Constructing and calculating Adams operations on topological modular forms

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Abstract

We construct stable Adams operations $\psi^n$ on the cohomology theory $\text{Tmf}$ of topological modular forms. This is done by extending the sheaf defining $\text{Tmf}$ to be functorial with respect to isogenies of generalised elliptic curves of invertible degree. These Adams operations are then calculated on the homotopy groups of $\text{Tmf}$ using a combination of descent spectral sequences, the theory of synthetic spectra, and Anderson duality.

Contents

Introduction 2

1 Background on log stacks 5
  1.1 Logarithmic structures on stacks ............................................. 5
  1.2 Our stacks .............................................................................. 6

2 Extending $O_{\text{top}}$ to isogenies of invertible degree 9
  2.1 Our site of isogenies of invertible degree ................................. 9
  2.2 Generalised elliptic cohomology theories ................................. 11
  2.3 Constructions using $p$-divisible groups .................................. 13
  2.4 Construction in the cuspidal neighbourhood ............................ 16
  2.5 Construction on the compactified moduli stack ....................... 22

3 Adams operations 25
  3.1 Generalities on endomorphisms of $\text{tmf}$ ............................... 26
  3.2 Anderson duality .................................................................... 32
  3.3 Proof of Theorem [B] .............................................................. 35
  3.4 A conjecture on dual endomorphisms ..................................... 37

A Appendix on the uniqueness of $O_{\text{top}}$ 38

References 40

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Introduction

Adams operations are some of the most utilised power operations in homotopy theory. This is exemplified by work of Adams counting the number of vector fields on spheres [Ada62], by Adams–Atiyah giving a “postcard sized” proof of the Hopf invariant one theorem [AA66], and by Quillen calculating the algebraic $K$-theory of finite fields [Qui72]. In these three examples, Adams operations arise as operators on real and complex topological $K$-theory, KO and KU. In this article we study Adams operations on another cohomology theory $Tmf$, called topological modular forms. One’s first guess at a construction of such operations on $Tmf$ might be to simply copy that for topological $K$-theory, however, there is a fatal flaw with this approach: the most common construction of Adams operations on $KO$ and $KU$ is performed by geometric methods, in particular by manipulating vector bundles, and no such geometric interpretation for the cohomology classes of $Tmf$ currently exists. Thus, to motivate our study of Adams operations on $Tmf$ in this article, let us first reimagine these operations on $KU$ through the lens of algebraic geometry.

There exists a sheaf of elliptic cohomology theories $\mathcal{O}^{\text{top}}$ on the moduli stack of generalised elliptic curves $\mathcal{M}_{\text{Ell}}$, and by definition the global sections of $\mathcal{O}^{\text{top}}$ yield $Tmf$. One can also view $KU$ in this algebro-geometric light: by restricting $\mathcal{O}^{\text{top}}$ to the cusp of $\mathcal{M}_{\text{Ell}}$ one obtains $\mathcal{O}^{\text{mult}}$ – a sheaf of cohomology theories with a particular section $\mathcal{O}^{\text{mult}}_{G_m}$ associated to the multiplicative group scheme $G_m$ over $\text{Spec} \mathbb{Z}$. With this description it is clear that $KU$ is acted upon by automorphisms of $G_m$, however, this only gives rise to the Adams operations $\psi^1$ and $\psi^{-1}$; the identity and complex conjugation. To obtain more Adams operations on $KU$ one might try to extend $\mathcal{O}^{\text{mult}}$ to be functorial with respect to isogenies on $G_m$ of invertible degree. This is not an unreasonable request, as $KU$ only remembers the formal group $G_m$ associated to $G_m$, and isogenies of invertible degree induce automorphisms on formal groups. Once this extended functorality is achieved, the Adams operations on $KU$ can be obtained by applying $\mathcal{O}^{\text{mult}}$ to the $n$-fold-multiplication map on $G_m$, which is an isogeny of invertible degree over $\mathbb{Z}[\frac{1}{n}]$.

The key to extending the functorality of $\mathcal{O}^{\text{mult}}$ is to start working in a $p$-complete setting for each prime $p$, and use the fact that isogenies of invertible degree induce isomorphisms on associated $p$-divisible groups. The $p$-completion of $\mathcal{O}^{\text{mult}}$ can then be obtained as the pullback along $\mathcal{M}_{G_m} \to \mathcal{M}_{BT_1}$ of another sheaf $\mathcal{O}^{\text{top}}_{BT_1}$ over the moduli stack $\mathcal{M}_{BT_1}$ of $p$-divisible groups of height 1. This description of the $p$-completion of $\mathcal{O}^{\text{mult}}$ makes it clear that the $n$-fold-multiplication map on $G_m$ over $\mathbb{Z}_p$ induces an automorphism on $p$-complete $K$-theory $KU_p$ for $n$ not divisible by $p$. The resulting automorphism is then precisely the $n$th Adams operation $\psi_n^*: KU_p \to KU_p$. One can then patch together these Adams operations for all primes $p$ not dividing $n$ with some rational information to obtain our desired operation $\psi_n: KU[\frac{1}{n}] \to KU[\frac{1}{n}]$.

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1 The sheaf $\mathcal{O}^{\text{top}}$ has a storied history, much of which can be found in [Beh19].
2 All of the above folklore can be found in [HL16], [Dav20] §2, and §2.
3 These operations are often called stable Adams operations, as they are defined as maps of spectra. These
This gives us a blueprint to define Adams operations on Tmf: first extend $\mathcal{O}^\text{top}$ to be functorial with respect to isogenies of generalised elliptic curves of invertible degree, and then define Adams operations using the $n$-fold-multiplication map. The following is stated in its precise form as Theorem 2.1.

**Theorem A.** The sheaf $\mathcal{O}^\text{top}$ defining $\text{Tmf}$ is functorial with respect to isogenies of generalised elliptic curves of invertible degree.

**Remark 0.1.** The above theorem answers a question of Hill–Lawson ([HL16, Q.1.3]) partially in the affirmative. Proposition 1.14 also confirms that our methods cannot extend $\mathcal{O}^\text{top}$ to be functorial with respect to all separable isogenies of generalised elliptic curves.

Using Theorem A, we obtain Adams operations $\psi^n : \text{Tmf}[\frac{1}{n}] \to \text{Tmf}[\frac{1}{n}]$ upon taking global sections of $\mathcal{O}^\text{top}$, as outlined for the case of topological $K$-theory above; see Definition 3.1. Once we have these operations, we collect some of their basic properties and calculations. Many elementary facts about these operations come straight from their definition, such as the existence of a collection of coherent homotopies $\psi^m \circ \psi^n \simeq \psi^{mn}$ between $E_8$-endomorphisms of $\text{Tmf}[\frac{1}{mn}]$, and the commutative diagram of $E_8$-rings

$$
\begin{array}{ccc}
\text{Tmf}[\frac{1}{n}] & \xrightarrow{\psi^n} & \text{Tmf}[\frac{1}{n}] \\
\downarrow & & \downarrow \\
\text{KO}[\frac{1}{n}] & \xrightarrow{\psi^n} & \text{KO}[\frac{1}{n}];
\end{array}
$$

see §3.1. The calculation of the operations $\psi^n$ on $\pi_* \text{Tmf}[\frac{1}{n}]$ proves to be less formal.

**Theorem B.** Given integers $k$ and $n$, then inside $\pi_k \text{Tmf}[\frac{1}{n}]$ we have the equality

$$
\psi^n(x) = \begin{cases} 
  x & x \in \mathfrak{Tors}_k \\
  n^{\lceil |x| \rceil} x & x \in \mathfrak{Free}_k,
\end{cases}
$$

where $\mathfrak{Tors}_k \subseteq \pi_k \text{Tmf}[\frac{1}{n}]$ is the subgroup of torsion elements, and $\mathfrak{Free}_k$ is the orthogonal subgroup of Notation 3.7.

Despite the similarity to the calculations of Adams operations on $\pi_* \text{KU}$ and $\pi_* \text{KO}$, the above theorem is significantly more difficult for us to prove.

**Outline**

In §1, we summarise the tools from classical algebraic geometry that we use to define and discuss the sheaf $\mathcal{O}^\text{top}$. This includes some background on logarithmic structures on Deligne–Mumford stacks and a list of the particular log stacks we are interested in.

In §2 we prove Theorem A in much the same way that Hill–Lawson construct $\mathcal{O}^\text{top}$ as a log étale sheaf on the moduli stack of generalised elliptic curves; see [HL16]. This is the only kind of Adams operations we will consider in this article, so let us forgo the adjective stable.
The key to obtaining our extra functorality is to use a theorem first stated by Lurie (see [BL10, Th.8.1.4] for the original reference and [Dav20, Th.0.0.5] for the proof), which allows us to construct pieces of $O^{top}$ with the desired additional functorality. An overview of the proof of Theorem A (also disguised as Theorem 2.1) can be found at the beginning of §2.

In §3 we use Theorem A to construct the Adams operations $\psi^n$ on $\text{Tmf}[\frac{1}{n}]$. Theorem B is then proved as follows:

1. First, we decompose the homotopy groups of $\text{Tmf}$ as $\mathbb{T}_{\text{tors}} \oplus \mathbb{F}_{\text{free}}$ as in Notation 3.7. Next, we show that endomorphisms of the connective cover $\tau_{\geq 0} \text{Tmf} = \text{tmf}$ preserve this decomposition; see Theorem 3.8. This is proven using calculations from motivic homotopy theory, although our arguments apply equally as well in the context of synthetic spectra or $\Gamma,S$-modules; see Remark 3.12. These general statements will be further capitalised on in future work [Dav], and also may be of independent interest.

2. We then calculate $\psi^n$ on $\mathbb{T}_{\text{tors}}$ in nonnegative degree using the $S$-linearity of $\psi^n$ and on $\mathbb{F}_{\text{free}}$ in nonnegative degrees using a rationalised algebraic computation; see the proof of Theorem 3.2.

3. To tackle the negative homotopy groups of $\text{Tmf}$, we appeal to its Anderson self-duality, originally proven by Stojanoska; see [Sto12]. This yields methods to calculate $\psi^n$ on $\pi_* \text{Tmf}[\frac{1}{n}]$ in negative degrees; see Corollary 3.13 and Lemma 3.26.

The above analysis of the Adams operations $\psi^n$ on $\pi_* \text{Tmf}[\frac{1}{n}]$ relies on some computations of the dual Adams operations $\check{\psi}^n$. This article ends by formulating a conjecture concerning the interaction of endomorphisms of self-dual $E_1$-rings and their dual endomorphisms, which we can validate in the cases of Adams operations acting on $KU$, $KO$, $\text{Tmf}$, and well as $\text{Tmf}(\Gamma)$ for particular congruence subgroups $\Gamma$; see §3.4.

In §A, we recall a statement known to experts about the uniqueness of the sheaf $O^{top}$.

**Past and future work** Operations on elliptic cohomology theories have been constructed by Baker [Bak90] and Ando [And00], and these include Adams operations. The Adams operations in this article can be seen as stable, homotopy coherent, and $E_{\infty}$-versions of those previously studied. As suggested in [Bak90] and [And00], one can also construct other endomorphisms on elliptic cohomology theories (and on $\text{Tmf}$ using Theorem A) such as Hecke operators. Hecke operators on $\text{Tmf}$ are more complicated than the above Adams operators for two related reasons: Firstly, a coherent construction of Hecke operators should involve a study of coherent transfers between sheaves of $E_{\infty}$-rings, for which clear foundations using the higher algebra of categories of spans has only recently emerged; see [Bar17, BH20, and EH20]. Secondly, Hecke operators on $\text{Tmf}$ are not multiplicative, so more elaborate arguments are necessary to show there are no exotic extensions at the primes 2 and 3. This will all be explored in forthcoming work [Dav].
Conventions

The language of $\infty$-categories will be used without explicit reference, so all categorical constructions and considerations will be of the $\infty$-categorical flavour. In particular, for a scheme $X$ and a finite group acting on $X$, we will write $X/G$ for what is sometimes called the stacky quotient. The only exception to the above are the $(2,2)$-categories mentioned in Definition 2.3. We implicitly use the equivalence of $\infty$-categories between $\text{CAlg}_\mathbb{Q}$ and of $E_\infty$-objects in the (unbounded) derived category $\mathcal{D}(\mathbb{Q})$; see [Lur17, Th.7.1.2.13]. Given a prime $p$, we will also write $\widehat{\mathcal{M}}$ for $\mathcal{M} \times \text{Spf} \mathbb{Z}_p$, where $\mathcal{M}$ is any stack. All of our sites are subsites of the small fppf site of a stack. All of our discrete rings will be commutative.

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1 Background on log stacks

As our main algebro-geometric object of interest is the moduli of generalised elliptic curves, we will freely speak of stacks and Deligne–Mumford stacks; see [Sta21, Tag 0ELS & Tag 03YO].

1.1 Logarithmic structures on stacks

Logarithmic structures on schemes have been studied by Illusie [Ill94] and Kato [Kat89], and on stacks by Olsson [Ols03] and Hill–Lawson [HL16].

Definition 1.1. A prelog structure on a Deligne–Mumford stack $X$ is an étale sheaf $\mathcal{M}$ of commutative monoids on $X$, together with a morphism $\alpha : \mathcal{M} \to \mathcal{O}_X$ of sheaves of commutative monoids, where $\mathcal{O}_X$ has the multiplicative monoidal structure. The pair $(X, \mathcal{M})$ is called a prelog stack. A morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ of prelog stacks is a morphism of Deligne–Mumford stacks $f : X \to Y$ and a morphism $\phi : f^{-1}N \to M$ of sheaves of commutative monoids on $X$, such that the diagram

\[
\begin{array}{ccc}
 f^{-1}N & \xrightarrow{f^{-1}\beta} & f^{-1}\mathcal{O}_X \\
 \downarrow \phi & & \downarrow \\
 M & \xrightarrow{\alpha} & \mathcal{O}_X
\end{array}
\]

commutes. A prelog structure $\mathcal{M}$ on $X$ is a log structure if the morphism of commutative monoids $\alpha : M \to \mathcal{O}_X$ induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times$. In this case, we will call such a pair $(X, \mathcal{M})$ a log stack. A morphism of log stacks is a morphism of underlying prelog stacks. Write $\text{Log DM}$ for the category of log stacks.
Log geometry interests us due to the fact that there is a notion of a log étale morphism of log stacks, which is more flexible than that of an étale morphism.

**Definition 1.2.** A morphism \((f, \phi): (X, M) \to (Y, N)\) of fine (see [HL16, Df.2.6]) log stacks is log étale if the following two conditions hold:

1. The morphism of Deligne–Mumford stacks \(f: X \to Y\) is locally of finite presentation;
2. For every commutative diagram of fine log stacks

\[
\begin{array}{ccc}
(T', L') & \longrightarrow & (X, M) \\
\downarrow (g, \psi) & & \downarrow (f, \phi) \\
(T, L) & \longrightarrow & (Y, N),
\end{array}
\]

where \(g\) is an underlying closed immersion of Deligne–Mumford stacks with defining ideal \(I\) square-zero and \(g^{-1}L \to L'\) is an isomorphism, the mapping space

\[
\text{Map}_{\text{Log DM}}^\text{log étale} ((T, L), (X, M))
\]

is contractible.

This is a mild generalisation of Kato’s definition [Kat89, (3.3)] of a log étale morphism of fine log schemes, and by [Kat89, Th.3.5] is equivalent to [HL16, Df.2.1.3].

**Proposition 1.3.** Let \(f: X \to Y\) and \(g: Y \to Z\) be morphisms of fine log stacks where \(g\) is log étale. Then \(f\) is log étale if and only if \(g \circ f\) is log étale. Moreover, if \(h: T \to Z\) is any other morphism of finite log stacks, then \(g': T \times_Z Y \to T\) is log étale.

**Proof.** The proof follows exactly as in the classical étale case [Sta21, Tag 02GO & Tag 02GW] and the case for log schemes [Kat89, §3].

It is not true that log étale morphisms are in general flat, however, in some situations we are interested in, this is true.

**Proposition 1.4 ([HL16, Pr.2.18]).** Let \((X, M)\) be a log stack, whose log structure is determined by a smooth divisor \(D\) on \(X\); see [HL16, §2.2]. Then any log étale morphism of log stacks \((Y, N) \to (X, M)\) is flat, and the log structure on \(Y\) is determined by a smooth divisor.

**Remark 1.5.** If \((X, M)\) is a log stack with log structure determined by a smooth divisor \(D\) on \(X\), then all morphisms in its log étale site \(\text{log DM}_{(X, M)}\) are flat. Indeed, by Proposition 1.3 all morphisms are log étale, and by Proposition 1.4 they are also flat.

**1.2 Our stacks**

Write \(\mathcal{M}_{\text{EI}}^{\text{sm}}\) for the moduli stack of smooth elliptic curves, and \(\mathcal{M}_{\text{El}}\) for its compactification, which has a moduli interpretation as the moduli stack of generalised elliptic curves; see [DR13], [Con07], or [Ces17]. Note that our generalised elliptic curves will always have irreducible geometric fibres, so either elliptic curves or Néron 1-gons.
**Definition 1.6.** Consider $\mathcal{M}_{\text{Ell}}$ as a log stack using the direct image log structure from the open inclusion $\mathcal{M}_{\text{Ell}}^{\text{sm}} \to \mathcal{M}_{\text{Ell}}$; see [HL16 §3.1]. In particular, the log structure on $\mathcal{M}_{\text{Ell}}$ is defined by the cusp $\mathcal{M}_{\text{Ell}}^{\text{sm}} \subset \mathcal{M}_{\text{Ell}}$. As mentioned in [HL16, Cor.3.4], this log structure on $\mathcal{M}_{\text{Ell}}$ is determined by a smooth divisor, hence $\mathcal{M}_{\text{Ell}}$ is fine and saturated, and Proposition 1.4 applies. The substack $\mathcal{M}_{\text{Ell}}^{\text{sm}} \subseteq \mathcal{M}_{\text{Ell}}$ is naturally equipped with a trivial log structure, however, there are other substacks of $\mathcal{M}_{\text{Ell}}$ with nontrivial log structures.

**Definition 1.7.** Define the *Tate moduli stack* as the quotient $\mathcal{M}_{\text{Tate}} = (\text{Spf } \mathbb{Z}[q])/C_2$, where $(q)$ generates the topology on $\mathbb{Z}[q]$ and $C_2$ acts by negation, so $q \mapsto -q$. Moreover, there is a map $\mathcal{M}_{\text{Tate}} \to \mathcal{M}_{\text{Ell}}$ associated to the *Tate curve*, a generalised elliptic curve over $\mathbb{Z}[q]$ defined by the formula

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

where

$$a_4(q) = -5 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n}, \quad a_6(q) = -\frac{1}{12} \sum_{n=1}^{\infty} \frac{(5n^3 + 7n^5)q^n}{1 - q^n}.$$ 

Giving $\mathcal{M}_{\text{Tate}}$ the log structure by the ideal $(q)$ ensures $\mathcal{M}_{\text{Tate}} \to \mathcal{M}_{\text{Ell}}$ is a morphism of log stacks.

When evaluated at $q = 0$, the smooth locus of $\mathcal{M}_{\text{Tate}}$ is isomorphic to $\mathbb{G}_m$, which motivates the following.

**Definition 1.8.** The *moduli stack of forms of $\mathbb{G}_m$* is defined as

$$\mathcal{M}_{\mathbb{G}_m} = (\text{Spec } \mathbb{Z})/C_2 = BC_2,$$

which comes with a natural closed immersion $\mathcal{M}_{\mathbb{G}_m} \to \mathcal{M}_{\text{Tate}}$ defined by setting $q = 0$.

In summary, we have the morphisms of log stacks

$$\mathcal{M}_{\text{Ell}}^{\text{sm}} \to \mathcal{M}_{\text{Ell}} \leftarrow \mathcal{M}_{\text{Tate}} \leftarrow \mathcal{M}_{\mathbb{G}_m}. \quad (1.9)$$

Note that $\mathcal{M}_{\text{Ell}}$ and $\mathcal{M}_{\text{Tate}}$ are the only stacks with nontrivial log structures above. To study the formal groups associated to generalised elliptic curves in a $p$-complete setting, we will use the moduli of $p$-divisible groups. For a fixed prime $p$, write $\mathcal{M}_{\text{BT}}$ for the *moduli stack of $p$-divisible groups* (also known as *Barsotti–Tate groups*) and $\mathcal{M}_{\text{BT}}^n$ for the substack of $p$-divisible groups of exact height $n$.

**Definition 1.10.** A *quasi $p$-divisible group* $\mathbf{G}$ over a ring $R$ is a collection of affine group schemes $\{\mathbf{G}_n\}_{n \geq 1}$ over $R$ and injective homomorphisms $i_n: \mathbf{G}_n \to \mathbf{G}_{n+1}$ such that the image of $i_n$ is $\mathbf{G}_{n+1}[p^n]$. A morphism $\mathbf{G} \to \mathbf{G}'$ is a collection of morphisms $\phi_n: \mathbf{G}_n \to \mathbf{G}'_n$ which commute with the structure maps. By definition, a $p$-divisible group is a quasi $p$-divisible group $\mathbf{G}$ such that each $\mathbf{G}_n$ is finite flat over $R$.

*Recall an injective morphism of affine groups schemes $X \to Y$ over $R$ is such that the morphism on global section $\phi_Y(Y) \to \phi_X(X)$ is surjective.*
The only reason we are interested in quasi-$p$-divisible groups is due to the following.

**Definition 1.11.** Given a generalised elliptic curve $E$ over a ring $R$, define the associated quasi $p$-divisible group $E[p^\infty]$ of $E$ to have $n$th level the $p^n$-torsion of the smooth locus $E^{\text{sm}}$,

$$E[p^\infty]_n = E^{\text{sm}}[p^n].$$

This operation is clearly functorial in morphisms of generalised elliptic curves over $R$. When restricted to smooth elliptic curves or Néron 1-gons $E$, this yields an ordinary $p$-divisible group of height 2 or 1; see [Tat67, §2]. In other words, we have morphisms of stacks

$$[p^\infty]: \mathcal{M}_{\text{Ell}}^{\text{sm}} \to \mathcal{M}_{\text{BT}}^p,$$

$$[p^\infty]: \mathcal{M}_m \to \mathcal{M}_{\text{BT}}^p.$$

Let $G$ be a $p$-divisible group over a ring $R$. Then if $R$ is $p$-complete there is a formal group $G^\circ$ associated to $G$ called its identity component; see [Lur18a, Th.2.0.8] for the construction of $(-)^\circ$ in this generality, and [Tat67, §2.2] for the inverse functor defined for connected $p$-divisible groups. This assignment is also compatible with the formal completion of abelian varieties, in the sense that after a base-change over $\text{Spf} \mathbb{Z}_p$, there is a morphism $[p^\infty]: \mathcal{M}_{\text{BT}}^p \to \mathcal{M}_{\text{FG}}$ of stacks which yields the diagrams of stacks,

\[
\begin{array}{ccc}
\mathcal{M}_{\text{Ell}}^{\text{sm}} & \xrightarrow{[p^\infty]} & \mathcal{M}_{\text{FG}} \\
\downarrow \downarrow \downarrow & \searrow \searrow & \downarrow \downarrow \downarrow \\
\mathcal{M}_{\text{BT}}^p & \rightarrow & \mathcal{M}_{\text{FG}} \\
\end{array}
\]

where $\mathcal{M}_{\text{FG}}$ is the moduli stack of formal groups; see [Nau07, §6]. The commutativity of the above diagrams follows along the lines of [Lur18a, Pr.7.4.1]; also see [Lur18a, §2.2.4].

**Definition 1.13.** An isogeny of generalised elliptic curves is a surjective homomorphism, and its degree is the local rank of its kernel; see [Ces17, Df.2.2.8].

The following is a key input for §2.

**Proposition 1.14.** Let $E \to E'$ be an isogeny of generalised elliptic curves over an affine formal scheme $\text{Spf} R$, where $R$ is $p$-complete. Then the induced morphism $E[p^\infty] \to E'[p^\infty]$ is an isomorphism of quasi $p$-divisible groups over $\text{Spf} R$ if and only if the degree of $E \to E'$ is invertible in $R$.

**Proof.** A quasi $p$-divisible group $G$ over $\text{Spf} R$ is determined by an inductive system of quasi $p$-divisible groups over $\text{Spec} R/I$, where $p \in I \subseteq R$ is the defining ideal of $\text{Spf} R$. Hence we may work over $\text{Spec} R$ where $p$ is nilpotent in $R$. As everything in sight is an étale sheaf, it suffices to check $E[p^\infty] \to E'[p^\infty]$ is an equivalence on geometric points; see [Sta21, Tag 03PU]. Hence we may further assume $R = \kappa$ is an algebraically closed field of characteristic $p$, and $E \to E'$ is either an isogeny of smooth elliptic curves or Néron 1-gons, which implies $E[p^\infty]$ and $E'[p^\infty]$ are ordinary $p$-divisible groups. For any $p$-divisible group $G = \{G_n\}_{n \geq 1}$ over $\kappa$, the formal $\kappa$-group scheme $G_\infty = \text{colim} G_n$ has $p^n$-torsion given by $G_n$, so it further suffices
to consider the induced morphism \( \text{colim } E[p^n] \to \text{colim } E'[p^n] \). It is a general fact that we have the following exact sequence of étale sheaves over \( \text{Spec } \kappa \),

\[
0 \to K_p \to \text{colim } E[p^n] \xrightarrow{[p^n]} \text{colim } E'[p^n] \to 0,
\]

where \( K_p \) is the component of the finite group scheme \( K = \text{ker}(E \to E') \) of maximal \( p \)-power order. The vanishing of \( K_p \) is then equivalent to the degree of \( E \to E' \) being invertible.

2 Extending \( \mathcal{O}^{\text{top}} \) to isogenies of invertible degree

In this section we prove the main technical statement of this article: that the sheaf \( \mathcal{O}^{\text{top}} \) defining \( \text{Tmf} \) is functorial with respect to isogenies of invertible degree of generalised elliptic curves.

**Theorem 2.1.** There is a log étale hypersheaf of \( E_8 \)-rings \( \mathcal{O}^{\text{top}} \) on the site \( \text{Isog}^{\text{log\'et}}_{\text{Ell}} \) of isogenies of generalised elliptic curves of invertible order of Definition 2.3, which when restricted to the affine log étale site of \( M_{\text{Ell}} \) is equivalent to \( \mathcal{O}^{\text{HL}} \) of [HL16, Th.5.17] as a functor into \( \text{CAlg}(\text{hSp}) \).

**Remark 2.2.** The original construction of \( \mathcal{O}^{\text{top}} \) goes back to unpublished work of Goerss–Hopkins–Miller, with the first published account due to Behrens [DFHH14, §12] and an alternative construction outlined in [Lur09, §4.3]. However, as is mentioned in Remark 1.4 of [DFHH14, §12] (which we will prove in Appendix A), the sheaf \( \mathcal{O}^{\text{top}} \) is unique as a diagram in \( \text{CAlg}(\text{hSp}) \), so our particular choice of model of \( \mathcal{O}^{\text{top}} \) is not important. In fact, more is true: the presheaf \( \mathcal{O}^{\text{top}} \) of \( E_8 \)-rings on the site of étale morphisms of affines \( \text{Spec } R \to M_{\text{Ell}} \) is uniquely defined up to the homotopy by the fact that it defines a natural generalised elliptic cohomology theory. This statement is folklore; see [Lur18a, Rmk.7.0.2] for a statement over \( M_{\text{Ell}}^{\text{sm}} \).

Our proof of Theorem 2.1 will closely follow the construction of Hill–Lawson, hence our use of the subscript HL. For the readers convenience, let us recall the morphisms of log stacks

\begin{align*}
M_{\text{Ell}}^{\text{sm}} &\to M_{\text{Ell}} \leftarrow M_{\text{Tate}} \leftarrow M_{G_m}.
\end{align*}

The proof of Theorem 2.1 proceeds as follows: First we construct \( \mathcal{O}^{\text{top}} \) on \( M_{\text{Ell}}^{\text{sm}} \) and \( M_{G_m} \) at each prime \( p \) using the derived moduli stack of oriented \( p \)-divisible groups (see Theorem 2.12), which we glue with rational information to obtain integral sheaves \( \mathcal{O}^{\text{sm}} \) and \( \mathcal{O}^{\text{mult}} \); see §2.3. This is the moment where the extra functoriality is obtained. Then we extend \( \mathcal{O}^{\text{mult}} \) from \( M_{G_m} \) to \( M_{\text{Tate}} \) by hand to obtain \( \mathcal{O}^{\text{Tate}} \), using explicit constructions to control logarithmic structures; see §2.4. Finally, we glue together \( \mathcal{O}^{\text{sm}} \) and \( \mathcal{O}^{\text{Tate}} \), first at each prime, then rationally, and finally integrally, to obtain our model for \( \mathcal{O}^{\text{top}} \) and Theorem 2.1.

2.1 Our site of isogenies of invertible degree

**Definition 2.3.** The prelocalised log étale site of isogenies of invertible degree over \( M_{\text{Ell}} \) is the \((2,2)\)-category \( \text{Isog}^{\text{log\'et}}_{\text{Ell}} \) defined by the data:

---

5This colimit is what some authors define as a \( p \)-divisible group.
• Objects are pairs \((X, E)\), consisting of a log stack \(X\) and a log étale morphism \(E: X \to \mathcal{M}_{\text{Ell}}\), defining a generalised elliptic curve \(E\) over \(X\).

• 1-morphisms are pairs \((f, \phi): (X, E) \to (X', E')\), consisting of a morphism of log stacks \(f: X \to X'\) and a morphism of generalised elliptic curves \(\phi: E \to f^*E'\) over \(X\), which we assume is an isogeny of invertible degree.

• 2-morphisms are morphisms \(\alpha: (f, \phi) \Rightarrow (g, \psi): (X, E) \to (X', E')\), given by a morphism \(\alpha: f^*E' \to g^*E'\) of generalised elliptic curves over \(X\) such that \(\alpha \circ \phi = \psi\), which we assume is an isogeny of invertible degree.

The log étale site of isogenies of invertible degree over \(\mathcal{M}_{\text{Ell}}\) is the initial \((2, 1)\)-category \(\text{Isog}^\log \text{ét}_{\text{Ell}}\) which receives a functor of \((2, 2)\)-categories from \(\text{Isog}^\log \text{ét}_{\text{Ell}}\). This can be obtained by formally inverting all of the 2-morphisms inside \(\text{Isog}^\log \text{ét}_{\text{Ell}}\). Given a single prime \(p\), denote the base-change of \(\text{Isog}^\log \text{ét}_{\text{Ell}}\) over \(\text{Spf} \mathbf{Z}_p\) by \(\text{Isog}^\log \text{ét}_{\text{Ell}, p}\). Similarly, associated to smooth elliptic curves, forms of the Tate curve, and forms of \(\mathbf{G}_m\), we define \((2, 1)\)-categories
\[
\text{Isog}^\log \text{ét}_{\text{Ell}, \text{sm}}, \quad \text{Isog}^\log \text{ét}_{\text{Tate}}, \quad \text{Isog}^\log \text{ét}_{\text{G}_m},
\]
as well as sites associated to their completions at a prime \(p\). Give the sites above either the log étale or étale topology, depending on their superscript, using the forgetful functor to the \((2, 1)\)-category of Deligne–Mumford stacks.

**Remark 2.4.** Define a wide \((2, 2)\)-subcategory \(\mathcal{C} \subseteq \text{Isog}^\log \text{ét}_{\text{Ell}}\) spanned by those 1-morphisms \((f, \phi)\) where \(\phi\) is an isomorphism of generalised elliptic curves. Then, any potential 2-morphisms \(\alpha\) are automatically also isomorphisms of elliptic curves in \(\mathcal{C}\), and we see \(\mathcal{C}\) is a \((2, 1)\)-category, hence its \((2, 1)\)-categorical localisation is equivalent to \(\mathcal{C}\). Furthermore, it is clear that \(\mathcal{C}\) is precisely the log étale site of Deligne–Mumford stacks over \(\mathcal{M}_{\text{Ell}}\), embedded into \(\text{Isog}^\log \text{ét}_{\text{Ell}}\).

Combining (1.9) and Definition 2.3 we obtain the functors of sites
\[
\text{Isog}^\log \text{ét}_{\text{Ell}, \text{sm}} \leftarrow \text{Isog}^\log \text{ét}_{\text{Ell}} \to \text{Isog}^\log \text{ét}_{\text{Tate}} \to \text{Isog}^\log \text{ét}_{\text{G}_m},
\]
as well as the analogous functors on \(p\)-completions.

The key step in our analysis here is to realise that when working with \(\text{Isog}^\log \text{ét}_{\text{Ell}, \text{sm}}\) or \(\text{Isog}^\log \text{ét}_{\text{G}_m}\) for a fixed prime \(p\), it suffices to work with \(p\)-divisible groups.

**Proposition 2.5.** Fixing a prime \(p\), the morphisms \([p^\mathbb{Z}]: \mathcal{M}_{\text{Ell}}^{\text{sm}} \to \mathcal{M}_{\text{BT}_p}^{\text{sm}}\) and \([p^\mathbb{Z}]: \mathcal{M}_{\text{G}_m} \to \mathcal{M}_{\text{BT}_p}^\mathbb{Z}\) induce functors
\[
[p^\mathbb{Z}]: \text{Isog}^\log \text{ét}_{\text{Ell}, \text{sm}} \to \text{fDM}/\bar{\mathcal{X}}_{\text{BT}_p}^{\text{sm}}, \quad [p^\mathbb{Z}]: \text{Isog}^\log \text{ét}_{\text{G}_m} \to \text{fDM}/\bar{\mathcal{X}}_{\text{BT}_p}^\mathbb{Z},
\]
where \(\text{fDM}\) is the category of formal Deligne–Mumford stacks considered as a subcategory of \(\text{Fun} (\text{CRing}, \mathcal{S})\), \(\text{CRing}\) is the category of discrete rings, and \(\mathcal{S}\) the \((\mathbb{X}^-)\) category of spaces.
Proof. There is a functor of $(2,2)$-categories
\[
[p^\infty]: Isog_{\text{Ell}}^{\text{log\-ét}} \rightarrow D; \quad (X,E) \mapsto (X,E[p^\infty])
\]
where $D$ is the $(2,2)$-category whose objects are pairs $(X,G)$ of a formal Deligne–Mumford stack and $G$ is a quasi $p$-divisible group over $X$, and 1- and 2-morphisms defined as in Definition 2.3, except without any constraints on the morphisms between quasi $p$-divisible groups. By Proposition 1.14, we see that all 2-morphisms of $C$ are sent to invertible 2-morphisms in $D$. Restricting to $x\text{M}_{\text{sm}}$ and $x\text{M}_{\text{Gm}}$, the above functor factors through $fDM\{\text{BT}\}_{\text{p\-\text{DIV}}} \subseteq D$, and this furthermore implies that $[p^\infty]$ also factors through $\text{Isog}_{\text{Ell,sm}}^{\text{log\-ét}}$ (resp. $\text{Isog}_{\text{Gm}}^{\text{log\-ét}}$).

The following is a standard fact regarding sheaves on the above sites.

Lemma 2.6. Let $C$ be one of the sites of Definition 2.3 and write $C^{\text{aff}}$ for the full subcategory spanned by affine objects. Then for any complete category $D$, the inclusion $i: C^{\text{aff}} \rightarrow C$ induces an equivalence of categories of $D$-valued sheaves
\[
i^*: \text{Shv}_D(C) \rightarrow \text{Shv}_D(C^{\text{aff}}).
\]
Proof. Using the “comparison lemma” of [Hoy14, Lm.C.3], which applies as $M_{\text{Ell}}$ is a qcqs Deligne–Mumford stack, we obtain the middle equivalence in the following chain of equivalences of categories:
\[
\text{Shv}_D(C) \sim \text{Shv}_D(\text{Shv}_S(C)) \sim \text{Shv}_D(\text{Shv}_S(C^{\text{aff}})) \rightarrow \text{Shv}_D(C^{\text{aff}}).
\]
The first and last equivalences follow by [Lur18b, Pr.1.3.1.7]; all of the inverses to the above equivalences are given by various right Kan extensions. The naturality of [Lur18b, Pr.1.3.1.7] show the above composite is equivalent to $i^*$.

2.2 Generalised elliptic cohomology theories

Let us collect some generalities about elliptic cohomology.

Definition 2.7. Consider a morphism $E: \text{Spec } R \rightarrow M_{\text{Ell}}$ defining a generalised elliptic curve $p: E \rightarrow \text{Spec } R$. An $E_\infty$-ring $E$ is a generalised elliptic cohomology theory for $E$ if the following hold:

1. $E$ is weakly 2-periodic, meaning $\Sigma^2 E$ is a locally free $E$-module of rank 1; see [Lur18a §4.1];
2. The groups $\pi_k E$ vanish for all odd integers $k$, hence $E$ has a complex orientation; see [Lur18a Ex.4.1.2];
3. There is a chosen isomorphism of rings $\pi_0 E \cong R$; and
4. There is a chosen isomorphism of formal groups $\hat{E} \cong G_\infty^{\text{pr}}$ over $R$, between the formal group of $E$ and the classical Quillen formal group of $E$. 

11
We say a collection of such $E$ is natural if the isomorphisms of parts 3-4 above are natural in a subcategory of affine objects of one of the categories of Definition 2.3.

Remark 2.8. Given a generalised elliptic cohomology theory $E$ for some $E: \text{Spec} R \to \mathcal{M}_{\text{Ell}}$, then there is an isomorphism of $R$-modules $\pi_{2k}E \simeq \omega^k_E$ for all $k \in \mathbb{Z}$. Here we write $\omega_E$ for the dualising line of the formal group $\hat{E}$, which is isomorphic to both $p_\ast \Omega^1_{E/\text{Spec} R}$ and the $R$-linear dual of the Lie algebra of $\hat{E}$; see [Lur18a §4.2] or [Hill86 §IV.4]. Indeed, in this case we see $\pi_0E$ is isomorphic (using the complex orientation) to the dualising line for the formal group $G_{\hat{E}}$, as $E$ is a complex periodic $E_8$-ring; see [Lur18a Ex.4.2.19]. By part 4 of Definition 2.7, we see $\pi_{2k}E \simeq \omega_k^E$ for all integers $k$; see [Lur18a Df.4.1.5].

Often we will construct our desired sheaves with Cartesian squares, in which case the following lemma is frequently used; see [HL16 Lm.4.5] and [LNN14 3.9].

Lemma 2.9. Let $E: \text{Spec} R \to \mathcal{M}_{\text{Ell}}$ be a morphism of stacks, and

$$
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{E}_0 \\
\downarrow & & \downarrow \\
\mathcal{E}_1 & \longrightarrow & \mathcal{E}_{01}
\end{array}
$$

be a a Cartesian diagram of $E_8$-rings such that $\pi_0E \simeq R$, and each $E_8$-ring $E_i$ is a natural generalised elliptic cohomology theory for $E|_{\text{Spec} \pi_0E_i}$ such that the complex orientations on $E_0$ and $E_1$ agree on $E_{01}$. Suppose that $\pi_0E_0 \oplus \pi_0E_1 \to \pi_0E_{01}$ is surjective. Then $E$ is uniquely a natural generalised elliptic cohomology theory for $E$. Moreover, the same holds when $p$-completed.

Proof. A Mayer–Vietoris sequence on homotopy groups reveals $E$ satisfies criterion 3 of Definition 2.7 and also that $\pi_{2k}E \simeq \omega^k_E$ for every integer $k$. This latter fact, and the assumption that $\mathcal{E}_0 \to \mathcal{E}_{01} \leftarrow \mathcal{E}_1$ are jointly surjective on $\pi_0$, imply that these maps are jointly surjective on $\pi_0$, which shows that $E$ satisfies criteria 2. This same Mayer–Vietoris sequence combined with the fact that each $\mathcal{E}_i$ is complex oriented and (2.10), gives $E$ complex orientation restricting to those on $\mathcal{E}_i$ and shows $E$ is weakly 2-periodic, giving us criterion 1. Using this complex orientation, one obtains the commutative diagram

$$
\begin{array}{ccc}
G_{\hat{E}}^{\mathbb{Q}_0} \otimes \pi_0E_0 & \leftarrow & G_{\hat{E}}^{\mathbb{Q}_0} \otimes \pi_0E_{01} \\
\downarrow \simeq & & \downarrow \simeq \\
\hat{E} \otimes \pi_0E_0 & \leftarrow & \hat{E} \otimes \pi_0E_{01}
\end{array}
$$

and the fact that $\pi_0$ of (2.10) yields a Cartesian diagram of discrete rings, we obtain an isomorphism of formal groups $G_{\hat{E}}^{\mathbb{Q}_0} \to \hat{E}$ over $R$ restricting to those given by (2.11). This yields criterion 4, and we are done. The “moreover” statement is the proven similarly.    

12
2.3 Constructions using \( p \)-divisible groups

Fix a prime \( p \), and let \( \mathcal{C}_{BT^n} \) be the subcategory of \( \text{Fun}(\text{CRing}, S)_{/\mathcal{M}_{BT^n}} \) spanned by those objects \( G : \mathcal{X} \rightarrow \mathcal{M}_{BT^n} \) where \( \mathcal{X} \) is a formal Deligne–Mumford stack of finite presentation over \( \text{Spf} \, \mathbb{Z}_p \) and \( G \) is a formally étale morphism; see [Dav20, Def.1.2.1]. Give this site the étale topology using the forgetful functor to the category of presheaves of spaces on discrete rings. This is a particular subsite of the site \( \mathcal{C}_{Z_p} \) of [Dav20, Def.1.1.3]; see [Dav20, Prop.1.1.6]. The following is then a simplification of [Dav20, Th.1.1.4].

**Theorem 2.12.** Let \( p \) be a prime and \( n \) a positive integer. Then there is an étale hypersheaf of \( E_8 \)-rings \( \mathcal{O}_{BT^n}^{\text{top}} \) on \( \mathcal{M}_{BT^n} \) such that for each affine \( G : \text{Spf} \, \mathcal{R} \rightarrow \mathcal{M}_{BT^n} \) in \( \mathcal{C}_{BT^n} \), the \( E_8 \)-ring \( \mathcal{O}_{BT^n}^{\text{top}}(G) = \mathcal{E} \) has the following properties:

1. \( \mathcal{E} \) is weakly 2-periodic;
2. The groups \( \pi_k \mathcal{E} \) vanish for all odd integers \( k \);
3. There is a chosen isomorphism of rings \( \pi_0 \mathcal{E} \simeq \mathcal{R} \); and
4. There is a chosen isomorphism of formal groups \( G^0 \simeq \hat{G}_E^{\mathbb{Q}_0} \) over \( \text{Spf} \, \mathcal{R} \), between the identity component of \( G \) and the classical Quillen formal group of \( \mathcal{E} \).

First we construct \( \mathcal{O}_{BT^n}^{\text{top}} \) on \( \mathcal{M}_{\text{Ell}}^{\text{sm}} \), starting with a \( p \)-complete statement.

**Proposition 2.13.** Fix a prime \( p \). There exists an étale hypersheaf of \( E_8 \)-rings \( \mathcal{O}_{\text{Ell}}^{\text{sm}} \) on the site \( \mathcal{Isog}_{\text{Ell}}^{\text{sm}} \) such that for an affine \( E : \text{Spec} \, \mathcal{R} \rightarrow \mathcal{M}_{\text{Ell}} \) in \( \mathcal{Isog}_{\text{Ell}}^{\text{sm}} \), the \( E_8 \)-ring \( \mathcal{O}_{\text{Ell}}^{\text{sm}}(\mathcal{R}) = \mathcal{E} \) defines a natural generalised elliptic cohomology theory for \( E \).

**Proof.** By Proposition 2.5 we obtain a functor

\[
[p^\mathcal{X}] : \mathcal{Isog}_{\text{Ell}}^{\text{et}} \rightarrow \text{fDM}_{/\mathcal{M}_{BT^n}}.
\]

By the Serre–Tate theorem (see [Dav20, Ex.1.2.7]), the above map factors through \( \hat{\mathcal{C}}_{BT^2} \), which yields a functor \( [p^\mathcal{X}] : \mathcal{Isog}_{\text{Ell}, \text{sm}}^{\text{et}} \rightarrow \hat{\mathcal{C}}_{BT^2} \). This functor clearly sends étale hypercovers to étale hypercovers, so we then define \( \mathcal{O}_{p}^{\text{sm}} \) as the composition

\[
\left( \mathcal{Isog}_{\text{Ell}, \text{sm}}^{\text{et}} \right)^{\text{op}} \xrightarrow{[p^\mathcal{X}]^{\text{op}}} \left( \hat{\mathcal{C}}_{BT^2} \right)^{\text{op}} \xrightarrow{\mathcal{O}_{BT^2}^{\text{top}}} \text{CAlg},
\]

and the rest follows by Theorem 2.12.

**Proposition 2.14.** There exists an étale hypersheaf of \( E_8 \)-rings \( \mathcal{O}_{\text{Ell}}^{\text{sm}} \) on the site \( \mathcal{Isog}_{\text{Ell}, \text{sm}}^{\text{et}} \) such that for an affine \( E : \text{Spec} \, \mathcal{R} \rightarrow \mathcal{M}_{\text{Ell}} \) in \( \mathcal{Isog}_{\text{Ell}, \text{sm}}^{\text{et}} \), the \( E_8 \)-ring \( \mathcal{O}_{\text{Ell}}^{\text{sm}}(\mathcal{R}) = \mathcal{E} \) defines a natural generalised elliptic cohomology theory for \( E \).
Proof. Let us first produce a rational sheaf \( \mathcal{O}^\text{sm}_Q \). By Lemma 2.6, it suffices to define \( \mathcal{O}^\text{sm}_Q \) on the subcategory of \( \text{Isog}^\text{ét}_{\text{Ell}, \text{sm}} \) spanned by affine objects. Following [HL16, Def.5.13], we define \( \mathcal{O}^\text{sm}_Q(R, E) \) for a pair \((R, E)\) by the \( \mathbb{Q}\)-cdga \( \omega^* \otimes \mathbb{Q} \), where \( \omega^* \) is the formal cdga defined by placing the invertible \( R \)-module \( \omega^\text{sm}_E \) in degree \( 2n \) for all \( n \in \mathbb{Z} \). This defines a étale hypersheaf as each of the discrete sheaves \( \omega^\text{sm}_E \) has this property. Using this definition, the functorality of \( \mathcal{O}^\text{sm}_Q \) with respect in the \((2,2)\)-category \( \text{Isog}^\text{log}^\text{ét}_{\text{Ell}} \) is clear, as isogenies of invertible degree of elliptic curves \( E \to E' \) over \( \mathbb{Q} \) induce morphisms \( \omega^\text{E}_E \to \omega^\text{E}_{E'} \) of line bundles. The functorality of \( \mathcal{O}^\text{sm}_Q \) with respect to the \((2,1)\)-category \( \text{Isog}^\text{ét}_{\text{Ell}} \) follows as isogenies of invertible degree induce nowhere zero morphisms of line bundles over rational affine schemes, which are always isomorphisms. As formal groups over \( \mathbb{Q} \) are uniquely isomorphic to the additive formal group via the logarithm, we see the above construction defines a natural generalised elliptic cohomology theory.

There is also a unique map \( \alpha \): \( \mathcal{O}^\text{sm}_Q \to (\prod_p \pi^p_\ast \mathcal{O}^\text{sm}_p)_Q \) such that \( \pi_\ast \) is rationalisation of the map into the product defined by tensoring \( \mathbb{Z} \to \mathbb{Z}_p \) with \( \omega^\ast_E \). Indeed, the rational argument in [DFHH14, §12.9] shows \( \mathcal{O}^\text{sm}_Q \) takes values in \( \mathbb{E}_\ast \text{-tmf}_Q \)-algebras and [HL16, Pr.3.12] states that the map \( \pi_\ast \text{tmf} \to \pi_\ast \mathcal{O}^\text{sm}_Q(R, E) \) is étale for all \( (R, E) \) inside \( \text{Isog}^\text{ét}_{\text{Ell}, \text{sm}} \). By [Lur17, Cor.7.5.4.6], the map induced by the functor \( \pi_\ast \) on mapping spaces

\[
\text{Map}_{\text{CAlg}_{\text{tmf}} Q} \left( \mathcal{O}^\text{sm}_Q(R, E), (\prod_p \pi^p_\ast \mathcal{O}^\text{sm}_p)_Q(R, E) \right) \xrightarrow{\sim} \text{Hom}_{\text{CAlg}_{\ast \text{-tmf}} Q} \left( \omega^*_E \otimes Q, (\prod_p \omega^*_E \otimes \mathbb{Z}_p) \otimes Q \right)
\]

is an equivalence for all \( (R, E) \), hence we obtain a natural map \( \alpha \) of sheaves of \( \mathbb{Q}\)-cdgas on \( \text{Isog}^\text{ét}_{\text{Ell}, \text{sm}} \). The sheaf \( \mathcal{O}^\text{sm} \) is then defined by the Cartesian diagram of étale sheaves

\[
\begin{array}{ccc}
\mathcal{O}^\text{sm} & \longrightarrow & \prod_p \pi^p_\ast \mathcal{O}^\text{sm}_p \\
\alpha \downarrow & & \downarrow \\
\mathcal{O}^\text{sm} & \longrightarrow & \left( \prod_p \pi^p_\ast \mathcal{O}^\text{sm}_p \right)_Q
\end{array}
\]

Lemma 2.9 ensures that \( \mathcal{O}^\text{sm} \) defines a natural generalised elliptic cohomology theory. \( \Box \)

Next we construct \( \mathcal{O}^\text{top} \) at the cusp; first over \( \text{Spf} \mathbb{Z}_p \), then over \( \text{Spec} \mathbb{Z} \).

**Proposition 2.15.** Fix a prime \( p \). There exists an étale hypersheaf of \( \mathbb{E}_\ast \text{-rings} \mathcal{O}^\text{mult}_p \) on the site \( \text{Isog}^\text{ét}_{\mathbb{G}_m} \) such that for an affine \( K \): \( \text{Spf} R \to \tilde{\mathcal{M}}_{\mathbb{G}_m} \) in \( \text{Isog}^\text{ét}_{\mathbb{G}_m} \), the \( \mathbb{E}_\ast \text{-ring} \mathcal{O}^\text{mult}_p(R) = K \) defines a natural generalised elliptic cohomology theory for \( K \).

**Remark 2.16.** We use the notation \( K \), as the above \( K \) is a form of \( K \)-theory à la Morava; see [Mor89]. In fact, by [Dav20, Pr.2.1.7] the value of \( \mathcal{O}^\text{mult} \) on the cover \( \text{Spf} \mathbb{Z}_p \to \tilde{\mathcal{M}}_{\mathbb{G}_m} \) is naturally equivalent to the \( p \)-completion of complex \( K \)-theory \( KU_p \), and the global sections \( \mathcal{O}^\text{mult}(\tilde{\mathcal{M}}_{\mathbb{G}_m}) \) is equivalent to the \( p \)-completion of real \( K \)-theory \( KO_p \) by [Dav20, Pr.2.1.17]. A similar remark holds for the integral statement of Proposition 2.18.
Proof of Proposition 2.13. Let us use the spectral deformation theory of [Lur18b, §17] and [Dav20, §1.2]. Writing $S_p$ for the $p$-completed sphere spectrum, $\text{Spf } S_p$ for formal spectral Deligne–Mumford stack, $\mathcal{M}_{G_m}$ for the formal spectral Deligne–Mumford stack $\text{Spf } S_p/C_2$ with respect to the trivial $C_2$-action, and $\mathcal{M}_{BT_1^p}$ for the base-change over $\text{Spf } S_p$ of spectral moduli stack of $p$-divisible groups of [Lur18a, Df.3.2.1] of height 1, we obtain morphism of presheaves of connective $E_{\infty}$-rings

$$\text{Spf } S_p \rightarrow \mathcal{M}_{G_m} \rightarrow \mathcal{M}_{BT_1^p}$$

(2.17)
corresponding to the multiplicative $p$-divisible group $\mu_{p^\infty}$ over $\text{Spf } S_p$ and the inversion automorphism. These composable maps lead to a (co)fibre sequence of cotangent complexes (see [Lur18b, Cor.17.2.5.3]) viewed as $p$-complete almost perfect spectra

$$q^* L_{\mathcal{M}_{G_m}/\mathcal{M}_{BT_1^p}} \rightarrow L_{\text{Spf } S_p/\mathcal{M}_{BT_1^p}} \rightarrow L_{\text{Spf } S_p/\mathcal{M}_{G_m}}$$

which we abbreviate to $q^* L_1 \rightarrow L_2 \rightarrow L_3$. Now $L_3$ vanishes as $q$ is étale, in fact, it is an affine étale cover recognising $\mathcal{M}_{G_m}$ as a formal spectral Deligne–Mumford stack. Furthermore, $L_2$ vanishes, as the composition (2.17) classifies the spectral universal deformation of $\mu_{p^\infty}$ over $\mathbb{F}_p$; see [Lur18a, Cor.3.1.19]. It follows that $q^* L_1$ vanishes as well, and as $q$ is an étale cover, it is faithfully flat, hence $L_1$ also vanishes in QCoh$(\mathcal{M}_{G_m})$. By [Dav20, Pr.1.2.19(4)] (also see [Dav20, Ex.1.2.27]), this implies that the map $\mathcal{M}_{G_m} \rightarrow \mathcal{M}_{BT_1^p}$ is formally étale as a morphism of presheaves of connective $E_{\infty}$-rings. Upon taking the restriction to discrete $E_{\infty}$-rings, we see that the morphism $\mathcal{M}_{G_m} \rightarrow \mathcal{M}_{BT_1^p}$ is formally étale (see [Dav20, Rmk.1.2.14]), hence the natural map

$$[p^\infty] \colon \text{Isog}_{G_m} \rightarrow \text{fDM}_{\mathcal{M}_{BT_1^p}}$$

factors through $\hat{\mathcal{M}}_{BT_1^p}$. The above morphisms sends étale hypersheaves to étale hypercovers, so we can set $\mathcal{O}_{\text{mult}} = \mathcal{O}_{\text{BT}_1^p, \mathcal{M}_{G_m}}$. Everything then follows from Theorem 2.12. \hfill \Box

Proposition 2.18. There exists an étale hypersheaf of $E_{\infty}$-rings $\mathcal{O}_{\text{mult}}$ on the site $\text{Isog}_{G_m}^{\text{et}}$ such that for an affine $K$: $\text{Spec } R \rightarrow \mathcal{M}_{G_m}$ in $\text{Isog}_{G_m}^{\text{et}}$, the $E_{\infty}$-ring $\mathcal{O}_{\text{mult}}(R) = K$ defines a natural generalised elliptic cohomology theory for $K$.

Proof. This follows the same outline of proof of Proposition 2.14 so in brief: Define $\mathcal{O}_{\text{mult}}^Q$ as the $\mathbb{Q}$-cdga $\mathcal{O}_{\text{mult}}^Q = \omega_{G_m}^{\text{et}}$ for each affine $\text{Spec } R \rightarrow \mathcal{M}_{G_m}$ in $\text{Isog}_{G_m}^{\text{et}}$. This étale hypersheaf $\mathcal{O}_{\text{mult}}^Q$ on $\text{Isog}_{G_m}^{\text{et}}$ defines a natural generalised elliptic cohomology theory. Appealing to [Lur17, Cor.7.5.4.6], we obtain a natural map $\mathcal{O}_{\text{mult}}^Q \rightarrow (\prod_p \pi_p^* \mathcal{O}_p)_{\mathbb{Q}}$, which allows us to define $\mathcal{O}_{\text{mult}}$ using a Hasse square. \hfill \Box

*These cotangent complexes are almost perfect (see [Lur17, Df.7.2.4.10]) by the same arguments made in the proof of [Dav20, Pr.1.1.6]. Notice $L_2$ and $L_3$ are $p$-complete by definition. Furthermore, as $L_{\mathcal{M}_{G_m}/\mathcal{M}_{BT_1^p}}$ is almost perfect, we see $q^* L_{\mathcal{M}_{G_m}/\mathcal{M}_{BT_1^p}}$ is almost perfect by [Lur18b, Cor.8.4.1.6], and almost perfect modules over $S_p$ are automatically $p$-complete by [Lur18a, Pr.7.3.5.7].
2.4 Construction in the cuspidal neighbourhood

The most intricate part in our proof of Theorem 2.1 is constructing $\mathcal{O}_{\text{Tate}}$.

**Proposition 2.19.** There is a log étale hypersheaf of $E_8$-rings $\mathcal{O}_{\text{Tate}}$ on $\text{Isog}^\log \Gamma_{\text{Tate}} = \mathcal{C}$ such that for an affine $T$: $\text{Spf} R \to \mathcal{M}_{\text{Tate}}^\log$ in $\mathcal{C}$, the $E_8$-ring $\mathcal{O}_{\text{Tate}}(R) = T$ defines a natural generalised elliptic cohomology theory for $T$.

To prove the above theorem, we will follow the outline given in [HL16, §5.1]: first we restrict to a subcategory $\mathcal{C}_{\text{aff}}$ of the affine objects of $\mathcal{C}$, and then we build $\mathcal{O}_{\text{Tate}}$ on $\mathcal{C}_{\text{aff}}$ by hand.

**Definition 2.20.** Let $\mathcal{C}_{\text{aff}}$ be the full subcategory of $\mathcal{C}$ spanned by pairs $(\text{Spf } R, T)$, where:

- $R$ is a connected étale $\mathbb{Z}_{(m)}[[q^{1/m}]]$-algebra, which is complete and separated with respect to the topology generated by $(q)$, and $R$ has log structure $\langle q^{1/m} \rangle$; and
- $T$ is the form of the Tate curve over $\text{Spf } R$ given by the composition $\text{Spf } R \to \text{Spf } \mathbb{Z}_{(m)}[[q^{1/m}]] \to \text{Spf } \mathbb{Z}_{(m)}[[q]] \to \mathcal{M}_{\text{Tate}}$.

As the form of the Tate curve $T$ is determined by the above $R$, we will often write elements of $\mathcal{C}_{\text{aff}}$ as $(R, m)$, or $(R, q^{1/m})$ when we want to make the integer $m$ visible. This integer $m$ will be called the log index of $(R, m)$. Moreover, the map $\mathcal{M}_{G_m} \to \mathcal{M}_{\text{Tate}}$ induces a functor $i: \mathcal{C}_{\text{aff}} \to \text{Isog}^\log \Gamma_{\text{G_m}}$ sending a pair $(R, m)$ which is log étale over $\mathcal{M}_{\text{Tate}}$ to the quotient $R = R/(q^{1/m})$, which is itself étale over $\mathcal{M}_{G_m}$; see [HL16, Df.5.2].

The following is analogous to the proof of Lemma 2.6 using [HL16, Cor.2.19].

**Proposition 2.21.** The inclusion $\mathcal{C}_{\text{aff}} \to \mathcal{C}_{\text{aff}}$ of Definition 2.20 induces an equivalence

$$\text{Shv}^{\log \Gamma}_{\text{CAlg}}(\mathcal{C}_{\text{aff}}) \to \text{Shv}^{\log \Gamma}_{\text{CAlg}}(\mathcal{C}_{\text{aff}}).$$

Furthermore, the same holds for hypersheaves.

**Remark 2.22.** Given an object $(R, m)$ of $\mathcal{C}_{\text{aff}}$, then we know that $R$ is an étale extension of $\mathbb{Z}[\frac{1}{m}][[q^{1/m}]]$. By [HL16, Rmk.2.20], we see that $R$ is determined up to nonunique isomorphism by the integer $m$ and the extension $\mathbb{Z}[\frac{1}{m}] \to R$ where $R = R/(q^{1/m})$. Such isomorphisms may differ by multiplication by an $m$th root of unity in $R$.

The remark above states that we can recover a pair $(R, m)$ from the pair $(\overline{R}, m)$ up to multiplication by an $m$th root of unity in $R$. This means the most natural way to reconstruct $(R, m)$ from $(\overline{R}, m)$ is to retain information about all the roots of unity in $R$. This leads us to the following definition.

**Definition 2.23** ([HL16, Df.5.3]). Define two presheaves of discrete commutative monoids $\mu$ and $A$ on $\mathcal{C}_{\text{std}}$ by the formulae

$$\mu(R, m) = \{\zeta \in R^\times \mid \zeta^m = 1\}, \quad A(R, m) = \mu(R, m) \times \left(\frac{1}{m} \cdot \mathbb{N}\right).$$
Remark 2.24. As in [HL16, Rmk.5.5], the data of a morphism $(R, m) \to (R', dm)$ inside $\mathcal{C}_{\text{aff}}_{\text{std}}$ is precisely the data of a morphism $(\overline{R}, G) \to (\overline{R'}, G')$ inside $\text{Isog}_{G_m}$ together with the dashed morphism of commutative monoids in the following diagram:

$$
\begin{align*}
\mu(R, m) \times q^N & \longrightarrow \mu(R', dm) \times q^N \\
\downarrow & \\
A(R, m) & \longrightarrow A(R', dm).
\end{align*}
$$

This is enough information now to construct our functor $\mathcal{E}_{\text{std}}^{\text{Tate}}$ on $\mathcal{C}_{\text{aff}}_{\text{std}}$ by hand.

Construction 2.25. We define a functor $\mathcal{E}_{\text{std}}^{\text{Tate}} : (\mathcal{C}_{\text{aff}}_{\text{std}})^{\text{op}} \to \text{CAlg}$ by the formula

$$
\mathcal{E}_{\text{std}}^{\text{Tate}}(R, m) = \left( i^* \mathcal{E}^{\text{mult}}(R, m) \otimes S[\frac{1}{m}, \mu_m(R)] \right)_{(q^N)}^{\wedge},
$$

where we note that there is an equivalence $i^* \mathcal{E}^{\text{mult}}(R, m) \simeq \mathcal{E}^{\text{mult}}(\overline{R})$ straight from Definition 2.20 and that $S[\frac{1}{m}, \mu_m(R)] \to \mathcal{E}^{\text{mult}}(\overline{R})$ is the essentially unique morphism of $E_{\mathfrak{X}}$-$\mathcal{S}[\frac{1}{m}]$-algebras recognising the canonical composition $\mathbb{Z}[\mu_m(R)] \to \overline{R} \to \overline{R}$. To see this, apply [Lur17, Cor.7.5.4.6] with $A = S[\frac{1}{m}], B = S[\frac{1}{m}, \mu_m(R)], C = \mathcal{E}^{\text{mult}}(\overline{R})$, note that $A \to B$ is étale as the order of $\mu_m(R)$ is invertible in $\pi_0 A$, and that the resulting map is continuous as $\mathbb{Z}[\mu_m(R)] \to \overline{R}$ is. The functorality of $\mathcal{E}_{\text{std}}^{\text{Tate}}$ follows from that of $\mathcal{E}^{\text{mult}}$ and Remark 2.24.

For the sake of the proof of Proposition 2.19 let us make a temporary definition.

Definition 2.27. Let $A$ and $B$ be two $\mathbb{E}_{\mathfrak{X}}$-rings with log structures on $\pi_0$ determined by a smooth divisor. We say a morphism $A \to B$ of $\mathbb{E}_{\mathfrak{X}}$-rings (together with a morphism $\pi_0 A \to \pi_0 B$ of affine log schemes, which we will often suppress) is $\pi_0$-$\text{log étale}$ if $A \to B$ is flat as a morphism of $\mathbb{E}_{\mathfrak{X}}$-rings, and $\pi_0 A \to \pi_0 B$ is a log étale morphism of discrete rings. We will say such a pair of datum is a $\pi_0$-$\text{log étale cover}$ if $A \to B$ is in addition faithfully flat; see [Lur18b, Df.B.6.1.1].

Let us warn the reader that the above definition is unconventional and will only be used in the proof of Proposition 2.19. There is a well-established study of log structures on $\mathbb{E}_{\mathfrak{X}}$-rings by Rognes, Sagave, and Schlichtkrull; see [Rog09] or [RSS15]. One should not compare our definition of $\pi_0$-$\text{log étale}$ with any of their work, or those found in [Lun20]. One reason why we want to discuss $\pi_0$-$\text{log étale}$ morphisms of $\mathbb{E}_{\mathfrak{X}}$-rings is the following.

Lemma 2.28. Let $A \to B \to C$ be two composable morphisms of $\mathbb{E}_{\mathfrak{X}}$-rings with log structures on $\pi_0$ determined by a smooth divisor, where $A \to B$ is $\pi_0$-$\text{log étale}$. Then $B \to C$ is $\pi_0$-$\text{log étale}$ if and only if $A \to C$ is $\pi_0$-$\text{log étale}$. Furthermore, $\pi_0$-$\text{log étale}$ morphisms are stable under base-change.

Proof. If $B \to C$ is $\pi_0$-$\text{log étale}$, it is clear that $A \to C$ is as well. Conversely, we see $\pi_0 B \to \pi_0 C$ is log étale by Proposition 1.3 and flat by Proposition 1.3. Finally, we have to check the natural map

$$
\pi_q B \otimes_{\pi_0 B} \pi_0 C \to \pi_q C
$$

17
is an isomorphism of $\pi_0C$-modules. This follows, as we have natural isomorphisms

\[
\pi_*B \otimes_{\pi_0B} \pi_0C \cong \pi_*A \otimes_{\pi_0A} \pi_0B \otimes_{\pi_0B} \pi_0C \cong \pi_*A \otimes_{\pi_0A} \pi_0C \cong \pi_*C,
\]

where the first and third isomorphisms come from the fact that $A \to B$ and $A \to C$ are $\pi_0$-log étale by assumption, respectively, and the second isomorphism is canonical. The base-change property is straightforward.

\[\square\]

Proof of Proposition 2.19. By Propositions 2.6 and 2.21 it suffices to define our desired hypersheaf $O^\text{Tate}$ on $C^\text{aff}$, which we will call $O_{\text{std}}^\text{Tate}$. Our desired presheaf $O_{\text{std}}^\text{Tate}$ can be defined by Construction 2.25, so it remains to show this presheaf $O_{\text{std}}^\text{Tate} = \mathcal{O}$ on $C_{\text{aff}}$ is indeed a log étale hypersheaf and defines a natural generalised elliptic cohomology theory. First we need some observations – let $(R, m) \to (R', dm)$ be an arbitrary morphism in $C^\text{aff}$.

1. Write $\overline{R} = R/(q_1^{\pm 1})$ and $\overline{R'} = R'/(q_1^{\pm 1})$. By [HL16, Df.5.2], the induced morphism $\overline{R} \to \overline{R'}$ lies in $\text{Isog}_{\text{ét}}^\text{\!Gm}$ and hence is étale.

2. Using Proposition 2.18 we obtain a morphism $\mathcal{R} = \mathcal{O}^\text{mult}(\overline{R}) \to \mathcal{O}^\text{mult}(\overline{R'}) = \mathcal{R}'$ of $E_8$-rings. We claim this is an étale morphism of $E_8$-rings. Indeed, by Proposition 2.18 this morphism induces the étale morphism $\overline{R} \to \overline{R'}$ on $\pi_0$, so it remains to show that the natural map of $\overline{R}$-modules

\[
\pi_*\mathcal{R} \otimes_{\pi_0\overline{R}} \pi_0\mathcal{R}' \to \pi_*\mathcal{R}'
\]

is an isomorphism. Consider the natural isomorphisms of $\overline{R}$-modules

\[
\pi_{2k}\mathcal{R} \otimes_{\pi_0\overline{R}} \pi_0\mathcal{R}' \cong \omega_{\mathcal{G}_{\overline{R}}^k} \otimes_{\mathcal{G}_{\overline{R}}} \overline{\mathcal{R}} \cong \omega_{\mathcal{G}_{\overline{R'}}^k} \cong \omega_{\mathcal{G}_{\overline{R'}}^k} \cong \pi_{2k}\mathcal{R}',
\]

where the first isomorphism comes from Proposition 2.18, the second from a standard base-change property for $\omega$ [Lur18a, Rmk.4.2.4], and the third from the fact that isogenies of invertible degree induce isomorphisms on associated $p$-divisible groups by Proposition 1.14 and hence an isomorphism on associated formal groups by (1.12).

3. Define a morphism of $E_8$-rings $\mathcal{R}[q^{1/\overline{m}}] \to \mathcal{R}'[q^{1/\overline{m}}]$ using the top cube in the diagram of
where the upper-front and upper-back faces are coCartesian, the upper-left face commutes as \( R \to R' \) is a log morphism (Remark 2.24), the vertical maps in the lower cubes are the canonical completion maps, and all other maps comes from the universal property of completion which also relies the maps above respecting the log-structures. Notice the above diagram produces Construction 2.25 applied to \( R \to R' \); see the bottom-right corner of (2.29). We claim the morphism

\[
\pi_0 \log \text{étale. Indeed, the natural map }
\]

\[
\pi_0 \mathcal{O}(R, m) \cong \mathcal{O}[\frac{1}{m}] \to \mathcal{O}'[\frac{1}{q \cdot m}] \cong \mathcal{O}(R', dm)
\]

is an isomorphism as \( \mathcal{R} \to \mathcal{R}' \) is étale, and all four vertical morphisms on the right are flat: the two in the upper-right base-change, and the two in the lower-right the completion maps of Noetherian (clear by \([\text{Lur}17, \text{Df.7.2.4.30}]\) \( \text{E}_x \)-rings are flat; see \([\text{Lur}18\text{H}, \text{Cor.7.3.6.9}]\). Taking \( \pi_0 \) of \( \mathcal{R}[\frac{1}{m}] \to \mathcal{R}'[\frac{1}{q \cdot m}] \) we obtain \( \overline{\mathcal{R}[\frac{1}{m}]} \to \overline{\mathcal{R}'[\frac{1}{q \cdot m}]} \), which by \([\text{HL16}, \text{Rmk.2.20}]\) is equivalent to the log map of rings \( (R, m) \to (R', dm) \), which we assumed to be log étale (and is furthermore flat by Remark 1.5).

\[
\text{E}_x \text{-rings}
\]
Take $R \to R^\bullet$ to be a log étale hypercover in $\mathcal{C}_{std}^{aff}$, which we can write diagrammatically as

$$
\begin{array}{c}
(R, m) \xrightarrow{\nu_0} (R^0, d_0 m) \\
\downarrow \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
It remains to prove Claim 2.33. Note that we have natural maps

\[ f_n : L_m \mathcal{O}(R^*) \to \mathcal{O}(L_m R^*) \]

which interchange the functor \( \mathcal{O} \) with the colimit defined by \( L_m \). Let us show these morphisms are equivalences by induction. To do this, let us proceed with a double induction on two integers \( m, n \geq 0 \) with our inductive hypothesis being that the natural map of \( \mathbf{E}_{\infty} \)-rings

\[ f_{m,n} : L_m \mathcal{O}(R^{*+n}) \to \mathcal{O}(L_m R^{*+n}) \]

is an equivalence, where \( R^{*+n} \) is the augmented semicosimplicial object given by precomposing \( R^* \) with (the opposite of) the functor

\[ \Delta_{s,+} \to \Delta_{s,+}, \quad [i] \mapsto [i + n], \]

which sends morphisms \( \phi : [i] \to [j] \) to \( \phi' : [i + n] \to [j + n] \) defined by \( k + n \mapsto \phi(k) + n \) for \( k \geq 0 \), and \( \phi'(i) = i \) for \(-1 \leq i \leq n - 1 \). The base case of \( m = n = 0 \) is clear as \( f_{0,0} \) is the identity. Similarly, we also see \( f_{0,n} \) is the identity for all \( n \geq 0 \). Inductively, let us assume that \( f_{m',n} \) is an equivalence for all \( m' \leq m \) and all \( n \geq 0 \), for some fixed \( m \). We are then required to show \( f_{m+1,n} \) is an equivalence for all \( n \geq 0 \). First we decompose the colimits \( M_n \) as done (dually for limits) in [Day20, (A.4.7)], which leads us to the natural equivalences

\[ L_{m+1} \mathcal{O}(R^{*+n}) \simeq \mathcal{O}(R^{m+n}) \otimes_{L_m \mathcal{O}(R^{*+n})} L_m \mathcal{O}(R^{*+n+1}), \quad L_{m+1} R^{*+n} \simeq R^{m+n} \otimes_{L_m R^{*+n}} L_m R^{*+n+1}, \]

where we note that the second expression above is discrete, as \( L_m R^{*+n} \to R^{m+n} \) is a log étale cover in \( \mathcal{C}^{\text{aff}}_{\text{std}} \) and so is flat. Using these natural equivalences, \( f_{m+1,n} \) is equivalent to

\[ f' : \mathcal{O}(R^{m+n}) \otimes_{L_m \mathcal{O}(R^{*+n})} L_m \mathcal{O}(R^{*+n+1}) \to \mathcal{O}(R^{m+n} \otimes_{L_m R^{*+n}} L_m R^{*+n+1}). \]

This morphism is an isomorphism on \( \pi_0 \), as from our inductive hypothesis and the analysis of \( \mathcal{O} \) above, we see that \( \pi_0 \) of \( L_m \mathcal{O}(R^{*+n}) \to \mathcal{O}(R^{m+n}) \) is equivalent to \( L_m R^{*+n} \to R^{m+n} \). It also follows from our inductive hypothesis and the construction of \( \mathcal{O} \), as well as a degenerate Tor-spectral sequence, that the homotopy groups of the \( \mathbf{E}_{\infty} \)-ring in both the domain and the codomain of \( f' \) are concentrated in even degrees. For every \( k \in \mathbb{Z} \) we then have natural isomorphisms of \( R \)-modules

\[ \pi_{2k} \left( \mathcal{O}(R^{m+n}) \otimes_{L_m \mathcal{O}(R^{*+n})} L_n \mathcal{O}(R^{*+n+1}) \right) \simeq R^{m+n} \otimes_{L_m R^{*+n}} \pi_{2k} \mathcal{O}(L_m R^{*+n+1}) \]

\[ \simeq R^{m+n} \otimes_{L_m R^{*+n}} \omega_{L_m R^{*+n+1}}^{\otimes k} \simeq \omega_{R^{m+n} \otimes_{L_m R^{*+n}} L_m R^{*+n+1}}^{\otimes k} \simeq \pi_{2k} \mathcal{O}(R^{m+n} \otimes_{L_m R^{*+n}} L_m R^{*+n+1}). \]

Indeed, the first isomorphism follows from our inductive hypothesis and the fact \( \mathcal{O} \) does not change \( \pi_0 \), the second comes from the construction of \( \mathcal{O} \), and the forth follows similarly. For the third isomorphism, we again appeal to the fact that the dualising line satisfies base-change.
for morphisms of rings, and the map induced by an isogeny of invertible degree is sent to an isomorphism of quasi-\( p \)-divisible, and hence also formal groups as this can be checked on geometric points, and hence isomorphisms on \( \omega \); see [Lur18a, Rmk.4.2.4] for the functorality in the ring and [Lur18a, Rmk.4.2.5] for the functorality in the formal group. It follows that \( f' \) is an isomorphism, and we have proven Claim 2.33, and finished our proof.

### 2.5 Construction on the compactified moduli stack

Writing \( \mathcal{O}_Tate \) for the \( \mathcal{O}_Tate \)-completion of \( \mathcal{O}_Tate \), we now glue together our \( \mathcal{O}_Tate \)-complete sheaves of Propositions 2.13, 2.15, and 2.19.

**Proposition 2.34.** Fix a prime \( p \). There is a log étale hypersheaf of \( \text{Ell}_\mathcal{O} \)-rings \( \mathcal{E} \) on \( \text{Isog} \) such that for an affine \( E: \text{Spf} \, R \rightarrow \text{Isog}_\mathcal{O} \), the \( \mathcal{O}_Tate \)-ring \( \mathcal{E}(R) = E \) defines a natural generalised elliptic cohomology theory for \( E \).

For the proof of the proposition above, we will need one more piece of preparation. Recall, the morphisms of log stacks

\[
\text{Isog} \xrightarrow{\text{log}} \text{Ell} \xrightarrow{\text{Tate}} \text{Isog}_\mathcal{O}
\]

and give the same name to the base-change of these morphisms over \( \text{Spf} \, \mathbb{Z}_p \).

**Proposition 2.35.** Fix a prime \( p \). There is an essentially unique morphism of hypersheaves of \( \text{Ell}_\mathcal{O} \)-algebras on \( \text{Isog} \) such that for an affine \( E: \text{Spf} \, R \rightarrow \text{Isog}_\mathcal{O} \), taking \( \pi_0 \) of the above morphism yields the canonical map of formal \( \mathbb{Z}_p \)-algebras

\[
R_{\text{sm}} \rightarrow (R_{\text{Tate}})^{\text{sm}},
\]

where \( R_{\text{sm}} \) is result of pulling back \( E \) against the inclusion \( \text{Isog}_\mathcal{O} \rightarrow \text{Isog}_\mathcal{O} \) and \( R_{\text{Tate}} \) is the result of pulling back \( E \) against the inclusion \( \text{Isog}_\mathcal{O} \rightarrow \text{Isog}_\mathcal{O} \).

**Proof.** This statement is essentially [HL16, Pr.5.9]. First, note that \( \mathcal{E}_p \) naturally takes values in \( \text{Ell}_\mathcal{O} \)-KO\( \mathcal{O} \)-algebras. Indeed, its global sections can be calculated as the \( C_2 \)-fixed points of \( \mathcal{O}_Tate \), which by Remark 2.16 and Construction 2.25 gives us the equivalences

\[
\mathcal{E}_p(\text{Spf} \, \mathbb{Z}_p[[q]])^{hC_2} \simeq \mathcal{E}_p(\text{Spf} \, \mathbb{Z}_p[[q]])^{hC_2} \simeq (\text{KO}[[q]])^{hC_2} \simeq \text{KO}[[q]].
\]

The \( \text{Ell}_\mathcal{O} \)-tmf-algebra structure is then the composite of the connective cover map \( \text{tmf} \rightarrow \text{Tmf} \) with the classical ‘evaluation at the cusp’ map \( \text{Tmf} \rightarrow \text{KO}[[q]] \) as defined in [HL16, Th.5.8]. When evaluated on an affine \( \text{Spf} \, R \rightarrow \text{Isog}_\mathcal{O} \), we obtain \( \text{Ell}_\mathcal{O} \)-tmf-algebras

\[
\mathcal{E}_p(\text{Spf} \, \mathbb{Z}_p[[q]])^{hC_2} \simeq \mathcal{E}_p(\text{Spf} \, \mathbb{Z}_p[[q]])^{hC_2} \simeq (\text{KO}[[q]])^{hC_2} \simeq \text{KO}[[q]].
\]

Both of these maps of stacks are affine, as the former is locally equivalent to inverting the discriminant \( \Delta \), and the latter is equivalent to completing at \( \Delta \).
Writing \( v_1 \in R \) for a lift of the Hasse invariant, we see that \( v_1 \) is invertible inside the right-hand \( \mathbf{E}_\infty \)-ring above, as nodal elliptic curves cannot be supersingular. This implies that our desired morphism of \( \mathbf{E}_\infty \)-tmf-algebras is equivalent to a natural morphism in the mapping space
\[
\text{Map}_{\text{CAlg}_{\text{tmf}}} (\mathcal{O}_p^{\text{sm}}(R^{\text{sm}})[v_1^{-1}], \mathcal{O}^{\text{Tate}}_p((R^{\text{sm}})^{\text{Tate}})). \tag{2.36}
\]
Again, as nodal elliptic curves cannot be supersingular, we see that the \( \mathbf{E}_\infty \)-tmf-algebras in the above mapping space are \( K(1) \)-local. It follows from [HL16, Pr.4.49] that the functor \( \pi_0 \) witnesses (2.36) as equivalent to the discrete set of morphisms of formal affine log schemes \( R^{\text{sm}} \to (R^{\text{Tate}})^{\text{sm}} \) over \( \mathcal{M}_{\text{Ell}} \). Hence our desired map is determined by the canonical map of \( \mathbb{Z}_p \)-algebras \( R^{\text{sm}} \to (R^{\text{Tate}})^{\text{sm}} \). Lemma [2.36] states that it suffices to construct this morphism on affines, so we are done.

**Proof of Proposition 2.34.** We define \( \mathcal{O}_p^{\text{top}} \) by the pullback diagram of log étale hypersheaves
\[
\begin{array}{ccc}
\mathcal{O}_p^{\text{top}} & \longrightarrow &  j^*_p \mathcal{O}^{\text{Tate}}_p \\
\downarrow & & \downarrow \\
 j^{\text{sm}}_* \mathcal{O}_p^{\text{sm}} & \longrightarrow &  j^{\text{sm}}_* j^* \mathcal{T}ate \mathcal{O}^{\text{Tate}}_p,
\end{array}
\]
where the right vertical morphism is the obvious unit, and the lower horizontal morphism is that of Proposition 2.35. Evaluating on some log étale \( \text{Spf } R \to \mathcal{M}_{\text{Ell}}^{\log} \) and taking \( \pi_0 \), the above diagram yields the commutative diagram of discrete \( R \)-modules
\[
\begin{array}{ccc}
R & \longrightarrow & R^{\Delta} \\
\downarrow & & \downarrow \\
\Delta^{-1} R & \longrightarrow & \Delta^{-1}(R^{\Delta}).
\end{array}
\]
It is now clear by the Artin–Rees theorem (also see [HL16, Pr.3.14]) that one can apply Lemma 2.9 to \( \mathcal{O}_p^{\text{top}} \), and have obtained our desired natural generalised elliptic cohomology theory.

The construction of the rational sheaf is the same.

**Proposition 2.37.** There is a log étale hypersheaf of \( \mathbf{E}_\infty \)-rings \( \mathcal{O}_Q^{\text{top}} \) on \( \text{Isog}_{\log}^{\text{ét}} \mathcal{E}ll_{\text{Ell}} = \mathcal{C} \) such that for an affine \( E : \text{Spec } R \to \mathcal{M}_{\text{Ell}}^{\log} \) in \( \mathcal{C} \), the \( \mathbf{E}_\infty \)-ring \( \mathcal{O}_Q^{\text{top}}(R) = \mathcal{E} \) defines a natural generalised elliptic cohomology theory for \( E \otimes Q \).

**Proof.** Define \( \mathcal{O}_Q^{\text{top}} \) by the Cartesian square of log étale hypersheaves of \( \mathbb{Q} \)-cdgas
\[
\begin{array}{ccc}
\mathcal{O}_Q^{\text{top}} & \longrightarrow & j^*_q \mathcal{O}^{\text{Tate}}_q \\
\downarrow & & \downarrow \\
 j^{\text{sm}}_* \mathcal{O}_Q^{\text{sm}} & \longrightarrow &  j^{\text{sm}}_* j^* \mathcal{T}ate \mathcal{O}^{\text{Tate}}_q,
\end{array}
\]
where $\mathcal{O}_Q^{\text{Tate}}$ is the rationalisation of $\mathcal{O}_Q^{\text{Tate}}$, and the bottom horizontal morphism exists as in the proof of Proposition 2.35 except now one references [HL16 Pr.3.12] and [Lur17 Cor.7.5.4.6] for the desired discreteness of mapping spaces. Moreover, this morphism can be chosen such that the following diagram commutes; see [HL16 Pr.5.12]:

\[
\begin{align*}
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
3 Adams operations

Let us now use Theorem 2.1 to construct some operations on $\text{Tmf}[\frac{1}{n}]^9$

**Definition 3.1.** For a generalised elliptic curve $E$ over a ring $R$, we write $[n] : E \to E$ for the quotient map $E \to E/H$ associated to the finite flat subgroup $H = E^{\text{sm}}[n] \subseteq E^{\text{sm}}$ using the identification $E/H \cong E$ in this case. Notice that if $n$ is invertible in $R$ then $[n] : E \to E$ is an isogeny of invertible degree, as on geometric points $[n] : E^{\text{sm}} \to E^{\text{sm}}$ is either of degree $n^2$ or $n$; see [SI86, Cor.III.6.4] or note that $\text{Spec} Z[t]/t^n = 1$. For every integer $n$ define the *Adams operation* $\psi^n : \text{Tmf}[\frac{1}{n}] \to \text{Tmf}[\frac{1}{n}]$ as the image under $O^{\text{top}}$ of the endomorphism of the universal generalised elliptic curve $E$ in $\text{Isog}_{\text{Ell}}^\log, \mathbb{Z}[\frac{1}{n}]$

$$ (\text{id}, [n]) : (\mathcal{M}_{\text{Ell}}[\mathbb{Z}[\frac{1}{n}]], \mathcal{E}) \to (\mathcal{M}_{\text{Ell}}[\mathbb{Z}[\frac{1}{n}]], \mathcal{E}). $$

Our main computational result is Theorem 3.2 which we prove in § 3.3, and repeat below for the readers convenience.

**Theorem 3.2.** Let $n$ be an integer. Then the Adams operation $\psi^n : \text{Tmf}[\frac{1}{n}] \to \text{Tmf}[\frac{1}{n}]$ is given on $\pi_\ast$ by

$$ \psi^n(x) = \begin{cases} n|x|_x & x \in \text{Free} \\ x & x \in \text{Tors}, \end{cases} $$

where $\text{Tors} \subseteq \pi_\ast \text{Tmf}[\frac{1}{n}]$ is the subgroup of torsion elements, and $\text{Free}$ is a particular orthogonal subgroup; see Notation 3.7.

If one needs too, one could also use the above theorem to calculate the effect of the Adams operations $\psi^n$ on $\text{tmf}[\frac{1}{n}]$ and $\text{TMF}[\frac{1}{n}]$, using the fact that the operations $\psi^n$ are multiplicative and we have equivalences of $\mathbf{E}_8$-rings

$$ \text{tmf}[\frac{1}{n}] \simeq \tau_{\geq 0} \text{Tmf}[\frac{1}{n}], \quad \text{TMF}[\frac{1}{n}] \simeq \text{Tmf}[\frac{1}{n}][\Delta^{-24}]. $$

The following properties follow straight from Definition 3.1.

**Proposition 3.3.** The operations $\psi^1$ and $\psi^{-1}$ are homotopic to the identity on $\text{Tmf}$. For any two integers $m$ and $n$, there are homotopies between the endomorphisms of $\mathbf{E}_8$-rings $\psi^m \circ \psi^n \simeq \psi^{mn}$ on $\text{Tmf}[\frac{1}{mn}]$.

**Remark 3.4.** Let us note that the homotopies between $\psi^m \circ \psi^n$ and $\psi^{mn}$ are coherent. By this we mean that there is a higher homotopy between the two homotopies comparing $\psi^k \circ \psi^m \circ \psi^n$ with $\psi^{kmn}$, for any integers $k, m, n$, and there are even higher homotopies comparing all higher compositions of Adams operations. As with the proof of Proposition 3.3 below, this also follows straight from the functorality of $O^{\text{top}}$.

---

9There is a natural map of $\mathbf{E}_8$-rings $\text{tmf}[\frac{1}{n}] = O^{\text{top}}(\mathcal{M}_{\text{Ell}})[\frac{1}{n}] \to O^{\text{top}}(\mathcal{M}_{\text{Ell}}[\mathbb{Z}[\frac{1}{n}]]$, and by [MM15, Th.7.2] it is an equivalence. Due to this fact, we will not distinguish between these two notions of localisation.
Proof of Proposition 3.3. All of the above follows from the functorality of $O^\text{top}$, but let us explain the identification of $\psi^{-1}$ in more detail. We learned the following argument from Lennart Meier. The trivial $C_2$-action $(\text{id, triv})$ of $(M_{\text{Ell}}, \mathcal{E})$ is homotopic to the $C_2$-action $(\text{id, [-1]})$ of $(M_{\text{Ell}}, \mathcal{E})$ in $\text{Isog}_{M_{\text{Ell}}}^{\text{log}}$, given by the invertible 2-morphism $[-1]: \text{id} \to \text{id}$. Applying $O^\text{top}$, we obtain a homotopy $\text{id} \simeq \psi^{-1}: \text{Tmf} \to \text{Tmf}$.

Recall that one can consider moduli stacks of generalised elliptic curves with level structures, which are Deligne–Mumford stacks $\mathcal{M}(\Gamma)$ over $M_{\text{Ell}, \mathbb{Z}[\frac{1}{N}]}$ for each subgroup $\Gamma$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $N \geq 1$; see [DR73 §IV.3] or [HL16 §3.6]. The morphisms $\mathcal{M}(\Gamma) \to \mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{N}]}$ are log étale and we denote $O^\text{top}(\mathcal{M}(\Gamma))$ by $\text{Tmf}(\Gamma)$; see [HL16 §6] or [Mei19 §2]. For the subgroups commonly used $\Gamma(N)$, $\Gamma_1(N)$, and $\Gamma_0(N)$ (see [KMS85 §3]), we write $\text{Tmf}(N)$, $\text{Tmf}_1(N)$, and $\text{Tmf}_0(N)$, respectively.

Definition 3.5. If $n$ and $N$ are coprime, we can define $n$-fold-multiplication maps on the universal curves $\mathcal{E}(\Gamma)$ over $\mathcal{M}(\Gamma)$, and we then define the Adams operation $\psi^n: \text{Tmf}(\Gamma) \to \text{Tmf}(\Gamma)$ as its image under $O^\text{top}$. Similarly, by restricting to the cuspidal substacks $\mathcal{M}_{G_m \times M_{\text{Ell}}} \mathcal{M}(\Gamma)$, we obtain $K$-theory spectra of level $\Gamma$, denoted by $K(\Gamma)$, and one can similarly construct Adams operations on $K(\Gamma)$. If $N = 1$, then by [Dav20 Pr.2.1.11] these are equivalent to the usual stable Adams operations on $KU$.

The following then also follows from the functorality of $O^\text{top}$.

Proposition 3.6. Given two coprime integers $n$ and $N$, and a subgroup $\Gamma \subseteq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we have the following commutative diagram of $E_\infty$-$\text{Tmf}[\frac{1}{mn}]$-algebras

$$
\begin{array}{ccc}
\text{Tmf}(\Gamma) & \xrightarrow{\psi^n} & \text{Tmf}(\Gamma) \\
\downarrow & & \downarrow \\
K(\Gamma) & \xrightarrow{\psi^n} & K(\Gamma).
\end{array}
$$

For example, taking $\Gamma = \Gamma_0(3)$, we obtain the commutative diagram of $E_\infty$-$\text{Tmf}$-algebras

$$
\begin{array}{ccc}
\text{Tmf}_0(3) & \xrightarrow{\psi^n} & \text{Tmf}_0(3) \\
\downarrow & & \downarrow \\
\text{KO}[\frac{1}{3}] & \xrightarrow{\psi^n} & \text{KO}[\frac{1}{3}].
\end{array}
$$

3.1 Generalities on endomorphisms of $\text{tmf}$

To prove Theorem B we will use some general facts about morphisms of $E_\infty$-rings. Recall the $E_\infty$-ring of connective topological modular forms $\text{tmf}$ can be defined as $\tau_{\geq 0} \text{Tmf}$.

---

10See [Mei19 Ex.6.12] for the same argument with respect to $M_{\text{Ell}}^{\text{sm}}$ with level structures. The added generality of level-structures makes the proof less opaque in the author’s opinion.
Notation 3.7. Let us define a decomposition of $\pi_* \text{tmf}$ into $\text{Tors} \oplus \mathfrak{Fr}$ree. The elements of $\text{Tors}$ are simply the torsion elements, which can also be interpreted as elements in strictly positive filtration in the descent spectral sequence; see [Bau08 §8]. The elements of $\mathfrak{Fr}$ree in nonnegative degree are then described in the following three cases:

- When 6 is inverted, $\mathfrak{Fr}$ree is multiplicatively generated by the classes $c_4$ and $c_6$ corresponding to the normalised Eisenstein series of weight 4 and 6, respectively, which are uniquely determined by the collapsing descent spectral sequence. In this case $\mathfrak{Fr}$ree $= \pi_* \text{tmf}[\frac{1}{6}]$.

- When localised at 3, $\mathfrak{Fr}$ree is multiplicatively generated by the classes $c_4, c_6, \{3\Delta\}, \{c_4\Delta\}, \{c_6\Delta\}, \{3\Delta^2\}, \{c_4\Delta^2\}, \{c_6\Delta^2\}, \Delta^3$; see [DFHH14].

- When localised at 2, $\mathfrak{Fr}$ree is multiplicatively generated by the classes
  
  $c_4, \{2c_6\}, \{8\Delta^{2i+1}\}, \{4\Delta^{4j+2}\}, \{2\Delta^4\}, \{c_4\Delta^{k+1}\}, \{2c_6\Delta^{k+1}\}, \Delta^8$,

  for $i \in \{0, 1, 2, 3\}$, $j \in \{0, 1\}$, and $k \in \{0, 1, 2, 3, 4, 5, 6\}$; see [DFHH14].

Let us define a splitting $\text{Tors} \oplus \mathfrak{Fr}$ree for $\pi_* \text{tmf}$ by using the above splitting of $\pi_* \text{tmf}$ in nonnegative degrees, setting $\text{Tors}$ to be the torsion subgroup of $\pi_* \text{tmf}$, and $\mathfrak{Fr}$ree is defined in negative degrees as follows:

- When 6 is inverted, the $\mathbb{Z}[\frac{1}{6}]$-module $\mathfrak{Fr}$ree is generated by elements of the form $[c_4^i c_6^j \Delta^k]$, for $i \leq -1$, $j \in \{0, 1\}$, and $k \leq -1$; see [Kon12] Th.3.1.

- When localised at 3, $\mathfrak{Fr}$ree is generated by elements of the form

  $[c_4^i c_6^j \Delta^{k-3l}], [c_4^{-1} c_6 \Delta^{-1-3l}], [\frac{1}{3} c_4^{-1} c_6 \Delta^{-2-3l}], [\frac{1}{3} c_4^{-1} c_6 \Delta^{-3(l+1)}]$,

  where $i \leq -1$, $j \in \{0, 1\}$, $k \in \{-3, -2, -1\}$, $j + k < 0$, and $l \geq 0$; see [Kon12] Th.4.1.

- When localised at 2, $\mathfrak{Fr}$ree is generated by elements of the form

  $[c_4^i (2c_6)^j \Delta^{k-8l}], [c_4^{-1} c_6 \Delta^{-1-3l}], [c_4^{-1} (2c_6^{(k+1)-2})^j \Delta^{k-8l}]$,

  where $i \leq -1$, $j \in \{0, 1\}$, $k \in \{-8, -7, \ldots, -1\}$, $l \geq 0$, and $c_2$ is the function which sends a nonzero integer $a$ the largest integer $b$ with $2^b | a$, and $c_2(0) = 3$; see [Kon12] Th.5.3.

\[^{11}\text{We define } c_4 \text{ as the unique class in } \pi_8 \text{tmf}_2 \text{ which is mapped to the well-defined } c_4 \text{ in } \pi_8 \text{tmf}_Q \text{ and which is also } \kappa\text{-torsion. Same goes for the classes } \{c_4\Delta^{k+1}\} \text{ below, which we furthermore take to be both } \kappa\text{- and } \pi\text{-torsion.}\]

\[^{12}\text{The notation below denotes an element in } E_{x,y} \text{-page of the descent spectral sequence. These elements define an unambiguous element in } \pi_* \text{tmf}_3 \text{ when the column on the } E_{x,y} \text{-page of the descent spectral sequence is empty above these elements. This is not the case for } [c_4^m \Delta^{m-n}] \in \pi_{-8n-24m-1} \text{tmf}_3 \text{ for positive } n \text{ and } m \text{ such that } -8n - 24m - 1 \equiv -49 \text{ modulo } 72, \text{ which we define as the product of two well-defined elements } c_4 [c_4^{m-1} \Delta^{m-n}] \text{. This is consistent with the multiplicative structure of } \pi_* \text{tmf}_3 \text{; see [Kon12] Th.4.1.}\]

\[^{13}\text{Similar to the 3-local case, there are extension problems in defining some of these classes in } \pi_* \text{tmf}_2 \text{ from classes which survive the descent spectral sequence. Any potentially ambiguous elements can be defined the product of either } c_4 \text{ or } c_4^2 \text{ with another well-defined element. Indeed, by abusing notation a little, for an element } [c_4^2 (2c_6)^j \Delta^{k-1}] \text{ is ambiguous notation for an element in } \pi_* \text{tmf}_2 \text{ for some negative } q = 8i + 12j + 24k - 1, \text{ then we define this element in homotopy as the product of } c_4 [c_4^{q-1} (2c_6)^j \Delta^k] \text{ if } q \text{ is congruent modulo } 72 \text{ to an}\]
Above, elements of the form \([c_4^i c_6^j \Delta^k]\) have degree \(8i + 12j + 24k - 1\).

To calculate the effect of endomorphisms of \(\text{tmf}\) on \(\pi_*\), we first would like to show that particular endomorphisms preserve the direct sum decomposition above. A reader with a mind for spectral sequences may notice that the following theorem is essentially a solution to potential extension problems arising from the descent spectral sequence for \(\text{tmf}\).

**Theorem 3.8.** If \(F: \text{tmf}(2) \to \text{tmf}(2)\) is a morphism of spectra, then \(F(\mathfrak{f}\text{ree}) \subseteq \mathfrak{f}\text{ree}\). Similarly, if \(F: \text{tmf}(3) \to \text{tmf}(3)\) is a morphism of spectra, then \(F(\mathfrak{f}\text{ree})_k \subseteq \mathfrak{f}\text{ree}_k\) in degrees \(k\) not congruent to 40 modulo 72. If \(F: \text{tmf}(3) \to \text{tmf}(3)\) is a morphism of algebra objects in \(\text{hSp}\), then \(F(\mathfrak{f}\text{ree}) \subseteq \mathfrak{f}\text{ree}\) holds in all degrees.

To prove the above statement, we will use the following general lemma, followed by a more specific argument at the prime 2.

**Lemma 3.9.** Let \(R\) be an algebra in \(\text{hSp}\), and \(A\) an \(R\)-algebra in \(\text{hSp}\), and suppose for some \(k \in \mathbb{Z}\) we have a decomposition of \(\pi_k A\) given by \(\text{Tors} \oplus \mathfrak{f}\text{ree}\), where the elements of \(\text{Tors}\) are precisely the \(\pi_0 R\)-torsion elements of \(\pi_k A\). Suppose that for each \(x \in \mathfrak{f}\text{ree}\), and each \(y \in \text{Tors}\), there is a \(z\) in the image of the unit \(\pi_* R \to \pi_* A\) such that \(zx' = 0\) for all \(x' \in \mathfrak{f}\text{ree}\) and the map of \(\pi_0 R\)-modules

\[
\pi_k A \ni \langle y \rangle \overset{z}{\rightarrow} \langle zy \rangle \subseteq \pi_{k+|z|} A
\]

is injective. Then for every \(R\)-module map \(F: A \to A\), the induced map on homotopy groups \(F: \pi_k A \to \pi_k A\) preserves the decomposition \(\text{Tors} \oplus \mathfrak{f}\text{ree}\).

**Proof.** Take an \(x \in \mathfrak{f}\text{ree}\) and write \(F(x) = x' + y\) where \(x' \in \mathfrak{f}\text{ree}\) and \(y \in \text{Tors}\) using the decomposition above. The hypotheses then lead us to the equalities

\[
0 = F(zx) = zF(x) = z(x' + y) = zy,
\]

where the second equality follows from the \(R\)-linearity of \(F\). The injectivity of \((3.10)\) leads us to the conclusion that \(y = 0\), and we are done. \(\square\)

**Lemma 3.11.** Given a map \(F: \text{tmf}(2) \to \text{tmf}(2)\) of spectra and an \(x \in \mathfrak{f}\text{ree}\) of degree congruent to 20, 40, 60, 68, 100, 116, 156, or 164 modulo 192, then \(F(x)\) also lies in \(\mathfrak{f}\text{ree}\).

**Proof.** Let us 2-complete everything in this proof without explicitly mentioning this fact in our notation. We will now consider the cellular (2-complete) \(\mathbb{C}\)-motivic stable category \(\text{Sp}_\mathbb{C}\). By [GIK18], there is a lax symmetric monoidal functor \(\Gamma_*: \text{Sp} \to \text{Sp}_\mathbb{C}\), and the motivic spectrum \(\mathfrak{m}\text{mf}\) of \(\mathbb{C}\)-motivic modular forms is defined as the image of \(\text{tmf}\) under this functor. There is also a functor \(\text{Sp}_\mathbb{C} \to \text{Sp}\) called Betti realisation, which corresponds to inverting a particular element in the set

\[
\{-37, -57, -61, -81, -87, -121, -133, -153, -157, -177\},
\]

and as the product \(c_3^{\overline{a}^2} \overline{2c_6^\prime \Delta^4}\) if \(q\) is congruent modulo 72 to an element in the set

\[
\{-49, -73, -145, -169\}.
\]
element \( \tau \in \pi_{0,-}S \) in the bigraded homotopy groups of the \( \mathbf{C} \)-motivic sphere. From [GIKR18] (also discussed in [Isa18]), the Betti realisation of \( \text{mmf} \) is \( \text{tmf} \).

Applying \( \Gamma \), to \( F : \text{tmf} \to \text{tmf} \) yields the endomorphism \( F_{\text{mot}} : \text{mmf} \to \text{mmf} \) of \( \mathbf{C} \)-motivic spectra, which we will now study as we studied endomorphisms of spectra in Lemma 3.9. Consider the \( \mathbf{C} \)-motivic Adams-Novikov spectral sequence for \( \text{mmf} \) (see [Isa09, §5]), which we will call the \emph{motivic descent spectral sequence}, or MDSS. Given an element \( x \in \pi_{*,*} \text{mmf} \) detected by a class in the zero line of the MDSS, we want to show that \( F_{\text{mot}}(x) \) is also detected by a class in the zero line of the MDSS; this spectral sequence is illustrated in [Isa09, p.271-3]. In our particular degrees, we see that \( F_{\text{mot}}(x) = x_0 + y \), where \( x_0 \) is detected by a class \( x'_0 \) lying in the zero line of the MDSS and \( y \) is detected by a class \( y' \) lying in strictly positive filtration in the MDSS. As the Betti realisation functor inverts the class \( \tau \in \pi_{0,-}S \), it suffices to consider the above equality up to \( \tau \)-torsion, hence, we may assume that \( y \) is \emph{not} \( \tau \)-torsion; this restricts us to the bullets in [Isa09, p.271-3], and allows us to ignore the lines given by powers of \( \eta \) (represented by \( h_1 \)). Note that multiplication by \( \nu \) (represented by the class \( h_2 \)) and \( \pi \) (represented by the class \( g \)) with any element in \( \pi_{*,*} \text{mmf} \) represented by a class in the zero line of the MDSS in degree not divisible by 24 is zero. Indeed, this is true on the \( E_2 \)-page \( E_2 \) of the descent spectral sequence for \( \text{tmf} \) (see [Bau08, §8]), and the \( E_2 \)-page of the MDSS is precisely \( E_2 = \otimes_{\mathbf{Z}[2]} \mathbf{Z}[2][\tau] \), where \( \mathbf{Z}[2][\tau] \) is the 2-localisation of the motivic cohomology of a point; see [Isa09, §5]. Let us now work case-by-case:

(20) Here we see \( y \) must lie in \( \mathbf{Z}/8\mathbf{Z}[\tau] \) generated by \( \kappa \) over \( \mathbf{Z}[\tau] \). However, multiplication by \( \pi \) (also denoted as \( g \) in [Isa09] and [Isa18]) on this group takes any \( y' \) with \( \tau \)-degree 0 to a nonzero element in degree 40. As multiplication by \( \pi \) on elements in the same bidegree as \( x'_0 \) is always zero, we then see (as in Lemma 3.9) that \( y \equiv 0 \) and hence \( F_{\text{mot}}(x) \equiv x_0 \) modulo \( \tau \)-torsion.

(40) Similar to the previous case, \( y \) lies in the \( \mathbf{Z}/4\mathbf{Z}[\tau] \) generated by \( \kappa^2 \), and multiplication by \( \nu \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in the group in degree 43, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.

(60) Similar to the previous cases, \( y \) lies in the \( \mathbf{Z}/4\mathbf{Z}[\tau] \) generated by \( \kappa^3 \), and multiplication by \( \nu \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in degree 63, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.

(68) Similar to the previous cases, \( y \) lies in the \( \mathbf{Z}/2\mathbf{Z}[\tau] \) generated by \( \nu \kappa \{ \nu^2 \} \), and an exotic multiplication by \( \nu \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in degree 71, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.

(100) Similar to the previous cases, \( y \) lies in the \( \mathbf{Z}/2\mathbf{Z}[\tau] \) generated by \( \kappa^5 \), and multiplication by \( \nu \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in the group in degree 103, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.

(116) Similar to the previous cases, \( y \) lies in the \( \mathbf{Z}/4\mathbf{Z}[\tau] \) generated by \( \kappa \{ 2^4 \} \), and multiplication by \( \pi \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in the group in degree 136, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.
(156) Similar to the previous cases, \( y \) lies in the \( \mathbb{Z}/2\mathbb{Z}[\tau] \) generated by \( \kappa^2\{q\Delta^4\} \), and multiplication by \( \nu \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in the group in degree 159, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.

(164) Similar to the previous cases, \( y \) lies in the \( \mathbb{Z}/2\mathbb{Z}[\tau] \) generated by \( \pi\{4\Delta^6\} \), and an exotic multiplication by \( \nu \) carries any \( y \) of \( \tau \)-degree zero to a nonzero element in the group in degree 63, hence as above we see \( y \equiv 0 \) modulo \( \tau \)-torsion.

When we localise at \( \tau \), we therefore obtain \( F_{\text{mot}}(x)[\tau^{-1}] = x_0 \), as elements on the zero line of the MDSS are never \( \tau \)-torsion. As the operation of inverting \( \tau \) takes us from \( F_{\text{mot}} : \text{mmf} \to \text{tmf} \) to \( F : \text{tmf} \to \text{tmf} \), we see that \( F \) preserves \( \text{free} \) in the degrees indicated above. \( \square \)

**Remark 3.12.** The use of motivic homotopy theory above is the most convenient for us to refer to right now, but a “purely topological” argument can be given by either referring to the work on \( \Gamma_*\text{S} \)-modules in [GIKR18] or synthetic spectra as in [Pst19]. Morally, this means that motivic homotopy theory is not necessary for this article, but a detailed analysis of the descent spectral sequence for \( \text{tmf} \) is, which is can equivalently be expressed using these tools.

We are now in the position to prove Theorem 3.8.

**Proof of Theorem 3.8.** For an endomorphism \( F : \text{tmf}(2) \to \text{tmf}(2) \) of spectra, the only degrees where there are nonzero elements of both \( \mathfrak{f}\text{ree} \) and \( \mathcal{T}\text{ors} \) are those congruent to one of

\[ 8, 20, 28, 32, 40, 52, 60, 68, 80, 100, 104, 116, 124, 128, 136, 148, 156, 164, \text{ modulo } 192. \]

In degrees congruent modulo 192 to 8, 28, 32, 52, 104, 124, 128, 136, and 148, we use Lemma 3.9 with \( R = \text{S} \), \( A = \text{tmf}(2) \), the decomposition of Notation 3.7 and \( z = \kappa \) for all \( y \in \mathcal{T}\text{ors} \) in these degrees. Similarly, in degrees congruent to 80 modulo 192, we can use \( z = \tau \) for all possible \( y \).

For \( k \) not covered above, we appeal to Lemma 3.11 which finishes the case of \( \text{tmf}(2) \).

For an endomorphism \( F : \text{tmf}(3) \to \text{tmf}(3) \), the only interesting degrees are those congruent to 20 or 40 modulo 72. For \( k = 20 \), we use Lemma 3.9 with \( R = \text{S} \), \( A = \text{tmf}(3) \), the decomposition of Notation 3.7 and \( y = \pm \beta^l \Delta^3l \) with \( l \geq 0 \) and \( z = \beta \), as we have \( yz = \pm \beta^3 \) which generates \( \pi_{8+10} \text{tmf}(3) \cong F_3 \). The case when \( k = 40 \) is dealt with in the multiplicative case by noting that there are no multiplicative generators for \( \pi_* \text{tmf} \) in this degree. \( \square \)

We can extend Theorem 3.8 to \( \text{Tmf}(p) \) for multiplicative endomorphisms.

**Corollary 3.13.** Given a prime \( p \) and a morphism \( F : \text{Tmf}(p) \to \text{Tmf}(p) \) of algebra objects in \( \text{hSp} \), then we have \( F(\mathfrak{f}\text{ree}) \subseteq \mathfrak{f}\text{ree} \).

**Remark 3.14.** Using a motivic analogue of \( \text{Tmf} \), as opposed to \( \text{tmf} \) as used to prove Lemma 3.11, the author has obtained a generalisation of Corollary 3.13 to morphisms of (2-local) spectra, which appear in [Dav].

**Proof of Corollary 3.13.** By Theorem 3.8 we only need to check if \( F(x) \in \mathfrak{f}\text{ree} \) for an element \( x \) in \( \mathfrak{f}\text{ree} \) of negative degree. If \( p \) is neither 2 nor 3 then we are immediately done – in this case \( \mathfrak{f}\text{ree} = \pi_* \text{Tmf}(p) \). Otherwise, we have two cases:
1. If $p = 3$, then we look at [Kon12] Figure 13 to see the only degrees where $\mathbb{T}_{\text{tors},0}$ and $\mathfrak{S}_{\text{ree},0}$ have nonzero elements are when $|x|$ is congruent to $-49$ modulo $72$. From the $\Delta^{-3}$-fold periodicity of [Kon12] Figure 13, it suffices to consider $x \in \pi_{-49} \text{tmf}_{(3)}$ inside $\mathfrak{S}_{\text{ree}}$, and this subgroup is generated by $[c_4^{-3}\Delta^{-1}]$. As $F$ is multiplicative and $\mathfrak{S}_{\text{ree},-57} = \pi_{-57} \text{tmf}_{(3)}$ by inspection, we then have the equations

$$F([c_4^{-3}\Delta^{-1}]) = F(c_4[c_4^{-4}\Delta^{-1}]) = F(c_4)F([c_4^{-4}\Delta^{-1}]) \in \mathfrak{S}_{\text{ree}},$$

and we are done.

2. If $p = 2$, we now look at [Kon12] Figure 27, make the same multiplicative arguments, use $\Delta^{-8}$-fold periodicity, and see the only degrees where $\mathbb{T}_{\text{tors},0}$ and $\mathfrak{S}_{\text{ree},0}$ are both nonzero are when $|x|$ is equal to $-57, -81, -153$, or $-177$. Luckily for us, we can apply Lemma 3.19 with $z = \nu$ to the cases $|x| \in \{-57, -153\}$ and $z = \pi$ to the cases $|x| \in \{-81, -177\}$. And we are done.

We have one more statement to make concerning torsion elements in $\pi_* \text{tmf}$.

**Proposition 3.15.** Let $x$ be a homogeneous element of $\mathbb{T}_{\text{tors}} \subseteq \pi_* \text{tmf}$, $\mathcal{P}$ a set of primes which we implicitly localize away from everywhere, and $F$: tmf $\to$ tmf a morphism of spectra. Furthermore, if $2$ is not contained in $\mathcal{P}$, then assume that for all $l \geq 0$, $F(\Delta^{8(l+1)})$ is congruent to $\Delta^{8(l+1)}F(1)$ modulo 2, and if $3$ is not contained in $\mathcal{P}$, then assume that for all $l \geq 0$, $F(\Delta^{3(l+1)})$ is congruent to $\Delta^{3(l+1)}F(1)$ modulo 3. Then we have the equality

$$F(x) = xF(1) \quad \in \pi_* \text{tmf}.$$  

**Proof.** There are three cases to consider: First, suppose $x$ is in the image of the Hurewicz morphism $\pi_* S \to \pi_* \text{tmf}$. In this case, as $F$ is $S$-linear, as all maps of spectra are, we obtain the equalities

$$F(x) =xF(1) = x \quad \in \pi_{|x|} \text{tmf}. $$

Second, suppose that there exists a $y \in \pi_* \text{tmf}$ in the Hurewicz image such that $xy$ is nonzero and also lies in the Hurewicz image. Moreover, assume that the multiplication-by-$y$-map

$$\pi_{|x|} \text{tmf} \ni \langle y \rangle \xrightarrow{\cdot y} \langle xy \rangle \subseteq \pi_{|xy|} \text{tmf} \quad (3.16)$$

is injective. In this case we have the equalities

$$xy = F(xy) = F(x)y,$$

which using the assumption that (3.16) is injective, implies that $F(x) = x$. Third, by inspecting the torsion of $\pi_* \text{tmf}$, one notices the only torsion left to discuss is of the form

$$\alpha \Delta^{3(l+1)}, \quad \eta \Delta^{8(l+1)}, \quad \nu \Delta^{8(l+1)}, \quad l \geq 0;$$

Here we are using $\alpha$ to denote a generator of $\pi_3 \text{tmf}_{(3)}$ and $\nu$ to denote a generator of $\pi_3 \text{tmf}_{(2)}$. 

31
this is clear by analysing the conjectural Hurewicz image in [DFHH14 §13], recently proven in [BMQ20] \(^{15}\). However, these cases are handled by the \(S\)-linearity of \(F\) and the mod 2 and 3 reductions of \(F(\Delta^{8(l+1)})\) and \(F(\Delta^{3(l+1)})\) assumed above. For example,

\[
F(\alpha \Delta^{3(l+1)}) = \alpha F(\Delta^{3(l+1)}) \equiv \alpha \Delta^{3(l+1)} F(1) \in F_3 \{\alpha \Delta^{3(l+1)}\} \subseteq \pi_{3+72(l+1)} \text{tmf}. \qed
\]

### 3.2 Anderson duality

To systematically study the negative homotopy groups of \(\text{tmf}\), we will use the following form of duality.

**Definition 3.17.** For an injective abelian group \(J\), we write \(I_J\) for the spectrum represented by the cohomology theory

\[
\text{Sp} \to \text{Ab}, \quad X \mapsto \text{Hom}_{\text{Ab}}(\pi_{-*} X, J).
\]

For a general abelian group \(A\), we take an injective resolution of the form \(0 \to A \to J_1 \to J_2\), which by functorality, yields a morphism of spectra \(I_{J_1} \to I_{J_2}\). The fibre of this morphism we denote by \(I_A\), and for a spectrum \(X\), we define the *Anderson dual of \(X\) to be the function spectrum\( I_A X = F(X, I_A)\).

From the definition above one can calculate

\[
\pi_{-*} I_J X \cong \text{Hom}_{\text{Z}}(\pi_{-*} X, J)
\]

for an injective abelian group \(J\). When \(A\) is a general abelian group, we obtain the following functorial exact sequence of abelian groups for all \(k \in \mathbb{Z}\)

\[
0 \to \text{Ext}^1_{\mathbb{Z}}(\pi_{-k-1} X, A) \to \pi_k I_A X \to \text{Hom}_{\mathbb{Z}}(\pi_{-k} X, A) \to 0,
\]

which noncanonically splits when \(A\) is a subring of \(\mathbb{Q}\). More basic facts about Anderson duality, such as the fact that the natural map \(X \to I_A I_A X\) is an equivalence when \(X\) has finitely generated homotopy groups, can be found in [Lur18b §6.6], under the guise of Grothendieck duality in spectral algebraic geometry. Anderson duality is of interest to us as many of the spectra we will study in this article are Anderson self-dual.

**Definition 3.19.** Let \(X\) be a spectrum and \(A\) an abelian group. We say that \(X\) is *Anderson self-dual* if there is an integer \(d\) and an equivalence of spectra

\[
\phi: \Sigma^d X \cong I_A X.
\]

We also want to define a stricter form of self-duality for ring spectra. Let \(R\) be an \(E_1\)-ring with \(\pi_0 R \simeq A\) such that \(\pi_{-*} R\) is a free \(A\)-module of rank one. We say an element \(D \in \pi_{-*} R\) *witnesses the Anderson self-duality* of \(R\) if the isomorphism \(\phi_D: \pi_{-*} R \to A\) sending \(D \mapsto 1\) which identifies \(D\) as an \(A\)-module generator of \(\pi_{-*} R\), lifts to an element \(D' \in \pi_{d} I_A R\) under the surjection of \(\text{mod}_{\text{E}_1} R\) whose representing map of left \(R\)-modules \(D': \Sigma^d R \to I_A R\) is an equivalence.

\(^{15}\)The entirety of [BMQ20] occurs 2-locally. Moreover, it is mentioned in their introduction that the 3-local statement of the image of \(\pi_{*} S \to \pi_{*} \text{tmf}\) in [DFHH14 §13] also holds by recent work of Shimomura [Shi].
Example 3.20. There are some famous examples of Anderson self-duality.

- The class $1 \in \pi_0 \text{KU}$ witnesses the Anderson self-duality of KU, ie,
  
  $1^\vee : \text{KU} \xrightarrow{\simeq} I_\mathbb{Z} \text{KU}$

  is an equivalence. This is originally due to Anderson [And69], and is an immediate consequence of the fact that $\text{Hom}_\mathbb{Z}(\pi_* \text{KU}, \mathbb{Z})$ is a free $\pi_*$ KU-module; see [HS14 p.3].

- The class $vu^{-1}_R \in \pi_{-4} \text{KO}$ witnesses the Anderson self-duality of KO, ie,
  
  $\left(vu^{-1}_R\right)^\vee : \Sigma^4 \text{KO} \xrightarrow{\simeq} I_\mathbb{Z} \text{KO}$

  is an equivalence. This result is also due to Anderson, and an accessible modern proof (with an eye towards spectral algebraic geometry) can be found in [HS14 Th.8.1].

- The class $D = [c_1^{-1}c_6 \Delta^{-1}] \in \pi_{-21} \text{Tmf}$ witnesses the Anderson self-duality of Tmf, ie,
  
  $D^\vee : \Sigma^{21} \text{Tmf} \xrightarrow{\simeq} I_\mathbb{Z} \text{Tmf}$

  is an equivalence. This result is due to Stojanoska; see [Sto12 Th.13.1] for the case with 2 inverted and [Sto14] where it is announced in general.

- The class $\frac{1}{\lambda_1 \lambda_2} \in \pi_{-9} \text{Tmf}(2)$ witnesses the Anderson self-duality of $\text{Tmf}(2)$, ie,
  
  $\left(\frac{1}{\lambda_1 \lambda_2}\right)^\vee : \Sigma^{9} \text{Tmf}(2) \xrightarrow{\simeq} I_{\mathbb{Z}[\frac{1}{2}]} \text{Tmf}(2)$

  is an equivalence. This is also due to Stojanoska; see [Sto12 Th.9.1].

- There are classes $D_m$ in $\pi_{l_m} \text{Tmf}_1(m)$, with $m$ and $l_m$ taking the values

  \[
  \begin{array}{cccccccccccc}
  m & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 11 & 14 & 15 & 23 \\
  l_m & 13 & 9 & 7 & 5 & 5 & 3 & 3 & 1 & 1 & 1 & -1
  \end{array}
  \]

  which witness the Anderson self-duality of these particular $\text{Tmf}_1(m)$, ie, the map

  $D_m^\vee : \Sigma^{l_m} \text{Tmf}_1(m) \xrightarrow{\simeq} I_{\mathbb{Z}[\frac{1}{m}]} \text{Tmf}_1(m)$

  is an equivalence. This result is due to Meier [Mei19 Th.5.14], where he also shows the above are the only $m \geq 2$ such that $\text{Tmf}_1(m)$ is Anderson self-dual.

Remark 3.21. The decompositions of Notation 3.7 were defined as to interact well with the Anderson self-duality of Tmf. Indeed, consider the short exact sequence

$0 \to \text{Ext}_\mathbb{Z}(\pi_{-k-22} \text{Tmf}, \mathbb{Z}) \to \pi_k \text{Tmf} \to \text{Hom}_\mathbb{Z}(\pi_{-k-21} \text{Tmf}, \mathbb{Z}) \to 0$

from (3.18) in the case of Tmf, which simplifies to

$0 \to \text{Ext}_\mathbb{Z}(\text{Tors}_{-k-22}, \mathbb{Z}) \to \text{Tors}_k \oplus \text{Free}_k \to \text{Hom}_\mathbb{Z}(\text{Free}_{-k-21}, \mathbb{Z}) \to 0$. 

33
Clearly the image of the Ext-group on the left is precisely $\text{Tor}_k$. Furthermore, if we invert 6 by writing $(\epsilon_k^i \Delta^k) \chi = [\epsilon_k^{i-1} \chi_6^{i} \Delta^{k-1}]$, one can define a splitting $s$ of the surjection above by sending a characteristic function $\chi_g$: $\mathfrak{Fr}_{-k-21} \to \mathbb{Z}^{\chi}$ for a generator $g$ of $\mathfrak{Fr}_{-k-21}$. By construction this shows $\mathfrak{Fr}_{-k}$ is precisely the image of $s$. One obtain similar results without inverting 6 by treating the above with a little more care.

Studying endomorphisms of Anderson self-dual spectra leads us to dual endomorphisms.

**Definition 3.22.** Let $A$ be an abelian group, $X$ an Anderson self-dual spectrum, and $F: X \to X$ an endomorphism of $X$. Define the dual endomorphism of $F$ as the composite

$$\tilde{F}: X \xrightarrow{\phi} \Sigma^{-d} I_A X \xrightarrow{\Sigma^{-d} I_A F} \Sigma^{-d} I_A X \xleftarrow{\phi} X.$$

Given $A$, $X$, and $F$ from the above definition, then the functorality of (3.23) yields the following commutative diagram of abelian groups with exact rows for all $k \in \mathbb{Z}$:

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Ext}^1_{\mathbb{Z}}(\pi_{-k-1-d} X, A) \\
& & \downarrow \text{Ext}^1_{\mathbb{Z}}(F, A) = F^\ast_1 \\
0 & \longrightarrow & \text{Hom}_{\mathbb{Z}}(\pi_{-k-d} X, A) \\
& & \downarrow \text{Hom}_{\mathbb{Z}}(F, A) = F^\ast_1 \\
\end{array}
$$

(3.23)

Our calculations of $\psi^n$ on $\text{Tmf}$ in negative degrees will rest upon explicit calculations of $\tilde{\psi}^n$ and using (3.23).

When working with 6 inverted, these also exists a kind of algebro-geometric duality on $\mathcal{M}_{\text{Ell}}$ called Serre duality. The following can be found in [Mei20, Ap.A] using the well-known identification of $\mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{6}]}$ with the weighted projective line $\mathcal{P}_{\mathbb{Z}[\frac{1}{6}]}(4, 6)$; see [Mei20, Ex.2.1].

**Theorem 3.24.** The dualising sheaf for $\mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{6}]}$ is $\omega^{-10}$. In particular, for any integer $k$ the natural cup product map

$$H^0(\mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{6}]}, \omega^k) \otimes H^1(\mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{6}]}, \omega^{-k-10}) \to H^1(\mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{6}]}, \omega^{-10})$$

is a perfect pairing of $\mathbb{Z}[\frac{1}{6}]$-modules.

Let us note that the stack $\mathcal{M}_{\text{Ell}}$ certainly has no Serre duality before inverting 6, which can be seen through the cohomology calculations of $\omega^k$ over $\mathcal{M}_{\text{Ell}}$ from [Kon12].

**Remark 3.25.** A simple consequence of the above theorem is that one can immediately see the $\mathbf{E}_X$-ring $\text{Tmf}[\frac{1}{6}]$ is Anderson self-dual. Indeed, as discussed on [Stol21, p.8], the Serre duality statement of Theorem 3.24 the calculation of $H^s(\mathcal{M}_{\text{Ell}, \mathbb{Z}[\frac{1}{6}]}, \omega^s)$ in [Kon12, §3], and a collapsing descent spectral sequence, immediately implies the Anderson self-duality of $\text{Tmf}[\frac{1}{6}]$ as in Example 3.20.

When 6 is inverted, dual endomorphisms on $\text{Tmf}$ defined using Anderson duality, can be computed directly using Serre duality.
Lemma 3.26. Let \( \mathcal{P} \) be a set of primes containing both 2 and 3 and implicitly localise everywhere away from \( \mathcal{P} \). If \( F: \text{Tmf} \to \text{Tmf} \) is a morphism of spectra, then one can compute \( \tilde{F} \) on \( \pi_* \text{Tmf} \) in negative degrees as the composite

\[
\tilde{F}: \pi_k \text{Tmf} \simeq H^1(\mathcal{M}_{\text{Ell}}, \omega^{-\frac{k-1}{2}-10}) \to F^* H^1(\mathcal{M}_{\text{Ell}}, \omega^{-\frac{k-1}{2}-10}) \simeq \pi_k \text{Tmf},
\]

where we have implicitly used the Serre duality isomorphism above. Moreover, one can compute \( \tilde{F} \) on \( \pi_* \text{Tmf} \) in nonnegative degrees as the composite

\[
\tilde{F}: \pi_k \text{Tmf} \simeq H^0(\mathcal{M}_{\text{Ell}}, \omega^{-\frac{k-1}{2}-10}) \to F^* H^0(\mathcal{M}_{\text{Ell}}, \omega^{-\frac{k-1}{2}-10}) \simeq \pi_k \text{Tmf},
\]

where again, we have implicitly used the Serre duality isomorphism.

Proof. This follows immediately from the definitions, as in this case the Anderson duality equivalence comes directly from Serre duality; see Remark 3.25.

3.3 Proof of Theorem [B]

Proof. Take an element \( x \in \pi_* \text{Tmf}[[\frac{1}{n}]] \) lying in \( \mathfrak{g} \text{ree} \). As the natural localisation map

\[
\pi_* \text{Tmf}[[\frac{1}{n}]] \to \pi_* \text{Tmf}[[\frac{1}{6n}]]
\]

is injective on the submodule \( \mathfrak{g} \text{ree} \) of \( \pi_* \text{Tmf}[[\frac{1}{n}]] \), and the morphism \( \psi^n \) preserves \( \mathfrak{g} \text{ree} \) by Corollary 3.13 then it suffices to invert 6 and work inside \( \pi_* \text{Tmf}[[\frac{1}{6n}]] \). We now have two cases:

1. First, suppose the degree of \( x \) is nonnegative. In this case, the descent spectral sequence collapses immediately (as 6 is inverted) and we see that \( \psi^n(x) = n^{\text{ceiling}} x \), as this what the \( n \)-fold-multiplication map induces on \( \omega \). The ceiling function in this case is unnecessary.

2. Next, suppose the degree of \( x \) is negative. In this case, we have to compute the morphism

\[
\psi^n: H^1(\mathcal{M}_{\text{Ell}, \mathbb{Z}[[\frac{1}{6n}]]}, \omega^k) \to H^1(\mathcal{M}_{\text{Ell}, \mathbb{Z}[[\frac{1}{6n}]]}, \omega^k)
\]

for all \( k < 0 \). This we can do with a calculation of the cohomology of the stack with graded structure sheaf \( (\mathcal{M}_{\text{Ell}, \mathbb{Z}[[\frac{1}{6n}]]}, \omega^*) \), which is equivalent to the weighted projective line \( \mathcal{P} \mathbb{Z}[[\frac{1}{6n}]](4,6) \); see [Mei20, Ex.2.1]. In this case we can use the fact that groups \( H^*(\mathcal{P} \mathbb{Z}[[\frac{1}{6n}]](4,6), \omega^*) \) are isomorphic to the groups \( H^*(\tilde{\mathcal{P}}(4,6), \mathcal{O}) \), where \( \tilde{\mathcal{P}}(4,6), \mathcal{O} \) is \( (\text{Spec } A - \{0\}, \mathcal{O}) \), where \( A = \mathbb{Z}[\frac{1}{6n}]c_4, c_6 \), together with the \( \mathbb{G}_m \)-action given by the gradings \( |c_4| = 4 \) and \( |c_6| = 6 \). Using the long exact sequence on cohomology induced by the expression \( \tilde{\mathcal{P}}(4,6) \subseteq \text{Spec } A \supseteq \{0\} \) [Har83, Exercise III.2.3], and the fact that \( R\Gamma_0(\text{Spec } A, \mathcal{O}) \) can be computed via the Koszul complex

\[
A \to A[\frac{1}{c_4}] \times A[\frac{1}{c_6}] \to A[\frac{1}{c_4 c_6}],
\]

35
we obtain the exact sequence

\[ 0 \to A \to H^0(\mathcal{P}(4,6), \mathcal{O}) \to 0 \to H^1(\mathcal{P}(4,6), \mathcal{O}) \to A/(c_4^0, c_6^0) \to 0. \]

Using this, we can explicitly calculate \( \psi^n \) on \( H^1(\mathcal{P}(4,6), \mathcal{O}) \cong A/(c_4^0, c_6^0) \) as

\[ \psi^n \left( \frac{1}{c_4^0 c_6^0} \right) = n^{-4i-6j} \frac{1}{c_4^0 c_6^0}, \]

which as \( \frac{1}{c_4^0 c_6^0} \) represents a class in \( \pi_* \text{Tmf}[\frac{1}{m}] \) of topological degree \(-8i-12j-1\), gives us the desired result.

Let us now consider an element \( x \in \pi_* \text{Tmf}[\frac{1}{m}] \) inside \( \mathbb{T_\text{ors}} \), and implicitly invert \( n \) for the rest of this proof. If \( x \) has nonnegative degree, then we can immediately apply Proposition 3.14 and we are done. Indeed, the hypotheses of that proposition apply as if 3 does not divide \( n \), then \( \psi^n(\Delta^{3l+1}) = n^{36l+1} \Delta^{3l+1} \) is congruent to \( \Delta^{3l+1} \) modulo 3, for all \( l \geq 0 \), and similarly if 2 does not divide \( n \) then \( \psi^n(\Delta^{3l+1}) \) is congruent to \( \Delta^{3l+1} \) modulo 2. If \( x \) is an element of \( \mathbb{T_\text{ors}} \) of negative degree, then we will consider \( \text{(3.23)} \) for \( \text{Tmf} \), which yields the commutative diagram of abelian groups for every integer \( k \)

\[\begin{array}{c}
0 \longrightarrow \text{Ext}^1_Z(\pi_{-k-22} \text{Tmf}) \longrightarrow \pi_k \text{Tmf} \longrightarrow \text{Hom}_Z(\pi_{-k-21} \text{Tmf}) \longrightarrow 0
\end{array}\]

\[\begin{array}{cc}
\downarrow (\psi^n)_1 & \downarrow \tilde{\psi}^n & \downarrow (\psi^n)_0 \\
0 \longrightarrow \text{Ext}^1_Z(\pi_{-k-22} \text{Tmf}) \longrightarrow \pi_k \text{Tmf} \longrightarrow \text{Hom}_Z(\pi_{-k-21} \text{Tmf}) \longrightarrow 0,
\end{array}\]

(3.27)

where all the Ext- and Hom-groups above have \( Z \) as a codomain. As \( \psi^n \) is an \( S \)-linear map, then we can detect the effect of \( \psi^n \) on \( \mathbb{T_\text{ors}} \) in \( \pi_* \text{Tmf} \) by the effect of \( (\psi^n)_1 \) on the above Ext-groups. To calculate \( \tilde{\psi}^n \) on elements in \( \mathbb{T_\text{ors}} \) of nonnegative degree, we would like to use Proposition 3.15, which first requires us to calculate some values of \( \psi^n \). In particular, notice that it will suffice to calculate \( \psi^n(\Delta^k) \), where

\[ k = \begin{cases} 
1 & \text{if 6 divides } n \\
3 & \text{if only 2 divides } n \\
8 & \text{if only 3 divides } n \\
24 & \text{if neither 2 nor 3 divides } n.
\end{cases} \]

By Theorem 3.8 the value of \( \tilde{\psi}^n(\Delta^k) \) will always remain inside \( \mathbb{F}_\text{ree} \), hence it suffices to do the calculation of \( \tilde{\psi}^n \) after inverting 6. We can then use Lemma 3.26 and the above calculations of \( \psi^n \) to obtain

\[ \tilde{\psi}^n(\Delta^k) = n^{-10-12k}\Delta^k. \]

We can now apply Proposition 3.15 to \( \text{(3.27)} \) to obtain a calculation of \( \psi^n(x) \) for \( x \in \mathbb{T_\text{ors}} \) of negative degree, and we are done.

As shown at the end of the above proof, we can actually calculate \( \tilde{\psi}^n \) for a range of elements in \( \pi_* \text{Tmf}[\frac{1}{n}] \).
Proposition 3.28. Let \( n \) be an integer. Then the effect of the dual Adams operation \( \tilde{\psi}^n \) on \( \pi_* \text{Tmf} \left[ \frac{1}{m} \right] \) is given by

\[
\tilde{\psi}^n(x) = \begin{cases} 
  n^{-10-\left\lfloor \frac{n}{x} \right\rfloor} & x \in \text{Free} \\
  n^{-10} x & x \in \text{Tors},
\end{cases}
\]

except for \( x \in \text{Free} \) in the following exceptional cases:

- 3 does not divide \( n \), \( |x| \geq 0 \), and \( |x| \equiv 40 \) modulo 72.
- 3 does not divide \( n \), \( |x| < 0 \), and \( |x| \equiv -49 \) modulo 72.
- 2 does not divide \( n \), \( |x| < 0 \), and \( |x| \equiv k \) modulo 192, where \( k \) is an element of \( \{-49, -73, -97, -121, -145, -169, -177\} \).

Remark 3.29. It is possible to include the above exceptional cases at the prime 2 by referring to the generalisation of Corollary 3.13 mentioned in Remark 3.14. This involves calculating the motivic homotopy groups of a \( \mathbb{C} \)-motivic spectrum we call \( \text{Mmf} \), which is not far out of the scope of the current article.

Proof of Proposition 3.28. Using Theorem 3.8, we can compute \( \tilde{\psi}^n \) for \( x \in \text{Free} \) in nonnegative degree after inverting 6, up to potential 3-torsion in degrees congruent to 40 modulo 72, hence our first exception. This computation with 6 inverted is then carried out using Theorem B and Lemma 3.26. For \( x \) in \( \text{Tors} \) in nonnegative degree, we can apply Proposition 3.15. For \( x \) in \( \text{Tors} \) in negative degree, we can look at (3.27). Finally, we have to deal with elements \( x \) of \( \text{Free} \) with negative degree. In this case, we work prime-by-prime:

1. If 3 does not divide \( n \), we use [Kon12, Diagram 13] to see that \( \tilde{\psi}^n(x) \in \text{Free} \) for \( |x| \) not congruent to \(-49 \) modulo 72 at the prime 3, hence our second exception.

2. If 2 does not divide \( n \), we use [Kon12, Diagram 27] to see that \( \tilde{\psi}^n(x) \in \text{Free} \) for \( |x| \) not congruent to \( k \) modulo 192, where \( k \) is an element of \( \{-37, -49, -57, -61, -73, -81, -97, -121, -133, -145, -153, -157, -169, -177\} \).

We can reduce this problematic set to that given in our third exception by applying Lemma 3.9 as follows: for \( k = -37 \) and \( z = \eta \), for \( k = -57 \) and \( z = \nu \), for \( k = -61 \) and \( z = \{\eta\Delta\} \), for \( k = -81 \) and \( z = \pi \), for \( k = -133 \) and \( z = \eta \), for \( k = -153 \) and \( z = \nu \), for \( k = -157 \) and \( z = \{\eta\Delta\} \), and for \( k = -177 \) and \( z = \pi \).

This finishes the proof.

3.4 A conjecture on dual endomorphisms

We end this article with a conjecture about the relation between endomorphisms of self-dual spectra, and their duals.
**Conjecture 3.30.** Let $R$ be an $E_1$-ring spectrum and write $A = \pi_0 R$. Suppose that there is a class $D \in \pi_{-d} R$ such that $D$ witnesses the Anderson self-duality of $R$. Then, for any endomorphism $F : R \to R$ of algebra objects in $hSp$ such that $F(D) = \lambda D$ for some $\lambda \in A$, the composites $F \circ F$ and $F \circ F$ are equivalent to multiplication by $\lambda$ on $\pi_* R$.

One can validate this conjecture in the following cases:

- For $KU[\frac{1}{n}]$ and $\psi^n$ one has $D = 1$ and $\lambda = 1$. In this case the above conjecture can be checked using (3.23).
- For $KO[\frac{1}{n}]$ and $\psi^n$ one has $D = n\psi^{-1}$ and $\lambda = n^{-2}$. In this case the above conjecture can be checked using (3.23) again. Furthermore, Heard–Stojanoska verified that in the stable homotopy category localised at the first Morava $K$-theory at the prime 2, there is a homotopy between $\psi^j$ and $\Sigma l^{-2} \psi^{j/l}$, where $l$ is a topological generator of $\mathbb{Z}_2^\wedge /\{\pm 1\}$; see [HS14, Lm.9.2].
- For $\text{Tmf}[\frac{1}{n}]$ and $\psi^n$, one has $D = [c_1^{-1} c_6 \Delta^{-1}]$ and $\lambda = n^{-10}$. In this case the above conjecture can be checked (in a range of degrees) using Theorem [3] and Proposition 3.28.
- For $\text{Tmf}(2)$ one can use Definition 3.3 to define $\psi^n$ for odd $n$, and then one has $D = \frac{1}{x_1 x_3}$ and $\lambda = n^{-4}$. Using the explicit computations of [Sto12 §7-9], one can verify the above conjecture. Indeed, as $\pi_* \text{Tmf}(2)$ is torsion-free (see [Sto12 §9]), one can immediately use a version of Lemma 3.26 for $\text{Tmf}(2)$.
- For the $\text{Tmf}_1(m)$ of Example 3.20 one has $\psi^n$ for $n$ coprime to $m$ by Definition 3.3 and $\lambda = n^{-\frac{\log m}{\log p}}$. In this case one can also verify the above conjecture in this case as well. Indeed, as $\pi_* \text{Tmf}_1(m)$ is torsion-free in this case (see [Mei19 §5.3]), we can again use a version of Lemma 3.26 for $\text{Tmf}_1(m)$.

**Remark 3.31.** Let us note a possible counter-example if we do not assume $F$ is multiplicative, mentioned to us by Lennart Meier. Consider $F = \text{id} + \psi^{-1}$ as an endomorphism of $KU$. Then $\lambda = 2$, however $F(u) = u - u = 0$ on the usual generator $u \in \pi_2 KU$, so Conjecture 3.30 cannot possibly hold in this case.

### A Appendix on the uniqueness of $\mathcal{O}^{\text{top}}$

The uniqueness of $\mathcal{O}^{\text{top}}$ is a well-known fact to experts; see [DFHH14, Rmk.12.1.4] and [Lur18a, Rmk.7.0.2]. Let us prove two simple uniqueness statements used in the proof of Theorem 2.1. Stronger uniqueness statements can be found in [Lur18a] Rmk.7.0.2.

**Proposition A.1.** The composition

$$h \mathcal{O}^{\text{top}} : \left( \text{Aff}^{\log \text{et}} / \mathcal{M}_{\text{Ell}} \right)^{\text{op}} \rightarrow \left( \text{LogDM}^{\log \text{et}} / \mathcal{M}_{\text{Ell}} \right)^{\text{op}} \rightarrow \text{CAlg} \rightarrow \text{CAlg}(hSp)$$

is characterised up equivalence by the fact that it defines a natural generalised elliptic cohomology theory; see Definition 2.7.
The following is essentially the argument found in [DFHH14, Rmk.12.1.4] by Behrens.

Proof. Write $\mathcal{O}$ for a presheaf of homotopy commutative ring spectra on $\mathcal{C} = \text{Aff}_{\mathcal{M}_{\text{Ell}}}^{\log\acute{e}t}$ defining a natural generalised cohomology theory. Given an object $E : \text{Spec} R \to \mathcal{M}_{\text{Ell}}$ of $\mathcal{C}$, the map $E$ is flat by Proposition [1.4] hence the composition

$$\text{Spec} R \xrightarrow{E} \mathcal{M}_{\text{Ell}} \xrightarrow{(\_)} \mathcal{M}_{\text{FG}}$$

is flat, and we see $\mathcal{O}(E)$ is therefore Landweber exact; see [Nau07, §5]. Furthermore, there are no phantom maps between Landweber exact cohomology theories (see [HS99, §2]), meaning the functor $\text{hSp} \to \text{CohomTh}$ from the stable homotopy category to the 1-category of cohomology theories is fully-faithful on the essential image of $\mathcal{O}$; the same also holds for the functor $\text{CAlg}(\text{hSp}) \to \text{MulCohomTh}$ from homotopy ring spectra to multiplicative cohomology theories. These facts imply that we can recover each $\mathcal{O}(E)$ in $\text{CAlg}(\text{hSp})$ from its homotopy groups $\pi_* \mathcal{O}(E)$ as an algebra over $\pi_* \text{MU}$ corresponding to the complex orientation of $\mathcal{O}(E)$, and even the whole diagram $\mathcal{O} : \mathcal{C}^{\text{op}} \to \text{CAlg}(\text{hSp})$ from its composition into $\text{MulCohomTh}_{\text{MU}^*(-)}$. As each $\mathcal{O}(E)$ is a natural generalised elliptic cohomology theory, there are natural equivalences of $\pi_* \text{MU}$-algebras

$$\pi_* \mathcal{O}(E) \simeq \omega_E^{\otimes *} \simeq \pi_* \mathcal{O}^{\text{top}}(R),$$

which induce a natural equivalence $\mathcal{O}(E)^*(\_) \simeq \mathcal{O}^{\text{top}}(R)^*(\_)$ between Landweber exact multiplicative cohomology theories. The naturality of such equivalences in $R$, and the fact that equivalences in $\text{CAlg}(\text{hSp})$ are detected on homotopy groups yields an equivalence of presheaves $\mathcal{O} \simeq \mathcal{O}^{\text{top}} : \mathcal{C}^{\text{op}} \to \text{CAlg}(\text{hSp})$.

The same proof works when we restrict $\mathcal{O}^{\text{top}}$ to the affines of $\text{Isog}_{\text{Ell}}^{\log\acute{e}t}$ using Remark [1.5] i.e., the same proof works with respect to greater functorality.

**Proposition A.2.** Writing $\text{Isog}_{\text{Ell}, \text{aff}}^{\log\acute{e}t}$ for the affine objects of $\text{Isog}_{\text{Ell}}^{\log\acute{e}t}$, then the composition

$$(\text{Isog}_{\text{Ell}, \text{aff}}^{\log\acute{e}t})^{\text{op}} \to (\text{Isog}_{\text{Ell}}^{\log\acute{e}t})^{\text{op}} \xrightarrow{\mathcal{O}^{\text{top}}} \text{CAlg} \to \text{CAlg}(\text{hSp})$$

is characterised up equivalence by the fact that it defines a natural generalised elliptic cohomology theory; see Definition [2.7].
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