Two-Body Dirac Equations from Relativistic Constraint Dynamics with Applications to QED and QCD Bound-States and to N-N Scattering

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1 Introduction

The formulation of relativistic two-body bound state wave equations and their relationship to quantum field theory began with the work by Eddington and Gaunt in 1928 [1]. However, the large variety of approaches attempted in recent years shows that this problem still has no generally agreed-upon solution. Perhaps for this reason, most recent field theory books have skirted this topic. In his recent text, Steven Weinberg states [2]: “It must be said that the theory of relativistic effects and radiative corrections in bound states is not yet in satisfactory shape.”

Of course, this topic is often presented as covered by the manifestly covariant Bethe-Salpeter equation obtained directly from relativistic quantum field theory. Over the years, however, many problems have turned up to impede its direct implementation, mostly related to the central role played in it by the relative time or energy [3]. These difficulties have led many authors to attempt reformulations.

We describe here a recent approach resulting from “Two-Body Dirac Equations” (emerging from Dirac’s Relativistic Constraint Dynamics) that does satisfy many of the requirements one would demand of a treatment of the relativistic two-body problem. We use its applications to QED bound states such as positronium, QCD quarkonia, and the nucleon-nucleon scattering problem to demonstrate the advantages of the approach.

2 The One-Body Dirac Equation

For a single body, the original Dirac equation for a single spin-one-half particle (nearly universally accepted) provides a successful bound state equation.
free Dirac equation \((\gamma \cdot p + m)\psi = 0\) serves as a relativistic version of Newton’s 1st law. When the four-vector substitution for electromagnetic interaction \(p_\mu \rightarrow p_\mu - A_\mu\), and the minimal mass substitution for scalar interaction \(m \rightarrow m + S\) are performed on it, one obtains a relativistic version of Newton’s 2nd law

\[
(\gamma \cdot (p - A) + m + S)\psi = 0.
\] (1)

The two-body Dirac equations of constraint dynamics successfully extend this one-body minimal coupling form to the interacting two-body system.

## 3 Two-Body Dirac Equations from Constraint Dynamics

In the 1970’s, Todorov, Kalb and Van Alstine, and Komar independently used Dirac’s constraint mechanics to attack the relativistic two-body problem for spinless particles \[4\]. By covariantly controlling the relative time variable these authors eliminated negative norm states as well as circumvented the no-interaction theorem of Currie, Jordan, and Sudarshan\[5\] that had discouraged further work in this area since the 1960’s. By combining constraint dynamics with particle supersymmetries, Crater and Van Alstine extended those works to pairs of spin one half particles to obtain two-body quantum bound state equations that correct not only defects in the Breit equation but those in the ladder approximation to the Bethe-Salpeter equation as well. \[6\] These Two-Body Dirac Equations of constraint dynamics possess a number of important features (some of which are unique) which provide an alternative formulation of fundamental field-theoretic results (while yielding standard perturbative spectra) and correct defects in phenomenological applications that result from patchwork rearrangement of interactions: They a) provide a three dimensional but covariant rearrangement of the Bethe-Salpeter equation, b) yield simple three-dimensional Schrödinger-like forms similar to their nonrelativistic counterparts, c) contain spin dependences determined naturally by their incorporation of Dirac’s one-body structures, d) contain well defined strong potential structures that pass the necessary test that they reproduce correct QED perturbative results when solved nonperturbatively, e) in phenomenological applications make unnecessary the ad hoc introduction of cutoff parameters generally used to avoid singular potentials and f) have relativistic potentials which may be related directly to the interactions of perturbative quantum field theory or (e.g. for QCD) may be introduced semiphenomenologically\[7\]. These equations provide a nonperturbative or strong-potential framework for extrapolating perturbative field theoretic results into the highly relativistic regime of bound light particles in a quantum mechanically well defined way.

### 3.1 World Vector and Scalar Interactions

The constraint formalism is embodied in a system of two coupled, compatible Dirac equations on a single wave-function. For particles interacting through
world vector and scalar interactions the two-body Dirac equations take the general minimal-coupling form

\[ S_1 \psi \equiv \gamma_5 (\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1 + \tilde{S}_1) \psi = 0 \]

\[ S_2 \psi \equiv \gamma_5 (\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2 + \tilde{S}_2) \psi = 0. \] (2)

The two equations are compatible in the sense that,

\[ [S_1, S_2] \psi = 0. \] (3)

This condition is satisfied as a result of the presence in these equations of spin supersymmetries, a relativistic 3rd law, and covariant restrictions on the relative time and energy. Its direct dynamical consequence is the automatic incorporation of correct spin-dependent recoil terms.

For the case where only vector interactions are present, the compatibility condition is most naturally satisfied by vector potentials in the “hyperbolic” momentum and spin-dependent forms

\[ \tilde{A}_1 = [1 - \cosh(\mathcal{G})] p_1 + \sinh(\mathcal{G}) p_2 - \frac{i}{2} (\partial \exp \mathcal{G} \cdot \gamma_2) \gamma_2 \] (4)

\[ \tilde{A}_2 = [1 - \cosh(\mathcal{G})] p_2 + \sinh(\mathcal{G}) p_1 + \frac{i}{2} (\partial \exp \mathcal{G} \cdot \gamma_1) \gamma_1. \] (5)

In this case, the compatibility condition enforces a “third law” with the two constituent potentials actually depending on only one invariant \(\mathcal{A}\) through the interaction function \(\mathcal{G}(\mathcal{A})\). The explicit form of \(\mathcal{G}(\mathcal{A})\) follows both from comparison with classical Wheeler-Feynman electrodynamics or with QED through leading term summation of ladder, cross-ladder and constraint diagrams [8]

\[ \mathcal{G}(\mathcal{A}) = -\frac{1}{2} \log(1 - \frac{2\mathcal{A}}{w}) ; \ w \text{ the total c.m. energy}. \] (6)

Compatibility also requires that the relative time be covariantly controlled through interactions depending only on \(x_\perp\), a covariant spacelike particle separation variable perpendicular to the total momentum \(\hat{P}\)

\[ \mathcal{A} = \mathcal{A}(x_\perp) \]

\[ x_\perp^\mu = x_\mu + \hat{P}^\mu (\hat{P} \cdot x) , \ \hat{P} \equiv \frac{P}{w} \text{ is a time-like unit vector}. \] (7)

For lowest order electrodynamics,

\[ \mathcal{A} = \mathcal{A}(x_\perp) = -\frac{\alpha}{r} ; r \equiv \sqrt{x_\perp^2}. \] (8)

For quark-models, we must include scalar potentials \(\tilde{S}_i\). When appearing with vector interactions they depend not only on two invariant mass potential
functions \( M_1(x_\perp), M_2(x_\perp) \), related to each other through one invariant function \( L(x_\perp) \) but also on the vector interaction through \( G(A(x_\perp)) \):

\[
\tilde{S}_1 = M_1 - m_1 - \frac{i}{2} \exp G(A) \gamma_2 \cdot \frac{\partial M_1}{M_2},
\]

\[
\tilde{S}_2 = M_2 - m_2 + \frac{i}{2} \exp G(A) \gamma_1 \cdot \frac{\partial M_2}{M_1},
\]

\[
M_1^2 - M_2^2 = m_1^2 - m_2^2 \implies M_1 = m_1 \cosh L + m_2 \sinh L \quad \text{RD LAW. (10)}
\]

The counterpart to the invariant \( A \) for scalar interactions is \( S \) with the form of \( L = L(S(x_\perp), A(x_\perp)) \) with \( m_w = m_1 m_2 / w \) from

\[
M_i^2 = m_i^2 + \exp G(A)(2m_w S + S^2) ; i = 1, 2.
\]

Retardative effects are already included through the c.m. energy dependences of the potential structures. Although the potential forms in these equations may seem unfamiliar, expansion of the resulting classical dynamics in \( 1/c \) around the nonrelativistic limit shows that it is canonically equivalent to order \( 1/c^2 \) to the dynamics generated by the corresponding single-quantum exchange in field theory.

### 3.2 Manifest Covariance and Well-Defined Quantum-Mechanical Behavior

For our applications we use a Pauli reduction to bring our equations to the covariant Schrödinger-like form (with \( p \) the relative momentum)

\[
(p^2 + \Phi_w(\sigma_1, \sigma_2, p_\perp, A(r), S(r)))\psi = b^2(w)\psi \quad \text{(12)}
\]

incorporating exact two-body relativistic kinematics through the eigenvalue \( b^2(w) = (w^4 - 2m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2) / 4w^2 \equiv \varepsilon_w^2 - m_w^2 \) in terms of \( m_w \) and \( \varepsilon_w = (w^2 - m_1^2 - m_2^2) / 2w \) respectively the mass and energy of the fictitious particle of relative motion. This Schrödinger-like equation is not only manifestly covariant but quantum mechanically well defined: one can solve it nonperturbatively in both QED and QCD bound state cases for which every term in the quasipotential \( \Phi_w(\sigma_1, \sigma_2, p_\perp, A(r), S(r)) \) is less singular than \(-1/4r^2\) (in contrast to all reductions of the Breit equation and many reductions of the Bethe-Salpeter equation). It involves at most 2 coupled wave equations but all portions of the 16 component wave function play essential roles in spectral calculations, either directly or through the strong potential structures that they generate when they are eliminated. The explicit forms of the spin dependent potentials that appear in the Pauli-form quasipotential \( \Phi_w \) are dictated by the interaction structure of the original two coupled Dirac equations and are not put in by hand.
The Schrödinger-like form of the Two-Body Dirac equations takes the minimal coupling form

\[(p^2 + (m_w + S)^2 - (\varepsilon_w - A)^2 + \Phi_{sp}(\sigma_1, \sigma_2, p_\perp, A(r), S(r)))\psi = 0\quad (13)\]

in which

\[
\Phi_{sp} = \Phi_{D1} \hat{r} \cdot p + \Phi_{D2} + \Phi_{SL} \cdot (\sigma_1 + \sigma_2) + \Phi_{SOD} L \cdot (\sigma_1 - \sigma_2) + \Phi_{SPO} L \cdot (\sigma_1 \times \sigma_2) \\
+ \Phi_{SS} \sigma_1 \cdot \sigma_2 + \Phi_{T} \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r} + \Phi_{DT} \hat{r} \cdot p \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r}. \quad (14)
\]

The $\Phi_i = \Phi_i(A, S, w)$ are not independent but are all determined in terms of $A, S$ through the Pauli reduction.

We have checked analytically and numerically that our strong potential terms do not lead to spurious results by solving them nonperturbatively to obtain agreement with the standard fine and hyperfine spectra of perturbative QED. For example, for the singlet positronium system with $A = -\alpha/r$ our fully coupled system of 16-component equations $S_1 \psi = S_2 \psi = 0$ is exactly solvable with total energy $w$

\[w = m(2+2/(1+\alpha^2/((l+1/2)^2-\alpha^2)^{1/2}-(l-1/2)^2)^{1/2})^{1/2} \pm m(2-\alpha^2/4 - 21\alpha^4/64)]_{\text{ground state}}\quad (15)
\]

Such validation should be required of all candidate equations for nonperturbative quark model calculations and other semiphenomenological applications when their quark-model kernels are replaced by ones appropriate for QED. Otherwise, how can one trust the short distance spectral contributions obtained when applied in QCD? No other approaches have yet fully passed this test. In fact Sommerer et al [10] have shown that all of the well-known quasipotential approaches (e.g. the Blankenbecler- Sugar equation, the Gross, and Kadeshevsky approaches) fail this crucial test even for the ground state. However, by varying parameters permitted by the non-uniqueness of the quasipotential approaches, Sommerer et al obtained a quasipotential model that does reproduce this ground state numerically. We have successfully extended the check in our equations and found numerical agreement with the perturbative results for a range of angular momentum and radial states and for unequal masses [9].

The invariant forms for $A$ or $S$ in our equations follow both from relativistic classical field theory and from the perturbative treatment of corresponding quantum field theories [9] through Sazdjian’s derivation of the constraint equations as a “quantum mechanical transform of the Bethe-Salpeter equation” [11].

4 Two-Body Dirac Equations in Meson Spectroscopy

4.1 The Adler-Piran Potential

We obtain a constraint version of the naive quark model for mesons from a covariant adaptation of a static quark potential due to Adler and Piran. These
authors use an effective non-linear field theory derived from QCD to find
\[ V_{AP}(r) = \Lambda(U(\Lambda r) + U_0) (= A + S). \]  
(16)

The original \( V_{AP} \) is nonrelativistic, appearing as the sum of world vector and scalar potentials in the nonrelativistic limit. \( V_{AP} \) incorporates a running coupling constant form in coordinate space at short distance \( \Lambda U(\Lambda r << 1) \sim 1/r \ln \Lambda r \) and includes linear confinement plus subdominant logarithm terms at long distance
\[ V_{AP}(r) = \Lambda(c_1 \Lambda r + c_2 \log(\Lambda r) + \frac{c_3}{\Lambda r} + \frac{c_4}{\Lambda r} + c_5), \quad \Lambda r \gg 1. \]  
(17)

When the nonrelativistic quark model is constructed with realistic potentials such as the Adler-Piran potential or Richardson potential it fails for light mesons. Beyond a certain limit, it gives meson masses that increase with decreasing quark mass. However, this effect is entirely absent from the light meson spectra produced by the Two-Body Dirac Equations with their relativistic kinematics and QCD-determined relativistic potentials.

4.2 Relativistic Naive Quark Model

We reinterpret the static \( V_{AP} \) covariantly by a) replacing the nonrelativistic \( r \) by \( \sqrt{x^2 + p^2} \) and b) parcelling out the static potential \( V_{AP} \) into the invariant functions \( A(r) \) and \( S(r) \) as follows
\[ A = \exp(-\beta r)[V_{AP} - \frac{c_1}{r}] + \frac{c_4}{r} + \frac{c_1 c_2}{r^2}, \quad S = V_{AP} + \frac{c_1 c_2}{r^2} - A. \]  
(18)

This partially phenomenological step ensures that at short distance the potential is strictly vector while at long distance the vector portion is strictly Coulombic with the confining portion at long distance (including subdominant portions) strictly scalar. Note that in our equations, once \( A \) and \( S \) have been determined, so are all the accompanying spin-dependent interactions. Here our approach is that of a naive quark model since we ignore flavor mixing and the effects of decays on the bound state energies.

4.3 Meson Spectroscopy

Our results are spectrally quite accurate, from the heaviest upsilonium states to the pion. Notable exceptions are light meson orbital and radial excitations and their spin-orbit splittings. We include only selected portions of the whole table here (see hep-ph-0208186 for the complete results). The parameter values we obtain from the best fit are \( m_b = 4.877 \), \( m_c = 1.507 \), \( m_s = 0.253 \), \( m_u = 0.0547 \), \( m_d = 0.0580 \) GeV along with \( \Lambda = 0.216 \), \( \Lambda U_0 = 1.865 \) GeV and \( \beta = 1.936 \). We find that the static Adler-Piran potential, having a close connection with quantum chromodynamics (QCD), gives a good fit to the mostly nonrelativistic bottomonium spectrum. We obtain 9.453, 9.842, 9.889, 9.921, 10.022 GeV for the ground state and first orbital and radial states.
of 9.460, 9.860, 9.892, 9.913, 10.023 GeV. For the B mesons the fit results from
the fact that for them our equations essentially reduce to the one-body Dirac
equation. We obtain for the $B$ mesons the results $5.273, 5.321, 5.368, 5.427$ GeV. The goodness of the fit to the charmed
quark mesons shows that the equations perform well in the semirelativistic re-
gion. The ground state $^1S_0$, $^3S_1$, and first orbital $^1P_1, ^3P_0, ^3P_1, ^3P_2$ and ra-
dial $^2S_0, ^2S_1$ excitations of $2.980, 3.097, 3.526, 3.415, 3.510, 3.596, 3.594, 3.686$ GeV match our theoretical values of $2.978, 3.129, 3.520, 3.507, 3.549, 3.610, 3.688$ GeV quite nicely. The same thing happens for the $D$ mesons for which we obtain the values of $1.866, 2.000, 1.976, 2.123$ GeV. The goodness of the accompanying fit to the lighter mesons (with the same two invariant potential functions used for the entire spectrum) is due to exact two-body relativistic kinematics combined with the minimal interaction and strong potential structures of our equations for vector and scalar potentials. For example, for the ground states $K(494), K^*(892), \phi(1019), \pi(140), \rho(767)$, we obtain $0.492, 0.910, 1.033, 0.144, 0.792$ GeV. The results for the light meson orbital and radial excitations and their spin-orbit splittings are mixed. For example the $b_1(1.231), a_0(1.450), a_1(1.230), a_2(1.318)$ meson fits of $1.392, 1.491, 1.568, 1.310$ are quite uneven and the radially excited $\pi(1.300)$ and $\rho(1.465)$ results of 1.536 and 1.775 GeV are quite far off the mark. On the other hand the results $1.319, 1.533, 1.493$ GeV for the $\phi$ orbital excitations are reasonable. In the fu-
ture, we plan to use a coupled channel formalism to investigate the origin of some of these problems.

The strength of the Two-Body Dirac approach is that using it, with just
two parametric functions $A$ and $S$ we are able to obtain an overall fit about
as good as that obtained by Godfrey and Isgur[13], who used six parametric functions, basically one for each type of spin dependence.

As a bonus, we find that the pion is a Goldstone boson in the sense that
$m_\pi(m_q \to 0) \to 0$ while the $\rho$ and excited $\pi$ have finite mass in this limit (see hep-ph/0208186 for complete plots). We have shown elsewhere that $\langle 0 | \partial_\mu J_\mu \pi \rangle = (M_1(0) + M_2(0)) = (m_1 + m_2) \langle 0 | Tr(\gamma_5 \psi) \rangle$ supporting our claim that our potential model incorporates aspects of a spontaneous breakdown of chiral symmetry.

5 Two-Body Dirac Equations in Nucleon-Nucleon
Scattering

The two-body Dirac equations of constraint dynamics provide a natural method
for extending nonrelativistic phenomenological treatments to the relativistic
domain with effective potentials determined from the standard field-theoretic
treatment of meson exchanges. This provides a severe test for the strong
potential terms that turned out to be essential for the treatment of QED
and QCD bound states. The mesons we include are the pseudoscalar mesons
$\pi(135), \eta(548), \eta'(952)$ the vector mesons $\rho(770), \omega(776), \phi(1020)$ and the scalar
mesons $\sigma(600), a_0(980), f_0(983)$. The $\pi, \rho,$ and the $a_0$ are isovector mesons while
the rest are isoscalar mesons. To incorporate these nine meson exchange forces into the two-body Dirac equations we need to generalize the interactions contained in our equations to include others beyond world-scalar and world-vector.

5.1 Two-Body Dirac Equations for General Covariant Interactions: The role of supersymmetry

The detailed forms of interaction in our equations are actually the consequences of supersymmetries in our interacting two-body system. As an illustration, we review the derivation of Eq. (2). Define theta matrices in terms of Dirac matrices

\[ \theta^\mu \equiv \sqrt{\frac{1}{2}} \gamma^5 \gamma^\mu, \quad \mu = 0, 1, 2, 3, \]

\[ \theta^5 \equiv \sqrt{\frac{1}{2}} \gamma^5. \]

In the “correspondence” limit in which the \( \theta \)'s become Grassmann variables, the Dirac equation becomes a constraint imposed on both bosonic (\( p \)) and fermionic (\( \theta, \theta^5 \)) variables:

\[ S_0 \psi \equiv (p \cdot \theta + m\theta^5) \psi = 0; \quad \implies S_0 \equiv (p \cdot \theta + m\theta^5) \approx 0. \quad \text{(19)} \]

For a single free particle, this “classical Dirac-Equation” constraint is supersymmetric[6], under the supersymmetry generated by

\[ p \cdot \theta + \sqrt{-p^2} \theta^5 \]

which however does not leave the position four-vector \( x \) invariant. However, it does leave the “zitterbewegungless” position variable \( \tilde{x}^\mu = x^\mu + i\theta^\mu \theta^5/m \) invariant. In the presence of scalar interaction \( M = m + S \), \( \tilde{x}^\mu \) becomes the ”self-referent” form \( \tilde{x}^\mu = x^\mu + i\theta^\mu \theta^5/M(\tilde{x}) \). The ”pseudoclassical” Dirac dynamics is then governed by the supersymmetric system of constraints

\[ S = p \cdot \theta + M(\tilde{x})\theta^5 \approx 0, \quad \frac{1}{i} \{S, S\} \equiv \mathcal{H} = p^2 + M^2(\tilde{x}) \approx 0. \quad \text{(20)} \]

Since \( \theta^2_5 = 0 \), the expansion of the self-referent form truncates so that \( M(\tilde{x}) = M(x) + i\partial M(x)/\theta^5/M(x) \). Upon quantization, Eqs. (20) then turn into the Dirac Equation and its standard square when the Grassmann variables become theta matrices while dynamical variables \( x \) and \( p \) become their operator counterparts. Thus the supersymmetry that preserves \( \tilde{x} \) is a natural feature of both the single-particle Dirac equation for the free case and its standard form for external scalar interaction.

For two particles we introduce interactions that preserve such a supersymmetry for each spinning particle through the replacement

\[ m_i \rightarrow M_i(x_1 - x_2) \rightarrow M_i(\tilde{x}_1 - \tilde{x}_2) \equiv \tilde{M}_i, \quad i = 1, 2. \quad \text{(21)} \]

The Grassmann Taylor expansions of the \( \tilde{M}_i \) truncate leading to [6]

\[ S_1 \psi = (\theta_1 \cdot p + \epsilon_1 \theta_1 \cdot \hat{P} + M_1 \theta_{31} - i\partial L \cdot \theta_2 \theta_{32} \theta_{31}) \psi = 0, \]

\[ S_2 \psi = (-\theta_2 \cdot p + \epsilon_2 \theta_2 \cdot \hat{P} + M_2 \theta_{32} + i\partial L \cdot \theta_1 \theta_{31} \theta_{32}) \psi = 0, \quad \text{(22)} \]

with the invariants \( M_i \) and \( L(x^\perp) \) related by Eq. (10). Eq. (2) becomes Eq. (22) when restricted to scalar interactions. The consequences of pseudoclassical supersymmetries are the extra spin-dependent recoil corrections to the ordinary one-body Dirac equations, essential for the compatibility of the two equations.
5.2 Hyperbolic Form of the Two-Body Dirac Equations for General Covariant Interactions

We introduce general interactions by recasting the minimal interaction forms of the two-body Dirac equations into ones that generalize the hyperbolic forms we encountered in the treatment of scalar interaction. In the scalar case, if we begin with the two constraints

\[
S_1\psi \equiv (S_{10\theta} \cosh(\Delta)+S_{20\theta} \sinh(\Delta))\psi = 0, \quad S_2\psi \equiv (S_{20\theta} \cosh(\Delta)+S_{10\theta} \sinh(\Delta))\psi = 0
\]  

we find that Eqs. (22) are equivalent to the linear combinations

\[
S_1\psi = [\cosh(\Delta)S_1 + \sinh(\Delta)S_2]\psi = 0, \quad S_2\psi = [\cosh(\Delta)S_2 + \sinh(\Delta)S_1]\psi = 0
\]

in which the interaction appears through the invariant matrix function \(\Delta = -\theta_3\theta_2 L(x)\) while the \(S_0 = (p_i \cdot \theta_i + m\theta_5)\) are free Dirac operators. The \(S_i\) constraints (and hence the \(S_i\) ) are a compatible pair for general \(\Delta\); \(\psi_2 = 0\) (and \(\psi_1 = 0\)) provided only that \(\Delta = \Delta(x)\). Consider the four polar and four axial interactions. For the polar interactions we find the forms \(\Delta(x) = -L(x)\theta_5 - G(x)\theta_1 \cdot \theta_2\) and \(G(x)\theta_2 \cdot \theta_3\) for time-like and space-like vector respectively and \(\Delta(x) = F(x)\theta_1 \cdot \theta_2\theta_3 - \theta_4\theta_3\) and \(\theta_2\theta_3\theta_4\) for the axial tensor. For the scalar interactions \(\psi = 0\) (and \(\psi = 0\)) provided only that \(\Delta = \Delta(x)\).

5.3 Strong Potential Forms for Nucleon-Nucleon Scattering

To represent meson exchanges, we require pseudoscalar interactions as well as vector and scalar. Thus we must use

\[
\Delta(x) = -L(x)\theta_5 - G(x)\theta_1 \cdot \theta_2 - C(x)/2.
\]

with the electromagnetic four-vector (Feynman gauge) condition \(J(x) = -G(x)\) relating the time and space-like components \((\theta_1 \cdot \theta_2 = \theta_1 \cdot \theta_2 - \theta \cdot \theta_\theta\).

Reduction of the coupled Dirac equations (24) to Schrödinger-like form for these combined interactions produces [4] for equal masses

\[
\Phi_w \rightarrow \Phi_S + \Phi_D + (\Phi_{SO} + \Phi_{SOT}\hat{\sigma}_2 \hat{\sigma}_2 \hat{\sigma}_2 \hat{\sigma}_2) L \cdot (\sigma_1 + \sigma_2) + \Phi_{SO} \sigma_1 \cdot \sigma_2 + \Phi_{SOT} \hat{\sigma}_2 \hat{\sigma}_2 \hat{\sigma}_2 \hat{\sigma}_2
\]
in which \( L, C, G \) fix all separate quasipotential pieces. Note that the quadratic nature of many of the strong potential terms (e.g. \( S, A^2, L', G', L'C' \) etc.) could lead to disastrous results for large coupling-constants.

Beyond the limitation to the above nine mesons, one must choose how the corresponding nine Yukawa potentials are included in the three invariant functions \( C, L, G = -J \). We structure our strong potential terms by assuming that in \( G = -\frac{1}{2} \log(1 - \frac{2A}{w}) \) we take

\[
A = \frac{w}{\pi} \arctan \left( \frac{2A}{w} \right), \quad A > 0, **
\]

** \( A = g_2 \bar{r}_1 \cdot \bar{r}_2 \exp(-m_\pi \bar{r}) + g_w \exp(-m_\omega \bar{r}) + g_\eta \exp(-m_\eta \bar{r}) \) (27)

For the invariant \( L \) we use Eq. (11) for \( S > 0 \) and

\[
L = -\frac{1}{2} \log(1 - \frac{2S}{w - 2A}); \quad S < 0, **
\]

** \( S = g_2 \exp(-m_\pi \bar{r}) - \frac{g_2}{\omega} \exp(-m_\omega \bar{r}) - g_\eta \bar{r}_1 \cdot \bar{r}_2 \exp(-m_\eta \bar{r}) \). (28)

The modifications (**) of our strong potential terms for large repulsive vector and large attractive scalar interactions lead to corresponding changes in the quasipotential portions. For the pseudoscalar invariant function \( C \) we use

\[
C = \frac{1}{w} \left[ g_2^2 \bar{r}_1 \cdot \bar{r}_2 \exp(-m_\pi \bar{r}) + g_2 g_\eta \exp(-m_\eta \bar{r}) + g_\eta \exp(-m_\eta \bar{r}) \right]. (29)
\]

We model effects of form factors by replacing \( r (=\sqrt{x^2}) \) by \( \bar{r} = \sqrt{r^2 + r_0^2} \). In addition we take into account that the vector mesons may have an anomalous “magnetic moment” type of coupling. The net effect is to include pairs of additional vector and scalar Yukawa interactions but with opposite signs.

For \(^3S_1\) \( n - p \) and \(^1S_0\) \( n - p \) scattering we obtain excellent phase shift fits in the energy range from 1 to 350 MeV [14]. Examination of some of the other scattering states shows, however, that the model needs improvement through inclusion of a) world tensor coupling, b) pseudovector coupling of the pseudoscalar mesons and c) the off mass shell effects of the vector meson couplings.

6 Concluding Remarks

Nonperturbative solution of our two-body Dirac equations (with all of their strong-potential structure) in QED has demonstrated that these relativistic wave equations reproduce the field-theoretic perturbative spectral results thereby increasing our confidence in their use whenever Coulomb-like potentials play a significant role in dynamics in field-theoretic or phenomenological application. They have been successfully applied in QCD with spectral results as good as those of the most popular approach but using just two invariant potential
functions (with Goldstone boson behavior as a bonus). Their application to $NN$-scattering necessitates their extension to include general covariant interactions (achieved through a natural hyperbolic structure present in them). Phase shift results obtained from them look promising. In the future, the covariant and local Schrödinger-like structure of the equations that make them simple to implement may allow us to combine them with formalisms originally developed for the nonrelativistic Schrödinger equation e.g. the microscopic theory of meson-meson scattering [15] and the unitarized quark model [10].

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