Integral and Theta Formulae for Solutions of $sl_N$
Knizhnik-Zamolodchikov Equation at Level Zero

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Abstract

The solutions of the $sl_N$ Knizhnik-Zamolodchikov (KZ) equations at level 0 are studied. We present the integral formula which is obtained as a quasi-classical limit of the integral formula for form factors of the $SU(N)$ invariant Thirring model due to F. Smirnov. A proof is given that those integrals satisfy $sl_N$ KZ equation of level 0. The relation of the integral formulae with the chiral Szegő kernel is clarified. As a consequence the integral formula with the special choice of cycles is rewritten in terms of the Riemann theta functions associated with the $Z_N$ curve. This formula gives a generalization of Smirnov’s formula for $sl_2$.

0 Introduction

In [14] F. Smirnov derived a curious theta formula for the solution of the $sl_2$ Knizhnik-Zamolodchikov (KZ) equation at level 0. The aim of this paper is to generalize Smirnov’s results to the case of $sl_N$.

Before giving a more detail of our results let us summarize the reason why we are interested in the level 0 case of the KZ equation.

The KZ equation was introduced in [5] as one of the fundamental equations characterizing the correlation functions of the Wess-Zumino-Witten (WZW) model in conformal field theory. For the affine Lie algebra $\hat{G}$ and its highest weight representations $V_1, \cdots, V_m$ the KZ equation has the form

$$(k + g) \frac{\partial F}{\partial \lambda_i} = \sum_{j \neq i} \frac{\Omega_{ij}}{\lambda_i - \lambda_j} F,$$

where $F$ is a $V_1 \otimes \cdots \otimes V_m$ valued function in $\lambda_1, \cdots, \lambda_m$, $\Omega_{ij}$ is the invariant tensor, with respect to the symmetric invariant bilinear form of $G$, acting on $i$-th and $j$-th tensor components, $g$ is the dual Coxeter number of $G$ and $k$ is a parameter. The number $k$ is called level. In the WZW models levels are positive integers which coincide with those of the integrable highest weight representation of $\hat{G}$.

The KZ equation acquires a new life from the study of the two dimensional integrable massive quantum field theories (IMQFT) and solvable lattice models (SLM). F. Smirnov formulated an axiom of locality for form factors and, for several models, obtained integral formulas of form factors [13]. In [12] the rational $q$ deformed KZ ($q$KZ) equation was found as a consequence of the axiom. Hence the moment the $q$KZ equation is invented the integral formula for the solution is constructed. It is important to note that the $q$KZ equation appeared in this context is of level 0.

Around the same time I. Frenkel and N. Reshetikhin developed a general theory of vertex operators for quantum affine algebras [3]. They derived a $q$KZ equation of general level as the equation satisfied by the highest-highest matrix element of the vertex operators. This theory was successfully applied to the study of SLMs [4]. Although, in this application to SLM, the building blocks are the vertex operators of positive integer levels, the form factors and the correlation functions are shown to satisfy the level 0 and level $-2 \times (\text{dual Coxeter number})$ $q$KZ equation respectively [5][6]. Thus the level 0 $q$KZ equation and their degenerations are of special importance in the context of IMQFT and SLM.
In order to understand the nature of form factors F. Smirnov studied the quasi-classical limit [2] [3]. He noticed that the period integral of the hyperelliptic curve $s^2 = f(z) = \prod_{j=1}^{2n} (z - \lambda_j)$ appears as the limit of the integral formula for the form factors of $SU(2)$ invariant Thirring model. Then in [4] he rewrites them in terms of Riemann theta functions as

$$f_{\epsilon_1, \cdots, \epsilon_{2n}}(\lambda_1, \cdots, \lambda_{2n}) = \zeta_{\Lambda}(\det A)^{-3} \Delta^{-3/4} \theta[e_{\Lambda}] (0)^4 \det \left( \partial_i \partial_j \log \theta[e_{\Lambda}] (0) \right)_{1 \leq i,j \leq g},$$

where $\Lambda = (\epsilon_1, \cdots, \epsilon_{2n})$ is the sequence of $\pm$, the number of $+$ being equal to the number of $-$, $\zeta_{\Lambda}$ a certain fourth root of unity, $e_{\Lambda}$ a nonsingular even half period corresponding to the partition $\{1, 2, \cdots, 2n\} = \{j | \epsilon_j = +\} \cup \{j | \epsilon_j = -\}$, $\Delta = \prod_{i<j} (\lambda_i - \lambda_j)$, $\partial_i = \partial/\partial z_i$, $g = n - 1$ the genus of the curve, $\{A_i, B_j\}$ a canonical homology basis and $A = (\int_A z^{j-1} dz/s)_{1 \leq i,j \leq 2n}$. The function $F = \sum f_{\epsilon_1, \cdots, \epsilon_{2n}} v_1 \otimes \cdots \otimes v_{2n}$ gives a solution to the KZ equation taking the value in $V^{\otimes 2n}$ with $V = C v_+ \oplus C v_-$ being the vector representation of $sl_2$.

Since the theta function of an algebraic curve is the tau function, modulo some factor, of a soliton equation, this result suggests an intimate relation of the level 0 KZ equation with the soliton equations. In spite of Smirnov’s effort on this problem [4] [5] this relation is not yet clearly understood.

Integral formulas are known for the solutions of the KZ equation with an arbitrary level associated with any Kac Moody Lie algebra [6] [11]. In [4] it is shown that those general integral formula has the exact forms as their integrands in the case of $sl_2$, level 0 and singlet solutions. Taking this fact into consideration is crucial to give a complete correspondence between general formulae at level 0 and the Smirnov type formulae in [4]. A completely analogous structure exists in the case of $sl_2$ rational qKZ equation [9]. Thus Smirnov type formula is related with a subtle structure of level 0. In the $sl_N$ case to find a similar structure to the $sl_2$ case in the formulae in [4] [11] is not yet succeeded.

One strategy to understand Smirnov type solutions more clearly will be to generalize it. This is the reason why we are interested in the generalization of the Smirnov’s results to the other types of Lie algebras than $sl_2$.

Now let us describe our results. In [3] the integral formula for form factors of the $SU(N)$ invariant Thirring model is obtained. It is a solution to the $sl_N$ rational qKZ equation of level zero. We take the quasi-classical limit of this integral formula. It is expressed as the determinant of the period integrals of a $Z_N$ curve. A $Z_N$ curve is a natural generalization of a hyperelliptic curve, which corresponds to $N = 2$. Roughly speaking the integral formula obtained in this manner should give a solution to the $sl_N$ KZ equation of level zero. From the mathematical point of view it is not very easy to prove rigorously that the asymptotics satisfies the KZ equation. On the other hand the formula for the quasi-classical limit is rather simple. Hence it is desirable and interesting to prove directly that it satisfies the KZ equation. We give a proof which is new even for the $sl_2$ case. Compared with the proof in the generic level case [7] [11] our proof looks more complicated. It will be related with the degenerate structure of Smirnov type solutions found in [3]. Since we have established a correct Smirnov type integral formula in the $sl_N$ case it is an interesting problem to get them from the formulae in [4] [11] in the spirit of [2].

We rewrite the integral formula in terms of theta functions on a $Z_N$ curve. A priori this is not a trivial task at all. In fact the following major problems are not obvious from the formula and arguments in the $sl_2$ case. The first one is what kind of rational periods parametrize the tensor components of the solution. The second one is whether we can expect the second order derivatives of the logarithm of theta functions or not in the $sl_N$ case. The first problem is resolved with the help of the Thomae formula for $Z_N$ curves which was discovered by Bershadsky and Radul [11]. Namely the tensor component is parametrized by certain non-singular $1/N$ or $1/2N$ periods introduced in [11]. The second problem is solved by finding a relation of the integrand of the integral formula with the Szegő kernel. In fact the product of Szegő kernels is related with the second order derivatives of the logarithm of theta functions by the formula due to Fay [2].

Now the present paper is organized in the following manner. In section 1 the integral formula is given. The theta formula is given in section 2. It is proved in section 3. In section 4 a proof is given that the integral formula satisfies the KZ equation and belongs to the trivial representation of $sl_N$. A derivation of fundamental relations among differential forms used in section 4 is given in appendix.
1 Integral Formulas

Let $sl_N$ be the simple Lie algebra of type $A_{N-1}$, $(\ , \ )$ the symmetric bilinear form on $sl_N$ given by $(X, Y) = \text{tr}(XY)$, $\{I_j\}$ a basis of $sl_N$ and $\{P^j\}$ the dual basis with respect to $(\ , \ )$. The invariant element $\Omega$ is given by

$$\Omega = \sum_j I_j \otimes P^j.$$ 

Let $V$ be the $N$ dimensional irreducible representation of $sl_N$ and $m$ a positive integer. The Knizhnik-Zamolodchikov (KZ) equation with values in the $Nm$ fold tensor product $V^\otimes Nm$ of $V$ is the differential equation for the $V^\otimes Nm$ valued function $F$

$$(k + N) \frac{\partial F}{\partial \lambda_i} = \sum_{j \neq i} \Omega_{ij} \frac{\partial F}{\lambda_i - \lambda_j},$$

where $\Omega_{ij}$ means the action of $\Omega$ on the $i$-th and $j$-th components of $V^\otimes Nm$, $k$ is a complex number called level. The explicit form of KZ equation in terms of the vector components is given in section 4.

Let $v_j = \bigotimes_{r=1}^j(0,\cdots,1,\cdots,0)$ in $C^N$, where 1 is on the $j$-th place. Then we have $V = \bigoplus_{j=1}^N C v_j$. We denote by $\Lambda = (\Lambda_1,\cdots,\Lambda_N)$ the ordered partition of $\{1,2,\cdots,Nm\}$ such that the number $|\Lambda_i|$ of the elements of $\Lambda_i$ is $m$ for any $i$. To an ordered partition $\Lambda$ we associate the weight zero vector $v_\Lambda$ of $V^\otimes Nm$ by

$$v_\Lambda = v_{k_1} \otimes \cdots \otimes v_{k_{Nm}},$$

where

$$i \in \Lambda_j \text{ if and only if } k_i = j.$$ 

The set of $\{v_\Lambda\}$ forms a base of the weight zero subspace of $V^\otimes Nm$.

The operators $\sum_{j \neq i}(\lambda_i - \lambda_j)^{-1}\Omega_{ij}$ in the right hand side of (2) commute with the action of $sl_N$. Thus it has a sense to consider the KZ equation for a function taking values in a fixed weight subspace of $V^\otimes Nm$. In this paper we exclusively consider the solution $F$ whose value is in the weight zero subspace of the tensor product $V^\otimes Nm$. Then we can define the component $f_\Lambda$ of $F$ by

$$F = \sum_{\Lambda} f_\Lambda v_\Lambda,$$

where the sum is over all ordered partition $\Lambda$.

We denote by $C$ the compact Riemann surface defined from the equation $s^N = f(z) = \prod_{i=1}^{Nm}(z - \lambda_i)$. It is called a $Z_N$ curve [1]. The genus $g$ of $C$ is given by $2g = (N - 1)(Nm - 2)$. For $\Lambda_r$ and $p \in \Lambda_r$ set

$$g^{(\Lambda_r)}(z) = \prod_{j \notin \Lambda_r}(z - \lambda_j), \quad g_{\Lambda_r}(z) = \prod_{j \in \Lambda_r}(z - \lambda_j), \quad g^{(p)}_{\Lambda_r}(z) = \prod_{j \notin \Lambda_r \cup \Lambda_r}(z - \lambda_j)$$

and define the meromorphic differential form $\mu^\Lambda_p(z)$ on $C$ by

$$\mu^\Lambda_p(z) = \frac{g^{(\Lambda_r)}(\lambda_p)g^{(p)}_{\Lambda_r}(z)}{(z - \lambda_p)^N} \, dz.$$ 

We set $L = (N - 1)m - 1$. Then we have

Theorem 1 Let $\{p_1,\cdots,p_L\}$ is an arbitrary subset of $\{1,2,\cdots,Nm\}$. Define

$$f(\lambda_1,\cdots,\lambda_{Nm})_\Lambda = \frac{\Delta_{\Lambda}^{N-1}}{\prod_{i<j}(\Lambda_i \Lambda_j)} \frac{\det(f_{ij}^{(p_1,\cdots,p_L)})_{1 \leq i,j \leq L}}{\Delta(p_1,\cdots,p_L)},$$

where $(\Lambda_i \Lambda_j) = \prod_{r \in \Lambda_i, s \in \Lambda_j}(\lambda_r - \lambda_s)$, $\Delta(p_1,\cdots,p_L) = \det(\lambda_{p_i}^{L-j})_{1 \leq i,j \leq L}$ and $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$. Then
The right hand side of (4) does not depend on the choice of \( \{p_1, \ldots, p_L\} \).

The function \( F \) given by (3) and (4) is a solution to the \( sl_N \) KZ equation of level zero for arbitrary set of \( L \) cycles \( \{\gamma_1, \ldots, \gamma_L\} \) on \( C \).

For any \( X \in sl_N \), \( XF = 0 \).

The first statement of Theorem 1 follows from another expression for \( f_\Lambda \). Let us set
\[
\zeta^\Lambda_j = \frac{dz}{s} \sum_{k=1}^{N} g_{\lambda_k}(z) \left[ \frac{d}{dz} g^{(\lambda_k)}(z) \right]_{0},
\]
where \([\ ]_0\) denotes the polynomial part of a Laurent polynomial. It is obvious that \( d/dz \) can be outside of the symbol \([\ ]_0\).

**Theorem 2** The function \( f_\Lambda \) given by (4) is also written as
\[
f(\lambda_1, \ldots, \lambda_{Nm})_\Lambda = \frac{\Delta_{N-1}}{\prod_{i<j}(\lambda_i \lambda_j)} \det(\int_{\gamma_i} \zeta^\Lambda_j)_{1 \leq i,j \leq L}.
\] (5)

We shall give some comments on the integral formula given here. In [13] F. Smirnov derived the integral formula of form factors of \( SU(N) \) invariant Thirring model which satisfy the deformed Knizhnik-Zamolodchikov(dKZ) equation on level zero. Scaling the rapidity variables \( \beta_j \) as \( \beta_j = \lambda_j/h \) and taking the quasi-classical limit \( h \to 0 \), we obtain the integral formula in Theorem 2 with some special choice of cycles \( \{\gamma_i\} \).

### 2 Theta Formula

We shall give another expression for the solution \( F \) given in Theorem 1. To give a precise statement we prepare necessary notations associated with the \( Z_N \) curve \( C \).[7] The \( N \)-cyclic automorphism \( \phi \) of \( C \) is defined by \( \phi: (z, s) \mapsto (z, \omega s) \), where \( \omega \) is the \( N \)-th primitive root of unity. There are \( Nm \) branch points \( Q_1, \ldots, Q_{Nm} \) whose projection to \( z \) coordinate are \( \lambda_1, \ldots, \lambda_{Nm} \). The basis of holomorphic 1-forms on \( C \) is given by
\[

w^{(\alpha)}_\beta = \frac{z^{1-1}dz}{s^\alpha} \quad 1 \leq \alpha \leq N-1, \quad 1 \leq \beta \leq \alpha m - 1.
\]

We fix a canonical homology basis \( \{\alpha_i, \beta_j\} \) whose intesection numbers are \( \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0 \), \( \alpha_i \cdot \beta_j = \delta_{ij} \). Let \( \Delta \) be a Riemann divisor for this choice of canonical basis. Let us define the divisor class \( D \) by \( D = NQ_1 \), which is independent of the choice of \( i \). To each ordered partition \( \Lambda \) we associate the divisor class \( e_\Lambda \)[6] by
\[
e_\Lambda \equiv \Lambda_1 + 2\Lambda_2 + \cdots + (N-1)\Lambda_{N-1} - D - \Delta,
\]
where for a subset \( S \) of \( \{1, 2, \ldots, Nm\} \) we set
\[
S = \sum_{j \in S} Q_j.
\]

The divisor class \( e_\Lambda \) is a \( 1/N \) period for \( N \) even and is a \( 1/2N \) period for \( N \) odd. We consider the index of \( \Lambda_j \) by modulo \( N \). In particular \( \Lambda_0 = \Lambda_N \).
Let \( \{ v_j(x) \} \) be the basis of the normalized abelian differentials of the first kind whose normalization is

\[
\int_{A_j} v_k(x) = 2\pi i \delta_{jk}.
\]

We set \( \tau_{jk} = \int_{B_j} v_k(x) \). Then the period matrix \( \tau = (\tau_{jk}) \) is symmetric and its real part is negative definite. The Jacobian variety \( J(C) \) of \( C \) is described as \( J(C) = \mathbb{C}^g/2\pi i \mathbb{Z}^g + \mathbb{Z}^g \tau \). For any element \( e \in \mathbb{C}^g \), there exist unique elements \( \delta, \epsilon \in \mathbb{R}^g \) such that

\[
e = \begin{bmatrix} \delta \\ \epsilon \end{bmatrix}_\tau = 2\pi i \epsilon + \delta \tau.
\]

We call \( \delta, \epsilon \) the characteristics of \( e \). The Riemann theta function with characteristics \( \delta, \epsilon \) is defined by

\[
\theta \left[ \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \right](z) = \sum_{m \in \mathbb{Z}^g} \exp \left( \frac{1}{2} (m + \delta) \tau (m + \delta)^t + (z + 2\pi i \epsilon)(m + \delta)^t \right).
\]

It satisfies the equation

\[
\theta \left[ \begin{bmatrix} \delta + m \\ \epsilon + n \end{bmatrix} \right](z) = \exp(2\pi i n \delta) \theta \left[ \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \right](z),
\]

for \( m, n \in \mathbb{Z}^g \). For an ordered partition \( \Lambda \) let us take a representative \( \bar{e}_\Lambda \in \mathbb{C}^g \) of \( e_\Lambda \) and let \( \bar{e}_\Lambda = \begin{bmatrix} \delta \\ \epsilon \end{bmatrix}_\tau \). Then the logarithmic derivatives

\[
\partial^\alpha \log \theta \left[ \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \right](z), \quad |\alpha| \geq 1,
\]

are independent of the choice of the representative \( \bar{e}_\Lambda \) by (6), where \( \alpha = (\alpha_1, \ldots, \alpha_g) \), \( |\alpha| = \alpha_1 + \cdots + \alpha_g \), \( \partial^\alpha = \partial_{\alpha_1}^1 \cdots \partial_{\alpha_g}^g \) and \( \partial_j = \partial/\partial z_j \). Hence we use the notation \( \partial^\alpha \log \theta[e_\Lambda](z) \) for those logarithmic derivatives for the sake of simplicity.

Let us define the connection matrix between \( \{ w^{(\alpha)}_\beta \} \) and \( \{ v_j(x) \} \) by

\[
v_j(x) = \sum_{\alpha, \beta} \sigma_{(\alpha, \beta)} w^{(\alpha)}_\beta.
\]

With the aid of this matrix we define the vector field on \( J(C) \) by

\[
D_\beta = \sum_{j=1}^g \sigma_{j(N-1)\beta} \partial_j, \quad 1 \leq \beta \leq L.
\]

Now we can state the theta formula.

**Theorem 3** For any subset \( \{ i_1 < \cdots < i_L \} \) of \( \{ 1, 2, \ldots, g \} \) we take the cycles \( \{ \gamma_j \} \) as \( \gamma_j = A_{i_j} \). Then the corresponding solution of the KZ equation in Theorem 1 is given by

\[
f(\lambda_1, \ldots, \lambda_{Nm})_\Lambda = c \frac{\Delta_{N-1}}{\prod_{i<j}(\Lambda_i A_j)} \det \left( \partial_{i_k} \log \theta[e_\Lambda](0) \right)_{1 \leq j, k \leq L}.
\]

where \( c \) is the overall constant independent of \( \lambda_i \)’s and \( \Lambda \).

Using the Thomae formula for \( \mathbb{Z}_N \) curves one can rewrite the \( \prod_{i<j}(\Lambda_i A_j) \) in terms of theta constants. The result is
**Theorem 4** For the same choice of cycles as in Theorem 3, we have
\[ f(\lambda_1, \cdots, \lambda_N) = C(\lambda)^{\zeta} \prod_{\sigma \in S_{N-1}} \theta[e_{\sigma}]^{(0)} \det (\partial_{i} D_{j} \log \theta[e_{\sigma}](0))_{1 \leq i, j \leq L}, \]
where \( \zeta \) is some \( N(N+1)!/3 \)-th root of unity, \( C(\lambda) \), which is independent of the partition \( \Lambda \), is given by
\[ C(\lambda) = c(\det A)^{-\frac{N}{2(N+1)}} \Delta^{-\frac{3}{4} \frac{(r+1)}{N+1}}. \]
Here \( c \) is a constant independent of \( \lambda_i \)'s and \( A = \det(\int_{A} w_{\alpha}(\beta)). \) For an element \( \sigma \) of the symmetric group \( S_{N-1} \) of degree \( N-1 \) we define
\[ \Lambda^{\sigma} = (\Lambda_0, \Lambda_{\sigma(1)}, \cdots, \Lambda_{\sigma(N-1)}). \]

**Remark.** If \( N = 2 \), then \( L = m-1 = g = (\sigma_{j(1k)}) = A^{-1}, \) where \( g \) is the genus of the hyperelliptic curve \( C. \) In particular \((i_1, \cdots, i_L) = (1, \cdots, g). \) From the matrix relation
\[ (\partial_{i} D_{j} \log \theta[e_{\sigma}](0)) = (\partial_{i} \partial_{j} \log \theta[e_{\sigma}](0)) A^{-1} \]
we have
\[ f_{\Lambda} = c' \zeta^{g} (\det A)^{-3} \Delta^{-3/4} \theta[e_{\sigma}](0)^{4} \det (\partial_{i} \partial_{j} \log \theta[e_{\sigma}](0))_{1 \leq i, j \leq g} \]
which is nothing but the Smirnov’s formula \( \text{(6)} \) for \( sl_2. \)

**3 Proof of the Theta Formula**

Let \( \tilde{C} \) be the universal covering space of \( C. \) We identify a holomorphic one forms on \( C \) with those on \( \tilde{C} \) which are invariant under the action of the fundamental group of \( C. \) We set \( v = (v_1, \cdots, v_g) \), the vector of the normalized differentials of the first kind. Recall that the chiral Szegő kernel defined by \( e_{\Lambda} \) is
\[ R(x, y|e_{\Lambda}) = \frac{\theta[e_{\Lambda}](y-x)}{\theta[e_{\Lambda}](0)E(x, y)} \quad x, y \in \tilde{C}, \]
where
\[ y - x = \int_{x}^{y} v, \]
the integral being taken in \( \tilde{C} \) and \( E(x, y) \) is the prime form \( \text{(5)} \). We remark that, as to the \( e_{\Lambda} \) dependence, \( R(x, y|e_{\Lambda}) \) depends only on the divisor class of \( e_{\Lambda}. \)

Let \( \omega(x, y) \) be the canonical symmetric differential, that is, \( \omega(x, y) \) is the section of the canonical bundle of \( C \times C, \) symmetric in \( x \) and \( y, \) has a double pole at \( x = y, \) has the vanishing \( A \) period in each of the variables and has some normalization (for more precise definition see \( \text{(5)} \)). Theorem 4 is a corollary of the following proposition.

**Proposition 1** For \( 1 \leq p \leq Nm \) we have
\[ \mu_{p}^{\Lambda} = Nf'(\lambda_{p})^{\frac{N_{p}-1}{2}} \omega(x, Q_{p}) + N^{2} \sum_{i=1}^{g} \sum_{\beta=1}^{L} \Lambda_{\beta}^{3-1} D_{\beta} \partial_{i} \log \theta[e_{\sigma}](0)v_{i}(x). \]
As in [3] the value of a (half)differential form at the branch point $Q_p$ is defined as the coefficient of $dt$ (or $\sqrt{dt}$ in the half differential case) in the expansion of the form in the local coordinate $t = (z - \lambda_p)^{1/N}$. Assuming this proposition let us first prove Theorem 3.

Proof of Theorem 3
Since the integral of $\omega(x, Q_p)$ along the cycle $A_i$ is zero for any $i$, we have

$$\int_{A_i} \mu_p^\Lambda = N^2 \sum_{\beta=1}^{L} \Lambda_{\beta}^{i-1} D_{\beta} \log \theta(e_{\Lambda})(0),$$

where we use the normalization condition of $\{v_j\}$. Thus we have

$$\det \left( \int_{A_{ij}} \mu_p^\Lambda \right)_{1 \leq j, k \leq L} = N^{2L} \det (\partial_{\beta j} D_{\beta} \log \theta(e_{\Lambda})(0))_{1 \leq j, \beta \leq L} \det (\Lambda_{\beta}^{i-1})_{1 \leq k, l \leq L} = (-1)^{k(L-1)} N^{2L} \det (\partial_{\beta j} D_{\beta} \log \theta(e_{\Lambda})(0))_{1 \leq j, \beta \leq L} \det (\Lambda_{\beta}^{L-1})_{1 \leq k, l \leq L} = (-1)^{k(L-1)} N^{2L} \det (\partial_{\beta j} D_{\beta} \log \theta(e_{\Lambda})(0))_{1 \leq j, \beta \leq L} \Delta(p_1, \cdots, p_L).$$

Substituting this equation into (4) we obtain the formula of Theorem 3. □

In order to prove Proposition 3 we first prove

**Proposition 2** For $1 \leq p \leq Nm$ we have

$$\mu_p^\Lambda = N f'(\lambda_p)^{\frac{N-1}{N}} R(x, Q_p| e_{\Lambda}) R(x, Q_p| -e_{\Lambda}). \quad (7)$$

For the proof of this proposition let us recall the following notation[7, 11]:

$$\mathcal{L} = \{-N - \frac{1}{2}, -N - \frac{1}{2} + 1, \cdots, N - \frac{1}{2}\},$$

$$q_l(i) = \frac{1 - N}{2N} + \left\{ \frac{l + i + N - 1}{2N} \right\},$$

where $l \in \mathcal{L} + \mathbb{Z}$, $i \in \mathbb{Z}$ and $\{a\} = a - [a]$ is the fractional part of $a \in \mathbb{Q}$.

In [3] we have proved

**Proposition 3** For an ordered partition $\Lambda$ we have

$$R(x, y| e_{\Lambda}) = \frac{1}{N} \sum_{i \in \mathcal{L}} f_i(x, \Lambda) f_{i}(y, \Lambda^+) \quad (8)$$

$$f_{\pm l}(x, \Lambda^\pm) = \prod_{i=1}^{Nm} (z(x) - \lambda_i)^{\pm q_l(i)} \sqrt{d(z(x)},$$

where $\Lambda^- = (\Lambda_0, \Lambda_{N-1}, \cdots, \Lambda_1)$, $\Lambda^+ = \Lambda$ and $k_i = j$ is determined by $i \in \Lambda_j$.

By Proposition 4 in [3], for $l = -(N - 1)/2 + j$ with $j \in \mathbb{Z}$, we have

$$\text{div} f_i(x, \Lambda^\pm) = \Lambda_{\pm 1+j} + 2\Lambda_{\pm 2+j} + \cdots + (N - 1)\Lambda_{\pm N+j} - \sum_{k=1}^{N} \infty^{(k)}. \quad (9)$$
where \( \{\infty^{(k)}\} \) are \( N \) infinity points. The index \( k \) of \( \Lambda_k \) is considered by modulo \( N \). The \( \Lambda_{r,j} \) is missing in the right hand side of (9). Let \( p \in \Lambda_e \), \( 0 \leq r \leq N - 1 \). We set \( y = Q_p \) in (8). By (3) only the term \(-l = -(N - 1)/2 + r \) in the sum of the right hand side of (8) is alive. Hence we have

\[
R(x, Q_p|e_{\Lambda}) = \frac{1}{N} \int \frac{f_{-\frac{N}{2} + r}(x, \Lambda) f_{-\frac{N}{2} + r}(Q_p, \Lambda^{-})}{\lambda_p - z(x)},
\]

where \( r^- = N - 1 - r \). Similarly

\[
R(x, Q_p| - e_{\Lambda}) = R(x, Q_p| e_{\Lambda^-}) = \frac{1}{N} \int \frac{f_{-\frac{N}{2} + r - 1}(x, \Lambda^-) f_{-\frac{N}{2} + r - 1}(Q_p, \Lambda^-) f_{-\frac{N}{2} + N - r}(Q_p, \Lambda)}{\lambda_p - z(x)}.
\]

Therefore

\[
R(x, Q_p|e_{\Lambda})R(x, Q_p|- e_{\Lambda}) = \frac{1}{N^2(z(x) - \lambda_p)} \left( f_{-\frac{N}{2} + r}(x, \Lambda) f_{-\frac{N}{2} + r - 1}(x, \Lambda^-) f_{-\frac{N}{2} + r}(Q_p, \Lambda^-) f_{-\frac{N}{2} + N - r}(Q_p, \Lambda) \right) \tag{10}
\]

Let us calculate the right hand side of (10).

**Lemma 1** The following equations hold:

\[
f_{-\frac{N}{2} + r}(Q_p, \Lambda^-) f_{-\frac{N}{2} + r}(Q_p, \Lambda) = \frac{N f'(\lambda_p)^{1/N}}{g_{\Lambda_e}(\lambda_p)}, \tag{11}
\]

\[
f_{-\frac{N}{2} + r - 1}(x, \Lambda) f_{-\frac{N}{2} + r - 1}(x, \Lambda^-) = \frac{q_{\Lambda_e}(z)}{s} dz. \tag{12}
\]

Proof. Let \( t = (z - \lambda_p)^{1/N} \) be the local coordinate around \( Q_p \). Then

\[
f_{-\frac{N}{2} + r}(x, \Lambda^-) = \sqrt{N} \prod_{j \neq p} (\lambda_p - \lambda_j)^{q_{-\frac{N}{2} + r}(N-k_j)} \sqrt{dt(1 + O(t^N))},
\]

\[
f_{-\frac{N}{2} + r + N - r}(x, \Lambda) = \sqrt{N} \prod_{j \neq p} (\lambda_p - \lambda_j)^{q_{-\frac{N}{2} + r + N - r}(k_j)} \sqrt{dt(1 + O(t^N))},
\]

where we use the relation \(-q(i) = q_{-i}(N - i) \). Therefore we need to calculate the number \( q_{-\frac{N}{2} + r}(N - k_j) + q_{-\frac{N}{2} + r + N - r}(k_j) \).

By a direct calculation we have

\[
q_{-\frac{N}{2} + r}(N - i) + q_{-\frac{N}{2} + r + N - r}(i) = \begin{cases} -1 + \frac{1}{N} & \text{if } i = r \mod N \\ \frac{1}{N} & \text{if } i \neq r \mod N. \end{cases}
\]

Hence

\[
f_{-\frac{N}{2} + r}(Q_p, \Lambda^-) f_{-\frac{N}{2} + r + N - r}(Q_p, \Lambda) = N \prod_{j \notin \Lambda_r} (\lambda_p - \lambda_j)^{1/N} \prod_{j \in \Lambda_e \setminus \{p\}} (\lambda_p - \lambda_j)^{-1 + 1/N}
\]

\[
= \frac{N f'(\lambda_p)^{1/N}}{\prod_{j \in \Lambda_e \setminus \{p\}} (\lambda_p - \lambda_j)}. \]
Similarly, using
\[ q_{-\frac{N-1}{2}+r}(i) + q_{-\frac{N-1}{2}+r-1}(N-i) = \begin{cases} 
1 - \frac{1}{N} & \text{if } i = r \mod N \\
-\frac{1}{N} & \text{if } i \neq r \mod N 
\end{cases} \]
we have
\[ f_{-\frac{N-1}{2}+r}(x, \Lambda) f_{-\frac{N-1}{2}+r-1}(x, \Lambda^-) = \prod_{j \in \Lambda \setminus \{z - \lambda_j\}} \frac{s}{dz}. \]

Thus the lemma is proved. \(\Box\)

Multiplying (11) and (12) and substituting it into (10) we obtain the equation (7).

\[ \Box \]

Recall the Fay’s formula ([2], Corollary 2.12, see also [7] section 4):
\[ R(x, y|e^\Lambda) R(x, y|-e^\Lambda) = \omega(x, y) + \sum_{i, j = 1}^{g} \frac{\partial^2 \log \left[ \theta_{e^\Lambda}(0) \prod_{i \leq j} (\lambda_{r_i} - \lambda_{r_j}) \right]}{\partial z_i \partial z_j} \theta_{e^\Lambda}(0) v_i(x) v_j(y). \] (13)

By calculation we have
\[ v_j(Q_p) = \frac{N}{f^i(\lambda_p) \prod_{\beta=1}^{N} \sigma_{j(N-1)\beta}^2 - 1}. \] (14)

Substituting (14) into (13) and using Proposition 3 we have the equation in Proposition 1.

\[ \Box \]

In order to prove Theorem 4 let us recall the Thomae formula for \(Z_N\) curves ([1], [7]):
\[ \theta_{[e^\Lambda]}(0)^{2N} = C_{\Lambda} (\det A)^N \prod_{i \leq j} (\Lambda_i \Lambda_j)^{2Nq(i, j) + N \mu}, \]
where \(C_{\Lambda}\) is a constant independent of \(\lambda_i\)’s and
\[ q(i, j) = \sum_{t \in \mathcal{C}} q_t(i) q_t(j), \quad \mu = \frac{(N-1)(2N-1)}{6N}, \]
\[ (\Lambda_i \Lambda_j) = \prod_{r \in \Lambda_i, s \in \Lambda_j} (\lambda_r - \lambda_s) \quad \text{for } i \neq j, \]
\[ (\Lambda_i \Lambda_i) = \prod_{r, s \in \Lambda_i, r < s} (\lambda_r - \lambda_s). \] (15)

The number \(q(i, j)\) depends only on \(|i - j|\). In particular \(q(0, 0) = q(i, i), q(i, j) = q(j, i)\) for any \(i\) and \(j\). We define the action of the symmetric group \(S_N\) of degree \(N\) on the set of ordered partitions by
\[ \Lambda^\sigma = (\Lambda_{\sigma(0)}, \cdots, \Lambda_{\sigma(N-1)}), \quad \sigma \in S_N. \]
The subgroup \(S_{N-1}\) acts on the index 1, 2, \cdots, \(N-1\) as we already defined.

**Proposition 4** For an ordered partition \(\Lambda\) we have
\[ \prod_{i \leq j} (\Lambda_i \Lambda_j) = C_{\tilde{\zeta}} (\det A) \prod_{\sigma \in \Lambda^\sigma} \Delta^\Lambda_{\tilde{\zeta}} \theta_{[e^\Lambda^\sigma]}(0)^{-\frac{N^2}{N+1}}, \]
where \(\tilde{\zeta}\) is some \(N(N+1)!/3\)-th root of unity and \(C\) is a constant independent of \(\lambda_i\)’s and \(\Lambda\).
Let us prove Proposition 4. By taking the product of theta function with the characteristics $e_{\Lambda^s}$ for all $s \in S_N$ we have

$$\prod_{s \in S_N} \theta[e_{\Lambda^s}](0)^{2N} = \left( \prod_{s \in S_N} C_{\Lambda^s} \right) \left( \det A \right)^{N!N} \prod_{s \in S_N} (\Lambda_{s(i)} \Lambda_{s(j)})^{2Nq(i,j) + N\mu}$$

$$= \pm \left( \prod_{s \in S_N} C_{\Lambda^s} \right) \left( \det A \right)^{N!N\mu} \prod_{i \leq j} (\Lambda_i \Lambda_j)^{2N} \sum_{s \in S_N} q(s(i),s(j)).$$

Set $\gamma = q(0,0)$ and

$$F := \prod_{i \leq j} (\Lambda_i \Lambda_j)^{2N} \sum_{s \in S_N} q(s(i),s(j))$$

$$= \prod_{i=0}^{N-1} (\Lambda_i \Lambda_j)^{N!2N\gamma} \prod_{i < j} (\Lambda_i \Lambda_j)^{2N} \sum_{s \in S_N} q(s(i),s(j)).$$

Since, for $i \neq j$,

$$\sum_{s \in S_N} q(s(i),s(j)) = 2 \cdot (N - 2)! \sum_{r \neq s} q(r, s),$$

we have

$$F = \prod_{i=0}^{N-1} (\Lambda_i \Lambda_j)^{N!2N\gamma} \prod_{i < j} (\Lambda_i \Lambda_j)^{4N \cdot (N - 2)!} \sum_{i < j} q(i,j).$$

Using $q(i, j) = q(i + 1, j + 1)$ and Lemma 10 in \[7\] we have

$$\sum_{i < j} q(i,j) = -\frac{N^2 - 1}{24}, \quad \gamma = \frac{N^2 - 1}{12N}.$$

Thus

$$F = \Delta \left( \frac{(N+1)(N-1)}{6} \right) \prod_{i < j} (\Lambda_i \Lambda_j)^{\frac{(N+1)N}{6}}.$$

From this we obtain

$$\prod_{s \in S_N} \theta[e_{\Lambda^s}](0)^{2N} = \pm \left( \prod_{s \in S_N} C_{\Lambda^s} \right) \left( \det A \right)^{N!N} \Delta \left( \frac{N^N(N-1)}{2} \prod_{i < j} (\Lambda_i \Lambda_j)^{\frac{(N+1)N}{6}}.\right.$$

Recall that the ordered partitions which are obtained from $\Lambda$ by the cyclic permutation of indices correspond to linear equivalent divisor $e_\Lambda \[7\]. Therefore

$$\prod_{s \in S_{N-1}} \theta[e_{\Lambda^s}](0)^{2N^2} = \pm \left( \prod_{s \in S_{N-1}} C_{\Lambda^s} \right) \left( \det A \right)^{N!N} \Delta \left( \frac{N^N(N-1)}{2} \prod_{i < j} (\Lambda_i \Lambda_j)^{\frac{(N+1)N}{6}}.\right.$$

Since $C_{\Lambda}^{2N}$ does not depend on $C_\Lambda$, we have the equation in Proposition 3. \[\Box\]

In this section we shall give a proof of Theorem 1 and Theorem 2.
3.1 Proof of Theorem 2

Theorem 2 follows from the following proposition.

Proposition 5 For any $p$

$$\sum_{j=1}^{L} \zeta(z)\lambda_{p}^{L-j} = \mu_{p}(z) + Nd\frac{z^{N-1}}{z - \lambda_{p}}.$$  

The proof of this proposition is totally similar to the case of sl$_{2}$ [14]. For the sake of making the paper self-contained we give a proof.

Proof. Let $p \in \Lambda_{r}$. It is sufficient to prove the following equation, the coefficient of $dz/s$:

$$\sum_{j=1}^{L} \sum_{k=1}^{N} g_{\lambda_{k}}(z) \left[ \frac{d}{dz} \left( \frac{g_{\lambda_{k}}(z)}{z^{L-j+1}} \right) \right] \lambda_{p}^{L-j} = \left( \frac{g_{\lambda_{p}}(\lambda_{p})}{z - \lambda_{p}} \right) - N \frac{f(z)}{z - \lambda_{p}} + \frac{(N-1)f'(z)}{z - \lambda_{p}}. \quad (16)$$

Since both hand sides are rational functions in $z$, it is sufficient to prove (16) for $|z|$ sufficiently large.

Let $t$ be a complex parameter. More generally we calculate the right hand side of (16) replaced $\lambda_{p}$ by $t$. We have

$$\sum_{j=1}^{L} \sum_{k=1}^{N} g_{\lambda_{k}}(z) \left[ \frac{d}{dz} \left( \frac{g_{\lambda_{k}}(z)}{z^{L-j+1}} \right) \right] t^{L-j} = \frac{d}{dz} \left[ \sum_{j=0}^{\infty} (tz^{-1})^{j} \frac{f(z)}{z^{L-j+1}} \right]_{0} = \frac{d}{dz} \frac{g_{\lambda_{p}}(z)}{z - t} \bigg|_{0} = \frac{1}{z - t} \frac{g_{\lambda_{p}}(z) - g_{\lambda_{p}}(t)}{z - t}. \quad (17)$$

Thus

$$\sum_{j=1}^{L} \sum_{k=1}^{N} g_{\lambda_{k}}(z) \left[ \frac{d}{dz} \left( \frac{g_{\lambda_{k}}(z)}{z^{L-j+1}} \right) \right] t^{L-j} \quad = -N \frac{f(z)}{(z - t)^{2}} + \frac{(N-1)f'(z)}{z - t} + \sum_{k=1}^{N} \frac{g_{\lambda_{k}}(z)g_{\lambda_{k}}(t)}{(z - t)^{2}}. \quad (17)$$

In this calculation we use

$$\sum_{k=1}^{N} g_{\lambda_{k}}(z) \frac{d}{dz} g_{\lambda_{k}}(z) = (N - 1)f'(z).$$

If we set $t = \lambda_{p}$, $p \in \Lambda_{r}$ in (17), then we get the right hand side of (16). $\square$

3.2 Some Notations

For an ordered partition $\Lambda$ let us denote by $\Lambda^{(ij)}$ the ordered partition which is obtained from $\Lambda$ by exchanging $i$ and $j$. For example if $\Lambda_{i} = \{(i-1)m + 1, \cdots, im\}$, $1 \leq i \leq N$, then $\Lambda_{1}^{(12)} = \Lambda$ (assuming $m \geq 2$), $\Lambda_{1}^{(1m+1)} = \{m + 1, 2, \cdots, m\}$, $\Lambda_{2}^{(1m+1)} = \{1, m + 2, \cdots, 2m\}$, $\Lambda_{k}^{(1m+1)} = \Lambda_{k}$ for $k \geq 3$.

In terms of the components $f_{\lambda_{i}}$, if $p \in \Lambda_{i}$, the KZ equation of level zero is

$$N \frac{\partial f_{\lambda_{i}}}{\partial \lambda_{p}} = \left((1 - \frac{1}{N}) \sum_{j \in \Lambda_{i}, j \neq p} \frac{1}{\lambda_{p} - \lambda_{j}} - \frac{1}{N} \sum_{j \notin \Lambda_{i}} \frac{1}{\lambda_{p} - \lambda_{j}} \right) f_{\lambda_{i}} + \sum_{j \notin \Lambda_{i}} \frac{1}{\lambda_{p} - \lambda_{j}} f_{\lambda_{(ij)}}.$$
If we define the function $\bar{f}_\Lambda$ by

$$\bar{f}_\Lambda = \Delta \frac{N}{N^2} f_\Lambda,$$

the KZ equation above is equivalent to

$$\partial \bar{f}_\Lambda \partial \lambda_p = -\frac{1}{N} \sum_{j \notin \Lambda_i} \bar{f}_\Lambda \frac{\lambda_p - \lambda_j}{\lambda_p - \lambda_j} + \frac{1}{N} \sum_{j \notin \Lambda_i} \frac{1}{\lambda_p - \lambda_j} \bar{f}_\Lambda \Lambda (p_j).$$  \hspace{1cm} (18)

For a nonnegative integer $r$ and a subset $\{p_1, \cdots, p_r\} \subset \{1, \cdots, Nm\}$, set

$$\begin{pmatrix} \Lambda & \cdots & \Lambda \\ p_1 & \cdots & p_r \end{pmatrix} = \det \left( \mu_{\Lambda_i}(z_j) \right)_{1 \leq i, j \leq r}.$$

Then Theorem \[\text{II}\] is equivalent to the following proposition.

**Proposition 6** Let

$$\bar{f}_\Lambda = \frac{\Delta(p_1, \cdots, p_L)^{-1}}{\prod_{i<j}(\Lambda_i \Lambda_j)} \begin{pmatrix} \Lambda & \cdots & \Lambda \\ p_1 & \cdots & p_L \end{pmatrix}.$$  \hspace{1cm} (19)

Then, modulo exact forms, $\bar{f}_\Lambda$ satisfies the equation \[18\] and $X \bar{f}_\Lambda = 0$ for any $X \in \mathfrak{sl}_N$.

Our aim is to prove this proposition. Let us set

$$\Lambda_r = \{i^r_1, \cdots, i^r_m\} \quad \text{for } 1 \leq r \leq N.$$

For the sake of simple exposition we shall prove the equation \[18\] for $p = i^N_m$. Other cases are similarly proved.

In the following sections we use the usual equality symbol $=$ for the equality modulo exact forms. We remark that all the modulo exact relations, which we use, follow from the relation in Proposition \[III\].

3.3 Fundamental Relations

Now let us give all the relations which we need for our purpose. For the sake of simplicity, in the formulas below, we denote $\lambda_i$ by $i$. For instance $i - j = \lambda_i - \lambda_j$. We set $K = \{i^r_j | r < N, (r, l) \neq (N - 1, m)\}$ and

$$A_r = \frac{\prod_{s=1}^{m-1} (i^N_m - i^r_s)}{\prod_{s=1}^{m} (i^N_m - i^r_s)}.$$

For $p \neq i^N_m$ and $1 \leq j \leq Nm$ we also set

$$\begin{pmatrix} p \\ j \end{pmatrix} = \begin{pmatrix} \Lambda(i^N_m, p) \\ j \end{pmatrix}$$

for the sake of simplicity. Now the relations are given as follows.

1. For $p \in \Lambda_r, r \neq N$

$$\frac{\partial}{\partial \lambda_{i^N_m}} \begin{pmatrix} \Lambda \\ p \end{pmatrix} = \left(1 - \frac{1}{N}\right) \frac{1}{i^N_m - p} \begin{pmatrix} \Lambda \\ p \end{pmatrix} - \frac{1}{N \cdot i^N_m - p} \prod_{j \notin \Lambda_r, j \neq i^N_m} \frac{p - j}{i^N_m - j} \begin{pmatrix} p \\ i^N_m \end{pmatrix}.$$  \hspace{1cm} (20)
2. For $l \neq l'$, $(r,l), (r,l') \neq (N,m)$,

$$
\left( \frac{\Lambda}{i^m_l} \right) = \left( \frac{\Lambda}{i^m_l} \right) + \frac{i_N^m - i_N^{l'}}{i_N^m - i_N^{l'}} \sum_{k=1}^{m-1} \prod_{j \notin \Lambda_r \cap j \neq i_N^m} (i_N^m_j - j) \prod_{i \neq j, l'} (i_N^m_{i^m_k} - i_N^{l'}) \left[ - \left( \Lambda \right) \left( \frac{\Lambda}{i_N^{l'}} \right) \right]. \quad (21)
$$

3. For $l \neq l'$,

$$
\left( \frac{1}{i_{l'}^m} \right)^{N-1}_{i_{l'}^m} = \left( \frac{1}{i_{l'}^m} \right)^{N-1}_{i_{l'}^m} + \left( \frac{1}{i_{l'}^m} \right)^{N-1}_{i_{l'}^m} \prod_{j \notin \Lambda_{N-1} \cap j \neq i_N^{l'}} (i_N^m_j - j) \prod_{i \neq j, l'} (i_N^m_{i^m_k} - i_N^{l'}) \left[ - \left( \Lambda \right) \left( \frac{\Lambda}{i_N^{l'}} \right) \right]. \quad (22)
$$

4. For $r \neq N$,

$$
\left( \frac{1}{i_{l'}^m} \right) = \sum_{k \in K, j \notin K, j \neq k} \prod_{j \notin K, j \neq k} \frac{i_N^m - j}{i_N^m - j} \left( \Lambda \right) \left( \frac{\Lambda}{k} \right) + \prod_{k=1}^{m-1} \prod_{j \notin K, j \neq k} \frac{i_N^m - j}{i_N^m - j} B(r,l,k) \left[ - \left( \Lambda \right) \left( \frac{\Lambda}{i_N^{l'}} \right) \right]. \quad (23)
$$

where we set

$$
B(r,l,k) = 1 - \frac{i_N^m - i_N^{l'}}{A_r} \prod_{s=1}^{m} \prod_{j \notin K, j \neq k} \frac{i_N^m - j}{i_N^m - j}.
$$

5. For $(r,l) \neq (N,m)$,

$$
\left( \frac{1}{i_{l'}^m} \right) = \sum_{k \in K, k \neq i_{l'}^m} \prod_{j \in K, j \neq T_i} \frac{i_N^m - j}{i_N^m - j} \left( \Lambda \right) \left( \frac{\Lambda}{i_N^m} \right) + \prod_{j \notin K, j \neq i_{l'}^m} \frac{i_N^m - j}{i_N^m - j} \left( \frac{\Lambda}{i_N^m} \right). \quad (24)
$$

The proofs of these relations are given in appendix.

### 3.4 Equation for $\tilde{f}_\Lambda$

Let us take $(p_1, \cdots, p_L) = (i_1^t, \cdots, i_{m-1}^t)$ in the expression of $\tilde{f}_\Lambda$. By differentiating the defining equation (15) in $\lambda_{i_N}$ we have

$$
\frac{\partial \tilde{f}_\Lambda}{\partial \lambda_{i_N}} = \sum_{j \notin \Lambda_N} \frac{-1}{i_N^m - j} \tilde{f}_\Lambda + \frac{\Delta(i_1^t, \cdots, i_{m-1}^t)^{-1}}{\prod_{t \leq u} (\Lambda_i \Lambda_u)} \frac{\partial}{\partial \lambda_{i_N}} \left( \frac{\Lambda}{i_N^m} \cdots \frac{\Lambda}{i_N^{m-1}} \right). \quad (25)
$$

Substituting (20) into (25) we have

$$
\frac{\prod_{t \leq u} (\Lambda_i \Lambda_u)}{\Delta(i_1^t, \cdots, i_{m-1}^t)^{-1}} \frac{\partial \tilde{f}_\Lambda}{\partial \lambda_{i_N}} = \left( - \frac{1}{i_N^m - i_N^{t-1}} - \frac{1}{N \sum_{j \notin K} \frac{1}{i_N^m - j}} \right) \left( \frac{\Lambda}{i_N^m} \cdots \frac{\Lambda}{i_N^{m-1}} \right) + \frac{-1}{N \sum_{j \notin K} \frac{1}{i_N^m - j}} \prod_{j \notin \Lambda_r \cap j \neq i_N^m} \frac{i_N^m - j}{i_N^m - j} \left( \frac{\Lambda}{i_N} \cdots \frac{\Lambda}{i_N^{m-1}} \right). \quad (26)
$$

Using (23) in the second term of (26) we obtain
The equation (28) follows from the residue theorem for the function

\[
(-N)^{N} \frac{\prod_{t<u}(\Lambda_t \Lambda_u)}{\Delta(i_1, \ldots, i_{N-1})} \frac{\partial \bar{f}_{\lambda}}{\partial \lambda_{i_m}} \\
= \left[ -\frac{m+3}{i_m - i_{N-1}} \right] + \sum_{l=1}^{m-1} \frac{2}{i_m - i_l} + \sum_{r=1}^{N-2} \frac{1}{i_m - i_r}
+ \sum_{r=1}^{N-2} \frac{A_r}{(i_m - i_r)^2} \prod_{s=1}^{m-1} (i_r - i_s) \right] \left( \Lambda_i \cdots \Lambda_i \right)
+ \sum_{r=1}^{N-2} \frac{A_r}{(i_m - i_r)^2} \prod_{s=1}^{m-1} (i_r - i_s)
+ \sum_{r=1}^{N-2} \frac{A_r}{(i_m - i_r)^2} \prod_{s=1}^{m-1} (i_r - i_s)
\times \left( \Lambda_i \cdots \Lambda_i \right),
\] (27)

where in the second term \((i_m - i_{N-1})\) is on the l-th position counted from \((\Lambda_i)\). In the derivation of (27) we have used

\[
\sum_{r=1}^{N-2} \frac{A_r}{(i_m - i_r)^2} \prod_{s=1}^{m-1} (i_r - i_s) \prod_{r \neq l} (i_r - i_s) = -N - 2 \frac{A_r}{(i_m - i_r)^2} \prod_{s=1}^{m-1} (i_r - i_s)
\]

which follows from

\[
\frac{i_r - i_{N-1}}{(i_m - i_r)(i_m - i_{N-1})} = \frac{1}{i_m - i_r} - \frac{1}{i_m - i_{N-1}}
\]

and

\[
\sum_{r=1}^{m} \frac{\prod_{s=1}^{m-1} (i_m - i_s)}{(i_m - i_r) \prod_{r \neq l} (i_r - i_s)} = \frac{\prod_{s=1}^{m-1} (i_r - i_s)}{i_m - i_s} = A_r^{-1}.
\] (28)

The equation (28) follows from the residue theorem for the function

\[
\frac{\prod_{s=1}^{m-1} (z - i_s)}{(z - i_m) \prod_{s=1}^{m} (z - i_s)}.
\]

This is the typical argument to prove an identity in the proof below.

### 3.5 KZ Equation

We shall rewrite the KZ equation in a similar manner to (27). We take \((p_1, \ldots, p_L) = (i_1, \ldots, i_{N-1})\) in the expression \((13)\) for \(f_{\lambda(\mu_1|\mu_2)}\) for any \(j \notin \Lambda_N\). Then the KZ equation substituted by \((19)\) is

\[
\frac{\prod_{t<u}(\Lambda_t \Lambda_u)}{\Delta(i_1, \ldots, i_{N-1})} \frac{\partial \bar{f}_{\lambda}}{\partial \lambda_{i_m}} = -\frac{1}{N} \sum_{j \notin \Lambda_N} \frac{1}{i_m - j} \left( \Lambda_i \cdots \Lambda_i \right)
+ \frac{1}{N} \sum_{r=1}^{N-1} \frac{1}{i_m - i_r} \prod_{t<u}(\Lambda_t \Lambda_u) \left( \Lambda_i \cdots \Lambda_i \right) (i_r \cdots i_{N-1})
\] (29)
Note the relations
\[ \frac{\prod_{k \leq u} (A_k A_u)}{\prod_{k < u} (A_k^{(i_{kn})} A_u^{(i_{m1})})} = - \frac{A_r}{i_m} \frac{\prod_{r=1}^{m-1} (i_r - i_s^N)}{\prod_{s \neq r} (i_r - i_s^N)} \] (30)

Thus we have
\[ \frac{\prod_{k \leq u} (A_k A_u)}{\Delta(i_1^N, \ldots, i_{m-1}^N)} \frac{\partial \bar{f}_\Lambda}{\partial \lambda_{i_m}^{(i_k^N)}} = - \frac{1}{N} \sum_{j \notin \Lambda_N} \frac{1}{i_m - j} \left( \frac{\Lambda}{i_1^N - i_1^N} \right) \]
\[ + \frac{1}{N} \sum_{r=1}^{m-1} \sum_{l=1}^{m-1} \frac{A_r}{i_m - i_l^N} \frac{\prod_{r=1}^{m-1} (i_r - i_s^N)}{\prod_{s \neq r} (i_r - i_s^N)} \left( i_r^N \right) \] (31)

By the equation (24), if \( r \neq N \) and \( i_r^N \neq i_m^N - 1 \), we have
\[ \left( \frac{i_r^N}{i_r^N} \right) = \prod_{j \in \Lambda_N \neq i_l^N} \left( \frac{i_r^N}{i_r^N} \right) \] (32)

Using the relation (23) we have, for \( r \leq N - 2 \),
\[ \left( \frac{\Lambda}{i_1^N} \right) = \prod_{k=1}^{m} \frac{i_m^N}{i_k^N} \frac{\prod_{j \neq i_l^N} (i_j^N - i_s^N)}{\prod_{j=1}^{m} (i_j^N - i_s^N)} \left( \frac{\Lambda}{i_1^N} \right) \]
\[ + \sum_{k=1}^{m-1} \prod_{j \in \Lambda_N \neq i_l^N} B(r, l, k) \left( \frac{\Lambda}{i_1^N} \right) \] (33)

and, for \( r \neq m \),
\[ \left( \frac{\Lambda}{i_1^N} \right) = \prod_{j \in \Lambda_N \neq i_l^N} \frac{i_m^N}{i_k^N} \frac{\prod_{j \neq i_l^N} (i_j^N - i_s^N)}{\prod_{j=1}^{m} (i_j^N - i_s^N)} \left( \frac{\Lambda}{i_1^N} \right) \]
\[ + \sum_{k=1}^{m-1} \prod_{j \in \Lambda_N \neq i_l^N} B(r, l, k) \left( \frac{\Lambda}{i_1^N} \right) \] (34)

If we substitute (32), (33) and (34) into (31) we have
\[
(\Delta)^{(-N)} \frac{\prod_{k \leq u} (A_k A_u)}{\Delta(i_1^N, \ldots, i_{m-1}^N)} \frac{\partial \bar{f}_\Lambda}{\partial \lambda_{i_m}^{(i_k^N)}} = \sum_{j \notin \Lambda_N} \frac{1}{i_m - j} \left( \frac{\Lambda}{i_1^N} \right) \]
\[ + \frac{A_{N-1}}{(i_m^N - i_m^N)^2} \prod_{s=1}^{m-1} \frac{i_m^N}{i_m^N - i_s^N} \left( \frac{\Lambda}{i_1^N} \right) \]
\[ + \sum_{r=1}^{N-2} \sum_{k=1}^{m} \frac{1}{i_m^N - i_k^N} \left( \frac{\Lambda}{i_1^N} \right) \]
\[ + \sum_{l=1}^{m-1} \frac{1}{i_m^N - i_l^N} \left( \frac{\Lambda}{i_1^N} \right) \]
Then (21) can be written as

$$\sum_{i_j \in \mathcal{K}} \frac{1}{(i_j^N - i_j^r)(i_j^r - i_j^{N-1})} \prod_{k=1}^{m-1} \frac{A_r}{i_j^N - i_j^r} \prod_{j \in \mathcal{K}, j \neq i_j^m} \frac{(i_j^r - j)}{i_j^N - i_j^r} B(r, l, k) \times \left( \Lambda_{i_j^r, i_j^r} \cdot \cdot \cdot \Lambda_{i_j^N, i_j^{N-1}} \right)$$

where as in the previous case $$\left( \Lambda_{i_j^r, i_j^r} \right)$$ etc. are all on the l-th place counted from $$\left( \frac{i_j^r}{i_j^r} \right)$$ etc. For the economy of space we set

$$\Omega(r, l, \Lambda, k) = \left( \Lambda_{i_j^r, i_j^r} \cdot \cdot \cdot \Lambda_{i_j^N, i_j^{N-1}} \right)$$

$$\Omega(r, l, N - 1, k) = \left( \Lambda_{i_j^r, i_j^r} \cdot \cdot \cdot \Lambda_{i_j^{N-1}, i_j^{N-1}} \right)$$

where the positions of $$\left( \Lambda_{i_j^r, i_j^r} \right)$$ and $$\left( \frac{i_j^{N-1}}{i_j^{N-1}} \right)$$ are as above. If $$r = N - 1$$ then $$i_j^r$$ should be replaced by $$i_j^{N-1}$$.

The forms

$$\left( \Lambda_{i_j^r, i_j^r} \right), \cdot \cdot \cdot , \left( \Lambda_{i_j^N, i_j^{N-1}} \right), \left( \frac{i_j^{m-1}}{i_j^{m-1}} \right), \cdot \cdot \cdot , \left( \frac{i_j^{N-1}}{i_j^{N-1}} \right)$$

are linearly independent in the cohomology group $$H^1(C, \mathbb{C})$$. Since we do not use this fact in this paper, we do not give a proof of it. But the fact helps to understand the strategy of the proof below. The right hand side of (27) is written using these forms only. We shall rewrite the right hand side of (35) in terms of these forms.

### 3.6 Reduction of Expressions of Fundamental Determinants

For $$1 \leq r \leq N - 1$$, $$I = \{s_1 < \cdot \cdot \cdot < s_p\}$$, $$J = \{t_1 < \cdot \cdot \cdot < t_q\}$$, $$p + q = m(r < N - 1)$$, $$p + q = m - 1(r = N - 1)$$, we set

$$\Omega_{IJ} = \left( \Lambda_{i_j^r, i_j^r} \cdot \cdot \cdot \Lambda_{i_j^N, i_j^{N-1}} \right) = \left( \Lambda_{i_j^r, i_j^r} \cdot \cdot \cdot \Lambda_{i_j^{N-1}, i_j^{N-1}} \right)$$

where $$\left( \Lambda_{i_j^r, i_j^r} \right)$$ is on the $$(r-1)m + 1$$-th position counted from $$\left( \frac{i_j^r}{i_j^r} \right)$$.

- The coefficient of $$\Omega_{IJ}$$ in $$\Omega(r, l, \Lambda, k)$$.

We assume $$r < N - 1$$. Let us denote this coefficient by $$\det_{IJ}^r$$. We set

$$F_{tu}^r = \frac{1}{i_t^N - i_t^r}, \quad G_t^r = \frac{1}{\prod_{j \notin \mathcal{K}, j \neq i_t^m} (i_j^r - j)}$$

and

$$A_{tu}^r = \frac{i_j^N - i_j^r}{i_j^N - i_j^{N-1}} \prod_{e \notin l, i_j^m, i_j^{N-1}, i_j^{N-1} = (i_j^r - j)} F_{tu}^r G_t^r$$

Then (21) can be written as

$$\left( \frac{i_j^r}{i_j^r} \right) = \left( \frac{i_j^r}{i_j^r} \right) + \sum_{k=1}^{m-1} A_{tk} \left[ \left( \frac{i_j^{N-1}}{i_j^{N-1}} \right) \right]. \quad (36)$$
Using this equation we have

\[
\begin{vmatrix}
1 & 0 & \cdots & A_{r1}^r & \cdots & A_{mm}^r \\
\vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & & \vdots \\
1 & A_{m1}^r & \cdots & A_{mm}^r \\
\end{vmatrix}
\]

\[
= \left( \frac{i_N^N - i_t^r}{i_m^N - i_m^{N-1}} \right)^{|J|} \prod_{u \in J} \left( \frac{\prod_{s \neq t}^{m} (i_{u}^N - i_{s}^r)}{\prod_{j \neq t \neq i, i_{m}^{N-1} = i_{u}^N - j)} \right) \prod_{t=1}^{m} G_t^r \\
\times \begin{vmatrix}
G_1^r & \cdots & 0 & F_{11}^r & \cdots & F_{1m}^r \\\n\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & G_t^r & 0 & \cdots & 0 \\
G_{t+1}^r & \vdots & 0 & \vdots & & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
G_m^r & F_{m1}^r & \cdots & F_{mm}^r \\
\end{vmatrix}
\]

The meaning of the determinant symbol of the matrix above is the following. The matrix consists of two matrices, say, the left matrix and the right matrix. We take I-th columns from the left matrix and J-th columns from the right matrix. Then form the determinant of the resulting matrix of degree $|I| + |J|$. We shall use similar notations from now on.

Notice that, by definition, $\det^2_{IJ} = 0$ unless $k \in I$.

- The coefficient of $\Omega^r_{IJ}$ in $\Omega(r, I, N - 1, k)$.

We assume $r < N - 1$. Let us denote this coefficient by $\det^2_{IJ}$. Then, again by [36],

\[
\det^2_{IJ} = \left( \frac{i_N^N - i_t^r}{i_m^N - i_m^{N-1}} \right)^{|J|-1} \prod_{u \in J \setminus \{k\}} \left( \frac{\prod_{s \neq t}^{m} (i_{u}^N - i_{s}^r)}{\prod_{j \neq t \neq i, i_{m}^{N-1} = i_{u}^N - j)} \right) \prod_{t=1}^{m} G_t^r \\
\times \begin{vmatrix}
G_1^r & \cdots & 0 & F_{11}^r & \cdots & F_{1k}^r \\\n\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & 0 & 0 \\
G_{t+1}^r & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
G_m^r & F_{m1}^r & \cdots & F_{mk}^r & \cdots & F_{mm}^r \\
\end{vmatrix}
\]

where

\[
A_{rk}^l = A_{rk}^l G_t^r.
\]

- The coefficient of $\Omega^N_{IJ}$ in $\Omega(N - 1, I, A, l)$.
This coefficient is denoted by $\det^l_{ij}$. Let us set, for $s \neq l$,

$$
C^l_{ss} = \frac{(i_s^{N-1} - i_s^{m-1})(i_s^{N} - i_s^{N-1})}{(i_s^{m} - i_s^{m-1})(i_s^{N} - i_s^{m-1})}, \quad C^l_{sl} = -\frac{(i_s^{N} - i_s^{N-1})(i_s^{N-1} - i_s^{N-1})}{(i_s^{m} - i_s^{m-1})(i_s^{N-1} - i_s^{m-1})} \prod_{j \notin \Lambda_{N-1}, \, j \neq i_s} \frac{i_s^{N-1} - j}{i_s^{N-1} - j},
$$

$$
D^l_{ss} = \frac{(i_s^{N} - i_s^{m-1})(i_s^{N-1} - i_s^{m-1})}{(i_s^{m} - i_s^{m-1})(i_s^{N-1} - i_s^{m-1})}, \quad D^l_{sl} = -C^l_{sl}.
$$

Then, (32) is written as

$$
\left( \begin{array}{c}
i_s^{N-1} \\
i_s^{N-1} 
\end{array} \right) = C^l_{ss} \left( \begin{array}{c}
\Lambda \n_s^{N-1} \\
\Lambda \n_s^{N-1} 
\end{array} \right) + C^l_{sl} \left( \begin{array}{c}
\Lambda i_s^{N-1} \\
\Lambda i_s^{N-1} 
\end{array} \right) + D^l_{ss} \left( \begin{array}{c}
i_s^{m-1} \\
i_s^{m-1} 
\end{array} \right) + D^l_{sl} \left( \begin{array}{c}
i_s^{m-1} \\
i_s^{m-1} 
\end{array} \right).
$$

(37)

Using (37) we have

$$
\det_{ij}^l =
\begin{vmatrix}
C^l_{11} & \cdots & C^l_{1l} & 0 & D^l_{11} & \cdots & D^l_{1l} & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & C^l_{l-1l-1} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & \cdots & C^l_{l+1l+1} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & C^l_{m-1l} & \cdots & \cdot & \cdot & \cdots & D^l_{m-1l} & \cdots & D^l_{m-1m-1}
\end{vmatrix}
$$

$\bullet$ The coefficient of $\Omega_{ij}^{N-1}$ in $\Omega(N - 1, l, N - 1, k)$.

This coefficient is denoted by $\det_{ij}^l$. Then

$$
\det_{ij}^l =
\begin{vmatrix}
C^l_{11} & \cdots & C^l_{1l} & 0 & D^l_{11} & \cdots & 0 & D^l_{1l} & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & C^l_{l-1l-1} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
0 & \cdots & C^l_{l+1l+1} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & C^l_{m-1l} & \cdots & \cdot & \cdot & \cdots & 0 & \cdots & D^l_{m-1l} & \cdots & D^l_{m-1m-1}
\end{vmatrix}
$$

In the right matrix 1 is in the $(l, k)$ component.

### 3.7 Comparison of Two Equations

Now let us calculate the reduced expression of the right hand side of (35) and compare it with (27). We shall calculate the coefficient of $\Omega_{ij}^l$ by dividing the case into nine as

(I) $r = N - 1, I = \emptyset, J = \{1, 2, \cdots, m - 1\}$,

(II) $r = N - 1, I = \{1, 2, \cdots, m - 1\}, J = \emptyset$ or $r < N - 1, I = \{1, 2, \cdots, m\}, J = \emptyset$,

(III) $r = N - 1, I = \{1, 2, \cdots, m - 1\}\setminus\{t\}, J = \{t\}$,

(IV) $r = N - 1, I = \{1, 2, \cdots, m - 1\}\setminus\{u\}, J = \{t\}, u \neq t$,

(V) $r < N - 1, I = \{1, 2, \cdots, m - 1\}\setminus\{u\}, J = \{t\}$,

(VI) $r = N - 1, I \cap J = \emptyset, |I| \geq 1, |J| \geq 2,$
(VII) \( r = N - 1, |I \cap J| = 1, |J| \geq 2 \),

(VIII) \( r = N - 1, |I \cap J| \geq 2 \),

(IX) \( r < N - 1, |J| \geq 2 \).

(I): The coefficient of \( \Omega_{IJ}^{N-1} \) with \( I = \phi, J = \{1, 2, \ldots, m - 1\} \).

Let us calculate the contribution from the term which contains \( \Omega(N-1, l, N-1, k) \) \( (1 \leq l \leq m - 1) \). We have, for \( k \neq l \),

\[
\det_{l,J}^{4} = -\frac{D_{kl}^{4}}{D_{kk}^{4}} \prod_{s \neq l}^{m-1} D_{ss}^{4} = \frac{i_{l}^{N-1} - i_{m-1}^{N} (i_{m}^{N} - i_{m-1}^{N})}{i_{k}^{N-1} - i_{m}^{N} (i_{m}^{N} - i_{k}^{N})} \prod_{s \neq l}^{m-1} \frac{i_{m}^{N} - i_{s}^{N-1}}{i_{l}^{N-1} - i_{s}^{N-1}} \prod_{j \notin \Lambda N-1, j \neq i_{k}^{N}}^{N-1} \frac{i_{k}^{N-1} - j}{i_{l}^{N-1} - j}.
\]

For \( k = l \), \( \det_{l,J}^{4} = \prod_{s \neq l}^{m-1} D_{ss}^{4} \). If we set \( k = l \) in the right hand side of

\[
\frac{D_{kl}^{4}}{D_{kk}^{4}} = \prod_{j \notin \Lambda N-1, j \neq i_{k}^{N}}^{N-1} \frac{i_{k}^{N-1} - j}{i_{l}^{N-1} - j},
\]

we have \(-1\). Thus the formula for \( \det_{l,J}^{4} \) given above is valid for all \( 1 \leq k, l \leq m - 1 \). The contribution to (38) is

\[
-A_{N-1} \prod_{s=1}^{m-1} \frac{(i_{m}^{N-1} - i_{s}^{N-1})}{(i_{m}^{N} - i_{m-1}^{N})} \sum_{l=1}^{m-1} \frac{(i_{m}^{N} - i_{l}^{N-1})^{m-3}}{\prod_{s \neq l}^{m-1} (i_{l}^{N-1} - i_{s}^{N-1})} \times \sum_{k=1}^{m-1} \frac{\prod_{s \neq k}^{m-1} (i_{k}^{N-1} - i_{s}^{N})}{(i_{m}^{N} - i_{k}^{N}) \prod_{s \neq k}^{m-1} (i_{k}^{N-1} - i_{s}^{N-1})} B(N-1, l, k).
\]

(38)

By the residue theorem we have

\[
\sum_{k=1}^{m-1} \frac{i_{m}^{N} - i_{k}^{N-1}}{A_{N-1}} \sum_{k=1}^{m-1} \frac{\prod_{s \neq k}^{m-1} (i_{k}^{N-1} - i_{s}^{N})}{(i_{m}^{N} - i_{k}^{N}) \prod_{s \neq k}^{m-1} (i_{k}^{N-1} - i_{s}^{N-1})} = \frac{1}{A_{N-1}} - \frac{\prod_{s=1}^{m-1} (i_{m}^{N-1} - i_{s}^{N})}{(i_{m}^{N} - i_{m-1}^{N}) \prod_{s=1}^{m-1} (i_{m}^{N} - i_{s}^{N-1})}.
\]

Thus, substituting the definition of \( B(N-1, l, k) \) to (38),

\[
(38) = A_{N-1} \prod_{s=1}^{m-1} \frac{(i_{m}^{N} - i_{s}^{N})^{m-3}}{\prod_{s \neq l}^{m-1} (i_{l}^{N-1} - i_{s}^{N-1})} \times \sum_{l=1}^{m-1} \frac{(i_{m}^{N} - i_{l}^{N-1})^{m-3}}{\prod_{s \neq l}^{m-1} (i_{l}^{N-1} - i_{s}^{N-1})}.
\]

Here we again use the residue theorem to evaluate the summation in \( l \). Hence the coefficient of \( \Omega \) in the right hand side of (38) is zero. This is the case for (27).

(II): The coefficient of \( \Omega_{IJ}^{r} = \begin{pmatrix} \Lambda & \cdots & \Lambda \\ i_{1}^{r} & \cdots & i_{m-1}^{r} \end{pmatrix} \) with \( r = N - 1, I = \{1, 2, \ldots, m-1\} \) and \( J = \phi \) or \( r < N - 1, I = \{1, 2, \ldots, m\} \) and \( J = \phi \).
(II-I) The contribution to \( \text{rhs} \) of (35) from the term which contains \( \Omega(r, l, \Lambda, k) \) \((1 \leq l \leq m)\).

We have

\[
\det_{lJ}^1 = \begin{cases} 
0 & k \neq l \\
1 & k = l
\end{cases}
\]

Hence the contribution to the rhs of (35) from the term containing these determinants is

\[
\sum_{r=1}^{N-2} \sum_{l=1}^m A_r \frac{\prod_{j \notin \Lambda_r, j \neq i_m^N, i_m^N-1} (i_l^r - j)}{\prod_{j \in K_r, j \neq i_l^r} (i_l^r - j)} = \sum_{r=1}^{N-2} \sum_{l=1}^m A_r \frac{\prod_{s \neq l} (i_l^r - i_s^N)}{\prod_{s \neq l} (i_l^r - i_s^N)}. \tag{40}
\]

(II-II) The contribution to (35) from the term which contains \( \Omega(N - 1, l, \Lambda, l) \) \((1 \leq l \leq m - 1)\).

We have

\[
\det_{lJ}^3 = \prod_{s \neq l} C_{ss} = \left( i_l^{N-1} - i_m^{N-1} \right) \prod_{s \neq l} \frac{i_l^N - i_s^N - 1}{i_l^N - i_s^N}.
\]

The contribution to the rhs of (35) from the terms containing these determinants is

\[
\prod_{s=1}^{m-1} (i_l^m - i_s^{N-1}) \sum_{i=1}^{m-2} \left( \prod_{s=1}^{m-2} (i_l^m - i_s^{N-1})^2 \prod_{s \neq l} (z - i_s^{N-1}) \right) \sum_{s=1}^{m-1} \frac{1}{i_l^m - i_s^{N-1}} \prod_{s \neq l} (i_l^m - i_s^{N-1}) = -\frac{m - 2}{m - 1} + \sum_{s=1}^{m-1} \frac{1}{i_l^m - i_s^{N-1}}. \tag{41}
\]

From (II) and (II-I) the coefficient of \( \Omega \) in the rhs of (35) is

\[
\sum_{j \notin \Lambda_N} \frac{1}{i_l^N - j} - \frac{m - 2}{m - 1} + \sum_{s=1}^{m-1} \frac{1}{i_l^m - i_s^{N-1}} + \sum_{r=1}^{N-2} \sum_{l=1}^m A_r \frac{\prod_{s \neq l} (i_l^r - i_s^N)}{\prod_{s \neq l} (i_l^r - i_s^N)} \prod_{s=1}^{m-1} (i_l^r - i_s^{N-1}) = -\frac{m + 3}{m - 1} + \sum_{s=1}^{m-1} \frac{2}{i_l^m - i_s^{N-1}} + \sum_{r=1}^{N-2} \sum_{l=1}^m \frac{1}{i_l^m - i_l^r} + \sum_{r=1}^{N-2} \sum_{l=1}^m A_r \frac{\prod_{s \neq l} (i_l^r - i_s^N)}{\prod_{s \neq l} (i_l^r - i_s^N)} \prod_{s=1}^{m-1} (i_l^r - i_s^{N-1}).
\]

This coincides with the coefficient of \( \Omega \) in the rhs of (27).

(III): The coefficient of \( \Omega_{lJ}^{-1} \) for which \( J = \{1, 2, \ldots, m - 1\} \setminus \{t\} \) and \( J = \{t\} \).

(III-I) The contribution to (35) from the term which contains \( \Omega(N - 1, l, \Lambda, l) \).

It is obvious that \( \det_{lJ}^3 = 0 \) for \( t = l \). We have, for \( t \neq l \),

\[
\det_{lJ}^3 = (-1)^{m+l+1} D_{lt} \prod_{s \neq t, l} C_{ss} = \prod_{s \neq t, l} C_{ss} \left( i_l^{N-1} - i_m^{N-1} \right) (i_l^N - i_s^N)^{m-3} \prod_{s \neq l} (i_l^N - i_s^{N-1}) \prod_{s \neq l} (i_l^N - i_s^{N-1}) = (-1)^{m+t+1} \left( i_l^{N-1} - i_m^{N-1} \right) (i_l^N - i_s^N)^{m-3} \prod_{s \neq l} (i_l^N - i_s^{N-1}) \prod_{s \neq l} (i_l^N - i_s^{N-1}). \tag{42}
\]
(III-II) The contribution from the term which contains $\Omega(N-1,l,N-1,l)$ $(1 \leq l \leq m-1)$. If $l \neq t$ then $\det_{l,t}^4 = 0$. We have, for $l = t$,

$$\det_{t,t}^4 = (-1)^{m+t+1} \prod_{s \neq t,}^{m-1} C_{ss}^t = (-1)^{m+t+1} \left( \frac{i_{t}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{t}^{N-1}} \right) \prod_{s \neq t}^{m-1} \frac{i_{s}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{s}^{N-1}}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$\frac{(-1)^{m+t+1} A_{N-1} (i_{t}^{N-1} - i_{m}^{N-1})^{m-3}}{(i_{t}^{N-1} - i_{m}^{N-1})^3 (i_{m}^{N-1} - i_{t}^{N-1})^{m-2}} \prod_{s \neq t}^{m-1} \frac{(i_{s}^{N-1} - i_{t}^{N-1})}{(i_{t}^{N-1} - i_{s}^{N-1})} \prod_{s \neq t}^{m-1} \frac{i_{s}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{s}^{N-1}} \prod_{j \neq A_{N-1}, j \neq i_{l}^{N-1}} \frac{i_{t}^{N-1} - j}{i_{t}^{N-1} - i_{l}^{N-1}}.$$

Note that the second term of (33) is the $t = l$ case of the summand of (22).

(III-III) The contribution from the term which contains $\Omega(N-1,l,N-1,k)$ with $1 \leq k,l \leq m-1$ and $k \neq l$.

If $k \neq t$ then $\det_{l,t}^4 = 0$. We have, for $k = t$,

$$\det_{t,t}^4 = (-1)^{m+t+1} C_{tt}^{(m-1)} \prod_{s \neq t,}^{m-1} C_{ss}^t = (-1)^{m+t+1} \left( \frac{i_{t}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{t}^{N-1}} \right) \prod_{s \neq t}^{m-1} \frac{i_{s}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{s}^{N-1}} \prod_{j \neq A_{N-1}, j \neq i_{l}^{N-1}} \frac{i_{t}^{N-1} - j}{i_{t}^{N-1} - i_{l}^{N-1}}.$$

The contribution to the rhs of (35) is

$$\frac{(-1)^{m+t+1} A_{N-1} \prod_{s \neq t}^{m-1} (i_{s}^{N-1} - i_{t}^{N-1}) \prod_{s \neq t}^{m-1} (i_{s}^{N-1} - i_{m}^{N-1})^{m-3}}{(i_{t}^{N-1} - i_{m}^{N-1})^3 (i_{m}^{N-1} - i_{t}^{N-1})^{m-2}} \prod_{s \neq t}^{m-1} \frac{(i_{s}^{N-1} - i_{t}^{N-1})}{(i_{t}^{N-1} - i_{s}^{N-1})} \prod_{s \neq t}^{m-1} \frac{i_{s}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{s}^{N-1}} \prod_{j \neq A_{N-1}, j \neq i_{l}^{N-1}} \frac{i_{t}^{N-1} - j}{i_{t}^{N-1} - i_{l}^{N-1}}.$$

In deriving (34) we use

$$\frac{i_{m}^{N-1} - i_{t}^{N-1}}{A_{N-1} \prod_{s \neq t}^{m-1} (i_{s}^{N-1} - i_{t}^{N-1})} = 0$$

which is a consequence of $t \neq l$. Note that the first term in (33) is the $l = t$ case of the summand of (44).

We add (32), (33), (34) and obtain

$$\frac{(-1)^{m+t+1} A_{N-1} \prod_{s \neq t}^{m-1} (i_{s}^{N-1} - i_{t}^{N-1})^{m-3}}{(i_{t}^{N-1} - i_{m}^{N-1})^3 (i_{m}^{N-1} - i_{t}^{N-1})^{m-2}} \prod_{s \neq t}^{m-1} \frac{(i_{s}^{N-1} - i_{t}^{N-1})}{(i_{t}^{N-1} - i_{s}^{N-1})} \prod_{s \neq t}^{m-1} \frac{i_{s}^{N-1} - i_{m}^{N-1}}{i_{m}^{N-1} - i_{s}^{N-1}} \prod_{j \neq A_{N-1}, j \neq i_{l}^{N-1}} \frac{i_{t}^{N-1} - j}{i_{t}^{N-1} - i_{l}^{N-1}} B(N-1,t,t) \times \sum_{l=1}^{m-1} \frac{(i_{l}^{N-1} - i_{m}^{N-1})^{m-3}}{(i_{t}^{N-1} - i_{m}^{N-1}) (i_{m}^{N-1} - i_{t}^{N-1})} \prod_{s \neq t}^{m-1} \frac{(i_{s}^{N-1} - i_{t}^{N-1})}{(i_{t}^{N-1} - i_{s}^{N-1})} \prod_{j \neq A_{N-1}, j \neq i_{l}^{N-1}} \frac{i_{t}^{N-1} - j}{i_{t}^{N-1} - i_{l}^{N-1}}.$$

In deriving (35) we use the identity

$$\sum_{l=1}^{m-1} \frac{(i_{l}^{N-1} - i_{m}^{N-1})^{m-3}}{(i_{t}^{N-1} - i_{m}^{N-1}) (i_{m}^{N-1} - i_{t}^{N-1})} = \frac{(i_{m}^{N-1} - i_{t}^{N-1})^{m-3}}{\prod_{s=1}^{m-1} (i_{t}^{N-1} - i_{s}^{N-1})}.$$
The equation (43) is nothing but the corresponding coefficient in the rhs of (27).

(IV): The coefficient of $\Omega_{IJ}^{N-1}$ with $I = \{1, 2, ..., m - 1\}\setminus\{u\}$ and $J = \{t\}$, $u \neq t$.

(IV-I) The contribution to (35) from the term which contains $\Omega(N - 1, l, \Lambda, l)$.

If $l = u$ then $\det_{IJ}^t = 0$. We assume $l \neq u$. Then $\det_{IJ}^t = 0$ for $t \neq l$, since $u$-th and $l$-th rows are proportional. Thus we assume $l = t$. We have

$$\det_{IJ}^t = (-1)^{u + m + 1} D_{ut}^t \prod_{s \neq u, t}^{m-1} C_{ss}^t$$

$$= (-1)^{u + m + 1} \left( \frac{i_l^{N-1} - i_m^{N-1}}{i_m^{N-1} - i_l^{N-1}} \right)^m \prod_{s \neq u, t}^{m-1} \frac{(i_{m}^{N-1} - i_{s}^{N-1})}{(i_{l}^{N-1} - i_{s}^{N-1})} \prod_{j \notin \Lambda_{N-1}, j \neq i_m}^{t} \frac{i_{u-1}^{N-1} - j}{i_{l}^{N-1} - j}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$\left(-1\right)^{u + m + 1} \left( \frac{i_l^{N-1} - i_m^{N-1}}{i_m^{N-1} - i_l^{N-1}} \right)^m \prod_{s \neq u, t}^{m-1} \frac{(i_{m}^{N-1} - i_{s}^{N-1})}{(i_{l}^{N-1} - i_{s}^{N-1})} \prod_{j \notin \Lambda_{N-1}, j \neq i_m}^{t} \frac{i_{u-1}^{N-1} - j}{i_{l}^{N-1} - j}. \quad (47)$$

(IV-II) The contribution to (35) from the term which contains $\Omega(N - 1, l, N - 1, l)$ with $1 \leq l \leq m - 1$.

If $l \neq t$ then $\det_{IJ}^t = 0$. We have, for $l = t$,

$$\det_{IJ}^t = (-1)^{u + m + 1} C_{ut}^t \prod_{s \neq u, t}^{m-1} C_{ss}^t$$

$$= (-1)^{u + m + 1} \left( \frac{i_l^{N-1} - i_m^{N-1}}{i_m^{N-1} - i_l^{N-1}} \right)^m \prod_{s \neq u, t}^{m-1} \frac{(i_{m}^{N-1} - i_{s}^{N-1})}{(i_{l}^{N-1} - i_{s}^{N-1})} \prod_{j \notin \Lambda_{N-1}, j \neq i_m}^{t} \frac{i_{u-1}^{N-1} - j}{i_{l}^{N-1} - j}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$\left(-1\right)^{u + m + 1} A_{N-1} \left( \frac{i_l^{N-1} - i_m^{N-1}}{i_m^{N-1} - i_l^{N-1}} \right)^m \prod_{s \neq u, t}^{m-1} \frac{(i_{m}^{N-1} - i_{s}^{N-1})}{(i_{l}^{N-1} - i_{s}^{N-1})} \prod_{j \notin \Lambda_{N-1}, j \neq i_m}^{t} \frac{i_{u-1}^{N-1} - j}{i_{l}^{N-1} - j}. \quad (48)$$

Note that the second term of this equation is the minus of (47).

(IV-III) The contribution to (35) from the term which contains $\Omega(N - 1, l, N - 1, k)$ with $1 \leq k, l \leq m - 1$ and $k \neq l$.

If $k \neq t$ then $\det_{IJ}^t = 0$. We have, for $k = t$,

$$\det_{IJ}^t = (-1)^{u + m} \frac{C_{ul}^t}{C_{uu}^t} \prod_{s \neq \ell}^{m-1} C_{ss}^t$$

$$= (-1)^{u + m + 1} \left( \frac{i_l^{N-1} - i_m^{N-1}}{i_m^{N-1} - i_l^{N-1}} \right)^m \prod_{s \neq u, t}^{m-1} \frac{(i_{m}^{N-1} - i_{s}^{N-1})}{(i_{l}^{N-1} - i_{s}^{N-1})} \prod_{j \notin \Lambda_{N-1}, j \neq i_m}^{t} \frac{i_{u-1}^{N-1} - j}{i_{l}^{N-1} - j}.$$

Here we understand $C_{ul}^t / C_{uu}^t = -1$ for $u = l$. This follows from the equation

$$\frac{C_{ul}^t}{C_{uu}^t} = -\frac{i_l^{N-1} - i_m^{N-1}}{i_m^{N-1} - i_u^{N-1}} \prod_{j \notin \Lambda_{N-1}, j \neq i_m}^{t} \frac{i_{u-1}^{N-1} - j}{i_{l}^{N-1} - j}.$$
The contribution to the rhs of (35) is

\[
(-1)^{u+m+1} A_{N-1} \prod_{s \neq u}^{m-1} (i_N^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1}, j \neq i_u^N} (i_u^{N-1} - j) \left(\frac{(i_N^{N-1} - i_t^{N-1})(i_N^{N-1} - i_s^{N-1})}{\prod_{j \in K, j \neq i_t^{N-1}} (i_t^{N-1} - j)}\right) \times \sum_{l=1}^{m-1} \frac{(-1)^{l+1} \prod_{s \neq l}^{m-1} (i_l^{N-1} - i_s^{N-1})}{\prod_{i \neq l}^{m-1} (i_l^{N-1} - i_s^{N-1})}.
\]

(49)

Note that the \(l = t\) term of this equation is precisely the first term of (48).

Thus, using (46), we have

\[
(47) + (48) + (49)
\]

\[
= (-1)^{u+m+1} A_{N-1} \prod_{s \neq u}^{m-1} (i_N^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1}, j \neq i_u^N} (i_u^{N-1} - j) \left(\frac{(i_N^{N-1} - i_t^{N-1})(i_N^{N-1} - i_s^{N-1})}{\prod_{j \in K, j \neq i_t^{N-1}} (i_t^{N-1} - j)}\right) \times \sum_{l=1}^{m-1} \frac{(-1)^{l+1} \prod_{s \neq l}^{m-1} (i_l^{N-1} - i_s^{N-1})}{\prod_{i \neq l}^{m-1} (i_l^{N-1} - i_s^{N-1})} \prod_{j \notin \Lambda_{N-1}, j \neq i_l^{N-1}} (i_l^{N-1} - j) \prod_{j \in K, j \neq i_l^{N-1}} (i_l^{N-1} - j).
\]

which coincides with the coefficient of \(\Omega_j^{N-1}\) in the rhs of (27).

(V): The coefficient of \(\Omega_j^{m-1}\) for which \(I = \{1, 2, ..., m\}\) \(\setminus\{u\}\) and \(J = \{t\}\), \(r < N - 1\).

(V-I) The contribution from the term which contains \(\Omega(r, t, \Lambda), l\).

If \(l = u\) then \(\det_{t,j}^l = 0\). We assume \(l \neq u\). Then

\[
\det_{t,j}^l = (-1)^{u+m} \frac{i_N^t - i_t^u}{i_N^t - i_m^{N-1}} \prod_{s \neq u}^{m} (i_t^{N-1} - i_s^{N-1}) \prod_{j \in \Lambda_{N-1}, j \neq i_u^N} (i_u^{N-1} - j) \prod_{s \neq u, s \neq l}^{m} G_s^r \prod_{s \neq u}^{m} G_s^r.
\]

The contribution to the rhs of (35) is

\[
\frac{(-1)^{u+m} A_{l} \prod_{s \neq u}^{m} (i_t^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1}, j \neq i_u^N} (i_u^{N-1} - j) \prod_{s \neq u, s \neq l}^{m} G_s^r \prod_{s \neq u}^{m} G_s^r}{(i_N^t - i_m^{N-1})} \sum_{i \neq u}^{m} \frac{1}{(i_N^t - i_i^t)(i_t^{N-1} - i_i^u)} \prod_{s \neq i}^{m} (i_i^{N-1} - i_s^{N-1}).
\]

(50)

(V-II) The contribution from the term which contains \(\Omega(r, l, \Lambda, k)\) with \(k \neq l\).

If \(l \neq u\) then \(\det_{t,j}^l = 0\). In fact if further \(k \neq u\) then \(k\)-th, \(l\)-th and \(u\)-th rows are proportional and if \(k = u\) then \(l\)-th row is a null vector. Thus we assume \(l = u\). Then

\[
\det_{t,j}^l = (-1)^{u+m-1} \frac{i_N^t - i_t^u}{i_N^t - i_m^{N-1}} \prod_{s \neq u}^{m} (i_t^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1}, j \neq i_u^N} (i_u^{N-1} - j) \prod_{s \neq u, s \neq l}^{m} G_s^r \prod_{s \neq u}^{m} G_s^r.
\]

\[
= (-1)^{u+m-1} \frac{i_N^t - i_t^u}{i_N^t - i_m^{N-1}} \prod_{s \neq u}^{m} (i_t^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1}, j \neq i_u^N} (i_u^{N-1} - j) \prod_{s \neq u, s \neq l}^{m} G_s^r \prod_{s \neq u}^{m} G_s^r.
\]

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The contribution to the rhs of (53) is
\[
(-1)^{u+m-1} A_r \prod_{s \neq u} (i_{t_s}^{N-1} - i_{s_s}^{N}) \prod_{j \notin A_r, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{N-1} - j) \\
\times \sum_{k \neq u} \frac{(i_k^{N} - i_{m}^{N-1}) \prod_{s=1}^{m-1} (i_k^{N} - i_s^{N})}{(i_k^{N} - i_{m}^{N-1})(i_k^{N} - i_{k}^{N-1})}.
\]
(51)

Note that the \(k = u\) term of this equation is equal to the minus of the \(l = u\) term in (50).

(V-III) The contribution from the term which contains \(\Omega(r, l, N - 1, k)\).

If \(k \neq t\), \(\det^2_{tJ} = 0\). We assume \(k = t\). Then \(\det^2_{tJ} = 0\) for \(l \neq u\), since \(l\)-th and \(u\)-th rows are proportional. Thus we assume \(l = u\). Then
\[
\det^2_{tJ} = (-1)^{u+m}.
\]

The contribution to the rhs of (53) is
\[
(-1)^{u+m} A_r \prod_{j \notin A_r, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{r} - j) \\
\frac{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1}) \prod_{i \in K, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{N-1} - j)}{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1})} B(r, u, t)
\]
\[
(-1)^{u+m} A_r \prod_{j \notin A_r, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{r} - j) \\
\frac{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1}) \prod_{i \in K, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{N-1} - j)}{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1})} B(r, u, t)
\]
\[
+ (-1)^{u+m} A_r \prod_{j \notin A_r, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{r} - j) \\
\frac{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1}) \prod_{i \in K, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{N-1} - j)}{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1})} B(r, u, t),
\]
(52)
where we have used
\[
\frac{1}{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1})} = \frac{1}{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1})} + \frac{1}{(i_{m}^{N} - i_{t_s}^{N}) (i_{m}^{N} - i_{t_s}^{N-1})}.
\]

Note that the first term of (52) is nothing but the coefficient of \(\Omega_{tJ}^{r}i^{u}_{j}\) in (47).

Let us calculate (50) + (51). We have
\[
(50) + (51) = (-1)^{u+m} A_r \prod_{s \neq u} (i_{t_s}^{N-1} - i_{s_s}^{N}) \prod_{j \notin A_r, j \neq i_{t_s}^{N}, i_{t_s}^{N-1}} (i_{t_s}^{N-1} - j) \\
\times \sum_{l=1}^{m} \left[ 1 + \frac{i_{t_s}^{r} - i_{m}^{N-1}}{i_{t_s}^{r} - i_{m}^{N-1}} \right] \frac{\prod_{s=1}^{m-1} (i_{t_s}^{r} - i_{s_s}^{N})}{\prod_{s \neq l} (i_{t_s}^{r} - i_{s_s}^{N})}.
\]

By the residue theorem
\[
\sum_{l=1}^{m} \frac{\prod_{s=1}^{m-1} (i_{t_s}^{r} - i_{s_s}^{N})}{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s=1}^{m-1} (i_{t_s}^{r} - i_{s_s}^{N})}
= \frac{\prod_{s=1}^{m-1} (i_{t_s}^{N-1} - i_{s_s}^{N})}{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s=1}^{m-1} (i_{t_s}^{N-1} - i_{s_s}^{N})}
+ \frac{\prod_{s=1}^{m-1} (i_{t_s}^{N-1} - i_{s_s}^{N})}{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s=1}^{m-1} (i_{t_s}^{N-1} - i_{s_s}^{N})}.
\]
\[
\sum_{k=1}^{m} \frac{(i_{k}^{r} - i_{m}^{N-1}) \prod_{s \neq k} (i_{t_s}^{r} - i_{s_s}^{N})}{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s \neq k} (i_{t_s}^{r} - i_{s_s}^{N})}
= \frac{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s \neq k} (i_{t_s}^{r} - i_{s_s}^{N})}{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s \neq k} (i_{t_s}^{r} - i_{s_s}^{N})}
+ \frac{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s \neq k} (i_{t_s}^{r} - i_{s_s}^{N})}{(i_{t_s}^{r} - i_{m}^{N-1}) \prod_{s \neq k} (i_{t_s}^{r} - i_{s_s}^{N})}.
\]
Hence

\[(50) + \{51\} = (-1)^{u+m} A_r \prod_{s \not= u} (i^{N-1}_t - i^r_s) \prod_{j \not\in \Lambda, j \not\in \{i^r_u - j\}} (i^r_u - j) \times \]

\[\left(\frac{i^N_m - i^r_u}{i^N_m - i^r_s}\right) \prod_{s \not= u} (i^{N-1}_t - i^r_s) \prod_{j \not\in \Lambda, j \not\in \{i^r_u - j\}} (i^r_u - j) \times \]

\[\left(\frac{\prod_{s=1}^{m-1} (i^N_m - i^N_s)}{(i^N_m - i^N_t)} \prod_{s \not= u} (i^{N-1}_t - i^N_s) \prod_{j \not\in \Lambda, j \not\in \{i^r_u - j\}} (i^r_u - j) \right) \times \]

\[\left(\frac{-1}{(i^r_u - i^{N-1}_m)} \right) \frac{(i^N_m - i^{N-1}_m)(i^N_m - i^{N-1}_t)}{(i^N_m - i^{N-1}_s)(i^N_m - i^{N-1}_t - j)} B(r, u, t).\]

This is the minus of the second term of (62). Hence

\[(50) + \{51\} + \{52\} = \text{the first term of } \{52\}\]

which is equal to the coefficient of \(\Omega^l_{r,J}\) in (67).

(6V): The coefficient of \(\Omega^l_{r,J}\) for which \(I \cap J = \emptyset, |I| \geq 1\) and \(|J| \geq 2\).

Let us set \(I = \{p_1, \ldots, p_u\}\) and \(J = \{q_1, \ldots, q_l\}\), \(u, t \leq m - 1, u + t = m - 1\).

(6V-I) The contribution from the term which contains \(\Omega(N - 1, l, \Lambda, l)\).

If \(l \not\in I\) then \(\det^3_{l,J} = 0\). We assume \(l \in I\). We have

\[\det^3_{l,J} = \text{sgn} \cdot \prod_{s \in I \setminus \{l\}} C^l_{ss} \prod_{s \in J} D^l_{ss} \times \]

\[\prod_{s \in I \setminus \{l\}} (i^{N-1}_t - i^r_u) \prod_{s \in J} (i^{N-1}_t - i^r_s) \sum_{l \in I} \frac{(i^N_m - i^N_s)^{u-1}(i^N_m - i^N_1)^{t-2}}{(i^{N-1}_m - i^N_1)^{m-2}} \prod_{s \not= l} (i^N_m - i^N_s).\]

(6V-II) The contribution from the term which contains \(\Omega(N - 1, l, N - 1, l)\) with \(1 \leq l \leq m - 1\).

If \(l \not\in J\), \(\det^4_{l,J} = 0\). We assume \(l \in J\). We have

\[\det^4_{l,J} = \text{sgn} \cdot \prod_{s \in I} C^l_{ss} \prod_{s \in J \setminus \{l\}} D^l_{ss} \times \]

\[\prod_{s \in I} (i^N_m - i^N_s) \prod_{s \in J \setminus \{l\}} (i^N_m - i^N_s) \sum_{l \in J} \frac{(i^N_m - i^N_s)^{u-1}(i^N_m - i^N_1)^{t-2}}{(i^{N-1}_m - i^N_1)^{m-2}} \prod_{s \not= l} (i^N_m - i^N_s).\]
Note that the sum of the second term of this equation and (53) is zero by the residue theorem and the conditions $u \geq 1$, $t \geq 2$.

(VI-III) The contribution from the term which contains $\Omega(N - 1, l, N - 1, k)$ with $1 \leq k, l \leq m - 1$, $k \neq l$.

If $k \notin J$ then $\det l_{IJ}^4 = 0$. We assume $k \in J$.

(VI-III-I) $l \in I$ case.

We have

$$
\det l_{IJ}^4 = -\text{sgn} \cdot C_{kl}^I \prod_{s \in I \setminus \{l\}} C_{ss}^I \prod_{s \notin J \setminus \{k\}} D_{ss}^I
$$

$$
= \text{sgn} \frac{(i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{(i_l N - i_m N - 1, m - 2)} \sum_{l \in I} \frac{(i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{\prod_{s \notin l} (i_l N - i_s N - 1)}
$$

$$
\times \prod_{j \notin A \setminus I \neq i N \neq i N} \frac{l_{k - 1} - j}{l_{k - 1} - j}.
$$

The contribution to the rhs of (55) is

$$
\text{sgn} \frac{A_{N - 1} \prod_{s \in I} (i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{(i_l N - i_m N - 1, m - 2)} \sum_{l \in I} \frac{(i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{\prod_{s \notin l} (i_l N - i_s N - 1)}
$$

$$
\times \prod_{k \notin K \setminus i N \neq i N} \frac{l_{k - 1} - j}{l_{k - 1} - j}.
$$

(VI-III-II) $l \in J$ case.

We have

$$
\det l_{IJ}^4 = -\text{sgn} \cdot D_{kl}^J \prod_{s \in I} C_{ss}^J \prod_{s \notin J \setminus \{k\}} D_{ss}^J
$$

$$
= -\text{sgn} \frac{(i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 1)}{(i_l N - i_m N - 1, m - 2)} \prod_{s \in I} (i_l N - i_s N - 1) \prod_{s \notin J \setminus \{k\}} (i_m N - i_s N - 1)
$$

$$
\times \prod_{j \notin A \setminus I \neq i N \neq i N} \frac{l_{k - 1} - j}{l_{k - 1} - j}.
$$

The contribution to the rhs of (55) is

$$
\text{sgn} \frac{A_{N - 1} \prod_{s \in I} (i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 1)}{(i_l N - i_m N - 1, m - 2)} \sum_{l \in J} \frac{(i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{\prod_{s \notin l} (i_l N - i_s N - 1)}
$$

$$
\times \prod_{k \notin K \setminus i N \neq i N} \frac{l_{k - 1} - j}{l_{k - 1} - j}.
$$

If we set $k = l$ in this equation, then it is equal to the first term of (54).

Thus we have

$$(53) + (54) + (55) + (56)$$

$$
= \text{sgn} \frac{A_{N - 1} \prod_{s \in I} (i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{(i_l N - i_m N - 1, m - 2)} \sum_{l = 1}^{m - 1} \frac{(i_l N - i_m N - 1, u) (i_m N - i_l N - 1, t - 2)}{\prod_{s \notin l} (i_l N - i_s N - 1)}
$$

$$
\times \prod_{k \notin J \setminus i N \neq i N} \frac{l_{k - 1} - j}{l_{k - 1} - j}.
$$

$$
= 0,
$$

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by applying the residue theorem to the summation in \( l \).

(VII): The coefficient of \( \Omega^{N-1}_{lJ} \) for which \( \{I \cap J\} = 1, |J\| \geq 2 \).

We set \( I = \{p_1 < \cdots < p_u\}, J = \{q_1 < \cdots < q_t\} (u + t = m - 1) \) and \( I \cap J = \{\tilde{k}\} \).

(VII-I) The contribution from the term which contains \( \Omega_{lJ} \) is

\[
\text{set } I = \{p_1 < \cdots < p_u\}, J = \{q_1 < \cdots < q_t\} (u + t = m - 1) \) and \( I \cap J = \{\tilde{k}\} \).

If \( k \neq l \) then \( \det^3_{lJ} = 0 \). In fact if \( k \neq l \), either the \( l \)-th row is a null vector or there exists a row proportional to the \( l \)-th row. We assume \( l = \tilde{k} \). Let us define \( v, w, y \) by \( p_v = q_u = l, I \cup J = \{1, 2, \ldots, m - 1\}\} \{y\}. Then we have

\[
\det^3_{lJ} = \text{sgn} \cdot D_{yl}^l \prod_{s \in I \setminus \{l\}} C_{ss}^l \prod_{s \in J \setminus \{l\}} D_{ss}^l
\]

\[
= \text{sgn} \cdot \frac{(i_l^{N-1} - i_m^{N-1})u(i_m^{N-1} - i_l^{N-1})l \prod_{s \in I \setminus \{l\}} (i_m^{N-1} - i_s^{N-1}) \prod_{s \in J \setminus \{l\}} (i_m^{N-1} - i_s^{N-1})}{(i_m^{N-1} - i_l^{N-1})m \prod_{s \neq \tilde{k}} (i_l^{N-1} - i_s^{N-1})}
\]

\[
\times \prod_{j \notin A_{N-1} \cup \{N\}} i_{j}^{N-1} - i_{\tilde{k}}^{N-1}.
\]

where \( \text{sgn} = \text{sgn}(p_1, \cdots, p_u, q_1, \cdots, y, \cdots, q_t) \), \( y \) being on the place of \( q_u \). The contribution to the rhs of \( (57) \) is

\[
-\text{sgn} \cdot \frac{(i_l^{N-1} - i_m^{N-1})u(i_m^{N-1} - i_l^{N-1})l \prod_{s \in I \setminus \{l\}} (i_m^{N-1} - i_s^{N-1}) \prod_{s \in J \setminus \{l\}} (i_m^{N-1} - i_s^{N-1})}{(i_m^{N-1} - i_l^{N-1})m \prod_{s \neq \tilde{k}} (i_l^{N-1} - i_s^{N-1})}
\]

\[
\times \prod_{j \notin A_{N-1} \cup \{N\}} i_{j}^{N-1} - i_{\tilde{k}}^{N-1}.
\]

(VII-II) The contribution from the term which contains \( \Omega(N - 1, l, N - 1, l) \) with \( 1 \leq l \leq m - 1 \). If \( k \neq l \) then \( \det^4_{lJ} = 0 \) by the same reason as (VII-I). We assume \( l = \tilde{k} \). Let us define \( v, w, y \) as in (VII-I). We have

\[
\det^4_{lJ} = -\text{sgn} \cdot C_{yl}^l \prod_{s \in I \setminus \{l\}} C_{ss}^l \prod_{s \in J \setminus \{l\}} D_{ss}^l
\]

\[
= \text{sgn} \cdot D_{yl}^l \prod_{s \in I \setminus \{l\}} C_{ss}^l \prod_{s \in J \setminus \{l\}} D_{ss}^l
\]

which is same as \( \det^3_{lJ} \) in (VII-I). The contribution to the rhs of \( (57) \) is

\[
-\text{sgn} \cdot \frac{A_{N-1}(i_l^{N-1} - i_m^{N-1})u(i_m^{N-1} - i_l^{N-1})l - 3 \prod_{i = 1}^{m-1} (i_l^{N-1} - i_s^{N-1}) \prod_{s \notin \{l\}} (i_l^{N-1} - i_s^{N-1})}{(i_m^{N-1} - i_l^{N-1})m \prod_{s \neq \tilde{k}} (i_l^{N-1} - i_s^{N-1})}
\]

\[
\times \prod_{j \notin A_{N-1} \cup \{N\}} i_{j}^{N-1} - i_{\tilde{k}}^{N-1}.
\]

The second term of this equation is equal to the minus of \( (57) \).
(VII-III) The contribution from the term which contains $\Omega(N-1, l, N-1, k)$ with $1 \leq k, l \leq m - 1$, $k \neq l$.

If $k \neq k$ then $\text{det}_{lJ}^k = 0$. In fact if $\bar{k} \neq k$ then $\bar{k}$-th column in the right matrix and that in the left matrix are proportional. We assume $k = \bar{k}$. Let us define $v, w, y$ by $p_v = q_w = k, I \cup J = \{1, 2, \cdots, m - 1\}\setminus y$.

(VII-III-I) $l \in I$ case.

We have

$$\text{det}_{lJ}^l = -\text{sgn} \cdot C_{yl}^l \prod_{s \in I \setminus \{l\}} D_{ss}^l \prod_{s \in J \setminus \{k\}} D_{ss}^l$$

$$= \text{sgn} \cdot \frac{(i_l^{N-1} - i_{m_l}^{N-1})^y(i_m^{N-1} - i_l^{N-1})^l \prod_{s \in I \setminus \{l\}} (i_m^{N-1} - i_{s_m}^{N-1}) \prod_{s \in J \setminus \{k\}} (i_l^{N-1} - i_s^{N-1})(i_m^{N-1} - i_{s_m}^{N-1})}{(i_m^{N-1} - i_{s_m}^{N-1})^m - 2 \prod_{s \neq l} (i_l^{N-1} - i_s^{N-1})}$$

The contribution to the rhs of (35) is

$$\text{sgn} \cdot A_{N-1} \prod_{s \in I \setminus \{k\}} (i_m^{N-1} - i_{s_m}^{N-1}) \prod_{s \in J \setminus \{k\}} (i_l^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1,i} \setminus \bar{i}_s^{m} \setminus \bar{i}_l^{N-1}} (i_l^{N-1} - j) \prod_{j \notin \Lambda_{N-1,i} \setminus \bar{i}_s^{m} \setminus \bar{i}_l^{N-1}} (i_l^{N-1} - j)$$

(VII-III-II) $l \in J$ case.

We have

$$\text{det}_{lJ}^l = -\text{sgn} \cdot D_{yl}^l \prod_{s \in I} C_{ss}^l \prod_{s \in J \setminus \{k\}} D_{ss}^l$$

$$= \text{sgn} \cdot \frac{(i_l^{N-1} - i_{m_l}^{N-1})^y(i_m^{N-1} - i_l^{N-1})^l \prod_{s \in I} (i_m^{N-1} - i_s^{N-1}) \prod_{s \in J \setminus \{k\}} (i_l^{N-1} - i_s^{N-1})(i_m^{N-1} - i_s^{N-1})}{(i_m^{N-1} - i_s^{N-1})^m - 2 \prod_{s \neq l} (i_l^{N-1} - i_s^{N-1})}$$

The contribution to the rhs of (35) is

$$\text{sgn} \cdot A_{N-1} \prod_{s \in I \setminus \{k\}} (i_m^{N-1} - i_{s_m}^{N-1}) \prod_{s \in J \setminus \{k\}} (i_l^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1,i} \setminus \bar{i}_s^{m} \setminus \bar{i}_l^{N-1}} (i_l^{N-1} - j) \prod_{j \notin \Lambda_{N-1,i} \setminus \bar{i}_s^{m} \setminus \bar{i}_l^{N-1}} (i_l^{N-1} - j)$$

Note that the first term of (58) is equal to the $l = \bar{k}$ term of (60).

(VII-III-III) $l = y$ case.

We have

$$\text{det}_{lJ}^l = \text{sgn} \cdot \prod_{s \in I} C_{ss}^l \prod_{s \in J \setminus \{k\}} D_{ss}^l$$

$$= \text{sgn} \cdot \frac{(i_l^{N-1} - i_{m_l}^{N-1})^y(i_m^{N-1} - i_l^{N-1})^l \prod_{s \in I} (i_m^{N-1} - i_s^{N-1}) \prod_{s \in J \setminus \{k\}} (i_l^{N-1} - i_s^{N-1})(i_m^{N-1} - i_s^{N-1})}{(i_m^{N-1} - i_s^{N-1})^m - 2 \prod_{s \neq l} (i_l^{N-1} - i_s^{N-1})}.$$
The contribution to the rhs of (35) is

$$\text{sgn} \cdot A_{N-1} \prod_{s \in I \setminus \{k\}} (i_m^N - i_s^{N-1}) \prod_{s \in J \setminus \{k\}} (i_{m}^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1} \setminus j \neq i_m^N} (i_j^{N-1} - j)$$

$$\times \frac{(i_m^{N-1} - i_s^{N-1})u^{-1}(i_m^{N-1} - i_s^{N-1})t^{-2}}{\prod_{s \neq y}(i_s^{N-1} - i_y^{N-1})}. \quad (61)$$

This equation coincides with that obtained from (57) or (58) by setting \(l = y\).

Now we have

$$(57) + (58) + (59) + (60) + (61)$$

$$= \text{sgn} \cdot A_{N-1} \prod_{s \in I \setminus \{k\}} (i_m^N - i_s^{N-1}) \prod_{s \in J \setminus \{k\}} (i_{m}^{N-1} - i_s^{N-1}) \prod_{j \notin \Lambda_{N-1} \setminus j \neq i_m^N} (i_j^{N-1} - j)$$

$$\times \sum_{l=1}^{m-1} \frac{(i_l^{N-1} - i_m^{N-1})u^{-1}(i_m^{N-1} - i_l^{N-1})t^{-2}}{\prod_{s \neq l}(i_l^{N-1} - i_s^{N-1})} = 0.$$

The last equality follows from the residue theorem.

(VIII): The coefficient of \(\Omega_{IJ}^{N-1}\) for which \(|I \cap J| \geq 2\).

(VIII-I) The contribution from the term which contains \(\Omega(N - 1, l, L, l)\) is zero.

In fact, \(\det I_{J}^3 = 0\) since at least one pair of common column is linearly dependent.

(VIII-II) The contribution from the term which contains \(\Omega(N - 1, l, N - 1, l)\) with \(1 \leq l \leq m - 1\) is zero by the same reason as (VIII-I).

(VIII-III) The contribution from the term which contains \(\Omega(N - 1, l, N - 1, k)\) with \(1 \leq k, l \leq m - 1, k \neq l\) is zero by the following reason. Since \(C_{kl}^l = -D_{kl}^l\) for \(k \neq l\), the common column except \(k\)-th column is linearly dependent. Hence \(\det I_{J}^l = 0\).

As a whole the coefficient of \(\Omega_{IJ}^{N-1}\) in the rhs of (53) is zero.

(IX): The coefficient of \(\Omega_{IJ}^{t}\) for which \(|I| \geq 2\).

Let us set \(I = \{p_1 < \cdots < p_u\}, J = \{q_1 < \cdots < q_v\}, \bar{I} = \{1, 2, \cdots, m\} \setminus I = \{\bar{p}_1 < \cdots < \bar{p}_u\}\) with \(u + t = m\).

(IX-I) The contribution from the term which contains \(\Omega(r, l, A, l)\).

If \(l \notin I\), \(\det I_{J}^l = 0\). We assume \(l \in I\). Then

$$\det I_{J}^l = (-1)^{\sum p_s + \frac{t}{2}(u + 1)} \prod_{s \in \bar{I}} G_s^r \text{det}(F_{p, q}) \left(\frac{i_m^{N} - i_l^{t}}{i_m^{N-1} - i_l^{t}}\right)^t \frac{E_l}{\prod_{s \in J}(i_s^{N-1} - i_l^{t})} \prod_{s=1}^{m} G_s^r$$

$$= \text{sgn} \cdot \frac{E_l D_{IJ}^l}{G^t(\bar{I})} \left(\frac{i_m^{N} - i_l^{t}}{i_m^{N-1} - i_l^{t}}\right)^t \prod_{s \in J}(i_s^{N-1} - i_l^{t}),$$

where \(\text{sgn} = (-1)^{\sum p_s + \frac{t}{2}(u + 1) + \frac{t}{2}(t + 1)} G^t(\bar{I}) = \prod_{s \in \bar{I}} G_s^r\) and

$$D_{IJ}^l = (-1)^{\frac{t}{2}(t + 1)} \text{det}(F_{p, q}) = \prod_{a < b} (i_{p_a}^{r} - i_{p_b}^{r}) \prod_{a < b} (i_{q_a}^{N-1} - i_{q_b}^{N-1}) \prod_{s \in I} \prod_{s' \in J}(i_{s}^{t} - i_{s'}^{N-1})$$

$$E_l = \prod_{y \in J} \left(\frac{\prod_{s=1}^{m}(i_y^{N-1} - i_s^{t})}{\prod_{s \neq i_m^N, i_m^{N-1}, i_y^{N-1}, i_s^{N-1}}(i_y^{N-1} - j)}\right).$$

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The contribution to the rhs of (35) is

\[
\frac{\text{sgn} \cdot A_i E_J D_{IJ}}{(i_m^N - i_m^{N-1})^t G(I)} \sum_{t \in \mathcal{I}} \frac{(i_m^N - i_t^r)^{t-2}}{\prod_{s \neq t} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r)}.
\]

(IX-II) The contribution from the term which contains \(\Omega(r, l, \Lambda, k)\) with \(k \neq l\).

If \(l \in I\) or \(k \notin I\) then \(\det^1_{IJ} = 0\). In fact if \(l \notin I\) then \(l\)-th column in the left matrix is a null vector and if \(k \notin I\) then \(k\)-th row is a null vector. We assume \(l \notin I\) and \(k \in I\). Let us set \(\tilde{I} = \{\tilde{p}_1 < \cdots < \tilde{p}_k\} = \tilde{I} \setminus \{l\} \cup \{k\}\) and \(p_i^* = \tilde{p}_i (\tilde{p}_i \neq l), p_k^* = k (\tilde{p}_k = l)\). Let us define \(v, w\) by \(p_l = k, p_w-1 < l < p_w\). Then

\[
\det^1_{IJ} = (-1)^{\sum_{i=1}^k p_i + \frac{1}{2} (u+1) + l - k + v - w} \frac{E_J G^r_{I}}{(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \cdot \det(F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
= (-1)^{\sum_{i=1}^k p_i + \frac{1}{2} (u+1) - 1} \frac{E_J G^r_{I}}{(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \cdot \det(F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
= (-1)^{-\sum_{i=1}^k p_i + \frac{1}{2} (u+1) - 1} \frac{E_J G^r_{I}}{(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \cdot \det(F_{\tilde{p}_1, \tilde{q}_1})
\]

where we use

\[
\det(F_{\tilde{p}_1, \tilde{q}_1}) = (-1)^{k - l + w - v - 1} \det(F_{\tilde{p}_1, \tilde{q}_1}).
\]

The contribution to the rhs of (35) is

\[
- \frac{\text{sgn} \cdot A_i E_J D_{IJ}}{G^r(T)(i_m^N - i_m^{N-1})^t} \sum_{t \in \mathcal{I}} \frac{(i_m^N - i_t^r)^{t-1}}{\prod_{s \neq t} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \times
\]

\[
\sum_{k \in \tilde{I}} \frac{(i_k^r - i_m^N)^{m-1}}{\prod_{s \in J}(i_k^r - i_s^l) \prod_{s \neq \tilde{I}} (i_k^r - i_s^l) \prod_{s \in J} (i_s^N - i_k^r)}.
\]

If we set \(k = 1\) in this equation then it equals to the minus of (32).

(IX-III) The contribution from the term which contains \(\Omega(r, l, N - 1, k)\).

If \(l \in I\) or \(k \notin J\), \(\det^2_{IJ} = 0\). In fact if \(l \in I\) then \(l\)-th column in the left matrix is zero and if \(k \notin J\) \(l\)-th row is zero. We assume \(l \in I\) and \(k \in J\). We define \(v, w\) here by \(q_v = k\) and \(p_w < l < p_w+1\). We note that

\[
\mathcal{P}\{s | \tilde{p}_s < l\} = l - 1 - w, \quad \mathcal{P}\{s | \tilde{q}_s < k\} = v - 1.
\]

Using these relations we have

\[
\det^2_{IJ} = (-1)^{\sum_{i=1}^k p_i + \frac{1}{2} (u+1) + l - w - v} \frac{E_J G^r_{I}}{(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \times
\]

\[
\prod_{s \in I} G^r_{s \setminus \{l\} \cup \{k\}} \cdot (F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
= \text{sgn} \frac{E_J D_{IJ}}{G^r(T)(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \times
\]

\[
\frac{\prod_{s \in \tilde{I}} (i_k^r - i_m^N)^{m-1}}{\prod_{s \in J}(i_k^r - i_s^l) \prod_{s \neq \tilde{I}} (i_k^r - i_s^l) \prod_{s \in J} (i_s^N - i_k^r)}
\]

\[
\times \frac{E_J \prod_{j \neq i_k^r, i_m^N - j \neq i_k^r} (i_k^N - 1 - j)}{\prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r}
\]

\[
\times \prod_{s \neq \tilde{I}} G^r_{s \setminus \{l\} \cup \{k\}} \cdot (F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
= \frac{\text{sgn} \cdot A_i E_J D_{IJ}}{(i_m^N - i_m^{N-1})^t G^r(I)} \sum_{t \in \mathcal{I}} \frac{(i_m^N - i_t^r)^{t-2}}{\prod_{s \neq t} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \times
\]

\[
\sum_{k \in \tilde{I}} \frac{(i_k^r - i_m^N)^{m-1}}{\prod_{s \in J}(i_k^r - i_s^l) \prod_{s \neq \tilde{I}} (i_k^r - i_s^l) \prod_{s \in J} (i_s^N - i_k^r)}
\]

\[
\times \prod_{s \neq \tilde{I}} G^r_{s \setminus \{l\} \cup \{k\}} \cdot (F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
= (-1)^{\sum_{i=1}^k p_i + \frac{1}{2} (u+1) - 1} \frac{E_J G^r_{I}}{(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \cdot \det(F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
= (-1)^{-\sum_{i=1}^k p_i + \frac{1}{2} (u+1) - 1} \frac{E_J G^r_{I}}{(i_m^N - i_m^{N-1})^t \prod_{s \neq I} (i_t^r - i_s^l) \prod_{s \in J} (i_s^N - i_t^r) \prod_{s \neq I} G_s^r} \cdot \det(F_{\tilde{p}_1, \tilde{q}_1})
\]

\[
\times \frac{\prod_{s \neq \tilde{I}} G^r_{s \setminus \{l\} \cup \{k\}} \cdot (F_{\tilde{p}_1, \tilde{q}_1})}{\prod_{s \neq \tilde{I}} G^r_{s \setminus \{l\} \cup \{k\}} \cdot (F_{\tilde{p}_1, \tilde{q}_1})}
\]

\[
= (-1)^{k - l + w - v - 1} \det(F_{\tilde{p}_1, \tilde{q}_1}).
\]

\[
30
\]
The contribution to the rhs of (35) is

\[
\begin{aligned}
&\frac{(-1)^{l-1} \text{sgn} \cdot A_{r} E_{\alpha} D_{IJ}}{G^{r}(I)(i_{m}^{N} - i_{m-1}^{N})} \sum_{l \in I} \left( i_{l}^{r} - i_{m}^{N-1} \right) \prod_{s \in I \cap \{l\}} \left( i_{l}^{r} - i_{s}^{N} \right) \\
&\times \prod_{k \in J} \left( i_{m}^{N} - i_{k}^{N-1} \right) \prod_{s \in J \setminus \{k\}} \left( i_{k}^{N-1} - i_{s}^{N-1} \right) \\
&\text{sgn} \cdot E_{\alpha} D_{IJ} \sum_{k \in J} \left( i_{k}^{N-1} - i_{s}^{N-1} \right) \prod_{s \in J \setminus \{k\}} \left( i_{k}^{N-1} - i_{s}^{N-1} \right) \\
&= \frac{(-1)^{l} \text{sgn} \cdot A_{r} E_{\alpha} D_{IJ}}{G^{r}(I)(i_{m}^{N} - i_{m-1}^{N})} \sum_{l \in I} \left( i_{l}^{r} - i_{m}^{N-1} \right) \prod_{s \in I \cap \{l\}} \left( i_{l}^{r} - i_{s}^{N} \right) \\
&\times \prod_{k \in J} \left( i_{m}^{N} - i_{k}^{N-1} \right) \prod_{s \in J \setminus \{k\}} \left( i_{k}^{N-1} - i_{s}^{N-1} \right) \\
&\quad \text{sgn} \cdot E_{\alpha} D_{IJ} \prod_{s \in J} \left( i_{s}^{N-1} - i_{l}^{N} \right)
\end{aligned}
\]

In deriving the last equation we use

\[
\begin{aligned}
&\sum_{l \in I} \frac{(i_{l}^{r} - i_{m}^{N-1})^{t-2}}{\prod_{s \in I \cap \{l\}} \left( i_{l}^{r} - i_{s}^{N} \right)} = \frac{(-1)^{l-1} \text{sgn} \cdot A_{r} E_{\alpha} D_{IJ}}{G^{r}(I)(i_{m}^{N} - i_{m-1}^{N})} \\
&\sum_{k \in J} \frac{\prod_{s \in J \setminus \{k\}} \left( i_{k}^{N-1} - i_{s}^{N-1} \right)}{\prod_{s \in J \setminus \{k\}} \left( i_{k}^{N-1} - i_{s}^{N-1} \right)} = \frac{\text{sgn} \cdot E_{\alpha} D_{IJ}}{G^{r}(I)(i_{m}^{N} - i_{m-1}^{N})} \prod_{s \in J} \left( i_{s}^{N-1} - i_{l}^{N} \right)
\end{aligned}
\]
we have
\[
Y = -\frac{\text{sgn} \cdot A_r E_j D_{ij}}{G^r(f)(i^{N}_m - i^{N}_m)^{t-1}} \sum_{i \in I} \frac{(i^{N}_m - i^{N}_m)^{t-1}}{(i^{N}_m - i^{N}_m)(i^{N}_k - i^{N}_k)} \prod_{s \in J \setminus \{I\}} (i^{N}_s - i^{N}_s)
\]
\[
+ \sum_{k \in J} \frac{(i^{N}_k - i^{N}_k)^{t-1}}{(i^{N}_k - i^{N}_k)(i^{N}_k - i^{N}_k)} \prod_{s \in I} (i^{N}_s - i^{N}_s).
\]
Using further \[60\] and
\[
\sum_{i \in I} \frac{(i^{N}_m - i^{N}_m)^{t-1}}{(i^{N}_m - i^{N}_m)(i^{N}_k - i^{N}_k)} \prod_{s \in J \setminus \{I\}} (i^{N}_s - i^{N}_s)
\]
we have \(Y = -W - Z - X.\) Thus
\[
(62) + (63) + (64) = 0. \tag{67}
\]
This completes the proof of (1) of Theorem 1.

### 3.8 Proof of (2) of Theorem 1

Let us prove the remaining part of Theorem 1. Let \(\{E_{ij}\}\) be the standard basis of \(gl_N\) where \(E_{ij}\) is the matrix unit with 1 in \(ij\) component. Set \(h_i = E_{i \cdot} - E_{i+1 \cdot -1}\) (\(1 \leq i \leq N - 1\)). By the definition of \(f = \sum f_{\Lambda} v_{\Lambda}\) it has weight zero, \(h_i f = 0\) for any \(i\). Hence it is sufficient to prove \(E_{ij} f = 0\) for any \(i \neq j\).

First we assume \(N \geq 3\). Then it is sufficient to prove
\[
E_{rN} f = 0, \quad r = 1, \ldots, N - 2. \tag{68}
\]
In fact by the following reason the proof for an arbitrary \(E_{ij}\) case is reduced to the above case. In our description of our basis of differential forms the index \(N\) and \(N - 1\) play a special role. For \(E_{ij}\) we replace the role of \(N\) by \(j\) and that of \(N - 1\) by \(k\) with \(k \neq i, j\). This is possible because \(N \geq 3\). Then the following proof is totally the same in this modified situation. Thus let us prove \[63\].

Let \(\Lambda = (\Lambda_1, \ldots, \Lambda_N)\) with \(\Lambda_j = (i^{N}_1, \ldots, i^{N}_m)\) (\(1 \leq j \leq m\)). We consider \(\Lambda' = (\Lambda'_1, \ldots, \Lambda'_N)\) with
\[
\Lambda'_j = \Lambda_j (j \neq r, N), \quad \Lambda'_r = (i^{N}_1, \ldots, i^{N}_m, i^{N}_N), \quad \Lambda'_N = (i^{N}_1, \ldots, i^{N}_{m-1}).
\]
Define \(v_{\Lambda'}\) in an obvious way. Then the coefficient of \(v_{\Lambda'}\) of \(E_{rN} f\) is
\[
f_{\Lambda} + \sum_{l=1}^{m} f_{\Lambda(l^{N}_1, i^{N}_m)}.
\]
Hence it is sufficient to prove
\[
\tilde{f}_{\Lambda} + \sum_{l=1}^{m} f_{\Lambda(l^{N}_1, i^{N}_m)} = 0. \tag{69}
\]

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We shall divide the case into two for the proof of (69).

(i): The coefficient of \( \Omega = \left( \frac{\Lambda}{i_1} \cdots \frac{\Lambda}{i_{N-1}} \right) \) of the left hand side of (69).

In (II) of the proof of (1) of Theorem 1 we have calculated the coefficient of \( \Omega \) in

\[
\frac{\Delta(i_1, \ldots, i_{N-1})}{\prod_{i < u}(\Lambda_i \Lambda_u)} \sum_{i=1}^{m} \frac{1}{i_N - i_i} f_{\Lambda(i_i^N)}.
\]

From the calculation there we can easily read off the coefficient of \( \Omega \) in

\[
\frac{\Delta(i_1, \ldots, i_{N-1})}{\prod_{i < u}(\Lambda_i \Lambda_u)} \sum_{i=1}^{m} \tilde{f}_{\Lambda(i_i^N)}.
\]

It is

\[-m \sum_{i=1}^{m} A_r \frac{m-1}{i_N - i_i} \prod_{s \neq i}^m (i_i^N - i_s^N) = -A_r \prod_{s=1}^m (i_N - i_s^N) = -1.\]

Thus the coefficient of \( \Omega \) in the lhs of (69) is 0, since

\[
\frac{\Delta(i_1, \ldots, i_{N-1})}{\prod_{i < u}(\Lambda_i \Lambda_u)} \tilde{f}_\Lambda = \Omega.
\]

(ii): The coefficient of \( \Omega_{f,J} \) with \(|J| \geq 1\) in the lhs of (69).

In (IX) we have proved that the coefficient of \( \Omega_{f,J} \) in

\[
\frac{\Delta(i_1, \ldots, i_{N-1})}{\prod_{i < u}(\Lambda_i \Lambda_u)} \sum_{i=1}^{m} \frac{1}{i_N - i_i} f_{\Lambda(i_i^N)}
\]

is zero. There the condition \(|J| \geq 2\) is used only when the residue theorem is applied. Taking care of it we can again easily read off the coefficient of \( \Omega_{f,J} \) of

\[
\frac{\Delta(i_1, \ldots, i_{N-1})}{\prod_{i < u}(\Lambda_i \Lambda_u)} \sum_{i=1}^{m} \tilde{f}_{\Lambda(i_i^N)}
\]

from the calculation in (IX). We used the notation \((\frac{\Lambda}{i_1}, \ldots, \frac{\Lambda}{i_{N-1}}), X, Y \) to denote the equation appeared there. We shall use the prime of them for the corresponding equation like \((\frac{\Lambda}{i_1}'), X', Y' \) etc. Then

\[
Z' = \frac{(-1)^l \text{sgn} \cdot A_r E_J D_{f,J}}{G'(\bar{I}) \prod_{s \in f}(i^N_s - i^N_{s'})} \sum_{J \subseteq \bar{I}} \prod_{s \in J}(i_N^N - i_s^N) \prod \left( i_N^N - i_s^N \right) \prod_{s \in J \setminus \{k\}} \prod_{s \in J \setminus \{l\}} \prod_{s \in J \setminus \{m\}} \prod_{s \in J \setminus \{i\}}
\]

\[
W' = -\frac{\text{sgn} \cdot E_J D_{f,J} \prod_{s \in J}(i_s^N - i_s^N)}{G'(\bar{I}) \prod_{s \in J}(i_N^N - i_s^N)}
\]

Also we have

\[
X' = \frac{\text{sgn} \cdot A_r E_J D_{f,J} \prod_{s \in J}(i_s^N - i_s^N)}{G'(\bar{I}) \prod_{s \in J}(i_N^N - i_s^N)} \sum_{J \subseteq \bar{I}} \prod_{s \in J}(i_N^N - i_s^N) \prod \left( i_N^N - i_s^N \right) \prod_{s \in J \setminus \{k\}} \prod_{s \in J \setminus \{l\}} \prod_{s \in J \setminus \{m\}} \prod_{s \in J \setminus \{i\}}
\]

and

\[
Y' = \frac{\text{sgn} \cdot E_J D_{f,J} \prod_{s \in J}(i_s^N - i_s^N)}{G'(\bar{I}) \prod_{s \in J}(i_N^N - i_s^N)}
\]

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where the right hand side is the binomial coefficient. The number

Thus the equation (69) is proved.

Hence

Since (71) is greater than 1 if

On the other hand the dimension of the vector space spanned by our integral formulae is less than the

In this paper we give integral and theta formulae for the solutions of

4 Discussion

In this paper we give integral and theta formulae for the solutions of $sl_N$ Knizhnik-Zamolodchikov (KZ) equations of level 0 with the value in the trivial representation in the tensor product of the vector representations of $sl_N$. The formula generalizes the Smirnov’s formula in the case of $sl_2$. We have found that the differential form $\mu^A$, which is a building block of the integral formula, is obtained by evaluating one of the variables to the branch point $Q_p$ in the product of chiral Szegő kernels. This is a key for the proof of the theta formula.

Let us discuss remaining problems and related subjects.

In $N = 2$ case it is conjectured that Smirnov type solutions span the singlet solution space $[16]$. On the other hand the dimension of the vector space spanned by our integral formulae is less than the multiplicity of the trivial representation in $V^\otimes Nm$ for $N \geq 3$ and $m \geq 2$. In fact the multiplicity is given by

\[
\text{mult}(0, V^\otimes Nm) = \frac{(Nm)!}{\prod_{k=0}^{N-1} \prod_{j=0}^{m-1} (m + k - j)}.
\]

On the other hand the dimension $D(N, m)$ of the vector space spanned by integral formulae satisfies

\[
D(N, m) \leq \text{I}(N, m) = \binom{Nm - 2}{(N - 1)m - 1},
\]

where the right hand side is the binomial coefficient. The number $Nm - 2$ is the dimension of an eigenspace of the $N$-cyclic automorphism $\phi$ on the first homology group of a $Z_N$ curve. Then

\[
\frac{\text{mult}(0, V^\otimes Nm)}{\text{I}(N, m)} = \frac{N - 1}{N - 1} \prod_{k=1}^{N - 1} \prod_{j=0}^{m - 2} \frac{(N - 1)(m - 1 - j) + k}{m - j + k}.
\]

Since

\[
(N - 1)(m - 1 - j) + k - (m - j + k) = (N - 2)(m - 2 - j) + \frac{N - 3}{N - 2},
\]

is greater than 1 if $N \geq 3$ and $m \geq 2$. Note that $\text{mult}(0, V^\otimes N) = \text{I}(N, 1) = 1$. For $N = 2$ we have

\[
\text{mult}(0, V^\otimes 2m) = \text{I}(2, m) - \binom{2m - 2}{m - 3},
\]

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where the second term in the right hand side comes from the Riemann’s bilinear identity [16].

This structure of solution space should be same in the qKZ case. To construct remaining solutions for both KZ and qKZ equations is an interesting and important problem. In the qKZ case to study a relation of these missing solutions with form factors is also interesting.

We still do not understand the relation between the integral formula given here and those given in [6, 10, 11] in the case of $sl_N, N \geq 3$. In $N = 2$ case the relation is given in [9]. If we understand this structure then it will help to find the missing solution discussed above.

The relation of the solution to the KZ equation of level 0 with a classical integrable system is still to be clarified. The relation with the Szegő kernel will give some hint to understand this problem since the Szegő kernel is related with the tau function of the KP hierarchy. Anyway it is true that we can introduce a Jacobian variable in the theta formula for the solutions to the KZ equation. Hence it is natural to ask what kind of equation governs the dependence on the Jacobian variables and what the zero value means for that equation.

Once we introduce the Jacobian variable we can ask what is the difference analogue, q analogue of the theta function? As to the abelian integral, Smirnov [16, 17] discussed its difference analogue.

Since the Smirnov type formula is related with the algebraic curves in the case of $sl_N$, it is interesting to study Smirnov type solutions for other type of Lie algebras and whether they are related with algebraic curves.

The determinantal structure of Smirnov type solution is still lacking an understanding from the representation theoretical viewpoint.

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A Appendix 1

In this section we give a derivation of the formula (20)-(24). We recall the definition of \( \binom{\Lambda}{p} \)

\[
\binom{\Lambda}{p} = \frac{g^{(\Lambda_r)}(p)g^{(p)}_{\Lambda_r}(z)dz}{(z-p)s} \quad p \in \Lambda_r,
\]

where as in the main text \( z - p \) means \( z - \lambda_p \) and \( g^{(\Lambda_r)}(p) \) means \( g^{(\Lambda_r)}(\lambda_p) \) etc.

(I). a derivation of the formula (20):

By differentiating the defining formula of \( \binom{\Lambda}{p} \) we have

\[
\frac{\partial}{\partial \lambda_i^N} \binom{\Lambda}{p} = -\frac{1}{p-i_m^N} \binom{\Lambda}{p} + \frac{1}{N} g^{(\Lambda_r)}(p)g^{(p)}_{\Lambda_r}(z)dz \quad (z-p)(z-i_m^N)s \quad (72)
\]

\[
= (1 - \frac{1}{N}) \frac{1}{i_m^N - p} \binom{\Lambda}{p} + \frac{1}{N} \frac{1}{i_m^N - p} g^{(\Lambda_r)}(p)g^{(p)}_{\Lambda_r}(z)dz \quad (73)
\]

\[
= (1 - \frac{1}{N}) \frac{1}{i_m^N - p} \binom{\Lambda}{p} - \frac{1}{N} \frac{1}{i_m^N - p} \prod_{j \notin \Lambda_r, j \notin i_m^N} \frac{p-j}{i_m^N - j} \binom{p}{i_m^N} \quad (74)
\]

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Here to obtain (73) from (72) we use
\[
\frac{1}{(z - p)(z - i_m^N)} = \frac{1}{i_m^N - p} \left[ -\frac{1}{z - p} + \frac{1}{z - i_m^N} \right],
\]
and to get (74) from (73) we use
\[
g_{\Lambda_r}^{(p)}(z) = g_{\Lambda_r}^{(i_m^N)}(z),
\]
\[
g_{\Lambda_r}^{(p)}(p) = \prod_{j \neq \Lambda_r, j \neq i_m^N} \frac{p - j}{i_m^N - j}.
\]

(II). a derivation of the formula (21):
Since
\[
\left( \Lambda \begin{array}{c} i' \\ i'' \end{array} \right) - \left( \begin{array}{c} i' \\ i'' \end{array} \right) = \frac{g_{\Lambda_r}^{(i')}(i'')}{(z - i''_s)} g_{\Lambda_r}^{(i')} g_{\Lambda_r}^{(i''_s)}(z) \frac{dz}{(z - i''_s)}
\]
we have
\[
\left( \Lambda \begin{array}{c} i' \\ i'' \end{array} \right) - \left( \begin{array}{c} i' \\ i'' \end{array} \right) = \frac{g_{\Lambda_r}^{(i')}(i'')}{(z - i''_s)} g_{\Lambda_r}^{(i')} g_{\Lambda_r}^{(i''_s)}(z) \frac{dz}{(z - i''_s)}
\]
\[
= \frac{i_m^N - i''_s}{i_m^N - i''_s} g_{\Lambda_r}^{(i')}(i'') g_{\Lambda_r}^{(i''_s)}(z) \frac{dz}{s},
\]
where we set
\[
g_{\Lambda_r}^{(i')}(i'') = \prod_{j \neq \Lambda_r, j \neq i', i''} (z - j).
\]
Hence
\[
g_{\Lambda_r}^{(i')}(i'') \frac{dz}{s} = \prod_{j \neq \Lambda_r, j \neq i', i''} (z - j).
\]

(75)

If we set \( r = N - 1 \) and \( l' = m \) in this equation we get
\[
\frac{g_{\Lambda_{N-1}}^{(i^{N-1})}}{s} \frac{dz}{s} = \prod_{k \neq}^{m-1} (i_k^{N-1} - i_k^{N-1}) \frac{1}{g_{\Lambda_{N-1}}^{(i^{N-1})}} \left( \begin{array}{c} \Lambda \\ i'' \end{array} \right) - \left( \begin{array}{c} i' \\ i'' \end{array} \right).
\]

(76)

Now let \( F(z) \) be an arbitrary polynomial of degree at most \( m - 2 \) then
\[
F(z) = \sum_{k=1}^{m-1} \frac{F(i_k^{N-1})}{\prod_{s \neq k} (i_k^{N-1} - i_s^{N-1})} g_{\Lambda_{N-1}}^{(i_k^{N-1}, i_s^{N-1})}(z).
\]

(77)

Thus we have
\[
\frac{g_{\Lambda_r}^{(i')}(i'')}{s} \frac{dz}{s} = \sum_{k=1}^{m-1} \frac{g_{\Lambda_r}^{(i_k')}(i_k^{N-1})}{\prod_{s \neq k} (i_k^{N-1} - i_s^{N-1})} \frac{g_{\Lambda_{N-1}}^{(i_k^{N-1}, i_s^{N-1})}}{s} (z).
\]

(78)

Substituting (74) and (76) into (78) we get (21). The relation (22) is a special case \( r = N - 1 \) of (21).
(III). Here we prove the formula

\[
\left( \begin{array}{c}
A \\
i^N_m
\end{array} \right) = \sum_{i'_j \in K, j \in K, j \neq i'_i} \prod_{i'_i \neq j} \frac{i^N_m - j}{i'_i - j} \left( \begin{array}{c}
A \\
i'_i
\end{array} \right) ,
\]

(79)

For the sake of simplicity we set \(i'_j = i_{(r-1)m+j}\). By Proposition \(3\) we have

\[
\left( \begin{array}{c}
A \\
i_i \\
i_L
\end{array} \right) = \left[ c_1^{(A)}, \cdots, c_L^{(A)} \right] A ,
\]

where \(A\) is the \(L\) by \(L\) matrix whose \(kl\) component \(A_{kl}\) is given by \(A_{kl} = \lambda_i^{k-1}\). Let \(d_{kl}\) be the \(kl\) cofactor of \(A\). Then we have

\[
c_k^{(A)} = \frac{1}{\det A} \sum_{j=1}^L \frac{(A)}{i_j} d_{kj}.
\]

Again by Proposition \(3\)

\[
\left( \begin{array}{c}
A \\
i^N_m
\end{array} \right) = \sum_{k=1}^L c_k^{(A)} \lambda_i^{k-1} = \frac{1}{\det A} \sum_{j=1}^L \sum_{k=1}^L \lambda_i^{k-1} d_{kj} \left( \begin{array}{c}
A \\
i_j
\end{array} \right) .
\]

(80)

Using the expansion of the Vandermon determinant in a column we have

\[
\frac{1}{\det A} \sum_{k=1}^L \lambda_i^{k-1} d_{kj} = \prod_{s \neq j} \frac{i^N_m - i_s}{i_j - i_s}.
\]

(81)

Substituting (81) into (80) we have (79).

The relation (24) is proved in an exactly similar manner.

(IX). a derivation of the formula (23):

The derivation is similar to (II). We have

\[
\left( \begin{array}{c}
A \\
i^N_m
\end{array} \right) - \left( \begin{array}{c}
i^N_m \\
i^N_m
\end{array} \right) = g^{(A_N)}(i^N_m) \frac{dz}{(z - i^N_m)s} \left[ g^{(i^N_m)}(z) - \frac{g^{(i^N_m)}(N_m)}{g^{(i^N_m)}(i^N_m)} g^{(i^N_m)}(z) \right] .
\]

(82)

Since the polynomial in \([\quad]\) is divided by \(z - i^N_m\) we can define the polynomial \(G^r(z)\) by

\[
G^r(z) = \frac{1}{z - i^N_m} \left[ g^{(i^N_m)}(N_m) - \frac{g^{(i^N_m)}(N_m)}{g^{(i^N_m)}(i^N_m)} g^{(i^N_m)}(z) \right].
\]

Then

\[
G^r(i^N_k - 1) = \prod_{s=1}^{m-1} \frac{i^N_k - i^N_s}{i^N_k - i^N_m} \left[ 1 - \prod_{s=1}^{m-1} \frac{i^N_k - i^N_s}{i^N_k - i^N_m} \prod_{s \neq l} \frac{i^N_k - i^N_l}{i^N_m - i^N_s} \right] .
\]

(83)

Using (83), (77) and (76) we have, from (82),

\[
\left( \begin{array}{c}
A \\
i^N_m
\end{array} \right) - \left( \begin{array}{c}
i^N_m \\
i^N_m
\end{array} \right) = \sum_{k=1}^{m-1} \prod_{j \in K, j \neq i'_i} \frac{i^N_k - j}{i^N_k - i^N_m} \left[ - \prod_{s=1}^{m-1} \frac{i^N_k - i^N_s}{i^N_k - i^N_m} \prod_{s \neq l} \frac{i^N_k - i^N_l}{i^N_m - i^N_s} \right] \left( \begin{array}{c}
A \\
i^N_k - 1
\end{array} \right) .
\]

(84)

Substituting (79) into (84) we have (23).

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