THE CORONA THEOREM AND BASS STABLE RANK
FOR $M(D(\sum_{i=1}^{k} a_i \delta_{\zeta_i}))$

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Abstract. In this paper, we prove the corona theorem for $M(D(\mu_k))$ in two different ways, where $\mu_k = \sum_{i=1}^{k} a_i \delta_{\zeta_i}$. Then we prove that the Bass stable rank of $M(D(\mu_k))$ is one.

1. Introduction

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc. Let $\mu$ be a nonnegative Borel measure on the boundary $\mathbb{T}$ of the unit disc. Let $\varphi_\mu$ be the harmonic function

$$\varphi_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The Dirichlet type space $D(\mu)$ is defined as the space of all analytic functions on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$$

is finite. For any $f \in D(\mu)$, $\|f\|_{D(\mu)}^2 := \|f\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$.

When $\mu = \frac{dt}{2\pi}$, $D(\frac{dt}{2\pi})$ is the Dirichlet space $D$.

Dirichlet type spaces were introduced by Richter in [4], when studying analytic two-isometries. In [6], Richter and Sundberg showed that if $f \in D(\delta_{\zeta})$, then

$$D_{\zeta}(f) = \int_{\mathbb{D}} |f'(z)|^2 \frac{1 - |z|^2}{|\zeta - z|^2} dA(z), \quad \zeta \in \mathbb{T}$$

which is a convenient tool in studying these spaces, where $D_{\zeta}(f) := \|f - \frac{f(\zeta)}{\zeta - \zeta}\|_{H^2(\mathbb{D})}^2$ is called the local Dirichlet integral of $f$ at $\zeta$. Thus,

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for any $f \in D(\mu)$, $\|f\|_{D(\mu)}^2 = \|f\|_{H^2(D)}^2 + \int_T D_\zeta(f) d\mu(\zeta) = \|f\|_{H^2(D)}^2 + \int_T \|\frac{f(z)}{z - \zeta}\|_{H^2(D)}^2 d\mu(\zeta)$.

In this paper, we will consider $\mu = \sum_{i=1}^k a_i \delta_{\zeta_i} := \mu_k$, where $a_i$’s are positive numbers, $\zeta_i$’s are in $T$. Let $M(D(\mu_k))$ be the space of multipliers of $D(\mu_k)$, that is

$$M(D(\mu_k)) = \{\phi \in D(\mu_k) : \phi f \in D(\mu_k), \forall f \in D(\mu_k)\}.$$ 

Also we will consider $D_{l^2}(\mu_k)$, or $\oplus_1^\infty D(\mu_k)$, which can be considered as $l^2$-valued $D(\mu_k)$ space.

Given $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$, we let $\Phi(z) = (\varphi_1(z), \varphi_2(z), \ldots)$. We use $M_\Phi$ to denote the (column) operator from $D(\mu_k)$ to $\oplus_1^\infty D(\mu_k)$ defined by

$$M_\Phi(f) = \{\varphi_j f\}_{j=1}^\infty \quad \text{for } f \in D(\mu_k).$$

The famous corona theorem goes back to Lennart Carleson. In 1962 Carleson [2] proved the absence of a corona in the maximal ideal space of $H^\infty(D)$ by showing that if $\{\varphi_1, \ldots, \varphi_n\}$ is a finite set of functions in $H^\infty(D)$ satisfying

$$\sum_{j=1}^n |\varphi_j(z)|^2 \geq \eta > 0, \quad z \in D, \quad \text{(Corona condition)}.$$ 

then there are functions $\{f_1, \ldots, f_n\} \subseteq H^\infty(D)$ with

$$\sum_{j=1}^n f_j(z) \varphi_j(z) = 1, \quad z \in D, \quad \text{(Bezout equation)}.$$ 

This is also equivalent to say that the unit disc is dense in the maximal ideal space of $H^\infty(D)$ in the weak* topology. Then it was shown that the corona theorem is also true in $M(D)$, the multiplier of the Dirichlet space $D$ (see Tolokonnikov [10], Xiao [15]). In this paper, we wish to prove the corona theorem for $M(D(\mu_k))$ in two ways. The first version is as follows:

**Theorem 1.1.** The set of multiplicative linear functionals consisting of evaluations at points of $D$ is dense in the maximal ideal space of $M(D(\mu_k))$.

By the standard Gelfand theory of Banach algebras Theorem 1.1 implies:

**Corollary 1.2.** The following are equivalent:
(i) $\varphi_1, \ldots, \varphi_n \in M(D(\mu_k))$ and there exists a $\eta > 0$ such that
\[
\sum_{j=1}^{n} |\varphi_j(z)|^2 \geq \eta > 0, \quad z \in \mathbb{D}.
\]

(ii) There are functions $b_1, \ldots, b_n \in M(D(\mu_k))$ such that
\[
\sum_{j=1}^{n} \varphi_j(z)b_j(z) = 1, \quad z \in \mathbb{D}.
\]

Also the corona theorem has been generalized to infinitely many functions in $H^\infty(D)$ and $M(D)$ (see Rosenblum \cite{7}, Tolokonnikov \cite{10} and Trent \cite{13}). The infinite version, given by Rosenblum \cite{7} and Tolokonnikov \cite{10}, can be formulate as follows (see Trent \cite{14}):

**Theorem 1.3.** Let $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^\infty(D)$. Suppose that
\[
0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1, \quad \text{for all } z \in \mathbb{D}.
\]

Then there exists $\{e_j\}_{j=1}^{\infty} \subseteq H^\infty(D)$ such that $\sum_{j=1}^{\infty} \varphi_j e_j = 1$ and $\sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \leq \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon}$, where $C_0$ is a constant.

Note that the pointwise hypothesis $\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ implies that the operator $T_\Phi$ defined on $H^2(D)$ in analogy to that of $M_\Phi$ is bounded and $\|T_\Phi\| = \sup_{z \in \mathbb{D}} (\sum_{j=1}^{\infty} |\varphi_j(z)|^2)^{1/2}$. Note that since $M(D(\mu_k)) = D(\mu_k) \cap H^\infty(D)$, the pointwise upper bound hypothesis will not be sufficient to conclude that $M_\Phi$ is bounded from $D(\mu_k)$ to $\oplus_1^{\infty} D(\mu_k)$. Thus, we will replace the assumption $\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ for $z \in \mathbb{D}$ by the condition $\|M_\Phi\| \leq 1$. Then we have the following theorem:

**Theorem 1.4.** Let $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$. Suppose that
\[
\|M_\Phi\| \leq 1 \quad \text{and} \quad 0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \quad \text{for all } z \in \mathbb{D}.
\]

Then there exists $\{b_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$ such that
(i) $\Phi(z)B(z)^\top = 1$ for all $z \in \mathbb{D}$, and
(ii) $\|M_B\| \leq \frac{1}{\epsilon^2} \left(2 + 16\|M_{B_{k-1}}\|^2\right)^{1/2}$, where $B_{k-1}$ is the solution for the corona theorem in $M(D(\mu_{k-1}))$.

We will use induction to prove Theorem \cite{11} and Theorem \cite{14}. In section 4, we show that the Bass stable rank of $M(D(\mu_k))$ is one. Throughout this paper, we use $C, C_1, C_2, \ldots$ for absolute constants.
Lemma 2.1. Let \( f \in D(\delta_1) \). Then

(i) \( f = f(1) + (z-1)g \) for some \( g \in H^2(\mathbb{D}) \) and \( D_1(f) = \|g\|_{H^2(\mathbb{D})}^2 \).

(ii) \( \lim_{r \to 1^-} f(r) = f(1) \).

(iii) \( |f(1)| \leq C\|f\|_{D(\delta_1)} \) (see \([11]\)).

Lemma 2.2. Let \( \varphi \in H^\infty(\mathbb{D}) \) and \( f \in D(\delta_\zeta) \). Then \( \varphi f \in D(\delta_\zeta) \) if and only if \( f(\zeta) = 0 \) or \( \varphi \in D(\delta_\zeta) \). Furthermore,

\[
D_\zeta(\varphi f) \leq 2(\|\varphi\|^2_\infty D_\zeta(f) + |f(\zeta)|^2 D_\zeta(\varphi))
\]

and

\[
|f(\zeta)|^2 D_\zeta(\varphi) \leq 2(\|\varphi\|^2_\infty D_\zeta(f) + D_\zeta(\varphi f)).
\]

If \( f(\zeta) = 0 \) then one even has \( D_\zeta(\varphi f) \leq \|\varphi\|^2_\infty D_\zeta(f) \), while the second inequality can be replaced with the trivial observation that the right-hand side is nonnegative.

Thus, by Lemma 2.2, we have \( M(D(\mu_k)) = D(\mu_k) \cap H^\infty(\mathbb{D}) \), where \( \mu_k = \sum_{i=1}^k a_i \delta_{\zeta_i} \). The norm in \( D(\mu_k) \cap H^\infty(\mathbb{D}) \) is defined by

\[
\|f\|_{D(\mu_k) \cap H^\infty(\mathbb{D})} = \|f\|_{D(\mu_k)} + \|f\|_\infty, \quad f \in D(\mu_k) \cap H^\infty(\mathbb{D}).
\]

We will use a similar idea as in Lemma 2.1 of \([4]\) to prove the corona theorem for \( M(D(\delta_1)) \).

For ease of notation, we let \( K := M(D(\delta_1)) = D(\delta_1) \cap H^\infty(\mathbb{D}) \), and \( K_0 := \{f \in K, f(1) = 0\} \). Note that \( K_0 \subset K \), and \( K_0 \) is a Banach algebra without identity.

Note that evaluation at \( z \in \mathbb{D} \cup \{1\} \) is a multiplicative linear functional on \( K_0 \) (if \( z = 1 \), then it is a trivial one). We have the following lemma.

Lemma 2.3. The set of multiplicative linear functionals consisting of evaluations at points of \( \mathbb{D} \) is dense in the set of all multiplicative linear functionals on \( K_0 \).

Proof. Let \( m \) be a non-zero multiplicative linear functional on \( K_0 \), then there exists a function \( g_0 \in K_0 \), such that \( m(g_0) \neq 0 \).

If \( f \in H^\infty(\mathbb{D}) \), define \( M(f) := \frac{m(f g_0)}{m(g_0)} \).

Claim: \( M \) is well-defined, and \( M \) is a non-zero multiplicative linear functional on \( H^\infty(\mathbb{D}) \).

If we assume that the claim holds, then by Carleson’s corona Theorem, there exists a net \( (\beta_i)_{i \in I} \) of point evaluations in \( \mathbb{D} \) that converges
to $M$ in the weak* topology of the maximal ideal space of $H^\infty(D)$. Note that $m$ is the restriction of $M$ to $K_0$:

$$M(f) = \frac{m(fg_0)}{m(g_0)} = \frac{m(f)m(g_0)}{m(g_0)} = m(f), f \in K_0.$$ 

Also the restriction of $(\beta_i)_{i \in I}$ gives a net of point evaluations in $D$ that converges to $m$ in the weak* topology on the dual space of $K_0$.

We are left to prove the claim: $f \in H^\infty(D), g_0 \in K_0$, so $fg_0 \in K$ by Lemma 2.2. Also $(fg_0)(1) = 0$, so $fg_0 \in K_0$, which implies $M$ is well-defined.

Clearly $M$ is linear, when $f \in H^\infty(D),

$$|M(f)| = |\frac{m(fg_0)}{m(g_0)}| \leq \frac{\|fg_0\|_K}{|m(g_0)|} \leq \frac{\|f\|_\infty \|g_0\|_\infty + \|f\|_\infty \|g_0\|_{D(\delta_1)}}{|m(g_0)|} = \frac{\|g_0\|_K}{|m(g_0)|} \|f\|_\infty,$$

so $M$ is a bounded functional on $H^\infty(D)$.

When $f, h \in H^\infty(D), m(fhg_0)m(g_0) = m(fhg_0g_0) = m(fg_0)m(hg_0)$, thus we get

$$M(fh) = \frac{m(fhg_0)}{m(g_0)} = \frac{[m(fg_0)m(hg_0)]/m(g_0)}{m(g_0)} = M(f)M(h).$$

Therefore the claim is proved. ■

Now, we can prove the following Theorem.

**Theorem 2.4.** The set of multiplicative linear functionals consisting of evaluations at points of $D \cup \{1\}$ is dense in the maximal ideal space of $K$.

**Proof.** Suppose $M$ is a non-zero multiplicative linear functional on $K$.

Let $m = M|_{K_0}$, then $m$ is a multiplicative linear functional on $K_0$. If $f \in K$, then $f - f(1) \in K_0$, so $M(f) = f(1) + m(f - f(1))$.

**Case 1.** If $m = 0$, then $M(f) = f(1)$, so $M$ is the point evaluation at 1.

**Case 2.** If $m \neq 0$, the by Lemma 2.3 there exists a net $(\beta_i)_{i \in I}$ of point evaluations in $D$ that converges to $m$ in the weak* topology on
the dual space of $K_0$. Therefore, for all $f \in K$,
\[
M(f) = f(1) + m(f - f(1)) = f(1) + \lim_{i \in I} \beta_i(f - f(1))
\]
\[
= f(1) + \lim_{i \in I} (f(\beta_i) - f(1))
\]
\[
= \lim_{i \in I} f(\beta_i) = (\lim_{i \in I} \beta_i)(f).
\]
Thus $M = \lim_{i \in I} \beta_i$, and this completes the proof. ■

Remark 2.5. For any $f \in K$, $0 < r < 1$, let $E_r(f) = f(r)$, then from Lemma 2.7 we have $f(r) \to f(1)$ as $r \to 1$. Thus $E_r \to E_1$ in the weak star topology of $K$ as $r \to 1$, which implies the set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D}$ is dense in the maximal ideal space of $K$.

2.2. In this subsection, we consider general $k \geq 1$. Let $\mu$ be a Borel measure in $\mathbb{T}$ with $\mu(\zeta) = 0$, where $\zeta \in \mathbb{T}$, and suppose that $\mathbb{D}$ is dense in the maximal ideal space of $M(D(\mu))$. Let $H := M(D(\mu)) \cap D(\delta_\zeta)$ and $H_0 := \{ f \in H, f(\zeta) = 0 \}$. Then we have:

Lemma 2.6. $H$ is a Banach algebra, $H_0 \subset H$ and $H_0$ is a Banach algebra without identity.

Proof. We only need to verify that $H$ is an algebra. Suppose $f, g \in H = M(D(\mu)) \cap D(\delta_\zeta)$, then $fg \in M(D(\mu))$. Also $f - f(\zeta) \in H$ implies $\frac{f - f(\zeta)}{z - \zeta} g \in H^2(\mathbb{D})$, thus
\[
f g = (z - \zeta) \left( \frac{f - f(\zeta)}{z - \zeta} g \right) + f(\zeta) g \in D(\delta_\zeta),
\]
and so $fg \in H$. ■

Lemma 2.7. The set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D}$ is dense in the maximal ideal space of $H_0$.

Proof. Let $m$ be a non-zero multiplicative linear functional on $H_0$, then there exists a function $g_0 \in H_0$, such that $m(g_0) \neq 0$.

If $f \in M(D(\mu))$, define $M(f) := \frac{m(fg_0)}{m(g_0)}$.

Claim: $M$ is well-defined, and $M$ is a non-zero multiplicative linear functional on $M(D(\mu))$.

The proof of the claim is similar to the argument in Lemma 2.3. Then there exists a net $(\beta_i)_{i \in I}$ of point evaluations in $\mathbb{D}$ that converges to $M$ in the Gelfand topology of the maximal ideal space of $M(D(\mu))$. Note that $m$ is the restriction of $M$ to $H_0$. Also the restriction of $(\beta_i)_{i \in I}$ gives a net of point evaluations in $\mathbb{D}$ that converges to $m$ in the weak* topology on the dual of $H_0$. ■
By the same argument as in Theorem 2.4 we have the following Proposition:

**Proposition 2.8.** The set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D}$ is dense in the maximal ideal space of $H$.

Now we can prove Theorem 1.1.

**Proof.** This clearly follows from Proposition 2.8 and induction. □

**Remark 2.9.** If we let $d\mu = \frac{dt}{2\pi}$, then $D(\frac{dt}{2\pi})$ is the Dirichlet space $D$. By Tolokonnikov [10], Xiao [15] we have the corona theorem in $M(D)$, then by Proposition 2.8 we also have the corona theorem in $M(D) \cap D(\delta_\zeta)$ for any $\zeta \in \mathbb{T}$.

3. INFINITE VERSION FOR $M(D(\mu_k))$

3.1. First, we consider $M(D(\delta_1))$.

The following Lemma can be derived from [13, Lemma 6] (see also [8]).

**Lemma 3.1.** Let $\{a_j\}_{j=1}^\infty \in l^2$ and $A = (a_1, a_2, \ldots) \in B(l^2, \mathbb{C})$. Then there exists an $\infty \times \infty$ matrix $Q_A$, such that the entries of $Q_A$ belong to the set $\{0, \pm a_j : j = 1, 2, \ldots\}$ and $Q_A$ satisfies

(a) range of $Q_A \subseteq$ kernel of $A$.
(b) $(AA^*)I - A^*A = Q_AQ_A^*$.
(c) If $\{d_j\}_{j=1}^\infty \in l^2$ and $D = (d_1, d_2, \ldots)$, then

$$(AD^\top)I - D^\top A = Q_AQ_D^\top.$$ 

We need one lemma before we prove the corona theorem for infinitely many functions in $M(D(\delta_1))$.

**Lemma 3.2.** Let $\{\phi_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$. Then

(i) $M_\Phi$ is a bounded operator if and only if $\sum_{j=1}^\infty \|\phi_j\|_{D(\delta_1)}^2$ and $\sup_{z \in \mathbb{D}} \sum_{j=1}^\infty |\phi_j(z)|^2$ are finite.
(ii) If $\|M_\Phi\| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^\infty |\phi_j(z)|^2$ for all $z \in \mathbb{D}$, then

$\Phi(1) = (\phi_1(1), \phi_2(1), \ldots) \neq 0$.

(iii) If $\|M_\Phi\| \leq 1$ and $f = \sum_{i=1}^\infty [\phi_i - \varphi_i(1)]\overline{\varphi_i(1)}$, then $f \in M(D(\delta_1))$ and $f(1) = 0$. 
Proof. (i): Suppose that $M_{\Phi}$ is bounded from $D(\delta_1)$ to $\oplus_{n=1}^{\infty} D(\delta_1)$ with $\|M_{\Phi}\| \leq 1$, then $\sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ (see [13]). Let $f = 1 \in D(\delta_1)$, then

$$
\sum_{j=1}^{\infty} \|\varphi_j\|^2_{D(\delta_1)} = \|M_{\Phi}f\|^2_{\oplus_{n=1}^{\infty} D(\delta_1)}
\leq \|M_{\Phi}\|^2 \|f\|_{D(\delta_1)} \leq 1.
$$

Conversely suppose $\sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ and $\sum_{j=1}^{\infty} \|\varphi_j\|^2_{D(\delta_1)} \leq 1$. Let $f \in D(\delta_1)$, suppose $f = f(1) + (z - 1)g$ for some $g \in H^2(\mathbb{D})$, then $D_1(f) = \|g\|^2_{H^2(\mathbb{D})}$ and

$$
\|M_{\Phi}f\|^2_{\oplus_{n=1}^{\infty} D(\delta_1)} = \sum_{j=1}^{\infty} \|\varphi_jf\|^2_{D(\delta_1)}
= \sum_{j=1}^{\infty} \|\varphi_jf\|^2_{H^2(\mathbb{D})} + \sum_{j=1}^{\infty} \|\varphi_jf - (\varphi_jf)(1)\|^2_{H^2(\mathbb{D})}
\leq \|f\|^2_{H^2(\mathbb{D})} + \sum_{j=1}^{\infty} \left[2\left\|\frac{\varphi_jf(1) - (\varphi_jf)(1)}{z - 1}\right\|^2_{H^2(\mathbb{D})} + 2\left\|\frac{\varphi_jg(1)}{z - 1}\right\|^2_{H^2(\mathbb{D})}\right]
\leq \|f\|^2_{H^2(\mathbb{D})} + 2|f(1)|^2 \sum_{j=1}^{\infty} D_1(\varphi_j) + 2\|g\|^2_{H^2(\mathbb{D})}
\leq 2\|f\|_{D(\delta_1)} + 2|f(1)|^2.
$$

Since $|f(1)| \leq C\|f\|_{D(\delta_1)}$ (see [11]), we conclude that $M_{\Phi}$ is bounded from $D(\delta_1)$ to $\oplus_{n=1}^{\infty} D(\delta_1)$.

(ii): Suppose $\{g_j\}_{j=1}^{\infty} \subseteq H^2(\mathbb{D})$ such that

$$
\varphi_j(z) = \varphi_j(1) + (z - 1)g_j(z), \quad \text{and} \quad D_1(\varphi_j) = \|g_j\|^2_{H^2(\mathbb{D})}, \ j = 1, 2, \ldots.
$$

Note that

$$
|\varphi_j(z)|^2 \leq |\varphi_j(1)|^2 + |z - 1|^2|g_j(z)|^2 + 2|\varphi_j(1)||z - 1||g_j(z)|^2
\leq (1 + \eta)|\varphi_j(1)|^2 + (1 + \frac{1}{\eta})|z - 1|^2|g_j(z)|^2,
$$
where \( \eta \) is any positive number. Then we have
\[
e^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq \sum_{j=1}^{\infty} (1 + \eta)|\varphi_j(1)|^2 + (1 + \frac{1}{\eta})|z - 1|^2|g_j(z)|^2
\]
\[
\leq \sum_{j=1}^{\infty} (1 + \eta)|\varphi_j(1)|^2 + (1 + \frac{1}{\eta})|z - 1|^2 \sum_{j=1}^{\infty} \|\varphi_j\|_{D(\delta_1)}^2
\]
\[
\leq \sum_{j=1}^{\infty} (1 + \eta)|\varphi_j(1)|^2 + (1 + \frac{1}{\eta})|z - 1|^2 \|g_j\|_{D(\delta_1)}^2 \text{ for all } z \in \mathbb{D},
\]
where in the last inequality we used part (i). Let \( z = r \to 1^- \) we get
\[
e^2 \leq \sum_{j=1}^{\infty} (1 + \eta)|\varphi_j(1)|^2 := (1 + \eta)|\Phi(1)|^2.
\]
Let \( \eta \to 0, \) we have \( |\Phi(1)|^2 = \sum_{j=1}^{\infty} |\varphi_j(1)|^2 \geq e^2, \) thus \( \Phi(1) = (\varphi_1(1), \varphi_2(1), \ldots) \neq 0.\)

(iii) Suppose \( \|M_\Phi\| \leq 1 \) and \( f = \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}, \) then \( f \in H^\infty(\mathbb{D}) \) and
\[
\|f\|_{D(\delta_1)}^2 = \|\sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}\|_{D(\delta_1)}^2
\]
\[
\leq \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 \|\varphi_i(1)\|^2
\]
\[
\leq 2 \left[ \sum_{i=1}^{\infty} \|\varphi_i\|^2_{D(\delta_1)} + \sum_{i=1}^{\infty} |\varphi_i(1)|^2 \right] \sum_{i=1}^{\infty} |\varphi_i(1)|^2
\]
\[
\leq 4,
\]
where in the last inequality we used part (i).

For any \( k \in \mathbb{N}, \) let \( f_k = \sum_{i=1}^{k} [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)} \). Then \( f_k \to f \in D(\delta_1), \) note that \( f_k(1) = 0 \) and point evaluation at 1 is continuous, we conclude that \( f(1) = 0. \)

Now we can prove the corona theorem for \( M(D(\delta_1)). \)

**Theorem 3.3.** Let \( \{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\delta_1)). \) Suppose that \( \|M_\Phi\| \leq 1 \) and
\[
0 < e^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \text{ for all } z \in \mathbb{D}. \]
Then there exists \( \{b_j\}_{j=1}^{\infty} \subseteq M(D(\delta_1)) \) such that
\[
(i) \Phi(z)B(z)^\top = 1 \text{ for all } z \in \mathbb{D}, \text{ and}
(ii) \|M_B\| \leq \frac{1}{e^2}(2 + 8C_0 \ln \frac{1}{e^2})^{1/2}.
\]
Proof. (i): By Theorem 1.3 there exists an $E \in H_\infty(D)$ such that
$$\Phi(z)E(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$
and
$$\|E\|^2_{H_\infty(D)} := \sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \leq \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2}.$$

Let $A = \Phi(z)$, $D = E(z)$ in Lemma 3.1, then
$$I - E(z)^\top \Phi(z) = Q\Phi(z)Q_{E(z)}^\top,$$
thus
$$I = E(z)^\top \Phi(1) + E(z)^\top (\Phi(z) - \Phi(1)) + Q\Phi(z)Q_{E(z)}^\top \Phi(1)^*.$$

Let $\Phi(1)^* = (\varphi_1(1), \varphi_2(1), \ldots)^\top$, then $|\Phi(1)|^2 = \Phi(1)\Phi(1)^*$ and
$$\Phi(1)^* = E(z)^\top |\Phi(1)|^2 + E(z)^\top [\Phi(z) - \Phi(1)]\Phi(1)^* + Q\Phi(z)Q_{E(z)}^\top \Phi(1)^*.$$

By Lemma 3.2 we have $\Phi(1) = (\varphi_1(1), \varphi_2(1), \ldots) \neq 0$, then from (3.2) we have
$$\Phi(1)^* = \frac{E(z)^\top |\Phi(1)|^2}{|\Phi(1)|^2} + \frac{E(z)^\top [\Phi(z) - \Phi(1)]\Phi(1)^*}{|\Phi(1)|^2} + Q\Phi(z)Q_{E(z)}^\top \Phi(1)^*,$$
therefore,
$$E(z)^\top + Q\Phi(z)Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2} = \frac{\Phi(1)^*}{|\Phi(1)|^2} - \frac{[\Phi(z) - \Phi(1)]\Phi(1)^*}{|\Phi(1)|^2}E(z)^\top.$$

Let $B(z)^\top = E(z)^\top + Q\Phi(z)Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2}$. From Lemma 3.1 we have
$$\Phi(z)B(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$
and
$$b_j(z) = \frac{\varphi_j(1)}{|\Phi(1)|^2} - \frac{\sum_{i=1}^{\infty} [\varphi_i(z) - \varphi_i(1)]\varphi_i(1)}{|\Phi(1)|^2} e_j(z), \quad j = 1, 2, 3, \ldots.$$

By Lemma 3.2 we have $f := \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)]\varphi_i(1) \in M(D(\delta_1))$ and $f(1) = 0$. Thus from Lemma 2.2 we have $b_j \in H_\infty(\mathbb{D}) \cap D(\delta_1) = M(D(\delta_1)), \quad j = 1, 2, \ldots$. 


(ii): Let \( f \in D(\delta_1) \), then

\[
\sum_{j=1}^{\infty} \|b_j f\|_{D(\delta_1)}^2 \\
\leq \frac{2}{|\Phi(1)|^2} \left[ \sum_{j=1}^{\infty} \|\varphi_j(1) f\|_{D(\delta_1)}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 \frac{\varphi_i(1)e_j f\|_{D(\delta_1)}^2}{|\varphi_i(1)|^2} \right] \\
\leq \frac{2}{|\Phi(1)|^2} \left[ |\Phi(1)|^2 \|f\|_{D(\delta_1)}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 |\Phi(1)|^2 \right] \\
= \frac{2}{|\Phi(1)|^2} \left[ \|f\|_{D(\delta_1)}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 \right].
\]

Note that

\[
(3.3) \\
\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 e_j f\|_{D(\delta_1)}^2 \\
= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{H^2(\mathbb{D})}^2 e_j f\|_{H^2(\mathbb{D})}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|\frac{\varphi_i - \varphi_i(1)}{z - 1}\|_{H^2(\mathbb{D})} e_j f\|_{H^2(\mathbb{D})}^2 \\
\leq \|E\|_{H^\infty(\mathbb{D})}^2 \sum_{i=1}^{\infty} \left( \|\varphi_i - \varphi_i(1)\|_{H^2(\mathbb{D})}^2 + \|\frac{\varphi_i - \varphi_i(1)}{z - 1}\|_{H^2(\mathbb{D})}^2 \right) e_j f\|_{D(\delta_1)}^2 \\
= \|E\|_{H^\infty(\mathbb{D})}^2 \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|_{D(\delta_1)}^2 e_j f\|_{D(\delta_1)}^2 \\
\leq 2\|E\|_{H^\infty(\mathbb{D})}^2 \left[ \sum_{i=1}^{\infty} \|\varphi_i f\|_{D(\delta_1)}^2 + \sum_{i=1}^{\infty} \|\varphi_i(1) f\|_{D(\delta_1)}^2 \right] \\
\leq 2\|E\|_{H^\infty(\mathbb{D})}^2 \left[ \|M\|^2 + |\Phi(1)|^2 \right] \|f\|_{D(\delta_1)}^2 \\
\leq 4\|E\|_{H^\infty(\mathbb{D})}^2 \|f\|_{D(\delta_1)}^2.
\]

Thus

\[
\sum_{j=1}^{\infty} \|b_j f\|_{D(\delta_1)}^2 \leq \frac{2}{|\Phi(1)|^2} \left[ \|f\|_{D(\delta_1)}^2 + 4\|E\|_{H^\infty(\mathbb{D})}^2 \|f\|_{D(\delta_1)}^2 \right],
\]

therefore
\[
\|M_B\| \leq \left[ \frac{2}{|\Phi(1)|^2} (1 + 4\|E\|_H^2) \right]^{1/2}
\leq \frac{1}{\varepsilon} (2 + \frac{8C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2})^{1/2},
\]
where in the last inequality we used $|\Phi(1)| \geq \varepsilon$ in the proof of Lemma 3.2.

\[\text{Remark 3.4.} \quad \text{From equation (3.1), we can get another corona solution} \]
\[D(z) = (d_1(z), d_2(z), \ldots), \text{ such that} \]
\[\sum_{j=1}^\infty \varphi_j(z)d_j(z) = 1, \quad z \in \mathbb{D}. \tag{3.4} \]

Suppose $|\varphi_1(1)| = \max\{|\varphi_j(1)|: j = 1, 2, \ldots\}$, let $d_1(z) = \frac{1}{\varphi_1(1)} - \frac{\varphi(z) - \varphi(1)}{\varphi_1(1)} e_1(z)$, $d_j(z) = -\frac{\varphi(z) - \varphi(1)}{\varphi_1(1)} e_j(z), j = 2, 3, \ldots$. Then (3.4) is satisfied and we have
\[\|M_D\| \leq \left[ \frac{2}{|\varphi_1(1)|^2} + 4\left( \frac{\|\varphi_1\|_{M(D(\delta_1))}^2}{|\varphi_1(1)|^2} + 1 \right) \frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2} \right]^{1/2}, \]
but in this case the bound of the corona solution depends on the chosen $\varphi_1$. It would be of interest to determine the best possible bound for the solution $B$ in terms of $\|M_\Phi\|$ and $\varepsilon$.

3.2. For general $k$, we use induction to prove Theorem 1.4.

\[\text{Proof.} \quad \text{The idea is the same as in Theorem 3.3. We sketch a proof here.} \]

If $k = 1$, then by Theorem 3.3, it is true.

Suppose $k = l \geq 1$, it is true.

If $k = l + 1$, note that $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_{l+1})) \subseteq M(D(\mu_l))$, by induction, there exists $\{e_j\}_{j=1}^\infty \subseteq M(D(\mu_l))$ such that
\[\Phi(z)E(z)^\top = 1 \quad \text{for} \quad z \in \mathbb{D}, \tag{3.5} \]
and
\[\|M_E\| \leq \frac{1}{\varepsilon} \left( 2 + 16\|M_{B_{l-1}}\|^2 \right)^{1/2}, \]

Following the same argument as in Lemma 3.2, we have $\Phi(\zeta_{l+1}) = (\varphi_1(\zeta_{l+1}), \varphi_2(\zeta_{l+1}), \ldots) \neq 0$ and
\[I = E(z)^\top \Phi(\zeta_{l+1}) + E(z)^\top (\Phi(z) - \Phi(\zeta_{l+1})) + Q_\Phi(z)Q_E(z). \]

Thus
\[b_j(z) = \frac{\varphi_j(\zeta_{l+1})}{|\Phi(\zeta_{l+1})|^2} - \frac{\sum_{i=1}^\infty [\varphi_i(z) - \varphi_i(\zeta_{l+1})]/\Phi(\zeta_{l+1})^2}{|\Phi(\zeta_{l+1})|^2} e_j(z) \in M(D(\mu_l)), \]

and
\[\Phi(z)E(z)^\top = 1 \quad \text{for} \quad z \in \mathbb{D}, \tag{3.5} \]

Following the same argument as in Lemma 3.2, we have $\Phi(\zeta_{l+1}) = (\varphi_1(\zeta_{l+1}), \varphi_2(\zeta_{l+1}), \ldots) \neq 0$ and
\[I = E(z)^\top \Phi(\zeta_{l+1}) + E(z)^\top (\Phi(z) - \Phi(\zeta_{l+1})) + Q_\Phi(z)Q_E(z). \]

Thus
\[b_j(z) = \frac{\varphi_j(\zeta_{l+1})}{|\Phi(\zeta_{l+1})|^2} - \frac{\sum_{i=1}^\infty [\varphi_i(z) - \varphi_i(\zeta_{l+1})]/\Phi(\zeta_{l+1})^2}{|\Phi(\zeta_{l+1})|^2} e_j(z) \in M(D(\mu_l)), \]
and $\Phi(z)B(z)^\top = 1$ for all $z \in \mathbb{D}$.

Now we estimate $\|M_B\|$. Let $f \in D(\mu_{t+1})$, then

$$\sum_{j=1}^{\infty} \|b_j f\|_{D(\mu_{t+1})}^2 \leq \frac{2}{|\Phi(\zeta_{t+1})|^2} \left[ \|f\|_{D(\mu_{t+1})}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{t+1})]e_j f\|_{D(\mu_{t+1})}^2 \right].$$

Suppose $\mu_{t+1} = \mu_t + \delta_{\zeta_{t+1}}$, note that using inequality (3.3) we have

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{t+1})]e_j f\|_{D(\mu_{t+1})}^2$$

$$\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{t+1})]e_j f\|_{D(\mu_{t+1})}^2 + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{t+1})]e_j f\|_{D(\delta_{\zeta_{t+1}})}^2$$

$$\leq \sum_{i=1}^{\infty} \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_t)}^2 + 4 \|E\|_{H_{\mathbb{D}}^\infty}^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\delta_{\zeta_{t+1}})}^2$$

$$\leq \|M_E\|^2 \left[ \|M\| + [\Phi(\zeta_{t+1})]^2 \right] \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_{t+1})}^2 + 4 \|E\|_{H_{\mathbb{D}}^\infty}^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\delta_{\zeta_{t+1}})}^2$$

$$\leq 4 \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_{t+1})}^2 + 4 \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\delta_{\zeta_{t+1}})}^2$$

$$= 8 \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_{t+1})}^2.$$

Thus

$$\sum_{j=1}^{\infty} \|b_j f\|_{D(\mu_{t+1})}^2 \leq \frac{2}{|\Phi(\zeta_{t+1})|^2} \left[ \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_{t+1})}^2 + 8 \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_{t+1})}^2 \right]$$

$$\leq \frac{1}{\varepsilon^2} \left( 2 + 16 \|M_E\|^2 \right) \|[\varphi_i - \varphi_i(\zeta_{t+1})]f\|_{D(\mu_{t+1})}^2,$$

and so $\|M_B\| \leq \frac{1}{\varepsilon} \left( 2 + 16 \|M_E\|^2 \right)^{1/2}$.

\section{Bass stable rank for $M(D(\sum_{i=1}^{k} a_i \delta_{\zeta_i}))$}

The notion of stable rank of a ring was introduced by Bass [1] to facilitate computations in algebraic K-theory. Let us recall the main definition.

\textbf{Definition 4.1.} Let $\mathcal{A}$ be any ring with identity 1. An $n$-tuple $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$ is called unimodular or invertible, if there exists
an n-tuple \( b = (b_1, \ldots, b_n) \in \mathcal{A}^n \) such that \( \sum_{i=1}^{n} a_i b_i = 1 \). The set of all invertible n-tuples is denoted by \( U_n(\mathcal{A}) \). An \((n+1)\)-tuple \( x = (x_1, \ldots, x_{n+1}) \in \mathcal{A}^{n+1} \) is called reducible, if there exists an n-tuple \( y = (y_1, \ldots, y_n) \) such that \( (x_1 + y_1 x_{n+1}, \ldots, x_n + y_n x_{n+1}) \) is invertible. The Bass stable rank of \( \mathcal{A} \) is the least \( n \) such that every invertible \((n+1)\)-tuple is reducible.

In recent years, the Bass stable rank has been studied by many authors in the setting of Banach algebras. Jones, Marshall and Wolff \[8\] showed that the Bass stable rank of the disc algebra \( A \) is one; Treil \[12\] proved that the Bass stable rank of \( H^\infty(\mathbb{D}) \) is one; and in \[3\], Mortini, Sasane, and Wick showed that the Bass stable rank of \( \mathbb{C} + BH^\infty \) and \( A_B \) are one as well. In this paper, we show that the Bass stable rank of \( M(D(\mu_k)) \) is also one, where \( \mu_k = \sum_{i=1}^{k} a_i \delta_{z_i} \).

First, we prove that the Bass stable rank of \( M(D(\delta_1)) = D(\delta_1) \cap H^\infty(\mathbb{D}) \) is one.

**Lemma 4.2.** The Bass stable rank of \( D(\delta_1) \cap H^\infty(\mathbb{D}) \) is one.

**Proof.** Let \((f, h)\) be a unimodular pair in \((D(\delta_1) \cap H^\infty(\mathbb{D}))^2\), i.e., there exists \((g_1, g_2) \in (D(\delta_1) \cap H^\infty(\mathbb{D}))^2\) such that \( fg_1 + hg_2 = 1 \). Then \( \inf_{z \in \mathbb{D}} |f(z)| + |h(z)| := \eta > 0 \).

**Case 1.** If \( f(1) \neq 0 \), then we claim \((f, (f-f(1))h)\) is unimodular.

In fact, if \( z \in \mathbb{D} \) is such that \(|f(z) - f(1)| \geq |f(1)| \), then \(|f(z)| + |(f(z) - f(1))h(z)| \geq |f(z)| + |f(1)| |h(z)| \geq \min\{1, |f(1)|\} \eta \).

If \( z \in \mathbb{D} \) is such that \(|f(z) - f(1)| \leq |f(1)| / 2 \), then \(|f(z)| = |f(z) - f(1) + f(1)| \geq |f(1)| - |f(z) - f(1)| \geq |f(1)| \), and so \(|f(z)| + |(f(z) - f(1))h(z)| \geq |f(z)| \geq |f(1)| / 2 | \).

Thus, \((f, (f-f(1))h)\) is unimodular. By Theorem 1 in \[12\], there is some element \( g \in H^\infty(\mathbb{D}) \) such that \( f + g((f-f(1))h) \) is invertible in \( H^\infty(\mathbb{D}) \). Note that \( g(f-f(1)) \in D(\delta_1) \cap H^\infty(\mathbb{D}) \), by the corona theorem for \( M(D(\delta_1)) \), we get that \( f + g((f-f(1))h) \) is also invertible in \( D(\delta_1) \cap H^\infty(\mathbb{D}) \).

**Case 2.** If \( f(1) = 0 \), then \( h(1) \neq 0 \), since \( \inf_{z \in \mathbb{D}} |f(z)| + |h(z)| := \eta > 0 \). We claim the pair \((f+h, h)\) is unimodular: By the corona theorem for \( M(D(\delta_1)) \), there exists \((g_1, g_2) \in (D(\delta_1) \cap H^\infty(\mathbb{D}))^2\) such that \( fg_1 + hg_2 = 1 \), so \((f+h)g_1 + h(g_2 - g_1) = 1 \), which implies \((f+h, h)\) is unimodular.

By Case 1, there exists some \( g \in D(\delta_1) \cap H^\infty(\mathbb{D}) \), such that \((f+h) + gh \) is invertible in \( D(\delta_1) \cap H^\infty(\mathbb{D}) \). Note that \((f+h) + gh = f + (1+g)h \), and \( 1 + g \in D(\delta_1) \cap H^\infty(\mathbb{D}) \), we are done. 

\[\blacksquare\]
Now we show the Bass stable rank of $M(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2})) = D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.

**Lemma 4.3.** The Bass stable rank of $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.

**Proof.** Let $(f, h)$ be a unimodular pair in $(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D}))^2$.

**Case 1.** $f(\zeta_2) \neq 0$. As in Lemma 4.2 we conclude that $(f, (f - f(\zeta_2))h)$ is unimodular. Then by Lemma 4.2, there exists some $g \in D(\delta_1) \cap H^\infty(\mathbb{D})$ such that $f + g([f - f(\zeta_2)]h)$ is invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$. Note that $g(f - f(\zeta_2)) \in D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$, by the corona theorem for $M(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}))$, we get $f + g([f - f(1)]h)$ is also invertible in $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$.

**Case 2.** $f(\zeta_2) = 0$. As in Lemma 4.2 we consider the pair $(f + h, h)$ and conclude that the Bass stable rank of $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.

For general $k$, by induction we obtain that the Bass stable rank of $M(D(\mu_k))$ is one.

**Theorem 4.4.** The Bass stable rank of $M(D(\mu_k))$ is one.

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