D-MODULES AND PROJECTIVE STACKS

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Abstract. We study twisted D-modules on weighted projective stacks. We determine for which values of the twist and the weight the global sections functor is an equivalence, thus, proving a version of Beilinson-Bernstein Localisation Theorem.

A key observation in the proof of Kazhdan-Lusztig Conjecture by Beilinson and Bernstein is that the (generalised) flag varieties $G/P$ are $D$-affine. This is known as Beilinson-Bernstein Localisation Theorem. So far these are the only known connected smooth projective $D$-affine varieties. In particular, Thomsen proves that a toric smooth projective $D$-affine variety must be a product of projective spaces [12]. On the other hand, Van den Bergh proves that weighted projective spaces are $D$-affine (they are singular) [13].

The goal of this paper is to re-examine the $D$-affinity of weighted projective spaces. Instead of looking at them as singular varieties, we consider them as stacks. We give necessary and sufficient criteria for a weighted projective stack to be $D$-affine. Our method of proof is also different: Van den Bergh uses Hodges-Smith Criterion for $D$-affinity [8], while we do a direct calculation.

In section 1 we make general observations about $D$-affinity on varieties. In section 2 we establish a technical framework for working with twisted D-modules on a smooth projective stack. In section 3 we use this framework to study D-modules on weighted projective stacks.

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1. D-modules on varieties

We work with a connected algebraic variety $X$ over an algebraically closed field $K$ of characteristics zero in this section. Let $\mathcal{O}_X$ be its sheaf of functions, $\mathcal{D}_X$ its sheaf of differential operators, $D(X) = \mathcal{D}_X(X)$ its global sections. We consider the category of quasicoherent $\mathcal{D}_X$-modules $\mathcal{D}_X$–Qcoh and the category of modules over the globally defined differential operators $D(X)$–Mod. They are connected by the global sections functor

$$\Gamma : \mathcal{D}_X$–Qcoh \to D(X)$–Mod.

$X$ is called D-affine if $\Gamma$ is an equivalence. Affine varieties are D-affine but the converse statement is not true: the generalised flag variety $G/P$ is a smooth projective D-affine variety [4]. In the light of this result, it is interesting to pose the following question.

**Question:** Classify connected smooth projective $D$-affine varieties.

It would be interesting to find other examples of such varieties besides $G/P$. Notice that any such example $X$ must have zero Hodge numbers $h^{0,m}(X)$ for $m > 0$ because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, hence, has no higher cohomology. A glimmering hope for settling this question is the result of Thomsen who classified smooth toric D-affine varieties [12]. Hereby we will explain that some other classes of varieties will not give new examples.

Recall that a variety $X$ is homogeneous if a connected algebraic (not necessarily linear) group $G$ acts transitively on $X$. For a complete variety $X$ it is equivalent to asking that the automorphism group of $X$ acts transitively on $X$ [10]. Such $X$ is necessarily smooth.

**Theorem 1.** Suppose $X$ is a homogeneous complete $D$-affine variety. Then $X$ is isomorphic to a generalised flag variety.

**Proof.** By Borel-Remmert Theorem [10], $X$ is a product of a partial flag variety and an abelian variety $A$. It remains to notice that $A$ is not D-affine because $R^{\dim A}(X, \mathcal{O}_A) \neq 0$ by Serre’s duality, unless $A$ is a point. This would imply that $R^{\dim A}(X, \mathcal{O}_X) \neq 0$ that is impossible because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module. Thus, $A$ is a point and $X$ is a generalised flag variety.

If $K = \mathbb{C}$ is the field of complex numbers, this result can be slightly improved.

**Theorem 2.** Suppose $X$ is a complex complete $D$-affine variety and the tangent sheaf $\mathcal{T}_X$ is generated by global sections. Then $X$ is isomorphic to a generalised flag variety.
Proof. Since $X$ is a complete algebraic variety, the global (algebraic) vector fields $\mathfrak{g} = \Gamma(\mathcal{T}_X)$ form a finite dimensional Lie algebra [11, p. 95]. Let $G$ be an analytic connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. The group $G$ locally acts on $X$ by the second Lie Theorem [1, p. 23]. Since $X$ is compact, each element $a \in \mathfrak{g}$ defines a one-parameter group $\gamma_a(t)$ of (global) diffeomorphisms of $X$ [1, p. 20]. Choosing a real basis $a_1, \ldots, a_k$ of $\mathfrak{g}$, we can extend the assignment
\[
\text{Exp}_G(t_1a_1) \cdot \text{Exp}_G(t_2a_2) \cdots \text{Exp}_G(t_ka_k) \mapsto \gamma_{a_1}(t_1)\gamma_{a_2}(t_2) \cdots \gamma_{a_k}(t_k)
\]
to a global (real) analytic action of $G$ on $X$ [1, p. 29].

Since $\mathcal{T}_X$ is generated by global sections, each point $x \in X$ lies in the interior of its orbit $G \cdot x$. Hence each point belongs to an open set, entirely within this point’s orbit. By connectedness there is only one orbit, hence, $X \cong G/H$ as analytic manifolds.

By Borel-Remmert Theorem [11 p. 101], there exists an abelian variety $A$ such that $X$ is an $A$-fibration over a generalised flag variety $Y$. If $A$ is a point, we are done. If $A$ is not a point, $R^{\dim A}(\mathcal{O}_A) \neq 0$ by Serre’s duality. Thus, the derived push-forward $R(X \to Y)_*(\mathcal{O}_X)$ has higher cohomology and so does $\mathcal{O}_X$. This is a contradiction. \(\square\)

Observe that $\mathcal{T}_X$ is not usually a $\mathcal{D}_X$-module. This would require a flat connection on $\mathcal{T}_X$ which is quite rare. For instance, abelian varieties admit a flat connection on $\mathcal{T}_X$ as well as any other variety with a trivial tangent sheaf. On the other hand, the only generalised flag variety with a flat connection on $\mathcal{T}_X$ is a point.

Corollary 3. If $X$ is complex complete $D$-affine variety and $\mathcal{T}_X$ is a $\mathcal{D}_X$-module, then $X$ is the point.

It would be interesting to extend Theorem 2 and Corollary 3 to varieties over an arbitrary algebraically closed field $\mathbb{K}$. Our proof does not work because we use analytic methods.

2. D-modules on smooth projective stacks

The theory of D-modules on stacks is known [5] but it is significantly simpler on a quotient stack. Let $Y$ be a smooth algebraic variety with an action of an algebraic group $G$. D-modules on the quotient stack $[X] = [Y/G]$ can be understood in terms of $G$-equivariant D-modules on $Y$.

We can define a quasicoherent $\mathcal{D}_{[X]}$-module as a quasicoherent $\mathcal{D}_Y$-module $M$ with a $G$-equivariant structure on the level of D-modules. Such a module is called a strongly equivariant D-module. A $\mathcal{D}_Y$-module $M$ with an $\mathcal{O}_Y$-module $G$-equivariant structure is sometimes called
a weakly equivariant D-module. The Lie algebra $\mathfrak{g}$ of $G$ acts on $M$ in two ways: via the differential of the action $\mathfrak{g} \to \mathcal{D}_Y$ and via the differential of the equivariant structure. An equivalent condition for a weakly equivariant D-module $M$ to be strong is that these two actions coincide.

There are different notions of a projective stack, for instance, a stack whose coarse moduli space is a projective variety. Here we use a more restrictive notion: a projective stack is a smooth closed substack of a weighted projective stack [14]. Let us spell it out. Let $V = \bigoplus V_k$ be a positively graded $n + 1$-dimensional $\mathbb{K}$-vector space. Naturally we treat it as a $\mathbb{G}_m$-module with positive weights by $\lambda \cdot v_k = \lambda^k v_k$ where $v_k \in V_k$. Let $Y$ be a smooth closed $\mathbb{G}_m$-invariant subvariety of $V \setminus \{0\}$. We define a projective stack as the stack $[X] = [Y/\mathbb{G}_m]$. The G.I.T.-quotient $X = Y/\mathbb{G}_m$ is the coarse moduli space of $[X]$.

Let us describe the category $\mathcal{O}_{[X]} - \text{Qcoh}$ of quasicoherent sheaves on $[X]$. Choose a homogeneous basis $e_i$ on $V$ with $e_i \in V_{d_i}$, $i = 0, 1, \ldots, n$. Let $x_i \in V^*$ be the dual basis. Then $\mathbb{K}[V] = \mathbb{K}[x_0, \ldots, x_n]$ possesses a natural grading with $\deg(x_i) = d_i$. Let $I$ be the defining ideal of $Y$. Since $Y$ is $\mathbb{G}_m$-invariant, the ideal $I$ and the ring

$$A := \mathbb{K}[Y] = \mathbb{K}[x_0, \ldots, x_n]/I$$

are graded. Both $X$ and $[X]$ can be thought of as the projective spectrum of $A$. The scheme $X$ is naturally isomorphic to the scheme theoretic Proj $A$. The stack $[X]$ is the Artin-Zhang projective spectrum $\text{Proj}_{AZ} A$ [3], i.e. its category of quasicoherent sheaves $\mathcal{O}_{[X]} - \text{Qcoh}$ is equivalent to the quotient category $A - \text{Grmod}/A - \text{Tors}$ where $A - \text{Grmod}$ is the category of $\mathbb{Z}$-graded $A$-modules, $A - \text{Tors}$ is its full subcategory of torsion modules.

Recall that

$$\tau(M) = \{m \in M \mid \exists N \forall k > N A_k m = 0\}$$

is the torsion submodule of $M$. $M$ is said to be torsion if $\tau(M) = M$. It can be seen as well that the torsion submodule of $M$ is the sum of all the finite dimensional submodules of $M$ since $A$ is connected.

Denote by

$$\pi_A : A - \text{Grmod} \to A - \text{Grmod}/A - \text{Tors}$$

the quotient functor. Since $A - \text{Grmod}$ has enough injectives and $A - \text{Tors}$ is dense then there exists a section functor

$$\omega_A : A - \text{Grmod}/A - \text{Tors} \to A - \text{Grmod}$$
which is right adjoint to $\pi_\mathbb{A}$ in the sense that
\[ \text{Hom}_{\mathbb{A}-\text{Grmod}}(N, \omega_\mathbb{A}(M)) \cong \text{Hom}_{\mathbb{A}-\text{Grmod}/\mathbb{A}-\text{Tors}}(\pi_\mathbb{A}(N), M). \]
Recall that $\pi_\mathbb{A}$ is exact, $\omega_\mathbb{A}$ is left exact and $\pi_\mathbb{A}\omega_\mathbb{A} \cong \text{Id}_{\mathbb{A}-\text{Grmod}/\mathbb{A}-\text{Tors}}$. We call $\omega_\mathbb{A}\pi_\mathbb{A}(M)$ the $\mathbb{A}$-saturation of $M$. We say that a module is $\mathbb{A}$-saturated if it is isomorphic to the saturation of a module. It can be seen from the adjunction that an $\mathbb{A}$-saturated module is torsion-free and is isomorphic to its own saturation. If $M$ and $N$ are $\mathbb{A}$-saturated, then being isomorphic in $\mathbb{A}-\text{Grmod}/\mathbb{A}-\text{Tors}$ is equivalent to being isomorphic in $\mathbb{A}-\text{Grmod}$.

We need a description of the global sections functor on $[X]$ in these terms:
\[ \Gamma : \mathcal{O}_{[X]}^{-\text{Qcoh}} \to \text{VS}_K, \quad \Gamma(M) = \omega_\mathbb{A}(M)_0. \]
In particular, if $M$ is an $\mathbb{A}$-saturated module then
\[ \Gamma(\pi_\mathbb{A}(M)) = M_0. \]

The sheaf $\mathcal{O}_{[X]}(k)$ is defined as $\pi_\mathbb{A}(\mathbb{A}[k])$ where $\mathbb{A}[k]$ is the shifted regular module and the grading is given by $\mathbb{A}[k]_m = \mathbb{A}_{k+m}$.

In particular, $\Gamma(\mathcal{O}_{[X]}(k)) = \mathbb{A}_k$ if $\mathbb{A}[k]$ is $\mathbb{A}$-saturated which is the case for polynomial rings of more than two variables [2]. A well-known example of a ring, not $\mathbb{A}$-saturated (as an $\mathbb{A}$-module), is the polynomial ring in one variable $\mathbb{A} = \mathbb{K}[x]$. Its $\mathbb{A}$-saturation is the Laurent polynomial ring $\mathbb{K}[x, x^{-1}]$ seen as an $\mathbb{A}$-module. Finally we will need the push-forward functor
\[ \pi_* : \mathcal{O}_{[X]}^{-\text{Qcoh}} \to \mathcal{O}_X^{-\text{Qcoh}}, \]
given by associating a sheaf on $X$ to a graded $\mathbb{A}$-module. In general, it is not an equivalence. For instance, $\mathcal{O}_{[X]}(1)$ is an invertible sheaf but $\mathcal{O}_X(1) \cong \pi_*(\mathcal{O}_{[X]}(1))$ is not invertible, in general [6].

Let us now describe the (twisted) $D_{[X]}$-modules. Let $\partial_i = \partial/\partial x_i$, $i = 0, 1, \ldots, n$. The Weyl algebra $D(V) = \mathbb{K}\langle x_0, \ldots, x_n, \partial_0, \ldots, \partial_n \rangle$ gets a grading from the $G_m$-action on $V$: $\text{deg}(x_i) = d_i$, $\text{deg}(\partial_i) = -d_i$. We define the reduced Weyl algebra as
\[ \mathbb{D} := \text{End}_{D(V)}(D(V)/\text{Id}(V)) \cong \mathrm{I}(\text{Id}(V))/\text{Id}(V) \]
where
\[ \mathrm{I}(\text{Id}(V)) = \{ w \in D(V) \mid w\text{Id}(V) \subseteq \text{Id}(V) \} \]
is the idealiser of $\text{Id}(V)$ in $D(V)$. Notice that $\mathbb{D}$ is graded: $I$ is graded, then $\text{Id}(V)$ is graded, then $\mathrm{I}(\text{Id}(V))$ is graded, and finally $\mathbb{D}$ is graded. Observe that $\mathbb{A}$ is a graded subalgebra of $\mathbb{D}$ since $\mathbb{K}[x]_i \subseteq \mathrm{I}(\text{Id}(V))$.

It is known that for $w \in D(V)$ [9, 15.5.9]
\[ w \in \text{Id}(V) \iff w(\mathbb{K}[x]) \subseteq I \quad \text{and} \quad w \in \mathrm{I}(\text{Id}(V)) \iff w(I) \subseteq I \]
where $w$ acts naturally on polynomials in $I$. This defines an algebra embedding $D \hookrightarrow \text{End}_\mathbb{K}(\mathbb{A})$ whose image lies in $D(Y)$, the ring of differential operators on $\mathbb{A}$.

**Proposition 4.** [9, 15.5.13] The map $\phi : D \to D(Y)$ is an isomorphism.

The element $\sum_i d_i x_i \partial_i$ belongs to the idealiser $\mathbb{I}(D(V))$. We call its image in $D$ the Euler field $E = \sum_i d_i x_i \partial_i + ID(V)$.

It belongs to $D_0$ and defines the grading of $D$ and its subalgebra $A$.

**Lemma 5.** Let $x \in D$. Then $x \in D_k$ if and only if $Ex - xE = kx$.

**Proof.** It suffices to check it on the generators:

$$Ex_i = \sum_j d_j x_j \partial_j x_i = x_i E + d_i x_i.$$  

Similarly,

$$E \partial_i = \partial_i E - d_i \partial_i.$$  

The Euler field can be used to define gradings on $D$-modules.

**Lemma 6.** Let $M$ be a $D$-module. The span $M'$ of all eigenvectors of the Euler field $E$ is a $\mathbb{K}$-graded $D$-submodule of $M$.

**Proof.** Let $m \in M^\lambda$, the $\lambda$-eigenspace of $E$. Using Lemma 5,

$$Ex_i m = x_i Em + d_i x_i m = (\lambda + d_i) x_i m,$$

so

$$x_i m \in M^{\lambda + d_i}.$$  

Similarly,

$$E \partial_i m = \partial_i Em - d_i \partial_i m = (\lambda - d_i) \partial_i m$$

and

$$\partial_i m \in M^{\lambda - d_i}.$$  

Let us fix $\lambda \in \mathbb{K}$. In general,

$$M \geq M' = \bigoplus_{\mu \in \mathbb{K}} M^\mu \geq M^{(\lambda)} := \bigoplus_{n \in \mathbb{Z}} M^{\lambda + n}.$$  

A $D$-module $M$ is called $\lambda$-Euler if $M = M^{(\lambda)}$. A $\lambda$-Euler $D$-module $M$ admits a canonical $\mathbb{Z}$-grading given by $M_k = M^{\lambda + k}$. The category of $\lambda$-Euler $D$-modules $D$-Grmod$^\lambda$ is a full subcategory of the category
of graded $\mathbb{D}$-modules $\mathbb{D}^{-\text{Grmod}}$. The full subcategory of the torsion (as $A$-modules) modules is denoted $\mathbb{D}^{-\text{Tors}}$. Notice as well that the torsion submodule of a graded $\mathbb{D}$-module is a graded $\mathbb{D}$-module and that if, moreover, it is $\lambda$-Euler, then the torsion submodule is $\lambda$-Euler too.

$\mathbb{D}^{-\text{Grmod}}$ is a locally small category. $\mathbb{D}^{-\text{Tors}}$ is a Serre subcategory of $\mathbb{D}^{-\text{Grmod}}$ which is closed under taking arbitrary direct sums. Therefore, $\mathbb{D}^{-\text{Tors}}$ is a localising subcategory of $\mathbb{D}^{-\text{Grmod}}$ and the quotient functor

$$\pi_{\mathbb{D}}^\lambda : \mathbb{D}^{-\text{Grmod}} \to \mathbb{D}^{-\text{Grmod}}/\mathbb{D}^{-\text{Tors}}$$

is exact and has a right adjoint section functor

$$\omega_{\mathbb{D}}^\lambda : \mathbb{D}^{-\text{Grmod}}/\mathbb{D}^{-\text{Tors}} \to \mathbb{D}^{-\text{Grmod}}.$$

It follows that we have

$$\text{Hom}_{\mathbb{D}^{-\text{Grmod}}} (N, \omega_{\mathbb{D}}^\lambda (M)) \cong \text{Hom}_{\mathbb{D}^{-\text{Grmod}}/\mathbb{D}^{-\text{Tors}}} (\pi_{\mathbb{D}}^\lambda (N), M).$$

**Theorem 7.** The category $\mathcal{D}_{[X]}^{-\text{Qcoh}}$ of quasicoherent $D$-modules on the stack $[X]$ is equivalent to the quotient category $\mathbb{D}^{-\text{Grmod}}^0/\mathbb{D}^{-\text{Tors}}^0$.

**Proof.** The category of $D$-modules on $\overline{Y}$ is just the category of $D(\overline{Y})$-modules since $\overline{Y}$ is affine. The category of weakly $\mathbb{G}_m$-equivariant $D$-modules on $\overline{Y}$ is $D(\overline{Y})^{-\text{Grmod}}$. The two actions of the Lie algebra of the multiplicative group $\mathbb{G}_m$ are given by the Euler element $E$ and by the grading. Thus, the category of strongly $\mathbb{G}_m$-equivariant $D$-modules on $\overline{Y}$ is the category of 0-Euler $D$-modules $D(\overline{Y})^{-\text{Grmod}}$.

By definition, the category $\mathcal{D}_{[X]}^{-\text{Qcoh}}$ is the category of strongly $\mathbb{G}_m$-equivariant $D$-modules on $Y$. Thus, taking sections on the open set $Y$ induces an exact functor

$$\Gamma(Y, \_ ) : \mathcal{D}_{[X]}^{-\text{Qcoh}} \to D(Y)^{-\text{Grmod}}$$

where $D(Y)$ is the ring of global differential operators on $Y$. Proposition 4 makes the global sections $\Gamma(Y, \mathcal{M})$ into a graded $\mathbb{D}$-module via the restriction map $\mathbb{D} \cong D(\overline{Y}) \to D(Y)$. This module is 0-Euler, because $\mathcal{M}$ is strongly equivariant. Thus, we obtain exact functors

$$\Gamma(Y, \_ ) : \mathcal{D}_{[X]}^{-\text{Qcoh}} \to \mathbb{D}^{-\text{Grmod}}^0$$

and

$$\pi_{\mathbb{D}}^0 \circ \Gamma(Y, \_ ) : \mathcal{D}_{[X]}^{-\text{Qcoh}} \to \mathbb{D}^{-\text{Grmod}}^0/\mathbb{D}^{-\text{Tors}}^0.$$

Let us examine the sheafification functor $\mathbb{D}^{-\text{Grmod}} : \mathcal{D}_{[X]}^{-\text{Qcoh}} \to \mathcal{D}_{[X]}^{-\text{Qcoh}}$. The sheafification of an object in $\mathbb{D}^{-\text{Tors}}^0$ is supported at 0. Hence objects in $\mathbb{D}^{-\text{Tors}}^0$ give the zero sheaf on $Y$. So it induces a functor on the quotient

$$\sim : \mathbb{D}^{-\text{Grmod}}^0/\mathbb{D}^{-\text{Tors}}^0 \to \mathcal{D}_{[X]}^{-\text{Qcoh}}$$
which is quasiinverse to $\pi^0_D \circ \Gamma(Y, \underline{\_})$. □

An inquisitive reader may observe that we have defined the category $D_{[X]}^\lambda$-Qcoh without defining the object $D_{[X]}^\lambda$. Later on we remedy this partially by constructing an object $D_{[X]}^\lambda$ for each $\lambda \in \mathbb{K}$ so that $D_{[X]} = \pi^0_D(D^0_{[X]})$. Let us define the category $D_{[X]}^\lambda$-Qcoh of twisted $D$-modules on $[X]$ as the quotient $D_{[X]}$-Grmod/$D_{[X]}$-Tors.$^\lambda$. It is possible to define the category internally and then prove a version of Theorem 7 but we see no value in doing it here.

Given a module $M$ in $D_{[X]}$-Grmod$^\lambda$, we call $\omega^\lambda_D \pi^\lambda_D(M)$ the $D^\lambda$-saturation of $M$. We say that a module is $D^\lambda$-saturated if it is isomorphic to the $D^\lambda$-saturation of a module. It can be seen from the adjunction that a $D^\lambda$-saturated module is torsion-free and is isomorphic to its own saturation.

We shall prove now that an $A$-saturated $\lambda$-Euler $\mathbb{D}$-module is automatically $D^\lambda$-saturated. This will make our forthcoming calculations easier.

**Lemma 8.** Let $M$ be a $\lambda$-Euler $\mathbb{D}$-module. Then the $D^\lambda$-saturation of $M$ is an $A$-submodule of its $A$-saturation.

**Proof.** We have a map

$$M \to \omega^\lambda_D \pi^\lambda_D(M)$$

in $D_{[X]}$-Grmod$^\lambda$ [2]. The kernel and cokernel of this map are torsion which implies that

$$\pi_A(\omega^\lambda_D \pi^\lambda_D(M)) \cong \pi_A(M).$$

From adjunction, this isomorphism is the image of a map in $A$-Grmod,

$$\phi : \omega^\lambda_D \pi^\lambda_D(M) \to \omega_A \pi_A(M).$$

We claim that this map is injective. Since $\pi_A(\phi)$ is an isomorphism then Ker$\phi$ is a torsion $A$-module. Consider $\mathbb{D}\text{Ker}\phi$ (which contains Ker$\phi$), it is a left $\mathbb{D}$-submodule of $\omega^\lambda_D \pi^\lambda_D(M)$. Take $m \in \text{Ker}\phi$ then there exists an integer $N$ such that

$$A_{\geq N} m = 0.$$

For any $d \in \mathbb{D}$ of order $k$ we have

$$A_{\geq N+k}(dm) \leq \mathbb{D}A_{\geq N} m = 0.$$

It follows that it is a torsion submodule of $\omega^\lambda_D \pi^\lambda_D(M)$ but $\omega^\lambda_D \pi^\lambda_D(M)$ is torsion-free. Hence Ker$\phi = 0$. □

An immediate corollary is the following:

**Corollary 9.** Any $A$-saturated $\lambda$-Euler $\mathbb{D}$-module is $D^\lambda$-saturated.
Let us give examples of objects in $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}$. The sheaf $\mathcal{O}_{[X]}(k)$ is an object in $\mathcal{D}^k_{[X]} - \text{Qcoh}$. We introduce

$$D^\lambda_{[X]} := \mathbb{D}/ \mathbb{D}(E - \lambda).$$

Another interesting object in $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}$ is

$$\mathcal{D}^\lambda_{[X]} := \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]}).$$

It plays the role of the sheaf of twisted differential operators, although $D^\lambda_{[X]}$ is not an algebra because $\mathbb{D}(E - \lambda)$ is not a two-sided ideal, in general. However, $E$ is a central element of $\mathbb{D}_0$, so

$$D^\lambda_{[X]0} := \mathbb{D}_0/ \mathbb{D}_0(E - \lambda)$$

is an algebra. It plays the role of the algebra of global sections of the twisted differential operators on $[X]$. $D^\lambda_{[X]}$ is a $\mathbb{D} - D^\lambda_{[X]0}$-bimodule.

In the next section the adjoint functors of global sections and localisation will play a role. This adjoint pair $(\Gamma, L)$ is defined as:

$$\Gamma : \mathcal{D}^\lambda_{[X]} - \text{Qcoh} \to D^\lambda_{[X]0} - \text{Mod}, \quad \Gamma(M) := \omega^\lambda_{\mathbb{D}}(M)_0 = \omega^\lambda_{\mathbb{D}}(M)\lambda,$$

$$L : D^\lambda_{[X]0} - \text{Mod} \to \mathcal{D}^\lambda_{[X]} - \text{Qcoh}, \quad L(N) := \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]0}} N).$$

The ways we defined our global sections functors for $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}$ and $\mathcal{O}_{[X]} - \text{Qcoh}$ are not necessarily equivalent. Yet we know that

$$\Gamma(\pi^\lambda_{\mathbb{D}}(M)) \leq \Gamma(\pi_\mathbb{A}(M))$$

as $\mathbb{A}$-modules for any $\lambda$-Euler $\mathbb{D}$-module $M$.

The exposition would be greatly simplified if restricting the section functor $\omega_\mathbb{A}$ to $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}$ were equivalent to $\omega^\lambda_{\mathbb{D}}$. This explains why we have different global sections functor for different $\lambda$ although geometrically only one is needed. However, to ensure that we obtain $\lambda$-Euler $\mathbb{D}$-modules and not just $\mathbb{A}$-modules we use $\omega^\lambda_{\mathbb{D}}$.

3. $\textbf{D}$-modules on weighted projective space

In this section we consider $Y = V \setminus \{0\}$, the punctured vector space of dimension at least 2 and $[X] = [Y/\mathbb{G}_m] = [\mathbb{P}(V)]$, the weighted projective stack. In this case $I = \{0\}, \mathbb{A} = \mathbb{K}[x_0, \ldots, x_n]$ where the degree of $x_i$ is $d_i > 0$ and $\mathbb{D} = \mathbb{K}(x_0, \ldots, x_n, \partial_0, \ldots, \partial_n)$ is the Weyl algebra. Without loss of generality, we assume that $0 < d_0 \leq d_1 \leq \ldots \leq d_n$.

Let us look at the $\mathbb{D}$-module $\Delta$ generated by the delta-function at zero $\delta = \delta_0(x_0, \ldots, x_n)$

$$\Delta = \mathbb{D}\delta \cong \mathbb{D}/(\mathbb{D}x_0 + \mathbb{D}x_1 + \ldots + \mathbb{D}x_n).$$
The linear map
\[ \mathbb{K}[\partial_0, \ldots, \partial_n] \to \Delta, \quad f(\partial_0, \ldots, \partial_n) \mapsto f(\partial_0, \ldots, \partial_n) \cdot \delta \]
is an isomorphism of vector spaces. If we identify \( \mathbb{K}[\partial_0, \ldots, \partial_n] \) with \( \Delta \) using this linear map, then \( \partial_i \) acts by multiplication and \( x_i \) acts by derivation \( \partial_j \mapsto -\delta_{i,j} \). In particular,
\[ E \cdot \delta = E \cdot 1 = \sum_j d_j x_j \cdot \partial_j = \sum_j -d_j = -(\sum_j d_j) \delta. \]
Hence, \( \Delta \) is \( k \)-Euler for each integer \( k \). Its canonical \( k \)-Euler grading is given by
\[ \delta \in \Delta^{-\sum_j d_j} = \Delta_{-k-\sum_j d_j}, \quad \partial_i \cdot \delta \in \Delta_{-k-d_i-\sum_j d_j}. \]

Let \( J = (x_0, \ldots, x_n) \triangleleft A \). If \( M \) is a \( D \)-module, \( \tau(M) = \{ m \in M \mid \exists k \ J^k m = 0 \} \) is its torsion \( D \)-submodule (a reader can easily verify that if \( J^k m = 0 \), then \( J^{k+1} \partial_i m = 0 \)). The torsion \( D \)-modules are those, supported set theoretically on the zero \( 0 \in V \). By Kashiwara’s theorem, any \( D \)-module supported at \( 0 \) is a direct sum of copies of \( \Delta \).

Let us introduce some notations. Suppose that \( M \) and \( N \) are two \( \mathbb{Z} \)-graded \( A \)-modules. We say that an \( A \)-module homomorphism \( f : M \to N \) has degree \( l \) if \( f(M_i) \subset N_{i+l} \) for all \( i \). Denote by \( \text{Hom}(M, N)_l \) the set of all degree \( l \) \( A \)-module homomorphisms and write \( \text{Hom}_A(M, N) = \bigoplus_{l \in \mathbb{Z}} \text{Hom}(M, N)_l \).

Now let \( \text{Ext}^q(M, N)_l \) be the derived functor of \( \text{Hom}(M, N)_l \) and write \( \text{Ext}_A^q(M, N) = \bigoplus_{l \in \mathbb{Z}} \text{Ext}_A^q(M, N)_l \).

Artin and Zhang prove \[2\] that for any graded \( A \)-module \( M \),
\[ \tau_h(M) \cong \lim_{\to} \text{Hom}_h(A/A_{\geq k}, M), \]
\[ R^1 \tau_h(M) \cong \lim_{\to} \text{Ext}_A^1(A/A_{\geq k}, M) \]
and that there exists a long exact sequence of \( A \)-modules
\[ 0 \to \tau_h(M) \to M \to \omega_h \pi_h(M) \to R^1 \tau_h(M) \to 0 \]
where \( \tau_h(M) \) and \( R^1 \tau_h(M) \) are torsion modules. This implies the following proposition.

**Proposition 10.** A \( \lambda \)-Euler \( D \)-module \( M \) is \( D^\lambda \)-saturated if it is torsion-free and \( \lim_{\to} \text{Ext}_A^1(A/A_{\geq k}, M) = 0 \).

The next lemma will prove primordial in the proof that \( \Gamma_{\lambda} L_{\lambda} \cong Id_{D[X]_{10}-\text{Mod}} \) for any \( \lambda \) and \( n \geq 2 \).
Lemma 11. For $n \geq 2$, $D^\lambda_{[X]}$ is $\mathcal{D}^\lambda$-saturated.

Proof. Recall that $D^\lambda_{[X]} = \mathcal{D}/\mathcal{D}(E - \lambda)$. It is easier to compute Ext groups by taking a projective resolution of the left argument than an injective one of the right argument. Since $\mathcal{A}/\mathcal{A}_{\geq 1} \cong K$, the first three terms of the Koszul resolution are given by

$$\cdots \rightarrow \bigoplus_{i_0 < i_1} \mathcal{A}(-d_{i_0} - d_{i_1}) \rightarrow \bigoplus_{i=0}^n \mathcal{A}(-d_i) \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}_{\geq 1} \rightarrow 0.$$ 

Take away $\mathcal{A}/\mathcal{A}_{\geq 1}$ and apply $\text{Hom}_\mathcal{A}(-, D^\lambda_{[X]})$ to the above exact sequence to get

$$0 \rightarrow D^\lambda_{[X]} \xrightarrow{\phi_1} \bigoplus_{i=0}^n D^\lambda_{[X]}(d_i) \xrightarrow{\phi_2} \bigoplus_{i_0 < i_1} D^\lambda_{[X]}(d_{i_0} + d_{i_1}) \rightarrow \cdots$$

where

$$\phi_1 : m \mapsto (x_i m)_{i=0}^n$$

and

$$\phi_2 : (m_i)_{i=0}^n \mapsto (x_{i_0} m_{i_1} - x_{i_1} m_{i_0})_{i_0 < i_1}.$$ 

It follows that

$$\text{Hom}_\mathcal{A}(\mathcal{A}/\mathcal{A}_{\geq 1}, D^\lambda_{[X]}) \cong \ker(\phi_1),$$

$$\text{Ext}_1^\mathcal{A}(\mathcal{A}/\mathcal{A}_{\geq 1}, D^\lambda_{[X]}) \cong \frac{\ker(\phi_2)}{\text{Im}(\phi_1)}.$$ 

The claim is that both $\text{Hom}_\mathcal{A}(\mathcal{A}/\mathcal{A}_{\geq 1}, D^\lambda_{[X]})$ and $\text{Ext}_1^\mathcal{A}(\mathcal{A}/\mathcal{A}_{\geq 1}, D^\lambda_{[X]})$ vanish.

Let us first compute $\text{Hom}_\mathcal{A}(\mathcal{A}/\mathcal{A}_{\geq 1}, D^\lambda_{[X]})$. Pick $m \in \ker(\phi_1)$, then for all $i$ we have $x_i m = 0$ where

$$m = m + \mathcal{D}(E - \lambda).$$

We can assume $m$ to be homogeneous, so

$$x_i m = p_i(E - \lambda)$$

for some homogeneous $p_i \in \mathcal{D}$. We want to show that $p_i \in x_i \mathcal{D}$. Suppose, for a contradiction, that it is not. Then we can write

$$p_i = x_i m' + f \partial^\beta + LT$$

where $m' \in \mathcal{D}$, $f \in \mathcal{K}[x_0, ..., x_n]$ is the highest term, free of $x_i$, $\beta$ the biggest power, and $LT$ are the lower terms using $\text{DegLex}$ for the ordering of the monomials in $\partial$. Without loss of generality we can assume $i \neq 0$. It follows that

$$x_i m = d_0 f x_0 \partial^\beta + m + LT$$
since $f \partial_0 \beta_0 = f x_0 \partial_0 \beta_0 + LT$. But $f x_0$ is not divisible by $x_i$ and we obtain a contradiction. Thus,

$$\text{Hom}_A(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]}) = 0.$$ 

Similarly, let us show that $\text{Ext}^1_A(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]})$ vanishes. To proceed, choose $(m_i)_{i=0}^n \in \text{Ker}(\phi_2)$. Then for all $i, j$ there exists a $\theta_{ij} \in D$ such that

$$x_i m_j = x_j m_i + \theta_{ij}(E - \lambda).$$ 

Write

$$m_j = x_j m_j' + f \partial_0 \beta + LT$$

where $m_j' \in D$, $f \in \mathbb{K}[x_0, ..., x_n]$ is the highest term, free of $x_j$, $\beta$ is the highest power and $LT$ are the lower terms using $\text{DegLex}$ for the ordering of the monomials in $\partial$. Let us, suppose for the sake of a contradiction, that $|\beta| \neq 0$. Then without loss of generality, we can assume that $\beta$ is the lowest among all the possible representatives of $m_j$. Write

$$\theta_{ij} = g \partial_0 \gamma (E - \lambda) + LT$$

where $g \in \mathbb{K}[x_0, ..., x_n]$ is the highest term, free of $x_j$. This implies that

$$x_i x_j m_j' + x_j f \partial_0 \beta + LT = x_j m_i + g \partial_0 \gamma (E - \lambda) + LT.$$ 

Again without loss of generality, suppose that $i, j \neq 0$. By comparing the highest terms, free of $x_j$, we get

$$x_i f \partial_0 \beta = d_0 g x_0 \partial_0 \beta_0 + e_0$$

with $|\gamma| = |\beta| - 1$. Therefore,

$$f \partial_0 \beta = d_0 \frac{g}{x_i} x_0 \partial_0 \beta_0 + \frac{g}{x_i} \partial_0 \gamma (E - \lambda) + LT.$$ 

So $m_j - f \partial_0 \beta$ is another representative of $m_j$ which has an index $\gamma$ lower than $\beta$, contrary to our hypothesis. Thus,

$$m_j = x_j m_j'$$

For all $i, j$ we now have

$$x_i x_j m_j' = x_i x_j m_i' + \theta_{ij}(E - \lambda)$$

which is equivalent to

$$x_i x_j (m_j' - m_i') = \theta_{ij}(E - \lambda).$$ 

By using the previous argument twice, for all $i, j$, we have

$$m_j' - m_i' \in D(E - \lambda).$$
Write
\[ \bar{m}^j := \frac{m_j}{m_i} = \frac{m_x}{m_y}. \]
for the residues of \( m^j \) and \( m_i \). Then for all \( i \),
\[ m_i = x_i \bar{m}. \]
Hence,
\[ \text{Ext}^1_k(A/A_{\geq 1}, D_{[X]}^\lambda) = 0. \]
To finish our proof, for each \( k \) we have a short exact sequence of \( A \)-modules:
\[ 0 \to A_{\geq k}/A_{\geq k+1} \to A/A_{\geq k+1} \to A/A_{\geq k} \to 0. \]
Now by applying \( \text{Hom}_k(A, D_{[X]}^\lambda) \) to this short exact sequence and by induction on \( k \), we conclude that for all \( k \):
\[ \text{Hom}^1_k(A/A_{\geq k}, D_{[X]}^\lambda) = 0, \]
\[ \text{Ext}^1_k(A/A_{\geq k}, D_{[X]}^\lambda) = 0. \]
Taking direct limit [2] it follows that
\[ \tau_k(D_{[X]}^\lambda) = 0, \quad \text{and} \quad \varprojlim \text{Ext}^1(A/A_{\geq k}, D_{[X]}^\lambda) = 0. \]
Hence \( D_{[X]}^\lambda \) is \( \mathbb{D}^\lambda \)-saturated by Proposition [10].

The condition on \( n \) in the last proof is necessary. We can prove that \( D_{[X]}^\lambda \) is not \( \mathbb{D}^\lambda \)-saturated for all \( \lambda \) when \( n = 1 \). For this, it suffices to notice that for \( \lambda = 0 \),
\[ (-d_1 \partial_1, d_0 \partial_0) \in \text{Ker}(\phi_2) \]
but
\[ (-d_1 \partial_1, d_0 \partial_0) \notin \text{Im}(\phi_1) \]
since \( d_0 x_0 \partial_0 = -d_1 x_1 \partial_1 + E \).

**Lemma 12.** Let \( n \geq 2 \). If \( \Gamma_\lambda \) is exact then \( \Gamma_\lambda L_\lambda \cong \text{Id}_{D_{[X]}^\lambda} \circ \text{Mod} \)

**Proof.** Let \( N \) be a \( D_{[X]}^\lambda \)-module. Take the first two terms of a free resolution of \( N \)
\[ P_1 \to P_0 \to N \to 0 \]
where \( P_i = \bigoplus_{j \in I_i} D_{[X]}^\lambda \) and \( I_i \) is an index set. Since both \( D_{[X]}^\lambda \circ D_{[X]}^\lambda \circ \pi_{[X]}^\lambda \) and \( \pi_{[X]}^\lambda \) are right exact functors, it follows that
\[ \Gamma_\lambda L_\lambda(P_1) \to \Gamma_\lambda L_\lambda(P_0) \to \Gamma_\lambda L_\lambda(N) \to 0 \]
is exact. We can compute the first two terms explicitly:

\[ \Gamma_L \lambda L( P_i ) = ( \omega_D^\lambda \pi_D^\lambda ( D_{[X]}^\lambda \otimes D_{[X]_0}^\lambda P_i ) )_0 \]
\[ = ( \omega_D^\lambda \pi_D^\lambda ( D_{[X]}^\lambda \otimes D_{[X]_0}^\lambda ( \bigoplus_{j \in I_i} D_{[X]}^\lambda ) )_0 \]
\[ \cong ( \omega_D^\lambda \pi_D^\lambda ( \bigoplus_{j \in I_i} D_{[X]}^\lambda )_0 )_0 \]
\[ \cong ( \omega_D^\lambda \pi_D^\lambda ( \bigoplus_{j \in I_i} D_{[X]}^\lambda ) )_0 \]

since the tensor product commutes with arbitrary direct sums and that \( D_{[X]}^\lambda \otimes D_{[X]_0}^\lambda D_{[X]}^\lambda \cong D_{[X]}^\lambda \). The category \( \mathbb{D} - \text{Grmod}^\lambda \) is locally noetherian. By a result of Gabriel, the section functor \( \omega_D^\lambda \) commutes with inductive limits and, in particular, with arbitrary direct sums [7, p. 379]. Moreover, \( \pi_D^\lambda \) is left adjoint to \( \omega_D^\lambda \), so \( \pi_D^\lambda \) commutes as well with arbitrary direct sums. This yields the following sequence of natural isomorphims:

\[ \Gamma_L \lambda L( P_i ) \cong ( \omega_D^\lambda \pi_D^\lambda ( \bigoplus_{j \in I_i} D_{[X]}^\lambda ) )_0 \]
\[ \cong ( \bigoplus_{j \in I_i} \omega_D^\lambda \pi_D^\lambda ( D_{[X]}^\lambda ) )_0 \]
\[ \cong ( \bigoplus_{j \in I_i} D_{[X]}^\lambda )_0 \]
\[ \cong \bigoplus_{j \in I_i} D_{[X]_0}^\lambda \]
\[ \cong P_i \]

since \( D_{[X]}^\lambda \) is \( \mathbb{D}^\lambda \)-saturated and that \( ( \_ )_0 \) commutes with arbitrary direct sums. Thus, we constructed a commutative diagram with exact rows:

\[ \begin{array}{c}
P_1 \rightarrow P_0 \rightarrow \Gamma_L \lambda L( N ) \rightarrow 0 \\
\downarrow \alpha \downarrow \beta \downarrow \gamma \\
P_1 \rightarrow P_0 \rightarrow N \rightarrow 0
\end{array} \]

where \( \alpha \) and \( \beta \) are isomorphisms, so \( \Gamma_L \lambda L( N ) \cong N \) is a natural isomorphism by the four lemma.

\[ \square \]

**Theorem 13.** Let \( \mathcal{A} \) be the \( \mathbb{Z}_{\geq 0} \)-span of all \( d_i \)-s. If \( \lambda \in \mathbb{K} \setminus ( - \sum_i d_i - \mathcal{A} ) \), then the global sections functor \( \Gamma_L : \mathbb{D}_{[X]}^\lambda - \text{Qcoh} \rightarrow \mathbb{D}_{[X]_0}^\lambda - \text{Mod} \)
is exact. In this case, $\Gamma_\lambda$ defines an equivalence between the quotient category $\mathcal{D}^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$ and $\mathcal{D}^{\lambda}_{[X]} - \text{Mod}$.

**Proof.** The category $\mathcal{D}^{\lambda}_{[X]} - \text{Qcoh}$ is the quotient category of the category of $\lambda$-Euler modules by the category of torsion modules. The canonical grading on a $\lambda$-Euler module $M$ is given by $M_k = M^{k+\lambda}$. The torsion modules are direct sums of $\Delta$. The global sections functor $\Gamma_\lambda$ is $\Gamma_\lambda : M \mapsto \omega^\lambda_D(M) = \omega^\lambda_D(M)^\lambda$.

We know that $\omega^\lambda_D$ is a left exact functor. Taking $\lambda$-eigenspaces is an exact functor, so we are left to prove that $\Gamma_\lambda$ is right exact. An epimorphism $f : M \to N$ induces the exact sequence

$$\omega^\lambda_D(M) \to \omega^\lambda_D(N) \to \text{coker}(\omega^\lambda_D(f)) \to 0$$

where $\text{coker}(\omega^\lambda_D(f))$ is a torsion $\mathbb{D}$-module. Taking the zeroth graded part, we get the exact sequence

$$\Gamma_\lambda(M) \to \Gamma_\lambda(N) \to \text{coker}(\omega^\lambda_D(f))_0 \to 0.$$

Our restriction on $\lambda$ provides that $\text{coker}(\omega^\lambda_D(f))_0 = 0$. Indeed, if $\lambda \not\in \mathbb{Z}$, then $\text{coker}(\omega^\lambda_D(f)) = 0$. If $\lambda \in \mathbb{Z}$, then $\text{coker}(\omega^\lambda_D(f)) = \oplus \Delta$ and $\text{coker}(\omega^\lambda_D(f))_0 = \oplus \Delta^\lambda$. Since the $E$-weights of $\Delta$ are $-\sum_i d_i - A$, $\text{coker}(\omega^\lambda_D(f))_0 = 0$. Hence $\Gamma_\lambda$ is exact.

The kernel $\text{Ker}\Gamma_\lambda$ is the full subcategory of $\mathcal{D}^{\lambda}_{[X]} - \text{Qcoh}$ whose objects are those $M$ without non-trivial global sections, i.e., with $\Gamma_\lambda(M) = 0$. Since $\Gamma_\lambda$ is exact, it is a Serre subcategory, and $\Gamma_\lambda$ descends to a functor

$$\tilde{\Gamma}_\lambda : \mathcal{D}^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda \to \mathcal{D}^{\lambda}_{[X]} - \text{Mod}.$$

and let

$$Q : \mathcal{D}^{\lambda}_{[X]} - \text{Qcoh} \to \mathcal{D}^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$$

be the quotient functor. We claim that $QL_\lambda$ is a quasiinverse of $\tilde{\Gamma}_\lambda$.

Now in one direction,

$$\tilde{\Gamma}_\lambda(QL_\lambda)(N) = (\tilde{\Gamma}_\lambda Q)L_\lambda(N)$$

$$= \Gamma_\lambda L_\lambda(N)$$

$$\cong N$$

since $\Gamma_\lambda$ is exact. Thus,

$$\tilde{\Gamma}_\lambda Q L_\lambda \cong \text{Id}_{\mathcal{D}^{\lambda}_{[X]} - \text{Mod}}.$$

In the opposite direction, we have a natural transformation

$$QL_\lambda \tilde{\Gamma}_\lambda \to \text{Id}_{\mathcal{D}^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda}.$$
Take an object $\tilde{M}$ in $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$. Then there exists an object $\mathcal{M}$ in $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}$ such that $\tilde{M} = Q(\mathcal{M})$. Hence,

$$QL_\lambda \Gamma_\lambda(\tilde{M}) = QL_\lambda \Gamma_\lambda(\mathcal{M})$$

$$= Q\pi^\lambda_D(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} (\omega^\lambda_D(\mathcal{M})))_0).$$

On a level of a $\lambda$-Euler module $M$ (with its canonical grading), the natural map

$$D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0 \rightarrow M$$

gives rise to the long exact sequence

$$0 \rightarrow K \rightarrow D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0 \rightarrow M \rightarrow N \rightarrow 0$$

where $K$ is its kernel and $N$ is its cokernel. Since $\pi^\lambda_D$ is exact,

$$0 \rightarrow \pi^\lambda_D(K) \rightarrow \omega^\lambda_D(M)_0 \rightarrow \omega^\lambda_D(M)_0 \rightarrow \pi^\lambda_D(N) \rightarrow 0$$

is a long exact sequence as well. If $M = \omega^\lambda_D(M)$, applying $\Gamma_\lambda$ yields

$$0 \rightarrow \Gamma_\lambda \pi^\lambda_D(K) \rightarrow \omega^\lambda_D(M)_0 \rightarrow \omega^\lambda_D(M)_0 \rightarrow \Gamma_\lambda \pi^\lambda_D(N) \rightarrow 0$$

since $\Gamma_\lambda \pi^\lambda_D(\omega^\lambda_D(M)) \cong \omega^\lambda_D(M)_0$ and $\Gamma_\lambda L_{\lambda} \cong \text{Id}_{D^\lambda_{[X]_0} - \text{Mod}}$ when $\Gamma_\lambda$ is exact. The middle map

$$\omega^\lambda_D(M)_0 \rightarrow \omega^\lambda_D(M)_0$$

is the identity map and hence an isomorphism. It follows that $\pi^\lambda_D(K)$ and $\pi^\lambda_D(N)$ are objects in $\text{Ker}(\Gamma_\lambda)$. Therefore,

$$\pi^\lambda_D(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} \omega^\lambda_D(M)_0) \rightarrow \pi^\lambda_D(\omega^\lambda_D(M))$$

is an isomorphism in $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$ and

$$QL_\lambda \Gamma_\lambda(\tilde{M}) \cong Q\pi^\lambda_D(\omega^\lambda_D(M))$$

$$\cong Q(\mathcal{M})$$

$$\cong \tilde{M}.$$ 

It follows that $QL_\lambda \Gamma_\lambda \cong I_{\mathcal{D}^\lambda_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda}$.

It remains to compute the kernel of $\Gamma_\lambda$. It can be non-zero. If $k \in \mathbb{Z}$, then $\mathcal{O}_{[X]}(k) = \pi^\lambda_D(\mathbb{A}[k])$ is a non-zero $\mathcal{D}^\lambda$-saturated (since it is $\mathbb{A}$-saturated [2]) object of $\mathcal{D}^\lambda_{[X]} - \text{Qcoh}$ because $1 \in \mathbb{A}_0 = \mathbb{A}[k]_{-k}$ and

$$E \cdot 1 = 0 = (-k + k) \cdot 1.$$

The global sections

$$\Gamma_k(\mathcal{O}_{[X]}(k)) = \mathbb{A}[-k]_0 = \mathbb{A}_k$$
are non-zero if and only if \( k \in \mathcal{A} \). Thus, if \( k \in \mathbb{Z} \setminus \mathcal{A} \), then \( \mathcal{O}_{[X]}(k) \) is a non-zero object of \( \text{Ker}\Gamma_k \).

Now let us assume that the greatest common divisor \( d \) of \( d_0, \ldots, d_n \) is greater than 1. It easily follows that

\[
\mathbb{D}_1 = \mathbb{D}_2 = \ldots = \mathbb{D}_{d-1} = 0.
\]

Let \( M \) be the \( \mathbb{K} \)-vector space with a basis of all formal monomials \( x_0^{a_0} \ldots x_n^{a_n}, a_i \in \mathbb{K} \). It is a \( \mathbb{D} \)-module under the following operations, defined on the monomials by

\[
x_i \cdot x_0^{a_0} \ldots x_n^{a_n} = x_0^{a_0} \ldots x_i^{a_i + 1} x_{i+1} \ldots x_n^{a_n},
\]

\[
\partial_i \cdot x_0^{a_0} \ldots x_n^{a_n} = a_i x_0^{a_0} \ldots x_i^{-1+a_i} x_{i+1} \ldots x_n^{a_n}.
\]

Given \( \lambda \in \mathbb{K} \), we consider the \( \mathbb{D} \)-submodule \( N = \mathbb{D} x_0^{(\lambda-1)/d_0} \). Since

\[
\mathcal{E} \cdot x_0^{(\lambda-1)/d_0} = d_0 x_0 \partial_0 \cdot x_0^{(\lambda-1)/d_0} = (\lambda - 1) x_0^{(\lambda-1)/d_0},
\]

the module \( N \) is \( \lambda \)-Euler and \( x_0^{(\lambda-1)/d_0} \in N^{\lambda-1} = N_{-1} \) in the canonical \( \lambda \)-Euler grading. Put \( \mathcal{N} = \pi^\lambda_{\mathbb{D}}(N) \). By definition of \( \mathcal{N} \), that it is torsion-free. So the long exact sequence \( [2] \)

\[
0 \to \tau^\lambda_{\mathbb{D}}(N) \to N \to \omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(N) \to R^1 \tau^\lambda_{\mathbb{D}}(N) \to 0
\]

reduces to the short exact sequence

\[
0 \to N \to \omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(N) \to R^1 \tau^\lambda_{\mathbb{D}}(N) \to 0.
\]

But \( R^1 \tau^\lambda_{\mathbb{D}}(N) \) is a torsion \( \mathbb{D} \)-module, hence it is a direct sum of copies of \( \Delta \). The \( \mathcal{E} \)-weights of \( N \) are congruent to \(-1\) modulo \( d \) and the \( \mathcal{E} \)-weights of the module \( \Delta \) are congruent to \( 0 \) modulo \( d \). It follows that the short exact sequence splits and

\[
\omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(N) \cong N \oplus R^1 \tau^\lambda_{\mathbb{D}}(N).
\]

Since \( \omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(N) \) is torsion free, \( \omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(N) \cong N \) and \( R^1 \tau^\lambda_{\mathbb{D}}(N) = 0 \). This means that \( N \) is \( \mathbb{D} \)-saturated and

\[
\Gamma_\lambda(\mathcal{N}) = N_0 = 0.
\]

Hence, \( \mathcal{N} \) is a non-zero object in \( \text{Ker}\Gamma_\lambda \). In all the other cases the kernel is trivial.

**Theorem 14.** Let us assume that the greatest common divisor \( \gcd_i(d_i) \)

is equal to 1. If \( \lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A} \), then \( \text{Ker}\Gamma_\lambda \) is a zero category.

**Proof.** Let \( m \) be the least common multiple of \( d_0, \ldots, d_n \). Suppose that \( \mathcal{M} \) is a non-zero object in \( \mathcal{D}^\lambda_{[X]} \cdot \text{Qcoh} \). Then \( M := \omega^\lambda_{\mathbb{D}}(\mathcal{M}) \) is a non-zero \( \lambda \)-Euler torsion-free \( \mathbb{D} \)-module. We need to show that \( M_0 \neq 0 \). Let us suppose that the contrary is true, i.e., \( M_0 = 0 \). We proceed to arrive at a contradiction via a sequence of claims.
Claim 1. \( M_{-mt} = 0 \) for any \( t \in \mathbb{Z}_{>0} \).

Proof of Claim: If \( a \in M_{-mt} \), then \( x_i^{mt/d_i} \cdot a = 0 \) for all \( i = 0, \ldots, n \) since it is an element of \( M_0 \). Hence, \( a \) generates a torsion \( \mathbb{D} \)-submodule of \( M \) but \( M \) is torsion-free. Hence \( a = 0 \).

Claim 2. \( M_{-mt+kd_i} = 0 \) for all \( i \) and \( 0 \leq k \leq \frac{mt}{d_i} \). In particular, \( M_{-kd_i} = 0 \) for all \( k \geq 0 \).

Proof of Claim: We proceed by induction. The case \( k = 0 \) is Claim 1. Assume that this is true for \( k \), and let us prove it for \( k+1 \). If \( -mt + (k+1)d_i = 0 \), then we are done. Otherwise, let us pick a non-zero element \( a \in M_{-mt+(k+1)d_i} \). It follows that \( \partial_i \cdot a \in M_{-mt+kd_i} \) which is zero by induction. Moreover, \( x_i^{-(k+1)+mt/d_i} \cdot a \in M_0 \) which is zero again. Since

\[
\left[ \partial_i, x_i^{-(k+1)+mt/d_i} \right] = \left( \frac{mt}{d_i} - (k + 1) \right) x_i^{-(k+2)+mt/d_i},
\]

we conclude that \( x_i^{-(k+2)+mt/d_i} \cdot a = 0 \). We can repeat this argument to conclude that \( x_i^{-(k+l)+mt/d_i} \cdot a = 0 \) for all positive \( l \) with \( \frac{mt}{d_i} - (k + l) \geq 0 \). In particular, \( a = x_i^0 \cdot a = 0 \). \( \square \)

Claim 3. If \( c_0, \ldots, c_k \) are positive integers and \( g \) is their greatest common divisor, then there exist integers \( r_0 \leq 0, r_1, \ldots, r_k \geq 0 \) such that \( r_0 c_0 + \ldots + r_k c_k = g \).

Proof of Claim: Let \( l \) be the least common multiple of \( c_0, \ldots, c_k \). By the Euclidean algorithm there exist integers \( s_0, \ldots, s_k \) such that

\[
s_0 c_0 + \ldots + s_k c_k = 1.
\]

Now we can add \( -\frac{l}{c_0} c_0 + \frac{l}{c_i} c_i = 0 \) for various \( i \) to this relations to get integers \( r_0, \ldots, r_k \) such that

\[
r_0 c_0 + \ldots + r_k c_k = 1
\]

and \( r_1, \ldots, r_k \geq 0 \). Inevitably, \( r_0 \leq 0 \). \( \square \)

Claim 4. For all integer \( b_0, \ldots, b_l \geq 0 \), \( M_{-(b_0d_0+\ldots+b_ld_l)} = 0 \).

Proof of Claim: We proceed by induction on \( l \). The base case \( l = 0 \) is Claim 2. Assume this is true for \( l-1 \). In particular, it is true if \( b_i = 0 \) for some \( i \).

Let \( g_l = \gcd (d_0, \ldots, d_l) \) and fix a positive integer \( k \). Consider a non-zero element \( a \in M_{-kg_l} \). There exist positive integers \( c_0, c_1, \ldots, c_l \) such that

\[
\partial_i^{c_0} \cdot a = \partial_i^{c_1} \cdot a = \ldots = \partial_i^{c_l} \cdot a = 0.
\]
Indeed, by Claim 3, there exist $r_i \leq 0$ and $r_0, \ldots, r_{i+1}, \ldots, r_l \geq 0$ such that

$$\sum_{i=0}^{l} r_i d_i = g_l$$

Now if $c_i = -kr_i \geq 0$, then

$$\partial^c_i \cdot a \in M_{-c_i d_i - k g_l} = M_{-k(\sum_{i=0}^{l} r_i d_i)} = 0,$$

by induction. Let us consider the Weyl algebra

$$\tilde{D} = \mathbb{K}\langle x_0, \ldots, x_l, \partial_0, \ldots, \partial_l \rangle$$

and its polynomial subalgebra $\tilde{A} = \mathbb{K}[\partial_0, \ldots, \partial_l]$. The $\tilde{A}$-module $\tilde{D}a$ is supported at zero, hence, it must be a direct sum of copies of $\tilde{\Delta} = \tilde{D}\delta(\partial_0, \ldots, \partial_l) \cong \mathbb{K}[x_0, \ldots, x_l]$. It follows that

$$x_0^{b_0} \cdots x_l^{b_l} \cdot a \neq 0 \text{ for all } b_0, \ldots, b_l \geq 0.$$

We want to determine for which $k$, we can find $b_0, \ldots, b_l \geq 0$ such that $x_0^{b_0} \cdots x_l^{b_l} \cdot a \in M_0 = 0$. We get a contradiction and hence $M_{-k g_l} = 0$ for such $k$. The condition is that

$$k g_l \in \mathbb{Z}_{\geq 0} d_0 + \mathbb{Z}_{\geq 0} d_1 + \ldots + \mathbb{Z}_{\geq 0} d_l.$$

In particular, it is true for $l = n$, i.e., $M_{-k} = 0$ for all $k \in A$. Now let us finish the proof of the theorem. By Schur’s Theorem there exists $K \geq 0$ such that $k \in A$ for all $k > K$, in particular, $M_{-k} = 0$ for all $k > K$. Thus, $M$ is supported at zero as a $\mathbb{K}[\partial_0, \ldots, \partial_n]$-module. By Kashiwara’s Theorem $M$ is a direct sum of copies of $A = \mathbb{K}[x_0, \ldots, x_n]$. If $\lambda \in \mathbb{K} \setminus \mathbb{Z}$ then $A$ is not $\lambda$-Euler. Thus, $M = 0$. Finally, if $\lambda \in \mathbb{Z}$ then $A$ is $\lambda$-Euler. Moreover, as a graded module $M$ is a direct sum of copies of $A[\lambda]$. Observe that $A[\lambda]_0 = A_{\lambda} \neq 0$ if and only if $\lambda \in A$. Thus, if $\lambda \in A$, then $M = 0$ as well.

Together with Theorem 13 this gives the following corollary.

**Corollary 15.** Let us suppose that $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup A$ and $\gcd(d_0, \ldots, d_n) = 1$. Then $\Gamma_{\lambda} : D_{X}^\lambda \text{-Qcoh} \to D_{[X]_0}^\lambda \text{-Mod}$ is an equivalence of categories.

A similar functor for varieties

$$\Gamma'_{\lambda} : D_{X}^\lambda \text{-Qcoh} \to D_{[X]_0}^\lambda \text{-Mod}$$

\[\text{1}\]The smallest such $K$ is called the Frobenius number. It is a NP-hard problem to find such $K$. There is no known closed formula that gives $K$ as a function of $d_0, \ldots, d_n$ for $n \geq 2$. 


is studied by Van den Bergh [13]. It is instructive to compare it with the push-forward functor
\[ \pi_* : \mathcal{D}^\lambda_{[X]} \text{Qcoh} \to \mathcal{D}^\lambda_X \text{Qcoh}. \]

The functors \( \Gamma'_\lambda \pi_* \) and \( \Gamma_\lambda \) are naturally equivalent, so we can conclude the final corollary.

**Corollary 16.** Let us suppose that \( \lambda \in \mathbb{K} \setminus \mathbb{Z} \cup \mathcal{A} \) and \( \gcd_{i\neq j}(d_i) = 1 \) for every \( j \) (the well-formedness condition). Then the push-forward functor \( \pi_* : \mathcal{D}^\lambda_{[X]} \text{Qcoh} \to \mathcal{D}^\lambda_X \text{Qcoh} \) is an equivalence of categories.

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