A Numeraire-free and Original Probability Based Framework for Financial Markets*

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Abstract

In this paper, we introduce a numeraire-free and original probability based framework for financial markets. We reformulate or characterize fair markets, the optional decomposition theorem, superhedging, attainable claims and complete markets in terms of martingale deflators, present a recent result of Kramkov and Schachermayer (1999, 2001) on portfolio optimization and give a review of utility-based approach to contingent claim pricing in incomplete markets.

2000 Mathematics Subject Classification: 60H30, 60G44.
Keywords and Phrases: Expected utility maximization, Fair market, Fundamental theorem of asset pricing (FTAP), Martingale deflator, Minimax martingale deflator, Optional decomposition theorem, Superhedging.

1. Introduction

A widely adopted setting for “arbitrage-free” financial markets is as follows: one models the price dynamics of primitive assets by a vector semimartingale, takes the saving account (or bond) as numeraire, and assumes that there exists an equivalent local martingale measure for the deflated price process of assets. According to the fundamental theorem of asset pricing (FTAP, for short), due to Kreps (1981) and Delbaen and Schachermayer (1994) if the deflated price process is locally bounded, this assumption is equivalent to the condition of “no free lunch with vanishing risk” (NFLVR for short). However, the property of NFLVR is not invariant under a change of numeraire. Moreover, under this setting, the market is “arbitrage-free” only for admissible strategies, the market may allow arbitrage for static trading

*The work was supported by the 973 project on mathematics of the Ministry of Science and Technology and the knowledge innovation program of the CAS. The author wishes to thank Dr. Jianming Xia for helpful comments.
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strategies with short-selling, and a pricing system using an equivalent local martingale measure may not be consistent with the original prices of some primitive assets. In order to remedy these drawbacks, Yan (1998) introduced the numeraire-free notions of “allowable strategy” and fair market. In this paper, we will further present a numeraire-free and original probability based framework for financial markets in a systematic way.

The paper is organized as follows: In Section 2, we introduce the semimartingale model, define the notion of martingale deflator. In Section 3, we reformulate Kramkov’s optional decomposition theorem in terms of martingale deflators, and give its applications to the superhedging of contingent claims and the characterizations of attainable claims and complete markets. In Section 4, we present a recent result of Kramkov and Schachermayer (1999, 2001) on optimal investment and give a review of utility-based approach to contingent claim pricing in incomplete markets.

2. Semimartingale model and basic concepts

We consider a security market model in which the uncertainty and information structure are described by a stochastic basis \((\Omega, \mathcal{F}, P; (\mathcal{F}_t))\) satisfying the usual conditions with \(\mathcal{F}_0\) being trivial. We call \(P\) the original (or objective) probability. It models the “real world” probability.

The market consists of \(d\) (primitive) assets whose price processes \((S^i_t)\), \(i = 1, \ldots, d\) are assumed to be non-negative semimartingales with initial values non-zero. We further assume that the process \(\sum_{i=1}^{d} S^i_0\) is strictly positive and that each \(S^i_t\) vanishes on \([T^i, \infty), \) where \(T^i(\omega) = \inf\{t > 0 : S^i_t(\omega) = 0, \text{ or } S^i_{t-}(\omega) = 0\}\) stands for the ruin time of the company issuing asset \(i\). We will see later that this latter assumption is automatically satisfied for a fair market, since any non-negative supermartingale satisfies this property. In the literature, it was assumed that all primitive assets have strictly positive prices.

Let \(S_t = (S^1_t, \ldots, S^d_t)\). Throughout the paper, we will use the following notation:

\[
S^*_t = \left(\sum_{i=1}^{d} S^i_0\right)^{-1} \sum_{i=1}^{d} S^i_t.
\]

By assumption, \(S^*_t\) is a strictly positive semimartingale. In the literature on mathematical finance, one often takes a primitive asset whose price never vanishes as numeraire. In our model, such a primitive numeraire asset may not exist. However, by our assumption on the model, we can always take \(S^*_t\) as numeraire.

2.1. Self-financing strategy

A trading strategy is an \(R^d\)-valued \(\mathcal{F}_t\)-predictable process \(\theta(t) = (\theta^1(t), \ldots, \theta^d(t))\), which is integrable w.r.t. the semimartingale \(S_t\). Here \(\theta^i(t)\) represents the numbers of units of asset \(i\) held at time \(t\). The wealth \(W_t(\theta)\) at time \(t\) of a trading strategy \(\theta\) is \(W_t(\theta) = \theta(t) \cdot S_t\), where \(a \cdot b\) denotes the inner product of two vectors.
A trading strategy $\theta$ is said to be self-financing, if

$$W_t(\theta) = W_0(\theta) + \int_0^t \theta(u) dS_u. \quad (2.1)$$

In this paper we use notation $\int_{0}^{t} H \cdot dX$ or $(H.X)_t$ to denote the integral of $H$ w.r.t. $X$ over the interval $(0,t]$. In particular, we have $(H.X)_0 = 0$.

The following theorem concerns a result on stochastic integrals of semimartingales, which represents an important property of self-financing strategies. It was given in Xia and Yan (2002).

**Theorem 2.1** Let $X$ be an $\mathbb{R}^d$-valued semimartingale and $H$ an $\mathbb{R}^d$-valued predictable process. If $H$ is integrable w.r.t. $X$ and $H_t \cdot X_t = H_0 \cdot X_0 + \int_{0}^{t} H_s dX_s$, \quad (2.2)

then for any real-valued semimartingale $y$, $H$ is integrable w.r.t. $yX$ and

$$y_t(H \cdot X)_t = y_0(H \cdot X)_0 + \int_{0}^{t} H_s d(yX)_s. \quad (2.3)$$

As a consequence of Theorem 2.1, we obtain the following

**Theorem 2.2** 1) For any given $\mathbb{R}^d$-valued $S$-integrable predictable process $\theta(t)$ and a real number $x$ there exists a real-valued predictable process $\theta^*(t)$ such that $\{\theta^*(t)1_d + \theta(t)\}$ is a self-financing strategy with initial wealth $x$, where $1_d$ is the $d$-dimensional vector $(1,1,\cdots,1)$.

2) A strategy $\theta$ is self-financing if and only if $d\tilde{W}_t(\theta) = \theta(t) d\tilde{S}_t$, where $\tilde{S}_t = S_t(S^*_t)^{-1}, \tilde{W}_t(\theta) = W_t(\theta)(S^*_t)^{-1}$.

**2.2. Fair market and fundamental theorem of asset pricing**

Now we consider a finite time horizon $T$. In Yan (1998), we introduced the notions of allowable strategy and fair market under assumption that all price processes of assets are strictly positive. The following definitions extend these notions to the present model.

**Definition 2.1** A strategy $\theta$ is said to be allowable, if it is self-financing and there exists a positive constant $c$ such that the wealth $W_t(\theta)$ at any time $t$ is bounded from below by $-cS^*_t$.

**Definition 2.2** A market is said to be fair if there exists a probability measure $Q$ equivalent to the original probability measure $P$ such that the deflated price process $(\tilde{S})$ is a (vector-valued) $Q$-martingale.

We call such a $Q$ an equivalent martingale measure for the market. Throughout the sequel we denote by $\mathcal{Q}$ the set of all equivalent martingale measures.

If the market is fair, the deflated wealth process of any allowable strategy is a local $Q$-martingale, and consequently, is also a $Q$-supermartingale, for all $Q \in \mathcal{Q}$.

By the main theorem in Delbaen and Schachermayer (1994), Yan (1998) obtained an intrinsic characterization of fair markets. This result can be regarded
as a numeraire-free version of the FTAP due to Kreps (1981) and Delbaen and Schachermayer (1994). The same result is valid for our more general model.

**Theorem 2.3** The market is fair if and only if there is no sequence \((\theta_n)\) of allowable strategies with initial wealth \(\theta\) such that \(W_T(\theta_n) \geq -\frac{1}{S^*_T} a.s.\), \(\forall n \geq 1\), and such that \(W_T(\theta_n) a.s.\) tends to a non-negative random variable \(\xi\) satisfying \(P(\xi > 0) > 0\).

**Remark** If we take \(S_t^*\) as numeraire and consider the market in deflated terms, the condition in Theorem 2.3 is the NFLVR condition introduced in Delbaen and Schachermayer (1994).

### 2.3. Martingale deflators

In principle, we can take any strictly positive semimartingale as a numeraire, and its reciprocal as a deflator.

**Definition 2.3** A strictly positive semimartingale \(M_t\) with \(M_0 = 1\) is called a martingale deflator for the market, if the deflated price processes \((S_i^*M_t)\), \(i = 1, \ldots, d\) are martingales under the original probability measure \(P\).

In the literature, such a deflator \(M\) is called “state price deflator”. Here we propose to name it as “martingale deflator”. A martingale deflator \(M\) is uniquely determined by its terminal value \(M_T\). In fact, we have \(M_t = (S_t^*)^{-1}E[M_T S_T^*|F_t]\).

In terms of martingale deflators, a market is fair if and only if there exists a martingale deflator for the market.

Assume that the market is fair. We denote by \(\mathcal{M}\) the set of all martingale deflators, and denote by \(\mathcal{Q}\) the set of all equivalent martingale measures, when \(S_t^*\) is taken as numeraire. Note that there exists a one-to-one correspondence between \(\mathcal{M}\) and \(\mathcal{Q}\). If \(M \in \mathcal{M}\), then \(\frac{dQ}{dP} = M_T S_T^*\) define an element \(Q\) of \(\mathcal{Q}\). If \(Q \in \mathcal{Q}\), then we can define an element \(M\) of \(\mathcal{M}\) with \(M_T = \frac{dQ}{dP}(S_T^*)^{-1}\). If \(\mathcal{M}\) (or \(\mathcal{Q}\)) contains only one element, the market is said to be complete. Otherwise, the market is said to be incomplete.

We will see in following sections that the use of martingale deflators instead of equivalent martingale measures has some advantages in handling financial problems.

### 3. Optional decomposition theorem and its applications

The optional decomposition theorem of Kramkov is a very useful tool in mathematical finance. It generalizes the classical Doob-Meyer decomposition theorem for supermartingales. This kind of decomposition was first proved by El Karoui and Quenez (1995), in which the process involved is the value process of a superhedging strategy for a contingent claim in an incomplete market modelled by a diffusion process. Kramkov (1996) extended this result to the general semimartingale setting, but under the assumption that the underlying semimartingale is locally bounded and the supermartingale to be decomposed is non-negative and locally bounded. Föllmer and Kabanov (1998) removed any boundedness assumption. But in both
papers, the theorem was formulated in the setting that there exists equivalent local martingale measures for the underlying semimartingales.

3.1. Optional decomposition theorem in terms of martingale deflators

Based on Theorem 2.1, Xia and Yan (2002) obtained the following version of the optional decomposition theorem in the equivalent martingale measure setting.

**Theorem 3.1**

Let $Y$ be a vector-valued semimartingale with non-negative components. Assume that the set $Q$ of equivalent martingale measures for $Y$ is nonempty. If $X$ is a local $Q$-supermartingale, i.e. local $Q$-supermartingale for all $Q \in Q$, then there exist an adapted, right continuous and increasing process $C$ with $C_0 = 0$, and a $Y$-integrable predictable process $\varphi$ such that

$$X = X_0 + \varphi.Y - C.$$  

Moreover, if $X$ is non-negative, then $\varphi.Y$ is a local $Q$-martingale.

The following theorem is a reformulation of Theorem 3.1 in terms of martingale deflators.

**Theorem 3.2**

Assume that the market is fair. We denote by $M$ the set of all martingale deflators. Let $X$ be a semimartingale. If $XM$ is a local supermartingale for all $M \in M$, then there exist an adapted, right continuous and increasing process $C$ with $C_0 = 0$, and an $S$-integrable predictable process $\varphi$ such that

$$X = X_0 + \varphi.S - C.$$  

Moreover, if $X$ is non-negative, then $(\varphi.S)M$ is a local martingale for all $M \in M$.

**Proof**

Let $\bar{S}_t = S_t(S^*_t)^{-1}$ and $\bar{X}_t = X_t(S^*_t)^{-1}$. Let $Q$ denote the set of all martingale measures for $\bar{S}$. Then $\bar{X}$ is a local $Q$-supermartingale. By Theorem 3.1 we have

$$\bar{X} = X_0 + \psi.\bar{S} - D,$$

where $D$ is an adapted, right continuous and increasing process with $D_0 = 0$. By Theorem 2.2 there exists a real-valued predictable process $\theta^*(t)$ such that $\{\theta^*(t)1_d + \psi(t)\}$ is a self-financing strategy with initial wealth $X_0$. Since $\sum_{i=1}^{d} S^*_i = \sum_{i=1}^{d} S^*_0$, we have $\theta^*(t)1_d.\bar{S} = 0$. Consequently,

$$X_0 + ((\theta^*1_d + \psi).S)_t = S^*_t(X_0 + ((\theta^*1_d + \psi).\bar{S})_t) = S^*_t(X_0 + (\psi.\bar{S})_t) = X_t + S^*_t D_t.$$  

Put $\varphi = (\theta^* - D_\pi)1_d + \psi$ and $C = S^*.D$, we get the desired decomposition.

3.2. Superhedging

By a contingent claim (or derivative) we mean a non-negative $\mathcal{F}_T$-measurable random variable. Let $\xi$ be a contingent claim. In general, one cannot find a self-financing strategy to perfectly replicate $\xi$. It is natural to raise the question: Does there exist an admissible strategy with the minimal initial value, called superhedging...
strategy, such that its terminal wealth is no smaller than the claim $\xi$. Here and henceforth, by an *admissible strategy* we mean a self-financing strategy with non-negative wealth process. For a market with diffusion model, this problem has been solved by El Karoui and Quenez (1995). For a general semimartingale model, it was solved by Kramkov (1996) and Föllmer and Kabanov (1998) using the optional decomposition theorem. The initial value of the superhedging strategy is called the *cost of superhedging* $\xi$. It can be considered as the “selling price” or “ask price” of $\xi$.

In a fair market setting, based on the corresponding result of Kramkov (1996), Xia and Yan (2002) proved the following result: if $\sup_{Q \in \mathcal{Q}} E_Q \left[ (S^*_T)^{-1} \xi \right] < \infty$, then the cost at time $t$ of superhedging the claim $\xi$ is given by

$$U_t = \text{esssup}_{Q \in \mathcal{Q}} S^*_t E_Q \left[ (S^*_T)^{-1} \xi \big| \mathcal{F}_t \right].$$

(3.1)

$U$ is the smallest non-negative $\mathcal{Q}$-supermartingale with $U_T \geq \xi$. In terms of martingale deflators, we can rewrite (3.1) as

$$U_t = \text{esssup}_{M \in \mathcal{M}} M^{-1}_t E \left[ M_T \xi \big| \mathcal{F}_t \right].$$

(3.2)

Using the optional decomposition theorem Föllmer & Leukert (2000) showed that the optional decomposition of a suitably modified claim gives a more realistic hedging (called *efficient hedging*) of a contingent claim. This result can be also reformulated in terms of martingale deflators.

### 3.3. Attainable claims and completeness of the market

Xia and Yan (2002) introduced the notions of regular and strongly regular strategies. We reformulate them in terms of martingale deflators.

**Definition 3.1** A self-financing strategy $\psi$ is said to be regular (resp. strongly regular), if for some (resp. for all) $M \in \mathcal{M}$, $W_t(\psi)M_t$ is a martingale. A contingent claim is said to be attainable if it can be replicated by a regular strategy.

By Theorem 3.2, one can easily deduce the following characterizations for attainable claims and complete markets.

**Theorem 3.3** Let $\xi$ be a contingent claim such that $\sup_{M \in \mathcal{M}} E \left[ \xi M_T \right] < \infty$. Then $\xi$ is attainable (resp. replicatable by a strongly regular strategy) if and only if the above supremum is attained by an $M^* \in \mathcal{M}$ (resp. $E[\xi M_T]$ doesn’t depend on $M \in \mathcal{M}$).

**Theorem 3.4** The market is complete if only if any contingent claim $\xi$ dominated by $S^*_T$ is attainable, or equivalently, $E[\xi M_T]$ doesn’t depend on $M \in \mathcal{M}$.

### 4. Portfolio optimization and contingent claim pricing

The portfolio optimization and contingent claim pricing and hedging are three major problems in mathematical finance. In a market where assets prices follow an exponential Lévy process, the portfolio optimization problem was studied in Kallsen...
In the general semimartingale model, for utility functions $U$ with effective domains $D(U) = \mathbb{R}_{+}$, the portfolio optimization problem was completely solved by Kramkov and Schachermayer (1999, 2001), henceforth K-S (1999, 2001). Bellini & Frittelli (2002) and Schachermayer (2002) studied the problem for utility functions $U$ with $D(U) = \mathbb{R}$. The relationship between portfolio optimization and contingent claim pricing was studied in Frittelli (2000) and Goll & Rüschendorf (2001), among others. In what concerning the problem of hedging contingent claims, we refer the reader to Schweizer (2001) for quadratic hedging, Föllmer & Leukert (2000) for efficient hedging, and Delbaen et al. (2001) for exponential hedging.

In this section, under our framework, we will present the main results of K-S (1999, 2001) and give a review of utility-based approach to contingent claim pricing.

### 4.1. Expected utility maximization

We consider an agent whose objective is to choose a trading strategy to maximize the expected utility from terminal wealth at time $T$. In the sequel, we only consider such a utility function $U : (0, \infty) \to \mathbb{R}$, which is strictly increasing, strictly concave, continuously differentiable and satisfies $\lim_{x \to 0} U'(x) = \infty$, $\lim_{x \to \infty} U'(x) = 0$. We denote by $I$ the inverse function of $U'$. The conjugate function $V$ of $U$ is defined as

$$V(y) = \sup_{x > 0} [U(x) - xy] = U(I(y)) - yI(y), \quad y > 0.$$  

For $x > 0$, we denote by $\mathcal{A}(x)$ the set of all admissible strategies $\theta$ with initial wealth $x$. For $x > 0$, $y > 0$, we put

$$\mathcal{X}(x) = \{ W(\theta) : \theta \in \mathcal{A}(x) \}, \quad \mathcal{X} = \mathcal{X}(1),$$

$$\mathcal{Y} = \{ Y \geq 0 : Y_0 = 1, \ X \text{ is a supermartingale } \forall X \in \mathcal{X} \}, \quad \mathcal{Y}(y) = y\mathcal{Y},$$

$$\mathcal{C}(x) = \{ g \in L^0(\Omega, \mathcal{F}_T, P) : 0 \leq g \leq X_T, \ \text{for some } X \in \mathcal{X}(x) \}, \quad \mathcal{C} = \mathcal{C}(1),$$

$$\mathcal{D}(y) = \{ h \in L^0(\Omega, \mathcal{F}_T, P) : 0 \leq h \leq Y_T, \ \text{for some } Y \in \mathcal{Y}(y) \}, \quad \mathcal{D} = \mathcal{D}(1).$$

The agent’s optimization problem is:

$$\widehat{\psi}(x) = \arg \max_{\psi \in \mathcal{A}(x)} E[U(W_T(\psi))].$$

To solve this problem we consider two optimization problems (I) and (II):

$$\widehat{X}(x) = \arg \max_{X \in \mathcal{X}(x)} E[U(X_T)]; \quad \widehat{Y}(y) = \arg \min_{Y \in \mathcal{Y}(y)} E[V(Y_T)].$$

Problem (II) is the dual of problem (I). Their value functions are

$$u(x) = \sup_{X \in \mathcal{X}(x)} E[U(X_T)], \quad v(y) = \inf_{Y \in \mathcal{Y}(y)} E[V(Y_T)].$$

The following theorem is the reformulation of the main results of K-S (1999, 2001) under our framework.
Theorem 4.1 Assume that there is a $\psi \in \mathcal{A}(1)$ such that $W_T(\psi) \geq K$ for a positive constant $K$ (e.g., $S_T^* \geq K$). If $v(y) < \infty, \forall y > 0$, then the value functions $u(x)$ and $v(y)$ are conjugate in the sense that

$$ v(y) = \sup_{x > 0} [u(x) - xy], \quad u(x) = \inf_{y > 0} [v(y) + xy], $$

and we have:

1. For any $x > 0$ and $y > 0$, both optimization problems (I) and (II) have unique solutions $\hat{X}(x)$ and $\hat{Y}(y)$, respectively.

2. If $y = u'(x)$, then $\hat{X}(x) = I(\hat{Y}(y))$ and the process $\hat{X}(x)\hat{Y}(y)$ is a martingale.

3. $v(y) = \inf_{M \in \mathcal{M}} E[V(yM_T)],$

**Proof** The proof is almost the same as that in K-S (1999, 2001). We indicate below main differences from K-S (1999, 2001). Obviously, $\mathcal{C}$ and $\mathcal{D}$ are convex sets. By Proposition 3.1 and a slight modification of Lemma 4.2 in K-S(1999), one can show that $\mathcal{C}$ and $\mathcal{D}$ are closed under the convergence in probability. For Items 1 and 2, as in Lemma 3.2 of K-S(1999) and Lemma 1 of K-S(2001), in order to prove the families $(V^- (h))_{h \in \mathcal{D}(y)}$ and $(U^+(g))_{g \in \mathcal{C}(x)}$ are uniformly integrable, we need to use a fact that $\mathcal{C}$ contains a positive constant. In our case, we have indeed $K \in \mathcal{C}$, since by assumption $K \leq W_T(\psi)$ for some $\psi \in \mathcal{A}(1)$. As for Item 3, according to Proposition 1 in K-S(2001) we only need to show $\hat{D} = \{M_T : M \in \mathcal{M}\}$ satisfies the following conditions:

- For any $g \in \mathcal{C}$, $\sup_{h \in \mathcal{D}} E[gh] = \sup_{h \in \mathcal{D}} E[gh]$
- $\hat{D} \subset \mathcal{D}$, $\hat{D}$ is convex and closed under countable convex combinations.

The first condition follows easily from (3.2), the second one is trivial.

4.2. Utility-based approach to contingent claim pricing

Assume that the market is fair. Let $\xi$ be a contingent claim such that $M_T\xi$ is integrable for some $M \in \mathcal{M}$. We put

$$ V_t = (M_t)^{-1} E [M_T \xi \mid \mathcal{F}_t]. $$

If we specify $(V_t)$ as the price process of an asset generated by $\xi$, then the market augmented with this derivative asset is still fair, because $M$ is still a martingale deflator for the augmented market. So we can define $(V_t)$ as a “fair price process” of $\xi$. This pricing rule is consistent with the original price processes of primitive assets. However, if the market is incomplete (i.e., the martingale deflator is not unique) we cannot, in general, define uniquely the fair price process of a contingent claim.

In deflated terms, pricing of contingent claims in an incomplete market consists in choosing a reasonable martingale measure. There are several approaches to make such a choice. A well-known one is the so-called “utility-based approach”. The basic idea of this approach is as follows. Assume that the representative agent in the market has preference represented by a utility function. In certain cases, the dual
optimization problem (II) may produce a so called minimax martingale measure (MMM for short).

Now under our framework we show how the expected utility maximization problem is linked by duality to a martingale deflator. Assume that the solution \( \hat{Y}(y) \) of the dual optimization problem (II) lies in \( y\mathcal{M} \). We put \( \hat{M}(y) = y^{-1}\hat{Y}(y) \). Then \( \hat{M}(y) \in \mathcal{M} \), and we have

\[
\hat{M}(y) = \arg \min_{M \in \mathcal{M}} E[V(yM_T)].
\]

We call \( \hat{M}(y) \) the minimax martingale deflator.

The following theorem gives a necessary and sufficient condition for the existence of the minimax martingale deflator.

**Theorem 4.2** Assume that there is a \( \psi \in \mathcal{A}(1) \) such that \( W_T(\psi) \geq K \) for a positive constant \( K \) (e.g., \( S^*_T \geq K \)), and that \( v(y) < \infty \) for all \( y > 0 \). Let \( x > 0 \) be the agent’s initial wealth and \( M^* \in \mathcal{M} \). In order that \( M^* \in \mathcal{M} \) is the minimax martingale deflator corresponding to the utility function \( U \) if and only if there exist \( y > 0 \) and \( X^* \in \mathcal{X}(x) \) such that \( X^*_T = I(yM^*_T) \) and \( E[M^*_T X^*_T] = x \). If it is the case, then \( X^* \) solves the optimization problem (I).

**Proof** We only need to prove the sufficiency of the condition. We have the following inequality

\[
U(I(z)) \geq U(w) + z[I(z) - w], \quad \forall w > 0, z > 0.
\]

If we replace \( z \) and \( w \) by \( yM^*_T \) and \( X^*_T \in \mathcal{X}(x) \) and take expectation w.r.t. \( P \), we get immediately that \( E[U(X^*_T)] \geq E[U(X_T)] \) for all \( X \in \mathcal{X}(x) \). This shows that \( X^* \) solves the optimization problem (I). On the other hand, since \( X^*_T = I(yM^*_T) \) and the assumption \( E[M^*_T X^*_T] = x \) implies that \( M^*X^* \) is a martingale, by Theorem 3.1, \( yM^* \) must solve the optimization problem (II). In particular, \( M^* \) is the minimax martingale deflator.

Now assume the minimax martingale deflator \( \hat{M}(y) \) exists. Let \( \xi \) be a contingent claim. If we use \( \hat{M}(y) \) to compute a fair price of \( \xi \) by (4.1), then it coincides with the fair price of Davis (1997), which is derived through the so-called “marginal rate of substitution” argument. In fact, the Davis’ fair price of \( \xi \) is defined by

\[
\hat{\pi}(\xi) = \frac{E[U'(\hat{X}_T(x))\xi]}{u'(x)}.
\]

Since \( y = u'(x) \) and \( U'(\hat{X}_T(x)) = \hat{Y}(y) \), we have \( \hat{\pi}(\xi) = E[\hat{M}_T(y)\xi] \).

Now we explain the economic meaning of Davis’ fair price of a contingent claim. Let \( \xi \) be a contingent claim with \( E[\hat{M}_T(y)\xi] < \infty \). Put \( \xi_t = (\hat{M}_t(y))^{-1}E[\hat{M}_T(y)\xi|\mathcal{F}_t] \).

We augment the market with derivative asset \( \xi \), and consider the portfolio maximization problem in the new market. Then it is easy to see that \( \hat{Y}(y) \) is still the solution of the dual optimization problem (II) in the new market. Consequently, the value function \( v \) and its conjugate function \( u \) remain unchanged. By Theorem 4.1, \( \hat{X}_T(x) \) solves again the optimization problem (I) in the new market. This shows that if the price of a contingent claim is defined by Davis’ fair price, no trade on
this contingent claim increases the maximal expected utility in comparison to an optimal trading strategy. This fact was observed in Goll and Rüschendorf (2001).

Note that in general the MMM (or minimax martingale deflator) depends on the agent’s initial wealth \( x \). This is a disadvantage of the utility-based approach to contingent claim pricing. However, for utility functions \( \ln x, \alpha \in (-\infty, 1) \setminus \{0\}, \alpha > 0 \), the MMM is independent of the agent’s initial wealth \( x \). This is due to the fact that the conjugate functions of the above utility functions are \( -\ln x - 1, -\frac{\alpha}{p} x^{\frac{\alpha}{p}}, -x + x \ln x \), respectively, and that \( E[dQ/dP] = 1 \) for any equivalent martingale measure \( Q \). Under our framework, the situation is a little different: for exponential utility function \( U(x) = -e^{-x} \), the minimax martingale deflator depends still on the agent’s initial wealth \( x \).

For \( U(x) = -e^{-x} \), the corresponding MMM is called the \textit{minimal entropy martingale measure}. We refer the reader to Frittelli (2000), Miyahara (2001) and Xia & Yan (2000) for studies on the subject. If \( U(x) = \ln x \), the minimax martingale deflator \( \hat{M} \), if it exists, is nothing but the reciprocal of the wealth process \( \hat{X}(1) \) of the growth optimal portfolio. Yan, Zhang & Zhang (2000) worked out explicit expressions for growth optimal portfolios in markets driven by a jump-diffusion-like process or by a Lévy process. See also Becherer (2001) for a study on the subject.

5. Concluding remarks

We have introduced a numeraire-free and original probability based framework for financial markets. This framework has the following advantages: Firstly, it permits us to formulate financial concepts and results in a numeraire-free fashion. Secondly, since the original probability models the “real world” probability, one can investigate the martingale deflators by statistical methods using market data. Thirdly, using martingale deflators to deal with problems of pricing and hedging as well as portfolio optimization is sometimes more convenient than the use of equivalent martingale measures. Lastly, our framework includes the traditional one with deflated terms as a particular case. In fact, if the price process of one primitive asset is the constant 1, our framework is reduced to the traditional one.

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