Dimensional regularization applied to nuclear systems

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We apply for the first time the dimensional regularization procedure to treat an ultraviolet divergence occurring in the framework of the nuclear many-body problem. We consider the second-order correction (beyond the mean-field approximation) to the equation of state of symmetric nuclear matter with a zero-range effective interaction. The unphysical ultraviolet divergence generated at second order by the zero range of the interaction is properly removed by the regularization technique and the regularized equation of state (mean-field + second-order contributions) is adjusted to a reference equation of state. One of the main practical advantages of this procedure, with respect to a cutoff regularization, is to provide a unique set of parameters for the adjusted effective interaction. This occurs because the regularized second-order correction does not contain any cutoff dependence. The encouraging results found in this work indicate that such an elegant and rigorous technique to remove the divergent terms and to generate regularized effective interactions is likely to be applied in future in the framework of beyond mean-field models for finite nuclei.

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Perturbation theories are employed in many domains of physics and regularization and renormalization techniques are adopted in cases where the inclusion of higher-order terms with respect to the leading contribution in the perturbative expansion generates divergences. This type of divergences are well known for instance in particle physics in the context of quantum field theories [1]. In several frameworks of many-body physics, such as in nuclear physics and in atomic physics, other types of divergences occur in mean-field-based models like the Bogoliubov–de Gennes or the Hartree–Fock–Bogoliubov theories if a zero-range interaction is employed in the pairing channel to treat a superfluid many-fermion system [2,3]. Apart from this specific case related to the pairing interaction, in the perturbative treatment of the many-body problem an ultraviolet divergence always appears if a zero-range interparticle interaction is used when higher-order terms are included beyond the mean-field approximation (which corresponds to the leading order in the perturbative solution of the Dyson equation). This unphysical divergence has been analyzed in a previous work [4].

We develop here a technique which is currently adopted in the context of quantum field theories, the so-called dimensional regularization, for the second-order correction beyond the mean-field approximation within a specific case of the nuclear many-body problem. As done in Ref. [5], we consider the equation of state (EoS) of symmetric nuclear matter with a contact interaction $g \delta (\vec{r}_1 - \vec{r}_2)$ where the coupling constant $g$ is density dependent and contains three parameters, $t_0$, $t_3$ and $\alpha$: $g(\rho) = t_0 + \frac{t_3}{3} \rho^\alpha$. This corresponds to the $t_0 - t_3$ model of the nuclear effective Skyrme interaction [6]. In Ref. [4], we have already analyzed in this specific model the nature of the divergence related to the second-order correction of the EoS. The divergent term has been calculated analytically and, by deriving its asymptotic expansion, it is possible to show that it goes linearly with the momentum $\Lambda$ which is introduced as a cutoff regulator. A cutoff regularization has been applied to absorb the divergence by means of a fit of parameters in the corrected mean-field + second-order EoS. In this work we treat the same divergence with a dimensional regularization technique which is for the first time applied to the nuclear many-body problem. The dimensional regularization technique is an elegant procedure which has the advantage of preserving symmetry laws and of including high-energy effects which are sharply discarded with a cutoff regularization. This technique has been introduced in the framework of the electroweak theory [7,8] and consists in replacing the dimension of the divergent integrals with a continuous variable $d$. The main idea is that, if an integral diverges in a given integer dimension, the result may be finite by replacing the integer dimension with a non integer $d$. One then performs a kind of analytic continuation in the dimension to return to the initial integer value [10]. The dimensional regularization eliminates power-law divergences, isolates logarithmic divergences and regularizes infrared divergences [11]. Together with the continuous variable $d$, a regulator $\epsilon$ (which is dimensionless) is introduced so that, when $\epsilon \to 0$, the dimension of the integral comes back to the initial integer value. In addition, an auxiliary scale $\mu$ is included to maintain the correct dimensions of the physical quantities which are calculated. After regularization, a renormalization is applied by a minimal subtraction procedure to remove the regulator $\epsilon$ which appears as a $1/\epsilon$ pole if the divergence is logarithmic. All the physical observables are independent of the auxiliary scale $\mu$ due to the renormalization group equation $\mu dS/d\mu = 0$, where $S$ is a generic observable.

Let us consider the second-order correction beyond the mean-field EoS in our $t_0 - t_3$ model for symmetric nuclear matter [5]. We write its generalized expression where we
introduce a continuous dimension $d$ in the integral and

\[ \frac{\Delta E(\rho)}{A} = \frac{6}{\mu^{3(d-3)}} \frac{\Omega_d^{d-3}}{(2\pi)^{3d}} \frac{g^2(\rho)}{\rho} \int d^d \bar{q} \int_{|\vec{k}_1| < k_F} d^d \vec{k}_1 \frac{d^d \vec{k}_2}{q^2 + q^2 (k_1 - k_2)}. \]  

(1)

In our case the regulator $\epsilon$ can be written in terms of $d$ as $\epsilon = 3 - d$. When $\epsilon \rightarrow 0$, $d$ returns to the integer value 3.

By making some manipulations and by using the Schwinger's proper time representation of Feynman integrals the following compact expression may be derived,

\[ \frac{\Delta E(\rho)}{A} = C_d(\mu, \rho) \frac{g^2(\rho)}{\rho} \int_{C_1} d^d \bar{q} \ J_d(q), \]  

(2)

where the domain of integration is

\[ C_1 = \left[ |\vec{k}_1| < k_F, |\vec{k}_2| < k_F, |\vec{k}_1 + \vec{q}|, |\vec{k}_2 - \vec{q}| > k_F \right] \]  

(3)

with $k_F = \left( \frac{3\pi^2}{2\rho} \right)^{1/3}$. The integrand in Eq. (2) can be written as

\[ J_d(q) = \int_0^\infty d\alpha \ e^{-\alpha q^2} \left[ I_d(\alpha, q) \right]^2, \]
\[ I_d(\alpha, q) = \int_{|\vec{k}| < k_F} d^d \vec{k} \ e^{-\alpha \vec{k} \cdot \vec{q}}, \]
\[ C_d(\mu, \rho) = -\frac{m}{\hbar^2} \frac{6}{\mu^{3(d-3)}} \frac{\Omega_d^{d-3}}{(2\pi)^{3d}} k_F^{3d-2}. \]  

(4)

The quantity $B$ is finite for $|q| < 2$ when $d \rightarrow 3$ and one can show that its value is equal to

\[ 32\pi^3 \left( \frac{38}{63} - \frac{92}{105} \log 2 \right). \]  

(8)

The quantity $A$ may be written as a function of $d$,

\[ A = \frac{\pi^d}{\Gamma(1 + \frac{d}{2})} T(d). \]

By using hypergeometric functions \[13], after straightforward but lengthy manipulations one can write

\[ T(d) = \frac{2^{d-1} \pi}{1 - \frac{d}{2}} 4 \tilde{F}_3 \left[ \begin{array}{c} 1 + d, 1, 2 - d \n 1, \frac{2 - d}{2} : 1 + d, \frac{4 - d}{2}, 1 \end{array} \right] - 1, \]  

\[ \lim_{d \rightarrow 3} T(d) = 16\pi \left( \frac{23}{35} + \frac{36}{35} \log 2 \right). \]  

(9)

$T(d)$ converges for $0 \leq d < 4$ (with a pole at $d = 4$) as shown in Fig. 1. For $d = 3$ the divergence has
been removed by the regularization (the value of \( T(3) \) is positive and finite).

\[
E_A(\rho) = \frac{3\hbar^2}{10\,m} \left( \frac{3\pi^2}{2} \rho \right)^{\frac{2}{3}} + \frac{3}{8}\rho\,g(\rho) + 64\,\pi^3 C_3(\rho)\,g^2(\rho) \left( \frac{1 + 2\log 2}{105} \right),
\]

(10)

One can notice that in Eq. (10) the auxiliary scale \( \mu \) (see Eq. (1)) has disappeared in the second-order correction. This occurs because in our case the divergence is not logarithmic (no 1/\( \epsilon \)-poles are found after regularization). That means that we do not need to renormalize after regularization. We observe also that, as expected, the power-law divergent terms have been removed and do not appear in the final expression. Finally, if we compare Eq. (10) with the asymptotic expression of the second-order correction previously obtained with a cutoff regulator (that is, the asymptotic expression of the result shown in Ref. [3]), we observe that the effect of applying the dimensional regularization technique is not only to remove the divergent term (the linear term in \( \Lambda \) is now absent), but also to renormalize the finite part: with the cutoff regularization, the numerical factor multiplying \( 64\pi^3 C_3(\rho)g^2(\rho) \) in the finite part is equal to \( \frac{1}{105} \left( 11 + 2\log 2 \right) \) whereas in Eq. (10) it is equal to \( \frac{1}{105} \left( 1 + 2\log 2 \right) \). This difference comes out from the subtraction procedure of the integrals \( I_1 \) and \( I_2 \) which renormalizes also the finite part. Notice that this subtractions must be done carefully and the domain of integration must be taken correctly as the intersection of \( C \) and \( |q| > 2 \, (|q| < 2) \) for \( I_1 \) (\( I_2 \)).

In the upper panel of Fig. 2 one can see the regularized second-order EoS (dashed line) with respect to a reference Skyrme mean-field EoS (full line). As in Ref. [3], the SkP [12] mean-field EoS has been chosen as a benchmark. The regularized second-order EoS has been calculated with the SkP parameters. We notice that the second-order correction at the saturation point is \( \sim -8 \) MeV, which is comparable to what obtained in Ref. [3] in the case of a cutoff calculation for a value of the momentum cutoff \( \Lambda = 1.5\,fm^{-1} \). We also notice that the profiles of the two second-order corrections (with a cutoff calculation and with the dimensional regularization) are quite different. The removal of the divergent part in the latter case leads to an equilibrium point located at 0.167 \( fm^{-3} \) which is close to the saturation point of matter. This was not the case for the cutoff calculation where the minimum was appreciably shifted by the second-order term. The shift was actually due to the density dependence of the integral which multiplied the factors \( C_3 \) and \( g^2 \). In the present case the density dependence of the second-order correction is contained only in the coefficients \( C_3 \) and \( g^2 \) and does not modify in practice the position of the equilibrium point which is provided for symmetric matter by the first-order contribution.

Since SkP is a phenomenological interaction where the parameters have been adjusted at the mean-field level, we have to readjust the parameters for the second-order and regularized EoS to have a reasonable curve for the energy. We choose as a reference for this fit the mean-field SkP-EoS as done in Ref. [3]. Upon fitting the regularized EoS, a unique set of Skyrme parameters (no cutoff dependence) is generated (parameters are listed in Table 1). The fit is performed on 15 points on the SkP mean-field curve up to a density of 0.30 \( fm^{-3} \). The procedure is a \( \chi^2 \) minimization with the following definition of \( \chi^2 \),

\[
\chi^2 = \sum_i \frac{(E_{\text{fitted}}(i) - E_{\text{ref}}(i))/\Delta E_i)^2}{2},
\]

The errors \( \Delta E_i \) in the denominator have been chosen equal to 1% of the reference SkP energies. The resulting \( \chi^2 \) per point

FIG. 1. \( T(d) \) as a function of the space-dimension \( d \).

By taking \( d = 3 \), the mean-field + second-order EoS finally reads
TABLE I. Refitted parameters compared with the SkP values for $t_0$, $t_3$ and $\alpha$.

|        | $t_0$ (MeV fm$^3$) | $t_3$ (MeV fm$^{3+3\alpha}$) | $\alpha$ |
|--------|---------------------|-------------------------------|----------|
| Skp    | -2931.70            | 18708.97                      | 0.16667  |
| New parameters | -2893.730       | 17977.603                     | 0.131    |

is equal to 0.547 indicating that, on average, the fitted points deviate less than 1% from the reference SkP-EoS. We can conclude that the fit is of very good quality. This value of the $\chi^2$ per point cannot be compared with the corresponding numbers found in Ref. [5] in the case of the cutoff regularization because a different expression of $\chi^2$ has been used in the two works.

FIG. 2. Upper panel: Regularized second-order (dashed line) and mean-field SkP (full line) EoS. Lower panel: Regularized and refitted second-order EoS (dashed line) compared with the SkP mean-field EoS (full line). The star indicates the position of the saturation point.

The second-order regularized and refitted EoS is shown in the lower panel of Fig. 2. The saturation point is located at 0.161 fm$^{-3}$. The curve is practically superposed to the mean-field SkP-EoS. The resulting pressure $P(\rho) = \rho^2 \frac{d^2}{d\rho^2} \left[ \frac{1}{A}(\rho) \right]$ and incompressibility modulus $K(\rho) = 9\rho^2 \frac{d^2}{d\rho^2} \left[ E(\rho) \right]$ are also shown in Figs. 3 and 4, respectively. We stress that these quantities do not enter explicitly in the fit. The agreement with the corresponding SkP values is good except at densities larger than $\sim 0.25$ fm$^{-3}$. The region around the saturation point has thus reasonable properties. Based on these encouraging results, work is presently in progress to extend the application of the dimensional regularization procedure to the whole Skyrme effective interaction by including also the velocity-dependent terms which are more strongly divergent [10]. This opens new perspectives for future applications of dimensional regularized Skyrme-type effective interactions adjusted at a beyond mean-field level. These interactions would be well adapted to be used in the framework of beyond mean-field approaches for finite nuclei.

FIG. 3. Pressure calculated at second order with the refitted parameters (dashed line) compared with the SkP-mean-field curve (full line).

FIG. 4. Incompressibility modulus calculated at second order with the refitted parameters (dashed line) compared with SkP-mean-field curve (full line).

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[1] Michael E. Peskin, Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Advanced Book Program (1995).
[2] G. Bruun, Y. Castin, R. Dum, and K. Burnett, Eur. Phys. J. D 7, 433 (1999).
[3] A. Bulgac and Y. Yu, Phys. Rev. Lett. 88, 042504 (2002).
[4] M. Grasso and M. Urban, Phys. Rev. A 68, 033610 (2003).
[5] K. Moghrabi, M. Grasso, G. Colò, and N. van Giai, Phys. Rev. Lett. 105, 262501 (2010).
[6] T.H.R. Skyrme, Philos. Mag. 1, 1043 (1956); Nucl. Phys. 9, 615 (1959); D. Vautherin and D.M. Brink, Phys. Rev. C 5, 626 (1972).
[7] Gerard’t Hooft and M.J.G. Veltman, Nucl. Phys. B 44, 189 (1972).
[8] Gerard’t Hooft, Nucl. Phys. B 61, 455 (1973).
[9] C.G. Bollini and J.J. Giambiagi, Nuovo Cimento B 12, 20 (1972).
[10] Kenneth G. Wilson, Phys. Rev. D 7, 29112926 (1973).
[11] Georges Leibbrandt, Rev. Mod. Phys. 47, 849 (1975).
[12] I. Mitra, A. DasGupta, and B. Dutta-Roy, Am. J. Phys. 66, 1101 1109 (1998).
[13] K. Adhikari and A. Ghosh, J. Phys. A 30, 6553 6564 (1997).
[14] J. Dobaczewski, et al., Nucl. Phys. A 422, 103 (1984).
[15] Larry C. Andrews, *Special Functions for Engineers and Applied Mathematics*, Macmillan Pub. Co. (1985).
[16] K. Moghrabi, M. Grasso, X. Roca-Maza, and G. Colò, submitted Phys. Rev. C.