COUNTING ORIENTATIONS OF GRAPHS WITH NO STRONGLY CONNECTED TOURNAMENTS

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Abstract. Let $S_k(n)$ be the maximum number of orientations of an $n$-vertex graph $G$ in which no copy of $K_k$ is strongly connected. For all integers $n, k \geq 4$ where $n \geq 5$ or $k \geq 5$, we prove that $S_k(n) = 2^{t_{k-1}(n)}$, where $t_{k-1}(n)$ is the number of edges of the $n$-vertex $(k-1)$-partite Turán graph $T_{k-1}(n)$, and that $T_{k-1}(n)$ is the only $n$-vertex graph with this number of orientations. Furthermore, $S_4(4) = 40$ and this maximality is achieved only by $K_4$.

1. Introduction

Let $G$ be a graph and $\vec{F}$ be an oriented graph. An orientation $\vec{G}$ of $G$ is $\vec{F}$-free if $\vec{G}$ contains no copy of $\vec{F}$. Given a family $\vec{F}$ of oriented graphs, denote by $D(G, \vec{F})$ the family of orientations of $G$ that are $\vec{F}$-free for every $\vec{F} \in \vec{F}$ and write $D(G, \vec{F}) = |D(G, \vec{F})|$. Finally, let

$$D(n, \vec{F}) = \max\{D(G, \vec{F}): |V(G)| = n\}. \quad (1)$$

An $n$-vertex graph that achieves equality in (1) is called $\vec{F}$-extremal.

The Turán graph, denoted by $T_{k-1}(n)$, is the $n$-vertex graph without copies of $K_k$ and maximum number of edges possible, which is the balanced complete $(k-1)$-partite graph. For simplicity, we write $t_{k-1}(n)$ for $|E(T_{k-1}(n))|$.

The problem of determining $D(n, \vec{F})$ has been solved by Alon and Yuster [1] for large $n$ when $\vec{F}$ consists of a single tournament. Recently, Araújo and the first and last authors [2] extended this result to every $n$ in the case where the forbidden tournament is the strongly connected triangle, here denoted by $C_{3\rightarrow}$. More precisely, these results are given as follows.

**Theorem 1.1.** For $n \geq 1$, we have $D(n, \{C_{3\rightarrow}\}) = \max\{2^{\lfloor n^2/4 \rfloor}, n!\}$. Furthermore, among all graphs $G$ with $n \geq 8$ vertices, $D(G, \{C_{3\rightarrow}\}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G$ is the Turán graph $T_2(n)$.

More recently, Bucić and Sudakov [6] determined $D(n, \{C_{2k+1}\})$ for strongly connected odd cycles $C_{2k+1}$ when $n$ is sufficiently large.

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Clearly, if $\mathcal{F}$ and $\mathcal{F}'$ are families of oriented graphs such that $\mathcal{F} \subseteq \mathcal{F}'$, then $D(n, \mathcal{F}) \leq D(n, \mathcal{F}')$ for any integer $n$. Thus, the result of Alon and Yuster determining $D(n, \{K_k\})$ for any fixed tournament $K_k$ on $k \geq 3$ vertices and sufficiently large $n$ immediately implies the following: for any nonempty family $\mathcal{F}$ of orientations of $K_k$, we also have $D(n, \mathcal{F}) = 2^{t_k-1(n)}$ for large $n$. More generally, if $\mathcal{F}$ is a family of tournaments (not necessarily with the same size) and $k$ is the minimum size of a tournament in $\mathcal{F}$, we must also have $D(n, \mathcal{F}) = 2^{t_k-1(n)}$ for large $n$. However, to the best of our knowledge, the only nontrivial family $\tilde{\mathcal{F}}$ for which $D(n, \tilde{\mathcal{F}})$ is known for all values of $n \geq 1$ is $\tilde{\mathcal{F}} = \{C_3^k\}$. We extend Theorem 1.1 to the family $\mathcal{F}_k$ of all strongly connected tournaments on $k \geq 4$ vertices. More precisely, we prove in Theorem 3.7 that for all integers $n$, $k \geq 4$ where $n \geq 5$ or $k \geq 5$, we have $D(n, \mathcal{F}_k) = 2^{t_k-1(n)}$ and we show that $T_{k-1}(n)$ is the only $\mathcal{F}_k$-extremal graph. For $n = k = 4$, we have $D(n, \mathcal{F}_4) = 40$ and $K_4$ is the only $\mathcal{F}_4$-extremal graph.

We remark that our result implies a similar one for any family $\mathcal{F}$ of complete graphs with at least $k$ vertices that contains $\mathcal{F}_k$. In fact, it is enough that $\mathcal{F}$ is a family of graphs with chromatic number at least $k$ for which $\mathcal{S}_k \subset \mathcal{F}$. More precisely, one can easily conclude that for $n \geq 5$ or $k \geq 5$ we have $D(n, \mathcal{F}) = 2^{t_k-1(n)}$ and that $T_{k-1}(n)$ is the only $\mathcal{F}$-extremal graph. As an application of this, consider the family $\mathcal{R}_k$ of all orientations of $K_k$ that are not transitive, so that $D(n, \mathcal{R}_k)$ is the maximum number of orientations of an $n$-vertex graph for which every copy of $K_k$ is transitively oriented. Since $\mathcal{S}_k \subset \mathcal{R}_k$, our result implies that for $n \geq 5$ or $k \geq 5$ we have $D(n, \mathcal{R}_k) = 2^{t_k-1(n)}$ and $T_{k-1}(n)$ is the unique $\mathcal{R}_k$-extremal graph (for $n = k = 4$, it is easy to see that $T_3(4)$ is the unique $\mathcal{R}_4$-extremal graph). Our results for $\mathcal{R}_k$ and $\mathcal{S}_k$ solve Problems 3 and 4 in [2].

In Section 2 we prove general results for orientations avoiding graphs from a family $\mathcal{F}$ of tournaments (see Lemmas 2.5 and 2.6). These results imply that if $G$ is an $\mathcal{F}$-extremal graph that is not complete and multipartite, then one can construct $\mathcal{F}$-extremal complete multipartite graphs $G_1$ and $G_2$ such that $|E(G_2)| < |E(G)| \leq |E(G_1)|$. We use these tools to prove our main result in Section 3. We finish with some open problems in Section 4.

2. Multipartite complete extremal graphs

In this section we obtain some results derived with the approach in [5], which in turn was influenced by the proof of Turán’s Theorem by Zykov Symmetrization. To highlight the similarities, we use the notation of [5] whenever is possible. Fix a family $\mathcal{F}$ of tournaments (not necessarily with the same number of vertices) on at least three vertices and fix a positive integer $n$. We prove that there is a complete multipartite $n$-vertex graph that is $\mathcal{F}$-extremal (see Lemmas 2.5 and 2.6).

Let $\vec{x}$ be a vector whose coordinates are indexed by a set $T$. Given $t \in T$, we denote by $x(t)$ the value of $x$ at coordinate $t$. Given $p \in (0, \infty)$, the $\ell_p$-norm of $\vec{x}$, denoted by $\|\vec{x}\|_p$, is given by
Moreover, for a sequence of vectors $\vec{x}_1, \ldots, \vec{x}_s$, each indexed by $T$, their pointwise product $\vec{y}$, i.e., the vector in which $y(t) = \prod_{k=1}^{s} x_k(t)$ for $t \in T$, is denoted by $\prod_{k=1}^{s} \vec{x}_k$.

Let $G$ be a graph and let $\vec{F}$ be a family of oriented graphs. Given a subgraph $H$ of $G$ and an $\vec{F}$-free orientation $\vec{H}$ of $H$, we denote by $c_F(G \mid \vec{H})$ the number of ways to orient the edges in $E(G) \setminus E(H)$ in order to extend $\vec{H}$ to an $\vec{F}$-free orientation of $G$. For simplicity, given $v \in V(G) \setminus V(H)$, if $H$ is an induced subgraph of $G$, then we write $c_F(v, \vec{H})$ for $c_F(G[V(H) \cup \{v\}], \vec{H})$. Similarly, given an edge $\{u, v\}$ of $E(G) \setminus E(H)$, we use $c_F(\{u, v\}, \vec{H})$ for the number of ways to orient the edges between $V(H)$ and $\{u, v\}$, and the edge $\{u, v\}$ (again avoiding $\vec{F}$), i.e., the edges in $E(G[V(H) \cup \{x, y\}] \setminus E(H)$.

We also define $\vec{v}_{H,F}$ as the vector indexed by the set $D(H, \vec{F})$ of all $\vec{F}$-free orientations of $H$, whose coordinate corresponding to an orientation $\vec{H}$ is given by $\vec{v}_{H,F}(\vec{H}) = c_F(v, \vec{H})$. When $\vec{F}$ is a family of tournaments, we obtain the following simple observation.

**Fact 2.1.** Let $\vec{F}$ be a family of tournaments. If $H$ is an induced subgraph of $G$ such that $S = V(G) \setminus V(H)$ is an independent set in $G$, and $\vec{H}$ is an $\vec{F}$-free orientation of $H$, then

$$c_F(G \mid \vec{H}) = \prod_{v \in S} c_F(v, \vec{H}).$$

We shall also use the following consequence of Hölder’s inequality.

**Lemma 2.2.** Let $\vec{x}_1, \ldots, \vec{x}_s$ be complex-valued vectors indexed by a set $T$. Then,

$$\left\| \prod_{k=1}^{s} \vec{x}_k \right\|_{1} \leq \prod_{k=1}^{s} \left\| \vec{x}_k \right\|_{s}.$$

Furthermore, equality holds if and only if, for every $i, j \in [s]$, there exists $\alpha_{i,j}$ with the property that $x_i(t) = \alpha_{i,j} x_j(t)$ for all $t \in T$.

We say that two non-adjacent vertices of a graph are *twins* if they have the same neighborhood. For the next lemma we consider the following operation: take an independent set $S$ of a graph $G$, select a particular vertex $v \in S$, delete all vertices in $S \setminus \{v\}$ and add $|S| - 1$ new twins of $v$. We show that there is a vertex $v \in S$ for which this operation produces a graph that has at least as many $\vec{F}$-free orientations as $G$.

**Lemma 2.3.** Let $\vec{F}$ be a family of tournaments, and let $G$ be a graph on $n$ vertices. If $S \subset V(G)$ is a non-empty independent set, then the following holds.

(a) if $v$ is a vertex in $S$ for which $\left\| \vec{v}_{G-S,F} \right\|_{|S|}$ is maximum among all vertices of $S$, then the graph $\vec{G}$ obtained from $G$ by replacing the vertices in $S \setminus \{v\}$ with $|S| - 1$ twins of $v$ is such that $D(\vec{G}, \vec{F}) \geq D(G, \vec{F})$; and

(b) if $G$ is $\vec{F}$-extremal, then $\vec{u}_{G-S,F} = \vec{u}_{G-S,F}$ for any vertices $u, w \in S$. 

Proof. Let \( \mathcal{F}, G, \) and \( S \) be as in the statement, and put \( H = G - S \) and \( s = |S| \). By Fact 2.1, the total number of \( \mathcal{F} \)-free orientations of \( G \) is given by

\[
D(G, \mathcal{F}) = \sum_{\vec{H} \in D(H, \mathcal{F})} c_\mathcal{F}(G \mid \vec{H}) = \sum_{\vec{H} \in D(H, \mathcal{F})} \prod_{u \in S} c_\mathcal{F}(u, \vec{H}) = \left\| \prod_{u \in S} \vec{u}_{H, \mathcal{F}} \right\|_1,
\]

(2)

where, in the last equality, we used the fact that every coordinate of \( \vec{u}_{H, \mathcal{F}} \) is non-negative.

Let \( v \) be a vertex in \( S \) for which \( \left\| \vec{u}_{H, \mathcal{F}} \right\|_s \) is maximum. By Lemma 2.2, we have

\[
\left\| \prod_{u \in S} \vec{u}_{H, \mathcal{F}} \right\|_1 \leq \prod_{u \in S} \left\| \vec{u}_{H, \mathcal{F}} \right\|_s \leq \left\| \vec{v}_{H, \mathcal{F}} \right\|_s.
\]

(3)

On the other hand, let \( \tilde{G} \) be the graph obtained from \( G \) by replacing the vertices in \( S \setminus \{v\} \) by \( s - 1 \) twins of \( v \). Then, we have:

\[
D(\tilde{G}, \mathcal{F}) = \sum_{\vec{H} \in D(H, \mathcal{F})} c_\mathcal{F}(v, \vec{H})^s = \left\| \vec{v}_{H, \mathcal{F}} \right\|_s.
\]

(4)

Therefore, combining (2), (3) and (4), we have \( D(\tilde{G}, \mathcal{F}) \geq D(G, \mathcal{F}) \). This proves (a).

Now, assume \( G \) is \( \mathcal{F} \)-extremal. Since \( D(\tilde{G}, \mathcal{F}) \geq D(G, \mathcal{F}) \), we have \( D(\tilde{G}, \mathcal{F}) = D(G, \mathcal{F}) \). This implies that both inequalities in (3) hold with equality, and hence for every vertex \( u \in S \), we must have \( \left\| \vec{u}_{H, \mathcal{F}} \right\|_s = \left\| \vec{v}_{H, \mathcal{F}} \right\|_s \). By the equality conditions of Lemma 2.2, together with the fact that all coordinates in our vectors are nonnegative, we have \( \vec{u}_{H, \mathcal{F}} = \vec{v}_{H, \mathcal{F}} \). This proves (b). \( \square \)

Corollary 2.4. Let \( \mathcal{F} \) be a family of tournaments, and let \( G \) be an \( \mathcal{F} \)-extremal graph. If \( u, v \in V(G) \) are non-adjacent, then the graph obtained from \( G \) by replacing \( v \) with a twin of \( u \) is also \( \mathcal{F} \)-extremal.

Proof. Let \( G_{uw} = G - \{u, v\} \). Since \( G \) is \( \mathcal{F} \)-extremal, by Lemma 2.3(b) with \( S = \{u, v\} \), we have \( \vec{u}_{G_{uw}, \mathcal{F}} = \vec{v}_{G_{uw}, \mathcal{F}} \). Therefore, by Lemma 2.3(a), replacing \( v \) with a twin of \( u \) does not change the number of \( \mathcal{F} \)-free orientations of the graph. \( \square \)

Note that a graph \( G \) is a complete multipartite graph if and only if, for any \( u, v, w \in V(G) \) such that \( \{u, v\}, \{u, w\} \notin E(G) \), we have \( \{v, w\} \notin E(G) \). By repeatedly applying Corollary 2.4, we show that there exists a complete multipartite graph on \( n \) vertices that is \( \mathcal{F} \)-extremal.

Lemma 2.5. Let \( \mathcal{F} \) be a family of tournaments and let \( G \) be an \( n \)-vertex \( \mathcal{F} \)-extremal graph. Then there exists an \( n \)-vertex complete multipartite graph \( G^* \) that is \( \mathcal{F} \)-extremal and satisfies \( |E(G^*)| \geq |E(G)| \).

Proof. Let \( G \) be an \( n \)-vertex \( \mathcal{F} \)-extremal graph. A vertex \( u \) is eccentric if there is a non-neighbor of \( u \) that is not its twin. Put \( G_0 = G \) and let \( G_0, \ldots, G_t \) be a maximal sequence of graphs such that, for \( i = 0, \ldots, t - 1 \), the graph \( G_{i+1} \) is obtained from \( G_i \) by picking
an eccentric vertex \( u \) of maximum degree in \( G_i \) and replacing every non-neighbor of \( u \) with a twin of \( u \). Thus, every vertex of \( G_{i+1} \) is either a neighbor or a twin of \( u \). Note that in \( G_{i+1} \) the vertex \( u \) is not eccentric. Moreover, if a vertex \( v \) is eccentric in \( G_{i+1} \), then \( v \) is also eccentric in \( G_i \), and hence \( G_{i+1} \) contains fewer eccentric vertices than \( G_i \), from which we conclude that the sequence \( G_0, \ldots, G_i \) is finite.

By construction, since we always pick a vertex \( u \) with maximum degree, \(|E(G_i)| \geq |E(G)|\). Then, by Corollary 2.4, \( G_i \) is \( \bar{F} \)-extremal. Moreover, every pair of non-adjacent vertices \( u \) and \( v \) in \( G_i \) are twins. Therefore, \( G_i \) is a multipartite complete graph.

The following result implies that any \( \bar{F} \)-extremal graph may also be turned into an \( \bar{F} \)-extremal multipartite graph by removing edges.

**Lemma 2.6.** Let \( \bar{F} \) be a family of tournaments, let \( G \) be an \( \bar{F} \)-extremal graph, and let \( u, v, w \) be distinct vertices of \( G \) such that \( \{u, v\}, \{u, w\} \notin E(G) \) and \( \{v, w\} \in E(G) \). Then, the graph obtained from \( G \) by deleting the edge \( \{v, w\} \) is \( \bar{F} \)-extremal. Furthermore, for \( H = G - \{u, v, w\} \) we have \( \bar{u}_{H,F} = \bar{w}_{H,F} = \bar{v}_{H,F} \).

**Proof.** Let \( \bar{F}, G, u, v, w \) be as in the statement. Let \( H = G - \{u, v, w\} \), and write \( H^x = G[V(H) \cup x] \) for \( x \in \{u, v, w\} \) and let \( G_{twins}^u (G_{twins}^v) \) be the graph obtained from \( H^u \) by adding twins \( u_1, u_2 \) of \( u \) (twins \( w_1, w_2 \) of \( w \)). By Corollary 2.4 applied twice, the graph \( G_{twins}^u \) is also \( \bar{F} \)-extremal. Therefore, \( D(G, \bar{F}) = D(G_{twins}^u, \bar{F}) \).

Applying Proposition 2.1 to \( G_{twins}^u \) with \( S = \{u, u_1, u_2\} \), we have

\[
D(G_{twins}^u, \bar{F}) = \sum_{H \in D(H, \bar{F})} c_{\bar{F}}(G_{twins}^u | H) = \sum_{H \in D(H, \bar{F})} c_{\bar{F}}(u, \bar{H})^3 = \|\bar{u}_{H,F}\|^3_3.
\]

By an analogous computation, we have \( D(G_{twins}^v, \bar{F}) = \|\bar{w}_{H,F}\|^3_3 \). However, since \( w \) is neighbor of \( u \) and \( v \), Corollary 2.4 cannot be applied to the graph \( G_{twins}^w \), so it is not possible to conclude whether \( G_{twins}^w \) is \( \bar{F} \)-extremal. But since \( G_{twins}^u \) is \( \bar{F} \)-extremal, we have

\[
\|\bar{w}_{H,F}\|^3_3 = D(G_{twins}^w, \bar{F}) \leq D(G_{twins}^u, \bar{F}) = \|\bar{u}_{H,F}\|^3_3.
\] (5)

Since there are no edges between \( u \) and \( \{v, w\} \), we can compute \( D(G, \bar{F}) \) as follows:

\[
D(G, \bar{F}) = \sum_{H \in D(H, \bar{F})} \left( c_{\bar{F}}(u, \bar{H}) c_{\bar{F}}(G - u | \bar{H}) \right) = \sum_{H \in D(H, \bar{F})} \left( c_{\bar{F}}(u, \bar{H}) \sum_{H^w \cup H} c_{\bar{F}}(v, H^w) \right),
\]

where the inner sum is taken over the \( \bar{F} \)-free orientations of \( H^w \) that extend a given \( \bar{F} \)-free orientation of \( H \), that is, over the orientations of the edges between \( w \) and \( H \), for which the resulting orientation is \( \bar{F} \)-free. By Lemma 2.3(b), since \( G \) is \( \bar{F} \)-extremal and \( \{u, v\} \notin E(G) \), we have \( \bar{v}_{H^w, F} = \bar{u}_{H^w, F} \), i.e., \( c_{\bar{F}}(v, H^w) = c_{\bar{F}}(u, H^w) \) for every \( H^w \). Finally, note that since \( \bar{F} \) is a family of tournaments and \( u \) and \( w \) are not neighbors, \( c_{\bar{F}}(u, H^w) \) does...
Let \( \vec{H} \) be an extension of \( \vec{H} \), so \( c_{\vec{F}}(u, \vec{H}^w) = c_{\vec{F}}(u, \vec{H}) \).
Therefore,

\[
D(G, \vec{F}) = \sum_{\vec{H} \in D(H, \vec{F})} \left( c_{\vec{F}}(u, \vec{H}) \sum_{\vec{H}^w \mid \vec{H}} c_{\vec{F}}(u, \vec{H}^w) \right) = \sum_{\vec{H} \in D(H, \vec{F})} \left( c_{\vec{F}}(u, \vec{H}) c_{\vec{F}}(u, \vec{H}^w) \sum_{\vec{H}^w \mid \vec{H}} 1 \right)
\]

\[
= \sum_{\vec{H} \in D(H, \vec{F})} c_{\vec{F}}(u, \vec{H})^2 c_{\vec{F}}(w, \vec{H})
\]

\[
\leq \left\| \vec{u}_{H,\vec{F}} \right\|_3^3 \left\| \vec{w}_{H,\vec{F}} \right\|_3\left\| \vec{w}_{H,\vec{F}} \right\|_3
\]

(6)

\[
\leq \left\| \vec{u}_{H,\vec{F}} \right\|_3^3.
\]

(7)

Note that (6) follows by Lemma 2.2, and (7) follows from (5). Finally, since \( D(G, \vec{F}) = D(G_{\text{twins}}^u, \vec{F}) = \left\| \vec{u}_{H,\vec{F}} \right\|_3^3 \), we must have equality in both (6) and (7), which in turn leads to \( \left\| \vec{u}_{H,\vec{F}} \right\|_3 = \left\| \vec{w}_{H,\vec{F}} \right\|_3 \). The equality condition in Lemma 2.2 implies that \( \vec{u}_{H,\vec{F}} = \vec{w}_{H,\vec{F}} \).

Analogously, \( \vec{u}_{H,\vec{F}} = \vec{v}_{H,\vec{F}} \).

Finally, let \( G^- \) be the graph obtained from \( G \) by deleting the edge \( \{v, w\} \). It follows that

\[
D(G_{\text{twins}}^u, \vec{F}) = \sum_{\vec{H}} c_{\vec{F}}(u, \vec{H})^3 = \sum_{\vec{H}} c_{\vec{F}}(u, \vec{H}) c_{\vec{F}}(v, \vec{H}) c_{\vec{F}}(w, \vec{H}) = D(G^-, \vec{F}).
\]

Therefore, \( G^- \) is \( \vec{F} \)-extremal. This concludes the proof. \( \square \)

An interesting consequence of the main results of this section (Lemmas 2.5 and 2.6) is that if \( G \) is an \( \vec{F} \)-extremal graph that is not complete multipartite, then one can construct \( \vec{F} \)-extremal complete multipartite graphs \( G_1 \) and \( G_2 \) such that \( |E(G_2)| < |E(G)| \leq |E(G_1)| \).

We finish this section proving that for some families \( \vec{F} \) of forbidden tournaments every \( \vec{F} \)-extremal graph is a multipartite complete graph. Given an oriented graph \( \vec{G} \), we write \( (v, w) \) to denote an edge oriented from \( v \) to \( w \) in \( \vec{G} \).

**Lemma 2.7.** Let \( \vec{F} \) be a family of tournaments with no source and let \( n \) be a positive integer. Then, every \( n \)-vertex \( \vec{F} \)-extremal graph is complete multipartite.

**Proof.** Let \( n \geq 4 \) be an integer and let \( \vec{F} \) as in the statement. Let \( G \) be an \( n \)-vertex \( \vec{F} \)-extremal graph and assume for a contradiction that \( G \) is not complete multipartite. Fix vertices \( u, v, w \) such that \( \{u, v\}, \{u, w\} \notin E(G) \) and \( \{v, w\} \in E(G) \).

Let \( H = G - \{u, v, w\} \), and \( H^x = G[V(H) \cup x] \) for \( x \in \{u, v, w\} \). From Lemma 2.6, we have \( \vec{u}_{H,\vec{F}} = \vec{w}_{H,\vec{F}} = \vec{v}_{H,\vec{F}} \), so for every orientation \( \vec{H} \) of \( H \) we have \( c_{\vec{F}}(u, \vec{H}) = c_{\vec{F}}(w, \vec{H}) = c_{\vec{F}}(v, \vec{H}) \). Note that since \( \vec{F} \) is a family of tournaments and \( u \) and \( w \) are not adjacent, for every extension of \( \vec{H} \) to an orientation \( \vec{H}^w \), we must have \( c_{\vec{F}}(u, \vec{H}^w) = c_{\vec{F}}(u, \vec{H}) \). Finally,
since \(u\) and \(v\) are not adjacent, by Lemma 2.3 (b), we have \(\vec{u}_{H^w, \vec{F}} = \vec{v}_{H^w, \vec{F}}\), that is, 
\(c_{\vec{F}}(u, \vec{H}^w) = c_{\vec{F}}(v, \vec{H}^w)\) for every orientation \(\vec{H}^w\) of \(H^w\). It follows that, for every \(\vec{F}\)-free extension \(\vec{H}^w\) of \(\vec{H}\), we must have 
\[c_{\vec{F}}(v, \vec{H}^w) = c_{\vec{F}}(v, \vec{H}).\] (8)

We will get a contradiction from this fact (which implies that this graph \(G\) cannot exist). We only need to find an orientation of \(\vec{H}\) and an extension of it to \(H^w\), which is \(\vec{F}\)-free and such that (8) does not hold. By the definition of \(\vec{F}\), we may start with a transitive orientation \(\vec{H}\) of \(H\), which can be extended to an \(\vec{F}\)-free orientation \(\vec{H}^w\) of \(H^w\) by orienting all edges \(\{x, w\}\) between \(w\) and \(x \in V(H)\) as \((w, x)\). Let \(H(v)\) and \(H^w(v)\) be the classes of all orientations that extend \(\vec{H}\) to \(H^v\) and \(\vec{H}^w\) to \(G - u\), respectively. We show that there is an injective mapping \(\phi: H(v) \to H^w(v)\) that is not surjective.

Given an orientation \(\vec{H}^v \in H(v)\), let \(\phi(\vec{H}^v)\) be the orientation of \(G - u\) that extends \(\vec{H}^w\) by orienting any edge \(e = \{v, x\}\) between \(v\) and \(x \in V(H)\) the same orientation as \(e\) in \(\vec{H}^v\) and by assigning the orientation \((w, v)\) to \(\{v, w\}\). The function \(\phi\) is clearly injective.

We claim that \(\phi(H(v)) \in H^w(v)\). To see why this is true, suppose that \(\phi(H(v))\) contains a tournament \(\vec{T}\) with no source. Clearly, this copy involves both \(v\) and \(w\), otherwise it would also occur in \(\vec{H}^v\), a contradiction. However, \(w\) is a source in \(\phi(H(v))\), so it cannot lie in \(\vec{T}\).

On the other hand, any transitive orientation of \(G - u\) where \(\{v, w\}\) is oriented \((v, w)\) and all edges \(\{x, v\}\) and \(\{y, w\}\) with \(x, y \in V(H)\) are oriented \((v, x)\) and \((w, y)\), respectively, must lie in \(H^w(v)\). However, it does not lie in \(\phi(H(v))\), as \(\phi\) always orients \(\{v, w\}\) as \((w, v)\). So \(\phi\) is not surjective and we have reached the desired contradiction. \(\square\)

## 3. Avoiding strongly connected tournaments

We start by giving some results concerning Hamilton cycles in oriented graphs. We refer to a directed Hamilton cycle (resp. directed path) in an oriented graph simply as Hamilton cycle (path). The following basic fact about Hamilton paths and cycles is useful in our proof.

**Fact 3.1.** Every tournament contains a Hamilton path and every strongly connected tournament contains a Hamilton cycle.

By using Fact 3.1 we can guarantee the existence of strongly connected tournaments of any length in strongly connected tournaments.

**Lemma 3.2.** Every strongly connected tournament \(\vec{K}\) contains a strongly connected tournament of order \(\ell\) for every \(3 \leq \ell \leq |V(\vec{K})|\).

**Proof.** Let \(\vec{K}\) be a strongly connected tournament. By Fact 3.1, \(\vec{K}\) contains a Hamilton cycle. The proof is by induction on \(n = |V(\vec{K})|\). Clearly, if \(n = 3\), then \(\vec{K}\) is a strongly connected orientation of \(K_3\), and the statement follows. Thus, we assume \(n \geq 4\). Let \(C = \)
Let \( K \) be a complete subgraph of a graph \( G \) and let \( \vec{K} \) be an \( \vec{S}_k \)-free orientation of \( K \). Given vertices \( u \) and \( v \) of \( G \), recall that we write \( c_{\vec{S}_k}(u, \vec{K}) \) for the number of orientations of the set of edges \( \{u, w\} \) with \( w \in K \) that extend \( \vec{K} \) keeping the \( \vec{S}_k \)-freeness of the orientation. In the next lemma we present a bound on \( c_{\vec{S}_k}(u, \vec{K}) \) for particular sizes of \( K \) and particular vertices \( u \).

**Lemma 3.3.** Let \( K \) be a complete subgraph of a graph \( G \) with \( x \in \{k-1, k, k+1\} \) vertices and let \( u \) be a vertex in \( V(G) \setminus V(K) \) that is adjacent to all vertices of \( K \). Then, for any orientation \( \vec{K} \) of \( K \), we have \( c_{\vec{S}_k}(u, \vec{K}) \leq (x - k + 4) \cdot 2^{k-3} \).

**Proof.** We assume \( x = k + 1 \) as the proof for the other values of \( x \) is analogous. Let \( K, G \) and \( u \) be as in the statement. Let \( \vec{K} \) be an orientation of \( K \) and note that by Fact 3.1 there is a Hamilton path \((v_1, \ldots, v_{k+1})\) in \( \vec{K} \). By Lemma 3.2, any orientation of the edges between \( u \) and \( v \) in \( \vec{K} \) that extends \( \vec{K} \) to an \( \vec{S}_k \)-free orientation cannot form a directed cycle of length \( k \). If \( \{u, v_1\} \) is oriented towards \( v_1 \), then the edges between any \( w \) in \( \{v_{k-1}, v_k, v_{k+1}\} \) and \( u \) must be oriented towards \( w \). Since the edges \( \{u, v_j\} \) with \( j \in \{2, \ldots, k-2\} \) can be oriented in two ways, there are at most \( 2^{k-3} \) such possible orientations. If \( \{u, v_1\} \) is oriented towards \( u \) and \( \{u, v_2\} \) is oriented towards \( v_2 \), then the edges between any \( w \) in \( \{v_k, v_{k+1}\} \) and \( u \) are oriented towards \( w \). Again, since the edges \( \{u, v_j\} \) with \( j \in \{3, \ldots, k-1\} \) can be oriented in two ways, there are at most \( 2^{k-3} \) such possible orientations. Analogously, if \( \{u, v_1\} \) and \( \{u, v_2\} \) are oriented towards \( u \) and \( \{u, v_3\} \) is oriented towards \( v_3 \), then there are at most \( 2^{k-3} \) such possible orientations. Finally, there are at most \( 2^{k-2} \) such possible orientations for which \( \{u, v_1\}, \{u, v_2\} \) and \( \{u, v_3\} \) are oriented towards \( u \). Therefore, \( c_{\vec{S}_k}(u, \vec{K}_x) \leq 2^{k-3} + 2^{k-3} + 2^{k-3} + 2^{k-2} = 5 \cdot 2^{k-3} \), as desired. \( \square \)

As before, let \( K \) be a complete subgraph of a graph \( G \) and let \( \vec{K} \) be an \( \vec{S}_k \)-free orientation of \( K \). Given an edge \( \{u, v\} \) of \( G \), similar to Lemma 3.3, we now provide a bound on \( c_{\vec{S}_k}((u, v), \vec{K}) \), where we recall that \( c_{\vec{S}_k}((u, v), \vec{K}) \) stands for the number of ways to extend \( \vec{S}_k \)-free orientations by orienting the edges in \( E(G[V(K) \cup \{u, v\}]) \setminus E(K) \).
Lemma 3.4. Let $K$ be a complete subgraph of a graph $G$ with $k − 1$ vertices and let $u$, $v$ be adjacent vertices in $V(G) \setminus V(K)$ that are adjacent to all vertices of $K$. Then, $c_{S_k}(\{u, v\}, \vec{K}) < 2 \cdot 3 \cdot 2^{k−5}$.

Proof. Let $K$, $G$ and $u$ be as in the statement. Let $\vec{K}$ be an orientation of $K$ and note that by Fact 3.1 there is a Hamilton path $(v_1, \ldots , v_{k−1})$ in $\vec{K}$. There are two possible orientations for the edge $\{u, v\}$. Let us estimate in how many ways one can orient the edges between $\vec{K}$ and $\{u, v\}$ without creating a strongly connected $K_k$. Suppose, without loss of generality, that $\{u, v\}$ is oriented towards $u$.

Note that by Lemma 3.2, any orientation of the edges between $u$ and $\vec{K}$ that extends $\vec{K}$ to an $\vec{S}_k$-free orientation cannot form a directed cycle of length $k$. Analogously to the proof of Lemma 3.3, if $\{u, v_1\}$ is oriented towards $v_1$, then the edge between any $v_{k−1}$ and $u$ must be oriented towards $v_{k−1}$, and the edges between any $w$ in $\{v_{k−2}, v_{k−1}\}$ and $v$ must be oriented towards $w$. Since the remaining edges from $u$ or $v$ to $K$ can be oriented in two ways, there are at most $2^{2−6}$ possible orientations. Similarly, there are at most $2^{2−5}$ possible orientations in which $\{u, v_1\}$ is oriented towards $u$ and $\{u, v_2\}$ is oriented towards $v_2$, and there are at most $2^{2−6}$ possible orientations in which $\{u, v_1\}$ and $\{u, v_2\}$ are oriented towards $u$, and $\{v, v_1\}$ is oriented towards $v_1$. Finally, there are at most $2^{2−5}$ possible orientations in which $\{u, v_1\}$ and $\{u, v_2\}$ are oriented towards $u$, and $\{v, v_1\}$ is oriented towards $v$. Therefore, $c_{S_k}(uv, \vec{K}) ≤ 2 \cdot (2^{2−6} + 2^{2−5} + 2^{2−6} + 2^{2−5}) = 2 \cdot 3 \cdot 2^{2−5}$.

The next result states that for $k ≥ 5$, at least half of the orientations of $K_k$ are strongly connected. Let $SC(G)$ denote the number of strongly connected orientations of $G$.

Lemma 3.5. Let $k ≥ 5$ be a positive integer. Then $SC(K_k) > 2^{(\frac{k}{2})−1}$.

Proof. The proof is by induction on $k$. Let $\{u_1, \ldots , u_k\}$ be the vertex set of $G = K_k$ and consider the complete graph $G' = G - u_k$ on $k − 1$ vertices. First, suppose $k = 5$. We prove that $SC(K_5) < 512$. It is easy to see that out of the 40 not strongly connected orientations of $G'$, there are 24 transitive orientations and 16 non-transitive orientations. Each of these non-transitive orientation contains a directed $C_3$, and there are 7 ways to extend this directed $C_3$ to obtain a strongly connected orientation of $G$. Noting that there are 24 strongly connected orientations of $G'$, we have

$$SC(K_5) = 24 \cdot 4 + 24 \cdot (2^4 - 2) + 16 \cdot 7 = 544 > 2^{(\frac{5}{2})−1}.$$ 

Now, let $k ≥ 6$ and suppose that $SC(K_{k−1}) > 2^{(\frac{k−1}{2})−1}$. Let $\vec{G}'$ be an orientation of $G'$. If $\vec{G}'$ is strongly connected, then we obtain a strongly connected orientation of $G$ if and only if the remaining edges are oriented so that $u_k$ has in degree and out degree at least 1. Therefore, $\vec{G}'$ can be extended in precisely $2^{k−1}−2$ ways to strongly connected orientation of $G$. Thus, we may assume that $\vec{G}'$ is not strongly connected. By Fact 3.1, $G'$ contains a Hamilton path $P$. We may assume, without loss of generality, that $P = (u_1, \ldots , u_{k−1})$. By
orienting \( \{u_1, u_k\} \) towards \( u_1 \) and \( \{u_{k-1}, u_k\} \) towards \( u_k \), and orienting the edges \( \{u_i, u_k\} \) in any direction, for \( i = 1, \ldots, k - 2 \), we obtain a strongly connected orientation of \( G \). Thus, there are at least \( 2^{k-3} \) strongly connected orientations of \( G \). Therefore, we have

\[
\text{SC}(K_k) \geq (2^{k-1} - 2) \cdot \text{SC}(K_{k-1}) + 2^{k-3} \cdot (2^{k-1} - \text{SC}(K_{k-1}))
\]

\[
= (2^{k-1} - 2 - 2^{k-3}) \cdot \text{SC}(K_{k-1}) + 2^{k-3} \cdot 2^{(k-1)}
\]

\[
> 2^{k-3} \cdot 2^{(k-1)} + 2^{k-2} \cdot 2^{(k-1)} - (2^{k-2} - 2 - 2^{k-3}) \cdot \text{SC}(K_{k-1})
\]

\[
= 2^{(k-1)} - 2 - 2^{k-3} \cdot 2^{(k-1)} - 1
\]

\[
> 2^{(k-2)} - 1,
\]

which concludes the proof. \( \square \)

Given a graph \( G \), we denote by \( \mathcal{S}_k(G) \) the family of \( \mathcal{S}_k \)-free orientations of \( G \) and we write \( S_k(G) = |\mathcal{S}_k(G)| \). Combining some of the previous results, we prove that cliques of size \( k \) and \( k + 1 \) are not \( \mathcal{S}_k \)-extremal.

**Corollary 3.6.** For \( k \geq 5 \) we have \( S_k(K_k) < 2^{k-1(k)} \) and for \( k \geq 4 \) we have \( S_k(K_{k+1}) < 2^{k-1(k+1)} \).

**Proof.** We start by showing that \( S_4(K_5) < 2^{6(5)} = 2^8 \). Every non-transitive orientation of \( K_5 \) contains a \( C_3^* \). We claim that there are precisely \( \binom{5}{3} \cdot 2 \cdot 6 \) such non-transitive orientations without a strongly connected \( K_4 \). In fact, there are \( \binom{5}{3} \) triangles in a \( K_5 \) and \( 2 \) strongly connected orientations of each triangle. It is not hard to see that there are at most \( 6 \) ways to orient the edges outside the triangle to obtain an orientation of \( K_5 \) with no strongly connected \( K_4 \). Since there are \( 120 \) transitive orientations of \( K_5 \), we obtain \( S_4(K_5) \leq 120 + \binom{5}{3} \cdot 2 \cdot 6 = 240 < 2^8 \).

Thus, assume \( k \geq 5 \). Note that a strongly connected orientation of \( K_{k-1} \) can be extended in only two (resp. six) ways to an orientation of \( K_k \) (resp. \( K_{k+1} \)) that does not contain a strongly connected \( K_k \). By Lemma 3.3, every orientation of \( K_{k-1} \) can be extended in at most \( 3 \cdot 2^{k-3} \) ways to an \( \mathcal{S}_k \)-free orientation of \( K_k \), which gives

\[
S_k(K_k) \leq 2 \cdot \text{SC}(K_{k-1}) + 3 \cdot 2^{k-3} (2^{(k-1)} - \text{SC}(K_{k-1})).
\]  

(9)

To obtain an estimate for \( S_k(K_{k+1}) \) we use Lemma 3.4, which implies that every orientation of \( K_{k-1} \) can be extended in at most \( 2 \cdot 3 \cdot 2^{2k-5} \) ways to an \( \mathcal{S}_k \)-free orientation of \( K_{k+1} \). Then,

\[
S_k(K_{k+1}) \leq 6 \cdot \text{SC}(K_{k-1}) + 2 \cdot 3 \cdot 2^{2k-5} (2^{(k-1)} - \text{SC}(K_{k-1})).
\]  

(10)

If \( k = 5 \), then, by Lemma 3.5, we have \( \mathcal{S}_5(K_5) = 2^{(5)} - \text{SC}(K_5) < 2^9 = 2^{4(5)} \). From (10), since \( \text{SC}(K_4) = 24 \), we obtain \( \mathcal{S}_5(K_6) \leq 6 \cdot 24 + 2 \cdot 3 \cdot 2^5 (2^6 - 24) = 7824 < 2^{13} = 2^{4(6)} \).
Now we assume that \( k \geq 6 \). From (9), using Lemma 3.5 and \( t_{k-1}(k) = t_{k-1}(k-1) + k - 2 = \binom{k-1}{2} + k - 2 \), we have

\[
S_k(K_k) \leq 2 \cdot SC(K_{k-1}) + 3 \cdot 2^{k-3} \left( 2^{\binom{k-1}{2}} - SC(K_{k-1}) \right)
= 3 \cdot 2^{k-3} 2^{\binom{k-1}{2}} - (3 \cdot 2^{k-3} - 2) \cdot SC(K_{k-1})
< 3 \cdot 2^{k-4} 2^{\binom{k-1}{2}} + 2^{\binom{k-1}{2}}
\leq 2^{k-2} 2^{\binom{k-1}{2}} = 2^{t_{k-1}(k)}.
\]

Analogously, from (10), since \( t_{k-1}(k+1) = t_{k-1}(k-1) + 2k - 3 = \binom{k-1}{2} + 2k - 3 \) we have

\[
S_k(K_{k+1}) \leq 6 \cdot SC(K_{k-1}) + 2 \cdot 3 \cdot 2^{2k-5} \left( 2^{\binom{k-1}{2}} - SC(K_{k-1}) \right)
\leq 3 \cdot 2^{2k-5} 2^{\binom{k-1}{2}} + 3 \cdot 2^{\binom{k-1}{2}}
\leq 2^{2k-3} 2^{\binom{k-1}{2}} = 2^{t_{k-1}(k+1)},
\]

which finishes the proof.

3.1. **Proof of the main result.** For simplicity, we write \( S_k(n) \) for \( D(n, \vec{S}_k) \), i.e., \( S_k(n) = \max \{ S_k(G) : G \text{ is an } n\text{-vertex graph} \} \), which stands for the maximum number of orientations of \( n \)-vertex graphs with no strongly connected copies of \( K_k \). The following theorem is the main result of this paper.

**Theorem 3.7.** Let \( n \geq k \geq 4 \) Then,

\[
S_k(n) = \begin{cases} 
40 & \text{if } n = k = 4, \\
2^{k-1}(n) & \text{if } n \geq 5 \text{ or } k \geq 5.
\end{cases}
\]

Furthermore, \( K_4 \) is the only \( \vec{S}_4 \)-extremal graph and, if \( n \geq 5 \) or \( k \geq 5 \), then the Turán graph \( T_{k-1}(n) \) is the only \( \vec{S}_k \)-extremal graph.

Before proving Theorem 3.7, recall that Lemma 2.7 implies that every \( \vec{F} \)-extremal graph is complete multipartite for the family \( \vec{F} \) of \( k \)-vertex tournaments with no source. Therefore, to prove Theorem 3.7 (for \( n \geq 5 \) or \( k \geq 5 \)), it is enough to show that every complete multipartite \( \vec{F} \)-extremal graph is the \((k-1)\)-partite Turán graph.

**Proof of Theorem 3.7.** The proof is by induction on \( n \). First, note that if \( n < k \) then any graph \( G \) of order \( n \) is \( K_k \)-free, and hence has \( 2^{|E(G)|} \) \( S_k \)-free orientations. Therefore, \( S_k(n) = 2^{\binom{n}{2}} = 2^{k-1}(n) \).

For \( n = k = 4 \) we used a simple computer program to verify that \( S_4(4) = 40 \) and that \( K_4 \) is the only \( \vec{S}_4 \)-extremal graph. Also with a (simple) computer program we verified
that $S_4(n) = 2^{t_3(n)}$ for $5 \leq n \leq 8$ and that $T_3(n)$ is the only $\bar{S}_4$-extremal graph. Thus, we assume that either $k = 4$ and $n \geq 9$, or $n \geq k \geq 5$.

Since no tournament in $\bar{S}_k$ contains a source, Lemma 2.7 implies that every $\bar{S}_k$-extremal graph is complete multipartite. Let $G$ be an $\bar{S}_k$-extremal $r$-partite graph and assume that $r \geq k$. First, suppose that $G$ contains a clique $K$ of size $k+1$. From Corollary 3.6, we have for $k \geq 4$ that $S_k(K) < 2^{t_{k-1}(k+1)}$. Note that, since $G$ is a complete multipartite graph, if $u \notin V(K)$, then $u$ is adjacent to either $k$ or $k+1$ vertices of $K$. Thus, by Lemma 3.3, we have $c_{\bar{S}_k}(u, K) \leq 5 \cdot 2^{k-3}$. Therefore, we have

$$
S_k(G) \leq S_k(K)(5 \cdot 2^{k-3})^{n-k-1}S_k(G \setminus K) < 2^{t_{k-1}(k+1)+\log 5+k-3(n-k-1)}S_k(G \setminus K).
$$

If $k = 4$ and $n - k - 1 \neq 4$, or $k \geq 5$ then, by the induction hypothesis, we have $S_k(G \setminus K) \leq 2^{t_{k-1}(n-k-1)}$. Therefore, from (11) and Proposition 4.5 we have

$$
S_k(G) < 2^{t_{k-1}(k+1)+\log 5+k-3(n-k-1)} \cdot S_k(G \setminus K) < 2^{t_{k-1}(n)},
$$

A contradiction with the fact that $G$ is $\bar{S}_k$-extremal. Now, suppose $k = 4$ and $n - k - 1 = 4$. In this case, $S_k(G \setminus K) \leq 40$, and hence, from (11) we have $S_4(G) < 2^{t_3(5)+4(\log 5+1)} \cdot S_k(G \setminus K)$, which implies

$$
S_4(G) < 2^{8+4(\log 5+1)+\log 40} < 2^{27} = 2^{t_3(9)},
$$
a contradiction. Therefore, no $\bar{S}_k$-extremal graph contains a clique with $k+1$ vertices.

Since $G$ contains no clique with $k+1$ vertices, we may and shall assume that $G$ is $k$-partite. We first deal with the case $k \geq 5$. If $n = k$, then $G \simeq K_k$, and hence, by Corollary 3.6, we have $S_k(G) < 2^{t_{k-1}(k)}$. Thus, we may assume that $n \geq k + 1$. Let $K$ be a clique of size $k$ in $G$. Since $G$ is a complete $k$-partite graph, if $u \notin V(K)$, then $u$ is adjacent to precisely $k-1$ vertices of $K$. Thus, by Lemma 3.3, we have $c_{\bar{S}_k}(u, K) \leq 3 \cdot 2^{k-3}$. Therefore, we have

$$
S_k(G) \leq S_k(K)(3 \cdot 2^{k-3})^{n-k} \cdot S_k(G \setminus K).
$$

Clearly, if $G \setminus K$ is $(k-1)$-partite, then $S_k(G \setminus K) \leq 2^{t_{k-1}(n-k)}$; and if $G \setminus K$ is not $(k-1)$-partite, then, by the induction hypothesis, we have $S_k(G \setminus K) \leq 2^{t_{k-1}(n-k)}$. Moreover, from Corollary 3.6, we have $S_k(K) < 2^{t_{k-1}(k)}$. Therefore, by Proposition 4.6 we have
\[ S_k(G) < 2^{t_{k-1}(k)+\log 3+k-3(n-k)} \cdot S_k(G \setminus K) \]
\[ \leq 2^{t_{k-1}(k)+\log 3+k-3(n-k)+t_{k-1}(n-k)} \]
\[ < 2^{t_{k-1}(n)}, \]
which contradicts the fact that \( G \) is \( S_k \)-extremal.

It remains deal with the case for \( k = 4 \) and \( n \geq 9 \) (where \( G \) contains no clique with \( k+1 \) vertices). For that, note that removing a copy of \( K_4 \) from \( G \) results in a graph that is not a copy of \( K_4 \). Then, by arguments analogous to the ones above (for \( k \geq 5 \), replacing Proposition 4.6 with Proposition 4.7 and using that \( S_4(K_4) = 40 \), we get \( S_4(G) < 2^{t_3(n)} \), getting the desired contradiction. \( \square \)

4. Final remarks and open problems

In this paper, given a fixed family \( \mathcal{F} \) of oriented graphs and a positive integer \( n \), we consider the quantity \( D(n, \mathcal{F}) \), the maximum number of \( \mathcal{F} \)-free orientations of any \( n \)-vertex graph. Given \( k \geq 4 \) and \( \mathcal{F} \in \{ \mathcal{S}_k, \mathcal{R}_k \} \), where \( \mathcal{S}_k \) and \( \mathcal{R}_k \) are, respectively, the families of \( k \)-vertex tournaments that are strongly connected and non-transitive, we determined \( D(n, \mathcal{F}) \) and the corresponding extremal \( n \)-vertex graphs for all values of \( n \).

For \( k = 3 \), we have \( \mathcal{S}_3 = \mathcal{R}_3 = \{ C_3^\circ \} \) and Theorem 1.1 determines the value of \( D(n, \{ C_3^\circ \}) \) for all values of \( n \).

This type of extremal problem was first investigated by Alon and Yuster [1], who showed that the Turán graph \( T_{k-1}(n) \) is the unique \( \{ \tilde{K}_k \} \)-extremal graph for sufficiently large \( n \). For more than 10 years, there had been no substantial contribution to this problem, but this has changed in the past year [2, 6]. This renewed interest in this problem comes at a time of many advances in a related problem about the number of distinct edge-colorings avoiding some fixed color-patterns, known as the Erdős-Rothschild problem (see, e.g., [3, 4, 7, 8, 9]).

In the remainder of this section, we raise a few questions whose answers might substantially improve our understanding of \( D(n, \mathcal{F}) \) and of the corresponding \( n \)-vertex \( \mathcal{F} \)-extremal graphs.

Given an integer \( k \geq 3 \) and a \( k \)-vertex tournament \( \tilde{K}_k \), define \( n_0(\tilde{K}_k) \) as follows: if \( T_{k-1}(n) \) is the unique \( \tilde{K}_k \)-extremal graph for every \( n \geq 1 \), we set \( n_0(\tilde{K}_k) = 1 \). Otherwise, set \( n_0(\tilde{K}_k) = n_0 \), where \( n_0 \) is an integer such that \( T_{k-1}(n) \) is the unique \( n \)-vertex \( \tilde{K}_k \)-extremal graph for every \( n \geq n_0 \), but \( T_{k-1}(n_0 - 1) \) is not the unique \( \tilde{K}_k \)-extremal graph with \( n_0 - 1 \) vertices. For the next problem, let \( \mathcal{O}_k \) be the set of all orientations of \( \mathcal{K}_k \) and define \( n_{\max}(k) = \max\{n_0(\tilde{K}_k) : \tilde{K}_k \in \mathcal{O}_k \} \).

**Problem 4.1.** Given an integer \( k \geq 3 \), determine \( n_{\max}(k) \) and characterize the orientations \( \tilde{K}_k \) for which \( n_0(\tilde{K}_k) = n_{\max}(k) \).
For $k = 3$, the answer to Problem 4.1 is known. The only orientation of $K_3$ other than $C_3^<$ is the transitive tournament $\bar{T}_3$. It is easy to see that $n_0(\bar{T}_3) = 1$ and Theorem 1.1 shows that $n_0(C_3^>) = 8$, so that $n_{\text{max}}(3) = 8$. However, the answer to Problem 4.1 for $k \geq 4$ remains open.

**Problem 4.2.** Determine families of oriented graphs $\vec{F}$ for which all $n$-vertex $\vec{F}$-extremal graphs are complete multipartite and verify whether these graphs are necessarily balanced.

In connection with Problem 4.2, Lemma 2.5 states that if $\vec{F}$ is an arbitrary family of tournaments, then, for every $n$, there exists an $n$-vertex $\vec{F}$-extremal graph that is complete multipartite. For specific families $\vec{F}$ of tournaments, Lemma 2.7 does more and establishes that all $\vec{F}$-extremal graphs are complete multipartite. However, neither proofs ensure balancedness, even though all $\vec{F}$-extremal graphs known so far are balanced complete multipartite graphs.$^*$

For the next problem given a graph $F$, we say an $n$-vertex graph $H$ is $F$-extremal if $|E(H)|$ is maximum among all $n$-vertex graphs that do not contain a copy of $F$.

**Problem 4.3.** Determine the graphs $F$ such that there exists an orientation $\vec{F}$ with the property that, for arbitrarily large values of $n$, there are $n$-vertex $\vec{F}$-extremal graphs that are not $F$-extremal.

As mentioned before, it is known that no complete graph satisfies the property described in Problem 4.3. However, if $F$ is the complete bipartite graph $K_{1,3}$ and $\vec{F}$ is the orientation where all edges are oriented towards its leafs, it is easy to see that, for any positive integer $n$ divisible by 4, a graph given by $n/4$ disjoint copies of $K_4$ admits $32^{n/4} = 2^{5n/4}$ distinct $\vec{F}$-free copies, while any $K_{1,3}$-extremal graph (which has maximum degree at most 2) admits at most $2^n$ orientations.

We present now one final open problem concerning the uniqueness of extremal graphs.

**Problem 4.4.** Determine the families of orientations $\vec{F}$ with the property that, for all $n \geq 1$, there is a unique $n$-vertex $\vec{F}$-extremal graph.

We have seen that the families $\vec{S}_k$ and $\vec{R}_k$ satisfy the property described in Problem 4.4. As a consequence of the proof of [1, Theorem 1.1], at least for large $n$, any family $\vec{F}$ of tournaments admits a unique $n$-vertex $\vec{F}$-extremal graph. However, this does not necessarily extend to all values of $n$. For instance, consider the family $\mathcal{U}_4$ of all orientations of $K_4$ that have no source. Then $K_4$ and $T_3(4)$ both admit 32 distinct $\mathcal{U}_4$-free orientations, and are $\mathcal{U}_4$-extremal graphs on four vertices. Since $\vec{S}_4 \subset \mathcal{U}_4$, Theorem 3.7 implies that $T_3(n)$ is the unique $n$-vertex $\mathcal{U}_4$-extremal graph for all $n \geq 5$.

$^*$At least in situations where the (Turán) extremal graphs of the graphs whose orientations are in $\vec{F}$ are complete multipartite graphs.
Proposition 4.5. Let

\[
\text{Proof.} \quad \text{First, let } n > k \geq 4. \text{ Then }
\]

\[
t_{k-1}(k+1) + (\log 5 + k - 3)(n - k - 1) + t_{k-1}(n - k - 1) < t_{k-1}(n).
\]

This, we have

\[
t_{k-1}(k+1) + (\log 5 + k - 3)(n - k - 1) + t_{k-1}(n - k - 1)
\]

\[
= \binom{k+1}{2} - 2 + (k+1)(n-k-1) + (\log 5 - 4)(n-k-1)
\]

\[
+ \binom{n-k-1}{2} - r'(\binom{q'+1}{2}) - (k-1 - r')\binom{q'}{2}
\]

\[
= \binom{n}{2} - 2 + (\log 5 - 4)(n-k-1) - r'(\binom{q'+1}{2}) - (k-1 - r')\binom{q'}{2}
\]

\[
= \binom{n}{2} - 2 + (\log 5 - 3)(n-k-1) - r'(\binom{q'+2}{2}) - (k-1 - r')\binom{q'+1}{2},
\]

where in the last equality we used that \(n-k-1 = r'(q'+1) + (k-1 - r')q'.\)
Since $k \geq 4$, we have

\[
\begin{align*}
\log 5 - 3(n - k - 1) &= (\log 5 - 3)((k - 1)q' + r') \\
&\leq (\log 5 - 3)(k - 1)q' \\
&\leq (\log 5 - 3)3q' \\
&< -2q',
\end{align*}
\]

and hence,

\[
\begin{align*}
\binom{n}{2} - 2 + (\log 5 - 3)(n - k - 1) - r'\left(\frac{q' + 2}{2}\right) - (k - 1 - r')\left(\frac{q' + 1}{2}\right) \\
< \binom{n}{2} - 2 - 2q' - r'\left(\frac{q' + 2}{2}\right) - (k - 1 - r')\left(\frac{q' + 1}{2}\right)
\end{align*}
\]

In what follows, we divide the proof according to $(q', r')$. If $(q', r') = (q - 1, r - 2)$, then $r \geq 2$ and we have

\[
\begin{align*}
\binom{n}{2} - 2 - 2q' - r'\left(\frac{q' + 2}{2}\right) - (k - 1 - r')\left(\frac{q' + 1}{2}\right) \\
&= \binom{n}{2} - (r' + 2)\left(\frac{q' + 2}{2}\right) - (k - 1 - r')\left(\frac{q' + 1}{2}\right) \\
&= \binom{n}{2} - r\left(\frac{q + 1}{2}\right) - (k - 1 - r)\left(\frac{q}{2}\right) \\
&= t_{k-1}(n).
\end{align*}
\]

If $(q', r') = (q - 2, k - 3)$, then $r = 0$ and we have

\[
\begin{align*}
\binom{n}{2} - 2 - 2q' - r'\left(\frac{q' + 2}{2}\right) - (k - 1 - r')\left(\frac{q' + 1}{2}\right) \\
&= \binom{n}{2} - 2 - 2q' - (k - 3)\left(\frac{q}{2}\right) - 2\left(\frac{q - 1}{2}\right) \\
&= \binom{n}{2} - (k - 1)\left(\frac{q}{2}\right) \\
&= t_{k-1}(n).
\end{align*}
\]

If $(q', r') = (q - 2, k - 2)$, then $r = 1$ and we obtain the following variation of (12).

\[
\begin{align*}
\log 5 - 3(n - k - 1) &= (\log 5 - 3)((k - 1)q' + r') \\
&\leq (\log 5 - 3)((k - 1)q' + (k - 2)) \\
&\leq (\log 5 - 3)(3q' + 2) < -2q' - 1
\end{align*}
\]
Therefore,
\[
\binom{n}{2} - 2 + (\log 5 - 3)(n - k - 1) - r'(\frac{q' + 2}{2}) - (k - 1 - r')(\frac{q' + 1}{2})
\]
\[
< \binom{n}{2} - 2 - 2q' - 1 - r'(\frac{q' + 2}{2}) - (k - 1 - r')(\frac{q' + 1}{2})
\]
\[
= \binom{n}{2} - 3 - 2q' - (k - 2)\left(\frac{q}{2}\right) - \left(\frac{q - 1}{2}\right)
\]
\[
= \binom{n}{2} + 1 - 2q' - (k - 2)\left(\frac{q}{2}\right) - \left(\frac{q - 1}{2}\right)
\]
\[
= \binom{n}{2} - (k - 1 - r)\left(\frac{q}{2}\right) - r\left(\frac{q + 1}{2}\right)
\]
\[
= t_{k-1}(n).
\]

\[\square\]

**Proposition 4.6.** Let \(n\) and \(k\) be positive integers such that \(n > k \geq 4\). Then
\[
t_{k-1}(k) + (\log 3 + k - 3)(n - k) + t_{k-1}(n - k) < t_{k-1}(n).
\]

**Proof.** First, let \(q\) and \(r\) be such that \(n = (k - 1)q + r\), where \(0 \leq r < k - 1\), and let \(q'\) and \(r'\) be such that \(n - k = (k - 1)q' + r'\), where \(0 \leq r' < k - 1\). Note that \(r' \equiv r - 1 \mod (k - 1)\), and hence \((q', r') \in \{(q - 2, k - 2), (q - 1, r - 1)\}\), where \(r' = k - 2\) if and only if \(r = 0\). Note that \(t_{k-1}(k) = \binom{k}{2} - 1\) and \(t_{k-1}(n - k) = \binom{n-k}{2} - r'(\frac{q' + 1}{2}) - (k - 1 - r')(\frac{q'}{2})\).

Thus, we have
\[
t_{k-1}(k) + (\log 3 + k - 3)(n - k) + t_{k-1}(n - k)
\]
\[
= \binom{k}{2} - 1 + k(n - k) + (\log 3 - 3)(n - k) + \binom{n - k}{2} - r'(\frac{q' + 1}{2}) - (k - 1 - r')(\frac{q'}{2})
\]
\[
= \binom{n}{2} - 1 + (\log 3 - 3)(n - k) - r'(\frac{q' + 1}{2}) - (k - 1 - r')(\frac{q'}{2})
\]

Since \(n - k = r'(q' + 1) + (k - 1 - r')q'\), we have
\[
\binom{n}{2} - 1 + (\log 3 - 3)(n - k) - r'(\frac{q' + 1}{2}) - (k - 1 - r')(\frac{q'}{2})
\]
\[
= \binom{n}{2} - 1 + (\log 3 - 2)(n - k) - r'(\frac{q' + 2}{2}) - (k - 1 - r')(\frac{q' + 1}{2})
\]

Since \(k \geq 4\), we have
\[
(\log 3 - 2)(n - k) = (\log 3 - 2)((k - 1)q' + r') \leq (\log 3 - 2)(k - 1)q' \leq (\log 3 - 2)3q' < -q',
\]

In what follows, we divide the proof according to \((q', r')\). If \((q', r') = (q - 1, r - 1)\), then \(r \geq 1\) and we have

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\[
\binom{n}{2} - 1 + (\log 3 - 2)(n - k) - r'(\binom{q' + 2}{2}) - (k - 1 - r')(\binom{q' + 1}{2})
< \binom{n}{2} - 1 - q' - r'(\binom{q' + 2}{2}) - (k - 1 - r')(\binom{q' + 1}{2})
= \binom{n}{2} - (r' + 1)(\binom{q' + 2}{2}) - (k - 1 - r' - 1)(\binom{q' + 1}{2})
= \binom{n}{2} - r(\binom{q + 1}{2}) - (k - 1 - r)(\binom{q}{2})
= t_{k-1}(n).
\]

If \((q', r') = (q - 2, k - 2)\), then \(r = 0\) and we have
\[
\binom{n}{2} - 1 - q' - r'(\binom{q' + 2}{2}) - (k - 1 - r')(\binom{q' + 1}{2})
= \binom{n}{2} - 1 - q - (k - 2)(\binom{q' + 2}{2}) - (\binom{q' + 1}{2})
= \binom{n}{2} - (k - 1)(\binom{q}{2})
= t_{k-1}(n).
\]

\[
\text{Proposition 4.7. Let } n \text{ be a positive integer such that } n \geq 9. \text{ Then}
\log 40 + (\log 3 + 1)(n - 4) + t_3(n - 4) < t_3(n).
\]

\[
\text{Proof. First, let } q \text{ and } r \text{ be such that } n = 3q + r, \text{ where } 0 \leq r < 3, \text{ and let } q' \text{ and } r'
\text{ be such that } n - 4 = 3q' + r', \text{ where } 0 \leq r' < 3. \text{ Note that } r' \equiv r - 1 \pmod{3}, \text{ and hence }
(q', r') \in \{(q - 2, 2), (q - 1, r - 1)\}, \text{ where } r' = 2 \text{ if and only if } r = 0. \text{ Note that }
t_3(n - 4) = \binom{n - 4}{2} - r'(\binom{q' + 1}{2}) - (3 - r')(\binom{q'}{2}). \text{ Thus, we have}
\log 40 + (\log 3 + 1)(n - 4) + t_3(n - 4)
= 6 + (\log 40 - 6) + (\log 3 - 3)(n - 4) + 4(n - 4) + \binom{n - 4}{2} - r'(\binom{q' + 1}{2}) - (3 - r')(\binom{q'}{2})
= \binom{n}{2} + (\log 40 - 6) + (\log 3 - 3)(n - 4) - r'(\binom{q' + 1}{2}) - (3 - r')(\binom{q'}{2})
\text{Since } n - 4 = r'(q' + 1) + (3 - r')q', \text{ we have}
\]
\[
\binom{n}{2} + (\log 40 - 6) + (\log 3 - 3)(n - 4) - r'\left(\binom{q' + 1}{2}\right) - (3 - r')\left(\binom{q'}{2}\right)
\]
\[
= \binom{n}{2} + (\log 40 - 6) + (\log 3 - 2)(n - 4) - r'\left(\binom{q' + 2}{2}\right) - (3 - r')\left(\binom{q' + 1}{2}\right)
\]
It is not hard to check that, since \(n \geq 9\), we have \(r' > 0\) or \(q' \geq 2\), and hence

\[
(\log 40 - 6) + (\log 3 - 2)(n - 4) = (\log 40 - 6) + (\log 3 - 2)(3q' + r') < -q' - 1,
\]
In what follows, we divide the proof according to \((q', r')\). If \((q', r') = (q - 1, r - 1)\), then \(r \geq 1\) and we have

\[
\binom{n}{2} + (\log 40 - 6) + (\log 3 - 2)(n - 4) - r'\left(\binom{q' + 2}{2}\right) - (3 - r')\left(\binom{q' + 1}{2}\right)
\]
\[
< \binom{n}{2} - 1 - q' - r'\left(\binom{q' + 2}{2}\right) - (3 - r')\left(\binom{q' + 1}{2}\right)
\]
\[
= \binom{n}{2} - (r' + 1)\left(\binom{q' + 2}{2}\right) - (2 - r')\left(\binom{q' + 1}{2}\right)
\]
\[
= \binom{n}{2} - r\left(\binom{q + 1}{2}\right) - (3 - r)\left(\binom{q}{2}\right)
\]
\[
= t_{k-1}(n).
\]
If \((q', r') = (q - 2, 2)\), then \(r = 0\) and we have

\[
\binom{n}{2} - (r' + 1)\left(\binom{q' + 2}{2}\right) - (2 - r')\left(\binom{q' + 1}{2}\right)
\]
\[
= \binom{n}{2} - 3\left(\binom{q' + 2}{2}\right)
\]
\[
= \binom{n}{2} - 3\left(\binom{q}{2}\right)
\]
\[
= t_{k-1}(n).
\]

\[\square\]
