Study of LG-Holling type III predator-prey model with disease in predator

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Abstract

In this article, a Leslie-Gower Holling type III predator-prey model with disease in predator has been developed from both biological and mathematical point of view. The total population is divided into three classes, namely, prey, susceptible predator and infected predator. The local stability, global stability together with sufficient conditions for persistence of the ecosystem near biologically feasible equilibria is thoroughly investigated. Boundedness and existence of the system are established. All the important analytical findings are numerically verified using program software MATLAB and Maple.

Keywords: Eco-epidemic model, Intra-specific competition, Local and global stability, Lyapunov function, Persistence.

1 Introduction

The predators and the preys carry a dynamic relationship among themselves. And for its universal existence and importance, this relationship is one of the dominant themes in theoretical ecology. Mathematical modelling is considered to be very useful tool to understand and analyze the dynamic behavior of predator-prey systems. Predator functional response on prey population is the major element in predator-prey interaction. It describes the number of prey consumed per predator per unit time for given quantities of prey and predator. The most important and useful functional responses are Lotka-Volterra functional responses such as Holling type I functional response, Holling type II functional response and three species population models with such functional responses are widely researched in ecological literature [24], [23], [21], [19]. There are also many research works on three species systems like two preys one predator [1], [20], [12], [10], tritrophic food chain [5], [16], [2] etc.

The Mathematical modelling of epidemics has become a very important subject

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of research after the seminal model of Kermack-MacKendrick (1927) on SIRS (susceptible-infected-removed-susceptible) systems. It describes the evolution of a disease which gets transmitted upon contact. Important studies have been carried out with the aim of controlling the effects of diseases and of developing suitable vaccination strategies [4], [22], [3]. Eco-epidemic research describes disease that spread among interacting populations, where the epidemic and demographic aspects are merged within one model. During the last decade, this branch of science is developing and studied by the authors in [4], [15], [27]. In the natural world, species do not exist alone. It is of more biological significance to study persistence-extinction threshold of each population in systems of two or more interacting species subjected to parasitism. In mathematical biology the predator prey systems and models for transmissible disease are major field of study in their own right. In the growing ecoepidemic literature and from early papers [11], disease mainly spreading in the prey are examined in [29], [8], [9], but in [28], [17], [18], the epidemics are assumed to affect the predators. The predator-prey model with modified Lesli-Gower Holling type II Scheme was introduced in [6], [14], [26]. The LG model with Holling type II response function with disease in predator is discussed in [25]. But no one pay the attention for the modified LG model with Holling type III response function for predation with disease in predator. Here we make an attempt to study the above said model with Holling type III response for predation and intra-specific competition among predators. The rest of the article is as follows. In Section 2, we explain the formulation of the model under consideration and its assumptions. Section 3 contains some preliminary results. In Section 4, we analyze the system behavior of the trivial equilibria. Also the model with intra specific competition is analyzed for the system behavior around axial and boundary equilibria in Section 4. In Section 5, local and global stability of the interior equilibria is analyzed. Section 6 contains persistence of the system. Numerical simulation has been carried out in Section 7 to support our analytical findings. The article comes to an end with a discussion of the results obtained in Section 8.

2 Mathematical model formulation

We make the following assumptions:

- The disease spreads only among the predators. Let $y$ denotes the susceptible predators and $z$ the infected ones. The total predator population is $n(t) = y(t) + z(t)$.

- The disease spreads with a simple mass action law (with the disease incidence $\theta > 0$). The prey population $x$ grows logistically with intrinsic growth rate $a_1 > 0$ and carrying capacity $a_1/b_1$ in the absence of predator population.

- We introduce intra-specific competition among the predator’s sound and infected sub-populations.
Holling type-III response mechanism is considered for predation. According to the above assumptions, we get the following model with non negative parameters

\[
\frac{dx}{dt} = a_1x - b_1x^2 - c_1x^2y \frac{pc_1x^2z}{k_1 + x^2} = f_1(x, y, z), \tag{2.1a}
\]
\[
\frac{dy}{dt} = a_2y - c_2y(y + z) \frac{k_2 + x}{k_2 + x} - \theta yz = f_2(x, y, z), \tag{2.1b}
\]
\[
\frac{dz}{dt} = \theta yz + a_3z - c_3z(y + z) \frac{k_2 + x}{k_2 + x} = f_3(x, y, z), \tag{2.1c}
\]
\[x(0) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0,
\]

where \(a_2, a_3(a_2 \geq a_3)\) are the per capita growth rates of each predator sub population. Thus from sick parents, the disease can be transmitted to their offspring. The parameter \(k_1\) represents the half saturation constant of the prey and \(k_2\) is the measure of alternative food. Hence the Jacobian matrix of the system (2.1) is

\[
J = (m_{ij}) \in \mathbb{R}^{3 \times 3}
\]

with entries

\[
m_{11} = a_1 - 2b_1x - 2c_1x^2y \frac{pc_1x^2z}{(x + k_1)^2}, \quad m_{12} = -c_1x^2 \frac{pc_1x^2z}{x + k_1}, \quad m_{13} = -c_1x^2 \frac{pc_1x^2z}{x + k_1},
\]
\[
m_{21} = c_2y(y + z) \frac{k_2}{(x + k_2)^2}, \quad m_{22} = a_2 - c_2(y + z) \frac{k_2}{x + k_2} - \theta z, \quad m_{23} = -c_2y \frac{k_2}{x + k_2},
\]
\[
m_{31} = c_3z(y + z) \frac{k_2}{(x + k_2)^2}, \quad m_{32} = \theta z - c_3z \frac{k_2}{x + k_2}, \quad m_{33} = \theta y + a_3 - c_3(y + z) \frac{k_2}{x + k_2}.
\]

Table 1: The set of model parameters and variables, dimension and their biological description.

| Variable or Parameter | Unit or Dimension | Description |
|-----------------------|-------------------|-------------|
| \(x\)                 | \(V\)             | Prey density |
| \(y\)                 | \(V\)             | Density of susceptible predator |
| \(z\)                 | \(V\)             | Density of infected predator |
| \(a_1\)               | \(T^{-1}\)        | Intrinsic growth rate of prey |
| \(a_2\)               | \(T^{-1}\)        | Intrinsic growth rate of susceptible predator |
| \(a_3\)               | \(T^{-1}\)        | Intrinsic growth rate of infected predator |
| \(b_1\)               | \(V^{-1}\)        | Intra-specific competition rate of prey |
| \(c_1\)               | \(T^{-1}\)        | Predation rate of susceptible predator |
| \(c_2\)               | \(T^{-1}\)        | Death rate due to intra-specific competition of susceptible predator |
| \(c_3\)               | \(T^{-1}\)        | Death rate due to intra-specific competition of infected predator |
| \(\theta\)            | \(V^{-1}\)        | Disease incidence rate |
| \(k_1\)               | \(V\)             | Half saturation constant of the prey |
| \(k_2\)               | \(V\)             | Measure of alternative food |
| \(p\)                 | Dimensionless      | Constant lies between 0 to 1 |
3 Preliminary results

3.1 Existence

Theorem 1. Every solution of the system (2.1) with initial conditions exists in the interval \((0, +\infty)\) and \(x(t) \geq 0, y(t) \geq 0, z(t) \geq 0\) for all \(t \geq 0\).

Proof. We have \(\frac{dx}{dt} = f_1(x, y, z), \frac{dy}{dt} = f_2(x, y, z), \frac{dz}{dt} = f_3(x, y, z)\). Integrating we get \(x(t) = x(0)e^{\int_0^t f_1(x,y,z)ds}, y(t) = y(0)e^{\int_0^t f_2(x,y,z)ds}, z(t) = z(0)e^{\int_0^t f_3(x,y,z)ds}\), where \(x(0) = x_0 > 0, y(0) = y_0 > 0, z(0) = z_0 > 0\). Since \(f_1, f_2, f_3\) are continuous function and hence locally Lipschitzian on \(R^3\), the solution with positive initial condition exists and unique on \((0, \xi)\) where \(0 < \xi < \infty\). Hence the theorem. \(\square\)

3.2 Boundedness

Theorem 2. All the solutions of the system which initiate in \(R^3_+\) are uniformly bounded.

Proof. Let us define a function \(\omega = x + y + z\). Therefore, we have

\[
\frac{d\omega}{dt} + \mu \omega = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} + \mu(x + y + z)
\]

\[
= [a_1x(1 - \frac{b_1}{a_1}x) + \mu x] - \frac{(c_1y + pc_1z)x^2}{x^2 + k_1} - \frac{(c_2y + c_3z)(y + z)}{x + k_2}
\]

\[
+ (a_2 + \mu)y + (a_3 + \mu)z
\]

\[
\leq (a_1 + \mu - b_1x)x + (a_2 + \mu)y + (a_3 + \mu)z
\]

\[
\leq \frac{(a_1 + \mu)^2}{4b_1} + (a_2 + \mu)y + (a_3 + \mu)z
\]

Hence we find \(l > 0\) such that \(\frac{d\omega}{dt} + \mu \omega \leq l, \forall t \in (0, t_0)\). Using the theory of differential inequality [7], we obtain \(0 < \omega(x, y, z) \leq \frac{l}{\mu}(1 - e^{-\mu t}) + \omega(0,0,0)e^{-\mu t}\) and for \(t_0 \rightarrow \infty\) we have \(0 < \omega \leq \frac{\omega}{\mu}\).

Hence all the solutions of the system that initiate in \(R^3_+\) are confined in the region \(\gamma = \{(x, y, z) \in R^3_+: \omega = \frac{\omega}{\mu} + \epsilon\}\) for any \(\epsilon > 0\) and for \(t\) large enough.

Hence the theorem. \(\square\)

3.3 Equilibrium points

The system of equations (2.1) has the equilibrium points \(E_0(0, 0, 0), E_1(0, 0, \frac{a_1k_1}{c_3}), E_2(0, \frac{a_2k_2}{c_2}, 0), E_3(0, y_3, z_3), E_4(0, 0, 0), E_5(x_5, y_5, 0), E_6(x_6, 0, z_6)\) and \(E_+(x_*, y_*, z_*)\).
The co-existence equilibrium is $E_1(x_*, y_*, z_*)$ where $y_* = \frac{-a_3 \theta x - a_2 k_2 \theta + a_2 c_1 - a_3 c_1}{\theta (a_2 + k_2 + c_2)}$, $z_* = \frac{-a_2 \theta x - a_2 k_2 \theta + a_2 c_1 - a_3 c_1}{\theta (a_2 + k_2 + c_2)}$, and $x_*$ is root of the following equation $P(x) = A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0 = 0$.

Here $A_4 = b_1 \theta^2$, $A_3 = b_1 k_2 \theta^2 - a_1 \theta^2 + b_1 c_2 \theta - b_1 c_3 \theta$, $A_2 = -a_1 k_2 \theta^2 + a_2 c_1 \theta + b_1 k_2 \theta^2 - a_1 c_2 \theta + a_1 c_3 \theta - a_3 c_1 \theta$, $A_1 = a_2 c_1 k_2 \theta + b_1 k_1 k_2 \theta^2 - a_1 k_1 \theta^2 - a_2 c_1 c_3 p + a_3 c_1 c_2 p - a_3 c_2 k_1 \theta - b_1 c_3 k_1 \theta + a_2 c_3 c_3 - a_3 c_1 \theta$, $A_0 = -a_1 k_1 k_2 \theta^2 - a_1 c_2 k_1 \theta + a_1 c_3 k_1 \theta$. We consider $P(x) = (\gamma_1 x^2 + \delta x + \phi_1)(\gamma_2 x^2 + \delta x + \phi_2)$.

**Case-1:** $P(x)$ has two real roots if either $\delta_1^2 - 4 \gamma_1 \phi_1 > 0$ or $\delta_2^2 - 4 \gamma_2 \phi_2 > 0$. For the set of parameters $a_1 = 4.5$, $a_2 = 3.8$, $a_3 = 0.005$, $b_1 = 0.075$, $k_1 = 50$, $k_2 = 160$, $c_1 = 2.8$, $c_2 = 1.97$, $c_3 = 1.95$, $\theta = 0.0937$, $p = 0.047$, there are two real roots $[x = 56.43479200, y = 3.837197569, z = 36.62450111]$ and $[x = -161.055378, y = -1006.972848, z = 1057.801828]$ in which first one is biologically feasible.

**Case-2:** $P(x)$ has four real roots if $\delta_1^2 - 4 \gamma_1 \phi_1 > 0$ and $\delta_2^2 - 4 \gamma_2 \phi_2 > 0$. For the set of parameters $a_1 = 5.0$, $a_2 = 7.8$, $a_3 = 1.5$, $b_1 = 0.0005$, $k_1 = 50$, $k_2 = 55$, $c_1 = 1.7$, $c_2 = 1.95$, $c_3 = 1.0$, $\theta = 0.0217$, $p = 0.73$, there are four real roots $[x = 0.5990751398, y = 161.9321592, z = 116.8375923], [x = -65.33582457, y = -1734.893676, z = 2108.504719], [x = 73.69033011, y = 33.0094773, z = 252.2068593], [x = 9933.742272, y = -67.78533292, z = 358.0409590]$ in which first and third one are biologically feasible.

For the equilibrium point $E_3(x_5, y_5, 0)$, $z_5 = 0$ gives $y_5 = \frac{a_2 (x_5 + k_2)}{c_2}$. Here $x_5$ is the root of $Q(x) = A_3 x^3 + A_2 x^2 + A_1 x + A_0 = 0$, where $A_3 = b_1 c_2$, $A_2 = -a_1 c_2 + a_2 c_1$, $A_1 = a_2 c_1 k_2 + b_1 c_2 k_1$, $A_0 = -a_1 c_2 k_1$. We consider $Q(x) = (\nu x + a_1)(\nu x + c_1)$.

**Case-1:** $Q(x)$ has only one real root if $\xi_1^2 - 4 \nu_1 \eta_1 < 0$, which yields $x_5 = -\frac{a_1}{\nu}$. For the set of parameters $a_1 = 4.5$, $a_2 = 3.8$, $a_3 = 0.005$, $b_1 = 0.075$, $k_1 = 100$, $k_2 = 160$, $c_1 = 2.8$, $c_2 = 1.97$, $c_3 = 1.95$, $\theta = 0.0937$, $p = 0.047$, there is only one real root $[x = 0.5159678886, y = 309.6247096, z = 0]$ which is biologically feasible.

**Case-2:** $Q(x)$ has three real roots if $\xi_1^2 - 4 \nu_1 \eta_1 > 0$. For the set of parameters $a_1 = 5.0$, $a_2 = 7.8$, $a_3 = 1.5$, $b_1 = 0.0005$, $k_1 = 50$, $k_2 = 55$, $c_1 = 1.7$, $c_2 = 1.95$, $c_3 = 1.0$, $\theta = 0.0217$, $p = 0.73$, there are three real roots $[x = 0.358509536, y = 411.2346427, z = 0], [x = -91.96262935, y = -274.5795323, z = 0], [x = -15165.53874, y = -1.122497163 \times 10^9, z = 0]$ in which first one is biologically feasible.

For the equilibrium point $E_6(x_6, 0, z_6)$, $y_6 = 0$ gives $z_6 = \frac{a_3 (x_6 + k_2)}{c_3}$. Here $x_6$ is the root of
\[ R(x) = A_3 x^3 + A_2 x^2 + A_1 x + A_0 = 0, \]
where \( A_3 = b_1 c_3, A_2 = -a_1 c_3 + a_3 c_1 p, A_1 = a_3 c_1 k_2 p + b_1 c_3 k_1, \) \( A_0 = -a_1 c_3 k_1. \)

We consider \( R(x) = (\mu x + \alpha)(\nu_2 x^2 + \xi_2 x + \eta_2). \)

**Case-1:** \( R(x) \) has only one real root if \( \xi_2^2 - 4 \nu_2 \eta_2 < 0, \) which yields \( x_0 = -\frac{\alpha}{\eta_2}. \)

For the set of parameters \( a_1 = 4.5, a_2 = 3.8, a_3 = 0.005, b_1 = 0.075, k_1 = 100, k_2 = 160, c_1 = 2.8, c_2 = 1.97, c_3 = 1.95, \theta = 0.0937, p = 0.047, \) there is only one real root \([x = 59.98394610, y = 0, z = 0.5640614003] \) which is biologically feasible.

**Case-2:** \( R(x) \) has three real roots if \( \xi_2^2 - 4 \nu_2 \eta_2 > 0. \)

For the set of parameters \( a_1 = 5.0, a_2 = 7.8, a_3 = 1.5, b_1 = 0.0005, k_1 = 50, k_2 = 55, c_1 = 1.7, c_2 = 1.05, c_3 = 1.0, \theta = 0.0217, p = 0.73, \) there are three real roots \([x = 2.657590193, y = 0, z = 86.48638529] \), \([x = 30.13036196, y = 0, z = 127.6955429] \), \([x = 6244.212048, y = 0, z = 9448.818072] \) which are biologically feasible.

## 4 System behaviour around boundary equilibria

### 4.1 Stability for \( E_0 \)

The characteristic equation for \( E_0 \) is given by

\[
\begin{vmatrix}
    a_1 - \lambda & 0 & 0 \\
    0 & a_2 - \lambda & 0 \\
    0 & 0 & a_3 - \lambda
\end{vmatrix} = 0.
\]

The equilibrium point \( E_0 \) has the eigenvalues \( a_1, a_2, a_3. \) All the eigenvalues are positive and it is unstable.

### 4.2 Stability for \( E_1 \)

The characteristic equation for \( E_1 \) is given by

\[
\begin{vmatrix}
    a_1 - \lambda & 0 & 0 \\
    0 & (a_2 c_3 - a_3 (c_2 + k_2 \theta)) - \lambda & 0 \\
    \frac{a_3}{c_3} & a_3 (-1 + \frac{b_3 \phi}{c_3}) & -a_3 - \lambda
\end{vmatrix} = 0.
\]

Since one of the eigenvalues of \( E_1 \) is \( a_1, \) which is always positive and so, \( E_1 \) is unstable.
4.3 Stability for $E_2$

The characteristic equation for $E_2$ is given by

$$
\begin{vmatrix}
    a_1 - \lambda & 0 & 0 \\
    \frac{a_2^2}{c_2} & -a_2 - \lambda & -a_2 - \frac{a_2 k_3 \theta}{c_2} \\
    0 & 0 & (a_3 - \frac{a_2 c_3}{c_2} + \frac{a_2 k_3 \theta}{c_2}) - \lambda
\end{vmatrix} = 0.
$$

Since one of the eigenvalues of $E_2$ is $a_1$, which is always positive and therefore, $E_2$ is unstable.

4.4 Stability for $E_3$

The characteristic equation of the equilibrium $E_3(0, y_3, z_3)$ is given by

$$
\begin{vmatrix}
    a_1 - \lambda & 0 & 0 \\
    \frac{c_2 y_3 (y_3 + z_3)}{k_2} & \left(a_2 - \frac{c_2 (2y_3 + z_3)}{k_2}\right) - z_3 \theta - \lambda & 0 \\
    \frac{c_1 z_3 (y_3 + z_3)}{k_2} & -\frac{c_1 z_3}{k_2} + z_3 \theta & \left(a_3 - \frac{c_1 (y_3 + 2z_3)}{k_2}\right) + y_3 \theta - \lambda
\end{vmatrix} = 0,
$$

where $y_3 = \frac{-a_2 k_2 \theta + a_2 c_3 - a_3 c_3}{\theta (k_2 + c_2 - c_3)}$ and $z_3 = \frac{-a_2 k_2 \theta + a_2 c_3 - a_3 c_3}{\theta (k_2 \theta + c_2 - c_3)}$. As eigenvalue $a_1$ for $E_3$ is always positive, the equilibrium point $E_3$ is unstable.

4.5 Stability for $E_4$

The characteristic equation for $E_4$ is given by

$$
\begin{vmatrix}
    -a_1 - \lambda & -\frac{a_2 c_1}{b_1 (\frac{c_2}{c_2} + k_1)} & -\frac{a_2 c_1 p}{b_1 (\frac{c_2}{c_2} + k_1)} \\
    0 & a_2 - \lambda & 0 \\
    0 & 0 & a_3 - \lambda
\end{vmatrix} = 0.
$$

Eigenvalues $a_2$ and $a_3$ are always positive of the equilibrium $E_4$. Hence $E_4$ is unstable.

4.6 Stability for $E_5(x_5, y_5, 0)$

At $E_5(x_5, y_5, 0)$, the Jacobian matrix for the system is given by

$$
J_5 = \begin{pmatrix}
    m_{11} & m_{12} & m_{13} \\
    m_{21} & m_{22} & m_{23} \\
    m_{31} & m_{32} & m_{33}
\end{pmatrix},
$$

where $m_{11} = -b_1 x_5 - \frac{c_1 x_5 y_5}{x_5^2 + k_1}$, $m_{12} = -\frac{c_1 x_5^2 y_5}{(x_5^2 + k_1)^2}$, $m_{13} = -\frac{pc_1 x_5^2}{x_5^2 + k_1}$.
\[
m_{21} = \frac{c_2 y^2}{(x + k_2)^2}, \quad m_{22} = -\frac{c_2 y^2}{x + k_2}, \quad m_{23} = -\frac{c_2 y^2}{x + k_2} - \theta y_5, \quad m_{31} = 0, \quad m_{32} = 0, \quad m_{33} = \theta y_5 + a_3 - \frac{c_3 y^2}{x + k_2}.
\]

The Characteristic equation for \( J_5 \) is given by
\[
(m_{33} - \lambda)\{(m_{11} - \lambda)(m_{22} - \lambda) - m_{21}m_{12}\} = 0
\]
\[
\Rightarrow (m_{33} - \lambda)\{\lambda^2 - (m_{11} + m_{22})\lambda + m_{11}m_{22} - m_{21}m_{12}\} = 0
\]
\[
\Rightarrow \lambda_{1,2} = \frac{m_{11} + m_{22} \pm \sqrt{(m_{11} + m_{22})^2 - 4m_{11}m_{22} - m_{21}m_{12}}}{2} \quad \text{and} \quad \lambda_3 = m_{33}.
\]

We choose \( m_{11} < 0 \) and \( m_{33} < 0 \). Then \( E_5 \) will be stable if
(i) \( b_1 x_5 + \frac{c_1 x_5 y_5}{x_5 + k_1} > \frac{2c_1 y_5^2}{(x_5 + k_1)^2} \),
(ii) \( \frac{c_3 y_5}{x_5 + k_2} > \theta y_5 + a_3 \).

### 4.7 Stability for \( E_6(x_6, 0, z_6) \)

At \( E_6(x_6, 0, z_6) \), the Jacobian matrix for the system is given by
\[
J_6 = \begin{pmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{pmatrix},
\]
where
\[
m_{11} = -b_1 x_6 - \frac{c_1 x_6^2 y_6}{x_6 + k_1} + \frac{2pc_1 x_6^2 y_6}{(x_6 + k_1)^2}, \\
m_{12} = -\frac{c_2 y_6^2}{x_6 + k_2}, \\
m_{13} = -\frac{pc_2 y_6^2}{x_6 + k_1}, \quad m_{21} = 0, \\
m_{22} = a_2 - \frac{c_3 z_6^2}{x_6 + k_2} - \theta z_6, \\
m_{23} = 0, \quad m_{31} = \frac{c_3 z_6^2}{x_6 + k_2}, \quad m_{32} = \theta z_6 - \frac{c_3 z_6^2}{x_6 + k_2}, \quad m_{33} = \frac{c_3 z_6^2}{x_6 + k_2}.
\]

The Characteristic equation for \( J_6 \) is given by
\[
(m_{22} - \lambda)\{(m_{11} - \lambda)(m_{33} - \lambda) - m_{31}m_{13}\} = 0
\]
\[
\Rightarrow (m_{22} - \lambda)\{\lambda^2 - (m_{11} + m_{33})\lambda + m_{11}m_{33} - m_{31}m_{13}\} = 0
\]
\[
\Rightarrow \lambda_{1,2} = \frac{m_{11} + m_{33} \pm \sqrt{(m_{11} + m_{33})^2 - 4m_{11}m_{33} - m_{31}m_{13}}}{2} \quad \text{and} \quad \lambda_3 = m_{22}.
\]

We choose \( m_{22} < 0 \) and \( m_{11} > 2 \). Then \( E_6 \) will be stable if
(i) \( b_1 x_6 + \frac{pc_1 z_6}{x_6 + k_1} > \frac{2pc_1 y^2}{(x_6 + k_1)^2} \),
(ii) \( \frac{c_3 z_6^2}{x_6 + k_2} + \theta z_6 > a_2 \).

### 5 System behaviour near the coexistence equilibrium \( E_*(x_*, y_*, z_*) \)

The entries for the Jacobian at \( E_*(x_*, y_*, z_*) \) are
\[
m_{11} = a_1 - 2b_1 x_* + \frac{2c_1 x_*^2 y_*}{(x_* + k_1)^2} + \frac{2pc_1 x_*^2 y_*}{(x_* + k_1)^2}, \\
m_{12} = -\frac{c_2 y_*^2}{x_* + k_2}, \\
m_{13} = -\frac{pc_2 y_*^2}{x_* + k_1}, \quad m_{21} = \frac{c_2 y_*(y_* + z_*)}{(x_* + k_2)^2}, \\
m_{22} = -\frac{c_2 y_5^2}{x_* + k_2}, \\
m_{23} = -\frac{c_2 y_5^2}{x_* + k_2} - \theta y_* , \quad m_{31} = \frac{c_3 z_*^2}{(x_* + k_2)^2}, \\
m_{32} = \theta z_* - \frac{c_3 z_*^2}{x_* + k_2}, \quad m_{33} = -\frac{c_3 z_*^2}{x_* + k_2}.
\]
5.1 Local Stability

The characteristic equation for $J_*$ is $\theta^3 + A_1\theta^2 + A_2\theta + A_3 = 0$, where

$A_1 = -m_{11} - m_{22} - m_{33},$

$A_2 = m_{11}m_{22} + m_{11}m_{33} + m_{22}m_{33} - m_{13}m_{31} - m_{23}m_{32} - m_{21}m_{12},$

$A_3 = m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33} + m_{13}m_{22}m_{31} - m_{11}m_{22}m_{33} - m_{12}m_{23}m_{31} - m_{13}m_{21}m_{32},$

$A_1A_2 - A_3 = -m_{11}^2m_{22} - m_{11}^2m_{33} - m_{22}^2m_{33} - m_{11}m_{22}^2 - m_{11}m_{33}^2 - m_{22}m_{33}^2 - 2m_{11}m_{22}m_{33} + m_{11}m_{13}m_{31} + m_{11}m_{12}m_{21} + m_{22}m_{12}m_{21} + m_{22}m_{23}m_{32} + m_{33}m_{23}m_{32} + m_{33}m_{13}m_{31} + m_{13}m_{23}m_{31} + m_{13}m_{21}m_{32}.$

Assume (i) $m_{11} < 0$, (ii) $m_{32} > 0$, (iii) $m_{11}m_{31} + m_{33}m_{31} + m_{21}m_{32} < 0$ or $m_{11}m_{12} + m_{22}m_{12} + m_{13}m_{32} > 0$ or $m_{22}m_{23} + m_{33}m_{23} + m_{13}m_{32} > 0$ and we have $A_1 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$. By Routh-Hurwitz criterion, the interior or co-existence equilibrium $E_*(x_*, y_*, z_*)$ is locally asymptotically stable.

5.2 Global Stability

**Theorem 3.** The co-existence equilibrium point $E_*$ is globally asymptotically stable if $P_1 > 0$, $P_2 > 0$ and $P_3 > 0$ where $P_1$, $P_2$, $P_3$ are defined latter.

**Proof.** Let us define the function $L(x, y, z) = L_1(x, y, z) + L_2(x, y, z) + L_3(x, y, z)$, where $L_1 = x - x_* - x_*\ln \frac{z}{z_*}$, $L_2 = y - y_* - y_*\ln \frac{y}{y_*}$, $L_3 = z - z_* - z_*\ln \frac{z}{z_*}$.

It is to be shown that $L$ is a Lyapunav function and $L$ vanishes at $E_*$ and it is positive for all $x, y, z > 0$. Hence $E_*$ represents its global minimum. We have

$$\frac{dL_1}{dt} = (x - x_*)\left(21 - b_1x - \frac{c_1yx}{x^2 + k_1} - \frac{pc_1xz}{x^2 + k_1}\right) = (x - x_*)\left(b_1x + \frac{c_1yx}{x^2 + k_1} + \frac{pc_1xz}{x^2 + k_1} - b_1x - \frac{c_1yx}{x^2 + k_1} - \frac{pc_1xz}{x^2 + k_1}\right) = (x - x_*)\left(\frac{c_1x(x - x_*)(x - k_1)(y + p_1z_1)}{(x^2 + k_1)(x^2 + k_1)} - b_1(x - x_*) - \frac{c_1x(y - y_*)}{x^2 + k_1}\right) - \frac{pc_1x(z - z_*)}{x^2 + k_1}.$$

$$\frac{dL_2}{dt} = (y - y_*)\left(22 - \frac{c_2(y + z)}{x + k_2} - \theta z\right) = (y - y_*)\left(\theta z + \frac{c_2(y + z)}{x + k_2} - \frac{c_2(y + z)}{x + k_2} - \theta z\right) = (y - y_*)\left[-\theta(z - z_*) + \frac{c_2(y + z_*)(x - x_*)}{(x + k_2)(x + k_2)} - \frac{c_2((y - y_*) + (z - z_*)}{x + k_2}\right].$$
\[
\frac{dL_3}{dt} = (z - z_\ast)(a_3 - \frac{c_3(y + z)}{x + k_2} + \theta y)
\]
\[
= (z - z_\ast)\left(-\theta z_\ast + \frac{c_3(y + z_\ast)}{x_\ast + k_2} - \frac{c_3(y + z)}{x + k_2} + \theta y\right)
\]
\[
= (z - z_\ast)\left[\theta(y - y_\ast) + \frac{c_3(y_\ast + z_\ast)(x - x_\ast)}{(x_\ast + k_2)(x + k_2)} - \frac{c_3((y - y_\ast) + (z - z_\ast))}{x + k_2}\right].
\]

We consider
\[
\frac{dL}{dt} = -\left[A(x - x_\ast)^2 + B(y - y_\ast)^2 + C(z - z_\ast)^2 + 2H(x - x_\ast)(y - y_\ast)\right.
\]
\[
+ 2F(y - y_\ast)(z - z_\ast) + 2G(z - z_\ast)(x - x_\ast)\bigg] = -V^TQV
\]

where \(V = \begin{pmatrix} (x - x_\ast), (y - y_\ast), (z - z_\ast) \end{pmatrix}^T\) and \(Q\) is symmetric quadratic form given by
\[
Q = \begin{pmatrix}
A & H & G \\
H & B & F \\
G & F & C
\end{pmatrix}
\]

with the entries that are functions only of the variable \(x\) and
\[
A = b_1 - \frac{c_1(y_\ast + p_2)(x - x_\ast)}{(x_\ast + k_1)(x + k_2)}, \quad B = \frac{c_2}{x + k_2}, \quad C = \frac{c_3}{x + k_2}, \quad F = \frac{c_4 + c_3}{2(x + k_2)}.
\]
\[
H = \frac{1}{2}\left[\frac{c_2}{x + k_1} - \frac{c_2(y_\ast + z_\ast)}{(x_\ast + k_1)(x + k_2)}\right], \quad G = \frac{1}{2}\left[\frac{c_2}{x + k_1} - \frac{c_2(y_\ast + z_\ast)}{(x_\ast + k_1)(x + k_2)}\right].
\]

If the matrix \(Q\) is positive definite, then \(\frac{dL}{dt} < 0\). So, all the principal minors of \(Q\), namely, \(P_1 \equiv A, \ P_2 \equiv AB - H^2, \ P_3 \equiv ABC + 2FGH - AF^2 - BG^2 - CH^2 = C(AB - H^2) + G(FH - BG) + F(GH - AF)\) to be positive, i.e., \(P_1 > 0, \ P_2 > 0, \ P_3 > 0\).

\[
\square
\]

6 Persistence

If a compact set \(D \subset \Omega = \{(x, y, z) : x > 0, y > 0, z > 0\}\) exists such that all solution of the system (2.1) eventually enter and remain in \(D\), the system is called persistent.

Proposition 4. The system (2.1) is persistent if

1. \(\frac{a_3c_3}{a_3} > (c_2 + k_2\theta)\),
2. \(a_3c_2 > a_2c_3\),

10
constant to be determined. We define  

\[
\Pi(0^0) = \frac{c2 + (k2 + x6)\theta}{c2(k1 + x5^2)}, \quad a1c3(k1 + x6^2) > x6(a3c1p(k2 + x6) + b1c3(k1 + x6^2)) \]

Proof. Let us consider the method of average Lyapunov function, see [13], considering a function of the form  

\[
V(x, y, z) = x^{\gamma1}y^{\gamma2}z^{\gamma3}, \quad \text{where} \quad \gamma1 = 1, 2, 3 \quad \text{are positive constant to be determined. We define}
\]

\[
\Pi(x, y, z) = \frac{\dot{V}}{V} = \gamma1\left(a1 - b1x - \frac{c1xy}{x^2 + k1} - \frac{pc1xz}{x^2 + k1}\right) + \gamma2\left(a2 - \frac{c2(y + z)}{x + k2} + \theta z\right) + \gamma3\left(a3 - \frac{c3(y + z)}{x + k2} + \theta y\right).
\]

We are to prove that this function is positive at each boundary equilibrium. We have  

\[
\Pi(0, 0, 0) = \gamma1a1 + \gamma2a2 + \gamma3a3 > 0 \quad \text{and} \quad \Pi(\frac{a2k2}{c2}, 0, 0) = \gamma2a2 + \gamma3a3 > 0.
\]

Here  

\[
\Pi(0, 0, \frac{a2k2}{c2}, 0) = a1\gamma1 + \frac{(a2c3 - a3c2 - a3k2\theta)}{c3}\gamma2 > 0 \quad \text{follows by condition 1.}
\]

With the condition 2, we have  

\[
\Pi(0, \frac{a2k2}{c2}, 0) = a1\gamma1 + \frac{(a2c3 - a3c2 - a3k2\theta)}{c3}\gamma3 > 0.\]

Also  

\[
\Pi(0, \frac{a2c3 - a3c2 - a3k2\theta}{\theta(k2\theta + c2 - c3)}, -\frac{a2c3 - a3c2 - a2k2\theta}{\theta(k2\theta + c2 - c3)}) = a1\gamma1 > 0.
\]

We find  

\[
\Pi\left(x5, \frac{a2(x5 + k2)}{c2}\right, 0) = \gamma1\left(a1 - \left(b1x5 + \frac{a2c1x5(k2 + x5)}{c2(k1 + x5^2)}\right)\right) + \gamma3\left(\frac{a2c2 - a2c3 + a2k2\theta + a2x5\theta}{c2}\right) > 0 \quad \text{by the condition 3,}
\]

\[
\Pi\left(x6, 0, \frac{a3(x6 + k2)}{c3}\right) = \gamma1\left(a1c3(k1 + x6^2) - x6(a3c1p(k2 + x6) + b1c3(k1 + x6^2))\right) + \gamma2\left(\frac{a2c3 - a3(c2 + (k2 + x6)\theta)}{c3}\right) > 0 \quad \text{by the condition 4.}
\]

Hence a suitable choice of  \(\gamma1, \gamma2, \gamma3\) is required to ensure  \(\Pi > 0\) at the boundary equilibria. Hence  \(V\) is an average Lyapunov function and thus the system (2.1) is persistent. \(\square\)
Analytical studies can never be completed without numerical verification of the derived results. In this section, we present computer simulations of some solutions of the system (2.1). Beside verification of our analytical findings, these numerical simulations are very important from practical point of view. We use four different set of numerical values for support of analytical results mentioned in Table 2.

Table 2: Set of parameter values for numerical simulations; $S \equiv$ Parameter sets

| S  | $a_1$ | $a_2$ | $a_3$ | $b_1$ | $k_1$ | $k_2$ | $c_1$ | $c_2$ | $c_3$ | $\theta$ | $p$ |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|-----|
| $S_1$ | 4.5   | 3.8   | 0.005 | 0.075 | 100   | 160   | 2.8   | 1.97  | 1.95  | 0.0937   | 0.047 |
| $S_2$ | 4.5   | 3.8   | 0.005 | 0.075 | 100   | 20    | 2.8   | 1.97  | 0.005 | 0.0937   | 0.047 |
| $S_3$ | 5.0   | 7.8   | 1.5   | 0.0005| 50    | 55    | 1.7   | 1.05  | 1.0   | 0.0217   | 0.73  |
| $S_4$ | 4.0   | 6.0   | 0.05  | 0.005 | 100   | 200   | 0.08  | 0.7   | 0.50  | 0.002537 | 0.93  |

Figure 1: Stability behaviour of model the system (2.1) around the equilibrium position $E^*$ with the initial conditions $x_0 = 50$, $y_0 = 40$, $z_0 = 80$ and the set of parameter values $S_1$, (a) Phase portrait, (b) Time series. Here $A_1 = 4.3463$, $A_2 = 2.7135$, $A_3 = 4.8600$, $A_1A_2 - A_3 = 6.9339$.

Figure 2: Stability behaviour of the system around the equilibrium position $E^*$, with the initial conditions $x_0 = 20$, $y_0 = 90$, $z_0 = 80$ and the set of parameter values $S_3$, (a) Phase portrait, (b) Time series.
Figure 3: Stability behaviour of model the system around the equilibrium position $E_5$ with the initial conditions $x_0 = 40$, $y_0 = 40$, $z_0 = 0$ and the set of parameter values $S_3$, (a) Phase portrait, (b) Time series. Here (i) $b_1x_5 + \frac{c_1x_5y_5}{x_5^2 + k_1} = 5.0000 > \frac{2c_1x_5^2y_5}{(x_5^2 + k_1)^2} = 0.02563$, (ii) $\frac{c_1y_5}{x_5^2 + k_2} = 12.6285 > \theta y_5 + a_3 = 10.4237$.

Figure 4: Stability behaviour of model the system around the equilibrium position $E_6$ with the initial conditions $x_0 = 100$, $y_0 = 20$, $z_0 = 300$ and the set of parameter values $S_3$, (a) Phase portrait, (b) Time series. Here (i) $b_1x_6 + \frac{c_2x_6z_6}{x_6^2 + k_1} = 5.0000 > \frac{2c_2x_6^2z_6}{(x_6^2 + k_1)^2} = 3.7557$, (ii) $\frac{c_2z_6}{x_6^2 + k_2} + \theta z_6 = 206.614 > a_2 = 7.8$.

Figure 5: Stability behaviour of model the system around the equilibrium position $E_7^*$ with the initial conditions $x_0 = 7$, $y_0 = 150$, $z_0 = 80$ and the set of parameter values $S_4$, (a) Phase portrait, (b) Time series.
Figure 6: Stability behaviour of model the system around the equilibrium position $E^I_{\ast}$ with the initial conditions $x_0 = 50$, $y_0 = 1450$, $z_0 = 80$ and the set of parameter values $S_4$, (a) Phase portrait, (b) Time series.

Figure 7: Stability behaviour of model the system around the equilibrium position $E_6$ with the initial conditions $x_0 = 50$, $y_0 = 10$, $z_0 = 80$ and the set of parameter values $S_2$, (a) Phase portrait, (b) Time series.

Figure 8: Stability behaviour of model the system around the equilibrium position $E_5$ with the initial conditions $x_0 = 100$, $y_0 = 200$, $z_0 = 0$ and the set of parameter values $S_2$, (a) Phase portrait, (b) Time series.
Table 3: Schematic representation of our analytical results for set of parameter values $S_1$

| Eq. pts. $E_0$ | Existence | Feasibility | Stability | Figure |
|----------------|-----------|-------------|-----------|--------|
| $E_0$ (0, 0, 0) | Feasible  | Unstable    | —         | —     |
| $E_1$ (0, 0.4102564103) | Feasible | Unstable | —         | —     |
| $E_2$ (0.3086294416, 0) | Feasible | Unstable | —         | —     |
| $E_3$ (0.5, 0.207637519, 0.3524000437) | Feasible | Unstable | —         | —     |
| $E_4$ (0, 0, 0) | Feasible  | Unstable | —         | —     |
| $E_5$ (0.5159678886, 0.3096247096, 0) | Feasible | Unstable | —         | —     |
| $E_{11}$ | Does not exist | — | —         | —     |
| $E_{12}$ | Does not exist | — | —         | —     |
| $E_{13}$ | Does not exist | — | —         | —     |
| $E_{14}$ | Does not exist | — | —         | —     |

Feasibility conditions are mentioned in section 5 Figure 1

Table 4: Schematic representation of our analytical results for set of parameter values $S_2$

| Eq. pts. $E_0$ | Existence | Feasibility | Stability | Figure |
|----------------|-----------|-------------|-----------|--------|
| $E_0$ (0, 0, 0) | Feasible  | Unstable    | —         | —     |
| $E_1$ (0, 0, 20) | Feasible | Unstable | —         | —     |
| $E_2$ (0, 38.57868020, 0) | Feasible | Unstable | —         | —     |
| $E_3$ (60, 0, 0) | Feasible  | Unstable    | —         | —     |
| $E_4$ (3.751381115, 45.81484682, 0) | Feasible | Stability conditions are mentioned in section 4.6 | Figure 8 |
| $E_{11}$ | Does not exist | — | —         | —     |
| $E_{12}$ | Does not exist | — | —         | —     |
| $E_{13}$ | Does not exist | — | —         | —     |
| $E_{14}$ | Does not exist | — | —         | —     |

Feasibility conditions are mentioned in section 4.7 Figure 7
Table 5: Schematic representation of our analytical results for set of parameter values $S_3$

| Eq. pts. | Existence | Feasibility | Stability | Figure |
|----------|-----------|-------------|-----------|--------|
| $E_0$    | (0, 0, 0) | Feasible    | Unstable  | —      |
| $E_1$    | (0, 0.82, 50000000) | Feasible    | Unstable  | —      |
| $E_2$    | (0, 408.54714286, 0) | Feasible    | Unstable  | —      |
| $E_3$    | (0.164, 37569596, 114.3012791) | Feasible    | Unstable  | —      |
| $E_4$    | (10000, 0, 0) | Feasible    | Unstable  | —      |
| $E_5$    | (0.3585095936, 411.234627, 0) | Feasible    | Stability conditions are mentioned in section 4.6 | Figure 3 |
| $E_6$    | (0.0, 0, 0) | Feasible    | Unstable  | —      |

Table 6: Schematic representation of our analytical results for set of parameter values $S_4$

| Eq. pts. | Existence | Feasibility | Stability | Figure |
|----------|-----------|-------------|-----------|--------|
| $E_0$    | (0, 0, 0) | Feasible    | Unstable  | —      |
| $E_1$    | (0, 0, 29) | Feasible    | Unstable  | —      |
| $E_2$    | (0, 171.425714, 0) | Feasible    | Unstable  | —      |
| $E_3$    | (0, 1637.974544, 44.24202326) | Feasible    | Unstable  | —      |
| $E_4$    | (800, 0, 0) | Feasible    | Unstable  | —      |
| $E_5$    | (3.14271393, 1741.237355, 0) | Feasible    | Unstable  | —      |
| $E_6$    | (41.15227929, 2067.019537, 0) | Feasible    | Unstable  | —      |
| $E_7$    | (618.562922, 7016.245604, 0) | Feasible    | Unstable  | —      |
| $E_8$    | (798.1394249, 0, 998.1394249) | Feasible    | Unstable  | —      |
| $E_9$    | (3.12713936, 1618.749669, 71.16589424) | Feasible    | Stability conditions are mentioned in section 5 | Figure 5 |
| $E_10$   | (756.7723630, 426.6171136, 1740.142126) | Feasible    | Stability conditions are mentioned in section 5 | Figure 6 |
| $E_11$   | (33.1228249, 1461.965361, 290.6548798) | Feasible    | Stability conditions are mentioned in section 5 | Figure 7 |
| $E_12$   | (0.0, 0, 0) | Feasible    | Unstable  | —      |

8 Conclusions and comments

In this paper, we have proposed and analyzed an eco-epidemiological model where only the predator population is infected by an infectious disease. Here we have
considered a modified Leslie-Gower and Holling type-III predator-prey model. We have divided the predator population into two sub classes: susceptible and infected. Then we study the behaviour of the system at various equilibrium points and their stability. The conditions for existence and stability of all the equilibria of the system have been given. The system (2.1) has eight equilibrium points: one trivial equilibrium $E_0$, three axial equilibrium points $E_1$, $E_2$, $E_4$, three planar equilibrium points $E_3$, $E_5$, $E_6$ and one coexistence equilibrium $E_\ast$. For our model: $E_i$, $i = 0, 1, 2, 4$ exist and are unstable for all times. $E_3$ exists if $a_3 k_2 \theta < a_2 c_3 + a_3 c_2 < a_2 k_2 \theta$ and $c_2 > c_3$ but unstable. The equilibrium point $E_5$ is locally asymptotically stable if $b_1 x_5 + \frac{c_1 x_5}{x_5^2 + k_1} > \frac{2a_3 x_5^3}{(x_5^2 + k_1)^2}$, $c_2 x_5 > \theta y_5 + a_3$. Also $E_6$ is locally asymptotically stable if $b_1 x_6 + \frac{c_1 x_6 z_6}{z_6^2 + k_1} > \frac{2p c_1 x_6^3}{(x_6^2 + k_1)^2}$, $c_2 z_6 + \theta z_6 > a_2$. The coexistence equilibrium point $E_\ast$ is locally as well as globally asymptotically stable under some conditions. Persistence of the system is also shown.

At last, we conclude that our eco-epidemic predator–prey model with infected predator exhibits very interesting dynamics. Here we have assumed Holling type III response mechanism for predation. So, we can refine the model considering other type of functional response. We can also consider the disease infection in the prey population, which can give us a very rich dynamics. There must be some time lag, called gestation delay. So, as part of future work to improve the model we can incorporate the gestation delay in our model to make it more realistic.

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