Kinetic theory in curved space: a first quantised approach

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We study the real time formalism of non-equilibrium many-body theory, in a first quantised language. We argue that on quantising the relativistic scalar particle in spacetime with Minkowski signature, we should study both propagations $e^{i(p^2-m^2)\lambda}$ and $e^{-i(p^2-m^2)\lambda}$ on the particle world line. The path integral needs regulation at the mass shell $p^2 = m^2$. If we regulate the two propagations independently we get the Feynman propagator in the vacuum, and its complex conjugate. But if the regulation mixes the two propagations then we get the matrix propagator appropriate to perturbation theory in a particle flux. This formalism unifies the special cases of thermal fluxes in flat space and the fluxes ‘created’ by Cosmological expansion, and also gives covariance under change of particle definition in curved space. We comment briefly on the proposed application to closed strings, where we argue that coherent fields and ‘exponential of quadratic’ particle fluxes must both be used to define the background for perturbation theory.
1. Introduction.

When we quantise a field in curved space, the notion of a particle becomes very curious. For example in an expanding Universe we have particle creation, which means that a spacetime which looks empty in terms of particle modes natural in the past, may look full of particles on using co-ordinates natural to the future. The Minkowski vacuum appears to have a particle flux for an accelerated observer. For black holes, imposing vacuum conditions at past null infinity gives Hawking radiation in the future, due to the time-dependent gravitational field of a collapsing object. ([1] and references therein.)

How ‘real’ are such particles, and more specifically, how do they affect the gravitational field? Beyond semi-classical approximations, such questions have traditionally been deferred to a time when a consistent theory is available with both matter and gravity quantised.

Strings provide such a consistent theory, so we wonder if the above questions have been implicitly answered in computing string amplitudes. One deals with closed strings (which include the graviton in their spectrum) in a first quantised language. The particle analogue of this approach is summing over all particle trajectories between initial and final points (with branching trajectories giving possible interactions). If we just take a free scalar particle and compute the two-point function by such a first quantised path integral, which notion of particle are we using?

The answer to this question is known [2]. If \( |0 >_{in} \) is the vacuum based on positive frequency modes at \( t = -\infty \) and \( |0 >_{out} \) for the modes based at \( t = \infty \) then

\[
\int D[\text{paths}(x \rightarrow x')] e^{iS} = _{out} < 0 | T[\phi(x')\phi(x)] | 0 >_{in} / _{out} < 0 | 0 >_{in} \tag{1.1}
\]

(The notion of ‘in’ and ‘out’ vacuua here does not necessarily require flat spacetime at \( t \rightarrow \pm \infty \).)

But we may not want such a hybrid ‘in-out’ expectation value for our two-point function. We may wish to use the notion of ‘in’ particles, which requires computing \( _{in} < 0 | T[\phi(x')\phi(x)] | 0 >_{in} \). Or we may have an ensemble of ‘in’ particles to start with, which implies a density matrix based on ‘in’ states (e.g. \( \rho \sim e^{-\beta n} | n >_{in} in < n | \) with propagator \( \text{Tr}\{\rho T[\phi(x')\phi(x)]\} \). How do we modify the first quantised path integral to obtain the requisite propagators, and carry out perturbation theory?

To handle non-equilibrium situations with general density matrices one uses the ‘real time’ formalism to develop a perturbation theory. If \( \rho \) is specified in terms of the states
at $t = -\infty$ then we evolve the fields from $t = -\infty$ to $t = \infty$ and back to $-\infty$ where we insert $\rho$ and take a trace. The propagator becomes a 2x2 matrix propagator, with $a = 1, 2$ labelling operators on the first and second parts of the above time path. With correspondingly generalised interaction vertices, one computes Feynman diagrams in the usual way to obtain the correlators in the many-body situation ($[3]$ and references therein).

The goal of this paper is to study this real-time formalism from a first quantised viewpoint, using the simple example of a scalar field. We require a covariant language for the density matrix (and time path) rather than a Hamiltonian language based on spacelike slices, because it is such a covariant language that we can extend to strings.

In brief, our approach and results are as follows. For flat space, we know that a density matrix of the form ‘exponential of linear in the field’ gives a coherent state. Shifting the field by its classical value removes such a part of $\rho$. The class of $\rho$ of the form ‘exponential of quadratic in the field’ is also special; for correlators with such $\rho$ the Wick decomposition holds $[4]$. The matrix propagator of the real-time formalism encodes such $\rho$ and the choice of time path for the perturbation theory. Departures from this ‘exponential of quadratic’ form of the density matrix gives ‘correlation kernels’, which are handled perturbatively as vertices analogous to the interaction vertices in the Lagrangian.

For the first quantised language, we argue that a careful quantisation of the relativistic scalar particle requires considering both the propagation $e^{i(p^2 - m^2)}$ and the propagation $e^{-i(p^2 - m^2)}$ on the world line. We may collect these two possibilities into a 2x2 matrix form, getting a world line Hamiltonian diag$[-(p^2 - m^2), (p^2 - m^2)]$. Near the mass shell $p^2 - m^2 = 0$ the path integral needs regulation. If we add $-i\epsilon I$ to the Hamiltonian ($I$ is the 2x2 identity matrix) then we get a diagonal matrix propagator, with $[1,1]$ and its complex conjugate as the first and second diagonal entries. But if we regulate instead by adding $-i\epsilon M$, where $M$ is a non-diagonal matrix, then we get the matrix propagator corresponding to an ‘exponential of quadratic’ particle flux.

We compute $M$ for a thermal distribution in Minkowski space, for the Niemi-Semenoff $[5]$ choice of time path, and for the ‘closed time path’ developed for non-equilibrium theory by Keldysh $[6]$, Schwinger $[7]$, and others. These two $M$ matrices are not the same, which reflects the fact that the matrix propagator depends on both $\rho$ and the choice of time path. For a curved space example, we consider a 1 + 1 spacetime with ‘sudden’ expansion. We compute $M$ to obtain the matrix propagator appropriate to perturbation theory with the ‘in’ vacuum; i.e. for $\rho = \in |0><0|_i$ and time path beginning and ending at past infinity.
Thus the ‘exponential of quadratic’ form of the density matrix, basic to perturbation theory, is naturally obtained in the first quantised formalism. Thermal fluxes in flat space, and the flux given by the Bogoliubov transformation due to spacetime expansion, are special cases of this form. Further, this class of density matrices is closed under change of the basis of functions used to define particles, so our formalism is covariant under such transformations.

We conclude with a discussion of the significance of our results for strings, which were the motivation for this study of the first quantised formalism. The usual first quantised path integrals for strings would give the analogue of \([L, L]\), which corresponds to specific boundary conditions at spacetime infinity. To handle phenomena involving particle fluxes, we propose extending the world sheet Hamiltonian to a 2x2 Hamiltonian as in the above particle case. Thus we would allow not only classical deformations of the background field (analogous to the ‘exponential of linear’ \(\rho\) in the particle case) but also particle flux backgrounds (corresponding to ‘exponential of quadratic’ density matrices). Studying \(\beta\)-function equations [8] for this extended theory should give not equations between classical fields but equations relating fields and fluxes. The latter kind of equation, we believe, would be natural to a description of quantised matter plus gravity.

fThe plan of this paper is as follows. Section 2 reviews finite temperature perturbation theory, and discusses the significance of ‘exponential of quadratic’ density matrices for the curved space theory. Section 3 translates the finite temperature results of flat space to first quantised language. Section 4 gives a curved space example. Section 5 is a summary and a discussion relating to strings.

2. Propagators in the presence of a particle flux.

2.1. Review of the real time formalism.

Consider a scalar field in Minkowski spacetime (metric signature+ − ...−)

\[
S = \int dx \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right) \tag{2.1}
\]

Suppose this field is at a temperature \(T = (\beta)^{-1}\). To take into account this temperature we evolve the field theory in imaginary time from \(t = 0\) to \(t = -i\beta\), and identify these
two time slices. In the ‘real time’ approach of Niemi and Semenoff we use instead the following path \( C_1 \) in complex \( t \) space to connect these two slices

\[
C_1: \quad I: \quad -\infty \to \infty \\
II: \quad \infty \to \infty - i\beta/2 \\
III: \quad \infty - i\beta/2 \to -\infty - i\beta/2 \\
IV: \quad -\infty - i\beta/2 \to -\infty - i\beta
\]

The concept of time ordered correlation functions is now replaced by ‘path ordered’ correlation functions, with \( t \) running along the above path \( C_1 \). One argues that the parts \( II, IV \) of \( C_1 \) can be ignored. Field operators on part \( I \) are labelled with the subscript 1 while on part \( II \) are labelled with the subscript 2. Thus for the two point function of the scalar field there are four possible combinations of subscripts, and the corresponding propagators are collected into a matrix:

\[
D(p) = \begin{pmatrix}
\frac{i}{p^2 - m^2 + i\epsilon} + 2\pi n(p)\delta(p^2 - m^2) & 2\pi[(n(p)(n(p) + 1)]^{1/2}\delta(p^2 - m^2) \\
2\pi[n(p)(n(p) + 1)]^{1/2}\delta(p^2 - m^2) & \frac{-i}{p^2 - m^2 - i\epsilon} + 2\pi n(p)\delta(p^2 - m^2)
\end{pmatrix}
\]

where

\[
n(p) = \left(e^{\beta|p_0|} - 1\right)^{-1}
\]

is the number density of particles for the free scalar field at temperature \((\beta)^{-1}\). We can write

\[
D(p) = U(p)D^0(p)U(p)
\]

with

\[
D^0(p) = \begin{pmatrix}
\frac{i}{p^2 - m^2 + i\epsilon} & 0 \\
0 & \frac{-i}{p^2 - m^2 - i\epsilon}
\end{pmatrix}
\]

and

\[
U(p) = \begin{pmatrix}
\sqrt{1 + n(p)} & \sqrt{n(p)} \\
\sqrt{n(p)} & \sqrt{1 + n(p)}
\end{pmatrix}
\]

Perturbation in the coupling \( \lambda \) is described by the following diagram rules. The external insertions are all of type 1. The vertices from the perturbation term must have either all legs of type 1 or all legs of type 2. Attach a factor \((-i\lambda)\) to the vertex in the former case, \((i\lambda)\) in the latter. The propagator is given by the matrix \((2.3)\) and can connect vertices of type 1 to vertices of type 2 at nonzero temperature. Summing all Feynman diagrams with these rules gives the correlation function at temperature \(\beta^{-1}\).
The power of the real time approach is that the density matrix need not be thermal, and the system need not be in equilibrium. Suppose the density matrix is specified at $t = -\infty$ in terms of the free field operators, so that the perturbation Hamiltonian will modify the distribution as time progresses. One uses the closed time path $C_2$ \[ 3, 4 \]:

$$
C_2 = \begin{align*}
I : & -\infty \to \infty \\
II : & \infty \to -\infty
\end{align*}
$$

(2.8)

for time evolution, and takes a trace after inserting $\rho$. Perturbation theory may be developed with the same rules as above, but using the matrix propagator appropriate to the contour $C_2$. In particular, if $\rho$ at $t = -\infty$ is of thermal form with temperature $\beta^{-1}$, then the matrix propagator is

$$
D(p) = \left( \frac{\frac{i}{p^2 - m^2 + i\epsilon}}{2\pi n(p) + \theta(p_0)} \delta(p^2 - m^2) + 2\pi n(p)\delta(p^2 - m^2) \right) \frac{2\pi [n(p) + \theta(-p_0)]\delta(p^2 - m^2)}{p^2 - m^2 + i\epsilon} + 2\pi n(p)\delta(p^2 - m^2)
$$

(2.9)

Perturbation theory based on the contours $C_1$ and $C_2$ are not equivalent, even for zero temperature \[ 4 \]. Contour $C_2$ corresponds to computing correlators with $\rho = \ket{0}\bra{0}$ inserted at $t = -\infty$. Consider the scalar field in $0+1$ spacetime dimensions, for simplicity. Let the only perturbation be the time dependent term $\mu \int dt \delta(t-t_0) : \phi^2(t) :$, and compute to first order the two point correlator for field insertions at $t_1 < t_0 < t_2$:

$$
\langle \phi(t_2)\phi(t_1) \rangle = 2\mu [\langle 0|\phi(t_2)\phi(t_0)|0 \rangle \langle 0|\phi(t_0)\phi(t_1)|0 \rangle + \langle 0|\phi(t_0)\phi(t_2)|0 \rangle \langle 0|\phi(t_0)\phi(t_1)|0 \rangle]
$$

(2.10)

With the contour $C_1$, for $\beta \to \infty$, we just get the first term on the RHS. The second term on the RHS comes from correcting the state at $t = \infty$ away from the vacuum $|0>$, to be such that it evolves from $|0>$ after suffering the perturbation at $t_0$. In short, the time path is significant because perturbation theory works in the interaction picture, while the physical situation is described in the Heisenberg picture. The second leg of the time path makes the state at $t = \infty$ the same as the state at $t = -\infty$ in the Heisenberg picture, which implies that perturbation corrections must be made along this path segment in the interaction picture.

Note that the corrections to $D^0(p)$ (eq. (2.6)) that give (2.3) or (2.9) are all on shell. This is a manifestation of the fact that these corrections arise from a flux of real particles. The expression $\delta(p^2 - m^2)$ does not have a Euclidean counterpart. (In Euclidean space one would consider $(p^2 + m^2)$ which is positive definite.) Thus temperature is a phenomenon
of Minkowski signature spacetime, where Green’s functions are subject to the addition of solutions of the homogeneous field equation.

To understand the origin of the on shell terms, let us consider the element $D_{11}$ in (2.3). We can decompose the free scalar field in Minkowski space into Fourier modes. Each quantised mode is a harmonic oscillator with frequency $\omega(p) = (p^2 + m^2)^{1/2}$. The two point function $< T[\phi(x_2)\phi(x_1)] >$ reduces, for each Fourier mode, to $< T[\hat{q}(t_2)\hat{q}(t_1)] >$ for each harmonic oscillator. Thus we focus on a single oscillator (we suppress its momentum label). Let $t_2 > t_1$. Then

$$< \hat{q}(t_2)\hat{q}(t_1) > \equiv \sum_n e^{-\beta(n+\frac{1}{2})}\omega < n|T[\hat{q}(t_2)\hat{q}(t_1)]|n > / \sum_n e^{-\beta(n+\frac{1}{2})}\omega$$

$$= \frac{< n+1 >}{2\omega} e^{-i\omega(t_2-t_1)} + \frac{< n >}{2\omega} e^{i\omega(t_2-t_1)}$$  \hspace{1cm} (2.11)

The first term on the RHS corresponds to the stimulated emission of a quantum at $t_1$ with absorption at $t_2$. The second term corresponds to the annihilation at time $t_1$ of one of the existing quanta in the thermal bath, and the subsequent transport of a hole from $t_1$ to $t_2$, where another particle is emitted to replace the one absorbed from the bath. The Fourier transform in time of (2.11) is

$$\int_{-\infty}^{\infty} dt e^{-i\omega t} < T[\hat{q}(t_2)\hat{q}(t_1)] > = \frac{i}{\omega r^2 - \omega^2 + i\epsilon} + 2\pi < n > \delta(\omega r^2 - \omega^2)$$  \hspace{1cm} (2.12)

which gives the matrix element $D_{11}$ in (2.3).

Thus the correction to $< T[\phi(x_2)\phi(x_1)] > = D_{11}(x_2, x_1)$ (and other elements of $D$) due to the particle flux is not an effect of interactions with the particles of the bath. This correction arises from the possible exchange of the propagating particle with identical real particles in the ambient flux. Using such a corrected propagator with the interaction vertices gives the interaction of the bath particles with the propagating particle.

2.2. ‘Exponential of quadratic’ density matrices.

Perturbation theory in the vacuum involves separating a free part which is described by propagators, and an interaction part which is described by vertices. In studying kinetic theory in Minkowski space, we also need to identify a free part and an interaction term. The free part involves specifying a density matrix $\rho$ of a special form, and a choice of time path [4]. These special $\rho$ are of the form ‘exponential of an expression with quadratic and linear terms in the field’ [4]. The linear term describes a coherent state, and gives a change
in the classical value of the field. Shifting the field by this classical value gets rid of this linear term, and we will always assume that this has been done. The quadratic part of the exponential implies a density matrix of the form

$$\rho = \prod_i e^{\alpha_i a_i^\dagger a_i^\vphantom{\dagger}} e^{-\beta_i a_i^\dagger a_i^\vphantom{\dagger}} e^{\gamma_i a_i^\dagger a_i^\vphantom{\dagger}}, \quad (2.13)$$

where the product is over different frequency modes, and $a_i, a_i^\dagger$ are the annihilation and creation operators for these modes. We will call the form (2.13) an ‘exponential of quadratic’ density matrix. The special role of density matrices (2.13) is due to the fact that Wick’s theorem extends to correlators computed with such $\rho$ \[4\]. Thus for operators $A_i$ linear in the field,

$$<A_1 \ldots A_n> \equiv \frac{1}{\text{Tr}\rho} \text{Tr}\{\rho A_1 \ldots A_n\} = \sum_{\text{permutations}} <A_{i_1} A_{i_2} \ldots A_{i_{n-1}} A_{i_n}> \quad (2.14)$$

We sketch a proof of (2.14) in the appendix.

The time path specifies where $\rho$ directly gives the particle flux. Thus a time path beginning and ending at $t = -\infty$ says that the density matrix (specified in the interaction picture) is the flux at $t = -\infty$, which implies that the flux at later times would be modified by perturbative corrections.

Perturbation theory may be developed for $\rho$ of the form (2.13) in the same manner as for the vacuum theory, with the replacement of propagators with matrix propagators as in the thermal examples of sec. 2.1. The matrix propagator encodes both the choice of $\rho$ and the choice of time path. Deviations from the form (2.13) of $\rho$ give rise to ‘correlation kernels’ which are three and higher point vertices arising from correlations encoded in the density matrix, rather than the interaction Hamiltonian \[4\].

We take the point of view that it is incorrect to construct a theory of quantised matter and gravity without including ‘exponential of quadratic’ density matrices in the possible backgrounds about which the perturbation will be developed. Suppose we choose a time co-ordinate $t$ and start with a distribution $e^{-\beta H}$, thermal for the Hamiltonian giving evolution in $t$. As the Universe expands, the distribution will not remain thermal, in general. Redshifting of wavelengths gives an obvious departure from thermal form if the field has a mass or is not conformally coupled. But even a massless conformally coupled field departs from thermal form if the time co-ordinate $t$ is not appropriately chosen \[10\]. However, the density matrix remains within the class (2.13), if it starts in this class, even
for a massive field. Indeed, we can describe such $\rho$ in a covariant fashion by choosing a set of global solutions to the wave-equation, and attaching operators $a_i, a_i^\dagger$ to pairs of functions $f_i, f_i^*$. $\rho$ gives the linear map (on this space of solutions) that must be made before identifying the two ends of the time path of the perturbation theory.

The class (2.13) of $\rho$ is closed under change of the basis of functions used to define creation and annihilation operators. For example an expanding Universe suffers particle creation, so that an initial vacuum state would be seen as filled with particles by an observer using ‘out’ frequency modes [1]. The density matrix in terms of ‘in’ modes is $|0\>_\text{in} |0\>_\text{in}$ which is obtained from (2.13) with $\alpha = \delta = 0, \beta \rightarrow \infty$. In terms of ‘out’ modes the Bogoliubov transformation gives the form $\rho \sim e^{b_1 a^\dagger} e^{b^* a} |0\>_\text{out} |0\>_\text{out}$ which is (2.13) with $\alpha = b, \gamma = b^*, \beta \rightarrow \infty$. The fact that in each case we are describing the state at the past time boundary is encoded in the time path, which starts at this boundary, and returns to it.

To summarise, the class of ‘exponential of quadratic’ $\rho$ is natural for defining the propagator. Considering this class unifies the special cases of thermal fluxes in flat space and the fluxes ‘created’ in spacetime expansion, and also gives covariance under change of the basis functions in spacetime used to define particles.

3. The first quantised formalism.

3.1. The regulator matrix.

The Feynman propagator for a scalar field in Minkowski space can be written as

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} = \int_{\tilde{\lambda}=0}^{\infty} d\tilde{\lambda} e^{i\tilde{\lambda}(p^2 - m^2 + i\epsilon)}$$

(3.1) can be used to express $G_F(p)$ in a first quantised language, with $p^2 = -\partial$. (See for example [11], [12].) The Hamiltonian on the world line is $-(p^2 - m^2)$, evolution takes place in a fictitious time for a duration $\tilde{\lambda}$, and this length $\tilde{\lambda}$ of the world line is summed over all values from 0 to $\infty$.

Let us write the matrix propagator (2.3) in a similar fashion

$$D_{NS}(p) = \int_0^\infty d\tilde{\lambda} e^{-i\tilde{\lambda}H - \epsilon\tilde{\lambda}M}$$

(3.2)
where

\[ H = \begin{pmatrix} -\left(p^2 - m^2\right) & 0 \\ 0 & \left(p^2 - m^2\right) \end{pmatrix}, \quad M = \begin{pmatrix} \frac{1 + 2n(p)}{\sqrt{n(p)(n(p) + 1)}} & -2\sqrt{n(p)(n(p) + 1)} \\ -2\sqrt{n(p)(n(p) + 1)} & \frac{1 + 2n(p)}{\sqrt{n(p)(n(p) + 1)}} \end{pmatrix} \] (3.3)

\[ n(p) \text{ is given by } (2.4). \] This matrix world line Hamiltonian has the following structure.

If we forget the term multiplying \( \epsilon \), then the \( a = 1 \) component of the vector state on the world line evolves as \( e^{i\tilde{\lambda}(p^2 - m^2)} \) while the \( a = 2 \) component evolves as \( e^{-i\tilde{\lambda}(p^2 - m^2)} \). To define the path integral we need the regulation from \( \epsilon \) at the mass shell \( p^2 - m^2 = 0 \). But the regulator matrix is not diagonal, for nonzero temperature. Thus transitions are allowed from state 1 to state 2. In the limit \( \epsilon \to 0^+ \), which we must finally take, these transitions occur only on the mass shell. (The absolute values of the entries in \( M \) are not significant, because \( \epsilon \) goes to 0, but the relative values are.)

Similarly we can write the matrix propagator (2.9) in the form (3.2) with \( H \) as in (3.3) but

\[ M = \begin{pmatrix} \frac{1 + 2n(p)}{\sqrt{n(p)(n(p) + 1)e^{\beta p_0/2}}} & -2\sqrt{n(p)(n(p) + 1)e^{-\beta p_0/2}} \\ -2\sqrt{n(p)(n(p) + 1)e^{\beta p_0/2}} & \frac{1 + 2n(p)}{\sqrt{n(p)(n(p) + 1)e^{-\beta p_0/2}}} \end{pmatrix} \] (3.4)

Again we have the \( a = 1, 2 \) states propagating on the world line with Hamiltonians \( \mp(p^2 - m^2) \). Transitions between \( a = 1, 2 \) are again of order \( \epsilon \), but are different from those in (3.4).

Looking at these examples the following picture emerges. To obtain the matrix propagator in a many-body situation (with ‘exponential of quadratic’ density matrices) we need to consider both evolutions \( e^{i\tilde{\lambda}H} \) and \( e^{i\tilde{\lambda}H} \) on the world line. A damping factor is needed to define the first quantised path integral, near the mass shell. But the regulator matrix \( M \) need not be diagonal, and it reflects both the ‘exponential of quadratic particle flux and the choice of time path. (For example, in the limit \( \beta \to \infty \) in (3.3) is diagonal, but in (3.4) it is not.)

3.2. Quantising the relativistic particle.

What is the origin of the two components \( a = 1, 2 \) of the state on the world line? We would like to offer the following heuristic ‘derivation’ as a more physical description of the matrix structure in (3.2).

The geometric action for a scalar particle is

\[ S = \int_{X_i}^{X_f} m ds = \int_{X_i}^{X_f} m(X^\mu,_{\tau} X_{\mu,\tau})^{1/2} d\tau \equiv \int L d\tau \] (3.5)
where \( \tau \) is an arbitrary parametrisation of the world line. The canonical momenta

\[
P_\mu = \frac{\partial L}{\partial X^{\mu,\tau}} = \frac{mX_{\mu,\tau}}{(X^{\mu,\tau} X_{\mu,\tau})^{1/2}}
\]  

(3.6)
satisfy the constraints

\[
P_\mu P_\mu - m^2 = 0
\]  

(3.7)

We choose the range of the parameter \( \tau \) as \([0, 1]\). Following the approach in [12], we impose

the constraint at each \( \tau \) through a \( \delta \)-function:

\[
\delta(p^2(\tau) - m^2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\lambda(\tau) e^{-i\lambda/2(p^2(\tau) - m^2)}
\]  

(3.8)

The path integral amplitude to propagate from \( X_i \) to \( X_f \) becomes

\[
N \int \frac{D[X]D[P]D[\lambda]}{\text{Vol[Diff]}} e^{i \int_0^1 d\tau [P_\mu X^{\mu,\tau}(\tau) - \lambda/2(p^2(\tau) - m^2)]}
\]  

(3.9)

where \( N \) is a normalisation constant, \( P_\mu X^{\mu,\tau} = m(X^{\mu,\tau} X_{\mu,\tau})^{1/2} \) is the original action

(3.5) and we have divided by the volume of the symmetry group, which which is related to \( \tau \)-diffeomorphisms in the manner discussed below. (The \( \delta \)-function constraint on the momenta and dividing by \( \text{Vol[Diff]} \) remove the two phase space co-ordinates redundant in the description of the particle path.)

There are two ways to consider the symmetry of the action (3.9). The action is invariant under

\[
S_1 : \quad \delta X^{\mu}(\tau) = h(\tau) P^{\mu}(\tau)
\]
\[
\delta P_\mu(\tau) = 0
\]
\[
\delta \lambda(\tau) = h(\tau),\tau
\]  

(3.10)

and

\[
S_2 : \quad \delta X^{\mu}(\tau) = \epsilon(\tau) \lambda(\tau) P^{\mu}(\tau)
\]
\[
\delta P_\mu(\tau) = 0
\]
\[
\delta \lambda(\tau) = (\epsilon(\tau) \lambda(\tau)),\tau
\]  

(3.11)

The difference between \( S_1 \) and \( S_2 \) is best seen by considering the finite transformations on \( \lambda \):

\[
S_1 : \quad \lambda'(\tau) = \lambda(\tau) + \frac{dh(\tau)}{d\tau}
\]
\[
S_2 : \quad \lambda'(\tau) = \frac{d\tau}{d\tau'}(\tau)\lambda(\tau)
\]  

(3.12)
Using $S_1$ we can gauge fix any function $\lambda(\tau)$ to any other function $\lambda_1(\tau)$, provided $\lambda$, $\lambda_1$ have the same value of

$$\int_0^1 \lambda(\tau) d\tau \equiv \Lambda$$

(3.14)

With $S_2$, $\lambda$ transforms as an einbein under the diffeomorphism $\tau \rightarrow \tau'(\tau)$. Note that for regular $\epsilon(\tau)$, $\lambda$ either changes sign for no $\tau$ or for all $\tau$. We take Diff as the group of regular diffeomorphisms connected to the identity; then we have only the former case. These diffeomorphisms cannot gauge-fix $\lambda(\tau)$ to any preassigned function $\lambda_1(\tau)$. We again have the restriction (3.14), where $\Lambda$ may now be interpreted as the length of the world line. This restriction is usually assumed to mean that the length of the world line is the only remaining parameter after gauge-fixing. What we find instead is that there is a discrete infinity of classes, each with one or more continuous parameters. One member of this class comes from configurations $\lambda(\tau)$ which are everywhere positive; this class can be gauge-fixed to have

$$\dot{\lambda}(\tau) = 0, \quad \int_0^1 d\tau \lambda(\tau) = \Lambda$$

(3.15)

with $0 < \Lambda < \infty$. Similarly, the set of everywhere negative $\lambda(\tau)$ can be gauge-fixed as in (3.15) but with $-\infty < \Lambda < 0$. Keeping the first class alone gives the Feynman propagator for particles, while the second gives its complex conjugate. But we also have for example the class of $\lambda(\tau)$ which are positive for $0 < \tau < \tau_1$, negative for $\tau_1 < \tau < 1$. The group of orientation preserving diffeomorphisms can gauge fix this to

$$\dot{\lambda}(\tau) = 0 \quad \text{for} \quad \tau \neq \tau_1, \quad \int_0^{\tau_1} d\tau \lambda(\tau) = \Lambda_1, \quad \int_{\tau_1}^1 d\tau \lambda(\tau) = \Lambda_2$$

(3.16)

with $0 < \Lambda_1 < \infty$, $-\infty < \Lambda_2 < 0$. We would like to identify this sector as the contribution to the amplitude to start with a state of type 1 and end with a state of type 2 (the off-diagonal element $D_{12}$ of the matrix propagator). Similarly, all sectors beginning and ending with $\Lambda > 0$ (thus having an even number of changes of the sign of $\Lambda$) contribute to $D_{11}$. We can add together these sectors for $D_{11}$ once we choose the factor to be attached to each change in the sign of $\Lambda$. Choosing this factor is equivalent to choosing the regulator matrix $M$, and an explicit summation of sectors reproduces the matrix propagator.

\footnote{A restriction to the range $(0, \infty)$ for $\Lambda$ can be naturally obtained using a Newton-Wigner formalism \cite{13}. Here the particle travels only forwards in the time co-ordinate $X^0$, thus it is not a co-variant approach.}
3.3. BRST formalism.

One might wonder if the more formal BRST quantisation of the relativistic particle would resolve the above issues about different possible quantisations. We follow the notation in [14]. We introduce the canonical conjugate $\pi$ for $\lambda ([\lambda, \pi] = i)$ and ghosts $(\eta^1, P_1), (\eta^2, P_2)$ $([\eta^i, P_j] = -i\delta^i_j)$ for the two constraints $\pi = 0$ and $\frac{1}{2}(P^2 - m^2) = 0$ respectively. The BRST charge

$$Q = \eta^1 \pi + \frac{\eta^2}{2}(P^2 - m^2)$$  \hspace{1cm} (3.17)

is nilpotent ($Q^2 = 0$), and gives the variations:

$$\delta X^\mu = -i[X^\mu, Q] = \eta^2 P^\mu \quad \delta P_\mu = 0$$
$$\delta \lambda = \eta^1 \quad \delta \pi = 0$$
$$\delta \eta^1 = 0 \quad \delta P_1 = -\pi \sim 0$$
$$\delta \eta^2 = 0 \quad \delta P_2 = -\frac{1}{2}(P^2 - m^2) \sim 0$$  \hspace{1cm} (3.18)

The equation of motion (obtained after gauge fixing) gives $\dot{\eta}^2 = \pi$, which agrees with (3.10).

But we can define another nilpotent BRST charge

$$Q' = \eta'^1 \pi' \lambda' + \frac{\eta'^2}{2}(P'^2 - m'^2)$$  \hspace{1cm} (3.19)

which generates the symmetry

$$\delta X'^\mu = \eta'^2 \lambda' P'^\mu \quad \delta P'_\mu = 0$$
$$\delta \lambda' = \lambda' \eta'^1 \quad \delta \pi' = -\eta'^1 \pi' \sim 0$$
$$\delta \eta'^1 = 0 \quad \delta P'_1 = -\lambda' \pi' \sim 0$$
$$\delta \eta'^2 = 0 \quad \delta P'_2 = -\frac{1}{2}(P'^2 - m'^2) \sim 0$$  \hspace{1cm} (3.20)

The equation of motion gives $\dot{\eta}'^2 = \eta'^1 \lambda'$, which suggests that we should identify the above symmetry with $S_2$.

The symmetries $Q$ and $Q'$ are related through the identifications

$$\pi = \pi' \lambda', \quad \lambda = log\lambda'$$  \hspace{1cm} (3.21)

all other primed variables equalling the unprimed ones. From (3.21) we find that $-\infty < \lambda < \infty$ corresponds to $0 < \lambda' < \infty$. If we perform a path integral with the primed variables
and sum over both positive and negative $\lambda'$ then we are summing over more than is being summed in the unprimed variable path integral.

We thus see sources of ambiguity on the quantisation of the relativistic particle working with a Fadeev-Popov approach in sec 3.2 and a BRST approach in sec. 3.3. In fact the action we start with, $(3.5)$, is itself ambiguous because of the two possible signs of the square root. The particle trajectory would keep switching in general between timelike and spacelike, and at each switch we have to choose afresh the sign of the real or imaginary quantity obtained in these two cases respectively. This suggests that the world line configuration should be described by the pair $\{X^\mu(\tau), \sigma(\tau)\}$ with $\sigma = \pm 1$ giving the choice of root. Evaluating the quadratic form of the action (given in $(3.9)$) classically we find the sign of $\lambda$ to be related to the sign of the square root chosen for $(3.5)$.

The above discussion suggests a close connection between the ambiguities found in three different approaches to the quantum relativistic particle, and it would be good to determine if they indeed are the same. For the rest of this paper we simply adopt as basic the picture of two complex conjugate propagations on the world line, with switching between them possible through the regulator matrix.

4. A curved space example: spacetime with expansion.

Consider the free scalar field ($(2.1)$ with $\lambda = 0$) propagating in $1 + 1$ spacetime with metric

$$ds^2 = C(\eta)[d\eta^2 - dx^2], \quad -\infty < \eta < \infty, \quad 0 \leq x < 2\pi$$

$$C(\eta) = A + B \tanh \kappa \eta, \quad A > B \geq 0$$

The conformal factor $C(\eta)$ tends to $A \pm B$ at $\eta \to \pm \infty$. The limit $\kappa \to \infty$ gives a step function for $C(\eta)$; the Universe jumps from scale factor $A - B$ to $A + B$ at $\eta = 0$. We will work in this limit to ensure simpler expressions.

For $\eta \to -\infty$ it is natural to expand $\phi$ as

$$\phi(\eta, x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_n^+}} (a_n e^{-i\omega_n^- \eta + inx} + a_n^\dagger e^{i\omega_n^- \eta - inx})$$

with

$$\omega_n^- = (n^2 + (A - B)m^2)^{1/2} > 0$$
We define the ‘in’ vacuum by

$$a_n|0_{\text{in}} = 0, \quad \text{for all } n$$  \hspace{1cm} (4.5)

Similarly, for $$\eta \to \infty$$ we write

$$\phi(\eta, x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_n^+}} (\tilde{a}_n e^{-i\omega_n^+ \eta + inx} + \tilde{a}_n^\dagger e^{i\omega_n^+ \eta - inx})$$  \hspace{1cm} (4.6)

with

$$\omega_n^+ = (n^2 + (A + B)m^2)^{1/2} > 0$$  \hspace{1cm} (4.7)

The ‘out’ vacuum is defined through

$$|0_{\text{out}} = 0, \quad \text{for all } n$$  \hspace{1cm} (4.8)

The ‘out’ vacuum does not equal the ‘in’ vacuum, even in the free theory:

$$|0_{\text{out}} = C_0 e^{b_0 a_0^\dagger a_0^\dagger} \prod_{n>0} C_n e^{b_n a_n^\dagger a_{-n}^\dagger} |0_{\text{in}}$$  \hspace{1cm} (4.9)

$$b_n = \frac{\omega_n^+ - \omega_n^-}{\omega_n^+ + \omega_n^-}, \quad C_n = (1 - b_n^2)^{1/2}, \quad n \geq 0.$$  \hspace{1cm} (4.10)

The Bogoliubov transformation (4.9) says that an ‘out’ observer will find particles in his frame as $$\eta \to \infty$$, if the ‘in’ observer sees a vacuum. Since the Bogoliubov transformation is given by the exponential of a quadratic in the field, we guess that our formalism developed in the preceding section should apply. In other words, the effect of the flux created by spacetime expansion can be incorporated by a change in the regulator matrix of the first quantised path integral. We demonstrate this explicitly in our example.

The first quantised path integral gives [2] (we denote the pair $$(\eta, x)$$ by $$z$$)

$$<z_2| \int_0^\infty d\lambda D[X] e^{-i \int_0^1 d\tau (1/2)(X^2/\lambda + m^2 \lambda)} |z_1> = \frac{\langle 0_{\text{out}}|T[\phi(z_2)\phi(z_1)]|0_{\text{in}}\rangle}{\langle 0_{\text{out}}|0_{\text{in}}\rangle}$$  \hspace{1cm} (4.11)

(We have integrated out $$p$$ in the phase space path integral.) It is not surprising that both vacuua appear in this quantity; after all the action and measure are covariantly given and should not distinguish the past or future as special. Using $$-i$$ in place of $$i$$ in the exponential gives $$\text{in} < 0|T[\phi(z_2)\phi(z_1)]|0_{\text{out}} /_{\text{in}} < 0|0_{\text{out}}.$$ $$(\bar{T}$$ denotes anti-time-ordering.)

In studying kinetic theory we typically wish to specify the density matrix in terms of the ‘in’ states, e.g. $$\rho = \sum e^{-\beta n}|n\rangle_{\text{in}} < n| / (\sum e^{-\beta n})$$ where $$|n\rangle_{\text{in}}$$ gives the occupation
number \(n\) state for some positive frequency mode at past infinity. For \(\beta \to \infty\), \(\rho \equiv \rho_0 = |0 >_{in} \in < 0\). We use \(\rho_0\) for our illustration; it should be straightforward to consider both expansion of spacetime and an initial exponential distribution of particles, by putting together the results of this section and the last section.

To develop a perturbation theory using \(\rho_0\) we need a real time contour running from \(\eta = -\infty\) to \(\eta = \infty\), and then back to \(\eta = -\infty\) where we insert \(\rho_0\) and take a trace to close the path. This perturbation theory requires a matrix propagator \(D\):

\[
D(z_2, z_1) = \begin{pmatrix}
|0 >_{in}|T[\phi(z_2)\phi(z_1)]|0 >_{in} \\
|0 >_{in}|\phi(z_2)\phi(z_1)|0 >_{in} \\
|0 >_{in}|T[\phi(z_2)\phi(z_1)]|0 >_{in}
\end{pmatrix}
\] (4.12)

Using the ‘matrix action’ \(i(-\Box - m^2)\sigma_3 - \epsilon I\) in the path integral (4.11) gives the matrix propagator \(D^0 = \text{diag}\{\text{out} < 0|T[\phi(z_2)\phi(z_1)]|0 >_{in} , |0 >_{in}|\tilde{T}[\phi(z_2)\phi(z_1)]|0 >_{out}\}\) We wish to change only the operator multiplying \(\epsilon\), and we wish to get (4.12).

Let us set up the calculation of Green’s functions in the first quantised formalism. We need eigenfunctions of the world line Hamiltonian:

\[
(\Box + m^2)\psi_s(\eta, x) = -H\psi_s(\eta, x) = s\psi_s(\eta, x)
\] (4.13)

The following is a complete set: \((-\infty < n < \infty)\)

\[
m^2 + \frac{n^2}{A + B} < s < m^2 + \frac{n^2}{A - B}:
\]

\[
\psi_{\nu_+, n}(\eta, x) = \frac{e^{inx}}{\sqrt{2\pi}}e^{-\tilde{\nu}_+\eta}, \quad \eta > 0
\]

\[
\frac{e^{inx}}{\sqrt{2\pi}}\left[\frac{1}{2}(1 + \frac{\tilde{\nu}_+}{\nu_-})e^{-i\nu_-\eta} + \frac{1}{2}(1 - \frac{\tilde{\nu}_+}{\nu_-})e^{i\nu_-\eta}\right], \quad \eta < 0
\]

\[
\tilde{\nu}_+ = - (A + B)(m^2 - s) - n^2]^{1/2} > 0, \quad \nu_- = [(A - B)(m^2 - s) + n^2]^{1/2} > 0
\] (4.14)

\[
-\infty < s < m^2 + \frac{n^2}{A + B}:
\]

\[
\psi_{\nu_+, n}(\eta, x) = \frac{e^{inx}}{\sqrt{2\pi}}e^{-i\nu_+\eta}, \quad \eta > 0
\]

\[
\frac{e^{inx}}{\sqrt{2\pi}}\left[\frac{1}{2}(1 + \frac{\nu_+}{\nu_-})e^{-i\nu_-\eta} + \frac{1}{2}(1 - \nu_+)e^{i\nu_-\eta}\right], \quad \eta < 0
\]

\[
\nu_+^2 = (A \pm B)(m^2 - s) + n^2, \quad \text{sign}(\nu_+) = \text{sign}(\nu_-)
\] (4.15)
These functions are normalised as
\[
(ψ_{ν_{+},n'}, ψ_{ν_{+},n}) = \int dη dxC(η)ψ_{ν_{+},n'}^*(η, x)ψ_{ν_{+},n}(η, x)
= δ_{n', n}π(A + B) \frac{υ_{+}^2 + \bar{υ}_{+}^2}{2υ_{-}υ_{+}} δ(υ_{+}' - υ_{+})
\]
\[
(ψ_{ν_{+},n'}, ψ_{ν_{+},n}) = δ_{n', n}π(A + B)[(υ_{+}' + υ_{+})^2 2υ_{-}υ_{+} δ(υ_{+}' - υ_{+}) + \frac{υ_{-}' - υ_{+}^2}{2υ_{-}υ_{+}} δ(υ_{+}' + υ_{+})]
\]
\[(4.16)
\]

Let us first recover the propagator \(\frac{2π}{A+B} n\) in this formalism. Let \(η', η > 0\). The range \(m^2 + \frac{n^2}{A+B} < s < m^2 + \frac{n^2}{A-B}\) gives the contribution
\[
\sum_{n} \int_{\bar{υ}_{+}, \bar{υ}_{+} = 0}^{\sqrt{2B/(A-B)|n|}} d\bar{υ}_{+} d\bar{υ}_{+} < η', x' | ψ_{ν_{+}}, n > < ψ_{ν_{+}}, n | \int_{0}^{∞} dλ e^{i(-s+iλ)λ} | ψ_{ν_{+}}, n > < ψ_{ν_{+}, n} | η, x > δ(υ_{+}' - υ_{+})[(A + B)π]^{-1} \frac{2υ_{-}υ_{+}}{υ_{+}' υ_{+}}
\]
\[
= \sum_{n} \frac{e^{in(x'-x)}}{2π} \int_{0}^{\sqrt{2B/(A-B)|n|}} dυ_{+} (i) e^{-υ_{+} (η'+η)} \frac{2υ_{-}υ_{+}}{υ_{+}' υ_{+}} \frac{1}{(υ_{+}'^2 - n^2 - m^2(A + B))}
\]
\[(4.17)
\]

Similarly the range \(-∞ < s < m^2 + \frac{n^2}{A+B}\) provides the contribution
\[
\sum_{n} \frac{e^{in(x'-x)}}{2π} \int_{-∞}^{∞} dυ_{+} \frac{i}{2π} \left[ e^{iυ_{+}(η'-η)} + \frac{υ_{+}' - υ_{+}}{υ_{+} + υ_{-}} e^{iυ_{+}(η'+η)} \right] \frac{1}{(υ_{+}'^2 - n^2 - m^2(A + B) + iε)}
\]
\[(4.18)
\]

There is a branch cut in the complex \(υ_{+}\) plane joining \(υ_{+} = ±i \sqrt{2B/(A-B)|n|}\). The \(υ_{+}\) integral in (4.19) has a discontinuous jump across this cut for the part multiplying \(e^{iυ_{+}(η'+η)}\). The contribution from (4.18) can be added to (4.19), however, with the identification \(υ_{+}' = iυ_{+}\). This results in a contour passing over the cut. Evaluating the resulting contour integrals one obtains the result
\[
D_{11}^{0}(η', x', η, x) = \sum_{n} \frac{e^{in(x'-x)}}{2π} \frac{1}{2ω_{n}^+} e^{-iω_{n}^+ |η'-η|} + \frac{1}{2ω_{n}^+ ω_{n}^- + ω_{n}^-} e^{-iω_{n}^-(η'+η)}, \text{ for } η, η' > 0
\]
\[(4.19)
\]

which may be readily verified in the operator language using (4.6) and (4.9).

In the flat space examples of section 3 energy-momentum conservation implied that the regulator matrix was diagonal in the basis of eigenfunctions of \(-(|T| + m^2)\) given by fixed \((p_0, p)\). In the time-dependent situation that we have now, energy is not conserved. The regulator matrix acts within the eigenspace of a fixed value of \(s\). Thus for each \(n\) we need
to consider a 4x4 matrix, which acts on the column vector $(\{f_{s,n}^1, f_{s,n}^2\}^+, \{f_{s,n}^1, f_{s,n}^2\}^-)$. Here the first pair of functions propagate on the world line as $e^{-i\sigma}$ while the second pair propagates as $e^{i\sigma}$. Within each type (+ or −) we have two linearly independent functions for any given $s$ near the mass shell $s = 0$. Since we wish to compute the propagator with the ‘in’ vacuum density matrix, we choose in this space the basis which at the mass shell becomes $(\omega^-_n > 0)$

$$f_n^1 = \frac{e^{inx}}{\sqrt{2\pi}}e^{-i\omega^-_n \eta}, \quad \eta < 0$$

$$= \frac{e^{inx}}{\sqrt{2\pi}}\left[\frac{1}{2}(1 + \frac{\omega^-_n}{\omega^+_n})e^{-i\omega^+_n \eta} + \frac{1}{2}(1 - \frac{\omega^-_n}{\omega^+_n})e^{i\omega^+_n \eta}\right], \quad \eta > 0 \quad (4.21)$$

$$f_n^2(\eta, x) = f_n^1(\eta, x)$$

We want the matrix propagator for the density matrix $\rho = |0 >_{in} in < 0|$ and the time path starting at $t = -\infty$, going to $t = \infty$ and then back to $t = -\infty$. A calculation similar to the above yields that the propagator (4.12) is obtained if the regulator matrix in the space of the $n$th Fourier mode $e^{inx}$ is

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 4B_n & 0 \\
\frac{-2}{1+4B_n^2} & \frac{4B_n}{1+4B_n^2} & \frac{1}{1+4B_n^2} & \frac{-2}{1+4B_n^2} \\
0 & \frac{4B_n}{1+4B_n^2} & \frac{1}{1+4B_n^2} & 0
\end{pmatrix} \quad (4.22)$$

where

$$B_n = \frac{1}{2}\frac{\omega^+_n - \omega^-_n}{\omega^+_n + \omega^-_n} \quad (4.23)$$

(In this computation we need to note that $\lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi \delta(x)$, but $\lim_{\epsilon \to 0} \frac{\epsilon}{(x+i\epsilon)^2} = 0$.)

5. Discussion.

We have taken the view that in a theory of quantised matter and gravity the background for perturbation theory should not only be a specification of classical values of fields, but also a specification of ‘exponential of quadratic’ particle fluxes. We know that these two different aspects of the background arise naturally in flat space kinetic theory. With curved space, it becomes natural to consider the kinetic theory and to not construct

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\(^2\) The regulator matrix needs to be defined only in the infinitesimal neighbourhood of the mass shell.
a ‘vacuum’ theory at all. The reason is that starting with physically acceptable conditions in the past, say, particle fluxes can be created in the co-ordinates natural in the future.

The usual first quantised approach of string theory, applied to the scalar particle, gives a ‘vacuum’ theory, where certain ‘in-out’ vacuum boundary conditions are chosen at temporal infinity. These boundary conditions appear unnatural for physical purposes, so we would like to be able to move to a more general class of states at the boundary. In particular we would like to be allowed a radiation flux at the past time boundary, as in the radiation dominated Cosmologies. We find that in the first quantised language there is a natural way to obtain this more general theory. Quantisation of the relativistic particle indicates that we should consider both propagations $e^{-iH\lambda}$ and $e^{iH\lambda}$ on the world line. The regulator matrix needed to regularise this path integral need not be diagonal in these two modes of propagation. Off-diagonal terms encode an ‘exponential of quadratic’ density matrix and a choice of time path for perturbation theory.

One special case of particle flux, the flux for constant temperature $\beta^{-1}$, may be studied without the real time formalism. One studies the theory on spacetime with time rotated to Euclidean signature and identifies $t$ with $t - i\beta$ [15]. But constant temperature is unnatural in a theory with gravity, as the particle density gives a gravitational field, which gives an acceleration of the scale factor of the Universe (unless we carefully balance the matter density with a Cosmological constant). The changing scale factor violates the constant temperature requirement needed for the Euclidean time trick to work.\footnote{Leblanc [16] studied the real time formalism for open and closed strings, for the case of constant temperature. The propagator was computed in the ‘thermo-field dynamics’ language, which used the Niemi-Semenoff time path, and so was given by (3.2), (3.3). For this time-independent situation amplitudes were computed and the Hagedorn temperature recovered. In [17] The imaginary time formulation was used to compute the Background field equations for the closed string at constant temperature.}

We should distinguish two different limits in which the physics of fluxes may be studied. One limit is where the collisions are so rapid that approximate thermal equilibrium is maintained at all times, and we need only let $\beta$ be a function of time. The other limit is that of kinetic theory, where we assume that collisions are rare; particle wavefunctions

\footnote{More general flux situations may be obtained by making a canonical transformation on the variables living on the world line and compactifying the new ‘time’ co-ordinate after analytic continuation to imaginary values. But such a description does not appear physically illuminating.}
evolve on the time-dependent background, and collisions between these particles are taken into account by perturbation theory. Our approach assumes the latter limit.

The limit \( \epsilon \to 0 \) implies that the effect of the regulator matrix \( M \) is felt only on-shell (i.e. for \( p^2 - m^2 = 0 \)). Equivalently, we may say that only world lines of infinite length (\( \tilde{\lambda} = \infty \)) see the regulator matrix. To see this, let \( M \) and \( M' \) be two different regulator matrices. We write

\[
D_{M'} = \lim_{L \to \infty} \lim_{\epsilon \to 0^+} \left\{ \left[ \int_0^L d\tilde{\lambda} e^{-iH\tilde{\lambda} - \epsilon M\tilde{\lambda}} + \int_0^\infty d\tilde{\lambda} e^{-iH\tilde{\lambda} - \epsilon M\tilde{\lambda}} \right] \\
+ \left[ \int_L^\infty d\tilde{\lambda} e^{-iH\tilde{\lambda} - \epsilon M'\tilde{\lambda}} - \int_L^\infty d\tilde{\lambda} e^{-iH\tilde{\lambda} - \epsilon M\tilde{\lambda}} \right] \\
+ \left[ \int_0^L d\tilde{\lambda} e^{-iH\tilde{\lambda} - \epsilon M'\tilde{\lambda}} - \int_0^L d\tilde{\lambda} e^{-iH\tilde{\lambda} - \epsilon M\tilde{\lambda}} \right] \right\}
\]

The first square bracket on the RHS is \( D_M \), the last vanishes with the indicated limits, while the second has support only on world lines of infinite length.

The analogue of the above statement for strings is that the effect of \( \rho \) is to give a contribution to the boundary of the moduli space of Riemann surfaces, where a homologically trivial or non-trivial cycle is pinched. (It is important to have Minkowski signature target space, and correspondingly a Minkowski signature world sheet, to allow the on-shell condition for the particle flux.) A \( \beta \)-function calculation for the string world sheet theory would have to take into account such pinches while considering the small handle contribution studied by Fishler and Susskind [18]. This calculation should yield a relation between the classical fields and the particle fluxes, rather than just among the classical fields giving the background.

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Appendix A. Wick theorem for ‘exponential of quadratic’ density matrices.

We wish to establish Wick’s theorem for density matrices of the form

\[ \rho = e^{\alpha a^\dagger a^\dagger} e^{-\beta a^\dagger a} e^{\gamma a a} \quad (A.1) \]

A string of creation and annihilation operators can be brought to normal ordered form in the same way as for the usual Wick theorem in the vacuum. What we need to show in addition is that

\[ \frac{1}{\text{Tr}\rho} \text{Tr}\{\rho a^\dagger \ldots a^\dagger a \ldots a\} = \sum \frac{\text{Tr}\{\rho a^\dagger a\}}{\text{Tr}\rho} \ldots \frac{\text{Tr}\{\rho a a\}}{\text{Tr}\rho} \quad (A.2) \]

where the RHS has a summation over all possible pairings of the \(a^\dagger, a\) operators on the LHS. We sketch below some of the steps involved in the derivation.

Note

\[ e^{\alpha a^\dagger a^\dagger} e^{-\beta a^\dagger a} = e^{-\beta a^\dagger a} e^{\alpha' a^\dagger a^\dagger} \quad (A.3) \]

with \(\alpha' = \alpha e^{2\beta}\). A straightforward calculation gives

\[ \text{Tr}\rho = (1 - e^{-\beta})^{-1}[1 - \frac{4\alpha\gamma}{(1 - e^{-\beta})^2}]^{-1/2} \quad (A.4) \]

which we may rewrite as

\[ \text{Tr}\rho = \text{Tr}\{e^{-\beta a^\dagger a} e^{\alpha' a^\dagger a^\dagger} e^{\gamma a a}\} = (1 - e^{-\beta})^{-1}[1 - \frac{4\alpha'\gamma}{(1 - e^{-\beta})^2}]^{-1/2} \quad (A.5) \]

From (A.1) we see that in (A.2) there must be either an even number \((2p)\) of \(a\) oscillators and an even number \((2q)\) of \(a^\dagger\) oscillators, or an odd number \((2p+1)\) of \(a\) and an odd number \((2q+1)\) of \(a^\dagger\) oscillators. Assume first that we have the former case. Then the LHS of (A.2) is obtained as

\[ \text{Tr}\{\rho(a^\dagger)^{(2q)}(a)^{2p}\} = \frac{1}{\text{Tr}\rho} (\partial_{\alpha'})^q (\partial_{\gamma})^p \text{Tr}\rho \quad (A.6) \]

(Partial derivatives are taken with \(\alpha', \beta, \gamma\) as independent variables, unless otherwise mentioned.) In particular,

\[ <a^\dagger a^\dagger> = \frac{1}{\text{Tr}\rho} \partial_{\alpha'} \text{Tr}\rho = 2\gamma e^{-2\beta}/K \equiv A \]

\[ <aa> = \frac{1}{\text{Tr}\rho} \partial_{\gamma} \text{Tr}\rho = 2\alpha' e^{-2\beta}/K \equiv B \quad (A.7) \]

\[ <a^\dagger a> = -\frac{1}{\text{Tr}\rho} \partial_{\beta}[\text{Tr}\rho]_{\alpha,\gamma} = e^{-\beta}(1 - e^{-\beta})/K \equiv C \]
where in computing $C$, $\partial_\beta$ is a partial derivative with $\alpha$, $\gamma$ held fixed. Here

$$K = (1 - e^{-\beta})^2 - 4\alpha' \gamma e^{-2\beta}$$

(A.8)

We find

$$\text{Tr} \rho = e^\beta [C^2 - AB]^{1/2} = K^{-1/2}$$

(A.9)

For fixed $\beta$

$$\partial_\alpha'A = 2A^2 \quad \partial_\gamma A = 2C^2$$
$$\partial_\alpha'B = 2C^2 \quad \partial_\gamma B = 2B^2$$
$$\partial_\alpha'C = 2AC \quad \partial_\gamma C = 2BC$$

(A.10)

Using the above formulae, we can establish (A.2) by induction. Suppose (A.2) holds with $2p$ operators ‘$a’$ and $2q$ operators ‘$a^\dagger’$. A typical term on the RHS would have the form $FA^{n_1}B^{n_2}C^{n_3}$, where $F$ is a constant and $n_1$, $n_2$, $n_3 \geq 0$. To establish the result for $2p$ operators ‘$a’$ and $2q + 2$ operators ‘$a^\dagger’$ we get for the LHS of (A.2):

$$\frac{1}{e^\beta(C^2 - AB)^{1/2}} \partial_\alpha' [e^\beta(C^2 - AB)^{1/2} FA^{n_1}B^{n_2}C^{n_3}] = F[A^{n_1+1}B^{n_2}C^{n_3} + 2n_1A^{n_1+1}B^{n_2}C^{n_3} + 2n_2A^{n_1}B^{n_2-1}C^{n_3} + 2n_3A^{n_1+1}B^{n_2}C^{n_3}]$$

(A.11)

The first term on the RHS of (A.11) gives the pairing of the two new operators $a^\dagger$ with each other. The second term gives the $n_1$ ways to choose an existing pair ($a^\dagger a^\dagger$) and to contract the new $a^\dagger$ operators with members of this pair instead. The third term corresponds to choosing an ($aa$) pair in the original expression and contracting the ‘$a’$ operators with the new ‘$a^\dagger’$ operators instead. The last term corresponds to exchanging the $a^\dagger$ in an existing $a^\dagger a$ pair with one of the new $a^\dagger$ operators. It is easily seen that this generates all the new terms required on the RHS of (A.2) for the induction to hold.

To work with the case of an odd number of $a$ and $a^\dagger$ operators we start with the expression $C \text{Tr} \rho + 2\gamma \partial_\gamma \text{Tr} \rho = \text{Tr} \{e^{-\beta a^\dagger a} e^{\alpha' a^\dagger a} a^\dagger a e^{\gamma aa}\}$ in place of $\text{Tr} \rho$, and proceed as above to introduce extra $a^\dagger a$ and $aa$ pairs in the induction.
References

[1] N.D. Birrell and P.C.W. Davies ‘Quantum fields in curved space’ (1982) Cambridge Univ. Press.
[2] H. Rumpf and H.K. Urbantke, Ann. Phys. 114 (1978) 332, H. Rumpf, Phys. Rev. D 24 (1981) 275, D 28 (1983) 2946.
[3] N.P. Landsman and Ch.G. Van Weert, Phys. Rep. 145 (1987) 141.
[4] P.A. Henning, Nucl. Phys. B 337 (1990) 547.
[5] A. Niemi and G. Semenoff, Ann. Phys. 152 (1981) 105, Nucl. Phys. B 230 (1984) 181.
[6] L.V. Keldysh, J.E.T.P. 20 (1965) 1018.
[7] J.S. Schwinger, J. Math. Phys. 2 (1961) 407.
[8] A. Sen, Phys. Rev. Lett. 55 (1985) 1846.
[9] J.P. Paz, in ‘Thermal Field Theories’ ed. H. Ezawa, T. Arimitsu and Y. Hashimoto (1991) Elsevier Science Publishers.
[10] G. Semenoff and N. Weiss, Phys. Rev. D 31 (1985) 689.
[11] G. Parisi, ‘Statistical Field Theory’ (1988) Addison Wesley.
[12] M. Kaku, ‘Introduction to Superstrings’ (1988) Springer-Verlag.
[13] J.B. Hartle and K.V. Kuchar, Phys. Rev. D 34 (1986) 2323.
[14] J. Govaerts, ‘Hamiltonian quantisation and constrained dynamics’ (1991) Leuven Univ. Press.
[15] B. Sathiapalan, Phys. Rev. Lett. 58 (1987) 1597, J.J. Attick and E. Witten, Nucl. Phys. B 310 (1988) 291.
[16] Y. Leblanc, Phys. Rev. D 36 (1987) 1780, D 37 (1988) 1547, D 39 (1989) 1139.
[17] M. Hellmund and J. Kripfganz, Phys. Lett. B241 (1990) 211.
[18] W. Fishler and L. Susskind, Phys. Lett. B171 (1986) 383, B173 (1986) 262.