Orthogonal Stability and Nonstability of a Generalized Quartic Functional Equation in Quasi-\(\beta\)-Normed Spaces

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In this work, we examine the generalized Hyers-Ulam orthogonal stability of the quartic functional equation in quasi-\(\beta\)-normed spaces. Moreover, we prove that this functional equation is not stable in a special condition by a counterexample.

1. Introduction

In this paper, \(\mathbb{R}\) and \(\mathbb{C}\) denote sets of all real numbers and complex numbers, respectively. In the fall of 1940, Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows:

Ulam’s question: let \((G_1, \ast), (G_2, \star)\) be two groups and \(d : G_2 \times G_2 \to [0, \infty)\) be a metric. Given \(\delta > 0\), does there exist \(\varepsilon > 0\) such that if a function \(g : G_1 \to G_2\) satisfies the inequality

\[
d(g(x \ast y), g(x) \star g(y)) \leq \delta,
\]

for all \(x, y \in G_1\), then there is a homomorphism \(h : G_1 \to G_2\) with

\[
d(g(x), h(x)) \leq \varepsilon \text{ for all } x \in G_1.
\]

In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. This result was generalized by Aoki [3] for additive mappings.

During the past few years, several mathematicians have published on various generalizations and applications of generalized Hyers-Ulam stability to a number of functional equations and mappings (see, for instance, [4–15]).

In [16], Xu et al. obtained the general solution and investigated the Ulam stability problem for the quintic functional equation in quasi-\(\beta\)-normed spaces via fixed point method. This method is different from the direct method, initiated by Hyers in [2]. And also, Eskandani et al. [17, 18] obtained the general solution for the mixed additive and quadratic functional equation and a cubic functional equation and established its generalized Hyers-Ulam stability in quasi-\(\beta\)-normed spaces.

The Ulam-type stability result for the quartic functional equation

\[
F(x_1 + 2x_2) + F(x_1 - 2x_2) + 6F(x_1) = 4[F(x_1 + x_2) + F(x_1 - x_2) + 6F(x_2)],
\]
was first developed by Rassias [19]. Subsequently, Sahoo and Chung [20] determined the general solution of (3) without assuming any regularity conditions on the unknown function. In fact, they proved that the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a solution of (3) if and only if \( f(x) = A(x, x, x, x) \), where the function \( A : \mathbb{R}^4 \rightarrow \mathbb{R} \) is symmetric and additive in each variable. Since the solution of (3) is even, we can rewrite (3) as

\[
f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y).
\]

(4)

Lee et al. [21] obtained the general solution of (4) and proved the Hyers-Ulam-Rassias stability of this equation. It is easy to show that the function \( f(x) = x^4 \) satisfies the functional equation (4), which is called a quartic functional equation, and every solution of the quartic functional equation is said to be a quartic function. In [22] Ravi et al. have investigated the generalized Hyers-Ulam product-sum stability of functional equations and have the following theorem.

**Theorem 1.** Let \( f : E \rightarrow F \) be a mapping which satisfies the inequality

\[
\|f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2(m^2 - 2)f(x) + 2f(y)\|_F \\
\leq \epsilon \left( \|x\|_E^p \|y\|_E^p + \|x\|_E^2 + \|y\|_E^2 \right),
\]

(5)

for all \( x, y \in E \) with \( x \perp y \), where \( \epsilon \) and \( p \) are constants with \( \epsilon > 0 \) and either \( m > 1 \), \( p < 1 \) or \( m < 1 \), \( p > 1 \) with \( m \neq 0 \), \( m \neq \pm 1 \), \( m \neq \pm \sqrt{2} \), and \( -1 \neq |m|^{p-1} < 1 \). Then, the limit \( \lim_{m \to \infty} m^{-2p}f(mx) \) exists for all \( x \in E \), and \( Q : E \rightarrow F \) is the unique orthogonally Euler-Lagrange quadratic mapping such that

\[
\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2|m^2 - m^2p|} \|x\|_E^{2p},
\]

(6)

for all \( x \in E \).

In 1982, Rassias [23] provided generalizations of the Hyers-Ulam stability theorem which allows the Cauchy difference controlled by a product of different powers of norm. And then, the result of the Rassias theorem has been generalized by Gavruta [24] by replacing the unbounded Cauchy difference by a generalized control function. Also, Rassias (see [23, 25–28]) solved the Ulam problem for different mappings. In addition, Ravi et al. considered the mixed product-sum of powers of norms control function [22]. Note that the mixed product-sum function was introduced by Ravi et al. in 2008-2009 ([22, 29–31]).

In this paper, we examine the generalized Hyers-Ulam orthogonal stability of the quartic functional equation as

\[
\phi \left( \sum_{d=1}^{m} v_d \right) = \sum_{1 \leq a < b < c < d \leq m} \phi(v_a + v_b + v_c + v_d) + (-m + 4) \sum_{i=1}^{m} \phi(v_i) + \frac{(m^2 - 7m + 12)}{2} \sum_{1 \leq a < b} \phi(v_a + v_b) \\
\hspace{1cm} + \frac{3}{6} \sum_{a=1}^{m} \phi(2v_a) - \frac{m^3 + 9m^2 - 26m + 120}{2} \sum_{a=1}^{m} \phi(v_a) + \frac{\phi(-v_a)}{2},
\]

(7)

where \( m \) is a positive integer with \( \mathbb{N} \setminus \{0, 1, 2, 3, 4\} \). It is easy to see that the function \( \phi(v) = av^4 \) is a solution of the functional equation (7).

2. Orthogonal Hyers-Ulam Stability

**Lemma 2** (see [32]. Let \( E \) and \( F \) be real vector spaces. If the mapping \( \phi : E \rightarrow F \) satisfies the functional equation (7) for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \perp v_j ; i \neq j, 1, 2, \ldots, m \), then \( \phi \) is quartic.

**Remark 3.** Let \( E \) be a linear space and \( \phi : \mathbb{R} \rightarrow F \) be a function satisfies (7). Then, the following two assertions hold:

1. \( \phi(r^4v) = r^4\phi(v) \) for all \( v \in \mathbb{R} \) and \( r \in \mathbb{Q} \) and \( k \) integers.
2. \( \phi(v) = v^4\phi(1) \) for all \( v \in \mathbb{R} \) if \( \phi \) is continuous.

Here, let us consider \( E \) to be a linear space over \( \mathbb{F} \) and \( F \) is a \((\beta, p)\)-Banach space with \( p\)-norm \( \|\cdot\|_F \).

Let \( K \) be the modulus concavity of \( \|\cdot\|_F \).

For our convenience, we use the abbreviation for a function \( \phi : E \rightarrow F \):

\[
\Delta \phi(v_1, v_2, \ldots, v_m) = \phi \left( \sum_{1 \leq a < b < c < d \leq m} v_a + v_b + v_c + v_d \right) \\
\hspace{1cm} - \sum_{1 \leq a < b < c < d \leq m} \phi(v_a + v_b + v_c + v_d) \\
\hspace{1cm} - \sum_{1 \leq a < b} \phi(v_a + v_b) \\
\hspace{1cm} + \sum_{a=1}^{m} \phi(2v_a) - \frac{m^3 + 9m^2 - 26m + 120}{2} \sum_{a=1}^{m} \phi(v_a) + \frac{\phi(-v_a)}{2},
\]

(8)

for all \( v_1, v_2, \ldots, v_m \in E \).
Theorem 4. Let a function \( \phi : E \to F \) which there exists \( \psi : E^m \to [0,\infty) \) such that

\[
\|\Delta \phi(v_1, v_2, \ldots, v_m)\|_F \leq \psi(v_1, v_2, \ldots, v_m), \quad v_1, v_2, \ldots, v_m \in E,
\]

(9)

with \( v_i \neq v_j, i \neq j \in 1, 2, \ldots, m \), and the contractively subadditive function \( \psi \) and a constant \( L \) fulfilling \( 2^{(1-\beta)}L < 1 \). Then, there exists a unique mapping \( Q_4 : E \to F \) which is quartic such that

\[
\|\phi(v) - Q_4(v)\|_F \leq \frac{K}{\sqrt{2^{4\beta} - (2L)^p}} \psi(v, 0, \ldots, 0),
\]

(10)

for all \( v \in E \).

Proof. Setting \((v_1, v_2, \ldots, v_m)\) by \((v, 0, \ldots, 0)\) in (9), we have

\[
\|\phi(2v) - 2^4 \phi(v)\|_F \leq \psi(v, 0, \ldots, 0),
\]

(11)

for all \( v \in E \). Replacing \( v \) in (11) by \( 2^mv \) and dividing by \( 2^{4(m+1)} \) in (11) we attain

\[
\|\phi(2^{m+1}v) - \phi(2^mv)\|_F \leq \frac{K}{2^{4(m+1)} \beta} \psi(2^mv, 0, \ldots, 0), \quad v \in E, m > 0.
\]

(12)

We have

\[
\|\phi(2^{m+1}v) - \phi(2^mv)\|_F^p \leq \sum_{i=1}^m \|\phi(2^{m+1}v) - \phi(2^mv)\|_F^p \leq \sum_{i=1}^m \frac{K}{2^{4(m+1)\beta}} \psi^p(2^mv, 0, \ldots, 0) \leq K \psi^p(v, 0, \ldots, 0) \left(2^{4m} \right)^{\beta p}.
\]

(13)

for all \( v \in E \) and \( m \geq i > 0 \). Clearly, \( F \) is complete, the Cauchy sequence \( \{\phi(2^mv)/2^m\} \) converges for every \( v \in E \). Next, we define a mapping \( Q_4 : E \to F \) by

\[
Q_4(v) = \lim_{m \to \infty} \frac{\phi(2^mv)}{2^m},
\]

(14)

for all \( v \in E \). Letting \( i = 0 \) and taking \( m \to \infty \) in (13), we obtain (10). Next, we want to prove that \( Q_4 \) is quartic. From (9) and (14) that

\[
K \Delta Q_4(v_1, v_2, \ldots, v_m)\|_F \leq \lim_{m \to \infty} \frac{K \Delta \phi(2^mv_1, 2^mv_2, \ldots, 2^mv_m)}{2^m}\|_F \leq \lim_{m \to \infty} K \frac{\psi(2^mv_1, 2^mv_2, \ldots, 2^mv_m)}{2^{4mp}} \psi(2^mv_1, 2^mv_2, \ldots, 2^mv_m) \leq \lim_{m \to \infty} K (2^{4m})^{\beta p} \psi^p(v_1, v_2, \ldots, v_m) = 0,
\]

(15)

for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \neq v_j, i \neq j \in 1, 2, \ldots, m \). Therefore, by Lemma 2, we conclude that \( Q_4 \) is quartic. Next, to show that the function \( Q_4 \) is unique.

Let us consider another quartic function \( R_4 : E \to F \) which fulfills the inequality (10) we get

\[
\|Q_4(v) - R_4(v)\|_F = \lim_{m \to \infty} \frac{1}{2^{4mp}} \|\phi(2^mv) - R_4(2^mv)\|_F \leq \lim_{m \to \infty} K \psi^p(v, 0, \ldots, 0) \left(2^{4m} \right)^{-\beta p} \leq \lim_{m \to \infty} K \psi^p(v, 0, \ldots, 0).
\]

(16)

This shows that \( Q_4 = R_4 \); therefore, \( Q_4 \) is unique mapping. This ends the proof of the theorem.

Corollary 5. If \( \beta = 1 \) and \( \tau \) be a positive real number and a function \( \phi : E \to F \) for which

\[
\|\Delta \phi(v_1, v_2, \ldots, v_m)\|_F \leq \tau,
\]

(17)

for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \neq v_j, i \neq j \in 1, 2, \ldots, m \). Then, there exists \( Q_4 : E \to F \) which is a unique quartic mapping that fulfills

\[
\|\phi(v) - Q_4(v)\|_F \leq \frac{K \tau}{\sqrt{2^{4\beta} - (2L)^p}}, \quad v \in E.
\]

(18)

The following theorem is obtained by replacing the expansive superadditive instead of the contractive subadditive in Theorem 4.

Theorem 6. Let a function \( \phi : E \to F \) in which exists a mapping \( \psi : E^m \to [0,\infty) \) such that

\[
\|\Delta \phi(v_1, v_2, \ldots, v_m)\|_F \leq \psi(v_1, v_2, \ldots, v_m),
\]

(19)

for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \neq v_j, i \neq j \in 1, 2, \ldots, m \), and the expansively superadditive function \( \psi \) and a constant \( L \) fulfilling \( 2^{4(\beta-1)}L < 1 \). Then, there exists a unique mapping \( Q_4 : E \to F \) which is quartic which fulfills
\[ \| \phi(v) - Q_4(v) \|_F \leq \frac{KL}{\sqrt{2^p - (2^p L)^p}} \psi(v, 0, \ldots, 0), \quad (20) \]

for all \( v \in E \).

With the upcoming theorems, we establish the stability of the equation \( (7) \) by using an idea of Gavrutina in [24].

**Theorem 7.** Let a mapping \( \psi : E^m \to [0, \infty) \) such that
\[ \lim_{m \to \infty} \frac{1}{2^m} \psi(2^m v_1, 2^m v_2, \ldots, 2^m v_m) = 0, \quad (21) \]
for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \perp v_j, i \neq j = 1, 2, \ldots, m \), and
\[ \tilde{\psi}_{Q_4}(v) = \sum_{a=0}^{\infty} \frac{K}{2^{a p}} \psi(2^a v, 0, \ldots, 0) < \infty, \quad v \in E. \quad (22) \]

If \( \phi : E \to F \) is a mapping which fulfills
\[ \| \Delta \phi(v_1, v_2, \ldots, v_m) \|_F \leq \psi(v_1, v_2, \ldots, v_m), \quad v_1, v_2, \ldots, v_m \in E, \quad (23) \]
with \( v_i \perp v_j, i \neq j = 1, 2, \ldots, m \), then there exists a unique mapping \( Q_4 : E \to F \) which is quartic which satisfies
\[ \| \phi(v) - Q_4(v) \|_F \leq \frac{K}{2^{4 p}} \psi_{Q_4}(v)^{1/p}, \quad (24) \]
for all \( v \in E \).

**Proof.** From equation \( (11) \) in Theorem 4, we get
\[ \| \phi(2v) - 2^4 \phi(v) \| \leq \psi(v, 0, \ldots, 0), \quad v \in E. \quad (25) \]
Replacing \( v \) through \( 2^m v \) in inequality \( (25) \) and dividing by \( 2^m v \), we obtain
\[ \| \phi(2^m v) - 2^m \phi(v) \|_{2^m} \leq \frac{K}{2^{4 p}} \psi(2^{m} v, 0, \ldots, 0), \quad v \in E, m > 0. \quad (26) \]

Already, we know that \( F \) is a \((\beta, p)\)-Banach space; we obtain
\[ \| \phi(2^{m+1} v) - \phi(v) \|_{2^{m+1}} \leq \frac{K}{2^{4 p}} \psi(2^{m+1} v, 0, \ldots, 0), \quad (27) \]
for all \( v \in E \) with \( m \geq i > 0 \). From inequalities \( (22) \) and \( (27) \) that the sequence \( \{ \phi(2^{m} v)/2^{4 m} \} \) is Cauchy in \( F \) for every \( v \in E \). We know that if \( F \) is complete, the sequence \( \{ \phi(2^{m} v)/2^{4 m} \} \) converges for every \( v \in E \). Now, we can define a map-

\[ Q_4(v) := \lim_{m \to \infty} \phi(2^m v), \]

for all \( v \in E \). Letting \( i = 0 \) and taking \( m \to \infty \) in \( (27) \), we obtain the result \( (24) \). The remaining proof is the same as the proof of Theorem 4.

**Theorem 8.** Let \( \psi : E^m \to [0, \infty) \) be a mapping such that
\[ \lim_{m \to \infty} 2^m \psi(2^m v_1, 2^m v_2, \ldots, 2^m v_m) = 0, \quad v_1, v_2, \ldots, v_m \in E, \quad (29) \]
with \( v_i \perp v_j, i \neq j = 1, 2, \ldots, m \), and
\[ \psi_Q(v, 0, \ldots, 0) = \sum_{a=0}^{\infty} 2^{4 a p} \psi(2^a v, 0, \ldots, 0) < \infty, \quad (30) \]
for all \( v \in E \). If \( \phi : E \to F \) fulfills
\[ \| \Delta \phi(v_1, v_2, \ldots, v_m) \|_F \leq \psi(v_1, v_2, \ldots, v_m), \quad v_1, v_2, \ldots, v_m \in E, \quad (31) \]
with \( v_i \perp v_j, i \neq j = 1, 2, \ldots, m \). Then, there exists a unique function \( Q_4 : E \to F \) which is quartic which fulfills
\[ \| \phi(v) - Q_4(v) \|_F \leq \frac{K}{2^4 p} \psi(v, 0, \ldots, 0), \quad v \in E. \quad (32) \]

**Proof.** From equation \( (11) \), we get
\[ \| \phi(2v) - 2^4 \phi(v) \| \leq \psi(v, 0, \ldots, 0), \quad v \in E. \quad (33) \]
Setting \( v \) by \( v/2^{m+1} \) in \( (33) \) and multiply by \( 2^4 m \), we have
\[ \| 2^m \phi(v/2^m) - 2^m \phi(v) \|_F \leq \psi(v/2^m, 0, \ldots, 0), \quad v \in E, m > 0, \quad (34) \]
we have
\[ \| 2^m \phi(v/2^{m+1}) - 2^m \phi(v/2^{m+1}) \|_F \leq \sum_{i=m}^{\infty} 2^{4 i p} \psi(v/2^{m+1}, 0, \ldots, 0), \quad v \in E, m \geq i > 0. \quad (35) \]

Then, we conclude from \( (42) \) and \( (34) \) that the sequence \( \{ 2^m \phi(v/2^m) \} \) is Cauchy in \( F \) for every \( v \in E \).

As \( F \) is complete, the sequence \( \{ 2^m \phi(v/2^m) \} \) converges for every \( v \in E \). Next, we define a mapping \( Q_4 : E \to F \) by
\[ Q_4(v) := \lim_{m \to \infty} 2^m \phi(v/2^m), \quad (36) \]
for all \( v \in E \). Letting \( i = 0 \) and taking \( m \to \infty \) in \( (34) \), we
obtain (32). The remaining proof is the same as the proof of Theorem 4.

**Corollary 9.** Let \( s, t \) be the positive real numbers such that \( s + t < 4\beta \) or \( s + t > 4 \). If a mapping \( \phi : E \to F \) satisfies the inequality

\[
\| \Delta \phi(v_1, v_2, \ldots, v_m) \|_F \leq \prod_{a=1}^{m} \| v_a \|_E^{(s+t)} + \sum_{a=1}^{m} \| v_a \|_E^{m(s+t)},
\]

(37)

for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \neq v_j, i \neq j = 1, 2, \ldots, m \), then there exists a unique quartic mapping \( Q_4 : E \to F \) which satisfies

\[
\| \phi(v) - Q_4(v) \|_F \leq \frac{K\|v\|^{(s+t)}}{\sqrt{12^{(m+1)} - 2^{(m+3)}t}}.
\]

(38)

for all \( v \in E \).

**Corollary 10.** Let \( s, t \) be the positive real numbers such that \( s + t < 4\beta \) or \( s + t > 4 \). If a mapping \( \phi : E \to F \) satisfies the inequality

\[
\| \Delta \phi(v_1, v_2, \ldots, v_m) \|_F \leq \prod_{a=1}^{m} \| v_a \|_E^{(s+t)} + \sum_{a=1}^{m} \| v_a \|_E^{m(s+t)},
\]

(39)

for all \( v_1, v_2, \ldots, v_m \in E \) with \( v_i \neq v_j, i \neq j = 1, 2, \ldots, m \), then the mapping \( \phi : E \to F \) is quartic.

### 3. Counterexample

Here, we proved the nonstability of equation (7) in a special condition by a counterexample which is a modified idea of Gajda [9].

**Example 11.** Let a mapping \( \phi : \mathbb{R} \to \mathbb{R} \) defined by

\[
\phi(v) = \sum_{m=0}^{\infty} \chi(2^m v^2), \quad \text{where}
\]

\[
\chi(v) = \begin{cases} 
\Theta v^4, & -1 < v < 1, \\
\Theta, & \text{otherwise},
\end{cases}
\]

then the function \( \phi : \mathbb{R} \to \mathbb{R} \) fulfills

\[
\| \Delta \phi(v_1, v_2, \ldots, v_m) \| \leq \left( \frac{-m^3 + 12m^2 - 53m + 198}{6} \right) \left( \frac{4096}{15} \right) \sum_{a=1}^{m} |v_a|^4,
\]

(42)

for all \( v_1, v_2, \ldots, v_m \in \mathbb{R} \), but there does not exist a quartic mapping \( Q_4 : \mathbb{R} \to \mathbb{R} \) such that

\[
\| \phi(v) - Q_4(v) \| \leq \varepsilon |v|^4, \quad v \in \mathbb{R},
\]

(43)

where \( \Theta \) and \( \varepsilon \) are constants.

**Proof.** Clearly, \( \phi \) is bounded by \((16/15)\Theta\) on \( \mathbb{R} \). If \( \sum_{a=1}^{m} |v_a|^4 \geq 1/2^4 \) or 0, then the left side of (29) is less than \(((−m^3 + 12m^2 - 53m + 198)/6)\Theta\), and thus, (29) is true.

Next, we assume that

\[
0 < \sum_{a=1}^{m} |v_a|^4 < \frac{1}{2^4},
\]

then there exists an integer \( i \) such that

\[
\sum_{a=1}^{m} |v_a|^4 < \frac{1}{2^4(1+i)}.
\]

So that \( 2^i |v_1| < 1/2, 2^i |v_2| < 1/2, \ldots, 2^i |v_m| < 1/2 \) and \( 2^m v_1, 2^m v_2, \ldots, 2^m v_m \in (-1, 1) \) for every \( m = 0, 1, 2, \ldots, i-1 \). For \( m = 0, 1, 2, \ldots, i-1 \),

\[
\chi \left( \sum_{a=1}^{m} 2^m v_a \right) - \sum_{1 \leq a < b < c < d \leq m} \chi(2^m (v_a + v_b + v_c + v_d))
\]

\[
- \left( -m + 4 \right) \sum_{1 \leq a < b < c < d \leq m} \chi(2^m (v_a + v_b + v_c + v_d))
\]

\[
- \left( m^2 - 7m + 12 \right) \sum_{a=1}^{m} \chi(2^m (v_a + v_b + v_c))
\]

\[
\sum_{a=1}^{m} \chi(2^m v_a) - \chi(2^m v_a)
\]

\[
= 0.
\]

(46)
\[
\sum_{n=1}^{m} \left( \frac{\lambda(2^{n} v_{n}) + \lambda(-2^{n} v_{n})}{2} \right) \\
\leq \sum_{n=1}^{m} \left( \frac{n^{3} + 12m - 53m + 198}{6} \right) \Theta \\
\leq \left( \frac{n^{3} + 12m - 53m + 198}{6} \right) \frac{2^{(1-i)}}{15} \Theta.
\]

(47)

It follows from (43) that

\[
|\Delta \phi(v_{1}, v_{2}, \ldots, v_{m})| \leq \left( \frac{n^{3} + 12m - 53m + 198}{6} \right) \frac{4096}{15} \Theta \left( \sum_{n=1}^{m} |v_{n}|^{4} \right),
\]

(48)

for all \( v_{1}, v_{2}, \ldots, v_{m} \in \mathbb{R} \). Thus, \( \phi \) satisfies (29) for all \( v_{1}, v_{2}, \ldots, v_{m} \in \mathbb{R} \) with \( v_{1}, v_{2}, \ldots, v_{m} \in \mathbb{R} \) with \( v_{1} \neq v_{2}, \ldots, v_{m} \).

Assume that there is a contrary mapping \( Q_{4} : \mathbb{R} \rightarrow \mathbb{R} \)

which is quartic which fulfills (42). We know that, for every \( \lambda \in \mathbb{R} \), \( \lambda \) is bounded and continuous and \( Q_{4} \) is bounded on any open interval containing the origin which is continuous at the origin.

In the view of Remark 3, \( Q_{4} \) must be \( Q_{4}(\lambda) = \lambda v^{4}, \lambda \in \mathbb{R} \). Thus, we have

\[
|\phi(\lambda) - (\lambda |a|) \lambda v^{4}| = \epsilon |a| \lambda v^{4}, \quad \lambda \in \mathbb{R}.
\]

(49)

But we can select an integer \( i \geq 0 \) with \( i \Theta > \epsilon + |a| \). If \( \lambda \in (0, 1/2^{i-1}) \), then \( 2^{n} \lambda \in (0, 1) \) for any \( m = 0, 1, \ldots, i - 1 \), and for \( \lambda \), we obtain

\[
\phi(\lambda) = \sum_{m=0}^{\infty} \lambda(2^{m} \lambda) \geq \sum_{m=0}^{i-1} \Theta(2^{m} \lambda)^{4} = i \Theta \lambda^{4} > (\epsilon + |a|) \lambda v^{4},
\]

(50)

which contradicts.

**Data Availability**

No data were used to support the findings of the study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All authors contributed equally to this work. And all the authors have read and approved the final version of the manuscript.

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