Exact power series solutions of the structure equations of the general relativistic isotropic fluid stars with linear barotropic and polytropic equations of state

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Abstract Obtaining exact solutions of the spherically symmetric general relativistic gravitational field equations describing the interior structure of an isotropic fluid sphere is a long standing problem in theoretical and mathematical physics. The usual approach to this problem consists mainly in the numerical investigation of the Tolman-Oppenheimer-Volkoff and of the mass continuity equations, which describes the hydrostatic stability of the dense stars. In the present paper we introduce an alternative approach for the study of the relativistic fluid sphere, based on the relativistic mass equation, obtained by eliminating the energy density in the Tolman-Oppenheimer-Volkoff equation. Despite its apparent complexity, the relativistic mass equation can be solved exactly by using a power series representation for the mass, and the Cauchy convolution for infinite power series. We obtain exact series solutions for general relativistic dense astrophysical objects described by the linear barotropic and the polytropic equations of state, respectively. For the polytropic case we obtain the exact power series solution corresponding to arbitrary values of the polytropic index \( n \). The explicit form of the solution is presented for the polytropic index \( n = 1 \), and for the indexes \( n = 1/2 \) and \( n = 1/5 \), respectively. The case of \( n = 3 \) is also considered. In each case the exact power series solution is compared with the exact numerical solutions, which are reproduced by the power series solutions truncated to seven terms only. The power series representations of the geometric and physical properties of the linear barotropic and polytropic stars are also obtained.

Keywords general relativistic fluid sphere; exact power series solutions; linear barotropic equation of state; polytropic equation of state

1 Introduction

Karl Schwarzschild was the first scientist to find the exact solution of the Einstein’s gravitational field equations describing the interior of a constant density compact astrophysical object in 1916 (Schwarzschild 1916). The search for exact solutions describing static neutral, charged, isotropic or anisotropic stellar type configurations has continuously attracted the interests of physicists and mathematicians. A wide range of analytical solutions of the gravitational field equations describing the interior structure of the static fluid spheres were found in the past 100 years (for reviews of the interior solutions of Einstein’s gravitational field equations see (Kramer et al. 1980; Delgaty & Lake 1998; Finch & Skea 1998)). Unfortunately, among these many found solutions, there are very few exact interior solutions of the field equations satisfying the required general physical conditions. The criteria for physical acceptability of an interior solution can be formulated as follows (Delgaty & Lake 1998): 1) the solutions must be integrated from the regular origin of the stars. 2) the pressure and the energy density be positive definite at the origin of the stars. 3) the pressure vanishes at the surface of the stars. 4) the pressure and the energy density be monotonically decreasing to the surface of the stars for all radius. 5) causality requirement is that the speed of sound cannot be faster than the speed of light inside the stars. 6) the interior metric should
be joined continuously with the exterior Schwarzschild metric. Note that in the field of static spherically symmetric fluid spheres, an important bound on the mass-radius ratio for stable general relativistic stars was obtained in Buchdahl (1959), given by $2GM/c^2 R \leq 8/9$, where $M$ is the mass of the star as measured by its external gravitational field, and $R$ is the boundary radius of the star. The Buchdahl bound was generalized to include the presence of the cosmological constant as well as higher dimensions and electromagnetic fields in (Mak et al. 2006; Burikham et al. 2013; 2016a,b).

In recent years, many exact solutions of the field equations describing the interior structure of the fluid stars have been found by assuming the existence of the anisotropic pressure (Maurya et al. 2015; Bhat 2015; Dev & Gleisen 2004, 2003; Mak et al. 2002a; Mak & Harko 2002b, c, 2003). Since there are three independent field equations representing the stellar model, after adding the anisotropy parameter to the model, one has more mathematical freedom, and hence it is easier to solve the field equations analytically. However, it may be unphysical to assume the existence of anisotropic stresses. For instance, in a compact star, although the radial pressure vanishes at the surface of the star, one still could postulate the tangential pressure to exist. While the latter does not alter the spherical symmetry, it may create some streaming fluid motions (Riazi et al. 2015). Thus, in order to obtain a realistic description of stellar interiors in the following we assume that the matter content of dense general relativistic can be described thermodynamically by the energy density $\rho (r)$ and the isotropic pressure $p (r)$. Therefore, from a mathematical point of view the isotropic stellar models are governed by the three field equations for four unknowns: the $tt$ and $rr$ components of the metric tensor $\exp [\nu (r)]$ and $\exp [\lambda (r)]$, the energy density $\rho (r)$, and the pressure $p (r)$ respectively. Thus, the general relativistic stellar problem is an underdetermined one. In order to close the system of field equations an equation of state must be imposed. Very recently, the isotropic pressure equation was reformulated as a Riccati equation. By using the general integrability condition for the Riccati equation proposed in Mak & Harko (2012; 2013a), an exact non-singular solution of the interior field equations for a fluid star expressed in the form of infinite series was obtained in Mak & Harko (2013b). The astrophysical analysis indicates that this power series solution can be used as a realistic model for static general relativistic high density objects, for example neutron stars.

In 1939, Tolman rewrote the isotropic pressure equation as the exact differential form involving the metric tensor components, subsequently leading him to obtain the eight analytical solutions of the field equations (Tolman 1939). However, in order to ensure not to violate the causality condition, in the present paper, we do not follow Tolman’s approach. Alternatively, we need one more constraint to close the system of the equations and to satisfy the causality requirement. Hence in the present paper we assume first that the matter energy density $\rho (r)$ and the thermodynamic pressure $p (r)$ obey the linear barotropic equation of state given by

$$p (r) = \gamma \rho (r) c^2, \quad (1)$$

where $\gamma$ is the arbitrary constant satisfying the inequality $0 \leq \gamma \leq 1$. A static interior solution of the field equations in isotropic coordinates with the equation of state (1) was presented in Mak & Harko (2005). The structure and the stability of relativistic stars with the equation of state (1) were studied in Chavanis (2008). An exact analytical solution describing the interior of a charged strange quark star satisfying the MIT bag model equation of state $3p = \rho c^2 - 4B$, where $B$ is a constant, was found in Mak & Harko (2004) under the assumption of spherical symmetry and the existence of a one-parameter group of conformal motions.

Numerical solutions of Einstein’s field equation describing static, spherically symmetric conglomerations of a photon gas, forming so-called photon stars, were obtained in Schmidt & Homan (2000). The solutions imply a back reaction of the metric on the energy density of the photon gas. In Mitra & Glendenning (2010) it was pointed out that a class of objects called Radiation Pressure Supported Stars (RPSS) may exist even in Newtonian gravity. Such objects can also exist in standard general relativity, and they are called "Relativistic Radiation Pressure Supported Stars" (RRPSS). The formation of RRPSSs can take place during the continued gravitational collapse. Irrespective of the details of the contraction process, the trapped radiation flux should attain the corresponding Eddington value at sufficiently large $z >> 1$. On the basis of Einstein’s theory of relativity, the principle of causality, and Le Chatelier’s principle, in Rhoades & Ruffini (1974) it was established that the maximum mass of the equilibrium configuration of a neutron star cannot be larger than $3.2 M_\odot$. To obtain this result it was assumed that for high densities the equation of state of matter is given by $p = \rho c^2$. The absolute maximum mass of a neutron star provides a decisive method of observationally distinguishing neutron stars from black holes.

There is a long history in the context of physics and astrophysics for the study of the polytropic equation of state, defined as (Horedt 2004)

$$p (r) = K \rho^\Gamma (r) , \quad (2)$$
Here \( K \) is the polytropic constant, and the adiabatic index \( \Gamma \) is defined as \( \Gamma = 1 + 1/n \), where \( n \) is the polytropic index. Using the polytropic equation of state, the physicists have investigated the properties of the astrophysical objects in Newtonian gravity. Note that \( K \) is fixed in the degenerate system for instance a white dwarf or a neutron star and free in a non-degenerate system. The hydrostatic equilibrium structure of a polytropic star is governed for spherical symmetry by the Lane-Emden equation [Horedt 2004]

\[
\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y^n = 0, 
\]

where the dimensionless variables \( y \) and \( x \) are defined as

\[
x^2 = \frac{4\pi G \rho_c}{(1+n)K}, \quad y^n = \frac{\rho}{\rho_c}, 
\]

respectively where \( \rho_c \) and \( r \) are the central density and the radius of the star, \( G \) is the Newtonian gravitational constant, and \( y \) is the dimensionless gravitational potential. The Lane-Emden Eq. (3) was first introduced by [Land 1870] and later studied by [Emden 1907], [Fowler 1930] and [Milne 1930], respectively. In order to ensure the regularity of the solution at the center of the sphere, Eq. (5) must be solved with the initial conditions given by

\[
y(0) = 1, \quad \left( \frac{dy}{dx} \right)_{x=0} = 0. 
\]

It is well-known that the exact analytical solutions of Eq. (3) can only be obtained for \( n = 0, 1, 5 \) [Horedt 2004, Chandrasekhar 2012]. However, not all solutions of Eq. (3) for \( n = 5 \) were known until the year 2012, when all real solutions of Eq. (3) for \( n = 5 \) were obtained in terms of Jacobian and Weierstrass elliptic functions [Mabti 2012]. Two integrable classes of the Emden-Fowler equation of the type \( z'' = A \chi^{-\lambda-2} z^n \) for \( \lambda = \frac{n-1}{2} \), and \( \lambda = n + 1 \) were discussed in [Mancas & Rosu 2016]. By using particular solutions of the Emden-Fowler equations both classes were reduced to the form \( \tilde{v} + a \tilde{v}' + b(\nu - \nu^n) = 0 \), where \( a, b \) depend only on \( \lambda \) and \( n \), respectively. For both cases the solutions can be represented in a closed parametric form, with some values of \( n \) yielding Weierstrass elliptic solutions. It is generally accepted that the power series method is one of the powerful techniques in solving ordinary differential equations. Thus the Lane-Emden Eq. (3) was solved by using a power series method in [Mohan & Al-Bayati 1980, Rooburgh & Stockman 1994, Hunter 2001, Nouh 2004], where the convergence of the solutions was also studied.

The polytropic equation of state has also been adopted to study the interior structure of the fluid stars in the framework of general relativity [Tooper 1964]. The solution of the gravitational field equations for relativistic static spherically symmetric stars in minimal dilatonic gravity using the polytropic equation of state was presented in [Fiziev & Marinov 2013]. The general formalism to model polytropic general relativistic stars with the anisotropic pressure was considered in [Herrera & Barreto 2013], and its stellar applications were also discussed. By solving the Tolman-Oppenheimer-Volkoff (TOV) equation, a class of compact stars made of a charged perfect fluid with the polytropic equation of state was analyzed in [Arbani et al. 2013]. Exact solutions of the Einstein-Maxwell equation with the anisotropic pressure and the electromagnetic field in the presence of the polytropic equation of state were obtained in [Mafa Takisa & Maharaj 2013]. Charged polytropic stars, and a generalization of the Lane-Emden equation was investigated in [Picanco et al. 2004]. Using the power series methodology, a new analytical solution of the TOV equation for polytropic stars was presented in [Nouh & Saeed 2013].

The divergence and the convergence of the power series solutions for the different values of the polytropic index \( n \) were also discussed. The gravitational field equations for the static spherically symmetric perfect fluid models with the polytropic equation of state can be written as two complementary 3 dimensional regular systems of ordinary differential equations on compact state space. Due to the highly nonlinear structure of the systems, it is difficult to solve them exactly, and thus they were analyzed numerically and qualitatively using the theory of dynamical systems in [Nilsson & Ugga 2004, Boehmmer & Harko 2010]. The three-dimensional perfect fluid stars with the polytropic equation of state, matched to the exterior three-dimensional black hole geometry of Bañados, Teitelboim and Zanelli were considered in [Sil 1999]. A new class of exact solutions for a generic polytropic index was found, and analyzed. The structure of the relativistic polytropic stars and the stellar stability analysis embedded in a chameleon scalar field was discussed in [Fiziev & Marinov 2015]. In [Lai & Xu 2004], a polytropic quark star model was suggested in order to establish a general framework in which theoretical quark star models could be tested by the astrophysical observations. Spherically symmetric static matter configurations with the polytropic equation of state for a class of \( f(R) \) models in Palatini formalism were investigated in [Olmo 2008], and it was shown that the surface singularities are not physical in the case of Planck scale modified Lagrangians.

It is the purpose of the present paper to study the interior structure of the general relativistic fluid stars
with the linear barotropic and the polytropic equations of state, and to obtain exact power series solutions of the corresponding equations. As a first step in our study we introduce the basic equation describing the interior mass profile of a relativistic star, and which we call the relativistic mass equation. This equation is obtained by eliminating the energy density between the mass continuity equation and the hydrostatic equilibrium equation. Despite its apparent mathematical complexity, the relativistic mass equation can be solved exactly for both linear barotropic and polytropic equations of state, by looking to its exact solutions as represented in the form of power series. In this way we obtain closed form representations of the coefficients we use the presented in the form of power series. In order to obtain closed form representations of the coefficients we use the Cauchy convolution of the power series. In this way we obtain the exact series solutions for relativistic spheres described by linear barotropic equations of state with arbitrary polytropic index \( n \), and for the polytropic equation of state described by arbitrary \( n \), we investigated independently, and the corresponding power series solution is also obtained. We compare the truncated power series solutions containing seven terms only with the exact numerical solution of the TOV and mass continuity equations. In all considered cases we find an excellent agreement between the power series solution, and the numerical one.

The present paper is organized as follows. The gravitational field equations, their dimensionless formulation and the basic relativistic mass equation are presented in Section 2. The definition of the Cauchy convolution for infinite power series is also introduced. The non-singular power series solution for fluid spheres described by a linear barotropic equation of state is presented in Section 3. The comparison between the exact numerical solution, and the comparison with the exact numerical solution is also performed. The case of the arbitrary polytropic index \( n \) is considered in Section 4. The power series solutions are compared with the exact numerical solutions for the cases \( n = 1/2, n = 1/5 \) and \( n = 3 \), respectively. We discuss our results and conclude our paper in Section 5. The power series solutions containing seven terms only are derived in Section 4.

2 The gravitational structure equations, dimensionless variables, and the relativistic mass equation

We start our study by writing down the gravitational field equations describing a static spherically symmetric general relativistic star, and presenting the corresponding structure equations for stellar type objects. In order to simplify the mathematical and the numerical formalism, we rewrite the basic equations in a set of dimensionless variables, and we obtain the basic nonlinear second order differential equation describing the mass distribution inside the relativistic stars.

### 2.1 Gravitational field equations and structure equations for compact spherically symmetric objects

The static and spherically symmetric metric for describing a gravitational relativistic sphere in Schwarzschild coordinates is given by the line element

\[
ds^2 = e^\gamma c^2 dt^2 - e^\lambda dr^2 - r^2 d\Omega^2, \tag{6}\]

where the metric components \( \nu \) and \( \lambda \) are function of radial coordinate \( r \), for simplicity we have denoted the quantity \( d\Omega^2 \) as \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). The Einstein’s gravitational field equations are

\[
R_i^k - \frac{1}{2} R \delta_i^k = \frac{8\pi G}{c^4} T_i^k, \tag{7}\]

where \( G \) is the Newtonian gravitational constant, and \( c \) is the speed of light, respectively. For an isotropic spherically symmetric matter distribution the components of the energy-momentum tensor are of the form

\[
T_i^k = (\rho c^2 + p) u_i u^k - p \delta_i^k, \tag{8}\]

where \( u^i \) is the four velocity, given by \( u^i = \delta_0^i \), and the quantities \( \rho (r) \) and \( p (r) \) are the energy density and the isotropic pressure, respectively. For any physically acceptable stellar models, we require that the energy density and the pressure must be positive and finite at all points inside the fluid spheres. By inserting Eqs. (8) and (9) into Eq. (7), the latter equations yield the Einstein’s gravitational field equations describing the interior of a static fluid sphere as (Landau & Lifshitz 1975)

\[
-\frac{1}{r^2} \frac{d}{dr} \left( re^{-\lambda} \right) + \frac{1}{r^2} = \frac{8\pi G}{c^2} \rho (r), \tag{9}\]

\[
e^{-\lambda} \frac{d\nu}{r} + e^{-\lambda} \frac{1}{r^2} = \frac{8\pi G}{c^4} p (r), \tag{10}\]

\[
e^{-\lambda} \left[ \frac{1}{2} \frac{d^2 \nu}{dr^2} + \frac{1}{4} \left( \frac{d\nu}{dr} \right)^2 - \frac{1}{4} \frac{d\nu}{dr} \frac{d\lambda}{dr} + \frac{1}{2r} \left( \frac{d\nu}{dr} - \frac{d\lambda}{dr} \right) \right] = \frac{8\pi G}{c^4} p (r). \tag{11}\]
The conservation of the energy-momentum tensor gives the relation
\[
dv{v}{r} = -\frac{2}{\rho(r)c^2 + p(r)} \frac{dp}{dr}. \tag{12}
\]
Eq. (9) can be immediately integrated to give
\[
e^{-\lambda} = 1 - \frac{2GM(r)}{c^2r}, \tag{13}
\]
where \( M(r) \) is the mass inside radius \( r \). An alternative description of the interior of the star can be given in terms of the TOV and of the mass continuity equations, which can be written as
\[
\frac{dp}{dr} = -\left(\frac{G}{c^2}\right) \left(\rho c^2 + p\right) \left(4\pi/c^2\right) r^3 + M, \tag{14}
\]
\[
\frac{dM}{dr} = 4\pi \rho r^2, \tag{15}
\]
respectively. The system of the structure equations of the star must be integrated with the initial and boundary conditions
\[
M(0) = 0, \ p(R) = 0, \tag{16}
\]
where \( R \) is the radius of the star, and together with an equation of state \( p = p(\rho) \).

2.2 Dimensionless form of the structure equations

By introducing a set of dimensionless variables \( \eta \) (dimensionless radial coordinate), \( \epsilon(\eta) \) (energy density), \( P(\eta) \) (pressure) and \( m(\eta) \) (mass), by means of the transformations
\[
\eta = R, \ \rho = \rho_c(\epsilon), \ p = \rho_c c^2 P(\eta), \ M = M^* m(\eta), \tag{17}
\]
where \( \rho_c \) is the central density, the TOV and the mass continuity equations take the form
\[
\frac{dP}{d\eta} = -\frac{\epsilon(\eta) + P(\eta)}{\eta^2 [1 - 2a m(\eta)/\eta]}, \tag{18}
\]
\[
\frac{dm}{d\eta} = \eta^2 \epsilon(\eta), \tag{19}
\]
respectively, where we have fixed the constants \( a \) and \( M^* \) by the relations
\[
a = \frac{4\pi G \rho_c R^2}{c^2}, \ M^* = 4\pi \rho_c R^3. \tag{20}
\]
In order to close the above system of equations the dimensionless form \( P = P(\epsilon) \) of the matter equation of state must also be given. Then the coupled system of Eqs. (18) and (19) must be solved with the initial and boundary conditions \( \epsilon(0) = 1, \ m(0) = 0, \) and \( \epsilon(1) = \epsilon_S \), where \( \epsilon_S \) is the value of the surface density of the star, and \( \eta = 1 \) is the value of the dimensionless radial coordinate \( \eta \) on the star’s surface. As a function of the parameter \( a \) the radius \( R \) and the total mass \( M_S \) of the star are given by the relations
\[
R = \sqrt{\frac{3}{a}} \sqrt{\frac{c}{4\pi G \rho_c}} = \sqrt{\frac{\pi}{10.3622} \frac{\rho_c}{10^{15} \text{g/cm}^3}} \ \text{km}, \tag{21}
\]
\[
M_S = a^{3/2} \frac{c^3}{\sqrt{4\pi G \rho_c}} m(1) = a^{3/2} \times 6.9910 \times \left(\frac{\rho_c}{10^{15} \text{g/cm}^3}\right)^{1/2} \times m(1) \ M_\odot, \tag{22}
\]
where the quantity \( M_\odot \) is the mass of the sun. For the mass-radius ratio of the star, we obtain
\[
\frac{GM_s}{c^2 R} = am(1). \tag{23}
\]

2.3 The relativistic mass equation and the Cauchy convolution

By eliminating the energy density \( \epsilon(\eta) \) between the equations (19) and (18) we obtain the following second order differential equations, which in the following we will call the relativistic mass equation,
\[
\eta \frac{d^2 m(\eta)}{d\eta^2} - 2 \frac{dm(\eta)}{d\eta} + a\left[ \eta^3 P \left(\frac{m'(\eta)}{\eta^2}\right) + m(\eta) \right] \left[ m'(\eta) + \eta^2 P \left(\frac{m'(\eta)}{\eta^2}\right) \right] = 0, \tag{24}
\]
where we have used the simple mathematical relation \( dP/d\eta = (dP/d\epsilon) (d\epsilon/d\eta) \), and we have denoted
\[
P' \left(\frac{m'(\eta)}{\eta^2}\right) = \frac{dP(\epsilon)}{d\epsilon} \big|_{\epsilon=m'(\eta)/\eta^2}.
\]
Equivalently, the relativistic mass equation takes the form
\[
\left[ 1 - \frac{2am(\eta)}{\eta} \right] \left[ \eta^2 \left(\frac{d^2 m(\eta)}{d\eta^2} - 2 \frac{dm(\eta)}{d\eta} \right) P' \left(\frac{m'(\eta)}{\eta^2}\right) + a\eta^2 \left(\frac{m'(\eta)}{\eta^2} + P \left(\frac{m'(\eta)}{\eta^2}\right)\right) \times \left( \frac{m(\eta)}{\eta} + \eta^2 P \left(\frac{m'(\eta)}{\eta^2}\right) \right) \right] = 0. \tag{25}
\]
Eq. (21) must be integrated with the initial conditions \( m(0) = 0 \), and \( m'(0) = 0 \), respectively, and together with the equation of state of the matter, \( P = P(\epsilon) = P \left( \frac{m'(\eta)}{\eta^2} \right) \). It is important to note that the point \( \eta = 0 \) is an ordinary point for Eq. (20). This is due to the fact that all coefficients in the equation take finite values at the origin. Thus, \( \lim_{\eta \to 0} m(\eta)/\eta = \lim_{\eta \to 0} \epsilon(\eta)\eta^3/\eta = 0 \), and \( \lim_{\eta \to 0} m'(\eta)/\eta^2 = \epsilon(0) = 1 \), respectively. Since the thermodynamic parameters of the star must be finite at the origin, it follows that \( P \left( \frac{m'(\eta)}{\eta^2} \right) \) and \( P' \left( \frac{m'(\eta)}{\eta^2} \right) \) are all finite at \( \eta = 0 \).

In the next Sections we will investigate the possibility of obtaining exact power series solutions of Eq. (21) for the linear barotropic and the polytropic equations of state. In order to obtain our solutions we will use the Cauchy convolution of the power series, defined as

\[
\text{Definition. Let}
\]

\[
f_1 = \sum_{i_1=0}^{\infty} a_{1,i_1} x^{i_1}, f_2 = \sum_{i_2=0}^{\infty} a_{2,i_2} x^{i_2},
\]

(26)

be \( s \) convergent power series, \( s \geq 2 \). Then we define the Cauchy product (convolution) of the \( s \) power series, \( s \geq 2 \), as

\[
f_1 \circ f_2 = \left( \sum_{i_1=0}^{\infty} a_{1,i_1} x^{i_1} \right) \left( \sum_{i_2=0}^{\infty} a_{2,i_2} x^{i_2} \right) = \sum_{i_1,i_2=0}^{\infty} a_{1,i_1} a_{2,i_2} x^{i_1+i_2} = \sum_{j_2=0}^{\infty} \left( \sum_{i_1=0}^{j_2} a_{1,i_1} a_{2,j_2-i_1} \right) x^{j_2} = \sum_{j_1=0}^{\infty} A_{2,j_2} x^{j_2},
\]

(27)

\[
A_{2,j_2} = \sum_{i_1=0}^{j_2} a_{1,i_1} a_{2,j_2-i_1},
\]

(28)

\[
f_1 \circ f_2 \circ f_3 = \left( \sum_{i_1=0}^{\infty} a_{1,i_1} x^{i_1} \right) \left( \sum_{i_2=0}^{\infty} a_{2,i_2} x^{i_2} \right) \left( \sum_{i_3=0}^{\infty} a_{3,i_3} x^{i_3} \right) = \sum_{i_1,i_2,i_3=0}^{\infty} a_{1,i_1} a_{2,i_2} a_{3,i_3} x^{i_1+i_2+i_3} = \sum_{j_3=0}^{\infty} A_{3,j_3} x^{j_3},
\]

(29)

(30)

(31)

(32)

3 Exact series solution of the relativistic mass equation for a linear barotropic fluid

As a first example of an exact power series solution of the relativistic mass Eq. (21) we consider the case of the linear barotropic equation of state \( p = \gamma \rho c^2 \). Using Eq. (17), we rewrite the equation of state (11) in the form

\[
P(\eta) = \gamma \epsilon(\eta), \gamma = \text{constant}, \gamma \in [0,1].
\]

(33)

Then the TOV Eq. (15) becomes

\[
\frac{d\epsilon(\eta)}{d\eta} = \gamma + 1 \frac{\epsilon(\eta)}{\gamma} \frac{\gamma \epsilon(\eta) \eta^3 + m(\eta)}{\eta^2 [1 - 2am(\eta)/\eta]}.
\]

(34)
3.1 Exact power series solution of the relativistic mass equation

By using the linear barotropic equation of state the relativistic mass Eq. (24) takes the form

\[ \eta \left(1 - \frac{2am}{\eta}\right) \frac{d^2m}{d\eta^2} + \left[ a \left(\frac{1}{\gamma} + 5\right) \frac{m}{\eta} - 2\right] \frac{dm}{d\eta} + a (1 + \gamma) \left(\frac{dm}{d\eta}\right)^2 = 0, \quad (35) \]

or, equivalently,

\[ \frac{d^2m}{d\eta^2} - 2 \frac{dm}{d\eta} - 2am \frac{d^2m}{d\eta^2} + a \frac{m}{\eta} \frac{dm}{d\eta} + \beta \left(\frac{dm}{d\eta}\right)^2 = 0, \quad (36) \]

where for simplicity we have introduced the coefficients \( \alpha \) and \( \beta \) defined as \( \alpha = a (1/\gamma + 5) \) and \( \beta = a (1 + \gamma) \), respectively.

Eqs. (35) or (36) must be solved with the initial conditions \( m(0) = 0 \), and \( (dm/d\eta)|_{\eta=0} = 0 \). Note that Eqs. (35) and (36) are not in the autonomous form, that is, the coefficients of the derivative \( dm/d\eta \) depend both on the mass function \( m(\eta) \) and the dimensionless radius \( \eta \). In order to solve Eq. (35) we will look for exact power series solution of the equation. Therefore we can state the following

**Theorem 1.** The relativistic mass equation (50) describing the interior of a star with matter content described by a linear barotropic equation of state \( P(\eta) = \gamma \epsilon(\eta) \), \( \gamma = \) constant, has an exact non-singular convergent power series solution of the form

\[ m(\eta) = \sum_{n=1}^{\infty} c_{2n+1} \eta^{2n+1}, \eta \leq 1. \quad (37) \]

with the coefficients \( c_{2n+1} \) obtained from the recursive relation

\[ c_{2n+1} = -\frac{a}{2 (n - 1) (2n + 1) \gamma} \times \sum_{i=1}^{n-1} (2n - 2i + 1) \left[ 2 \gamma (\gamma + 3) i - 4 \gamma n + \gamma^2 + 6 \gamma + 1 \right] c_{2i+1} c_{2n-2i+1}, n \geq 2. \quad (38) \]

**Proof.** In the following we will look for a convergent power series solution of Eq. (35), by choosing \( m(\eta) \) in the form given by Eq. (37). Then it is easy to show the relations \( \frac{dm}{d\eta} = \sum_{n=1}^{\infty} (2n+1) c_{2n+1} \eta^{2n} \), and \( \frac{d^2m}{d\eta^2} = 2 \sum_{n=1}^{\infty} n (2n+1) c_{2n+1} \eta^{2n-1} \), respectively. For the product of two power series we will use the Cauchy convolution, so that

\[ \left( \sum_{i=0}^{\infty} a_i \eta^i \right) \left( \sum_{j=0}^{\infty} b_j \eta^j \right) = \sum_{i,j=0}^{\infty} a_i b_j \eta^{i+j} = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} a_i b_{n-i} \right) \eta^n, \quad (39) \]

Thus,

\[ \left( \frac{dm}{d\eta} \right)^2 = \sum_{i,j=1}^{\infty} (2i + 1) (2j + 1) c_{2i+1} c_{2j+1} \eta^{2i+2j} = \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{n} (2i + 1) (2n - 2i + 1) c_{2i+1} c_{2n-2i+1} \right] \eta^{2n}, \quad (40) \]

\[ \frac{m}{\eta} \frac{dm}{d\eta} = \sum_{i,j=1}^{\infty} c_{2i+1} (2j + 1) c_{2j+1} \eta^{2i+2j} = \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{n} (2n - 2i + 1) c_{2i+1} c_{2n-2i+1} \right] \eta^{2n} \quad, \quad (41) \]

\[ \frac{m}{\eta} \frac{d^2m}{d\eta^2} = \sum_{i,j=1}^{\infty} c_{2i+1} c_{2j+1} \eta^{2i+2j} = \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{n} 2 (n - i) (2n - 2i + 1) c_{2i+1} c_{2n-2i+1} \right] \eta^{2n}, \quad (42) \]

Hence by substituting these results into Eq. (35) gives immediately

\[ \sum_{n=1}^{\infty} \left\{ 2 (n - 1) (2n + 1) c_{2n+1} + \sum_{i=1}^{n} \left[ -4a (n - i) (2n - 2i + 1) c_{2i+1} c_{2n-2i+1} + \alpha (2n - 2i + 1) c_{2i+1} c_{2n-2i+1} + \beta (2i + 1) \times (2n - 2i + 1) c_{2i+1} c_{2n-2i+1} \right] \right\} \eta^{2n} = 0, \quad (43) \]

where we have transformed all the products of the power series by using the Cauchy convolution. After using the definitions of \( \alpha \) and \( \beta \), we obtain

\[ \sum_{n=1}^{\infty} \left\{ 2 (n - 1) (2n + 1) \gamma c_{2n+1} + a \sum_{i=1}^{n} (2n - 2i + 1) \left[ 2 \gamma (\gamma + 3) i + \gamma^2 + 2 \gamma (3 - 2n) + 1 \right] c_{2i+1} c_{2n-2i+1} \right\} \eta^{2n} = 0, n \geq 2. \quad (44) \]
By solving the above equation for the coefficients $c_{2n+1}$ gives the recursive relation \(38\) for the coefficients of the series representation of the mass function. This ends the proof of Theorem 1.

For the values of the coefficients $c_{2n+1}$ we obtain the following explicit expressions

$$c_5 = -\frac{3a(\gamma + 1)(5\gamma + 1)}{10\gamma} c_3,$$

$$c_7 = \frac{3a^2(\gamma + 1)(3\gamma + 1)(5\gamma + 2 + 3\gamma + 1)}{140\gamma^2} c_3,$$

$$c_9 = -\frac{a^3(\gamma + 1)(3\gamma + 1)}{2520\gamma^3} \left(945\gamma^4 + 864\gamma^3 + 618\gamma^2 + 200\gamma + 61\right) c_3,$$

$$c_{11} = \frac{a^4(\gamma + 1)(3\gamma + 1)}{18480\gamma^4} \left(85050\gamma^6 + 91665\gamma^5 + 80892\gamma^4 + 38832\gamma^3 + 17936\gamma^2 + 42393\gamma + 1258\right) c_3,$$

$$c_{13} = -\frac{a^5(\gamma + 1)(3\gamma + 1)}{12012000\gamma^5} \left(7016625\gamma^8 + 8057475\gamma^7 + 7978905\gamma^6 + 4456683\gamma^5 + 2486451\gamma^4 + 839697\gamma^3 + 346075\gamma^2 + 61953\gamma + 22952\right) c_3,$$

$$c_{15} = \frac{a^6(\gamma + 1)(3\gamma + 1)}{5045040000\gamma^6} \left(3831077250\gamma^{10} + 4428596025\gamma^9 + 4702427055\gamma^8 + 2757559491\gamma^7 + 1705375683\gamma^6 + 636216069\gamma^5 + 311382965\gamma^4 + 72456873\gamma^3 + 36302375\gamma^2 + 3752022\gamma + 2703152\right) c_3,$$

By estimating the energy density at the center of the star $\eta = 0$ gives $\epsilon(0) = 1 = 3c_3$, which fixes the value of the constant $c_3$ as

$$c_3 = \frac{1}{3}.$$ 

By inserting Eq. \(37\) into Eq. \(13\), we obtain the metric tensor component $e^{-\lambda}$ as

$$e^{-\lambda(\eta)} = 1 - 2a \frac{m(\eta)}{\eta} = 1 - 2a \sum_{n=1}^{\infty} c_{2n+1}\eta^{2n}.$$ 

By substituting Eq. \(38\) into Eq. \(12\), then the latter equation can be integrated to yield

$$e^{\nu(\eta)} = e^{\nu(0)} \left[\frac{\epsilon(\eta)}{\epsilon(0)}\right]^{-\frac{2}{1+\gamma}},$$

where $e^{\nu(0)}$ is the value of the metric coefficient at the center of the star, and $\epsilon(0) = 1$. With the help of Eq. \(51\) we rewrite Eq. \(54\) in the form

$$e^{\nu(\eta)} = e^{\nu(0)} \left[\sum_{n=1}^{\infty} (2n + 1) c_{2n+1}\eta^{2n-2}\right]^{-\frac{2}{1+\gamma}}, \gamma \neq -1.$$ 

Thus the interior line element for a fluid sphere satisfying a linear barotropic equation of state takes the form

$$ds^2 = c^2 e^{\nu(0)} \left[\sum_{n=1}^{\infty} (2n + 1) c_{2n+1}\eta^{2n-2}\right]^{-\frac{2}{1+\gamma}} \frac{dt^2}{1 - 2a \sum_{n=1}^{\infty} c_{2n+1}\eta^{2n}} - r^2 d\Omega^2, \gamma \neq -1.$$ 

At the surface of the barotropic matter distribution $\epsilon(1) = \epsilon_S = \rho_S/\rho_c = \text{constant}$, where $\rho_S = \rho(R)$ is the density of the barotropic fluid distribution on the boundary separating the two phases. Thus we obtain $e^{\nu(R)} = e^{\nu(0)} \left(\rho_S/\rho_c\right)^{-2\gamma/(1+\gamma)}$. Moreover, it follows that on the boundary $\eta = 1$ of the barotropic component the coefficients $c_{2n+1}$ must satisfy the condition

$$\sum_{n=1}^{\infty} (2n + 1) c_{2n+1} = \epsilon_S.$$ 

For the $e^{-\lambda}$ metric component at the star surface we obtain $e^{-\lambda} = 1 - 2GM_S/c^2R$.

In order to test the accuracy of our power series solution we consider the cases $\gamma = 1/3$ and $\gamma = 1$, respectively, corresponding to the radiation fluid ($\gamma = 1/3$), and stiff fluid ($\gamma = 1$) equations of state, respectively. The comparisons between the series solution of the

Using Eqs. \(19\) and \(37\), we obtain the energy density of the matter inside a general relativistic star described by a linear barotropic equation of state as

$$\epsilon(\eta) = \sum_{n=1}^{\infty} (2n + 1) c_{2n+1}\eta^{2n-2}.$$ 

(51)
Variation of the dimensionless mass $m(\eta)$ (left figure) and dimensionless energy density $\epsilon(\eta)$ for the radiation fluid star, with $\gamma = 1/3$, and for the stiff fluid equation of state star, with $\gamma = 1$, respectively. The dashed curve represents the numerical solution of the TOV and continuity Eqs. (15) and (19), while the solid and dotted curves represent the power series solution of the relativistic mass equation, truncated to $n = 7$. For the radiation fluid star the surface density is fixed at $\epsilon_S = 0.351165$, corresponding to $a = 0.9876$. For the stiff fluid star $\epsilon_S = 0.441812$, and $a = 0.76389$.

TOV and continuity equations, obtained via the solution of the relativistic mass equation, and the exact numerical solution, computed by numerically integrating the coupled system of Eqs. (15) and (19) is represented in Fig. 1.

To numerically integrate Eqs. (15) and (19) we have used the NDSolve command of the Mathematica software [Wolfram 2003], which finds solutions iteratively, and by using the default setting of Automatic for AccuracyGoal and PrecisionGoal. The power series solution has been truncated to seven terms only. Overall, even with this small number of terms, the power series solution gives a good approximation of the exact solution obtained by numerical integration of the structure equations of the linear barotropic relativistic star.

For the radiation fluid star we have adopted the values $a = 0.9876$, giving a surface density $\epsilon_S = \epsilon(1) = 0.351165$, with the total dimensionless mass obtained as $m(1) = 0.175377$. The physical parameters of this stellar model are given by

$$R = 10.2978 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ km}, \quad (57)$$

$$M_S = 1.2033 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ M}_\odot. \quad (58)$$

For the stiff fluid star, with $\gamma = 1$, $a = 0.76389$, giving a surface density of $\epsilon_S = 0.441812$, and a total dimensionless mass of $m(1) = 0.201045$. The global parameters of this high density star model can be obtained as

$$R = 9.05665 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ km}, \quad (59)$$

$$M_S = 0.93838 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ M}_\odot. \quad (60)$$

3.2 Matching with a constant density atmosphere

Now we match the interior metric of the fluid sphere with matter content satisfying a linear barotropic equation of state to the metric corresponding to a constant density atmosphere, with matter density $\rho = \rho_S$ is constant, and pressure $p_c(r)$, respectively. This metric is matched on the star’s surface with the exterior Schwarzschild metric, given by

$$ds^2 = c^2 \left( 1 - \frac{2GM_{tot}}{c^2 R_{tot}} \right) dt^2 - \frac{1}{1 - 2GM_{tot}/c^2 R_{tot}} dr^2 - r^2 d\Omega^2, \quad (61)$$

where $M_{tot} = M_S + M_c$ and $R_{tot} = R + R_c$ are the total mass and radius of the star, including both the linear barotropic and the constant density components. We assume that the metric functions $g_{tt}$, $g_{rr}$ and $\partial g_{tt}/\partial r$ are all continuous at both the contact region between the barotropic and constant density matter, as well as at the vacuum boundary surface of the star. In the constant density region we obtain first

$$m_c(r) = 4\pi \rho_S \int_{R}^{r} r'^2 dr' = \frac{4\pi \rho_S}{3} \left(r^3 - R^3\right), \quad \frac{R}{R_t} \leq r \leq R_{tot}, \quad (62)$$

$$e^{-\lambda(r)} = 1 - \frac{2G \left[M_S + 4\pi \rho_S \left(r^3 - R^3\right)/3\right]}{c^2 r}, \quad \frac{R}{R_t} \leq r \leq R_{tot}. \quad (63)$$
The continuity of $\lambda$ at $r = R_{\text{tot}}$, $e^{-\lambda}(R) = 1 - 2GM_{\text{tot}}/c^2 R_{\text{tot}}$ fixes the value of the surface density of the linear barotropic region as

$$\rho_S = \frac{3(M_{\text{tot}} - M_S)}{4\pi(R_{\text{tot}}^2 - R^3)}.$$  \hfill (64)

For the total mass of the star $M_{\text{tot}}$ from Eqs. (63) and (65) we obtain

$$M_{\text{tot}} = M_S + \frac{4\pi}{3} \rho_S (R_{\text{tot}}^3 - R^3).$$  \hfill (65)

In the constant density region Eq. (12) can be integrated to give

$$e^{\nu(r)} = \frac{C}{[\rho_S c^2 + p_c (r)]^2}, \quad R \leq r \leq R_{\text{tot}}.$$  \hfill (66)

For $r = R$ we have $p_c(R) = \gamma \rho_S c^2$, $e^{\nu(R)} = 1 - 2GM_S/c^2 R$, giving for the integration constant $C$ the value $C = (1 - 2GM_S/c^2 R) (1 + \gamma)^2 (\rho_S c^2)^2$, respectively. Thus we obtain

$$e^{\nu(r)} = \left(1 - \frac{2GM_S}{c^2 R}\right)(1 + \gamma)^2 \left(\frac{\rho_S c^2}{\rho_S c^2 + p_c (r)}\right)^2, \quad R \leq r \leq R_{\text{tot}}.$$  \hfill (67)

On the surface of the star $p_c(R_{\text{tot}}) = 0$, and therefore

$$\frac{2GM_{\text{tot}}}{c^2 R_{\text{tot}}} = 1 - (1 + \gamma)^2 \left(1 - \frac{2GM_S}{c^2 R}\right).$$  \hfill (68)

Eq. (68) gives the total mass-total radius ratio of the star, once the mass, radius and equation of state of the core described by a linear barotropic equation of state are known.

In the next Section, we shall consider power series solutions of the relativistic mass equation for polytropic fluids.

4 Exact power series solutions of the relativistic mass equation for polytropic stars

Polytropic models play an important role in the galactic dynamics and in the theory of stellar configuration and evolution (Chandrasekhar 2012). In particular, polytropic models with $n = 1$ can be used to model Bose-Einstein Condensate dark matter (Boehmer & Harko 2007), and Bose-Einstein Condensate stars (Chavanis & Harko 2012), respectively. For a polytropic system, the interior structure of the compact objects can be described by an equation of state of the form

$$p(r) = K \rho^{1 + \frac{1}{n}}(r),$$  \hfill (69)

where $p(r)$ and $\rho(r)$ are the pressure and the energy density respectively, while $K$ and $n$ are constants. The constant $n$ is called the polytropic index. In galactic dynamics $n > 1/2$, and no polytropic stellar system can be homogeneous (Binney & Tremaine 1987). In the case of the theory of stellar structure and evolution, in general, $n$ ranges from 0 to $\infty$ (Chandrasekhar 2012; Kippenhahn & Weigert 1990). Similarly to the previous Section, with the help of Eqs. (63) and (67), we obtain the polytropic equation of state in a dimensionless form given by

$$P(\eta) = k \epsilon^{1+1/n}(\eta),$$  \hfill (70)

where we have denoted the constant $k$ as $k = K \rho^{1/n}_{c}/c^2$.

By inserting Eq. (70) into Eq. (12), then the latter can be integrated to give

$$e^{\nu(\eta)} = e^{\nu(0)} \left[\frac{1 + k \epsilon^{1/n}(\eta)}{1 + k}\right]^{-2(1+n)}.$$  \hfill (71)

On the surface of the polytropic star, corresponding to $\eta = 1$, the metric tensor coefficient (71) must be matched with the Schwarzschild line element, thus giving

$$e^{\nu(1)} = e^{\nu(0)} \left[\frac{1 + k \epsilon^{1/n}(\eta)}{1 + k}\right]^{-2(1+n)} = 1 - \frac{2GM_S}{c^2 R},$$  \hfill (72)

where $M_S$ and $R$ are the mass and the radius of the star, respectively. A vanishing surface energy density $\epsilon(1) = 0$ would give, for $n > 0$, the value of the metric tensor coefficient at the center of the star as

$$e^{\nu(0)} = \frac{1}{(1 + k)^{2(1+n)}} \left(1 - \frac{2GM_S}{c^2 R}\right).$$  \hfill (73)

By assuming that at the surface of polytropic star the density (and the pressure) does not vanish, and that $\epsilon(1) = \epsilon_S \neq 0$, the surface density is determined by the equation

$$\epsilon_S^{1/n} = \frac{1}{k} \left\{\left[\frac{1 + k}{e^{\nu(0)}} \left(1 - \frac{2GM_S}{c^2 R}\right)\right]^{1/n} - 1\right\}.$$  \hfill (74)

By setting again $M^*$ and $a$ as $M^* = 4\pi \rho_c R^3$ and $a = 4\pi G \rho_c R^2/c^2$, respectively, and with the help of Eq. (70), the TOV Eq. (18) gives a differential equation for $\epsilon(\eta)$

$$\frac{d\epsilon}{d\eta} = - \frac{a\epsilon(\eta)}{k (n + 1)} \times \frac{[e^{-1/n}(\eta) + k] \left(k\eta^3 \epsilon^{1+1/n}(\eta) + m(\eta)\right)}{\eta^2 \left[1 - 2am(\eta)/\eta\right]}.$$  \hfill (75)
By inserting Eq. (19) into Eq. (75), the latter gives the differential equation for the relativistic mass function $m(\eta)$ as

$$
\eta^2 \left[ 1 - 2a \frac{m(\eta)}{\eta} \right] \frac{d^2 m}{d\eta^2} + \frac{an}{n+1} \eta^k \left( \frac{1}{\eta^2} \frac{dm}{d\eta} \right) + \frac{1}{\eta^2} \left( \frac{d}{d\eta} \right)^2 m(\eta) - 2\eta \frac{dm}{d\eta} = 0. \quad (76)
$$

In the following, we consider first that $n = 1$, and we show that for this case a power series solution of the relativistic mass equation does exist, by explicitly constructing it. As a next step in our study we will consider the power series solution of the relativistic mass equation for arbitrary $n$.

4.1 The case $n = 1$

For $n = 1$, which corresponds to a polytropic equation of state of the form $p \propto \rho^2$, Eq. (76) takes the form

$$
2\eta^2 \left[ 1 - 2a \frac{m(\eta)}{\eta} \right] \frac{d^2 m}{d\eta^2} + (9am(\eta) - 4\eta) \frac{dm}{d\eta} + a\eta \left( \frac{d}{d\eta} \right)^2 m(\eta) = 0. \quad (77)
$$

For mathematical convenience, we rewrite Eq. (77) in the form

$$
\eta^2 \frac{d^2 m}{d\eta^2} - 2\eta \frac{dm}{d\eta} + \frac{2k}{2k} \eta^2 m - 2a m \frac{d^2 m}{d\eta^2} + \frac{9a}{2} \frac{dm}{d\eta} + \frac{a}{2k} \eta^k \left( \frac{d}{d\eta} \right)^2 m + \frac{ak}{2} \eta \left( \frac{d}{d\eta} \right)^2 \frac{dm}{d\eta} = 0. \quad (78)
$$

We will look again for a power series solution of Eq. (77), and therefore our results can be summarized in the following

**Theorem 2.** The relativistic mass equation (74) describing the interior physical and geometrical properties of a general relativistic polytropic fluid sphere with polytropic index $n = 1$ has an exact non-singular power series solution $m(\eta) = \sum_{l=1}^{\infty} c_{2l+1} \eta^{2l+1}$, with coefficients $c_{2l+1}$, $l = 1, 2, \ldots, \infty$ given by the recursive relation

$$
c_{2l+3} = -\frac{a}{2l(2l+3)} \left[ \frac{1}{2k} c_{2l+1} - \sum_{i=1}^{l} (4l - 5i - 1) \times \right.

(2l - 2i + 3) c_{2i+1} c_{2l-2i+3} + \frac{k}{2} \sum_{i=1}^{l} \sum_{j=1}^{l-i+1} (2i + j + 1) (2l - 2i - 2j + 5) \times

(2l + 1) (2j + 1) c_{2l+1} c_{2l+1} \left. \right]. l \in [1, \infty). \quad (79)
$$

**Proof.** By inserting the power series representation of the mass, given by Eq. (37), into Eq. (78), the latter becomes

$$
2 \sum_{l=1}^{\infty} (2l+1) (l-1) c_{2l+1} \eta^{2l+1} + \frac{a}{2k} \sum_{l=1}^{\infty} c_{2l+1} \eta^{2l+3} -

4a \sum_{i,j=1}^{\infty} j (2j+1) c_{2i+1} c_{2j+1} \eta^{2i+2j+1} +

9a \sum_{i,j=1}^{\infty} (2j+1) c_{2i+1} c_{2j+1} \eta^{2i+2j+1} -

\frac{a}{2} \sum_{i,j=1}^{\infty} (2i+1) (2j+1) c_{2i+1} c_{2j+1} \eta^{2i+2j+1} +

\frac{ak}{2} \sum_{i,j,h=1}^{\infty} (2i+1) (2j+1) (2h+1) c_{2i+1} \times

c_{2j+1} c_{2h+1} \eta^{2i+2j+2h+1} = 0. \quad (80)
$$

In the first sum in Eq. (80), the term corresponding to $l = 1$ identically vanishes. Hence we can replace in the first sum $l$ by $l+1$. In the terms containing $\eta^{2i+2j+1}$ we use the Cauchy convolution of the power series, and takes $2i + 2j + 1 = 2l + 3$, or $i + j = l + 1$. The last
term in Eq. (80) can be transformed as follows
\[
\eta^{2i+2j+2h-1} = \sum_{i,j,h=1}^{\infty} (2i + 1) (2j + 1) (2h + 1) c_{2i+1} c_{2j+1} c_{2h+1} \times \\
\sum_{i=1}^{\infty} (2i + 1) \eta^{2i-1} \times \\
\left[ \sum_{j,h=1}^{\infty} (2j + 1) (2h + 1) c_{2j+1} c_{2h+1} \eta^{2j+2h} \right] = \\
\sum_{i=1}^{\infty} (2i + 1) c_{2i+1} \eta^{2i-1} \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} (2j + 1) (2r - 2j + 1) c_{2j+1} c_{2r-2j+1} \eta^{2r} = \\
\sum_{i=1}^{\infty} (2i + 1) c_{2i+1} \eta^{2i-1} \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} (2j + 1) (2r - 2j + 1) c_{2i+1} c_{2j+1} c_{2r-2j+1} \eta^{2r}.
\]

Therefore it follows that the coefficients \( c_{2l+1} \) must satisfy the algebraic condition
\[
\sum_{i=1}^{\infty} \left[ 2l (2l + 3) c_{2l+3} + \frac{a}{2k} c_{2l+1} - \\
\frac{a}{2k} \sum_{i=1}^{l} (4l - 5i - 1) (2l - 2i + 3) c_{2i+1} c_{2l-2i+3} + \\
\frac{ak}{2} \sum_{i=1}^{l} \sum_{j=1}^{l-i+1} (2i + 1) (2j + 1) (2l - 2i - 2j + 5) c_{2i+1} c_{2j+1} c_{2l-2i-2j+5} \right] \eta^{2l+3} = 0.
\]

By solving the above equation for the coefficients \( c_{2l+3} \) gives the stated recursive relationship. This ends the proof of Theorem 2.

The condition \( \epsilon(0) = \sum_{l=1}^{\infty} (2l + 1) c_{2l+1} \eta^{2l-2} \big|_{\eta=0} = 1 \) fixes the coefficient \( c_3 \) in the series expansion (79) as \( c_3 = 1/3 \). Then Eq. (79) gives for the coefficients \( c_{2l+1} \), \( l = 1, \ldots, 7 \), the values
\[
c_5 = -\frac{a(k + 1)(3k + 1)}{60k},
\]
\[
c_7 = \frac{a^2(k + 1)(3k + 1)(45k^2 - 2k + 3)}{10080k^2},
\]
\[
c_9 = -\frac{a^3(k + 1)(3k + 1)}{362880k^3} \left( 525k^4 - 62k^3 + 82k^2 - 30k + 1 \right).
\]

\[
c_{11} = \frac{a^4(k + 1)(3k + 1)}{63866880k^4} \left( 33075k^6 - 6630k^5 + 8947k^4 - 3964k^3 + 1777k^2 - 206k + 1 \right),
\]
\[
c_{13} = -\frac{a^5(k + 1)(3k + 1)}{49816166400k^5} \left( 983275k^8 - 2813850k^7 + 3948156k^6 - 2032438k^5 + 1202938k^4 - 489670k^3 + 106892k^2 - 3178k + 3 \right),
\]
\[
c_{15} = \frac{a^6(k + 1)(3k + 1)}{41845557977600k^6} \left( 3277699425k^{10} - 1226770650k^9 + 1791820419k^8 - 1054681792k^7 + 715276538k^6 - 389612036k^5 + 164352038k^4 - 47978464k^3 + 3637621k^2 - 34210k + 7 \right).
\]

\[\ldots\]

It is interesting to compare the solution of the relativistic mass equation (83) with the exact solution of the Newtonian Lane-Emden equation (3), corresponding to \( n = 1 \). By combining the non-relativistic hydrostatic equilibrium equation with the mass continuity equation we obtain
\[
\frac{d}{dr} \left( \rho \frac{dp}{dr} \right) = -4\pi r^2 \rho.
\]

By taking into account the polytropic equation of state with index \( n = 1 \), \( p = K \rho^2 \), Eq. (89) gives
\[
\frac{d}{dr} \left( \rho \frac{dp}{dr} \right) = -\frac{2\pi G}{K} r^2 \rho.
\]

With the help of the transformations \( r = R \eta, \rho = \rho_c \epsilon, K = kc^2/\rho_c \), and by taking into account the definition of \( a \), Eq. (90) becomes
\[
\frac{d}{d\eta} \left( \eta^2 d\rho/d\eta \right) = -\frac{a}{2k^2} \epsilon. \tag{91}
\]

Eq. (91) has the non-singular solution
\[
\epsilon(\eta) = \left( \frac{2k}{a} \right)^{1/4} \sin \left[ \left( \frac{a/2k}{1/4} \right) \eta \right], \tag{92}
\]
satisfying the condition \( \epsilon(0) = 1 \). The mass distribution of the Newtonian \( n = 1 \) polytrope is given by

\[
m(n) = \frac{\sqrt{2}}{(a/k)^{3/4}} \left( 2^{1/4} \sin \left[ \left( a/2k \right)^{1/4} n \right] \right) - \eta \left( a/k \right)^{1/4} \cos \left[ \left( a/2k \right)^{1/4} n \right] \quad (93)
\]

The comparison of the exact numerical solution of the TOV and mass continuity equations, and the power series solution of the relativistic mass equations is presented in Fig. 2.

For the value of the dimensionless parameter \( k \) we have adopted the value \( k = 0.1 \). As one can see from the Figures, there is an excellent agreement between the Figures, there is an excellent agreement between the power series and the numerical solutions, respectively. The surface energy density is fixed at \( \epsilon_S = 0.0385648 \), which gives for the coefficient \( a \) the numerical value \( a = 1.2145 \). The physical parameters of the \( n = 1 \) general relativistic star can be obtained as

\[
R = 11.4196 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ km}, \quad (94)
\]

\[
M_S = 1.07496 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} M_\odot. \quad (95)
\]

On the other hand, as expected, the Newtonian non-relativistic solutions of the Lane-Emden equation give a poor description of the interior properties of dense general relativistic stars.

5 The case of the arbitrary polytropic index \( n \)

In order to obtain a convergent power series solution of Eq. (76) for arbitrary \( n \) we will make use of the following

**Theorem 3 [Chang and Mott] (Chang & Mott 1975).** If \( F(x) \) is an arbitrary function of \( x \), differentiable at \( x = 0 \), and \( A(z) = \sum_{j=1}^{\infty} a_j z^j \), then \( G(z) = F[A(z)] = \sum_{k=0}^{\infty} g_k z^k \), where \( g_k = \sum_{h=0}^{\infty} f_h a_h k \), where \( f_h = \frac{1}{h!} \frac{d^h}{dx^h} F(x) \bigg|_{x=0} \), and \( a_h k \), where \( a_h k = \frac{1}{k!} \frac{d^h}{dx^h} A(z) \bigg|_{z=0} \), follows.

**Proof.** By Taylor series expansion \( F(x) = \sum_{h=0}^{\infty} f_h x^h \), and \( A(z) = \sum_{j=1}^{\infty} a_j z^j \). Then it immediately follows

\[
G(z) = F[A(z)] = \sum_{k=0}^{\infty} f_h a_h k = \sum_{k=0}^{\infty} \left( \sum_{h=0}^{k} f_h a_h k \right) z^k = \sum_{k=0}^{\infty} g_k z^k. \quad (96)
\]

With the use of **Theorem 3** we can now formulate the following

**Theorem 4.** The relativistic mass Eq. (76), describing the structure of polytropic general relativistic stars with arbitrary polytropic index \( n \), has an exact power series solution \( m(\eta) = \sum_{l=1}^{\infty} c_{2l+1} \eta^{2l+1} \), with the coefficients of the power series satisfying the recursive relation

\[
c_{2l+3} = -\frac{a}{2l(2l+3)} \sum_{h=1}^{l} \left( (2l-2h+3) \times \left[ \frac{h}{(k+1)n+1} + 4 \right] + \frac{(k+1)n}{k(n+1)} - 4l \right) c_{2h+1} c_{2l-2h+3} + \frac{n}{n+1} \sum_{i=1}^{l} (2l-2h-2i+3) \left[ (2l+1) k \alpha^+ \right] c_{2i+1} c_{2l-2h-2i+3}, \quad (97)
\]

where

\[
\alpha^{-}_{2j} = \frac{1}{(2j)!} \left[ \frac{d^{2j}}{d\eta^{2j}} \left( \sum_{r=1}^{\infty} (2r+1) c_{2r+1} \eta^{2r-2} \right) \right]^{\pm 1/n} \bigg|_{\eta=0}, \quad j = 1, 2, 3, \ldots \quad (98)
\]

**Proof.** As a first step in our proof we rewrite Eq. (66) in the form

\[
\eta^2 \frac{d^2 m}{d\eta^2} - 2\eta \frac{dm}{d\eta} - 2a \eta m \frac{d^2 m}{d\eta^2} + \frac{a n}{n+1} \eta \left( \frac{dm}{d\eta} \right)^2 + \frac{a(5n+4)}{n+1} \frac{dm}{d\eta} + \frac{a n}{n+1} \frac{1}{\eta^2} \left( \frac{dm}{d\eta} \right)^{1/n} \frac{dm}{d\eta}^2 + \frac{a n}{n+1} \left( \frac{1}{\eta^2} \frac{dm}{d\eta} \right)^{-1/n} \frac{dm}{d\eta} = 0. \quad (99)
\]

We will look again for a power series solution of Eq. (99), by choosing \( m(\eta) \) in the form \( m(\eta) = \sum_{l=1}^{\infty} c_{2l+1} \eta^{2l+1} \). Then we obtain immediately

\[
\left( \frac{1}{\eta^2} \frac{dm}{d\eta} \right)^{\pm 1/n} = \left( \sum_{r=1}^{\infty} (2r+1) c_{2r+1} \eta^{2r-2} \right)^{\pm 1/n} \bigg|_{\eta=0}.
\]

By direct checking it can be shown that

\[
\left[ \frac{d^j}{d\eta^j} \left( \sum_{r=1}^{\infty} (2r+1) c_{2r+1} \eta^{2r-2} \right) \right]^{\pm 1/n} \bigg|_{\eta=0} = 0, \quad j = 1, 3, 5, \ldots. \quad (100)
\]
and thus
\[
\left( \frac{1}{\eta^2} \frac{dm}{d\eta} \right)^{\pm 1/n} = 1 + \sum_{j=1}^{\infty} \alpha_{2j}^\pm \eta^{2j},
\]
(102)

where
\[
\alpha_{2j}^\pm = \frac{1}{(2j)!} \left[ \frac{d^{2j}}{d\eta^{2j}} \left( \sum_{n=1}^{\infty} (2r+1) c_{2r+1} \eta^{2r-2} \right) \right] \left|_{\eta=0} \right. ,
\]
(103)

\( j = 1, 2, 3, \ldots \)

Therefore it follows that
\[
\frac{an}{n+1} K \left( \frac{1}{\eta^2} \frac{dm}{d\eta} \right)^{\pm 1/n} \left( \frac{dm}{d\eta} \right)^2 = \frac{an}{n+1} k \eta^{2h} \sum_{h=1}^{\infty} \alpha_{2h}^\pm \eta^{2h} \times \sum_{i,j=1}^{\infty} (2i+1)(2j+1) c_{2i+1} c_{2j+1} \eta^{2i+2j+1} = \frac{an}{n+1} K \left( \frac{dm}{d\eta} \right)^2 + \frac{an}{n+1} k \sum_{h=1}^{\infty} \alpha_{2h}^\pm \eta^{2h} \times \sum_{r=1}^{\infty} \sum_{i=1}^{r} (2i+1)(2r-2i+1) c_{2i+1} c_{2r-2i+1} \eta^{2r+1} = \frac{an}{n+1} K \left( \frac{dm}{d\eta} \right)^2 + \frac{an}{n+1} k \sum_{h=1}^{\infty} \alpha_{2h}^\pm \eta^{2h} \times \sum_{r=1}^{\infty} \sum_{i=1}^{r} (2i+1)(2r-2i+1) c_{2i+1} c_{2r-2i+1} \eta^{2r+1} \left( 2i+1 \right) \left( 2l-2h-2i+3 \right) c_{2i+1} c_{2l-2h-2i+3} \eta^{2i+3},
\]
(104)

\[
\frac{an}{K(n+1)} \left( \frac{1}{\eta^2} \frac{dm}{d\eta} \right)^{-1/n} \frac{dm}{d\eta} = \frac{an}{K(n+1)} \frac{dm}{d\eta} + \frac{an}{K(n+1)} \sum_{h=1}^{\infty} \alpha_{2h}^- \eta^{2h} \sum_{i,j=1}^{\infty} (2j+1) c_{2i+1} c_{2j+1} \eta^{2i+2j+1} = \frac{an}{K(n+1)} \frac{dm}{d\eta} + \frac{an}{K(n+1)} \sum_{h=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{2h}^- \eta^{2h} \left[ \sum_{r=1}^{\infty} \sum_{i=1}^{r} (2i+1)(2r-2i+1) c_{2i+1} c_{2r-2i+1} \right] \eta^{2r+1} = \frac{an}{K(n+1)} \frac{dm}{d\eta} + \frac{an}{K(n+1)} \sum_{h=1}^{\infty} \sum_{i=1}^{\infty} \alpha_{2h}^- \eta^{2h} \left( 2l-2h-2i+3 \right) c_{2i+1} c_{2l-2h-2i+3} \eta^{2i+3}. \]
(105)

Then we successively obtain
\[
\eta^2 \frac{d^2m}{d\eta^2} - 2 \eta \frac{dm}{d\eta} = \sum_{l=1}^{\infty} 2l(2l+3) c_{2l+3} \eta^{2l+3}, \quad \text{(106)}
\]
\[
-2 \eta \frac{d^2m}{d\eta^2} = -4a \sum_{l=1}^{\infty} \left[ \sum_{h=1}^{l} \sum_{i=1}^{l-h+1} (2l-2h+3) c_{2h+1} c_{2l-2h-2i+3} \right] \eta^{2l+3} \text{(107)}
\]
\[
\frac{an(k+1)}{n+1}\eta \left(\frac{dm}{d\eta}\right)^2 = \frac{an(k+1)}{n+1} \sum_{l=1}^{\infty} \sum_{h=1}^{l} \left[ (2l+1)(2l+2h+3) \right] c_{2l+1} c_{2l-2h+3} \eta^{2l+3}. \tag{108}
\]

By substituting all the above results in Eq. (109) we obtain

\[
\sum_{l=1}^{\infty} \sum_{h=1}^{l} (2l+1)(2l+2h+3) c_{2l+1} c_{2l-2h+3} \eta^{2l+3} = 0. \tag{110}
\]

From Eq. (110) the recursive relation between the coefficients \(c_{2l+1}\) immediately follows, and thus we obtain Eq. (97). This ends the proof of Theorem 4.

The values of the first seven coefficients of the exact power series solution of the general relativistic stars with arbitrary polytropic index are presented in Appendix A.

5.1 Applications: Polytropic stars with index \(n = 1/2\), \(n = 1/5\), and \(n = 3\)

In the following we present some direct applications of Theorem 4, by comparing the estimations obtained from the exact general power series solution of Eq. (76), as given by Theorem 4, with the solution obtained by numerically integrating the structure equations of the star.

5.1.1 The case \(n = 1/2\)

As a first application of the exact power series solution of the relativistic mass equation for polytropic stars we present in detail the case \(n = 1/2\). Then Eq. (73) is given by

\[
3\eta^2 \left[ 1 - 2\eta \left(\frac{m}{\eta}\right) \right] \left(\frac{dm}{d\eta}\right)^2 + [13am(\eta) - 6\eta] \left(\frac{dm}{d\eta}\right)^2 + \eta \left(\frac{dm}{d\eta}\right) + \frac{a}{\eta} \left(\frac{dm}{d\eta}\right) = 0. \tag{111}
\]

Then the coefficients of the general power series solution of Eq. (111) of the form \(m(\eta) = \sum_{l=1}^{\infty} c_{2l+1} \eta^{2l+1}\) can be obtained immediately from Theorem 4, and are given by

\[
c_5 = -\frac{a(k+1)(3k+1)}{90k}, \tag{112}
\]

\[
c_7 = \frac{a^2(k+1)(3k+1)(20k^2 - 11k - 1)}{11340k^2}, \tag{113}
\]

\[
c_9 = -\frac{a^3(k+1)(3k+1)}{2755620k^3} \left( 220k^4 - 1131k^3 + 620k^2 + 45k + 9 \right), \tag{114}
\]

\[
c_{11} = \frac{a^4(k+1)(3k+1)}{1818709200k^4} \left( 496125k^6 - 304515k^5 + 26342k^4 - 106991k^3 - 58k^2 - 3582k - 279 \right), \tag{115}
\]

\[
c_{13} = -\frac{a^5(k+1)(3k+1)}{2127889764000k^5} \left( 212837625k^8 - 148281300k^7 + 160686045k^6 - 102260088k^5 + 42372329k^4 - 4403892k^3 + 2118955k^2 + 305712k + 17478 \right), \tag{116}
\]

\[
c_{15} = \frac{a^6(k+1)(3k+1)}{2010855826980000k^6} \left( 7747289550k^{10} - 59686119675k^9 + 74882398185k^8 - 58288209489k^7 + 3683114753k^6 - 15322471612k^5 + 315124375k^4 - 100240522k^3 - 186941375k^2 - 21156579k - 962217 \right). \tag{117}
\]
The comparison of the results obtained by numerically solving the structure equations of the $n = 1/2$ polytropic general relativistic star and of the results obtained from the power series solution, truncated at $l = 7$, are presented in Fig. 3.

For the surface density of the star we have adopted the value $\epsilon_S = 0.209255$, which gives for $a$ the value $a = 0.76593$. The maximum value of the dimensionless mass $m(1)$ is $m(1) = 0.189948$. The physical parameters of the star are given by

$$ R = 9.06873 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ km}, \quad (118) $$

$$ M_S = 0.890139 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} M_\odot. \quad (119) $$

### 5.1.2 The case $n = 1/5$

For $n = 1/5$, the relativistic mass Eq. [25] or Eq. [70] becomes

$$ 6\eta^2 \left[ 1 - 2a \frac{m(\eta)}{\eta} \right] \left( \frac{dm}{d\eta} \right)^4 \frac{d^2m}{d\eta^2} + [25am(\eta) - 12\eta] \times \left( \frac{dm}{d\eta} \right)^5 + a\eta \left( \frac{dm}{d\eta} \right)^6 + \frac{ak}{\eta^3} \left( \frac{dm}{d\eta} \right)^{11} + \frac{a}{k} \eta^{10} m(\eta) = 0. \quad (120) $$

We will not present here the explicit expressions of the coefficients $c_{2l+1}$ of the exact series solution of Eq. [120]. The comparison of the exact numerical solution of Eq. [120] and of the power series solution, truncate at $l = 7$ is presented in Fig. 3. For the surface density of the star we obtain the value $\epsilon_S = 0.703388$, corresponding to $a = 0.45734$. The total dimensionless mass of the star is $m(l) = 0.28339$, giving for the physical parameters of the star the values

$$ R = 7.00764 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} \text{ km}, \quad (121) $$

$$ M_S = 0.612749 \times \left( \frac{\rho_c}{10^{15} \text{ g/cm}^3} \right)^{-1/2} M_\odot. \quad (122) $$

### 5.1.3 The case $n = 3$

A polytrope of the order of $n = 3$, corresponding to the equation of state $p = \text{constant} \times \rho^{4/3}$ is known as the Eddington approximation [Eddington 1926]. From an astrophysical point of view it corresponds to a wholly radiative star. Thus, for example, the $n = 3$ polytrope is used to model the astrophysical properties of our Sun [Bahcall & Ulrich 1988]. For $n = 3$ the relativistic mass equation takes the form

$$ \frac{d^2m}{d\eta^2} - \frac{2}{3} \frac{dm}{d\eta} + \frac{3a}{4k} \left[ m + k\eta^3 \left( \frac{m}{\eta^2} \right)^{4/3} \right] \left( \frac{m}{\eta^2} \right)^{4/3} = 0, \quad (123) $$

where we have used the relation $dP/d\epsilon = (4/3)k\epsilon^{1/3}$, giving $P' \left( \frac{m}{\eta^2} \right) = (4k/3) \left( \frac{m}{\eta^2} \right)^{1/3}$.

The coefficients of the power series solutions of Eq. [123] are given by

$$ c_5 = \frac{a(k + 1)(3k + 1)}{40k}, \quad (124) $$

$$ c_7 = \frac{a^2(k + 1)(3k + 1)(105k^2 + 34k + 19)}{13440k^6}, \quad (125) $$

$$ c_9 = \frac{-a^3(k + 1)(3k + 1)}{8709120k^3} \left( 2425k^4 + 10374k^3 + 7390k^2 + 470k + 619 \right), \quad (126) $$

$$ c_{11} = \frac{a^4(k + 1)(3k + 1)}{5109350400k^4} \left( 5457375k^6 + 2399670k^5 + 2059023k^4 + 82268k^3 + 244509k^2 - 58914k + 17117 \right), \quad (127) $$

$$ c_{13} = \frac{-a^5(k + 1)(3k + 1)}{797058662400k^5} \left( 3421774125k^8 + 1377441450k^7 + 1441718820k^6 - 295035785k^5 + 227366374k^4 - 89104202k^3 + 35760180k^2 - 13335878k + 1208293 \right), \quad (128) $$
Fig. 3 Variation with respect to the dimensionless radial coordinate $\eta$ of the dimensionless mass $m(\eta)$ (left panel), and of the dimensionless energy density $\epsilon(\eta)$ (right panel) for a polytropic stars with indexes $n = 1/2$, $n = 1/5$, and $n = 3$, respectively. For the constant $k$ in all cases we have adopted the value $k = 0.1$. In all cases the long dashed curves represent the numerical solutions of the TOV and mass continuity Eqs. (19) and (76). The power series solutions of the relativistic mass equation, truncated to $l = 7$, are represented by a solid curve ($n = 1/2$), a dotted curve ($n = 1/5$), and by a short dashed curve ($n = 3$), respectively.

$$c_{15} = \frac{a^9(k + 1)(3k + 1)}{40171756584960000k^8} \left(7161773243625k^{10} + 2400911008650k^9 + 3276987869295k^8 - 339209275608k^7 + 684186777306k^6 - 32774628204k^5 + 171863068046k^4 - 75604400072k^3 + 27410294125k^2 - 5942307478k + 267910291 \right). \quad (129)$$

In Fig. 3 we compare the power series solution for the polytropic index $n = 3$, truncated to seven terms, with the exact numerical solution. For $a$ we have adopted the value $a = 0.687329$, giving a surface density $\epsilon_S = 0.319079$. The total dimensionless mass of the star is $m(1) = 0.173226$. The physical parameters of the compact general relativistic object described by the $n = 3$ polytrope can be obtained as

$$R = 8.59081 \times \left(\frac{\rho_c}{10^{15} \text{ g/cm}^3}\right)^{-1/2} \text{ km}, \quad (130)$$

$$M_S = 0.69008 \times \left(\frac{\rho_c}{10^{15} \text{ g/cm}^3}\right)^{-1/2} M_\odot. \quad (131)$$

6 Conclusions and final remarks

In the present paper we have obtained exact power series solutions of the mass continuity and hydrostatic equilibrium equation describing the structure of general relativistic stars. In order to obtain the solutions we have formulated the second order differential equation describing the mass profile of the stars. The relativistic mass equation admits exact, convergent and non-singular, power series solutions for both the linear barotropic and polytropic equations of state. We have obtained the power series solutions for arbitrary values of $\gamma$ for the linear barotropic equation of state $p = \gamma \rho c^2$, and for the general case of the arbitrary polytropic index $n$. We have compared in detail our exact results with the results obtained by numerically integrating the gravitational field equations, by considering the cases $\gamma = 1/3$, $\gamma = 1$, and $n = 1, 1/2, 1/5, 3$, respectively. By truncating our power series to only seven terms we can basically reproduce the numerical results for the mass and density distribution of the general relativistic stars described by linear barotropic and polytropic equations of state. The power series solution are non-singular at the center of the star, and they can be extended up to the vacuum boundary/surface of the dense matter distribution. Due to the adopted equations of state the physical requirements for the acceptability of the solutions are automatically satisfied. Thus, the speed of sound $c_s = \sqrt{\partial p/\partial \rho} = \sqrt{\gamma}c$ is a constant inside the star, and for $\gamma \in [0, 1]$ satisfies the constraint $c_s \leq c$.

For the polytropic stars we obtain

$$c_s = \sqrt{k(n + 1)/n} \rho_c^{1/2n} = c_s(\rho_c) \epsilon^{1/2n}, \quad (132)$$

where we have denoted

$$c_s(\rho_c) = \rho_c^{1/2n} \sqrt{k(n + 1)/n}. \quad (133)$$
Using the relation $\epsilon (\eta) = \sum_{i=1}^{\infty} (2i + 1) c_{2i+1} \eta^{2i-2}$ then we find

$$c_s = c_s \left( \rho_c \right) \left[ \sum_{i=1}^{\infty} (2i + 1) c_{2i+1} \eta^{2i-2} \right]^{1/2n} \leq c. \quad (134)$$

The power series solutions of the Newtonian Lane-Emden Eq. (3) have been intensively studied in the astrophysical and mathematical literature (Mohan & Al-Bayati 1980; Roxburgh & Stockman 1999; Hunter 2001; Nouh 2004). The series solutions are represented as $\theta = \sum_{k=0}^{\infty} a_k \xi^{2k}$ and $\theta^n = \sum_{k=0}^{\infty} b_k \xi^{2k}$, respectively, with $a_0 = b_0 = 1$ (Nouh 2004). One can define the radius of convergence of these series as the distance from $\xi = 0$ to the closest singularity of $\theta(\xi)$ in the complex $\xi$-plane. Non-linear ordinary differential equations, such as the LaneEmden equation for $n > 1$, can have two kinds of singularities, fixed and movable (Nouh 2004). The LaneEmden equation for polytropic index $n > 1$ and its $n \to \infty$ limit, corresponding to the limit of the isothermal sphere equation, are singular at some negative value of the radius squared (Nouh 2004). It is this singularity that prevents real power series solutions about the center to converge to the outer surface once the condition $n > 1.9121$ is satisfied. However, as shown in Nouh (2004), an Euler transformation gives power series that do converge up to the outer radius. Moreover, the Euler-transformed series converge significantly faster than the series obtained in Roxburgh & Stockman (1999), which are limited to finite radii whenever $n > 5$ by a complex conjugate pair of singularities. Series solutions for polytropic stars by using the Euler transform were constructed in Nouh (2004), so that longer than 60-term series are needed for the outer regions of $n > 3$ polytropic Newtonian stars, while 120-term and 300-term series are needed to obtain the function $\theta(\xi)$ to seven decimal place accuracy all the way from the center to the surface of the compact object for $n = 3.5$ and $n = 4$, respectively (Nouh 2004). In this context we would like to point out that the power series solutions of the relativistic mass equation can be extended to the boundary of the considered stars, and only seven terms are required to reproduce the numerical solutions with a high precision.

Although the numerical solutions of the structure equations of spherically symmetric static general relativistic stars can be obtained numerically in a very efficient, simple and accurate way, we must point out that power series represent one of the most powerful methods of mathematical analysis. The use of power series is no less convenient than the use of elementary functions, especially when solutions of differential equations are to be studied numerically. In the case of the approach based on the relativistic mass equation an important advantage of a power series solution is that it gives the value of the mass and energy density inside the star as a recurrent power series in the radial coordinate $r$, since the dimensionless variable $\eta = r/R$. Consequently, we can predict the physical and geometrical parameters of the star at any radius directly. Moreover, power series analytical solutions describing the interior of compact general relativistic objects usually offers deeper insights into their physical and geometrical properties, thus offering the possibility of a better understanding of the structure of dense stars.

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The first seven coefficients of the power series solution of the relativistic mass equation for
arbitrary polytropic index \( n \)

The first seven coefficients \( c_{2l+1} \), \( l = 1, 2, \ldots, 7 \) describing the solution of the relativistic mass equation for a
general relativistic polytropic star with arbitrary polytropic index \( n \in \mathbb{R} \) are given by

\[
c_3 = \frac{1}{3}, \quad (A1)
\]

\[
c_5 = \frac{a(k+1)(3k+1)n}{30k(n+1)}, \quad (A2)
\]

\[
c_7 = \frac{a^2(k+1)(3k+1)n}{2520k^2(1+n)^2} \left[ k^2(30n+15) + k(18n-20) + 8n - 5 \right], \quad (A3)
\]

\[
c_9 = \frac{a^3(1+k)(1+3k)n}{408240k^3(1+n)^3} \left[ 315k^4(6n^2+7n+2) + 6k^3(288n^2-241n-140) + \\
2k^2(618n^2-809n+560) + 10k(40n^2-123n+56)+122n^2-183n+70 \right], \quad (A4)
\]

\[
c_{11} = \frac{a^4(k+1)(3k+1)n}{179625600k^4(n+1)^4} \left[ 14175k^6(24n^3+46n^2+29n+6) + 90k^5(4074n^3-1727n^2- \\
4402n-1260) + k^4\left(323568n^3-412518n^2+330915n+160650\right) + 4k^3 \\
(38832n^3-139547n^2+106520n-50400) + k^2\left(71744n^3-256154n^2+418725n-154350\right) + \\
18k\left(942n^3-5847n^2+6490n-2100\right) + 5032n^3-12642n^2+10805n-3150 \right], \quad (A5)
\]

\[
c_{13} = \frac{a^5(k+1)(3k+1)n}{70053984000k^5(n+1)^5} \left[ 467775k^8\left(120n^4+326n^3+329n^2+146n+24\right) + 1350k^7 \times \\
(47748n^4+2516n^3-77423n^2-55548n-11088) + 180k^6\left(354618n^4-343910n^3+ \\
333411n^2+518150n+124740\right) + 6k^5\left(5942244n^4-22880944n^3+17820615n^2- \\
11274200n+4851000\right) + 2k^4\left(9945804n^4-43963854n^3+95782105n^2-52438750n+ \\
17740800\right) + 2k^3\left(3358788n^4-27242948n^3+54777285n^2-60621700n+18711000\right) + \\
20k^2\left(138430n^4-873842n^3+2695909n^2-2399130n+679140\right) + k\left(495624n^4- \\
5988304n^3+11652870n^2-8520800n+2217600\right) + 183616n^4-663166n^3+ \\
915935n^2-574850n+138600 \right], \quad (A6)
\]
\[ c_{15} = \frac{a^6(k + 1)(3k + 1)n}{88268019840000k^6(n + 1)^6} \left[ 42567525k^{10} \left( 720n^5 + 2556n^4 + 3604n^3 + 2521n^2 + 874n + 120 \right) + 28350k^9 \left( 1249692n^5 + 686860n^4 - 2577633n^3 - 3421416n^2 - 1539028n - 240240 \right) + 135k^8 \left( 278662344n^5 - 141122820n^4 + 282670258n^3 + 825130367n^2 + 466600470n + 79879800 \right) + 36k^7 \left( 612790998n^5 - 2472456073n^4 + 1527229265n^3 - 1345472610n^2 - 187749300n - 399399000 \right) + 6k^6 \left( 2273834244n^5 - 10928033994n^4 + 29030321910n^3 - 15039510755n^2 + 7751107700n + 3006003000 \right) + 4k^5 \left( 1272432138n^5 - 12749058409n^4 + 32186860166n^3 - 50094344110n^2 + 21616077000n - 5381376000 \right) + 2k^4 \left( 1245531860n^5 - 10585913054n^4 + 47107629124n^3 - 68141956465n^2 + 56246234100n - 14777763000 \right) + 4k^3 \left( 144913746n^5 - 2616936275n^4 + 9372735679n^3 - 19092417410n^2 + 14124980100n - 3552549000 \right) + 5k^2 \left( 58083800n^5 - 526446912n^4 + 2878464312n^3 - 4529906063n^2 + 2903305230n - 685284600 \right) + 6k \left( 5002696n^5 - 154497310n^4 + 462711169n^3 - 560651580n^2 + 316735300n - 70070000 \right) + 21625216n^5 - 103178392n^4 + 200573786n^3 + 199037015n^2 + 101038350n - 21021000 \right]. \]