1 Introduction

Let $R$ and $S$ be rings. A ring morphism $\varphi: R \to S$ is said to be local if, for every $r \in R$, $r$ is invertible in $R$ whenever $\varphi(r)$ is invertible in $S$. For instance, if $R$ is a ring and $I$ is a two-sided ideal of $R$ contained in the Jacobson radical of $R$, the canonical projection $R \to R/I$ is a local morphism. Conversely, the kernel of every local morphism $R \to S$ is contained in the Jacobson radical of $R$. We will denote by $J(R)$ the Jacobson radical of any ring $R$.

In Algebraic Geometry and Commutative Algebra, local morphisms are defined as the ring morphisms $\varphi: R \to S$, between local commutative rings $(R, M)$ and $(S, N)$, for which $\varphi(M) \subseteq N$. This definition coincides with ours in the case of $R$ and $S$ local.

In this spirit, Cohn considered local morphisms $R \to S$ when $R, S$ are not necessarily commutative and $S$ is a division ring. It is easily seen that if a ring $R$ has a local morphism into a division ring, then $R$ is a local ring.

Recall that a ring $R$ is called semilocal if $R/J(R)$ is a semisimple Artinian ring. The aim of this paper is to prove that under weak finiteness assumptions on an object $A$ of a Grothendieck category $\mathcal{C}$, the endomorphism ring $\text{End}_\mathcal{C}(A)$ of $A$ is semilocal. We prove that...
these rings $\text{End}_C(A)$ are semilocal making use of suitable ring homomorphisms which we show to be local morphisms.

It is known that endomorphism rings of artinian modules over an arbitrary ring [CD, Corollary 6], or of finitely generated modules over a semilocal commutative ring, or of finite-rank torsion-free modules over a commutative valuation domain or a semilocal commutative principal ideal domain [Wa, Lemma 2.3, Theorems 5.2 and 5.4] are semilocal. A number of other examples of modules with a semilocal endomorphism ring are given in [HS]. In this paper, we extend these results from the category $\text{Mod-}R$ of right $R$-modules to an arbitrary Grothendieck category $C$, and, also in the case in which $C = \text{Mod-}R$, we obtain new classes of modules whose endomorphism rings are semilocal. The advantage of knowing that a module has a semilocal endomorphism ring lies in the fact that modules with a semilocal endomorphism ring have a very good behavior as far as direct sums are concerned: they cancel from direct sums, satisfy the $n$-th root property, have only finitely many direct summands up to isomorphism, and have only finitely many direct-sum decompositions up to isomorphisms in the sense of the Krull-Schmidt theorem [K1 § 4.2]. Moreover, classes of modules with semilocal endomorphism rings give rise to Krull monoids [F2, Theorem 3.4]. This implies that though modules with semilocal endomorphism rings do not have uniqueness of direct-sum decomposition up to isomorphism, the direct-sum decompositions of these modules have a very regular geometrical pattern. Conversely, every finitely generated Krull monoid arises in this way from a finitely generated module over a noetherian commutative semilocal ring [Wi].

After a first introductory section with the main elementary properties of local morphisms (Section 2), we prove in Section 3 that every finitely presented module over a semilocal ring has a semilocal endomorphism ring (Theorem 3.3). This is one of the main results of the paper, and generalizes the previously known fact that every finitely generated module over a commutative semilocal ring has a semilocal endomorphism ring. We give an example of a finitely generated module over a noncommutative semilocal ring whose endomorphism ring is not semilocal (Example 3.5).

In Section 4, we show that local morphisms arise naturally in the construction of the spectral category $\text{Spec-}C$ of an arbitrary Grothendieck category $C$. The spectral category is obtained from $C$ inverting all essential monomorphisms [GO], and there is a natural functor $P: C \to \text{Spec-}C$. If $A$ is an object of $C$, there is a close relation between the fact that the ring morphism $\varphi_A: \text{End}_C(A) \to \text{End}_{\text{Spec-}C}(A)$ induced by the functor $P$ is local and that fact that every monomorphism $A \to A$ is an isomorphism. This allows us to generalize [HS, Theorem 3(1)]. In particular, a corollary of this is that endomorphism rings of artinian modules are semilocal.

In Section 5, we consider finitely copresented objects, that is, the objects $A$ of a Grothendieck category $C$ for which there exists an exact sequence $0 \to A \to L_0 \to L_1 \to 0$ with $L_0$ injective and both $L_0$ and $L_1$ of finite Goldie dimension. For a finitely copresented object $A$, there is a local morphism $\text{End}_C(A) \to \text{End}_{\text{Spec-}C}(A) \times \text{End}_{\text{Spec-}C}(L_1)$ (Theorem 5.3). As a corollary, the endomorphism ring of a finitely copresented object is semilocal. For instance, this shows that finite-rank torsion-free modules over any semilocal commutative noetherian
domain $R$ of Krull dimension 1 have semilocal endomorphism rings (Corollary 5.9), a fact which was previously known only under the stronger condition of $R$ semilocal commutative principal ideal domain.

In Section 6, we dualize the construction of spectral category, obtaining a category $C'$ inverting all superfluous epimorphisms of a Grothendieck category $C$. There is a natural functor $F: C \rightarrow C'$. As the dual of a Grothendieck category $C$ is not a spectral category in general, the additive category $C'$ obtained in this way is not necessarily a Grothendieck category. We consider the ring morphism $\psi_A: \text{End}_C(A) \rightarrow \text{End}_{C'}(A)$ induced by the functor $F$ for every object $A$ of $C$. This morphism $\psi_A$ is local when $A$ has finite dual Goldie dimension and every epimorphism $A \rightarrow A$ in $C$ is an isomorphism (Proposition 6.3). For an arbitrary object $A$ of $C$, the ring morphism $(\varphi_A, \psi_A): \text{End}_C(A) \rightarrow \text{End}_{\text{Spec-}C}(A) \times \text{End}_{C'}(A)$ turns out to be local. This leads to a generalization of [HS, Theorem 3, (2) and (3)].

In the last section, we apply the results about the functor $F$ obtained in Section 6 to objects with a projective cover. For every exact sequence $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$ where $P \rightarrow A$ is a projective cover, there is a local morphism $\text{End}_C(A) \rightarrow \text{End}_{C'}(A) \times \text{End}_{C'}(K)$ (Theorem 7.2). Thus if both $A$ and $K$ have finite dual Goldie dimension, then the endomorphism ring of $A$ is semilocal.

Our rings are associative and have an identity, and modules are unital.

## 2 Local morphisms

In the next lemma we collect some basic properties of local morphisms. If $\varphi: R \rightarrow S$ is a ring morphism, we shall denote by $M_n(\varphi): M_n(R) \rightarrow M_n(S)$ the ring morphism induced by $\varphi$ between the rings of $n \times n$ matrices with entries in $R$ and $S$ respectively.

**Lemma 2.1** Let $\varphi: R \rightarrow S$, $\psi: S \rightarrow T$ be ring morphisms.

1. If $\varphi$ is local, then $\ker(\varphi) \subseteq J(R)$ [FH, Lemma 3.1].
2. If $\varphi$ is onto and local, then $\varphi(J(R)) = J(S)$ and the induced morphism $M_n(\varphi): M_n(R) \rightarrow M_n(S)$ is local for every $n > 1$ [FH, Lemma 3.1].
3. If $\varphi$ and $\psi$ are local morphisms, then $\psi \circ \varphi$ is local.
4. If $\psi \circ \varphi$ is a local morphism, then $\varphi$ is local.

Local morphisms can be characterized in terms of endomorphisms between cyclic projective modules:

**Lemma 2.2** Let $\varphi: R \rightarrow S$ be a ring morphism. The following statements are equivalent:

1. The morphism $\varphi$ is local.
2. If $f: P \rightarrow P$ is an endomorphism of a cyclic projective right $R$-module $P$ such that $f \otimes_R S$ is invertible, then $f$ is invertible.
3. If $g: Q \rightarrow Q$ is an endomorphism of a cyclic projective left $R$-module $Q$ such that $S \otimes_R g$ is invertible, then $g$ is invertible.

Most of our examples of local morphisms will satisfy stronger properties also. To avoid confusion, it is interesting to keep in mind the following examples.
Examples 2.3  (1) For any ring $R$, the canonical projection $\pi: R \to R/J(R)$ is a local morphism.

(2) If $D$ is a division ring, the ring embedding $\varphi: R = \left( \begin{array}{cc} D & D \\ 0 & D \end{array} \right) \to S = M_2(D)$ is local. Let $e_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), e_2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$ and $x = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$. Left multiplication by $x$ induces a morphism $f: e_2 R \to e_1 R$ such that $M_n(f \otimes_R S)$ is invertible for all $n \geq 1$, but $f$ is not invertible. Notice that, in view of (1), another local morphism of $R$ is given by the natural projection $R \to R/J(R) \cong D \times D$.

(3) It is not true in general that $\varphi: R \to S$ local implies that the induced morphism between the matrix ring $M_n(\varphi): M_n(R) \to M_n(S)$ is local for all $n \geq 2$. An example in which this fails is given in [FH, p. 189].

In Section 6, we shall recall the definition of the dual Goldie dimension $\text{codim}(A)$ of an object $A$ of an arbitrary Grothendieck category. It is a non-negative integer or $\infty$. Now we only recall that if a ring $R$ is semilocal if and only if $\text{codim}(R_R)$ is finite, if and only if $\text{codim}(R_R)$ is finite. In this case, $\text{codim}(R_R) = \dim(R/J(R))$ of the semisimple module $R/J(R)$ (cf. [FH] Proposition 2.43).

The following deep result by Rosa Camps and Warren Dicks (see [CD] Theorem 1 or [FH] Theorem 4.2) characterizes semilocal rings in terms of local morphisms. We will use it throughout the paper.

**Theorem 2.4** If $R \to S$ is a local morphism between arbitrary rings $R$ and $S$, then $\text{codim}(R) \leq \text{codim}(S)$. In particular, a ring $R$ is semilocal if and only if there exists a local morphism of $R$ into a semilocal ring, if and only if there exists a local morphism of $R$ into a semisimple artinian ring.

In general, little can be said about rings having local morphisms to arbitrary products of division rings or to products of rings of matrices over division rings. If $R$ is a commutative ring with maximal spectrum $\text{Max}(R)$, then the morphism $R \to \prod_{M \in \text{Max}(R)} R/M$ given by $r \mapsto (r + M)$ is local. This result can be extended to the noncommutative setting taking as spectrum the set of primitive ideals of $R$. In some sense, the Camps-Dicks Theorem characterizes semilocal rings as those having “finite spectrum”. Notice that from Theorem 2.4 it will follow that for every ring $R$ there exists a local morphism of $R$ into a von Neumann regular right self-injective ring.

If $\varphi: R \to S$ is a local morphism with $S$ semilocal, it is not clear which relation there is between $R/J(R)$ and $S/J(S)$, apart from the fact that $\text{codim}(R) \leq \text{codim}(S)$, that is, $\dim(R/J(R)) \leq \dim(S/J(S))$ (cf. Example 2.3 and Theorem 2.4). In the following Proposition, whose proof is modelled by the proof of [CM] Lemma 3.2, we analyze the case in which $S/J(S)$ is a finite direct product of division rings. We show that the induced morphisms $M_n(\varphi): M_n(R) \to M_n(S)$ are also local, which is not true for arbitrary rings [FH] p. 189].
Proposition 2.5 Let $\varphi: R \to S$ be a local morphism. Assume that $S/J(S) \cong D_1 \times \cdots \times D_k$, where $D_i$ is a division ring for every $i = 1, \ldots, k$. For $i = 1, \ldots, k$, let $\tau_i: R \to D_i$ denote the composition of $\varphi$ with the projection $S \to D_i$. Then:

(1) There exist $m \leq k$ and $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$ such that $R/J(R) \cong D_{i_1} \times \cdots \times D_{i_m}$, where $D_j$ is a division subring of $D_j$ for every $j = 1, \ldots, m$.

(2) The ring $R$ has exactly $m$ maximal ideals, and these are the ideals $\ker(\tau_{i_j})$ for $j = 1, \ldots, m$. Hence,

$$(\tau_{i_1}, \ldots, \tau_{i_m}): R \to D_{i_1} \times \cdots \times D_{i_m}$$

is a local morphism with kernel $J(R)$.

(3) The induced ring morphism $M_n(\varphi): M_n(R) \to M_n(S)$ is local for every $n \geq 1$.

Proof. Let $\pi: S \to S/J(S)$ denote the canonical projection. By Lemma 2.1(3 and 4), the morphism $\pi \circ \varphi$ is local, and $M_n(\varphi)$ is local if and only if $M_n(\pi \circ \varphi)$ is local for any $n \geq 1$. Thus, to prove the Proposition, we may assume that $S = D_1 \times \cdots \times D_k$.

By Lemma 2.1(1), $\ker(\varphi) \subseteq J(R)$. The inclusion $\epsilon: \varphi(R) \hookrightarrow S$ is a local morphism. Hence, for any $n \geq 1$, the morphism $M_n(\epsilon)$ is local if and only if the morphism $M_n(\varphi)$ is local. Thus, to prove the Proposition, we may assume that $R$ is a subring of $S = D_1 \times \cdots \times D_k$ such that the embedding $\varphi: R \hookrightarrow D_1 \times \cdots \times D_k$ is local.

If $k = 1$, $R$ is a division subring of $D_1$ and (1), (2), (3) hold trivially. Now we shall proceed by induction on $k$. Assume $k > 1$.

If $R$ has a nontrivial idempotent $e$, then $e$ is central because all idempotents of $D_1 \times \cdots \times D_k$ are central. Therefore there is a partition of $\{1, \ldots, k\}$ into two nonempty subsets $I, J$ such that the embeddings

$$\varphi|_eR: eR \to eS = \prod_{i \in I} D_i$$

and

$$\varphi|_{(1-e)R}: (1-e)R \to (1-e)S = \prod_{j \in J} D_j$$

are local morphisms. By the inductive hypothesis, the Proposition holds for $eR$ and $(1-e)R$, so it holds for $R = eR \times (1-e)R$. Therefore we may assume that $R$ has no nontrivial idempotent.

For any element $r \in R$, set

$$\text{supp}(r) = \{ i \mid i = 1, \ldots, k, \tau_i(r) \neq 0 \}.$$ 

Let $d$ be the function of $R$ into the set of nonnegative integers defined by $d(r) = |\text{supp}(r)|$, so that $d$ is a nonzero function. Let $\ell$ be the least nonzero value of $d$. If $\ell = k$, then we can choose any $i \in \{1, \ldots, k\}$ and we get that $\tau_i: R \to D_i$ is local, so the Proposition follows from the case $k = 1$. Assume $\ell < k$.

Let $r \in R$ be an element such that $d(r) = \ell$. Suppose that there exists $t \in R$ such that $1 - tr$ is not invertible. Then $tr \neq 0$, so that $\text{supp}(tr) \subseteq \text{supp}(r)$ implies $\text{supp}(tr) = \text{supp}(r)$. As $\text{supp}(r) \cup \text{supp}(1 - tr) = \{1, 2, \ldots, k\}$ and $1 - tr$ is not invertible, that is, $\text{supp}(1 - tr) \neq \{1, 2, \ldots, k\}$, it follows that $\text{supp}(r) \nsubseteq \text{supp}(1 - tr)$. This implies that $\text{supp}(r(1 - tr)) = \text{supp}(r) \cap \text{supp}(1 - tr) \nsubseteq \text{supp}(r)$. Hence, by the choice of $r$, $r = rt^r$. But then $tr$ is idempotent and, as $R$ has no nontrivial idempotent and $tr \neq 0$, it follows that
$tr = 1$, which is impossible because $d(r) = \ell < n$. This shows that $1 - tr$ is invertible for every $t \in R$, so that $r \in J(R)$.

Let

$$I = \{ x \in R \mid \text{supp}(x) \subseteq \text{supp}(r) \} = \{ x \in R \mid \text{supp}(x) = \text{supp}(r) \} \cup \{ 0 \}.$$ 

Note that $I$ is a two-sided ideal of $R$ which is contained in $J(R)$ by our previous argument. Set $K = \{ 1, \ldots, n \} \setminus \text{supp}(r)$. The local embedding $\varphi$ induces an injective ring morphism $\varphi: R/I \to \prod_{k \in K} D_k$. Let us prove that $\varphi$ is local. If $t \in R$ is such that $\varphi(t + I)$ is invertible, then $K \subseteq \text{supp}(t)$, so that $\text{supp}(t) \cup \text{supp}(r) = \{ 1, \ldots, n \}$. As $\text{supp}(t) \cap \text{supp}(r) = \text{supp}(tr)$ must have either 0 or $\ell$ elements, it follows that either $\text{supp}(t) \cap \text{supp}(r) = \emptyset$ or $\text{supp}(r) \subseteq \text{supp}(t)$. If $\text{supp}(t) \cap \text{supp}(r) = \emptyset$, then $\varphi(t + r)$ is invertible. If $\text{supp}(r) \subseteq \text{supp}(t)$, then $\text{supp}(t) = \{ 1, \ldots, n \}$, so that $\varphi(t)$ is invertible. Hence either $\varphi(t + r)$ or $\varphi(t)$ is invertible, so that either $t + r$ or $t$ is invertible in $R$. In both cases $t + I$ is invertible in $R/I$. Thus $\varphi$ is a local morphism. By the inductive hypothesis, claims (1), (2) and (3) hold for $R/I$. Therefore they hold for $R$, because $I \subseteq J(R)$. ■

For further reference, we specialize Proposition 2.5 to the case $k = 2$.

**Corollary 2.6** Let $\varphi: R \to D_1 \times D_2$ be a local morphism where $D_1$ and $D_2$ are division rings. For $i = 1, 2$, let $\tau_i: R \to D_i$ be the composition of $\varphi$ with the projection $D_1 \times D_2 \to D_i$. Then there are two possibilities:

1. either $R$ is local, and there exists $i \in \{ 1, 2 \}$ such that $\tau_i$ is a local morphism. In this case the maximal ideal of $R$ is $\ker(\tau_i)$;
2. or $R/J(R) \cong D'_1 \times D'_2$, where $D'_i$ is a division subring of $D_i$ for $i = 1, 2$. Moreover, $J(R) = \ker \varphi$ and the two maximal ideals of $R$ are $\ker(\tau_1)$ and $\ker(\tau_2)$.

We conclude this section with a result that is easy but very useful in producing examples of modules whose endomorphism ring is semilocal.

**Proposition 2.7** Let $R \to S$ be a ring morphism, and let $M_S$ be an $S$-module with $\text{End}(M_R)$ semilocal. Then $\text{End}(M_S)$ is semilocal.

**Proof.** Since $S$-module endomorphisms of $M_S$ are $R$-module endomorphisms, there is an embedding $\text{End}(M_S) \to \text{End}(M_R)$, which is clearly a local morphism. The Proposition follows from Theorem 2.4. ■

### 3 Finitely presented modules over semilocal rings

We begin this section with a known result, of which we give an elementary proof using the notion of local morphism studied in this paper.

**Proposition 3.1** Every finitely generated module over a commutative semilocal ring has a semilocal endomorphism ring.
Proof. Let $M_R$ be a finitely generated module over a commutative semilocal ring $R$ and let $\text{End}(M_R)$ be its endomorphism ring. Consider the canonical mapping $\varphi : \text{End}(M_R) \to \text{End}(M_R/M_RJ(R))$. This mapping $\varphi$ is a local morphism. To see it, let $f$ be an endomorphism of $M_R$ with $\varphi(f)$ an automorphism of $M_R/M_RJ(R)$. By Nakayama’s Lemma, $f$ must be an epimorphism. Then $f$ must be also injective by [V1, Proposition 1.2]. This proves that $\varphi$ is local. But $\text{End}(M_R/M_RJ(R))$ is the endomorphism ring of a finitely generated module over the ring $R/J(R)$, which is a direct product of finitely many fields. Thus $\text{End}(M_R/M_RJ(R))$ is semilocal, so that $\text{End}(M_R)$ is semilocal by Theorem 2.4.

Combining Proposition 3.1 and Proposition 2.7 we get the following extension of Proposition 3.1 [Wa, Lemma 2.3].

**Proposition 3.2** Let $R$ be a semilocal commutative ring and $S$ a (not necessarily commutative) $R$-algebra. If $M_S$ is any $S$-module such that $M_R$ is finitely generated, then the endomorphism ring of $M_S$ is semilocal.

Now we show that Proposition 3.1 can be extended to semilocal rings not necessarily commutative, provided we consider finitely presented modules only.

**Theorem 3.3** The endomorphism ring of a finitely presented module over a semilocal ring is a semilocal ring.

Proof. Let $R$ be a semilocal ring, $M$ a finitely presented right $R$-module and $\text{End}(M)$ its endomorphism ring.

Step 1. The Theorem holds under the additional hypothesis that there exists an exact sequence $0 \to K \xrightarrow{\iota} F \to M \to 0$, where $F$ denotes a finitely generated free $R$-module, $K$ is a submodule of $FJ(R)$ and $\iota : K \to F$ denotes the inclusion.

For every endomorphism $g$ of a right $R$-module $A$, we shall denote by $g$ the endomorphism of the module $A/AJ(R)$ induced by $g$.

If $f \in \text{End}(M)$, then there exist an endomorphism $f_0$ of $F$ and an endomorphism $f_1$ of $K$ making the diagram

\[
\begin{array}{cccccc}
0 & \to & K & \xrightarrow{\iota} & F & \to & M & \to & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
0 & \to & K & \xrightarrow{\iota} & F & \to & M & \to & 0
\end{array}
\]

commute.

We claim that the endomorphism $\overline{f_1}$ of $K/KJ(R)$ does not depend on the choice of the lifting $f_0$ of $f$. In order to prove the claim, let $f_0'$ be another lifting of $f$ and $f_1'$ the corresponding restriction to $K$. Then $(f_0 - f_0')(F) \subseteq K$, so that there exists a morphism $g : F \to K$ with $f_0 - f_0' = \iota g$. Thus $g(K) \subseteq g(FJ(R)) \subseteq KJ(R)$. Therefore $(f_1 - f_1')(K) = (f_0 - f_0')(K) = \iota g(K) \subseteq KJ(R)$. That is, $\overline{f_1} - \overline{f_1'} = f_1 - f_1' = 0$. This proves the claim.

Define $\psi : \text{End}(M) \to \text{End}(M/MJ(R)) \times \text{End}(K/KJ(R))$ by $\psi(f) = (\overline{f}, \iota^* f)$ for every $f \in \text{End}(M)$. This is a well defined mapping by the claim, and it is clearly a ring morphism. We shall now prove that $\psi$ is local. Let $f \in \text{End}(M)$ be an endomorphism of $M$ with $\psi(f)$...
invertible. Let \( f_0 \in \text{End}(F) \) be a lifting of \( f \) and \( f_1 : K \to K \) be the restriction of \( f_0 \) to \( K \). As \( \psi(f) = (\overline{f}, \overline{f_1}) \) is invertible, \( f \) must be surjective by Nakayama’s Lemma and \( \ker f \subseteq MJ(R) \). Similarly, as \( \overline{f_1} \) is invertible, \( f_1 \) must be surjective and \( \ker f_1 \subseteq KJ(R) \).

Since \( f \) is surjective, it follows that \( f_0(F) + K = F \), hence \( f_0 \) also must be surjective by Nakayama’s Lemma. Thus \( f_0 \) must be a splitting epimorphism, because \( F \) is projective, so that \( F \cong F \oplus \ker f_0 \). In particular, \( \ker f_0 \) is a finitely generated \( R \)-module, and the finitely generated semisimple modules \( F/FJ(R) \) and \( F/F(R) \oplus \ker f_0/\ker f_0J(R) \) are isomorphic, so that \( \ker f_0/\ker f_0J(R) = 0 \), from which \( \ker f_0 = 0 \). This proves that \( f_0 \) is an automorphism. Since \( f_1 \) is surjective, that is, \( f_0(K) = K \), it follows that \( f_0^{-1}(K) = K \). Therefore \( f \) is injective. This proves that \( \psi \) is a local morphism. By Theorem [2], the ring \( \text{End}(M) \) is semilocal.

**Step 2.** For every simple \( R \)-module \( S \) there exists a finitely presented \( R \)-module \( N \) such that \( S \cong N/NJ(R) \).

As \( R \) is semilocal, \( S \) is isomorphic to a direct summand of \( R/J(R) \), so that there exists an isomorphism \( \varphi : S \oplus T \to R/J(R) \) for some \( R/J(R) \)-module \( T \). The \( R/J(R) \)-module \( T \) is cyclic. Let \( t \) denote a generator of \( T \), and let \( r \in R \) be such that \( \varphi(t) = r + J(R) \). Then \( N = R/rR \) has the required property, because

\[
N/NJ(R) \cong (R/rR)/(R/rR)J(R) \cong (R/rR)/(J(R) + rR/rR) \\
\cong R/\langle J(R) + rR \rangle \cong (R/J(R))/(J(R) + rR/J(R)) \\
\cong (R/J(R))/r(R/J(R)) = (R/J(R))/(r + J(R))(R/J(R)) \cong S.
\]

**Step 3.** For every finitely generated \( R \)-module \( M \), there exists a finitely presented \( R \)-module \( N \) such that \( M/MJ(R) \oplus N/NJ(R) \) is a free \( R/J(R) \)-module.

The \( R \)-module \( M/MJ(R) \) is finitely generated and semisimple, and \( R/J(R) \) contains a direct summand isomorphic to every simple \( R \)-module. Therefore \( M/MJ(R) \) is isomorphic to a direct summand of \( (R/J(R))^n \) for some nonnegative integer \( n \). Thus there exist simple \( R \)-modules \( S_1, \ldots, S_m \) with \( M/MJ(R) \oplus S_1 \oplus \cdots \oplus S_m \cong (R/J(R))^n \). By Step 2, there exist finitely presented \( R \)-modules \( N_1, \ldots, N_m \) with

\[
M/MJ(R) \oplus N_1/N_1J(R) \oplus \cdots \oplus N_m/N_mJ(R) \cong (R/J(R))^n.
\]

The module \( N = N_1 \oplus \cdots \oplus N_m \) has the required properties.

**Step 4.** Every finitely presented \( R \)-module \( M \) has a semilocal endomorphism ring.

By Step 3, there exists a finitely presented \( R \)-module \( N \) such that

\[
M/MJ(R) \oplus N/NJ(R) \cong (R/J(R))^n
\]

for some \( n \geq 0 \). Let \( F \) be the free \( R \)-module \( R^n \), so that there exists a surjective morphism of \( R \)-modules \( F \to M/MJ(R) \oplus N/NJ(R) \) with kernel \( FJ(R) \). Thus there exists a surjective morphism of \( R \)-modules \( F \to M \oplus N \) whose kernel \( K \) is contained in \( FJ(R) \). By Step 1, the finitely presented \( R \)-module \( M \oplus N \) has a semilocal endomorphism ring. As direct summands of modules with semilocal endomorphism rings have semilocal endomorphism rings [2, Proposition 1.13], the module \( M \) also has a semilocal endomorphism ring.
Remark 3.4 We have made the proof of Theorem 3.3 as self-contained as possible, but in the rest of the paper we will develop and refine the ideas and the techniques we have met in the proof. Step 1 in the proof of Theorem 3.3 is a consequence of Theorem 7.3 because finitely generated modules over a semilocal ring have finite dual Goldie dimension. The remaining part of the proof of Theorem 3.3 is devoted to showing the somewhat interesting fact that every finitely presented module over a semilocal ring is a direct summand of a finitely presented module with a projective cover.

In Example 3.5 we shall show that there exist finitely generated modules over semilocal rings whose endomorphism ring is not semilocal. Thus Proposition 3.1 cannot be extended to arbitrary semilocal rings, and Theorem 3.3 cannot be extended to arbitrary finitely generated modules.

Recall that a semiperfect ring is a semilocal ring whose idempotents can be lifted modulo the Jacobson radical. A semiprimary ring is a semilocal ring whose Jacobson radical is nilpotent, and a right perfect ring is a semilocal ring whose Jacobson radical is right T-nilpotent. Björk proved that finitely presented right modules over a semiprimary ring have a semiprimary endomorphism ring [B2, Theorem 4.1]. This result was reproved and extended by Schofield [Sc, Theorem 7.18] and Rowen [R, Corollary 11]. Their results show that a finitely presented right module over a right (or left) perfect ring has a right (left, respectively) perfect endomorphism ring. Wiegand constructed plenty of examples of finitely generated modules over local (in particular, semiperfect) commutative noetherian rings whose endomorphism rings are semilocal but not semiperfect [Wi].

Our next example is a variation of [B1, Example 2.1, p. 127]. It shows that the endomorphism ring of finitely generated modules over semiprimary rings need not be semilocal.

Example 3.5 Let $K$ be a field with a non-onto endomorphism $\alpha : K \to K$. Let $K_0 = \alpha(K)$. Let $K V$ be a non-zero $K$-vector space. View $K V$ as a $K$-$K$-bimodule taking the scalar product by $K$ as left action and setting as right action $v \cdot k = \alpha(k)v$ for every $v \in V$ and every $k \in K$.

Let $R = \begin{pmatrix} K & K V_K \\ 0 & 0 \end{pmatrix}$. Then $J(R) = \begin{pmatrix} 0 & K V_K \\ 0 & 0 \end{pmatrix}$, $R/J(R) \cong K \times K$ and $J(R)^2 = 0$, so that $R$ is semiprimary. Fix $a \in K \setminus K_0$ and $0 \neq w \in V$. Consider the right ideal

$$I = \sum_{n \geq 0} \begin{pmatrix} 0 & a^n w \\ 0 & 0 \end{pmatrix} R = \begin{pmatrix} 0 & K_0[a]w \\ 0 & 0 \end{pmatrix}$$

of $R$. Then $E := \text{End}_R(R/I) \cong I/I$, where $I$ is the idealizer of $I$ in $R$, that is, $I = \{ r \in R \mid rI \subseteq I \}$.

Let $\begin{pmatrix} k_1 & v \\ 0 & k_2 \end{pmatrix} \in I$. As

$$\begin{pmatrix} k_1 & v \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & k_1 w \\ 0 & 0 \end{pmatrix} \in I,$$
we deduce that $\mathcal{I} = \begin{pmatrix} K_0[a] & 0 \\ 0 & V \end{pmatrix}$. Hence $E/J(E) \cong K_0[a] \times K$.

If we choose $K$, $\alpha$ and $\alpha$ such that $\alpha$ is transcendental over $K_0$, then $K_0[a] \times K$ is not semisimple artinian. Hence, $E$ is not semilocal.

If in Example 3.3 one considers the trivial extension of $K$ by $V$ instead of $R$, that is,

$$K \times V = \left\{ \begin{pmatrix} k & v \\ 0 & k \end{pmatrix} \mid k \in K \text{ and } v \in V \right\},$$

then one obtains an example of a cyclic module over the local ring $K \times V$ whose endomorphism ring is not semilocal.

### 4 Spectral Categories

In this section, we shall recall some results about spectral categories that will be used in the following section. Spectral categories were introduced by Gabriel and Oberst, see [GO] or [St, Ch. V, §7]. For a Grothendieck category $\mathcal{C}$, the spectral category of $\mathcal{C}$, denoted by $\text{Spec-} \mathcal{C}$, is the category with the same objects as $\mathcal{C}$ and, for objects $A$ and $B$ of $\mathcal{C}$, with $\text{Hom}_{\text{Spec-}\mathcal{C}}(A,B) = \lim_{\rightarrow} \text{Hom}_{\mathcal{C}}(A',B)$, where the direct limit is taken over the downwards directed family of essential subobjects $A'$ of $A$. There is a left exact canonical functor $P: \mathcal{C} \to \text{Spec-} \mathcal{C}$, which is the identity on objects and takes $f \in \text{Hom}_{\mathcal{C}}(A,B)$ to its canonical image in $\text{Hom}_{\text{Spec-}\mathcal{C}}(A,B)$. This functor $P$ induces a ring morphism $\varphi_A: \text{End}_{\mathcal{C}}(A) \to \text{End}_{\text{Spec-}\mathcal{C}}(A)$ for every object $A$ of $\mathcal{C}$.

**Remark 4.1** The kernel of $\varphi_A$ is the ideal $I_A$ of all $f \in \text{End}_{\mathcal{C}}(A)$ with kernel essential in $A$.

For every object $A$ of $\mathcal{C}$, let $E(A)$ denote the injective envelope of $A$ in $\mathcal{C}$. Then $\text{End}_{\text{Spec-}\mathcal{C}}(A) \cong \text{End}_{\mathcal{C}}(E(A))/J(\text{End}_{\mathcal{C}}(E(A)))$ is a von Neumann regular right self-injective ring.

**Remark 4.2** If $A$ is an injective object, the morphism $\varphi_A: \text{End}_{\mathcal{C}}(A) \to \text{End}_{\text{Spec-}\mathcal{C}}(A)$ is the canonical projection of $\text{End}_{\mathcal{C}}(A)$ onto $\text{End}_{\mathcal{C}}(E(A))/J(\text{End}_{\mathcal{C}}(E(A))) \cong \text{End}_{\text{Spec-}\mathcal{C}}(A)$. Therefore $\varphi_A$ is a local morphism for every injective object $A$ (Example 2.3(1)).

Recall that an object $A$ of a Grothendieck category is said to be *directly finite* if it is not isomorphic to a proper direct summand of itself.

**Proposition 4.3** Let $A$ be an object in a Grothendieck category $\mathcal{C}$. If every monomorphism $A \to A$ is an isomorphism, then $\varphi_A: \text{End}_{\mathcal{C}}(A) \to \text{End}_{\text{Spec-}\mathcal{C}}(A)$ is a local morphism. Conversely, if $\varphi_A$ is a local morphism and $E(A)$ is directly finite, then every monomorphism $A \to A$ is an isomorphism.
Proof. Assume that every monomorphism \( A \rightarrow A \) is an isomorphism. Let \( f \in \text{End}_C(A) \). If \( \varphi_A(f) \) is invertible, then any extension \( \overline{f} : E(A) \rightarrow E(A) \) of \( f \) is a monomorphism. Thus \( f \) is a monomorphism and, hence, an isomorphism. Conversely, let \( \varphi_A \) be a local morphism and \( E(A) \) directly finite. If \( f : A \rightarrow A \) is a monomorphism, then \( f \) extends to a monomorphism \( \overline{f} : E(A) \rightarrow E(A) \). As \( E(A) \) is directly finite, \( \overline{f} \) is an automorphism. Thus \( \varphi_A(f) \) is invertible. Since \( \varphi_A \) is local, \( f \) must be an isomorphism.

Proposition 4.4 The following conditions are equivalent for an object \( A \) of a Grothendieck category \( C \) and a nonnegative integer \( n \).

1. \( A \) has finite Goldie dimension \( n \).
2. \( P(A) \) is an object of finite length \( n \) in \( \text{Spec}-C \).
3. \( \text{End}_{\text{Spec}-C}(A) \) is a semisimple artinian ring of Goldie dimension \( n \).

Proof. (1) \( \Rightarrow \) (2). If \( A \) has finite Goldie dimension \( n \), then \( P(A) \cong P(E(A)) \) is a semisimple object in \( \text{Spec}-C \) of composition length \( n \) [St, p. 133].

(2) \( \Rightarrow \) (3). Every object of finite length in a spectral category is semisimple, hence it has a semisimple artinian endomorphism ring.

(3) \( \Rightarrow \) (1). Assume \( \text{End}_{\text{Spec}-C}(A) \cong \text{End}_C(E(A))/J(\text{End}_C(E(A))) \) is semisimple artinian. Then \( \text{End}_C(E(A)) \) is semiperfect, thus \( E(A) \) decomposes into a finite direct sum of injective indecomposable subobjects. Therefore \( E(A) \), hence \( A \), has finite Goldie dimension.

We shall denote the Goldie dimension of \( A \) by \( \text{dim}(A) \). We conclude the section with a slight generalization of [HS, Theorem 3(1)].

Corollary 4.5 Let \( A \) be an object in a Grothendieck category \( C \). Assume that \( A \) has finite Goldie dimension and that every monomorphism \( A \rightarrow A \) is an isomorphism. Then \( \text{End}_C(A) \) is semilocal.

Proof. By Proposition 4.4, the ring \( \text{End}_{\text{Spec}-C}(A) \) is semisimple artinian, and, by Proposition 4.3, \( \varphi_A \) is a local morphism. The statement follows as an application of Theorem 2.4.

From Corollary 4.5 and for \( C = \text{Mod}-R \), \( R \) any ring, one obtains that every artinian module has a semilocal endomorphism ring. For a different example, let \( R \) be a commutative ring of Krull dimension 0, that is, such that every prime ideal is maximal. Let \( M_R \) be a finitely generated module of finite Goldie dimension. Then \( \text{End}(M_R) \) is semilocal [V2].

5 Finitely copresented objects

In all this section, \( C \) will denote a Grothendieck category. An object \( A \) of \( C \) is said to be finitely copresented if there is an exact sequence in \( C \)

\[ 0 \rightarrow A \rightarrow L_0 \rightarrow L_1 \rightarrow 0, \]
with $L_0$ injective, and both $L_0$ and $L_1$ of finite Goldie dimension.

**Lemma 5.1** The following statements are equivalent for an object $A$ of a Grothendieck category $\mathcal{C}$.

1. The object $A$ is finitely copresented.
2. There is an exact sequence
   
   \[ 0 \to A \to E_0 \to E_1 \]
   
   with $E_0$ and $E_1$ injective objects of finite Goldie dimension and $A \to E_0$ an essential monomorphism.
3. The object $A$ is the kernel of a morphism between injective objects of finite Goldie dimension.

**Proof.** (1) $\Rightarrow$ (2) Assume there is an exact sequence

\[ 0 \to A \to L_0 \to L_1 \to 0, \]

with $L_0$ injective, $\dim(L_0) < \infty$ and $\dim(L_1) < \infty$. Then $L_0$ has an injective envelope $E_0$ of $A$ as a direct summand, and the sequence

\[ 0 \to A \to E_0 \to L_1 \]

is exact. Now substitute $L_1$ by its injective envelope $E_1$.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial. ■

Following the notation introduced in the previous section, let $P: \mathcal{C} \to \text{Spec-} \mathcal{C}$ denote the canonical functor of $\mathcal{C}$ into its spectral category.

**Lemma 5.2** Let $A$ be an object of $\mathcal{C}$ and let $L_0$ be its injective envelope. Consider the exact sequence $0 \to A \to L_0 \to L_1 \to 0$, so that every $f \in \text{End}_\mathcal{C}(A)$ extends to an endomorphism $f_0$ of $L_0$, which induces an endomorphism $f_1$ of $L_1$. Then $P(f_1)$ depends only on $f$ and not on the choice of the extension $f_0$ of $f$.

**Proof.** Let $f'_0$ be another extension of $f$ and $f'_1$ the corresponding endomorphism of $L_1$. Then $f_0 - f'_0$ induces a morphism $g: L_0/A \to L_0$. The inverse image $g^{-1}(A)$ is essential in $L_0/A$ because $A$ is essential in $L_0$. Therefore the endomorphism $f_1 - f'_1$ of $L_1$ induced by $f_0 - f'_0$ has an essential kernel. That is, $P(f_1 - f'_1) = P(f_1) - P(f'_1) = 0$. ■

By Lemma 5.2 for every object $A$ of $\mathcal{C}$ there is a ring morphism

\[ \chi: \text{End}_\mathcal{C}(A) \to \text{End}_{\text{Spec-} \mathcal{C}}(A) \times \text{End}_{\text{Spec-} \mathcal{C}}(L_1) \]

defined by $\chi(f) = (P(f), P(f_1))$.

**Theorem 5.3** The ring morphism $\chi$ is local for every object $A$ of $\mathcal{C}$.
Proof. Let \( f \in \text{End}_C(A) \) and assume \( \chi(f) \) invertible. Let \( f_0 \in \text{End}_C(L_0) \) be an extension of \( f \), and let \( f_1: L_1 \to L_1 \) be the induced endomorphism of \( L_1 \), so that we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & A & \to & L_0 & \to & L_1 & \to & 0 \\
\downarrow f & & \downarrow f_0 & & \downarrow f_1 & & & \\
0 & \to & A & \to & L_0 & \to & L_1 & \to & 0.
\end{array}
\]

As \( P(f) \) and \( P(f_1) \) are invertible, the morphisms \( f \) and \( f_1 \) must be essential monomorphisms. But \( P(A) \) is canonically isomorphic to \( P(L_0) \) and \( P(f) \) is an isomorphism in \( \text{Spec}-C \), so that \( P(f_0) \) is an isomorphism in \( \text{Spec}-C \). By Remark 4.2, the morphism \( f_0 \) of \( C \) is an isomorphism. The Snake Lemma gives an exact sequence

\[
0 = \ker f_1 \to \text{coker } f \to \text{coker } f_0 = 0,
\]

so that \( \text{coker } f = 0 \), i.e., \( f \) is also an epimorphism.  

From Theorem 5.3 it follows that for every ring \( R \) there exists a local morphism of \( R \) into a von Neumann regular right self-injective ring.

**Theorem 5.4** Let \( A \) be a finitely copresented object of a Grothendieck category \( C \). Then \( \text{End}_C(A) \) is a semilocal ring. Moreover, if \( L_0 \) denotes the injective envelope of \( A \), then \( \text{codim}(\text{End}_C(A)) \leq \dim(A) + \dim(L_0/A) \).

**Proof.** By Proposition 4.4, the ring \( \text{End}_{\text{Spec-}C}(A) \times \text{End}_{\text{Spec-}C}(L_0/A) \) is semisimple artinian and its Goldie dimension is \( \dim(A) + \dim(L_0/A) \). Now apply Theorem 5.3 and Theorem 2.4 to conclude.  

In the case in which \( C = \text{Mod-}R \), Theorem 5.4 becomes

**Corollary 5.5** Let \( M \) be a finitely copresented right module over an arbitrary ring \( R \). Then \( \text{End}_C(M) \) is a semilocal ring. Moreover, if \( L_0 \) denotes the injective envelope of \( M \), then \( \text{codim}(\text{End}_R(M)) \leq \dim(M) + \dim(L_0/M) \).

We say that a module \( M \) is *quotient finite dimensional* if every homomorphic image of \( M \) has finite Goldie dimension.

**Lemma 5.6** Let \( N \) be a submodule of a module \( M \). If both \( N \) and \( M/N \) are quotient finite dimensional, then \( M \) is quotient finite dimensional.

**Proof.** Let \( P \) be a submodule of \( M \). We must show that \( M/P \) has finite Goldie dimension. As \( M/(N + P) \) has finite Goldie dimension, there exist injective modules \( E_1, \ldots, E_n \) of Goldie dimension 1 and a homomorphism \( f: M \to E_1 \oplus \cdots \oplus E_n \) with \( \ker f = N + P \). Now \( N + P/P \cong N/N \cap P \) has finite Goldie dimension. Hence there exist injective modules \( E_{n+1}, \ldots, E_m \) of Goldie dimension 1 and a homomorphism \( g: N + P \to E_{n+1} \oplus \cdots \oplus E_m \) with \( \ker g = P \). The homomorphism \( g \) extends to a homomorphism \( h: M \to E_{n+1} \oplus \cdots \oplus E_m \).
Notice that \((N + P) \cap \ker h = P\). Consider the homomorphism \((f, h): M \to E_1 \oplus \cdots \oplus E_n \oplus E_{n+1} \oplus \cdots \oplus E_m\). Then \(\ker (f, h) = \ker f \cap \ker h = (N + P) \cap \ker h = P\). Therefore \(M/P\) has finite Goldie dimension. ■

Corollary 5.7 A direct sum of finitely many quotient finite dimensional modules is quotient finite dimensional.

From Lemma 5.1 and Corollary 5.5, we obtain:

Corollary 5.8 Every submodule of a quotient finite dimensional injective module has a semilocal endomorphism ring.

For instance, let \(R\) be a commutative noetherian semilocal domain of Krull dimension 1, and let \(Q\) be the field of fractions of \(R\). By [M Theorem 1 p. 571], the \(R\)-module \(Q/R\) is artinian, so that \(Q\) is a quotient finite dimensional injective \(R\)-module. By Corollary 5.7 all \(Q^n\) are quotient finite dimensional injective modules, so that their submodules, that is, torsion-free modules of finite rank, have semilocal endomorphism rings. Applying Proposition 2.4, we get the following corollary, which generalizes a result proved by Warfield only for the case in which \(R\) is a commutative semilocal principal ideal domain (cf. [Wa, Theorem 5.2]).

Corollary 5.9 Let \(R\) be a commutative noetherian semilocal domain of Krull dimension 1 and let \(S\) be an \(R\)-algebra. Let \(M_S\) be an \(S\)-module that is torsion-free of finite rank as an \(R\)-module. Then \(\text{End}(M_S)\) is semilocal.

More generally, we have shown that if \(R\) is a commutative integral domain, the field of fractions \(Q\) of \(R\) is a quotient finite dimensional \(R\)-module, \(S\) is an \(R\)-algebra and \(M_S\) is an \(S\)-module that is torsion-free of finite rank as an \(R\)-module, then \(M_S\) has a semilocal endomorphism ring. For the case of \(R\) a valuation domain, this is [Wa, Theorem 5.4].

We shall now give a further extension of [Wa, Theorem 5.4] to the noncommutative setting. Recall that a right module \(M\) is uniserial if its lattice of submodules is linearly ordered by set inclusion, that is, if for any submodules \(N\) and \(P\) of \(M\) either \(N \subseteq P\) or \(P \subseteq N\). A module is serial if it is a direct sum of uniserial submodules.

Corollary 5.10 Let \(E\) be an injective serial right module of finite Goldie dimension over an arbitrary (not necessarily commutative) ring. Then the endomorphism ring of every submodule of \(E\) is semilocal.

Proof. The module \(E\) is a direct sum of uniserial submodules, necessarily finitely many because \(E\) has finite Goldie dimension. Thus \(E\) is quotient finite dimensional by Corollary 5.7. Now apply Corollary 5.8. ■

We conclude this section with an application of Theorem 5.4 to a category \(C\) that is not a category Mod-\(R\).
Corollary 5.11 Let $R$ be a ring. Let $E_0, E_1$ be direct sums of $n,m$ indecomposable pure-injective right $R$-modules, respectively. Let $f: E_0 \rightarrow E_1$ be a morphism whose kernel $M$ is pure in $E_0$ and whose image $f(E_0)$ is pure in $E_1$, so that the pure-injective envelopes of $M$ and $f(E_0)$ are direct sums of $r \leq n$ and $s \leq m$ indecomposable pure-injective right $R$-modules, respectively. Then $\text{End}_R(M)$ is a semilocal ring and $\text{codim}(\text{End}_R(M)) \leq 2r+s-n$.

Proof. Let $R\text{-mod}$ denote the category of finitely presented left $R$-modules, and let $\mathcal{F} := \text{Add}(R\text{-mod}, \text{Ab})$ denote the category of additive functors from $R\text{-mod}$ to the category $\text{Ab}$ of abelian groups. The assignment $X \mapsto X \otimes_R -$ defines a functor $\Phi: \text{Mod}-R \rightarrow \mathcal{F}$, which is a full and faithful. Moreover, $\Phi$ sends pure-injective objects of $\text{Mod}-R$ to injective objects of the Grothendieck category $\mathcal{F}$ and pure-exact sequences of $\text{Mod}-R$ to exact sequences of $\mathcal{F}$ (cf. [11, Theorem B.16] or [11, § 1.6]).

Therefore $\Phi$ sends the pure-exact sequences $0 \rightarrow M \rightarrow E_0 \rightarrow f(E_0) \rightarrow 0$ and $0 \rightarrow f(E_0) \rightarrow E_1$ to the exact sequences $0 \rightarrow \Phi(M) \rightarrow \Phi(E_0) \rightarrow \Phi(f(E_0)) \rightarrow 0$ and $0 \rightarrow \Phi(f(E_0)) \rightarrow \Phi(E_1)$. Thus the sequence $0 \rightarrow \Phi(M) \rightarrow \Phi(E_0) \rightarrow \Phi(E_1)$ is exact, i.e., the functor $\Phi(M)$ is the kernel of the morphism $\Phi(f): \Phi(E_0) \rightarrow \Phi(E_1)$ between injective objects of $\mathcal{F}$. Notice that if $E$ is a direct sums of $t$ indecomposable pure-injective right $R$-modules, the object $\Phi(E)$ has Goldie dimension $t$ in $\mathcal{F}$, because, by Proposition 4.4, the Goldie dimension of $\Phi(E)$ in $\mathcal{F}$ is equal to the Goldie dimension of $\text{End}_{\text{Spec}-\mathcal{F}}(\Phi(E)) \cong \text{End}_\mathcal{F}(\Phi(E))/J(\text{End}_\mathcal{F}(\Phi(E))) \cong \text{End}_R(E)/J(\text{End}_R(E))$. This shows that $\Phi(M)$ is a finitely copresented object in $\mathcal{F}$ (Lemma 5.1).

Theorem 5.24 implies that $\text{End}_\mathcal{F}(\Phi(M)) \cong \text{End}_R(M)$ is semilocal of dual Goldie dimension $\leq \dim(\Phi(M)) + \dim(F/\Phi(M))$, where $F$ denotes the injective envelope of $\Phi(M)$. Thus $F = \Phi(P)$, where $P$ denotes the pure-injective envelope of $M$. But $\dim(\Phi(M)) = \dim(F) = \dim(\Phi(P)) = r$ and $\dim(F/\Phi(M)) = \dim(\Phi(P)/\Phi(M)) = \dim(\Phi(P/M)) = \dim(\Phi(E_0/M)) - \dim(\Phi(E_0/P)) = \dim(\Phi(f(E_0)))/n - r = s - n + r.$

6 The dual construction

The construction of the spectral category can be dualized. For a Grothendieck category $\mathcal{C}$, consider the category $\mathcal{C}'$ with the same objects as $\mathcal{C}$ and, for objects $A$ and $B$ of $\mathcal{C}$, with $\text{Hom}_{\mathcal{C}'}(A, B) = \varprojlim_2 \text{Hom}_{\mathcal{C}}(A, B/B')$, where the direct limit is taken over the upwards directed family of superfluous (= small = inessential) subobjects $B'$ of $B$. There is a canonical functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ which is the identity on objects.

More formally, assume that $\mathcal{C}$ is any abelian category and let $S$ be the system of all its superfluous epimorphisms (epimorphisms with superfluous kernel), that is, the epimorphisms $s: A \rightarrow B$ such that, for every subobject $A'$ of $A$, $s(A') = B$ implies $A' = A$. It is easily seen that if $s: A \rightarrow B$ and $t: B \rightarrow C$ are epimorphisms, then $ts$ has superfluous kernel if and only if both $s$ and $t$ have superfluous kernels. Moreover, every co-angle

\[
\begin{array}{c}
A \xrightarrow{s} A' \\
\downarrow s \\
B
\end{array}
\]
has a pushout

\[
\begin{array}{ccc}
A & \xrightarrow{g} & A' \\
\downarrow s & & \downarrow s' \\
B & \xrightarrow{f} & B'
\end{array}
\]

and if \(s\) is a superfluous epimorphism, then \(s'\) is a superfluous epimorphism, because if \(k: K \to A\) is the kernel of \(s\), then the kernel of \(s'\) is the image of \(gk: K \to A'\). Thus \(S\) is a left-calculable multiplicative system of morphisms in \(C\) [\(\text{[4]}\) p. 152]. Assume that the abelian category \(C\) has a set of generators, so that it is locally small and colocally small. Consider, for every object \(B\) of \(C\), the category \(B/S\) whose objects are the pairs \((s, C)\) with \(s: B \to C\) a superfluous epimorphism in \(C\) and whose morphisms \(f: (s, C) \to (s', C')\) are the morphisms \(f: C \to C'\) in \(C\) with \(sf = s'\). Then \(B/S\) has a small cofinal subcategory, because it is sufficient to consider the pairs \((s, C)\) where \(s: B \to C\) ranges in a set of representatives of quotient objects of \(B\) in the colocally small category \(C\). Under these conditions, the category \(C' = C_S\) of additive fractions of \(C\) relative to \(S\) exists [\(\text{[4]}\) Theorem 4.1.4]. It has the same objects as \(C\) and, for objects \(A\) and \(B\) of \(C\), \(\text{Hom}_{C'}(A, B)\) is the inductive limit of the abelian groups \(\text{Hom}_C(A, C)\) where \((s, C)\) ranges in \(B/S\), that is, the inductive limit of the functor \(\text{Hom}_C(A, -): B/S \to \text{Ab}\). The morphisms in \(C'\) are usually denoted as fractions \((s/f): A \to B\), where \((s, C)\) is an object of \(B/S\) and \(f: A \to C\) is a morphism in \(C\). This category \(C'\) can also be constructed by passing to the dual category of \(C\). Let \(C\) be an abelian category with a set of generators. Then the dual category \(C^0\) of \(C\) is a locally small abelian category, the superfluous epimorphisms of \(C\) become the essential monomorphisms in \(C^0\), so that \(S\) is a right-calculable system in \(C^0\) [\(\text{[4]}\) Corollary 4.2.2] and it is possible to construct \(C' = C^0_S\). Notice that the category \(C^0\) is locally small but does not satisfy the hypothesis of [\(\text{[4]}\) Theorem 4.2.5], so that \(C' = C^0_S\) is not necessarily a spectral category.

The category \(C'\) defined in this way can be far from being spectral also in the case of a Grothendieck category \(C\). For instance, if \(C\) is the category \(\text{Ab}\) of abelian groups, and \(\mathbb{Z}\) is the abelian group of integers, then \(\mathbb{Z}\) does not have non-zero superfluous subobjects in \(\text{Ab}\), so that the endomorphism ring of the object \(\mathbb{Z}\) in the category \(C'\) is the ring \(\mathbb{Z}\), while in spectral categories endomorphism rings are always von Neumann regular right self-injective rings. Nevertheless we are only interested in the ring morphisms \(\psi_A: \text{End}_C(A) \to \text{End}_C(A)\) induced by the functor \(F\) for every object \(A\) of \(C\). The kernel of \(\psi_A\) is the ideal \(K_A\) of all \(f \in \text{End}_C(A)\) whose image is a superfluous subobject of \(A\).

For instance, let \(R\) be a ring, \(C = \text{Mod}-R\), \(P\) a finitely generated projective right \(R\)-module and \(\text{End}_R(P)\) its endomorphism ring. Then \(\text{End}_{C'}(P) \cong \text{End}_R(P)/J(\text{End}_R(P))\) [\(\text{[4]}\) Proposition 17.11]. More generally, if \(N\) is a finitely generated right \(R\)-module with a projective cover \(P\), then \(\text{End}_{C'}(N) \cong \text{End}_R(P)/J(\text{End}_R(P))\). Hence, if \(N/NJ(R)\) is projective as an \(R/J(R)\)-module, \(\text{End}_{C'}(N) \cong \text{End}_R(N/NJ(R))\) [\(\text{[4]}\) Corollary 17.12].

We state the following elementary lemma for later reference.

**Lemma 6.1** Let \(f: A \to B\) be a morphism in a Grothendieck category \(C\). Then:

(1) The morphism \(F(f)\) is an isomorphism if and only if \(f\) is a superfluous epimorphism.

(2) If \(B\) is projective and \(F(f)\) is an isomorphism, then \(f\) is an isomorphism.
Recall that the dual Goldie dimension $\text{codim}(A)$ of an object $A$ of a Grothendieck category $\mathcal{C}$ is the Goldie dimension of the dual lattice of the lattice $\mathcal{L}(A)$ of all subobjects of $A$ [DF § 2.8]. An object $A$ of $\mathcal{C}$ is couniform if $\text{codim}(A) = 1$, that is, if it is uniform in the dual category of $\mathcal{C}$ [DF p. 184]. Equivalently, a non-zero object $A$ of $\mathcal{C}$ is couniform if and only if the sum of any two proper subobjects of $A$ is a proper subobject of $A$, if and only if every proper subobject of $A$ is superfluous, if and only if whenever $f: A' \to A$ and $g: A'' \to A$ are morphisms in $\mathcal{C}$ and the coproduct morphism $f \oplus g: A' \oplus A'' \to A$ is an epimorphism, at least one of the morphisms $f$ and $g$ is an epimorphism. We have the following

**Lemma 6.2** Let $U$ and $V$ be couniform objects of a Grothendieck category $\mathcal{C}$. Then:

1. $\text{End}_\mathcal{C}(F(U))$ is a division ring.
2. $\text{Hom}_\mathcal{C}(F(U), F(V)) \neq 0$ if and only if there exist proper subobjects $U'$ of $U$ and $V'$ of $V$ with $U/U'$ isomorphic to $V/V'$, if and only if $F(U)$ is isomorphic to $F(V)$.

**Proof.** Every morphism $F(U) \to F(V)$ is represented by a morphism $f: U \to V/V'$ for some proper subobject $V'$ of $V$. Also, the image of such an $f: U \to V/V'$ is zero in $\text{Hom}_\mathcal{C}(F(U), F(V))$ if and only if $f$ is not an epimorphism in $\mathcal{C}$. For $U = V$, it follows that every non-zero element of $\text{End}_\mathcal{C}(F(U))$ is an isomorphism, which proves (1). If $\text{Hom}_\mathcal{C}(F(U), F(V)) \neq 0$, then there is an epimorphism $f: U \to V/V'$ for some proper subobject $V'$ of $V$. Thus the kernel $\ker(f) \to U$ of $f$ is a proper subobject of $U$ and $U/\ker(f)$ is isomorphic to $V/V'$. If $U', V'$ are proper subobjects of $U, V$ respectively with $U/U'$ isomorphic to $V/V'$, then there is an epimorphism $f: U \to V/V'$ and its image in $\text{Hom}_\mathcal{C}(F(U), F(V))$ is an isomorphism. The rest is clear. ■

**Proposition 6.3** Let $A$ be an object of finite dual Goldie dimension in a Grothendieck category $\mathcal{C}$. Then:

1. The ring $\text{End}_\mathcal{C}(F(A))$ is semisimple artinian of Goldie dimension $= \text{codim}(A)$.
2. If $f \in \text{End}_\mathcal{C}(A)$, the morphism $F(f)$ is invertible if and only if $f$ is an epimorphism.
3. If every epimorphism $A \to A$ in $\mathcal{C}$ is an isomorphism, then $\psi_A: \text{End}_\mathcal{C}(A) \to \text{End}_\mathcal{C}(A)$ is a local morphism.

**Proof.** (1) Since $\text{codim}(A) = n$ is finite, $A$ has a superfluous subobject $K$ with $A/K = U_1 \oplus \cdots \oplus U_n$, where $U_i$ is a couniform object for every $i = 1, \ldots, n$. As $F(A)$ is isomorphic to $F(A/K)$ in $\mathcal{C}'$, we may assume that $A$ is a finite direct sum of couniform objects $U_1, \ldots, U_n$. Statement (1) is now a consequence of Lemma 5.1.

(2) Let $f \in \text{End}_\mathcal{C}(A)$. By Lemma 6.2(1), $F(f)$ is invertible if and only if $f$ is a superfluous epimorphism. Since $A$ has finite dual Goldie dimension, all epimorphisms $A \to A$ have superfluous kernels.

(3) is a consequence of (2). ■

Recall that in Remark 4.1, we denoted by $I_A$ the kernel of $\varphi_A: \text{End}_\mathcal{C}(A) \to \text{End}_{\text{Spec-}\mathcal{C}}(A)$, that is, the ideal of all endomorphisms of $A$ with essential kernel, and that we denote by $K_A$ the kernel of $\psi_A$, that is, the ideal of all endomorphisms of $A$ with superfluous image. In Proposition 6.4, we put together the ring morphisms $\varphi_A$ and $\psi_A$ to obtain a local morphism:
**Proposition 6.4** Let \( A \) be an object in a Grothendieck category \( C \). Then the ring morphism

\[(\varphi_A, \psi_A): \text{End}_C(A) \to \text{End}_{\text{Spec-}C}(A) \times \text{End}_{C'}(A),\]

defined by \( f \mapsto (P(f), F(f)) \) for every \( f \in \text{End}_C(A) \), is local. The kernel of this ring morphism is \( I_A \cap K_A \), that is, the set of all \( f \in \text{End}_C(A) \) with essential kernel and superfluous image.

**Proof.** Assume that \( f \in \text{End}_C(A) \) is such that \( P(f) \) and \( F(f) \) are invertible. In general, \( P(f) \) is invertible if and only if \( f \) is an essential monomorphism. By Lemma 6.1(1), if \( F(f) \) is invertible, then \( f \) is an epimorphism. Hence, \( f \) is invertible. \( \blacksquare \)

**Corollary 6.5** Let \( A \) be an object of a Grothendieck category \( C \). Assume that \( A \) has finite Goldie dimension \( n \) and finite dual Goldie dimension \( m \). Then \( \text{End}_C(A) \) is semilocal and \( \text{codim}(\text{End}_C(A)) \leq n + m \).

**Proof.** By Propositions 4.4 and 6.3(1), the rings \( \text{End}_{\text{Spec-}C}(A) \) and \( \text{End}_{C'}(A) \) are semisimple artinian of Goldie dimension \( \text{dim}(A) \) and \( \text{codim}(A) \), respectively. Thus \( \text{End}_C(A) \) is semilocal and \( \text{codim}(\text{End}_C(A)) \leq n + m \) by Theorem 2.4 and Proposition 6.4. \( \blacksquare \)

If \( A \) is a uniform object, then \( I_A = \{ f \in \text{End}_C(A) \mid f \) is not a monomorphism} and \( \text{End}_{\text{Spec-}C}(A) \) is a division ring. If \( A \) is a couniform object, then \( K_A = \{ f \in \text{End}_C(A) \mid f \) is not an epimorphism} and \( \text{End}_{C'}(A) \) is a division ring. Therefore, if \( A \) is both uniform and couniform, the local ring morphism \( (\varphi_A, \psi_A) \) maps \( \text{End}_C(A) \) into the direct product of two division rings. From Corollary 2.6 and Proposition 6.4 we recover the basic results on the endomorphism ring of bimoniform modules [F1 Theorem 9.1] that we extend to the context of Grothendieck categories.

**Corollary 6.6** Let \( A \) be an object of a Grothendieck category \( C \). Assume that \( A \) is uniform and couniform. Then there are two possibilities:

1. either the ideals \( I_A \) and \( K_A \) are comparable and, in this case, \( \text{End}_C(A) \) is local with maximal ideal \( I_A + K_A \), or
2. the ideals \( I_A \) and \( K_A \) are not comparable, \( \text{End}_C(A)/J(\text{End}_C(A)) \) is the product of two division rings, and \( I_A, K_A \) are the two maximal ideals of \( \text{End}_C(A) \).

7 **Objects with a projective cover**

In all this section, \( C \) will denote a Grothendieck category. Now we shall apply the results of the previous section about the functor \( F: C \to C' \) to objects with a projective cover.

**Lemma 7.1** Let \( A \) be an object of \( C \) with a projective cover \( \pi: P \to A \), and let \( K \to P \) be the kernel of \( \pi \). Let \( f \in \text{End}_R(A) \), so that \( f \) lifts to an endomorphism \( f_0 \) of \( P \), which restricts to an endomorphism \( f_1 \) of \( K \). Then \( F(f_1) \) depends only on \( f \) and not on the choice of the lifting \( f_0 \) of \( f \).
Proof. Let $f'_0$ be another lifting of $f$ and $f'_1$ the corresponding restriction to $K$. As $\pi \circ (f_0 - f'_0) = 0$, the difference $f_0 - f'_0 \colon P \to P$ factors through the kernel $\iota \colon K \to P$ of $\pi$, that is, $f_0 - f'_0 = \iota g$ for a suitable morphism $g \colon P \to K$. As $K$ is superfluous in $P$, its image $g(K)$ is superfluous in $K$. Therefore the image of the restriction $f_1 - f'_1 \colon K \to K$ of $g$ to $K$ is superfluous in $K$. That is, $F(f_1 - f'_1) = F(f_1) - F(f'_1) = 0$. 

By Lemma 7.1 there is a ring morphism

$$\Phi \colon \text{End}_C(A) \to \text{End}_{C'}(A) \times \text{End}_{C'}(K)$$

defined by $\Phi(f) = (F(f), F(f_1))$ for every $f \in \text{End}_C(A)$.

Theorem 7.2 Let $A$ be an object of a Grothendieck category $C$. Suppose that there exists a projective cover $\pi \colon P \to A$. Then the ring morphism $\Phi$ is local.

Proof. Let $K \to P$ be the kernel of $\pi$. Let $f \in \text{End}_C(A)$ be such that $\Phi(f)$ is invertible. Let $f_0 \in \text{End}_C(P)$ be a lifting of $f$, and let $f_1 \colon K \to K$ be the restriction of $f_0$ to $K$, so that we have a commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & K & \to & P & \to & A & \to & 0 \\
& & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\
0 & \to & K & \to & P & \to & A & \to & 0.
\end{array}
$$

As $F(f)$ and $F(f_1)$ are invertible, the morphisms $f$ and $f_1$ must be epimorphisms by Lemma 6.1(1). We must prove that $f$ is a monomorphism. As $F(P)$ and $F(A)$ are canonically isomorphic via $F(\pi)$ and $F(f)$ is an isomorphism, it follows that $F(f_0)$ is an isomorphism in $C'$. From Lemma 6.1(2), we get that $f_0$ must be an isomorphism in $C$. The Snake Lemma gives an exact sequence $0 = \ker f_0 \to \ker f \to \coker f_1 = 0$. Hence $\ker f = 0$, as we wanted to prove.

Theorem 7.3 Let $A$ be an object of a Grothendieck category $C$. Suppose that there exists a projective cover $\pi \colon P \to A$. Let $K \to P$ be the kernel of $\pi$ and assume that both $A$ and $K$ have finite dual Goldie dimension. Then $\text{End}_C(A)$ is a semilocal ring. Moreover,

$$\text{codim}(\text{End}_C(A)) \leq \text{codim}(A) + \text{codim}(K).$$

Proof. By Proposition 6.3 the ring $\text{End}_{C'}(A) \times \text{End}_{C'}(K)$ is semisimple artinian of dual Goldie dimension $\text{codim}(A) + \text{codim}(K)$. Now apply Theorems 7.2 and 2.4 to conclude.

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