SCHUBERT INDUCTION

RAVI VAKIL

ABSTRACT. We describe a Schubert induction theorem, a tool for analyzing intersections on a Grassmannian over an arbitrary base ring. The key ingredient in the proof is the Geometric Littlewood-Richardson rule of [V2].

Schubert problems are among the most classical problems in enumerative geometry of continuing interest. As an application of Schubert induction, we address several longstanding natural questions related to Schubert problems, including: the “reality” of solutions; effective numerical methods; solutions over algebraically closed fields of positive characteristic; solutions over finite fields; a generic smoothness (Kleiman-Bertini) theorem; and monodromy groups of Schubert problems. For example, we show that all Schubert problems for all Grassmannians are enumerative over the real numbers, completely answering the classical “reality question” for Schubert problems. These methods conjecturally extend to the flag variety.

CONTENTS

1. Introduction 1
2. Questions and answers 2
3. The main theorem, and its proof 7
4. Applications 11
5. Galois/monodromy groups of Schubert problems 11
References 21

1. INTRODUCTION

The Schubert induction theorem (Theorem 3.5) is a tool for studying intersections on a Grassmannian over an arbitrary base, by deformations. One motivation for such a result are Schubert problems, among the most classical problems in enumerative geometry of continuing interest. It is perhaps surprising that many natural questions about Schubert problems remain open. We describe these questions, and give some answers, in Section 2. Most applications conjecturally extend to the flag variety (Sect. 2.11). The theorem is stated and proved in Section 3 and the applications are shown in Sections 4 and 5.

Date: Saturday, April 12, 2003.

1991 Mathematics Subject Classification. Primary 14M15, 14N15; Secondary 14N10, 14C17, 14P99, 14Q10, 14G15, 14G27.

Partially supported by NSF Grant DMS–0228011, an AMS Centennial Fellowship, and an Alfred P. Sloan Research Fellowship.
1.1. Notation and philosophy. Fix a Grassmannian \( G(k, n) = \mathbb{G}(k-1, n-1) \) over a base field (or ring) \( K \). If \( \alpha \) is a partition, let \( \Omega_\alpha \in A^*(G(k, n)) \) denote the corresponding Schubert class. Let \( \Omega_\alpha(F) \) be the closed Schubert variety with respect to the flag \( F \). Let \( \Omega_\alpha(F) \subset G(k, n) \times Fl(n) \) be the universal Schubert variety.

Let \( \pi_i : G(k, n) \times Fl(n)^m \rightarrow G(k, n) \times Fl(n) \) denote the projection, where the projection to \( Fl(n) \) is from the \( i \)th \( Fl(n) \) of the domain. We will make repeated use of the following diagram.

\[
\pi_1^* \Omega_{\alpha_1}(F^1) \cap \pi_2^* \Omega_{\alpha_2}(F^2) \cap \cdots \cap \pi_m^* \Omega_{\alpha_m}(F^m) \rightarrow G(k, n) \times Fl(n)^m
\]

Questions about Schubert problems often reduce to questions about the morphism \( S \), and in particular about \( S^{-1}(p) \) where \( p \) is a general point of \( Fl(n)^m \). Schubert induction involves studying this question by specializing \( p \) through a carefully chosen sequence of codimension 1 degenerations, ending with \( p \) being a totally degenerate point (where the \( m \) flags coincide). In each application, the property in question will behave well under these degenerations. The flavor of the statement is that if something is true for \( m = 1 \), then it is true for all \( m \). This philosophy is described in detail in Section 3.1.

As an informal but illustrative example, consider the statement “there is a means of combinatorially computing the number of preimages of \( p \)”. Then Schubert induction in this case amounts to the checker-tournament method of solving Schubert problems described in [V2, Sect. 3.12]. The base case is the trivial statement “there is one point in the zero-dimensional Schubert variety”.

1.2. Acknowledgments. I am grateful to F. Sottile for advice and discussions. His philosophy is clearly present in this paper. In particular, the phrase “Schubert induction” appeared first in [S6, Sect. 1], although with a slightly different meaning. I thank A. Buch and A. Knutson for introducing me to this subject. I also thank B. Sturmfels for suggesting that [V2] might have numerical applications (see Sect. 2 Question 5), D. Allcock for discussions on the symmetric group, B. Poonen for advice on the Chebotarev density theorem, and W. Fulton for improving the exposition of the main result. I am grateful to H. Derksen for pointing out (using the theory of quivers) that the Galois/monodromy group of Schubert problems is sometimes not the full symmetric group, and for producing explicit examples.

2. Questions and answers

Given a partition \( \alpha \), the condition (i.e. element of \( A^*(G(k, n)) \)) corresponding to \( \Omega_\alpha(F) \) is called a Schubert condition. A Schubert problem is the following:
2.1. Schubert problem. Given \( m \) Schubert conditions \( \Omega_\alpha(F^i) \) with respect to general flags \( F^i \) (1 ≤ \( i \) ≤ \( m \)) whose total codimension is \( \dim G(k, n) \), what is the cardinality of their intersection?

In other words, how many \( k \)-planes satisfy various linear algebraic conditions with respect to \( m \) general flags? (Or: what is the cardinality of \( S^{-1}(F^1, \ldots, F^m) \) for general \( (F^1, \ldots, F^m) \subseteq Fl(n)^m \)?) This is the natural generalization of the classical problem: how many lines in \( \mathbb{P}^3 \) meet four (fixed) general lines? The points of intersection are called the solutions of the Schubert problem. (For clarity’s sake, we say that the number of solutions is the answer to the Schubert problem.) An immediate (if imprecise) follow-up is: What can one say about the solutions?

For example, if \( K = \mathbb{C} \), the answer to the Schubert problems for \( m = 3 \) are precisely the Littlewood-Richardson coefficients \( c_{\gamma, \beta} \).

Suppose the base field is \( K \), and \( \alpha_1, \ldots, \alpha_m \) are given such that \( \dim (\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m}) = 0 \). The corresponding Schubert problem is said to be enumerative over \( K \) if there are \( m \) flags \( F^1, \ldots, F^m \) defined over \( K \) such that \( S^{-1}(F^1, \ldots, F^m) \) consists of \( \deg (\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m}) \) (distinct) \( K \)-points.

2.2. The answer to this problem over \( \mathbb{C} \) is the prototype of the program in enumerative geometry. By the Kleiman-Bertini theorem [Kl], the Schubert conditions intersect transversely, i.e. at a finite number of reduced points. Hence the problem is reduced to one about the intersection theory of the Grassmannian. The intersection ring (the Schubert calculus) is known, using other interpretations of the Littlewood-Richardson coefficients in combinatorics or representation theory.

Yet many natural questions remain:

2.3. Reality questions. The classical “reality question” for Schubert problems [F1, p. 55], [F2, Ch. 13], [FP, Sect. 9.8] is:

**Question 1.** Are all Schubert problems enumerative over \( \mathbb{R} \)?

See [S1] [S6] for this problem’s history. The case \( G(1, n) \) (and \( G(n - 1, n) \)) is trivially linear algebra. Sottile proved the result for \( G(2, n) \) (and \( G(n - 2, n) \)) for all \( n \), [S2], and for all problems involving only Pieri classes [S5], see [S3] for further discussion. The case \( G(2, n) \), as well as that of conics, also follows from [V1].

This question can be fully answered with Schubert induction.

2.4. **Proposition.** — All Schubert problems for all Grassmannians are enumerative over \( \mathbb{R} \). Moreover, for a fixed \( m \), there is a set of \( m \) flags that works for all choices of \( \alpha_1, \ldots, \alpha_m \).

This argument carries through with \( \mathbb{R} \) replaced by any field satisfying the implicit function theorem, such as \( \mathbb{Q}_p \).
As noted in [V2, Sect. 4.11(g)], Eisenbud’s suggestion that the deformations of the Geometric Littlewood-Richardson rule are a degeneration of that arising from the osculating flag to a rational normal curve, along with this proposition, would imply that the Shapiro-Shapiro conjecture is true asymptotically. (See [EG] for the proof in the case $k = 2$.)

2.5. Enumerative geometry in positive characteristic. Enumerative geometry in positive characteristic is almost a stillborn field, because of the failure of the Kleiman-Bertini theorem. (Examples of the limits of our understanding are plane conics and cubics in characteristic 2 [Vn, Ber].) In particular, the Kleiman-Bertini Theorem fails in positive characteristic for all $G(k, n)$ that are not projective spaces (i.e. $1 < k < n - 1$) — Kleiman’s counterexample [Kl, ex. 9] for $G(2, 4)$ easily generalizes. D. Laksov and R. Speiser have developed a sophisticated characteristic-free theory of transversality [L, Sp, LSp1, LSp2], but it does not apply in this case [S7, Sect. 5].

**Question 2.** Are Schubert problems enumerative over an algebraically closed field of positive characteristic?

To answer this question, we first answer a logically prior one:

**Question 3.** Is there any patch to the failure of the Kleiman-Bertini theorem on Grassmannians?

A related natural question is:

**Question 4.** Are Schubert problems enumerative over finite fields?

We now answer all three questions. The appropriate replacement of Kleiman-Bertini is the following. We say a morphism $f : X \rightarrow Y$ is generically smooth if there is a dense open set $V$ of $Y$ and a dense open set $U$ of $f^{-1}(V)$ such that $f$ is smooth on $U$. If $X$ and $Y$ are varieties and $f$ is dominant, this is equivalent to the condition that the function field of $X$ is separably generated over the function field of $Y$.

2.6. Generic smoothness theorem. — The morphism $S$ is generically smooth. More generally, if $Q \subset G(k, n)$ is a subvariety such that $Q \cap \Omega_{\alpha}(F) \rightarrow Fl(n)$ is generically smooth for all $\alpha$, then

$$Q \cap \pi_{1}^{*}\Omega_{\alpha_{1}}(F) \cap \pi_{2}^{*}\Omega_{\alpha_{2}}(F) \cap \cdots \cap \pi_{m}^{*}\Omega_{\alpha_{m}}(F) \rightarrow Fl(n)^{m}$$

is as well.

This begs the following question: is the only obstruction to the Kleiman-Bertini theorem for $G(k, n)$ that suggested by Kleiman, i.e. whether the variety in question intersects a general translate of all Schubert varieties transversely? More precisely, is it true that for all $Q_1$ and $Q_2$ such that $Q_i \cap \Omega_{\alpha}(F) \rightarrow Fl(n)$ is generically smooth for all $\alpha$, and $i = 1, 2$, it follows that

$$Q_1 \cap \sigma(Q_2) \rightarrow PGL(n)$$

is also generically smooth, where $\sigma \in PGL(n)$?

Theorem 2.6 answers Question 3, and leads to answers to Questions 2 and 4:
2.7. Corollary. —

(a) All Schubert problems are enumerative for algebraically closed fields.

(b) For any prime $p$, there is a positive density of points $P$ defined over finite fields of characteristic $p$ where $S^{-1}(P)$ consists of $\text{deg} (\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m})$ distinct points. Moreover, for a fixed $m$, there is a positive density of points that works for all choices of $\alpha_1, \ldots, \alpha_m$.

Part (a) follows as usual (see Sect. 2.2). If $\dim (\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m}) = 0$, then Theorem 2.6 implies that $S$ is generically separable (i.e. the extension of function fields is separable). Then (b) follows by applying the Chebotarev density theorem for function fields to

$$\coprod_{\alpha_1, \ldots, \alpha_m} \pi_1^{\ast} \Omega_{\alpha_1} (F_1) \cap \pi_2^{\ast} \Omega_{\alpha_2} (F_2) \cap \cdots \cap \pi_m^{\ast} \Omega_{\alpha_m} (F_m) \longrightarrow FL(n)^m$$

(see for example [E, Lemma 1.2], although all that is needed is the curve case, e.g. [FJ, Sect. 5.4]).

Sottile has proved transversality for intersection of codimension 1 Schubert varieties [S7], and P. Belkale has recently proved transversality in general, using his proof of Horn’s conjecture [Bel, Thm. 0.9].

2.8. Effective numerical solutions (over $\mathbb{C}$) to all Schubert problems for all Grassmannians. Even over the complex numbers, questions remain.

Question 5. Is there an effective numerical method for solving Schubert problems (i.e. calculating the solutions to any desired accuracy)?

The case of intersections of “Pieri classes” was dealt with in [HSS]. For motivation in control theory, see for example [HV]. In theory, one could numerically solve Schubert problems using the Plucker embedding; however, this is unworkable in practice.

Schubert induction leads to an algorithm for effectively numerically finding all solutions to all Schubert problems over $\mathbb{C}$. The method will be described in [SVV], and the reasoning is sketched in Section 4.3.

2.9. Galois or monodromy groups of Schubert problems. The Galois or monodromy group of an enumerative problem measures three (related) things:

(a) (geometric) As the conditions are varied, how do the solutions permute?

(b) (arithemetic) What is the field of definition of the solutions, given the field of definition of the flags?

(c) (algebraic) What is the Galois group of the field extension of the “variety of solutions” over the “variety of conditions” (see (1))? 

The modern study of such problems was initiated by J. Harris in [H]; the connection between (a) and (c) is made there. The connection to (b) is via the Hilbert irreducibility theorem, as the target of $S$ is rational.

Question 6. What is the Galois group of a Schubert problem?
We partially answer this question. There is an explicit combinatorial criterion that implies that a Schubert problem has Galois group “at least alternating” (i.e. if there are \(d\) solutions, the group is \(A_d\) or \(S_d\)). This criterion holds over an arbitrary base ring. To prove it, we will discuss useful methods for analyzing Galois groups via degenerations. The criterion is quite strong, and seems to apply to all but a tiny proportion of Schubert problems. For example:

**2.10. Theorem.** — The Galois group of any Schubert problem on the Grassmannians \(G(2, n)\) \((n \leq 16)\) and \(G(3, n)\) \((n \leq 9)\) is either alternating or symmetric.

A short Maple program applying the criterion to a general Schubert problem is available upon request from the author.

One might reasonably expect that the Galois group of a Schubert problem is always the full symmetric group. However, this not the case. To our knowledge, the first examples are due to H. Derksen. In Section\[5.12\] we describe the smallest example (involving 4 flags in \(G(4, 8)\)), and determine (using the explicit checker criterion) that the Galois action is that of \(S_4\) on order 2 subsets of \(\{1, 2, 3, 4\}\). In Section\[5.14\] we give a family of examples with \(\left(\begin{array}{c}N \\ K\end{array}\right)\) solutions, with Galois group \(S_N\), and action corresponding to the \(S_N\)-action on order \(K\) subsets of \(\{1, \ldots, N\}\).

We also describe three-flag examples (i.e. corresponding to Littlewood-Richardson coefficients) with similar behavior (Sect.\[5.15\]). Littlewood-Richardson coefficients interpret structure coefficients of the ring of symmetric functions as the cardinality of some set. These three-flag examples show that the set has further structure, i.e. the objects are not indistinguishable. (More correctly, pairs of objects are not indistinguishable; this corresponds to failure of two-transitivity.)

This family of examples was independently found by Derksen. From his quiver-theoretic point of view, the smallest member of this family (in \(G(6, 12)\)) corresponds to the extended Dynkin diagram of \(E_6\), and the smallest member of the other family (in \(G(4, 8)\)) corresponds to the extended Dynkin diagram of \(D_4\).

**2.11. Flag varieties.** Conjecture 4.9 of [V2] would imply that the results of this paper except for those on Galois/monodromy groups apply to all Schubert problems on flag manifolds. In particular, as the conjecture is verified for \(n \leq 5\) [V2 Prop. 4.10], the results all hold in this range. For example:

**2.12. Proposition.** — All Schubert problems for \(Fl(n)\) are enumerative over any algebraically closed field or any field with an implicit function theorem (e.g. \(\mathbb{R}\)) for \(n \leq 5\). For a fixed \(m\), there is a set of \(m\) flags that works for all choices of \(\alpha_1, \ldots, \alpha_m\).

(The generalizations of the other statements in this paper are equally straightforward.)

We note that in the case of triple intersections where the answer is 1, Knutson has shown that the solution to the problem can be obtained by using spans and intersections of the linear spaces in the three flags [K], see also Purbhoo’s result [P].
3. The main theorem, and its proof

3.1. The key observation. Suppose \( f : X \to Y \) is a proper morphism of irreducible varieties that we wish to show has some property \( P \). We will require that \( P \) satisfy several conditions, including that it depend only on dense open subsets of the target (condition (A)). An example of such a property is “\( f \) is generically finite, and there is a Zariski-dense subset \( U \) of real points of \( Y \) for which \( f^{-1}(p) \) consists of \( \deg f \) real points for all \( p \in U \).”

Suppose \( D \) is a Cartier divisor of \( Y \) such that \( D \times X Y \) is reduced, and \( D \times X Y \to D \) has property \( P \). For good choices of \( P \) (call this condition on \( P \) (C)), such as the example, this implies that \( f \) has property \( P \).

This motivates the following inductive approach. Suppose

\[
X_0 = X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_s
\]

is a sequence of inclusions, where \( X_{i+1} \) is a Cartier divisor of \( X_i \). Suppose \( Y_{i,j} \) (\( 1 \leq i \leq s \), \( 1 \leq j \leq J_i \)) is a subvariety of \( Y \) such that \( f \) maps \( Y_{i,j} \) to \( X_i \), and \( Y_{i,j} \to X_i \) is proper, and for each \( 0 \leq i < s \), \( 1 \leq j \leq J_i \),

\[
Y_{i,j} \times_{X_i} X_{i+1} = \bigcup_{j' \in I_{i,j}} Y_{i+1,j'}
\]

for some \( I_{i,j} \subset J_{i+1} \), where each \( Y_{i+1,j'} \) appears with multiplicity one.

If

- \( Y_{i+1,j'} \to X_{i+1} \) has \( P \) for all \( j' \in I_{i,j} \) implies \( \bigcup_{j' \in I_{i,j}} Y_{i+1,j'} \to X_{i+1} \) has \( P \) (condition (B)), and
- \( Y_{s,j} \to X_s \) has \( P \) for all \( j \in J_s \) (condition (D), the base case for the induction),

then we may conclude that \( f : X \to Y \) has \( P \). (Note that \( Y \times_X X_s \to X_s \) may be badly behaved; hence the need for the inductive approach.)

The main result of this paper is that this process may be applied to the morphism \( S \). The key ingredient is the Geometric Littlewood-Richardson rule.

For some applications, we will need to refine the statement slightly. For example, to obtain lower bounds on monodromy groups, we will need the fact that \( I_{i,j} \) never has more than two elements.

3.2. Sketch of the Geometric Littlewood-Richardson rule \([V2]\). The key ingredient in the proof of the Schubert induction theorem \(3.5\) is the Geometric Littlewood-Richardson rule, which we sketch here.

The variety \( Fl(n) \times Fl(n) \) is stratified by the locally closed subvarieties with fixed numerical data. For each \((a_{ij})_{i,j \leq n} \), the corresponding subvariety is \( \{(F, F') : \dim F_i \cap F'_j = a_{ij} \} \). We denote such numerical data by the configuration \( \bullet \) (normally interpreted as a partition), and the corresponding locally closed subvariety by \( X_\bullet \).
The variety $G(k, n) \times Fl(n) \times Fl(n)$ is the disjoint union of “two-flag Schubert varieties”, locally closed subvarieties with specified numerical data. For each $(a_{ij}, b_{ij})_{i,j \leq m}$, the corresponding subvariety is $\{(F, F', V) : \dim F_i \cap F'_j = a_{ij}, \dim F_i \cap F'_j \cap V = b_{ij}\}$. We denote the data of the $(b_{ij})$ by $\circ$, so the locally closed subvarieties are indexed by the configuration $\circ\bullet$. Denote the corresponding two-flag Schubert variety by $X_{\circ\bullet}$. (Warning: the closure of a two-flag Schubert variety need not be a union of two-flag Schubert varieties [V2 Cor. 3.13(a)], so this is not a stratification.)

There is a specialization order $\bullet_{\text{init}} \ldots \bullet_{\text{final}}$ in the Bruhat order, corresponding to partial factorizations of the longest word [V2 Sect. 2.2]. If $\bullet \neq \bullet_{\text{final}}$ is in the specialization order, then let $\bullet_{\text{next}}$ be the next term in the order. We have $X_{\bullet_{\text{next}}} \subset X_{\bullet_{\text{init}}}$, $\dim X_{\bullet_{\text{next}}} = \dim X_{\bullet_{\text{init}}} - 1$, $X_{\bullet_{\text{init}}}$ is dense in $Fl(n) \times Fl(n)$, and $X_{\bullet_{\text{final}}}$ is the diagonal in $Fl(n) \times Fl(n)$.

There is a subset of configurations $\circ\bullet$, called mid-sort, where $\bullet$ is in the specialization order.

3.3. Geometric Littlewood-Richardson Rule, inexplicit form (cf. [V2 Sect. 3]). —

(i) For any two partitions $\alpha_1, \alpha_2$, $\pi_1^* \Omega_{\alpha_1}(F^1) \cap \pi_2^* \Omega_{\alpha_2}(F^2) = X_{\circ\bullet_{\text{init}}}$ for some mid-sort $\circ\bullet_{\text{init}}$, or $\alpha_2, \pi_1^* \Omega_{\alpha_1}(F^1) \cap \pi_2^* \Omega_{\alpha_2}(F^2) = \emptyset$.

(ii) For any mid-sort $\circ\bullet_{\text{final}}$, $\overline{X_{\circ\bullet_{\text{final}}}} = \pi_1^* \Omega_{\alpha} \cap \Delta (= \pi_2^* \Omega_{\alpha} \cap \Delta)$ for some $\alpha$, where $\Delta$ is the pullback to $G(k, n) \times Fl(n) \times Fl(n)$ of the diagonal $X_{\bullet_{\text{final}}}$ of $Fl(n) \times Fl(n)$.

(iii) For any mid-sort $\circ\bullet$ with $\bullet \neq \bullet_{\text{final}}$, consider the diagram [V2 equ. (1)]

\[
\begin{array}{ccc}
X_{\circ\bullet} & \xrightarrow{\text{open}} & X_{\circ\bullet} \\
\downarrow & & \downarrow \\
X_{\bullet} & \xrightarrow{\text{open}} & X_{\bullet} \cup X_{\bullet_{\text{next}}} & \xrightarrow{\text{closed}} & X_{\bullet_{\text{next}}}.
\end{array}
\]

The closures of $X_{\circ\bullet}$ are taken in $G(k, n) \times X_{\bullet}$ and $G(k, n) \times (X_{\bullet} \cup X_{\bullet_{\text{next}}})$ respectively, and the Cartier divisor $D_X$ is defined by fibered product. There are one or two mid-sort configurations (depending on $\circ\bullet$), denoted by $\circ_{\text{swap}} \bullet_{\text{next}}$ and/or $\circ_{\text{stay}} \bullet_{\text{next}}$, such that $D_X = \overline{X_{\circ_{\text{swap}} \bullet_{\text{next}}}} \cup \overline{X_{\circ_{\text{stay}} \bullet_{\text{next}}}}$ or $\overline{X_{\circ_{\text{stay}} \bullet_{\text{next}}}} \cup \overline{X_{\circ_{\text{swap}} \bullet_{\text{next}}}}$.

There is a more precise version of this rule describing the mid-sort $\circ\bullet$, and $\circ_{\text{swap}} \bullet_{\text{next}}$ and $\circ_{\text{stay}} \bullet_{\text{next}}$ (see [V2 Sect. 3]). For almost all applications here this version will suffice, but the precise definition of mid-sort, $\circ_{\text{swap}} \bullet_{\text{next}}$ and $\circ_{\text{stay}} \bullet_{\text{next}}$ will be implicitly required for the Galois/monodromy results of Section 5.

3.4. Statement of Main theorem. — Fix $Q \subset G(k, n)$, and define $S = S(\alpha_1, \ldots, \alpha_{m-1}) \subset G(k, n) \times Fl(n)^{m-1}$ by

\[
S := (Q \times Fl(n)^{m-1}) \cap \pi_1^* \Omega_{\alpha_1}(F^1) \cap \cdots \cap \pi_{m-1}^* \Omega_{\alpha_{m-1}}(F^{m-1}).
\]

Then $S$ is irreducible, and the projection to $B := Fl(n)^{m-1}$ has relative dimension $\dim Q - \sum |\alpha_i|$. (This follows easily by constructing $S$ as a fibration over $Q$.)
Let $P$ be a property of morphisms depending only on dense open subsets of the target, i.e. if $U \subset Y$ is a dense open subset, then $f : X \to Y$ has $P$ if and only if $f|_{f^{-1}(U)}$ has $P$ (call this condition (A)). For such $S \to B$, and any mid-sort $\odot$, let $\rho_1$ and $\rho_2$ be the two projections from $B \times (Fl(n) \times Fl(n))$ onto its factors. Using (2), construct

\begin{align*}
\rho_1^* S \cap \rho_2^* X_{\odot} & \xrightarrow{\text{open}} \rho_1^* S \cap \rho_2^* X_{\odot} \xrightarrow{\text{closed}} \rho_1^* S \cap \rho_2^* D_X \xrightarrow{\rho_1^* S \cap \rho_2^* X_{\odot \odot \odot}} \rho_1^* S \cap \rho_2^* X_{\odot \odot \odot}
\end{align*}

As in (2), $\overline{X_{\odot}}$ is the closure of $X_{\odot}$ in the appropriate space; $\rho_2^* \overline{X_{\odot}}$ is the pullback of $\overline{X_{\odot}}$ from $X_\bullet$ or $X_\bullet \cup X_{\bullet \text{next}}$, and similarly for the other terms of the top row. The upper right should be interpreted as

$$\rho_1^* S \cap \rho_2^* \overline{X_{\odot \odot \odot}}, \quad \rho_1^* S \cap \rho_2^* \overline{X_{\odot \odot \odot}}, \quad \text{or} \quad \rho_1^* S \cap \rho_2^* \overline{X_{\odot \odot \odot}},$$

as in the Geometric Littlewood-Richardson rule (3).

3.5. Schubert induction theorem. — Suppose that for any such $S \to B$ and any mid-sort $\odot$, (B) if $(\dag \dag \dag)$ has $P$, then $(\dag \dag \dag)$ has $P$, and (C) if $(\dag \dag)$ has $P$, then $(\dag)$ has $P$. If the projection

\begin{align*}
(Q \times Fl(n)) \cap \Omega_\alpha(F.) & \longrightarrow Fl(n)
\end{align*}

has $P$ for all partitions $\alpha$ (the “base case” of the Schubert induction), then the projection

\begin{align*}
(Q \times Fl(n)^m) \cap \pi_1^* \Omega_{\alpha_1} (F.1) \cap \cdots \cap \pi_m^* \Omega_{\alpha_m} (F.m) & \longrightarrow Fl(n)^m
\end{align*}

has $P$ for all $m, \alpha_1, \ldots, \alpha_m$.

In particular (taking $Q = G(k, n)$) if the projection

$$\Omega_\alpha(F.) \longrightarrow Fl(n)$$

has $P$ for all $\alpha$ (condition (D)), then the projection

$$S : \pi_1^* \Omega_{\alpha_1} (F.1) \cap \cdots \cap \pi_m^* \Omega_{\alpha_m} (F.m) \longrightarrow Fl(n)^m$$

has $P$.

Proof. We show that

\begin{align*}
(Q \times Fl(n)^{m-1} \times X_\bullet) \cap \pi_1^* \Omega_{\alpha_1} (F.1) \cap \cdots \cap \pi_{m-1}^* \Omega_{\alpha_{m-1}} (F.\alpha_{m-1}) \cap \rho^* \overline{X_{\odot}} & \longrightarrow Fl(n)^{m-1} \times X_\bullet
\end{align*}

(where $\rho$ is the projection to $X_\bullet$) has $P$ for all $m$ and mid-sort $\odot$, by induction on $(m, \bullet)$, where $(m_1, \bullet_1)$ precedes $(m_2, \bullet_2)$ if $m_1 < m_2$, or $m_1 = m_2$ and $\bullet_1 < \bullet_2$ in the specialization order.
Base case $m = 1$, $\bullet = \bullet_{\text{final}}$. By the Geometric Littlewood-Richardson rule \(3.5\) (ii),

\[
(Q \times X_{\bullet_{\text{final}}}) \cap \rho^* \overline{X}_{\bullet_{\text{final}}} \longrightarrow X_{\bullet_{\text{final}}}
\]

(i.e., \(5\))

\[
(Q \times X_{\bullet_{\text{final}}}) \cap \pi_1^\ast \Omega_{\alpha} \cap \Delta \longrightarrow X_{\bullet_{\text{final}}}
\]

\[
(Q \times Fl(n)) \cap \Omega_{\alpha}(F) \longrightarrow Fl(n)
\]

has $P$ by \(5\).

**Inductive step, case $\bullet \neq \bullet_{\text{final}}$.** By the inductive hypothesis,

\[
(Q \times Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}) \cap \pi_1^\ast \Omega_{\alpha_1}(F_1) \cap \cdots \cap \pi_{m-1}^\ast \Omega_{\alpha_{m-1}}(F_{m-1}) \cap \rho^* \overline{X}_{\bullet_{\text{init}}} \rightarrow Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}
\]

and/or

\[
(Q \times Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}) \cap \pi_1^\ast \Omega_{\alpha_1}(F_1) \cap \cdots \cap \pi_{m-1}^\ast \Omega_{\alpha_{m-1}}(F_{m-1}) \cap \rho^* \overline{X}_{\bullet_{\text{swap_{next}}}} \rightarrow Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}
\]

have $P$. Then an application of (B) and (C) shows that \(6\) has $P$ as well.

**Inductive step, case $\bullet = \bullet_{\text{final}}, m > 1$.** Suppose $\overline{X}_{\bullet_{\text{final}}} = \pi_1^\ast \Omega_{\alpha} \cap \Delta$ and $\pi_1^\ast \Omega_{\alpha_m-1}(F_m) \cap \pi_2^\ast \Omega_{\alpha_m}(F') = \overline{X}_{\bullet_{\text{init}}}$ (using the Geometric Littlewood-Richardson rule \(3.5\) (ii) and (i) respectively). Then

\[
(Q \times Fl(n-1) \times X_{\bullet}) \cap \pi_1^\ast \Omega_{\alpha_1}(F_1) \cap \cdots \cap \pi_{m-1}^\ast \Omega_{\alpha_{m-1}}(F_{m-1}) \cap \rho^* \overline{X}_{\bullet_{\text{init}}} \longrightarrow Fl(n)^{m-1} \times X_{\bullet}
\]

\[
(Q \times Fl(n) \times X_{\bullet}) \cap \pi_1^\ast \Omega_{\alpha_1}(F_1) \cap \cdots \cap \pi_{m-1}^\ast \Omega_{\alpha_{m-1}}(F_{m-1}) \cap \rho^* \overline{X}_{\bullet_{\text{init}}} \longrightarrow Fl(n)^{m}
\]

which has $P$ as (by (A))

\[
(Q \times Fl(n)^{m-2} \times X_{\bullet}) \cap \pi_1^\ast \Omega_{\alpha_1}(F_1) \cap \cdots \cap \pi_{m-2}^\ast \Omega_{\alpha_{m-2}}(F_{m-2}) \cap \rho^* \overline{X}_{\bullet_{\text{init}}} \longrightarrow Fl(n)^{m-2} \times X_{\bullet_{\text{init}}}
\]

has $P$ by the inductive hypothesis.

For some applications, we will need a slight variation.

**3.6. Schubert induction theorem, bis.** — Suppose $P$ satisfies conditions (A–C). If

\[
\bigwedge_{\alpha \colon \dim Q_{-|\alpha|} = 0} (Q \times Fl(n)) \cap \Omega_{\alpha}(F) \longrightarrow Fl(n)
\]

has $P$ then

\[
\bigwedge_{\alpha_1, \ldots, \alpha_m \colon \dim Q_{-\sum |\alpha_i|} = 0} (Q \times Fl(n)^m) \cap \pi_1^\ast \Omega_{\alpha_1}(F_1) \cap \cdots \cap \pi_m^\ast \Omega_{\alpha_m}(F_m) \longrightarrow Fl(n)^m
\]

has $P$ for all $m$.

The proof is identical to that of Theorem \(3.5\).
4. Applications

We now verify the conditions (A–C) for several $P$ to prove the results claimed in Section 2.

4.1. Positive characteristic: Proof of Proposition 2.6. Let $P$ be the property that the morphism $f$ is generically smooth. Then $P$ clearly satisfies (A–C) (note that the relative dimensions of $(† - † † †)$ are the same, and that $X_{\text{stay}\text{next}}$ and $X_{\text{swap}\text{next}}$ are disjoint), and the Schubert induction hypothesis (D); apply Theorem 3.5. □

4.2. Reality: Proof of Proposition 2.4. Let $P$ be the property that there is a Zariski-dense subset $U$ of real points of $Y$ for which $S^{-1}(p)$ consists of $\deg S$ real points for all $p \in U$. Clearly $P$ satisfies (A–C) and the Schubert induction hypothesis (D). Apply Theorem 3.6. □

As mentioned earlier, the same argument applies to any field satisfying the implicit function theorem, such as $\mathbb{Q}_p$.

4.3. Numerical solutions. Informally, this application corresponds to applying Theorem 3.6 to the property of generically finite morphisms $f : X \hookrightarrow G(k, n) \times Y \rightarrow Y$ “for each point of $Y$ whose preimage is a finite number of points, there is an effective algorithm for numerically finding these points”. Condition (C) corresponds to the fact that if $|f^{-1}(y)| = \deg f$ and the points of $f^{-1}(y)$ can be numerically calculated, then by the implicit function theorem, the points of $f^{-1}(y')$ can be numerically calculated for all $y'$ such that $\dim f^{-1}(y') = 0$. This idea will be developed in [SVV].

5. Galois/monodromy groups of Schubert problems

5.1. We recall the “checker tournament” algorithm [V2 Sect. 3.12] for solving Schubert problems. We begin with $m$ partitions, and we make a series of moves. Each move consists of one of the following.

(i) Take two partitions, and begin a checker game if possible, else end the tournament.
(ii) Translate a completed checker game back to a partition.
(iii) Make a move in an ongoing checker game.

These parallel (i)–(iii) of Theorem 3.3. When one partition and no checker games are left, the tournament is complete. At step (iii), the checker tournament may bifurcate (if both a “stay” and a “swap” are possible), and both branches must be completed.

This answer to the Schubert problem can be interpreted as a creating a directed tree, where the vertices correspond to partially completed checker tournament. Each vertex
has in-degree 1 (one immediate ancestor) except for the root (corresponding to the original Schubert problem), and out-degree (number of immediate descendants) between 0 and 2. The graph is constructed starting with the root, and for each vertex that is not a completed checkergame, a choice may be made (depending on (iii)) which may lead to a bifurcation. Vertices corresponding to a single partition and no checkergames are called *leaves*. (There may be other vertices with out-degree 0, arising from (i); these are not leaves.) The answer is the number of leaves of the tree.

The answer is of course independent of the choices made; in the description of Sect. 3.12, and in the proof of Theorems 3.5 and 3.6 each checkergame was chosen to be completed before the next was begun.

5.2. *Theorem.* — Suppose we are given a Schubert problem such that there is a directed tree as above, where each vertex with out-degree two satisfies either

(a) there are a different number of leaves on the 2 branches, or
(b) there is one leaf on each branch.

Then the Galois group of the Schubert problem is at least alternating.

5.3. *Specialization of monodromy.* To prove the theorem, we will examine how Galois groups behave under specialization.

We say a generically finite morphism \( f : X \to Y \) is *generically separable* if the corresponding extension of function fields is separable. Define the Galois group \( \text{Gal}_f \) of a generically finite and separable (i.e. generically étale) morphism to be the Galois group of the Galois closure of the corresponding extension of function fields.

5.4. *Remark: The complex case.* To motivate later statements over an arbitrary ground ring, we first consider the complex case. Suppose

\[
\begin{array}{c}
W \ar{d}^Y \\
X \ar{l}^Z
\end{array}
\]

is a fiber diagram of complex schemes, where the vertical morphisms are proper generically finite degree \( d \); \( W, X, \) and \( Z \) are irreducible varieties; \( Z \) is Cartier in \( X \); \( X \) is regular in codimension 1 along \( Z \); and \( Y \) is reduced. Then \( \text{Gal}_{W \to X} \) can be interpreted as an element of \( S_d \) by fixing a point of \( X \) with \( d \) preimages, and considering loops in the smooth locus of \( X \) based at that point, and their induced permutations of the preimages.

(a) If \( Y \) is irreducible, then by interpreting \( \text{Gal}_{Y \to Z} \) by choosing a general base point of \( Z \) and elements of the fundamental group of the smooth part of \( Z \) generating the Galois group, we have constructed an inclusion \( \text{Gal}_{Y \to Z} \hookrightarrow \text{Gal}_{W \to X} \). In particular, if the first group is at least alternating, then so is the second.
(b) If $Y$ has two components $Y_1$ and $Y_2$, which each map generically finitely onto $Z$ with degrees $d_1$ and $d_2$ respectively (so $d_1 + d_2 = d$), then the same construction produces a subgroup $H$ of $\text{Gal}_{Y_1 \to Z} \times \text{Gal}_{Y_2 \to Z}$ which surjects onto $\text{Gal}_{Y_i \to Z}$ (for $i = 1, 2$), and an injection of $H$ into $\text{Gal}_{W \to X}$ (via the induced inclusion $S_{d_1} \times S_{d_2} \hookrightarrow S_d$).

Then a purely group-theoretical argument (Prop. 5.7) relying on Goursat’s lemma will show that if $\text{Gal}_{Y_i \to Z}$ is at least alternating ($i = 1, 2$), $W$ is connected (so $\text{Gal}_{W \to X}$ is transitive), and $d_1 \neq d_2$ or $d_1 = d_2 = 1$, then $\text{Gal}_{W \to X}$ is at least alternating as well.

5.5. The general case. With this complex intuition in hand, we prove Remarks 5.4(a) and (b) over an arbitrary ring. Suppose $k_1 \subset k_2$ is a separable degree $d$ field extension. Choose an ordering $x_1, \ldots, x_d$ of the $k_1$-valued points (over $\text{Spec } k_1$) of $\text{Spec } k_2$. If $g : X \to Y$ is a generically finite separable (i.e. generically étale) morphism, define the “Galois scheme” $\text{GalSch}_g$ by

\[
\text{deg } g \cdot \Delta
\]

where $\Delta$ is the “big diagonal”. Recall that the Galois group of $k_1 \subset k_2$ can be interpreted as a subgroup of $S_d$ as follows: $\sigma$ is in the Galois group if and only if $(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ is in the same component of $\text{GalSch}_{\text{Spec } k_2 \to \text{Spec } k_1}$ as $(x_1, \ldots, x_d)$.

To understand how this behaves in families, let $R$ be a discrete valuation ring with function field $K$ and residue field $k$. Suppose the following is a fiber diagram

\[
\begin{array}{ccc}
X_K & \xleftarrow{\text{open}} & X_R & \xrightarrow{\text{closed}} & X_k \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec } K & \xleftarrow{\text{open}} & \text{Spec } R & \xrightarrow{\text{closed}} & \text{Spec } k
\end{array}
\]

where $X_K$ is irreducible, $X_k$ is reduced, and the vertical morphisms are finite and separable (and hence étale, using $X_k$ reduced).

After choice of algebraic closures, there is a bijection from the $\overline{K}$-valued points of $X_K$ with the $\overline{k}$-valued points of $X_k$.

By observing that each component of $\text{GalSch}_{X_k \to \text{Spec } k}$ lies in a unique irreducible component of $\text{GalSch}_{X_R \to \text{Spec } R}$ (as $\text{GalSch}_{X_R \to \text{Spec } R} \to \text{Spec } R$ is étale along $X_k$), we see that Remarks 5.4(a) and (b) hold in general (by applying these comments to $\mathcal{O}_{X,Z}$).

5.6. Proof of Theorem 5.2. Each vertex $v$ corresponds to a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{f_v} & B \times G(k, n) \\
\downarrow & & \downarrow \\
B & \xrightarrow{f_v} & B
\end{array}
\]

where $X$ is irreducible, and $B$ is a product of flag varieties (one for each partition) and strata $X_{\sigma}$ (one for each checkergame in process). The morphism $f_v$ is generically finite and separable, and its degree is the answer to the corresponding enumerative problem.
We label each vertex \( v \) with the number of leaves on that branch (i.e. with \( \deg f_v \)), and with the Galois group \( \text{Gal}_v \) of that problem. We prove that \( \text{Gal}_v = A_{\deg f_v} \) or \( S_{\deg f_v} \) for all \( v \) by induction on \( v \). (This is a slight generalization of Schubert induction.)

If \( v \) is a leaf, the result is trivial.

Suppose next that \( v \) is a vertex with one descendant \( w \), and \( \text{Gal}_w \) is at least alternating. If the move from \( v \) to \( w \) is of type (i) or (ii), then the morphism \( f_v \) is the same as \( f_w \), so the result holds. If the move from vertex \( v \) is of type (iii) (so exactly one of \{stay, swap\} is possible), then \( G_v \) is at least alternating by Remark 5.4(a).

Next, suppose \( v \) has two immediate descendants, so we are in case (iii) and both “stay” and “swap” are possible. If one branch has no leaves, then \( G_v \) is at least alternating by Remark 5.4(a), so assume otherwise. As \( X \) is irreducible, \( \text{Gal}_v \) is transitive. By Remark 5.4(b) and the group theoretic calculation of Proposition 5.7, \( \text{Gal}_v \) is at least alternating, and the inductive step is complete.

Thus by induction the root vertex has at least alternating Galois group, completing the proof of the Theorem 5.2. \( \square \)

5.7. Proposition. — Suppose \( G \) is a transitive subgroup of \( S_{m+n} \) such that \( G \cap (S_m \times S_n) \) contains a subgroup \( H \) such that the projection of \( H \) to \( S_m \) (resp. \( S_n \)) is either \( A_m \) (\( m \geq 4 \)) or \( S_m \) (resp. \( A_n \) for \( n \geq 4 \), or \( S_n \)).

(a) If \( m \neq n \), then \( G = A_{m+n} \) (\( m+n \geq 4 \)) or \( S_{m+n} \).
(b) If \( m = n = 1 \) then \( G = S_2 \).

Note that if \( m = n \), then
\[
\{ e, (1, n+1)(2, n+2) \cdots (n, 2n) \} \times (S_{\{1, \ldots, n\}} \times S_{\{n+1, \ldots, 2n\}})
\]
is a subgroup of \( S_{2n} \), whose intersection with \( S_{\{1, \ldots, n\}} \times S_{\{n+1, \ldots, 2n\}} \) surjects onto each of its factors.

Proof. Part (b) is trivial, so we prove (a). Assume without loss of generality that \( n > m \).

Recall Goursat’s lemma: if \( H \subset G_1 \times G_2 \), such that \( H \) surjects onto both factors, then there are normal subgroups \( N_i \triangleleft G_i \) (\( i = 1, 2 \)) and an isomorphism \( \phi : G_1/N_1 \cong G_2/N_2 \) such that \((g_1, g_2) \in H \) if and only if \( \phi(g_1N_1) = g_2N_2 \).

We first show that if \( G \) is a transitive subgroup of \( S_{m+n} \) (\( m > m \geq 3 \)) containing \( A_m \times A_n \), then \( G \) contains any 3-cycle and hence \( A_{m+n} \). Color the numbers 1 through \( m \) red and \( m+1 \) through \( m+n \) green. Any monochromatic 3-cycle lies in \( A_m \times A_n \) and hence \( G \). Suppose \( \tau \) is any element of \( G \) sending a green number to a red position. (i) If there are two numbers of each color in the positions of one color (say, red) then the conjugate of a 3-cycle in \( A_m \) by \( \tau \) will be a 3-cycle \( \alpha \) of 1 red and 2 green objects, and the conjugate of a different 3-cycle in \( A_m \) by \( \tau \) will be a 3-cycle \( \alpha \) of 1 green and 2 red objects. Similarly, (ii) if there is at least one number of each color in the positions of both colors, we can find a
conjugate $\alpha$ of a 3-cycle in $A_m$ or $A_n$ by $\tau$ that is a 3-cycle $\alpha$ of 1 red and 2 green objects, and the conjugate of a 3-cycle in $A_n$ or $A_m$ by $\tau$ that is a 3-cycle $\alpha$ of 1 green and 2 red objects. By conjugating $\alpha$ and $\beta$ further by elements of $A_m \times A_n$, we can obtain any non-monochromatic 3-cycle. Now $\tau$ falls into case (i) and/or (ii), or $n = m + 1$ and $\tau$ sends all red objects to green positions, and all but one green object to red positions. Suppose $p$ is the green position containing the green object in $\tau$, and $\sigma$ is a 3-cycle in $A_n$ moving $p$. Then $\tau^{-1} \sigma \tau$ is a permutation where exactly one red object is sent to a green position, and vice versa, and we are in case (ii). Thus in all cases $G$ contains $A_{m+n}$, as desired.

We now deal with the case $m, n \geq 3$. By Goursat’s lemma, $G$ must contain $A_m \times A_n$. (For example, if the projections of $H$ to $S_m$ and $S_n$ are surjective, then $H$ arises from isomorphic quotients $S_n/N_m \cong S_n/N_n$. Then $(N_m, N_n) = (S_m, S_n)$ or $(N_m, N_n) = (A_m, A_n)$; in both cases $A_m \subset N_m$ and $A_n \subset N_n$.) Then apply the previous paragraph.

For the remaining cases, it is straightforward to see (using Goursat) that (i) if the image of $H$ is $A_n$ (resp. $S_n$) and $m = 1$, then $A_{m+n} \subset G$ (resp. $G = S_{m+n}$), and (ii) if $H$ surjects onto $A_n$ ($n \geq 4$) or $S_n$ and $m = 2$, then $A_{m+n} \subset G$. □

5.8. Remark. We note for use in Section 5.12 that if $n = 1$ and the projection to $S_m$ is surjective, then the same argument shows that $G = S_{m+n}$.

5.9. Applying Theorem 5.2. Theorem 5.2 is quite strong, and can be checked with a naive computer program. For example, it implies that all Schubert problems for $G(2, n)$ for $n \leq 16$ are at least alternating. It also implies that all but a tiny handful of Schubert problems for Grassmannians of dimension less than 20 are at least alternating; we will describe these exceptions.

For $k > 1$, the criterion will fail for the Schubert problem $(1)^{k^2}$ on $G(k, 2k)$: the first degeneration (i.e. the first vertex with out-degree 2) will correspond to

$$(1)^{k^2} = (2)(1)^{k^2} + (1, 1)(1)^{k^2}$$

and the two branches will have the same number of leaves by symmetry. More generally, if $1 \leq m < k$ and $(m, k) \neq (1, 2)$, the criterion will fail for the Schubert problem

$$(m, \ldots, m)(1)^{k^2-m^2}$$
on $G(k, 2k)$ for the same reason.

Figure 1. The two counterexamples of $G(3, 6)$
On $G(3, 6)$, the only counterexamples are of this sort, when $m = 1$ and 2, shown in Figure 1. By “embedding” these problems in larger problems, these trivially induce counterexamples in larger Grassmannians; for example, Figure 2 is a counterexample in $G(3, 7)$ that is really an avatar of the second example in $G(3, 6)$. We call counterexamples in $G(k, n)$ not arising in this way, i.e. involving only subpartitions not meeting the right column and bottom row of the rectangle, primitive counterexamples.

Then $G(3, 7)$ has only three counterexamples, shown in Figure 3, and the counterexamples in $G(4, 7)$ are given by the transposes of these. The Grassmannian $G(3, 8)$ has six counterexamples, shown in Figure 4 and $G(3, 9)$ has 13 counterexamples, shown in Figure 5.

All of these exceptions can be excluded with the following, slightly stronger criterion.

5.10. Theorem. — Suppose we are given a Schubert problem such that there is a directed tree as above, where each vertex with two immediate descendants satisfies (a) or (b) of Theorem 5.2 or (c) there are $m \neq 6$ leaves on each branch, and it is known that the corresponding Galois group is two-transitive.

Then the Galois group of the Schubert problem is at least alternating.
In particular, to show that the Galois group is \((n - 2)\)-transitive, it often suffices to show that it is two-transitive.

As with Theorem 5.2, the proof reduces to the following variation of Proposition 5.7.

5.11. Proposition. — Suppose \(G\) is a two-transitive subgroup of \(S_{2m}\) \((m \neq 6)\) such that \(G \cap (S_m \times S_m)\) contains a subgroup \(H\) such that the projection of \(H\) to both factors \(S_m\) is either \(A_m\) \((m \geq 4)\) or \(S_m\). Then \(G = A_{2m}\) or \(S_{2m}\).

The proof is similar to that of Proposition 5.7 and is omitted.

If \(m = n = 6\), D. Allcock has pointed out that the Mathieu group \(M_{12}\) can be expressed as a subgroup of \(S_{12}\) such that

\[
M_{12} \cap (S_6 \times S_6) = \{(g, \sigma(g)) : g \in S_6\}
\]

where \(\sigma\) is an outer automorphism of \(S_6\). Thus Proposition 5.11 cannot be extended to \(m = 6\).

We say two vertices \(v, w\) in a directed tree (as in Sect. 5.1) are equivalent if they are connected by a chain of edges \(v_1 \to v_2 \to \cdots \to v_s\) \((v_1, v_s) = (v, w)\) or \((w, v)\) and \(\deg v = \deg w\) (and hence \(= \deg v_i\) for all \(i\)). In each of the cases \(G(3, n)\) \((6 \leq n \leq 9)\) given above, it is possible to find such a tree satisfying Theorem 5.10(a)–(c), where the vertices of type (c) are equivalent to vertices corresponding to Schubert problems (i.e. corresponding to a set of partitions, with no checkergames-in-progress), and to show by ad hoc means that these Schubert problems are two-transitive. (The details are omitted; this method should not be expected to be workable in general.) Hence all Schubert problems for these Grassmannians have Galois group at least alternating.
The Grassmannian $G(4, 8)$ has only 31 Schubert problems where the criterion of Theorem 5.2 does not apply (not shown here). Each of these cases may be reduced to checking that a certain Schubert problem is two-transitive. As we shall see in the next Section, in one of these cases two-transitivity does not hold!

5.12. Galois groups of Schubert problems needn’t be the full symmetric group, or alternating.

5.13. Derksen’s example in $G(4, 8)$. One of the 31 examples in $G(4, 8)$ described above has a Galois group that is not at least alternating (and hence is not two-transitive by our earlier discussion): the Schubert problem of Figure 6. This example (and the existence of Schubert problems with non-full Galois group) is due to H. Derksen. By Theorem 2.10, this is the smallest example of a Schubert problem with a Galois group smaller than alternating.

5.14. A family of examples generalizing Derksen’s. Derksen’s example can be generalized to produce other examples of smaller-than-expected Galois groups, where the Galois action is that of $S_N$ acting on the order $K$ subsets of $\{1, 2, \ldots, N\}$, as follows. The Schubert
problem of Figure 8 in $G(2K, 2N)$ has $\binom{N}{K}$ solutions. Given four general flags in $G(2, 2N)$, the auxiliary problem of Figure 9 has $N$ solutions, corresponding to $N$ transverse 2-planes $V_1, \ldots, V_N$ in $G(2, 2N)$. By repeated applications of Remark 5.8, the Galois group of the auxiliary Schubert problem is $S_N$. The subspace $V_{i_1} + \cdots + V_{i_K}$ ($1 \leq i_1 < \cdots < i_K \leq N$) is a solution to the original problem of Figure 8. Hence the original problem exhibits the desired behavior.

\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{figure8.png}
\caption{A Schubert problem in $G(2K, 2N)$ with $\binom{N}{K}$ solutions and Galois group $S_N$}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{figure9.png}
\caption{An auxiliary problem}
\end{figure}

The only statements in the previous paragraph that are nontrivial to verify are (i) the enumeration of solutions to the Schubert problem, and (ii) the fact that the Galois group of the auxiliary problem is $S_N$. Both are easiest to see in terms of puzzles. (See [KTW] for a definition of puzzles, and the appendix to [V2] for the bijection between checkers and puzzles.)

Part (i) is the number of ways of filling in the puzzle of Figure 10 (where the blocks of 1’s are all of size $K$, and the blocks of 0’s are all of size $N - K$), which reduces to Figure 11. After trying the puzzle, the reader will quickly see that the number of solutions is $\binom{N}{K}$. The solutions correspond to the choice of labels on segment $A$ — there will be $N - K$ 0’s and $K$ 1’s, and each order appears in precisely one completed puzzle.

To construct the directed tree for part (ii), note that the order of the first checker game corresponds to filling in the top half of the puzzle of Figure 10 (and hence Figure 11) row by row; the directed graph corresponds to the tree of choices made while completing the puzzle in this order. Applying this in the case $K = 1$, the puzzles of the previous paragraph show that the tree is of the desired form.

It is interesting (but inessential) to note more generally that the tree for $\binom{N}{K}$ (call it of type $(N, K)$) can be interpreted in terms of Pascal’s triangle as follows. The two branches at the first branch point have $\binom{N-1}{K-1}$ and $\binom{N-1}{K}$ leaves, and the two corresponding directed trees are of type $(N - 1, K - 1)$ and $(N - 1, K)$ respectively. Thus Theorem 5.2 fails to apply because of vertices of type $(2N'', N'')$, corresponding to the central terms in Pascal’s triangle.
In how many ways can you fill in the puzzle with pieces of the form \( \begin{array}{c} 0 \ \Delta \ 1 \\ \ \ 1 \ \Delta \ 1 \end{array} \)?

**Figure 10.** The puzzle corresponding to Figure 8

As with the previous family, to prove this, first count solutions using checkers or puzzles. The puzzle is shown in Figure 13 which again reduces to Figure 11 (without the equatorial cut). Next, fix three general flags. Consider the analogous problem with \( K = 1 \). There are \( N \) solutions, corresponding to \( N \) transverse 3-spaces \( V_1, \ldots, V_N \). The Galois group is \( S_N \) by Remark 5.8 as the tree is identical to that of the previous section. The sum of any \( K \) of these 3-spaces is a solution to the original Schubert problem (with respect to the same three flags). Thus the Galois group of the original problem is \( S_N \) as desired.

**5.15. A similar family of three-flag examples.** We now exhibit a family of three-flag examples with behavior similar to that of the previous section. The Schubert problem of Figure 12 in \( G(3K, 3N) \) has \( \binom{N}{K} \) solutions and Galois group \( S_N \), where the action is that of \( S_N \) on order \( K \) subsets of \( \{1, \ldots, N\} \).

**Figure 11.** The puzzle corresponding to Figure 8 partially completed
Figure 12. A Schubert problem in $G(3K, 3N)$ with $\binom{N}{K}$ solutions and Galois group $S_N$

Figure 13. The puzzle corresponding to Figure 12

References

[Bel] P. Belkale, Geometric proofs of Horn and saturation conjectures, preprint 2002, math.AG/0208107v2.
[Ber] A. Berg, Enumerative geometry for plane cubic curves in characteristic 2, Compositio Math. 111 (1998), no. 2, 123–147.
[E] T. Ekedahl, An effective version of Hilbert’s irreducibility theorem, Séminaire de Théorie des Nombres, Paris 1988–1989, 241–249, Progr. Math., 91, Birkhäuser Boston, Boston, MA, 1990.
[EG] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, Ann. Math. 155 (2002), no. 1, 105–129.
[FJ] M. Fried and M. Jarden, Field Arithmetic, Springer-Verlag, Berlin, 1986.
[F1] W. Fulton, Introduction to intersection theory in algebraic geometry, Regional Conf. Series in Math. 54, Amer. Math. Soc., Providence, 1984.
[F2] W. Fulton, Intersection Theory, Springer-Verlag, New York, 1980.
[FP] W. Fulton and P. Pragacz, Schubert varieties and degeneracy loci, Lecture Notes in Math. 1689, Springer-Verlag, Berlin, 1998.
[H] J. Harris, Galois groups of enumerative problems, Duke Math. J. 46 (1979), no. 4, 685–724.
[HSS] B. Huber, F. Sottile, and B. Sturmfels, Numerical Schubert calculus, J. Symbolic Comput. 26 (1998), no. 6, 767–788.
[HV] B. Huber and J. Verschelde, Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control, SIAM J. Control Optim. 38 (2000), 1265–1287.
[K] S. Kleiman, The transversality of a general translate, Compositio Math. 28 (1974), 287–297.
[K] A. Knutson, personal communication.
A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of \( GL(n) \) tensor products II: Puzzles give facets of the L-R cone*, preprint 2001, [math.CO/0107011](http://arxiv.org/abs/math.CO/0107011), J. Amer. Math. Soc., to appear.

D. Laksov, *Deformations of determinantal schemes*, Compositio Math. 30 (1975), 287–297.

D. Laksov and R. Speiser, *Transversality criteria in any characteristic*, Enumerative geometry (Sitges, 1987), Lectures Notes in Math. 1436, Springer-Verlag, 1990, pp. 139–150.

D. Laksov and R. Speiser, *Transversality criteria in any characteristic*, Pacific J. Math. 156 (1992), 307–328.

K. Purbhoo, preprint.

F. Sottile, *Enumerative geometry for real varieties*, Algebraic Geometry, Santa Cruz, 1995, J. Kollár et al ed., Proc. Sympos. Pure Math. vol. 56, Part 2, Amer. Math. Soc., 1997.

F. Sottile, *Enumerative geometry for the real Grassmannian of lines in projective space*, Duke Math. J., 87 (1997), 59–85.

F. Sottile, *Real enumerative geometry and effective algebraic equivalence*, J. Pure and App. Alg. 117 and 118 (1997), 601–615.

F. Sottile, *Pieri’s formula via explicit rational equivalence*, Can. J. Math, 46 (1997), 1281–1298.

F. Sottile, *The special Schubert calculus is real*, ERA of the Amer. Math. Soc. 5 (1999), 35–39.

F. Sottile, *Some real and unreal enumerative geometry for flag manifolds*, Michigan Math. J. (Fulton volume) 48 (2000), 573–592.

F. Sottile, *Elementary transversality in the Schubert calculus in any characteristic*, Michigan Math. J., to appear.

F. Sottile, R. Vakil, and J. Verschelde, *Effective solutions to all Schubert problems*, work in progress.

R. Speiser, *Transversality theorems for families of maps*, Algebraic Geometry (Sundance 1986), A. Holme and R. Speiser eds., Lectures Notes in Math. 1311, Springer-Verlag, 1988, pp. 235–252.

I. Vainsencher, *Conics in characteristic 2*, Compositio Math. 36 (1978), no. 1, 101–112.

R. Vakil, *The enumerative geometry of rational and elliptic curves in projective space*, J. Reine Angew. Math. (Crelle) 529 (2000), 101–153.

R. Vakil, *A geometric Littlewood-Richardson rule*, preprint 2003, [math.AG/0302294](http://arxiv.org/abs/math.AG/0302294) submitted for publication.

**DEPT. OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD CA 94305–2125**

*E-mail address: vakil@math.stanford.edu*