = 0$. Then we have

\[
\Sigma M(\oplus \rho(f)) = \sum_{i=j}^n M_i(\rho(f_i)) = M_{j-1}(\rho(f_{j-1})) + \sum_{i=j}^n M_i(\rho(\mu_{j-1}(f_i'))) + \sum_{i=j}^n M_i(\rho(f_i''')).
\]
The last summand is equal to \( \Sigma M(\oplus \rho((0,\ldots,0,f^n_j,\ldots,f^n_n))) \), hence it vanishes by the inductive assumption. Moreover, by the commutativity conditions in the definition of \( C^p \), we have \( M_i(\rho(\mu_{j-1}(f^n_j))) = M_{j-1}(\rho(\mu(f^n_j))) = M_{j-1}(\rho(x_i f^n_j)) \). Therefore \( \Sigma M(\oplus \rho(f)) = M_{j-1}(\rho(\sum_{i=j}^n x_i f^n_j)) = M_{j-1}(\rho(0)) = 0 \).

The commutativity of the right square holds by the construction of \( \beta \) and the surjectivity of \( B \) is a direct consequence of that of \( \oplus \rho \), and \( \Sigma M \) and of the commutativity of the right square.

To complete the construction of our diagram, we now prove that \( \ker(\beta) \) is equal to \( I_{A,R+1} \). By the commutativity of the two squares and the surjectivity of \( \Sigma \mu_{I,R} \), it follows that \( I_{A,R+1} \) is contained in \( \ker(\beta) \). To prove the reverse inclusion we observe that \( I_{A,R}, I_{A,R+1} , \ker(\beta) \) depend functorially on \( A \) in the sense that if \( A \to B \) is a morphism of \( k \)-algebras, if \( L_A \in \{ I_{A,R}, I_{A,R+1}, \ker(\beta) \} \) is one of these three \( A \)-modules, then \( L_B = L_A \otimes_A B \). Then, we may check that for each maximal ideal \( m \), \( (\ker(\beta)/I_{A,R+1}) \otimes_A A_m = 0 \). In other words, we may replace \( A \) with \( A_m \) and suppose that \( A \) is local with maximal ideal \( m \).

The \( A \)-module \( \ker(\beta) \) is finitely generated as a kernel of a map between finitely generated free modules ([1, Exercise 12, p.32]). Thus \( \ker(\beta)/I_{A,R+1} \) is finitely generated and, by Nakayama, we may even suppose that \( A \) is a field. When \( A \) is a field, the inclusion \( I_{A,R+1} \subset \ker(\beta) \) is an equality if \( \dim I_{A,R+1} \geq \dim \ker(\beta) \) as vector spaces. Since \( \text{codim}(I_{A,R}, S_R) = p(R) \), Macaulay’s maximal growth Theorem (Theorem 2.3) gives the inequality \( \text{codim}(I_{A,R+1}, S_{A,R+1}) \leq p(R)^{- T_{\text{hm}}^{2.5} \geq p(R+1) = \text{codim}(\ker(\beta), S_{A,R+1}) \). This completes the proof of the equality \( I_{A,R+1} = \ker(\beta) \).

To conclude, we may associate to the data \( (\rho,M) \in \mathcal{H}_{C^p}(A) \) the pair \( (I_{A,R} = \ker(\rho), I_{A,R+1} = \ker(\beta)) \). This association depends functorially on \( A \). From the functorial description of the Hilbert scheme in Theorem 2.2, this corresponds to a morphism \( \pi : C^p \to \text{Hilb}^p_g \), which is the required morphism.

Next, we characterize the morphisms \( \text{Spec}(A) \to \text{Hilb}^p_g \) which factorize through \( C^p \).

**Proposition 3.7.** Let \( \varphi : \text{Spec}(A) \to \text{Hilb}^p_g \) be a morphism defined by a pair of \( A \)-modules \( I_{A,R}, I_{A,R+1} \) in the functorial description of Theorem 2.2. Then \( \varphi \) factorizes through \( \pi : C^p \to \text{Hilb}^p_g \) if and only if \( S_{A,R}/I_{A,R} \) and \( S_{A,R+1}/I_{A,R+1} \) are free.

**Proof.** Let \( \varphi : \text{Spec}(A) \to \text{Hilb}^p_g \) be a morphism which factorizes. Then the \( A \)-modules \( I_{A,R}, I_{A,R+1} \) can be recovered from the morphism \( \text{Spec}(A) \to C^p \) using the diagram (3.1). It is shown in Proposition 3.6 that the rows of the diagram (3.1) are exact. Therefore, \( S_{A,R}/I_{A,R} \simeq A_{p(R)} \) and \( S_{A,R+1}/I_{A,R+1} \simeq A_{p(R+1)} \) are free.

Conversely, suppose that the quotients are free. Choose isomorphisms \( \rho' : S_{A,R}/I_{A,R} \to A_{p(R)} \) and \( \beta' : S_{A,R+1}/I_{A,R+1} \to A_{p(R+1)} \) and let \( \rho \) and \( \beta \) be their lifts to \( S_{A,R} \) and \( S_{A,R+1} \).

\[
\begin{array}{ccccccc}
0 & \to & I_{A,R} & \hookrightarrow & S_{A,R} & \xrightarrow{\rho} & A_{p(R)} & \to & 0 \\
\downarrow{\mu_{i,I,R}} & & \downarrow{\mu_{i,R}} & & \downarrow{M_i} & & \downarrow{M_i} & & \downarrow{M_i} \\
0 & \to & I_{A,R+1} & \hookrightarrow & S_{A,R+1} & \xrightarrow{\beta} & A_{p(R+1)} & \to & 0
\end{array}
\]

(3.2)

By construction, the two rows are exact. Then, by diagram chasing we define the dash arrow \( M_i := \beta \circ \mu_{i,I,R} \circ \rho^{-1} \) that makes the diagram commutative. Moreover, as \( \mu_{i,t} \) and \( \mu_{j,t} \) are simply the multiplication by \( x_t \) and \( x_j \) respectively, we have \( M_j \circ \rho \circ \mu_{i,R-1} = \beta \circ \mu_{j,R} \circ \mu_{i,R} \circ \mu_{i,R-1} = \beta \circ \mu_{i,R} \circ \mu_{j,R-1} = M_i \circ \rho \circ \mu_{j,R-1} \).
If $M$ and $\Sigma M$ are defined from the $M_i$ as in Notation 3.1, we claim that $(\rho, M) \in H_{C^p}(A)$. For this it remains to observe that the surjectivity of $\Sigma M$ follows by that of $\Sigma \mu_R$ and $\beta$.

The functorial datum $(\rho, M) \in H_{C^p}(A)$ corresponds to a map $\text{Spec}(A) \to C^p$. To check that this is a factorization of the morphism $\text{Spec}(A) \to \text{Hilb}^{p}_{\mathbb{P}^n}$, we need to recover $I_{A,R}$ from $(\rho, M)$. This is a consequence of Proposition 3.6.

**Theorem 1.** The morphism $\pi : C^p \to \text{Hilb}^{p}_{\mathbb{P}^n}$ is a $GL_{p(R)} \times GL_{p(R+1)}$ principal bundle over the Hilbert scheme $\text{Hilb}^{p}_{\mathbb{P}^n}$.

**Proof.** Let $\text{Spec}(A)$ be an open affine subscheme of $\text{Hilb}^{p}_{\mathbb{P}^n}$ over which the universal quotients $S_{A,R}/I_{A,R}$ and $S_{A,R+1}/I_{A,R+1}$ are free $A$-modules. We choose and fix a basis for each quotient. The proposition being local on the Hilbert scheme, it suffices to prove that $\pi^{-1}(\text{Spec}(A))$ is isomorphic to $\text{Spec}(A) \times GL_{p(R)} \times GL_{p(R+1)}$.

To do so, we proceed functorially and, for each $k$-algebra $B$, we identify the morphisms $\psi \in \text{Hom}(\text{Spec}(B), \pi^{-1}(\text{Spec}(A)))$ with the morphisms $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \text{Hom}(\text{Spec}(B), \text{Spec}(A) \times GL_{p(R)} \times GL_{p(R+1)})$.

First, suppose we have $\varphi$ as above. Since $S_{A,R}/I_{A,R}$ and $S_{A,R+1}/I_{A,R+1}$ are free, the quotients $S_{B,R}/I_{B,R}$ and $S_{B,R+1}/I_{B,R+1}$ obtained by pullback from the morphism $\varphi_1 \in \text{Hom}(\text{Spec}(B), \text{Spec}(A))$ are still free, with bases obtained as the images of the initial bases via the morphisms $S_{A,R}/I_{A,R} \to S_{B,R}/I_{B,R}$ and $S_{A,R+1}/I_{A,R+1} \to S_{B,R+1}/I_{B,R+1}$.

Using Proposition 3.7, the morphism $\varphi_1$ can be lifted to a morphism $\psi_1 \in \text{Hom}(\text{Spec}(B), \pi^{-1}(\text{Spec}(A)))$. Using the action of $GL_{p(R)} \times GL_{p(R+1)}$ from Proposition 3.5 and $(\varphi_2, \varphi_3) \in \text{Hom}(\text{Spec}(B), GL_{p(R)} \times GL_{p(R+1)})$, we get $\psi = (\varphi_2, \varphi_3)(\psi_1)$.

Suppose reciprocally, that we have $\psi \in \text{Hom}(\text{Spec}(B), \pi^{-1}(\text{Spec}(A)))$. Let $\varphi_1 = \pi \circ \psi$. By the same construction as above, we obtain a basis for $S_{B,R}/I_{B,R}$, thus an identification $\theta : S_{B,R}/I_{B,R} \simeq B^{p(R)}$. We have the following diagram

$$
\begin{array}{ccc}
0 & \to & I_{B,R} \\
\downarrow & & \downarrow \\
0 & \to & I_{B,R} \\
\end{array}
\begin{array}{ccc}
\varphi_1 & \to & S_{B,R} \\
\phi & \to & B^{p(R)} \\
\psi & \to & B^{p(R)} \\
\end{array}
\begin{array}{ccc}
0 & \to & 0 \\
\end{array}
\begin{array}{ccc}
0 & \to & 0 \\
\end{array}
$$

where:

- both lines are exact,
- the morphism $\rho$ is defined by the morphism $\psi \in \text{Hom}(\text{Spec}(B), C^p)$ using the functorial description of $C^p$ (Definition 3.2),
- the first line is defined by $\rho$ and its exactness,
- $\rho$ is the composition $S_{B,R} \to S_{B,R}/I_{B,R} \xrightarrow{\theta} B^{p(R)}$,
- the second line is defined by $\rho'$ and its exactness,
- the kernels of $\rho'$ and $\rho$ coincide because of the identity $\varphi_1 = \pi \circ \psi$,
- the first two vertical arrows are identity maps,
- the map $\varphi_2$ is an isomorphism defined by diagram chasing as $\rho' \circ \rho^{-1}$.

We define $\varphi_3$ in a similar way to $\varphi_2$, replacing degree $R$ with $R + 1$. We have thus constructed a morphism $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \text{Hom}(\text{Spec}(B), \text{Spec}(A) \times GL_{p(R)} \times GL_{p(R+1)}).$
4. Plücker coordinates

In this section we exploit the quiver construction done in the previous ones and get linear equations for the Hilbert scheme. Specifically, we prove that the Plücker coordinates on the Hilbert scheme can be obtained by explicit formulas that only involve the entries of matrices associated to the maps $M$ and $\rho$ in the quiver. From this we get the linear equations by simple algebraic manipulations.

Recall that there are two conventions for the Plücker coordinates, which give different signs \cite[eq. 1.6]{10}. The next propositions recall the basics about Grassmannians. They introduce the notations that we need and they precise our sign convention for the Plücker coordinates.

We denote by $G^r_V$ the Grassmannian of codimension $r$ spaces of a vector space $V$. If $E = (e_1, \ldots, e_q)$ is an (ordered) basis of $V$, for every $k$-algebra $A$, we also denote by $E$ the corresponding basis $(e_i \otimes 1_A)$ of the free $A$-module $V_A := V \otimes_k A$.

Recall that a morphism $\text{Spec}(A) \to G^r_V$ is functorially defined by an inclusion of $A$-modules $W_A \subset V_A$ such that the quotient $V_A/W_A$ is locally free of rank $r$. The Plücker coordinates of such a morphism $f \in \text{Hom}(\text{Spec}(A), G^r_V)$ with $V_A/W_A$ free are defined as follows.

**Definition 4.1.** Let $f \in \text{Hom}(\text{Spec}(A), G^r_V)$ such that $V_A/W_A$ is free of rank $r$ with basis $F$. Let $N \in M_{r,q}(A)$ be the matrix with columns $N_{1,1}, \ldots, N_{q,1}$ corresponding to the canonical morphism $V_A \to V_A/W_A$ with respect to the bases $E$ and $F$. Consider a multi-index $(i_1, \ldots, i_r)$ with $1 \leq i_j \leq q$ for all $j$. The Plücker coordinate $P_{i_1,\ldots,i_r} \in A$ of the morphism $\text{Spec}(A) \to G^r_V$ is by definition the determinant $P_{i_1,\ldots,i_r} = \text{det}(N_{i_1,1}, \ldots, N_{i_r,1})$. It is well defined up to multiplication by an invertible constant depending on the basis $F$.

Equivalently, $P_{i_1,\ldots,i_r} = (e_{i_1} \otimes 1_A) \wedge \cdots \wedge (e_{i_r} \otimes 1_A) \in \Lambda^r(V_A/W_A) \cong A$.

For $(i_1, \ldots, i_r)$ with $1 \leq i_1 < i_2 \cdots < i_r \leq q$, we choose an indeterminate $x_{i_1,\ldots,i_r}$ and we consider the projective space $\mathbb{P} = \text{Proj}(k[X_{i_1,\ldots,i_r}])$ of dimension $(q-1)$.

The Plücker embedding $P : G^r_V \to \mathbb{P} = \text{Proj}(k[X_{i_1,\ldots,i_r}])$ is the embedding characterized by the following: if $f \in \text{Hom}(\text{Spec}(A), G^r_V)$ is such that $V_A/W_A$ is free of rank $r$, then $P \circ f \in \text{Hom}(\text{Spec}(A), \mathbb{P})$ is described in coordinates by $X_{i_1,\ldots,i_r} = P_{i_1,\ldots,i_r}$.

Starting from a morphism $f : \text{Spec}(A) \to C^n$, we define $f_R \in \text{Hom}(\text{Spec}(A), G^{p(R)}_{S_R})$ and $f_{R+1} \in \text{Hom}(\text{Spec}(A), G^{p(R+1)}_{S_{R+1}})$ by the following compositions:

\begin{align}
(4.1) \quad f_R : \text{Spec}(A) \to C^n &\to \text{Hilb}^p_{\mathbb{P}^n} \to G^{p(R)}_{S_R} \\
(4.2) \quad f_{R+1} : \text{Spec}(A) \to C^n &\to \text{Hilb}^p_{\mathbb{P}^n} \to G^{p(R+1)}_{S_{R+1}}.
\end{align}

The vector spaces $S_R$ and $S_{R+1}$ are considered with their natural bases of monomials (ordered for instance lexicographically). We consider the Plücker coordinates with respect to these bases, i.e. we consider Plücker coordinates $P_{z_1,\ldots,z_{p(R)}}$ and $P_{v_1,\ldots,v_{p(R+1)}}$ with $z_i$ monomials in $S_R$ and $v_j$ monomials in $S_{R+1}$.

The next proposition describes the Plücker coordinates of $f_R$ and $f_{R+1}$.

**Proposition 4.2.** With the above notations, the Plücker coordinates $P_{z_1,\ldots,z_{p(R)}}$ of $f_R$ are the maximal minors of $\rho$. The Plücker coordinates $P_{v_1,\ldots,v_{p(R+1)}}$ of $f_{R+1}$ are the maximal minors of $\Sigma M \circ \bigoplus_{\rho=0}^p \rho$. More specifically, if for each monomial $v_i \in S_{R+1}$, we choose a monomial $z_{t(i)} \in S_R$ and a variable $x_{j(i)}$ such that $v_i = x_{j(i)} z_{t(i)}$ and set...
\( \tilde{z}_i = (0, \ldots, 0, z_{t(i)}, 0, \ldots, 0) \in (S_{A,R})^{n+1} \), where \( z_{t(i)} \in S_{A,R} \) is located at position \( j(i) \) so that \( \Sigma \mu(\tilde{z}_i) = \mu_{j(i)}(z_{t(i)}) = v_i \), then \( P_{v_1, \ldots, v_{p(R+1)}} \) is the determinant of the matrix whose \( i \)-th column is \( C_i := (\Sigma M \circ \oplus \rho)(\tilde{z}_i) \).

**Proof.** From our constructions and the proof of 3.6, we have the two following diagrams with exact lines and commutative squares.

\[
0 \rightarrow I_{A,R} \rightarrow S_{A,R} \xrightarrow{\rho} A^{p(R)} \rightarrow 0
\]

\[
0 \rightarrow (I_{A,R})^{n+1} \rightarrow (S_{A,R})^{n+1} \xrightarrow{\oplus_{i=0}^{n} \rho} (A^{p(R)})^{n+1} \rightarrow 0
\]

\[
0 \rightarrow I_{A,R+1} \rightarrow S_{A,R+1} \xrightarrow{\beta} A^{p(R+1)} \rightarrow 0
\]

Using the functorial description of the Grassmannian, the morphism \( f_R \) is described by the inclusion \( I_{A,R} \subset S_{A,R} \). The first line shows that the Plücker coordinates in degree \( R \) are given by the maximal minors of \( \rho \).

The morphism \( f_{R+1} \) is described by the inclusion \( I_{A,R+1} \subset S_{A,R+1} \). The last line shows that the Plücker coordinates in degree \( R + 1 \) are given by the maximal minors of \( \beta \). Since \( \Sigma \mu \) is surjective and sends the monomial basis of \( (S_{A,R})^{n+1} \) to the monomial basis of \( S_{A,R+1} \), the maximal minors of \( \beta \) coincide with the maximal minors of \( \beta \circ \Sigma \mu = \Sigma M \circ \oplus \rho \).

More specifically, if for each monomial \( v_i \in S_{A,R+1} \), we choose a monomial \( \tilde{z}_i \), as described in the statement, then \( \beta(v_i) = \beta(\Sigma \mu(\tilde{z}_i)) = (\Sigma M \circ \oplus \rho)(\tilde{z}_i) \). The Plücker coordinate \( P_{v_1, \ldots, v_{p(R+1)}} \) is the determinant build with the \( \beta(v_i) \) as columns, so the second equality of the proposition follows. \( \square \)

By Proposition 4.2, the minors of \( M_i \circ \rho \) are Plücker coordinates on the Hilbert scheme and they vanish for a non-constant Hilbert polynomial. Indeed, suppose for simplicity that \( \text{Im}(M_i \circ \rho) \) is a free module. In the composition \( S_{A,R} \xrightarrow{\rho} A^{p(R)} \xrightarrow{M_i} A^{p(R+1)} \), the rank of the image \( \text{Im}(M_i \circ \rho) \) is at most the rank \( p(R) \) of the space in the middle. Since \( p \) is non-constant, \( p(R+1) > p(R) \geq \text{rank}(M_i \circ \rho) \), so that all minors of \( M_i \circ \rho \) of order \( p(R+1) \) vanish.

The following Lemma 4.3 is a functorial extended version of this remark. More specifically, if \( m_i \in S_R \), \( n_i \in S_{R+1} \) are monomials, if we apply the Lemma 4.3 to \( \ell = x_i \), \( z_i = m_i \) and \( v_i = n_i \) and to an open immersion \( f : \text{Spec}(A) \rightarrow \text{Hilb}_{p\rho}^{p} \), we obtain the vanishing of the Plücker coordinate corresponding to the monomials \( x_im_1, x_im_2, \ldots, x_im_{p(R)+1}, n_1, \ldots, n_{p(R+1)-p(R)-1} \) on \( \text{Spec}(A) \). Since \( A \) may run through an open cover of the Hilbert scheme, we get the vanishing of the Plücker coordinate on the Hilbert scheme. Playing with the same lemma and different choices for \( \ell \), we will get the equations \( E(m, n, 2) = 0 \) of the following Theorem 4.5.

When \( p \) is non-constant, by Theorem 2.5 we have \( p(R+1) \geq p(R) + 1 \). By convention, if \( p(R+1) = p(R) + 1 \), we assume that any list \( v_1, v_2, \ldots, v_{p(R+1)-p(R)-1} \) is empty.

**Lemma 4.3.** Suppose that \( p \) is not constant. Let \( f \in \text{Hom}(\text{Spec}(A), \text{Hilb}_{p\rho}^{p}) \) be such that the universal quotients \( S_{A,R}/I_{A,R} \) and \( S_{A,R+1}/I_{A,R+1} \) are free. Let \( \beta : \Lambda^{p(R+1)} S_{A,R+1} \rightarrow \Lambda^{p(R+1)} (S_{A,R+1}/I_{A,R+1}) \simeq A \).

Then, for every choice of \( z_1, \ldots, z_{p(R)+1} \in S_R, v_1, \ldots, v_{p(R+1)-p(R)-1} \in S_{R+1} \) and \( \ell \in S_1 \), the element \( e = \ell z_1 \wedge \ell z_2 \wedge \cdots \wedge \ell z_{p(R)+1} \wedge v_1 \wedge \cdots \wedge v_{p(R+1)-p(R)-1} \in \Lambda^{p(R+1)} S_{A,R+1} \) verifies \( \beta(e) = 0 \).
Proof. Since the quotients are free, \( f \) can be lifted to a morphism \( \text{Spec}(A) \to C^p \) according to Proposition 3.7. Then, by the functorial description of \( C^p \) of Definition 3.2, we obtain morphisms \( \rho : S_{A,R} \to A^{p(R)} \) and \( \beta : S_{A,R+1} \to A^{p(R+1)} \) and the diagram (3.2) commutative, with exact rows. We can define a similar diagram corresponding to the multiplication by \( \ell \): If \( \ell = a_0 x_0 + \cdots + a_n x_n \) with \( a_i \in A \), we set \( \mu_\ell := a_0 \mu_0 + \cdots + a_n \mu_n \) and \( M_\ell := a_0 M_0 + \cdots + a_n M_n \) and get the following commutative diagram with exact rows.

\[
\begin{array}{cccc}
0 & \to & I_{A,R} & \hookrightarrow & S_{A,R} & \xrightarrow{\rho} & A^{p(R)} & \to & 0 \\
& & \downarrow \mu_\ell, I_{R} & & \downarrow \mu_\ell, R & & \downarrow & & . \\
0 & \to & I_{A,R+1} & \hookrightarrow & S_{A,R+1} & \xrightarrow{\beta} & A^{p(R+1)} & \to & 0
\end{array}
\]

By hypothesis, \( S_{A,R}/I_{A,R} \simeq A^{p(R)} \), hence \( \Lambda^{p(R)+1}(S_{A,R}/I_{A,R}) \simeq \Lambda^{p(R)+1}A^{p(R)} = 0 \). Therefore, \( \rho(z_1) \wedge \rho(z_2) \cdots \wedge \rho(z_{p(R)+1}) = 0 \) and \( M_\ell(\rho(z_1)) \wedge M_\ell(\rho(z_2)) \cdots \wedge M_\ell(\rho(z_{p(R)+1})) = 0 \).

By the commutativity of (4.3) we deduce \( \beta(\ell z_1) \wedge \beta(\ell z_2) \cdots \wedge \beta(\ell z_{p(R)+1}) = 0 \), hence \( \beta(z_1) \wedge \beta(z_2) \cdots \wedge \beta(z_{p(R)+1}) = 0 \), \( \wedge \beta(\ell z_{p(R)+1}) = 0 \), namely \( \overline{\beta}(e) = 0 \). \( \square \)

Definition 4.4. Let \( m = (m_1, \ldots, m_{p(R)+1}) \) be a tuple of monomials in \( S_R \), \( n = (n_1, \ldots, n_{p(R+1)-p(R)-1}) \) be a tuple of monomials in \( S_{R+1} \) and \( \underline{x} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{p(R)+1}}) \) be a tuple of variables. Let \( P_{\underline{m},\underline{n},\underline{x}} = x_{i_1} m_1 \cdots \wedge x_{i_{p(R)+1}} m_{p(R)+1} \wedge n_1 \cdots \wedge n_{p(R+1)-p(R)-1} \) be the corresponding Plücker coordinate on \( \mathbb{G}^{p(R+1)}_{S_{R+1}} \). The permutations \( \sigma \) of \( \{1, \ldots, p(R)+1\} \) act on the set of \( \underline{x} \) by \( \sigma.\underline{x} = (x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(p(R)+1)}}) \) and we denote by \( O_\underline{x} \) the orbit of \( \underline{x} \).

We define

\[
E(\underline{m},\underline{n},\underline{x}) := \sum_{\underline{y} \in O_\underline{x}} P_{\underline{m},\underline{n},\underline{y}}.
\]

We define \( \mathcal{H}_{\underline{m},\underline{n},\underline{x}} \subset \mathbb{G}^{p(R+1)}_{S_{R+1}} \subset \mathbb{P}^{N+1} \) as the locus defined by the linear equation \( E(\underline{m},\underline{n},\underline{x}) = 0 \). Finally, we let \( H \subset \mathbb{G}^{p(R+1)}_{S_{R+1}} \) the locus cut out by all the hyperplanes.

In symbols, \( H = \cap \mathcal{H}_{\underline{m},\underline{n},\underline{x}} \subset \mathbb{G}^{p(R+1)}_{S_{R+1}} \) where the intersection runs through all the possible choices for \( \underline{m},\underline{n},\underline{x} \) as above.

Theorem 4.5. Suppose that \( p \) is not constant. Then the Hilbert scheme \( \text{Hill}_p^{p(R)} \subset \mathbb{G}^{p(R+1)}_{S_{R+1}} \) is a subscheme of \( H \).

Proof. We check the inclusion in any hyperplane \( \text{Hill}_p^{p(R)} \subset \mathcal{H}_{\underline{m},\underline{n},\underline{x}} \) locally using a covering of the Hilbert scheme by open subschemes \( i_A : \text{Spec}(A) \to \text{Hill}_p^{p(R)} \) such that the quotients \( S_{A,R}/I_{A,R} \) and \( S_{A,R+1}/I_{A,R+1} \) are free.

Let \( L = a_0 x_0 + \cdots + a_n x_n \) where \( a_i \) are indeterminates. Let \( e(a_0, \ldots, a_n) := \overline{\beta}(L m_1 \wedge L m_2 \wedge \cdots \wedge L m_{p(R)+1} \wedge n_1 \wedge \cdots \wedge n_{p(R+1)-p(R)-1}) \) in \( (\Lambda^{p(R+1)}S_{A,R+1}/I_{A,R+1})[a_0, \ldots, a_n] \simeq A[a_0, \ldots, a_n] \). Lemma 4.3 applied with \( z_i = m_i, v_i = n_i, f = i_A \) shows that \( e(a_0, \ldots, a_n) \) vanishes when we replace the indeterminates \( a_i \) with scalars \( \lambda_i \in k \). Since \( k \) is infinite, \( e(a_0, \ldots, a_n) = 0 \). We expand \( e \) as a polynomial in the indeterminates \( a_i \). The coefficient of the monomial \( a_{i_1} a_{i_2} \cdots a_{i_{p(R)+1}} \) is precisely \( E(\underline{m},\underline{n},\underline{x}) \) with \( \underline{x} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{p(R)+1}}) \), thus it vanishes. \( \square \)
5. Generic linear forms

The goal of this section is to define precisely the conductor \((\mathcal{I}_{R+1} : L)\) for \(L = a_0x_0 + \cdots + a_nx_n\) the generic linear form and for a family \(\mathcal{I}_{R+1}\) of vector spaces \(I_{R+1} \subset S_{R+1}\). We give the base change formulas for this conductor.

We defined in notation 2.1 the vector space \(S_1\) as a \(k\)-vector space of linear forms. In the following, we need to consider \(S_1\) as a scheme, in particular we want to consider generic linear forms in \(S_1\). For this, recall that if \(V\) is a \(k\)-vector space endowed with its natural Zariski topology, with dual space \(V^*\), then the scheme \(t(V)\) canonically associated to \(V\) is \(\text{Spec}(\text{Sym}(V^*))\) where \(\text{Sym}(V^*)\) denotes the symmetric \(k\)-algebra over \(V^*\) ([18, II, Prop.2.6]).

Applying the above construction in our context, we consider the \(k\)-vector space \(V = (x_0, \ldots, x_n)\). Let \(a_0, \ldots, a_n\) be the basis of \(V^*\) dual to \(x_0, \ldots, x_n\), and let \(S_1 = \text{Spec}(k[a_0, \ldots, a_n])\). This is compatible with the previous definition of \(S_1\) in the sense that the \(k\)-points of \(\text{Spec}(k[a_0, \ldots, a_n])\) are canonically identified with the linear forms \(\lambda_0x_0 + \cdots + \lambda_nx_n\), with \((\lambda_0, \ldots, \lambda_n) \in k^{n+1}\). In other words, we use the same symbol \(S_1\) for the scheme and for the underlying variety of \(k\)-points which is the vector space of linear forms considered previously.

Let \(z : Z \to S_1 \times G_{S_{R+1}}\). The morphism \(pr_1 \circ z : Z \to S_1\) is defined by the pullbacks \(s_i \in H^0(\mathcal{O}_Z)\) of the variables \(a_i \in k[a_0, \ldots, a_n]\), or equivalently by a linear form \(l = s_0x_0 + \cdots + s_nx_n \in H^0(\mathcal{O}_Z) \otimes S_1\) with coefficients \(s_i \in H^0(\mathcal{O}_Z)\). By universal property of the Grassmannian, the morphism \(pr_2 \circ z : Z \to G_{S_{R+1}}^{p(R+1)}\) is defined by the sheaf \(\mathcal{I}_{Z,R+1} = z^* \circ pr_2^*\mathcal{I}_{R+1} \subset \mathcal{O}_Z \otimes S_{R+1}\) where \(\mathcal{I}_{R+1} \subset \mathcal{O}_{G_{S_{R+1}}^{p(R+1)}} \otimes S_{R+1}\) is the universal sheaf on the Grassmannian. We consider the sequence of sheaves on \(Z\):

\[
\mathcal{O}_Z \otimes S_R \xrightarrow{\varphi_l} \mathcal{O}_Z \otimes S_{R+1} \xrightarrow{q} (\mathcal{O}_Z \otimes S_{R+1})/\mathcal{I}_{Z,R+1}
\]

where the map \(\varphi_l\) is the multiplication by \(l \in H^0(\mathcal{O}_Z) \otimes S_1\) and \(q\) is the natural quotient.

**Definition 5.1.** We define

\[(\mathcal{I}_{R+1} : l)_z \subset \mathcal{O}_Z \otimes S_R\]

or by abuse of notation

\[(\mathcal{I}_{R+1} : l)_Z \subset \mathcal{O}_Z \otimes S_R\]

the kernel of \(q \circ \varphi_l\).

**Notation 5.2.** We denote by \(L\) the generic point of \(S_1\), i.e. the ideal \((0) \subset k[a_0, \ldots, a_n]\). Let \(k(L) = k(a_0, \ldots, a_n)\) be the residual field of \(L \subset S_1\). For any \(k\)-scheme \(X\), we denote \(X_L = X \times_k \text{Spec}(k(L))\). As in the above definition, the inclusion of \(L\) in \(S_1\) is an embedding \(i_L : L \simeq \text{Spec}(k(L)) \to S_1\) functorially defined by the linear form \(a_0x_0 + \cdots + a_nx_n \in k(L) \otimes S_1\) with coefficients \(a_i \in k(L)\). We write \(L = a_0x_0 + \cdots + a_nx_n\) as a shortcut for this functorial definition of \(i_L\). If \(z : Z \to G_{S_{R+1}}^{p(R+1)}\) is a morphism, we consider the morphism \(z_L = z \times i_L : Z \times \text{Spec}(k(L)) \to G_{S_{R+1}}^{p(R+1)} \times S_1\).

Suppose that we want to make the local computation of \((\mathcal{I}_{R+1} : l)_{z_L}\) when \(Z = \text{Spec}(A)\) is affine. Then, according to definition 5.1, \(l = \sum a_ix_i \in A(a_i) \otimes S_1 = H^0(\mathcal{O}_{Z_L}) \otimes S_1\). In particular, \(l\) may be identified with \(L = \sum a_ix_i \in k(a_i) \otimes S_1\) through the embedding \(k(a_i) \otimes S_1 \to A(a_i) \otimes S_1\). In other words, for local computations, we use for \(l\) the formal expression \(\sum a_ix_i\) of \(L\). Our notation emphasizes this identification as follows:

\[(\mathcal{I}_{R+1} : L)_{z_L} := (\mathcal{I}_{R+1} : l)_{z_L} \subset \mathcal{O}_{Z_L} \otimes S_R\]
This notation reminds us that locally \((I_{R+1} : L)_{Z_L}\) is just the set of elements \(e\) such that 
\[ eL = e(a_0x_0 + \cdots + a_nx_n) \in I_{R+1} \]

The base change properties in the computation of \((I_{R+1} : l)\) are given by the following proposition, which says that the pullback of the conductor is included in the conductor of the pullback, with simplifications in some particular cases.

**Proposition 5.3** (Base change for generic conductors). Let \(W \xrightarrow{w} Z \xrightarrow{g} S_1 \times \mathbb{G}^{p(R+1)}_{S_{R+1}}\) and let \((I_{R+1} : l)_W \subset O_W \otimes S_R\) and \((I_{R+1} : l)_Z \subset O_Z \otimes S_R\) be the corresponding conductors. Let \(w^\#(I_{R+1} : l)_Z \subset O_W \otimes S_R\) be the image of \(w^*(I_{R+1} : l)_Z\) in \(w^*O_Z \otimes S_R = O_W \otimes S_R\). Then \(w^\#(I_{R+1} : l)_Z \subset (I_{R+1} : l)_W\). If \(w\) is flat, the inclusion is an equality. If \(O_Z \otimes S_R/(I_{R+1} : l)_Z\) is locally free, then \(w^*(I_{R+1} : l)_Z = w^\#(I_{R+1} : l)_Z\).

**Proof.** We consider the exact sequence defining \((I_{R+1} : L)_Z\):

\[ 0 \to (I_{R+1} : L)_Z \to O_Z \otimes S_R \to (O_Z \otimes S_{R+1})/I_{Z,R+1} \]

Pulling back to \(W\), we obtain the sequence

\[ w^*(I_{R+1} : L)_Z \to (O_W \otimes S_R)/(I_{W,R+1}) \]

with \(f \circ g = 0\). Thus the kernel \((I_{R+1} : L)_W\) of \(f\) contains \(Im(g) = w^\#(I_{R+1} : L)_Z\). If \(w\) is flat, the second sequence is exact, hence the equality. If \(O_Z \otimes S_R/(I_{R+1} : l)_Z\) is locally free, then pulling back to \(W\) the sequence \(0 \to (I_{R+1} : l)_Z \to O_Z \otimes S_R \to (O_Z \otimes S_R)/(I_{R+1} : l)_Z \to 0\) shows that \(g\) is injective, so that \(w^*(I_{R+1} : L)_Z \simeq w^\#(I_{R+1} : L)_Z\). \(\square\)

6. INTERPRETATION OF THE EQUATIONS

The linear equations \(E(m, n, z)\) defining \(H\) in Theorem 4.5 are the result of algebraic computations. In this section, we give a geometric interpretation of \(H\).

For the closed points, the geometric interpretation says that \(I_{R+1} \in \mathbb{G}^{S_{R+1}}_{p(R+1)}\) is a point of \(H \subset \mathbb{G}^{S_{R+1}}_{p(R+1)}\) iff the conductor \((I_{R+1} : l) \subset S_R\) has codimension \(p(R)\) for a general linear form \(l\), where \((I_{R+1} : l) := \{f \in S_R, lf \in I_{R+1}\}\).

The set of \(l\) which are suitable depends on \(I_{R+1}\). To organize these vector spaces \((I_{R+1} : l)\) as a family over \(H\), we need some \(l\) which works uniformly for all \(I_{R+1}\); we use the generic linear form \(L\) from the previous section. We prove that \((O_{H_L} \otimes S_R)/(I_{R+1} : L)\) is locally free of rank \(p(R)\) on a nice open subscheme \(U_{H,L} \subset H_L\) (Proposition 6.7). These constructions are functorial (Proposition 6.9).

We recall that an \(A\)-module is locally free of constant rank \(i\) iff the Fitting ideals satisfy \(Fitt_i(M) = A\) and \(Fitt_{i-1}(M) = 0\), where \(Fitt\) denotes the Fitting ideal [6, Proposition 20.8].

**Lemma 6.1.** Let \(M\) be an \(A\)-module of finite type. The following conditions are equivalent:

- \(Fitt_{i-1}M = A\),
- \(\Lambda^i M = 0\).

**Proof.** The Fitting ideals are functorial [6, Cor. 20.5]: If \(Fitt_{i-1}M = A\), then for every maximal ideal \(m \subset A\), \(Fitt_{i-1}M_m = (Fitt_{i-1}M)_m = A_m\). By [6, Prop. 20.6], the localization \(M_m\) is generated by \(i - 1\) elements \(e_1, \ldots, e_{i-1}\). It follows that \(\Lambda^i M_m = 0\) for every \(m\), thus \(\Lambda^i M = 0\).
Conversely, if \( \text{Fitt}_{i-1}M \neq A \), then there exists a maximal ideal \( \mathfrak{m} \) which contains it. Thus \( 0 = \text{Fitt}_{i-1}M(A/\mathfrak{m}) = \text{Fitt}_{i-1}(M \otimes A/\mathfrak{m}) \subset A/\mathfrak{m} \). This means that \( M \otimes (A/\mathfrak{m}) \) is an \( A/\mathfrak{m} \)-vector space of dimension at least \( i \). Let \( m_1, \ldots, m_i \) be elements in \( M \) whose classes are linearly independent in \( M \otimes A/\mathfrak{m} \). Then \( A^iM \neq 0 \) since the projection of \( m_1 \wedge \cdots \wedge m_i \) to \( \Lambda^iM \otimes A/\mathfrak{m} = \Lambda^i(M \otimes A/\mathfrak{m}) \) does not vanish.

The next proposition characterizes \( H \subset \mathbb{G}_{S_{R+1}}^{p(R+1)} \) as the maximal locus \( Z \subset \mathbb{G}_{S_{R+1}}^{p(R+1)} \) such that the sheaf \( (\mathcal{O}_{Z_L} \otimes S_R)/(I_{R+1} : L)_{Z_L} \) has some trivial Fitting ideal.

**Proposition 6.2.** Let \( p \) be a non-constant Hilbert polynomial. Let \( z : Z \to \mathbb{G}_{S_{R+1}}^{p(R+1)} \) The following conditions are equivalent:

1. The morphism \( z \) factorizes through \( H \subset \mathbb{G}_{S_{R+1}}^{p(R+1)} \).
2. \( \text{Fitt}_{p(R)}((\mathcal{O}_{Z_L} \otimes S_R)/(I_{R+1} : L)_{Z_L} = \mathcal{O}_{Z_L} \).

**Proof.** The problem is local on \( Z \) and we replace \( Z \) by an affine scheme \( \text{Spec}(B) \). Similarly we may suppose that \( z : Z = \text{Spec}(B) \to \mathbb{G}_{S_{R+1}}^{p(R+1)} \) factorizes through an open subscheme \( \text{Spec}(A) \hookrightarrow \mathbb{G}_{S_{R+1}}^{p(R+1)} \) such that the universal sheaf \( \mathcal{I}_{\text{Spec}(A),R+1} \subset \mathcal{O}_{\text{Spec}(A)} \otimes S_{R+1} \) is identified with a free \( A \)-module \( I_{A,R+1} \subset S_{A,R+1} \).

In this local context, we identify the sheaf \( (I_{R+1} : L)_{Z_L} \) with its set of global sections \( (I_{B \otimes k(L),R+1} : L) \subset S_{B \otimes k(L),R} \) and we want the factorization through \( H \) to be equivalent to the condition \( \text{Fitt}_{p(R)}S_{B \otimes k(L),R}/(I_{B \otimes k(L),R+1} : L) = B \otimes k(L) \).

The closed subscheme \( z^{-1}(H) \subset \text{Spec}(B) \) is the locus where the equations \( z^*E(m, n, x) = 0 \) hold, i.e. for all monomials \( m_i \in S_R, n_i \in S_{R+1} \), the element \( Lm_1 \wedge Lm_2 \cdots \wedge Lm_{p(R)+1} \wedge n_{p(R)+2} \wedge \cdots \wedge n_{p(R)+1} \in \Lambda^{p(R)}((S_{B,R+1}/I_{B,R+1}) \otimes k(L)) \) vanishes. By linearity, for all \( z_1, \ldots, z_{p(R)+1} \in S_{B,R} \), for all \( z_{p(R)+2} \cdots z_{p(R)+1} \in S_{B,R+1} \), the element \( e = Lz_1 \wedge Lz_2 \cdots Lz_{p(R)+1} \wedge z_{p(R)+2} \wedge \cdots \wedge z_{p(R)+1} \in \Lambda^{p(R)}((S_{B,R+1}/I_{B,R+1}) \otimes k(L)) \) verifies \( e = 0 \).

Since this is true for all \( z_i \), the vanishing of the elements \( e \) is equivalent to the vanishing of \( Lz_1 \wedge Lz_2 \cdots Lz_{p(R)+1} \) in \( \Lambda^{p(R)+1}((S_{B,R+1}/I_{B}) \otimes k(L)) \) for all \( z_i \in S_{B,R} \).

According to [6, Prop. A.2.2.d], the two maps

\[
\begin{align*}
\wedge^{p(R)+1}S_{B \otimes k(L),R} & \xrightarrow{\varphi_L} \wedge^{p(R)+1}((S_{B,R+1}/I_{B}) \otimes k(L)) \\
z_1 \wedge \cdots \wedge z_{p(R)+1} & \mapsto Lz_1 \wedge \cdots \wedge Lz_{p(R)+1}
\end{align*}
\]

and

\[
\wedge^{p(R)+1}S_{B \otimes k(L),R} \rightarrow \wedge^{p(R)+1}(S_{B \otimes k(L),R}/(I_{B \otimes k(L),R+1} : L))
\]

have the same kernel \( (I_{B \otimes k(L),R+1} : L) \wedge \wedge^{p(R)}S_{B \otimes k(L),R} \).

By the above, \( z \) factorizes through \( H \) iff \( z_1 \wedge z_2 \cdots \wedge z_{p(R)+1} \) is in the Kernel of \( \varphi_L \) for all \( z_i \in S_{B,R} \), which is equivalent to the vanishing of \( z_1 \wedge \cdots \wedge z_{p(R)+1} \) in \( \wedge^{p(R)+1}(S_{B \otimes k(L),R}/(I_{B \otimes k(L),R+1} : L)) \). We conclude by Lemma 6.1 that this may be reformulated as \( \text{Fitt}_{p(R)}S_{B \otimes k(L),R}/(I_{B \otimes k(L),R+1} : L) = B \otimes k(L) \).

**Definition 6.3.** If \( p \) is a constant Hilbert polynomial, we define \( H \subset \mathbb{G}_{S_{R+1}}^{p(R+1)} \) to be \( H = \mathbb{G}_{S_{R+1}}^{p(R+1)} \).

**Proposition 6.4.** Proposition 6.2 remains valid for a constant polynomial \( p \) when \( H \) is defined by Definition 6.3.
Proof. We use the notations from the proof of Proposition 6.2. All morphisms \( z \) factorize through \( H \). To prove that all corresponding Fitting ideals are trivial, we need by Lemma 6.1 to check the vanishing of \( z_1 \wedge \cdots \wedge z_{p(R)+1} \) in \( \wedge^{p(R)+1}(S_{B \otimes k(L)} \otimes (I_{B \otimes k(L)} \otimes R_{p(R)+1} : L)) \) or equivalently to check that the map \( \varphi_L \) is zero. This vanishing is true since \( (S_{B,R+1}/IB) \) is free of rank \( p(R+1) = p(R) \), thus the target space \( \wedge^{p(R)+1}((S_{B,R+1}/IB) \otimes k(L)) \) of \( \varphi_L \) is zero. \( \square \)

**Definition 6.5.** A morphism \( f : X \rightarrow Y \) is called schematically dominant if the smallest closed subscheme \( Z \subset Y \) such that \( f \) factorizes through \( Z \) is \( Z = Y \), or equivalently if the morphism of sheaf \( f^* : O_Y \rightarrow f_*(O_X) \) is injective.

**Lemma 6.6.** Let \( X \) be a scheme over \( k \) locally of finite type and \( X_L = X \times_k \text{Spec}(k(L)) \). If \( c : \text{Spec}(k) \rightarrow X \) is a closed point, we denote by \( c_L : \text{Spec}(k(L)) \rightarrow X_L \) the corresponding \( k(L) \)-point in \( X_L \). If \( u : U_{X,L} \hookrightarrow X_L \) is an open embedding such that \( U_{X,L} \) contains all the points \( c_L \), then \( u \) and \( pr_1 : U_{X,L} \rightarrow X \) are schematically dominant morphisms.

Proof. The problem is local and we may suppose that \( X = \text{Spec}(A) \) is affine. We argue by contradiction and we suppose that \( u^* : A \otimes k(L) \rightarrow O_{U_{X,L}}(U_{X,L}) \) vanishes on a non-zero element \( f \in A \otimes k(L) \). Let \( m \in \text{Spec}(A) \) be a maximal ideal and let \( c = c_m : \text{Spec}(k) \rightarrow \text{Spec}(A) \) be the corresponding morphism. We choose \( m \) so that the localization \( f_m \in A_m \otimes k(L) \) remains non zero. We decompose \( f_m \) as \( f_m = \sum \alpha_i \otimes l_i \) where the \( l_i \in k(L) \) are linearly independent over \( k \) and \( \alpha_i \neq 0 \). Let \( t_i \in \mathbb{N} \) such that \( \alpha_i \in m^{t_i} \setminus m^{t_i+1} \). By the Theorem of Krull [1, Theorem 10.17], \( t_i \) is well defined. Let \( t = \min(t_i) \). Then \( f_m = f_1 + f_2 \) with \( f_1 = \sum_{t_i = t} \alpha_i \otimes l_i \) and \( f_2 = \sum_{t_i > t} \alpha_i \otimes l_i \).

Let \( U_m = U_{X,L} \times_{X_L} \text{Spec}(A_m \otimes k(L)) \). Then \( u_m : U_m \hookrightarrow \text{Spec}(A_m \otimes k(L)) \) is an open embedding and \( U_m \) contains \( c_L \). In particular, there exists a fundamental open set \( D = \text{Spec}((A_m \otimes L)_g) \subset U_m \) defined by the non-vanishing locus of a function \( g \in A_m \otimes k(L) \) such that \( c_L \in D \), i.e. \( g \notin m \otimes k(L) \). Thus the class \( \bar{g} \in A/m \otimes k(L) \simeq k(L) \) is invertible. Let \( \bar{h} \) be the inverse. Then \( gh = 1 + e \) with \( e \in m \otimes k(L) \).

Let \( j : D \rightarrow U_m \rightarrow U_{X,L} \) be the composition and consider the maps:

\[
A \otimes k(L) \xrightarrow{u^*} O_{U_{X,L}}(U_{X,L}) \xrightarrow{j^*} (A_m \otimes k(L))_g.
\]

Since \( j^*u^*(f) = 0 \), we get \( g^*f_m = 0 \) for some \( s \). Thus

\[
0 = h^*g^*f_m = (1 + e)^*(f_1 + f_2) = \sum_{t_i = t} \alpha_i \otimes l_i + d,
\]

with \( d \in m^{t+1} \otimes k(L) \). It follows that \( \alpha_i \in m^{t+1} \) if \( t_i = t \). This is a contradiction. Thus \( u^* \) is injective. Since \( pr_1^* \) is the composition of \( u^* \) with the injection \( A \rightarrow A \otimes k(L) \), \( pr_1^* \) is injective too. \( \square \)

The next proposition is the main result of this section. It is the functorial statement corresponding to the fact that a closed point \( I_{R+1} \in \mathbb{G}^{p(R+1)}_{S_{R+1}} \) lies in \( H \) iff the conductor \( (I_{R+1} : l) \subset S_R \) has codimension \( p(R) \) for a general linear form \( l \).

**Proposition 6.7.** Let \( z : Z \rightarrow H \subset \mathbb{G}^{p(R+1)}_{S_{R+1}} \), with \( Z \) locally of finite type. There exists an open subscheme \( U_{Z,L} \subset Z_L = Z \times \text{Spec}(k(L)) \) such that

- \( U_{Z,L} \) contains all the points \( c_L \) with \( c \) closed point of \( Z \).
• The restriction of the sheaf $(\mathcal{O}_{Z_L} \otimes S_R)/(\mathcal{I}_{R+1} : L)_{Z_L}$ to $U_{Z_L} \subset Z_L$ is locally free of rank $p(R)$ and induces a morphism $h_{Z,L} : U_{Z,L} \to \mathbb{G}^{p(R)}_{S_R}$.
• The projection $\text{pr}_1 : U_{Z,L} \to Z$ and the embedding $U_{Z,L} \hookrightarrow Z_L$ are schematically dominant.

**Lemma 6.8.** Let $(A, \mathfrak{m})$ be a local noetherian ring, let $\varphi : A^\ast \to A'$ be a $A$-module morphism, and $\psi : (A/\mathfrak{m})^\ast \to (A/\mathfrak{m})^\ast$ the reduction of $\varphi \mod \mathfrak{m}$. If $\text{rank}(\psi) \geq r$ then $\text{Fitt}_{r-1}(A^\ast/\text{Ker}(\varphi)) = 0$.

**Proof.** Let $p_{i_1,\ldots,i_r} : A^\ast \to A^r$, $(x_1,\ldots,x_l) \mapsto (x_{i_1},\ldots,x_{i_r})$ and $q_{i_1,\ldots,i_r} : (A/\mathfrak{m})^\ast \to (A/\mathfrak{m})^r$ the reduction of $p_{i_1,\ldots,i_r} \mod \mathfrak{m}$. By hypothesis, there exist indexes $i_1,\ldots,i_r$ such that $q_{i_1,\ldots,i_r} \circ \psi$ is surjective. By Nakayama, $p_{i_1,\ldots,i_r} \circ \varphi : A^\ast \to A^r$ is surjective. Since $\text{Ker}(p_{i_1,\ldots,i_r} \circ \varphi) \supset \text{Ker}\varphi$, we obtain

$$\text{Fitt}_{r-1}A^\ast/\text{Ker}\varphi \subset \text{Fitt}_{r-1}A^\ast/\text{Ker}(p_{i_1,\ldots,i_r} \circ \varphi) = \text{Fitt}_{r-1}A^r = 0.$$ 

\[\square\]

**Proof of Proposition 6.7.** Let $c : \text{Spec}(k) \to Z$ be a closed point. Let $l \in S_{k(l),1}$ be a linear form corresponding to a (possibly non-closed) point in $S_1$ with residual field $k(l)$. We denote by $c_l : \text{Spec}(k(l)) \to Z \times \text{Spec}(k(l)) = Z_1$ the corresponding point.

The morphism $h_{Z,L}$ follows from the universal property of the Grassmannian if we prove that $\mathcal{O}_{Z_L} \otimes S_R/(\mathcal{I}_{R+1} : L)_{Z_L}$ is locally free of rank $p(R)$ on some open subscheme $U_{Z,L} \subset Z_L$. For any open subscheme $U_{Z,L}$ containing the points $c_L$, the assertions on the schematically dominant morphisms follow from Lemma 6.6. Thus it suffices to prove the local freeness of $\mathcal{O}_{Z_L} \otimes S_R/(\mathcal{I}_{R+1} : L)_{Z_L}$ on a neighborhood of any fixed $c_L$.

By Propositions 6.2 and 6.4, the Fitting ideal of order $p(R)$ of $(\mathcal{O}_{Z_L} \otimes S_R)/(\mathcal{I}_{R+1} : L)_{Z_L}$ is the unit ideal on any neighborhood of $c_L$. According to [6, Prop. 20.8], it remains to prove that the Fitting ideal of order $p(R) - 1$ vanishes on a neighborhood $V_c$ of $c_L \in Z_L$ to obtain the local freeness of $\mathcal{O}_{Z_L} \otimes S_R/(\mathcal{I}_{R+1} : L)_{Z_L}$. According to Theorem 2.4, there exists an open set $U \subset S_1 \times \mathbb{G}^{p(R+1)}_{S_{R+1}}$ such that for every $(l, c) \in U$, then $l \neq 0$ and the restriction of $I_{c,R+1}$ on the hyperplane $P_l$ defined by $l$ has codimension at most $p(R+1)<_{R>}$. In symbols, with $T = k[x_1,\ldots,x_r]$, $\text{codim}(\frac{I_{c,R+1}}{(I_{c,R+1} \cap l)_T} \subset T_{R+1}) \leq p(R+1)<_{R>}$. We define $U_H$ and $U_Z$ by the following cartesian diagram, where the second line is the pullback of the first line by the open immersion $U \hookrightarrow S_1 \times \mathbb{G}^{p(R+1)}_{S_{R+1}}$.

$$
\begin{array}{ccc}
S_1 \times Z & \xrightarrow{\text{Id}_{S_1}} & S_1 \times H \\
\uparrow & & \uparrow \\
U_Z & \rightarrow & U_H \\
\end{array}
$$

Let $p \in Z$ and $l \in S_1$ be $k$-points such that $(l,p) \in U_Z$. Consider the sequence

$$S_R \xrightarrow{q_l} S_{R+1} \xrightarrow{q} S_{R+1}/I_{z(p),R+1},$$

so that $\text{Ker}(q \circ q_l) \simeq (I_{z(p),R+1} : l)$. Then

$$\text{rank}(q \circ q_l) = \dim S_R + \dim T_{R+1} - \dim I_{z(p),R+1} + (\dim I_{z(p),R+1} - \dim T_{R+1} - \dim l(I_{z(p),R+1} : l)) \geq p(R+1) - p(R+1) = p(R).$$

Lemma 6.8 shows that the germ of $\text{Fitt}_{p(R)-1}\frac{\mathcal{O}_Z \otimes S_R}{(I_{R+1} : l)_Z}$ at every point $(p,l) \in U_Z$, ie $\text{Fitt}_{p(R)-1}\frac{\mathcal{O}_{U_Z} \otimes S_R}{(I_{R+1} : l)_{U_Z}} = 0$. Let $U_{Z,c} = U_Z \cap (S_1 \times \{c\}) \subset S_1 \times Z$. The sequence
Consider the following diagram:

\[ U_{Z,L} \xrightarrow{h_{Z,L}} \mathbb{G}_{S_R}^{p(R)} \]
\[ \downarrow \text{pr}_1 \]
\[ Z \xrightarrow{h} H \xrightarrow{l} \mathbb{G}_{S_{R+1}}^{p(R+1)} \]
\[ \uparrow \text{pr} \]
\[ \text{Hilb}_{\mathbb{P}^n}^p \]

The following conditions are equivalent:

(1) There is a factorization \( h_{Z,L} = h \circ \text{pr}_1 \) of \( h_{Z,L} : U_{Z,L} \xrightarrow{pr_1} Z \xrightarrow{h} \mathbb{G}_{S_R}^{p(R)} \) through \( Z \).
Lemma 7.2. Let $\varphi : \text{Spec}(A) \rightarrow \text{Hilb}^{p}_{B}$ and $\psi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be morphisms of $k$-schemes. Let $i_{R} : \text{Hilb}^{p}_{B} \rightarrow \mathbb{G}^{p}_{S_{R}}$ and $i_{R+1} : \text{Hilb}^{p}_{B} \rightarrow \mathbb{G}^{p(R+1)}_{S_{R+1}}$ be the embeddings of Theorem 2.2. Then $(I_{R,R+1} : S_{1}) = (I_{A,R+1} : S_{1}) \otimes B$.

Proof. It follows from Theorem 2.2 and its proof ([19, C4.2,C.27,C.29]) that $i_{R} \circ \varphi$ is defined by $(I_{A,R+1} : S_{1})$ and $i_{R} \circ \varphi \circ \psi$ is defined by $(I_{B,R+1} : S_{1})$. By universal property of the Grassmannian, the pullback $i_{R} \circ \varphi \circ \psi$ of $i_{R} \circ \varphi$ is defined by $(I_{A,R+1} : S_{1}) \otimes B$. □

Proof of Proposition 7.1. If 2) is true, we consider the embedding $i_{R} : \text{Hilb}^{p}_{B} \rightarrow \mathbb{G}^{p}_{S_{R}}$ of Theorem 2.2: it is defined by the universal ideal sheaf $i_{R}^{*}I_{R} = (i^{*}l^{*}I_{R+1} : S_{1})$. The map $i_{R} \circ j \circ pr_{1} : U_{Z,L} \rightarrow \mathbb{G}^{p}_{S_{R}}$ is induced by the sheaf

$$pr_{1}^{*}j^{*}i_{R}^{*}I_{R} = pr_{1}^{*}j^{*}(i^{*}l^{*}I_{R+1} : S_{1}) \overset{Lemma7.2}{=} (pr_{1}^{*}j^{*}i^{*}l^{*}I_{R+1} : S_{1}) = (pr_{1}^{*}b^{*}l^{*}I_{R+1} : S_{1})$$

whereas $h_{Z,L} : U_{Z,L} \rightarrow \mathbb{G}^{p}_{S_{R}}$ is by construction induced by the sheaf $(I_{R+1} : L)_{Z,L} \otimes O_{U_{Z,L}}$. If a section $e$ of $O_{U_{Z,L}} \otimes S_{R}$ satisfies $x_{i}e \in pr_{1}^{*}b^{*}l^{*}I_{R+1} \subset O_{U_{Z,L}} \otimes S_{R+1}$ for every variable $x_{i}$ then by $O_{U_{Z,L}}$-linearity $(a_{0}x_{0} + \cdots + a_{n}x_{n})e \in pr_{1}^{*}b^{*}l^{*}I_{R+1}$. Algebraically, this remark corresponds to the inclusion of sheaves on $U_{Z,L}$:

$$(I_{R+1} : L)_{Z,L} \otimes O_{U_{Z,L}} \supset (pr_{1}^{*}b^{*}l^{*}I_{R+1} : S_{1})$$

The quotients $S_{R} \otimes O_{U_{Z,L}}/(I_{R+1} : L)_{Z,L} \otimes O_{U_{Z,L}}$ and $S_{R} \otimes O_{U_{Z,L}}/(pr_{1}^{*}b^{*}l^{*}I_{R+1} : S_{1})$ are locally free of the same rank $p(R)$ hence it follows from the displayed inclusion that they are equal. We conclude that the maps $h_{Z,L}$ and $i_{R} \circ j \circ pr_{1}$ defined by these sheaves are equal, hence the factorization of $h_{Z,L}$.

Suppose conversely that 1) is true. Then the sheaf

$$(I_{R+1} : L)_{Z,L} \otimes O_{U_{Z,L}} = pr_{1}^{*}(b^{*}l^{*}I_{R+1} : L) \overset{Lemma6.9}{=} (pr_{1}^{*}b^{*}l^{*}I_{R+1} : L)$$

defining $h_{Z,L}$ coincides with $pr_{1}^{*}h^{*}I_{R}$. Since $i$ is an embedding, the problem of factorizing $b$ is local so we may suppose that both $Z$ and $U_{Z,L}$ are affine with rings $\Gamma(Z)$ and $\Gamma(U_{Z,L})$. Let $J_{R} := H^{0}(h^{*}I_{R}) \subset \Gamma(Z) \otimes S_{R}$ and $J_{R+1} := H^{0}(b^{*}l^{*}I_{R+1}) \subset \Gamma(Z) \otimes S_{R+1}$. The inclusion

$$LJ_{R} \subset J_{R+1} \otimes k(L) \subset \Gamma(Z) \otimes k(L) \otimes S_{R+1}$$

holds. Indeed, since the embedding $U_{Z,L} \rightarrow Z_{L}$ is dominant by Proposition 6.7, we need to check that it holds after pullback to $U_{Z,L}$, i.e. we need to check

$$Lpr_{1}^{*}J_{R} \subset pr_{1}^{*}J_{R+1}$$

which is true since $pr_{1}^{*}h^{*}I_{R} = (pr_{1}^{*}b^{*}l^{*}I_{R+1} : L)$. The set of linear forms $l \in S_{1}$ satisfying $lJ_{R} \subset J_{R+1} \otimes k(l)$ is a closed locus in $S_{1}$. Since this locus contains the generic point $L$, any linear form $l$ satisfies $lJ_{R} \subset J_{R+1} \otimes k(l)$. In particular $S_{1}J_{R} \subset J_{R+1}$. The pair $(J_{R}, J_{R+1})$ then defines a morphism $j : Z \rightarrow \text{Hilb}^{p}_{B}$. Since $l \circ b$ and $l \circ i \circ j$ are both defined by $J_{R+1}$, and since $l$ is an embedding, we get $b = i \circ j$. □

The Plücker coordinates of definition 4.1 are the maximal minors of a surjective morphism. Our situation is different since $(I_{R+1} : L)$ is the kernel of a non-surjective morphism. The various determinants that we compute naturally in this context form a
superabundant family of Plücker coordinates: some determinants are Plücker coordinates while others are multiple of Plücker coordinates. The formal definition is the following.

**Definition 7.3.** Let \( I \subset O_X \) be a sheaf of modules on a scheme \( X \) such that \( O_X/I \) is locally free of rank \( l \) on an open subscheme \( D \subset X \). Let \( f : D \to \mathbb{G}^l_{k^n} \) be the corresponding Grassmannian morphism.

A superabundant family of Plücker coordinates for \( I \) or for \( f \) is a tuple \( (p_i(j)) \) satisfying the following conditions:

- \( \forall i, j, p_i(j) \in \mathcal{O}_X(X) \) is a global section of \( \mathcal{O}_X \)
- \( j \) runs through a set \( J \),
- \( i = (i_1, \ldots, i_l) \) runs through the set \( \{1, \ldots, n\}^l \),
- \( \forall x \in D, \exists j_x \in J \) such that the germs of \( (p_i(j_x)) \in \mathcal{O}_{X,x} \) are Plücker coordinates for the localized morphism \( f_x : \text{Spec}(\mathcal{O}_{X,x}) \to \mathbb{G}^l_{k^n} \).
- \( \forall x \in D, \forall j \in J, \exists b_j \in \mathcal{O}_{X,x} \) such that \( \forall i, p_i(j) = b_j p_i(j_x) \) locally in \( \mathcal{O}_{X,x} \).

In intuitive terms, for each index \( j \) fixed, we get a set of functions \( p_i(j) \) which is a candidate to be a set of Plücker coordinates. When we conduct a local study around a point \( x \), one index \( j_x \) corresponds to a set of Plücker coordinates. The functions associated to an other index \( j \) are multiple of the Plücker coordinates associated to \( j_x \). The multiplication constant is denoted by \( b_j \) for simplicity (although it depends on \( j \) and \( x \)) as there will be no confusion.

**Proposition 7.4.** Let \( f : \text{Spec}(A) \to \mathbb{G}^l_{k(L)} \) be a morphism to the Grassmannian defined by an exact sequence of \( A \)-modules \( 0 \to K \to A^n \to A^n/K \to 0 \), with \( A^n/K \) locally free of rank \( l \). Suppose that \( K \) is the kernel of a (possibly not surjective) morphism of \( A \)-modules \( g : A^n \to A^N \) with \( N \geq l \). Let \( e_1, \ldots, e_n, f_1, \ldots, f_N \) be the canonical bases of \( A^n \) and \( A^N \). Let

\[
p_i(j) = g(e_{i_1}) \wedge g(e_{i_2}) \wedge \cdots \wedge g(e_{i_l}) \wedge f_{j_1} \wedge \cdots \wedge f_{j_{N-l}} \in \wedge^l A^N \cong A
\]

be the minor of order \( l \) of the matrix of \( g \) with columns in \( i = (i_1, \ldots, i_l) \) and rows not in \( j = (j_1, \ldots, j_{N-l}) \). Then the elements \( p_i(j) \in A \) form a superabundant family of Plücker coordinates for \( f \). If \( \alpha : A^P \to A^N \) is surjective, and \( m_1, \ldots, m_P \) is the basis of \( A^P \), the elements

\[
q_i(j) = g(e_{i_1}) \wedge g(e_{i_2}) \wedge \cdots \wedge g(e_{i_l}) \wedge \alpha(m_{j_1}) \wedge \cdots \wedge \alpha(m_{j_{N-l}}) \in \wedge^l A^N \cong A
\]

form a superabundant family of Plücker coordinates for \( f \).

**Proof.** Being a set of superabundant Plücker coordinate is a local property, thus we may suppose that \( A \) is local with maximal ideal \( m \). Let \( j_0 = (j_1, \ldots, j_{N-l}) \in \{1, \ldots, N\}^{N-l} \) be a tuple of distinct elements. Let \( t = (t_1, \ldots, t_l) \subset \{1, \ldots, N\}^{N-l} \) be a complementary tuple of \( j_0 \) in \( \{1, \ldots, N\} \). We choose \( j_0 \) such that the rows \( L_{t_1}, \ldots, L_{t_l} \) of the matrix of \( g \otimes A/m \) has rank \( l \). Let \( pr_{j_0} : A^N \to A^l \) be the projection on the components number \( (t_1, \ldots, t_l) \) so that \( pr_{j_0} \circ (g \otimes A/m) : (A/m)^n \to (A/m)^l \) is surjective. By Nakayama, the map \( pr_{j_0} \circ g : A^n \to A^l \) is surjective. The kernel \( K_{j_0} \subset A^n \) of \( pr_{j_0} \circ g \) contains the kernel \( K \) of \( g \). The quotient \( Q = K_{j_0}/K \) satisfies \( Q \otimes A/m = 0 \) by hypothesis. Thus it follows by Nakayama again that \( K_{j_0} = K \). From the exact sequence

\[
0 \to K \to A^n_{m} \xrightarrow{pr_{j_0} \circ g} A^l \to 0,
\]
we deduce that the Plücker coordinates of $f$ are the determinants of $pr_{j_0} \circ g$. The determinant for the tuple of columns $i = (i_1, \ldots, i_l)$ coincides with $p_i(j_0)$, up to a sign depending on $j_0$.

Thus $\{p_i(j_0), i \in \{1, \ldots, n\}\}$ is a set of Plücker coordinates for $f$. It remains to show that $p_i(j)$ is a multiple of $p_i(j_0)$ to show that $\{p_i(j)\}$ is a family of superabundant Plücker coordinates. Let $j = (j_1, \ldots, j_{N-1}) \in \{1, \ldots, N\}^{N-1}$ be a tuple of distinct elements. With the same notation as above, the kernel $K_j$ contains $K = K_{j_0}$, hence we have the following diagram, where $s$ is obtained by diagram chasing:

$$0 \to K \to A^n \xrightarrow{pr_{j_0} \circ g} A^l \to 0$$

The composition formula for determinants and the commutative triangle of this diagram show that $p_i(j) = b_{j,j_0}p_i(j_0)$, where $b_{j,j_0} = \pm \det(s)$.

Finally, we may expand the elements $q_i(j)$ on the elements $p_i(j)$ as $q_i(j) = \sum_k c_{jk}p_i(k) = d_jp_i(j_0)$ with $d_j = (\sum_k c_{jk}b_{k,j_0})$, where the coefficients $c_{jk}$ depend on $j$ and on the matrix of $\alpha$. Since $\alpha$ is surjective, the $A$-module generated by $\{q_i(j), j \in \{1, \ldots, P\}^{N-1}\}$ contains $p_i(j_0)$, $d_j$ is not in $m$ for some $j$.

It follows that the elements $q_i(j)$ form a superabundant family of Plücker coordinates for $f$: one index $j$ corresponds locally to a Plücker coordinate and the other indexes correspond to a multiple of a Plücker coordinate.

Proposition 7.5. Superabundant Plücker coordinates for $h_{Z,L}$. Let $i_Z : Z = \text{Spec}(A) \hookrightarrow H$ be a $k$-morphism. For every tuple of monomials $i = (m_1, \ldots, m_{p(R)}) \in S_R$, for every tuple of monomials $j = (n_1, \ldots, n_{p(R+1)-p(R)}) \in S_{R+1}$, let

$$P_i(j) = Lm_1 \wedge \cdots \wedge Lm_{p(R)} \wedge n_1 \cdots \wedge n_{p(R+1)-p(R)} \in (\wedge^{p(R+1)}S_{A_L,R+1}/I_{A_L,R+1}) \simeq A_L.$$  

Then $P_i(j)$ is a superabundant set of Plücker coordinates for the morphism $h_{Z,L} : U_{Z,L} \to \mathbb{G}^{p(R)}_{S_R}$ of Proposition 6.7.

Proof. The morphism $h_{Z,L}$ is defined by (the restriction on $U_{Z,L}$ of the universal sheaf $(I : L)_{Z_L} \subset S_{A_L,R}$, which is by definition the kernel of $g : S_{A_L,R} \to S_{A_L,R+1}/I_{A_L,R+1}$, where $g$ is the multiplication by $L$. To compute the superabundant Plücker coordinates, we apply Proposition 7.4 with $g$ as above and the projection $\alpha : S_{A_L,R+1} \to S_{A_L,R+1}/I_{A_L,R+1}$. □

Remark 7.6 (Cross product remark). A $k(L)$-point of $\mathbb{P}^n$ is a $k$-point when quadratic cross product equations hold. For instance, the point $P = (3a_1 + 2a_0a_2 : 6a_1 + 4a_0a_2 : 9a_1 + 6a_0a_2) = (1 : 2 : 3) \in \mathbb{P}^2$ is a $k$-point. The coefficients $(3 : 6 : 9)$ and $(2 : 4 : 6)$ of the graded parts are proportional. This is measured by determinants of order 2.

The factorization of $h_{Z,L}$ is an incarnation of computations with coefficients in $k$. Hence it can be measured algebraically by quadratic determinantal equations as in the above example. This is shown formally in the next proposition.

We use the notation $k[L] = \text{Sym}(S_1^*) \simeq k[a_0, \ldots, a_n]$ so that $S_1 = \text{Spec}(k[L])$ and $k(L) = \text{frac}(k[L])$.

Proposition 7.7. Let $A$ be a $k$-algebra and $A_L = A \otimes k(L)$. Let $e : U_{A,L} \hookrightarrow \text{Spec}(A_L)$ be a schematically dominant open embedding such that the first projection $pr_1 : U_{A,L} \to \text{Spec}(A)$ is schematically dominant. Let $h_{A,L} : U_{A,L} \to \mathbb{G}^n_{k^s}$ be a morphism to a Grassmannian. Let $(p_i(j))_{j \in J} \subset A \otimes k[L]$ such that the restrictions $e^*(p_i(j)) \in O_{U_{A,L}}(U_{A,L})$
form a superabundant set of Plücker coordinates for $h_{A.L}$. For $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, let $p_i^\alpha(j) \in A$ be the coefficient of $a_0^{\alpha_0} \cdots a_n^{\alpha_n}$ of $p_i(j)$. Then the following two conditions are equivalent

1. There exists a morphism $h : \text{Spec}(A) \to \mathbb{G}^l_k$, factorizing $h_{A.L}$ i.e. $h_{A.L} = h \circ p_{r_1}$
2. For all $i_1, i_2, j_1, j_2, \alpha_1, \alpha_2$, the equality $p_{i_1}^{\alpha_1}(j_1)p_{i_2}^{\alpha_2}(j_2) = p_{i_2}^{\alpha_2}(j_1)p_{i_1}^{\alpha_1}(j_2) \in A$ holds.

Proof. Since $p_{r_1}$ is schematically dominant, the existence of a factorization $h$ is a local problem on $\text{Spec}(A)$. The equality $p_{i_1}^{\alpha_1}(j_1)p_{i_2}^{\alpha_2}(j_2) = p_{i_2}^{\alpha_2}(j_1)p_{i_1}^{\alpha_1}(j_2)$ can be checked locally too. Hence, restricting $\text{Spec}(A)$ if necessary, we may suppose that there exists $j_0$ such that $(p_i(j_0)) \in A \otimes k[L]$ restrict on $U_{A,L}$ to a set of Plücker coordinates of $h_{A.L}$.

Suppose that 1) is true. Let $(p_i) \in A$ be a set of Plücker coordinates for $h$. Since the Plücker coordinates do not vanish simultaneously and are defined up to a multiplicative constant, we may localize and suppose that there exists $i_0$ such that $p_{i_0} = 1 \in A$. Since $h_{A.L} = h \circ p_{r_1}$, then $p_{i_1}^*(p_{i_0}(j_0)) = e^*(p_{i_0}(j_0))$ for some $b_j \in \mathcal{O}_{U_{A.L}}(U_{A.L})$. The case $i = i_0$ shows that $b_j = e^*(p_{i_0})$. The equality $e^*(p_i(j)) = e^*(p_{i_0}(j))$ follows. The map $e^* : A \to \mathcal{O}_{U_{A.L}}(U_{A.L})$ is injective since $e$ is dominant. Identifying the homogeneous components, we obtain $p_i^\alpha(j) = p_{i_0}^\alpha(j_i) \in A$ since $p_{i_0}(j_0) \in A$. The equality $p_{i_1}^{\alpha_1}(j_1)p_{i_2}^{\alpha_2}(j_2) = p_{i_2}^{\alpha_2}(j_1)p_{i_1}^{\alpha_1}(j_2)$ follows.

Conversely, suppose that 2) holds. Since the Plücker coordinates do not vanish simultaneously, for any maximal ideal $m$ in $A$, there exists $i_0$ and $\alpha_0$ such that the class of $p_{i_0}^{\alpha_0}(j_0)$ in $A/m$ is non-zero. Localizing, we may suppose that $p_{i_0}^{\alpha_0}(j_0) \in A$ is invertible. Let $i_G : \mathbb{G}^l_k \to \mathbb{P}(l)^{-1}$ be the Plücker embedding. Let $\hat{h} : \text{Spec}(A) \to \mathbb{P}(l)^{-1}$ be the morphism defined by the Plücker coordinates $p_i^{\alpha_i}(j_0) \in A$. Let $d_\alpha = \frac{p_i^{\alpha_i}(j_0)}{p_{i_0}^{\alpha_0}(j_0)} \in A$ and let $d = \sum_{\alpha=\alpha_0,\ldots,\alpha_n} d_\alpha a_0^{\alpha_0} \cdots a_n^{\alpha_n} \in A_L$. The Plücker coordinates $e^*(p_i(j))$ of $h_{A.L}$ are a multiple of the Plücker coordinates $pr_1^*(p_i^{\alpha_i}(j_0)) = e^*(p_i^{\alpha_i}(j_0))$ for $h \circ pr_1$ since $p_{i_0}(j_0) = dp_i^{\alpha_i}(j_0)$. Thus $i_G \circ h_{A.L} = \hat{h} \circ pr_1$. Since $h \circ pr_1$ factorizes through $\mathbb{G}^l_k$, and since $pr_1$ is schematically dominant, it follows that $Im(\hat{h}) \subset \mathbb{G}^l_k$, or more precisely there is a factorization $\hat{h} = i_G \circ h$, with $h : \text{Spec}(A) \to \mathbb{G}^l_k$. The factorization $h_{A.L} = h \circ pr_1$ follows. \hfill $\square$

**Remark 7.8.** The quadratic equations measure the proportionality of the graded parts of $(\mathcal{I}_{R+1} : L)$ with respect to the variables $a_i$ of the generic linear form $L = a_0x_0 + \cdots + a_nx_n$. This is how we discovered them and this gives meaning to the quadratic equations of the Hilbert scheme. However, once the equations are known (using the present paper) or guessed by whatever global method, it is possible to check them locally using only usual (non-generic) linear forms. Several technical details involving the generic $L$ are then avoided. This approach has been followed in an independent version (arxiv, version 1).

**Definition 7.9.** Let $\underline{m} = (m_1, \ldots, m_{p(R)})$ be a tuple of monomials in $S_R$ and $\underline{n} = (n_1, \ldots, n_{p(R+1)-p(R)})$ be a tuple of monomials in $S_{R+1}$. Let $\underline{x} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{p(R)}})$ be a tuple of $p(R)$ variables. Let $P_{\underline{m},\underline{n}} \underline{x} = x_{i_1}m_{1} \land \cdots \land x_{i_{p(R)}}m_{p(R)} \land n_1 \land \cdots \land n_{p(R+1)-p(R)}$ be the corresponding Plücker coordinate on $S_{(R+1)}^L$. The permutations $\sigma$ of $\{1, \ldots, p(R)\}$ act on the set of $\underline{x}$ by $\sigma.\underline{x} = (x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(p(R)}})$ and we denote by $O_{\underline{x}}$ the orbit of $\underline{x}$. We
They vanish since the 4 monomials $F_{m,n}(x)$ defined the Hilbert scheme over the algebraic closure $k$.

**Theorem 2.** The Hilbert scheme is the subscheme of $Hilb$ defined by the quadratic equations $F(m_1, n_1, x_1)F(m_2, n_2, x_2) = F(m_2, n_1, x_1)F(m_1, n_2, x_2)$.

**Proof.** We check that these global equations define the Hilbert scheme on each affine open subscheme $Z = \text{Spec}(A)$ of $H$. By Proposition 7.1, the Hilbert scheme in $Z$ is the locus where the morphism $h_{Z,L} : U_{Z,L} \to \mathbb{G}^{p(R)}_S$ factorizes. By Proposition 7.7, this factorization locus is defined by quadratic equations in the superabundant Plücker coordinates. Finally, the superabundant Plücker coordinates have been computed in Propositions 7.5 and 7.10. 

**8. Extensions of the theorem**

The goal of this section is to discuss the hypotheses of Theorem 1.3. We prove that:

- Theorem 1.3 (proved so far for $k$ algebraically closed) is true for any field,
- the bound on $R$ is sharp.

**Proposition 8.1.** Theorem 1.3 holds for any base field $k$.

**Proof.** We consider the embedding of the Hilbert scheme $Hilb^p_{k} \subset \mathbb{P}^{N-1}_k$ over $k$. Let $\overline{k}$ be the algebraic closure of $k$. Consider the inclusion $T = k[y_0, \ldots, y_{N-1}] \subset \overline{T} = \overline{k}[y_0, \ldots, y_{N-1}]$ where $T$ is the homogeneous coordinate ring of $\mathbb{P}^{N-1}_k$. According to [16, Prop. 1.3.10], $Hilb^p_{\overline{k}} = Hilb^p_{k} \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$. It follows that any set of equations in $T$ which define the Hilbert scheme $Hilb^p_{k} \subset \mathbb{P}^{N-1}_k$ over $\overline{k}$ are equations defining the Hilbert scheme $Hilb^p_{\overline{k}} \subset \mathbb{P}^{N-1}_k$ over $\overline{k}$. The equations of Theorem 1.3 have coefficients in $k$, and define the Hilbert scheme over the algebraic closure $\overline{k}$, hence the equations define the Hilbert scheme over $\overline{k}$. 

**Proposition 8.2.** It is not possible to replace the condition $R \geq r$ with $R \geq r - 1$ in Theorem 1.3.

**Proof.** Let us consider the Hilbert scheme parameterizing subschemes in $\mathbb{P}^2$ with polynomial $p(t) = t + 2$: an example of such schemes is the disjoint union of a line and a point. The Gotzmann number of $p$ is $r = 2$ and we choose $R = r - 1 = 1$.

When $m, n$ are fixed and $x$ varies freely, the linear forms $E(m, n, x)$ are the coefficients of the expansion of $Lm_1 \wedge \cdots \wedge Lm_{p(R) + 1} \wedge n_1 \wedge \cdots \wedge n_{p(R) - p(R) - 1} = Lm_1 \wedge \cdots \wedge Lm_4$.

They vanish since the 4 monomials $m_i$ of degree $R = 1$ are dependent.

Similarly, the linear forms $F(m, n, x)$ are the coefficients of the expansion of $Lm_1 \wedge \cdots \wedge Lm_3 \wedge n_1$. The only possibly non-zero linear forms $F(m, n, x)$ are
those with $m = [x_0, x_1, x_2]$ (up to permutation). Hence, each quadratic equation $F(m_1, n_1, x_1)F(m_2, n_2, x_2) - F(m_2, n_1, x_1)F(m_1, n_2, x_2)$ vanishes: if $m_1 = m_2$ (up to permutation), the vanishing is obvious and if $m_1 \neq m_2$, at least one factor in each addend is zero.

Therefore, in the present case, the equations described in Theorem 1.3 with $R = r - 1$ reduce to the set of Plücker ones, so that the scheme they describe is the whole Grassmannian $\text{Gr}_2^{S_2}$. The Hilbert scheme $\text{Hilb}_{2S_2}^2$ can be embedded in this Grassmannian, but as a proper subscheme: for instance $(k[x_0, x_1, x_2]/(x_0^2, x_1^2, x_2, x_1x_2))_2$ is a $k$-point of the Grassmannian that does not belong to $\text{Hilb}_{2S_2}^2$ since the corresponding scheme is empty. □

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