On Non– Efficiency of Quantum Computer

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Let $E$ be the energy used by the quantum computer to perform the computation, $t_c$ the total computation time including the preparation of an input and the measurement of an output state and $C$ the complexity of the problem defined as a minimal number of logical steps needed to solve it. We advocate as a plausible hypothesis a previously proposed inequality motivated by the Heisenberg energy-time uncertainty principle which has the form $E_c t_c \gg \hbar c$. This hypothesis is supported by the following explicit examples of quantum operations and computations: preparation of an input $n$ q-bit state, two Hamiltonian versions of the Grover’s search algorithm, a model of a ”quantum telephone directory”, a quantum-optical device which can factorize any number and a network used in Shor’s algorithm.

$I. \ \text{INTRODUCTION}$

The standard description of a quantum computer is the following. We have a quantum system with $N = 2^n$ orthogonal states (computational basis) which can store $n = \log_2 N$ bits of information typically realized as a collection of $n$ 2-level subsystems (q-bits).

The process of computation is divided into three stages, first an initial (input) state is prepared, then a quantum algorithm is performed which is realized as a sequence of $\mathcal{G}$ unitary transformation called quantum gates and finally, the output state is measured. The number $\mathcal{G}$ of involved quantum gates is called sometimes a quantum complexity and one assumes, in analogy to classical digital computers, that the physical time needed to achieve a given task is proportional to $\mathcal{G}$. If $\mathcal{G}$ is polynomial in $n$ the algorithm is called efficient. The remarkable result of Shor [1] had an enormous impact on the whole field of quantum information and quantum computing [2]. He constructed a quantum algorithm as a sequence of polynomial in $\log_2 N$ number of unitary transformations which factorizes numbers smaller than $N$ into primes. It is known that the logical complexity $C$ of the problem measured by a number of logical steps needed to solve it grows exponentially with $n = \log_2 N$ for the case of factorization. The possibility of efficient factorization of large numbers by quantum computers must frighten the experts responsible for the safety of information transmission.

The reasoning outlined above possesses several drawbacks. First of all the idea that the physical time of computation $t_c$ is proportional to the complexity $C$

$$t_c \sim C \ \text{or} \ t_c \sim \mathcal{G} \quad (1)$$

is only true for the existing digital computers which are ensembles of controlled bistable elements which correspond to Boolean logical values 0,1 and can literary mimic logical operations. Other theoretically conceivable classical computers like, for example, ballistic computer of Fredkin and Toffoli [3] (hard spheres colliding with each other and with fixed reflective barriers) need not satisfy (1). By rescaling masses, distances and initial velocities we can have in principle for a fixed $N$ arbitrarily short $t_c$ and arbitrarily low energy store needed. Obviously, atomic structure puts limits on the rescaling and classical chaos together with friction make the idea of ballistic computers impractical.

The number of quantum gates $\mathcal{G}$ which form a quantum algorithm is not unique and does not determine the real physical time of computation. The very idea of quantum gate is not natural in quantum mechanical context. Quantum evolution is continuous in time and governed by a (possibly time-dependent) Hamiltonian which describes the energy of a system. In particular by increasing the energy level spacing of the computer’s Hamiltonian we can speed up its time evolution. Hence the physical efficiency of computation should be given in terms of the product of a characteristic energy and time (”action parameter”) rather than the computation time alone. A dynamical character and finite duration of the input state preparation procedure and the output state measurement process are also important. As we shall see studying concrete examples there is no clear separation between the preparation-measurement processes and the action of quantum algorithm.

As many authors noticed the decoherence effects due to the interaction with an environment are the main practical obstacles for the operation of quantum computer. As the decoherence typically grows in a nonlinear way with the energy level spacing this can produce an optimal splitting of the action parameter into time and energy. This problem is studied in details in [4] and will be not discussed here.

$II. \ \text{HEISENBERG ENERGY-TIME RELATIONS}$

One of the interpretations of the Heisenberg energy-time uncertainty relation

$$\Delta E \cdot \Delta t \geq \hbar \quad (2)$$
is that a quantum state with spread in energy $\Delta E$ takes
time at least $\Delta t = \pi h / 2 \Delta E$ to evolve to an orthogonal
and hence distinguishable state [5]. Hence, it is quite
natural to investigate the
quantum mechanical limitations on the action parameter $E_c \cdot t_c$ where $E_c$ is an energy store needed for the
quantum computation and $t_c$ is a total computation time
including the preparation of an input state and
the measurement of an output. Following [6] we can propose the inequality of the form

$$E_c \cdot t_c \gg \hbar C_q$$  \hspace{1cm} (3)

where $C_q$ is a not yet defined ”quantum complexity” of
the problem. The ”much larger” symbol in (3) takes
into account the probabilistic character of quantum algorithms which should run a certain number of times to
achieve a given level of confidence. Although is it not
formulated explicitly in

the form of the inequality (3) the general believe among
the experts in quantum computation is that the quantum complexity is equal to the minimal number of quantum
gates

$$C_q = \mathcal{G}.$$  \hspace{1cm} (4)

In [6] the following hypothesis supported by a single ex-
ample has been proposed

$$C_q \equiv C$$  \hspace{1cm} (5)

what means essentially that there is no quantum complexity. In particular, if the problem needs an exponen-
tial in the input bit size number of logical steps the quan-
tum computer needs also an exponential time or an expo-
ential energy store. It is still better than for the existing
digital computers which need both but nevertheless the quantum computation is not practically efficient.

### III. EXAMPLES

In order to advocate the hypothesis (5) we discuss sev-
eral examples of quantum algorithms or their essential parts.

#### A. State preparation

Consider $n$ two level system ($q$-bits) with a standard computational basis which consists of products of two $q$
bit states $|0\rangle$ and $|1\rangle$. To prepare an arbitrary input state from this basis starting from a certain fixed initial
state we need to rotate some of the 2-level system in a
fixed time $t_c$ to the orthogonal states. Therefore according
to the presented interpretation of (1) we need on the
average the energy $E_c = (n/2) \pi \hbar / 2 t_c$ what gives in this case

$$C_q \approx n.$$  \hspace{1cm} (6)

The similar result can be obtained for the quantum measurement of an output state. As writing or reading of $n$
bit messages involves $n$ logical steps we have here $C_q = C$

#### B. Grover’s search algorithms

We shall analyze two different Hamiltonian realizations of the Grover’s search algorithm [7]. In the original for-
mulation every element of the database is represented by
a state of the standard computational basis of the $n$
$q$-bit system. One of this states – a searched one– is denoted by $|x\rangle$ and we have also a certain standard initial state
$|in\rangle$ which is usually a uniform superposition of computa-
tional basis \{ $|\phi_j\rangle; j = 1, 2, ..., N = 2^n$ \}

$$|in\rangle = (N)^{-1/2} \sum_{j=1}^{N} |\phi_j\rangle, \quad <s|in\rangle = N^{-1/2}.$$  \hspace{1cm} (7)

These two states are hidden in the dynamics of the sys-
tem described either by the repeated unitary transforma-
tions or the continuous time Hamiltonian evolution. The
first Hamiltonian proposed in [8] has form

$$H_1 = E(|x\rangle\langle x| + |in\rangle\langle in|)$$  \hspace{1cm} (8)

where $E$ is an energy scale. The Hamiltonian acts es-
sentially on the two dimensional subspace of the Hilbert
space and the corresponding energy difference is equal to

$$\hbar \omega_N = EN^{-1/2}.$$  \hspace{1cm} (9)

One can easily compute that we need a time

$$t_N = \pi N^{1/2} / 2E$$  \hspace{1cm} (10)

to reach from the initial state $|in\rangle$ to the searched one $|x\rangle$. So the quantum complexity of this stage of searching is of the order $\hbar \omega_N t_N \approx 1$. Adding preparation and measurement processes we obtain

$$C_q \approx n.$$  \hspace{1cm} (11)

The second Hamiltonian used with the different rescaling in [9] reads

$$H_2 = iE(|x\rangle\langle x| - |in\rangle\langle in|).$$  \hspace{1cm} (12)

Again the problem is essentially two dimensional with the energy difference

$$\hbar \omega_N = 2E + o(1/N).$$  \hspace{1cm} (12)

The time needed to reach the state $|x\rangle$ can be estimated by

$$t_N = \pi / 4E + o(1/N).$$  \hspace{1cm} (13)

Here again adding state preparation and measurement we obtain (11).
In the literature on the Grover’s algorithm it is claimed that its quantum complexity is $\sqrt{N}$ which is compared with the classical complexity of the problem claimed to be equal to $N/2$. Both statements are incorrect. We have just computed the quantum complexity of the problem equal to $n = \log_2 N$. The classical analogon of Grover’s search algorithm is not finding an item in a randomly ordered phone book but rather a search for a one heavier ball among $N$ otherwise indistinguishable ones. The later problem can be solved in $\log_2 N$ steps.

C. Quantum telephone directory

We discuss now a true quantum analog of a random telephone directory. It is again a $n$ q-bit system with a computational basis $\{\phi_j; j = 1, 2, \ldots, N = 2^n\}$. We fix an initial state to be $\phi_1$ and propose the following time dependent Hamiltonian

$$H(t) = H_0 + V \cos(\Omega t)$$

where

$$H_0 = \sum_{j=1}^{N} E_j |\phi_j><\phi_j|$$

$V$ is a (randomly chosen) weak perturbation and $\Omega$ is a tunable frequency. The energies $E_j \geq 0$ are not degenerated and provide labels for the states $\phi_j$ (we put $E_1 = 0$). To find a state labeled by $E_j$ we tune the frequency to the value $\Omega_j = E_j/h$. The time dependent first-order perturbation calculus [10] gives us the probability of excitation of the state $\phi_k$

$$p_k(t) = 2|<\phi_1|V|\phi_k>|^2 \frac{\sin^2 \left\{ \frac{1}{2} (E_k - E_j)t/h \right\}}{(E_k - E_j)^2}.$$  

It follows from the formula (16) that we have to wait for a time at least of the order

$$t_j \approx \frac{h}{|E_j - E_k|}$$

to be sure that the searched state $\phi_j$ has been prepared with a much larger probability than the other neighboring state $\phi_k$. Therefore, on the average we obtain the computation time

$$t_c \approx hN/E_{max}$$

where $E_{max} = \max\{E_j\}$. Then as $E_c \approx E_{max}$ we see again that the quantum complexity coincides with the classical one.

D. Quantum device factorizing numbers

This model has been introduced in [6] but we briefly discuss it again for the sake of completeness. In fact this model is very similar to the previous one.

A resonant cavity supports radiation modes with the frequencies being the logarithms of prime numbers times a fixed frequency unit $\omega$

$$\omega_q = \omega \log q \ , \ q = 2, 3, 5, 7, 11, 13, \ldots$$

The second quantization Hamiltonian of the electromagnetic field

$$H = \hbar \omega \sum_q (\log q) \ a_q^+ a_q$$

has nondegenerated eigenvalues being proportional to the logarithms of all natural numbers

$$H \psi_N = E_N \psi_N \ , \ E_N = \hbar \omega \log N \ , \ N = 1, 2, 3, \ldots$$

The structure of $\psi_N$ reveals the factorization of $N$ into prime numbers

$$\psi_N \sim (a_{q_1}^+)^{m_1} (a_{q_2}^+)^{m_2} \ldots (a_{q_r}^+)^{m_r} \psi_1$$

where

$$N = (q_1)^{m_1} (q_2)^{m_2} \ldots (q_r)^{m_r}$$

and $\psi_1$ is a vacuum state. The eq. (22) means that we have $m_1$ photons of the frequency $\omega \log q_1$, $m_2$ photons of the frequency $\omega \log q_2$, ..., and $m_r$ photons of the frequency $\omega \log q_r$. Therefore, transferring a given energy portion $\hbar \omega \log N$ to the empty cavity and then opening the cavity and counting photons in different modes we obtain the factorization of $N$. It can be done similarly to the previous example perturbing the system periodically in time with a tunable frequency $\Omega$ and selecting $\Omega = \omega \log N$.

As the energy level spacing around $E_N = \hbar \omega \log N$ is $\delta E_N \approx \hbar \omega/N$ it follows from the analogue of the formula (16) that we have to wait for a time at least of the order

$$t_c \approx N\omega^{-1}$$

to select a proper state $\psi_N$. The energy used equals $E_c = \hbar \omega \log N$ and once again the quantum complexity essentially coincides with the classical one.

E. Shor’s algorithm

The Shor’s factorization algorithm is rather complicated but for our purposes we need only a part of it – the so-called phase shift computation. We follow here the simple presentation in [11]. The phase shift computation consists in applying to the initial state of a $q$ bit system

$$|in> = 2^{-n/2}(|0> + |1>) \otimes \cdots \otimes (|0> + |1>)$$

(25)
a $n$-gates unitary operation

$$U = U_2^0 \otimes U_2^1 \otimes \cdots \otimes U_2^n$$  \hspace{1cm} (26)

where

$$U \left| 0 > > |0 > , \ U \left| 1 > = e^{-i\alpha} |1 > , \ \alpha \in [0, 2\pi) \right. \hspace{1cm} (27)$$

The unitary $U$ can be realized as $\exp\{-iHt_n/\hbar\}$ with the following Hamiltonian

$$H = \hbar \omega \sum_{k=0}^{n-1} 2^k \langle 1 > < 1 | \rangle_k$$  \hspace{1cm} (28)

acting for a time $t_n = \alpha/\omega$. The averaged energy in the state $|in >$ grows exponentially with $n$ and is given by

$$E_n = \langle in | H | in > = \left( \frac{\hbar \omega}{2} \right) \sum_{k=0}^{n-1} 2^k = \hbar \omega (2^n - 1/2).$$  \hspace{1cm} (29)

For the total factorization procedure $t_c >> t_n$ and $E_c >> E_n$. Taking for $\alpha$ its average value $\pi$ we have $E_c t_c >> \hbar 2^n = \hbar N$ in agreement with the hypothesis (5).

IV. CONCLUSION

Although the general proof would be very desirable the analysis of the presented examples, in particular the powerful Shor’s algorithm, provides a strong evidence for the Heisenberg-like bound (3)(5) on the efficiency of quantum computations.

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[1] P. Shor, in Proceedings of the 35th Annual Symposium on Foundations of Computer Science, edited by S. Goldwasser (IEEE Press, New York, 1994), pp. 56–65.
[2] H.-K. Lo, S. Popescu, and T. Spiller (eds.) Introduction to Quantum Computation and Information, (World Scientific, Singapore, 1998).
[3] C. H. Bennet, Int.J.Theor.Phys. 21, 905 (1982)
[4] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki (in preparation)
[5] Y. Aharonov and D. Bohm, Phys.Rev. 122, 1649 (1961)
[6] R. Alicki, e-print quant-ph/0006018
[7] L.K. Grover, Phys.Rev.Lett. 79, 325 (1997)
[8] E. Farhi and S. Gutmann, Phys.Rev. A 57, 2403 (1998)
[9] M. Mussinger, A. Delgado and G. Alber, e-print quant-ph/0003141
[10] A. Messiah, Quantum Mechanics , (North-Holland, Amsterdam, 1962).
[11] R. Cleve et.al., e-print quant/9903061