3-Total Edge Product Cordial Labeling for Stellation of Square Grid Graph

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1. Introduction and Definitions

Let G be a simple graph with vertex set V(G) and edge set E(G). An edge labeling $\delta: E(G) \rightarrow \{0, 1, \ldots, p - 1\}$, where $p$ is an integer, induces a vertex labeling $\delta^*: V(G) \rightarrow \{0, 1, \ldots, p - 1\}$ defined by $\delta^*(v) = \delta(e_1)\delta(e_2)\cdots\delta(e_p) \mod p$, where $e_1, e_2, \ldots, e_p$ are edges incident to $v$. The labeling $\delta$ is said to be $p$-total edge product cordial (TEPC) labeling of G if $|\delta(e_1) + \delta(e_2) - \delta(e_3) + \delta(e_4)| \leq 1$ for every $i, j$, $0 \leq i \leq j \leq p - 1$, where $e_1(i)$ and $e_2(j)$ are numbers of edges and vertices labeled with integer $i$, respectively. In this paper, we have proved that the stellation of square grid graph admits a 3-total edge product cordial labeling.

Considerable amount of work have been done on cordial labeling. For latest results, see [3–10]. A vertex labeling $\delta: V(G) \rightarrow \{0, 1\}$ induces an edge labeling $\delta^*: E(G) \rightarrow \{0, 1\}$ defined by $\delta^*(xy) = \delta(x)\delta(y)$ which is called product cordial labeling if $|v_0(0) - v_1(1)| \leq 1$ and $|e_0(0) - e_1(1)| \leq 1$, where $v_0(0)$ and $v_1(1)$ represent the number of vertices that are labeled 0 and 1, respectively. While $e_0(0)$ and $e_1(1)$ represent the number of edges labeled with 0 and 1, respectively. The concept named product cordial labeling was first presented by Sundaram et al. [11]. A variation in the cordial theme, namely, edge product cordial labeling and a TEPC labeling was introduced by Vaidya and Barasara [12, 13].
and vertices labeled with integer $i$, respectively. Azaizheh et al. [14] introduced the concept of $p$-TEPC labeling. A graph $G$ that admits a $p$-TEPC labeling is called a $p$-TEPC graph. Baca et al. [15] investigated the 3-TEPC labeling of carbon nanotube networks. Ahmad et al. [16] showed that the grid graph $P_m \square P_n$ admits a 3-TEPC labeling. Ahmad et al. [3] proved that the hexagonal grid $H_m^r$ admits 3-TEPC labeling. Javed and Jamil [17] proved that the Rhombic grid $R_{m}^n$ is 3-TEPC for $m, n \geq 1$.

Let $P_n$ denote a path graph on $n$ vertices. A rectangular grid is an $m \times n$ lattice graph and is obtained by taking the Cartesian product of $P_m$ with $P_n$. The graph of rectangular grid is denoted by $L(m, n)$ and has $n$ and $m$ squares in each row and column respectively. It is easy to observe that rectangular grid $L(m, n)$ has $mn$ vertices and $mn - m - n + 1$ edges. The stellation of $L(m, n)$ is obtained by adding a vertex in each face of $L(m, n)$ and then joining this vertex to each vertex of the respective face. We denote the stellation of $L(m, n)$ by $G_{m}^n$, as shown in Figure 1. In this paper, we show that the graph $G_{m}^n$ admits 3-TEPC labeling.

2. Main Results

Let $m, n \geq 1$ and $G_{m}^n$ be stellation of rectangular grid containing $m$ rows and $n$ columns. Observe that $G_{m}^n$ has $2mn + m + n + 1$ vertices and $6mn + m + n$ edges. We use the notations $G_1 \oplus G_2$ for gluing the graph $G_1$ with $G_2$ vertically. Similarly, $G_1 \oplus G_2$ represent gluing $G_1$ with $G_2$ horizontally.

If we have a labeled segment or labeled graph $H$ and we rotate it by 90 degree in clockwise direction, then we will denote it by $\overline{H}$.

Theorem 1. For $m \geq 1$, the graph $G_{m}^1$ is 3-TEPC.

Proof. The 3-TEPC labeling of $G_{m}^1$ is shown in Figure 2. Similarly, the 3-TEPC labeling of $G_{m}^3$ and the labeled segment $S_1^3$ are shown in Figure 3. The segment $S_1^3$ has the property that open edges are assigned labeled 1. Hence, if we glue the segment $S_1^3$ with itself vertically, then it will not change the vertex labels in $S_1^3 \oplus S_1^3 = 2S_1^3$. Observe that the labels 0, 1, and 2 are used 10 times in the segment $S_1^3$. Table 1 shows the multiplicity of numbers 0, 1, and 2, respectively, used in the labeled graph $G_{m}^3$ for $m = 1, 2, 3$.

Case (i). $m = 3r, r \geq 1$.

To construct labeled graph $G_{m}^r$, we will use the labeled segments $S_1^3$. First, glue $r - 1$ copies of labeled segment $S_1^3$ vertically that is $S_1^3 \oplus S_1^3 \oplus \cdots \oplus S_1^3 = (r - 1)S_1^3$. Finally, glue vertically the label segment $G_{m}^1$ to the open edges of $(r - 1)S_1^3$ to get labeled graph $G_{m}^r$, that is,

$$G_{m}^r = \begin{bmatrix} (r - 1)S_1^3 \\ \Phi_v \\ G_{m}^1 \end{bmatrix}.$$

In the labeled graph $G_{m}^r$, the multiplicity of 0, 1, and 2 is $10r + 1$ exactly.

Case (ii): $m = 3r + 1, r \geq 1$.
To construct the labeled graph $G_m^{2}$, we glue $r$ copies of the labeled segment $S_1^3$ and then finally glue $G_1^2$ vertically. That is,

$$G_m^{2} = \left[ \begin{array}{c} rS_1^3 \\ \Phi_v \\ G_1^2 \end{array} \right].$$

(2)

In the labeled graph $G_m^{2}$, the multiplicity of 0 is $10r + 5$, whereas the multiplicity of 1 and 2 is $10r + 4$.

Case (iii): $m = 3r + 2$, $r \geq 1$.

We obtain the labeled graph $G_m^{2}$ by gluing $r$ times the labeled segment $S_1^3$ and finally gluing $G_1^2$ in vertical direction. That is,

$$G_m^{2} = \left[ \begin{array}{c} rS_1^3 \\ \Phi_v \\ G_1^2 \end{array} \right].$$

(3)

In the labeled graph $G_m^{2}$, the multiplicity of 0 is $10r + 7$, whereas the multiplicity of 1 and 2 is $10r + 8$.

#### Theorem 2.

For $m \geq 1$, the graph $G_m^{2}$ is 3-total edge product cordial.

**Proof.** Observe that the graphs $G_1^2$ and $G_1^1$ are isomorphic and the 3-total edge cordial labeling of $G_1^2$ is given in Figure 2. Therefore, $G_1^2$ is 3-TEPC. The 3-total edge product cordial labeling of the graphs $G_2^2$ and $G_2^1$ is given in Figures 4 and 5, respectively. Table 2 shows the multiplicity of numbers 0, 1, and 2 used in $G_1^1$ and $G_2^1$.

Figure 6 depicts the labeled segment $S_2^3$, which has the property that open edges are assigned labeled 1 and each number 0, 1, and 2 is used 18 times.

Case (i): $m = 3r$, $r \geq 1$.

To construct labeled graph $G_m^{2}$, we will use the labeled segments $S_2^3$. First, we glue $r - 1$ copies of labeled segment $S_2^3$ vertically, that is, $S_2^3 \oplus S_2^3 \oplus \cdots \oplus S_2^3 := (r - 1)S_2^3$. Since the open edges of $S_2^3$ are labeled with 1, therefore, this gluing process does not change the label of other vertices of $(r - 1)S_2^3$. Finally, we glue vertically the label segment $G_2^2$ to the open edges of $(r - 1)S_2^3$ to get labeled graph $G_m^{2}$. That is,

$$G_m^{2} = \left[ \begin{array}{c} (r - 1)S_2^3 \\ \Phi_v \\ G_2^3 \end{array} \right].$$

(4)

In the labeled graph $G_m^{2}$, the multiplicity of 0 is $18r + 1$, whereas the multiplicity of 1 and 2 is $18r + 2$.

Case (ii): $m = 3r + 1$, $r \geq 1$.

To construct the labeled graph $G_m^{2}$, we glue $r$ copies of the labeled segment $S_2^3$ and then finally glue $G_2^2$ vertically. That is,

$$G_m^{2} = \left[ \begin{array}{c} rS_2^3 \\ \Phi_v \\ G_2^2 \end{array} \right].$$

(5)

In the labeled graph $G_m^{2}$, the multiplicity of 0 is $18r + 7$ whereas the multiplicity of 1 and 2 is $18r + 8$.

Case (iii): $m = 3r + 2$, $r \geq 1$.

The labeled graph $G_m^{2}$ can be obtained by gluing $r$ times the labeled segment $S_2^3$ and then gluing $G_2^2$ in vertical direction. That is,

$$G_m^{2} = \left[ \begin{array}{c} rS_2^3 \\ \Phi_v \\ G_2^2 \end{array} \right].$$

(6)

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**Table 2: The multiplicity of 0, 1, and 2 in $G_m^{2}$, for $m = 2, 3$.**

| $m$   | $e_0(0)$  | $e_1(0)$  | $e_2(0)$  | $e_0(2)$  | $e_1(2)$  | $e_2(2)$  |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| 2     | 13        | 14        | 14        | 19        | 20        | 20        |
| 3     | 19        | 20        | 20        |           |           |           |

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In the labeled graph $G_m^3$, the multiplicity of 0 is $18r + 13$, whereas the multiplicity of 1 and 2 is $18r + 14$. □

**Theorem 3.** The graph $G_m^3$ is 3-TEPC for $m \geq 1$.

**Proof.** Observe that the graphs $G_1^3$ and $G_3^3$ are isomorphic. Similarly, the graphs $G_1^3$ and $G_3^3$ are also isomorphic. The 3-TEPC labeling of $G_1^3$ and $G_3^3$ are given in Figures 3 and 5, respectively. The 3-TEPC labeling of $G_3^3$ is shown in Figure 7. In the labeled graph $G_m^3$, the multiplicity of 0 is 29, whereas the multiplicity of 1 and 2 is 28.

Figure 8 shows the labeled segment $S_3^3$ which has the property that open edges are assigned with label 1 and each number 0, 1, and 2 appears 26 times.

Case (i): $m = 3r$, $r \geq 1$.

To construct labeled graph $G_m^3$, we use the labeled segment $S_3^3$. First, glue $r - 1$ copies of labeled segment $S_3^3$ vertically, that is, $S_3^3 \oplus S_3^3 \oplus \cdots \oplus S_3^3 = (r - 1)S_3^3$. Since the open edges of $S_3^3$ are labeled with 1, therefore, this gluing process does not change the label of other vertices of $(r - 1)S_3^3$. Finally, glue vertically the label segment $G_3^3$ to the open edges of $(r - 1)S_3^3$ to get labeled graph $G_m^3$. That is,

$$G_m^3 = \begin{bmatrix} (r - 1)S_3^3 \\ \oplus_v \\ G_3^3 \end{bmatrix}. \quad (7)$$

In the labeled graph $G_m^3$, the multiplicity of 0 is $26r + 3$, whereas the multiplicity of 1 and 2 is $26r + 2$.

Case (ii): $m = 3r + 1$, $r \geq 1$.

To construct the labeled graph $G_m^3$, we glue $r$ copies of the labeled segment $S_3^3$ vertically and then finally glue $G_3^3$ vertically. That is,

$$G_m^3 = \begin{bmatrix} rS_3^3 \\ \oplus_v \\ G_3^3 \end{bmatrix}. \quad (8)$$

In the labeled graph $G_m^3$, the multiplicity of 0, 1, and 2 is $26r + 11$.

Case (iii): $m = 3r + 2$, $r \geq 1$.

We obtain the labeled graph $G_m^3$ by gluing $r$ times the labeled segment $S_3^3$ vertically and then finally glue $G_3^3$ in vertical direction. That is,

$$G_m^3 = \begin{bmatrix} rS_3^3 \\ \oplus_v \\ G_3^3 \end{bmatrix}. \quad (9)$$

In the labeled graph $G_m^3$, the multiplicity of 0 is $26r + 19$, whereas the multiplicity of 1 and 2 is $26r + 20$. □
Theorem 4. The graph $G_n^m$ is 3-TEPC for $m, n \geq 1$.

Proof. To construct the labeled graph of $G_n^m$, and to examine its 3-TEPC labeling, we introduced a new segment $R_3$. This segment has 17 open edges which are labeled with the number 1 and multiplicity of 0, 1, and 2 is 24. The labeled segment $R_3^1$ is shown in Figure 9.

Case 1: $m = 3r, r \geq 1$.

First, we glue the segment $R_3^1$ vertically $r - 1$ times, that is, $R_3^1 \oplus R_3^1 \oplus \cdots \oplus R_3^1 = (r - 1)R_3^1$. Since the open edges in the segment are labeled with number 1, it follows that gluing these segments do not change the vertex labels in the segment $(r - 1)R_3^1$. Finally, we glue the segment $S_3^1$ in the vertical direction. This gives a new segment $X$ and is defined as

$$X = \begin{bmatrix} (r - 1)R_3^1 \\ R_3^1 \\ S_3^1 \end{bmatrix}. \tag{10}$$

Note that the labels of open edges of $X$ are 1 and multiplicity of each number 0, 1, and 2 is $24r + 2$.

Subcase 1: $n = 3s, s \geq 1$.

First, we glue $s - 1$ times the segment $X$ horizontally and finally glue the labeled segment $G_3^m$ horizontally to obtain the labeled graph $G_n^m$. That is,

$$G_n^m = [(s - 1)X \oplus hG_3^m]. \tag{11}$$

Subcase 2: $n = 3s + 1, s \geq 1$.

First, we glue $s$ times the segment $X$ horizontally and finally glue the labeled segment $G_3^m$ horizontally with $sX$ to obtain the labeled graph $G_n^m$. That is,

$$G_n^m = [sX \oplus hG_3^m]. \tag{12}$$

Subcase 3: $n = 3s + 2, s \geq 1$.

First, we glue $s$ times the segment $X$ horizontally and finally glue the labeled segment $G_3^m$ horizontally with $sX$ to obtain the labeled graph $G_n^m$. That is,

$$G_n^m = [sX \oplus hG_2^m]. \tag{13}$$

Case 2: when $m = 3r + 1, r \geq 1$.

First, we glue the segment $R_3^1$ vertically $r$ times, that is, $R_3^1 \oplus R_3^1 \oplus \cdots \oplus R_3^1 = rR_3^1$. Then, we glue the segment $S_3^1$ in the vertical direction. This gives us a new segment $Y$ defined as

$$Y = \begin{bmatrix} rR_3^1 \\ R_3^1 \\ S_3^1 \end{bmatrix}. \tag{14}$$

Note that the labels of open edges of $Y$ are 1 and multiplicity of each number 0, 1, and 2 is $24r + 10$.

Subcase 1: $n = 3s, s \geq 1$.

First, we glue $s - 1$ times the segment $Y$ horizontally and finally glue the labeled segment $G_3^m$ horizontally with $(s - 1)Y$ to obtain the labeled graph $G_n^m$. That is,

$$G_n^m = [(s - 1)Y \oplus hG_3^m]. \tag{15}$$

Subcase 2: $n = 3s + 1, s \geq 1$.

First, we glue $s$ times the segment $Y$ horizontally and finally glue the labeled segment $G_3^m$ horizontally with $sY$ to obtain the labeled graph $G_n^m$. That is,

$$G_n^m = [sY \oplus hG_3^m]. \tag{16}$$

Subcase 3: $n = 3s + 2, s \geq 1$.

First, we glue $s$ times the segment $Y$ horizontally and finally glue the labeled segment $G_3^m$ horizontally with $sY$ to obtain the labeled graph $G_n^m$. That is,

$$G_n^m = [sY \oplus hG_2^m]. \tag{17}$$

Case 3: $m = 3r + 2, r \geq 1$.

First, we glue the segment $R_3^1$ vertically $r$ times, that is, $R_3^1 \oplus R_3^1 \oplus \cdots \oplus R_3^1 = rR_3^1$. Then, we glue in the vertical direction of the segment $S_3^1$. This gives us a new segment $Z$ defined as

$$Z = \begin{bmatrix} rR_3^1 \\ R_3^1 \\ S_3^1 \end{bmatrix}. \tag{18}$$

Note that the labels of open edges of $Z$ are 1 and multiplicity of each number 0, 1, and 2 is $24r + 18$.

Subcase 1: $n = 3s, s \geq 1$.

First, we glue $s - 1$ times the segment $Y$ horizontally and finally glue the labeled segment $G_3^m$ horizontally with $(s - 1)Z$ to obtain the labeled graph $G_n^m$. That is,
In this paper, we constructed 3-TEPC labeling for the stellation of square grid graph $G_m^n$. For every $m \geq 1$ and every $n \geq 1$, we proved that $G_m^n$ is 3-TEPC.

3. Conclusion

In this paper, we constructed 3-TEPC labeling for the stellation of square grid graph $G_m^n$. For every $m \geq 1$ and every $n \geq 1$, we proved that $G_m^n$ is 3-TEPC.

## Data Availability

No data were used to support the findings of the study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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