Recent results on complex Cartan spaces

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Abstract

In this paper, we first provide an updated survey of the geometry of complex Cartan spaces. New characterizations for some particular classes of complex Cartan spaces are pointed out, e.g. Landsberg-Cartan, strongly Berwald-Cartan and others. We introduce the Cartan-Randers spaces which offer examples of Berwald-Cartan and strongly Berwald-Cartan spaces. Then, we investigate the complex geodesic curves of a complex Cartan space, using the image by Legendre transformation ($L$—duality) of complex geodesic curves of a complex Finsler space. Assuming the weakly Kähler condition for a complex Cartan space, we establish that its complex geodesic curves derive from Hamilton-Jacobi equations. Also, by $L$—duality, we introduce the correspondent notion of the projectively related complex Finsler metrics, on the complex Cartan spaces. Various descriptions of the projectively related complex Cartan metrics are given. As applications, the projectiveness of a complex Cartan-Randers metric and the locally projectively flat complex Cartan metrics are analyzed.

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1 Introduction

The study of geometry of the holomorphic cotangent bundle, endowed with a complex Hamiltonian, has been deepened in some previous works of the second author [16, 17]. By analogy with the real case, where remarkable results are known ([15, 8, 10]), the geometry achieved here is called complex Hamilton geometry. The particular context in which the complex Hamiltonian is homogeneous on the fibre, is known as complex Cartan geometry.

The approach of the complex Cartan spaces has been justified by the existence of a pseudo-distance, on the dual holomorphic bundle, highlighted
by S. Kobayashi in [13]. Using the equivalence method, J. Faran studied in [12] the complex Cartan spaces, (which he calls Finsler-Hamilton spaces), with constant holomorphic curvature. He also gave some characterizations of the dual Kobayashi metric.

On the other side, as we well know, the Hamiltonian Mechanics can be retrieved via the Lagrangian Mechanics, by so called Legendre transformation. This problem was already extended to the complex case ([16]), the study of geometric objects on the holomorphic cotangent bundle $T^*M$, via the complex Legendre transformation, defined on the holomorphic tangent bundle $T'M$, being called $\mathcal{L}-$ dual process. By $\mathcal{L}-$ duality, it is shown that the dual Kobayashi metric is exactly the $\mathcal{L}-$ dual of the well-known Kobayashi metric on $T'M$, ([16, 17]).

The $\mathcal{L}-$ dual process seems a satisfactory technique for the investigation of the geometry of complex Cartan spaces, using corresponding notions from complex Finsler spaces, for which comprehensive results are known, ([1]-[7], [16]). But, a more advertent analysis of previous results, obtained by $\mathcal{L}-$ dual process, induced us to come back to some ideas which we reformulated then. More exactly, the $\mathcal{L}-$ dual of the vertical natural frame $\frac{\partial}{\partial \eta^k}$ on $T_u^*(T'M)$ is identified with the frame that is obtained by lifting the subscripts of the vertical natural frame $\frac{\partial}{\partial \bar{\zeta}^k}$ on $T_u^*(T'M)$, only in the purely Hermitian case. This leads us to a lot of difficulties and some new ideas, which we discuss and solve in the present paper.

The paper is organized as follows. After a short survey of complex Cartan spaces in our own notation, (Section 2), we extend some results about classes of complex Cartan spaces obtained in [6]. To the Chern-Cartan complex nonlinear connection, with local coefficients

$$N_{ji} = -h_{jk} \frac{\partial h_{\bar{k}l}}{\partial z^i} \bar{\zeta}_l,$$

we associate a complex linear connection of Berwald type $B:=(N_{ji}, B^i_{jk}, B_{jk}^i, 0, 0)$, which is not of $(1,0)$ - type or metrical compatible. Here, we prove that the conditions: $B\Gamma$ is horizontal metrical compatible and $B\Gamma$ is of $(1,0)$ - type are equivalent (Theorem 3.2) and, we call such a space Landsberg-Cartan. Also, we obtain that any Landsberg-Cartan space with weakly Kähler-Cartan property is a Kähler-Cartan space, (Theorem 3.3). The complex Berwald-Cartan spaces (i.e., the spaces with $B^i_{jk}(z)$) are Landsberg-Cartan. We show that any purely Hermitian complex Cartan space is a complex Berwald-Cartan space. The complex Berwald-Cartan spaces which are weakly Kähler-Cartan are called strongly Berwald-Cartan spaces and they are contained in the class of Kähler-Cartan spaces, (Corollary 3.1). All these results are described in Section 3.

In Section 4, we introduce the Cartan-Randers metrics $\tilde{C} = \alpha + |\beta|$, where $\alpha = \sqrt{a^{ij}(z)\zeta_i \zeta_j}$ is a purely Hermitian complex Cartan metric on the comp-
plex manifold \( M \) and \( |\beta| \) is obtained by \( \beta = b^i \zeta_i, \ b^i := a^j \zeta_i b_j(z), \) with \( b_i(z) \) the local coefficients of a differential \((1, 0)-\) form on \( M \). Complex Cartan-Randers metrics are remarkable, they represent the medium in which Hermitian geometry properly interferes with complex Cartan geometry. Theorem 4.2 and Corollary 4.1 report on the necessary and sufficient conditions for a complex Cartan-Randers metric to be a Berwald-Cartan metric or strongly Berwald-Cartan metric. The existence of complex Cartan-Randers spaces with Berwald-Cartan and strongly Berwald-Cartan properties is attested by some explicit examples.

The problem of the complex Cartan spaces obtained as image of the complex Finsler spaces, via complex Legendre transformation is described in Section 5. First of all, we deduce the correct form of the \( L- \) dual of vertical natural frame \( \partial \partial \eta_k \), called the nonholonomic vertical frame, (Theorems 5.1, 5.2). Also, we determine the \( L- \) dual of Chern-Finsler complex linear connection and by \( L- \) duality the weakly Kähler-Finsler property is sent in weakly Kähler-Cartan property. The problem of the complex geodesics curves of a complex Cartan spaces is also investigated by \( L- \) dual process. The image \( \sigma^*(s) \), by \( L- \) duality of the complex geodesics curve \( \sigma(s) \) of a complex Finsler spaces is obtained, (Theorem 5.3) and, in the weakly Kähler-Cartan case \( \sigma^*(s) \) is a solution of the Hamilton-Jacobi equations. \( \sigma^*(s) \) is called the complex geodesic curve of a complex Cartan space and its equations can be rewritten in a more significant form as in Theorem 5.4.

The projectively related complex Cartan spaces are approached by \( L- \) duality, too. Two complex Cartan metrics \( \tilde{C} \) and \( C \) on a common underlying manifold \( M \), obtained by \( L- \) duality, are called projectively related if any complex geodesic curve, in the sense describe above, of the first is also a complex geodesic curve for the second and vice versa. This means that between the functions \( \tilde{N}_k \) and \( N_k \) there is a so-called projective change \( \tilde{N}_k = N_k + B_k + Q \zeta_k \), where \( Q \) is a smooth function on \( \tilde{T}^* \tilde{M} \) with complex values and \( B_k := h_{sk} \Theta^s - h_{sk} \Theta^{*s}, \) (Theorem 5.5). Finally, considering a Cartan-Randers metric \( \tilde{C} = \alpha + |\beta|, \) we prove that \( \tilde{C} \) can be the image by \( L- \) duality of a complex Finsler metric only if it is purely Hermitian. Then, we find the necessary and sufficient conditions under which \( \tilde{C} \) and \( \alpha \) are projectively related. Also, the locally projectively flat complex Cartan metrics are pointed out, (Corollary 5.3).

2 Preliminaries

Geometry of real Finsler spaces is already one classic today, ([1] [9] [14] [18], etc.). During the last years, we remark a significant progress in the study
of complex Finsler geometry, ([11]-[12], [14, 15], etc.). Also, the study of Cartan spaces (real and complex) is enthralling, ([15]-[16], etc.)

Let \(M\) be a \(n\) - dimensional complex manifold and \(z = (z^k)_{k=1,n}\) be complex coordinates in a local chart. The complexified of the real tangent bundle \(T_C M\) splits into the sum of holomorphic tangent bundle \(T'M\) and its conjugate \(T''M\). The bundle \(T'M\) is itself a complex manifold and the coordinates in a local chart will be denoted by \(u = (z^k, \eta^k)_{k=1,n}\). These are changed into \((z^k, \eta^k)_{k=1,n}\) by the rules \(z^k = z^k(z)\) and \(\eta^k = \frac{\partial z^k}{\partial z^k} \eta^j\), \(\text{rank}(\frac{\partial z^k}{\partial z^j}) = n\). The dual of \(T'M\) is denoted by \(T''M\). On the manifold \(T''M\), a point \(u^*\) is characterized by the coordinates \(u^* = (z^k, \zeta^k)_{k=1,n}\), and a change of these has the form \(z^k = z^k(z)^*\) and \(\zeta^k = \frac{\partial z^k}{\partial z^j} \zeta_j^*\), \(\text{rank}(\frac{\partial z^k}{\partial z^j}) = n\). Here and further, we use the notation with star for the partial derivatives with respect to \(z\), on \(T''M\), only to distinguish them from those on \(T'M\).

**Definition 2.1.** A complex Cartan space is a pair \((M,C)\), where \(C : T''M \rightarrow \mathbb{R}^+\) is a continuous function satisfying the conditions:

i) \(H := C^2\) is smooth on \(T''M := T''M \setminus \{0\}\);

ii) \(C(z, \zeta) \geq 0\), the equality holds if and only if \(\zeta = 0\);

iii) \(C(z, \lambda \zeta) = |\lambda|C(z, \zeta)\) for \(\forall \lambda \in \mathbb{C}\);

iv) the Hermitian matrix \((h^{\bar{z}i}(z, \zeta))\) is positive definite, where \(h^{\bar{z}i} := \frac{\partial^2 H}{\partial \zeta_i \partial \zeta^j}\) is the fundamental metric tensor.

Equivalently, the condition iv) means that the indicatrix is strongly pseudo-convex.

Consequently, from iii) we have \(\frac{\partial H}{\partial \zeta_k} \zeta_k = \frac{\partial H}{\partial \zeta_k} \bar{z}_k = H\), \(\frac{\partial h^{\bar{z}i}}{\partial \zeta_k} \zeta_k = \frac{\partial h^{\bar{z}i}}{\partial \zeta_k} \bar{z}_k = 0\) and \(H = h^{\bar{z}i} \bar{z}_j \zeta_i\). An usual example of complex Cartan space is so called purely Hermitian complex Cartan space, this means that \(h^{\bar{z}i} = h^{\bar{z}i}(z)\).

We say that a function \(f\) on \(T''M\) is \((p, q)\)-homogeneous with respect to the coordinate \(\zeta = (\zeta_k)\) iff \(f(z^k, \lambda \zeta_k) = \lambda^p \lambda^q f(z^k, \zeta_k)\), for any \(\lambda \in \mathbb{C}\). For instance, \(H := C^2\) is a \((1, 1)\) - homogeneous function.

Roughly speaking, the geometry of a complex Cartan space consists in the study of the geometric objects of the complex manifold \(T''M\) endowed with the Hermitian metric structure defined by \(h^{\bar{z}i}\). Therefore, further the first step is the study of sections in the complexified tangent bundle of \(T''M\), which is decomposed in the sum \(T_C(T''M) = T'(T''M) \oplus T''(T''M)\).

Let \(VT''M \subset T'(T''M)\) be the vertical bundle, which has the vertical distribution \(V_{u^*}(T''M)\), locally spanned by \(\{\frac{\partial}{\partial \zeta_k}\}\). A complex nonlinear connection, briefly (c.n.c.), on \(T''M\) is a supplementary subbundle in \(T'(T''M)\) of \(V(T''M)\), i.e., \(T'(T''M) = H(T''M) \oplus V(T''M)\). The horizontal distribution \(H_{u^*}(T''M)\) is locally spanned by \(\{\frac{\delta}{\delta z^i}\}\), where \(\frac{\delta^*}{\delta z^i} = \frac{\delta^n}{\delta z^i} + N_{jk} \frac{\partial}{\partial \zeta_j}\).
and functions $N_{jk}$ are the coefficients of the (c.n.c.) on $T^*M$. The pair \( \{ \delta_k^* := \frac{\partial}{\partial z^k}, \delta^k := \frac{\partial}{\partial z^k} \} \) will be called the adapted frame of the (c.n.c.), which obey the change rules $\delta_k^* = \frac{\partial z^j}{\partial z^k} \delta_j^*$ and $\delta^k = \frac{\partial z^k}{\partial z^j} \delta^j$. By conjugation everywhere we have obtained an adapted frame $\{ \delta_k^*, \delta^k \}$ on $T^*_u(T^*M)$. The dual adapted frames are $\{ d^* z^k, \delta \zeta_k = d \zeta_k - N_{kj} d z^j \}$ and $\{ d^* z^k, \bar{\delta} \bar{\zeta}_k \}$.

A Hermitian connection $D$, of (1,0)− type is so called Chern-Cartan connection (cf. [16], in brief $C-C$ connection, and it is locally given by the following coefficients

\[
N_{ji} = -h_{jk} \frac{\partial^* h_{kl}}{\partial z^i} \zeta_l \quad H^i_{jk} := h^{\bar{m}i} (\delta_{\bar{m}} h_{jm}) \quad V^i_{jk} := -h_{jm} (\partial^k h^{\bar{m}i}), \tag{2.1}
\]

and $H^i_{jk} = V^i_{jk} = 0$, where here and hereinafter $\delta_k^*$ is the adapted frame of the $C-C$ (c.n.c.), $h_{jk} h_{kl} = \delta_j^* \delta_l^*$ and $D_k^* \delta^* = H^i_{jk} \delta_i$, $D_{\bar{\delta}}^* \delta^* = -H^i_{kj} \delta_i$, $D_{\bar{\delta}}^* \delta^* = V^i_{jk} \delta_i$, $D_{\bar{\delta}}^* \delta^* = -V^i_{jk} \delta_i$, etc. Moreover, we have

\[
H^i_{jk} = \bar{\partial}^i N_{jk} \quad H^i_{jk} \zeta_l = N_{ij}. \tag{2.2}
\]

Denoting by $|’|$, $|\bar{'}|$, $|’\bar{'}|$ and $\bar{’}|$, the $h^*-$, $v^*-$, $\bar{h}^*-$, $\bar{v}^*-$ covariant derivatives with respect to $C-C$ connection, respectively, it results $h^*_{jk} = h^*_{jk} = h^*_{jk}| = h^*_{jk}| = 0$, i.e. $C-C$ connection is $h^*$ and $v^*$ metrical.

For more details on complex Cartan spaces, see [16]. Further on, in order to simplify the writing, we use a bar over indices to denote the complex conjugation of the variables or of the frames, e.g., $\zeta_k := \bar{\zeta}_k$ or $\partial^k := \bar{\partial}^k$.

### 3 Complex Landsberg-Cartan spaces

In [16] we investigated some classes of complex Cartan spaces. A complex Cartan space $(M, C)$ is called strongly Kähler-Cartan iff $T^i_{jk} = 0$, Kähler-Cartan iff $T^i_{jk} \zeta_0 = 0$ and weakly Kähler-Cartan iff $T^i_{jk} \zeta_0 = 0$, where $T^i_{jk} := H^i_{jk} - H^i_{kj}$ is the $h-$torsion and $\zeta^i := h^{\bar{m}i} \zeta_{\bar{m}}$. But, the notions of strongly Kähler and Kähler coincide, as in complex Finsler geometry (11). Also, in the particular case of a purely Hermitian complex Cartan metric all those nuances of Kähler-Cartan are same with $\frac{\partial h_{\bar{m}n}}{\partial z^j} = \frac{\partial h_{\bar{m}n}}{\partial z^k}$. The space $(M, C)$ is called Berwald-Cartan iff the coefficients $H^i_{jk}$ depend only on the position $z$.

**Theorem 3.1.** ([16]) Let $(M, C)$ be a $n$ - dimensional complex Finsler space. Then the following assertions are equivalent:

i) $(M, C)$ is a Berwald-Cartan;

ii) $N_{ij}$ are holomorphic in $\zeta$;
Examples of complex Berwald-Cartan metrics are provided first by the class of purely Hermitian complex Cartan metrics. Indeed, in this case the local coefficients of $C - C$ (c.n.c.), $N_{ji} = -h_{jk}(z)\frac{\partial^* h_{ik}(\zeta)}{\partial z^i}\zeta_l$ are holomorphic in $\zeta$.

Another example of Berwald-Cartan metric is given by the function

$$C^2 = H(z, w; \zeta, \upsilon) := e^{2\sigma} \left( |\zeta|^{4} + |\upsilon|^{4} \right)^{\frac{1}{2}},$$

with $\zeta, \upsilon \neq 0$, (3.1) on $\mathbb{C}^2$, where $\sigma(z, w)$ is a real valued function and $|\zeta|^2 := \zeta_i \zeta_i$, $\zeta \in \{\zeta, \upsilon\}$, $i = 1, 2$. In (3.1) we relabeled the usual local coordinates $z^1, z^2, \zeta^1, \zeta^2$ as $z, w, \zeta, \upsilon$, respectively. Direct computation leads to

$N_{11} = -\frac{\partial \sigma}{\partial z} \zeta; \ N_{12} = -\frac{\partial \sigma}{\partial w} \zeta; \ N_{21} = -\frac{\partial \sigma}{\partial z} \upsilon; \ N_{22} = -\frac{\partial \sigma}{\partial w} \upsilon,$

which attest that $N_{ji}, i, j = i = 1, 2$, are holomorphic in $\zeta$ and $\upsilon$.

Two complex nonlinear connections (Chern-Finsler and the canonical) play a significant role in complex Finsler geometry, (see [3, 5]). The first induces a complex spray and, the derivative of the local coefficients of this spray leads to the second. It would be expected (taking into account the $L$–dual process which will be described later) that similar things happen in complex Cartan geometry. So, for a complex Cartan space, we can take the canonical (c.n.c.) besides the $C - C$ (c.n.c.) as follows. The local coefficients of the canonical (c.n.c.) on $(M, C)$ are defined by $N_{ji} := (\partial^k N_{ji})\zeta_k$. But, the $(1, 0)$ - homogeneity with respect to the variables $\zeta = (\zeta_k)$ and $\check{\zeta} = (\check{\zeta}_k)$ of $N_{ji}$, (i.e., $(\partial^k N_{ji})\zeta_k = N_{ji}$ and $(\partial^k N_{ji})\zeta_k = 0$), implies $\bar{N}_{ji} = N_{ji}$. Therefore, in complex Cartan geometry only the $C - C$ (c.n.c.) from (2.1) is available.

Nevertheless, we associate to the $C - C$ (c.n.c.) another complex linear connection of Berwald type

$$B \Gamma := \left( N_{ji}, B^i_{jk} := \partial^i N_{jk}, B^i_{jk} := (\partial^i N_{ks})\zeta^s \zeta_j, 0, 0 \right).$$

$B \Gamma$ is neither $h^* -$ nor $v^* -$ metrical, (for more details see [16]). Moreover, we have the following properties:

$$B^i_{jk} = H^i_{jk}; \ (\partial^m H^i_{jk})\zeta_m = 0; \ (\partial^m H^i_{jk})\zeta_m = 0; \ (\partial^i N_{ks})\zeta^k = 0,$$

and their conjugations.
Definition 3.1. Let \((M, C)\) be a n - dimensional complex Cartan space. 
\((M, C)\) is called complex Landsberg-Cartan space if \(\Gamma\) is \(h^*\) – metrical, (i.e., 
\(h^*_{||k} = h^*_{||k} = 0\), where “\(\parallel\)" is \(h^*\) – covariant derivatives with respect to \(\Gamma\)).

Theorem 3.2. Let \((M, C)\) be a n - dimensional complex Cartan space. Then 
\((M, C)\) is a complex Landsberg-Cartan space if and only if \(\Gamma\) is of \((1, 0)\) - type.

Proof. We know that \(h^*\) – covariant derivatives with respect to \(\Gamma\) of 
\(h^*_{||k} = \delta_k h^*_{j} - h^*_{ji} B^i_{lk} - h^*_{mi} B^i_{mk}.\) Since \(B^i_{jk} = H^i_{jk}\) and \(C - C\) connection is metrical, then \(h^*_{||k} = -h^*_{mi} B^i_{mk}.\) Thus, \(h^*_{||k} = 0\) if and only if \(B^i_{mk} = 0\), i.e., \(\Gamma\) is of \((1, 0)\) - type.

Note that any complex Berwald-Cartan space is a complex Landsberg-Cartan space.

Theorem 3.3. Let \((M, C)\) be a n - dimensional complex Cartan space. \((M, C)\) is a Kähler-Cartan space if and only if it is a Landsberg-Cartan space with weakly Kähler-Cartan property.

Proof. Since \((\partial^\iota N_{ks})\zeta^k = 0\), it is easy to check that
\[
(\partial^\iota N_{ks})\zeta^k = \partial^\iota[(N_{ks} - N_{sk})\zeta^k] - (N_{ks} - N_{sk})h^{is}.
\] (3.2)

First, we suppose that \(C\) is a Kähler-Cartan metric. Then, (3.2) implies 
\((\partial^\iota N_{ks})\zeta^k = 0\), and so \(B^i_{mk} = 0\).

Conversely, if \(C\) is a Landsberg-Cartan metric with weakly Kähler-Cartan property, then (3.2) becomes \((N_{ks} - N_{ks})h^{is} = 0\), which gives \(N_{ks} = N_{sk}\). This completes the proof.

Definition 3.2. Let \((M, C)\) be a n - dimensional complex Cartan space. \((M, C)\) is called strongly Berwald-Cartan space if it is weakly Kähler-Cartan and \(H^i_{jk}(z)\).

Obviously, the strongly Berwald-Cartan spaces define a subclass of the Berwald-Cartan spaces. Moreover, we can prove.

Corollary 3.1. Let \((M, C)\) be a n - dimensional complex Cartan space. If 
\((M, C)\) is strongly Berwald-Cartan then it is a Kähler-Cartan space.

Proof. Due to the weakly Kähler-Cartan property, we have \((N_{jk} - N_{kj})\zeta^j = 0\). Now, differentiating this with respect to \(\zeta_m\), and using \(\partial^\iota N_{jk} = 0\), it results \((N_{jk} - N_{kj})h^{mj} = 0\). From here, we obtain \(N_{jk} = N_{kj}\), i.e., the space is Kähler-Cartan.
4 Complex Cartan-Randers metrics

On the complex manifold $M$ we consider

$$a := a_{ij}(z)dz^i \otimes d\bar{z}^j$$

a Hermitian positive metric with its inverse $a^{ij}(z)$ and $b = b_i(z)dz^i$ a differential $(1, 0)$--form. Then, $\alpha^2(z, \zeta) := a^{ij}(z)\zeta_j \zeta_i$, $(\alpha : T^*M \to \mathbb{R}^+)$, defines a purely Hermitian complex Cartan metric on $M$. Also, denoting by $b^i := a^{ji}(z)b_j$, we construct a function $\tilde{C} : T^*M \to \mathbb{R}^+$,

$$\tilde{C}(z, \zeta) := \alpha + |\beta|,$$

where

$$|\beta|^2 = \beta \bar{\beta} \text{ with } \beta(z, \zeta) = b_i(z)\zeta^i.$$ 

A direct computation gives

$$\tilde{h}^{ji} := \partial^2 \tilde{C}^2 = \frac{\tilde{C}_a}{\alpha} a^{ji} - \frac{\tilde{C}_\beta}{2|\beta|} b^jb^i + \frac{1}{2\tilde{C}} \tilde{\zeta}^j \tilde{\zeta}^i,$$

where

$$\zeta^i := \partial^i \alpha^2; \tilde{\zeta}^i := \frac{\tilde{C}_\alpha}{\alpha} \zeta^i + \frac{\tilde{C}_\beta}{|\beta|} b^i.$$ 

Applying Proposition 2.2, from [7], it results.

**Proposition 4.1.** Corresponding to the function (4.1), we have

1) $\tilde{h}_{ij} = \alpha \frac{\partial^2 \tilde{C}^2}{\partial \zeta^i \partial \zeta_j} = \frac{\tilde{C}}{\alpha} a^{ij} + \frac{|\beta|(|\alpha||b|^2 + |\beta|)}{\tilde{C}^2} \zeta_i \zeta_j - \frac{\alpha^3}{\tilde{C}\gamma} b_i b_j - \frac{\alpha}{\tilde{C}\gamma} (\bar{\beta} \zeta_i b_j + \beta b_i \zeta_j)$;

2) $\det\left(\tilde{h}^{ji}\right) = \left(\frac{\tilde{C}}{\alpha}\right)^n \frac{\gamma}{2|\beta|} \det\left(a^{ji}\right)$,

where $||b||^2 := a^{ji}b_i b_j; \gamma := \tilde{C}^2 + \alpha^2(||b||^2 - 1)$.

Having formula for $\det\left(\tilde{h}^{ji}\right)$, we can say that $\tilde{h}^{ji}$ is positive definite if and only if $\gamma > 0$, for any $\zeta \in \tilde{T}^*M$. Also, it is obvious that the function $\tilde{C} = \alpha + |\beta|$ satisfies the conditions i) -- iii) from Definition 2.1. So, we have proved the following result.

**Theorem 4.1.** The function (4.1) with $\gamma > 0$ is a complex Cartan metric.

Further on, the function (4.1) with $\gamma > 0$ is called a complex Cartan-Randers metric and the pair $(M, \alpha + |\beta|)$ is a complex Cartan-Randers space.

Note that a complex Cartan-Randers metric can be purely Hermitian if and only if $\alpha^2||b||^2 = |\beta|^2$. Since any purely Hermitian metric is a complex Berwald-Cartan metric, we now focus on non-purely Hermitian complex Cartan-Randers metrics.
Once obtained the metric tensor of a complex Cartan-Randers space, it is a technical computation to get the expression of the coefficients $\tilde{N}_{ji}$ of Chern-Cartan (c.n.c.), with respect to the metric \((\ref{eq:metric})\). After a lot of trivial calculus, we obtain a simplified writing for these

$$
\tilde{N}_{ji} = \tilde{N}_{ji} - \frac{1}{\gamma} \left( \frac{\partial b_j^\alpha}{\partial z^i} \xi^\gamma - \frac{\alpha \beta}{|\beta|} \frac{\partial b^\beta}{\partial z^i} b^\gamma \right) \xi_j - \frac{\beta}{|\beta|} k_j^\gamma \frac{\partial b^\gamma}{\partial z^i},
$$

(4.3)

where $\xi_j := \bar{\beta} \xi_j + \alpha^2 b^\gamma$, $k_j^\gamma := \gamma \alpha a_{j\gamma} + \frac{\alpha |b^\gamma| + |\beta|}{\gamma} \xi_j \xi^\gamma - \frac{\alpha \beta}{|\beta|} b_{j\gamma} \xi^\gamma$ and $\tilde{N}_{ji} := -a_{j\gamma} \frac{\partial a_{i\gamma}}{\partial z^i} \xi_j$.

**Theorem 4.2.** Let \((M, \tilde{\mathcal{C}})\) be a connected non-purely Hermitian complex Cartan-Randers space. Then, \((M, \tilde{\mathcal{C}})\) is a Berwald-Cartan space if and only if $\delta^a_\gamma | \beta |= 0$, where $\delta^a_\gamma$ is the adapted frame corresponding to $\tilde{N}_{ji}$.

**Proof.** If \((M, \tilde{\mathcal{C}})\) is Berwald then $N_{jk} = H_j^k(z) \xi_j$, which means that $N_{jk}$ are homogeneous polynomials in $\xi_i$ of first degree. Thus, using \((\ref{eq:metric})\) we have

$$
\begin{align*}
\alpha |\beta| \{ & 2(\tilde{N}_{ji} - \tilde{N}_{ji}) \\
+ \frac{\alpha}{|\beta|} \frac{\partial a_{j\gamma}}{\partial z^i} [a_{j\gamma}(\alpha^2 |b|^2 + |\beta|^2) + |b|^2 \xi_j \xi^\gamma - \alpha^2 b^\gamma b_{\gamma} - (\bar{\beta} \xi_j b^\gamma + \bar{\beta} b_{\gamma} \xi_j)] \\
+ (\alpha^2 |b|^2 + |\beta|^2)(\tilde{N}_{ji} - \tilde{N}_{ji}) + \frac{\alpha^2 b^\gamma}{|\beta|^2} \xi_j + \beta \frac{\partial b^\gamma}{\partial z^i} \xi_j = 0,
\end{align*}
$$

which contains an irrational part and a rational one. Thus, we obtain

$$
\begin{align*}
-\frac{\beta}{|\beta|} \frac{\partial b^\gamma}{\partial z^i}[a_{j\gamma}(\alpha^2 |b|^2 + |\beta|^2) + |b|^2 \xi_j \xi^\gamma - \alpha^2 b^\gamma b_{\gamma} - (\bar{\beta} \xi_j b^\gamma + \bar{\beta} b_{\gamma} \xi_j)] \\
= 2|\beta|^2(\tilde{N}_{ji} - \tilde{N}_{ji}) \text{ and} \\
-\frac{\bar{\beta}}{|\beta|^2} \frac{\partial b^\gamma}{\partial z^i} + \frac{\partial b^\gamma}{\partial z^i} \xi_j - \frac{\alpha^2}{|\beta|^2} (\bar{\beta} \xi^\gamma \frac{\partial b^\gamma}{\partial z^i} - \beta \frac{\partial b^\gamma}{\partial z^i} \xi_j)b_j - 2\alpha^2 \beta \frac{\partial b^\gamma}{\partial z^i} a_{j\gamma}
\end{align*}
$$

= (\alpha^2 |b|^2 + |\beta|^2)(\tilde{N}_{ji} - \tilde{N}_{ji}).

Contracting the above relations with $b_j^\gamma$ and $\xi^\gamma$, it results

$$
(\tilde{N}_{ji} - \tilde{N}_{ji}) b^\gamma = 0;
$$

(4.4)

$$
\begin{align*}
2|\beta|^2(\tilde{N}_{ji} - \tilde{N}_{ji}) \xi^\gamma + 2\alpha^2 \beta \frac{\partial b^\gamma}{\partial z^i} (|b|^2 \xi_j - \bar{\beta} b_{\gamma}) &= 0; \\
(\alpha^2 |b|^2 + |\beta|^2) \xi_j \frac{\partial b^\gamma}{\partial z^i} - \frac{\beta^2}{|\beta|^2} (\alpha^2 |b|^2 - |\beta|^2) \frac{\partial b^\gamma}{\partial z^i} \xi_j + 2\alpha^2 \beta \frac{\partial b^\gamma}{\partial z^i} b_j &= 0; \\
(\alpha^2 |b|^2 + |\beta|^2)(\tilde{N}_{ji} - \tilde{N}_{ji}) \xi^\gamma + 2\alpha^2 (\bar{\beta} \xi^\gamma \frac{\partial b^\gamma}{\partial z^i} + \beta \frac{\partial b^\gamma}{\partial z^i} \xi_j) &= 0.
\end{align*}
$$
Adding the second and the third relations from (4.4), we obtain
\[2|\beta|^2(\tilde{N}_{ji} - \bar{N}_{ji})\zeta^j + (\alpha^2|b|^2 + |\beta|^2)(\tilde{\beta}\zeta \frac{\partial b}{\partial \zeta^j} + \bar{\beta}\zeta \frac{\partial \bar{b}}{\partial \zeta^j} + \zeta) = 0.\]

This, together with the fourth equation from (4.4), implies \((\tilde{N}_{ji} - \bar{N}_{ji})\zeta^j = 0\) and \(\bar{\beta}\zeta \frac{\partial b}{\partial \zeta^j} + \beta \frac{\partial \bar{b}}{\partial \zeta^j} + \zeta = 0\). The last condition can be rewritten as \(\delta_k^a|\beta| = 0\).

Conversely, if \(\delta_k^a|\beta| = 0\), by derivation with respect to \(\zeta\) and then with \(\zeta_\bar{m}\), we deduce
\[b_\bar{m}^a a^\bar{r} b_{\bar{r}} \frac{\partial b}{\partial z^i} + b^i \frac{\partial b}{\partial z^j} = 0.\]

The last relation gives
\[\frac{\partial^* b^m}{\partial z^i} = -\frac{\bar{\beta}}{|\beta|^2} b^m \zeta^j \frac{\partial^* b}{\partial \zeta^j},\]
which substituted into (4.3) implies \(\tilde{N}_{ji} = \bar{N}_{ji}\) and so, \(\tilde{N}_{ji}\) are holomorphic in \(\zeta_i\), i.e. the space is Berwald.

**Corollary 4.1.** Let \((M, \tilde{C})\) be a connected non-purely Hermitian complex Cartan-Randers space. Then, \((M, \tilde{C})\) is a strongly Berwald-Cartan space if and only if \(\delta_k^a|\beta| = 0\) and \(\alpha\) is Kähler.

**Proof.** It follows by Theorem 4.2. \(\square\)

In the remaining part of this section we come with some examples of Cartan-Randers metrics which are Berwald-Cartan or strongly Berwald-Cartan.

**Example 4.1.** We consider \(\Delta = \{(z_1, z_2) := (z, w) \in \mathbb{C}^2, |w| < |z| < 1\}\) and \((\zeta_1, \zeta_2) := (\zeta, \upsilon) \in \mathbb{T}'^*\Delta\). We choose \(\beta = b^i(z, w)\zeta_i, i = 1, 2,\) with
\[b^1 = \frac{w}{|z|^2 - |w|^2}; \quad b^2 = -\frac{z}{|z|^2 - |w|^2}.\] (4.5)

The function \(\alpha := \sqrt{a^{ij}\zeta_i\zeta_j}; i, j = 1, 2,\) where
\[a^{11} = \frac{1}{(1 - |z|^2)^2} + b^1 b^1; \quad a^{12} = b^1 b^2; \quad a^{22} = b^2 b^2;\] (4.6)
defines a purely Hermitian complex Cartan metric on \(\Delta\). Using these tools, we obtain the complex Cartan-Randers metric
\[\tilde{C} = \alpha + |\beta| = \sqrt{\frac{|\zeta|^2}{(1 - |z|^2)^2} + |\beta|^2 + |\beta|},\] (4.7)
with \(|b|^2 = 1, b_1 = 0\) and \(b_2 = -\frac{|z|^2 - |w|^2}{|z|^2 - |w|^2}\). Some computations lead us to the conclusion that the metric (4.7) is Berwald-Cartan, that is,
\[ a \delta_1^*|\beta| = \bar{\beta} \xi^2 \frac{\partial^* b_2}{\partial z^i} + \beta \frac{\partial^* b_2}{\partial z^i} \bar{\xi} = \bar{\beta} \xi^2 \frac{\partial^* b_2}{\partial z^i} + \beta \left( \frac{\partial^* b_2}{\partial z^i} \bar{\xi} + \frac{\partial^* b_2}{\partial z^i} \bar{\eta} \right) = 0 \]
\[ a \delta_2^*|\beta| = \bar{\beta} \xi^2 \frac{\partial^* b_2}{\partial w} + \beta \frac{\partial^* b_2}{\partial w} \bar{\xi} = \bar{\beta} \xi^2 \frac{\partial^* b_2}{\partial w} + \beta \left( \frac{\partial^* b_2}{\partial w} \bar{\xi} + \frac{\partial^* b_2}{\partial w} \bar{\eta} \right) = 0. \]

Note that the metric (4.7) is not one strongly Berwald-Cartan because \( \alpha \) from (4.6) is not a Kähler metric.

**Example 4.2.** On \( M = \mathbb{C}^2 \) we set the purely Hermitian metric

\[ \alpha^2 = e^{z_1 + \bar{z}_1} |\xi_1|^2 + e^{z_2 + \bar{z}_2} |\xi_2|^2 \]  

and we choose \( \beta = e^{z_2} \xi_2 \). Then, \( |\beta|^2 = e^{z_2 + \bar{z}_2} |\xi_2|^2 \) and so, \( b_1 = b^1 = 0, b_2 = e^{-z_2}, b^2 = e^{z_2} \) and \( ||b|| = 1 \).

With these tools we construct a complex Cartan-Randers metric

\[ \tilde{C} = \sqrt{e^{z_1 + \bar{z}_1} |\xi_1|^2 + e^{z_2 + \bar{z}_2} |\xi_2|^2} + \sqrt{e^{z_2 + \bar{z}_2} |\xi_2|^2}, \]  

(4.9)

which is non purely Hermitian, and \( \det(\tilde{h}^{ji}) = \frac{\tilde{C}^4}{2 \alpha^2 |\beta^2|} \det(a^{ji}) > 0, i, j = 1, 2. \)

The metric (4.9) is Berwald-Cartan. Indeed,

\[ a \delta_1^*|\beta| = \bar{\beta} \xi^2 \frac{\partial^* b_2}{\partial z^i} + \beta \frac{\partial^* b_2}{\partial z^i} \xi_2 = 0, i = 1, 2. \]

Moreover, due to Theorem 4.2 we have

\[ \tilde{N}_11 = b_1 = \xi_1; \tilde{N}_12 = b_2 = 0; \tilde{N}_21 = b^1 = 0; \tilde{N}_22 = \xi_2, \]

which attest that the metric (4.8) is Kähler. Thus, by Corollary 4.1, (4.9) is a strongly Berwald-Cartan metric. Note that the above example can be generalized to a class of strongly Berwald-Cartan metrics, taking on \( M = \mathbb{C}^n \),

\[ \alpha^2 = \sum_{k=1}^{n} e^{z_k + \bar{z}_k} |\xi_k|^2. \]

For \( \beta \) we can choose for instance \( \beta = e^{z_k} \eta^k \), where \( k = 1, n. \)

**5 The \( \mathcal{L} \)– duality between complex Finsler and complex Cartan spaces**

Another way to describe the complex Cartan spaces is given by the correspondence between the various geometrical objects on a complex Finsler space \((M, F)\) and those of a complex Cartan space \((M, C)\), via the complex Legendre transformation (the \( \mathcal{L} \)– dual process), [16].
5.1 A nonholonomic vertical frame

In our next approach we need another vertical frame besides \( \dot{\mathbf{x}}^k = \frac{\partial}{\partial \xi^k} \) and its conjugate. Denoting \( \zeta^k := \frac{\partial H}{\partial \xi^k} = \dot{\mathbf{x}}^k \zeta_j \), they change as \( \zeta^k = \frac{\partial \mathbf{H}_j}{\partial \eta^j} \zeta^l \). Corresponding to the new variables \( \zeta^k \), further on we construct a frame, denoted by \( \frac{\partial}{\partial \zeta^k} \).

With respect to the vertical natural frame \{\( \dot{\mathbf{x}}^k, \dot{\mathbf{H}}^k \)} on \( VT^*M \oplus \mathbb{R}^*M \), \( \frac{\partial}{\partial \zeta^k} \) can be decomposed as

\[
\frac{\partial}{\partial \zeta^k} := A_{km} \dot{\mathbf{x}}^m + h_{km} \dot{\mathbf{H}}^m, \tag{5.1}
\]

where the tensor \( A_{km} \) can be found. We require the conditions \( \frac{\partial C}{\partial \zeta^k} = \delta_k^l \) and \( \frac{\partial C}{\partial \zeta^k} = 0 \) and thus, it results

\[
A_{kr} h^{jr} = 0; \quad A_{kr} = -h_{rj} h_{km} h^{jm} = A_{rk}, \tag{5.2}
\]

where \( h^{jr} := \frac{\partial^2 H}{\partial \zeta^j \partial \xi^r} = (\dot{\mathbf{H}} \dot{\mathbf{x}}^j) \zeta^r \), which implies

\[
h_{rj} h^{jm} h^{sr} = 0; \quad (\dot{\mathbf{H}} A_{kr}) \zeta^r = 0; \quad (\dot{\mathbf{x}}^s A_{kr}) \zeta^r = h_{km} h^{ms}. \tag{5.3}
\]

Lemma 5.1. Let \( (M, \mathcal{C}) \) be a complex Cartan space. If the tensor \( A_{km} \) satisfies the conditions \( \{5.2\} \), then \( A_{kr} = \frac{\partial^2 H}{\partial \zeta^k \partial \zeta^r} \) and \( h_{kr} = \frac{\partial^2 H}{\partial \zeta^k \partial \zeta^r} \).

Proof. Due to \( \{5.1\} \) and \( \{5.3\} \), we have

\[
\frac{\partial H}{\partial \xi^k} = A_{kr} \dot{\mathbf{H}} \dot{\mathbf{x}}^r + h_{km} \dot{\mathbf{H}} \dot{\mathbf{x}}^m = A_{kr} \zeta^r + \zeta_k
\]

\[
\frac{\partial^2 H}{\partial \zeta^k \partial \zeta^r} = \frac{\partial}{\partial \zeta^r} \left( \frac{\partial H}{\partial \xi^k} \right) = (A_{rj} \dot{\mathbf{H}} + h_{rj} \dot{\mathbf{H}}) (A_{kl} \zeta^l + \zeta_k)
\]

\[
A_{rj} (\dot{\mathbf{H}} A_{kl}) \zeta^l + A_{rj} A_{kl} \dot{\mathbf{H}} \dot{\mathbf{x}}^l + h_{rj} \dot{\mathbf{H}} A_{kl} \zeta^l + h_{rj} \dot{\mathbf{H}} A_{kl} \dot{\mathbf{x}}^l = 2 A_{rk} + A_{rj} (\dot{\mathbf{H}} A_{kl}) \zeta^l + h_{rj} \dot{\mathbf{H}} A_{kl} \zeta^l = A_{rk}.
\]

Also, it results

\[
\frac{\partial^2 H}{\partial \zeta^k \partial \zeta^r} = \frac{\partial}{\partial \zeta^r} \left( \frac{\partial H}{\partial \xi^k} \right) = (A_{rj} \dot{\mathbf{H}} + h_{rj} \dot{\mathbf{H}}) (A_{kl} \zeta^l + \zeta_k)
\]

\[
A_{rj} (\dot{\mathbf{H}} A_{kl}) \zeta^l + A_{rj} A_{kl} \dot{\mathbf{H}} \dot{\mathbf{x}}^l + h_{rj} (\dot{\mathbf{H}} A_{kl}) \zeta^l + h_{rj} A_{kl} \dot{\mathbf{H}} \dot{\mathbf{x}}^l + h_{kr} = A_{rj} h_{km} h_{lj} \dot{\mathbf{x}}^m \dot{\mathbf{x}}^l + h_{kr} = h_{kr}, \text{ which completes the proof.}
\]

Now, we consider the Hessian matrix \( \mathcal{H}_1 = \begin{pmatrix} h_{kl} & h_{kp} \\ h_{ls} & h_{lp} \end{pmatrix} \) on \( T_{\mathcal{C}}(T^*M) \), of the complex Cartan metric \( H = H(z^k, \zeta_k) \). But, the complex Cartan metric \( H \) can be seen as a function of \( (z^k, \zeta_k) \) and so, its Hessian matrix on \( T_{\mathcal{C}}(T^*M) \) is \( \mathcal{H}_2 = \begin{pmatrix} h_{kj} \\ h_{km} \end{pmatrix} \), where \( h_{kj} := \frac{\partial^2 H}{\partial \zeta^k \partial \zeta^r} \).
Theorem 5.1. Let \((M, C)\) be a complex Cartan space. Then, \(\{\frac{\partial}{\partial \zeta^k}, \frac{\partial}{\partial \bar{\zeta}^k}\}\) is a vertical frame on \(VT^*M \oplus \overline{VT^*M}\), with
\[
\frac{\partial}{\partial \zeta^k} := h_{km} \dot{\eta}^m + h_{km} \dot{\bar{\eta}}^m, \tag{5.4}
\]
if and only if \(H_1 H_2 = H_2 H_1 = I_{2n}\).

Proof. This is immediate, taking into account that the condition \(H_1 H_2 = H_2 H_1 = I_{2n}\) is equivalent with (5.2).

Moreover, after some computation, we obtain the expression of \(\dot{\zeta}^k\), with respect to the frame \(\{\frac{\partial}{\partial \zeta^k}, \frac{\partial}{\partial \bar{\zeta}^k}\}\),
\[
\dot{\zeta}^k = h_{kl} \frac{\partial}{\partial \zeta^l} + h_{\bar{k}l} \frac{\partial}{\partial \bar{\zeta}^l}, \tag{5.5}
\]
under assumption \(H_1 H_2 = H_2 H_1 = I_{2n}\).

In view of (5.4) this vertical frame is said to be nonholonomic, because it depends on the tensors \(h_{km}\) and \(h_{km}\).

Lemma 5.2. Under assumptions (5.2), we have that \(\frac{\partial H}{\partial \zeta^k} \zeta^k = \frac{\partial H}{\partial \bar{\zeta}^k} \bar{\zeta}^k = H, h_{rk} \zeta^r = h_{rk} \bar{\zeta}^r = 0\) and \(h_{ij}\) is \((1, -1)\) - homogeneous with respect to the variables \(\zeta = (\zeta_k)\).

Proof. By (5.4) we have, \(\frac{\partial H}{\partial \zeta^k} \zeta^k = h_{km} \frac{\partial H}{\partial \zeta^m} \zeta^m + h_{km} \frac{\partial H}{\partial \bar{\zeta}^m} \bar{\zeta}^m\). Due to (5.2), the first term is vanishing and the second is \(\frac{\partial H}{\partial \zeta^m} \zeta^m = H\). So, \(\frac{\partial H}{\partial \zeta^k} \zeta^k = H\), and by conjugation, \(\frac{\partial H}{\partial \bar{\zeta}^k} \bar{\zeta}^k = H\).

Now, \(h_{rk} \zeta^r = \frac{\partial^2 H}{\partial \zeta^k \partial \zeta^r} \zeta^r = \frac{\partial}{\partial \zeta^k} (\frac{\partial H}{\partial \zeta^r} \zeta^r) - \frac{\partial H}{\partial \zeta^k} \zeta^r = 0\).

Using again (5.2), it results \((\dot{\zeta}^k h_{ij}) \zeta_k = h_{ij}\) and \((\dot{\bar{\zeta}}^k h_{ij}) \bar{\zeta}_k = -h_{ij}\), which complete our claim.

5.2 \(\mathcal{L}\)– dual process

Let \((M, F)\) be a complex Finsler, where \(F : T' M \to \mathbb{R}^+\) is a continuous function satisfying the conditions:
\begin{itemize}
  \item[i)] \(L := F^2\) is smooth on \(\tilde{T'M} := T'M \setminus \{0\}\);
  \item[ii)] \(F(z, \eta) \geq 0\), the equality holds if and only if \(\eta = 0\);
  \item[iii)] \(F(z, \lambda \eta) = |\lambda| F(z, \eta)\) for \(\forall \lambda \in \mathbb{C}\);
  \item[iv)] the Hermitian matrix \((g_{ij}(z, \eta))\) is positive definite, where \(g_{ij} := \frac{\partial^2 L}{\partial \eta^i \overline{\eta^j}}\) is the fundamental metric tensor.
\end{itemize}
We consider the adapted frame \( \{ \delta_k := \frac{\delta}{\partial z^k} = \frac{\partial}{\partial x^k} - N_k^l \frac{\partial}{\partial y^l}, \hat{\delta}_k := \frac{\partial}{\partial \eta^k} \} \) of Chern-Finsler (c.n.c.), where \( N_k^l(z, \eta) = g_{lm} \frac{\partial g_{nk}}{\partial y^m} \eta^l \), and the Chern-Finsler connection (in brief \( C = \hat{F} \) connection), whose local coefficients are (see [16])

\[
L^i_{jk} = g^{i\ell} \delta_k g_{\ell j} = \hat{\delta}_j N^i_k ; \quad C^i_{jk} = g^{i\ell} \hat{\delta}_k g_{\ell j},
\]

and \( L^i_{jk} = C^i_{jk} = 0 \). We recall that in [11]’s terminology, the complex Finsler space \((M, F)\) is Kähler iff \( T^i_{jk} \eta^j = 0 \) and weakly Kähler iff \( g_{\eta^j} T^i_{jk} \eta^j = 0 \), where \( T^i_{jk} := L^i_{jk} - L^j_{ik} \).

The Chern-Finsler (c.n.c.) does not generally derive from a spray, but it always determines a complex spray with the local coefficients \( G^i = \frac{1}{2} N^i_j \eta^j \).

In [16] the complex Legendre transformation was introduced, i.e. a local diffeomorphism \( \Phi \times \Phi \) with \( \Phi : U \subset T'M \to \bar{U}^* \subset T'^*M, \Phi(z^k, \eta^k) = (z^k, \hat{\delta}_k L) \), and \( \Phi : \bar{U} \subset T''M \to U^* \subset T'^*M, \Phi(z^k, \bar{\eta}^k) = (z^k, \bar{\delta}_k L) \). For simplicity, hereinafter the complex Legendre transformation is denoted only by \( \Phi \) and the distinction between the open sets \( U \) and \( \bar{U} \) is not specified, but we have assumed that it is defined, as above, on whole \( T_C M \). The properties obtained by \( \Phi \) or by \( \Phi^{-1} \) are called \( \mathcal{L} \)—dual one to another. Also, we can assume that in any point of \( T_C M \) there are local charts which are sent by \( \mathcal{L} \)—duality in local charts on \( T_C^2 M \). We keep the notations from [16], (p.163), namely \( \ast \) is the image of various geometric objects by \( \Phi \), and \( \ast' \) is their image by \( \Phi^{-1} \).

Now, setting the locally tangent map \( d\Phi : T_C(T'M) \to T_C(T'^*M) \) and \( d\bar{\Phi} : T_C(T''M) \to T_C(T'^*M) \), we determine the conditions under which \( d\Phi \) sends the complex tangent vectors in \( T'M \) into the complex tangent vectors in \( T'^*M \), such that the image by complex Legendre transformation, of a complex Finsler space \((M, F)\) is locally a complex Cartan space \((M, C)\), and conversely, i.e.,

\[
(L(z^k, \eta^k))^* = H(z^k, \zeta_k) ; \quad (H(z^k, \zeta_k))^o = L(z^k, \eta^k),
\]

with

\[
\frac{\partial L}{\partial z^i} = -\frac{\partial H}{\partial z^i} ; \quad (\eta^k)^* = \hat{\delta}^k H ; \quad (\zeta^k)^o = \hat{\delta}_k L.
\]

Let \( G = \begin{pmatrix} g_{kj} & g_{j\bar{m}} \\ g_{k\bar{n}} & g_{\bar{m}\bar{n}} \end{pmatrix} \) be the Hessian matrix on \( T_C(T'M) \), of a complex Finsler metric \( L(z^k, \eta^k) \), where \( g_{jk} := \frac{\partial^2 L}{\partial y^j \partial \eta^k} \) and \( g_{jk} := \frac{\partial^2 L}{\partial y^j \partial \bar{\eta}^k} \) is the metric tensor.

Since the Jacobi matrix of \( \Phi \) is \( \begin{pmatrix} \delta_{jk} & \frac{\partial L}{\partial \eta^k \partial \bar{\eta}^j} \\ 0 & g_{j\bar{m}} \end{pmatrix} \), then \( d\Phi \) sends \( \begin{pmatrix} \frac{\partial}{\partial z^k} \hat{\delta}_k \end{pmatrix} \)
regulate has locally det \( L^- \) given by (5.7). The dual of \( \dot{\mathcal{C}} \) is locally regulate \( F \) and so, \((\dot{k})^* = \dot{k}H\), lead to

\[
\begin{align*}
g_{k\dot{k}}\dot{r}^j + g_{ik}\dot{h}^k = \delta^j_i; & \quad g_{ik}\dot{h}^m + g_{ik}\dot{h}^b = 0, \\
g_h^k + h_k^m = \delta^i_j; & \quad g_h^k + h_{ik}\dot{\eta}^m = (\dot{\eta})^* = 0. 
\end{align*}
\] (5.10)

which by \((\eta^k)^* = \dot{k}H\), we find that the image by \( \Phi \) of the frame \( \dot{k} \) is the vertical frame (5.5), i.e. \((\dot{k})^* = \frac{\partial}{\partial z^k}\), which together (5.7), yields

\[
\begin{align*}
(g_{kj})^* & = h_{kj} ; \quad (g_{jr})^* = h_{jr} ; \quad (g_{ks})^* = h_{ks} ; \quad (g_{pk})^* = h_{pk}, \\
(\dot{k})^*(\frac{\partial L}{\partial z^i}) & = -\frac{\partial}{\partial \zeta^k}(\frac{\partial^* H}{\partial \zeta^i}) = -h_{ki}\dot{\eta}^k(\frac{\partial^* H}{\partial \zeta^i}) - h_{km}\dot{\eta}^m(\frac{\partial^* H}{\partial \zeta^i}); \\
(\dot{k})^*(\frac{\partial L}{\partial z^i}) & = -\frac{\partial}{\partial \zeta^k}(\frac{\partial^* H}{\partial \zeta^i}) = -h_{ik}\dot{\eta}^k(\frac{\partial^* H}{\partial \zeta^i}) - h_{ik}\dot{\eta}^i(\frac{\partial^* H}{\partial \zeta^i}).
\end{align*}
\] (5.11)

Moreover, using (5.9) and \( C - F \) (c.n.c.), \( N_i^k = g^{mk}\frac{\partial g_{im}}{\partial \eta^m} \eta^i = g^{mk}\frac{\partial^2 L}{\partial z^i \partial \eta^m} \), it results

\[
\frac{\partial}{\partial z^i} - N_i^k(\dot{k})^* = \frac{\partial^*}{\partial z^i} + \dot{N}_{ki}\dot{k}^k,
\]

with \( \dot{N}_{ki} := \frac{\partial^2 L}{\partial z^i \partial \eta^m} - g_{jk}\dot{N}_{i}^j \). Next, due to (5.11), we obtain that the image by \( \mathcal{L}^- \) duality of \( \dot{N}_{ki} \) is \( C - C \) (c.n.c.), i.e. \((\dot{N}_{ki})^* = N_{ki} \) and so, \((\dot{k})^* = \delta^k_i\).
Proposition 5.1. The $\mathcal{L}$—dual of the $C - F$ connection is a connection $D\Gamma^*$ with local coefficients

$$
H'^{ij}_{jk} = H^i_{jk}; \ H'^{i}_{jk} = 0;
$$

$$
V'^{i}_{j} = -h^{kr}h_{r\bar{m}}h_{j\bar{s}}(\partial_{\bar{m}}h_{\bar{s}i}) ; \ V'^{i}_{j} = h^{kr}h_{ri}V'^{i}_{j} = h_{j\bar{s}}(\partial_{\bar{k}}h_{\bar{s}i}).
$$

Proof. Indeed,

$$
H'^{i}_{jk} := (L^i_{jk})^* = [g^T_{ik}(\delta_k g_{jr})] = \delta^i_{kr}h_{r\bar{m}}h_{j\bar{s}}(\partial_{\bar{m}}h_{\bar{s}i}) = \delta^i_{kr}h_{j\bar{s}}(\partial_{\bar{k}}h_{\bar{s}i}).
$$

Due to (5.7) and (5.13), the image by $\mathcal{L}$—duality of the last relation is

$$
0 = -\eta \delta^i H_{jk} + (N_{jk} - N_{kj})\zeta^i.
$$

Since $N_{ki}$ are local coefficients of $C - C$ (c.n.c.), then $\frac{\delta^i H_{jk}}{\partial z^k} = 0$ and so,

$$(N_{jk} - N_{kj})\zeta^i = 0,$$

which gives our claim.

Proposition 5.2. Let $M$ be a complex manifold with the $\mathcal{L}$—dual metrics $F$ and $C$ given by (5.7). If $F$ is a weakly Kähler metric, then $C$ is a weakly Kähler metric, too.

Proof. The weakly Kähler property of $F$ can be rewritten as

$$
0 = g_{kA}(\tilde{L}_{jk} - \tilde{L}_{kj})\eta^l\eta^j = \eta^l\frac{\partial L}{\partial z^k} - \tilde{N}_{kJ}\eta^j = -(\frac{\partial^2 H}{\partial z^k} + \tilde{N}_{k}\eta^j) + (\tilde{N}_{jk} - \tilde{N}_{kj})\eta^j.
$$

Due to (5.7) and (5.13), the image by $\mathcal{L}$—duality of the last relation is

$$
0 = -\delta^i H_{jk} + (N_{jk} - N_{kj})\zeta^i.
$$

5.3 Geodesic curves of a complex Cartan space

Let $\sigma : [a, b] \to \tilde{T}\tilde{M}$ be a parametrized curve which, in a local chart of $\tilde{T}\tilde{M}$, is given by $\sigma(s) = (z^k(s), \eta^k(s))$, $s \in [a, b], k = 1, n$, where $\eta^k(s) = \frac{d^k}{ds}$ is a tangent vector to the curve $(z^k(s))$ on $M$.

In $[1]$'s sense, $\sigma(s)$ is a complex geodesic curve on a complex Finsler space $(M, F)$ if and only if it is solution of the equations

$$
d^2z^k ds^2 + 2G^k(z(s), \frac{dz}{ds}) = \Theta^k(z(s), \frac{dz}{ds}) \quad ; \quad k = 1, n, \tag{5.12}
$$

where $\Theta^k = 2g^{jk}(\delta_j L) = g^{mk}g_{jl}(L^l_{nm} - L^l_{mn})\eta^j\eta^n$. Note that $\Theta^k$ is vanishing if and only if the space $(M, F)$ is weakly Kähler.

With notation $T^k := \frac{d^k}{ds} \frac{dz^k}{ds}$, the equations (5.12) become

$$
T^k = \Theta^k \quad ; \quad k = 1, n. \tag{5.13}
$$
Let us consider \((M,F)\) a locally regulate complex Finsler space and \((M,C)\) is the complex Cartan space, obtained by \(\mathcal{L}\)-dual process. Further on, our goal is to determine the image by \(\mathcal{L}\)-duality of the complex geodesic curve \(\sigma(s)\). The first question is how to map a curve from \(T'M\) into a curve on \(T''M\) or, more precisely, on \(T'''M\)?

By definition, a curve on \(T''M\) is a map
\[
s\to \sigma^*(s) = (z^k(s), \zeta_k(s)),
\]
where \(\zeta_k(s)\) are the components of \((1,0)\)-form. We know the isomorphism between the tangent and cotangent spaces, via a metric tensor, but this is defined on \(M\).

On a complex Cartan space \((M,C)\), with the metric tensor \(h^{jk}(z, \zeta)\), we can consider the tangent vector (from \(T_{C,u}(T'rM)\)) to a curve \(\sigma^*(s)\), given by \(X = \left(\frac{dz^k}{ds}, \zeta_k = h^{jk}\zeta_m(s)\right)\). When \(\zeta^k = \frac{dz^k}{ds}\), that is \(\sigma^*(s) = \sigma(s) = (z^k(s), \zeta_k = h_{km}\frac{dz^m}{ds})\), we say that \(\sigma^*(s)\) is the image by \(\mathcal{L}\)-duality of the curve \(\sigma(s) = (z^k(s), \eta^k(s))\) from \(T'M\), where \(\eta^k(s) = \frac{dz^k}{ds}\). It is clear that \(\zeta^k = \frac{dz^k}{ds}\) are the components of a tangent vector to the curve \(\sigma^*(s)\), which is \(\mathcal{L}\)-dual of \(\sigma(s)\). Making an excessive use of notation, we write \(\sigma^*(s) : = [\sigma(s)]^*\), which in a local chart of \(\mathcal{T}'M\), is \(z^k = z^k(s), \zeta_k = \zeta_k(s), k = 1, n\).

For two \(\mathcal{L}\)-dual curves, we have
\[
(T^k)^* = (\Theta^k)^* \quad ; \quad k = 1, n,
\]
with \(\frac{ds^k}{ds} = [\eta^k(s)]^* = \zeta^k(s)\).

Taking into account \([5.7]\) and Proposition 5.2, we obtain
\[
\Theta^k : = (\Theta^k)^* = h^{mk}(N_{\bar{r}\bar{m}} - N_{\bar{m}\bar{r}})\zeta^r,
\]
which is \((1,1)\)-homogeneous with respect to the variables \(\zeta = (\zeta_k)\).

Moreover, the space \((M,C)\) is weakly Kähler iff \(\Theta^k = 0\).

In order to obtain \((T^k)^*\), we can rewrite \(\frac{dz^k}{ds}\) as follows:
\[
\frac{d}{ds} \left(\frac{dz^k}{ds}\right) = \frac{d}{ds} \eta^k = \frac{d}{ds} \left(g^{mk}\eta_m\right) =
\]
\[
= \left[\frac{\partial g^{mk}}{\partial z^r} \frac{dz^r}{ds} + \frac{\partial g^{mk}}{\partial \bar{z}^r} \frac{d\bar{z}^r}{ds} + (\frac{\partial H}{\partial z^r}) \frac{d\eta^r}{ds} + (\frac{\partial H}{\partial \bar{z}^r}) \frac{d\bar{\eta}^r}{ds}\right] \eta_m + g^{mk} \frac{d\eta_m}{ds} =
\]
\[
= g^{mk} \left[\frac{\partial g^{mk}}{\partial z^r} \frac{dz^r}{ds} + \frac{\partial g^{mk}}{\partial \bar{z}^r} \frac{d\bar{z}^r}{ds} + (\frac{\partial H}{\partial z^r}) \frac{d\eta^r}{ds} + (\frac{\partial H}{\partial \bar{z}^r}) \frac{d\bar{\eta}^r}{ds}\right] \eta_m + g^{mk} \frac{d\eta_m}{ds} =
\]
\[
= -N^r \frac{dz^r}{ds} - g^{mk} \frac{\partial z^r}{\partial \bar{z}^p} \frac{dz^p}{ds} + g^{mk} \frac{\partial z^r}{\partial z^p} \frac{dz^p}{ds} + \frac{dz^r}{ds} + g^{mk} \frac{d\eta_m}{ds}.
\]

This implies,
\[
\frac{d}{ds} \left(\frac{dz^k}{ds}\right) + N^k \frac{dz^k}{ds} = -g^{nk} \left(\frac{\partial^2 H}{\partial z^p \partial z^p} \frac{dz^p}{ds} + g^{r\bar{r}} \frac{d\eta^r}{ds} \frac{d\eta^r}{ds}\right)
\]
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and by $L$– duality, it leads to

$$(T^k)^* = -h^{ik}[N_{\bar{\imath}k} \frac{d\bar{\imath}}{ds} + h_{\bar{\imath}k}(T^\bar{\imath})^* - \frac{d\zeta_{\bar{\imath}}}{ds}].$$

(5.17)

Now, substituting (5.14) and (5.15) into (5.17), it results

$$\Theta^{*k} = -h^{ik}[N_{\bar{\imath}k} \frac{d\bar{\imath}}{ds} + h_{\bar{\imath}k}\Theta^{*\bar{\imath}} - \frac{d\zeta_{\bar{\imath}}}{ds}].$$

(5.18)

This is equivalent with

$$\Theta^{*k} + h^{ik} h_{\bar{\imath}k}\Theta^{*\bar{\imath}} = h^{ik} [\frac{d\zeta_{\bar{\imath}}}{ds} - N_{\bar{\imath}k} \frac{d\zeta^j}{ds}] + h^{ik} (N_{\bar{\imath}k} - N_{\imath\bar{\imath}}) \frac{d\zeta^j}{ds}$$

which leads to $\frac{d\zeta_{\bar{\imath}}}{ds} - N_{\bar{\imath}k} \frac{d\zeta^j}{ds} = h_{\bar{\imath}k}\Theta^{*\bar{\imath}}$. Thus, we have proved.

**Theorem 5.3.** Let $M$ be a complex manifold with the $L$– dual metrics $F$ and $C$ given by (5.7). If $\sigma(s)$ is a complex geodesic curve on the complex Finsler space $(M, F)$ then its image by $L$– duality $\sigma^*(s)$ satisfies the equations

$$\frac{dz_k}{ds} = \zeta^k; \quad \frac{d\zeta_k}{ds} - N_{jk} \frac{dz^j}{ds} = h_{jk}\Theta^{*j}; \quad k = \bar{1}, \bar{n}.$$  

(5.19)

Moreover, if $(M, F)$ is weakly Kähler, then $\sigma^*(s)$ satisfies

$$\frac{dz_k}{ds} = \zeta^k; \quad \frac{d\zeta_k}{ds} - N_{jk} \frac{dz^j}{ds} = 0; \quad k = \bar{1}, \bar{n}.$$  

(5.20)

It is natural to ask if $\sigma^*(s)$ is a geodesic curve for $C$. The answer is provided below. We start with the fact that the image by $L$– duality of the Euler-Lagrange equations for $\sigma(s)$ are the Hamilton-Jacobi equations for $\sigma^*(s)$,

$$\frac{dz_k}{ds} = \frac{\partial H}{\partial \zeta_k}; \quad \frac{d\zeta_k}{ds} = -\frac{\partial H}{\partial z^k}; \quad k = \bar{1}, \bar{n}.$$  

(5.21)

But the equations (5.20) are equivalent with (5.21). Indeed, the first equation from (5.21) is $\frac{dz_k}{ds} = h^{ik} \zeta_{\bar{m}} = \zeta^k$. The second equation (5.21) can be rewritten as $\frac{d\zeta_k}{ds} = -\frac{\partial}{\partial z^k} (h^{mk} \zeta_{\bar{m}})$. This is equivalent with $\frac{d\zeta_k}{ds} = -\frac{\partial h^{mk}}{\partial z^k} \zeta_{\bar{m}}$, which leads to $\frac{d\zeta_k}{ds} - N_{jk} \zeta^j = 0$. So, under assumption of weakly Kähler for
Theorem 5.4. Let \((M, C)\) be a complex Cartan space. Then \(\sigma^s(s)\) is a complex geodesic curve for \(C\) if and only if

\[
\frac{dz^k}{ds} = \zeta^k; \quad \frac{d^2 z^k}{ds^2} + H_{jl}^k \zeta^l = \Theta^k; \quad k = 1, n. \quad (5.22)
\]

Proof. We suppose that \(\sigma^s(s)\) is a complex geodesic for \(C\), i.e., it satisfies (5.19). Differentiating the equation \(\frac{dz^k}{ds} = \zeta^k\) with respect to \(s\), it results

\[
\frac{d^2 z^k}{ds^2} = \frac{d\zeta^k}{ds} = \frac{d(h^{mk}\zeta_m)}{ds} = [\frac{\partial h^{mk}}{\partial z^l}\frac{dz^l}{ds} + \frac{\partial h^{mk}}{\partial \bar{z}^l}\frac{d\bar{z}^l}{ds} + (\partial h^{mk})\frac{d\zeta_l}{ds}]\zeta_m + h^{mk}\frac{dz_m}{ds}.
\]

We obtain

\[
\frac{d^2 z^k}{ds^2} = \frac{\partial h^{mk}}{\partial z^l}\frac{dz^l}{ds} + \frac{\partial h^{mk}}{\partial \bar{z}^l}\frac{d\bar{z}^l}{ds} + (\partial h^{mk})\frac{d\zeta_l}{ds} \zeta_m + h^{mk}\frac{dz_m}{ds}.
\]

Note that when we say complex geodesic curve for \(C\), we simply mean the curves which are the image by \(\mathcal{L}\)–duality of a complex geodesic curve on \((M, F)\).

Conversely, we suppose that \(\sigma^s(s)\) is solution of (5.22). As above calculus, we obtain

\[
\frac{d^2 z^k}{ds^2} + H_{jl}^k \zeta^l = h^{kl}(\frac{d\zeta_l}{ds} - N_{jl}\frac{dz^l}{ds}) + h^{kl}h_{l}\Theta^l \Theta^k. \quad (5.23)
\]

Now, using (5.22) and (5.2), (5.23) leads to

\[
h^{kl}(\frac{d\zeta_l}{ds} - N_{jl}\zeta^l - h_{jl}\Theta^l) + h^{mk}(\frac{d\zeta_m}{ds} - N_{lm}\frac{dz^m}{ds} - h_{lm}\Theta^m) = 0. \quad (5.24)
\]
Denoting with \( S_l := \frac{d\zeta_l}{ds} - N_{jl}\zeta^j - h_{jl}\theta^j \), the equation (5.24) becomes
\[
h^{kl} S_l + h^{\bar{m}k} S_{\bar{m}} = 0,
\]
which yields \( h^{kl} h^{im} h_{il} - h^{km} S_m = 0 \). Since \( h^{kl} h^{im} h_{il} = 0 \), then \( h^{km} S_m = 0 \), and so, \( S_m = 0 \), i.e., \( \sigma^*(s) \) is a complex geodesic curve for \( \mathcal{C} \).

5.4 Projectively related complex Cartan metrics

In [5] we investigated the projectively related complex Finsler metrics. Namely, the complex Finsler metrics \( F \) and \( \tilde{F} \) are called projectively related if they have the same complex geodesic curves as point sets. This means that for any complex geodesic curves: \( \sigma_1 = \sigma_1(s) \) of \( (M,F) \), (given by (5.12) or equivalently (5.13)), and \( \sigma_2 = \sigma_2(\tilde{s}) \) of \( (M,\tilde{F}) \), (given by \( \frac{dz^k}{dt} \), \( \frac{d\tilde{s}}{dt} \)), then \( \sigma_1 \) and \( \sigma_2 \) must represent the same set of points. To achieve this, we compare the above mentioned equations, making the same parameter \( t \).

The equations of geodesic curve \( \sigma_1 \) in the parameter \( t \), with \( \frac{dt}{ds} > 0 \), are not preserved because it is transformed in \( \frac{dz^k}{dt} \), \( \frac{d\tilde{s}}{dt} \).

\[
[T^k(t) - \Theta^k(t)] \left( \frac{dt}{ds} \right)^2 = T^k(s) - \Theta^k(s) - \frac{dz^k}{dt} \frac{d^2 t}{ds^2} = -\frac{dz^k}{dt} \frac{d^2 t}{ds^2},
\]
where \( T^k(t) := \frac{d^2 z^k}{dt^2} + N^k(t) \frac{dz^k}{dt} \), with \( N^k(t) := N^k(z, \frac{dz}{dt}) \) and \( \Theta^k(t) := \Theta^k(z(t), \frac{dz}{dt}) \) (for more details see [5]).

Similar equations are obtained for \( \sigma_2(t(\tilde{s})) \). Subtracting these equations, we obtained that the spray coefficients of two projectively related complex Finsler metrics are linked by \( \tilde{G}^k = G^k + B^k + P_\eta^k \), where \( B^k = \frac{1}{2}(\tilde{\Theta}^k - \Theta^k) \) and \( P(z, \eta) \) is a smooth function on \( \tilde{T}\mathcal{M} \).

Based on these, we introduce by \( \mathcal{L}^{-} \)- duality the correspondent notion on the \( \mathcal{L}^{-} \)- dual complex Cartan spaces.

Let \( M \) be the complex manifold with \( F \) and \( \tilde{F} \) projectively related Finsler metrics. By (5.7), we obtain two Cartan metrics \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \), which are the images by \( \mathcal{L}^{-} \)- duality of \( F \) and \( \tilde{F} \), respectively. Having in mind the notion of complex geodesic curve on \( (M,\tilde{\mathcal{C}}) \) as being a \( \mathcal{L}^{-} \)- dual of a complex geodesic curve on \( (M,F) \), introduced in the preview section, we give.

**Definition 5.1.** The complex Cartan metrics \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \) on the manifold \( M \), which are the images by \( \mathcal{L}^{-} \)- duality of the complex Finsler metrics \( F \) and \( \tilde{F} \), respectively, are called projectively related if they have the same complex geodesic curves as point sets.
Our next goal is to find the image by $\mathcal{L}$– duality of the equations (5.26).
For this, a similar calculus as in (5.16), yields
\[
T^k(s) = -g^{\hat{s}k}[\hat{N}_{\hat{r}r}(s)\frac{d\hat{z}^r}{ds} + g_{\tilde{r}\tilde{n}}\tilde{T}^\tilde{r}(s) - \frac{d\eta_{\tilde{n}}}{ds}]
\]
\[
= -g^{\hat{s}k}[\hat{N}_{\hat{r}r}(t)\frac{d\hat{z}^r}{dt} + g_{\tilde{r}\tilde{n}}\tilde{T}^\tilde{r}(t) - \frac{d\eta_{\tilde{n}}}{dt}] \left(\frac{dt}{ds}\right)^2
\]
\[
= T^k(t) \left(\frac{dt}{ds}\right)^2,
\]
which together with (5.26) leads to
\[
- g^{\hat{s}k}[\hat{N}_{\hat{r}r}(t)\frac{d\hat{z}^r}{dt} + g_{\tilde{r}\tilde{n}}\tilde{T}^\tilde{r}(t) - \frac{d\eta_{\tilde{n}}}{dt}] - \Theta^k(t) = -\frac{d\hat{z}^k}{ds} \frac{d^2t}{ds^2} \left(\frac{dt}{ds}\right)^2. 
\]
(5.27)
Now, setting the image by $\mathcal{L}$– duality of the equation (5.27), it results the equations of the complex geodesic curve $\sigma^*_{1} = \sigma^*_{1}(t(s))$ in the parameter $t$
\[
- h^{\hat{s}k}\{N_{\hat{r}r}\frac{d\hat{z}^r}{dt} + h_{\tilde{r}\tilde{n}}\Theta^\tilde{r}(t) - \frac{d\eta_{\tilde{n}}}{dt}\} - \Theta^k(t)
\]
\[
= - \frac{d\hat{z}^k}{dt} \frac{d^2t}{ds^2} \left(\frac{dt}{ds}\right)^2
\]
and $\frac{d\hat{z}^k}{dt} = [\eta^k(t)]^* = \zeta^k(t)$, which are equivalent with
\[
\frac{d\zeta_{\hat{n}}}{dt} - N_{\hat{r}r}\frac{d\hat{z}^r}{dt} = h_{\tilde{r}\tilde{n}}\Theta^\tilde{r}(t) - (h_{\tilde{r}\tilde{n}}\frac{d\hat{z}^r}{dt} + h_{k\tilde{n}}\frac{d\hat{z}^k}{dt}) \frac{d^2t}{ds^2} \left(\frac{dt}{ds}\right)^2
\]
(5.28)
and $\frac{d\hat{z}^k}{dt} = \zeta^k(t)$. Taking the conjugation of (5.28) and then using $h_{jk}\eta^j = h_{jk}\frac{d\eta^j}{dt} = 0$ (from Lemma 5.2), we obtain that the equations (5.27) in parameter $t$ are
\[
\frac{d\zeta_{k}}{dt} - N_{jk}\zeta^j(t) - h_{jk}\Theta^j(t) = -\zeta_k \frac{d^2t}{ds^2} \left(\frac{dt}{ds}\right)^2, \; k = 1, n, 
\]
(5.29)
and $\zeta^k(t) = \frac{d\hat{z}^k}{dt}$.
Note that, by the transformation of the parameter $t = t(s)$, with $\frac{dt}{ds} > 0$, the equations of (5.29) are not preserved.
Corresponding to the complex Cartan metric $\tilde{\mathcal{C}}$, on the same manifold $\tilde{M}$, we have the coefficients $\tilde{N}_{jk}$ of Chern-Cartan (c.n.c.), $\tilde{\zeta}^j := \frac{\partial H}{\partial \zeta^j}$ and the
functions $\tilde{\Theta}^*$. Also, we suppose that $\sigma_2^* = \sigma_2^*(\tilde{s})$ is a complex geodesic curve of $(M, \tilde{C})$, where $\tilde{s}$ is the parameter corresponding to $\tilde{C}$. Now, assuming that the same parameter $t$ is transformed by $t = t(\tilde{s})$ as above, we obtain

$$
\frac{d\zeta_k}{dt} - \tilde{N}_{jk}\tilde{\zeta}^j(t) - \tilde{h}_{jk}\tilde{\Theta}^{*j}(t) = -\zeta_k \frac{d^2t}{ds^2} \left( \frac{dt}{ds} \right)^2, \ k = \overline{1,n}. \quad (5.30)
$$

If $C$ and $\tilde{C}$ are projectively related, then $\sigma_1^*$ and $\sigma_2^*$ represent the same set of points and, the difference between corresponding equations from (5.29) and (5.30) gives

$$
\tilde{N}_{jk}\tilde{\zeta}^j + \tilde{h}_{jk}\tilde{\Theta}^{*j} - N_{jk}\zeta^j - h_{jk}\Theta^{*j} = \zeta_k \frac{d^2t}{ds^2} \left( \frac{dt}{ds} \right)^2 - \frac{d^2t}{ds^2} \left( \frac{dt}{ds} \right)^2, \ k = \overline{1,n}. \quad (5.31)
$$

With the notations: $\tilde{N}_k := \tilde{N}_{jk}\tilde{\zeta}^j$ and $N_k := N_{jk}\zeta^j$, (5.31) can be rewritten more generally as

$$
\tilde{N}_k + \tilde{h}_{jk}\tilde{\Theta}^{*j} = N_k + h_{jk}\Theta^{*j} + Q\zeta_k, \ k = \overline{1,n}, \quad (5.32)
$$

where $Q$ is a smooth function on $T^nM$, with complex values.

Denoting by $B_k := \tilde{h}_{jk}\tilde{\Theta}^{*j} - h_{jk}\Theta^{*j}$, the homogeneity properties of the functions $h_{jk}$ and $\Theta^{*j}$ give $(\partial^k B_i)\zeta_k = 2B_i$ and $(\partial^k B_i)\tilde{\zeta}_k = 0$. Moreover, the relations (5.32) become

$$
\tilde{N}_k = N_k + B_k + Q\zeta_k. \quad (5.33)
$$

Now, we use their homogeneity properties, going from $\zeta_k$ to $\lambda\zeta_k$. Thus, differentiating in (5.33) with respect to $\zeta_k$ and $\tilde{\zeta}_k$ and then setting $\lambda = 1$, we obtain

$$
B_k = -(\tilde{\partial}^* Q)\zeta_k \tilde{\zeta}_k \quad \text{and} \quad B_k = [(\tilde{\partial}^* Q)\zeta_k - Q]\zeta_k \quad (5.34)
$$

and so,

$$
(\tilde{\partial}^* Q)\zeta_k + (\tilde{\partial}^* Q)\tilde{\zeta}_k = Q, \quad (5.35)
$$

which means that $Q(z^k, \mu\zeta_k) = \mu Q(z^k, \zeta_k)$, for any $\mu \in \mathbb{R}$.

**Lemma 5.3.** Between the coefficients $\tilde{N}_k$ and $N_k$ corresponding to the metrics $C$ and $\tilde{C}$ on the manifold $M$ there are the relations $\tilde{N}_k = N_k + B_k + Q\zeta_k$, for any $k = \overline{1,n}$, where $Q$ is a smooth function on $T^nM$ with complex values, if and only if $\tilde{N}_k = N_k + (\tilde{\partial}^* Q)\zeta_k$, $B_k = -(\tilde{\partial}^* Q)\zeta_k$, for any $k = \overline{1,n}$, and $(\tilde{\partial}^* Q)\zeta_k + (\tilde{\partial}^* Q)\tilde{\zeta}_k = Q$.  

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From above considerations, we obtain.

**Lemma 5.4.** If the complex Cartan metrics $C$ and $\tilde{C}$ on the manifold $M$ are projectively related, then there is a smooth function $Q$ on $T^*M$ with complex values, satisfying $(\partial^r Q)\zeta_r + (\partial^\bar{r} Q)\zeta_{\bar{r}} = Q$, such that

$$\tilde{N}_k = N_k + (\partial^r Q)\zeta_r \zeta_k \text{ and } B_k = -(\partial^r Q)\zeta_r \zeta_k; \quad k = 1, n. \quad (5.36)$$

Conversely, under assumption that $\sigma^* = \sigma^*(s)$ is a complex geodesic curve of $(M, C)$, we show that the complex Cartan metric $\tilde{C}$ with the coefficients $\tilde{N}_{jk}$ of Chern-Cartan (c.n.c.) and the functions $\tilde{\zeta}_k = \partial_{\tilde{H}} \zeta_k$, given by (5.32) is projectively related to $C$, where $Q$ is a smooth function on $\tilde{T}'^*M$ with complex values. This means that there is a parametrization $\tilde{s} = \tilde{s}(s)$, with $\frac{ds}{d\tilde{s}} > 0$, such that $\sigma^* = \sigma^*(\tilde{s}(s))$ is a geodesic of $(M, \tilde{C})$.

If there is a parametrization $\tilde{s} = \tilde{s}(s)$, then it yields

$$\frac{d\tilde{\zeta}_k}{d\tilde{s}} = \tilde{N}_{rk}(\tilde{s}) + h_{rk}\Theta^{rr}(\tilde{s}) = -\tilde{\zeta}_k \frac{d^2 \tilde{s}}{d\tilde{s}^2} \left( \frac{1}{\frac{ds}{d\tilde{s}}} \right); \quad k = 1, n. \quad (5.37)$$

Now, using (5.32), it results

$$\frac{d\tilde{\zeta}_k}{d\tilde{s}} = \tilde{N}_{rk}(\tilde{s}) + \tilde{h}_{rk}\tilde{\Theta}^{rr}(\tilde{s}) + \left[ Q(\tilde{s}) - \frac{d^2 \tilde{s}}{d\tilde{s}^2} \left( \frac{1}{\frac{ds}{d\tilde{s}}} \right)^2 \right] \zeta_k; \quad k = 1, n.$$

So, $\sigma^* = \sigma^*(\tilde{s}(s))$ is a geodesic of $(M, \tilde{C})$ if and only if

$$[Q(\tilde{s}) - \frac{d^2 \tilde{s}}{d\tilde{s}^2} \left( \frac{1}{\frac{ds}{d\tilde{s}}} \right)^2] \zeta_k = 0; \quad k = 1, n. \quad (5.37)$$

Since $\zeta_k \neq 0$, it results $Q(\tilde{s}) \left( \frac{d\tilde{s}}{ds} \right)^2 = \frac{d^2 \tilde{s}}{ds^2}$. Due to (5.35), it leads to

$$Q(s) \frac{d\tilde{s}}{ds} = \frac{d^2 \tilde{s}}{ds^2}. \quad (5.38)$$

Denoting by $u(s) := \frac{d\tilde{s}}{ds}$, we have $\frac{d^2 \tilde{s}}{ds^2} = \frac{du}{ds}$ and so, $Q(s)u = \frac{du}{ds}$. We obtain $u = ae^{\int Q(s)ds}$. From here, it results that there is

$$\tilde{s}(s) = a \int e^{\int Q(s)ds} ds + b,$$

where $a, b$ are arbitrary constants.

Corroborating all above results we have proven.

**Theorem 5.5.** Let $C$ and $\tilde{C}$ be complex Cartan metrics on the manifold $M$. Then $C$ and $\tilde{C}$ are projectively related if and only if there is a smooth function $Q$ on $T^*M$ with complex values, such that

$$\tilde{N}_k = N_k + B_k + Q\zeta_k; \quad k = 1, n. \quad (5.39)$$
As a consequence of Lemma 5.3 we have the following.

**Corollary 5.1.** Let $C$ and $\hat{C}$ be the complex Cartan metrics on the manifold $M$. $C$ and $\hat{C}$ are projectively related if and only if there is a smooth function $Q$ on $\tilde{T}^n M$ with complex values, such that $\tilde{N}_k = N_k + (\tilde{\partial}^* Q)\zeta_r \zeta_k$, $B_k = - (\tilde{\partial}^* Q)\zeta_r \zeta_k$, for any $k = 1, n$, and $(\tilde{\partial}^* Q)\zeta_r + (\tilde{\partial}^* Q)\zeta_f = Q$.

The relations (5.39) between the functions $\tilde{N}_k$ and $N_k$ of the projectively related complex Cartan metrics $C$ and $\hat{C}$ will be called projective change.

**Theorem 5.6.** Let $C$ and $\hat{C}$ be the complex Cartan metrics on the manifold $M$, which are projectively related. If either $C$ or $\hat{C}$ is weakly Kähler-Cartan then, $B_k = 0$ and the projective change is $\tilde{N}_k = N_k + Q\zeta_k$, where $Q$ is a $(0, 1)$-homogeneous function.

**Proof.** We assume that $\tilde{N}_k = N_k + (\tilde{\partial}^* Q)\zeta_r \zeta_k$, $B_k = - (\tilde{\partial}^* Q)\zeta_r \zeta_k$ and $(\tilde{\partial}^* Q)\zeta_r + (\tilde{\partial}^* Q)\zeta_f = Q$.

If $\hat{C}$ is weakly Kähler then $\Theta^{*s} = 0$ and so, $\tilde{h}_{sk} \hat{\Theta}^{*s} = -(\tilde{\partial}^* Q)\zeta_r \zeta_k$, which contracted by $\zeta^k$, implies $(\tilde{\partial}^* Q)\zeta_r = 0$. This leads to $B_k = \tilde{h}_{sk} \hat{\Theta}^{*s} = 0$.

**Theorem 5.7.** Let $C$ be complex Euclidean metric on a domain $D$ from $C^n$ and $\hat{C}$ another complex Cartan metric on $D$. Then $C$ and $\hat{C}$ are projectively related if and only if $\tilde{N}_k = Q\zeta_k$ and $\tilde{h}_{sk} \hat{\Theta}^{*s} = 0$, where $Q = -\frac{1}{H} \frac{\partial^i \tilde{h}}{\partial \zeta^i}$.

**Proof.** We suppose that $C$ and $\hat{C}$ are projectively related. But, the complex Euclidean metric $\bar{C} := |\zeta|^2 = \sum_{k=1}^n \zeta_k \zeta_k$ is Kähler, (it has $\Theta^{*s} = 0$) with $N_k = 0$. By these assumptions and taking into account Theorem 5.6 it results $\tilde{N}_k = Q\zeta_k$ and $\tilde{h}_{sk} \hat{\Theta}^{*s} = 0$. Further on, contracting with $\zeta^k$ the relation $\tilde{N}_k = Q\zeta_k$ we obtain $Q = -\frac{1}{H} \frac{\partial^i \tilde{h}}{\partial \zeta^i}$. The converse is obvious.

This section is completed with two applications.

We first wish to find if a complex Cartan-Randers metric $\bar{C} = \alpha + |\beta|$ can be obtained by $L$-duality from a complex Finsler metric. We do not expect the metric $\bar{C} = \alpha + |\beta|$ comes from a complex Finsler-Randers metric. Below, we show a degree more. The existence of the nonholonomic vertical frame $\left\{ \frac{\partial}{\partial \zeta^k}, \frac{\partial}{\partial \zeta^f} \right\}$ is conditioned by the relations (5.2). Thus, we can prove the following result.

**Theorem 5.8.** Let $(M, \bar{C})$ be a complex Cartan-Randers space. Then, $\bar{C}$ is image by $L$-duality of a complex Finsler metric on the same manifold $M$ if and only if $\bar{C}$ is a purely Hermitian complex Cartan-Randers metric.
Proof. Corresponding to the metric $\tilde{C} = \alpha + |\beta|$, by direct computation it results
$$\tilde{h}^{ji} = \frac{\partial^2 \tilde{c}^2}{\partial \zeta_j \partial \zeta_i} = \frac{2\alpha |\beta|}{2\alpha^2} \zeta^j - \frac{\partial \alpha}{\partial \zeta_j} \bar{b}^j \left( \frac{1}{\alpha} \zeta^j - \frac{\partial \beta}{\partial \zeta_j} b^j \right).$$

Now, we suppose that $\tilde{C}$ is image by $L-$ duality of a complex Finsler metric. Then, $\tilde{C}$ must satisfy (5.2). The second relation from (5.2) leads to $\tilde{h}_{ij} = \frac{2\alpha |\beta|}{2\alpha^2} \left( (|\alpha| |b||^2 + |\beta|) \zeta_i - \frac{\partial \alpha}{\partial \zeta_i} \bar{b}_i \right) \left( (|\alpha| |b||^2 + |\beta|) \zeta_j - \frac{\partial \beta}{\partial \zeta_j} b_j \right)$. Thus, the first condition from (5.2) gives $\alpha^2 |\beta|^2 = |\beta|^2$, i.e., $\tilde{C}$ is a purely Hermitian metric.

The converse is obvious. Indeed, if $\tilde{C}$ is a purely Hermitian metric then $\tilde{h}^{ji} = \tilde{h}_{ij} = 0$, which proves our claim. \qed

Therefore, we can to discuss only the projectiveness of the purely Hermitian complex Cartan-Randers metric $\tilde{C} = (1 + ||b||)\alpha$.

Since $\tilde{C}$ is a complex Berwald-Cartan metric, then $\tilde{N}_{ji} = \tilde{N}_{ji}$ which contracted with $\tilde{c}^i := (1 + ||b||)^2 \zeta^i$ implies
$$\tilde{N}_i = \tilde{N}_i (1 + ||b||)^2. \quad (5.40)$$

So, we obtain.

**Corollary 5.2.** The metrics $\tilde{C} = (1 + ||b||)\alpha$ and $\alpha$ are projectively related if and only if $\tilde{N}_i = P \zeta_i$, where $P = \frac{(2 + ||b|| ||b||)}{\alpha^2} \frac{\partial \alpha^2}{\partial \zeta_i} \zeta_i$.

Proof. We suppose that $\tilde{C} = (1 + ||b||)\alpha$ and $\alpha$ are projectively related. Since $\tilde{C}$ and $\alpha$ are purely Hermitian, then there is a smooth function $Q$ on $\tilde{T}^s M$ with complex values, such that $\tilde{N}_i = \tilde{N}_i + Q \zeta_i$. But, using (5.40), it results $Q \zeta_i = (2 + ||b|| ||b||) \tilde{N}_i$. Contracting the last relation with $\zeta_i$, we obtain $Q = \frac{(2 + ||b|| ||b||)}{\alpha^2} \frac{\partial \alpha^2}{\partial \zeta_i} \zeta_i$, which prove the direct claim.

Conversely, if $\tilde{N}_i = P \zeta_i$ then taking into account (5.40), we obtain
$$\tilde{N}_i = \tilde{N}_i + (2 + ||b|| ||b||) \tilde{N}_i = \tilde{N}_i + (2 + ||b|| ||b||) P \zeta_i = \tilde{N}_i + Q \zeta_i,$$
where $Q := (2 + ||b|| ||b||) P = \frac{(2 + ||b|| ||b||)}{\alpha^2} \frac{\partial \alpha^2}{\partial \zeta_i} \zeta_i$ is a smooth function $Q$ on $\tilde{T}^s M$ with complex values. This completes our proof. \qed

Moreover, the metric $\tilde{C} = (1 + ||b||)\alpha$ is projectively related with the complex Euclidean metric $C$ on a domain $D$ if and only if $\alpha$ is projectively related with $C$.

The second application refers to the locally projectively flat complex Cartan metrics. Let $\tilde{C}$ be a locally Minkowski complex Cartan metric on the underlying manifold $M$. Corresponding to the metric $\tilde{C}$ there exist in any
point the local charts in which we have $\tilde{N}_k = 0$ and $\tilde{\Theta}^{ss} = 0$ because, in such local charts, the fundamental metric tensor $\tilde{h}^{\mu\nu}$ depends only on $\zeta$.

Also, we consider $\mathcal{C}$ another complex Cartan metric on the complex manifold $M$. Note that, we have assumed that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are the images by $\mathcal{L}$—duality of the locally regulate complex Finsler metrics $F$ and $\tilde{F}$ on $M$, respectively.

The complex Finsler metrics $\mathcal{C}$ will be called \textit{locally projectively flat} if it is projectively related to the locally Minkowski metric $\tilde{\mathcal{C}}$.

\textbf{Corollary 5.3.} $\mathcal{C}$ is locally projectively flat if and only if $h_{sk}\Theta^{ss} = 0$ and $N_k = -Q\zeta_k$, where $Q = \frac{2}{c^2} \frac{\partial^2 C}{\partial z_i^2} z^i$.

\textit{Proof.} It follows by Theorems 5.5 and 5.6. \hfill $\Box$

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