Superradiant stability of the Kerr black holes

Jia-Hui Huang, Wen-Xiang Chen, and Zi-Yang Huang

Institute of quantum matter,
School of Physics and Telecommunication Engineering,
South China Normal University,
Guangzhou 510006, China

We study the superradiant stability of the system of a Kerr black hole and a massive scalar perturbation. It was proved previously that this system is superradiantly stable when $\mu \geq \sqrt{2}m\Omega_H$, where $\mu$ is the proper mass of the scalar, $m$ is the azimuthal number of the scalar mode, and $\Omega_H$ is the angular velocity of the Kerr black hole horizon. Our study is a complementary work of this result. We analytically prove that in the complementary parameter region $\mu < \sqrt{2}m\Omega_H$, when the parameters of scalar perturbation and Kerr black hole satisfy two simple inequalities, $\omega < \frac{\mu}{\sqrt{2}}$, $\frac{r^+}{r^-} < 0.802$, the system is also superradiantly stable.

I. INTRODUCTION

Black holes are important and peculiar objects predicted by general relativity. Aspects of black hole physics have been studied extensively. One interesting phenomenon is the superradiant scattering of black holes[1–5], e.g., when a charged bosonic wave is impinging upon a charged rotating black hole, the wave is amplified by the black hole if the wave frequency $\omega$ obeys

$$\omega < m\Omega_H + e\Phi,$$

where $e$ and $m$ are the charge and azimuthal number of the bosonic wave mode, $\Omega_H$ is the angular velocity of black hole horizon and $\Phi$ is the electromagnetic potential of the black hole horizon. This amplification is the superradiant scattering, which was studied long time ago [6–12], and has broad applications in various areas of physics(for a recent review, see[4]). Through the superradiant process, the rotational energy or electromagnetic energy of a black hole can be extracted. Due to the existence of superradiant modes, a black hole bomb mechanism was proposed by Press and Teukolsky[13]. If there is a mirror between the black hole horizon and space infinity, the amplified wave can be scattered back and forth and grows exponentially, which leads to the superradiant instability of the background black hole geometry [14–16]. Superradiant (in)stability of various kinds of black holes have been studied extensively in the literature.

For charged Reissner-Nordstrom(RN) black holes, it has been proved that they are superradiantly stable against charged massive scalar perturbation[17–20]. The reason is that when the superradiant modes exist in such a system of a RN black hole with a charged massive scalar wave, there is no effective trapping potential/mirror outside the black hole horizon, which reflects the superradiant modes back and forth [18, 19]. After introducing a mirror-like mechanism, RN black hole will become superradiantly unstable. When a mirror or a cavity is imposed outside a charged RN black hole horizon, this black hole is superradiantly unstable in certain parameter spaces [15, 16, 21–23]. Charged black holes in curved backgrounds, such as anti-de Sitter/de Sitter(AdS/dS) space, are proved to be superradiantly unstable because these backgrounds provide natural mirror-like boundary conditions[24–28]. There is

*Electronic address: zhanfeng.mai@gmail.com
a similar case for stringy RN black holes. The stringy RN black hole is shown to be superradiantly stable against charged massive scalar perturbation\cite{29}. But if a mirror is introduced, superradiant modes are supported and the stringy RN black hole becomes superradiantly unstable \cite{30–32}. It is also found that extra coupling between the scalar field and the gravity can result in superradiant instability of RN/RN-AdS black holes \cite{33, 34}.

For rotating Kerr black holes, if the incoming scalar perturbation has a nonzero mass, this mass term will act as a natural mirror and lead to superradiant instability of Kerr black holes when the parameters of the Kerr black holes and the scalar fields are in certain parameter spaces\cite{35–43}. Beyond the massive scalar perturbations, superradiant instability of Kerr black holes that are impinged upon by a massive vector field is also discussed \cite{44, 45}. The superradiant instability of rotating black holes in curved space, such as Kerr-AdS black holes, has also been reported \cite{46–52}.

Although there has been so much study on superradiance of rotating black holes, even Kerr black hole is not investigated thoroughly. In \cite{39}, the author proved that a Kerr black hole is superradiantly stable under massive scalar perturbation when

\[
\mu \geq \sqrt{2}m\Omega_H, \tag{2}
\]

where \(\mu\) and \(m\) are the mass and azimuthal number of the incident scalar wave and \(\Omega_H\) is the angular velocity of the Kerr black hole horizon. In a following work \cite{42}, a stronger bound on the stability regime of the Kerr-black-hole-massive-scalar system was derived.

In order for a complete discussion of the superradiant stability of Kerr-black-hole-massive-scalar system, it is interesting to investigate the superradiance property of the system in the complementary parameter spaces of the previous result in \cite{39}. Explicitly, in the present paper, we will find the parameter regions of a Kerr black hole and a massive scalar perturbation system with \(\mu < \sqrt{2}m\Omega_H\), where the system is superradiantly stable. The scattering of massive scalar field in a Kerr black hole background can be described by a Schrodinger-like radial equation with an effective potential. In order for the instability of the system, there are two key conditions should be satisfied. One is that the parameters of the black hole and the scalar field should satisfy superradiance condition, i.e., there exist superradiance modes. The second one is that there is mirror-like condition outside the horizon of the black hole. In our case, if there is a potential well outside the black hole horizon, it will act as a reflecting mirror and may lead to instability of the system. So when there is no potential well outside the black hole horizon for the superradiance modes, the Kerr-black-hole-massive-scalar system will be superradiantly stable.

The paper is organized as follows. In Section 2, we provide a simple introduction of the Kerr-black-hole-massive-scalar system and the angular part of the equation of motion. In Section 3, we derive the Schrodinger-like radial equation and the effective potential for the scalar perturbation in Kerr background. This effective potential is the main object as mentioned above. In Section 4, we derive the superradiantly stable parameter regions for the system based on the effective potential. Finally, we give a summary and discussion in Section 5.

II. KERR BLACK HOLE AND A MASSIVE SCALAR PERTURBATION

Kerr black hole describes a stationary and axially symmetric spacetime geometry. The metric of the 4-dimensional Kerr black hole in Boyer-Lindquist coordinates is \cite{53–55}(we take \(G = h = c = 1\))

\[
ds^2 = \frac{\Delta}{\rho^2}(dt - \rho^2d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[adt - (r^2 + a^2)d\phi]^2, \tag{3}
\]

where

\[
\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2\theta. \tag{4}
\]

\(M\) is the mass of the black hole and \(a\) is the angular momentum per unit mass of the black hole. The inner and outer horizons of the Kerr black hole are

\[
r_{\pm} = M \pm \sqrt{M^2 - a^2}, \tag{5}
\]
and it is obvious that
\[ r_+ + r_- = 2M, \quad r_+ r_- = a^2. \] (6)

The angular velocity of the Kerr black hole horizon is
\[ \Omega_H = \frac{a}{r_+^2 + a^2}. \] (7)

The dynamics of a massive scalar field perturbation \( \Psi \) with proper mass \( \mu \) in the Kerr black hole background is described by the covariant Klein-Gordon equation
\[ (\nabla^\nu \nabla_\nu - \mu^2) \Psi = 0. \] (8)

This equation of motion has been studied for a long time. The solution of this equation with definite frequency can be decomposed as [10, 56–60]
\[ \Psi(t, r, \theta, \phi) = \sum_{l,m} \mathcal{R}_{lm}(r) S_{lm}(\theta) e^{im\phi} e^{-i\omega t}, \] (9)

where \( l \) is the spherical harmonic index, \( m \) is the azimuthal harmonic index with \( -l \leq m \leq l \) and \( \omega \) is the angular frequency of the scalar mode. \( S_{lm}(\theta) \) is the scalar spheroidal harmonics satisfying the angular equation of motion
\[ \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{dS_{lm}}{d\theta}) + [\lambda_{lm} + a^2(\mu^2 - \omega^2)\sin^2 \theta - \frac{m^2}{\sin^2 \theta}] S_{lm} = 0, \] (10)

where \( \lambda_{lm} \) is an angular eigenvalue [64]. When \( a^2(\mu^2 - \omega^2) > 0 \), the eigenfunction \( S_{lm} \) is called prolate; when \( a^2(\mu^2 - \omega^2) < 0 \), the eigenfunction \( S_{lm} \) is called oblate.

Although the angular equation (10) has been studied for a long time, an explicitly analytic expression of the general eigenvalue \( \lambda_{lm} \) is still lacking [59]. In various limit cases, we know asymptotic analytic expressions for the eigenvalues. When \( |a^2(\mu^2 - \omega^2)| \ll 1 \), the leading order of the angular eigenvalue for both prolate and oblate cases is \( l(l+1) \). This is obvious since the angular equation is approaching to Legendre equation in this case. When \( |a^2(\mu^2 - \omega^2)| \gg 1 \), the asymptotic behaviors of prolate and oblate eigenvalues are remarkably different. The asymptotic eigenvalues for the prolate and oblate cases are respectively,
\[ \lambda_{lm} \sim -a^2(\mu^2 - \omega^2); \quad \lambda_{lm} \sim 2q_{lm} a \sqrt{\mu^2 - \omega^2}, \] (11)

where \( q_{lm} \) is some numerical constant [59]. In our discussion, the prolate case is important. According to the reference [59], one key result about the prolate angular eigenvalue is
\[ \lambda_{lm} > l(l+1) - a^2(\mu^2 - \omega^2), \] (12)

which is an important bound for our following discussion.

III. THE RADIAL EQUATION OF MOTION AND EFFECTIVE POTENTIAL

The radial equation of (8) obeyed by \( R_{lm}(r) \) is given by
\[ \Delta \frac{d}{dr} (\Delta \frac{dR_{lm}}{dr}) + UR_{lm} = 0, \] (13)

where
\[ U = [\omega(r^2 + a^2) - ma]^2 + \Delta \{2ma\omega - \mu^2(r^2 + a^2) - \lambda_{lm}\}. \] (14)

In order to study the superradiant modes of black holes to the massive perturbation, the asymptotic solutions of the radial equation near the horizon and infinity are considered under appropriate boundary conditions. Defining the
tortoise coordinate $r_*$ by equation $\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}$ and a new radial function $\tilde{R}_{lm} = \sqrt{1 + a^2} R_{lm}$, the above radial wave equation can be rewritten as

$$\frac{d^2 \tilde{R}_{lm}}{dr_*^2} + \tilde{U} \tilde{R}_{lm} = 0. \quad (15)$$

The asymptotic behaviors of $\tilde{U}$ at $r_* \to \pm \infty$ are as follows:

$$\tilde{U} \to (\omega - m \Omega H)^2, \quad r_* \to -\infty (r \to r_+); \quad \quad (16)$$

$$\tilde{U} \to \omega^2 - \mu^2, \quad r_* \to +\infty (r \to +\infty). \quad (17)$$

The physical boundary conditions that we are interested in are ingoing wave at the horizon ($r_* \to -\infty$) and bound states (exponentially decaying modes) at spatial infinity ($r_* \to +\infty$). Then the asymptotic solutions of the radial wave equation are the following

$$r \to +\infty (r_* \to +\infty), \quad \tilde{R}_{lm} \sim e^{-\sqrt{\mu^2 - \omega^2} r_*}, \quad (18)$$

$$r \to r_+ (r_* \to -\infty), \quad \tilde{R}_{lm} \sim e^{-i(\omega - m \Omega H) r_*}. \quad (19)$$

It is easy to see that in order to get the decaying modes we need following condition

$$\omega^2 < \mu^2. \quad (20)$$

In order to analyse the superradiant stability of the Kerr black hole, we define a new radial wavefunction $\varphi = \Delta^{1/2} R$. The radial equation (13) is transformed into a Schrodinger-like wave equation with effective potential $V_1$,

$$\frac{d^2 \varphi}{dr^2} + (\omega^2 - V_1) \varphi = 0, \quad V_1 = \omega^2 - \frac{U + M^2 - a^2}{\Delta^2}. \quad (21)$$

In order to see if there exist a trapping potential outside the horizon, we should analyze the shape of the effective potential $V_1$. From the following asymptotic behaviors of the potential $V_1$:

$$V_1 (r \to \infty) \to \mu^2 - \frac{4M\omega^2 - 2M\mu^2}{r} + O\left(\frac{1}{r^2}\right), \quad (22)$$

$$V_1 (r \to r_+) \to -\infty, V_1 (r \to r_-) \to -\infty, \quad (23)$$

$$V_1' (r \to \infty) \to \frac{4M\omega^2 - 2M\mu^2}{r^3} + O\left(\frac{1}{r^3}\right). \quad (24)$$

So when

$$\omega^2 < \frac{\mu^2}{2}, \quad (25)$$

then

$$V_1' (r \to \infty) < 0. \quad (26)$$

This means that there is no potential wells when $r \to \infty$ and the black hole may be superradiantly stable, so (25) is one important basic inequality. In the next section, we will find the regions of the parameter space where there is only one extreme outside the event horizon $r_+$ for effective potential $V_1$, no trapping well exists, which is separated from the horizon by a potential barrier, and the Kerr black holes are superradiantly stable.
IV. THE SUPERRADIANT STABILITY ANALYSIS

In this section, we will determine the regions of the parameter space where the system of Kerr black hole and massive scalar is superradiantly stable. We define a new variable $z$, $z = r - r_-$. Then the explicit expression of the derivative of the effective potential $V_1$ is

$$V_1'(r) = V_1'(z) = \frac{Ar^4 + Br^3 + Cr^2 + Dr + E}{-\Delta^3} = \frac{A_1z^4 + B_1z^3 + C_1z^2 + D_1z + E_1}{-\Delta^3};$$  \hspace{1cm} (27)

$$A_1 = A_1; B_1 = B + 4r_-A_1; C_1 = C + (3r_-)B_1 + (6r_-^2)A_1;$$

$$D_1 = D + (4r_-^3)A_1 + (3r_-^2)B_1 + (2r_-)C_1;$$

$$E_1 = E + r_-^4A_1 + r_-^3B_1 + r_-^2C_1 + r_-D_1;$$

$$A_1 = 2M(\mu^2 - 2\omega^2),$$

$$B_1 = -16Mr_-\omega^2 - \mu^2r_-^2 + 2\lambda_{lm} + 3\mu^2r_-^2 + 2a^2\mu^2,$$

$$C_1 = -24Mr_-\omega^2 + 6\omega am(r_- + r_+ - 3(r_+ - r_-)(a^2\mu^2 + r_-^2\mu^2 + \lambda_{lm}),$$

$$D_1 = -16M^3r_-^3\omega^2 + 4aMm(5r_- - r_+)\omega + (r_+ - r_-)^2(\mu^2a^2 + \lambda_{lm} - 1) + a^2(\mu^2(r_+ - r_-)^2 - 4m^2),$$

$$E_1 = 4M(a^4 - r_+^4)\omega^2 + 8Mma(r_-^2 - a^2)\omega + 2(r_+ - r_-)(M^2 + m^2a^2 - a^2).$$

We denote the numerator of the derivative of the effective potential $V_1$ by $f(z)$, which is a quartic polynomial in $z$. Whether there is a trapping well outside the horizon can be analyzed through the property of roots of equation $f(z) = 0$. The four roots for equation $f(z) = 0$ are $z_1, z_2, z_3$ and $z_4$. According to Vieta theorem, we have the following relations,

$$z_1z_2z_3z_4 = \frac{E_1}{A_1},$$

$$z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = \frac{-D_1}{A_1},$$

$$z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = \frac{C_1}{A_1},$$

$$z_1 + z_2 + z_3 + z_4 = \frac{-B_1}{A_1}.$$

Based on the asymptotic behaviors of the effective potential at the inner and outer horizons and infinity (22),(23), one can deduce that equation $V_1'(z) = 0$(or $f(z) = 0$) has at least two positive real roots when $z > 0$. These two positive roots are denoted by $z_1, z_2$, namely

$$z_1 > 0, \hspace{1cm} z_2 > 0.$$  \hspace{1cm} (40)

In the following, we will find a parameter region of the system where there are only two positive roots for the equation $V_1'(z) = 0$, i.e., $z_3, z_4$ are both negative. Then there is no trapping potential well acting as a mirror outside the horizon of the Kerr black hole. The system is superradiantly stable in this region.

Under the condition (25), $A_1 > 0$. Taking $E_1$ as a quadratic function of $\omega$, one can show that $E_1 > 0$ without subtlety. $E_1$ can be written as follows,

$$E_1(\omega) = 4M(a^4 - r_+^4)\omega^2 + 8Mma(r_-^2 - a^2)\omega + 2(r_+ - r_-)(M^2 + m^2a^2 - a^2).$$

(41)
The discriminant of $E_1$ can be calculated directly as following,
\[ \Delta_{E_1} = -4r^2 (r_+ - r_-)^4 (r_+ + r_-)^2, \]  
which is obviously negative, $\Delta_{E_1} < 0$. The coefficient of $\omega^2$ in $E_1$ is obviously larger than 0. One can conclude that $E_1 > 0$ for any $\omega$. Then according to (36), we have
\[ z_1 z_2 z_3 z_4 > 0. \]
As we mentioned above, $z_1$ and $z_2$ have been denoted as two positive roots of the derivative of the effective potential. From the above equation, we find that $z_3$ and $z_4$ can only be both positive or both negative, if they are real roots.
Next we turn to focus on the following equation in detail,
\[ z_1 + z_2 + z_3 + z_4 = -\frac{B_1}{A_1}. \]
Because $A_1 > 0$, we want to identify a parameter region where $B_1 > 0$, then the roots $z_3, z_4$ are both negative. Using the condition of the eigenvalue of our angular equation given in (12), we have
\[ B_1 > -(6r_+ r_- + 8r_+^2) \omega^2 + 2l(l + 1) + \mu^2 (3r_+^2 - r_+^2). \]
Now we define a quadratic function with respect to $\omega$ as follows:
\[ g(\omega) = -(6r_+ r_- + 8r_+^2) \omega^2 + 2l(l + 1) + \mu^2 (3r_+^2 - r_+^2). \]
Identifying a parameter region where $B_1 > 0$ is transformed into identifying the parameter region where $g(\omega) > 0$. We will do this in the following.
As mentioned in the introduction, we focus on discussing Kerr black holes and massive scalar perturbation system in the regime $\mu < \sqrt{2} m \Omega_H$, i.e., $\mu^2 < \frac{2 m^2 a^2}{(r_+ + a)^2}$. It is easy to see that if $g(\omega) > 0$ is true for some real $\omega$, the following inequality should be satisfied first,
\[ 2l(l + 1) + \mu^2 (3r_+^2 - r_+^2) > 0. \]
Using the conditions $l(l + 1) > m^2$ and $\mu^2 < \frac{2 m^2 a^2}{(r_+ + a)^2}$, we have
\[ 2l(l + 1) - \mu^2 r_+^2 + 3 \mu^2 r_-^2 > 2m^2 - \mu^2 r_+^2 + 3 \mu^2 r_-^2 > \mu^2 r_+^2 \left( \frac{(r_+^2 + a^2)^2}{r_+^2 - r_-^2} - 1 \right) + 3 \mu^2 r_-^2. \]
It is worth to pointing out that
\[ \frac{(r_+^2 + a^2)^2}{r_+^2 - r_-^2} = (1 + \frac{r_-}{r_+})(1 + \frac{r_+}{r_-}) > 1. \]
So we conclude that the following inequality
\[ 2l(l + 1) - \mu^2 r_+^2 + 3 \mu^2 r_-^2 > 0, \]
could be always satisfied in our Kerr black hole and massive scalar perturbation system.
From our discussion above we have known that there must exist two real roots for $g(\omega)$ with respect to $\omega$. Taking $\omega_0$ as the positive real root of $g(\omega)$, then we have
\[ \omega_0^2 = \frac{2l(l + 1) + \mu^2 (3r_+^2 - r_+^2)}{8r_+^2 + 6 r_+ r_-}. \]
When $0 < \omega < \omega_0$, one can obtain $g(\omega) > 0$. Remember that in our system the parameters of massive scalar perturbation satisfy (25). So we will have $g(\omega) > 0$, if $\omega_0^2 > \frac{a^2}{2},$ i.e.,
\[ \frac{2l(l + 1) + \mu^2 (3r_+^2 - r_+^2)}{8r_+^2 + 6 r_+ r_-} - \frac{\mu^2}{2} > 0. \]
Considering the relations $\mu < \sqrt{2}m\Omega_H$ and $l(l + 1) > m^2$, the above inequality can be transformed into

$$1 + \lambda - 2\lambda^2 - \lambda^3 > 0,$$

(53)

where $\lambda = \frac{r_-}{r_+} \in (0, 1]$, which is only related to the parameters of the Kerr black hole.

There exist three real roots in the cubic equation $1 + \lambda - 2\lambda^2 - \lambda^3 = 0$ and here we list them as

$$\lambda_1 = \frac{1}{3}(-2 + 2\sqrt{7}\cos\frac{\theta}{3}),$$

$$\lambda_2 = \frac{1}{3}(-2 - \sqrt{7}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right)),$$

$$\lambda_3 = \frac{1}{3}(-2 - \sqrt{7}\left(\cos\frac{\theta}{3} - \sqrt{3}\sin\frac{\theta}{3}\right)),$$

(54)

where $\theta = \arccos(-\frac{1}{2\sqrt{7}})$. Note that $\lambda_1 \approx 0.802 > 0$, $\lambda_2 = -0.555 < 0$, $\lambda_3 = -2.247 < 0$. One could plot the curve for the cubic function $1 + \lambda - 2\lambda^2 - \lambda^3$ with respect to $\lambda$, see Fig.(1), from which one can see $1 + \lambda - 2\lambda^2 - \lambda^3 > 0$ when $\lambda \in (0, \lambda_1)$.

![FIG. 1: Here we plot the curve of $1 + \lambda - 2\lambda^2 - \lambda^3$ in the regime $\lambda \in [-3, 3/2]$. Actually we just need to focus on the interval $0 < \lambda < 1$. From this curve one can obviously see that $1 + \lambda - 2\lambda^2 - \lambda^3 > 0$ when $\lambda \in (0, \lambda_1)$.](image)

So we prove that when the parameters of the Kerr black hole satisfy the following inequality,

$$0 < \frac{r_-}{r_+} < \frac{1}{3}\left(-2 + 2\sqrt{7}\cos\frac{\theta}{3}\right), \quad \theta = \arccos(-\frac{1}{2\sqrt{7}}),$$

(55)

we have $B_1 > g(\omega) > 0$. Then there is no trapping potential well for the effective potential of the radial equation of motion and the Kerr black hole and scalar perturbation system is superradiantly stable.

V. CONCLUSION

In the present paper, we investigate the superradiant stability of a system with a Kerr black hole and massive scalar perturbation. Our discussion is focusing on models where the mass scalar perturbation is smaller than an upper bound related with the angular velocity of the Kerr black hole horizon, i.e., $\mu < \sqrt{2}m\Omega_H$, and is a complementary study of previous work.

The equation of motion of the scalar perturbation in the Kerr black hole background is separated into angular and radial parts. In our discussion, the spheroidal angular equation is prolate and we choose a general bound for the eigenvalue of this equation. The radial equation can be transformed into a Schrodinger-like equation and the effective potential is important for the stability analysis. We find that when the parameters of scalar perturbation satisfy $\omega < \frac{\mu}{\sqrt{2}}$ and the parameters of Kerr black hole satisfy $\frac{r_-}{r_+} < 0.802$, there is no potential well outside the black hole horizon acting as a mirror and the system is superradiantly stable.
Our study in this paper treats the Kerr black hole as a background geometry and only considers the dynamics of the free scalar perturbation. In order to get a more refined result of the superradiant behavior of the Kerr black hole and scalar perturbation system, one further step is to study the coupled nonlinear equations of motion of the scalar perturbation and black hole. Recently, such nonlinear evolution of superradiant process has been studied in several models [22, 25, 44, 61, 62], and has found some exciting applications in astrophysical physics. Another interesting extension is to study a model with nonlinear self-interacting scalar perturbation. It is pointed out that when we take self-interaction into account for the scalar perturbation, superradiant behavior of the system will be different and non-linear scalar hairs will exist[63]. It will be interesting to investigate the detailed superradiant behavior of a system consisting of a Kerr black hole and a scalar field with self-interaction.

Acknowledgements:

Z.F.M thanks Professor H. Lu for useful discussion. J.H.H. is supported by the Natural Science Foundation of Guangdong Province (No.2016A030313444).

[1] C.A. Manogue, Annals of Phys.181, 261 (1988).
[2] W. Greiner, B. Muller, J. Rafelski, Quantum Electrodynamics of Strong Fields, Springer-Verlag, Berlin, 1985.
[3] V. Cardoso, O.J.C. Dias, J.P.S. Lemos, S. Yoshida, Phys. Rev. D 70, 044039(2004); Phys. Rev. D 70, 049903 (2004)(Erratum).
[4] R. Brito, V. Cardoso and P. Pani, Lect. Notes Phys. 906, pp.1 (2015).
[5] R. Brito, V. Cardoso and P. Pani, Class. Quant. Grav. 32, no. 13, 134001 (2015).
[6] R. Penrose, Revista Del Nuevo Cimento,1,252 (1969).
[7] D. Christodoulou, Phys. Rev. Lett. 25, 1596 (1970).
[8] C. W. Misner, Phys. Rev. Lett. 28,994 (1972).
[9] Y. B. Zeldovich, JETP Lett. 14, 180 (1971).
[10] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Astrophys. J. 178, 347 (1972).
[11] J. D. Bekenstein, Phys. Rev. D 7, 949 (1973).
[12] T. Damour, N. Deruelle and R. Ruffini, Lett. Nuovo Cim. 15, 257 (1976).
[13] W.H. Press, S.A. Teukolsky, Nature (London) 238, 211 (1972).
[14] V. Cardoso, O. J. C. Dias, J. P. S. Lemos and S. Yoshida, Phys. Rev. D 70, 044039 (2004), Erratum: [Phys. Rev. D 70, 049903 (2004)].
[15] C. A. R. Herdeiro, J. C. Degollado and H. F. Rnarsson, Phys. Rev. D 88, 063003 (2013).
[16] J. C. Degollado and C. A. R. Herdeiro, Phys. Rev. D 89, no. 6, 063005 (2014).
[17] S. Hod, Phys. Lett. B 713, 505 (2012).
[18] J. H. Huang and Z. F. Mai, Eur. Phys. J. C 76, no. 6, 314 (2016).
[19] S. Hod, Phys. Rev. D 91, no. 4, 044047 (2015).
[20] L. Di Menza and J.-P. Nicolas, Class. Quant. Grav. 32, no. 14, 145013 (2015).
[21] R. Li, J. K. Zhao and Y. M. Zhang, Commun. Theor. Phys. 63, no. 5, 569 (2015).
[22] N. Sanchis-Gual, J. C. Degollado, P. J. Montero, J. A. Font and C. Herdeiro, Phys. Rev. Lett. 116, no. 14, 141101 (2016).
[23] O. Fierro, N. Grandi and J. Oliva, Class. Quant. Grav. 35, no. 10, 105007 (2018).
[24] M. Wang and C. Herdeiro, Phys. Rev. D 89, no. 8, 084062 (2014).
[25] P. Bosch, S. R. Green and L. Lehner, Phys. Rev. Lett. 116, no. 14, 141102 (2016).
[26] Y. Huang, D. J. Liu and X. Z. Li, Int. J. Mod. Phys. D 26, no. 13, 1750141 (2017).
[27] P. A. Gonzalez, E. Papantonopoulos, J. Saavedra and Y. Vasquez, Phys. Rev. D 95, no. 6, 064046 (2017).
[28] Z. Zhu, S. J. Zhang, C. E. Pellicer, B. Wang and E. Abdalla, Phys. Rev. D 90, no. 4, 044042 (2014), Addendum: [Phys. Rev. D 90, no. 4, 049904 (2014)].
[29] R. Li, Phys. Rev. D 88, 127901 (2013).
[30] R. Li and J. Zhao, Eur. Phys. J. C 74, no. 9, 3051 (2014).
[31] R. Li and J. Zhao, Phys. Lett. B 740, 317 (2015).
[32] R. Li, Y. Tian, H. b. Zhang and J. Zhao, Phys. Lett. B 750, 520 (2015).
[33] T. Kolyvaris, M. Koukouvaou, A. Machattou and E. Papantonopoulos, Phys. Rev. D 98, no. 2, 024045 (2018).
[34] E. Abdalla, B. Cuadros-Melgar, R. D. B. Fontana, J. de Oliveira, E. Papantonopoulos and A. B. Pavan, Phys. Rev. D 99, no. 10, 104065 (2019).
[35] M. J. Strafuss and G. Khanna, Phys. Rev. D 71, 024034 (2005).
[36] R. A. Konoplya and A. Zhidenko, Phys. Rev. D 73, 124040 (2006).
[37] V. Cardoso, S. Chakrabarti, P. Pani, E. Berti and L. Gualtieri, Phys. Rev. Lett. 107, 241101 (2011).
[38] S. R. Dolan, Phys. Rev. D 87, no. 12, 124026 (2013).
[39] S. Hod, Phys. Lett. B 708, 320 (2012).
[40] S. Hod, Phys. Lett. B 736, 398 (2014).
[41] A. N. Aliev, JCAP 1411, no. 11, 029 (2014).
[42] S. Hod, Phys. Lett. B 758, 181 (2016).
[43] J. C. Degollado, C. A. R. Herdeiro and E. Radu, Phys. Lett. B 781, 651 (2018).
[44] W. E. East and F. Pretorius, Phys. Rev. Lett. 119, no. 4, 041101 (2017).
[45] W. E. East, Phys. Rev. D 96, no. 2, 024004 (2017).
[46] V. Cardoso and O. J. C. Dias, Phys. Rev. D 70, 084011 (2004).
[47] V. Cardoso, O. J. C. Dias, G. S. Hartnett, L. Lehner and J. E. Santos, JHEP 1404, 183 (2014).
[48] C. Y. Zhang, S. J. Zhang and B. Wang, JHEP 1408, 011 (2014).
[49] O. Delice and T. Durgut, Phys. Rev. D 92, no. 2, 024053 (2015).
[50] A. N. Aliev, Eur. Phys. J. C 76, no. 2, 58 (2016).
[51] M. Wang and C. Herdeiro, Phys. Rev. D 93, no. 6, 064066 (2016).
[52] H. R. C. Ferreira and C. A. R. Herdeiro, Phys. Rev. D 97, no. 8, 084003 (2018).
[53] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
[54] I. Bredberg, T. Hartman, W. Song and A. Strominger, JHEP 1004, 019 (2010).
[55] S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford University Press, New York, 1983.
[56] S. A. Teukolsky, Phys. Rev. Lett. 29, 1114 (1972).
[57] S. A. Teukolsky, Astrophys. J. 185, 635 (1973).
[58] D. R. Brill, P. L. Chrzanowski, C. Martin Pereira, E. D. Fackerell and J. R. Ipser, Phys. Rev. D 5, 1913 (1972).
[59] E. Berti, V. Cardoso and M. Casals, Phys. Rev. D 73, 024013 (2006) Erratum: [Phys. Rev. D 73, 109902 (2006)].
[60] S. Hod, Phys. Lett. B 746, 365 (2015).
[61] P. M. Chesler and D. A. Lowe, Phys. Rev. Lett. 122, no. 18, 181101 (2019).
[62] V. Cardoso, O. J. C. Dias, G. S. Hartnett, M. Middleton, P. Pani and J. E. Santos, JCAP 03, 043(2018).
[63] J. P. Hong, M. Suzuki and M. Yamada, arXiv:1907.04982 [gr-qc].
[64] Compared with the similar parameter $\alpha \omega m$ in [59], $\lambda \omega m = \alpha \omega m - \alpha^2 (\mu^2 - \omega^2)$. 
