Symmetry group analysis of relativistic heat conducting fluids

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Abstract. The Lie symmetry group for 1+1 dimensional relativistic heat-conducting fluid is calculated for two different theories, Eckart and Israel-Stewart and a comparison between the group-invariant solutions has been made. Both fluids were founded to be physical acceptable in the sense that during the evolution of the fluid there are three velocity solutions that are decreasing exponentially for particular choices of the initial conditions.

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1. Introduction

Relativistic fluid dynamics provide a simple but intuitive description of many physical systems. The hot and compressed nuclear matter behaves like a compressible fluid and fluid dynamical effects are observed in high energy heavy ion reactions [1]. Supernova or neutron star may be another situation where the relativistic heat-conducting fluid gives a satisfactory description.

The standard theory of the relativistic dissipative fluid proposed by Eckart [2] and a similar one by Landau and Lifshitz [3] was analysed by Hiscock and Lindblom [4, 5, 6, 7] and it was found to be unstable, acausal and ill posed in the linear regime near equilibrium. A more complicated extended hydrodynamic theory has been proposed by Israel and Stewart [8, 9, 10] which is free of the most part of the troubles of the standard theory.

The Hiscock and Lindblom studies were dedicated to causality, stability, hyperbolicity and to the connection between them. In [7] the solution discussion was restricted to spatially homogeneous fluid. This paper will improve the dimensional restriction considering the evolution equations for the plane symmetric motion (1+1 dimensional) of the relativistic heat-conducting fluid and propose a complementary analysis dedicated to group-invariant solutions study.

Symmetry analysis is a systematic and accurate way to obtain solutions of differential equations. The Lie symmetry group method is well known but the major obstacle in the application of this method to partially differential equations is the large number of tedious calculations usually involved.

We begin our analysis by the calculation of the Lie symmetry group for both Eckart and Israel-Stewart fluids using, for the equation of state, the high temperature limit of the ideal gas, where $p = \rho/3 = nkT$. This constraint allows us to eliminate from the fluid equations two dependent variables, in particular, we substitute the temperature $T$ and the pressure $p$ in favour of the energy density $\rho$ and the number density $n$. Next, we make a systematic classification of the solutions constructing the optimal system of transformations. The investigation of the solutions is completed by solving the reduced system of equations; we present both analytical and numerical solutions.

Section 2 will display the system of equations of the relativistic dissipative fluid in its general form eqs. (1) and in the particular case of heat-conducting theory eqs. (4). The application of the Lie symmetry group method to differential equations and the corresponding Lie algebra of the fluid equations (4) are presented in Section 3. Section 4 is devoted to integrability conditions and invariants, Section 5 to the classification of the solutions and Section 6 to group invariant solutions analysis. Possible applications of the group-invariant solutions to high energy collisions, miscellaneous comments and final conclusions are discussed in Section 7.
2. Dissipative relativistic fluid mechanics

In the particle frame, used by Eckart, the energy tensor $T^{\alpha\beta}$ and the particle flux vector $N^\alpha$ take the form [8]

$$
T^{\alpha\beta} = \rho u^\alpha u^\beta + (p + \sigma) (g^{\alpha\beta} + u^\alpha u^\beta) + q^\alpha u^\beta + q^\beta u^\alpha + \sigma^{\alpha\beta}
$$

$$
N^\alpha = n u^\alpha
$$

where $u^\alpha$ is a unit time-like vector field which may be thought of as the four-velocity of the fluid, $g^{\alpha\beta}$ is the Minkowski metric diag(-1,1,1,1), $\rho$ the energy density of the fluid, $p$ the thermodynamical pressure, $n$ the number density, $q^\alpha$ is the heat flow, $\sigma$ the bulk stress and $\sigma^{\alpha\beta}$ the shear stress.

The heat conducting case is obtained by setting $\sigma = 0$ and $\sigma^{\alpha\beta} = 0$. In this particular condition the heat flow obeys the following phenomenological law [8]

$$
q^\alpha = -\kappa T (g^{\alpha\beta} + u^\alpha u^\beta) \left( T^{-1} \partial_\beta T + u^\mu \partial_\mu u_\beta + \tilde{\beta}_1 u^\mu \partial_\mu q_\beta \right)
$$

where beside the usual thermal conductivity $\kappa$ and temperature $T$ we have a new phenomenological coefficient $\tilde{\beta}_1$ [8]. Setting $\tilde{\beta}_1 = 0$ we obtain the standard Eckart heat conducting fluid theory.

The conservation laws:

$$
\partial_\alpha T^{\alpha\beta} = 0, \quad \partial_\alpha N^\alpha = 0
$$

and the equation (2) form the complete system of equations of the general theory of relativistic dissipative fluids.

If we write the four-velocity in the form $u^\alpha = (\cosh \Psi, \sinh \Psi, 0, 0)$, the heat flow vector, which is orthogonal to $u^\alpha$, will be $q^\alpha = q (\sinh \Psi, \cosh \Psi, 0, 0)$ and the system of equations of the fluid becomes:

$$
\begin{align*}
\sinh \Psi \partial_x n - \cosh \Psi \partial_t n + n \cosh \Psi \partial_x \Psi - n \sinh \Psi \partial_t \Psi &= 0 \\
\cosh \Psi \partial_t \rho - \sinh \Psi \partial_x \rho + \sinh \Psi \partial_t q - \cosh \Psi \partial_x q + [(p + \rho) \sinh \Psi + 2q \cosh \Psi] \partial_t \Psi - [(p + \rho) \cosh \Psi + 2q \sinh \Psi] \partial_x \Psi &= 0 \\
\cosh \Psi \partial_x \rho - \sinh \Psi \partial_t \rho + \sinh \Psi \partial_x q - \cosh \Psi \partial_t q - [(p + \rho) \cosh \Psi + 2q \sinh \Psi] \partial_t \rho + [(p + \rho) \sinh \Psi + 2q \cosh \Psi] \partial_x \rho &= 0 \\
T^{-1} (\cosh \Psi \partial_x T - \sinh \Psi \partial_t T) + \tilde{\beta}_1 (\sinh \Psi \partial_x q - \cosh \Psi \partial_t q) + \sinh \Psi \partial_x \Psi - \cosh \Psi \partial_t \Psi + \frac{\kappa}{\kappa T} &= 0
\end{align*}
$$

We will supplement the eqs. (4) by an equation of state (EOS) for the fluid and a thermodynamic expression for $\tilde{\beta}_1$. Choosing the high temperature limit of an ideal gas, the equation of state is

$$
p = \rho/3 = nkT
$$

Finally, we take the thermal conductivity to be constant and the second-order thermodynamic coefficient $\tilde{\beta}_1$ to be given by $\tilde{\beta}_1 = 5\lambda/4p$; the parameter $\lambda$ will be 1 for the Israel-Stewart theory and 0 for the Eckart theory.

The equation (5) allows us to eliminate $T$ and $p$ in favour of $\rho$ and $n$, obtaining finally a closed set of evolution equations for four variables ($\Psi, n, \rho, q$).
3. Symmetry group of transformations and its Lie algebra

The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transforms solutions of the system to other solutions. Let $\mathcal{S}$ be a system of differential equations. A symmetry-group of the system $\mathcal{S}$ is a local group of transformations $\mathcal{G}$ acting on an open subset $\mathcal{M}$ of the space of independent and dependent variables of the system with the property that whenever $u=f(x)$ is a solution of $\mathcal{S}$, and whenever $g \cdot f$ is defined for $g \in \mathcal{G}$, then $u = g \cdot f(x)$ is also a solution of the system.

The symmetry group infinitesimal generator is defined by:

$$\tilde{\mathcal{V}} = \tau \partial_t + \xi \partial_x + \Phi \partial_y + \Sigma \partial_n + \Gamma \partial_p + \Omega \partial_q$$

(6)

and the first order prolongation of $\tilde{\mathcal{V}}$ is:

$$pr^{(1)}\tilde{\mathcal{V}} = \xi \partial_x + \tau \partial_t + \Phi \partial_y + \Sigma \partial_n + \Gamma \partial_p + \Omega \partial_q + \Phi^x \partial_{y_x} + \Phi^t \partial_{y_t} + \Sigma^x \partial_{n_x} + \Sigma^t \partial_{n_t} + \Gamma^x \partial_{p_x} + \Gamma^t \partial_{p_t} + \Omega^x \partial_{Q_x} + \Omega^t \partial_{Q_t}$$

(7)

where, for example,

$$\Phi^x = D_x(\Phi - \xi \Psi_x - \tau \Psi_t) + \xi \Psi_{xx} + \tau \Psi_{xt}$$

(8)

and $D_x \Phi = \Phi_x + \Phi \Psi_x + \Phi_n n_x + \Phi \rho \rho_x + \Phi_q q_x$ is the total derivative and $\Phi_x = \partial_x \Phi$, etc.

Suppose $\Delta_\nu(x,u^{(n)}) = 0$, $\nu = 1,...,l$, is a system of differential equations, where $u^{(n)} = (\Psi, n, \rho, q, \Psi_x, \Psi_t, ..., q_{tt})$. If $\mathcal{G}$ is a local group of transformations acting on $\mathcal{M}$ and $pr^{(n)}\tilde{\mathcal{V}} \left[ \Delta_\nu(x,u^{(n)}) \right] = 0$, $\nu = 1,...,l$, whenever $\Delta(x,u^{(n)}) = 0$, for every infinitesimal generator $\tilde{\mathcal{V}}$ of $\mathcal{G}$, then $\mathcal{G}$ is a symmetry group of the system.

The standard procedure\footnote{A good description can be found in \cite{11}} is based on finding the infinitesimal coefficient functions $\xi, \tau, \Phi, \Sigma, \Gamma$ and $\Omega$. Substituting the general formulae for $\Phi^x, \Sigma^x, \text{etc.}$ and equating the coefficients of various monomials in the first and second order partial derivatives of $\Psi, n, \rho$ and $q$, we find the defining equations. We wish to determine all possible coefficient functions $\xi, \tau, \Phi, \Sigma, \Gamma$ and $\Omega$ by solving the defining equations system so that the corresponding one-parameter group $\exp(\varepsilon \tilde{\mathcal{V}})$ will be the symmetry group of the equations\footnote{A good description can be found in \cite{11}}. These solutions are:

**Eckart**: $\tau = c_1 + tc_4$, $\xi = c_2 + xc_4$, $\Phi = 0$, $\Sigma = -nc_4$, $\Gamma = \rho c_3$, $\Omega = qc_3$

**Israel – Stewart**: $\tau = c_1 + tc_4$, $\xi = c_2 + xc_4$, $\Phi = 0$, $\Sigma = 0$, $\Gamma = \rho c_3$, $\Omega = qc_3$

(9)

where $c_i$ are constants. The basis of the corresponding Lie algebra are:

**Eckart**: $V_1 = \partial_t$, $V_2 = \partial_x$, $V_3 = \rho \partial_p + q \partial_q$, $V_4 = t \partial_t + x \partial_x - n \partial_n$

**Israel – Stewart**: $V_1 = \partial_t$, $V_2 = \partial_x$, $V_3 = \rho \partial_p + q \partial_q$

(10)
4. Solvable group and invariants

Because we have the Lie algebra of the equations (4) we want to know if the general solution of the system of differential equations can be found by quadratures. This thing is possible if the Lie group is solvable. The requirement for solvability is equivalent to the existence of a basis \( \{ V_1, \ldots, V_r \} \) of Lie algebra \( g \) such that

\[
[V_i, V_j] = \sum_{k=1}^{j-1} c_{ij}^k V_k
\]

whenever \( i < j \). Looking at the commutator table of the Lie algebra we will see that the requirement of solvability is fulfilled for both Eckart and Israel-Stewart theories.

**Table 1.** Commutator table for the Eckart and Israel-Stewart algebra

|    | \( V_1 \) | \( V_2 \) | \( V_3 \) | \( V_4 \) |
|----|---------|---------|---------|---------|
| \( V_1 \) | 0       | 0       | \( V_1 \) | 0       |
| \( V_2 \) | 0       | 0       | \( V_2 \) | 0       |
| \( V_3 \) | -\( V_1 \) | -\( V_2 \) | 0       | 0       |
| \( V_4 \) | 0       | 0       | 0       | 0       |

We use the method of characteristics to compute the invariants of the Lie algebra hoping that the reduced system, which can be obtained using the invariants of the group, will help us to solve the system of equations (4). An \( n \)-th order differential invariant of a group \( G \) is a smooth function depending on the independent and dependent variables and their derivatives, invariant on the action of the corresponding \( n \)-th prolongation of \( G \) [11]. Suppose that we have the following generator:

\[
V_i = \tau_i \partial_t + \xi_i \partial_x + \Phi_i \partial_y + \Sigma_i \partial_n + \Gamma_i \partial_\rho + \Omega_i \partial_q
\]

A local invariant \( \zeta \) of \( V_i \) is a solution of the linear, homogeneous first order partial differential equation:

\[
V_i(\zeta) = \tau_i \partial_t \zeta + \xi_i \partial_x \zeta + \Phi_i \partial_y \zeta + \Sigma_i \partial_n \zeta + \Gamma_i \partial_\rho \zeta + \Omega_i \partial_q \zeta = 0
\]

The classical theory of such equations shows that the general solution of equation (13) can be found by integrating the corresponding characteristic system of differential equations, which is

\[
\frac{dt}{\tau_i} = \frac{dx}{\xi_i} = \frac{d\Psi}{\Phi_i} = \frac{dn}{\Sigma_i} = \frac{d\rho}{\Gamma_i} = \frac{dq}{\Omega_i}
\]

Doing this integration we get, in this case, five invariants; we now re-express the next generator of Lie algebra in terms of these five invariants and then we perform another integration. We continue this calculation until we re-express and integrate the last generator; at this point we obtain a set of invariants that represent the system of independent invariants of this group. The system of invariants can be used to reduce the order of the original equations - constructing the reduced order system of equations. Doing this one can hope to find simple equations that can be integrated (for example [11]).
5. Classification of group-invariant solutions

A solution of the system of partial differential equations is said to be $\mathcal{G}$-invariant if it is unchanged by all the group transformations in $\mathcal{G}$. In general, to each s-parameter subgroup $\mathcal{H}$ of the full symmetry group $\mathcal{G}$ of a system of differential equations, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions of the system. We need an effective systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other solution can be derived. Since the elements $g \in \mathcal{G}$ not in the subgroup $\mathcal{H}$ will transform an $\mathcal{H}$-invariant solution to some other group-invariant solution, only those solutions not so related need to be listed in our optimal system. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list (conjugacy map: $h \rightarrow ghg^{-1}$). Let $\mathcal{G}$ be a Lie group with Lie algebra $\mathfrak{g}$; for each $v \in \mathfrak{g}$, the adjoint vector $ad\ v$ at $w \in \mathfrak{g}$ is $ad\ v|_w = [w, v] = -[v, w]$. If $v \in \mathfrak{g}$ generates the one-parameter subgroup $\mathcal{H} = \{\exp(\varepsilon v) : \varepsilon \in \mathbb{R}\}$, then $Ad\ g(v)$ will generate the conjugate one-parameter subgroup $g\mathcal{H}g^{-1}$. Now we can construct the adjoint representation $Ad\ \mathcal{G}$ of the Lie group by summing the Lie series

$$Ad(\exp(\varepsilon v))w = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (ad\ v)^n(w) = w - \varepsilon[v, w] + \frac{\varepsilon^2}{2}[v, [v, w]] - \ldots$$

obtaining the adjoint table.

| $Ad$ | $V_1$ | $V_2$ | $V_3 - \varepsilon V_1$ | $V_4$ |
|------|-------|-------|------------------------|-------|
| $V_1$ | $V_1$ | $V_2$ | $V_3 - \varepsilon V_1$ | $V_4$ |
| $V_2$ | $V_1$ | $V_2$ | $V_3 - \varepsilon V_2$ | $V_4$ |
| $V_3$ | $e^\varepsilon V_1$ | $e^\varepsilon V_2$ | $V_3$ | $V_4$ |
| $V_4$ | $V_1$ | $V_2$ | $V_3$ | $V_4$ |

| $Ad$ | $V_1$ | $V_2$ | $V_3 - \varepsilon V_1$ |
|------|-------|-------|------------------------|
| $V_1$ | $V_1$ | $V_2$ | $V_3 - \varepsilon V_1$ |
| $V_2$ | $V_1$ | $V_2$ | $V_3 - \varepsilon V_2$ |
| $V_3$ | $e^\varepsilon V_1$ | $e^\varepsilon V_2$ |

Finding the optimal system is a well known standard method [11]; we have found an optimal system of one-dimensional ($a \in \mathbb{R}$) sub-algebras to be that spanned by:

**Eckart**

1) $\partial_x$
2) $\partial_t$
3) $x\partial_x + t\partial_t - n\partial_n$
4) $\partial_t + a\partial_x$
5) $x\partial_x + t\partial_t - n\partial_n + a\rho\partial_{\rho} + aq\partial_{q}$
6) $\rho\partial_{\rho} + q\partial_{q} + a\partial_x + \partial_t$
7) $\rho\partial_{\rho} + q\partial_{q} + \partial_x$
8) $\rho\partial_{\rho} + q\partial_{q} + \partial_t$
9) $\rho\partial_{\rho} + q\partial_{q}$

**Israel – Stewart**

$\partial_x$
$\partial_t$
$x\partial_x + t\partial_t - n\partial_n$
$\partial_t + a\partial_x$

(16)
6. Group-invariant solutions study

Each symmetry of the optimal system of transformations \([\mathcal{G}]\) will be treated separately.

1) The fluid is spatially homogeneous and because the reduced system is very simple the fluid equations can be immediately integrated analytically. The Figure 1 shows that, for any initial state, the Eckart theory is unstable: as \(t \to \infty\) then \(v \to 1 \Rightarrow T \to \infty, n \to 0, \rho = 3p \to 3E_0\) and \(q \to -2E_0\). The same thing happens, for the Israel-Stewart theory, only if the initial velocity is higher than the critical value 0.76 but it can be stable for initial values \(v < 0.76\), where \(v = \tanh \Psi\). This spatially homogeneous case was also studied by Hiscock and Lindblom [7] and they found the critical value around \(v_c = 0.51\).

![Figure 1. Israel-Stewart and Eckart velocity solution, where \(t\) is \(\frac{4kN_0}{\chi}t\).](image)

2) Also in this stationary case, the reduced system is very easy to integrate and the space dependence of the velocity can be observed in the Figure 2. For the Eckart fluid, if the initial velocity is higher then 0.77 then as \(x \to \infty\) we have \(v \to 1 \Rightarrow T \to \infty, n \to 0, \rho = 3p \to 3E_0, q \to -2E_0\). If we correlate this stationary behaviour with the previous result, we can comment that the stationary solution will increase in time for any initial Eckart velocity conditions; the Stewart-Israel fluid has the chance to decrease when \(t \to \infty, x \to \infty\) for a initial velocity smaller that 0.76.

![Figure 2. Israel-Stewart and Eckart velocity solution, where \(x\) is \(\frac{4kN_0}{\chi}x\).](image)

3) The invariants of the characteristic equation \(dx/x = dt/t = -dn/n\) are \(\alpha = n \cdot t, y = x/t\) and for the Eckart fluid the reduced system of equations take
the following form:

\[ \alpha^{-1} \alpha_y = -\frac{d}{dy} [\ln (\sinh \Psi + y \cosh \Psi)] \Rightarrow \alpha = N_0 (\sinh \Psi + y \cosh \Psi)^{-1} \]

\[ q = \frac{\rho_y}{6\Psi_y} \frac{(1 + y^2) \cosh(2\Psi) + 2(-1 + y^2 + y \sinh(2\Psi))}{1 - y^2} \]

\[ 3q_y + 4\rho \Psi_y + \rho_y \frac{2y \cosh(2\Psi) + (1 + y^2) \sinh(2\Psi)}{1 - y^2} = 0 \tag{17} \]

\[ \rho_y - \frac{3kN_0}{\chi} \frac{q}{\sinh \Psi + y \cosh \Psi} + 2\rho \frac{1 + [(1 + y^2) \cosh(2\Psi) + 2y \sinh(2\Psi)]}{(1 + y^2) \sinh(2\Psi) + 2y \cosh(2\Psi)} \Psi_y = 0 \]

where \( \Psi_y = \frac{d\Psi}{dy} \), etc.. The system of equations can be decoupled and the behaviour of the velocity numerical solution can be observed in the first picture of the Figure 3.

\[ T_0 = \sqrt{\frac{1}{k}} \]

\[ \frac{\chi}{kN_0} = \frac{2}{3} \rho \]

\[ v = \tanh \left[ \ln \left( \sqrt{\chi C_1/(kN_0)} \tanh \left( \sqrt{\chi C_1/(kN_0)}(y - C_2) \right) \right) \right] \]

\[ n = N_0 \sqrt{(1 + v)/(1 - v)}, \quad \rho_y = 0, \quad q = \frac{2\chi}{3kN_0} \rho \frac{v_y}{(1 + \nu)^2} \tag{18} \]

The velocity solution is plotted in the second picture of the Figure 3. This \( t + x \) dependence is typical for a wave type solution but as \( y = t + x \to \infty \) we arrive again to wrong physical situation: \( v \to 1 \Rightarrow n \to \infty, T \to 0 \). The choice \( a = -1 \) does not affect the physics, it can change the \( x \) axes direction or it can scale the space coordinate.

For the Israel-Stewart fluid the reduced system is much more complicated and we didn’t succeed to solve the equations.

4) For this optimal transformation the invariant is \( y = t - ax \); if we choose \( a = -1 \) the reduced system can be easily decoupled and for the Eckart fluid the solutions are:

\[ v = \tanh \left[ \ln \left( \sqrt{\chi C_1/(kN_0)} \tanh \left( \sqrt{\chi C_1/(kN_0)}(y - C_2) \right) \right) \right] \]

\[ n = N_0 \sqrt{(1 + v)/(1 - v)}, \quad \rho_y = 0, \quad q = \frac{2\chi}{3kN_0} \rho \frac{v_y}{(1 + \nu)^2} \tag{18} \]

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The Israel-Stewart reduced system is also integrable but we found complex velocity solution.
5) The invariants are \( \beta = n \cdot x, y = x/t, w = \rho \cdot t^a, \theta = q/\rho \). After some calculations, the reduced system of equations becomes:

\[
\beta = N_0 y / (\sinh \Psi + y \cosh \Psi)
\]

\[
w^{-1} w_y = -\left\{ \left[ \ln \left( \sinh \Psi + y \cosh \Psi \right) \cosh \Psi + y \sinh \Psi \right] \right\}_y + \frac{(a - 2) \sinh \Psi}{\cosh \Psi + y \sinh \Psi} + \frac{a(1 + y) \sinh 2\Psi}{\cosh \Psi + y \sinh \Psi} + 2 \frac{y (y^2 - 1) \Psi_y - 3a}{(6(y^2 - 1) \Psi_y - 3a)}
\]

\[
[1 + \frac{3kN_0 \chi}{(\sinh \Psi + y \cosh \Psi)(\cosh \Psi + y \sinh \Psi)(6(y^2 - 1) \Psi_y - 3a)}]^{-1}
\]

\[
\theta = \frac{w^{-1} w_y (2 y \cosh 2 \Psi + (1 + y^2) \sinh 2 \Psi + 2(y^2 - 1) + a ((1 + y) \cosh 2 \Psi + 2y))}{(6(y^2 - 1) \Psi_y - 3a)}
\]

\[
\frac{w_y}{w} (y \sinh 2 \Psi + \cosh 2 \Psi - 2 - 3y \theta) - 3y \theta_y + a(\sinh 2 \Psi - 3 \theta) - 2(2y + 3 \theta) \Psi_y = 0
\]

The system of equations can be decoupled and solved numerically. The behaviour of the numerical velocity solution is the same as that was founded for the third optimal transformation; of course, more precise numerical investigations are also needed.

6) In this case the invariants are \( y = x \pm at, \sigma = \rho \exp(-t), \alpha = q/\rho \) and the reduced system is:

\[
n_y (\sinh \Psi - a \cosh \Psi) + n (\cosh \Psi - a \sinh \Psi) \Psi_y = 0
\]

\[
\sigma^{-1} \sigma_y (a \cosh \Psi - \sinh \Psi + a \alpha \sinh \Psi - \alpha \cosh \Psi) + \sigma_y (a \sinh \Psi - \cosh \Psi)
\]

\[
+ \cosh \Psi + \alpha \sinh \Psi + [4/3 (a \sinh \Psi - \cosh \Psi) + 2\alpha (a \cosh \Psi - \sinh \Psi)] \Psi_y = 0
\]

\[
\sigma^{-1} \sigma_y (1/3 \cosh \Psi - 1/3a \sinh \Psi + \alpha \sinh \Psi - a \alpha \cosh \Psi)
\]

\[
+ \sigma_y (\sinh \Psi - a \cosh \Psi) - 1/3 \sinh \Psi - \alpha \cosh \Psi
\]

\[
+ [4/3 (\sinh \Psi - a \cosh \Psi) + 2\alpha (\cosh \Psi - a \sinh \Psi)] \Psi_y = 0
\]

\[
\cosh \Psi (\rho^{-1} \rho \sigma^{-1} \alpha - n^{-1} n_y) - \sinh \Psi [\rho^{-1} \rho (a \sigma^{-1} \alpha + 1) - n^{-1} n_y]
\]

\[
+ (\sinh \Psi - a \cosh \Psi) \Psi_y + 3\alpha k / \kappa = 0
\]

We can decouple the equations and in the end we obtain the following equation:

\[
\Psi_{yy} + k_1 \Psi^2_y + k_2 f (\Psi) \Psi_y + k_3 g (\Psi) = 0
\]

(21)

where \( k_{1,2,3} \) are constants and \( f (\Psi), g (\Psi) \) are known functions of \( \Psi \). The equation (21) is integrable [2]; because the integration algorithm is very long we will present only the final form of the equation that can be integrated:

\[
d\Psi(y)/dy = -t \xi^t (t) \exp (C_2 \Psi)
\]

(22)

where \( \xi = C_4 \exp [(2 - C_2) \Psi] \) and \( C_4 \) are constants; \( t \), which is a function of \( \Psi \) via \( \xi \), has to be found from the equation \( t^2 \xi''(t) + C_4 \xi(t) = 0 \). Because the analytical solution formula is very long and complicated we don’t write it here, we prefer to mention that it has the same exponentially increasing trouble founded before.

For the 7),8) and 9) transformation we can’t reduce the fluid system of equations.
7. Summary and outlook

The Lie symmetry group method was applied to two different relativistic heat-conducting fluid theories, the Eckart theory and the Israel-Stewart theory, and a comparison between them has been made. We may summarise the results as follows:

- applying the Lie symmetry group method to both fluids, we found that the Lie algebra of the Eckart theory has an additional transformation, namely the space, time and particle density dilatation ($V_4$).
- a systematic classification of the solutions is done by constructing the optimal system of transformations.
- solving all the Eckart’s reduced systems of equations, we found that during the evolution of the fluid, there are, beside the exponentially increasing velocities, also solutions with good physical behaviour; we mention that in the 1 dimensional approximation, for the Eckart theory, there are only exponentially increasing solutions.
- due to a different (more complicate) phenomenological law, the optimal transformation system of the Israel-Stewart theory is trivial and as a consequence the reduced systems of equations are not very much simplified but the transformation invariants; anyway, we found good physical behaviour for the spatially homogeneous and stationary solutions.

Due to the fact that each fluid has a different phenomenological law, the corresponding symmetry groups is different. We considered here only two values for the parameter $\lambda$ that leads us to some well known fluids, but performing the same kind of analysis for other $\lambda$ values, we could shed some light on the domain in which the relativistic heat-conducting fluid theory is well behaved. Taking into account the bulk stress and the shear stress, the phenomenological law becomes more general but then the fluid equations will be more difficult to solve and in spite of the tremendous work needed, it is possible to obtain a very simple symmetry group that does not help very much on the reduced system integration. It is important to point out that we are looking to find the simplest well behaved fluid theory in order to be solved much easier. Anyway, it is an open subject that needs some attention.

Because the 1+1 dimensional approximation is appropriate to describe the collision of two highly Lorentz contracted heavy ions, further analysis of the relativistic fluid, with or without viscosities, with different $\lambda$ values, etc., are very important because, for some years, any theoretical predictions can be compared with the experimental data that were taken at AGS or CERN. The concept of a quark-gluon plasma (QGP) predicted for heavy ion collisions [13] has been the driving force for the experimental and theoretical studies of hadronic matter at high energy densities. A lot of theoretical work has been done in order to use the hydrodynamical models in heavy ion physics and several models based on relativistic hydrodynamics exist, the reader is reffered to Refs. [4, 14, 15].
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