A simple derivation of Lorentz self-force

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Abstract
We derive the Lorentz self-force for a charged particle in arbitrary non-relativistic motion by averaging the retarded fields. The derivation is simple and at the same time pedagogically accessible. We obtain the radiation reaction for a charged particle moving in a circle. We pin down the underlying concept of mass renormalization.

Keywords: self-force, radiation reaction, renormalization

1. Introduction

The electromagnetic field goes to infinity at the position of a point charge. The electrostatic field at the position of the charged particle

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \xrightarrow{r \to 0} \infty$$

(1)

and the self-energy of the point charge in the rest frame of the charged particle

$$U = \frac{\varepsilon_0}{2} \int \vec{E}^2 \, \text{d}^4 \vec{r} = \frac{1}{32\pi\varepsilon_0} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \, d\phi \int_0^r \frac{q^2}{r^2} \, dr = -\frac{q^2}{8\pi\varepsilon_0} \left[ \frac{1}{r} \right]_{r=0}^{r=\infty} \to \infty$$

(2)

turn out divergent. The concept of the point-like (structureless or dimensionless object) charge may be an idealization. The measurement of the anomalous magnetic moment of an electron based on the quantum field theoretic calculations leads to the following upper bound on the size of the electron [1]: \( l \leq 10^{-17} \) cm. Thus, a charged particle might be considered as an extended object with a finite size.

In order to circumvent the problem of divergence of the field at the point charge, it is plausible either: (1) to consider the averaged value of the field in the suitably small closed region surrounding the point charge as the value of the field under consideration at the position of the point charge or (2) to think of a charged particle as an extended object having a small dimension with a charge distribution. Our derivation of the self-force is based on the former consideration.
If an extended charged particle moves with non-uniform velocity, the charge elements comprising such charge distribution begin to exert forces on one another. However, these forces do not cancel out due to retardation, giving rise to a net force known as the self-force. Thus, the radiating extended charged particle experiences a self-force which acts on the charge particle itself. The Lorentz self-force \([2]\) arising due to a point charge conceived as a uniformly charged spherical shell of radius \(s\) is given by

\[
\vec{F}_{\text{self}} = \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0 c^3 s} \ddot{\vec{v}}(t) + \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0 c^3} \dot{\vec{v}}(t) + O(s) \quad \text{with} \quad \left| \vec{s} \right| = s
\]  

(3)

where,

- the quantity \(\frac{1}{3} \frac{q^2}{4\pi\varepsilon_0 c^3 s}\) in the first term stands for electromagnetic mass and becomes divergent as \(s \to 0^+\),
- the second term represents the radiation reaction and is independent of the dimension of the charge distribution and
- the third term corresponds to the first finite size correction and is proportional to the radius of the shell \(s\).

The usual method \([2]\) of computing the self-force (3) involves rather cumbersome calculation. Boyer \([3]\) has obtained the expression for the self-force using the averaged value of the retarded field for a charged particle in uniform circular motion. Boyer’s derivation of the self-force for charged particle in uniform circular motion involves an unsophisticated calculation. In this article, we derive the expression for the self-force for a point charge in arbitrary non-relativistic motion by averaging the retarded field in a rather neat and sophisticated way. Our derivation of the self-force unlike Boyer’s derivation pertaining to the specific context, leads to the general (non-relativistic) expression for the self-force. In the following section, we shall define the self-force in terms of the averaged retarded field over the surface of the spherical shell.

2. The self-force

For our purpose, we shall consider an average field on the surface of a spherical shell of radius \(s\) due to a point charge \(q\) sitting at the center of the shell. An average field \([4]\) over the surface of a spherical shell of radius \(s\) is defined by

\[
\overline{\vec{E}}_q(t) = \frac{1}{4\pi s^2} \int_\Sigma dA \overline{E}_q(\vec{r}, t)
\]

(4)

where \(\Sigma\) is the surface of the spherical shell of radius \(s\). We now define the self-force as

\[
\vec{F}_{\text{self}} = q \lim_{r \to 0} \overline{\vec{E}}_q(t) = q\overline{\vec{E}}_{\text{self}}
\]

(5)

where the field \(\overline{\vec{E}}(\vec{r}, t)\) depends upon the position and motion of the charge particle at the retarded time. The field due to an accelerated charged particle is the sum of the velocity fields as well as the acceleration fields:

\[
\overline{\vec{E}}(\vec{r}, t) = \overline{\vec{E}}^{\text{Vol}}(\vec{r}, t) + \overline{\vec{E}}^{\text{Acc}}(\vec{r}, t).
\]

(6)

1 For a rigorous definition and the criteria as to how to choose the small surface surrounding the charge see \([4]\).
The average field contribution from the velocity fields \( E_{V}^{\text{vel}}(t) \) on the surface of a spherical shell due to a charged particle at its center gets filtered out of \( E_{q}(t) \) because for each spatial point (say \((s_{x}, s_{y}, s_{z})\)) on the surface there exists a corresponding point \((-s_{x}, -s_{y}, -s_{z})\) on the surface. However, the average field contribution from the acceleration fields \( E_{q}\) \( \text{Acc}(t) \) over the surface of a spherical shell due to a charged particle situated at the center of the shell turns out to be nonzero because the acceleration fields involve a term \( \vec{s} \cdot \vec{a} \), which is even in the vector \( \vec{s} \). Moreover, \( E_{q}\) \( \text{Acc}(t) \) leads to radiation reaction as well as a term associated with the electromagnetic contribution to the mass. We shall now obtain the derivation of the self-force in the following section.

### 3. Radiation reaction from a point charge in arbitrary non-relativistic motion

In order to obtain the expression for the self-force, we begin with the field [2] \( p 664 \) due to a point charge located at \( \vec{r}_{q}(t) \) at time \( t \) in an arbitrary motion which is given by

\[
\vec{E}(\vec{r}, t) = \left[ \frac{q}{4\pi \varepsilon_{0}} \left( \frac{\vec{s} - \frac{\vec{s}}{c^2}}{\vec{s} - \frac{\vec{s}}{c^2}} \right) \left( 1 - \frac{\vec{v}}{c} \right) + \frac{\vec{s} \times \left( \vec{s} - \frac{\vec{s}}{c^2} \right)}{4\pi \varepsilon_{0}} \left( \vec{s} - \frac{\vec{s}}{c^2} \right) \right]_{\vec{r} = \vec{r}_{q}(t)}
\]

where,

\[
\vec{s} = \vec{r} - \vec{r}_{q}(t_{\text{Ret}}) = \left( x - x_{q}(t_{\text{Ret}}) \right) \hat{i} + \left( y - y_{q}(t_{\text{Ret}}) \right) \hat{j} + \left( z - z_{q}(t_{\text{Ret}}) \right) \hat{k},
\]

and \( t_{\text{Ret}} = t - s/c \). The quantities \( \vec{s}, \vec{v} \) and \( \vec{a} \) in the square brackets are evaluated at the retarded time \( t_{\text{Ret}} \). The first term represents the velocity fields \( \vec{E}_{Q}^{\text{vel}}(\vec{r}, t) \) and is independent of the acceleration of the point charge. The second term represents acceleration fields \( \vec{E}_{Q}^{\text{Acc}}(\vec{r}, t) \) and exists only when \( \vec{a} \neq 0 \). Thus, for \( \vec{a} = 0 \), there will be no radiation. For sufficiently small speed of the point charge (in the limit \( v/c \to 0 \)) the field takes the form

\[
\vec{E}(\vec{r}, t) = \frac{q}{4\pi \varepsilon_{0}} \left[ \vec{s} \times \left( \vec{s} - \vec{a} \right) \right] \left( \frac{1}{c^2} \right) + \frac{q}{4\pi \varepsilon_{0}} \left[ \vec{s} \times \left( \vec{s} - \vec{a} \right) \right] \left( \frac{1}{c^2} \right)
\]

\[
= \frac{q}{4\pi \varepsilon_{0}} \left[ \vec{s} \times \left( \vec{s} - \vec{a} \right) \right] \left( \frac{1}{c^2} \right) + \frac{q}{4\pi \varepsilon_{0}} \left[ \vec{s} \times \left( \vec{s} - \vec{a} \right) \right] \left( \frac{1}{c^2} \right)
\]

The velocity and corresponding acceleration of the point charge are given by

\[
\vec{v} = \dot{\vec{r}}_{q}(t_{\text{Ret}}) = \dot{x}_{q}(t_{\text{Ret}}) \hat{i} + \dot{y}_{q}(t_{\text{Ret}}) \hat{j} + \dot{z}_{q}(t_{\text{Ret}}) \hat{k}
\]

\[
\vec{a} = \ddot{\vec{r}}_{q}(t_{\text{Ret}}) = \ddot{x}_{q}(t_{\text{Ret}}) \hat{i} + \ddot{y}_{q}(t_{\text{Ret}}) \hat{j} + \ddot{z}_{q}(t_{\text{Ret}}) \hat{k}
\]
Now, the average field over the surface of the sphere (as shown in figure 1) is

$$
\vec{E}_q(t) = \frac{1}{4\pi s^2} \int \int d\theta d\phi s^2 \sin \theta \vec{E} (\vec{r}, t)
= \frac{q}{4\pi \varepsilon_0} \frac{1}{4\pi s^2} \int \int d\theta d\phi \sin \theta \left[ c^2 \vec{s} + \vec{s} \left( \vec{s}, \vec{a} \right) - \vec{s} \vec{a} \right]
$$

(11)

where $\vec{s} \left( \vec{s}, \vec{a} \right) = \left[ \vec{s} \left( s, a_x + s, a_y + s, a_z \right) \right]$

$$
= \left( s^i a_i + s^i s_j a_j + s^i s_k a_k \right) \hat{i} + \left( s^i s_j a_i + s^i a_i + s^i s_j a_j \right) \hat{j}
+ \left( s^i s_j a_i + s^i s_k a_i + s^i a_k \right) \hat{k}.
$$

(12)

The vector $\vec{s}$ in the spherical polar coordinates is given by

$$
\vec{s} = (s, s, s) = (s \sin \theta \cos \phi, s \sin \theta \sin \phi, s \cos \theta).
$$

(13)

We can show that

$$
\int \int \sin \theta d\theta d\phi = 0
$$

(14)

$$
\int \int s_i s_j \sin \theta d\theta d\phi = \frac{4\pi}{3} s^2 \delta_{ij}
$$

(15)

where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise with $i, j = x, y, z$. The average field now becomes

$$
\vec{E}_q(t) = \frac{q}{4\pi \varepsilon_0 c^2} \left[ \frac{1}{3} \hat{a} \left( t_{ret} \right) - \hat{a} \left( t_{ret} \right) \right] = \frac{q}{4\pi \varepsilon_0 c^2} \left[ -\frac{2}{3} \hat{a} \left( t_{ret} \right) \right].
$$

(16)

We note that the velocity fields contribution to $\vec{E}_q(t)$ vanishes for a charged particle in arbitrary non-relativistic motion. In fact, the vanishing contribution of the velocity fields apparently brings out the error in Boyer’s [3] calculation for the contribution of the velocity fields. In the limit $s \to 0^+$, we get
\[ \tilde{a} (t_{\text{Rel}}) = \tilde{a} (t - s/c) = \tilde{a} (t) - \frac{s}{c} \tilde{a} (t) + O (s^2). \]  

(17)

Now,

\[ \vec{E}_s (t) = \frac{2}{3} \frac{q}{4 \pi \epsilon_0 c^2} \vec{a} (t) + \frac{2}{3} \frac{q}{4 \pi \epsilon_0 c^2} \vec{d} (t) + O (s). \]  

(18)

Thus, the self-field \( \vec{E}_{\text{Self}} \) yields

\[ \vec{E}_{\text{Self}} = \lim_{s \to 0^+} \vec{E}_s (t) = - \lim_{s \to 0^+} \left( \frac{2}{3} \frac{q}{4 \pi \epsilon_0 c^2} \right) \tilde{a} (t) + \frac{2}{3} \frac{q}{4 \pi \epsilon_0 c^2} \tilde{d} (t). \]  

(19)

Now, the self-force for the point charge limit is given as

\[ \vec{F}_{\text{Self}} = q \vec{E}_{\text{Self}} = - \lim_{s \to 0^+} \left( \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2} \right) \tilde{a} (t) + \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2} \tilde{d} (t), \]  

(20)

which is the same as equation (3) in the limit \( s \to 0^+ \). The self-force can be expressed in terms of \( r_q \) and \( \theta_q \) variables as

\[ \vec{F}_{\text{Self}} = - \frac{2}{3} \frac{q}{4 \pi \epsilon_0 c^2} \left[ \left( \vec{r}_q - r_q \vec{u}_\theta \right) \vec{r}_\phi + \left( r_q \vec{u}_\theta + 2r_q \vec{u}_\phi \right) \vec{u}_\phi \right] \]

\[ + \frac{2}{3} \frac{q}{4 \pi \epsilon_0 c^2} \left[ \left( - 2 r_q \vec{u}_\phi - 3 r_q \vec{u}_\theta \right) \vec{r}_\phi + \left( r_q \vec{u}_\theta + 3 r_q \vec{u}_\phi \right) \vec{u}_\phi - r_q \vec{u}_{\phi \theta} \right] \]  

(21)

We shall now study the following illustrative examples pertaining to radiation reaction.

3.1. Radiation reaction from a charged particle executing simple harmonic motion

Consider a charged particle \( q \) of mass \( m \) executing simple harmonic motion along the X-axis with frequency \( \omega \), its displacement from equilibrium is

\[ x_q (t) = x_0 \sin \omega t \]

and its acceleration is

\[ a_q (t) = \ddot{x} = - x_0 \omega^2 \sin \omega t. \]

The charged particle having nonzero acceleration will radiate and will therefore experience the self-force (please see equation (A.2)) given by

\[ \vec{F}_{\text{Oscillator}} = \lim_{s \to 0^+} \left( - \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2 s} \right) \left[ - x_0 \omega^2 \sin \omega \left( t - \frac{s}{c} \right) \right] \]

\[ = \lim_{s \to 0^+} \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2 x_0 \omega^2} \sin \omega t - \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2 x_0 \omega^3} \cos \omega t. \]  

(22)

Thus, the self-force is the sum of the finite piece \( - \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2} x_0 \omega^3 \cos \omega t \) and the divergent piece \( \left( \lim_{s \to 0} \frac{2}{3} \frac{q^2}{4 \pi \epsilon_0 c^2} x_0 \omega^3 \sin \omega t \right) \).
3.2. Radiation reaction from a charged particle moving in a circle

Suppose a charged particle is moving in a circle of radius $R$ with uniform angular speed $\omega$ (as shown in figure 2). The charge particle will experience the centripetal force $-m\omega^2 R \hat{r}$ acting towards the center. Now,

$$\vec{s}(t_{\text{ret}}) = (x - R \cos \omega t_{\text{ret}}) \hat{i} + (y - R \sin \omega t_{\text{ret}}) \hat{j} + \hat{k}. \quad (23)$$

The velocity and acceleration are

$$\vec{v}(t_{\text{ret}}) = -R\omega \cos \omega t_{\text{ret}} \hat{i} + R\omega \sin \omega t_{\text{ret}} \hat{j} \quad (24)$$

$$\vec{a}(t_{\text{ret}}) = -R\omega^2 \cos \omega t_{\text{ret}} \hat{i} - R\omega^2 \sin \omega t_{\text{ret}} \hat{j}. \quad (25)$$

The acceleration $\vec{a}(t_{\text{ret}})$ (please see the appendix) in the limit $s \to 0^+$ yields,

$$\vec{a}(t_{\text{ret}}) = -R\omega^2 \cos \omega \left( t - \frac{s}{c} \right) \hat{i} - R\omega^2 \sin \omega \left( t - \frac{s}{c} \right) \hat{j} \quad (26)$$

$$= -R\omega^2 \left[ \hat{r}_q - \frac{\omega \hat{s}}{c} \right].$$

The self-force is given by

$$\vec{F}_{\text{Self}}^{\text{Circle}} = \lim_{s \to 0^+} \frac{2}{3} \frac{q^2 R}{4\pi \varepsilon_0 c^3 s} \omega^2 \hat{r}_q = \frac{2}{3} \frac{q^2 R}{4\pi \varepsilon_0 c^3 \omega^2} \hat{q}. \quad (27)$$

The self-force experienced by the charged particle picks up both the tangential component which is responsible for the radiation reaction as well as the radial component which displays the singular behaviour in the limit $s \to 0^+$. 

Figure 2. A charged particle moving in a circle of radius $R$. 

Eur. J. Phys. 35 (2014) 055006 A Haque
4. Mass renormalization

The equation of motion for a radiating charged particle is given as

\[
\begin{align*}
\vec{m}_B \ddot{\vec{r}} &= \vec{F}_{\text{Ext}} + \vec{F}_{\text{Self}} \\
&= \vec{F}_{\text{Ext}} - \left( \frac{2}{3} \frac{q^2}{4 \pi \varepsilon_0 c^3} \right) \ddot{\vec{v}}(t) + \frac{2}{3} \frac{q^2}{4 \pi \varepsilon_0 c^3} \dddot{\vec{v}}(t) \\
&= \vec{F}_{\text{Ext}} - m_{\text{Em}} \ddot{\vec{v}}(t) + \frac{2}{3} \frac{q^2}{4 \pi \varepsilon_0 c^3} \dddot{\vec{v}}(t)
\end{align*}
\]  

(28)

where \( m_B \) corresponds to the mass of the charged particle that is not associated with the radiation reaction and is called bare mass. The bare mass \( m_B \) refers to the physical phenomena at arbitrary short distance surrounding the point charge. The bare mass is not directly related to quantities that one measures at finite spatial length from the charged particle. However, \( m_{\text{Em}} = \frac{2}{3} \frac{q^2}{4 \pi \varepsilon_0 c^3} \), defined as the electromagnetic mass, arises due to the presence of the electromagnetic field. The electromagnetic mass \( m_{\text{Em}} \) is divergent for the point charge \((s \to 0^+)\). Now, we may rewrite equation (28) as

\[
(m_B + m_{\text{Em}}) \ddot{\vec{v}}(t) = \vec{F}_{\text{Ext}} + \frac{2}{3} \frac{q^2}{4 \pi \varepsilon_0 c^3} \dddot{\vec{v}}(t).
\]  

(29)

In order to tame the divergence, the process of renormalization is implemented as follows: since a point charge causes an infinite electromagnetic mass, we assume it to be \((+\infty + m_B)\) so its bare mass must be postulated to be minus infinite \((-\infty)\) so as to render the observable physical (renormalized) mass

\[
m_B = m_B + m_{\text{Em}} = m_B + \frac{2}{3} \frac{q^2}{4 \pi \varepsilon_0 c^3} = \text{Finite}
\]  

(30)

finite. This shift is known as mass renormalization. The bare mass and the electromagnetic mass are themselves not physical observables. In the case of a charged particle executing simple harmonic motion, the divergence piece of the self-force \( m_{\text{Em}} \omega^2 \sin \omega t \) acts away from the equilibrium position. Whereas for the charged particle moving in a circle, the divergent piece of the self-force appears in the form of the centrifugal force \( m_{\text{Em}} \omega^2 \vec{r} \). To have a sensible theory, these infinities are made to absorb via mass renormalization to obtain the physically observable mass.

5. Conclusion

We derive the expression for the self-force for a charged particle in arbitrary non-relativistic motion in a rather neat and sophisticated way than that presented by Boyer [3] in the specific context of a charged particle in uniform circular motion. We discuss illustrative examples pertaining to radiation reaction and obtain explicitly the divergence pieces in the expressions of their respective self-forces. We discuss the concept of mass renormalization which implements the renormalization prescription as to how to tame the divergence.
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Appendix A. Calculation of $\vec{a}(t_{\text{Ret}})$ for a charge particle moving in a circle

In the limit $s \to 0^+$, we have

\[ \cos\left(\omega t - \frac{\omega s}{c}\right) = \cos \omega t \cos \frac{\omega s}{c} + \sin \frac{\omega s}{c} \sin \omega t \approx \cos \omega t + \frac{\omega s}{c} \sin \omega t \]  
(A.1)
\[ \sin\left(\omega t - \frac{\omega s}{c}\right) = \sin \omega t \cos \frac{\omega s}{c} - \cos \omega t \sin \frac{\omega s}{c} \approx \sin \omega t - \frac{\omega s}{c} \cos \omega t. \]  
(A.2)

The unit vectors $\hat{r}_q(t_{\text{Ret}})$ and $\hat{\theta}_q(t_{\text{Ret}})$ in the limit $s \to 0^+$ may be expressed as

\[ \hat{r}_q(t_{\text{Ret}}) = \cos\left(\omega t - \frac{\omega s}{c}\right) \hat{i} + \sin\left(\omega t - \frac{\omega s}{c}\right) \hat{j} \approx \hat{r}_q(t) - \frac{\omega s}{c} \hat{\theta}_q(t) \]  
(A.3)
\[ \hat{\theta}_q(t_{\text{Ret}}) = -\sin\left(\omega t - \frac{\omega s}{c}\right) \hat{i} + \cos\left(\omega t - \frac{\omega s}{c}\right) \hat{j} \approx \frac{\omega s}{c} \hat{r}_q(t) + \hat{\theta}_q(t). \]  
(A.4)

The velocity $\dot{\vec{r}}_q(t_{\text{Ret}})$ is

\[ \dot{\vec{r}}_q(t_{\text{Ret}}) = R\omega \dot{\hat{\theta}}_q(t_{\text{Ret}}) = R\omega \dot{\hat{\theta}}_q\left(t - \frac{s}{c}\right) \]
\[ = R\omega \left[ \frac{\omega s}{c} \hat{r}_q(t) + \hat{\theta}_q(t) \right] + O\left(s^2\right). \]  
(A.5)

The acceleration $\vec{a}(t_{\text{Ret}})$ reads

\[ \vec{a}(t_{\text{Ret}}) = \ddot{\vec{r}}_q(t_{\text{Ret}}) = -R\omega^2 \hat{r}_q(t_{\text{Ret}}) \]
\[ = -R\omega^2 \left[ \hat{r}_q(t) - \frac{\omega s}{c} \hat{\theta}_q(t) \right] + O\left(s^2\right). \]  
(A.6)

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