Universality of determinantal point processes associated with Bergman kernels

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Abstract

In this article, we study determinantal point processes whose kernel is the Burgman kernel of a high power of a positive Hermitian holomorphic line bundle over a compact Kähler manifold. We first provide a simple proof of a theorem by Berman [Ber18] giving the scaling limit of such processes, using a near-diagonal expansion of the Bergman kernel by Bleher, Shiffman and Zelditch [BSZ00a, BSZ00b], then as an application we recover another result by Berman [Ber09] on the convergence of empirical measures to an equilibrium measure.

Keywords—Bergman kernel, Determinantal point processes, Kähler geometry, universality.

1 Introduction

Determinantal point processes (DPP) are a class of random configurations on a space $E$, whose points are guided by a repulsive interaction. They were first introduced by Macchi [Mac75] as a general model, but many specific examples have been studied earlier, such as some classical models in random matrix theory [Wig55, Dys62a, Dys62b, Gin65]. One of the first motivations comes from quantum mechanics, where the wavefunction of a particle is described by $\psi : E \to \mathbb{C}$ and its squared modulus is interpreted (up to renormalization) as a probability density with respect to a reference measure on $X$. Given an orthonormal family $(\psi_1, \ldots, \psi_N)$ of such wavefunctions, assuming that each one represents a fermion, one can define a $N$-particle wavefunction by taking the Slater determinant as follows:

$$\Psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det(\psi_i(x_j)).$$

(1)

The squared norm of this wavefunction is again interpreted (up to renormalization) as a probability density, the joint density of the $N$ fermions. If we denote by $X_1, \ldots, X_N$ their positions in $E$, the point process $\sum_{i=1}^N \delta_{X_i}$ is a determinantal point process (DPP), i.e. for any $1 \leq n \leq N$ its $n$-point correlation functions are given by

$$\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{1 \leq i,j \leq n},$$

(2)

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where the function $K : E \times E \to \mathbb{C}$ is the reproducing kernel of $L^2(E)$

$$K(x, y) = \sum_{i=1}^{N} \psi_i(x) \overline{\psi_i(y)}. \quad (3)$$

Determinantal processes arise in many mathematical contexts, from the eigenvalues of random matrices to zeros of Gaussian analytic functions (see [HKPV09] for a review). Recently, many applications of determinantal point processes were recently discovered in the following fields: wireless networks [MS14], machine learning [KT12] or numerical integration [BH20]. In this paper, we study determinantal point processes on compact Kähler manifolds. Consider a compact complex manifold $M$ of dimension $d$ and a holomorphic line bundle $L \to M$, and assume that there is a smooth Hermitian metric $h$ on $L$ such that the curvature form

$$\Theta_h = -\frac{\partial \overline{\partial}}{\log h}$$

is positive. In this case, the form $\omega = \frac{i}{2} \Theta_h$ is a Kähler form and $(M, \omega)$ a Kähler manifold; the form $\frac{1}{2} d\omega^d$ defines a Riemannian volume form and corresponds to a measure $d\nu_M$ on $M$. We can consider the line bundle $L^k = L \otimes \mathbb{C}^k$, endowed with the metric $h^k$, and we denote by $H^0(M, L^k)$ the finite-dimensional vector space of its holomorphic sections. From a physical point of view, in the context of integer quantum Hall effect, it corresponds to the state space of fermions on $M$ under a magnetic field of strength $\omega$ – see [DK10] for instance. The inner product on fibers corresponding to $h^k$ yields a $L^2$ inner product on $H^0(M, L^k)$ and becomes a Hilbert space. Given an orthonormal basis $(s_i)$ of this space, one can define the $N$-particle wavefunction ($N$ being the dimension of the space)

$$\Psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det(s_i(x_m)), \quad (4)$$

which is the exact equivalent of (1), with a noticeable difference: the determinant here does not involve directly the multiplication of complex numbers, but the tensor products of sections. As in the general case, one can consider the reproducing kernel $B_k$ of $H^0(M, L^k)$, called the Bergman kernel, and defined by

$$B_k(x, y) = \sum_{i=1}^{N} s_i(x) \otimes \overline{s_i(y)}. \quad (5)$$

As we will see, taking the norm of the $N$-particle wavefunction defined in (4) will define a density on $M^N$ with respect to $d\nu_M^\otimes N$ which corresponds to a DPP with kernel $B_k$.

1.1 Presentation of results

Our main result is a scaling limit of the correlation functions: we obtain that this limit does not depend on the metric. We say that a set of local coordinates $\varphi : x \in U \mapsto (z_1, \ldots, z_d) \in \mathbb{C}^d$ on an open set $U \subset M$ is normal for $x \in U$ if $\varphi(x) = (0, \ldots, 0)$ and if

$$\omega(x) = \frac{i}{2} \sum_{i=1}^{d} dz_i \wedge d\overline{z_i}.$$
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**Theorem 1.1.** Let $M$ be a compact complex manifold of dimension $d$, and $(L,h)$ be a Hermitian holomorphic line bundle over $M$ such that $\omega = \frac{i}{2} \Theta_h$ is a Kähler form that induces a measure $dv_M$ on $M$. Set $N_k = \dim H^0(M, L^k)$ and consider an orthonormal basis $(S_1, \ldots, S_{N_k})$ of $H^0(M, L^k)$ for the $L^2$ inner product

$$ (s_1, s_2)_{L^2(dv_M), h^k} = \int_M h^{-k}_x s_1(x), s_2(x) dv_M(x). $$

Let $(X_1, \ldots, X_{N_k})$ be a random family on $M^d$ with joint density

$$ \frac{1}{N_k!} \| \det(S_i(z_j)) \|_{h^k} dv_M^\otimes N_k(z_1, \ldots, z_{N_k}). $$

Then, the following assertions hold:

(i) $(X_1, \ldots, X_{N_k})$ is a determinantal point process with kernel $B_k$, the Bergman kernel associated to $H^0(M, L^k)$.

(ii) As $k \to \infty$, its correlation functions admit the following scaling limit on normal coordinates:

$$ \frac{1}{k^{nd} \rho_n}\left(\frac{u_1}{\sqrt{k}}, \ldots, \frac{u_n}{\sqrt{k}}\right) = \det(B_\infty(u_i, u_j)) + O\left(\frac{1}{\sqrt{k}}\right), \quad \forall u, v \in \mathbb{C}^d, $$

where $B_\infty$ is the kernel defined locally by

$$ B_\infty(u, v) = \frac{1}{(\pi)^d} e^{u \cdot \pi - \frac{1}{2} |u|^2 - \frac{1}{2} |v|^2}, \quad \forall u, v \in \mathbb{C}^d. $$

The kernel $B_\infty$ is a universal kernel obtained from the asymptotic expansion of Bergman kernels, obtained by Bleher, Shiffman, Zelditch [BSZ00b]. In their article and the companion paper [BSZ00a], they prove a similar result in the case where $(X_1, \ldots, X_{N_k})$ are zeros of random holomorphic sections of $L$; although the models are different in general, it is interesting that they show a kind of similar universality, and we will detail it in Section 1.2. A well-known determinantal point process with kernel $B_\infty$ in dimension 1 is the infinite Ginibre ensemble, which will also be introduced in Section 1.2. More recently, Berman proved a slightly more general result, by using pluripotential theory.

**Theorem 1.2** ([Ber18], Theorem 1.1). Let $L$ be a holomorphic line bundle over a compact complex manifold $M$ of dimension $d$, $h$ be a Hermitian metric on $L$ with associated weight $\phi$ and a finite measure $\mu$ on $M$. Assume that $\phi$ is $C^{1,1}_{\text{loc}}$ and that the volume form $\frac{1}{\pi} d\omega$ is continuous. Let $x$ be a fixed point in the weak bulk and take normal local coordinates $z$ centered at $x$ and a normal trivialization of $L$. Then,

$$ \lim_{k \to \infty} k^{-d} B_k\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) = \frac{\det \lambda}{\pi^d} e^{(\lambda z, w)} \quad (6) $$

in the $C^\infty$-topology on compact subsets of $\mathbb{C}_z^d \times \mathbb{C}_w^d$. In particular, the 2-point correlation function has the following scaling asymptotics

$$ \lim_{k \to \infty} k^{-2d} \rho_2\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) = \left(\frac{\det \lambda}{\pi^d}\right)^2 e^{-\lambda|z-w|^2} \quad (7) $$

uniformly on compacts of $\mathbb{C}_z^d \times \mathbb{C}_w^d$. 

Although Theorem 1.1 is a particular case of this result, we find it interesting to highlight the Kähler case for two main reasons:

(i) The emergence of infinite Ginibre ensemble provides a link between complex geometry and random matrix theory;

(ii) This setting might be easier to understand for probabilists, as many of them are not familiar with pluripotential theory\(^1\). Indeed, as we shall see in Section 4, our proof does not involve any geometric or analytic tools other than the asymptotic expansion of Bergman kernel.

The following result, initially due to Berman with weaker assumptions as well [Ber09], can be found as a consequence of Theorem 1.1.

**Corollary 1.3.** Let \((X_1, \ldots, X_N)\) be as in Theorem 1.1, and \(\mu\) be the probability measure \((\int_M dv_M)^{-1} dv_M\). As \(k \to \infty\), the empirical measures \(\hat{\mu}_k = \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{X_i}\) converge in probability to \(\mu\), with respect to the weak topology.

As explained in [Ber09], this convergence result offers for instance an alternative of Fujita’s approximation theorem [Fuj94] on the volume of big line bundles\(^2\). Berman’s proof does not rely directly on the determinantal structure of the point process, but rather on pluripotential theory, in particular the Monge–Ampère measure \((\omega e^{\phi})^{n}\) associated to the equilibrium potential \(\phi_e\) of a smooth potential \(\phi\). Again, our proof avoids this theory and is here purely probabilistic.

### 1.2 Comparison with other models

Our framework fits in a general attempt to find a geometric counterpart of well-known random algebraic models, such as random matrices or random polynomials. Perhaps the most prominent example is the study of zeros of random holomorphic sections of large line bundles over complex manifolds, initiated by Bleher, Shiffman and Zelditch [SZ99, BSZ00a, BSZ00b] and continued by Bayraktar, Coman, Marinescu and others (see [BCHM18] for a recent survey), which generalizes results about the zeros of random polynomials. Our approach, following in particular the works of Berman [Ber14, Ber18], rather corresponds to the field of orthogonal ensembles, a particular case of determinantal point process that we describe in Section 2.1. All these correspondences are recapitulated in Fig. 1.

| Algebraic model | Geometric model |
|-----------------|-----------------|
| Zeros of random polynomials | Zeros of random holomorphic sections |
| Orthogonal ensembles | DPP with Bergman kernel |

**Figure 1:** Algebraic vs geometric frameworks.

In the next paragraphs, we will briefly introduce some models in these fields to show their similarities and their differences.

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\(^1\)Note that the recent survey by Dujardin [Duj20] aims to fill this gap.

\(^2\)See Section 2.3 for a definition of the volume of a line bundle.
1.2.1 Random matrices and orthogonal ensembles

In random matrix theory, many classical models satisfy an orthogonal or a unitary distributional invariance. For such random matrix, its distribution can often be transported to its eigenvalues, and their associated point process is simple as long as they are distinct almost-surely. For instance, the Ginibre ensemble of rank $N$ is the point process of the eigenvalues of complex Gaussian matrices. The distribution of such matrices is given by

$$\pi^{-N^2} e^{-\text{Tr}(AA^*)} dA,$$

which is equivalent to say that the entries of $A$ are complex random variables $A_{jk} = x_{jk} + iy_{jk}$, with $(x_{jk}, y_{jk})$ independent real random variables with distribution $\mathcal{N}(0,1)$.

The joint distribution of the eigenvalues $(\xi_1, \ldots, \xi_N)$ of a complex gaussian matrix is given by

$$\frac{\pi^{-N}}{N!(N-1)! \cdots 1!} \prod_{1 \leq i < j \leq N} |z_j - z_i|^2 e^{-\sum_{k=1}^{N} |z_k|^2}.$$

(9)

The polynomials $(\phi_j)_{0 \leq j \leq N-1}$ defined as $\phi_j(x) = \frac{x^j}{\sqrt{j!}}$ are orthonormal in $L^2(C, \lambda_0)$ with $\lambda_0 = \mathcal{N}(0, I_2)$, and it is clear that the density (9) can be rewritten

$$\frac{1}{N!} |\det(\phi_{j-1}(z_k))|^2 \lambda_0(dz_1) \cdots \lambda_0(dz_N).$$

According to a classical result on determinantal point processes (see Theorem 2.3), the process $\mu_N^{\text{Gin}} = \sum_{j=1}^{N} \delta_{\xi_j}$ is determinantal with kernel

$$K_N^{\text{Gin}}(z, w) = \sum_{j=0}^{N-1} \phi_j(z) \overline{\phi_j(w)} = \sum_{j=0}^{N-1} \frac{1}{j!} (z \overline{w})^j.$$

If we let $N$ go to infinity, we obtain another determinantal process on $C$, namely the infinite Ginibre ensemble. Its kernel is simply

$$B_\infty(z, w) = \frac{1}{\pi} e^{z \overline{w} - \frac{1}{2} |z|^2 - \frac{1}{2} |w|^2}$$

(10)

with respect to the Lebesgue measure $dm(z)$, or equivalently

$$K^{\text{Gin}}_\infty(z, w) = e^{z \overline{w}}$$

with respect to the Gaussian measure $\mathcal{N}(0, I_2)$. We recognize in (10) the universal kernel described in Theorem 1.1 for a compact Riemann surface: it could be said that the determinantal point processes with kernel $B_k$ look locally like a $d$-dimensional infinite Ginibre ensemble, when $k$ goes to infinity.

It is known since [Gin65] that the Ginibre ensemble admits a scaling limit: for $N \in \mathbb{N}$, let us denote by $\nu_N^{\text{Gin}} = \sum_{j=1}^{N} \frac{\delta_{\xi_j}}{\sqrt{N}}$ the associated rescaled point process and $\hat{\nu}_N^{\text{Gin}} = \frac{1}{N} \nu_N^{\text{Gin}}$ its empirical measure.

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3see for instance [For10, Prop.15.1.1].
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Theorem 1.4 (Strong circular law). Almost-surely, the sequence \( \nu_N^{\text{Gin}} \) of empirical measures of the Ginibre ensemble converge in distribution to the uniform measure on the unit disk \( D \).

The proof of this theorem relies mainly on a moment estimation as well as the following remark: the density (with respect to Lebesgue measure) of an eigenvalue, which is

\[
p_1(z) = \frac{1}{\pi} K_N^{\text{Gin}}(z, z) e^{-|z|^2} = \frac{1}{\pi} e^{-|z|^2} \sum_{j=0}^{N-1} \frac{|z|^{2j}}{j!},
\]

only depends on \( R = |z| \), and converges to \( \pi^{-1} \) when \( R < N \) and 0 when \( R > N \). The strong circular law is an analog of Corollary 1.3 in the context of random matrix theory, and we will also state another equivalent for zeros of random polynomials and zeros of random holomorphic sections.

Let us also mention, in a somewhat different direction, another approach of determinantal point processes in random matrix theory which can be related to noninteracting fermions, due to Cunden, Mezzadri and O’Connell [CMO18]. In their paper, they consider a system of \( N \) noninteracting fermions on \( \mathbb{C} \) confined in a potential \( V \), which is a determinantal point process with kernel

\[
K_N^V(x, y) = \sum_{k=0}^{N-1} \overline{\psi}_k(x) \psi_k(y),
\]

where the functions \( \psi_k \) are the first \( N \) eigenfunctions of the single-particle Schrödinger operator \(-\Delta + V\). If the support of the potential is the unit circle \( S^1 \), identified with the real interval \([0, 2\pi]\) with several possible boundary conditions, then the corresponding kernel \( K_N^V \) is in fact the correlation kernel of the Haar measure on a compact classical group. For instance, in the case of periodic boundary conditions, one retrieves the kernel

\[
\frac{1}{2\pi} \sum_{|k| \leq N} e^{ik(y-x)}
\]

which is the correlation kernel of \( U(2N + 1) \). We do not know yet if there is such a group-theoretic approach of free fermions on a Kähler manifold confined by a potential, but it would be interesting to investigate.

1.2.2 Random polynomials

Consider a random complex polynomial \( P_k = \sum_{j=0}^k c_j z^j \) of degree \( k \), where \( c = (c_0, \ldots, c_k) \) is a random vector of \( \mathbb{C}^{k+1} \). Let \( Z_{P_k} = \{\zeta_1, \ldots, \zeta_k\} \) be the set of its zeros, counted with multiplicity: it is not a priori a simple point process, unless the zeros are all distinct. It is natural to ask how its distribution is related to the law of \( c \). Let us assume for instance that all coefficients are i.i.d. and follow the complex standard Gaussian law: it corresponds to Kac polynomials in the literature. Using Vieta’s formulas and
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a change of variable, one finds that $(\zeta_1, \ldots, \zeta_k)$ has the following density with respect to the Lebesgue measure on $\mathbb{C}^k$:

$$\frac{1}{Z_k} \prod_{\ell < m} |z_\ell - z_m|^2$$

The Vandermonde determinant on the numerator ensures that $(\zeta_1, \ldots, \zeta_k)$ are almost-surely pairwise distinct, hence they form a simple point process, which is actually not determinantal. Its correlation functions can be found in [HKPV09] for instance.

Denote by $dv_C$ the standard Euclidean volume form on $\mathbb{C}$, which corresponds to the Lebesgue measure on $\mathbb{R}^2$:

$$dv_C(z) = \frac{i}{2} dz \wedge d\bar{z}.$$  

For any connected open set $U \subset \mathbb{C}$, the Bergman space is defined by

$$A^2(U) = \mathcal{O}(U) \cap L^2(U, dv_C),$$

i.e. the space of square-integrable holomorphic functions on $U$. The linear form $e_z : A^2(U) \to \mathbb{C}$ which maps $f$ to $f(z)$ is continuous, hence from Riesz–Markov theorem, for any $z \in \mathbb{C}^d$ there exists $k_z \in A^2(U)$ such that for any $f \in A^2(U)$,

$$f(z) = \int_U f(w) k_z(w) dv_C(w). \quad (12)$$

The Bergman kernel is the integral kernel of (12), and it has an explicit form when $U = D_R$ is a disk of radius $R$ in $\mathbb{C}$:

$$K_{D_R}(z, w) = \frac{R^2}{\pi (R^2 - z\bar{w})^2},$$

as well as $\Omega = \mathbb{C} \setminus D_R$:

$$K_{\Omega}(z, w) = \frac{R^2}{\pi (R^2 z\bar{w} - 1)}.$$

**Theorem 1.5.** Almost-surely, the sequence $(Z_{P_k})_{k \leq 1}$ of zeros of Kac polynomials converges in distribution to a determinantal process with kernel $K_{\mathbb{D}}$ where $\mathbb{D}$ is the unit disk.

This result is in fact a consequence of the two following results.

**Lemma 1.6** ([HKPV09], Lemma 2.2.3). Let $(f_n)$ be a sequence of holomorphic functions on a locally compact polish space $M$. If $\sum_n |f_n(z)|^2$ converges uniformly on every compact subset of $M$ and if $(c_n)$ is a sequence of i.i.d. standard complex Gaussian random variables, then a.s. $\sum_n c_n f_n(z)$ converges uniformly on every compact and defines a random gaussian analytic function with covariance kernel

$$K(z, w) = \sum_n f_n(z) \overline{f_n(w)}.$$  

\footnote{This so-called Bergman kernel is exactly the equivalent, on the complex plane, of the Bergman kernel of compact Kähler manifolds. The main difference is that the kernel on $U \subset \mathbb{C}$ has infinite rank because the Bergman space has infinite dimension, whereas the kernel on a compact manifold $M$ has finite rank.}
**Theorem 1.7** ([PV05], Theorem 1). Let \((c_n)\) be a sequence of i.i.d. standard complex Gaussian random variables, and \(f : z \mapsto \sum_{n=0}^{\infty} c_n z^n\) be a random function on \(\mathbb{D}\). Then the set \(Z_f \cap \mathbb{D}\) of the zeros of \(f\) inside the unit disk is determinantal with kernel \(K_D\).

As opposed to the previous determinantal point processes we have seen, this one is not exactly an orthogonal ensemble in the sense of Theorem 2.3, because of the infinite rank of the Bergman kernel \(K_D\). The following result, from Shepp and Vanderbei [SV95], provides another limit that can be related to Corollary 1.3 or to the strong circular law for the Ginibre ensemble.

**Theorem 1.8.** The expectation of the empirical measure of zeros of Kac polynomials converge to the uniform measure on the unit circle as \(k \to \infty\).

Theorem 1.5 has been recently generalized to many other cases by Butez and García-Zelada [BGZ22] using the model of jellium. If \(\nu\) is a rotationally-invariant measure on \(\mathbb{C}\) such that

\[
\int_{\mathbb{C}} |\log|z|| d\nu(z) < \infty,
\]

a jellium associated to \(\nu\) corresponds to a system of \(N\) interacting particles at equilibrium whose positions \(x_1, \ldots, x_N \in \mathbb{C}\) have distribution

\[
\frac{1}{Z_N} \prod_{i<j} |x_i - x_j|^2 e^{-2(N+1) \sum_i V^\nu(x_i)} d\nu_{\mathbb{C}}(x_1, \ldots, x_N),
\]

where the potential \(V^\nu : \mathbb{C} \to \mathbb{R}\) is defined by

\[
V(z) = \int_{\mathbb{C}} \log|z - w| d\nu(w).
\]

**Theorem 1.9** ([BGZ22], Theorem 1.3). Let \((x_1, \ldots, x_N)\) be a jellium associated to \(\nu\). Let \(U\) be a connected component of \(\mathbb{C} \setminus \text{supp}(\nu)\) and suppose that it is either an open disk or the complement of a closed disk. Then the sequence of point processes \\{\(x_k\) such that \(x_k \in U\)\} converges weakly, as \(N \to \infty\), to a determinantal point process with kernel \(B_U\).

They related this model to zeros of random polynomials defined by

\[
P_N(z) = \sum_{k=0}^{N} a_k R_{k,N}(z),
\]

where \((a_k)\) is a family of i.i.d. random variables and \((R_{k,N})_{0 \leq k \leq N}\) is an orthonormal basis of \(\mathbb{C}_N[X]\) for the inner product

\[
\langle P, Q \rangle = \int_{\mathbb{C}} P(z)\overline{Q}(z) e^{-2N V^\nu(z)} d\nu(z).
\]

**Theorem 1.10** ([BGZ22], Theorem 1.7). Assume that \((P_N)_{N}\) is a sequence of random polynomials of the form (14), such that almost-surely \(a_0 \neq 0\) and \(\mathbb{E}[\log(1 + |a_0|)] < \infty\). Let \(U\) be a connected component of \(\mathbb{C} \setminus \text{supp}(\nu)\) and suppose that it is either a disk or the complement of a closed disk.
If \( U = D_R \) for \( R > 0 \), then
\[
\{ z_k \text{ such that } z_k \in U \} \Rightarrow R \cdot (Z_f \cap D),
\]
where \( f \) is the random analytic function defined on \( U \) by
\[
f(z) = \sum_{k=0}^{\infty} a_k z^k.
\]

If \( U = C \setminus \overline{D_R} \) for \( R > 0 \), then
\[
\{ z_k \text{ such that } z_k \in U \} \Rightarrow R \cdot (\tilde{Z}_f \cap D),
\]
where \( \tilde{f} \) is the random analytic function defined on \( U \) by
\[
\tilde{f}(z) = f(z-1).
\]

This theorem, paired with Lemma 1.6, implies that if \( U = D_R \) and \((a_k)\) are i.i.d. standard complex Gaussian random variables, then set \( \{ z_k \text{ such that } z_k \in U \} \) converges weakly to a determinantal process with kernel \( K_{D_R} \).

Let us mention two universality results that look like Theorem 1.1, due to Bleher and Di: let \( P_k \) be a random polynomial of the form
\[
P_k(x) = \sum_{j=0}^{k} \sqrt{\binom{k}{j}} c_j x^j,
\]
where \( c_j \) are i.i.d. standard Gaussian random variables [BD97] or i.i.d. random variables with zero mean and unit variance such that the characteristic function of \( c_j \) satisfies a few conditions [BD04]. The correlations functions of the real zeros of \( P_k \) verify the following scaling limit:
\[
\lim_{k \to \infty} \frac{1}{k^{d/2}} \rho_n(x + u_1 \sqrt{k}, \ldots, x + u_n \sqrt{k}) = \rho^\infty_n(u_1, \ldots, u_n),
\]
where the limit \( \rho^\infty_n \) is “universal”. More results about universality can be found in [TV15].

### 1.2.3 Random holomorphic sections

Let \( M \) be a compact complex manifold of dimension \( d \), \( L \to M \) a holomorphic line bundle with a smooth Hermitian metric \( h \) such that \( \omega = \frac{i}{2} \Theta_h \) is a Kähler form. For a large integer \( k \), one considers the Hilbert space \( H^0(M, L^k) \) and endows it with the following probability measure:
\[
d\mu(s) = \frac{1}{\pi^{N_k}} e^{-|c|^2} dc_1 \cdots dc_{N_k},
\]
for any section \( s = \sum_{j=1}^{N_k} c_j S_j^{(k)} \in H^0(M, L^k) \) decomposed on an orthonormal basis \( (S_\ell^{(k)}) \).

As in the case of polynomials, we denote by \( Z_s \) the set of zeros of the section \( s \); we also denote by \( \rho_n^{(k)} \) the \( n \)-point correlation function of the random set \( Z_{s_1} \cap \cdots \cap Z_{s_m} \), where \( s_1, \ldots, s_m \) are \( m \) independent random sections of \( L^k \) with measure \( \mu \).

**Theorem 1.11** ([BSZ00a, BSZ00b]). Let \( M \) be a compact complex manifold with the Kähler form \( \omega = \frac{i}{2} \Theta_h \) corresponding to a Hermitian holomorphic line bundle \( (L, h) \). Let \( S = H^0(M, L^k)^n \), for \( n \geq 1 \), endowed with the standard Gaussian measure \( \mu \). Then the
n-point correlation functions $\rho^{(k)}_{n,m}$ satisfy the following universality in normal coordinates around $z \in M$:

$$\frac{1}{k^{nd}} \rho^{(k)}_{n,m}(\frac{u_1}{\sqrt{k}}, \ldots, \frac{u_n}{\sqrt{k}}) = K_{n,m,d}(u_1, \ldots, u_n) + O(N^{-\frac{1}{2}}),$$

(16)

where the function $K_{n,m,d}$ is a rational function expressed using the infinite Ginibre kernel $B_{\infty}(u,v)$ and its partial derivatives.

This result is the counterpart of Theorem 1.1 and is based on similar tools as our proof. An analog of Corollary 1.3 is given by the following result by Shiffman and Zelditch.

**Theorem 1.12 ([SZ99], Theorem 1.1).** Under the same assumptions as in Theorem 1.11, for $\mu$-almost all $s = \{s_k\} \in S$,

$$\frac{1}{k} Z_{s_k} \longrightarrow \omega,$$

(17)

weakly in the sense of measures.

Many generalizations of this theorem occurred since then, some of which are given in [Zel18] for so-called quantum ergodic sections, and some others in [BCHM18]. In a quite different direction, we can also mention the recent paper [ALF22] by Ancona and Le Floch, where they consider the zeros of $T_k s_k$, where $s_k$ is a Gaussian random holomorphic section of $L^k$ and $T_k$ is a Berezin–Toeplitz operator, which are quantum observable operators corresponding to classical observables $f \in C^\infty(M, L^k)$.

**Theorem 1.13 ([ALF22], Theorem 1.9).** Let $f \in C^\infty(M, \mathbb{R})$ be a smooth function vanishing transversally and let $T_k$ be a Berezin–Toeplitz operator with principal symbol $f$. Then, as $k \rightarrow \infty$,

$$\frac{1}{k} \mathbb{E}[Z_{T_k s}] \longrightarrow \frac{\omega}{2\pi}$$

weakly in the sense of currents, and

$$\mathbb{E}[Z_{T_k s}] - \frac{k \omega}{2 \pi} \rightarrow \frac{i}{2 \pi} \partial \bar{\partial} \log f^2$$

weakly in the sense of currents.

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2 Background

In this section, we introduce all tools and concepts needed for Theorem 1.1: the framework of determinantal point processes on one hand, with a stress on orthogonal ensembles, and the complex geometric framework of Bergman kernels on Kähler manifolds on the other hand.

2.1 Determinantal point processes

There have been many extensive presentations of determinantal point processes in the early 2000s [Sos00, Lyo03, Joh05, BO05, HKPV09] and most of what we will present here, as well as lots of further results, can be found in these references.

Point processes can be described in various ways: as random measures, as random ensembles, or also as random tuples. Each approach has its own interest, and we will present them and explain their interplay. We will start with the (probably) most formal definition.

**Definition 2.1** (Simple point process as random measures). Let $E$ be a locally compact Polish space endowed with its Borel $\sigma$-algebra $\mathcal{B}(E)$. A simple discrete measure on $E$ is a measure $\mu$ on $E$ defined by

$$\mu = \sum_{i \in I} \delta_{x_i},$$

where $(x_i)_{i \in I}$ is a family of distinct elements of $E$ with no accumulation point, indexed by a finite or countable set $I$. We denote by $\mathcal{M}_d(E)$ the set of simple discrete measures on
If \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, a *simple point process* on \(E\) is a random variable \(\mu : \Omega \to \mathcal{M}_d(E)\).

**Definition 2.2**: (Simple point process as a random configuration). Let \(E\) be a locally compact Polish space endowed with its Borel \(\sigma\)-algebra \(\mathcal{B}(E)\), \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and \(N\) be a positive integer. A *configuration* on \(E\) is a locally finite subset of \(E\), and the set of configurations on \(E\) is denoted by \(\text{Conf}(E)\). A *simple point process* on \(E\) is a random configuration \(X : \Omega \to \text{Conf}(E)\).

**Definition 2.3**: (Simple point process as a random tuple). Let \(E\) be a locally compact Polish space endowed with its Borel \(\sigma\)-algebra \(\mathcal{B}(E)\), \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and \(N\) be a positive integer. A *simple point process of size* \(N\) on \(E\) is a family \((X_1, \ldots, X_N)\) of random variables \(X_i : \Omega \to E\) which are pairwise distinct \(\mathbb{P}\)-almost-surely.

Note that the last restriction is more restrictive than the others: in this case, the number of points in the process is fixed and set to \(N\). In general, if \(\mu\) is a simple point process on \(E\), for any \(\omega \in \Omega\), \(\mu(\omega) = \mu(\omega)\) is a simple discrete measure and its total mass \(\mu(E)\) is the integer random variable

\[
\mu(E) : \omega \mapsto \int_E \mu_\omega(dx),
\]

which corresponds to the total number of points of the process in \(E\). More generally, for any \(B \in \mathcal{B}(E)\), \(\mu(B)\) denotes the total number of points of the process that belong to \(B\).

The equivalence between Definitions 2.1 and 2.2 follows from the fact that \(\{\text{Conf}(E) \to \mathcal{M}_d(E) : \sum_{x \in X} \delta_x\} \) is a bijection, and the condition of locally finiteness of the configuration \(\{x_i, i \in I\}\) is equivalent to the absence of accumulation points. Now, let us restrict to the subset \(\text{Conf}_N(E) \subset \text{Conf}(E)\) of subsets of \(E\) with cardinal \(N\). The symmetric group \(\mathfrak{S}_N\) acts on \(E^N\) by permutation of coordinates, and if we denote by

\[
\Delta_N = \bigcup_{i \neq j} \{(x_1, \ldots, x_N) \in E^N : x_i = x_j\}
\]

the union of diagonals of \(E^N\), the map

\[
\begin{align*}
E^N &\setminus \Delta_N \quad \to \quad \text{Conf}_N(E) \\
(x_1, \ldots, x_N) &\quad \mapsto \quad \{x_1, \ldots, x_N\}
\end{align*}
\]

descends to a bijection \((E^N \setminus \Delta_N)/\mathfrak{S}_N \simeq \text{Conf}_N(E)\). This bijection sets the correspondence between Definition 2.3 and 2.2 in the case of a simple process on \(E\) which has cardinal \(N\) almost-surely.

In the following, we will denote by \(\mu\) a point process\(^5\) seen as a random measure, \(X = \{X_1, \ldots, X_{\mu(E)}\}\) the same process seen as a random configuration, and \((X_1, \ldots, X_{\mu(E)})\) the

\(^5\)From now on, all point processes will be supposed simple, hence we will omit it for the sake of conciseness.
process seen as a random tuple, whenever it makes sense. There are several observables of interest for point processes, and we will recall them. In fact, we already have described the simplest one, which is the number of particles in a Borel set \( B \subset E \): it is the random variable

\[
\mu(B) : \begin{cases} 
\Omega & \rightarrow \mathbb{N} \\
\omega & \mapsto \int_{B} \mu_{\omega}(dx) = \#(B \cap X(\omega)).
\end{cases}
\]

The other main observables are symmetric functions \( \rho_n : E^n \rightarrow \mathbb{R} \), for \( n \in \mathbb{N} \), called the correlation functions.

**Definition 2.4.** Let \( N \geq n \geq 1 \) be two integers, and \( \mu \) be a simple point process on \( E \) such that \( \mu(E) = N \) almost-surely. Consider \( (X_1, \ldots, X_N) \) a family of random variables on \( E \) such that \( \mu = \sum_{i=1}^{N} \delta_{X_i} \) almost-surely. The \( n \)-th *factorial moment measure* of \( \mu \) is the measure \( \mu^{(n)} \) on \( E^n \) defined as

\[
\mu^{(n)} = \mathbb{E} \left[ \sum_{1 \leq i_1, \ldots, i_n \leq N} \delta_{X_{i_1}, \ldots, X_{i_n}} \right].
\]  

(19)

Note that we have \( \mu^{(1)} = \mu \). If \( \lambda \) is a Radon measure on \( E \) and \( \mu^{(N)} \) is absolutely continuous with respect to \( \lambda^{\otimes N} \), then the \( N \) point *correlation function* of \( \mu \) is the Radon–Nikodym derivative

\[
\rho_n(x_1, \ldots, x_n) = \frac{d\mu^{(n)}}{d\lambda^{\otimes n}}(x_1, \ldots, x_n).
\]  

(20)

Let us be more explicit about the factorial moment measures and correlation functions. If \( B_1, \ldots, B_n \) are Borel subsets of \( E \), then

\[
\mu^{(n)}(B_1 \times \cdots \times B_n) = \sum_{1 \leq i_1, \ldots, i_n \leq N, \ i_1 \neq \cdots \neq i_n} \mathbb{P}[X_{i_1} \in B_1, \ldots, X_{i_n} \in B_n].
\]

In particular, for any \( B = B_1 \times \cdots \times B_N \) Borel subset of \( E^N \),

\[
\mu^{(N)}(B) = N! \mathbb{P}(X \subset B).
\]

It is also possible to describe the correlation functions in terms of test functions. Indeed, if \( f : E^n \rightarrow \mathbb{R} \) is a measurable function, then

\[
\int_{E^n} f(x_1, \ldots, x_n) \rho_n(x_1, \ldots, x_n) d\lambda^{\otimes n}(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_n \leq N} \sum_{\sigma \in S_n} \mathbb{E}[f(X_{\sigma(i_1)}, \ldots, X_{\sigma(i_n)})].
\]

In practice, if a point process is described as a random family of points, the following result is helpful to compute correlation functions from the joint density of the random points.

**Proposition 2.1 ([Joh05]).** Let \( N \geq 1 \) be an integer, and \( X \) a point process on \( E \) such that \( (X_1, \ldots, X_N) \) has density \( f \) with respect to \( \lambda^{\otimes N} \), where \( f : E^N \rightarrow \mathbb{R}_+ \) is a symmetric function. Then for all \( n \geq 1 \), the correlation functions of \( X \) exist and they satisfy

\[
\rho_n(x_1, \ldots, x_n) = \begin{cases} 
\frac{1}{(N-n)!} \int_{E^{N-n}} f(x_1, \ldots, x_N) \lambda^{\otimes N-n}(dx_{n+1} \cdots dx_N) & \text{if } n \leq N; \\
0 & \text{if } n > N.
\end{cases}
\]  

(21)
For $1 \leq n \leq N$, the $n$-point correlation function correspond is therefore, up to a constant factor, to the marginal distribution of the subfamily $(X_1, \ldots, X_n)$ of random variables. In particular, the 1-point correlation function gives, up to a factor $N$, the probability density function of one point of the process, and the $N$-point correlation function is exactly the joint density function of all points.

**Definition 2.5.** A point process $\mu$ on $E$ is **determinantal** of kernel $K : E^2 \to \mathbb{C}$ if for any $n \geq 1$, the $n$-point correlation function satisfies

$$\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{1 \leq i, j \leq n}. \quad (22)$$

Depending on the kernel $K$, there may or not exist such a determinantal point process (often shortened as DPP); the following theorem provides a sufficient condition on $K$ for its existence.

**Theorem 2.2 ([Sos00]).** If $K$ is the kernel of a hermitian locally trace class operator on $L^2(E, \lambda)$, then there exists a determinantal point process with kernel $K$ if and only if $\text{Spec}(K) \subset [0, 1]$. If the DPP exists, then it is unique.

A particular class of determinantal point processes associated to a hermitian kernel is the class of **orthogonal ensembles**. They are related to reproducing kernels on finite-dimensional subspaces of $L^2(E, \lambda)$. One may find many results about them in [Bor99] and [BS03], in particular the following theorem.

**Theorem 2.3.** Let $(\phi_i)_{1 \leq i \leq N}$ be an orthonormal family of $L^2(E, \lambda)$. The kernel

$$K_N : \left\{ \begin{array}{ccc} E^2 & \to & \mathbb{C} \\ (x, y) & \mapsto & \sum_{i=1}^{N} \phi_i(x)\overline{\phi_i(y)} \end{array} \right. \quad (23)$$

is a reproducing kernel on the subspace $H$ of $L^2(E, \lambda)$ generated by $\phi_1, \ldots, \phi_N$. Moreover, if $X_1, \ldots, X_N$ are random variables on $E$ of joint distribution

$$p(x_1, \ldots, x_N) = \frac{1}{N!} \det(\phi_i(x_j))_{1 \leq i, j \leq N}^2 \lambda(dx_1) \cdots \lambda(dx_N), \quad (24)$$

then the simple point process $\mu = \sum_{i=1}^{N} \delta_{X_i}$ is determinantal with kernel $K_N$. It is called the **orthogonal ensemble of kernel** $K_N$, and $K_N$ is called the Christoffel–Darboux kernel on $E$ with respect to $\lambda$.

In most known cases, the first way to understand the large $N$ behaviour of orthogonal ensembles is to compute the limit of $K_N$ (up to rescaling) when $N$ tends to infinity. For instance, if $(P_i)$ is a family of orthogonal polynomials for the measure $\lambda$, then $(\phi_i)$ is an orthonormal family of $L^2(E, \lambda)$ with $\phi_i = \frac{P_i}{\|P_i\|}$. If $E = \mathbb{R}$ the associated kernel $K_N$ satisfies the celebrated Christoffel–Darboux formula [Sze75, Thm 3.2.2]:

$$K_N(x, y) = \frac{a_{N-1}}{a_N} \frac{\phi_{N-1}(x)\phi_{N-1}(y) - \phi_{N-1}(x)\phi_N(y)}{x - y}, \quad (25)$$

where $a_i$ is the leading coefficient of the polynomial $\phi_i$. An analog holds as well when $E = \mathbb{C}$ and $\text{supp}(\lambda) = S^1$. In classical cases (Jacobi, Hermite, Laguerre), this formula is one of the main ingredients to study the asymptotics of $K_N$ when $N$ tends to infinity. Unfortunately, there is no such formula for Kähler manifold. Our approach will instead be based on direct asymptotic results on the Bergman kernel obtained in the last 20 years using advanced complex geometry techniques.
2.2 Complex manifolds and Hermitian line bundles

The aim of this section is to recall a few facts about complex manifolds for non-specialists (mostly probabilists), and it does not include any particular novelty. Good references for this content are for instance [Huy05] for an introduction, [Dem] for a more analytic point of view, or [GH94] for a more algebraic-geometric point of view.

A complex manifold is a topological space $M$ endowed with a family $(U_i, \phi_i)_{i \in I}$ of open subsets $U_i \subset M$ and homeomorphisms $\phi_i : U_i \to \mathbb{C}^d$ such that, if $U_i \cap U_j \neq \emptyset$, then

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a biholomorphism between subsets of $\mathbb{C}^d$. Similarly, one defines a $d$-dimensional smooth real manifold by replacing $\mathbb{C}^d$ by $\mathbb{R}^d$ and biholomorphism by smooth diffeomorphism. In particular, if $M$ is a $d$-dimensional complex manifold, then it is also a 2$d$-dimensional real manifold using the bijection $\mathbb{C} \simeq \mathbb{R}^2$. A holomorphic line bundle over a compact complex manifold $M$ is a complex manifold $L$ endowed with a holomorphic map $\pi : L \to M$ such that any fiber $L_x = \pi^{-1}(\{x\})$ is a one-dimensional complex vector space for $x \in M$, with the following extra condition: there exists an open covering $(U_i)_{i \in I}$ of $M$ and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}$ commuting with the projection $\pi$ and so that the induced map $L_x \cong \mathbb{C}$ is $\mathbb{C}$-linear.

If $L \to M$ is a holomorphic line bundle with the associated open covering $(U_i, \psi_i)$, then the functions $\psi_i$ are called trivialization functions and the $\mathbb{C}$-linear functions

$$\gamma_{ji}(x) : z \mapsto (\psi_j \circ \psi_i^{-1})(x, z)$$

are called the transition functions for any $i, j$ and any $x \in U_i \cap U_j$. A holomorphic section of $L \to M$ is a holomorphic function $s : M \to L$ such that $\pi \circ s = \text{id}_M$. The space of holomorphic sections of $L$ is denoted by $H^0(M, L)$ and is a finite-dimensional complex vector space. A section $s \in H^0(M, L)$ is entirely characterized by local holomorphic functions $f_i : U_i \to \mathbb{C}$ that satisfy the compatibility conditions

$$f_j(x) = \gamma_{ji}(x)f_i(x), \quad \forall x \in U_i \cap U_j. \tag{26}$$

For instance, for any $i \in I$ set $f_i = \text{pr}_2 \circ \psi_i \circ s_{|U_i}$, where $\text{pr}_2 : U_i \times \mathbb{C} \to \mathbb{C}$ is the projection on the second coordinate: these functions satisfy (26). By construction, we have then

$$\psi_i \circ s(x) = (x, f_i(x)), \quad \forall x \in U_i. \tag{27}$$

Another way of dealing with sections is to use local frames: if $e^{(i)}$ is a nonvanishing section of $L$ over $U_i$, then there are holomorphic functions $f_i : U_i \to \mathbb{C}$ such that

$$s(x) = f_i(x)e^{(i)}(x), \quad \forall x \in U_i.$$ 

In this case, the compatibility condition (26) implies

$$f_j(x)e^{(j)}(x) = \gamma_{ji}(x)f_i(x)e^{(j)}(x) = f_i(x)e^{(i)}(x), \quad \forall x \in U_i \cap U_j,$$

so that $f_i, f_j, e^{(i)}$, and $e^{(j)}$ are related through $\gamma_{ji}$. If two line bundles $\pi_1 : L_1 \to M$ and $\pi_2 : L_2 \to M$ are isomorphic, then so are $H^0(M, L_1)$ and $H^0(M, L_2)$: assuming that for
any $x \in M$ there is an isomorphism $\Phi_x : (L_1)_x \simeq (L_2)_x$, then we have the isomorphism $\Phi : H^0(M, L^1) \to H^0(M, L^2)$ defined by

$$\Phi \circ s(x) = \Phi_x(s(x)), \forall s \in H^0(M, L^1), \forall x \in M.$$ 

If $L_1$ and $L_2$ are two holomorphic line bundles over $M$, we have then the canonical isomorphism

$$H^0(M, L_1 \otimes L_2) \cong H^0(M, L_2 \otimes L_1). \tag{28}$$

It also holds for the external tensor product: if $L_1, \ldots, L_n$ are line bundles over $M$ and $\text{pr}_i : M^n \to M$ is the projection onto the $i$-th coordinate, then the external tensor bundle $L_1 \boxtimes \cdots \boxtimes L_n$ is the line bundle over $M^n$ defined by

$$(\text{pr}_1^*L_1) \otimes \cdots \otimes (\text{pr}_n^*L_n).$$

We have the canonical isomorphism (of finite-dimensional Hilbert spaces)

$$H^0(M^n, L_1 \boxtimes \cdots \boxtimes L_n) \cong H^0(M, L_1) \otimes \cdots \otimes H^0(M, L_n), \tag{29}$$

and the canonical isomorphism (28) generalizes, for any $\sigma \in \mathfrak{S}_n$, into

$$H^0(M^n, L_1 \boxtimes \cdots \boxtimes L_n) \cong H^0(M^n, L_{\sigma(1)} \boxtimes \cdots \boxtimes L_{\sigma(n)}). \tag{30}$$

### 2.3 Kähler potentials

Let $M$ be a closed compact complex manifold of dimension $d$ and $L \to M$ be a holomorphic line bundle. A standard procedure of geometric quantization is to consider the space $H^0(M, L^k)$ of holomorphic sections of $L^k$ as the space of quantum states. In the context of integer quantum Hall effect (IQHE), it corresponds to particles in the lowest Landau level (LLL) under a uniform magnetic field of strength $k$ [Kle16].

A Hermitian structure on $L$ is the data of a Hermitian inner product $h_x$ on each fiber $L_x$, for $x \in M$. The inner product $h$ is called a Hermitian metric. If $h$ is a smooth Hermitian metric on $L$ and $\mu$ is a finite measure on $M$, the space $H^0(M, L^k)$ can be equipped with the following Hermitian inner product:

$$\langle s_1, s_2 \rangle_{L^2(\mu), h^k} = \int_M h^k_x(s_1(x), s_2(x))d\mu(x). \tag{31}$$

Let us explain how this expression translates in local coordinates. If $(U_i)$ is an open covering on $M$ and $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ are associated local trivialization functions, a section $s \in H^0(M, L)$ can be locally identified to holomorphic functions $f_i : U_i \to \mathbb{C}$ according to (27). The Hermitian metric $h$ then reads on $U_i$

$$h_x(s^{(1)}(x), s^{(2)}(x)) = \langle \phi_i \circ s^{(1)}(x), \phi_i \circ s^{(2)}(x) \rangle_i = f_i^{(1)}(x)\overline{f_i^{(2)}(x)}e^{-\phi_i(x)}, \tag{32}$$

where $\phi_i : U_i \to \mathbb{R}$ is the local weight of the inner product $\langle \cdot, \cdot \rangle_i$ on $\mathbb{C}$ corresponding to $h$ on $U_i$. Using a local frame, we can also write

$$h_x(s^{(1)}(x), s^{(2)}(x)) = f_i^{(1)}(x)\overline{f_i^{(2)}(x)}\|e^{(i)}(x)\|^2_{h^k},$$
and one recovers (32) by setting $\phi_i(x) = -2 \log \| e^{(i)}(x) \|_{h^k}$.

Analogously, a section $f \in H^0(M, L^k)$ can be described by local functions $f_i : U_i \to \mathbb{C}$: if $e^{(i)}$ is a local frame of $L$ over $U_i$, then $(e^{(i)})^{\otimes k}$ is a local frame of $L^k$ and a section $s \in H^0(M, L^k)$ satisfies

$$s(x) = f_i(x)(e^{(i)})^{\otimes k}(x), \ \forall x \in U_i.$$ 

The metric $h^k$ on $L^k$ induced by $h$ corresponds then to the local weight

$$-2 \log \| (e^{(i)}(x))^{\otimes k}\|_{h^k}^2 = -2 \log \| e^{(i)}(x) \|_{h^k}^{2k} = k\phi_i(x).$$

**Proposition 2.4.** For any $i, j$ and any $x \in U_i \cap U_j$,

$$\phi_i(x) = \phi_j(x) - 2 \log |\gamma_{ji}(x)|. \quad (33)$$

**Proof.** Let $s^{(1)}$ and $s^{(2)}$ be two holomorphic sections of $L^k$. Using (32), we have on one hand,

$$h^k_x(s^{(1)}(x), s^{(2)}(x)) = f_i^{(1)}(x)\overline{f_i^{(2)}(x)}e^{-k\phi_i(x)},$$

and on the other hand

$$h^k_x(s^{(1)}(x), s^{(2)}(x)) = f_j^{(1)}(x)\overline{f_j^{(2)}(x)}e^{-k\phi_j(x)}.$$ 

If we denote $\gamma_{ji} : U_i \cap U_j \to \mathbb{C}^*$ the transition function from $L|_{U_i} \otimes L|_{U_j}$, then the transition function from $L^k|_{U_i} \otimes L^k|_{U_j}$ is $\gamma_{ji}^k$ and we have for $\ell \in \{1, 2\}$ and for any $x \in U_i \cap U_j$

$$f_j^{(\ell)}(x) = \gamma_{ji}^k(x)f_i^{(\ell)}(x).$$

It follows that for any $x \in U_i \cap U_j$,

$$f_i^{(1)}(x)f_i^{(2)}(x)e^{-k\phi_i(x)} = f_j^{(1)}(x)f_j^{(2)}(x)|\gamma_{ji}(x)|^2e^{-k\phi_j(x)}.$$ 

As it holds true for any two sections $s^{(1)}$ and $s^{(2)}$, the proposition is proved.

An immediate consequence is that for any $s^{(1)}, s^{(2)} \in H^0(M, L^k)$, the inner product $h^k(s^{(1)}, s^{(2)})$ defines a global function on $M$, which justifies the definition of the inner product (31). In particular, the associated norm is also globally defined: for any $x \in U_i \cap U_j$,

$$\|s\|_{h^k}^2(x) = |f_i(x)|^2e^{-k\phi_i(x)} = |f_j(x)|^2e^{-k\phi_j(x)}.$$ 

The curvature form $\Theta_h$ of the metric $h$ is defined locally by $\Theta_h = -\partial\overline{\partial}\phi$ for a local weight $\phi$, and one can turn it into a real $(1, 1)$-form $\omega = \frac{i}{2}\Theta_h$. We said that $\omega$ is a Kähler form if it is closed, i.e. $d\omega = 0$, where $d = \partial + \overline{\partial}$, and $\phi$ is called a Kähler potential for $\omega$. Let us insist that the Kähler potential is only locally defined most of the time; a necessary and sufficient condition for the Kähler potential to be global is given by the global $\partial\overline{\partial}$-lemma [Huy05, Cor. 3.2.10].

If we denote by $N_k$ the dimension of $H^0(M, L^k)$, then the volume of $L$ is defined by

$$\operatorname{vol}(L) = \lim_{k \to \infty} \frac{d!}{kd} N_k, \quad (34)$$
and $L$ is big if its volume is positive. It follows from the asymptotic Riemann–Roch theorem [MM07, Thm 1.7.1] and the Kodaira–Serre vanishing theorem [MM07, Thm 1.5.6] that for $k$ large enough,

$$N_k = \frac{k^d}{d!} \int_M \frac{1}{\pi^d} \omega^d + o(k^d),$$  \hfill (35)

hence the volume of $L$ satisfies

$$\text{vol}(L) = \frac{1}{\pi^d} \int_M \omega^d.$$  \hfill (36)

In particular, we see that the total volume of $M$ with respect to $dv_M$ is $\frac{\pi^d \text{vol}(L)}{d!}$.

### 2.4 Tensor product and duality

Let $M$ be a complex manifold of dimension $d$. If $L \to M$ is a holomorphic line bundle defined by the transition functions $\gamma_{ji}$ endowed with a Hermitian metric $h$, then its dual bundle $L^* \to M$ is defined by $L_x^* = (L_x)^*$ and by the transition functions $\gamma_{ij}^{-1}$. From the Riesz representation theorem we know that for any $v \in L_x^*$ there exists a unique $\varphi \in L_x$ such that for all $w \in L_x$

$$(\varphi, w) = h_x(w, v).$$

For any finite-dimensional Hilbert space $E$, there is a canonical isomorphism of Hilbert spaces $L_x \otimes E \otimes \overline{L_x} \cong E$, given by the contraction

$$u_x \otimes e \otimes v_x \mapsto h_x(u_x, v_x)e.$$  

By induction, for any $\sigma \in S_n$ we obtain the canonical isomorphism

$$L_{x_1} \otimes \overline{L}_{x_{\sigma(1)}} \otimes \cdots \otimes L_{x_n} \otimes \overline{L}_{x_{\sigma(n)}} \cong \mathbb{C}.$$  \hfill (37)

In particular, any tensor product of the form

$$s^{(1)}(x_1) \otimes \overline{s^{(1)}(x_{\sigma(1)})} \otimes \cdots \otimes s^{(n)}(x_n) \otimes \overline{s^{(n)}(x_{\sigma(n)})}$$

can be canonically identified with the product of functions $h_{x_1}(s^{(1)}(x_1), s^{(\sigma^{-1}(1))}(x_1)) \cdots h_{x_n}(s^{(n)}(x_n), s^{(\sigma^{-1}(n))}(x_n))$.

Moreover, if $e_U$ is a local nonvanishing section of $L$ on $U$ and $\phi$ is a local weight of $h$ on $U$, then for any $x, y \in U$, $e_U(x) \otimes \overline{e_U(y)}$ can be identified with $e^{-\frac{1}{2}(\phi(x)+\bar{\phi}(y))}$ by contraction: we have the canonical isomorphisms $L_x \otimes \overline{L_y} \cong \text{Hom}(L_x, L_y) \cong \mathbb{C}$ and

$$\|e_U(x) \otimes \overline{e_U(y)}\|^2_{h_k} = e^{-k(\phi(x)+\bar{\phi}(y))}.$$  

---

6 We assume finite dimensions so that the algebraic tensor product of Hilbert spaces is automatically a Hilbert space.

7 The second isomorphism is canonical because the bases of $L_x$ and $L_y$ are given in the same trivialization.
2.5 The Bergman kernel and its determinant

Let $M$ be a complex manifold of dimension $d$ and $(L, h)$ be a Hermitian holomorphic line bundle over $M$ such that $\omega = -\frac{i}{2} \partial \bar{\partial} \log h$ is a Kähler form. Consider for any integer $k$ the space $\mathcal{C}^\infty(M, L^k)$ of smooth sections of $L^k$; it can be endowed with the inner product (31) introduced in Section 2.3, and its completion for such inner product is $L^2(M, L^k)$. Let us start by recalling the Schwartz kernel theorem.

**Theorem 2.5** ([MM07], Theorem B.2.7). Let $E$, $F$ be two vector bundles on $M$ and $A : \mathcal{C}^\infty_0(M, E) \to \mathcal{D}'(M, F)$ be a linear continuous operator. There exists a unique distribution $K \in \mathcal{D}'(M^2, F \otimes E^*)$, called the Schwartz kernel distribution, such that

$$ (Au, v) = (K, v \otimes u), \forall u \in \mathcal{C}^\infty_0(M, E^*), \forall v \in \mathcal{C}^\infty_0(M, F). $$

Moreover, for any volume form $d\mu$, the Schwartz kernel of $A$ is represented by a smooth kernel $K \in \mathcal{C}^\infty(M^2, F \otimes E^*)$, called the Schwartz kernel of $A$ with respect to $d\mu$, such that

$$ (Au)(x) = \int_M K(x, y)u(y)d\mu(y), \forall u \in \mathcal{C}^\infty_0(M, E). $$

Finally, $A$ can be extended as a linear continuous operator $A : \mathcal{D}'_0(M, E) \to \mathcal{C}^\infty_0(M, F)$ by setting

$$ (Au)(x) = (u(\cdot), K(x, \cdot)), \forall x \in M, \forall u \in \mathcal{D}'_0(M, E). $$

**Definition 2.6.** The Bergman projection is the orthogonal projection

$$ P_k : L^2(M, L^k) \rightarrow H^0(M, L^k), $$

and the Schwartz kernel $B_k$ of $P_k$ with respect to $d\mu$ is called the Bergman kernel.

Note that the kernel $B_k$ is in particular the reproducing kernel of the Hilbert space $H^0(M, L^k)$ endowed with the inner product (31), as a consequence of (39). It is the geometric version of the Christoffel–Darboux kernel introduced in Section 2.1. Although it does not appear in the notation $B_k$, the Bergman kernel does depend on the choice of the inner product, thus on the geometric structure induced by $h$.

**Proposition 2.6.** Let $(s_\ell)$ be an orthonormal basis of $H^0(M, L^k)$. The Bergman kernel can be written

$$ B_k(x, y) = \sum_{\ell=1}^{N_k} s_\ell(x) \otimes \overline{s_\ell(y)} \in L^k_x \otimes L^k_y, $$

In particular, on the diagonal, it defines a global function

$$ B_k(x, x) = \sum_{\ell=1}^{N_k} \|s_\ell(x)\|^2_{L^k_x}. $$

**Proof.** By unicity of the reproducing kernel of the Hilbert space $H^0(M, L^k)$, we only need to prove that the kernel defined in (42) is reproducing on $H^0(M, L^k)$. We have

$$ \int_M \left( \sum_{\ell=1}^{N_k} s_\ell(x) \otimes \overline{s_\ell(y)} \right) s_m(y) d\mu_M(y) = \sum_{\ell=1}^{N_k} \int_M (s_\ell(x) \otimes \overline{s_\ell(y)})(1 \otimes s_m(y)) d\mu_M(y). $$
The RHS can be identified, according to Section 2.4, to
\[ \sum_{\ell=1}^{N_k} \int_M h_k^\ell(s_m(y), s_\ell(y)) dv_M(y)s_\ell(x), \]
and by orthonormality we obtain
\[ \int_M B_k(x, y)s_m(y) dv_M(y) = s_\ell(x). \]
We can extend this by linearity to get the reproducing property for all elements of \( H^0(M, L^k) \). Equation (43) follows from similar arguments.

The reproducing property of the Bergman kernel has a few direct consequences. For instance,
\[ \int_M B_k(x,x) dv_M(x) = \sum_{\ell} \| s_\ell \|_{L^k}^2 = N_k, \quad (44) \]
and for any \( x, z \in M \)
\[ \int_M B_k(x,y)B_k(y,z) dv_M(y) = B_k(x,z). \quad (45) \]

In the last equation, the product \( B_k(x,y)B_k(y,z) \) can be interpreted as follows: we have \( B_k(x,y) \in L^k_x \otimes L^k_y \cong \text{Hom}(L^k_y, L^k_x) \) and \( B_k(y,z) \in L^k_y \otimes L^k_z \cong \text{Hom}(L^k_z, L^k_y) \), so that the product is the composition of homomorphisms, and \( B_k(x,z) \in \text{Hom}(L^k_x, L^k_z) \cong L^k_x \otimes L^k_z \) as expected.

**Remark 2.1.** In Theorem 1.1, in order to claim that such process is determinantal with kernel \( B_k \), we need to make a clear statement about how to define its determinant\(^8\). Indeed, the determinant \( \det(B_k(x_i, x_j))_{1 \leq i,j \leq n} \), denoted by \( \det_n(B_k(x_i, x_j)) \), can be formally written as:
\[ \det_n(B_k(x_i, x_j)) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \bigotimes_{i=1}^{n} B_k(x_i, x_{\sigma(i)}), \quad (46) \]
which becomes, thanks to Proposition 2.6:
\[ \det_n(B_k(x_i, x_j)) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \sum_{i_1, \ldots, i_n=1}^{N_k} \bigotimes_{j=1}^{n} s_{i_j}(x_j) \otimes \overline{s_{i_{\sigma(j)}}}(x_{\sigma(j)}). \quad (47) \]
However these equations do not really make sense, because we are adding tensors of different vector spaces. At best, using canonical isomorphisms (30), one could rearrange the terms to define the determinant by
\[ \det_n(B_k(x_i, x_j)) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \sum_{i_1, \ldots, i_n=1}^{N_k} \bigotimes_{j=1}^{n} s_{i_j}(x_j) \otimes \overline{s_{i_{\sigma-(j)}}}(x_j), \quad (48) \]
\(^8\)In the reference paper [Ber18], Berman treats the Bergman kernel formally as if it was an integral kernel on \( \mathbb{C}^d \), which is only locally true. The main goal of this section is to ensure that the expected properties still hold on a global level, as there are no other references doing it, at least to our knowledge.
so that $\det_n(B_k(x_i, x_j))$ is interpreted as a section of $(L_k \otimes \overline{L})^{2n} \rightarrow M^n$. We propose another direction, based on the discussion of Section 2.4: in the RHS of (47), each tensor can be identified to a product of scalar products, and $\det(B_k(x_i, x_j))_{1 \leq i, j \leq n}$ can then be defined as a function $M^n \rightarrow \mathbb{C}$ using the canonical isomorphism (37):

$$\det_n(B_k(x_i, x_j)) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \sum_{i_1, \ldots, i_n=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{\sigma^{-1}(j)}}(x_j)).$$

(49)

It would also be tempting to use the cycle decomposition of permutations to define the determinant as follows:

$$\det_n(B_k(x_i, x_j)) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{c \text{ cycle of } \sigma} B_k(x_{k_1}, x_{k_2}) \cdots B_k(x_{k_m}, x_{k_1}),$$

but this definition is again problematic because it would still lead to the summation of elements of different vector spaces.

The determinant of the Bergman kernel satisfies the following inductive property, well-known for Christoffel–Darboux kernels (see see [Dei99, Lemma 5.27] or [Meh04, Thm 5.1.4] for instance).

**Lemma 2.7.** Let $B_k$ be the Bergman kernel of $H^0(M, L^k)$. For any $n < N_k$, and any $(x_1, \ldots, x_n) \in M^n$,

$$\int_M \det_{n+1}(B_k(x_i, x_j)) dv_M(x_{n+1}) = (N_k - n) \det_n(B_k(x_i, x_j)).$$

(50)

**Proof.** Let us start by integrating (49) with respect to the last variable:

$$\int_M \det_{n+1}(B_k(x_i, x_j)) dv_M(x_{n+1}) = \sum_{\sigma \in S_{n+1}} \varepsilon(\sigma) \sum_{i_1, \ldots, i_{n+1}=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{\sigma^{-1}(j)}}(x_j))$$

$$\times \int_M h_{x_{n+1}}^k(s_{i_{n+1}}(x_{n+1}), s_{i_{\sigma^{-1}(n+1)}}(x_{n+1})) dv_M(x_{n+1}).$$

By orthogonality we get

$$\int_M \det_{n+1}(B_k(x_i, x_j)) dv_M(x_{n+1}) =$$

$$\sum_{\sigma \in S_{n+1}} \varepsilon(\sigma) \sum_{i_1, \ldots, i_{n+1}=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{\sigma^{-1}(j)}}(x_j)) \delta_{i_{n+1}, i_{\sigma^{-1}(n+1)}}.$$

(51)

Let us set, for any $\tau \in S_n$,

$$P_{\tau} = \sum_{i_1, \ldots, i_n=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{\tau^{-1}(j)}}(x_j)).$$


so that $\det_n(B_k(x_i, x_j)) = \sum_{\tau} \varepsilon(\tau) P_\tau$. If $\sigma(n + 1) = n + 1$, we simply have

$$\sum_{i_1, \ldots, i_{n+1}=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{n-1(j)}}(x_j)) \delta_{i_{n+1},i_{n-1(n+1)}} = N_k P_{\sigma'},$$

where $\sigma' \in \mathcal{S}_n$ is the restriction of $\sigma$ to $\{1, \ldots, n\}$; in particular $\varepsilon(\sigma') = \varepsilon(\sigma)$. Otherwise, there exists $j_0 \leq n$ such that $\sigma(j_0) = n + 1$, and we can separate the sum over $i_{n+1}$ from the rest, leaving

$$\sum_{i_1, \ldots, i_{n}=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{n-1(j)}}(x_j)) \delta_{i_{n+1},j_0} = \sum_{i_1, \ldots, i_{n}=1}^{N_k} \prod_{j=1}^{n} h_{x_j}^k(s_{i_j}(x_j), s_{i_{n-1(j)}}(x_j)) = P_\tau,$$

where $\tau \in \mathcal{S}_n$ is defined by $\tau(i) = \sigma(i)$ for $i \neq j_0$ and $\tau(j_0) = \sigma(n + 1)$. Note that $\varepsilon(\tau) = -\varepsilon(\sigma)$. Integrating all these results (there are $n$ case of $\tau$ and one case of $\sigma'$) in (51), we obtain

$$\int_M \det_{n+1}(B_k(x_i, x_j))dv_M(x_{n+1}) = (N_k - n) \sum_{\tau \in \mathcal{S}_n} \varepsilon(\tau) P_\tau = (N_k - n) \det_n(B_k(x_i, x_j)),$$

which concludes the proof. \hfill \Box

Since the seminal work of Tian [Tia90] on the Bergman kernel of projective varieties, and its generalization by Zelditch [Zel98] and Catlin [Cat99], the Bergman kernel has been known to admit an asymptotic expansion. We refer to [MM07] for a list of general results about this topic, which is still active after more than 20 years [Hou22, HS22, Kor18]. We will need the following version.

**Theorem 2.8** ([BSZ00b], Theorem 3.1). Let $(L, h)$ be a positive Hermitian line bundle on a $d$-dimensional compact complex manifold $M$ with Kähler form $\omega = \frac{i}{2} \Theta_h$. Let $z_0 \in M$ and choose normal local coordinates in a neighborhood of $z_0$. Then the Bergman kernel $B_k$ of $H^0(M, L^k)$ satisfies, as $k \to \infty$:

$$k^{-d} B_k\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) = B_\infty(u, v) + O(k^{-\frac{1}{2}}). \quad (52)$$

### 3 Free fermions on Kähler manifolds

In the integer quantum Hall effect (IQHE), one assumes that the lowest Landau level is filled with $N_k$ free fermions. If they lie in a Kähler manifold $(M, \omega)$ with a positive Hermitian line bundle $(L, h)$ such that $\omega = -\frac{i}{2} \Theta_h$, the state space of a particle in the lowest Landau level is $H^0(M, L^k)$. Under the assumption of a constant magnetic field of strength $k \in \mathbb{N}$, this space is endowed with the inner product (31). If $(s_\ell)$ is an orthonormal basis of $H^0(M, L^k)$, the $N_k$-particle wavefunction, corresponding to the completely filled LLL, is given by their Slater determinant

$$\Psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N_k!}} \det(s_{\ell}(x_m))_{1 \leq \ell, m \leq N_k}.$$
To be precise, this determinant is in fact a holomorphic section of the external tensor power bundle $(L^k)^{\otimes N_k} \rightarrow M^{N_k}$, and $H^0(M^{N_k}, (L^k)^{\otimes N_k})$ is endowed with the inner product

$$\langle s^{(1)} \otimes \cdots s^{(1)}_{N_k}, s^{(2)} \otimes \cdots s^{(2)}_{N_k} \rangle_{k^*}(x_1, \ldots, x_{N_k}) = \prod_{i=1}^{N_k} (h^k)_{x_i}(s^{(1)}_{x_i}(x_i), s^{(2)}_{x_i}(x_i)), \quad (53)$$

that induces a $L^2$ inner product $\langle \cdot, \cdot \rangle_{L^2(dv_M^{\otimes N_k}), h^k}$ with respect to $\mu^{\otimes N_k}$ and $h^k$. The squared norm of $\Psi$ defines a probability density function on $M^{N_k}$ with respect to $dv_M^{\otimes N_k}$:

$$p(x_1, \ldots, x_{N_k}) = \|\Psi(x_1, \ldots, x_N)\|^2_{h^k}. \quad (54)$$

The main result of this section is the following theorem.

**Theorem 3.1.** Let $(X_1, \ldots, X_N)$ be a family of random variables with joint density $(54)$ with respect to $dv_M^{\otimes N_k}$. It is a determinantal process on $(M, dv_M)$ with kernel $B_k$.

Before giving the proof, let us remark that this result is immediate for $(x_1, \ldots, x_{N_k})$ belonging to an open subset $U \subset M$ where the line bundle is trivial. Indeed, if $e_U$ is a local nonvanishing section on $U$ such that $\|e_U\|_{h^k}^2 = e^{-k\phi}$ and

$$s^{(i)}(x) = f^{(i)}(x)e_U(x), \quad \forall 1 \leq i \leq N, \quad \forall x \in U,$$

then for any $x_1, \ldots, x_N \in U^N$,

$$p(x_1, \ldots, x_N) = \frac{1}{N!} |\det(f^{(i)}(x_j))|^2 e^{-k\sum \phi(x_i)}$$

and in this case one can use Theorem 2.3. It is in fact sufficient for studying the scaling limit of correlation functions, but for Theorem 3.1 we need to ensure that the determinantal property is global and does not depend on a choice of local frame. Consider for instance $N = 2$, $U_1, U_2 \subset M$ such that $U_1 \cap U_2 = \emptyset$ and for $i, j \in \{1, 2\}$,

$$s^{(i)}(x) = f^{(i)}_{x_j}(x)e_{U_j}(x), \quad \forall x \in U_j,$$

then for $x_1 \in U_1$ and $x_2 \in U_2$ one has

$$\|\det(s^{(i)}(x_j))\|^2_{h^k} = (f^{(1)}_{x_1}(x_1)f^{(2)}_{x_2}(x_2) - f^{(1)}_{x_2}(x_2)f^{(2)}_{x_1}(x_1))e^{-k\phi_1(x_1) - k\phi_2(x_2)}.$$

It cannot be interpreted anymore as a determinant of functions, and the proof needs to be treated at the level of sections. However, as we shall see, the algebraic tricks used to prove Theorem 2.3 still apply. Theorem 3.1 is mentioned by Berman in a similar setting [Ber18, Lemma 5.1], but the proof is only suggested, as the author considers it as a “formal consequence of [the reproducing property of Bergman kernel]”. Although it is true (to some extent), we find it important to show the details, especially considering that the determinant of the Bergman kernel has a specific definition, see (49).

**Proof of Theorem 3.1.** Let us compute the correlation functions of $(X_1, \ldots, X_N)$. According to Proposition 2.1, $\rho_n(x_1, \ldots, x_n) = 0$ when $n > N_k$ and

$$\rho_n(x_1, \ldots, x_n) = \frac{N_k!}{(N_k - n)!} \int_{M^{N_k-n}} p(x_1, \ldots, x_{N_k}) dv_M^{\otimes N_k-n}(x_{n+1}, \ldots, x_N)$$
when \( n \leq N_k \).

Let us start with the extremal case \( n = N_k \). We have

\[
\rho_{N_k}(x_1, \ldots, x_N) = N_k! p(x_1, \ldots, x_{N_k}),
\]

so that

\[
\rho_{N_k}(x_1, \ldots, x_{N_k}) = \| \sum_{\sigma \in S_{N_k}} \varepsilon(\sigma) s^{(\sigma(1))}(x_1) \otimes \cdots \otimes s^{(\sigma(N_k))}(x_{N_k}) \|_{h^k}^2,
\]

and by sesquilinearity we obtain

\[
\rho_{N_k}(x_1, \ldots, x_{N_k}) = \sum_{\sigma, \tau \in S_{N_k}} \varepsilon(\sigma) \varepsilon(\tau) \prod_{i=1}^{N_k} h^k(\sigma^{(i)})(x_i), \sigma^{(i)}(x_i)). \tag{55}
\]

We also have, according to (49):

\[
\det(B_k(x_i, x_j)) = \sum_{\sigma \in S_{N_k}} \varepsilon(\sigma) \prod_{\ell=1}^{N_k} h^k_{x_{\ell}}(s^{(\ell)})(x_\ell), s^{(\ell^{-1}(\ell))}(x_\ell)).
\]

Now, if there are \( \ell \neq m \) such that \( i_\ell = i_m \), then

\[
\sum_{\sigma} \varepsilon(\sigma) \prod_{\ell} h^k_{x_{\ell}}(s^{(\ell)})(x_\ell), s^{(\ell^{-1}(\ell))}(x_\ell)) = 0
\]

because for each \( \sigma \) the product is the same as \( \hat{\sigma} = \sigma \circ (i_\ell, i_m) \), and \( \varepsilon(\sigma \circ (i_\ell, i_m)) = -\varepsilon(\sigma) \), so that both terms cancel each other. It follows that the sum over \( i_1, \ldots, i_{N_k} \) is equivalent to a sum over \( \tau \in S_{N_k} \), and we get

\[
\det(B_k(x_i, x_j)) = \sum_{\sigma, \tau \in S_{N_k}} \varepsilon(\sigma) \prod_{\ell=1}^{N_k} h^k_{x_{\ell}}(s^{(\tau(\ell)})(x_\ell), s^{(\tau^{-1}(\ell))}(x_\ell)).
\]

If we do the following index change on the sum: \( (\sigma, \tau) \mapsto (\tau, \tau \circ \sigma^{-1}) \), we get:

\[
\det(B_k(x_i, x_j)) = \sum_{\sigma, \tau \in S_{N_k}} \varepsilon(\sigma) \varepsilon(\tau) \prod_{\ell=1}^{N_k} h^k_{x_{\ell}}(s^{(\sigma (\ell))}(x_\ell), s^{(\tau (\ell))}(x_\ell)).
\]

If we combine this equality with (55), we obtain the expected result.

Now let us turn to the case \( n < N_k \). We will prove it by induction on \( n \), showing that for any \( \ell < N_k \) one goes from \( \rho_{\ell+1} \) to \( \rho_{\ell} \) the same way as one goes from \( \det_{\ell+1}(B_k(x_i, x_j)) \) to \( \det_{\ell}(B_k(x_i, x_j)) \). By Proposition 2.1,

\[
\rho_{\ell}(x_1, \ldots, x_\ell) = \frac{N_k!}{(N_k - \ell)!} \int_{M^{N_k-\ell}} p(x_1, \ldots, x_{N_k}) dv_{M}^{\otimes N_k-\ell} (x_{\ell+1}, \ldots, x_{N_k})
\]

and

\[
\rho_{\ell+1}(x_1, \ldots, x_{\ell+1}) = \frac{N_k!}{(N_k - \ell - 1)!} \int_{M^{N_k-\ell-1}} p(x_1, \ldots, x_{N_k}) dv_{M}^{\otimes N_k-\ell-1} (x_{\ell+2}, \ldots, x_{N_k}).
\]
We deduce that
\[ \rho_\ell(x_1, \ldots, x_\ell) = \frac{1}{N_k - \ell} \int_M \rho_{\ell+1}(x_1, \ldots, x_{\ell+1}) d\nu_M(x_{\ell+1}). \]

By assumption,
\[ \rho_{\ell+1}(x_1, \ldots, x_{\ell+1}) = \det_{\ell+1}(B_k(x_i, x_j))_{1 \leq i, j \leq \ell+1}, \]
and according to Lemma 2.7 we have
\[ (N_k - \ell) \det_{\ell}(B_k(x_i, x_j)) = \int_M \det_{\ell+1}(B_k(x_i, x_j)) d\nu_M(x_{\ell+1}), \]
which concludes the proof. \(\square\)

4 Proofs of the main results

In this section we prove the main results of this paper. We start with Theorem 1.1.

Proof of Theorem 1.1. From Theorem 3.1, we already know that the process is determinantal with kernel \(B_k\). Let us fix \(x \in M\), and choose a set of normal coordinates \((x_1, \ldots, x_d) \in C^d\) on \(U \ni x\) and a corresponding holomorphic trivialization \(e_U\) such that \(\|e_U\|_{L^2}^2 = e^{-k\phi}\). In these local coordinates, there are holomorphic functions \(f_\ell : C^d \to C\) for all \(1 \leq \ell \leq N_k\) such that
\[ s_\ell(u) = f_\ell(u)e_U(u), \forall 1 \leq \ell \leq N_k, \forall u = (u_1, \ldots, u_d) \in C^d. \]

According to (42), the Bergman kernel reads
\[ B_k(u, v) = \sum_{\ell=1}^{N_k} f_\ell(u) \overline{f_\ell(v)} e_U(u) \otimes \overline{e_U(v)}. \]

We can identify \(e_U(u) \otimes \overline{e_U(v)}\) to \(e^{-\frac{k}{2}(\phi(u)+\phi(v))}\) as in Section 2.4, and the kernel becomes a Christoffel–Darboux kernel on \(C^d\):
\[ B_k(u, v) = \sum_{\ell=1}^{N_k} f_\ell(u) \overline{f_\ell(v)} e^{-\frac{k}{2}(\phi(u)+\phi(v))}, \forall u, v \in C^d, \]
and its determinant is unambiguously defined, as opposed to the global case discussed in Remark 2.1. For any \((u_1, \ldots, u_n) \in (C^d)^n,\)
\[ \rho_n(u_1 \sqrt{k}, \ldots, u_n \sqrt{k}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} B_k(u_i \sqrt{k}, u_{\sigma(i)} \sqrt{k}). \]

From Theorem 2.8 we have
\[ \rho_n(u_1 \sqrt{k}, \ldots, u_n \sqrt{k}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) k^{nd} \prod_{i=1}^{n} \left( B_{\infty}(u_i, u_{\sigma(i)}) + O(k^{-\frac{1}{2}}) \right). \]
When we expand the product on the RHS, every term other than \( \prod_{i=1}^{n} B_{\infty}(u_i, u_{\sigma(i)}) \) becomes \( O(k^{-\frac{d}{2}}) \) with \( \ell \geq 1 \), and there are \( 2^n - 1 \) such terms:

\[
k^{-nd} \rho_n\left(\frac{u_1}{\sqrt{k}}, \ldots, \frac{u_n}{\sqrt{k}}\right) = \sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) \left( \prod_{i=1}^{n} (B_{\infty}(u_i, u_{\sigma(i)}) + (2^n - 1)O(k^{-\frac{1}{2}})) \right).
\]

We conclude the proof by expanding it further, yielding

\[
k^{-nd} \rho_n\left(\frac{u_1}{\sqrt{k}}, \ldots, \frac{u_n}{\sqrt{k}}\right) = \det(B_{\infty}(u_i, u_j)) + C_n O(k^{-\frac{1}{2}}),
\]

where \( C_n \) is a constant that depends on \( n \) but not on \( k \), so that it can be absorbed in the big \( O \).

Now we turn to the proof of Corollary 1.3.

**Proof of Corollary 1.3.** To get the convergence in probability, we can prove that for any extraction function \( \psi : \mathbb{N} \rightarrow \mathbb{N} \), there exists an extraction function \( \kappa \) such that, with probability 1, the sequence of random measures \( \hat{\mu}_{\psi(k)} \) converges in distribution to \( \mu \). Furthermore, by the Portmanteau theorem [Bil99, Thm 2.1], it is equivalent to

\[
\liminf_{k} \hat{\mu}_{\psi(k)}(G) \geq \mu(G), \text{ for all open } G \subset M.
\]  

(56)

As \( M \) is a compact metric space, it admits a countable base \( (U_i)_{i \in \mathbb{N}} \) of open subsets. Let us show that for any \( i \in \mathbb{N} \), \( \hat{\mu}_k(U_i) \) converges in probability to \( \mu(U_i) \). For any \( i \), we have first

\[
\mathbb{E}[\hat{\mu}_k(U_i)] = \frac{1}{N_k} \int_{U_i} B_k(x, x)dv_M(x) = \frac{1}{N_k} \int_{U_i} \rho_1(x)dv_M(x).
\]

According to Theorem 1.1, we have for any \( x \in M \)

\[
\rho_1(x) = k^d B_{\infty}(0, 0) + O(k^{d-\frac{1}{2}}) = \frac{k^d}{\pi^d} + O(k^{d-\frac{1}{2}}),
\]

and we have by (35) that \( N_k = \frac{k^d}{\pi^d} (\int_M dv_M + o(1)) \), so that

\[
\frac{1}{N_k} \rho_1(x) \xrightarrow{k \to \infty} \frac{1}{\int_M dv_M}, \forall x \in M.
\]  

(57)

It implies the convergence in distribution of \( \frac{1}{N_k} B_k(x, x)dv_M(x) \) to the measure \( \mu \). In particular, for all \( i \) the sequence of random variables \( \hat{\mu}_k(U_i) \) converges in distribution to \( \mu(U_i) \), and this convergence also holds in probability because \( \mu(U_i) \) is constant.

We are now able to prove (56). If \( G \) is an arbitrary open set of \( M \), there exists \( I \subset \mathbb{N} \) such that \( G = \bigcup_{i \in I} U_i \). Take an extraction function \( \psi : \mathbb{N} \rightarrow \mathbb{N} \). From the probability convergence of \( \hat{\mu}_k(B_i) \) for \( i \in I \), one deduces by diagonal extraction that there exists a subsequence \( \hat{\mu}_{\psi(k)} \) such that for all \( i \in I \),

\[
\hat{\mu}_{\psi(k)}(B_j) \xrightarrow{a.s.} \mu(B_j).
\]

Hence, almost-surely,

\[
\liminf_k \hat{\mu}_{\psi(k)}(G) \geq \sup_{n \in \mathbb{N}} \left( \liminf_k \hat{\mu}_{\psi(k)}(B_{jn}) \right),
\]

and the right-hand side is in fact \( \mu(G) \), which yields (56).
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