NONUNIQUENESS OF CONFORMAL METRICS WITH CONSTANT $Q$-CURVATURE

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ABSTRACT. We establish several nonuniqueness results for the problem of finding complete conformal metrics with constant (fourth-order) $Q$-curvature on compact and noncompact manifolds of dimension $\geq 5$. Infinitely many branches of metrics with constant $Q$-curvature, but without constant scalar curvature, are found to bifurcate from Berger metrics on spheres and complex projective spaces. These provide examples of nonisometric metrics with the same constant negative $Q$-curvature in a conformal class with negative Yamabe invariant, echoing the absence of a Maximum Principle. We also discover infinitely many complete metrics with constant $Q$-curvature conformal to $S^m \times \mathbb{R}^d$, $m \geq 4$, $d \geq 1$, and $S^m \times \mathbb{H}^d$, $2 \leq d \leq m - 3$; which give infinitely many solutions to the singular constant $Q$-curvature problem on round spheres $S^n$ blowing up along a round subsphere $S^k$, for all $0 \leq k < (n - 4)/2$.

1. Introduction

The study of fourth-order conformal invariants of a Riemannian manifold $(M, g)$ of dimension $n \geq 3$ naturally leads to the definition of $Q$-curvature:

$$Q_g = \frac{1}{2(n-1)} \Delta_g \text{scal}_g - \frac{2}{(n-2)^2} \| \text{Ric}_g \|_2^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \text{scal}_g^2,$$

where $\Delta_g u = -\text{div}_g(\nabla u)$ is the (nonnegative) Laplacian operator. Following the seminal works of Branson [Bra85] and Paneitz [Pan08], there has been great interest in understanding geometric and analytic properties of $Q_g$. This remains a very active pursuit, as evidenced by developments even just over the last 3 years [GHL16, GM15, HY15, HY16b, LY16, LY17, Lin15], see [HY16a] for a survey.

Analogously to the Yamabe problem, a central question is whether $Q_g$ can be made constant by using conformal deformations of $(M, g)$, which corresponds to a fourth-order elliptic PDE (2.2) on the conformal factor. Despite substantial progress regarding the existence of solutions, to our knowledge, the issue of uniqueness (or lack thereof) has only been inspected in a small number of geometric settings. Grunau, Ould Ahmedou, and Reichel [GOAR08] established the existence of a continuum of radially symmetric solutions on hyperbolic space $\mathbb{H}^n$, $n \geq 5$, with ODE techniques. Using an involved perturbation argument and Mazzeo’s microlocal analysis of elliptic edge operators, Li [Li13, Li14] obtained a continuum of solutions conformal to a perturbation of Poincaré-Einstein metrics. A blowing up sequence of solutions (with one bubble) was built by Wei and Zhao [WZ13] on spheres $S^n$, $n \geq 5$, with a certain non conformally flat metric, implying the existence of infinitely many solutions in that conformal class; see also [HR04, QR06].

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Nevertheless, in consonance with the plethora of results for the Yamabe problem, the scope of nonuniqueness in the constant $Q$-curvature problem ought to be much richer, both geometrically and topologically. The purpose of the present paper is to confirm this, by exhibiting extensive nonuniqueness phenomena on a wide class of compact and noncompact manifolds, using variational bifurcation theory and other topological methods. We restrict ourselves to manifolds of dimension $n \geq 5$, as the low-dimensional cases $n = 3$ and especially $n = 4$ require a separate discussion.

We begin by establishing a criterion (Theorem 3.4) to detect bifurcations along families $g_t$ of metrics with constant $Q$-curvature and constant scalar curvature that admit horizontally Einstein Riemannian submersions with minimal fibers. This criterion is well-suited to metrics $g_t$ that form a canonical variation, i.e., are obtained by rescaling the vertical directions of a submersion by a factor $0 < t < +\infty$. We show that bifurcations for the constant $Q$-curvature problem are ubiquitous among such families near the degenerate limits $t = 0$ and $t = +\infty$, see Propositions 3.6 and 3.7. A convenient framework to generate examples is provided by homogeneous fibrations, see Section 4. Although our results yield many more examples (see e.g. Proposition 4.1), to simplify the exposition, we now only state our findings on the so-called Hopf bundles, where $g_t$ are often referred to as Berger metrics:

**Theorem A.** There exists an infinite sequence of bifurcating branches of metrics on $M$ with constant $Q$-curvature, but nonconstant scalar curvature, that issue from the Berger metrics $g_t$ as $t \searrow 0$ and/or $t \nearrow +\infty$, according to the table below.

| Hopf bundle $F \rightarrow M \rightarrow B$ | Infinitely many bifurcations as $t \searrow 0$ | Infinitely many bifurcations as $t \nearrow +\infty$ |
|---------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $S^1 \rightarrow S^{2q+1} \rightarrow CP^q$ | no                                           | if $q \geq 6$                                  |
| $S^3 \rightarrow S^{4q+3} \rightarrow HP^q$ | if $q \geq 1$                                | if $q \geq 2$                                  |
| $CP^1 \rightarrow CP^{2q+1} \rightarrow HP^q$ | if $q \geq 2$                                | if $q \geq 3$                                  |
| $S^7 \rightarrow S^{15} \rightarrow S^8(1/2)$ | yes                                          | yes                                           |

Some remarkable facts about the constant $Q$-curvature problem on closed manifolds can be observed with the above result. First, one obtains global examples, e.g., on odd-dimensional spheres, of metrics that have constant $Q$-curvature but do not have constant scalar curvature. Second, nonuniqueness of constant $Q$-curvature may take place on conformal classes with negative Yamabe invariant; as is the case of conformal classes of Berger metrics $g_t$ for sufficiently large $t$, since $\text{scal}_{g_t} \searrow -\infty$ as $t \nearrow +\infty$ if $\dim F \geq 2$. Third, some of these Berger metrics $g_t$, with large $t$, simultaneously have $\text{scal}_{g_t} < 0$ and $Q_{g_t} < 0$, see Appendix A; and nonisometric conformal metrics with the same constant (negative) $Q$-curvature can exist even in this regime. For comparison, recall that a metric with constant negative scalar curvature is unique in its conformal class by the Maximum Principle. Clearly, this argument is not applicable to the constant $Q$-curvature problem due to its fourth-order nature.

Leaving the realm of closed manifolds, the constant $Q$-curvature problem can also be posed on manifolds with boundary or noncompact manifolds. On the latter, the natural boundary condition is completeness of the metric. For instance, this is
trivially satisfied by metrics that descend to a compact quotient $M/\Gamma$; we call these periodic solutions with period $\Gamma$. Our second main result exploits the abundance of discrete cocompact groups $\Gamma$ on symmetric spaces to find infinitely many periodic solutions with different periods, reflecting a truly noncompact phenomenon:

**Theorem B.** Let $(C, g)$ be a closed manifold with constant scalar curvature and $(N, h)$ be a simply-connected symmetric space of noncompact or Euclidean type, such that $(C \times N, g \oplus h)$ has dimension $\geq 5$, $\text{scal} \geq 0$ and $Q > 0$ but $Q \neq 0$. Then $(C \times N, g \oplus h)$ has infinitely many nonhomothetic periodic conformal metrics with constant positive $Q$-curvature.

An immediate consequence of the above is the existence of infinitely many complete metrics with constant $Q$-curvature conformal to the standard product metrics on $S^n \times \mathbb{R}^d$ for all $m \geq 4$, $d \geq 1$, and on $S^n \times \mathbb{H}^d$ for all $2 \leq d \leq m - 3$. Using the stereographic projection, it is easy to see that $S^{n-1} \times \mathbb{R}$ is conformally equivalent to both $S^n \setminus \{\pm p\}$ and $\mathbb{R}^n \setminus \{0\}$, endowed with their (incomplete) constant curvature metrics. Thus, there are also infinitely many solutions to the constant $Q$-curvature problem on $S^n \setminus \{\pm p\} = S^n \setminus S^0$ and $\mathbb{R}^n \setminus \{0\}$, for all $n \geq 5$.

A higher codimension version of this argument shows that, for all $1 \leq k < n$, $S^{n-k-1} \times \mathbb{H}^{k+1}$ is conformally equivalent to $S^n \setminus S^k$ endowed with the (incomplete) round metric, see [BPS16]. Therefore, another consequence of Theorem B is that:

**Corollary C.** There are infinitely many complete metrics with constant positive $Q$-curvature on $S^n \setminus S^k$, $0 \leq k < \frac{n-4}{2}$, conformal to the round metric.

It would be interesting to determine whether $k < \frac{n-4}{2}$ is the maximal range of dimensions in which nonuniqueness occurs. Notably, it follows from a result of Chang, Hang, and Yang [CHY04, Thm 1.2] that if periodic solutions to the constant $Q$-curvature problem exist on $S^n \setminus \Lambda$ with $Q > 0$ and $\text{scal} > 0$, then $\dim \Lambda < \frac{n-4}{2}$. Recall that, by the above conformal equivalence $S^n \setminus S^k \cong S^{n-k-1} \times \mathbb{H}^{k+1}$, $1 \leq k < n$, the pullback of the standard product metric gives a (trivial) solution with

$$Q = \frac{n}{8} (n^2 - 4n(k+1) + 4k(k+2)) \quad \text{and} \quad \text{scal} = (n-1)(n-2k-2),$$

and note that $Q > 0$ if and only if $k < \frac{n-4}{2}$ or $k > \frac{n}{2}$, while $\text{scal} > 0$ if and only if $k < \frac{n-2}{2}$. Furthermore, $k < \frac{n-4}{2}$ is precisely the range of dimensions $k$ of limit sets of Kleinian groups associated to locally conformally flat closed manifolds for which Qing and Raske [QR06] establish $C^\infty$-compactness of the space of metrics with constant positive $Q$-curvature and positive scalar curvature.

### 1.1. About the proofs.

Although the methods used to prove Theorems A and B are adapted from nonuniqueness results for the Yamabe problem, the technical hurdles to implement them are substantially more challenging in the (fourth-order) constant $Q$-curvature problem.

The Yamabe parallel to Theorem A is the main result in [BP13a], whose proof uses classical variational bifurcation criteria [SW00, Kie12] that rely on computing the Morse index of a metric $g$ as a critical point of the total scalar curvature functional, also called Hilbert-Einstein functional. This can be accomplished by analyzing the spectrum of the Laplacian $\Delta_g$. Meanwhile, computing the Morse index of a critical point of the total $Q$-curvature functional (2.3) requires analyzing the spectrum of the Paneitz operator

$$P_g \psi = \Delta^2_g \psi + \frac{4}{n-2} \text{div}_g (\text{Ric}_g (\nabla \psi, e_i) e_i) - \frac{n^2 - 4n + 8}{2(n-1)(n-2)} \text{div}_g (\text{scal}_g \nabla \psi) + \frac{n-4}{2} Q_g \psi.$$
Inspired by a simplification due to Otoba and Petean [OP] of the techniques in [BP13a], we overcome this difficulty by finding an appropriate geometric framework that, while not imposing too many topological restrictions, reduces $P_g$ on basic functions to a quadratic polynomial on $\Delta_g^{-1}$. This facilitates the required spectral analysis, and leads to Theorem 3.4, of which Theorem A is a special instance.

Theorem B and Corollary C are, in turn, the $Q$-curvature doppelgängers of [BP18, Thm 1.1, Cor. 1.2], proved by playing the Aubin inequality against the volume growth along an infinite tower of finite-sheeted coverings. While the volume estimate side of the argument is the same, the existence result that produces new solutions at each step is a recent breakthrough of Hang and Yang [HY16b], involving a reverse Aubin-type inequality for a new conformal invariant, see Proposition 2.2.

1.2. Organization of the paper. In Section 2, we provide an overview of the constant $Q$-curvature problem, including its variational aspects and related conformal invariants. The main bifurcation criterion (Theorem 3.4) is proved in Section 3, together with its consequences for degenerating canonical variations (Propositions 3.6 and 3.7). Section 4 contains the Lie theoretic apparatus used to produce examples of homogeneous fibrations to which the above bifurcation criteria apply, along with the proof of Theorem A. The proof of Theorem B is given in Section 5. Finally, explicit formulae for the $Q$-curvature of Berger metrics are provided in Appendix A.

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2. Preliminaries on the constant $Q$-curvature problem

In this section, we recall the variational formulation of the constant $Q$-curvature problem, including first and second variations, as well as some Yamabe-type invariants, an Aubin-type inequality and an existence result of Hang and Yang [HY16b].

2.1. Paneitz operator and $Q$-curvature. Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 5$. Recall that the $Q$-curvature of the metric $g$ is defined as

$$Q_g = \frac{1}{2(n-2)} \Delta_g \text{scal}_g - \frac{2}{(n-2)^2} \left\| \text{Ric}_g \right\|^2 + \frac{n^2-4n^2+16n-16}{8(n-1)^2(n-2)^2} \text{scal}_g^2,$$

where $\Delta_g u = -\text{div}_g(\nabla u)$ is the (nonnegative) Laplace operator on $(M, g)$. The Paneitz operator $P_g$ is defined using a local $g$-orthonormal frame $(e_i)_{i=1}^n$, as

$$P_g \psi = \Delta_g^2 \psi + \frac{4}{n-2} \text{div}_g(\text{Ric}_g(\nabla \psi, e_i)e_i) - \frac{n^2-4n+8}{2(n-1)(n-2)} \text{div}_g(\text{scal}_g \nabla \psi) + \frac{n-4}{2} Q_g \psi.$$

Fix a background metric $g_0$ in $M$ and denote by $[g_0]$ its conformal class. Writing conformal metrics $g \in [g_0]$ as $g = u \frac{\sqrt{-h}}{h} g_0$, where $u : M \to \mathbb{R}$, $u > 0$, the Paneitz operator satisfies the covariance property that for any $\psi : M \to \mathbb{R}$,

$$P_g \psi = u^{-\frac{n+4}{2}} P_{g_0}(u \psi).$$

Thus, $Q_g = \frac{2}{n-2} P_{g_0}(1) = \frac{2}{n-2} u^{-\frac{n+4}{4}} P_{g_0}(u)$, and the following expression holds for the $Q$-curvature in terms of the Paneitz operator:

$$P_{g_0} u = \frac{4}{n-4} Q_g u^{\frac{n+4}{n-4}}.$$  \hspace{1cm} (2.1)

Therefore, the constant $Q$-curvature equation for the metric $g = u \frac{\sqrt{-h}}{h} g_0$ reads:

$$P_{g_0} u = \lambda u^{\frac{n+4}{n-4}}, \quad \lambda \in \mathbb{R}.$$  \hspace{1cm} (2.2)
In particular, it follows by elliptic regularity that constant $Q$-curvature metrics (in a smooth conformal class) are smooth.

2.2. Variational setup. For the remainder of this section, suppose that $M$ is closed. Consider the (normalized) total $Q$-curvature functional

\begin{equation}
Q: [g_0] \to \mathbb{R}, \quad Q(g) = \text{Vol}(M, g) \frac{4-n}{4} \int_M Q_g \text{vol}_g.
\end{equation}

Note that $Q$ is invariant under homotheties, i.e., $Q(\mu g) = Q(g)$ for all $\mu > 0$. Using (2.1), it is easy to see that $g = u^\frac{4}{4-n} g_0$ satisfies:

\[
\frac{n-4}{2} Q(g) = \left( \int_M u^{\frac{4}{4-n}} \text{vol}_{g_0} \right)^{\frac{n-4}{4}} \int_M u P_{g_0} u \text{vol}_{g_0} = \frac{E_{g_0}(u)}{\|u\|^2_{L^{\frac{4n}{n-4}}(M, \text{vol}_{g_0})}},
\]

where $E_{g_0}$ is the quadratic functional associated to the Paneitz operator $P_{g_0}$,

\[
E_{g_0}(u) = \int_M u P_{g_0} u \text{vol}_{g_0}.
\]

The first variation of the functional $Q$ can be computed as follows [GM15]:

\begin{equation}
d Q(g): T_{[g_0]} C^\infty(M) \to \mathbb{R}, \quad d Q(g) \phi = \frac{n-4}{4} \int_M (Q_g - \overline{Q}_g) \phi \text{vol}_g,
\end{equation}

where $\overline{Q}_g$ is the mean value of $Q_g$. Thus, $g \in [g_0]$ is a critical point of $Q$ if and only if it has constant $Q$-curvature.

Remark 2.1. By scale-invariance, constant $Q$-curvature metrics can also be characterized as critical points of the functional $E_{g_0}$ in $[g_0]$ subject to the constraint

\[
\|u\|_{L^{\frac{4n}{n-4}}(M, \text{vol}_{g_0})} = 1.
\]

It is easy to recover the constant $Q$-curvature equation (2.2) as the Euler–Lagrange equation of this constrained variational problem, and compute the value $Q_g$ of the $Q$-curvature of $g = u^\frac{4}{4-n} g_0$ in terms of the Lagrange multiplier associated to the critical point $u$.

If $g$ has constant $Q$-curvature, the second variation $d^2 Q(g)$ is represented by the 4th order Jacobi operator obtained by linearizing (2.2), namely:

\begin{equation}
J_g \psi = \frac{n+4}{4} Q_g \psi.
\end{equation}

This is a Fredholm operator $J_g: C^{j+1,\alpha}(M) \to C^j,\alpha(M)$ for all $j \geq 0$, which is symmetric with respect to the $L^2$-inner product.

2.3. Yamabe-type invariants. Recall that the Yamabe invariant of $(M, g_0)$ is:

\begin{equation}
Y(M, g_0) = \inf_{u \in C^\infty(M) \setminus \{0\}} \frac{\int_M u L_{g_0} u \text{vol}_{g_0}}{\|u\|^2_{L^{\frac{4n}{n-4}}(M, \text{vol}_{g_0})}}
\end{equation}

where $L_{g_0} = 4 \frac{n-2}{n-4} \Delta_{g_0} + \text{scal}_{g_0}$ is the conformal Laplacian.

Analogous conformal invariants have been defined for the Paneitz operator and $Q$-curvature. First, in analogy with the Yamabe invariant of a conformal class:

\begin{equation}
Y_4(M, g_0) = \inf_{u \in C^\infty(M) \setminus \{0\}} \frac{E_{g_0}(u)}{\|u\|^2_{L^{\frac{4n}{n-4}}(M, \text{vol}_{g_0})}}.
\end{equation}
It is well-known that the infimum in (2.6) is always attained at some positive function \( u > 0 \), and the corresponding conformal metric \( g = u^{\frac{4}{n-2}} g_0 \) has constant scalar curvature. However, unlike the second order case, if a minimizer \( u \) exists for the right hand side of (2.7), it need not be positive and hence there may be no conformal metric associated to it. Thus, it is natural to also define:

\[
Y_4^+(M, g_0) = \inf_{u \in C^\infty_0(M) \setminus \{0\}} \frac{E_g(u)}{\|u\|^2_{L^\infty(M, \text{vol}_{g_0})}} = \inf_{g \in [g_0]} Q(g).
\]

Clearly, \( Y_4^+(M, g_0) \geq Y_4(M, g_0) \), and these invariants coincide in some special cases. For instance, if \( n = \dim M \geq 6 \) and there exists \( g \in [g_0] \), with \( \text{scal}_g > 0 \) and \( Q_g > 0 \), then \( Y_4(M, g_0) = Y_4^+(M, g_0) \), and the (positive) infimum in (2.7) is attained by a positive function \( u \) such that \( u^{\frac{4}{n-2}} g_0 \) has positive constant \( Q \)-curvature, and everywhere positive scalar curvature [GHL16].

Lastly, suppose \( Y(M, g_0) > 0 \) and \( Q_{g_0} \) is almost positive, that is, \( Q_{g_0} \geq 0 \) everywhere and \( Q_{g_0} > 0 \) at some point. Although it is not known whether this implies \( Y_4(M, g_0) > 0 \), in this situation \( \ker P_{g_0} = \{0\} \), and the Green’s function \( G_{P_{g_0}} \) is everywhere positive on \( M \times M \). The inverse of \( P_{g_0} \) is the integral operator

\[
G_{P_{g_0}} f(p) = \int_M G_{P_{g_0}}(p, q) f(q) \text{vol}_{g_0}(q).
\]

A new conformal invariant was introduced by Hang and Yang [HY16b] in this context:

\[
\Theta_4(M, g_0) = \sup_{f \in L^\infty(M, g_0) \setminus \{0\}} \frac{\int_M f G_{P_{g_0}} f \text{vol}_{g_0}}{\|f\|^2_{L^\infty(M, \text{vol}_{g_0})}}.
\]

In some sense, \( \Theta_4(M, g_0) \) is comparable with the reciprocal \( 1/Y_4^+(M, g_0) \). The advantages of considering this quantity are that if a maximizer \( f \) for (2.9) exists, then \( f \) is smooth, does not change sign, and the conformal metric \( f^{\frac{4}{n-2}} g_0 \) has constant \( Q \)-curvature. Moreover, \( \Theta_4(M, g_0) \) can be used to prove the following existence result and reverse Aubin-type inequality [HY16b, Thm. 1.4, Lem. 2.1].

**Proposition 2.2.** If \( (M, g_0) \) is a closed Riemannian manifold of dimension \( n \geq 5 \), with \( Y(M, g_0) > 0 \), and \( Q_{g_0} \) almost positive, then

\[
\Theta_4(M, g_0) = \frac{2}{n-4} \sup_{g \in [g_0]} \frac{\int_M Q_g \text{vol}_g}{\|Q_g\|^2_{L^\infty(M, \text{vol}_g)}}.
\]

The supremum in (2.9) is attained at some smooth function \( f \in C^\infty(M) \), and the conformal metric \( f^{\frac{4}{n-2}} g_0 \) has constant \( Q \)-curvature. Moreover,

\[
\Theta_4(M, g_0) \geq \Theta_4(S^n, g_{\text{round}}),
\]

with equality if and only if \( (M, g_0) \) is conformally equivalent to \( (S^n, g_{\text{round}}) \).

---

1 If \( \dim M \geq 6 \), the existence of \( g \in [g_0] \) with with \( \text{scal}_g > 0 \) and \( Q_g > 0 \) is proved in [GHL16] to be equivalent to \( Y(M, g_0) > 0 \) and \( P_{g_0} > 0 \).
3. Bifurcation criteria using Riemannian submersions

We now define a notion of bifurcation for families of metrics with constant Q-curvature, and establish sufficient conditions for this phenomena to take place on the total space of Riemannian submersions, as its fibers are collapsed or dilated.

**Definition 3.1.** Let \( M \) be a closed manifold of dimension \( n \geq 5 \), and let \( g_t \), \( t \in [a, b] \), be a 1-parameter family of Riemannian metrics on \( M \) with constant Q-curvature. We say that \( t_* \in (a, b) \) is a bifurcation instant for the family \( g_t \) if there exists a sequence \( t_k \) in \( [a, b] \) and a sequence of nonconstant smooth positive functions \( u_k : M \to \mathbb{R} \) such that:

(i) \( \lim_{k \to \infty} t_k = t_* \),

(ii) the conformal metric \( u_k^{-2} \tilde{g}_{t_k} \) has constant Q-curvature for all \( k \),

(iii) \( \lim_{k \to \infty} u_k = 1 \) in the Sobolev space \( W^{2,2}(M) \).

The collection of metrics \( u_k^{-2} \tilde{g}_{t_k} \) is called a bifurcating branch for the family \( g_t \).

**3.1. Branch regularity.** Ellipticity of the constant Q-curvature problem can be used to improve the convergence of bifurcating solutions \( u_k \) in Definition 3.1:

**Proposition 3.2.** Assume that \( [a, b] \ni t \mapsto g_t \) is continuous in the \( C^s \)-topology, with \( s \geq 4 \), and let \( \tilde{g}_{t_k} = u_k^{-2} \tilde{g}_{t_k} \) be a bifurcating branch for the family \( g_t \). Then:

(i) \( \lim_{k \to \infty} Q_{\tilde{g}_{t_k}} = Q_{\tilde{g}_{t_*}} \),

(ii) \( \lim_{k \to \infty} u_k = 1 \) in the Whitney \( C^\infty \)-topology.

In particular, \( \tilde{g}_{t_k} \) tends to \( \tilde{g}_{t_*} \) in the \( C^s \)-topology, and its scalar curvature function tends to \( \text{scal}_{\tilde{g}_{t_*}} \) in the \( C^{s-2} \)-topology.

**Proof.** Since the metrics \( \tilde{g}_{t_k} \) converge to \( \tilde{g}_{t_*} \) in the \( C^4 \)-topology, one obtains:

\[
\lim_{k \to \infty} Q_{\tilde{g}_{t_k}} = Q_{\tilde{g}_{t_*}}, \quad \text{and} \quad \lim_{k \to \infty} \text{scal}_{\tilde{g}_{t_k}} = \text{scal}_{\tilde{g}_{t_*}} \quad \text{in the \( C^2 \)-topology.}
\]

The coefficients of the differential operator \( P_{\tilde{g}_{t_k}} \) tend to those of \( P_{\tilde{g}_{t_*}} \) in the \( C^{s-3} \)-topology; in particular, they converge uniformly. Moreover, a simple integration by parts argument shows that the quadratic form \( u \mapsto \int_M u P_{\tilde{g}_{t_k}} u \text{ vol}_{\tilde{g}_{t_k}} \) is continuous in the \( W^{2,2} \)-topology. Since \( u_k \) converges to 1 in \( W^{2,2}(M) \), using the Sobolev embedding \( W^{2,2}(M) \hookrightarrow L^{\frac{2n}{n-2}}(M) \), we have that \( \text{Vol}(M, \tilde{g}_{t_k}) = \int_M u_k^{\frac{2(n-2)}{n-4}} \text{vol}_{\tilde{g}_{t_k}} \) converges to \( \text{Vol}(M, \tilde{g}_{t_*}) \). Using these observations and (2.1), it follows that

\[
Q_{\tilde{g}_{t_k}} = \frac{2}{n-2} \text{Vol}(M, \tilde{g}_{t_k})^{-1} \int_M u_k P_{\tilde{g}_{t_k}} u_k \text{ vol}_{\tilde{g}_{t_k}},
\]

converges to \( Q_{\tilde{g}_{t_*}} \), verifying (i).

Using the constant Q-curvature equation, we have that

\[
\Delta^2_{\tilde{g}_{t_k}} u_k = -\frac{1}{n-2} \text{div}_{\tilde{g}_{t_k}} (\text{Ric}_{\tilde{g}_{t_k}} (\nabla u_k, e_i) e_i)
+ \frac{n^2 - 4n + 8}{2(n-1)(n-2)} \text{div}_{\tilde{g}_{t_k}} (\text{scal}_{\tilde{g}_{t_k}} \nabla u_k) - \frac{n-4}{2} Q_{\tilde{g}_{t_k}} u_k + \frac{n-4}{2} Q_{\tilde{g}_{t_k}} u_k^\frac{n+4}{n-4};
\]

As observed above, all coefficients of the differential operators appearing in (3.1) converge uniformly. Moreover, using the Sobolev embedding \( W^{2,2}(M) \hookrightarrow L^{\frac{2n}{n-2}}(M) \),
Theorem 3.4. Let \( \alpha \) be a standard elliptic estimates (see for instance [GT01, Thm. 9.14, p. 240]) imply that \( u_k \) converge in \( W^{4,p}(M) \), hence (ii) follows from a standard bootstrap argument. \( \square \)

3.2. Main bifurcation criterion. In order to search for bifurcations, we restrict to families of metrics that admit a very special type of submersion defined as follows.

**Definition 3.3.** A Riemannian submersion \( \pi: (M, g_M) \to (B, g_B) \) is **horizontally Einstein** if the Ricci tensor of \( (M, g_M) \) splits as \( \text{Ric}_{g_M} = \text{Ric}_H \oplus \text{Ric}_V \) pointwise according to the splitting \( T_y M = H_p \oplus V_p \) into horizontal and vertical subspaces, and \( \text{Ric}_H = \kappa \pi^* (g_B) \), where the constant \( \kappa \) is called the **horizontal Einstein constant**.

Our key detection tool for bifurcation is the following:

**Theorem 3.4.** Let \( M \) be a closed manifold with \( n = \dim M \geq 5 \), and

\[
\pi_t: (M, g_t) \to (B, g_B), \quad t \in [t_* - \varepsilon, t_* + \varepsilon],
\]

be a 1-parameter family of horizontally Einstein Riemannian submersions with minimal fibers. Assume that \( g_t \) has constant scalar curvature and constant \( Q \)-curvature for all \( t \), and denote by \( \kappa_t \), the horizontal Einstein constant of \( \pi_t \). Given an eigenvalue \( \lambda > 0 \) of the Laplacian \( \Delta_{g_B} \) of the base manifold \( (B, g_B) \), set

\[
(3.2) \quad \alpha_t = \frac{(n^2 - 4n + 8) \text{scal}_{g_t} - 8 \kappa_t(n - 1)}{4(n - 1)(n - 2)} \quad \text{and} \quad \beta_t = -2 Q_{g_t},
\]

and assume that

\[
\frac{1}{\pi} \lambda^2 + \alpha_t, \lambda + \beta_t = 0 \quad \text{and} \quad \alpha_t' \lambda + \beta_t' \neq 0,
\]

where \( \alpha_t' = \frac{d}{dt} \alpha_t |_{t = t_*} \) and \( \beta_t' = \frac{d}{dt} \beta_t |_{t = t_*} \). Then \( t_* \) is a bifurcation instant for the family \( g_t \) of constant \( Q \)-curvature metrics. If, in addition, \( \lambda \neq \frac{\text{scal}_{g_t}}{n - 1} \), then the constant \( Q \)-curvature metrics in the bifurcation branch that are sufficiently close to \( g_{t_*} \) do not have constant scalar curvature.

**Proof.** The strategy we employ is inspired by a bifurcation criterion for the Yamabe problem on Riemannian submersions due to Otoba and Petean [OP], which simplifies earlier results by the first and second named authors [BP13a, BP13b].

Given a smooth function \( v: B \to \mathbb{R} \), denote by \( \overline{v} = v \circ \pi_t \). Functions of the form \( \overline{v} \) are called **basic functions**. Similarly, given a vector field \( V \in \mathcal{X}(B) \), denote by \( \overline{V} \in \mathcal{X}(M) \) its horizontal lift, i.e., the unique vector field on \( M \) which is \( \pi_t \)-related to \( V \). From the definition of Riemannian submersion, the gradient \( \nabla \overline{v} \) with respect to \( g_t \) is the horizontal lift of the gradient \( \nabla v \) with respect to \( g_B \). Since the fibers of \( \pi_t \) are minimal, \( \text{div}_{g_t} \overline{V} \) is the horizontal lift of \( \text{div}_{g_B} V \). In particular, \( \Delta_{g_t} \overline{V} \) is the horizontal lift of \( \Delta_{g_B} v \); and \( \Delta_{g_t}^2 \overline{V} \) is the horizontal lift of \( \Delta_{g_B}^2 v \). Using that \( g_t \) is horizontally Einstein, \( \text{div} \left( \sum_i \text{Ric}_{g_t}(\nabla_{\overline{V}}, e_i) e_i \right) \) is the horizontal lift of \( -\kappa_t \Delta_{g_t} v \).

Consider the family \( P_t \) of fourth-order linear differential operators on \( B \) given by:

\[
P_t v = \Delta_{g_B}^2 v + 2 \alpha_t \Delta_{g_B} v - \frac{n-4}{n-2} \beta_t v.
\]

Using the observations above, it follows that the horizontal lift of \( P_t v \) is \( P_{g_t} \overline{v} \). Thus, for a positive smooth function \( v: B \to \mathbb{R} \), the conformal metric \( \overline{v}^{\frac{4}{n-2}} g_t \) has constant \( Q \)-curvature equal to \( C \) if and only if \( v \) satisfies:

\[
P_t v = \frac{n-4}{2(n-2)} C v^{\frac{n+4}{n-4}}.
\]
Observe that (3.3) is an elliptic equation with subcritical exponent, since \( \dim B < n \).
Clearly, (3.3) is the Euler–Lagrange equation for critical points in \( C^\infty(B) \) of the quadratic functional \( E_t(v) = \int_B v P_t v \, \text{vol}_B \) subject to the constraint:\(^2\)

\[
\|v\|_{L^{2n/(n+4)}(B, \text{vol}_B)} = \text{const.}
\]

We may choose the value of this constant equal to \( \text{Vol}(B, g_B)^{\frac{n+4}{4n}} \) for all \( t \), in such way that (3.5) is satisfied by the constant function 1.

Recalling the characterization of constant Q-curvature metrics in \([g_0]\) as constrained critical points (Remark 2.1), we conclude that horizontality is a natural constraint for the constant Q-curvature problem.\(^3\) In other words, conformal metrics \( g = v^2 g_0 \) such that \( Q_g \) is constant, where \( v \) is a basic function, are precisely the critical points of the restriction of the total Q-curvature functional (2.3) to the subset of basic functions. In particular, given such a conformal factor \( v \), denoting by \( v: B \to \mathbb{R} \) the corresponding function on the base so that \( v = v \circ \pi_t \), the second variation of the quadratic functional \( E_t \) at the critical point \( v \) is given by the restriction of the Jacobi operator \( J_g \) defined in (2.5). We thus define the 1-parameter family of Jacobi operators:

\[
J_t \phi = \frac{1}{2} P_t \phi - \frac{n+4}{4} Q_{g_t} \phi, \quad \phi \in C^\infty(B).
\]

Consider the restriction of \( J_t \) to the tangent space to the sphere (3.5) at \( v = 1 \), which consists of functions \( \phi: B \to \mathbb{R} \) such that \( \int_B \phi \, \text{vol}_B = 0 \). Note that this space is invariant under \( J_t \), and hence the eigenvalues of the restriction are exactly the eigenvalues of \( J_t \) with nonconstant eigenfunctions. Since \( J_t = \frac{1}{2} \Delta_{g_B}^2 + \alpha_t \Delta_{g_B} + \beta_t \) is a polynomial in \( \Delta_{g_B} \), its eigenvalues are:

\[
\frac{1}{2} \lambda^2 + \alpha_t \lambda + \beta_t,
\]

where \( \lambda \) is an eigenvalue of \( \Delta_{g_B} \). As the spectrum of \( \Delta_{g_B} \) is discrete, so is the spectrum of \( J_t \) for all \( t \). The assumptions in the statement imply that the constant function 1 is a degenerate critical point of \( E_t \), subject to the constraint (3.5), that its Morse index jumps at \( t = t_* \), and that 1 is a nondegenerate critical point of \( E_{t_*+\varepsilon} \) for \( \varepsilon > 0 \) sufficiently small. The conclusion that \( t_* \) is a bifurcation instant then follows from standard variational bifurcation results, see e.g. [SW90, Kie12].

As to the last claim, \( t_* \) is not a bifurcation instant for the constant scalar curvature problem if \( \lambda \neq \frac{\text{scal}_{g_*}}{n} \), see [dLPZ12, Cor. 4]. Thus, metrics sufficiently close to \( g_{t_*} \) and nonhomothetic to some \( g_t \) do not have constant scalar curvature. \( \square \)

**Remark 3.5.** It follows from the above proof that conformal factors associated with bifurcating solutions detected by Theorem 3.4 are basic, i.e., constant along the fibers of the submersion \( \pi_t \). Thus, the conformal deformations of \((M, g_t)\) producing other metrics with constant Q-curvature only involve horizontal directions.

---

\(^2\)Recall that the fibers of a Riemannian submersion with minimal fibers have constant volume, and denote by \( \nu_t \) the \( g_{t_*} \)-volume of the fibers of \( \pi_t \). Thus, for any continuous map \( v: B \to \mathbb{R} \),

\[
\|v\|_{L^{2n/(n+4)}(M, \text{vol}_{g_t})} = \nu_t^{\frac{n+4}{4n}} \|v\|_{L^{2n/(n+4)}(B, \text{vol}_B)}.
\]

In particular, \( C^\infty(B) \ni v \to \nu \subset C^\infty(M) \) maps \( L^{2n/(n+4)} \)-spheres to \( L^{2n/(n+4)} \)-spheres.

\(^3\)This can also be proved directly since, under these assumptions, \( dQ(g_0) f \) vanishes whenever \( f: M \to \mathbb{R} \) is a function with zero average on the fibers of the submersion.
3.3. Infinite bifurcations from asymptotic behavior. We now specialize to 1-parameter families $\pi_t: (M,g_t) \to (B,g_B)$ of Riemannian submersions obtained rescaling the vertical space of a fixed Riemannian submersion $\pi: (M,g) \to (B,g_B)$ using the parameter $0 < t < +\infty$. Such a family $\pi_t$ is called the canonical variation of $\pi$ in Besse [Bes08, §9 G]. We give sufficient conditions for the above criterion (Theorem 3.4) to apply along each element $t_\lambda$ of sequences that either accumulate at 0 or $+\infty$. Geometrically (in Gromov-Hausdorff sense), these correspond respectively to families $(M,g_t)$ of metrics with constant $Q$-curvature that bifurcate infinitely many times either as they collapse to the base $(B,g_B)$ or as they degenerate to a sub-Riemannian limit. We henceforth systematically ignore other bifurcation instants $t_*$ that may exist at a bounded distance away from 0 or $+\infty$.

Let $\pi: (M,g) \to (B,g_B)$ be a Riemannian submersion with totally geodesic fibers isometric to $(F,g_F)$, and denote by $\mathcal{H}$ and $\mathcal{V}$ the corresponding horizontal and vertical distributions, so that $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$ is a $g$-orthogonal direct sum for all $p \in M$. Assume that $(B,g_B)$ and $(F,g_F)$ are Einstein manifolds, that is, there exist $\Lambda_B, \Lambda_F \in \mathbb{R}$ such that

$$\text{(3.7)} \quad \text{Ric}_B = \Lambda_B \delta g_B \quad \text{and} \quad \text{Ric}_F = \Lambda_F g_F.$$ 

Set $n = \dim M$ and $l = \dim F$, and note that $n \geq l$. Following Besse [Bes08, §9.33], denote by $(X_i)_{i=1}^{n-l}$ and $(U_j)_{j=1}^{l}$ $g$-orthonormal bases of $\mathcal{H}$ and $\mathcal{V}$ respectively, and let

$$\text{(3.8)} \quad (A_X, A_Y) := \sum_{i=1}^{n-l} g(A_X X_i, A_Y X_i) = \sum_{j=1}^{l} g(A_X U_j, A_Y U_j)$$

and

$$\text{(3.9)} \quad (A_U, A_V) := \sum_{i=1}^{n-l} g(A_X U, A_X V),$$

where $A$ is the Gray-O’Neill tensor $A_Z W = (\nabla_{Z_i} W_j)_H + (\nabla_{Z_i} W_j)_V$, $Z, W \in T_pM$. Suppose there exist $\zeta, \eta \in \mathbb{R}$ such that, for all $X, Y \in \mathcal{H}$ and $U, V \in \mathcal{V}$,

$$\text{(3.10)} \quad (A_X, A_Y) = \zeta g(X,Y) \quad \text{and} \quad (A_U, A_V) = \eta g(U,V).$$

Under the above hypotheses, the canonical variation $g_t = t \delta g_V \oplus \delta g_H$, $t > 0$, is a metric such that $\pi_t: (M,g_t) \to (B,g_B)$ is a Riemannian submersion with totally geodesic fibers isometric to $(F,t g_f)$, see [Bes08, Prop. 9.68]. Moreover, the Ricci tensor $\text{Ric}_t$ of $(M,g_t)$ satisfies, see [Bes08, Prop. 9.70]:

$$\text{Ric}_t(X,Y) = (\Lambda_B - 2\zeta) g(X,Y),$$

$$\text{Ric}_t(U,V) = (\Lambda_F + \eta t^2) g(U,V),$$

$$\text{Ric}_t(X,V) = 0,$$

for all $X, Y \in \mathcal{H}$ and $U, V \in \mathcal{V}$, where $\eta t = \zeta(n-l)$, see [Bes08, §9.37]. In particular, $(M,g_t)$ has constant scalar curvature

$$\text{(3.11)} \quad \text{scal}_t = \frac{l \Lambda_F}{t} + \Lambda_B (n-l) - \eta lt,$$

as well as constant $Q$-curvature

$$\text{(3.12)} \quad Q_t = \frac{2(n-l) (\Lambda_B - 2\zeta)^2}{(n-2)^2} - \frac{2l}{(n-2)^2} \left( \frac{\Lambda_F}{t} + \eta t \right)^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \left( \frac{\Lambda_F}{t} + (n-l) \Lambda_B - \eta lt \right)^2.$$
Clearly, the Riemannian submersion $\pi_t$ is horizontally Einstein, with horizontal Einstein constant $\kappa_t = \Lambda_B - 2\ell t$.

Using the above notation, we now establish conditions for the existence of infinitely many bifurcation instants accumulating at 0 and $\infty$.

**Proposition 3.6.** Suppose that one of the following dimensional assumptions holds:

- (D1) $5 \leq n \leq 8$ and $l \geq 3$, or
- (D2) $n \geq 9$ and $l \geq 2$.

If $\Lambda_F > 0$, then there exists a sequence $\{t_\lambda\}$ accumulating at 0 of bifurcation instants for the family $g_\lambda$. Moreover, for $t_\lambda$ sufficiently small, the bifurcating metrics sufficiently close to $g_{t_\lambda}$ do not have constant scalar curvature.

**Proof.** The functions $\alpha_t$ and $\beta_t$ defined in (3.2) can be computed explicitly using (3.11) and (3.12). From these computations, asymptotically as $t \downarrow 0$, we have

$$\alpha_t^2 - 2\beta_t \sim \frac{(n^4 + 64n - 64)l - 128(n-1)^2l \Lambda_F^2}{16(n-1)^2(n-2)^2} + O \left( \frac{1}{t} \right).$$

Direct inspection shows that, under any of the dimensional assumptions (D1) or (D2), the numerator in the above leading term satisfies

$$a(n, l) = (n^4 + 64n - 64)l - 128(n-1)^2l > 0$$

and hence $\alpha_t^2 - 2\beta_t \nearrow +\infty$ as $t \downarrow 0$. Thus, for $t > 0$ sufficiently small, the equation

$$\frac{1}{4} \lambda^2 + \alpha_t \lambda + \beta_t = 0$$

has two real solutions $\lambda^+_t = -\alpha_t \pm \sqrt{\alpha_t^2 - 2\beta_t}$. Moreover, asymptotically as $t \downarrow 0$,

$$\lambda^+_t \sim \frac{-n^2 - 4n + 8l + \sqrt{a(n, l) \Lambda_F}}{4(n-1)(n-2)} + O(1).$$

Either (D1) or (D2) implies that the above leading coefficient is positive and, since $\Lambda_F > 0$, we have that $\lambda^+_t \nearrow +\infty$ as $t \downarrow 0$. Thus, for arbitrarily large $\lambda \in \text{Spec}(\Delta_B)$, there exists a sufficiently small $t_\lambda > 0$ such that $\lambda$ is a solution of (3.14) with $t = t_\lambda$. Since $\text{Spec}(\Delta_B)$ is unbounded, the corresponding sequence $\{t_\lambda\}$ accumulates at 0.

A direct computation shows that, asymptotically as $t \downarrow 0$,

$$\alpha_t' \lambda^+_t + \beta_t' \sim \frac{a(n, l) - (n^2 - 4n + 8l)^{\frac{1}{2}} \sqrt{a(n, l) \Lambda_F}}{16(n-1)^2(n-2)^2} + O \left( \frac{1}{t^2} \right).$$

The above leading coefficient is positive whenever one of the dimensional assumptions are satisfied, so $\alpha_t' \lambda^+_t + \beta_t' \nearrow +\infty$ as $t \downarrow 0$. Thus, Theorem 3.4 implies that $t_\lambda$ is a bifurcation instant for $g_\lambda$, provided $\lambda \in \text{Spec}(\Delta_B)$ is sufficiently large.

Finally, in order to prove the last claim, observe that asymptotically as $t \downarrow 0$,

$$\lambda^+_t - \frac{\text{scal}_t}{n-1} \sim \frac{\sqrt{a(n, l) - n^2l} \Lambda_F}{4(n-1)(n-2)} + O(1).$$

The above leading coefficient is nonzero if and only if $n \neq \frac{3}{4} + 1$, which is again satisfied because of the dimensional assumptions. Thus, for $t_\lambda$ sufficiently small, $\lambda \neq \frac{\text{scal}_t}{n-1}$ and hence the last claim also follows from Theorem 3.4. \qed

**Proposition 3.7.** Suppose that one of the dimensional assumptions (D1) or (D2) in Proposition 3.6 holds, or else that

- (D3) $n \geq 21$ and $l = 1$. 


Moreover, assume that $\zeta > 0$, $\eta > 0$, and
\begin{equation}
\frac{\eta}{\zeta} > \frac{8(n-1)\sqrt{n-l}}{\sqrt{(n^3 - 4n^2 + 16n - 16)t^2 - 16(n-1)^2}}.
\end{equation}
Then there exists a sequence $\{t_\lambda\}$ that diverges to $+\infty$ of bifurcation instants for the family $g_t$. Moreover, for $t_\lambda$ sufficiently large, the bifurcating metrics sufficiently close to $g_{t_\lambda}$ do not have constant scalar curvature.

Proof. Computing $\alpha_t$ and $\beta_t$ explicitly, it follows that asymptotically as $t \nearrow +\infty$,
\begin{equation}
\alpha_t^2 - 2\beta_t \sim \frac{a(n,l)\eta^2 + b(n,l)\eta\zeta + c(n,l)\zeta^2}{16(n-1)^2(n-2)^2}t^2 + O(t),
\end{equation}
where $a(n,l)$ is defined in $(3.13)$ and
\begin{equation}
b(n,l) = -32l(n^3 - 5n^2 + 12n - 8), \quad c(n,l) = -512(n-1)^2(n-l - \frac{1}{2}).
\end{equation}
Routine computations show that any of the dimensional assumptions (D1), (D2), or (D3) implies:
\begin{equation}
a(n,l) > 0, \quad b(n,l) < 0, \quad c(n,l) < 0,
\end{equation}
and hence $\delta(n,l) = b(n,l)^2 - 4a(n,l)c(n,l) > 0$. Consider the homogeneous polynomial
\begin{equation}
q_1(\eta, \zeta) = a(n,l)\eta^2 + b(n,l)\eta\zeta + c(n,l)\zeta^2.
\end{equation}
Since $\delta(n,l) > 0$, there is a factorization
\begin{equation}
\frac{q_1(\eta, \zeta)}{\zeta^2} = a(n,l)\left(\frac{\eta}{\zeta} - \rho_--\rho_+ight)\left(\frac{\eta}{\zeta} - \rho_++\rho_-ight),
\end{equation}
where $\rho_\pm = \frac{-b(n,l)\pm \sqrt{b(n,l)}}{2a(n,l)}$. Thus, by $(3.17)$, we have that $q_1(\eta, \zeta) > 0$ if and only if $\frac{\eta}{\zeta} < \rho_-$ or $\frac{\eta}{\zeta} > \rho_+$. The latter condition is satisfied due to $(3.15)$, since
\begin{equation}
\frac{8(n-1)\sqrt{n-l}}{\sqrt{(n^3 - 4n^2 + 16n - 16)t^2 - 16(n-1)^2}} > \rho_+
\end{equation}
under any of the dimensional assumptions (D1), (D2), or (D3). Therefore, we have $\alpha_t^2 - 2\beta_t \nearrow +\infty$ as $t \nearrow +\infty$. Routine computations imply that
\begin{equation}
\lambda_1^+ = -\alpha_t + \sqrt{\alpha_t^2 - 2\beta_t} \sim \frac{(n^2 - 4n + 8)\eta - 16(n-1)\zeta + \sqrt{q_1(\eta, \zeta)}}{4(n-1)(n-2)}t + O(1)
\end{equation}
and hence $\lambda_1^+ \nearrow +\infty$ as $t \nearrow +\infty$ if and only if $(n^2 - 4n + 8)\eta - 16(n-1)\zeta + \sqrt{q_1(\eta, \zeta)} > 0$. This holds by our assumptions, because $\frac{16(n-1)}{(n^3 - 4n^2 + 16n - 16)} < \rho_+$. Thus, for arbitrarily large $\lambda \in \text{Spec}(\Delta_B)$, there exists a sufficiently large $t_\lambda > 0$ such that $\lambda$ is a solution of $(3.14)$ with $t = t_\lambda$. Since $\text{Spec}(\Delta_B)$ is unbounded, the corresponding sequence $\{t_\lambda\}$ diverges to $+\infty$. Moreover, similar computations show that $(3.15)$ implies $\alpha_t^+ + \beta_t^+ \searrow -\infty$ as $t \nearrow +\infty$. Thus, Theorem 3.4 implies that $t_\lambda$ is a bifurcation instant for $g_{t_\lambda}$, provided $\lambda \in \text{Spec}(\Delta_B)$ is sufficiently large.

Finally, the last claim follows from the fact that $\text{scal} \searrow -\infty$ as $t \nearrow +\infty$ since $\eta > 0$, hence $\lambda_1^+ \searrow -\frac{\text{scal}}{n-1} \nearrow +\infty$. In particular, for $t_\lambda$ sufficiently large, $\lambda \neq \frac{\text{scal}}{n-1}$ and hence the claim also follows from Theorem 3.4. □
Remark 3.8. Assumption (3.15) is stated (for convenience) in terms of the ratio of \( \eta \) and \( \zeta \), which are quantities related to the Gray-O’Neill \( A \)-tensor of the submersion \( F \to M \to B \), see (3.9). Nevertheless, (3.15) is a purely dimensional restriction, that is, it only depends on the dimensions of \( F \) and \( B \) and not on the \( A \)-tensor of the bundle itself, since \( \frac{\eta}{\xi} = \frac{n-l}{l} \), see [Bes08, §9.37].

Remark 3.9. Submersions with 1-dimensional fibers are included in Proposition 3.7, but excluded altogether from Proposition 3.6, because \( l = 1 \) implies \( \Lambda_F = 0 \). Under these conditions, \( \alpha_t^2 - 2\beta_t \) remains bounded as \( t \searrow 0 \), so there can be at most finitely many bifurcation instants for \( t \) near 0.

4. Homogeneous examples with two isotropy summands

In this section, we use homogeneous spaces to provide a wealth of examples of Riemannian submersions to which the results (Propositions 3.6 and 3.7) from the previous section apply. Given a triple of compact Lie groups \( H \subset K \subset G \), define

\[
\pi: G/H \to G/K, \quad \pi(gH) = gK.
\]

Denote by \( h \subset t \subset g \) the corresponding Lie algebras. Endow \( g \) with a bi-invariant inner product, and let \( m \) and \( p \) be complements such that the direct sums

\[
g = t \oplus m \quad \text{and} \quad t = h \oplus p
\]

are orthogonal. There is a bijective correspondence between \( \text{Ad}(K) \)-invariant inner products on \( m \) and homogeneous (\( G \)-invariant) Riemannian metrics on \( G/K \); and analogously for \( \text{Ad}(H) \)-invariant inner products on \( p \) and homogeneous (\( K \)-invariant) Riemannian metrics on \( K/H \). The orthogonal direct sum of two such inner products is an inner product on \( m \oplus p \) that corresponds to a homogeneous (\( G \)-invariant) Riemannian metric \( g \) on \( G/H \) for which (4.1) is a Riemannian submersion with totally geodesic fibers isometric to \( K/H \), see [Bes08, Thm. 9.80]. Clearly, \( m \) and \( p \) are respectively identified with the horizontal and vertical spaces of this submersion. Assume that \( m \) and \( p \) are inequivalent irreducible \( \text{Ad}(H) \)-representations. Then, by Schur’s Lemma, the \( \text{Ad}(H) \)-invariant pairings (3.8) are multiples of the metric, i.e., (3.9) holds, and the Ricci tensor \( \text{Ric}_{G/H}: m \oplus p \to m \oplus p \) is block diagonal. Thus, (3.7) holds as a consequence of (3.9) and [Bes08, Prop. 9.70].

Proposition 4.1. Let \( H \subset K \subset G \) be compact Lie groups as above, and assume that \( m \) and \( p \) are inequivalent irreducible \( \text{Ad}(H) \)-representations, \( K \) is not Abelian, and that \( n = \dim G/H \) and \( l = \dim K/H \) satisfy either (D1) or (D2) in Proposition 3.6. Then there exists an infinite sequence of branches of constant \( Q \)-curvature inhomogeneous metrics bifurcating from the (canonical variation) family \( g_t \) of \( G \)-invariant metrics on \( G/H \) as \( t \searrow 0 \). On each bifurcating branch, infinitely many of these inhomogeneous metrics do not have constant scalar curvature.

Proof. As observed above, (3.7) and (3.9) hold, since \( m \) and \( p \) are inequivalent irreducible \( \text{Ad}(H) \)-representations. Furthermore, as the fiber \( F = K/H \) is a compact homogeneous space, its Einstein constant satisfies \( \Lambda_F \geq 0 \). Equality holds if and only if \( K/H \) is a flat manifold, which would imply \( K \) is Abelian. Thus, \( \Lambda_F > 0 \) and hence the entire statement follows from Proposition 3.6. □

Triples of compact Lie groups \( H \subset K \subset G \) such that \( m \) and \( p \) are inequivalent irreducible \( \text{Ad}(H) \)-representations, as in Proposition 4.1, were classified by Dickinson and Kerr [DK08], with corrections by He [He12]. This classification requires the
further assumptions\textsuperscript{4} that $G$ is simple and simply-connected or $G = \text{SO}(n)$, and $H$ is connected, and produces several dozen examples including many infinite families.

4.1. Hopf bundles. A well-known and geometrically interesting subclass of homogeneous spaces $G/H$ as in (4.1) with two isotropy summands are the Hopf bundles:

| Table 1. The Hopf bundles and corresponding Lie groups |
| --- |
| $F \rightarrow M \rightarrow B$ | $G$ | $K$ | $H$ |
| (i) $S^1 \rightarrow S^{2q+1} \rightarrow \mathbb{C}P^q$ | $\text{SU}(q+1)$ | $\text{SU}(q)$ | $\text{U}(q)$ |
| (ii) $S^3 \rightarrow S^{4q+3} \rightarrow \mathbb{H}P^q$ | $\text{Sp}(q+1)$ | $\text{Sp}(q)$ | $\text{Sp}(q)$ |
| (iii) $CP^1 \rightarrow CP^{2q+1} \rightarrow \mathbb{H}P^q$ | $\text{Sp}(q+1)$ | $\text{Sp}(q)$ | $\text{Sp}(q)$ |
| (iv) $S^7 \rightarrow S^{15} \rightarrow S^8(1/2)$ | $\text{Spin}(9)$ | $\text{Spin}(8)$ | $\text{Spin}(7)$ |

It is immediate to verify that Proposition 4.1 applies to the above Hopf bundles, provided the dimensional restrictions are satisfied; i.e., (i) is excluded, (ii) for all $q \geq 1$, (iii) for all $q \geq 2$, and (iv). Furthermore, Proposition 3.7 also applies to the above examples, provided that the dimensional restrictions, including (3.15), are satisfied. This is the case on (i) for all $q \geq 10$, on (ii) for all $q \geq 3$, on (iii) for all $q \geq 4$, and on (iv). Indeed, (3.15) can be evaluated using the following table:

| Table 2. Some invariants of the Hopf bundles, see (3.7) and (3.9). |
| --- |
| $M$ | $(n, l)$ | $\zeta$ | $\eta$ | $\Lambda_F$ | $\Lambda_B$ |
| (i) $S^{2q+1}$ | $(2q + 1, 1)$ | 1 | 2$q$ | 0 | 2$q + 2$ |
| (ii) $S^{4q+3}$ | $(4q + 3, 3)$ | 3 | 4$q$ | 2 | 4$q + 8$ |
| (iii) $CP^{2q+1}$ | $(4q + 2, 2)$ | 2 | 4$q$ | 4 | 4$q + 8$ |
| (iv) $S^{15}$ | $(15, 7)$ | 7 | 8 | 6 | 28 |

Direct computations show that the dimensional requirements on families (i), (ii), and (iii) can be further relaxed; that is, bifurcation occurs as $t \nearrow +\infty$ even in some cases where (3.15) is violated.\textsuperscript{5} Namely, the conclusion of Proposition 3.7 still holds on (i) if $6 \leq q \leq 9$, on (ii) if $q = 2$, and on (iii) if $q = 3$, as can be verified using the computations in the Appendix A to evaluate $\alpha_t$ and $\beta_t$, proceeding as in the proof of Proposition 3.7. Note that, in these cases, bifurcating branches issuing from $g_t$, with $t$ sufficiently large, provide examples of nonuniqueness of constant $Q$-curvature metrics in conformal classes $[g_t]$ with $Y(M, g_t) < 0$ and $Y^+_q(M, g_t) < 0$.

\textsuperscript{4}These assumptions are not needed for our results.

\textsuperscript{5}Given the nature of the estimates where (3.15) is used, it should not be surprising that this condition is only sufficient and not necessary.
5. Nonuniqueness of Conformal Metrics with Constant Q-Curvature

The problem of finding metrics with constant Q-curvature in a given conformal class can also be studied on noncompact Riemannian manifolds. In this situation, the geometrically natural boundary condition to impose is completeness of the metric \( g = u^4 \tilde{g}_0 \), which translates to an appropriate growth rate for the conformal factor \( u \). Although the constant Q-curvature equation remains (2.2), variational formulations such as (2.3) are no longer available. To circumvent this issue and establish nonuniqueness results also in this context, we combine infinite towers of coverings and the reverse Aubin inequality (2.11), in an argument inspired by nonuniqueness results for the Yamabe problem on noncompact manifolds [BP18].

**Proposition 5.1.** Let \( (M_0, g_0) \) be a closed Riemannian manifold of dimension \( n \geq 5 \), and let \( (M_\infty, g_\infty) \to (M_0, g_0) \) be a Riemannian covering whose group of deck transformations has infinite profinite completion. Suppose \( Q_{g_\infty} \) is almost positive and \( Y(M_0, g_0) \geq 0 \). Then, the conformal class \( [g_\infty] \) contains infinitely many pairwise nonhomothetic complete metrics with constant positive Q-curvature.

**Proof.** It follows from the assumption on the group of deck transformations that there exists an infinite tower of Riemannian coverings

\[
(M_\infty, g_\infty) \to \cdots \to (M_k, g_k) \to \cdots \to (M_2, g_2) \to (M_1, g_1) \to (M_0, g_0),
\]

where \( M_k \to M_0 \) is an \( \ell_k \)-sheeted covering, with \( \ell_k \geq 2 \) and \( \ell_k \not\sim \infty \) as \( k \to \infty \), see [BP18, Lemma 3.6]. Since \( (M_k, g_k) \) are locally isometric to \( (M_0, g_0) \), also \( Q_{g_k} \) is almost positive. Using an observation of Aubin, see Akutagawa and Neves [AN07, Lemma 3.6], we have that \( Y(M_k, g_k) > 0 \) for all \( k \geq 1 \). Thus, by Proposition 2.2,

\[
\Theta(M_k, g) = \frac{\int_{M_k} Q_g \text{vol}_g}{\|Q_g\|^2 L_{\frac{n+2}{n+4}}(M_k, \text{vol}_{g_k})}
\]

attains its maximum in each conformal class \( [g_k] \) at some metric \( \tilde{g}_k \in [g_k] \) for all \( k \geq 1 \). Denote by \( h_k \) the pullback to \( M_k \) of the metric \( \tilde{g}_1 \), and note that \( h_k \in [g_k] \). Since \( q_1 = Q_{\tilde{g}_1} \) is (a positive) constant and \( h_k \) is locally isometric to \( \tilde{g}_1 \), then \( Q_{h_k} = q_1 \) for all \( k \geq 1 \), and we have:

\[
\Theta(M_k, h_k) = \frac{\int_{M_k} Q_{h_k} \text{vol}_{h_k}}{\left( \int_{M_k} Q_{h_k}^{\frac{n}{n-4}} \text{vol}_{h_k} \right)^{\frac{n+4}{4}}} = q_1^{-1} \text{Vol}(M_1, \tilde{g}_1)^{-\frac{4}{n-4}} \left( \frac{\ell_k}{\ell_1} \right)^{-\frac{\ell_1}{n-4}} \searrow 0, \text{ as } k \to \infty.
\]

Thus, \( \frac{2}{n-4} \Theta(M_k, h_k) < \Theta_4(S^n, g_{\text{round}}) \) if \( k \geq k_0 \) for some \( k_0 \geq 1 \) sufficiently large, and hence \( h_k \) is not a maximizer of \( \Theta_4 \) in \( [g_{h_k}] \). By Proposition 2.2, there exists a maximizer of \( \Theta_4 \) in \( [h_{k_0}] \), not homothetic to \( h_{k_0} \). The pullbacks to \( M_\infty \) of this maximizer and \( h_{k_0} \) are two nonhomothetic complete metrics with constant positive Q-curvature in \( [g_\infty] \). Infinitely many such metrics can be produced by iterating this procedure, replacing \( (M_1, \tilde{g}_1) \) with \( (M_{k_0}, g_{k_0}) \). \( \square \)

We are now ready to prove Theorem B in the Introduction, using Proposition 5.1.

**Proof of Theorem B.** By classical result of Borel [Bor63], every symmetric space \( N \) of noncompact type admits irreducible compact quotients \( \Sigma = N/\Gamma \), and the same
is true for the Euclidean space $\mathbb{R}^d$. Let us fix such a compact quotient $(\Sigma, h_{\Sigma})$ with the induced locally symmetric metric. Since $\pi_1(\Sigma)$ is infinite and residually finite, it has infinite profinite completion. The conclusion follows from Proposition 5.1 with $(M_\infty, g_\infty) = (C \times N, g \oplus h)$, and $(M_0, g_0) = (C \times \Sigma, g \oplus h_{\Sigma})$, which clearly has nonnegative Yamabe invariant and almost positive $Q$-curvature. □

Appendix A. The $Q$-curvature of Berger metrics

For the convenience of the reader, we now provide explicit formulae for the $Q$-curvature of the canonical variation $g_t = t g_V + g_\mathcal{H}$, $t > 0$, for each of the Hopf bundles listed in Table 1. These metrics are commonly referred to as Berger metrics. Further details about such metrics can be found in [Bes08, §9.81-§9.85] and [BP13a].

We begin by recalling that, since $(S^n, g_{\text{round}})$ has Ricci tensor $\text{Ric} = (n - 1)1d$, its scalar curvature is $\text{scal} = n(n - 1)$ and its $Q$-curvature is $Q = \frac{n}{8}(n^2 - 4)$.

A.1. Berger spheres $(S^{2q+1}, g_t)$. Consider the Hopf bundle (i) in Table 1 and the canonical variation $g_t$, where $(S^{2q+1}, g_t)$ is the unit round sphere. The Ricci tensor of $(S^{2q+1}, g_t)$ has eigenvalues

\begin{align*}
2qt & \quad \text{with multiplicity 1,} \\
2q + 2 - 2t & \quad \text{with multiplicity 2q,}
\end{align*}

hence the Riemannian submersion $\pi_t : (S^{2q+1}, g_t) \to CP^n$ is horizontally Einstein, with $\kappa_t = 2q + 2 - 2t$, and furthermore one can explicitly compute:

\begin{align*}
\| \text{Ric}_{g_t} \|^2 &= (2qt)^2 + 2q(2q + 2 - 2t)^2, \\
\text{scal}_{g_t} &= 2q(2q + 2 - t),
\end{align*}

\begin{align*}
Q_{g_t} &= -\frac{2}{(2q-1)^2} \| \text{Ric}_{g_t} \|^2 + \frac{(2q+1)^2 - 4(2q+1)^2 + 16(2q+1) - 16}{8(2q-1)^2} \text{scal}_{g_t}^2 \\
&= \frac{8q^3 - 64q^2 - 106q - 3}{8(2q-1)^2} - 2q^4 + 4q^3 - 46q^2 - 45q - 3 - t - \frac{(2q^2 + 3q + 1)(2q - 3)}{2(2q - 1)^2}.
\end{align*}

A simple analysis of the above coefficients shows the following asymptotic behavior:

\[ \lim_{t \to 0} Q_{g_t} < +\infty, \text{ for all } q \geq 1, \quad \text{and} \quad \lim_{t \to +\infty} Q_{g_t} = \begin{cases} -\infty, & \text{for all } q \leq 9 \\ +\infty, & \text{for all } q \geq 10. \end{cases} \]

A.2. Berger spheres $(S^{4q+3}, g_t)$. Consider the Hopf bundle (ii) in Table 1 and the canonical variation $g_t$, where $(S^{4q+3}, g_t)$ is the unit round sphere. The Ricci tensor of $(S^{4q+3}, g_t)$ has eigenvalues

\begin{align*}
\frac{2}{t} + 4qt & \quad \text{with multiplicity 3,} \\
4q + 8 - 6t & \quad \text{with multiplicity 4q},
\end{align*}

and hence the Riemannian submersion $\pi_t : (S^{4q+3}, g_t) \to HP^n$ is horizontally Einstein Riemannian submersion with $\kappa_t = 4q + 8 - 6t$, and one can explicitly compute:

\begin{align*}
\| \text{Ric}_{g_t} \|^2 &= 12\left(\frac{2}{t} + 4qt\right)^2 + 16q(2q + 4 - 3t)^2, \\
\text{scal}_{g_t} &= 2 \left(\frac{4}{t} + 8q(q + 2) - 6qt\right),
\end{align*}
\[ Q_{g_t} = -\frac{2}{(4q+1)^2} \| \text{Ric}_{g_t} \|^2 + \frac{(4q+3)^2 - 4(4q+3)^2 + 16(4q+3) - 16}{8(4q+2)^3} \text{scal}_{g_t}^2 \\
= \frac{3(4q-1)^2(12q+5)}{8(4q+1)^2(4q+1)^3} t^2 + \frac{(64q^3 + 80q^2 + 76q + 23)(6q^2 - 12q) + 8}{16q^3(4q+1)^2} t^2 \\
+ 1024t^2 + 5376q^6 + 9008q^5 + 4656q^4 - 3600q^3 - 5100q^2 - 1439t^2 \\
- \frac{(64q^3 + 80q^2 - 52q^2 - 105q - 32)(12q^2 + 24q)}{2(4q+1)^2(4q+1)^2} t + \frac{(48q^3 - 40q^2 - 169q - 64)(12q^2 + 2q)}{2(4q+1)^2(2q+1)^2} t^2. \]

Again, a simple analysis of coefficients shows that:

\[ \lim_{t \to 0} Q_{g_t} = +\infty, \quad \text{for all } q \geq 1, \quad \text{and} \quad \lim_{t \to +\infty} Q_{g_t} = \begin{cases} -\infty, & \text{for all } q \leq 2 \\ +\infty, & \text{for all } q \geq 3 \end{cases}. \]

### A.3. Berger metrics \((CP^{2q+1}, g_t)\)

Consider the Hopf bundle (iii) in Table 1 and the canonical variation \(g_t\), where \((CP^{2q+1}, g_{t1})\) is the Fubini-Study metric. The Ricci tensor of \((CP^{2q+1}, g_t)\) has eigenvalues

\[ g_t + 4qt \quad \text{with multiplicity } 2, \]
\[ 4q + 8 - 4t \quad \text{with multiplicity } 4q \]

and hence the Riemannian submersion \(\pi_t: (CP^{2q+1}, g_t) \to \mathbb{P}^q\) is horizontally Einstein Riemannian submersion with \(\kappa_t = 4q + 8 - 4t\), and one can explicitly compute:

\[ \| \text{Ric}_{g_t} \|^2 = \frac{32}{q^2} + 64q(q^2 + 4q + 5) - 128(q^2 + 2q)t + 32q(q + 2)t^2, \]
\[ \text{scal}_{g_t} = 2\left(\frac{4}{q^2} + 4qt\right) + 4q(4q + 8 - 4t), \]
\[ Q_{g_t} = \frac{8(4q^2 - 6q - 1)}{q(4q+1)^2} t + \frac{16(8q^4 + 20q^3 + 14q^2 + 13q + 2)}{q(4q+1)^2} t^2 \]
\[ + \frac{8(16q^6 + 72q^5 + 92q^4 + 2q^3 - 61q^2 - 42q - 6)}{q(4q+1)^2} t^2 \\
+ 16 \left(1 - \frac{8q^4 + 20q^3 + 14q^2 + 13q + 2}{(4q+1)^2} + \frac{2}{q}\right) t - 4 \left(1 - \frac{8q^3 + 4q^2 + 6q + 1}{(4q+1)^2} + \frac{2}{q}\right) t^2. \]

As before, a simple analysis of coefficients shows that:

\[ \lim_{t \to 0} Q_{g_t} = \begin{cases} -\infty, & \text{if } q = 1 \\ +\infty, & \text{for all } q \geq 2 \end{cases}, \quad \text{and} \quad \lim_{t \to +\infty} Q_{g_t} = \begin{cases} -\infty, & \text{for all } q \leq 3 \\ +\infty, & \text{for all } q \geq 4 \end{cases}. \]

### A.4. Berger spheres \((S^{15}, g_t)\)

Consider the Hopf bundle (iv) in Table 1 and the canonical variation \(g_t\), where \((S^{15}, g_{t1})\) is the unit round sphere. The Ricci tensor of \((S^{15}, g_t)\) has eigenvalues

\[ g_t + 8t \quad \text{with multiplicity } 7, \]
\[ 28 - 14t \quad \text{with multiplicity } 8, \]

and hence the Riemannian submersion \(\pi_t: (S^{15}, g_t) \to S^8(1/2)\) is horizontally Einstein Riemannian submersion with \(\kappa_t = 28 - 14t\), and one can explicitly compute:

\[ \| \text{Ric}_{g_t} \|^2 = \frac{252}{t^2} + 6944 - 6272t + 2016t^2, \]
\[ \text{scal}_{g_t} = \frac{42}{t^2} - 56t + 224, \]
\[ Q_{g_t} = \frac{20559}{15352} t^2 + \frac{32388}{169} t + \frac{54358}{169} - \frac{30640}{169} t + \frac{1366}{169} t^2. \]

Note that

\[ \lim_{t \to 0} Q_{g_t} = +\infty, \quad \text{and} \quad \lim_{t \to +\infty} Q_{g_t} = +\infty. \]
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