Canonical Connection on a Class of Riemannian Almost Product Manifolds

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Abstract. The canonical connection on a Riemannian almost product manifolds is an analogue to the Hermitian connection on an almost Hermitian manifold. In this paper we consider the canonical connection on a class of Riemannian almost product manifolds with nonintegrable almost product structure.

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1 Introduction

On an Hermitian manifold $(M, J, g)$ there exists an unique linear connection $D$ with a torsion tensor $T$ such that $DJ = Dg = 0$ and $T(x, Jy) = T(Jx, y)$ for all vector fields $x, y$ on $M$. This is the Hermitian connection of the manifold ([4], [5], [1]). The group of the conformal transformations of the metric $g$ generates the conformal group of the transformations of $D$. Analogously to the Hermitian connection on an almost Hermitian manifold, V. Mihova in [7] find the canonical connection on a Riemannian almost product manifold.

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano [10]. In [8] A. M. Naveira gives a classification of these manifolds with respect to the covariant differentiation of the almost product structure. Having in mind the results in [8], M. Staikova and K. Gribachev give in [9] a classification of the Riemannian almost product manifolds with zero trace of the almost product structure.

In the present work we consider the canonical connection on the manifolds of the class $W_3$ from the classification in [9].
2 Preliminaries

Let \((M, P, g)\) be a Riemannian almost product manifold, i.e. a differentiable manifold \(M\) with a tensor field \(P\) of type \((1, 1)\) and a Riemannian metric \(g\) such that

\[ P^2 x = x, \quad g(Px, Py) = g(x, y) \]  
\[ \tag{1} \]

for arbitrary \(x, y\) of the algebra \(\mathfrak{X}(M)\) of the smooth vector fields on \(M\). Obviously \(g(Px, y) = g(x, Py)\).

Further \(x, y, z, w\) will stand for arbitrary elements of \(\mathfrak{X}(M)\).

In this work we consider Riemannian almost product manifolds with \(\text{tr} P = 0\). In this case \((M, P, g)\) is an even-dimensional manifold.

If \(\dim M = 2n\) then the associated metric \(\tilde{g}\) of \(g\), determined by \(\tilde{g}(x, y) = g(Px, Py)\), is an indefinite metric of signature \((n, n)\). Since \(\tilde{g}(Px, Py) = \tilde{g}(x, y)\), the manifold \((M, P, \tilde{g})\) is a pseudo-Riemannian almost product manifold.

The classification in [9] of Riemannian almost product manifolds is made with respect to the tensor field \(F\) of type \((0,3)\), defined by

\[ F(x, y, z) = g((\nabla_x P)y, z), \]  
\[ \tag{2} \]

where \(\nabla\) is the Levi-Civita connection of \(g\). The tensor \(F\) has the following properties:

\[ F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z). \]  
\[ \tag{3} \]

The basic classes of the classification in [9] are \(W_1\), \(W_2\) and \(W_3\). Their intersection is the class \(W_0\) of the Riemannian \(P\)-manifolds, determined by the condition \(F(x, y, z) = 0\) or equivalently \(\nabla P = 0\). In the classification there are include the classes \(W_1 \oplus W_2\), \(W_1 \oplus W_3\), \(W_2 \oplus W_3\) and the class \(W_1 \oplus W_2 \oplus W_3\) of all Riemannian almost product manifolds.

In the present work we consider manifolds from the class \(W_3\). This class is determined by the condition

\[ \mathfrak{S}_{x,y,z} F(x, y, z) = 0, \]  
\[ \tag{4} \]

where \(\mathfrak{S}_{x,y,z}\) is the cyclic sum by \(x, y, z\). This is the only class of the basic classes \(W_1\), \(W_2\) and \(W_3\), where each manifold (which is not Riemannian \(P\)-manifold) has a nonintegrable almost product structure \(P\). This means that in \(W_3\) the Nijenhuis tensor \(N\), determined by

\[ N(x, y) = (\nabla_x P)y - (\nabla_{Px} P)y + (\nabla_y P)x - (\nabla_{Py} P)x, \]  
\[ \tag{5} \]

is non-zero.
In [9] it is introduced an associated tensor \( N^* \) by
\[
N^* (x, y) = (\nabla_x P) y + (\nabla_{P x} P) y + (\nabla_{y P} P) x. \tag{6}
\]
It is proved that the condition (4) is equivalent to \( N^* (x, y) = 0 \).
Further, manifolds of the class \( W_3 \) we call Riemannian \( W_3 \)-manifolds.
As it is known the curvature tensor field \( R \) of a Riemannian manifold with metric \( g \) is determined by
\[
R(x, y) z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z
\]
and the corresponding tensor field of type (0, 4) is defined as follows
\[
R(x, y, z, w) = g_4 R(x, y) z, w. \tag{7}
\]
Let \( (M, P, g) \) be a Riemannian almost product manifold and \( \{e_i\} \) be a basis of the tangent space \( T_p M \) at a point \( p \in M \). Let the components of the inverse matrix of \( g \) with respect to \( \{e_i\} \) be \( g_{ij} \). If \( \rho \) and \( \tau \) are the Ricci tensor and the scalar curvature, then \( \rho^* \) and \( \tau^* \), defined by
\[
\rho^* (y, z) = g_{ij} R(e_i, y, z, P e_j) \quad \text{and} \quad \tau^* = g_{ij} \rho^* (e_i, e_j),
\]
are called an associated Ricci tensor and an associated scalar curvature, respectively. We denote \( \tau^{**} = g_{ij} g_{ks} R(e_i, e_k, P e_s, P e_j) \).

The square norm of \( \nabla P \) is defined by
\[
\| \nabla P \|^2 = g^{ij} g^{ks} g((\nabla_{e_i} P) e_k, (\nabla_{e_j} P) e_s). \tag{8}
\]
Obviously \( \| \nabla P \|^2 = 0 \) iff \((M, P, g)\) is a Riemannian \( P \)-manifold. In [9] it is proved that if \((M, P, g)\) is a Riemannian \( W_3 \)-manifold then
\[
\| \nabla P \|^2 = -2g^{ij} g^{ks} g((\nabla_{e_i} P) e_k, (\nabla_{e_j} P) e_s) = 2 (\tau - \tau^{**}). \tag{9}
\]
A tensor \( L \) of type (0, 4) with the properties
\[
L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \tag{10}
\]
\[
\xi L(x, y, z, w) = 0 \quad (\text{the first Bianchi identity})
\]
is called a curvature-like tensor. Moreover, if the curvature-like tensor \( L \) has the property
\[
L(x, y, P z, P w) = L(x, y, z, w), \tag{11}
\]
we call it a Riemannian \( P \)-tensor.
If the curvature tensor \( R \) on a Riemannian \( W_3 \)-manifold \((M, P, g)\) is a Riemannian \( P \)-tensor, i.e. \( R(x, y, P z, P w) = R(x, y, z, w) \), then \( \tau^{**} = \tau \). Therefore \( \| \nabla P \|^2 = 0 \), i.e. \((M, P, g)\) is a Riemannian \( P \)-manifold.

### 3 Natural connection on Riemannian almost product manifolds

Let \( \nabla' \) be a linear connection with a tensor \( Q \) of the transformation \( \nabla \rightarrow \nabla' \) and a torsion tensor \( T \), i.e.
\[
\nabla'_{x} y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y].
\]
The corresponding (0,3)-tensors are defined by
\[ Q(x, y, z) = g(Q(x, y, z), T(x, y, z) = g(T(x, y, z)). \tag{12} \]

The symmetry of the Levi-Civita connection implies
\[ T(x, y) = Q(x, y) - Q(y, x), \tag{13} \]
\[ T(x, y) = -T(y, x). \tag{14} \]

A partial decomposition of the space \( T \) of the torsion tensors \( T \) of type (0,3) (i.e. \( T(x, y, z) = -T(y, x, z) \)) is valid on a Riemannian almost product manifold \((M, P, g)\):
\[ T = T_1 \oplus T_2 \oplus T_3 \oplus T_4, \]
where \( T_i (i = 1, 2, 3, 4) \) are invariant orthogonal subspaces \([7]\). For the projection operators \( p_i \) of \( T \) in \( T_i \) is established:
\[ p_1(x, y, z) = \frac{1}{8} \left\{ 2T(x, y, z) - T(y, z, x) - T(z, x, y) - T(Pz, x, Py) + T(Py, z, Px) + T(z, Px, Py) - 2T(Px, Py, z) + T(Py, Pz, x) + T(z, Px, Pz) - T(y, Pz, Px) \right\}, \]
\[ p_2(x, y, z) = \frac{1}{8} \left\{ 2T(x, y, z) + T(y, z, x) + T(z, x, y) + T(Pz, x, Py) - T(Py, z, Px) - T(z, Px, Py) - 2T(Px, Py, z) - T(Py, Pz, x) - T(z, Px, Pz) + T(y, Pz, Px) \right\}, \]
\[ p_3(x, y, z) = \frac{1}{4} \left\{ T(x, y, z) + T(Px, Py, z) - T(Pz, x, y) - T(x, Py, Pz) \right\}, \]
\[ p_4(x, y, z) = \frac{1}{4} \left\{ T(x, y, z) + T(Px, Py, z) + T(Pz, x, y) + T(x, Py, Pz) \right\}. \]

A linear connection \( \nabla' \) on a Riemannian almost product manifold \((M, P, g)\) is called a natural connection if \( \nabla'P = \nabla'g = 0 \). The last conditions are equivalent to \( \nabla'g = \nabla'\tilde{g} = 0 \). If \( \nabla' \) is a linear connection with a tensor \( Q \) of the transformation \( \nabla \rightarrow \nabla' \) on a Riemannian almost product manifold, then it is a natural connection iff the following conditions are valid:
\[ F(x, y, z) = Q(x, y, Pz) - Q(x, Py, z), \tag{15} \]
\[ Q(x, y, z) = -Q(x, z, y). \] (16)

Let \( \Phi \) be the \((0,3)\)-tensor determined by
\[ \Phi(x, y, z) = g\left( \nabla x y - \nabla x y, z \right), \] (17)
where \( \nabla \) is the Levi-Civita connection of the associated metric \( \tilde{g} \).

**Theorem 3.1** ([7]). A linear connection with the torsion tensor \( T \) on a Riemannian almost product manifold \((M, P, g)\) is natural iff
\[ 4p_1(x, y, z) = -\Phi(x, y, z) + \Phi(y, z, x) - \Phi(x, P y, P z) - \Phi(y, P z, P x) + 2\Phi(z, P x, P y), \] (18)
\[ 4p_3(x, y, z) = -g(N(x, y), z) = -2 \{ \Phi(z, P x, P y) + \Phi(x, y, z) \}. \] (19)

In [9] it is proved that the both basic tensors \( F \) and \( \Phi \) on a Riemannian almost product manifold \((M, P, g)\) are related as follows:
\[ \Phi(x, y, z) = \frac{1}{2} \{-F(P z, x, y) + F(x, y, P z) + F(y, P z, x)\}, \] (20)
\[ F(x, y, z) = \Phi(x, y, P z) + \Phi(x, z, P y). \] (21)

If \((M, P, g)\) is a Riemannian \( W_3 \)-manifold then (3), (4) and (20) imply
\[ \Phi(x, y, z) = -F(x, P y, z) - F(y, P x, z), \] (22)
which is equivalent to
\[ \Phi(x, y, z) = -F(P z, x, y). \] (23)

**Theorem 3.2.** For a natural connection with a torsion tensor \( T \) on a Riemannian \( W_3 \)-manifold \((M, P, g)\), which is not Riemannian \( P \)-manifold, the following properties are valid
\[ p_1 = 0, \quad p_3 \neq 0. \] (24)

**Proof.** From equalities (23), (18), (3), (4) we get \( p_1 = 0 \). If we suppose \( p_3 = 0 \) then (18) implies \( N = 0 \). Because of the last condition and \( N^{*} = 0 \), the manifold \((M, P, g)\) becomes a Riemannian \( P \)-manifold, which is a contradiction. Therefore, \( p_3 \neq 0 \) is valid. \( \square \)
4 Canonical connection on Riemannian $\mathcal{W}_3$-manifolds

Definition 4.1 ([7]). A natural connection with torsion tensor $T$ on a Riemannian almost product manifold $(M, P, g)$ is called a canonical connection if

$$T(x, y, z) + T(y, z, x) + T(Px, y, Pz) + T(y, Pz, Px) = 0.$$  \hspace{1cm} (25)

In [7] it is shown that (25) is equivalent to the condition

$$p_2 = p_4 = 0,$$ \hspace{1cm} (26)

i.e. to the condition $T \in T_1 \oplus T_3$. The same paper shows that on every Riemannian almost product manifold $(M, P, g)$ there exists an unique canonical connection $\nabla'$, and it is determined by

$$g(\nabla'_x y, z) = g(\nabla_x y, z) + \frac{1}{4} \{ \Phi(x, y, z) - 2\Phi(z, x, y) - \Phi(x, Py, Pz) \}.$$ \hspace{1cm} (27)

For the torsion tensor $T$ of this connection it is valid

$$T(x, y, z) = \frac{1}{4} \{ \Phi(y, z, x) - \Phi(z, x, y) + \Phi(y, Pz, Px) + \Phi(Pz, x, Py) \}.$$ \hspace{1cm} (28)

By virtue of (28) and (25) we obtain the following property for a Riemannian $\mathcal{W}_3$-manifold

$$T(Px, y) = -PT(x, y).$$ \hspace{1cm} (29)

Then the torsion tensor $T$ of the canonical connection on a Riemannian $\mathcal{W}_3$-manifold has the properties:

$$T(x, y, z) = -T(y, x, z), \quad T(Px, y, z) = T(x, Py, z) = -T(x, y, Pz).$$ \hspace{1cm} (30)

From Theorem 3.2 and condition (26) we obtain immediately the following

Theorem 4.1. For the torsion tensor $T$ of the canonical connection on a Riemannian $\mathcal{W}_3$-manifold the equality $T = p_3$ is valid, i.e. $T \in T_3$. \hspace{1cm} $\square$

Equalities (25) and (27) imply the following

Proposition 4.2. The canonical connection $\nabla'$ on a Riemannian $\mathcal{W}_3$-manifold $(M, P, g)$ is determined by

$$\nabla'_x y = \nabla_x y + \frac{1}{4} \{ - (\nabla_y P) Px + (\nabla_{Py} P) x - 2 (\nabla_x P) Py \}.$$ \hspace{1cm} (31)

\hspace{1cm} $\square$
Let $\nabla'$ be the canonical connection on a Riemannian $W_3$-manifold $(M, P, g)$. According to (31), for the tensor $Q$ of the transformation $\nabla \to \nabla'$ we have

$$Q(x, y) = \frac{1}{4} \left\{ - (\nabla_y P) P x + (\nabla P_y) P x - 2 (\nabla_x P) P y \right\}. \quad (32)$$

Then

$$T(x, y) = -\frac{1}{2} \{ (\nabla_x P) P y + (\nabla P_x) P y \}. \quad (33)$$

Hence, having in mind $N^* = 0$, (2) and (12), we obtain

$$T(x, y, z) = -\frac{1}{2} \{ F(x, P y, z) + F(P x, y, z) \}. \quad (34)$$

The equalities (32), (12) and (2) imply

$$Q(x, y, z) = -\frac{1}{4} \left\{ F(y, P x, z) - F(P y, x, z) + 2F(x, P y, z) \right\}. \quad (35)$$

Hence, because of (3) and (4), we conclude that

$$Q(x, y, z) = -Q(y, x, z) - F(P z, x, y). \quad (36)$$

**Theorem 4.3.** Let $\tau'$ and $\tau$ be the scalar curvatures for the canonical connection $\nabla'$ and the Levi-Civita connection $\nabla$, respectively, on a Riemannian $W_3$-manifold. Then

$$\tau' = \tau + \frac{1}{8} \| \nabla P \|^2. \quad (37)$$

**Proof.** According to (11) and (31), for a Riemannian almost product manifold we have $g^{ij} F(P z, e_i, e_j) = 0$. Then, from (36), after contraction by $x = e_i$, $y = e_j$, we obtain

$$g^{ij} Q(e_i, e_j, z) = 0. \quad (38)$$

Because of $\nabla g^{ij} = 0$ (for the Levi-Civita connection $\nabla$) and (35), we get

$$g^{ij} (\nabla_x Q) (e_i, e_j, z) = 0. \quad (39)$$
It is known that for the curvature tensors $R'$ and $R$ of $\nabla'$ and $\nabla$, respectively, the following is valid:

$$R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) + Q(x, Q(y, z), w) - Q(y, Q(x, z), w).$$

Then from (16) and (12) it follows that

$$R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) - g(Q(x, w), Q(y, z)) + g(Q(y, w), Q(x, z)).$$

(40)

for a Riemannian almost product manifold $(M, P, g)$.

Using a contraction by $x = e_i$, $w = e_j$ in (40) and combining (16), (38) and (39), we find that the Ricci tensors $\rho'$ and $\rho$ for $\nabla'$ and $\nabla$ satisfy

$$\rho'(y, z) = \rho(y, z) + g^{ij}(\nabla_{e_i} Q)(y, z, e_j) + g^{ij}g(Q(y, e_j), Q(e_i, z)).$$

(41)

Similarly, after a contraction by $y = e_k$, $z = e_s$ in (41) and according to (39), we obtain

$$\tau' = \tau + g^{ij}g^{ks}g(Q(e_k, e_j), Q(e_i, e_s)),$$

(42)

for the scalar curvatures $\tau'$ and $\tau$ for $\nabla'$ and $\nabla$. The equalities (42) and (32) imply

$$g^{ij}g^{ks}g(Q(e_k, e_j), Q(e_i, e_s)) = \frac{1}{16}g^{ij}g^{ks}g(A_{jk}, A_{si})$$

(43)

for a Riemannian $\mathcal{W}_3$-manifold $(M, P, g)$, where

$$A_{jk} = - (\nabla_{e_k} P) P e_k + (\nabla_{P e_k} P) e_k - 2 (\nabla_{e_k} P) P e_j.$$

From (43), (1), (7) and (8) we get

$$g^{ij}g^{ks}g(Q(e_k, e_j), Q(e_i, e_s)) = \frac{1}{8}\|\nabla P\|^2.$$

The last equality and (42) imply (37).

Corollary 4.4. A Riemannian $\mathcal{W}_3$-manifold is a Riemannian $P$-manifold if and only if the scalar curvatures for the canonical connection and the Levi-Civita connection are equal.
5 Canonical connection with Riemannian $P$-tensor of curvature on a Riemannian $W_3$-manifold

The curvature tensor $R'$ of a natural connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ satisfies property (9), according to (40). Since $\nabla'P = 0$, the property (11) is also valid. Therefore, $R'$ is Riemannian $P$-tensor iff the first Bianchi identity (10) is satisfied. On the other hand, it is known ([3]) that for every linear connection $\nabla'$ with a torsion $T$ and a curvature tensor $R'$ the following equality (the first Bianchi identity) is valid

$$\mathfrak{S}_{x,y,z} R'(x, y, z) = \mathfrak{S}_{x,y,z} \left\{ (\nabla'_x T) (y, z) + T(T(x, y), z) \right\}.$$ 

Since we have $\mathfrak{S} \nabla'g = 0$, the last equality implies

$$\mathfrak{S}_{x,y,z} R'(x, y, z, w) = \mathfrak{S}_{x,y,z} \left\{ (\nabla'_x T) (y, z, w) + T(T(x, y), z, w) \right\}.$$ 

Thus, $R'$ satisfies (10) iff

$$\mathfrak{S}_{x,y,z} \left\{ (\nabla'_x T) (y, z, w) + T(T(x, y), z, w) \right\} = 0.$$  (44)

This leads to the following

**Lemma 5.1.** The curvature tensor for the natural connection $\nabla'$ with a torsion $T$ on a Riemannian almost product manifold is a Riemannian $P$-tensor iff (44) is valid. □

We substitute $Pz$ for $z$ and $Pw$ for $w$ in (44). Hence, according to (30), we obtain

$$(\nabla'_x T) (y, z, w) - (\nabla'_y T) (z, x, w) + (\nabla'_Pz T) (x, y, Pw)$$

$$+ T(T(x, y), z, w) + T(T(y, Pz), x, w) + T(T(Pz, x), y, Pw) = 0.$$ 

We add the last equality to (44), and substitute $Px$ for $x$ and $Pw$ for $w$ in the result. Then, using (30), we get

$$(\nabla'_x T) (x, y, z) - (\nabla'_Pz T) (x, Py, w)$$

$$+ 2T(T(y, z), x, w) + 2T(T(z, x), y, w) = 0.$$  (45)

We substitute $Py$, $Pz$ for $y$, $z$, respectively, and we apply (30). We subtract the obtained equality from (45) and we reapply (30). This leads to

$$T(T(z, x), y, w) = 0.$$
Hence, (34) and (3) imply
\[ F(Py, w, T(z, x)) = -T(y, w, T(z, x)), \]
and from (2) and (12) we obtain
\[ g\left(T(x, z), T(y, w) - (\nabla Py) w\right) = 0. \tag{46} \]

Since, according to (33) and (2), we have
\[ T(y, w) = -\frac{1}{2}\left\{ (\nabla y) Pw + (\nabla Py) w\right\}, \]
the following equality is valid
\[ T(y, w) + (\nabla Py) w = -\frac{1}{2}\left\{ (\nabla y) Pw - (\nabla Py) w\right\}. \]
Thus, using (46), we arrive at the following

**Theorem 5.2.** Let \((M, P, g)\) be a Riemannian \(W_3\)-manifold, whose canonical connection has a Riemannian \(P\)-tensor of curvature. Then the following equality is valid
\[ g\left((\nabla x) Pz + (\nabla Px) z, (\nabla Py) w - (\nabla y) Pw\right) = 0. \]

\[ \square \]

6  Canonical connection with parallel torsion on a Riemannian \(W_3\)-manifold

In this section we consider a canonical connection \(\nabla'\) with parallel torsion \(T\) (i.e. \(\nabla'T = 0\)) on a Riemannian \(W_3\)-manifold \((M, P, g)\).

According to the Hayden theorem (2)
\[ Q(x, y, z) = \frac{1}{2}\left\{ T(x, y, z) - T(y, z, x) + T(z, x, y)\right\}. \tag{47} \]
Combining this with (13), (15), (35), leads to the following

**Proposition 6.1.** Let \(\nabla'\) be a natural connection on a Riemannian almost product manifold \((M, P, g)\). Then the tensors \(T\), \(Q\) and \(F\) are parallel or non-parallel at the same time with respect to \(\nabla'\).

\[ \square \]
Let $\nabla'$ be a natural connection with parallel torsion on a Riemannian almost product manifold $(M, P, g)$. According to Proposition 6.1 we have $\nabla' Q = 0$. Then, having in mind the formula for the covariant derivative of $Q$, we obtain

$$x Q(y, z, w) - Q(\nabla_x y, z, w) - Q(y, \nabla_x z, w) - Q(y, z, \nabla_x w) = 0.$$  \hfill (48)

Since $Q$ is the tensor of the deformation $\nabla \rightarrow \nabla'$, applying the formula for the covariant derivative of $Q$ with respect to $\nabla$ and equalities (12) and (13), we obtain the following

**Lemma 6.2.** Let $R'$ be the curvature tensor for a natural connection $\nabla'$ with a parallel torsion $T$ on a Riemannian almost product manifold $(M, P, g)$. Then the following equality is valid

$$R'(x, y, z, w) = R(x, y, z, w) + Q(T(x, y), z, w)$$

$$+ g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)).$$  \hfill (49)

Let $(M, P, g)$ be a Riemannian $\mathcal{W}_3$-manifold whose canonical connection $\nabla'$ has a parallel torsion $T$. Then, according to (16), (36) and (2), we have

$$Q(T(x, y), z, w) = g(Q(z, w), T(x, y)) - g((\nabla P_w P) z, T(x, y)).$$

The last equality and Lemma 6.2 imply

**Theorem 6.3.** Let $(M, P, g)$ be a Riemannian $\mathcal{W}_3$-manifold whose canonical connection $\nabla'$ has a parallel torsion $T$. Then for the curvature tensor $R'$ of $\nabla'$ we obtain

$$R'(x, y, z, w) = R(x, y, z, w)$$

$$+ g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w))$$

$$+ g(Q(z, w), T(x, y)) - g((\nabla P_w) z, T(x, y)).$$  \hfill (50)

Because of (38) we have $g^{ij} Q(e_i, e_j) = 0$. Then, from (50) via a contraction by $x = e_i, w = e_j$, we get

$$\rho'(y, z) = \rho(y, z) - g^{ij} g(Q(e_i, z), Q(y, e_j))$$

$$+ g^{ij} g(Q(z, e_j), T(e_i, y)) - g^{ij} g((\nabla P_{e_j}) z, T(e_i, y)),$$  \hfill (51)

where $\rho'$ and $\rho$ are the Ricci tensors for $\nabla'$ and $\nabla$, respectively.
Combining (12), (36), (30), (4), (13), (3) and (2), we obtain
\[
g(Q(z, e_j), T(e_i, y)) = g(Q(e_j, z), Q(y, e_i)) - g(Q(e_j, z), Q(e_i, y)) + g((\nabla e_j P) z, T(e_i, y)) + g((\nabla P e_j) z, T(e_i, y)).
\] (52)

We get the following equality from (51) and (52):
\[
\rho'(y, z) = \rho(y, z) - g^{ij} g(Q(e_j, z), Q(e_i, y)) + g^{ij} g((\nabla P e_j) e_i, T(e_i, y)).
\] (53)

A contraction by \(y = e_k, z = e_s\) leads to
\[
\tau' = \tau - g^{ij} g^{ks} g(Q(e_j, e_s), Q(e_i, e_k)) + g^{ij} g^{ks} g((\nabla P e_s) e_j, T(e_i, e_k)),
\] (54)
where \(\tau'\) and \(\tau\) are the respective scalar curvatures for \(\nabla'\) and \(\nabla\).

Using (31), (8) and (7), we get
\[
g^{ij} g^{ks} g(Q(e_j, e_s), Q(e_i, e_k)) = \frac{1}{4} \|\nabla P\|^2.
\] (55)

From (8) and \(2T(e_i, e_j) = - (\nabla e_i P) e_k + (\nabla P e_i) e_k\) we have
\[
g^{ij} g^{ks} g((\nabla P e_s) e_j, T(e_i, e_k)) = \frac{1}{2} \|\nabla P\|^2.
\] (56)

Then, (54), (55) and (56) imply
\[
\tau' = \tau + \frac{1}{4} \|\nabla P\|^2.
\]

From the last equality and (37) we obtain the following

**Theorem 6.4.** Let \((M, P, g)\) be a Riemannian \(W_3\)-manifold whose canonical connection \(\nabla'\) has a parallel torsion \(T\). Then \((M, P, g)\) is Riemannian \(P\)-manifold.

\(\square\)

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