Ultrametric pseudodifferential operators and wavelets for the case of non homogeneous measure

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Abstract

A family of orthonormal bases of ultrametric wavelets in the space of quadratically integrable with respect to arbitrary measure functions on general (up to some topological restrictions) ultrametric space is introduced.

Pseudodifferential operators (PDO) on the ultrametric space are investigated. We prove that these operators are diagonal in the introduced bases of ultrametric wavelets and compute the corresponding eigenvalues.

Duality between ultrametric spaces and directed trees is discussed. In particular, a new way of construction of ultrametric spaces by completion of directed trees is proposed.

1 Introduction

In the present paper we continue to develop the analysis of pseudodifferential operators on ultrametric spaces, following Π.

We consider for an ultrametric space $X$ the directed tree $\mathcal{T}(X)$ of balls in $X$, and consider the partial order on $X \cup \mathcal{T}(X)$ defined by inclusion of balls and inclusion of points of $X$ into balls. We consider on the space $X$ a ($\sigma$–additive and possessing a countable or finite basis) measure $\nu$ of a general form and investigate pseudodifferential operators of the form

$$Tf(x) = \int T^{(\text{sup}(x,y))}(f(x) - f(y))d\nu(y)$$

acting in the space $L^2(X, \nu)$ of quadratically integrable complex valued functions. The integration kernel $T^{(I)}$ is a function on the tree $\mathcal{T}(X)$ and $I = \text{sup}(x, y)$ is defined by the partial order on $X \cup \mathcal{T}(X)$.

We introduce the orthonormal basis of ultrametric wavelets in the space $L^2(X, \nu)$, which diagonalizes the pseudodifferential operator $T$, and compute the corresponding eigenvalues.

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Also we discuss the duality between ultrametric spaces and directed trees. In particular, we propose a new construction of ultrametric space as a completion of a directed tree $\mathcal{T}$ (i.e. of a tree with a direction, or defined in the special way partial order) with respect the the metric, defined by the direction.

The present paper develops the approach of the paper [1], where the particular case of the analysis ultrametric pseudodifferential operators of the type (i.e. of a tree with a direction, or defined in the special way partial order) was considered, for which the measure $\nu$ was chosen in the special way, for which the maximal subballs in an ultrametric ball have the equal measure. In the present paper we consider more general case, for which the measures of the balls can be arbitrary positive numbers. We call this case the case of non homogeneous measure. Moreover, in [1] the more standard construction of ultrametric space as the set of classes of equivalence of decreasing infinitely continued paths in the directed tree was applied.

Investigation of ultrametric pseudodifferential operators was started [2] with the introduction of the Vladimirov operator of $p$–adic fractional derivation, see [3] for detailed exposition. Different problems of $p$–adic analysis and $p$–adic mathematical physics were considered in [4]–[8].

The Vladimirov operator can be diagonalized by the $p$–adic Fourier transform. Also there exist bases of eigenvectors with compact support [3]. In paper [9] the basis of $p$–adic wavelets in the space $L^2(Q_p)$ of quadratically integrable complex valued functions on the field of $p$–adic numbers was introduced and it was shown that this basis is a basis of eigenvectors of the Vladimirov operator. In papers [10], [11] the construction of the wavelet basis of [9] was generalized onto more general local fields and groups.

In paper [12] the family of pseudodifferential operators in the space $L^2(Q_p)$, diagonal in the basis of $p$–adic wavelets, but not diagonalizable by the Fourier transform, was constructed, and the corresponding eigenvalues were computed. Further generalization of this result onto the case of pseudodifferential operators in $L^2(Q_p)$ of more general form was performed in [13], [14]. In paper [11] theory of ultrametric wavelets and PDO related to general ultrametric spaces was developed (in less general case, compared to the present paper).

Theory of ultrametric pseudodifferential operators has physical applications. In papers [15], [16] it was shown that the Parisi matrix, which describes replica symmetry breaking is a discrete analogue of some $p$–adic pseudodifferential operator. In papers [17], [18] relation between ultrametric diffusion and dynamics of macromolecules was discussed. For the review of the results of $p$–adic mathematical physics see [3]. In particular, one can mention applications to string theory [19], [20], and mathematical models of biology and cognitive science [5], [21].

The present paper has the following structure.

In Section 2 build the structure of a directed tree on the set $\mathcal{T}(X)$ of balls in an ultrametric space $X$.

In Section 3 we introduce the family of directed trees and define ultrametric on the trees from this family.

In Section 4 we perform completion of the trees with respect to the introduced ultrametric and discuss the properties of the corresponding ultrametric spaces.
In Section 5 we construct orthonormal bases of ultrametric wavelets in the spaces of quadratically integrable functions on the ultrametric spaces under consideration.

In Section 6 we introduce pseudodifferential operators acting on complex valued functions on ultrametric spaces, show that these operators are diagonal in the bases of ultrametric wavelets, and compute the corresponding eigenvalues.

2 Directed tree \( \mathcal{T}(X) \) of balls in ultrametric space

In the present Section we discuss relation between ultrametric space \( X \) and directed tree \( \mathcal{T}(X) \) of balls in \( X \). Let us give the necessary definitions.

**Definition 1** An ultrametric space is a metric space with the ultrametric \( |xy| \) (where \( |xy| \) is called the distance between \( x \) and \( y \)), i.e. the function of two variables, satisfying the properties of positivity and non degeneracy

\[
|xy| \geq 0, \quad |xy| = 0 \implies x = y;
\]

symmetricity

\[
|xy| = |yx|;
\]

and the strong triangle inequality

\[
|xy| \leq \max(|xz|, |yz|), \quad \forall z.
\]

Consider a complete ultrametric space \( X \), satisfying the following properties:

1) The set of all balls of nonzero diameter in \( X \) is no more than countable;

2) For any decreasing sequence of balls \( \{D^{(k)}\} \), \( D^{(k)} \supset D^{(k+1)} \), diameters of the balls tend to zero;

3) Any ball is a finite union of maximal subballs.

Property 2 imply the following condition:

For any two balls \( I, J \) in \( X \) (of non zero diameter) and any sequence of balls \( \{D^{(k)}\} \), for which \( I \subset D^{(k)} \subset J \) for all \( k \), the sequence \( \{D^{(k)}\} \) must be finite.

**Proposition 2** Complete ultrametric space, satisfying the properties 1, 2, 3 above, is locally compact.

Here the topology is generated by the ultrametric.

**Proof** To prove that \( X \) is locally compact (i.e. any ball in \( X \) is compact) consider a sequence \( \{x_k\} \) in ball \( D \) in \( X \). Then, if the ball \( D \) is not minimal, it contains a subball \( D' \), which contains infinite subsequence of \( \{x_k\} \). Repeating this procedure, we obtain decreasing sequence of balls with the diameter tending to zero, where each of the balls contains infinite subsequence of \( \{x_k\} \). Therefore \( \{x_k\} \) has a limiting point in \( D \). Since the topology on \( X \) has a countable base, this implies the local compactness.

The next proposition gives the example of \( \sigma \)-additive measure on \( X \).
Proposition 3 Consider on the complete ultrametric space $X$, satisfying properties 1, 2, 3 the measure $\mu$, satisfying for each ball $D_I$ the following property:

Measures of all the balls $D_{I_j}$, which are maximal subballs in $D_I$, are equal.

Then the measure $\mu$ is $\sigma$–additive and has a countable or finite basis of balls.

Proof It is easy to see that the considered above condition defines the measure $\mu$ up to multiplication by a constant.

Then $\sigma$–additivity follows from local compactness of $X$ (in the same way as for the Lebesgue measure) [23]. The countable or finite basis for $\mu$ is given by no more than countable set of balls in $X$. This finishes the proof of the proposition.

The next definition is the standard definition of a partially ordered set.

Definition 4 Partially ordered set is a set with partially defined order, i.e. for an arbitrary pair $(x, y)$ of elements of the set these elements $x, y$ either incomparable or $x > y$, or $x < y$, and, moreover:

1) An element can not be greater or smaller than itself;
2) If $x > y$ and $y > z$, then $x > z$ (transitivity).

If any two elements of the partially ordered set are comparable, then this set is called completely ordered.

A supremum sup $S$ of the subset $S$ in the partially ordered set is a minimal element of the partially ordered set, which is greater or equal to all the elements of the subset $S$. If any finite subset of the partially ordered set has the unique supremum, then this partially ordered set is called directed (and the partial order is called a direction).

Definition 5 A graph is a pair of sets $\{I\}$ (the set of vertices), $\{i\}$ (the set of links), where each of these sets is finite or countable. Moreover, each link $i$ is a pair of different vertices $I_0, I_1$ (called the beginning and the end of the link).

An infinite path in a graph is an injection of the set of natural numbers into the graph, such that numbers with the difference one correspond to the neighbor (connected by a link) vertices of the graph. A path of the length $N$ in a graph is an injection of the set $1, \ldots, N$ into the graph, such that the numbers with the difference one correspond to the neighbor (connected by a link) vertices of the graph. In the following we will not specialize, finite or infinite path we consider if this is clear from the context. By the applied definition all the considered paths have an orientation.

The image of 1, defined by the path in the graph, we will call the beginning of the path. For the finite path of length $N$ the vertex, which is the image of $N$, will be called the end of the path. A graph is called connected if for arbitrary two vertices $I, J$ there exists a finite path with the beginning in $I$ and the end in $J$. A cycle in the graph is a pair of different finite paths, for which the beginnings and the ends coincide.
A tree is a connected graph without cycles. Consider an arbitrary tree $T$ (finite or infinite), such that the number of links incident to any of the vertices is finite.

Assume that the tree $T$ is partially ordered. If non maximal vertex $I$ is incident to $p_I + 1$ links, we will say that branching index of the vertex $I$ is equal to $p_I$. If maximal vertex $I$ is incident to $p_I$ links, we will say that branching index of the vertex $I$ is equal to $p_I$. Equivalently, branching index of a vertex $I$ in partially ordered tree is the number of vertices $I_j$: $I_j < I$, $|II_j| = 1$, where the distance between vertices of the tree is the number of edges in the path connecting these vertices.

For partially ordered trees $T$ consider the following property:

**Property 1**  
In any finite path there exists the unique maximal vertex.

In particular, all neighbor (connected by a link) vertices are comparable.

For ultrametric space $X$ consider the set $T(X)$, which contains all the of balls in $X$ of nonzero diameter, and the balls of zero diameter which are maximal subbals in balls of nonzero diameter. On the set $T(X)$ there is the natural partial order: $I < J$ if for the corresponding balls $I \subset J$. Since for any two balls in an ultrametric space there exists the unique minimal ball, containing these two balls (which the supremum of the balls), the mentioned partial order will be a direction.

The set $T(X)$ has the following structure of graph: vertices of the graph $T(X)$ are balls in $X$, two vertices $I$ and $J$ are connected by a link if one is a subball in the other, say $I \supset J$, and between $I$ and $J$ there are no other elements of $T(X)$ (i.e. $J$ is a maximal subball in $I$).

Prove that the graph $T(X)$ will be a tree with finite branching indices. The properties (1), (3) of the ultrametric space $X$ imply that $T(X)$ has no more than countable number of vertices, and all branching indices are finite.

Since $T(X)$ is directed and by the property (2) the graph $T(X)$ is connected.

Assume we have a cycle in the graph $T(X)$. Take a minimal element $I$ of the cycle, and consider the balls $J, K$ in $X$ lying at the cycle and $I \subset J, I \subset K$. By ultrametricity of the space $X$, for any three balls $I, J, K$ in $X$, such that $I \subset J, I \subset K$, the balls $J$ and $K$ will be comparable: either $J \subset K$ or $K \subset J$. Therefore the cycle does not exist and the graph $T(X)$ is a tree.

Then, it is easy to see that the branching index of the tree $T(X)$ can not be equal to 1 for any vertex, and for the maximal vertex (if it exists) the branching index can not be equal to zero (if $X$ contains at least two points). Moreover, balls of nonzero diameter in $X$ correspond to vertices of branching index $\geq 2$ in $T(X)$, and the balls of zero diameter which are maximal subbals in balls of nonzero diameter correspond to vertices of branching index 0 in $T(X)$.

Let us prove that the direction in $T(X)$ satisfies the Property 1. Consider $I, J$ in $T(X)$ and $K = \sup(I, J)$. Consider the paths $IJ, IK, JK$ and the vertex

$$L = IJ \cap IK \cap JK$$
Then \( L \geq I, L \geq J \) and \( L \leq K \) which implies \( L = K \), i.e. the supremum \( K \) lies at the path \( IJ \). Moreover, it is easy to see that the paths \( IK \) and \( JK \) are completely ordered, which implies the Property 1.

We proved the following theorem.

**Theorem 6** The set \( \mathcal{T}(X) \) which contains all the of balls of nonzero diameter, and the balls of zero diameter which are maximal subballs in balls of nonzero diameter in a non–trivial (containing at least two points) ultrametric space \( X \), satisfying properties (1), (2), (3) above with the partial order, defined by inclusion of balls, is a directed tree where all neighbor vertices are comparable.

Branching index for vertices of this tree may take finite integer non–negative values not equal to one, and the maximal vertex (if exists) has the branching index \( \geq 2 \). Balls of nonzero diameter in \( X \) correspond to vertices of branching index \( \geq 2 \) in \( \mathcal{T}(X) \), and the balls of zero diameter which are maximal subballs in balls of nonzero diameter correspond to vertices of branching index \( 0 \) in \( \mathcal{T}(X) \).

Moreover, the direction in \( \mathcal{T}(X) \) satisfies the Property 1.

Consider the set \( X \cup \mathcal{T}(X) \), where we identify the balls of zero diameter from \( \mathcal{T}(X) \) with the corresponding points in \( X \). We call \( \mathcal{T}(X) \) the tree of balls in \( X \), and \( X \cup \mathcal{T}(X) \) the extended tree of balls. One can say that \( X \cup \mathcal{T}(X) \) is the set of all the balls in \( X \), of nonzero and zero diameter.

Introduce the structure of a directed set on \( X \cup \mathcal{T}(X) \). At the tree \( \mathcal{T}(X) \) this structure is already defined, and the relations of order with points of \( X \) are introduced as follows.

**Definition 7** Any two points of the ultrametric space \( X \) are incomparable. The relation of order between the points of \( X \) and vertices of the tree \( \mathcal{T}(X) \) are defined as follows: if \( x \in X \) and \( I \in \mathcal{T}(X) \), then \( x \leq I \) if and only if \( x \in I \).

This implies the following lemma.

**Lemma 8** The partial order on \( X \cup \mathcal{T}(X) \), introduced in definition 7, is a direction. This direction can be described as follows.

The supremum

\[
\sup(x, y) = I
\]

of points \( x, y \in X \) is the minimal ball \( I \), containing the both points.

Analogously, for \( J \in \mathcal{T}(X) \) and \( x \in X \) the supremum

\[
\sup(x, J) = I
\]

is the minimal ball \( I \), which contains the ball \( J \) and the point \( x \).

This construction extends the notion of supremum of two vertices of the tree \( \mathcal{T}(X) \) (since the ball \( I = \sup(J, K) \) corresponds to the minimal ball \( I \), which contains the balls \( J \) and \( K \)).

Introduce the structure of an ultrametric space on the extended tree \( X \cup \mathcal{T}(X) \).
Definition 9 For $I, J$ in $X \cup \mathcal{T}(X)$ and $I \neq J$ the distance $|IJ|$ is the diameter of the ball $K \in X \cup \mathcal{T}(X)$, $K = \sup(I, J)$, for $I$ in $X \cup \mathcal{T}(X)$ the distance $|II| = 0$.

In particular, the tree $\mathcal{T}(X)$ will be (in general, incomplete) ultrametric space. Moreover, a measure $\nu$ on ultrametric space $X$ induces the measure (which we also denote by $\nu$) on the extended tree $X \cup \mathcal{T}(X)$: the measure of $I \in \mathcal{T}(X)$ we put equal to the measure of the corresponding ball in $X$.

3 Ultrametric on directed trees

In the present Section we discuss ultrametric on directed trees. The next theorem describe a family of equivalent ultrametrics on a directed tree (we call two metrics on the same space equivalent, if the both define the same set of balls).

Theorem 10 If $F(I)$ is a positive increasing function on a directed tree $\mathcal{T}$, then the formula

$$|AB| = F(\sup(A, B)), \quad A \neq B, \quad |AA| = 0, \quad \forall A, B \in \mathcal{T} \quad (2)$$

defines on the tree $\mathcal{T}$ the ultrametric (i.e. it is non negative, equal to zero only for $A = B$, symmetric with respect to permutation of $A$ and $B$, and satisfies the strong triangle inequality):

$$|AB| \leq \max(|AC|, |BC|), \quad \forall A, B, C \in \mathcal{T}$$

All the ultrametrics defined in this way are equivalent.

Proof To prove that $|AB|$ is an ultrametric, it is sufficient to prove that $|AB|$ satisfies the strong triangle inequality (the other conditions, necessary for ultrametricity, are obvious).

Consider vertices $A, B, C$. Let $I = \sup(A, B), J = \sup(B, C), K = \sup(A, C)$.

Since the both vertices $I$ and $K$ are larger than $A$, these vertices are comparable (otherwise we would have in the tree $\mathcal{T}$ the cycle consisting of the two different paths $AI\sup(I, K), AK\sup(I, K)$). Analogously, vertices $I$ and $J$ are comparable; also vertices $J$ and $K$ are comparable. Therefore $I, J, K$ is a completely ordered set.

Two variants are possible: either $I = J = K$, or there are some non coinciding vertices in this set. If $I = J = K$, then by (2)

$$|AB| = |AC| = |BC|$$

and the strong triangle inequality is satisfied.

Take $I > J$, i.e. $\sup(A, B) > \sup(B, C)$. Then $\sup(A, C) = \sup(A, B)$, i.e. $I = K$. Thus by (2)

$$|AB| = |AC| > |BC|$$

and the strong triangle inequality is satisfied.
Analogously, with other choices of order on the set $I, J, K$ we obtain the strong triangle inequality.

Since the structure of the set of balls, defined in this way, depends only on the direction on the tree and does not depend on the function $F$, ultrametrics, defined by different $F$, are equivalent.

This finishes the proof of the theorem.

**Lemma 11** *Partial order in a tree $\mathcal{T}$, satisfying Property 1, is a direction.*

*Proof* Let us prove that, if Property 1 is satisfied, then for an arbitrary finite set of vertices there exists the unique supremum (i.e. the partial order is a direction). It is sufficient to prove this statement for the case of a pair of vertices, since if for any two vertices $A, B$ there exists the unique supremum $\sup(A, B)$, and we choose an arbitrary vertex $C$, then

\[ \sup(A, B, C) = \sup(C, \sup(A, B)) \]

Therefore the existence of the unique supremum for any pair of vertices implies the existence and uniqueness of $\sup(A, B, C)$. Analogously one can argue for the case of arbitrary finite number of vertices.

Property 1 implies that for the maximal vertex $C$ at the path $AB$ the paths $AC$ and $CB$ are completely ordered sets.

For vertices $A, B$ we consider the path $AB$ which connects these vertices and prove, that the unique maximal vertex $C$ at this path is the supremum for $A, B$, and, moreover, this supremum is uniquely defined.

Let $D$ be some vertex, greater than $A, B$. Consider the path which connects $A$ and $D$. By Property 1 this path contains the maximal vertex, which greater than $A$ and $D$. Since $D > A$, this maximal vertex coincides with $D$. Analogously, vertex $D$ is the maximal vertex at the path which connects $B$ and $D$. Moreover, the paths $AD$ and $BD$ are completely ordered.

Consider now the paths $AB, AD, BD$. Since we consider a tree, there exists the unique vertex $E$, which belongs to all the three paths. Vertex $E$ satisfies the inequalities

\[ E > A, \quad E > B, \quad E < D. \]

Therefore $E = C$. This shows that $C$ is the unique supremum for $A$ and $B$, and the partial order is a direction.

This finishes the proof of the lemma.

**Definition 12** *We call two paths in the tree $\mathcal{T}$ equivalent, if they coincide starting from some vertex (and therefore either both are infinite or finish at the same vertex). The path from the equivalence class $x$, which begins in vertex $A$, we denote $Ax$.***

The next lemma shows that there exists one–to–one correspondence between directions in the tree $\mathcal{T}$, satisfying the Property 1, and equivalence classes of paths in $\mathcal{T}$. This gives the constructive way to describe directions on trees, satisfying the Property 1.
Lemma 13 Let us fix some equivalence class $x$ of paths in the tree $T$. Introduce the following partial order in the tree $T$. We say that $A < B$, where $A, B$ are vertices of the tree $T$, if there exists the path from the equivalence class $x$, such that $A, B$ lie at this path, and $A < B$ in the sense of the order at the path (any path is a completely ordered set in the natural sense: a vertex is smaller if it closer to the beginning of the path).

This partial order satisfies the Property 1 and therefore is a direction in $T$.

Moreover, any direction in the tree $T$, satisfying the Property 1, can be defined by this procedure (i.e. for any direction of this kind there exists some equivalence class $x$ of paths which defines this direction).

Proof Let vertices $A, B$ be connected by a link. Then, since the tree $T$ does not contain cycles, either $B \in Ax$, or $A \in Bx$, and vertices $A, B$ are comparable.

Now let $A, B$ be arbitrary vertices. Consider the paths $Ax, Bx$. These paths and the path $AB$ intersect in the unique vertex $C$. By construction $C > A, C > B$, and $C$ is the unique maximal vertex at the path $AB$. Therefore the introduced partial order satisfies the Property 1 and by lemma 11 is a direction.

Conversely, consider a direction on the tree $T$, which satisfies the Property 1. Take vertex $A$ and consider the neighbor vertices (connected to $A$ by links). By the Property 1 this set of vertices (which contain $A$ and the neighbor vertices) contains the unique maximal vertex $A_1$. Again, consider the set of vertices, which are the neighbors of $A_1$ (and $A_1$ itself), and take the maximal vertex $A_2$. Repeating this procedure, we get the increasing path $AA_1A_2\ldots$ (which may be finite or infinite).

Starting from some vertex $B$, in the analogous way we obtain the increasing path $BB_1B_2\ldots$ Since by the Property 1 the path $AB$ contains the unique maximal vertex $C = \sup(A, B)$, then

$$AB \subset AA_1A_2\ldots \bigcup BB_1B_2\ldots$$

and the paths $AA_1A_2\ldots, BB_1B_2\ldots$ coincide starting from vertex $C$. Thus these two paths are in the same equivalence class.

This finishes the proof of the lemma.

Introduce now the ultrametric on the directed tree $T$ (where the direction satisfies the Property 1), which we call the standard. This example of ultrametric was discussed in [1].

Let us put into correspondence to a link of the tree $T$ the branching index of the largest vertex of the link (this definition is correct since any two vertices, connected by a link, are comparable). Link is increasing, if the end of the link is larger than the beginning, and is decreasing in the opposite case.

Definition 14 Consider the directed tree $T$, where the direction satisfies the Property 1. Let all the branching indices of vertices of the tree are finite and not equal to one, the maximal vertex either does not exist or has branching index $\geq 2$. Fix an arbitrary vertex $R$ of the tree (we will call this vertex the root of the tree).
Let $A, B, A \neq B$ be vertices of the tree $T$ and $I = \sup(A, B)$, let $R$ be the root of the tree $T$. Let us define the distance $|AB|$ between vertices of the tree as the product of branching indices of the links along the finite directed path $RI$

$$RI = I_0 \ldots I_N, \quad I_0 = R, \quad I_N = I$$

in the degrees $\pm 1$, where the branching indices of the increasing links $I_j, I_{j+1}$ are in the degree $+1$, and the branching indices of decreasing links are in the degree $-1$:

$$|AB| = \prod_{j=0}^{N-1} p_{I_j, I_{j+1}} \varepsilon_{I_j, I_{j+1}}$$

(3)

where $\varepsilon_{I_j, I_{j+1}} = 1$ for $I_j < I_{j+1}$, $\varepsilon_{I_j, I_{j+1}} = -1$ for $I_j > I_{j+1}$.

If vertices $A, B$ coincide then the distance between them we put equal to zero.

By theorem [10] the introduced distance is an ultrametric.

4 Absolute and completed tree

In the present Section we consider completions of directed trees with respect to ultrametric, defined by theorem [10] of the previous Section. This gives a constructive way to build ultrametric spaces. This construction is similar to the construction of real (and $p$–adic) numbers by completion of rational numbers.

Definition 15 For the directed tree $T$ (where the direction satisfies the Property 1) consider the set $\tilde{X}(T)$, which is the completion of $T$ with respect to the ultrametric, defined by definition [12]. The space $\tilde{X}(T)$ we call the completed tree, corresponding to the tree $T$.

Consider also the set $X(T) = \tilde{X}(T) \setminus (T \setminus T_{min})$ (where $T_{min}$ is the set of minimal vertices in $T$), i.e. $X(T)$ is $\tilde{X}(T)$, where all the vertices of the tree $T$ are subtracted, besides the minimal vertices. The space $X(T)$ we call the absolute of the tree $T$.

Sets $\tilde{X}(T)$ and $X(T)$ are complete ultrametric spaces.

Here we understand subtraction of the tree from its completion as follows: vertex $A$ is identified with the equivalence class of sequences of vertices, coinciding with $A$, starting from some element.

Definition [15] is equivalent to the standard definition of the absolute of the tree, see [22] (more definitely, to the absolute without one point, corresponding to the equivalence class of increasing paths). Let us call a path in the tree infinitely continued, if the path is either infinite or finish at vertex with branching index 0.

Lemma 16 The set $\tilde{X}(T)$ is in one to one correspondence with the set of classes of equivalence of decreasing paths in $T$. The set $X(T)$ is in one to one correspondence with the set of classes of equivalence of infinitely continued decreasing paths in $T$.
Proof We put into correspondence to a decreasing path in the tree $T$ the sequence of vertices as follows: if the path is infinite, the sequence is the sequence of vertices at the path; if the path is finite, the sequence is the sequence of vertices at the path, extended by the infinite sequence where the term is the last vertex of the path.

Applying this construction to a class of equivalence of decreasing paths, we get a set of fundamental sequences from the same equivalence class with respect to the ultrametric (2). Thus the set of equivalence classes of decreasing paths is a subset in $\tilde{X}(T)$.

Formula (2) implies that any fundamental sequence in $T$ with metric (2) contains a subsequence which coincide, starting from some term, with a subsequence of the sequence, defined by some decreasing path. Therefore, the set $\tilde{X}(T)$ is equivalent to the set of all equivalence classes of decreasing paths.

Analogously, an equivalence class of infinitely continued decreasing paths corresponds to some point of the absolute $X(T)$, and any equivalence class of sequences of vertices of the tree, corresponding to some point of the absolute, contains an equivalence class of infinitely continued decreasing paths. This finishes the proof of the lemma.

The introduced completed tree $\tilde{X}(T)$ coincides with $X \cup T(X)$, where $X = X(T)$. This shows the duality between complete ultrametric spaces, satisfying conditions (1), (2), (3) of Section 2, and directed trees with finite branching index $\neq 1$, the direction satisfying the Property 1, and where the maximal vertex (if exists) has branching index $\geq 2$.

5 Ultrametric wavelets

In the present and next Sections we consider complete ultrametric space $X$, satisfying the properties 1, 2, 3. Consider a $\sigma$–additive possessing a countable or finite basis of balls positive measure $\nu$ on ultrametric space $X$.

Build a basis in the space $L^2(X, \nu)$ of quadratically integrable with respect to the measure $\nu$ functions, which we will call the basis of ultrametric wavelets.

Denote $V_I$ the space of functions on $X$, generated by characteristic functions of the maximal subballs in the ball of nonzero radius $D_I$. Correspondingly, we denote $V^0_I$ the subspace of codimension 1 in $V_I$ of functions with zero mean with respect to the measure $\nu$. The proof of the following lemma is straightforward.

Lemma 17 Spaces $V^0_I$ for different $I$ are orthogonal in $L^2(X, \nu)$.

We introduce in the space $V^0_I$ some orthonormal basis $\{\psi_{IJ}\}$, where the number of vectors in the basis is obviously less or equal to $p_I - 1$. The next theorem shows how to construct the orthonormal basis in $L^2(X, \nu)$, taking the union of bases $\{\psi_{IJ}\}$ in spaces $V^0_I$ over all non minimal $I$.

Theorem 18 1) Let the ultrametric space $X$ contains an increasing sequence of embedded balls with infinitely increasing measure. Then the set of functions $\{\psi_{IJ}\}$,
where $I$ runs over all non minimal vertices of the tree $T(X)$ is an orthonormal basis in $L^2(X, \nu)$.

2) Let for the ultrametric space $X$ there exists the supremum of measures of the balls, which is equal to $A$. Then the set of functions $\{\psi_{IJ}, A^{-\frac{1}{2}}\}$, where $I$ runs over all non minimal vertices of the tree $T(X)$ is an orthonormal basis in $L^2(X, \nu)$.

The introduced in the present theorem basis we call the basis of ultrametric wavelets. For the case when the measure $\nu$ is defined in the special way: the measure of a ball is equal to its diameter, this theorem (and the results of the next Section on diagonalization of ultrametric PDO) was obtained in [1].

**Proof**  By lemma 17 the described in the statement of the theorem functions are orthonormal.

To prove the totality we use the Parseval identity. Since the set of characteristic functions of all the balls $D_I$ is total in $L^2(X, \nu)$, to prove the totality it is sufficient to prove the Parseval identity only for characteristic functions $\chi_I(x)$.

Consider the characteristic function $\chi_I$ of the ball $D_I$, satisfying the condition $\nu(D_I) > 0$ (and therefore $\chi_I \neq 0$ in $L^2(X, \nu)$). Expand the characteristic function $\chi_I(x)$ over the wavelets. It is sufficient to consider wavelets $\psi_{IJ}$ with $J > I$.

Denote $P_V$ the orthogonal projection onto $V$ in $L^2(X, \nu)$. Then

$$P_{V^0} = P_V - P_{\chi_J}$$

Consider the vector

$$\tilde{\chi}_I = \sum_{J \in T} P_{V^0} \chi_I = \sum_{J > I} (P_{V_J} - P_{\chi_J}) \chi_I = \sum_{J > I} (P_{\chi_{J-1, I}} - P_{\chi_J}) \chi_I$$

where $(J - 1, I)$ is the (uniquely defined) maximal vertex, which is less than $J$ and larger than $I$. The above vector is expanded into the series over the orthogonal vectors. Compute the square of the length $\tilde{\chi}_I$:

$$\|\tilde{\chi}_I\|^2 = \sum_{J > I} \| (P_{\chi_{J-1, I}} - P_{\chi_J}) \chi_I \|^2$$

Since for $J \geq I$

$$P_{\chi_J} \chi_I = \frac{\nu(D_I)}{\nu(D_J)} \chi_J$$

(this expression is correct since $\nu(D_J) > 0$ for $J \geq I$), we get

$$\| (P_{\chi_{J-1, I}} - P_{\chi_J}) \chi_I \|^2 = \left( \frac{\nu(D_I)}{\nu(D_{J-1, I})} \chi_{J-1, I} - \frac{\nu(D_I)}{\nu(D_J)} \chi_J \right)^2 =$$

$$= \left( \frac{\nu(D_I)}{\nu(D_{J-1, I})} - \frac{\nu(D_I)}{\nu(D_J)} \right)^2 \nu(D_{J-1, I}) + \left( \frac{\nu(D_I)}{\nu(D_{J-1, I})} \right)^2 \left( \nu(D_J) - \nu(D_{J-1, I}) \right) =$$

$$= \nu^2(D_I) \left[ \frac{1}{\nu(D_{J-1, I})} - \frac{1}{\nu(D_J)} \right]$$
This implies
\[ \|\tilde{\chi}_I\|^2 = \nu^2(D_I) \sum_{J > I} \left[ \frac{1}{\nu(D_{J-1,I})} - \frac{1}{\nu(D_J)} \right] = \nu^2(D_I) \lim_{J \to \infty, J > I} \left[ \frac{1}{\nu(D_I)} - \frac{1}{\nu(D_J)} \right] \]
where the limit at the RHS is the limit of the expression in square brackets for the sequence of increasing \( J \), which begins from \( I \).

In the case 1 formula (4) implies the Parseval identity for \( \chi_I \).

In the case 2 we get for (4)
\[ \nu^2(D_I) \left[ \frac{1}{\nu(D_I)} - \frac{A}{A} \right] \]

Since in this case to prove the totality we have to add to the expression above the term corresponding to the contribution of the normed constant:
\[ \nu^2(D_J) \]
we again obtain the Parseval identity, which finishes the proof of the theorem.

6 Diagonalization of ultrametric PDO

In the present Section we study the ultrametric pseudodifferential operator (or the PDO) of the form
\[ Tf(x) = \int T^{(\sup(x,y))}(f(x) - f(y))d\nu(y) \]
Here \( T^{(I)} \) is some complex valued function on the tree \( \mathcal{T}(X) \). Thus the structure of this operator is determined by the direction on \( X \cup \mathcal{T}(X) \).

**Theorem 19** Let the following series converge absolutely:
\[ \sum_{J > R} T^{(J)}(\nu(D_J) - \nu(D_{J-1,R})) < \infty \] (5)
Then the operator
\[ Tf(x) = \int T^{(\sup(x,y))}(f(x) - f(y))d\nu(y) \]
has the dense domain in \( L^2(X, \nu) \), and is diagonal in the basis of ultrametric wavelets from the theorem \[18\]
\[ T\psi_{Ij}(x) = \lambda_I \psi_{Ij}(x) \]
with the eigenvalues:
\[ \lambda_I = T^{(I)}(\nu(D_I) + \sum_{J > I} T^{(J)}(\nu(D_J) - \nu(D_{J-1,I}))) \]
(7)
and is self-adjoint if \( T^{(I)} \) is real valued function.
Here \((J - 1, I)\) is the maximal vertex which is less than \( J \) and larger than \( I \).
Also the operator \( T \) kills constants.
Proof Consider the action of the operator onto the wavelet $\psi_{Ij}$:

$$T\psi_{Ij}(x) = \int T^{(\sup(x,y))} (\psi_{Ij}(x) - \psi_{Ij}(y)) d\nu(y)$$

Consider the following cases.

1) Let $x$ does not lie at the ball $D_I$. Then

$$T\psi_{Ij}(x) = -T^{(\sup(x,I))} \int \psi_{Ij}(y) d\nu(y) = 0$$

2) Let $x \in D_I$. Denote $\mu(D_I)$ the diameter of the ball $D_I$. Then

$$T\psi_{Ij}(x) = \left( \int_{|x| > \mu(D_I)} + \int_{|x| = \mu(D_I)} + \int_{|x| < \mu(D_I)} \right) T^{(\sup(x,y))} (\psi_{Ij}(x) - \psi_{Ij}(y)) d\nu(y) =$$

$$= \left( \int_{|x| > \mu(D_I)} + \int_{|x| = \mu(D_I)} \right) T^{(\sup(x,y))} (\psi_{Ij}(x) - \psi_{Ij}(y)) d\nu(y) =$$

$$= \psi_{Ij}(x) \int_{|x| > \mu(D_I)} T^{(\sup(x,y))} d\nu(y) + \int_{|x| = \mu(D_I)} T^{(\sup(x,y))} (\psi_{Ij}(x) - \psi_{Ij}(y)) d\nu(y) =$$

$$= \psi_{Ij}(x) \int_{|x| > \mu(D_I)} T^{(\sup(x,y))} d\nu(y) + T^{(l)} \nu(D_I) \psi_{Ij}(x)$$

To prove the last identity let us compute for $\psi \in V_0$ the integral

$$\int_{|x| = \mu(D_I)} T^{(\sup(x,y))} (\psi(x) - \psi(y)) d\nu(y) = T^{(l)} \int_{|x| = \mu(D_I)} (\psi(x) - \psi(y)) d\nu(y) =$$

$$= T^{(l)} \sum_{j=0}^{p_l-1} \chi_{I_j}(x) \left[ \int_{D_I \setminus D_{I_j}} (\psi(x) - \psi(y)) d\nu(y) \right] =$$

$$= T^{(l)} \sum_{j=0}^{p_l-1} \chi_{I_j}(x) \left[ \psi(x)(\nu(D_I) - \nu(D_{I_j})) - \int_{D_I \setminus D_{I_j}} \psi(y) d\nu(y) \right] =$$

$$= T^{(l)} \sum_{j=0}^{p_l-1} \chi_{I_j}(x) \left[ \psi(x)(\nu(D_I) - \nu(D_{I_j})) + \int_{D_{I_j}} \psi(y) d\nu(y) \right] = T^{(l)} \nu(D_I) \psi(x)$$

Here $D_{I_j}$ are the maximal subballs in $D_I$.

We get

$$T\psi_{Ij}(x) = \lambda_I \psi_{Ij}(x)$$

where

$$\lambda_I = T^{(l)} \nu(D_I) + \int_{|x| > \mu(D_I)} T^{(\sup(x,y))} d\nu(y)$$

For $J > I$

$$\int_{|x| = \mu(D_J)} d\nu(y) = \nu(D_J) - \nu(D_{J-1,I})$$

Since any two increasing paths in a directed tree coincide starting from some vertex, condition (5) provides convergence of the integral $\int_{|x| > \mu(D_I)} T^{(\sup(x,y))} d\nu(y)$.
This implies
\[ \lambda_I = T^{(I)} \nu(D_I) + \sum_{J>I} T^{(J)}(\nu(D_J) - \nu(D_{J-1,I})) \]

Proof that the operator \( T \) kills constants is straightforward. This finishes the proof of the theorem.

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