The problem of the natural oscillations of a rectangle in the micropolar theory of elasticity

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Abstract. We consider the plane problem of natural harmonic oscillations of a rectangle with
mixed boundary conditions within the framework of the linear theory of micropolar elasticity.
The exact analytical solution of the problem under consideration was obtained by the new
method. A parametric analysis of the obtained exact solution is carried out according to which
there is a "size effect". It consists in an increase in micropolar effects with a decrease in the
size of the rectangle compared with the classical case. The proposed method can be developed
for the case of other boundary conditions and for the three-dimensional case.

1. Introduction

Now it is clearly acknowledged that mechanical properties of many modern materials need be
described by means of generalized continua models. For this purpose, the Cosserat model is
most suitable, in which each particle of a continuous medium is endowed with six degrees of
freedom [1]. This model provides a useful description of medium with a complex microstructure
like polycrystalline, composite materials, granular and powderlike materials, masonries, cellular
or porous media and foams, bones, bioceramics, soils etc. At this time the micropolar elasticity
theory based on the Cosserat model is described in tens of books [2, 3] etc. This theory has
found applications in construction of various generalized models for beams, plates, and shells.

In this paper we study a dynamic problem in the linear theory of the micropolar elasticity for
isotropic homogeneous and centrally symmetric medium with small deformations [2]. In this case
the micropolar medium is determined by six independent elastic moduli. Despite the sufficient
development of the micropolar elasticity theory foundations only a small number of authors have
obtained exact analytical solutions of static and dynamic problems of micropolar elastic bodies
with finite sizes. A small number of known analytical solutions suit for creation of methods to
identify a micropolar elastic constants from laboratory measurements. Smith [4], Gauthier and
Jahsman [5], Reddy and Venkatasubramanian [6], Ieşan and Chirita [7], Chirita [8], Gauthier [9]
obtained some exact solutions for circular cylinders. These solutions have been extended to
isotropic hollow cylinders by Taliercio [10].

The first attempt to determine all six elastic constants was made by Gauthier and Jahsman
[5, 11]. The method used was based on the analytical solutions to the problems of a torsion of
a circular cylinder and bending of a rectangular plate. There are successful works of Lakes et al. on experimental determination of the micropolar elastic constants of materials with internal microstructures such as bones, polymeric foams, and metallic foams (see [12, 13] etc.).

From exact analytical solutions Chauhan [14] derived the effect of an increasing macroscopic stiffness for a semicircular ring of rectangular cross section bent by transverse radial shear resultants. Singh [15] obtained solutions for micropolar elastic domains with spherical boundaries. Meladze [16] obtained solutions to the main problems of micropolar elastic ball equilibrium in terms of absolutely and uniformly converging series. Kulesh [17] analytically solved the following one and twodimensional problems in the framework of the micropolar elasticity theory: shear deformation of elastic infinite plane layer (plate); torsion of a ring rigidly fixed at the external contour due to rotation of the inner one; deformation of a plane washer caused by a rigid displacement of the internal contour relative to the external one; the Kirsch problem on unilateral extension of a plate weakened by a circular hole.

For a parallelepiped and a rectangle a lot of exact analytical solutions are known. Khomassiridze [18] showed that in the case of so-called consistent boundary conditions on the surface of a parallelepiped, Dirichlet or Neumann problems arise for the divergence of the displacement vector. Developing this result Grigor’ev [19, 20, 21] stated the basics of the method for solving equilibrium problems.

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We pointed out above all known papers in which there are the exact analytical solutions for static problems of the micropolar elasticity in bounded domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. In this paper we present new exact analytical solution and its numerical implementations as well as parametric analysis. We consider the plane problem of the natural oscillations of a rectangle with mixed boundary conditions when tangential components of stresses, normal components of displacement and rotation vector are given on the boundary.

2. Formulation of the problem

The equations of micropolar elastic harmonic oscillations in the plane-strain state for the displacement vector and the rotation vector [2]

$$\mathbf{u} = u_1(x_1, x_2)i + u_2(x_1, x_2)j, \quad \omega = \omega(x_1, x_2)k,$$

have the form

$$(\mu + \alpha)\triangle u_1 + (\lambda + \mu - \alpha)(\nabla \cdot \mathbf{u})_1 + 2\alpha \omega_2 + \rho \tau^2 u_1 = 0,$$

$$(\mu + \alpha)\triangle u_2 + (\lambda + \mu - \alpha)(\nabla \cdot \mathbf{u})_2 - 2\alpha \omega_1 + \rho \tau^2 u_2 = 0,$$

$$B\triangle \omega + 2\alpha(u_{2,1} - u_{1,2}) - 4\alpha \omega + j \tau^2 \omega = 0, \quad \mathbf{r} \in P,$$

where $\lambda, \mu$ are Lame constants; $\alpha, \beta, \gamma, \varepsilon$ are physical constants of the material in the framework of a micropolar elastic medium; $B \equiv \gamma + \varepsilon; \tau$ is the oscillation frequency; $\rho$ is the mass density and $\mathbf{j}$ is the inertia coefficient responsible for the rotational property of medium particles. Hereinafter in the index, a coma with a variable means the partial derivative w.r.t this variable.

The equations of the considered oscillations for the components of the stress tensor and moment tensor are of the form [2]

$$\sigma_{11,1} + \sigma_{21,2} + \rho \omega^2 u_1 = 0, \quad \sigma_{12,1} + \sigma_{22,2} + \rho \omega^2 u_2 = 0, \quad \sigma_{21} - \sigma_{12} + j \omega^2 = 0. \quad (3)$$

The appearance of moment stresses is associated with the appearance of internal rotations. Therefore, it is natural to suggest that these moments do work only on internal rotations, and force stresses only on displacements. Then the dependence between the components of the stress tensor and moment stresses and the components of the displacement vector and the rotation vector takes the form (Hookes law) [2]

$$\sigma_{11} = \lambda \nabla \cdot \mathbf{u} + 2\mu u_{1,1}, \quad \sigma_{22} = \lambda \nabla \cdot \mathbf{u} + 2\mu u_{2,2}, \quad \sigma_{33} = \lambda \nabla \cdot \mathbf{u},$$
$$\sigma_{12} = (\mu + \alpha)u_{2,1} + (\mu - \alpha)u_{1,2} - 2\alpha \omega, \quad \sigma_{21} = (\mu + \alpha)u_{1,2} + (\mu - \alpha)u_{2,1} + 2\alpha \omega, \quad (4)$$

$$\mu_{13} = B\omega_{1}, \quad \mu_{23} = B\omega_{2}, \quad \mu_{31} = (\gamma - \varepsilon)\omega_{1}, \quad \mu_{32} = (\gamma - \varepsilon)\omega_{2}. \quad (4)$$

So, the problem of determining the natural oscillations \((u, \omega, \sigma, \mu)\) for a rectangle \(P \equiv \{r = (x_{1}, x_{2}) : 0 < x_{i} < a_{i} \equiv const, \ i = 1, 2\}\) of micropolar elastic material \((\rho, \lambda, \mu, \alpha, B, j)\) is investigated \((1)-(4)\), the rectangle is placed in a smooth hard pit with sizes that match the rectangle sizes

$$\sigma_{12}|_{x_{1}=a_{1},0} = u_{1}|_{x_{1}=a_{1},0} = \omega|_{x_{1}=a_{1},0} = 0, \quad \sigma_{21}|_{x_{2}=a_{2},0} = u_{2}|_{x_{2}=a_{2},0} = \omega|_{x_{2}=a_{2},0} = 0. \quad (5)$$

### 3. New analytical method

In this section, we give further development of the method formed by the authors in articles \([19, 21]\), based on the work of \([18]\). We are looking for a solution from the function class \(u_{1}, u_{2} \in C^{3}(P) \cap C^{3}(\overline{P})\) and \(\omega \in C^{4}(P) \cap C^{2}(\overline{P})\). We introduce a vector auxiliary function

$$f \equiv (\mu + \alpha)\nabla \times u. \quad (6)$$

The following four statements can be proved.

**Statement 1.** Let \(u\) be a solution to problem \((2), (5)\). Then the auxiliary function \((6)\) satisfies the following two Sturm-Liouville problems

$$(\Delta + \lambda_{1}^{2})f_{1} = 0, \quad r \in P \quad (\Delta + \lambda_{1}^{2})f_{2} = 0, \quad r \in P \quad (7)$$

$$f_{1}|_{x_{1}=a_{1},0} = f_{1}|_{x_{2}=a_{2},0}, \quad f_{2}|_{x_{1}=a_{1},0} = f_{2}|_{x_{2}=a_{2},0},$$

where

$$\lambda_{1}^{2} + \lambda_{2}^{2} = \frac{(\mu + \alpha)j + j\sigma}{B(\mu + \alpha)} - 4\alpha \mu, \quad \lambda_{1}^{2} \lambda_{2}^{2} = \frac{\rho j^{2} (j^{2} - 4 \alpha)}{B(\mu + \alpha)}. \quad (8)$$

**Statement 2.** Let the auxiliary function \(f\) \((6), (7)\) be known. Then the component of the rotation vector \(\omega(x_{1}, x_{2})\) satisfies the problem in the form

$$\Delta \omega + \frac{j\tau^{2} - 4 \alpha}{B} \omega = -\frac{2 \alpha}{\alpha + \mu} f_{3}, \quad r \in P \quad (9)$$

$$\omega|_{x_{1}=a_{1},0} = \omega|_{x_{2}=a_{2},0} = 0.$$

We also introduce a scalar auxiliary function

$$f \equiv (\mu + \alpha)\nabla \cdot u,$$

for which the following is true.

**Statement 3.** Let \(u\) be a solution to the problem \((2), (5)\). Then the auxiliary function \((9)\) satisfies the Sturm-Liouville problem in the form

$$\Delta f + \frac{\rho j^{2}}{\lambda + 2 \mu} f = 0, \quad r \in P \quad (10)$$

$$f_{1,1}|_{x_{1}=a_{1},0} = 0, \quad f_{2,1}|_{x_{2}=a_{2},0} = 0.$$
Statement 4. Let the auxiliary function $f$ (9) and the component of the rotation vector $\omega(x_1, x_2)$ be found. Then the components of the displacement vector $u$ are solutions of the following problems
\[
\triangle u_1 + \frac{\rho r^2}{\mu + \alpha} u_1 = -\frac{1}{\mu + \alpha} \left( \frac{\lambda + \mu - \alpha}{\lambda + 2\mu} f_{1,1} + 2\alpha \omega_{1,2} \right), \quad r \in P
\]
\[u_1|_{x_1=a_1,0} = u_{1,2}|_{x_2=a_2,0} = 0,\]
and
\[
\triangle u_2 + \frac{\rho r^2}{\mu + \alpha} u_2 = -\frac{1}{\mu + \alpha} \left( \frac{\lambda + \mu - \alpha}{\lambda + 2\mu} f_{2,2} + 2\alpha \omega_{1,1} \right), \quad r \in P
\]
\[u_2|_{x_2=a_2,0} = u_{2,1}|_{x_1=a_1,0} = 0.\]
Thus, the solution of the problem of the natural oscillations of a micropolar elastic rectangle with mixed boundary conditions (1)-(5) is reduced to the sequential solution of problems in Statements 1-4. These problems are solved by standard methods of mathematical physics, for example, by the method of separation of variables [22].

So, the eigenvalues of the problems of Statement 1 have the form
\[
\lambda_1^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \lambda_2^2 = \pi^2 \left( \frac{p^2}{a^2} + \frac{q^2}{b^2} \right), \quad (10)
\]
where $m, n = 1, 2, 3, ..., p, q = 1, 2, 3, ...$. Then from (8), (10) we find the eigenoscillations
\[
\tau_f^2 = \tau_f^2 \equiv \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) + \frac{p^2}{a^2} + \frac{q^2}{b^2} \left( \frac{\pi^2 B (\mu + \alpha)}{(\mu + \alpha) j + \rho B} + \frac{4\alpha\mu}{(\mu + \alpha) j + \rho B} \right). \quad (11)
\]
where $\tau_f^2$ satisfies the matching conditions
\[
\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \left( \frac{p^2}{a^2} + \frac{q^2}{b^2} \right) = \frac{\rho r^2(\pi^2 - 4\alpha)}{\pi^2 B (\mu + \alpha)}, \quad \tau_T > 2\sqrt{\frac{\alpha}{j}}. \quad (12)
\]
Next, respectively Statements 2-4 we find
\[
\tau_\omega = \sqrt{\frac{j\tau_f^2 - 4\alpha}{B}}, \quad \tau_f = \sqrt{\frac{\lambda + 2\mu}{\rho} \tau_T}, \quad \tau_u = \sqrt{\frac{\mu + \alpha}{\rho} \tau_T}. \quad (13)
\]
Thus, there are three ”varieties” of frequencies (11)-(13) of natural oscillations of the studied problem (1)-(5). As compared with the case of the classical medium, the frequencies $\tau_\omega$ in (13) arise which are responsible for the micropolar behavior of the medium. Moreover, the boundary conditions on the rectangle that placed in a smooth hard pit with sizes that match the rectangle sizes (5) do not allow the traditional method of dividing the displacement vector into a potential and solenoid component to be applied.

Solving the corresponding boundary value problems in the Statements 1-4, one can write out the forms of natural oscillations $(u, \omega, \sigma, \mu)$ corresponding to the natural frequencies (13).

4. Parametric analysis
In this section we give numerical implementations of representations for the quantities obtained in the previous sections associated with the foamy polyuteran considered by Lakes [23]. For such a material we have $\rho=340\ \text{kg/m}^3$, $\lambda = 416\ \text{Pa}$, $\mu = 104\ \text{Pa}$, $\alpha = 4.33\ \text{Pa}$, $B = 45.3\ \text{N}$, microstructure size $r \sim 0.0011\ \text{m}$, $j = 1.76 \cdot 10^{-3}\ \text{kg/m}$.

In the next page Table 1 shows the numerical values of the natural oscillations frequency calculated according to the representations (11)-(13). With increasing rectangle size, the influence of micropolar effects decreases sharply.
Table 1. Natural oscillation frequency depending on the size of the rectangle

| a(m) | b(m) | m | n | p | q | $\tau_f$ (Hz) | $\tau_\omega$ (Hz) | $\tau_f$ (Hz) | $\tau_\omega$ (Hz) |
|------|------|---|---|---|---|--------------|------------------|--------------|------------------|
| 0.003 | 0.002 | 1 | 2 | 1 | 1 | 8.9·10$^5$ | 2734.14 | 1.21·10$^9$ | 5.07·10$^8$ |
| 0.01 | 0.016 | 1 | 2 | 1 | 1 | 2.4·10$^5$ | 421.4 | 3.25·10$^8$ | 1.35·10$^8$ |
| 0.1 | 0.102 | 1 | 1 | 8 | 9 | 1.98·10$^5$ | 46.6 | 2.69·10$^8$ | 1.18·10$^8$ |
| 1 | 1.002 | 1 | 1 | 70 | 96 | 1.98·10$^5$ | 4.7 | 2.69·10$^8$ | 1.12·10$^8$ |
| 10 | 10.015 | 1 | 1 | 640 | 1000 | 1.98·10$^5$ | 0.47 | 2.68·10$^8$ | 1.11·10$^8$ |

5. Conclusions
Unlike the classical theory of elasticity there are three ”varieties” of frequencies (11)-(13) of natural harmonic oscillations of a rectangle, the rectangle is placed in a smooth hard pit with sizes that match the rectangle sizes within the framework of the linear theory of micropolar elasticity (1)-(5). With increasing rectangle size, the influence of micropolar effects decreases sharply (”size effect”). The proposed method can be developed for the case of other boundary conditions and for the three-dimensional case of harmonic oscillations of a rectangular parallelepiped.

References
[1] Cosserat E and Cosserat F 1909 Théorie des Corps Déformables (Paris: Herman et Fils)
[2] Nowacki W 1986 Theory of Asymmetric Elasticity (Warsaw: PWN Polish Scientific Publishers)
[3] Eremeyev V A, Lebedev L P and Altenbach H 2013 Foundations of Micropolar Mechanics (Heidelberg: Springer)
[4] Smith A C 1970 Recent Advances in Engineering Science vol 5 ed A Eringen (Heidelberg: Springer) pp. 129–37
[5] Gauthier R and Jahsman W 1975 Trans ASME Ser. E J. Eng. Mech. 42 (2) 369–74
[6] Reddy G V K and Venkatasubramanian N K 1976 Int. J. Eng. Sci. 14 1047–57
[7] Iesan D and Chirita S 1979 Int. J. Eng. Sci. 17 573–86
[8] Chirita S 1981 Int. J. Eng. Sci. 19 845–53
[9] Gauthier R D 1982 Experimental investigations on micropolar media Mechanics of Micropolar Media (Singapore: World Scientific) pp 395–463
[10] Taliercio A 2010 Mech. Res. Commun. 34 406–11
[11] Gauthier R and Jahsman W 1976 J. Appl. Mech. Trans ASME. 43 502–3
[12] Yang J and Lakes R 1981 J. Appl. Mech. Trans ASME. 15 91–8
[13] Yang J and Lakes R 1982 J. Biomech. 103 275–9
[14] Chauhan R S 1969 Int. J. Eng. Sci. 23 77–87
[15] Singh S J 1975 Gerl. Beitr. Geophys. 7 895–903
[16] Meladze R V 1987 Proceedings of the Institute of Applied Mathematics. I.N. Vekua 84 55–66 (in Russian)
[17] Kulesh M A, Matveenko V P and Shardakov I N 2003 ZAMM-Z. Angew. Math. Me. 23 (4) 238–48
[18] Khomasuridze N G 1972 Study of Some Equations of Mathematical Physics (Tbilisi, Georgian S.S.R.: Tbilisi State Univer.) pp 123–47
[19] Grigorev Yu M 1992 Model. Mekh. 6 (23) 21–6 (in Russian)
[20] Grigorev Yu M 2007 Vestnik YuGU 4 (4) 19–26 (in Russian)
[21] Grigorev Yu M and Gavrilieva A A 2019 Continuum Mech. Therm. 31 (6) 1699–718
[22] Budak B M, Tikhonov A N and Samarsky A A 1980 Collection of Problems in Mathematical Physics (Moscow: Nauka)
[23] Lakes R 1995 Experimental methods for study of Cosserat elastic solids and other generalized continua Continuum Models for Materials with Micro-Structure ed H Mühlhaus (New York: J. Wiley & Sons, Inc.) pp 1–22