MASS-TEMPERATURE RELATION OF GALAXY CLUSTERS: A THEORETICAL STUDY

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ABSTRACT

Combining conservation of energy throughout the nearly spherical collapse of galaxy clusters with the virial theorem, we derive the mass-temperature relation for X-ray clusters of galaxies, \(T = CM^{2/3}\). The normalization factor \(C\) and the scatter of the relation are determined from first principles, with the additional assumption of an initial Gaussian random field. We are also able to reproduce the recently observed break in the \(M-T\) relation at \(T \sim 3\) keV, based on the scatter in the underlying density field for a low-density \(\Lambda\)CDM cosmology. Finally, by combining observational data of high-redshift clusters with our theoretical formalism, we find a semiempirical temperature-mass relation that is expected to hold at redshifts up to unity with less than 20% error.

Subject headings: cosmology: theory — galaxies: clusters: general — galaxies: halos — large-scale structure of universe — X-rays: galaxies: clusters

1. INTRODUCTION

The abundance of clusters of galaxies provides one of the strongest constraints on cosmological models (Peebles, Daly, & Juszkiewicz 1989; Bahcall & Cen 1992; White, Efstathiou, & Frenk 1996; Eke, Cole, & Frenk 1996; Okubk, Bartlett, & Blanchard 1997; Bahcall, Fan, & Cen 1997; Pen 1998; Cen 1998; Henry 2000; Wu 2000), with an uncertainty on the amplitude of density fluctuations of about 10% on a ~30 h\(^{-1}\) Mpc scale. Theoretically, it is often desirable to translate the mass of a cluster, predicted by either analytic theories such as Press-Schechter (1974) theory or \(N\)-body simulations, to the temperature of the cluster, which is directly observed. Simple arguments based on virialization density suggest that \(T \propto M^{2/3}\), where \(T\) is the temperature of a cluster within a certain radius (e.g., the virial radius) and \(M\) is the mass within the same radius. However, the proportionality coefficient has not been self-consistently determined from first principles, although numerical simulations have frequently been used to calibrate the relation (e.g., Evrard, Metzler, & Navarro 1996, hereafter EMN96; Bryan & Norman 1998; Thomas et al. 2001).

We note that the results from different observational methods of mass measurements are not consistent with one another or with the simulation results (e.g., Horner, Mushotzky, & Scharf 1999, hereafter HMS99; Neumann, & Arnaud 1999; Nevalainen, Markewich, & Forman 2000; Finoguenov, Reiprich, & Böhringer 2001, hereafter FRB01). In general, X-ray mass estimates are about 80% lower than the predictions of hydro simulations. Figure 1 compares X-ray cluster observational data with the best-fit line to EMN96 simulation results (FRB01). On the other hand, mass estimates from galaxy velocity dispersion seem to be consistent with simulation results (HMS99). The error in the gravitational lensing mass measurements is still too large to distinguish between these two (Hjorth, Okubk, & van Kampen 1998).

Another recent observational finding is the possible existence of a break in the \(M-T\) relation. By using resolved temperature profiles of X-ray clusters observed by ASCA, FRB01 have investigated the \(T-M\) relation in the low-mass end and find that \(M \propto T^{-2}\), compared to \(M \propto T^{-3/2}\) at the high-mass end. Suggestions have been made to explain this behavior by attributing it to the effect of formation redshift (FRB01), cooling (Muanwong et al. 2001), and heating (Bialek, Evrard, & Mohr 2001) processes.

In this paper we use conservation of energy for an almost spherically collapsing region to derive the \(M-T\) relation. In \S 2 we find the initial and final energy of the cluster. Section 3 constrains various factors that enter the normalization of \(M-T\) relation, via statistical methods, simulations, and observational input. Section 4 considers predictions of our model and comparison with observational and simulation results. In \S 5, we discuss the limitations of our approach and justify some of the approximations. Section 6 concludes the paper.

2. CONSERVATION OF ENERGY

2.1. Initial Kinetic and Potential Energy of a Protocluster

We begin by deriving the kinetic energy of the protocluster. We can write velocity as a function of gravitational potential \(\phi_i\) (e.g., Padmanabhan 1993),

\[
v = H_i x - \frac{2}{3 H_i} \nabla \phi_i ,
\]

at the initial time in the linear regime, where \(H_i\) is the Hubble constant at the initial time. There is a small dependence on the initial density parameter \(\Omega_i\) in equation (1), which we ignore since initially it is very close to unity and the difference will be in second-order terms that we ignore for the protocluster. Then the kinetic energy is given by

\[
K_i = \frac{1}{2} \int \rho v^2 d^3 x = \frac{1}{2} \rho_i \int \left( 1 + \delta \right) H_i x - \frac{2}{3 H_i} \nabla \phi_i \right)^2 \nabla \phi_i \right) d^3 x .
\]

Keeping the terms up to the linear order, we obtain

\[
K_i = \frac{1}{2} \rho_i \int \left( H_i^2 x^2 + \frac{2}{3} \nabla^2 \phi_i x^2 - \frac{2}{3} x \cdot \nabla \phi_i \right) d^3 x .
\]
In deriving equation (3), we have used the Poisson equation,
\[ \nabla^2 \phi_i = 4\pi G \rho_i \delta_i , \]
(4)
to substitute for \( \delta_i \), where \( \rho_i \) is the initial mean density of the universe. For equation (3) we then use Gauss’s theorem to make the third term similar to the second, at the expense of a surface term:
\[ K_i = \frac{1}{2} \rho_i \int \left( \frac{4}{3} \chi^2 + \frac{4}{3} \chi^2 \nabla^2 \right) \phi_i d^3 x - \frac{1}{2} \rho_i \oint \chi^2 \nabla \phi_i \cdot da . \]
(5)
Assuming that the deviations from spherical symmetry are not important at the boundary of the protocluster, we find \( \nabla \phi_i \) in the second term in equation (5) as a function of \( \delta_i \):
\[ \nabla \phi_i = \frac{\rho}{G} \frac{\partial M}{R_i^2} = \frac{\rho}{G} \int \delta_i d^3 x , \]
(6)
where \( G \) is the gravitational constant and \( R_i \) is the boundary radius of the initial protocluster, which leads to
\[ K_i = \frac{4\pi G \rho_i^2}{3 \Omega} \int \left[ (1 + 2 \delta_i) \chi^2 - R_i^2 \delta_i \right] d^3 x , \]
(7)
where we have used equation (4) to substitute back for \( \phi_i \), and \( \Omega_i \equiv 8 \pi G \rho_i / (3H_i^2) \). Let us now find an expression for the gravitational potential energy of the protocluster. Using its definition, we have
\[ U_i = - \frac{\rho_i^2}{2} \int \left[ 1 + \delta_i \chi(x_1) \right] d^3 x_1 \int \frac{d^3 x_2}{|x_1 - x_2|} . \]
(8)
Keeping the terms to the first order and using the symmetry under interchange of \( x_1 \) and \( x_2 \), we arrive at
\[ U_i = - \frac{\rho_i^2}{2} \int \left[ 1 + 2 \delta_i \chi(x_1) \right] d^3 x_1 \int \frac{d^3 x_2}{|x_1 - x_2|} , \]
(9)
which, taking the second integral in a spherical volume, gives
\[ U_i = - \frac{4\pi G}{3} \rho_i^2 \int \left( 1 + 2 \delta \left( \frac{3R_i^2 - x^2}{4} \right) \right) d^3 x . \]
(10)
Adding equations (7) and (10) gives the total initial energy
\[ E_i = \frac{4\pi G}{3} \rho_i^2 \left[ \frac{4\pi}{5} (1 - \Omega_i) R_i^2 - \frac{5}{2} \int \delta \chi(R_i^2 - x^2) d^3 x \right] , \]
(11)
to first order. Defining \( \bar{x}, \delta_i, \) and \( B \) as
\[ \bar{x} \equiv \frac{x}{R_i} \quad \delta_i \equiv \delta_i + \frac{3}{5} (\Omega_i - 1) , \]
\[ B \equiv \int_0^{R_i} \delta \chi(1 - \bar{x}^2) d^3 \bar{x} , \]
(12)
equation (11) is simplified to
\[ E_i = - \frac{10\pi G}{3} \rho_i^2 R_i^2 B . \]
(13)
The integral in the definition of \( B \) is in fact a three-dimensional integral, and the limits denote that the integration domain is the unit sphere. Note that \( \delta_i \) can be considered as the density perturbation to a flat (i.e., \( \Omega_m = 1 \)) universe for which energy vanishes. Another aspect of this statement is that both terms in the definition of scale as \( \Omega \) in the early (matter-dominated) universe, and so does \( \delta_i \) itself. For a flat universe, the first term dominates at high redshift, since the second term scales as \( a^3 \).

2.2. Energy of a Virialized Cluster

According to the virial theorem, the sum of the total energy \( E_f \) of a virialized cluster and its kinetic energy \( K_f \) vanishes. However, nonvanishing pressure at the boundary of the cluster can significantly modify the virial relation. Integrating the equation of hydrostatic equilibrium, we have
\[ K_f + E_f = 3P_{\text{ext}} V , \]
(14)
where \( P_{\text{ext}} \) is the pressure on the outer boundary of the virialized region (i.e., virial radius), and \( V \) is the volume. For now, we assume that the surface term is related to the final potential energy \( U_f \) by
\[ 3P_{\text{ext}} V = - \nu U_f . \]
(15)
We consider the coefficient \( \nu \) and its possible mass dependence in § 3.4. For a system of fully ionized gas plus dark matter, the virial equation (14) with equation (15) leads to
\[ - \left( \frac{1 + \nu}{1 - \nu} \right) E_f = K_f + \frac{3}{2} M_{\text{DM}} \sigma_v^2 + \frac{3}{2} M_{\text{gas}} kT , \]
(16)
where \( \sigma_v \) is the mass-weighted mean one-dimensional velocity dispersion of dark matter particles, \( M_{\text{DM}} \) is the total dark matter mass, \( k \) is the Boltzmann constant, \( \mu = 0.59 \) is the
mean molecular weight, and $m_p$ is the proton mass. Assuming that the ratio of gas to dark matter mass in the cluster is the same as that of the universe as a whole and $f$ is the fraction of the baryonic matter in the hot gas, we get

$$K_f = \frac{3\beta_{\text{spec}} M k T}{2\mu m_p} \left[ 1 + \left( \frac{f^2}{\beta_{\text{spec}}^2} - 1 \right) \frac{\Omega_b}{\Omega_m} \right],$$  \hspace{1cm} (17)

with $\beta_{\text{spec}} = \sigma_v^2/(kT/\mu m_p)$. Hydrodynamic simulations indicate that $\beta_{\text{spec}} \approx 1$. For simplicity, we define $\tilde{\beta}_{\text{spec}}$ as

$$\tilde{\beta}_{\text{spec}} = \beta_{\text{spec}} \left[ 1 + \left( \frac{f^2}{\beta_{\text{spec}}^2} - 1 \right) \frac{\Omega_b}{\Omega_m} \right].$$  \hspace{1cm} (18)

Thus, equation (16) reduces to

$$K_f = \frac{3\tilde{\beta}_{\text{spec}} M k T}{2\mu m_p}. \hspace{1cm} (19)$$

Assuming energy conservation (i.e., $E_i = E_f$) and combining this result with equations (13) and (16) leads to the temperature as a function of initial density distribution,

$$kT = \frac{5\mu m_p}{8\pi \tilde{\beta}_{\text{spec}}} \left[ 1 + \frac{\nu}{1 - \nu} \right] \frac{1}{H_t^2 R_i^2 B}.$$

In § 2.3 we find an expression for $H_t^2 R_i^2$ in terms of cluster mass $M$ and the initial density fluctuation spectrum.

2.3. Virialization Time

Defining $e$ as the energy of a test particle with unit mass, which is at the boundary of the cluster $R_i$ of mass $M$ initially,

$$e = \frac{v_i^2}{2} - \frac{GM}{R_i}.$$

The collapse time $t$ can be written as

$$t = \frac{2\pi GM}{(-2\epsilon)^{3/2}}. \hspace{1cm} (22)$$

Following the top-hat model, we assume that the collapse time of the particle is approximately the same as the time necessary for the particle to be virialized. Choosing $t$ to be the time of observation and assuming the mass $M$ interior to the test particle is virialized at $t$, and by combing equations (1), (6), and (22), we find $e$ as function of initial density distribution and relate it to the collapse time:

$$-2e = \frac{5}{4\pi} H_t^2 R_i^2 \int_0^1 \delta(\bar{\delta})d\bar{\delta} \bar{\delta} = \frac{2\pi GM}{t} \left( \frac{2\mu m_p}{\rho_i} \right)^{2/3}.$$

Using $M = (4/3) \pi R_i^3 \rho_i$ and the Friedmann equations, we obtain

$$A \equiv \int_0^1 \delta(\bar{\delta})d\bar{\delta} \bar{\delta} = \frac{2}{5} \left( \frac{3\pi^4}{8} \right)^{1/3} \left( \frac{\rho_i}{G\rho} \right)^{1/3}. \hspace{1cm} (24)$$

2.4. M-T Relation

Combining equations (20), (13), and (24), we arrive at the cluster temperature-mass relation:

$$kT = \frac{\mu m_p}{2\tilde{\beta}_{\text{spec}}} \frac{1 + \nu}{1 - \nu} \left( \frac{2\pi GM}{t} \right)^{2/3} \frac{B}{A}. \hspace{1cm} (25)$$

Note that although $B$ and $A$ are both functions of the initial moment $t_i$, since both are proportional to the scale factor $a$, the ratio is a constant; the derived $T$-$M$ relation (eq. [25]) does not depend on the adopted initial time, as expected. As a specific example, in the spherical top-hat model in which the density contrast is assumed to be constant, this ratio $B/A$ is $2/5$. Let us gather all the unknown dimensionless factors in $\tilde{Q}$:

$$\tilde{Q} = \left( \frac{\tilde{\beta}_{\text{spec}}}{0.9} \right)^{-1} \left( \frac{1 + \nu}{1 - \nu} \right) \left( \frac{B}{A} \right) \left( H_t \right)^{-2/3}. \hspace{1cm} (26)$$

Then, inserting the numerical values, equation (25) reduces to

$$kT = (6.62 \text{ keV}) \left( \frac{M}{10^{15} h^{-1} M_\odot} \right)^{2/3}, \hspace{1cm} (27)$$

or equivalently,

$$M = 5.88 \times 10^{13} \tilde{Q}^{-1.5} \left( \frac{kT}{1 \text{ keV}} \right)^{1.5} h^{-1} M_\odot, \hspace{1cm} (28)$$

where $H = 100 h$ km s$^{-1}$ Mpc$^{-1}$ is the Hubble constant. This result can be compared with the EMN96 simulation results,

$$M_{200} = (4.42 \pm 0.56) \times 10^{13} \left( \frac{kT}{1 \text{ keV}} \right)^{1.5} h^{-1} M_\odot. \hspace{1cm} (29)$$

To convert the $M_{500}$ masses of EMN96 to $M_{200}$, we have used the observed scaling of the mass with density contrast $M_5 \propto \delta_{200}^{0.246}$ (HMS99), which is consistent with the NFW profile (Navarro, Frenk, & White 1997) for simulated dark matter halos as well as observations (e.g., Tyson, Kochanski, & dell’Antonio 1998) in the relevant range of radii.

3. NUMERICAL FACTORS, SCATTER AND UNCERTAINTIES

In this section we try to use different analytical methods as well as the results of dark matter simulations and observed gas properties, available in the literature, to constrain the numerical factors that appear in $\tilde{Q}$ (eq. [26]).

3.1. $\beta_{\text{spec}}$ and the Gas Fraction

So far we have made no particular assumption about the gasdynamics or its history, and so we rely on the available observational results to constrain gas properties. Here $\beta_{\text{spec}}$ is defined as the ratio of kinetic energy per unit mass of dark matter to the thermal energy of gas particles. This ratio is typically of the order of unity, though different observational and theoretical methods lead to different values.

The hydrodynamic simulation results usually point to a larger value of $\beta_{\text{spec}}$. For example, Thomas et al. (2001) find $\beta_{\text{spec}} = 0.94 \pm 0.03$. On the other hand, observational data point to a slightly lower value of $\beta_{\text{spec}}$. Observationally, there is as yet no direct way to accurately measure the velocity dispersion of dark matter particles in the cluster, and one must assume that the velocity distribution of galaxies follows that of dark matter or adopt a velocity bias. Under the assumption of no velocity bias, Girardi et al. (1998) find it to be $0.88 \pm 0.04$. Girardi & Mezzetti (2001) study $\beta_{\text{spec}}$ for a sample of high-redshift clusters and do not find any evidence for redshift dependence. From the theoretical point of view, the actual value of $\beta_{\text{spec}}$ might be substantially different from the observed number, because $^{3}$ With the exception of any heating/cooling of the gas being negligible with respect to the gravitational energy of the cluster.
the velocity and density of galaxies do not necessarily follow those of the dark matter, which could have resulted in some nonnegligible selection effects. Unknown sources of heating, such as due to gravitational energy on small scales, which is often substantially underestimated in simulations due to limited resolutions, or baryonic processes like supernova feedback, may affect the value of $\beta_{\text{spec}}$ as well.

Hydrodynamic simulations show that only a small fraction of the baryons contribute to galaxy formation in large clusters (e.g., Blanton et al. 2000), and so $f$ is close to 1. Observationally, we quote Bryan (2000), who compiled different observations for the cluster mass fraction in gas and galactic components with $f = 1 - 0.26(T/10 \text{ keV})^{-0.35}$, (30) albeit with a large scatter in the relation. Inserting equation (30) into equation (18), we see that the correction to $\beta_{\text{spec}} \sim 0.9$ is less than 5% for all feasible cosmological models that are dominated by nonbaryonic dark matter. In what follows, unless mentioned otherwise, we adopt the value $\beta_{\text{spec}} = 0.9$, and absorb any correction into the overall normalization of the $T$-$M$ relation.

3.2. $B/A$: Single Central Peak Approximation and Freeze-Out Time

In the original top-hat approximation (Gunn & Gott 1972), which has been extensively used in the literature, the initial density distribution is assumed to be constant inside $R_c$, which leads to $B/A = 2/5$. Shapiro, Iliev, & Raga (1999) and Iliev & Shapiro (2001) have extended the original treatment of top-hat spherical perturbation to a more self-consistent case with a truncated isothermal sphere final density distribution including the surface pressure term (see §3.4). Here we consider the general case with an arbitrary density profile of a single-density peak. But before going further, let us separate out the term due to space curvature in the definition of $B/A$. Using the definitions of $B$ and $A$ in equations (12) and (24), and Friedmann equations to insert for $\rho_i$ in terms of the present-day cosmological parameters, we get

$$
\frac{B}{A} = \frac{b}{a} + \left(\frac{b}{a} - \frac{2}{5}\right) (1 - \Omega_m - \Omega_\Lambda) \left(\frac{H(t)}{\Omega_m \pi} \right)^{2/3},
$$

(31)

where $\Omega_m$ and $\Omega_\Lambda$ are the density parameters due to nonrelativistic matter and the cosmological constant, respectively, at the time of observation, and $a$ and $b$ are $^4$ the same as $A$ and $B$ with $\delta_i$ replaced by $\delta$, in their definitions, equations (24) and (12).

Assuming that the initial density profile has a single, spherically symmetric peak and assuming a power law for the initial linear correlation function at the cluster scale, we can replace $\delta(x)$ by $\xi(x)/\delta(0)$,

$$
\xi(x) = \left(\frac{r_{0i}}{x}\right)^{3+n},
$$

(32)

where $n$ is the index of the density power spectrum (Peebles 1980) and $r_{0i}$ is the correlation length. This gives

$$
\frac{B}{A} = \left(\frac{1}{1 - n/2}\right) \left[1 + \frac{3}{5} \frac{(n + 3)(1 - \Omega_m - \Omega_\Lambda)}{(n + 3)(1 - \Omega_m - \Omega_\Lambda)} \left(\frac{H(t)}{\Omega_m \pi} \right)^{2/3}\right],
$$

(33)

where all the quantities are evaluated at the age of the observed cluster. Note that for models of interest the physically plausible range for $n$ is ($-3, 0$). One can see that the second term in equation (33) is indeed proportional to $t^{2/3}$, and so for an open universe it dominates for large time (after curvature domination). Noting that in equation (25), the temperature is proportional to $t^{-2/3}(B/A)$, so when the second term dominates, the $T$-$M$ relation will no longer evolve with time. This indicates that in an open universe, the cluster formation freezes out after a certain time. The presence of freeze-out time is independent of the central peak approximation, since the ratio $b/a$ only depends on the statistics of the initial fluctuations at high redshifts, where there is very little dependence on cosmology. It is interesting to note that in the case of $n = -3$, the ratio has no dependence on cosmology and there is no freeze-out even in low-density universes. This is an interesting case in which linear theory does not apply, because all scales become nonlinear at the same time and the universe is inhomogeneous on all scales.

Voit (2000) uses a different method to obtain exactly the same result. As we argue next, both treatments ignore cluster mergers.

3.3. $B/A$: Multiple Peaks and Scatter

The single-peak approximation discussed in §3.2 ignores the presence of other peaks in the initial density distribution. In hierarchical structure formation models, the mass of a cluster grows with time through mergers as well as accretion. This means that multiple peaks may be present within $R_c$ and may suppress the effect of the central peak.

Assuming Gaussian statistics for initial density fluctuations, we can find the statistics of $b/a$. Note that using equation (24), we can fix the value $A$ (and hence $a$) for a given mass and virialization time. Thus, the problem reduces to finding the statistics of $b$ (or $B$) for fixed $a$. Under the assumption of a power-law spectrum (see Appendix), calculations give

$$
\langle b \rangle > \frac{4(1 - n)}{(n - 5)(n - 2)},
$$

(34)

where

$$
\Delta b = \frac{16 \pi^2 \pi^{n/2}}{(5 - n)(2 - n)} \left[\frac{n + 3}{n(7 - n)(n - 3)}\right]^{1/2} \left(\frac{r_{0i}}{R_c}\right)^{(n + 3)/2},
$$

(35)

which, inserting into equation (31), yields

$$
\langle B / A \rangle = \frac{4(1 - n)}{(n - 5)(n - 2)} \times \left[1 - \frac{n(n + 3)}{10(1 - n)} \left(1 - \Omega_m - \Omega_\Lambda\right) \left(\frac{H(t)}{\Omega_m \pi} \right)^{2/3}\right],
$$

(36)

$$
\Delta B / A = \tau(n) \left(\frac{M}{M_{0L}}\right)^{-(n + 3)/6} \left(\frac{\Omega_m H(t)}{\pi}\right)^{2/3},
$$

(37)

where $D(t)$ is the growth factor of linear perturbations, normalized to $(1 + z)^{-1}$ for large redshift, and

$$
M_{0L} = \frac{4}{3} \pi \rho_0 r_{0i}^3,
$$

(38)

$^4$ a should not be confused with the cosmological scale factor.
where $\zeta_{1}(r)$ is the linearly evolved correlation function at the present time, with $r_{0L}$ being the correlation length, and

$$\tau(n) \equiv \frac{20 \times 2^{-n/2}}{(5 - n)(2 - n)}\left[\frac{n + 3}{n(7 - n)(n - 3)}\right]^{1/2}. \quad (39)$$

Figure 2 shows the dependence of $B/A$ on $n$. The upper dotted curve shows the result of the single-peak approximation (eq. [33]), while three lower curves show the multiple-peak calculation described above (eq. [36]) and its approximation (eq. [33]), while three lower curves show the multi-

undergo a large number of mergers for large values of $n$. This implies that clusters dominate and the distribution becomes close to homogeneous (top-hat) on large scales. This implies that clusters undergo a large number of mergers for large values of $n$. Interestingly, as $n$ approaches zero, small peaks dominate and the distribution becomes close to homogeneous (top-hat) on large scales. This implies that clusters undergo a large number of mergers for large values of $n$. Interestingly, in this case the ratio $B/A$ again approaches $2/5$, the value for the top-hat case. We use the multiple-peak approximation in our subsequent calculations.

It is worth mentioning that the ratio of the cosmology-dependent term in the average value of $\langle B/A \rangle$ to the constant term, in the multiple-peak calculation, is small. For example, for $\Omega_m = 0.3$ and $n = -1.5$, this ratio is about 0.07. This implies that the freeze-out time is large comparing to the current age of the universe for feasible open cosmological models, and consequently $\langle B/A \rangle$ is largely determined by the spectral index of the underlying linear power spectrum $n$.

### 3.4. $v$: Surface Term and Its Dependence on the Final Equilibrium Density Profile

As discussed in § 2, corrections to the virial relation due to finite surface pressure change the $M-T$ relation (eq. [25]). Shapiro et al. (1999) and Iliev & Shapiro (2001) have previously also taken into account the surface pressure term in their treatment of the truncated isothermal sphere equilibrium structure with a top-hat initial density perturbation, and have found results in good agreement with simulations. Voit (2000) uses a NFW profile for the final density distribution to constrain the extra factor and finds that for typical concentration parameters $c \equiv r_{200}/r_s \sim 5$, $(y + 1)/(y - 1)$ is $\sim 2$. We investigate this correction, $v$, for a given concentration parameter $c$. Let us assume that the density profile is given by

$$\rho(r) = \rho_s f(r/r_s), \quad (40)$$

where $\rho_s$ is a characteristic density, $r_s$ the scale radius, and $f$ is the density profile. For the NFW profile,

$$f_{\text{NFW}}(x) = \frac{1}{x(1 + x)^2}. \quad (41)$$

Moore et al. (1999) have used simulations with higher resolutions to show that the central density profile is steeper than the one already probed by low-resolution simulations such as those used by NFW, yielding the Moore profile

$$f_{\text{Moore}}(x) = \frac{1}{x^{1.5}(1 + x^{1.5})}. \quad (42)$$

For a given $f(x)$, the mass of the cluster is

$$M = 4\pi \rho_s r_s^3 \int g(x) dx, \quad (43)$$

The gravitational energy of the cluster is given by

$$U = -\int \frac{GM \, dM}{r} = -16\pi^2 G \rho_s^2 r_s^4 \int f(x)g(x)x \, dx. \quad (44)$$

To find the surface pressure, we integrate the equation of hydrostatic equilibrium,

$$\nabla P = \rho g = -\frac{GM \rho r}{r^3}. \quad (45)$$

This leads to

$$P_{\text{ext}} = 4\pi G \rho_s^2 r_s^4 \int_0^\infty f(x)g(x)x^{-2} \, dx, \quad (46)$$

which, by the definition of $v$ (eq. [15]), gives

$$v(c, f) \equiv -\frac{3P_{\text{ext}}}{U} = \frac{c^3}{c + 1} \int_0^\infty f(x)g(x)x^{-2} \, dx. \quad (47)$$

Note that $v(c, f)$ is a function of both $c$ and the density profile $f$.

If we define $Q(c, f)$ as

$$Q(c, f) \equiv \frac{(1 + v)}{(1 + v)} f = \frac{B}{A(Ht)^{2/3}}. \quad (48)$$

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This is of course valid in the case of isotropic velocity dispersion profile. As an approximation, we neglect any correction due to this possible anisotropy.
equation (25) can be written as

\[ kT = \left( \frac{\mu m_p}{2\beta_{\text{spec}}} \right) (2\pi G \text{H})^{2/3} Q(c,f), \]  

(49)

or, in numerical terms,

\[ kT = (6.62 \text{ keV}) Q(c,f) \left( \frac{M}{10^{15} \text{ h}^{-1} M_{\odot}} \right)^{2/3} \]

(50)

for $\beta_{\text{spec}} = 0.9$. Note that with this choice of $\beta_{\text{spec}}$, the definition of $Q$ is equivalent to that of $\tilde{Q}$ in equation (26).

Figure 3 shows the average value of $y$ for different cosmologies, using equations (36) and (51), for $n = -1.5$.

Since all three parameters, $\psi(c,f)$, $\psi_y(c,f)$, and $Q(c,f)$, are functions of both $c$ and $f$, one can express any one of them as a function of another, for fixed $f$. Figure 4 shows the normalization factor $Q$ as functions of $y$. The dashed curves are for the NFW profile and the solid curves for the Moore et al. (1999) profile. The crosses with error bars show various simulation and observational results (see Table 1). Note that even if we relax the conservation of energy, one can still use $Q(y)$ to find the $T$-$M$ relation, using the value of $y$ obtained from its definition, equation (49), with the corrected energy.

An important feature of the behavior of $Q(y)$ is the presence of a minimum in or close to the region of physical interest. As a result, $Q(y)$ has a very weak dependence on the history of the cluster; for example, the largest variation in $Q$ is about 3%. This is probably why simulations do not show significant cosmology dependence, ~5% (e.g., EMN96; Mathiesen 2001).

Another way of stating this property is that the heat capacity of the cluster is very small. It is well known that the heat capacity of gravitationally bound systems such as stars is negative. Yet we know that if nongravitating gas is bound by an external pressure, its heat capacity is positive. In the case of clusters, the interplay of external accretion pressure and gravitational binding energy causes it to vanish. It is only after the freeze-out time in an extreme open universe (low $\Omega_m$, no cosmological constant) that the heat capacity becomes negative, similar to an ordinary gravitationally bound system.

### 3.5. Concentration Parameter $c$

We point out that by using conservation of energy, one can constrain the concentration parameter and subsequently the surface correction for a given density profile. A typical density profile is specified by two parameters: a characteristic density $\rho_c$ and scale radius $r_s$. If we know the mass (e.g., $M_{200}$) and total energy of the cluster, we can fix these two parameters. The concentration parameter is then fixed by $\rho_c$ and the critical density of the universe. To show the procedure, let us rederive the $T$-$M$ relation for a known density profile. Combining equations (16) and (19) gives

\[ kT = - \frac{2\mu m_p}{3\beta_{\text{spec}} M} \left( \frac{1 + y}{1 - y} \right) E_f \]  

(51)

**TABLE 1**

| $Q$     | Method                  | Reference |
|---------|-------------------------|-----------|
| 1.12 ± 0.02  | analytic                | 1         |
| 1.21 ± 0.06  | hydro simulation        | 2         |
| 1.05 ± 0.13  | hydro simulation        | 3         |
| 1.22 ± 0.11  | hydro simulation        | 4         |
| 1.32 ± 0.17  | optical mass estimate   | 5         |
| 1.70 ± 0.20  | resolved temperature profile | 5       |
| 1.70 ± 0.18  | resolved temperature profile | 6       |

* These rows only include clusters hotter than 3 keV.

**REFERENCES.** (1) This paper; (2) Thomas et al. 2001; (3) Bryan & Norman 1998; (4) Evrard et al. 1996; (5) Horner et al. 1999; (6) Finoguenov et al. 2001.
Note that equation (51) only depends on the properties of the virialized cluster and is independent of its history. Defining \( y \) as
\[
y \equiv \frac{4E}{3M} (2\pi GMH_0)^{-2/3},
\]
equation (51) reduces to
\[
kT = \frac{\mu m_p}{2\beta_{\text{spec}}} \left( \frac{1 + v}{1 - v} \right) \left( \frac{2\pi GM}{t_0} \right)^{2/3} [(H_0 t_0)^{2/3} y].
\]
Comparing this result with equation (25), we see that
\[
y = \frac{B}{A(Ht)^{2/3}},
\]
only if the energy is conserved (i.e., assuming \( E_f \) in eq. (49) is equal to \( E_i \)). On the other hand, by combining equation (52) with equations (43), (44), and (47) and the virial theorem of equation (14), \( y \) can be written as a function of \( c \), for a fixed density profile \( f \):
\[
y(c, f) = \frac{\Delta_{1/3} (1 - v) c \int_0^\infty f(x)g(x)x \, dx}{3\pi^{2/3} g^2(c)},
\]
where we have assumed the boundary of the virialized region to be the radius at which the average density is \( \Delta_c \) times the critical density of the universe (which is usually chosen to be 200), and \( v \) is a function of \( c \) and \( f \) in equation (47). Equation (55) fixes the concentration parameter \( c \) as a function of \( y \) for a fixed density profile \( f \), which in turn is determined by equation (54).

The concentration parameter is fixed by the cosmology (\( y \) parameter) as shown in Figure 5. This relation can be well fitted by
\[
\log_{10} c = -0.17 + 1.2y,
\]
for the NFW profile, accurate to 5% in the range \( 0 < y < 1 \).

Let us now consider the evolution of \( c \). We know that in an expanding universe \( \Omega_m \) decreases with time. Comparing with Figure 4, we see that in a flat \( \Lambda \)CDM universe \( y \) is decreasing with time, while in an open/Einstein–de Sitter universe, it is almost constant. Then, equation (56) implies that \( c \) is a decreasing function of time (increasing function of redshift) in a \( \Lambda \)CDM universe, while it does not significantly evolve in an OCDM universe. As an example, the concentration parameter in an Einstein–de Sitter universe is about 40% larger than that of a flat \( \Lambda \)CDM universe with \( \Omega_m = 0.25 \). This is consistent with the results of NFW, who find an increase of about 35% for \( c \) as a function of mass in units of nonlinear mass scale.

NFW simulations show a weak dependence of the concentration parameter on mass, \( c \propto M^{-0.1} \). We see that our concentration parameter does not depend on mass. However, its scatter is larger for small masses and so is marginally consistent with the simulation results (see Fig. 6). Assuming that this discrepancy is only a consequence of the nonspherical shape of the original protocluster, in the next section we attempt to modify the value of \( y \) to match the simulation results.

3.6. Corrections for Initial Nonsphericity

In this section we try to incorporate the effects due to the nonspherical shape of the initial protocluster into our formalism. Unlike previous sections, the calculations of this section are not very rigorous and should be considered as an estimate of the actual corrections. In particular, these approximations lose accuracy if there are large deviations from sphericity, which, as we see, is the case for the low-mass end of the \( M-T \) diagram.

We assume that nonsphericity comes in through a modifying factor \( 1 + \mathcal{N} \) that only depends on the initial
Note from equation (54), see that, although the Eke, Navarro, & Steinmetz (2001) fitting formula shows the same comparison for a de Sitter universe. We see that the introduction of nonsphericity parameter as a function of mass in an Einstein (2001) (Fig. 6). Figure 7 shows the modified concentration equation (56) to get the modified concentration parameter and its systematic error at the cluster mass range is indeed comparable to its difference with our prediction.

If we try to write down a perturbative expansion for , the lowest order terms will be quadratic, since there is no rotationally invariant first-order term. Moreover, keeping in mind that the gravitational dynamics is dominated by the large-scale structure of the object, as an approximation, we keep the lowest value. Since is only a translation of the sphere and does not change its geometry, the lowest nonvanishing multipoles are for , and the only rotation-invariant expression is

where we absorbed any constant factor in the definition of . The next simplifying assumption is that the are Gaussian variables, with amplitudes proportional to the amplitude of the density fluctuations at the cluster mass scale. Mathematically, this is motivated by the fact that the concentration parameter predicted by simulations is closer to our prediction for spherical protoclusters, at large-mass end, where amplitude is smaller.

The physical motivation is that since the density fluctuations decrease with scale, more massive clusters tend to deviate less from sphericity. Choosing Gaussian statistics for is only a simplifying assumption to carry out the calculations. Then it is easy to see that

and then using the definition of , assuming that it is a small correction, we get

Note from equation (54), , and are given in equation (37). We have also assumed that and are statistically independent variables. In the next step, we define the amplitude of ,

The numerical value of is fixed by plugging into equation (56) to get the modified concentration parameter and comparing this result with the simulations of Thomas et al. (2001) (Fig. 6). Figure 7 shows the modified concentration parameter as a function of mass in an Einstein–de Sitter universe. We see that the introduction of nonsphericity results in the cluster concentration parameter being a decreasing function of cluster mass with a scatter that also decreases with mass, in accord with simulations. Figure 8 shows the same comparison for an LCDM cosmology with the Eke, Navarro, & Steinmetz (2001) fitting formula. We see that, although was obtained by fitting the Einstein–de Sitter simulations, our prediction is marginally consistent with the LCDM simulations as well.

3.7. Scatter in M–T Relation

Since is linear in , it also has a Gaussian probability distribution function (PDF):

where and are related to equations (36) and (37), by equation (54). As we mentioned above, the variation of for the average values of is negligible. However, the scatter in the value of can be large, especially in the low-mass end (see eq. [37]), so the scatter in might become significant. In what follows we only consider the NFW profile, since it is extensively considered in the literature. We find that the behavior of can be fitted by

where

The error in this fitting formula is less than 3% in the range . Inserting this into equation (63) leads to the PDF of ,

where

Eke et al. (2001) fitting formula was made for a lower mass range, and its systematic error at the cluster mass range is indeed comparable to its difference with our prediction.
Figure 8.—Modified concentration parameter vs. mass for a ΛCDM cosmology with Ω_m = 0.3, Ω_λ = 0.7, n = −1.5, and σ_8 = 0.9. The dashed line is for the Eke et al. (2001) fitting formula with C_5 = 23, and the shaded area is the 68% likelihood region.

Figure 9 shows three different examples of the PDF obtained here. It is clear that the scatter in Q is asymmetric. In fact, since the average value of y is close to the minimum of equation (57), the scatter in y shifts the average value of Q upward systematically. In the limit of large Δy, where this

Figure 9.—Probability distribution function of normalization for three different y and Δy. The PDF vanishes for Q < Q_0 = 1.11. The large value of y = 1.0 is possible if one includes nonspherical corrections (§ 3.6).

Figure 10.—Our result with shifted normalization for an Einstein–de Sitter universe vs. FRB01 data. The shaded area indicates the 68% confidence level region (§ 3.7).

shift is significant, P(Q) is approximately

\[ P(Q) dQ = \frac{2Q dQ}{\sqrt{2\pi \Delta y^2 N(Q^2 - Q_0^2)}} \times \exp \left[ -\frac{(Q^2 - Q_0^2)}{2N \Delta y^2} \right] \Theta(Q - Q_0) , \]  

where Θ is the Heaviside step function.

Although the assumption of Gaussianity is not strictly valid for y, that includes nonspherical corrections, we can

Figure 11.—Our result with shifted normalization for an OCDM universe vs. FRB01 data. The shaded area indicates the 68% confidence level region (§ 3.7).
still, as an approximation, use the above expressions for the PDF by replacing $\bar{y}$ and $\Delta y$ by $\bar{y}_\nu$ and $\Delta y_\nu$. It is also easy to find the average of $Q^2$,

$$\langle Q^2 \rangle = Q_0^2 + N([\bar{y} - y_0]^2 + \Delta y^2)$$.

As an example, for $\bar{y} \sim y_0$, while $\Delta y = 0.5$ gives only about 16% systematic increase in $Q$, $\Delta y = 1.0$ leads to $\sim 60\%$ increase.

3.8. Observed and Average Temperatures

The temperature found in § 2.4 is the density-weighted temperature of the cluster averaged over the entire cluster. However, the observed temperature, $T_f$, can be considered as a flux-weighted spectral temperature averaged over a smaller, central region of the cluster. The two temperatures may be different, due to the presence of inhomogeneities in temperature. We use the simulation results by Mathiesen & Evrard (2001) to relate these two temperatures and refer the reader to their paper for their exact definitions and the discussion of the effects that lead to this difference:

$$T = T_f[1 + (0.22 \pm 0.05) \log_{10} T_f(\text{keV}) - (0.11 \pm 0.03)] .$$

(68)

This correction changes the mass-temperature relation from $M \propto T^{1.5}$ to $M \propto T^{1.64}$ for the observed X-ray temperatures. We use this correction in converting the observed X-ray temperature to virial temperature in Figures 1 and 10–12.

4. Predictions versus Observations

4.1. Power Index

It is clear from equation (49) that we arrive at the usual $M \propto T^{1.5}$ relation that is expected from simple scaling arguments and that is consistent with the numerical simulations (e.g., EMN96 or Bryan & Norman 1998). On the other hand, the observational $\beta$-model mass estimates lead to a steeper power index, in the range 1.7–1.8. Although originally interpreted as an artifact of the $\beta$-model (HMS99), the same behavior was seen for masses estimated from a resolved temperature profile (FRB01). FRB01 carefully analyzed the data and interpret this behavior as a bent in the $M$-$T$ at low temperatures. This was confirmed by Xu, Jin, & Wu (2001), who found the break at $T_X = 3$–4 keV.

As discussed in § 3.7, the asymmetric scatter in $Q$ introduces a systematic shift in the $M$-$T$ relation. For large values of $\Delta y$, all of the temperatures are larger than the value given by the scaling relation of equation (54) for average value of $y$. This scatter increases for smaller masses (see eq. [37]); hence, smaller clusters are hotter than the scaling prediction. As a result, the $M$-$T$ relation becomes steeper in the low-mass range as $Q$ increases, while the intrinsic scatter of the data is also getting larger. Indeed, increased scatter is also observed in the FRB01 data (Fig. 1 and Figs. 10–12 to compare with our prediction), but they interpret it as the effect of different formation redshifts. We address this interpretation in § 5.

4.2. Normalization

As we discussed in § 3.4, our normalization (i.e., $Q$ in eq. [50]) is rather stable with respect to variations of cosmology and the equilibrium density profile. Table 1 compares this value with various observational and simulation results. We see that our analytical method is consistent with the hydro-simulation results, indicating validity of our method, since both have a similar physics input and the value of $\beta_{\text{spec}}$ used here was obtained from simulations. On the other hand, X-ray mass estimates lead to normalizations about 50% higher than our result and simulations. The result of the optical mass estimates quoted is marginally consistent with our result. Assuming that this is true would imply that optical masses are systematically higher than X-ray masses by $\sim 80\%$. Lewis et al. (1999) have compared optical and X-ray masses for a sample of 14 clusters and found, on the contrary, a systematic difference of less than 10%. The optical masses used in the work of Lewis et al. (1999) were all derived by the CNOC group (Canadian Network for Observational Cosmology; Carlberg et al. 1996), while the Girardi et al. (1998) data, used in HMS99 analysis quoted above, was compiled from different sources and has larger scatter and unknown systematic errors. In fact, HMS99 excluded a number of outliers to get the correct slope, and the original data had even larger scatter (see their Fig. 1). Therefore, it may be that the systematic error in the optical result of Table 1 is much larger and so in agreement with other observations.

As discussed in § 3.1, one possible source for the difference between theoretical and observational normalizations is that the values for $\beta_{\text{spec}}$ are different in the two cases because of systematic selection effects. Also, intriguingly, Bryan & Norman (1998) show that there is a systematic increase in the value of $\beta_{\text{spec}}$ obtained while increasing the resolution of the simulations. Whether this is a significant and/or real effect for even higher resolutions is not clear to us.

However, the fact that the slope is unchanged indicates that the missing process is probably happening at small scales and so relates the intermediate-scale temperature to the small-scale flux-weighted spectral temperature by a constant factor, independent of the large-scale structure of the cluster. In this case the actual value of $\beta_{\text{spec}}$ must be $\sim 0.6$. 
Figures 10–12 show the prediction of our model, shifted downward to fit the observational data in the massive end, versus the observational data of FRB01 using the resolved temperature profile and corrected as discussed in § 3.8. The correction due to initial nonsphericity (§ 3.6) is included in the theoretical plot. The value of \( \sigma_b \), which enters \( \Delta B/A \) (eq. [37]) through \( M_{0L} \), is fixed by cluster abundance observations (e.g., Bahcal & Fan 1998). We see that while an Einstein–de Sitter cosmology underestimates the scatter in the low-mass end, a typical low-density OCDM cosmology overestimates it. On the other hand, a typical ΛCDM cosmology is consistent with the observed scatter. Interestingly, this is consistent with various other methods, in particular cosmic microwave background + SNe Ia results, which point to a low-density flat universe (de Bernardis et al. 2000; Balbi et al. 2000; Riess et al. 1998).

4.3. Evolution of M-T Relation

As we discussed in § 3.4, the value of our normalization has a weak dependence on cosmology. Going back to equation (49), we see that the time dependence of the M-T relation is simply \( M \propto h^{-1}T^{1.5} \). Assuming a constant value of \( \beta_{\text{spec}} \), this formula can potentially be used to measure the value of \( h \) at high redshifts and so constrain the cosmology.

Schindler (1999) has compiled a sample of 11 high-redshift clusters (0.3 < \( z < 1.1 \)) from the literature with measured isothermal β-model masses. In these estimates, the gas is assumed to be isothermal and to have the density profile

\[
\rho_\text{g}(r) = \rho_\text{g}(0) \left[ 1 + \left( \frac{r}{r_c} \right)^2 \right]^{-\frac{3}{2}\beta_{\text{fit}}}.
\]

Then, the mass in overdensity \( \Delta_c \) is given by

\[
M \approx \left( \frac{3H^2\Delta_c}{2G} \right)^{-1/2} \left( \frac{3\beta_{\text{fit}}}{G\mu m_p} \right)^{3/2} kT. \tag{70}
\]

Comparing this with equation (49), and neglecting the difference between virial and X-ray temperatures, we get

\[
Q \approx (3\pi)^{-2/3}(\beta_{\text{spec}}/\beta_{\text{fit}})\Delta_c^{1/3}. \tag{71}
\]

In the last two equations, we have ignored \( r_c \) with respect to the radius of the virialized region, which introduces less than 3% error. This allows us to find the normalization \( Q \) from the value of \( \beta_{\text{fit}} \) (independent of cosmology in this case).

Assuming \( \beta_{\text{spec}} = 0.9 \), equation (71) gives the value of \( Q \) for a given \( \beta_{\text{fit}} \). Figure 13 shows the value of \( Q \) versus redshift for the Schindler (1999) sample and also the FRB01 resolved temperature method for low-redshift clusters. This result is consistent with no redshift dependence, and the best fit is

\[
\log Q = 0.23 \pm 0.01 \text{(systematic)} \pm 0.04 \text{(random)}
\]

+ (0.09 \pm 0.04)z. \tag{72}

The combination of this result and equation (50) gives

\[
kT = (11.2 \pm 1.1 \text{ keV}) e^{0.21 \pm 0.09z} \left( \frac{M}{10^{15}h^{-1}M_\odot} \right)^{2/3}, \tag{73}
\]

which relates the X-ray temperature of galaxy clusters to their masses, which does not depend on the theoretical uncertainties with regard to the normalization coefficient of the M-T relation in this range of redshifts. The systematic error in this result is less than 5%, while the random scatter can be as large as 20%. This result is valid for \( T_X > 4 \text{ keV} \), since below this temperature the systematic shift due to random scatter becomes significant (§ 3.7). It is easy to see that this threshold moves to lower temperatures in high redshifts in our formalism.

Note that the more realistic interpretation of a possible evolution in observed \( Q \), obtained above, is that in equation (71), \( Q \) remains constant (Fig. 5), and instead, \( \beta_{\text{spec}} \) varies with time. However, there will be virtually no difference with respect to the M-T relation, and, moreover, the weak redshift dependence in equation (73) shows that \( \beta_{\text{spec}} \) is indeed almost constant.

5. DISCUSSION

In this section we discuss the validity of different approximations adopted throughout this paper.

In the calculation of the initial and final energy of the cluster, we ignored any contribution from the vacuum energy. In fact, we know that the cosmological constant in the Newtonian limit can be considered as ordinary matter with a constant density. If the cosmological constant does not change with time, then its effect can be considered as a conservative force, and so energy is conserved. However, in both the initial state and the final equilibrium state of the cluster, the density of the cosmological constant is much smaller than the density of matter, and hence its contribution is negligible. This does not hold for quintessence models of the vacuum energy, since \( \Lambda \) changes with time and energy is not conserved. This can be used as a potential method to distinguish these models from a simple cosmological constant.

Let us make a simple estimate of the importance of this effect. We expect the relative contribution of a varying
cosmological constant be maximum when the density of the protocluster is minimum. This happens at the turnover radius, which is almost twice \( r_s \), the virial radius, in the top-hat approximation. Then the contribution to the energy due to the vacuum energy would be
\[
\frac{\delta E}{E} \sim \frac{(4\pi/3)\Lambda(t_0/2)(2r_s)^3}{M} = \frac{8\, \Lambda(t_0/2)}{200\, \rho_s(t_0) r_s} = \frac{8}{200} \frac{H^2(t_0/2)}{H^2(t_0)} \Omega_\Lambda \left( \frac{t_s}{t_0} \right)^2 \approx 0.1 . \tag{74}
\]
This gives about 10% correction to \( y \) or about 15% correction to \( c \). We see that the effect is still small, but in principle observable if we have a large sample of clusters with measured concentration parameters.

A systematic error in our results might have been introduced by replacing the density distribution by an averaged radial profile that underestimates the magnitude of gravitational energy and so the temperature. In addition, as shown by Thomas et al. (2000), most of the clusters in simulations are either steeper or shallower than the NFW profile at large radii. Since our normalization is consistent with simulation results, we think that these effects do not significantly alter our predictions.

Let us now compare our results with that of Voit (2000), who made the first such analytic calculation (although we were not aware of his elegant work until the current work was near completion), which is not based on the top-hat initial density perturbation and uses the same ingredients to obtain the \( M-T \) relation. First, as noted in § 3.3, his result, which is equivalent to the central-peak approximation to find \( B/A \), ignores the possibility of mergers. As we see in Figure 2, the value of \( B/A \) and so \( y \) is about 50% larger in the single-peak case than the multiple-peak case. Although it does not change the normalization very much, it overestimates the concentration parameter (Fig. 5) in our formalism. However, as pointed out by the referee, our formalism for finding the value of \( c \) is based on the assumption of isotropic velocity dispersion (eq. [45]), and hence may not be directly compared to Voit (2000), who does not make such an assumption. Voit (2000) has also neglected the cosmology dependence of the surface correction, \((1 + v)/(1 - v)\), which gives a false cosmology dependence to the normalization. This is inconsistent with hydro simulations (e.g., EMN96; Mathiesen 2001) and does not give the systematic shift in the lower mass end, if we assume that it does not have a nongravitational origin.

Finally, we comment on the interpretation of the scatter/bent in the lower mass end as being merely due to different formation redshifts, as suggested by FRB01. Although different formation redshifts can certainly produce this scatter/bent, it is not possible to distinguish it from scatter in initial energy of the cluster or its initial nonsphericity, in our formalism. On the other hand, the FRB01 prescription, which assumes constant temperature after the formation time, is not strictly true, because of the ongoing accretion of matter even after the cluster is formed. So, as argued by Mathiesen (2001) using simulation results, the formation time might not be an important factor, whereas the effect can be produced by scatter in initial conditions of the protocluster.

6. CONCLUSION

We combine conservation of energy with the virial theorem to derive the mass-temperature relation of the clusters of galaxies and obtain the following results:

1. Simple spherical model gives the usual relation \( T = CM^{1/3} \), with the normalization factor \( C \) being in excellent agreement with hydro simulations. However, both our normalization and that from hydro simulations are about 50% higher than the X-ray mass estimate results. This is probably due to our poor understanding of the history of the cluster gas.

2. Nonsphericity introduces an asymmetric, mass-dependent scatter (the lower the mass, the larger the scatter) for the \( M-T \) relation, and thus alters the slope at the low-mass end (\( T \sim 3 \text{ keV} \)). We can reproduce the recently observed scatter/bent in the \( M-T \) relation in the lower mass end for a low-density CDM cosmology, while Einstein–de Sitter/ OCDM cosmologies under/overestimate this scatter/bent. We conclude that the behavior at the low-mass end of the \( M-T \) diagram can be used to constrain cosmological models.

3. We point out that the concentration parameter of the cluster and its scatter can be determined by our formalism. The concentration parameter determined using this method is marginally consistent with simulation results, which provides a way to find nonspherical corrections to the \( M-T \) relation by fitting our concentration parameter to the simulation results.

4. Our normalization has a very weak dependence on cosmology and formation history. This is consistent with simulation results.

5. We find a mass-temperature relation (eq. [73]) for clusters of galaxies, based on the observations calibrated by our formalism, which can be used to find masses of galaxy clusters from their X-ray temperature in the range of redshift \( 0 < z < 1.1 \) with an accuracy of 20%. This is a powerful tool for finding the evolution of the mass function of clusters, using their temperature function.

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APPENDIX

STATISTICS OF \( b/a \)

The terms \( a \) and \( b \) are defined as
\[
a = \int_0^1 \delta(\tilde{x}) d^3x , \quad b = \int_0^1 (1 - x^2)\delta(\tilde{x}) d^3x . \tag{A1}
\]
Assuming a Gaussian statistics for the linear density field, the probability distribution function for $a$ and $b$ takes a Gaussian form,

$$P(a, b) da db = \frac{1}{2\pi L} \exp \left[ -\frac{1}{2L} (\langle b^2 \rangle a^2 + \langle a^2 \rangle b^2 + 2\langle ab \rangle ab) \right] da db ,$$

$$L = \langle a^2 \rangle \langle b^2 \rangle - \langle ab \rangle^2 . \quad (A2)$$

Then, for fixed $a$, we have

$$\langle b \rangle = \frac{\langle ab \rangle}{\langle a^2 \rangle} , \quad (A3)$$

$$\Delta b = \sqrt{\frac{L}{\langle a^2 \rangle}} . \quad (A4)$$

To find the quadratic moments of $a$ and $b$, we assume a power-law linear correlation function,

$$\xi_2(r) = \langle \delta(x) \delta(x + r) \rangle = \left( \frac{r_{0i}}{r} \right)^{3+n} . \quad (A5)$$

The moments become:

$$\langle a^2 \rangle = 8\pi^2 \left( \frac{r_{0i}}{R_i} \right)^{3+n} F_{00}(n), \quad (A6)$$

$$\langle ab \rangle = 8\pi^2 \left( \frac{r_{0i}}{R_i} \right)^{3+n} [F_{00}(n) - F_{02}(n)], \quad (A7)$$

$$\langle b^2 \rangle = 8\pi^2 \left( \frac{r_{0i}}{R_i} \right)^{3+n} [F_{00}(n) - 2F_{02}(n) + F_{22}(n)], \quad (A8)$$

where $F_{ml}(n)$ is defined as

$$F_{ml}(n) = \frac{1}{8\pi} \int x_1^m x_2^l |x_1 - x_2|^{-1-n} d^3 x_1 d^3 x_2 , \quad (A9)$$

where the integral is taken inside the unit sphere. Taking the angular parts of the integral, this reduces to

$$F_{ml} = \frac{1}{n+1} \int_0^1 \int_0^1 d x_1 d x_2 x_1^{m+1} x_2^{l+1} |x_1 - x_2|^{-1-n} - |x_1 + x_2|^{-1-n} . \quad (A10)$$

Then, taking this integral for the relevant values of $m$ and $l$, and inserting the result into equations (A6)–(A8) and subsequently equations (A3)–(A4) gives

$$\langle b \rangle = \frac{4(1-n)}{(n-5)(n-2)} , \quad (A11)$$

with

$$\Delta b = \frac{16\pi^2 n/2}{(5-n)(2-n)} \left[ \frac{n+3}{n(n-3)} \right]^{1/2} \left( \frac{r_{0i}}{R_i} \right)^{(n+3)/2} . \quad (A12)$$

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