Sudden Death of Genuine Tripartite Entanglement

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(Dated: December 26, 2022)

Here we examine the entanglement dynamics of an unusual but Schrödinger-based decoherence process that leads, after a finite time, to a non-analytic end-stage [Yu and Eberly, Phys. Rev. Lett. 93, 140404 (2004)]. This abrupt ending occurs even though the individual qubits themselves, in contrast to their entanglement, continue to evolve deterministically smoothly and analytically. However, many details remain a mystery. The most important unknown element is whether the speed of ESD arrival can be controlled. In fact, ESD can occur almost instantaneously, certainly after only a small fraction of the decoherence time $T_1$ or $T_2$, if the initial state is appropriate. Here we address the challenge to expose the initial conditions that are certain to have non-analytic vanishing entanglement at any moment as their end stage. We address this challenge with the assistance of a newly discovered 3-qubit genuine entanglement measure [Xie and Eberly, Phys. Rev. Lett. 127, 040403 (2021)].

In the presence of decoherence, entanglement dynamics can be found to obey a process labeled ESD or Early Stage Decoherence. ESD is a quantum process [1, 2]. The entanglement’s evolution is deterministic in the same way as Schrödinger theory is deterministic. And ESD is an “early” event in the sense that there is a specific finite time when the entanglement drops to the value zero. It does this abruptly and with non-zero slope, i.e., non-analytically. These combined features are unexpectedly dramatic and final, so Entanglement Sudden Death is an alternative term for ESD. Its initial conditions and its speed of arrival are addressed here.

Since its discovery, ESD has been described in many examples [3–7], as well as been multiply observed experimentally [8–11], but to determine the speed of ESD onset, and to catalog the variety of initial conditions leading to ESD have remained challenging.

This challenge originates from an associated mystery—ESD has never been predicted to affect entanglement of more than just two qubits. This is notable because scalable entangled systems provide significant platforms for quantum information and computing [12–15]. Entanglement can also play a more helpful dynamical role if the number of entangled parties is increased. For example, three-party entanglement is more able than two-party entanglement in the quantum task of teleportation [16]. Thus, whether or not ESD is possible for three or more participating parties may be a practical matter as well as a theoretical puzzle.

Various attempts have been made to study the ESD process with multiple parties engaged (see examples in [3, 11, 17–19]), however, an important and subtle requirement was initially not accounted for and then only gradually recognized (see Ma et al. [20]). This requirement insists that a feature now labeled “genuine” must be taken into account. It is a feature noticed by Dür, Vidal, and Cirac [21], and they proved that the concept of “genuine entanglement” is available for only two classes of three-qubit states, commonly called the GHZ class and the W class. Their familiar representative states are

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$ \hspace{1cm} (1)

$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle).$$ \hspace{1cm} (2)

In taking account of this requirement, the question of interest becomes: Will genuine multipartite entanglement also be subject to sudden death?

Attention to that question logically starts with the simplest three-qubit version of GME. But until recently there was no known way to quantify GME for three qubits (except see [20]). Closing this challenging gap between two and three qubits has recently been possible, leading via a geometric approach [22] to the discovery of a genuine tripartite entanglement measure for pure states, called concurrence Fill. For ESD this is not yet enough because ESD occurs only for mixed states. Therefore, an extension of the pure-state GME Fill measure is needed for mixed-state entanglement.

The extension we need is available via convex-hull construction, sometimes called the “entanglement of creation”, quantifying the resource needed to create a given entangled state [23]. The construction is given by an optimization formula

$$F(\rho) = \inf_{\{p_i, \psi_i\}} \sum_i p_i F(\psi_i),$$ \hspace{1cm} (3)

where the infimum is taken over all possible decompositions $\rho = \sum p_i |\psi_i\rangle \langle \psi_i|$, and where $F(\psi_i)$ is the new Fill measure [24] for pure states. A mixed state $\rho$ contains no genuine entanglement if and only if $F(\rho) = 0$.

One cannot predict the number of terms appearing in the sum of Eq. (3). In principle, it requires infinitely many parameters to fully characterize the optimization. However, Röthlisberger, Lehmann, and Loss proposed...
(see \([25]\)) that, for a rank-\(r\) density matrix \(\rho\), the result is already very accurate when we make a cutoff chosen so that the number of the pure-state terms in the decomposition Eq. \((3)\) is no greater than \(r + 4\). The total number of parameters is then given by \(r(r + 8)\). In our own experience, we have tested that the \(r + 4\) cutoff indeed provides enough accuracy for the ESD problem, even when the system’s entanglement is very small—on the order of \(10^{-3}\), as will be shown later. For the parametrization, we will use the Euler-Hurwitz angles, a method introduced in Ref. \([25]\).

However, the above Cutoff approach is limited when the rank of the mixed state \(\rho\) is large, say \(r = 8\), when there are in total 128 parameters in the optimization \([3]\). In this case, we find that the direct evaluation of Eq. \((3)\) becomes extremely difficult even numerically. An alternative solution was given by Eisert, Brandao, and Audenaert \([26]\). Noticing the convex property of Eq. \((3)\), they applied the Legendre transform twice to obtain

\[
F(\rho) = \sup_{X} \inf_{\psi} \left\{ \text{Tr} \left[ X (\rho - |\psi\rangle \langle \psi|) \right] + F(\psi) \right\},
\]

\(\text{(4)}\)

where the interior infimum is over all possible three-qubit pure states \(\psi\) with 14 parameters, while the exterior supremum is over all possible Hermitian matrices \(X\). At first glance, Eq. \((4)\) is much more complicated than \((3)\), with both “sup” and “inf” engaged. However, the infinitely many possible decompositions \(\{p_i, \psi_i\}\) in \((3)\) are eliminated. Furthermore, the dimension of the Hermitian \(X\) space for the exterior supremum, normally 64 for three-qubit systems, can be reduced to a rather small number by the symmetry of the state \(\rho\), which greatly simplifies the numerical task, as was shown by Ryu, Lee, and Sim \([27]\). As an example of the method, the Fill measure for the mixture \(\rho(s) = s|\text{GHZ}\rangle \langle \text{GHZ}| + (1 - s)|W\rangle \langle W|\) is demonstrated in Fig. \(1\). The analytic solution for the mixture \(s = 5s^2 - 4s + 8/9\), provided by Lohmayer et al. \([28]\). For this state, the dimension of the exterior supremum for \(X\) in \((4)\) is reduced to 8 (compared to 64) by symmetries of the state.

The above MinMax approach in Eq. \((4)\), on the other hand, has its own limit. When the symmetry of the mixed state \(\rho\) cannot effectively reduce the external dimension of \(X\), the method is less efficient. In this work, to study three-qubit ESD, we will use the different approaches for different states. When a mixed state is given, we will clarify which approach we are using. The Nelder-Mead algorithm in Fortran \([29]\) is applied to calculate \(F(\rho)\) in both approaches Eq. \((3)\) and \((4)\).

With the entanglement measure fixed, we are ready to answer our ESD question by examining the dynamics of a three-qubit system. For the simplest model that engages mixed states, we couple our three qubits to three additional cavity-qubits as a reservoir of decoherence. The system-qubits are labeled by \(A\), \(B\), and \(C\), while the cavity-qubits are labeled by \(a\), \(b\), and \(c\). We allow the six qubits to interact with each other only in pairs, e.g., \(A\) with \(a\), and the other pairs similarly, in the following way, using \(A\) and \(a\) for example:

\[
|1\rangle_A |0\rangle_a \xrightarrow{t/\tau} p(t)|1\rangle_A |0\rangle_a + q(t)|0\rangle_A |1\rangle_a,
\]

\(\text{(5)}\)

where \(p(t) = \sqrt{e^{-t/\tau}}\) and \(q(t) = \sqrt{1 - e^{-t/\tau}}\) are time-dependent coefficients. The interaction is known as amplitude-damping for zero-temperature spontaneous emission of two-level atoms, where an excited atom in the state \(|1\rangle\) eventually emits a photon and evolves into the ground state \(|0\rangle\). In our example, the three cavity-qubits always start from the state \(|000\rangle\)\(_{abc}\). Although the three system-qubits \(ABC\) have no direct interactions among themselves, they are initially prepared in a genuinely entangled pure state, to be determined below separately. After an infinite time, the system-qubits evolve into the product state \(|000\rangle\)\(_{ABC}\) having no entanglement. The key question is whether entanglement can vanish earlier, within a finite time.

We first prepare the initial state for the system as

\[
|\text{GHZ}(\theta)\rangle = \cos \theta |111\rangle + \sin \theta |000\rangle,
\]

\(\text{(6)}\)

a generalized form of the GHZ state, which is then subjected to the amplitude damping channel Eq. \((1)\). The tripartite system’s state immediately becomes mixed due to interaction with the cavity-qubits \(abc\). During evolution, the rank of the system’s density matrix \(\rho\) is \(r = 8\), so the Cutoff approach Eq. \((3)\) is less efficient. However, \(\rho\) has strong symmetries such that the exterior dimension of \(X\) in \((4)\) can be reduced to 6. Therefore, for this state, we choose the MinMax approach Eq. \((4)\) to evaluate the genuine entanglement numerically for the system-qubits \(ABC\).
The results are provided in Fig. 2, having chosen three different values of cosθ: 1/√2 (blue), 0.88 (orange), and 0.17 (red). In the left panel Fig. 2(a), the curves are plotted with a linear vertical scale. The existence of ESD can be observed more clearly when we convert to a logarithmic vertical scale, as shown in right panel Fig. 2(b). The red curve is a straight line in the log scale, corresponding to the usual exponential decay on the linear scale. This indicates that there is no ESD when cosθ = 0.17. However, the blue and orange curves in the log scale abruptly reach negative infinity before t/τ = 0.65 and t/τ = 0.4, respectively. Their constant negative infinities in the log scale correspond to constant zero values for those two curves in the linear scale Fig. 2(a). We believe that this is the first indication of ESD for genuine multipartite entanglement.

A side remark can be made: the initial GHZ(θ) mixed state for the system-qubits ABC in (6) remains what is called an X-form density matrix [30] throughout the amplitude damping interactions with the cavity-qubits abc. A separate GME measure, called genuine multipartite concurrence and labeled GMC [20] was earlier found by Hashemi-Rafsanjani et al. [31] to obey analytic formulas for X-matrices, even if they are mixed states. Next, we apply their results to verify our result of 3-qubit ESD.

For the GHZ(θ) initial state (6), the GMC measure for the system-qubits as a function of t/τ is

\[ \text{GMC} = \max \left[ 0, \ p^{3}(t) \sin \theta - 6p^{3}(t)q^{3}(t) \cos^{2} \theta \right], \]  

with which it can be verified that ESD begins at the time specified by:

\[ | \tan \theta | = 3 \left( 1 - e^{-t/\tau} \right)^{3/2}. \]

Since the value on the right-hand side is in the range \([0, 3]\), it implies that, in order for the state (6) to exhibit ESD, one must choose \( | \tan \theta | < 3 \), or equivalently \( | \cos \theta | > 1/\sqrt{10} \approx 0.316 \). It was pointed out in [24] that Fill and GMC are inequivalent GME measures. What matters is that they always give the same answer to the question whether or not a state is genuinely entangled. Therefore, when GMC exhibits ESD, so does Fill. Based on this information, we find that our results duplicate the ESD predictions that are made using the GMC measure. For example, when \( \cos \theta = 1/\sqrt{2} \) or 0.88, which are both greater than 1/\sqrt{10}, we find ESD exhibited. When \( \cos \theta = 0.17 \), ESD does not appear.

This is the first ESD observation for a three-qubit system, and we can now answer the “looking ahead” question: What are the initial conditions that will lead to ESD? For our amplitude-damping model Eq. (4), we reach the answer by preparing the system’s initial state as any one of the natural “partners” of the generalized GHZ(θ) state. Specifically, we examine the following generalized W-type initial states for the system

\[ | W(\theta) \rangle = \cos \theta |000\rangle + \frac{1}{\sqrt{2}} \sin \theta (|100\rangle + |001\rangle), \]

\[ | \overline{W}(\theta) \rangle = \cos \theta |011\rangle + \frac{1}{\sqrt{2}} \sin \theta (|101\rangle + |110\rangle). \]  

The \( W(\theta) \) and \( \overline{W}(\theta) \) states correspond to single-excitation and double-excitation cases of the W class. They are connected by local spin-flip operations, but they have different dynamics when subjected to the amplitude damping channel [5]. The parameter \( \theta \) creates bias on the first qubit. The density matrices for the initial \( W(\theta) \) and \( \overline{W}(\theta) \) states are no longer of X-state form, so one can no longer apply the GMC formula in [31] to predict the existence of ESD. What’s more, the symmetries of the density matrices are not strong enough in the sense that the exterior dimension of \( X \) in Eq. (4) can only be reduced to 12, which makes the MinMax approach less efficient (thus, inaccurate results). On the other hand, the density matrices have a relatively small rank \( r = 5 \). For this reason, we choose the Cutoff approach to evaluate...
the entanglement for the initial states \( W \).

The left and center panels of Fig. 3 show results of the initial \( W(\theta) \) states with \( \cos \theta = 1/\sqrt{3} \) (blue), 0.9 (orange), and 0.1 (red), where Fig. 3(a1) is plotted on a linear vertical scale and Fig. 3(a2) is shown on a log scale. As can be seen, the three log-scale curves are all straight lines, signaling smoothly exponential decay. Thus, there is no ESD when the initial state is a \( W(\theta) \) state. The center two panels of Fig. 4 show results of the \( \overline{W}(\theta) \) states with \( \cos \theta = 1/\sqrt{3} \) (blue), 0.9 (orange), and 0.1 (red). Again, there is only smooth decay and no ESD exhibited for the \( \overline{W}(\theta) \) states.

A question can arise: In Fig. 3, (a2) and (b2) only show the absence of ESD at times earlier than \( t/\tau = 3 \) and \( t/\tau = 2 \). How do we know that ESD cannot occur at a much later time, say \( t/\tau = 10 \) or even \( t/\tau = 100 \)? We point out that the intrinsic time scale of our system is \( \tau \), as can be seen from the evolution equations \( \xi \). Therefore, as we vary the \( \cos \theta \) value for these states, we expect that ESD, if present, will have to produce an abrupt logarithmic drop at some time not far from \( t/\tau \approx 1 \). Clearly, no such ESD behavior is observed for the three \( \cos \theta \) values we chose. The same point is made clearer in the following discussion.

Focus on the questions: What is so important that allows the GHZ(\( \theta \)) state Eq. \( \xi \) to exhibit ESD? What is missing in the \( W(\theta) \) and \( \overline{W}(\theta) \) states so that they cannot also exhibit ESD? Look back at Fig. 2. When \( \cos \theta \) has a larger value, that is, when the \( |111\rangle \) component in the initial superposition has a larger amplitude, the ESD arrives earlier. The orange line with \( \cos \theta = 0.88 \) has ESD earlier than the blue line with \( \cos \theta = 1/\sqrt{2} \). When \( \cos \theta \) is smaller than the threshold (as was explained earlier, in the three-qubit case, the threshold is \( 1/\sqrt{10} \)), ESD arrives rather late—even later than infinite time—ESD will never happen, as is the case for the red curve with \( \cos \theta = 0.17 \).

We can point out a competing process that is hidden in Eq. \( \xi \). The \( |111\rangle \) component is a predictor for ESD, while the \( |000\rangle \) component leads to no ESD. In more detail, if \( |111\rangle \) dominates, or equivalently, when \( \cos \theta \) is larger than the \( 1/\sqrt{10} \) threshold, then ESD occurs. If \( |000\rangle \) dominates on the other hand, or equivalently \( \cos \theta \) is smaller than the threshold, there is no ESD. No such competition from \( |000\rangle \) is ever seen for the initial \( W(\theta) \) and \( \overline{W}(\theta) \) states, no matter how one tunes the parameter \( \theta \) in Eq. \( \xi \). This again identifies the \( |111\rangle \) state as the true predictor for ESD.

To further verify the importance of \( |111\rangle \), we go one step further. We speculate that, even when \( |111\rangle \) is superposed with any other state, the process of ESD will remain. To verify this conjecture, we superpose \( |111\rangle \)
with the all-symmetric state $|W\rangle$ in (2), as is given by
\begin{equation}
|\Sigma(\theta)\rangle = \cos \theta |111\rangle + \sin \theta |W\rangle.
\end{equation}

Here, the rank of the density matrices during evolution is $r = 8$. For the same reason as in the generalized GHZ state (6), we choose the MinMax approach for this state. The exterior dimension of $X$ in (4) is now reduced to 10 due to the density matrix’s symmetries.

The results are shown in the right two panels of Fig. 3. The competing process is quite obvious, even though we don’t know the exact threshold value for $\cos \theta$ in this case. The orange line with $\cos \theta = \sqrt{0.9}$ shows ESD that starts quite early, while the ESD for the blue line with $\cos \theta = \sqrt{0.5}$ arrives later. When $\cos \theta = \sqrt{0.7}$ as for the red line, no ESD is exhibited, since a straight line appears in the log scale. The results add evidence that it is |111⟩ in the initial state that allows ESD to occur.

Now, what if we take the state |111⟩ alone as the initial state? The answer is quite obvious. Since |111⟩ is an unentangled product state, the dynamics of entanglement will remain constantly zero. Alternatively, we can say that ESD is so strong that it already occurs from the beginning, at $t/\tau = 0$.

In this work, we examined the dynamics of genuine tripartite entanglement for three qubits subject to mixing via individual amplitude damping channels, when the initial state is selected to be different pure states. We provided the first example of sudden death of genuine tripartite entanglement. In particular, we found that the key to observe ESD is the necessary presence of |111⟩ in the initial state. More specifically, |111⟩ has to dominate in the initial superposition. Or equivalently, the amplitude of |111⟩ has to be greater than a specific threshold, which is determined by an extra state superposed with it.

Our work focuses on three-qubit systems, and one can find that the same conclusion is seen in examples of two-qubit ESD, when the entanglement measure is easily calculated using Wootters’ concurrence formula (22). Then the initial state acting as consistent ESD predictor is |11⟩, and we speculate that the conclusion is universal—ESD is inevitable for all any-number qubit systems, given initial dominance of the any-number all-excited state |111 · · · 11⟩.

It is also interesting to know of cases when other channels are applied: depolarizing channel, dephasing channel, etc., since two-qubit ESD has been observed for these channels previously. An interesting geometric illustration of the effects of different environments on ESD was studied in (8). ESD in qudit systems will also be interesting to study.

We thank Prof. W.A. Coish, Prof. L. Davidovich, Prof. D.F.V. James, Prof. X.-F. Qian, and Prof. T. Yu for valuable discussions. Financial support was provided by National Science Foundation Grants No. PHY-1501589 and No. PHY-1539859 (INSPIRE), and a competitive grant from the University of Rochester. Calculations were performed on the BlueHive supercomputing cluster at the University of Rochester.
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