Quantum counterparts of three-dimensional real Lie algebras over harmonic oscillator

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Abstract: Operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of three-dimensional (3D) real Lie algebras in Bianchi classification. The Jacobi operators of the quantum algebras are found.

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1. Introduction

In Hamiltonian formalism, a mechanical system is described by the canonical variables $q^i, p_i$ and their time evolution is prescribed by the Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad (1)$$

Thus, from the algebraic point of view, mechanical systems can be represented by linear operators, i.e. by linear maps $V \rightarrow V$ of a vector space $V$. As a generalization of this one can pose the following question [7]: how can the time evolution of the linear operations (multiplications) $V^\otimes n \rightarrow V$ be described?

The algebraic operations (multiplications) can be seen as an example of the operadic variables [2]. If an operadic system depends on time, one can speak about operadic dynamics [7]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics by using the Gerstenhaber brackets instead of the commutator bracketing in the Lax equation (2). The idea is very simple: if $L : V^\otimes n \rightarrow V$ is an $n$-ary operation, then we can modify the the Lax (2) equation, by replacing the commutator bracketing on the r.h.s. of (2) by the Gerstenhaber brackets (see Sec. 2). Using the Gerstenhaber brackets is natural, because these brackets satisfy the graded Jacobi identity and if $n = 1$, then the Gerstenhaber brackets coincide with the ordinary commutator bracketing. Thus, the time evolution of the operadic variables may be given by

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the operadic Lax equation.

In Refs. [8–10], the low-dimensional binary operadic Lax representations for the harmonic oscillator were constructed.

In the present paper, the operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of the three-dimensional (3D) real Lie algebras in Bianchi classification (see Tab. 3). The Jacobi operators of the quantum algebras are found. It turns out that the quantum algebras $I^b, II^b, VI^b, IV^b, V^b, VIII^b$ are Lie algebras and the quantum algebras $V^h, IV^h, VII^h, III^h_{n=1}, VII^h_{n+1}$ are anomalous in the sense that their Jacobi identities are violated.

2. Endomorphism operad and Gerstenhaber brackets

Let $K$ be a unital associative commutative ring, $V$ be a unital $K$-module, and $\mathcal{E}_V^n \equiv \text{End}_V^n \equiv \text{Hom}(V^\otimes n, V)$ $(n \in \mathbb{N})$. For an operation $f \in \mathcal{E}_V^n$, we refer to $n$ as the degree of $f$ and often refer (when it does not cause confusion) $f$ instead of deg $f$. For example, $(-1)^f \equiv (-1)^n$, $\mathcal{E}_V^n \equiv \mathcal{E}_V^0$ and $o_i \equiv o_n$. Also, it is convenient to use the reduced degree $|f| \equiv n - 1$. Throughout this paper, we assume that $\otimes \equiv \otimes_K$.

**Definition 2.1 (endomorphism operad [2]).**

For $f \otimes g \in \mathcal{E}_V^I \otimes \mathcal{E}_V^J$, define the partial compositions

$$f \circ_i g \equiv (-1)^{|g|i} f \circ (id_V^{|g|} \otimes g \otimes id_V^{0(|f| - i)}) \in \mathcal{E}_V^{I+|g|},$$

$$0 \leq i \leq |f|.$$

The sequence $\mathcal{E}_V \equiv \{\mathcal{E}_V^n\}_{n \in \mathbb{N}}$, equipped with the partial compositions $o_i$, is called the endomorphism operad of $V$.

**Definition 2.2 (total composition [2]).**

The total composition $\bullet : \mathcal{E}_V^I \otimes \mathcal{E}_V^J \rightarrow \mathcal{E}_V^{I+|J|}$ is defined by

$$f \bullet g \equiv \sum_{i=0}^{|f|} f \circ_i g \in \mathcal{E}_V^{I+|J|}, \quad || | = 0.$$

The pair $\text{Com} \mathcal{E}_V \equiv \{\mathcal{E}_V, \bullet\}$ is called the composition algebra of $\mathcal{E}_V$.

**Definition 2.3 (Gerstenhaber brackets [2]).**

The Gerstenhaber brackets $[\cdot, \cdot]$ are defined in $\text{Com} \mathcal{E}_V$ as a graded commutator by

$$[f, g] \equiv f \bullet g - (-1)^{|f||g|} g \bullet f = (-1)^{|f||g|}[g, f], \quad ||, || = 0.$$

The commutator algebra of $\text{Com} \mathcal{E}_V$ is denoted as $\text{Com}^{-} \mathcal{E}_V \equiv \{\mathcal{E}_V, [\cdot, \cdot]\}$. One can prove (e.g. [2]) that $\text{Com}^{-} \mathcal{E}_V$ is a graded Lie algebra. The Jacobi identity reads

$$(-1)^{|f||g|}[f, [g, h]] + (-1)^{|g||h|}[g, [h, f]] + (-1)^{|h||f|}[h, [f, g]] = 0.$$

3. Operadic Lax equation and the harmonic oscillator

Assume that $K \equiv \mathbb{R}$ or $K \equiv \mathbb{C}$ and operations are differentiable. Dynamics in operadic systems (operadic dynamics) may be introduced by

**Definition 3.1 (operadic Lax pair [7]).**

Allow a classical dynamical system to be described by the Hamiltonian system (1). An operadic Lax pair is a pair $(L, M)$ of operations $L, M \in \mathcal{E}_V$, such that the Hamiltonian system (1) may be represented as the operadic Lax equation

$$\frac{dL}{dt} = [M, L] \equiv M \cdot L - (-1)^{|M||L|} L \cdot M.$$

The pair $(L, M)$ is also called an operator Lax representation of the operadic Hamiltonian system (1). Evidently, the degree constraints $|M| = |L| = 0$ give rise to ordinary Lax equation (2) [1, 6]. In this paper we assume that $|M| = 0$.

The Hamiltonian of the harmonic oscillator (HO) is

$$H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2).$$

Thus, the Hamiltonian system of HO reads

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q. \quad (3)$$

If $\mu$ is a linear algebraic operation we can use the above Hamilton equations to obtain

$$\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q} \frac{dq}{dt} + \frac{\partial \mu}{\partial p} \frac{dp}{dt} = \rho \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu].$$

Therefore, we get the following linear partial differential equation for $\mu(q, p)$:

$$\rho \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu]. \quad (4)$$

By integrating (4) one can get collections of operations called the operadic (Lax representation for) harmonic oscillator. Since the general solution of a partial differential equation depends on arbitrary functions, these representations are not uniquely determined.
4. 3D binary anti-commutative operadic Lax representations for the harmonic oscillator

**Lemma 4.1.**

Matrices

\[ L = \begin{pmatrix} p & \omega q & 0 \\ \omega q & -p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M = \frac{\omega}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

give a 3D Lax representation for the harmonic oscillator.

**Definition 4.1 (quasi-canonical coordinates).**

Denoting by \( H \) the Hamiltonian of the harmonic oscillator define its quasi-canonical coordinates \( A_\pm \) by the relations

\[
\begin{align*}
A_+^2 + A_-^2 &= 2\sqrt{2}H, \\
A_+^2 - A_-^2 &= 2p, \\
A_+A_- &= \omega q.
\end{align*}
\]  

(5)

Note that \( A_\pm \) can not be simultaneously zero. One can also see that the second and the third relations imply the first one in (5).

---

**Theorem 4.1 ([10]).**

Let \( C_\nu \in \mathbb{R} \ (\nu = 1, \ldots, 9) \) be arbitrary real-valued parameters, such that

\[
C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2 + C_6^2 \neq 0. \tag{6}
\]

Let \( M \) be defined as in Lemma 4.1, and

\[
\begin{align*}
\mu_{11} &= \rho_{12} = \mu_{13} = \mu_{21} = \mu_{22} = \mu_{23} = 0, \\
\mu_{14} &= \mu_{15} = \mu_{16} = \mu_{24} = \mu_{25} = \mu_{26} = 0, \\
\mu_{34} &= -\mu_{35} = C_2 \rho - C_6 \omega q - C_4, \\
\mu_{35} &= -\mu_{36} = C_2 \rho - C_6 \omega q + C_4, \\
\mu_{43} &= -\mu_{44} = C_2 \omega q + C_6 p - C_4, \\
\mu_{44} &= -\mu_{45} = C_2 \omega q - C_6 p - C_4, \\
\mu_{53} &= -\mu_{55} = C_2 \omega q + C_6 p + C_4, \\
\mu_{55} &= -\mu_{56} = C_2 \omega q - C_6 p + C_4.
\end{align*}
\]  

(7)

be the structure constants of the multiplication \( \mu : V \otimes V \to V \) in a 3D real vector space \( V \). Then \((\mu, M)\) is a 3D anti-commutative binary operadic Lax pair for HO.

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5. Initial conditions and dynamical deformations

Specify the coefficients \( C_\nu \) in Theorem 4.1 by the initial conditions

\[
\mu|_{t=0} = \rho, \quad p|_{t=0} = p_0, \quad q|_{t=0} = 0.
\]

Denoting \( E \equiv H|_{t=0} \), the latter together with (5) yield the initial conditions for \( A_\pm \):

\[
\begin{align*}
\left\{ \begin{align*}
A_+^2 + A_-^2 &= 2\sqrt{2}E, \\
A_+^2 - A_-^2 &= 2p_0, \\
A_+A_- &= 0,
\end{align*} \right. \quad &\iff \begin{align*}
p_0 > 0, \\
A_+|_{t=0} &= 0, \\
A_-|_{t=0} &= 0, \\
A_+^2|_{t=0} &= 2p_0, \quad &A_-^2|_{t=0} &= -2p_0.
\end{align*}
\end{align*}
\]

In what follows assume that \( p_0 > 0 \) and \( A_+|_{t=0} = \sqrt{2p_0} \). The other cases can be treated similarly. Note that in this case \( E = \sqrt{2p_0} \).

From (7) we get the following linear system:

\[
\begin{align*}
\dot{\mu}_{12} &= C_2 \rho - C_4, & \quad \dot{\mu}_{11} &= C_3 \rho - C_1, & \quad \dot{\mu}_{13} &= C_5 \sqrt{2p_0}, \\
\dot{\mu}_{22} &= -C_6 \sqrt{2p_0}, & \quad \dot{\mu}_{21} &= C_4 \rho + C_1, & \quad \dot{\mu}_{23} &= C_6 \sqrt{2p_0}, \\
\dot{\mu}_{32} &= -C_5 \sqrt{2p_0}, & \quad \dot{\mu}_{31} &= C_4 \rho - C_1, & \quad \dot{\mu}_{33} &= C_5 \sqrt{2p_0}. 
\end{align*}
\]  

(8)
One can easily check that the unique solution of the latter system with respect to $C_\nu$ ($\nu = 1, \ldots, 9$) is

\[
\begin{align*}
C_1 &= \frac{1}{2} \left( \hat{\mu}_{23}^2 - \hat{\mu}_{31}^1 \right), \\
C_2 &= \frac{1}{2 p_0} \left( \hat{\mu}_{13}^2 + \hat{\mu}_{12}^3 \right), \\
C_3 &= \frac{1}{2 p_0} \left( \hat{\mu}_{23}^1 + \hat{\mu}_{13}^2 \right), \\
C_4 &= \frac{1}{2} \left( \hat{\mu}_{13}^2 - \hat{\mu}_{23}^1 \right), \\
C_5 &= \frac{1}{2 p_0} \hat{\mu}_{12}^3, \\
C_6 &= -\frac{1}{2 p_0} \hat{\mu}_{23}^1, \\
C_7 &= \frac{1}{\sqrt{2 p_0}} \hat{\mu}_{13}^2, \\
C_8 &= -\frac{1}{\sqrt{2 p_0}} \hat{\mu}_{12}^3, \\
C_9 &= \frac{\sqrt{2}}{\sqrt{2 p_0}} \hat{\mu}_{23}^1.
\end{align*}
\]

**Remark 5.1.**
Note that the parameters $C_\nu$ have to satisfy condition (6) to get the operadic Lax representations.

**Definition 5.1.**
If $\nu = \mu$, then the multiplication $\hat{\mu}$ is called dynamically rigid (over HO). Otherwise $\mu$ is called a dynamical deformation of $\hat{\mu}$ (over HO).

### 6. Bianchi classification of 3d real Lie algebras

We use the Bianchi classification of the 3D real Lie algebras given in [4, 5]. The structure equations of the 3-dimensional real Lie algebras can be presented as follows:

\[
[e_1, e_2] = -ae_2 + n^3 e_1, \quad [e_2, e_3] = n^1 e_1, \quad [e_3, e_1] = n^2 e_2 + ae_3.
\]

The values of the parameters $a, n^1, n^2, n^3$ and the corresponding structure constants are presented in Tab. 1.

**Table 1.** 3d real Lie algebras in Bianchi classification. Here $a > 0$.

| Bianchi type | $a$ | $n^1$ | $n^2$ | $n^3$ | $\hat{\mu}_{12}^3$ | $\hat{\mu}_{13}^2$ | $\hat{\mu}_{23}^1$ | $\hat{\mu}_{21}^3$ | $\hat{\mu}_{31}^2$ | $\hat{\mu}_{32}^1$ |
|-------------|-----|-------|-------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| I           | 1   | 0     | 0     | 0     | 0              | 0              | 0               | 0               | 0               | 0               |
| II          | 1   | 0     | 1     | 0     | 0              | 0              | 1               | 0               | 0               | 0               |
| VII         | 0   | 1     | 1     | 0     | 0              | 1              | 0               | 0               | 0               | 1               |
| VI          | 0   | 1     | -1    | 0     | 0              | 1              | 0               | 0               | 0               | -1              |
| IX          | 0   | 1     | 1     | 0     | 0              | 1              | 0               | 0               | 0               | 1               |
| VIII        | 0   | 1     | 1     | -1    | 0              | -1             | 1               | 0               | 0               | 0               |
| V           | 1   | 0     | 0     | 0     | -1             | 0              | 0               | 0               | 0               | 1               |
| IV          | 1   | 0     | 0     | 1     | -1             | 1              | 0               | 0               | 0               | 0               |
| $\text{VII}_a$ | $a$ | 0     | 1     | 1     | 0              | -a             | 1               | 0               | 0               | 1               |
| $\text{III}_{a-1}$ | $a$ | 0     | 1     | -1    | 0              | -1             | -1              | 0               | 0               | 0               |
| $\text{V}_{a+1}$ | $a$ | 0     | 1     | -1    | 0              | -a             | -1              | 0               | 0               | 0               |

The Bianchi classification is, for instance, used to describe the spatially homogeneous spacetimes of dimension $3+1$. In particular, the Lie algebra $\text{VII}_a$ is very interesting for the cosmological applications because it is related to the Friedmann–Robertson–Walker metric. One can find more details in Refs. [3–5].

### 7. Dynamical deformations of 3d real Lie algebras

By using the structure constants of the 3D Lie algebras in the Bianchi classification, Theorem 4.1 and relations (8) one can propose that evolution of the 3D real Lie algebras can be prescribed as given in Tab. 2.
Table 2. Evolution of 3d real Lie algebras. Here \( p_0 = \sqrt{\frac{\mu}{2}} \).

| Dynamical Bianchi type | \( \mu_{12}^1 \) | \( \mu_{12}^2 \) | \( \mu_{12}^3 \) | \( \mu_{23}^1 \) | \( \mu_{23}^2 \) | \( \mu_{23}^3 \) | \( \mu_{31}^1 \) | \( \mu_{31}^2 \) | \( \mu_{31}^3 \) |
|------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| I'                    | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| II'                   | 0               | 0               | 0               | \( \frac{\mu + p_0}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) | 0               | \( \frac{\mu + q}{2p_0} \) | 0               | 0               |
| VII'                  | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| VI'                   | 0               | 0               | 0               | \( \frac{\mu}{p_0} \) | \( \frac{\mu}{q_0} \) | 0               | \( \frac{\mu}{q_0} \) | 0               | \( \frac{\mu}{p_0} \) |
| IX'                   | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| VIII'                 | 0               | 0               | \( -1 \)       | 1               | 0               | 0               | 0               | 1               | 0               |
| V'                    | \( \frac{A_1}{\sqrt{2p_0}} \) | \( -\frac{A_1}{\sqrt{2p_0}} \) | 0               | 0               | 0               | \( -\frac{A_1}{\sqrt{2p_0}} \) | 0               | 0               | \( \frac{A_1}{\sqrt{2p_0}} \) |
| IV'                   | \( \frac{A_1}{\sqrt{2p_0}} \) | \( -\frac{A_1}{\sqrt{2p_0}} \) | 1               | 0               | 0               | \( -\frac{A_1}{\sqrt{2p_0}} \) | 0               | 0               | \( \frac{A_1}{\sqrt{2p_0}} \) |
| VII'\(_n\)           | \( \frac{A_1}{\sqrt{2p_0}} \) | \( -\frac{A_1}{\sqrt{2p_0}} \) | \( -1 \)       | \( \frac{\mu - p_0}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) | 0               | \( \frac{\mu - p_0}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) |
| VII'\(_{n+1}\)       | \( \frac{A_1}{\sqrt{2p_0}} \) | \( -\frac{A_1}{\sqrt{2p_0}} \) | \( -1 \)       | \( \frac{\mu - p_0}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) | 0               | \( \frac{\mu - p_0}{2p_0} \) | \( \frac{\mu + q}{2p_0} \) |

**Theorem 7.1 (dynamically rigid algebras).** The algebras I, VII, VIII, and IX are dynamically rigid over the harmonic oscillator.

**Proof.** This is evident from Tabs. 1 and 2.

**Theorem 7.2 (dynamical Lie algebras [11]).** The algebras II', IV', V', V', VI', VII'\(_n\), VI'\(_{n+1}\), and VII'\(_n\) are Lie algebras.

Thus, we can see that the evolution of these algebras are generated by the harmonic oscillator, because their multiplications depend on the canonical and quasi-canonical coordinates of the harmonic oscillator.

**8. Quantum algebras from the Bianchi classification**

By using the structure constants of the 3D Lie algebras in Bianchi classification (Tab. 1), operadic Lax representations (Theorem 4.1) of the harmonic oscillator and relations (8) we found the evolution (dynamical deformations) of these algebras generated by the harmonic oscillator (see Tab. 2).

Now, by using the dynamically deformed Bianchi classification (Tab. 2), we can propose the canonically quantized counterparts of the 3D Lie algebra in the Bianchi classification (see Tab. 3).

Here we use the Schrödinger picture, *i.e.* the quantum operators \( \hat{q}, \hat{p}, \hat{H} \) and \( \hat{A}_a \) do not depend on time. Following the canonical quantization prescription, the quantum canonical coordinates satisfy the canonical commutation relations

\[
[\hat{q}, \hat{\tilde{q}}] = 0 = [\hat{p}, \hat{\tilde{p}}], \quad [\hat{\tilde{p}}, \hat{\tilde{q}}] = \frac{\hbar}{i},
\]

while the quasi-canonical coordinates would satisfy (cf. (5)) the constraints

\[
\hat{\lambda}_+^2 + \hat{\lambda}_-^2 = 2\sqrt{2\hat{H}}, \quad \hat{\lambda}_+^2 - \hat{\lambda}_-^2 = 2\hat{p},
\]

\[
\hat{A}_a\hat{\lambda}_- + \hat{A}_a\hat{\lambda}_+ = 2\omega\hat{q}. \quad (9)
\]

Let us study the Jacobi identities for the quantum algebras from Tab. 3. Denoting \( \hat{\mu} \equiv [\cdot, \cdot]_h \), the quantum **Jacobi operator** is defined by

\[
\hat{J}_h(x; y; z; x)_h = [x, y]_h + [y, z]_h + [z, x]_h + [x, y]_h = \hat{J}_h(x; y; z)e_1 + \hat{J}_h(x; y; z)e_2 + \hat{J}_h(x; y; z)e_3.
\]
By direct calculations one can see that

\[ J^h_k(x; y; z) = \hat{J}^h_k(x; y; z) = J^h_k(x; y; z) = 0. \]

**Theorem 8.3 (anomalous quantum algebras of the first type).**

The algebras \( \mathbb{V}^h \) and \( V^h \) are non-Lie algebras.

**Proof.** By direct calculations one can see that

\[ \hat{J}^h_k(x; y; z) = 0 = \hat{J}^h_k(x; y; z) = J^h_k(x; y; z), \]

where the scalar triple product of the vectors \( x, y, z \) is denoted by

\[ (x, y, z) = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}. \]

**Theorem 8.4 (anomalous quantum algebras of the second type).**

The algebras \( \mathbb{III}^{h}_{a=1} \), \( \mathbb{VI}^{h}_{a=1} \), and \( VII^{h}_{a} \) are non-Lie algebras.

**Proof.** Let

\[ \hat{\xi}_\pm \equiv \omega \hat{q} \hat{A}_\pm \pm (\hat{p} \mp \hat{p}_0) \hat{A}_\pm. \]

Then, by direct calculations one can check that for the algebras \( VII^{h}_{a} \) and \( \mathbb{VI}^{h}_{a=1} \) the Jacobi operator coordinates are

\[ \hat{J}^h_k(x; y; z) = \frac{-a(x, y, z)}{\sqrt{2\hat{p}_0}} \hat{\xi}_+. \]

\[ \hat{J}^h_k(x; y; z) = \frac{-a(x, y, z)}{\sqrt{2\hat{p}_0}} \hat{\xi}_-. \]

and for the algebra \( \mathbb{III}^{h}_{a=1} \) one has the same formulae with \( a = 1 \).

More closely these formulae are studied in [12].

### 9. Concluding remarks

In the present paper, the operadic Lax representations for the harmonic oscillator were used to construct the quan-
tum counterparts of 3D real Lie algebras in Bianchi classification (see Tab. 3). The Jacobi operators of the quantum algebras were found. It turned out that the quantum algebras $I^\hbar, II^\hbar, VII^\hbar, VI^\hbar, IX^\hbar, VIII^\hbar$ are Lie algebras and the quantum algebras $V^\hbar, IV^\hbar, VII_{a=1}^\hbar, III_{a>0}^\hbar, VI_{a=1}^\hbar$ are anomalous in the sense that their Jacobi identities are violated.

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