Fused Mackey functors

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Abstract: Let $G$ be a finite group. In [5], Hambleton, Taylor and Williams have considered the question of comparing Mackey functors for $G$ and biset functors defined on subgroups of $G$ and bifree bisets as morphisms.

This paper proposes a different approach to this problem, from the point of view of various categories of $G$-sets. In particular, the category $G$-set of fused $G$-sets is introduced, as well as the category $\mathcal{S}(G)$ of spans in $G$-set. The fused Mackey functors for $G$ over a commutative ring $R$ are defined as $R$-linear functors from $R\mathcal{S}(G)$ to $R$-modules. They form an abelian subcategory $\text{Mack}^f_R(G)$ of the category of Mackey functors for $G$ over $R$.

The category $\text{Mack}^f_R(G)$ is equivalent to the category of conjugation Mackey functors of [5]. The category $\text{Mack}^f_R(G)$ is also equivalent to the category of modules over the fused Mackey algebra $\mu^f_R(G)$, which is a quotient of the usual Mackey algebra $\mu_R(G)$ of $G$ over $R$.

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1. Introduction

This note is devoted to the frequently asked question of comparing Mackey functors for a single finite group $G$ with biset functors defined only on subgroups of $G$ and left-right free bisets as morphisms. The answer to this question has already been given by Hambleton, Taylor and Williams ([5]), but in a rather computational and non canonical way (in particular, in Section 7, the definition of the functor $j_\bullet$ requires the choice of sets of representatives of orbits of any finite $G$-set).

The present paper makes a systematic use of Dress definition ([3]) and Lindner definition ([6]) of Mackey functors, to avoid these non canonical choices. This leads to the definition of the category of fused $G$-sets (Section 3), and the category of fused Mackey functors (Section 4) for a finite group $G$, which is equivalent to the category of “conjugation invariant Mackey functors” of [5]. This category is also equivalent to the category of modules over the fused Mackey algebra, introduced in Section 5.

2. Conjugation bisets revisited

2.1. First a notation : when $G$ is a finite group, and $X$ is a finite $G$-set, let $G$-set$_X$ denote the category of (finite) $G$-sets over $X$: its objects are pairs $(Y, b)$ consisting of a finite $G$-set $Y$, and a morphism of $G$-sets $b : Y \to X$. A
morphism \( f : (Y, b) \to (Z, c) \) in \( G\text{-}\text{set}_{\downarrow X} \) is a morphism of \( G \)-sets \( f : Y \to Z \) such that \( c \circ f = b \).

There is an obvious notion of disjoint union in \( G\text{-}\text{set}_{\downarrow X} \), and the corresponding Grothendieck group is called the Burnside group over \( X \). It will be denoted by \( \mathcal{B}(G X) \), or \( \mathcal{B}(X) \) when \( G \) is clear from the context.

Similarly, when \( G \) and \( H \) are finite groups, and \( U \) is a \((G, H)\)-biset, one can define the category \((G, H)\text{-}\text{biset}_{\downarrow U}\) of \((G, H)\)-bisets over \( U \), and the Burnside group \( \mathcal{B}(G U H) \) of \((G, H)\)-bisets over \( U \).

### 2.2

When \( H \) is a subgroup of \( G \), and \( Y \) is an \( H \)-set, induction from \( H \)-sets to \( G \)-sets is an equivalence of categories from \( H\text{-}\text{set}_{\downarrow Y} \) to \( G\text{-}\text{set}_{\downarrow \text{Ind}_G^H Y} \). A quasi-inverse equivalence is the functor sending the \( G \)-set \((X, a)\) over \( \text{Ind}_G^H Y \) to \((a^{-1}(1 \times H Y), a)\) (see [2] Lemma 2.4.1). In particular \( \mathcal{B}(H Y) \cong \mathcal{B}(G \text{Ind}_G^H Y) \).

### 2.3

Now an observation: when \( H \) and \( K \) are subgroups of \( G \), the conjugation \((K, H)\)-bisets defined in Section 6 of [5] are exactly those over the biset \( K G_H \) (the set \( G \) on which \( K \) and \( H \) act by multiplication), i.e. the \((K, H)\)-bisets \( U \) for which there exists a biset morphism \( U \to K G_H \).

Indeed, a conjugation \((K, H)\)-biset \( U \) is a bifree \((K, H)\)-biset isomorphic to a disjoint union of bisets of the form \((K \times H)/S\), where \( S \) is a subgroup of \( K \times H \) of the form

\[
S_{g,A} = \{(g x, x) \mid x \in A\}
\]

where \( A \) is a subgroup of \( H \), and \( g \) is an element of \( G \) such that \( g A \leq K \).

For such a transitive biset \((K \times H)/S\), the map

\[
\forall (k, h) S \in (K \times H)/S, \; (k, h) S \mapsto kgh^{-1}
\]

is a morphism of \((K, H)\)-bisets.

Conversely, let \( U \) be a \((K, H)\)-biset for which there exists a biset morphism \( \alpha : U \to K G_H \). Then for any \( u \in U \), the stabilizer \( S_u \) of \( u \) in \( K \times H \) is the subgroup

\[
S_u = \{(k, h) \in K \times H \mid k \cdot u \cdot h^{-1} = u\}
\]

of \( K \times H \). Then if \( (k, h) \in S_u \),

\[
\alpha(k \cdot u) = k \alpha(u) = \alpha(u \cdot h) = \alpha(u) h
\]

Let \( A_u \) denote the projection of \( S_u \) into \( H \), and set \( g_u = \alpha(u) \). It follows that \( S_u \subseteq S_{g_u A_u} \).
Conversely, if \((k, h) \in S_{g_u, A_u}\), then \(k = g_u h\), and there exists some \(x \in K\) such that \((x, h) \in S_u\), since \(h \in A_u\). Thus \(x \cdot u \cdot h^{-1} = u\), from which follows that \(\alpha(x \cdot u) = x g_u = \alpha(u \cdot h) = g_u h\), hence \(x = g_u h = k\), and \(S_u = S_{g_u, A_u}\). Observation 2.3 follows.

2.4. In other words, conjugation \((K, H)\)-bisets form a category \(\text{Conj}^G_{K,H}\), and there is a forgetful functor \(\Phi : (K, H)\text{-biset} \downarrow_{K,G} \rightarrow \text{Conj}^G_{K,H}\) sending \((U,a)\) to \(U\). This functor is full, preserves disjoint unions, and moreover it induces a surjection on the corresponding sets of isomorphism classes. This means that \(\Phi\) induces a surjective group homomorphism (still denoted by \(\Phi\)) from \(\mathcal{B}(K,G)\) to the Grothendieck group \(\mathcal{B}^G_{K,H}\) of conjugation \((K,H)\)-bisets.

2.5. If \(H, K\) and \(L\) are subgroups of \(G\), if \((U,a)\) is a \((K,H)\)-biset over \(K,G,H\) and \((V,b)\) is an \((L,K)\)-biset over \(L,G,K\), the composition \((V,b) \circ (U,a)\) is the \((L,H)\)-biset over \(L,G,H\) defined by the following diagram:

\[
\begin{array}{ccc}
V & 
\xrightarrow{b} & U
\\
\downarrow{b} & & \downarrow{a}
\\
L \times_K G_H & & G \times_K G
\\
\end{array}
\]

where \(\mu\) is multiplication in \(G\). This composition is associative, and additive with respect to disjoint unions. Hence it induces a composition

\[
\widehat{\circ} : \mathcal{B}(L \times_K G) \times \mathcal{B}(K \times G) \rightarrow \mathcal{B}(L \times_K G).
\]

Hence, one can define a category \(\widehat{\mathcal{B}}(G)\) whose objects are the subgroups of \(G\), and such that \(\text{Hom}_{\mathcal{B}(G)}(H,K) = \mathcal{B}(K \times G)\), for subgroups \(H\) and \(K\) of \(G\). Composition is given by \(\circ\), and the identity morphism of the subgroup \(H\) of \(G\) in the category \(\widehat{\mathcal{B}}(G)\) is the class of the biset \((i_H H, i_H)\), where \(i_H : H \rightarrow G\) is the inclusion map from \(H\) to \(G\).

Since the functor \(\Phi\) maps the composition \(\circ\) to the composition of bisets, and the identity morphism of \(H\) in \(\widehat{\mathcal{B}}(G)\) to the identity biset \(i_H H\), one can extend \(\Phi\) to a functor \(\widehat{\mathcal{B}}(G) \rightarrow \mathcal{B}(G)\), which is the identity on objects.

In other words, the category \(\mathcal{B}(G)\) introduced in Section 3 of \([5]\) is the quotient of the category \(\widehat{\mathcal{B}}(G)\) obtained by identifying morphisms which have the same image by \(\Phi\).

2.6. By the above Remark 2.2 when \(H\) and \(K\) are subgroups of \(G\), there is
a group isomorphism

\[ B(KG_H) \cong B(\text{Ind}_{K \times H}^{G \times G}(KG_H)) \]

(with the usual identification of \((K, H)\)-bisets with \((K \times H)\)-sets). Now the biset \(KG_H\) is actually the restriction to \((K \times H)\) of the \((G, G)\)-biset \(G\). By the Frobenius reciprocity, it follows that

\[
\text{Ind}_{K \times H}^{G \times G}(KG_H) \cong \text{Ind}_{K \times H}^{G \times G} \text{Res}_{K \times H}^{G \times G}(G \times G) \cong (\text{Ind}_{K \times H}^{G \times G} \bullet) \times G \times G,
\]

where \(\bullet\) is a set of cardinality 1. Since \(\text{Ind}_{K \times H}^{G \times G} \bullet \cong (G/K) \times (G/H)\), it follows (after switching \(G/H\) and \(G\)) that

\[
\text{Ind}_{K \times H}^{G \times G}(KG_H) \cong (G/K) \times G \times (G/H),
\]

where the \((G, G)\)-biset structure of the right hand side is given by

\[
\forall (a, b, x, y, g) \in G^5, \quad a \cdot (xK, g, yH) \cdot b = (axK, agb, b^{-1}yH).
\]

2.7. It should now be clear that the additive completion \(\hat{\mathcal{B}}_\bullet(G)\) is equivalent to the category whose objects are finite \(G\)-sets, where for any two finite \(G\)-sets \(X\) and \(Y\)

\[
\text{Hom}_{\hat{\mathcal{B}}_\bullet(G)}(X, Y) = \mathcal{B}(G(Y \times G \times X)_G),
\]

the \((G, G)\)-biset structure on \((Y \times G \times X)\) being given as above by

\[
\forall (a, b, g, x, y) \in G^3 \times X \times Y, \quad a \cdot (y, g, x) \cdot b = (ay, agb, b^{-1}x).
\]

Keeping track of the composition \(\circ\) along the above isomorphism shows that the composition in the category \(\hat{\mathcal{B}}_\bullet(G)\) can be defined by linearity from the following: if \(X, Y,\) and \(Z\) are finite \(G\)-sets, if

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & V \\
\downarrow & & \downarrow \text{d} \\
G & \xrightarrow{e} & Y
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
U & \xrightarrow{c} & V \\
\downarrow & & \downarrow \text{a} \\
Y & \xrightarrow{b} & X
\end{array}
\]

are \((G, G)\)-bisets over \((Z \times G \times Y)\) and \((Y \times G \times X)\), respectively, their composition is given by the following \((G, G)\)-biset over \((Z \times G \times X)\)

\[
(V \times_{d,c} U)/G
\]

are finite \(G\)-sets, if
where \( V \times_{d,c} U \) is the pullback of \( V \) and \( U \) over \( Y \), i.e. the set of pairs \((v, u) \in V \times U\) with \( d(v) = c(u)\), and \((V \times_{d,c} U)/G\) the set of orbits of \( G \) on it for the action given by \((v, u) \cdot g = (vg, g^{-1}u)\). This makes sense because \( d(v \cdot g) = g^{-1}d(v) = g^{-1}c(u) = c(g^{-1} \cdot u)\) if \( d(v) = c(u)\). The map \((\gamma, \beta, \alpha)\) is given by

\[
(\gamma, \beta, \alpha)((v, u)) = (f(v), e(v)b(u), a(u))
\]

2.8. The functor \( \Phi : \hat{\mathcal{B}}(G) \to \mathcal{B}(G) \) extends uniquely to an additive functor \( \Phi_* : \hat{\mathcal{B}}_*(G) \to \mathcal{B}_*(G) \), and the category \( \mathcal{B}_*(G) \) is the quotient of \( \hat{\mathcal{B}}_*(G) \) obtained by identifying morphisms which have the same image by \( \Phi_* \). Clearly, two morphisms \( f, g \in \text{Hom}_{\hat{\mathcal{B}}_*(G)}(X, Y) \) are identified if and only if \( f - g \) is in the kernel of the group homomorphism

\[
\phi : \mathcal{B}(G(Y \times G \times X)G) \to \mathcal{B}(G(Y \times X)G)
\]

induced by the correspondence

\[
\begin{array}{ccc}
Y & \xrightarrow{c} & U \\
\downarrow{b} & & \downarrow{a} \\
G & \to & X
\end{array} \quad \leftrightarrow \quad \begin{array}{ccc}
Y & \xrightarrow{c} & U \\
\downarrow{b} & & \downarrow{a} \\
G & \to & X
\end{array}
\]

on bisets. In other words, a morphism \( f \) in \( \hat{\mathcal{B}}_*(G) \) gives the zero morphism in \( \mathcal{B}_*(G) \) if and only if it belongs to \( \text{Ker} \phi \).

2.9. Now the \((G, G)\)-biset \( _G G \) is isomorphic to \( \text{Ind}_{\Delta(G)}^{G \times G} \cdot \), where \( \Delta(G) \) is the diagonal subgroup of \( G \times G \). It follows that there is an isomorphism of \((G, G)\)-bisets

\[
Y \times G \times X \cong \text{Ind}_{\Delta(G)}^{G \times G}(Y \times X)
\]

Hence, by Remark 2.2 again, since \( \Delta(G) \cong G \),

\[
\mathcal{B}(G(Y \times G \times X)G) \cong \mathcal{B}(G(Y \times X))
\]

where \( G(Y \times X) \) is the usual cartesian product with diagonal \( G \)-action. More precisely, this isomorphism is induced by the correspondence

\[
\begin{array}{ccc}
Y & \xrightarrow{c} & U \\
\downarrow{b} & & \downarrow{a} \\
G & \to & X
\end{array} \quad \leftrightarrow \quad \begin{array}{ccc}
Y & \xrightarrow{c} & U \\
\downarrow{b} & & \downarrow{a} \\
G & \to & X
\end{array}
\]

\[b^{-1}(1)\]
It is then easy to check that the composition of

\[
\begin{array}{c}
V \\
\downarrow^f \\
Z
\end{array}
\quad \text{and} \quad
\begin{array}{c}
U \\
\downarrow^c \\
Y
\end{array}
\begin{array}{c}
\downarrow^d \\
G
\end{array}
\begin{array}{c}
\downarrow^e \\
Y
\end{array}
\begin{array}{c}
\downarrow^a \\
X
\end{array}
\]

corresponds to the usual pullback diagram

\[
\begin{array}{c}
e^{-1}(1) \times_{d,c} b^{-1}(1)
\end{array}
\]

\[
\begin{array}{c}
\text{and}
\end{array}
\]

\[
\begin{array}{c}
e^{-1}(1)
\end{array}
\begin{array}{c}
\downarrow^f \\
Z
\end{array}
\begin{array}{c}
\downarrow^d \\
Y
\end{array}
\begin{array}{c}
\downarrow^e \\
Y
\end{array}
\begin{array}{c}
\downarrow^b \\
G
\end{array}
\begin{array}{c}
\downarrow^c \\
X
\end{array}
\begin{array}{c}
\downarrow^a \\
X
\end{array}
\]

In other words, the category \(\mathcal{B}_G(G)\) is equivalent to the category \(S(G)\) whose objects are the finite \(G\)-sets, where

\[
\text{Hom}_{S(G)}(X, Y) = B(G(Y \times X)),
\]

and composition is induced by pullback. It has been shown by Lindner (see also [2]) that the additive functors on this category are precisely the Mackey functors for \(G\).

**2.10.** It remains to keep track of identifications by \(\Phi\), i.e. to start with a morphism \(f \in \text{Hom}_{S(G)}(X, Y)\), to lift it to

\[
f^+ \in \text{Hom}_{\mathcal{B}_G(G)}(X, Y) = B(G(Y \times G \times X)_G),
\]

and see when \(f^+\) lies in \(\text{Ker}\phi\). Now \(f\) is represented by a difference of two \(G\)-sets over \(G(Y \times X)\) of the form

\[
\begin{array}{c}
Z \\
\downarrow^b \\
Y
\end{array}
\begin{array}{c}
\downarrow^a \\
X
\end{array}
\quad - \quad
\begin{array}{c}
Z' \\
\downarrow^b' \\
Y
\end{array}
\begin{array}{c}
\downarrow^a' \\
X
\end{array}
\]

By induction from \(\Delta(G)\) to \(G \times G\), the \(G\)-set on the left hand side lifts to the following \((G \times G)\)-set over \((G \times G)(Y \times G \times X)\)

\[
\begin{array}{c}
G \times Z \\
\downarrow^\gamma \\
Y
\end{array}
\begin{array}{c}
\downarrow^\beta \\
G
\end{array}
\begin{array}{c}
\downarrow^\alpha \\
X
\end{array}
\]

6
where the \((G \times G)\)-actions on \(G \times Z\) and \(Y \times G \times X\) are given respectively by \((s,t) \cdot (g,z) = (s\cdot g \cdot t^{-1}, zt)\) and \((s,t) \cdot (y,g,x) = (sy \cdot g \cdot t^{-1}, tx)\), and where \((\gamma, \beta, \alpha)(g,z) = (gb(z), g, a(z))\).

Similarly the \(G\)-set \((Z', (b', a'))\) lifts to \((G \times Z', (\gamma', \beta', \alpha'))\).

Now \(f^+\) is in \(\text{Ker} \, \phi\) if and only if there is an isomorphism

\[
\begin{array}{ccc}
G \times Z & \xrightarrow{\gamma} & Y \\
\alpha & & \\
G \times Z' & \xrightarrow{\gamma'} & Y \\
\beta' & & \\
\end{array}
\]

of \((G \times G)\)-sets over \(Y \times X\). Since \((g,z) = g \cdot (1, z)\) for any \((g,z) \in G \times Z\), it follows that \(\theta\) is a map from \(G \times Z\) to \(G \times Z'\) of the form

\[
(g,z) \mapsto (gu(z), v(z)) ,
\]

where \(u\) is a map from \(Z\) to \(G\) and \(v\) is a map from \(Z\) to \(Z'\). Now for any \((s,t) \in G \times G\), the equality

\[
\theta((s,t) \cdot (g,z)) = (s,t) \cdot \theta((g,z))
\]

gives

\[
(s\cdot g \cdot t^{-1})u(tz), v(tz) = (sgu(z)t^{-1}, tv(z)) .
\]

This is equivalent to

\[
u(tz) = 'u(z) \quad \text{and} \quad v(tz) = tv(z) .
\]

This means that \(u\) is a morphism of \(G\)-sets from \(Z\) to \(G^c\), which is the set \(G\) with \(G\)-action by conjugation, and \(v\) is a morphism of \(G\)-sets.

Moreover \(\theta\) is a bijection if and only if \(v\) is.

Finally \(\theta\) is an morphism of \((G,G)\)-bisets over \(Y \times X\) if and only if \(\alpha' \circ \theta = a\) and \(\gamma' \circ \theta = \gamma\), i.e. equivalently if

\[
a' \circ v = a \quad \text{and} \quad gu(z) \cdot b' \circ v(z) = g \cdot b(z)
\]

for any \((g,z) \in G \times Z\). In other words

\[
a = a' \circ v \quad \text{and} \quad b = u \cdot (b' \circ v)
\]

where, for any map \(w : Z \rightarrow Y\), the map \(u * w : Z \rightarrow Y\) is defined by \((u * w)(z) = u(z) \cdot w(z)\). The map \(u * w\) is a map of \(G\)-sets if \(u : Z \rightarrow G^c\).
and \( w : Z \to Y \) are. Note that \( w' = u \ast w \) if and only if \( w = \tilde{u} \ast w' \), where \( \tilde{u} : Z \to G^c \) is defined by \( \tilde{u}(z) = u(z)^{-1} \).

It follows that \( f \) maps to the zero morphism in \( B(G) \) if and only if there exists \( u : Z \to G^c \) and an isomorphism \( v : Z \to Z' \) such that
\[
 a' \circ v = a \quad \text{and} \quad b' \circ v = u \ast b ,
\]
But then \( v \) is an isomorphism
\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node (X) at (0,0) {X};
  \node (Z) at (-2,2) {Z};
  \node (Y) at (2,2) {Y};
  \node (U) at (0,4) {Y};
  \draw[->] (Z) to (X);
  \draw[->] (Z) to (Y);
  \draw[->] (Y) to (U);
  \draw[->] (Z) to (U);
  \node at (-1,1) {v};
\end{tikzpicture}
\end{array}
\]
of \( G \)-sets over \( Y \times X \), and \( f \) is also represented by the difference
\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node (X) at (0,0) {X};
  \node (Z) at (-2,2) {Z};
  \node (Y) at (2,2) {Y};
  \node (U) at (0,4) {Y};
  \draw[->] (Z) to (X);
  \draw[->] (Z) to (Y);
  \draw[->] (Y) to (U);
  \draw[->] (Z) to (U);
  \node at (-1,1) {v};
\end{tikzpicture}
\end{array}
\]
since \( a' \circ v = a \) and \( b' \circ v = u \ast b \). These are the morphisms in the category \( S(G) \) that vanish in \( B_\ast(G) \). In other words:

**2.11. Theorem:** Let \( G \) be a finite group. Let \( S(G) \) denote the quotient category of \( S(G) \) defined by setting, for any two finite \( G \)-sets \( Y \) and \( Y \)
\[
\text{Hom}_{S(G)}(X,Y) = \mathcal{B}(G)(Y \times X))/K(Y,X) ,
\]
where \( K(Y,X) \) is the subgroup generated by the differences
\[
(2.12) \begin{array}{c}
\begin{tikzpicture}[scale=0.8]
  \node (X) at (0,0) {X};
  \node (Z) at (-2,2) {Z};
  \node (Y) at (2,2) {Y};
  \node (U) at (0,4) {Y};
  \draw[->] (Z) to (X);
  \draw[->] (Z) to (Y);
  \draw[->] (Y) to (U);
  \draw[->] (Z) to (U);
  \node at (-1,1) {v};
\end{tikzpicture}
\end{array}
\]
where \( a : Z \to X \), \( b : Z \to Y \), and \( u : Z \to G^c \) are morphisms of \( G \)-sets.
Then the functor \( \Phi_\ast \) induces an equivalence of categories \( S(G) \cong B_\ast(G) \).
Since the difference \( 2.12 \) factors as
\[
\begin{array}{ccc}
Z & \xrightarrow{b} & Z \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\text{Id}} & Z
\end{array}
\quad \circ \quad \begin{array}{ccc}
Z & \xrightarrow{\text{Id}} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\text{Id}} & Z
\end{array}
\quad \circ \quad \begin{array}{ccc}
Z & \xrightarrow{\text{Id}} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{a} & X
\end{array}
\]
the morphism vanishing in \( S(G) \) are generated in the category \( S(G) \) by the morphisms of the form
\[
\begin{array}{ccc}
Z & \xrightarrow{\text{Id}} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\text{Id}} & Z
\end{array}
\quad \circ \quad \begin{array}{ccc}
Z & \xrightarrow{\text{Id}} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\text{Id}} & Z
\end{array}
\quad \circ \quad \begin{array}{ccc}
Z & \xrightarrow{\text{Id}} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\text{Id}} & Z
\end{array}
\]

2.13. It follows that the additive functors from \( S(G) \) to the category of abelian groups are exactly those Mackey functors (in the sense of Dress) such that for any \( G \)-set \( Z \) and any \( u : Z \to G^c \), the morphism \( M_s(u \circ \text{Id}) \) is equal to the identity map of \( M(Z) \).

This condition is additive with respect to \( Z \), since the map \( u \circ \text{Id}_Z \) maps each \( G \)-orbit of \( Z \) to itself. Hence these functors are exactly the functors for which the map \( M_s(u \circ \text{Id}) \) is the identity map of \( M(G/H) \), for any subgroup \( H \) of \( G \) and any \( u : G/H \to G^c \). Such a map is of the form \( gH \mapsto gcH \), where \( c \in C_G(H) \). The map \( u \circ \text{Id} : G/H \to G/H \) is the map \( gH \mapsto gcH \).

Translated in terms of the usual definition of Mackey functors, this map expresses the action of \( c \) on \( M(H) = M(G/H) \). This shows that additive functors from \( S(G) \) to abelian groups are exactly the Mackey functors for the group \( G \) such that, for any \( H \leq G \), the centralizer \( C_G(H) \) acts trivially on \( M(H) \). These are the “conjugation invariant Mackey functors” introduced in \([5]\).

3. Fused \( G \)-sets

Let \( Z \) be any (finite) \( G \)-set. The multiplication \( (u, v) \mapsto u \circ v \) endows the set \( \text{Hom}_{G\text{-set}}(Z, G^c) \) with a group structure. Moreover, for any finite \( G \)-set \( X \), this group acts on the left on the set \( \text{Hom}_{G\text{-set}}(Z, X) \), via \( (u, f) \mapsto u \circ f \).

This action is compatible with the composition of morphisms: if \( Y \) is a finite \( G \)-set, if \( u : Z \to G^c \) and \( v : Y \to G^c \) are morphisms of \( G \)-sets, then for any morphisms of \( G \)-sets \( f : Z \to Y \) and \( g : Y \to X \), one checks easily that
\[
(3.1) \quad (v \circ g) \circ (u \circ f) = (u \circ (v \circ f)) \circ (g \circ f) .
\]
3.2. Notation: Let $G\text{-set}$ denote the category of fused $G$-sets: its objects are finite $G$-sets, and for any finite $G$-sets $Z$ and $Y$

$$\text{Hom}_{G\text{-set}}(Z,Y) = \text{Hom}_{\text{G-set}}(Z, G^c) \setminus \text{Hom}_{G\text{-set}}(Z, Y).$$

The composition of morphisms in $G\text{-set}$ is induced by the composition of morphisms in $G\text{-set}$.

3.3. Remark: For any $G$-set $Y$, set $Y^I = Y \times G^c$. This notation is chosen to evoke a path object in homotopy theory (cf. [4] Section 4.12). There is a natural morphism $p : Y^I \to Y \times Y$, defined by $p(y,g) = (y,gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y,1)$, for $y \in Y$. The composition $p \circ i$ is equal to the diagonal map $Y \to Y \times Y$.

Two morphisms $a, b : Z \to Y$ in $G\text{-set}$ are equal in the category $G\text{-set}$ if and only if the morphism $(a,b) : Z \to Y \times Y$ factors as

\[
\begin{array}{ccc}
Z & \xrightarrow{(a,b)} & Y \times Y \\
\downarrow \varphi & & \downarrow p \\
Y^I & \xrightarrow{\phi} & Y \times Y
\end{array}
\]

for some morphism of $G$-sets $\varphi : Z \to Y^I$.

3.4. Remark: It follows from 3.1 that the map $u \mapsto u \ast \text{Id}_Z$ is a group antihomomorphism from $\text{Hom}_{G\text{-set}}(Z, G^c)$ to the group of $G$-automorphisms of $Z$. Hence a morphism $f : Z \to Y$ in the category $G\text{-set}$ is an isomorphism if and only if any of its representatives $f : Z \to Y$ in $G\text{-set}$ is an isomorphism.

3.5. Weak pullbacks of fused $G$-sets. Disjoint union of $G$-sets is a coproduct in $G\text{-set}$. There is also a weak version of pullback in $G\text{-set}$: let

\[
\begin{array}{ccc}
X & \xleftarrow{a} & Y \\
\uparrow \downarrow & \Rightarrow & \downarrow \uparrow \\
Z & \xrightarrow{b} & Y
\end{array}
\]

be a commutative diagram in $G\text{-set}$, where underlines denote the images in $G\text{-set}$ of morphisms in $G\text{-set}$. This means that $a \circ \underline{c} = b \circ \underline{d}$, i.e. that there
exists \( u \in \text{Hom}_{G\text{-set}}(T, G^c) \) such that
\[
 b \circ d = u \ast (a \circ c) .
\]
But \( u \ast (a \circ c) = a \circ (u \ast c) \). It follows that there is a unique morphism 
\( e \in \text{Hom}_{G\text{-set}}(T, X \times_{a,b} Y) \) such that the diagram
\[
\begin{array}{ccc}
T & \rightarrow & X \times_{a,b} Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
\]
is commutative in \( G\text{-set} \), where \( p : X \times_{a,b} Y \rightarrow X \) and \( q : X \times_{a,b} Y \rightarrow Y \) are the canonical morphisms from the pullback \( X \times_{a,b} Y \). In other words, the diagram
\[
(3.6)
\]
is commutative in \( G\text{-set} \).

But still \( (X \times_{a,b} Y, p, q) \) need not be a pullback in \( G\text{-set} \), since the morphism \( e \) making Diagram 3.6 commutative is generally not unique, as \( e \) itself depends on the choice of \( u \). Moreover, the lifts \( a \) and \( b \) of \( a \) and \( b \) to \( G\text{-set} \) are not unique: it should be noted however that if \( a' = v \ast a \) and \( b' = w \ast b \) are other lifts of \( a \) and \( b \), respectively, where \( v \in \text{Hom}_{G\text{-set}}(X, G^c) \) and \( w \in \text{Hom}_{G\text{-set}}(Y, G^c) \), then the map 
\[
f : (x, y) \mapsto (v(x)x, w(y)y)
\]
is an
isomorphism of $G$-sets from $X \times_{a',b'} Y$ to $X \times_{a,b} Y$, such that the diagram

is commutative in $G$-set. Since $a' = a$, $b' = b$, $v \ast \text{Id} = \text{Id}$, and $w \ast \text{Id} = \text{Id}$, this yields a commutative diagram

in $G$-set, and $f$ is an isomorphism. This shows that the weak pullback $X \times_{a,b} Y$ only depends on $a$ and $b$ in the category $G$-set. For this reason, it may be denoted by $X \times_{a,b} Y$.

3.7. **Spans of fused $G$-sets.** Recall (cf. [9], [1] for the general definition) that if $X$ and $Y$ are finite $G$-sets, then a span $\Lambda_{Z,a,b}$ over $X$ and $Y$ in the category $G$-set is a diagram of the form

where $Z$ is a finite $G$-set and $a$, $b$ are morphisms in the category $G$-set. Two spans $\Lambda_{Z,a,b}$ and $\Lambda_{Z',a',b'}$ over $X$ and $Y$ are equivalent if there exists an isomorphism $f : Z \to Z'$ in $G$-set, such that the diagram

12
is commutative. The set of equivalence classes of spans of fused $G$-sets over $X$ and $Y$ is an additive monoid, where the addition is defined by disjoint union (i.e. $\Lambda_{Z_1,a_1,b_1} + \Lambda_{Z_2,a_2,b_2} = \Lambda_{Z_1 \cup Z_2,a_1 \cup a_2,b_1 \cup b_2}$). The corresponding Grothendieck group is isomorphic to $\text{Hom}_{\mathcal{S}(G)}(Y,X)$.

It should be noted that even if there is no pullback construction in the category $G$-\textbf{set}, the isomorphism classes of spans in $G$-\textbf{set} can still be composed by weak pullback, and this induces the composition of morphisms in $\mathcal{S}(G)$.

4. Fused Mackey functors

4.1. Definition : Let $R$ be a commutative ring. Let $R\mathcal{S}(G)$ (resp. $R\mathcal{S}(G)$) denote the $R$-linear extension of the category $\mathcal{S}(G)$ (resp. $\mathcal{S}(G)$), defined as follows:

- The objects of $R\mathcal{S}(G)$ and $R\mathcal{S}(G)$ are finite $G$-sets.
- For finite $G$ sets $X$ and $Y$,
  \[ \text{Hom}_{R\mathcal{S}(G)}(X,Y) = R \otimes \mathbb{Z} \text{Hom}_{\mathcal{S}(G)}(X,Y), \]
  \[ \text{Hom}_{R\mathcal{S}(G)}(X,Y) = R \otimes \mathbb{Z} \text{Hom}_{\mathcal{S}(G)}(X,Y). \]

- Composition of morphisms is induced by the pullback in $G$-\textbf{set} (resp. the weak pullback in $G$-\textbf{set}).

A Mackey functor for $G$ over $R$ in the sense of Lindner ([6]) is an $R$-linear functor from $R\mathcal{S}(G)$ to the category $R\text{-Mod}$ of $R$-modules.

Similarly, a fused Mackey functor for $G$ over $R$ is an $R$-linear functor from $R\mathcal{S}(G)$ to $R\text{-Mod}$. A morphism of fused Mackey functors is a natural transformation of functors. Fused Mackey functors for $G$ over $R$ form a category denoted by $\text{Mack}_R^f(G)$.

The following is an equivalent definition of fused Mackey functors, à la Dress:

4.2. Definition : Let $R$ be a commutative ring. A fused Mackey functor for the group $G$ over $R$ is a bivariant $R$-linear functor $M = (M^+, M_*)$ from $G$-\textbf{set} to $R\text{-Mod}$ such that:
1. For any finite $G$-sets $X$ and $Y$, the maps
\[ M(X) \oplus M(Y) \xrightarrow{(M_*{\Lambda}_X, M_*{\Lambda}_Y)} M(X \sqcup Y) \]
induced by the canonical inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ are mutual inverse isomorphisms.

2. If
\[ X \times \underline{\underline{Z}} Y \]
\[ \begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow{a} & & \downarrow{b} \\
Z & \xleftarrow{q} & \underline{Z} \\
\end{array} \]
is a weak pullback diagram in $G$-set, then $M^*(a)M_*(b) = M_*(p)M^*(q)$.

A morphism of fused Mackey functors is a natural transformation of bivariant functors.

The category $\text{Mack}_R^f(G)$ can be viewed as a full subcategory of the category $\text{Mack}_R(G)$ of Mackey functors for $G$ over $R$. In the case $R = \mathbb{Z}$, this category is equivalent to the category of conjugation invariant Mackey functors introduced in [5].

The inclusion functor $\text{Mack}_R^f(G) \hookrightarrow \text{Mack}_R(G)$ has a left adjoint:

4.3. Definition: Let $M$ be a Mackey functor for $G$ over $R$, in the sense of Lindner, i.e. an $R$-linear functor $RS(G) \to R\text{-Mod}$. When $X$ is a finite $G$-set, set
\[ M^f(X) = M(X)/\sum_{Z,a,u} \text{Im}(M(\Lambda_{a,\text{Id}_Z}) - M(\Lambda_{u*a,\text{Id}_Z})) , \]
where the summation runs through triples $(Z,a,u)$ consisting of a finite $G$-set $Z$, and morphisms of $G$-sets $a : Z \to X$ and $u : Z \to G^c$, and $\Lambda_{a,\text{Id}_Z}$ denotes the span
\[ \begin{array}{ccc}
Z & \xrightarrow{a} & X \\
\downarrow{\text{Id}_Z} & & \downarrow{\text{Id}_Z} \\
X & & Z \\
\end{array} \]
of $G$-sets.
4.4. Proposition: Let $R$ be a commutative ring, and $G$ be a finite group. 

1. Let $M$ be a Mackey functor for $G$ over $R$. The correspondence 

$$X \mapsto M^f(X)$$ 

is a fused functor $M^f$ for $G$ over $R$.

2. The correspondence $\mathcal{F} : M \mapsto M^f$ is a functor from $\text{Mack}_R(G)$ to $\text{Mack}_R^f(G)$, which is left adjoint to the inclusion functor 

$$\mathcal{I} : \text{Mack}_R^f(G) \hookrightarrow \text{Mack}_R(G).$$

Moreover $\mathcal{F} \circ \mathcal{I}$ is isomorphic to the identity functor of $\text{Mack}_R^f(G)$.

Proof: For Assertion 1, to prove that $M^f$ is a Mackey functor, observe that if $\Lambda_{Z,a,b}$ is a span of finite $G$-sets of the form 

$$\begin{array}{ccc}
    Z & \xrightarrow{a} & X \\
    \downarrow{b} & & \downarrow{X} \\
    Y & \xrightarrow{} & Y
\end{array}$$ 

and $u : Z \to G^c$ is a morphism of $G$-sets, then 

$$\Lambda_{Z,a,b} - \Lambda_{Z,u*a,b} = (\Lambda_{Z,a,\text{id}_Z} - \Lambda_{Z,u*a,\text{id}_Z}) \circ \Lambda_{Z,\text{id}_Z,b}.$$

It follows that the $R$-module 

$$\sum_{Z,a,u} \text{Im}(M(\Lambda_{a,\text{id}_Z}) - M(\Lambda_{u*a,\text{id}_Z}))$$

is equal to the sum 

$$\sum_{Z,a,b,u} \text{Im}(M(\Lambda_{a,b}) - M(\Lambda_{u*a,b})).$$

In other words, it is equal to the image by $M$ of the $R$-submodule $K_R(X,Y)$ of $\text{Hom}_{R\text{S}(G)}(Y,X)$ generated by the morphisms $\Lambda_{a,b} - \Lambda_{u*a,b}$, i.e. to the kernel of the quotient morphism 

$$\text{Hom}_{R\text{S}(G)}(Y,X) \to \text{Hom}_{R\text{S}(G)}(Y,X).$$

This shows that $K_R$ is an ideal in the category $R\text{S}(G)$. So if $M$ is an $R$-linear functor $R\text{S}(G) \to R\text{-Mod}$, the correspondence 

$$X \mapsto M^f(X) = M(X) / \sum_{f \in K_R(X,Y)} \text{Im}(f)$$
is an $R$-linear functor from the quotient category $R\mathcal{S}(G)$ to $R\text{-Mod}$.

Assertion 2 is straightforward: first it is clear that $F \circ I$ is isomorphic to the identity functor, since $N^{I} = N$ when $N$ is a fused Mackey functor. This isomorphism $F \circ I \cong \text{Id}_{\text{Mack}^{f}(G)}$ provides the counit of the adjunction. Next for any Mackey functor $M$, there is a projection morphism $M \rightarrow I F(M)$, and this yields the unit of the adjunction.

4.5. Remark : Assertion 2 shows that $\text{Mack}^{f}(G)$ is a reflective subcategory of $\text{Mack}^{f}(G)$ (cf. [7], Chapter IV, Section 3).

4.6. Remark : If the Mackey functor $M$ is given in the sense of Dress, then for any finite $G$-set $X$

$$M^{f}(X) = M(X)/\sum_{u:Z \rightarrow G^{c}} \text{Im}(M_{*}(a) - M_{*}(u * a)) ,$$

where $Z$ is a finite $G$-set, and $a, u$ are morphisms of $G$-sets.

4.7. Corollary :

1. If $P$ is a projective Mackey functor, then $P^{f}$ is projective in the category $\text{Mack}^{f}(G)$.

2. The category $\text{Mack}^{f}(G)$ has enough projective objects. More precisely, if $N$ is a fused Mackey functor, and $\theta : P \rightarrow I(N)$ is an epimorphism in $\text{Mack}^{f}(G)$ from a projective Mackey functor $P$, then $F(\theta) : P^{f} \rightarrow N$ is an epimorphism in $\text{Mack}^{f}(G)$.

Proof : Assertion 1 follows from the fact that $F$ is left adjoint to the exact functor $I$. Assertion 2 is then straightforward.

5. The fused Mackey algebra

When $G$ is a finite group, set $\Omega_{G} = \bigsqcup_{H \leq G} G/H$, and let $RB_{\Omega_{G}}$ denote the Dress construction for the Burnside functor $RB$ over the ring $R$. Recall that $RB_{\Omega_{G}}$, as a Mackey functor in the sense of Dress, is obtained by precomposition of $RB$ with the endofunctor $X \mapsto X \times \Omega_{G}$ of $G$-set.

Also recall (cf. [2] Lemma 7.3.2 and Proposition 4.5.1) that the functor $RB_{\Omega_{G}}$ is a progenerator of the category $\text{Mack}^{f}(G)$, and that the algebra $\text{End}_{\text{Mack}^{f}(G)}(B_{\Omega_{G}}) \cong B(\Omega_{G}^{2})$ is isomorphic to the Mackey algebra $\mu_{R}(G)$ of $G$ over $R$, introduced by Thévenaz and Webb ([8]).
It follows from Corollary 4.7 that the functor \((RB\Omega_G)^f\) is a progenerator in the category \(\text{Mack}^f_R(G)\). Hence this category is equivalent to the category of modules over the algebra \(\text{End}_{\text{Mack}^f_R(G)}((RB\Omega_G)^f)\).

### 5.1. Definition

The fused Mackey algebra of \(G\) over \(R\) is the algebra

\[
\mu^f_R(G) = \text{End}_{\text{Mack}^f_R(G)}((RB\Omega_G)^f) .
\]

### 5.2. Lemma

Let \(X\) be a finite \(G\)-set. Then \((RB_X)^f\) is isomorphic to the Yoneda functor \(\text{Hom}_{\text{R}\mathbf{S}(G)}(X, -)\).

**Proof:** Denote by \(\mathcal{Y}_X\) the Yoneda functor \(\text{Hom}_{\text{R}\mathbf{S}(G)}(X, -)\). For any fused Mackey functor \(N\) for \(G\) over \(R\)

\[
\text{Hom}_{\text{Mack}^f_R(G)}((RB_X)^f, N) \cong \text{Hom}_{\text{Mack}^f_R(G)}(RB_X, \mathcal{I}(N)) \\
\cong \mathcal{I}(N)(X) \cong N(X) \\
\cong \text{Hom}_{\text{Mack}^f_R(G)}(\mathcal{Y}_X, N) .
\]

The lemma follows, since all these isomorphisms are natural.

### 5.3. Theorem

The fused Mackey algebra \(\mu^f_R(G)\) is isomorphic to the quotient of the algebra \(\text{RB}(\Omega_G^2) \cong \mu_R(G)\) by the \(R\)-module generated by differences of the form

\[
\Omega_G \xrightarrow{b} Z \xrightarrow{a} \Omega_G \quad \Omega_G \xrightarrow{a \ast b} \Omega_G \xrightarrow{a} \Omega_G ,
\]

where \(a, b : Z \to \Omega_G\) and \(u : Z \to G^c\) are morphisms of \(G\)-sets.

**Proof:** This follows from Lemma 5.2, since the quotient in the theorem is precisely \(\text{End}_{\text{R} \mathbf{S}(G)}(\Omega_G)\).

### 5.4. Remark

One can deduce from this theorem that the fused Mackey algebra \(\mu^f_R(G)\) is always free of finite rank as an \(R\)-module, and this rank does not depend on the commutative ring \(R\). More precisely, Thévenaz and Webb have shown (Proposition 3.2) that the Mackey algebra \(\mu_R(G)\) has
an $R$-basis consisting of elements of the form

$$t^H_K c_{g,K} r^L_K,$$

where $(H, L, g, K)$ runs through a set of representatives of 4-tuples consisting of two subgroups $H$ and $L$ of $G$, and element $g$ of $G$, and a subgroup $K$ of $H \cap gL$, for the equivalence relation $\equiv$ given by

$$(H, L, g, K) \equiv (H', L', g', K') \Leftrightarrow \begin{cases} H = H', L = L', \\ \text{and} \\ \exists h \in H, \exists l \in L, g' = hgl, K' = hK \end{cases}.$$ 

Similarly, the quotient algebra $\mu^f_R(G)$ of $\mu_R(G)$ has a basis consisting of the images of the elements $t^H_K c_{g,K} r^L_K$, where $(H, L, g, K)$ runs through a set of representatives of 4-tuples as above, modulo the relation $\equiv^f$ defined by

$$(H, L, g, K) \equiv^f (H', L', g', K') \Leftrightarrow \begin{cases} H = H', L = L', \\ \text{and} \\ \exists h \in H, \exists l \in L, \exists x \in C_G(K), \\ g' = hxgl, K' = hK \end{cases}.$$ 

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