Upper bounds and asymptotic expansion for Macdonald’s function and the summability of the Kontorovich-Lebedev integrals

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ABSTRACT
Uniform upper bounds and the asymptotic expansion with an explicit remainder term are established for the Macdonald function $K_\tau (x)$. The results can be applied, for instance, to study the summability of the divergent Kontorovich-Lebedev integrals in the sense of Jones. Namely, we answer affirmatively a question (cf. [Ehrenmark U. Summability experiments with a class of divergent inverse Kontorovich-Lebedev transforms. Comput Math Appl. 2018;76(1):141–154.]) whether these integrals converge for even entire functions of the exponential type in a weak sense.

1. Upper bounds of the Lebedev type
In the theory of the Kontorovich-Lebedev transform (KL-transform) [1], whose inverse involves the integration with respect to the pure imaginary index of the modified Bessel function or Macdonald function $K_\tau (x)$ it is important to know its asymptotic behaviour at infinity by the index for a bounded positive argument, or to have an upper bound, being valued for arbitrary real parameter $\tau$ and positive argument $x$. For instance, to investigate mapping properties in Lebesgue spaces of the Kontorovich-Lebedev operator we oftenly use the so-called Lebedev inequality (cf. [1, p. 219])

$$|K_\tau (x)| \leq A \frac{x^{-1/4}}{\sqrt{\sinh(\pi \tau)}}, \quad x, \tau > 0,$$

(1.1)

where $A > 0$ is the absolute constant which will be defined below. As far as the author is aware, such bounds are unknown for the Mehler-Fock, index Whittaker, Olevskii transforms, Lebedev’s transform with the product of the modified Bessel functions among the others index transformations [1], and will be established in the forthcoming paper. Moreover, Lebedev conjectured in 1951 in his Habilitation Thesis more tight than (1.1) bounds
by the index
\[ |K_{i\tau}(x)| \leq A \frac{x^{1/4}}{\sqrt{\tau \sinh(\pi \tau)}}, \quad 0 < x \leq 1 \ (x \geq 1), \ \tau > 0. \] (1.2)

Our goal will be to establish such kind of upper bounds being valued for arbitrary positive parameters in the index and argument. To do this, we will appeal to the bounds for Bessel function of the first kind [2], namely to estimate the value
\[ c_v = \sup_{x \geq 0} \sqrt{x} |J_v(x)|, \quad v \geq -\frac{1}{2}. \] (1.3)

The classical Szegö result says that \( c_v = \sqrt{2/\pi} \) for \( |v| \leq 1/2 \). For \( v > 0 \) Olenko proved the following inequality
\[ c_v \leq b \left( v^{1/3} + \frac{\alpha}{v^{1/3}} + \frac{3\alpha^2}{10v} \right)^{1/2}, \] (1.4)
where \( b = 0.674885 \ldots \) is the Landau constant and \( \alpha = 1.855757 \ldots \).

Let us give an explicit value for \( A \) in the Lebedev inequality (1.1). To proceed this, we will consider the following integral (cf. [3], Vol. II, Entry 2.12.14.2)
\[ K_{i\tau}^2(x) = \frac{\pi}{\sinh(\pi \tau)} \int_0^\infty J_0(2x \sinh t) \sin(2\tau t) \, dt. \] (1.5)

Hence via elementary substitutions in the integral (1.5), and since the modified Bessel function \( K_{i\tau}(x) \), \( x, \tau > 0 \) is real-valued, we have the estimate
\[
K_{i\tau}^2(x) \leq \frac{\pi}{\sinh(\pi \tau)} \int_0^\infty |J_0(2x \sinh t)| \, dt = \frac{\pi}{\sinh(\pi \tau)} \int_0^\infty |J_0(2xy)| \, \frac{dy}{(y^2 + 1)^{1/2}}
\]
\[
\leq \frac{\pi}{(2x)^{1/2} \sinh(\pi \tau)} \int_0^\infty \frac{dy}{y^{1/2}(y^2 + 1)^{1/2}}.
\]

The latter supremum is \( c_0 = \sqrt{2/\pi} \) via (1.2). Therefore, calculating a simple beta-integral (see [3], Vol. I, Entry 2.2.4.24), we derive the Lebedev inequality (1.1) with an explicit constant
\[ |K_{i\tau}(x)| \leq \frac{\Gamma(1/4)}{\sqrt{2}} \frac{x^{-1/4}}{\sqrt{\sinh(\pi \tau)}}, \quad x, \tau > 0, \] (1.6)
where \( \Gamma(z) \) is the Euler gamma function [3], Vol. III. Now we will derive a family of Lebedev’s upper bounds, appealing to the integral in [3], Vol. II, Entry 2.12.4.28, namely
\[ K_{v-\rho+1}(x) = \left( \frac{2}{x} \right)^{\rho-1} \Gamma(\rho) \int_0^\infty \frac{y^{v+1} J_v(xy)}{(1 + y^2)^{\rho}} \, dy, \] (1.7)
where \( x > 0, \ -1 < v < 2\Re\rho - 1/2 \). In fact, letting \( \rho = v - \mu + 1 + i\tau, \ \mu < (v + 1/2)/2, \) we have from (1.4), (1.7)
\[
|K_{\mu+i\tau}(x)| \leq 2^v \mu^{-1} c_v |\Gamma(v - \mu + 1 + i\tau)| x^{v - \mu - 1/2} \int_0^\infty \frac{y^{v+1/2}}{(1 + y^2)^{\mu+1}} \, dy
\]
\[
= 2^v \mu^{-1} c_v \frac{\Gamma((v + 3/2)/2) \Gamma((v - 2\mu + 1/2)/2)}{\Gamma(v - \mu + 1)} |\Gamma(v - \mu + 1 + i\tau)| x^{v - \mu - 1/2},
\]
i.e. we derive the following inequality
\[ |K_{\mu+i\tau}(x)| \leq 2^{v-\mu-1} c_v \frac{\Gamma((v + 3/2)/2)\Gamma((v - 2\mu + 1/2)/2)}{\Gamma(v - \mu + 1)} \times \left| \Gamma\left(v - \mu + 1 + i\tau\right)\right| x^{\mu-v-1/2}, \quad x, \tau > 0. \tag{1.8} \]

When \( \tau \to +\infty \) the gamma function in (1.8) behaves as \( O(\tau^{v-\mu+1/2} \exp(-\pi \tau/2)) \). Consequently, letting \( \mu = 0 \) in (1.8), and taking into account values of \( c_v \) in (1.4), it gives after a slight simplification the following upper bounds
\[ |K_{i\tau}(x)| \leq \frac{\Gamma(v + 1/2)}{\Gamma(v + 1)} \left| \Gamma(v + 1 + i\tau)\right| x^{v-1/2}, \quad -\frac{1}{2} < v \leq \frac{1}{2}, \tag{1.9} \]
\[ |K_{i\tau}(x)| \leq b \left(v^{1/3} + \frac{\alpha}{v^{1/3}} + \frac{3\alpha^2\sqrt{10v}}{10v}\right)^{1/2} \frac{\Gamma(v + 1/2)}{\Gamma(v + 1)} \times \left| \Gamma(v + 1 + i\tau)\right| x^{v-1/2}, \quad v > 0. \tag{1.10} \]

If \( v = 0 \) in (1.9), it yields the modified Lebedev inequality
\[ |K_{i\tau}(x)| \leq \sqrt{\frac{\pi^2 \tau}{x \sinh(\pi \tau)}}, \quad x, \tau > 0. \tag{1.11} \]

Let \( v = -1/2 + \delta, \quad 0 < \delta < 1 \). Hence, inequality (1.9) implies
\[ |K_{i\tau}(x)| \leq \frac{\Gamma(\delta)}{\Gamma(\delta + 1/2)} \left| \Gamma\left(\frac{1}{2} + \delta + i\tau\right)\right|, \quad x, \tau > 0. \tag{1.12} \]

Let \( \mu = v < 1/2 \). Then we find from (1.8)
\[ |K_{v+i\tau}(x)| \leq \Gamma((v + 3/2)/2)\Gamma((1/2 - v)/2) \sqrt{\frac{\tau}{2x \sinh(\pi \tau)}}, \quad x, \tau > 0. \tag{1.13} \]

One can put \( v = -1/2 \) in (1.13). This yields immediately uniform upper bounds for the kernels \( \Re K_{1/2+i\tau}(x), \quad \Im K_{1/2+i\tau}(x) \) of the Lebedev-Skalskaya transforms [1]. Namely, we have, for instance for the \( \Re \)-transform
\[ \left| \Re K_{1/2+i\tau}(x) \right| = \left| \Re K_{-1/2-i\tau}(x) \right| = \left| \Re K_{-1/2+i\tau}(x) \right| \]
\[ \leq \left| K_{-1/2+i\tau}(x) \right| \leq \sqrt{\frac{\pi^2 \tau}{2x \sinh(\pi \tau)}}, \quad i.e. \quad \left| \Re K_{1/2+i\tau}(x) \right| \leq \sqrt{\frac{\pi^2 \tau}{2x \sinh(\pi \tau)}}, \quad x, \tau > 0, \tag{1.14} \]
i.e. (as well as for the \( \Im \)-transform)
\[ \left| \Im K_{1/2+i\tau}(x) \right| \leq \sqrt{\frac{\pi^2 \tau}{2x \sinh(\pi \tau)}}, \quad x, \tau > 0. \tag{1.15} \]
The case \( \mu = 1, \nu \geq 3/2 \) in (1.8) gives the inequality
\[
|K_{1+i\tau}(x)| \leq 2^{\nu-2} c_{\nu} \frac{\Gamma((\nu + 3/2)/2)\Gamma((\nu - 3/2)/2)}{\Gamma(\nu)} \left| \Gamma\left(\nu + i\tau\right) \right| x^{1/2-\nu}. \tag{1.16}
\]

Taking into account properties for the modified Bessel functions [2], Vol. II, namely, the equality
\[
\tau K_{i\tau}(x) = x \text{ Im } K_{1+i\tau}(x), \tag{1.17}
\]
we obtain from (1.16)
\[
|K_{i\tau}(x)| \leq 2^{\nu-2} c_{\nu} \frac{\Gamma((\nu + 3/2)/2)\Gamma((\nu - 3/2)/2)}{\tau \Gamma(\nu)} \left| \Gamma\left(\nu + i\tau\right) \right| x^{3/2-\nu}. \tag{1.18}
\]

If \( \nu = 3/2 + \delta, \delta > 0 \), we have (compare with (1.12))
\[
|K_{i\tau}(x)| \leq 2^{3/2+\delta} c_{\nu} \frac{\Gamma(3 + \delta/2)\Gamma(\delta/2)}{\tau \Gamma(3/2 + \delta)} \left| \Gamma\left(3/2 + \delta + i\tau\right) \right| x^{-\delta}, \quad x, \tau > 0. \tag{1.19}
\]

Further, returning to the unproved Lebedev bound (1.2), we recall integral (1.5) and the derivative of the Bessel function \( J_0(z) \) to write it in terms of the iterated improper integral
\[
\frac{1}{\pi} \sinh(\pi \tau) K^2_{i\tau}(x) = - \lim_{T \to \infty} \int_0^T \sin(2\tau t) \int_0^{2x \sinh t} J_1(y) \, dy \, dt. \tag{1.20}
\]

Interchanging the order of integration in (1.20) owing to the uniform convergence by \( t \in [0, T] \), it reads
\[
\frac{1}{\pi} \sinh(\pi \tau) K^2_{i\tau}(x) = \lim_{T \to \infty} \int_0^T \sin(2\tau t) \left[ 1 - \int_0^{2x \sinh t} J_1(y) \, dy \right] dt
\]
\[
= \lim_{T \to \infty} \left[ \frac{1 - \cos(2\tau T)}{2\tau} - \int_0^T J_1(y) \int_0^T \sin(2\tau t) \, dt \, dy \right].
\]

Hence the integration with respect to \( t \) and simple change of variables yield
\[
\frac{1}{\pi} \sinh(\pi \tau) K^2_{i\tau}(x) = \lim_{T \to \infty} \left[ \frac{1 - \cos(2\tau T)}{2\tau} + \frac{x}{\tau} \int_0^{T/(2x)} J_1(2xy) \left[ \cos(2\tau T) - \cos\left(2\tau \log\left(y + (y^2 + 1)^{1/2}\right)\right) \right] dy \right]
\]
\[
= \lim_{T \to \infty} \frac{1}{2\tau} \left[ 1 - \cos(2\tau T) J_0(T) - 2x \int_0^{T/(2x)} J_1(2xy) \cos\left(2\tau \log\left(y + (y^2 + 1)^{1/2}\right)\right) dy \right]. \tag{1.21}
\]
Passing to the limit, we find the following representation

\[ \frac{\tau}{\pi} \sinh(\pi \tau) K_{\tau}^2(x) = 1 - 2x \int_0^\infty I_1(2xy) \times \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) dy. \]  

(1.22)

In order to proceed further estimations we will employ the equality for Bessel function from [4]

\[ J_\nu(z) = \left( \frac{2}{\pi z} \right)^{1/2} \left( \cos \left( z - \frac{\pi}{2} \nu - \frac{\pi}{4} \right) \left( \sum_{n=0}^{N-1} (-1)^n \frac{a_{2n}(\nu)}{z^{2n}} + R_{2N}(z, \nu) \right) \right. \]

\[ - \sin \left( z - \frac{\pi}{2} \nu - \frac{\pi}{4} \right) \left( \sum_{n=0}^{M-1} (-1)^n \frac{a_{2n+1}(\nu)}{z^{2n+1}} - R_{2M+1}(z, \nu) \right), \]  

(1.23)

where

\[ a_n(\nu) = (-1)^n \frac{\cos(\pi \nu)}{2^n n! \pi} \Gamma \left( n + \frac{1}{2} + \nu \right) \Gamma \left( n + \frac{1}{2} - \nu \right), \]  

(1.24)

and the remainder is given by the integral [4, formula (23)]

\[ R_N(z, \nu) = (-1)^{[N/2]} \left( \frac{2}{\pi} \right)^{1/2} \frac{\cos(\pi \nu)}{z^N \pi} \int_0^\infty \frac{t^{N-1/2} e^{-t} K_\nu(t)}{1 + (t/z)^2} dt, \]  

(1.25)

provided \(|\text{Re}\nu| < N - 1/2, \ |\arg z| < \pi/2\). Therefore we write by virtue of (1.4), (1.22), (1.23) and (1.25)

\[ \frac{\tau}{\pi} \sinh(\pi \tau) K_{\tau}^2(x) \leq 1 + 2b \sqrt{2x} \left( 1 + \alpha + \frac{3\alpha^2}{10} \right)^{1/2} \]

\[ + \left( \frac{4x}{\pi} \right)^{1/2} \left| \int_1^\infty \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y}} \right| \]

\[ + \left( \frac{4x}{\pi} \right)^{1/2} \left[ \sum_{n=1}^{N-1} \frac{\Gamma \left( 2n + \frac{3}{2} \right) \Gamma \left( 2n - \frac{1}{2} \right)}{4^n (2n)! (2n - 1/2)} \sum_{n=0}^{M-1} \frac{\Gamma \left( 2n + \frac{5}{2} \right) \Gamma \left( 2n + \frac{1}{2} \right)}{4^n (2n + 1)! (4n + 1)} \right] \]

\[ + \frac{2\sqrt{2x}}{\pi^2} \int_1^\infty \int_0^\infty \frac{ye^{-t} K_1(t)}{y^2 + t^2} \left[ \left( \frac{t}{y} \right)^{2N-1/2} + \left( \frac{t}{y} \right)^{2M+1/2} \right] dt dy, \]
where \( x, \tau > 0, M, N \in \mathbb{N} \). The latter double integral can be estimated via Entry 8.4.23.3 in [3], Vol. III and an elementary inequality. Thus we get finally

\[
\begin{align*}
\frac{\tau}{\pi} \sinh(\pi \tau) K_{2\tau}^2(x) &\leq 1 + 2b\sqrt{2x} \left( 1 + \alpha + \frac{3\alpha^2}{10} \right)^{1/2} \\
&+ \frac{4\sqrt{x}}{\pi} \left[ \sum_{n=1}^{N-1} 2^{2n-1} \Gamma \left( 2n - \frac{1}{2} \right) B \left( 2n + \frac{3}{2}, 2n - \frac{1}{2} \right) \\
&+ \sum_{n=0}^{M-1} 4^n \Gamma \left( 2n + \frac{1}{2} \right) B \left( 2n + \frac{5}{2}, 2n + \frac{1}{2} \right) \\
&+ \frac{2\sqrt{x}}{\pi^2} \left[ 4^N \Gamma \left( 2N - \frac{3}{2} \right) B \left( 2N - \frac{3}{2}, 2N + \frac{1}{2} \right) \\
&+ 4^{M-1} \Gamma \left( 2M - \frac{1}{2} \right) B \left( 2M - \frac{1}{2}, 2M + \frac{3}{2} \right) \right] \\
&+ \left( \frac{4x}{\pi} \right)^{1/2} \left| \int_{1}^{\infty} \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y}} \right|,
\end{align*}
\]

(1.26)

where \( B(z, w) \) is the Euler beta function [3]. The main goal is to estimate the integral in (1.26). To do this, we fix a positive \( \delta \) and split the integral as follows

\[
\begin{align*}
\int_{1}^{\infty} \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y}} &= \left( \int_{1}^{(1+\tau)^{2\delta}} + \int_{(1+\tau)^{2\delta}}^{\infty} \right) \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y}}.
\end{align*}
\]

Then

\[
\left| \int_{1}^{(1+\tau)^{2\delta}} \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y}} \right| \\
\leq \int_{1}^{(1+\tau)^{2\delta}} \frac{dy}{\sqrt{y}} = 2 \left[ (1 + \tau)^{\delta} - 1 \right].
\]

Concerning the second integral, we have, owing to the integration by parts,

\[
\begin{align*}
\int_{(1+\tau)^{2\delta}}^{\infty} \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y}} &= \frac{(1 + \tau)^{-\delta}}{2x} \cos \left( 2x(1 + \tau)^{2\delta} - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( (1 + \tau)^{2\delta} + ((1 + \tau)^{4\delta} + 1)^{1/2} \right) \right) \\
&- \tau \int_{(1+\tau)^{2\delta}}^{\infty} \cos \left( 2xy - \frac{\pi}{4} \right) \sin \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{\sqrt{y(y^2 + 1)}} \\
&- \frac{1}{4x} \int_{(1+\tau)^{2\delta}}^{\infty} \cos \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + (y^2 + 1)^{1/2} \right) \right) \frac{dy}{y^{3/2}},
\end{align*}
\]
and, accordingly,
\[
\left| \int_{(1+\tau)^{2\delta}}^\infty \sin \left( 2xy - \frac{\pi}{4} \right) \cos \left( 2\tau \log \left( y + \left( y^2 + 1 \right)^{1/2} \right) \right) \frac{dy}{\sqrt{y}} \right| \\
\leq \frac{1}{x} (1 + 2\tau) (1 + \tau)^{-\delta}.
\]

Thus, combing with (1.26), we establish the following inequality for the Kontorovich-Lebedev kernel \((x, \tau > 0, \delta > 0, M, N \in \mathbb{N})\)
\[
|K_{ir}(x)| \leq \frac{\sqrt{\pi}}{\sqrt{\tau \sinh(\pi \tau)}} \left[ \frac{2}{\sqrt{\pi x}} (1 + 2\tau) (1 + \tau)^{-\delta} \\
+ \frac{4\sqrt{x}}{\pi^2} \sum_{n=1}^{N-1} 2^{2n-1} \Gamma \left( 2n - \frac{1}{2} \right) B \left( 2n - \frac{1}{2}, 2n - \frac{1}{2} \right) \\
+ \sum_{n=0}^{M-1} 4^n \Gamma \left( 2n + \frac{1}{2} \right) B \left( 2n + \frac{5}{2}, 2n + \frac{1}{2} \right) \\
+ \frac{2\sqrt{x}}{\pi^2} \left[ 4 \Gamma \left( 2N - \frac{1}{2} \right) B \left( 2N - \frac{3}{2}, 2N + \frac{1}{2} \right) \\
+ 4 \Gamma \left( 2M - \frac{1}{2} \right) B \left( 2M - \frac{3}{2}, 2M + \frac{1}{2} \right) \right]^{1/2}. \tag{1.27}
\]

Finally in this section, we show that the integral (see [3], Vol. II, Entry 2.16.3.6)
\[
K_{ir}^2(x) = 2 \int_1^\infty \frac{K_{2ir}(2xy)}{(y^2 - 1)^{1/2}} \, dy \tag{1.28}
\]
provides a set of the so-called iterated Lebedev type upper bounds. In fact, owing to (1.6), we find from (1.28)
\[
K_{ir}^2(x) \leq \frac{(2/x)^{1/4} \Gamma(1/4)}{\sqrt{\sinh(2\pi \tau)}} \int_1^\infty \frac{y^{-1/4}}{(y^2 - 1)^{1/2}} \, dy \\
= \frac{2^{-3/2} \Gamma^2(1/8) x^{-1/4}}{\sqrt{\sinh(2\pi \tau)}},
\]
i.e. we derive the following inequality
\[
|K_{ir}(x)| \leq \frac{\Gamma(1/8)}{2^{3/4}} \frac{x^{-1/8}}{\sinh^{1/4}(2\pi \tau)}, \quad x, \tau > 0. \tag{1.29}
\]

Now, returning to (1.28), we apply (1.29) to get the estimate
\[
K_{ir}^2(x) \leq \frac{(2/x)^{1/8} \Gamma(1/8)}{\sinh^{1/4}(4\pi \tau)} \int_1^\infty \frac{y^{-1/8}}{(y^2 - 1)^{1/2}} \, dy = \frac{2^{-7/4} \Gamma^2(1/16) x^{-1/8}}{\sinh^{1/4}(4\pi \tau)}.
\]
Hence it yields the inequality
\[ |K_{it}(x)| \leq \frac{\Gamma(1/16)}{2^{7/8}} \frac{x^{-1/16}}{\sinh^{1/8}(4\pi \tau)}, \quad x, \tau > 0. \quad (1.30) \]

Continuing to apply the bound which is obtained on the \(n\)-th step to integral (1.28), we observe the sequence of the Lebedev type inequalities
\[ |K_{it}(x)| \leq \frac{\Gamma \left( \frac{2^{-n-1}}{2^{1-2^{-n}}} \right)}{\sqrt{x} \sinh \left( \frac{2^n \pi \tau}{2} \right)} ^{-2^{-n}}, \quad n \in \mathbb{N}, \ x, \tau > 0. \quad (1.31) \]

### 2. Uniform asymptotic expansion

In this section, we propose a new approach to establish a uniform asymptotic expansion for the kernel \(K_{it}(x)\), comparing with [1, Section 1.2], and give an explicit remainder term with the corresponding error estimate. We start, appealing to its relation with the modified Bessel function of the first kind
\[ K_{it}(x) = \frac{\pi}{2i \sinh(\pi \tau)} \left[ I_{-it}(x) - I_{it}(x) \right], \quad x, \tau > 0, \quad (2.1) \]
where
\[ I_{v}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+v}}{k! \Gamma(k+v+1)}. \quad (2.2) \]

Hence, employing the reflection formula for gamma function, we write (2.1) in the form
\[
K_{it}(x) = \frac{\Gamma(it)}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k-it} \Gamma(1-it)}{k! \Gamma(k-it+1)} + \frac{\Gamma(-it)}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k+it} \Gamma(1+it)}{k! \Gamma(k+it+1)}
\]
\[
= \Re \left[ \Gamma(it) \left( \frac{x}{2} \right)^{-it} \right] + \frac{\Gamma(it)}{2} \sum_{k=1}^{\infty} \frac{(x/2)^{2k-it} \Gamma(1-it)}{k! \Gamma(k-it+1)}
\]
\[
+ \frac{\Gamma(-it)}{2} \sum_{k=1}^{\infty} \frac{(x/2)^{2k+it} \Gamma(1+it)}{k! \Gamma(k+it+1)}. \quad (2.3)
\]

Then, appealing to the simple beta-integral, we obtain the equality
\[
K_{it}(x) = \Re \left[ \Gamma(it) \left( \frac{x}{2} \right)^{-it} \right] + \frac{\Gamma(it)}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k-it+2}}{(k+1)! k!} \int_{0}^{1} y^{-it} (1-y)^k \ dy
\]
\[
+ \frac{\Gamma(-it)}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k+it+2}}{(k+1)! k!} \int_{0}^{1} y^{it} (1-y)^k \ dy. \quad (2.3)
\]

Interchanging the order of integration and summation via the absolute and uniform convergence and using (2.2), we derive after simple substitutions
\[ K_{it}(x) = \Re \left[ \Gamma(it) \left( \frac{x}{2} \right)^{-it} \left[ 1 + x^{2it} \int_{0}^{x} (x^2 - y^2)^{-it} I_{1} (y) \ dy \right] \right]. \quad (2.4) \]
The integral in (2.4) can be treated via integration by parts and the use of the differential relation for the modified Bessel function of the first kind $I_\nu(z)$ [3, Vol. II]

$$\frac{d}{dz} [z^{-\nu} I_\nu(z)] = z^{-\nu} I_{\nu+1}(z). \quad (2.5)$$

Indeed, we find after $N$ times consecutive integration by parts

$$x^{2i\tau} \int_0^x (x^2 - y^2)^{1-i\tau} I_1(y) \frac{dy}{y} = x^2 \frac{\nu}{N-1} + \nu I_\nu + \frac{1}{2N} \int_0^x (x^2 - y^2)^{N-i\tau} I_{N+1}(y) \frac{dy}{y^{N/2}}, \quad (2.6)$$

where $(a)_n$ is the Pochhammer symbol. Thus we derive the following key formula for the asymptotic expansion of the Kontorovich-Lebedev kernel by the index

$$K_{i\tau}(x) = \Re \left[ \Gamma(i\tau) \left( \frac{x}{2} \right)^{i\tau} \left[ 1 + \sum_{m=1}^{N} \frac{(x/2)^{2m}}{m! (1 - i\tau)_m} \right. 
+ \frac{x^{2i\tau}}{2^N (1 - i\tau)_N} \int_0^x (x^2 - y^2)^{N-i\tau} I_{N+1}(y) \frac{dy}{y^{N/2}} \right] \right], \quad x, \tau > 0, \ N \in \mathbb{N}_0. \quad (2.7)$$

Precisely, we have

**Theorem 2.1:** Let $N$ be a non-negative integer and $x \in (0, X], \ X > 0$. Then the modified Bessel function $K_{i\tau}(x)$ has the following asymptotic expansion

$$K_{i\tau}(x) = \sqrt{\frac{2\pi}{\tau}} e^{-\pi \tau/2} \left[ \cos \left( \tau \log \left( \frac{2\tau}{ex} \right) - \frac{\pi}{4} \right) + R_N(\tau) \right], \quad \tau \to +\infty, \quad (2.8)$$

where the remainder term is given explicitly

$$R_N(\tau) = \Re \left[ \exp \left( i \left( \tau \log \left( \frac{2\tau}{ex} \right) - \frac{\pi}{4} \right) \right) \left[ r(\tau) + \left( 1 + r(\tau) \right) \right. 
\times \left[ \sum_{m=1}^{N} \frac{(x/2)^{2m}}{m! (1 - i\tau)_m} \right. 
+ \frac{x^{2i\tau}}{2^N (1 - i\tau)_N} \int_0^x (x^2 - y^2)^{N-i\tau} I_{N+1}(y) \frac{dy}{y^{N/2}} \right] \right], \quad (2.9)$$

and

$$r(\tau) = \exp \left( \int_0^\infty e^{-it} \left[ \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] dt \right) - 1. \quad (2.10)$$
Moreover, the remainder term \( R_N(\tau) \) has the following upper bound

\[
|R_N(\tau)| \leq \frac{1}{\tau} \left[ \frac{e^{1/(6\tau_0)}}{6} + \left( \tau_0 + \frac{e^{1/(6\tau_0)}}{6} \right) \left[ \exp \left( \frac{X^2}{4\tau_0} \right) + \left( \frac{X^2}{2\tau_0} \right)^N \left[ \frac{I_N(X)}{X^N} - \frac{1}{2^N N!} \right] \right] \right], \tag{2.11}
\]

where \( \tau \geq \tau_0 > 0 \).

**Proof:** In fact, appealing to the Stirling formula for gamma function [5] for the pure imaginary argument, we have the equality

\[
\Gamma(i\tau) \left( \frac{X}{2} \right)^{-i\tau} = \sqrt{\frac{2\pi}{\tau}} \exp \left( -\frac{\pi \tau}{2} + i \left( \tau \log \left( \frac{2\tau}{e} \right) - \frac{\pi}{4} \right) \right) [1 + r(\tau)], \tag{2.12}
\]

where \( r(\tau) \) is defined by (2.10). Then, substituting the right-hand side of (2.11) in (2.7) and taking the real part, we get (2.8). In order to prove inequality (2.11), we use the known bound for the remainder \( r(\tau) \) [5], namely,

\[
|r(\tau)| \leq \frac{e^{1/(6\tau_0)}}{6} - 1, \tag{2.13}
\]

where \( \tau \) is bounded away from zero, i.e. \( \tau \geq \tau_0 > 0 \). Hence, from (2.13) we derive the estimate

\[
|r(\tau)| \leq \frac{1}{6\tau} \sum_{k=0}^{\infty} \frac{(6\tau_0)^{-k}}{(k+1)!} \leq \frac{e^{1/(6\tau_0)}}{6\tau}.
\]

Therefore, we have from (2.2), (2.5) and (2.9)

\[
|R_N(\tau)| \leq |r(\tau)| + (1 + |r(\tau)|)
\]

\[
\times \left[ \sum_{m=1}^{N} \frac{(X/2)^{2m}}{m! |(1-i\tau)m|} + \frac{1}{2^N N! |(1-i\tau)_N|} \int_0^X \left( X^2 - y^2 \right)^N I_{N+1} \left( \frac{\tau y}{\sqrt{\tau_0}} \right) \frac{dy}{y^{N+1}} \right]
\]

\[
\leq \frac{1}{\tau} \left[ \frac{e^{1/(6\tau_0)}}{6} + \left( \tau_0 + \frac{e^{1/(6\tau_0)}}{6} \right) \left[ \sum_{m=1}^{N} \frac{(X/2\sqrt{\tau_0})^{2m}}{m!} \right] \right]
\]

\[
+ \frac{1}{(2\tau_0)^N} \int_0^X \left( X^2 - y^2 \right)^N I_{N+1} \left( \frac{\tau y}{\sqrt{\tau_0}} \right) \frac{dy}{y^{N+1}} \right]
\]

\[
\leq \frac{1}{\tau} \left[ \frac{e^{1/(6\tau_0)}}{6} + \left( \tau_0 + \frac{e^{1/(6\tau_0)}}{6} \right) \left[ \exp \left( \frac{X^2}{4\tau_0} \right) + \left( \frac{X^2}{2\tau_0} \right)^N \left[ \frac{I_N(X)}{X^N} - \frac{1}{2^N N!} \right] \right] \right].
\]

This proves (2.11) and completes the proof of Theorem 2.1. ■
3. Summability of the KL-integrals in the Jones sense

As it is suggested in [6, Section 5], we will consider the following Kontorovich-Lebedev integrals in the Jones sense [7]

\[ f(x, a) \equiv \lim_{\varepsilon \to 0^+} \int_0^\infty e^{-\varepsilon \tau^2} \left[ f_1(\tau) \cosh \left( \frac{\pi}{2} + a \right) \tau 
+ f_2(\tau) \tau \sinh \left( \frac{\pi}{2} + a \right) \tau \right] K_{ir}(x) \, d\tau, \]  

(3.1)

where \( x > 0, a \) is a parameter and \( f_1(\tau), f_2(\tau) \) are even entire functions of the exponential type. Our goal is to prove the existence of the limit (3.1) in a weak sense under the restriction \( \Re a \in [0, \pi/2) \).

First we observe that when \( -\pi < \Re a < 0 \) the limit (3.1) exists, appealing to Theorem 1 and Lebedev’s type upper bounds above via the absolute and uniform convergence. Second, it is enough to consider the real case of \( a \in [0, \pi/2) \).

We begin with a key example of (3.1).

**Theorem 3.1:** Let \( x > 0, a \in [0, \pi/2) \). The following limit holds

\[ \lim_{\varepsilon \to 0^+} \int_0^\infty e^{-\varepsilon \tau^2} \left[ \left( \frac{\pi}{2} + a \right) \tau \right] K_{ir}(x) \, d\tau = \frac{\pi}{2} \exp (x \sin a), \]  

(3.2)

where the convergence is understood in the sense of the generalized Mellin transform (cf. [8]), i.e.

\[ \lim_{\varepsilon \to 0^+} \langle I_\varepsilon(x, a), e^{-x x^{s-1}} \rangle = \frac{\pi}{2} \langle e^{x \sin a}, e^{-x x^{s-1}} \rangle, \quad \Re s > 0. \]  

(3.3)

**Proof:** In fact, since \( e^{-x I_\varepsilon(x, a)x^{s-1}} \in L_1(\mathbb{R}^+), e^{-x I_\varepsilon} \) represents a regular generalized function. Hence we write, using the conventional Mellin transform,

\[ \int_0^\infty e^{-x I_\varepsilon(x, a)x^{s-1}} \, dx = \int_0^\infty e^{-x x^{s-1}} \int_0^\infty e^{-\varepsilon \tau^2} \cosh \left( \frac{\pi}{2} + a \right) \tau \times K_{ir}(x) \, d\tau \, dx. \]  

(3.4)

The interchange of the order of integration in (3.4) is allowed for each \( \varepsilon > 0 \) due to the absolute convergence of the iterated integral. Then, appealing to the Mellin transform (cf. Entry 8.4.23.3 in [3, Vol. III])

\[ \int_0^\infty e^{-x} K_{ir}(x)x^{s-1} \, dx = 2^{-s} \sqrt{\pi} \frac{\Gamma(s + i\tau)\Gamma(s - i\tau)}{\Gamma(s + 1/2)}, \quad \Re s > 0, \]  

(3.5)

we obtain

\[ \int_0^\infty e^{-x I_\varepsilon(x, a)x^{s-1}} \, dx = \frac{2^{-s} \sqrt{\pi}}{\Gamma(s + 1/2)} \int_0^\infty e^{-\varepsilon \tau^2} \cosh \left( \frac{\pi}{2} + a \right) \tau \times \Gamma(s + i\tau)\Gamma(s - i\tau) \, d\tau. \]  

(3.6)

Now, one can pass to the limit when \( \varepsilon \to 0^+ \) under the integral sign on the right-hand side of (3.6) via Abel’s test and Stirling’s asymptotic formula for the gamma function. Hence the
integral by \( \tau \) is calculated in [1, formula (1.104)], namely,

\[
\int_0^\infty \cosh \left[ \left( \frac{\pi}{2} + a \right) \tau \right] \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau = \frac{\pi \Gamma(2s)}{2^{2s}} \left[ \cos \left( \frac{\pi}{4} + \frac{a}{2} \right) \right]^{-2s}.
\] (3.7)

Therefore, employing the duplication formula for gamma function, we find from (3.6), (3.7)

\[
\lim_{\varepsilon \to 0+} \int_0^\infty e^{-x} I_\varepsilon(x, a)x^{s-1} \, dx = \frac{\pi}{2} \Gamma(s) [1 - \sin a]^{-s}.
\]

Finally, employing the Euler integral for the gamma function, we establish (3.2) and complete the proof of Theorem 3.1.

Now we are ready to prove the main result of this section.

**Theorem 3.2:** Let \( x > 0, \ a \in [0, \pi/2) \) and \( \psi_j(\tau), j = 1, 2 \) are even entire functions of the exponential type \( b_j \in [0, (1 - \sin a)/(2e)), j = 1, 2 \). Then the limit (3.1) has the value

\[
f(x, a) = \frac{\pi}{2} \left[ \psi_1(D_a) + D_a \psi_2(D_a) \right] \{e^x \sin a\},
\] (3.8)

where \( \psi_j(D_a), j = 1, 2 \) are functions of the differential operator \( D_a \equiv d/da \), acting on \( e^{x \sin a} \) and the convergence is understood in the sense of the generalized Mellin transform

\[
\lim_{\varepsilon \to 0+} \langle f_\varepsilon(x, a), e^{-x}x^{s-1} \rangle \rightleftharpoons \frac{\pi}{2} \left[ \psi_1(D_a) + D_a \psi_2(D_a) \right] \{e^x \sin a\}, e^{-x}x^{s-1}, \ \text{Re} \ s > 0.
\] (3.9)

**Proof:** Indeed, functions \( \psi_j(\tau), j = 1, 2 \) are represented in terms of the Taylor series

\[
\psi_j(\tau) = \sum_{n=0}^\infty c_{2n,j} \tau^{2n}, \quad j = 1, 2,
\]

where the coefficients \( c_{2n,j} \) satisfy the Cauchy estimates

\[
|c_{2n,j}| < \left( \frac{eb_j}{2n} \right)^{2n}, \quad n > N, j = 1, 2.
\] (3.10)

Hence by the same arguments as in the proof of Theorem 2, we recall (3.1), (3.5) to obtain

\[
\int_0^\infty e^{-x}f_\varepsilon(x, a)x^{s-1} \, dx = \frac{2^{-s} \sqrt{\pi}}{\Gamma(s+1/2)} \int_0^\infty e^{-\varepsilon \tau^2} \left[ \psi_1(\tau) \cosh \left( \frac{\pi}{2} + a \right) \right] \tau \\
+ \psi_2(\tau) \tau \sinh \left( \frac{\pi}{2} + a \right) \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau.
\] (3.11)

The interchange of the order of integration on the left-hand side of is guaranteed for each positive \( \varepsilon \) via the absolute convergence of the corresponding iterated integral. Then in the
same manner one passes to the limit when $\varepsilon \to 0^+$ due to the convergence of the integral

$$\int_0^\infty \left[ \psi_1(\tau) \cosh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \right. + \left. \psi_2(\tau) \tau \sinh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \right] \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau$$

under conditions of the theorem. Therefore,

$$\lim_{\varepsilon \to 0^+} \int_0^\infty e^{-x_{\varepsilon}(x, a)x^{s-1}} \, dx = \frac{2^{-s} \sqrt{\pi}}{\Gamma(s + 1/2)} \int_0^\infty \left[ \psi_1(\tau) \cosh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \right. + \left. \psi_2(\tau) \tau \sinh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \right] \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau. \quad (3.12)$$

Meanwhile, returning to (3.7) and differentiating with respect to parameter $a$, we get the value

$$\int_0^\infty \tau \sinh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau$$

$$= \frac{\pi \Gamma(2s + 1)}{2^{2s+1}} \left[ \cos\left(\frac{\pi}{4} + \frac{a}{2}\right) \right]^{-2s-1} \sin\left(\frac{\pi}{4} + \frac{a}{2}\right), \quad (3.13)$$

since the latter integral converges uniformly for $0 \leq a \leq \pi/2 - a_0$, $a_0 \in (0, \pi/2]$. The right-hand side of (3.12) can be treated by substitution of the series (3.9) and using the differentiation with respect to a parameter. In fact, it yields

$$\sum_{n=0}^\infty c_{2n,1} \int_0^\infty \tau^{2n} \cosh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau$$

$$+ \sum_{n=0}^\infty c_{2n,2} \int_0^\infty \tau^{2n+1} \sinh\left(\left(\frac{\pi}{2} + a\right) \tau\right) \Gamma(s + i\tau) \Gamma(s - i\tau) \, d\tau,$$

where the interchange of the order of integration and summation and further differentiation under the integral sign can be justified, employing inequality (3.10), the representation of the product of gamma functions by the Mellin transform from Entry 8.4.23.1 in [3, Vol. II]

$$\Gamma(s + i\tau) \Gamma(s - i\tau) = 2^{2(1-s)} \int_0^\infty K_{2i\tau}(x)x^{2s-1} \, dx, \quad \Re s > 0 \quad (3.14)$$

and the inequality for the Macdonald function (see [1, formula (1.100)])

$$|K_{i\tau}(x)| \leq e^{-\delta \tau} K_0(x \cos \delta), \quad x, \tau > 0, \delta \in \left(0, \frac{\pi}{2}\right). \quad (3.15)$$
Precisely, we derive for a big enough positive integer \( N \) and \( \delta \in (1/2(a + \pi/2 + \max(b_1, b_2)), \pi/2) \)

\[
\sum_{n=N+1}^{\infty} |c_{2n,1}| \int_0^\infty \tau^{2n} \cosh \left( \frac{\pi}{2} + a \right) \tau \Gamma(s + i\tau) \Gamma(s - i\tau) \ d\tau + \sum_{n=N+1}^{\infty} |c_{2n,2}| \int_0^\infty \tau^{2n+1} \sinh \left( \frac{\pi}{2} + a \right) \tau \Gamma(s + i\tau) \Gamma(s - i\tau) \ d\tau < 2^{3-2\text{Re} s} \int_0^\infty K_0(x \cos \delta) x^{2\text{Re} s-1} \ dx \sum_{n=N+1}^{\infty} \frac{(2n)!}{e^{2n}} \frac{b_2}{b_1} \left( \frac{b_1}{2\delta-a-\pi/2} \right)^{2n} \to 0, \quad N \to \infty.
\]

Hence we find from (3.7), (3.12) and (3.13) the equalities

\[
\lim_{\varepsilon \to +0} \int_0^\infty e^{-\varepsilon f(x,a)x^{s-1}} \ dx = \frac{2^{-s} \sqrt{\pi}}{\Gamma(s + 1/2)} \sum_{n=0}^{\infty} c_{2n,1} \frac{d^{2n+1}}{da^{2n+1}} \int_0^\infty \cosh \left( \frac{\pi}{2} + a \right) \tau \Gamma(s + i\tau) \Gamma(s - i\tau) \ d\tau + \sum_{n=0}^{\infty} c_{2n,2} \frac{d^{2n+1}}{da^{2n+1}} \int_0^\infty \cosh \left( \frac{\pi}{2} + a \right) \tau \Gamma(s + i\tau) \Gamma(s - i\tau) \ d\tau
\]

\[
= \frac{\pi}{2} \frac{\Gamma(s)}{2} \left[ \sum_{n=0}^{\infty} c_{2n,1} \frac{d^{2n}}{da^{2n}} [1 - \sin a]^{-s} + \sum_{n=0}^{\infty} c_{2n,2} \frac{d^{2n+1}}{da^{2n+1}} [1 - \sin a]^{-s} \right]
\]

\[
= \frac{\pi}{2} \left[ \psi_1(D_a) + D_a \psi_2(D_a) \right] \int_0^\infty \exp (-x(1 - \sin a)) x^{s-1} \ dx, \quad D_a \equiv \frac{d}{da}. \quad (3.16)
\]

The differentiation with respect to \( a \) under the integral sign of the latter integral in (3.15) and the action of the operator \( \psi_1(D_a) + D_a \psi_2(D_a) \) inside this integral we will motivate, as above, owing to the uniform convergence of the integrals for derivatives. To do this, we
appeal to the Hoppe formula [9] to write the $2n$th derivative of $\exp(x \sin a)$ in the form
\[
\frac{d^{2n}}{da^{2n}} [e^{x \sin a}] = e^{x \sin a} \sum_{k=0}^{2n} \frac{(-1)^k x^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} [\sin a]^{k-j} \frac{d^{2n}}{da^{2n}} [\sin a]^j
\]
\[\quad = e^{x \sin a} (-1)^n \sum_{k=0}^{2n} \frac{(-1)^k x^k}{k!} \sum_{j=0}^{k} \frac{(1-1)^j \binom{k}{j}}{(2i)^j} [\sin a]^{k-j} \sum_{r=0}^{j} (-1)^r \binom{j}{r} e^{i(j-2r)a} (j - 2r)^n.\]

Hence
\[
\left| \frac{d^{2n}}{da^{2n}} [e^{x \sin a}] \right| \leq e^{x \sin a} \sum_{k=0}^{2n} \frac{x^k}{k!} \sum_{j=0}^{2j} \frac{(-1)^j \binom{k}{j} (2i)^j}{j!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \left( e^{(j-2r)a} (j - 2r)^n \right).
\]

Then for $\text{Re } s = \gamma > 0$ and some $n_0 \in \mathbb{N}$
\[
\int_0^\infty \sum_{n=n_0+1}^\infty \left| c_{2n,1} \right| \left| \frac{d^{2n}}{da^{2n}} [e^{x \sin a}] \right| e^{-x \gamma x^{-1}} dx
\]
\[\quad \leq (1 - \sin a)^{-\gamma} \Gamma(\gamma) \sum_{n=n_0+1}^\infty \left| c_{2n,1} \right| (2n)^2n \sum_{k=0}^{2n} \frac{2k^r(y)_{k}}{k!(1 - \sin a)^k}
\]
\[\quad < (1 - \sin a)^{-\gamma} \Gamma(\gamma) \sum_{n=n_0+1}^\infty (eb_1)^2n \left[ \sum_{k=0}^{n} \frac{2k^r(y)_{2k}}{(2k)!(1 - \sin a)^{2k}} + \sum_{k=0}^{n-1} \frac{2k^{r+1}(y)_{2k+1}}{(2k+1)!(1 - \sin a)^{2k+1}} \right]
\]
\[\quad \times \left[ \sum_{k=0}^{\infty} \frac{2k^r(y)_{2k}}{(2k)!(1 - \sin a)^{2k}} \sum_{n=k}^{\infty} (eb_1)^2n + \sum_{k=0}^{\infty} \frac{2k^{r+1}(y)_{2k+1}}{(2k+1)!(1 - \sin a)^{2k+1}} \sum_{n=k}^{\infty} (eb_1)^2(n+1) \right]
\]
\[\quad = \frac{(1 - \sin a)^{-\gamma} \Gamma(\gamma)}{1 - (eb_1)^2} \left[ \sum_{k=0}^{\infty} \frac{(2eb_1)^2k^r(y)_{2k}}{(2k)!(1 - \sin a)^{2k}} + 2(eb_1)^2 \sum_{k=0}^{\infty} \frac{(2eb_1)^2k^{r+1}(y)_{2k+1}}{(2k+1)!(1 - \sin a)^{2k+1}} \right] < \infty
\]

when $0 \leq b_1 < (1 - \sin a0)/(2e)$, $0 \leq a \leq a_0 < \pi/2$. In the same manner we justify the action of the operator $D_a^2\psi_2(D_a)$ inside the integral in (3.15). Thus we arrive at (3.9) and complete the proof of Theorem 3.2.

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