Quantum Key Distribution with Qubits Encoded in Qutrit

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We present a novel one-way quantum key distribution protocol based on 3-dimensional quantum state, a qutrit, that encodes two qubits in its 2-dimensional subspaces. The qubits hold the classical bit information that has to be shared between the legitimate users. Alice sends such a qutrit to Bob where he decodes one of the qubit and measures it along the random Pauli basis. This scheme has higher secure key rate at longer transmission distance than the standard BB84 protocol.

Quantum key distribution (QKD) allows two legitimate users, namely Alice and Bob, to share secure key for cryptographic purpose. The security of the shared key is guaranteed by the laws of quantum physics. Besides its unconditional security, a promising QKD protocol needs to meet higher key generation rate as well as longer transmission distance. Since its first proposal in 1984 the protocol BB84 [1] has been scrutinized and proven secure under a vast range of eavesdropping strategies [2–5]. There are also other protocols [6–8], merited by their own account, that enhance the secure key generation rate and transmission distance beyond the BB84 protocol.

However, efforts are also made on BB84 protocol for increasing its throughput. For example, decoy state BB84 [8] increases the transmission distance by securing it against the photon number splitting (PNS) attack. Using integrated optics based on planar lightwave circuit technology [11] increases the key generation rate of BB84 protocol implemented with time-bin qubits. Along this direction, we propose a one-way scheme that utilizes 3-dimensional quantum state, a qutrit, for performing QKD. Performance of qutrit based protocols have been studied before in [12–15]. The main feature of our scheme is that the qutrit encodes two BB84 qubits in its 2-dimensional subspaces. By saying BB84 qubits we address the qubits originally prepared along the random Pauli basis for performing the protocol BB84. In order to extract the classical bit information one has to first decode the qubits from the qutrit and then measure it along the correct basis.

The advantage of this scheme is that Eve, also Bob, cannot decode the qubits with certainty. As a result, eavesdropping strategies assisted with quantum memory [16] may not be fully effective for stealing complete bit of information. This implies that the practical implementation of our scheme using weak coherent source can have comparatively higher mean photon number, $\mu$, and higher key rate at longer transmission distance than the standard BB84 scheme. Additionally, using single photon pulses may drain less information to Eve during individual attacks [17].

In the following, we will describe our scheme by featuring its encoding and decoding procedure by adopting the technique given in [13]. This will follow a comparison of the performance of the proposed protocol with standard BB84 protocol, while they are under collective attacks.

Let us start the description of our protocol, see Fig. 1 by considering two qubit states $|\psi_a\rangle, (|0\rangle_a + e^{i\varphi_a}|1\rangle_a)/\sqrt{2}$ and $|\psi_b\rangle = (|0\rangle_b + e^{i\varphi_b}|1\rangle_b)/\sqrt{2}$, that span the 2-dimensional Hilbert spaces $H_a$ and $H_b$, respectively. The basis of the joint state $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$ are labeled as: $|i\rangle_a |j\rangle_b \rightarrow |3j - i\rangle$. Let $|\phi\rangle = (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}$ be a qutrit state in $H$. It is possible to encode the two qubits in the qutrit by performing a projective measurement on the joint state $|\phi\rangle \otimes |\psi\rangle$ using the projector

$$\Pi = \sum_{i=0}^{2} |i\rangle \langle i| \otimes |i\rangle \langle i|.$$ (1)

The resultant qutrit state, after the unitary operation $|i\rangle \otimes |j\rangle \rightarrow |i\rangle \otimes |j - i\rangle$, is

$$|\Phi\rangle = \frac{1}{\sqrt{3}} ((|0\rangle + e^{i\varphi_a}|1\rangle + e^{i\varphi_b}|2\rangle).$$ (2)

We will study the performance of the protocol, as well as the security against collective attacks, by comparing it to the BB84 protocol. The qubit transmission distance is increased by designing a protocol that uses a qutrit to encode two BB84 qubits and to share a secure key, using quantum circuit technology.
By projecting the qutrit $|\Phi\rangle$ onto a 2-dimensional subspace the encoded qubits can be retrieved with probability 2/3. The respective projectors are:

$$\Pi_1 = |0\rangle\langle 0| + |1\rangle\langle 1| \quad \text{and} \quad \Pi_2 = |1\rangle\langle 1| + |2\rangle\langle 2|.$$  (3)

And the resultant qubit states are,

$$|\psi\rangle_i = \frac{\Pi_i |\Phi\rangle}{\sqrt{\langle \Phi | \Pi_i |\Phi\rangle}} = \left\{ \begin{array}{ll} \frac{1}{\sqrt{2}} (|0\rangle + e^{i\varphi_a} |1\rangle) & : i = 1 \\ \frac{1}{\sqrt{2}} (|1\rangle + e^{i\varphi_b} |2\rangle) & : i = 2 \end{array} \right.$$  (4)

In the above equation global phases are ignored.

Now, consider Alice prepares two qubits in one of the eigen states of Pauli $Z$ and $X$ operators, by randomly choosing the value of $\varphi_a$ and $\varphi_b$ from the set $S = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. She then sends them to Bob one after other where he measures them, randomly, along Pauli $Z$ or $X$ basis. This is the standard BB84 protocol, executing twice for the distribution of secret key between the legitimate users. Here, we would like to examine how the qubit state results from the encoding process can be used for performing such a secret key distribution if Alice would have prepared the initial qubits in eigen states of Pauli $Z$ and $X$ operators. We call this scheme as ‘Qutrit’ QKD, hereafter. An experimental demonstration of encoding and decoding procedure is given in [19]. However, for the simplicity and thence practicality, we can assume that a qutrit state prepared in Eq. (2) inherently encodes qubits in its subspace. This assumption helps Alice to starts the protocol straight from the qutrit $|\Phi\rangle$ by obviating the encoding process. Alice chooses the values of $\varphi_a$ and $\varphi_b$ from the set $S$ such that qubits, $|\psi_a\rangle$ and $|\psi_b\rangle$, in the subspaces are in Pauli eigen states. She then sends the qutrit to Bob through a quantum channel. Upon receiving the qutrit, Bob randomly selects the projectors $\Pi_1$ or $\Pi_2$ and decodes one of the qubits. He then measures the qubit along the random Pauli $Z$ or $X$ basis and registers the measurement outcome together with the order (say, first or second) of the qubit. This creates his raw key with probability 2/3. Bob fails to decode the qubit with probability 1/3 and the protocol fails. During key sifting process, Bob announces which qubit he has decoded and on which basis he has measured it. If the basis matches for the respective qubit then Alice agrees to keep the key bit for further process such as error correction and privacy amplification. Otherwise, they discard it. This procedure exactly follows the classical part of BB84. Since our proposed qutrit protocol shares many features of the BB84 protocol, we would like to compare the performance of both of the protocols in terms of the secure key rate.

For the practical reason we consider the protocols are implemented using weak coherent source. Such a source can easily be realized using only standard semiconductor laser and calibrated attenuators. The photon number statistics of a weak coherent source follows Poisson distribution. Therefore, the probability of finding $n$ photons in the quantum state generated by Alice is $P(n, \mu) = \mu^ne^{-\mu}/n!$, where $\mu$ is the mean photon number per signal. Accordingly, with certain probability, a weak coherent source emits signals that contain more than one photon. Such multi-photon pulses breach the security of the QKD protocol because an eavesdropper gains partial or complete information on the shared key without revealing her presence to the legitimate users. Probability of multi photon pulses can be made arbitrarily small by choosing $\mu < 1$. But very low value of $\mu$ results in overall reduction in signal generation rate. Therefore, there is a trade-off between the value of $\mu$ chosen by Alice and secure key rate of the protocols, for each transmission distance. For example, at higher transmission distance, Eve may replace the lossy quantum channel with a lossless channel and performs PNS attack on the multi-photon pulses. She takes advantage on the attenuation of the lossy channel by blocking as much as single photon pulses such that the total signal detected by bob remains constant. Therefore, for transmission thorough a lossy channel, value of $\mu$ must be chosen smaller than that of an ideal lossless channel. We can see that the proposed qutrit protocol allows Alice to choose comparatively higher $\mu$ than the BB84 protocol and that in turn considerably enhance the key generation rate at higher transmission distances.

In the following we compare the performance of the BB84 and Qutrit protocols by quantifying the achievable secret key rate $K$ while considering the protocols are under collective attacks [5]. During this attack, Eve independently attacks each quantum signal sent by Alice. She can store her ancillary states in quantum memory and wait for the classical post-processing and adopt a best measurement strategy to extract the information. Following an approach similar to that in [4] we will now consider the security aspect of the protocol while they are under collective attack. The secret key rate of a QKD protocol can be defined as

$$K = P_{\text{accept}}P_{\text{sift}}R_{\text{raw}}[1 - h(Q) - I_E]).$$  (5)

In the above equation, $P_{\text{accept}}$ depends on the protocol implementation and is the probability that a detector click contributes to the raw key. $P_{\text{sift}} = 1/2$ accounts for the fraction of raw key discarded due to wrong measurement basis. $R_{\text{raw}} = (R_{\text{sig}} + 2p_d(1 - R_{\text{sig}}))$ is the raw key rate in which $R_{\text{sig}} = 1 - e^{-\mu\tau_{G_b}\eta} \approx \mu \Gamma_q \Gamma_b \eta$ is the total detector clicks due to photons that survived the attenuation $\Gamma_q$ of the quantum channel and $\Gamma_b$ of Bob’s apparatus with detectors of average efficiency $\eta$ and dark count probability $p_d$. The above approximation holds for the mean photon number per pulse $\mu < 1$. The channel attenuation related to transmission distance $l$ as $\Gamma_q = 10^{-\alpha l/10}$, where $\alpha$ is the attenuation coefficient. $I_E$ is the amount of information eavesdropped by Eve. $h(Q)$is the amount of information leaked to Eve during error correction procedure and $h(.)$is the Shannon entropy. The qber $Q$ is a function of $\mu$ and is defined as

$$Q = p_d(1 - R_{\text{sig}})/R_{\text{raw}} + Q_{\text{opt}}.$$  (6)
The first term in the above equation is the error rate due to dark counts of Bob’s detectors. The constant $Q_{opt}$ accounts for the errors from optical misalignment of Bob’s apparatus. We attribute all sources of error to Eve.

During collective attacks, Eve learns the number of photon $n$ present in the weak coherent pulse sent by Alice and adopt a best attacking strategy that maximize her information on the final secure key shared by the users. On single photon pulses, Eve can gain information at the expenses of introducing as error $\varepsilon_1$. The amount of information she obtains on single photon pulse is $I_{E,1} = h(\varepsilon_1)$ \[4\]. For example, she can perform simple intercept and resend attack that creates an error with probability $1/4$ and gain $1/2$ a bit of information. For multi-photon pulses, $n \geq 2$, the optimal attack is PNS attack during which Eve forwards one photon to Bob and keeps the rest in her quantum memory. She makes no error and gain complete bit of information, i.e., $\varepsilon_{n>2} = 0$ and $I_{E,n\geq2} = 1$. However, Eve learns zero information while Alice sends vacuum pulse and Bob gets detection event due to detector dark counts. Let us consider the parameter $Y_n = R_n / R_{raw}$ be the probability that Bob gets a sifted key from $n$-photon pulse sent by Alice with probability $P_A(n)$ . Here, $R_n = P_{sift} P_A(n) f_n$ and $f_n$ is the probability that Eve forwards a single photon to Bob for $n$-photon pulse. Eve can optimize the value of $f_n$ for the total detection rate $R_{raw}$. The overall information exposed to Eve is

$$I_E = \max \left[ Y_1 h(\varepsilon_1) + (1 - Y_0 - Y_1) \right] = 1 - \min \left[ Y_0 + Y_1 [1 + h(\varepsilon_1)] \right].$$

(7)

In order to maximizing her information, Eve’s optimal attack should be compatible with the measured parameters available to the legitimate users: i.e., key rate $R_{raw}$ and the qber $Q = Y_1 \varepsilon_1$. This is achieved by minimizing $Y_1$ in Eq. (7) by setting $f_0 = 0$ and $f_{n\geq2} = 1$. Therefore, we can rewrite Eq. (7) for the protocol BB84 as

$$I_E^{BB84} = 1 - Y_1 [1 - h(\varepsilon_1)],$$

(8)

where,

$$Y_1 = 1 - Y_0 - Y_{n\geq2} = 1 - P_{sift} P_A(n \geq 2) / R_{raw}.$$  

(9)

The corresponding achievable secret key rate for the protocol BB84 is

$$K_{BB84} = R_{raw} \{ Y_1 [1 - h(\varepsilon_1)] - h(Q) \} / 2.$$  

(10)

Here we have taken $P_{accept} = 1$. It is true for the implementation with polarization coding of the BB84 protocol. However, we can take the same value for the phase encoding scheme as well \[20\].

Now, let us consider information eavesdropped on qutrit protocol during collective attack. An attack against qutrit protocol said to be successful only when both Eve and Bob decode the qubit from the same subspace. On single photon qutrit pulse Eve decodes one of the qubits with probability $2/3$ and performs her measurement. She then forwards a new qutrit to Bob that encodes two new qubits in its subspaces. The quantum state of the qubit in the subspace under attack corresponding to Eve’s measurement result. On the other hand, since Eve has no information on the state of the qubit which was not under her attack, she prepares the other qubit in random Pauli basis. On failure of decoding, she sends a vacuum pulse to Bob. One may argue that empty pulses reduce Bob’s overall detection rate but it is not the case: since Eve can optimize the function $f_n$ compatible with Bob’s detection rate. Therefore, the probability that Eve and Bob decodes the qubit from the same subspace becomes $1/2$. For each successful decoding, Eve gets $I_{E,1} = h(\varepsilon_1)$ bits of information. Here, as mentioned earlier, $\varepsilon_1$ is the error introduced by Eve. On the other hand, if Bob decodes the qubit from the subspace which was not under attack, not only Eve gets zero information but also she introduce an error with probability $1/2$. Therefore, on average, Eve gets $h(\varepsilon_1)/2$ bits of information and creates $(\varepsilon_1 + 1/2)/2$ bits of error.

There is another strategy at Eve’s disposal: send a qubit to Bob corresponding to the decoded subspace rather than a qutrit. This lowers Bob’s detection probability in the second subspace to $1/2$ out of which half of the detections contribute to errors. Bob attributes the reduction in the detection probability to the channel loss though, Eve compensates it by tuning the transfer function $f_n$. On average, Eve gains $2h(\varepsilon_1)/3$ bits of information at the expense of $2\varepsilon_1/3 + 1/6$ bits of error. It can be seen that, this strategy gives comparatively higher information gain to Eve at lower error probability. Therefore the optimal attacking strategy for Eve on single photon pulse is forward qubit rather than qutrit, to Bob.

PNS attack on multi-photon qutrit pulses is described as follows. After the basis revelation by the users, Eve projects the qutrit onto exactly the same subspace from which Bob had decoded his qubit. This happens with a probability $2/3$ and she measures the decoded qubits along the correct basis and gains full information. Therefore, for two-photon pulses Eve gets $2/3$ bit of information, contrary to one bit of information in PNS attack against BB84 protocol. For three-photon pulses, Eve gets $8/9$ bit of information which can be approximated to 1 bit of information. Moreover, at very low value of mean photon number the contribution of three-photon pulses are negligible. By setting the values of $f_0 = 0$ and $f_{n\geq2} = 1$ for maximizing her information and the qber $Q = Y_1 (2\varepsilon_1/3 + 1/6)$, we can rewrite Eq. (7) for the qutrit protocol as

$$I_E^{qutrit} = 1 - Y_1 [1 - 2h(\varepsilon_1)/3] - Y_2/3,$$

(11)

where, $Y_1$ is same as that defined in Eq. (9) and

$$Y_2 = P_{sift} P_A(n = 2) / R_{raw}.$$  

(12)

Finally, the secure key rate of the qutrit protocol is

$$K_{qutrit} = R_{raw} \{ Y_1 [1 - 2h(\varepsilon_1)/3] + Y_2/3 - h(Q) \} / 3.$$  

(13)
FIG. 2: Key rate $K$ vs transmission distance $L$. The figure shows the comparison of Eq. (10) and Eq. (13) in terms of transmission distance. The solid and dashed lines are for qutrit and BB84 protocol, respectively. The parameters used for this plot are: $p_d=10^{-5}, \eta=10\%, Q_{opt}=0.5\%, \Gamma_b=0.5$. The qutrit protocol gives secure transmission distance of 69 km compare to 52km of BB84 protocol.

with $P_{\text{accept}}=2/3$ as the probability of successful decoding of qubit from the qutrit.

The Fig. 2 shows the comparison of secure key rate of BB84 and qutrit protocol in terms of transmission distance. The mean photon number $\mu$ is numerically optimized and used in Eq. (10) and Eq. (13) for maximizing the respective secure key rate for each transmission distance. It can be seen that, despite of imperfect qubit decoding probability, the qutrit protocol gives same key rate as BB84 at shorter distance. More importantly, it gives comparatively higher key rate at longer distance, which is the linchpin of the qutrit protocol. Under the same experimental conditions, maximum secure transmission distance of 69km is obtained for qutrit protocol which is remarkably higher than 52 km of BB84 protocol.

In conclusion, we have presented a one-way protocol based on qutrit for secure key distribution between two legitimate users. The information encoded in the qubit subspaces of the qutrit make the protocol highly tolerable to photon number splitting attack; thanks to non-unity qubit decoding probability. Additionally, it is more robust against attacks on single photon pulses. This is a promising feature and may encourage an experimental realization of the proposed protocol in the near future.

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