A UNIVERSAL ENVELOPING FOR $L_\infty$-ALGEBRAS.

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Abstract. For any $L_\infty$-algebra $L$ we construct an $A_\infty$-algebra structure on the symmetric coalgebra $Sym_*^c(L)$ and prove that this structure satisfies properties generalizing those of the usual universal enveloping algebra. These properties follow from an invariant contracting homotopy one the cobar construction of an exterior coalgebra and its relation to combinatorics of permutahedra and semistandard Young tableaux.

1. Introduction

The purpose of this article is to generalize the universal enveloping from Lie to $L_\infty$-algebras. One candidate is well-known, cf. [6]: the cobar construction $\Omega C(L)$ of the Cartan-Chevalley-Eilenberg coalgebra $C(L)$. In fact, for a DG Lie algebra $L$ there exists a surjective quasi-isomorphism of DG algebras $\Omega C(L) \to U(L)$ (and even of DG Hopf algebras). Of course, $\Omega C(L)$ is much larger than $U(L)$: on the level of vector spaces the former is isomorphic to tensor algebra $T^* \Lambda^*(L)$ on the exterior coalgebra $\Lambda^*(L)$, while the latter by PBW theorem is isomorphic to the symmetric coalgebra $Sym^*(L)$.

The DG algebra $\Omega C(L)$ also makes sense for a general $L_\infty$-algebra $L$ and works well enough as a universal enveloping if we deal with DG algebras up to quasi-isomorphism. In other situations, one would like to have some structure on $Sym^*(L)$ generalizing the usual universal enveloping. Since $A_\infty$-algebras relate to associative algebras as $L_\infty$-algebras to Lie algebras, it is natural to expect that $Sym^*(L)$ should be an $A_\infty$-algebra. To construct it, first consider $L$ as a DG vector space (= DG Lie algebra with trivial bracket). Then $C(L)$ turns into the exterior coalgebra $\Lambda^*(L)$ (if we ignore the homological grading) and the universal enveloping into the symmetric algebra $Sym^*(L)$. Passing from $\Omega \Lambda^*(L)$ to $\Omega C(L)$ amounts to perturbing the differential on the tensor algebra and the standard techniques of homological perturbation theory, cf. e.g. [5], give an $A_\infty$-structure on $Sym^*(L)$. After the first draft of the present paper has been completed, it was pointed out to the author that a similar strategy (but using filtrations instead of perturbation theory) was used in [13] to prove a PBW-type theorem.

However, the functorial properties of such $A_\infty$-structure will depend on a homotopy contracting $\Omega \Lambda^*(L)$ onto $Sym^*_c(L)$. For example, when $L$ is a finite dimensional vector space in degree zero, one needs the homotopy to be $GL(L)$-invariant.

This motivates a closer study of $\Omega \Lambda^*(V)$ for a DG vector space $V$. In Section 3 we prove an isomorphism of complexes, cf. Theorem 1:

\begin{equation}
\Omega \Lambda^*(V) \simeq k \oplus \bigoplus_{n \geq 1} \left( V^{\otimes n} \otimes k[\Sigma_n] C_*(P_n) \right)
\end{equation}

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where $\Sigma_n$ is the symmetric group and $C_*(P_n)$ is the complex computing the cell homology of the $n$-th permutedhedron $P_n$.

Recall, e.g. [15], that $P_n$ may be defined as the convex hull of the orbit of $(1, 2, \ldots, n) \in \mathbb{R}^n$ under the permutation action of $\Sigma_n$. The $k$-dimensional faces of $P_n$ are labeled by ordered partitions $\{1, \ldots, n\} = \psi_1 \cup \ldots \cup \psi_{n-k}$ of $\{1, \ldots, n\}$ into a disjoint union of its $(n-k)$ subsets, e.g. the vertices of $P_n$ correspond to permutations in $\Sigma_n$. In these terms, an element $(v_1 \otimes \ldots \otimes v_n) \otimes (\psi_1, \ldots, \psi_{n-k})$ on the right hand side of (1), corresponds to $\pm \otimes_{j=1}^{n-k} (\lambda \in \psi_j, v_j)$ on the left hand side. Of course, it is agreement with differentials and contractibility of $P_n$ which are important in (1). Informally, the differentials agree since codimension one sub-faces of a face $\{\psi_1, \ldots, \psi_{n-k}\} \subset P_n$ are obtained by breaking a single $\psi_i$ into a pair of disjoint subsets, which also corresponds to the standard coproduct in $\Lambda^*(V)$. Permutahedra were studied in [12] (where they are denoted by $C(n-1)$) precisely in relation to the loop spaces, but the algebraic statement (1) appears to be new. It leads to a choice of a contracting homotopy for $\Omega \Lambda^*(V) \to \text{Sym}^*(V)$ which is functorial in $V$ - but not quite canonical.

In Section 2 we construct the universal enveloping $U(L)$ and prove that it has several properties generalizing those of the classical universal enveloping. In particular, Theorem 3 shows that $U(L)$ is a sort of “homotopy Hopf algebra” even though the operadic meaning of our construction, e.g. the precise relation to the “operadic” universal enveloping algebras of Lada and Markl, cf. [8], remains unclear at the moment. In particular, we prove that the diagonal of $\text{Sym}_c^*(L)$ is a strict morphism of $A_\infty$-algebras if $\text{Sym}^*(L) \otimes \text{Sym}^*(L)$ is identified with $\text{Sym}^*(L \oplus L)$. Although such “$A_\infty$ tensor product” is extremely natural in the present context, its relation to the Saneblidze-Umble diagonal, cf. [15], or the diagonal on the W-construction of the associative operad, cf. [10], still needs to be clarified. The correspondence $L \mapsto U(L)$ falls short of being a functor: we only prove that $U(\psi) \circ U(\phi) = U(\psi \circ \phi)$ if one of the $L_\infty$-morphisms $\psi, \phi$ is strict, and give an example showing that this fails in general. In Theorem 4 we generalize the classical complex $\Lambda^*(L) \otimes U(L)$ and prove a derived equivalence between $C(L)$ and $U(L)$ (i.e. a version of the BGG correspondence). In Theorem 4 we show that appropriate categories of $A_\infty$-modules over $U(L)$ and $L_\infty$-modules over $L$, are equivalent. While the derived equivalence between $C(L)$ and $U(L)$ and the functor $F$ from $U(L)$-modules to $L$-modules are relatively easy to obtain, the inverse functor $G$ from $L$-modules to $U(L)$-modules depends on Theorems 1 and 3 in an essential way. By a recent spectacular result of Merkulov, cf. [11], a homotopy Lie bialgebra structure on $L$ induces a homotopy bialgebra structure on $\text{Sym}^*(L)$, defined via some non-explicit operadic maps. It would be interesting to see if Theorem 4 allows one to describe the latter structure along the lines of Kazhdan-Etingof.

2. The Universal Enveloping

2.1. Notations and standard definitions. We consider complexes of vector spaces $k$ over a field of characteristic zero. We use cohomological grading, to be denoted by superscripts, in which differentials have degree $+1$. If $V$ is a complex, its suspension $sV$ is defined by $(sV)^p = V^{p+1}$, $d(sV) = -s(dV)$. In particular $\deg(sV) = \deg V - 1$. All tensor products are over $k$ unless indicated otherwise. Throughout this paper we
use the Koszul sign rule

\[(F \otimes G)(a \otimes b) = (-1)^{\deg G \cdot \deg a} F(a) \otimes G(b)\]

If \(V\) is a graded vector space \(\text{Sym}^*(V) = \oplus_{k \geq 0} \text{Sym}^k(V)\) will stand for its graded symmetric tensors, i.e. \(\text{Sym}^k(V)\) is the space of vectors in \(V^\otimes k\) which are invariant with respect to the \textit{graded} action of the symmetric group \(S_k\) (i.e. whenever two odd elements are permuted this leads to a change of sign). If we disregard the grading and assume that \(V\) has only even vectors (resp. only odd vectors) this will become the usual space of symmetric (resp. antisymmetric) tensors. Note that \(\text{Sym}^*(V)\) has standard structures of a commutative algebra \(\text{Sym}^*_0(V)\) and a cocommutative coalgebra \(\text{Sym}^*_1(V)\). We will also denote \(\Lambda^*(V) = \text{Sym}^*(sV)\) - observe the \textit{shift of grading} involved.

Let \(L\) be a DG Lie algebra with differential \(l_1\) and the bracket \(l_2 : L^\otimes 2 \to L\). Its \textit{Cartan-Chevalley-Eilenberg construction} \(C(L)\) is the DG coalgebra \(\Lambda^*_c(L)\) with the differential \(\delta_C = c_1 + c_2\) defined as follows. Let \(s^\otimes n : L^\otimes n \to (sL)^\otimes n\) be the obvious degree \((-n)\) isomorphism and set

\[c_1 = -s \ l_1 \ s^{-1} : \Lambda^1(L) \to \Lambda^1(L); \quad c_2 = s \ l_2(s^\otimes 2)^{-1} : \Lambda^2(L) \to \Lambda^1(L)\]

extending these maps to \(\Lambda^*_c(sL)\) as coderivations. Then \(\delta^2_C = 0\) follows from \(l_2^2 = 0\), the Leibniz Rule and the Jacobi Identity for \(l_2\). If \(\delta\) is a general differential on \(\Lambda^*_c(L)\) which is a coderivation, we can consider compositions \(c_n : \Lambda^n(L) \to \Lambda^*_c(L) \xrightarrow{\delta} \Lambda^*_c(L) \to \Lambda^1(L)\) and define \(l_n : L^\otimes n \to L\) via

\[c_n = (-1)^n s \ l_n(s^\otimes n)^{-1}\]

Then \(\{l_n\}_{n \geq 1}\) give \(L\) the structure of an \(L_\infty\)-algebra, cf. [8]. If \(\phi : (\Lambda^*_c(L), \delta) \to (\Lambda^*_c(L'), \delta')\) is a degree zero morphism of DG coalgebras, in a similar way we get a sequence of degree \(1 - i\) maps \(\phi_i : \Lambda^i(L) \to \Lambda^1(L')\). The sequence \(\{\phi_i\}_{i \geq 1}\) (or, equivalently, the original morphism \(\phi\)) is called an \(L_\infty\)-morphism from \(L\) to \(L'\).

Let \(A = k \oplus \bar{A}\) be an augmented DG algebra with differential \(m_1\) and product \(m_2\). Its \textit{reduced cobar construction} \(B(A)\) is the tensor coalgebra \(T^*_c(sar{A})\) with the coproduct

\[\Delta_B[a_1, \ldots, a_n] = 1 \otimes [a_1, \ldots, a_n] + [a_1, \ldots, a_n] \otimes 1 + \sum_{i=1}^{n-1} [a_1, \ldots, a_i] \otimes [a_{i+1}, \ldots, a_n],\]

and the similar differential \(\delta_B = b_1 + b_2:\)

\[b_1 = -s \ m_1 s^{-1} : sar{A} \to sar{A}; \quad b_2 = s \ m_2(s^\otimes 2)^{-1} : (sar{A})^\otimes 2 \to sar{A}.\]

Then \(b_1\) and \(b_2\) extend uniquely to \(B(A)\) as coderivations and \(\delta_B^2 = 0\) follows from \(m_2^2 = 0\), the Leibniz Rule and associativity of \(m_2\).

Again, one can consider a general differential \(\delta_B\) on \(T^*_c(sar{A})\) which is a coderivation, and obtain operations \(m_n : A^\otimes n \to A\) by first considering

\[b_n : (sar{A})^\otimes n \to BA \xrightarrow{\delta_B} BA \to sar{A}\]

and then writing

\[b_n = (-1)^{n} s \ m_n(s^\otimes n)^{-1}.\]
The resulting operations \( \{ m_n \} \) for \( n \geq 1 \) give \( A \) a structure of an \( A_\infty \)-algebra, cf. [7]. Since we use the reduced bar construction, \( A \) is automatically strictly unital, i.e.,

\[
m_n(v_1, \ldots, v_n) = 0; \text{ if } n \geq 3 \text{ and } v_i = 1 \text{ for some } i
\]

and \( m_2(v, 1) = m_2(1, v) = v \). A DG coalgebra morphism \( f : (BA, \delta_B) \to (BA', \delta_B') \) gives a sequence of degree \( (1 - i) \) maps \( f_i : A^\otimes i \to A' \), called an \( A_\infty \)-morphism from \( A \) to \( A' \). Again, for reduced bar constructions such a morphism is automatically strictly unital: \( f_i = 0 \) if \( i \geq 2 \) and one of its arguments is equal to 1 in \( A \).

Finally, let \( C = k \oplus \mathcal{C} \) be a coaugmented DG coalgebra. Its reduced cobar construction is a DG algebra \( \Omega(C) = T_\circ(s^{-1}\mathcal{C}) \) with the differential \( \delta_\Omega = \omega_1 + \omega_2 \) where \( \omega_1 \) and \( \omega_2 \) are obtained from the differential on \( C \) and the reduced coproduct \( \Delta : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \), respectively, using the same pattern \( (w_1, w_2) \) are extended from \( s^{-1}\mathcal{C} \) to \( \Omega(C) \) as derivations. If \( C \) is cocommutative the DG algebra \( \Omega(C) \) also has a shuffle coproduct \( \Delta_\Omega : \Omega(C) \to \Omega(C) \otimes \Omega(C) \) defined on \( s^{-1}\mathcal{C} \subset \Omega(C) \) by

\[
\Delta_\Omega(u) = u \boxtimes 1 + 1 \boxtimes u
\]

and extended to \( \Omega(C) \) multiplicatively. Thus, \( \Omega(C) \) becomes a DG bialgebra (the fact that \( \delta_\Omega \) is also a coderivation uses cocommutativity of \( C \)).

2.2. Universal enveloping: case of Lie algebras and the general plan. Let \( L \) be a DG Lie algebra and \( U(L) \) its universal enveloping algebra. One way - perhaps a little exotic - to construct \( U(L) \) is as follows. The natural projection \( C(L) = \Lambda^\ast(L) \to \Lambda^1(L) \) induces a DG-bialgebra morphism of \( \Omega C(L) \) to \( U(L) \). By Theorem 22.9 and the first equality on page 290 in [4], it is also a quasi-isomorphism. In Section 3 we essentially re-prove this assertion.

We can turn this property inside out and use as a definition. First, consider \( L \) with the same differential but trivial Lie bracket. The above construction gives a quasi-isomorphism of DG algebras \( \Omega \Lambda_\circ(L) \to 
\]

sym_\circ(L) \). Bringing back the original bracket on \( L \) will deform the differential on \( \Lambda^\ast(L) \), and therefore the differential on \( \Omega \Lambda_\circ(L) \). The general machinery of perturbation theory, see [5] and Section 2.3 below, gives a new DG algebra structure on \( \text{sym}_\ast(L) \) and a multiplicative projection from \( \Omega C(L) \) onto \( \text{sym}_\ast(L) \) which is still a quasi-isomorphism. In Theorem 3 (v) we prove that the new structure on \( \text{sym}_\ast(L) \) is precisely the universal enveloping \( U(L) \) (identified by PBW theorem with \( \text{sym}_\ast(L) \) as a coalgebra).

This approach also gives a recipe for a general \( L_\infty \)-algebra \( L \), since an \( L_\infty \)-structure also gives a perturbation of the differential on \( \Lambda^\ast(L) \) and we can carry out a similar procedure of adjusting the product on \( \text{sym}_\ast(L) \). By loc. cit. such adjustment in general leads to an \( A_\infty \)-structure on \( \text{sym}_\ast(L) \). As the procedure depends on a choice of homotopy on \( \Omega \Lambda_\circ(L) \) our construction will be based on the following result.

**Theorem 1.** For a complex \( V \) set \( A(V) = \Omega \Lambda_\circ(V) \), \( E(V) = \text{sym}_\ast(V) \). Let \( f_V : A(V) \to E(V) \) be the multiplicative extension of the projection \( s^{-1} \Lambda^\geq 1(V) \to V \), and \( g_V : E(V) \to A(V) \) the map given by composition of natural embeddings

\[
\text{sym}_n(V) \hookrightarrow V^\otimes n \hookrightarrow T^\ast(V) \hookrightarrow T^\ast(s^{-1} \Lambda_\circ(V)) = \Omega \Lambda_\circ(V)
\]
From this we produce a contraction of the free tensor coalgebra in Theorem 1, which gives a contraction \( f \) which satisfies
\[
1 - g f = dh + h d; \quad f g = 0; \quad g f = 0; \quad h f = h g = h h = 0
\]
and is functorial in the following sense: for every morphism of complexes \( \phi : V \to W \) the natural induced map \( A(V) \to A(W) \) fits into commutative diagram
\[
\begin{array}{ccc}
A(V) & \longrightarrow & A(W) \\
\downarrow h & & \downarrow h \\
A(V) & \longrightarrow & A(W)
\end{array}
\]
Moreover, one can choose \( h \) to commute with the algebra anti-involution \( \iota_\Omega \) on \( \Omega \Lambda^*(V) \) which acts by \(-1\) on the space of generators \( s^{-1} \Lambda^*(V) \).

The proof of this theorem is given in Section 3. We will see that such a homotopy \( h \) (or rather a system of homotopies \( V \mapsto h \)) is not unique but its choice depends on purely combinatorial data that has nothing to do with \( V \).

### 2.3. Universal enveloping: construction and first properties.

Let \((L, \{l_i\}_{i \geq 1})\) be an \( L_\infty \)-algebra. First consider the complex \((L, l_1)\) and set \( V = L \) in Theorem 1, which gives a contraction \((f_L, g_L, h_L)\) from \( A(L) = \Omega \Lambda^*(L) \) to \( E(L) = \text{Sym}^*_{\Lambda}(L) \) and hence a contraction \((f'_L, g'_L, h'_L)\) from \( sA(L) \) to \( sE(L) \) given by
\[
f'_L = s f_L s^{-1}, \quad g'_L = s g_L s^{-1}, \quad h'_L = -s h_L s^{-1}.
\]
From this we produce a contraction of the free tensor coalgebra \( T^*_c(sA(L)) \) onto the free tensor coalgebra \( T^*_c(sE(L)) \) (recall that \((\cdot)\) denotes the augmentation ideal). On the \( n \)-th tensor components set
\[
(2) \quad F = (f'_L)^{\otimes n}, \quad G = (g'_L)^{\otimes n}, \quad H = \sum_{t=1}^{n} (g'_L f'_L)^{\otimes (t-1)} \otimes h'_L \otimes 1^{\otimes (n-t)}.
\]
It follows from the definitions that
\[
1 - GF = dH + Hd; \quad FH = 0; \quad HG = 0; \quad HH = 0
\]
Denote by \( \delta_L^c \) and \( d_L^c \) the differentials of the two tensor coalgebras, respectively. By definition \( BA(L) \) differs from \( T^*_c(sA(L)) \) only in its differential, given by
\[
\delta_L = \delta_L^c + t_L (3)
\]
where \( t_L \) is the part that encodes the product on the tensor algebra \( A(L) \) and \( t_L \) is the perturbation which encodes the \( L_\infty \)-structure on \( L \), cf. Section 2.1. The contraction \((F, G, H)\) can be adjusted to work with \( \delta_L \) using the following Basic Perturbation Lemma, cf. [3]:

**Lemma 2.** Let \((M, d_M), (N, d_N)\) be two complexes and consider a contraction
\[
F : N \to M; \quad G : M \to N; \quad H : N \to N
\]
which satisfies
\[
(4) \quad FG = 1_N, \quad 1_M - GF = d_N H + H d_N, \quad FH = 0, \quad HH = 0, \quad HG = 0.
\]
Given a new differential \( d_N + t \) on \( N \) such that \((tH)\) is locally nilpotent (i.e. for any element \( n \in N \) there is a positive integer \( k(n) \) such that \((tH)^{k(n)}(n) = 0\)) so that the infinite sum

\[
X = t - tHt + tH^2t - \ldots
\]
is well-defined; the formulas

\[
F_t = F(1 - XH); \quad G_t = (1 - HX)G; \quad H_t = H - HXH; \quad (d_M)_t = d_M + FXG
\]
give a contraction of the complex \((N, d_N + t)\) to the complex \((M, (d_M)_t)\) which satisfies equations similar to (4).

Applying this result to the perturbation (3) we get a new contracting homotopy

\[
F_L = F_{t_L + t_t}, \quad G_L = G_{t_L + t_t}, \quad H_L = H_{t_L + t_t}
\]
from \( BOC(L) \) to \( T^*_\omega(sE(L)) \) with its new differential \( d_L = (d_{sL})_{t_L + t_t} \). Moreover, the new contraction agrees with the coalgebra structures: since \( F,G \) are morphisms of coalgebras, \( H \) is a coalgebra homotopy and \( t = t_L + t_t \) is a coalgebra perturbation, i.e.:

\[
\Delta_B H = (H \otimes 1 + GF \otimes H)\Delta_B; \quad \Delta_B t = (t \otimes 1 + 1 \otimes t)\Delta_B,
\]
the new differential \( d_L \) is again a coderivation, \( F_L, G_L \) are morphisms of DG coalgebras and \( H_L \) is a coalgebra homotopy, cf. [5].

**Definition.**

i. Denote by \( U(L) \) the vector space \( E(L) = \text{Sym}^*(L) \) with the \( A_\infty \)-structure \( \{m_i\}_{i \geq 2} \) given by the above coalgebra differential \( d_L \) on \( T^*_\omega(sE(L)) \). Then \( (T^*_\omega(sE(L)), d_L) \) is the cobar construction \( BU(L) \) of \( U(L) \).

ii. If \( L,M \) are two \( L_\infty \)-algebras and \( \phi : C(L) \to C(M) \) is an \( L_\infty \)-morphism, cf. [8], let \( U(\phi) = F_M B\Omega(\phi) G_L : BU(L) \to BU(M) \).

iii. If \( \phi : C(L) \to C(M) \) and \( \psi : C(M) \to C(N) \) are two \( L_\infty \)-morphisms, set \( H(\phi, \psi) = F_N B\Omega(\psi) H_M B\Omega(\phi) G_L : BU(L) \to BU(N) \).

**Theorem 3.** Let \( \phi : C(L) \to C(M) \) be an \( L_\infty \)-morphism of \( L_\infty \)-algebras \( L,M \) and \( \phi_1 : L \to M \) be its first component. Then

i. \( U(\phi) \) is an \( A_\infty \)-morphism from \( U(L) \) to \( U(M) \) and its first component \( U(\phi_1) : U(L) = \text{Sym}^*(L) \to \text{Sym}^*(M) = U(M) \) is given by symmetrization of \( \phi_1 \).

ii. If \( \phi : L \to M \) is a strict morphism of \( L_\infty \)-algebras, i.e. \( \phi_i = 0 \) for \( i \geq 2 \), then the same holds for \( U(\phi) \), i.e. \( U(\phi)_i = 0 \) for \( i \geq 2 \).

iii. The standard coproduct \( \Delta : \text{Sym}^*(L) \to \text{Sym}^*(L) \otimes \text{Sym}^*(L) \) is a strict morphism of \( A_\infty \)-algebras, if the latter is given an \( A_\infty \)-structure via the natural isomorphism

\[
\text{Sym}^*(L) \otimes \text{Sym}^*(L) \simeq \text{Sym}^*(L \oplus L).
\]

iv. If \( \phi : C(L) \to C(M) \) and \( \psi : C(M) \to C(N) \) are two \( L_\infty \)-morphisms then \( U(\psi \circ \phi) - U(\psi) \circ U(\phi) = d_{U(N)} H(\phi, \psi) + H(\phi, \psi) d_{U(L)} : BU(L) \to BU(N) \).

Moreover, if at least one of the morphisms \( \phi, \psi \) is strict, then \( H(\phi, \psi) = 0 \).

v. If the 2-truncation \( (L, l_1, l_2) \) is a DG Lie algebra then \( (U(L), m_1, m_2) \) is a DG algebra isomorphic to the usual universal enveloping of \( (L, l_1, l_2) \).
Let $\iota: U(L) \to U(L)$ be the linear involution that corresponds to the action of $(-1)^k$ on $\text{Sym}^k(L)$. Then
\[ m_n \circ \iota^{\otimes n} = \iota \circ m_n \circ \omega_n \]
where $\omega_n$ is the permutation $\{1, \ldots, n\} \to \{n, \ldots, 1\}$. In other words, $\iota$ is a strict morphism $U(L) \to U(L)^{\text{op}}$, where $(-)^{\text{op}}$ is the opposite $A_\infty$-structure.

Let $n \geq 2$ and $v_1, \ldots, v_n \in L \subset U(L)$. Let $\text{Alt}(v_1 \otimes \ldots \otimes v_n)$ be the graded antisymmetrization of $v_1 \otimes \ldots \otimes v_n$. Then
\[ m_n(\text{Alt}(v_1 \otimes \ldots \otimes v_n)) = l_n(v_1, \ldots, v_n). \]

Proof. Parts (i) - (iv). In this proof we deal with several $L_\infty$-algebras and it helps to re-denote the operators in (2) by $F^0_M, G^0_M, H^0_M$ and similarly for $M, N$. To prove (i) first observe that $F_M$ and $G_L$ are DG coalgebra morphisms by [5] and $B\Omega(\phi)$ is a DG coalgebra morphism since $\phi$ itself is a DG coalgebra morphism. Therefore $U(\phi): BU(L) \to BU(M)$ is a DG coalgebra morphism encoding an $A_\infty$-morphism $U(L) \to U(M)$. To compute the first component we need to evaluate $U(\phi)$ on $v \in \overline{U(L)} \subset BU(L)$. But, for such an element, all terms in $F_M, G_L$ which involve perturbation of the differentials on $B\Omega(C)(L), B\Omega(C)(M)$, are identically zero, therefore $U(\phi)(v) = F^0_M B\Omega(\phi) G^0_L(v)$ and the latter map is precisely given by the symmetrization $S\text{ym}(\phi)$ of $\phi$. To prove (ii) we observe that for a strict morphism $\phi$ one has $H^0_M B\Omega(\phi) = B\Omega(\phi) H^0_L$ by Theorem 1. Using the explicit formulas of the Basic Perturbation Lemma,
\[ F_M = F^0_M(1 - X_M H^0_M); \quad G_L = (1 - H^0_L X_L) G^0_L; \quad H_M = H^0_M(1 - X_M H^0_M) \]
and the side conditions $H^3_M H^0_M = 0, F^0_M H^0_M = 0, H^0_L G^0_L = 0$ we obtain
\[ F_M \circ B\Omega(\phi) \circ G_L = F^0_M \circ B\Omega(\phi) \circ G^0_L = B\text{Sym}(\phi) \]
Part (iii) is an immediate application of (ii) to the diagonal map $L \to L \oplus L, x \mapsto x \oplus x$ which is a strict morphism of $L_\infty$-algebras. Finally, the left hand side in (iv) by definition is equal to
\[ F_N B\Omega(\psi)(1 - G_M F_M) B\Omega(\phi) G_L = F_N B\Omega(\psi)(\delta_N H_M + H_M \delta_N) B\Omega(\phi) G_L \]
and the assertion follows since $F_N, B\Omega(\psi), B\Omega(\phi)$ and $G_L$ are morphisms of complexes. To prove the vanishing we observe that, by Theorem 1, $H^3_M B\Omega(\phi) = B\Omega(\phi) H^3_L$ if $\phi$ is strict, and similarly for $\psi$. Now the side conditions and the formulas for $F, G, H$ finish the proof.

Part (v). First we assume that $L$ is a Lie algebra, i.e. all $l_i$ vanish for $i \geq 3$. The $A_\infty$-structure on $E(L) = \text{Sym}^*(L)$ is given by the following differential on $T_c^*(sE(L))$:
\[ d_L = d^0_L + F^0_L \left( \sum_{i \geq 0} (-1)^i \left( (t_\mu + t_L) H^0_L \right)^{i} \right)(t_\mu + t_L) G^0_L \]
To simplify this expression we first introduce a “geometric grading” on $\Omega A^*_N(L)$ by declaring that elements of $s^{-1}N^k(L)$ have degree $(k - 1)$, and extending to $\Omega A^*_N(L)$ multiplicatively (we can declare that $k \subset \Omega A^*_N(L)$ has degree $(-1)$ but that will not be used in the proof). From the point of view Lemma 6 in Section 3, this grading corresponds to dimension of the cells of permutahedra. We extend it to $B\Omega A^*_N(L)$ in the obvious way (again, setting to $(-1)$ on the constants).
Then $t_L$ vanishes on elements of geometric degree 0 since those elements are products of linear symmetric tensors, and the bracket $l_2$ encoded by $t_L$ needs two inputs. 

Since the image of $G^*_L$ belongs to the degree 0 part we will have $t_L G^*_L = 0$. Also, the proof of Theorem 1, cf. Section 3.2, implies that $H^*_L$ increases the geometric degree by 1, $t_L$ decreases by 1, $t_\mu$ preserves it, while $F^*_L$ vanishes on elements of positive degree. Consequently, the above formula for the deformed differential simplifies to 

$$d_L = d^*_L + F^*_L \left( \sum_{i \geq 0} (-1)^i (t_L H^*_L)^i \right) t_\mu G^*_L.$$

The terms responsible for a multiple product $m_n : U(L)^{\otimes n} \to U(L)$ contain $t_\mu$ exactly $(n - 1)$ times, thus the differential on $U(L)$ is the same on $\text{Sym}^*(L)$ and all $m_n$ with $n \geq 3$ vanish. So $U(L)$ is a DG algebra. Denoting the usual symmetric product in $E(L) = \text{Sym}^*(L)$ by $*$, we also see that for $x, y \in \text{Sym}^*(L)$ homogeneous in the geometric grading:

$$m_2(x, y) = x * y + \text{terms of lower geometric degree}.$$

Therefore, the subspace $L \subset U(L)$ generates $U(L)$ as an algebra. For $v, u \in L$ an explicit computation shows

$$m_2(v, u) = v * u + \frac{1}{2} l_2(v, u).$$

Denote for a moment by $U^{cl}(L)$ the classical universal enveloping. The last formula gives a surjective DG algebra morphism $U^{cl}(L) \to U(L)$ which is easily seen to be an isomorphism by an inductive argument involving natural filtrations on both algebras.

Next, we assume that the higher products $l_i, i \geq 3$ of $L$ are not necessarily zero. Then the perturbation $\delta_L = \delta^*_L + t_\mu + t_L$ can be split as $(\delta^*_L + t_\mu + t_L^{(2)}) + (t_L - t_L^{(2)})$ where $t_L^{(2)}$ is the term coming from the bracket $l_2$. The expression in the first parenthesis has square zero since by assumption $(L, l_1, l_2)$ is a DG Lie algebra. Thus, setting $t_1 = t_\mu + t_L^{(2)}$, $t_2 = t_L - t_L^{(2)}$ we see that both $\delta^*_L + t_1$ and $\delta^*_L + t_1 + t_2$ satisfy the condition of Lemma 2. By direct computation one can show that in such a setting one always has $(d_M)_{l_1 + l_2} = ((d_M)_{l_1})_{l_2}$ and similarly for $F, G$ and $H$. 

Thus the $A_\infty$-structure of $U(L)$ corresponds to the perturbation of $F_{l_1}, G_{l_1}, H_{l_1}$ and $d_{l_1}$ by $t_2$. In particular, the differential of $BU(L)$ is given by

$$d_{l_1} + F_{l_1} \left( \sum_{i \geq 0} (-1)^i (t_2 H_{l_1})^i \right) t_2 G_{l_1}.$$

Evaluating the second term on $sU(L) \subset BU(L)$ and $sU(L) \otimes sU(L) \subset BU(L)$ will give zero for the following reasons. Firstly, for $x \in sU(L)$ we have $G_{l_1}(x) = G^*_L(x)$ since $t_1 G^*_L(x) = 0$. But then $t_2 G_{l_1}(x) = t_2 G^*_L(x) = 0$ since $t_2$ vanishes on terms of geometric degree $\leq 1$. Secondly, for $x_1, x_2 \in sU(L)$ by a similar computation

$$G_{l_1}(x_1 \otimes x_2) = \left[ \sum_{i \geq 0} (-1)^i (H^*_L t_2^{(2)})^i \right] H^*_L (G^*_L(x_1) \otimes G^*_L(x_2)).$$

Since $H$ increases the geometric degree by 1 and $t_2^{(2)}$ decreases it by 1, the above expression has degree 1, hence $t_2$ vanishes on it. So the differential and the product of $U(L)$ are the same as for the 2-truncation $(L, l_1, l_2)$, which proves (v).
To prove (vi) for \( n \geq 3 \) consider a similar anti-involution \( t_\Omega : \Omega C(L) \rightarrow \Omega C(L)^{op} \) of Theorem 1. Let \( \tilde{\omega} \) be a linear involution on \( BU(L) \) which acts by \( \omega_n \) on \( (sU(L))^\otimes n \) and use the same notation for the corresponding involution on \( B\Omega C(L) \). Denote by \( \pi : BU(L) \rightarrow \overline{U(L)} \) projection onto the first component. Also, let \( Bt, Bt_\Omega \) be the linear involutions on the bar constructions which act by \( s^{n_1} s^{n_2} (s^{n_3})^{-1}, s^{n_4} s^{n_5} (s^{n_6})^{-1} \) on the \( n \)-th tensor components, respectively. Since \( \omega_n s^{\otimes n} = ( -1 )^{n(n-1)/2} s^{\otimes n} \omega_n : (\overline{U(L)})^{\otimes n} \rightarrow (s\overline{U(L)})^{\otimes n} \), we need to show that

\[
( -1 )^{n(n-1)/2} \pi(F^2 L X_L G^2 L)(Bt \tilde{\omega}) = (Bt \tilde{\omega}) \pi(F^2 L X_L G^2 L)
\]
on \((s\overline{U(L)})^{\otimes n} \). By Section 5.2 in the appendix \( X_L \) is a sum of several terms of the form

\[
( -1 )^s a_1 \ldots a_s t_\mu
\]
where each \( a_i \) is either \((t_L H^2_L)\) or \((t_\mu H^2_L)\). If such a term is to give a nonzero contribution to the expression above, the operator \( t_\mu \) should be used exactly \((n - 1)\) times, since we need to get from \((sU(L))^{\otimes n} \) to \( s\overline{U(L)} \). It is easy to see that

\[
(Bt \tilde{\omega}) F^2 L = F^2 L(Bt_\Omega \tilde{\omega}); \quad (Bt_\Omega \tilde{\omega}) G^2 L = G^2 L(Bt \tilde{\omega})
\]
and that \((Bt_\Omega \tilde{\omega})\) commutes with the operators \( t_L \) and \( H^2_L \). Now what we need to prove follows from the following formula, easily checked by direct computation:

\[
(Bt_\Omega \tilde{\omega}) t_\mu = ( -1 )^{i-1} t_\mu (Bt_\Omega \tilde{\omega}) : (s\overline{U(L)})^{\otimes i} \rightarrow (s\overline{U(L)})^{\otimes (i-1)}.
\]
For \( n = 2 \) the same argument works for \((m_2 - *)\) where \(*\) is the usual product on \( Sym^*(L) \). Since \(*\) is commutative, the assertion holds for \( m_2 \) as well. For \( n = 1 \), the differential on \( U(L) \) is the same as on \( Sym^*(L) \) and the statement holds again.

Finally, (vii) is a restatement of Theorem 3 (i) below and its proof will be given there. \( \square \)

**Example.** We give an example when \( U(\psi \circ \phi) \neq U(\psi) \circ U(\phi) \) regardless of the choice of \( h_V \) in Theorem 1. To that end, assume that \( L, M, N \) are vector spaces placed in homological degrees 0, \(-1\), \(-3\), respectively and that they have have trivial differentials and brackets (i.e. \( C(L), C(M), C(N) \) reduce to graded exterior algebras with trivial differentials). Then \( \phi \) is a sequence of degree zero linear maps \( \phi_i : \Lambda^i(L) \rightarrow (sM) \) and for degree reasons only \( \phi_2 \) can be non-zero. Since differentials and brackets vanish, any degree zero linear map \( \phi_2 : \Lambda^2(L) \rightarrow sM \) gives an \( L_\infty \)-morphism. Similarly, for \( \psi \) only the component \( \psi_2 : \Lambda^2(M) \rightarrow sN \) can be non-zero and any such degree zero linear map will do. Note that \( \phi_2 \) is skew-symmetric and \( \psi_2 \) is symmetric (since \( M \) is an odd vector space). Then \( \rho = \psi \circ \phi \) can only have a non-trivial component \( \rho_4 : \Lambda^4(L) \rightarrow N \) given, up to normalizing factor, by

\[
\rho(v_1, v_2, v_3, v_4) = 
\]

\[
= \psi(\phi(v_1, v_2), \phi(v_3, v_4)) - \psi(\phi(v_1, v_3), \phi(v_2, v_4)) + \psi(\phi(v_1, v_4), \phi(v_2, v_3)).
\]

In notation of the proof of Theorem 2(iv), we want to show that \( F_N B\Omega(\psi)(1 - G_M F_M) B\Omega(\phi) G_L \neq 0 \) which would imply that \( U(\rho) \neq U(\psi \circ U(\phi)) \). To distinguish between the tensor products in \( B(...) \) and \( \Omega(...) \) we denote the first one by \( \otimes \) and the second by \( \otimes \). Observe that in our case \( F_N = F^2_N \) and \( F_M = F^3_M \).
Consider now $Z = v_1 \boxtimes v_2 \boxtimes v_3 \boxtimes v_4 \in (sL) \boxtimes \subset BU(L)$. Then by Basic Perturbation
Lemma, $G_L(Z) = (1 - (H_1^2)t + (H_1^2t)^2 - (H_1^2t)^3)G_2^1$ where $t$ is the coderivation of
$B\Omega C(L)$ which encodes the product structure of $\Omega C(L)$. By explicit computation,
the terms of $G_L(Z)$ which can give a non-zero contribution into $B\Omega(\phi)G_L(Z)$ are

$$
-\frac{1}{4}(v_1 \wedge v_2) \boxtimes (v_3 \wedge v_4) + c (v_1 \wedge v_2 \wedge v_3 \wedge v_4) \in B\Omega C(L)
$$

where $c$ is a certain constant. All other terms will involve a $\boxtimes$-factor of either $v_i$ or $v_i \wedge v_j \wedge v_k$, and $B\Omega(\phi)$ applied to them is zero since $\phi_2$ must have two inputs. Up to
a factor,

$$
B\Omega(\phi)(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = \phi_2(v_1, v_2) \star \phi_3(v_3, v_4) - \phi_2(v_1, v_3) \star \phi_2(v_2, v_4) + \phi_2(v_1, v_4) \star \phi_2(v_2, v_3)
$$

where $\star$ is the product in $C(M)$. The last expression is in the kernel of $F_M$ therefore
applying $F_N B\Omega(\psi)(1 - G_M F_M)$ will give (a multiple of) $\rho(v_1, v_2, v_3, v_4)$. On the other hand, applying $(1 - G_M F_M)B\Omega(\phi)$ to $(v_1 \wedge v_2) \boxtimes (v_3 \wedge v_4)$ we get

$$(1 - G_M F_M)(\phi_2(v_1, v_2) \boxtimes \phi_3(v_3, v_4)) = (H_2^1t')(\phi_2(v_1, v_2) \boxtimes \phi_2(v_3, v_4))$$

where $t'$ is the coderivation of $B\Omega C(M)$ which encodes the product of $\Omega C(M)$. Then

$F_N B\Omega(\psi)(1 - G_M F_M)B\Omega(\phi)G_L(Z) = c' \psi_2(\phi_2(v_1, v_2), \phi_2(v_3, v_4)) + c'' \rho(v_1, v_2, v_3, v_4)$

where $c'$ and $c''$ are constants, and $c' \neq 0$. Since the first term is not antisymmetric in $v_1, v_2, v_3, v_4$, the sum cannot be zero and $U(\psi \circ \phi) \neq U(\psi) \circ U(\phi)$ on $v_1 \boxtimes v_2 \boxtimes v_3 \boxtimes v_4$.

We have even proved that $U(\psi) \circ U(\phi) : U(L) \to U(N)$ does not arise from any $L_\infty$-morphism $L \to N$.

2.4. Universal enveloping: categories of modules. Recall that $U(L)$ denotes
the vector space $Sym^*(L)$ with the $A_\infty$-structure constructed in the previous subsection. The next theorem deals with the notion of a generalized twisted cochain and
the functors defined by it, see appendix. Part (iii) asserts a BGG-type equivalence
between two derived categories, $DU(L)$ and $DC(L)$. The derived category $DU(L)$ is
obtained by localizing the category $Mod_{\infty}(U(L))$ of strictly unital $A_\infty$-modules over
$U(L)$ and strictly unital morphisms (= the full subcategory of DG-comodules over
$BU(L)$ which are free as comodules), at the class of quasi-isomorphisms. The derived
category $DC(L)$ is obtained by localizing the category $Comod_c(C(L))$ of complete
counital DG-comodules over $C(L)$, by the class of weak equivalences (i.e. morphisms
which induce a quasi-isomorphism on the bar construction). See Chapter 2 in [9] and
Section 3.2 in [2] for more details.

Theorem 4. The universal enveloping $U(L)$ has the following properties:

i. the composition $\tau : C(L) \to L \to U(L)$ is a generalized twisted cochain;
ii. the complex $C(L) \otimes_k U(L)$ is quasi-isomorphic to $k$ and the DG algebra mor-

phism $\Omega C(L) \to \Omega BU(L)$ induced by $\tau$, is a quasi-isomorphism;
iii. the functors $M \mapsto M \otimes_{C(L)}$ and $N \mapsto N \otimes_{C(L)} U(L)$ induce mutually inverse

equivalences of the derived categories $DC(L)$ and $DU(L)$.
Proof. To prove (i), start with the composition
\[ C(L) \to B\Omega C(L) \overset{F_L}{\to} BU(L). \]
Since it is a DG coalgebra morphism, by 5.3 in the appendix, its projection onto \( U(L) \) is a generalized twisted cochain \( C(L) \to U(L) \). It is easy to check that it coincides with \( \tau \). Part (ii) is known when \( L \) is an abelian and the general case follows by perturbation lemma as in the construction before Theorem 3. Alternatively, for the first assertion we could first replace \( U(L) \) with \( \tau \) perturbation lemma as in the construction before Theorem 3. Alternatively, for the second assertion we could first replace \( U(L) \) by \( \Omega C(L) \) where the corresponding results are again well known, cf. [4], and then pass from \( \Omega C(L) \) to \( U(L) \) using the strategy of [1]; while the second assertion is entirely similar to the case of Lemma 6 in [2]. Part (iii) follows from (ii) as in Section 3.3. of [2], see also [9] for the associative case. \( \square \)

We can also construct a pair of functors relating \( L \)-modules to \( U(L) \)-modules, see appendix for the definitions. Let \( \text{Mod}(L) \) be the category of \( L_{\infty} \)-modules over \( L \) and \( L_{\infty} \)-morphisms (= the category of DG comodules over \( C(L) \) which are free as \( C(L) \)-comodules). By the appendix, we can also view an \( L \)-module structure on \( M \) as a twisted cochain \( \tau : C(L) \to End(M) \). The corresponding DG coalgebra map \( C(L) \to BEnd(M) \) admits a canonical factorization
\[ C(L) \to B\Omega C(L) \to BEnd(M) \]
since we can extend \( \tau \) to a DG algebra map \( \Omega C(L) \to End(M) \) and then apply the bar construction. Therefore, composing with \( G_L : BU(L) \to B\Omega C(L) \) we get a DG-coalgebra map \( BU(L) \to BEnd(M) \), i.e. a strictly unital \( A_{\infty} \)-module structure on \( M \). This defines a functor
\[ \mathcal{G} : \text{Mod}(L) \to \text{Mod}_{\infty}(U(L)) \]
In the other direction, we start with a DG coalgebra morphism \( BU(L) \to BEnd(M) \) and then composing with the canonical map \( C(L) \to B\Omega C(L) \) and \( F_L : B\Omega C(L) \to BU(L) \) we get a DG coalgebra map \( C(L) \to BEnd(M) \), i.e. a twisted cochain \( C(L) \to End(M) \) which gives \( M \) a structure of an \( L_{\infty} \)-module over \( L \). This defines a functor
\[ \mathcal{F} : \text{Mod}_{\infty}(U(L)) \to \text{Mod}(L). \]
Observe that in both cases the underlying vector space does not change.

Theorem 5. The above functors \( \mathcal{G}, \mathcal{F} \) are mutually inverse equivalences.

Proof. In one direction, suppose we start with an \( A_{\infty} \)-module structure on \( M \) given by \( BU(L) \to BEnd(M) \). Applying \( \mathcal{G}\mathcal{F} \) amounts to considering the composition
\[ BU(L) \overset{G_L}{\to} B\Omega C(L) \overset{F_L}{\to} BU(L) \to BEnd(M). \]
Since the composition of the first two arrows is identity, we conclude that the identity map on \( M \) gives an isomorphism of \( A_{\infty} \)-modules \( \mathcal{G}\mathcal{F}(M) \) and \( M \).

In the other direction, suppose we start with a twisted cochain \( C(L) \to End(M) \) and construct \( B\Omega C(L) \to BEnd(M) \) as above. The \( L_{\infty} \)-module corresponding to \( \mathcal{F}\mathcal{G}(M) \) is obtained from a DG coalgebra morphism
\[ C(L) \to B\Omega C(L) \overset{F_L}{\to} BU(L) \overset{G_L}{\to} B\Omega C(L) \to BEnd(M) \]
In view of \( G_L F_L = 1 - \delta_L H_L - H_L \delta_L \) it suffices to show that the composition
\[
C(L) \rightarrow B\Omega C(L) \xrightarrow{\delta L H_L + H_L \delta L} B\Omega C(L) \rightarrow B\text{End}(M)
\]
is zero. That in its turn would follow from the vanishing of
\[
C(L) \rightarrow B\Omega C(L) \xrightarrow{H_L} B\Omega C(L).
\]
But the latter holds since \( h_L \) vanishes on \( s^{-1}C(L) \subset \Omega C(L) \) by its construction, see Section 3.2 (the homotopy \( H_n \) vanishes on the top-dimensional cell of the permutahedron \( P_n \)). Thus, the identity on \( M \) also gives an isomorphism of \( L_\infty \)-modules \( M \) and \( \mathcal{F}_G(M) \), which finishes the proof. \( \square \)

2.5. An example: toric complete intersections. The following example had originally motivated our study of \( L_\infty \)-algebras. See [2] for details. Let \( X \subset \mathbb{P}^\Sigma \) be a complete intersection in a toric variety defined by a fan \( \Sigma \). Then \( X \) has a “homogeneous coordinate ring” \( S(X) = \text{Sym}^*(V)/J \), a quotient of a polynomial ring by an ideal generated by a regular sequence of polynomials \( W_1, \ldots, W_m \). For a general toric variety \( S(X) \) will be graded by a finitely generated abelian group \( A(X) \) and \( W_1, \ldots, W_m \) will be homogeneous in this grading (but not the usual grading of \( \text{Sym}^*(V) \)). One can always assume that \( W_1, \ldots, W_m \) have no constant or linear terms. In this setting, define the “Koszul dual” of \( S(X) \) as the Yoneda algebra \( E(X) = \text{Ext}_{S(X)}^*(k,k) \) with its natural \( A_\infty \)-structure, cf. [7], defined in general up to \( A_\infty \)-homotopy.

Introducing formal degree 2 variables \( z_1, \ldots, z_m \) which span a vector space \( U \) we can define an \( L_\infty \)-algebra \( L = s^{-1}V^\vee \oplus U \) by viewing the formal sum \( W = \sum W_i(s z_i) \) as a differential on \( C(L) \simeq \text{Sym}^*_{\mathcal{C}L}(V^\vee) \otimes \Lambda^*_U(U) \), if we agree that \( W_j \) act by differential operators on \( \text{Sym}^*(V^\vee) \).

It was shown in [2] that the Koszul dual \( E(X) \) may be identified with the universal enveloping \( U(L) \) (loc. cit.) uses Koszul type-resolutions instead of \( \Omega C(L) \) which still leads to the same \( A_\infty \)-structure, perhaps after an adjustment of a contracting homotopy). The interpretation in terms of \( \text{Ext} \) groups also follows from Theorem 3 (ii).

3. A homotopy on the cobar construction

3.1. Permutahedra. Let \( n \geq 1 \) and \( 1 \leq d \leq n \) and set \( P(n,d) \) to be the set of ordered partitions of \( \{1, \ldots, n\} \) which have \( d \) parts. Equivalently, any such partition can be viewed as a surjective map \( \psi: \{1, \ldots, n\} \rightarrow \{1, \ldots, d\} \): setting \( \psi_i = \psi^{-1}(i) \subset \{1, \ldots, n\}, 1 \leq i \leq d \) we get an ordered partition \( \psi_1 \cup \ldots \cup \psi_d \). As mentioned in the introduction, \( P(n,d) \) labels the set of \( d \)-dimensional faces of the \((n-1)\)-dimensional polytope \( P_n \). To consider the homology complex of \( P_n \) define an orientation of
\[
\psi: \{1, \ldots, n\} \rightarrow \{1, \ldots, d\}
\]
as an equivalence class of orderings on each subset \( \psi_i \), such that two orderings are equivalent if they differ by an even permutation of \( \{1, \ldots, n\} \). We choose the orientation given by the natural increasing ordering on \( \psi_j \) and denote by \([\psi_1] \ldots [\psi_d]\) the corresponding oriented cell.
Let $C_*(P_n)$ be the homology complex of $P_n$ with grading inverted to ensure that differential has degree +1 (thus, $C_*(P_n)$ is concentrated in degrees $-n+1, \ldots, 0$). The notation $\psi = [\psi_1 | \ldots | \psi_d]$ allows to reduce most of the signs below to the Koszul sign rule if we assume that the symbol $| \cdot$ has degree (+1) and each of the elements in $\psi_i$ degree (-1).

The differential of $C_*(P)$, cf. [15], is given by:

$$\partial[\psi_1 | \ldots | \psi_d] = \sum_{1 \leq k \leq d \atop M \subset \nu_k} (-1)^{\nu_k, M} [\psi_1 | \ldots | \psi_{k-1}] M | \psi_k \setminus M | \psi_{k+1} | \ldots | \psi_d].$$

The sign is

$$(-1)^{\psi, M} = (-1)^{m_1 + \ldots + m_{k-1} + (k-1) + \# M (-1)^{\sigma}}$$

where $m_i = \# \psi_i$ and $\sigma_M$ is the unshuffle that takes $\psi_k$ to $[M | \psi_k \setminus M]$ (again, taken with the natural increasing ordering). The symmetric group $\Sigma_n$ acts from the left on each $P(n, d)$ and on $C_*(P_n)$:

$$\sigma[\psi_1 | \ldots | \psi_d] = \pm[\sigma(\psi_1) | \ldots | \sigma(\psi_d)].$$

where the sign is (+1) if the ordering induced from $\psi$ by $\sigma$ is equivalent to the increasing ordering, and (-1) otherwise. In addition, $C_*(P_n)$ has an involution

$$\nu_n[\psi_1 | \psi_2 | \ldots | \psi_{d-1} | \psi_d] = (-1)^{n(d-1) + \sum_{i<j} i \psi_j} (\sum_{i} m_i) \psi_i \psi_{i-1} \ldots \psi_2 | \psi_1]$$

which commutes with the differential and the $\Sigma_n$-action. Therefore, we actually have a $\Sigma_n \times \mathbb{Z}_2$-action on $C_*(P)$.

Define a bilinear map $\Theta : V^\otimes n \times C_*(P_n) \to \Omega \Lambda^*_c(sV)$ by

$$\Theta(v_1 \otimes \ldots \otimes v_n, [\psi_1 | \ldots | \psi_d]) = (-1)^{(n-d)(\sum_{i} \deg v_i)} s^{-1}(s^\otimes m) [v_1 \otimes \ldots \otimes v_n] \cdot \sigma$$

where $\sigma$ is the permutation $\{1, \ldots, n\} \to [\psi_1 | \ldots | \psi_d]$ and each $s^{-1}(s^\otimes m)$ is viewed as a map $V^\otimes m \to s^{-1}(\Lambda^m(V))$, $u_1 \otimes \ldots \otimes u_m \mapsto \pm s^{-1}(s u_1 \wedge \ldots \wedge s u_m)$ with the sign determined by the Koszul rule.

Lemma 6. The map $\Theta$ induces an isomorphism of complexes

$$\Omega \Lambda^*_c(V) \simeq k \otimes \bigoplus_{n \geq 1} V^\otimes n \otimes_{k[\Sigma_n]} C_*(P_n)$$

which takes $\iota_\Omega$ to $1 \otimes \bigoplus_{n \geq 1} (1 \otimes v_n)$.

Proof. Since $\Theta$ maps $(v_1 \otimes \ldots \otimes v_n) \otimes [\psi_1 | \ldots | \psi_d]$ to $\pm \otimes_{i=1}^d s^{-1}(\bigwedge_{i \in \psi} s u_i)$, it is clearly well-defined, i.e. indeed descends to a tensor product over $k[\Sigma_n]$, and surjective.

To prove injectivity, consider all faces of $P_n$ which correspond to ordered partitions $\psi = [\psi_1 | \ldots | \psi_d]$ with fixed $m_i = \# \psi_i$. Denoting $m = (m_1, \ldots, m_d)$ we see that the set of such faces is a single $\Sigma_n$-orbit of

$$\psi_m = [1, \ldots, m_1] (m_1 + 1), \ldots, (m_2 + m_1) \ldots (m_1 + \ldots + m_{d-1} + 1), \ldots, n]$$
If orientations are taken into account, it becomes clear that the line $k \cdot \psi_m \subset C_*(P_n)$ is isomorphic to the sign representation $\rho_m$ of the stabilizer $\Sigma_m = \Sigma_{m_1} \times \cdots \times \Sigma_{m_d} \subset \Sigma_n$. Therefore, the $\Sigma_n$-submodule
\begin{equation}
M_m = \bigoplus_{\{\psi \neq \psi_m, \forall i\} \subset \Sigma_n} k \cdot \psi \subset C_*(P_n)
\end{equation}
is the induced representation $\rho |_{\Sigma_n}^{\Sigma_m_n}$. Therefore, $V_{\pm} \otimes \Sigma_m \rho_m |_{\Sigma_n}^{\Sigma_m_n}$ can be identified with the set of vectors in $V_{\pm} \otimes \Sigma_m$ which acts by the sign representation, i.e. precisely with $\otimes_{i=1}^d s^{-1} \Lambda^{m_i}(V) \subset \Omega \Lambda^{*_n}(V)$, so $\Theta$ is indeed a bijection.

The assertion about involutions follows from the definitions.

To prove that $\Theta$ commutes with the differentials, we work up to signs (leaving the signs to the motivated reader). To simplify notation also assume that $V$ has zero differential (the general case is quite similar). Then denoting by $v(\psi) = s^{-1}(\bigwedge_{i \in \psi} s v_i) \in s^{-1} \Lambda^*(V)$ for any $\psi \subset \{1, \ldots, n\}$ and $v_1, \ldots, v_n \in V$, we have
\[
\Theta(1 \otimes \partial) \left( (v_1 \otimes \cdots \otimes v_n) \otimes [\psi_1] \cdots [\psi_d] \right) = \Theta \left( \sum_{1 \leq k \leq d \atop \varnothing \leq \varnothing \subseteq \varnothing_k} \pm [\psi_1] \cdots [M|\psi_k \setminus M] \cdots [\psi_d] \right) = \\
= \sum_{1 \leq k \leq d \atop \varnothing \leq \varnothing \subseteq \varnothing_k} \pm v(\psi_1) \otimes \cdots \otimes v(M) \otimes v(\psi_k \setminus M) \otimes \cdots \otimes v(\psi_d) = \\
= \delta_\Omega \Theta \left( (v_1 \otimes \cdots \otimes v_n) \otimes [\psi_1] \cdots [\psi_d] \right)
\]
This finishes the proof. \hfill \qed

3.2. Proof of Theorem 1.

Proof. Since $P_n$ is a convex polyhedron, the complex $C_*(P_n)$ has cohomology $k$ in degree 0, and zero everywhere else. Let $F_n : C_*(P_n) \to k$, $G_n : k \to C_*(P_n)$, be the natural $\Sigma_n \times \mathbb{Z}_2$-equivariant projection and embedding, respectively (where $k$ is viewed as a trivial $\Sigma_n \times \mathbb{Z}_2$-module). Since $\text{char } k = 0$, we can find a $\Sigma_n \times \mathbb{Z}_2$-equivariant contracting homotopy $H_n : C_*(P_n) \to C_*(P_n)$. It is well known that we can also assume the side conditions:
\[
H_n G_n = 0, \quad F_n H_n = 0, \quad H_n H_n = 0
\]
(if the first two identities are not satisfied then replace $H_n$ by $H'_n = (1 - G_n F_n) H_n (1 - G_n F_n)$, then if the last identity is not satisfied, replace $H'_n$ by $H''_n = H'_n \partial H'_n$). Using the decomposition of Lemma 6, set
\[
h_V = 0 \oplus \bigoplus_{n \geq 1} (1 \otimes G_n)
\]
By $\Sigma_n \times \mathbb{Z}_2$-equivariance of $H_n$, it follows that $h_V$ is a homotopy contracting $\Omega \Lambda^*_n(V)$ to
\[
k \oplus \bigoplus_{n \geq 1} \left( V_{\pm} \otimes k[\Sigma_m] \right) = \text{Sym}^*(V)
\]
and that $h_V$ commutes with the anti-involution $\iota_\Omega$ as well. \hfill \qed
3.3. Relation with semistandard tableaux. Our original approach to Theorem 1 was based on the equivalent language of semistandard tableaux. The main advantage of using permutahedra is better compatibility with the involution \( \iota \) on \( \Omega L_{\infty}(V) \). On the other hand, semi-standard tableaux give an explicit decomposition of \( \Omega L_{\infty}(V) \) into irreducible \( GL(V) \)-modules (e.g. when \( V \) is a finite dimensional vector space in homological degree 0). These results (perhaps known to experts in combinatorics) are not used in this paper, and the proof is left to the interested reader.

Denote by \( S^\lambda \) the irreducible Specht module corresponding to a partition \( \lambda \) on \( n \), cf. e.g. [14], and recall that its multiplicity in \( M_m \cong \rho_m \uparrow_{\Sigma n}^{\Sigma \lambda} \), cf. (5), can be computed as the number of column-semistandard tableaux \( T \) with content \( m \), cf. Theorem 2.11.2 in loc. cit. Thus, Lemma 6 above will give a decomposition of \( \Omega L_{\infty}(V) \) in terms of Schur complexes. It takes some additional effort to make it compatible with the differential. We use the same notation \( \lambda \) for the corresponding Young diagram.

Choose a \( \lambda \)-tableau \( T \), i.e. a bijective map \( \{ \lambda \} \to \{ 1, \ldots, n \} \) where \( \{ \lambda \} \) is the set of cells in \( \lambda \). Let \( C_T, R_T \subset \Sigma_n \) be the column stabilizer and row stabilizer, respectively, i.e. those permutations which preserve values in the columns, resp. rows of \( T \). Setting

\[
c_T = \sum_{\sigma \in C_T} \sigma; \quad r_T = \sum_{\sigma \in R_T} (-1)^{\sigma} \sigma; \quad e_T = c_T r_T^{-1}
\]

we can define the Schur complex \( S^T(V) = (V \otimes n) e_T \) for any complex of vector spaces \( V \).

Now suppose that \( T \) is standard, i.e. the values increase in rows and columns. Set

\[ J_T = \{ i \mid 1 \leq i \leq n - 1, \text{ and } T^{-1}(i) \text{ is strictly above } T^{-1}(i + 1) \} \subset \{ 1, \ldots, n \} \]

For any subset \( J \subset J_T \) with \( p \) elements consider the unique weakly increasing surjective map

\[ \zeta_J : \{ 1, \ldots, n \} \to \{ 1, \ldots, n - p \} \]

such that \( J = \{ i \mid \zeta_J(i) = \zeta_J(i + 1) \} \). Then the composition

\[ T_J : \{ \lambda \} \xrightarrow{T} \{ 1, \ldots, n \} \xrightarrow{\zeta_J} \{ 1, \ldots, n - p \} \]

is a column-semistandard tableau, i.e. the values increase weakly in the columns and strictly in the rows. It is easy to see that any column-semistandard tableau \( U : \{ \lambda \} \to \{ 1, \ldots, n - p \} \) has the form \( T_J \) for unique \( T \) and \( J \subset J_T \).

**Theorem 7.** One has a direct sum decomposition

\[
\Omega L_{\infty}(V) \cong k \oplus \bigoplus_{\lambda} (C_T \otimes S^T(V))
\]

where \( C_T \) is a combinatorial complex spanned in degree \((-p)\) by \( T_J \) with \( J \subset J_T, \#J = p \) and differential given by

\[
\partial(T_J) = \sum_{j \in J} (-1)^{\#X(J,j)} T_{J \setminus \{j\}}; \quad X(J,j) = \{ i \mid 1 \leq i \leq j - 1, i \notin J \}
\]
To write the isomorphism explicitly, for $J \subset J_T$ let $m(J) = (m(J)_1, \ldots, m(J)_{n-p})$ with $m(J)_i = \zeta(J)(i)$ and $\sigma(m(J)) \in k[\Sigma_n]$ the average of all elements in the corresponding subgroup $\Sigma_{m(J)} \subset \Sigma_n$. Then for $u \in S^T(V) = V^\otimes_p e_T$ we set

$$(T_j \otimes u) \mapsto \frac{1}{m(J)_1! \ldots m(J)_{n-p}!} \pi_J(u \sigma(m(J))) \in \Omega^*_C(V)$$

where $\pi_J$ is induced by the projection (combined with (de)suspensions)

$$V^\otimes_n \to s^{-1} \Lambda^m(J)_1(V) \otimes \ldots \otimes s^{-1} \Lambda^m(J)_{n-p}(V)$$

The complex $C_T$ is isomorphic to the standard Koszul complex on the vector space spanned by elements of $J_T$, hence it admits a homotopy

$$h_T(T_j) = \frac{1}{\#J_T} \sum_{j \in (J_T \setminus J)} (-1)^{\# \Omega(J)} T_{(J \cup j)}.$$

Setting $h_V = \sum_T h_T \otimes 1$ in terms of the decomposition of Theorem 7, we get an explicit functorial homotopy as in Theorem 1. But to ensure that $h_V$ commutes with $\iota_1$, we may have to replace it by $h'_V = \frac{1}{2} (h_V + \iota_1 h_V \iota_1)$ and this has no apparent meaning in terms of semistandard tableaux.

4. Appendix: twisted cochains and $L_\infty, A_\infty$-modules

A degree $+1$ map $\tau : \widetilde{C} \to A$ for a coaugmented DG coalgebra $C$ and an augmented DG algebra $A$, is called a twisted cochain if $\tau$ satisfies

$$\tau d_C + d_A \tau = \mu \circ (\tau \otimes \tau) \circ \Delta$$

where $\Delta : \widetilde{C} \to \widetilde{C} \otimes \widetilde{C}$ is the reduced coproduct of $C$ and $\mu$ is the product in $A$. This conditions guarantees that both the canonical coalgebra morphism $C \to BA$ and the canonical algebra morphism $\Omega C \to A$, induced by $\tau$, commute with differentials. For a general strictly unital $A_\infty$-algebra $A$, the canonical coalgebra morphism $C \to BA$ induced by $\tau$, commutes with differentials precisely when the following generalized twisted cochain condition holds, cf. [1]:

$$\tau d_C + d_A \tau = \sum_{i \geq 2} \mu_i \circ \tau^{\otimes i} \circ \Delta^{(i)}$$

where $\mu_i$ are the products in $A$ and $\Delta^{(i)} : \widetilde{C} \to \widetilde{C}^{\otimes i}$ is the iteration of the reduced coproduct.

If $L$ is an $L_\infty$-algebra then an $L_\infty$-module structure on a vector space $M$ is defined by choosing a differential $d$ on $C(L) \otimes M$ which makes it a DG-comodule over $C(L)$. This differential encodes maps $\Lambda^k(L) \otimes M \to M$ which satisfy a series of quadratic identities arising from $d^2 = 0$. It follows from the definitions that the same structure is also encoded by a twisted cochain $C(L) \to \text{End}(M)$. Similarly, $A_\infty$-modules over an $A_\infty$-algebra $A$ are encoded either by comodule differentials on $BA \otimes M$ or twisted cochains $BA \to \text{End}(M)$.

If $\tau$ is a generalized twisted cochain and $N$ is a DG comodule over $C$, denote by $N \otimes_{\tau} A$ the tensor product $N \otimes A$ with the differential

$$\delta = \delta_N \otimes 1 + 1 \otimes \delta_A + \sum_{s \geq 2} (1 \otimes m_s)(1 \otimes \tau^{\otimes (s-1)} \otimes 1)(\Delta^{(s)} \otimes 1)$$
where \( m_s \) is the \( s \)-th product in \( A \) and \( \Delta_N^{(s)} : N \to N \otimes C^{\otimes(s-1)} \) is the iterated reduced coaction map. The infinite sum makes sense if \( N \) is cocomplete, i.e. \( N = \bigcup_i \text{Ker}(\Delta^{(i)}) \).

On the other hand, if \( M \) is an \( A_\infty \)-module over \( A \) with action maps \( m_s^M : M \otimes A^{\otimes(s-1)} \to M \) then denote by \( M \otimes \tau \) the tensor product \( M \otimes C \) with the differential
\[
\delta = \delta_M \otimes 1 + 1 \otimes \delta_C + \sum_{s \geq 2} (m_s^M \otimes 1)(1 \otimes \tau^{\otimes(s-1)} \otimes 1)(1 \otimes \Delta^{(s)}).
\]

See Section 3 in [2] on how to define these functors on morphisms, and other details.

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