On Natural Deduction for Herbrand Constructive Logics II: Curry-Howard Correspondence for Markov’s Principle in First-Order Logic and Arithmetic

Federico Aschieri* and Matteo Manighetti†

Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10/104, 1040, Vienna, Austria

Abstract

Intuitionistic first-order logic extended with a restricted form of Markov’s principle is constructive and admits a Curry-Howard correspondence, as shown by Herbelin. We provide a simpler proof of that result and then we study intuitionistic first-order logic extended with unrestricted Markov’s principle. Starting from classical natural deduction, we restrict the excluded middle and we obtain a natural deduction system and a parallel Curry-Howard isomorphism for the logic. We show that proof terms for existentially quantified formulas reduce to a list of individual terms representing all possible witnesses. As corollary, we derive that the logic is Herbrand constructive: whenever it proves any existential formula, it proves also an Herbrand disjunction for the formula. Finally, using the techniques just introduced, we also provide a new computational interpretation of Arithmetic with Markov’s principle.

1998 ACM Subject Classification F.4.1 Proof Theory

Keywords and phrases Markov’s Principle, first-order logic, natural deduction, Curry-Howard

1 Introduction

Markov’s Principle was introduced by Markov in the context of his theory of Constructive Recursive Mathematics (see [15]). Its original formulation is tied to Arithmetic: it states that given a recursive function \( f : \mathbb{N} \to \mathbb{N} \), if it is impossible that for every natural number \( n \), \( f(n) \neq 0 \), then there exists a \( n \) such that \( f(n) = 0 \). Markov’s original argument for justifying it was simply the following: if it is not possible that for all \( n \), \( f(n) \neq 0 \), then by computing in sequence \( f(0), f(1), f(2), \ldots \), one will eventually hit a number \( n \) such that \( f(n) = 0 \) and will effectively recognize it as a witness.

Markov’s principle is readily formalized in Heyting Arithmetic as the axiom scheme

\[ \neg \neg \exists \alpha \forall \alpha \neg P \rightarrow \exists \alpha \forall \alpha P \]

where \( P \) is a primitive recursive predicate [14]. When added to Heyting Arithmetic, Markov’s principle gives rise to a constructive system, that is, one enjoying the disjunction and the existential witness property [13] (if a disjunction is derivable, one of the disjoints is derivable)

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too, and if an existential statement is derivable, so it is one instance of it). Furthermore, witnesses for any provable existential formula can be effectively computed using either Markov’s unbounded search and Kleene’s realizability [9] or much more efficient functional interpretations [7, 3].

1.1 Markov’s Principle in First-Order Logic

The very shape of Markov’s principle makes it also a purely logical principle, namely an instance of the double negation elimination axiom. But in pure logic, what exactly should Markov’s principle correspond to? In particular, what class of formulas should \( P \) be restricted to? Since Markov’s principle was originally understood as a constructive principle, it is natural to restrict \( P \) as little as possible, while maintaining the logical system as constructive as possible. As proven by Herbelin [8], it turns out that asking that \( P \) is propositional and with no implication \( \rightarrow \) symbols guarantees that intuitionistic logic extended with such a version of Markov’s principle is constructive. The proof of this result employs a Curry-Howard isomorphism based on a mechanism for raising and catching exceptions. As opposed to the aforementioned functional interpretations of Markov’s principle, Herbelin’s calculus is fully isomorphic to an intuitionistic logic: there is a perfect match between reduction steps at the level of programs and detour eliminations at the level of proofs. Moreover, witnesses for provable existential statements are computed by the associated proof terms. Nevertheless, as we shall later show, the mechanism of throwing exceptions plays no role during these computations: intuitionistic reductions are entirely enough for computing witnesses.

A question is now naturally raised: as no special mechanism is required for witness computation using Herbelin’s restriction of Markov’s principle, can the first be further relaxed so that the second becomes stronger as well as computationally and constructively meaningful? Allowing the propositional matrix \( P \) to contain implication destroys the constructivity of the logic. It turns out, however, that Herbrand constructivity is preserved. An intermediate logic is called Herbrand constructive if it enjoys a strong form of Herbrand’s Theorem [5, 4]: for every provable formula \( \exists \alpha A \), the logic proves as well an Herbrand disjunction

\[
A[m_1/\alpha] \lor \ldots \lor A[m_k/\alpha]
\]

So the Markov principle we shall interpret in this paper is

\[
\text{MP} : \neg \neg \exists \alpha P \rightarrow \exists \alpha P \quad (P \text{ propositional formula})
\]

and show that when added to intuitionistic first-order logic, the resulting system is Herbrand constructive. This is the most general form of Markov’s principle that allows a significant constructive interpretation: we shall show how to non-trivially compute lists of witnesses for provable existential formulas thanks to an exception raising construct and a parallel computation operator. MP can also be used in conjunction with negative translations to compute Herbrand disjunctions in classical logic, something which is not possible with Herbelin’s form of Markov’s principle.

1.2 Restricted Excluded Middle

The Curry-Howard correspondence we present here is by no means an ad hoc construction, only tailored for Markov’s principle. It is a simple restriction of the Curry-Howard correspondence for classical first-order logic introduced in [4], where classical reasoning is formalized by the excluded middle inference rule:
\[
\Gamma, a : \forall x \ Q \vdash u : C \\
\Gamma, a : \exists x \neg Q \vdash v : C \\
\Gamma \vdash u \parallel v : C \\
\text{EM}
\]

It is enough to restrict the conclusion \(C\) of this rule to be a simply existential statement and the \(Q\) in the premises \(\forall x \ Q, \exists x \neg Q\) to be propositional. We shall show that the rule is intuitionistically equivalent to \(\text{MP}\). With our approach, strong normalization is just inherited and the transition from classical logic to intuitionistic logic with \(\text{MP}\) is smooth and natural.

1.3 Markov’s Principle in Arithmetic

We shall also provide a computational interpretation of Heyting Arithmetic with \(\text{MP}\). The system is constructive and witnesses for provable existential statements can be computed. This time, we shall restrict the excluded middle as formalized in \([2]\) and we shall directly obtain the desired Curry-Howard correspondence. As a matter of fact, the interpretation of \(\text{MP}\) in Arithmetic ends up being a simplification of the methods we use in first-order logic, because the decidability of atomic formulas greatly reduces parallelism and eliminates case distinction on the truth of atomic formulas.

1.4 Plan of the Paper

In Section \([2]\) we provide a simple computational interpretation of first-order intuitionistic logic extended with Herbelin’s restriction of Markov’s principle. We also show that the full Markov principle \(\text{MP}\) cannot be proved in that system. In Section \([3]\) we provide a Curry-Howard correspondence for intuitionistic logic with \(\text{MP}\), by restricting the excluded middle, and show that the system is Herbrand constructive. In Section \([4]\) we extend the Curry-Howard to Arithmetic with \(\text{MP}\) and show that the system becomes again constructive.

2 Herbelin’s Restriction of Markov’s Principle

In \([8]\) Herbelin introduced a Curry-Howard isomorphism for an extended intuitionistic logic. By employing exception raising operators and new reduction rules, he proved that the logic is constructive and can derive the axiom scheme

\[\text{HMP} : \neg\neg\exists \alpha P \rightarrow \exists \alpha P \quad (P \text{ propositional and } \rightarrow \text{ not occurring in } P)\]

Actually, Herbelin allowed \(P\) also to contain existential quantifiers, but in that case the axiom scheme is intuitionistically equivalent to \(\neg\neg\exists \alpha_1 \ldots \exists \alpha_n \rightarrow \exists \alpha_1 \ldots \exists \alpha_n P\), again with \(P\) propositional and \(\rightarrow\) not occurring in \(P\). All of the methods of our paper apply to this case as well, but for avoiding trivial details, we keep the present \(\text{HMP}\).

Our first goal is to show that \(\text{HMP}\) has a simpler computational interpretation and to provide a straightforward proof that, when added on top of first-order intuitionistic logic, \(\text{HMP}\) gives rise to a constructive system. In particular, we show that the ordinary Prawitz reduction rules for intuitionistic logic and thus the standard Curry-Howard isomorphism \([13]\) are enough for extracting witnesses for provable existential formulas. The crucial insight, as we shall see, is that \(\text{HMP}\) can never actually appear in the head of a closed proof term having existential type. It thus plays no computational role in computing witnesses; it plays rather a logical role, in that it may be used to prove the correctness of the witnesses.

To achieve our goal, we consider the usual natural deduction system for intuitionistic first-order logic \([12, 13]\), to which we add \(\text{HMP}\). Accordingly, we add to the associated lambda calculus the constants \(M_P : \neg\neg\exists \alpha P \rightarrow \exists \alpha P\). The resulting Curry-Howard system is called \(\text{IL + HMP}\) and is presented in fig. \(\[\text{1}\] The reduction rules for \(\text{IL + HMP}\) presented in fig. \(\[\text{2}\] are
just the ordinary ones of lambda calculus. On the other hand, $\mathcal{MP}$ has no computational content and thus no associated reduction rule. Of course, the strong normalization of $\text{IL} + \text{HMP}$ holds by virtue of the result for standard intuitionistic Curry-Howard.

**Theorem 1.** The system $\text{IL} + \text{HMP}$ is strongly normalizing

**Grammar of Untyped Proof Terms**

$t, u, v := x | tu | tm | \lambda x u | (t, u) | u_0 | u_1 | u_0(u) | t[x, u, y, v] | (m, t) | t[(\alpha, x), u] | H^{\bot \rightarrow P} | \mathcal{MP}$

where $m$ ranges over terms of the first-order language of formulas $L$, $x$ over proof-term variables, $\alpha$ over first-order variables.

**Contexts** With $\Gamma$ we denote contexts of the form $x_1 : A_1, \ldots, x_n : A_n$, where each $x_i$ is a proof-term variable, and $x_i \neq x_j$ for $i \neq j$.

**Axioms**

\[ \Gamma, x : A \vdash x : A \quad \Gamma \vdash \mathcal{MP} : \lnot \lnot \exists \alpha P \rightarrow \exists \alpha P \quad \Gamma \vdash H^{\bot \rightarrow P} : \bot \rightarrow P \]

**Conjunction**

\[ \Gamma \vdash u : A, \Gamma \vdash t : B \quad \Gamma \vdash \langle u, t \rangle : A \land B \]

\[ \Gamma \vdash u : A \land B \quad \Gamma \vdash u_\pi_0 : A \quad \Gamma \vdash u_\pi_1 : B \]

**Implication**

\[ \Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A \quad \Gamma \vdash tu : B \]

\[ \Gamma, x : A \vdash u : B \quad \Gamma \vdash \lambda x u : A \rightarrow B \]

**Disjunction Introduction**

\[ \Gamma \vdash u : A \quad \Gamma \vdash u : B \quad \Gamma \vdash u : A \lor B \]

\[ \Gamma \vdash u : A, \Gamma \vdash t : B \quad \Gamma, x : A \vdash t : C \quad \Gamma, y : B \vdash t : C \quad \Gamma \vdash u[t : B] : C \]

**Disjunction Elimination**

\[ \Gamma \vdash u : A \land B \quad \Gamma \vdash u_\pi_0 : A \quad \Gamma \vdash u_\pi_1 : B \]

**Universal Quantification**

\[ \Gamma \vdash u : \forall \alpha A \quad \Gamma \vdash \lambda u \pi_0 : A[\alpha/u] \quad \Gamma \vdash \lambda u \pi_1 : \forall \alpha A \]

where $\alpha$ is any term of the language $L$ and $\alpha$ does not occur free in any formula $B$ occurring in $\Gamma$.

**Existential Quantification**

\[ \Gamma \vdash u : A[m/\alpha] \quad \Gamma \vdash u : \exists \alpha A \quad \Gamma \vdash u : \exists \alpha A \quad \Gamma \vdash \langle [\alpha, x], t \rangle : C \]

where $\alpha$ is not free in $C$ nor in any formula $B$ occurring in $\Gamma$.

**Figure 1** Term Assignment Rules for IL + HMP

**Reduction Rules for IL**

| Reduction Rule | Description |
|----------------|-------------|
| $(\lambda x u)t \rightarrow u[t/x]$ | |
| $(\lambda x u)m \rightarrow u[m/\alpha]$ | |
| $(\alpha, u_0)\pi_1 \rightarrow u_i$, for $i=0,1$ | |
| $t_i(u)[x_1, x_2, t_2] \rightarrow t_i[u/x_i]$, for $i=0,1$ | |
| $(m, u)[(\alpha, x), v] \rightarrow v[m/\alpha][u/x]$ | |

for each term $t$ of $\mathcal{L}$

**Figure 2** Reduction Rules for IL + HMP

As we shall see in Theorem 4, the reason why HMP cannot be appear in the head of a closed proof term having existential type is that its premise $\lnot \lnot \exists \alpha P$ is never classically valid, let alone provable in intuitionistic logic.

**Proposition 2.** Assume that the symbol $\rightarrow$ does not occur in the propositional formula $P$. Then $\lnot \lnot \exists \alpha P$ is not classically provable.
Proof. We provide a semantical argument. \( \neg \exists \alpha P \) is classically provable if and only if it is classically valid and thus if and only if \( \exists \alpha P \) is classically valid. For every such a formula, we shall exhibit a model falsifying it. Consider the model \( \mathfrak{M} \) where every \( n \)-ary predicate is interpreted as the empty \( n \)-ary relation. We show by induction on the complexity of the formula \( P \) that \( P^{\mathfrak{M}} = \bot \) for every assignment of individuals to the free variables of \( P \), and therefore \( (\exists \alpha P)^{\mathfrak{M}} = \bot \).

- If \( P \) is atomic, then by definition of \( \mathfrak{M} \), we have \( P^{\mathfrak{M}} = \bot \) for every assignment of the variables.
- If \( P = P_1 \land P_2 \), then since by induction \( P_1^{\mathfrak{M}} = \bot \) and \( P_2^{\mathfrak{M}} = \bot \)
- If \( P = P_1 \lor P_2 \), then since by induction \( P_1^{\mathfrak{M}} = \bot \) and \( P_2^{\mathfrak{M}} = \bot \)

In order to derive constructivity of \( \text{IL} + \text{HMP} \), we shall just have to inspect the normal forms of proof terms. Our main argument, in particular, will use the following well-known syntactic characterization of the shape of proof terms.

▶ Proposition 3 (Head of a Proof Term). Every proof-term of \( \text{IL} + \text{HMP} \) is of the form

\[
\lambda z_1 \ldots \lambda z_n, r u_1 \ldots u_k
\]

where

- \( r \) is either a variable or a constant or a term corresponding to an introduction rule: \( \lambda x.t \), \( \lambda \alpha.t \), \( \langle t_1, t_2 \rangle \), \( \iota(t) \), \( (m, t) \)
- \( u_1 \ldots u_k \) are either proof terms, first order terms, or one of the following expressions corresponding to elimination rules: \( \pi_i \), \( [x, w_1, y, w_2] \), \( [\alpha, x].t \).

Proof. Standard. ▶

We are now able to prove that \( \text{IL} + \text{HMP} \) is constructive.

▶ Theorem 4 (Constructivity of \( \text{IL} + \text{HMP} \)).

1. If \( \text{IL} + \text{HMP} \vdash t : \exists \alpha A \), and \( t \) is in normal form, then \( t = (m, u) \) and \( \text{IL} + \text{HMP} \vdash u : A[m/\alpha] \).
2. If \( \text{IL} + \text{HMP} \vdash t : A \lor B \) and \( t \) is in normal form, then either \( t = \iota_0(u) \) and \( \text{IL} + \text{HMP} \vdash u : A \) or \( t = \iota_1(u) \) and \( \text{IL} + \text{HMP} \vdash u : B \).

Proof.

1. By Proposition 3, \( t \) must be of the form \( ru_1 \ldots u_k \). Let us consider the possible forms of \( r \).
   - Since \( t \) is closed, \( r \) cannot be a variable.
   - We show that \( r \) cannot be \( M_P \). If \( r \) were \( M_P : \neg \neg \exists x P \to \exists \alpha P \) for some \( P \), then \( \text{IL} + \text{MP} \vdash u_1 : \neg \neg \exists \alpha P \). Since \( \text{IL} + \text{HMP} \) is contained in classical logic, we have that \( \neg \neg \exists \alpha P \) is classically provable. However we know from Proposition 2 that this cannot be the case, which is a contradiction.
   - We also show that \( r \) cannot be \( \rho^P \). Indeed, if \( r \) were \( \rho^P \) for some \( P \), then \( \text{IL} + \text{MP} \vdash u_1 : \bot \), which is a contradiction.
   - The only possibility is thus that \( r \) is one among \( \lambda x.t \), \( \lambda \alpha.t \), \( \langle t_1, t_2 \rangle \), \( \iota(t) \), \( (m, t) \). In this case, \( k \) must be 0 as otherwise we would have a redex. This means that \( t = r \) and thus \( t = (m, u) \) with \( \text{IL} + \text{HMP} \vdash u : A(m) \).
Finally, we prove that $IL + HMP$ is not powerful enough to express full Markov’s principle $MP$. Intuitively, the reason is that $IL + HMP$ is a constructive system and thus cannot be strong enough to interpret classical reasoning. This would indeed be the case if $IL + HMP$ proved $MP$, an axiom which complements very well negative translations.

**Proposition 5.** $IL + HMP \not\vdash MP$.

**Proof.** Suppose for the sake of contradiction that $IL + HMP \vdash MP$. Consider any proof in classical first-order logic of a simply existential statement $\exists \alpha P$. By the Gödel-Gentzen negative translation (see [14]), we can then obtain an intuitionistic proof of $\neg \neg \exists \alpha P^N$, where $P^N$ is the negative translation of $P$, and thus $IL + HMP \vdash \exists \alpha P^N$. By Theorem 4 there is a first-order term $m$ such that $IL + HMP \vdash P^N[m/\alpha]$. Since $P^N[m/\alpha]$ is classically equivalent to $P[m/\alpha]$, we would have a single witness for every classically valid simply existential statement. But this is not possible: consider for example the first-order language $\mathcal{L} = \{P, a, b\}$ and the formula $F = (P(a) \lor P(b)) \rightarrow P(\alpha)$ where $P$ is an atomic predicate. Then the formula $\exists \alpha F$ is classically provable, but there is no term $m$ such that $F[m/\alpha]$ is valid, let alone provable:

- it cannot be $m = a$, as it is shown by picking a model where $P$ is interpreted as the set \{a\}
- it cannot be $m = b$, because we can interpret $P$ as the set \{b\}.

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**3 Full Markov Principle and Restricted Excluded Middle in First-Order Logic**

In this section we describe the natural deduction system and Curry-Howard correspondence $IL + EM_1^-$, which arise by restricting the excluded-middle in classical natural deduction [4]. This computational system is based on delimited exceptions and a parallel operator. We will show that on one hand full Markov principle $MP$ is provable in $IL + EM_1^-$ and, on the other hand, that $IL + MP$ derives all of the restricted classical reasoning that can be expressed in $IL + EM_1^-$, so that the two systems are actually equivalent. Finally, we show that the system $IL + EM_1^-$ is Herbrand constructive and that witnesses can effectively be computed.

All of the classical reasoning in $IL + EM_1^-$ is formally restricted to negative formulas.

**Definition 6 (Negative, Simply Universal Formulas).** We denote propositional formulas as $P_1, \ldots, P_n, Q, R, \ldots$. We say that a propositional formula is negative whenever $\lor$ does not occur in it. Formulas of the form $\forall \alpha_1 \ldots \forall \alpha_n P$, with $P$ negative, will be called simply universal.

In order to computationally interpret Markov’s principle, we consider the rule $EM_1^-$, which is obtained by restricting the conclusion of the excluded middle $EM_1$ [4, 2] to be a simply existential formula,

$$
\Gamma, a : \forall \alpha P \vdash u : \exists \beta Q \quad \Gamma, a : \exists \alpha \neg P \vdash v : \exists \beta Q \\
\Gamma \vdash u \|_a v : \exists \beta Q \quad EM_1^-
$$
where both $P$ and $Q$ are negative formulas. This inference rule is complemented by the axioms:

\[
\Gamma, a : \forall \alpha P \vdash H^\forall_\alpha P : \forall \alpha P
\]

\[
\Gamma, a : \exists \alpha \neg P \vdash W^\exists_\alpha \neg P : \exists \alpha \neg P
\]

These last two rules correspond respectively to a term making a *Hypothesis* and a term waiting for a *Witness* and these terms are put in communication via $EM^\neg_1$. A term of the form $H^\forall_\alpha P m$, with $m$ first-order term, is said to be *active*, if its only free variable is $a$: it represents a raise operator which has been turned on. The term $u \parallel_a v$ supports an exception mechanism: $u$ is the ordinary computation, $v$ is the exceptional one and $a$ is the communication channel. Raising exceptions is the task of the term $H^\forall_\alpha P$, when it encounters a counterexample $m$ to $\forall \alpha P$; catching exceptions is performed by the term $W^\exists_\alpha \neg P$. In first-order logic, however, there is an issue: when should an exception be thrown? Since the truth of atomic predicates depends on models, one cannot know. Therefore, each time $H^\forall_\alpha P$ is applied to a term $m$, a new pair of parallel independent computational paths is created, according as to whether $P[m/\alpha]$ is false or true. In one path the exception is thrown, in the other not, and the two computations will never join again. To render this computational behaviour, we add the rule $EM_0$ of propositional excluded middle over negative formulas

\[
\frac{\Gamma, a : \neg P \vdash u : A \quad \Gamma, a : P \vdash v : A}{\Gamma \vdash u \mid v : A} \quad EM_0
\]

even if in principle it is derivable from $EM^\neg_1$; we also add the axiom

\[
\Gamma, a : P \vdash H^P : P
\]

We call the resulting system $IL + EM^\neg_1$ (fig. 3) and present its reduction rules in fig. 4 they just form a restriction of the system $IL + EM$ described in [4]. The reduction rules are in fig. 4 and are based on the following definition, which formalizes the raise and catch mechanism.

**Definition 7 (Exception Substitution).** Suppose $v$ is any proof term and $m$ is a term of $L$. Then:

1. If every free occurrence of $a$ in $v$ is of the form $W^\exists_\alpha P_a$, we define

\[
v[a := m]
\]

as the term obtained from $v$ by replacing each subterm $W^\exists_\alpha P_a$ corresponding to a free occurrence of $a$ in $v$ by $(m, H^P[m/\alpha])$.

2. If every free occurrence of $a$ in $v$ is of the form $H^\forall_\alpha P_a m$, we define

\[
v[a := m]
\]

as the term obtained from $v$ by replacing each subterm $H^\forall_\alpha P_a m$ corresponding to a free occurrence of $a$ in $v$ by $H^P[m/\alpha]$.

As we anticipated, our system is capable of proving the full Markov Principle $MP$ and thus its particular case $HMP$.

**Proposition 8 (Derivability of $MP$).** $IL + EM^\neg_1 \vdash MP$
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Grammar of Untyped Proof Terms

\[ t, u, v ::= x | tu | tm | \lambda x u | (t, u) | u\pi_0 | u\pi_1 | t_0(u) | t_1(u) | t[[x, u, v]] | (m, t) | t[[\alpha, x], u] \]

where \( m \) ranges over terms of \( \mathcal{L} \), \( x \) over proof-term variables, \( \alpha \) over first-order variables, \( a \) over hypothesis variables, \( A \) is either a negative formula or simply universal formula with a negative propositional matrix, and \( P \) is negative.

We assume that in the term \( u ||_a v \) there is some formula \( P \), such that \( a \) occurs free in \( u \) only in subterms of the form \( B^0_\alpha P \) and \( a \) occurs free in \( v \) only in subterms of the form \( W^1_\alpha P \), and the occurrences of the variables in \( P \) different from \( \alpha \) are free in both \( u \) and \( v \).

**Contexts** With \( \Gamma \) we denote contexts of the form \( x_1 : A_1, \ldots, x_n : A_n \), where each \( x_i \) is a proof-term variable, and \( x_i \neq x_j \) for \( i \neq j \).

**Axioms**

\[ \Gamma, x : A \vdash x : A \quad \Gamma, \alpha A : B^0_\alpha A \vdash \forall \alpha A \quad \Gamma, a : \exists \alpha P \vdash W^1_\alpha P : \exists \alpha P \]

**Conjunction**

\[ \Gamma \vdash t : B \quad \Gamma \vdash t : B \quad \Gamma \vdash \langle t, u \rangle : A \land B \]

**Implication**

\[ \Gamma \vdash u : A \quad \Gamma \vdash t : B \quad \Gamma \vdash u \land t : B \]

**Disjunction Introduction**

\[ \Gamma \vdash t : A \lor B \quad \Gamma \vdash u : B \quad \Gamma \vdash \langle t, u \rangle : A \lor B \]

**Disjunction Elimination**

\[ \Gamma \vdash u : A \lor B \quad \Gamma, x : A \vdash u : B \quad \Gamma, x : A \vdash u : B \]

**Universal Quantification**

\[ \Gamma \vdash u : \forall \alpha A \quad \Gamma \vdash u : A \quad \Gamma \vdash u \land \alpha u : \forall \alpha A \]

where \( m \) is any term of the language \( \mathcal{L} \) and \( \alpha \) does not occur free in any formula \( B \) occurring in \( \Gamma \).

**Existential Quantification**

\[ \Gamma \vdash u : A[m/\alpha] \quad \Gamma \vdash u : A \quad \Gamma \vdash u \land \exists \alpha A \quad \Gamma, x : A \vdash t : C \]

where \( \alpha \) is not free in \( C \) nor in any formula \( B \) occurring in \( \Gamma \).

**Equivalence Rules**

\[ \Gamma, a : \exists \alpha P \vdash u : C \quad \Gamma, a : \exists \alpha P \vdash v : C \]

**Equivalence Rules**

\[ \Gamma, a : \exists \alpha P \vdash u : \exists \beta P \quad \Gamma, a : \exists \alpha P \vdash v : \exists \beta Q \]

**Figure 3** Term Assignment Rules for IL + EM_1^−

**Proof.** First note that with the use of EM_0, we obtain that IL + EM_1^− \vdash P \lor \neg P for any atomic formula \( P \). Therefore IL + EM_1^− can prove any propositional tautology, and in particular IL + EM_1^− \vdash P \lor Q \leftrightarrow \neg \neg (P \land \neg Q) for any propositional formulas \( P, Q \), thus proving that each propositional formula is equivalent to a negative one.

Consider now any instance \( \neg \neg \exists \alpha Q \rightarrow \exists \alpha Q \) of MP. Thanks to the previous observation, we obtain

\[ IL + EM_1^− \vdash (\neg \neg \exists \alpha Q \rightarrow \exists \alpha Q) \leftrightarrow (\neg \neg \exists \alpha P \rightarrow \exists \alpha P) \]

for some negative formula \( P \) logically equivalent to \( Q \). The following formal proof shows that IL + EM_1^− \vdash \neg \neg \exists \alpha P \rightarrow \exists \alpha P. \]
Reduction Rules for IL

$$(\lambda x.u)t \mapsto u[t/x] \quad (\lambda \alpha.u)m \mapsto u[m/\alpha]$$

$$(u_0, u_1)\pi_i \mapsto u_i, \text{ for } i=0,1$$

$$u_i[u[x_1.t_1, x_2.t_2] \mapsto t_i[u/x_i], \text{ for } i=0,1$$

$$(m, u)((\alpha, x).v) \mapsto v[m/\alpha][u/x], \text{ for each term } m \text{ of } L$$

Permutation Rules for EM_0

$$(u | v)w \mapsto uw | vw$$

$$(u | v)\pi_i \mapsto u\pi_i | v\pi_i$$

$$(u | v)[x.w_1, y.w_2] \mapsto u[x.w_1, y.w_2 | v[x.w_1, y.w_2]$$

$$(u | v)[(\alpha, x).w] \mapsto u[(\alpha, x).w] | v[(\alpha, x).w]$$

Reduction Rules for EM_1^−

$$u \parallel_a v \mapsto u, \text{ if } a \text{ does not occur free in } u$$

$$u \parallel_a v \mapsto v[a := m] | (u[a := m] \parallel_a v), \text{ whenever } u \text{ has some active subterm } H^\alpha_m$$

**Figure 4** Reduction Rules for IL + EM_1^−

\[
\frac{[\forall \alpha \neg P]\text{EM}_1^-}{\neg P} \quad \frac{[P]_\exists}{\exists} \quad \frac{[\neg \neg P]_\exists \quad \neg P}{\text{EM}_0} \\
\frac{[\neg \exists \alpha P]_2}{\neg \exists \alpha P} \quad \frac{\exists}{P} \quad \frac{[\exists \alpha \neg P]\text{EM}_1^-}{P} \quad \frac{P}{\exists \alpha P} \quad \frac{\exists \alpha P}{\text{EM}_1^-} \quad \frac{\exists \alpha P}{\neg \exists \alpha P \to \exists \alpha P} \quad (2)
\]

Finally, this implies IL + EM_1^− ⊢ ¬¬∃α Q → ∃α Q.

Conversely, everything which is provable within our system can be proven by means of first-order logic with full Markov principle.

**Theorem 9.** If IL + EM_1^− ⊢ F, then IL + MP ⊢ F.

**Proof.** We just need to show that IL + MP can prove the rules EM_1^− and EM_0. For the case of EM_0, note that IL + MP ⊢ ¬¬P → P for all propositional formulas P, thanks to MP. Since for every propositional Q we have IL + MP ⊢ ¬¬(Q ∨ ¬Q), we obtain IL + MP ⊢ Q ∨ ¬Q, and therefore IL + MP can prove EM_0 by mean of an ordinary disjunction elimination.

In the case of EM_1^−, if we are given the proofs of $\exists \alpha C$ in IL + MP, the following derivation shows a proof of $\exists \alpha C$ in IL + MP.
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As in [4], all of our main results about witness extraction are valid not only for closed terms, but also for quasi-closed ones, which are those containing only pure universal assumptions.

Definition 10 (Quasi-Closed terms). An untyped proof term \( t \) is said to be quasi-closed, if it contains as free variables only hypothesis variables \( a_1, \ldots, a_n \), such that each occurrence of them is of the form \( \forall \vec{\alpha} P_i a_i \), where \( \forall \vec{\alpha} P_i \) is simply universal.

\[ \text{IL} + \text{EM}^-_1 \] with the reduction rules in figure fig. 4 enjoys the Subject Reduction Theorem, as a particular case of the Subject Reduction for \( \text{IL} + \text{EM} \) presented in [4].

Theorem 11 (Subject Reduction). If \( \Gamma \vdash t : C \) and \( t \mapsto u \), then \( \Gamma \vdash u : C \).

No term of \( \text{IL} + \text{EM}^-_1 \) gives rise to an infinite reduction sequence [4].

Theorem 12 (Strong Normalization). Every term typable in \( \text{IL} + \text{EM}^-_1 \) is strongly normalizing.

We now update the characterization of proof-terms heads given in Proposition 3 to the case of \( \text{IL} + \text{EM}^-_1 \).

Theorem 13 (Head of a Proof Term). Every proof term of \( \text{IL} + \text{EM}^-_1 \) is of the form:

\[ \lambda z_1 \ldots \lambda z_n . ru_1 \ldots u_k \]

where

- \( r \) is either a variable \( x \), a constant \( H^P \) or \( H^{\forall \alpha \alpha} A \) or \( \neg \exists \alpha \neg P \) or an excluded middle term \( u \parallel v \) or \( u \lnot v \) or a term corresponding to an introduction rule \( \lambda x.t, \lambda \alpha.t \), \( \langle t_1,t_2 \rangle \), \( \iota_i(t) \), \( (m,t) \)
- \( u_1, \ldots, u_k \) are either lambda terms, first order terms, or one of the following expressions corresponding to elimination rules: \( \pi_i \), \( [x.w_1,y.w_2] \), \( \langle (\alpha,x),t \rangle \).

Proof. Standard.

We now study the shape of the normal terms with the most simple types.

Proposition 14 (Normal Form Property). Let \( P, P_1, \ldots, P_n \) be negative propositional formulas, \( A_1, \ldots, A_m \) simply universal formulas. Suppose that

\[ \Gamma = z_1 : P_1, \ldots, z_m : P_n, a_1 : \forall \alpha_1 A_1, \ldots, a_m : \forall \alpha_m A_m \]

and \( \Gamma \vdash t : \exists \alpha P \) or \( \Gamma \vdash t : P \), with \( t \) in normal form and having all its free variables among \( z_1, \ldots, z_n, a_1, \ldots, a_m \). Then:
1. Every occurrence in $t$ of every term $H_{\alpha_i}^{\kappa_i,A_i}$ is of the active form $H_{\alpha_i}^{\kappa_i,A_i}m$, where $m$ is a term of $\mathcal{L}$.

2. $t$ cannot be of the form $u \parallel_a v$.

Proof. We prove 1. and 2. simultaneously and by induction on $t$. There are several cases, according to the shape of $t$:

- $t = (m, u)$. We immediately get 1. by induction hypothesis applied to $u$, while 2. is obviously verified.
- $t = \lambda x u$. We immediately get 1. by induction hypothesis applied to $u$, while 2. is obviously verified.
- $t = \langle u, v \rangle$. We immediately get 1. by induction hypothesis applied to $u$, while 2. is obviously verified.
- $t = u | v$. We immediately get the thesis by induction hypothesis applied to $u$ and $v$, while 2. is obviously verified.
- $t = u \parallel_a v$. We show that this is not possible. Note that $a$ must occur free in $u$, otherwise $t$ is not in normal form. Since $\Gamma, a : \forall \alpha A \vdash u : \exists \alpha P$, we can apply the induction hypothesis to $u$, and obtain that all occurrences of hypothetical terms must be active; in particular, this must be the case for the occurrences of $H_{\alpha_i}^{\kappa_i,A_i}$, but this is not possible since $t$ is in normal form.
- $t = H_{\alpha_i}^{\kappa_i,A_i}$. This case is not possible, for $\Gamma \vdash t : \exists \alpha P$ or $\Gamma \vdash t : \mathcal{L}$.
- $t = H_{\alpha_i}^{\kappa_i,A_i}$. In this case, 1. and 2. are trivially true.
- $t$ is obtained by an elimination rule and by Theorem [13] we can write it as $r t_1 t_2 \ldots t_n$. Notice that in this case $r$ cannot correspond to an introduction rule neither a term of the form $u \parallel_a v$, because of the induction hypothesis, nor $u \parallel v$, because of the permutation rules and $t$ being in normal form; moreover, $r$ cannot be $H_{\beta}^{\kappa_i,P}$, otherwise $b$ would be free in $t$ and $b \neq a_1, \ldots, a_n$. We have now two remaining cases:

1. $r = x_i$ (resp. $r = H_{\alpha_i}^{\kappa_i}$). Then, since $\Gamma \vdash x_i : \mathcal{P}_i$ (resp. $\Gamma \vdash H_{\beta}^{\kappa_i} : \mathcal{P}$), we have that for each $i$, either $t_i$ is $\pi_b$ or $\Gamma \vdash t_i : \mathcal{Q}$, where $\mathcal{Q}$ is a negative propositional formula. By induction hypothesis, each $t_i$ satisfies 1. and also 2. is obviously verified.

2. $r = H_{\alpha_i}^{\kappa_i,A_i}$. Then, $t_1$ is $m_1$ for some closed term of $\mathcal{L}$. Let $A_i = \forall \gamma_1 \ldots \forall \gamma_l Q$, with $Q$ propositional, we have that for each $i$, either $t_i$ is a closed term $m_i$ of $\mathcal{L}$ or $t_i$ is $\pi_j$ or $\Gamma \vdash t_i : \mathcal{R}$, where $\mathcal{R}$ is a negative propositional formula. By induction hypothesis, each $t_i$ satisfies 1. and thus also 2. is obviously verified.

If we omit the parentheses, we will show that every normal proof-term having as type an existential formula can be written as $v_0 | v_1 | \ldots | v_n$, where each $v_i$ is not of the form $u | v$; if for every $i$, $v_i$ is of the form $(m_i, u_i)$, then we call the whole term an Herbrand normal form, because it is essentially a list of the witnesses appearing in an Herbrand disjunction. Formally:

**Definition 15 (Herbrand Normal Forms).** We define by induction a set of proof terms, called Herbrand normal forms, as follows:

- Every normal proof-term $(m, u)$ is an Herbrand normal form;
- if $u$ and $v$ are Herbrand normal forms, $u \parallel v$ is an Herbrand normal form.
Our last task is to prove that all quasi-closed proofs of any existential statement \( \exists \alpha A \) include an exhaustive sequence \( m_1, m_2, \ldots, m_k \) of possible witnesses. This theorem is stronger than the usual Herbrand theorem for classical logic \([3]\), since we are stating it for any existential formula and not just for formulas with a single and existential quantifier.

**Theorem 16 (Herbrand Disjunction Extraction).** Let \( \exists \alpha A \) be any closed formula. Suppose \( \Gamma \vdash t : \exists \alpha A \) in IL + EM\( _1^- \) for a quasi closed term \( t \), and \( t \rightarrow^* t' \) with \( t' \) in normal form. Then \( \Gamma \vdash t' : \exists \alpha A \) and \( t' \) is an Herbrand normal form

\[
(m_0, u_0) \mid (m_1, u_1) \mid \ldots \mid (m_k, u_k)
\]

Moreover, \( \Gamma \vdash A[m_1/\alpha] \lor \cdots \lor A[m_k/\alpha] \).

**Proof.** By the Subject Reduction Theorem \([11]\), \( \Gamma \vdash t' : \exists \alpha A \). We proceed by induction on the structure of \( t' \). According to Theorem \([13]\) we can write \( t' \) as \( r u_1 \ldots u_n \). Note that since \( t' \) is quasi closed, \( r \) cannot be a variable \( x \); moreover, \( r \) cannot be a term \( \mathbb{B}^p \) or \( \mathbb{B}_0^\alpha B \), otherwise \( t' \) would not have type \( \exists \alpha A \), nor a term \( \mathbb{W}^{\exists \alpha P} \), otherwise \( t' \) would not be quasi closed. \( r \) also cannot be of the shape \( u [a] v \), otherwise \( \Gamma \vdash u [a] v : \exists \alpha Q \), for some negative propositional \( Q \), but from Proposition \([14]\) we know that this is not possible. By Theorem \([13]\) we are now left with only two possibilities.

1. \( r \) is obtained by an introduction rule. Then \( n = 0 \), otherwise there is a redex, and thus the only possibility is \( t' \rightarrow t = (n, u) \) which is an Herbrand Normal Form.
2. \( r = u [a] v \). Again \( n = 0 \), otherwise we could apply a permutation rule; then \( t' \rightarrow t = u [a] v \), and the thesis follows by applying the induction hypothesis on \( u \) and \( v \).

We have thus shown that \( t' \) is an Herbrand normal form

\[
(m_0, u_0) \mid (m_1, u_1) \mid \ldots \mid (m_k, u_k)
\]

Finally, we have that for each \( i \), \( \Gamma_i \vdash u_i : A[m_i/\alpha] \), for the very same \( \Gamma_1 \) that types \( (m_i, u_i) \) of type \( \exists \alpha A \) in \( t' \). Therefore, for each \( i \), \( \Gamma_i \vdash u_i^+ : A[m_1/\alpha] \lor \cdots \lor A[m_k/\alpha] \), where \( u_i^+ \) is of the form \( t_{i_1} \ldots t_{i_k}(u_i) \ldots \). We conclude that

\[
\Gamma \vdash u_0^+ \mid u_1^+ \mid \ldots \mid u_k^+ : A[m_1/\alpha] \lor \cdots \lor A[m_k/\alpha]
\]

\[\blacktriangleleft\]

### 4 Markov’s Principle in Arithmetic

The original statement of Markov’s principle refers to Arithmetic and can be formulated in the system of Heyting Arithmetic HA as

\[\neg \neg \exists \alpha P \rightarrow \exists \alpha P, \text{ for } P \text{ atomic}\]

By adapting IL + EM\( _1^- \) to Arithmetic, following \([2]\), we will now provide a new computational interpretation of Markov’s principle. Note first of all that propositional formulas are decidable in intuitionistic Arithmetic HA; therefore we will not need the rule EM\( _0^- \) and the parallelism operator. For the very same reason, we can expect the system HA + EM\( _1^- \) to be constructive and the proof to be similar to the one of Herbrand constructivity for IL + EM\( _1^- \). In this section indeed we will give such a syntactic proof. We could also have used the realizability interpretation for HA + EM\( _1^- \) introduced in \([2]\) (see \([11]\)).
4.1 The system $\text{HA} + \text{EM}_1$

We will now introduce the system $\text{HA} + \text{EM}_1$. We start by defining the language:

Definition 17 (Language of $\text{HA} + \text{EM}_1$). The language $\mathcal{L}$ of $\text{HA} + \text{EM}_1$ is defined as follows.

1. The terms of $\mathcal{L}$ are inductively defined as either variables $\alpha, \beta, \ldots$ or $0$ or $S(t)$ with $t \in \mathcal{L}$.

2. There is one symbol $P$ for every primitive recursive relation over $\mathbb{N}$; with $P^\bot$ we denote the symbol for the complement of the relation denoted by $P$. The atomic formulas of $\mathcal{L}$ are all the expressions of the form $P(t_1, \ldots, t_n)$ such that $t_1, \ldots, t_n$ are terms of $\mathcal{L}$ and $n$ is the arity of $P$. Atomic formulas will also be denoted as $P, Q, P_i, \ldots$ and $P(t_1, \ldots, t_n)^\bot := P^\bot(t_1, \ldots, t_n)$.

3. The formulas of $\mathcal{L}$ are built from atomic formulas of $\mathcal{L}$ by the connectives $\lor, \land, \to, \forall, \exists$ as usual, with quantifiers ranging over numeric variables $\alpha^\mathbb{N}, \beta^\mathbb{N}, \ldots$.

The system $\text{HA} + \text{EM}_1$ in fig. 5 extends the usual Curry-Howard correspondence for $\text{HA}$ with our rule $\text{EM}_1$ and is a restriction of the system introduced in [2]. The purely universal arithmetical axioms are introduced by means of Post rules, as in Prawitz [12].

As we anticipated, there is no need for a parallelism operator. Therefore $\text{EM}_1$ introduces a pure delimited exception mechanism, explained by the reduction rules in fig. 6: whenever we have a term $u \parallel a v$ and $P^\alpha_m a$ appears inside $u$, we can recursively check whether $P[m/\alpha]$ holds, and switch to the exceptional path if it doesn’t; alternatively, if it does hold we can remove the instance of the assumption. When there are no free assumptions relative to a left in $u$, we can forget about the exceptional path.

Similarly to the previous sections, we extend the characterization of the proof-term heads to take into account the new constructs.

Theorem 18 (Head of a Proof Term). Every proof term of $\text{HA} + \text{EM}_1$ is of the form:

$$\lambda z_1 \ldots \lambda z_n. ru_1 \ldots u_k$$

where

- $r$ is either a variable $x$, a constant $W^\alpha_0 P, W^\alpha_0 P, \tau$ or $R$, an excluded middle term $u \parallel v$, or a term corresponding to an introduction rule $\lambda x.t, \lambda \alpha.t, \langle t_1, t_2 \rangle, \iota_i(t), (m, t)$
- $u_1, \ldots, u_k$ are either lambda terms, first order terms, or one of the following expressions corresponding to elimination rules: $\pi_i, [x.w_1, y.w_2], (\alpha, x).t$

The new system proves exactly the same formulas that can be proven by making use of Markov’s principle in Heyting Arithmetic.

Theorem 19. For any formula $F$ in the language $\mathcal{L}$, $\text{HA} + \text{MP} \vdash F$ if and only if $\text{HA} + \text{EM}_1 \vdash F$

Proof. The proof is identical as the one in the previous section.

$\text{HA} + \text{EM}_1$ with the reduction rules in figure fig. 4 enjoys the Subject Reduction Theorem [2] [11].

Theorem 20 (Subject Reduction). If $\Gamma \vdash t : C$ and $t \mapsto u$, then $\Gamma \vdash u : C$. 

Grammar of Untyped Terms

\[ t, u, v ::= x \mid tu \mid tm \mid \lambda x u \mid \lambda u \mid \langle t, u \rangle \mid u\pi_0 \mid u\pi_1 \mid t_1(u) \mid t[x,u,y,v] \mid (m,t) \mid tl[\alpha,x].u \]

where \( m \) ranges over terms of \( \mathcal{L} \), \( x \) over variables of the lambda calculus and \( a \) over \( \text{EM}_1 \) hypothesis variables. Moreover, in terms of the form \( u \parallel_v \) there is a \( P \) such that all the free occurrences of \( a \) in \( u \) are of the form \( H^\alpha_0 P \) and those in \( v \) are of the form \( W^\alpha_0 P \).

Contexts

With \( \Gamma \) we denote contexts of the form \( 1, \ldots, e_n : A_n \), where \( e_i \) is either a proof-term variable \( x, y, z \ldots \) or a \( \text{EM}_1 \) hypothesis variable \( a, b, \ldots \)

Axioms

\[ \Gamma, x : A \vdash x : A \quad \Gamma, a : \forall a^\alpha P \vdash H^\alpha_0 P : \forall a^\alpha P \quad \Gamma, a : \exists a^\beta^\alpha P \vdash W^\alpha_0 P : \exists a^\beta^\alpha P \]

Conjunction

\[ \frac{\Gamma \vdash u : A \quad \Gamma \vdash v : B}{\Gamma \vdash \langle u,v \rangle : A \land B} \quad \frac{\Gamma \vdash u : A \quad \Gamma \vdash v : B}{\Gamma \vdash \lambda x u : A \lor B} \]

Implication

\[ \frac{\Gamma \vdash t : A \rightarrow B}{\Gamma \vdash tu : B} \quad \frac{\Gamma \vdash u : A}{\Gamma \vdash u} \]

Disjunction Introduction

\[ \frac{\Gamma \vdash u : A \quad \Gamma \vdash v : B}{\Gamma \vdash u \lor v : A \lor B} \quad \frac{\Gamma \vdash u : A}{\Gamma \vdash u \lor v : A \lor B} \]

Disjunction Elimination

\[ \frac{\Gamma \vdash u : A \lor B \quad \Gamma, x : A \vdash w_1 : C \quad \Gamma, y : B \vdash w_2 : C}{\Gamma \vdash u[x,w_1,y,w_2] : C} \]

Universal Quantification

\[ \frac{\Gamma \vdash u : \forall a^\alpha A}{\Gamma \vdash u[m,\alpha] : A[m/\alpha]} \quad \frac{\Gamma \vdash u : A}{\Gamma \vdash \lambda a u : \forall a^\alpha A} \]

where \( m \) is any term of the language \( \mathcal{L} \) and \( \alpha \) does not occur free in any formula \( B \) occurring in \( \Gamma \).

Existential Quantification

\[ \frac{\Gamma \vdash u : A[\alpha/\alpha] \quad \Gamma \vdash u : \exists a^\beta^\alpha A}{\Gamma \vdash u \parallel_v : \exists a^\beta^\alpha A} \quad \frac{\Gamma \vdash u : A \quad \Gamma, x : A \vdash t : C}{\Gamma \vdash u[\alpha,x].t \parallel_v : C} \]

where \( \alpha \) is not free in \( C \) nor in any formula \( B \) occurring in \( \Gamma \).

Induction

\[ \frac{\Gamma \vdash u : A(0) \quad \Gamma \vdash v : \forall a^\alpha A(\alpha) \rightarrow A(S(\alpha))}{\Gamma \vdash Ruvm : A[m/\alpha]} \]

Post Rules

\[ \frac{\Gamma \vdash u : A \quad \Gamma \vdash u_1 : A_1 \quad \Gamma \vdash u_2 : A_2 \quad \cdots \quad \Gamma \vdash u_n : A_n}{\Gamma \vdash u : A} \]

where \( A_1, A_2, \ldots, A_n, A \) are atomic formulas of \( \text{HA} \) and the rule is a Post rule for equality, for a Peano axiom or for a classical propositional tautology or for booleans and if \( n > 0 \), \( u = ru_1 \ldots u_n \), otherwise \( u = \text{True} \).

\[ \frac{\Gamma, a : \forall a P \vdash u : \exists \beta Q \quad \Gamma, a : \exists a \rightarrow P \vdash v : \exists \beta Q}{\Gamma \vdash u \parallel_v : \forall \beta Q \quad (P \text{ atomic, } Q \text{ negative propositional})}^{\text{EM}_1} \]

\[ \text{Figure 5 Term Assignment Rules for } \text{HA + EM}_1 \]

No term of \( \text{HA + EM}_1 \) gives rise to an infinite reduction sequence \([\square]\).
Reduction Rules for HA

\[ (\lambda x. u)t \mapsto u[t/x] \quad (\lambda x. u)m \mapsto u[m/\alpha] \]

\[ (u_0, u_1)\pi_i \mapsto u_i, \text{ for } i=0,1 \]

\[ u_i[u_1, t_1, x_2, t_2] \mapsto t_i[u/x_i], \text{ for } i=0,1 \]

\[ (m, u)[(\alpha, x).v] \mapsto v[m/\alpha][u/x], \text{ for each term } m \text{ of } L \]

\[ Ru(v \pi_0) \mapsto v \quad Ru(v \pi_1) \mapsto \pi \]

Reduction Rules for \( EM_1^- \)

\[ u \parallel_a v \mapsto u, \text{ if } a \text{ does not occur free in } u \]

\[ u \parallel_a v \mapsto v[a := n], \text{ if } \lambda^\alpha_P n \text{ occurs in } u \text{ and } P[n/\alpha] = \text{False} \]

\[ (\lambda^\alpha_P n) \mapsto \text{True} \text{ if } P[n/\alpha] \text{ is closed and } P[n/\alpha] \equiv \text{True} \]

\[ \text{Figure 6 Reduction Rules for HA + EM}_1^{-} \]

\[ \textbf{Theorem 21 (Strong Normalization). Every term typable in } HA + EM_1^{-} \text{ is strongly normalizing.} \]

4.2 HA + EM_1^- is Constructive

We can now proceed to prove the constructivity of the system, that is the disjunction and existential properties. We will do this again by inspecting the normal forms of the proof terms; the first thing to do is adapting Proposition 14 to HA + EM_1^-.

\[ \textbf{Proposition 22 (Normal Form Property). Let } P, P_1, \ldots P_n \text{ be negative propositional formulas, } A_1, \ldots A_m \text{ simply universal formulas. Suppose that} \]

\[ \Gamma = z_1 : P_1, \ldots z_n : P_n, a_1 : \forall \alpha_1 A_1, \ldots a_m : \forall \alpha_m A_m \]

and \( \Gamma \vdash t \vdash \exists \alpha P \text{ or } \Gamma \vdash t : P, \text{ with } t \text{ in normal form and having all its free variables among } \]

\[ z_1, \ldots z_n, a_1, \ldots a_m. \text{ Then:} \]

1. Every occurrence in } t \text{ of every term } \lambda^\alpha_P A_i \text{ is of the active form } \lambda^\alpha_P A_i. m, \text{ where } m \text{ is a term of } L \]

2. } t \text{ cannot be of the form } u \parallel_a v. \]

\[ \textbf{Proof. The proof is identical to the proof of Proposition 14. We just need to consider the following additional cases:} \]

- } t = rt_1 t_2 \ldots t_n. \text{ Then } \Gamma \vdash t_i : Q_i \text{ for some atomic } Q_i \text{ and for } i = 1 \ldots n; 1. \text{ holds by applying the inductive hypothesis to the } t_i, \text{ while 2. is obviously verified.} \]

- } t = R t_1 \ldots t_n. \text{ This case is not possible, otherwise since } t_3 \text{ is a numeral and thus } t \text{ would not be in normal form.} \]
Thanks to this, we can now state the main theorem. The proof of the existential property is the same as the one for Theorem 16: we just need to observe that since we don’t have a parallelism operator in HA + EM₁⁻, every Herbrand disjunction will consist of a single term. The disjunction property will follow similarly.

**Theorem 23** (Constructivity of HA + EM₁⁻).

- If HA + EM₁⁻ ⊢ t : ∃αA, then there exists a term t’ = (n, u) such that t ⊢ t’ and HA + EM₁⁻ ⊢ u : A[n/α].
- If HA + EM₁⁻ ⊢ t : A ∨ B, then there exists a term t’ such that t ⊢ t’ and either t’ = u₀(u) and HA + EM₁⁻ ⊢ u : A, or t’ = u₁(u) and HA + EM₁⁻ ⊢ u : B.

**Proof.** For both cases, we start by considering a term t’ such that t ⊢ t’ and t’ is in normal form. By the Subject Reduction Theorem 11 we have that HA + EM₁⁻ ⊢ t’ : ∃αA (resp. HA + EM₁⁻ ⊢ t’ : A ∨ B). By Theorem 13 we can write t’ as rt₁...tn. Since t’ is closed, r cannot be a variable x or a term η₀P or η₁P; moreover it cannot be r, otherwise the type of t’ would have to be atomic, and it cannot be R, otherwise the term would not be in normal form. r also cannot have been obtained by EM₁⁻, otherwise HA + EM₁⁻ ⊢ r : ∃αP, for P atomic and r = t₁[α]t₂; but this is not possible due to Proposition 22. Therefore, r must be obtained by an introduction rule. We distinguish now the two cases:

- HA + EM₁⁻ ⊢ t’ : ∃αB. Since the term is in normal form, n has to be 0, that is t’ = r and r = (n, u); hence also HA + EM₁⁻ ⊢ u : A(n).
- HA + EM₁⁻ ⊢ t’ : A ∨ B. Then either t’ = u₀(u), and so HA + EM₁⁻ ⊢ u : A, or t’ = u₁(u), and so HA + EM₁⁻ ⊢ u : B.

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