Incidence of nonextensive thermodynamics in temporal scaling at Feigenbaum points

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Abstract

Recently, in Phys. Rev. Lett. 95, 140601 (2005), P. Grassberger addresses the interesting issue of the applicability of $q$-statistics to the renowned Feigenbaum attractor. He concludes there is no genuine connection between the dynamics at the critical attractor and the generalized statistics and argues against its usefulness and correctness. Yet, several points are not in line with our current knowledge, nor are his interpretations. We refer here only to the dynamics on the attractor to point out that a correct reading of recent developments invalidates his basic claim.

Key words: transition to chaos, Feigenbaum attractor, $q$-statistics, $q$-phase transitions
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1 Introduction

A sharp disagreement and refutation has been published recently [1] on the contention, based on many studies [2] - [23], that the generalized statistical-mechanical scheme known as nonextensive statistical mechanics [24] is pertinent to the dynamical properties at the so-called onset of chaos in dissipative low-dimensional iterated maps. The ‘dialogue’ between advocates and opponents of the appropriateness of this new formalism, here referred to as $q$-statistics, to this old problem has been hampered by the lack of both: i) an all-inclusive explanation of how and why the new formalism relates to the aforementioned dynamics, and ii) an objective appraisal of the published results on the subject as they have developed in time and in depth. Whereas a

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definite rationalization may be hastened by the ongoing accumulation of information from current research, unbalanced judgments are sustained by the prevalent belief that, after the historic and extensive work on the transitions to chaos carried out a few decades ago, there are no major unknown features left to be revealed, nor intrinsic value in new approaches. Here we attempt to break this impasse by explaining as plainly as possible the manner in which $q$-statistics manifests at the transitions to chaos, and by indicating how it relates to previous approaches, such as the ‘thermodynamic formalism’ for nonlinear dynamics as adapted by Mori and colleagues [25] to the study of this kind of attractor.

When the control parameter of a nonlinear one-dimensional map is set at the threshold between periodic and chaotic motion an unusual and complicated dynamics arises, whose features have been explored long ago [26]-[29] and described recently in greater detail [16] - [23]. For both the period doubling and the quasiperiodic routes to chaos the transition from periodic to chaotic behavior is mediated by the appearance of a multifractal ‘critical’ attractor with geometrical properties already known for a couple of decades [30]-[32]. Critical attractors have a vanishing Lyapunov coefficient $\lambda_1$ and a sensitivity to initial conditions $\xi_t$ that does not converge to any single-valued function but instead displays a fluctuating pattern that grows as a power law in time $t$ [25]. Trajectories within such a critical attractor show self-similar temporal structures, they preserve memory of their previous locations and do not have the mixing property of truly chaotic trajectories [25]. A special version of the thermodynamic formalism for deterministic chaos [31], [33]-[37] was adapted long ago [25] to study the dynamics at critical attractors and quantitative results were obtained that provided a first understanding, particularly about the envelope of the fluctuating $\xi_t$ and the occurrence of a so-called ‘$q$-phase’ dynamical phase transition [25].

Below we describe the scaling properties of the fluctuating sensitivity $\xi_t(x_{in})$ at a multifractal critical attractor as well as those of its associated spectra of generalized Lyapunov coefficients $\lambda(x_{in})$ (where $x_{in}$ stands for the initial position of trajectories). We illustrate these properties and their connection to the $q$-statistical expressions by considering mainly the specific case of the period doubling onset of chaos, i.e. the so-called Feigenbaum attractor. We explain that the dynamics at the attractor consists of families of dynamical $q$-phase transitions and discuss the ensuing relationship between the thermodynamic and the $q$-statistical formalisms.

We refer to various points raised in Ref. [1] as a means of clarifying our current understanding of this problem [38]. We chose this format in part because subsequent critical opinions [39] on our developments [16] - [23] appear to be based on a cursory and unquestioning belief that the arguments in Ref. [1] are indisputable and its views definitive. However, as we demonstrate below,
the contents in Ref. [1] suffer shortcomings that seem to stem from overconfidence and poor reading of the studies cited there, particularly [16] - [23]. Another reason for adopting an extended ‘comment’ layout is that the customary channels for response have been discouraged or barred by the editors of the journal and magazine where the criticisms have been published, preventing an informed, in-depth and balanced debate on this issue. We would wish that this propagation of swiftly-taken and disingenuous judgements be counterbalanced by standard objective reasoning.

In Section 2 we sum up the features of $q$-statistics in relation to the dynamics we discuss. In Section 3 we explain the structure of $\xi_t$ for the fluctuating dynamics at the Feigenbaum attractor. We point out there that this dynamical structure is distinguished by a built-in ‘aging’ or ‘waiting time’ scaling law. In Section 4 we describe the relationship between the thermodynamic approach and $q$-statistics as applied to critical attractors. We indicate that the fixed values for the Tsallis’ $q$ index correspond to the values that Mori’s field variable $q$ takes at the dynamical phase transitions, i.e. $q_{\text{trans}} = q$. In Section 5 we demonstrate the condition for the linear growth of the $q$-entropy $S_q$, i.e. the occurrence of $q$-phase transitions. We also prove the identity between the rates of $q$-entropy growth and the $q$-generalized Lyapunov coefficients. For clarity of presentation the previous sections focus on the properties that link only the two dominant scaling regions of the multifractal attractor, its most crowded and most sparse. In Section 6 all the scaling regions of the multifractal are considered, it is shown how $\xi_t$ is obtained in terms of the discontinuities of Feigenbaum’s trajectory scaling function $\sigma$, and it is observed that the entire dynamics is made of a hierarchical family of pairs of $q$-phase transitions. In Section 7 we refer to the parallelisms in the expression of $q$-statistics in the other two routes to chaos displayed by low-dimensional maps, via quasiperiodicity and via intermittency. In Section 8 we summarize the main results.

2 $q$-statistics for critical attractors

First of all it is indispensable to stipulate what (according to the author) is intended by the manifestation of $q$-statistics in the dynamics at a critical attractor. This can be summarized in two related properties. The first concerns the (finite time) sensitivity to initial conditions $\xi_t$, defined as

$$\xi_t(x_{in}) \equiv \lim_{\Delta x_0 \to 0} \left| \frac{\Delta x_t}{\Delta x_{in}} \right|, \quad (1)$$
where $\Delta x_{in}$ is the initial separation of two trajectories and $\Delta x_t$ that at time $t$. This quantity is expected to involve expressions of the form

$$\xi_t(x_{in}) = \exp_q[\lambda_q(x_{in})t], \quad (2)$$

where $q$ is the entropic index, $\lambda_q(x_{in})$ is the $q$-generalized Lyapunov coefficient, and $\exp_q(x) \equiv [1 - (q-1)x]^{1/(q-1)}$ is the $q$-exponential function. The second property relates to ‘temporal extensivity’ of entropy production [40] at critical attractors [2], [20], [22], [23]. That is, linear growth with time $t$ of the entropy associated to an ensemble of trajectories. The fitting expression for the rate of entropy production $K_q(x_{in})$ to be used is thought to be given by

$$K_q(x_{in})t = S_q(t,x_{in}) - S_q(0,x_{in}), \quad (3)$$

where

$$S_q \equiv \sum_i p_i \ln_q p_i^{-1} = 1 - \sum_i W_i p_i^q, \quad (4)$$

is the Tsallis entropy [24] and where $p_i(t)$ is the distribution of the trajectories in the ensemble at time $t$ given that they were distributed initially within a small interval around $x_{in}$. Recall that $\ln_q y \equiv (y^{1-q} - 1)/(1-q)$ is the inverse of $\exp_q(y)$ and notice the explicit dependence of all quantities on the initial position $x_{in}$.

Several comments are in order:

i) It is anticipated that the index $q$ in Eq. (2) takes a well defined value determined by the basic attractor properties such as its universal constants. For a multifractal critical attractor there may be a distinct discrete family of such values for $q$.

ii) Because of the memory retention of the trajectories the dependence on $x_{in}$ in Eq. (2) (or Eq. (3)) does not disappear for sufficiently large $t$. Notice that the usual large $t$ limiting condition is not present in the definition of $\xi_t$ in Eq. (2) nor in Eq. (3).

iii) Since $\xi_t$ fluctuates according to a deterministic pattern (given by the specific route to chaos studied) the variable $t$ in Eq. (2) cannot run through all positive integers sequentially but only through specific infinite subsets. However, and as seen below, all times $t$ appear in Eq. (2) when $x_{in}$ is varied.

iv) The standard exponential form for $\xi_t$ with the ordinary $\lambda_1$ is obtained from Eq. (2) when $q \to 1$. This limit is brought about when the control parameter
in the map is shifted from its value at the chaos threshold to a value that corresponds to either a periodic or a chaotic attractor. For these cases the fluctuations in $\xi_t$ die out and the Lyapunov spectra $\lambda_q(x_{in})$ collapses into the single number $\lambda_1$ independent of $x_{in}$ for $t$ large.

v) Depending on the value of $q$ and the sign of $\lambda_q(x_{in})$, the $q$-exponential sensitivity can grow or decrease asymptotically as a power law with $t$, but it can also grow or decrease faster than an ordinary exponential. As we see below, for a multifractal critical attractor Eq. (2) is obtained from an exact transformation of a pure power law [19], [20], but a faster than exponential $q$-sensitivity is rigorously obtained for the critical attractor at a tangent bifurcation [18].

vi) The value of the index $q$ that makes the rate $K_q$ a time independent constant is expected to be the same value appearing in Eq. (2).

vii) The identity $K_q(x_{in}) = \lambda_q(x_{in})$ is supposed to hold. In the limit $q \rightarrow 1$ the familiar identity $K_1 = \lambda_1$ [32] is recovered (where the rate of entropy production $K_1$ is given by $K_1 t = S_1(t) - S_1(0)$ and $S_1 = -\sum_i p_i \ln p_i$). Note that in this identity the rate $K_1$ is not the so-called KS entropy [30]-[32] but a closely related quantity. See Section 5.

## 3 Fluctuating dynamics at the Feigenbaum attractor

Let us consider now properties specific to the Feigenbaum map $g(x)$, obtained from the fixed point equation $g(x) = \alpha g(g(x/\alpha))$ with $g(0) = 1$ and $g'(0) = 0$, and where $\alpha = -2.5029079...$ is one of Feigenbaum’s universal constants [30]. For expediency we shall from now on denote the absolute value $|\alpha|$ by $\alpha$. Numerically, the properties of $g(x)$ can be conveniently obtained from the logistic map

$$ f_{\mu,2}(x) = 1 - \mu x^2, \quad -1 \leq x \leq 1, \quad (5) $$

with $\mu = \mu_\infty = 1.401155189...$. A first important issue that merits amplification and explanation lies behind Grassberger’s comment that his Eq. (3) in [1], here rewritten as

$$ \xi_t(x_{in}) = \alpha^k \quad (6) $$

holds only for special values of time, e.g. $t = 2^k - 1$, $k = 0, 1, ...$ when $x_{in} = 1$. Not quite, there are many other time sequences all of which satisfy Eq. (6) **exactly** [20]. To see this consider the trajectory $x_n$ with initial position $x_{in} = 0$, plotted in Fig. 1 in a way that makes evident the structure of the time
sequences and the power law property that their positions share. It can be observed there that the positions \( x_{n(k,l)} \) of the sequences with times \( n(k,l) = (2l + 1)2^k \), each obtained by running through \( k = 0, 1, 2, \ldots \) for a fixed value of \( l = 0, 1, 2, \ldots \) fall along straight diagonal lines. The starting elements of these sequences are the positions \( x_{2l+1} \). To be precise, the entire attractor can be decomposed into this family of position sequences as every natural number \( n \) appears in one of them. Actually, the complete fluctuating sensitivity \( \xi_t(x_{in} = 1) \) can be decomposed into a family of identical power laws given by Eq. (6) with \( t = n - (2l+1) \) [20]. The sequence for \( l = 0 \) belongs to the envelope of \( \xi_t \) while those for other values of \( l \) are shifted replicas that describe interior values of \( \xi_t \). This feature offers an enormous simplification to the study of the dynamics on the attractor. Time rescaling of the form \( t/t_w \) with \( t_w = 2l + 1 \) makes all power law sequences collapse into a single one in a manner analogous to the so-called ‘aging’ property of ‘glassy’ dynamics where \( t_w \) is a ‘waiting time’ [41], [42].

\[ x_{n(k,l)} \]

Fig. 1. Absolute values of positions in logarithmic scales of the first 1000 iterations for the trajectory of the logistic map at the onset of chaos \( \mu_\infty \) with initial condition \( x_{in} = 0 \). The numbers correspond to iteration times. The power-law decay shared by the time sequences mentioned in the text can be clearly appreciated.

The family of power laws in Eq. (6) can be rewritten [20], with the use of the identity

\[ \alpha^k \equiv \left(1 + \frac{t}{2l + 1}\right)^{\frac{\ln \alpha}{\ln 2}}, \]  

(7)
with \( t = (2l + 1)2^k - 2l - 1 \), as the family of \( q \)-exponentials,

\[
\xi_t(x_{in}) = \exp_q[\lambda^{(t)}_q t],
\]

(8)

all with the same value of \( q \). Above, \( q = 1 - \ln 2 / \ln \alpha \) and \( \lambda^{(t)}_q = (2l + 1)^{-1} \ln \alpha / \ln 2 \) with \( t = n - 2l - 1 \), when \( x_{in} = 1 \), or with \( t = n = (2l + 1)2^k \) when \( x_{in} \) is any of the positions \( x_{2l+1} \) in Fig. 1, in all cases running through \( k = 0, 1, \ldots \) while \( l \) is fixed to a given value \( l = 0, 1, \ldots \) Alternatively, Eq. (8) can be written as the single \( q \)-exponential

\[
\xi_t(x_{in}) = \exp_q[\lambda^{(t)}_q t].
\]

(9)

So, contrary to the statement in [1] - that his Eq. (3) is not a scaling law because it only holds for special values of \( t \) - Eq. (6), and its equivalent forms Eqs. (8) or (9), in fact correspond to a more complex form of a scaling law, one that fits the fluctuating dynamics at the attractor, and referred here as a ‘two-time’ scaling law. Below we make clear that the positions of these sequences for large \( k \) belong to one scaling region within the multifractal, its most sparse region.

A persuasive reason for considering a \( q \)-exponential as a further way to write a power law is the presence of a time scale factor: the generalized Lyapunov coefficient \( \lambda^{(t)}_q \) that appears multiplying \( t \) in \( \xi_t \). This useful quantity (hidden in the power law) can be immediately ‘read’ from the anomalous \( \xi_t \) just as the ordinary Lyapunov coefficient \( \lambda_1 \) is read from the exponential \( \xi_t \) of chaotic dynamics. It is important to stress that it is in the scaling limit (large \( n \) or \( k \) that \( \xi_t \) becomes \( \alpha^k \), and it is this which transforms (exactly) into the \( q \)-exponential \( \xi_t \). Evidently, reference and use of this property involves more than a mere choice between a power law and a \( q \)-exponential representation of the sensitivity. The beauty of a scale-free power law is there to enjoy. As for generality, all of these properties apply with only minor changes to the period-doubling transition to chaos in unimodal maps with extremum of order \( z > 1 \) [22]. The derivation of Eq. (6) for \( z > 1 \) is given in [21] so there is no need to reproduce it in [1].

So, what is the advantage in studying the nonmixing trajectories at the Feigenbaum attractor in terms of the above-explained power-law network structure for \( \xi_t \) instead of the time and position averages described in [1]? Clearly, the dynamical organization within the attractor is hard to resolve from a simple time evolution: starting from an arbitrary position \( x_{in} \) on the attractor and recorded at every \( t \). What is observed are strong fluctuations with a scrambled structure that persist in time. Conversely, unsystematic averages over \( x_{in} \) and/or \( t \) would rub out details of the multiscale properties. However, if
specific initial positions with known location within the multifractal are chosen, and subsequent positions are observed only at pre-selected times, e.g. $t = 2^k - 1$, when the trajectories visit another region of choice, a well-defined $q$-exponential sensitivity $\xi_t$ appears, with $q$ and the Lyapunov spectrum $\lambda(x_{in})$ fixed by the attractor universal constants.

4 Thermodynamic approach and $q$-statistics

We now address the connection between the thermodynamic approach as employed by Mori and colleagues [25] to study dynamical phase transitions at critical attractors and the $q$-statistical properties of the Feigenbaum attractor. In particular: i) We establish the equivalence of the generalized Lyapunov coefficients present in the two methods. ii) We show the link between the occurrence of dynamical phase transitions and the existence of unique values for the entropic index $q$. iii) We demonstrate the linear growth of the Tsallis entropy at the dynamical phase transitions and the identity between the entropy growth rates with their corresponding $q$-Lyapunov coefficients.

4.1 Generalized Lyapunov coefficients and $q$-deformation

We make use of the scaling features of $\xi_t$ just described to examine Grassberger’s advice in [1] that Eq. (4) in [1] gives the natural generalization of the Lyapunov coefficient $\lambda_1$. Indeed, a closely related prescription,

$$\lambda_t(x_{in}) \equiv \frac{1}{\ln t} \ln \left| \frac{dg(t)(x_{in})}{dx_{in}} \right|,$$

was initially adopted in Refs. [27], [29]. It differs however from Eq. (4) in [1] in that no time average is taken. As a function of consecutive values of $t$, $\lambda_t(x_{in})$ does not converge to any constant but does undergo recurrence. This recurrence suggests inspection of $\lambda_t(x_{in})$ for values of $t$ along the time sequences described in the previous section. A straightforward calculation shows that $\lambda_t(x_{in} = 1)$ as defined in Eq. (10) is exactly given by

$$\lambda_t(x_{in} = 1) = \frac{1}{t} \ln_q \left| \frac{dg(t)(x_{in})}{dx_{in}} \right|_{x_{in} = 1} = \frac{\ln \alpha}{(2l + 1) \ln 2} = \lambda_q^{(l)},$$

where $t$, as before, runs through the sequences $t = (2l+1)2^k - 2l - 1$, $k = 0, 1, \ldots$, with $l = 0, 1, \ldots$ fixed. So, the earlier definition for the generalized Lyapunov coefficient is equivalent to that given for the same quantity by the $q$-statistics.
The meaning of the index $q$ is given above. It is the degree of ‘$q$-deformation’ of the ordinary logarithm that makes $\lambda_t$ finite for large $t$. It is not difficult to corroborate that this result is valid for general $x_{in}$ on the attractor.

### 4.2 Dynamical partition function

Eqs. (10) and (11) suggest a broader connection between the thermodynamic formalism and the $q$-statistical features found for the dynamics at the Feigenbaum attractor. Indeed, it has been shown in [22], [23] that the physical origin of the existence of a distinct value (or values) for the entropic index $q$ is linked to the occurrence of dynamical phase transitions, of Mori’s $q$-phase type [25], which connect qualitatively different regions of the attractor.

The thermodynamic formalism is built on the dynamical partition function

$$ Z(n, q) \equiv \int d\lambda \ P(\lambda, t) \ W(\lambda, t)^{1-q}, \quad (12) $$

where the weight $W(\lambda, t)$ was assumed [25] to have the form $W(\lambda, t) = \exp(\lambda t)$ for chaotic attractors and $W(\lambda, t) = t^\lambda$ for the attractor at the onset of chaos, with $\lambda$ a generalized Lyapunov coefficient (whose dependence on $t$ and $x_{in}$ is not written explicitly). The variable $q$ plays the role of a ‘thermodynamic field’ like the external magnetic field in a thermal magnet or, alternatively, $1 - q$ can be thought analogous to the inverse temperature. We note that $W(\lambda, t)$ can be written in both cases as

$$ W(\lambda, t) = \left| \frac{df(t)(x_{in})}{dx_{in}} \right|, \quad (13) $$

where $f(x)$ is the iterated map under consideration, that is,

$$ \lambda_t(x_{in}) = B(t) \ln \left| \frac{df(t)(x_{in})}{dx_{in}} \right| \quad (14) $$

with $B(t) = t^{-1}$ for chaotic attractors and $B(t) = (\ln t)^{-1}$ for the fluctuating sensitivity at the critical attractor [25]. As mentioned, for chaotic attractors the fluctuations of $\lambda_t(x_{in})$ die out as $t \to \infty$ and this quantity becomes the ordinary Lyapunov coefficient $\lambda_1$ independent of the initial position $x_{in}$. For critical attractors $\lambda_t(x_{in})$ maintains its dependence on $t$ and $x_{in}$ but as already seen it is a well-defined constant provided $t$ takes values along each time sequence, or if the two-time scaling property (Eq. (9)) is summoned.
For critical attractors the density distribution for the values of \( \lambda, t \gg 1 \), \( P(\lambda, t) \), is written in the form [25]

\[
P(\lambda, t) = t^{-\psi(\lambda)} P(0, t),
\]

(15)

where \( \psi(\lambda) \) is a concave spectrum of the fluctuations of \( \lambda \) with minimum \( \psi(0) = 0 \) and is obtained as the Legendre transform of the ‘free energy’ function \( \phi(q) \), defined as \( \phi(q) \equiv -\lim_{t \to \infty} \ln Z(t, q)/\ln t \). The generalized Lyapunov coefficient \( \lambda(q) \) is given by \( \lambda(q) \equiv d\phi(q)/dq \) [25]. The functions \( \phi(q) \) and \( \psi(\lambda) \) are the dynamic counterparts of the Renyi dimensions \( D(q) \) and the spectrum \( f(\tilde{\alpha}) \) that characterize the geometric structure of the attractor [31], [32]. For non hyperbolic attractors like those at the onset of chaos the functions \( \phi(q) \) and \( \psi(\lambda) \) that are used to describe the dynamics are independent of \( D(q) \) and \( f(\tilde{\alpha}) \).

\subsection*{4.3 \( q \)-phase transitions and Tsallis \( q \) index}

As with thermal 1st order phase transitions, a \( q \)-phase transition is indicated by a section of linear slope \( m_c = 1 - q \) in the spectrum (free energy) \( \psi(\lambda) \) and consequently a discontinuity at \( q = q \) in the Lyapunov function (order parameter) \( \lambda(q) \). For the Feigenbaum attractor a single \( q \)-phase transition was numerically determined [29] and found to occur approximately at a value around \( m_c = -(1 - q) \approx -0.7 \). It was pointed out in Ref. [27], [29] that this value would actually be \( m_c = -(1 - q) = -\ln 2/\ln \alpha = -0.7555... \)

From the knowledge we have gained on \( q \)-generalized Lyapunov coefficients for this attractor, e.g. Eq. (11), we can determine the free energy \( \psi(\lambda) \). But to do this we need to extend our results a bit further. We note that \( \lambda_t(x_{in}) \) in Eq. (11), corresponds to trajectories that originate in the most crowded region of the attractor, e.g. \( x_{in} = 1 \), and terminate at its most sparse region, \( x_n = (2l+1)^2k \approx 0 \), large \( k \). Naturally, these trajectories when observed at times \( n = (2l + 1)^2k \) grow apart from each other as \( k \) increases (\( \lambda_t(x_{in}) > 0 \)). When the inverse situation is considered, e.g. \( x_{in} = 0 \), with final positions \( x_n = (2l+1)^2k+1 \approx 1 \), one obtains [22]

\[
\xi_t(x_{in}) = \exp_Q[\lambda_Q^{(l)}t],
\]

(16)

with \( Q = 2 - q = 1 + \ln 2/\ln \alpha \) and

\[
\lambda_t(x_{in} = 0) = -\frac{2\ln \alpha}{(2l+1)\ln 2} = \lambda_Q^{(l)},
\]

(17)
where \( t = (2l + 1)2^k - 2l, \ k = 0, 1, \ldots, \ l = 0, 1, \ldots \) Above, of course, \( \lambda_t(0) < 0 \) since the trajectories when observed at times \( (2l + 1)2^k + 1 \) come progressively close to each other. Notice that the change in sign of \( \lambda_t \) and the relation \( Q = 2 - q \) match the property of inversion of the \( q \)-exponential, \( \exp_q(x) = 1/\exp_{2-q}(-x) \). Also, trivially, when the initial and final positions of trajectories belong to the same region within the attractor, e.g. \( x_1 = 1 \) and \( x_n = (2l + 1)2^k + 1 \simeq 1 \), one has \( \xi_t = 1 \) with \( \lambda_q = 0 \).

From the results for \( \lambda_q^{(l)} \) and \( \lambda_Q^{(l)} \) the Lyapunov function \( \lambda(q) \) and spectrum \( \psi(\lambda) \) are constructed [22]

\[
\lambda(q) = \begin{cases} 
\lambda_q^{(0)}, & -\infty < q \leq q, \\
0, & q < q < Q, \\
\lambda_Q^{(0)}, & Q \leq q < \infty,
\end{cases}
\]  \hspace{1cm} (18)

and

\[
\psi(\lambda) = \begin{cases} 
(1 - Q)\lambda, & \lambda_Q^{(0)} < \lambda < 0, \\
(1 - q)\lambda, & 0 < \lambda < \lambda_q^{(0)},
\end{cases}
\]  \hspace{1cm} (19)

with \( \lambda_q^{(0)} = \ln \alpha / \ln 2 \simeq 1.323 \) and \( \lambda_Q^{(0)} = -2\lambda_q^{(0)} \simeq -2.646 \). The constant slopes of \( \psi(\lambda) \) represent the \( q \)-phase transitions associated to trajectories linking two regions of the attractor, \( x_1 \simeq 1 \) and \( x \simeq 0 \), and their values \( 1 - q \) and \( q - 1 \) correspond to the Tsallis index \( q \) obtained from \( \xi_t \). The slope \( q - 1 \simeq -0.7555 \) coincides with that initially detected in Refs. [28], [27].

The significance of the above exercise is that it reveals the physical reason for the existence of a well-defined value of the entropic index \( q \) in critical attractor dynamics. This is the occurrence of a \( q \)-phase transition. The value of the field \( q \) at which the transition takes place is precisely \( q \) (and the same of course for the inverse process, \( q_{trans} = Q \)).

5 Temporal extensivity of the \( q \)-entropy

As we have seen, along each time sequence \( n(k, l) = (2l + 1)2^k, \ l \) fixed, the sensitivity \( \xi_t \) with \( x_m = 1 \) is given by the \( q \)-exponential in Eq. (8), and consequently we can express \( \lambda_t(x_m) \) in terms of the \( q \)-deformed logarithm in Eq. (11). Therefore for times of the form \( t = n - (2l + 1) \) we have

\[
P(\lambda, t)W(\lambda, t) = \delta(\lambda - \lambda_q^{(l)}) \exp_q(\lambda_q^{(l)}t),
\]  \hspace{1cm} (20)
and
\[ Z(t, q) = W(\lambda_q^{(l)}, t)^{1-q} = \left[ 1 + (1-q)\lambda_q^{(l)}t \right]^{(1-q)/(1-q)} . \]  

Notice that in Eq. (21) \( q \) is a running variable while \( q = 1 - \ln 2 / \ln \alpha \) is fixed.

Our next step is to consider the uniform probability distribution, for fixed \( \lambda_q^{(l)} \) and \( t \), given by \( p_i(\lambda_q^{(l)}, t) = W^{-1} \), \( i = 1, ..., W \), where \( W \) is the integer nearest to (a large) \( W \). A trajectory starting at \( x_{in} = 1 \) visits a new site \( x_n \) belonging to the time sequence \( n(k,l) \), \( l \) fixed, every time the variable \( k \) increases by one unit and never repeats one. Each time this event occurs phase space is progressively covered with an interval of length \( \Delta x_{k,l} = x_{(2l+1)2^k} - x_{(2l+1)2^{k+1}} > 0 \). The difference of the logarithms of the times between any two such consecutive events is the constant \( \ln 2 \), \( n \) large, whereas the logarithm of the corresponding phase space distance covered \( \Delta x_{k,l} \) is the constant \( \ln \alpha \). We define \( W_k \) to be the total phase space distance covered by these events up to time \( n = (2l + 1)2^k \).

In logarithmic scales this is a linear growth process in space and time from which the constant probability \( W_k^{-1} \) of the uniform distribution \( p_i \) is defined. In ordinary phase space \( x \) and time \( t \) we obtain instead \( W_t^{-1} = \exp(-\lambda_q^{(l)}t) \). We then have that
\[ Z(t, q) = \sum_{i=1}^{W} [p_i(t)]^q = 1 + (1-q)S_q(t), \]

where \( S_q = \ln_q W \) is the Tsallis entropy for \( p_i \).

The usefulness of the \( q \)-statistical approach is now evident when we recall from our discussion above that the dynamics on the critical attractor displays as its main feature a \( q \)-phase transition and that at this transition the field variable \( q \) takes the specific value \( q_{trans} = q \). Comparison of Eqs. (21) and (22) indicates that at each such transition both \( Z \) and \( S_q \) grow linearly with time along the sequences \( n(k,l) \). In addition, \( \lambda_q^{(l)} \) can be determined from \( S_q/t = \lambda_q^{(l)} \). Thus, with the knowledge we have gained, a convenient procedure for determining the relevant quantities for the fluctuating dynamics at a critical multifractal attractor could take advantage of the above properties. Curiously, the same features have been observed for \( S_q \) with \( q = 0 \) associated to trajectories (with linear \( \xi_t \)) in a conservative two-dimensional map with vanishing ordinary Lyapunov coefficients [43].

Further, the identity \( S_q/t = \lambda_q^{(l)} \) (also derived in [22]) between the rate of \( q \)-entropy change and the generalized \( \lambda \) is not the identity
\[ t^{-1}(S_1(t) - S_1(0)) = t^{-1} \ln \left| \frac{dg^{(l)}(x_{in})}{dx_{in}} \right| \]

12
in [1] (the zero identity for \( t \to \infty \)) but refers to \( \lambda_t(x_{in}) \) as above. Yes, it considers an instantaneous entropy rate but it is comparable in the sense of [44] to the \( q \)-generalized KS entropy studied in [12]. The identity \( S_q/t = \lambda_q^{(l)} \) holds for \( t \to \infty \) as the interval length (around \( x_{in} \)) vanishes. It fluctuates, but as explained, we look for the detailed dependence on both \( x_{in} \) and \( t \). The invariant density is not smooth and is built up by placing an initial condition at each point \( x_{in} \) on the attractor. In contrast to the chaotic case there is not one identity but many, and the claim in [1] that averages are needed for applications of Pesin’s identity seems ineffectual for nonmixing trajectories. Our results may not be insignificant as these coincide (when \( l = 0 \)) with those in [12] where the \( q \)-KS entropy was considered. On the contrary, the entropies \( H_n^q \) in [1] from symbolic dynamics do not sense (i.e. do not depend on) the universal \( \alpha \) and/or the nonlinearity \( z \).

6 Hierarchical family of \( q \)-phase transitions

With reference to the ‘rich zoo’ of values for the index \( q \) alluded in [1], there is a well-defined family [22] of these within the attractor that deserves to be explained. This family is determined by the discontinuities of Feigenbaum’s trajectory scaling function \( \sigma \) (which measures the convergence of positions in period \( 2^k \) orbits as \( k \to \infty \)) [30] and are all expressed in terms of the universal constants of unimodal maps (for general nonlinearity \( z > 1 \)). Eqs. (8) and (16) and the values of \( q \) and \( 2 - q \) given there are obtained from the largest discontinuity in \( \sigma \), that is itself related to the most crowded and most sparse regions in the attractor. The other discontinuities lead to expressions for \( \xi_t \) similar to Eqs. (8) and (16). There is a corresponding family of pairs of Mori’s \( q \)-phase transitions, each associated to orbits with common starting and finishing positions at specific locations of the attractor. As before, the special values for \( q \) in \( \xi_t \) are equal to those of the field variable \( q \) in Mori’s formalism at which the transitions occur [22]. Since the amplitudes of the discontinuities of \( \sigma \) diminish rapidly, there is a hierarchical structure and consideration only of the dominant discontinuity provides a sound description of the dynamics. When all discontinuities are considered it becomes apparent that the dynamics on the critical attractor is constituted in its entirety by the infinite but discrete set of \( q \)-phase transitions. We reproduce below some basic expressions [22].

The trajectory scaling function \( \sigma \) is obtained as the \( k \to \infty \) limit of

\[
\sigma_k(j) = \frac{d_{k+1,j}}{d_{k,j}},
\]

where the numbers \( d_{k,j} \) are the so-called diameters that measure the bifurcation forks that form the period-doubling cascade sequence [30]. The diameters
are determined from the positions of the 'superstable' periodic orbits of lengths $2^k$, i.e. the $2^k$-cycles that contain the point $x = 0$ at $\mu_k < \mu_\infty$ [30]. This function has finite (jump) discontinuities at all rationals of the form $y_j = j/2^{t+1}$, and we denote the values of $\sigma$ before and after them (omitting the subindex $k$) as $\sigma(y_j^-) = 1/\alpha_j^-$ and $\sigma(y_j^+) = 1/\alpha_j^+$, respectively. The numbers $\alpha_j^\pm$ are universal constants. For the largest discontinuity $y_j = 0$ one has $\alpha_0^- = \alpha_2^-$ and $\alpha_0^+ = -\alpha_2$.

The strategy employed [22] in determining $\xi_t$ from $\sigma$ is to choose the initial and the final separation of the trajectories to be the diameters $\Delta x_{in} = d_{k,j}$ and $\Delta x_t = d_{k,j+t}$, $t = 2^k - 1$, respectively. So, $\xi_t(x_{in}), x_{in} = x_{in}(j)$, is obtained as

$$\xi_t(x_{in}) = \lim_{k \to \infty} \left| \frac{d_{k,j+t}}{d_{k,j}} \right|. \quad (25)$$

Notice that in the $k \to \infty$ limit $\Delta x_{in} \to 0, t \to \infty$ and the $2^k$-supercycle becomes the onset of chaos (the $2^\infty$-supercycle). Then, for each discontinuity of $\sigma$ at $y_j$, $\xi_t(x_{in})$ can be written as [22]

$$\xi_{t,y_j} \approx \left| \frac{\sigma_k(y_j^-)}{\sigma_k(y_j^+)} \right|^k, \quad t = 2^k - 1, \quad k \gg 1 \quad (26)$$

For the inverse process, starting at $\Delta x_{in} = d_{k,j+t} = -d_{k,j-1}$ and ending at $\Delta x_t = d_{k,j} = -d_{k,j-1+t}$, with $t = 2^k + 1$ one obtains [22]

$$\xi_{t,y_j} \approx \left| \frac{\sigma_k(y_j^+)}{\sigma_k(y_j^-)} \right|^k, \quad t = 2^k + 1, \quad k \gg 1 \quad (27)$$

Therefore

$$\xi_{t,y_j} = \left| \frac{\alpha_j^-}{\alpha_j^+} \right|^k, \quad \xi_{t,y_j} = \left| \frac{\alpha_j^-}{\alpha_j^+} \right|^{-k}$$

and, similarly to Eq. (6), the sensitivities in Eq. (28) can be re-expressed as $q$-exponentials with $q$-indexes

$$q = 1 - \frac{\ln 2}{\ln \left| \frac{\alpha_j^-}{\alpha_j^+} \right|}, \quad Q = 2 - q = 1 + \frac{\ln 2}{\ln \left| \frac{\alpha_j^-}{\alpha_j^+} \right|}, \quad (29)$$

respectively. All other consequences described in the previous sections follow in a straightforward manner.
It is interesting to note that discrete infinite families of values for the index \( q \) arise in different contexts, such as in the proposed \( q \)-generalization of the ordinary and Lévy-Gnedenko central limit theorems [45].

As a check on the leading role of the most crowded and sparse regions of the attractor in determining its dynamics we consider a two-scale Cantor set approximation of the multifractal attractor of fractal dimension \( d_f \). Following a standard procedure [31] for a trajectory \( x_0, x_1, ..., x_{2^k+1-1}, k \) large, on the attractor we cover it with a set of intervals of lengths \( l_i^{(k)} = |x_i - x_{i+2^k}| \), \( i = 0, ..., 2^k - 1 \). The smallest length scale \( l_{\text{min}}^{(k)} \sim \alpha^{-2k} \) is observed in the neighborhood of \( x = 1 \) while the largest \( l_{\text{max}}^{(k)} \sim \alpha^{-k} \) occurs near \( x = 0 \). Trajectories starting in the vicinity of \( x = 1 \) in the multifractal visit the vicinity of \( x = 0 \) at times of the form \( t = 2^k - 1 \). In the two-scale set the equivalent trajectories of duration \( t = 2^k - 1 \) starting at positions assigned to the scale \( l_{\text{min}}^{(k)} \) visit positions that correspond to the scale \( l_{\text{max}}^{(k)} \) a number of times equal to \( d_f t \). For these times \( |dx_i/dx_0| = l_{\text{max}}^{(k)}/l_{\text{min}}^{(k)} \), otherwise \( |dx_i/dx_0| = 1 \). Use of this in the time-averaged expansion rate [1]

\[
\beta \ln t \equiv 1/t \sum_{i=1}^{t=2^k-1} \ln |dx_i/dx_0|, \quad k \gg 1 \tag{30}
\]

leads to

\[
\beta \ln t = d_f \ln \frac{l_{\text{max}}^{(k)}}{l_{\text{min}}^{(k)}}, \quad k \gg 1 \tag{31}
\]

or

\[
\beta = d_f \ln \alpha/\ln 2. \tag{32}
\]

As \( d_f \simeq 0.5388 \) [30] we obtain \( \beta \simeq 0.7131 \) which falls (given the number of digits considered in the value of \( d_f \)) within 0.033\% of the numerical value reported for \( \beta \) in [1]. Eq. (32) is indeed a very simple relation linking the expansion rate constant \( \beta \) and the universal constant \( \alpha \).

7 \textit{q-statistics and the other routes to chaos}

With reference to the generality of our results for the Feigenbaum attractor, and besides all unimodal maps of arbitrary nonlinearity \( z > 1 \), there is a very similar picture obtained for another important multifractal critical attractor. This corresponds to the quasiperiodic route to chaos, recently studied [23] in
the framework of the familiar golden-mean (gm) onset of chaos in the critical circle map [30], [32]. The dynamics on the attractor (a fat fractal [46]) is more involved than that for the Feigenbaum case but there is a strong parallelism that can be glimpsed through the following equivalences when only the most crowded and most sparse attractor regions are considered [23]:

i) The time sequences along which the sensitivity to initial conditions exhibits the power law scaling $\xi_t(x_{in} = 1) = \alpha^k$ are $n = (l - m)F_k + mF_{k-2}$, where $F_k$ is the Fibonacci number of order $k$, each sequence is obtained by running through $k = 1, 2, 3, \ldots$ for fixed values of $l = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots, l - 1$. Notice that the sequences now depend on one additional index. As before there is a time shift $t = n - t_w$, here with $t_w = l - m + mw_{gm}^2$, linking the two scales $t$ and $n$ and with the property that rescaling of the form $t/t_w$ makes all power law sequences collapse into a single one. The number $\alpha$ is now the absolute value of the universal constant $\alpha_{gm} \simeq -1.288575$ obtained as the scaling factor that satisfies the fixed-point map equation $g(x) = \alpha_{gm}g(\alpha_{gm}g(x/\alpha_{gm}^2))$.

ii) The role of the period doubling time scale factor $2^k$ is now taken (asymptotically) by $w_{gm}^{-k}$ where $w_{gm}^{-1} = (\sqrt{5} - 1)/2$ is the golden mean.

iii) The sensitivity is given by $q$-exponentials as in Eqs. (8) and (16) with $q = 1 + \ln w_{gm}/2\ln \alpha$, $\lambda_q^{(l,m)} = 2\ln \alpha/(l - m + mw_{gm}^2)\ln w_{gm}$, $Q = 2 - q$ and $\lambda_Q^{(l,m)} = -2\lambda_q^{(l,m)}$.

iv) A pair of $q$-phase transitions take place at $q = q$ and at $q = Q = 2 - q$, that correspond to switching starting and finishing orbital positions. And, as before, linear growth (or reduction) of the Tsallis entropy $S_q$ occurs when $q = q$ (or $q = Q$) with rates $K_q^{(l,m)} = \lambda_q^{(l,m)}$ (or $K_Q^{(l,m)} = \lambda_Q^{(l,m)}$).

More generally, use of the trajectory scaling function $\sigma$ for the golden mean quasiperiodic attractor reveals an infinite family of pairs of $q$-phase transitions, with a hierarchical structure, each member with properties like in i-iv above [23]. As before the dominant behavior arises from the most crowded and most sparse regions of the multifractal.

Furthermore, it is significant to call attention to the fact that the fixed-point map solution of $g(x) = \alpha g(g(x/\alpha))$ for the tangent bifurcations of unimodal maps of general nonlinearity $z > 1$, the third route to chaos, is rigorously given by a $q$-exponential map. See [17], [18]. This feature leads immediately to a $q$-exponential $\xi_t$ with $q = 3/2$ for all $z > 1$ [18]. The tangent bifurcations display weak insensitivity to initial conditions, i.e. power-law convergence of orbits when at the left-hand side ($x < x_c$) of the point of tangency $x_c$. However at the right-hand side ($x > x_c$) of the bifurcation there is a ‘super-strong’ sensitivity to initial conditions, i.e. a sensitivity that grows faster than exponential [18]. The two different behaviors can be couched as a $q$-phase transition with
indexes $q$ and $2-q$ for the two sides of the tangency point. Also a two time or aging scaling property like that in Eq. (9) holds for this critical attractor [47]. The sensitivity $\xi_t$ is dependent on the initial position $x_{in}$ or, equivalently, on its waiting time $t_w$; the closer $x_{in}$ is to the point of tangency the longer $t_w$ but the sensitivity of all trajectories fall on the same $q$-exponential curve when plotted against $t/t_w$. It is worthwhile to underscore that in this case the $q$-exponential is not an alternative way to express a power law but the exact function that describes the faster than exponential increase of $\xi_t$.

8 Conclusions

Thus, we have explained in detail the dynamics at the Feigenbaum and other critical attractors. In all cases the fluctuating sensitivity to initial conditions has the form of infinitely many interlaced $q$-exponentials that fold into a single one with use of the two-time scaling $t/t_w$ property. More precisely, there is a hierarchy of such families of interlaced $q$-exponentials; an intricate (and previously unknown) state of affairs that befits the rich scaling features of a multifractal attractor. Therefore, the comment in the abstract of Ref. [1] that ”the... behavior at the Feigenbaum point based on non-extensive thermodynamics... can be easily deduced from well-known properties of the Feigenbaum attractor” is somewhat an overstatement.

Summing up, the incidence of $q$-statistics in the temporal scaling at the Feigenbaum attractor is verified; the $q$-exponential sensitivity is explained; the origin of the index $q$ is made clear; and the equality between $q$-generalized Lyapunov coefficients and $q$-entropy growth rates is demonstrated. A similar corroboration applies to the other two known routes to chaos, those via quasiperiodicity and via intermittency.

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