Two-loop computation of a finite volume running coupling on the lattice

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In pure SU(2) gauge theory we compute the two-loop coefficient in the relation between the lattice bare coupling and the running coupling defined through the Schrödinger functional. This result is required to relate the latter to the \( \overline{\text{MS}} \)-coupling in our programme to compute \( \alpha_s \). In addition it allows us to implement O(a) improvement of the Schrödinger functional to two-loop order. The two-loop \( \beta \)-function is verified in a perturbative computation on the lattice, and the behavior of expansions in the standard and in the Parisi-improved bare couplings are investigated beyond one loop.

1. CONTEXT OF THE CALCULATION

In this contribution we report on a perturbative 2-loop computation of the running coupling constant \( g(L) \), defined through the Schrödinger functional \[ \Gamma \], in terms of the lattice bare coupling \( g_0 \). Such a result is required as part of our project of computing \( \alpha_s \) for QCD. The strategy there is to first numerically construct the continuum limit of \( g \) for some large values of \( L^{-1} \) in physical units. These are then connected with the more conventional \( g_{\overline{\text{MS}}} \) by renormalized perturbation theory. This is expected to be a well-behaved application of the series, as the scales \( L^{-1} \) and \( q \) are of the same order and can be chosen well above the nonperturbative scales. With growing statistical accuracy, especially on parallel computers, the 2-loop coefficient in this conversion is required to achieve a reasonable balance and control of the systematic errors from truncating the series. The connection between \( \overline{\Gamma} \) and \( g_{\overline{\text{MS}}} \) is established by expressing both of them in terms of the lattice bare coupling \( g_0 \). In this way, apart from avoiding problems of a direct connection via dimensional regularization in the presence of a finite background field, we have the additional benefit of getting the two-loop O(a) improvement coefficient to speed up the continuum limit \[ 3 \]. Here we present the results of a first such computation of \( \overline{\Gamma} \) for SU(2) pure gauge theory \[ 4 \]. For the connection with \( g_{\overline{\text{MS}}} \) see Peter Weisz’s contribution to these proceedings.

2. FINITE VOLUME RUNNING COUPLING

The definition of \( \overline{\Gamma} \) from the Schrödinger functional is discussed in more detail in \[ 1 \] and we are brief here. The effective action \( \Gamma \) is given by the path integral over SU(2) gauge fields \( U(x, \mu) \) on a lattice with extension \( L \) in all four directions,

\[
e^{-\Gamma} = \int D[U] \ e^{-S(U)}. \tag{1}
\]

While space is periodic, the fixed boundary values in the time direction, \( U(x, k)|_{x^0=0} = \exp\{\eta \tau_3 / iL\} \) and \( U(x, k)|_{x^0=L} = \exp\{(\pi - \eta) \tau_3 / iL\} \), induce an \( \eta \)-dependent abelian background field. The action \( S \) is the standard Wilson action modified for the surfaces at \( x^0 = 0, L \),

\[
S[U] = \frac{1}{g_0^2} \sum_p w(p) \text{tr}\{1 - U(p)\}, \tag{2}
\]

where the sum runs over all oriented plaquettes \( p \). The weight \( w(p) \) is unity except for the electromagnetic plaquettes touching the fixed field boundary, where it equals

\[
c_t(g_0) = 1 + c_t^{(1)} g_0^2 + c_t^{(2)} g_0^4 + \cdots. \tag{3}
\]

Its tuning allows for the suppression of O(a) artifacts arising from the presence of the boundaries. Following Symanzik, we perturbatively determine the coefficients in

\[
c_t(g_0) = 1 + c_t^{(1)} g_0^2 + c_t^{(2)} g_0^4 + \cdots. \tag{4}
\]
The response
\[ \tilde{g}^2(L) = \frac{\Gamma_0}{\Gamma_0^\eta=\pi/4} \]
defines \( \tilde{g} \), which runs with the finite system size \( L \) as the only scale in the problem. Here \( \Gamma \) denotes \( \eta \)-derivatives, and \( \Gamma_0 \) is the exactly known classical limit \((g_0 \to 0)\) of \( \Gamma \).

3. PERTURBATIVE CALCULATION

A suitable gauge fixing with the introduction of ghosts and a systematic expansion of the fluctuations of the \( U \)-field around the induced background lead to a well-defined lattice perturbation expansion. In particular, all modes are quadratically damped with the present boundary conditions. In the resulting expansion
\[ \tilde{\Gamma}^2(L) = g_0^2 + m_1(L/a)g_0^4 + m_2(L/a)g_0^6 + \cdots, \]
the coefficient \( m_1 \) was given in and, at 2 loop order, \( m_2 \) is reported here and in more detail in . Evaluation of (the \( \eta \)-dependence of) the order of 10 vacuum 2-loop diagrams is required for \( m_2 \). The computation is involved, because all propagators have to be computed numerically with the finite background field in place, and because the 3- and 4-gluon vertices have a large number (~100) of terms. Hence the diagrams are summed numerically for \( L/a \leq 32 \). To get to this size on workstations in a reasonable time, all symmetries were used eventually to reduce the number of terms. On smaller lattices they were however checked to hold first. As further error checks independence of a continuous gauge fixing parameter was verified, and the two authors of carried out independent calculations for the smaller lattices before optimizing one of the codes.

The resulting column of values of \( m_2 \) vs. \( L/a \) was analyzed as described in . The asymptotic result is
\[ m_1 = 2b_0 \ln(L/a) + 0.20235 + O(a^2/L^2) \]
\[ m_2 = m_1^2 + 2b_1 \ln(L/a) + 0.01607 + O(a^2/L^2). \]
Here \( b_0, b_1 \) are the universal coefficients of the \( \beta \)-function
\[ b_0 = \frac{11}{24\pi^2}, \quad b_1 = \frac{17}{96\pi^4}. \]
The value of \( b_1 \) was confirmed in our analysis of \( m_2(L/a) \) to about 1 part in \( 10^3 \), and to our knowledge this is the first such check on the lattice. The lattice artifacts of \( O(a) \) have been canceled in \( m_1, m_2 \). This requirement fixes the improvement coefficients
\[ c_s^{(1)} = -0.0543(5), \quad c_s^{(2)} = -0.0115(5). \]

4. EXPANSION IN THE BARE COUPLING AT TWO LOOPS

As mentioned, the result of the previous section is needed to connect \( \tilde{g} \) and \( g_0 \) via \( g_0 \). As a byproduct, we have the opportunity to study the quality of the lattice perturbation expansion in \( g_0 \) for a physical quantity. Eqs. (5), (6), (8) can be used to relate \( \tilde{g} \) at a scale \( a/s, s = O(1) \), in the continuum with the bare lattice coupling \( g_0 \) associated with the cutoff \( a \) in the regularization in use. On the one hand, no large logarithms are expected here as scales of the same order are connected. On the other hand, the expansion involves non-universal lattice quantities and cannot be systematically improved for fixed physical scale by approaching the continuum limit. The precise derivation of the relation, although simple in result, is slightly tricky to argue . One first uses (5) in a regime close to the continuum limit, where \( g_0^3, g_0^5 \ln(L/a) \) are small in the expansion, and \( a^2/L^2 \) is negligible. The resulting \( \tilde{\Gamma}^2(L) \) is re-expanded in terms of \( \tilde{\Gamma}^2(a/s) \) by using continuum perturbation theory. This procedure is formally equivalent to dropping powers of \( a/L \) in (5), (8) and then putting \( a/L = s \) under the logarithms.

In terms of \( \alpha(q) = \tilde{\Gamma}^2(L)/(4\pi), L = q^{-1} \), and \( \alpha_0 = g_0^2/(4\pi) \) one gets for \( s = 1 \)
\[ \alpha(a^{-1}) = \alpha_0 + 2.543 \alpha_0^3 + 9.00 \alpha_0^3 + O(\alpha_0^4). \]
The rather large coefficients here are consistent with the experience, that \( g_0 \) used in this way is not a good expansion parameter. An alternative way of using the series is to fix \( s \) such that there is no 1-loop term, and this leads to
\[ \alpha(8.83a^{-1}) = \alpha_0 + 1.287(1)\alpha_0^3 + O(\alpha_0^4). \]
Note that this choice simultaneously produces a small 2-loop coefficient, which is nontrivial and
leaves the chance, that the series may be better behaved with this rather large scale shift $s = 8.83$. Recently the use of modified bare couplings has become popular to try to amend the bare series \cite{ref6}. A simple proposal is \cite{ref7}

$$\tilde{\alpha}_0 = \alpha_0 / P,$$  

where $P$ is the average plaquette in infinite volume. Using the known series for $P$, one finds analogously to (12)

$$\alpha(1.17 a^{-1}) = \tilde{\alpha}_0 + 0.951(1)\tilde{\alpha}_0^3 + O(\tilde{\alpha}_0^4).$$  

(14)

Here the 2-loop coefficient is reasonable also, and the scale factor is much closer to one.

We now want to compare the expansions (12) and (14) with nonperturbative results for $\gamma$. We consider $\beta = 2.85$, where $\alpha_0$ and $\tilde{\alpha}_0$ are available, and the scale $a_{2.85}^{-1}$ is about 8 GeV if the SU(2) potential is matched with nature \cite{ref3}. The expansions produce $\gamma$ at $1.17 a_{2.85}^{-1}$ and $8.83 a_{2.85}^{-1}$, which, for the comparison, we both evolve to $10 a_{2.85}^{-1}$ with the numerically controlled $\beta$-function of $\gamma$ and negligible error. At this scale $\gamma^2$ itself is known numerically \cite{ref4}, and all values are collected in table 1. The 2-loop results are only off by about 1 and 2.5 error margins for $\tilde{\alpha}_0$ and $\alpha_0$ respectively.

| $\alpha$ method | \hline
| 0.1135(8) | nonperturbative |
| 0.1098 | 1-loop in $\alpha_0$ |
| 0.1115 | 2-loop in $\alpha_0$ |
| 0.1110 | 1-loop in $\tilde{\alpha}_0$ |
| 0.1128 | 2-loop in $\tilde{\alpha}_0$ |

When using the 1-loop term to fix the scale for which to apply bare perturbation theory, we find both a reasonably sized 2-loop term and approximate agreement with the “exact” result. Perhaps a bit surprisingly, this works for the ordinary bare lattice coupling in a qualitatively similar fashion as for $\tilde{\alpha}_0$, with just the scale factor being rather large. One has to be cautious, however, that this experience depends on the renormalized quantity (here $\gamma$) that is computed and need not be universally true.

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