Non-perturbative instanton corrections to the moduli space geometry of type IIA string theory compactified on a Calabi-Yau space are derived and found to contain order $e^{-1/g_s}$ contributions, where $g_s$ is the string coupling. The computation reduces to a weighted sum of supersymmetric extremal maps of strings, membranes and fivebranes into the Calabi-Yau space, all three of which enter on equal footing. It is shown that a supersymmetric 3-cycle is one for which the pullback of the Kähler form vanishes and the pullback of the holomorphic three-form is a constant multiple of the volume element. Quantum mirror symmetry relates the sum in the IIA theory over supersymmetric, odd-dimensional cycles in the Calabi-Yau space to a sum in the IIB theory over supersymmetric, even-dimensional cycles in the mirror.
1. Introduction

Progress in string theory hinges on gaining a better understanding of non-perturbative effects. Recently there has been substantial progress in understanding these effects in the context of four-dimensional string theories with $N = 2$ spacetime supersymmetry. In this paper we will describe how non-perturbative corrections modify the geometry of $D = 4$, $N = 2$ string theories. Our results also bear on the role played by extended objects other than strings in the theory which has – perhaps misleadingly – come to be known as string theory.

The basic idea is as follows. IIA string theory contains membrane, fourbrane and sixbrane solitons\(^1\) which carry Ramond-Ramond (RR) charges, while the IIB theory contains string, threebrane and fivebrane solitons which carry RR charges[1]. In the context of Calabi-Yau compactification, euclidean $p$-branes can wrap around non-trivial $(p + 1)$-cycles in the Calabi-Yau space\(^2\). The supersymmetric, minimal action configuration is an instanton, whose effects can be computed using standard techniques. We shall find corrections which are independent of $g_s$, the string coupling constant, and are just the standard worldsheet instanton corrections. New corrections which behave as $e^{-1/g_s}$ are also found. The existence of such effects was predicted in general from the growth of string perturbation theory by Shenker\(^8\), and in a context close to the present one by Witten\(^9\).

In $D = 4$, $N = 2$ supergravity, there are both hypermultiplets and vector multiplets, but supersymmetry forbids them from talking to one another at the level of the low-energy effective action[10]. The dilaton ($e^{\phi} = g_s$) lives in a hypermultiplet. It follows that there can be no non-perturbative corrections to the vector multiplet geometry. However there can be, and we shall see that there are, non-perturbative corrections to the geometry of the hypermultiplet moduli space.

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1 The standard, if confusing, terminology is that a $p$-brane is a $(p + 1)$-dimensional extended object, a membrane is a twobrane and a string is a onebrane.

2 Wrapping modes of $p$-branes have been discussed in a variety of related contexts in ref. $[2,3,4,5,6,7]$. 
In the IIA theory, hypermultiplets arise from the odd cohomology (i.e. $H^3$) of the Calabi-Yau, while in the IIB theory they arise from the even cohomology (i.e. $H^0$, $H^2$, $H^4$ and $H^6$). In the IIA theory, we show that $e^{-1/g_s}$ corrections arise from membrane instantons which wrap odd-dimensional cycles (i.e. 3-cycles) in the Calabi-Yau, and therefore depend on the hypermultiplet moduli which governs the size and shape of those cycles. In the IIB theory, $e^{-1/g_s}$ corrections arise from string, threebrane and fivebrane instantons which wrap even (two-, four- and six-dimensional) cycles in the Calabi-Yau – which for the IIB theory are the ones associated to hypermultiplets. The existence of odd-(even-)dimensional $p$-branes in the IIA (IIB) theory is in perfect harmony with the fact that only hypermultiplet geometry can be corrected. Thus the low-energy supergravity theory “knows” about non-perturbative corrections.

In Yang-Mills theory, an instanton anti-instanton pair can be smoothly deformed to the vacuum. Cluster decomposition can then be used to determine the weighting of instanton sectors. No such procedure is known for string theory or quantum gravity in general, so some guesswork is involved in the rules stated herein for inclusion of non-perturbative effects. However they are severely constrained by consistency and other considerations.

A first constraint is provided by mirror symmetry, which relates the IIA theory compactified on a Calabi-Yau $\mathcal{X}$ to the IIB theory on the mirror $\tilde{\mathcal{X}}$ of $\mathcal{X}$. If this is to hold at the quantum level, then the sum over odd cycles in the IIA theory on $\mathcal{X}$ must equal (in appropriate coordinates on the moduli spaces) the sum over even cycles in the IIB theory on $\tilde{\mathcal{X}}$. The existence of such a relation seems mathematically surprising.

The second check, which is being investigated in [11], concerns conifold singularities in the hypermultiplet moduli space and is discussed in section 6.

Ordinary mirror symmetry provides a check on the usual formulae for worldsheet instanton corrections [12] as was spectacularly demonstrated in ref. [13]. Second quantized mirror symmetry provides an analogous check of the formulae
herein. Proposed examples [14,15], relate $N = 2$ heterotic and type II string compactifications. On the heterotic side, the dilaton lies in a vector multiplet and so hypermultiplet geometry cannot be renormalized. The tree-level hypermultiplet geometry is exact and sums up type II RR instantons [15]. A preliminary analysis indicates that the appearance of $e^{-1/g_s}$ can be understood in this way for dualities involving $K3$ fibrations [16].

This paper is organized as follows. In preparation for the analysis of the IIA theory, in section 2 we analyze membrane instantons in Calabi-Yau compactification in the notationally simpler case of $D = 11$ supergravity. In 2.1 conditions for a supersymmetric map, the analog of a holomorphic map for 3-cycles, is derived. In 2.2 we give an explicit example of a supersymmetric map. In 2.3 it is shown that such maps saturate a Bogomol’nyi bound relating the membrane volume to the period of the holomorphic three-form. In 2.4 we briefly review the quaternionic geometry of the hypermultiplet moduli space. In 2.5 we construct membrane vertex operators (analogs of the usual string vertex operators), count zero-modes and present formulae for the quantum corrections to the geometry coming from supersymmetric 3-cycles. In 2.6 we argue that such supersymmetric maps also count extremal black holes in the IIB string theory. In section 3 we obtain formulae in IIA string theory by compactifying the eleven-dimensional theory to ten dimensions on a circle. In this process a new type of membrane instanton, described in 3.2, arises in which one leg of the membrane wraps the circle while the other two wrap a Calabi-Yau 2-cycle. An important check on our results is that this reproduces the standard formula for classical worldsheet instanton corrections. The issue of multiple covers is raised but not resolved in 3.3. Fivebrane instantons are discussed in section 4. In section 5 the IIB string is discussed, and it is argued that a sum over supersymmetric maps into even-dimensional Calabi-Yau cycles is related by mirror symmetry to the sum over odd-cycles in the IIA theory. The remaining steps to obtain a precise mathematical form of this relation are outlined. In section 6 we briefly address the problem of conifold singularities in the hypermultiplet moduli space. We conclude with a discussion on the role of $p$-branes in string theory in
section 7. Our conventions are in an appendix.

The derivation of the corrections from supersymmetric maps of membranes into 3-cycles in sections 2 and 3 was greatly aided by the prior construction [17] of the covariant supermembrane action. Such actions have not been constructed for the $p$-branes of the IIB theory. Our formulae in section 4 are accordingly somewhat preliminary and schematic. Indeed many aspects of instanton corrections for both IIA and IIB theories – such as multiple covers, topological sums and zero modes – are not settled herein. Further details and analysis will be presented elsewhere.

2. Non-Perturbative Effects in D=11 Supergravity

In this section we consider Calabi-Yau compactification to five dimensions of eleven-dimensional supergravity. We will see that the geometry of this five-dimensional theory is non-perturbatively corrected by instantons, corresponding to supermembranes wrapping around 3-cycles of the Calabi-Yau space.

2.1. Membrane Instantons

The bosonic part of the euclidean action of $D = 11$ supergravity [18,19] is

$$S_{11} = \frac{1}{2\pi^2 \ell^9} \int d^{11}x \sqrt{g} \left[ -\hat{R} + \frac{1}{48}(dC)^2 \right] + \frac{i}{12\pi^2 \ell^2} \int C \wedge dC \wedge dC. \quad (2.1)$$

In this expression, $\hat{g}$ is the spacetime metric (the hat denotes eleven-dimensional quantities), $C$ is a three-form potential and $\ell$ is the eleven-dimensional Planck length. The simplest way to analyze the membrane instanton solutions of this theory is to go directly to an effective description on scales large relative to the thickness of the membrane, in which the membrane is described by an effective worldbrane action. The form of this action is completely fixed by supersymmetry
\[ S_3 = \frac{1}{\ell^3} \int d^3\sigma \sqrt{h} \left[ \frac{1}{2} h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \tilde{g}_{MN} - \frac{1}{2} - i\bar{\Theta} \Gamma^a \nabla_a \Theta \right. \\
+ \left. \frac{i}{3!} \epsilon^{\alpha\beta\gamma} \epsilon_{MNP} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P + \ldots \right]. \] (2.2)

Here \( X^M(\sigma) \), with \( M, N = 1, \ldots, 11 \), describes the membrane configuration, \( \Theta \) is an eleven-dimensional Dirac spinor and \( h_{\alpha\beta} \) with \( \alpha, \beta = 1, 2, 3 \), is an auxiliary worldbrane metric with euclidean signature. In this, and all subsequent expressions only the leading order terms in powers of the fermi fields are displayed. The above action is a generalization of the Green-Schwarz action for superstrings [25], to a two dimensional extended object. We henceforth adopt units in which \( \ell = 1 \).

The equation of motion of the auxiliary worldbrane metric sets it equal to the induced metric

\[ h_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N \tilde{g}_{MN}. \] (2.3)

In the following, we will consider field configurations where \( dC = 0 \), which is required for supersymmetric compactifications [26]. The first term of the action then reduces to the volume of the membrane, and the \( X \) equation of motion requires extremization of the volume.

The global fermionic symmetries act on the membrane fields as

\[ \delta_\epsilon \Theta = \epsilon, \]
\[ \delta_\epsilon X^M = i\epsilon \Gamma^M \Theta, \] (2.4)

where \( \epsilon \) is a constant anticommuting eleven-dimensional spinor. The theory is invariant under local fermionic transformations, the so-called \( \kappa \) symmetries, which

---

3 The construction of the membrane action built on earlier work of Hughes, Liu and Polchinski [20]. The membrane soliton solution was derived in ref. [21] and the quantization condition for the membrane tension is discussed in ref. [22]. A good review article is ref. [23]. In the euclidean version presented here, there is a doubling of the fermionic components (because the Majorana condition cannot be imposed in eleven-dimensional euclidean space), so only half of the components of \( \Theta \) should be integrated over, as in the four-dimensional discussion in ref. [24].
act on the fields as
\[
\delta_{\kappa}\Theta = 2P_+\kappa(\sigma), \\
\delta_{\kappa}X^M = 2i\bar{\Theta}\Gamma^M P_+\kappa(\sigma),
\]
where $\kappa$ is a $D = 11$ spinor and $P_\pm$ are the projection operators \cite{20}:
\[
P_\pm = \frac{1}{2} \left(1 \pm \frac{i}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \Gamma_{MNP}\right). \tag{2.6}
\]
These obey
\[
P^2_\pm = P_\pm, \\
P_+P_- = 0, \tag{2.7}
\]
\[
P_+ + P_- = 1.
\]

A general bosonic membrane configuration $X(\sigma)$ breaks all the global supersymmetries generated by $\epsilon$. Unbroken supersymmetries remain if and only if (2.4) can be compensated for by a $\kappa$-transformation, i.e. there exists a spinor $\kappa(\sigma)$ such that
\[
\delta_{\epsilon}\Theta + \delta_{\kappa}\Theta = \epsilon + 2P_+\kappa(\sigma) = 0. \tag{2.8}
\]
Applying $P_-$ to both sides one finds that this requires
\[
P_-\epsilon = 0. \tag{2.9}
\]

We are interested in the case for which six dimensions are compactified on a Calabi-Yau manifold, and the remaining are flat (this has been studied in ref. [27,28,29,30] among others). In complex coordinates the metric is of the mixed

\footnote{4 The factor of $i$, which does not appear in ref. [23], arises in euclidean space.}
form $\tilde{g}_{m\bar{n}}$ with $m, n = 1, 2, 3$. A Calabi-Yau manifold admits a nowhere vanishing holomorphic $(3, 0)$ form

$$\Omega = \frac{1}{3!} \Omega_{mnp}(X) dX^m \wedge dX^n \wedge dX^p,$$

(2.10)

and a Kähler form $\tilde{J}_{m\bar{n}} = i\tilde{g}_{m\bar{n}}$. There are eight covariantly constant spinors $\epsilon$, associated to the global supersymmetries which remain unbroken by the Calabi-Yau compactification.

To analyze the condition (2.9) further, let $\epsilon_+ = (\epsilon_-)^*$ (in an imaginary representation of the gamma matrices) be two covariantly constant six-dimensional spinors with opposite chirality, where $\epsilon_+$ is defined to have positive chirality. The normalization can be chosen so that:

$$\gamma_{mnp} \epsilon_+ = e^{-K} \Omega_{mnp} \epsilon_-,$$

$$\gamma_{mnp} \epsilon_+ = 2i \tilde{J}_{m[n} \gamma_{p]} \epsilon_+,$$

$$\gamma_{m} \epsilon_+ = 0,$$

(2.11)

where we have introduced

$$K = \frac{1}{2} (K_V - K_H),$$

(2.12)

with the Kähler potential on the moduli space of complex structures given by

$$K_H = - \log \left( i \int_X \Omega \wedge \bar{\Omega} \right),$$

(2.13)

and the potential on the Kähler moduli space

$$K_V = - \log \left( \frac{4}{3} \int_X \tilde{J} \wedge \tilde{J} \wedge \tilde{J} \right).$$

(2.14)

Define

$$\epsilon_\theta = e^{i\theta} \epsilon_+ + e^{-i\theta} \epsilon_-,$$

(2.15)

and let $\epsilon$ in eq. (2.9) be of the form $\epsilon_\theta \lambda$, with $\lambda$ a five-dimensional spinor. One
then finds that $P_\epsilon \epsilon_\theta = 0$ implies

$$
\frac{i}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p e^{i\theta} e^{-K} \Omega_{mnp} \epsilon_- e^{-i\theta} \epsilon_- + e^{-i\theta} \epsilon_- + c.c. = 0. 
$$

(2.16)

The spinors $\epsilon_-$ and $\gamma_{\bar{p}} \epsilon_-$ are linearly independent since they transform differently under the holonomy group. Therefore, the third term cannot be cancelled by anything else and must vanish on its own. This requires that the pullback of the Kähler form on to the membrane vanishes:

$$
\partial_\alpha X^m \partial_\beta \bar{X}^n \bar{J}_{m\bar{n}} = 0. 
$$

(2.17)

The remaining terms can cancel if and only if the phase and magnitude of the first are constant. This cancellation requires:

$$
\partial_\alpha X^m \partial_\beta X^n \partial_\gamma \bar{X}^p \Omega_{mnp} = e^{-i\varphi} e^K \epsilon_\alpha \epsilon_\beta \epsilon_\gamma, 
$$

(2.18)

where $\varphi = 2\theta + \pi/2$. This equation states that the membrane volume element is proportional to the pullback of $\Omega$. Given (2.17) and (2.18), a spinor of the form $P_+ \epsilon$ is covariantly constant and obeys (2.9). Hence (2.17) and (2.18) are the necessary and sufficient conditions for a map $X(\sigma)$ to be supersymmetric.

2.2. A SUPERSYMMETRIC 3-CYCLE IN THE QUINTIC

It is easy to see that a flat membrane in $\mathbb{R}^{10}$ is supersymmetric. A more nontrivial example of a solution of eqs. (2.17) and (2.18), suggested to us by G. Moore [31] and by E. Witten [32], is as follows.

---

5 We are grateful to E. Witten and S. -T. Yau for very helpful discussions on these conditions.
The equation
\[ \sum_{i=1}^{5} (X^i)^5 = 0, \] (2.19)
defines a quintic hypersurface in \( \mathbb{CP}^4 \). The Ricci-flat metric \( \hat{g} \) on the quintic has the isometry
\[ X^i \rightarrow DX^i. \] (2.20)

\( D \) leaves fixed a three-dimensional submanifold \( C_3 \) on which all the \( X^i \)'s are real. We now show that \( C_3 \) is a supersymmetric 3-cycle.

Since \( D \) preserves the metric but reverses the orientation, it acts on the Kähler form associated to \( \hat{g} \) as
\[ \hat{J} \rightarrow - \hat{J}. \] (2.21)

On the other hand the pullback of \( \hat{J} \) onto the fixed surface \( C_3 \) must be invariant. This is only possible if the pullback vanishes. It follows that (2.17) is satisfied.

To see that the pullback of \( \Omega \) is the induced volume form on \( C_3 \), we note that the holomorphic 3-form can be written as
\[ \Omega = \frac{dY^1 \wedge dY^2 \wedge dY^3}{(Y^4)^4}, \] (2.22)
where we have introduced the inhomogeneous coordinates:
\[ Y^k = \frac{X^k}{X^5} \quad \text{with} \quad k = 1, \ldots, 4. \] (2.23)

The norm of \( \Omega \) is
\[ \| \Omega \|^2 = \frac{1}{3!} \Omega_{mnp} \bar{\Omega}^{mnp} = \frac{1}{\hat{g}(Y^4)^8}, \] (2.24)
where \( \hat{g} \) is the determinant of \( \hat{g}_{\bar{m}\bar{n}} \). Since \( \Omega \) is covariantly constant, this norm is a
constant
\[ \| \Omega \|^2 = 8e^{2\kappa}. \quad (2.25) \]

It follows that
\[ \widehat{g} = \frac{e^{-2\kappa}}{8|Y^4|^8}. \quad (2.26) \]

When the pullback of \( \widehat{J} \) vanishes, the induced metric is
\[ h_{\alpha\beta} = 2\partial_\alpha Y^m \partial_\beta \bar{Y}^n \widehat{g}_{mn} \quad (2.27) \]

and, using (2.26) and reality of \( Y^4 \) on \( C_3 \)
\[ \sqrt{h} = | \det(\partial Y) | \frac{e^{-\kappa}}{|Y^4|^4}. \quad (2.28) \]

Substituting into eq. (2.22) and pulling back to \( C_3 \) one finds
\[ \frac{1}{3!} d\sigma^\alpha \wedge d\sigma^\beta \wedge d\sigma^\gamma \partial_\alpha Y^m \partial_\beta \bar{Y}^n \partial_\gamma Y^p \Omega_{mnp} = e^{i\omega} \frac{e^{-\kappa}}{\sqrt{h}} \sigma^1 \wedge \sigma^2 \wedge \sigma^3, \quad (2.29) \]

for some phase \( \omega \). Therefore eq. (2.18) is satisfied.

\subsection*{2.3. A Bogomol'nyi Bound}

Eqs. (2.17) and (2.18) imply that the membrane has minimized its volume. To see this consider
\[ \int d^3\sigma \sqrt{h} \bar{e}_\theta P^\dagger_\theta P_\theta - e_\theta \geq 0, \quad (2.30) \]

with \( P_\_ \) constructed from six-dimensional gamma matrices. Using the relation \( P^\dagger_\_ P_\_ = P_\_ \) the inequality becomes
\[ 2\widehat{V}_3 \geq e^{-\kappa} \left( e^{i\varphi} \int \Omega + e^{-i\varphi} \int \bar{\Omega} \right), \quad (2.31) \]

where \( \varphi \) is defined below eq. (2.18). By adjusting \( \varphi \) to maximize the right hand
side one finds
\[ \hat{V}_3 \geq e^{-K} \left| \int \Omega \right|. \] (2.32)

On the other hand the left hand side of (2.30) vanishes if and only if \( P_{\epsilon_0} = 0 \), so this bound is saturated if and only if the maps obey the supersymmetry conditions (2.17) and (2.18).

2.4. Quaternionic Geometry and the Five-Dimensional Action

We wish to compute the instanton corrections to the low-energy effective action in five dimensions. In this subsection we recall the relevant features of this action. It contains both hypermultiplets and vector multiplets. \( N = 2 \) supersymmetry forbids neutral couplings (for the lowest dimension terms) between these multiplets. The moduli space \( \mathcal{M} \) is a direct product of the moduli space of hypermultiplets \( \mathcal{M}_\mathcal{H} \) and the moduli space of vector multiplets \( \mathcal{M}_\mathcal{V} \):

\[ \mathcal{M} = \mathcal{M}_\mathcal{H} \times \mathcal{M}_\mathcal{V}. \] (2.33)

Since the instanton action scales with the size of the Calabi-Yau space, which is parametrized by a hypermultiplet, instanton corrections can affect only the hypermultiplet geometry.

As shown in ref. [35] in the context of \( N = 2, D = 4 \) supergravity, the \( 4n \) scalars of \( n \) hypermultiplets parametrize a quaternionic geometry, with holonomy group \( Sp(n) \cdot Sp(1) \) and a non-vanishing \( Sp(1) \) connection \( 7 \). The Riemann tensor of this geometry can be obtained from:

\[ R_{ijkl} \gamma^{l}_{CI} \gamma^{k}_{BJ} = \epsilon_{CB} R_{ijIJ} + \epsilon_{IJ} R_{ijCB}, \] (2.34)

where \( C, B = 1, 2, I, J = 1, \ldots, 2n \) and \( i, j = 1, \ldots, 4n \); \( R_{ijAB} \) and \( R_{ijIJ} \) are the \( Sp(1) \) and \( Sp(n) \) curvatures respectively and \( \gamma^{l}_{A} \) are covariantly constant functions.

---

6 The factor of \( e^{K_V/2} \) is absent in a related bound discussed in ref. [34,6] because of rescaling of the four-dimensional metric.

7 An explicit construction of the quaternionic manifolds that arise at tree level in type II string theories compactified on Calabi-Yau manifolds has been carried out in ref. [36].
of the $4n$ scalars that satisfy identities similar to those of the Dirac gamma matrices. The $Sp(1)$ connection can be written in the form

$$\mathcal{R}_{ijAB} = \varsigma^2 \left( \gamma_i A \Gamma_j B^I - \gamma_j AI \Gamma_i B^I \right)$$

(2.35)

while the $Sp(n)$ connection is given by

$$\mathcal{R}_{ijIJ} = \varsigma^2 \left( \gamma_i A \Gamma_j A^J - \gamma_j AI \Gamma_i A^J \right) + \gamma_i A L \Gamma_j A^K \mathcal{R}_{IJKL},$$

(2.36)

where $\mathcal{R}_{IJKL}$ is totally symmetric in its indices and $\varsigma^2$ is proportional to the five-dimensional Newton’s constant.

In the following we shall focus on corrections to the four-fermi coupling, given by [37]

$$\int d^5 x \sqrt{\hat{g}} (\bar{\chi}^I \chi^J) (\bar{\chi}^K \chi^L) \mathcal{R}_{IJKL},$$

(2.37)

where $\chi^I$ is the fermionic component of the hypermultiplet. We note that, since the spinor $\chi^I$ has four real (anticommuting) components, and $\mathcal{R}_{IJKL}$ is symmetric, there is a unique invariant coupling of four fermions to the $Sp(n)$ curvature.

In the context of Calabi-Yau compactification, there are $h_{21} + 1$ hypermultiplets, one for each pair of harmonic 3-forms. The five-dimensional massless fermions $\chi^I$ arise from the gravitino zero-modes proportional to:

$$\Psi_{0M}^I = d_M^I \Gamma_{NP}^\alpha (\epsilon_+ + \epsilon_-)$$

(2.38)

where $M, N = 1, \ldots, 6$ and $d^I$ is a symplectic basis of real harmonic three-forms, which obey:

$$\int d^I \wedge d^J = \epsilon^{IJ},$$

(2.39)

with $\epsilon^{IJ}$ the invariant antisymmetric tensor of $Sp(h_{21} + 1)$. 

13
2.5. Membrane Zero-Modes, Vertex Operators and the Quantum Corrected Geometry

Membrane fermion zero-modes are generated by supersymmetries which are unbroken by the Calabi-Yau compactification, but broken by the presence of the supermembrane. For the membrane instanton there are four such Nambu-Goldstone zero-modes, given by

$$\Theta_0(\sigma) = \epsilon_0(X(\sigma)) \quad (2.40)$$

where

$$P_+ \epsilon_0 = 0. \quad (2.41)$$

In general there may be additional zero-modes. This will indeed be the case if the instanton is part of a continuous family, since bosonic zero-modes must come with fermionic superpartners. In this paper we consider only the minimal case of four zero-modes.

We will compute the contribution of membrane instantons to the low-energy five-dimensional effective action. Since there are four fermion zero-modes, the simplest quantity to compute is the correction to the couplings (2.37) of four space-time fermions. Corrections to other terms then follow from supersymmetry.

The low-energy effective field theory describing a membrane soliton coupled to eleven-dimensional supergravity contains bilinear couplings of the space-time fermions to membrane fermions. In particular there is a coupling to the gravitino

$$\int d^3 \sigma \sqrt{h} \Psi_M(X(\sigma)) \mathcal{V}^M(\sigma), \quad (2.42)$$

where $\mathcal{V}^M$ is the membrane vertex operator$^8$. This operator is uniquely fixed by

---

$^8$ The usual string vertex operators can be derived in a similar manner by treating the fundamental string as a soliton.
coordinate and \( \kappa \)-invariance as

\[
\mathcal{V}^M = \left( h^{\alpha \beta} \partial_\alpha X^M \partial_\beta X^N \Gamma_N - \frac{i}{2} \epsilon^{\alpha \beta \gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \Gamma_{NP} \right) \Theta, \tag{2.43}
\]

to leading order in the fermionic variable \( \Theta \), at zero momentum, and up to an overall normalization which can be deduced from formulae in ref. [17] (we suppress normalization constants in subsequent equations).

The vertex operator associated to the five-dimensional field \( \chi^I \) is then obtained from the zero-mode wave function (2.38):

\[
\mathcal{V}^I = \overline{\Psi}^I_0 \mathcal{V}^M, \tag{2.44}
\]

and there is a corresponding coupling \( \overline{\chi}^I \mathcal{V}_I \). This coupling leads to a non-vanishing instanton contribution to the correlator \( \langle (\overline{\chi}_I \chi_J) (\overline{\chi}_K \chi_L) \rangle \) (where indices are lowered with \( \epsilon_{IJK} \)), corresponding to an instanton-induced correction to the effective action proportional to

\[
\Delta \mathcal{C}_3 \mathcal{R}_{IJKL} = \langle (\overline{\chi}_I \chi_J) (\overline{\chi}_K \chi_L) \rangle_{\text{Instanton}}, \tag{2.45}
\]

The four \( \Theta \) zero-modes can be absorbed by pulling down four powers of \( \overline{\chi}^I \mathcal{V}_I \) from the action. The four left over space-time fermions are then contracted with the four fermions in eq. (2.45). One therefore finds that the space-time correlator (2.45) is reduced to a correlator in the three-dimensional field theory on the membrane

\[
\langle (\overline{\chi}_I \chi_J) (\overline{\chi}_K \chi_L) \rangle_{\text{Instanton}} = \langle (\overline{\mathcal{V}}_I \mathcal{V}_J) (\overline{\mathcal{V}}_K \mathcal{V}_L) \rangle_{\text{Instanton}}, \tag{2.46}
\]

which has just the right number of \( \Theta \)'s to soak up the four fermion zero-modes on the membrane. To evaluate this, we substitute the four zero-mode wave functions...
(2.40) into the vertex operators:

$$\int d^3\sigma \sqrt{h} V^I = \int d^3\sigma \sqrt{h} e^{\alpha_\beta \gamma} \partial_\alpha X^M \partial_\beta X^P \partial_\gamma X^Q d^I_{MRS} (\bar{\epsilon}_+ + \bar{\epsilon}_-) \Gamma^{RS} \Gamma_{PQ} (\epsilon_+ + \epsilon_-).$$

(2.47)

Using the identities of the appendix and (2.17) one finds

$$\int d^3\sigma \sqrt{h} V_I = \int_{C_3} d_I,$$

(2.48)

where $C_3$ is the homology class of the instanton. Weighting by the bosonic membrane action for the instanton one thereby obtains

$$\Delta_{C_3} R_{IJKL} = N e^{-e^{-\kappa} \int_{C_3} \Omega - i \int_{C_3} C} \int_{C_3} d_I \int_{C_3} d_J \int_{C_3} d_K \int_{C_3} d_L.$$

(2.49)

The prefactor $N$ here and hereafter is inserted to remind the reader that we have not computed the normalization factors, which also includes a determinant here. This formula indicates that membrane instantons give non-perturbative quantum corrections to the modular geometry similar to the non-perturbative classical $\alpha'$ corrections of string instantons [12]. To obtain the full corrections one must sum contributions of the type (2.49) over all supersymmetric cycles, suitably modified to account for the possible effects of additional fermion or boson zero-modes as well as multiple covers of the Calabi-Yau 3-cycle by the membrane.

### 2.6. Relation to Four-Dimensional Extremal Black Holes

The results of the previous sections can be used to partially fill a gap in the discussion of ref. [6]. In that work, it was argued that the threebranes of the type IIB theory can wrap around 3-cycles and appear as four-dimensional black holes; this is in contrast to the wrapping modes of IIA twobranes which give instantons. Four dimensional supersymmetry implied a bound on the black hole mass of the form

$$M \geq e^{\kappa \Omega / 2} | \int \Omega |.$$

(2.50)

On the other hand, the threebrane in ten dimensions has a fixed mass per unit three-volume. Thus one expects $M \propto V_3$, where $V_3$ is the volume of the threebrane.
The minimal threebrane volume should be bounded by the period of $\Omega$. This will indeed be the case if the conditions for a supersymmetric IIB threebrane soliton are the same as for a supersymmetric IIA twobrane instanton. Verification of this will require a better understanding of the threebrane dynamics, but we expect that solutions of (2.17) and (2.18) count extremal black holes in the IIB theory as well as IIA instantons.

3. Non-Perturbative Effects in Type IIA String Theory

In this section non-perturbative effects in type IIA string theory compactified to four dimensions on a Calabi-Yau manifold are derived. These arise from RR instantons corresponding to the twobrane solution of ref. [1] wrapping around a 3-cycle of the Calabi-Yau space, and correct the geometry of the hypermultiplet moduli space.

The low-energy effective action of the type IIA string theory is the non-chiral $N = 2$ and $D = 10$ supergravity, which was originally derived [38] by reduction of $D = 11$ supergravity. The effective action of the membrane solitons of ref. [1], is simply the supermembrane action (2.2). Instanton effects in the IIA theory can thus be derived by $S^1$ compactification of the eleven-dimensional analysis of the previous section. The extra $S^1$ leads to a second type of membrane instanton, in which the membrane wraps around the $S^1$ and a Calabi-Yau 2-cycle. In accord with the observations that IIA strings are compactified $D = 11$ membranes [39,5] and that eleven dimensional supergravity is strongly-coupled IIA string theory [9], we shall see that in this case our formulae reduce to standard worldsheet instanton corrections. Membranes which wrap Calabi-Yau 3-cycles lead to new non-perturbative string effects which behave as $e^{-1/g_s}$. 
3.1. IIA Membrane Instantons

Reduction to $D = 10$ proceeds by periodically identifying

$$X^{11} \simeq X^{11} + 1,$$  \hfill (3.1)

in eq. (2.2). The $D = 11$ spinor $\Theta$ is decomposed into two $D = 10$ spinors $\Theta_{\pm}$ obeying $\Gamma_{11} \Theta_{\pm} = \pm \Theta_{\pm}$. The eleven-dimensional metric $\hat{g}$ is related to the ten-dimensional string metric $g$ and dilaton $\phi$ by \[38,9\]

$$d\hat{s}_{11}^2 = e^{4\phi/3} (dx_{11} + A_m dx^m)^2 + e^{-2\phi/3} ds_{10}^2,$$  \hfill (3.2)

where $A$ is a one-form. The eleven-dimensional supergravity action then reduces to

$$S_{10} = \frac{1}{2\pi^2} \int d^{10}x \sqrt{\hat{g}} \left[ e^{-2\phi} \left( -R - 4 (\nabla \phi)^2 + \frac{1}{12} (dB)^2 \right) + \frac{1}{4} (dA)^2 + \frac{1}{48} J^2 \right]$$

$$+ \frac{i}{4\pi^2} \int dC \wedge dC \wedge B,$$  \hfill (3.3)

where $B$ is the 2-form obtained by the reduction $C = dx^{11} \wedge B$ and $J = dC + A \wedge dB$ and our units are $\ell^2 = 2\pi \alpha' = 1$. For membranes which wrap Calabi-Yau 3-cycles, the instanton corrections to hypermultiplet geometry are obtained simply substituting the appropriate field redefinitions into eq. (2.49). In particular

$$\hat{V}_6 = \frac{1}{3!} \int \hat{J} \wedge \hat{J} \wedge \hat{J},$$  \hfill (3.4)

in eq. (2.49) is the volume of the Calabi-Yau space as measured by the eleven-dimensional supergravity metric. This is related to the string frame volume by

$$\hat{V}_6 = \frac{1}{g_s^2} V_6.$$  \hfill (3.5)

Eq. (2.49) then becomes

$$\Delta_C \mathcal{R}_{IJKL} = N e^{-\frac{1}{2} \hat{V}_6} e^{-\kappa [\hat{f}_3 \Omega] - i \int \hat{f}_3 C} \int_{\hat{C}_3} d_1 \int_{\hat{C}_3} d_1 \int_{\hat{C}_3} d_1 \int_{\hat{C}_3} d_1,$$  \hfill (3.6)

where it is understood that the $\mathcal{K}$ in this expression is constructed from the string
Kähler form $\tilde{J}$ rather than $\hat{J}$. The fact that non-perturbative corrections in string theory should behave like $e^{-1/g_s}$ was predicted by Shenker [8] from an analysis of the growth of the perturbation expansion. It is satisfying to see this can be realized in an explicit calculation, as suggested in ref. [9].

3.2. IIA String Instantons

There are also membrane instantons which wrap the $S^1$ and a Calabi-Yau 2-cycle, i.e.

$$X^{11} = \sigma_3,$$  \hspace{1cm} (3.7)

while $X^m, X^\bar{m}$ are functions of $\sigma_1$ and $\sigma_2$. These instantons will correct the moduli space geometry of the vector multiplets. A slight modification of the analysis of the previous section is required to find the supersymmetric maps. Using eq. (3.7) the membrane action (2.2) reduces to the Green-Schwarz string action [39]

$$S_2 = \frac{1}{2} \int d^2 \sigma \sqrt{h} \left[ h^{\alpha \beta} \partial_{\alpha} X^M \partial_{\beta} X^N g_{MN} + i \epsilon^{\alpha \beta} B_{MN} \partial_{\alpha} X^M \partial_{\beta} X^N + \text{(fermi)} \right].$$ \hspace{1cm} (3.8)

Note that the factors of $e^\phi$ cancel when the action is written in terms of the string metric. The $\kappa$-transformations of $\Theta$ reduce to

$$\delta_{\kappa \pm} \Theta_\pm = 2P_{\pm} \kappa_\pm,$$ \hspace{1cm} (3.9)

where

$$P_{\pm} = \frac{1}{2} \left( 1 \pm \frac{i}{2} \epsilon^{\alpha \beta} \partial_{\alpha} X^M \partial_{\beta} X^N \Gamma_{MN} \right).$$ \hspace{1cm} (3.10)

Repeating the analysis of the previous section, one finds that a global supersymmetry transformation is equivalent to a $\kappa$-transformation if and only if

$$\bar{\partial} X^m = 0 \hspace{1cm} \text{or} \hspace{1cm} \partial X^{\bar{m}} = 0,$$ \hspace{1cm} (3.11)

where $\bar{\partial}$ is the antiholomorphic worldsheet exterior derivative. Eq. (3.11) is of course the usual holomorphy condition for worldsheets instantons.
The Bogomol’nyi bound for the induced area is now expressed in terms of the Kähler form as
\[ V_2 \geq \int J, \quad (3.12) \]
and it is saturated by states satisfying (3.11).

The vertex operators of the ten-dimensional theory can be easily obtained from a double reduction of eq. (2.43)
\[ \mathcal{V}_M^\pm = \left( h^{\alpha\beta} \mp i\epsilon^{\alpha\beta} \right) \partial_\alpha X^M \partial_\beta X^P \Gamma_P \Theta_\pm \quad \text{with} \quad M, P = 1, \ldots, 10. \quad (3.13) \]
It is straightforward to show that (3.13) is invariant under $\kappa$-transformations. The solutions of the six dimensional Dirac equation which lead to vector multiplets will now be written in terms of the harmonic $(1, 1)$ forms
\[ u_I^M = b_{I M N} \Gamma^N (\epsilon_+ + \epsilon_-) \quad \text{with} \quad I = 1, \ldots, h_{11}, \quad (3.14) \]
and $M, N = 1, \ldots, 6$. One finds that the associated zero momentum vertex operators are given by
\[ \int d^2 \sigma \sqrt{h} \mathcal{V}_I = \int_{\mathcal{C}_2} b_I. \quad (3.15) \]
The result for the four-point correlation function of the fermionic operators (3.13) is therefore
\[ \Delta_{\mathcal{C}_2} \mathcal{F}_{IJKL} = N e^{-\int_{\mathcal{C}_2} J - i \int_{\mathcal{C}_2} B} \int_{\mathcal{C}_2} b_I \int_{\mathcal{C}_2} b_J \int_{\mathcal{C}_2} b_K \int_{\mathcal{C}_2} b_L. \quad (3.16) \]
For vector multiplets, the four-fermi coupling is the derivative of the “Yukawas” $\mathcal{F}_{IJK}$ with respect to the modulus $z^I$, where
\[ J + iB = z^I b_I. \quad (3.17) \]
Integrating (3.16) once with respect to the modulus, one recovers the standard formula for worldsheet instanton corrections to Yukawas. This provides an important
check on our derivations (for the case of simple covers and minimal fermion zero modes).

Our analysis also gives insight into the relation between $N = 2, D = 4$ and $N = 1, D = 5$ supergravity. For hypermultiplets, there is no difference in the four- and five-dimensional geometries. However for vector multiplets there is a marked difference. In five dimensions the geometry is completely determined by the constants $C_{IJK}$ [37], which for the case of Calabi-Yau compactified $D = 11$ supergravity are just the cubic intersection form [33]. In four dimensions, however, the $C_{IJK}$ are functions of the moduli, which characterize a special geometry [10]. This is in perfect harmony with the fact that membrane instantons cannot correct $D = 5$ vector multiplet geometry because there are no 3-cycles associated to the vector multiplet geometry. However such 3-cycles appear upon $S^1$ compactification from $D = 5$ to $D = 4$, and non-perturbative corrections can and do arise. Thus we see again that the low-energy supergravity theories “know” about non-perturbative corrections!

3.3. A Puzzle

It was discovered in ref. [39] that, when eleven-dimensional supergravity is $S_1$-compactified to IIA supergravity, a supermembrane with one leg wound around the $S_1$ becomes the IIA string. This observation was used in the previous subsection to reduce certain eleven-dimensional membrane instantons to IIA string instantons.

However one may also consider a membrane for which one leg winds not once but $n$ times around the $S_1$. This leads to ten-dimensional strings with $n$ times the minimal string tension and axion charge. Such objects cannot be incorporated into IIA string theory.

The rules for inclusion of such configurations must be determined by consistency. One possible out is to regard the $n$-wound membrane as $n$ fundamental IIA strings on top of one another rather than as a fundamental charge $n$ object. Superficially this is similar to the situation encountered in ref. [6], in which charge
4. Fivebrane Instantons

In addition to strings and membranes, type II string theories contain fivebrane solitons, which are described by exact conformal field theories \[2,41,42\]. Euclidean fivebranes wrapping around a Calabi-Yau space lead to non-perturbative corrections to hypermultiplet geometry which in principle might be computed analogously to string and membrane instanton corrections. However, while the static gauge worldsheet field content was derived in ref. \[42\], the covariant action has not been constructed. Fortunately there is a much simpler way to proceed. Explicitly the neutral four-dimensional field configuration is

\[\begin{align*}
H_{\mu\nu\lambda} &= -2\epsilon_{\mu\nu\lambda}\rho\nabla_{\rho}\phi, \\
g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu}, \\
e^{-2\phi} &= e^{-2\phi_0} + \frac{1}{2\pi r^2}
\end{align*}\]  

where \(H = dB\). The internal six-dimensional geometry is unaffected by the fivebrane instanton, so standard four-dimensional instanton methods can be used to
compute corrections from (4.1). The instanton action is\(^9\)

\[ S_6 = \frac{1}{\pi g_s^2} + \frac{ia}{\pi}, \]  

(4.2)

where the axion field \(a\) is related to \(H\) by

\[ da = \frac{1}{2} e^{-2\phi} \star H. \]  

(4.3)

This instanton has four zero-modes obtained from supersymmetry, for which the dilatino wave functions are proportional to

\[ \chi^D = \sqrt{\phi} \eta_+ \]  

(4.4)

\(\chi^D\) is the linear combination of the \(\chi^I\)'s corresponding to the dilatino. This leads to instanton corrections to the coupling of four dilatinos

\[ \Delta_{C_6} R_{DDDD} = e^{-1/\pi g_s^2 - ia/\pi} + c.c. \]  

(4.5)

Similar corrections arise in \(D = 11\) supergravity and IIB string theory. In the latter case, the formulae must be modified when the second RR dilaton has non-zero expectation value. Furthermore, there is a second fivebrane \([42]\) which couples to the RR Kalb-Ramond field.

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\(^9\) The \(a\) dependence of the action arises from a boundary term which must be added to the action to fix the asymptotic value of \(a\). A very similar case is treated in ref. \([43]\). The strange-looking factor of \(1/\pi\) is the price we pay for using units in which the string and membrane tensions are unity.
5. Mirror Symmetry and the IIB String

At string tree level, mirror symmetry implies an exact equivalence between the IIA string theory compactified on a Calabi-Yau space $X$ (with $N_V = h_{11}$ vector multiplets and $N_H = h_{21}+1$ hypermultiplets) and IIB string theory (with $N_V = h_{21}$ and $N_H = h_{11}+1$) on the mirror $\tilde{X}$ of $X$. It is natural to suppose that these theories are also equivalent at the quantum level. In particular the instanton corrections to hypermultiplet geometry should be the same in each case.

While the IIA theory contains RR membranes and fourbranes the IIB theory contains RR strings, threebranes and fivebranes. There are generically no interesting fourbrane instantons which would have to wrap 5-cycles. Thus quantum mirror symmetry will relate the sum (2.49) over 3-cycles in the IIA theory on $X$ to a sum over even-cycles in the IIB theory on $\tilde{X}$.

The threebrane solution was discovered in ref. [1], and its static gauge field content was found in ref. [44] to be an abelian $D=4, N=4$ vector multiplet. Unfortunately the covariant action is as yet unknown so some guesswork will be needed to deduce the IIB analogue of eq. (2.49). The bosonic part of the covariant action for $F = 0$ is presumably

$$S_4 = \int d^4 \sigma \sqrt{h} \left( \frac{1}{2} h^{\alpha \beta} \partial_{\alpha} X^M \partial_{\beta} X^N e^{-\phi} g_{MN} - 1 + \frac{i}{4!} e^{\alpha \beta \gamma \delta} D_{MNPQ} \partial_{\alpha} X^M \partial_{\beta} X^N \partial_{\gamma} X^P \partial_{\delta} X^Q \right), \quad (5.1)$$

where $D$ is a closed rank four antisymmetric potential. Note the factor of $e^{-\phi}$ which arises because, as follows from formulae in ref. [1] $^{10}$ the action per unit four-volume of a charge one threebrane is inversely proportional to the string coupling.

The real part of the action is proportional to the four-volume of the threebrane.

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$^{10}$ The threebrane geometry was computed in the Einstein frame in ref. [1], the factor of $1/g_s$ is seen after rescaling to the (fundamental) string frame.
It is not hard to show that the volume satisfies the bound
\[ V(\Sigma_4) \geq \frac{1}{2} \int J \wedge J. \]  
(5.2)

This bound is saturated if the map \( X(\sigma) \) is holomorphic
\[ \bar{\partial}X^m(\sigma, \bar{\sigma}) = 0, \]  
(5.3)

where we have introduced complex coordinates on the threebrane. We presume that the configurations which satisfy (5.3) preserve half of the supersymmetries. In addition there are abelian gauge fields on the worldbrane. The structure of the ten-dimensional supersymmetry transformations suggest that self-dual gauge configurations will break some, but not all, of the supersymmetries preserved by (5.3). Since more supersymmetries are broken by such configurations, there will be more than four fermion zero-modes. Such configurations might still contribute to the four-fermi couplings but will not be further considered.

If the threebrane instanton breaks half the supersymmetries, there will be four Goldstino zero-modes and corrections to four-fermi couplings. In analogy with the string and the membrane case, the natural expression for these corrections is
\[ \Delta_{\mathcal{C}_4} \mathcal{R}_{IJKL} = N e^{-\frac{1}{2g_s} \int_{\mathcal{C}_4} J^{\wedge J-i} f_{\mathcal{C}_4}^D} \int_{\mathcal{C}_4} f_I \int_{\mathcal{C}_4} f_J \int_{\mathcal{C}_4} f_K \int_{\mathcal{C}_4} f_L. \]  
(5.4)

\( \mathcal{C}_4 \) is a holomorphic map and \( f_I \) is the four form Hodge dual to the \( \Gamma \)'th \((1,1)\) class. Of course with no restriction on topology (5.4) will be hard to evaluate. It is possible that, as in the string case, an analysis of zero-modes will lead to a restriction on the topology.

This is not the end of the story. IIB string theory contains a second soliton string [45,46] which is related to the fundamental IIB string by an \( SL(2, \mathbb{Z}) \) duality transformation. Worldsheet instantons of the soliton string will give corrections to hypermultiplet geometry similar to those of fundamental strings, except with two important differences:
a) the string tension and therefore the instanton action differs by a factor of $1/g_s$.

b) it couples to the second Kalb-Ramond field, which we shall denote by $\tilde{B}$, arising in the RR sector.

The corrections are:

$$\Delta_c \mathcal{R}_{IJKLM} = N g_s^{-1} \int_{c_2} J^{-i} \int_{c_2} \tilde{B} \int_{c_2} b_I \int_{c_2} b_J \int_{c_2} b_K \int_{c_2} b_L.$$ \hspace{1cm} (5.5)

$SL(2, \mathbb{Z})$ invariance implies that a genus $g$ instanton is weighted by $(g_s)^{-2g}$, but it also implies that $g \geq 0$ instantons will not correct hypermultiplet geometry.

Finally there is a second fivebrane, which couples to the dual of the RR Kalb-Ramond field strength, and has an action which, in the string frame, is greater by a factor of $g_s$ than the NS-NS fivebrane. Therefore it corrects the four-dilatino coupling by:

$$\Delta_c \mathcal{R}_{DDDD} = N e^{-1/\pi g_s}.$$ \hspace{1cm} (5.6)

There is presumably also some exponential dependence on other members of the dilaton hypermultiplet (this could also arise in eqs. (5.4) and (5.5)) which we have not analyzed.

Quantum mirror symmetry then relates the IIB sums of $e^{-1/g_s}$ corrections (5.4), (5.5) and (5.6), to the IIA sum (2.49). It may be useful at this point to summarize the main steps required to obtain a precise mathematical formula. On the IIA side, we must understand when (if ever) there are more than four zero-modes, and determine the contributions of such 3-cycles. On the IIB side, we must verify (5.4), presumably by first constructing the covariant threebrane action. The zero-mode and multiple cover problems must then be analyzed. There is also the possibility that contributions to (5.4) are weighted by some power of $g_s^\chi$, where $\chi$ is the Euler character of the surface. Phases associated with the signature of the
surface may also appear. There may be ambiguities in relating the coordinates on the two moduli spaces. Finally one hopes to evaluate this sums and check the predictions in special cases or limits. Clearly much remains to be done. We hope to report on these issues elsewhere.

6. Conifold Singularities in the Hypermultiplet Moduli Space

The moduli space of a Calabi-Yau space generically contains conifolds at which the moduli space metric becomes singular. If the moduli lie in vector multiplets, this singularity is resolved by charged BPS states which become massless at the conifold \[6,7\]. Such a resolution can not occur for Calabi-Yau moduli which lie in hypermultiplets because there are no associated charges, BPS bounds or massless states. However, the consistency of string theory demands some resolution of the hypermultiplet singularities. Since hypermultiplet geometry is not protected by a non-renormalization theorem, it is natural to suppose that the singularity is eliminated by quantum corrections.

Some insight into the problem can be gained by considering a field theory analog. In particular \(D = 4, N = 2\) SU(2) Yang-Mills theory contains a conifold singularity in the vector multiplet moduli space which is resolved by light monopoles. After compactification to \(D = 3\) on a circle, the monopoles become infinitely massive (due to long-range fields) and leave the Hilbert space. Therefore they can no longer resolve the singularity. This singularity is actually in a hypermultiplet geometry, since in \(D = 3\) vector multiplets dualize to hypermultiplets. However, the \(D = 3\) theory contains instantons corresponding to \(D = 4\) monopoles orbiting the compactification circle. The instanton sum contains a logarithmic divergence which seems to cancel the logarithm of the four dimensional theory \[47,48\], although the details have not all been worked out. Thus in the field theory example it appears that quantum corrections resolve the singularity.

This analogy is in fact quite close due to the following observation \[49,50\]. Compactification of the IIA string theory on a product of a Calabi-Yau space \(\mathcal{X}\)
with a circle of radius $R$ is equivalent to compactification on the mirror $\tilde{X}$ of $X$ and a circle of radius $1/R$. In $D = 4$, exchanging $X$ with $\tilde{X}$ interchanges vector multiplets and hypermultiplets. Since hypermultiplet bosons are all scalars, compactification from $D = 4$ to $D = 3$ does not change the hypermultiplet geometry. The hypermultiplet geometry for the IIA theory on $X$ can therefore be found by compactifying the IIA theory on $\tilde{X}$ on a circle of radius $R$ and then taking $R$ to zero. Hence exactly the same mechanism which eliminates hypermultiplet singularities in the $D = 3$ field theory example should eliminate them in $D = 4$ type II string compactifications. This remains to be verified or understood in detail.

7. Fundamental $p$-Branes?

A striking feature of our calculation is the democratic fashion in which all the $p$-branes enter. At the non-perturbative level, the role played by strings does not seem special. Indeed, a glance at the brane-scan \cite{23,51} reveals that strings occupy a relatively undistinguished position, compared with the maximally symmetric positions occupied by the membrane and fivebrane. Recent work \cite{7} in which strings smoothly transform into twobranes or threebranes further threatens the privileged role of strings in string theory.

One might conclude from this that type II “string theory” is really a theory of “fundamental” membranes and/or fivebranes \cite{5,17,52}. However this is not our view. Rather we feel that string theory can not be fundamentally defined as a theory of extended objects of any kind – including strings. The various different extended objects have calculational utility in different regions of the moduli space. For example we have seen that membranes are useful for understanding aspects of strongly coupled, eleven-dimensional IIA theory \cite{9} while strings are not. What is special about strings is that string diagrams can be ordered by genus. This leads to a systematic perturbation expansion of e.g. graviton-graviton scattering in certain regions of the moduli space. It is doubtful that such calculation will be possible
in the near future using membranes or fivebranes. But this does not mean that strings are more “fundamental”, they are just more useful.

What then is “string theory”? We do not know, but in any case it does not seem to be fundamentally defined as a theory of strings.

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Appendix

Our notation and conventions are as follows:

- $\Gamma^N$ with $N = 1, \ldots, 11$ are the euclidean space-time gamma matrices and

  $$\Gamma_\alpha = \partial_{\alpha}X^M \Gamma_M \quad \text{with} \quad \alpha = 1, 2, 3$$

  are their projections onto the worldbrane. Using eq. (2.3) they satisfy

  $$\{\Gamma_\alpha, \Gamma_\beta\} = 2h_{\alpha\beta}$$

- We use the notation

  $$\Gamma^{M_1 \ldots M_n} = \Gamma_{[M_1 \Gamma_{M_2} \ldots \Gamma_{M_n}]},$$

  where the square bracket implies a sum over $n!$ terms with an $1/n!$ prefactor.

- Six-dimensional gamma matrices are chosen to satisfy $\gamma_m^* = -\gamma_m = \gamma_m^T$.

- $\epsilon_{\alpha\beta\gamma}$ transforms as a tensor under membrane diffeomorphisms.

- For a spinor $\psi$ we define $\bar{\psi} = (\psi^*)^T$ in euclidean space.

- $n$-forms are defined with a factor $1/n!$, so for example $J = \frac{1}{2} J_{MN} dX^M \wedge dX^N$.

- The following identity is useful to check the $\kappa$-symmetry of our membrane vertex operator

  $$\Gamma_\alpha P_\pm = \pm \frac{i}{2} \epsilon_{\alpha\beta\gamma} \Gamma_\beta \Gamma_\gamma P_\pm.$$ 

- To prove eq. (2.48) the following identities are useful

  $$\bar{\eta}_+ \gamma^{rs} \gamma_{pq} \eta_+ = \bar{J}_p \gamma_s J_q,$$

  $$\bar{\eta}_+ \gamma^{rs} \gamma_{pq} \eta_+ = 8J^r_p J^s_q.$$

  where $\eta_+$ and $\eta_-$ are two covariantly constant six-dimensional spinors with opposite chirality and unit norm.
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