Irreducible characters with bounded root Artin conductor

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Abstract
In this work, we prove that the best possible lower bound for the Artin conductor is exponential in the degree.

1 Introduction
Let $K$ be an algebraic number field such that $K/Q$ is Galois and let $\chi$ be the character of a linear representation of $\text{Gal}(K/Q)$. We denote by $f_\chi$ the Artin conductor of $\chi$. In [8], Odlyzko found lower bounds for $f_\chi$ by applying analytic methods to the Artin $L$-function. We have improved Odlyzko’s lower bounds in [5] by using explicit formulas for Artin $L$-functions. In particular, if $\chi$ is an irreducible character of $\text{Gal}(K/Q)$ and by assuming that $\chi\overline{\chi}$ satisfies the Artin conjecture, we obtained

$$f_\chi^{1/\chi(1)} \geq 4.73(1.648)^{\frac{(a_\chi-b_\chi)^2}{\chi(1)^2}} e^{-(13.34/\chi(1))^2},$$

where $a_\chi$ and $b_\chi$ are nonnegative integers giving the $\Gamma$-factors of the completed Artin $L$-function. Namely, $a_\chi+b_\chi = \chi(1)$ and $a_\chi-b_\chi = \chi(\sigma)$, with $\sigma \in \text{Gal}(K/Q)$ the complex conjugation. This bound is even better when we assume that $L(s,\chi\overline{\chi})$ satisfies the Generalized Riemann Hypothesis. We have to point out that, throughout this article, no additional hypothesis are needed.

A natural question now is how far from being optimal these bounds are. This problem has been studied for the discriminant of a number field. If $n_0 = r_1 + 2r_2$, let $d_n$ be the minimal discriminant of the field $K$ with degree $n$ such that $n$ is a multiple of $n_0$ and $r_1(K)$ and $r_2(K)$ are in the same ratio as $r_1, r_2$. Let $\alpha(r_1, r_2) = \lim \inf_{n \to \infty} d_n^{1/n}$.

Martinet considered number fields with infinite 2-class field towers and proved that [6]

$\alpha(0,1)<93$ and $\alpha(1,0)<1059$.

In this work, we follow this idea and consider a number field $K$ with infinite $p$-class field tower for some prime $p$. Under some technical conditions on $K$, we find an upper bound (depending only on $K$) for the root Artin conductor of the irreducible characters of $\text{Gal}(K_n/Q)$ (given by $f_\chi^{1/\chi(1)}$), where $K_n$ is the Hilbert $p$-class field of $K_{n-1}$ with $K_0 = K$. 

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This work is organized as follow. In §2, we propose a technique obtained from the Clifford’s theory which is useful to classify the irreducible characters of \( \text{Gal}(K_n/Q) \) in terms of a certain normal subgroup. This characterization is convenient in order to obtain upper bounds for root Artin conductors. In §3, we conclude that there exists an infinite sequence \( \{ \chi_n \} \) of irreducible Artin characters with \( \chi_n(1) \to \infty \) and such that \( f_{\chi_n}^{1/\chi_n(1)} \leq C \), where \( C > 0 \) is an effective computable constant. In §4, we apply the results obtained in §2 and §3 to the number field \( K = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1}, \sqrt{2}, \sqrt{-23}) \). This field was found by Martinet in [6] and has infinite 2-class field tower and lowest known discriminant. In particular, we prove that for each \( n \geq 1 \) it is possible to find an irreducible character of \( \text{Gal}(K_n/Q) \) with large degree and

\[
f_{\chi}^{1/\chi(1)} \leq C, \text{ where } C \leq 11^4 \cdot 2^{15} \cdot 23.
\]

2 Irreducible characters of large degree

In this section, we develop a technique to classify the irreducible characters of groups with a normal subgroup of prime index. Also, by using a result from [4], we obtain conditions that ensure the existence of irreducible characters of large degree. We believe these results are of independent interest.

Let us consider a finite group \( G \) and a normal subgroup \( H \) of \( G \). We denote the set of irreducible characters of \( G \) by \( \text{Irr}(G) \). If \( \chi \) and \( \theta \) are characters of \( G \) and \( H \) respectively, we denote the restriction of \( \chi \) to \( H \) by \( \text{Res}_H^G \chi \) and the induced character of \( \theta \) to \( G \) by \( \text{Ind}_H^G \theta \). If \( \theta \in \text{Irr}(H) \), we define the conjugate character to \( \theta \) in \( G \) by \( \theta^g : H \to \mathbb{C}, \text{ where } \theta^g(h) = \theta(ghg^{-1}). \) The inertia group of \( \theta \) in \( G \) is given by

\[
I_G(\theta) = \{ g \in G : \theta^g = \theta \}.
\]

\( G \) acts on \( \text{Irr}(H) \) by conjugation and \( I_G(\theta) \) is the stabilizer of \( \theta \) under this action. The next result of Clifford will be the main argument allowing us to give a classification of the irreducible characters of \( G \).

**Theorem 1.** (Clifford, [3, p. 253]) Let \( H \) be a normal subgroup of \( G \) and \( \theta \in \text{Irr}(H), \chi \in \text{Irr}(G) \) such that \( \theta \) is an irreducible constituent of \( \text{Res}_H^G \chi \), with \( \langle \text{Res}_H^G \chi, \theta \rangle = e > 0 \). Suppose that \( \theta = \theta^{g_1}, \theta^{g_2}, \ldots, \theta^{g_k} \) are the distinct conjugates of \( \theta \) in \( G \). Assume also that

\[
G = \bigcup_{j=1}^{t} I_G(\theta)g_j, \text{ with } t = [G : I_G(\theta)].
\]

Then,

(a) \( \text{Res}_H^G \text{Ind}_H^G \theta = |I_G(\theta)/H| \sum_{j=1}^{t} \theta^{g_j} \).

(b) \( \langle \text{Ind}_H^G \theta, \text{Ind}_H^G \theta \rangle = |I_G(\theta)/H|. \text{ In particular, } \text{Ind}_H^G \theta \in \text{Irr}(G) \text{ if and only if } I_G(\theta) = H. \)
(c) $\text{Res}_{H}^{G} \chi = e \sum_{j=1}^{t} \theta^{g_{j}}$. In particular,

$$\chi(1) = et\theta(1) \quad \text{and} \quad \langle \text{Res}_{H}^{G} \chi, \text{Res}_{H}^{G} \chi \rangle = e^{2}t.$$  

Also, $e^{2} \leq |I_{G}(\theta)/H|$ and $e^{2}t \leq |G/H|$.

In order to ensure the existence of a sequence of irreducible characters of growing degrees, we prove the following corollary which is a consequence of Clifford’s Theorem which is given as an exercise in [3] p. 98.

**Corollary 2.** Let $G$ be a group with a chain of normal subgroups

$$1 = H_{0} \trianglelefteq H_{1} \trianglelefteq H_{2} \ldots \trianglelefteq H_{n} = G$$

such that $H_{i}/H_{i-1}$ is non abelian for $i = 1, \ldots, n$. Then, there exists an irreducible character $\phi$ of $G$, such that $\phi(1) \geq 2^{n}$.

**Proof.** Let us prove it by induction on $n$. For $n = 1$, note that since $H_{1} \cong H_{1}/H_{0}$ is non abelian there exists a character $\psi_{1} \in \text{Irr}(H_{1})$ with $\psi_{1}(1) \geq 2$.

Let us see the case $n = 2$, which illustrates how to proceed in the general case. Let $\psi_{2} \in \text{Irr}(H_{2})$ such that $\psi_{1}$ is an irreducible constituent of $\text{Res}_{H_{1}}^{H_{2}} \psi_{2}$ (take for example $\psi_{2} \in \text{Ind}_{H_{1}}^{H_{2}} \psi_{1}$). From Theorem 1, it follows that

$$\text{Res}_{H_{1}}^{H_{2}} \psi_{2} = e \sum_{i=1}^{t} \psi_{1,i},$$

where $\psi_{1,1} = \psi_{1}, \ldots, \psi_{1,t}$ are the distinct conjugates of $\psi_{1}$ in $H_{2}$, $e = \langle \text{Res}_{H_{1}}^{H_{2}} \psi_{2}, \psi_{1} \rangle$ and $t = [H_{2} : I_{H_{2}}(\psi_{1})]$.

If $t = 1$, then $H_{2} = I_{H_{2}}(\psi_{1})$ and $\text{Res}_{H_{1}}^{H_{2}} \psi_{2} = e\psi_{1}$. Hence,

(a) $\psi_{2}(1) = e\psi_{1}(1) \geq 2$.

(b) For all $\beta \in \text{Irr}(H_{2}/H_{1})$, the character $\psi_{2}\beta$ belongs to $\text{Irr}(H_{2})$ (see [3] Theorem 19.5]). As $H_{2}/H_{1}$ is non abelian, we choose $\beta_{2} \in \text{Irr}(H_{2}/H_{1})$ such that $\beta_{2}(1) \geq 2$. Then, $\phi = \psi_{2}\beta_{2} \in \text{Irr}(H_{2})$ and $\phi(1) = \psi_{2}(1)\beta_{2}(1) \geq 2^{2}$.

If $t \geq 2$, we just take $\phi = \psi_{2}$ so, $\phi(1) = et\psi_{1}(1) \geq 2\psi_{1}(1) \geq 2^{2}$.

Now, suppose that it is true for every $m < n$. Then there is $\psi_{m} \in \text{Irr}(H_{m})$ such that $\psi_{m}(1) \geq 2^{m}$. Let us choose $\psi_{m+1} \in \text{Irr}(H_{m+1})$ such that $\psi_{m}$ is an irreducible constituent of $\text{Res}_{H_{m}}^{H_{m+1}} \psi_{m+1}$. As in [1],

$$\text{Res}_{H_{m}}^{H_{m+1}} \psi_{m+1} = e \sum_{i=1}^{t} \psi_{m,i},$$

where $\psi_{m,1} = \psi_{m}, \ldots, \psi_{m,t}$ are the conjugates of $\psi_{m}$ in $H_{m+1}$, $e = \langle \text{Res}_{H_{m+1}}^{H_{m+1}} \psi_{m+1}, \psi_{m} \rangle$ and $t = [H_{m+1} : I_{H_{m+1}}(\psi_{m})]$.

If $t = 1$, then
(a) $\psi_{m+1}(1) \geq 2^m$;

(b) $\psi_{m+1}\beta \in \text{Irr}(H_{m+1})$, with $\beta \in \text{Irr}(H_{m+1}/H_m)$. Let us choose $\beta$ such that $\beta(1) \geq 2$, then $\phi = \beta \psi_{m+1} \in \text{Irr}(H_{m+1})$ and $\phi(1) = \beta(1)\psi_{m+1}(1) \geq 2^{m+1}$.

If $t \geq 2$, we take $\phi = \psi_{m+1}$ and $\phi(1) \geq et\psi_m(1) \geq 2^{m+1}$.

Now we state the following result which is crucial for the proof of Theorem 14.

**Proposition 3.** Let $H$ be a normal subgroup of a finite group $G$. Let $\theta \in \text{Irr}(H)$. Then there exists $\rho \in \text{Irr}(G)$ such that:

(i) $\rho(1) \geq [G : H]^{-1/2}\theta(1)$,

(ii) $\langle \text{Ind}^G_H \theta, \rho \rangle = a \geq 1$.

**Proof.** By the Frobenius reciprocity formula, we get

$$\langle \text{Ind}^G_H \theta, \text{Ind}^G_H \theta \rangle = \langle \theta, \text{Res}^G_H (\text{Ind}^G_H \theta) \rangle.$$

Let us recall that

$$\text{Res}^G_H (\text{Ind}^G_H \theta)(h) = \sum_{s \in S} \theta_s(h),$$

where $S$ is a system of representative classes of $G/H$ and the character $\theta_s(h) = shs^{-1}$, with $h \in H$. Because $\theta \in \text{Irr}(H)$, the $\theta_s$ are also irreducible. In particular, it is possible to prove that

$$\langle \theta, \text{Res}^G_H (\text{Ind}^G_H \theta) \rangle_H \leq \#S = [G : H].$$

If we write

$$\text{Ind}^G_H \theta = \sum_{\rho \in T} a_{\rho} \rho,$$

where $T = \{ \rho \in \text{Irr}(G) : \langle \text{Ind}^G_H \theta, \rho \rangle = a_{\rho} \geq 1 \}$ it verify that

$$\langle \text{Ind}^G_H \theta, \text{Ind}^G_H \theta \rangle = \sum_{\rho \in T} a_{\rho}^2.$$

Thus, we get

$$\sum_{\rho \in T} a_{\rho}^2 \leq [G : H],$$

which implies

(a) $a_{\rho} \leq \sqrt{[G : H]}$ for each $\rho \in T$,

(b) $\#T \leq [G : H]$. 

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Observe that
\[ [G : H] \theta(1) = \text{Ind}_{G}^{H} \theta(1) = \sum_{\rho \in T} a_{\rho} \rho(1). \]

If we take \( \rho_0 \in T \) of maximal degree, we get
\[ [G : H] \theta(1) \leq \left[ \frac{G}{H} \right]^{3/2} \rho_0(1), \]
so
\[ \rho_0(1) \geq \left[ \frac{G}{H} \right]^{-1/2} \theta(1). \]

We say that an irreducible character \( \theta \) of \( H \) is extendible to \( G \) if there is an irreducible character \( \chi \) of \( G \) such that \( \text{Res}_{G}^{H} \chi = \theta \). The following result gives us a criterion to decide when a character is extendible.

**Theorem 4.** (Gallagher, [1, p. 225]) Let \( G \) be a finite group with a normal subgroup \( H \) of prime index \( q \) in \( G \). If \( \theta \in \text{Irr}(H) \) is invariant in \( G \) (i.e. \( I_{G}(\theta) = G \)), then \( \theta \) is extendible to \( G \).

**Lemma 5.** Suppose that \( G \) is a finite group with a normal subgroup \( H \) such that \( [G : H] = q \), where \( q \) is a prime number. If \( \theta \in \text{Irr}(H) \), then the inertia group of \( \theta \) is either,

(i) \( I_{G}(\theta) = G \), or

(ii) \( I_{G}(\theta) = H \).

**Proof.** Since \( H \) is a normal subgroup of \( G \), \( H \subset I_{G}(\theta) \). So if we put \( t = [G : I_{G}(\theta)] \), then \( t \mid [G : H] \). Therefore \( t = 1 \) or \( t = q \). \( \square \)

**Theorem 6.** Under the conditions of Lemma 5, let \( \chi \) be an irreducible character of \( G \). Then, either

(i) \( \text{Res}_{H}^{G} \chi = \theta \), for some \( \theta \in \text{Irr}(H) \) or

(ii) \( \chi = \text{Ind}_{H}^{G} \theta \), for some \( \theta \in \text{Irr}(H) \).

**Proof.** Let \( \chi \in \text{Irr}(G) \) and take \( \theta \in \text{Irr}(H) \) an irreducible constituent of \( \text{Res}_{H}^{G} \chi \). Then \( \text{Res}_{H}^{G} \chi = e \sum_{i=1}^{t} \theta_{i} \), where \( \theta_{i} \) are the conjugates of \( \theta \) and \( \langle \text{Res}_{H}^{G} \chi, \theta \rangle = e > 0 \). Let us fix \( \theta = \theta_1 \). If \( t = 1 \), then \( I_{G}(\theta) = G \) and by the Theorem 4, \( \theta \) is extendible to \( G \). On the other hand, we know that \( [G : H] = q \) so \( e^2 \mid [G : H] = q \), so \( e^2 = 1 \). Therefore, \( e = 1 \) and \( \text{Res}_{H}^{G} \chi = \theta \), so we have (i).

If \( t = q \) then \( I_{G}(\theta) = H \) and, from Theorem 1, \( \text{Ind}_{H}^{G} \theta \in \text{Irr}(G) \). By Frobenius reciprocity,
\documentclass{article}
\usepackage{amsmath,amssymb}
\begin{document}
\begin{align*}
\langle \text{Ind}_{H}^{G} \theta, \chi \rangle_G & = \langle \theta, \text{Res}_{H}^{G} \chi \rangle_H = \langle \text{Res}_{H}^{G} \chi, \theta \rangle = \bar{e} = e.
\end{align*}
Since \( \text{Ind}_{H}^{G} \theta \) and \( \chi \) are irreducible and \( \theta \) is a constituent of \( \text{Res}_{H}^{G} \chi \), it follows that \( e = 1 \) and so \( \text{Ind}_{H}^{G} \theta = \chi \).

\section{Estimation for the root Artin conductor of irreducible characters of \( G_n \)}

Let \( L/M \) be a Galois extension and \( \chi \) be the character of a linear representation of \( \text{Gal}(L/M) \). The Artin conductor attached to \( \chi \) is given by the ideal
\[
f_{\chi} = \prod_{p|\infty} p^{f_{\chi}(p)},
\]
where
\[
f_{\chi}(p) = \frac{1}{|G_0|} \sum_{j \geq 0} (|G_j|\chi(1) - \chi(G_j))
\]
and \( G_i \) is the \( i \)-th ramification group of the local extension \( L_b/M_p \) with \( b \) a prime over \( p \) and \( \chi(G_j) = \sum_{g \in G_j} \chi(g) \).

It is well known that if \( L \) is an unramified extension of \( M \), then \( f_{\chi} \) is the trivial ideal.

Then, in order to find a family of irreducible representations with bounded root Artin conductor, let us consider a number field \( K \) with infinite \( p \)-class field tower for some prime \( p \). Let \( K_n \) be the Hilbert \( p \)-class field of \( K_{n-1} \) with \( K_0 = K \) and \( G_n = \text{Gal}(K_n/Q) \).

The main objective of this section is to prove that, under some conditions over \( K \) and applying the results of the previous section, there exists an upper bound for the root Artin conductor of the irreducible characters of \( G_n \). This bound depends only on the base field \( K \). In addition, we obtain that for each \( n > 1 \) it is possible to find an irreducible character of \( G_n \) with degree increasing with \( n \).

\textbf{Proposition 7.} Let \( K \) be a Galois extension of \( \mathbb{Q} \) with infinite \( p \)-class field tower, for some prime \( p \). Suppose that \( K \) has a subfield \( \bar{k} \) satisfying the following conditions:

(a) \( \bar{k} \) is Galois over \( \mathbb{Q} \).

(b) \( [\bar{k} : \mathbb{Q}] = q \), with \( q \) a prime number.

Let \( \chi \in \text{Irr}(G_n) \), where \( G_n = \text{Gal}(K_n/Q) \). If \( \bar{H}_n = \text{Gal}(K_n/\bar{k}) \), then either

(i) \( \text{Res}_{\bar{H}_n}^{G_n} \chi = \theta \), for some \( \theta \in \text{Irr}(\bar{H}_n) \), or

(ii) \( \chi = \text{Ind}_{\bar{H}_n}^{G_n} \theta \), for some \( \theta \in \text{Irr}(\bar{H}_n) \).

\textbf{Proof.} The proof follows directly from Theorem \ref{thm} with \( G = G_n \) and \( H = \bar{H}_n \). \qed

\end{document}
Proposition 8. Let $K$ be a number field with infinite $p$-class field tower for some prime $p$. If $T_n=\text{Gal}(K_n/K)$, then for each $n \geq 1$ there exists $\phi \in \text{Irr}(T_n)$ such that

$$\phi(1) > 2^{\frac{n-1}{2}}.$$ 

Proof. Let us consider the following chain of subgroups. If $n$ is even, we take for $1 \leq j \leq \frac{n}{2}$:

\[
\begin{align*}
H_0 &= \{1\}, \\
H_1 &= \text{Gal}(K_n/K_{n-2}), \quad H_1/H_0 \cong H_1 \\
H_2 &= \text{Gal}(K_n/K_{n-4}), \quad H_2/H_1 \cong \text{Gal}(K_{n-2}/K_{n-4}) \\
&\vdots \\
H_j &= \text{Gal}(K_n/K_{n-2j}), \quad H_j/H_{j-1} \cong \text{Gal}(K_{n-2(j-1)}/K_{n-2j}) \\
&\vdots \\
H_{\frac{n}{2}} &= T_n = \text{Gal}(K_n/K), \quad H_{\frac{n}{2}}/H_{\frac{n}{2}-1} \cong \text{Gal}(K_2/K).
\end{align*}
\]

If $l < i-1$ then $K_i/K_1$ is a non abelian group, so by Corollary 2, there exists $\phi \in \text{Irr}(T_n)$ with $\phi(1) \geq 2^\frac{n}{2} > 2^{\frac{n-1}{2}}$.

If $n$ is odd, for $j < \frac{n-1}{2}$ we take $H_j$ and $H_j/H_{j-1}$ as in the even case. For $j = (n-1)/2$ we take $H_{\frac{n-1}{2}} = T_n$ and $H_{\frac{n-1}{2}}/H_{\frac{n-1}{2}-1} \cong \text{Gal}(K_3/K)$. Hence, there exists $\phi \in \text{Irr}(G)$ such that $\phi(1) > 2^{\frac{n-1}{2}}$. 

\[\square\]

Corollary 9. Let $G_n$ be as in Proposition 7. Then for each $n > 1$, there exists $\chi \in \text{Irr}(G_n)$ such that

$$\chi(1) > 2^{\frac{n-1}{2}}.$$ 

Proof. Note that if $T_n = \text{Gal}(K_n/K)$ has an irreducible character $\theta$ with $\theta(1) > 2^{\frac{n-1}{2}}$, then there exists $\chi \in \text{Irr}(G)$ with $\chi(1) > 2^{\frac{n-1}{2}}$. In fact, let $\theta \in \text{Irr}(T_n)$ with $\theta(1) > 2^{\frac{n-1}{2}}$ and choose $\chi \in \text{Irr}(G_n)$ such that $\theta$ is an irreducible constituent of $\text{Res}_{T_n}^{G_n} \chi$. By Theorem 7, $\chi(1) = et\theta(1)$, where $e = \langle \text{Res}_{T_n}^{G_n} \chi, \theta \rangle$ and $t = [G_n : I_\theta(\theta)]$. As $e, t \geq 1$, then $\chi(1) \geq \theta(1) > 2^{\frac{n-1}{2}}$. 

\[\square\]

Now, we obtain upper bounds for the root Artin conductor of irreducible characters of $G_n$.

Theorem 10. Assume $G_n$ as in Proposition 7 and $\chi \in \text{Irr}(G_n)$.

(i) If $\text{Res}_{H_n}^{G_n} \chi = \theta$, for some $\theta \in \text{Irr}(H_n)$ then

$$f^{1/\chi(1)} \leq |D_{k/Q}|N_{k/Q}(f_\theta)^{1/\theta(1)}.$$
(ii) If $\chi = \text{Ind}_{H_n}^{G_n} \theta$, for some $\theta \in \text{Irr}(\tilde{H}_n)$ then

$$f^{1/\chi(1)}_\chi = |D_{k/Q}|^{1/q} N_{k/Q}(f_\theta)^{1/q\theta(1)}.$$ 

**Proof.** In the first case, we have $\chi(1) = \theta(1)$ and

$$\text{Ind}_{H_n}^{G_n} \theta = \sum_{i=1}^{q} \psi_i(1) \cdot \chi \psi_i,$$

where $\text{Irr}(G_n/\tilde{H}_n) = \{\psi_1, \psi_2, \ldots, \psi_q\}$ (see [3, Theorem 19.5]). Since $G_n/\tilde{H}_n$ is isomorphic to the abelian group $\mathbb{Z}/q\mathbb{Z}$, it follows that $\text{Ind}_{H_n}^{G_n} \theta = \sum_{i=1}^{q} \chi \psi_i$. The Artin conductor of this induced character is, on the one hand,

$$f_{\text{Ind}_{H_n}^{G_n} \theta} = |D_{k/Q}|^{\theta(1)} N_{k/Q}(f_\theta),$$

where the ideal $f_\theta$ is the Artin conductor of $\theta$. On the other hand, assuming that $\psi_1$ is the trivial character,

$$f_{\text{Ind}_{H_n}^{G_n} \theta} = f \sum_{i=1}^{q} \chi \psi_i = f \chi \prod_{i=2}^{q} f \chi \psi_i.$$

Now, combining these expressions we get

$$f_\chi = |D_{\tilde{k}/Q}|^{\theta(1)} N_{\tilde{k}/Q}(f_\theta) \cdot \left( \prod_{i=2}^{q} f \chi \psi_i \right)^{-1},$$

so

$$f^{1/\chi(1)}_\chi \leq |D_{\tilde{k}/Q}|^{1/q} N_{\tilde{k}/Q}(f_\theta)^{1/q\theta(1)}.$$ 

In the second case,

$$\chi(1) = [G_n : \tilde{H}_n] \theta(1) = q \theta(1)$$

and we can see that the root Artin conductor of $\chi$ is given by the expression

$$f^{1/\chi(1)}_\chi = |D_{\tilde{k}/Q}|^{1/q} N_{\tilde{k}/Q}(f_\theta)^{1/q\theta(1)}.$$ 

\[Q.E.D.\]

In order to obtain a bound for the root Artin conductors, we need the following result.

**Lemma 11.** Assume $K_n$ and $K$ as in the Proposition [7]. Let $p$ be a prime in $\tilde{k}$, with $b$ primes over $p$ in $K_n$ and $K$ respectively. Let $G_i(K_{n,b}/K_p)$ and $G_i(K_q/K_p)$ be the $i$-th ramification groups of the local extensions $K_{n,b}/\tilde{k}_p$ and $K_q/\tilde{k}_p$. Then, for $i \geq 0$

(a) $G_i(K_{n,b}/K_q) = G_i(K_{n,b}/\tilde{k}_p) \cap G(K_{n,b}/K_q) = \{1\}$

(b) $|G_i(K_{n,b}/\tilde{k}_p)| = |G_i(K_q/\tilde{k}_p)|$. 

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The proof of this lemma follows directly from properties of higher ramification groups (see for example [7, p.177-180]) and by the fact that \( K_n/K \) is an unramified extension.

**Corollary 12.** There is an infinite sequence \( \{ \chi_n \}_{n \in \mathbb{N}} \) of irreducible Artin characters with \( \chi_n(1) \to \infty \) and with
\[
f_{\chi_n(1)}^{1/\chi_n(1)} \leq C,
\]
where \( C > 0 \) is an effective computable constant.

**Proof.** By the Corollary 9 and Theorem 10, we know that for each \( n \) there is an irreducible character \( \chi_n \) of \( G_n \) with \( \chi_n(1) \to \infty \) and
\[
f_{\chi_n(1)}^{1/\chi_n(1)} \leq |D_{\tilde{k}/Q}|N_{\tilde{k}/Q}(f_\theta)^{\theta(1)},
\]
for some \( \theta \in \text{Irr}(\tilde{H}_n) \). By the properties of the higher ramification groups stated in Lemma 11 and considering that the primes ramifying in \( K \) are the only ones that appears in \( N_{\tilde{k}/Q}(f_\theta) \), it is possible to find a constant \( T > 0 \) depending only on the base field \( K \), such that \( N_{\tilde{k}/Q}(f_\theta) \leq T^{\theta(1)} \). Hence,
\[
f_{\chi_n(1)}^{1/\chi_n(1)} \leq |D_{\tilde{k}/Q}|T := C.
\]

**Remark 13.** As the referee pointed out, it is possible to avoid the hypothesis about the degree of \( \tilde{k}/Q \) and obtain the same type of bounds for the asymptotic behavior of \( f_{\chi_n(1)}^{1/\chi_n(1)} \). This is accomplished in Theorem 14 below.

**Theorem 14.** Let \( K \) be a Galois extension of \( \mathbb{Q} \) with infinite \( p \)-class field tower. Let \( m = [K : \mathbb{Q}] \). Then there exists an infinite sequence \( \{ \chi_n \}_{n \in \mathbb{N}} \) of irreducible Artin characters such that \( \chi_n(1) \to \infty \) and and
\[
f_{\chi_n(1)}^{1/\chi_n(1)} \leq |D_{K/Q}|^{m/2}.
\]

**Proof.** Let \( G_n = \text{Gal}(K_n/\mathbb{Q}) \) and \( T_n = \text{Gal}(K_n/K) \). We can choose \( \theta_n \in \text{Irr}(T_n) \) as in Proposition 8. Then, by the Proposition 3 there exists \( \chi_n \in \text{Irr}(G_n) \) such that \( \langle \text{Ind}_{H_n}^{G_n} \theta_n, \chi_n \rangle = a \geq 1 \) and with \( \chi_n(1) > m^{-1/2}\theta_n(1) \), so
\[
\chi_n(1) > m^{-1/2}2^{n-1}.
\]
Hence, by the properties of the Artin conductor we get
\[
f_{\chi_n}^a \leq f_{\text{Ind}_{H_n}^{G_n} \theta_n} = |D_{K/Q}|^{\theta_n(1)},
\]
and therefore,
\[
f_{\chi_n(1)}^{1/\chi_n(1)} \leq f_{\chi_n}^{a/\chi_n(1)} \leq |D_{K/Q}|^{m/2}.
\]
4 Number fields with infinite 2-class field tower

Golod and Shafarevich \cite{2} proved that a number field \( K \) has an infinite \( p \)-class field tower if the \( p \)-rank of the class group of \( K \) is large enough. In this case,

\[
\alpha(r_1, r_2) \leq |D_K|^{1/[K:\mathbb{Q}]},
\]

where \( D_K \) is the discriminant of \( K \).

In addition, Martinet has constructed a number field with infinite Hilbert class field towers and lowest known root discriminant and proved that

\[
\alpha(0,1) < 93 \quad \text{and} \quad \alpha(1,0) < 1059.
\]

In particular, he found that \( K = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1}, \sqrt{2}, \sqrt{-23}) \) has infinite 2-class field tower. Since \( \tilde{k} = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1}) \) is a subfield of \( K \) of degree 5 over \( \mathbb{Q} \), \( K \) satisfies the conditions of the Theorem \[10\].

The discriminant of \( \tilde{k} \) is

\[
|D_{\tilde{k}/\mathbb{Q}}| = 14641 = 11^4
\]

and the only rational primes that ramify in \( K \) are 2, 11 and 23. Using PARI/GP \cite{9}, we can estimates the sizes of the higher ramification groups. Thus, we get the upper bound

\[
N_{\tilde{k}/\mathbb{Q}}(f_\theta) \leq (2^{15}23)^{\theta(1)}.
\]

With this estimation, it follows the explicit result:

**Corollary 15.** For each \( n \geq 1 \), there exists a irreducible character \( \chi_n \) such that \( \chi_n(1) \to \infty \) and

\[
f_{1/\chi_n(1)} \leq C,
\]

where \( C \leq 11^4 \cdot 2^{15} \cdot 23 \).

An open problem now is to improve the constant \( C \).

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**References**

[1] P. X. Gallagher, *Group characters and normal Hall subgroups*, Nagoya Math. J. **21** (1962), 223–230.
[2] E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272.

[3] B. Huppert, *Character theory of finite groups*, De Gruyter, Berlin-New York, 1998.

[4] I. Martin Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, 2006.

[5] A. Pizarro - Madariaga, *Lower bounds for the Artin conductor*, Mathematics of Computation **80** (2011), no. 273, 539–561.

[6] J. Martinet, *Tours de corps de classes et estimations de discriminants*, Inventiones Mathematicae **44** (1978), 65–73.

[7] J. Neukirch, *Algebraic number theory*, Springer-Verlag, Berlin, 1999.

[8] A.M. Odlyzko, *On conductors and discriminants*, Proc. Sympos., Univ. Durham **1** (1977), 377–407.

[9] The PARI Group, Bordeaux, *PARI/GP version 2.7.1*, 2014, available from [http://pari.math.u-bordeaux.fr/](http://pari.math.u-bordeaux.fr/)