Coupling of Tachyons and Discrete States in $c = 1$ 2-D Gravity

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Abstract

All the three point couplings involving tachyons and/or discrete states are obtained in $c = 1$ two-dimensional (2-D) quantum gravity by means of the operator product expansion (OPE). Cocycle factors are found to be necessary in order to maintain the analytic structure of the OPE, and are constructed explicitly both for discrete states and for tachyons. The effective action involving tachyons and discrete states is worked out to summarize all of these three point couplings.
1 Introduction

Recent advances in the matrix model [1, 2] for the nonperturbative treatment of two-dimensional (2-D) quantum gravity has prompted much progress with the continuum approach by means of the Liouville theory. In spite of the nonlinear dynamics of the Liouville theory, a method based on conformal field theory has now been sufficiently developed to understand the results of the matrix model and to offer in some cases a more powerful method for computing various quantities [3, 4]. In particular, one can now calculate not only partition functions [5, 6] but also correlation functions, using the procedure of analytic continuation [7-11].

So far only conformal field theories with central charge $c \leq 1$ have been successfully coupled to quantum gravity. The $c = 1$ case is the richest and the most interesting. It has been observed that $c = 1$ quantum gravity can be regarded effectively as a critical string theory in two dimensions, since the Liouville field zero mode provides an additional “time-like” dimension besides the obvious single spatial dimension given by the zero mode of the $c = 1$ matter [12]. The center of mass motion of the string provides a physical scalar particle. In the usual bosonic string theory at the critical dimension ($D = 26$), the scalar particle has negative squared mass and is called tachyon. In the present case of a noncritical string, the scalar particle becomes massless but is still called “tachyon” following the usual terminology of critical string theory. Since there are no transverse directions, the continuous (field) degrees of freedom are exhausted by the tachyon field. However, it has been noted that there exist other discrete degrees of freedom in the $c = 1$ matter coupled to the 2-D quantum gravity [13-18]. It has been pointed out that the symmetry group relevant to the dynamics of these discrete states in the $c = 1$ quantum gravity is that of the area preserving diffeomorphisms whose generators fall into representations of SU(2) [19, 20]. By using the SU(2) symmetry, Klebanov and Polyakov have recently worked out the three point interactions of the discrete states and have proposed an effective action for these discrete states in the case of integer spins [21].
The purpose of this paper is to obtain all possible three point couplings of tachyons and discrete states in $c = 1$ quantum gravity, and to write down an effective action representing all of these couplings. Using the operator product expansion (OPE), we have found all possible three point couplings involving tachyons and/or discrete states: couplings among three tachyons, and couplings among two tachyons and a single discrete state in addition to those among three discrete states which were already worked out [21]. We have also found that cocycle factors are needed to maintain analyticity of the OPE and have constructed them explicitly. An effective action is worked out for the three point coupling of discrete states even in the presence of half odd integer spins. We have also found an effective action representing the three point coupling involving tachyons.

In the next section, tachyon and discrete state vertex operators are constructed as physical states of the $c = 1$ quantum gravity. In sect. 3, the cocycle factors are shown to be necessary to maintain proper analytic behaviour of the OPE, and they are explicitly constructed. The OPEs involving tachyons are worked out in sect. 4. In sect. 5, an effective action is constructed for the discrete states when there are half odd integer spins beside integer spins. An effective action for couplings involving tachyons is obtained in sect. 6.

While we are writing this paper, we have received papers [22] where the operator algebra involving the discrete states and the tachyon operators is discussed. Part of our results has some overlap with theirs, although our method is different from theirs.
2 Tachyons and Discrete States

We consider the $c = 1$ conformal matter realized by a single scalar field (string variable) $X$ coupled to two-dimensional quantum gravity. In the conformal gauge $g_{\alpha\beta} = e^{\phi} \hat{g}_{\alpha\beta}$ with a reference metric $\hat{g}_{\alpha\beta}$, the Liouville field $\phi$ represents all the local degrees of freedom of the metric. The dynamics of $X$ and $\phi$ can be described by the following action on a surface with boundary $[3, 4, 23]$

$$S[\hat{g}, X, \phi] = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{\hat{g}} \left( \hat{g}^{\alpha\beta} \partial_{\alpha} X \partial_{\beta} X + \hat{g}^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi - 2\sqrt{\alpha'} \hat{R} \phi \right) + \frac{4\alpha' \mu e^{-2\phi/\sqrt{\alpha'}}}{\pi\sqrt{\alpha'}} \int d\hat{s} \left( -\hat{k} \phi + \sqrt{\alpha'} \lambda e^{-\phi/\sqrt{\alpha'}} \right),$$

(2.1)

where $\alpha'$ is the Regge slope parameter, $\hat{R}$ the scalar curvature, $\hat{k}$ the geodesic curvature along the boundary and $d\hat{s}$ the line element of the boundary with respect to the reference metric $\hat{g}_{\alpha\beta}$. We have rescaled the Liouville field $\phi$. The action (2.1) can be regarded as describing a critical string theory in a two-dimensional target space-time with “time” $\phi$ and a spatial coordinate $X$ [12]. Since the large negative values of the Liouville field $\phi$ are suppressed by the cosmological term even for a small cosmological constant $\mu$, the zero mode of $\phi$ is effectively limited to a finite length proportional to $\ln \mu$. Amplitudes generally have contributions proportional to $\ln \mu$ which are called bulk or resonant amplitudes [14]. In the following, we will consider only the bulk correlation functions, for which the “energy” and the momentum conjugate to $\phi$ and $X$ respectively are conserved. To compute such correlation functions we can put $\mu = \lambda = 0$. Consequently we shall use the action without the cosmological terms in the following, and compute the correlation functions by means of the conformal field theory technique with free field realizations [7-11, 15].

There are two types of physical operators. The open string vertex operators are given by line integrals of primary fields with boundary conformal weight one along the boundary, while the closed string vertex operators are given by surface integrals of primary fields with conformal weight $(1, 1)$. It is convenient to set
\[ \alpha' = 4 \] (1) when we discuss the closed (open) string vertex operators. With this convention the integrands of the closed string vertex operators can be constructed by combining the holomorphic operator and the anti-holomorphic operator, both of which have the same form as those of the open string vertex operators.

Let us first consider the open string vertex operators. The simplest field for such operators is the gravitationally dressed tachyon vertex operator with momentum \( p \)

\[
\Psi_p^{(\pm)}(z) = e^{ipX(z)} e^{(\pm p - 1)\phi(z)}. \tag{2.2}
\]

The requirement that conformal weight be unity gives two solutions \( \pm p - 1 \) for the Liouville energy. The upper (lower) sign is called positive (negative) chirality [14]. One should note that the momentum \( p \) can take arbitrary real values if the boson \( X \) is noncompact. For higher levels there are non-trivial primary fields only when the momentum is an integer or a half odd integer. They are primary fields for the “discrete states” [13, 14]. They form SU(2) multiplets and can be constructed as [19, 21]

\[
\Psi_{J,m}^{(\pm)}(z) = \sqrt{(J + m)!} (2J)! (J - m)! \oint \frac{du_J^{-m}}{2\pi i} H_-(u_{J-m}) \cdots \oint \frac{du_1}{2\pi i} H_-(u_1) \Psi_J^{(\pm)}(z), \tag{2.3}
\]

where \( J = \frac{1}{2}, 1, \cdots; m = -J, -J + 1, \cdots, J \) and \( \Psi_J^{(\pm)}(z) \) is the tachyon operator (2.2) with the momentum \( p = J \). The integrals are along closed contours surrounding a point \( z \) with \( |u_i| > |u_j| \) for \( i > j \). The field \( H_-(z) \) corresponds to the lowering operator of the SU(2) quantum numbers and is one of the SU(2) currents

\[
H_{\pm}(z) = e^{\pm iX(z)} = \pm \Psi_{1,\pm 1}^{(+)}(z), \quad H_3(z) = \frac{1}{2} i \partial X(z) = -\frac{1}{\sqrt{2}} \Psi_{1,0}^{(+)}(z). \tag{2.4}
\]

The quantum numbers \( J \) and \( m \) correspond to the “spin” and the magnetic quantum number in SU(2). Actually, the fields \( \Psi_{J,m}^{(\pm)} \) with \( m = \pm J \) are not higher level
operators but tachyon operators (2.2) at integer or half odd integer momenta $\pm J$.

$$
\Psi^{(\pm)}_{J,J}(z) = \Psi^{(\pm)}_{J}(z), \quad \Psi^{(\pm)}_{J,-J}(z) = (-1)^J(2J-1)\Psi^{(\mp)}_{-J}(z).
$$

(2.5)

The sign factor in the second equation is a consequence of the definition (2.3) which respects the usual relative sign convention of states with different $m$ [24]. It should be noted that the type of discrete state $(+)$ or $(-)$ is opposite to the chirality of the corresponding tachyon for the case of lowest magnetic quantum number $m = -J$.

Strictly speaking, the expression (2.3) is correct only for integer $J$ as we will see in the next section. We will explain how to modify eq. (2.3) to give a correct expression for half odd integer $J$ when we discuss cocycle operators in the next section.

### 3 Discrete State OPE and Cocycle

In ref. [21] the OPEs of the fields for discrete states (2.3) were obtained using the SU(2) symmetry. Here we make a remark on the analytic property of the OPEs. The OPE of two vertex operators in eq. (2.3) at $z$ and $w$ gives a result different in sign depending on the ordering of the two vertex operators. Hence the OPE is not analytic in the complex coordinates $z$ at $|z| = |w|$, even if we use the radial ordering of the two vertex operators as usual in conformal field theory.

To be more precise, let us consider the OPE of $\Psi^{(+)}_{J_1,m_1}(z)$ and $\Psi^{(+)}_{J_2,m_2}(w)$. The conservation of momentum and the Liouville energy follows from the zero mode dependence of $X$ and $\phi$. Hence one immediately finds that the OPE of the two operators gives the operator $\Psi^{(+)}_{J_1+J_2-1,m_1+m_2}(w)$ as the only possible operator with conformal weight one. When $|z| > |w|$, the product of the two operators gives

$$
\frac{1}{z-w}F_{J_1,m_1,J_2,m_2}\Psi^{(+)}_{J_1+J_2-1,m_1+m_2}(w)
$$

with a certain coefficient $F_{J_1,m_1,J_2,m_2}$. On the other hand, when $|w| > |z|$, it gives

$$
\frac{1}{z-w}F'_{J_1,m_1,J_2,m_2}\Psi^{(+)}_{J_1+J_2-1,m_1+m_2}(w)
$$

with a different coefficient $F'_{J_1,m_1,J_2,m_2}$. By using explicit representations (2.3), it can be
shown that they differ by a sign

\[ F_{J_1, m_1, J_2, m_2} = (-1)^{2J_1(J_2-m_2-1)+2J_2(J_1-m_1-1)} F'_{J_1, m_1, J_2, m_2}. \]  

(3.1)

Therefore, the coefficient function of the OPE is not analytic at \(|z| = |w|\). It is desirable to obtain OPEs with analytic coefficients since the techniques of conformal field theories make full use of the analyticity. The analytic OPE can be achieved by multiplying a correction factor to the operators (2.3) as in the vertex operator construction of the affine Kac-Moody algebra [25].

The correction factor can be constructed as follows. Let us consider a two-dimensional lattice

\[ \Lambda = \left\{ \alpha = \sqrt{2}(m, J - 1) \mid m, J \in \frac{1}{2}\mathbb{Z}, J - m \in \mathbb{Z} \right\}. \]  

(3.2)

with a Lorentzian inner product

\[ \alpha_1 \cdot \alpha_2 = 2m_1m_2 - 2(J_1 - 1)(J_2 - 1), \quad \alpha_1, \alpha_2 \in \Lambda. \]  

(3.3)

The lattice \(\Lambda\) is an even lattice, i.e. \(\forall \alpha \in \Lambda, \alpha \cdot \alpha \in 2\mathbb{Z}\) and therefore it is an integral lattice: \(\forall \alpha, \beta \in \Lambda, \alpha \cdot \beta \in \mathbb{Z}\). Furthermore, the lattice \(\Lambda\) is self-dual \(\Lambda = \Lambda^*\), where \(\Lambda^* = \{x \mid \forall \alpha \in \Lambda, \alpha \cdot x \in \mathbb{Z}\}\) is the dual lattice of \(\Lambda\). Using these notations the sign factor in eq. (3.1) can be written as \((-1)^{\alpha_1 \cdot \alpha_2}\). The effect of the correction factor must cancel this sign factor.

To construct the correction factor we need a cocycle \(\varepsilon(\alpha, \beta) = \pm 1\) \((\alpha, \beta \in \Lambda)\) which satisfies

\[ \varepsilon(\alpha, \beta) = (-1)^{\alpha \cdot \beta} \varepsilon(\beta, \alpha), \]

\[ \varepsilon(\alpha, \beta)\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \beta + \gamma)\varepsilon(\beta, \gamma). \]  

(3.4)

The cocycle \(\varepsilon(\alpha, \beta)\) for \(\Lambda\) can be obtained as follows. Consider a three-dimensional
Lorentzian cubic lattice with the metric $(1, -1, 1)$ for diagonal components

\[ \Lambda_3 = \left\{ \tilde{\alpha} = \sum_{i=1}^{3} n^i \tilde{e}_i \mid n^i \in \mathbb{Z} \right\} \quad (3.5) \]

with basis vectors

\[ \tilde{e}_1 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{1} \right), \quad \tilde{e}_2 = \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{1} \right), \quad \tilde{e}_3 = \left( 0, -\sqrt{2}, 1 \right). \quad (3.6) \]

The inner product is given by

\[ \tilde{e}_i \cdot \tilde{e}_j = \begin{cases} +1 & (i = j = 1, 2) \\ 0 & (i \neq j) \\ -1 & (i = j = 3). \end{cases} \quad (3.7) \]

The two-dimensional lattice $\Lambda$ can be identified with a sublattice of $\Lambda_3$

\[ \Lambda \sim \left\{ \tilde{\alpha} = \sum_{i=1}^{3} n^i \tilde{e}_i \mid n^i \in \mathbb{Z}, \ n_1 + n_2 + n_3 = 0 \right\}. \quad (3.8) \]

For the cubic lattice $\Lambda_3$ it is easy to obtain the cocycle factor by using the Clifford algebra associated with the lattice [25]. The cocycle factor of $\Lambda$ can be obtained from that of $\Lambda_3$ by restricting it to the sublattice (3.8). Thus we find the following choice of the cocycle factor for $\Lambda$

\[ \varepsilon(\alpha_1, \alpha_2) = (-1)^{2s_1 J_1(s_2 J_2 - m_2 - 1)} \quad (3.9) \]

for the two-vector corresponding to the type $s_i(= \pm)$ discrete states

\[ \alpha_i = \sqrt{2} (m_i, s_i J_i - 1), \quad i = 1, 2. \quad (3.10) \]

It is easy to see that eq. (3.9) indeed satisfies the cocycle conditions (3.4).
With this cocycle factor we can construct the correction factor which is called the cocycle operator \[25\]

\[
c_\alpha = \sum_{\beta \in \Lambda} \varepsilon(\alpha, \beta) |\beta\rangle \langle \beta|,
\]

where $|\beta\rangle$ is an eigenstate of the energy and the momentum with an eigenvalue $\beta/\sqrt{2}$ as in eq. (3.2). Then the corrected operators

\[
\Psi'_J(z) = \Psi_J(z) c_\alpha, \quad \alpha = \sqrt{2} (m, sJ - 1)
\]

satisfy the OPEs which are analytic in the complex $z$ plane. The fields in eqs. (2.3) and (2.4) should also be replaced by corrected operators with cocycle operators included. Then the expression (2.3) is correct for half odd integer $J$ as well as for integer $J$. We find that after an appropriate rescaling the corrected operators (3.12) satisfy the same OPEs as those given in ref. [21]. The non-trivial OPEs are given by

\[
\begin{align*}
\tilde{\Psi}'_{J_1, m_1}^{(+)}(z) \tilde{\Psi}'_{J_2, m_2}^{(+)}(w) & \sim \frac{1}{z - w} (J_2 m_1 - J_1 m_2) \tilde{\Psi}'_{J_1 + J_2 - 1, m_1 + m_2}^{(+)}(w), \\
\tilde{\Psi}'_{J_1, m_1}^{(+)}(z) \tilde{\Psi}'_{J_1 + J_3 - 1, -m_1 + m_3}^{(-)}(w) & \sim \frac{1}{z - w} (-J_1 m_3 - J_3 m_1) \tilde{\Psi}'_{J_3, m_3}^{(-)}(w).
\end{align*}
\]

(3.13)

Other OPEs have no singular term. We have used rescaled fields

\[
\begin{align*}
\tilde{\Psi}'_{J, m}^{(+)}(z) &= \tilde{N}(J, m) \Psi_{J, m}^{(+)}(z), \\
\tilde{\Psi}'_{J, m}^{(-)}(z) &= (-1)^{J(2J-1)+J-m} \left[ \tilde{N}(J, m) \right]^{-1} \Psi_{J, m}^{(-)}(z),
\end{align*}
\]

(3.14)

where

\[
\tilde{N}(J, m) = (2J - 1)! \sqrt{\frac{J}{2}} N(J, m), \quad N(J, m) = \left[ \frac{(J + m)!(J - m)!}{(2J - 1)!} \right]^{\frac{1}{2}}.
\]

(3.15)
4 Tachyon OPE

We shall now generalize these results for the OPE to include the tachyon operator (2.2). Since the discrete state OPE (3.13) contains tachyon states with integer or half odd integer momenta as in eq. (2.5), it is clear that the tachyon OPE should also have cocycle factors in order to maintain analyticity. Although the construction of cocycle factors for a tachyon vertex operator with arbitrary momentum is difficult or perhaps ill-defined, we can construct cocycle factors restricted for our purposes. As we see below, we need to discuss the tachyon OPE in the case of two momenta adding up to integer or half odd integer values. Let us define the integer or half odd integer part $J$ of momentum $p$ as

$$J = \begin{cases} 
n & \text{for } n < p < n + \frac{1}{4} \\
n + \frac{1}{2} & \text{for } n + \frac{1}{4} < p < n + \frac{3}{4} \\
n + 1 & \text{for } n + \frac{3}{4} < p < n + 1 \end{cases} \quad (4.1)$$

for $n \in \mathbb{Z}$. From this definition we immediately find

$$J_1 + J_2 = p_1 + p_2 \quad \text{if} \quad p_1 + p_2 \in \mathbb{Z} \quad \text{or} \quad \mathbb{Z} + \frac{1}{2}. \quad (4.2)$$

We shall define the integer or half odd integer part $J$ for $p = n + \frac{1}{4}$ or $p = n + \frac{3}{4}$ as the limit of the above cases. Therefore two momenta adding up to integer or half odd integer actually correspond to the limit of one of the momenta approaching from above and the other from below. We can now associate a two-vector $\alpha$ on the lattice for the tachyon with momentum $p$ whose integer or half odd integer part is $J$ and whose chirality is $s$

$$\alpha = \sqrt{2} (J, sJ - 1). \quad (4.3)$$

These two-vectors belong to the lattice (3.2), and allow us to use the same cocycle factor $\varepsilon(\alpha, \beta)$ in eq. (3.9). Since the cocycle factor satisfies the same algebra as
before, it satisfies the cocycle condition. We can now define the cocycle operator for the tachyon with momentum $p$ using the corresponding two-vector $\alpha$ as

$$c_p = \sum_{\beta \in \Lambda} \varepsilon(\alpha, \beta) |\beta + \alpha - \hat{p}\rangle \langle \beta|, \quad \hat{p} = \sqrt{2} (p, sp - 1), \quad (4.4)$$

where the two-vector $\hat{p}$ is to cancel the effect of multiplying the zero mode part of the vertex operator $e^{ipX_0 + (sp - 1)\phi_0}$. More precisely speaking, we can add any fixed momentum $\bar{p}$ to all of the momentum eigenstates appearing in the cocycle operator. Precisely just as for the ordinary lattice for various Lie algebras [25], this momentum $\bar{p}$ just specifies the conjugacy class of momenta: the conjugacy class in the present case is defined by the relation that two momenta add up to give half integer values. We can choose $0 < \bar{p} < \frac{1}{4}$. In fact, the cocycle operators are only effective for two momenta from the same conjugacy class. With this cocycle operator, we can construct the corrected operator

$$\Psi_p^{(s)}(z) = \Psi_p^{(s)}(z)c_\alpha, \quad (4.5)$$

which can be shown to satisfy the OPEs with analytic coefficient functions.

From the conservation of energy and momentum we find that only four non-trivial OPEs are possible:

$$\begin{align*}
\Psi_{p_1}^{(+)}(z) \Psi_{p_2}^{(+)}(w) &\sim \frac{1}{z - w} F_{p_1p_2}^{(+)} J_{J_3,1-J_3}(w) \quad (J_3 = -p_1 - p_2 + 1), \\
\Psi_{p_1}^{(-)}(z) \Psi_{p_2}^{(-)}(w) &\sim \frac{1}{z - w} F_{p_1p_2}^{(-)} J_{J_3,1-J_3}(w) \quad (J_3 = p_1 + p_2 + 1), \\
\tilde{\Psi}_{J_1,J_1-1}^{(+)}(z) \Psi_{p_2}^{(+)}(w) &\sim \frac{1}{z - w} G_{J_1,J_1-1}^{(+)} \Psi_{p_3}^{(+)}(w) \quad (p_3 = J_1 - 1 + p_2), \\
\tilde{\Psi}_{J_1,J_1-1}^{(+)}(z) \Psi_{p_2}^{(-)}(w) &\sim \frac{1}{z - w} G_{J_1,J_1-1}^{(-)} \Psi_{p_3}^{(-)}(w) \quad (p_3 = 1 - J_1 + p_2).
\end{align*} \quad (4.6)$$

Since the discrete states with $m = \pm J$ are actually tachyons, the above OPEs contain the cases involving three tachyons and without any discrete state. We find that the above OPEs exhaust all possible three point couplings involving tachyons.

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We have to compute the coefficients $F$’s and $G$’s. The coefficient in the third OPE in eq. (4.6) can easily be obtained using the representation (2.3) for $\Psi^{(+)}_{J_{\pm 1}}$ and directly evaluating the OPE. The coefficient in the last OPE in eq. (4.6) can be evaluated similarly using the representation

$$
\Psi^{(+)}_{J_{1-1}}(z) = (-1)^{J_{1-1}} \frac{1}{\sqrt{2J}} \oint \frac{du}{2\pi i} H_+(u) \Psi^{(-)}_{-J_{1-1}}(z),
$$

which can be derived from eq. (2.3). In this way we find

$$
G^{(+)}_{J_{1p_2}} = \frac{\Gamma(1 + 2p_3)}{2 \Gamma(2p_3)} = (-1)^{2J_1} \frac{\Gamma(1 - 2p_2)}{2 \Gamma(-2p_2)} = -\tilde{N}(p_3, p_3) \tilde{N}(p_2, p_2)^{-p_2},
$$

$$
G^{(-)}_{J_{1p_2}} = (-1)^{J_{2(2J_{-1})+J_{3(2J_{-1})}+1}} \frac{\Gamma(1 + 2p_2)}{2 \Gamma(2p_3)} = (-1)^{J_{2(2J_{-1})+J_{3(2J_{-1})}+2J_1}} \frac{\Gamma(1 - 2p_2)}{2 \Gamma(-2p_2)} = (-1)^{J_{2(2J_{-1})+J_{3(2J_{-1})}+1}} \tilde{N}(p_2, p_2)^{-p_3},
$$

where $\tilde{N}(p, p) = \frac{1}{2} \Gamma(1 + 2p)$. One should note that the sign factors due to the cocycle $(-1)^{2J_1}$ and $(-1)^{2J_{2(2J_{-1})}}$ for the first and the second equations respectively have been included in the above formulas.

To obtain the coefficient of the first OPE in eq. (4.6), we apply the operator $\oint \frac{du}{2\pi i} H_-(u)$ to both hand sides of the equation, where the integration contour surrounds both of $z$ and $w$. The left hand side of the OPE becomes

$$
\oint \frac{du}{2\pi i} H_-(u) \Psi^{(+)}_{p_1}(z) \Psi^{(+)}_{p_2}(w) \sim \frac{1}{z - w} \oint \frac{dx}{2\pi i} x^{-2p_1} (1 + x)^{-2p_2} \Psi^{(-)}_{J_{3-1}}(w),
$$

where we have changed the integration variable to $x = (u - z)/z$ and the $x$-integration surrounds 0 and $-1$. The right hand side of the OPE becomes proportional to

$$
\oint \frac{du}{2\pi i} H_-(u) \Psi^{(-)}_{J_{31-3}}(w) = \sqrt{2J_3} \Psi^{(-)}_{J_{31-3}}(w).
$$

Therefore, by evaluating the integral in eq. (4.9) we obtain the coefficient $F^{(+)}$. The coefficient of the second OPE in eq. (4.6) can be obtained similarly by applying
We find

\[ F_{p_1 p_2}^{(+)} = (-1)^{2 J_1} \frac{\Gamma(1 - 2 p_1)}{2 \Gamma(2 p_2)} = (-1)^{2 J_2 - 1} \frac{\Gamma(1 - 2 p_2)}{2 \Gamma(2 p_1)} \]

\[ = (-1)^{2 J_1} \left[ N(p_1, p_1) \tilde{N}(p_2, p_2) \right]^{-1} \frac{\pi p_1 p_2}{2 \sin(2 \pi p_1)}, \]

\[ F_{p_1 p_2}^{(-)} = (-1)^{J_1 (2 J_1 - 1) + J_2 (2 J_2 - 1) - 2 J_1} \frac{\Gamma(1 + 2 p_2)}{2 \Gamma(-2 p_1)} \]

\[ = (-1)^{J_1 (2 J_1 - 1) + J_2 (2 J_2 - 1) + 2 J_2 - 1} \frac{\Gamma(1 + 2 p_1)}{2 \Gamma(-2 p_2)} \]

\[ = (-1)^{J_1 (2 J_1 - 1) + J_2 (2 J_2 - 1) + 2 J_1 - 1} \tilde{N}(p_1, p_1) \tilde{N}(p_2, p_2) \frac{2 \sin(2 \pi p_1)}{\pi}. \]

We have included the sign factor due to the cocycle \((-1)^{2 J_1}\) and \((-1)^{2 J_1 (2 J_2 - 1)}\) for the first and the second equations respectively.

### 5 Effective Action for Discrete States

The coefficients of the OPE determine the three-point correlation functions of the physical operators. From SL(2, R) symmetry the three-point function of the fields takes the form

\[ \langle \tilde{\Psi}_{J_1, m_1}^{(+)}(z_1) \tilde{\Psi}_{J_2, m_2}^{(+)}(z_2) \tilde{\Psi}_{J_3, m_3}^{(-)}(z_3) \rangle = c(J_1, m_1; J_2, m_2; J_3, m_3) \]

\[ = \frac{c(J_1, m_1; J_2, m_2; J_3, m_3)}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}. \]  

Therefore the three-point function of the integrated physical operators is given by the coefficient \(c(J_1, m_1; J_2, m_2; J_3, m_3)\). Using the first OPE of eq. (3.13) in eq. (5.1), we find that the coefficient is given by a product of the OPE coefficient and the two-point correlation function. Let us obtain the two-point function. From the SU(2) and SL(2, R) symmetries it takes the form

\[ \langle \tilde{\Psi}_{J_1, m_1}^{(+)}(z) \tilde{\Psi}_{J_2, m_2}^{(-)}(w) \rangle = \langle J_1 J_2 m_1 m_2|00\rangle c(J_1, J_2) \frac{1}{(z - w)^2} \]

\[ = \frac{(-1)^{J_2 - m_1}}{\sqrt{2 J_2 + 1}} c(J_1, J_2) \delta_{J_1, J_2} \delta_{m_1, -m_2} \frac{1}{(z - w)^2}. \]  

The coefficient \(c(J_1, J_2)\) can be determined by explicitly evaluating the two-point
function for \( J_1 = m_1 = J_2 = -m_2 \). Thus we find

\[
\langle \Psi^{(+)}_{J_1,m_1}(z) \Psi^{(-)}_{J_2,m_2}(w) \rangle = \delta_{J_1,J_2} \delta_{m_1,-m_2} \frac{(-1)^{J_2(2J_2-1)+J_2+m_2}}{(z-w)^2}.
\]  

(5.3)

Using eq. (5.3) we find the coefficient of the three-point function is given by

\[
c(J_1, m_1; J_2, m_2; J_3, m_3) = -(J_2m_1 - J_1m_2) \delta_{J_1+J_2-1,J_3} \delta_{m_1+m_2+m_3,0}.
\]

(5.4)

The result of the correlation functions can be summarized by the effective action, which reproduces the three-point function \( c(J_1, m_1; J_2, m_2; J_3, m_3) \) of the integrated physical operators. Introducing a variable \( g^{(s)}_{J,m} \) \((s = \pm)\) for each discrete state, the cubic terms of the effective action determined by the OPEs (3.13) are [21]

\[
S_3 = \frac{g_o}{2} \sum_{J_1,m_1,J_2,m_2,A,B,C} (J_2m_1 - J_1m_2) f^{ABC} g^{(-)A}_{J_1,J_2-1,-m_1-m_2} g^{(+)B}_{J_1,m_1} g^{(+)C}_{J_2,m_2} \int d\phi,
\]

(5.5)

where we have introduced the Chan-Paton index \( A \) in the adjoint representation of some Lie algebra and open string coupling constant \( g_o \).

In ref. [21] it was shown that the terms in the cubic interaction (5.5) which depend only on the integer modes \( g^{(s)}_{J,m} \) \((J, m \in \mathbb{Z})\) can be written in a compact form by introducing a scalar field on \( \mathbb{R} \times S^2 \)

\[
\Phi_0(\phi, \theta, \varphi) = \sum_{s,A,J,m} T^A g^{(s)A}_{J,m} M^s(J,m) D^J_{m0}(\varphi, \theta, 0) e^{(sJ-1)\phi}.
\]

(5.6)

Here, \( T^A \) are the representation matrices of the Lie algebra and \( D^J_{m0} \) are components of the SU(2) rotation matrix [24]

\[
D^J_{m m'}(\varphi, \theta, \psi) = \langle Jm | e^{-i\varphi J_z} e^{-i\theta J_y} e^{-i\psi J_z} | Jm' \rangle,
\]

(5.7)

\[ 0 \leq \varphi, \psi < 2\pi, \quad 0 \leq \theta \leq \pi. \]
The coefficients $M^s(J, m)$ are chosen to be

$$
M^+ (J, m) = \frac{N(J, m)N(J, 0)}{J}, \quad M^- (J, m) = \frac{(-1)^m}{4\pi} \frac{J(2J + 1)}{N(J, m)N(J, 0)}.
$$

(5.8)

In terms of the field $\Phi_0$ the effective action can be written as

$$
S_3^{(1)} = \frac{1}{3} ig_o \int d\phi e^{2\phi} \int d\theta d\varphi \epsilon^{ij} \text{Tr} \left( \Phi_0 \frac{\partial \Phi_0}{\partial x^i} \frac{\partial \Phi_0}{\partial x^j} \right),
$$

(5.9)

where $x^i = (\theta, \varphi)$.

We shall generalize this construction to the terms depending on half odd integer modes as well as integer modes. We introduce two spinor fields $\Phi^{\frac{1}{2}}$ and $\Phi^{\frac{-1}{2}}$ on $\mathbb{R} \times S^2$ for half odd integer modes $g^{(s)A}_{J, m}$ ($J, m \in \mathbb{Z} + \frac{1}{2}$)

$$
\Phi_{\mu}(\phi, \theta, \varphi) = \sum_{s, A, J, m} T^A g^{(s)A}_{J, m} M^s_{\mu}(J, m) D^J_m(\varphi, \theta, 0) e^{(sJ-1)\phi} \quad (\mu = \pm \frac{1}{2}),
$$

(5.10)

where

$$
M^+_{\mu}(J, m) = \frac{N(J, m)N(J, \frac{1}{2})}{J + \frac{1}{2}}, \quad M^-_{\mu}(J, m) = \frac{(-1)^{m+\mu}}{4\pi} \frac{2J(2J + 1)}{N(J, m)N(J, \frac{1}{2})},
$$

(5.11)

Note that $\Phi^{\frac{1}{2}}$ and $\Phi^{\frac{-1}{2}}$ have the same coefficients $g^{(s)A}_{J, m}$ and therefore are not independent. In order to write down the effective action in terms of these fields we need covariant derivatives on $S^2$ acting on spinor fields $\Phi_{\mu}$. They are given by

$$
\nabla_\pm = \mp \partial_\theta - \frac{1}{\sin \theta} (i \partial_\varphi - \mu \cos \theta)
$$

(5.12)

when acting on $\Phi_{\mu}$. They act on the rotation matrix as raising and lowering
operators:

\[ \nabla \pm D_{\mu}^{J} m(\varphi, \theta, 0) = \sqrt{(J \mp \mu)(J \pm \mu + 1)} D_{m \mu \pm 1}^{J}(\varphi, \theta, 0). \quad (5.13) \]

We also need an integration formula for the rotation matrix [24]

\[
\int d\theta d\varphi \sin \theta \ D_{m_{1} m_{2} m_{3}}^{J_{1} J_{2} J_{3}} (\varphi, \theta, 0) \ D_{m_{1} m_{2} m_{3}}^{J_{1} J_{2} J_{3}} (\varphi, \theta, 0) \\
= 2\pi \delta_{\mu_{1} + \mu_{2} + \mu_{3}, 0} \int d\theta d\varphi \sin \theta \ D_{m_{1} m_{2} m_{3}}^{J_{1} J_{2} J_{3}} (\varphi, \theta, 0) \ D_{m_{1} m_{2} m_{3}}^{J_{1} J_{2} J_{3}} (\varphi, \theta, 0) \\
= 8\pi^{2} \delta_{m_{1} + m_{2} + m_{3}, 0} \delta_{\mu_{1} + \mu_{2} + \mu_{3}, 0} \begin{pmatrix} J_{1} & J_{2} & J_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} \begin{pmatrix} J_{1} & J_{2} & J_{3} \\ \mu_{1} & \mu_{2} & \mu_{3} \end{pmatrix},
\]

where \( \begin{pmatrix} J_{1} & J_{2} & J_{3} \\ m_{1} & m_{2} & m_{3} \end{pmatrix} \) is the 3j-symbol. Using eqs. (5.13) and (5.14) the effective action can be written as

\[
S_{3}^{(2)} = g_{o} \int d\phi e^{2\phi} \int_{S^{2}} d\theta d\varphi \sin \theta \ \text{Tr} \left( \Phi_{0} \left[ \nabla_{-} \Phi_{1/2}, \nabla_{+} \Phi_{-1/2} \right] \right).
\quad (5.15)
\]

The sum of eqs. (5.9) and (5.15) gives the complete cubic terms for the discrete states (5.5).

We can also construct the effective action for the closed string. Similarly to the open string case (5.1), we can obtain the three-point correlation function of discrete state

\[
\Psi_{J, m, m'}^{(s)} (z, \bar{z}) = \Psi_{J, m}^{(s)} (z) \Psi_{J, m'}^{(s)} (\bar{z}).
\quad (5.16)
\]

The result can be summarized by the effective action. Introducing a closed string variable \( g_{J, m, m'}^{(s)} \) \( s = \pm \) for each discrete state, the cubic terms of the effective action determined by the OPEs (3.13) are [21]

\[
S_{3,c} = -\frac{g_{c}}{2} \sum_{J_{1}, m_{1}, m_{1}' J_{2}, m_{2}, m_{2}'} \left( J_{2} m_{1} - J_{1} m_{2} \right) \left( J_{2} m_{1}' - J_{1} m_{2}' \right) \\
\times g_{J_{1} + J_{2} - 1, \ldots, m_{2} - m_{2}'} (-) g_{J_{1}, m_{1}, m_{1}'} (+) g_{J_{2}, m_{2}, m_{2}'} (+) \int d\phi,
\]

where \( g_{c} \) is the closed string coupling constant. Following ref. [21] we define a
scalar field on $\mathbb{R} \times S^3$

$$\Phi_c(\phi, \theta, \varphi, \psi) = \sum_{s, J, m, m'} g^{(s)}_{J, m, m'} n^s(J, m, m') D^J_{mm'}(\varphi, \theta, \psi) e^{(sJ-1)\phi}. \quad (5.18)$$

Using eq. (5.14) one obtains the closed string action as [21]

$$S_{3,c} = \frac{g_c}{3!} \int d\phi e^{2\phi} \frac{1}{8\pi^2} \int_{S^3} d\theta d\varphi d\psi \sin \theta \Phi_c^3. \quad (5.19)$$

In obtaining the above result, we find the normalization factor in eq. (5.18)

$$n^+(J, m, m') = \sqrt{(J + m)!(J - m)!(J + m')!(J - m')!}, \quad (2J - 1)!,$$

$$n^-(J, m, m') = (-1)^{m-m'+1} \frac{J(J + 1)(2J + 1)(2J - 1)!}{\sqrt{(J + m)!(J - m)!(J + m')!(J - m')!}. (5.20)}$$

6 Effective Action involving Tachyons

Since the noncompact case can be obtained by taking the infinite radius limit, we can consider, without loss of generality, a string compactified on a circle with a radius $R$

$$X \sim X + 2\pi R = X + 2\pi a \sqrt{\alpha'}, \quad (6.1)$$

where $a$ is the reduced radius in unit of the self dual radius $\sqrt{\alpha'}$. Both discrete momenta and winding numbers should be taken into account in the case of the closed string. We normalize discrete momenta $p$ and winding numbers $w$

$$p = \frac{n}{2a}, \quad w = \frac{ma}{2}, \quad n, m \in \mathbb{Z} \quad (6.2)$$

to take integer or half odd integer values at the self dual radius $a = 1$. It is also convenient to define the left and right moving momenta as linear combinations
\[ p_L = \frac{p + w}{2}, \quad p_R = \frac{p - w}{2}. \] (6.3)

At the self-dual radius \( a = 1 \), the discrete momenta and the winding numbers are restricted to those corresponding to discrete states. In order to have couplings with the discrete states, we shall take the radius of the compact boson to be a fractional multiple of the self-dual radius

\[ a = \frac{N}{M}, \quad N, M \in \mathbb{Z}. \] (6.4)

Noncompact \( X \) can be obtained simply by taking the limit of infinite radius \( R = \sqrt{\alpha' N/M} \to \infty \). Hence we do not lose any generality.

Here we shall discuss the open string case by taking only the right or left moving part of the closed string. As in the previous section we obtain the three-point functions involving tachyons from the OPEs (4.6)

\[ \langle \Psi_{p_1}^{(+)}(z_1) \Psi_{p_2}^{(+)}(z_2) \tilde{\Psi}_{J_3,m_3}^{(+)}(z_3) \rangle = \frac{c^{(+)}(p_1; p_2; J_3, m_3)}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}, \]
\[ \langle \Psi_{p_1}^{(-)}(z_1) \Psi_{p_2}^{(-)}(z_2) \tilde{\Psi}_{J_3,m_3}^{(+)}(z_3) \rangle = \frac{c^{(-)}(p_1; p_2; J_3, m_3)}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}, \] (6.5)

where the coefficients, which are equal to the integrated correlation functions are

\[ c^{(+)}(p_1; p_2; J_3, m_3) = \delta_{1-p_1-p_2,J_3} \delta_{m_3,J_3-1} (-1)^{2J_2} \frac{\Gamma(1-2p_2)}{2 \Gamma(2p_1)}, \]
\[ c^{(-)}(p_1; p_2; J_3, m_3) = \delta_{1+p_1+p_2,J_3} \delta_{m_3,1-J_3} \times (-1)^{J_1(2J_1-1)+J_2(2J_2-1)+J_3+1} \frac{\Gamma(1+2p_2)}{2 \Gamma(-2p_1)}. \] (6.6)

From these correlation functions we find that there are two terms in the action
for a three point coupling involving tachyons in the open string

\[ S_{T,o} = S_{T,o}^{(+)} + S_{T,o}^{(-)}. \]  

(6.7)

The action for tachyons with positive chirality is given in terms of variables \( g_{p}^{(+)}A \), and

\[ S_{T,o}^{(+)} = \frac{g_{o}}{2} \sum_{A,B,C} f^{ABC} \sum_{n_{1} \in \mathbb{Z}} \sum_{J_{3}=1/2}^{\infty} (-1)^{2J_{2}+1} \frac{\Gamma(1-2p_{2})}{2 \Gamma(2p_{1})} g_{p_{1}}^{(+)A} g_{p_{2}}^{(+)}B g_{J_{3},J_{3}-1}^{(+)} \int d\phi, \]

\[ p_{1} = \frac{n_{1}}{2a}, \quad p_{2} = -p_{1} - J_{3} + 1. \]  

(6.8)

One should note that the tachyon coupling \( g_{o} \) is the same as the discrete state coupling in eq. (5.5). The action for tachyons with negative chirality is given in terms of variables \( g_{p}^{(-)}A \), and

\[ S_{T,o}^{(-)} = \frac{g_{o}}{2} \sum_{A,B,C} f^{ABC} \sum_{n_{1} \in \mathbb{Z}} \sum_{J_{3}=1/2}^{\infty} (-1)^{J_{1}(2J_{1}-1)+J_{2}(2J_{2}-1)+2J_{1}} \]

\[ \times \frac{\Gamma(1+2p_{2})}{2 \Gamma(-2p_{1})} g_{p_{1}}^{(-)A} g_{p_{2}}^{(-)}B g_{J_{3},1-J_{3}}^{(+)} \int d\phi, \]

\[ p_{1} = \frac{n_{1}}{2a}, \quad p_{2} = -p_{1} + J_{3} - 1. \]  

(6.9)

Since the natural mode function for the tachyon is given by the momentum eigenstate, we introduce a tachyon field for positive and negative chirality

\[ T^{A}(\phi, X) = T^{(+)}A(\phi, X) + T^{(-)}A(\phi, X), \]

\[ T^{(+)}A(\phi, X) = \sum_{n_{1} \in \mathbb{Z}} \frac{1}{\sqrt{4\pi a}} \Gamma(1-2p_{2}) e^{ip_{X}} e^{(p-1)\phi} g_{p_{2}}^{(+)}A g_{p_{1}}^{(-)A}, \]  

(6.10)

\[ T^{(-)}A(\phi, X) = \sum_{n_{1} \in \mathbb{Z}} \frac{1}{\sqrt{4\pi a}} (-1)^{J(2J-1)} \Gamma(1+2p_{2}) e^{ip_{X}} e^{(-p-1)\phi} g_{p_{2}}^{(-)A} g_{p_{1}}^{(-)A}. \]

To write down the action we need another field constructed from the same coeffi-
By rewriting the variables \( g_{p}^{(\pm)A} \) in terms of the field \( T^{(\pm)A} \) and \( \tilde{T}^{(\pm)A} \), we can obtain the coordinate representation of the tachyon action (6.7)

\[
S_{T,o}^{(+)} = -\frac{1}{8\pi i g_o} \sum_{A,B,C} f^{ABC} \int d\phi e^{2\phi} \int_{-2\pi a}^{2\pi a} dX \times \left[ T^{(+)A}(\phi, X + 2\pi) - T^{(+)A}(\phi, X - 2\pi) \right] \tilde{T}^{(+)B}(\phi, X) \Psi^{C}(\phi, X),
\]

\[
S_{T,o}^{(-)} = -\frac{1}{8\pi i g_o} \sum_{A,B,C} f^{ABC} \int d\phi e^{2\phi} \int_{-2\pi a}^{2\pi a} dX \times \left[ \tilde{T}^{(-)A}(\phi, X + 2\pi) - \tilde{T}^{(-)A}(\phi, X - 2\pi) \right] T^{(-)B}(\phi, X) \Psi^{C}(\phi, X),
\]

where we have introduced the \( X \) representation of the discrete states

\[
\Psi^{A}(\phi, X) = \sum_{s=\pm} \Psi^{(s)A}(\phi, X),
\]

\[
\Psi^{(s)A}(\phi, X) = \sum_{J=1/2}^{\infty} \sum_{m=-J}^{J} e^{imX} e^{(sJ-1)\phi} g_{J,m}^{(s)A}.
\]

Actually only the discrete states with \( s = + \) and \( m = \pm(J - 1) \) contribute to the action.

Now we shall discuss the closed string case by combining the left and right moving part. Since the Liouville zero mode is effectively noncompact for the bulk amplitudes, the Liouville energy has to be common to left and right movers. If
the left moving and right moving momenta are equal, the tachyon corresponds to the center of mass motion and has discrete momenta \( p \) but without winding numbers. We shall denote the field for such a mode as \( g_p^{(s,s)} \) with \( s \) being the common chirality of left and right movers. If the left moving and right moving momenta have opposite sign, the tachyon has winding numbers \( w \) but without discrete momenta. We shall denote the field for such a mode as \( g_w^{(s,-s)} \) with \( s \) being the chirality of left movers.

From the OPEs (4.6), we find that there are two terms in the action for three point couplings involving tachyons

\[
S_{T,c} = S_{T,c}^{cm} + S_{T,c}^{wd}. \tag{6.14}
\]

The action for tachyons with discrete momenta (corresponding to the center of mass motion) is given by

\[
S_{T,c}^{cm} = -\frac{g_c}{2} \sum_{n_1 \in \mathbb{Z}} \sum_{J_3=1/2}^{\infty} \sum_{s=\pm} \left[ \frac{\Gamma(1-2sp_2)}{2\Gamma(2sp_1)} \right]^2 g_{p_1}^{(s,s)} g_{p_2}^{(s,s)} g_{J_3,s(J_3-1),s(J_3-1)} \int d\phi,
\]

\[
p_1 = \frac{n_1}{2a}, \quad p_2 = -p_1 - s(J_3 - 1). \tag{6.15}
\]

One should note that the tachyon coupling \( g_c \) is the same coupling as the discrete state coupling in eq. (5.17). The action for tachyons with winding numbers is given by

\[
S_{T,c}^{wd} = \frac{g_c}{2} \sum_{n_1 \in \mathbb{Z}} \sum_{J_2=1/2}^{\infty} \sum_{s=\pm} (-1)^{J_1(2J_1+1)+J_2(2J_2+1)+2J_3} \times \left[ \frac{\Gamma(1-2sp_2)}{2\Gamma(2sp_1)} \right]^2 g_{p_1}^{(s,-s)} g_{p_2}^{(s,-s)} g_{J_3,s(J_3-1),-s(J_3-1)} \int d\phi,
\]

\[
p_1 = \frac{n_1 a}{2}, \quad p_2 = -p_1 - s(J_3 - 1). \tag{6.16}
\]

Exactly analogous to the open string case, we introduce the closed string
tachyon field for center of mass motion

\[
T^{\text{cm}}(\phi, X) = \sum_{s=\pm} T^{\text{cm}(s)}(\phi, X),
\]

\[
T^{\text{cm}(s)}(\phi, X) = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{4\pi a}} \frac{\Gamma(1 - 2sp) \Gamma(2sp)}{\Gamma(2sp)} e^{ipX} e^{(sp-1)\phi} g_{p=\pm}^{(s,s)}.
\]  

(6.17)

where \(X\) is the center of mass coordinate (multiplied by \(2/\sqrt{\alpha'}\)). Similarly we introduce the closed string tachyon field for the winding modes

\[
T^{\text{wd}}(\phi, X_r) = \sum_{s=\pm} T^{\text{wd}(s)}(\phi, X_r),
\]

\[
T^{\text{wd}(s)}(\phi, X_r) = \sum_{n \in \mathbb{Z}} \sqrt{\frac{a}{4\pi}} (-1)^{J(2J+1)} \frac{\Gamma(1 - 2sp) \Gamma(2sp)}{\Gamma(2sp)} e^{ip_r X_r} e^{(sp-1)\phi} g_{p_r=\pm}^{(s,s)}.
\]  

(6.18)

We also need a field in the \(X\) representation for the discrete states of closed string

\[
\Psi^{\text{cm}}_c(\phi, X) = \sum_{s=\pm} \Psi^{\text{cm}(s)}_c(\phi, X), \quad \Psi^{\text{wd}}_c(\phi, X_r) = \sum_{s=\pm} \Psi^{\text{wd}(s)}_c(\phi, X_r),
\]

\[
\Psi^{\text{cm}(s)}_c(\phi, X) = \sum_{J=1/2}^{\infty} \sum_{m=-J}^{J} (-1)^{2J} e^{imX} e^{(sJ-1)\phi} g_{J,m,m}^{(s)}, \quad (6.19)
\]

\[
\Psi^{\text{wd}(s)}_c(\phi, X_r) = \sum_{J=1/2}^{\infty} \sum_{m=-J}^{J} (-1)^{2J} e^{imX_r} e^{(sJ-1)\phi} g_{J,m,-m}^{(s)}.
\]

Actually only the discrete states with \(s = +\) and \(m = \pm(J - 1)\) contribute to the action.

From eq. (6.14) we find the coordinate representation of the cubic terms in the effective action involving tachyons

\[
S^{\text{cm}}_{T,c} = \frac{g_c}{8} \int d\phi e^{2\phi} \int_{-2\pi a}^{2\pi a} dX T^{\text{cm}}(\phi, X) T^{\text{cm}}(\phi, X) \Psi^{\text{cm}}_c(\phi, X),
\]

\[
S^{\text{wd}}_{T,c} = \frac{g_c}{8} \int d\phi e^{2\phi} \int_{-2\pi/a}^{2\pi/a} dX_r T^{\text{wd}}(\phi, X_r) T^{\text{wd}}(\phi, X_r) \Psi^{\text{wd}}_c(\phi, X_r).
\]  

(6.20)

We have succeeded to write the effective action involving tachyons in a space
with two flat variables $\phi$ and $X$ or $X_r$. On the other hand, the effective action for discrete states is written down in a space with $\mathbb{R} \times S^2 \ (\mathbb{R} \times S^3)$ for the open (closed) string. It is possible to consider any convenient space to construct the coordinate representation of an action. However, it is certainly desirable to use the same space and the same field for the same dynamical degree of freedom to write down various pieces of an effective action. Since the momentum conservation in the $X$ representation corresponds precisely to the magnetic quantum number conservation in the angular representation, we have tried to use the angular coordinates $\theta, \varphi$ and $\psi$ on the sphere to describe the tachyon field. However, we were not quite successful, apparently because the natural wave function of the tachyon is the momentum eigenstate in the $X$ representation.

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