INTERNALITY, TRANSFER, AND INFINITESIMAL MODELING OF INFINITE PROCESSES

EMANUELE BOTTAZZI AND MIKHAIL G. KATZ

Abstract. A probability model is underdetermined when there is no rational reason to assign a particular infinitesimal value as the probability of single events. Pruss claims that hyperreal probabilities are underdetermined. The claim is based upon external hyperreal-valued measures. We show that internal hyperfinite measures are not underdetermined. The importance of internality stems from the fact that Robinson’s transfer principle only applies to internal entities. We also evaluate the claim that transferless ordered fields (surreals, Levi-Civita field, Laurent series) may have advantages over hyperreals in probabilistic modeling. We show that probabilities developed over such fields are less expressive than hyperreal probabilities.

Contents

1. Introduction 2
2. Archimedean and otherwise models of Pruss spinners 3
  2.1. Discrete models of the spinners 5
  2.2. Spinner models based on the Lebesgue measure 6
  2.3. Hyperfinite models of the spinners 7
  2.4. Hidden assumptions on infinitesimal models of spinners 9
  2.5. Refining the models by means of additional constraints 10
3. Are infinitesimal probabilities underdetermined? 12
  3.1. The first theorem of Pruss 13
  3.2. The Ω-limit axiom 14
  3.3. The second theorem of Pruss 15
  3.4. Internal probability measures are not underdetermined 16
4. Probability measures on transferless fields 17
  4.1. Probability measures on the surreal numbers 18

Date: August 27, 2020.

2010 Mathematics Subject Classification. Primary 03H05; Secondary 03H10, 00A30, 26E30, 26E35, 28E05, 60A05, 01A65.

Key words and phrases. Infinitesimals; hyperreals; hyperfinite measures; internal entities; probability; regularity; axiom of choice; saturated models; underdetermination; non-Archimedean fields.
1. Introduction

Alexander Pruss (AP) claims in [28] that hyperreal probabilities are underdetermined, meaning that, given a model, there is no rational reason to assign a particular infinitesimal value as the probability of a single event. AP’s underdetermination claim hinges upon the following:

1. examples of uniform processes that allegedly do not allow for a uniquely defined infinitesimal probability for singletons, and
2. a pair of theorems asserting that for every hyperreal-valued probability measure there exist uncountably many others that induce the same decision-theoretic preferences.

In [12] we presented our main arguments highlighting some hidden biases in the representation of some infinite processes such as lotteries, coin tosses, and other infinite processes.

In Section 2 we analyze some Archimedean and non-Archimedean models of Prussian spinners (rotating pointers). In particular, we will show that hyperfinite models of the spinners are not underdetermined. Moreover, we will discuss additional constraints that may narrow down the choice of infinitesimal probabilities, as proposed by AP himself in [28], Section 3.3 as well as 3.5 (“Other putative constraints”).

We show that the additional probabilities introduced in AP’s theorems are all external; see Section 3. Since external functions do not obey the transfer principle of Robinson’s framework, these additional measures are inferior to internal ones when it comes to modeling using hyperreal fields. Thus AP’s external measures are parasitic in the sense of Clendinnen ([14], 1989) \footnote{See [12], note 2.} and fail to establish underdetermination. In the light of our analysis, AP’s theorems can be framed as a warning against the use of external probabilities that do not satisfy the transfer principle. Thus, among the additional constraints that, in AP’s words, “may help narrow down the choice of infinitesimal probabilities” ([28, Section 3.3]), the first choice should be the internal constraint; see Sections 2.5 and 3.4 as well as [12], Section 3.2. AP speculates that
[An] approach in terms of imprecise probabilities, where a family of hyperreal-valued probability functions is assigned as in Sect. 6.1.2 of Benci et al. (2018), may well be able to escape the underdetermination worries. ([28, Section 1]; emphasis in the original)

AP’s wording implies that a single hyperreal-valued probability is unable to escape such ‘worries.’ However, we show that a single, internal hyperreal-valued probability does escape Prussian worries.

AP also speculated that other non-Archimedean fields may bestow advantages as compared to Robinson’s framework. In Section 4 we argue that, contrary to AP’s speculation, infinitesimal probabilities in other familiar non-Archimedean extensions of the real numbers are less expressive than real-valued probabilities. This is due to the absence of a uniform way of extending most real functions to these fields so as to preserve their elementary properties, and to the limitations of the measure theory currently developed for these fields. From the viewpoint of Robinson’s framework, such limitations are due to the unavailability of a transfer principle for these fields.

2. ARCHIMEDEAN AND OTHERWISE MODELS OF PRUSS SPINNERS

We analyze AP’s non-Archimedean models of the spinners, and show how hyperfinite non-Archimedean models avoid the claimed underdetermination problem. Starting from some of these hyperfinite models, it is also possible to define an infinitesimal probability with a standard sample space and taking values in a field of hyperreals $^\ast \mathbb{R}$, thus addressing AP’s original model (see Section 2.3). In this section we examine one particular example of the failure of AP’s underdetermination charge for hyperreal probabilities. Our rebuttal of his claim that every non-Archimedean probability is underdetermined appears in Section 3.

AP provides the following example (see [28, Section 3]) of a pair of uniform processes with a common sample space:

1. the first process is a spinner that is designed to stop uniformly at an angle $\theta \in [0, 360)$;
2. the second process is a spinner that is designed to stop uniformly at an angle $\theta \in [0, 360)$, but once it stops, the angle obtained is doubled.

AP then shows that the outcome of both spinners can be represented by a uniform probability over the real interval $[0, 360)$. Having introduced this Archimedean model, AP alleges that the description of the spinners by means of a hyperreal-valued probability depends upon the underlying mechanism governing each spinner. He concludes that
we cannot simply specify an infinitesimal [probability] by saying that it is whatever is the probability of a uniform process hitting a particular point. For what that probability is — assuming it is infinitesimal and not zero — depends on details of the process that go beyond uniformity. [28, Section 3.2]

To refute AP’s claim, we will show that his conclusion is only due to inappropriate choices he makes in the non-Archimedean description of the uniform process, and that indeed appropriate non-Archimedean models do not depend upon irrelevant details.

We observe that AP appears to extrapolate to every non-Archimedean model some properties of the continuous model obtained from the Lebesgue measure. For instance, he assumes that in every infinitesimal description of the spinners, the sample space $S$ must have the property that if $x \in S$, then also $\frac{x}{2} \in S$. This property is satisfied if $S$ is an interval of $\mathbb{Q}$ or of $\mathbb{R}$, but the spinners can also be described by means of a hyperfinite set $S_H$ that is discrete in an appropriate sense. The representations of the uniform processes by means of a hyperfinite sample space $S_H$ enable us to specify a unique uniform probability measure over $S_H$ that is also regular (see also Section 2.3). Thus for such hyperreal probabilities no underdetermination occurs. In Sections 3.2 and 3.3 we argue that this result applies also to non-Archimedean probabilities that satisfy the $\Omega$-limit axioms of Benci et al., since these probabilities can be obtained as the restriction of suitable hyperfinite measures.

Consequently, in his intuitive arguments for his underdetermination claim AP relies on additional hypotheses on the non-Archimedean models that are not satisfied by many relevant hyperreal probabilities.

In Archimedean mathematics, the two spinners discussed by AP can be described by a family of discrete models and by a continuous model, as we explain in Sections 2.1 and 2.2. In Sections 2.3 and 2.4 we show how internal hyperfinite models of AP’s spinners avoid the alleged drawbacks. Finally, in Section 2.5 we show how properly defined hyperfinite measures can approximate to varying degrees other properties of the models based upon the Cantor–Dedekind representation of the continuum.

---

1We have already argued that a given physical process can be given distinct mathematical representations; see [12], Section 2.3. Thus we reject the claim that changing the sample space, as we have proposed here, changes also the underlying process.
2.1. **Discrete models of the spinners.** The discrete models for the first spinner can be obtained as follows. Let \( n \in \mathbb{N} \) and define
\[
S_n = \{0, \ldots, \frac{k\pi}{n}, \ldots, \frac{(2n-1)\pi}{n}\} \subseteq [0, 2\pi).
\]
One can imagine the set \( S_n \) as representing the points on the circle obtained from rotations by integer multiples of the angle \( \frac{\pi}{n} \). A uniform probability over \( S_n \) is given by \( P_n(A) = \frac{|A|}{|S_n|} = \frac{|A|}{2n} \). The probability measure \( P_n \) has the following four properties:
- **Tot** \( P_n \) is total, in the sense that it is defined on the powerset \( \mathcal{P}(S_n) \) of the sample space \( S_n \);
- **Un** \( P_n \) is uniform in the sense that for every \( A, B \subseteq S_n \) with \( |A| = |B| \), one has also \( P_n(A) = P_n(B) \);
- **Reg** \( P_n \) is regular;
- **Sy** \( P_n \) is not symmetric with respect to rotations by an arbitrary real angle, since if \( \theta \in \mathbb{R} \) in general \( \frac{k\pi}{n} + \theta \) mod \( 2\pi \) \( \not\in S_n \).

In addition, as a consequence of property **Un** \( P_n \), the measure \( P_n \) satisfies the following discrete symmetry condition:
- **Sy** \( P_n \) symmetric with respect to rotations by multiples of \( \frac{\pi}{n} \), i.e. if \( 0 \leq \theta \leq 2n - 1 \) is an integer, then
\[
P\left(\left\{\frac{k_1\pi}{n}, \ldots, \frac{k_j\pi}{n}\right\}\right) = P\left(\left\{\frac{k_1\pi}{n} + \theta \frac{\pi}{n} \text{ mod } 2\pi, \ldots, \frac{k_j\pi}{n} + \theta \frac{\pi}{n} \text{ mod } 2\pi\right\}\right).
\]

The discrete models \( Q_n \) for the second spinner can be obtained from the measure \( P_{2n} \) as follows:
\[
Q_n(\{x\}) = P_{2n}(\{\frac{x}{2}\}) + P_{2n}(\{\frac{x}{2} + \pi\}) = 2\frac{1}{4n} = \frac{1}{2n} = P_n(\{x\})
\]
for all \( x \in S_n \). From the previous equation it is readily seen that \( Q_n \) agrees with \( P_n \) over \( S_n \), even though it is obtained from another description of the uniform process. As a consequence, \( Q_n \) has the properties **Un**, **Reg**, **Sy** \( \mathbb{R} \) and **Sy** \( n \) already discussed for \( P_n \).

As already observed by Bascelli et al. ([2], 2014), with a choice of some sufficiently large \( n \), the models \( P_n \) and \( Q_n \) might already be sufficient for many practical purposes, since

all physical quantities can be entirely parametrized by the usual rational numbers alone, due to the intrinsic limits of our capability to measure physical quantities.

[2, p. 853]

---

3By **regular** we mean that the measure \( P_n \) assigns probability 0 only to the empty event. This should not be confused with the notion of **regular measure**, i.e. of a measure \( \nu \) such that \( \nu(A) = \sup\{\nu(F) : F \subseteq A \text{ is compact and measurable}\} = \inf\{\nu(G) : G \supseteq A \text{ is open and measurable}\} \).
This position is shared by Herzberg [21, Section 1]. Nevertheless, we agree with the common mathematical practice that it is more convenient to introduce some level of idealisation for the description of $P_n$ and of $Q_n$, especially as $n$ becomes very large.

2.2. Spinner models based on the Lebesgue measure. A typical idealisation is the use of continuous models based upon the real numbers. A continuous model for the first spinner possessing the following property, $\text{Un}$, of uniformity:

$\text{Un}$ if two intervals have the same length, then they have the same probability

is obtained by using the Lebesgue measure $\mu$ over the real interval $[0, 2\pi)$. In this model, events are the measurable subsets of $[0, 2\pi)$ and the probability of an event $A$ is defined as $P_\mu(A) = \frac{\mu(A)}{\mu([0, 2\pi))} = \frac{\mu(A)}{2\pi}$.

Notice also that the choice of $[0, 2\pi)$ as the sample space is arbitrary. One could equally well measure angles with arc degrees instead of radians. In this case, the sample space would be the real interval $[0, 360)$ and the probability of an event $A$ would be defined as $P_{360}(A) = \frac{\mu(A)}{\mu([0, 360))} = \frac{\mu(A)}{360}$. Despite the arbitrariness of the choice of the sample space, one obtains the compatibility conditions

$$P_{360}(A) = P_\mu \left( \{ x \in [0, 2\pi) : \frac{360}{2\pi} x \in A \} \right)$$

for all measurable sets $A \subseteq [0, 360)$, and

$$P_\mu(B) = P_{360} \left( \{ x \in [0, 360) : \frac{2\pi}{360} x \in B \} \right)$$

for all measurable sets $B \subseteq [0, 2\pi)$. From now on, we will refer mostly to the probability measure $P_\mu$, but the discussion remains valid also for $P_{360}$. The probability measure $P_\mu$ has different properties from its discrete counterparts $P_n$. Namely, it has the following three properties:

$\text{Un}$ $P_\mu$ is uniform;
$\neg \text{Reg}$ $P_\mu$ is not regular;
$\text{Sy}_\mathbb{R}$ $P_\mu$ is symmetric with respect to rotations by an arbitrary real angle.

Observe that we have not taken a position on whether the continuous models satisfy $\text{Tot}$ or $\neg \text{Tot}$: if one rejects the full Axiom of Choice, then the Lebesgue measure can be total; see Solovay ([35], 1970). However, by assuming a sufficiently strong choice principle it is possible to prove that there are sets that are not Lebesgue measurable, so that $P_\mu$ would not be total. Recall also that without some weak form of the Axiom of Choice it is not possible to prove that the Lebesgue measure is countably additive. For more details, we refer to Fremlin ([20], 2008) and to Bottazzi et al. ([11], 2019, Section 5).
Notice also that for all $n \in \mathbb{N}$, and for all $h < k \leq 2n$, we have

\[ P_\mu \left( \left[ \frac{h \pi}{n}, \frac{k \pi}{n} \right] \right) = P_n \left( \left\{ \frac{h \pi}{n}, \ldots, \frac{(k-1) \pi}{n} \right\} \right). \]

The previous equality can be interpreted as a coherence between $P_\mu$ and $P_n$ over arcs with endpoints in $S_n$.

Similarly, the second spinner can be modeled by using the Lebesgue measure. The probability of an event $A$ is

\[ Q_\mu(A) = \frac{\mu(\{x: 2x \mod 2\pi \in A\})}{\mu([0,2\pi])} = \frac{\mu(\{x/2: x \in A\})}{2\pi} + \frac{\mu(\{x/2 + \pi: x \in A\})}{2\pi}. \]

Since the Lebesgue measure is translation invariant, we have the equality

\[ \frac{\mu(\{x/2: x \in A\})}{2\pi} = \frac{\mu(\{x/2 + \pi: x \in A\})}{2\pi}, \]

from which we deduce

\[ Q_\mu(A) = 2 \frac{\mu(\{x/2: x \in A\})}{2\pi} = 2P_\mu(\{x/2: x \in A\}) = P_\mu(A). \]

The discrete models and the continuous model capture different aspects of the spinners. The choice of which model to use in specific circumstances ought to reflect the relevant properties of each situation. For instance, one should take into account that, as is well known, the properties $\text{Un}$, $\text{Reg}$ and $\text{Sy}_R$ cannot be simultaneously satisfied by any real-valued measure defined over an infinite set $\Omega$. This imposes some limitations on the scope of $P$. For instance, it is not possible to define from $P$ a conditional probability with respect to every measurable event $B$. This occurs since the common definition of conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$ requires $B$ to have positive measure.

2.3. Hyperfinite models of the spinners. The continuous model based upon the Lebesgue measure is not the only possible idealisation of the two discrete models for the spinners. By using hyperfinite representations, it is possible to define models that satisfy $\text{Tot}$ and $\text{Reg}$, and that approximate to varying degrees the properties $\text{Un}$ and $\text{Sy}_R$.

A simple model of the first spinner is obtained by taking as sample space the set $S_n$ for some infinitely large hypernatural $n^5$. The corresponding probability measure $P_n$ has the properties $\text{Tot}$, $\text{Un}_n$, $\text{Sy}_R$.

\[ ^5\text{Such models exist thanks to the transfer principle of Robinson’s framework. Moreover, transfer ensures that the properties of the finite models discussed in Section 2.1 are shared by the hyperfinite models presented here.} \]
EMANUELE BOTTAZZI AND MIKHAIL G. KATZ

Reg. \(\neg \text{Sy}_\mathbb{R} \), \(\text{Sy}_n\) described for the discrete models of the spinner.\(^6\)
If \(n = m!\) for some infinite hypernatural \(m\), then \(\text{Sy}_n\) implies \(\text{Sy}_{\mathbb{Q}}\),
that is symmetry with respect to rotations by any rational angle. As observed for instance by Benci et al. [3, p. 5], \(P_n\) cannot satisfy the
properties \(\text{Un} \) and \(\text{Sy}_\mathbb{R}\); however it has the coherence property
\[
\text{Co} \quad P_n(\{A \cap S_n\}) \approx P_n(A) \quad \text{for each measurable } A \subseteq [0, 2\pi)\]
that implies the weak form of uniformity
\[
P_n([a, b)) \approx P_n([c, d)) \quad \text{whenever } b - a = d - c,
\]
and the weak form of symmetry
\[
P_n(\{A \cap S_n\}) \approx P_n(\{x + \theta \mod 2\pi : x \in A\} \cap S_n)
\]
for every measurable \(A \subseteq [0, 2\pi)\) and for every real \(\theta \in [0, 2\pi)\).

As already discussed in the Archimedean discrete models, the second spinner can be represented by the hyperfinite probability \(Q_n\) for an
infinite hypernatural \(n\). This probability measure shares the properties \(\text{Tot}, \text{Un}_n, \neg \text{Sy}_\mathbb{R}, \text{Sy}_n\) and \(\text{Co}\) with the first spinner; moreover one
has \(Q_n(\{x\}) = P_n(\{x\})\), as in the Archimedean models.

Recall that the continuous description of the spinners based upon
the real Lebesgue measure could be formulated with different choices
of the sample space (e.g., the interval \([0, 2\pi)\) or the interval \([0, 360)\))
but it is independent of such choices in the sense expressed by equations (2.1) and (2.2). For the hyperfinite measures \(P_n\) we have a similar
property, expressed by the following compatibility condition: whenever \(\gcd(m, n) = \min\{m, n\} = n\), we have
\[
(2.3) \quad P_n(A) = P_m(\{x \in S_m : \frac{m}{n}x \mod 2\pi \in A\})
\]
for all \(A \subseteq S_n\).

Equality (2.3) is the discrete counterpart of equations (2.1) and
(2.2). The main differences between the continuous equations and
the hyperfinite one is that in the real case there is a bijection be-
tween the sets \([0, 2\pi)\) and \([0, 360)\) given by the map \(x \mapsto \frac{360}{2\pi}x\), while
in the discrete case (either finite or hyperfinite) there is no bijection

\(^6\)Property \(\text{Tot}\) should be interpreted as follows: \(P_n\) is defined on every internal
set \(A \subseteq S_n\). Similarly, property \(\text{Un}_n\) should be interpreted as follows: for every internal \(A, B \subseteq S_n\) with \(|A| = |B|\), one has also \(P_n(A) = P_n(B)\). The symbol \(|A|\)
denotes the element of \(^*\mathbb{N}\) corresponding to the internal cardinality of the internal
set \(A\).

\(^7\)Here \(\approx\) denotes the relation of infinite proximity, i.e., the relation of being infinitely
close.
between the sets $S_n$ and $S_m$ unless $n = m$. On the other hand, if $\gcd(m, n) = \min\{m, n\} = n$, then there is a $\frac{m}{n}$-to-one correspondence between $S_m$ and $S_n$; together with properties $Un_n$ and $Un_m$ of the measures $P_n$ and $P_m$, this is sufficient to entail the relation (2.3).

2.4. Hidden assumptions on infinitesimal models of spinners. In his Section 3.2, AP seems to suggest that in Robinson’s framework every pair of probability measures $P$ and $Q$ for the first and the second spinner, respectively, satisfy the following conditions:

(Pr 1) $P$ and $Q$ have the same sample space;
(Pr 2) $P$ and $Q$ should assign the same probability to singletons;
(Pr 3) $Q(\{x\}) = 2P(\{x/2\})$.

From properties (Pr 2) and (Pr 3) AP obtains his underdetermination claim [28, Section 3.2].

However, as we have seen in the discussion of the discrete models, if one wishes to retain property (Pr 1) in a hyperfinite setting, the requirement (Pr 3) must be replaced by a subtler condition. In particular, for the hyperfinite probabilities $P_n$ and $Q_n$ defined on $S_n$ we have seen that $Q_n$ should be defined in terms of $P_{2n}$ as follows:

\[(3') \quad Q_n(\{x\}) = P_{2n}(\{\frac{x}{2}\}) + P_{2n}(\{\frac{x}{2} + \pi \mod 2\pi\}) = 2P_{2n}(\{\frac{x}{2}\}),\]

since for some $x \in S_n$ the number $x/2$ does not belong to the sample space $S_n$ but does belong to $S_{2n}$. We have already argued that definition (3') implies that $Q_n(\{x\})$ is indeed equal to $P_n(\{x\})$ for every $x \in S_n$. This example shows how internal hyperfinite models of the two spinners are sufficient to refute AP’s claim that the infinitesimal probability of a pair of uniform processes on the same outcome space allegedly “depends on details of the process that go beyond uniformity” [28, Section 3.2].

A similar rebuttal applies to AP’s description of the pair of uniform lotteries over $\mathbb{N}$ in [28, Section 4.1].

2.5. Refining the models by means of additional constraints. AP notes that

---

8We remark that there is no internal bijection between $S_n$ and $S_m$. If one drops the requirement that the bijection must be internal, then it is possible to find external bijections witnessing that the external cardinality of $S_n$ is equal to the external cardinality of $S_m$ whenever both $n$ and $m$ are infinite. However, the existence of external bijections between $S_n$ and $S_m$ has no bearing our argument, that is instead based upon the internal cardinality. For a discussion on the limited relevance of external objects and functions in hyperreal models see Section 3 and note 22, as well as [12], Sections 3.2 and 3.4.
There are potential types of constraints on the choice of infinitesimals that may help narrow down the choice of infinitesimal probabilities. One obvious constraint is formal: the axioms of finitely additive probability. A second type of constraint is match between the extended real probabilities for a problem and the corresponding classical real-valued probabilities. (28, Section 3.3; emphasis added)

Thus, AP envisions the possibility of introducing additional constraints that may narrow down the choice of infinitesimal probabilities. He even evokes more specifically the possibility of exploring a “match” between extended-real probabilities and classical real-valued probabilities. To practitioners of mathematics in Robinson’s framework it would be natural to interpret such a “match” in terms of the constraint of being internal. By envisioning the possibility of introducing further constraints, AP opens the door to introducing the internal condition. The internal condition would in any case be the obvious first choice for a practitioner of mathematics in Robinson’s framework. It is a constraint that AP failed to consider. Similar shortcomings of Elga’s analysis in (17, 2004) were signaled by Herzberg (21, 2007).

In addition, it is possible to introduce further constraints. As an example, we propose two different hyperfinite representations of the uniform spinners. Hyperfinite entities, being internal, satisfy the transfer principle. Moreover, each of the following hyperfinite models will have various properties that improve upon \( \text{Tot}, \text{Un}_n, \text{Reg}, \neg \text{Sy}_\mathbb{R} \) and \( \text{Co} \) of the measure \( P_n \).

The main result of Benci et al. [3] entails that there exists

- an algebra of Lebesgue measurable sets \( \mathcal{B} \subseteq \mathcal{P}([0,2\pi)) \) such that for every \( A \in \mathcal{B} \), either \( A = \emptyset \) or \( P_\mu(A) \neq 0 \);
- a hyperfinite set \( \Omega \subseteq \ast[0,2\pi) \) such that for every \( x \in [0,2\pi) \), \( \ast x \in \Omega \);
- a hyperfinite probability measure \( P_\Omega \) over \( \Omega \)

with the properties \( \text{Tot}, \text{Un}_\Omega, \text{Reg}, \neg \text{Sy}_\mathbb{R}, \text{Co} \) and the following additional property:

\( \text{Un}_{\mathcal{B}} \) \( P_\Omega \) is uniform over sets of \( \mathcal{B} \), i.e., \( P_\Omega(\ast A \cap \Omega) = P_\Omega(\ast B \cap \Omega) \) whenever \( A, B \in \mathcal{B} \) and \( P_\mu(A) = P_\mu(B) \).

9Internal sets are elements of (\( \ast \)-extensions of) classical sets; see 12, Section 3.1. Robinson’s book (29, 1966) deals with internal entities systematically.

10An interesting and nontrivial choice is the ring of finite unions of intervals of the form \([a, b)\). For more details, see Benci et al. (3, 2015, p. 43).
As a consequence of $\mathbf{Un}|\Omega|$, $P_{\Omega}$ satisfies $\mathbf{Un}_n$ for all $n \in \mathbb{N}$. Moreover, the property $\mathbf{Un}_\Omega$ implies that $P_{\Omega}$ is symmetric with respect to rotations by an arbitrary real angle, but only on the nonstandard extensions of sets in $\mathcal{B}$. For more details, we refer to [3] and [1, Section 3], where the example of the Lebesgue measure is discussed in detail.

The hyperfinite measure $P_{\Omega}$ can also be used to define a non-Archimedean probability over the sample space $[0, 2\pi)$ by setting

$$P(A) = P_{\Omega}(^*A \cap \Omega) \quad \text{for each} \quad A \subseteq [0, 2\pi).$$

The hypothesis that for every $x \in [0, 2\pi)$, $^*x \in \Omega$ ensures that $P_{\Omega}(x) = \frac{1}{|\Omega|}$ for every $x \in [0, 2\pi)$. Thus the measure $P$ of (2.4) is regular and uniform in the sense that it assigns the same nonzero probability to singletons.

In Section 3.2 we argue that non-Archimedean probabilities that satisfy the Omega-limit axiom of Benci et al. can be obtained in a similar way, i.e., by restricting a suitable hyperfinite probability. In this external measure over $[0, 2\pi)$, the choice for the probability of a singleton is uniquely determined from the underlying hyperfinite set $\Omega$, and it is not arbitrary as argued by AP.

By using an earlier result by Wattenberg [36], it is possible to obtain another hyperfinite set $\Omega_H$ and a hyperfinite probability measure $P_{H}$ over $\Omega_H$ that has an additional advantage. Namely, it is coherent not only with the Lebesgue measure, but also with the Hausdorff $t$-measures.

Recall that the Hausdorff outer measure of order $t$ of a set $A \subseteq \mathbb{R}$ is defined as

$$\mathcal{H}^t(A) = \lim_{\delta \to 0} \left\{ \sum_{n \in \mathbb{N}} \lambda(I_n)^t : A \subseteq \bigcup_{n \in \mathbb{N}} I_n, \lambda(I_n) < \delta \right\}.$$  

The Hausdorff measure $H^t$ is the restriction of the outer measure $\mathcal{H}^t$ to the $\sigma$-algebra of measurable subsets of $\Omega$.

The probability measure $P_H$ over $\Omega_H$ has the properties $\mathbf{Tot}$, $\mathbf{Un}_{|\Omega_H|}$, $\mathbf{Reg}$, $\neg \mathbf{Sy}_\mathbb{R}$ as well as the following property:

$\mathbf{Co}_H$ if $B$ is a Borel set with finite nonzero Hausdorff $t$-measure for some real $t \in [0, +\infty)$, then for every $H^t$-measurable set $A \subseteq \Omega_H$ the sets $A_H = ^*A \cap \Omega_H$ and $B_H = ^*B \cap \Omega_H$ satisfy the relations

$$\frac{H^t(A \cap B)}{H^t(B)} \approx \frac{P_H(A_H \cap B_H)}{P_H(B_H)} = P_H(A_H|B_H).$$

Notice that, by taking $t = 1$ and $B = [0, 2\pi)$, $\mathbf{Co}_H$ implies $\mathbf{Co}$. 
Each of the probability measures $P_n$, $P_\Omega$ and $P_H$ described above models different aspects of the spinner. In particular, they are all uniform, regular and total probability measures; moreover they approximate to varying degrees the properties of the real continuous model based upon the Lebesgue measure or upon the Hausdorff measure.

Furthermore, the probability $P_\Omega$ shows how the critique that non-Archimedean probabilities do not preserve intuitive symmetries, presented by AP in [28, Section 3.3], can be addressed by means of a suitable hyperfinite model.

Meanwhile, the probability $P_H$ shows that hyperreal-valued probability measures can be used simultaneously to represent the uncountably many Hausdorff $t$-measures. The strength of these kinds of hyperfinite models is not discussed by AP, who only considers the coherence of a non-Archimedean measure with a single real-valued measure.

3. ARE INFINITESIMAL PROBABILITIES UNDERDETERMINED?

In Section 2 we presented an analysis of AP’s non-Archimedean modeling. Now we turn to his underdetermination theorems. AP’s first theorem is based upon the following construction. Let $P$ be a probability function with values in a non-Archimedean ordered extension of $\mathbb{R}$. AP sets

$$P_\alpha(A) = \text{St}(P(A)) + \alpha(P(A) - \text{St}(P(A)))$$

for every $\alpha > 0$.\(^{11}\) AP’s first theorem expresses the fact that $P_\alpha$ is infinitely close to $P$ and satisfies the same inequalities as $P$ does. This result can be interpreted as the fact that the probabilities $P$ and $P_\alpha$ induce the same comparisons between events.

AP’s second theorem asserts a similar result for probability measures that satisfy the $\Omega$-limit axiom of Benci et al. In this case, starting from a probability that satisfies the $\Omega$-limit axiom (see Section 3.2) and from an automorphism $\phi$ of $^*\mathbb{R}$ that fixes only the standard real numbers, it is possible to define a new probability

$$P_\phi = \phi \circ P$$

that yields the same comparisons between events as $P$, and that still satisfies the $\Omega$-limit axiom.

Both results are technically correct, but what AP fails to mention is that, if $P$ is internal, then the additional probabilities $P_\alpha$ and $P_\phi$ are all external whenever they differ from $P$, as we show in Sections 3.1 and 11. AP assumes in addition that $\alpha \in \mathbb{R}$. However his results are still valid for every positive finite $\alpha \in ^*\mathbb{R}$.
Thus, they are badly behaved and do not satisfy the transfer principle of Robinson’s framework (see [12], Section 3.2) and are therefore parasitic in the sense of Clendinnen [14].

The significance of transfer and related principles both in the current practice of non-Archimedean mathematics based upon Robinson’s framework (see [12], Sections 3.1 and 3.2) and in the historical development of mathematical theories with infinitesimals (see [12], Section 3.3) is sufficient reason to recast AP’s theorems as a warning against the use of external probabilities in hyperreal modeling. Thus a careful analysis of Prussian theorems enables a meaningful criterion for the rejection of AP’s underdetermination charge.

For the purposes of the discussion that follows, recall that the transfer principle entails that any internal probability measure on a hyperfinite sample space Ω is hyperfinitely additive, i.e., that for every internal \(A \subseteq \Omega\), one has \(P(A) = \sum_{\omega \in A} P(\{\omega\})\). Consequently, if a probability measure is not hyperfinitely additive, then it is not internal.

3.1. The first theorem of Pruss. We begin our analysis with AP’s first theorem. AP is not clear on the domain of \(P\); here we will assume that \(P\) is an internal probability measure defined over a hyperfinite set \(\Omega\) and that \(P(\{\omega\}) \approx 0\) for all \(\omega \in \Omega\). This hypothesis is not restrictive, since these measures are general enough to represent every non-atomic real-valued probability measure [3, Theorem 2.2, p. 6].

We define \(P_\alpha\) as in (3.1). Then the following theorem holds.

**Theorem 1.** Let \(\Omega\) be a hyperfinite set and let \(P: \mathcal{P}(\Omega) \to \mathbb{R}\) be an internal probability measure that satisfies \(P(\{\omega\}) \approx 0\) for all \(\omega \in \Omega\). If \(\alpha \neq 1\) then \(P_\alpha\) is external.

The significance of such externality can be appreciated in light of the following fact. If \(\{X_i: i < H\}\) is an internal sequence of sets of infinite hyperfinite length \(H\), and \(P\) an internal measure, then the sum \(\sum_i P(X_i)\) is well defined in \(\mathbb{R}\); but if \(P\) is not internal then generally speaking \(\sum_i P(X_i)\) cannot be reasonably defined at all.

**Proof of Theorem 1.** Since \(P\) is internal, it is hyperfinitely additive. As a consequence,

\[
\sum_{\omega \in \Omega} P_\alpha(\{\omega\}) = \sum_{\omega \in \Omega} \alpha P(\{\omega\}) = \alpha \sum_{\omega \in \Omega} P(\{\omega\}) = \alpha.
\]

\[\text{See note 12}\]

\[\text{A real-valued probability function is called non-atomic if and only if it assigns measure 0 to every singleton. If a real-valued probability measure \(P\) can be decomposed into a sum of a non-atomic measure \(P_{\text{na}}\) and a discrete measure \(P_d\), then our Theorem 1 can still be applied to the hyperfinite representatives of \(P_{\text{na}}\).}\]
Since $P_\alpha(\Omega) = 1$, the probability $P_\alpha$ is hyperfinitely additive if and only if $\alpha = 1$, so that if $\alpha \neq 1$ then $P_\alpha$ cannot be hyperfinitely additive and, as a consequence, it is external. □

3.2. The $\Omega$-limit axiom. A similar argument refutes AP’s interpretation of his second theorem; see Section 3.3. Before showing that the probability measures $P_\phi$ obtained by AP are external, it will be convenient to recall the $\Omega$-limit axiom of Benci et al. [5] and some of its consequences. In this subsection, $\Omega$ will be a set of classical mathematics. Define also $\Lambda = \{ A \subseteq \Omega : |A| \in \mathbb{N} \}$. Thus if $\lambda \in \Lambda$, then $\lambda$ is a finite set.

The $\Omega$-limit is a notion of limit governed by the following definition.

**Definition 1.** Let $\Omega$ be an infinite set and $F$ an ordered field $F \supset \mathbb{R}$. An $\Omega$-limit in $F$ is a correspondence that associates to every function $f : \Lambda \to \mathbb{R}$, an element of $F$, denoted by $\lim_{\lambda \uparrow \Omega} f$, in such a way that the following properties hold:

1. if there is a $\lambda \in \Lambda$ with $f(\lambda) = c \in \mathbb{R}$ for every $\lambda \supseteq \lambda$, then $\lim_{\lambda \uparrow \Omega} f(\lambda) = c$;
2. for every $f, g : \Lambda \to \mathbb{R}$, one has
   - $\lim_{\lambda \uparrow \Omega} (f(\lambda) + g(\lambda)) = \lim_{\lambda \uparrow \Omega} f(\lambda) + \lim_{\lambda \uparrow \Omega} g(\lambda)$, and
   - $\lim_{\lambda \uparrow \Omega} (f(\lambda) \cdot g(\lambda)) = \lim_{\lambda \uparrow \Omega} f(\lambda) \cdot \lim_{\lambda \uparrow \Omega} g(\lambda)$.

It is possible to obtain an $\Omega$-limit by a suitable ultrapower construction. If one defines $F = \mathbb{R}^\Lambda / \mathcal{U}$, where $\mathcal{U}$ is a fine and free ultrafilter over $\Lambda$, then $F$ is a field of hyperreal numbers. An $\Omega$-limit over $F$ can then be obtained by setting $\lim_{\lambda \uparrow \Omega} f(\lambda) = [f]_U$.

A probability function $P : \mathcal{P}(\Omega) \to *\mathbb{R}$ satisfies the $\Omega$-limit axiom if and only if there exists an $\Omega$-limit such that $P(A) = \lim_{\lambda \uparrow \Omega} P(A|\lambda)$ for every $A \subseteq \Omega$.

Using the $\Omega$-limit axiom it is possible to define a notion of infinite sum for $P$. Thus, for every $A \subseteq \Omega$, the sum $\sum_{\omega \in A} P(\omega)$ is defined as $\lim_{\lambda \uparrow \Omega} \left( \sum_{\omega \in A \cap \lambda} P(\omega) \right)$; see also [6, p. 6].

Since a probability that satisfies the $\Omega$-limit axiom is defined over a classical set $\Omega$, the non-Archimedean probabilities that satisfy the $\Omega$-limit axiom are external [6, p. 25]. However, this is only due to the choice by Benci et al. of working with a sample space that is not internal. In fact, we will now show that non-Archimedean probabilities that

---

14 An ultrafilter $\mathcal{U}$ over $\Lambda$ is fine whenever for every $\lambda \in \Lambda = \{ A \subseteq \Omega : |A| \in \mathbb{N} \}$, one has $\{ L \subseteq \Lambda : \lambda \in L \} \in \mathcal{U}$. Such ultrafilters were referred to as adequate in earlier literature; see e.g., Kanovei–Reeken [23, p. 143]. A free ultrafilter is an ultrafilter that does not have a $\subseteq$-least element.
satisfy the Ω-limit axiom can be obtained as the restriction of suitable internal hyperfinite probabilities. Let $\Omega \subseteq \mathbb{R}$. We define the set
\begin{equation}
\Omega_\Lambda = \left\{ \lim_{\lambda \uparrow \Omega} f(\lambda) : \Lambda \xrightarrow{f} \mathbb{R} \text{ and } f(\lambda) \in \lambda \right\}.
\end{equation}
Then $\Omega_\Lambda$ is an internal hyperfinite set that represents $\Omega$. Moreover, it is possible to define an internal, uniform probability measure
\begin{equation}
P : \mathcal{P}(\Omega_\Lambda) \to \mathbb{R}
\end{equation}
by setting $P(A) = \frac{|A|}{|\Omega_\Lambda|}$ for every internal $A \subseteq \Omega_\Lambda$. With this definition, one has
\[ P(A) = P(A \cap \Omega_\Lambda) = \frac{|A \cap \Omega_\Lambda|}{|\Omega_\Lambda|} \]
for every $A \subseteq \Omega$. A proof of these statements can be obtained from the proof of Theorem 2.2 in \cite{3} as well as \cite[Section 3.6]{6}. Observe also that, if $\phi$ is an automorphism of $^{*}\mathbb{R}$, then $P_\phi(A) = \phi \left( P(A \cap \Omega_\Lambda) \right)$.

3.3. The second theorem of Pruss. As we showed in Section 3.2, non-Archimedean probabilities that satisfy the Ω-limit are restrictions of hyperfinite internal measures. We can now state our result concerning AP’s probabilities $P_\phi$. First, recall that in \cite[Section 3.6, Proposition 1]{28} AP proves that there are uncountably many nontrivial automorphisms of $^{*}\mathbb{R}$ that fix $\mathbb{R}$. However, these automorphisms are all external.

**Proposition 1.** If $\phi : ^{*}\mathbb{R} \to ^{*}\mathbb{R}$ is a field automorphism, then either $\phi$ is the identity or $\phi$ is external.

**Proof.** It is well known that the only field automorphism of $\mathbb{R}$ is the identity. Thus, by transfer, the only internal field automorphism of $^{*}\mathbb{R}$ is the identity. \hfill $\square$

Our Proposition 1 already suggests that probability measures $P_\phi$ are external whenever they do not coincide with $P$. However, we will prove such a result explicitly, establishing our counterpart to AP’s second theorem.

**Theorem 2.** Let $P : \mathcal{P}(\Omega) \to ^{*}\mathbb{R}$ be a probability measure that satisfies the Ω-limit axiom and let $\Omega_\Lambda$ and $P$ be defined as in (3.3) and (3.4). Suppose also that $\phi : ^{*}\mathbb{R} \to ^{*}\mathbb{R}$ is a field automorphism. Then the measure $P_\phi = \phi \circ P$ is internal if and only if the restriction of $\phi$ to the range of $P$ is the identity.

\[^{15}\text{This hypothesis can be avoided by extending the notion of Ω-limit as shown by Bottazzi \cite[Sections 4.2 and 4.3]{8}.}\]
Proof. Recall that the range of the measure $\overline{P}$ is the hyperfinite set $S = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k}{n}, \ldots, 1\}$, where $n = |\Omega|$. Let $\psi$ be the restriction of $\phi$ to the range of $\overline{P}$. If $\psi$ is the identity, then $\overline{P}_\phi = \overline{P}$.

Suppose $\psi$ is an internal automorphism different from the identity. Then the set $\{0, \frac{1}{\psi(n)}, \frac{2}{\psi(n)}, \ldots, \frac{k}{\psi(n)}, \ldots, 1\}$ is internal. Multiplying every element of this set by $\psi(n)$, we would obtain that the set $\{0, 1, 2, \ldots, \psi(k), \ldots, \psi(n)\}$ is internal, as well. Let $k$ be the least number such that $\psi(k) \neq k$. Then $\psi(k - 1) = k - 1$, so $\psi(k) = \psi(k - 1 + 1) = \psi(k - 1) + \psi(1) = k - 1 + 1 = k$, a contradiction.

Thus $\psi$ is necessarily external (or the identity). It remains to show that $\psi \circ \overline{P}$ is external. Since $\overline{P}$ is an internal mapping of the hyperfinite set $\overline{\mathbb{P}}(\Omega)$ onto $S = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, it has an internal right inverse $\chi : S \to \overline{\mathbb{P}}(\Omega)$ such that $\overline{P} \circ \chi$ is the identity on $S$. If $\psi \circ \overline{P}$ were internal, then $\psi \circ \overline{P} \circ \chi = \psi$ would be internal, contrary to our hypothesis. □

3.4. Internal probability measures are not underdetermined.

Our Theorems 1 and 2 indicate that the underdetermination issue raised by AP is present only whenever one considers external probability measures in addition to the internal ones. Significantly, such external probability measures do not exist in Nelson’s Internal Set Theory (see [12], Section 3.2). Notice also that working with internal probability measures is not restrictive, since, as we have shown, non-Archimedean functions that satisfy the $\Omega$-limit can be obtained as the restriction of suitable internal hyperfinite probabilities.

It should be noted that we are not claiming that external measures are not useful for hyperreal models. In fact, the external non-Archimedean probabilities by Benci et al. and the Loeb measures, that are obtained from suitable internal measures, play a relevant role for the development of infinitesimal probabilities and for the hyperreal measure theory, respectively. \footnote{Recall that the Loeb measure of an internal measure $\mu$ is obtained by composing $\mu$ with the (external) standard part function. By the Caratheodory extension theorem, the resulting pre-measure is then extended to a measure defined on an external $\sigma$-algebra, that is usually called the Loeb measure associated to $\mu$.} What we showed is that the external probabilities proposed by AP do not have even basic properties such as hyperfinite additivity. As a consequence, these external probabilities are clearly inferior to their internal counterparts.

In conclusion, once a hyperfinite sample space $\Omega$ is determined, there is only one internal, uniform and regular probability measure $P$ over $\Omega$. 

As a consequence, internal uniform probability measures with hyperfinite support are not *underdetermined*, contrary to what was alleged by Pruss.\(^{17}\)

Indeed, Pruss’ pair of theorems can be interpreted as a warning against the use of external measures in hyperreal probabilities. As already noted in Section 2.5, Pruss envisions the possibility of introducing additional constraints. The discussion above suggests that the very first constraint one should envision is the *natural* (and even *obvious*, to a practitioner of mathematics in Robinson’s framework) constraint of being *internal*.

In an analysis of the phenomenon of underdetermination, Clendinnen speaks of “empirically equivalent alternatives [that are] parasitic on [the original] theory” [14, p. 63], an apt description of Pruss’ external measures of the form (3.1) and (3.2).

4. **Probability measures on transferless fields**

Pruss claims that some extensions of the real numbers with infinitesimals, such as the surreal numbers, the fields of Laurent series or the Levi-Civita field, might provide a non-arbitrary choice of infinitesimals for the development of non-Archimedean probabilities.

Due to the absence of a transfer principle, already discussed in [12, Section 3.4], there are some obstacles to developing a probability theory over these non-Archimedean extensions of the real numbers. We will discuss two examples in detail in Sections 4.1 and 4.2.

4.1. **Probability measures on the surreal numbers.** Pruss suggests that Conway’s surreal numbers (usually denoted \(\text{No}\)) might be suitable for the development of infinitesimal probabilities, since

> Whatever you can do with hyperreals, you can do with surreals, since any field of hyperreals can be embedded in the surreals. [28, Section 2]

Pruss seeks support for such a claim in Ehrlich [16]. In fact, the result [16, Theorem 20] entails that in the von Neumann–Bernays–Gödel set theory with global choice, there exists a unique (up to an isomorphism) structure \((\mathbb{R}, \mathbb{R}^*, \ast)\) such that \(\mathbb{R}^*\) satisfies Keisler’s axioms for hyperreal number systems and \(\mathbb{R}^*\) is a proper class isomorphic to the class of surreal numbers.

\(^{17}\)This argument and the possibility of uniquely specifying a hyperreal field, as discussed in Section 2.1 of [12], refute Pruss’ claim that hyperreal probabilities are underdetermined both in the choice of a specific hyperreal field and in the choice of the infinitesimal probability of singletons.
Ehrlich observes, furthermore, that every field of hyperreal numbers is isomorphic to an initial segment of the surreal numbers:

Since every real-closed ordered field is isomorphic to an initial subfield of $\mathbb{No}$, the underlying ordered field of any hyperreal number system is likewise isomorphic to an initial subfield of $\mathbb{No}$. For example, the familiar ultrapower construction of a hyperreal number system as a quotient ring of $\mathbb{R}^N$ (modulo a given nonprincipal ultrafilter on $\mathbb{N}$) is isomorphic to $\mathbb{No}(\omega_1)$ assuming [the continuum hypothesis] . . . Similarly, if we assume there is an uncountable inaccessible cardinal, $\omega_\alpha$ being the least, then $\mathbb{No}(\omega_\alpha)$ is isomorphic to the underlying ordered field in the hyperreal number system employed by Keisler in his *Foundations of Infinitesimal Analysis*. [16, Section 9, p. 35]

Indeed, it is only the isomorphisms mentioned by Ehrlich that endow the surreal numbers (or their initial segments) with the transfer principle of Robinson’s framework. In addition, the $\ast$-map allows for the extension of functions in a way that the simplicity hierarchical structure of the surreal numbers is not yet able to handle.

If one refrains from exploiting the transfer principle of Robinson’s framework, then the current developments of analysis over the surreal field are fairly limited. For instance, it is still an open problem to define an integral over $\mathbb{No}$, despite some limited results obtained in the last two decades (Costin et al. [15], 2015; Fornasiero [19], 2004). As a consequence, it is not currently possible to develop a measure theory, let alone a probability theory, over the surreal numbers.

4.2. **Probability measures on the Levi-Civita field.** At the other end of the spectrum, Pruss advocates the use of the “elegantly small” Levi-Civita field $\mathcal{R}$. This extension of the real numbers was introduced by Levi-Civita in [24, 25]. It is the smallest non-Archimedean ordered field extension of the field $\mathbb{R}$ of real numbers that is both real closed and complete in the order topology. For an introduction, we refer to Lightstone–Robinson [26] and to Berz–Shamseddine [34].

---

18 It should be noted that the property of being an ordered field is a small fraction of the properties of the reals required to develop any substantial calculus. The surreals lack many relevant properties even with regard to their collection of natural numbers; see [12], note 28.

19 As well as with the natural numbers and all the associated structure over $\langle \mathbb{R}; \mathbb{N} \rangle$ necessary to develop calculus and measure theory.

20 As opposed to other forms of completeness.
The Levi-Civita field is defined as the set
\[ R = \{ x \in \mathbb{R}^Q : \forall q \in \mathbb{Q} \text{ supp}(x) \cap (-\infty, q] \text{ is finite} \} \]
together with the pointwise sum, and the product defined by the formula
\begin{equation}
(x \cdot y)(q) = \sum_{q_1 + q_2 = q} x(q_1) \cdot y(q_2).
\end{equation}
If one defines the element \( d \in R \) by posing
\[ d(q) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1, \end{cases} \]
then every number in \( R \) can be written as a formal sum \( x = \sum_{q \in \mathbb{Q}} a_q d^q \). In this sum the set \( Q(x) = \{ q \in \mathbb{Q} : a_q \neq 0 \} \) has the property that the intersection \( Q(x) \cap (-\infty, q] \) is finite for every \( q \in \mathbb{Q}^+ \). Elements of \( \mathbb{R} \) can be identified with those elements in \( R \) whose support is a subset of the singleton \( \{0\} \). The field \( R \) can be linearly ordered\(^{22}\) and is sequentially complete in the order topology, but due to the presence of infinitesimal elements it is totally disconnected\(^{23}\) (the topological properties of \( R \) are discussed in detail by Shamseddine\(^{31}\)).

If one considers the language \( L \) of ordered fields\(^{24}\) then the real numbers \( \mathbb{R} \), fields of hyperreal numbers and the Levi-Civita field \( R \)

---

\(^{21}\)Pruss incorrectly states that only finitely many of the \( a_q \) are nonzero\(^{28}\) Section 2).

\(^{22}\)It is customary to define the order on \( R \) in a way that the number \( d \) is an infinitesimal.

\(^{23}\)This property is shared by every non-Archimedean extension of \( \mathbb{R} \). However, fields of hyperreal numbers of Robinson’s framework overcome this limitation by working with internal sets and functions. For instance, in both the Levi-Civita field and in \( \ast \mathbb{R} \) the series \( \sum_{n \in \mathbb{N}} r \) is not defined unless \( r = 0 \), but a sum of hyperfinitely many copies of \( r \in \ast \mathbb{R} \) is well-defined. Similarly, in the Levi-Civita field one should take into account that the function \( f \) defined by
\[ f(x) = \begin{cases} 1 & \text{if } x \approx 0 \\ 0 & \text{if } x \neq 0 \end{cases} \]
is differentiable at every point in \( R \), but it does not satisfy e.g. the Intermediate Value Theorem or the Mean Value Theorem. Meanwhile, the counterpart of this function in Robinson’s framework is external, so it does not provide a counterexample to the Intermediate Value Theorem or to the Mean Value Theorem.

\(^{24}\)Namely \( L = \{ +, -, 0, \cdot, -1, 1, < \} \), where \( + \) and \( \cdot \) are binary functions, \( - \) and \( -1 \) are unary functions, 0 and 1 are constant symbols, and \( < \) is a binary relation.
are $L$-structures which are models of the model-complete\textsuperscript{25} $L$-theory of real closed ordered fields [13, 27]. However, the Levi-Civita field is not elementarily equivalent to $\mathbb{R}$, and the only result analogous to the transfer principle between $\mathbb{R}$ and $\mathcal{R}$ is fairly limited. In particular, only locally analytic functions can be extended from real closed intervals to closed intervals in $\mathcal{R}$ in a way that preserves elementary properties [9, Sections 3, 4].

Consider now the uniform measure $m$ on the Levi-Civita field studied by Berz–Shamseddine (33, 2003), Shamseddine (32, 2012), and Bottonazzi (10, 2020). A set is measurable with respect to $m$ if it can be approximated arbitrarily well by intervals, in analogy with the Lebesgue measure. However, due to the properties of the topology of the Levi-Civita field, the measure $m$ turns out to be rather different from the Lebesgue measure over $\mathbb{R}$. For instance, in $\mathcal{R}$ the complement of a measurable set is not necessarily measurable. As a consequence, the family of measurable sets is not even an algebra.

As with the Lebesgue measure, the family of measurable functions is obtained from a family of simple functions. For the Lebesgue measure, these are the step functions; however a similar choice in the Levi-Civita field would lead to a very narrow class of measurable functions. A more fruitful choice for the Levi-Civita field is to define the simple functions as the analytic functions over a closed interval.

Regardless of the choice of simple functions, Shamseddine and Berz proved that any measurable function on a set $A \subseteq \mathcal{R}$ is locally simple almost everywhere on $A$ [33, Proposition 3.4, p. 379]. As a consequence, an absolutely continuous probability measure over $\mathcal{R}$ can only have a locally analytic density function; similarly, the only real probability functions that can be extended to probability functions over the Levi-Civita field are locally analytic. A partial converse of this negative result shows that measurable functions are not expressive enough to approximate with an infinitesimal error a real probability that is not locally analytic at any point of its domain [10, Proposition 3.16].

We now turn our attention to discrete probability measures defined over the Levi-Civita field. We note that this field has no notion of hyperfiniteness or of sum over sets that are not countable. Moreover, if $h \in \mathcal{R}$ is an infinitesimal, the limit $\lim_{n \to \infty} \sum_{i=0}^{n} h = \lim_{n \to \infty} nh$ is not defined, since the sequence $\{nh\}_{n \in \mathbb{N}}$ does not converge in $\mathcal{R}$. This is a significant difference with hyperreal-valued measures, where one

\textsuperscript{25}A theory in a language $L$ is model-complete if for every pair $M$ and $N$ of models of the theory, if there is an embedding $i: M \hookrightarrow N$, then the embedding is elementary. An embedding is elementary whenever for every first-order formula $\phi$ in the language $L$, $M \models \phi(a_1, \ldots, a_n)$ if and only if $N \models \phi(i(a_1), \ldots, i(a_n))$. 
can sum a internal hyperfinite family of infinitesimal terms and obtain a well-defined hyperreal result.

As a consequence, whenever $\Omega \subseteq \mathcal{R}$ is not a finite set, there cannot be a regular probability measure $m$ over $\Omega$ that assigns the same probability to every $\omega \in \Omega$. In this regard, $\mathcal{R}$-valued discrete probability measures do not offer any improvement upon real-valued discrete probability measures.

These properties of the uniform measure and of the discrete probability measures over the Levi-Civita field impose serious limitations on the use of the Levi-Civita field as the target space of a probability measure, be it absolutely continuous, discrete, or a combination of the two.

4.3. **Probability measures in other fields with infinitesimals.**

The Hahn field, obtained from functions $x \in \mathbb{R}^\mathbb{Q}$ with well-ordered support with the pointwise sum and the product defined as in formula (4.1), shares some properties with the Levi-Civita field. Nevertheless, there is no measure theory over the Hahn field yet, even though some preliminary results on the convergence of power series were obtained by Flynn and Shamseddine in ([18], 2019).

Kaiser recently developed a uniform measure over a class of non-Archimedean real closed fields ([22], 2018). However, this measure is only defined for semialgebraic sets. This condition is even more restrictive than the one imposed on the measurable sets in the Levi-Civita field (see Section 4.2). Note also that the resulting measure is only finitely additive.

The picture that emerges is very different from the one sketched by Pruss. Robinson’s hyperreal fields have many advantages over other non-Archimedean extensions of $\mathbb{R}$, as already realized by Fraenkel (see [12], Section 3.3), and it is unlikely that the gap between these theories will be closed any time soon.

5. **Conclusion**

In his *Synthese* paper [23], Pruss claims that the infinitesimal probabilities of Robinson’s framework for analysis with infinitesimals are *underdetermined*. His claim hinges upon

1. some models of infinitesimal probabilities, and
2. a pair of theorems that entail the existence of uncountably many infinitesimal probabilities that yield the same decision-theoretic comparisons as the original one.
In Section 2 we addressed issue (1) by showing that proper hyperfinite models avoid the underdetermination problem. In Section 3 we focused on Prussian theorems mentioned in item (2), and proved that all of the additional infinitesimal probabilities obtained by Pruss are external. As a consequence, once a hyperfinite sample space has been chosen, there is only one internal probability measure over it. The results of Sections 2 and 3 suggest that the underdetermination critique of Pruss is limited to external entities. Thus, working with internal sets and functions in Robinson’s framework dissolves the underdetermination objection. While recognizing that it may be possible to narrow down the choice of infinitesimal probability using additional constraints, Pruss fails to consider the natural internal constraint. Pruss could respond by arguing that internality is not the kind of criterion he had in mind; however, the point remains that internality is such an obvious choice that it should have been addressed one way or another. The fact that the issue is not examined in his paper constitutes a serious shortcoming of his analysis.

A would-be critic of Robinson’s framework could then retreat to an even more limited objection to the effect that the choice of the sample space is underdetermined. However, in the discussion of the hyperfinite models we showed the following:

- this choice is underdetermined in the Archimedean case, as well; however, this underdetermination is not problematic, since different models (be they Archimedean or hyperfinite) are compatible, as discussed in Sections 2.2 and 2.3.
- it is also possible to specify additional criteria for the choice of a hyperfinite model: for instance, it is possible to preserve rotational symmetry on a non-trivial family of sets or coherence with uncountably many real-valued measures, as shown in Section 2.5. Thus the possibility of working with hyperfinite models that improve upon the properties of the Archimedean ones should be regarded as an advantage of hyperreal probabilities.

In his critique, Pruss also suggests that other fields with infinitesimals are more suitable for the development of infinitesimal probabilities. However, in Sections 3.3 and 3.4 we showed that this claim ignores the Klein–Fraenkel criteria for the usefulness of a theory with infinitesimals. In addition, in Section 4 we showed that probabilities...
taking values in the surreal numbers and in the Levi-Civita field are less expressive than real-valued probabilities, mainly due to the absence of a transfer principle for these structures. Moreover, the absence of a comprehensive transfer principle makes such fields vulnerable to Theorem 1 of Pruss. In contrast to hyperreal fields, transferless fields do not possess a notion of internality and are thus unable to escape underdetermination.

ACKNOWLEDGMENTS

We are grateful to Karel Hrbacek and Vladimir Kanovei for insightful comments on earlier versions that helped improve our article, and to anonymous referees for constructive criticism. The influence of Hilton Kramer (1928–2012) is obvious.

REFERENCES

[1] Albeverio, Sergio, Raphael Høegh-Krohn, Jens Erik Fenstad, and Tom Lindstrom [1986]: Nonstandard Methods in Stochastic Analysis and Mathematical Physics. Pure and Applied Mathematics, 122. Academic Press, Orlando, FL.

[2] Bascelli, Tiziana, Emanuele Bottazzi, Frederik Herzberg, Vladimir Kanovei, Karin Katz, Mikhail Katz, Tahl Nowik, David Sherry, and Steven Shnider [2014]: ‘Fermat, Leibniz, Euler, and the gang: The true history of the concepts of limit and shadow’, Notices of the American Mathematical Society 61, no. 8, 848–864. See http://www.ams.org/notices/201408/rnoti-p848.pdf and https://arxiv.org/abs/1407.0233

[3] Benci, Vieri, Emanuele Bottazzi, and Mauro Di Nasso [2014]: ‘Elementary numerosity and measures’, Journal of Logic and Analysis 6, Paper 3, 14 pp.

[4] Benci, Vieri, Emanuele Bottazzi, and Mauro Di Nasso [2015]: ‘Some applications of numerosities in measure theory’, Rendiconti Lincei-Matematica E Applicazioni 26, no. 1, 37–48.

[5] Benci, Vieri, Leon Horsten, and Sylvia Wenmackers [2013]: ‘Non-Archimedean probability’, Milan Journal of Mathematics 81, 121–151.

[6] Benci, Vieri, Leon Horsten, and Sylvia Wenmackers [2018]: ‘Infiniteesimal probabilities’, British Journal for the Philosophy of Science 69, 509–552.

[7] Berarducci, Alessandro, Philip Ehrlich, and Salma Kuhlmann [2017]: ‘Mini-workshop: Surreal numbers, surreal analysis, Hahn fields and derivations’, Oberwolfach Reports 13, no. 4, 3313–3372.

[8] Bottazzi, Emanuele [2012]: ‘Ω-Theory: Mathematics with Infinite and Infinitesimal Numbers’, Master thesis, University of Pavia, Italy.

[9] Bottazzi, Emanuele [2018]: ‘A transfer principle for the continuation of real functions to the Levi-Civita field’, p-Adic Numbers, Ultrametric Analysis, and Applications, 10, no. 3, 179–191.

[10] Bottazzi, Emanuele [2020]: ‘Spaces of measurable functions on the Levi-Civita field’, Indagationes Mathematicae 31, no. 4, 650–694. See https://doi.org/10.1016/j.indag.2020.06.005 and https://arxiv.org/abs/1909.10069
[11] Bottazzi, Emanuele, Vladimir Kanovei, Mikhail Katz, Thomas Mormann, and David Sherry [2019]: ‘On mathematical realism and the applicability of hyperreals’, Mat. Stud. 51, no. 2, 200–224. See http://dx.doi.org/10.15330/ms.51.2.200-224 and https://arxiv.org/abs/1907.07040
[12] Bottazzi, Emanuele, and Mikhail Katz [2020]: ‘Infinite lotteries, spinners, and the applicability of hyperreals,’ Philosophia Mathematica, to appear.
[13] Chang, Chen Chung, and H. Jerome Keisler [1971]: Model Theory. Appleton Century Crofts, New York.
[14] Clendinnen, F. John [1989]: ‘Realism and the underdetermination of theory’, Synthese 81, 63–90.
[15] Costin, Ovidiu, Philip Ehrlich, and Harvey Friedman [2015]: ‘Integration on the surreals: a conjecture of Conway, Kruskal and Norton’, preprint. See https://arxiv.org/abs/1505.02478
[16] Ehrlich, Philip [2012]: ‘The absolute arithmetic continuum and the unification of all numbers great and small’, The Bulletin of Symbolic Logic 18, no. 1, 1–45. http://dx.doi.org/10.2178/bsl/1327328438
[17] Elga, Adam [2004]: ‘Infinitesimal chances and the laws of nature’, Australasian Journal of Philosophy 82, 67–76.
[18] Flynn, Darren, and Khodr Shamseddine [2019]: ‘On the topological structure of a Hahn field and convergence of power series’, Indagationes Mathematicae (N.S.) 30 (2019), no. 5, 773–795.
[19] Fornasiero, Antongiulio [2004]: ‘Integration on Surreal Numbers’, PhD thesis, The University of Edinburgh.
[20] Fremlin, David [2008]: Measure theory. Vol. 5. Set-theoretic Measure Theory. Part II. Torres Fremlin, Colchester.
[21] Herzberg, Frederik [2007]: ‘Internal laws of probability, generalized likelihoods and Lewis’ infinitesimal chances—a response to Adam Elga’, British Journal for Philosophy of Science 58, no. 1, 25–43.
[22] Kaiser, Tobias [2018]: ‘Lebesgue measure and integration theory on nonarchimedean real closed fields with archimedean value group’, Proceedings of the London Mathematical Society (3) 116, no. 2, 209–247.
[23] Kanovei, Vladimir, and Michael Reeken [2004]: Nonstandard Analysis, Axiomatically. Springer Monographs in Mathematics, Berlin, Springer.
[24] Levi-Civita, Tullio [1892]: ‘Sugli infiniti ed infinitesimi attuali quali elementi analitici’, Atti Ist. Veneto di Sc., Lett. ed Art. 7a (4), p. 1765.
[25] Levi-Civita, Tullio [1898]: ‘Sui numeri transfiniti’, Rend. Acc. Lincei 5a (7), 91–113.
[26] Lightstone, Albert Harold, and Abraham Robinson [1975]: Nonarchimedean Fields and Asymptotic Expansions. North-Holland Mathematical Library, Vol. 13. North-Holland Publishing, Amsterdam–Oxford; American Elsevier Publishing, New York.
[27] D’Mello, Clayton, and William Weiss [2000]: Fundamentals of Model Theory. See http://at.yorku.ca/i/a/a/i/10.htm
[28] Pruss, Alexander [2018]: ‘Underdetermination of infinitesimal probabilities’, Synthese, online first at https://doi.org/10.1007/s11229-018-02064-x
[29] Robinson, Abraham [1966]: Non-standard Analysis. North-Holland Publishing, Amsterdam.
[30] Robinson, Abraham [1979]: Selected papers of Abraham Robinson. Vol. II. Nonstandard analysis and philosophy. Edited and with introductions by W. A. J. Luxemburg and S. Körner. Yale University Press, New Haven, Conn.

[31] Shamseddine, Khodr [2010]: ‘On the topological structure of the Levi-Civita field’, Journal of Mathematical Analysis and Applications 368, no. 1, 281–292.

[32] Shamseddine, Khodr [2013]: ‘New results on integration on the Levi-Civita field’, Indagationes Mathematicae 24, no. 1, 199–211.

[33] Shamseddine, Khodr, and Martin Berz [2003]: ‘Measure theory and integration on the Levi-Civita field’, Contemporary Mathematics 319, 369–388.

[34] Shamseddine, Khodr, and Martin Berz [2010]: ‘Analysis on the Levi-Civita field, a brief overview’, Contemporary Mathematics 508, 215–237.

[35] Solovay, Robert [1970]: ‘A model of set-theory in which every set of reals is Lebesgue measurable’, Annals of Mathematics (2) 92, 1–56.

[36] Wattenberg, Frank [1977]: ‘Nonstandard measure theory. Hausdorff measure’, Proceedings of the American Mathematical Society 65, no. 2, 326–331.

E. Bottazzi, Department of Civil Engineering and Architecture, University of Pavia, Via Adolfo Ferrata 3, 27100 Pavia, Italy, orcid 0000-0001-9680-9549
E-mail address: emanuele.bottazzi@unipv.it, emanuele.bottazzi.phd@gmail.com

M. Katz, Department of Mathematics, Bar Ilan University, Ramat Gan 5290002 Israel, orcid 0000-0002-3489-0158
E-mail address: katzmik@macs.biu.ac.il