POLYTOPES WITH MASS LINEAR FUNCTIONS II: THE 4-DIMENSIONAL CASE

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Abstract. This paper continues the analysis begun in Polytopes with mass linear functions, Part I of the structure of smooth moment polytopes $\Delta \subset t^*$ that support a mass linear function $H \in t$. As explained there, besides its purely combinatorial interest, this question is relevant to the study of the homomorphism $\pi_1(T^n) \to \pi_1(\text{Symp}(M,\omega))$ from the fundamental group of the torus $T^n$ to that of the group of symplectomorphisms of the $2n$-dimensional symplectic toric manifold $(M,\omega)$ associated to $\Delta$.

In Part I, we made a general investigation of this question and classified all mass linear pairs $(\Delta, H)$ in dimensions up to three. The main result of the current paper is a classification of all 4-dimensional examples. Along the way, we investigate the properties of general constructions such as fibrations, blowups and expansions (or wedges), describing their effect both on moment polytopes and on mass linear functions.

We end by discussing the relation of mass linearity to Shelukhin’s notion of full mass linearity. The two concepts agree in dimensions up to and including 4. However full mass linearity may be the more natural concept when considering the question of which blow ups preserve mass linearity.

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Date: May 18, 2011.
2000 Mathematics Subject Classification. 14M25,52B20,53D99,57S05.
Key words and phrases. simple polytope, Delzant polytope, center of gravity, toric symplectic manifold, mass linear function, Hamiltonian group, symplectomorphism group.

First author partially supported by NSF grant DMS 0905191, and second by NSF grant DMS 0707122.
4. Introduction

1.1. Statement of main results. This paper continues the analysis begun in [8] of the structure of smooth polytopes $\Delta$ that support an essential mass linear function $H$. As we show there, besides its purely combinatorial interest, this question is relevant to the study of the homomorphism $\pi_1(T) \to \pi_1(\text{Symp}(M_\Delta, \omega_\Delta))$ from the fundamental group of the torus $T$ to that of the group of symplectomorphisms of the symplectic toric manifold $(M_\Delta, \omega_\Delta, T)$ associated to $\Delta$. The paper [6] describes other applications, such as understanding when a product manifold of the form $(M \times S^2, \omega + \sigma)$ has more than one toric structure.

In [8] (from now on called Part I), we made a general investigation of the properties of mass linear functions and classified all essential mass linear pairs $(\Delta, H)$ in dimensions up to three. The main result of the current paper is a classification of all 4-dimensional examples. We also develop new tools for understanding the topological properties of symplectic toric manifolds.

Before stating our results we shall remind the reader of some of the basic concepts introduced in Part I; more details can be found there.

Let $t$ be a real vector space with integer lattice $t \mathbb{Z} \subset t$. Let $t^*$ denote the dual space and let $\langle \cdot , \cdot \rangle : t \times t^* \to \mathbb{R}$ denote the natural pairing. A (convex) polytope $\Delta \subset t^*$ is the bounded intersection of a finite set of affine half-spaces. In this paper, we shall always write $\Delta$ in the form

\begin{equation}
\Delta = \bigcap_{i=1}^{N} \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \},
\end{equation}

where the outward conormals $\eta_i$ lie in $t$ and the support numbers $\kappa_i$ lie in $\mathbb{R}$ for all $1 \leq i \leq N$. We always assume that $\Delta$ has a nonempty interior, and that the affine span of each facet $F_i := \Delta \cap \{ x \in t^* \mid \langle \eta_i, x \rangle = \kappa_i \}$ is a hyperplane. Further, we assume that $\Delta$ is smooth, that is, for each vertex $v$ of $\Delta$ the primitive outward conormals to the facets which meet at $v$ form a basis for the integral lattice $t_2$ of $t$. In particular, a smooth polytope is simple, that is, $\dim t$ facets meet at every vertex.

Given a polytope $\Delta \subset t^*$, let $c_\Delta$ denote its center of mass, considered as a function of the support constants $\kappa$. An element $H \in t$ is said to be mass linear on $\Delta = \Delta(\kappa)$ if the
function \( \tilde{H} : \kappa' \mapsto \langle H, c_\Delta(\kappa') \rangle \)
is linear for all \( \kappa' \) near \( \kappa \); cf. \cite[Definition 1.2 and Lemma 2.3]{I}. In this case there are real
numbers \( \beta_i \), called the coefficients of \( H \) such that \( \langle H, c_\Delta(\kappa') \rangle = \sum \beta_i \kappa'_i \)
for all \( \kappa' \) near \( \kappa \).

To explain the important distinction between essential and inessential mass linear functions, we introduce an equivalence relation on the facets. Following \cite[Definition 1.12]{I} (and \cite[Corollary 3.5 and Remark 1.6]{I}), we say that two distinct facets \( F_i \) and \( F_j \) are equivalent, denoted \( F_i \sim F_j \), exactly if there is an integral affine transformation of \( \Delta(\kappa) \) that takes \( F_i \) to \( F_j \) and is robust, in the sense that it persists when \( \kappa \) is perturbed. Let \( \mathcal{I} \) denote the set of
equivalence classes of facets. We say that \( H \in \mathfrak{t} \) is inessential iff
\[
H = \sum \beta_i \eta_i, \quad \text{where } \beta_i \in \mathbb{R} \forall i \text{ and } \sum_{i \in I} \beta_i = 0 \forall I \in \mathcal{I}.
\]

Otherwise, we say that \( H \) is essential. By Proposition \ref{prop:essential_inequivalence}, every inessential function is mass linear.

As an example consider the standard \( k \)-simplex \( \Delta_k \), that is,
\[
\Delta_k = \left\{ x \in \mathbb{R}^k \mid 0 \leq x_i \forall i \text{ and } \sum_{i=1}^k x_i = 1 \right\}.
\]
Any pair of facets of \( \Delta_k \) is equivalent, and so every \( H \in \mathfrak{t} \) is inessential.

To understand the implications of these definitions, consider the symplectic toric manifold
\((M_, \omega_\Delta, \Phi_\Delta)\) with moment image \( \Phi_\Delta(M_\Delta) = \Delta \). Let \( \text{Symp}(M_\Delta, \omega_\Delta) \) denote the group
of symplectomorphisms of \((M_\Delta, \omega_\Delta)\), and let \( \text{Isom}(M_\Delta, \omega_\Delta) \) denote the group of Kähler
isometries, that is, the subgroup of symplectomorphisms that also preserve the canonical
complex structure on \( M \). As we showed in \cite[§1.2]{I}, if the circle \( \Lambda_H \) generated by \( H \in \mathbb{t}_\mathbb{Z} \) has
finite order in \( \pi_1(\text{Symp}(M_\Delta, \omega_\Delta)) \), then \( H \) is mass linear. Moreover, \( \Lambda_H \) has finite order in
\( \pi_1(\text{Isom}(M_\Delta, \omega_\Delta)) \) exactly if \( H \) is inessential. Finally, if there are no essential mass linear
functions on \( \Delta \), then the natural map \( \pi_1(\text{Isom}(M, \omega)) \to \pi_1(\text{Symp}(M, \omega)) \) is an injection.
For more details, see \cite[§1.2]{I}.

Most polytopes do not admit nonzero mass linear functions. We showed in Part I that
in dimension two the only ones that do are the triangle, the parallelogram, and trapezoids,
corresponding respectively to the projective plane \( \mathbb{C}P^2 \), the product \( S^2 \times S^2 \) and the different
Hirzebruch surfaces (\( S^2 \)-bundles over \( S^2 \)). Moreover, in each case all mass linear functions
are inessential.

In dimension three, although there are more examples of polytopes with mass linear
functions (see Proposition \ref{prop:mass_linear}), there are very few with essential mass linear functions.
To describe these, we need the notion of “bundle”, which is given in Definition \ref{def:bundle} below.
One key fact about bundles is that if a smooth polytope \( \Delta \) is a bundle over \( \Delta \) with fiber \( \hat{\Delta} \), then the corresponding toric manifold \( M_\Delta \) is a bundle over \( M_\hat{\Delta} \) with fiber \( M_{\hat{\Delta}} \); see \cite[Remark 5.2]{I}. In Part I, we showed that every smooth 3-dimensional polytope which admits

\footnote{In fact, McDuff shows in \cite[§4]{6} that \( H \) is mass linear precisely if the rational cohomology ring of the toric bundle \( M_\Delta \times_{\Lambda_H} S^3 \to S^2 \) is isomorphic to the product ring \( H^*(M_\Delta) \otimes H^*(S^3) \).}
an essential mass linear function is a $\Delta_2$ bundle over $\Delta_1$. Since the moment image of $\mathbb{C}P^n$ is $\Delta_2$, this implies that, $M_\Delta$ is a $\mathbb{C}P^2$ bundle over $\mathbb{C}P^1$.

The analogous statement in dimension 4 is more complicated because there is a much greater variety of examples. Correspondingly, we need to introduce new terminology. Blowups are defined in Definition 2.4.1. As the name suggests, if $\Delta'$ is the blowup of a polytope $\Delta$, then the corresponding toric manifold $M_{\Delta'}$ is the blowup of $M_\Delta$; see Remark 2.4.4 (i). Double expansions are defined in Definition 2.3.5. By [I, Remark 5.4], a polytope $\Delta$ that is an expansion of $\tilde{\Delta}$ corresponds to a toric manifold that is a nonsingular pencil with fibers $M_{\tilde{\Delta}}$. Thus if $\Delta$ is a double expansion, the corresponding toric manifold is a “double pencil”.

Additionally, fix $H \in t$ and a polytope $\Delta \subset t^\ast$. We say that a facet is symmetric (or $H$-symmetric), if $\langle H, c_{\Delta}(\kappa) \rangle$ does not change when that facet is moved. Otherwise, we say that the facet is asymmetric (or $H$-asymmetric). Further, we say that a facet is pervasive if it meets all other facets.

Finally, a face of $\Delta$ is a (nonempty) intersection of some collection of facets of $\Delta$; a $k$-face is a face of dimension $k$. We denote the faces of $\Delta$ by $F_\Delta$. Then we say that the face $F_I$ is symmetric if the facet $F_i$ is symmetric for each $i \in I$.

**Theorem 1.1.1.** Let $H \in t$ be an essential mass linear function on a smooth 4-dimensional polytope $\Delta \subset t^\ast$. There exists a smooth 4-dimensional polytope $\bar{\Delta} \subset t^\ast$ so that either:

(a) $H$ is an essential mass linear function on $\bar{\Delta}$ and at least one of the following statements is true:
   (a1) $\bar{\Delta}$ is a $\Delta_3$ bundle over $\Delta_1$,
   (a2) $\bar{\Delta}$ is a $\Delta_1$ bundle over a polytope which is a $\Delta_2$ bundle over $\Delta_1$,
   (a3) $\bar{\Delta}$ is a $\Delta_2$ bundle over a polygon $\tilde{\Delta}$;

(b) $H$ is inessential on $\bar{\Delta}$, the polytope $\bar{\Delta}$ is the double expansion of a polygon $\tilde{\Delta}$, and the asymmetric facets are the four base-type facets.

Moreover, $\Delta := \bar{\Delta}(m)$ can be obtained from $\bar{\Delta} := \bar{\Delta}(0)$ by a series of blowups. For each $k = 1, \ldots, m$, the polytope $\bar{\Delta}(k)$ is obtained from $\bar{\Delta}(k - 1)$ by blowing up either along a symmetric 2-face or along an edge of the form $F_{ij} \cap \bar{G} := F_i \cap F_j \cap \bar{G}$, where $\bar{G}$ is a symmetric facet of $\bar{\Delta}(k - 1)$, $F_{ij} \cap \bar{G}$ intersects every asymmetric facet, and $\gamma_i + \gamma_j = 0$. Here $\gamma_i$ is the coefficient of the support number of the facet $F_i$ in the linear function $\langle H, c_{\bar{\Delta}} \rangle$.

Combining this with the results of [I, §1.2], we obtain the following corollary.

**Corollary 1.1.2.** Let $(M, \omega)$ be an 8-dimensional symplectic toric manifold. The natural map

$$\pi_1(\text{Isom}(M, \omega)) \to \pi_1(\text{Symp}(M, \omega))$$

is injective unless $M$ is a very special blowup of either a double pencil or a bundle. Moreover the bundle either has $\mathbb{C}P^2$ or $\mathbb{C}P^3$ as its fiber or has a $\mathbb{C}P^2$ bundle over $\mathbb{C}P^1$ as its base.

The following results elaborate the statement of Theorem 1.1.1.

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2For most polytopes $\bar{\Delta}$, only one of these statements is true. The one exception is $\Delta_2 \times \Delta_1$ bundles over $\Delta_1$, which belong in case (a2) and case (a3).
• We give a complete description of all the essential mass linear functions \( H \) on polytopes \( \Delta \) satisfying conditions (a1), (a2), and (a3) in Corollary 3.1.4, Proposition 3.1.6, and Proposition 3.1.8, respectively. The inessential functions on a polytope \( \Delta \) satisfying condition (b) are described in Lemma 3.2.1.

• By Lemma 2.4.8 and Proposition 2.4.10, a mass linear function on a polytope will still be mass linear if the polytope is blown up by the types of blowups described above; moreover, an essential function will still be essential on the blowup. In fact, the linear function \( \langle H, c(\Sigma(k)) \rangle \) is unchanged under blowup: the exceptional divisor of each blowup is symmetric and so has zero coefficient, while the coefficients of the other facets remain the same.

• We explain in Proposition 3.2.2 exactly which blowups on the polytope \( \Delta \) described in (b) convert \( H \) from an inessential to an essential mass linear function.

Combining the above results, we can draw a number of conclusions.

1. In each case of part (a) of Theorem 1.1.1 some, but not all, polytopes \( \Delta \) of the given form support essential mass linear functions. In case (a1) and (a2) one can take the bundle \( \Delta \) to be generic. However, in case (a3), while the base \( \hat{\Delta} \) can be any polygon except a triangle, the bundle itself must satisfy some very special conditions that have a topological interpretation; see Proposition 3.1.12. Similarly, in case (b) the polygon \( \Delta \) can be anything except a triangle.

2. For any essential mass linear function of type (a), the polytope \( \Delta \) has between 3 and 7 asymmetric facets; see Remark 3.1.14. In all cases \( \Delta \) can have 3 asymmetric facets. However, it is only possible to have 4 or 6 asymmetric facets in the case (a1), and the only case with 7 asymmetric facets is the product of \( \Delta_1 \) with a \( \Delta_2 \) bundle over \( \Delta_1 \). Further in case (b) \( \Delta \) always has 4 asymmetric facets. Since, by Lemma 2.4.7, blowing down the facets of a polytope with a mass linear function does not change the number of asymmetric facets, the original polytope \( \Delta \) has the same number of asymmetric facets.

3. If the polytope \( \Delta \) has more than four asymmetric facets then in all cases \( \Delta = \Delta \), that is, no blowups are possible. Therefore, every polytope that supports an essential mass linear function with more than 6 asymmetric facets is the product of \( \Delta_1 \) with a \( \Delta_2 \) bundle over \( \Delta_1 \); the only examples with 6 asymmetric facets are \( \Delta_3 \) bundles over \( \Delta_1 \).

   If it has three or four asymmetric facets then the situation is more complicated. However, by Proposition 2.4.10, the edge blowups described in Theorem 1.1.1 are only possible when there are four asymmetric facets. For more details see Remark 3.1.15.

4. In all cases \( \sum \gamma_i = 0 \); see Corollary 5.2.5 and Remark 5.2.6.

5. The mass linear functions on a given polytope \( \Delta \) form a vector space \( V_{ML} \), with a subspace \( V_{iness} \) consisting of inessential functions. If \( \Delta \) is 4-dimensional, then the quotient \( V_{ML}/V_{iness} \) has dimension at most 1 unless \( \Delta \) is a \( \Delta_3 \) bundle over \( \Delta_1 \), in which case it has dimension at most 2.
Remark 1.1.3. Suppose that $H$ is a mass linear function on $\Delta$ and that $\overline{\Delta}$ is obtained from $\Delta$ by any sequence of blowdowns. Then Lemma 2.4.7 shows that $H$ is mass linear on $\overline{\Delta}$. Therefore, when classifying mass linear functions one may assume that the underlying polytope is minimal, i.e., that none of its facets can be blown down.

The results in §5.1 show that Theorem 1.1.1 still holds with the additional requirement that $\overline{\Delta}$ is minimal as long as we omit the last two sentences, which restrict the kinds of blowups allowed. For further details, see Remark 5.1.16 (i).

On the other hand, allowing arbitrary blowups does not allow us to reduce the list of examples; the results of §5.1 also show that there exist minimal polytopes of each type. In fact, for any sufficiently large $N$ there exists a minimal polytope $\overline{\Delta}$ with $N$ facets which satisfies the conditions of case (a3) or (b) of Theorem 1.1.1; see Propositions 5.1.12 and 5.1.15. In contrast, the polytopes $\overline{\Delta}$ described in (a1) and (a2) have 6 and 7 facets, respectively.

Remark 1.1.4. (i) Many of our constructions and intermediate results are valid for all simple polytopes. We wrote much of Part I in this generality, although our main classification theorem was only for smooth polytopes. In this paper we assume throughout that $\Delta$ is smooth.

(ii) We have not explicitly written down a list of all the 4-dimensional smooth polytopes that admit nonzero mass linear functions because the answer is too messy to be very enlightening. However, this information is easy to extract from our paper. On the one hand, Propositions 2.1.1 and 2.3.3 allow us to classify polytopes which admit inessential mass linear functions with at most three asymmetric facets. On the other hand, Propositions 4.1.2 and 4.1.4 classify all 4-dimensional smooth polytopes which admit essential mass linear functions with exactly three asymmetric facets, and Proposition 4.2.4, 4.2.6, 4.3.1, and 4.3.4 classify all 4-dimensional smooth polytopes which admit mass linear functions with at least four asymmetric facets.

We end this section with an example, which demonstrates how blowups can transform an inessential function into an essential mass linear function. Hence, it is an example of case (b) of Theorem 1.1.1 and not case (a1).

Example 1.1.5. Let $\overline{\Delta} \subset (\mathbb{R}^4)^*$ be the $\Delta_3$ bundle over $\Delta_1$ with conormals

$$\eta_1 = (-1, 0, 0, 0), \; \eta_2 = (0, -1, 0, 0), \; \eta_3 = (0, 0, -1, 0), \; \eta_4 = (1, 1, 1, 0),$$

$$\alpha_1 = (0, 0, 0, -1), \; \text{and} \; \alpha_2 = (-1, -1, 0, 1).$$

The polytope $\overline{\Delta}$ is also the double expansion of the trapezoid with conormals $(-1, 0), \; (0, -1), \; (1, 1), \; \text{and} \; (-1, -1)$ along its two parallel facets.

Denote by $F_i$ and $G_j$ the facets with conormals $\eta_i$ and $\alpha_j$, respectively. By Lemma 2.1.9 two facets are equivalent exactly if the conormals of all the other facets lie in a 3-dimensional subspace. Hence, $F_1 \sim F_2 \not\sim F_3 \sim F_4$, and so the function

$$H := \eta_1 - \eta_2 - \eta_3 + \eta_4$$
is inessential on \( \overline{\Delta} \). By Proposition 2.1.1, this implies that \( H \) is mass linear on \( \overline{\Delta} \); in fact, 
\[
\langle H, c_\Delta(\kappa) \rangle = \kappa_1 - \kappa_2 - \kappa_3 + \kappa_4.
\]

Now consider the blowup \( \Delta \) of \( \overline{\Delta} \) along the edge \( F_{24} \cap G_1 \). This has a new facet \( G_0 \) (the exceptional divisor) with conormal 
\[
\alpha_0 = \eta_2 + \eta_4 + \alpha_1 = (1, 0, 1, -1).
\]
None of the facets \( F_i \) are equivalent in \( \Delta \). However, Proposition 2.4.10 and (1.2) together imply that \( H \) is still mass linear on \( \Delta \). Therefore, \( H \) is essential on \( \Delta \).

The corresponding toric manifold \( M_\Delta \) is a \( \mathbb{C}P^3 \) bundle over \( \mathbb{C}P^1 \); in fact, it is the projectivization of the vector bundle \( O(-1) \oplus O(-1) \oplus O \oplus O \to \mathbb{C}P^1 \). The toric manifold \( M_\Delta \) is the blowup of \( M_\Delta \) along a line in one of the fibers.

1.2. Proof of Theorem 1.1.1. This section explains the proof of Theorem 1.1.1. Since we use results from part I without comment, readers might find it useful look over the beginning of §2.1 where we summarize its main results.

We divide the proof of Theorem 1.1.1 into four steps.

Step 1: Theorem 1.1.1 holds if \( \Delta \) has a nonpervasive asymmetric facet.

Proof. Let \( H \in t \) be an essential mass linear function on a 4-dimensional polytope \( \Delta \subset t^* \). If one of the asymmetric facets \( F \) is not pervasive, then Proposition 4.3.4 implies that \( \Delta \) is a bundle of one of the three types mentioned in part (a).

The proof of Proposition 4.3.4 relies on the fact, proved in [I, Corollary A.8], that if all the facets of \( \Delta \) are asymmetric then \( \Delta \) is combinatorially equivalent to a product of simplices. Since all facets are pervasive, it must therefore be combinatorially equivalent to \( \Delta_4 \) or \( \Delta_2 \times \Delta_2 \). Moreover, because \( \Delta \) is smooth, in the latter case Lemma 2.1.13 implies that it is a \( \Delta_2 \) bundle over \( \Delta_2 \). On the other hand, if \( \Delta \) has at least one symmetric facet \( G \), then Proposition 2.1.5 implies that the restriction of \( H \) to \( G \) is mass linear. By the 3-dimensional classification, this implies that there are only a few possibilities for \( G \). The proof is completed by analyzing these.

Step 2: Theorem 1.1.1 holds if \( \Delta \) has more than four asymmetric facets.

Proof. By Step 1, we may assume that all the asymmetric facets are pervasive. Hence, Proposition 4.3.1 implies that \( \Delta \) is either \( \Delta_4 \) or a \( \Delta_2 \) bundle over \( \Delta_2 \). But \( \Delta_4 \) has no essential mass linear functions. Therefore we are in case (a3). (In fact, this case does not occur for essential \( H \); see Corollary 3.1.9.)

The proof of Proposition 4.3.1 relies on the fact, proved in [I, Corollary A.8], that if all the facets of \( \Delta \) are asymmetric then \( \Delta \) is combinatorially equivalent to a product of simplices. Since all facets are pervasive, it must therefore be combinatorially equivalent to \( \Delta_4 \) or \( \Delta_2 \times \Delta_2 \). Moreover, because \( \Delta \) is smooth, in the latter case Lemma 2.1.13 implies that it is a \( \Delta_2 \) bundle over \( \Delta_2 \). On the other hand, if \( \Delta \) has at least one symmetric facet \( G \), then Proposition 2.1.5 implies that the restriction of \( H \) to \( G \) is mass linear. By the 3-dimensional classification, this implies that there are only a few possibilities for \( G \). The proof is completed by analyzing these.

Step 3: Theorem 1.1.1 holds if \( \Delta \) has four asymmetric facets.

Proof. By Step 1, we may assume that all the asymmetric facets are pervasive. If their conormals are linearly dependent, then by Proposition 4.2.4 \( \Delta \) is the blowup of a \( \Delta_3 \) bundle
over $\Delta_1$ by a series of blowups of the type described in Theorem 1.1.1. If $H$ is essential on $\Delta$ then we are in case (a1); if $H$ is inessential on $\Delta$ then by Proposition 5.3.7 we are in the special case of (b) in which the double expansion is along two parallel edges of a quadrilateral $\Delta$ (as in Example 1.1.5). On the other hand, if their conormals are linearly independent then we are in case (b) by Proposition 4.2.6.

In each case, the classification is established by exploiting the classification of polygons and 3-dimensional polytopes with four asymmetric facets to analyze the set of symmetric facets of $\Delta$; see for example Lemmas 4.2.2, 4.2.3 and 4.2.6.

Step 4: Completion of the proof.

Proof. It remains to consider the case when $\Delta$ has fewer than four asymmetric facets. By Proposition 2.1.8, $\Delta$ must have exactly three asymmetric facets. If their conormals are linearly dependent, then Proposition 4.1.2 implies that $\Delta$ itself is a $\Delta_2$ bundle over a polygon and the asymmetric facets are the fiber facets; hence, we are in case (a3). If their conormals are linearly independent, then Proposition 4.1.4 implies either that the triple intersection $F_{123}$ is empty, we are in case (a2), and the asymmetric facets correspond to the facets of $\Delta_2$, or that $F_{123}$ is nonempty, we are in case (a1), and three of the four fiber facets are asymmetric. In all these cases we analyze the structure of $\Delta$ by exploiting the fact that the symmetric faces of smallest dimension are 2-dimensional triangles with edges $F_1 \cap g, F_2 \cap g,$ and $F_3 \cap g$.

Remark 1.2.1. As we explain above, the proof of Theorem 1.1.1 depends on the number of asymmetric facets. In particular, as is shown in Steps 3 and 4, the arguments needed if there are four pervasive asymmetric facets are very different from those needed if there are three. However, these cases are not so distinct as they might seem. By Corollary 3.1.4, a generic $\Delta_3$ bundle over $\Delta_1$ admits essential mass linear functions with either three or four pervasive asymmetric facets. As the proof above shows, in case (a2) and (a3) the polytope necessarily has three pervasive asymmetric facets, and in case (b) it has four.

1.3. Questions and comments. Many results and techniques used in this paper extend to higher dimensions. However, the proof of Theorem 1.1.1 relies on a very detailed result about polygons (Lemma 2.5.6) as well as the description in Proposition 2.1.15 below of all 3-dimensional polytopes that have nonzero mass linear functions. Since higher dimensional polytopes are not yet so well understood, one cannot expect such a complete classification in higher dimensions. Additionally, to make further progress with our current methods we would first need to answer the following question since, as explained in Step 2 of §1.2 above, this is the basis of our inductive argument: once one has a symmetric facet $G$ one can analyze the structure of the mass linear pair $(\Delta, H)$ by using information on the lower dimensional pair $(G, H|_G)$.

Question 1.3.1. If $\Delta$ has a mass linear function such that all facets are asymmetric, is the equivalence relation on the facets of $\Delta$ nontrivial?

If the answer were yes, then by Lemma 2.1.2 there would be an inessential $H'$ such that $H - H'$ has a symmetric facet. Further Proposition 2.3.3 would imply that $\Delta$ must be either...
a bundle over a simplex or an expansion. In fact, it seems quite likely that in this situation \( \Delta \) must be combinatorially equivalent to a product of simplices, and hence, by an extension of [I, Lemma 4.10], an iterated simplex bundle. See Corollary 4.3.6 for the 4-dimensional case.

Together with Timorin, we found a purely combinatorial argument that showed Question 1.3.1 has a positive answer in dimensions \( \leq 4 \); see [I, Appendix]. As pointed out in [6, Lemma 2.4], this argument does not extend to higher dimensions. However, Chen [1] showed that the answer is again positive for 5-dimensional polytopes with at most 9 facets.

Question 1.3.1 seems hard, though very interesting. An easier task would be to analyze properties of particular kinds of polytopes. The obvious examples of polytopes with a mass linear function for which all facets are asymmetric are products of simplices. As we showed in [I, Theorem 1.20] these polytopes have several other interesting characterizations: they are the only polytopes for which every \( H \in \mathfrak{t} \) is mass linear, and also the only polytopes such that for each equivalence class \( I \) of facets the intersection \( F_I := \cap_{i \in I} F_i \) is empty. The latter condition implies that \( \Delta \) has no singleton facets, i.e., that \( |I| > 1 \) for all equivalence classes \( I \). Polytopes with this property are analyzed, though not fully classified, in the proof of [6, Lemma 3.7]; they are a particular kind of expansion. Here are some questions.

**Question 1.3.2.** If \( \Delta \) is an expansion (but not a bundle over a simplex), can it support a mass linear function for which all facets are asymmetric? Which polytopes support an \((n - 1)\)-dimensional family of mass linear functions, where \( n := \dim \Delta \)?

Note that by Propositions 2.2.3 and 3.1.1 every \( \Delta_1 \)-bundle over \( \Delta_{n-1} \) and every \( \Delta_{n-1} \) bundle over \( \Delta_1 \) has an \((n - 1)\)-dimensional family of mass linear functions, while, by Corollary 3.1.9generic \( \Delta_2 \) bundles over \( \Delta_2 \) have only a 2-dimensional family of mass linear functions, and these are all inessential. Moreover, Proposition 3.1.1 shows that each generic \( \Delta_{n-1} \) bundle over \( \Delta_1 \) has mass linear functions for which every facet is asymmetric, while by Proposition 2.1.7 and Corollary 3.1.9generic bundles of the other two types must have symmetric fiber facets. In this paper we do not study mass linear functions on \( \Delta_s \) bundles over \( \Delta_{n-s} \) for general \( s \).

We show in Corollary 4.3.5 that the only 4-dimensional polytope that has a mass linear function with 8 asymmetric facets is the product \((\Delta_1)^4\). In fact it is easy to see that a mass linear function on an \( n \)-dimensional polytope has at most \( 2n \) asymmetric facets. This holds because by [I, Proposition A.2] every asymmetric facet \( F \) is powerful, i.e. it is connected to every vertex of \( \Delta \setminus F \) by an edge.

**Question 1.3.3.** If \( H \) is a mass linear function on a smooth \( n \)-dimensional polytope \( \Delta \) with \( 2n \) asymmetric facets, must \( \Delta \) be the product \((\Delta_1)^n\)?

Another interesting question concerns which blowups preserve mass linearity. It is easy to see that mass linearity is destroyed if one blows up along a face \( f \) that does not meet all asymmetric facets; see Lemma 2.4.7. Lemma 2.4.12 is another straightforward result showing that if \( \Delta' \) is the blowup of the polytope \( \Delta \) along a face \( f \) that lies in all asymmetric facets then every inessential function on \( \Delta \) is inessential on \( \Delta' \). However, it is not clear whether this condition on \( f \) is sufficient for mass linearity to be preserved.
**Question 1.3.4.** Suppose that $H$ is a mass linear function on $\Delta$ with asymmetric facets $F_j, j \in J$. Let $\Delta'$ be the blowup of $\Delta$ along the face $f = F_I$ where $J \subseteq I$. Is $H$ mass linear on $\Delta'$? More generally, describe all blowups that preserve mass linearity.

In dimension 4, besides the blowup operations described in Theorem 1.1.1 two others occur during the classification proof; namely, blowing up at a vertex or edge that meets all asymmetric facets. Corollary 5.3.3 shows that the vertex blowup preserves mass linearity in any dimension. However, if one blows up along an edge $e = F_I$, additional conditions are needed. One of these is very natural, namely that $\sum_{i \in I} \gamma_i = 0$. (By Remark 5.2.6 this holds if $J \subseteq I$ as in Question 1.3.4.) Corollary 5.3.5 shows that in dimension 4 this extra condition suffices. Our proof also suggests that the natural framework in which to consider the effect of blowing up may not be the set of mass linear functions, but rather the set of fully mass linear functions that we now discuss.

Our analysis of the image of $\pi_1(T)$ in $\pi_1(\text{Symp}(M,\omega))$ is based on the properties of Weinstein’s action homomorphism $A_\omega$; see [I, §5]. In [9], Shelukhin defined a series of related homomorphisms that allow one to formulate properties, in principle stronger than mass linearity, that must be satisfied whenever the loop $\Lambda_H$ generated by an integral $H \in t_\mathbb{Z}$ contracts in $\text{Symp}(M,\omega)$. These are discussed further in [6, §4] where we called Shelukhin’s conditions full mass linearity; see also §5.2 below.

**Question 1.3.5.** Does every mass linear function satisfy Shelukhin’s additional conditions?

The results of this paper imply that the answer is yes in dimensions $\leq 4$; see Proposition 5.2.2. However, although the mass linear condition seems to be very strong, it is not clear whether it is equivalent to full mass linearity in higher dimensions. If not, many of the above questions might be better investigated for fully mass linear functions.

**Organization of the paper.** Section 2 begins with a review of the results from Part I that we use most often. It then describes in detail some general ways to construct polytopes, namely bundles, expansions, and blowing up and down. In each case, we describe the behavior of mass linear functions under these operations. We also develop criteria for recognizing when a facet can be blown down (Lemma 2.5.3) and for recognizing when a polytope is a double expansion (Lemma 2.3.7).

In section 3 we construct all the (essential) mass linear functions on the polytopes described in Theorem 1.1.1. §3.1 gives detailed information on the three kinds of bundles in case (a) of Theorem 1.1.1 while §3.2 discusses double expansions, showing precisely how blowing up an inessential function on a double expansion can convert it into an essential function.

Section 4 finishes the proof of Theorem 1.1.1, thus showing that our list of examples is complete. §4.1 deals with the case when there are three asymmetric facets, and §4.2 with the case of four asymmetric and pervasive facets. These arguments are quite different, because by Proposition 2.1.5 the symmetric 2-faces are triangles in the first case and are rectangles in the second. The final subsection §4.3 discusses the case when there are more than four asymmetric facets.
The last section contains a variety of further results.  §5.1 explains exactly when the polytopes $\Delta$ in Theorem 1.1.1 are minimal. In §5.2 we use Theorem 1.1.1 to show that in dimensions $\leq 4$ every mass linear function is fully mass linear. Finally, in §5.3 we consider the question of which blowup operations preserve mass linearity.

Acknowledgements. Both authors are very grateful to MSRI for its hospitality in Spring 2010; the first author also thanks the Simons Foundation for its support via an Eisenbud Professorship.

2. Constructions

After a review of basic results, this section describes in detail some general ways to construct polytopes: bundles, expansions, blowups, and blowdowns. We also analyze certain natural mass linear functions on each type of polytope.

2.1. Review of basic results. For the convenience of the reader we begin by assembling the results from Part I that will be used most often in this paper; in particular, we describe all smooth polytopes of dimension at most three that have mass linear functions. In the process, we give the definition of a bundle. Many of the results quoted below are valid for simple polytopes; however we restrict to the smooth case for simplicity. Thus, even if it is not stated explicitly, we assume that every polytope is smooth.

We begin by noting that the definition of mass linearity given in §1.1 is slightly different from, but equivalent to, the definition used in Part I. Given a smooth polytope $\Delta \subset t^*$, the chamber $C_{\Delta}$ of $\Delta := \Delta(\kappa)$ is the connected component that contains $\kappa$ of the set of all $\kappa' \in \mathbb{R}^N$ such that $\Delta(\kappa')$ is smooth. Note that, for every $\kappa' \in C_\Delta$, the polytope $\Delta(\kappa')$ is analogous to $\Delta$, that is, for all $I \subset \{1, \ldots, N\}$ the intersection $\bigcap_{i \in I} F_i'$ is empty exactly if the intersection $\bigcap_{i \in I} F_i$ is empty. In §1.1, we gave a local definition of mass linearity, that is, we only required the function $\widehat{H}$ which takes $\kappa'$ to $\langle H, c_{\Delta}(\kappa') \rangle$ to be linear for $\kappa'$ in some open neighborhood of $\kappa$. In contrast, in Part I we required the function to be linear on all of $C_{\Delta}$. However, as we show in [I, Lemma 2.3], these two definitions are equivalent because $\widehat{H}$ is always a rational function. A similar remark applies to the definition of equivalent facets; see [I, Corollary 3.5].

One extremely useful fact – which follows quite easily from the definitions – is that inessential functions are mass linear.

Proposition 2.1.1 (I, Proposition 1.18). Fix $H \in t$ and a polytope $\Delta \subset t^*$. Let $I$ denote the set of equivalence classes of facets of $\Delta$. If $H$ is inessential, write

$$H = \sum \beta_i \eta_i,$$

where $\beta_i \in \mathbb{R}$ and $\sum_{i \in I} \beta_i = 0$ for all $I \in I$.

Then

$$\langle H, c_{\Delta}(\kappa) \rangle = \sum \beta_i \kappa_i.$$ 

It is straightforward to use the formula above to show that we can reduce the number of asymmetric facets if some of them are equivalent.
Lemma 2.1.2 (I, Lemma 3.19). Let $H \in \mathfrak{t}$ be a mass linear function on a polytope $\Delta \subset \mathfrak{t}^*$. If $F_1, \ldots, F_m$ are equivalent facets, there exists an inessential function $H' \in \mathfrak{t}$ so that the mass linear function $\tilde{H} = H - H'$ has the following properties:

- For all $i < m$, the facet $F_i$ is $\tilde{H}$-symmetric.
- For all $i > m$, the facet $F_i$ is $\tilde{H}$-symmetric iff it is $H$-symmetric.

Note that, in general, even if $H = \sum \beta_i \eta_i$ is mass linear, $\langle H, c_\Delta(\kappa) \rangle$ need not equal $\sum \beta_i \kappa_i$ since the $\beta_i$ are not uniquely determined by $H$. In contrast, the next lemma shows that the coefficients of a mass linear function $H \in \mathfrak{t}$ always determine the function $H$ itself.

Lemma 2.1.3 (I, Lemma 2.6). Fix $H \in \mathfrak{t}$ and a polytope $\Delta \subset \mathfrak{t}^*$. If $\langle H, c_\Delta(\kappa) \rangle = \sum \beta_i \kappa_i$, then $H = \sum \beta_i \eta_i$.

In part I, the proof of the lemma above relies on the following fact.

Remark 2.1.4. Given a polytope $\Delta = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa_i \} \subset \mathfrak{t}^*$ and $\xi \in \mathfrak{t}^*$, consider $\Delta' = \Delta + \xi$, the translate of $\Delta$ by $\xi$. Then

$$
\Delta' = \bigcap_{i=1}^N \{x \in \mathfrak{t}^* \mid \langle \eta_i, x \rangle \leq \kappa'_i \}, \quad \text{where} \quad \kappa'_i = \kappa_i + \langle \eta_i, \xi \rangle \quad \forall \ i.
$$

We continue with some very useful results about symmetric faces, taken from [I, §2.3]. They imply that if $H$ is a mass linear function on $\Delta$ and $(\Delta, H)$ has a symmetric face $f$ then the pair $(f, H|_f)$ is also mass linear. (Here, we consider the face $f$ as a polytope in $P(f) \subset \mathfrak{t}^*$, the smallest affine plane containing $f$.) Hence one can use knowledge of the structure of the lower dimensional $(f, H|_f)$ to analyse $(\Delta, H)$.

Proposition 2.1.5 (I, Lemma 2.7 and Proposition 2.9). Let $H \in \mathfrak{t}$ be a mass linear function on a polytope $\Delta \subset \mathfrak{t}^*$. Let $f$ be a symmetric face of $\Delta$. Then the following hold:

- $\langle H, c_f(\kappa) \rangle = \langle H, c_\Delta(\kappa) \rangle$ for all $\kappa \in C_\Delta$, where $c_f$ denotes the center of mass of $f$ in $P(f)$.
- The restriction of $H$ to $f$ is mass linear.
- Intersection induces a one-to-one correspondence between the asymmetric facets of $\Delta$ and the asymmetric facets of $f$.
- The coefficient of the support number of a facet $F$ in $\langle H, c_\Delta \rangle$ is the coefficient of the support number of $f \cap F$ in $\langle H, c_f \rangle$.

Remark 2.1.6. In fact, it is not hard to prove the following slightly stronger claims:

- If $H$ is inessential on $\Delta$ then it is inessential on $f$ (but not conversely).
- $H$ is mass linear on $\Delta$ exactly if the restriction of $H$ to $f$ is mass linear.

On the other hand, asymmetric facets have special properties. Recall that a facet $F$ of a polytope $\Delta \subset \mathfrak{t}^*$ is called pervasive if it has nonempty intersection with every other facet of $\Delta$. Further, we say that $F$ is flat if there is a hyperplane in $\mathfrak{t}$ that contains the conormal of every other facet (other than $F$ itself) that meets $F$. 
Proposition 2.1.7 (I, Proposition 2.11). Let \( H \in \mathfrak{t} \) be a mass linear function on a polytope \( \Delta \subset \mathfrak{t}^* \). Then every asymmetric facet is pervasive or flat (or both).

The following statement combines [I, Lemma 2.13] with results from [I, §4.2].

Proposition 2.1.8. Let \( H \in \mathfrak{t} \) be a nonzero mass linear function on a polytope \( \Delta \subset \mathfrak{t}^* \). Then \( \Delta \) has at least two asymmetric facets. Moreover, if it has exactly two, then they are equivalent and \( H \) is inessential.

Next, we give another characterization of the equivalence relation on the facets.

Lemma 2.1.9 (I, Lemma 3.7). Let \( \Delta \subset \mathfrak{t}^* \) be a smooth polytope. Given a subset \( I \subset \{1, \ldots, N\} \), we have \( F_i \sim F_j \) for all \( i \) and \( j \) in \( I \) exactly if the plane \( \nu \subset \mathfrak{t} \) spanned by the outward conormals \( \eta_k \) for \( k \notin I \) has codimension \( |I| - 1 \). Moreover, in this case the linear combination \( \sum_{i \in I} c_i \eta_i \) lies in \( \nu \) if and only if \( c_i = c_j \) for all \( i \) and \( j \).

Remark 2.1.10. In particular, \( F_i \sim F_j \) exactly if there is a vector \( \xi \in \mathfrak{t}^* \) that is parallel to all the facets except \( F_i \) and \( F_j \). This condition is very easily tested. The vector \( \xi \) is preserved by the corresponding reflection symmetry; in Masuda [5] it is called a root. Since \( \Delta \) is smooth, there is also a homological interpretation of this equivalence relation: by [I, Remark 5.8], \( F_i \sim F_j \) exactly if the corresponding submanifolds \( \Phi^{-1}(F_i) \) and \( \Phi^{-1}(F_j) \) are homologous in \( H_{2n-2}(M_\Delta) \).

Our next aim is to describe all the 2- and 3-dimensional polytopes that have mass linear functions. Since many of these polytopes are bundles, we start with the formal definition. Two polytopes \( \Delta \) and \( \Delta' \) are said to be combinatorially equivalent if there exists a bijection of facets \( F_i \leftrightarrow F'_i \) so that \( \cap_{i \in I} F_i \neq \emptyset \) exactly if \( \cap_{i \in I} F'_i \neq \emptyset \).

Definition 2.1.11. Let \( \Delta = \bigcap_{j=1}^{\tilde{N}} \{ x \in \mathfrak{t}^* \mid \langle \eta_j, x \rangle \leq \tilde{\kappa}_j \} \) and \( \tilde{\Delta} = \bigcap_{i=1}^{\tilde{N}} \{ y \in \mathfrak{t}^* \mid \langle \tilde{\eta}_i, y \rangle \leq \tilde{\kappa}_i \} \) be smooth polytopes. We say that a smooth polytope \( \Delta \subset \mathfrak{t}^* \) is a bundle with fiber \( \Delta \) over the base \( \tilde{\Delta} \) if there exists a short exact sequence

\[
0 \to \tilde{\mathfrak{t}} \to \mathfrak{t} \to \mathfrak{t} \to 0
\]

so that the following hold:

- \( \Delta \) is combinatorially equivalent to the product \( \tilde{\Delta} \times \tilde{\Delta} \).
- If \( \tilde{\eta}_j' \) denotes the outward conormal to the facet \( \tilde{F}_j' \) of \( \tilde{\Delta} \) which corresponds to \( \tilde{F}_j \subset \tilde{\Delta} \), then \( \tilde{\eta}_j' = \iota(\tilde{\eta}_j) \) for all \( 1 \leq j \leq \tilde{N} \).
- If \( \tilde{\eta}_i' \) denotes the outward conormal to the facet \( \tilde{F}_i' \) of \( \tilde{\Delta} \) which corresponds to \( \tilde{F}_i \subset \tilde{\Delta} \), then \( \pi(\tilde{\eta}_i') = \tilde{\eta}_i \) for all \( 1 \leq i \leq \tilde{N} \).

The facets \( \tilde{F}_1', \ldots, \tilde{F}_{\tilde{N}}' \) will be called fiber facets, and the facets \( \tilde{F}_1', \ldots, \tilde{F}_{\tilde{N}}' \) will be called base facets.

Observe that if \( \Delta \) is such a bundle then the faces \( \tilde{F}_i' := \cap_{j \in I} \tilde{F}_i' \) of \( \Delta \) corresponding to the vertices \( \tilde{F}_i \) of the base \( \tilde{\Delta} \) are all affine equivalent. In contrast, the faces \( \tilde{F}_j' \) of \( \Delta \) corresponding to the vertices \( \tilde{F}_j \) of the fiber \( \tilde{\Delta} \) may not be affine equivalent, but they
are analogous, that is, we may identify the affine plane $P(\bar{F}_j')$ with $\hat{\mathbf{t}}^*$ so that there is combinatorial equivalence between $\bar{F}_j'$ and $\hat{\Delta}$ in which corresponding facets are parallel; see [I, §1.1].

To help the reader understand this rather complicated definition, here is a recognition lemma, which explains how to identify a given polytope $\Delta$ as a bundle with fiber $\bar{\Delta}$ and base $\hat{\Delta}$. The proof is elementary and is left to the reader.

**Lemma 2.1.12.** A smooth polytope $\Delta$ is a bundle over $\hat{\Delta}$ with fiber $\bar{\Delta}$ exactly if all the following conditions hold:

- $\Delta$ is combinatorially equivalent to the product $\bar{\Delta} \times \hat{\Delta}$.
- The conormals $\bar{n}_j'$ to the fiber facets $\bar{F}_j'$ lie in a dim $\bar{\Delta}$ subspace.
- There is a vertex $\bar{F}_I$ of $\bar{\Delta}$ so that the face $\bar{F}_I'$ of $\Delta$ is analogous to $\bar{\Delta}$.
- There is a vertex $\bar{F}_J$ of $\bar{\Delta}$ so that the face $\bar{F}_J'$ of $\Delta$ is analogous to $\bar{\Delta}$.

Later, we will also need the following result from Part I; it explains why the smooth case is easier than the general one.

**Lemma 2.1.13** (I, Lemma 4.10). Let $\Delta$ be a smooth polytope which is combinatorially equivalent to $\Delta_k \times \Delta_n$. Then $\Delta$ is either a $\Delta_k$ bundle over $\Delta_n$, or a $\Delta_n$ bundle over $\Delta_k$.

Finally here are some detailed results about mass linear pairs in dimensions 2 and 3.

**Proposition 2.1.14** (I, Proposition 4.2 and Corollary 4.3). Let $H \in \mathbf{t}$ be a nonzero mass linear function on a smooth polygon $\Delta \subset \mathbf{t}^*$. Then one of the following statements holds:

- $\Delta$ is the simplex $\Delta_2$; at most one edge is symmetric.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_1$; the base facets are the asymmetric edges.
- $\Delta$ is the product $\Delta_1 \times \Delta_1$; each edge is asymmetric.

In any case, $H$ is inessential. Moreover, if two edges $F_i$ and $F_j$ do not intersect then $\gamma_i + \gamma_j = 0$, where $\gamma_k$ is the coefficient of the support number of $F_k$ in the linear function $\langle H, c_\Delta \rangle$.

**Proposition 2.1.15** (I, Theorem 1.4, Proposition 4.14, and Lemma 4.15). Let $H \in \mathbf{t}$ be a mass linear function on a smooth 3-dimensional polytope $\Delta \subset \mathbf{t}^*$. If $\Delta$ has more than two asymmetric facets, then one of the following statements holds:

- $\Delta$ is the simplex $\Delta_3$.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_2$; the base facets are the asymmetric facets.
- $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$; if either base facet is asymmetric then both are.
- $\Delta$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$; the base facets are the asymmetric facets.
- $\Delta$ is the product $\Delta_1 \times \Delta_1 \times \Delta_1$; every facet is asymmetric.

Moreover, $H$ is inessential unless $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$. Finally,

$$\sum_{i=1}^{N} \gamma_i = 0,$$

where $\gamma_i$ is the coefficient of the support number of the facet $F_i$ in the linear function $\langle H, c_\Delta \rangle$. 

2.2. Bundles. In this subsection, we give a new way to construct mass linear functions on bundles. More precisely, we show that there is a one-to-one correspondence between (essential) mass linear functions on the base and (essential) mass linear functions with symmetric fiber facets on the bundle. Let \( \Delta \subset t^* \) be a \( \hat{\Delta} \subset \hat{t}^* \) bundle over \( \hat{\Delta} \subset \hat{t}^* \) and consider the associated short exact sequences

\[
0 \to \hat{t} \xrightarrow{\iota} t \xrightarrow{\pi} \hat{t}^* \to 0 \quad \text{and} \quad 0 \to \hat{t}^* \xrightarrow{\iota^*} t^* \xrightarrow{\pi^*} \hat{t}^* \to 0.
\]

We begin our discussion with the following elementary but useful lemma.

**Lemma 2.2.1.** Let \( \Delta \subset t^* \) be a \( \hat{\Delta} \subset \hat{t}^* \) bundle over \( \hat{\Delta} \subset \hat{t}^* \).

(i) Two base facets \( \hat{F}'_i \) and \( \hat{F}'_j \) of \( \hat{\Delta} \) are equivalent exactly if the corresponding facets \( \hat{F}_i \) and \( \hat{F}_k \) of \( \hat{\Delta} \) are equivalent.

(ii) If two fiber facets of \( \hat{\Delta} \) are equivalent then the corresponding facets of the fiber are equivalent; the converse need not hold.

(iii) A base facet \( \hat{F}'_i \) of \( \hat{\Delta} \) is never equivalent to a fiber facet \( \hat{F}'_j \).

**Proof.** To prove (i), note that (2.1) implies that the image of \( \pi^*: \hat{t}^* \to t^* \) is the annihilator of \( \iota(\hat{t}) \subset t \); moreover,

\[
\langle \hat{\eta}', \pi^*(\hat{\xi}) \rangle = \langle \pi(\hat{\eta}'), \hat{\xi} \rangle = \langle \hat{\eta}, \hat{\xi} \rangle \quad \forall \ \hat{\xi} \in \hat{t} \text{ and } \forall \ i.
\]

Hence, the claim follows from Lemma 2.1.9.

The first part of (ii) is easy. To illustrate the second, consider the \( \Delta_2 \) bundle over \( \Delta_1 \) with polytope \( Y \) as in Equation (3.1). The fiber facets are not all equivalent unless \( a_1 = a_2 = 0 \).

To prove (iii), note that since \( \hat{\Delta} \) is compact, the outward conormals to all but one facet of \( \hat{\Delta} \) still span \( \hat{t} \). Since the same holds for \( \hat{\Delta} \), the claim follows from (2.1) and Lemma 2.1.9 \( \Box \)

We are now ready – after one last definition – to state the main result of this subsection.

**Definition 2.2.2.** Let \( \hat{H} \in \hat{t} \) be a mass linear function on \( \hat{\Delta} \); write \( \langle \hat{H}, c_\Delta \rangle = \sum \beta_i \hat{\kappa}_i \). The lift of \( \hat{H} \) to \( \Delta \) is

\[ H = \sum \beta_i \hat{\eta}'_i \in t. \]

Note that, by Lemma 2.1.3, \( \hat{H} = \sum \beta_i \hat{\eta}_i \); hence, \( \pi(H) = \hat{H} \).

**Proposition 2.2.3.** Let \( \Delta \subset t^* \) be a \( \hat{\Delta} \subset \hat{t}^* \) bundle over \( \hat{\Delta} \subset \hat{t}^* \). Then the following hold:

(i) If \( H \in t \) is mass linear on \( \Delta \) and the fiber facets are symmetric, then \( \hat{H} = \pi(H) \) is mass linear on \( \hat{\Delta} \). More specifically, if \( \langle H, c_\Delta \rangle = \sum \beta_i \hat{\kappa}_i \), then \( \langle \hat{H}, c_\Delta \rangle = \sum \beta_i \hat{\kappa}_i \).

(ii) Conversely, if \( \hat{H} \in \hat{t} \) is mass linear on \( \hat{\Delta} \), then the lift of \( \hat{H} \) to \( \Delta \) is mass linear on \( \Delta \) and the fiber facets are symmetric.

(iii) In the cases described above, \( H \) is inessential on \( \Delta \) exactly if \( \hat{H} \) is inessential on \( \hat{\Delta} \).

In particular, \( \pi \) induces a one-to-one correspondence between essential mass linear functions on \( \Delta \) with symmetric fiber facets and essential mass linear functions on \( \Delta \).
Figure 2.1. Slicing $\Delta$ by the “sections” $Y_\alpha$

Proof. Fix any $H \in t$ and let $\tilde{H} = \pi(H)$. Given $\alpha \in \iota^*(\Delta) \subset \tilde{t}^*$, consider the slice $Y_\alpha$ of $\Delta$ defined by

$$Y_\alpha := \{ y \in \Delta \mid \iota^*(y) = \alpha \}.$$

The $Y_\alpha$’s form a family of parallel polytopes that are analogous to the base polytope; see Figure 2.1. More precisely, fix $x \in (\iota^*)^{-1}(\alpha) \subset t^*$ and define an isomorphism $j_x: \tilde{t}^* \to (\iota^*)^{-1}(\alpha)$ by $j_x(y) = \pi^*(y) + x$. Since $Y_\alpha = \bigcap_{i=1}^N \{ z \in t^* \mid \langle \tilde{\eta}_i, z \rangle \leq \tilde{\kappa}_i \} \cap (\iota^*)^{-1}(\alpha)$,

$$\tilde{\Delta}_x := j_x^{-1}(Y_\alpha) = \bigcap_{i=1}^N \{ y \in \tilde{t}^* \mid \langle \tilde{\eta}_i, y \rangle \leq \tilde{\kappa}_i^x \},$$

where $\tilde{\kappa}_i^x = \tilde{\kappa}_i' - \langle \tilde{\eta}_i', x \rangle$.

By definition, this polytope $\tilde{\Delta}_x$ is analogous to $\tilde{\Delta}$, that is, its support numbers $\tilde{\kappa}_i^x := (\tilde{\kappa}_i^x)$ lie in the chamber $C_{\tilde{\Delta}}$. Hence, $\langle \tilde{H}, c_{\tilde{\Delta}_x} \rangle = \langle \tilde{H}, c_{\tilde{\Delta}}(\tilde{\kappa}^x) \rangle$. Therefore,

$$\langle H, c_Y(\tilde{\kappa}') \rangle = \langle \tilde{H}, c_{\tilde{\Delta}}(\tilde{\kappa}^x) \rangle + \langle H, x \rangle, \quad \text{where} \quad \tilde{\kappa}_i^x = \tilde{\kappa}_i' - \langle \tilde{\eta}_i', x \rangle.$$  

First, assume that $H \in t$ is mass linear on $\Delta$ and that the fiber facets are symmetric; write $\langle H, c_{\tilde{\Delta}} \rangle = \sum \beta_i \tilde{\kappa}_i$. Then $H = \sum \beta_i \tilde{\eta}_i$ by Lemma 2.1.3 and so $\langle H, x \rangle = \sum \beta_i \langle \tilde{\eta}_i', x \rangle$. Choose $\alpha$ to be a vertex of the polytope $\iota^*(\Delta)$, which is analogous to the fiber $\tilde{\Delta}$. Then $Y_\alpha$ is the intersection of the corresponding fiber facets, and hence is a symmetric face of $\Delta$. Thus, by Proposition 2.1.5, $\langle H, c_Y(\tilde{\kappa}') \rangle = \sum \beta_i \tilde{\kappa}_i'$. Hence, substituting in Equation (2.2) we find that

$$\langle \tilde{H}, c_{\tilde{\Delta}}(\tilde{\kappa}^x) \rangle = \sum \beta_i \tilde{\kappa}_i^x.$$  

This proves (i).

Conversely, assume that $\tilde{H} \in \tilde{t}$ is mass linear on $\tilde{\Delta}$; write $\langle \tilde{H}, c_{\tilde{\Delta}} \rangle = \sum \beta_i \tilde{\kappa}_i$. Let $H = \sum \beta_i \hat{\eta}_i$ be the lift of $\tilde{H}$. By Lemma 2.1.3, $\tilde{H} = \sum \beta_i \tilde{\eta}_i$, and so $\pi(H) = \tilde{H}$. Therefore, Equation (2.2) implies that for all $\alpha \in \iota^*(\Delta)$, $\langle H, c_Y(\tilde{\kappa}') \rangle = \sum \beta_i \tilde{\kappa}_i'$. Since $\Delta$ is the union of such $Y_\alpha$, this immediately implies that $\langle H, c_{\tilde{\Delta}} \rangle = \sum \beta_i \tilde{\kappa}_i'$. This proves (ii).

Finally, (iii) follows immediately from Proposition 2.1.1 and Lemma 2.2.1. 

$\square$
In particular, every bundle over a simplex has many inessential functions with symmetric fiber facets. In [I, §3.3] we used this fact to prove the following result.

**Proposition 2.2.4** (I, Proposition 3.22). Let \( H \in \mathfrak{t} \) be a mass linear function on a polytope \( \Delta \subset \mathfrak{t}^* \) which is a bundle over the simplex \( \Delta_k \). Then we can write \( H = H' + \bar{H} \), where

- \( H' \) is inessential and the fiber facets are \( H' \)-symmetric, and
- \( \bar{H} \) is mass linear and the base facets are \( \bar{H} \)-symmetric.

Part (i) of the next proposition is Corollary 3.24 from Part I. The second part then follows easily from the proposition above, just as in the proof of Proposition 3.25 in Part I.

**Proposition 2.2.5.** Let \( H \in \mathfrak{t} \) be a mass linear function on a polytope \( \Delta \subset \mathfrak{t}^* \).

(i) If \( F \) is an asymmetric facet that is not pervasive, then \( \Delta \) is an \( F \)-bundle over \( \Delta_1 \).

(ii) We can write \( H = H' + \bar{H} \), where

- \( H' \) is inessential and the pervasive facets are \( H' \)-symmetric, and
- \( \bar{H} \) is mass linear and the nonpervasive facets are \( \bar{H} \)-symmetric.

The next lemma explores what happens when we assume that the base facets are symmetric; cf. Proposition 2.2.3. We will not need it in this paper.

**Lemma 2.2.6.** Let \( \Delta \subset \mathfrak{t}^* \) be a \( \bar{\Delta} \subset \bar{\mathfrak{t}}^* \) bundle over \( \bar{\Delta} \subset \bar{\mathfrak{t}}^* \). Then the following hold:

(i) If \( H \in \mathfrak{t} \) is a mass linear function on \( \Delta \) and the base facets are symmetric, then \( H \in \iota(\mathfrak{t}) \) and \( \bar{H} = \iota^{-1}(H) \) is mass linear on \( \bar{\Delta} \). More specifically, if \( \langle H, c_\Delta \rangle = \sum \beta_i \kappa_i' \), then \( \langle \bar{H}, c_{\bar{\Delta}} \rangle = \sum \beta_i \bar{\kappa}_i \).

(ii) In contrast, even if \( \bar{H} \in \bar{\mathfrak{t}} \) is mass linear (and inessential) on \( \bar{\Delta} \), \( H = \iota(\bar{H}) \) may not be mass linear on \( \Delta \).

(iii) In case (i) above, if \( H \) is inessential on \( \Delta \) then \( \bar{H} \) is inessential on \( \bar{\Delta} \).

**Proof.** To prove (i) above, first note that by Lemma 2.1.3, \( H \) lies in the span of the fiber facets, that is, \( H = \iota(\bar{H}) \) for some \( \bar{H} \in \bar{\mathfrak{t}} \). Moreover, let \( f \) be the face formed by intersecting any \( k = \dim \bar{\Delta} \) base facets. Then, under the natural identification (as affine spaces) of \( P(f) \) with \( \mathfrak{t}^* \), \( f \) is analogous to \( \bar{\Delta} \) and \( H \) restricts to \( \bar{H} \). Since \( f \) is symmetric, the first claim now follows from Proposition 2.1.5.

To prove (ii) let \( \Delta \) be a nontrivial \( \Delta_1 \)-bundle over some base polytope \( \hat{\Delta} \). Every nonzero element \( \bar{H} \in \bar{\mathfrak{t}} \) is mass linear (and inessential) on \( \Delta_1 \). So assume that \( H = \iota(\bar{H}) \in \mathfrak{t} \) is mass linear on \( \Delta \). By Proposition 2.1.8 \( \Delta \) has at least two \( H \)-asymmetric facets. On the other hand, let \( F \) be a fiber facet. Since the bundle is not trivial, \( F \) is not flat, and so Proposition 2.1.7 implies that \( F \) is symmetric. Therefore, by Proposition 2.1.5 the restriction of \( H \) to \( F \) is a mass linear function with at least two asymmetric facets; in particular, the restriction of \( H \) to \( F \) is not constant. But this is impossible because \( H = \iota(\bar{H}) \) is constant on \( F \) by construction.

---

Proposition 3.22 in [I] has a slightly different statement, but its proof clearly establishes this stronger claim.
Note finally that if $H = \iota(\tilde{H})$ is inessential and the base facets are symmetric, then Proposition 2.1.1 and Lemma 2.2.1 imply that $\tilde{H}$ is inessential on $\tilde{\Delta}$; this proves (iii). □

2.3. Expansions. We now describe a class of polytopes -- $k$-fold expansions[4] -- which have inessential mass linear functions. Overall, these polytopes are very similar to bundles over the simplex $\Delta_k$, except that in this case the "base" facets all intersect. As we proved in Part I, these two classes of polytopes are the only ones which admit nonzero inessential functions.

**Definition 2.3.1.** Let $\tilde{\Delta} = \bigcap_{j=1}^{\tilde{N}} \{x \in \tilde{t}^* | \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j \}$ be a smooth polytope. Given a natural number $k$, a polytope $\Delta \subset t^*$ is the $k$-fold expansion of $\tilde{\Delta}$ along the facet $\tilde{F}_1$ if there is an identification $t = \tilde{t} \oplus \mathbb{R}^k$ so that

$$\Delta = \bigcap_{j=2}^{\tilde{N}} \{x \in t^* | \langle \tilde{\eta}_j, 0 \rangle \leq \tilde{\kappa}_j \} \cap \bigcap_{i=1}^{k+1} \{x \in t^* | \langle \tilde{\eta}_i, x \rangle \leq \tilde{\kappa}_i \},$$

where

$$\tilde{\eta}_i = (0, -e_i) \text{ and } \tilde{\kappa}_i = 0 \text{ for all } 1 \leq i \leq k, \quad \tilde{\eta}_{k+1} = (\tilde{\eta}_1, \sum e_i) \text{ and } \tilde{\kappa}_{k+1} = \tilde{\kappa}_1.$$

We shall call the facet $\tilde{F}_j'$ of $\Delta$ with outward conormal $(\tilde{\eta}_j, 0)$ the fiber-type facet (associated to $\tilde{F}_j$) for all $j > 1$ and the facets $\tilde{F}_i$ with outward normals $\tilde{\eta}_i$ the base-type facets.

![Figure 2.2](image.png)

**Figure 2.2.** (a) is the 1-fold expansion of the shaded polygon along $f$; (b) is the 2-fold expansion of the heavy line at the vertex $v$

It is easy to check that $\Delta$ is smooth.

**Remark 2.3.2.** (i) The base-type facets are pervasive; in fact, the face $\bigcap_{i \neq n} \tilde{F}_i$ can be identified with $\tilde{\Delta}$ for all $n \in \{1, \ldots, k+1\}$. Similarly, the face $\bigcap_{i=1}^{k+1} \tilde{F}_i$ can be identified with $\tilde{F}_1$. In particular, for any $J \subset \{2, \ldots, \tilde{N}\}$ and $n \in \{1, \ldots, k+1\}$, the face $\tilde{F}_J' \cap \left( \bigcap_{i \neq n} \tilde{F}_i \right)$

---

[4] In the combinatorial literature this construction is known as a wedge; cf. Haase and Melnikov [3].
$\Delta$ is empty exactly if $\tilde{F}_J \subset \tilde{\Delta}$ is empty, and the face $\tilde{F}_J' \cap (\bigcap_{i=1}^{k+1} \tilde{F}_i) \subset \Delta$ is empty exactly if $\tilde{F}_J \cap \tilde{F}_1 \subset \tilde{\Delta}$ is empty.

(ii) The base-type facets $\hat{F}_1, \ldots, \hat{F}_{k+1}$ are clearly equivalent. By Lemma 2.1.9 two fiber-type facets $\tilde{F}_i'$ and $\tilde{F}_j'$ of $\Delta$ are equivalent exactly if the corresponding facets $\tilde{F}_i$ and $\tilde{F}_j$ of $\Delta$ are equivalent. Similarly, a fiber-type facet $\tilde{F}_i'$ is equivalent to the base-type facets exactly if $\tilde{F}_i \sim \tilde{F}_1$.

Conversely, if a polytope has equivalent facets, it is either an expansion or a bundle over a simplex.

**Proposition 2.3.3** (I, Proposition 3.17). Let $\Delta \subset t^*$ be a smooth polytope. Let $I \in I$ be an equivalence class of facets and define $I' := I \setminus \{n\}$ for some $n \in I$.

(i) If $F_I = \emptyset$, then $\Delta$ is a $F_{I'}$ bundle over $\Delta|_{I'}$ with base facets $\{F_i\}_{i \in I}$.

(ii) If $F_I \neq \emptyset$, then $\Delta$ is the $(|I| - 1)$-fold expansion of $F_{I'}$ along $F_I = F_n \cap F_{I'}$ with base-type facets $\{F_i\}_{i \in I}$.

**Remark 2.3.4.** In most ways, mass linear functions on $k$-fold expansions behave like mass linear functions on bundles over the simplex $\Delta_k$. For example, since the base-type facets are equivalent there is a one-to-one correspondence between mass linear functions on $\Delta$ with symmetric fiber facets and mass linear functions on $\Delta_k$, and all such functions are inessential; cf. Proposition 2.2.3. Similarly, as in Lemma 2.2.6 if $H \in t$ is a mass linear function on $\Delta$ with symmetric base-type facets, then there exists a mass linear $\widetilde{H} \in \tilde{t}$ such that $\iota(\widetilde{H}) = H$, where $\iota: \tilde{t} \to t$ is the natural inclusion. In contrast, just as for bundles, even if $\widetilde{H}$ is mass linear on $\tilde{\Delta}$, $\iota(\widetilde{H})$ may not be mass linear on $\Delta$; see Example 3.1.16.

However, there are some significant differences between these two cases. Most notably, Remark 2.3.2 (ii) implies that $\widetilde{H} \in \tilde{t}$ is inessential on $\tilde{\Delta}$ exactly if $H = \iota(\widetilde{H})$ is inessential on $\Delta$ and the base-type facets are symmetric. By Lemma 2.2.6 (ii) the corresponding statement is not true for bundles. (Contrast Remark 2.3.2 (ii) with Lemma 2.2.1 (ii).) These differences arise because expansions correspond to very special bundles. In fact, Example 2.4.3 shows how to convert a $k$-fold expansion into a bundle over $\Delta_k$ by blowing up; but the converse operation is not usually possible.

Let $\tilde{\Delta} \subset \tilde{t}$ be a smooth polytope. If we first take the 1-fold expansion of $\tilde{\Delta}$ along a facet $\tilde{F}_1$, and then take the 1-fold expansion of the resulting polytope along one of the base-type facets, we simply obtain the 2-fold expansion of $\tilde{\Delta}$ along $\tilde{F}_1$. (By repeating this process, we can obtain the $k$-fold expansion.) However, if instead we expand the 1-fold expansion of $\tilde{\Delta}$ along the fiber-type facet associated to $\tilde{F}_2$, we get something new: a double expansion.

**Definition 2.3.5.** Let $\tilde{\Delta} = \bigcap_{j=1}^{\tilde{N}} \{x \in \tilde{t}^* \mid \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j \}$ be a smooth polytope. A polytope $\Delta \subset t^*$ is the **double expansion** of $\tilde{\Delta}$ along the facets $\tilde{F}_1$ and $\tilde{F}_2$ if there is an identification...
t = \tilde{t} \oplus \mathbb{R}^2$ so that
\[
\Delta = \bigcap_{j=3}^{\tilde{N}} \{ x \in t^* \mid \langle \tilde{\eta}_j, 0 \rangle, x \leq \tilde{\kappa}_j \} \cap \bigcap_{i=1}^{4} \{ x \in t^* \mid \langle \tilde{\eta}_i, x \rangle \leq \tilde{\kappa}_i \}, \quad \text{where}
\]
\[
\tilde{\eta}_1 = (0, -e_1), \quad \tilde{\eta}_2 = (\tilde{\eta}_1, e_1), \quad \tilde{\eta}_3 = (0, -e_2), \quad \tilde{\eta}_4 = (\tilde{\eta}_2, e_2),
\]
\[
\tilde{\kappa}_1 = \tilde{\kappa}_3 = 0, \quad \tilde{\kappa}_2 = \tilde{\kappa}_1, \quad \text{and} \quad \tilde{\kappa}_4 = \tilde{\kappa}_2.
\]

We shall call the facet $\tilde{F}_j$ of $\Delta$ with outward conormal $(\tilde{\eta}_j, 0)$ the fiber-type facet (associated to $\tilde{F}_j$) for all $j > 2$, the facets $\tilde{F}_1$ and $\tilde{F}_2$ the base-type facets (associated to $\tilde{F}_1$), and the facets $\tilde{F}_3$ and $\tilde{F}_4$ the base-type facets (associated to $\tilde{F}_2$).

Note that the order of the expansions does not matter; if we expand first along $\tilde{F}_2$ and then along the fiber-type facet associated to $\tilde{F}_1$, the resulting polytope is isomorphic to $\Delta$ under the transformation that interchanges the last two coordinates of $\tilde{t} \oplus \mathbb{R}^2$. Here are a few properties which will be useful later.

**Remark 2.3.6.** (i) Fix $k \subset \{3, \ldots, \tilde{N}\}$. If the facet $\tilde{F}_k$ of $\Delta$ intersects both $\tilde{F}_1$ and $\tilde{F}_2$, then – applying Remark 2.3.2 (i) twice – the face $\tilde{F}_{ij} \cap \tilde{F}_k' := \tilde{F}_i \cap \tilde{F}_j \cap \tilde{F}_k'$ of $\Delta$ intersects all the base-type facets for any $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Conversely, if $\tilde{F}_k$ does not intersect both $\tilde{F}_1$ and $\tilde{F}_2$, then $\tilde{F}_k'$ will not intersect both $\tilde{F}_{12}$ and $\tilde{F}_{34}$. A fortiori, the face $\tilde{F}_{ij} \cap \tilde{F}_k'$ will not intersect all the base-type facets for any $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

(ii) Similarly, applying Remark 2.3.2 (ii) twice, the base-type facets $\tilde{F}_1$ and $\tilde{F}_2$ are equivalent, as are the base-type facets $\tilde{F}_3$ and $\tilde{F}_4$. Moreover, these facets are all equivalent to each other exactly if the facets $\tilde{F}_1$ and $\tilde{F}_2$ are equivalent.

Finally we show how to recognize double expansions.

**Lemma 2.3.7.** Let $F_1, \ldots, F_4$ be facets of a smooth polytope $\Delta \subset t^*$. If $F_1 \sim F_2$, $F_3 \sim F_4$, $F_{12} \neq \emptyset$, and $F_{34} \neq \emptyset$, then $\Delta$ is the double expansion of $F_{13}$ along $F_2 \cap F_{13}$ and $F_4 \cap F_{13}$ with base-type facets $F_1, \ldots, F_4$.

**Proof.** By Proposition 2.3.3, $\Delta$ is the 1-fold expansion of $F_1$ along $F_2 \cap F_1$. Clearly, the fact that $F_3$ is equivalent to $F_4$ implies that $F_{13}$ is equivalent to $F_{14}$. By Remark 2.3.2 (i), the fact that $F_{34} \neq \emptyset$ implies that $F_{13} \cap F_{14} = F_{34} \cap F_1 \neq \emptyset$. Therefore, Proposition 2.3.3 also implies that $F_1$ is the 1-fold expansion of $F_{13}$ along $F_4 \cap F_{13}$. The claim follows immediately. \hfill \square

2.4. Blowing up. In this section, we show how to construct new polytopes by blowing up faces of polytopes. We also consider how this operation affects mass linear functions. We begin with the definition of blowup.

**Definition 2.4.1.** Let $\Delta = \bigcap_{i=1}^{\tilde{N}} \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ be a smooth polytope. Given a face $f = F_1$ of codimension at least 2 and $\varepsilon > 0$, let $\eta_0' := \sum_{i \in I} \eta_i$ and $\kappa_0' := \sum_{i \in I} \kappa_i - \varepsilon$. The polytope
\[
\Delta' = \Delta \cap \{ x \in t^* \mid \langle \eta_0', x \rangle \leq \kappa_0' \}
\]
is the blowup of $\Delta$ along $f$ provided that $\epsilon$ is sufficiently small that $\langle \eta_0', v \rangle < \kappa_0'$ for all vertices $v \in \Delta$ which do not lie on $f$.

It is easy to check that $\Delta'$ is smooth. The facet $F_0'$ corresponding to $\eta_0'$ is called the exceptional divisor; there is a natural one-to-one correspondence between the facets $F_j$ of $\Delta$ and the remaining facets $F_j' = F_j \cap \Delta'$ of $\Delta'$.

**Remark 2.4.2.** The exceptional divisor is a $\Delta_{|I|-1}$ bundle over $f = F_I$ with fiber facets $F_i' \cap F_0'$ for $i \in I$; the base facets are the facets of $f$. Moreover, $f$ is the only face of $\Delta$ which is “lost”. Hence if $K \subset \{1, \ldots, N\}$ does not contain $I$, then $\bigcap_{k \in K \cup \{0\}} F_k' \neq \emptyset$ exactly if $\bigcap_{k \in K} F_k \neq \emptyset$; similarly, $\bigcap_{k \in K} F_k' \neq \emptyset$ exactly if $\bigcap_{k \in K} F_k \neq \emptyset$.

The following example demonstrates the very close connection between bundles and expansions.

**Example 2.4.3.** Suppose that $\Delta$ is the $k$-fold expansion of $\tilde{\Delta}$. Let $\Delta'$ be the blowup of $\Delta$ along the face $f = \bigcap_{i=1}^{k+1} \tilde{F}_i$. It is straightforward to check directly that $\Delta'$ is a $\tilde{\Delta}$ bundle over $\Delta_k$ and that the base facets are $\tilde{F}_1 \cap \Delta', \ldots, \tilde{F}_{k+1} \cap \Delta'$; this justifies our terminology.

In the next remark we show that the blowup of a polytope $\Delta$ along a face $f$ corresponds to the usual geometric blowup of the toric manifold $M_{\Delta}$ along a submanifold $M_f$, and give a geometric interpretation of the preceding example.

**Remark 2.4.4.** (i) Let $\Delta \subset t^*$ be a smooth polytope, and let $\Delta'$ be the blowup of $\Delta$ along a face $f = \bigcap_{i \in I} F_i$; assume that $I = \{1, \ldots, k\}$; we will use the notation of the definition above. Construct the associated toric manifolds $M_\Delta = U/K_C$ and $M_{\Delta'} = U'/K_{C'}$ as in [I, Remark 5.1], where $U \subset \mathbb{C}^N$ and $U' \subset \mathbb{C}^{N+1}$; identify $\mathbb{C}^{N+1}$ with $\mathbb{C} \times \mathbb{C}^N$. Since $\eta_0' = \sum_{i \in I} \eta_i$, $\lambda = e_0 - \sum_{i \in I} e_i$ lies in $t$; define $\Lambda: S^1 \to K$ by $\Lambda(\exp(t)) = \exp(t\lambda)$. Moreover, the intersection of $K'$ with the inclusion $(S^1)^N \to (S^1)^{N+1}$ is $K$; hence we can write $K' = K \times \Lambda$. It is easy to check that the map $f: U' \to U$ defined by $f(z_0, \ldots, z_N) = (z_0^{-1} z_1, \ldots, z_0^{-1} z_k, z_{k+1}, \ldots, z_N)$ is surjective and induces a well defined map of toric manifolds. If $z_i = 0$ for all $i \in I$, the preimage $f^{-1}(z)$ is isomorphic to $\mathbb{C}^k \setminus \{0\}$; otherwise, the preimage is a single $\Lambda$ orbit. Therefore, $f$ induces a surjective holomorphic map $\tilde{f}: M_{\Delta'} \to M_{\Delta}$ which collapses $\Phi_{\Delta}^{-1}(F_0')$ to $\Phi_{\Delta}^{-1}(f)$, but is otherwise a homeomorphism.

(ii) By [I, Remark 5.4] the toric manifold $M_{\Delta}$ corresponding to the 1-fold expansion $\Delta$ of $\tilde{\Delta}$ along $\tilde{F}_1$ can be thought of as a nonsingular symplectic pencil with fibers $M_{\tilde{X}}$ and axis $\tilde{M}_{\tilde{F}_1}$. Thus Example 2.4.3 shows that when we blow up this axis we get a toric bundle.

The next lemma explains how blowing up affects the facet equivalence relation.

**Lemma 2.4.5.** Let $\Delta'$ be the blowup of a polytope $\Delta$ along a face $F_I$.

(i) Given facets $F_i$ and $F_j$ of $\Delta$, the corresponding facets $F_i'$ and $F_j'$ of $\Delta'$ are equivalent exactly if $F_i$ is equivalent to $F_j$ and the pair $\{i, j\}$ is either contained in $I$ or disjoint from $I$.

(ii) The exceptional divisor $F_0'$ is not equivalent to any other facet.
Proof. If $F_i$ and $F_j$ are not equivalent, claim (i) is clear. So assume that $F_i$ and $F_j$ are equivalent. By Lemma 2.1.9 the subspace $V \subset t$ spanned by the conormals $\eta_k$ for $k \notin \{i, j\}$ has codimension 1; moreover, the sum $\eta_i + \eta_j$ lies in $V$. Hence, if both $i$ and $j$ are in $I$, then $\eta_0 = \eta_i + \eta_j \in V$, and if neither are in $I$ then again $\eta_0 \in V$. Hence in these cases $F_i'$ and $F_j'$ are equivalent. In contrast, if only one is in $I$ then $\eta_0 \notin V$, and so $F_i'$ and $F_j'$ are not equivalent.

Now consider (ii). If $F_0'$ were equivalent to $F_k'$ then by Lemma 2.1.9 the subspace of $t$ spanned by the outward conormals to all the facets of $\Delta$ except $F_k$ would have codimension 1. But this is impossible when $\Delta$ is compact. □

We are now ready to analyze the behavior of mass linear functions under blowups. Our arguments use the elementary fact that the volume and $H$-moment

$$V(\Delta) := \int_\Delta 1 \quad \text{and} \quad \mu_H(\Delta) := \int_\Delta H(x)$$

of the polytope $\Delta$ with respect to the affine volume form are additive when $\Delta$ is decomposed as a sum $\Delta' \cup W$. In other words $V(\Delta) = V(\Delta') + V(W)$ and $\mu_H(\Delta) = \mu_H(\Delta') + \mu_H(W)$. Note also that $\mu_H(\Delta) = \langle H, c_\Delta \rangle V(\Delta)$.

Since the facets of $\Delta$ are a subset of the facets of $\Delta'$, we may think of $\langle H, c_\Delta \rangle$ as a function on an open subset $U$ of $\mathbb{R}^{N+1}$ — a function which does not depend on the support number $\kappa_0'$ of the exceptional divisor. We will say that $\langle H, c_\Delta \rangle$ and $\langle H, c_{\Delta'} \rangle$ are equal if they agree on a nonempty open subset of the form $U \cap C_{\Delta'}$. In this case, the exceptional divisor is symmetric and $H$ is mass linear on $\Delta$ exactly if it is mass linear on $\Delta'$; moreover, if they are mass linear the coefficient of the support number of a facet $F_1$ in $\langle H, c_\Delta \rangle$ is the coefficient of the support number of $F_1 \cap \Delta'$ in $\langle H, c_{\Delta'} \rangle$. Similarly, we may think of $\langle H, c_W \rangle$ as a function on an open subset of $\mathbb{R}^{N+1}$ which does not depend on the support numbers of any of the facets of $\Delta$ which do not intersect $f$.

**Lemma 2.4.6.** Fix $H \in t$. Let $\Delta'$ be the blowup of a polytope $\Delta \subset t^*$ along a face $f$ and write $\Delta = \Delta' \cup W$. Assume that two of the three functions $\langle H, c_\Delta \rangle$, $\langle H, c_{\Delta'} \rangle$, and $\langle H, c_W \rangle$ are equal. Then all three functions are equal; in particular, $H$ is mass linear on $\Delta$ exactly if $H$ is mass linear on $\Delta'$.

**Proof.** Since the $H$-moment is additive,

$$\langle H, c_\Delta \rangle V(\Delta) = \langle H, c_{\Delta'} \rangle V(\Delta') + \langle H, c_W \rangle V(W).$$

Since $V(\Delta) = V(\Delta') + V(W)$ the three functions $\langle H, c_\Delta \rangle$, $\langle H, c_{\Delta'} \rangle$, and $\langle H, c_W \rangle$ must agree on some nonempty open set, and hence, as explained at the beginning of §2.1 on $C_{\Delta'}$. □

We first describe what happens when $H$ is mass linear on a polytope $\Delta'$ that is a blowup.

**Lemma 2.4.7.** Let $H \in t$ be a mass linear function on a polytope $\Delta'$ that is the blowup of a polytope $\Delta$ along a face $f$. The following hold.

(i) The exceptional divisor $F_0'$ is symmetric.

(ii) $H$ is mass linear on $\Delta$ and $\langle H, c_\Delta \rangle = \langle H, c_{\Delta'} \rangle$.

(iii) The face $f$ meets every asymmetric facet.
(iv) If $H$ is inessential on $\Delta'$, then it is inessential on $\Delta$.

Proof. Decompose $\Delta$ as $\Delta' \cup W$ where $W$ is a $\Delta|_I$-bundle over $f = F_I$ with a fiber of size $\epsilon = \sum_{i \in I} \kappa_i - \kappa'_0 > 0$. Let $V_W$ and $V'$ denote the volume of $W$ and $\Delta'$, respectively; similarly, let $\mu_W$ and $\mu'$ denote the $H$-moment of $W$ and $\Delta'$ as in Equation (2.3). At $\epsilon = 0$, the partial derivatives $\frac{\partial \mu_W}{\partial \kappa'_0}$ and $\frac{\partial \mu_W}{\partial \kappa'_0}$ both vanish since $V_W$ and $\mu_W$ are polynomial functions with a factor $\epsilon^k$, where $k > 1$. By the additivity of the volume and moment, this implies that $\frac{\partial \mu'}{\partial \kappa'_0}$ and $\frac{\partial \mu'}{\partial \kappa'_0}$ also both vanish at $\epsilon = 0$. Finally, since $\mu' = \langle H, c_{\Delta'} \rangle V'$, this implies that $\frac{\partial (H, c_{\Delta'})}{\partial \kappa'_0}$ vanishes at $\epsilon = 0$. Therefore, since $H$ is mass linear on $\Delta'$, $F'_0$ is symmetric.

This proves (i).

Now fix $\kappa \in C_\Delta$. Since $F'_0$ is symmetric, $\langle H, c_{\Delta'}(\kappa'_0, \kappa) \rangle$ does not depend on the support number $\kappa'_0$ as long as $(\kappa'_0, \kappa)$ lies in $C_{\Delta'}$. In fact, since the center of mass is a continuous function of the support numbers, the same claim holds as long as $(\kappa'_0, \kappa)$ lies in the closure of $C_{\Delta'}$. Moreover, if $\kappa'_0 = \sum_{i \in I} \kappa_i$ then $P(F'_0)$ intersects $\Delta$ at exactly $f$, and the polytopes $\Delta'(\kappa'_0, \kappa)$ and $\Delta(\kappa)$ coincide. Therefore $\langle H, c_{\Delta}(\kappa) \rangle = \langle H, c_{\Delta'}(\kappa'_0, \kappa) \rangle$ is a linear function of $\kappa$. The claims in (ii) follow immediately.

Since the symmetric facet $F'_0$ meets all asymmetric facets of $\Delta'$ by Proposition 2.1.5, the face $f$ does as well by Remark 2.4.2. This proves (iii).

Claim (iv) follows immediately from Lemma 2.4.5.

We are now ready to consider the question of which blowups preserve mass linearity. The simplest case is symmetric blowup, that is, blowing up along a symmetric face.

Lemma 2.4.8. Let $H \in \mathfrak{t}$ be a mass linear function on a polytope $\Delta \subset \mathfrak{t}^*$. Let $\Delta'$ be the blowup of $\Delta$ along a symmetric face $f$. Then the following hold.

(i) $H$ is mass linear on $\Delta'$ and $\langle H, c_{\Delta} \rangle = \langle H, c_{\Delta'} \rangle$.

(ii) $H$ is essential on $\Delta$ exactly if it is essential on $\Delta'$.

Proof. Let $f = F_I$, and let $\eta_i$ be the outward conormal to $F_i$ for all $i$. Then $\Delta = \Delta' \cup W$, where $W$ is a $\Delta|_I$ bundle over $f$. The outward normals to the fiber facets of $W$ are $\{\eta_i\}_{i \in I}$ and $-\sum_{i \in I} \eta_i$. The outward normals to its base facets are the outward normals to the facets of $\Delta$ that restrict to facets of $f$. Since $f$ is symmetric, the restriction of $H$ to $f$ is mass linear with the same coefficients. Hence, by Lemma 2.1.3, the restriction of $H$ to $W$ is the lift of the restriction of $H$ to $f$. Therefore, Proposition 2.1.1 implies that $H|_W$ is mass linear with the same coefficients on $W$, that is, $\langle H, c_{\Delta} \rangle = \langle H, c_{\Delta'} \rangle$. Thus (i) follows from Lemma 2.1.6.

Since $F_i$ is symmetric for all $i \in I$, Proposition 2.1.1 implies that every inessential $H$ has the form $H = \sum_{j \not\in I} \beta_j \eta_j$. Therefore to prove (ii) it suffices to recall that, by Lemma 2.1.5, $F_k$ and $F_\ell$ are equivalent facets of $\Delta$ for some $k$ and $\ell$ not in $I$ exactly if $F_k'$ and $F_\ell'$ are equivalent facets of $\Delta'$; moreover, the exceptional divisor $F'_0$ is not equivalent to any other facet.

\footnote{Here we use the fact that locally mass linear functions are globally mass linear: cf. [I, Lemma 2.3]. Thus $\langle H, c_{\Delta} \rangle$ is a linear function of $(\kappa'_0, \kappa)$ throughout the chamber $C_{\Delta'}$.}
We can also blow up faces which are not symmetric, but then the situation is more complicated. We first describe the scenario that is most relevant to the 4-dimensional classification. It turns out to be an important mechanism for creating new essential mass linear functions, since $H$ may be essential on $\Delta'$ even if it is inessential on $\Delta$. See Example 1.1.5 for an easy instance of this process, and Propositions 3.2.2 and 5.3.7 for a more extended discussion.

**Definition 2.4.9.** Let $H$ be a mass linear function on a polytope $\Delta$ with asymmetric facets $F_1, \ldots, F_k$. We say that a blowup of $\Delta$ is of type $(F_{ij}, g)$ if it is the blowup of $\Delta$ along the edge $F_{ij} \cap g$, where $g$ is a symmetric 3-face, $F_{ij} \cap g$ intersects every asymmetric facet, and $\gamma_i + \gamma_j = 0$. Here $\gamma_k$ is the coefficient of the support number of $F_k$ in the linear function $(H, c_\Delta)$.

**Proposition 2.4.10.** Let $H \in \mathfrak{t}$ be a mass linear function on a smooth polytope $\Delta \subset \mathfrak{t}^*$. Let $\Delta'$ be a blowup of $\Delta$ of type $(F_{ij}, g)$. The following hold.

(i) $\Delta$ has zero, two, or four asymmetric facets.
(ii) $H$ is mass linear on $\Delta'$ and $(H, c_\Delta) = (H, c_{\Delta'})$.
(iii) If $H$ is essential on $\Delta$, then it is essential on $\Delta'$; otherwise, $H$ is essential on $\Delta'$ exactly if $F_i \not\sim F_j$ and there are four asymmetric facets.

**Proof.** Label the facets of $\Delta$ so that the two facets that intersect the edge $F_{12} \cap g$ are $F_3$ and $F_4$, and so that $g = \cap_{j=1}^n G_j$. Let $\eta_i$ denote the outward conormal to $F_i$ and let $\alpha_j$ denote the outward conormal to $G_j$. Since the edge $F_{12} \cap g$ intersects every asymmetric facet, each facet except possibly $F_1, \ldots, F_4$ is symmetric.

By Proposition 2.1.5, $\langle H, c_{g(\kappa)} \rangle = \langle H, c_{\Delta}(\kappa) \rangle$ for all $\kappa \in C_\Delta$. In particular, the restriction of $H$ to $g$ is mass linear. Thus, Proposition 2.1.5 implies that $\sum_{i=1}^4 \gamma_i = 0$. Since $\gamma_1 + \gamma_2 = 0$, this implies that $\gamma_3 + \gamma_4 = 0$. Therefore, $\gamma_1$ and $\gamma_2$ (respectively $\gamma_3$ and $\gamma_4$) are either both zero or both nonzero. This proves (i).

If $F_1$ and $F_2$ are symmetric, claim (ii) follows from Lemma 2.4.8. Hence, we may assume that $F_1$ and $F_2$ are asymmetric facets. By Proposition 2.1.5, intersection induces a one-to-one correspondence between the asymmetric facets of $\Delta$ and the asymmetric facets of $g$. Therefore, $F_1 \cap g$ and $F_2 \cap g$ are asymmetric facets of $g$ and $F_1 \cap g$ intersects every asymmetric facet of $g$. Hence, Lemma 2.4.11 below implies that $F_1 \cap g$ and $F_2 \cap g$ are equivalent facets of $g$.

We claim that $\eta_3, \eta_4, \alpha_1, \ldots, \alpha_{n-3}$, and $\eta_1 + \eta_2$ all lie in a hyperplane of $\mathfrak{t}$. To see this, observe that the smallest affine plane $P(g) \subset \mathfrak{t}^*$ containing the face $g$ is

$$P(g) = \bigcap_{j=1}^{n-3} \{ x \in \mathfrak{t}^* \mid \langle \alpha_j, x \rangle = \kappa_j \},$$

and hence may be identified with the dual to the quotient of $\mathfrak{t}$ by the span $V_\alpha$ of the $\alpha_j$. (This is explained in more detail at the beginning of [I, §2].) Let $\pi : \mathfrak{t} \to \mathfrak{t}/V_\alpha$ denote the projection. Then the claim will follow if we can show that the vectors $\pi(\eta_1) + \pi(\eta_2), \pi(\eta_3)$, and $\pi(\eta_1) + \pi(\eta_2)$ span a hyperplane in $\mathfrak{t}/V_\alpha$. But by Lemma 2.1.9 this follows from the fact that $F_1 \cap g$ and $F_2 \cap g$ are equivalent facets of $g$. 


Now note that $\Delta = \Delta' \cup W$, where $W$ is a $\Delta_{n-1}$ bundle over $\Delta_1$. The outward conormals to the fiber facets of $W$ are $\eta_1, \eta_2, \alpha_1, \ldots, \alpha_{n-3}$, and $-\eta_1 - \eta_2 - \sum_i \alpha_i$; the outward conormals to the base facets are $\eta_3$ and $\eta_4$. Therefore, the facets of $W$ with conormals $\eta_3$ and $\eta_4$ are equivalent. Moreover, since $\eta_3, \eta_4, \alpha_1, \ldots, \alpha_{n-3}$, and $\eta_1 + \eta_2$ lie in a hyperplane, the facets of $W$ with conormals $\eta_1$ and $\eta_2$ are equivalent. By Lemma 2.1.3 $H = \sum_{i=1}^4 \gamma_i \eta_i$. Since $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 = 0$, $H$ is inessential on $W$; hence by Proposition 2.1.1

$$\langle H, c_W \rangle = \sum \gamma_i \kappa_i = \langle H, c_\Delta \rangle.$$ 

Claim (ii) now follows from Lemma 2.4.6.

Since the first part of claim (iii) is a special case of Lemma 2.4.7 (iv), we may assume that $H$ is inessential on $\Delta$. By Proposition 2.1.1 this implies that every asymmetric facet must be equivalent to at least one other asymmetric facet. Moreover, recall that $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 = 0$. Hence, if $F_1 \sim F_2$ or if any of the facets $F_1, \ldots, F_4$ are symmetric, then the following statements are both true.

- $F_1 \sim F_2$ or $\gamma_1 = \gamma_2 = 0$,
- $F_3 \sim F_4$ or $\gamma_3 = \gamma_4 = 0$.

Hence, $H$ is inessential on $\Delta'$ by Lemma 2.4.5 (i). In contrast, if $\gamma_1 \neq 0$ and $F_1 \not\sim F_2$ the same lemma implies that $F_1$ is not equivalent to any other asymmetric facet. Claim (iii) follows immediately. \qed

Here is the auxiliary lemma used above.

**Lemma 2.4.11.** Let $H \in t$ be a mass linear function on a smooth 3-dimensional polytope $\Delta \subset t^*$. If $F_1$ and $F_2$ are asymmetric facets and the edge $F_{12}$ meets every asymmetric facet, then $F_1$ and $F_2$ are equivalent facets of $\Delta$.

**Proof.** By Proposition 2.1.15 we see that there are only three possibilities:

- $\Delta$ has exactly two asymmetric facets;
- $\Delta$ is the simplex $\Delta_3$; or
- $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$, $F_1$ and $F_2$ are fiber facets, and the third fiber facet is symmetric.

If $\Delta$ has exactly two asymmetric facets, $F_1$ and $F_2$, then $F_1 \sim F_2$ by Proposition 2.1.8.

If $\Delta$ is the simplex $\Delta_3$, then all the facets are equivalent. Therefore it remains to consider the third case. By Proposition 2.2.4 there is an inessential function $H'$ so that the $\tilde{H}$-asymmetric facets are exactly $F_1$ and $F_2$, where $\tilde{H} := H - H'$. By Proposition 2.1.8 this implies that $F_1$ and $F_2$ are equivalent. \qed

As we mentioned above, blowups of the form considered in Proposition 2.4.10 may convert an inessential function on $\Delta$ to an essential function on the blowup $\Delta'$. The next result shows that this is not possible if we blow up along a face that is contained in every asymmetric facet.

**Lemma 2.4.12.** Fix $H \in t^*$ and let $\Delta'$ be the blowup of a polytope $\Delta$ along a face $F_I$ that is contained in every $H$-asymmetric facet. Then $H$ is inessential on $\Delta$ exactly if it is inessential on $\Delta'$.
Proof. If $H$ is inessential on $\Delta$, this follows easily from Proposition 2.1.1 and Lemma 2.4.5 (i). The converse is a special case of Lemma 2.4.7 (iv). □

Further examples of blowups that preserve mass linearity are given in §5.3. For example, we show that blowing up a polytope at a vertex that meets every asymmetric facet preserves mass linearity; see Corollary 5.3.3

2.5. Blowing down. Although one can always blow up a smooth polytope along a facet of codimension at least 2 to obtain a new smooth polytope, it is not so easy to decide when this process can be reversed. This subsection explores general conditions under which this is possible. Explicit 4-dimensional examples may be found in §5.1

We say that a smooth polytope $\Delta$ can be blown down along a facet $F_0$ if $\Delta$ is the blowup of a smooth polytope $\Delta'$ along some face $\mathcal{F}$, and $F_0$ is the exceptional divisor. In this case, the polytope $\Delta'$ is obtained from $\Delta$ by moving the hyperplane $P(F_0)$ outwards (i.e. increasing its support number $\kappa_0$) until it no longer intersects the intersection of the remaining half spaces. The facet $F_0$ must be a bundle whose fiber is a simplex $\Delta_k$. As $\kappa_0$ increases, the sizes and relative positions of the fiber and the base facets of $F_0$ changes. If the outward conormal to $F_0$ is a positive multiple of the sum of the fiber facets, the size of the fiber facet will decrease as we move $P(F_0)$ outwards. The transition from $\Delta$ to $\Delta'$ is a blowdown if $\Delta'$ is smooth and if during this movement of $P(F_0)$ there is precisely one value of $\kappa_0$ for which $P(F_0)$ intersects a vertex of $\Delta$. What is crucial is that the size of the fiber shrinks to zero before any new intersections of the base facets of $F_0$ are created.

It is easy to check that an edge of a smooth 2-dimensional polygon can be blown down exactly if the outward conormal to that edge is the sum of the outward conormals to the two adjacent edges. In higher dimensions, the situation is somewhat more complicated.

Proposition 2.5.1. Let $\Delta = \bigcap_{i=0}^N \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ be a smooth polytope; denote the facets by $F_0, \ldots, F_N$. Fix $I \subset \{1, \ldots, N\}$. Then the polytope $\Delta = \bigcap_{i=1}^N \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ is smooth and $\Delta$ is the blowup of $\Delta$ along $\mathcal{F}_I := \bigcap_{i \in I} P(F_i) \cap \Delta$ with exceptional divisor $F_0$ exactly if

(i) The facet $F_0$ is a $\Delta_{|I|=1}$ bundle with fiber facets $\{ F_i \cap F_0 \}_{i \in I}$ and base facets $\{ F_j \cap F_0 \}_{j \in J}$ for some $J \subset \{1, \ldots, N\}$.

(ii) $\eta_0 = \sum_{i \in I} \eta_i$.

(iii) Given $K \subset J$, if $F_K = \emptyset$ then $\mathcal{F}_K := \bigcap_{k \in K} P(F_k) \cap \Delta = \emptyset$.

Proof. We have already seen that if $\Delta$ is smooth and $\Delta$ is the blowup of $\Delta$ along $\mathcal{F}_I$ with exceptional divisor $F_0$, then (i), (ii) and (iii) hold; see Remark 2.4.2.

To prove the converse, first note that since $\Delta$ is compact, the positive span of the $\eta_i$ is all of $t$. By assumption (ii), this implies that the positive span of the $\eta_i$ for $i \geq 1$ is also all of $t$, and so $\Delta$ is compact.

Next, consider a “new” vertex $\bar{v}$ of $\Delta$, that is, a vertex which satisfies $\langle \eta_0, \bar{v} \rangle > \kappa_0$ and hence does not lie in $\Delta$. Write $\bar{v} = \mathcal{F}_I \cap \mathcal{F}_K$, where $I' \subset I$ and $K \cap I = \emptyset$. Since the facet $F_K$ is not empty and $\mathcal{F}_k = P(F_k) \cap \Delta$ is connected, the intersection $\mathcal{F}_k \cap F_0 = F_k \cap F_0$ is not empty for any $k \in K$. Hence, $K \subset J$, and so by assumption (iii) the face $F_K$ is also
nonempty. Since $F_K$ is connected, this implies that $F_K \cap F_0 = F_K \cap F_0$ is not empty. By assumption (i), this implies that $F_{I \setminus \{i\}} \cap F_K \cap F_0$ is a vertex of $\Delta$ for all $i \in I$. Since $\Delta$ is simple, this implies that $|I| + |K| = n$. Since there must be at least $n$ facets through $\overline{v}$, it also implies that $|I'| = |I|$. Hence $I' = I$ and also $\overline{v}$ is a simple vertex. Since $\Delta$ is smooth, the vectors $\{\eta_j\}_{j \in I \setminus \{i\}}, \{\eta_k\}_{k \in K}$ and $\eta_0$ span the lattice $t \mathbb{Z}$ for all $i \in I$. By part (ii), this means that the vectors $\{\eta_j\}_{j \in I}$ and $\{\eta_k\}_{k \in K}$ also span the lattice, that is, that $v$ is a smooth vertex.

Since $\Delta$ is smooth, and every “new” vertex is smooth, $\overline{\Delta}$ is also smooth. Finally, since $\Delta$ is compact and every new vertex lies on $F_I$, $F_I$ is not empty.

Remark 2.5.2. In some cases, the polytope $\Delta$ can be blown down along the facet $F_0$ for some values of $\kappa \in \mathcal{C}_\Delta$, but not for other values $\kappa' \in \mathcal{C}_\Delta$; see Figure 2.3. This is because condition (iii) may depend on $\kappa$.

![Figure 2.3](image1.png)

**Figure 2.3.** (b) is the blowup of (a) along $e$. When the top facet is moved down as in (c), the facet $F_0'$ no longer blows down.

![Figure 2.4](image2.png)

**Figure 2.4.** (b) is the blowdown of (a) along $F_0$ with $I = \{4,5\}$; (c) is the blowdown with $I = \{2,3\}$. 
Another possibility is that the blowdown of $\Delta$ along $F_0$ depends on the choice of $\kappa$. For example, suppose that $\Delta = \bigcap_{i=0}^5 \{ x \in \mathbb{R}^3 \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ where
\[
\kappa = (1, 0, 0, \lambda, 0, 2), \quad \lambda > 1; \text{ see Figure 2.4}
\]
In this case, $F_0$ can be viewed as a $\Delta_1$ bundle over $\Delta_1$ in two ways – either the fiber facets are $F_{20}$ and $F_{30}$, or the fiber facets are $F_{40}$ and $F_{50}$; so a priori we can take $I = \{2, 3\}$ or $I = \{4, 5\}$. Either way, condition (ii) is also satisfied. If $\lambda > 2$, then condition (iii) also holds if we take $I = \{4, 5\}$; so $\Delta$ is the blowup of $\overline{\Delta}$ along the (non empty) face $F_{23}$. Conversely, if $\lambda < 2$, then $\Delta$ is the blowup of $\overline{\Delta}$ along the (non empty) face $F_{23}$. Finally, if $\lambda = 2$, then condition (iii) is not satisfied in either case. In fact, it is easy to see that $\overline{\Delta}$ is not a simple polytope.

In practice, we will not directly prove that condition (ii) of Proposition 2.5.1 holds; instead, we will use the following technical lemma which allows us to reduce to the simpler case of lower dimensional polytopes.

**Lemma 2.5.3.** Let $\Delta = \bigcap_{i=0}^N \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ be a smooth polytope. Assume that $F_0$ is a $\Delta_{|I|-1}$ bundle with fiber facets $\{ F_i \cap F_0 \}_{i \in I}$. Also assume that there exists $L \subset \{1, \ldots, N\}$ so that the face $F_L$ is the blowup of a smooth polytope $\overline{F}_L$ along the face $\bigcap_{i \in I} P(F_i) \cap \overline{F}_L$ with exceptional divisor $F_0 \cap F_L$. Then condition (ii) of Proposition 2.5.1 is satisfied.

*Proof.* By the definition of a $\Delta_{|I|-1}$ bundle, $\sum_{i \in I} \eta_i$ is constant when restricted to $P(F_0)$, that is, $\sum_{i \in I} \eta_i = c \eta_0$ for some real number $c$. Since $F_0 \cap F_L$ is not empty and $\Delta$ is simple, $\eta_0$ is nonconstant when restricted to $P(F_L)$. Since $F_L$ is the blowup of a smooth polytope $\overline{F}_L$ along the face $\bigcap_{i \in I} P(F_i) \cap \overline{F}_L$ with exceptional divisor $F_0 \cap F_L$, $\eta_0 - \sum_{i \in I} \eta_i$ is constant on $P(F_L)$. Therefore, $(1 - c) \eta_0$ is also constant on $P(F_L)$. Since $\eta_0$ is nonconstant on $P(F_L)$, this implies that $c = 1$. \hfill $\square$

**Remark 2.5.4.** Conversely, let $\Delta'$ be the blowup of a polytope $\Delta$ along a face $F_I$ with exceptional divisor $F_0'$. Let $F_L'$ be a face of $\Delta'$ that meets $F_0'$. If $|I \setminus (I \cap L)| \geq 2$, then $F_L' := F_L \cap \Delta'$ is the blowup of $F_L$ along the face $F_I \cap F_L$ with exceptional divisor $F_0' \cap F_L$.

In most of the cases we consider, condition (iii) of Proposition 2.5.1 is extremely straightforward to check. However, for the third case of Lemma 4.2.3 we will need the following lemma.

**Lemma 2.5.5.** Let $\Delta = \bigcap_{i=0}^N \{ x \in t^* \mid \langle \eta_i, x \rangle \leq \kappa_i \}$ be a smooth 4-dimensional polytope. Assume that $F_0$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$ with base facets $F_i \cap F_0, \ldots, F_4 \cap F_0$, and that $F_{ij} := F_i \cap F_j$ is not empty for any pair $1 \leq i < j \leq 4$. Then condition (iii) of Proposition 2.5.1 is satisfied.

*Proof.* It follows immediately from the assumptions that (iii) holds for all $K \subset \{1, 2, 3, 4\}$ with at most two elements. By renumbering, we may assume that $F_{12} \cap F_0$ and $F_{34} \cap F_0$ are empty. Since $F_{12}$ is not empty and $\overline{F}_{12} := P(F_1) \cap P(F_2) \cap \overline{\Delta}$ is connected, this implies that $\overline{F}_{12} \subset \Delta$. Hence, condition (iii) is satisfied for all $K \subset \{1, 2, 3, 4\}$ which contain $\{1, 2\}$. A similar argument shows that condition (iii) is satisfied for all $K \subset \{1, 2, 3, 4\}$ which contain $\{3, 4\}$.
\{3,4\}. Since every subset of \{1, 2, 3, 4\} with more than 2 elements contains one of these pairs, this completes the proof.

We end this section by considering blowdowns of polygons. This process is well understood; every smooth polygon with more than four edges can be blown down to a trapezoid; see [2]. We shall need the following more precise version of this result.

**Lemma 2.5.6.** Let $\Delta$ be a smooth convex 2-dimensional polygon with more than four edges.

(i) If $e$ and $e'$ are parallel edges, then there exists an edge which is not equal to $e$ or $e'$, which is not adjacent to $e$, and which can be blown down.

(ii) Let $e, e', e''$ be adjacent edges with outward conormals $\alpha, \alpha'$, and $\alpha''$, respectively. If $\alpha'$ is not a positive linear combination of $\alpha$ and $\alpha''$, then there is an edge which is not equal to $e, e'$ or $e''$ which can be blown down.

**Proof.** We begin with the first claim, following the proof in [2]. Let $e = e_1, e_2, \ldots, e_k = e'$ be a sequence of edges in $\Delta$ with outward conormals $\alpha_1, \ldots, \alpha_k$, respectively. Since $\Delta$ has more than four edges, we may assume that $k > 3$. Since $\alpha_1$ and $\alpha_2$ form an integral basis, we may write $\alpha_j = -a_j\alpha_1 + b_j\alpha_2$ for each $j$, and set $c_j = a_j + b_j$. Since for each $j$ there is an integer $d_j$ such that $\alpha_j = \frac{1}{d_j}(\alpha_{j-1} + \alpha_{j+1})$, we see that $c_j = \frac{1}{d_j}(c_{j-1} + c_{j+1})$. Note that $d > 0$ since $e_1$ and $e_k$ are parallel. Hence $c_3 = 1 + d_2 \geq 2$ and $c_k = 1$. It follows that there exists $\ell \geq 3$ with $c_\ell > c_{\ell+1}$ and $c_\ell \geq c_{\ell-1}$. In this case, $d_\ell$ must be 1, and so $e_\ell$ can be blown down.

To prove (ii), note first that every smooth convex polygon with more than three edges is the blowup of a trapezoid and so must have (at least) two edges which are parallel. Our assumptions imply that $e'$ together with extensions of the two edges $e, e''$ form a triangle. It follows that $e'$ must be parallel to another edge. We can now apply the first part.

### 3. Examples of essential mass linear functions

In this section we give examples of (essential) mass linear functions on polytopes. We consider two basic types of examples: bundles and blowups of double expansions. The examples that we consider include $\Delta_2$ bundles over $\Delta_1$. By Proposition 2.1.15, this implies that in this section we construct every essential mass linear function on a smooth polytope of dimension at most 3. More importantly, the examples we consider include all the types of polytopes described in Theorem 1.1.1. Therefore, we also construct every essential mass linear function on a smooth 4-dimensional polytope.

The results in this section are not needed for the proof of the main theorem since that gives necessary rather than sufficient conditions for mass linearity. In fact, except for Corollary 4.3.2 and several remarks), this section and §4 are completely independent.

#### 3.1. Essential mass linear functions on bundles

In this subsection, we find all essential mass linear functions on each of the bundles described in part (a) of Theorem 1.1.1.
To begin, consider a polytope $Y$ that is $\Delta_k$ bundle over $\Delta_1$. By definition there exists $a = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$, an identification of $t$ with $\mathbb{R}^{k+1}$, and $\kappa \in \mathbb{R}^{k+3}$ so that

\[(3.1) \quad Y = \bigcap_{i=1}^{k+3} \{ x \in (\mathbb{R}^{k+1})^* \mid \langle x, \eta_i \rangle \leq \kappa_i \}, \quad \text{where} \]

\[\eta_i = -e_i \quad \text{for all } 1 \leq i \leq k, \quad \eta_{k+1} = \sum_{i=1}^{k} e_i, \quad \eta_{k+2} = -e_{k+1}, \quad \text{and} \quad \eta_{k+3} = e_{k+1} + \sum_{i=1}^{k} a_i e_i.\]

Here, $e_1, \ldots, e_{k+1}$ is the standard basis for $\mathbb{R}^{k+1}$. The fiber facets are $F_1, \ldots, F_{k+1}$. Conversely, (3.1) describes a $\Delta_k$ bundle over $\Delta_1$ exactly if $\kappa \in \mathcal{C}_a$, where

\[\mathcal{C}_a = \left\{ \kappa \in \mathbb{R}^{k+3} \mid \sum_{i=1}^{k+3} \kappa_i > 0 \text{ and } \sum_{i=1}^{k} a_i \kappa_i + \kappa_{k+2} + \kappa_{k+3} \geq \max(0, a_1, \ldots, a_k) \sum_{i=1}^{k+3} \kappa_i \right\}.\]

**Proposition 3.1.1.** Let $Y$ be the $\Delta_k$ bundle over $\Delta_1$ associated to $a \in \mathbb{R}^k$ as in (3.1) above; set $a_{k+1} = 0$. Then $H \in t$ is a mass linear function on $Y$ exactly if

\[H = \sum_{i=1}^{k+3} \gamma_i \eta_i, \quad \text{where} \quad \gamma_{k+2} + \gamma_{k+3} = \sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} a_i \gamma_i = 0.\]

In this case, $\langle H, c_Y \rangle = \sum_{i=1}^{k+3} \gamma_i \kappa_i$. Moreover, $H$ is inessential exactly if

\[\sum_{a_i = \alpha} \gamma_i = 0 \quad \forall \alpha \in \mathbb{R},\]

where the sum is over $i \in \{1, \ldots, k+1\}$ such that $a_i = \alpha$.

**Remark 3.1.2.** (i) Because $\sum_{i=1}^{k+3} \eta_i = 0$ and $\eta_{k+2} + \eta_{k+3} = \sum_{i \leq k} a_i \eta_i$, each $H \in t$ can be written as $\sum_{i=1}^{k+3} \gamma_i \eta_i$ where $\gamma_{k+2} + \gamma_{k+3} = \sum_{i=1}^{k} \gamma_i = 0$. Therefore the most significant condition on $H$ above is that $\sum_{i \leq k} a_i \gamma_i = 0$. Note that this holds for all $H$ exactly if $a_1 = \cdots = a_k = 0$, that is, exactly if $Y$ is the product $\Delta_1 \times \Delta_k$. Moreover, in this case every $H \in t$ is inessential. (More generally, by [1, Theorem 1.20], the only polytopes for which all vectors $H \in t$ are mass linear are products of simplices.) Otherwise, $Y$ admits a $k$-dimensional family of mass linear functions and the inessential mass linear functions form a subspace of dimension $k + 2 - |\{a_1, \ldots, a_k, 0\}|$.

(ii) The polytope $Y$ is smooth exactly if $a \in \mathbb{Z}^k$. In this case, the corresponding toric manifold $M_Y$ is the $\mathbb{C}P^k$ bundle over $\mathbb{C}P^1$ associated to the action

\[e^{i\theta} \cdot [z_1 : \cdots : z_{k+1}] = [e^{-a_1 i \theta} z_1 : \cdots : e^{-a_k i \theta} z_k : z_{k+1}].\]

The polytope $Y$ is determined up to translation by two constants, namely $\lambda := \sum_{i=1}^{k+3} \kappa_i$ and $h := \sum_{i=1}^{k} a_i \kappa_i + \kappa_{k+2} + \kappa_{k+3}$; cf. the proof of Lemma 3.1.3 below. Note that $\lambda$ determines the “size” of the fiber, while $h$ determines that of (one section of) the base.

(iii) At first glance, the restrictions on $H$ in Proposition 3.1.1 may seem mysterious; we will now give a geometric motivation. Suppose that $H$ is mass linear and write $\langle H, c_Y \rangle =$
Lemma 3.1.3. Let \( Y \) be the \( \Delta_k \) bundle over \( \Delta_1 \) associated to \( a \in \mathbb{R}^k \) as in (3.1) above. Let \( H = \sum_{i=1}^{k+1} \gamma_i \eta_i \), where \( \sum_{i=1}^{k+1} \gamma_i = 0 \). Then \( H \) is mass linear on \( Y \) if and only if
\[
\sum_{i=1}^{k} \gamma_i a_i = 0; \quad \text{in this case,} \quad \langle H, cy \rangle = \sum_{i=1}^{k+1} \gamma_i \kappa_i.
\]

Proof. As a first step, fix \( \kappa_1 = \cdots = \kappa_k = 0 \) and \( \kappa_{k+2} = 0 \), and let \( \kappa_{k+1} = \lambda \) and \( \kappa_{k+3} = h \). Let \( \Delta_k^1 \subset \mathbb{R}^k \) denote the \( k \)-simplex described by the inequalities
\[
x_i \geq 0 \quad \text{for all} \quad 1 \leq i \leq k \quad \text{and} \quad \sum_{i=1}^{k} x_i \leq \lambda.
\]

An elementary calculation shows that for any non-negative integers \( i_1, \ldots, i_k \),
\[
\int_{\Delta_k^1} x_{i_1} x_{i_2} \cdots x_{i_k} = \frac{i_1! i_2! \cdots i_k! \lambda^{I+k}}{(I+k)!}, \quad \text{where} \quad I = \sum_{j=1}^{k} i_j.
\]

Here by convention \( 0! = 1 \). Furthermore, both here and elsewhere we integrate with respect to the standard measure \( dx_1 \cdots dx_k \) on \( \mathbb{R}^k \). Since \( Y \) is a \( \Delta_k^1 \) bundle over \( \Delta_1 \), \( Y \) has volume
\[
V = \int_{\Delta_k^1} \left( h - \sum_{i=1}^{k} a_i x_i \right) = \frac{(k+1)h\lambda^k - \left( \sum_{i=1}^{k} a_i \right) \lambda^{k+1}}{(k+1)!}.
\]

For \( j \neq k + 1 \), the moment \( \mu_j \) of \( Y' \) along the \( x_j \) axis is
\[
\mu_j = \int_{\Delta_k^1} \left( h x_j - \sum_{i=1}^{k} a_i x_i x_j \right) = \frac{(k+2)h\lambda^{k+1} - \left( a_j + \sum_{i=1}^{k} a_i \right) \lambda^{k+2}}{(k+2)!}.
\]

Let \( c_j := \mu_j / V \) denote the \( j \)'th component of the center of mass. For \( j \neq k + 1 \),
\[
c_j = \frac{\lambda}{k+2} \frac{h(k+2) - \lambda \left( a_j + \sum_{i=1}^{k} a_i \right)}{h(k+1) - \lambda \sum_{i=1}^{k} a_i}.
\]
Since $\sum_{i=1}^{k+1} \gamma_i = 0$, a straightforward calculation shows that

$$\langle H, cy \rangle = \sum_{i=1}^{k} (\gamma_{k+1} - \gamma_i) c_i = \lambda \left( \gamma_{k+1} + \frac{\lambda \sum_{i=1}^{k} \gamma_i a_i}{(k+2)(h(k+1) - \lambda \sum_{i=1}^{k} a_i)} \right).$$

This is linear function of $h$ and $\lambda$ exactly if $\sum_{i=1}^{k} \gamma_i a_i = 0$. Hence, if $H$ is mass linear, this sum must be zero.

To prove the converse, assume that $\sum_{i=1}^{k} \gamma_i a_i = 0$. Given $\kappa \in C_a$, note that by Remark 2.1.4

$$Y(\kappa) = Y(0, \ldots, 0, \lambda, 0, h) - (\kappa_1, \ldots, \kappa_k, \kappa_{k+2}),$$

where

$$\lambda = \sum_{i=1}^{k} \kappa_i \quad \text{and} \quad h = \sum_{i=1}^{k} a_i \kappa_i + \kappa_{k+2} + \kappa_{k+3}.$$

Hence, (3.3) implies that

$$\langle H, cy(\kappa) \rangle = \langle H, cy(0, \ldots, 0, \lambda, 0, h) \rangle - \langle H, (\kappa_1, \ldots, \kappa_k, \kappa_{k+2}) \rangle = \sum_{i=1}^{k+1} \kappa_i \gamma_i.$$

This completes the proof. $\square$

We are now ready to prove our first main proposition.

**Proof of Proposition 3.1.1.** As explained in Remark 3.1.2 (i) above, every $H \in t$ can be written uniquely as $H = \sum_{i=1}^{k+3} \gamma_i \eta_i$, where $\gamma_{k+2} + \gamma_{k+3} = \sum_{i=1}^{k+1} \gamma_i = 0$. By Lemma 2.1.9 $F_{k+2}$ and $F_{k+3}$ are equivalent. Hence, $\gamma_{k+2} \eta_{k+2} + \gamma_{k+3} \eta_{k+3}$ is inessential, and so by Proposition 2.1.1

$$\langle \gamma_{k+2} \eta_{k+2} + \gamma_{k+3} \eta_{k+3}, cy(\kappa) \rangle = \gamma_{k+2} \kappa_{k+2} + \gamma_{k+3} \kappa_{k+3}.$$

On the other hand, by Lemma 3.1.3 $\bar{H} = \sum_{i=1}^{k+1} \gamma_i \eta_i$ is mass linear exactly if $\sum_{i=1}^{k} a_i \gamma_i = 0$, in which case

$$\langle \bar{H}, cy \rangle = \sum_{i=1}^{k+1} \gamma_i \kappa_i.$$

The first two claims follow immediately.

To establish the conditions under which $H$ is inessential, note first that since $\gamma_{k+2} \eta_{k+2} + \gamma_{k+3} \eta_{k+3}$ is inessential, $H$ is inessential exactly if $\bar{H}$ is inessential. Further, Lemma 2.1.9 implies that for each pair $\{i, j\} \subset \{1, \ldots, k+1\}$, we have $F_i \sim F_j$ exactly if $a_i = a_j$. Therefore the equivalence classes of the relation $\sim$ on $\{F_1, \ldots, F_{k+1}\}$ are precisely the sets $\{F_i : a_i = \alpha\}$. Since $\bar{H}$ can be written uniquely as $\bar{H} = \sum_{i=1}^{k+1} \gamma_i \eta_i$, where $\sum_{i=1}^{k+1} \gamma_i = 0$, it is inessential exactly if $\sum_{a_i = \alpha} \gamma_i = 0$ for each $\alpha$. $\square$

Proposition 3.1.1 immediately gives all essential mass linear functions for $\Delta_3$ bundles over $\Delta_1$, the polytopes $\Delta$ in case (a1) of Theorem 1.1.1.
Corollary 3.1.4. Let $Y$ be the $\Delta_3$ bundle over $\Delta_1$ associated to $a \in \mathbb{R}^3$ as in (3.1) above; set $a_4 = 0$. Then $H \in \mathfrak{t}$ is a mass linear function on $Y$ exactly if

$$H = \sum_{i=1}^{6} \gamma_i \bar{\eta}_i,$$

where

$$\gamma_5 + \gamma_6 = \sum_{i=1}^{4} \gamma_i = \sum_{i=1}^{3} a_i \gamma_i = 0.$$

In this case, $\langle H, c_Y \rangle = \sum_{i=1}^{6} \gamma_i \kappa_i$. Moreover, if $a_i \neq a_j$ for all $1 \leq i < j \leq 4$, then $H$ is inessential exactly if $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, and so there is a 1-dimensional subspace of inessential functions in the 3-dimensional space of mass linear functions. If $a_i = a_j$ for only one such pair \{i, j\}, there is a 2-dimensional family of inessential functions – for example, if $a_1 = a_2$ but $a_i \neq a_j$ for all $2 \leq i < j \leq 4$ then $H$ is inessential exactly if $\gamma_3 = \gamma_4 = 0$. If more than one such equality holds, then every $H$ is inessential.

Next we consider the second class of polytopes in Theorem 1.1.1.

Definition 3.1.5. A 121-bundle is a smooth polytope $Z$ that is a $\Delta_1$ bundle over a polytope $Y$ which is itself a $\Delta_2$ bundle over $\Delta_1$.

Given a 121-bundle $Z$, it is easy to see that we may identify $t$ with $\mathbb{R}^4$ so that its outward conormals are

$$\bar{\eta}_0 = (1, 0, 0, 0), \quad \bar{\eta}_1 = (-1, 0, 0, 0), \quad \eta'_2 = (0, -1, 0, 0), \quad \eta'_5 = (0, 0, -1, 0), \quad \eta'_6 = (a_1, a_2, a_3, 1)$$

for some $a \in \mathbb{Z}^3$ and some integer $d \geq 0$. Here the fiber $\Delta_1$ of $Z$ lives in the first coordinate direction, and $Y$ is the $\Delta_2$ bundle over $\Delta_1$ associated to $(a_2, a_3)$ and with conormals equal to the projections of the $\eta'_j$ for $2 \leq j \leq 6$ onto the last three coordinates.

Proposition 3.1.6. Let $Z$ be a 121-bundle as in (3.4) above. Then $H \in \mathfrak{t}$ is a mass linear function on $Z$ exactly if

$$H = \gamma_0 \bar{\eta}_0 + \gamma_1 \bar{\eta}_1 + \sum_{i=2}^{6} \gamma_i \eta'_i,$$

where

$$\gamma_0 + \gamma_1 = d \gamma_0 = a_1 \gamma_0 = 0 \quad \text{and} \quad \gamma_2 + \gamma_3 + \gamma_4 = a_2 \gamma_2 + a_3 \gamma_3 = \gamma_5 + \gamma_6 = 0.$$

In this case $\langle H, c_Z \rangle = \sum_{i=0}^{6} \gamma_i \kappa_i$. If $a_2 a_3 (a_2 - a_3) \neq 0$, then $H$ is inessential exactly if $\gamma_2 = \gamma_3 = \gamma_4 = 0$; otherwise, every $H$ is inessential.

Proof. Since $Z$ is a bundle with fiber $\Delta_1$, the fiber facets $\tilde{F}_0$ and $\tilde{F}_1$ are not pervasive. If $d \neq 0$ or if $a_1 \neq 0$, then these facets also are not flat. Therefore, if $H \in \mathfrak{t}$ is mass linear then Proposition 2.1.7 implies that $\tilde{F}_0$ and $\tilde{F}_1$ are symmetric. On the other hand, if $d = a_1 = 0$ then $Z = \Delta_1 \times Y$, and so we can also view $Z$ as a $Y$ bundle over $\Delta_1$ with base facets $\tilde{F}_0$ and $\tilde{F}_1$. Therefore, Proposition 2.2.4 implies that $H$ is mass linear exactly if $H = \gamma_0 \bar{\eta}_0 + \gamma_1 \bar{\eta}_1 + \tilde{H}$, where $\gamma_0 + \gamma_1 = 0$ and $\tilde{H}$ is a mass linear function on $Z$ so that $\tilde{F}_0$ and $\tilde{F}_1$ are symmetric. Therefore, the result is an immediate consequence of Proposition 2.2.3 and Proposition 3.1.1. \(\square\)
The following corollary is immediate.

**Corollary 3.1.7.** Let $Z$ be a 121-bundle that admits a mass linear function so that every facet is asymmetric. Then $Z = \Delta_1 \times Y$, where $Y$ is a $\Delta_2$ bundle over $\Delta_1$.

We now consider the third class of polytopes in Theorem 1.1.1. Let $\Delta \subset t^*$ be a smooth $\Delta_2$ bundle over a 2-dimensional polygon $\Delta \subset \mathbb{R}^2$. We aim to find all essential mass linear functions on $\Delta$.

First, we describe $\Delta$ in more detail. Let $\Delta_1 \subset \mathbb{R}^2$ be a smooth $\Delta_2$ bundle over a 2-dimensional polygon $\Delta \subset \mathbb{R}^2$. We aim to find all essential mass linear functions on $\Delta$.

Moreover, $\Delta$ is a simplex, then $\Delta$ is a 2-dimensional family of mass linear functions, and the fiber facets are symmetric for every mass linear function.

**Proof.** If $\Delta$ is a simplex, then $P(0, 0, r_3) \neq 0$ unless $r_3 = 0$. Therefore, both claims follow immediately from the proposition above. \hfill $\Box$

In contrast, many $\Delta_2$ bundles over quadrilaterals admit essential mass linear functions. For example, let $\Delta$ be a generic $\Delta_1 \times \Delta_2$ bundle over $\Delta_1$. As we mentioned in the introduction, this implies that $\Delta$ can either be viewed as a 121-bundle or as a $\Delta_2$ bundle over
a trapezoid. Hence, we can use either Proposition 3.1.6 or Proposition 3.1.8 to show that \( \Delta \) admits essential mass linear functions. In fact, if \( W \) is a generic \( \Delta_2 \) bundle over \( \Delta_1 \), then \( \Delta_1 \times W \) admits essential mass linear functions with 7 asymmetric facets. We spell out the details here because this example is rather special; as explained in the remarks (2) and (3) after Theorem 1.1.1 no other 4-dimensional polytopes admit an essential mass linear function with more than 6 asymmetric facets.

**Corollary 3.1.10.** Let \( H \in t \) be an essential mass linear function on a polytope \( \Delta \subset t^* \) which is a \( \Delta_2 \) bundle over a polygon \( \Delta_c \). If \( \Delta \) has more than six asymmetric facets, then \( \Delta = \Delta_1 \times Y \), where \( Y \) is a \( \Delta_2 \) bundle over \( \Delta_1 \).

**Proof.** By Proposition 3.1.8 this is impossible unless \( \Delta \) admits an inessential function with four asymmetric facets. Hence, \( \Delta = \Delta_1 \times \Delta_1 \). We may assume that \( \hat{\eta}_1 = (-1,0), \hat{\eta}_2 = (0,-1), \hat{\eta}_3 = (1,0) \) and \( \hat{\eta}_4 = (1,0) \). Then \( P(0,0, r_3, r_4) = r_3 r_4 \), so that \( P(0,0, r_3, r_4) = 0 \) exactly if \( r_3 = 0 \) or \( r_4 = 0 \), that is, exactly if \( (b_1^3, b_2^3) = (0,0) \) or \( (b_1^4, b_2^4) = (0,0) \).

The proof of Proposition 3.1.8 is based on the following lemma.

**Lemma 3.1.11.** Let \( \Delta \) be the \( \Delta_2 \) bundle over \( \Delta_c \) defined above. Let \( P(\hat{\gamma}_1, \ldots, \hat{\gamma}_k) \) be the polynomial which gives the area of \( \Delta_c \) for all \( \hat{\gamma} \in C\Delta_c \). Let \( H = \sum_{i=1}^3 \gamma_i \eta_i \), where \( \sum_{i=1}^3 \gamma_i = 0 \). If \( H \) is not zero, then \( H \) is mass linear on \( \Delta \) exactly if there are real numbers \( r_3, \ldots, r_k \) so that

(i) \( (b_1^i, b_2^i) = r_i(\gamma_2, -\gamma_1) \) for all \( i \in \{3, \ldots, k\} \), and

(ii) either \( P(0,0, r_3, \ldots, r_k) = 0 \) or \( \gamma_1 \gamma_2 \gamma_3 = 0 \).

In this case, \( \langle H, c\Delta_c \rangle = \sum_{i=1}^3 \gamma_i \kappa_i \), where \( \kappa_i \) is the support number of the fiber facet \( F_i \).

**Proof.** Let \( G_i \) be a base facet. Since \( \Delta_c \) is smooth, there is an integer \( m_i \) so that

\[
(3.7) \quad m_i \hat{\eta}_i = \hat{\eta}_{i-1} + \hat{\eta}_{i+1},
\]

where as usual we interpret the \( i \) in cyclic order. The facet \( G_i \) is a \( \Delta_2 \) bundle over \( \Delta_1 \) with fiber facets \( F_1 \cap G_i, F_2 \cap G_i, \) and \( F_3 \cap G_i \), and base facets \( G_{i-1} \cap G_i \) and \( G_{i+1} \cap G_i \). As such, it is determined by a pair of integers \((a_1^i, a_2^i)\). One can check that

\[
(3.8) \quad (a_1^i, a_2^i) = (b_1^{i-1}, b_2^{i-1}) - m_i (b_1^i, b_2^i) + (b_1^{i+1}, b_2^{i+1}).
\]

If \( H \in t \) is mass linear on \( \Delta \), then by Proposition 2.1.5 it must also be mass linear on \( G_i \). Hence, Lemma 3.1.3 implies that \( \gamma_1 a_1^i + \gamma_2 a_2^i = 0 \) for all \( i \). Since \( (b_1^1, b_2^1) = (b_1^2, b_2^2) = (0,0) \) by assumption, this and Equation (3.8) together imply that \( \gamma_1 b_1^i + \gamma_2 b_2^i = 0 \) for all \( i \). Therefore, since \( H \neq 0 \), there is a constant \( r_i \) for each \( 1 \leq i \leq k \) such that

\[
(3.9) \quad (b_1^i, b_2^i) = r_i(\gamma_2, -\gamma_1);
\]

note that \( r_1 = r_2 = 0 \).

So now assume that (3.9) holds for all \( i \). As in Lemma 3.1.3 it is convenient first to consider the case that the support numbers of the fiber facets \( F_1 \) and \( F_2 \) are 0. Let \( \lambda \) denote the support number of \( F_3 \) and \( \hat{\kappa}_i \) denote the support number of the base facet \( G_i \). To get the volume \( V \) of \( \Delta \) we integrate over the simplex \( \Delta_2^3 \) the function which gives the area
of the intersection of $\Delta$ with the 2-plane where $x_1$ and $x_2$ are constant. Each such section is affine equivalent to the base polygon $\hat{\Delta}$ with structural constants $\hat{\kappa}_i = \hat{\kappa}_i - r_i(\gamma_2 x_1 - \gamma_1 x_2)$. Since $P$ is a homogeneous quadratic function of the support numbers, the area of this section is

$$P(\hat{\kappa}_i - r_i(\gamma_2 x_1 - \gamma_1 x_2)) = Q_0 + Q_1(\gamma_2 x_1 - \gamma_1 x_2) + Q_2(\gamma_2 x_1 - \gamma_1 x_2)^2,$$

where each $Q_d$ is a polynomial of degree $d$ in the $r_i$ and of degree $2-d$ in the $\hat{\kappa}_i$; moreover $Q_2$ is $P(r_1, \ldots, r_k)$. Therefore, by [3.2]

$$V = \int_{\hat{\Delta}_2^3} Q_0 + Q_1(\gamma_2 x_1 - \gamma_1 x_2) + Q_2(\gamma_2 x_1 - \gamma_1 x_2)^2$$

$$= \frac{\lambda^2}{12} \left( 6Q_0 + 2(\gamma_2 - \gamma_1)Q_1\lambda + (\gamma_2^2 - \gamma_2\gamma_1 + \gamma_1^2)Q_2\lambda^2 \right).$$

Similarly, the moment $\mu_1$ along the $x_1$ axis is

$$\mu_1 = \int_{\hat{\Delta}_2^3} Q_0 x_1 + Q_1(\gamma_2 x_1 - \gamma_1 x_2)x_1 + Q_2(\gamma_2 x_1 - \gamma_1 x_2)^2 x_1$$

$$= \frac{\lambda^3}{120} \left( 20Q_0 + (10\gamma_2 - 5\gamma_1)Q_1\lambda + (6\gamma_2^2 - 4\gamma_2\gamma_1 + 2\gamma_1^2)Q_2\lambda^2 \right).$$

By symmetry the moment $\mu_2$ along the $x_2$ axis is given by interchanging $\gamma_2$ and $-\gamma_1$. As before, a straightforward (though tedious) calculation shows that

$$\langle H, c_\Delta \rangle = (\gamma_3 - \gamma_1)\frac{\mu_1}{V} + (\gamma_3 - \gamma_2)\frac{\mu_2}{V}$$

$$= \lambda \left( \gamma_3 + \frac{\gamma_1\gamma_2\gamma_3 Q_2\lambda^2}{5(6Q_0 + 2(\gamma_2 - \gamma_1)Q_1\lambda + (\gamma_2^2 - \gamma_2\gamma_1 + \gamma_1^2)Q_2\lambda^2)} \right).$$

This is a linear function exactly if $\gamma_1\gamma_2\gamma_3 = 0$ or $Q_2 = 0$.

Together, these two paragraphs imply that if $H$ is mass linear on $\Delta$, then $\langle b_i^1, b_i^2 \rangle = r_i(\gamma_2, -\gamma_1)$ for all $i$ and either $P(0, 0, r_3, \ldots, r_k) = 0$ or $\gamma_1\gamma_2\gamma_3 = 0$. It remains to show that if $H$ is mass linear, then we must have $\langle H, c_\Delta \rangle = \sum_{i=1}^3 \gamma_i \kappa_i$. We calculated $\langle H, c_\Delta \rangle = \lambda \gamma_3$ above in the special case when $\kappa_1 = \kappa_2 = 0$ and $\kappa_3 = \lambda$. Just as at the end of the proof of Lemma [3.1.3] the general case follows by using Remark [2.1.4].

We are now ready to complete the proof.

**Proof of Proposition 3.1.8.** If $\hat{\Delta} = \Delta_2$, then by Proposition [2.2.4] we can write $H' = \hat{H} + \bar{H}$, where

- $\hat{H}$ is inessential and the fiber facets are $\hat{H}$-symmetric, and
- $\bar{H}$ is mass linear and the base facets are $\bar{H}$- symmetric.

On the other hand, if $\hat{\Delta}$ contains more than three edges, then the base facets are the nonpervasive facets. Hence, in this case the same claim follows from Proposition [2.2.5]. It then follows from Proposition [2.2.3] that $\hat{H}$ is the lift of an inessential function on $\Delta$. □
Now consider $\widetilde{H}$. Since all its asymmetric facets are fiber facets, Lemma \[3.1.11\] implies that $\widetilde{H}$ lies in the span of the conormals to the fiber facets. Since $\sum_{i=1}^{3} \gamma_i = 0$, this means that there are constants $\gamma_i$ so that $\widetilde{H} = \sum_{i=1}^{3} \gamma_i \eta_i$, where $\sum_{i=1}^{3} \gamma_i = 0$. Therefore, by Lemma \[3.1.11\] (i) and (ii) hold. Moreover $H$ is inessential exactly if $\widetilde{H}$ is. If $\gamma_1 \gamma_2 \gamma_3 \neq 0$ then $H$ is inessential exactly if the three fiber facets $F_1, F_2, F_3$ are equivalent. But by \[3.6\] this happens only if all $b_j = 0$, that is, if the bundle $\Delta \to \hat{\Delta}$ is trivial. \hfill $\Box$

We now consider the topological implication of Proposition \[3.1.8\].

**Proposition 3.1.12.** Let $\Delta$ be a $\Delta_2$-bundle over a polygon $\hat{\Delta} \subset \mathbb{R}^2$, and let $M_\Delta$ and $M_{\hat{\Delta}}$ denote the associated toric manifolds. Then $\Delta$ admits an essential mass linear function exactly if there exist integers $\gamma_1$ and $\gamma_2$, and a $T_{\hat{\Delta}}$-equivariant principal $S^1$-bundle $L$ over $M_{\hat{\Delta}}$ such that

(i) the (ordinary) Euler class $\chi \in H^2(M_{\hat{\Delta}}; \mathbb{Z})$ of $L$ is not trivial but has vanishing square,

(ii) $\gamma_1 \gamma_2 (\gamma_2 - \gamma_1) \neq 0$, and

(iii) $M_\Delta$ is $T_{\hat{\Delta}}$-equivariantly diffeomorphic to $L \times_{S^1} \mathbb{C}P^2$, where $S^1$ acts on $\mathbb{C}P^2$ by $\lambda \cdot [z_1 : z_2 : z_3] = [\lambda^{\gamma_2} z_1 : \lambda^{-\gamma_1} z_2 : z_3]$.

**Proof.** We may assume that $\Delta$ is described by \[3.5\] and \[3.6\], where $(b_1, b_2) = (b_1', b_2') = 0$ and where $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the outward conormals to $\Delta$. Let $P$ be the polynomial which gives the area of $\Delta$. By Proposition \[3.1.8\] $\Delta$ admits an essential mass linear function exactly if there exist real numbers $\gamma_1, \gamma_2$ and $r_1, \ldots, r_k$ such that

(a) the $r_i$'s are not all zero but $P(0, 0, r_3, \ldots, r_k) = 0$,

(b) $\gamma_1 \gamma_2 (\gamma_2 - \gamma_1) \neq 0$, and

(c) $(b_1, b_2) = r_i (\gamma_2, -\gamma_1)$ for all $i$.

Further, by multiplying $\gamma_1$ and $\gamma_2$ by a suitable constant, we may assume that $\gamma_1$ and $\gamma_2$ are mutually prime integers, so that each $r_1$ is also in $\mathbb{Z}$.

By \[I, \text{Remark 5.2}\], $M_\Delta$ is a $\mathbb{C}P^2$ bundle over $M_{\hat{\Delta}}$. More specifically, identify $M_{\hat{\Delta}}$ with the symplectic quotient $\mathbb{C}^k/\tilde{K}$ for a suitable subtorus $\tilde{K} \subset (S^1)^k$, and let $(S^1)^2$ act on $\mathbb{C}P^2$ by $\lambda \cdot [z_1 : z_2 : z_3] = [\lambda_1 z_1 : \lambda_2 z_2 : z_3]$. Then $M_\Delta$ is the $\mathbb{C}P^2$ bundle associated to the homomorphism $\rho: \tilde{K} \to (S^1)^2$ given by

$$\rho(\exp(x)) = \exp \left( \sum x_i b_i' + \sum x_i b_2' \right) \quad \text{for all} \quad x = (x_1, \ldots, x_k) \in \mathfrak{e} \subset \mathbb{R}^k.$$

Next observe that the torus $(S^1)^k$ acts on $M_{\hat{\Delta}}$ via its quotient $T_{\hat{\Delta}} = (S^1)^k/\tilde{K}$. Moreover, there is a one-to-one correspondence between $(S^1)^k$ equivariant principal $S^1$-bundles over $M_{\hat{\Delta}}$, representations of $(S^1)^k$, and $k$-tuples $r \in \mathbb{Z}^k$. Hence, $M_\Delta$ is the $\mathbb{C}P^2$ bundle associated to an equivariant principal $S^1$ bundle over $M_{\hat{\Delta}}$ exactly if there exist integers $\gamma_1, \gamma_2$, and $r_1, \ldots, r_k$ such that $(b_1, b_2) = r_i (\gamma_2, -\gamma_1)$ for all $i$. In this case, $S^1$ acts on $\mathbb{C}P^2$ by $\lambda \cdot [z_1 : z_2 : z_3] = [\lambda^{\gamma_2} z_1 : \lambda^{-\gamma_1} z_2 : z_3]$. 


Thus conditions (ii) and (iii) in the proposition are equivalent to conditions (b) and (c). To complete the proof we must show that condition (i) is equivalent to (a). This is accomplished in Lemma 3.1.13 below.

**Lemma 3.1.13.** Let \( \widehat{\Delta} \subset \widehat{t}^n \) be an \( n \)-dimensional polytope with facets \( F_1, \ldots, F_k \); let \( M_{\widehat{\Delta}} = \mathbb{C}^k \slash \widehat{K} \) be the associated toric manifold. Given \( r \in \mathbb{Z}^k \), let \( \chi \) be the Euler class of the principal \( S^1 \)-bundle associated to the induced homomorphism from \( \widehat{K} \subset (S^1)^k \) to \( S^1 \). Let \( P(\widehat{\kappa}) \) be the polynomial which gives the volume of \( \widehat{\Delta}(\widehat{\kappa}) \) for all \( \widehat{\kappa} \in C_{\widehat{\Delta}} \). Then

\[
(3.10) \quad P(r_1, \ldots, r_k) = 0 \iff \int_{M_{\widehat{\Delta}}} \chi^n = 0.
\]

Moreover, if \( \bigcap_{i=1}^n F_i \neq \emptyset \) and \( r_1 = \cdots = r_n = 0 \), then \( \chi = 0 \) exactly if \( r_i = 0 \) for all \( n < i \leq k \).

**Proof.** Fix \( \widehat{\kappa} \in C_{\widehat{\Delta}} \). There exists a symplectic form \( \omega \) on \( M_{\widehat{\Delta}} \) with moment map \( \widehat{\Phi}: M_{\widehat{\Delta}} \to \widehat{t}^n \) such that \( \widehat{\Phi}(M_{\widehat{\Delta}}) = \widehat{\Delta}(\widehat{\kappa}) \). On the one hand, the symplectic form \( \omega \) represents the cohomology class \( \sum_i \kappa_i X_i \), where \( X_i \in H^2(M_{\widehat{\Delta}}) \) represents the Poincaré dual to the compact submanifold \( \widehat{\Phi}^{-1}(F_i) \) for all \( 1 \leq i \leq k \). On the other hand, since \( M_{\widehat{\Delta}} \) is a toric manifold the Duistermaat-Heckman measure on \( \widehat{t}^n \) is given by Lebesgue measure on \( \widehat{\Delta}(\widehat{\kappa}) \) and vanishes outside \( \widehat{\Delta}(\widehat{\kappa}) \). (Recall that the Duistermaat-Heckman measure is the pushforward of the Liouville measure \( \frac{1}{n!} \omega^n \) on \( M \) under the moment map.) Therefore,

\[
P(\widehat{\kappa}) = \frac{1}{n!} \int_{M_{\widehat{\Delta}}} \omega^n = \frac{1}{n!} \int_{M_{\widehat{\Delta}}} \left( \sum_i \kappa_i X_i \right)^n \forall \widehat{\kappa} \in C_{\widehat{\Delta}}.
\]

Since both sides are polynomials, and since \( \chi = \sum r_i X_i \), the first claim follows.

The second claim holds because the outward conormals to \( F_1, \ldots, F_n \) form a basis for the lattice in \( \widehat{t} \). Hence, by the standard Stanley-Reisner presentation for the cohomology ring of a toric manifold, \( H^2(M_{\widehat{\Delta}}; \mathbb{Z}) \) is freely generated by \( X_{n+1}, \ldots, X_k \).

This completes our discussion of bundles that support essential mass linear functions. We end this section with some supplementary results. First, we determine the number of asymmetric facets for each of the polytopes \( \Delta \) described in case (a) of Theorem 1.1.1.

**Remark 3.1.14.** Let \( H \in t \) be an essential mass linear function on a polytope \( \Delta \).

- If \( \Delta \) is a \( \Delta_3 \) bundle over \( \Delta_1 \), as in case (a1), then at least 3 of the 4 fiber facets are asymmetric; if one base facet is asymmetric then both are. Thus the number of asymmetric facets can be anywhere between 3 and 6.

- If \( \Delta \) is a 121-bundle and the conormals to the three pervasive facets are linearly independent then the three pervasive facets are asymmetric; the two fiber facets are symmetric unless \( \Delta \) is the product \( \Delta_1 \times Y \), where \( Y \) is a \( \Delta_2 \) bundle over \( \Delta_1 \), in which case they may both be asymmetric; finally, if one of the remaining two facets is asymmetric then both are. Thus there are 3, 5, or 7 asymmetric facets, with 7 impossible unless \( \Delta = \Delta_1 \times Y \).
• If $\Delta$ is a $\Delta_2$ bundle over a polygon $\hat{\Delta}$ as in case (a3), the three fiber facts are asymmetric. The base facets are symmetric unless $\hat{\Delta}$ is a $\Delta_1$ bundle over $\Delta_1$. In that case, two base facets that correspond to equivalent facets of $\hat{\Delta}$ may both be asymmetric. As in the previous case, there can be 3, 5, or 7 asymmetric facets, with 7 impossible unless $\Delta = \Delta_1 \times Y$, where $Y$ is a $\Delta_2$ bundle over $\Delta_1$.

The first two claims follow trivially from Corollary 3.1.4 and Proposition 3.1.6. In order to see that the last claim follows from Propositions 2.1.1, 2.1.14 and 3.1.8, and Corollary 3.1.10, remember that there is no essential mass linear function on any $\Delta_2$ bundle over $\Delta_2$ by Corollary 3.1.9.

By Lemma 2.4.8 and Proposition 2.4.10, an essential mass linear function on a polytope will still be essential and mass linear if the polytope is blown up by either of the two types of blowups described in Theorem 1.1.1 – blowups along symmetric 2-faces and blowups of type $(F_{ij}, G)$. Moreover, the conclusions above allow us to analyze the ways that the bundles $\Delta$ listed in case (a) of Theorem 1.1.1 can be blown up in these ways. Therefore, we can now find all essential mass linear functions of the type described in case (a) of Theorem 1.1.1.

Remark 3.1.15. Let $H \in \mathfrak{t}$ be an essential mass linear function on a polytope $\Delta$, where $\Delta$ is one of the polytopes described in case (a) of Theorem 1.1.1.

(i) If $\Delta$ has exactly three asymmetric facets, then it must have symmetric 2-faces (which can be blown up). Otherwise, $\Delta$ does not have any symmetric 2-faces.

(ii) By Proposition 2.4.10 blowups of type $(F_{ij}, G)$ are not possible unless $\Delta$ has four asymmetric facets. However, the bundles in case (a) do not have four asymmetric facets unless $\Delta$ is a $\Delta_3$ bundle over $\Delta_1$, all four fiber facets are asymmetric, and the base facets are symmetric. In this case, blowups of this type are possible exactly if there exist fiber facets $F_i$ and $F_j$ so that $\gamma_i + \gamma_j = 0$, where $\gamma_k$ is the support number of $F_k$ in $\langle c_\Delta, H \rangle$.

(iii) By Lemma 2.4.8 and Proposition 2.4.10 any polytope $\Delta'$ obtained from $\Delta$ by a sequence of blowups of these types will itself have symmetric 2-faces. Type $(F_{ij}, G)$ blowups of $\Delta'$ are possible exactly if we are in the situation described in part (ii) above.

Finally, we can now construct an example in which mass linearity is destroyed by an expansion, as promised in Remark 2.3.4.

Example 3.1.16. Let $Y \subset (\mathbb{R}^3)^*$ be the $\Delta_2$ bundle over $\Delta_1$ associated to $a \in \mathbb{R}^2$ as in (3.1). Assume that $a_1 a_2 (a_2 - a_1) \neq 0$, that is, that none of the fiber facets are equivalent. By Proposition 3.1.1

$$\vec{H} = a_2 \eta_1 - a_1 \eta_2 + (a_2 - a_1) \eta_3 = (a_1 - 2a_2, 2a_1 - a_2, 0) \in \mathbb{R}^3.$$ 

is an essential mass linear function on $Y$.

Let $\Delta \subset \mathfrak{t}^*$ be the 1-fold expansion of $Y$ along the base facet with conormal $(0, 0, -1)$, and let $H = (a_1 - 2a_2, 2a_1 - a_2, 0, 0) \in \mathfrak{t}$ be the image of $\vec{H}$ under the natural inclusion.
Then $\Delta$ is a $\Delta_2$ bundle over $\Delta_2$; the outward conormals to the base facets are

$$\hat{\gamma}_1 = (a_1, a_2, 1, 0), \quad \hat{\gamma}_2 = (0, 0, 0, -1), \quad \text{and} \quad \hat{\gamma}_3 = (0, 0, -1, 1).$$

By Corollary 3.1.9 every mass linear function on $\Delta$ is inessential. Hence, since no fiber facet of $\Delta$ is equivalent to any other facet, every mass linear function on $\Delta$ has the form $H' = \sum_{i=1}^{3} \gamma_i \hat{\gamma}_i$, where $\sum \gamma_i = 0$. Therefore, $H$ is not mass linear.

### 3.2. Blowups of double expansions

In this subsection, we find all essential mass linear functions of the type described in case (b) of Theorem 1.1.1 To begin, we classify mass linear functions on double expansions with symmetric fiber-type facets, showing that they are all inessential.

**Lemma 3.2.1.** Let $\Delta \subset \mathfrak{t}^*$ be the double expansion of a polytope $\tilde{\Delta}$ along facets $\tilde{F}_1$ and $\tilde{F}_2$. Let $F_1$ and $F_2$ ($F_3$ and $F_4$) be the base-type facets associated to $F_1$ (respectively, $F_2$). Let $H \in \mathfrak{t}$ be a mass linear function on $\Delta$ with symmetric fiber-type facets. Then $H$ is inessential, and

$$H = \sum_{i=1}^{4} \gamma_i \hat{\gamma}_i,$$

where

1. $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 0$ if $\tilde{F}_1 \sim \tilde{F}_2$, and
2. $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 = 0$ if $\tilde{F}_1 \not\sim \tilde{F}_2$.

Conversely, any function of this form is inessential.

**Proof.** By Remark 2.3.6 (ii), $\tilde{F}_1 \sim \tilde{F}_2 \sim \tilde{F}_3 \sim \tilde{F}_4$ if $\tilde{F}_1 \sim \tilde{F}_2$, while $\tilde{F}_1 \sim \tilde{F}_2 \not\sim \tilde{F}_3 \sim \tilde{F}_4$ if $\tilde{F}_1 \not\sim \tilde{F}_2$. Thus, the last statement is clear. Moreover, by Lemma 2.1.2 applied twice, there exists an inessential function $H'$ so that at most two facets are $(H-H')$-asymmetric. Therefore $H$ is inessential by Proposition 2.1.8. Since, by hypothesis, the only asymmetric facets are $F_1, \ldots, F_4$, this implies that it has the given form.

The following proposition clarifies exactly which of the blowup operations allowed in Theorem 1.1.1 are needed in order for $H$ to become essential. We restrict to the 4-dimensional case, though the result can be extended to higher dimensions without too much difficulty. Blowups of type $(F_{ij}, g)$ are defined in Definition 2.4.9. Note that in the 4-dimensional case a symmetric 3-face $g$ is just a symmetric facet $G$.

**Proposition 3.2.2.** Let $\Delta \subset \mathfrak{t}^*$ be the double expansion of a smooth polygon $\tilde{\Delta}$ along edges $\tilde{F}$ and $\tilde{F}'$, and let $H \in \mathfrak{t}$ be a mass linear function on $\Delta$ such that the fiber-type facets are the symmetric facets. Let $F_1$ and $F_2$ ($F_3$ and $F_4$) be the base-type facets associated to $F$ (respectively, $F'$). Consider a polytope $\Delta'$ that is obtained from $\Delta$ by a sequence of blowups, where each blowup is either along a symmetric face or of type $(F_{ij}, G)$. Then $H$ is essential on $\Delta'$ exactly if one of the following occurs.

1. $\tilde{F} \not\sim \tilde{F}'$ and one of the blowups is of type $(F_{ij}, G)$, where $i \in \{1, 2\}$ and $j \in \{3, 4\}$.
2. $\tilde{F} \sim \tilde{F}'$ and there exists $\{i, j, k\} \subset \{1, 2, 3, 4\}$ such that one of the blowups is of type $(F_{ij}, G)$ and another is of type $(F_{ik}, G')$.

Moreover, in either case there exists a blowup $\Delta'$ of this type so that $H$ is essential exactly if $|\gamma_1| = |\gamma_2| = |\gamma_3| = |\gamma_4|$, the polygon $\tilde{\Delta}$ is not a triangle, and it contains an edge $\tilde{e}$ with endpoints $\tilde{e} \cap \tilde{F}$ and $\tilde{e} \cap \tilde{F}'$. 


Proof. As before, by Lemma 2.4.8 and Proposition 2.4.10, $H$ is mass linear on each intermediate blowup, the exceptional divisors are all symmetric, and the coefficients $\gamma_k$ remain constant under blowup. Moreover, $H$ is inessential on $\Delta$ by Lemma 3.2.1.

Assume first that $F \not\sim \tilde{F}'$. (In particular, $\Delta$ cannot be a triangle.) By Remark 2.3.6 (ii), this implies that $F_1 \sim F_2 \not\sim F_3 \sim F_4$. If all blowups are along symmetric faces, then $H$ is inessential by Lemma 2.4.8. Similarly, Proposition 2.4.10 and Lemma 2.4.5 imply that that $H$ remains inessential under any blowup of type $(F_{ij}, G)$ if $\{i, j\} \in \{1, 2\}$ or $\{3, 4\}$. but is essential on $\Delta'$ after a blowup of type $(F_{ij}, G)$ with $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

Since $H$ is inessential $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 = 0$, and so blowups of the latter type are not allowed unless $|\gamma_1| = |\gamma_2| = |\gamma_3| = |\gamma_4|$. So assume that this equation holds. Given an edge $\tilde{e}$ of $\Delta$, consider the corresponding symmetric fiber-type facet $G$ of $\Delta$. By Remark 2.3.6 (i), if $\tilde{e}$ has endpoints $\tilde{e} \cap \tilde{F}$ and $\tilde{e} \cap \tilde{F}'$, then $F_{ij} \cap G$ intersects every base-type facet for every $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Thus, we are allowed to blow up along the edge $F_{ij} \cap G$ for some such $\{i, j\}$, and this blowup makes $H$ essential. Conversely, assume that $\Delta$ does not contain any edge $\tilde{e}$ with endpoints $\tilde{e} \cap \tilde{F}$ and $\tilde{e} \cap \tilde{F}'$. Then Remark 2.3.6 (i) implies that $F_{ij} \cap G$ does not meet every asymmetric facet for any $i \in \{1, 2\}$, $j \in \{3, 4\}$, and symmetric facet $G$. Moreover, the allowed blowups cannot create a new symmetric facet $G_0$ so that $F_{ij} \cap G_0$ intersects every asymmetric facet. To see this, let $\Delta'$ be the blowup of $\Delta$ along a face $f$ with exceptional divisor $G_0$. If $F_{ij} \cap G_0$ intersects every asymmetric facet, then Remark 2.4.2 implies that $F_{ij} \cap f$ intersects every asymmetric facet. Since $f$ must lie on at least one symmetric facet, this is impossible. Therefore, the function $H$ remains inessential on all allowed blowups of $\Delta$.

Now assume that $\tilde{F} \sim \tilde{F}'$. By Remark 2.3.6 (ii), $F_1 \sim F_2 \sim F_3 \sim F_4$. Lemma 2.4.8 and Proposition 2.4.10 together imply that $H$ is essential on $\Delta'$ exactly if there exists $\{i, j, k\} \subset \{1, 2, 3, 4\}$ such that one of the blowups is of type $(F_{ij}, G)$ and another is of type $(F_{ik}, G')$.

Since $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 0$, blowups of these types are not allowed unless $|\gamma_1| = |\gamma_2| = |\gamma_3| = |\gamma_4|$; so assume that this equation holds. If we do perform these two blowups, then one of them is along an edge $F_{ij} \cap G$ with $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Just as in the previous case, this implies that $\Delta$ contains an edge $\tilde{e}$ with endpoints $\tilde{e} \cap \tilde{F}$ and $\tilde{e} \cap \tilde{F}'$. If $\Delta \not\sim \Delta_2$, then it is clear that there exists another edge $\tilde{e}' \not\sim \tilde{e}$ with endpoint $\tilde{e}' \cap \tilde{F}$, so that these two blowups are possible. Therefore, to finish the proof it remains to check that if $\Delta = \Delta_2$ suitable blowups are not possible. Since $\Delta$ is then a 4-simplex it has one symmetric facet, which we call $G$. By renumbering the $F_i$ if necessary, we may assume that the first blowup is along $F_{12} \cap G$; let us call this blowup $\Delta''$ and the exceptional divisor $G'$. By Remark 2.4.2, $F_{12} \cap G$ and $F_{34} \cap G'$ are both empty in $\Delta''$. Therefore, neither $F_{ij} \cap G$ nor $F_{ij} \cap G'$ intersect every asymmetric facet for any $i \in \{1, 2\}$ and $j \in \{3, 4\}$. As we have already seen, this remains true under all further allowed blowups. Hence we cannot blow up in such a way to make $H$ essential. \hfill $\square$

Remark 3.2.3. Let $\Delta$ and $H$ be as in Proposition 3.2.2 above, and assume that the conditions described in its last sentence hold.
(i) If $\tilde{\Delta}$ has four edges (the fewest possible), then $\tilde{F}$ and $\tilde{F}'$ are either parallel or equivalent (or both). If they are parallel, then $\Delta$ is a $\Delta_3$-bundle over $\Delta_1$ with fiber facets $F_1, \ldots, F_4$; if we write $\Delta$ as in (3.1), then $a_1 = a_2$ and $a_3 = 0$. See, for instance, Example 1.1.5. In this case, the mass linear functions that arise when we think of $\Delta$ as a double expansion are special cases of those considered in Corollary 3.1.4. In contrast, if $\tilde{F}$ and $\tilde{F}'$ are equivalent, the polytope $\Delta$ is a $\Delta_1$-bundle over $\Delta_3$ with base facets $F_1, \ldots, F_4$.

(ii) $\Delta$ has a symmetric 2-face (which can be blown up) exactly if $\tilde{\Delta}$ has more than four edges. Blowups of type $(F_{ij}, G)$ are always possible; indeed, they are required.

The final remark in this subsection will be relevant to our discussion in §5.1 of the minimality of the polytope $\Delta$; cf. Remark 1.1.3.

Remark 3.2.4. Again let $\Delta$ and $H$ be as in Proposition 3.2.2 but now assume that $\tilde{F}$ and $\tilde{F}'$ are adjacent edges. Then the base-type facets intersect in a vertex $F_{1234}$ and the blowup $\Delta''$ of $\Delta$ at $F_{1234}$ is the double expansion of $\tilde{\Delta''}$ along $\tilde{F} \cap \tilde{\Delta''}$ and $\tilde{F}' \cap \tilde{\Delta''}$, where $\tilde{\Delta''}$ itself is the blowup of $\Delta$ at the vertex $\tilde{F} \cap \tilde{F}'$. In particular, $\tilde{\Delta''}$ is not a triangle and the exceptional divisor meets the edges $\tilde{F} \cap \tilde{\Delta''}$ and $\tilde{F}' \cap \tilde{\Delta''}$. Moreover, by Lemma 2.4.12 $H$ is an inessential function on $\Delta''$. Hence, there exist a blowup $\Delta'$ of $\Delta''$ of the type described in Proposition 3.2.2 such that $H$ is essential on $\Delta'$ exactly if $|\gamma_1| = |\gamma_2| = |\gamma_3| = |\gamma_4|$: cf. Proposition 5.1.14.

4. 4-DIMENSIONAL POLYTOPES

In this section, we establish the propositions used in §1.2 to prove Theorem 1.1.1. It follows that the examples constructed in the previous section, together with their blowups, are the only essential mass linear functions on smooth 4-dimensional polytopes. The first two subsections analyze polytopes with three or four pervasive asymmetric facets, while the third considers the remaining cases.

4.1. Three asymmetric facets. This subsection analyzes mass linear functions on 4-dimensional polytopes with exactly three asymmetric facets. Our first main result, Proposition 4.1.2, addresses the case that the conormals to these asymmetric facets are linearly dependent; the case that the conormals are linearly independent is considered in Proposition 4.1.4. Many of the results in this subsection are valid in all dimensions. In particular, our first lemma implies that whenever their are exactly three asymmetric facets, each one is pervasive.

Lemma 4.1.1. Fix $H \in t$. Let $\Delta \subset t^\ast$ have exactly three asymmetric facets $F_1, F_2$, and $F_3$. Then every symmetric face intersects $F_{12}$, $F_{13}$, and $F_{23}$ (and hence $F_1$, $F_2$, and $F_3$). Moreover, every symmetric face contains a 2-dimensional triangular symmetric subface.

Proof. Every symmetric face contains a symmetric face $g$ which is minimal in the sense that it does not properly contain another symmetric face. By Proposition 2.1.5, $g$ is a polytope with exactly three facets, $F_1 \cap g, F_2 \cap g$, and $F_3 \cap g$. This is only possible if $g$ is a 2-dimensional triangle, and so it intersects $F_{12}$, $F_{13}$, and $F_{23}$. \qed
We first assume that the conormals to the three asymmetric faces are linearly dependent.

**Proposition 4.1.2.** Let \( H \in \mathfrak{t} \) be a mass linear function on \( \Delta \subset \mathfrak{t}^* \) with exactly three asymmetric facets \( F_1, F_2, \) and \( F_3 \) with linearly dependent conormals. Then \( \Delta \) is a \( \Delta_2 \) bundle over the face \( F_{12} \), and the base facets are the symmetric facets.

**Proof.** Since the outward conormals to the \( F_i \) are linearly dependent the triple intersection \( F_{123} \) must be empty. Thus, Lemma 4.1.1 implies that \( \Delta \) is combinatorially equivalent to the product \( \Delta_2 \times F_{12} \). Hence, by Lemma 2.1.12, \( \Delta \) is a \( \Delta_2 \) bundle over the face \( F_{12} \) with fiber facets \( F_1, F_2, \) and \( F_3 \). \( \square \)

We next assume that the conormals to the three asymmetric facets \( F_1, F_2, \) and \( F_3 \) are linearly independent. Then the three affine planes \( P(F_i) \) which contain the asymmetric facets intersect in an affine subspace \( \ell_{123} \) that contains the (possibly empty) face \( F_{123} \). Define a graph \( \Gamma \) as follows: its vertices \( V \) are the vertices in \( F_{12} \setminus F_{123} \) and its edges \( E \) are the edges of \( F_{12} \) that have both endpoints in \( V \) and are not parallel to \( \ell_{123} \).

**Lemma 4.1.3.** Fix \( H \in \mathfrak{t} \). Let \( \Delta \subset \mathfrak{t}^* \) be a smooth polytope with exactly three asymmetric facets \( F_1, F_2, \) and \( F_3 \) with linearly independent conormals.

(i) Let \( Y \) be a symmetric 3-face of \( \Delta \) that contains two symmetric 2-faces, and assume that \( Y \cap F_{12} \) is not parallel to \( \ell_{123} \). Then \( Y \) is a \( \Delta_1 \) bundle over \( \Delta_2 \), and the symmetric facets are the fiber facets.

(ii) If the associated graph \( \Gamma \) is connected then \( F_1 \sim F_2 \sim F_3 \).

**Proof.** Let \( Y \) be a symmetric 3-face that contains two symmetric 2-faces, and assume that \( Y \cap F_{12} \) is not parallel to \( \ell_{123} \). We now apply Lemma 4.1.1. Since the edge \( F_{12} \cap Y \) meets both symmetric faces of \( Y \), \( F_{123} \cap Y = \emptyset \). Hence, \( Y \) is combinatorially equivalent to \( \Delta_1 \times \Delta_2 \), where the symmetric faces are triangular. Since \( Y \cap F_{12} \) is not parallel to \( \ell_{123} \), the conormals to the \( F_i \) remain linearly independent when restricted to \( P(Y) \). Hence, Lemma 2.1.13 implies that \( Y \) is a \( \Delta_1 \) bundle over \( \Delta_2 \); the symmetric facets are the fiber facets. This proves (i).

Since \( \Delta \) is simple, Lemma 4.1.1 implies that intersection with \( F_{12} \) induces a one-to-one correspondence between the set of symmetric 2-faces and the vertex set \( V \) of \( \Gamma \). It also induces a one-to-one correspondence between the set of symmetric 3-faces \( Y \) that contain two symmetric 2-faces so that \( Y \cap F_{12} \) is not parallel to \( \ell_{123} \), and the edge set \( E \) of \( \Gamma \). Moreover, in this case claim (i) implies that the two symmetric 2-faces are parallel. Hence, two symmetric 2-faces \( X \) and \( X' \) are parallel if the vertices \( X \cap F_{12} \) and \( X' \cap F_{12} \) lie in the same component of \( \Gamma \).

If \( \Gamma \) is connected, this implies that all symmetric 2-faces are parallel. By Lemma 4.1.1 every symmetric facet must contain a symmetric 2-face, so this implies that the conormals to all the symmetric facets lie in a codimension 2 subspace. Hence, by Lemma 2.1.9 the three asymmetric facets are equivalent. \( \square \)

We now specialize to the 4-dimensional case. The definition of a 121-bundle may be found in Definition 3.1.5.
Proposition 4.1.4. Let $H \in t$ be an essential mass linear function on a smooth 4-dimensional polytope $\Delta \subset t^*$ with exactly three asymmetric facets $F_1, F_2, \text{and } F_3$ with linearly independent conormals. Then there exists a smooth polytope $\overline{\Delta} \subset t^*$ so that:

- $H$ is an essential mass linear function on $\overline{\Delta}$.
- One of the following statements is true:
  - $\overline{\Delta}$ is a $\Delta_3$ bundle over $\Delta_1$, and the base facets and one fiber facet of $\overline{\Delta}$ are the symmetric facets.
  - $\overline{\Delta}$ is a 121-bundle and the nonpervasive facets of $\overline{\Delta}$ are the symmetric facets.
- $\Delta$ can be obtained from $\overline{\Delta}$ by a series of blowups along symmetric 2-faces.

Proof. The 2-dimensional polygon $F_{12}$ has at most two edges parallel to $\ell_{123}$. If it has at most one such edge, then the associated graph $\Gamma$ is connected, and so $F_1 \sim F_2 \sim F_3$ by part (ii) of Lemma 4.1.3. By Lemma 2.1.2, this implies that there exists an inessential function $H' \in t$ so that the function $\widetilde{H} = H - H'$ has at most one asymmetric facet. By Proposition 4.1.4, this implies that $H$ is inessential. Therefore $F_{12}$ has two edges parallel to $\ell_{123}$. We now consider the following cases.

Case (a): $F_{12}$ has exactly four edges and $F_{123} \neq \emptyset$.

The edge $F_{123}$ is parallel to $\ell_{123}$. Let $G_1 \cap F_{12}$, $G_2 \cap F_{12}$, and $G_3 \cap F_{12}$ be the remaining edges of $F_{12}$, where each $G_i$ is a symmetric facet and $G_1 \cap F_{12}$ is parallel to $\ell_{123}$. Then the conormals to $F_1$, $F_2$, $F_3$, and $G_1$ are linearly dependent and the intersections $G_1 \cap F_3 \cap F_{12}$ and $G_2 \cap G_3 \cap F_{12}$ are empty, but the remaining edges of $F_{12}$ do intersect. Hence, Lemma 4.1.1 implies that $\Delta$ is combinatorially equivalent to $\Delta_3 \times \Delta_1$, where $G_2 \cap G_3$ and $G_1 \cap F_{123}$ are both empty. Hence, by Lemma 2.1.12, $\Delta$ is a $\Delta_3$ a bundle over $\Delta_1$; the fiber facets are $F_1, F_2, F_3$, and $G_1$.

Case (b): $F_{12}$ has exactly four edges and $F_{123} = \emptyset$.

Let $G_1 \cap F_{12}$, $G_2 \cap F_{12}$, $G_3 \cap F_{12}$, and $G_4 \cap F_{12}$ be the edges of $F_{12}$, where each $G_i$ is a symmetric facet and $G_1 \cap F_{12}$ and $G_2 \cap F_{12}$ are parallel to $\ell_{123}$. Then the intersections $G_1 \cap G_2 \cap F_{12}$ and $G_3 \cap G_4 \cap F_{12}$ are empty, but the remaining edges of $F_{12}$ do intersect. Lemma 4.1.1 implies that $\Delta$ is combinatorially equivalent to $\Delta_1 \times \Delta_1 \times \Delta_2$, where $G_{12}, G_{34}$ and $F_{123}$ are empty.

Let $\eta_i$ and $\alpha_j$ denote the outward conormals to $F_i$ and $G_j$, respectively. Since $G_1 \cap F_{12}$ and $G_2 \cap F_{12}$ are parallel to $\ell_{123}$, $\alpha_1$ and $\alpha_2$ both lie in the subspace spanned by $\eta_1, \eta_2$, and $\eta_3$. Moreover, applying part (i) of Lemma 4.1.3 to $G_3$, we find that its two symmetric faces $G_{13}$ and $G_{23}$ are parallel. If $G_1$ and $G_2$ are not parallel, this implies that $\alpha_3$ lies in the plane spanned by $\alpha_1$ and $\alpha_2$. Hence, by the claim above, $\alpha_3$ lies in the subspace spanned by $\eta_1, \eta_2$, and $\eta_3$. But $G_3$ is not parallel to $\ell_{123}$, so this is impossible. Hence, $G_1$ is parallel to $G_2$.

Therefore, by Lemma 2.1.12, $\Delta$ is a $\Delta_1$ bundle over the polytope $G_1$. Moreover, since $G_1 \cap F_{12}$ is parallel to $\ell_{123}$, $G_1$ itself is a $\Delta_2$ bundle over $\Delta_1$ with fiber facets $F_1 \cap G_1$, $F_2 \cap G_1$, and $F_3 \cap G_1$.

Case (c): The general case.
If $F_{12}$ has four edges, the result follows from (a) or (b). So assume that $F_{12}$ has more than four edges. The face $F_{12}$ is a 2-dimensional smooth polygon with two edges which are parallel to $\ell_{123}$. With the possible exception of one of these parallel edges, every edge has the form $G \cap F_{12}$, where $G$ is a symmetric facet. Therefore, by part (i) of Lemma 2.5.6 there exist a symmetric facet $G$ so that the edge $G \cap F_{12}$ can be blown down in $F_{12}$, is not parallel to $\ell_{123}$, and is adjacent to $G' \cap F_{12}$ and $G'' \cap F_{12}$, where $G'$ and $G''$ are also symmetric facets.

We claim that $\Delta$ is the blow-up of a smooth polytope $\overline{\Delta}$ along the face $\overline{G'} \cap \overline{G''}$ with exceptional divisor $G$. To prove this, we check the three conditions of Proposition 2.5.1.

First, by part (i) of Lemma 4.1.3, $G$ is a $\Delta_1$ bundle over $\Delta_2$ with fiber facets $G' \cap G$ and $G'' \cap G$ and base facets $F_1 \cap G$, $F_2 \cap G$, and $F_3 \cap G$, so condition (i) holds. Second, the previous paragraph implies that $F_{12}$ is the blowup of a smooth polytope $F'_{12}$ along the vertex $P(G') \cap P(G'') \cap F'_1$, with exceptional divisor $G \cap F_{12}$, and so by Lemma 2.5.3 condition (ii) also holds. Finally, if $F_{123} \neq \emptyset$, then condition (iii) holds trivially. On the other hand, if $F_{123} = \emptyset$, then $\ell_{123}$ is a line in $P(F_{12})$ which does not intersect $F_{12}$ and is parallel to two of its edges. Since $G \cap F_{12}$ is not parallel to $\ell_{123}$, the polygon obtained from blowing down $G \cap F_{12}$ in $F_{12}$ will not intersect $\ell_{123}$. Hence, $\overline{\Delta}$ also will not intersect $\ell_{123}$, that is, $\overline{F}_{123}$ is empty.

By Lemmas 2.4.7 and 2.4.8 $H$ is an essential mass linear function on $\overline{\Delta}$ and $\langle H, e_{\overline{\Delta}} \rangle = \langle H, e_{\Delta} \rangle$. The result now follows by induction.

4.2. Four asymmetric facets. We now analyze mass linear functions on 4-dimensional polytopes with exactly four asymmetric facets, each of which is pervasive. As before, we first consider the case that the conormals to these asymmetric facets are linearly dependent, and then the case that they are linearly independent; see Proposition 4.2.4 and Proposition 4.2.6.

We begin by considering the combinatorics.

Let $H \in \mathfrak{t}$ be a mass linear function on a polytope $\Delta$ with exactly four asymmetric facets $F_1, F_2, F_3$, and $F_4$. Proposition 2.1.5 implies that each symmetric 2-face $g$ has exactly four asymmetric edges: $F_1 \cap g, \ldots, F_4 \cap g$. Moreover $H$ is mass linear on $g$. Therefore, Proposition 2.1.4 implies that $g$ has no symmetric edges. Hence, for exactly two of the six pairs $\{i, j\} \subset \{1, 2, 3, 4\}$, the intersection $F_{ij} \cap g$ is empty; we will refer to this set of two pairs as the rectangle order of the face.

Suppose first that $\Delta$ is 3-dimensional. Then Proposition 2.1.15 implies that $\Delta$ has at most two symmetric facets; moreover, the following are true.

(i) If $\Delta$ has no symmetric facets, then $\Delta$ is the simplex $\Delta_3$.
(ii) If $\Delta$ has one symmetric facet then $\Delta$ is a $\Delta_2$ bundle over $\Delta_1$, and the symmetric facet is a fiber facet.
(iii) If $\Delta$ has two symmetric facets then $\Delta$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$, and the symmetric facets are fiber facets.

Now assume that $\Delta$ is 4-dimensional, and let the symmetric subspace $S \subset \Delta$ be the union of the symmetric facets. By Proposition 2.1.5 each 3-dimensional symmetric facet $G$ is a polytope with four asymmetric facets and $H$ is a mass linear function on $G$. By the discussion above, this implies that $G$ intersects at most two other symmetric facets.
Lemma 4.2.2. Let $G$ be a 4-dimensional polytope with exactly four asymmetric facets $F_1, \ldots, F_4$. Assume, for example, that $F_1$ and $F_2$ are opposite in the rectangle order of $G$. If $F_1 \cap G$ is empty, then $F_3 \cap G$ is not, and $F_1 \cap G$ and $F_2 \cap G$ are the triangular faces of $G$. Conversely, if $F_3 \cap G$ is empty then $F_1 \cap G$ is not, and $F_3 \cap G$ and $F_4 \cap G$ are the triangular faces. In contrast, if $G$ intersects two other symmetric facets, then both symmetric 2-faces of $G$ have the same rectangle order, and $G$ is determined combinatorially by this rectangle order. This proves the following result.

Lemma 4.2.1. Let $H \in t$ be a a mass linear function on a 4-dimensional polytope $\Delta \subset t^*$ with exactly four asymmetric facets. Let $C$ be a connected component of the symmetric subspace $S \subset \Delta$. Then every 2-face in $C$ has the same rectangle order.

The analysis above also implies that, after renumbering the $F_i$ if necessary, each component $C$ of $S$ has one of the following four types.

(a) The component $C$ contains only one symmetric facet.
(b) The component $C$ contains two symmetric facets $G$ and $G'$ that each intersect only one other symmetric facet, and $F_1 \cap G = F_3 \cap G' = \emptyset$. The remaining symmetric facets in $C$ each intersect two symmetric facets.
(c) The component $C$ contains two symmetric facets $G$ and $G'$ that each intersect only one other symmetric facet, and $F_2 \cap G = F_2 \cap G' = \emptyset$. The remaining symmetric facets in $C$ each intersect two symmetric facets.
(d) Every symmetric facet in $C$ intersects two other symmetric facets.

Lemma 4.2.2. Let $H \in t$ be a a mass linear function on a 4-dimensional polytope $\Delta \subset t^*$ with exactly four asymmetric facets $F_1, \ldots, F_4$, each of which is pervasive.

(i) If $F_{1234} = \emptyset$, the symmetric subspace $S \subset \Delta$ has two components. Otherwise, it has one component.
(ii) Each of the four triple intersections $F_{ijk}$ is nonempty.
(iii) Each component of the symmetric subspace $S \subset \Delta$ has type (a) or (b) above.

Proof. To begin, consider the 2-dimensional polygon $F_{ij}$ for any $i \neq j$; let $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. By assumption, the asymmetric facets are pervasive, so $F_{ij}$ cannot be empty. If the intersections $F_k \cap F_{ij}$ and $F_{\ell} \cap F_{ij}$ are nonempty, they are edges of this polygon. All other edges lie in $S \cap F_{ij}$. Since every 2-dimensional polygon has at least three edges, the set $S \cap F_{ij}$ cannot be empty. Therefore, after possibly switching $k$ and $\ell$, there are four possibilities:

1. $F_k \cap F_{ij} = F_{\ell} \cap F_{ij} = \emptyset$; this implies that $S \cap F_{ij}$ is homeomorphic to a circle.
2. $F_k \cap F_{ij} \neq \emptyset$ but $F_{\ell} \cap F_{ij} = \emptyset$; this implies that $S \cap F_{ij}$ is homeomorphic to a line segment and both its ends are adjacent to $F_k \cap F_{ij}$.
3. $F_k \cap F_{ij} \neq \emptyset$ and $F_{\ell} \cap F_{ij} \neq \emptyset$ but $F_k \cap F_{\ell} \cap F_{ij} = F_{1234} = \emptyset$; this implies that $S \cap F_{ij}$ has two components, each component is homeomorphic to a line segment, and each has one end adjacent to $F_k \cap F_{ij}$ and the other to $F_{\ell} \cap F_{ij}$.
Then and that $F$ be a symmetric facet. Assume that the edge $\cap \Delta$ Let Lemma 4.2.3. showed to be impossible at the beginning of the proof. So there exists another component proves (iii).

If we saw above, this is impossible. Hence there are no components of type (c). Similarly, no polygon $F_{ij}$ can intersect the space of symmetric facets in at least two components, one of which is homeomorphic to a circle. Therefore, there are no components of type (d). This proves (iii).

Since each component of $\mathcal{S}$ has type (a) or (b) its intersection with each polygon $F_{ij}$ is homeomorphic to a line segment, the ends of which are adjacent to different edges. Therefore cases (1) and (2) for $\mathcal{S} \cap F_{ij}$ are impossible. Statement (ii) follows immediately. Finally, if $F_{1234} = \emptyset$ we are in case (3), and if it is not we are in case (4). This proves (i).

We will use the next lemma to identify facets that can be blown down.

**Lemma 4.2.3.** Let $H \in \mathfrak{t}$ be a mass linear function on a 4-dimensional smooth polytope $\Delta \subset \mathfrak{t}^*$ with exactly four asymmetric facets $F_1, \ldots, F_4$, each of which is pervasive. Let $G$ be a symmetric facet. Assume that the edge $G \cap F_{13}$ of the polygon $F_{13}$ can be blown down, and that $F_1$ and $F_3$ are not opposite in the rectangle order of any symmetric 2-face of $G$. Then $\Delta$ is the blowup of a smooth polytope $\overline{\Delta} \subset \mathfrak{t}^*$ along a face $f$ with exceptional divisor $G$. Moreover, one of the following holds:
(i) $G$ has no symmetric facets and $f$ is the vertex $F_{1234}$.
(ii) $G$ has exactly one symmetric facet $G' \cap G$, and $f$ is the edge $\overline{G'} \cap F_{ij}$, where $F_i$ and $F_j$ are opposite in the rectangle order of $G' \cap G$. Moreover, $f$ intersects each $F_k$.
(iii) $G$ has two symmetric facets $G' \cap G$ and $G'' \cap G$, and $f = \overline{G'} \cap \overline{G''}$.

Proof. By Proposition 2.1.5 $G$ itself is a 3-dimensional polytope with exactly four asymmetric facets $F_1 \cap G, \ldots, F_4 \cap G$, and the restriction of $H$ to $G$ is mass linear. As before, Proposition 2.1.15 implies that there are only three possibilities:

Case (i): $G$ has no symmetric facets; it is a simplex with facets $F_1 \cap G, \ldots, F_4 \cap G$.

Condition (i) of Proposition 2.5.1 is clearly satisfied. Since the edge $G \cap F_{13}$ of the polygon $F_{13}$ can be blown down, and $G \cap F_{13}$ is adjacent to $F_2 \cap F_{13}$ and $F_4 \cap F_{13}$, $F_{13}$ is the blowup of a smooth polygon $F_{13}'$ along the vertex $P(F_2) \cap P(F_4) \cap F_{13}'$. Hence, Lemma 2.5.3 implies that condition (ii) of Proposition 2.5.1 is also satisfied. Finally, condition (iii) of Proposition 2.5.1 is trivial in this case. Hence, the claim follows by Proposition 2.5.1.

Case (ii): $G$ has one symmetric facet $G' \cap G$; it is a $\Delta_2$ bundle over $\Delta_1$ and $G' \cap G$ is a fiber facet.

Since $F_1$ and $F_3$ are not opposite in the rectangle order of $G' \cap G$, one is a base facet and one is a fiber facet. Hence, we may renumber so that $G$ is a $\Delta_2$ bundle over $\Delta_1$ with fiber facets $F_1 \cap G, F_2 \cap G$, and $G' \cap G$ and base facets $F_3 \cap G$ and $F_4 \cap G$. In particular, condition (i) of Proposition 2.5.1 is satisfied. Since the edge $G \cap F_{13}$ of the polygon $F_{13}$ can be blown down, and $G \cap F_{13}$ is adjacent to $F_2 \cap F_{13}$ and $G' \cap F_{13}$, $F_{13}$ is the blowup of a smooth polygon $F_{13}'$ along the vertex $P(F_2) \cap P(G') \cap F_{13}'$. Hence, Lemma 2.5.3 implies that condition (ii) of Proposition 2.5.1 is also satisfied. Since each asymmetric facet is pervasive, $F_{34} \neq \emptyset$. Hence, condition (iii) of Proposition 2.5.1 is satisfied. Therefore, Proposition 2.5.1 implies that $\Delta$ is the blowup of a smooth polytope $\Delta \subset t^*$ along the edge $\overline{G'} \cap F_{12}$ with exceptional divisor $G$. Finally, Remark 2.4.2 implies that $\overline{G'} \cap F_{12}$ intersects both $F_3$ and $F_4$.

Case (iii): $G$ has two symmetric facets $G' \cap G$ and $G'' \cap G$; it is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$ with fiber facets $G' \cap G$ and $G'' \cap G$ and base facets $F_1 \cap G, \ldots, F_4 \cap G$.

Clearly, condition (i) of Proposition 2.5.1 is satisfied. Since the edge $G \cap F_{13}$ of the polygon $F_{13}$ can be blown down, and $G$ is adjacent to $G' \cap F_{13}$ and $G'' \cap F_{13}$, $F_{13}$ is the blowup of a smooth polygon $F_{13}'$ along the vertex $P(G') \cap P(G'') \cap F_{13}'$. Hence, Lemma 2.5.3 implies that condition (ii) of Proposition 2.5.1 is satisfied. Finally, since each asymmetric facet is pervasive, $F_{ij} \neq \emptyset$ for all pairs $1 \leq i < j \leq 4$. Hence, Lemma 2.5.5 implies that condition (iii) of Proposition 2.5.1 is satisfied. Hence, the claim follows by Proposition 2.5.1.

We can now prove our first main proposition in this subsection.

Proposition 4.2.4. Let $H \in t$ be a mass linear function on a smooth 4-dimensional polytope $\Delta \subset t^*$ with exactly four asymmetric facets $F_1, \ldots, F_4$ with linearly dependent conormals. Assume that each asymmetric facet is pervasive. Then there exists a smooth polytope $\overline{\Delta} \subset t^*$ so that
• $H$ is a mass linear function on $\Delta$.
• $\Delta$ is a $\Delta_3$ bundle over $\Delta_1$, and the base facets of $\Delta$ are the symmetric facets.
• $\Delta$ can be obtained from $\Delta$ by a series of blowups. Each blowup is either along a symmetric 2-face or along an edge of type $(F_{ij}, \mathcal{G})$ for some symmetric facet $\mathcal{G}$ of $\Delta$.

Moreover, if $H$ is inessential on $\Delta$ then the polytope $\Delta$ is the double expansion of a trapezoid along two parallel edges and the asymmetric facets are the base-type facets.

Proof. Since the conormals to the $F_i$ are linearly dependent, $F_{1234} = \emptyset$. By part (i) of Lemma 4.2.2, this implies that the symmetric subspace $S \subset \Delta$ has two components. Moreover, by Lemma 4.2.1, the rectangle order is the same on every symmetric 2-face in each component of $S$. Hence, after possibly renumbering, $F_1$ and $F_3$ are not opposite on the rectangle order of any symmetric 2-face.

If both components of $S$ contain a single symmetric facet, then each symmetric facet is a 3-simplex. Since the two symmetric facets don’t intersect, $F_{1234} = \emptyset$, and the conormals to $F_1, \ldots, F_4$ lie in a hyperplane, this implies that $\Delta$ is a $\Delta_3$ bundle over $\Delta_1$, and the base facets are the symmetric facets.

So assume on the contrary that at least one component of $S$ contains more than one symmetric facet. Since $F_1$ and $F_3$ are not opposite in the rectangle order of any symmetric 2-face, every symmetric facet intersects $F_{13}$. Moreover, by part (ii) of Lemma 4.2.2, the triple intersections $F_2 \cap F_{13}$ and $F_4 \cap F_{13}$ are not empty. Since there are at least 3 symmetric facets, this implies that the smooth convex polygon $F_{13}$ has more than four edges. Also, since the $\eta_i$ are linearly dependent, the two edges $F_2 \cap F_{13}$ and $F_4 \cap F_{13}$ are parallel. Hence, by part (i) of Lemma 2.5.6 there exists a symmetric facet $G$ of $\Delta$ so that the edge $G \cap F_{13}$ of $F_{13}$ can be blown down; moreover, $G \cap F_{13}$ is not adjacent to both $F_2 \cap F_{13}$ and $F_3 \cap F_{13}$. Hence, $G$ has at least one symmetric 2-face $g$. Since $F_1$ and $F_3$ are not opposite in the rectangle order of any symmetric face of $G$, we may renumber so that $F_1$ and $F_2$ are opposite on the rectangle order of $g$. By Proposition 2.1.5, $g$ has four asymmetric edges $F_1 \cap g, \ldots, F_4 \cap g$ and the restriction of $H$ to $g$ is mass linear. Since $F_1 \cap F_2 \cap g = \emptyset$ and $F_3 \cap F_4 \cap g = \emptyset$, Proposition 2.1.14 implies that $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 = 0$, where $\gamma_i$ is the coefficient of the support number of $F_i$ in the linear function $\langle H, c_G \rangle$. Lemma 4.2.3 implies that $\Delta$ can be obtained from a smooth polytope $\Delta$ by blowing up along a face $f$, where $f$ is either a 2-face of the form $\overline{G'} \cap \overline{G''}$ where $G'$ and $G''$ are symmetric facets, or an edge of the form $f = F_{12} \cap \overline{G'}$ or $f = F_{34} \cap \overline{G'}$, where $G'$ is a symmetric facet; moreover, in the latter case, $f$ intersects each $F_i$. By Lemma 2.4.7, $\langle H, c_G \rangle = \langle H, c_\Delta \rangle$. The first claim now follows by induction.

Finally, assume that $H$ is inessential on $\Delta$. Since the asymmetric facets are the four fiber facets, each fiber facet is equivalent to at least one other fiber facet. It is straightforward to check that this implies that $\Delta$ is the double expansion of a trapezoid along the two parallel sides. (Alternatively, $\Delta$ is a double expansion by Proposition 5.3.7.)

To deal with the case when the asymmetric facets are linearly independent, we need one final technical lemma.
**Lemma 4.2.5.** Let $H \in t$ be a mass linear function on a 4-dimensional smooth polytope $\Delta \subset t^*$ with exactly four asymmetric facets $F_1, \ldots, F_4$ with linearly independent conormals. If $F_1$ and $F_2$ are opposite in the rectangle order of each symmetric 2-face, then $F_1 \sim F_2$.

**Proof.** First, note that by Proposition 2.1.5 every symmetric facet $G$ is itself a 3-dimensional smooth polytope with exactly four asymmetric faces $F_1 \cap G, \ldots, F_4 \cap G$, and the restriction of $H$ to $G$ is mass linear.

Let $\eta_i$ be the outward conormal to $F_i$. If $G$ has no symmetric facets, then by Proposition 2.1.15 $G$ is a 3-simplex with with facets $F_1 \cap G, \ldots, F_4 \cap G$. Hence, $\eta_1 + \eta_2 + \eta_3 + \eta_4$ is a multiple of $\alpha$, the outward conormal to $G$. Since the $\eta_i$ are linearly independent, this implies that $\alpha$ is a multiple of $\eta_1 + \eta_2 + \eta_3 + \eta_4$.

Otherwise, the component of $S$ which contains $G$ contains more than one symmetric facet. By part (iii) of Lemma 4.2.2 we may label these symmetric facets $G_1, \ldots, G_k$ so that $G_i \cap G_{i+1} \neq \emptyset$ for all $i$ but otherwise $G_i \cap G_j = \emptyset$, and so that $F_{12} \cap G_1 = \emptyset = F_{34} \cap G_k$. Let $\alpha_i$ denote the outward conormal to $G_i$. The polytope $G_1$ has one symmetric facet $G_2 \cap G_1$; moreover, $F_{12} \cap G_1 = \emptyset$. Hence by Proposition 2.1.15 $G$ is a $\Delta_2$ bundle over $\Delta_1$ with fiber facets $F_3 \cap G_1$, $F_4 \cap G_1$, and $G_2 \cap G_1$. Hence $\alpha_2 + \eta_3 + \eta_4$ is a multiple of $\alpha_1$, and so $\alpha_2$ lies in the plane spanned by $\eta_3 + \eta_4$ and $\alpha_1$. If $k > 2$, the polytope $G_2$ has two symmetric facets $G_1 \cap G_2$ and $G_3 \cap G_2$. Hence, by Proposition 2.1.15 $G$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$ with fiber facets $G_1 \cap G_2$ and $G_3 \cap G_2$. Hence, $\alpha_1 + \alpha_3$ is a multiple of $\alpha_2$, and so $\alpha_3$ lies in the plane spanned by $\alpha_1$ and $\alpha_2$, and hence in the plane spanned by $\eta_3 + \eta_4$ and $\alpha_1$. Continuing in this way, $\alpha_j$ lies in the plane spanned by $\eta_3 + \eta_4$ and $\alpha_1$ for all $j$. Since $F_{34} \cap G_k = \emptyset$, a similar argument shows that that $\alpha_j$ lies in the plane spanned by $\eta_1 + \eta_2$ and $\alpha_k$ for all $j$. Since the $\alpha_j$ are not all parallel and the $\eta_i$ are linearly independent, this implies that $\alpha_j$ lies in the plane spanned by $\eta_1 + \eta_2$ and $\eta_3 + \eta_4$ for all $j$.

In short, the conormal to every symmetric facet lies in the plane spanned by $\eta_1 + \eta_2$ and $\eta_3 + \eta_4$. Thus, the conormal to every facet except $F_1$ and $F_2$ lies in the hyperplane spanned by $\eta_1 + \eta_2$, $\eta_3$, and $\eta_4$. By Lemma 2.1.9, this implies that $F_1$ is equivalent to $F_2$. \hfill $\square$

**Proposition 4.2.6.** Let $H \in t$ be a mass linear function on a 4-dimensional smooth polytope $\Delta \subset t^*$ with exactly four asymmetric facets $F_1, \ldots, F_4$ with linearly independent conormals. Assume that every asymmetric facet is pervasive. Then there exists a smooth polytope $\overline{\Delta} \subset t^*$ so that:

- $H$ is an inessential mass linear function on $\overline{\Delta}$.
- $\overline{\Delta}$ is the double expansion of a smooth polygon $\Delta$, and the fiber-type facets are the symmetric facets.
- $\Delta$ can be obtained from $\overline{\Delta}$ by a series of blowups. Each blowup is along a symmetric 2-face or is of type $(F_{ij}, G)$.

**Proof.** Let $\eta_1, \ldots, \eta_4$ be the outward conormals to $F_1, \ldots, F_4$.

**Case (a):** Every symmetric 2-face has the same rectangle order

Assume, for example, that $F_1$ and $F_2$ are opposite in every such face. Lemma 4.2.5 implies that $F_1$ and $F_2$ are equivalent. Similarly, $F_3$ and $F_4$ are equivalent. Since $F_1, \ldots, F_4$ are
pervasive, Lemma 2.3.7 implies that $\Delta$ is a double expansion, and $F_1, \ldots, F_4$ are the base-type facets. Hence, by Lemma 3.2.1 $H$ is inessential, and the proposition holds with $\overline{\Sigma} = \Delta$.

**Case (b): The general case**

By Lemma 4.2.1 every 2-face in each component of the symmetric subspace has the same rectangle order. Hence, if $S$ has one component, we are in Case (a). Therefore, by Lemma 4.2.2 we may assume that $F_{1234} = \emptyset$ and that $S$ has two components; moreover, none of the triple intersections $F_{ijk}$ are empty. Consider the fan associated to the polytope $\Delta$. Since $F_{ijk} \neq \emptyset$, the set of non-negative linear combinations of $\eta_i, \eta_j, \text{ and } \eta_k$ is a convex cone in this fan for each triple $\{i, j, k\} \subset \{1, 2, 3, 4\}$. Deleting the union $B$ of these cones divides $t$ into two open regions. Since $\eta_1, \ldots, \eta_4$ are linearly independent, one of these regions is the open cone $C := \left\{ \sum_{i=1}^{4} a_i \eta_i \mid a_i > 0 \right\}$; denote the other by $C'$. Each cone in the fan lies entirely in the closure of one of these regions. On the other hand, since $F_{1234} = \emptyset$, the cone $C$ itself does not lie in the fan, and so there must be another facet whose outward conormal lies in $C$. Hence this boundary $B$ divides the rays of the fan corresponding to the symmetric facets into two nonempty sets, which must correspond to the two components of $S$. Let $C$ be the component of $S$ corresponding to the symmetric facets whose conormal lies in $C$; let $C'$ denote the other component.

If either component has only one symmetric facet $G$ then $G$ is a simplex, and so it has no symmetric 2-faces. Therefore, we are in Case (a). So assume on the contrary that $C$ contains more than one symmetric facet. After renumbering, we may assume that $F_1$ and $F_3$ are opposite in the rectangle order of the symmetric 2-faces in the component $C$, and $F_1$ and $F_2$ are opposite in the rectangle order of the symmetric 2-faces in $C'$. Since both components contain more than one symmetric facet, by Lemma 4.2.2 each component is of type (b). Hence, the edges of the 2-dimensional smooth polytope $F_{12}$ are (in order) $F_3 \cap F_{12}, F_1 \cap F_{12}, \ldots, G_k \cap F_{12}, F_3 \cap F_{12}, F_4 \cap F_{12}$, where $G_1, \ldots, G_k$ are the symmetric facets in $C$, and $G'$ is one of the end symmetric facets in $C'$. Moreover, restricting to the plane containing $F_{12}$, the fact that the conormal to $G_i$ is contained in $C$ for all $i$ implies that the outward conormals to the edges $G_i \cap F_{12}$ are all positive linear combinations of the outward conormals to $F_3 \cap F_{12}$ and $F_4 \cap F_{12}$. Hence the outward conormal to $G' \cap F_{12}$ cannot be. Therefore, by part (ii) of Lemma 2.5.6 there is at least one edge $G_i \cap F_{12}$ that can be blown down in $F_{12}$.

Note that $G_i$ has at least one symmetric 2-face $g$. Since $F_{13} \cap g = \emptyset$ and $F_{24} \cap g = \emptyset$, Proposition 2.1.14 implies that $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4 = 0$. Since $F_1$ and $F_2$ are not opposite in the rectangle order of the symmetric faces of $G_i$, Lemma 4.2.3 implies that $\Delta$ can be obtained from a smooth polytope $\overline{\Sigma}$ by blowing up along a face $f$, where $f$ is either a symmetric 2-face of the form $\overline{G}_{i-1} \cap \overline{G}_{i+1}$, or an edge of the form $\overline{F}_{13} \cap \overline{G}'$ or $\overline{F}_{24} \cap \overline{G}'$, where $G'$ is a symmetric face. By Lemma 2.4.7 $\langle H, c_{\overline{\Sigma}} \rangle = \langle H, c_\Delta \rangle$. The result now follows by induction.

**4.3. More than four asymmetric facets.** To finish the proof of Theorem 1.1.1 outlined in [4.1.2] we now analyze mass linear functions on 4-dimensional polytopes with more than four asymmetric facets, each of which is pervasive. As we see in Corollary 4.3.2 this case
does not occur for an essential $H$. Additionally, we classify polytopes which admit essential mass linear functions with nonpervasive asymmetric facets.

**Proposition 4.3.1.** Let $H \in \mathfrak{t}$ be a mass linear function on a 4-dimensional smooth polytope $\Delta \subset \mathfrak{t}^*$ with more than four asymmetric facets. Assume that every asymmetric facet is pervasive. Then one of the following statements is true:

1. $\Delta$ is the four-simplex $\Delta_4$, or
2. $\Delta$ is a $\Delta_2$ bundle over $\Delta_2$.

**Proof.** Assume first that every facet is asymmetric. By [I, Corollary A.8] this implies that $\Delta$ is combinatorially equivalent to the product of simplices. Since every facet is pervasive, $\Delta$ is either a 4-simplex or is combinatorially equivalent to $\Delta_2 \times \Delta_2$. In the second case, by Lemma 2.1.13, $\Delta$ is a $\Delta_2$ bundle over $\Delta_2$.

Therefore, we may assume that $\Delta$ has at least one symmetric facet. Label the asymmetric facets $F_1, \ldots, F_k$. Proposition 2.1.5 implies that every symmetric face has $k$ asymmetric facets and that the restriction of $H$ to this face is mass linear. Since Proposition 2.1.14 implies that a 2-dimensional polygon with a mass linear function has at most four asymmetric edges, there are no symmetric 2-faces. Hence, no symmetric facets of $\Delta$ intersect, and each symmetric facet has no symmetric faces. Thus, Proposition 2.1.15 implies that there are only two possibilities with $k > 4$.

**Case (a):** $\Delta$ has five asymmetric facets, and each symmetric facet is a $\Delta_2$ bundle over $\Delta_1$.

Let $S$ denote the set of symmetric facets. Since every symmetric facet is a $\Delta_2$ bundle over $\Delta_1$, there are $5|S|$ 2-dimensional faces, $9|S|$ edges, and $6|S|$ vertices that do lie on a symmetric facet. Since the five asymmetric facets are pervasive there are ten 2-dimensional faces that do not lie on any symmetric facet. Let $E$ and $V$ be the sets of edges and vertices, respectively, that do not lie on any symmetric facet. Since the Euler characteristic of $\Delta$ is 0,

$$5 + |S| - 10 - 5|S| + |E| + 9|S| - |V| - 6|S| = 0,$$

and hence $|E| = 5 + |S| + |V|$.

Each vertex in $V$ lies on four edges in $E$, and each vertex that lies on a symmetric facet lies on exactly one edge in $E$. Since exactly two vertices lie on each edge, $2|E| = 4|V| + 6|S|$. Combined, these equations yield $|V| + 2|S| = 5$. Since by assumption $S \neq \emptyset$, this implies that $|S| = 1$ or 2.

If $|S| = 1$, then $\Delta$ has 6 facets. It is well known that any $n$-dimensional polytope with $n+2$ facets, such as $\Delta$, is a product of two simplices; for a proof in the current setting see [10] Prop 1.1.1]. Since $\Delta$ has at most one facet that is not pervasive, this means that $\Delta$ is combinatorially equivalent to $\Delta_2 \times \Delta_2$. By Lemma 2.1.13 this implies that $\Delta$ is a $\Delta_2$ bundle over $\Delta_2$.

So assume instead that there are two symmetric facets, $G$ and $G'$. Then $|V| = 1$ and $|E| = 8$. Since no edge in $E$ can connect two vertices in the same symmetric facet, and $G$ and $G'$ each have 6 vertices, there must be four edges that join $G$ to $G'$ and two edges that join each to the vertex in $V$. By renumbering, we may assume that $F_{1234} \neq \emptyset$, that the edges $F_{123}$ and $F_{124}$ intersect $G$ but not $G'$, and that $F_{234}$ and $F_{134}$ intersect $G'$ but not $G$. 


Since both $G$ and $G'$ are $\Delta_2$ bundles over $\Delta_1$, this is only possible if the fiber facets of $G$ are $F_1 \cap G$, $F_2 \cap G$, and $F_3 \cap G$, and the base facets $F_3 \cap G$ and $F_4 \cap G$. Similarly, the fiber facets of $G'$ are $F_3 \cap G'$, $F_4 \cap G'$, and $F_5 \cap G'$, and the base facets are $F_1 \cap G'$ and $F_2 \cap G'$. This implies that the remaining four edges in $E$ are $F_{135}, F_{145}, F_{235}$, and $F_{245}$.

Now let $\eta_i$ denote the outward conormal to the facet $F_i$, and $\alpha$ and $\alpha'$ denote the outward conormals to $G$ and $G'$, respectively. Since $F_{1234} \neq \emptyset$ and $\Delta$ is smooth, there is a change of basis so that $\eta_1 = (1, 0, 0, 0), \eta_2 = (0, 1, 0, 0), \eta_3 = (0, 0, 1, 0)$, and $\eta_4 = (0, 0, 0, 1)$. Since the fiber facets of $G$ are $F_1 \cap G$, $F_2 \cap G$, and $F_3 \cap G$, we must have $\eta_1 + \eta_2 + \eta_3 = A\alpha$ for some integer $A$. Similarly, $\eta_3 + \eta_4 = A'\alpha'$, for some integer $A'$. Since $\eta_1 + \eta_2 \neq \eta_3 + \eta_4$, we may assume without loss of generality that $A' \neq 0$. The polygon $F_{12}$ has only three edges: $F_3, F_4, F_5$. Hence, it is the standard 2-simplex, that is, $\alpha = (x, y, -1, -1)$ for some integers $x$ and $y$. Therefore, $\eta_5 = (Ax - 1, Ay - 1, -A, -A)$. Since $A' \neq 0$,

$$\alpha' = \frac{1}{A'}(Ax - 1, Ay - 1, 1 - A, 1 - A).$$

Thus, the facets $F_3$ and $F_4$ are equivalent. By Lemma 2.1.2 this implies that there exists an inessential function $H'$ so that the mass linear function $H := H - H'$ has the following property: $F_3$ is $\overline{H}$-symmetric, but $F_1, F_2$, and $F_5$ are $\overline{H}$-asymmetric. Since $F_3$ is pervasive by hypothesis, and $\Delta$ has seven facets, $F_3$ has six facets. Moreover, by Proposition 2.1.5 $\overline{H}$ is mass linear on $F_3$ with at least three asymmetric facets. Thus we may apply Proposition 2.1.5 to $F_3$. Both the polytopes on this list with six facets are combinatorially equivalent to $\Delta_1 \times \Delta_1 \times \Delta_1$. But we saw above that $F_3 \cap G$ is a base facet of $G$ and so is a triangle. This is impossible. Hence this case also does not occur.

**Case (b): $\Delta$ has six asymmetric facets, and each symmetric facet is $\Delta_1 \times \Delta_1 \times \Delta_1$.**

Let $G$ be a symmetric facet. Consider the slices $Q^\lambda$ through $\Delta$ parallel to $G$. More precisely, consider how the the parallel planes $P(F_i) \cap Q^\lambda$ and $P(F_j) \cap Q^\lambda$ associated to opposite faces of the box come together as we move the slice through $\Delta$. If one (or two) of the pairs of planes come together before the other pairs (or pair), then the remaining facets will not intersect. This contradicts the claim that they are pervasive. So assume that the three pairs of planes come together at the same time. Since $\Delta$ is simple this point cannot lie in the polytope, that is, it must be cut off by some symmetric facet. But then none of the opposite pairs intersect, which again contradicts the claim that they are pervasive. Thus, this case does not occur.

This has a number of corollaries.

**Corollary 4.3.2.** In the situation of Proposition 4.3.1 every mass linear function on $\Delta$ is inessential.

**Proof.** By Corollary 3.1.9 all mass linear functions on $\Delta_2$ bundles over $\Delta_2$ are inessential. Similarly, every $H \in t$ is inessential on $\Delta_4$. 

**Corollary 4.3.3.** Let $H \in t$ be a mass linear function on a 4-dimensional smooth polytope $\Delta \subset t$. There is an inessential function $H' \in t$ such that $H - H'$ has at most 4 asymmetric facets.
Proof. By Proposition 2.2.5 (ii), we may assume without loss of generality that every asymmetric facet is pervasive. Hence, Proposition 4.3.1 implies that if \( \Delta \) has more than four asymmetric facets then either \( \Delta \) is the four-simplex \( \Delta_4 \), or it is a \( \Delta_2 \) bundle over \( \Delta_2 \). In the first case, every \( H \in t \) is inessential; in the second case, the result follows from Proposition 2.2.3. (Alternately, \( H \) is inessential by Corollary 4.3.2.) □

We now consider the final case.

**Proposition 4.3.4.** Let \( H \in t \) be a mass linear function on a smooth 4-dimensional polytope \( \Delta \subset t^* \). Assume that at least one asymmetric facet is not pervasive. If \( \Delta \) has at most four asymmetric facets, \( H \) is inessential. If \( \Delta \) has more than four asymmetric facets, then one of the following statements is true:

- \( \Delta \) is a \( \Delta_3 \) bundle over \( \Delta_1 \);
- \( \Delta \) is a 121-bundle;
- \( \Delta \) is a \( \Delta_2 \) bundle over a polygon which is a \( \Delta_1 \) bundle over \( \Delta_1 \); or
- \( \Delta \) is a \( \Delta_1 \) bundle over the product \( (\Delta_1)^3 \) and \( H \) is inessential; moreover, either the base facets are the asymmetric facets or every facet is asymmetric and \( \Delta = (\Delta_1)^4 \).

Proof. By the first part of Proposition 2.2.5, \( \Delta \) is a bundle over \( \Delta_1 \). Therefore, by Proposition 2.2.4, we can write \( H = H' + \tilde{H} \), where \( H' \) is an inessential function, the fiber facets are \( H' \)-symmetric, and the two base facets are \( \tilde{H} \)-symmetric.

If at most four facets are \( H \)-asymmetric, then at most two facets are \( \tilde{H} \)-asymmetric. Hence, \( \tilde{H} \) (and thus \( H \)) is inessential by Proposition 2.1.8.

On the other hand, if at least five facets are \( H \)-asymmetric, then at least three facets are \( \tilde{H} \)-asymmetric. Therefore, by Proposition 2.1.5, \( \tilde{H} \) is mass linear on \( F \) with at least three asymmetric facets. Hence, we may apply Proposition 2.1.5 to \( F \). If \( F = \Delta_3 \), \( F \) is a \( \Delta_1 \) bundle over \( \Delta_2 \), or \( F \) is a \( \Delta_2 \) bundle over \( \Delta_1 \) then we are clearly in one of the first three cases listed above. So suppose that \( F \) is a \( \Delta_1 \) bundle over \( \Delta_1 \times \Delta_1 \) and the base facets of \( F \) are the \( \tilde{H} \)-asymmetric facets. Repeating the argument above for each asymmetric facet, we see that \( \Delta \) is a \( \Delta_1 \) bundle over \( (\Delta_1)^3 \) and the base facets of \( \Delta \) are the \( H \)-asymmetric facets. Similarly, if \( F \) is \( (\Delta_1)^3 \) and every facet of \( F \) is \( \tilde{H} \)-asymmetric, then \( \Delta = (\Delta_1)^4 \) and every facet of \( \Delta \) is \( H \)-asymmetric by a similar argument. In either case, Proposition 2.2.5 (ii) implies that \( \Delta \) supports no essential mass linear functions. □

**Corollary 4.3.5.** Let \( H \in t \) be a mass linear function on a 4-dimensional smooth polytope \( \Delta \subset t^* \). The polytope \( \Delta \) has at most eight asymmetric facets. If it has eight then \( \Delta \) is the hypercube \( \Delta_1 \times \Delta_1 \times \Delta_1 \times \Delta_1 \), and \( H \) is inessential. Moreover, if it has exactly seven asymmetric facets then \( \Delta \) is the product \( \Delta_1 \times Y \), where \( Y \) is a \( \Delta_2 \) bundle over \( \Delta_1 \).

Proof. Assume that \( \Delta \) has at least seven asymmetric facets. Proposition 4.3.1 shows that there exists an asymmetric facet which is not pervasive. Hence Proposition 4.3.4 implies that \( \Delta \) is a 121-bundle, a \( \Delta_2 \) bundle over a polygon, or \( (\Delta_1)^4 \). Therefore, the claim follows immediately from Corollaries 3.1.7 and 3.1.10. □

**Corollary 4.3.6.** Let \( H \in t \) be a mass linear function on a 4-dimensional smooth polytope \( \Delta \subset t^* \) so that every facet is asymmetric. Then one of the following is true:
• $\Delta$ is the four-simplex $\Delta_4$,
• $\Delta$ is a $\Delta_3$ bundle over $\Delta_1$,
• $\Delta$ is a $\Delta_2$ bundle over $\Delta_2$
• $\Delta$ is the product $\Delta_1 \times Y$, where $Y$ is a $\Delta_2$ bundle over $\Delta_1$, or
• $\Delta$ is the product $(\Delta_1)^4$.

Proof. By assumption, $\Delta$ must have at least five asymmetric facets. If every asymmetric facet is pervasive, the claim follows immediately from Proposition 4.3.1. On the other hand, if there exists a asymmetric facet that is not pervasive, then Proposition 4.3.4 implies that $\Delta$ is a $\Delta_3$ bundle over $\Delta_1$, a 121-bundle, a $\Delta_2$ bundle over a polygon, or the product $(\Delta_1)^4$. Therefore, the claim follows immediately from Corollaries 3.1.7 and 3.1.10. $\square$

5. Further results

This section contains several results that are not needed for the proof of the main theorem. §5.1 addresses the question of which of the polytopes $\Delta$ in Theorem 1.1.1 are minimal. Then in §5.2 we use Theorem 1.1.1 to show that in dimensions $\leq 4$ every mass linear function is fully mass linear. Finally, in §5.3 we discuss the question of which blowup operations preserve mass linearity, considering both vertex and edge blowups. Additionally, we show that a vertex blowup never converts an essential function to an essential one. On the other hand, edge blowups of type $(F_{ij},g)$ may do this, but only if the underlying polytope is a double expansion.

5.1. Minimality. We now consider which of the polytopes $\Delta$ in Theorem 1.1.1 are minimal. In particular, we show that in most cases we can blow down $\Delta$ in the two allowed ways to obtain a minimal polytope $\overline{\Delta}$; the exceptions occur in cases (a2) and (b).

We begin with a useful technical lemma.

Lemma 5.1.1. Let $F$ and $F'$ be (distinct) equivalent facets of a polytope $\Delta$. Then neither $F$ nor $F'$ can be blown down.

Proof. Let $\eta$ and $\eta'$ be the outward conormals to $F$ and $F'$, respectively. By Lemma 2.1.9, there exists a vector $\xi \in t^*$ that is parallel to all the other facets. Since $\Delta$ is compact, $\langle \eta, \xi \rangle$ and $\langle \eta', \xi \rangle$ have opposite signs. Therefore, $\eta$ cannot be written as the positive sum of outward conormals without including $\eta$ itself. $\square$

As we show in the next proposition, polytopes of type (a1) which admit an essential mass linear function are all minimal.

Proposition 5.1.2. Let $\Delta$ be a $\Delta_3$ bundle over $\Delta_1$ that admits an essential mass linear function. Then $\Delta$ is minimal.

Proof. Let $\Delta$ be the $\Delta_3$ bundle over $\Delta_1$ associated to $a \in \mathbb{R}^3$ as in (3.1). By Lemma 5.1.3 below, some facet of $\Delta$ can be blown down exactly if $\sum_{j=1}^3 a_j e_j$ is the conormal to one of the fiber facets, that is, exactly if $a$ is $(-1,0,0), (0,-1,0), (0,0,-1)$, or $(1,1,1)$. By Corollary 3.1.4, none of these bundles admit essential mass linear functions. $\square$
Lemma 5.1.3. Let $\Delta$ be a $\Delta_3$ bundle over $\Delta_1$. The base facets cannot be blown down, and a fiber facet $F$ with outward conormal $\eta$ can be blown down exactly if $\eta = \alpha_1 + \alpha_2$, where $\alpha_1$ and $\alpha_2$ are the outward conormals to the base facets. In that case $\Delta$ is the blowup of a 4-simplex.

Proof. Since the base facets are equivalent, the first claim follows from Lemma 5.1.1.

Let $G_1$ and $G_2$ be the base facets, and let $F_1, \ldots, F_4$ be the fiber facets with outward conormals $\eta_1, \ldots, \eta_4$. Note that, for example, $F_1$ is a $\Delta_2$ bundle over $\Delta_1$ with fiber facets $F_{12}, F_{13}$, and $F_{14}$ and with base facets $G_1 \cap F_1$ and $G_2 \cap F_1$. Therefore, by Proposition 2.5.1 $F_1$ cannot be blown down unless either $\eta_1 = \eta_2 + \eta_3 + \eta_4$ or $\eta_1 = \alpha_1 + \alpha_2$.

Since the first equation does not hold, let us assume that $\eta_1 = \alpha_1 + \alpha_2$. Then the two facets $G_1 \cap F_1$ and $G_2 \cap F_1$ of $F_1$ are parallel, and so, we can also view $F_1$ as a (trivial) $\Delta_1$ bundle over $\Delta_2$ with fiber facets $G_1 \cap F_1$ and $G_2 \cap F_1$. Thus, conditions (i) and (ii) of Proposition 2.5.1 are both satisfied. Finally, since $F_{234} \subset \Delta$ is nonempty, condition (iii) is vacuous. Therefore the claims in the first sentence hold. The last statement holds because $\Delta_4$ is the only 4-dimensional polytope with 5 facets.

In contrast, as we show in the next lemma, there exist polytopes of type (a2) that admit essential mass linear functions but are not minimal. Note that, in most cases, the blowup described below is not one of the two types allowed in the main theorem.

Lemma 5.1.4. Let $\Delta \subset \mathbb{R}^t$ be a $\Delta_3$ bundle over $\Delta_1$ with fiber facets $F_1, \ldots, F_4$. Let $H \in \mathfrak{t}$ be a mass linear function on $\Delta$ such that $F_1$ is symmetric. The blowup $\Delta'$ of $\Delta$ along the edge $F_{234}$ is a 121-bundle; (see Definition 3.1.5). Moreover, $H$ is a mass linear function on $\Delta'$. Finally, $H$ is inessential on $\Delta'$ exactly if it is inessential on $\Delta$.

Proof. The first claim is easy. To prove the second, decompose $\Delta$ as $\Delta' \cup W$, where $W$ is also a $\Delta_3$ bundle over $\Delta_1$. Because the exceptional divisor is parallel to $F_1$, the polytope $W$ is analogous to $\Delta$. Since $F_1$ is symmetric, this implies that $\langle H, c_\Delta \rangle = \langle H, c_W \rangle$. Thus $H$ is mass linear on $c_\Delta$ by Lemma 2.4.6.

Finally, by Lemma 2.2.1(iii), the base facets of $\Delta$ are not equivalent to any fiber facet. Hence, Lemma 2.4.5 implies that two asymmetric facets are equivalent in $\Delta$ exactly if the corresponding facets of $\Delta'$ are equivalent, and the exceptional divisor $F'_{00}$ is not equivalent to any other facet. By the definition of inessential and Proposition 2.1.1 this proves the last claim.

We next show that all other polytopes $\Delta'$ of type (a2) that admit essential mass linear functions are minimal. We write $\Delta'$ as in (3.4) and denote by $F'_i$ the facet with outward conormal $\eta'_i$, and by $\tilde{F}_j$ the facet with outward conormal $\bar{\eta}_j$.

Proposition 5.1.5. Let $H$ be an essential mass linear function on a 121-bundle $\Delta'$ that is the blowup of another polytope $\Delta$. Then $H$ is an essential mass linear function on $\Delta$ and $\Delta$ is a $\Delta_3$ bundle over $\Delta_1$. Further, exactly three of the fiber facets of $\Delta$ are asymmetric, and the blowup is along the intersection of those three facets.

Proof. Write $\Delta'$ as in Equation (3.4). By Proposition 2.2.5 we may subtract an inessential function from $H$ to get an essential function $\tilde{H}$ such that the four nonpervasive facets are
\( \bar{H} \)-symmetric, but each pervasive facet is \( \bar{H} \)-symmetric exactly if it is \( H \)-symmetric. Since the fiber facets \( \bar{F}_0 \) and \( \bar{F}_1 \) are \( \bar{H} \)-symmetric, Proposition 2.2.3 implies that \( \bar{H} \) is the lift of an essential mass linear function on the base of \( \Delta' \). Hence, since the base of \( \Delta' \) is the \( \Delta_2 \) bundle over \( \Delta_1 \) associated to \( (a_2, a_3) \), Proposition 3.1.1 implies that \( a_2a_3(a_2 - a_3) \neq 0 \) and the three non-pervasive facets \( F_2', F_3', \) and \( F_4' \) are \( \bar{H} \)-asymmetric, and hence \( H \)-asymmetric. Therefore by Lemma 5.1.6 below, \( d = 1 \), \( \Delta \) is a \( \Delta_3 \) bundle over \( \Delta_1 \), and the blowup is along the intersection of the three fiber facets of \( \Delta \) corresponding to \( F \). Moreover the blowup is along three of its fiber facets.

Remark 5.1.7. \( \Delta \) is mass linear on \( \Delta \), and all the claims in the last sentence hold. Finally, \( H \) is essential on \( \Delta \) by Lemma 5.1.4. □

Lemma 5.1.6. Suppose that the 121-bundle \( \Delta' \) of Equation \( (3.4) \) is the blowup of a polytope \( \Delta \). If \( a_2a_3(a_2 - a_3) \neq 0 \), then \( d = 1 \) and \( \Delta \) is the \( \Delta_3 \) bundle over \( \Delta_1 \) associated to \( (a_1, a_2, a_3) \). Moreover the blowup is along the intersection of the three fiber facets of \( \Delta \) corresponding to the facets \( F_1', F_3', \) and \( F_4' \) of \( \Delta' \).

Proof. Since \( F_0' \sim F_0' \), neither facet can be blown down by Lemma 5.1.1. Next, fix \( i \in \{2, 3, 4\} \), and observe that \( F_i' \) is a \( \Delta_1 \) bundle over a trapezoid. The three non-intersecting pairs of facets of \( F_i' \) are \( \bar{F}_0 \cap F_i' \) and \( \bar{F}_1 \cap F_i' \), \( F_{ij}' \) and \( F_{ik}' \) where \( \{i, j, k\} = \{2, 3, 4\} \), and \( F_{ii}' \) and \( F_{i6}' \). It is easy to check that \( \eta_i' \neq \bar{\eta}_0 + \bar{\eta}_1 \) and \( \eta_i' \neq \eta_j' + \eta_k' \). Further, because \( a_2a_3(a_2 - a_3) \neq 0 \), we also have \( \eta_i' \neq \eta_5' + \eta_6' \). Hence, Proposition 2.5.1 implies that \( F_i' \) cannot be blown down.

So fix \( j \in \{0, 1\} \), and assume that \( \bar{F}_j \) can be blown down. Note that \( \bar{F}_j \) is a \( \Delta_2 \) bundle over \( \Delta_1 \) with fiber facets \( \bar{F}_i' \cap \bar{F}_j, i = 2, 3, 4 \) and base facets \( \bar{F}_3' \cap \bar{F}_j \) and \( \bar{F}_6' \cap \bar{F}_j \). Moreover, since \( a_2a_3(a_2 - a_3) \neq 0 \), \( \bar{\eta}_j \neq \eta_5' + \eta_6' \). Therefore, Proposition 2.5.1 implies that \( \Delta' \) is the blowup of \( \Delta \) along the edge \( F_{234} \), which (is not empty and) meets \( F_5' \) and \( F_6' \). (Here, \( F_i \) is the facet of \( \Delta \) such that \( F_i' = F_i \cap \Delta' \).) In particular, \( \bar{\eta}_j = \eta_2' + \eta_3' + \eta_4' \). Since \( d \geq 0 \) by assumption, this implies that \( j = 0 \) and \( d = 1 \). Therefore, \( \Delta \) is the \( \Delta_3 \) bundle over \( \Delta_1 \) associated to \( (a_1, a_2, a_3) \), and the blowup is along three of its fiber facets. □

Remark 5.1.7. Lemma 5.1.6 shows that if \( a_2a_3(a_2 - a_3) \neq 0 \), then the 121-bundle \( \Delta' \) of Equation (3.4) is minimal unless \( d = 1 \). In other words \( \Delta' \) cannot be blown down unless the sum of the outward conormals to its three pervasive facets is the outward conormal to a fiber facet. However, this condition is not sufficient because condition (iii) in Proposition 2.5.1 may fail for certain values of \( \kappa \). Here the base facets of \( \bar{F}_0 \) are given by its intersection with \( F_3' \) and \( F_6' \), and these facets may intersect when we remove \( \bar{F}_0 \); Figure 2.3 illustrates a similar 3-dimensional situation in which \( \Delta_2 \) is replaced by \( \Delta_1 \).

We next consider polytopes of type (a3), that is, \( \Delta_2 \) bundles over polygons. As we show below, every polytope of type (a3) that admits an essential mass linear function can be obtained from a minimal polytope of type (a3) that admits an essential mass linear function by a series of blowups along symmetric 2-faces. However, these minimal polytopes may have arbitrarily many facets.

Lemma 5.1.8. Let \( \Delta \) be a \( \Delta_2 \) bundle over a polygon \( \hat{\Delta} \). Let \( G_1, G_2, \ldots, G_N \) be the base facets of \( \hat{\Delta} \), and let \( \alpha_j \) be the outward conormal to \( G_j \) for all \( j \). Assume that the edges of \( \hat{\Delta} \)
corresponding to \( G_j \) and \( G_{j+1} \) are adjacent for all \( j \). (We interpret the \( G_i \) in cyclic order.) Then \( G_i \) can be blown down exactly if \( \alpha_i = \alpha_{i-1} + \alpha_{i+1} \). In this case, \( \Delta \) is the blowup of a polytope \( \Delta \) along the face \( P(G_{i+1}) \cap P(G_{i-1}) \cap \Delta \).

Proof. Let \( F_1, F_2, \) and \( F_3 \) be the fiber facets of \( \Delta \). The facet \( G_i \) is a \( \Delta_2 \) bundle over \( \Delta_1 \) with fiber facets \( F_1 \cap G_i, F_2 \cap G_i, \) and \( F_3 \cap G_i \), and with base facets \( G_{i-1} \cap G_i \) and \( G_{i+1} \cap G_i \).

Assume first that \( \alpha_i \neq \alpha_{i+1} + \alpha_{i-1} \). Since also \( \alpha_i \neq 0 = \eta_1 + \eta_2 + \eta_3 \), Proposition \[2.5.1\] implies that \( G_i \) cannot be blown down.

So assume instead that \( \alpha_i = \alpha_{i+1} + \alpha_{i-1} \). In this case, we can also consider \( G \) as a (trivial) \( \Delta_1 \) bundle over \( \Delta_2 \), and condition (ii) in Proposition \[2.5.1\] is clearly satisfied. Moreover, since \( P(F_1) \cap P(F_2) \cap P(F_3) = \emptyset \) and since \( F_K \neq \emptyset \) for any \( K \subset \{1, 2, 3\} \) condition (iii) is also satisfied. Hence the claim follows from Proposition \[2.5.1\]. \( \Box \)

Remark 5.1.9. Note that the edge of \( \Delta \) associated to \( G_i \) can be blown down exactly if \( \alpha_i = \alpha_{i-1} + \alpha_{i+1} \), that is the vector \( \alpha_i - \alpha_{i-1} - \alpha_{i+1} \) lies in the span of the fiber conormals. The condition \( \alpha_i = \alpha_{i-1} + \alpha_{i+1} \) given above is stronger; it also implies that the bundle \( \Delta \to \Delta \) is trivial when restricted to \( G_i \).

Lemma 5.1.10. Let \( H \in \mathfrak{t} \) be an essential mass linear function on a polytope \( \Delta \subset \mathfrak{t}^* \) that is a \( \Delta_2 \) bundle over a polygon. Then no fiber facets can be blown down.

Proof. Since \( H \) is essential, Proposition \[3.1.8\] implies that the fiber facets are all asymmetric. Hence, the claim follows from part (i) of Lemma \[2.4.7\]. \( \Box \)

Proposition 5.1.11. Let \( H \in \mathfrak{t} \) be an essential mass linear function on a polytope \( \Delta' \subset \mathfrak{t}^* \) that is a \( \Delta_2 \) bundle over a polygon \( \Delta' \). Then there exists a minimal polytope \( \Delta \) so that \( H \) is essential on \( \Delta \) and \( \Delta' \) can be obtained from \( \Delta \) by a series of blowups along symmetric 2-faces; moreover, \( \Delta \) is a \( \Delta_2 \) bundle over a polygon \( \Delta' \).

Proof. Let \( G_1, \ldots, G_k \) be the base facets of \( \Delta' \). Assume that they are labelled so that the edges of \( \Delta' \) corresponding to \( G_i \) and \( G_{i+1} \) are adjacent for all \( i \), where \( G_{k+1} = G_1 \). Let \( \alpha_i \) be the outward conormal to \( G_i \) for all \( i \).

First assume that \( \alpha_i \neq \alpha_{i-1} + \alpha_{i+1} \) for all \( i \). By Lemma \[5.1.8\], this implies that none of the base facets can be blown down. By Lemma \[5.1.10\] the fiber facets cannot be blown down either. Therefore \( \Delta' \) is minimal and the claim holds with \( \Delta = \Delta' \).

So assume instead that \( \alpha_i = \alpha_{i-1} + \alpha_{i+1} \) for some \( i \). By Lemma \[5.1.8\] this implies that there exists a polytope \( \Delta \) which is a \( \Delta_2 \) bundle over a polygon \( \Delta \) so that \( \Delta' \) can be obtained from \( \Delta \) by blowing up along the intersection of two base facets. By Proposition \[3.1.8\] since \( H \) is essential the fiber facets of \( \Delta' \) are also asymmetric. Hence, by part (ii) of Lemma \[2.4.7\] \( H \) is mass linear on \( \Delta \) and fiber facets of \( \Delta \) are also asymmetric. Hence, by Proposition \[3.1.8\] \( H \) is essential on \( \Delta \). By Corollary \[3.1.9\] this is impossible if \( \Delta \) has three edges. Since this implies that \( \Delta' \) has at least five edges, there are no nonzero inessential mass linear functions on \( \Delta' \) by Proposition \[2.1.14\]. Hence, Proposition \[3.1.8\] and Lemma \[2.4.7\] (ii) imply that the base facets of \( \Delta \) are symmetric. Thus, the blowup is along a symmetric 2-face, as required. The result now follows by induction. \( \Box \)
Proposition 5.1.12. For any $N \geq 7$, there exists a minimal polytope $\Delta$ that has $N$ facets, is a $\Delta_2$ bundle over a polygon $\hat{\Delta}$, and admits an essential mass linear function.

Proof. Start with the polygon $\Delta_2$ with facets $e_1, e_2$, and $e_3$ with outward conormals $(-1, 0), (0, -1)$, and $(1, 1)$, respectively. Then form a polygon $\hat{\Delta}$ with $k = N - 3 \geq 4$ sides by first blowing up along $e_{13} = e_1 \cap e_3$, and every subsequent time blowing up along the intersection of the new exceptional divisor and $e_1$. Then $\hat{\Delta}$ has edges $e_1, \ldots, e_k$, where $e_4, e_5, \ldots, e_k$ are the edges that are formed by the successive blowup operation. (Thus they are labelled in order of adjacency.) Let $P(\kappa_1, \ldots, \kappa_k)$ be the polynomial which gives the area of $\hat{\Delta}(\hat{k})$ for all $\hat{k} \in \mathcal{C}_{\hat{\Delta}}$. The blowup which introduces $e_j$ for $j \geq 4$ is performed by cutting out a triangle with affine side length $\kappa_{j-1} - \kappa_j + \kappa_1$. Hence

$$P(0, 0, \kappa_3, \ldots, \kappa_k) = \frac{1}{2} (\kappa_3)^2 - \frac{1}{2} \sum_{j=4}^{k} (\kappa_{j-1} - \kappa_j + \kappa_1)^2.$$

Let $r_1 = r_2 = 0$, $r_3 = \cdots = r_{k-1} = 1$ and $r_k = 2$. Fix integers $\gamma_1$ and $\gamma_2$ such that $\gamma_1 \gamma_2 (\gamma_1 + \gamma_2) \neq 0$, and define

$$(b_1^i, b_2^i) = r_i (\gamma_2, -\gamma_1) \in \mathbb{Z}^2 \quad \text{for all } 1 \leq i \leq k.$$

These can be used to construct a polytope $\Delta$ which is a $\Delta_2$ bundle over $\hat{\Delta}$ with the outer conormals to the fiber and base facets given by Equations (3.5) and (3.6), respectively. Since $P(0, 0, r_3, \ldots, r_k) = 0$ and $\gamma_1 \gamma_2 (\gamma_1 + \gamma_2) \neq 0$, $\Delta$ admits an essential mass linear function by Proposition 3.1.8.

Finally, by Equations (3.6) and (5.1), and Lemma 5.1.8 the base facet of $\Delta$ associated to the edge $e_i$ of $\hat{\Delta}$ cannot be blown down unless the edge $e_i$ itself can be blown down and $r_i = r_{i-1} + r_{i+1}$. (Here, as always, we use cyclic order on the edges.) However, if $k = 5$ then $e_1$ and $e_3$ are the only edges of $\hat{\Delta}$ that can be blown down; if $k \neq 5$ only $e_k$ can be blown down. Since $r_k = 2 \neq 1 = r_{k-1} + r_1$ and $r_1 = 0 \neq 2 = r_k + r_2$ for all $k$, this implies that none of the base facets of $\Delta$ can be blown down. The claim then follows from Lemma 5.1.10.

Finally, we consider polytopes of type (b), that is, double expansions of polygons. As we see below, every polytope of type (b) can be obtained from a minimal polytope of type (b) by a series of blowups. As in the previous case, these minimal polytopes may have arbitrarily many facets.

Lemma 5.1.13. Let $\Delta'$ be the double expansion of a polygon $\tilde{\Delta}'$ with edges $\tilde{F}_1, \ldots, \tilde{F}_k$ along the edges $\tilde{F}_1$ and $\tilde{F}_2$.

(i) If $\tilde{F}_i$ cannot be blown down for any $i > 2$, then $\Delta'$ is minimal.

(ii) In contrast, if $\tilde{F}'_i$ is the blowup of a polygon $\tilde{\Delta}$ with exceptional divisor $\tilde{F}_i$ for some $i > 2$, let $\Delta$ be the double expansion of $\tilde{\Delta}$ along the edges $P(\tilde{F}_1) \cap \tilde{\Delta}$ and $P(\tilde{F}_2) \cap \tilde{\Delta}$.

Then $\Delta'$ is the blowup of $\Delta$ along a face of one of the following three types:

---

6We do not assume that the edges are labelled in order of adjacency.
the intersection of two fiber-type facets.
- the intersection of a fiber-type facet with either $P(\widetilde{F}_1)$ or $P(\widetilde{F}_3)$.
- the vertex $\cap_{i=1}^4 P(\widetilde{F}_i)$.

Here, $\widetilde{F}_1$ and $\widetilde{F}_2$ ($\widetilde{F}_3$ and $\widetilde{F}_4$) are the base-type facets of $\Delta'$ associated to $\widetilde{F}_1$ (respectively, $\widetilde{F}_2$).

Proof. Let $\iota : \mathbb{R}^2 \to \mathbb{R}^4$ be inclusion into the first two coordinates, and let the outward conormals to the edges $\tilde{F}_1, \ldots, \tilde{F}_k$ of the polygon $\Delta'$ be $\tilde{a}_1, \ldots, \tilde{a}_k$. Then the conormal to the fiber-type facet $\tilde{F}_j$ of $\Delta'$ associated to $\tilde{F}_j$ is $\alpha_j = \iota(\tilde{a}_j)$ for $j > 2$, and the conormals to the base-type facets $\tilde{F}_1, \ldots, \tilde{F}_4$ are

$$\eta_1 = (0, 0, -1, 0), \quad \eta_2 = \iota(\tilde{a}_1) + (0, 0, 1, 0), \quad \eta_3 = (0, 0, 0, -1), \quad \eta_4 = \iota(\tilde{a}_2) + (0, 0, 0, 1).$$

By Remark 2.3.6 (ii) and Lemma 5.1.1, none of the base-type facets can be blown down. Now fix $j > 2$, and assume that the fiber-type facet $\tilde{F}_j$ can be blown down. Let $\tilde{F}_k$ and $\tilde{F}_\ell$ be the edges of $\Delta$ that meet $\tilde{F}_j$.

Assume first that $k$ and $\ell$ are both greater than 2. Then $\tilde{F}_j$ is a $\Delta_1$ bundle over $\Delta_1 \times \Delta_1$, with opposite base facets given by its intersections with $\tilde{F}_1$ and $\tilde{F}_2$, and with $\tilde{F}_3$ and $\tilde{F}_4$, and with fiber facets $\tilde{F}_{jk}$ and $\tilde{F}_{j\ell}$. Since $\tilde{a}_j \neq \tilde{a}_1$, we have $\alpha_j \neq \eta_1 + \eta_2$; similarly, $\alpha_j \neq \eta_3 + \eta_4$. Therefore, by Proposition 2.5.1, $\alpha_j$ must be equal to $\alpha_k + \alpha_\ell$, and so $\tilde{F}_j$ blows down in $\Delta'$.

Next suppose that $k = 1$ and $\ell > 2$. Then $\tilde{F}_j$ is a $\Delta_2$ bundle over $\Delta_1$ with fiber facets $\tilde{F}_1 \cap F_j, \tilde{F}_2 \cap F_j, F_k \cap F_j$, and base facets $\tilde{F}_3 \cap F_j$ and $\tilde{F}_4 \cap F_j$. Since the equation $\alpha_j = \eta_3 + \eta_4$ is never satisfied, Proposition 2.5.1 implies that $\alpha_j = \eta_1 + \eta_2 + \alpha_k$. Hence $\tilde{a}_j = \tilde{a}_1 + \tilde{a}_k$ so that $\tilde{F}_j$ blows down in $\Delta'$.

Finally suppose that $k = 1$ and $\ell = 2$. Then $\tilde{F}_j$ is a 3-simplex with facets $\tilde{F}_i \cap F_j$ for $i = 1, \ldots, 4$. Therefore, in this case Proposition 2.5.1 implies that $\sum_{i=1}^4 \eta_i = \alpha_j$. Once again, this implies that $\tilde{F}_j$ blows down in $\Delta'$. This proves (i).

To prove (ii) it remains to check that in each of the three cases considered above $\Delta'$ is the blowup of $\Delta$ along an appropriate face. We leave this to the reader. \hfill \Box

Proposition 5.1.14. Let $\Delta' \subset t'$ be the double expansion of a polygon. Then $\Delta'$ can be obtained from a minimal polytope $\Delta$ that is also the double expansion of a polygon by a series of blowups. Moreover, if $H \in \mathfrak{t}$ is an inessential function on $\Delta'$ and the asymmetric facets are the base-type facets, then

- $H$ is also an inessential function on $\Delta$,
- the asymmetric facets of $\Delta$ are the base-type facets, and
- each blowup is either one of the two types permitted in Theorem 1.1.1 or is at the vertex formed by the intersection of the four asymmetric facets.

Proof. The first claim is immediate from Lemma 5.1.13. The second claim follows from Lemmas 3.2.1 and 5.1.13. \hfill \Box

Proposition 5.1.15. For any $N \geq 5$ there exists a minimal polytope $\Delta \subset t'$ that has $N$ facets, is the double expansion of a polygon $\Delta$, and admits an inessential function $H \in \mathfrak{t}$ so
that the asymmetric facets are the base-type facets. Moreover, when \( N = 6 \) or \( N \geq 8 \) we can choose \( \Delta \) so that \( H \) is essential on a polytope \( \Delta' \subset t^* \) that can be obtained from \( \Delta \) by a sequence of blowups, where each blowup is either along a symmetric face or of type \((F_{ij}, G)\). When \( N \) is 5 or 7 the previous statement holds only if we also allow blowups at the point where the four base-type facets intersect.

Proof. Start with the simplex \( \Delta_2 \) with facets \( e_1, e_2, \) and \( e_3 \). Then form a polygon \( \widetilde{\Delta} \) with \( k = N - 2 \) by first blowing up along \( e_{13} = e_1 \cap e_3 \) and every subsequent time blowing up along the intersection of the new exceptional divisor and \( e_1 \) as in the proof of Proposition 5.1.12. If \( k = 5 \) (and \( N = 7 \)) let \( \Delta \) be the double expansion of \( \Delta' \) along \( e_5 \) and \( e_1 \); otherwise let \( \Delta \) be the double expansion of \( \Delta' \) along \( e_k \) and \( e_{k-2} \). If \( k = 5 \), then \( e_1 \) and \( e_5 \) are the only edges of \( \Delta \) that can be blown down; if \( k = 3 \) then no edge can be blown down; otherwise, \( e_k \) is the only edge that can be blown down. Therefore, \( \Delta \) is minimal by Lemma 5.1.13 and it admits an inessential mass linear function so that the asymmetric facets are the four base-type facets by Lemma 3.2.1. This proves the first claim.

To prove the second claim, observe that if \( k \neq 3, 5 \) then Proposition 3.2.2 implies that there is a blowup \( \Delta' \) of \( \Delta \) with the required properties. However, this argument does not work when \( k = 5 \) since we expanded along adjacent edges \( e_1, e_5 \) in order to make \( \Delta \) minimal. In fact every polygon \( \widetilde{\Delta} \) with 5 edges is a blowup of a trapezoid and so always has two adjacent edges that can be blown down. Therefore, if \( k = 3 \) or 5 Proposition 5.3.7 implies that there cannot be a blowup \( \Delta' \) of \( \Delta \) with the required properties. On the other hand, if we allow vertex blowups then we can find such a blowup for \( k = 3 \) or 5 by Remark 3.2.4. This proves the final claim.

We are now ready to summarize the results of this subsection.

**Remark 5.1.16.** (i) In this section, we have shown that each polytope \( \overline{\Delta} \) described in case (a) of Theorem 1.1.1 is the blowup of a minimal polytope of type (a), and that each polytope \( \overline{\Delta} \) described in case (b) is the blowup of a minimal polytope of type (b). More specifically, Proposition 5.1.2 shows that polytopes of type (a1) with an essential mass linear function are minimal, while Proposition 5.1.11 shows that a polytope of type (a3) that admits an essential mass linear function can be obtained from a minimal polytope with the same properties by a series of blowups along symmetric 2-faces. In contrast, by Proposition 5.1.5 a polytope of type (a2) with an essential mass linear function may be the blowup of a minimal polytope along the intersection of the three asymmetric fiber facets – not one of the types permitted in Theorem 1.1.1. Finally, if we assume that \( H \) is inessential on \( \overline{\Delta} \), that the polytope \( \overline{\Delta} \) is the double expansion of a polygon, and that the asymmetric facets are the base-type facets as in case (b), then we may conclude from Proposition 5.1.14 that \( \overline{\Delta} \) is the blowup of a minimal polytope with the same properties. However, one blowup may be along the vertex formed by the intersection of the four asymmetric base-type facets, which is not one of the types permitted in Theorem 1.1.1.

(ii) By Remark 5.1.7 in the case (a2) necessary and sufficient conditions for \( \overline{\Delta} \) to be minimal depend on \( \kappa \). But this is not true in the cases (a3) and (b). In case (a3) the facets \( F_1, F_2, \) and \( F_3 \) have a very simple intersection pattern, which forces the troublesome condition (iii)
in Proposition 2.5.1 to hold. In case (b), on the other hand, there is no choice for the relevant components of \( \kappa \): in an expansion the components of \( \kappa \) corresponding to the base facets are determined by the structural constants of the fiber.

5.2. Full mass linearity. We now discuss a strengthening of the mass linear condition proposed by Shelukhin in [9]: namely, “fully mass linear.” In this subsection, we prove that every mass linear function on a polytope of dimension at most four also satisfies this stronger condition.

As we mentioned in the discussion just before Question 1.3.5, if \( H \) generates a loop that lies in the kernel of the homomorphism \( \pi_1(T) \to \pi_1(\text{Symp}(M_\Delta, \omega_\Delta)) \), then \( H \) is fully mass linear. Therefore the class of fully mass linear functions may be more natural than the class of mass linear functions.

**Definition 5.2.1** (Shelukhin [9]). Let \( \Delta \subset t^* \) be an \( n \)-dimensional polytope, and for \( k = 0, \ldots, n \) denote by \( B_k \) the barycenter (center of mass) of the union of the \( k \)-dimensional faces of \( \Delta \). Thus \( B_n(\Delta) = c_\Delta \) and \( B_0(\Delta) \) is the average of the vertices of \( \Delta \). Then \( H \in t \) is said to be **fully mass linear** on \( \Delta \) if

\[
\langle H, B_k(\Delta) \rangle = \langle H, B_n(\Delta) \rangle \quad \text{for all } k = 0, \ldots, n.
\]

Further, we say that \( H \in t \) is **generated by the vector** \( \xi_H \in t^* \) if

\[
\langle H, c_\Delta(\kappa) \rangle = \sum_{i=1}^N \langle \eta_i, \xi_H \rangle \kappa_i.
\]

The coordinates of \( B_0(\Delta) \) are linear functions of \( \kappa \). Thus, if \( \langle H, B_n(\Delta) \rangle = \langle H, B_0(\Delta) \rangle \) then \( H \) is mass linear on \( \Delta \); in particular, every fully mass linear function is mass linear. A priori, the converse may not hold. For example, the three barycenters \( B_0(\Delta), B_1(\Delta) \) and \( B_2(\Delta) \) of a generic trapezoid are distinct. However, we do not know any of counterexamples, and it does hold in dimension at most four.

**Proposition 5.2.2.** Every mass linear function on a polytope of dimension at most four is fully mass linear.

The converse also holds for inessential functions.

**Lemma 5.2.3.** Every inessential function \( H \in t \) on a polytope \( \Delta \subset t^* \) is fully mass linear and is generated by some vector \( \xi_H \).

**Proof.** It suffices to consider the case when \( H = \eta_i - \eta_j \) where \( F_i \sim F_j \). By Remark 2.1.10 since \( F_i \sim F_j \) there is a vector \( \xi_H \in t^* \) that is parallel to all facets except for \( F_i \) and \( F_j \). It follows easily that the affine reflection in the plane \( H = \eta_i - \eta_j = 0 \) preserves \( \Delta \). Thus all the barycenters \( B_k(\Delta) \) lie on the plane \( H = \text{const} \) that is fixed by this reflection. Moreover, the integrality conditions on the conormal vectors \( \eta_i \) imply that \( \langle \xi, \eta_i \rangle = -\langle \xi, \eta_j \rangle = 1 \); see part I, Lemma 3.4. Hence \( H \) is generated by \( \xi_H \). \( \square \)

We prove Proposition 5.2.2 by showing that all the mass linear pairs \( (\Delta, H) \) described in Proposition 2.1.15 and Theorem 1.1.1 are fully mass linear. The following basic result is taken from McDuff [6].
Lemma 5.2.4. Let $\Delta \subset t^*$ be an $n$-dimensional polytope and fix $H \in t$.

(i) The function $H$ is mass linear exactly if
$$\langle H, B_0(\Delta) \rangle = \langle H, B_{n-1}(\Delta) \rangle = \langle H, B_n(\Delta) \rangle.$$ 

(ii) There is a vector $\xi_H \in t^*$ that generates $H$ exactly if
$$\langle H, B_0(\Delta) \rangle = \langle H, B_{n-2}(\Delta) \rangle = \langle H, B_n(\Delta) \rangle.$$ 

Proof. Part (i) is proved in [6, Proposition 4.7], while (ii) holds by [6, Remark 4.10]. □

Corollary 5.2.5. If $H \in t$ is mass linear on a polytope $\Delta \subset t^*$ then $\sum \gamma_i = 0$, where $\langle H, c_\Delta(\kappa) \rangle = \sum \gamma_i \kappa_i$.

Proof. By Lemma 5.2.4 if $H$ is mass linear then $\langle H, B_{n-1}(\Delta) \rangle = \langle H, c_\Delta(\kappa) \rangle$. It is shown in [6, Lemma 4.5] that this implies that $\sum \gamma_i = 0$. □

Remark 5.2.6. Since inessential functions have the property that $\sum \gamma_i = 0$ one could prove this corollary by induction on the dimension provided that Question 1.3.1 has a positive answer. For then, after subtracting an inessential function, we can assume that every mass linear $H$ has a symmetric facet $G$ and use the fact that the coefficients of $H$ are the same as those for $H|_G$. This is the approach taken in Part I to prove $\sum \gamma_i = 0$ in dimension 3 (cf. Proposition 2.1.15), and by [I, Theorem A.9] it works also in dimension 4.

Lemma 5.2.7. Let $H \in t$ be a mass linear function on an $n$-dimensional polytope $\Delta \subset t^*$, where $\Delta$ is a $\Delta_k$ bundle over $\Delta_1$, a 121-bundle, or a $\Delta_2$-bundle over a polygon $\Delta$. Then
$$\langle H, B_{n-2}(\Delta) \rangle = \langle H, B_n(\Delta) \rangle.$$ 

Proof. By Lemma 5.2.4, it suffices to show that $H$ is generated by some $\xi_H \in t$. Moreover, recall from Lemma 5.2.3 that every inessential function $H' \in t$ is generated by some vector in $t$. Hence, we may subtract any convenient inessential function. We now check case by case that there is a suitable vector $\xi_H$.

First suppose that $\Delta$ is a $\Delta_k$ bundle $Y$ over $\Delta_1$, as in (3.1). By Proposition 3.1.1 after possibly subtracting an inessential function, we may assume that $H = \sum_{i=1}^{k+1} \gamma_i \eta_i$, where $\sum_{i=1}^{k+1} \gamma_i = \sum_{i=1}^{k} a_i \gamma_i = 0$; moreover, $\langle H, c_\Delta(\kappa) \rangle = \sum_{i=1}^{k+1} \gamma_i \kappa_i$. Therefore, if
$$\xi_H := -(\gamma_1, \ldots, \gamma_k, 0),$$
then $\langle H, c_\Delta(\kappa) \rangle = \sum_{i=1}^{k+3} \langle \eta_i, \xi_H \rangle \kappa_i$, that is, $H$ is generated by $\xi_H$.

Next, let $\Delta$ is a 121-bundle, as in (3.4). By Proposition 3.1.6, we may assume that $H = \sum_{i=2}^{4} \gamma_i \eta_i$, where $\gamma_2 + \gamma_3 + \gamma_4 = a_2 \gamma_2 + a_3 \gamma_3 = 0$; moreover, $\langle H, c_\Delta(\kappa) \rangle = \sum_{i=2}^{4} \gamma_i \kappa_i$. Therefore, $H$ is generated by $\xi_H = (0, -\gamma_2, -\gamma_3, 0)$.

Finally, let $\Delta$ be a $\Delta_2$ bundle over a polygon $\Delta$, as in (3.6). By Proposition 3.1.8, we may assume that $H = \sum_{i=1}^{3} \gamma_i \eta_i$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $b_1^j \gamma_1 + b_2^j \gamma_2 = 0$ for each edge $e_j$ of $\tilde{\Delta}$; moreover, $\langle H, c_\Delta(\kappa) \rangle = \sum_{i=1}^{3} \gamma_i \kappa_i$. Hence, $H$ is generated by $\xi_H = (-\gamma_1, -\gamma_2, 0)$. □

Corollary 5.2.8. Every mass linear function on a polytope of dimension at most three is fully mass linear.
Proof. By Lemma 5.2.3, every inessential function is fully mass linear. Moreover, by Propositions 2.1.14 and 2.1.15, the only polytope of dimension at most three that supports an essential mass linear function $H$ is a $\Delta_2$ bundle over $\Delta_1$. Hence, the claim follows from part (i) of Lemma 5.2.4 and the first case of Lemma 5.2.7. □

Given a set of edges $E$ of $\Delta$, we denote by $B_1(E)$ the corresponding barycenter.

Lemma 5.2.9. Let $H \in t$ be a mass linear function on an $n$-dimensional polytope $\Delta$. Let $E$ be the set of edges that lie on at least one symmetric facet. Assume that $H|_f$ is fully mass linear on $f$ for every symmetric face $f$. Then

$$\langle H, B_1(E) \rangle = \langle H, B_n(\Delta) \rangle.$$

Proof. Let $f$ be any symmetric $k$-face. Since $\langle H, B_1(f) \rangle = \langle H, B_k(f) \rangle$ by assumption, Proposition 2.1.5 implies that $\langle H, B_1(f) \rangle = \langle H, B_n(\Delta) \rangle$. By induction on $k$, this implies that $\langle H, B_1(\mathcal{E}_f) \rangle = \langle H, B_n(\Delta) \rangle$, where $\mathcal{E}_f$ is the set of edges that lie on $f$ but do not lie on any smaller symmetric face. The result follows immediately. □

Lemma 5.2.10. Let $H \in t$ be a mass linear function on an $n$-dimensional polytope $\Delta \subset t^*$. If $\Delta$ is a $\Delta_k$ bundle over $\Delta_1$, a 121-bundle, or a $\Delta_2$-bundle over a polygon $\hat{\Delta}$, then

$$\langle H, B_1(\Delta) \rangle = \langle H, B_n(\Delta) \rangle.$$

Proof. As before, Lemma 5.2.3 implies that we may subtract any convenient inessential function.

First suppose that $\Delta$ is a $\Delta_k$ bundle $Y$ over $\Delta_1$, as in (3.1). By Proposition 3.1.1, after possibly subtracting an inessential function, we may assume that

$$H = \sum_{i=1}^{k+1} \gamma_i \eta_i, \quad \text{where} \quad \sum_{i=1}^{k+1} \gamma_i = \sum_{i=1}^k a_i \gamma_i = 0;$$

moreover $\langle H, c_\Delta \rangle = \sum_{i=1}^{k+1} \gamma_i \kappa_i$. In particular, in coordinates we have

$$H = (\gamma_{k+1} - \gamma_1, \cdots, \gamma_{k+1} - \gamma_k, 0).$$

Divide the edges of $Y$ into two groups $\mathcal{E}_1$ and $\mathcal{E}_2$, where $\mathcal{E}_1$ consists of those edges that lie in one of the base facets, and $\mathcal{E}_2$ consists of the remaining edges, which are parallel to the last coordinate axis $e_{k+1}$. The base facets are symmetric. Therefore, by Lemma 5.2.9,

$$\langle H, B_1(\mathcal{E}_1) \rangle = \langle H, B_n(\Delta) \rangle.$$

(5.3)

Let $\lambda = \sum_{i=1}^{k+1} \kappa_i$ and $h = \sum_{i=1}^k a_i \kappa_i + \kappa_{k+2} + \kappa_{k+3}$. There are $k + 1$ edges in $\mathcal{E}_2$, one over $v_0 = -(\kappa_1, \ldots, \kappa_k, \kappa_{k+2})$ of length $h$, and one over the vertex $v_i$ at $v_0 + \lambda e_i$ of length $h + a_i \lambda$ for all $1 \leq i \leq k$. Therefore,

$$B_1(\mathcal{E}_2) = v_0 + \frac{\lambda}{h(k + 1) + \lambda \sum a_i} (h + a_1 \lambda, \ldots, h + a_k \lambda, x),$$
for some constant $x$. Thus,

$$(5.4) \quad \langle H, B_1(\mathcal{E}_2) \rangle = \sum_{i=1}^{k+1} \gamma_i \kappa_i = \langle H, B_n(\Delta) \rangle.$$ 

Together, (5.3) and (5.4) imply that $\langle H, B_1(\Delta) \rangle = \langle H, B_n(\Delta) \rangle$, as required.

Next, suppose that $\Delta$ is a 121-bundle as in (3.4). By Proposition 3.1.6, after possibly subtracting an inessential function, we may assume that the $\Delta$ has only three asymmetric facets: $F_2', F_3'$, and $F_4'$. Since these facets do not intersect, every edge of $\Delta$ lies on at least one symmetric facet. Moreover, the restriction $H|_f$ is fully mass linear on every symmetric face $f$ by Corollary 5.2.8. Therefore, the claim follows immediately from Lemma 5.2.9.

Finally, assume that $\Delta$ is a $\Delta_2$ bundle over $\bar{\Delta}$. By Proposition 3.1.8, after possibly subtracting an inessential function we may again assume that $\Delta$ has only three asymmetric facets, and that these facets do not intersect. The argument follows exactly as above. □

Lemma 5.2.11. Let $H$ be a totally mass linear function on a polytope $\Delta$ and suppose that $\Delta'$ is a blowup of $\Delta$ either of type $(F_{ij}, g)$, or along a symmetric face $f$ such that $H|_f$ is inessential. Then $H$ is totally mass linear on $\Delta'$.

Proof. Let $\Delta'$ be the blow up of $\Delta$ along a face $f = F_I$. Write $\Delta = \Delta' \cup W$, where $F'_0 := W \cap \Delta'$ is the exceptional divisor. Then $W$ is a $\Delta|_f$ bundle over $f$.

We claim that

1. $\langle H, c_W(\kappa) \rangle = \langle H, c_{\Delta}(\kappa) \rangle$, and
2. the restriction of $H$ to $W$ is inessential.

If $f$ is symmetric, we saw in the proof of Lemma 2.4.8 that $\langle H, c_f(\kappa) \rangle = \langle H, c_{\Delta}(\kappa) \rangle$ and the restriction of $H$ to $W$ is the lift of $H|_f$ from $f$ to $W$. Hence, both claims follow from Proposition 2.2.3. On the other hand, if $\Delta'$ is a blowup of type $(F_{ij}, g)$, then both claims are explicitly proved in the penultimate paragraph of Proposition 2.4.10.

In either case, by Proposition 2.1.5 and Lemma 2.4.6 claim (1) implies that

$$\langle H, B_n(\Delta') \rangle = \langle H, B_n(\Delta) \rangle = \langle H, B_n(W) \rangle = \langle H, B_{n-1}(F'_0) \rangle.$$ 

Moreover, since $F'_0$ is symmetric, claim (2) and Remark 2.1.6 imply that $H$ is also inessential on $F'_0$, and so $H$ is fully mass linear on both $W$ and $F'_0$ by Lemma 5.2.4. Hence, the equation above implies that

$$\langle H, B_n(\Delta') \rangle = \langle H, B_1(\Delta) \rangle = \langle H, B_1(W) \rangle = \langle H, B_1(F'_0) \rangle.$$ 

In either case, let us first consider the effect of blowing up on $\langle H, B_1(\Delta) \rangle$. Each edge of $\Delta$ which does not meet $f$ is an edge of $\Delta' \setminus F'_0$. The edges of $\Delta \setminus f$ which meet $f$ are cut by the hyperplane $P(F'_0)$ containing $F'_0$ into two pieces, one of which is an edge of $W \setminus F'_0$, and the other an edge of $\Delta' \setminus F'_0$. Finally, each edge of $f$ is an edge of $W \setminus F'_0$. The remaining edges of $W$ and $\Delta'$ lie in $F'_0$. Therefore

$$V_1(\Delta) + 2V_1(F'_0) = V_1(\Delta') + V_1(W),$$
where $V_1(X)$ denotes the sum of the lengths of the edges of $X$. Moreover, by the additivity of the $H$-moment, we have

$$
\langle H, B_1(\Delta) \rangle V_1(\Delta) + 2\langle H, B_1(F'_0) \rangle V_1(F'_0) = \langle H, B_1(\Delta') \rangle V_1(\Delta') + \langle H, B_1(W) \rangle V_1(W).
$$

Since $V_1(\Delta') \neq 0$, the last three displayed equations above imply that $\langle H, B_1(\Delta') \rangle = \langle H, B_1(\Delta') \rangle$. A nearly identical argument – but with $i$-faces instead of edges – implies that $\langle H, B_{i}(\Delta') \rangle = \langle H, B_{i}(\Delta') \rangle$ for all $i$.

**Proof of Proposition 5.2.2.** This holds by combining Theorem 1.1.1 with Lemmas 5.2.3, 5.2.7, 5.2.10 and 5.2.11. Note that we can always apply Lemma 5.2.11 because Proposition 2.1.14 implies that the restriction of $H$ to a symmetric 2-face is inessential. \qed

### 5.3. Blowups and mass linearity.

We end with a general discussion about the effect of blowing up on mass linearity. Lemma 2.4.7 shows that if a mass linear function on a polytope remains mass linear on its blowup along a face $f$, the face $f$ must meet all the asymmetric facets. The following example shows that this condition is not sufficient.

**Example 5.3.1.** Let $\Delta_4$ be the 4-simplex and let $H = \sum_{i=1}^{5} \gamma_i \eta_i$, where $\sum_{i=1}^{5} \gamma_i = 0$ and $\eta_1, \ldots, \eta_5$ are the outward conormals to the facets of $\Delta_4$. The blowup $\Delta'$ of $\Delta_3$ along the edge $F_{123}$ is a $\Delta_2$ bundle over $\Delta_2$ with base facets $F'_1, F'_2,$ and $F'_3$, where $F'_i = F_i \cap \Delta'$ for all $i$. Then $H$ is mass linear on $\Delta_4$ and $F_{123}$ meets all asymmetric facets. However, by Proposition 3.1.8 $H$ is mass linear on $\Delta'$ exactly if $\gamma_1 + \gamma_2 + \gamma_3 = 0$.

In the above example it is enough to add the condition $\gamma_1 + \gamma_2 + \gamma_3 = 0$. However, to get a general result we need yet more conditions.

**Lemma 5.3.2.** Let $H \in t$ be mass linear on a polytope $\Delta \subset t^*$. Let $\Delta'$ be the blowup of $\Delta$ along a face $F_I$ which meets every asymmetric facet of $\Delta$, and assume that $\sum_{i \in I} \gamma_i = 0$. Write $\Delta = \Delta' \cup W$; if $F_I$ is a simplex and $H$ is mass linear on $W$, then $H$ is mass linear on $\Delta'$.

**Proof.** We aim to show that $\langle H, c_W \rangle$ is equal to $\langle H, c_\Delta \rangle$. The result then follows from Lemma 2.4.6.

Let $\{F_\ell\}_{\ell \in L}$ be the set of facets that meet $F_I$, and let $\eta_\ell$ be the outward conormal to $F_\ell$ for all $\ell$. Since by hypothesis every asymmetric facet meets $F_I$, we may write $\langle H, c_\Delta \rangle = \sum_{\ell \in L} \gamma_\ell \kappa_\ell$. Because $\sum_{\ell \in L} \gamma_\ell = 0$ by Corollary 5.2.5, our hypotheses imply that

$$
\sum_{i \in I} \gamma_i = \sum_{j \in J} \gamma_j = 0,
$$

where $J = L \setminus I$.

There is a facet $F'_0$ of $W$ with outward conormal $\eta_0 = -\sum_{i \in I} \eta_i$ (corresponding to the exceptional divisor in $\Delta'$); the remaining facets of $W$ are $\{F_\ell \cap W\}_{\ell \in L}$. Since $H$ is mass linear on $W$, we may write $\langle H, c_W \rangle = \gamma_0 \kappa_0 + \sum_{\ell \in L} \gamma_\ell \kappa_\ell$. Additionally, observe that $W$ is a $\Delta_{|I|}$ bundle over $F_I$; cf. Remark 2.4.2 Its fiber facets are $F_0$ and $\{F_\ell \cap W\}_{i \in I}$; its base...
facets are \( \{ F_j \cap W \}_{j \in J} \). Therefore there is precisely one linear relation among the conormals \( \{ \eta_\ell \}_{\ell \in L} \); it has the form

\[
\sum_{j \in J} \eta_j = \sum_{i \in I} a_i \eta_i.
\]

Moreover, by Proposition 2.2.4 and Corollary 5.2.5 we have

\[
\gamma_0 + \sum_{i \in I} \gamma'_i = \sum_{j \in J} \gamma'_j = 0.
\]

Next, note that by Lemma 2.1.3, \( H = \sum_{\ell \in L} \gamma_\ell \eta_\ell \) and \( H = \gamma'_0 \eta_0 + \sum_{\ell \in L} \gamma'_\ell \eta_\ell \). Hence, since \( \eta_0 = -\sum_{i \in I} \eta_i \), if we use Equation 5.5 to write \( \eta_0 = -\sum_{j \neq j_0} \eta_j + \sum_{i \in I} a_i \eta_i \) for some \( j_0 \in J \), we see that

\[
\sum_{i \in I} (\gamma_i + a_i \gamma_{j_0}) \eta_i + \sum_{j \in J \setminus \{ j_0 \}} (\gamma_j - \gamma_{j_0}) \eta_j = \sum_{i \in I} (\gamma_i' - \gamma_0 + a_i \gamma_{j_0}') \eta_i + \sum_{j \in J \setminus \{ j_0 \}} (\gamma_j' - \gamma_{j_0}') \eta_j.
\]

Since the vectors \( \{ \eta_\ell \}_{\ell \in L \setminus \{ j_0 \}} \) are linearly independent, this implies that \( \gamma_j - \gamma_{j_0} = \gamma'_j - \gamma'_{j_0} \) for all \( j \in J \), and \( \gamma_i + a_i \gamma_{j_0} = \gamma_i' - \gamma_0 + a_i \gamma_{j_0}' \) for all \( i \). Since \( \sum_{j \in J} \gamma_j = 0 = \sum_{j \in J} \gamma_j' \), the first equation implies that \( \gamma_j = \gamma'_j \) for all \( j \in J \). Hence, since \( \sum_{i \in I} \gamma_i = 0 = \gamma'_0 + \sum_{i \in I} \gamma'_i \), the second equation implies that \( \gamma'_0 = 0 \) and \( \gamma_i = \gamma'_i \) for all \( i \in I \). Therefore \( \langle H, c_W \rangle = \langle H, c_\Delta \rangle \) as claimed.

The difficulty now is to understand when the restriction of \( H \) to \( W \) is mass linear. Here is a simple example.

**Corollary 5.3.3.** Let \( H \) be a mass linear function on \( \Delta \) and \( v \) any vertex of \( \Delta \). Then \( H \) is mass linear on the blowup \( \Delta' \) of \( \Delta \) at \( v \) exactly if \( v \) lies on all asymmetric facets of \( \Delta \). Moreover, if \( H \) is inessential on \( \Delta \) then it is inessential on the blowup.

**Proof.** If \( F_I \) is a vertex then \( W \) is a simplex so that all \( H \) are mass linear on \( W \). Moreover, the equality \( \sum_{i \in I} \gamma_i = 0 \) holds by Corollary 5.2.5. Therefore if \( v \) lies on all asymmetric facets \( H \) is mass linear on \( \Delta' \) by Lemma 5.3.2. The converse follows from Lemma 2.4.7. This proves the first claim. The second follows from Lemma 2.4.12.

Now consider the case when \( F_I \) is an edge that meets every asymmetric facet, and assume that \( \sum_{i \in I} \gamma_i = 0 \). Then \( W \) is a \( \Delta_{|I|} \) bundle over \( \Delta_I \), and so Proposition 3.1.1 implies that \( H \) is mass linear on \( W \) (and hence \( \Delta' \)) exactly if \( \sum_{i \in I} a_i \gamma_i = 0 \). It turns out that this condition, which involves the relative slope of the two facets transverse to \( F_I \), is satisfied whenever \( H \) is generated by \( \xi_H \) in the sense of Definition 5.2.1.

**Lemma 5.3.4.** Let \( H \in t \) be a mass linear function on a polytope \( \Delta \subset t^* \). Let \( \Delta' \) be the blowup of \( \Delta \) along an edge \( F_I \) which meets every asymmetric facet of \( \Delta \), and assume that \( \sum_{i \in I} \gamma_i = 0 \). If \( H \) is generated by some \( \xi_H \in t^* \), then \( H \) is mass linear on \( \Delta' \), and is generated on \( \Delta' \) by the same vector \( \xi_H \).
Proposition 5.3.7. Let $\eta_{n+1} = (a_1, \ldots, a_{n-1}, 1)$ for some $a \in \mathbb{Z}^{n-1}$. Moreover, these facets, together with the facet $F_0$ corresponding to the exceptional divisor of $\Delta'$, are the facets of $W$.

By Lemma 2.1.3, $H = \sum_{i \leq n+1} \gamma_i \eta_i$. By assumption, $\gamma_i = \langle \eta_i, \xi_H \rangle$ for all $i$; therefore, $\xi_H = (-\gamma_1, \ldots, -\gamma_{n-1}, -\gamma_n)$. Moreover, since $\sum_{i \leq n-1} \gamma_i = 0$ by hypothesis and $\sum_i \gamma_i = 0$ by Corollary 5.2.5, we have

$$0 = \gamma_n + \gamma_{n+1} = \langle \eta_n + \eta_{n+1}, \xi_H \rangle = -\sum_{i \in I} \gamma_i a_i.$$

Thus $\langle H, c_W \rangle = \sum_{i=1}^{n+1} \gamma_i \kappa_i$ by Proposition 3.1.1. Hence Lemma 5.3.2 implies that $\langle H, c_{\Delta'} \rangle = \sum_{i=1}^{n+1} \gamma_i \kappa_i$. The result follows. \hfill $\square$

Corollary 5.3.5. Let $H$ be mass linear on a 4-dimensional polytope $\Delta$ and let $\Delta'$ be its blowup along an edge $F_{123}$ that meets all asymmetric facets. Then $H$ is mass linear on $\Delta'$ if $\gamma_1 + \gamma_2 + \gamma_3 = 0$.

Proof. By Proposition 5.2.2 in dimensions $\leq 4$ every mass linear function is fully mass linear. Therefore, by Lemma 5.2.4(ii) every mass linear $H$ is generated by some $\xi_H$. \hfill $\square$

Remark 5.3.6. To go further with this question one would obviously need to understand more about mass linear functions on the polytopes $W$. One could also consider the question of which blowups preserve full mass linearity. For example, if one blows up at a vertex then $W$ is a simplex and all linear functions on a simplex are inessential and hence fully mass linear. In this case the proof of Lemma 5.2.11 adapts to show that a vertex blow up preserves full mass linearity. We leave further discussion of such questions for the future.

Another interesting question concerns which blowups convert inessential functions to essential ones. We end by showing (in any dimension) that if a blow up of type $(F_{ij}, g)$ has this property, then the underlying polytope is a double expansion.

Proposition 5.3.7. Let $H \in \mathfrak{t}$ be an inessential mass linear function on a smooth polytope $\overline{\Delta} \subset \mathfrak{t}^*$. Assume that $H$ is an essential mass linear function on a polytope $\Delta$ that is obtained from $\overline{\Delta}$ by a series of blowups. Moreover, assume that each blowup is either along a symmetric face or of type $(F_{ij}, g)$. Then

- $\overline{\Delta}$ is the double expansion of a smooth polytope $\tilde{\Delta}$.
- The four base-type facets are the asymmetric facets.

Proof. By Lemma 2.4.8(i) and Proposition 2.4.10(ii), $H$ is mass linear on each intermediate blowup, the exceptional divisors are all symmetric, and the coefficients $\gamma_k$ remain constant under blowup. Since $H$ is essential on $\Delta$ but not on $\overline{\Delta}$, there exists a polytope $\Delta'$ in the sequence so that $H$ is inessential on $\Delta'$ but essential on the blowup. Lemma 2.4.8(ii) implies that this blow up must be of the form $(F_{ij}, g')$. Moreover, Proposition 2.4.10(iii) implies that $\Delta'$ has exactly four asymmetric facets and that $F_1' \not\sim F_2'$. Since $H$ is inessential on $\Delta'$, we may label the asymmetric facets so that $F_1 \sim F_2$ and $F_3 \sim F_4$. Hence, $i \in \{1, 2\}$
and \( j \in \{3, 4\} \). Since \( F'_{ij} \cap \mathcal{F} \) meets every asymmetric facet, this implies that \( F_{12} \neq \emptyset \) and \( F_{34} \neq \emptyset \). Therefore, the claim follows from Lemma 2.3.7.

\[\Box\]

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