Klein-Gordon particles in Som-Raychaudhuri cosmic string spacetime with space-like dislocation: vorticity-energy and charge-energy correlations

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Abstract: We argue that only through some reliable and admissible quantum mechanical treatments, one may understand and explore the effects of gravitational fields (generated by the corresponding spacetime structures) on the energy levels of relativistic and non-relativistic quantum particles. In the current proposal, we intend to clean up and correct the quantum mechanical mess (like additional quantization recipes or effective potential parametric correlations provided by the biconfluent Heun series/polynomials) injected into the literature and introduce proper treatments so that the effects of the gravitational fields on the quantum mechanical particles are made clear. Here, moreover, we consider position-dependent mass KG-particles in Som-Raychaudhuri cosmic string spacetime with space-like screw dislocation. New concepts like vorticity-energy and charge-energy correlations emerge in the process as consequences of the gravitational field effects on the KG-particles spectra. We support our findings with a brute-force-evidence that would clean up the injected-into-the-literature quantum mechanical mess for good.

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I. INTRODUCTION

The first cosmological solutions, with rotating matter, of Einstein’s field equations of gravitation were suggested by the Gödel spacetime metrics [1]. A generalization of the Gödel-type spacetime metrics is introduced and shown to provide new solutions for gravity and supergravity theories is different dimensions [2, 3]. It has been explicitly verified that the Gödel-type spacetime metrics with both flat and non-flat backgrounds always have closed time-like curves or null like curves [4]. Rebouças and Tiomno [5] have shown that all spacetime (ST)-homogeneous Gödel-type metrics characterized by the vorticity \( \Omega = \pm |\Omega| \) can be transformed (with \( h = c = G = 1 \) units) in cylindrical coordinates [6–10] to

\[
 ds^2 = - \left( dt + \alpha \Omega \frac{\sinh^2(\tilde{\mu} r)}{\tilde{\mu}^2} d\varphi \right)^2 + \alpha^2 \frac{\sinh^2(2\tilde{\mu} r)}{4 \tilde{\mu}^2} d\varphi^2 + dr^2 + dz^2. \tag{1}
\]

Where \( \alpha = 1 - 4\tilde{\mu} \) is the disclination parameter (or the deficit angle of the conical geometry), and \( \tilde{\mu} \) is the linear mass density of the string. It should be noted, moreover, that \( 0 < \alpha < 1 \) is used in general relativity with cosmic string (\( \tilde{\mu} > 0 \)), \( \alpha = 1 \) corresponds to Minkowski flat spacetime, and \( \alpha > 1 \) is used in the geometric theory of defects in

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condensed matter ($\mu < 0$). Yet, at the limit $\mu \to 0$ of the Gödel spacetime metric $[1]$ we obtain the ST-homogeneous Som-Raychaudhuri (SR) solution

$$ds^2 = -(dt + \alpha \Omega r^2 d\varphi)^2 + \alpha^2 r^2 d\varphi^2 + dr^2 + dz^2$$

(2)

of the Einstein field equations $[11]$. It is interesting to know that the original Gödel solutions $[1]$ are obtained for $\mu^2 = \Omega^2/2$ and $\alpha = 1$, the anti-de Sitter one $[12]$ for $\mu^2 = 0$ and $\alpha = 1$, and the Minkowski one for $\Omega = 0$ and $\alpha = 1$. In the current study, however, we shall also consider a space-like dislocation so that the ST-homogeneous SR-metric with space-like dislocation reads

$$ds^2 = -(dt + \alpha \Omega r^2 d\varphi)^2 + \alpha^2 r^2 d\varphi^2 + dr^2 + (dz + \delta d\varphi)^2,$$

(3)

where $\delta$ denotes space-like dislocation parameter. Hence, the covariant and contravariant metric tensors associated with the Som-Raychaudhuri spacetime with space-like dislocation are given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & -\alpha \Omega r^2 & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha \Omega r^2 & 0 & \left[\alpha^2 r^2 (1 - \Omega^2 r^2) + \delta^2\right] & \delta \\ 0 & 0 & \delta & 1 \end{pmatrix} \iff g^{\mu\nu} = \begin{pmatrix} (\Omega^2 r^2 - 1) & 0 & \frac{\Omega \delta}{\alpha} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\Omega}{\alpha} & 0 & \frac{1}{\alpha^2 r^2} & -\frac{\delta}{\alpha^2 r^2} \\ \frac{\Omega \delta}{\alpha} & 0 & -\frac{\delta}{\alpha^2 r^2} & \left(1 + \frac{\delta^2}{\alpha^2 r^2}\right) \end{pmatrix},$$

(4)

where $\det (g) = -\alpha^2 r^2$. The Gödel-type ST-homogeneous SR-spacetime with space-like dislocation metric $[31]$ is the core of the current proposal.

However, in classical mechanics, the introduction of Mathews and Lakshmanan $[13]$ position-dependent mass (PDM) oscillator has inspired studies on PDM-systems. Few years later, von Roos $[14]$ has introduced PDM-Schrödinger Hamiltonian in quantum mechanics. Such two prominent PDM-models have subsequently attracted research interest over the last few decades both in quantum and classical mechanics (e.g., $[15–31]$). Hereby, it has been emphasized that the PDM notion is a metaphor notion that is manifested by coordinate transformation/deformation $[26,31]$. Such coordinate transformation/deformation would necessarily and effectively change the form of the canonical momentum in classical mechanics and the momentum operator in quantum mechanics. In classical mechanics, for example, the canonical momentum $p = m_o \dot{u}$ for a particle of rest mass $m_o$. Which, under the coordinate transformation

$$u = \int \sqrt{g(x)} dx = \sqrt{Q(x)} x \iff \sqrt{g(x)} = \sqrt{Q(x)} \left[1 + \frac{Q'(x)}{2Q(x)} x\right],$$

(5)

would imply that $\dot{u} = \sqrt{g(x)} \dot{x}$ and hence the canonical momentum is now given by $p(x) = m_o \dot{u} = m_o \sqrt{g(x)} \dot{x}$. Where, $g(x)$ and $Q(x)$ are positive-valued scalar multipliers. One would then rewrite the PDM canonical momentum as $p(x) = M(x) \dot{x}$, where $M(x) = m_o \sqrt{g(x)}$. This what renders the mass to be metaphorically and effectively position-dependent. In quantum mechanics on the other hand, the PDM-momentum operator is shown $[26]$ to be given, in $\hbar = 2m_o = 1$ units, by

$$\hat{p}_j (x) = -i \left(\partial_j - \frac{\partial g(x)}{4g(x)} \right); \ j = 1, 2, 3,$n$$

(6)

and consequently yields, in its most simplistic one-dimensional form, the von Roos $[14]$ PDM kinetic energy operator

$$\hat{T} = \left(\frac{\hat{p}_x (x)}{\sqrt{g(x)}}\right)^2 = -g(x)^{-1/4} \partial_x g(x)^{-1/2} \partial_x g(x)^{-1/4},$$

(7)

where $\theta$ denotes space-like dislocation parameter.
which is known in the literature as Mustafa and Mazharimousavi’s ordering. In short, moreover, for quantum mechanical particles under effective/metaphoric PDM settings, the textbook momentum operator $\hat{p} = -i\partial_x$ should be replaced by the PDM momentum operator of (6). So should be the case for relativistic PDM quantum particles [28–30]. Yet, in the relativistic Dirac and Klein-Gordon (KG) wave equations the assumption that the rest mass energy term $m_0 \rightarrow m_0 + m(x) + S(x)$ would give PDM settings [32–36] should be abandoned, for it just redefines the Lorentz scalar potential $S(x) \rightarrow m(x) + S(x)$ and does not introducing PDM settings as a manifestation of coordinate transformation/deformation. PDM KG-particles moving in the background of SR-spacetime with space-like dislocation form the complementary part of the current proposal.

We may argue, herewith, that only through some reliable and admissible quantum mechanical treatments, one may understand and explore the effects of gravitational fields (generated by the corresponding spacetime structures) on the energy levels of relativistic and non-relativistic quantum particles. In the current proposal, we intend to clean up and correct the quantum mechanical mess injected into the literature (e.g., [36, 38, 39, 41–47] and related references cited therein) and introduce proper treatments so that the effects of the gravitational fields on the quantum mechanical particles are made clear. New concepts, like vorticity and charge energy-correlations, emerge in the process as consequences of the gravitational field effects on the KG-particles spectra in particular and relativistic particles in general. Moreover, we consider PDM KG-particles in Som-Raychaudhuri cosmic string spacetime with space-like screw dislocation [3].

The organization of the current proposal is in order. In section 2, we revisit and discuss KG-particles in SR-type cosmic string spacetime with space-like dislocation. We observe that, as a consequence of SR-spacetime structure, KG-oscillators are formed and analysed in details. Where, we discuss the emerging spacetime associated degeneracies (STAD) (i.e., we found that part of the energy levels for a given radial quantum number collapse into the corresponding $S$-state, namely, states with the magnetic quantum number $\ell = 0$) and vorticity-energy correlations between KG-oscillators that may serve for quantum entanglement. Such analysis is reflected, in section 3, on PDM KG-oscillators in SR-type cosmic string spacetime with space-like dislocation. Therein, we discuss the spacetime associated (STAD) and charge-energy correlations. In section 4, we discuss and report on some PDM KG-oscillators in a linear confinement in SR-type cosmic string spacetime with space-like dislocation. Wherein, we elaborate on the biconfluent Heun series/polynomials approach and discuss the what should be classified as quantum mechanical redundant conditions. Such redundant condition have created quantum mechanical mess through the claim/believe that it introduces further quantization recipes and/or correlates the effective potential parameters. We clean up this mess and put the gravitational field effects on quantum particles (in general) on the correct track. We devote section 5 for our concluding remarks.

II. KG-PARTICLES IN SR-TYPE COSMIC STRING SPACETIME WITH SPACE-LIKE SCREW DISLOCATION

The KG-equation for a spin-0 particle in a 4-vector $A_\mu$ and a Lorentz scalar potentials $S(r)$ in the Gödel SR-type spacetime with space-like dislocation [3] is given by

$$\frac{1}{\sqrt{-g}} \tilde{D}_\mu \left( \sqrt{-gg^{\mu\nu}} \tilde{D}_\nu \Psi \right) = (m_0 + S(r))^2 \Psi, \tag{8}$$
where \( m_\circ \) represents the rest mass energy of the KG-particle, \( e \) is its charge, \( S (r) \) is the Lorentz scalar potential, and the gauge-covariant derivative is given by \( \tilde{D}_\mu = D_\mu + F_\mu = \partial_\mu + F_\mu - ieA_\mu \). This would in turn allow us to cast equation (8) as

\[
\left\{ (\Omega^2 r^2 - 1) \tilde{D}_t^2 - \frac{2\Omega}{\alpha} \tilde{D}_t \tilde{D}_\phi + \frac{2\Omega \delta}{\alpha} \tilde{D}_t \tilde{D}_x + \frac{1}{r} \tilde{D}_x r \tilde{D}_r \\
+ \frac{1}{\alpha^2 r^2} \tilde{D}_\phi^2 - \frac{2\delta}{\alpha^2 r^2} \tilde{D}_\phi \tilde{D}_x + \left( 1 + \frac{\delta^2}{\alpha^2 r^2} \right) \tilde{D}_x^2 - (m_\circ + S (r))^2 \right\} \Psi = 0. \tag{9}
\]

Let us assume that \( A_\mu = (0, 0, A_\phi, 0) \), \( F_\mu = (0, F_r, 0, 0) \); \( F_r = g' (r) / 4g (r) \), where \( g (r) \) is a positive valued scalar multiplier that introduces PDM settings \( 6 \). We now define our covariant derivatives as

\[
\tilde{D}_t = \partial_t, \quad \tilde{D}_r = \partial_r + F_r, \quad \tilde{D}_\phi = \partial_\phi - ieA_\phi, \quad \tilde{D}_x = \partial_x,
\]

(10)

to imply

\[
\left\{ (\Omega^2 r^2 - 1) \partial_t^2 - \frac{2\Omega}{\alpha} (\partial_\phi - ieA_\phi) + \frac{2\Omega \delta}{\alpha} \partial_t \partial_\phi + \frac{1}{r} (\partial_r + F_r) \partial_r - F_r (\partial_r - F_r) + \frac{1}{\alpha^2 r^2} (\partial_\phi - ieA_\phi)^2 \\
- \frac{2\delta}{\alpha^2 r^2} (\partial_\phi - ieA_\phi) \partial_x + \left( 1 + \frac{\delta^2}{\alpha^2 r^2} \right) \partial_x^2 - (m_\circ + S (r))^2 \right\} \Psi = 0 \tag{11}
\]

Then, we use the wave function

\[
\Psi (t, r, \phi, z) = \exp (i [\ell \phi + k z - Et]) \psi (r) = \exp (i [\ell \phi + k z - Et]) \frac{R (r)}{\sqrt{r}}, \tag{12}
\]

and the assumptions

\[
A_\phi = -\frac{\alpha B_\circ}{2} r^2 + \frac{\phi B_\circ}{2\pi} \Rightarrow eA_\phi = -\frac{\tilde{B}_\circ}{2} r^2 + \bar{\phi}; \quad \bar{\phi} = \frac{e \phi B_\circ}{2\pi}, \quad \tilde{B}_\circ = \alpha e B_\circ, \tag{13}
\]

where \( \phi B_\circ \) is an internal quantum magnetic flux that admits constant values and \( B_\circ \) is the external magnetic field strength \( 10, 37, 38 \). Under such settings, one obtains

\[
\left\{ \partial_t^2 - \frac{(L^2 - 1/4)}{r^2} - \Omega^2 r^2 - M (r) - 2m_\circ S (r) - S (r)^2 + \lambda \right\} R (r) = 0, \tag{14}
\]

with

\[
M (r) = -\frac{3}{16} \frac{g' (r)}{g (r)} \frac{2}{4} \left( g'' (r) / 4r g (r) \right) \Omega^2 = \left( \Omega E + \frac{e B_\circ}{2} \right)^2, \tag{15}
\]

\[
\lambda = E^2 - 2 \left( \Omega E + \frac{e B_\circ}{2} \right) L - (k^2 + m_\circ^2) \Omega = \ell - \bar{k}, \quad \hat{\ell} = \frac{\ell - \bar{\ell}}{\alpha}, \quad \hat{k} = \delta, \quad \hat{L} = \left( \hat{\ell} - \hat{k} \right)^2, \tag{16}
\]

and \( \ell = 0, \pm 1, \pm 2, \cdots \) is the magnetic quantum number. Clearly, we find that KG-oscillators are introduced by the very nature of the Gödel SR-type cosmic string spacetime with space-like dislocation.

At this point, one observes that the second term of \( \lambda \) in \( 16 \) plays a critical role in removing the degeneracies associated with \( L = \mp |L| \), even when the magnetic field is off (i.e. when \( B_\circ = 0 \)). This would suggest that whilst the Gödel SR-type cosmic string spacetime manifestly plays a the delicate role of a magnetic field (in lifting the degeneracies associated with \( L = \pm |L| \)), it introduces a different type of degeneracies (i.e., spacetime associated degeneracies STAD that are related to \( \Omega E \) for a given \( L \), where the vorticity \( \Omega = \pm |\Omega| = \Omega_\pm \) and the relativistic particles energies \( E = \pm |E| = E_\pm \)). On the other hand, the central repulsive/attractive core \( (L^2 - 1/4) / r^2 \)
includes within the effect of the space-like dislocation in the form of a shift in the irrational magnetic quantum number $\tilde{\ell} \rightarrow \tilde{\ell} - \tilde{\delta} k$. Therefore, the effect of the gravitational field, introduced by the spacetime structure, on the non-relativistic/relativistic quantum system should be treated with diligence. An illustrative example is given in the sequel.

A. KG-oscillators in SR-type cosmic string spacetime with space like dislocation and vorticity-energy correlations

Let us consider the most simplistic and elementary model of (14) with $B_0 = 0 = \tilde{\phi} = S(r)$ and $g(r) = 1$ (i.e., KG-particles with constant mass) to obtain

$$\left\{ \partial_r^2 - \frac{(L^2 - 1/4)}{r^2} - \omega^2 r^2 + \lambda \right\} R(r) = 0, \quad \lambda = E^2 - 2 \mathcal{L} \Omega E - k^2 - m^2_\circ, \quad \omega^2 = \Omega^2 E^2. \quad (17)$$

This effective two-dimensional textbook oscillator would immediately imply that

$$\lambda = 2 |\omega| (2n_r + |\mathcal{L}| + 1), \quad R(r) \sim r^{|\mathcal{L}|+1/2} \exp \left( -\frac{|\omega| r^2}{2} \right) L_{n_r}^{|\mathcal{L}|} (|\omega| r^2), \quad (18)$$

and hence

$$E^2 - 2 \mathcal{L} \Omega E - (k^2 + m^2_\circ) = 2 |\Omega E| (2n_r + |\mathcal{L}| + 1). \quad (19)$$

It is obvious that $|\omega| = |\Omega E| \geq 0$ is a vital condition so that the radial wave function is finite and square integrable, which in turn introduces two sets of eigenvalues and eigenfunctions corresponding to $|\Omega E| = \Omega_\pm E_\pm$ and $|\Omega E| = -\Omega_\mp E_\mp$. That is,

$$E^2_{\pm} - 2 \Omega_\pm E_\pm N_\pm - (k^2 + m^2_\circ) = 0 \quad \text{for} \quad |\Omega E| = \Omega_\pm E_\pm, \quad (20)$$

and

$$E^2_{\pm} - 2 \Omega_\mp E_\pm N_- - (k^2 + m^2_\circ) = 0 \quad \text{for} \quad |\Omega E| = -\Omega_\mp E_\pm. \quad (21)$$

where

$$N_\pm = 2n_r + |\mathcal{L}| \pm \mathcal{L} + 1, \quad (22)$$

The two sets of energies can be treated in two ways. The first of which is based on the possible combinations of $|\Omega E|$ so that

$$E_{\pm,1} = \pm |\Omega| N_\pm \pm \sqrt{\Omega^2 N^2_\pm + m^2 + k^2} \Rightarrow \begin{cases} E_{+,1} = + |\Omega| N_+ + \sqrt{\Omega^2 N^2_+ + m^2 + k^2} \\ E_{-,1} = - |\Omega| N_- - \sqrt{\Omega^2 N^2_- + m^2 + k^2} \end{cases}, \quad (23)$$

for $\omega = +\Omega_\pm E_\pm$, and

$$E_{\pm,2} = \pm |\Omega| N_\pm \pm \sqrt{\Omega^2 N^2_\pm + m^2 + k^2} \Rightarrow \begin{cases} E_{+,2} = + |\Omega| N_+ + \sqrt{\Omega^2 N^2_+ + m^2 + k^2} \\ E_{-,2} = - |\Omega| N_- - \sqrt{\Omega^2 N^2_- + m^2 + k^2} \end{cases}, \quad (24)$$
FIG. 1: The energy levels for KG-oscillators in SR-type spacetime of (19) are plotted with $\alpha = 1/2$, $m_0 = k = 1$, $\delta = 0$ (a) for $\delta = 0$ (i.e., without space-like dislocation), $\ell = 0$ and $n_r = 0, 1, 2, 3$, (b) for $\delta = 0$, $n_r = 3$ and $\ell = 0, \pm 1, \pm 2, \pm 3$, and (c) for $\delta = 1$ (i.e., with space-like dislocation), $n_r = 3$ and $\ell = 0, \pm 1, \pm 2, \pm 3$.

for $\omega = -\Omega \phi \pm E_\pm$. Whereas, the second way is to report the energies based on the vorticity $\Omega = \Omega_\pm = \pm |\Omega|$ so that the energies (20) and (21) yield

$$E_{\pm,\Omega_+} = \pm |\Omega| \mathcal{N}_\pm \pm \sqrt{\Omega^2 \mathcal{N}_\pm^2 + m_0^2 + k^2}, \quad (25)$$

and

$$E_{\pm,\Omega_-} = \pm |\Omega| \mathcal{N}_\pm \pm \sqrt{\Omega^2 \mathcal{N}_\pm^2 + m_0^2 + k^2}. \quad (26)$$

This seems to be a more interesting way to express the related energy levels. Where, one may clearly observe vorticity-energy correlations between $E_{\pm,\Omega_+}$ and $E_{\pm,\Omega_-}$ so that, $E_{+,\Omega_+} (\mathcal{L} = \mp |\mathcal{L}|) = E_{+,\Omega_-} (\mathcal{L} = \pm |\mathcal{L}|)$ and $E_{-,\Omega_+} (\mathcal{L} = \mp |\mathcal{L}|) = E_{-,\Omega_-} (\mathcal{L} = \pm |\mathcal{L}|)$ for a given radial quantum number $n_r$. Such correlations may very well serve quantum entanglement between the KG-particles at hand. One should also notice that the energies for $E_{\pm,\Omega_+}$ are represented by the first and second quarters, whereas those for $E_{\pm,\Omega_-}$ are represented by the third and fourth quarters of Figures 1. We also notice that in each quarter (i.e., for each branch of the four energy branches) $\mathcal{N}_+ = 2n_r + 1$ for $\forall \mathcal{L} = -|\mathcal{L}|$ and $\mathcal{N}_- = 2n_r + 1$ for $\forall \mathcal{L} = +|\mathcal{L}|$. This would make all states with a given $n_r$ collapse into the corresponding $s$-state (i.e., $\ell = 0$ state) as shown in Figures 1(a), and 1(b) for $\delta = 0$, and to the corresponding $p_\pm$-states (i.e., $\ell = 1$ state) as shown in Figure 1(c). In general, for a given $n_r$ all states collapse into the corresponding $(\ell = \delta k)$-state in each quarter of the energy branches. This kind of degeneracies is called spacetime associated degeneracies (STAD).

Nevertheless, the results in (23) and (24), or the more sophisticated vorticity-energy correlations (25) and (26), are anticipated to represent new insights on the effect of gravitational fields on the quantum mechanical properties of relativistic particles (here, KG-particles) that may flourish research interest in the study of different aspects related to such findings.
III. PDM KG-OSCILLATORS IN SR-TYPE COSMIC STRING SPACETIME WITH SPACE-LIKE DISLOCATION

In this section, we consider a set of PDM KG-oscillators in the SR-type cosmic string spacetime described by (14), (15), and (16). We report, in what follows, vorticity-energy and charge-energy correlations.

A. PDM KG-oscillators and vorticity-energy correlations

A power-law type positive valued scalar multiplier \( g(r) = A r^\nu \Rightarrow M(r) = v^2/16r^2 \), with \( S(r) = 0 = V(r) \) and \( B_\nu = 0 \), one obtains

\[
\left\{ \partial_r^2 - \frac{\left( \ell^2 - 1/4 \right)}{r^2} - \omega^2 r^2 + \lambda \right\} R(r) = 0; \quad \omega^2 = \Omega^2 E^2,
\]

where

\[
\lambda = E^2 - eB_\nu \mathcal{L} - (k^2 + m_\nu^2), \quad \ell^2 = (\ell - \delta k)^2 + \frac{v^2}{16}.
\]

Under such settings, with \( \ell^2 = (\ell - \delta k)^2 + \frac{v^2}{16} \), one obtains

\[
\lambda = 2 |\omega| \left( 2n_r + |\mathcal{L}| + 1 \right), \quad R(r) \sim r^{|\mathcal{L}|+1/2} \exp \left( -\frac{|\omega| r^2}{2} \right) L_{n_r}^{|\mathcal{L}|} (|\omega| r^2),
\]

and consequently one obtains two sets of energies

\[
E_{\pm,1} = \pm |\Omega| \hat{N}_+ \pm \sqrt{\Omega^2 \hat{N}_+^2 + m^2 + k^2} \Rightarrow \begin{cases} E_{+,1} = \pm |\Omega| \hat{N}_+ + \sqrt{\Omega^2 \hat{N}_+^2 + m^2 + k^2} \\ E_{-,1} = \pm |\Omega| \hat{N}_- - \sqrt{\Omega^2 \hat{N}_-^2 + m^2 + k^2} \end{cases},
\]

for \( \omega = +\Omega \pm E_\pm \), and

\[
E_{\pm,2} = \pm |\Omega| \hat{N}_- \pm \sqrt{\Omega^2 \hat{N}_-^2 + m^2 + k^2} \Rightarrow \begin{cases} E_{+,2} = \pm |\Omega| \hat{N}_- + \sqrt{\Omega^2 \hat{N}_-^2 + m^2 + k^2} \\ E_{-,2} = \pm |\Omega| \hat{N}_- - \sqrt{\Omega^2 \hat{N}_-^2 + m^2 + k^2} \end{cases},
\]

for \( \omega = -\Omega \mp E_\pm \). Where,

\[
\hat{N}_\pm = \left( 2n_r + |\mathcal{L}| \mp |\mathcal{L}| + 1 \right).
\]

However, in terms of the vorticities we may rearrange such energies so that

\[
E_{\pm,\Omega_+} = \pm |\Omega| \hat{N}_\pm \pm \sqrt{\Omega^2 \hat{N}_\pm^2 + m_\nu^2 + k^2},
\]

and

\[
E_{\pm,\Omega_-} = \pm |\Omega| \hat{N}_\pm \pm \sqrt{\Omega^2 \hat{N}_\pm^2 + m_\nu^2 + k^2}.
\]

Again, one clearly observes the vorticity-energy correlations between \( E_{\pm,\Omega_+} \) and \( E_{\pm,\Omega_-} \) so that, \( E_{+,\Omega_+} (\mathcal{L} = \mp |\mathcal{L}|) = E_{+,\Omega_-} (\mathcal{L} = \mp |\mathcal{L}|) \) and \( E_{-,\Omega_+} (\mathcal{L} = \pm |\mathcal{L}|) = E_{-,\Omega_-} (\mathcal{L} = \pm |\mathcal{L}|) \) for a given radial quantum number \( n_r \). Notably, the STADs in each branch of the energy levels are now lifted by the PDM settings as can be observed in \( \hat{N}_\pm \) of (32).
B. PDM KG-oscillators and charge-energy correlations

A charged PDM KG-particle moving in a non-rotating Gödel SR-type cosmic string spacetime (i.e., no vorticity, hence $\Omega = 0$, and metric (2) reduces into being just a cosmic string spacetime metric) with space-like dislocation, in $S(r) = 0 = V(r)$, and under the effect of the magnetic and Aharonov-Bohm flux fields of (13). In this case and for a positive valued scalar multiplier $g(r) = \exp(2\xi r^2) \Rightarrow M(r) = \xi^2 r^2 + 2\xi; \xi \geq 0$, we get

$$\partial^2_r - \frac{L^2}{r^2} - \tilde{\Omega}^2 r^2 + \hat{\lambda} R(r) = 0; \tilde{\Omega} = \sqrt{\left(\frac{eB_o}{2}\right)^2 + \xi^2},$$ \hspace{1cm} (35)

This would result in

$$\hat{\lambda} = E^2 - eB_o L - (k^2 + m^2_o + 2\xi) = 2|\tilde{\Omega}|(2n_r + |L| + 1).$$ \hspace{1cm} (36)

and consequently one obtains

$$E = \pm \sqrt{k^2 + m^2_o + 2\xi + eB_o L + |e| B_o \sqrt{1 + \frac{4\xi^2}{e^2 B_o^2}}(2n_r + |L| + 1)}.$$ \hspace{1cm} (37)

Notably, with

$$\tilde{N}_\pm = \sqrt{1 + \frac{4\xi^2}{e^2 B_o^2}}(2n_r + |L| + 1) \pm L$$ \hspace{1cm} (39)

two sets of energies are obtained

$$E_{\pm, +ve} = \pm \sqrt{m^2 + k^2 + 2\xi + |e| B_o \tilde{N}_+},$$ \hspace{1cm} (40)

for positively (+ve) charged PDM KG-particle, i.e., $e = +|e|$, and

$$E_{\pm, -ve} = \pm \sqrt{m^2 + k^2 + 2\xi + |e| B_o \tilde{N}_-},$$ \hspace{1cm} (41)

for negatively (−ve) charged PDM KG-particle, i.e., $e = -|e|$. We clearly observe that yet another type of correlations is introduced here, to be called, hereinafter, charge-energy correlations, $E_{\pm, +ve}(L = \pm |L|) = E_{\pm, -ve}(L = \mp |L|)$. Again we see that the PDM settings lift the STADs in each branch of the energy levels. However, the STADs reappear when $\xi = 0$ (i.e., $\tilde{N}_\pm = 2n_r + 1$ for $\forall L = \mp |L|$) and that would make all states with a given $n_r$ collapse into the corresponding $S$-state (i.e., $\ell = 0$ state).

IV. PDM KG-OSCILLATORS IN A LINEAR CONFINEMENT IN SR-TYPE COSMIC STRING SPACETIME WITH SPACE-LIKE DISLOCATION

Let us consider a charged PDM KG-particle in a SR-type cosmic string spacetime metric with space-like dislocation, in a linear Lorentz scalar potential $S(r) = \kappa r$ [33, 40]. we shall also assume that Lorentz vector potential $V(r) = 0$,
FIG. 2: The energy levels for PDM KG-oscillators in linear confinement in SR-type spacetime with space-like dislocation of [17] are plotted with \( \alpha = 1/2, m_0 = k = \tilde{\phi} = \delta = 1, \) and \( \nu = 4 \) (a) for \( \ell = 0, \kappa = 1, \) and \( n_r = 0, 1, 2, 3, \) (b) for \( \kappa = 1, n_r = 0 \) and \( \ell = 0, \pm 1, \pm 2, \) and (c) for \( \kappa = 1, n_r = 3 \) and \( \ell = 0, \pm 2, \pm 3. \)

the positive valued scalar multiplier \( g (r) = A r^\nu \Rightarrow M (r) = \nu^2 / 16 r^2, \) and \( B_0 = 0 \) (no magnetic field). In this case \( [14] \) reads

\[
\left\{ \partial_r^2 - \left( \frac{\tilde{L}^2 - 1/4}{r^2} \right) - \tilde{\omega}^2 r^2 - 2m_0 \kappa r + \lambda \right\} R (r) = 0, \tag{42}
\]

where

\[
\lambda = E^2 - 2\Omega E L - (k^2 + m_0^2), \quad \tilde{\omega}^2 = \omega^2 + \kappa^2, \quad \tilde{L}^2 = (\tilde{\ell} - \tilde{\delta} k)^2, \quad \tilde{L}^2 = L^2 + \frac{\nu^2}{16}, \tag{43}
\]

The two-dimensional Schrödinger-like quantum model \( [42] \) admits a solution in the form of

\[
R (r) = \sqrt{\pi} \psi (r) \sim r^{1/2} \exp \left( - \frac{\tilde{\omega} r^2}{2} - \frac{\tilde{\kappa}}{2 |\tilde{\omega}|} r \right) H_B \left( \alpha', \beta', \gamma', \delta', \sqrt{\tilde{\omega}r} \right), \tag{44}
\]

where \( \tilde{\kappa} = 2m_0 \kappa, \)

\[
\alpha' = 2 |\tilde{L}|, \quad \beta' = \frac{\tilde{\kappa}}{|\tilde{\omega}|^{3/2}}, \quad \gamma' = \frac{\lambda}{|\tilde{\omega}|} + \frac{\tilde{\kappa}}{4 |\tilde{\omega}|^3}, \quad \delta' = 0, \tag{45}
\]

and \( H_B \left( \alpha', \beta', \gamma', \delta', \sqrt{\omega r} \right) \) is the biconfluent Heun series. In order for the biconfluent Heun series to become a polynomial of degree \( n \geq 0, \) one has to apply the condition that \( \gamma' = 2 (n + 1) + \alpha' \tag{38, 39, 40} \) This condition would imply that

\[
\lambda = 2 |\tilde{\omega}| \left( n + |\tilde{L}| + 1 \right) - \frac{\tilde{\kappa}^2}{4 \tilde{\omega}^2} = 2 |\tilde{\omega}| \left( 2n_r + |\tilde{L}| + 1 \right) - \frac{\kappa^2}{4 |\omega|^2}, \tag{46}
\]

where \( n = 2n_r; \) \( n_r = 0, 1, 2, \cdots, \) so that when \( \kappa = 0 \) the eigenvalues of the pure oscillator of \( [18] \) are recovered. Therefore, \( n = 2n_r \) represents the correlation between the truncation order \( n \) and the radial quantum number \( [28, 30] \) (to be shown in the sequel subsection). This result would allow us to obtain

\[
E^2 - 2\Omega E L - (k^2 + m_0^2) = 2 |\tilde{\omega}| \sqrt{1 + \frac{\kappa^2}{\Omega^2 E^2}} (2n_r + |\tilde{L}| + 1) - \frac{m_0^2 \kappa^2}{\Omega^2 E^2 + \kappa^2}. \tag{47}
\]

It is obvious that this result retrieves those in \( [23] \) and \( [24] \) when the linear confinement is switched off, i.e., \( \kappa = 0. \)
FIG. 3: The energy levels for PDM KG-oscillators in linear confinement in SR-type spacetime with space-like dislocation of (47) are plotted with \( m = k = \phi = 1 \), and \( \nu = 4 \) (a) for \( \alpha = \frac{1}{2} \), \( n_r = 1 \), \( \ell = 1 \), \( \delta = 1 \), and \( \kappa = 0, 2, 5, 7, 10 \), (b) for \( \alpha = \frac{1}{2} \), \( \kappa = 1 \), \( n_r = 1 \), \( \ell = 1 \), and \( \delta = 0, \pm 2, \pm 5 \), and (c) for \( \kappa = 1 = \delta \), \( n_r = 1 \) and \( \alpha = 0.1, 0.2, 0.4, 0.7, 10 \).

This is in fact the natural tendency of any physically admissible and viable solution of a more general problem. The corresponding energies in (47) are plotted in Figures 2 and 3. Where, the vorticity-energy correlation remains as 
\[ E^+ \pm \Omega^+ (|L| = \mp |L|) = E^- \pm \Omega^- (|L| = \pm |L|), \]
where \( E^\pm \Omega^\pm \) are represented by the first and second quarters, whereas \( E^\pm \Omega^\mp \) are represented by the third and fourth quarters of Figures 2 and 3.

A. Validity and/or admissibility of the biconfluent Heun series/polynomials conditions

At this point, it is unavoidably inviting to question the quantum mechanical validity and/or admissibility of the biconfluent Heun series/polynomials conditions for the PDM KG-oscillators in a linear confinement (i.e., the Schrödinger-like model in (42)). In so doing, we substitute
\[ R(r) = N r^{|\mathcal{L}|+1/2} \exp \left( -\frac{|\tilde{\omega}| r^2}{2} - \frac{\tilde{\kappa}}{2 |\tilde{\omega}|} r \right) H \left( \sqrt{|\tilde{\omega}|} r \right); \]
\[ H \left( \sqrt{|\tilde{\omega}|} r \right) = H(\rho) = \sum_{j=0}^{\infty} A_j \rho^j, \]
(48)
in (42) to obtain \( A_1 = \tilde{\kappa}/2 |\tilde{\omega}|^{3/2} \), \( A_0 = 1 \), and
\[ A_{j+2} (j+2) \left( j+2 \right| \tilde{\mathcal{L}} \right| + 2) = A_{j+1} \left[ A_1 \left( 2j+2 \right| \tilde{\mathcal{L}} \right| + 3) + A_j \left( 2j - \frac{\tilde{\lambda}}{|\tilde{\omega}|} \right) ; j = 0, 1, 2, \cdots, \]
(49)
where
\[ \tilde{\lambda} = \tilde{\lambda} + \frac{\tilde{\kappa}^2}{4 \tilde{\omega}^2} - 2 |\tilde{\omega}| \left( 1 + \left| \tilde{\mathcal{L}} \right| \right). \]

Next, the biconfluent Heun series should be terminated at some \( j = n \) to become a polynomial of degree \( n \geq 0 \) to secure finiteness and square integrability of the corresponding radial wavefunctions. Then, one would require that for \( \forall j > n \) we have \( A_{n+1} = 0 = A_{n+2} \) to imply
\[ A_n \left( 2n - \frac{\tilde{\lambda}}{|\tilde{\omega}|} \right) = 0 \implies \tilde{\lambda} = 2 |\tilde{\omega}| n \implies \tilde{\lambda} = 2 |\tilde{\omega}| (n + \left| \tilde{\mathcal{L}} \right| + 1) - \frac{m_0^2 \tilde{\kappa}^2}{\tilde{\omega}^2}; n \geq 0. \]

(51)
This would allow us to cast our (48) as

\[ R(r) = N r^{n+1/2} \exp \left( -\frac{|\tilde{\omega}| r^2}{2} - \frac{\tilde{\kappa}}{2 |\tilde{\omega}|} r \right) \sum_{j=0}^{n} A_j \left( \sqrt{|\tilde{\omega}|} r \right)^j. \]  

(52)

Nevertheless, it has become a traditional practice for some authors to claim and/or believe that the condition \( A_{n+1} = 0 \) should be used again to find out additional \( n \)-dependent quantization recipes or correlations between the effective potential parameters (e.g., [42–46] and references cited therein). Hereby, the three terms recursion relation (49) is rewritten (following e.g., [42–46]) as

\[ A_{j+1} \left( j + 1 \right) \left( j + 2 \right) \left| \tilde{\omega} \right| + 1 \right] = A_{j} \left( 2j + 2 \left| \tilde{\omega} \right| + 1 \right) + A_{j-1} \left( 2 \left| \tilde{\omega} \right| + \left| j - 1 - n \right| \right); \quad A_{-1} = 0, \]  

(53)

so that for \( \forall j > n \) we have \( A_{n+1} = 0 \) to imply that

\[ A_{n} \left[ A_{1} \left( 2n + 2 \left| \tilde{\omega} \right| + 1 \right) \right] = 2 \left| \tilde{\omega} \right| A_{n-1}; \quad n \geq 0, \quad A_{-1} = 0. \]  

(54)

Which, for \( n = 0, A_0 = 1 \) and \( A_1 = \tilde{\kappa}/2 \left| \tilde{\omega} \right|^{3/2} \neq 0 \), gives

\[ A_0 \left( 1 + 2 \left| \tilde{\omega} \right| \right) = 2 \left| \tilde{\omega} \right| A_{-1} = 0 \Rightarrow 1 + 2 \left| \tilde{\omega} \right| = 0 \Rightarrow \left| \tilde{\omega} \right| = -\frac{1}{2}. \]  

(55)

This result is neither mathematically nor physically an acceptable one. Although straightforward, there is no need to continue for \( n = 1, 2, \cdots \), therefore. Of course, it could be mathematically interesting to know that the coefficients of \( A_{n+1} = 0 \) are obvious polynomials of degree \( n + 1 \) in \( A_1 = \tilde{\kappa}/2 \left| \tilde{\omega} \right|^{3/2} \) (e.g., [42–47]). However, one should be aware that

(i) \( A_{n+1} \) it is not involved in our solution (52), and hence whatever it entails does not belong to our acceptable quantum mechanical solution.

(ii) Re-writing (49) in as (53) and using the condition \( A_{n+1} = 0 \) again, yields mathematically and/or physically unacceptable results (like: \( \left| \tilde{\omega} \right| = -1/2 \) in our case [53], \( |\gamma| = -1/2 \) in Fernández [46], and \( |m| = -1/2 \) in Verčin [47] (line 1 of his (18) for his \( k = 0 \) and \( b = 0 \), and the angular momentum quantum number \( \ell = -1 \) in the case of Fernández [45]).

(iii) Yet, in the results of Medeirosa and de Mello [48] (section 4.3), we notice that things are drastically catastrophic in the sense that if the linear confinement is set off (e.g., \( \eta_L = 0 \) in their (40)), then their \( \tilde{\omega}_{1,m} \) of their (52) takes the value \( \tilde{\omega}_{1,m} = 0 \). In this case, although they still have the harmonic oscillator potential term their reported energies collapse into that of free particle \( E_{k,m,n} = \pm \sqrt{k^2 + M^2} \).

Next, we now show that the truncation order \( n \) is correlated with the radial quantum number \( n_r = 0, 1, 2, \cdots \) through \( n = 2n_r \). In so doing, we recall (42) and adjust our \( \tilde{L} \) so that \( \tilde{L}^2 - 1/4 = 0 \) and cast it as

\[ \left\{ \partial_{\rho}^2 - \tilde{\omega}^2 \rho^2 + \tilde{\Lambda} \right\} R(\rho) = 0, \]  

(56)

where,

\[ \rho = r + \eta; \quad \eta = \frac{m_\kappa}{\tilde{\omega}^2}; \quad \tilde{\Lambda} = \tilde{\lambda} + \frac{m_\kappa^2 \kappa^2}{\tilde{\omega}^2}. \]  

(57)
This is obviously a two-dimensional radial shifted-oscillator model that admits exact textbook eigenvalues given by
\[ \tilde{\Lambda} = 2 |\tilde{\omega}| \left( 2n_r + \frac{3}{2} \right) \implies \lambda = 2 |\tilde{\omega}| \left( 2n_r + \frac{3}{2} \right) - \frac{m^2 \kappa^2}{\tilde{\omega}^2}. \] (58)

If we compare (51), for \(|\tilde{\mathcal{L}}| = 1/2\), with (58) we obtain that \( n = 2n_r \). This would, in turn, provide a brute-force evidence on the correctness of the correlation suggested by Mustafa [28–30] and used in (46). In a crystal clear manner, moreover, it suggests beyond doubt that there is no additional \( n \)-dependent quantization recipes or correlations between the effective potential parameters (as claimed in e.g., [42–48]).

All arguments given above should lead us to one conclusion. As long as quantum mechanics is in point, using the condition \( A_{n+1} = 0 \) more than one time yields catastrophic results that are neither physically nor mathematically acceptable. Therefore, the repeated usage of this condition should be labeled, hereinafter, as a quantum mechanically redundant condition. This redundant condition would render the biconfluent Heun series/polynomials approach quantum mechanically useful and reliable, even with its status of being a conditionally exact solution. This is what the biconfluent Heun series/polynomials approach can, quantum mechanically, safely offer, at its best.

V. CONCLUDING REMARKS

In this work, we have revisited KG-particles in SR-type cosmic string spacetime with space-like dislocation and discussed the detailed spectroscopic structure through the assumption that the vorticity of the SR-type spacetime is may take two possible values, \( \Omega = \pm |\Omega| \). This assumption has allowed us to express the energy levels in terms of the vorticity and hence vorticity-energy correlations between \( E_{\pm,\Omega_+} \) of (25) and \( E_{\pm,\Omega_-} \) of (26) (i.e., \( E_{+,\Omega_+} (\mathcal{L} = \mp |\mathcal{L}|) = E_{+,\Omega_-} (\mathcal{L} = \pm |\mathcal{L}|) \) and \( E_{-,\Omega_+} (\mathcal{L} = \mp |\mathcal{L}|) = E_{-,\Omega_-} (\mathcal{L} = \pm |\mathcal{L}|) \) ) emerged in the process as a new interesting phenomenon that may very well serve quantum entanglement (between KG-oscillators in particular and KG-particles in general). Under such settings, one obtains four branches of energy levels that are introduced by the term \( |\Omega E| \geq 0 \) of (19) as \( |\Omega E| = \Omega_{\pm} E_{\pm} \) and \( |\Omega E| = -\Omega_{\pm} E_{\pm} \). Yet, we have observed that in each branch of the energy levels, the irrational quantum number \( N_{\pm} \) of (22) manifestly suggests that \( N_+ = 2n_r + 1 \) for \( \forall \mathcal{L} = -|\mathcal{L}| \) and \( N_- = 2n_r + 1 \) for \( \forall \mathcal{L} = +|\mathcal{L}| \). This have made all states with a given \( n_r \) degenerate with the corresponding \( S \)-state (i.e., \( \ell = 0 \) state) as shown in Figures 1(a), 1(b), and 1(c). We have called this kind of degeneracies as spacetime associated degeneracies (STAD). Based on such findings above, we have discussed PDM KG-oscillators and reported not only vorticity-energy correlations but also charge-energy correlations. We have also considered PDM KG-oscillators in a linear confinement and reported, through Figures 2 and 3, the corresponding vorticity-energy correlations. To the best of our knowledge, such results and findings have never been discussed elsewhere.

Moreover, in connection with PDM KG-oscillators in a linear confinement in SR-type cosmic string spacetime with space-like dislocation, we have given this problem some particular attention. This problem is discussed in [42, 48] (and related references therein). We have provided a brute-force-evidence that there is no additional quantization, related to the condition \( A_{n+1} = 0 \), as claimed in [42, 48] or correlations between the effective potential parameters as in [45, 47]. Therefore, as long as quantum mechanics is in point, using the condition \( A_{n+1} = 0 \) more than one time yields catastrophic results that are neither physically nor mathematically acceptable, as documented in section 4. The repeated usage of this condition should be labeled, hereinafter, as a quantum mechanically redundant condition. This would, at least, render the biconfluent Heun series/polynomials approach quantum mechanically useful and reliable,
even with its classification of being a conditionally exact solution. That is, we have to accept the fact that this is what the biconfluent Heun series/polynomials approach can, quantum mechanically, safely offer, at its best.

Finally, such findings are anticipated to provide new insights on the effects of the gravitational fields (introduced by the spacetime structure at hand) on the quantum mechanical properties of relativistic particles in general, which in turn may flourish research interest in the study of different aspects in this direction.

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