ON THE SYMPLECTIC FILLINGS OF STANDARD REAL PROJECTIVE SPACES

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Abstract. We prove, in a geometric way, that the standard contact structure on \( \mathbb{R}P^{2n-1} \) is not Liouville fillable for \( n \geq 3 \) and odd. We also prove for all \( n \) that semipositive fillings of such contact structures are always simply connected. Finally we give yet another proof of the Eliashberg–Floer–McDuff theorem on the diffeomorphism type of the symplectically aspherical fillings of the standard contact structure on \( S^{2n-1} \).

1. Introduction

The standard contact structure \( \xi \) on \( S^{2n-1} \) is described in coordinates by the equation

\[
\xi = \ker \sum_{j=1}^{n} (x_j \, dy_j - y_j \, dx_j).
\]

Geometrically, \( \xi_p \) is the unique complex hyperplane in \( T_p S^{2n-1} \) for every \( p \in S^{2n-1} \). The antipodal involution of \( S^{2n-1} \) preserves \( \xi \), and therefore induces a contact structure on \( \mathbb{R}P^{2n-1} \) which we still denote by \( \xi \). The disc bundle of the line bundle \( \mathcal{O}_{\mathbb{P}^{n-1}}(-2) \) on \( \mathbb{C}P^{n-1} \) is a strong symplectic filling of \((\mathbb{R}P^{2n-1}, \xi)\). On the other hand, \( \mathbb{R}P^{2n-1} \) cannot be the boundary of a \( 2n \)-dimensional manifold with the homotopy type of an \( n \)-dimensional CW complex if \( 2n - 1 \geq 5 \); see [3, Section 6.2]. This implies that a real projective space of dimension at least 5 does not admit any Weinstein fillable contact structure. Our main result is the following.

Theorem 1.1. The standard contact structure on \( \mathbb{R}P^{2n-1} \) admits no symplectically aspherical fillings for \( n > 1 \) and odd. In particular, it is not Liouville fillable.

These are the first examples of strongly but not Liouville fillable contact structures in high dimension. Examples in dimension three were given by the first author in [1] using Heegaard Floer homology. In contrast with the high dimensional situation, the standard contact structure on \( \mathbb{R}P^{3} \) is the canonical contact structure on the unit cotangent bundle of \( S^{2} \) and therefore is Weinstein fillable.

After a preliminary version of our result (originally for \( \mathbb{R}P^{5} \) only) was announced, Zhou proved in [11] that \((\mathbb{R}P^{2n-1}, \xi)\) is not Liouville fillable if \( n \neq 2k \). He also proves similar nonfillability results for some other links of cyclic quotient singularities.

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Zhou’s proof uses advanced properties of symplectic cohomology; in contrast our proof is more direct, as it relies on the analysis of how a certain moduli space of holomorphic spheres can break, in the spirit of McDuff’s classification of symplectic fillings of $\mathbb{RP}^3$ in [7].

The strategy is the following. The standard contact structure $\xi$ on $\mathbb{RP}^{2n-1}$ admits a contact form whose Reeb orbits are the fibres of the Hopf fibration $\mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$. If $(W, \omega)$ is a strong symplectic filling of $(\mathbb{RP}^{2n-1}, \xi)$, by a symplectic reduction of $\partial W$ (informally speaking, by replacing $\partial W$ with its quotient by the Reeb flow) we obtain a closed symplectic manifold $(\overline{W}, \omega)$ with a codimension two symplectic submanifold $W_\infty \cong \mathbb{CP}^{n-1}$ (corresponding to the quotient of $\partial W$) such that $\overline{W} \setminus W_\infty$ is symplectomorphic to $\text{int}(W)$; that is, $\overline{\omega}|_{\overline{W} \setminus W_\infty} = \omega|_{\text{int}(W)}$. The normal bundle of $W_\infty$ is isomorphic to $\mathcal{O}_\mathbb{P}^{n-1}(2)$.

We fix a point and a hyperplane in $W_\infty$, and we consider the moduli space of holomorphic spheres in $\overline{W}$ which are homotopic to a projective line and pass both through the point and the hyperplane. We prove by topological considerations that if the compactification of that moduli space contains only nodal curves with at most two irreducible components each of which intersect $W_\infty$ nontrivially, then some of these nodal curves will be composed of two spheres that represent identical homology classes up to torsion. This implies in particular that the homology class of a projective line in $W_\infty$ is the double of some homology classes in $\overline{W}$ up to torsion. This implies in particular that the homology class of a projective line in $W_\infty$ is the double of some homology classes in $\overline{W}$ up to torsion.

If $n$ is odd this is a contradiction because the first Chern class of a line is $n + 2$; only at this step we use the hypothesis on the parity of $n$. This implies that there is either a nodal holomorphic sphere in $\overline{W}$ in the homology class of a line of $W_\infty$ with at least three connected components or a nodal holomorphic sphere with an irreducible component which is disjoint from $W_\infty$. Since a nodal sphere intersects $W_\infty$ in exactly two points, in either case at least one irreducible component must lie entirely in $\text{int}(W)$, which therefore is not symplectically aspherical.

If $(W, \omega)$ is not symplectically aspherical we lose control on the compactification of the moduli space, which is not surprising, given that $(\mathbb{RP}^{2n-1}, \xi)$ does admit spherical fillings. However, if $W$ is semipositive (and maybe even more generally, using some abstract perturbation scheme) we still have enough control to be able to draw conclusions about the fundamental group of $W$.

**Theorem 1.2.** If $(W, \omega)$ is a semipositive symplectic filling of $(\mathbb{RP}^{2n-1}, \xi)$, then $W$ is simply connected.

If we apply the same techniques to a symplectically aspherical filling of the standard contact structure on $S^{2n-1}$ we obtain that the filling must be diffeomorphic to the ball, a result originally due to Eliashberg, Floer and McDuff. This is, at least, the fifth proof, after the original one in [8], a very similar one in [10], the one in [5] using moduli spaces of holomorphic discs with boundary on a family of LOB’s, and the one in [1]. The proof given here is close to the original one, but uses a different compactification of the filling and is slightly simpler.

2. **The moduli space of lines**

2.1. **The smooth stratum.** By the Weinstein neighbourhood theorem, $W_\infty$ has a tubular neighbourhood that is symplectomorphic to a neighbourhood of the zero
Lemma 2.1. \( \mathcal{M}(p_0, H_\infty) \) has expected dimension \( 2n - 2 \) and \( \mathcal{M}_z(p_0, H_\infty) \) has expected dimension \( 2n \), where \( \dim W = 2n \).

Proof. The decomposition (1) gives \( \langle c_1(TW), [\ell] \rangle = n + 2 \). The expected dimension of \( \mathcal{M}(p_0, H_\infty) \) is

\[
\text{vir-dim } \mathcal{M}(p_0, H_\infty) = 2\langle c_1(TW), [\ell] \rangle + 2n + 4 - 2n - 4 - 6 = 2n - 2.
\]

The first two terms compute the index of the linearised Cauchy-Riemann operator, the third is the contribution of two extra marked points, the fourth and the fifth come from the condition that the marked points be mapped to \( p_0 \) and \( H_\infty \), and the last is the dimension of the biholomorphism group of the sphere. \( \square \)

The main reason for keeping the almost complex structure integrable near \( W_\infty \) is to have positivity of intersection between \( W_\infty \) and \( J \)-holomorphic spheres. This fact makes our moduli space particularly well behaved, as the following lemma shows.

Lemma 2.2. All \( J \)-holomorphic spheres of \( \mathcal{M}(p_0, H_\infty) \) are simply covered and are either lines in \( W_\infty \) or intersect \( W_\infty \) transversely in exactly two points.

Proof. Since the algebraic intersection between \( W_\infty \) and \( \ell \) is 2, positivity of intersection implies that a sphere of \( \mathcal{M}(p_0, H_\infty) \) is either contained in \( W_\infty \), in which case it is a line and therefore simply covered, or it intersect \( W_\infty \) with total multiplicity two. Since the constraints force two distinct intersection points, positivity of intersection implies that they are the only ones and that they each have multiplicity one. \( \square \)

Proposition 2.3. For a generic almost complex structure \( J \in \mathcal{J} \) the moduli spaces \( \mathcal{M}(p_0, H_\infty) \) and \( \mathcal{M}_z(p_0, H_\infty) \) are smooth manifolds of dimension \( 2n - 2 \) and \( 2n \) respectively.
Proof. The $J$-holomorphic spheres of $\mathcal{M}(p_0, H_\infty)$ which are contained in the neighbourhood of $W_\infty$ where $J$ is integrable correspond to holomorphic sections of $\mathcal{O}_{\mathbb{P}^{n-1}}$ and therefore admit a decomposition of the restriction of $J$ as in Equation (1). Since the decomposition is into positive holomorphic line bundles, those spheres are Fredholm regular for every almost complex structures $J \in \mathcal{J}$ because the Cauchy-Riemann operator on a positive line bundle over $\mathbb{C}P^1$ is surjective by Serre duality; see [9, Lemma 3.3.1).

All other $J$-holomorphic spheres of $\mathcal{M}(p_0, H_\infty)$ are Fredholm regular for a generic $J \in \mathcal{J}$ because they are simply covered and intersect the region where $J$ is generic. Moreover, the pointwise constraints cut out $\mathcal{M}(p_0, H_\infty)$ transversely for a generic $J$: for spheres near $W_\infty$ this is an explicit computation, and for all other spheres of $\mathcal{M}(p_0, H_\infty)$ it follows from [9, Theorem 3.4.1] and [9, Remark 3.4.8]. Therefore $\mathcal{M}(p_0, H_\infty)$ is a smooth manifold of the dimension predicted by Lemma 2.1. The corresponding statements for $\mathcal{M}_z(p_0, H_\infty)$ follow from those for $\mathcal{M}(p_0, H_\infty)$. \qed

2.2. The compactified moduli space. Let $\overline{\mathcal{M}}(p_0, H_\infty)$ and $\overline{\mathcal{M}}_z(p_0, H_\infty)$ be the Gromov compactifications of $\mathcal{M}(p_0, H_\infty)$ and $\mathcal{M}_z(p_0, H_\infty)$ respectively, and let $\mathcal{M}$ be the forgetful map. We denote $\mathcal{M}^{\text{red}}(p_0, H_\infty) = \overline{\mathcal{M}}(p_0, H_\infty) \setminus \mathcal{M}(p_0, H_\infty)$ and $\mathcal{M}_z^{\text{red}}(p_0, H_\infty) = \overline{\mathcal{M}}_z(p_0, H_\infty) \setminus \mathcal{M}_z(p_0, H_\infty)$.

Lemma 2.4. If $\overline{\mathcal{W}} \setminus W_\infty$ is symplectically aspherical, then every nodal sphere of $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ has exactly two irreducible components, one of which intersects $W_\infty$ only at $p_0$ and the other one which intersects $W_\infty$ only at a point of $H_\infty$. Both components are simply covered and their intersection with $W_\infty$ is transverse.

Proof. None of the irreducible components of nodal spheres in $\overline{\mathcal{M}}(p_0, H_\infty)$ is contained in $W_\infty$ because any bubble component needs to have positive symplectic area strictly smaller than the symplectic area of $\ell$, but the homology class of $\ell$ has the smallest positive symplectic area in $\mathbb{C}P^{n-1}$. Then by positivity of intersection with $W_\infty$ a nodal sphere must intersect $W_\infty$ in at most two points. Moreover, if $\overline{\mathcal{W}} \setminus W_\infty$ is symplectically aspherical, every irreducible component must intersect $W_\infty$. This implies that there are exactly two irreducible components and the intersection of each with $W_\infty$ has multiplicity one. Therefore both components are simply covered. \qed

This lemma implies that we have enough topological control on the nodal curves to show that they have smooth moduli spaces.

Lemma 2.5. The moduli space $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ is a smooth manifold of dimension $2n - 4$. The forgetful map $\mathcal{M}_z^{\text{red}}(p_0, H_\infty) \to \mathcal{M}^{\text{red}}(p_0, H_\infty)$ is a locally trivial fibration with fibre $S^2 \vee S^2$.

Proof. By Lemma 2.4 the irreducible components of the nodal spheres of the moduli space $\mathcal{M}^{\text{red}}(p_0, H_\infty)$ are simply covered and intersect the region where the almost complex structure can be made generic. Then the statement follows from [9, Theorem 6.2.6]. \qed
What we have shown so far about \( \overline{M}_z(p_0, H_\infty) \) is enough to show that the image of the evaluation is a pseudocycle in \( \overline{W} \) (see [21 Section 6.5]). While pseudocycles are good enough for certain degree arguments, as for example in Section 3, in the proof of our main theorem we will need a differentiable structure on the compactified moduli spaces. The reason is that our proof is based on studying the properties of the homology class \( [ev^{-1}(\ell)] \) in \( \overline{M}_z(p_0, H_\infty) \), and we have found no better way to show that \( ev^{-1}(\ell) \) is well-behaved than by the implicit function theorem. Since there is no reason to expect that \( \ell \) can be made disjoint from the image of \( \mathcal{M}^{\text{red}}_z(p_0, H_\infty) \) by the evaluation map, we need a smooth structure on the whole compactified moduli space, and not only in its irreducible part. Luckily our moduli space is simple enough that standard gluing theory (as explained, for example, in [21]) already produces a smooth structure.

In the rest of the section we will sketch the construction of a \( C^1 \)-structure, which in turn can be promoted to a smooth structure by a classical result in differential topology; see for example [22, Section 2.2, Theorem 2.10].

In order to endow \( \overline{M}(p_0, H_\infty) \) and \( \overline{M}_z(p_0, H_\infty) \) with the structure of a \( C^1 \)-manifold, we exhibit them as union of two open manifolds which are patched together by a diffeomorphism of class \( C^1 \). The two patches are the irreducible strata \( \mathcal{M}(p_0, H_\infty) \) and \( \mathcal{M}_z(p_0, H_\infty) \) on one side, and suitable fibrations over the reducible strata \( \mathcal{M}^{\text{red}}_z(p_0, H_\infty) \) and \( \mathcal{M}^{\text{red}}_z(p_0, H_\infty) \) on the other hand. The fibres of those fibrations are, roughly speaking, spaces of gluing parameters. We are going to define them momentarily.

The first step in our construction is to introduce local gauge fixing conditions to simplify the presentation of the moduli spaces, so that they almost become spaces of parametrised \( J \)-holomorphic spheres. Identify the neighbourhood of \( W_\infty \) in \( \overline{W} \) with a neighbourhood of the 0-section of \( \mathcal{O}_{\mathbb{P}^n-1}(2) \) as already discussed above. The hyperplane \( H_\infty \) is the 0-set of a holomorphic section \( \sigma \) in \( \mathcal{O}_{\mathbb{P}^n-1}(1) \), and it follows that \( \sigma^2 \) is a section of \( \mathcal{O}_{\mathbb{P}^n-1}(2) \) that has a zero of order two along \( H_\infty \). Multiplying \( \sigma^2 \) with a small constant, we can assume that its image lies in an arbitrarily small neighbourhood of the 0-section. Its graph is a \( J \)-holomorphic hypersurface in \( \overline{W} \) that we will call \( \tilde{W}_\infty \). In particular \( \tilde{W}_\infty \cap W_\infty = H_\infty \) and \( TW_\infty|H_\infty = TW_\infty|H_\infty \).

Then every sphere of \( \mathcal{M}(p_0, H_\infty) \) which is not contained in \( W_\infty \) intersects \( W_\infty \) in two points: one in \( H_\infty \) and one in \( \tilde{W}_\infty \setminus H_\infty \).

Let \( \tilde{\mathcal{M}}(p_0, H_\infty) \) be the open subset of \( \mathcal{M}(p_0, H_\infty) \) consisting of those spheres which are not contained in \( W_\infty \) and \( \tilde{\mathcal{M}}_z(p_0, H_\infty) \) the corresponding open subset of \( \mathcal{M}_z(p_0, H_\infty) \). By the discussion in the previous paragraph, we can fix a parametrisation for every element in \( \tilde{\mathcal{M}}(p_0, H_\infty) \) identifying this moduli space with the set of \( J \)-holomorphic maps \( u: S^2 \to \overline{W} \) whose image is homotopic to \( \ell \) but not contained in \( W_\infty \), and such that \( u(0) = p_0, u(1) \in \tilde{W}_\infty \setminus H_\infty \) and \( u(\infty) \in H_\infty \).

We also denote by \( \tilde{\mathcal{M}}^{\text{red}}_z(p_0, H_\infty) \) the set of pairs of \( J \)-holomorphic maps \((u^0, u^\infty)\), with \( u^0, u^\infty: S^2 \to \overline{W} \), such that

\[
\begin{align*}
u^0(0) &= p_0, & u^0(\infty) &= u^\infty(0), & u^\infty(\infty) &\in H_\infty, \\
u^0(1) &\in \tilde{W}_\infty \setminus H_\infty, & |d_\infty u^\infty| &= 1
\end{align*}
\]

and the image of the “connected sum map” \( u_0 \# u^\infty: S^2 \# S^2 \simeq S^2 \to \overline{W} \) is homotopic to \( \ell \). Here \( |d_\infty u^\infty| \) denotes the norm of the differential of \( u^\infty \) at \( \infty \in S^2 \) computed with respect to the round metric on \( S^2 \) and the metric induced by \( J \) and
\(\varpi\) on \(\mathcal{W}\). Only the component \(u^0\) meets \(\mathcal{W}_\infty \setminus H_\infty\), because \(u^1\) already intersects \(\mathcal{W}_\infty\) in \(H_\infty\).

The group of complex numbers of modulus 1 acts on \(\tilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty)\) by
\[
\theta \cdot (u^0, u^\infty) = (u^0, u^\infty(\theta^{-1}.))
\]
and the quotient by this action is \(\mathcal{M}^{\text{red}}(p_0, H_\infty)\). This implies that the projection \(\tilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty) \to \mathcal{M}^{\text{red}}(p_0, H_\infty)\) is a principal \(S^1\)-bundle which need not be trivial.

The second step is to define the fibrations over \(\mathcal{M}^{\text{red}}(p_0, H_\infty)\) and \(\mathcal{M}^{\text{red}}(p_0, H_\infty)\) which give one of the two patches. Let \(\pi: S^2 \times S^2 \to S^2\) be the rational map \(\pi(x, y) = y/x\), which is not defined at the points \((0, 0)\) and \((\infty, \infty)\). If we make \(S^1\) act on \(S^2 \times S^2\) by \(\theta \cdot (x, y) = (x, \theta y)\) and on \(S^2\) by \(\theta \cdot w = \theta w\), then \(\pi\) is \(S^1\)-equivariant. Let \(\mathfrak{X}\) be the smooth variety obtained by blowing up \(S^2 \times S^2\) at \((0, 0)\) and \((\infty, \infty)\). The action of \(S^1\) on \(S^2 \times S^2\) induces an action on \(\mathfrak{X}\), and \(\pi\) extends to a smooth \(S^1\)-equivariant map \(\pi: \mathfrak{X} \to S^2\).

We denote by \(D_\epsilon\) the disc with centre in \(0\) and radius \(\epsilon\) in \(\mathbb{C} \subset S^2\) and \(\mathfrak{X}_\epsilon = \pi^{-1}(D_\epsilon)\). We define \(E_\epsilon = \tilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty) \times_{S^1} D_\epsilon\) and \(X_\epsilon = \tilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty) \times_{S^1} \mathfrak{X}_\epsilon\).

Both \(E_\epsilon\) and \(X_\epsilon\) are fibre bundles over \(\mathcal{M}^{\text{red}}(p_0, H_\infty)\) and there is a bundle map
\[
\begin{array}{ccc}
X_\epsilon & \xrightarrow{\varpi} & E_\epsilon \\
\downarrow & & \downarrow \\
\mathcal{M}^{\text{red}}(p_0, H_\infty) & & \mathcal{M}^{\text{red}}(p_0, H_\infty)
\end{array}
\]

The zero section \(E_0\) of \(E\) is, of course, diffeomorphic to \(\mathcal{M}^{\text{red}}(p_0, H_\infty)\). Let us denote \(X_0 = \pi^{-1}(0)\) and \(X_0 = \varpi^{-1}(E_0)\). We observe that \(X_0\) is diffeomorphic to \(\mathcal{M}^{\text{red}}(p_0, H_\infty)\). In fact \(X_0 = \tilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty) \times_{S^1} \mathfrak{X}_0\) and \(\mathfrak{X}_0 = \{y = 0\} \cup \{x = \infty\}\), so we can identify the sphere \(\{y = 0\}\) with the domain of \(u^0\) and the sphere \(\{x = \infty\}\) with the domain of \(u^\infty\).

The third, and last, step is the definition of the gluing maps between the two patches. Let \(\hat{E}_\epsilon\) denote \(E_\epsilon\) with the zero section removed, and \(\hat{X}_\epsilon\) the preimage of \(\hat{E}_\epsilon\). We can identify \(\hat{E}_\epsilon \cong [e^{-1}, +\infty) \times \mathcal{M}^{\text{red}}(p_0, H_\infty)\), and therefore standard gluing theory (see for example [9] Chapter 10) yields a \(C^1\)-embeddings \(g: \hat{E}_\epsilon \to \tilde{\mathcal{M}}(p_0, H_\infty): \) if \((r, (u^0, u^\infty)) \in [e^{-1}, +\infty) \times \tilde{\mathcal{M}}^{\text{red}}(p_0, H_\infty) \cong \hat{E}_\epsilon\), then \(g(r, (u^0, u^\infty)) \in \tilde{\mathcal{M}}(p_0, H_\infty)\) is the \(J\)-holomorphic sphere obtained by gluing \(u^0\) and \(u^\infty\) with gluing parameter \(R = \sqrt{r}\); see [9] Section 10.1].

We can also define a \(C^1\)-embedding \(\varrho: \hat{X}_\epsilon \to \tilde{\mathcal{M}}_\epsilon(p_0, H_\infty)\) such that the diagram
\[
\begin{array}{ccc}
\hat{X}_\epsilon & \xrightarrow{\varpi} & \hat{E}_\epsilon \\
\downarrow & & \downarrow \\
\tilde{\mathcal{M}}_\epsilon(p_0, H_\infty) & \xrightarrow{f} & \tilde{\mathcal{M}}(p_0, H_\infty)
\end{array}
\]
commutes. To define \(\varrho\) it is enough to identify \(\varpi^{-1}(e)\) with the domain of \(g(e)\) for all \(e \in \hat{E}_\epsilon\). We recall that \(g(e)\) is obtained by deforming a preglued map \(p(e)\) with the same domain, so it is enough to identify (in a smooth way) \(\varpi^{-1}(e)\) with the domain of the preglued map \(p(e)\), whose construction we sketch now. Denote
\[ \epsilon = [(u^0, u^\infty), t] \] with \( t \in \mathbb{D} \) and \((u^0, u^\infty) \in \tilde{\mathcal{M}}^{\text{red}}(\rho_0, H_\infty) \), and \( S_t = \pi^{-1}(t) \). We define \( \tilde{p}(u^0, u^\infty, t) : S_t \to \overline{W} \) by
\[
\tilde{p}(u^0, u^\infty, t)(x, y) = \begin{cases} 
    u^0(x) & \text{if } |x| < \frac{1}{\sqrt{|t|}} \\
    u^\infty(y) & \text{if } |x| > \frac{2}{\sqrt{|t|}}
\end{cases}
\]
and in the region \( \{ (x, y) \in S_t : \frac{1}{2\sqrt{|t|}} \leq |x| \leq \frac{2}{\sqrt{|t|}} \} \) we interpolate between \( u^0 \) and \( u^\infty \) while remaining close to \( u^0(\infty) = u^\infty(0) \). It is possible to choose the interpolation compatibly with the \( S^1 \)-actions on \( \tilde{\mathcal{M}}^{\text{red}}(\rho_0, H_\infty) \) and \( X_\epsilon \), so that the map \( \tilde{p}(u^0, u^\infty, t) \) induces a well defined map \( p(\epsilon) : W^{-1}(\epsilon) \to \overline{W} \). If we choose a representative \((u^0, u^\infty, t)\) of \( \epsilon \) where \( t \) is real positive and we define \( R = \frac{1}{\sqrt{|t|}} \), we see that the pregluing \( p(\epsilon) \) is the same as the pregluing defined in \( \cite{9} \) Section 10:2], up to a holomorphic change of coordinates and the introduction of a constant \( \delta \), which is necessary for the gluing estimates, but does not change in any significant way the geometric picture we have described.

Combining \( \cite{9} \) Theorem 6.2.6] with the discussion above we obtain the following structural result for the moduli spaces we are interested in.

**Proposition 2.6.** The moduli spaces \( \overline{\mathcal{M}}(\rho_0, H_\infty) \) and \( \overline{\mathcal{M}}_2(\rho_0, H_\infty) \) are closed and orientable \( C^1 \)-manifolds and there is a \( C^1 \)-map
\[ \overline{\mathcal{f}} : \overline{\mathcal{M}}_2(\rho_0, H_\infty) \to \overline{\mathcal{M}}(\rho_0, H_\infty) \]
which forgets the marked point.

While \( \overline{\mathcal{M}}(\rho_0, H_\infty) \) is not a priori connected, since we have not ruled out that a \( J \)-holomorphic sphere could be homotopic to a line \( \ell \subset W_\infty \) but not homotopic through \( J \)-holomorphic spheres, we can assume without loss of generality that \( \overline{\mathcal{M}}(\rho_0, H_\infty) \) is connected by restricting our attention to the connected component which contains a line \( \ell \subset W_\infty \).

3. **Proof of the main theorem**

3.1. **Degree of the evaluation map.** Let \( \text{ev} : \overline{\mathcal{M}}_2(\rho_0, H_\infty) \to \overline{W} \) be the evaluation map at the free marked point.

**Lemma 3.1.** There is an open subset \( U \subset \overline{W} \) such that every \( J \)-holomorphic sphere of \( \overline{\mathcal{M}}(\rho_0, H_\infty) \) passing through a point of \( U \) belongs to \( \mathcal{M}(\rho_0, H_\infty) \) and its image is contained in the neighbourhood of \( W_\infty \) on which \( J \) is integrable.

*Proof.* Choose a point \( q_0 \in W_\infty \setminus H_\infty \) such that \( q_0 \neq \rho_0 \). The unique line \( \ell_0 \) in \( W_\infty \) passing through \( q_0 \) and \( \rho_0 \) also intersects \( H_\infty \), and therefore determines an element of \( \mathcal{M}(\rho_0, H_\infty) \). Moreover, any sphere of \( \mathcal{M}(\rho_0, H_\infty) \) passing through \( q_0 \) intersects \( W_\infty \) in three points, and therefore must be contained in it, so it is equal to \( \ell_0 \).

Since none of the nodal spheres passes through \( q_0 \), and since \( \overline{\mathcal{M}}^{\text{red}}(\rho_0, H_\infty) \) is compact, there is a neighbourhood \( U \) of \( q_0 \) in \( \overline{W} \) such that \( \text{ev}^{-1}(U) \subset \overline{\mathcal{M}}_2(\rho_0, H_\infty) \).

After possibly reducing the size of \( U \), we can assume that every \( J \)-holomorphic sphere of \( \mathcal{M}(\rho_0, H_\infty) \) passing through \( U \) is contained in the neighbourhood of \( W_\infty \) on which \( J \) is integrable. Suppose on the contrary that there is a sequence \([u_n] \) of elements of \( \mathcal{M}(\rho_0, H_\infty) \) and a sequence of points \( q_n \in \overline{W} \) converging to \( q_0 \) such that the image of \( u_n \) contains \( q_n \), but is not contained in some fixed neighbourhood
of $W_{\infty}$. Then by Gromov compactness there is a subsequence of $[u_n]$ converging to a (possibly nodal) $J$-holomorphic sphere of $\overline{\mathcal{M}}(p_0, H_{\infty})$ passing through $q_0$ and not contained in the fixed neighbourhood of $W_{\infty}$. This is a contradiction because the only element of $\overline{\mathcal{M}}(p_0, H_{\infty})$ passing through $q_0$ is $\ell_0$, which is contained in $W_{\infty}$. □

**Lemma 3.2.** The evaluation map $\text{ev}: \overline{\mathcal{M}}_z(p_0, H_{\infty}) \to W$ has degree one.

**Proof.** Let $U$ be the neighbourhood defined in Lemma 3.1. We will show that $\# \text{ev}^{-1}(q) = 1$ for every $q \in U$.

Since all $J$-holomorphic spheres passing through $U$ are contained in the neighbourhood where $J$ is integrable, we can pretend we are working in the total space of $\mathcal{O}_{\mathbb{C}P^n}(2)$. Given $q \in U$, let $\overline{q}$ be its projection to $\mathbb{C}P^n \cong W_{\infty}$. Any $J$-holomorphic sphere of $\mathcal{M}(p_0, H_{\infty})$ passing through $q$ projects to the unique line $\ell_q$ in $W_{\infty}$ passing through $p_0$ and $\overline{q}$. The sphere itself corresponds then to a section of $\mathcal{O}_{\mathbb{C}P^n}(2)|_{\ell_q} \cong \mathcal{O}_{\mathbb{C}P^1}(2)$ which vanishes at $p_0$ and at $p_{\infty} = \ell_q \cap H_{\infty}$. The space of sections of $\mathcal{O}_{\mathbb{C}P^1}(2)$ vanishing at $p_0$ and $p_{\infty}$ has complex dimension one, and thus there is a unique such section for any point $q$ in the fibre of $\mathcal{O}_{\mathbb{C}P^1}(2)$ over $\overline{q}$.

This shows that $\# \text{ev}^{-1}(q) = 1$ for every $q \in U$, and since $U$ is open, by Sard’s theorem it contains a regular value of the evaluation map. This proves that ev has degree one. □

It is important to have a degree one map because such maps induce surjections in homology. More generally, we have the following lemma.

**Lemma 3.3.** Let $f: X \to Y$ be a smooth map between closed oriented smooth manifolds of the same dimension. Assume that $f$ has degree $d$, and let $S \subset Y$ be a compact, oriented $k$-dimensional submanifold that is transverse to $f$.

Then it follows that $S' := f^{-1}(S)$ has an induced orientation and, with that orientation, we have the equality

$$f_*([S']) = d[S]$$

in $H_k(Y; \mathbb{Z})$.

**Proof.** A submanifold $S$ is transverse to a map $f$ if, for every $y \in S$ and $x \in f^{-1}(y)$ we have $T_y S \oplus d_x f(T_x X) = T_y Y$. This property implies that

- $S' = f^{-1}(S)$ is a compact submanifold of $X$, and
- $df$ defines an isomorphism between the normal bundle of $S'$ and the normal bundle of $S$.

The orientations of $S$ and $Y$ determine an orientation of the normal bundle of $S$. This in turn induces an orientation of the normal bundle of $S'$ via $df$ which, combined with the orientation of $X$, induces the orientation of $S'$.

Let $f_S: S' \to S$ be the restriction of $f$. The condition on the normal bundles implies that the regular values of $f_S$ are also regular values of $f$. If $y$ is a regular value of $f_S$, then

$$\deg(f_S) = \sum_{x \in f_S^{-1}(y)} \text{sign}(d_x f_S)$$

$$\deg(f) = \sum_{x \in f^{-1}(y)} \text{sign}(d_x f).$$
Since $f^{-1}(y) = f^{-1}(y)$ by the definition of $f$ and sign$(d_x f) = \text{sign}(d_x f)$ because $df$ is an orientation preserving isomorphism between the normal bundles, we obtain $\deg(f) = \deg(f) = d$.

Now we consider the commutative diagram

$$
\begin{array}{ccc}
H_k(S'; \mathbb{Z}) & \xrightarrow{(f_\ast)} & H_k(S; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_k(X; \mathbb{Z}) & \xrightarrow{f_\ast} & H_k(Y; \mathbb{Z})
\end{array}
$$

where the vertical arrows are induced by the inclusions. The fundamental class of $S'$ is mapped by $(f_\ast)_p$ to $\deg(f)$ times the fundamental class of $S$. The homology classes $[S']$ and $[S]$ are the images of the fundamental classes of $S'$ and $S$ in $H_k(X; \mathbb{Z})$ and $H_k(Y; \mathbb{Z})$ respectively, and therefore $f_\ast[S'] = \deg(f)[S] = d[S]$. □

3.2. Decomposition of the line. The following lemma is a warm up which illustrates how to derive topological implications from Lemma 3.2.

Lemma 3.4. The moduli space $M_z(p_0, H_\infty)$ is not compact.

Proof. The moduli space $M_z(p_0, H_\infty)$ is an $S^2$-bundle over $M(p_0, H_\infty)$ with two distinguished sections $ev^{-1}(p_0)$ and $ev^{-1}(H_\infty)$. Then $M_z(p_0, H_\infty) \setminus ev^{-1}(H_\infty)$ retracts onto $ev^{-1}(p_0)$. This implies that

$$
ev_* : H_k(M_z(p_0, H_\infty) \setminus ev^{-1}(H_\infty); \mathbb{Z}) \rightarrow H_k(W; \mathbb{Z})$$

is trivial whenever $k > 0$.

Let $\ell \subset W$ be an embedded sphere which is homologous to a line in $W_\infty$ but disjoint from $H_\infty$. It is possible to find such a sphere because $H_\infty$ has codimension 4 in $W$, but, in general, $\ell$ will not be holomorphic. We perturb $\ell$ to be transverse to the evaluation map and we denote $ev^{-1}(\ell)$ by $\ell'$. If $M_z(p_0, H_\infty)$ is compact, $ev_*([\ell']) = [\ell]$ by Lemma 3.2. Since $\ell \cap H_\infty = 0$, it follows that $\ell'$ does not intersect $ev^{-1}(H_\infty)$. The previous paragraph implies then that $[\ell] = ev_*([\ell']) = 0$. This is a contradiction because $\ell$ is homologous to a symplectic sphere, and therefore is nontrivial in homology. □

Lemma 3.4 tells us thus that $M_z(p_0, H_\infty)$ is nonempty. We decompose it into connected components

$$M_{z0}(p_0, H_\infty) = M_1(p_0, H_\infty) \cup \cdots \cup M_N(p_0, H_\infty)$$

and, correspondingly, we decompose the moduli space with a free marked point into connected components

$$M_{z0}(p_0, H_\infty) = M_1(p_0, H_\infty) \cup \cdots \cup M_N(p_0, H_\infty).$$

Each $M_z(p_0, H_\infty)$ is an $S^2 \vee S^2$-bundle over $M_z(p_0, H_\infty)$ with three distinguished sections: one, denoted $S_0^{(i)}$, where the free marked point is mapped to $p_0$, one, denoted $S_0^{(i)}$, where the free marked point is mapped to $H_\infty$, and one, denoted $S_0^{(i)}$, where the free marked point lies on the node. Therefore we can see each $M_z(p_0, H_\infty)$ as the union of two sphere bundles $N_0^{(i)}$ and $N_1^{(i)}$ over $M_z(p_0, H_\infty)$

\[^1\text{Strictly speaking ghost bubbles appear in these three cases and we tacitly contract them. We ignore this technical complication as it has no topological consequence.}\]
oriented submanifolds which intersect transversely, 

Lemma 3.5. we denote by

Given homology classes

where the summand

Theorem (see [2, Theorem 5.11] for its cohomological form),

classes which are equal up to torsion.

Proof. According to Equation (2), we can represent

fibres of

passing through

representing the fibres of

and to

when the free marked point lies in the domain of the irreducible component passing through

We denote the homology classes representing the fibres of

and of

by

respectively.

Our aim is to show that there is a nodal curve in the compactification of

that is composed of two holomorphic spheres representing homology classes which are equal up to torsion.

A nodal curve in

is composed of two holomorphic spheres that are fibres of

for

so that

in

The pull-back of the symplectic form

is cohomologically nontrivial on the fibres of

for any

Therefore, by the Leray-Hirsch Theorem (see [2] Theorem 5.11 for its cohomological form),

where the summand

is generated by a fibre of

In the next two lemmas we use the Leray-Hirsch Theorem to gain homological information on the evaluation maps, and on the components of the nodal curves. Given homology classes

and

of complementary degrees (in the same manifold), we denote by

their intersection product. If

and

are represented by closed, oriented submanifolds which intersect transversely,

is the algebraic count of intersection points.

Lemma 3.5. The map

is trivial for every

Proof. By Equation (2) every class

is written as the sum of a class in

and a multiple of the class of the fibre. Since

is mapped to

we obtain

and thus

. By Lemma 2.4

while

so that

Let

be the restriction of

be a line that lies in

and perturb it (inside

) to make it transverse to

Then

which, by Lemma 3.3 satisfies

with

Using that

and

commute with the corresponding inclusions, we also obtain

According to Equation (2), we can represent

as

Lemma 3.6. If

then

modulo torsion.

Proof. Let

be now a line that lies in

and perturb it (inside

) to make it transverse to

Then

is a smooth submanifold of

which, by Lemma 3.3 satisfies

with

Using that

and

commute with the corresponding inclusions, we also obtain

According to Equation (2), we can represent

as


for some $d \in \mathbb{Z}$ and $c \in H_2(S^{(i)}; \mathbb{Z})$. This combined with Lemma 3.5 shows that $d \text{ ev}_* (A^{(i)}_x) = \kappa_i [\ell]$, and by intersecting with $W_\infty$ we obtain $d = 2\kappa_i$ so that

$$\kappa_i (2 \text{ ev}_* (A^{(i)}_x) - [\ell]) = 0.$$  

\[\square\]

The next step is to show that $\text{deg} (\text{ev}_x^{(i)}) \neq 0$ for at least one $i \in \{1, \ldots, N\}$. This will be the goal of the next lemmas.

**Lemma 3.7.** Let $X$ be a compact oriented $n$-dimensional manifold containing two closed oriented submanifolds $S$ and $Y$. Suppose that $\dim S + \dim Y = n$, that $\dim Y \geq 2$, and that $S$ and $Y$ intersect transversely.

Then there is an oriented submanifold $S'$ that is homologous to $S$ and that intersects $Y$ transversely in exactly $|[S] \cdot [Y]|$ many points.

More precisely, we can choose an arbitrarily small neighbourhood of $Y$ such that $S$ and $S'$ agree outside this neighbourhood. Furthermore, given a compact subset $Y' \subset X$ that is disjoint from $S$, and that intersects $Y$ in a codimension $2$ submanifold, we can additionally assume that $S'$ is also disjoint from $Y'$.

**Proof.** We obtain $S'$ by attaching certain $1$-handles to $S$.

If the number of intersection points of $S$ and $Y$ does not agree with $|[S] \cdot [Y]|$, then there needs to be a pair of intersection points $\{x_-, x_+\} \subset S \cap Y$ of opposite sign. Choose an embedded path $\gamma$ in $Y$ with end points $x_-$ and $x_+$ that avoids any other intersection point in $S \cap Y$ and also $Y' \cap Y$, if such a $Y'$ has been chosen.

Identify a tubular neighbourhood of $Y$ with the normal bundle of $Y$, and assume that $S$ corresponds in this neighbourhood to the fibres of the normal bundle over the points in $S \cap Y$. Note that the normal bundle is naturally oriented by the orientations of $Y$ and $X$.

The restriction of the disk bundle over $\gamma$ is a solid cylinder $D^k \times [0, 1]$ such that $D^k \times \{0, 1\}$ is a neighbourhood of $\{x_-, x_+\}$ in $S$. The solid cylinder is naturally oriented, and the orientation of $S$ at $\{x_-, x_+\}$ is equal to the boundary orientation of $D^k \times [0, 1]$.

Remove $D^k \times \{0, 1\}$ from $S$, and glue instead the tube $(\partial D^k) \times [0, 1]$ along the boundary of the two holes that we have created in $S$ (abstractly this corresponds to performing an index $1$ surgery). This yields, after smoothing, an oriented closed manifold $S'$ that agrees outside the chosen tubular neighbourhood of $Y$ with $S$. We can do this construction also avoiding $Y'$ if necessary. Note that $S' \cap Y = (S \cap Y) \setminus \{x_-, x_+\}$, and that $S$ and $S'$ are homologous, because $|S' - [S]| = \partial (D^k \times [0, 1])$.

By repeating this construction as often as necessary, we can cancel all pairs of intersection points of opposite sign until all points in $S' \cap Y$ have the same sign. This then implies as desired that $\#(S \cap Y) = |[S] \cdot [Y]|$. \[\square\]

**Lemma 3.8.** Let $\ell$ be a surface in $\overline{W}$ that is transverse to the evaluation map $\text{ev}$. Denote the oriented submanifold $\text{ev}^{-1}(\ell)$ in $\overline{M}_z(p_0, H_\infty)$ by $\ell'$.

If the intersection product $|[\ell']| \cdot [N^{(i)}_\infty]$ is trivial for all $i = 1, \ldots, N$, then it follows that $\ell$ is null-homologous.

**Proof.** Generically, $\ell$ is disjoint from $H_\infty$, so we may assume that $\ell'$ does not intersect $\text{ev}^{-1}(H_\infty)$, and after a further perturbation we can assume that $\ell'$ is transverse to $N^{(i)}_\infty$ without changing the homology class of $\ell'$.

If $|[\ell']| \cdot [N^{(i)}_\infty] = 0$, we can apply Lemma 3.7 to find a surface $\ell''$ in $\overline{M}_z(p_0, H_\infty)$ that is homologous to $\ell'$ and that does not have any intersection points either with
Lemma 3.10. There exists an $\ell \in \overline{W}$ that represents the homology class of a line in $\overline{W}_\infty$. Furthermore, since this modification has been performed in an arbitrarily small neighbourhood of $\mathcal{N}_\infty^{(i)}$, we may assume that we have not created any new intersection points with one of the other components $\mathcal{N}_\infty^{(j)}$ for $j \neq i$.

Thus, if the intersection product $[\ell'] \cdot [\mathcal{N}_\infty^{(i)}]$ is trivial for all $i = 1, \ldots, N$, we obtain by successively applying this construction for each $i$ a surface $\ell''$ in $\overline{M}(p_0, H_\infty)$ with $[\ell''] = [\ell']$ that does not intersect any of the $\mathcal{N}_\infty^{(i)}$ or $\text{ev}^{-1}(H_\infty)$.

We then have that $\text{ev}_\ast([\ell'']) = 0$ as in the proof of Lemma 3.4 because $\overline{M}(p_0, H_\infty) \setminus (\bigcup_i \mathcal{N}_\infty^{(i)} \cup \text{ev}^{-1}(H_\infty))$ retracts to $\text{ev}^{-1}(p_0)$, but due to Lemma 3.3 we see that $\text{ev}_\ast([\ell'']) = [\ell]$. Since $[\ell''] = [\ell']$, it follows that $[\ell] = 0$. □

Lemma 3.9. Let $\ell \subset \overline{W}$ be a surface that is transverse to the evaluation map $\text{ev}$ and that represents the homology class of a line in $W_\infty$. Then it follows for $\ell' = \text{ev}^{-1}(\ell)$ that

$$[\ell'] \cdot [\mathcal{N}_\infty^{(i)}] = \text{deg}(\text{ev}_\infty^{(i)}) \cdot$$

Proof. Let $y \in H_\infty$ be a regular value of $\text{ev}_\infty^{(i)}$ for all $i = 1, \ldots, N$, and let $\ell_0$ be a line in $W_\infty$ intersecting $H_\infty$ transversely at $y$. It follows that $\ell_0$ is transverse to $\text{ev}|_{\mathcal{N}_\infty^{(i)}}$ at $y$, that is, for every $x \in \text{ev}^{-1}(y) \cap \mathcal{N}_\infty^{(i)}$ we have

$$T_y\ell_0 \oplus dx \text{ ev}(T_x\mathcal{N}_\infty^{(i)}) = T_y\overline{W},$$

because the nodal $J$-holomorphic spheres in $\overline{M}(p_0, H_\infty)$ are all transverse to $W_\infty$.

By construction $(\text{ev}_\infty^{(i)})^{-1}(y) = (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(\ell_0)$. If $x \in (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(\ell_0)$, we define $\text{sign}(x) = +1$ if the equality of Equation (3) preserves the orientation, and $\text{sign}(x) = -1$ otherwise. Then $\text{sign}(x) = \text{sign}(dx |_{\mathcal{N}_\infty^{(i)}})$ because $dx$ ev is complex linear in the extra direction $T_x\mathcal{N}_\infty^{(i)} / T_x\mathcal{S}_\infty^{(i)}$.

Now let $\ell$ be a small perturbation of $\ell_0$ which is transverse to ev. By Equation (3) we can assume that the perturbation is supported away from $y$ and that no new intersection points between $\text{ev}^{-1}(\ell)$ and $\mathcal{N}_\infty^{(i)}$ are created. Then

$$[\mathcal{N}_\infty^{(i)}] \cdot [\text{ev}^{-1}(\ell)] = \sum_{x \in (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(\ell_0)} \text{sign}(x) = \sum_{x \in (\text{ev}|_{\mathcal{N}_\infty^{(i)}})^{-1}(y)} \text{sign}(dx |_{\mathcal{N}_\infty^{(i)}}) = \text{deg}(\text{ev}_\infty^{(i)}) \cdot$$

Lemma 3.10. There exists an $i \in \{1, \ldots, N\}$ such that $\text{deg}(\text{ev}_\infty^{(i)}) \neq 0$.

Proof. Let $\ell$ be a surface in $\overline{W}$ that is transverse to the evaluation map $\text{ev}$ and that represents the homology class of a line in $W_\infty$. Denote $\text{ev}^{-1}(\ell)$ by $\ell'$ byLemma 3.9 $\text{deg}(\text{ev}_\infty^{(i)}) = [\ell'] \cdot [\mathcal{N}_\infty^{(i)}]$. Thus, if $\text{deg}(\text{ev}_\infty^{(i)})$ were 0 for every $i = 1, \ldots, N$, it would follow from Lemma 3.3 that $[\ell] = 0$. But this is impossible because the symplectic form evaluates positively on $[\ell]$. □

After all this preparation, the proof of Theorem 1.1 is falling at our feet like a ripe fruit.

Proof of Theorem 1.1. By Lemma 3.10 there exists an $i \in \{1, \ldots, N\}$ such that $\text{deg}(\text{ev}_\infty^{(i)}) \neq 0$. Then, by Lemma 3.6 $[\ell] = 2 \text{ ev}_\ast([\mathcal{A}_\infty^{(i)}])$ modulo torsion. If we evaluate the first Chern class of $T \overline{W}$ on $[\ell]$ we obtain

$$n + 2 = \langle c_1(T \overline{W}), [\ell] \rangle = 2 \langle c_1(T \overline{W}), \text{ev}_\ast([\mathcal{A}_\infty^{(i)}]) \rangle,$$

where $\mathcal{A}_\infty^{(i)}$ is the moduli space of $J$-holomorphic spheres in $\overline{W}$. □
which is a contradiction when \( n \) is odd. \( \square \)

4. Fundamental group of semipositive fillings

In this section let \( (W, \omega) \) be a semipositive filling of \( (\mathbb{RP}^{2n-1}, \xi) \). We recall that \( (W, \omega) \) is semipositive if every class \( A \) in the image of the Hurewicz homomorphism \( \pi_2(W) \to H_2(W; \mathbb{Z}) \) satisfying the conditions \( \langle \omega, A \rangle > 0 \) and \( \langle c_1(TW), A \rangle \geq 3 - n \) also satisfies \( \langle c_2(TW), A \rangle \geq 0 \). See [9, Definition 6.4.1].

We use the same compactification \( (\overline{W}, \omega) \) and the same set of almost complex structures \( J \) as in the previous sections, but now that \( (W, \omega) \) does not need to be symplectically aspherical we cannot assume anymore that \( \overline{M}(p_0, H_\infty) \) is a manifold or that its elements have no irreducible component contained completely inside \( \overline{W} \setminus W_{\infty} \). The irreducible components which intersect \( W_{\infty} \) must be simply covered because the intersections are simple, and therefore are Fredholm regular for a generic almost complex structure \( J \in J \), but the irreducible components which are contained in \( \overline{W} \setminus W_{\infty} \) can be multiply covered. However according to [9, Theorem 6.6.1] the image of \( M_{\text{red}}^0(p_0, H_\infty) \) under the evaluation map is contained in the union of images of finitely many compact codimension two smooth manifolds for a generic \( J \in J \) because the irreducible components intersecting \( W_{\infty} \) are Fredholm regular and the irreducible components contained in \( \overline{W} \setminus W_{\infty} \) are controlled by semipositivity. In particular \( \overline{W} \setminus \text{ev}(M_{\text{red}}^0(p_0, H_\infty)) \) is open, dense and connected. Moreover the restriction of the evaluation map

\[
ev: \overline{M}(p_0, H_\infty) \setminus \text{ev}^{-1}\left(\text{ev}(M_{\text{red}}^0(p_0, H_\infty))\right) \to \overline{W} \setminus \text{ev}(M_{\text{red}}^0(p_0, H_\infty))
\]

is proper by Gromov compactness, and therefore its degree is well defined. Then Lemma 3.2 can be rephrased as follows.

**Lemma 4.1.** If \( (W, \omega) \) is semipositive and \( y \in \overline{W} \setminus \text{ev}(M_{\text{red}}^0(p_0, H_\infty)) \) is a regular value of \( \text{ev} \), then

\[
\sum_{x \in \text{ev}^{-1}(y)} \text{sign}(d_x \text{ev}) = 1.
\]

In particular, \( \text{ev}: \overline{M}_x(p_0, H_\infty) \to \overline{W} \) is surjective.

If we apply the argument of Lemma 3.4 to a 1-dimensional submanifold of \( W \) we obtain the following result.

**Lemma 4.2.** If \( (W, \omega) \) is a semipositive symplectic filling of \( (\mathbb{RP}^{2n-1}, \xi) \), then the inclusion \( \iota: \mathbb{RP}^{2n-1} \to W \) induces a surjective map \( \pi_1(\mathbb{RP}^{2n-1}) \to \pi_1(W) \).

**Proof.** Recall that \( \overline{W} \setminus W_{\infty} \) is equal to \( W \setminus \partial W \). Instead of proving that \( \pi_1(\partial W) \) maps surjectively onto \( \pi_1(W) \), we can equivalently show that for a sufficiently small neighbourhood \( U_\epsilon \) of \( W_{\infty} \), \( \pi_1(U_\epsilon \setminus W_{\infty}) \) is surjective in \( \pi_1(\overline{W} \setminus W_{\infty}) \).

Choose a base point \( b \) for \( \pi_1(\overline{W} \setminus W_{\infty}) \) that lies in the neighbourhood \( U \) of Lemma 3.2 and use \( b' = \text{ev}^{-1}(b) \) as the base point for \( \pi_1(M_x(p_0, H_\infty)) \).

We can represent every element of \( \pi_1(\overline{W} \setminus W_{\infty}) \) by a smooth embedding

\[
\gamma: S^1 \to W
\]

that avoids \( \text{ev}(M_{\text{red}}^0(p_0, H_\infty)) \) by a codimension argument and that is transverse to the evaluation map. Using the fact that \( \text{ev} \) is a diffeomorphism of \( U \) onto its image and arguing as in point (i) of the proof of [9, Lemma 2.3] we obtain a loop
\( \Gamma: S^1 \to M_z(p_0, H_\infty) \) such that \( \Gamma(1) = b' \) and \( \text{ev}_*(\Gamma) = [\gamma] \) in \( \pi_1(\overline{W} \setminus W_\infty) \). Furthermore \( \Gamma \) does not intersect any singular stratum or \( \text{ev}^{-1}(W_\infty) \).

We can isotope \( M_z(p_0, H_\infty) \setminus \text{ev}^{-1}(H_\infty) \) into an arbitrarily small neighbourhood of \( \text{ev}^{-1}(p_0) \) by pushing the marked point in every holomorphic sphere from \( \infty \) towards \( 0 \). This isotopy restricts to \( M_z(p_0, H_\infty) \setminus \text{ev}^{-1}(W_\infty) \), so that \( \Gamma \) is homotopic in \( M_z(p_0, H_\infty) \setminus \text{ev}^{-1}(W_\infty) \) to a loop in a neighbourhood of \( \text{ev}^{-1}(p_0) \).

Then it follows that \( [\gamma] \) is homotopic in \( \overline{W} \setminus W_\infty \) to a loop that lies in an arbitrarily small neighbourhood of \( p_0 \) and \( \pi_1(U_\epsilon \setminus W_\infty) \to \pi_1(\overline{W} \setminus W_\infty) \) is surjective. \( \square \)

Combining this with the argument found in [9, Section 6.2] we obtain the main result of this section.

**Theorem 4.3.** Any semipositive symplectic filling of \( (\mathbb{R}\mathbb{P}^{2n-1}, \xi) \) is simply connected.

**Proof.** Let \( (W, \omega) \) be a semipositive symplectic filling of \( (\mathbb{R}\mathbb{P}^{2n-1}, \xi) \). By Lemma 4.2 the map \( \pi_1(\partial W) \to \pi_1(W) \) induced by the inclusion \( \iota: \partial W \hookrightarrow W \) is surjective so that \( \pi_1(W) \) is either trivial or isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

In the latter case \( \iota \) induces an isomorphism between the fundamental groups, and thus

\[ \iota^*: H^1(W; \mathbb{Z}/2\mathbb{Z}) \to H^1(\partial W; \mathbb{Z}/2\mathbb{Z}) \]

is also an isomorphism. Let \( \alpha \in H^1(W; \mathbb{Z}/2\mathbb{Z}) \) be the nontrivial element. Then \( \iota^*\alpha \in H^1(\partial W; \mathbb{Z}/2\mathbb{Z}) \) is also nontrivial and, since \( H^*(\mathbb{R}\mathbb{P}^{2n-1}; \mathbb{Z}/2\mathbb{Z}) \) is generated as an algebra by the nontrivial element of degree one, \( (\iota^*\alpha)^{2n-1} \) is the nontrivial element of \( H^{2n-1}(\partial W; \mathbb{Z}/2\mathbb{Z}) \).

By the naturality of the cup product \( (\iota^*\alpha)^{2n-1} = \iota^*(\alpha^{2n-1}) \). However

\[ \iota_*: H_{2n-1}(\partial W; \mathbb{Z}/2\mathbb{Z}) \to H_{2n-1}(W; \mathbb{Z}/2\mathbb{Z}) \]

is trivial, and consequently \( \iota^*: H^{2n-1}(W; \mathbb{Z}/2\mathbb{Z}) \to H^{2n-1}(\partial W; \mathbb{Z}/2\mathbb{Z}) \) is also trivial by duality because we are working over a field. This contradicts \( \iota^*(\alpha^{2n-1}) \neq 0 \) and therefore shows that \( W \) is simply connected. \( \square \)

5. Yet another proof of the Eliashberg-Floer-McDuff theorem

In this section we apply the constructions of this article to the symplectic fillings of the standard contact structure \( \xi \) on \( S^{2n-1} \). This will lead to small changes in the meaning of the notation. If \( (W, \omega) \) is a symplectic filling of \( (S^{2n-1}, \xi) \) and we perform symplectic reduction of its boundary, we obtain a closed symplectic manifold \( (\overline{W}, \overline{\omega}) \) with a codimension two symplectic submanifold \( W_\infty \cong \mathbb{CP}^{n-1} \) whose normal bundle is isomorphic to \( O_{p_{n-1}}(1) \). We choose an almost complex structure \( J \) on \( \overline{W} \) which is integrable near \( W_\infty \) and generic elsewhere. Let \( p_0 \in W_\infty \) be a point; we denote by \( M(p_0) \) the moduli space of unparametrised \( J \)-holomorphic spheres in \( \overline{W} \) that are homotopic to a line in \( W_\infty \) and pass through \( p_0 \). If \( \ell \) is a line in \( W_\infty \), then

\[ T\overline{W}\| \ell \cong O_{p_{1}}(2) \oplus \underbrace{O_{p_{1}}(1) \oplus \cdots \oplus O_{p_{1}}(1)}_{n-1} \].

Since \( [\ell]: [W_\infty] = 1 \) all elements of \( M(p_0) \) are simply covered, and therefore \( M(p_0) \) is a smooth manifold by the analogue of Proposition 2.3. Let \( M_z(p_0) \) is the moduli space obtained by adding a free marked point to the elements of \( M(p_0) \). A Riemann-Roch calculation gives \( \dim M(p_0) = 2n - 2 \) and \( \dim M_z(p_0) = 2n \).
Lemma 5.1. If \((W, \omega)\) is symplectically aspherical, then \(\mathcal{M}_z(p_0)\) is compact.

Proof. As the algebraic intersection between a line with \(W_\infty\) is one, any nodal \(J\)-holomorphic curve representing the homology class of a line must have an irreducible component in \(\mathcal{W} \setminus W_\infty \cong W\).

Lemma 3.2 still holds with the minimal necessary modifications, and therefore the evaluation map \(ev: \mathcal{M}_z(p_0) \to \mathcal{W}\) has degree one.

Lemma 5.2. If \((W, \omega)\) is a symplectically aspherical filling of \((S^{2n-1}, \xi)\), then \(H_*(W; \mathbb{Z}) = 0\)

Proof. The moduli space \(\mathcal{M}_z(p_0)\) is an \(S^2\)-bundle over \(\mathcal{M}(p_0)\) and \(ev^{-1}(p_0)\) is a section. Let \(\widetilde{W}_\infty\) be a \(J\)-holomorphic hypersurface of \(\mathcal{W}\) contained in the neighbourhood of \(W_\infty\) where \(J\) is integrable and obtained as the graph of a section of \(\mathcal{O}_{S^{2n-1}}(1)\). We choose \(\widetilde{W}_\infty\) such that \(p_0 \notin \widetilde{W}_\infty\): then \(ev^{-1}(\widetilde{W}_\infty)\) is a section of \(\mathcal{M}_z(p_0)\) which is disjoint from \(ev^{-1}(p_0)\). The map

\[
\begin{align*}
ev_*: H_*(\mathcal{M}_z(p_0) \setminus ev^{-1}(\widetilde{W}_\infty); \mathbb{Z}) & \to H_*(\mathcal{W} \setminus \widetilde{W}_\infty; \mathbb{Z}) \\
& \cong H_*(W; \mathbb{Z})
\end{align*}
\]

is surjective by Lemma 3.3. That lemma, strictly speaking, is about homology classes represented by submanifolds, but there are several ways to extend it to general homology classes.

On the other hand \(\mathcal{M}_z(p_0) \setminus ev^{-1}(\widetilde{W}_\infty)\) retracts onto \(ev^{-1}(p_0)\), and therefore the map \(\square\) is trivial for \(* > 0\).

The proof of Lemma 1.2 works with the obvious modifications more or less unchanged for fillings of \((S^{2n-1}, \xi)\), and therefore \(W\) is simply connected. Then the \(h\)-cobordism theorem implies the following corollary.

Corollary 5.3 (Eliashberg–Floer–McDuff). If \((W, \omega)\) is a symplectically aspherical filling of \((S^{2n-1}, \xi)\), then \(W\) is diffeomorphic to the ball \(D^{2n}\).

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