On the distribution of extrema for a class of Lévy processes

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\textbf{Abstract}

Suppose $X_t$ is either a regular exponential type Lévy process or a Lévy process with a bounded variation jumps measure. The distribution of the extrema of $X_t$ play a crucial role in many financial and actuarial problems. This article employs the well known and powerful Riemann-Hilbert technique to derive the characteristic functions of the extrema for such Lévy processes. An approximation technique along with several examples is given.

\textbf{Keywords:} Principal value integral; Hölder condition; Padé approximant; continued fraction; Fourier transform; Hilbert transform.

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\section{1. Introduction}

Suppose $X_t$ be a one-dimensional, real-valued, right continuous with left limits (càdlàg), and adapted Lévy process, starting at zero. Suppose also that the corresponding jumps measure, $\nu$, is defined on $\mathbb{R}\setminus\{0\}$ and satisfies $\int_{\mathbb{R}} \min\{1, x^2\} \nu(dx) < \infty$. Moreover, suppose the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter $q$ that is independent of the Lévy process.

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$X_t$, and that $\tau(0) = \infty$. The extrema of the Lévy process $X_t$ are defined to be

$$M_q = \sup\{X_s : s \leq \tau(q)\};$$

$$I_q = \inf\{X_s : s \leq \tau(q)\}.$$  \hspace{1cm} (1)

The Wiener-Hopf factorization method is a technique that can be used to study the characteristic function of $M_q$ and $I_q$. The Wiener-Hopf method has been used to show that:

(i) The random variables $M_q$ and $I_q$ are independent (Kuznetsov; 2009b and Kypriano; 2006 Theorem 6.16);

(ii) The product of their characteristic functions is equal to the characteristic function of the Lévy process $X_t$ (Bertoin; 1996 page 165); and

(iii) The random variable $M_q$ ($I_q$) is infinitely divisible, positive (negative), and has zero drift (Bertoin; 1996, page 165).

Supposing that the characteristic function of a Lévy process, $X_t$, can be decomposed as a product of two functions, one of which is the boundary values of a of a function that is analytic and bounded in the complex upper half-plane (i.e., $\mathbb{C}^+ = \{\lambda : \lambda \in \mathbb{C} \text{ and } Im(\lambda) \geq 0\}$) and the other of which is the boundary values of a function that is analytic and bounded in the complex lower half-plane (i.e., $\mathbb{C}^- = \{\lambda : \lambda \in \mathbb{C} \text{ and } Im(\lambda) \leq 0\}$), we then have that the characteristic functions of $M_q$ and $I_q$ can be determined explicitly. The required decomposition can be obtained explicitly if, for example, the characteristic function of the Lévy process is a rational function. Furthermore, there is a very general existence result for such decompositions, based on the theory of singular integrals (specifically Sokhotskyi-Plemelj integrals). Lewis & Mordecki (2005) considered a Lévy process $X_t$ which has negative jumps distributed according to a mixed-gamma family of distributions and has an arbitrary positive jumps measure. They established that such a process has a characteristic function which can be decomposed as a product of a rational function and a more or less arbitrary function, and that these functions are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Recently, they provided
an analogous result for a Lévy process whose positive jumps measure is given by a mixed-gamma family of distributions and whose negative jumps measure has an arbitrary distribution, more detail can be found in Lewis & Mordecki (2008).

Unfortunately, in the majority of situations, the characteristic function of the process is not a rational function nor can be explicitly decomposed as a product of two analytic functions in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \). Of course, there is a general theory allowing the characteristic functions of \( M_q \) and \( I_q \) to be expressed in terms of a Sokhotskyi-Plemelj integral (see Equation 2). This provides an existence result, but presents some difficulties in numerical work due to slow evaluation and numerical problems caused by singularities in the complex plane that are near the contour used in the integral. To overcome these problems, approximation methods may be considered.

Roughly speaking, the Wiener-Hopf factorization technique attempts to find a function \( \Phi \) that is analytic, bounded, and complex-valued except for a prescribed jump discontinuity on the real line within the complex plane. The radial limits at the real line, denoted \( \Phi^\pm \), satisfy \( \Phi^+(\omega)\Phi^-(\omega) = g(\omega) \), where \( \omega \in \mathbb{R} \) and \( g \) is a given function with certain conditions (\( g \) is a zero index function which satisfies the Hölder condition). The radial limits provide the desired decomposition of \( g \) into a product of boundary value functions that was alluded to above. The Wiener-Hopf factorization technique can be extended to a more general setting and is then also known as the Riemann-Hilbert method. The Riemann-Hilbert method is theoretically well developed and it is often more convenient to work with than the Wiener-Hopf technique, see Kucerovsky & Payandeh (2009) for more detail. The Riemann-Hilbert problem has proved remarkably useful in solving an enormous variety of model problems in a wide range of branches of physics, mathematics, and engineering. Kucerovsky, et al. (2009) employed the Riemann-Hilbert problem to solve a statistical decision problem. More precisely, using the Riemann-Hilbert problem, they established the mle estimator under absolute-error loss function is a generalized Bayes estimator for a wide class of location family of distributions.
This article considers the problem of finding the distributions of the extrema of a Lévy process whose (i) either its corresponding jumps measure is a finite variation measure or is the regular exponential Lévy type process.; and (ii) its corresponding stopping time \( \tau(q) \) is either a geometric or an exponential distribution with parameter \( q \) independent of the Lévy process \( X_t \) where \( \tau(0) = \infty \).

Then, it develops a procedure in terms of the well known and powerful Riemann-Hilbert technique, to solve the problem of finding the characteristic functions of \( M_q \) and \( I_q \). A remark has been made that is helpful in the situation where such characteristic functions cannot be found explicitly.

Section 2 collects some essential elements which are required for other sections. Section 3 states the problem of finding the characteristic functions for the distribution of the extrema in terms of a Riemann-Hilbert problem. Then, in that section is derived an expression for such characteristic functions in terms of the Sokhotskyi-Plemelj integral. A remark that is helpful in situations where such characteristic functions cannot be found explicitly is made, and several examples are given.

2. Preliminaries

Now, we collect some lemmas which are used later.

The index of an analytic function \( h \) on \( \mathbb{R} \) is the number of zero minus number of poles of \( h \) on \( \mathbb{R} \), see Payandeh (2007, chapter 1), for more technical detail. Computing the index of a function is usually a key step to determine the existence and number of solutions of a Riemann-Hilbert problem. We are primarily interested in the case of zero index.

The Sokhotskyi-Plemelj integral of a function \( s \) which satisfies the Hölder condition and it is defined by a principal value integral, as follows.

\[
\phi_s(\lambda) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{s(x)}{x-\lambda} \, dx, \quad \text{for } \lambda \in \mathbb{C}. \tag{2}
\]

The following are some well known properties of the Sokhotskyi-Plemelj integral, proofs can be found in Ablowitz & Fokas (1990, chapter 7), Gakhov (1990, chapter 2), and Pandey (1996, chapter
4), among others.

**Lemma 1.** The radial limit of the Sokhotskyi-Plemelj integral of \(s\), given by \(\phi_s^\pm(\omega) = \lim_{\lambda \to \omega \pm i0^\pm} \phi_s(\lambda)\) can be represented as:

i) **jump formula.** i.e., \(\phi_s^\pm(\omega) = \pm s(\omega)/2 + \phi_s(\omega)\), where \(\omega \in \mathbb{R}\);

ii) \(\phi_s^\pm(\omega) = \pm s(\omega)/2 + H_s(\omega)/(2i)\), where \(H_s(\omega)\) is the Hilbert transform of \(s\) and \(\omega \in \mathbb{R}\).

The Riemann-Hilbert problem is the function-theoretical problem of finding a single function which is analytic separately in \(\mathbb{C}^+\) and \(\mathbb{C}^-\) (called sectionally analytic) and having a prescribed jump discontinuity on the real line. The following states the homogeneous Riemann-Hilbert problem which one deals with in studying the characteristic functions of \(M_q\) and \(I_q\).

**Definition 1.** The homogeneous Riemann-Hilbert problem, with zero index, is the problem of finding a sectionally analytic function \(\Phi\) whose the upper and lower radial limits at the real line, \(\Phi^\pm\), satisfy

\[
\Phi^+(\omega) = g(\omega)\Phi^-(\omega), \quad \text{for } \omega \in \mathbb{R},
\]

(3)

where \(g\) is a given continuous function satisfying a Hölder condition on \(\mathbb{R}\). Moreover, \(g\) is assumed to have zero index, to be non-vanishing on \(\mathbb{R}\), and bounded above by 1.

A homogeneous Riemann-Hilbert problem always has a family of solutions if no restrictions on growth at infinity are posed. But a unique solution can be obtained with further restrictions. Solutions vanishing at infinity are the most common restriction considered in mathematical physics and in engineering applications, see Payandeh (2007, chapter 1), for more detail. With these restrictions, the solutions of the homogeneous Riemann-Hilbert problem are given by

\[
\Phi^\pm(\lambda) = \exp\{\pm \phi_{\ln(g)}(\lambda)\}, \quad \text{for } \lambda \in \mathbb{C}
\]

where \(\phi_{\ln(g)}\) stands for the Sokhotskyi-Plemelj integral, given by \(\text{2} \) of \(\ln(g)\).
In this paper, we need to solve a homogeneous Riemann-Hilbert problem (also known as a Wiener-Hopf factorization problem) with

$$\Phi^+(\omega)\Phi^-(\omega) = g(\omega), \ \omega \in \mathbb{R}, \quad (4)$$

where $g$ is a given, zero index function which satisfies the Hölder condition and $g(0) = 1$. For convenience in presentation, we will simply call the above homogeneous Riemann-Hilbert problem a Riemann-Hilbert problem. The following provides solutions for the above Riemann-Hilbert problem. We begin with what we term the Resolvent Equation for Sokhotsky-Plemelj integrals.

**Lemma 2.** The Sokhotsky-Plemelj integral of a function $f$ satisfies

$$\phi_f(\lambda) - \phi_f(\mu) = (\lambda - \mu)\phi_{\frac{x}{x-\lambda}}(\mu),$$

for $\lambda$ and $\mu$ real or complex.

**Proof.** In general,

$$(x - \lambda)^{-1} - (x - \mu)^{-1} = (\lambda - \mu)(x - \mu)^{-1}(x - \lambda)^{-1}.$$  

Then, see Dunford & Schwartz (1988), we have an equation of Cauchy integrals, where $\Gamma = \mathbb{R}$:

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(x)}{x-\lambda} dx - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(x)}{x-\mu} dx = \frac{\lambda - \mu}{2\pi i} \int_{\Gamma} \frac{f(x)}{(x-\mu)(x-\lambda)} dx.$$

The above is valid only for $\lambda$ and $\mu$ not on the real line. However, by Lemma 2 the values of $\phi_f$ on the real line are obtained by averaging the limit from above, $\phi^+_f$, and the limit from below, $\phi^-_f$. We thus obtain the stated equation in all cases.

**Lemma 3.** Suppose $\Phi^\pm$ are sectionally analytic functions satisfying the Riemann-Hilbert problem given by (4). Moreover, suppose that $g$ is a zero index function satisfies the Hölder condition and $g(0) = 1$. Then,

$$\Phi^\pm(\lambda) = \exp\{\pm\phi_{\ln g}(\lambda) \mp \phi_{\ln g}(0)\}, \ \lambda \in \mathbb{C}$$

where $\phi_{\ln g}$ stands for the Sokhotskyi-Plemelj integration of $\ln g$.  

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Proof. By taking logarithm from both sides, the above equation can be rewritten as

\[ \ln \Phi^+(\omega) - (-\ln \Phi^-(\omega)) = \ln g(\omega). \]

Since \( \ln g(0) = 0 \), the above equation does not satisfy the non-vanishing condition of the standard Riemann-Hilbert problem. One may handle this by dividing both sides by \( \omega \) (Gakhov; 1990) suggested this kind of modification to extend the domain of the Riemann-Hilbert method). Now, we have

\[ \frac{\ln \Phi^+(\omega)}{\omega} - \frac{(-\ln \Phi^-(\omega))}{\omega} = \frac{\ln g(\omega)}{\omega}. \]

The above equation meets all conditions for the usual solution of the additive Riemann-Hilbert problem by Sokhotskyi-Plemelj integrals, and therefore, the solutions of our Riemann-Hilbert problem (equation 4) are

\[ \Phi^\pm(\lambda) = \exp\{\pm \frac{\lambda}{2\pi i} \int_{\mathbb{R}} \ln g(x)/x - \lambda \, dx\}, \lambda \in \mathbb{C}. \]

Lemma 2 with \( f = \ln g \) gives

\[ \phi_{\ln g}(\lambda) - \phi_{\ln g}(\mu) = (\lambda - \mu)\phi_{\ln g}(\mu). \]

Letting \( \lambda \) go to zero from above, in the complex plane, and using the fact that \( \ln g(0) = 0 \), Lemma 2 lets us conclude that

\[ \phi_{\ln g}(0) - \phi_{\ln g}(\mu) = -\mu \phi_{\ln g}(\mu). \]

Substituting this into the above equation for \( \Phi^\pm \) gives our claimed result. \( \square \)

The following explores some properties of the above lemma.

Remark 1. Using the jump formula. One can conclude that

\[ \Phi^\pm(\omega) = \sqrt{g(\omega)} \exp\{\pm \frac{i}{2} (H_{\ln g}(0) - H_{\ln g}(\omega)) \}, \]

where \( H_{\ln g} \) stands for the Hilbert transform of \( \ln g \).
The following explores Carlemann’s technique for obtaining solutions of the Riemann-Hilbert problem directly rather than using the Sokhotskyi-Plemelj integrations.

**Remark 2.** (Carlemann’s technique) If \( g \) can be decomposed as a product of two sectionally analytic functions \( g^+ \) and \( g^- \), respectively in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \). Then, solutions of the Riemann-Hilbert problem are \( \Phi^+ \equiv g^+ \) and \( \Phi^- \equiv g^- \).

Carlemann’s method amounts to solution by inspection. The most favorable situation for the Carlemann’s method is the case where \( g \) is a rational function. In the case that approximation solutions are required, Kucerovsky & Payandeh (2009) suggested approximating \( g \) with a rational function obtained from a Padé approximant or a continued fraction expansion.

The Paley-Wiener theorem is one of the key elements for restating problem of finding the characteristic functions of extrema in a Lévy process as a Riemann-Hilbert problem, as in equation (4). The theorem is stated below, a proof may be found in Dym & McKea (1972, page 158).

**Theorem 1.** (Paley-Wiener) Suppose \( s \) is a function in \( L^2(\mathbb{R}) \), then the following are equivalent:

i) The real-valued function \( s \) vanishes on the left half-line.

ii) The Fourier transform \( s \), say, \( \hat{s} \) is holomorphic on \( \mathbb{C}^+ \) and the \( L^2(\mathbb{R}) \)-norms of the functions \( x \mapsto \hat{s}(x + iy_0) \) are uniformly bounded for all \( y_0 \geq 0 \).

**Definition 2.** (Mixed gamma family of distributions) A nonnegative random variable \( X \) is said to be distributed according to a mixed gamma distribution if its density function is given by

\[
p(x) = \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} c_{kj} \frac{\alpha_k^j x^{j-1}}{(j-1)!} e^{-\alpha_k x}, \quad x \geq 0,
\]

where \( c_{kj} \) and \( \alpha_k \) are positive values where \( \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} c_{kj} = 1 \).

The following explores some properties of the characteristic function of the above, a proof can be found in Bracewell (2000, page 433), and Lewis & Mordecki (2005), among others.
Lemma 4. The characteristic function of a distribution (or equivalently the Fourier transform of its density function), say \( \hat{p} \), has the following properties:

i) \( \hat{p} \) is a rational function if and only if the density function belongs to the class of mixed gamma family of distributions, given by [5].

ii) \( \hat{p}(\omega) \) is a Hermition function, i.e., the real part of \( \hat{p} \) is even function and the imaginary part odd function;

iii) \( \hat{p}(0) = 1 \); and the norm of \( \hat{p}(\omega) \) bounded by 1.

3. Main results

Suppose that \( X_t \) is a one-dimensional real-valued Lévy process starting at \( X_0 = 0 \) and defined by a triple \((\mu, \sigma, \nu)\): the drift \( \mu \in \mathbb{R} \), volatility \( \sigma \geq 0 \), and the jumps measure \( \nu \) is given by a nonnegative function defined on \( \mathbb{R} \setminus \{0\} \) satisfying \( \int_{\mathbb{R}} \min\{1, x^2\} \nu(dx) < \infty \). The Lévy-Khintchine representation states that the characteristic exponent \( \psi \) (i.e., \( \psi(\omega) = \ln(\mathbb{E}(\exp(i\omega X_1))) \), \( \omega \in \mathbb{R} \)) can be represented by

\[
\psi(\omega) = i\mu \omega - \frac{1}{2} \sigma^2 \omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x I_{[-1,1]}(x)) \nu(dx), \quad \omega \in \mathbb{R}.
\]

(6)

Now, we explore some properties of the two expressions \( q(q - \psi(\omega))^{-1} \) and \( (1 - q)(1 - q\psi(\omega))^{-1} \), \( \omega \in \mathbb{R} \) that will play an essential rôle in the rest of this section.

Lemma 5. The Lévy process \( X_t \) has a jumps measure \( \nu \) that satisfies \( \int_{\mathbb{R}\setminus[-1,1]} |x|^\varepsilon \nu(x) < \infty \), for some \( \varepsilon \in (0,1) \). Then

i) \( q(q - \psi(\omega))^{-1} \) satisfies the Hölder condition;

ii) \( (1 - q)(1 - q\exp\{-\psi(\omega)\})^{-1} \) satisfies the Hölder condition.
Proof. A proof of part (i) may be found in Kuznetsov (2009a) and the proof of part (ii) is a minor variation of the proof of part (i). □

The above condition on the jumps measure \( v \), (i.e., \( \exists \varepsilon \in (0, 1) \), such that \( \int_{\mathbb{R}\setminus[-1,1]} |x|^\varepsilon v(x) < \infty \)) is a very mild restriction and many Lévy processes, such as all stable processes, meet this property. It only excludes cases which the jumps measure has extremely heavy tail (behaves like \( |x|^{-1}/(\ln |x|)^2 \) for large enough \( x \)), see Kuznetsov (2009a) for more detail.

The following recalls the definition of a very useful class of Lévy processes.

Definition 3. A Lévy process \( X_t \) is said to be of regular exponential type, if its corresponding characteristic exponent is analytic and continuous in a strip about the real line.

Loosely speaking, a Lévy process \( X_t \) is a regular Lévy process of exponential type (RLPE) if its jumps measure has a polynomial singularity at the origin and decays exponentially at infinity, see Boyarchenko & Levendorskiï (1999, 2002a-c). The majority of classes of Lévy processes used in empirical studies in financial markets satisfy the conditions given above (i.e., \( \psi \) is analytic and continuous in a stripe about the real line). Brownian motion, Kou’s model (Kou, 2002); hyperbolic processes (Eberlein & Keller, 1995, Eberlein et al, 1998, and Barndorff-Nielsen, et al, 2001); normal inverse gaussian processes and their generalization (Barndorff-Nielsen, 1998 and Barndorff-Nielsen & Levendorskiï 2001); extended Koponen’s family (Koponen, 1995 and Boyarchenko & Levendorskiï, 1999) are examples of the regular Lévy process of exponential type. While the variance gamma processes (Madan et al, 1998) and stable Lévy processes are two important exceptions, see Cardi (2005) for more detail.

Lemma 6. The Lévy process \( X_t \) has a analytic and continuous characteristic exponent on the real line \( \mathbb{R} \) either one the following conditions are hold:

i) \( X_t \) has a jumps measure \( v(dx) \) with bounded variation (i.e., \( \int_{-1}^{1} x v(dx) < \infty \)).

ii) \( X_t \) is a regular Lévy process of exponential type (see Definition 3).
Proof. For part (i), observe that the characteristic exponent for the bounded variation jumps measure $\nu(dx)$ is given by

$$
\psi(\omega) = i\mu\omega - \int_\mathbb{R} (e^{i\omega x} - 1 - i\omega x I_{[-1,1]}(x))\nu(dx)
$$

$$
= i\mu\omega - 1 - i\omega \int_{[-1,1]} xv(dx) + \int_{(-\infty,0]} e^{i\omega x} v^-(dx) + \int_{(0,\infty)} e^{i\omega x} v^+(dx),
$$

see Bertoin (1996). From the fact that $\nu(dx)$ is a bounded variation jumps measure, one can conclude that three first terms are analytic on $\mathbb{R}$. A double application of the Paley-Wiener Theorem shows that two last terms are, respectively, analytic and bounded in $\mathbb{C}^-$ and $\mathbb{C}^+$. Therefore, these terms are analytic on $\mathbb{R} = \mathbb{C}^- \cap \mathbb{C}^+$. The proof of part (ii) follows from Definition. □

Lemma 7. Suppose the Lévy process $X_t$ either is a regular exponential type or has a bounded variation jumps measure $\nu$. Then,

i) letting the geometric stopping time be $\tau(q)$, with parameter $q$ ($q \neq 1$), the function $(1 - q)(1 - q \exp\{-\psi(\omega)\})^{-1}$ has zero index on the real line;

ii) for exponential stopping time $\tau(q)$ with constant rate $q$ ($q > 0$), the function $(q)(q - \psi(\omega))^{-1}$ has zero index on the real line.

Proof. Firstly, observe that the functions $q(q-\psi(\omega))^{-1}$ and $(1-q)(1-q \exp\{-\psi(\omega)\})^{-1}$ have no zero on $\mathbb{R}$. They may have a zero at $\pm\infty$. Moreover, equations $q - \psi(\omega) = 0$ and $1 - q \exp\{-\psi(\omega)\} = 0$ are, respectively, equivalent to $E(\exp\{-i\omega X_1\}) = \exp\{q\}$ and $E(\exp\{-i\omega X_1\}) = q$. Since $q$ is positive, real valued, and $E(\exp\{-i\omega X_1\})$ is a Hermitian function, these equations have no solutions on $\mathbb{R}$. Moreover, from Lemma observe that two functions $q(q-\psi(\omega))^{-1}$ and $(1-q)(1-q \exp\{-\psi(\omega)\})^{-1}$ are analytic and bounded on the real line. The desired proof comes from the above observations along with the fact that the index of an analytic function is the number of zeros minus number of poles within the contour (Gakhov; 1990). □

The extrema of a Lévy process play a crucial role in determining many aspects of a Lévy process,
see Mordecki (2003), Renming & Vondraček (2008), Dmytro (2004), and Albrecher, et al. (2008), among many others.

The following theorem addresses the question of how the problem of finding the characteristic functions of the distribution of the extrema can be restated in term of a Riemann-Hilbert problem.

**Theorem 2.** Suppose $X_t$ is a Lévy process whose stopping time $\tau(q)$ has either a geometric or an exponential distribution with parameter $q$ independent of the Lévy process $X_t$ and $\tau(0) = \infty$. Moreover, suppose that

A1) its jumps measure $\nu$ satisfies $\int_{\mathbb{R}\setminus[-1,1]} |x|^{ \epsilon} \nu(dx) < \infty$, for some $\epsilon > 0$;

A2) either its jumps measure $\nu$ is of bounded variation or $X_t$ is a regular exponential type Lévy process.

Then, the characteristic functions of $M_q$ and $I_q$, say $\Phi^+_q$ and $\Phi^-_q$, respectively, satisfy

i) the Riemann-Hilbert problem $\Phi^+_q(\omega)\Phi^-_q(\omega) = q(q - \psi(\omega))^{-1}$, $\omega \in \mathbb{R}$, whenever $\tau(q)$ has an exponential distribution with parameter $q$ ($q > 0$), has a unique solution

$$
\Phi^+_q(\omega) = \sqrt{q/(q - \psi(\omega))} \exp\{\pm \frac{i}{2}(H_{\ln(q-\psi)}(\omega) - H_{\ln(q-\psi)}(0))\}, \omega \in \mathbb{R};
$$

ii) the Riemann-Hilbert problem $\Phi^+_q(\omega)\Phi^-_q(\omega) = (1 - q)(1 - q\psi(\omega))^{-1}$, $\omega \in \mathbb{R}$, whenever $\tau(q)$ has a geometric distribution with parameter $q$ ($q \neq 1$), has a unique solution

$$
\Phi^+_q(\omega) = \sqrt{(1 - q)/(1 - q \exp{-\psi(\omega)})} \exp\{\pm \frac{i}{2}(H_{\ln(1-\exp{-\psi})}(\omega) - H_{\ln(1-\exp{-\psi})}(0))\}, \omega \in \mathbb{R}.
$$

**Proof.** To establish the desired result observe that: (1) the characteristic function of the Lévy process $X_t$ can be uniquely decomposed as a product of two characteristic functions of the supremum and infimum of the process, see Cardi (2005, pages 43–4), for more detail; (2) the functions $M_q$ and
$I_q$ attain, respectively, nonnegative and nonpositive values. Therefore, a double application of the Paley-Wiener theorem (Theorem 1) shows $\Phi^+_q$ and $\Phi^-_q$, respectively, are sectionally analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$; (3) The two expressions $q(q - \psi(\omega))^{-1}$ and $(1 - q)(1 - q \exp\{-\psi(\omega)\})^{-1}$ satisfy a Hölder condition (see Lemma 5) and have zero index (see Lemma 7); (4) The characteristic function of the Lévy process $X_t$ is $q(q - \psi(\omega))^{-1}$, for an exponential distribution stopping time $\tau(q)$ (see Cardi; 2005, page 26) and $(1 - q)(1 - q \exp\{-\psi(\omega)\})^{-1}$, for a geometric stopping $\tau(q)$ (see Cardi; 2005, page 25). The above observations along with Remark 1 complete the proof. □

The following examples provide application of the above results for several Lévy processes.

Example 1. Lewis & Mordecki (2008) considered Lévy process $X_t$ with an exponential stopping time $\tau(q)$ and a jumps measure $\nu$ given by $\nu(dx) = \nu^-(dx)I_{(\omega,0)}(x) + \lambda p(x)I_{(0,\infty)}(x)dx$ where $p$ is the mixed gamma density function given by Equation 2 with $0 < \alpha_1 < Re(\alpha_2) \leq \cdots \leq Re(\alpha_v)$. They established that an expression $q(q - \psi(\omega))^{-1}$, $(\lambda \in \mathbb{C})$: (i) has zeros at $i\alpha_1, i\alpha_2, \cdots, i\alpha_v$, respectively, with order $n_1, n_2, \cdots, n_v$ in $\mathbb{C}^-$ (ii) has poles at $i\beta_1(q), i\beta_2(q), \cdots, i\beta_\mu(q)$, respectively, with multiplicities $m_1(q), m_2(q), \cdots, m_\mu(q)$ in $\mathbb{C}^-$. Using these observations, one may decompose an expression $q(q - \psi(\omega))^{-1}$, $\lambda \in \mathbb{C}$, as a product of two analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, say respectively, $\rho^+_q$ and $\rho^-_q$, i.e., $q(q - \psi(\omega))^{-1} = \rho^+_q(\omega)\rho^-_q(\omega)$, where $\rho^+_q(\omega) = q(q - \psi(\omega))^{-1}\prod_{j=1}^{\mu(q)}(\lambda - i\beta_j(q))^{m_j(q)}\prod_{k=1}^{v}(\lambda - i\alpha_k)^{-n_k}$, $\rho^-_q(\omega) = \prod_{k=1}^{v}(\omega - i\alpha_k)^{n_k}\prod_{j=1}^{\mu(q)}(\omega - i\beta_j(q))^{-m_j(q)}$, and $\lambda \in \mathbb{C}$. Now using Remark 2 one may verify Lewis & Mordecki (2008)'s finding which $\Phi^\pm_q \equiv \rho^\pm_q$.

Similar results have been established for Lévy process $X_t$ which has a mixed gamma negative jumps measure and an arbitrary positive jumps, see Lewis & Mordecki (2005) for more details.

Example 2. Consider the $\alpha$–stable process $X_t$ having an exponential stopping time $\tau(q)$ and a jumps measure $\nu(dx) = c_1 x^{-1-\alpha}I_{(0,\infty)}(x)dx + c_2 |x|^{-1-\alpha}I_{(\omega,0)}(x)dx$, where $\alpha \in (0, 1) \cup (1, 2)$. Donkey (1987) studied distribution of $M_q$, and $I_q$. Since, the characteristic exponent of the process is $\psi(\omega) = (c_1 + c_2)|\omega|^{\alpha}\{(c_1 + c_2) - i(c_1 - c_2)\text{sgn}(\omega)\tan(\pi\alpha/2)\} + i\omega\eta$, where $\eta$ is a real-valued constant, and $\omega \in \mathbb{R}$. An expression $q(q - \psi(\omega))^{-1}$, $\lambda \in \mathbb{C}$, is a rational function. Therefore, one readily can
be found two rational functions $\rho^+_q$ and $\rho^-_q$ which are analytic, respectively, in $\mathbb{C}^+$ and $\mathbb{C}^-$ and $q(q - \psi(\lambda))^{-1} = \rho^+_q(\lambda)\rho^-_q(\lambda)$ $\lambda \in \mathbb{C}$. Therefore, $\Phi^\pm_q \equiv \rho^\pm_q$, which verifies Doney’s observation.

The following remark suggests an approximation technique to find the characteristic functions of $M_q$ and $I_q$, approximately, whenever they cannot be found explicitly.

**Remark 3.** In the situation where function $q/(q - \psi(\omega))$ (or $(1 - q)(1 - q \exp\{-\psi(\omega)\})^{-1}$) cannot be explicitly decompose as a product of two sectionally analytic functions in $\mathbb{C}^+$ and $\mathbb{C}^-$, we suggest to replace such function by a rational function which is obtained from a Padé approximant or a continued fraction expansion and uniformly converges to the original function. An application of Carlemann’s method leads to an approximation solution for the characteristic functions of $M_q$ and $I_q$.

The following example represents a situation where the characteristic functions of $M_q$ and $I_q$ apparently cannot be found explicitly.

**Example 3.** Kuznetsov (2009b) considered a compound Poisson process with a jumps measure $\nu(dx) = \exp\{\alpha x\} \text{sech}(x) dx$ and an exponential stopping time $\tau(q)$. He showed the characteristic exponent for such compound Poisson is given by

$$\psi(\omega) = \frac{\pi}{\cos(\pi \alpha/2)} - \frac{\pi}{\cosh(\pi(\omega - i\alpha)/2)}, \omega \in \mathbb{R}.$$ 

He established that, in $\mathbb{C}$, an expression $q(q - \psi(\cdot))^{-1}$ can be, uniformly, approximated by product $\rho^+_q(\cdot)\rho^-_q(\cdot)$, where

$$\rho^+_q(\lambda) = \prod_{n=0}^{\infty} \left(1 - \frac{i \lambda}{4n+1+\eta}\right) \left(1 - \frac{i \lambda}{4n+3+\eta}\right)$$

$$\rho^-_q(\lambda) = \prod_{n=0}^{\infty} \left(1 + \frac{i \lambda}{4n+1+\eta}\right) \left(1 + \frac{i \lambda}{4n+3+\eta}\right),$$

where $\lambda \in \mathbb{C}$ and $\eta = 2/\pi \arccos(q/(q + \pi \sec(\alpha \pi/2)))$. Therefore, approximate solutions for $\Phi^\pm_q$ are $\rho^\pm_q$, more detail can be found in Kuznetsov (2009b).
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