The Apéry Numbers As a Stieltjes Moment Sequence

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The Apéry sequence \[ A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \cdots. \]

From the reference (or a CAS\footnote{SumTools\{Hypergeometric\}[Zeilberger](binomial(n,k)^2*binomial(n+k,k)^2,n,k,En);}) we find that it satisfies the recurrence
\[
(n + 1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)A_n + n^3 A_{n-1} = 0, \quad (1)
\]
\[
A_0 = 1, \quad A_1 = 5.
\]

We will show that the sequence \((A_n)\) is a Stieltjes moment sequence. In fact:

**Theorem 1.** There is \(c > 0\) and a positive Lebesgue integrable function \(\varphi\) such that
\[
A_n = \int_0^c x^n \varphi(x) \, dx
\]
for \(n = 0, 1, 2, \cdots\).

**Definition 2.** We say \(\varphi\) is the moment density function for \((A_n)\).

**Notes**
I have tried to make the argument as short as possible. This means many asides and variations have been removed.

Some of the proofs may be done using a computer algebra system (CAS). I used Maple 2015. These are the sort of thing that—until 1980 or later—would have been done by paper-and-pencil computation. I have added some of the Maple as footnotes.

This result arose from a question asked by Alan Sokal. It was posted on the MathOverflow discussion board \footnote{arXiv:2005.10733v1 [math.CA] 21 May 2020}. Pietro Majer provided the idea to use the differential equation.
Notation 3. We will use these values.

\[
\begin{align*}
\tau &= 1 + \sqrt{2} \approx 2.4142 \\
c &= \tau^4 = 17 + 12\sqrt{2} \approx 33.9705 \\
c_0 &= \tau^{-4} = \frac{1}{c} = 34 - c = 17 - 12\sqrt{2} \approx 0.0294
\end{align*}
\]

The Differential Equation

We proceed with a discussion of this third-order holonomic Fuchsian ODE:

\[
x^2(x^2 - 34x + 1)u'''(x) + 3x(2x^2 - 51x + 1)u''(x) \\
+(7x^2 - 112x + 1)u'(x) + (x - 5)u(x) = 0.
\] (DE3)

We consider \(x\) a complex variable, and sometimes consider solutions in the complex plane.

Differential equation [DE3] has four singularities: \(\infty, 0, c_0, c\). They are all regular singular points. Series solutions exist adjacent to each of them. From the Frobenius “series solution” method\(^2\)\(^3\) we may describe these series solutions:

**Proposition 4.** The general solution of [DE3] near the complex singular point \(\infty\) has the form

\[
A \left( \frac{1}{x} + o(x^{-1}) \right) + B \left( \frac{\log x}{x} + o(x^{-1}) \right) + C \left( \frac{(\log x)^2}{x} + o(x^{-1}) \right)
\]

as \(x \to \infty\), for complex constants \(A, B, C\). The general solution of [DE3] near the singular point \(0\) has the form

\[
A (1 + o(1)) + B (\log x + o(1)) + C ((\log x)^2 + o(1))
\]

as \(x \to 0\), for complex constants \(A, B, C\). The general solution of [DE3] near the singular point \(c_0\) has the form

\[
A \left( 1 - \frac{240 + 169\sqrt{2}}{48} (x - c_0) + O(|x - c_0|^2) \right)
\]

\[
+ B \left( (x - c_0)^{1/2} + O(|x - c_0|^{3/2}) \right) + C \left( (x - c_0) + O(|x - c_0|^2) \right)
\]

as \(x \to c_0\), for complex constants \(A, B, C\). The general solution of [DE3] near the singular point \(c\) has the form

\[
A \left( 1 - \frac{240 - 169\sqrt{2}}{48} (x - c) + O(|x - c|^2) \right)
\]

\[\text{dsolve(DE3, u(x), series, x=c);}\]
\[ + B \left( (x-c)^{1/2} + O(|x-c|^{3/2}) \right) + C \left( (x-c) + O(|x-c|^2) \right) \]

as \(x \to c\), for complex constants \(A, B, C\).

**Corollary 5.** If \(u(x)\) is any solution of (DE3) on \((0,c_0)\) or on \((c_0,c)\), then \(u(x)\) has at worst logarithmic singularities. So \(u(x)\) is (absolutely, Lebesgue) integrable.

**Notation 6.** Four particular solutions of (DE3) will be named for use here:

- Solution \(u_\infty(x) = \frac{1}{x} + o(x^{-1})\) as \(x \to \infty\), defined in the complex plane cut on the real axis interval \([0,c]\).
- Solution \(u_0(x) = 1 + o(1)\) as \(x \to 0^+\), defined for \(0 < x < c_0\).
- Solution \(v_0(x) = \log x + o(1)\) as \(x \to 0^+\), defined for \(0 < x < c_0\).
- Solution \(v_2(x) = (c-x)^{1/2} + O(|x-c|^{3/2})\) as \(x \to c^-\), defined for \(c_0 < x < c\).

**Proposition 7.** The Maclaurin series for \(u_0(x)\) is the generating function for the Apéry sequence:

\[ u_0(x) = \sum_{n=0}^{\infty} A_n x^n, \quad |x| < c_0. \]

**Proof.** This may be checked by your CAS. The recurrence \(4\) converted to a differential equation \(5\) yields (DE3). Of course the radius of convergence extends to the nearest singularity at \(c_0\).

**Corollary 8.** \(u_0(x) > 0\) for \(0 < x < c_0\).

Determining the signs of \(v_0\) and \(v_2\) will be more difficult.

**Proposition 9.** The Laurent coefficients for \(u_\infty(z)\) are the Apéry numbers:

\[ u_\infty(z) = \sum_{n=0}^{\infty} \frac{A_n}{z^{n+1}}, \quad |z| > c. \]

**Proof.** Check that if \(u(x)\) is a solution of (DE3), then \(w(z) = u(1/z)/z\) is also a solution of (DE3). Matching the boundary conditions, we get

\[ u_\infty(z) = \frac{1}{z} u_0 \left( \frac{1}{z} \right). \]

Apply Prop. 7.

**Note:** In general, for other similar sequences that can be handled in this same way:

(a) the generating function for the sequence, and

(b) the moment density function for the sequence satisfy _different_ differential equations.

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4 Rec:=((n+1)^3*Q(n+1)-(36*n^3+51*n^2+27*n+5)*Q(n)+n^3*Q(n-1));
5 gfun[rectodiffeq]({Rec,Q(0)=1,Q(1)=5},Q(n),u(x));
Figure 1: $u_\infty(x)$

Figure 2: $u_0(x)$ and $v_0(x)$

Figure 3: $v_2(x)$
The Function $\varphi$

Series solution $u_\infty$ of (DE3) is meromorphic and single-valued near $\infty$. It continues analytically to the complex plane with a cut on the interval $[0, c]$ of the real axis. We will still use the notation $u_\infty$ for that continuation. Since the Laurent coefficients are all real, we have

$$u_\infty(z) = u_\infty(\overline{z})$$

near $\infty$, and therefore on the whole domain. In particular, $u_\infty(z)$ is real for $z$ on the real axis (except the cut, of course). Define upper and lower values on the cut $0 < x < c$:

$$u_\infty(x + i\delta) = \lim_{\delta \to 0^+} u_\infty(x + i\delta), \quad u_\infty(x - i\delta) = \lim_{\delta \to 0^+} u_\infty(x - i\delta).$$

Then from (2) we have

$$u_\infty(x - i\delta) = u_\infty(x + i\delta), \quad 0 < x < c.$$  \hspace{1cm} (3)

**Notation 10.**

$$\varphi(x) = \frac{1}{2\pi i} \left( u_\infty(x - i0) - u_\infty(x + i0) \right).$$

Function $u_\infty$ in the upper half plane extends analytically to a solution in a neighborhood of $(0, c_0)$, and similarly $u_\infty$ in the lower half plane. Thus $\varphi(x)$ restricted to $(0, c_0)$ is a solution of (DE3), since it is a linear combination of solutions. In the same way, $\varphi(x)$ restricted to $(c_0, c)$ is a solution of (DE3).

See Figure 4; an enlargement shows the behavior near the singular point $c_0$. We will see that $\varphi$ has square root asymptotics near the right endpoint $c$ (Prop. 25) and logarithmic asymptotics near the left endpoint 0 (Prop. 26).

![Figure 4: Moment density function $\varphi(x)$](image)

**Proposition 11.** The Apéry numbers satisfy

$$A_k = \int_0^c x^k \varphi(x) \, dx, \quad k = 0, 1, 2, \cdots.$$
Proof. Fix a nonnegative integer \( k \). For \( \delta > 0 \), let \( \Gamma_{\delta} \) be the contour in the complex plane at distance \( \delta \) from \([0, c]\), as in Figure 5 (Two line segments and two semicircles; traced counterclockwise.) Now \( u_{\infty} \) has at worst logarithmic singularities, so we have this limit:

\[
\lim_{\delta \to 0^+} \oint_{\Gamma_{\delta}} z^k u_{\infty}(z) \, dz = \int_0^c x^k (u_{\infty}(x - i0) - u_{\infty}(x + i0)) \, dx.
\]

On the other hand, \( z^k u_{\infty}(z) \) is analytic on and outside the contour \( \Gamma_{\delta} \), except at \( \infty \) where it has an isolated singularity with residue \( A_k \). Therefore

\[
\oint_{\Gamma_{\delta}} z^k u_{\infty}(z) \, dz = 2\pi i A_k.
\]

Thus

\[
A_k = \int_0^c \frac{x^k}{2\pi i} (u_{\infty}(x - i0) - u_{\infty}(x + i0)) \, dx.
\]

Figure 5: Contour \( \Gamma_{\delta} \)

What remains to be proved: \( \varphi \) is nonnegative on \((0, c)\) (Cor. 27). From (3) we know that \( \varphi(x) \) is real on \((0, c)\).

Heun General Functions

Some of the basic solutions in Notation 6 may be represented in terms of Heun functions. The Heun functions are described in [7, 8, 9].
Definition 12. Let complex parameters $a, q, \alpha, \beta, \gamma, \delta, \varepsilon$ be given satisfying $a \neq 0$, \(\alpha + \beta + 1 = \gamma + \delta + \varepsilon\), \(\delta \neq 0\), and \(\gamma \neq 0, -1, -2, \ldots\). Define the Heun general function

$$H_n\left(\frac{a}{q} \left| \begin{array}{c} \alpha, \\ \gamma, \\ \delta \end{array} \right| z \right) = \sum_{n=0}^{\infty} p_n z^n,$$

where the Maclaurin coefficients satisfy initial conditions

$$p_0 = 1, \quad p_1 = \frac{q}{a\gamma},$$

and recurrence

$$R_n p_{n+1} - (q + Q_n) p_n + P_n p_{n-1} = 0,$$

with

$$R_n = a(n+1)(n+\gamma),$$

$$Q_n = n((n-1+\gamma)(1+a) + a\delta + \varepsilon),$$

$$P_n = (n-1+\alpha)(n-1+\beta).$$

This function satisfies the Heun general differential equation

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right) w'(z) + \frac{\alpha \beta z - q}{z(z-1)(z-a)} w(z) = 0.$$  

(Consult the references, or use your CAS to go from the recurrence to the differential equation.) This DE has singularities at \(\infty, 0, 1, a\); all regular singular points. Convergence of the series extends to the nearest singularity, so the radius of convergence in (4) is \(\min\{1, |a|\}\).

Proposition 13. Within the radius of convergence:

$$u_0(x) = H_n\left(\frac{a_2}{q_4} \left| \begin{array}{c} 1/2, \\ 1 \end{array} \right| \frac{1}{c} \right)^2$$

$$v_2(x) = \frac{(x - c_0)(c - x)^{1/2}}{c - c_0} \frac{H_n\left(\frac{a_1}{q_1} \left| \begin{array}{c} 3/2, \\ 3/2 \end{array} \right| 1 - c_0 x \right)}{H_n\left(\frac{a_1}{q_2} \left| \begin{array}{c} 1/2, \\ 1 \end{array} \right| 1 - c_0 x \right)} 
\cdot H_n\left(\frac{a_1}{q_2} \left| \begin{array}{c} 1/2, \\ 1 \end{array} \right| 1 - c_0 x \right),$$

$$u_\infty(z) = \frac{1}{z} H_n\left(\frac{a_2}{q_4} \left| \begin{array}{c} 1/2, \\ 1 \end{array} \right| \frac{c}{z} \right)^2,$$

where

$$a_1 = 1 - c_0^2 = -576 + 408\sqrt{2} \approx 0.9991$$

$$a_2 = c^2 = 577 + 408\sqrt{2} \approx 1153.9991$$

\footnote{HeunG(a,q,alpha,beta,gamma,delta,z)}
\[ q_1 = -\frac{1317}{4} + 234\sqrt{2} \approx 1.676 \]
\[ q_2 = \tau^{-1}(1 + c_0) = -42 + 30\sqrt{2} \approx 0.4264 \]
\[ q_4 = \frac{5c}{2} = \frac{85}{2} + 30\sqrt{2} \approx 84.93 \]

**Proof.** In each case verify that it satisfies the differential equation\(^7\) and boundary properties\(^8\) that specify the solution. \(\square\)

### All Coefficients Positive

In some cases we can determine that all Maclaurin coefficients of a Heun general function
\[ H_n \left( \begin{array}{c|cc} a & \alpha, \beta \\ q & \gamma \\ \end{array} ; z \right) \]
are positive. When that is true, then in particular this function will be positive and increasing and convex on \((0, R)\) where \(R = \min\{1, |a|\}\) is the radius of convergence.

**Lemma 14.** All Maclaurin coefficients are positive in
\[ H_n \left( \begin{array}{c|cc} a_1 & 3/2, 3/2 \\ q_1 & 1 \\ \end{array} ; z \right), \]
where \(a_1 = -576 + 408\sqrt{2}\) and \(q_1 = -\frac{1317}{4} + 234\sqrt{2}\).

**Proof.** Let \(p_n\) be the Maclaurin coefficients. Then
\[ R_n p_{n+1} - (q_1 + Q_n)p_n + P_n p_{n-1} = 0, \]
with
\[ R_n = a_1 (n+1)(n+\frac{3}{2}) \]
\[ Q_n = n((n+\frac{3}{2})(1+a_1)+a_1+\frac{3}{2}) \]
\[ P_n = (n+\frac{1}{2})^2. \]

Write \(r_n = p_n/p_{n-1}\) and rearrange:
\[ r_{n+1} = \frac{q_1 + Q_n}{R_n} - \frac{P_n}{R_n} \frac{1}{r_n}. \]
Recall that \(|a_1| < 1\); we expect \(r_n \to 1/a_1\). We claim: if
\[ n \geq 45 \quad \text{and} \quad 1 - \frac{1}{10n} < r_n < \frac{1}{a_1}, \]
\[ \text{subs(u(x)=v2,DE3): simplify(%)}; \]
\[ \text{MultiSeries[series](v2,x=c,2)}; \]
then also
\[
1 - \frac{1}{10(n + 1)} < r_{n+1} < \frac{1}{a_1},
\]
Once the claim is proved, all that remains is checking that \( p_0, \ldots, p_{45} \) are positive, and
\[
1 - \frac{1}{450} < r_{45} < \frac{1}{a_1}.
\]
By induction we conclude that \( r_n > 0 \) for all \( n \geq 45 \). So \( p_n \) with \( n > 45 \) is a product of positive numbers
\[p_{45} r_{46} r_{47} r_{48} \cdots r_n,\]
so \( p_n > 0 \).

Proof of the claim. Since
\[
r \mapsto \frac{q_1 + Q_n}{R_n} - \frac{P_n}{R_n} \frac{1}{r}
\]
is an increasing function, we need only check
\[
1 - \frac{1}{10(n + 1)} < \frac{q_1 + Q_n}{R_n} - \frac{P_n}{R_n} \frac{1}{a_1} < \frac{1}{a_1}
\]
and
\[
1 - \frac{1}{10(n + 1)} < \frac{q_1 + Q_n}{R_n} - \frac{P_n}{R_n} \frac{1 - \frac{1}{10n}}{r_n} < \frac{1}{a_1}
\]
where \( n \geq 45 \). Your CAS can be used for this. \(\Box\)

A warning for the computations. If you do this using 20-digit arithmetic—as I did at first—you may erroneously conclude that it is false. You may see negative coefficients. With exact arithmetic, we find that \( r_{45} \) involves integers with more than 100 digits. To compare \( \sqrt{2} \) to a rational number with 100-digit numerator and denominator, there are two methods: we can square those 100-digit numbers, or we can use a decimal value of \( \sqrt{2} \) accurate to more than 100 places. Of course a modern CAS can do either.

Lemma 15. All Maclaurin coefficients are positive in
\[
H_n \left( \begin{array}{c} a_1 \\ q_2 \end{array} \right) \left( \begin{array}{cc} 1/2 & \frac{1}{z} \\ 1 & 1 \end{array} \right),
\]
where \( a_1 = -576 + 408\sqrt{2} \) and \( q_2 = -42 + 30\sqrt{2} \).

Proof. The proof is similar to Lemma 14. Let \( p_n \) be the coefficients, and \( r_n = p_n/p_{n-1} \). Then
\[
r_{n+1} = \frac{q_2 + Q_n}{R_n} - \frac{P_n}{R_n} \frac{1}{r_n},
\]
\[
\text{where } a_1 = -576 + 408\sqrt{2} \text{ and } q_2 = -42 + 30\sqrt{2}.
\]
with

\[ R_n = a_1(n + 1)(n + \frac{1}{2}), \]
\[ Q_n = n((n - \frac{1}{2})(1 + a_1) + a_1 + \frac{3}{2}), \]
\[ P_n = n^2. \]

We claim: If

\[ n \geq 18 \quad \text{and} \quad 1 - \frac{1}{4n} < r_n < \frac{1}{a_1}, \]

then also

\[ 1 - \frac{1}{4(n + 1)} < r_{n+1} < \frac{1}{a_1}. \]

The remainder of the proof is similar to Lemma 14.

\[ \Box \]

**Proposition 16.** \( v_2(x) > 0 \) for \( c_0 < x < c \).

**Proof.** By Lemma 14, all Maclaurin coefficients of

\[ H_n \left( \begin{array}{c|ccc} a_1 & 3/2, & 3/2 \; 3/2 ; & 1 \; z \\ q_1 & 3/2 ; & 1 \end{array} \right) \]

are positive. It has radius of convergence \( a_1 = 1 - c_0^2 \), so

\[ H_n \left( \begin{array}{c|ccc} a_1 & 3/2, & 3/2 \; 3/2 ; & 1 \; 1 - c_0 x \\ q_1 & 3/2 ; & 1 \end{array} \right) > 0 \]

for all \( x \) with \( c_0 < x < c \). By Lemma 15, all Maclaurin coefficients of

\[ H_n \left( \begin{array}{c|ccc} a_1 & 1, & 1 \; 1 \; z \\ q_2 & 1/2 ; & 1 \end{array} \right) \]

are positive. Again,

\[ H_n \left( \begin{array}{c|ccc} a_1 & 1, & 1 \; 1 \; 1 - c_0 x \\ q_2 & 1/2 ; & 1 \end{array} \right) > 0 \]

for all \( x \) with \( c_0 < x < c \). Also

\[ \frac{(x - c_0)(c - x)^{1/2}}{c - c_0} \]

is positive on \((c_0, c)\). The product of three positive factors is \( v_2(x) \) on \((c_0, c)\), so \( v_2(x) > 0 \). \( \Box \)
Hypergeometric Function

Some Heun functions can be expressed in terms of hypergeometric $2F_1$ functions [11 Chap. 2–3]. Here, we will use only one of them.

**Definition 17.** $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} \frac{z^n}{27^n}$

**Lemma 18.** (a) $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ has radius of convergence 1. (b) For $0 < z < 1$, we have $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right) > 1$. (c) As $\delta \to 0^+$,

$$2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \delta\right) = -\frac{\sqrt{3}}{2\pi} \log \delta + \frac{3\sqrt{3} \log 3}{2\pi} + o(1).$$

**Proof.** (a) Ratio test.
(b) All Maclaurin coefficients are positive, and the constant term is 1.
(c) Due to Gauss (or perhaps Goursat?), see [11 Thm. 2.1.3],

$$2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \delta\right) = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \left[ \log \frac{1}{3} - 2\gamma - \psi\left(\frac{1}{3}\right) - \psi\left(\frac{2}{3}\right) \right] + o(1)$$

$$= -\frac{\sqrt{3}}{2\pi} \log \delta + \frac{3\sqrt{3} \log 3}{2\pi} + o(1).$$

Here $\gamma$ is Euler’s constant and $\psi$ is the digamma function. Use [11 Thm. 1.2.7] to evaluate the digamma of a rational number.

**Lemma 19.** Let the degree 1 Taylor polynomial for $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ at $z_0 = 1/(2^{3/2})$ be $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right) = S_0 + S_1 \cdot (z - z_0) + o(|z - z_0|)$ as $z \to z_0$. Then

$$S_0 \cdot (3S_1 + \sqrt{2}S_0) = \frac{3^{3/2}2^{1/2}}{\pi}.$$
\[
\lambda_2(x) = \frac{x^3 + 30x^2 - 24x + 1 + (x^2 - 7x + 1)\sqrt{x^2 - 34x + 1}}{2(x + 1)^3}
\]

(See Figures 6 and 7)

**Lemma 21.** For \(0 < x < c_0\), we have \(\mu(x) > 1\), \(\mu_2(x) > 1\), \(0 < \lambda(x) < 1\), and \(0 < \lambda_2(x) < 1\).

**Proof.** Elementary inequalities.

**Lemma 22.** As \(x \to 0^+\),

\[
\mu(x) \binom{2F1}{1/3, 2/3; 1; \lambda(x)} = 1 + \frac{5}{2}x + O(x^2),
\]

\[
\mu_2(x) \binom{2F1}{1/3, 2/3; 1; \lambda_2(x)} = \frac{-\sqrt{3}}{\pi \sqrt{2}} \log x + o(1).
\]

The second one indeed has constant term zero.

**Proof.** Compute (as \(z \to 0\) and \(x \to 0\)):

\[
\mu(x) = 1 + \frac{5}{2}x + O(x^2)
\]

\[
\lambda(x) = 27x^2 + O(x^3)
\]

\[
\binom{2F1}{1/3, 2/3; 1; z} = 1 + \frac{2}{9}z + O(z^2)
\]

\[
\binom{2F1}{1/3, 2/3; 1; \lambda(x)} = 1 + 6x^2 + O(x^3)
\]
Figure 7: $\lambda$ (bottom) and $\lambda_2$ (top)

\[
\mu(x) \ {}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right) = 1 + \frac{5}{2}x + O(x^2)
\]

For the second one, we apply Lemma 18(c). As $x \to 0$:

\[
\mu_2(x) = \sqrt{2} - \frac{7}{\sqrt{2}}x + O(x^2)
\]

\[
\lambda_2(x) = 1 - 27x + O(x^2)
\]

\[
{}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda_2(x)\right) = -\frac{\sqrt{3}}{2\pi} \log(27x) + \frac{3\sqrt{3} \log 3}{2\pi} + o(1)
\]

\[
= -\frac{\sqrt{3}}{2\pi} \log 27 + \frac{3\sqrt{3} \log 3}{2\pi} + o(1)
\]

\[
\mu_2(x) \ {}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda_2(x)\right) = -\frac{\sqrt{3}}{\sqrt{2\pi}} \log x + o(1)
\]

Proposition 23.

\[
u_0(x) = \mu(x)^2 \ {}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right)^2,
\]

\[
v_0(x) = -\frac{2\pi}{\sqrt{3}(x+1)} \ {}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right) \ {}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda_2(x)\right),
\]

\[
u_{\infty}(z) = \frac{1}{z^\mu} \ {}_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda\left(\frac{1}{z}\right)\right)^2.
\]
Proof. Note: $\mu(x)\mu_2(x) = \sqrt{2}/(x + 1)$. Verify that these expressions satisfy \[DE3\] as usual. Then verify the asymptotics using Lemma 22.

How were these formulas found? The first one is from Mark van Hoeij [10, A005259]; I do not know how he found it. But then it is natural to try the other square root, since that will still satisfy the same differential equation.

Proposition 24. $v_0(x) < 0$ for $0 < x < c_0$.

Proof. For $0 < x < c_0$: By Lemma 21, $0 < \lambda(x) < 1$, so by Lemma 18(b), $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right) > 0$. Similarly, $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda_2(x)\right) > 0$.

The Two Endpoints

Proposition 25. On interval $(c_0, c)$ we have exactly $\varphi(x) = v_2(x)/(2^{5/4}\tau^4\pi^2)$.

Proof. We examine the solution $u_\infty(x)$ of \[DE3\] on the interval $(c, +\infty)$. As $\delta \to 0^+$, the Frobenius series solution shows that

$$u_\infty(c + \delta) = A + B\sqrt{\delta} + C\delta + O(\delta^{3/2})$$

for some real constants $A, B, C$; we will have to evaluate the constant $B$ below. Following (5) around the point $c$ by a half-turn in either direction, we get

$$u_\infty(c - \delta - i0) = A + B(-i)\sqrt{\delta} - C\delta + O(\delta^{3/2})$$

$$u_\infty(c - \delta + i0) = A + Bi\sqrt{\delta} - C\delta + O(\delta^{3/2})$$

$$\varphi(c - \delta) = \frac{1}{2\pi i} (u_\infty(c - \delta - i0) - u_\infty(c - \delta + i0))$$

$$= \frac{0A - 2Bi\sqrt{\delta} + 0C\delta}{2\pi i} + O(\delta^{3/2})$$

$$= \frac{-B}{\pi} \sqrt{\delta} + O(\delta^{3/2}).$$

Therefore $\varphi(x) = (-B/\pi)v_2(x)$ on $(c_0, c)$.

On interval $(c, +\infty)$, we have

$$u_\infty(x) = \frac{1}{x^\mu} \left(\frac{1}{x}\right)^2 2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda\left(\frac{1}{x}\right)\right)^2.$$

Argument $\lambda(1/x)$ stays inside the unit disk, so no analytic continuation is required. Now $\lambda(c_0) = \lambda_2(c_0) = 1/(2^{3/2}\tau)$, called $z_0$ in Lemma 19. Let $S_0, S_1$ also be as in Lemma 19. As $\delta \to 0^+$,

$$\frac{1}{c + \delta} = \frac{1}{\tau^4} + O(\delta)$$

$$\mu\left(\frac{1}{c + \delta}\right) = \frac{\tau}{2^{1/4}3^{1/2}} - \frac{1}{4\cdot 3\cdot \tau}\sqrt{\delta} + O(\delta)$$

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\[
\lambda \left( \frac{1}{c + \delta} \right) = \frac{1}{2^{3/2} \tau} - \frac{\sqrt{3}}{2^{9/4} \tau^2} \sqrt{\delta} + O(\delta)
\]
\[
t _2 F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; \lambda \left( \frac{1}{c + \delta} \right) \right) = S_0 - \frac{\sqrt{3}}{2^{9/4} \tau^2} S_1 \sqrt{\delta} + O(\delta)
\]
\[
u_\infty (c + \delta) = \frac{S_0^2}{3 \sqrt{2} \tau^2} - \frac{S_0 (3S_1 + \sqrt{2} S_0)}{2^{7/4} 3 \sqrt{2} \tau^4} \sqrt{\delta} + O(\delta)
\]
\[
u_\infty (c + \delta) = \frac{S_0^2}{3 \sqrt{2} \tau^2} - \frac{1}{2^{5/4} \sqrt{4 \pi} \tau^4} \sqrt{\delta} + O(\delta).
\]
So we get \( B = -1/(2^{5/4} \tau^4 \pi) \).

**Proposition 26.** On interval \((0, c_0)\) we have exactly \( \varphi(x) = -6v_0(x)/\pi^2 \).

*Proof.* We examine the solution \( u_\infty(x) \) of (DE3) on the interval \((-\infty, 0)\). As \( \delta \to 0^+ \), the Frobenius series solution shows that
\[
u_\infty (-\delta) = A + B \log \delta + C (\log \delta)^2 + o(1) \tag{6}
\]
for some real constants \( A, B, C \); we will have to evaluate the constants \( B \) and \( C \) below. Following (6) around the point 0 by a half-turn in either direction, we get
\[
u_\infty (\delta - i0) = A + B (\log \delta + i \pi) + C (\log \delta + i \pi)^2 + o(1)
\]
\[
u_\infty (\delta + i0) = A + B (\log \delta - i \pi) + C (\log \delta - i \pi)^2 + o(1)
\]
\[
\varphi(\delta) = \frac{1}{2 \pi i} (u_\infty (\delta - i0) - u_\infty (\delta + i0))
\]
\[
= \frac{2B i \pi + 4 C i \pi \log \delta}{2 \pi i} + o(1)
\]
\[
= B + 2C \log \delta + o(1)
\]
Therefore \( \varphi(x) = Bu_0(x) + 2Cv_0(x) \) on \((0, c_0)\).

On interval \((-\infty, 0)\), we have
\[
u_\infty (x) = \frac{1}{x} \mu \left( \frac{1}{x} \right)^2 \ _2 F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; \lambda \left( \frac{1}{x} \right) \right)^2.
\]
Argument \( \lambda(1/x) \) stays inside the unit disk, so this is an easy analytic continuation of \( u_\infty \). As \( \delta \to 0^+ \),
\[
\frac{1}{-\delta} = -\frac{1}{\delta} + O(1)
\]
\[
\mu \left( \frac{1}{-\delta} \right)^2 = \delta + O(\delta^2)
\]
\[
\lambda \left( \frac{1}{-\delta} \right) = 1 - 27 \delta^2 + O(\delta^3)
\]
So by Lemma 18(c),
\[
\begin{align*}
_{2}F_{1}\left(\frac{1}{3},\frac{2}{3};1;\lambda\left(\frac{1}{-\delta}\right)\right) &= \frac{-\sqrt{3}}{2\pi} 2\log \delta + o(1) \\
u_{\infty}(-\delta) &= \frac{-3}{\pi^2} (\log \delta)^2 + o(1).
\end{align*}
\]
Thus we get \( B = 0 \) and \( C = -3/\pi^2 \).

**Corollary 27.** The moment density \( \varphi \) may be written
\[
\varphi(x) = \begin{cases} 
\frac{-6}{\pi^2} v_0(x), & 0 < x < c_0, \\
\frac{1}{2^{5/4+\frac{1}{4}\pi^2}} v_2(x), & c_0 \leq x \leq c.
\end{cases}
\]
It is positive on \((0, c_0) \cup (c_0, c)\).

**Proof.** For \( 0 < x < c_0 \), by Prop. 24 \( v_0(x) < 0 \). For \( c_0 < x < c \), by Prop. 16 \( v_2(x) > 0 \).

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