On the Cauchy Problem of 3D Nonhomogeneous Navier-Stokes Equations with Density-Dependent Viscosity and Vacuum

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Abstract

We consider the global existence and large-time asymptotic behavior of strong solutions to the Cauchy problem of the three-dimensional nonhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity and vacuum. We establish some key a priori exponential decay-in-time rates of the strong solutions. Then after using these estimates, we also obtain the global existence of strong solutions in the whole three-dimensional space, provided that the initial velocity is suitably small in the $\dot{H}^{\beta}$-norm for some $\beta \in (1/2, 1]$. Note that this result is proved without any smallness conditions on the initial density. Moreover, the density can contain vacuum states and even have compact support initially.

Keywords: nonhomogeneous Navier-Stokes equations; density-dependent viscosity; global strong solution; exponential decay-in-time; vacuum.

Math Subject Classification: 35Q35; 76N10.

1 Introduction

The nonhomogeneous incompressible Navier-Stokes equations (24) read as follows:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho)d) + \nabla P &= 0, \\
\text{div} u &= 0.
\end{align*}
\]
Here, \( t \geq 0 \) is time, \( x \in \Omega \subset \mathbb{R}^3 \) is the spatial coordinates, and the unknown functions \( \rho = \rho(x, t), \ u = (u^1, u^2, u^3)(x, t) \), and \( P = P(x, t) \) denote the density, velocity, and pressure of the fluid, respectively; The deformation tensor is defined by
\[
d = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right], \tag{1.2}
\]
and the viscosity \( \mu(\rho) \) satisfies the following hypothesis:
\[
\mu \in C^1[0, \infty), \quad \mu(\rho) > 0. \tag{1.3}
\]

We consider the Cauchy problem of (1.1) with \((\rho, u)\) vanishing at infinity and the initial conditions:
\[
\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x), \quad x \in \mathbb{R}^3, \tag{1.4}
\]
for given initial data \( \rho_0 \) and \( m_0 \).

There are lots of literatures on the mathematical study of nonhomogeneous incompressible flow. In particular, the system (1.1) with constant viscosity has been considered extensively. On the one hand, in the absence of vacuum, the global existence of weak solutions and the local existence of strong ones were established in Kazhikov [4,22]. Ladyzhenskaya-Solonnikov [23] first proved the global well-posedness of strong solutions to the initial boundary value problems in both 2D bounded domains (for large data) and 3D ones (with initial velocity small in suitable norms). Recently, the global well-posedness results with small initial data in critical spaces were considered by many people (see [1,10,11,17] and the references therein). On the other hand, when the initial density allows to vanish, the global existence of weak solutions is proved by Simon [29]. The local existence of strong solutions was obtained by Choe-Kim [8] (for 3D bounded and unbounded domains) and Lü-Xu-Zhong [25] (for 2D Cauchy problem) under some compatibility conditions. Recently, for the Cauchy problem in the whole 2D space, Lü-Shi-Zhong [26] obtained the global strong solutions for large initial data. For the 3D case, under some smallness conditions on the initial velocity, Craig-Huang-Wang [9] proved the following interesting result.

\textbf{Proposition 1.1 (9)} Let \( \Omega = \mathbb{R}^3 \). For positive constants \( \bar{\rho} \) and \( \mu \), assume that \( \mu(\rho) \equiv \mu \) in (1.1) and the initial data \((\rho_0, m_0)\) satisfy
\[
\begin{aligned}
0 \leq \rho_0 \leq \bar{\rho}, & \quad \rho_0 \in L^{3/2}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \\
u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \cap D^1_{0,\sigma}(\mathbb{R}^3) \cap D^{2,2}(\mathbb{R}^3), & \quad m_0 = \rho_0 \nu_0,
\end{aligned} \tag{1.5}
\]
and the compatibility condition
\[
-\mu \Delta \nu_0 + \nabla P_0 = \rho_0^{1/2} g, \quad \text{in} \quad \mathbb{R}^3, \tag{1.6}
\]
for some \((P_0, g) \in D^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\). Then, there exists some positive constant \( \varepsilon \) depending only on \( \bar{\rho} \) such that there exists a unique global strong solution to the Cauchy problem (1.1) (1.3) provided \( \|\nu_0\|_{\dot{H}^{1/2}} \leq \mu \varepsilon \). Moreover, the following large time decay rate holds for \( t \geq 1 \),
\[
\|\nabla u(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq \tilde{C} t^{-1/2}, \tag{1.7}
\]
where \( \tilde{C} \) depends on \( \bar{\rho}, \mu \), and \( \|\rho_0^{1/2} \nu_0\|_{L^2(\mathbb{R}^3)} \).
When it comes to the case that the viscosity \( \mu(\rho) \) depends on the density \( \rho \), it is more difficult to investigate the global well-posedness of system (1.1) due to the strong coupling between viscosity coefficient and density. In fact, allowing the density to vanish initially, Lions [24] first obtained the global weak solutions whose uniqueness and regularity are still open even in two spatial dimensions. Later, Desjardins [12] established the global weak solution with higher regularity for 2D case provided that the viscosity \( \mu(\rho) \) is a small perturbation of a positive constant in \( L^\infty \)-norm. Recently, some progress has been made on the well-posedness of strong solutions to (1.1) (see [2,3,7,19,20,27,30] and the reference therein). In particular, on the one hand, when the initial density is strictly away from vacuum, Abidi-Zhang [2] obtained the global strong solutions in whole 2D space under smallness conditions on \( \| \rho_0 \|_{L^2} \) and \( \| \rho_0 \|_{L^\infty} \). On the other hand, for the 3D case, they [3] obtained the global strong solutions on both \( \| u_0 \|_{L^2} \| \nabla u_0 \|_{L^2} \) and \( \| \mu(\rho_0) \|_{L^\infty} \). However, as pointed by Huang-Wang [20], the methods used in [20,30] depend heavily on the boundedness of the domains and little is known for the global well-posedness of strong solutions to the Cauchy problem (1.1)–(1.4) with density-dependent viscosity and vacuum.

Before stating the main results, we first explain the notations and conventions used throughout this paper. Set \( \int f dx \equiv \int_{\mathbb{R}^3} f dx \).

Moreover, for \( 1 \leq r \leq \infty, k \geq 1, \) and \( \beta > 0, \) the standard homogeneous and inhomogeneous Sobolev spaces are defined as follows:

\[
\begin{align*}
L^r(\mathbb{R}^3) & = W^{k,r}(\mathbb{R}^3), \quad W^{k,r}(\mathbb{R}^3) = H^k = W^{k,2}, \\
\| \cdot \|_{{B_1} \cap {B_2}} & = \| \cdot _{B_1} + \| \cdot _{B_2}, \text{ for two Banach spaces } B_1 \text{ and } B_2, \\
D^{k,r} & = D^{k,r}(\mathbb{R}^3) = \{ v \in L^1_{\text{loc}}(\mathbb{R}^3) | \nabla^k v \in L^r(\mathbb{R}^3) \}, \\
D^1 & = \{ v \in L^6(\mathbb{R}^3) | \nabla v \in L^2(\mathbb{R}^3) \}, \\
C_{0,\sigma}^\infty & = \{ f \in C^\infty_0 | \text{ div } f = 0 \}, \quad D_{0,\sigma}^1 = C_{0,\sigma}^\infty \text{ closure in the norm of } D^1, \\
\check{H}^\beta & = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R} | \| f \|_{\check{H}^\beta}^2 = \int | \xi |^{2\beta} | \hat{f}(\xi) |^2 d\xi < \infty \},
\end{align*}
\]

where \( \hat{f} \) is the Fourier transform of \( f \).

Our main result can be stated as follows:

**Theorem 1.2** For constants \( \bar{\rho} > 0, q \in (3, \infty), \) and \( \beta \in (\frac{1}{2}, 1] \), assume that the initial data \((\rho_0, m_0)\) satisfy

\[
0 \leq \rho_0 \leq \bar{\rho}, \quad \rho_0 \in L^{3/2} \cap H^1, \quad \nabla \mu(\rho_0) \in L^q, \quad u_0 \in \check{H}^\beta \cap D_{0,\sigma}^1, \quad m_0 = \rho_0 u_0.
\]  

Then for

\[
\mu \triangleq \min_{\rho \in [0, \bar{\rho}]} \mu(\rho), \quad \bar{\mu} \triangleq \max_{\rho \in [0, \bar{\rho}]} \mu(\rho), \quad M \triangleq \| \nabla \mu(\rho_0) \|_{L^q},
\]

there exists some small positive constant \( \varepsilon_0 \) depending only on \( q, \beta, \bar{\rho}, \mu, \bar{\mu}, \| \rho_0 \|_{L^{3/2}}, \) and \( M \) such that if

\[
\| u_0 \|_{\check{H}^\beta} \leq \varepsilon_0,
\]  

(1.9)
the Cauchy problem (1.1) - (1.4) admits a unique global strong solution \((\rho, u, P)\) satisfying that for any \(0 < \tau < T < \infty\) and \(p \in [2, p_0)\) with \(p_0 \triangleq \min\{6, q\}\),

\[
\begin{aligned}
0 \leq \rho &\in C([0,T]; L^{3/2} \cap H^1), \quad \nabla \mu(\rho) \in C([0,T]; L^q), \\
\nabla u &\in L^\infty(0,T; L^2) \cap L^\infty(\tau, T; W^{1,p_0}) \cap C([\tau,T]; H^1 \cap W^{1,p}), \\
P &\in L^\infty(\tau, T; W^{1,p_0}) \cap C([\tau,T]; H^1 \cap W^{1,p}), \\
\sqrt{\rho} u_t &\in L^2(0,T; L^2) \cap L^\infty(\tau, T; L^2), \\
\nabla u_t &\in L^\infty(\tau, T; L^2) \cap L^2(\tau, T; L^{p_0}), \\
(\rho u_t)_{t} &\in L^2(\tau, T; L^2).
\end{aligned}
\]  

(1.10)

Moreover, it holds that

\[
\sup_{0 \leq t < \infty} \|\nabla \rho\|_{L^2} \leq 2\|\nabla \rho_0\|_{L^2}, \quad \sup_{0 \leq t < \infty} \|\nabla \mu(\rho)\|_{L^q} \leq 2\|\nabla \mu(\rho_0)\|_{L^q},
\]  

(1.11)

and that there exists some positive constant \(\sigma\) depending only on \(\|\rho_0\|_{L^{3/2}}\) and \(\mu\) such that for all \(t \geq 1\),

\[
\|\nabla u_t(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{H^{1/2} \cap W^{1,p_0}}^2 + \|P(\cdot, t)\|_{H^{1/2} \cap W^{1,p_0}}^2 \leq Ce^{-\sigma t},
\]  

(1.12)

where \(C\) depends only on \(q, \beta, \rho, \|\rho_0\|_{L^{3/2}}, \mu, \bar{\mu}, M, \|\nabla u_0\|_{L^2}\), and \(\|\nabla \rho_0\|_{L^2}\).

As a direct consequence, our method can be applied to the case that \(\mu(\rho) \equiv \mu\) is a positive constant and obtain the following global existence and large-time behavior of the strong solutions which improves slightly those due to Craig-Huang-Wang [9] (see Proposition (1.1)).

**Theorem 1.3** For constants \(\bar{\rho} > 0\) and \(\mu > 0\), assume that \(\mu(\rho) \equiv \mu\) in (1.1) and the initial data \((\rho_0, u_0)\) satisfy (1.5) except \(u_0 \in D^{2,2}\). Then, there exists some positive constant \(\varepsilon\) depending only on \(\bar{\rho}\) such that there exists a unique global strong solution to the Cauchy problem (1.1) - (1.4) satisfying (1.10) with \(p_0 = 6\) provided \(\|u_0\|_{H^{1/2}} \leq \mu \varepsilon\). Moreover, it holds that

\[
\sup_{0 \leq t < \infty} \|\nabla \rho\|_{L^2} \leq 2\|\nabla \rho_0\|_{L^2},
\]  

(1.13)

and that there exists some positive constant \(\sigma\) depending only on \(\|\rho_0\|_{L^{3/2}}\) and \(\mu\) such that for \(t \geq 1\),

\[
\|\nabla u_t(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{H^{1/2} \cap W^{1,6}}^2 + \|P(\cdot, t)\|_{H^{1/2} \cap W^{1,6}}^2 \leq Ce^{-\sigma t},
\]  

(1.14)

where \(C\) depends only on \(\bar{\rho}, \mu, \|\rho_0\|_{L^{3/2}}, \|\nabla u_0\|_{L^2}, \text{ and } \|\nabla \rho_0\|_{L^2}\).

A few remarks are in order.

**Remark 1.1** To the best of our knowledge, the exponential decay-in-time properties (1.12) in Theorem 1.2 are new and somewhat surprising, since the known corresponding decay-in-time rates for the strong solutions to system (1.1) are algebraic even for the constant viscosity case [4, 9] and the homogeneous case [6, 10, 21, 28]. Moreover, as a direct consequence of (1.11), \(\|\nabla \rho(\cdot, t)\|_{L^2}\) remains uniformly bounded with respect to \(t\) which is new even for the constant viscosity case (see [9] or Proposition (1.1)).
Remark 1.2 It should be noted here that our Theorem 1.2 holds for any function \( \mu(\rho) \) satisfying (1.3) and for arbitrarily large initial density with vacuum (even has compact support) with a smallness assumption only on the \( \dot{H}^\beta \)-norm of the initial velocity \( u_0 \) with \( \beta \in (1/2, 1] \), which is in sharp contrast to Abidi-Zhang [3] where they need the initial density strictly away from vacuum and the smallness assumptions on both \( \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \) and \( \|\mu(\rho_0) - 1\|_{L^\infty} \).

Remark 1.3 For our case that the viscosity \( \mu(\rho) \) depends on \( \rho \), in order to bound the \( L^p \)-norm of the gradient of the density, we need the smallness conditions on the \( \dot{H}^\beta \)-norm (\( \beta \in (1/2, 1] \)) of the initial velocity. However, it seems that our conditions on the initial velocity may be optimal compared with the constant viscosity case considered by Craig-Huang-Wang [9] where they proved that the system (1.1) is globally wellposed for small initial data in the homogeneous Sobolev space \( \dot{H}^{1/2} \) which is similar to the case of homogeneous Navier-Stokes equations (see [13]). Note that for the case of initial-boundary-value problem in 3D bounded domains, Huang-Wang [20] and Zhang [30] impose smallness conditions on \( \|\nabla u_0\|_{L^2} \). Furthermore, in our Theorems 1.2 and 1.3, there is no need to imposed additional initial compatibility conditions, which is assumed in [9, 20, 30] for the global existence of strong solutions.

Remark 1.4 It is easy to prove that the strong-weak uniqueness theorem [24, Theorem 2.7] still holds for the initial data \((\rho_0, u_0)\) satisfying (1.8) after modifying its proof slightly. Therefore, our Theorem 1.2 can be regarded as the uniqueness and regularity theory of Lions’s weak solutions [24] with the initial velocity suitably small in the \( \dot{H}^\beta \)-norm.

Remark 1.5 In [7], Cho-Kim considered the initial boundary value problem in 3D bounded smooth domains. In addition to (1.8), assuming that the initial data satisfy the following compatibility conditions

\[-\text{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 = \rho_0^{1/2}g\]

for some \((P_0, g) \in H^1 \times L^2\), it is shown (2) that the local-in-time strong solution \((\rho, u)\) satisfies

\[\rho u_t \in C([0, T]; L^2).\]  

(1.15)

However, to obtain (1.15), it seems difficult to follow the proof of (1.15) as in [7]. Indeed, in our Proposition 3.7, we give a complete new proof to show that \( \rho u_t \in H^1(\tau, T; L^2) \) (for any \( 0 < \tau < T < \infty \)) which directly implies

\[\rho u_t \in C([\tau, T]; L^2).\]  

(1.16)

In fact, (1.16) is crucial for deriving the time-continuity of \( \nabla u \) and \( P \), that is (see (1.10)),

\[\nabla u, P \in C([\tau, T]; H^1 \cap W^{1,p}).\]  

(1.17)

We now make some comments on the analysis in this paper. To extend the local strong solutions whose existence is obtained by Lemma 2.1 globally in time, one needs to establish global a priori estimates on smooth solutions to (1.1)–(1.4) in suitable higher norms. It turns out that as in the 3D bounded case [20, 30], the key ingredient here is to get the time-independent bounds on the \( L^1(0, T; L^\infty) \)-norm of \( \nabla u \) and then the
In this section we shall enumerate some auxiliary lemmas.

2 Preliminaries

Assume that \( \rho_0, u_0 \) satisfies (1.8) except \( u_0 \in H^\beta \). Then there exist a small time \( T_0 > 0 \) and a unique strong solution \( (\rho, u, P) \) to the problem (1.1)–(1.4) in \( \mathbb{R}^3 \times (0, T_0) \) satisfying (1.10).

The following regularity results on the Stokes equations will be useful for our derivation of higher order a priori estimates.

Lemma 2.2 For positive constants \( \mu, \bar{\mu} \), and \( q \in (3, \infty) \), in addition to (1.3), assume that \( \mu(\rho) \) satisfies

\[
\nabla \mu(\rho) \in L^9, \quad 0 < \mu \leq \mu(\rho) \leq \bar{\mu} < \infty.
\]

Then, if \( F \in L^{6/5} \cap L^r \) with \( r \in [2q/(q + 2), q] \), there exists some positive constant \( C \) depending only on \( \mu, \bar{\mu}, r, \) and \( q \) such that the unique weak solution \( (u, P) \in D_{0,\sigma}^1 \times L^2 \) to the following Cauchy problem

\[
\begin{align*}
- \text{div}(2\mu(\rho)u) + \nabla P &= F, \quad x \in \mathbb{R}^3, \\
\text{div}u &= 0, \quad \quad \quad x \in \mathbb{R}^3, \\
u(x) &\to 0, \quad \quad \quad \quad \quad |x| \to \infty,
\end{align*}
\]

satisfies

\[
\|\nabla u\|_{L^2} + \|P\|_{L^2} \leq C\|F\|_{L^{6/5}},
\]
\[
\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \leq C\|F\|_{L^r} + C\|\nabla \mu(\rho)\|_{L^q}^{\frac{q(2r-6)}{6(r-3)}} \|F\|_{L^{6/5}}. \quad (2.4)
\]

Moreover, if \( F = \text{div} v \) with \( v \in L^2 \cap L^\tilde{r} \) for some \( \tilde{r} \in (6q/(q+6), q] \), there exists a positive constant \( C \) depending only on \( \mu, \tilde{\mu}, q \), and \( \tilde{r} \) such that the unique weak solution \((u, P) \in D_{0,\sigma}^1 \times L^2 \) to (2.2) satisfies

\[
\|\nabla u\|_{L^2 \cap L^q} + \|P\|_{L^2 \cap L^\tilde{r}} \leq C\|g\|_{L^2 \cap L^\tilde{r}} + C\|\nabla \mu(\rho)\|_{L^\tilde{q}}^{\frac{3\tilde{q}(\tilde{r}-2)}{2\tilde{q}(q-3)}} \|g\|_{L^2}. \quad (2.5)
\]

**Proof.** First, multiplying (2.2) by \( u \) and integrating by parts, we obtain after using (2.2)_2 that

\[
2 \int \mu(\rho)|d|^2 dx = \int F \cdot u dx \leq \|F\|_{L^{6/5}} \|u\|_{L^6} \leq C\|F\|_{L^{6/5}} \|\nabla u\|_{L^2},
\]

which together with (2.1) yields

\[
\|\nabla u\|_{L^2} \leq C\mu^{-1} \|F\|_{L^{6/5}}, \quad (2.6)
\]
due to

\[
2 \int |d|^2 dx = \int \|\nabla u\|^2 dx. \quad (2.7)
\]

Furthermore, it follows from (2.2)_1 that

\[
P = -(-\Delta)^{-1} \text{div} F - (-\Delta)^{-1} \text{div} \text{div}(2\mu(\rho)d),
\]

which together with the Sobolev inequality and (2.7) gives

\[
\|P\|_{L^2} \leq \|(-\Delta)^{-1} \text{div} F\|_{L^2} + \|2\mu(\rho) d\|_{L^2} \leq C\|F\|_{L^{6/5}} + C\|\nabla u\|_{L^2}.
\]

Combining this with (2.6) leads to (2.3).

Next, we rewrite (2.2)_1 as

\[
-\Delta u + \nabla \left( \frac{P}{\mu(\rho)} \right) = \frac{F}{\mu(\rho)} + \frac{2d \cdot \nabla \mu(\rho)}{\mu(\rho)} - \frac{P \nabla \mu(\rho)}{\mu(\rho)^2}. \quad (2.8)
\]

Applying the standard \( L^p \)-estimates to the Stokes system (2.3) (2.2)_2 (2.2)_3 yields that for \( r \in [2q/(q+2), q] \),

\[
\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \leq \|\nabla^2 u\|_{L^r} + C\left\|\nabla \left( \frac{P}{\mu(\rho)} \right) \right\|_{L^r} + C\left\|\frac{P \nabla \mu(\rho)}{\mu(\rho)^2}\right\|_{L^r} \\
\leq C\|F\|_{L^r} + C\|2d \cdot \nabla \mu(\rho)\|_{L^r} + C\left\|\frac{P \nabla \mu(\rho)}{\mu(\rho)^2}\right\|_{L^r} \\
\leq C\|F\|_{L^r} + C\|\nabla \mu(\rho)\|_{L^q}^2 \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^r} \|\nabla P\|_{L^2}^2 \|\nabla P\|_{L^r} \\
+ C\|\nabla \mu(\rho)\|_{L^q}^2 \|\nabla P\|_{L^2} \|\nabla P\|_{L^r} \|\nabla P\|_{L^r} \|\nabla P\|_{L^r} \|\nabla P\|_{L^r} \\
\leq C\|F\|_{L^r} + C\|\nabla \mu(\rho)\|_{L^q} \|\nabla u\|_{L^2} + \|P\|_{L^2} + \frac{1}{2} \left( \|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \right),
\]

which combined with (2.3) yields (2.4).
Finally, we will prove (2.5). Multiplying (2.2) by $u$ and integrating by parts lead to

$$
4 \int \mu(\rho)|d|^2 dx = -2 \int g \cdot \nabla u dx \leq \mu \|
abla u\|^2_{L^2} + C\|g\|^2_{L^2},
$$

which together with (2.7) gives

$$
\|\nabla u\|_{L^2} \leq C\|g\|_{L^2}. \tag{2.9}
$$

It follows from (2.2) that

$$
P = -(-\Delta)^{-1}\text{divdiv} g - (-\Delta)^{-1}\text{divdiv}(2\mu(\rho)d),
$$

which implies that for any $p \in [2, \tilde{r}]$,

$$
\|P\|_{L^p} \leq C(p)\|
abla u\|_{L^p} + C(p)\|g\|_{L^p}. \tag{2.10}
$$

In particular, this combined with (2.9) shows

$$
\|P\|_{L^2} + \|\nabla u\|_{L^2} \leq C\|g\|_{L^2}. \tag{2.11}
$$

Next, we rewrite (2.2) as

$$
-\Delta u + \nabla \left( \frac{P}{\mu(\rho)} \right) = \text{div} \left( \frac{g}{\mu(\rho)} \right) + \tilde{G}, \tag{2.12}
$$

where

$$
\tilde{G} \triangleq \frac{g \cdot \nabla \mu(\rho)}{\mu(\rho)^2} + \frac{2d \cdot \nabla \mu(\rho)}{\mu(\rho)} - \frac{P \nabla \mu(\rho)}{\mu(\rho)^2}
$$

satisfies for any $\varepsilon > 0$,

$$
\|\tilde{G}\|_{L^{\frac{2(q-2)}{q(\tilde{r}-1)}}} \leq \varepsilon (\|g\|_{L^2} + \|\nabla u\|_{L^2} + \|P\|_{L^2}) + C(\varepsilon)\|
abla \mu(\rho)\|_{L^2} (\|g\|_{L^2} + \|\nabla u\|_{L^2} + \|P\|_{L^2}). \tag{2.13}
$$

Using (2.12) and (2.2)3, we have

$$
\|\nabla u\|_{L^p} \leq C\|\nabla \times u\|_{L^p}
$$

$$
= C\|(-\Delta)^{-1}\nabla \times \text{div}(g(\mu(\rho))^{-1}) + (-\Delta)^{-1}\nabla \times \tilde{G}\|_{L^p}
$$

$$
\leq C\|g\|_{L^p} + C\|\tilde{G}\|_{L^{\frac{2q}{2(q-2)}}},
$$

which together with (2.10) yields

$$
\|\nabla u\|_{L^p} + \|P\|_{L^p} \leq C\|g\|_{L^p} + C\|\tilde{G}\|_{L^{\frac{2q}{2(q-2)}}}.
$$

Combining this, (2.13), and (2.11) gives (2.5). The proof of Lemma 2.2 is finished. □
3 A Priori Estimates

In this section, we will establish some necessary a priori bounds of local strong solutions \((\rho, u, P)\) to the Cauchy problem \((1.1) - (1.4)\) whose existence is guaranteed by Lemma 2.1. Thus, let \(T > 0\) be a fixed time and \((\rho, u, P)\) be the smooth solution to \((1.1) - (1.4)\) on \(\mathbb{R}^3 \times (0, T]\) with smooth initial data \((\rho_0, u_0)\) satisfying \((1.5)\).

We have the following key a priori estimates on \((\rho, u, P)\).

**Proposition 3.1** There exists some positive constant \(\varepsilon_0\) depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, \bar{\mu}, \|\rho_0\|_{L^{3/2}}, M\) such that if \((\rho, u, P)\) is a smooth solution of \((1.1) - (1.4)\) on \(\mathbb{R}^3 \times (0, T]\) satisfying

\[
\sup_{t \in [0,T]} \|\nabla \mu(\rho)\|_{L^3} \leq 4M, \quad \int_0^T \|\nabla u\|_{L^2}^2 dt \leq 2\|u_0\|_{H^{3/2}}^2, \tag{3.1}
\]

the following estimates hold

\[
\sup_{t \in [0,T]} \|\nabla \mu(\rho)\|_{L^3} \leq 2M, \quad \int_0^T \|\nabla u\|_{L^2}^4 dt \leq \|u_0\|_{H^{3/2}}^2, \tag{3.2}
\]

provided \(\|u_0\|_{H^{3/2}} \leq \varepsilon_0\).

Before proving Proposition 3.1 we establish some necessary a priori estimates, see Lemmas 3.2, 3.5.

We start with the following time-weighted estimates on the \(L^\infty(0,\min\{1,T\}; L^2)\)-norm of the gradient of velocity.

**Lemma 3.2** Let \((\rho, u, P)\) be a smooth solution to \((1.1) - (1.4)\) satisfying \((3.1)\). Then there exists a generic positive constant \(C\) depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, M\) such that

\[
\sup_{t \in [0,\zeta(T)]} \left(t^{1-\beta}\|\nabla u\|_{L^2}^2\right) + \int_0^{\zeta(T)} t^{1-\beta}\|\rho^{1/2} u_t\|_{L^2}^2 dt \leq C\|u_0\|_{H^{3/2}}^2, \tag{3.3}
\]

where \(\zeta(t)\) is defined by

\[
\zeta(t) \triangleq \min\{1, t\}.
\]

**Proof.** First, standard arguments \((24)\) imply that

\[
0 \leq \rho \leq \bar{\rho}, \quad \|\rho\|_{L^{3/2}} = \|\rho_0\|_{L^{3/2}}. \tag{3.4}
\]

Next, for fixed \((\rho, u)\), we consider the following linear Cauchy problem for \((w, \tilde{P})\):

\[
\begin{aligned}
\rho w_t + \rho u \cdot \nabla w - \text{div} (\mu(\rho) [\nabla w + (\nabla w)^T]) + \nabla \tilde{P} &= 0, \quad x \in \mathbb{R}^3, \\
\text{div} w &= 0, \quad x \in \mathbb{R}^3, \\
w(x,0) &= w_0, \quad x \in \mathbb{R}^3.
\end{aligned} \tag{3.5}
\]

It follows from Lemmas 2.2, (3.4), (3.1), (3.4), and the Garliardo-Nirenberg inequality that

\[
\|\nabla w\|_{H^1} + \|\tilde{P}\|_{H^1} \leq C \left(\|\rho w_t + \rho u \cdot \nabla w\|_{L^2} + \|\rho w_t + \rho u \cdot \nabla w\|_{L^{6/5}}\right)
\leq C(\bar{\rho}^{1/2} + \|\rho\|_{L^{3/2}}^{1/2}) \left(\|\rho^{1/2} w_t\|_{L^2} + \bar{\rho}^{1/2}\|u \cdot \nabla w\|_{L^2}\right)
\leq C\|\rho^{1/2} w_t\|_{L^2} + C\|\nabla u\|_{L^2} \|\nabla w\|_{L^2} + \|\nabla^2 w\|_{L^2}^{1/2}
\leq C\|\rho^{1/2} w_t\|_{L^2} + C\|\nabla u\|_{L^2}^2 \|\nabla w\|_{L^2} + \frac{1}{2} \|\nabla^2 w\|_{L^2},
\]

where \(C\) is a generic positive constant depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, M\).
which directly yields that
\[
\|\nabla w\|_{H^1} + \|\tilde{P}\|_{H^1} + \|\rho w_t + \rho u \cdot \nabla w\|_{L^{6/5}\cap L^2} \\
\leq C\|\rho^{1/2} w_t\|_{L^2} + C\|\nabla u\|_{L^2}^2 \|\nabla w\|_{L^2}. 
\] (3.6)

Multiplying (3.5) by \( w_t \) and integrating the resulting equality by parts lead to
\[
\frac{1}{4} \frac{d}{dt} \int \mu(\rho) |\nabla w + (\nabla w)^T|^2 \, dx + \int \rho |w_t|^2 \, dx \\
= -\int \rho u \cdot \nabla w \cdot w_t \, dx + \frac{1}{4} \int \mu(\rho) u \cdot \nabla |\nabla w + (\nabla w)^T|^2 \, dx \\
\leq \tilde{\rho}^{1/2} \|\rho^{1/2} w_t\|_{L^2} \|u\|_{L^6} \|\nabla w\|_{L^3} + C \mu \|u\|_{L^6} \|\nabla w\|_{L^3} \|\nabla^2 w\|_{L^2} \\
\leq C \|\rho^{1/2} w_t\|_{L^2} \|\nabla u\|_{L^2} \|\nabla w\|_{L^2}^{1/2} \|\nabla^2 w\|_{L^2}^{1/2} + C \|\nabla u\|_{L^2} \|\nabla w\|_{L^2}^{1/2} \|\nabla^2 w\|_{L^2}^{3/2} \\
\leq \frac{3}{4} \|\rho^{1/2} w_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla w\|_{L^2}^2,
\] (3.7)

where in the last inequality one has used (3.6). This combined with Grönwall’s inequality and (3.1) yields
\[
\sup_{t \in [0,\zeta(T)]} \int |\nabla w|^2 \, dx + \int_0^{\zeta(T)} \int \rho |w_t|^2 \, dx \, dt \leq C \|w_0\|_{L^2}^2. \tag{3.8}
\]

Furthermore, multiplying (3.7) by \( t \) leads to
\[
\frac{d}{dt} \left( \int \mu(\rho) |\nabla w + (\nabla w)^T|^2 \, dx \right) + \int \rho |w_t|^2 \, dx \\
\leq C t \|\nabla w\|_{L^2}^2 \|\nabla u\|_{L^2}^4 + C \|\nabla w\|_{L^2}^2. 
\]

Combining this with Grönwall’s inequality and (3.1) shows
\[
\sup_{t \in [0,\zeta(T)]} t \int |\nabla w|^2 \, dx + \int_0^{\zeta(T)} t \int \rho |w_t|^2 \, dx \, dt \leq C \|w_0\|_{L^2}^2, \tag{3.9}
\]

where one has used the following simple fact
\[
\sup_{t \in [0,\zeta(T)]} \|\rho^{1/2} w_t\|_{L^2}^2 + \int_0^{\zeta(T)} \|\nabla w\|_{L^2}^2 \, dt \leq C \|w_0\|_{L^2}^2,
\]

which can be obtained by multiplying (3.5) by \( w \) and integrating by parts.

Hence, the standard Stein-Weiss interpolation arguments (see [5]) together with (3.8) and (3.9) imply that for any \( \theta \in [\beta,1] \),
\[
\sup_{t \in [0,\zeta(T)]} t^{1-\theta} \int |\nabla w|^2 \, dx + \int_0^{\zeta(T)} t^{1-\theta} \int \rho |w_t|^2 \, dx \, dt \leq C(\theta) \|w_0\|_{H^\theta}^2. \tag{3.10}
\]

Finally, taking \( w_0 = u_0 \), the uniqueness of strong solutions to the linear problem (3.5) implies that \( w \equiv u \). The estimate (3.3) thus follows from (3.10). The proof of Lemma 3.2 is finished. \( \Box \)

As an application of Lemma 3.2 we have the following time-weighted estimates on \( \|\rho^{1/2} u_t\|_{L^2}^2 \) for small time.
Lemma 3.3 Let \((\rho, u, P)\) be a smooth solution to (1.1)–(1.3) satisfying (3.1). Then there exists a generic positive constant \(C\) depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, \mu_0\|L^{3/2}\), and \(M\) such that

\[
\sup_{t \in [0, T]} \left( t^{2-\beta} |\rho|^{1/2} u_t |L^2| \right) + \int_0^T t^{2-\beta} |\nabla u_t |^2 L^2 dt \leq C \|u_0\|^2_{H^\beta}.
\] (3.11)

Proof. First, operating \(\partial_t\) to (1.1) yields that

\[
\rho u_{tt} + pu \cdot \nabla u_t - \text{div}(2\mu(\rho) du) + \nabla P_t = -\rho u_t - (pu) \cdot \nabla u + \text{div}(2(\mu(\rho))_t d).
\] (3.12)

Multiplying the above equality by \(u_t\), we obtain after using integration by parts and (1.1) that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int 2\mu(\rho) |du|^2 dx
\]

\[
= -2 \int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot (u \cdot \nabla u \cdot u_t) dx
\]

\[
- \int \rho u_t \cdot \nabla u \cdot u_t dx + 2 \int (u \cdot \nabla \mu(\rho)) d \cdot u_t dx \triangleq \sum_{i=1}^4 J_i.
\] (3.13)

Now, we will use the Gagliardo-Nirenberg inequality, (3.1), and (3.4) to estimate each term on the right hand of (3.13) as follows:

\[
|J_1| + |J_3| \leq C(\|\rho^{1/2} u_t |L^1\| \nabla u_t |L^2\| u_t |L^6\| + C(\|\rho^{1/2} u_t |L^3\| \nabla u |L^2\| u_t |L^6\|
\]

\[
\leq C(\|\rho^{1/2} u_t |L^1\| \nabla u_t |L^2\| \|\nabla u \|^3 u_t |L^6\|)
\]

\[
\leq |1| 4 \mu \|\nabla u_t |L^2\| + C(\|\rho^{1/2} u_t |L^2\| \nabla u |L^2\|)
\] (3.14)

\[
|J_2| = \left| \int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \right|
\]

\[
\leq C \int \rho |u_t| |u_t| \left( |\nabla u|^2 + |u| |\nabla u|^2 \right) dx + \int \rho |u_t|^2 |\nabla u_t| dx
\]

\[
\leq C \|u_t |L^6\| \|u_t |L^6\| \left( |\nabla u|^2 + |u| |\nabla u|^2 \right) + C \|u_t |L^6\| \|\nabla u_t |L^2\|
\]

\[
\leq C \|\nabla u_t |L^2\| \nabla^2 u |L^2\| \|\nabla u_t |L^2\|
\]

\[
\leq 1 \frac{1}{8 |\mu|} \|\nabla u_t |L^2\|^2 + C \|\nabla u |L^2\| \|\nabla u |L^2\|
\] (3.15)

and

\[
|J_4| \leq C \|\nabla \mu(\rho) |L^6\| \|u_t |L^6\| \|\nabla u_t |L^2\| \nabla u |L^2\|
\]

\[
\leq C(q, M) \|u_t |L^6\| \|\nabla u |L^6\| \|\nabla u_t |L^2\| \|\nabla u_t |L^2\| \|u_t |L^6\| \|\nabla u |L^2\|
\]

\[
\leq 1 \frac{1}{8 |\mu|} \|\nabla u_t |L^2\|^2 + C \|\nabla u |L^2\| \|\nabla u |L^2\|^2 + C \|\nabla u |L^2\|
\] (3.16)

Substituting (3.14)–(3.16) into (3.13) gives

\[
\frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx
\]

\[
\leq C \left( \|\rho^{1/2} u_t |L^2\| + \|\nabla u_t |L^2\| \right) \|\nabla u_t |L^2\| + C \|\nabla u |L^2\| \|\nabla u |L^2\|^3 + C \|\nabla u |L^2\|
\]

\[
\leq C \|\rho^{1/2} u_t |L^2\| \|\nabla u |L^2\| + C \|\rho^{1/2} u_t |L^2\| \|\nabla u |L^2\| + C \|\nabla u |L^2\| + C \|\nabla u |L^2\|
\] (3.17)
where in the last inequality one has used
\[
\|\nabla u\|_{H^1} + \|P\|_{H^1} + \|\rho(u_t + u \cdot \nabla u)\|_{L^{6/5} \cap L^2} \\
\leq C \left( \|\rho^{1/2} u_t\|_{L^2} + \|\nabla u\|^2_{L^2} \right), \tag{3.18}
\]
which can be obtained by taking \(w \equiv u\) in (3.6). It thus follows from (3.17) and (3.3) that for \(t \in (0, \zeta(T))\),
\[
\frac{d}{dt} \int \rho|u_t|^2 \, dx + \mu \int |\nabla u|^2 \, dx \\
\leq C \|\rho^{1/2} u_t\|_{L^2}^2 \left( \|\nabla u\|^4_{L^2} + \|\rho^{1/2} u_t\|_{L^2} \|\nabla u\|_{L^2} \right) \\
+ C t^{3(\beta-1)} \|\nabla u\|^4_{L^2} + C \|\nabla u\|^2_{L^2}. \tag{3.19}
\]

Since (3.3) implies
\[
\int_0^{\zeta(T)} \|\rho^{1/2} u_t\|_{L^2} \|\nabla u\|_{L^2} \, dt \\
\leq C \sup_{0 \leq t \leq \zeta(T)} \left( t^{1-\beta} \|\nabla u\|_{L^2} \right) \left( \int_0^{\zeta(T)} t^{1-\beta} \|\rho u_t\|_{L^2} \, dt \right)^{1/2} \left( \int_0^{\zeta(T)} t^{3(\beta-2)} \, dt \right)^{1/2} \\
\leq C \|u_0\|_{H^\beta}^2,
\]
we multiply (3.19) by \(t^{2-\beta}\) and use Grönwall’s inequality, (3.1), and (3.3) to obtain (3.11). The proof of Lemma 3.3 is finished. \(\square\)

Next, we will prove the following exponential decay-in-time estimates on the solutions for large time, which plays a crucial role in our analysis.

**Lemma 3.4** Let \((\rho, u, P)\) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then for
\[
\sigma \triangleq 3\mu/(4\|\rho_0\|_{L^{3/2}}), \tag{3.20}
\]
there exists a generic positive constant \(C\) depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, \|\rho_0\|_{L^{3/2}}\), and \(M\) such that
\[
\sup_{t \in [0, T]} e^{\sigma t} \|\rho^{1/2} u\|_{L^2}^2 + \int_0^T e^{\sigma t} \int |\nabla u|^2 \, dx \, dt \leq C \|u_0\|_{H^\beta}^2, \tag{3.21}
\]
\[
\sup_{t \in [\zeta(T), T]} e^{\sigma t} \int |\nabla u|^2 \, dx + \int_{\zeta(T)}^T e^{\sigma t} \int \rho |u_t|^2 \, dx \, dt \leq C \|u_0\|_{H^\beta}^2, \tag{3.22}
\]
\[
\sup_{t \in [\zeta(T), T]} e^{\sigma t} \int \rho |u_t|^2 \, dx + \int_{\zeta(T)}^T e^{\sigma t} \int |\nabla u|^2 \, dx \, dt \leq C \|u_0\|_{H^\beta}^2, \tag{3.23}
\]
and
\[
\sup_{t \in [\zeta(T), T]} e^{\sigma t} \left( \|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 \right) \leq C \|u_0\|_{H^\beta}^2. \tag{3.24}
\]

**Proof.** First, multiplying (1.1)2 by \(u\) and integrating by parts lead to
\[
\frac{1}{2} \frac{d}{dt} \|\rho^{1/2} u\|_{L^2}^2 + \int 2\mu(\rho)|d|^2 \, dx = 0. \tag{3.25}
\]
It follows from the Sobolev inequality [14, (II.3.11)], (3.1), and (2.7) that
\[ \|\rho^{1/2}u\|_{L^2}^2 \leq \|\rho\|_{L^{3/2}} \|u\|_{L^6}^2 \leq \frac{4}{3} \|\rho_0\|_{L^{3/2}} \|\nabla u\|_{L^2}^2 \leq \sigma^{-1} \int 2\mu(\rho)|d|^2dx, \] (3.26)
with \(\sigma\) is defined as in (3.20). Putting (3.26) into (3.25) yields
\[ \frac{d}{dt}\|\rho^{1/2}u\|_{L^2}^2 + \sigma\|\rho^{1/2}u\|_{L^2}^2 + \int 2\mu(\rho)|d|^2dx \leq 0, \]
which together with Grönwall’s inequality gives
\[ \sup_{t\in[0,T]} e^{\sigma t}\|\rho^{1/2}u\|_{L^2}^2 + \int_0^T e^{\sigma t}\int |\nabla u|^2dxdt \leq C\|\rho_0^{1/2}u_0\|_{L^2}^2 \leq C\|\rho_0\|_{L^{3/2}}^2 \leq C\|u_0\|_{H^3}^2, \] (3.27)
due to \(\beta \in (1/2,1] \).

Next, similar to (3.7), we have
\[ \frac{d}{dt}\int 2\mu(\rho)|d|^2dx + \int \rho|u_t|^2dx \leq C\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2, \] (3.28)
which combined with Grönwall’s inequality, (3.27), (3.3), and (3.1) gives (3.22).

Furthermore, multiplying (3.17) by \(e^{\sigma t}\), we obtain (3.23) after using Grönwall’s inequality, (3.11), (3.1), (3.21), and (3.22).

Finally, it follows from (3.15), (3.22), and (3.23) that (3.24) holds. The proof of Lemma 3.4 is completed.

We will use Lemmas 3.2–3.4 to prove the following time-independent bound on the \(L^1(0,T;L^\infty)\)-norm of \(\nabla u\) which is important for obtaining the uniform one (with respect to time) on the \(L^\infty(0,T;L^q)\)-norm of the gradient of \(\mu(\rho)\).

**Lemma 3.5** Let \((\rho,u,P)\) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a generic positive constant \(C\) depending only on \(q,\beta,\bar{\rho},\bar{\mu},\|\rho_0\|_{L^{3/2}},\) and \(M\) such that
\[ \int_0^T \|\nabla u\|_{L^\infty}dt \leq C\|u_0\|_{H^3}. \] (3.29)

**Proof.** First, it follows from the Gagliardo-Nirenberg inequality that for any \(p \in [2,\min\{6,q\}],\)
\[ \|\rho^{1/2}u_t + \rho u \cdot \nabla u\|_{L^p} \leq C\|\rho^{1/2}w_1\|_{L^2}^{6p/(4p-6)} \|\rho^{1/2}w_4\|_{L^6}^{2p/(3p-6)} + C\|u\|_{L^p} \|\nabla u\|_{L^{6p/(4p-6)}}, \]
\[ \leq C\|\rho^{1/2}w_1\|_{L^2}^{6p/(4p-6)} \|\nabla u_t\|_{L^2}^{3p-6} + C\|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^p}^{4p-6}, \]
which together with (2.4) and (3.18) gives
\[ \|\nabla^2 u\|_{L^p} + \|\nabla P\|_{L^p} \leq C\|\rho u_t + \rho u \cdot \nabla u\|_{L^{6/5}} \|\nabla u_t\|_{L^2}^{2p/(3p-6)} + C\|\nabla u\|_{L^2}^{6p-6} \] (3.30)
\[ + \frac{1}{2}\|\nabla^2 u\|_{L^p} + C\|\rho^{1/2}w_1\|_{L^2} + C\|\nabla u\|_{L^2}^2. \]
Then, setting
\[ r \triangleq \frac{1}{2} \min \left\{ q + 3, \frac{3(5 - 2\beta)}{3 - 2\beta} \right\} \in \left( 3, \min \left\{ q, \frac{6}{3 - 2\beta} \right\} \right), \tag{3.31} \]
one derives from the Sobolev inequality and (3.30) that
\[ \| \nabla u \|_{L^\infty} \leq C\| \nabla u \|_{L^2} + C\| \nabla^2 u \|_{L^r} \]
\[ \leq C\| \nabla u \|_{L^2} + C\| \rho^{1/2} u_t \|_{L^2} + C\| \rho^{1/2} u_t \|_{L^2}^{6-r} \| \nabla u_t \|_{L^2}^{3r-6} \]  
\[ + C\| \nabla u \|_{L^2}^{6(\beta-1)/3} . \tag{3.32} \]

On the one hand, it follows from (3.33) and (3.11) that for \( t \in (0, \zeta(T)) \),
\[ \| \nabla u \|_{L^\infty} \leq C\| u_0 \|_{\tilde{H}^\beta} t^{\frac{\beta-2}{2}} + C\| u_0 \|_{\tilde{H}^\beta} t^{\frac{\beta-2}{2}} \left( t^{2-\beta} \| \nabla u_t \|_{L^2}^2 \right)^{\frac{3r-6}{4r}} \]
\[ + C\| u_0 \|_{\tilde{H}^\beta} t^{2r(\beta-1)/3} + C\| \nabla u \|_{L^2}^4 , \]
which together with (3.11), (3.11), and (3.31) gives
\[ \int_0^{\zeta(T)} \| \nabla u \|_{L^\infty} dt \]
\[ \leq C\| u_0 \|_{\tilde{H}^\beta} + C\| u_0 \|_{\tilde{H}^\beta} \left( \int_0^1 \frac{t^{2(\beta-1)r}}{\| \nabla u_t \|_{L^2}^{2r}} dt \right)^{\frac{r+6}{4r}} \left( \int_0^1 t^{2-\beta} \| \nabla u_t \|_{L^2}^2 dt \right)^{\frac{3r-6}{4r}} \]  
\[ \leq C\| u_0 \|_{\tilde{H}^\beta} . \tag{3.33} \]

On the other hand, using \( (3.32) \), \( (3.22) \), and \( (3.23) \), we obtain that for \( t \in [\zeta(T), T] \),
\[ \| \nabla u \|_{L^\infty} \leq C\| \rho^{1/2} u_t \|_{L^2} + C\| \nabla u_t \|_{L^2} + C\| \nabla u \|_{L^2} + C\| \nabla u_t \|_{L^2}^6 \]
\[ \leq C\| u_0 \|_{\tilde{H}^\beta} e^{-\sigma t/2} + C\| \nabla u_t \|_{L^2} , \]
and thus
\[ \int_{\zeta(T)}^T \| \nabla u \|_{L^\infty} dt \leq C\| u_0 \|_{\tilde{H}^\beta} + C \left( \int_{\zeta(T)}^T e^{-\sigma t} dt \right)^{1/2} \left( \int_{\zeta(T)}^T e^{\sigma t} \| \nabla u_t \|_{L^2}^2 dt \right)^{1/2} \]  
\[ \leq C\| u_0 \|_{\tilde{H}^\beta} . \tag{3.34} \]

Combining this with (3.33) gives (3.29) and finishes the proof of Lemma 3.5. \( \Box \)

With Lemmas 3.4, 3.5 at hand, we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Since \( \mu(\rho) \) satisfies
\[ (\mu(\rho))_t + u \cdot \nabla \mu(\rho) = 0 , \]
standard calculations show that
\[ \frac{d}{dt} \| \nabla \mu(\rho) \|_{L^q} \leq q \| \nabla u \|_{L^\infty} \| \nabla \mu(\rho) \|_{L^q} , \tag{3.35} \]
which together with Grönwall’s inequality and (3.29) yields
\[ \sup_{t \in [0, T]} \| \nabla \mu(\rho) \|_{L^q} \leq \| \nabla \mu(\rho_0) \|_{L^q} \exp \left\{ q \int_0^T \| \nabla u \|_{L^\infty} dt \right\} \]
\[ \leq \| \nabla \mu(\rho_0) \|_{L^q} \exp \left\{ C\| u_0 \|_{\tilde{H}^\beta} \right\} \]
\[ \leq 2 \| \nabla \mu(\rho_0) \|_{L^q} , \tag{3.36} \]
provided
\[ \|u_0\|_{H^\beta} \leq \varepsilon_1 \triangleq C^{-1} \ln 2. \quad (3.37) \]

Moreover, it follows from (3.3) and (3.22) that
\[
\int_0^T \|\nabla u\|_{L^2}^2 dt \leq \sup_{t \in [0, \zeta(T)]} \left( t^{1-\beta} \|\nabla u\|_{L^2}^2 \right)^2 \int_0^{\zeta(T)} t^{2\beta-2} dt
\]
\[ + \sup_{t \in [\zeta(T), T]} \left( e^{\sigma_t} \|\nabla u\|_{L^2}^2 \right)^2 \int_{\zeta(T)}^T e^{-2\sigma_t} dt \leq C \|u_0\|_{H^\beta}^4 \leq \|u_0\|_{H^\beta}^2, \quad (3.38) \]

provided
\[ \|u_0\|_{H^\beta} \leq \varepsilon_2 \triangleq C^{-1/2}. \quad (3.39) \]

Choosing \( \varepsilon_0 \triangleq \min\{1, \varepsilon_1, \varepsilon_2\} \), we directly obtain (3.2) from (3.36)–(3.39). The proof of Proposition 3.1 is finished. \( \square \)

The following Lemma 3.6 is necessary for further estimates on the higher-order derivatives of the strong solution \((\rho, u, P)\).

**Lemma 3.6** Let \((\rho, u, P)\) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant \(C\) depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, M, \\|\rho_0\|_{L^{3/2}}, \text{and} \\|\nabla u_0\|_{L^2}\) such that for \(p_0 \triangleq \min\{6, q\}\),
\[
\sup_{t \in [0, T]} e^{\sigma_t} \left( \|\nabla u\|_{L^2}^2 + \zeta \|\nabla u\|_{H^1}^2 + \zeta \|P\|_{H^1}^2 \right) + \int_0^T \zeta e^{\sigma_t} \|\nabla u_t\|_{L^2}^2 dt
\]
\[ + \int_0^T e^{\sigma_t} \left( \|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 + \zeta \|\nabla u\|_{W^{1, p_0}}^2 + \zeta \|P\|_{W^{1, p_0}}^2 \right) dt \leq C. \quad (3.40) \]

**Proof.** First, multiplying (3.28) by \(e^{\sigma_t}\), we get after using Grönwall’s inequality, (3.21), and (3.2) that
\[
\sup_{t \in [0, T]} e^{\sigma_t} \|\nabla u\|_{L^2}^2 + \int_0^T e^{\sigma_t} \|\rho^{1/2} u_t\|_{L^2}^2 dt \leq C. \quad (3.41) \]

Combining this with (3.17) gives
\[
\frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx \leq C \|\rho^{1/2} u_t\|_{L^2}^4 + C \|\nabla u\|_{L^2}^2,
\]
which along with Grönwall’s inequality, (3.41), and (3.21) implies that
\[
\sup_{t \in [0, T]} \zeta e^{\sigma_t} \|\rho^{1/2} u_t\|_{L^2}^2 + \int_0^T \zeta e^{\sigma_t} \|\nabla u_t\|_{L^2}^2 dt \leq C. \quad (3.42) \]

Combining this, (3.18), and (3.41) gives
\[
\sup_{t \in [0, T]} \zeta e^{\sigma_t} \left( \|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 \right) + \int_0^T e^{\sigma_t} \left( \|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 \right) dt \leq C. \quad (3.43) \]

Finally, it follows from (3.18), (3.30), (3.41), and (3.4) that for \(p_0 \triangleq \min\{6, q\}\),
\[
\|\nabla u\|_{H^1 \cap W^{1, p_0}} + \|P\|_{H^1 \cap W^{1, p_0}} \leq C \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2}, \quad (3.44) \]
which together with (3.42) and (3.21) implies

\[ \int_0^T \zeta e^{\sigma t} \left( \| \nabla u \|^2_{W^{1,p_0}} + \| P \|^2_{W^{1,p_0}} \right) \, dt \leq C. \]

This combined with (3.41)–(3.43) gives (3.40) and completes the proof of Lemma 3.6.

The following Proposition 3.7 is concerned with the estimates on the higher-order derivatives of the strong solution \((\rho, u, P)\) which in particular imply the continuity in time of both \(\nabla^2 u\) and \(\nabla P\) in the \(L^2 \cap L^p\)-norm.

**Proposition 3.7** Let \((\rho, u, P)\) be a smooth solution to (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant \(C\) depending only on \(q, \beta, \bar{\rho}, \bar{\mu}, \mu_0, L_0, M, \| \nabla u_0 \|_{L^2}, \) and \(\| \nabla \rho_0 \|_{L^2}\) such that for \(p_0 \equiv \min\{6, q\} \) and \(q_0 \equiv 4q/(q - 3),

\[
\sup_{t \in [0,T]} \zeta^{q_0} e^{\sigma t} \left( \| \nabla u \|^2_{W^{1,p_0}} + \| P \|^2_{W^{1,p_0}} + \| \nabla u_t \|^2_{L^2} \right) + \int_0^T \zeta^{q_0 + 1} e^{\sigma t} \left( \| (\rho u_t) \|_{L^2}^2 + \| \nabla u_t \|^2_{L^p_0} + \| P_t \|^2_{L^2 \cap L^{p_0}} \right) \, dt \leq C. \tag{3.45}
\]

**Proof.** First, similar to (3.32) and (3.36), we have

\[ \sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^2} \leq 2 \| \nabla \rho_0 \|_{L^2}, \tag{3.46} \]

which together with the Sobolev inequality and (3.41) gives

\[
\| \rho_t \|_{L^2 \cap L^{3/2}} = \| u \cdot \nabla \rho \|_{L^2 \cap L^{3/2}} \leq C \| \nabla \rho \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{H^1}^{1/2} \leq C \| \nabla u \|_{H^1}^{1/2}. \tag{3.47}
\]

Next, it follows from (3.12) that \(u_t\) satisfies

\[
\begin{cases}
- \text{div}(2\mu(\rho)u_t) + \nabla P_t = \tilde{F} + \text{div} g, \\
\text{div} u_t = 0,
\end{cases}
\]

with

\[ \tilde{F} \triangleq -\rho u_{tt} - \rho u \cdot \nabla u_t - \rho_t u_t - (\rho u)_t \cdot \nabla u, \quad g \triangleq -2u \cdot \nabla \mu(\rho)u. \]

Hence, one can deduce from Lemma 2.2 and the Sobolev inequality that

\[ \| \nabla u_t \|_{L^2 \cap L^{p_0}} + \| P_t \|_{L^2 \cap L^{p_0}} \leq C \| \tilde{F} \|_{L^6 \cap L^{3/2} \cap L^{3p_0}} + C \| g \|_{L^2 \cap L^{p_0}}. \tag{3.48} \]

Using (3.1), (3.21), (3.37), (3.40), and (3.41), we get by direct calculations that

\[
\begin{align*}
\| \tilde{F} \|_{L^6 \cap L^{3/2} \cap L^{3p_0}} & \leq C \| \rho \|_{L^{3/2} \cap L^{3p_0}} \| \rho \|_{L^{3/2} \cap L^{3p_0}} \| u \|_{L^{\infty}} \| \nabla u_t \|_{L^2} \\
& \quad + C \| \rho_t \|_{L^2 \cap L^{3/2}} \left( \| u_t \|_{L^6 \cap L^{3p_0}} + \| \nabla u \|_{H^1} + \| \nabla u \|_{H^1} \| \nabla u \|_{W^{1,p_0}} \right) \\
& \quad + C \| \rho \|_{L^2 \cap L^{p_0}} \| u_t \|_{L^6} \| \nabla u \|_{L^6} \\
& \leq C \| \rho u_{tt} \|_{L^2} + \varepsilon \| \nabla u_t \|_{L^{p_0}} + C(\varepsilon) \| u_t \|_{L^2} (1 + \| \nabla u \|_{H^1}^{3/2} + C \| \nabla u \|_{H^1}^{5/2},
\end{align*}
\]
and that
\[
\|g\|_{L^2 \cap L^{p_0}} \leq C\|\nabla \mu (\rho)\|_{L^q} \|u\|_{L^6, \cap L^\infty} \|\nabla u\|_{L^2 \cap L^\infty}
\leq C\|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} + C\|\nabla u\|_{H^1}^2,
\]  
(3.50)
where in the second inequality one has used the following simple fact
\[
\|\nabla u\|_{L^\infty} \leq C\|\nabla u\|_{H^1} \|W^{-1, p_0}\| \leq C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2},
\]  
(3.51)
due to the Sobolev inequality and (3.44). Then, putting (3.49) and (3.50) into (3.48), we obtain after choosing \(\varepsilon\) suitably small that
\[
\|\nabla u_t\|_{L^2 \cap L^{p_0}} + \|P_t\|_{L^2 \cap L^{p_0}}
\leq C\|\sqrt{\rho u_t}\|_{L^2} + C\|\nabla u_t\|_{L^2} \left(1 + \|\nabla u\|_{H^1}^2\right) + C\|\nabla u\|_{H^1} + C\|\nabla u\|_{H^1}^3.
\]  
(3.52)
Now, multiplying (3.12) by \(u_{tt}\) and integrating the resulting equality by parts lead to
\[
\int \rho|u_{tt}|^2 dx + \frac{d}{dt} \int \mu(\rho)|dt|^2 dx
= \int \text{div}(\mu(\rho)u)|dt|^2 dx - \int \rho(u \cdot \nabla u + u_t \cdot \nabla u) \cdot u_{tt} dx - \int \rho u_t^4 u_{tt}^4 dx
- \int \rho u \cdot \nabla u^4 u_t^4 dx - 2 \int \partial_i(\partial_k(\mu(\rho)dt)|u_{tt}^4 dx \triangleq \sum_{i=1}^{5} I_i.
\]  
(3.53)
We will use (3.40), (3.52), and the Sobolev inequality to estimate each term on the righthand side of (3.53) as follows:

First, it follows from (3.1), (3.40), and (3.52) that
\[
|I_1| \leq C\|u\|_{L^\infty} \|\nabla \mu (\rho)\|_{L^q} \|\nabla u_t\|_{L^2}^{\frac{2(q(p_0 - q_0) - 2p_0)}{q(p_0 - 2)}} \|\nabla u_t\|_{L^2}^{\frac{q(p_0 - 2)}{q(p_0 - 2)}}
\leq \varepsilon\|\nabla u_t\|_{L^{p_0}}^2 + C(\varepsilon)\|\nabla u\|_{L^6, \cap L^\infty} \|\nabla u_t\|_{L^2}^2
\leq C\varepsilon\|\sqrt{\rho u_t}\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2
+ C(\varepsilon)\|\nabla u\|_{H^1} + C(\varepsilon)\|\nabla u\|_{H^1}^6,
\]  
(3.54)
where in the last inequality we have used
\[
\frac{q(p_0 - 2)}{p_0 q - p_0 - 2q} \in [1, q_0].
\]
Next, Hölder’s inequality gives
\[
|I_2| \leq \varepsilon \int \rho|u_{tt}|^2 dx + C(\varepsilon)\|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2.
\]  
(3.55)
Then, direct calculations show
\[
I_3 = -\frac{1}{2} \frac{d}{dt} \int \rho_t|u_t|^2 dx + \int (\rho u^4_t) \partial_i u_t^4 u_t^4 dx
\leq -\frac{1}{2} \frac{d}{dt} \int \rho_t|u_t|^2 dx + C\|\rho\|_{L^6} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6}^2
+ C\|\rho_t\|_{L^2} \|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6}
\leq -\frac{d}{dt} \int \rho u \cdot \nabla u_t^4 u_t^4 dx + C(\varepsilon) \|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2 \|\nabla u_t\|_{L^2}^2
+ \varepsilon \int \rho|u_{tt}|^2 dx + C(\varepsilon)\|\nabla u\|_{H^1}^2 + C(\varepsilon)\|\nabla u\|_{H^1}^6,
\]  
(3.56)
where in the last inequality one has used (3.47) and (3.52).

Next, it follows from (1.1) and (3.47) that

\[ I_4 = -\frac{d}{dt} \int \rho_t u \cdot \nabla u^i u_t^i dx + \int (\rho u^i)_t \partial_t u \cdot \nabla u^j u_t^j dx + \int \rho_t (u \cdot \nabla u^j) u_t^j dx \]

\[ = -\frac{d}{dt} \int \rho_t u \cdot \nabla u^i u_t^i dx + \int \rho_t u^i (u \cdot \nabla u^j) u_t^j \partial_t u_t^i dx + \int \rho_t (u \cdot \nabla u^j) u_t^j dx \]

\[ \leq -\frac{d}{dt} \int \rho_t u \cdot \nabla u^i u_t^i dx + C \|u_t\|_{L^6} \|\nabla u\|_{L^6}^2 (\|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}) \]

\[ + C \|\rho_t\|_{L^2} \|\nabla u\|_{L^6} \|\nabla u\|_{H^1} \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2}^2 \|\nabla u_t\|_{H^2}^2 \]

\[ + \varepsilon \int |u_t|^2 dx + C(\varepsilon) \|\nabla u\|_{H^1} \|\nabla u\|_{H^1}^6 . \]

Finally, direct calculations lead to

\[ I_5 = -2 \frac{d}{dt} \int \partial_t (u^k \partial_k \mu(\rho) d_t^i) u_t^i dx - 2 \int \partial_t (u^k \mu(\rho) \partial_k d_t^i) u_t^i dx \]

\[ + 2 \int \partial_t (u^k \partial_k (\partial_t d_t^i)) u_t^i dx \]

\[ = 2 \frac{d}{dt} \int u^k \partial_k (u^i) d_t^i \partial_t u^i dx + 2 \int (\mu(\rho) u^k \partial_k d_t^i) \partial_t u_t^i dx \]

\[ - 2 \int (\partial_t u^k \mu(\rho) d_t^i) \partial_k u^i dx - 2 \int u^k \partial_t (\mu(\rho) d_t^i) \partial_k u_t^i dx \]

\[ - 2 \int u^k (\partial_t (\partial_t d_t^i)) \partial_k u_t^i dx \]

\[ \triangleq 2 \frac{d}{dt} \int u^k \partial_k (\mu(\rho) d_t^i) \partial_i u^i dx + \sum_{i=1}^{4} I_{5,i}. \]

We estimate each \( I_{5,i} (i = 1, \ldots, 4) \) as follows:

First, integration by parts gives

\[ I_{5,1} = 2 \int (\mu(\rho) u^k) \partial_k \partial_i u^i dx + 2 \int \mu(\rho) u^k \partial_k (d_t^i) \partial_i u_t^i dx \]

\[ = -2 \int u \cdot \nabla \mu(\rho) u^k \partial_k \partial_i u^i dx + 2 \int \mu(\rho) u^k \partial_k \partial_i \partial_i u_t^i dx \]

\[ - \int \div(\mu(\rho) u) |d_t|^2 dx \]

\[ \leq C \|u\|_{L_{\infty}}^2 \|\nabla \mu(\rho)\|_{L^2} \|\nabla u\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2 \]

\[ + C \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2}^2 + I_1 \]

\[ \leq C \varepsilon \|u\|_{L^2}^2 + C(\varepsilon) \|\nabla u\|_{H^1} + \|\nabla u_t\|_{L^2} \]

\[ + C(\varepsilon) \|\nabla u\|_{H^1} \|\nabla u_t\|_{H^1}^2 , \]

where in the last inequality we have used (3.40), (3.41), (3.52), and (3.54).
Then, it follows from (3.11) and (3.51) that
\[ |I_{5,2}| \leq C\|u\|_{L^\infty} \|\nabla \mu(\rho)\|_{L^q} \|\nabla u\|_{L^q/\delta - 3} \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \]
\[ + C\|\nabla u\|_{L^\infty} \|\nabla u_t\|_{L^2}^2 \]
\[ \leq C\|\nabla u\|_{H^1, W^{1,\infty}} (\|\nabla u\|_{L^2}^3 + C\|\nabla u_t\|_{L^2}^2) \]
\[ \leq C\|\nabla u\|_{H^1}^3 + C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2) \|\nabla u_t\|_{L^2}^2. \]

Similarly, combining Hölder’s inequality and (3.44) leads to
\[ |I_{5,3}| \leq C\|u_t\|_{L^6} \|\nabla u_t\|_{L^2} (\|\nabla \mu(\rho)\|_{L^q} \|\nabla u\|_{L^q/\delta - 3} + \|\nabla^2 u\|_{L^3}) \]
\[ \leq C\|\nabla u_t\|_{L^2}^2 (\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2}). \] (3.61)

Finally, using (1.1)2 and (1.1)3, we obtain after integrating by parts that
\[ I_{5,4} = -2 \int u^k \partial_j P_t \partial_k u_t^j dx - 2 \int u^k (\rho u_t^j + \rho u \cdot \nabla u^j) \partial_k u_t^j dx \]
\[ = 2 \int \partial_j u^k P_t \partial_k u_t^j dx - 2 \int u^k \rho u_t^j \partial_k u_t^j dx \]
\[ - 2 \int u^k (\rho u_t^j + (\rho u \cdot \nabla u^j)) \partial_k u_t^j dx \]
\[ \leq C\|\nabla u\|_{L^6} \|P_t\|_{L^3} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2} \]
\[ + C\|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\rho_t\|_{L^2} (\|u_t\|_{L^\infty} + \|\nabla u\|_{H^1} \|\nabla u\|_{L^\infty}) \]
\[ + C\|\nabla u\|_{L^\infty} \|\nabla u_t\|_{L^2} (\|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}) \|\nabla u\|_{H^1} \]
\[ \leq C\varepsilon \int \rho |u_t|^2 dx + C(\varepsilon)(1 + \|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^4) \|\nabla u_t\|_{L^2}^2 \]
\[ + C(\varepsilon) \|\nabla u\|_{H^1}^2 + C(\varepsilon) \|\nabla u\|_{H^1}^6, \]

where in the last inequality one has used (3.52) and (3.47).

Substituting (3.53)–(3.63) into (3.53), we get after choosing \( \varepsilon \) suitably small that
\[ \frac{d}{dt} \int \mu(\rho)|d_t|^2 dx + \Psi'(t) + \frac{1}{2} \int \rho |u_t|^2 dx \]
\[ \leq C(1 + \|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^6) \|\nabla u\|_{H^1}^2 + C \|\nabla u\|_{H^1}^6, \] (3.63)

where
\[ \Psi(t) \triangleq - \int \rho u \cdot \nabla u_t^j u_t^j dx - \int \rho u_t \cdot \nabla u^j u_t^j dx + 2 \int u^k \partial_k \mu(\rho) d_t^j \partial_j u_t^j dx \]
satisfies
\[ |\Psi(t)| \leq C\|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} \|\nabla u\|_{H^1} + C \|\rho_t\|_{L^2} \|u_t\|_{L^6} \|u_t\|_{L^5} \|\nabla u\|_{L^6} \]
\[ + C \|\nabla \mu(\rho)\|_{L^q} \|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1}^2 \]
\[ \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 \|\nabla u\|_{H^1}^2 + C \|\nabla u\|_{H^1}^4, \] (3.64)
due to (3.1) and (3.47).

Then, multiplying (3.63) by \( \zeta_0 e^{\eta t} \) and noticing that (3.40) gives
\[ \zeta_0 (1 + \|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^6) \|\nabla u_t\|_{L^2}^2 \leq C \zeta_0^{\eta + 1} \|\nabla u_t\|_{L^2}^4 + C \zeta \|\nabla u_t\|_{L^2}^2, \]
we get after using Grönwall’s inequality, (3.64), (3.40), and (3.42) that
\[
\sup_{0 \leq t \leq T} \zeta q_0 e^{\sigma t} \| \nabla u_t \|^2_{L^2} + \int_0^T \zeta q_0 e^{\sigma t} \int_\rho |u_{tt}|^2 dx dt \leq C. \tag{3.65}
\]
Furthermore, it follows from (3.47) and (3.40) that
\[
\| (\rho u_t)_t \|^2_{L^2} \leq C \| \nabla u \|_{H^1} \| \nabla u_t \|^2_{L^2 \cap L^{p_0}} + C \| \rho^{1/2} u_{tt} \|^2_{L^2},
\]
which together with (3.65), (3.44), (3.52), and (3.40) gives (3.45) and thus completes the proof of Proposition 3.7. \(\square\)

4 Proofs of Theorems 1.2 and 1.3

With all the a priori estimates in Section 3 at hand, we are now in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. First, by Lemma 2.1, there exists a \(T^* > 0\) such that the Cauchy problem (1.1)-(1.4) has a unique local strong solution \((\rho, u, P)\) on \(\mathbb{R}^3 \times (0, T^*)\).

It follows from (1.8) that there exists a \(T_1 \in (0, T^*)\) such that (3.1) holds for \(T = T_1\).

Next, set
\[
T^* \triangleq \sup \{ T \mid (\rho, u, P) \text{ is a strong solution on } \mathbb{R}^3 \times (0, T) \text{ and (3.1) holds} \}. \tag{4.1}
\]
Then \(T^* \geq T_1 > 0\). Hence, for any \(0 < \tau < T \leq T^*\) with \(T\) finite, one deduces from (3.40) and (3.45) that
\[
\nabla u, \ P \in C([\tau, T]; L^2) \cap C(\overline{\mathbb{R}^3 \times [\tau, T]}), \tag{4.2}
\]
where one has used the standard embedding
\[
L^\infty(\tau, T; H^1 \cap W^{1,p_0}) \cap H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^2) \cap C(\overline{\mathbb{R}^3 \times [\tau, T]}).
\]
Moreover, it follows from (3.1), (3.4), (3.40), and [24] Lemma 2.3 that
\[
\rho \in C([0, T]; L^{3/2} \cap H^1), \quad \nabla \mu(\rho) \in C([0, T]; L^q). \tag{4.3}
\]
Thanks to (3.41) and (3.45), the standard arguments yield that
\[
\rho u_t \in H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^2),
\]
which together with (1.2) and (4.3) gives
\[
\rho u_t + \rho u \cdot \nabla u \in C([\tau, T]; L^2). \tag{4.4}
\]
Since \((\rho, u)\) satisfies (2.8) with \(F \equiv \rho u_t + \rho u \cdot \nabla u\), we deduce from (1.1), (4.2), (4.3), (1.4), and (3.45) that
\[
\nabla u, \ P \in C([\tau, T]; D^1 \cap D^{1,p}), \tag{4.5}
\]
for any \(p \in [2, p_0)\).

Now, we claim that
\[
T^* = \infty. \tag{4.6}
\]
Otherwise, $T^* < \infty$. Proposition 3.1 implies that (3.2) holds at $T = T^*$. It follows from (4.2), (4.3), and (4.5) that
\[
(\rho^*, u^*)(x) = (\lim_{t \to T^*} (\rho, u)(x, t))
\]
satisfies
\[
\rho^* \in L^{3/2} \cap H^1, \quad u^* \in D^{1,0}_{0,\sigma} \cap D^{1,p}
\]
for any $p \in [2, p_0)$. Therefore, one can take $(\rho^*, \rho^* u^*)$ as the initial data and apply Lemma 2.1 to extend the local strong solution beyond $T^*$. This contradicts the assumption of $T^*$ in (4.1). Hence, (4.6) holds. We thus finish the proof of Theorem 1.2 since (1.11) and (1.12) follow directly from (3.46) and (3.45), respectively.

**Proof of Theorem 1.3**: With the global existence result at hand (see Proposition 1.1), one can modify slightly the proofs of Lemma 3.4 and (3.46) to obtain (1.13) and (1.14).

\[\blacksquare\]

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