A systematic approach to realising quantum filters for high-precision measurements using network synthesis theory

Joe Bentley, Hendra Nurdin, Yanbei Chen, and Haixing Miao

1 Institute for Gravitational Wave Astronomy, School of Physics and Astronomy, University of Birmingham, Birmingham B15 2TT, United Kingdom
2 School of Electrical Engineering and Telecommunications, University of New South Wales, Sydney 2052, Australia
3 Theoretical Astrophysics 350-17, California Institute of Technology, Pasadena, California 91125, USA

(Dated: February 21, 2020)

We develop a systematic approach to the realisation of active quantum filters directly from their frequency-domain transfer functions, utilising a set of techniques developed by the quantum control community. This opens the path to the development of new types of active quantum filters for high-precision measurements. As an illustration, the approach is applied to realise an all-optical unstable filter with broadband anomalous dispersion, proposed for enhancing the quantum-limited sensitivity of laser interferometric gravitational-wave detectors.

Introduction — In high-precision measurements, our understanding of physics is predominantly limited by quantum noise, arising due to the fundamental quantum fluctuations of the probing fields [1–4]. This is particularly true for advanced gravitational-wave (GW) detectors such as Advanced LIGO [5], Advanced Virgo [6] and KAGRA [7], as well as next-generation detectors [8–11]. Quantum and classical noises are also limiting factors in quantum optomechanical experiments [12–21] and searches for new physics [22–33]. To achieve a maximal signal-to-noise ratio, we need to engineer the frequency-dependent response of the measurement devices. Quantum (optical) filters are designed for this purpose, the simplest example of which is an optical cavity. Such a device can enhance the signal response in resonant detection schemes, and also shape how the quantum noises enter the system, e.g., the frequency-dependent squeezing scheme proposed for laser interferometric GW detectors [34–41].

Until now, formulating a physical realisation of a given quantum filter required a combination of intuition and prior experience. Constraints on the dynamics of active and passive physically realisable quantum systems have previously been established [42–45]. In this paper, we present an approach to systematically realising quantum filters for high-precision measurements directly from their frequency-domain transfer functions. This technique builds upon a general formalism for describing linear stochastic quantum networks and the synthesis of such networks, recently developed by the quantum control community [46–55]. It has powerful implications on how active quantum filters are designed, making the realisation of filters with arbitrarily complicated frequency responses a possibility.

As an illustration, our approach will be used to derive an all-optical realisation of a highly non-trivial anomalous dispersion filter, known as an unstable filter [56]. Such a device can be used to broaden the bandwidth of an advanced GW detector without sacrificing its peak sensitivity [56–70]. The system is unusual because it seemingly violates the Kramers-Kronig relations which imply that a stable anomalous-dispersion filter without absorption violates causality, however since this system is dynamically unstable this restriction does not apply [71–74]. Specifically, it has a frequency-domain transfer function given by

\[ G(s) = \frac{s - s_0}{s + s_0}, \]

where \( s \equiv i\Omega \) and \( s_0 = \gamma_{\text{neg}} \) is a characteristic frequency quantifying the anomalous (negative) dispersion. The systematic procedure for finding its physical realisation is illustrated in Fig. 1, starting from the transfer function, inferring a physically realisable state-space model, and then developing the physical realisation in terms of optical components.

Systematic approach — The process to find a physical realisation from a given set of transfer functions is general to multi-input multi-output lossless linear quantum systems; losses and other noise sources can be added later by augmenting the state space.

![Flowchart showing the steps in constructing the physical realisation of an anomalous dispersion filter starting from its frequency-domain transfer function.](image-url)
Our starting point is the frequency-domain transfer function matrix, defined as ratio of the Laplace transform of the system outputs \( \hat{y}(s) \) to its inputs \( \hat{u}(s) \):

\[
\mathbf{G}(s) = C(-sI - A)^{-1}B + D, \tag{2}
\]

\[
\hat{y}_i(s) = \sum_j G_{ij}(s)\hat{u}_j(s),
\]

where \((A,B,C,D)\) are the system matrices as defined below, and \(I\) is the identity matrix. The Laplace transform is defined as \( f(s) = \int_0^\infty e^{-st}f(t)dt \). If the \( G_{ij}(s) \) are given in pole-zero form (i.e. as the ratio of two polynomials in \( s \)) then a state-space representation can be found [75–79]:

\[
\dot{x} = A\hat{x} + B\hat{u}, \tag{3}
\]

\[
\hat{y} = C\hat{x} + D\hat{u}. \tag{4}
\]

Here \( \hat{x} \in \mathbb{L}^{2n \times 1} \) (\( \mathbb{L} \) being the space of linear operators on the relevant Hilbert space \( \mathcal{H} \)) is a vector of conjugate operator pairs representing the internal \( n \) degrees of freedom of the system, \( \hat{u} \in \mathbb{L}^{2m \times 1} \) is the vector of \( m \) system inputs, and \( \hat{y} \in \mathbb{L}^{2m \times 1} \) is the vector of \( m \) system outputs. The matrix \( A \in \mathbb{C}^{2n \times 2n} \) describes the internal dynamics of the system, \( B \in \mathbb{C}^{2n \times 2m} \) describes the coupling of the input into the system, \( C \in \mathbb{C}^{2m \times 2n} \) describes the coupling of the system to the output, and \( D \in \mathbb{C}^{2m \times 2m} \) describes the “direct-feed” of the input into the output. \((A,B,C,D)\) are together called the system matrices.

The system is called physically realisable if under the symplectic evolution of the system the commutation relations are preserved [47]:

\[
\forall i,j \quad \text{d}[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{y}_i(t), \hat{y}_j(t')] = \delta(t-t')\delta_{ij}, \tag{5}
\]

where the differential is treated using the quantum Itô rule, meaning that the cross-products of the differentials of the operators must be calculated [80–82]. The conditions on the system matrices for all evolutions to preserve these commutation relations are found by using Eqs. (3) and (4) to calculate \( \text{d}\hat{x}_i \) in Eq. (5) for a time period \( dt \). For an \( n \) degree-of-freedom system described using complex mode operators such that the state is \( \hat{x} = (\hat{a}_1, \hat{\alpha}_1^*; \ldots; \hat{a}_n, \hat{\alpha}_n^*)^T \) the constraints on the system matrices are given by

\[
AJ + JA^\dagger + BJB^\dagger = 0, \tag{6}
\]

\[
JC^\dagger + BJD^\dagger = 0, \tag{7}
\]

where \( J = \text{diag}(1,-1; \ldots; 1,-1) \) [83]. See Appendix A of [48] for a proof of these constraints.

However, the conventional procedure outlined in Refs. [75, 77] for transforming the transfer function to a state-space model may lead to an \((A',B',C',D')\) that does not satisfy Eqs. (6) and (7) and therefore not be physically realisable. We find a step allowing us to transform \((A',B',C',D')\) to a physically realisable counterpart. It is achieved by looking for a Hermitian matrix \( X \) that obeys the following constraints:

\[
A'X + X(A')^\dagger + B'J(B')^\dagger = 0, \tag{8}
\]

\[
X(C')^\dagger + B'J(D')^\dagger = 0. \tag{9}
\]

This matrix \( X \) can be written in the form of a similarity transformation \( X = TJT^\dagger \). It can then be applied via \( A = T^{-1}A'T, B = T^{-1}B', C = C'T, D = D' \) to find the physically realisable state-space \((A,B,C,D)\). The existence of \( T \) is guaranteed by the sympletic condition imposed on any physically realisable transfer matrix \( \mathbf{G}(s) \) and direct-feed matrix \( D \) [84]:

\[
\mathbf{G}^\dagger(s^*)J\mathbf{G}(-s) = J, \quad DJD^\dagger = J. \tag{10}
\]

After obtaining the physically realisable \((A,B,C,D)\), as shown in [47] for an \( n \) degree-of-freedom system, there is a one-to-one correspondence between \((A,B,C,D)\) and a generalised open oscillator quantified by \((S,L,H)[50–52]\). The scattering matrix \( S \) describes the transformation of the input fields through a passive network. The coupling operator \( \hat{L} = \hat{K}\hat{x} \) where \( \hat{K} \in \mathbb{C}^{m \times 2n} \) describes the coupling between the input and output fields and the internal degrees of freedom. The internal system Hamiltonian \( \hat{H} = \hat{x}^\dagger\hat{R}\hat{x} \) where \( \hat{R} = \hat{R}^\dagger \in \mathbb{R}^{2n \times 2n} \) describes the internal system dynamics. Specifically we have

\[
S = \hat{D}, \quad K = \hat{D}^{-1}C, \quad \hat{R} = i\hat{J}Jk = A - A^\dagger J, \tag{11}
\]

where \( R \) is derived in the supplementary material. The total Hamiltonian is then given by, [52]

\[
\hat{H}_{\text{tot}} = \hat{x}^\dagger\hat{R}\hat{x} + i(\hat{L}^\dagger\hat{\alpha}^1 + \hat{L}\hat{\alpha}^1), \tag{12}
\]

where the input fields \( \hat{u} \) are pre-processed by the static passive network described by \( S \), and the output fields are given by

\[
\hat{y} = \hat{K}\hat{x} + \hat{u}. \tag{13}
\]

Note that systems consisting of more than one internal degree-of-freedom must first be sub-divided into separate one degree-of-freedom systems coupled via direct interaction Hamiltonians via the main synthesis theorem proved in [52]. These systems can then be systematically realised by connecting the individual one degree-of-freedom systems in series, and overlapping them accordingly.[85]

**Illustrative example: an unstable filter** — Since the transfer function shown in Eq. (1) is first order in frequency \( s \), only one internal degree of freedom is required, so that \( \hat{x} \) has two elements: \( \hat{x}_1 = \hat{a}, \hat{x}_2 = \hat{\alpha}^1 \), and similarly for \( \hat{u} \) and \( \hat{y} \). In terms of a transfer-function matrix, Eq. (1) can be written as

\[
\mathbf{G}(s) = \begin{bmatrix} s & s_0 \\ s & s_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{14}
\]

To simplify the notation, we define a dimensionless \( s \) (and the corresponding time) which is normalised with respect
to \( s_0/2 = \gamma_{\text{neg}}/2 \) (a factor of 2 for convenience), namely
\[ s \to (s_0/2)s. \] A corresponding state-space model is given by
\[ \begin{bmatrix} \dot{a} \\ \dot{\tilde{a}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ \tilde{a} \end{bmatrix} + \begin{bmatrix} \dot{\hat{a}} \\ \dot{\tilde{\hat{a}}} \end{bmatrix}, \tag{15} \]
\[ \begin{bmatrix} \dot{y} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a \\ \tilde{a} \end{bmatrix} + \begin{bmatrix} \hat{u} \\ \tilde{\hat{u}} \end{bmatrix}. \tag{16} \]

In this case, one can find that the matrix \( T \) which transforms the above state-space model to the physically realisable one is given by
\[ T = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{17} \]

The resulting state-space model is
\[ A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad D = I, \tag{18} \]
which obey Eqs. (6) and (7) by construction.

Eq. (11) can now be used to calculate the scattering matrix, input-output coupling, and internal Hamiltonian for the unstable filter. We have
\[ S = I, \quad K = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad R = 0. \tag{19} \]

This implies that there is no input scattering with \( S = I, \) and \( L = \gamma_{\text{neg}}/2 \), and there is no detuning or internal squeezing of the cavity mode with \( R = 0. \)

As shown in Ref. [52] and earlier in Ref. [86], a coupling operator of the form \( L = \beta_{\text{ab}} \gamma_{\text{neg}} \) can be realised by indirectly coupling the mode \( \hat{a} \) to the external continuum fields \( \hat{u} \) and \( \hat{\tilde{u}} \) via an auxiliary mode \( \hat{b} \) through a non-degenerate parametric amplification process. This auxiliary mode \( \hat{b} \) will later be adiabatically eliminated.

Applying to our case, we construct the system shown in Fig. 1. It simply consists of two tuned cavities coupled via a \( \chi^{(2)} \) non-linear crystal, labelled OPO (optical parametric oscillator), pumped by a classical pump field, labelled pump. One of the cavities is coupled to the external fields. Specifically, we have
\[ \hat{H}_{\text{ab}} = -\hbar \sqrt{s_0} \gamma_{\text{neg}} (\hat{a} \hat{b}^\dagger + \hat{a}^\dagger \hat{b}), \tag{20} \]
\[ \hat{H}_{\text{ext}} = -\hbar \sqrt{s_0} (\hat{b} \hat{\tilde{u}}^\dagger - \hat{b}^\dagger \tilde{\hat{u}}). \tag{21} \]

The interaction Hamiltonian \( \hat{H}_{\text{ab}} \) describes the coupling of both cavity modes \( \hat{a} \) and \( \hat{\tilde{a}} \) via the OPO. As shown in the supplemental material, the coupling rate \( \sqrt{s_0} \gamma_{\text{neg}} \) is equal to \( rc/(2L_b) \), where \( r \) is the single-pass squeezing factor of the crystal and \( L_b \) is the length of the auxiliary cavity. As an order of magnitude estimate for implementation in a laser interferometer with arm length of \( L_{\text{arm}} = 4 \text{ km} \) (where \( s_0 \equiv \gamma_{\text{neg}} \equiv c/L_{\text{arm}} \) [56]), the required squeezing factor is
\[ r = 7.7 \times 10^{-5} \sqrt{\frac{T_b}{100 \text{ ppm}} \frac{L_b}{24 \text{ cm}} \frac{4 \text{ km}}{L_{\text{arm}}}}. \tag{22} \]

![FIG. 2. Required total \( \hat{a} \) cavity loss \( \epsilon_a \) as a function of ratio of \( \hat{a} \) cavity length to arm cavity length \( L_a/L_{\text{arm}} \) for the cases where the noise power contribution at \( \Omega = 0 \) due to \( \tilde{n}_a \) is a tenth of that of the signal power (blue line) and a hundred (orange line).](image)
to achieve low noise as a function of a cavity length is shown in Fig. 2. As we can see, given an interferometer arm length of $L_{\text{arm}} = 4 \text{ km}$, a loss per unit length of $\epsilon_a/L_{\text{arm}} = 25 \text{ ppm m}^{-1}$ is required to achieve a 1/10 noise contribution, which is already achievable with state-of-the-art optics [88, 89].

Acknowledgements — We would like to thank Rana Adhikari, Denis Martynov, Naoki Yamamoto, LIGO AIC, and QNWG for fruitful discussions. J.B. is supported by STFC and School of Physics and Astronomy at the University of Birmingham. J.B. and H.M. acknowledge the additional support from the Birmingham Institute for Gravitational Wave Astronomy. H.M. has also been supported by UK STFC Ernest Rutherford Fellowship (Grant No. ST/M005844/11). Y.C. is supported by the Simons Foundation (Award Number 568762), and the National Science Foundation, through Grants PHY-1708212 and PHY-1708213.

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Supplemental Material for “A systematic approach to realising quantum filters for high-precision measurements using network synthesis theory”

Joe Bentley, Hendra Nurdin, Yanbei Chen, and Haixing Miao

R MATRIX IN COMPLEX OPERATOR NOTATION

In this section the \(R\) matrix shown in Eq. (11) of the main text will be transformed from the real-quadrature form in Ref. [S1] to the complex ladder operator form.

The Hamiltonian in the real-quadrature form is given by

\[
\hat{H} = \hat{x}^\dagger R \hat{x},
\]

where \(\hat{x} = (\hat{q}_1, \hat{p}_1; \ldots; \hat{q}_n, \hat{p}_n)^T\) are the real quadrature operators. The relation between \(R_r\) and the dynamical matrix \(A_r\) in the state-space model is given uniquely by,

\[
R_r = \frac{1}{4} (-\Theta A_r + A_r^\dagger \Theta),
\]

where,

\[
\Theta = \text{diag}(\Theta_1, \ldots, \Theta_1) \in \mathbb{R}^{2n \times 2n},
\]

and,

\[
\Theta_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

The complex ladder operators are related to the real quadrature operators by \(\hat{x} = (\hat{a}_1, \hat{a}_1^\dagger; \ldots; \hat{a}_n, \hat{a}_n^\dagger) = U \hat{x}_r\), where,

\[
U = \text{diag}(U_1, \ldots, U_1) \in \mathbb{C}^{2n \times 2n},
\]

where,

\[
U_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix},
\]

is the unitary transformation that converts from the real quadrature operators \(\hat{q}, \hat{p}\) to the complex ladder operators \(\hat{a}, \hat{a}^\dagger\).

Note that we can write \(\Theta = -iU^\dagger JU\), and that the relation between the dynamical matrix in the real quadrature picture and the complex ladder operators is given by \(A = U^\dagger A_r U\), and recall that \(U\) is unitary. Substituting these facts into the expression for \(\hat{H}\) we get \(\hat{H} = \hat{x}^\dagger R \hat{x}\) where,

\[
R = \frac{i}{4} (JA - A^\dagger J).
\]

Where \(J\) is defined in the main text.

RELATING THE COUPLING RATE TO THE SINGLE-PASS SQUEEZING FACTOR

To compare the coupling rate \(\sqrt{s_0 r}\) to the single-pass amplification factor \(r\), we look at the degenerate case of the interaction Hamiltonian given in Eq. (18) of the main text,

\[
\hat{H}_{\text{deg}} = -\hbar \sqrt{s_0 r} (\hat{a}^\dagger)^2 + \hat{a}^2).
\]

Solving the equation of motion in the frequency domain, the resulting input-output relation for the amplitude quadrature \(\hat{a}_1\) in the two-photon formalism [S2, S3] is

\[
\hat{a}_1^{\text{out}}(\Omega) = \frac{\gamma + \sqrt{s_0 r} + i\Omega}{\gamma - \sqrt{s_0 r} - i\Omega} \hat{a}_1^{\text{in}}(\Omega).
\]

We can derive the same input-output relation by propagating the continuum field through the cavity with a nonlinear crystal, and obtain

\[
\hat{a}_1^{\text{out}}(\Omega) = \frac{-\sqrt{R} + e^{2r} e^{2i\Omega L/c}}{1 - \sqrt{R} e^{2r} e^{2i\Omega L/c}} \hat{a}_1^{\text{in}}(\Omega).
\]

Assuming \(T \equiv 1 - R, r, \Omega L/c \ll 1\), we can make the Taylor expansion of the above equation to the leading order of these small dimensionless quantities:

\[
\hat{a}_1^{\text{out}}(\Omega) \approx \frac{T/2 + 2r + 2i\Omega L/c \hat{a}_1^{\text{out}}(\Omega)}{T/2 - 2r - 2i\Omega L/c \hat{a}_1^{\text{in}}(\Omega)}.
\]

Eq. (S9) and Eq. (S11) become identical when

\[
\gamma \equiv \frac{T c}{4L}, \quad r = 2\sqrt{s_0 r} \frac{L}{c},
\]

which is the mapping used in the main text.

INCLUDING LOSSES INTO THE ANALYSIS

In this section, we show how the effects of optical loss is included in the analysis for the realistic implementation. The optical losses in the mirrors of both cavities will introduce quantum white noise vacuum processes [S4–S6], \(\hat{n}_a, \hat{n}_b\), which are coupled to modes \(\hat{a}\) and \(\hat{b}\) respectively via transmissivities \(T_a, T_b\). This results in extra terms added to the Heisenberg equations of motion for the two modes,

\[
\dot{\hat{b}} = -\gamma_b \hat{b} + \sqrt{2\gamma_b} \hat{n}_b + \frac{i}{\hbar} [\hat{H}_{\text{tot}}, \hat{b}],
\]

\[
\dot{\hat{a}} = -\gamma_a \hat{a} + \sqrt{2\gamma_a} \hat{n}_a + \frac{i}{\hbar} [\hat{H}_{\text{tot}}, \hat{a}],
\]
where $H_{\text{tot}}$ is the total Hamiltonian derived in the main text. The noise coupling constants for the $\hat{a}$ cavity and $\hat{b}$ cavity respectively are given by:

$$
\gamma_a^\epsilon = \epsilon_a c/(4L_a), \quad \gamma_b^\epsilon = \epsilon_b c/(4L_b), \quad \text{(S15)}
$$

where $\epsilon_a$ and $\epsilon_b$ are the optical losses described by cavity respectively. The loss from the non-linear crystal couples identically to the mirror loss into both cavities, and so can be included in $\epsilon_a, \epsilon_b$.

Solving the Heisenberg equations of motion in the frequency domain, we found that the noise contribution from the auxiliary cavity loss $\hat{n}_b$ is much smaller than the contribution from the $\hat{a}$ cavity loss $\hat{n}_a$ by a factor:

$$
\frac{\Omega^2 \gamma_b^\epsilon}{\gamma_{\text{neg}} \gamma_a^\epsilon} \ll 1, \quad \text{(S16)}
$$

assuming $\gamma_a^\epsilon \approx \gamma_b^\epsilon$, and $\Omega \ll \gamma_{\text{neg}}$, $\Omega \ll \gamma$, a result also found in the optomechanical case explored in [S7], in which the filter cavity takes the role of the auxiliary cavity mode $\hat{b}$ and the mechanical oscillator takes the role of the main cavity mode $\hat{a}$. However in our case the main cavity loss is due to vacuum and is not thermally driven, and so is effectively at zero temperature. The phase noise due to the thermal fluctuation of the non-linear crystal [S8] is negligible as there is almost no carrier power in either cavity.

**ALTERNATIVE TOPOGRAPHY**

Here we show an alternative topography for the realisation shown in Fig.1 of the main text. The system consists of a linear coupled cavity. We call the cavity with the nonlinear crystal in it the active cavity and the other the passive cavity. The length of the passive cavity $L_1$ differs from the length $L_2$ of the active cavity so that they have different mode spacings. The two modes $\hat{a}$ and $\hat{b}$ in this case belong to the same longitudinal modes of the active cavity but separated by one free spectral range. The passive cavity acts as a compound mirror with frequency-dependent effective phase $\phi_{\text{eff}}(\Omega)$ and transmissivity $T_{\text{eff}}(\Omega)$, the former shifting the resonances of the active cavity by $\omega_a$ and $\omega_b$ for the $\hat{a}$ and $\hat{b}$, and the latter imparting different bandwidths for the two modes, denoted $\gamma_a = T_{\text{eff}}(\omega_a)c/(4L_2)$ and $\gamma_b = T_{\text{eff}}(\omega_b)c/(4L_2)$ respectively. The non-linear crystal pump frequency is set to $\omega_p$ where $\omega_p/2$ is between the two modes $\hat{a}$ and $\hat{b}$. To make $\hat{b}$ satisfy the adiabatic condition, we require $\gamma_b \gg \Omega$, while to ensure good performance we require $\gamma_a \ll \gamma_{\text{neg}}$. Both bandwidths can be independently controlled by changing the relative lengths of the two cavities.

![Alternative Topography Diagram](attachment:alternative_topography.png)

**FIG. S1.** Optical diagram and relevant frequencies of the alternative topography, consisting of a non-linear crystal and two linear cavities with the crystal in only one cavity.

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