Comparison of Relativistic Nucleon-Nucleon Interactions

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Abstract

We investigate the difference between those relativistic models based on interpreting a realistic nucleon-nucleon interaction as a perturbation of the square of a relativistic mass operator and those models that use the method of Kamada and Glöckle to construct an equivalent interaction to add to the relativistic mass operator. Although both models reproduce the phase shifts and binding energy of the corresponding non-relativistic model, they are not scattering equivalent. The example of elastic electron-deuteron scattering in the one-photon-exchange approximation is used to study the sensitivity of three-body observables to these choices. Our conclusion is that the differences in the predictions of the two models can be understood in terms of the different ways in which the relativistic and non-relativistic $S$-matrices are related. We argue that the mass squared method is consistent with conventional procedures used to fit the Lorentz-invariant cross section as a function of the laboratory energy.
I. INTRODUCTION

Realistic relativistic quantum mechanical models of interacting nucleons are important for modeling medium and high-energy reactions involving nuclear targets. The simplest generalization of the non-relativistic two-body Hamiltonian is

\[ h = \frac{k^2}{m} + V \quad \rightarrow \quad M = 2\sqrt{k^2 + m^2} + V'. \]  

In this case the mass operator, \( M \), replaces the non-relativistic center of momentum Hamiltonian, \( h \). Bakamjian and Thomas [1] have determined sufficient conditions on \( V' \) in (1) so \( M \) can be interpreted as the mass operator of an equivalent relativistic quantum model.

In the non-relativistic case, interactions are determined by fitting parameters in the interaction so that the solutions of the dynamical equations reproduce the existing bound-state and scattering observables. This is a non-trivial task that has led to the construction of a number of realistic interactions [2] [3] [4]. Schiavilla [5] has repeated this analysis, by refitting parameters in an Argonne V14 interaction, to determine a realistic relativistic interaction, \( V' \). It is tedious to have to refit parameters for every new interaction.

A direct way to utilize existing non-relativistic interactions, without refitting the interaction, is preferable. Two methods exist that can be used to directly formulate relativistic models based on realistic non-relativistic interactions. Historically the first of these methods was introduced by Coester, Pieper, and Serduke [6] [7] [8]. In their construction an interaction is added to the square of the non-interacting mass operator. The interaction is chosen so the dynamical equations are equivalent to the non-relativistic equations. The result is that the eigenfunctions of the relativistic model can be trivially obtained from those of the corresponding non-relativistic model. The second method was introduced by Kamada and Glöckle [9]. This utilizes an \( S \)-matrix preserving unitary transformation, called a scattering equivalence [10] [11] [8], that relates the two operators in (1). The transformation involves a simple rescaling and a renormalization that ensures unitarity. Both methods lead to invariant \( S \)-matrices that are identical to the non-relativistic \( S \)-matrix.

In this paper we investigate the difference between the mass-squared method and the Kamada-Glöckle method by comparing both the two-body dynamics and elastic electron-deuteron scattering, which is the simplest three-body reaction.

The surprising conclusion of our analysis is that even though both methods give the same invariant scattering matrix as the corresponding non-relativistic model, they do not lead to identical predictions, even at the two-body level. Specifically, these two methods do not lead to scattering equivalent theories.

We also consider differences in the predictions for elastic electron-deuteron form factors based on these two models. While normally a change in the two-body interaction would lead to a corresponding change in the strong current, in typical calculations the consistency of the dynamics and the strong current is normally limited to ensuring current conservation and current covariance.

The comparisons in this paper are limited to understanding the quantitative sensitivity of the physical observables to the two different models of the two-body dynamics, keeping everything else the same. Specifically we compare the calculated observables using the deuteron eigenstates constructed from the mass-squared and Kamada-Glöckle methods.
based on the same realistic nucleon-nucleon interaction. In both calculations the current is
treated in an identical impulse-like approximation, using identical nucleon form factors and
an identical form of the dynamics. It is impossible to rigorously determine if the difference
in these three-body observables are differences that would arise if the two methods were
two-body scattering equivalent, or if they are due to the violations of the scattering equiva-
cence. However, the result of the above comparison is that there are small but measurable
differences in the two predictions. The parameter that controls the scale of these differences
is the same one that controls the scale of the violation of the scattering equivalence.

For the electron-scattering calculations the coupling to the electron is treated in the one-
photon-exchange approximation. The relativistic models are formulated in Dirac’s point
form of the dynamics. The hadronic current operators are modeled using the point-from
spectator approximation [12]. The input nucleon current matrix elements are constructed
from the Gari-Krümpelmann parameterization [13]. The specific choice of realistic interac-
tion, form of dynamics, or nucleon form factors is not central to our analysis.

The notion of scattering equivalence is necessary to understand the issues involved in
our analysis. Unitary transformations preserve probabilities in quantum mechanics. The
conventional interpretation of this observation is that the physical predictions of a quantum
theory do not depend on the choice of Hilbert space basis.

There is a more subtle interpretation of this result that is best understood in the context
of scattering theory. The S-matrix measures the difference between the free and interacting
Hamiltonian that can be observed asymptotically. Scattering observables are preserved
under unitary transformations provided the same unitary transformation is used to transform
both the free and interacting Hamiltonians.

A class of unitary transformations, called scattering equivalences, have the property that
if they are applied to the full Hamiltonian without applying them to the free Hamiltonian,
they give the same scattering matrix elements and bound state observables as the original
Hamiltonian. They are not meant to transform representations of free particles. In this
sense they have a different interpretation than a standard unitary transformation. They
parameterize the insensitivity of the asymptotic \( S \)-matrix observables to certain short-range
features of the interactions.

Scattering equivalences are unitary transformations \( W \) that satisfy the asymptotic condition
[11][8][14]

\[
\lim_{t \to \pm \infty} \| (I - W) e^{-iH_0 t} \Phi \| = 0.
\]

Examples of scattering equivalences include operators of the form

\[
W = \frac{I + iC}{I - iC}
\]

where \( C \) is any compact Hermitian operator.

In a field like nuclear physics, where model interactions are determined by fitting to \( S \)-
matrix and bound-state observables, realistic model Hamiltonians can at best be determined
up to an overall scattering equivalence. The ambiguity in the choice of representation of the
interaction is particularly confusing in the case of relativistic models, where there are many
scattering equivalent ways to interpret a given non-relativistic interaction. For example,
the kernel of a single two-body interaction defines different operators in each of Dirac’s [15]
forms of the dynamics. The operators differ by whether they commute with the total linear momentum, the total four velocity, or the light-front components of the four momentum. The resulting two-body models are scattering equivalent, but lead to different three-body predictions. Even within a single form of dynamics, there are scattering equivalences [16] that relate different natural choices of two-body interactions. In all of these cases, the different scattering equivalent models have the same two-body bound state and scattering observables. They can be made to have the same three-body observables by introducing additional three-body interactions [14]. If, as is customary, three-body interactions are not included, then different scattering equivalent two-body models give different three-body predictions.

In what follows a brief description of each model is given. Differences in the $S$-matrix, wave functions, and elastic electron-deuteron observables are analyzed.

II. THE MASS-SQUARED METHOD:

In this section the mass-squared method is described for the case of two particles with equal masses. The case of unequal masses is treated in [17].

Let $V_{nr}$ be a non-relativistic nucleon-nucleon interaction defined by a kernel of the form

$$
\langle \mathbf{p}, k, j, \sigma, l, s | V_{nr} | \mathbf{p}', k', j', \sigma', l', s' \rangle = \delta(k - k') \delta_{jj'} \delta_{\sigma\sigma'} \langle k, l, s | v_{nn} \parallel k', l', s' \rangle,
$$

(4)

where $k$ is the magnitude of the relative momentum. The physics is contained in the reduced kernel

$$
\langle k, l, s | v_{nn} \parallel k', l', s' \rangle.
$$

(5)

Define a model relativistic mass operator:

$$
M^2 := 4\left( k^2 + m^2 \right) + 4mV_{m^2};
$$

(6)

where $V_{m^2}$ is a point-form interaction defined by

$$
\langle \mathbf{v}, k, j, \sigma, l, s | V_{m^2} | \mathbf{v}', k', j', \sigma', l', s' \rangle
$$

$$
= \delta(\mathbf{v} - \mathbf{v}') \delta_{jj'} \delta_{\sigma\sigma'} \langle k, l, s | \hat{v}_{nn} \parallel k', l', s' \rangle.
$$

(7)

The interaction kernel of $V_{m^2}$ is identical to the non-relativistic kernel (4). In this expression $\mathbf{v}$ is the velocity of the non-interacting two-body system, $k$ is related to the invariant mass of the non-interacting system by $M^2_0 = 4\left( k^2 + m^2 \right)$, and $j$ is the spin of the non-interacting two-body system.

The two-body eigenvalue problem,

$$
M^2 | \psi \rangle = \lambda^2 | \psi \rangle,
$$

(8)
is mathematically equivalent to the non-relativistic Schrödinger equation

\[ H_{nr} |\psi\rangle = \left( \frac{k^2}{m} + V_{nr} \right) |\psi\rangle = -\epsilon_{nr} |\psi\rangle. \]  

(9)

The eigenfunctions are identical functions. Because of the relation

\[ M^2 = 4m(H_{nr} + m), \]  

(10)

the eigenvalues of these equations are related by

\[ \lambda^2 = (2m - \epsilon_r)^2 = 4m(-\epsilon_{nr} + m). \]  

(11)

It follows that for the deuteron

\[ \frac{\epsilon_r - \epsilon_{nr}}{\epsilon_r} = \frac{\epsilon_r}{4m} \sim \frac{1}{2000}, \]  

(12)

which implies that the relativistic and non-relativistic binding energies are identical to within experimental uncertainties.

The two-body mass operator (6) also leads to the same cross section as in the non-relativistic model (9). To see this note that the Lorentz invariant relativistic cross section is given by

\[
\left( \frac{d\sigma}{d\Omega d\epsilon} \right)_r
= \frac{(2\pi)^4}{v_r} |\langle f^+ | M - M_0 | i \rangle|^2 k^2 \frac{dk}{d\epsilon_r}
= \frac{(2\pi)^4}{2k/\omega} |\langle f^+ | M - M_0 | i \rangle|^2 k^2 \frac{\omega}{2k}
= (2\pi)^4 |\langle f^+ | \frac{M^2 - M_0^2}{4\omega} | i \rangle|^2 \left( \frac{\omega}{2} \right)^2
= (2\pi)^4 |\langle f^+ | V_{nr} \frac{m}{\omega} | i \rangle|^2 \left( \frac{\omega}{2} \right)^2
= \frac{(2\pi)^4}{2k/m} |\langle f^+ | V_{nr} | i \rangle|^2 k^2 \frac{m}{2k}
= \frac{(2\pi)^4}{v_{nr}} |\langle f^+ | V_{nr} | i \rangle|^2 k^2 \frac{dk}{d\epsilon_{nr}};
\]

(13)

where the relativistic and non-relativistic velocities are
\[ v_r = \frac{2k}{\omega}, \quad \omega = \sqrt{k^2 + m^2}, \quad v_{nr} = \frac{2k}{m}; \quad (14) \]

and the relativistic and non-relativistic kinematic internal energies are

\[ \epsilon_r = 2\omega, \quad \text{and} \quad \epsilon_{nr} = 2m + \frac{k^2}{m}. \quad (15) \]

The calculation in (13) shows that the relativistic cross section based on \( M^2 \) is identical to the non-relativistic cross section derived from the non-relativistic Hamiltonian (9).

This shows that the relativistic invariant cross section (with the relativistic velocity and phase space factor) is identical to the non-relativistic cross section (with the non-relativistic velocity and phase space factor). The two-body eigenstates of the non-relativistic Hamiltonian (9) are also eigenstates of the relativistic mass operator (6).

In this case, because the factors \( k^2 \) in equations (6) and (9) are identified, the scattering matrix elements are identified as functions of invariant relative momenta \( k \):

\[ S_r(k) = S_{nr}(k). \quad (16) \]

These momenta have different interpretations in the relativistic and non-relativistic cases when they are expressed in terms of single nucleon degrees of freedom. Both correspond to the same asymptotic observable, the momentum of a single particle in the center of momentum frame of the non-interacting system.

Ignoring the difference in relativistic and non-relativistic binding energy, the mass squared model is scattering equivalent to the non-relativistic model. There is no need to refit interaction parameters.

### III. THE KAMADA-GLÖCKLE INTERACTION

The Kamada-Glöckle method is an alternate method that is used to make relativistic quantum models using a realistic nucleon-nucleon model as input.

The Kamada-Glöckle method defines the operator \( q \) by

\[ 2\sqrt{m^2 + k^2} := 2m + \frac{q^2}{m} = F(q). \quad (17) \]

Equation (17) can be solved for \( q(k) \) and \( k(q) \):

\[ q^2 = m(2\sqrt{m^2 + k^2} - 2m); \quad (18) \]

\[ k^2 = q^2 \left( 1 + \frac{q^2}{4m^2} \right). \quad (19) \]

Kamada and Glöckle also define \( h(q) \) by the equation

\[ k^2 dk = h^2(q)q^2 dq \quad (20) \]
which can be solved to give

\[ h^2(q) = \frac{k}{q} \left( 1 + \frac{q^2}{2m^2} \right). \]  

(21)

Defining the operator \( V_{kg} \) by

\[
\langle \mathbf{v}, k, j, \sigma, l, s | V_{kg} | \mathbf{v}', k', j', \sigma', l', s' \rangle = \delta(\mathbf{v} - \mathbf{v}') \delta_{jj'} \delta_{\sigma\sigma'} \frac{1}{h(q(k))} \langle k, l, s || \psi^j_{mn} || k', l', s' \rangle \frac{1}{h(q(k'))} \]  

(22)

where \( \langle k, l, s || \psi^j_{mn} || k', l', s' \rangle \) is the reduced interaction (4) Kamada and Glöckle show that the operator

\[ M = 2\sqrt{k^2 + m^2} + V_{kg} - 2m \]  

(23)

has the exact same deuteron binding energy as predicted by the non-relativistic model. In addition, they show that the \( S \)-matrix elements predicted in their model and the non-relativistic model, as a function of

\[ \epsilon = 2\sqrt{m^2 + k^2} - 2m = \frac{q^2}{m}, \]  

(24)

are identical. Thus, these two models are scattering equivalent.

The relativistic wave functions and transition matrix elements can be obtained directly from the corresponding non-relativistic quantities:

\[ \phi^j(k, l, s) = \frac{\psi^j(q(k), l, s)}{h(q(k))} \]  

(25)

and

\[
T^j_r(k, l, s; \epsilon; k', l', s') := \frac{1}{h(q(k))} T^j_{nr}(q(k), l, s; \epsilon; q(k'), l', s') \frac{1}{h(q(k'))}. \]  

(26)

This has the advantage that the solution of the relativistic scattering and eigenvalue problems do not have to be recalculated.

With this method, the non-relativistic and relativistic scattering matrix elements are identified as identical functions of the non-relativistic and relativistic invariant center of momentum energy:

\[ S_r(\epsilon) = S_{nr}(\epsilon); \]  

(27)

where

\[ \epsilon = \frac{q^2}{m} + 2m = 2\sqrt{k^2 + m^2}. \]  

(28)

Thus, the Kamada-Glöckle model is also scattering equivalent to the non-relativistic model.
IV. DISCUSSION

Each of the methods discussed above provides a simple prescription for interpreting existing realistic non-relativistic models as equivalent relativistic models. In the mass-squared case a local interaction is added to the square of the free mass operator and in the Kamada-Glöckle case a non-local operator is added to the mass operator.

Because each model has a relativistic scattering matrix that is equivalent to the non-relativistic scattering matrix, one might expect that the two relativistic models would be scattering equivalent. This is not true. This is because in the mass-squared case the $S$-matrix elements are identified as functions of $k$ while in the Kamada-Glöckle case they are identified as function of the invariant energy.

In order to get a quantitative understanding of the difference in the $S$-matrix elements of these two models, note that both models solve the Schrödinger equation:

$$\left(\frac{q^2}{m} + V_{nr}\right)|\phi\rangle = \epsilon_{12}|\phi\rangle = \left(\frac{k^2}{m} + V_{m^2}\right)|\phi\rangle = \epsilon_{12}|\phi\rangle.$$  \hspace{1cm} (29)

Although the operators are different, the internal wave functions are the non-relativistic wave function as a function of $q$ or $k$ respectively. It follows that the corresponding scattering matrix elements are related by

$$S_{KG}(\epsilon(k_{KG})) = S_{nr}(\epsilon_{nr}(q)) = S_{m^2}(k_{m^2}) \quad \text{for} \quad k_{m^2} = q.$$ \hspace{1cm} (30)

The problem is that at point where all of the $S$-matrix elements agree, we have $k_{m^2} = q \neq k_{KG}$. On the other hand, $k_{KG}$ and $k_{m^2}$ represent the same asymptotic observable in the two relativistic models, which means the the physics requires

$$S_{m^2}(k_{m^2}) = S_{KG}(k_{KG}) \quad \text{for} \quad k_{m^2} = k_{KG}$$ \hspace{1cm} (31)

which contradicts (30).

To show this explicitly note by definition

$$k^2_{KG} = q^2(1 + q^2/4m^2).$$ \hspace{1cm} (32)

The point where the mass squared and Kamada-Glöckle $S$-matrices agree corresponds to $k_{m^2} = q$. Substituting $k^2_{m^2}$ for $q^2$ in the above equation gives the following relation between $k_{m^2}$ and $k_{KG}$:

$$k^2_{KG} = k^2_{m^2}(1 + k^2_{m^2}/4m^2).$$ \hspace{1cm} (33)

Since $k_{KG} > k_{m^2}$ at the point where the $S$-matrix elements agree, this implies that the $KG$ cross sections fall off slower than the mass-squared cross sections when considered as functions of $k$.

There is nothing theoretically wrong with either of these methods; nevertheless they are inconsistent with each other and cannot both be consistent with experiment.

The key to resolving this problem was originally noted by Breit [19]. He correctly pointed out that the laboratory cross section is a Lorentz invariant function of a Lorentz invariant
variable, the laboratory energy. Specifically, the laboratory energy $\epsilon_{\text{lab}}$ can be expressed in a manifestly invariant manner by:

$$p_1 \cdot p_2 = -\epsilon_{\text{lab}}m = -m^2 - 2k^2;$$

or

$$\epsilon_{\text{lab}} = 2\frac{k^2}{m} + m, \quad k^2 = \frac{m}{2}(\epsilon_{\text{lab}} - m).$$

This indicates that the experimentally determined $S$-matrix and cross sections are Lorentz invariant functions of the invariant $\epsilon_{\text{lab}}$ or equivalently $k^2$. The key point is that there is no such thing as the experimentally determined non-relativistic cross section.

"Non-relativistic" potentials are determined by adjusting the parameters of the potential until the scattering eigenstates of

$$H = \frac{k^2}{m} + V$$

reproduce the experimentally measured $S(k^2) = S(\epsilon_{\text{lab}})$. The analysis of the mass-squared method in equation (13) shows that the non-relativistic and relativistic formulas for the cross section lead to the same invariant quantity when $k^2$ is identified with the asymptotic relative momentum in the center of momentum frame.

If this same data were used to determine the Kamada-Gl"ockle interaction, it would be appropriate to determine the interaction $V_{\text{KG}}$ directly by comparing to the Lorentz invariant cross section as a function of $k^2$, as was done by Schiavilla [5]. The identification (17) defines $q$ by

$$q^2 = m(2\sqrt{m^2 + k_{\text{KG}}^2} - 2m) = m(2\sqrt{m^2 + \frac{m}{2}(\epsilon_{\text{lab}} - m) - 2m}).$$

This suggests that a Kamada-Gl"ockle interaction constructed from a non-relativistic model fit to the $S$-matrix as a function of laboratory energy will not be consistent with experiment at high energies. In practice, the differences are negligible as long as $k^2/4m^2$ is small. For momenta where this quantity is large there are questions about the interpretation of non-relativistic models.

Although the wave functions in the two models are not observable, they are shifted in the same manner as the $S$-matrix. The rescaling and renormalization of the wave function is responsible for all of the differences in our computations of the electron scattering observables. The result is that the Kamada-Gl"ockle wave functions will be spread out more in momentum space than the mass-squared deuteron wave functions.
The factor $k^2/4m^2$ in (33) defines the scale of the shift of the S-matrix or wave function at a given value of $k$ that results from the two different interpretations of the $k$ variables. This factor also defines the scale at which the two models fail to be scattering equivalent and the scale on how much the the factor $h(q^2)$ in (21) differs from unity. This suggests that the differences in the wave functions of the two models are not simply due to an underlying scattering equivalence. Instead, they are completely determined by the difference in the momentum dependence of the S-matrices. In the case of the electron-scattering predictions, using these wave functions in the same point-form spectator approximation leads to differences in the elastic observables. While this in not a true test of the three-body sensitivity, this comparison is consistent with the way that models of the hadronic current are tested.

Figures 1 and 2 show the momentum space $s$ and $d$ components of the deuteron wave functions for the mass-squared and Kamada-Glöckle models based on the same Argonne V18 potential. These are plotted as functions of the variable $k$, which is a function of invariant mass of two free particles. In the mass-squared case this is also interpreted as the non-relativistic relative momentum, while in the Kamada-Glöckle method the corresponding “non-relativistic” relative momentum, $q$, is smaller. As discussed above, this tends to spread out the momentum space Kamada-Glöckle wave functions relative to the mass-squared wave functions when plotted as functions of $k$. This is seen in the plots of the $s$ and $d$-state wave deuteron wave functions in Figures 1 and 2.

Figure 1 shows the reduced $s$-state wave function and Figure 2 shows the reduced $d$-state wave function. The solid line in both figures corresponds to the mass-squared model while the dashed line corresponds to the Kamada-Glöckle model. Both figures show the predicted shift of the Kamada-Glöckle wave function to the right of the mass-squared wave function.

The shift is due to the relationship between $k$ and $q$ in (12) rather than due to the factor $h(q)$, which is a compensating factor that simply ensures that the change of variables is unitary. Because this shift is minimal when $q^2/4m^2 \ll 1$, the two wave functions look similar for momenta out to about $5$fm$^{-1}$ ($q^2/4m^2 \sim .25$).

We conclude that the difference in these two models is due to the difference in how the non-relativistic invariant energy is interpreted relativistically. Specifically with the mass-squared method, $k^2$ is identified as a relativistic relative momentum while in the Kamada-Glöckle method the relativistic and non-relativistic rest energies are identified. These differences explain how the models fail to be scattering equivalent. Nevertheless, both models provide reasonable relativistic interpretations of non-relativistic models. They differ on scales where the non-relativistic interpretation is in question.

These differences have implications for the three-body problem. Because current matrix elements can couple high and low momentum components of the wave functions, the possibility of observable differences between these two approaches at momentum transfers below 1(GeV)$^2$ exists. This is particularly relevant in the point-form spectator approximation, where the momentum transfer seen by a constituent nucleon is larger than the momentum seen by the deuteron [12].

The calculations shown in Figures 3, 4, and 5 are the deuteron elastic electron scattering observables $A(Q^2)$, $B(Q^2)$, and $T_{20}(Q^2)$ respectively in the point-form spectator approximation [12]. In the point-form spectator approximation matrix elements of the transverse current,
\[ \tilde{J}^\mu(0) := J^\mu(0) - q^\mu \frac{q \cdot J(0)}{q^2}, \]

are identified with matrix elements of the one-body operator,

\[ \tilde{J}^\mu(0) = \tilde{J}^\mu_n(0) + \tilde{J}^\mu_p(0), \]

in the Breit frame. All of the other current matrix elements are determined by current conservation and current covariance. The one-body currents are determined using Gari-Krümpelmann [13] single-nucleon form factors. The dotted curves are the result of using the Argonne V18 interaction to construct a Kamada-Glöckle interaction and solid curves are the result of using the same interaction to construct a mass-squared interaction.

The results show small, but measurable differences, in \( A(Q^2) \) at momentum transfers near 1 (Gev)\(^2\). The observables \( B(Q^2) \) exhibit differences in the two predictions that extend below the scales where differences are expected. The data in Figure 3 is from refs. [20] [21] [22] [23] while the data in Figure 3 is from [24] [25] [23] [26].

The results for the tensor polarization, \( T_{20}(Q^2) \), are also similar. Differences in the predictions of the two methods again extend below \( q^2 = 1(\text{GeV})^2 \). The data are from [27] [28] [24] [31] [30] [31].

We conclude that the difference in these two methods is due to differences in the interpretation of the relative momentum variable that appears in the non-relativistic Hamiltonian. The relevant rescaling is given in (33). The variable \( k^2/4m^2 \) defines a scale where the different interpretations of the momenta become significant. This is the scale on which the two methods fail to be scattering equivalent and is the scale that defines the shift in the momentum space wave functions. The mass-squared method is consistent with parameterizations of the Lorentz invariant cross section as a function of the projectile energy in the laboratory frame. For a fixed current model, the three-body calculations are sensitive to the differences in the wave functions. The model calculations show that these differences can have a measurable impact at much lower momentum transfers. To consistently interpret the Kamada-Glöckle model relativistically, the non-relativistic phase shifts would have to be refit. This only impacts the high momentum structure of the model.

These differences are shown to have measurable implications in elastic electron-deuteron scattering. In the realistic model presented, these differences lead to measurable effects in the observables \( B(Q^2) \) and \( T_{20}(Q^2) \) for momentum transfers as low as a few tenths of a (GeV)\(^2\), far lower than the scales where differences are expected.

We conclude that the mass-squared and Kamada-Glöckle methods define different dynamical models and that the mass-squared method is the one that is consistent with conventional methods for determining experimental cross sections. Quantitatively, the differences in these models are small, but they have dynamical consequences at surprisingly low momentum transfers in the three-body system.

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1. FIG. 1. The momentum space deuteron s-wave wave function for a relativistic point-form dynamics based on the mass squared model (solid line) and the Kamada-Glöckle model (dashed line).

2. FIG. 2. The momentum space deuteron d-wave wave function for a relativistic point-form dynamics based on the mass squared model (solid line) and the Kamada-Glöckle model (dashed line).

3. FIG. 3. $A(Q^2)$ for the mass squared interaction (solid line) and the Kamada-Glöckle model (dashed line). Both calculation use Argonne V18 as the input interaction, Gari-Krümpelmann [13] form factors, and the point-form spectator approximation. Data is from [20] [21] [22] [23].

4. FIG. 4. $B(Q^2)$ for the mass squared interaction (solid line) and the Kamada-Glöckle model (dashed line). Both calculation use Argonne V18 as the input interaction, Gari-Krümpelmann [13] form factors, and the point-form spectator approximation. Data is from [24] [25] [23] [26].

5. FIG. 5 $T_{20}(Q^2)$ for the mass squared interaction (solid line) and the Kamada-Glöckle model (dashed line). Both calculation use Argonne V18 as the input interaction, Gari-Krümpelmann [13] form factors, and the point-form spectator approximation. Data is from [27] [28] [29] [30] [31].
