

Tropicalization of Del Pezzo Surfaces

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Abstract

We determine the tropicalizations of very affine surfaces over a valued field that are obtained from del Pezzo surfaces of degree 5, 4 and 3 by removing their \((-1)\)-curves. On these tropical surfaces, the boundary divisors are represented by trees at infinity. These trees are glued together according to the Petersen, Clebsch and Schl"afli graphs, respectively. There are 27 trees on each tropical cubic surface, attached to a bounded complex with up to 73 polygons. The maximal cones in the 4-dimensional moduli fan reveal two generic types of such surfaces.

1 Introduction

A smooth cubic surface \(X\) in projective 3-space \(\mathbb{P}^3\) contains 27 lines. These lines are characterized intrinsically as the \((-1)\)-curves on \(X\), that is, rational curves of self-intersection \(-1\). The tropicalization of an embedded surface \(X\) is obtained directly from the cubic polynomial that defines it in \(\mathbb{P}^3\). The resulting tropical surfaces are dual to regular subdivisions of the size 3 tetrahedron. These come in many combinatorial types [15, §4.5]. If the subdivision is a unimodular triangulation then the tropical surface is called smooth (cf. [15, Prop. 4.5.1]). Alternatively, by removing the 27 lines from the cubic surface \(X\), we obtain a very affine surface \(X^0\). In this paper, we study the tropicalization of \(X^0\), denoted \(\text{trop}(X^0)\), via the embedding in its intrinsic torus [11]. This is an invariant of the surface \(X\). The \((-1)\)-curves on \(X\) now become visible as 27 boundary trees on \(\text{trop}(X^0)\). This distinguishes our approach from Vigeland’s work [27] on the 27 lines on tropical cubics in \(\mathbb{TP}^3\). It also highlights an important feature of tropical geometry [17]: there are different tropical models of a single classical variety, and the choice of model depends on what structure one wants revealed.

Throughout this paper we work over a field \(K\) of characteristic zero that has a non-archimedean valuation. Examples include the Puiseux series \(K = \mathbb{C}\{\{t\}\}\) and the \(p\)-adic numbers \(K = \mathbb{Q}_p\). We use the term cubic surface to mean a marked smooth del Pezzo surface \(X\) of degree 3. A tropical cubic surface is the intrinsic tropicalization \(\text{trop}(X^0)\) described above. Likewise, tropical del Pezzo surface refers to the tropicalization \(\text{trop}(X^0)\) for degree \(\geq 4\). Here, the adjective “tropical” is used solely for brevity, instead of the more accurate “tropicalized” used in [15]. We do not consider non-realizable tropical del Pezzo surfaces, nor tropicalizations of surfaces defined over a field \(K\) with positive characteristic.

The moduli space of cubic surfaces is four-dimensional, and its tropical version is the four-dimensional Naruki fan. This was constructed combinatorially by Hacking, Keel and Tevelev [11], and it was realized in [21, §6] as the tropicalization of a very affine variety \(\mathfrak{Y}^0\),
obtained from the Yoshida variety $Y$ in $\mathbb{P}^{39}$ by intersecting with $(K^*)^{39}$. The Weyl group $W(E_6)$ acts on $Y$ by permuting the 40 coordinates. The maximal cones in $\text{trop}(Y^0)$ come in two $W(E_6)$-orbits. We here compute the corresponding cubic surfaces:

**Theorem 1.1.** There are two generic types of tropical cubic surfaces. They are contractible and characterized at infinity by 27 metric trees, each having 10 leaves. The first type has 73 bounded cells, 150 edges, 78 vertices, 135 cones, 189 flaps, 216 rays, and all 27 trees are trivalent. The second type has 72 bounded cells, 148 edges, 77 vertices, 135 cones, 186 flaps, 213 rays, and three of the 27 trees have a 4-valent node. (For more data see Table 1.)

Here, by cones and flaps we mean unbounded 2-dimensional polyhedra that are affinely isomorphic to $\mathbb{R}^2_{\geq 0}$ and $[0, 1] \times \mathbb{R}_{\geq 0}$ respectively. The characterization at infinity is analogous to that for tropical planes in [12]. Indeed, by [12, Theorem 4.4], every tropical plane $L$ in $\mathbb{TP}^{n-1}$ is given by an arrangement of $n$ boundary trees, each having $n-1$ leaves, and $L$ is uniquely determined by this arrangement. Viewed intrinsically, $L$ is the tropicalization of a very affine surface, namely the complement of $n$ lines in $\mathbb{P}^2$. Theorem 1.1 offers the analogous characterization for the tropicalization of the complement of the 27 lines on a cubic surface.

Tropical geometry has undergone an explosive development during the past decade. To the outside observer, the literature is full of conflicting definitions and diverging approaches. The text books [15, 17] offer some help, but they each stress one particular point of view.

An important feature of the present paper is its focus on the unity of tropical geometry. We shall develop three different techniques for computing tropical del Pezzo surfaces:

- Cox ideals, as explained in Section 2;
- fan structures on moduli spaces, as explained in Section 3;
- tropical modifications, as explained in Section 4.

The first approach uses the Cox ring of $X$, starting from the presentation given in [26]. Propositions 2.1 and 2.2 extend this to the universal Cox ideal over the moduli space. For any particular surface $X$, defined over a field such as $K = \mathbb{Q}(t)$, computing the tropicalization is a task for the software gf\text{fan} [13]. In the second approach, we construct del Pezzo surfaces from fibers in the natural maps of moduli fans. Our success along these lines completes the program started by Hacking et al. [11] and further developed in [21, §6]. The third approach is to build tropical del Pezzo surfaces combinatorially from the tropical projective plane $\mathbb{TP}^2$ by the process of tropical modifications in the sense of Mikhalkin [16]. It mirrors the classical construction by blowing up points in $\mathbb{P}^2$. All three approaches yield the same results. Section 5 presents an in-depth study of the combinatorics of tropical cubic surfaces and their trees, including an extension of Theorem 1.1 that includes all degenerate surfaces.

We now illustrate the rich combinatorics in our story for a del Pezzo surface $X$ of degree 4. Del Pezzo surfaces of degree $d \geq 6$ are toric surfaces, so they naturally tropicalize as polygons with $12-d$ vertices [17, Ch. 3]. On route to Theorem 1.1, we prove the following for $d = 4, 5$:

**Proposition 1.2.** Among tropical del Pezzo surfaces of degree 4 and 5, each has a unique generic combinatorial type. For degree 5, this is the cone over the Petersen graph. For degree 4, the surface is contractible and characterized at infinity by 16 trivalent metric trees, each with 5 leaves. It has 9 bounded cells, 20 edges, 12 vertices, 40 cones, 32 flaps, and 48 rays.
To understand degree 4, we consider the 5-regular Clebsch graph in Figure 1. Its 16 nodes are the \((-1)\)-curves on \(X\), labelled \(E_1, \ldots, E_5, F_{12}, \ldots, F_{45}, G\). Edges represent intersecting pairs of \((-1)\)-curves. In the constant coefficient case, when \(K\) has trivial valuation, the tropicalization of \(X\) is the fan over this graph. However, over fields \(K\) with non-trivial valuation, trop\((X^0)\) is usually not a fan, but one sees the generic type from Proposition 1.2. Here, the Clebsch graph deforms into a trivalent graph with 48 = 16 \(\cdot\) 3 nodes and 72 = 40 + 32 edges, determined by the color coding in Figure 1. Each of the 16 nodes is replaced by a trivalent tree with five leaves. Incoming edges of the same color (red or blue) form a cherry (= two adjacent leaves) in that tree, while the black edge connects to the non-cherry leaf.

Corollary 1.3. For a del Pezzo surface \(X\) of degree 4, the 16 metric trees on its tropicalization trop\((X^0)\), obtained from the \((-1)\)-curves on \(X\), are identical up to relabeling.

Proof. Moving from one \((-1)\)-curve on \(X\) to another corresponds to a Cremona transformation of the plane \(\mathbb{P}^2\). Each \((-1)\)-curve on \(X\) has exactly five marked points arising from its intersections with the other \((-1)\)-curves. Moreover, the Cremona transformations preserve the cross ratios among the five marked points on these 16 \(\mathbb{P}^1\)'s. From the valuations of all the various cross ratios one can read off the combinatorial trees along with their edge lengths, as explained in e.g. [15, Proposition 6.5.1] or [21, Example 5.2]. We then obtain the following relabeling rules for the leaves on the 16 trees, which live in the circular nodes of Figure 1.

We start with the tree \(G\) whose leaves are labeled \(E_1, E_2, E_3, E_4, E_5\). For the specific
example in Figure 1, this is the caterpillar tree \( \{\{E_1, E_4\}, E_3, \{E_2, E_5\}\} \). Now, given any trivalent tree for \( G \), the tree \( F_{ij} \) is obtained by relabeling the five leaves as follows:

\[
E_i \mapsto E_j, \quad E_j \mapsto E_i, \quad E_k \mapsto F_{lm}, \quad E_l \mapsto F_{km}, \quad E_m \mapsto F_{kl}.
\]

(1.1)

Here \( \{k, l, m\} = \{1, 2, 3, 4, 5\}\setminus\{i, j\} \). The tree \( E_i \) is obtained from the tree \( G \) by relabelling

\[
E_i \mapsto G \quad \text{and} \quad E_j \mapsto F_{ij} \quad \text{where} \ j \neq i.
\]

(1.2)

This explains the color coding of the graph in Figure 1.

The bounded complex of \( \text{trop}(X^0) \) is shown in Figure 2. It consists of a central rectangle, with two triangles attached to each of its four edges. There are 12 vertices, four vertices of the rectangle, labeled \( S \), and eight pendant vertices, labeled \( T \). To these 12 vertices and 20 edges, we attach the flaps and cones, according to the deformed Clebsch graph structure. The link of each \( S \) vertex in the surface \( \text{trop}(X^0) \) is the Petersen graph (Figure 3), while the link of each \( T \) vertex is the bipartite graph \( K_{3,3} \). The bounded complex has 16 chains \( \text{TST} \) consisting of two edges with different colors. These are attached by flaps to the bounded parts of the 16 trees. The Clebsch graph (Figure 1) can be recovered from Figure 2 as follows: its nodes are the \( \text{TST} \) chains, and two chains connect if they share precisely one vertex. Out at infinity, \( T \) vertices attach along cherries, while \( S \) vertices attach along non-cherry leaves. Each such attachment between two of the 16 trees links to the bounded complex by a cone.

2 Cox Ideals

We study del Pezzo surfaces over \( K \) of degrees 5, 4 and 3. Such surfaces \( X \) are obtained from \( \mathbb{P}^2 \) by blowing up 4, 5 or 6 points in general position, and we obtain moduli by varying these points. From an algebraic perspective, it is convenient to represent \( X \) by its \textit{Cox ring}

\[
\text{Cox}(X) = \bigoplus_{\mathcal{L} \in \text{Pic}(X)} H^0(X, \mathcal{L}).
\]

(2.1)
The Cox ring of a del Pezzo surface \( X \) was first studied by Batyrev and Popov [4]. We shall express this ring explicitly as a quotient of a polynomial ring over the ground field \( K \):

\[
\text{Cox}(X) = K \left[ x_C : C \text{ is a } (-1)-\text{curve on } X \right] \text{ modulo an ideal } I_X \text{ generated by quadrics.}
\]

The number of variables \( x_C \) in our three polynomial rings is 10, 16 and 27 respectively. The ideal \( I_X \) is the \textit{Cox ideal} of the surface \( X \). It was conjectured already in [4] that the ideal \( I_X \) is generated by quadrics. This conjecture was proved in several papers, including [25, 26].

The Cox ring encodes all maps from \( X \) to a projective space. Such a map is given by the \( N \)-graded subring \( \text{Cox}(X)[L] = \bigoplus_{m \geq 0} H^0(X, mL) \) for a fixed line bundle \( L \in \text{Pic}(X) \).

The image of the map \( X \to \text{Proj}(\text{Cox}(X)[L]) \) is contained in the projective space \( \mathbb{P}^N \), where \( N = \dim(H^0(X, L)) - 1 \), provided \( \text{Cox}(X)[L] \) is generated in degree 1. This applies to both the anticanonical map and to the blow-down map to \( \mathbb{P}^2 \).

In what follows, we give explicit generators for all relevant Cox ideals \( I_X \). Some of this is new and of independent interest. The tropicalization of \( X^0 \) we seek is defined from the ideal \( I_X \). So, in principle, we can compute \( \text{trop}(X^0) \) from \( I_X \) using the software \textsf{gfan} [13]. Recall that \( X^0 \) denotes the very affine surface obtained from \( X \) by removing all \((-1)\)-curves.

![Figure 3: The tropical del Pezzo surface \( \text{trop}(M_{0,5}) \) is the cone over the Petersen graph.](image)

**Del Pezzo Surfaces of Degree 5**

Consider four general points in \( \mathbb{P}^2 \). This configuration is projectively unique, so there are no moduli. The surface \( X \) is the moduli space \( \overline{M}_{0,5} \) of rational stable curves with five marked points, see for example [14]. The very affine variety \( X^0 \) is simply \( M_{0,5} \), the moduli space of rational curves with five distinct marked points. It is the complement of a hyperplane arrangement whose underlying matroid is the graphical matroid of the complete graph \( K_4 \). The Cox ideal is the \textit{Plücker ideal} of relations among 2 \times 2-minors of a 2 \times 5-matrix:

\[
I_X = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}, \ p_{12}p_{35} - p_{13}p_{25} + p_{15}p_{23}, \ p_{12}p_{45} - p_{14}p_{25} + p_{15}p_{24}, \ p_{13}p_{45} - p_{14}p_{35} + p_{15}p_{34}, \ p_{23}p_{45} - p_{24}p_{35} + p_{25}p_{34} \rangle.
\]

The affine variety of \( I_X \) in \( \mathbb{K}^{10} \) is the \textit{universal torsor} of \( X \), now regarded over the algebraic closure \( \mathbb{K} \) of the given valued field \( K \). From the perspective of blowing up \( \mathbb{P}^2 \) at 4 points, the ten variables (representing the ten \((-1)\)-curves) fall in two groups: the four exceptional fibers, and the six lines spanned by pairs of points. For example, we may label the fibers by

\[
E_1 = p_{15}, \ E_2 = p_{25}, \ E_3 = p_{35}, \ E_4 = p_{45}.
\]
and the six lines by

\[ F_{12} = p_{34}, \, F_{13} = p_{24}, \, F_{14} = p_{23}, \, F_{23} = p_{14}, \, F_{24} = p_{13}, \, F_{34} = p_{12}. \]

The Cox ideal \( I_X \) is homogeneous with respect to the natural grading by the Picard group \( \text{Pic}(X) = \mathbb{Z}^5 \). In Plücker coordinates, this grading is given by setting \( \deg(p_{ij}) = e_i + e_j \), where \( e_i \) represents the \( i \)th standard basis vector in \( \mathbb{Z}^5 = \text{Pic}(X) \). This translates into an action of the torus \((\mathbb{K}^*)^5 = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{K}^*\) on the universal torsor in \( \mathbb{K}^{10} \). We remove the ten coordinate hyperplanes in \( \mathbb{K}^{10} \), and we take the quotient modulo \((\mathbb{K}^*)^5\). The result is precisely the very affine del Pezzo surface we seek to tropicalize:

\[ X^0 = M_{0,5} \subset (\mathbb{K}^*)^{10}/(\mathbb{K}^*)^5. \quad (2.2) \]

The 2-dimensional balanced fan \( \text{trop}(X^0) \) is the Bergman fan of the graphical matroid of \( K_4 \). It is known from [2] that this is the cone over the Petersen graph. This is also easy to check directly with \textit{gfan} on \( I_X \). This fan is also the moduli space of 5-marked rational tropical curves, that is, 5-leaf trees with lengths on the two bounded edges (cf. [15, §4.3]).

**Del Pezzo Surfaces of Degree 4**

Consider now five general points in \( \mathbb{P}^2 \). There are two degrees of freedom. The moduli space is our previous del Pezzo surface \( M_{0,5} \). Indeed, fixing five points in \( \mathbb{P}^2 \) corresponds to fixing a point \( (p_{12}, \ldots, p_{45}) \) in \( M_{0,5} \), using Cox-Plücker coordinates as in (2.2). Explicitly, if we write the five points as a \( 3 \times 5 \)-matrix then the \( p_{ij} \) are the Plücker coordinates of its kernel. Replacing \( K \) with the previous Cox ring, we may consider the \textit{universal del Pezzo surface} \( \mathcal{Y} \). The \textit{universal Cox ring} is the quotient of a polynomial ring in \( 26 = 10 + 16 \) variables:

\[ K[\mathcal{Y}] = \text{Cox}(M_{0,5})[E_1, E_2, E_3, E_4, E_5, F_{12}, F_{13}, \ldots, F_{45}, G]/I_\mathcal{Y}. \quad (2.3) \]

As before, \( E_i \) represents the exceptional divisor over point \( i \), and \( F_{ij} \) represents the line spanned by points \( i \) and \( j \). The variable \( G \) represents the conic spanned by the five points.

**Proposition 2.1.** Up to saturation with respect to the product of the 26 variables, the \textit{universal Cox ideal} \( I_\mathcal{Y} \) for degree 4 del Pezzo surfaces is generated by the following 45 trinomials:

| Base Group | \( p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \) | \( p_{12}p_{35} - p_{13}p_{25} + p_{15}p_{23} \) | \( p_{12}p_{45} - p_{14}p_{25} + p_{15}p_{24} \) | \( p_{13}p_{45} - p_{14}p_{35} + p_{15}p_{34} \) | \( p_{23}p_{45} - p_{24}p_{35} + p_{25}p_{34} \) |
|------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| Group 1    | \( F_{23}F_{45} - F_{24}F_{35} + F_{25}F_{34} \) | \( p_{23}p_{45}F_{24}F_{35} - p_{24}p_{35}F_{23}F_{34} - GE_1 \) | \( p_{24}p_{35}F_{23}F_{45} - p_{25}p_{34}F_{23}F_{45} - GE_1 \) |
| Group 2    | \( F_{13}F_{45} - F_{14}F_{35} + F_{15}F_{34} \) | \( p_{13}p_{45}F_{14}F_{35} - p_{14}p_{35}F_{13}F_{34} - GE_2 \) | \( p_{14}p_{35}F_{13}F_{45} - p_{15}p_{34}F_{13}F_{45} - GE_2 \) |
| Group 3    | \( F_{12}F_{45} - F_{14}F_{25} + F_{15}F_{24} \) | \( p_{12}p_{45}F_{14}F_{25} - p_{14}p_{25}F_{12}F_{45} - GE_3 \) | \( p_{14}p_{25}F_{12}F_{45} - p_{15}p_{24}F_{12}F_{45} - GE_3 \) |
| Group 4    | \( F_{12}F_{35} - F_{13}F_{25} + F_{15}F_{23} \) | \( p_{12}p_{35}F_{13}F_{25} - p_{13}p_{25}F_{12}F_{35} - GE_4 \) | \( p_{13}p_{25}F_{12}F_{35} - p_{15}p_{23}F_{13}F_{35} - GE_4 \) |
| Group 5    | \( F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23} \) | \( p_{12}p_{34}F_{13}F_{24} - p_{13}p_{24}F_{12}F_{34} - GE_5 \) | \( p_{13}p_{24}F_{12}F_{34} - p_{14}p_{23}F_{13}F_{24} - GE_5 \) |
| Group 1’ | $p_{25}F_{12}E_2-p_{35}F_{13}E_3+p_{45}F_{11}E_4$ | $p_{24}F_{12}E_2-p_{34}F_{13}E_3+p_{45}F_{15}E_5$ |
|---------|---------------------------------|---------------------------------|
|         | $p_{23}F_{12}E_2-p_{34}F_{14}E_4+p_{35}F_{15}E_5$ | $p_{23}F_{13}E_3-p_{24}F_{14}E_4+p_{25}F_{15}E_5$ |
| Group 2’| $p_{15}F_{12}E_1-p_{35}F_{23}E_3+p_{45}F_{24}E_4$ | $p_{14}F_{12}E_1-p_{34}F_{23}E_3+p_{45}F_{25}E_5$ |
|         | $p_{13}F_{12}E_1-p_{34}F_{24}E_4+p_{35}F_{25}E_5$ | $p_{13}F_{23}E_3-p_{14}F_{24}E_4+p_{15}F_{25}E_5$ |
| Group 3’| $p_{15}F_{13}E_1-p_{35}F_{23}E_2+p_{45}F_{34}E_4$ | $p_{14}F_{13}E_1-p_{34}F_{23}E_2+p_{45}F_{35}E_5$ |
|         | $p_{12}F_{13}E_1-p_{34}F_{24}E_3+p_{25}F_{35}E_5$ | $p_{12}F_{23}E_2-p_{13}F_{24}E_3+p_{15}F_{35}E_5$ |
| Group 4’| $p_{15}F_{14}E_1-p_{25}F_{24}E_2+p_{35}F_{34}E_3$ | $p_{13}F_{14}E_1-p_{23}F_{24}E_2+p_{35}F_{45}E_5$ |
|         | $p_{12}F_{14}E_1-p_{23}F_{24}E_3+p_{25}F_{45}E_5$ | $p_{12}F_{24}E_2-p_{13}F_{24}E_3+p_{15}F_{45}E_5$ |
| Group 5’| $p_{14}F_{15}E_1-p_{24}F_{25}E_2+p_{34}F_{35}E_3$ | $p_{13}F_{15}E_1-p_{23}F_{25}E_2+p_{34}F_{45}E_4$ |
|         | $p_{12}F_{15}E_1-p_{23}F_{35}E_3+p_{24}F_{45}E_4$ | $p_{12}F_{25}E_2-p_{13}F_{35}E_3+p_{14}F_{45}E_4$ |

Proposition 2.1 will be derived later in this section. For now, let us discuss the structure and symmetry of the generators of $I_Y$. We consider the 5-dimensional demicube, here denoted $D_5$. This is the convex hull of the following 16 points in the hyperplane $\{a_0 = 0\} \subset \mathbb{R}^6$:

$$\{(1, a_1, a_2, a_3, a_4, a_5) \in \{0, 1\}^6 : a_1 + a_2 + a_3 + a_4 + a_5 \text{ is odd}\}. \quad (2.4)$$

The group of symmetries of $D_5$ is the Weyl group $W(D_5)$. It acts transitively on (2.4). There is a natural bijection between the 16 variables in the Cox ring and the vertices of $D_5$:

$$E_1 \leftrightarrow (1, 1, 0, 0, 0, 0), \quad E_2 \leftrightarrow (1, 0, 1, 0, 0, 0), \ldots, \quad E_5 \leftrightarrow (1, 0, 0, 0, 0, 1),$$

$$F_{12} \leftrightarrow (1, 0, 0, 1, 1, 1), \quad F_{13} \leftrightarrow (1, 0, 1, 0, 0, 0), \quad \ldots, \quad F_{45} \leftrightarrow (1, 1, 1, 1, 0, 0),$$

$$G \leftrightarrow (1, 1, 1, 1, 1, 1). \quad (2.5)$$

This bijection defines the grading via the Picard group $\mathbb{Z}^6$. We regard the $p_{ij}$ as scalars, so they have degree 0. Generators of $I_Y$ that are listed in the same group have the same $\mathbb{Z}^6$ degrees. The action of $W(D_5)$ on the demicube $D_5$ gives the action on the 16 variables.

Consider now a particular smooth del Pezzo surface $X$ of degree 4 over the field $K$, so the $p_{ij}$ are scalars in $K$ that satisfy the Plücker relations in the Base Group. The universal Cox ideal $I_Y$ specializes to the Cox ideal $I_X$ for the particular surface $X$. That Cox ideal is minimally generated by 20 quadrics, two per group. The surface $X^0$ is the zero set of the ideal $I_X$ inside $(\mathbb{K}^*)^{16}/(\mathbb{K}^*)^6$. The torus action is obtained from (2.5), in analogy to (2.2).

**Proof of Proposition 1.2.** We computed $\text{trop}(X^0)$ by applying `gfan` [13] to the ideal $I_X$. If $K = \mathbb{Q}$ with the trivial valuation then the output is the cone over the Clebsch graph in Figure 1. This 5-regular graph records which pairs of $(-1)$-curves intersect on $X$. This also works over a field $K$ with non-trivial valuation. The software `gfan` uses $K = \mathbb{Q}(t)$. If the vector $(p_{12}, \ldots, p_{55})$ tropicalizes into the interior of an edge in the Petersen graph then $\text{trop}(X^0)$ is the tropical surface described in Proposition 1.2. Each node in Figure 1 is replaced by a trivalent tree on 5 nodes according to the color coding explained in Section 1. The surface $\text{trop}(X^0)$ can also be determined by tropical modifications, as in Section 4. □

The same tropicalization method works for the universal family $Y^0$. Its ideal $I_Y$ is given by the 45 polynomials in 26 variables listed above, and $Y^0$ is the zero set of $I_Y$ in the 15-dimensional torus $(\mathbb{K}^*)^{10}/(\mathbb{K}^*)^5 \times (\mathbb{K}^*)^{16}/(\mathbb{K}^*)^6$. The tropical universal del Pezzo surface $\text{trop}(Y^0)$ is a 4-dimensional fan in $\mathbb{R}^{26}/\mathbb{R}^{11}$. We compute it by applying `gfan` to the universal
The Gröbner fan structure on $\operatorname{trop}(I_3)$ has $f$-vector $(76, 630, 1620, 1215)$. It is isomorphic to the Naruki fan described in [21, Table 5] and discussed further in Section 3.

**Del Pezzo Surfaces of Degree 3 (Cubic Surfaces)**

There exists a cuspidal cubic through any six points in $\mathbb{P}^2$. See e.g. [20, (4.4)] and [21, (6.1)]. Hence any configuration of six points in $\mathbb{P}^2$ can be represented by the columns of a matrix

$$D = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\
d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & d_6^3
\end{pmatrix},$$

The maximal minors of the matrix $D$ factor into linear forms,

$$[ijk] = (d_i - d_j)(d_i - d_k)(d_j - d_k)(d_i + d_j + d_k), \quad (2.6)$$

and so does the condition for the six points to lie on a conic:

$$[\text{conic}] = [134][156][235][246] - [135][146][234][256] = (d_1 + d_2 + d_3 + d_4 + d_5 + d_6) \cdot \prod_{1 \leq i < j \leq 6}(d_i - d_j). \quad (2.7)$$

The linear factors in these expressions form the root system of type $E_6$. This corresponds to an arrangement of 36 hyperplanes in $\mathbb{P}^5$. Similarly, the arrangement of type $E_7$ consists of 63 hyperplanes in $\mathbb{P}^6$, as in [20, (4.4)]. To be precise, for $m = 6, 7$, the roots for $E_m$ are

$$d_i - d_j \quad \text{for} \quad 1 \leq i < j \leq m,$$

$$d_i + d_j + d_k \quad \text{for} \quad 1 \leq i < j < k \leq m,$$

$$d_{i_1} + d_{i_2} + \cdots + d_{i_6} \quad \text{for} \quad 1 \leq i_1 < i_2 < \cdots < i_6 \leq m. \quad (2.8)$$

Linear dependencies among these linear forms specify a matroid of rank $m$, also denoted $E_m$.

The moduli space of marked cubic surfaces is the 4-dimensional Yoshida variety $\mathcal{Y}$ defined in [21, §6]. It coincides with the subvariety $\mathcal{Y}^0$ of $\mathbb{P}^{26}/(\mathbb{K}^*)^{11}$ cut out by the 45 trinomials in Proposition 2.1. This is the embedding of $\mathcal{Y}^0$ in its intrinsic torus, as in [21, Theorem 6.1], and it differs from the embedding of $\mathcal{Y}^0$ into $\mathbb{K}^{40}/(\mathbb{K}^*)$ referred to in Section 3 below.

The universal family for cubic surfaces is denoted by $\mathcal{G}^0$. This is the open part of the Göpel variety $\mathcal{G} \subset \mathbb{P}^{134}$ constructed in [20, §5]. The base of this 6-dimensional family is the 4-dimensional $\mathcal{Y}^0$. The map $\mathcal{G}^0 \to \mathcal{Y}^0$ was described in [11]. Thus the ring $\mathcal{K}[\mathcal{Y}]$ in (2.3) is the natural base ring for the universal Cox ring for degree 3 surfaces.

At this point it is essential to avoid confusing notation. To aim for a clear presentation, we use the uniformization of $\mathcal{Y}$ by the $E_6$ hyperplane arrangement. Namely, we take $R = \mathbb{Z}[d_1, d_2, d_3, d_4, d_5, d_6]$ instead of $\mathcal{K}[\mathcal{Y}]$ as the base ring. We write $\mathcal{X}$ for the universal cubic surface over $R$. The universal Cox ring is a quotient of the polynomial ring over $R$ in 27 variables, one for each curve on the cubic surface. Using variable names as in [26, §5], we write

$$\operatorname{Cox}(\mathcal{X}) = R[E_1, E_2, \ldots, E_6, F_{12}, F_{13}, \ldots, F_{56}, G_1, G_2, \ldots, G_6]/I_\mathcal{X}. \quad (2.9)$$

This ring is graded by the Picard group $\mathbb{Z}^7$, similarly to (2.5). The role of the 5-dimensional demicube $D_5$ is now played by the 6-dimensional Gosset polytope with 27 vertices, here also denoted by $E_6$. The symmetry group of this polytope is the Weyl group $W(E_6)$.  

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Proposition 2.2. Up to saturation by the product of all 27 variables and all 36 roots, the universal Cox ideal $I_X$ is generated by 270 trinomials. These are clustered by $\mathbb{Z}^7$-degrees into 27 groups of 10 generators, one for each line on the cubic surface. For instance, the 10 generators of $I_X$ that correspond to the line $G_1$ involve the 10 lines that meet $G_1$. They are

\[
\begin{align*}
(d_3-d_4)(d_1+d_3+d_4)E_2F_{12} - (d_2-d_4)(d_1+d_2+d_4)E_3F_{13} + (d_2-d_3)(d_1+d_2+d_3)E_4F_{14}, \\
(d_3-d_5)(d_1+d_3+d_5)E_2F_{12} - (d_2-d_5)(d_1+d_2+d_5)E_3F_{13} + (d_2-d_3)(d_1+d_2+d_3)E_5F_{15}, \\
(d_3-d_6)(d_1+d_3+d_6)E_2F_{12} - (d_2-d_6)(d_1+d_2+d_6)E_3F_{13} + (d_2-d_3)(d_1+d_2+d_3)E_6F_{16}, \\
(d_4-d_5)(d_1+d_4+d_5)E_2F_{12} - (d_2-d_5)(d_1+d_2+d_5)E_4F_{14} + (d_2-d_4)(d_1+d_2+d_4)E_5F_{15}, \\
(d_4-d_6)(d_1+d_4+d_6)E_2F_{12} - (d_2-d_6)(d_1+d_2+d_6)E_4F_{14} + (d_2-d_4)(d_1+d_2+d_4)E_6F_{16}, \\
(d_5-d_6)(d_1+d_5+d_6)E_2F_{12} - (d_2-d_6)(d_1+d_2+d_6)E_5F_{15} + (d_2-d_5)(d_1+d_2+d_5)E_6F_{16}, \\
(d_4-d_5)(d_1+d_4+d_5)E_3F_{13} - (d_3-d_5)(d_1+d_3+d_5)E_4F_{14} + (d_3-d_4)(d_1+d_3+d_4)E_5F_{15}, \\
(d_4-d_6)(d_1+d_4+d_6)E_3F_{13} - (d_3-d_6)(d_1+d_3+d_6)E_4F_{14} + (d_3-d_4)(d_1+d_3+d_4)E_6F_{16}, \\
(d_5-d_6)(d_1+d_5+d_6)E_3F_{13} - (d_3-d_6)(d_1+d_3+d_6)E_5F_{15} + (d_3-d_5)(d_1+d_3+d_5)E_6F_{16}, \\
(d_5-d_6)(d_1+d_5+d_6)E_4F_{14} - (d_4-d_6)(d_1+d_4+d_6)E_5F_{15} + (d_4-d_5)(d_1+d_4+d_5)E_6F_{16}.
\end{align*}
\]

The remaining 260 trinomials are obtained by applying the action of $W(E_6)$. The variety defined by $I_X$ in $\mathbb{P}^5 \times (\mathbb{K}^*)^{27}/(\mathbb{K}^*)^7$ is 6-dimensional. It is the universal family $X^0$.

Proof of Propositions 2.1 and 2.2. We consider the prime ideal in [20, §6] that defines the embedding of the Göpel variety $G$ into $\mathbb{P}^{134}$. By [20, Theorem 6.2], $G$ is the ideal-theoretic intersection of a 35-dimensional toric variety $T$ and a 14-dimensional linear space $L$. The latter is cut out by a canonical set of 315 linear trinomials, indexed by the 315 isotropic planes in $(\mathbb{P}_2)^6$. Pulling these linear forms back to the Cox ring of $T$, we obtain 315 quartic trinomials in 63 variables, one for each root of $E_7$. Of these 63 roots, precisely 27 involve the unknown $d_7$. We identify these with the $(-1)$-curves on the cubic surface via

\[
d_i - d_7 \mapsto E_i, \quad d_i + d_j + d_7 \mapsto F_{ij}, \quad -d_j + \sum_{i=1}^7 d_i \mapsto G_j. \tag{2.10}
\]

Moreover, of the 315 quartics, precisely 270 contain a root involving $d_7$. Their images under the map (2.10) are the 270 Cox relations listed above. Our construction ensures that they generate the correct Laurent polynomial ideal on the torus of $T$. This proves Proposition 2.2.

The derivation of Proposition 2.1 is similar, but now we use the substitution

\[
d_i - d_6 \mapsto E_i, \quad d_i + d_j + d_6 \mapsto F_{ij}, \quad \sum_{i=1}^6 d_i \mapsto G.
\]

We consider the 45 quartic trinomials that do not involve $d_7$. Of these, precisely five do not involve $d_6$ either. They translate into the five Plücker relations for $M_{0,5}$. With this identification, the remaining 40 quartics translate into the ten groups listed after Proposition 2.1.

Remark 2.3. The relations (2.10) are not unique. The non-uniqueness comes from both the symmetry of $E_7$ and the choice of $\pm$ sign for each variable $E_i, F_{ij}, G_j$. For the rest of this paper, we fix the relations (2.10). Other choices give the same result up to symmetry.
We now fix a $K$-valued point in the base $Y^0$, by replacing the unknowns $d_i$ with scalars in $K$. In order for the resulting surface $X$ to be smooth, we require $(d_1 : d_2 : d_3 : d_4 : d_5 : d_6)$ to be in the complement of the 36 hyperplanes for $E_6$. The corresponding specialization of $I_X$ is the Cox ideal $I_X$ of $X$. Seven of the ten trinomials in each degree are redundant over $K$. Only three are needed to generate $I_X$. Hence, the Cox ideal $I_X$ is minimally generated by 81 quadrics in the $E_i, F_{ij}$ and $G_i$. Its variety is the surface $X^0 = V(I_X) \subset (\bar{K}^*)^{27}/(\bar{K}^*)^7$.

**Proposition 2.4.** Each of the marked 27 trees on a tropical cubic surface has an involution.

**Proof.** Every line $L$ on a cubic surface $X$ over $K$, with its ten marked points, admits a double cover to $\mathbb{P}^1$ with five markings. The preimage of one of these marked points is the pair of markings on $L$ given by two other lines forming a tritangent with $L$. Tropically, this gives a double cover from the 10-leaf tree for $L$ to a 5-leaf tree with leaf labelings given by these pairs. The desired involution on the 10-leaf tree exchanges elements in each pair. \hfill $\square$

For instance, for the tree that corresponds to the line $L = G_1$, the involution equals

$$E_2 \leftrightarrow F_{12}, \ E_3 \leftrightarrow F_{13}, \ E_4 \leftrightarrow F_{14}, \ E_5 \leftrightarrow F_{15}, \ E_6 \leftrightarrow F_{16}. \,$$

Indeed, this involution fixes the 10 Cox relations displayed in Proposition 2.2. The induced action on the trees corresponding to the tropicalization of the lines can be seen in Figures 4 and 5, where the involution reflects about a vertical axis. The corresponding 5-leaf tree is the tropicalization of the line in

$$\text{Proj}(K[E_2F_{12}, E_3F_{13}, E_4F_{14}, E_5F_{15}, E_6F_{16}]) \, \simeq \, \mathbb{P}^4$$

that is the intersection of the 10 hyperplanes defined by the polynomials in Proposition 2.2.

We aim to compute $\text{trop}(X^0)$ by applying $\text{gfan}$ to the ideal $I_X$. This works well for $K = \mathbb{Q}$ with the trivial valuation. Here the output is the cone over the Schl"afli graph which records which pairs of $(-1)$-curves intersect on $X$. This is a 10-regular graph with 27 nodes. However, for $K = \mathbb{Q}(t)$, our $\text{gfan}$ calculations did not terminate. Future implementations of tropical algorithms will surely succeed; see also Conjecture 5.3. To get the tropical cubic surfaces, and to prove Theorem 1.1, we used the alternative method explained in Section 3.

### 3 Sekiguchi Fan to Naruki Fan

In the previous section we discussed the computation of tropical del Pezzo surfaces directly from their Cox ideals. This worked well for degree 4. However, using the current implementation in $\text{gfan}$, this computation did not terminate for degree 3. We here discuss an alternative method that did succeed. In particular, we present the proof of Theorem 1.1.

The successful computation uses the following commutative diagram of balanced fans:

\[
\begin{array}{ccc}
\text{Berg}(E_7) & \longrightarrow & \text{trop}(\mathcal{G}^0) \\
\downarrow & & \downarrow \\
\text{Berg}(E_6) & \longrightarrow & \text{trop}(\mathcal{Y}^0)
\end{array}
\]  

(3.1)
The moduli fans

Lemma 3.1. The Bergman fans of the rank m matroid defined by the (36 resp. 63) linear forms listed in (2.8). Thus, Berg(E_6) is a tropical linear space in $\mathbb{T}\mathbb{P}^{35}$, and Berg(E_7) is a tropical linear space in $\mathbb{T}\mathbb{P}^{62}$. Coordinates are labeled by roots.

The list (2.8) fixes a choice of injection of root systems $E_6 \hookrightarrow E_7$. This defines coordinate projections $\mathbb{R}^{36} \to \mathbb{R}^{36}$ and $\mathbb{T}\mathbb{P}^{62} \to \mathbb{T}\mathbb{P}^{35}$, namely by deleting coordinates with index 7. This projection induces the vertical map from Berg(E_7) to Berg(E_6) on the left in (3.1).

On the right in (3.1), we see the 4-dimensional Yoshida variety $Y \subset \mathbb{P}^{39}$ and the 6-dimensional Göpel variety $G \subset \mathbb{P}^{134}$. Explicit parametrizations and equations for these varieties were presented in [20, 21]. The corresponding very affine varieties $G^0 \subset (K^*)^{35}/K^*$ and $Y^0 \subset (K^*)^{40}/K^*$ are moduli spaces of marked del Pezzo surfaces [11]. Their tropicalizations trop($G^0$) and trop($Y^0$) are known as the Sekiguchi fan and Naruki fan, respectively. The modular interpretation in [11] ensures the existence of the vertical map trop($G^0$) $\to$ trop($Y^0$). This map is described in [11, Lemma 5.4], in [21, (6.5)], and in the proof of Lemma 3.1 below.

The two horizontal maps in (3.1) are surjective and (classically) linear. The linear map $G \to Y$ is given by the 135 $\times$ 63 matrix $A$ in [20, §6]. The corresponding toric variety is the object of [20, Theorem 6.1]. The map Berg(E_6) $\to$ trop($Y^0$) is given by the 40 $\times$ 36-matrix in [21, Theorem 6.1]. We record the following computational result. It refers to the natural simplicial fan structure on Berg(E_m) described by Ardila et al. in [3].

Lemma 3.1. The Bergman fans of $E_6$ and $E_7$ have dimensions 5 and 6. Their f-vectors are

\[
\begin{align*}
    f_{\text{Berg}(E_6)} &= (1, 750, 17679, 105930, 219240, 142560), \\
    f_{\text{Berg}(E_7)} &= (1, 6091, 315399, 3639804, 14982660, 24607800, 13721400).
\end{align*}
\]

The moduli fans trop($Y^0$) and trop($G^0$) have dimensions 4 and 6. Their f-vectors are

\[
\begin{align*}
    f_{\text{trop}(Y^0)} &= (1, 76, 630, 1620, 1215), \\
    f_{\text{trop}(G^0)} &= (1, 1065, 27867, 229243, 767025, 1093365, 547155).
\end{align*}
\]

Proof. The f-vector for the Naruki fan trop($Y^0$) appears in [21, Table 5]. For the other three fans, only the numbers of rays (namely 750, 6091 and 1065) were known from [21, §6]. The main new result in Lemma 3.1 is the computation of all 57273155 cones in Berg(E_7). The fans Berg(E_6) and trop($G^0$) are subsequently derived from Berg(E_7) using the maps in (3.1).

We now describe how $f_{\text{Berg}(E_7)}$ was found. We did not use the theory of tubings in [3]. Instead, we carried out a brute force computation based on [10] and [19]. Recall that a point lies in the Bergman fan of a matroid if and only if the minimum is obtained twice on each circuit. We computed all circuits of the rank 7 matroid on the 63 vectors in the root system $E_7$. That matroid has precisely 100662348 circuits. Their cardinalities range from 3 to 8. This furnishes a subroutine for deciding whether a given point lies in the Bergman fan.

Our computations were mostly done in sage [24] and java. We achieved speed by exploiting the action of the Weyl group $W(E_7)$ given by the two generators in [20, (4.2)]. The two matrices derived from these two generators using [20, (4.3)] act on the space $\mathbb{R}^7$ with coordinates $d_1, d_2, \ldots, d_7$. This gives subroutines for the action of $W(E_7)$ on $\mathbb{R}^{63}$, e.g. for deciding whether two given sequences of points are conjugate with respect to this action.
Remark 3.2. The (reduced) Euler characteristic of the link of $\text{Berg}(E_7)$ is the alternating sum of the entries of the f-vector of this Bergman fan. We see from Lemma 3.1 that this is

$$1 - 6091 + 315399 - 3639804 + 14982660 - 24607800 + 13721400 = 765765 = 1 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 17.$$  

This is the product of all exponents of $W(E_7)$, thus confirming the prediction in [20, (9.2)].

The Naruki fan $\text{trop}(\mathcal{Y}^0)$ is studied in [21, §6]. Under the action of $W(E_6)$ through $\text{Berg}(E_6)$, it has two classes of rays, labelled type (a) and type (b). It also has two $W(E_6)$-orbits of maximal cones: there are 135 type (aaaa) cones, each spanned by four type (a) rays, and 1080 type (aaab) cones, each spanned by three type (a) rays and one type (b) ray.
The map \( \text{trop}(G^0) \rightarrow \text{trop}(Y^0) \) tropicalizes the morphism \( G^0 ightarrow Y^0 \) between very affine \( K \)-varieties of dimension 6 and 4. That morphism is the universal family of cubic surfaces. In order to tropicalize these surfaces, we examine the fibers of the map \( \text{trop}(G^0) \rightarrow \text{trop}(Y^0) \). The next lemma concerns the subdivision of \( \text{trop}(Y^0) \) induced by this map. By definition, this is the coarsest subdivision such that each cone in \( \text{trop}(G^0) \) is sent to a union of cones.

**Lemma 3.3.** The subdivision induced by the map \( \text{trop}(G^0) \rightarrow \text{trop}(Y^0) \) is the barycentric subdivision on type (aaaa) cones. For type (aaab) cones, each cone in the subdivision is a cone spanned by the type (b) ray and a cone in the barycentric subdivision of the (aaa) face. Thus each (aaaa) cone is divided into 24 cones, and each (aaab) cone is divided into 6 cones.

**Proof.** The map \( \pi : \text{trop}(G^0) \rightarrow \text{trop}(Y^0) \) can be defined via the commutative diagram (3.1): for \( x \in \text{trop}(G^0) \), take any point in its preimage in \( \text{Berg}(E_7) \), then follow the left vertical map and the bottom horizontal map to get \( \pi(x) \) in \( \text{trop}(Y^0) \). It is well-defined because the kernel of the map \( \text{Berg}(E_7) \rightarrow \text{trop}(G^0) \) is contained in the kernel of the composition \( \text{Berg}(E_7) \rightarrow \text{Berg}(E_6) \rightarrow \text{trop}(Y^0) \). With this, we can compute the image in \( \text{trop}(Y^0) \) of any cone in \( \text{trop}(G^0) \). For each orbit of cones in \( \text{trop}(Y^0) \), pick a representative \( \sigma \), and examine all cones in \( \text{trop}(G^0) \) that map into \( \sigma \). Their images reveal the subdivision of \( \sigma \).

![Figure 4: The 27 trees on tropical cubic surfaces of type (aaaa)](image)

Lemma 3.3 shows that each (aaaa) cone of the Naruki fan \( \text{trop}(Y^0) \) is divided into 24 subcones, and each (aaab) cone is divided into 6 subcones. Thus, the total number of cones in the subdivision is \( 24 \times 135 + 6 \times 1080 = 9720 \). For the base points in the interior of a cone, the fibers are contained in the same set of cones in \( \text{trop}(G^0) \). The fiber changes continuously as the base point changes. Therefore, moving the base point around the interior of a cone simply changes the metric but not the combinatorial type of marked tropical cubic surface.

**Corollary 3.4.** The map \( \text{trop}(G^0) \rightarrow \text{trop}(Y^0) \) has at most two combinatorial types of generic fibers up to relabeling.

**Proof.** We fixed an inclusion \( E_6 \hookrightarrow E_7 \) in (2.8). The action of \( \text{Stab}_{E_6}(W(E_7)) \) on the fans is compatible with the entire commutative diagram (3.1). Hence, the fibers over two points...
that are conjugate under this action have the same combinatorial type. We verify that the 9720 cones form exactly two orbits under this action. One orbit consists of the cones in the type (aaaa) cones, and the other consists of the cones in the type (aaab) cones. Therefore, there are at most two combinatorial types, one for each orbit.

\[ \times 12 \]

\[ \times 3 \]

Figure 5: The 27 trees on tropical cubic surfaces of type (aaab)

We can now derive our classification theorem for tropical cubic surfaces.

**Proof of Theorem 1.1.** We compute the two types of fibers of \( \pi : \text{trop}(G^0) \to \text{trop}(Y^0) \). In what follows we explain this for a cone \( \sigma \) of type (aaaa). The computation for type (aaab) is similar. Let \( r_1, r_2, r_3, r_4 \) denote the rays that generate \( \sigma \). We fix the vector \( x = r_1 + 2r_2 + 3r_3 + 4r_4 \) that lies in the interior of a cone in the barycentric subdivision.

The fiber \( \pi^{-1}(x) \) is found by an explicit computation. First we determine the directions of the rays. They arise from rays of trop(\( G^0 \)) that are mapped to zero by \( \pi \). There are 27 such ray directions in \( \pi^{-1}(x) \). These are exactly the image of the 27 type \( A_1 \) rays in Berg(\( E_7 \)) that correspond to the roots in \( E_7 \setminus E_6 \). We label them by \( E_i, F_{ij}, G_j \) as in (2.10). Next, we compute the vertices of \( \pi^{-1}(x) \). They are contained in 4-dimensional cones \( \sigma' = \text{pos}\{R_1, R_2, R_3, R_4\} \) with \( x \in \pi(\sigma') \). The coordinates of each vertex in \( \mathbb{TP}^{134} \) is computed by solving \( y_1 \pi(R_1) + y_2 \pi(R_2) + y_3 \pi(R_3) + y_4 \pi(R_4) = x \) for \( y_1, y_2, y_3, y_4 \).

The part of the fiber contained in each cone in trop(\( G^0 \)) is spanned by the vertices and the \( E_i, F_{ij}, G_j \) rays it contains. Iterating through the list of cones and looking at this data, we get a list that characterizes the polyhedral complex \( \pi^{-1}(x) \). In particular, that list verifies that \( \pi^{-1}(x) \) is 2-dimensional and has the promised f-vector. For each of the 27 ray directions \( E_i, F_{ij}, G_j \), there is a tree at infinity. It is the link of the corresponding point at infinity \( \pi^{-1}(x) \subset \mathbb{TP}^{134} \). The combinatorial types of these 27 trees are shown in Figure 4. The metric on each tree can be computed as follows: the length of a bounded edge equals the lattice distance between the two vertices in the corresponding flap.

The surface \( \pi^{-1}(x) \) is homotopy equivalent to its bounded complex. We check directly that the bounded complex is contractible. This can also be inferred from Theorem 4.4. \( \square \)
Remark 3.5. We may replace \( x = r_1 + 2r_2 + 3r_3 + 4r_4 \) with a generic point \( x = x_1r_1 + x_2r_2 + x_3r_3 + x_4r_4 \), where \( x_1 < x_2 < x_3 < x_4 \). This lies in the same cone in the barycentric subdivision, so the combinatorics of \( \pi^{-1}(x) \) remains the same. Repeating the last step over the field \( \mathbb{Q}(x_1, x_2, x_3, x_4) \) instead of \( \mathbb{Q} \), we write the length of each bounded edge in the 27 trees in terms of the parameters. Each length either equals \( x_i \) or \( x_i - x_j \) for some \( i, j \).

4 Tropical modifications

In Section 2 we computed tropical varieties from polynomial ideals, along the lines of the book by Maclagan and Sturmfels [15]. We now turn to tropical geometry as a self-contained subject in its own right. This is the approach presented in the book by Mikhalkin and Rau [17]. Central to that approach is the notion of tropical modification. In this section we explain how to construct our tropical del Pezzo surfaces from the plane \( \mathbb{R}^2 \) by modifications. This leads to proofs of Proposition 1.2 and Theorem 1.1 purely within tropical geometry.

Tropical modification is an operation that relates topologically different tropical models of the same variety. This operation was first defined by Mikhalkin in [16]; see also [17, Chapter 5]. Here we work with a variant known as open tropical modifications. These were introduced in the context of Bergman fans of matroids in [23]. Brugallé and Lopez de Medrano [7] used them to study intersections and inflection points of tropical plane curves.

We fix a tropical cycle \( Y \in \mathbb{R}^n \), as in [17]. An open modification is a map \( p : Y' \to Y \) where \( Y' \subset \mathbb{R}^{n+1} \) is a new tropical variety to be described below. One should think of \( Y' \) as being an embedding of the complement of a divisor in \( Y \) into a higher-dimensional torus.

Consider a piecewise integer affine function \( g : Y \to \mathbb{R} \). The graph
\[
\Gamma_g(Y) = \{(y, g(y)) \mid y \in Y\} \subset \mathbb{R}^{n+1}
\]
is a polyhedral complex which inherits weights from \( Y \). However, it usually not balanced. There is a canonical way to turn \( \Gamma_g(Y) \) into a balanced complex. If \( \Gamma_g(Y) \) is unbalanced around a codimension one face \( E \), then we attach to it a new unbounded facet \( F_E \) in direction \(-e_{n+1}\). (We here use the max convention, as in [17]). The facet \( F_E \) can be equipped with a unique weight \( w_{F_E} \in \mathbb{Z} \) such that the complex obtained by adding \( F_E \) is balanced at \( E \). The resulting tropical cycle is \( Y' \subset \mathbb{R}^{n+1} \). By definition, the open modification of \( Y \) given by \( g \) is the map \( p : Y' \to Y \), where \( p \) comes from the projection \( \mathbb{R}^{n+1} \to \mathbb{R}^n \) with kernel \( \mathbb{R}e_{n+1} \).

The tropical divisor \( \text{div}_Y(g) \) consists of all points \( y \in Y \) such that \( p^{-1}(y) \) is infinite. This is a polyhedral complex. It inherits weights on its top-dimensional faces from those of \( Y' \). A tropical cycle is effective if the weights of its top-dimensional faces are positive. Therefore, the cycle \( Y' \) is effective if and only if \( Y \) and the divisor \( \text{div}_Y(g) \) are effective. Given a tropical variety \( Y \) and an effective divisor \( \text{div}_Y(g) \), we say the tropical modification \( p : Y' \to Y \) is along \( \text{div}_Y(g) \). See [16, 17, 23] for basics concerning cycles, divisors and modifications.

Open tropical modifications are related to re-embeddings of classical varieties as follows. Fix a very affine \( K \)-variety \( X \subset (\bar{K})^n \) and \( Y = \text{trop}(X) \subset \mathbb{R}^n \). Given a polynomial function \( f \in K[X] \), let \( D \) be its divisor in \( X \). Then \( X \setminus D \) is isomorphic to the graph of the restriction of \( f \) to \( X \setminus D \). In this manner, the function \( f \) gives a closed embedding of \( X \setminus D \) into \((\bar{K})^n\).

For the next proposition we require the tropicalization of a variety to be locally irreducible.
Let $y$ be a point in a tropical variety $Y$, then
\[
\text{Star}_y(Y) = \{ y' \mid \exists \epsilon > 0 \text{ s.t. } \forall 0 < \delta < \epsilon : y + \epsilon y' \in Y \},
\]
is a balanced tropical fan with weights inherited from $Y$. A tropical variety $Y$ is \textit{locally irreducible} if at every point $y \in Y$, we have that $\text{Star}_y(Y)$ is not a proper union of two tropical varieties, taking weights into consideration.

\textbf{Proposition 4.1.} Let $X \subset (\bar{K}^*)^n$ be a very affine variety. For a function $f \in K[X]$, let $D$ be the divisor $\text{div}_X(f)$, and let $X' = X \setminus D \subset (\bar{K}^*)^{n+1}$ denote the graph of $X$ along $f$ as described above. Let $Y = \text{trop}(X) \subset \mathbb{R}^n$ and $Y' = \text{trop}(X') \subset \mathbb{R}^{n+1}$. Suppose that $Y$ is locally irreducible, then there exists a piecewise integer affine function $g : Y \to \mathbb{R}$ such that $\text{div}_Y(g) = \text{trop}(D)$ and the coordinate projection $Y' \to Y$ is the open modification of $Y$ along that divisor.

\textit{Proof.} The coordinate projection $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ takes $Y'$ onto $Y$, since tropicalization acts coordinate-wise. We claim that the fiber over a point $y \in Y$ is either a single point or a half-line in the $-e_{n+1}$ direction. The fiber, $p^{-1}(y)$ is 1-dimensional and closed in $Y'$. Let $y'$ be an endpoint of a connected component of $p^{-1}(y)$. Then $p(\text{Star}_y(Y'))$ has the same dimension as $Y$. Since otherwise, $\text{Star}_y(Y')$ contains a space of linearity in the direction $e_{n+1}$ and $y'$ cannot be an endpoint of the fiber. If the one dimensional fiber $p^{-1}(y)$ contains two endpoints $y_1$ and $y_2$ then $Y$ must be reducible at $y$; it can be split into more than one component coming from the projection of $p(\text{Star}_{y_1}(Y'))$ and $p(\text{Star}_{y_2}(Y'))$. Therefore, $p^{-1}(y)$ consists of either a single point, a line, or a half line. However, since $f \in K[X]$ is a regular function, the fiber of a point $y \in Y$ cannot be unbounded in the $+e_{n+1}$ direction. Thus the only possibilities are that $p^{-1}(y)$ is a single point or a half line in the $-e_{n+1}$ direction.

Finally, we obtain the piecewise integer affine function $g$ by taking $g(y) = p^{-1}(y)$ for $y \in Y \setminus \text{trop}(D)$ and then extending by continuity to the rest of $Y$. Then $Y'$ is the modification along the function $g$ described above. \hfill \Box

Any two tropical rational functions $g$ and $g'$ that define the same tropical divisor on $Y$ must differ by a map which is integer affine on $Y$, see [1, Remark 3.6]. This leads to the following corollary.

\textbf{Corollary 4.2.} \textit{Under the assumptions of Proposition 4.1, the tropicalization of $X' = X \setminus D \subset \mathbb{R}^{n+1}$ is determined uniquely by those of $D$ and $X$, up to an integer affine map.}

In general, trop$(X')$ is not determined by the tropical hypersurface of $f \in K[X]$, as the tropicalization of the divisor $D = \text{div}_X(f)$ may differ from the tropical stable intersection of trop$(X')$ and that tropical hypersurface. Examples 4.2 and 4.3 of [7] demonstrate both this and that Proposition 4.1 can fail without the locally irreducibility hypothesis.

Suppose now that $X' \subset (\bar{K}^*)^{n+k}$ is obtained from $X \subset (\bar{K}^*)^n$ by taking the graph of a list of $k \geq 2$ polynomials $f_1, f_2, \ldots, f_k$. This gives us a sequence of projections
\[
X' = X_k \to X_{k-1} \to \cdots \to X_2 \to X_1 \to X_0 = X,
\]
(4.1)
where $X_i \subset (\bar{K}^*)^{n+i}$ is obtained from $X$ by taking the graph of $(f_1, \ldots, f_i)$. We further get a corresponding sequence of projections of the tropicalizations:
\[
\text{trop}(X') = \text{trop}(X_k) \to \text{trop}(X_{k-1}) \to \cdots \to \text{trop}(X_0) = \text{trop}(X).
\]
(4.2)
Figure 6: The tropical divisors in Example 4.3. The positions of \( \text{trop}(G_1 \cap H_1) \) in \( \text{trop}(X_1) \) for three choices of \( a \) are marked on the downward purple edge. For \( a = 1 \) we get \( M_{0,5} \).

We may ask if it is possible to recover \( \text{trop}(X') \subseteq \mathbb{R}^{n+k} \) just from \( \text{trop}(X) \) and the \( k \) tropical divisors \( \text{trop}(D_i) \) considered in \( \text{trop}(X) \). The answer is “yes” in the special case when \( \text{trop}(X) = \mathbb{R}^n \) and the arrangement of divisors \( \text{trop}(D_i) \) are each locally irreducible and intersect properly, meaning the intersection of any \( l \) of these divisors has codimension \( l \). However, in general, iterating modifications to recover \( \text{trop}(X') \) is a delicate procedure. In most cases, the outcome is not solely determined by the configuration of tropical divisors in \( \text{trop}(X) \), even if the divisors intersect pairwise properly. We illustrate this by deriving the degree 5 del Pezzo surface \( \text{trop}(M_{0,5}) \).

**Example 4.3.** This is a variation on [23, Example 2.29]. Let \( X = (\mathbb{K}^*)^2 \) and consider the functions \( f(x) = x_1 - 1, g(x) = x_2 - 1 \) and \( h(x) = ax_1 - x_2 \), for some constant \( a \in \mathbb{K}^* \) with \( \text{val}(a) = 0 \). Denote \( \text{div}_X(f) \) by \( F \), and analogously for \( G \) and \( H \). The tropicalization of each divisor is a line through the origin in \( \mathbb{R}^2 \). The directions of \( \text{trop}(F), \text{trop}(G), \) and \( \text{trop}(H) \) are \((1,0),(0,1),\) and \((1,1)\) respectively. Let \( X' \subseteq (\mathbb{K}^*)^5 \) denote the graph of \( X \) along the three functions \( f, g, \) and \( h \), in that order. This defines a sequence of projections,

\[
X' \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X = (\mathbb{K}^*)^2.
\]

Here, \( X_2 = \{(x_1, x_2, x_1 - 1, x_2 - 1)\} \subseteq (\mathbb{K}^*)^4 \). The tropical plane \( \text{trop}(X_2) \) contains the face \( \sigma = \{0\} \times \{0\} \times (-\infty, 0] \times (-\infty, 0] \), corresponding to points with \( \text{val}(x_1) = \text{val}(x_2) = 0 \). Let \( H_2 \) denote the graph of \( f \) and \( g \) restricted to \( H \). This is a line in 4-space, namely,

\[
H_2 = \{(x_1, ax_1, x_1 - 1, ax_1 - 1)\} \subseteq X_2 \subseteq (\mathbb{K}^*)^4.
\]

The tropical line \( \text{trop}(H_2) \) depends on the valuation of \( a - 1 \). It can be determined from

\[
\text{trop}(G_1 \cap H_1) = \{(0,0,-\text{val}(\frac{1}{a} - 1))\}.
\]

Here, \( H_1, G_1 \) denote the graph of \( f \) restricted to \( H \) and \( G \), respectively. Figure 6 shows the possibilities for \( \text{trop}(G_1 \cap H_1) \) in \( \text{trop}(X_1) \), and Figure 7 shows \( \text{trop}(H_2) \cap \sigma \) in \( \text{trop}(X_2) \).

We can prescribe any value \( v \in (0, \infty) \) for the valuation of \( \frac{1}{a} - 1 \), for instance by taking \( \frac{1}{a} = 1 + tv \) when \( K = \mathbb{C}\{t\} \). In these cases, the tropical plane \( \text{trop}(X') \) is not a fan. However,
it becomes a fan when \( v \) moves to either endpoint of the interval \([0, -\infty)\). For instance, \( v = 0 \) happens when the constant term of \( \frac{1}{a} \) is not equal to 1 and \( \text{trop}(X') \) is the fan obtained from \( \mathbb{R}^2 \) by carrying out the modifications along the pull-backs of the tropical divisors on the left in Figure 6. See Definition 2.16 of [23] for pull-backs of tropical divisors. The other extreme is when \( a = 1 \). Here, \( F, G, H \) are concurrent lines in \((\overline{K}^*)^2\), and \( \text{trop}(H_2) \) contains a ray in the direction \( e_3 + e_4 \). Upon modification, we obtain the fan over the Petersen graph in Figure 3. This is the tropicalization of the degree 5 del Pezzo surface in (2.2). Thus beginning from the tropical divisors \( \text{trop}(F), \text{trop}(G), \text{trop}(H) \) in \( \mathbb{R}^2 \), we recover \( \text{trop}(M_{0,5}) \) if we know that they represent tropicalizations of concurrent lines in \((\overline{K}^*)^2\).

The open tropical modification described above represents the tropicalization of the very affine variety \( M_{0,5} \). In this case the compactification of \( M_{0,5} \) which produces the del Pezzo surface of degree 5 is indeed the tropical compactification given by the fan \( \text{trop}(M_{0,5}) \), (see [15, §6.4] for an introduction to tropical compactifications). There is no direct connection between open tropical modifications and birational transformations, the link depends a choice of compactification of the very affine variety. Upon removing divisors one can find more interesting compactifications of the complement. For example, \((K^*)^2\) cannot be compactified to a del Pezzo surface of degree less than 6, but upon deleting the three divisors above one can compactify the complement to a del Pezzo surface of degree 5.

We now explain how this extends to a del Pezzo surface \( X \) of degree \( d \leq 4 \). As before, we write \( X^0 \) for the complement of the \((-1)\)-curves in \( X \). Then \( X' = X^0 \) is obtained from \((\overline{K}^*)^2\) by taking the graphs of the polynomials \( f_1, \ldots, f_k \) of the curves in \((\overline{K}^*)^2\) that give rise to \((-1)\)-curves on \( X \). More precisely, fix \( p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1), p_4 = (1 : 1 : 1) \), and take \( p_5, \ldots, p_{9-d} \) to be general points in \( \mathbb{P}^2 \). If \( d = 4 \) then there is only one extra point \( p_5 \), we have \( k = 8 \) in (4.1), and \( f_1, \ldots, f_8 \) are the polynomials defining

\[
F_{14}, F_{15}, F_{24}, F_{25}, F_{34}, F_{35}, F_{45}, G. \tag{4.3}
\]

For \( d = 3 \), there are two extra points \( p_5, p_6 \) in \( X \), we have \( k = 18 \), and \( f_1, \ldots, f_{18} \) represent

\[
F_{14}, F_{15}, F_{16}, F_{24}, F_{25}, F_{26}, F_{34}, F_{35}, F_{36}, F_{45}, F_{46}, F_{56}, G_1, G_2, G_3, G_4, G_5, G_6. \tag{4.4}
\]

We write \( P_i = \text{trop}(p_i) \in \mathbb{T}\mathbb{P}^2 \) for the image of the point \( p_i \) under tropicalization. The tropical points \( P_1, P_2, \ldots \) are in \textit{general position} if any two lie in a unique tropical line, these lines are distinct, any five lie in a unique tropical conic, and these conics are distinct in \( \mathbb{T}\mathbb{P}^2 \).
A configuration in general position for \( d = 4 \) is shown in Figure 8. Our next result implies that the colored Clebsch graph in Figure 1 can be read off from Figure 8 alone. For \( d = 3 \), in order to recover the tropical cubic surface from the planar configuration, the points \( P_i \) must satisfy further genericity assumptions, to be revealed in the proof of the next theorem.

**Theorem 4.4.** Fix \( d \in \{3, 4, 5\} \) and points \( p_1, \ldots, p_{9-d} \) in \( \mathbb{P}^2 \) whose tropicalizations \( P_i \) are sufficiently generic in \( \mathbb{T}\mathbb{P}^2 \). The tropical del Pezzo surface \( \text{trop}(X^0) \) can be constructed from \( \mathbb{T}\mathbb{P}^2 \) by a sequence of open modifications that is determined by the points \( P_1, \ldots, P_{9-d} \).

**Proof.** The sequence of tropical modifications we use to go from \( \mathbb{R}^2 \) to \( \text{trop}(X^0) \) is determined if we know, for each \( i \), the correct divisor on each \((-1)\)-curve \( C \) in the tropical model \( \text{trop}(X_i) \). Then, the preimage of \( C \) in the next surface \( \text{trop}(X_{i+1}) \) is the modification \( C' \) of the curve \( C \) along that divisor. By induction, each intermediate surface \( \text{trop}(X_i) \) is locally irreducible, since it is obtained by modifying a locally irreducible surface along a locally irreducible divisor. With this, Theorem 4.4 follows from Proposition 4.1, applied to both the \( i \)-th surface and its \((-1)\)-curves. The case \( d = 5 \) was covered in Example 4.3. From the metric tree that represents the boundary divisor \( C \) of \( X^0 \) we can derive the corresponding trees on each intermediate surface \( \text{trop}(X_i) \) by deleting leaves. Thus, to establish Theorem 4.4, it suffices to prove the following claim: the final arrangement of the (16 or 27) metric trees on the tropical del Pezzo surface \( \text{trop}(X^0) \) is determined by the locations of the points \( P_i \) in \( \mathbb{T}\mathbb{P}^2 \).

Consider first the case \( d = 4 \). The points \( P_4 \) and \( P_5 \) determine an arrangement of plane tropical curves (4.3) as shown in Figure 8. The conic \( G \) through all five points looks like an “inverted tropical line”, with three rays in directions \( P_1, P_2, P_3 \). By the genericity assumption, the points \( P_4 \) and \( P_5 \) are located on distinct rays of \( G \). These data determine a trivalent metric tree with five leaves, which we now label by \( E_1, E_2, E_3, E_4, E_5 \). Namely, \( P_4 \) forms a cherry together with the label of its ray, and ditto for \( P_5 \). For instance, in Figure 8, the cherries on the tree \( G \) are \( \{E_1, E_4\} \) and \( \{E_2, E_5\} \), while \( E_3 \) is the non-cherry leaf. This is precisely the tree sitting on the node labeled \( G \) in Figure 1.
edges of the tree $G$ are the distances from $P_4$ resp. $P_5$ to the unique vertex of the conic $G$ in $\mathbb{R}^2$. Thus the metric tree $G$ is easily determined from $P_4$ and $P_5$. The other 15 metric trees can also be determined in a similar way from the configuration of points and curves in $\mathbb{R}^2$ and by performing a subset of the necessary modifications. Alternatively, we may use the transition rules (1.1) and (1.2) to obtain the other 15 trees from $G$. This proves the above claim, and hence Theorem 4.4, for del Pezzo surfaces of degree $d = 4$.

Consider now the case $d = 3$. Here the arrangement of tropical plane curves in $\mathbb{R}^2 \subset \mathbb{TP}^2$ consists of three lines at infinity, $F_{12}, F_{13}, F_{23}$, nine straight lines, $F_{14}, F_{15}, \ldots, F_{36}$, three honest tropical lines, $F_{45}, F_{46}, F_{56}$, three conics that are “inverted tropical lines” $G_4, G_5, G_6$, and three conics with one bounded edge, $G_1, G_2, G_3$. Each of these looks like a tree already in the plane, and it gets modified to a 10-leaf tree, like to ones in Figures 4 and 5. We claim that these labeled metric trees are uniquely determined by the positions of $P_4, P_5, P_6$ in $\mathbb{R}^2$.

Consider one of the 9 straight lines in our arrangement, say, $F_{14}$. If the points $P_4, P_5, P_6$ are generically chosen, 7 of the 10 leaves on the tree $F_{ij}$ can be determined from the diagram in $\mathbb{R}^2$. These come from the 7 markings on the line $F_{14}$ given by $E_1, E_4, F_{23}, F_{25}, F_{26}, F_{35}, F_{36}$. The markings $E_1$ and $F_{23}$ are the points at infinity, the marking $E_4$ is the location of point $P_4$, and the markings $F_{25}, F_{26}, F_{35}, F_{36}$ are the points of intersection with those lines. Under our hypothesis, these 7 marked points on the line $F_{14}$ will be distinct. With this, $F_{14}$ is already a metric caterpillar tree with 7 leaves. The three markings which are missing are $G_1, G_4$ and $F_{56}$. Depending on the positions of $P_4, P_5, P_6$, the intersection points of these three curves with the line $F_{14}$ may coincide with previously marked points. Whenever this happens, the position of the additional marking on the tree $F_{14}$ can be anywhere on the already attached leaf ray. Again, the actual position of the point on that ray may be determined by performing modifications along those curves. Alternatively, we use the involution given in Corollary 2.4. The involution on the ten leaves of the desired tree $F_{14}$ is

$$E_1 \leftrightarrow G_4, \quad E_4 \leftrightarrow G_1, \quad F_{23} \leftrightarrow F_{56}, \quad F_{25} \leftrightarrow F_{36}, \quad F_{26} \leftrightarrow F_{35}.$$  

Since the involution exchanges each of the three unknown leaves with one of the seven known leaves, we can easily construct the final 10-leaf tree from the 7-leaf caterpillar.

A similar argument works the other six lines $F_{ij}$, and the conics $G_4, G_5, G_6$. In these cases, 8 of the 10 marked points on a tree are determined from the arrangement in the plane, provided the choice of points is generic. Finally, the conics $G_1, G_2, G_3$ are dual to subdivisions of lattice parallelograms of area 1. They may contain a bounded edge. Suppose no point $P_j$ lies on the bounded edge of the conic $G_i$, then the positions of all 10 marked points of the tree are visible from the arrangement in the plane. If $G_i$ does contain a marked point $P_j$ on its bounded edge, then the tropical line $F_{ij}$ intersects $G_i$ in either a bounded edge or a single point with intersection multiplicity 2, depending on the dual subdivision of $G_i$. In the first case the position of the marked point $F_{ij}$ is easily determined from the involution; the distance from a vertex of the bounded edge of $G_i$ to the marked point $F_{ij}$ must be equal to the distance from $P_j$ to the opposite vertex of the bounded edge of $G_i$.

If $G_i \cap F_{ij}$ is a single point of intersection multiplicity two, then $P_j$ and $F_{ij}$ form a cherry on the tree $G_i$, which is invariant under the involution. We claim that this cherry attaches to the rest of the tree at a 4-valent vertex. The involution on the 10-leaf tree can also be seen as a tropical double cover from our 10-leaf tree to a 5-leaf tree, $h : T \to t$, where the 5-leaf
tree $t$ is labeled with the pair of markings interchanged by the involution. As mentioned in Corollary 2.4, this double cover comes from the classical curve in the del Pezzo surface $X$. In particular, the double cover locally satisfies the tropical translation of the Riemann-Hurwitz condition [5, Definition 2.2]. In our simple case of a degree 2 map between two trees, this local condition for a vertex $v$ of $T$ is 

$$\deg(v) - d_{h,v}(\deg(h(v)) - 2) - 2 \geq 0,$$

where $\deg$ denotes the valency of a vertex, and $d_{h,v}$ denotes the local degree of the map $h$ at $v$. Suppose the two leaves did not attach at a four valent vertex, then they form a cherry, this cherry attaches to the rest of the tree by an edge $e$ which is adjacent to another vertex $v$ of the tree. The Riemann-Hurwitz condition is violated at $v$, since $\deg(v) = \deg(h(v)) = 3$ and $d_{h,v} = 2$.

We conclude that the tree arrangement can be recovered from the position of the points $P_1, P_2, \ldots$ in $\mathbb{R}^2$. Therefore it is also possible to recover the tropical del Pezzo surface $\text{trop}(X^0)$ by open modifications. In each case, we recover the corresponding final 10 leaf tree from the arrangement in $\mathbb{T} \mathbb{P}^2$ plus our knowledge of the involution in Corollary 2.4.

**Remark 4.5.** Like in the case $d = 4$, knowledge of transition rules among the 27 metric trees on $\text{trop}(X^0)$ can greatly simplify their reconstruction. We give such a rule in Proposition 5.2.

In this section we gave a geometric construction of tropical del Pezzo surfaces of degree $d \geq 3$, starting from the points $P_1, \ldots, P_{9-d}$ in the tropical plane $\mathbb{T} \mathbb{P}^2$. The lines and conics in $\mathbb{T} \mathbb{P}^2$ that correspond to the $(-1)$-curves are transformed, by a sequence of open tropical modifications, into the trees that make up the boundary of the del Pezzo surface. Knowing these well-specified modifications of curves ahead of time allows us to carry out a unique sequence of open tropical modifications of surfaces, starting with $\mathbb{R}^2$. In each step, going from right to left in (4.1), we modify the surface along a divisor given by one of the trees.

This gives a geometric construction for the bounded complex in a tropical del Pezzo surface: it is the preimage under (4.1) of the bounded complex in the arrangement in $\mathbb{R}^2$. For instance, Figure 2 is the preimage of the parallelogram and the four triangles in Figure 8.

The same modification approach can be used to construct (the bounded complexes of) any tropical plane in $\mathbb{T} \mathbb{P}^n$ from its tree arrangement. This provides a direct link between the papers [12] and [23]. That link should be useful for readers of the text books [15] and [17].

## 5 Tropical Cubic Surfaces and their 27 Trees

This section is devoted to the combinatorial structure of tropical cubic surfaces. Throughout, $X$ is a smooth del Pezzo surface of degree 3, without Eckhart points, and $X^0$ the very affine surface obtained by removing the 27 lines from $X$. Recall that an Eckhart point is an ordinary triple point in the union of the $(-1)$-curves. Going well beyond the summary statistics of Theorem 1.1, we now offer an in-depth study of the combinatorics of the surface $\text{trop}(X^0)$.

We begin with the construction of $\text{trop}(X^0)$ from six points in $\mathbb{T} \mathbb{P}^2$, as in Section 4. The points $P_5$ and $P_6$ are general in $\mathbb{R}^2 \subset \mathbb{T} \mathbb{P}^2$. The first four points are the coordinate points

$$P_1 = (0 : -\infty : -\infty), \ P_2 = (-\infty : 0 : -\infty), \ P_3 = (-\infty : -\infty : 0), \ P_4 = (0 : 0 : 0).$$

(5.1)

Theorem 4.4 tells us that $\text{trop}(X^0)$ is determined by the locations of $P_5$ and $P_6$ when the points are generically chosen. There are two generic types, namely (aaaa) and (aaab), as
shown in Figures 4 and 5. This raises the question of how the type can be decided from the positions of $P_5$ and $P_6$. To answer that question, we shall use tropical convexity [15, §5.2]. There are five generic types of tropical triangles, depicted here in Figures 11 and 12. The unique 2-cell in such a tropical triangle has either 3, 4, 5 or 6 vertices. Two of these have 4 vertices, but only one type contains a parallelogram. That is the type which gives (aaaa).

**Theorem 5.1.** Suppose that the tropical cubic surface constructed as in Theorem 4.4 has one of the two generic types. Then it has type (aaaa) if and only if the 2-cell in the tropical triangle spanned by $P_4, P_5$ and $P_6$ is a parallelogram. In all other cases, it has type (aaab).

Note that the condition that the six points $P_i$ are in general position is not sufficient to imply that the tropical cubic surface is generic. In some cases, the corresponding point in the Naruki fan $\text{trop}(\mathcal{Y})$ will lie on the boundary of the subdivision induced by the map from $\text{trop}(\mathcal{G})$, as described in Section 3 and below. If so, the tropical cubic surface is degenerate.

**Proof of Theorem 5.1.** The tree arrangements for the two types of generic surfaces consist of distinct combinatorial types, i.e. there is no overlap in Figures 4 and 5. Therefore, when the tropical cubic surface is generic, it is enough to determine the combinatorial type of a single tree. We do this for the conic $G_1$. Given our choices of points (5.1) in $TP^2$, the tropical conic $G_1$ is dual to the Newton polygon with vertices $(0,0), (1,0), (0,1)$, and $(1,1)$. The triangulation has one interior edge, either of slope 1 or of slope $-1$. We claim the following:

The tropical cubic surface $\text{trop}(X)$ has type (aaaa) if and only if the following holds:

1. The bounded edge of the conic $G_1$ has slope $-1$ and contains a marked point $P_j$, or
2. the bounded edge of the conic $G_1$ has slope 1 and contains a marked point $P_j$, and the other two points $P_j, P_k$ lie on opposite sides of the line spanned by the bounded edge.

To show this, we follow the proof of Theorem 4.4. For each configuration of $P_4, P_5, P_6$ on the conic $G_1$, we draw lines with slope 1 through these points. These are the tropical lines $F_{14}, F_{15}, F_{16}$. Each intersects $G_1$ at one further point. These are the images of $E_4, E_5, E_6$ under the tree involution, i.e. the points labeled $F_{14}, F_{15}, F_{16}$ on the tree $G_1$. Together with $E_2, E_3, F_{12}$ and $F_{13}$ lying at infinity of $\mathbb{TP}^2$, we can reconstruct a tree with 10 leaves. Then, we can identify the type of the tree arrangement. We did this for all possible configurations up to symmetry. Some of the results are shown in Figures 9 and 10. The claim follows.

To derive the theorem from the claim, we must consider the tropical convex hull of the points $P_4, P_5, P_6$ in the above cases. As an example, the 2-cells of the tropical triangle corresponding to the trees in Figures 9 and 10 are shown in Figures 11 and 12 respectively. The markings of $G_1$ producing a type (aaaa) tree always give parallelograms. Finally, if the marking of a conic produces a type (aaab) tree then the 2-cell may have 3, 4, 5, or 6 vertices. However, if it has 4 vertices, then it is a trapezoid with only one pair of parallel edges.

We next discuss some relations among the 27 boundary trees of a tropical cubic surface $X$. Any pair of disjoint $(-1)$-curves on $X$ meets exactly five other $(-1)$-curves. Thus, two 10-leaf trees $T$ and $T'$ representing disjoint $(-1)$-curves have exactly five leaf labels in common. Let $t$ and $t'$ denote the 5-leaf trees constructed from $T$ and $T'$ as in the proof of Proposition 2.4. Thus $T$ double-covers $t$, and $T'$ double-covers $t'$. Given a subset $E$ of the leaf labels of a tree $T$, we write $T|_E$ for the subtree of $T$ that is spanned by the leaves labeled with $E$.

**Proposition 5.2.** Let $T$ and $T'$ be the trees corresponding to disjoint $(-1)$-curves on a cubic surface $X$, and $E$ the set of five leaf labels common to $T$ and $T'$. Then $t = T'|_E$ and $t' = T|_E$.

**Proof.** The five lines that meet two disjoint $(-1)$-curves $C$ and $C'$ define five points on $C$ and five tritangent planes containing $C'$. The cross-ratios among the former are equal to the cross-ratios among the latter modulo $C'$, see [18, Section 4]. The proposition follows because the metric trees can be derived from the valuations of all the various cross ratios.

Proposition 5.2 suggests a combinatorial method for recovering the entire arrangement of 27 trees on trop($X^0$) from a single tree $T$. Namely, for any tree $T'$ that is disjoint from
Figure 12: The tropical triangles formed by points on $G_1$ as in Figure 10, giving type (aaaa).

We can recover both $t'$ and $T'|_E$. Moreover, for any of the 10 trees $T_i$ that are disjoint from both $T$ and $T'$, with labels $E_i$ common with $T$, we can determine $T'|_{E_i}$ as well. Then $T'$ is an amalgamation of $t'$, $T'|_E$, and the 10 subtrees $T'|_{E_i}$. This amalgamation process is reminiscent of a tree building algorithm in phylogenetics known as quartet puzzling [9].

We next examine tropical cubic surfaces of non-generic types. These surfaces are obtained from non-generic fibers of the vertical map on the right in (3.1). We use the subdivision of the Naruki fan $\text{trop}(\mathcal{Y}_0)$ described in Lemma 3.3. There are five types of rays in this subdivision. We label them (a), (b), (a$_2$), (a$_3$), (a$_4$). A ray of type (a$_k$) is a positive linear combination of $k$ rays of type (a). The new rays (a$_2$), (a$_3$), (a$_4$) form the barycentric subdivision of an (aaaa) cone. With this, the maximal cones in the subdivided Naruki fan are called (aa$a_2$a$_3$a$_4$) and (aa$a_2$a$_3$b). They are known as the generic types (aaaa) and (aaab) in the previous sections. A list of all 24 cones, up to symmetry, is presented in the first column of Table 1.

The fiber of $\text{trop}(\mathcal{G}_0) \to \text{trop}(\mathcal{Y}_0)$ over any point in the interior of a maximal cone is a tropical cubic surface. However, some special fibers have dimension 3. Such fibers contain infinitely many tropical cubic surfaces, including those with Eckhart points. Removing such Eckhart points is a key issue in [11]. We do this by considering the stable fiber, i.e. the limit of the generic fibers obtained by perturbing the base point by an infinitesimal. Alternatively, the tree arrangement of the stable fiber is found by setting some edge lengths to 0 in Remark 3.5. We computed representatives for all stable fibers. Our results are shown in Table 1.

We explain the two simplest non-trivial cases. The 36 type (a) rays in the Naruki fan are

Figure 13: The bounded complex of the tropical cubic surface of type (a)
in bijection with the 36 positive roots of $E_6$. Figure 13 shows the bounded cells in the stable fiber over the (a) ray corresponding to root $r = d_1 + d_3 + d_5$. It consists of six triangles sharing a common edge. The two shared vertices are labeled by $P$ and $Q$. Recall the identification of the roots of $E_6$ involving $d_7$ with the 27 $(-1)$-curves from (2.10). Then, considering $E_i$, $F_{ij}$ and $G_i$ as roots of $E_6$, exactly 15 of them are orthogonal to $r$. The other 12 roots are

$$E_1, F_{35}; \quad E_3, F_{15}; \quad E_5, F_{13}; \quad F_{24}, G_6; \quad F_{26}, G_4; \quad F_{46}, G_2.$$  

(5.2)

These form a Schlafli double six. The 36 double six configurations on a cubic surface are in bijection with the 36 positive roots of $E_6$. Each of the six pairs forms an $A_2$ subroot system with $d_1 + d_3 + d_5$. The non-shared vertices in the (a) surface are labeled by these pairs.

The 12 rays labeled by (5.2) emanate from $Q$, and the other 15 rays emanate from $P$. Each other vertex has 7 outgoing rays, namely its labels in Figure 13 and the 5 roots orthogonal to both of these. Figure 14 shows the resulting $27 = 12 + 15$ trees at infinity.

The 40 type (b) rays in the Naruki fan are in bijection with the type $A_2^{23}$ subroot systems

Figure 16: The 27 trees on the tropical cubic surface of type (b)
Table 1: All combinatorial types of tropical cubic surfaces

| Type | #cones in moduli | Vertices | Edges | Rays | Triangles | Squares | Flaps | Cones |
|------|------------------|----------|-------|------|-----------|---------|-------|-------|
| (a)  | 36               | 8        | 13    | 69   | 0         | 0       | 0     | 135   |
| (a_2) | 270              | 20       | 37    | 108  | 14        | 4       | 81    | 135   |
| (a_3) | 540              | 37       | 72    | 144  | 24        | 12      | 117   | 135   |
| (a_4) | 1620             | 59       | 118   | 177  | 36        | 24      | 150   | 135   |
| (b)  | 40               | 12       | 21    | 81   | 10        | 0       | 54    | 135   |
| (aa_2) | 540             | 23       | 42    | 114  | 13        | 7       | 87    | 135   |
| (aa_3) | 1620            | 43       | 82    | 156  | 22        | 18      | 129   | 135   |
| (aa_4) | 540             | 68       | 133   | 195  | 33        | 33      | 168   | 135   |
| (a_2a_3) | 1620        | 43       | 82    | 156  | 22        | 18      | 129   | 135   |
| (a_2a_4) | 810           | 71       | 138   | 201  | 32        | 36      | 174   | 135   |
| (a_3a_4) | 540           | 68       | 133   | 195  | 33        | 33      | 168   | 135   |
| (ab)  | 360             | 26       | 48    | 123  | 16        | 7       | 96    | 135   |
| (a_2b) | 1080            | 45       | 86    | 162  | 24        | 18      | 135   | 135   |
| (a_3b) | 1080            | 69       | 135   | 198  | 34        | 33      | 171   | 135   |
| (aa_2a_3) | 3240        | 46       | 87    | 162  | 21        | 21      | 135   | 135   |
| (aa_2a_4) | 1620        | 74       | 143   | 207  | 31        | 39      | 180   | 135   |
| (aa_3a_4) | 1620        | 74       | 143   | 207  | 31        | 39      | 180   | 135   |
| (a_2a_3a_4) | 1620    | 74       | 143   | 207  | 31        | 39      | 180   | 135   |
| (aa_2b) | 2160            | 48       | 91    | 168  | 23        | 21      | 141   | 135   |
| (aa_3b) | 3240            | 75       | 145   | 210  | 32        | 39      | 183   | 135   |
| (a_2a_3b) | 3240        | 75       | 145   | 210  | 32        | 39      | 183   | 135   |
| (aa_2a_3a_4) | 3240    | 77       | 148   | 213  | 30        | 42      | 186   | 135   |
| (aa_2a_3b) | 6480        | 78       | 150   | 216  | 31        | 42      | 189   | 135   |

in \(E_6\). Figure 15 illustrates the stable fiber over a point lying on the ray corresponding to

\[
\begin{align*}
d_1 - d_3, & \quad d_1 + d_2 + d_5, \quad d_2 + d_3 + d_5, \\
2d_2 - d_5, & \quad d_2 + d_4 + d_6, \quad d_4 + d_5 + d_6, \\
2d_4 - d_6, & \quad d_1 + d_3 + d_4, \quad d_1 + d_3 + d_6. 
\end{align*}
\]

(5.3)

This is the union of three type A_2 subroot systems that are pairwise orthogonal. The bounded complex consists of 10 triangles. The central triangle \(P_1P_2P_3\) has 3 other triangles attached to each edge. The 9 pendant vertices are labeled with the roots in (5.3). The 3 vertices in the triangles attached to the same edge are labeled with 3 roots in a type A_2 subroot system.

Each of \(P_1, P_2\) and \(P_3\) is connected with 9 rays, labeled with the roots in \(E_7 \setminus E_6\) that are orthogonal to a type A_2 subroot system in (5.3). Each of the other vertices is connected with 6 rays. The labels of these rays are the roots in \(E_7 \setminus E_6\) that are orthogonal to the label of that vertex but are not orthogonal to the other two vertices in the same group.

All of the 27 trees are isomorphic, as shown in Figure 16. In each tree, the 10 leaves are partitioned into 10 = 4 + 3 + 3, by orthogonality with the type A_2 subroot systems in (5.3). The bounded part of the tree is connected by two flaps to two edges containing the same \(P_i\).
We close this paper with a brief discussion of open questions and future directions. One obvious question is whether our construction can be extended to del Pezzo surfaces of degree $d = 2$ and $d = 1$. In principle, this should be possible, but the complexity of the algebraic and combinatorial computations will be very high. In particular, the analogues of Theorem 4.4 for 7 and 8 points in $\mathbb{T}P^2$ are likely to require rather complicated genericity hypotheses.

For $d = 4$, we were able compute the Naruki fan $\text{tr}op(Y^0)$ without any prior knowledge, by just applying the software $\text{gfan}$ to the 45 trinomials in Proposition 2.1. We believe that the same will work for $d = 3$, and that even the tropical basis property [15, §2.6] will hold:

**Conjecture 5.3.** The 270 trinomial relations listed in Proposition 2.2 form a tropical basis.

This paper did not consider embeddings of del Pezzo surfaces into projective spaces. However, it would be very interesting to study these via the results obtained here. For cubic surfaces in $\mathbb{P}^3$, we should see a shadow of Table 1 in $\mathbb{T}P^3$. Likewise, for complete intersections of two quadrics in $\mathbb{P}^4$, we should see a shadow of Figures 1 and 2 in $\mathbb{T}P^4$. One approach is to start with the following tropical modifications of the ambient spaces $\mathbb{T}P^3$ resp. $\mathbb{T}P^4$. Consider a graded component in (2.1) with $L$ very ample. Let $N + 1$ be the number of monomials in $E_i, F_{ij}, G_k$ that lie in $H^0(X, L)$. The map given by these monomials embeds $X$ into a linear subspace of $\mathbb{P}^N$. The corresponding tropical surfaces in $\mathbb{T}P^N$ should be isomorphic to the tropical del Pezzo surfaces constructed here. In particular, if $L = -K$ is the anticanonical bundle, then the subspace has dimension $d$, and the ambient dimensions are $N = 44$ for $d = 3$, and $N = 39$ for $d = 4$. In the former case, the 45 monomials (like $E_1F_{12}G_2$ or $F_{12}F_{34}F_{56}$) correspond to Eckhart triangles. In the latter case, the 40 monomials (like $E_1E_2F_{12}G$ or $E_1F_{12}F_{13}F_{45}$) are those of degree $(4, 2, 2, 2, 2)$ in the grading (2.5). The tropicalizations of these combinatorial anticanonical embeddings, $X \subset \mathbb{P}^3 \subset \mathbb{P}^{44}$ for $d = 3$ and $X \subset \mathbb{P}^4 \subset \mathbb{P}^{39}$ for $d = 4$, should agree with our surfaces here. This will help in resolving remaining issues surrounding the excess of lines in tropical cubic surfaces. Examples of the superabundance of tropical lines on generic smooth tropical cubic hypersurfaces were first found by Vigeland [27] and these examples were later considered in [6] and [8].

One last consideration concerns cubic surfaces defined over $\mathbb{R}$. A cubic surface equipped with a real structure induces another involution on the 27 metric trees corresponding to real $(-1)$-curves. These trees already come partitioned by combinatorial type, depending on the type of tropical cubic surface. One could ask which trees can result from real lines, and whether the tree arrangement reveals Segre’s partition of real lines on cubic surfaces into hyperbolic and elliptic types [22]. For example, for the (aaaa) and (aaab) types, if the involution on the trees from the real structure is the trivial one, then the trees with combinatorial type occurring exactly three times always correspond to hyperbolic real lines.

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