Stability of twisted states in the continuum Kuramoto model

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Abstract

We study a nonlocal diffusion equation approximating the dynamics of coupled phase oscillators on large graphs. Under appropriate assumptions, the model has a family of steady state solutions called twisted states. We prove a sufficient condition for stability of twisted states with respect to perturbations in the Sobolev and BV spaces. As an application, we study stability of twisted states in the Kuramoto model on small-world graphs.

1 Introduction

The Kuramoto model (KM) of coupled phase oscillators provides an important framework for studying collective dynamics in a variety of systems across physics and biology [2, 10]. To formulate the KM on a sequence of graphs, we first review the relevant background from the graph theory. Let \( \Gamma_n = (V(\Gamma_n), E(\Gamma_n)) \) be a graph on \( n \) nodes, \( V(\Gamma_n) = \{1, 2, \ldots, n\} =: [n] \). The pairs of connected nodes form the edge set of \( \Gamma_n, E(\Gamma_n) \). The KM on the following two graph sequences will be used below to motivate the analysis and to illustrate the results. The \( k \)-nearest-neighbor graph, \( C_{n,k} \), \( n \geq 2 \), \( 0 \leq k \leq \lfloor n/2 \rfloor \), our first example, can be geometrically constructed by arranging \( n \) nodes around a circle and connecting each node to its \( k \) neighbors from each side. Thus, \( C_{n,k} = ([n], E(C_{n,k})) \) such that for \( 1 \leq i \neq j \leq n \)

\[ \{i, j\} \in E(C_{n,k}) \text{ if } d_n(i, j) := \min\{|j - i|, 1 - |i - j|\} \leq k. \]

The small-world graph, \( S_{n,k,p} \) with \( n \) and \( k \) as above and \( p \in [0, 1] \) is obtained from \( C_{n,k} \) by replacing each edge with probability \( p \) (independently from other edges) by a random edge [11]. There are several variants of small-world graphs, which differ in technical details of how the random edges are selected (see [11, 8]). In this paper, we follow [6], where small-world graphs are interpreted as \( W \)-random graphs. Specifically, for a given \( n \gg 1 \), \( k \in [\lfloor n/2 \rfloor] \) and \( p \in (0, 1] \), \( S_{n,k,p} = ([n], E(S_{n,k,p})) \) such that

\[ \text{Prob}(\{i, j\} \in E(S_{n,k,p})) = \begin{cases} 1 - p, & d(i, j) \leq k, \\ p, & \text{otherwise}. \end{cases} \]
Figure 1: a) The limiting graphon for \( \{C_n, \lfloor rn \rfloor\} \) is a \( \{0,1\} \)–valued function on the unit square, whose support is shown in black. The limiting graphon of \( \{S_n, \lfloor rn \rfloor, p\} \) is equal to \( 1 - p \) over the black region and \( p \) otherwise. b) A 2–twisted state for a KM with twenty oscillators.

Let us scale the number of neighbors \( k = \lfloor rn \rfloor \) for \( r \in (0,1/2] \). The resultant graph sequences \( \{C_n, \lfloor rn \rfloor\} \) and \( \{S_n, \lfloor rn \rfloor, p\} \) have well-defined asymptotic behavior in the limit as \( n \to \infty \). In fact, \( \{C_n, \lfloor rn \rfloor\} \) and \( \{S_n, \lfloor rn \rfloor, p\} \) are convergent sequences of dense graphs \cite{3}. The asymptotic properties of convergent graph sequences are captured by a symmetric function on a unit square called a graphon. The graphons corresponding to the graph limits of \( \{C_n, \lfloor rn \rfloor\} \) and \( \{S_n, \lfloor rn \rfloor, p\} \) are piecewise constant functions, which are explained in Fig. 1a.

The KM of coupled identical phase oscillators on the graph sequence \( \{\Gamma_n\} \) is defined as follows

\[
\dot{u}_n^\omega = \omega + \frac{1}{n} \sum_{j: \{i,j\} \in E(\Gamma_n)} \sin(2\pi(u_n^\omega - u_m^\omega)), \quad i \in [n],
\]

where \( u_n \) and \( \omega \) stand for the phase and the intrinsic frequency of the oscillator \( i \) respectively. By switching to the rotating frame of coordinates, we eliminate \( \omega \) in (1.1). Thus,

\[
\dot{u}_n = \frac{1}{n} \sum_{j: \{i,j\} \in E(\Gamma_n)} \sin(2\pi(u_n - u_m)), \quad i \in [n].
\]

If \( \Gamma_n \) is a Cayley graph on a cyclic group, \cite{1,2} has a family of steady-state solutions

\[
u^{(q,c)}_n = \left( \frac{qi}{n} + c \right) \bmod 1, \quad i \in [n],
\]

where \( q \) is an integer between \( -n + 1 \) and \( n - 1 \) and \( c \in \mathbb{R} \) \cite{1} (see Fig. 1b). If \( q = 0 \) then \( u^{(0,c)}_n := (c,c,\ldots,c) \) is a spatially homogeneous solution. For \( q \neq 0 \), \( u^{(q,c)}_n, i \in [n] \) wind around the circle \( |q| \) times. In the original frame of coordinates, \( u_n^\omega = u_n^{(q,c)} + \omega t \) form uniformly twisted travelling waves. Hence, \( u_n^{(q,c)} \) are called twisted states. It was pointed out in \cite{12} that the stability of twisted states depends on the connectivity of \( \{\Gamma_n\} \). For instance, for the KM on \( \{C_{n,k}\} \) the stability depends on the number of neighbors \( k \). Similarly, stability of the twisted states can be linked to the network topology in the KM on many other families of graphs including Cayley, Erdős–Rényi, and small–world graphs \cite{7,4}. 

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The KM on convergent graph sequences like \(\{C_{n,[nr]}\}\) and \(\{S_{n,[nr]},p\}\), for large \(n\) can be approximated by the nonlocal diffusion equation
\[
\partial_t u(x,t) = \int_I W(x,y) \sin(2\pi(u(y,t) - u(x,t)))dy, \quad I := [0,1],
\]
where \(u(x,t)\) now describes the phases of the continuum of oscillators distributed over \(I\). The constructions of the continuum limits for the KM on \(\{C_{n,k}\}\) and \(\{S_{n,k,p}\}\) are explained in [5, 6, 7]. Here, we go over a few details that are relevant to the stability problem considered in this work.

For \(\{C_{n,[nr]}\}\) or \(\{S_{n,[nr]},p\}\), the limiting graphon is given by \(W(x,y) = K(y-x)\), where \(K: \mathbb{R} \to \mathbb{R}\) is a 1-periodic function. Specifically, for \(\{S_{n,[nr]},p\}\), we have \(W(x,y) = K_{r,p}(y-x)\), with \(K_{r,p}\) defined on the interval \((-1/2, 1/2]\) by
\[
K_{r,p}(x) = \begin{cases} 
1 - p, & \text{if } |x| \leq r, \\
p, & \text{otherwise},
\end{cases}
\]
and extended to \(\mathbb{R}\) by periodicity. For \(\{C_{n,[nr]}\}\) we have \(W(x,y) = K_r(y-x) := K_{r,0}(y-x)\). Thus, the continuum limit for the KM on these two graph sequences can be rewritten as
\[
\partial_t u(x,t) = \int_I K(y-x) \sin(2\pi(u(y,t) - u(x,t)))dy
\]
where \(K: \mathbb{R} \to \mathbb{R}\) is a given 1-periodic function. Likewise \(u(x,t)\) is 1-periodic with respect to \(x\).

If \(K\) is an even function, as in the case of \(K = K_{r,p}\), the continuum model (1.6) has a family of steady-state solutions
\[
u^{(q,c)}(x) := (qx + c) \mod 1, \quad q \in \mathbb{Z}, c \in \mathbb{R},
\]
i.e., the continuous twisted states; we henceforth assume that \(K\) is even.

We are interested in stability of twisted states (1.7). The following sufficient condition for the linear stability of the continuous twisted states was derived in [12]:
\[
\lambda(q,m) := \bar{K}(m+q) - 2\tilde{K}(q) + \tilde{K}(q-m) \leq \lambda_q < 0, \quad \exists \lambda_q, \forall m \in \mathbb{N},
\]
where \(\bar{K}(m) = \int I K(x) \cos(2\pi mx)dx\). In this work, we prove that (1.8), in fact, implies nonlinear stability in \(H_{per}^1\) (the usual Sobolev space of once weakly differentiable 1-periodic functions) or \(BV_{per}\) (the space of 1-periodic functions of bounded variation). Specifically we have the following theorem.

**Theorem 1** Suppose that \(K\) is an even function in \(L_{per}^2\). Let \(X = H_{per}^1\) or \(BV_{per}\). Suppose that condition (1.8) holds. Then there exists \(\delta > 0\), \(b > 0\) and \(C > 0\) such that \(\|u_0 - u - u^{(q,c)}\|_X \leq \delta\) implies
\[
\sup_{t \geq 0} e^{bt}\|u(\cdot , t) - \mu - u^{(q,c)}(\cdot)\|_X \leq C\|u_0 - \mu - u^{(q,c)}\|_X.
\]
Here \(\mu := \int_I (u_0(y) - u^{(q,c)}(y))dy\) and \(u(x,t)\) is the solution of (1.6) with \(u(x,0) = u_0(x)\).
The proof of Theorem 1 is based on a general stability theorem which we state and prove in Section 2 (specifically, see Theorem 2 below). This general stability theorem is a more or less classical “linear implies nonlinear stability” theorem of the sort used to prove asymptotic stability of fixed points in ordinary differential equations. The main technical difficulty in this work is showing that (1.6) and its \(q\)-twisted states satisfy the hypotheses of the general stability theorem.

The nonlocal equation (1.6) does not possess the strong smoothening property, which facilitates stability analysis of spatial structures in parabolic equations, a closely related class of models \[1\]. Solutions of the initial value problem for (1.6) may have poor spatial regularity if the kernel \(K\) and the initial data are not smooth enough (cf. \[5\] Theorem 3.3). Therefore, along with stability with respect to sufficiently regular perturbations from \(H^1_{\text{per}}\), we feel it is important to understand the stability of \(q\)-twisted states with respect to rough perturbations. To this end, in the second part of the paper, we study stability of the \(q\)-twisted states to perturbations from \(BV_{\text{per}}\). The Sobolev inequalities and the natural connection of \(H^1_{\text{per}}\) to Fourier series make much of our stability analysis in \(H^1_{\text{per}}\) pretty straightforward; after all, the linear stability condition (1.8) is a condition on the Fourier coefficients of \(K\). On the other hand, Fourier analysis for \(BV_{\text{per}}\) functions is not as simple as it is in \(H^1_{\text{per}}\) and this carries with it several challenges, of which the main one is showing that the linearization of (1.6) about a \(q\)-twisted states generates an appropriately contractive semigroup. Addressing all of these technical considerations takes place in Section 3 which contains the proof of Theorem 1.

We conclude this paper with the application to stability of twisted states in the continuum KM on small-world graphs in Section 4.

### 2 Linear implies nonlinear stability

Theorem 1 is a consequence of the following abstract stability result:

**Theorem 2** Suppose that \(X\) is a Banach space. Suppose there exist \(C_1, C_2 > 0, \rho > 0\) and \(b > 0\) such the following hold.

(i) \(L\) is a bounded linear map from \(X\) to itself.

(ii) The uniformly continuous semigroup generated by \(L\), given by \(e^{Lt}\), has, for all \(t > 0\), \(\|e^{Lt}\|_{X \to X} \leq C_1 e^{-bt}\).

(iii) \(N\) is a continuous map from the ball of radius \(\rho\) in \(X\), denoted \(B_{\rho}(0)\), back into \(X\).

(iv) \(N(0) = 0\).

(v) For all \(f, g \in B_{\rho}(0)\) we have \(\|N(f) - N(g)\|_X \leq C_2 (\|f\|_X + \|g\|_X) \|f - g\|_X\).

Then there exists \(\delta \in (0, \rho]\) and \(C_3 > 0\) such that the following hold for all \(\xi_0 \in B_\delta(0) \subset X\).

(a) There exists a unique \(\xi \in C^1(\mathbb{R}^+; X)\) for which \(\xi(0) = \xi_0\) and \(\partial_t \xi = L\xi + N(\xi)\) for all \(t > 0\).
Since \( f, g \in Y \), we have the conclusion (a).

**Proof of Theorem 2**

Let

\[
Y := \left\{ f \in C(\mathbb{R}^+; X) : \sup_{t \geq 0} e^{bt} \|f(t)\|_X =: \|f\|_Y < \infty \right\}.
\]

Note that \( Y \), equipped with \( \| \cdot \|_Y \), is a Banach space. Let

\[ \gamma := \max \left\{ \frac{b}{4C_1C_2}, 1, \rho \right\} \quad \text{and} \quad \delta := \max \left\{ \frac{3\gamma}{4C_1}, \rho \right\} \]

and fix \( \xi_0 \in X \) with \( \|\xi_0\|_X < \delta \). Finally put \( r := \frac{4}{3} C_1\|\xi_0\|_X \). Note that \( r < \gamma \).

For \( f \in Y \) with \( \|f\|_Y < \gamma \) define:

\[
\Psi(f) := e^{Lt}\xi_0 + \int_0^t e^{L(t-s)}N(f(s))ds.
\]

We claim that \( \Psi \) maps \( B_r := \{ f \in Y : \|f\|_Y < r \} \) back into itself and satisfies a contraction estimate on this set. If the claim is true then Banach’s fixed point theorem implies that \( \Psi \) has a unique fixed point in \( B_r \) which we denote \( \xi \). Differentiation of \( \xi = \Psi(\xi) \) with respect to \( t \) shows that \( \partial_t \xi = L\xi + N(\xi) \). This also shows that \( \xi \in C^1(\mathbb{R}^+; X) \). Moreover \( \xi(0) = \Psi(\xi)|_{t=0} = \xi_0 \). Thus we have the conclusion (a). Membership of \( \xi \in Y \) implies the conclusion (b).

Thus we need only establish the claim. Assume that \( f, g \in B_r \). The semigroup estimate (ii) and estimate (v) for \( N \) give:

\[
\|\Psi(f) - \Psi(g)\|_Y \leq \sup_{t \geq 0} e^{bt} \left\| \int_0^t e^{L(t-s)}[N(f(s)) - N(g(s))]ds \right\|_X \\
\leq C_1 \sup_{t \geq 0} e^{bt} \int_0^t e^{-b(t-s)}\|N(f(s)) - N(g(s))\|_X ds \\
\leq C_1 C_2 \sup_{t \geq 0} \int_0^t e^{-bs} (\|f(s)\|_X + \|g(s)\|_X) \|f(s) - g(s)\|_X ds \\
= b^{-1} C_1 C_2 (\|f\|_Y + \|g\|_Y) \|f - g\|_Y.
\]

(2.1)

Since \( f, g \in B_r \) and \( r < \gamma \) this implies \( \|\Psi(f) - \Psi(g)\|_Y \leq 2b^{-1} C_1 C_2 \gamma \|f - g\|_Y \). The definition of \( \gamma \) above then gives:

\[
\|\Psi(f) - \Psi(g)\|_Y \leq \frac{1}{2} \|f - g\|_Y.
\]

(2.2)

Thus \( \Psi \) satisfies a contraction estimate on \( B_r \).
To show that $\Psi$ maps $B_r$ into itself, we first use the semigroup property (ii) and the fact that $N(0) = 0$ to get:

$$\|\Psi(0)\|_Y = \sup_{t \geq 0} e^{bt} \|e^{Lt} \xi_0\|_X \leq C_1 \|\xi_0\|_X. \quad (2.3)$$

Then we use (2.1) in combination with this to get:

$$\|\Psi(f)\|_Y \leq \|\Psi(f) - \Psi(0)\|_Y + \|\Psi(0)\| \leq b^{-1} C_1 C_2 \|f\|_Y^2 + C_1 \|\xi_0\|_X \quad (2.4)$$

Since $\|f\|_Y < r < \gamma \leq b/4 C_1 C_2$ we have: $\|\Psi(f)\|_Y < \frac{1}{4} r + C_1 \|\xi_0\|_X$. Then we use the definition of $r$ to conclude that $\|\Psi(f)\|_Y < \frac{1}{4} r + \frac{3}{4} r = r$. Thus $\Psi$ maps $B_r$ to $B_r$ and we are done.

3 Proof of Theorem 1

Fix $q \in \mathbb{Z}^+$ and $c \in \mathbb{R}$. Suppose that $u(x,t) = u^{(q,c)}(x) + \eta(x,t)$ where $u(x,t)$ solves (1.6) and $\eta(x,t)$ is periodic (with period 1) in $x$. A routine computation show that $\eta(x,t)$ solves:

$$\partial_t \eta = \Phi(\eta) \quad \text{where} \quad \Phi(\eta)(x) := \int_I K(y-x) \sin(2\pi[q(y-x) + \eta(y) - \eta(x)])dy. \quad (3.1)$$

The following technical lemma is proved in Section 3.3.

**Lemma 3** Let $K \in L^2_{\text{per}}$. Let $X$ be either $BV_{\text{per}}$ or $H^1_{\text{per}}$. Then the map

$$\Phi(\eta)(x) := \int_I K(y-x) \sin(2\pi[q(y-x) + \eta(y) - \eta(x)])dy$$

is $C^\infty$ from $X$ to itself. Moreover each of its derivatives is uniformly Lipschitz on any bounded subset of $X$. Lastly $\Phi'(0) =: L$ where

$$L\eta(x) := 2\pi \int_I K(y-x) \cos(2\pi q(y-x))(\eta(y) - \eta(x))dy.$$ 

Note that this lemma implies that the initial value problem for (3.1) is well-posed in $BV_{\text{per}}$ and $H^1_{\text{per}}$.

Next we have

**Lemma 4** Let $K \in L^2_{\text{per}}$ be an even function. Then for any $\eta \in BV_{\text{per}}$ or $H^1_{\text{per}}$ we have

$$\int_I \Phi(\eta)(x)dx = \int_I L\eta(x) = 0.$$
Proof of Lemma 4: We prove the result only for \( \Phi \) as the result for \( L \) goes along the same lines. It is clear that

\[
\int_I \Phi(\eta)(x)dx = \int_I \int_I K(y-x) \sin(2\pi[q(y-x) + \eta(y) - \eta(x)])dydx
= \int_I \int_I K(x-y) \sin(2\pi[q(x-y) + \eta(x) - \eta(y)])dxdy.
\]  

(3.2)

Since \( K \) is even and sine is odd, applying Fubini’s theorem to the above gives:

\[
\int_I \Phi(\eta)(x)dx = -\int_I \int_I K(y-x) \sin(2\pi[q(y-x) + \eta(y) - \eta(x)])dydx = -\int_I \Phi(\eta)(x)dx.
\]

(3.3)

This implies that \( \int_I \Phi(\eta)(x)dx = 0 \).

This last lemma implies that a solution \( \eta(x,t) \) of (3.1) meets

\[
\int_I \eta(x,t)dx = \int_I \eta(x,0)dx =: \mu
\]

for all \( t \). Now set

\[
\xi(x,t) := \eta(x,t) - \mu.
\]

Clearly \( \int_I \xi(x,t)dx = 0 \) for all \( t \). Also observe that \( \partial_t \xi = \Phi(\xi) \) since \( \partial_t \xi = \partial_t \eta \) and \( \xi(x,t) - \xi(y,t) = \eta(x,t) - \eta(y,t) \).

If we put

\[
N(\xi) := \Phi(\xi) - L\xi
\]

then clearly

\[
\partial_t \xi = L\xi + N(\xi).
\]

(3.4)

If we can establish properties (i)-(v) in Theorem 2 for \( L \) and \( N \) as defined above, then the conclusions of that theorem immediately imply our main result, Theorem 1. To be clear, we will establish these properties on the spaces

\[
H^1_{\text{per},0} := \left\{ f \in H^1_{\text{per}} : \int_I f(x)dx = 0 \right\}
\]

and

\[
BV_{\text{per},0} := \left\{ f \in BV_{\text{per}} : \int_I f(x)dx = 0 \right\}.
\]

These are closed subspaces of Banach spaces and thus are themselves Banach spaces with the appropriate inherited norm. Lemma 4 implies that \( L \) and \( N \) are well defined maps on these spaces. Thus (3.4) is well-posed on either of these spaces.
3.1 Estimates for \( N \)

The estimates for \( N \) are relatively simple, given Lemma 3. In particular, that lemma immediately implies property (iii) of Theorem 2 holds for any \( \rho > 0 \). Moreover, by construction \( N'(0) = \Phi'(0) - L = 0 \), which is to say that property (iv) is satisfied. Property (v) follows from an invocation of the “difference of squares” estimate:

**Lemma 5** Suppose that \( X \) is a Banach space and \( U \subset X \) is open, convex and contains 0. Suppose that \( N: U \to X \) is \( C^{1,1} \)—that is to say, \( N' \) is uniformly Lipschitz continuous on \( U \). Furthermore suppose that \( N'(0) = 0 \). Then there exists \( C > 0 \) such that for all \( f, g \in U \) we have

\[
\|N(f) - N(g)\|_X \leq C (\|f\|_X + \|g\|_X) \|f - g\|_X.
\]

**Proof of Lemma 5** Fix \( f, g \in U \). Since \( N \) is differentiable we have by the fundamental theorem of calculus and chain rule:

\[
N(f) - N(g) = \int_0^1 \frac{d}{dt} (N(g + t(f - g))) \, dt = \int_0^1 N'(g + t(f - g)) (f - g) \, dt.
\]

Note that convexity of \( U \) implies \( g + t(f - g) \in U \) for all \( t \in [0,1] \). Since \( N'(0) = 0 \) we have

\[
N(f) - N(g) = \int_0^1 [N'(g + t(f - g)) - N'(0)] (f - g) \, dt.
\]

Since \( N' \) is uniformly Lipschitz on \( U \) (with constant \( C_L \geq 0 \), say), we have by the triangle inequality:

\[
\|N(f) - N(g)\|_X \leq C_L \int_0^1 \|g + t(f - g)\|_X \|f - g\|_X \, dt \leq C (\|f\|_X + \|g\|_X) \|f - g\|_X.
\]

That completes the proof. \( \Box \)

Thus we have established properties (iii)-(v) for \( N \) in Theorem 2.

3.2 Estimates for \( L \)

Property (i) in Theorem 2 is implied by Lemma 3. Next observe that

\[
L \xi = K^{(q)} \ast \xi - \gamma \eta
\]

where the “\( \ast \)” denotes the usual periodic convolution (i.e. if \( f \) and \( g \) are 1—periodic then \( f \ast g(x) := \int_I f(y - x)g(y) \, dy \)). Above

\[
K^{(q)}(x) := 2\pi K(x) \cos(2\pi qx) \quad \text{and} \quad \gamma := 2\pi \int_I K(y) \cos(2\pi y) \, dy.
\]

We will get a formula for the semigroup \( e^{Lt} \) by means of the Fourier series. For periodic functions \( f(x) \), let

\[
\hat{f}(k) := \int_I f(x)e^{2\pi ikx} \, dx
\]
be the coefficients of its Fourier series. The Fourier inversion theorem implies

\[ f(x) = \sum_{k} \hat{f}(k)e^{-2\pi ikx} \]

where the equality is in the sense of \( L^2_{\text{per}} \).

The convolution theorem gives

\[ \hat{(Lf)}(k) = \lambda(k)\hat{f}(k) \]

where

\[ \lambda(k) := 2\pi \hat{K}^{(q)}(k) - \gamma. \]

Thus we can define the semigroup \( e^{Lt} \) by

\[ e^{Lt}f(x) := \sum_{k \neq 0} e^{\lambda(k)t - 2\pi ikx} \hat{f}(k). \quad (3.5) \]

That is to say

\[ \hat{e^{Lt}}f(k) = e^{\lambda(k)t}\hat{f}(k). \]

Condition (1.8) implies that there exists \( b \in \mathbb{R} \) such that

\[ \text{Re} \lambda(k) = \text{Re}(2\pi \hat{K}^{(q)}(k) - \gamma) \leq -b < 0 \quad (3.6) \]

for all \( k \neq 0 \). We are considering mean zero functions and so the \( k = 0 \) mode is excluded, in any case. Which is to say that the spectrum of \( L \) is in the left half plane and thus the system is spectrally stable.

### 3.2.1 Semigroup estimates in \( H^1_{\text{per},0} \)

The semigroup estimates here are simple to establish, as the semigroup is a Fourier multiplier. By Plancheral’s theorem we have, for \( f \in H^1_{\text{per},0} \),

\[ \|e^{Lt}f\|_{H^1_{\text{per}}}^2 = \sum_{k \neq 0} (1 + k^2) \left| e^{\lambda(k)t} \right|^2 \left| \hat{f}(k) \right|^2 = \sum_{k \neq 0} (1 + k^2)e^{2\text{Re} \lambda(k)t} \left| \hat{f}(k) \right|^2. \]

Using (3.6) gives, for \( t > 0 \):

\[ \|e^{Lt}f\|_{H^1_{\text{per}}}^2 \leq \sum_{k \neq 0} (1 + k^2)e^{-2bt} \left| \hat{f}(k) \right|^2 = e^{-2bt}\|f\|_{H^1_{\text{per}}}^2. \]

Thus we have property (ii) for \( X = H^1_{\text{per},0} \). Given the validity of Lemma 3, this concludes the proof of Theorem 1 for \( X = H^1_{\text{per},0} \).
3.2.2 Semigroup estimates in $BV_{\text{per},0}$

Suppose that $f \in BV_{\text{per}}$. Recall that

$$\|f\|_{BV_{\text{per}}} := \int_I |f(x)|\,dx + V(f)$$

where

$$V(f) := \sup_{P \in \mathcal{P}(I)} \sum_P |f(x_i) - f(x_{i-1})|$$

and $\mathcal{P}(I)$ is the set of ordered partitions of $I$. It is not true that Fourier multipliers with bounded symbols define bounded maps on $BV_{\text{per}}$. But $e^{L_t}$ does, as the following computations show.

First observe that

$$e^{\lambda(k)t} = e^{-\gamma t} + e^{-bt} \left( e^{(2\pi \hat{K}(q)(k) - \hat{\gamma}) t} - e^{-\hat{\gamma} t} \right).$$

In the above $\hat{\gamma} := \gamma - b \geq 0$. Note that (3.6) implies that $\Re(2\pi \hat{K}(q)(k) - \hat{\gamma}) \leq 0$ for all $k$. The fundamental theorem of calculus gives:

$$\hat{M}(k,t) := e^{(2\pi \hat{K}(q)(k) - \hat{\gamma}) t} - e^{-\hat{\gamma} t} = -t \int_0^\hat{\gamma} e^{(2\pi \hat{K}(q)(k) - \hat{\gamma}) s} e^{-st} \,ds.$$  

We assume $t > 0$. Note that since $\hat{K}(q)$ may be complex-valued, the integral above is computed along the line segment in $\mathbb{C}$ connecting $\hat{\gamma}$ to $\hat{\gamma} - 2\pi \hat{K}(q)(k)$ which is in the right half plane. The “ML-inequality” gives:

$$\left| \hat{M}(k,t) \right| \leq |t| \left| 2\pi \hat{K}(q)(k) \right|.$$  

We have $K \in L^2_{\text{per}}$ and thus so is $K(q)$. Therefore letting

$$M(x,t) := \sum_k \hat{M}(k,t) e^{-2\pi ikx}$$

defines an $L^2_{\text{per}}$ function by virtue of the preceding estimate. Of course $\|M(t)\|_{L^2_{\text{per}}} \leq C|t|\|K(q)\|_{L^2_{\text{per}}}.$

Since $e^{L_t}$ is defined by (3.5), the decomposition $e^{\lambda(k)t} = e^{-\gamma t} + e^{-bt} \hat{M}(k,t)$ gives, via the convolution theorem, the following formula for the semigroup:

$$e^{L_t}f = e^{-\gamma t}f + e^{-bt}M(\cdot,t) * f.$$  

Now fix $b' \in (0,b)$. We have

$$\|e^{L_t}f\|_{BV_{\text{per}}} \leq e^{-\gamma t}\|f\|_{BV_{\text{per}}} + e^{-bt}\|M(\cdot,t)\|_{L^2_{\text{per}}} \|f\|_{BV_{\text{per}}} \leq Cb'e^{-b't}\|f\|_{BV_{\text{per}}}.$$  

Thus we have property (ii) for $X = BV_{\text{per},0}$. Given the validity of Lemma 3 this concludes the proof of Theorem 1 for $X = BV_{\text{per},0}$. 

10
3.3 Properties of $\Phi$

This section will ultimately establish Lemma 3. First we have:

**Lemma 6** Let $X = H^1_{\text{per}}$ or $BV_{\text{per}}$. Then the maps

$$\Xi(\eta)(x) = \cos(2\pi \eta(x)) \quad \text{and} \quad S(\eta)(x) = \sin(2\pi \eta(x))$$

are $C^\infty$ from $X$ to itself. Moreover each derivative is uniformly Lipschitz on any bounded subset of $X$.

**Proof of Lemma 6** When $X = H^1_{\text{per}}$, $S$ and $\Xi$ are Nemitskii substitution operators and the result is classical [9]. And so we only prove this result for $X = BV_{\text{per}}$.

First note that

$$\|S(\eta)\|_{BV_{\text{per}}} = \int_I |\sin(\eta(x))|dx + \sup_{P \in \mathcal{P}} \sum_{P} |\eta(x_i) - \eta(x_{i-1})|$$

Since $|\sin(\theta) - \sin(\phi)| \leq 2|\theta - \phi|$ for any $\theta, \phi$, the above gives:

$$\|S(\eta)\|_{BV_{\text{per}}} \leq C \int_I |\eta(x)|dx + C \sup_{P \in \mathcal{P}} \sum_{P} |\eta(x_i) - \eta(x_{i-1})| = C\|\eta\|_{BV_{\text{per}}}. $$

Likewise

$$\|\Xi(\eta)\|_{BV_{\text{per}}} \leq C + C\|\eta\|_{BV_{\text{per}}}. $$

Using the trigonometry identities $\sin(\theta) - \sin(\phi) = 2\sin((\theta - \phi)/2)\cos((\theta + \phi)/2)$ and $\cos(\theta) - \cos(\phi) = -2\sin((\theta-\phi)/2)\sin((\theta+\phi)/2)$, together with the algebra estimate $\|fg\|_{BV_{\text{per}}} \leq C\|f\|_{BV_{\text{per}}} \|g\|_{BV_{\text{per}}}$, we get

$$\|S(\eta) - S(\xi)\|_{BV_{\text{per}}} + \|\Xi(\eta) - \Xi(\xi)\|_{BV_{\text{per}}} \leq C (1 + \|\eta\|_{BV_{\text{per}}} + \|\xi\|_{BV_{\text{per}}}) \|\eta - \xi\|_{BV_{\text{per}}}. $$

Which is to say that both $S$ and $\Xi$ are Lipschitz continuous on all of $BV_{\text{per}}$ and the Lipschitz constant can be taken uniformly on any bounded subset.

Next we claim that that $S'(\eta)h(x) = 2\pi \Xi(\eta)h(x)$. Fix $\eta(x), h(x) \in BV_{\text{per}}$. The fundamental theorem of calculus implies

$$\Delta(x) := S(\eta + h)(x) - S(\eta)(x) - 2\pi \Xi(\eta)h(x) = -4\pi^2 \int_0^{h(x)} \int_0^s \sin(2\pi (\eta(x) + r)) dr ds. \quad (3.7)$$

If we can show $\|\Delta\|_{BV_{\text{per}}} \leq C(1 + \|\eta\|_{BV_{\text{per}}})\|h\|_{BV_{\text{per}}}^2$ we will have established the claim.

Elementary estimates show

$$|\Delta(x)| = |S(\eta + h)(x) - S(\eta)(x) - 2\pi \Xi(\eta)h(x)| \leq C|h(x)|^2$$
for some constant $C > 0$, independent of both $h$ and $\eta$. This implies
\[
\|\Delta\|_{L^1_{\text{per}}} = \|S(\eta + h) - S(\eta) - 2\pi\Xi(\eta)h\|_{L^1_{\text{per}}} \leq C\|h\|^2_{L^\infty_{\text{per}}} \leq C\|h\|_{BV_{\text{per}}}^2. \tag{3.8}
\]

Next we estimate
\[
V(\Delta) = \sup_{P \in P} \sum_P |\Delta(x_i) - \Delta(x_{i-1})|.
\]

Then (3.7) implies
\[
|\Delta(x_i) - \Delta(x_{i-1})| = 4\pi^2 \left| \int_0^{h(x_i)} \int_0^s \sin(2\pi(\eta(x_i) + \tau))d\tau ds - \int_0^{h(x_{i-1})} \int_0^s \sin(2\pi(\eta(x_{i-1}) + \tau))d\tau ds \right|.
\]

Adding zero, the triangle inequality and elementary properties of the integral, give:
\[
|\Delta(x_i) - \Delta(x_{i-1})| = 4\pi^2 \left| \int_0^{h(x_i)} \int_0^s \sin(2\pi(\eta(x_i) + \tau))d\tau ds \right|
+ 4\pi^2 \left| \int_0^{h(x_{i-1})} \int_0^s [\sin(2\pi(\eta(x_i) + \tau)) - \sin(2\pi(\eta(x_{i-1}) + \tau))] d\tau ds \right|. \tag{3.9}
\]

Since $|\sin(\theta)| \leq 1$ for any $\theta$ the first term is bounded by
\[
C |h(x_i) - h(x_{i-1})| |h(x_i) + h(x_{i-1})| \leq C\|h\|_{L^\infty_{\text{per}}} |h(x_i) - h(x_{i-1})| \leq C\|h\|_{BV_{\text{per}}} |h(x_i) - h(x_{i-1})|.
\]

The constant $C > 0$ is independent of $h, \eta, i$ or $P$. And since $|\sin(\theta) - \sin(\phi)| \leq |\theta - \phi|$ for any $\theta, \phi$ the second term in (3.9) is bounded by
\[
C |h(x_i)|^2 |\eta(x_i) - \eta(x_{i-1})| \leq C\|h\|_{BV_{\text{per}}}^2 |\eta(x_i) - \eta(x_{i-1})|
\]

The constant $C > 0$ is independent of $h, \eta, i$ or $P$.

Thus we have
\[
V(\Delta) \leq C\|h\|_{BV_{\text{per}}} \sup_{P \in P} \left( |h(x_i) - h(x_{i-1})| + \|h\|_{BV_{\text{per}}} |\eta(x_i) - \eta(x_{i-1})| \right)
\leq C\|h\|_{BV_{\text{per}}} V(h) + C\|h\|_{BV_{\text{per}}}^2 V(\eta) \leq C(1 + \|\eta\|_{BV_{\text{per}}})\|h\|_{BV_{\text{per}}}^2. \tag{3.10}
\]

Adding this to (3.8) gives
\[
\|\Delta\|_{BV_{\text{per}}} \leq C(1 + \|\eta\|_{BV_{\text{per}}})\|h\|_{BV_{\text{per}}}^2
\]

Thus we have, for any $\eta, \xi \in BV(I)$
\[
S'(\eta)\xi = 2\pi\Xi(\eta)\xi \tag{3.11}
\]

In an almost identical fashion we can show
\[
\Xi'(\eta)\xi = -2\pi S(\eta)\xi \tag{3.12}
\]

The fact that $S$ and $\Xi$ are uniformly Lipschitz on any bounded subset of $BV_{\text{per}}$, together with (3.11) and (3.12) and the fact that $BV_{\text{per}}$ is an algebra, is sufficient to conclude that $S'$ and $\Xi'$ are
In this section, we apply Theorem 1 to establish stability of certain 4–twisted states is interpreted as stability with respect to perturbations in $H^1$ the linear stability results obtained for these models in [12, 7]. Throughout this section, stability

| limit of the KM on the nearest–neighbor and small–world graphs (cf. (1.4), (1.5)), thus extending

Theorem 4 implies that the product of two smooth maps is smooth. Thus we see that $\Phi(\eta)$ is $C^\infty$ since it is a composition of smooth maps. Finally the algebra estimates $\|fg\|_{BV_{per}} \leq C\|f\|_{BV_{per}}\|g\|_{BV_{per}}$ and $\|fg\|_{H^1_{per}} \leq C\|f\|_{H^1_{per}}\|g\|_{H^1_{per}}$ imply that the product of two smooth maps is smooth. Thus we see that $\Phi(\eta)$ is $C^\infty$ since it is a linear combination of products and compositions of smooth maps. All of its derivatives are uniformly Lipschitz on any bounded subsets since its constituent parts are.

4 Examples

In this section, we apply Theorem 1 to establish stability of certain $q$–twisted states in the continuum limit of the KM on the nearest–neighbor and small–world graphs (cf. (1.4), (1.5)), thus extending the linear stability results obtained for these models in [12, 7]. Throughout this section, stability of twisted states is interpreted as stability with respect to perturbations in $H^1_{per}$ and $BV_{per}$.

For convenience, we rewrite the continuum KM on small-world graphs (cf. (1.4), (1.5))

\begin{equation}
\partial_t u(x, t) = \int_I K_{r,p}(y - x) \sin(2\pi(u(y, t) - u(x, t)))dy.
\end{equation}
Note that for \( p = 0 \) \((K_{r,0}(\cdot) = K_r(\cdot))\), (4.1) contains the KM on the nearest-neighbor graphs as a special case.

The following theorem shows that \( q\)-twisted states are stable provided \( r > 0 \) and \( p \geq 0 \) are sufficiently small.

**Theorem 7** Let \( u^{(q)} \) be a \( q\)-twisted state solution of (4.1).

a) \( u^{(0)} \) is stable for any \( r \in (0, 1/2) \) and \( p \in [0, 1/2) \).

b) For \( q \in \mathbb{N} \) there exist \( r_q \in (0, 1/2) \) and \( p_q \in (0, 1/2) \) such that \( u^{(q)} \) is a stable steady-state solution of (4.1) for any \( r \in (0, r_q) \) and \( p \in [0, p_q) \).

We precede the proof of Theorem 7 with the following auxiliary lemma, whose proof is given at the end of this section.

**Lemma 8** Let

\[
 f(x) := \frac{\sin(x)}{x} \quad \text{and} \quad g(x, y) := f(y + x) - 2f(y) + f(y - x).
\]

There exist \( X_1, Y_1 > 0 \) and \( \delta < 0 \) such that the following are true when \( |y| \leq Y_1 \).

a) \( g(x, y) = 0 \) implies \( x = 0 \).

b) \( g(x, y) \leq 0 \) for all \( x \).

c) \( g(x, y) \leq \delta \) for \( |x| \geq X_1 \).

**Proof.** (Theorem 7) We prove the statements of the theorem for \( p = 0 \) in part A) below and then in part B) to show that these results continue to hold for small positive \( p \).

A) Suppose \( p = 0 \). First, we specialize the stability condition (1.8) for the KM (4.1). Using (1.5), we find

\[
 \tilde{K}_r(m) = \int_{-r}^{r} \cos(2\pi ms)ds = 2rf(2\pi mr), \ m \in \mathbb{Z},
\]

and rewrite (1.8) for the model at hand

\[
 \lambda(q, m) = 2rg(m\rho, q\rho) \leq \lambda_q < 0, \forall m \in \mathbb{N},
\]

where \( \rho := 2\pi r \).

We consider \( q = 0 \) first. Let \( 0 < \rho < \pi \) be arbitrary but fixed. In this case,

\[
 g(m\rho, 0) = 2(f(m\rho) - 1). 
\]
By Lemma 8,
\[ g(m\rho,0) \leq \delta < 0, \quad \text{for} \quad m \geq m_1 := \lceil X_1/\rho \rceil. \]
Thus, (4.3) holds with
\[ \lambda_0 = (2r)^{-1} \max \left\{ \delta, \max_{1 \leq i \leq m_1} 2(f(m\rho) - 1) \right\} < 0. \]

Next, suppose \( q \in \mathbb{N} \) is fixed. The quantity we are interested in is
\[ \sup_{m \in \mathbb{N}} g(m\rho,q\rho). \]
Let \( \rho_q := 2Y_1/q \) and \( \delta_q := \max_{x \in [\rho_q,X_1]} g(x,Y_1) \).

The first two conditions from the Lemma 8 imply that \( \delta_q < 0 \), since we have \( \rho_q > 0 \) and \( g(x,Y_1) \) is a continuous function of \( x \). Let \( \lambda_q := \max\{\delta_q,\delta\}/2r < 0 \). Then we have
\[ \lambda(q,m) \leq \sup_{m \geq 1} 2rg(m\rho,q\rho) \leq \sup_{x \geq \rho_q} 2rg(x,Y_1) \leq \lambda_q < 0. \]
(4.4)

Thus, (4.3) holds for all \( 0 < \rho < \rho_q \).

**B)** Let \( 0 < p < 1/2 \) and \( 0 < r < 1/2 \). For
\[ K_{r,p}(x) = (1 - 2p)1_{(-r,r)}(x) + p1_{(-1/2,1/2)}(x) \]
we compute
\[ \tilde{K}_{r,p}(m) = (1 - 2p)\tilde{K}_r(m) + p\delta_{m,0}, \]
where \( 1_A \) is the characteristic function of \( A \), \( \delta_{m,n} \) is the Kronecker delta, and \( \tilde{K}_r \) is given in (4.2). Thus, (1.8) becomes
\[ \lambda(q,m;p) = (1 - 2p)\lambda(q,m) + p\delta_{q,0}, \]
(4.5)

where \( \lambda(q,m) \) is given in (4.3). Stability of the 0-twisted state follows from the analysis in part **A**. For \( q > 0 \), using (4.4), we conclude that
\[ \lambda(q,m;p) \leq (1 - 2p)\lambda_q + p\delta_{m,q} \leq \lambda_q/2 \quad \forall \quad m \in \mathbb{N}, \]
provided \( 0 < p < -\lambda_q/(2(1 - 2\lambda_q)) \).

□

Proof. (Proof Lemma 8) Note that \( g(x,y) \) is even in both \( x \) and \( y \) so we only need to consider \( x, y > 0 \).

Let \( Y_{11} = \pi/2 \). Since \( f(y) \) is strictly decreasing on the interval \( (0,\pi) \) we know that \( f(y) \geq f(\pi/2) = 2/\pi \) when \( |y| \leq Y_{11} \). Thus \( |y| \leq Y_{11} \) implies
\[ g(x,y) \leq f(y + x) + f(y - x) - 4/\pi \quad \text{for all} \quad x. \]
Next let $X_1 := 5\pi/2$. If $0 < y < Y_{11} = \pi/2$ then clearly $x > X_1$ gives $x > 2y$. Which means that

$$x + y > x - y > x/2.$$ 

We know that

$$|f(x)| \leq 1/|x| \quad \text{for all } x \neq 0.$$ 

Thus

$$|f(x + y)| \leq 1/|x + y| \leq 2/|x| \leq 2/X_1 = 4/5\pi,$$

$$|f(y - x)| \leq 1/|x - y| \leq 2/|x| \leq 2/X_1 = 4/5\pi.$$ 

Thus we have

$$|x| \geq X_1 \text{ and } |y| \leq Y_{11} \implies g(x, y) \leq 8/5\pi - 4/\pi = -12/5\pi < 0. \quad (4.6)$$ 

Next we compute the Taylor series for $g(x, y)$ at $(0, 0)$. It is:

$$g(x, y) = x^2 \left( -\frac{1}{3} + \frac{1}{60} x^2 + \frac{1}{10} y^2 \right) + O(|x, y|^4).$$ 

Clearly $-\frac{1}{3} + \frac{1}{60} x^2 + \frac{1}{10} y^2 < 0$ for $\|(x, y)\|$ sufficiently small. Thus we can conclude that there exists $\rho_1 > 0$ such that

$$g(x, y) \leq 0 \quad \text{for } \|(x, y)\| \leq \rho_1 \text{ with equality only if } x = 0. \quad (4.7)$$ 

Next note that $g(x, 0) = 2f(x) - 2f(0) = 2f(x) - 2$. The fact that $|f(x)| < 1$ for all $x \neq 0$ means that $g(x, 0) < 0$ for all $x \neq 0$. Which means that there exists $\mu < 0$ such that $\sup_{x \in [\rho_1, X_1]} g(x, 0) = \mu$. Since $g(x, y)$ is a smooth function and $[\rho_1, X_1]$ is compact in $\mathbb{R}^2$, we can conclude that there is open neighborhood of $[\rho_1, X_1]$ where $g(x, y)$ remains strictly negative. That it to say there exists $\rho_2 > 0$ such that

$$\sup_{x \in [\rho_1 - \rho_2, X_1 + \rho_2], |y| \leq \rho_2} g(x, y) \leq \mu/2 < 0. \quad (4.8)$$ 

Now put $Y_1 := \min \{Y_{11}, \rho_1, \rho_2\}$. If $|y| \leq Y_1$ and $|x| \geq X_1$ then (4.6) gives us conclusion (c). If $|y| \leq Y_1$ and $|x| \leq X_1$ then (4.7) and (4.8) give us conclusions (a) and (b).

\[\square\]

5 Conclusion

Stability of a family of steady state solutions, such as twisted states of the nonlocal equation (1.6), provides valuable insights into the structure of the phase space and can be used for studying more complex dynamical regimes. Analysis of twisted states in the Kuramoto model of coupled phase oscillators has helped to understand better the link between the structure and dynamics in complex networks of interacting dynamical systems. It reveals a subtle relation between the fine properties of the network topology and stability of steady state solutions in coupled systems of nonlinear differential equations [12, 4, 7].
For the continuum model \( \text{(1.6)} \), which approximates the dynamics of the Kuramoto model on large graphs \([5, 6]\), Wiley, Strogatz, and Girvan found an elegant condition determining the stability of the twisted states in this model \([12]\). The condition relies on a set of linear inequalities in terms of the Fourier coefficients of the kernel of the integral operator in \( \text{(1.6)} \). It applies to the Kuramoto model on a variety of Cayley and random graphs \([4]\), including the small-world graphs \([7]\). It is well-known that in infinite-dimension systems like \( \text{(1.6)} \), linear stability does not automatically imply the nonlinear stability. In this work, we studied under what conditions and in what sense twisted states are stable if the linear stability condition from \([12]\) holds. We found that for the problem at hand the linear stability implies nonlinear stability with respect to perturbations in \( H_{\text{per}}^1 \) and \( BV_{\text{per}} \). The latter result is important, because solutions of the initial value problems for \( \text{(1.6)} \) in general have poor spatial regularity and, thus, it is natural to consider stability with respect to rough perturbations. The results of this work complement the linear stability analysis in \([12]\) and provide a method for studying stability of spatial patterns in the continuum Kuramoto system and related model.

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References

[1] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981.

[2] Y. Kuramoto, Cooperative dynamics of oscillator community, Progress of Theor. Physics Supplement (1984), 223–240.

[3] L. Lovász, Large networks and graph limits, AMS, Providence, RI, 2012.

[4] G. S. Medvedev and X. Tang, Stability of twisted states in the Kuramoto model on Cayley and random graphs, J. Nonlinear Sci. 25 (2015), no. 6, 1169–1208. MR 3415044

[5] G.S. Medvedev, The nonlinear heat equation on dense graphs and graph limits, SIAM J. Math. Anal. 46 (2014), no. 4, 2743–2766. MR 3238494

[6] ______., The nonlinear heat equation on W-random graphs, Arch. Ration. Mech. Anal. 212 (2014), no. 3, 781–803. MR 3187677

[7] ______., Small-world networks of Kuramoto oscillators, Physica D 266 (2014), 13–22.

[8] N.E.J. Newman and D.J. Watts, Renormalization group analysis of the small-world network model, Phys. Lett. A 263 (1999), 341–346.

[9] M. Renardy and R. Rogers, An Introduction to Partial Differential Equations, Texts in Applied Mathematics 13 (2nd ed). Springer-Verlag, New York, 2004.

[10] Steven Strogatz, Sync, Hyperion Books, New York, 2003. MR 2394754
[11] D.J. Watts and S.H. Strogatz, *Collective dynamics of small-world networks*, Nature 393 (1998), 440–442.

[12] D.A. Wiley, S.H. Strogatz, and M. Girvan, *The size of the sync basin*, Chaos 16 (2006), no. 1, 015103, 8. MR 2220552 (2007e:37016)