Isochronous solutions of Einstein’s equations and their Newtonian limit

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It has been recently demonstrated that it is possible to construct isochronous cosmologies, extending to general relativity a result valid for non-relativistic Hamiltonian systems. In this paper we review these findings and we discuss the Newtonian limit of these isochronous spacetimes, showing that it reproduces the analogous findings in the context of non-relativistic dynamics.

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1. Introduction

It has been shown [1,2] that, for any general autonomous dynamical system \( D \), it is possible to manufacture another autonomous dynamical system \( ˜D \), featuring two additional arbitrary positive parameters \( T \) and \( ˜T \) with \( T > ˜T \) and possibly also two additional dynamical variables, in such a way that: (1) For the same variables of the original system \( D \) the new system \( ˜D \) yields, over the arbitrarily long time interval \( ˜T \), a dynamical evolution which mimics arbitrarily closely, or possibly even identically, that yielded by the original system \( D \). (2) The system \( ˜D \) is isochronous: all its solutions are completely periodic with an arbitrarily assigned period \( T > ˜T \) for any initial data.

Moreover it has been shown [1,2] that, if the dynamical system \( D \) is a many-body problem characterized by a (standard, autonomous) Hamiltonian \( H \) which is translation-invariant (i. e., it features no external forces), other (also autonomous) Hamiltonians \( ˜H \) characterizing modified many-body problems can be manufactured
which feature the same dynamical variables as $H$ (i.e., in this case there is no need to introduce two additional dynamical variables) and which yield a time evolution arbitrarily close, or even identical, to that yielded by the original Hamiltonian $H$ over the arbitrarily assigned time $\tilde{T}$, while being isochronous with the arbitrarily assigned period $T$, of course with $T > \tilde{T}$.

Let us emphasize that the class of Hamiltonians $H$ for which this result is valid is quite general. In particular it includes the standard Hamiltonian system describing an arbitrary number $N$ of point particles with arbitrary masses moving in a space of arbitrary dimensions $d$ and interacting among themselves via potentials depending arbitrarily from the interparticle distances (including the possibility of multiparticle forces), being therefore generally valid for any realistic many-body problem, hence encompassing most of non-relativistic physics. This result is moreover true, mutatis mutandis, in a quantal context.

Recently these results, which have been obtained in the context of classical Hamiltonian mechanics [1,2], have been extended to a general relativity and cosmological context [3,4,5]. The main feature of these results is to point out the possibility to provide alternative descriptions of the relevant physics which are equivalent locally in time (for arbitrarily long time intervals) to the phenomenology being described by the more standard approaches but predict instead a cyclic behavior (of course over longer time periods).

In our previous paper [3] we have exploited this idea in the context of cosmology, and have shown that, for any non-degenerate, homogeneous, isotropic and spatially flat metric $g_{\mu\nu}$ satisfying Einstein’s equations and providing a model of the universe at large scales, it is possible to find a different (also homogeneous, isotropic and spatially flat but degenerate) metric solution $\tilde{g}_{\mu\nu}$ which is locally (in time) diffeomorphic to $g_{\mu\nu}$ and is periodic with an arbitrary period $T > \tilde{T}$ in the time coordinate $t$; but it is degenerate at an infinite, discrete sequence of times $t_n = t_0 \pm nT/2$, $n = 0, 1, 2, \ldots$. We interpreted these metrics as corresponding to isochronous cosmologies [3].

We emphasize that, due to the diffeomorphic correspondence between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ locally in time—for time intervals of order $\tilde{T}$—these two metrics give the same physics locally in time, and therefore there is no way to distinguish them using observations local in time; which is essentially the same finding valid in the context of the Hamiltonian systems considered in [1,2] and tersely recalled above. We stress that, in the general relativistic context considered in this paper, locally in time refers to time intervals smaller than the period $T$ of the isochronous metric yet of cosmological scale (billions of years). Indeed, the statement that the two metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ give the same physics locally in time means that this property is valid over time intervals smaller than, but of the same order as $T$, while any difference between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ can be measured only through observations which last more than $T$. Obviously, observations over a time period larger than $T$ cannot be considered humanly feasible, so two cosmologies which can only be distinguished via such
experiments are, in the context of human science, indistinguishable; even though the corresponding cosmologies may be de facto quite different, for instance one featuring a Big Bang, the other being cyclic but singularity free for all time.

In fact, while being cyclic on time scales larger than $\bar{T}$, the metric $\bar{g}_{\mu\nu}$ may yield over the time interval $\bar{T}$ just the same cosmology which characterize the metric $g_{\mu\nu}$ of the standard $\Lambda$-CDM cosmological model (see for instance [7] for a review), consistently with all observational tests [8,9,10,11,12,13,14,15,16,17,18]. Furthermore, it has been shown that $\bar{g}_{\mu\nu}$ can be manufactured to be geodesically complete [3,4] and therefore singularity-free so that the geodesic motion as well as all physical quantities described by scalar invariants are always well defined, and the Big Bang singularity may be avoided—even when some of the phenomenological observations can nevertheless be interpreted as remnants of a past Big Bang, which however may never be actually attained by the metric $\bar{g}_{\mu\nu}$, neither in the past nor in the future.

In [4] it has been pointed out that the realization of the isochronous spacetimes is not restricted to homogeneous, isotropic and spatially flat metrics, but it can be easily extended to any synchronous metric. Therefore our result is quite general, since any metric can be written in synchronous form by a diffeomorphic change of coordinates. Furthermore, it has been shown that isochronous spacetimes can be realized in two different ways: the first by means of isochronous metrics that are degenerate at a discrete set of instants $t_n$ when the time reversals occur; the second via non-degenerate metrics featuring a jump in their first derivatives at the inversion times $t_n$, which then implies a distributional contribution in the stress-energy tensor at $t_n$, see the discussion in [4].

A remark on the terminology: since the geodesics of the isochronous metrics are spirals [3,4] of the form $x(\lambda) = [\lambda, \bar{V}(\lambda)]$ with $\bar{V}(\lambda)$ periodic, we write below that isochronous metrics correspond to spiralling geometries.

Let us clarify that, being degenerate, such isochronous metrics are considered unacceptable in orthodox general relativity, although they are quite consistent with the foundational view underling general relativity that the basis of cosmology is the geometry of spacetime rather than the coordinate system—i. e., the metric—used to describe that spacetime. In this context a spiraling geometry should be acceptable; indeed, if it entails no singularity of the metric but only a degeneracy, it should perhaps be considered more acceptable than a geometry entailing singularities such as those associated with the occurrence of a Big Bang and/or a Big Crunch. In any case the acceptability or not of a class of metrics should be ultimately decided by experiment, not by a priori assumptions (for instance, that the metric be non-degenerate); if such experiments are unfeasible, there is no justifica-

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*Since periodicity is not an invariant concept (it is not invariant under a redefinition of time), it is usually preferred to talk of cyclic instead of periodic solutions of Einstein’s equations. For cyclic models in standard general relativity see [2].

*A spacetime is singularity-free if it is geodesically complete, i.e. if its geodesics can be always past- and future-extended [19,20].
tion to exclude in principle the consideration of such metrics. Hence, our point of view that a spiraling geometry of spacetime should be considered acceptable even if it requires a relaxation of the requirement that the metric be locally Minkowskian everywhere. On the other hand let us also note that an alternative description of a spacetime characterized by a cyclic evolution is also compatible with non-degenerate metrics, but at the cost of introducing distributional energy-momentum tensors: see [4].

In this paper we investigate the weak field (or Newtonian) limit of isochronous spacetimes. In particular we show that in this limit every isochronous metric reduces to a solution of an isochronous dynamical system $\tilde{D}$ that is obtained from a Newtonian $N$-body system through the techniques [1,2] which have been introduced in the context of autonomous dynamical systems. In fact, our purpose and scope is to provide a bridge among the results valid respectively in the general relativity and in the classical mechanics contexts, by revisiting in the new context [3,4,5] the standard derivation of Newtonian gravitational physics from general relativity.

This paper is organized as follows: in section 2 we review some mathematical aspects of isochronous metrics, which shall be useful for the study of the Newtonian limit; this treatment includes an extended review of previous findings [3,4,5], which we considered appropriate given the somewhat "unpleasant"—yet in our opinion cogent—implications of these findings, which do imply a substantial limitation of all cosmological theories based on general relativity. In section 3 we investigate the weak field limit of our treatment of general relativity and thereby derive the corresponding Newtonian framework. Finally, we conclude with some general considerations about our findings, as reported in the last section.

2. Isochronous solutions of Einstein’s equations

In this section we review the properties of the isochronous metrics discussed in [3,4,5]. In what follows Latin indexes such as $k, h, i, \ldots$ run over the integers from 1 to 3 and Greek indexes such as $\alpha, \beta, \gamma, \ldots$ run over the integers from 0 to 3. Let us start from a given metric tensor $g_{\mu\nu}(y)$ in a coordinate system $y$ defined by

$$ds^2 = g_{\mu\nu}(y) \, dy^\mu \, dy^\nu,$$

which is a solution of the Einstein’s equations

$$G_{\mu\nu}(y) = \frac{8\pi G}{c^4} T_{\mu\nu}(y),$$

where $G_{\mu\nu}(y) = R_{\mu\nu}(y) - R(y) \, g_{\mu\nu}(y)/2$ is the Einstein’s tensor constructed with the metric $g_{\mu\nu}(y)$, and $T_{\mu\nu}(y)$ is the energy-momentum tensor in the reference system $y$.

As in the case of non relativistic Hamiltonian many body systems, the trick to construct isochronous solutions is based on the introduction of a "periodic change of time". In the context of general relativity, this is realized formally by introducing
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the following ”change of coordinates”,

\[ dy^0 = b(x^0) \, dx^0, \quad dy^k = dx^k, \]

where the function \( b(x^0) \) is periodic with an \textit{arbitrarily assigned} period \( T, b(x^0 + T) = b(x^0) \), having moreover a vanishing mean value, so that the function \( \tau(x^0) \) defined by

\[ \tau(x^0) \equiv \int_0^{x^0} b(z) \, dz \]

is also periodic with period \( T \), that is \( \tau(x^0 + T) = \tau(x^0) \). As a consequence of these assumptions, the function \( b(x^0) \) must vanish at an infinite set of time-like hypersurfaces \( \Sigma_n \) corresponding to the times \( x^0 = x^0_n \) where \( b(x^0_n) \) changes its sign; thus \( \Sigma_0 \) is not a global, but only a local diffeomorphism.

Let us consider a different metric \( \tilde{g}_{\mu\nu}(x) \) in the coordinate system \( x \), defined by

\[ ds^2 = b(x^0)^2 g_{00}(y(x)) \, (dx^0)^2 + 2 b(x^0) g_{0k}(y(x)) \, dx^0 dx^k + g_{kh}(y(x)) \, dx^k dx^k \equiv \]

\[ \equiv \tilde{g}_{\mu\nu}(x) \, dx^\mu dx^\nu, \]

where \( g_{\alpha\beta}(y(x)) = g_{\alpha\beta}(y^0 = \tau(x^0), y^k = x^k) \). Such metric will be a solution of the equations

\[ \tilde{G}_{00}(x) = b(x^0)^2 G_{00}(y(x)) = b(x^0)^2 \frac{8\pi G}{c^4} T_{00}(y(x)) = \frac{8\pi G}{c^4} \tilde{T}_{00}(x), \]

\[ \tilde{G}_{0k}(x) = b(x^0)^2 G_{0k}(y(x)) = b(x^0)^2 \frac{8\pi G}{c^4} T_{0k}(y(x)) = \frac{8\pi G}{c^4} \tilde{T}_{0k}(x), \]

\[ \tilde{G}_{kh}(x) = G_{kh}(y(x)) = \frac{8\pi G}{c^4} \tilde{T}_{kh}(y(x)) = \frac{8\pi G}{c^4} \tilde{T}_{kh}(x), \]

where \( \tilde{G}_{\mu\nu}(x) = \tilde{R}_{\mu\nu}(x) - \tilde{R}(x) \tilde{g}_{\mu\nu}(x)/2 \) is the Einstein’s tensor constructed with the metric \( \tilde{g}_{\mu\nu}(x) \), and \( \tilde{T}_{\mu\nu}(x) \) is defined by \( \Box \). From \( \Box \) it is evident that the metric \( \Box \) satisfies Einstein’s equations everywhere except at the hypersurfaces \( \Sigma_n \), where it is degenerate, and where all the quantities containing the inverse tensor \( \tilde{g}^{\mu\nu} \), as the affine connection \( \tilde{\Gamma}^\alpha_{\beta\gamma} \) and the Ricci and Riemann curvature tensors, are not defined. However we note that, even if it is degenerate on the hypersurfaces \( \Sigma_n \), the curvature tensor \( \tilde{g}_{\mu\nu}(x) \) is continuous and differentiable at \( \Sigma_n \), provided \( g_{\mu\nu}(y) \) is differentiable on the hypersurfaces \( \Sigma_n \) defined by \( y^0 = y^0_0 \equiv \tau(x_n) \). Moreover, the scalar curvature \( \tilde{R}(x) = R(y(x)) \) is also continuous on \( \Sigma_n \), provided \( R(y) \) is regular on \( \Sigma_n \). This resembles the divergence of the Schwarzschild metric on the Schwarzschild horizon, where the metric tensor diverges but the Ricci scalar curvature \( \tilde{R} \) remains finite, and the singularity in the metric tensor does not imply a physical singularity on the horizon. However, the situation here is different in the sense that the metric \( \tilde{g}_{\mu\nu}(x) \) is differentiable but degenerate.
on $\Sigma_n$, and it has to be interpreted in the framework of degenerate solutions of Einstein’s equations (see the discussion in \cite{4}). Indeed, the metric \cite{5} does not entail any physical singularity on $\Sigma_n$, since all the physical quantities (such as, for instance, the Ricci scalar $\tilde{R}$) remain finite there, but it corresponds to a metric tensor which is periodic in time and becomes degenerate at the infinite set $\Sigma_n$. The degeneracy of the metric tensor implies the failure of the equivalence principle on $\Sigma_n$, in its formulation that states that the signature of the metric tensor must be Minkowskian. However, it is a matter of discussion whether this fact could have observable effects in an isochronous world, where the dynamics of all the universe, including that of its components, is periodic.

Degenerate metrics have been studied in the past, motivated by the consideration of the so called signature-changing metrics \cite{21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39}, which give a classical realization of the change of signature in quantum cosmology conjectured by Hartle and Hawking \cite{40,41,42,43,44}. The classical change of signature for a homogeneous, isotropic and spatially flat universe is realized by a metric tensor defined by the following line element,

$$ds^2 = N(t) \, dt^2 - a(t)^2 \, d\vec{x}^2,$$

the lapse function $N(t)$ being a continuous function changing sign at some time $t_0$ where the metric \cite{4} is degenerate. Note that this situation is physically different for that of the metric tensor \cite{5}, where the lapse function $N(t) = b^2(t)$ vanishes on $\Sigma_n$, but it never becomes negative.

The inclusion of degenerate metrics as \cite{5} and \cite{7} in the context of general relativity does not by itself entail well defined physical implications, which also depend on the prescriptions defining the behavior at the points of degeneracy of the metric: different prescriptions yield different physical theories \cite{24}. In particular different generalizations based on different prescriptions give different junction conditions on the degeneracy hypersurfaces, see for instance the discussion in \cite{23,24}. In fact "one can also consider discontinuities in the extrinsic curvature" \cite{23} on the degeneracy hypersurface of a degenerate solution of Einstein’s equations, "but attempts to relate these to a distributional matter source at the boundary require some form of field equations valid on the surface" \cite{24}, and this would require some special prescription on what Einstein’s equations are on the degeneracy hypersurfaces. Thus, if one accepts degenerate solutions of Einstein’s equations as physically acceptable spacetimes, there is no reason to infer that a discontinuity of the extrinsic curvature on a degeneracy surface implies that the stress energy tensor associated to such a metric tensor must have a distributional form there.

One might object that the isochronous metric \cite{5} entails a discontinuity of the extrinsic curvature $K_{ab}$ on the hypersurfaces $\Sigma_n$, which, in the context of Einstein’s theory, means that the energy-momentum tensor has distributional character on $\Sigma_n$. The point here is that the construction which implies this relation is no longer valid for isochronous metrics of the type \cite{6}, since the unitary normal vectors to the hypersurfaces $\Sigma_n$ do not exist because the metric is degenerate on $\Sigma_n$. 

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To discuss this point, let us recall the Darmois formalism [45], which defines the junction condition under which two different metric tensors can be joined on some hypersurface. Let us consider a hypersurface $\Sigma$ dividing the spacetime in two regions $V^+$ and $V^-$. The condition that the two metrics $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$ in the two regions $V^+$ and $V^-$ must satisfy in order to join smoothly on $\Sigma$ is that they must be the same on both sides of $\Sigma$ together with their first derivatives, see for instance Eq.(3.7.7) in [19], that is

$$g^{(+)}_{\mu\nu}|_{\Sigma} = g^{(-)}_{\mu\nu}|_{\Sigma},$$

(8a)

$$g^{(+)}_{\mu\nu,\sigma}|_{\Sigma} = g^{(-)}_{\mu\nu,\sigma}|_{\Sigma}.$$  

(8b)

From (8) it is therefore evident that the isochronous metric (5) satisfies such junction conditions on the hypersurfaces $\Sigma_n$, where it is infinitely differentiable.

In Einstein’s gravity (which implies a restriction to non-degenerate metrics) such a condition is usually expressed in an invariant way by use of the extrinsic curvature, which is diffeomorphism-invariant. Actually the second equation in (8) is expressed in terms of the derivatives $g_{\mu\nu,\sigma}$ of the metric tensor, which are not themselves tensors. Therefore (8) is more conveniently recast in tensor form as

$$g^{(+)}_{\mu\nu}|_{\Sigma} = g^{(-)}_{\mu\nu}|_{\Sigma}, \quad K^{(+)}_{ab}|_{\Sigma} = K^{(-)}_{ab}|_{\Sigma},$$

(9)

where $K_{ab}$ is the extrinsic curvature of $\Sigma$ defined as

$$K_{ab} = n_{\alpha;\beta} e^{\alpha}_a e^{\beta}_b,$$

(10)

with $n^\alpha$ the unitary normal vector to $\Sigma$ and $e^\alpha_a$ three unitary tangent vectors to $\Sigma$, so that $g_{\alpha\beta} = n^\alpha n_\beta = e^{a\alpha} e^b_\beta \delta^{ab}$, where $n_\alpha = g^{\alpha\beta} n^\beta$ and $e^{a\alpha} = g^{\alpha\beta} e^{b}_\beta$, and $g|_{\Sigma \Sigma} \equiv g_{\alpha\beta} e^{a\alpha}_a e^{b}_b$ is the induced metric over $\Sigma$. Note that (8) is more general than (9), since it is valid even where the extrinsic curvature does not exist. In this context a discontinuity of the extrinsic curvature on the hypersurface $\Sigma$ is associated with a singularity of the stress-energy tensor, which turns out to have distributional character (corresponding to the presence of thin shells) on $\Sigma$, acquiring a distributional contribution given by

$$T^{\text{distr}}_{\alpha\beta} = \delta_{\Sigma} S_{ab}|_{\Sigma} e^{a\alpha}_a e^{b}_b,$$

(11)

where $\delta_{\Sigma}$ is the delta function with support on the hypersurface $\Sigma$, and

$$S_{ab}|_{\Sigma} \equiv \frac{1}{8\pi} \left( [K_{ab}]|_{\Sigma} - [K]|_{\Sigma} \tilde{g}|_{\Sigma ab} \right),$$

(12)

with

$$[K_{ab}]|_{\Sigma} \equiv K_{ab}|_{\Sigma^+} - K_{ab}|_{\Sigma^-}, \quad [K]|_{\Sigma} \equiv [K_{ab}]|_{\Sigma} g^{ab}|_{\Sigma}.$$  

(13)

However, this result is valid in Einstein’s theory, which assumes that the metric tensor has Minkowskian signature, and therefore is non-degenerate. The case of the isochronous metric $\tilde{g}$ given in (5) is different, since $\tilde{g}$ is differentiable on $\Sigma_n$ and
therefore it satisfies the junction conditions in the form (8); however, since $n^a$ is not defined on $\Sigma_n$, the junction conditions cannot be recast in the form (9). Therefore, for degenerate metrics, the formalism which leads to (11) does not apply, and one is not allowed to conclude that a discontinuity of the extrinsic curvature implies that the stress energy tensor has distributional character on $\Sigma_n$.

What is more, delta functions with support on degeneracy hypersurfaces have no significant effect in covariant integrals [36,39], due to the fact that the covariant volume element $dx^4\sqrt{-\tilde{g}}$ associated to a degenerate metric is null on its degeneracy hypersurface, because the determinant $\tilde{g}$ vanishes there. For instance, for the metric (5) one has that $\int dx^4\sqrt{-\tilde{g}}\delta(t-t_n)f(x) = 0$, for any integrable function $f(x)$. Hence delta-like distributions centered on the degeneracy hypersurfaces $t = t_n$ disappear from integrals (see also the discussion in [36]).

So, the question of how to include degenerate metrics in general relativity is moot. In order to formulate a generalized gravitational theory which includes such metrics, it is convenient to focus directly on the Einstein’s equations characterizing the geometry of spacetime, which themselves admit degenerate metrics as their solutions. If, however, one considers desirable to derive this theory from a variational principle, a possibility is to assume the standard Einstein-Hilbert gravitational action of general relativity, such that the total action of gravitation plus the matter (nongravitational) fields is

$$S_{Tot} = \int_M \sqrt{-\tilde{g}} \left( -\frac{1}{2k} \tilde{R} + L_{Mat} \right) d^4x ,$$

(14)

where $k = 8\pi G/c^4$, $G$ is the gravitational constant, $c$ the speed of light, $M$ is the manifold defining the spacetime and $L_{Mat}$ is the Lagrangian density of matter fields; and to add the requirement that, for a metric tensor which is degenerate on some hypersurface $\Sigma$, the variation of the metric $\tilde{g}_{\mu\nu}$ vanish on $\Sigma$, i.e. $\delta\tilde{g}_{\mu\nu}|_\Sigma = 0$.

With such assumption the variation of the action (14) is

$$\delta S_{Tot} = \frac{1}{2} \int_{M/\Sigma} \sqrt{-\tilde{g}} \left( -\frac{1}{k} \tilde{G}^{\mu\nu} + \tilde{T}^{\mu\nu} \right) d^4x ,$$

(15)

where $\tilde{G}^{\mu\nu}$ is the Einstein tensor and $\tilde{T}^{\mu\nu}$ is the stress-energy tensor of matter fields, which gives the standard equations of general relativity in the region where the metric tensor is non-degenerate.

To bypass the problems related to degenerate signature-changing metrics, it has been proposed [21] to consider a different, non-degenerate but discontinuous, realization of the classical change of signature, as given by the metric

$$ds^2 = f(\tau) d\tau^2 - \alpha(\tau)^2 d\vec{x}^2,$$

(16)

with a discontinuous lapse function $f(\tau) = 1$ for $\tau > \tau_0$ and $f(\tau) = -1$ for $\tau < \tau_0$. It has been shown [21] that in this case it is possible to introduce a smooth (generalized) orthonormal reference frame which allows a variational derivation of Einstein’s equations. In such case, the Darmois formalism [45] applies, in such a
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way that the discontinuity of the extrinsic curvature on a hypersurface is related to a distributional stress energy tensor \(^3\). This is achieved by adding a surface term to the Einstein-Hilbert action, in such a way that the gravitational action becomes

\[
S_g = \int_{M/\Sigma} \sqrt{-g} R \, dt \, dx^d + \oint_{\Sigma} \sqrt{-g} K \, d\Sigma,
\]

where \(M\) is the manifold defining the spacetime, \(\Sigma\) is the boundary of \(M\) where the metric tensor is discontinuous, \(R\) is the Ricci scalar curvature and \(K\) the extrinsic curvature of \(\Sigma\). The addition of such a boundary term is always necessary when one considers manifolds with boundaries \(^4\). Incidentally we note that this action cannot be used in the case of degenerate metrics, because in this case the degeneracy surface \(\Sigma\) has no unitary normal vector hence the extrinsic curvature \(K\) does not exist.

The two realizations of the classical change of signature, the continuous and degenerate one, see (7), and the discontinuous one, see (16), are locally but not globally diffeomorphic, since they are related by a change of the time variable \(dt = d\tau/\sqrt{|N(t)|}\) which is not defined at the time of signature change \(t_0\) when \(N(t) = 0\). Hence they represent two different spacetimes. However, except for the instants when \(N(t)\) vanishes, they are locally diffeomorphic and thus they describe—excepts at those instants—the same physics.

As in the case of signature-changing metrics, it is also possible to consider a realization of isochronous cosmologies different from (5), via non-degenerate continuous metrics featuring a finite jump of their first derivatives at an infinite, discrete set of equispaced times. Such a realization is given by the formal change of time \(d\hat{\tau} = b(x^0)dx^0\) in (5), corresponding to \(dy^0 = C(\hat{\tau}) \, d\hat{\tau}\) in (1), with \(C(\hat{\tau}) = 1\) for \(2nT < \hat{\tau} < (2n + 1)T/2\) and \(C(\hat{\tau}) = -1\) for \((2n + 1)T/2 < \hat{\tau} < (2n + 1)T\) and \(n\) integer, giving

\[
d\tilde{s}^2 = g_{00}(y(\hat{\tau})) \, (d\hat{\tau})^2 + 2g_{0k}(y(\hat{\tau})) \, d\hat{\tau} dy^k + g_{kk}(y(\hat{\tau})) \, dy^k dy^k \equiv \tilde{g}_{\mu\nu}(x) \, dx^\mu \, dx^\nu.
\]

The metric (18) is isochronous and it is continuous but not differentiable at the set of times \(\hat{\tau}_n = nT\), and it entails a discontinuity of the extrinsic curvature at this infinite set of times. Since, as in the case of (16), it is possible to introduce a smooth (generalized) orthonormal reference frame for (18), the Darmois formalism \(^5\) applies and therefore one concludes that the discontinuity of the extrinsic curvature on \(\hat{\tau}_n\) is related to a distributional stress energy tensor \(^3\).

At this point, we must emphasize that the two metrics (5) and (18) are two different realizations of an isochronous spacetime, since they are locally, but not globally, diffeomorphic via the change of variable \(d\tilde{\tau} = b(x^0)dx^0\), which is singular at \(\Sigma_n\). In particular, both (5) and (18) are locally (but not globally) diffeomorphic to the Einsteinian metric (11), and therefore they give an identical dynamics in any region of spacetime where \(x^0 \neq x_0^0\) or \(\hat{\tau} \neq \hat{\tau}_n\). The main difference is that the isochronous evolution given by (5) entails the introduction of degenerate metrics and the failure of the equivalence principle at \(\Sigma_n\), and the non-degenerate realization
of the isochronous dynamics given by (18) implies the existence of a distributional contribution to the stress-energy tensor. In both cases the evolution of the universe is locally the same as that described by Einstein’s equations, implying that it is not possible to discriminate by experiments local in time between a cyclic and a noncyclic evolution of the universe.

A last consideration concerns the geodesic completeness of the isochronous metric (5). In fact isochronous spacetimes can be manufactured to be geodesically complete, that is singularity free. This happens for instance if the Einsteinian metric (1) is singularity free for \( y^0 \in \mathcal{Y}^0 \), where the interval \( \mathcal{Y}^0 \) is defined as the image of the real line \( \mathbb{R} \) through the function \( \tau(x^0) \), in formulas \( \mathcal{Y}^0 : \{ y^0 = \tau(x^0) \} \), \( \forall x^0 \in \mathbb{R} \). An example of singularity free isochronous metric has been discussed in [3,4,5], obtaining an isochronous realization of the homogeneous and spatially flat Friedmann-Robertson-Walker metric with no big bang singularity.

To derive the properties of the geodesics of the metric (5) one simply notes that these geodesics can be obtained from those of (1). In fact the geodesic equation for (1) is

\[
\frac{d^2y^\beta}{d\lambda^2} g_{\alpha\beta} + \Gamma_{\alpha\beta\gamma} \frac{dy^\beta}{d\lambda} \frac{dy^\gamma}{d\lambda} = g_{\beta\alpha} \frac{dy^\beta}{d\lambda} \frac{1}{\tilde{L}},
\]

(19)

where \( \Gamma_{\alpha\beta\gamma} \) are the Christoffel symbols constructed with the metric \( g_{\mu\nu} \), \( \lambda \) is a parameter used to parameterize the geodesic and \( \tilde{L} = \sqrt{(dy^\alpha/d\lambda)(dy^\beta/d\lambda)g_{\alpha\beta}} \) for timelike geodesics, \( \tilde{L} = \sqrt{-(dx^\alpha/d\lambda)(dx^\beta/d\lambda)\tilde{g}_{\alpha\beta}} \) for spacelike geodesics (an analogous treatment applies to null geodesics).

In the same way, the geodesics of (5) satisfy the equation

\[
\frac{d^2x^\beta}{d\tilde{\lambda}^2} \tilde{g}_{\beta\alpha} + \tilde{\Gamma}_{\beta\alpha\gamma} \frac{dx^\beta}{d\tilde{\lambda}} \frac{dx^\gamma}{d\tilde{\lambda}} = \tilde{g}_{\beta\alpha} \frac{dx^\beta}{d\tilde{\lambda}} \frac{1}{\tilde{\tilde{L}}},
\]

(20)

where again \( \tilde{\Gamma}_{\beta\alpha\gamma} \) are the Christoffel symbols constructed with \( \tilde{g}_{\mu\nu} \), \( \tilde{\lambda} \) is a parameter and \( \tilde{\tilde{L}} = \sqrt{(dx^\alpha/d\lambda)(dx^\beta/d\lambda)\tilde{g}_{\alpha\beta}} \) for timelike geodesics, \( \tilde{\tilde{L}} = \sqrt{-(dx^\alpha/d\lambda)(dx^\beta/d\lambda)\tilde{g}_{\alpha\beta}} \) for spacelike geodesics.

Note that (20) is adequate to define the geodesics even where the metric \( \tilde{g}_{\alpha\beta} \) is not invertible. Also note that we have not chosen \( d\lambda \) to coincide with the line element \( ds \), since in the case of the isochronous metric (5) the line element vanishes at \( \Sigma_n \) and therefore is not a good parameter for the geodesics. Moreover, when \( \lambda \) is not identified with \( s \), the equations (11) for \( \alpha = 0 \) reduce to an identity; indeed the functions \( y^0(\lambda) \) and \( x^0(\lambda) \) can be chosen arbitrarily. This arbitrariness reflects the fact that the action

\[
\mathcal{L} = \int \sqrt{\tilde{g}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} d\lambda,
\]

(21)

from which the geodesic equation is deduced, is invariant under the reparameterization \( \lambda \to f(\lambda) \), and therefore the arbitrariness in the choice of \( \lambda \) implies the arbitrariness in the choice of \( y^0(\lambda) \) and \( x^0(\lambda) \). However, when \( \lambda \) is forced to co-
incide with \( s \), the functional form of \( y^0(\lambda) \) and \( x^0(\lambda) \) is fixed. We will use this property of the geodesic equation in the next section.

It is straightforward to show that the geodesics of (1) are given by

\[
x(\lambda) = [\lambda, \vec{y}(\tau(\lambda))],
\]

(22)

provided that the geodesics of (1) have the expression

\[
y(\lambda) = [\lambda, \vec{y}(\lambda)].
\]

(23)

It is therefore evident that if \( \vec{y}(\tau(t)) \) is nonsingular, the geodesics (22) are always definite, and (1) is geodesically complete, thus singularity free, and this happens if the region of the spacetime (1) such that \( y^0 \in \mathcal{Y}^0 \) is singularity free.

As we have already emphasized in the introduction, since \( \tau(\lambda) \) is periodic the geodesics (22) are spirals, and we refer to this feature when we say that isochronous metrics correspond to spiralling spacetimes.

### 3. Newtonian limit of the isochronous solutions

Let us start this section by discussing the Newtonian limit for a system of point-like bodies in the framework of general relativity, see [47] for a review of the Newtonian limit in general relativity.

Let us first consider the metric (1). We make the hypothesis that the gravitational field is weak and the bodies under consideration are moving slowly. Moreover, we assume that the metric tensor is almost Minkowskian, so that

\[
\eta_{\mu\nu} + h_{\mu\nu}(y),
\]

(24)

where \( \eta_{\mu\nu} \) is the Minkowski metric, and the weak field and slow motion approximations read respectively

\[
|h_{\mu\nu}| \ll 1 \quad \frac{dy^k_i}{d\lambda_i} \ll \frac{dy^0_i}{d\lambda_i}, \quad k = 1, 2, 3.
\]

(25)

Solving the linearized Einstein's equations one has [47]

\[
h_{00}(y) \simeq -2 G \sum_i \left[m_i \frac{\theta(|\vec{y} - \vec{y}_i|)}{|\vec{y} - \vec{y}_i|}\right],
\]

(26)

where

\[
y_i(\lambda_i) = [y^0_i(\lambda_i), \vec{y}_i(\lambda_i)]
\]

(27)

is the geodesic trajectory of the \( i \)-th point-like particle of mass \( m_i \), parameterized as a function of the (unphysical) parameter \( \lambda_i \), and \( \theta \) is a "smoothened" version of the step function such that \( \theta(z) \simeq 1 \) for \( z > 0 \) and \( \theta(z) \simeq 0 \) for \( z < 0 \) and \( \lim_{z \to 0+} \theta(z)/z = 0 \).
The relation between $h_{00}$ and the Newtonian gravitational potential $\phi (y)$ is

$$
\phi (y_i) = - \sum_{k \neq i} \frac{m_i}{|y_i - \vec{y}_k|},
$$

(28)

The geodesic equations of motion (19) for the point-like objects of mass $m_i$ and trajectory $y_i(\lambda_i)$ read

$$
d\frac{y_i^\alpha}{d\lambda_i} \gamma_{\alpha \beta \gamma}(y_i) \frac{d y_i^\beta}{d\lambda_i} \frac{d y_i^\gamma}{d\lambda_i} = g_{\beta \gamma}(y_i) \frac{d y_i^\beta}{d\lambda_i} \frac{d y_i^\gamma}{d\lambda_i},
$$

(29)

where $L(y_i) = \sqrt{\frac{d y_i^\alpha}{d\lambda_i} \gamma_{\alpha \beta \gamma}(y_i) \frac{d y_i^\beta}{d\lambda_i} \frac{d y_i^\gamma}{d\lambda_i}}$ since we are considering massive objects, i.e. space-like geodesics.

In the weak field and the slow motion approximations, and assuming that the gravitational field is slowly varying in time, so that $\Gamma_{000} = \frac{h_{00,0}}{2} \simeq 0$ and $\Gamma_{i00} = h_{0i,0} - \frac{h_{00,1}}{2} \simeq -\frac{h_{00,1}}{2}$, from (29) one has

$$
\frac{d^2 y_i^0}{d\lambda_i^2} + \Gamma_{000} \left( \frac{d y_i^0}{d\lambda_i} \right)^2 \simeq \frac{d^2 y_i^0}{d L(y_i)} \frac{d L(y_i)}{d\lambda_i},
$$

(30)

$$
- \frac{d^2 y_i^k}{d\lambda_i^2} + \Gamma_{i00} \left( \frac{d y_i^0}{d\lambda_i} \right)^2 \simeq - \frac{d^2 y_i^k}{d\lambda_i^2} + \frac{1}{2} \frac{\partial h_{000}(y_i)}{\partial y_i^k} \left( \frac{d y_i^0}{d\lambda_i} \right)^2 \simeq - \frac{d y_i^k}{d\lambda_i} \frac{d L(y_i)}{d\lambda_i},
$$

(31)

where, above and hereafter, $k = 1, 2, 3$.

Using (29) one has

$$
L(y_i) \simeq \left| \frac{d y_i^0}{d\lambda_i} \right| \Rightarrow \frac{1}{L(y_i)} \frac{d L(y_i)}{d\lambda_i} \simeq \left| \frac{d y_i^0}{d\lambda_i} \right|^{-1} \frac{d y_i^0}{d\lambda_i} \left( \frac{d y_i^0}{d\lambda_i} \right)^{-1} \frac{d^2 y_i^0}{d\lambda_i^2},
$$

(32)

so that (30) becomes an identity, hence the functional form of $y_i^0(\lambda_i)$ is not fixed. As we have already noted, this is due to the arbitrariness of the action (21) under the reparameterization $\lambda \to f(\lambda)$. Furthermore, (31) becomes

$$
\frac{d^2 y_i^k(y_i^0)}{d y_i^0} \frac{d y_i^0}{d\lambda_i^2} = - \frac{1}{2} \frac{\partial h_{000}(y_i)}{\partial y_i^k} \left( \frac{d y_i^0}{d\lambda_i} \right)^2.
$$

(33)

Since the functional form of $y_i^0(\lambda_i)$ is not fixed by the dynamics, being $\lambda_i$ an unphysical parameter, one has the freedom to choose $y_i^0 = \lambda_i$, so that (33) reduces to

$$
\frac{d^2 y_i^k(y_i^0)}{d y_i^0} \frac{d y_i^0}{d\lambda_i^2} = \frac{1}{2} \frac{\partial h_{000}(y_i)}{\partial y_i^k} = - \left[ \nabla_{y_i} \phi (y_i) \right]^{(k)},
$$

(34)
where the Newtonian potential $\phi(y_i)$ is given by

$$\phi(y_i) = \frac{1}{2} h_{00}(y_i) = -G \sum_{k \neq i} \left[ m_i \frac{1}{|y_i - \hat{y}_k|} \right].$$

Finally, since time is absolute in Newtonian gravity, we set $y^0_i = \lambda_i = t$ in (34) for all the point-like masses $m_i$, obtaining Newton’s equations

$$\frac{d^2 y_j^k(t)}{dt^2} = -\partial y^i \phi(y_i).$$

At this point we can analyze the Newtonian limit in the corresponding isochronous metric (5). Following our prescription (3), we can write the isochronous realization of the weak field and slow motion limit as

$$\tilde{g}_{00} = b^2 (x_0) \ (\eta_{00} + h_{00}) , \quad \tilde{g}_{0i} = b(x_0) \ (\eta_{0i} + h_{0i}) , \quad \tilde{g}_{ij} = \eta_{ij} + h_{ij},$$

where $h_{\alpha\beta} = h_{\alpha\beta}(y_i(x_i))$.

The geodesic equations (20) for the $i$-th point-like mass read

$$\frac{d^2 x^0_i}{d\lambda^2_i} \tilde{g}_{00}(x_i) + \tilde{\Gamma}_{00\gamma}(x_i) \frac{dx^0_i}{d\lambda_i} \frac{dx^\gamma_i}{d\lambda_i} = \tilde{g}_{0\alpha}(x_i) \frac{dx^\alpha_i}{d\lambda_i} \frac{1}{L(x_i)} \frac{d\tilde{L}(x_i)}{d\lambda_i},$$

where $\tilde{L}(x_i) = \sqrt{\tilde{g}_{00}(x_i) \tilde{g}_{0\alpha}(y_i)}$ for massive point-like objects. More explicitly, the geodesic equations (20) for the space and time components $x^\alpha$ are

$$\tilde{g}_{00} \frac{d^2 x^0_i}{d\lambda^2_i} + \tilde{\Gamma}_{00\gamma}(x_i) \frac{dx^0_i}{d\lambda_i} \frac{dx^\gamma_i}{d\lambda_i} = b^2 (x_0) \left[ \frac{d^2 x^0_i}{d\lambda^2_i} + \frac{dx^0_i}{d\lambda_i} \frac{1}{L(x_i)} \frac{dL(x_i)}{d\lambda_i} \right],$$

and

$$\tilde{g}_{ij} \frac{d^2 x^j_i}{d\lambda^2_i} + \tilde{\Gamma}_{ij\gamma}(x_i) \frac{dx^j_i}{d\lambda_i} \frac{dx^\gamma_i}{d\lambda_i} = -\frac{dx^k_i}{d\lambda_i} \frac{1}{L(x_i)} \frac{dL(x_i)}{d\lambda_i},$$

where $b'(x^0_i) \equiv db(x^0_i)/dx^0_i$. Since in the limit (25) one has

$$\tilde{L}(x_i) \simeq \left| b(x_0) \frac{d x^0_i}{d\lambda_i} \right| \Rightarrow \frac{1}{L(x_i)} \frac{d\tilde{L}(x_i)}{d\lambda_i} \simeq \left( \frac{d x^0_i}{d\lambda_i} \right)^{-1} \left( \frac{b'(x^0_i)}{b(x^0_i)} \left( \frac{d x^0_i}{d\lambda_i} \right)^2 + \frac{d^2 x^0_i}{d\lambda^2_i} \right),$$

it is easy to recognize that (38) is an identity, as expected. On the other hand (39) becomes

$$-\frac{d^2 x^k_i}{d\lambda^2_i} + b^2 (x_0) \left( \frac{d x^0_i}{d\lambda_i} \right)^2 \frac{\partial_{x^k} h_{00}}{2} =$$

$$= \frac{d x^k_i}{d\lambda_i} \left( \frac{d x^0_i}{d\lambda_i} \right)^{-1} \left[ b'(x^0_i) \left( \frac{d x^0_i}{d\lambda_i} \right)^2 + \frac{d^2 x^0_i}{d\lambda^2_i} \right].$$
Since $x^0(\lambda)$ is arbitrary, we choose $x^0 = \lambda$ so that the last equation becomes

$$
\frac{d^2 x^k_i}{d\lambda^2} - \frac{d x^k_i b'(x^0_i)}{d\lambda_i b(x^0_i)} = -b^2 \left( x^0_i \right) \frac{\partial x^k_i}{\partial x^0_i} \frac{h_{00}(y_i(x_i))}{2}.
$$

(43)

We already know that the geodesics of the metric (5), which now is in the form (37), are related to those of the metric (1) through the relations (22-23). Thus, it is easy to verify that the solutions of (43) are given by

$$
x^k_i(x^0_i) = y^k_i(\tau(x^0_i)),
$$

(44)

where we recall that $\tau(x^0_i) \equiv \int b(x^0)dx^0$ (see (4)), provided that $y^k_i(y^0_i)$ is a solution of equations (34). Again, we set $x^0_i = \lambda_i = t$, so that (43) reads

$$
\frac{d^2 x^k_i}{dt^2} - \frac{d x^k_i b'(t)}{dt b(t)} = -b^2(t) \frac{\partial x^k_i}{\partial x^0_i} \frac{h_{00}(y_i(x_i))}{2},
$$

(45)

while the isochronous trajectories (44) of the point-like masses becomes

$$
x^k_i(t) = y^k_i(\tau(t))
$$

(46)

Therefore the dynamics of $N$ point-like bodies in an isochronous spacetime in the Newtonian limit is given by (46), so that it is itself isochronous, as expected. More precisely, the dynamics given by the isochronous metric (5) in the Newtonian limit (37) corresponds to the isochronous realization of the Newtonian $N$-body problem (34-35) by means of the fictitious change of coordinate (3), according to the standard procedure introduced in [1,2] in the context of non-relativistic dynamical systems.

4. Final remarks

As in the case of the treatments of non-relativistic dynamical systems, including $N$-body problems with Newtonian equations of motion ("accelerations equal forces") [12], the findings we recently reported [34] have an unpalatable connotation: they indicate that the hope to ascertain which is the correct description of our cosmos is doomed by the possibility to identify quite different cosmologies—for instance displaying or not displaying the peculiar feature to be isochronous, or featuring the effects of a Big Bang without actually experiencing it—which are however indistinguishable by any, reasonably conceivable, human experiment [34]. The unpleasant feeling caused by this notion might perhaps resemble that experienced by physicists educated before the discovery of quantum mechanics, when they were confronted by the notion that it is actually impossible to measure simultaneously with unlimited accuracy the position and the velocity of a moving particle. However the recognition that the microworld is governed by quantum rather than classical mechanics provided enormous payoffs, while the recognition of our inability to ever being able to determine what is the correct description of our cosmos does not seem likely to open the way to interesting developments in our understanding of the make-up of
the Universe. On the other hand it is not in the scientific ethos to hide unpalatable theoretical developments: *amicus Plato, sed magis amica veritas* must remain our motto.

In this spirit, we considered worthwhile to pursue these investigations by providing a critical review of our previous findings relevant in a cosmological context \[3\] and by ascertaining their connection with the analogous findings valid for Hamiltonian systems and in particular for classical many-body problems characterized by equations of motion of Newtonian type ("accelerations equal forces") \[12\]. The results—as reported above—are not surprising, but we nevertheless hope that they will be found of interest by members of both communities, those focussed on general relativity and cosmology as well as those focussed on Hamiltonian systems and classical mechanics.

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