Integrated solutions of non-densely defined semilinear integro-differential inclusions: existence, topology and applications

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Abstract. Given a linear closed but not necessarily densely defined operator $A$ on a Banach space $E$ with nonempty resolvent set and a multivalued map $F: I \times E \to E$ with weakly sequentially closed graph, we consider the integro-differential inclusion

\[
\dot{u} \in Au + F\left(t, \int_0^t u(s) \, ds \right) \quad \text{on } I, \quad u(0) = x_0.
\]

We focus on the case when $A$ generates an integrated semigroup and obtain existence of integrated solutions if $E$ is weakly compactly generated and $F$ satisfies

\[\beta(F(t, \Omega)) \leq \eta(t)\beta(\Omega) \quad \text{for all bounded } \Omega \subset E,\]

where $\eta \in L^1(I)$ and $\beta$ denotes the De Blasi measure of noncompactness. When $E$ is separable, we are able to show that the set of all integrated solutions is a compact $R_\delta$-subset of the space $C(I, E)$ endowed with the weak topology. We use this result to investigate a nonlocal Cauchy problem described by means of a nonconvex-valued boundary condition operator. We also include some applications to partial differential equations with multivalued terms are.

Bibliography: 26 titles.

Keywords: convergence theorem, De Blasi measure of noncompactness, integrated semigroup, integrated solution, $R_\delta$-set, semilinear integro-differential inclusion.

§ 1. Introduction and notation

The aim of this paper is to study the following integro-differential inclusion in the Banach space $E$:

\[
\begin{aligned}
\dot{u}(t) &\in Au(t) + F\left(t, \int_0^t u(s) \, ds \right) \quad \text{on } I := [0, T], \\
u(0) &= x_0,
\end{aligned}
\]

where $A: D(A) \subset E \to E$ is a closed linear operator, $F: I \times E \to E$ is a multivalued perturbation and $x_0 \in E$ is given.
The generic type of the semilinear differential inclusion, given by
\[
\begin{aligned}
\dot{u}(t) &\in Au(t) + F(t, u(t)) \quad \text{on } I, \\
u(0) &= x_0,
\end{aligned}
\] (2)
where \(A\) is the infinitesimal generator of a 0-times integrated semigroup has been thoroughly examined in the literature. The theoretical methods which apply to semilinear differential inclusions of the type (2) and methods using measures of noncompactness can be found in the monograph [13].

As we know, the domain \(D(A)\) of a strongly continuous semigroup generator must be dense in \(E\). However, a concept introduced in the 1980s by Arendt [1] allows us to extend the theory to the case of abstract Cauchy problems with operators that do not satisfy the Hille-Yosida conditions. The main idea behind this notion can be summarized as follows. Let \(\{U_t\}_{t \geq 0}\) be a \(C_0\)-semigroup on \(E\). Then \(S(t) := \int_0^t U(s) \, ds\) defines a family \(\{S(t)\}_{t \geq 0}\) of bounded operators having the following three properties:

(i) \(S(0) = 0\);
(ii) \(t \mapsto S(t)\) is strongly continuous;
(iii) \(S(s)S(t) = \int_0^s (S(r + t) - S(t)) \, dr\).

We call an integrated semigroup an operator family satisfying (i)–(iii) (for more information about these ideas, please refer to [14], [17] and [23]). The generator \(A\) of an integrated semigroup \(\{S(t)\}_{t \geq 0}\) gives an example of such a linear operator that does not meet the Hille-Yosida conditions.

In order to find a solution \(u \in C(I, D(A))\), which is differentiable and satisfies
\[
\begin{aligned}
\dot{u}(t) &= Au(t) + f(t) \quad \text{on } I, \\
u(0) &= x_0,
\end{aligned}
\] (3)
we usually have to impose a lot of smoothness both on \(x_0\) (\(x_0 \in D(A), x_0 \in D(A^2)\)) and on \(f\), either in the form of temporal regularity (that is, \(f \in W^{1,p}(I, E)\)) or spatial regularity (that is, \(f(t)\) is assumed to belong to \(D(A)\) almost everywhere on \(I\)). Without these additional regularity assumptions problem (3) has to be viewed in a generalized sense, suggested by the formal integration of both sides of (3). In this case we are dealing with integral solutions in the sense of Da Prato and Sinestrari (see [5]):
\[
u(t) = x_0 + A \int_0^t u(s) \, ds + \int_0^t f(s) \, ds, \quad t \in I;
\]
which means, in particular, that \(\int_0^t u(s) \, ds \in D(A)\).

The authors of [18] obtained solutions of the initial value problem (2) in the latter sense under the following assumptions:

(i) \(A\) is the generator of a locally Lipschitz continuous nondegenerate exponentially bounded integrated semigroup \(\{S(t)\}_{t \geq 0}\), whose derivative \(\{S'(t)\}_{t \geq 0}\) forms an equicontinuous semigroup;
(ii) the set-valued perturbation term $F$ is a convex compact valued upper-Carathéodory multimap that has the usual sublinear growth and is condensing with respect to the Hausdorff measure of noncompactness.

It is easy to deduce that if an integral solution of (3) exists then necessarily $x_0 \in \overline{D}(A)$. If we want to relax the smoothness condition on $x_0$ even more, we can integrate (3) twice. The latter approach motivates the following definition (cf. [23], Definition 6.4).

**Definition.** An integrated solution of problem (1) is a continuous function $u: I \to E$ such that

\[
\begin{cases}
\int_0^t u(s) \, ds \in D(A), \\
u(t) \in tx_0 + A \int_0^t u(s) \, ds + \int_0^t (t-s)F(s,u(s)) \, ds \quad \text{for } t \in I,
\end{cases}
\]

where the last integral on the right is understood in the sense of Aumann.

The main results in our paper are theorems regarding the existence of integrated solutions of problem (1) and a topological characterization of the solution set, in the situation where the operator $A$ is a generator of a nondegenerate exponentially bounded integrated semigroup and the multivalued perturbation term has weakly sequentially closed graph. To avoid compactness assumptions, the weak topology and the notion of the De Blasi measure of noncompactness are employed. Exploiting Theorem 2.8 from [16], on the behaviour of the measure of noncompactness $\beta$ with respect to integration, has allowed us to formulate the results also in the context of non-reflexive Banach spaces.

In §2 we present some important generalizations, from a technical point of view, of the result known in the literature as the convergence theorem. Section 3 contains the main results of the paper mentioned above (Theorem 5 and Theorem 7). Consequences of the geometric structure of the set of integrated solutions to the Cauchy problem (1) described previously have been collected in §4 in the form of theorems and examples illustrating the use of Theorem 7.

Now we introduce some notation which will be used in this paper.

Let $(E,|\cdot|)$ be a Banach space, $E^*$ its normed dual and $\sigma(E,E^*)$ its weak topology. If $M$ is a subset of a Banach space $E$, we let $(M,w)$ denote the topological space $M$ furnished with the relative weak topology of $E$.

The normed space of bounded linear operators $S: E \to E$ is denoted by $\mathcal{L}(E)$. Given $S \in \mathcal{L}(E)$, $\|S\|_{\mathcal{L}}$ is the norm of $S$. For any $\varepsilon > 0$ and $A \subset E$, $B(A,\varepsilon)$ ($D(A,\varepsilon)$) stands for an open (closed) $\varepsilon$-neighbourhood of the set $A$ (also $D_C(0,R)$ represents the ball in the space of continuous functions). If $x \in E$ we put $\text{dist}(x,A) := \inf\{ |x - y| : y \in A \}$. In addition, for two nonempty closed bounded subsets $A$ and $B$ of $E$, the symbol $h(A,B)$ stands for the Hausdorff distance from $A$ to $B$, that is, $h(A,B) := \max\{ \sup\{ \text{dist}(x,B) : x \in A \}, \sup\{ \text{dist}(y,A) : y \in B \} \}$.

We use the notation for functional spaces, such as $C(I,E)$, $L^p(I,E)$, $L^\infty(I,E)$, $H^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, in the commonly accepted meaning. The symbols $\|\cdot\|$ and $\|\cdot\|_p$ represent norms in the space $C(I,E)$ and $L^p(I,E)$, respectively.

Given a metric space $X$, a set-valued map $F: X \to E$ assigns to any $x \in X$ a nonempty subset $F(x) \subset E$; $F$ is (weakly) upper semicontinuous if the small
inverse image $F^{-1}(A) = \{x \in X : F(x) \subset A\}$ is open in $X$ whenever $A$ is (weakly) open in $E$. We say that $F : X \to E$ is upper hemicontinuous if for each $x^* \in E^*$, the function $\sigma(x^*, F(\cdot)) : X \to \mathbb{R} \cup \{+\infty\}$ is upper semicontinuous (as an extended real function), where $\sigma(x^*, F(x)) = \sup_{y \in F(x)} \langle x^*, y \rangle$.

We have the following characterization: a map $F : X \to E$ with convex values is weakly upper semicontinuous and has weakly compact values if and only if, given a sequence $(x_n, y_n)$ in the graph $Gr(F)$ of the map $F$ with $x_n \xrightarrow{n \to \infty} x$, there is a subsequence $y_{k_n} \xrightarrow{n \to \infty} y \in F(x)$ ($\xrightarrow{}$ denotes weak convergence).

Let $H^*(\cdot)$ denote the Alexander-Spanier cohomology functor with coefficients in the field of rational numbers $\mathbb{Q}$ (see [22]). We say that a nonempty topological space $X$ is acyclic if the reduced cohomology $\tilde{H}^q(X)$ is 0 for any $q \geq 0$. A nonempty compact metric space $X$ is an $R_\delta$-set if it is the intersection of a decreasing sequence of compact contractible metric spaces. In particular, $R_\delta$-sets are acyclic.

An upper semicontinuous map $F : E \to E$ is called acyclic if it has compact acyclic values. A set-valued map $F : E \to E$ is admissible (in the sense of Definition 40.1 in [10]) if there is a Hausdorff topological space $\Gamma$ and two continuous functions $p, q : \Gamma \to E$ such that $F(x) = q(p^{-1}(x))$ for every $x \in E$, where $p$ is a surjective perfect map with acyclic fibres. Clearly, every acyclic map is admissible. Moreover, the composition of admissible maps is admissible (see [10], Theorem 40.6).

A real function $\beta$ defined on the family of bounded subsets $\Omega$ of $E$ defined by the formula

$$\beta(\Omega) := \inf\{\varepsilon > 0 : \Omega \text{ has a weakly compact } \varepsilon\text{-net in } E\},$$

is called the De Blasi measure of noncompactness. Recall that $\beta$ is a measure of noncompactness in the sense of general definition, provided $E$ is endowed with the weak topology. It is easy to see that the measure of noncompactness $\beta$ is regular, monotone, nonsingular, semi-additive, algebraically semi-additive and invariant under translation (see [6]).

We recall the following results for the reader on account of their practical importance. The first is a weak compactness criterion in $L^p(\Omega, E)$; it originates from [24].

**Theorem 1** (see [24], Corollary 9). Let $(\Omega, \Sigma, \mu)$ be a finite measure space with $\mu$ a nonatomic measure on $\Sigma$. Let $A$ be a uniformly $p$-integrable subset of $L^p(\Omega, E)$ with $p \in [1, \infty)$. Assume that, for almost all $\omega \in \Omega$, the set $\{f(\omega) : f \in A\}$ is relatively weakly compact in $E$. Then $A$ is relatively weakly compact.

The next theorems are two well-known results in topological fixed point theory, which are essential to our proofs.

**Theorem 2** (see [8], Theorem 7.4). Let $X$ be an absolute extensor for the class of compact metrizable spaces and let $F : X \to X$ be an admissible map such that $F(X)$ is contained in a compact metrizable subset of $X$. Then $F$ has a fixed point.

**Theorem 3** (see [12], Theorem 5.2.18). If $M$ is a nonempty compact convex subset of a locally convex space $E$ and $F : M \to M$ is a convex compact-valued upper semicontinuous set-valued map, then $F$ has a fixed point.
§ 2. The convergence theorem

In the case of upper hemicontinuous maps the following theorem is an analogue of a relation binding upper semicontinuous set-valued maps and semi limits (cf. [3], Proposition 1.4.7).

**Theorem 4.** Let $F: E \rightarrow E$ be a closed convex valued upper hemicontinuous multimap. Then

$$ y \in \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{x' \in B(x, \delta)} B(F(x'), \varepsilon) \iff (x, y) \in \text{Gr}(F). \quad (5) $$

**Proof.** The ‘only if’ part is basically obvious. It follows from the fact that $(x, y) \in \text{Gr}(F)$ if and only if $y \in \limsup_{x' \rightarrow x} F(x')$, where the latter is the upper limit in the sense of Painlevé-Kuratowski. This limit is evidently contained in the intersection

$$ \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \overline{\text{co}} \bigcup_{x' \in B(x, \delta)} B(F(x'), \varepsilon). $$

Fix $y \in \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \overline{\text{co}} \bigcup_{x' \in B(x, \delta)} B(F(x'), \varepsilon)$. Let $x^* \in E^*$. By the definition of upper hemicontinuity

$$ \forall \varepsilon > 0 \ \exists \delta > 0 \ \sigma(x^*, F(B(x, \delta))) < \sigma(x^*, F(x)) + \varepsilon. $$

Thus,

$$ \inf_{\varepsilon > 0} \inf_{\delta > 0} \sigma(x^*, F(B(x, \delta))) \leq \inf_{\varepsilon > 0} (\sigma(x^*, F(x)) + \varepsilon). $$

The latter property implies that

$$ \langle x^*, y \rangle \leq \sigma \left( x^*, \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{x' \in B(x, \delta)} B(F(x'), \varepsilon) \right) \leq \inf_{\varepsilon > 0} \inf_{\delta > 0} \sigma(x^*, \overline{\text{co}} F(B(x, \delta))) $$

$$ = \inf_{\varepsilon > 0} \inf_{\delta > 0} \sigma(x^*, F(B(x, \delta))) \leq \inf_{\varepsilon > 0} (\sigma(x^*, F(x)) + \varepsilon) $$

$$ \leq \sigma(x^*, F(x)) + |x^*| \inf_{\varepsilon > 0} \varepsilon = \sigma(x^*, F(x)). $$

Since $F$ has closed convex values, it means that $y \in F(x)$, that is, $(x, y) \in \text{Gr}(F)$.

The theorem is proved.

Let $(I, \mathcal{L}(I), \ell)$ denote the Lebesgue measure space. The following property of upper hemicontinuous multimaps with closed convex values is a key tool, though strictly technical, used in the proofs of results regarding differential inclusions.

**Corollary 1** (Pliš Convergence Theorem). Let $F: E \rightarrow E$ be a closed convex-valued upper hemicontinuous multimap. Assume that functions $f_n, f: I \rightarrow E$ and $g_n, g: I \rightarrow E$ are such that

$$ g_n(t) \xrightarrow{E_{n \rightarrow \infty}} g(t) \quad \text{a.e. on } I \quad (6) $$

and

$$ f_n(t) \in \overline{\text{co}} B(F(B(g_n(t), \varepsilon_n)), \varepsilon_n) \quad \text{a.e. on } I, \quad \text{where } \varepsilon_n \rightarrow 0^+ \text{ as } n \rightarrow \infty. \quad (7) $$

If one of the following conditions holds

(i) \( f_n \xrightarrow[n \to \infty]{L^1(I,E)} f \);

(ii) \( f_n \) and \( f \) are weakly \( \ell \)-measurable and

\[
\forall J \in \mathcal{L}(I) \quad (D) \int_J f_n \, d\ell \xrightarrow[n \to \infty]{} (D) \int_J f \, d\ell,
\]

where \((D) \int_J f \, d\ell\) is the Dunford integral of \( f \) over \( J \);

(iii) \( f_n(t) \xrightarrow[n \to \infty]{E} f(t) \) almost everywhere on \( I \);

(iv) \( f(t) \in \overline{w\text{-}\limsup}_{n \to \infty} \{f_n(t)\} \) almost everywhere on \( I \), where

\[
\overline{w\text{-}\limsup}_{n \to \infty} A_n := \left\{ x \in E : x = w\text{-}\lim_{n \to \infty} x_n, \ x_n \in A_n, \ k_1 < k_2 < \cdots \right\},
\]

\[
\{A_n\}_{n=1}^{\infty} \subset 2^E \setminus \emptyset;
\]

(v) \( f(t) \in \bigcap_{n=1}^{\infty} \overline{\cup_{n=1}^{\infty} \{f_n(t)\}} \) almost everywhere on \( I \);
then \( f(t) \in F(g(t)) \) almost everywhere on \( I \).

Proof. The statement can be deduced directly from assumption (i), as has been done many times in the past (cf. the classic reference [2]). The implication between assumption (i) and (ii) can be easily justified under the additional assumption that \( E^* \) has the Radon-Nikodym property. The convergence \( f_n \xrightarrow[n \to \infty]{L^1(I,E)} f \) means that for every \( g \in L^\infty(I,E^*) \),

\[
\int_I \langle g(t), f_n(t) \rangle \, dt \xrightarrow[n \to \infty]{} \int_I \langle g(t), f(t) \rangle \, dt.
\]

For every \( J \in \mathcal{L}(I) \) and \( x^* \in E^* \) define \( g := x^*1_J \in L^\infty(I,E^*) \). Then

\[
\left\langle x^*, (D) \int_J f_n \, d\ell \right\rangle \xrightarrow[n \to \infty]{} \left\langle x^*, (D) \int_J f \, d\ell \right\rangle.
\]

Consequently,

\[
(D) \int_J f_n \, d\ell \xrightarrow[n \to \infty]{} (D) \int_J f \, d\ell.
\]

Condition (iii) implies condition (v). Assume that there exist \( n_0 \in \mathbb{N}, x^*_0 \in E^* \) and a subset \( J \in \mathcal{L}(I) \) such that \( \ell(J) > 0 \) and

\[
\langle x^*_0, f(t) \rangle > \sup_{m \geq n_0} \langle x^*_0, f_m(t) \rangle
\]

for every \( t \in J \). The set \( J \) has the form of a countable union of sets

\[
J_k := \left\{ t \in J : \langle x^*_0, f(t) \rangle > \sup_{m \geq n_0} \langle x^*_0, f_m(t) \rangle + \frac{1}{k} \right\}.
\]

The sets \( J_k \) are clearly measurable, since the function

\[
I \ni t \mapsto \langle x^*_0, f(t) \rangle - \sup_{m \geq n_0} \langle x^*_0, f_m(t) \rangle - \frac{1}{k} \in \mathbb{R}
\]

is \( \ell \)-measurable. Moreover, there must be a set \( J_{k_0} \) such that \( \ell(J_{k_0}) > 0 \). Now, observe that

\[
\langle x^*_0, (D) \int_{J_{k_0}} f \, d\ell \rangle = \int_{J_{k_0}} \langle x^*_0, f(t) \rangle \, dt > \int_{J_{k_0}} \langle x^*_0, f_m(t) \rangle \, dt + \frac{\ell(J_{k_0})}{k_0}
\]

\[
= \langle x^*_0, (D) \int_{J_{k_0}} f_m \, d\ell \rangle + \frac{\ell(J_{k_0})}{k_0}
\]
for every $m \geq n_0$. In view of (ii) we have
\[
\left\langle x^*_0, (D) \int_{J_{k_0}} f \, dl \right\rangle \geq \left\langle x^*_0, (D) \int_{J_{k_0}} f \, dl \right\rangle + \frac{\ell(J_{k_0})}{k_0},
\]
which is a contradiction. Thus,
\[
\forall n \geq 1 \quad \forall x^* \in E^* \quad \langle x^*, f(t) \rangle \leq \sup_{m \geq n} \langle x^*, f_m(t) \rangle \quad \text{a.e. on } I,
\]
that is, $f(t) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{m=n}^{\infty} \{f_m(t)\}$ almost everywhere on $I$.

Of course, (iii) implies (iv) and (iv) implies (v).

Fix $t \in I$ such that (6), (7) and (v) are satisfied simultaneously. Take $\varepsilon > 0$ and $\delta > 0$. In view of (6) there is $n \in \mathbb{N}$ such that $B(g_m(t), \varepsilon_m) \subset B(g(t), \delta)$ and $\varepsilon_m < \varepsilon$ for $m \geq n$. From (7) it follows that
\[
\overline{\text{co}} \bigcup_{m=n}^{\infty} \{f_m(t)\} \subset \overline{\text{co}} \bigcup_{m=n}^{\infty} \overline{\text{co}} B(B(g_m(t), \varepsilon_m)), \varepsilon_m)
\]
\[
\subset \overline{\text{co}} \bigcup_{m=n}^{\infty} B(B(g_m(t), \varepsilon_m)), \varepsilon_m) \subset \overline{\text{co}} B(B(g(t), \delta)), \varepsilon).
\]
Hence
\[
f(t) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{m=n}^{\infty} \{f_m(t)\} \subset \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{x \in B(g(t), \delta)} B(F(x), \varepsilon).
\]
Applying Theorem 4 we see that $f(t) \in F(g(t))$. Corollary 1 is proved.

**Corollary 2.** Let $F: E \to E$ be a closed convex-valued multimap satisfying:
\[
x_n \xrightarrow{E} x \quad \Rightarrow \quad \lim_{n \to \infty} \sigma(x^*, F(x_n)) \leq \sigma(x^*, F(x)) \quad \text{for all } x^* \in E^*. \quad (8)
\]
Assume that functions $f_n, f: I \to E$ and $g_n, g: I \to E$ are such that
\[
g_n(t) \xrightarrow{E} g(t) \quad \text{a.e. on } I \quad (9)
\]
and
\[
f_n(t) \in \overline{\text{co}} B(F(g_n(t)), \varepsilon_n) \quad \text{a.e. on } I, \quad \text{where } \varepsilon_n \to 0^+ \text{ as } n \to \infty. \quad (10)
\]
If the following condition holds:
\[
f(t) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{m=n}^{\infty} \{f_m(t)\} \quad \text{a.e. on } I, \quad (11)
\]
then $f(t) \in F(g(t))$ almost everywhere on $I$. 

Proof. Let $x_n \xrightarrow{n \to \infty} x$. Then $x_n \xrightarrow{E, n \geq N} x$ and

$$\lim \sup_{n \geq N} \sigma(x^*, F(x_n)) \leq \sigma(x^*, F(x))$$

for every $x^* \in E^*$ and $N \geq 1$, in view of (8). Therefore,

$$\forall x^* \in E^* \quad \sup_{N \geq 1} \inf_{n \geq N} \sup_{m \geq n} \sigma(x^*, F(x_m)) \leq \sigma(x^*, F(x)). \quad (12)$$

Take $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $\varepsilon_m < \varepsilon$ for $m \geq N$. From (10) it follows that

$$\overline{co} \bigcup_{m=N}^{\infty} \{f_m(t)\} \subset \overline{co} \bigcup_{m=N}^{\infty} \overline{co} B(F(g_m(t)), \varepsilon_m) \subset \overline{co} \bigcup_{m=N}^{\infty} B(F(g_m(t)), \varepsilon).$$

Hence

$$f(t) \in \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \bigcup_{\varepsilon > 0} \overline{co} \bigcup_{n \geq N} \bigcup_{m=n}^{\infty} B(F(g_m(t)), \varepsilon)$$

and eventually

$$f(t) \in \bigcap_{\varepsilon > 0} \bigcap_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{m=n}^{\infty} B(F(g_m(t)), \varepsilon) \quad \text{a.e. on } I.$$

Take $x^* \in E^*$. From (9) and (12) it follows that

$$\langle x^*, f(t) \rangle \leq \sigma \left( x^*, \bigcap_{\varepsilon > 0} \bigcap_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{m=n}^{\infty} B(F(g_m(t)), \varepsilon) \right)$$

$$\quad \leq \inf_{\varepsilon > 0} \sup_{N \geq 1} \inf_{n \geq N} \sup_{m \geq n} \sigma(x^*, B(F(g_m(t)), \varepsilon))$$

$$\quad \leq \sup_{N \geq 1} \inf_{n \geq N} \sup_{m \geq n} \sigma(x^*, F(g_m(t))) + \inf_{\varepsilon > 0} \varepsilon |x^*| \leq \sigma(x^*, F(g(t))).$$

Consequently, $f(t) \in F(g(t))$ almost everywhere on $I$.

The corollary is proved.

§ 3. The existence and the topology of solutions

The remainder of the article rests on the following hypotheses:

(A1) $A$: $D(A) \to E$ is a generator of a nondegenerate integrated semigroup \{S(t)\}$_{t \geq 0}$ such that $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$ with suitable constants $M > 0$ and $\omega \in \mathbb{R}$;

(A2) $A$: $D(A) \to E$ satisfies (A1) and the generated semigroup \{S(t)\}$_{t \geq 0}$ is equicontinuous;

(F1) for every $(t, x) \in I \times E$ the set $F(t, x)$ is nonempty and convex;

(F2) the map $F(\cdot, x)$ has a strongly measurable selection for every $x \in E$;

(F3) the graph $\text{Gr}(F(t, \cdot))$ is sequentially closed in $(E, w) \times (E, w)$ for almost all $t \in I$;
(F_4) $F$ satisfies the following growth condition:

$$\limsup_{r \to +\infty} r^{-1} \int_I \sup_{|x| \leq r} \|F(t, x)\| \, dt < M^{-1} e^{-\omega T},$$

where $M$ and $\omega$ are exactly the same constants as in (A_1);

(F_5) there is a function $\eta \in L^1(I, \mathbb{R})$ such that for all bounded $\Omega$ in $E$ and for almost all $t \in I$ the following inequality holds:

$$\beta(F(t, \Omega)) \leq \eta(t) \beta(\Omega).$$

Remark 1. A linear operator $A$ is called a generator of an integrated semigroup if

there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and there exists a strongly continuous exponentially bounded family $\{S(t)\}_{t \geq 0}$ of bounded operators such that $S(0) = 0$ and $(\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t) \, dt$ for $\lambda > \omega$. An integrated semigroup $\{S(t)\}_{t \geq 0}$ is called nondegenerate if $\bigcap_{t \geq 0} \ker S(t) = \{0\}$.

Remark 2. By condition (F_4) we mean implicitly that the map $I \ni t \mapsto \sup_{|x| \leq r} \|F(t, x)\| \in \mathbb{R}_+$ is bounded for every $r > 0$. The upper integral of a bounded (but not necessarily measurable) function $f: I \to \mathbb{R}_+$ is

$$\bar{\int}_I f(t) \, dt := \inf \left\{ \int_I g(t) \, dt : g \in L^1(I), f(t) \leq g(t) \text{ a.e. on } I \right\}.$$

Remark 3. If the integrated semigroup $\{S(t)\}_{t \geq 0}$ is exponentially stable in the sense that $\|S(t)\| \leq e^{-\omega t}$ for $t \geq 0$ with $\omega > 0$, then assumption (F_4) takes the form

$$\limsup_{r \to +\infty} r^{-1} \int_I \sup_{|x| \leq r} \|F(t, x)\| \, dt < 1.$$

Let $N_F: C(I, E) \to L^1(I, E)$ be the Nemytskii operator corresponding to $F$, that is,

$$N_F(u) := \{ w \in L^1(I, E) : w(t) \in F(t, u(t)) \text{ for a.a. } t \in I \}.$$

Remark 4. Under hypotheses (F_1)–(F_5) the Nemytskii operator $N_F$ is a nonempty weakly upper semicontinuous set-valued map that is convex and weakly compact-valued (cf., for instance, [21], Proposition 1).

We also define the Volterra integral operator $V: L^1(I, E) \to C(I, E)$ by the formula

$$V(f)(t) := \int_0^t S(t - s) f(s) \, ds \quad \text{for } t \in I. \quad (13)$$

Lemma 1. Assume (A_1) holds. Then the integral operator $V: L^1(I, E) \to C(I, E)$ defined by (13) is a bounded linear monomorphism with $\|V\|_{\mathcal{L}(L^1, C)} \leq Me^{\omega T}$. 
Proof. Apply Theorem 6.5 from [23] and the very definition of an integrated solution to the inhomogeneous Cauchy problem (3). The estimate of the norm $\|V\|_{\mathcal{L}(L^1,C)}$ follows straightforwardly. The lemma is proved.

It is important from a methodological point of view to realize that the solution set $S_F(x_0)$ of all integrated solutions to problem (1) coincides with the fixed point set $\text{Fix}(H)$ of the operator $H : C(I,E) \to C(I,E)$ defined by $H := S(\cdot)x_0 + V \circ N_F$. Indeed, if $u \in \text{Fix}(H)$ then $u = S(\cdot)x_0 + V(f)$ for some $f \in N_F(u)$. Thanks to Theorem 6.5 in [23] we know that $u$ belongs to $S_F(x_0)$. Suppose then that $u \in S_F(x_0)$. This means that

$$u(t) = tx_0 + A \int_0^t u(s) \, ds + \int_0^t (t-s)f(s) \, ds$$

for some $f \in N_F(u)$. The inhomogeneous Cauchy problem (3) has a unique integrated solution $x$ given by the formula: $x = S(\cdot)x_0 + V(f)$. This again follows from Theorem 6.5 in [23]. Since $u$ is also a solution to (3), it means that $u = S(\cdot)x_0 + V(f)$, that is, $u \in \text{Fix}(H)$.

Recall that the space $E$ is called \textit{weakly compactly generated} (WCG) if there is a weakly compact set $K$ in $E$ such that $E = \overline{\text{span}}(K)$.

\textbf{Lemma 2.} Let $E$ be a WCG space. Assume (A2), (F1) and (F3)–(F5). Then the solution set $S_F(x_0)$ is weakly compact in $C(I,E)$.

\textbf{Proof.} We claim that there are \textit{a priori} bounds for $S_F(x_0)$. Indeed, assume that for every $n \geq 1$ there exists $x_n \in S_F(x_0)$ such that $|x_n| > n$. Let $x_n = S(\cdot)x_0 + V(f_n)$ for some $f_n \in N_F(x_n)$. By (F4) we have

$$1 \leq \lim_{n \to \infty} \frac{\|x_n\|}{n} \leq \lim_{n \to \infty} \frac{\|S(\cdot)x_0 + V(f_n)\|}{n} \leq \lim_{n \to \infty} \frac{Me^{\omega T}|x_0| + \|V\|_{\mathcal{L}(L^1,C)}\|f_n\|}{n}$$

$$\leq \lim_{n \to \infty} \frac{Me^{\omega T}|x_0|}{\|x_n\|} + Me^{\omega T} \lim_{n \to \infty} \frac{\sup_{|x| \leq \|x_n\|} \|F(t,x)\|^{+}}{\|x_n\|} < 1.$$ 

Hence the claim is validated.

The solution set $S_F(x_0)$ is strongly equicontinuous in $C(I,E)$. Indeed, take an arbitrary $u \in S_F(x_0)$. Then $u = S(\cdot)x_0 + V(f)$ for some $f \in N_F(u)$. Let $g \in L^1(I)$ be such that $\sup_{|x| \leq \|S_F(x_0)\|} \|F(t,x)\|^{+} \leq g(t)$ almost everywhere on $I$. As we can see,

$$|u(t) - u(\tau)| \leq |S(t)x_0 - S(\tau)x_0| + \int_0^t S(t-s)f(s) \, ds - \int_0^\tau S(\tau-s)f(s) \, ds$$

$$\leq |S(t)x_0 - S(\tau)x_0| + \int_0^\tau \|S(t-s) - S(\tau-s)\| \mathcal{L}g(s) \, ds$$

$$+ Me^{\omega T} \int_\tau^t g(s) \, ds. \quad (14)$$

From Lebesgue’s dominated convergence theorem and assumption (A2) it follows that

$$\lim_{t \to \tau} \sup_{u \in S_F(x_0)} |u(t) - u(\tau)| = 0.$$
Now choose an arbitrary \((x_n)_{n=1}^\infty \subset S_F(x_0)\). Let \(f_n \in N_F(x_n)\) be such that \(x_n = S(\cdot)x_0 + V(f_n)\). It is easy to see that the strong equicontinuity of \(\{x_n\}_{n=1}^\infty\) implies the continuity of the function \(I \ni t \mapsto \beta((x_n(t))_{n=1}^\infty) \in \mathbb{R}_+\) (cf. [15]). From Theorem 2.8. in [16] it follows that

\[
\beta((x_n(t))_{n=1}^\infty) = \beta(\{S(t)x_0 + V(f_n(t))\}_{n=1}^\infty) \leq \beta\left(\left\{\int_0^t S(t-s)f_n(s)\,ds\right\}_{n=1}^\infty\right)
\]

\[
\leq \int_0^t \|S(t-s)\| \beta(\{f_n(s)\}_{n=1}^\infty)\,ds \leq Me^{\omega T} \int_0^t \eta(s) \beta(\{x_n(s)\}_{n=1}^\infty)\,ds
\]

for \(t \in I\). By Gronwall’s inequality, \(\beta((x_n(t))_{n=1}^\infty) = 0\) for every \(t \in I\). In particular, \(\beta(\{f_n(t)\}_{n=1}^\infty) = 0\) almost everywhere on \(I\). The family \(\{f_n\}_{n=1}^\infty\) is uniformly integrable since

\[
\lim_{\ell(J) \to 0} \sup_{n \geq 1} \int_J |f_n(t)|\,dt \leq \lim_{\ell(J) \to 0} \int_J \sup_{|x| \leq \|S_F(x_0)\|^+} \|F(t, x)\|^+\,dt
\]

\[
\leq \lim_{\ell(J) \to 0} \int_J g(t)\,dt = 0
\]

for some \(g \in L^1(I)\). In view of Theorem 1 we can extract a subsequence, again denoted by \((f_n)_{n=1}^\infty\), such that \(f_n \xrightarrow{n \to \infty} f\).

Observe that conditions \((F_1)\) and \((F_5)\) together with the hypothesis regarding \(w\)-\(w\) sequential closedness of \(\text{Gr}(F(t, \cdot))\) imply (8). Put \(x := S(\cdot)x_0 + V(f) = \wlim_{n \to \infty} S(\cdot)x_0 + V(f_n)\). Then \(x_n \xrightarrow{C(I,E)} x\). In particular, \(x_n(t) \xrightarrow{n \to \infty} x(t)\) for each \(t \in I\). Therefore, assumptions (9), (10) and (11) of Corollary 2 are satisfied (the implication \(f_n \xrightarrow{n \to \infty} f \Rightarrow (11)\) was proved earlier). Consequently, \(f \in N_F(x)\) and \(x \in S(\cdot)x_0 + V \circ N_F(x)\), that is, \(x \in S_F(x_0)\).

The lemma is proved.

The main result regarding the existence of integrated solutions to the initial value problem (1) is contained in the following.

**Theorem 5.** Let \(E\) be a WCG space. Assume that hypotheses \((A_1)\) and \((F_1)-(F_5)\) are satisfied. Then the solution set \(S_F(x_0)\) of the Cauchy problem (1) is nonempty.

**Proof.** Assume that \(u_n \xrightarrow{C(I,E)} u\). Then, in particular, \(\sup_{t \in I} \beta(\{u_n(t)\}_{n=1}^\infty) = 0\). Let \(f_n \in N_F(u_n)\) for \(n \geq 1\). Observe that the family \(\{f_n\}_{n=1}^\infty\) is uniformly integrable since the set \(\{u_n\}_{n=1}^\infty\) is bounded. (The latter follows by analogy with (15).) Taking account of the fact that \(\beta(\{f_n(t)\}_{n=1}^\infty) \leq \eta(t) \beta(\{u_n(t)\}_{n=1}^\infty) = 0\) almost everywhere on \(I\), we infer that \(f_n \xrightarrow{n \to \infty} f\) by passing to a subsequence if necessary; this results from Theorem 1. Moreover, the assumptions of Corollary 2 are satisfied and the conclusion that \(f \in N_F(u)\) follows. Since \(S(\cdot)x_0 + V(f_n) \in H(u_n)\) and \(S(\cdot)x_0 + V(f_n) \xrightarrow{n \to \infty} S(\cdot)x_0 + V(f) \in H(u)\), we conclude that the restriction \(H: (X, w) \to (C(I,E), w)\) is a convex upper semicontinuous compact-valued map for every weakly compact set \(X \subset C(I,E)\).
It is easy to show that the operator $H$ has an invariant ball $D_C(0, R)$. Assume, on the contrary, that for any $n \geq 1$ there exist $\|u_n\| \leq n$ and $v_n \in H(u_n)$ such that $\|v_n\| > n$. Then

$$1 \leq \lim_{n \to \infty} \frac{\|v_n\|}{n} \leq \lim_{n \to \infty} \frac{\|S(\cdot)x_0 + V(N_F(u_n))\|}{n}$$

$$\leq \lim_{n \to \infty} \frac{Me^{\omega T}x_0 + \|V\|\|N_F(u_n)\|}{n}$$

$$\leq \lim_{n \to \infty} \frac{Me^{\omega T}x_0}{n} + M e^{\omega T} \lim_{n \to \infty} \frac{\sup_{|x| \leq n} \|F(t, x)\| + \|V\|L \|N_F(u_n)\|}{n}$$

$$< 1$$

by (F.4).

Assume that the radius $R > 0$ is such that $H(D_C(0, R)) \subset D_C(0, R)$. Fix $\hat{x} \in D_C(0, R)$ and define

$$\mathcal{A} := \{X \in 2^{D_C(0, R)} \setminus \{\emptyset\} : X \text{ is closed and convex, and } \overline{co}(\{\hat{x}\} \cup H(X)) \subset X\}.$$

Then the intersection $X_0 := \bigcap_{X \in \mathcal{A}} X$ is nonempty and

$$\overline{co}(\{\hat{x}\} \cup H(X_0)) \subset \bigcap_{X \in \mathcal{A}} \overline{co}(\{\hat{x}\} \cup H(X)) \subset X_0.$$

Since $\overline{co}(\{\hat{x}\} \cup H(X_0)) \in \mathcal{A}$, we have $X_0 \subset \overline{co}(\{\hat{x}\} \cup H(X_0))$. The equality $X_0 = \overline{co}(\{\hat{x}\} \cup H(X_0))$ follows from this.

We claim that $X_0$ is weakly compact in $C(I, E)$. Since

$$\varphi(L) := \sup_{t \in I} e^{-Lt} \int_0^t e^{Ls} \eta(s) ds \xrightarrow{L \to +\infty} 0,$$

we can always pick a constant $L_0 > 0$ so that $Me^{\omega T} \varphi(L_0) < 1$. Let $\beta_{L_0}$ be a set function defined on the family of all bounded subsets of $C(I, E)$, given by the formula:

$$\beta_{L_0}(X) := \max \left\{ \sup_{t \in I} e^{-L_0 t} \beta(D(t)) : D \subset X \text{ denumerable} \right\}.$$

Clearly, $\beta_{L_0}$ is a nonsingular measure of noncompactness on $C(I, E)$. Therefore, we can always choose a subset $\{u_n\}_{n=1}^{\infty} \subset H(X_0)$ in such a way that

$$\sup_{t \in I} e^{-L_0 t} \beta(\{u_n(t)\}_{n=1}^{\infty}) = \beta_{L_0}(H(X_0)) = \beta_{L_0}(X_0).$$
Let \( u_n = S(\cdot)x_0 + V(f_n) \) with \( f_n \in N_F(v_n) \) and \( v_n \in X_0 \) for \( n \geq 1 \). Making use of Theorem 2.8 in [16] we can derive the following estimate:

\[
\sup_{t \in I} e^{-L_0 t} \beta(\{u_n(t)\}_{n=1}^\infty) = \sup_{t \in I} e^{-L_0 t} \beta(\{S(t)x_0 + V(f_n(t))\}_{n=1}^\infty)
\]

\[
\leq \sup_{t \in I} e^{-L_0 t} \int_0^t \|S(t-s)\| \beta(\{f_n(s)\}_{n=1}^\infty) \, ds
\]

\[
\leq e^{-L_0 t} M e^{\omega T} \int_0^t \eta(s) \beta(\{v_n(s)\}_{n=1}^\infty) \, ds
\]

\[
\leq M e^{\omega T} \left( \sup_{t \in I} e^{-L_0 t} \int_0^t e^{L_0 s} \eta(s) \, ds \right) \sup_{t \in I} e^{-L_0 t} \beta(\{v_n(t)\}_{n=1}^\infty)
\]

\[
\leq M e^{\omega T} \varphi(L_0) \sup_{t \in I} e^{-L_0 t} \beta(\{u_n(t)\}_{n=1}^\infty).
\]

Hence, \( \Gamma := \sup_{t \in I} e^{-L_0 t} \beta(\{u_n(t)\}_{n=1}^\infty) = 0 \). Consider an arbitrary sequence \( (x_n)_{n=1}^\infty \subset H(X_0) \). For each \( n \geq 1 \), there is \( v_n \in X_0 \) and \( w_n \in N_F(v_n) \) such that \( x_n = S(\cdot)x_0 + V(w_n) \). Condition (F_5) implies

\[
e^{-L_0 t} \beta(\{w_n(t)\}_{n=1}^\infty) \leq e^{-L_0 t} \eta(t) \beta(\{v_n(t)\}_{n=1}^\infty)
\]

\[
\leq \eta(t) \sup_{t \in I} e^{-L_0 t} \beta(\{v_n(t)\}_{n=1}^\infty) \leq \eta(t) \Gamma.
\]

Thus, \( \beta(\{w_n(t)\}_{n=1}^\infty) = 0 \) almost everywhere on \( I \). At the same time \( \{w_n\}_{n=1}^\infty \) is uniformly integrable. Consequently, we can assume, passing to a subsequence if necessary, that \( w_n \xrightarrow{L^1(I,E)} w \). The latter implies that \( x_n = S(\cdot)x_0 + V(w_n) \xrightarrow{C(I,E)} S(\cdot)x_0 + V(w) \), that is, the set \( H(X_0) \) is relatively weakly compact (rwc). The weak compactness of \( \overline{\co(\{x\} \cup H(X_0))} \) follows by the Krein-Šmulian theorem. Therefore, \( X_0 \) is weakly compact.

Summing up, by virtue of Theorem 3 we deduce that the convex compact-valued upper semicontinuous set-valued map \( H: (X_0, w) \rightarrow (X_0, w) \) has at least one fixed point. This fixed point constitutes a solution to the Cauchy problem (1).

Theorem 5 is proved.

**Theorem 6** (Leray-Schauder continuation theorem for weak topologies). Let \( E \) be a weakly normal \( w^* \)-separable Banach space and let \( U \subset E \) be weakly open. Assume that \( F: (U^w, w) \rightarrow (E, w) \) is sequentially upper semicontinuous with weakly compact convex values and that there exists \( x_0 \in U \) such that the following conditions hold

\[ \Sigma := \{ x \in U^w : \exists \lambda \in [0, 1] \ x \in (1 - \lambda)x_0 + \lambda F(x) \} \text{ is rwc}, \quad (17) \]

\[ X \subset U^w, \quad X \subset \overline{\co(\{x_0\} \cup F(X))} \implies X \text{ is rwc}, \quad (18) \]

\[ \forall x \in U^w \setminus U, \quad \forall \lambda \in [0, 1] \ x \notin (1 - \lambda)x_0 + \lambda F(x). \quad (19) \]

Then \( \text{Fix}(F) \neq \emptyset \).

**Proof.** Since the proof is straightforward and differs only slightly from the reasoning contained in [19], Theorem 3.2, we only sketch it. Since \( E \) is weakly angelic and \( \Sigma \) is sequentially weakly closed (due to the regularity of \( F \)), the latter must be
closed in the weak topology of \( E \). Moreover, \( \Sigma \subset U \) thanks to (19), and \( (U^w, w) \) is a normal subspace of the weakly normal space \( E \). Let \( \theta: (U^w, w) \rightarrow [0, 1] \) be a Urysohn mapping joining the closed disjoint sets \( \Sigma \) and \( U^w \setminus U \), that is, \( \theta|_{\Sigma} \equiv 1 \) and \( \theta|_{U^w \setminus U} \equiv 0 \). Put \( D := \overline{co}(\{x_0\} \cup F(U^w)) \). Now define an auxiliary map \( \widehat{F}: D \rightarrow D \) by

\[
\widehat{F}(x) := \begin{cases} 
(1 - \theta(x))x_0 + \theta(x)F(x) & \text{for } x \in D \cap U, \\
\{x_0\} & \text{for } x \in D \setminus U.
\end{cases}
\]

Clearly, \( \widehat{F}: (D, w) \rightarrow (D, w) \) is sequentially upper semicontinuous and has weakly compact convex values. Consider \( X = \overline{co}(\{x_0\} \cup \widehat{F}(X)) \). Observe that

\[
X \cap U \subset \overline{co}(\{x_0\} \cup \widehat{F}(X)) = \overline{co}(\{x_0\} \cup F(X \cap U)).
\]

By (18), \( X \cap U \) is rwc. In view of the Krein-Šmulian theorem \( X \) is weakly compact. In other words, the map \( \widehat{F} \) satisfies the following Mönch type condition

\[
X \subset D, \quad X = \overline{co}(\{x_0\} \cup \widehat{F}(X)) \implies X \text{ is weakly compact.} \quad (20)
\]

Define

\[
\mathcal{A} := \{X \in 2^D \setminus \{\emptyset\}: X \text{ is closed convex and } \overline{co}(\{x_0\} \cup \widehat{F}(X)) \subset X\}.
\]

Then the intersection \( X_0 := \bigcap_{X \in \mathcal{A}} X \) is nonempty and \( X_0 = \overline{co}(\{x_0\} \cup \widehat{F}(X_0)) \).

By (20), \( X_0 \) is weakly compact. Furthermore, it is \( \widehat{F} \)-invariant. Observe that \( (X_0, w) \) is a compact metrizable space due to the \( w^* \)-separability of \( E \). Thus, the set-valued map \( \widehat{F}: (X_0, w) \rightarrow (X_0, w) \) is compact admissible. Since \( (X_0, w) \) is also an acyclic absolute extensor for the class of metrizable spaces (Dugundji’s theorem), the map \( \widehat{F} \) possesses a fixed point by virtue of Theorem 7.4 in [8]. Obviously, it is also a fixed point for \( F \).

Theorem 6 is proved.

**Corollary 3.** Let \( E \) be a separable Banach space. Assume that hypotheses \((A_1), (F_1)-(F_3)\) and \((F_5)\) are satisfied. Assume also that, instead of \((F_4)\), the following growth condition holds

\[
\|F(t, x)\| \leq \mu(t)(1 + |x|) \quad \text{a.e. on } I \quad \text{for every } x \in E \quad \text{with } \mu \in L^1(I). \quad (21)
\]

Then the solution set \( S_F(x_0) \) of the Cauchy problem (1) is nonempty.

**Proof.** Thanks to the exponential boundedness of the semigroup \( \{S(t)\}_{t \geq 0} \) and the growth condition (21) the set \( \Sigma := \{x \in C(I, E): \exists \lambda \in [0, 1] \; x \in \lambda H(x)\} \) is bounded (simply use Gronwall’s inequality). Moreover, almost analogous reasoning to that carried out in the proof of Theorem 5 (the paragraph in which we showed that \( X_0 \) is a weakly compact subset of \( C(I, E) \)) proves that \( \Sigma \) is rwc. Fix a radius \( R > 0 \) such that \( \Sigma \subset D_C(0, R) \) and \( \xi \in \mathcal{E} := C(I, E)^* \) such that \( \|\xi\|_{\mathcal{E}} \leq 1 \). Then \( \Sigma \subset \xi^{-1}([-R, R]) \). Define \( U := \xi^{-1}([-R - 1, R + 1]) \). Then \( U \) is a weakly open neighbourhood of zero such that \( \Sigma \cap U^w \setminus U = \emptyset \). In particular the Leray-Schauder
boundary condition (19) is satisfied. In the proof of Theorem 5 we have also shown that

\[ X \subset C(I, E) \text{ bounded and } X \subset \overline{cO}(\{0\} \cup H(X)) \implies X \text{ is rwc.} \]

Hence, the Mönch type condition (18) is also satisfied. Since \( H: (\overline{U^w}, w) \to (C(I, E), w) \) is a convex weakly compact-valued map that is sequentially upper semicontinuous and \( C(I, E) \) is separable, \( \text{Fix}(H) \) is nonempty in view of Theorem 6.

The corollary is proved.

The topology of integrated solutions expresses itself in the following structure theorem, formulated in the context of a separable Banach space \( E \).

**Theorem 7.** Let \( E \) be a separable Banach space. Assume that \( A: D(A) \to E \) satisfies \( (A_2) \) and \( F: I \times E \to E \) satisfies \( (F_1)-(F_5) \). Then the solution set of the Cauchy problem (1) is \( R_δ \).

**Remark 5.** As far as we know, the topological assumption about the separability of the space \( E \) is indispensable to our proof. Applying Theorem 2.8 in [16] requires us to assume that \( E \) is a weakly compactly generated (WCG) Banach space. On the other hand the metrization theorem (see [7], Proposition 3.107) holds for \( E^* \) being \( w^* \)-separable. However, in view of Theorem 13.3 in [7] a WCG space \( E \) with \( w^* \)-separable dual \( E^* \) must be compactly generated.

**Remark 6.** The assumption regarding the equicontinuity of the semigroup \( \{S(t)\}_{t \geq 0} \) is not in fact excessively restrictive. This is still a weaker requirement than assuming that \( A \) satisfies the Hille-Yosida condition, which characterizes generators of locally Lipschitz continuous integrated semigroups.

**Proof of Theorem 7.** If the Banach space \( E \) is separable, then the topological dual \( E^* \) furnished with the \( w^* \)-topology \( \sigma(E^*, E) \) is also separable. Suppose that \( \{x_n^*\}_{n=1}^\infty \) is a countable \( \sigma(E^*, E) \)-dense subset of the unit sphere in \( E^* \). Using this sequence, we can define a metric \( d \) on \( E \) in the following way:

\[ d(x, y) := \sum_{n=1}^{\infty} 2^{-n} |\langle x_n^*, x-y \rangle|. \]  

Clearly, this \( d \)-metric topology is weaker than the weak topology \( \sigma(E, E^*) \) on \( E \). Moreover, the \( d \)-metric topology and the weak topology coincide on the weakly compact subsets of \( E \) (cf. [7], Proposition 3.107).

We claim that there is a nonempty weakly compact convex set \( X \subset C(I, E) \) that has the following property:

\[ S(t)x_0 + \int_0^t S(t-s) \overline{cO} F(s, \overline{X(s)}) \, ds \subset X(t) \quad \text{for every } t \in I. \]  

Let \( X_0 = D_C(0, R) \) and \( X_n = \overline{Y_n} \), where \( R > 0 \) is such that \( \|S_F(x_0)\|^+ \leq R \). Put

\[ Y_n = \left\{ y \in C(I, E) : y(t) \in S(t)x_0 + \int_0^t S(t-s) \overline{cO} F(s, \overline{X_{n-1}(s)}) \, ds \text{ for } t \in I \right\}. \]
It is easy to see that the sets $X_n$ are well-defined, nonempty, bounded, convex and equicontinuous (equicontinuity follows by (14), the remaining properties are justified in [21], Theorem 6). Moreover, $S_F(x_0) \subset X := \bigcap_{n=0}^{\infty} X_n$.

Using the Castaing representation for the Hausdorff continuous multimap $t \mapsto X_n(t)$, we may write $X_n(t) = \{u_k(t)\}_{k=1}^{\infty}$ with $\{u_k(t)\}_{k=1}^{\infty} \subset Y_n(t)$. Bearing in mind that $\lim_{t \to \infty} \varphi(L) = 0$, where $\varphi$ is a mapping given by (16), we choose $L_0 > 0$ so that $M e^{\omega T} \varphi(L_0) < 1$. Let $u_k = S(\cdot) x_0 + V(f_k)$ for some $f_k \in N_{\varphi F}(X_{n-1}(\cdot))$. In view of Theorem 2.8 in [16], we have

$$\sup_{t \in I} e^{-L_0 t} \beta(X_n(t)) = \sup_{t \in I} e^{-L_0 t} \beta(\{u_k(t)\}_{k=1}^{\infty}) = \sup_{t \in I} e^{-L_0 t} \beta(\{S(t) x_0 + V(f_k)(t)\})$$

$$\leq \sup_{t \in I} e^{-L_0 t} M e^{\omega t} \int_0^t \beta(\{f_k(s)\}_{k=1}^{\infty}) ds \leq M e^{\omega t} \sup_{t \in I} e^{-L_0 t} \int_0^t \eta(s) \beta(X_{n-1}(s)) ds$$

$$\leq M e^{\omega t} \sup_{t \in I} e^{-L_0 t} \int_0^t e^{L_0 s} \eta(s) ds \sup_{t \in I} e^{-L_0 t} \beta(X_{n-1}(t))$$

$$= M e^{\omega t} \varphi(L_0) \sup_{t \in I} e^{-L_0 t} \beta(X_{n-1}(t)).$$

Clearly, $\sup_{t \in I} e^{-L_0 t} \beta(X_n(t)) \xrightarrow{n \to \infty} 0$, which means that $\sup_{t \in I} e^{-L_0 t} \beta(X(t)) = 0$.

Since $\beta(X(I)) = \beta(\bigcup_{t \in I} X(t)) = \sup_{t \in I} \beta(X(t))$ (by Lemma 2 in [15]), the topological subspace $(X(I), \sigma(E, E^*))$ is metrizable by $d$ (defined by (22)). We claim that $X$ is contained in a compact subspace of the space $C(I, (X(I), d))$ furnished with the topology of uniform convergence. Take $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and every $t, \tau \in I$ we have

$$\sup_{x \in X} \sum_{n=n_0}^{\infty} 2^{-n} |\langle x_n^*, x(t) - x(\tau) \rangle| < \frac{\varepsilon}{2}.$$ 

On the other hand, there exists $\delta_i > 0$ for $i \in \{1, \ldots, n_0 - 1\}$ such that for every $t, \tau \in I$ with $|t - \tau| < \delta_i$ we have

$$2^{-i} \sup_{x \in X} |\langle x_i^*, x(t) - x(\tau) \rangle| \leq \sup_{x \in X} |\langle x_i^*, x(t) - x(\tau) \rangle| < \frac{\varepsilon}{2} (n_0 - 1).$$

Hence

$$\sup_{x \in X} d(x(t), x(\tau))$$

$$\leq \sup_{x \in X} \sum_{n=1}^{n_0-1} 2^{-n} |\langle x_n^*, x(t) - x(\tau) \rangle| + \sup_{x \in X} \sum_{n=n_0}^{\infty} 2^{-n} |\langle x_n^*, x(t) - x(\tau) \rangle| < \varepsilon$$

for every $t, \tau \in I$ such that $|t - \tau| < \delta = \min_{1 \leq i \leq n_0 - 1} \delta_i$. In other words, $X$ is equicontinuous with respect to $d$. At the same time, the cross-section $X(t)$ is relatively compact in $(X(I), d)$ for every $t \in I$. Consequently, $X$ is relatively compact in $C(I, (X(I), d))$, by virtue of Ascoli’s theorem.

Observe that the inclusion mapping $i: C(I, (X(I), d)) \hookrightarrow (C(I, E), w)$ is continuous. Therefore, $X$ is relatively weakly compact in $C(I, E)$. In fact, $X$ is weakly
compact since it is weakly closed. Consequently, $X(I)$ is weakly compact as well. Property (23) easily follows from the fact that

$$\left\{ y \in C(I, E): y(t) \in S(t)x_0 + \int_0^t S(t-s)\overline{\Theta} F(s, X(s)) \, ds \text{ for } t \in I \right\} \subset Y_n$$

for every $n \geq 1$.

For $A \subset E$, let $\text{dist}(x, A) := \inf_{y \in A} d(x, y)$. We denote the $d$-metric projection on the subset $\overline{X}(t)$ by $P: I \times E \to E$, that is,

$$P(t, x) := \left\{ y \in \overline{X}(t): d(x, y) = \text{dist}(x, X(t)) \right\}.$$

Since $\overline{X}(t)$ is $d$-compact, $P(t, x)$ must be nonempty. Relying on the weak compactness of the set $X$ constructed above, we define an auxiliary multimap $\overline{F}: I \times E \to E$ by the formula

$$\overline{F}(t, x) := \overline{\Theta} F(t, P(t, x)).$$

Property (23) plays a key role in proving that $S_{\overline{F}}(x_0) = S_F(x_0)$. It can be shown that $\overline{F}$ satisfies conditions $(F_1)$–$(F_5)$. Clearly, $\overline{F}$ is integrably bounded and the map $\overline{F}(t, \cdot)$ is weakly compact almost everywhere on $I$.

Properties $(F_2)$ and $(F_3)$ require some commentary. First, observe that the metric space $(E, d)$ is separable. Since $t \mapsto X(t)$ is Hausdorff continuous in the norm topology of $E$, $X(\cdot) : I \to (E, d)$ is measurable and $I \ni t \mapsto \text{dist}(x, X(t)) + 1/n \in \mathbb{R}$ is continuous. Thus, $G_n: I \to (E, d)$ given by $G_n(t) := \left\{ y \in E: d(x, y) \leq \text{dist}(x, X(t)) + 1/n \right\}$ is weakly measurable. Notice that

$$P(t, x) = X(t) \cap \bigcap_{n=1}^{\infty} G_n(t).$$

In view of Theorem 4.1 in [11], the set-valued map $P(\cdot, x): I \to (E, d)$ is measurable. Consequently, the codomain restriction $P(\cdot, x): I \to (X(I), d)$ of $P(\cdot, x)$ constitutes a measurable multimap. Since $(X(I), d)$ is a Polish space, there exists a measurable $p_x: I \to (X(I), d)$ such that $p_x(t) \in P(t, x)$ for $t \in I$ (see [11], Theorem 5.1). Consider a sequence $(p_n: I \to X(I))_{n=1}^{\infty}$ of simple functions such that $d(p_n(t), p_x(t)) \underset{n \to \infty}{\to} 0$ almost everywhere on $I$, that is, $p_n(t) \xrightarrow{E} p_x(t)$ almost everywhere on $I$. In accordance with $(F_2)$, there exists a measurable function $f_n: I \to (E, |\cdot|)$ such that $f_n(t) \in F(t, p_n(t))$ almost everywhere on $I$. In view of $(F_4)$ the family $(f_n)_{n=1}^{\infty}$ is uniformly integrable. By $(F_5)$ the cross-section $\{f_n(t)\}_{n=1}^{\infty}$ is relatively weakly compact in $E$. Therefore, $(f_n)_{n=1}^{\infty}$ is relatively weakly compact in $L^1(I, E)$, by Theorem 1. Assume that $F_n \overset{L^1(I, E)}{\underset{n \to \infty}{\rightharpoonup}} f$, passing to a subsequence if necessary. According to Corollary 2, $f(t) \in F(t, p_x(t))$ almost everywhere on $I$. Hence, $\overline{F}(\cdot, x)$ has a measurable selection.

Let $x_n \overset{E}{\rightharpoonup} x$. Fix $x^* \in E^*$. Obviously, there exists $z_n \in P(t, x_n)$ such that

$$\sigma(x^*, F(t, P(t, x_n))) = \sigma(x^*, F(t, z_n)) \quad \text{for } n \geq 1.$$
From the very definition of $P$ follows that there is a subsequence $z_{k_n} \xrightarrow{E} z \in P(t,x)$. Thus,
\[
\lim_{n \to \infty} \sigma(x^*, F(t, P(t, x_{k_n}))) = \lim_{n \to \infty} \sigma(x^*, F(t, z_{k_n})) \leq \sigma(x^*, F(t, z)) \\
\leq \sigma(x^*, F(t, P(t, x))) = \sigma(x^*, \tilde{F}(t, x))
\]
and eventually
\[
\lim_{n \to \infty} \sigma(x^*, \tilde{F}(t, x_n)) \leq \sigma(x^*, \tilde{F}(t, x)).
\]
The latter means that $\tilde{F}$ satisfies (F\(_3\)).

Now we define a set-valued approximation $F_n : I \times E \to E$ of the map $\tilde{F}$ in a routine manner, that is,
\[
F_n(t, x) := \sum_{y \in E} \psi^n_y(x) \, \overline{co} \, \tilde{F}(t, B_d(y, 2r_n)),
\]
where $r_n := 3^{-n}$, $B_d(y, 2r_n)$ is the ball considered in the metric space $(E, d)$ and the family $\{\psi^n_y : (E, d) \to [0, 1]\}_{y \in E}$ is a locally Lipschitz partition of unity whose supports form a locally finite covering inscribed into the covering $\{B_d(y, r_n)\}_{y \in E}$ of the space $(E, d)$. Moreover, for every $n \geq 1$ define a mapping $f_n : I \times E \to E$ in the following way:
\[
f_n(t, x) := \sum_{y \in E} \psi^n_y(x)g_y(t) \in F_n(t, x),
\]
where $g_y$ is a measurable selection of $\tilde{F}(\cdot, y)$.

If $H_n : C(I, E) \to C(I, E)$ is an operator given by $H_n := \sigma(\cdot)x_0 + V \circ N_{F_n}$, then the topological space $(\text{Fix}(H_n), \sigma(C(I, E), C(I, E)^*))$ is d-compact metrizable. Indeed, observe that $\emptyset \neq S_F(x_0) = S_{\tilde{F}}(x_0) \subset \text{Fix}(H_k)$, by Theorem 5 and (23). Let $(u_n)_{n=1}^{\infty} \subset \text{Fix}(H_k)$. Then $u_n = S(\cdot)x_0 + V(f_n)$, where $f_n \in N_{F_k}(u_n)$. Recall that $F_k(t, x) \subset \overline{co} \tilde{F}(t, B(x, 3r_k))$. Therefore,
\[
|f_n(t)| \leq \|F_k(t, u_n(t))\|^+ \leq \|\overline{co} \tilde{F}(t, B(u_n(t), 3r_k))\|^+ \leq \|F(t, X(t))\|^+
\]
almost everywhere on $I$ and
\[
\lim_{\ell(J) \to 0} \sup_{n \geq 1} \int_J |f_n(t)| \, dt \leq \lim_{\ell(J) \to 0} \int_J \sup_{|x| \leq \|X\|^+} \|F(t, x)\|^+ \, dt \leq \lim_{\ell(J) \to 0} \int_J g(t) \, dt = 0
\]
for some $g \in L^1(I)$. On the other hand,
\[
\beta(\{f_n(t)\}_{n=1}^{\infty}) \leq \beta(\overline{co} \tilde{F}(t, B(\{u_n(t)\}_{n=1}^{\infty}, 3r_k))) \leq \beta(F(t, X(t))) \leq \eta(t)\beta(X(t))
\]
for almost all $t \in I$. In view of Theorem 1 the sequence $(f_n)_{n=1}^{\infty}$ is relatively weakly compact in $L^1(I, E)$. Hence we may assume, passing to a subsequence if necessary, that $f_n \xrightarrow{n \to \infty} L^1(I, E) \to f$. As a result, $u_n \xrightarrow{n \to \infty} u := S(\cdot) + V(f)$. We would be done, if we could only demonstrate that $f \in N_{F_k}(u)$. Consider $x_n \xrightarrow{n \to \infty} x$. 

Since \( \overline{co} \tilde{F}(t, B(y, 2r_k)) \subset \overline{co} F(t, X(t)) \) and \( F(t, \cdot) \) is quasicompact in the weak topology, the map \( F_k \) has weakly compact values by the Krein–Šmulian theorem. Hence there exists \( y_n \in F_k(t, x_n) \) such that \( \sigma(x^*, F_k(t, x_n)) = \langle x^*, y_n \rangle \) for \( n \geq 1 \) and some fixed \( x^* \in E^* \). Since \( \overline{co} F(t, X(t)) \) is also weakly compact, \( y_n \xrightarrow{n \to \infty} y \), up to a subsequence. Moreover, for every \( m \geq 1 \) there exists \( z_m \in \text{co}\{y_n\}_{n=m}^{\infty} \) such that \( z_m \xrightarrow{E \to m} y \). From the very definition of \( F_k \) it follows that there exists \( \gamma > 0 \) such that for all \( x_1, x_2 \in X(I) \)

\[
  h(F_k(t, x_1), F_k(t, x_2)) \leq \sum_{y \in E} \left| \psi^n_y(x_1) - \psi^n_y(x_2) \right| \| \overline{co} \tilde{F}(t, B_d(y, 2r_k)) \|^+
  \leq \gamma \| F(t, X(t)) \|^+ d(x_1, x_2)
\]

almost everywhere on \( I \). Therefore,

\[
  \lim_{n \to \infty} \inf_{z \in F_k(t, x)} |y_n - z| \leq \lim_{n \to \infty} h(F_k(t, x_n), F_k(t, x)) = 0,
\]

that is,

\[
  \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ \ y_n \in B(F_k(t, x), \varepsilon).
\]

Thus,

\[
  \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m \geq N \ \ z_m \in B(F_k(t, x), \varepsilon)
\]

and eventually \( y \in D(F_k(t, x), \varepsilon) \) for every \( \varepsilon > 0 \). This means that condition (8) in Corollary 2 is met, since

\[
  \limsup_{n \to \infty} \sigma(x^*, F_k(t, x_n)) = \lim_{n \to \infty} \langle x^*, y_n \rangle = \langle x^*, y \rangle \leq \sigma(x^*, F_k(t, x))
\]

Now, since

\[
  \begin{cases}
    u_n(t) \xrightarrow{E \to n} u(t) & \text{for } t \in I,
    \\
    f_n \xrightarrow{L^1(I, E)} f,
    \\
    f_n(t) \in F_k(t, u_n(t)) & \text{a.e. on } I,
  \end{cases}
\]

we finally deduce that \( f(t) \in F_k(t, u(t)) \) for almost all \( t \in I \). Therefore, \( \text{Fix}(H_k) \) is weakly compact and forms a \( d \)-compact metrizable subspace of the separable space \( C(I, E) \).

It is easy to see that \( S_F(x_0) = \bigcap_{n=1}^{\infty} \text{Fix}(H_n) \). The inclusion \( \subset \) is self-evident, since \( \tilde{F}(t, x) \subset F_n(t, x) \). So, take \( u \in \bigcap_{n=1}^{\infty} \text{Fix}(H_n) \). Suppose that \( u = S(\cdot)x_0 + V(f_n) \) with \( f_n \in N_{F_n}(u) \). In a similar manner to previously we can prove that the sequence \( (f_n)_{n=1}^{\infty} \) is relatively weakly compact in \( L^1(I, E) \). Thus we can assume, passing to a subsequence if necessary, that \( S(\cdot)x_0 + V(f_n) \xrightarrow{C(I, E) \to n} S(\cdot)x_0 + V(f) \). Consider a subsequence \( (f_{k_n}(t))_{n=1}^{\infty} \) such that \( f_{k_n}(t) \xrightarrow{E \to n} z \). Fix \( x^* \in E^* \). There exists \( z_n \in P(t, B(u(t), 3r_k))^{w} \) such that \( \sigma(x^*, F(t, P(t, B(u(t), 3r_k)))^{w}) = \sigma(x^*, F(t, z_n)) \) for \( n \geq 1 \). Since \( z_n = w-\lim_{n \to \infty} y^n_m \) with \( y^n_m \in P(t, B(u(t), 3r_k)) \), there is \( w^n_m \in B(u(t), r_k) \) such that \( y^n_m \in P(t, w^n_m) \). The diagonalization procedure means
we can extract a subsequence \((y_{m_n})_{n=1}^\infty\), that satisfies \(d(z_n, y_{m_n}) < 1/n\). The strong convergence \(u_{m_n} \xrightarrow{E} u(t)\) means, passing to a subsequence if necessary, that \(y_{m_n} \xrightarrow{n \to \infty} y \in P(t, u(t))\). Hence \(z_n \xrightarrow{n \to \infty} y \in P(t, u(t))\). Considering that \(F(t, \cdot)\) is weakly sequentially upper hemicontinuous we obtain

\[
\langle x^*, z \rangle = \lim_{n \to \infty} \langle x^*, f_k(t) \rangle \leq \liminf_{n \to \infty} \sigma(x^*, \tilde{F}(t, B(u(t), 3r_k)))
\]

\[
= \liminf_{n \to \infty} \sigma(x^*, F(t, 3r_k)))
\]

\[
\leq \liminf_{n \to \infty} \sigma(x^*, F(t, P(u(t), 3r_k))))
\]

\[
= \limsup_{n \to \infty} \sigma(x^*, F(t, z_n)) \leq \sigma(x^*, F(t, y)) \leq \sigma(x^*, F(t, P(u(t))))
\]

\[
= \sigma(x^*, \tilde{F}(t, u(t))).
\]

Hence \(z \in F(t, u(t))\), which means that \(w^* \limsup_{n \to \infty} \{f_n(t)\} \subset F(t, u(t))\). In view of Proposition 2.3.31 in [9] it is clear that \(f(t) \in F(t, u(t))\) almost everywhere on \(I\). Thus \(u = S(\cdot) x_0 + V(f) \in S(\cdot)x_0 + (V \circ N_F)(u)\), proving that \(\cap_{n=1}^\infty \text{Fix}(H_n) \subset S_F(x_0)\).

Fix \(k \geq 1\). Observe that for each \(x \in \text{Fix}(H_k)\) there exists exactly one \(f \in N_{F_k}(x)\) such that \(x = S(\cdot) x_0 + V(f)\). This follows directly from the fact that \(x\) as an integrated solution has the form \(x(t) = tx_0 + A \int_0^t x(s) ds + \int_0^t (t-s)f(s) ds\) for \(t \in I\). Since the single-valued map \(f_k: I \times E \to E\) satisfies conditions (F1)–(F5), the integral equation

\[
u(t) = S(t)x_0 + \int_0^\tau S(t-s) f(s) ds + \int_\tau^t S(t-s) f_k(s, u(s)) ds \quad \text{for } t \in [\tau, T],
\]

with \(f \in N_F(x)\), possesses a solution (this can easily be justified by analyzing the proof of Theorem 5 carefully).

It is worth noting that for any weakly compact subset \(C \subset E\) there exists a constant \(\gamma_C > 0\) such that

\[
|f_k(t, x_1) - f_k(t, x_2)| \leq \gamma_C \|F(t, X(t))\|_+ d(x_1, x_2)
\]

for all \(x_1, x_2 \in C\) and for almost all \(t \in I\). If \(u_1\) and \(u_2\) are two solutions of equation (24), then

\[
|u_1(t) - u_2(t)| \leq \int_\tau^t \|S(t-s)\|_\mathcal{L}|f_k(s, u_1(s)) - f_k(s, u_2(s))| d(s)\]

\[
\leq \int_\tau^t \|S(t-s)\|_\mathcal{L}\gamma_{u_1(t)} u_2(t) g(s) d(u_1(s), u_2(s)) ds
\]

\[
\leq M e^{\omega T} \gamma_{u_1(t)} u_2(t) \int_\tau^t g(s) |u_1(s) - u_2(s)| ds,
\]

where \(g \in L^1(I)\) is such that \(g(t) \geq \sup_{|x| \leq \|X\|_+} \|F(t, x)\|_+\) almost everywhere on \(I\). Thus, equation (24) has a unique solution \(u(\tau, x(\tau))\).
Let $h : [0, 1] \times \text{Fix}(H_k) \rightarrow \text{Fix}(H_k)$ be the homotopy given by the formula

$$h(\lambda, x)(t) := \begin{cases} x(t), & t \in [0, \lambda T], \\ u[\lambda T; x(\lambda T)](t), & t \in [\lambda T, T]. \end{cases}$$

No need to emphasize that $h$ is well defined. Observe that $h(0, x) = u_0$ for all $x \in \text{Fix}(H_k)$, where

$$u_0(t) = S(t)x_0 + \int_0^t S(t - s)f_k(s, u_0(s)) \, ds, \quad t \in I.$$ 

At the same time $h(1, \cdot) = id_{\text{Fix}(H_k)}$. Assume that $(x_n)_{n=1}^{\infty} \subset \text{Fix}(H_k)$ and $(\lambda_n)_{n=1}^{\infty} \subset [0, 1]$ are such that $x_n \frac{\text{Fix}(H_k)}{n \rightarrow \infty} x$ and $\lambda_n \frac{\text{Fix}(H_k)}{n \rightarrow \infty} \lambda$. For definiteness, let $\lambda_n \nrightarrow \lambda$. There are two cases to consider: $t < \lambda T$ and $t \geq \lambda T$. If $t < \lambda T$, then we are simply dealing with the convergence $h(\lambda_n, x_n)(t) \xrightarrow{E_n \rightarrow \infty} x(t) = h(\lambda, x)(t)$. Suppose then that $t \geq \lambda T$, $x_n = S(\cdot)x_0 + V(f_n)$, $x = S(\cdot)x_0 + V(f)$, $u_n := h(\lambda_n, x_n)$, $u := h(\lambda, x)$ and $\{x_n^*\}_{n=1}^{\infty}$ is a $w^*$-dense subset of the dual $E^*$. From Theorem 1 it follows easily that $f_n \xrightarrow{n \rightarrow \infty} L^1(I, E) g$, up to a subsequence. Thus, $x_n \xrightarrow{C(I, E)} S(\cdot)x_0 + V(g)$ and consequently $V(f) = V(g)$. Eventually, $f_n \xrightarrow{n \rightarrow \infty} L^1(I, E) f$, by Lemma 1. Observe that

$$\int_0^{\lambda T} S(t - s)f_n(s) \, ds \xrightarrow{n \rightarrow \infty} \int_0^{\lambda T} S(t - s)f(s) \, ds$$

and

$$\int_{\lambda_n T}^{\lambda T} S(t - s)f_n(s) \, ds \xrightarrow{n \rightarrow \infty} 0,$$

Therefore,

$$\int_0^{\lambda_n T} S(t - s)f_n(s) \, ds \xrightarrow{n \rightarrow \infty} \int_0^{\lambda T} S(t - s)f(s) \, ds. \quad (25)$$

Notice that $K := \bigcup_{n=1}^{\infty} u_n(I)^w \cup u(I)$ is weakly compact. Estimates on the segment $[\lambda T, T]$ have the following form:

$$\beta(\{u_n(t)\}_{n=1}^{\infty}) = \beta\left(\{S(t)x_0 + \int_0^{\lambda_n T} S(t - s)f_n(s) \, ds + \int_{\lambda_n T}^{t} S(t - s)f_k(s, u_n(s)) \, ds\}_{n=1}^{\infty}\right)$$

$$\leq Me^{\omega T} \int_0^{\lambda_n T} \beta(\{f_n(s)\}_{n=1}^{\infty}) \, ds + Me^{\omega T} \int_{\lambda_n T}^{t} \beta(f_k(s, \{u_n(s)\}_{n=1}^{\infty})) \, ds$$

$$\leq Me^{\omega T} \int_0^{t} \eta(s)\beta(X(s)) \, ds = 0$$
and

$$\sup_{n \geq 1} |u_n(t) - u_n(\tau)|$$

$$\leq |S(t)x_0 - S(\tau)x_0| + \sup_{n \geq 1} \int_0^{\lambda_n T} \|S(t-s) - S(\tau-s)\|_F |f_n(s)| \, ds$$

$$+ \sup_{n \geq 1} \int_0^{\lambda_n T} \|S(t-s) - S(\tau-s)\|_F |f_k(s,u_n(s))| \, ds$$

$$+ \sup_{n \geq 1} \int_{\tau}^{t} \|S(t-s)\|_F |f_k(s,u_n(s))| \, ds$$

$$\leq |S(t)x_0 - S(\tau)x_0| + \int_0^{\tau} \|S(t-s) - S(\tau-s)\|_F g(s) \, ds$$

$$+ Me^{\omega T} \int_{\tau}^{t} g(s) \, ds,$$

where $g \in L^1(I)$ satisfies $g(t) \geq \sup_{|x| \leq \|x\|_+} \|F(t,x)\|_+^+$ almost everywhere on $I$. Thus, the family $\{u_n\}_{n=1}^\infty$ is equicontinuous and $\beta(\bigcup_{n=1}^\infty u_n(I)) = \sup_{t \in I} \beta(\{u_n(t)\}_{n=1}^\infty) = 0$. Weak compactness of $K$ now follows. In this connection, for every $n \geq 1$ and for almost all $t \in I$ we have

$$|f_k(t,u_n(t)) - f_k(t,u(t))| \leq \gamma_K \|F(t,X(t))\|_+^+ d(u_n(t),u(t)). \quad (26)$$

For every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{m=m_0}^{\infty} 2^{-m} |\langle x_m^*, u_n(t) - u(t) \rangle| \leq \frac{\varepsilon}{3}.$$

By virtue of (25) and (26) we can choose an $N \in \mathbb{N}$ such that for $m \in \{1, \ldots, m_0-1\}$ and $n \geq N$ we obtain

$$|\langle x_m^*, u_n(t) - u(t) \rangle|$$

$$\leq \left| \langle x_m^*, \int_0^{\lambda_n T} S(t-s)f_n(s) \, ds - \int_0^{\lambda T} S(t-s)f(s) \, ds \rangle \right|$$

$$+ \left| \langle x_m^*, \int_{\lambda_n T}^{t} S(t-s)f_k(s,u_n(s)) \, ds - \int_{\lambda T}^{t} S(t-s)f_k(s,u(s)) \, ds \rangle \right|$$

$$\leq \frac{\varepsilon}{3} + |x_m^*| \int_{\lambda_n T}^{t} \|S(t-s)\|_F |f_k(s,u_n(s))| \, ds$$

$$+ |x_m^*| \int_{\lambda T}^{t} \|S(t-s)\|_F |f_k(s,u_n(s)) - f_k(s,u(s))| \, ds$$

$$\leq \frac{\varepsilon}{3} + Me^{\omega T} \left( \int_{\lambda_n T}^{\lambda T} g(s) \, ds + \gamma_K \int_{\lambda T}^{t} g(s) \, d(u_n(s),u(s)) \, ds \right)$$

$$\leq \frac{2}{3} \varepsilon + Me^{\omega T} \gamma_K \int_{\lambda T}^{t} g(s) \, d(u_n(s),u(s)) \, ds,$$
where \( g \in L^1(I) \) satisfies \( g(t) \geq \sup_{|x| \leq \|X\|} \|F(t,x)\|^+ \) almost everywhere on \( I \). Hence

\[
d(u_n(t), u(t)) = \sum_{m=1}^{\infty} 2^{-m} |\langle x^*_m, u_n(t) - u(t) \rangle| \leq \frac{\varepsilon}{3} + \sum_{m=1}^{m_0-1} 2^{-m} |\langle x^*_m, u_n(t) - u(t) \rangle|
\]

\[
\leq \frac{\varepsilon}{3} + \sum_{m=1}^{m_0-1} 2^{-m} \left( \frac{2}{3} \varepsilon + Me\omega T \gamma K \int_{\lambda T}^{t} g(s) d(u_n(s), u(s)) \, ds \right)
\]

\[
\leq \varepsilon + Me\omega T \gamma K \int_{\lambda T}^{t} g(s) d(u_n(s), u(s)) \, ds
\]

for \( n \geq N \). Finally,

\[
d(u_n(t), u(t)) \leq \varepsilon \exp(Me\omega T \gamma K \|g\|_1),
\]

that is, \( d(u_n(t), u(t)) \xrightarrow{n \to \infty} 0 \) for \( t \geq \lambda T \). So, we have shown that \( h(\lambda_n, x_n)(t) \xrightarrow{E \to E \gamma K} h(\lambda, x)(t) \) for \( t \in I \). Since

\[
\sup_{n \geq 1 \atop t \in I} |h(\lambda_n, x_n)(t)| \leq Me\omega T (|x_0| + \|g\|_1),
\]

with \( g \in L^1(I) \) such that \( g(t) \geq \sup_{|x| \leq \|X\|} \|F(t,x)\|^+ \) for almost all \( t \in I \), the latter implies the weak convergence \( h(\lambda_n, x_n) \xrightarrow{C(I,E)} h(\lambda, x) \). This means that \( h \) is a continuous mapping with respect to the relative weak topology of \( \text{Fix}(H_k) \).

Summing up, the solution set \( S_F(x_0) \) can be represented in the form of the intersection of a decreasing sequence of compact contractible metric spaces \( \text{Fix}(H_n), d \).

Theorem 7 is proved.

**Corollary 4.** Suppose that (21) holds. Under the assumptions of Theorem 7, with the exclusion of condition (F4), the solution set of the Cauchy problem (1) is \( R_\delta \).

### §4. Applications

Theorem 7, formulated in §3, on the geometric structure of the solution set \( S_F(x_0) \), will allow us to employ an approach imitating the method of the operator of translation along the trajectories to demonstrate the existence of integrated solutions to the nonlocal Cauchy problem. Consider, therefore, the following boundary value problem:

\[
\begin{aligned}
\dot{x}(t) &\in Ax(t) + F\left(t, \int_0^t x(s) \, ds \right) \quad \text{on} \ I, \\
x(0) &\in G(x),
\end{aligned}
\]

(27)

where \( G: C(I,E) \to E \). By applying the approach mentioned above, we have been able to prove the following.
Theorem 8. Let $E$ be a separable Banach space. Assume that $A: D(A) \to E$ is a generator of a nondegenerate equicontinuous integrated semigroup $\{S(t)\}_{t \geq 0}$ such that $\|S(t)\|_\infty \leq e^{\omega t}$ for $t \geq 0$. Assume further that $F: I \times E \to E$ satisfies (F1)-(F5). Let $G: C(I, E) \to E$ be a set-valued operator whose restriction $G|_X: (X, w) \to (E, w)$ to any weakly compact subset $X \subset C(I, E)$ is an admissible map. If $G$ satisfies

\[(G_1) \quad \beta(G(\Omega)) \leq \beta(\Omega(T)) \text{ for bounded } \Omega \subset C(I, E); \]
\[(G_2) \quad \exists a, d > 0 \quad \exists R > 0 \quad \forall |x| \geq R \quad \|G(x)\|_+ \leq a\|x\|^\alpha + d \text{ for some } \alpha \in (0, 1) \text{ and } \]
\[\omega T + \|\eta\|_1 < 1; \quad (28)\]

then the nonlocal Cauchy problem (27) has an integrated solution.

Proof. Define $P: E \to E$ by $P = G \circ S_F$, where $S_F: E \to C(I, E)$ is the solution set map given by

$$S_F(x_0) := \{ x \in C(I, E) : x \text{ is an integrated solution of (1)} \}.$$ 

We will show that there exists $R > 0$ such that $P(D(0, R)) \subset D(0, R)$. Suppose not. Then there exist elements $x_n \in E$ and $y_n \in P(x_n)$ such that $|x_n| \leq n$ and $|y_n| > n$ for $n \geq 1$. If $y_n \in G(u_n)$, then either $(u_n)_{n=1}^{\infty}$ is bounded or $\|u_n\| \to +\infty$. In the first case there must be a radius $\hat{R} > 0$ such that $G(\{u_n\}_{n=1}^{\infty}) \subset D(0, \hat{R})$. Thus,

$$1 \leq \limsup_{n \to \infty} \frac{|y_n|}{n} \leq \limsup_{n \to \infty} \frac{\|G(u_n)\|_+}{n} \leq \lim_{n \to \infty} \frac{\hat{R}}{n} = 0,$$

which is a contradiction. Assume that $\|u_n\| \to +\infty$ and $\lim_{n \to \infty} \frac{|x_n|}{\|u_n\|} = 0$. Then

$$1 = \limsup_{n \to \infty} \frac{\|u_n\|}{\|u_n\|} \leq e^{\omega T} \left( \limsup_{n \to \infty} \frac{|x_n|}{\|u_n\|} + \limsup_{n \to \infty} \frac{1}{\|u_n\|} \int_I \sup_{\|x\| \leq \|u_n\|} \|F(t, x)\|_+ dt \right) < 1,$$

a contradiction. Suppose then that $\|u_n\| \to +\infty$ and that there exist $\varepsilon > 0$ and $(k_n)_{n=1}^{\infty}$ such that $|x_{k_n}| > \varepsilon\|u_{k_n}\|$ for $n \geq 1$. Using hypothesis (G2) we obtain

$$1 \leq \limsup_{n \to \infty} \frac{|y_{k_n}|}{k_n} \leq \limsup_{n \to \infty} \frac{\|G(u_{k_n})\|_+}{k_n} \leq \limsup_{n \to \infty} \frac{a\varepsilon^{-\alpha}(k_n)^\alpha + d}{k_n} = 0.$$

In other words, there must be a ball $D(0, \hat{R})$ invariant under the Poincaré-type operator $P$.

Consider a sequence $x_n \xrightarrow{n \to \infty} x$. Let $u_n \in S_F(x_n)$ be such that $u_n = S(\cdot)x_n + V(f_n)$. Clearly, the sequence of solutions $(u_n)_{n=1}^{\infty}$ possesses a priori bounds. Hence

$$|f_n(t)| \leq \|F(t, u_n(t))\|_+ \leq b(t)(1 + |u_n(t)|) \leq b(t) \left(1 + \sup_{n \geq 1} \|u_n\| \right)$$
for each \( n \geq 1 \) and for almost all \( t \in I \). At the same time

\[
\beta\left(\{u_n(t)\}_{n=1}^{\infty}\right) = \beta\left(\{S(t)x_n + V(f_n)(t)\}_{n=1}^{\infty}\right) \\
\leq \|S(t)\|_\mathcal{F} \beta\left(\{x_n\}_{n=1}^{\infty}\right) + \int_0^t \|S(t-s)\|_\mathcal{F} \beta\left(\{f_n(s)\}_{n=1}^{\infty}\right) \, ds \\
\leq \int_0^t \eta(s) \beta\left(\{u_n(s)\}_{n=1}^{\infty}\right) \, ds.
\]

Hence, \( \sup_{t \in I} \beta\left(\{u_n(t)\}_{n=1}^{\infty}\right) = 0 \) and eventually \( \beta\left(\{f_n(t)\}_{n=1}^{\infty}\right) = 0 \) almost everywhere on \( I \). By virtue of Theorem 1 we may assume, passing to a subsequence if necessary, that \( f_n \xrightarrow{L^1(I,E)} f \). Obviously, \( S(t)x_n \xrightarrow{E} S(t)x \). Consequently,

\[
u_n(t) = S(t)x_n + V(f_n)(t) \xrightarrow{E} S(t)x + V(f)(t) =: u(t) \quad \text{for each } t \in I.
\]

Since \( \sup_{n \geq 1} \|u_n\| < \infty \), the latter means that \( u_n \xrightarrow{C(I,E)} u \). Since the hypotheses of the convergence theorem are met (cf. Corollary 2), we deduce that \( f \in \mathcal{N}_F(u) \). On that account \( u \in S_F(x) \). So the operator \( S_F : (X, w) \rightarrow (C(I,E), w) \) is a weakly compact-valued map that is upper semicontinuous for each fixed relatively weakly compact subset \( X \subset E \). Now we can apply the structure theorem (Theorem 7) to get admissibility of the Poincaré-type operator \( P : (X, w) \rightarrow (E,w) \) (remember that the composition of two admissible maps is still admissible).

Now we repeat the reasoning contained in the proof of Theorem 5. Fix \( \widehat{x} \in D(0, R) \) and define

\[
\mathcal{A} := \{X \in 2^{D(0,R)} \setminus \{\emptyset\} : X \text{ is closed convex and } \overline{\mathcal{B}}(\{\widehat{x}\} \cup P(X)) \subset X\}.
\]

Then the intersection \( X_0 := \bigcap_{X \in \mathcal{A}} X \) is nonempty and possesses the following form: \( X_0 = \overline{\mathcal{B}}(\{\widehat{x}\} \cup P(X_0)) \). We will show that \( X_0 \) is weakly compact in \( E \). Let \( u_n = S(\cdot)x_n + V(f_n) \) with \( f_n \in \mathcal{N}_F(u_n) \) and \( x_n \in X_0 \). Put

\[
\Delta(\Omega) := \{D \in 2^\Omega \setminus \{\emptyset\} : D \text{ is countable}\}.
\]

In view of Theorem 2.8 in [16] we have

\[
\beta\left(\{u_n(t)\}_{n=1}^{\infty}\right) = \beta\left(\{S(t)x_n + V(f_n)(t)\}_{n=1}^{\infty}\right) \\
\leq \beta\left(\{x_n\}_{n=1}^{\infty}\right) + \int_0^t S(t-s) f_n(s) \, ds \\
\leq \|S(t)\|_\mathcal{F} \beta\left(\{x_n\}_{n=1}^{\infty}\right) + \int_0^t \|S(t-s)\|_\mathcal{F} \beta\left(\{f_n(s)\}_{n=1}^{\infty}\right) \, ds \\
\leq e^{\omega t} \max_{D \in \Delta(\Omega)} \beta(D) + \int_0^t e^{\omega(t-s)} \eta(s) \beta\left(\{u_n(s)\}_{n=1}^{\infty}\right) \, ds.
\]

Defining the right-hand side of the above inequality by \( \rho \), we see that

\[
\rho'(t) = \omega \rho(t) + \eta(t) \beta\left(\{u_n(t)\}_{n=1}^{\infty}\right) \leq (\omega + \eta(t)) \rho(t)
\]

almost everywhere on \( I \). Solving of this differential inequality leads to

\[
\beta\left(\{u_n(t)\}_{n=1}^{\infty}\right) \leq \rho(t) \leq \max_{D \in \Delta(\Omega)} \beta(D) \exp\left(\omega t + \int_0^t \eta(s) \, ds\right)
\]
for \( t \in I \). Using the latter and \((G_1)\) we obtain
\[
\max_{D \in \Delta(X_0)} \beta(D) = \max_{D \in \Delta(P(X_0))} \beta(D) \leq \max_{D \in \Delta(S_F(X_0))} \beta(G(D)) \leq \max_{D \in \Delta(S_F(X_0))} \beta(D(T)) \leq \max_{D \in \Delta(X_0)} \beta(D) e^{\omega T+\|\eta\|_1}.
\]
In view of \((28)\), \(\max_{D \in \Delta(X_0)} \beta(D) = 0\). In accordance with the Eberlein-Šmulian theorem the set \(X_0\) must be weakly compact.

Summing up, the admissible operator \(P : X_0 \to X_0\) from the convex subset \(X_0\) of the locally convex space \((E, w)\) to the compact metrizable subset of \(X_0\) has at least one fixed point, by virtue of Theorem 2. This fixed point constitutes a solution to the boundary value problem \((27)\).

Theorem 8 is proved.

**Corollary 5.** Assume that an operator \(G : C(I, E) \to E\) has a weakly sequentially closed graph, acyclic values, maps bounded sets into relatively weakly compact sets and satisfies the sublinear growth condition \((G_2)\). Then, taking the remaining hypotheses of Theorem 8 into account (apart from condition \((G_1)\) of course), the nonlocal Cauchy problem \((27)\) has a solution.

**Remark 7.** Corollary 5 emphasizes the advantage of Theorem 8 over Theorem 2.2 in [4], at least in the context of a separable Banach space and nondegenerate integrated semigroups. It would not be possible to weaken the assumption regarding the topology of the values of the boundary condition operator without using the structure theorem, that is, Theorem 7.

Now we turn our attention to the following multivalued wave equation:
\[
\begin{align*}
\Box u(t, x) &= f_2(t, x) + \Delta \int_0^t f_1(s, x) \, ds, \\
f_1(t, x) &\in \left[ h_1^1(t, x, \int_{\mathbb{R}^n} k_1^1(t, y) \, dy), h_2^1(t, x, \int_{\mathbb{R}^n} k_1^2(t, y) \, dy) \right], \\
f_2(t, x) &\in \left[ h_1^2(t, x, \int_{\mathbb{R}^n} k_2^1(t, y) u(t, y) \, dy), h_2^2(t, x, \int_{\mathbb{R}^n} k_2^2(t, y) u(t, y) \, dy) \right]
\end{align*}
\]
(29)
on \(I \times \mathbb{R}^n\), subject to the Cauchy condition
\[
\begin{align*}
\partial_t u(0, x) &= \dot{u}_2 & \text{on } \mathbb{R}^n, \\
u(0, x) &= \ddot{u}_1 & \text{on } \mathbb{R}^n,
\end{align*}
\]
(30)
where \(\Box\) is the d’Alembertian, and \(k_i^1(t, \cdot) \in L^1(\mathbb{R}^n)\) and \(k_i^2(t, \cdot) \in L^2(\mathbb{R}^n)\) for almost all \(t \in I\) and \(i = 1, 2\).

Let \(\langle \cdot, \cdot \rangle\) denote the inner product in \(L^2(\mathbb{R}^n)\). By a weak solution of problem \((29), (30)\) we mean a function \(w \in C(I, L^2(\mathbb{R}^n))\) such that for every \(v \in H^2(\mathbb{R}^n)\) the function \(\langle w(\cdot), v \rangle \in W^{2,1}(I)\) and
\[
\begin{align*}
\frac{d^2}{dt^2} \langle w(t), v \rangle &= \langle w(t), \Delta v \rangle + \langle f_2(t), v \rangle + \left\langle \int_0^t f_1(s) \, ds, \Delta v \right\rangle & \text{a.e. on } I, \\
\frac{d}{dt} \langle w(t), v \rangle |_{t=0} &= \langle \dot{u}_2, v \rangle, \\
w(0) &= \ddot{u}_1
\end{align*}
\]
(31)
for some functions $f_1, f_2 \in L^1(I, L^2(\mathbb{R}^n))$ such that

\[
\begin{cases}
    h_1^1(t, x, \int_{\mathbb{R}^n} k_1(t, y) dy) \leq f_1(t, x) \leq h_2^1(t, x, \int_{\mathbb{R}^n} k_2(t, y) dy), \\
    h_1^2(t, x, \int_{\mathbb{R}^n} k_1(t, y)(w(t, y) - \hat{u}_1(y)) dy) \leq f_2(t, x) \\
    \leq h_2^2(t, x, \int_{\mathbb{R}^n} k_2(t, y)(w(t, y) - \hat{u}_1(y)) dy)
\end{cases}
\]

for almost all $t \in I$ and almost all $x \in \mathbb{R}^n$. Let $\mathcal{F}(\hat{u}_1, \hat{u}_2)$ denote the set of all weak solutions of the problem (29), (30).

Our hypotheses on $h_j^i : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are the following:

(h$_1$) for any $u \in L^2(\mathbb{R}^n)$ there exist $v, w \in L^1(I, L^2(\mathbb{R}^n))$ such that

\[
\begin{cases}
    h_1^1(t, x, \int_{\mathbb{R}^n} k_1(t, y) dy) \leq v(t, x) \leq h_2^1(t, x, \int_{\mathbb{R}^n} k_2(t, y) dy), \\
    h_1^2(t, x, \int_{\mathbb{R}^n} k_2(t, y) u(y) dy) \leq w(t, x) \leq h_2^2(t, x, \int_{\mathbb{R}^n} k_2(t, y) u(y) dy)
\end{cases}
\]

for almost all $t \in I$ and almost all $x \in \mathbb{R}^n$;

(h$_2$) for almost all $t \in I$, for almost all $x \in \mathbb{R}^n$ and for every $z \in \mathbb{R}$ the function $h_j^i(t, x, \cdot)$ is lower semicontinuous while $h_j^i(t, x, \cdot)$ is upper semicontinuous;

(h$_3$) for $j = 1, 2$ there exist $b_1, b_2 \in L^1(I)$ and $c_1 : I \times \mathbb{R}^n \to \mathbb{R}$, $c_2 : I \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that

\[
\begin{cases}
    \sup_{|z| \leq \|k_j^i(t, \cdot)\|_1} |h_j^1(t, x, z)| \leq c_1(t, x), \\
    \sup_{|z| \leq \|k_j^i(t, \cdot)\|_{2r}} |h_j^2(t, x, z)| \leq c_2(t, x, r)
\end{cases}
\]

and

\[
\begin{cases}
    \int_{\mathbb{R}^n} c_1^2(t, x) dx \leq b_1^2(t), \\
    \int_{\mathbb{R}^n} c_2^2(t, x, r) dx \leq b_2^2(t)(1 + r)^2
\end{cases}
\]

for every $r > 0$, for almost all $t \in I$ and for almost all $x \in \mathbb{R}^n$.

**Theorem 9.** If hypotheses (h$_1$)–(h$_3$) hold, then for every $\hat{u}_1, \hat{u}_2 \in L^2(\mathbb{R}^n)$ the solution set $\mathcal{F}(\hat{u}_1, \hat{u}_2)$ is acyclic in the space $C(I, L^2(\mathbb{R}^n))$ endowed with the weak topology.

**Proof.** Let

\[
E := L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \quad D(A) := H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n),
\]

\[
E_0 := H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \quad D(A_0) := H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n).
\]

Assume that the Hilbert space $E$ is equipped with the norm

\[
\|(x, y)\|_E := \left(\|x\|_2^2 + \|y\|_2^2\right)^{1/2},
\]
and $E_0$ with
\[ \|(x, y)\| := (\|x\|_2^2 + \langle \nabla x, \nabla x \rangle_{L^2(\mathbb{R}^n, \mathbb{R}^n)} + \|y\|_2^2)^{1/2}. \]

The linear operator $A: D(A) \to E$, given by $A(u_1, u_2) := (u_2, \Delta u_1)$, generates an exponentially bounded nondegenerate integrated semigroup \( \{S(t)\}_{t \geq 0} \) on $E$ such that
\[ S(t)(\hat{u}_1, \hat{u}_2) = \left( \int_0^t w(s) \, ds, w(t) - \hat{u}_1 \right), \]
where $w \in C^2([0, \infty), L^2(\mathbb{R}^n))$ satisfies
\[
\begin{align*}
\frac{d^2}{dt^2} \langle w(t), v \rangle &= \langle w(t), \Delta v \rangle, \\
\frac{d}{dt} \langle w(t), v \rangle|_{t=0} &= \langle \hat{u}_2, v \rangle, \\
w(0) &= \hat{u}_1
\end{align*}
\]
for every $v \in H^2(\mathbb{R}^n)$ (see [23], Theorem 7.1). This is a consequence of the fact that the part $A_0: D(A_0) \to E_0$ of $A$ generates a strongly continuous semigroup \( \{T_0(t)\}_{t \geq 0} \) on $(E_0, \| \cdot \|)$, satisfying $\|T_0(t)\|_\mathcal{L} \leq e^{2t}$ (see [20], Ch. 7, Theorem 4.5).

We claim that the resolvent set $\rho(A)$ contains $(2, \infty)$. For every $(f_1, f_2) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ there exists a unique $(u_1, u_2) \in D(A)$ such that
\[
\begin{align*}
u_1 - \lambda u_1 &= f_1, \\
u_2 - \lambda \Delta u_1 &= f_2
\end{align*}
\]
for every real $\lambda \neq 0$ (see [20], Ch. 7, Lemma 4.3). Taking this into consideration, we are able to estimate:
\[
\begin{align*}
\|(f_1, f_2)\|_E^2 &= \langle u_1 - \lambda u_2, u_1 - \lambda u_2 \rangle + \langle u_2 - \lambda \Delta u_1, u_2 - \lambda \Delta u_1 \rangle \\
&= \|u_1\|_E^2 + \lambda^2 \|u_2\|_E^2 - \lambda \langle u_1, u_2 \rangle - \lambda \langle \Delta u_1, u_2 \rangle - \lambda \langle u_2, \Delta u_1 \rangle + \lambda^2 \|\Delta u_1\|_E^2 \\
&\geq \|u_1\|_E^2 + \|u_2\|_E^2 - 2\lambda \langle u_1, u_2 \rangle = \|(u_1, u_2)\|_E^2 - 2\lambda \langle u_1, u_2 \rangle \geq (1 - \lambda) \|(u_1, u_2)\|_E^2 \\
&\geq (1 - 2\lambda)^2 \|(u_1, u_2)\|_E^2
\end{align*}
\]
for $\lambda \in (0, 1/2)$. In other words, for every $\lambda \in (0, 1/2)$ and $f \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ there exists a unique $u \in D(A)$ such that $u - \lambda Au = f$ and
\[
\|u\|_E \leq (1 - 2\lambda)^{-1} \|f\|_E. \tag{32}
\]
Since $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ is dense in $E$ and the operator $A$ is closed, (32) implies that $\text{Im}(\lambda + A) = E$ for $\lambda > 2$, that is, $(2, \infty) \subset \rho(A)$. From (32) it follows also that
\[
\|R_\lambda\|_\mathcal{L} \leq \frac{1}{\lambda - 2} \quad \text{for } \lambda > 2, \tag{33}
\]
with $R_\lambda := (\lambda + A)^{-1}$. 
For every \((u_1, u_2) \in D(A_0)\) the norm \(|| \cdot ||\) has the following bound:

\[
||(u_1, u_2)||^2 = ||u_1||^2_2 + \int_{\mathbb{R}^n} \langle \nabla u_1(x), \nabla u_1(x) \rangle_R^n \, dx + ||u_2||^2_2
\]

\[
= ||u_1||^2_2 - \langle \Delta u_1, u_1 \rangle + ||u_2||^2_2 \leq ||u_1||^2_2 + ||\Delta u_1||_2 ||u_1||_2 + ||u_2||^2_2
\]

\[
\leq ||u_1||^2_2 + \frac{1}{2} ||u_1||^2_2 + \frac{1}{2} ||\Delta u_1||^2_2 + ||u_2||^2_2
\]

\[
\leq \frac{3}{2} (||u_1||^2_2 + ||u_2||^2_2) + \frac{1}{2} (||u_2||^2_2 + ||\Delta u_1||^2_2)
\]

\[
= \frac{3}{2} ||(u_1, u_2)||^2_E + \frac{1}{2} ||A(u_1, u_2)||^2_E.
\]

Hence

\[
||(u_1, u_2)||_E \leq ||(u_1, u_2)|| \leq \sqrt{2} (||(u_1, u_2)||_E + ||A(u_1, u_2)||_E).
\]

That being said, for every initial value \(x \in D(A_0)\) there exists a unique solution \(u \in C^1(\mathbb{R}_+, D(A_0))\) of the abstract Cauchy problem

\[
\begin{align*}
\frac{du}{dt} &= Au(t), \\
u(0) &= x,
\end{align*}
\]

satisfying

\[
||u(t)||_E \leq ||u(t)|| \leq e^{2t} ||x|| \leq \sqrt{2} e^{2t} (||x||_E + ||Ax||_E).
\]

For \(\lambda > 2\), by (33) the function \(w(t) := R_\lambda u(t)\) is a solution of (34) with

\[
||w(t)||_E \leq \sqrt{2} e^{2t} (||R_\lambda x||_E + ||AR_\lambda x||_E)
\]

\[
\leq \sqrt{2} e^{2t} (||R_\lambda x||_E + \lambda ||R_\lambda x||_E + ||x||_E)
\]

\[
\leq \sqrt{2} \left(1 + \frac{\lambda}{\lambda - 2} + 1\right) e^{2t} ||x||_E.
\]

Let \(v(t) := \int_0^t u(s) \, ds\) be the integrated solution. Then \(v(t) = \lambda \int_0^t w(s) \, ds - w(t) + R_\lambda x\). Moreover, the operator \(A\) generates an integrated semigroup \(\{S(t)\}_{t \geq 0}\), given by \(S(t)x = v(t)\) for \(x\) taken from the dense subspace \(D(A_0)\) of the space \(E\) (see [17], Theorem 4.2). Therefore,

\[
||S(t)x||_E \leq \lambda \int_0^t ||w(s)||_E \, ds + ||w(t)||_E + ||R_\lambda x||_E
\]

\[
\leq (1 + \lambda) \sqrt{2} \left(1 + \frac{\lambda}{\lambda - 2} + 1\right) e^{2t} ||x||_E + \frac{1}{\lambda - 2} ||x||_E \leq \frac{\sqrt{2} \lambda (2\lambda + 1)}{\lambda - 2} e^{2t} ||x||_E
\]

for every \(x \in E\) and \(\lambda > 2\). Eventually, we obtain the following exponential bound for the semigroup \(\{S(t)\}_{t \geq 0}\):

\[
||S(t)x||_E \leq \sqrt{2} \inf_{\lambda \in (2, \infty)} \frac{\lambda (2\lambda + 1)}{\lambda - 2} e^{2t} ||x||_E = \sqrt{2} (4\sqrt{5} + 9) e^{2t} ||x||_E.
\]
This semigroup is also equicontinuous, since
\[
\|S(t) - S(\tau)\|_{\mathcal{E}} = \sup_{\|\hat{u}_1, \hat{u}_2\|_{E} \leq 1} \|(S(t) - S(\tau))(\hat{u}_1, \hat{u}_2)\|_{E} \\
\leq \int_{\tau}^{t} \|w(s)\|_{2} ds + \|w(t) - w(\tau)\|_{2}.
\]
Define \(F_1: I \to L^{2}(\mathbb{R}^{n})\) and \(F_2: I \times L^{2}(\mathbb{R}^{n}) \to L^{2}(\mathbb{R}^{n})\) by the formulae
\[
F_1(t) := \begin{cases} 
\{ v \in L^{2}(\mathbb{R}^{n}) : h_1^1(t, x, \int_{\mathbb{R}^{n}} k_1^1(t, y) dy) \\
\leq v(x) \leq h_2^1(t, x, \int_{\mathbb{R}^{n}} k_2^1(t, y) dy) \}\text{ a.e. on } \mathbb{R}^{n}, 
\end{cases}
\]
and
\[
F_2(t, u) := \begin{cases} 
\{ v \in L^{2}(\mathbb{R}^{n}) : h_1^2(t, x, \int_{\mathbb{R}^{n}} k_1^2(t, y) u(y) dy) \\
\leq v(x) \leq h_2^2(t, x, \int_{\mathbb{R}^{n}} k_2^2(t, y) u(y) dy) \}\text{ a.e. on } \mathbb{R}^{n}, 
\end{cases}
\]
Let \(F: I \times E \to E\) be a map given by \(F(t, (u_1, u_2)) := F_1(t) \times F_2(t, u_2)\). Assume that the mapping \(F\) forms a multivalued perturbation of the abstract semilinear integro-differential inclusion (1). To be able to apply Corollary 4 we need to verify that the mapping \(F\) satisfies the sublinear growth condition (21). Notice also that \(F\) is concerned, we have verified it above. Hypotheses (F1) and (F2) follow immediately from assumption (h1).

Take \((u_1, u_2) \in E\) and \((f_1, f_2) \in F(t, (u_1, u_2))\). Then
\[
|f_1(x)| \leq \max \left\{ \left| h_1^1(t, x, \int_{\mathbb{R}^{n}} k_1^1(t, y) dy) \right|, \left| h_2^1(t, x, \int_{\mathbb{R}^{n}} k_2^1(t, y) dy) \right| \right\} \leq c_1(t, x),
\]
\[
|f_2(x)| \leq \max_{1 \leq i \leq 2} \left\{ \left| h_i^2(t, x, \int_{\mathbb{R}^{n}} k_i^2(t, y) u_2(y) dy) \right| \right\} \leq c_2(t, x, \|u_2\|_{2})
\]
and
\[
\begin{cases} 
\|f_1\|_{2} \leq b_1(t) & \text{ a.e. on } I, \\
\|f_2\|_{2} \leq b_2(t)(1 + \|u_2\|_{2}) & \text{ a.e. on } I.
\end{cases}
\]
Whence
\[
\|F(t, (u_1, u_2))\|_{2}^{+} \leq b_1(t) + b_2(t)(1 + \|u_2\|_{2}) \leq (b_1(t) + b_2(t))(1 + \|(u_1, u_2)\|_{E}).
\]
In other words, \(F\) satisfies the sublinear growth condition (21). Notice also that the multimap \(F(t, \cdot)\) is completely continuous (almost everywhere on \(I\)), that is, it maps bounded sets into relatively weakly compact sets (remember that \(L^{2}(\mathbb{R}^{n})\) is reflexive).

It remains to verify assumption (F3). Assume that \((u^k_1, u^k_2) \xrightarrow{E_{k \to \infty}} (u_1, u_2)\) and \((f^k_1, f^k_2) \xrightarrow{E_{k \to \infty}} (f_1, f_2)\) with \((f^k_1, f^k_2) \in F(t, (u^k_1, u^k_2))\) for \(k \geq 1\). Observe that
for $k \geq 1$

$$f^k_2(x) \in \left[ h^2_1(t, x, \int_{\mathbb{R}^n} k^2_j(t, y)u^k_j(y) dy), h^2_2(t, x, \int_{\mathbb{R}^n} k^2_j(t, y)u^k_j(y) dy) \right]$$
a.e. on $\mathbb{R}^n$

and

$$z_j^k := \int_{\mathbb{R}^n} k_j^2(t, y)u^k_j(y) dy \xrightarrow{k \to \infty} z_j := \int_{\mathbb{R}^n} k_j^2(t, y)u^k_2(y) dy$$

for a.a. $t \in I$ and $j = 1, 2$.

Thus,

$$\overline{\bigcup}_{m=k}^{\infty} \{ f^m_2(x) \} \subset \left[ \inf_{m \geq k} h^2_1(t, x, z^k_1), \sup_{m \geq k} h^2_2(t, x, z^k_2) \right]$$

and, by (h$_2$),

$$\bigcap_{k=1}^{\infty} \overline{\bigcup}_{m=k}^{\infty} \{ f^m_2(x) \} \subset \left[ \sup_{k \geq 1} \inf_{m \geq k} h^2_1(t, x, z^k_1), \inf_{k \geq 1} \sup_{m \geq k} h^2_2(t, x, z^k_2) \right]$$

$$\subset \left[ h^2_1(t, x, z_1), h^2_2(t, x, z_2) \right]$$

for almost all $t \in I$, for almost all $x \in \mathbb{R}^n$. Since $f_2(x) \in \bigcap_{k=1}^{\infty} \overline{\bigcup}_{m=k}^{\infty} \{ f^m_2(x) \}$ almost everywhere on $\mathbb{R}^n$ (cf. Corollary 1), we obtain $f_2 \in F_2(t, u_2)$. Notice that $F_1(t)$ is a weakly closed subset of $L^2(\mathbb{R}^n)$. Hence, $f_1 \in F_1(t)$. Consequently, the graph of $F(t, \cdot)$ is sequentially closed in $(E, w) \times (E, w)$ for almost all $t \in I$.

Owing to Corollary 4, we can see that the set $S_F(\bar{u}_1, \bar{u}_2)$ of all integrated solutions to the problem

$$\begin{cases} 
\dot{u}(t) \in Au(t) + F\left(t, \int_0^t u(s) \, ds\right) \quad &\text{on } I, \\
\quad u(0) = (\bar{u}_1, \bar{u}_2) 
\end{cases} \quad (35)$$

forms an $R_\delta$ subset of the space $C(I, E)$ furnished with the weak topology. Consider a projection $\Pi: S_F(\bar{u}_1, \bar{u}_2) \to C(I, L^2(\mathbb{R}^n))$ given by $\Pi(u_1, u_2) := u_2$.

Let $(u_1, u_2) \in S_F(\bar{u}_1, \bar{u}_2)$. Put $w := u_2 + \dot{u}_1$. Observe that $w \in C(I, L^2(\mathbb{R}^n))$. Moreover, since

$$\begin{cases} 
\begin{aligned}
u_1(t) &= t\bar{u}_1 + \int_0^t u_2(s) \, ds + \int_0^t (t-s)f_1(s) \, ds, \\
u_2(t) &= t\bar{u}_2 + \Delta \int_0^t u_1(s) \, ds + \int_0^t (t-s)f_2(s) \, ds,
\end{aligned} & t \in I,
\end{cases}$$

for some $(f_1, f_2) \in N_F((u_1, u_2))$, for every $v \in H^2(\mathbb{R}^n)$ we have

$$\langle w(t), v \rangle = \langle \dot{u}_1, v \rangle + t\langle \dot{u}_2, v \rangle + \left\langle \Delta \int_0^t u_1(s) \, ds, v \right\rangle + \left\langle \int_0^t (t-s)f_2(s) \, ds, v \right\rangle.$$
Hence
\[
\frac{d}{dt} \langle w(t), v \rangle = \langle \dot{u}_2, v \rangle + \langle u_1(t), \Delta v \rangle + \left\langle \int_0^t f_2(s) \, ds, v \right\rangle \\
= \langle \ddot{u}_2, v \rangle + t \langle \dot{u}_1, \Delta v \rangle + \int_0^t \left\langle u_2(s) + \int_0^s f_1(\tau) \, d\tau, \Delta v \right\rangle \, ds + \int_0^t \langle f_2(s), v \rangle \, ds;
\]
which means that \( w \) satisfies (31). On the other hand, if we assume that \( w \) is a weak solution of problem (29), (30), then \( u := (u_1, u_2) \), with
\[
\begin{cases}
  u_1(t) := t \dot{u}_1 + \int_0^t u_2(s) \, ds + \int_0^t (t - s) f_1(s) \, ds, & t \in I, \\
  u_2(t) := w(t) - \dot{u}_1, & t \in I,
\end{cases}
\]
gives an integrated solution of problem (35). Indeed, by integrating twice we obtain
\[
\langle w(t), v \rangle = \langle \dot{u}_1, v \rangle + \langle \dot{t} \ddot{u}_2, v \rangle + \left\langle \int_0^t u_1(s) \, ds, \Delta v \right\rangle + \left\langle \int_0^t (t - s) f_2(s) \, ds, v \right\rangle
\]
for \( t \in I \). As \( \Delta \) is self-adjoint on \( L^2(\mathbb{R}^n) \), we have \( \int_0^t (u_1(s), u_2(s)) \, ds \in D(A) \) and
\[
u_2(t) = t \ddot{u}_2 + \Delta \int_0^t u_1(s) \, ds + \int_0^t (t - s) f_2(s) \, ds, \quad t \in I,
\]
with \( f_2 \in N_{f_2}(u_2) \). Thus, \( (u_1, u_2) \in S_F((\dot{u}_1, \ddot{u}_2)) \).

As a matter of fact, we have shown that \( \dot{u}_1 + \Pi(S_F(\dot{u}_1, \ddot{u}_2)) = \mathcal{S}(\dot{u}_1, \ddot{u}_2) \). It is easy to see that the mapping \( \Pi \colon (S_F(\dot{u}_1, \ddot{u}_2), w) \to (\Pi(S_F(\dot{u}_1, \ddot{u}_2)), w) \) is continuous, surjective and proper. Moreover, a careful look at the set
\[
\{(u_1, u_2) \in S_F(\dot{u}_1, \ddot{u}_2) : u_2 = v \}
\]
reveals that it is essentially an \( R_\delta \)-type set. In practice, this means that the fibre \( \Pi^{-1}(\{v\}) \) is an acyclic subset of the space \( (C(I, E), w) \). Therefore, \( \Pi \) is a Vietoris mapping and \( \bar{H}^\ast((S_F(\dot{u}_1, \ddot{u}_2), w)) \approx \bar{H}^\ast((\Pi(S_F(\dot{u}_1, \ddot{u}_2)), w)) \) (in view of the Vietoris-Begle mapping theorem for the Alexander-Spanier cohomology functor; see [22], Theorem 6.9.15). Clearly, the solution set \( \mathcal{S}(\dot{u}_1, \ddot{u}_2) \) must be an acyclic subset of \( (C(I, L^2(\mathbb{R}^n)), w) \).

Theorem 9 is proved.

Now consider the following initial boundary value problem defined on \( I \times \mathbb{R} \):
\[
\begin{cases}
  \frac{\partial}{\partial t} u(t, x) - \sum_{j=0}^k a_j D^j u(t, x) = U(t) u(t, \cdot)(x) + h(t, x) & \text{in } I \times \mathbb{R}, \\
  u(0, x) = \ddot{u}(x) & \text{on } \mathbb{R}, \\
  \|h(t, \cdot)\|_2 \leq r(t, u(t, \cdot)) & \text{on } I.
\end{cases}
\]

Let \( E \) denote the complex Hilbert space \( L^2(\mathbb{R}, \mathbb{C}) \). Our hypotheses on the mappings \( r \colon I \times E \to \mathbb{R}_+ \) and \( U(t) \colon E \to E \) are the following:
Theorem 10. Assume that hypotheses $(U_1)$ and $(r_1)-(r_3)$ are satisfied. Suppose that $a_k \neq 0$ and $a_j (-i)^{3j} \in \mathbb{R}$ for $j = 0, \ldots, k$. If $\sup_{x \in \mathbb{R}} \Re(p(x)) < \infty$, then for every $\hat{u} \in L^2(\mathbb{R})$ the set $\mathcal{M}(\hat{u})$ of weak solutions to problem (36) forms an $R_\delta$ subset of the space $C(I, L^2(\mathbb{R}, \mathbb{C}))$ endowed with the weak topology.

Proof. Consider the differential operator $A: D(A) \to E$ given by $Af := \sum_{j=0}^{k} a_j D^j f$, defined on

$$D(A) := \left\{ f \in E : \sum_{j=0}^{k} a_j D^j f \in E \text{ distributionally} \right\}.$$ 

Since $a_k \neq 0$, it follows that $D(A) = H^k(\mathbb{R})$ (see [26], Theorem 10.14). The assumption that $a_j (-i)^{3j} \in \mathbb{R}$ for $j = 0, \ldots, k$ means that the differential operator $A$ is self-adjoint on $E$ (cf. [26], Theorem 10.12). In view of Theorem 4.1 in [14] the operator $A$ generates a norm-continuous integrated semigroup $\{S(t)\}_{t \geq 0}$ on the space $E$, given by

$$S(t)f := \frac{1}{\sqrt{2\pi}} \hat{\phi}_t * f,$$
where $\phi_t(x) := \int_0^t e^{p(x)s} ds$ and $\sim$ denotes the inverse of the Fourier transformation. Easy calculations show that

$$
\|\phi_t\|_2^2 \leq \int_{-\infty}^{-L_0} \frac{|e^{p(x)t} - 1|^2}{|p(x)|^2} dx + \int_{-L_0}^{L_0} \frac{|e^{p(x)t} - 1|^2}{|p(x)|^2} dx + \int_{L_0}^{\infty} |e^{p(x)t} - 1|^2 dx
$$

$$
\leq 2 \int_{-\infty}^{-L_0} \frac{16e^{2\omega t}}{|a_k x^k|^2} dx + \int_{-L_0}^{L_0} t^2 e^{2\omega t} dx + \int_{L_0}^{\infty} \frac{16e^{2\omega t}}{|a_k x^k|^2} dx
d$$

$$
= \left( \frac{32(L_0)^{-2k+1}}{(2k-1)|a_k|^2} + 2L_0 t^2 \right) e^{2\omega t}.
$$

For $|x| \geq L_0$ we have

$$
\left| \frac{p'(x)}{p(x)} \right| = \frac{\left| \sum_{j=1}^{k} ja_j x^{j-1} \right|}{|p(x)|} \leq 2 \sum_{j=1}^{k} j |a_j| \frac{|x^{j-1}|}{|a_k x^k|}
$$

$$
= 2 \sum_{j=1}^{k} j |a_j| \frac{|x^{k-j+1}|}{|a_k|} \leq \frac{2R}{|a_k|} \sum_{j=1}^{k} j = \frac{k(k+1)R}{|a_k| |x|}.
$$

Thus,

$$
\left\| \frac{d}{dx} \phi_t \right\|_2^2 \leq 2 \left( \int_{-\infty}^{-L_0} \left| \frac{p'(x)}{p(x)} \right| t e^{p(x)t} \left| \phi_t(x) \right|^2 dx + \int_{-L_0}^{L_0} \left| \frac{p'(x)}{p(x)} \right| \left| \phi_t(x) \right|^2 dx \right)
$$

$$
\leq 2 \int_{|x| \geq L_0} t^2 e^{2\omega t} \left( \frac{k(k+1)R}{|a_k|^2 |x|^2} \right)^2 dx + \int_{-L_0}^{L_0} \left| \frac{p'(x)}{p(x)} \right| t e^{p(x)t} \left| \phi_t(x) \right|^2 dx
$$

$$
\leq 4 \left( t^2 \left( \frac{k(k+1)R}{|a_k|} \right)^2 L_0^{-1} + L_0 t^2 \sup_{|x| \leq L_0} \left| \frac{p'(x)}{p(x)} \right| \right) e^{2\omega t}.
$$

As a result,

$$
\|\phi_t\|_{L^1_2} \leq \left( \frac{32(L_0)^{-2k+1}}{(2k-1)|a_k|^2} + 2L_0 t^2 \right)
$$

$$
+ 4t^2 \left( \frac{k(k+1)R}{|a_k|} \right)^2 L_0^{-1} + 4L_0 t^2 \sup_{|x| \leq L_0} \left| \frac{p'(x)}{p(x)} \right| \right)^{1/2} e^{\omega t}. \quad (38)
$$

Applying Lemma 4.4 from [14], (37) and (38) we obtain the following exponential bound for our semigroup:

$$
\|S(t)f\|_2 \leq \frac{1}{\sqrt{2\pi}} \|\tilde{\phi}_t\|_1 \|f\|_2 \leq \|\phi_t\|_{L^1_2} \|f\|_2 \leq M e^{\omega t} \|f\|_2 . \quad (39)
$$

As a result, assumption (A2) is met.

Define a multimap $F: I \times E \n rightarrow E$ by the formula

$$
F(t, u) := U(t)u + \left\{ v \in E : \|v\|_2 \leq r(t, u) \right\}.
$$
From \((U_1)\) and \((r_1)\) it follows straightforwardly that \(F\) satisfies \((F_1)\) and \((F_2)\). Moreover,

\[
\|F(t, u)\|_2^2 \leq \|U(t)\|_{\mathcal{F}} \|u\|_2 + r(t, u) \leq (b(t) + \|U(t)\|_{\mathcal{F}})(1 + \|u\|_2) \quad \text{a.e. on } I,
\]

that is, \((21)\) holds. The set \(\bigcup_{u \in \Omega} \{v \in E: \|v\|_2 \leq r(t, u)\}\) is relatively weakly compact for almost all \(t \in I\) and any bounded \(\Omega \subset E\), since \(E\) is reflexive. Hence

\[
\beta(F(t, \Omega)) \leq \beta(U(t)\Omega) + \beta \left( \bigcup_{u \in \Omega} \{v \in E: \|v\|_2 \leq r(t, u)\} \right) 
\leq \|U(t)\|_{\mathcal{F}} \beta(\Omega) \quad \text{a.e. on } I.
\]

When it comes to condition \((F_3)\), assume that \(u_k \xrightarrow{E \ n \rightarrow \infty} u\) and \(g_k \xrightarrow{E \ n \rightarrow \infty} g\), where \(g_k \in F(t, u_k)\) for \(k \geq 1\). Suppose that \(g_k = U(t)u_k + f_k\). Observe that

\[
f_k = g_k - U(t)u_k \xrightarrow{E \ k \rightarrow \infty} g - U(t)u
\]

and \(\|g - U(t)u\|_2 \leq \liminf_{k \rightarrow \infty} \|f_k\|_2 \leq \limsup_{k \rightarrow \infty} r(t, u_k) \leq r(t, u)\), by \((r_2)\).

Therefore, the weak limit point \(g = U(t)u + g - U(t)u \in F(t, u)\).

By virtue of Corollary 4, we know that the set \(S_F(\hat{u})\) of all integrated solutions of the Cauchy problem \((1)\) is nonempty \(R_\delta\) in the space \(C(I, E)\) endowed with the weak topology.

Take \(u \in \mathcal{S}(\hat{u})\) and \(v \in H^k(\mathbb{R}^n)\). Since \(\langle u(\cdot), v\rangle \in W^{1,1}(I)\), one has

\[
\langle u(t), v \rangle = \langle \hat{u}, v \rangle + \left( \int_0^t u(s) \, ds, Av \right) + \left( \int_0^t (t-s)U(s)u(s) + h(s) \, ds, v \right).
\]

Since \(A\) is self-adjoint on \(E\) and \(a_k \neq 0\), \(\int_0^t u(s) \, ds \in D(A)\) follows. Thus,

\[
u(t) = \hat{u} + A \int_0^t u(s) \, ds + \int_0^t (t-s)U(s)u(s) + h(s) \, ds.
\]

Observe that \(U(\cdot)u(\cdot) \in L^1(I, E)\) for \(u \in C(I, E)\) and \((U_1)\) holds. Define \(f: I \rightarrow E\) by \(f(t) := U(t)u(t) + h(t)\) for \(t \in I\). Then, \(f \in N_F(u)\). Consequently, \(u \in S_F(\hat{u})\).

Arguing in the opposite direction, fix \(u \in S_F(\hat{u})\). Then there is \(f \in N_F(u)\) such that \(f(t) = U(t)u(t) + h(t)\) with \(h \in L^1(I, E)\) satisfying \(\|h(t)\|_2 \leq r(t, u(t))\) almost everywhere on \(I\). For any \(v \in H^k(\mathbb{R}^n)\) we obtain

\[
\langle u(t), v \rangle = \langle \hat{u}, v \rangle + \left( \int_0^t u(s) \, ds, Av \right) + \left( \int_0^t (t-s)U(s)u(s) + h(s) \, ds, v \right) 
\]

\[
= t\langle \hat{u}, v \rangle + \int_0^t \langle u(s), Av \rangle \, ds + \int_0^t \left( \int_0^s U(\tau)u(\tau) + h(\tau) \, d\tau, v \right) \, ds
\]

on \(I\). Thus

\[
\frac{d}{dt} \langle u(t), v \rangle = \langle \hat{u}, v \rangle + \langle u(t), Av \rangle + \left( \int_0^t U(s)u(s) + h(s) \, ds, v \right)
\]

almost everywhere on \(I\), that is, \(u \in \mathcal{S}(\hat{u})\). Summing up, \(\mathcal{S}(\hat{u}) = S_F(\hat{u})\), and the required result follows.

Theorem 10 is proved.
Remark 8. In view of Theorem 4.4.1 in [25] the operator A, defined above, generates a $C_0$-semigroup of isometries. The mild solution to the problem (36) for $\tilde{u} \in L^2(\mathbb{R}) = \overline{D(\bar{A})}$ coincides with the solution in the sense of Da Prato-Sinestrari, that is, a continuous map $u \in C(I, E)$ such that

$$
\begin{align*}
&\begin{cases}
  u(t) = \tilde{u} + A \int_0^t u(s) \, ds + \int_0^t U(s) \int_0^s u(\tau) \, d\tau + h(s) \, ds \\
  \|h(t)\|_2 \leq r\left(t, \int_0^t u(s) \, ds\right)
\end{cases} \\
&\text{on } t \in I,
\end{align*}
$$

(40)

(cf. [5], Proposition 12.4). On the other hand, the mild solution can be thought of as the weak solution in the sense of Ball to the initial boundary value problem (36), that is, a continuous map $u: I \to E$ such that for every $v \in D(A^*)$, $\langle u(\cdot), v \rangle$ is absolutely continuous and

$$
\begin{align*}
&\begin{cases}
  \frac{d}{dt} \langle u(t), v \rangle = \langle u(t), \sum_{j=0}^k a_j D^j v \rangle + \langle U(t) \int_0^t u(s) \, ds + h(t), v \rangle \\
  u(0) = \tilde{u},
\end{cases} \\
&\|h(t)\|_2 \leq r\left(t, \int_0^t u(s) \, ds\right) \\
&\text{a.e. on } I,
\end{align*}
$$

(41)

on $I$.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset with regular boundary $\Gamma$. Consider the following nondensely defined semilinear feedback control system:

$$
\begin{align*}
&\begin{cases}
  \frac{\partial x}{\partial t} - \Delta x = U(t)x(t, \cdot)(z) \cdot u(t, z) \\
  x|_{I \times \Gamma} = 0, \\
  x(0, z) = x_0(z) \\
  u(t, z) \in U(t, z, x(t, z))
\end{cases} \\
&\text{a.e. on } I \times \Omega,
\end{align*}
$$

(41)

where $\tilde{u}(t) := u(t, \cdot)$.

The feedback set-valued map $U: I \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ meets the conditions:

1. the map $U$ has nonempty closed convex values;
2. the map $U(\cdot, \cdot, x(\cdot))$ is $\mathcal{L}(I) \otimes \mathcal{B}(\overline{\Omega})$-measurable for every $x \in C(\overline{\Omega})$;
3. $\|U(t, z, x)\| \leq \xi(t)(1 + |z|)$ a.e. on $I$ for any $(z, x) \in \overline{\Omega} \times \mathbb{R}$, where $\xi \in L^1(I, \mathbb{R}_+)$;
4. $\sup_{y \in U(t, z_2, x_2)} |y_1 - y_2| \leq k(t)|z_1 - z_2|$ for $z_1, z_2 \in \overline{\Omega}$, $x_1, x_2 \in \mathbb{R}$ and for almost all $t \in I$ with $k \in L^1(I, \mathbb{R}_+)$;

By an integrated solution $x$ of the problem (41) we mean a function $x \in C(I, C(\overline{\Omega}))$ such that

$$
x(t) = tx_0 + \Delta \int_0^t x(s) \, ds + \int_0^t (t - s) U(s)x(s) \cdot u(s) \, ds \\
\text{on } I,
$$

with $u \in L^1(I, C(\overline{\Omega}))$ such that $u(t)(z) \in U(t, z, x(t)(z))$ almost everywhere on $I \times \Omega$. 
Theorem 11. Assume that \( \{U(t)\}_{t \in I} \subset \mathcal{L}(C(\Omega)) \), \( U(\cdot)x \) is measurable for every \( x \in C(\Omega) \) and \( \|U(\cdot)\|_\mathcal{L} \in L^\infty(I) \). Under conditions \((U_1)-(U_4)\) the set \( \mathcal{F}(x_0) \) of integrated solutions of the feedback control system \((41)\) is \( R_5 \) for every \( x_0 \in C(\Omega) \).

Proof. Put \( E := C(\Omega), C_0(\Omega) := \{u \in E : u = 0 \text{ on } \Gamma\} \) and \( Au := \Delta u \) with

\[
D(A) := \{u \in C_0(\Omega) : \Delta u \in E \text{ distributionally}\}.
\]

In view of Proposition 14.6 in [5], \( A \) generates a contraction analytic semigroup

\[
e^{At} = \frac{1}{2\pi i} \int_{+C} e^{\lambda t} R(\lambda, A) \, d\lambda
\]
on \( E \), where \( +C \) is a suitable oriented path in the complex plane. Observe that \( \overline{D(A)} = C_0(\Omega) \subset E \). From Theorem 10.2 in [5] we know that

\[
R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t} e^{At} x \, dt
\]
for each \( x \in E \) and \( \lambda \in \mathbb{C} \) with Re \( \lambda > 0 \). The formula \( S(t) := \int_0^t e^{A\tau} \, d\tau \) for \( t \geq 0 \), defines a strongly continuous exponentially bounded family \( \{S(t)\}_{t \geq 0} \subset \mathcal{L}(E) \) such that \( S(0) = 0 \). In view of Theorem 10.1 in [5], the semigroup \( \{S(t)\}_{t \geq 0} \) is nondegenerate. Clearly, \( (0, \infty) \subset \rho(A) \) and \( R(\lambda, A) = \lambda \int_0^{\infty} e^{-\lambda t} S(t) \, dt \). In other words, operator \( A \) is the generator of an equicontinuous integrated semigroup \( \{S(t)\}_{t \geq 0} \).

Now, \( U(t, \cdot, x(\cdot)) \) is lower semicontinuous, \( U(\cdot, \cdot, x(\cdot)) \) is \( \mathcal{L}(I) \otimes \mathcal{B}(\Omega) \)-measurable and \((I, \mathcal{L}(I), \ell)\) is \( m \)-projective, and so the map \( U(\cdot, \cdot, x(\cdot)) \) admits a Carathéodory-Castaing representation \( (u_n^x)_{n=1}^\infty \), that is, \( U(t, z, x(z)) = \{u_n^x(t, z)\}_{n=1}^\infty \).

Let \( F : I \times E \to E \) be such that \( F(t, x) := U(t) x \cdot \text{co}(\{u^x_n(t, \cdot)\}_{n=1}^\infty) \), with ‘\( \cdot \)’ being the multiplication in the ring \( C(\Omega) \). Since the maps \( U(\cdot) : I \to E \) and \( I \ni t \mapsto u_n^x(t, \cdot) \subset E \) are measurable, the set-valued map \( t \mapsto \{U(t) x \cdot u_n^x(t, \cdot)\}_{n=1}^\infty \) is measurable. Hence, the convex envelope map \( t \mapsto \text{co}(\{U(t) x \cdot u_n^x(t, \cdot)\}_{n=1}^\infty) \) is measurable as well. Finally, the map \( F(\cdot, x) \) satisfies assumption \((F_2)\).

From \((U_4)\) it follows that the family \( \bigcup_{x \in E} \{u_n^x(t, \cdot)\}_{n=1}^\infty \) is equicontinuous. Combined with \((U_3)\) this means that the map \( \bigcup_{x \in M} \{u_n^x(t, \cdot)\}_{n=1}^\infty : E \to E \) is compact for almost all \( t \in I \). It is routine to check that \( \beta(K, M) \leq \|K\| + \beta(M) \) provided that \( K \subset E \) relatively compact and \( M \subset E \) bounded. Thus,

\[
\beta(F(t, M)) \leq \beta \left( \text{co}(U(t) M \cdot \left( \bigcup_{x \in M} \{u_n^x(t, \cdot)\}_{n=1}^\infty \right)) \right) \\
\leq \left\| \bigcup_{x \in M} \{u_n^x(t, \cdot)\}_{n=1}^\infty \right\|^+ \|U(t)\|_{\mathcal{F}} \beta(M) \leq \xi(t)(1 + \|\Omega\|^+)\|U(t)\|_{\mathcal{F}} \beta(M)
\]
almost everywhere on \( I \), by \((U_3)\). Hence \((F_5)\) holds with \( \eta := (1 + \|\Omega\|^+)\|U(\cdot)\|_{\mathcal{F}} \).

Since \( U(t, \cdot, \cdot) \) is upper hemicontinuous, the sequential upper semicontinuity of the map \( F(t, \cdot) : (E, w) \to (E, w) \) is a straightforward consequence of the Riesz-Markov representation theorem and the convergence theorem.
Observe that
\[ \|F(t, x)\| + \|U(t)\|_{\mathcal{X}} \leq (1 + \|\Omega\|) \|U(t)\|_{\mathcal{X}} (1 + \|x\|) \]

by (U₃). Hence, (21) is met. Equivalently, we can rewrite the feedback control problem (41) as (1) with \( A \) and \( F \) as above. Since the assumptions of Corollary 4 are satisfied, the result follows. Theorem 11 is proved.

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Received 26/SEP/19 and 20/MAR/21