Small-signal Stability of Grid-tied Inverter Networks

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Abstract

This paper considers the small-signal stability of electrical networks composed dominantly of three-phase grid-following inverters. We identify a suitable time-scale decomposition for the inverter dynamics and, using singular perturbation theory, we obtain an analytic sufficient condition for the small-signal stability of the network. In contrast to the alternative of performing an eigenvalue analysis of the full-order network dynamics, our analytic sufficient condition has the benefit of reducing computational complexity and yielding insights on the role of network topology and constitution as well as inverter filter and control parameters on small-signal stability. Our numerical analysis for an inverter network with radial topology validates the approach and illustrates that, in a wide parametric regime, our analytic condition coincides with the exact stability threshold.

Key words: networks of inverters, dynamical system analysis, stability analysis

1 Introduction

Problem description and motivation The ongoing shift from fossil-fuel-driven synchronous generators to power-electronics-interfaced renewable energy is leading to changes in how power grids are modeled, analyzed, and controlled. While synchronous generators are generally rated at several hundreds of MVA and installed on the transmission backbone, power electronics inverters are distributed across both transmission and distribution subsystems and are generally much smaller in capacity. Moreover, synchronous generators have large rotating masses that absorb supply-demand fluctuations and limit frequency excursions during transients, whereas inverters typically have very different dynamics—attributable dominantly to their digital controllers—and they possess no moving parts. In summary, future grids will have a highly distributed architecture as inverters assume a more prominent role, and this will concomitantly necessitate the development of compatible models and analysis approaches to ensure stability and reliability.

Broadly speaking, there are two main modes of control for the inverters in power networks. The first is grid-forming mode, in which terminal voltages of the inverters are regulated with the understanding that the inverters dictate the voltage magnitude and frequency in the network. Prior works in the literature on grid-forming controls include design of distributed PI controllers for secondary frequency control and power sharing in inductive networks with radial topology [33,34] and meshed topology [31], stability analysis using the virtual oscillator model [8,35], model-order reduction using singular perturbations [21], and designing controllers for a parallel network of identical grid-forming inverters using passivity [36]. The second mode of control is grid-following, in which inverters inject currents while synchronized to
the voltage at their terminals which is assumed to be set externally, i.e., they act as voltage-following current sources. Essentially, grid-tied inverters are typically grid-following in nature, while inverters installed in islanded microgrid settings are typically grid-forming. In both cases, the underlying control architecture for inverters consists of several inner- and outer-feedback loops [18]. Several aspects of stability of grid-tied networks of inverters have been studied in the literature [12,19,20]. The fault analysis for networks of grid-following inverters in the distribution grid is studied in [22]. In [24], a suitable filter and controller design for network of grid-following inverters is proposed. The paper [26] introduce a suitable scaling method for analysis and design of networks of grid-following inverters. In [29], model-order reduction and small-signal stability of networks of grid-following inverters is studied.

Renewable energy resources such as solar and wind constitute a considerable portion of electricity generation in the US power grid, and it is expected that they will be the fastest-growing source of electricity generation for at least several years [1]. With synchronous generators presumably continuing to form the mainstay of electricity generation infrastructure for some time to come, there will be a dramatic increase in the penetration of inverters into the power grid. For instance, in Oahu, Hawaii, nearly 800,000 micro-inverters connected to photovoltaic panels have recently been added to the grid, producing as much power as the state’s largest conventional power plant, Konkar [10]. To facilitate analysis of complex networks with multiple grid-following inverters, we propose a framework that leverages singular perturbation methods to obtain an analytic, computationally light-weighted condition for small-signal stability. We focus on three-phase distribution networks, where the voltage at one node is set (with value externally determined based on the operation of the bulk power system), and a subset of the remaining nodes are connected to three-phase grid-following inverters. Our analytically-driven sufficient condition for small-signal stability in this setting clearly demarcates the role of the network (topology and constitution) and pertinent inverter dynamics (filter and controller parameters). Furthermore, it addresses the high dimensionality of the underlying dynamics (e.g., the inverter model we study has 13 dynamical states) which renders traditional numerical approaches to be unwieldy. Through numerical case studies, we demonstrate applications of the result to network design and real-time stability assessment. The inverter dynamical model that we examine is prototypical and similar to the inverter structures used in the literature. For instance in [22,23,24,25,26,28,29,37], an inverter consists of a current controller, a voltage controller, a power controllers, a PLL, and an LCL filter is studied. In the grid-following mode, the current controller of the inverter is active. In the grid-forming mode, however, the voltage controller is active applying the frequency and voltage droop controls.

Most of the prior results on stability of inverter-based systems have typically focused on simplified models that neglect inner control loops, which underpin fast dynamics [32] (see the discussions in [5]). Some exceptions are [2,40], where stability of full-order inverter models have been studied in parallel grid-connected networks. However, these studies are restricted to the parallel networks of inverters and are not applicable to networks with general topologies. In some cases, detailed inverter models have been considered in general networks, but the system stability has only been studied for small networks using numerical eigenvalue analysis. For instance, small-signal stability is analyzed in the literature using eigenvalue analysis for IEEE 37 bus system with 7 inverters in [28,29], for a radial network consists of 3 inverters in [23], and for a single-machine-single-inverter network in [17]. Understandably, while numerical eigenvalue analysis of the linearized network is indeed a reasonable strategy, this approach comes with significant drawbacks. First, the large size of the network combined with the high dimensionality of the inverter model are likely to emerge as a major computational bottleneck in system analysis. Furthermore, eigenvalue analysis does not reveal the role of critical network attributes on system stability, insights, which if formalized can facilitate analysis and design of microgrids. Addressing the limitations of the previous efforts above, our solution strategy yields an analytic condition for stability that is agnostic to system size and clearly brings out the role of the network topology and pertinent system parameters.

Finally, on a tangential note, it must be acknowledged that there is a wide body of work on stability assessment of synchronous generator and grid-forming inverter systems, and this includes approaches that have applied model-order reduction using singular perturbation analysis. For instance, in [6,11], a model-reduction approach based on singular perturbation is proposed to study stability and control of grid-forming inverters. In [21], singular perturbation is applied to obtain a hierarchy of reduced-order models for inverters in the grid-forming mode. In [29], a suitable time-scale decomposition for a class of inverters is identified and an iterative scheme for model order reduction is proposed. We refer the interested readers to [30] for a survey on singular perturbation methods and to [32] for a survey on application of singular perturbation in stability and control of inverters.

**Contribution** We make several contributions to the study of small-signal stability of grid-following inverter networks. First, we show that adopting a static model for inverters (where they are assumed to be fixed sources of active and reactive power) and neglecting fast dynamics induced by the inverter controllers, may lead to erroneous conclusions regarding network stability. This underscores the importance of acknowledging a full-order
model for stability analysis of large networks. Next, we uncover a correspondence between the equilibrium points of the inverter-network dynamics and the solutions of the algebraic power-flow equations. As the main contribution of this paper, we then find an analytic sufficient condition for small-signal stability. We start by presenting a dimensionless transcription of the inverter network dynamics which leads to the identification of a physically meaningful parametrization of the inverters. We show that reasonable assumptions on the parameter space will result in a time-scale decomposition of the system. Using singular perturbation analysis, we propose an analytic sufficient condition which guarantees small-signal stability over a given parametric regime. As a unique contribution, we emphasize that the dimensionless form of the network equations as well as the regularity of the singular perturbation problem (i.e., existence of isolated quasi-steady state manifolds) are critical steps in a rigorous time-scale analysis. Over this specified parametric regime, our analytic sufficient condition can also be interpreted as a lower bound on the stability threshold of the network and allows us to check system stability with minimal computational complexity. This approach also sheds light on the role of network topology, power injection/demand profiles, and inverter filter and control parameters on stability. Numerical simulations are used to substantiate our results for an illustrative network. Finally, we compare the small-signal stability analysis in this paper with existing approaches in literature. In power network literature, small-signal stability of systems is usually studied using eigenvalue analysis for the full-order model (e.g., see [23]) or for the reduced-order models (e.g., see the survey [32]). Comparing to the eigenvalue analysis for the full-order system, our sufficient condition not only reduces the computational complexity of the analysis but also provides insights about the role of network variables in small-signal stability. Moreover, while most of the existing approaches in the literature only provide frameworks or iterative schemes for model-reduction (see [29]), our analysis provides an explicit reduced-order model and an analytic sufficient condition for small-signal stability of the system. In the special case of resistive networks with purely active power injections, this sufficient condition provides a scalable and computationally efficient method to check the small-signal stability of the system.

Paper organization In Section 2, we present the dynamical model for a class of three-phase grid-following inverters. In Section 3, we derive an equivalent dimensionless description for a grid-tied network of inverters and loads and we study the equilibrium points of the system. In Section 4, we introduce a physically meaningful parametrization of the inverters and provide a sufficient condition for existence of a locally exponentially stable equilibrium point for a grid-tied network of inverters. Finally, in Section 5, we illustrate some applications of the theoretical results in network design and stability assessment.

Notation

Vectors and matrices We denote the set of real numbers by ℜ, the set of complex numbers by ℂ, the set of complex numbers with negative real part by ℂ−, the set of binary n-tuples by ℤn2, and the n-dimensional torus by ℌn. We define i = √−1. We identify the complex plane ℂ with the real plane ℜ2. For a complex number v = v1 + iv2 ∈ ℂ, the norm of v is |v| = √v12 + v22 and the argument of v, arg(v), is the angle between v and the positive imaginary axis. We denote the identity matrix of dimension n by In, the n-column vector of zeros with 0n, and the n-column vector of ones with 1n. For a matrix A ∈ ℂn×m, we denote ∞-norm of A by ‖A‖∞ = max{‖A‖1, ‖A‖2} = maxj∑ni=1 |aij|. For a matrix A, with real eigenvalues, we denote the maximum and minimum eigenvalues of A by λmax(A) and λmin(A), respectively. A real symmetric matrix A is positive definite if all its eigenvalues are positive. For two real symmetric matrices A, B ∈ ℜn×n we write A ≻ B if A − B is positive definite. A real square matrix A ∈ ℜn×n is Metzler if all its off-diagonal entries are non-negative. For two square matrices A ∈ ℜn×n and B ∈ ℜm×m, the tensor product of A and B is denoted by A ⊗ B. For a vector x ∈ ℂn, we denote diag(x) by [x].

From n-complex variables to 2n-real variables While it is more conventional to describe the power flow equations in polar form, in this paper we focus on the Cartesian form of these equations. The reason is that the polar form of the power flow equations contains complex conjugation, which if appears in differential equation models can cause some regularity issues. In this section, we restrict our treatment to vectors and operators on the complex plane ℂ and the real plane ℜ2. These structures can be extended to the n-dimensional complex and real spaces using block diagonal matrices. For every complex Z = X + iY ∈ ℂ, the associated real variable in the real plane is denoted by z = (x, y)′ ∈ ℜ2. We define the matrix J ∈ ℜ2×2 by J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, the matrix H ∈ ℜ2×2 by H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, the rotation matrix by the angle θ ∈ ℜ1 by R(θ) = \begin{pmatrix} \cos(θ) & \sin(θ) \\ -\sin(θ) & \cos(θ) \end{pmatrix}. Let u ∈ ℜ2. We define two matrix-valued operators D : ℜ2 → ℜ2×2 and D′ : ℜ2 → ℜ2×2 by

D(u) = \begin{pmatrix} u_1 & u_2 \\ u_2 & -u_1 \end{pmatrix}, \quad D′(u) = \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix}.
Using $D$ and $D'$, for given average power $v$ and current $i$, one can represent the apparent power as $s = D(v)i = D'(i)v$. Let $V \in \mathbb{C}^n$ and $v \in \mathbb{R}^n$ be the associated real vector, then we define $\|v\|_\infty = \|V\|_\infty$.

**Algebraic graph theory** We denote an undirected weighted graph by a triple $G = (N, E, A)$, where $N = \{1, 2, \ldots, n\}$ is the set of nodes and $E \subseteq N \times N$ is the set of edges. The matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix. For every node $i \in V$, the degree of the node is given by $d_i = \sum_{j=1}^{n} a_{ij}$. For a fixed orientation on $G$, the incidence matrix of the graph $G$ is denoted by $B \in \mathbb{R}^{n \times m}$. The Laplacian for the graph $G$ is defined by $L = D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

**Power systems** We assume that the voltage of the grid is given by the time-varying function $v_g(t) = V_g \exp(i\theta_g(t))$, where $V_g$ is the amplitude of the grid voltage, $\theta_g(t)$ is the phase of the grid voltage, and $\omega_g(t)$ is the angular frequency. For a grid-tied network of inverters, one can define two different rotating frames. The first frame, which is usually referred as the *global* DQ-frame, is the rotating frame which is attached to the bulk grid’s voltage and rotates with the grid’s frequency $\omega_g(t)$. The second frame, which is usually referred as the *local* dq-frame, is attached to each individual inverter’s terminal voltage vector. For a three-phase quantity $x$, we denote the dq-frame representation of $x$ by $x_{dq} = (x_d, x_q)^\top$ and the DQ-frame representation of $x$ by $x_{DQ} = (x_D, x_Q)^\top$. These two representations are related as follows: $x_{dq} = R(\delta)x_{DQ}$, where $\delta$ is the angle between the dq-frame and the DQ-frame.

### 2 Model of a decoupled inverter

In this section, we briefly overview the dynamics of the class of grid-following 3-phase inverters examined in this work. For a detailed description of the model, we refer interested readers to [26,28]. The model captures all relevant AC-side dynamics, and is composed of a: i) phase-locked loop (PLL), ii) power controller, iii) current controller, and iv) LC output filter. An illustrative block diagram is given in Fig. 1. The PLL consists of a low-pass filter with cut-off frequency $\omega_c$, a PI controller with PI gains $k_p$ and $k_i$, and a PI controller, respectively. The dynamics of the PLL are given by:

$$\dot{v}_{PLL} = \omega_c v_{PLL} - v_{PLL},$$

$$\dot{\phi}_{PLL} = -v_{PLL},$$

$$\delta = -k_p v_{PLL} + 2\pi f_{PLL} \phi_{PLL},$$

where $v_{PLL}$ and $\phi_{PLL}$ denote states of the low-pass filter and PI controller, respectively. $\delta$ is the output of the PI controller, and $v_{od}$ is the $d$-component of the output voltage of the inverter. The frequency of the PLL loop is defined as $\omega_{PLL} = \omega_c(t) + \delta$, where $\omega_c(t)$ is the grid frequency. The power controller consists of two low-pass filters with cut-off frequency $\omega_c$ and two PI controllers with gains $k_p$ and $k_i$. The pertinent dynamics are given by:

$$\dot{s}_{avg} = \omega_c(s - s_{avg}),$$

$$\gamma_{ldq} = k_p H(s - s_{avg}) + k_i \int H(s - s_{avg})dt,$$

where $s_{avg} = [p_{avg}, q_{avg}]^\top$ collects the states of the low-pass filters, $\gamma_{ldq}$ capture the outputs of the PI controllers (these are the references for the current controller), $s^* = [p^*, q^*]_t$ collects the active- and reactive-power reference set points, and $s = [p, q]^\top$ is the vector collecting instantaneous active- and reactive-power outputs (measured at the point of common coupling):

$$s = \frac{3}{2} D(v_{odq}) \dot{v}_{odq} = \frac{3}{2} \left( v_{od} \dot{v}_{od} + v_{oq} \dot{v}_{oq} \right).$$

The current controller consists of two PI controllers with gains $k_p$ and $k_i$, with outputs to be the references for the inverter voltage at the switching terminals $v_{idq}$:

$$\dot{v}_{idq} = k_p \left( \dot{v}_{idq} - i_{dq} \right) + k_i \int (\dot{v}_{idq} - i_{dq}) dt \quad (3)$$

Since the switching period is typically much shorter than the filter and controller time constants, we assume that $v_{idq} = v_{id}$. The dynamics of the LC filter are given by:

$$\dot{i}_{ldq} = \frac{1}{L_f} (v_{idq} - v_{odq}) - \omega_{PLL}(\dot{J}_{ldq})$$

$$\dot{v}_{odq} = \frac{1}{C_f} (v_{idq} - i_{odq}) - \omega_{PLL}(\dot{J}_{odq}).$$

Finally, we introduce two new variables

$$\phi_s := \int (s^* - s_{avg}) dt, \quad \gamma_{dq} := \int (\dot{v}_{idq} - i_{ldq}) dt,$$

that will aid in subsequent developments. The inner- and outer-loop control architecture examined here is ubiquitous, see, e.g., [22,23,24,25,26,28,29,37], where similar models are utilized.

So far, we presented the governing dynamics for each component of a grid-following inverter in the local dq-frame. Since the goal of this paper is to study a network of inverters, we write the dynamical system of the inverters in the global DQ-frame. To this end, we introduce the DQ-variables:

$$s_{DQ} = R(\delta)s_{dq}, \quad i_{DQ} = R(\delta)i_{ldq}, \quad v_{oDQ} = R(\delta)v_{odq}.$$
For every vector $y$ with time-varying entries, the time derivatives in the global DQ-frame and local dq-frame are related by

$$
\dot{y}_{DQ} - JR(-\delta)\dot{y}_{DQ} = R(-\delta)\dot{y}_{dq}.
$$

Leveraging this identity, we can write the dynamical model for the grid-following inverter described previously in the global DQ-frame as below:

$$
\dot{x} = f(x) + g(x)\dot{y}_{DQ} + Cs^*,
$$

where $x = (v_{PLL}, \delta, \phi_s, s_{avg}, \gamma_{DQ}, \dot{i}_{DQ}, v_{DQ})^T \in \mathbb{R}^{13}$ captures states of the inverter in the global DQ reference frame, $s^* = (p^*, q^*)^T \in \mathbb{R}^2$ captures the references for active and reactive power, $f : \mathbb{R}^{13} \rightarrow \mathbb{R}^{13}$ is the drift vector field, and $g : \mathbb{R}^{13} \rightarrow \mathbb{R}^{13 \times 2}$, and $C \in \mathbb{R}^{13 \times 2}$ are control vector fields. The mappings $f$, $g$ and matrix $C$ follow from the dynamics governing the current controller, the power controller, the PLL, and the LC filter outlined previously.

### 3 Grid-tied Network of Inverters

In this section, we derive the dynamical system model governing the grid-tied network of inverters and loads and study the equilibrium points of the system. We model the network using an undirected, connected, complex-weighted graph $G$ with node set (buses) $N$, edge set (branches) $E \subseteq N \times N$, and the symmetric matrix-valued edge weights (admittances) $a_{kj} = a_{jk} = (R_k I_2 + \omega_L L_{kj} J)^{-1}$, for every $(k, j) \in E$. Suppose $B$ is the incidence matrix of $G$. Associated to the matrix-weighted graph, $G$, we define the nodal admittance matrix by $Y = (B \otimes I_2)A(B \otimes I_2)^T \in \mathbb{R}^{2|N| \times 2|N|}$, where

$$
A \in \mathbb{R}^{2|E| \times 2|E|} \text{ is given by } A = \text{blkdiag}(a_{jk}).
$$

There are three types of nodes in the network: we have one grid bus with time-varying voltage $v_g(t) = V_g \exp(i\theta_g(t))$ denoted by 0. The time-varying frequency of the grid is given by $\omega_g(t) = \dot{\theta}_g(t)$. The time-varying nature of the grid frequency is due to the changes in the bulk power grid generations as well as the disturbances in the system. We assume that the controllers on the transmission side of the grid impose the following constraints on the frequency of the grid:

$$
\omega_L \leq \omega_g(t) \leq \omega_u \lim_{t \to \infty} \omega_g(t) = \omega_u^*.
$$

where $\omega_L$ and $\omega_u$ are the admissible lower and upper bounds for grid frequency and $\omega_u^*$ is the nominal frequency of the grid (e.g., 60 Hz in US). Moreover, we assume that the grid voltage can be measured at all the times (for example using a PMU) and this information is communicated with all the inverters in the network. We have $n \geq 1$ inverter buses collected in the set $N_I$, and $m$ load buses collected in the set $N_L$. Without loss of generality, we assume that $N_I = \{1, \ldots, n\}$ and $N_L = \{n+1, \ldots, n+m\}$ such that $N = \{0\} \cup N_I \cup N_L$. The partition $N = \{0\} \cup N_I \cup N_L$ induces the following partitions for $Y$:

$$
Y = \begin{bmatrix}
Y_{00} & Y_{01} & Y_{0L} \\
Y_{10} & Y_{11} & Y_{1L} \\
Y_{L0} & Y_{L1} & Y_{LL}
\end{bmatrix}.
$$

The complex power injection (demand) at inverter $k \in N_I$ (load $k \in N_L$) is denoted $s_k = p_k + iq_k \in C$. We also establish the following convention: for a given variable (parameter) $y$ corresponding to the inverter, we define vector $y = (y_1, \ldots, y_n)^T$, where $y_k$ is the associated variable (parameter) for the $k$th inverter.
Inverter model. Using (5), the governing dynamics for all inverters in the network are given by:

\[ \dot{x} = F(x) + G(x)i_{DQ} + Cs^*, \]

where, in \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^{13n} \), \( x_k \) captures all the dynamic states for the \( k \)th inverter, \( F(x) = (f_1^T(x_1), \ldots, f_n^T(x_n))^T \), \( G(x) = \text{diag} (g_1(x_1), \ldots, g_n(x_n)) \), and \( C = \text{diag} (C_1, \ldots, C_n) \), where \( f_k \) is the drift vector field and \( g_k \) and \( C_k \) are control vector fields of inverter \( k \).

Load and line models. Let \( v_k \) be the \( k \)th load voltage and \( i_k \) be the current demand of the \( k \)th load. We collect the nodal voltages of the loads in vector \( v_L = (v_{n+1}, \ldots, v_{n+m})^T \) and the current demands of the loads in vector \( i_L = (i_{n+1}, \ldots, i_{n+m})^T \). In this paper, we consider the constant impedance model for loads. Suppose that \( R_L \in \mathbb{R}^m \) is the vector of resistances of the loads and \( L_L \in \mathbb{R}^m \) is the vector of inductances of the loads. Then the load dynamics are captured by

\[ ([L_L] \otimes I_2)i_L = (-[R_L] \otimes I_2 - \omega_\theta[L_L] \otimes J)i_L + v_L. \]

Suppose that the vector of line resistances and line inductances are denoted by \( R_\xi \in \mathbb{R}^{2|\xi|} \) and \( L_\xi \in \mathbb{R}^{2|\xi|} \), respectively and we denote the nodal current injections by \( i = (i_g, -i_L, i_{DQ})^T \) and nodal voltages by \( v = (v_g, v_L, v_{DQ})^T \). Then the governing dynamics for the transmission lines are [32, Equation 4.10]:

\[ ([L_\xi] \otimes I_2)\xi_{DQ} = (-[R_\xi] \otimes I_2 - \omega_\theta[R_\xi] \otimes J)\xi_{DQ} + (B^T \otimes I_2)v, \]

where \( \xi_{DQ} \in \mathbb{R}^{2|\xi|} \) is the vector of current flows in the lines. Then we have \( i = (B \otimes I_2)\xi_{DQ} \). We will assume that the line and load characteristics are such that they can be modeled with the dynamics ignored, i.e., in steady state. This follows either for purely resistive lines (and loads) or lines for which the \( R/X \) ratios are high: both of which are valid assumptions for distribution networks that we examine in this work [16,32]. With this, the relationship between nodal voltages and current injections is captured by

\[ i = Y^T v, \]

\[ i_L = [z_L]^{-1}v_L, \]

where \([z_L] = [R_L] \otimes I_2 + \omega_\theta[L_L] \otimes J\).

3.1 Model for the Dynamical System of Grid-tied Network of Inverters

The differential equations governing the network are (6), (7), and (8). Define

\[ Y_{red} := Y_{11} - Y_{IL} (Y_{LL} + [z_L]^{-1})^{-1} Y_{LI}, \]

\[ Y_{0g} := Y_{10} - Y_{IL} (Y_{LL} + [z_L]^{-1})^{-1} Y_{LO}, \]

\[ w := -Y_{red}^{-1}Y_{0g}. \]

Then, using Kron reduction [9], one can show

\[ i_{DQ} = Y_{red}v_{DQ} + Y_{0g}v_g. \]

Combining (6) and (10), the grid-tied inverter-network dynamics are:

\[ \dot{x} = F(x) + G(x)i_{DQ} + Cs^*, \]

where \( i_{DQ} = Y_{red}v_{DQ} + Y_{0g}v_g \).

Remark 1 For a purely resistive network, similar to (9), one can define real-valued \( L_{red}, L_{0g} \), and \( u = L_{red}^{-1}L_{0g} \) such that:

\[ Y_{red} = L_{red} \otimes I_2, \quad Y_{0g} = L_{0g} \otimes I_2, \quad w = u \otimes I_2. \]

Dimensionless transcription. In this section, we transcribe the differential equations (11) in a dimensionless format. For each inverter, we denote the nominal power injection by \( s_{\text{nom}} \), following which, we introduce the dimensionless variables:

\[ \bar{V}_{\text{PLL}} := \frac{v_{PLL}}{V_g}, \quad \bar{\phi}_{\text{PLL}} := \frac{k_{p,\text{PLL}}}{V_g} \phi_{\text{PLL}}, \quad \bar{s}_{\text{avg}} := \frac{s_{\text{avg}}}{s_{\text{nom}}}, \]

\[ \bar{\phi}_s := \frac{V_g k_i \phi_s}{s_{\text{nom}}}, \quad \bar{s}^* := \frac{s^*}{s_{\text{nom}}}, \quad \bar{\gamma}_{\text{DQ}} := \frac{k_i \gamma_{\text{DQ}}}{V_g}, \]

\[ \bar{\omega}_{\text{DQ}} := \frac{V_g \omega_{\text{DQ}}}{s_{\text{nom}}}, \quad \bar{i}_{\text{DQ}} := \frac{i_{\text{DQ}}}{V_g}, \quad \bar{v}_{\text{DQ}} := \frac{V_{\text{ref}}}{s_{\text{nom}}}. \]

We also isolate different time-constants of different system components as follows:

(i) The time constants corresponding to the low pass filter of the PLLs are \( \tau_{P,\text{PLL}} = \omega_{\text{PLL}}^{-1} \) and \( \tau'_{P,\text{PLL}} = (V_g k_{p,\text{PLL}})^{-1} \);

(ii) The integral time of the PI controller in the PLL is \( T_{\text{PLL}} = k_{p,\text{PLL}} \omega_{\text{PLL}}^{-1} \);

(iii) The time constant corresponding to the low-pass filter of the power controller is \( \tau_s = \omega_s^{-1} \);

(iv) The time constant for steady-state tracking in the power controller is \( \tau' = (V_s k_i)^{-1} \);

(v) The integral time of the PI controller in the power controller is \( T_s = \frac{k_p}{k_i} \).
The time constants for the LC filter are \( \tau_L = V_c^2(k_L s_{\text{nom}})^{-1} \).

The integral time of the PI controller in the current controller is \( T_s = \frac{k_p}{k_i} \).

The time constants for the LC filter are \( \tau_L = L_i s_{\text{nom}} V_c^{-2} \) and \( \tau_c = C_i V_c^2(s_{\text{nom}})^{-1} \).

Finally, we define the dimensionless network parameters:

\[
\hat{Y}_{\text{red}} := Y_g^2 s_{\text{nom}}^{-1} Y_{\text{red}}, \quad \hat{Y}_o := Y_g^2 s_{\text{nom}}^{-1} Y_{\text{o}}.
\]

Then, the dimensionless grid-tied inverter-network dynamics are:

\[
\begin{align*}
\hat{v}_{\text{PLL}} &= [\tau_{\text{PLL}}]^{-1}(\hat{v}_{\text{od}} - \hat{v}_{\text{PLL}}) , \\
\hat{\phi}_{\text{PLL}} &= -[T_{\text{PLL}}]^{-1}\hat{\phi}_{\text{PLL}} , \\
\hat{\delta} &= [\tau_{\text{PLL}}]^{-1}(\hat{\phi}_{\text{PLL}} - \hat{v}_{\text{PLL}}) , \\
\hat{s}_{\text{avg}} &= [\tau_s \otimes I_2]^{-1}(\hat{s} - \hat{s}_{\text{avg}}) , \\
\hat{\phi}_s &= [\tau_s \otimes I_2]^{-1}(\hat{s} - \hat{s}_{\text{avg}}) , \\
\hat{\gamma}_D &= [\tau_c \otimes I_2]^{-1}(\hat{I}_{\text{DQ}} - \hat{s}_{\text{avg}}) = \hat{\gamma}_D , \\
\hat{\gamma}_D &= \hat{\gamma}_D , \\
\hat{\gamma}_D &= \hat{\gamma}_D , \\
\hat{\gamma}_D &= \hat{\gamma}_D , \\
\hat{\gamma}_D &= \hat{\gamma}_D.
\end{align*}
\]

3.2 Reference-tracking trajectories of the Dimensionless Grid-tied Inverter-network Dynamics

We show that the reference-tracking trajectories of the dimensionless grid-tied inverter network dynamics (12) are in correspondence with the solutions of the following power-flow equations:

\[
\frac{3}{2} \mathcal{D}(\hat{v}_{\text{oDQ}}) \hat{I}_{\text{oDQ}} = \hat{s}^* , \\
\hat{I}_{\text{oDQ}} = \hat{Y}_{\text{red}} \hat{v}_{\text{oDQ}} + \hat{Y}_o.
\]

Lemma 2 For a given reference-power injection \( \hat{s}^* \) to the inverters, the following statements are equivalent:

(i) \( (\hat{v}_{\text{oDQ}}^*, \hat{I}_{\text{oDQ}}^*)^\top \in \mathbb{R}^{3n} \) is a solution for the power-flow equations (13);

(ii) for every \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_2^n \), the \( (\hat{x}_\alpha^*, \hat{I}_{\text{oDQ}}^*)^\top \) is a reference-tracking trajectory of the dimensionless grid-tied inverter-network dynamics (12), where \( \hat{x}_\alpha^* \in \mathbb{R}^{13n} \) is given by

\[
\hat{x}_\alpha^* = (\hat{v}_{2n}, \alpha^* \varpi, (-1)^{\alpha_1^*} \hat{\phi}_1^*, \hat{s}^*, \hat{\gamma}_{\text{DQ}}, \hat{\gamma}_{\text{DQ}}, \hat{v}_{\text{oDQ}}^*)^\top ,
\]

with

\[
\hat{\delta}^* = -\arg(\hat{v}_{\text{oDQ}}^*) , \\
\hat{\phi}_s = R(\hat{\delta}^*) \hat{H}_{\text{DQ}}^* , \\
\hat{\gamma}_{\text{DQ}}^* = \hat{v}_{\text{oDQ}}^*.
\]

Proof. Regarding (ii) \( \implies \) (i), from the power-controllers’ dynamics in (12), it is easy to see that if \( (\hat{x}_\alpha^*, \hat{I}_{\text{oDQ}}^*)^\top \) is a reference-tracking trajectory, then we have \( \hat{s} = \hat{s}^* \). This implies that \( (\hat{v}_{\text{oDQ}}^*, \hat{I}_{\text{oDQ}}^*)^\top \) should satisfy the power-flow equations (13).

Regarding (i) \( \implies \) (ii), suppose that \( (\hat{v}_{\text{oDQ}}^*, \hat{I}_{\text{oDQ}}^*)^\top \) is a solution to the power flow equations (13). Then from the PLL dynamics in (12), we have \( \hat{v}_{\text{PLL}} = 0_n \) and \( \hat{v}_{\text{od}} = 0_n \). Note that, \( \hat{v}_{\text{od}} = 0_n \), can be written in the trigonometric form

\[
v_{\text{oDQ}}^* \cos(\delta_{k^*}) + v_{\text{oDQ}} \sin(\delta_{k^*}) = 0.
\]

This implies that, for every \( k \in \{1, ..., n\} \), there exists \( \alpha_k \in \mathbb{Z}_2 \) such that \( \delta_{k^*} = -\arg(v_{\text{oDQ}}) + \alpha_k \pi \). From the power-controller dynamics in (12), we have \( \hat{s}_{\text{avg}} = \hat{s}^* \).

Finally, one can find the value of \( \hat{v}_{\text{oDQ}}, \hat{I}_{\text{oDQ}} \), and \( \hat{\gamma}_{\text{DQ}} \) by solving the remaining equations in (12). 

Remark 3 (i) The power-flow equations (13) have been studied extensively in the literature and many sufficient conditions for existence and uniqueness of solutions have been developed; see, e.g., [3,7,38].

Lemma 17 in Appendix C restates a result from [38] pertaining to uniqueness that is leveraged in subsequent results.

(ii) If the frequency of the grid is constant and equal to the nominal grid frequency (i.e., \( \omega(t) = \omega_k^* \)), then \( (\hat{x}_\alpha^*, \hat{I}_{\text{oDQ}}^*)^\top \) given in Lemma 2(ii) is an equilibrium point of the dimensionless grid-tied inverter-network dynamics (12).
4 Stability Analysis of the Dimensionless Grid-tied Inverter-network Dynamics

In bulk power-systems dynamics literature, it is commonplace to assume that the dynamics of the grid-following inverters are much faster than the dynamics of grid-forming inverters and synchronous machines [32]. This assumption justifies the use of a static model for grid-following inverters such that the inverter nodes are considered to be sources of constant (active and reactive) power. Subsequently, the network operation is described by the following power-flow equations:

\[ 0_2 = \hat{s}^* - \frac{3}{2} D(\hat{v}_{oDQ})\hat{\nu}_{oDQ}. \]  

Quite obviously, the static representation (14) does not capture any notion of stability. The following example shows that the internal dynamics of the inverters can induce instabilities, even if the power-flow equations in (14) admit a solution.

Example 4 (Instabilities Induced by Inverter Dynamics) Consider the radial grid-connected network consisting of 30 identical inverters (a sketch is provided in Fig. 2). Suppose the inverters have uniform reference power injections \( p = p \in \mathbb{R}^{30} \), each line has the resistance \( R = 10^{-2} \Omega \) and inductance \( L = 10^{-5} \) H, and the grid voltage is \( v_g = i(120\sqrt{2}) \) V(peak) with constant frequency \( \omega_k = 60 \) Hz.

(i) Static model for inverters: One can surmise that if the inverter power injections satisfy

\[ \hat{p} \parallel_{c, \infty} \hat{v}_{\text{red}}^{-1} \leq \frac{3}{8}, \]  

then, there exists a unique high-voltage, low-current solution for the power-flow equations (14) (see Lemma 17).

(ii) Dynamic model for inverters: We use the dynamic model (5) for the inverters (parameters of the inverters are given in the fourth column of Table A.2). Then, the governing equations for the network are the ones in (12), and condition (15) in Lemma 2 guarantees the existence of a family of equilibrium points \( (\hat{x}^*_0, \hat{r}^*_{oDQ})^T \). Linearizing the system (12), we study local stability of the equilibrium point \( (\hat{x}^*_0, \hat{r}^*_{oDQ})^T \).

Figure 3 plots the maximum real part of the eigenvalues of the linearized system (12) around the equilibrium point \( (\hat{x}^*_0, \hat{r}^*_{oDQ})^T \) as a function of active-power injection. The red vertical line is the threshold of the power injection for which the power-flow equations admit a unique solution (obtained from (15)). Notice that there are power injections for which a solution of the power-flow equations exists, however, the corresponding equilibrium point \( (\hat{x}^*_0, \hat{r}^*_{oDQ})^T \) is not stable.

Fig. 3. Shaded region indicates power injections that admit a unique high-voltage non-stressed power flow solution that are, however, not small-signal stable.

Having established the importance of acknowledging inverter dynamics in stability assessment, we now proceed to outline the main result of this paper.

4.1 Small-signal stability via time-scale separation

In this section, we focus on the small-signal stability of the dimensionless grid-tied inverter-network dynamics (12). Due to high-dimensionality and nonlinearity of the dynamic model, studying small-signal stability is not analytically tractable. Therefore, it is a reasonable goal to reduce the model order. To that end, we identify a physically meaningful parametrization of the inverters, and we show that under suitable assumptions, this parametrization leads to a time-scale decomposition of the system (12) which simplifies analysis. We begin by parametrization of time constants of different components of the system.

Definition 5 (Singular perturbation parameter.)

For the dimensionless grid-tied inverter-network dynamics (12), we define the singular perturbation parameter

\[ \epsilon := \max\{\|\tau_{PLL}\|_{\infty}, \|\tau'_{PLL}\|_{\infty}, \|\tau_{s}\|_{\infty}, \|T_{s}\|_{\infty}, \|T_{PLL}\|_{\infty}, \|\tau'_{LC}\|_{\infty}, \|\tau_{LC}\|_{\infty}, \|\tau'_{LC}\|_{\infty}, \|T_{c}\|_{\infty} \}. \]  

Remark 6 (i) Definition 5 establishes a physically meaningful time-scale separation of the components of an inverter when \( \epsilon \ll 1 \). In this case,
\[
\|z^*\|_\infty, \|z_{\text{LC}}\|_\infty, \|z'_{\text{LC}}\|_\infty \leq \varepsilon^2 \text{ which means that the current controller and the LC filter of the inverter are the fastest components of the inverter. Moreover, } \|z_{\text{PLL}}\|_\infty, \|z'_{\text{PLL}}\|_\infty, \|z_s\|_\infty \leq \varepsilon \text{ which means that the PLL and averaging-part of the power controller (i.e., } \tilde{s}_{\text{avg}}) \text{ are slower than the current controller and the LC filter but they are faster than the steady-state power-tracking controller (i.e., } \tilde{\phi}_s) \text{ dynamics.}
\]

(ii) The assumption \( \varepsilon \ll 1 \) is realistic in practice. For instance, it holds by taking \( \varepsilon = 0.1 \) for the parameters in Table A.2 for our work, and it also applies for the parameters used in papers [23,28,29].

(iii) For a different set of parameters and control architectures, one can conceivably identify a different singular perturbation parameter and time-scale decomposition. However, we expect the general nature of the stability result that follows to be similar.

**Theorem 7 (Small-signal Stability)** Consider the dimensionless grid-tied inverter-network dynamics (12) with states in \( \mathbb{R}^{13n} \) and reference power injections \( \tilde{s}^* \in \mathbb{R}^{2n} \). The following hold:

(i) if

\[
\left\| D'(\tilde{w})\tilde{y}_{\text{red}}^{-1}(D'(\tilde{w}))^{-1}D'(\tilde{s}^*) \right\|_{C,\infty} \leq \frac{3}{2} \tilde{s},
\]

then, there is a unique solution \( (\tilde{v}_{\text{ODQ}}^*, \tilde{\eta}_{\text{ODQ}}^*)^T \) for the power-flow equations (13) and family of reference-tracking trajectories \( (\tilde{x}_{\tilde{s}}, \tilde{\eta}_{\text{ODQ}}^*)^T, \alpha \in \mathbb{Z}_2^1, \) for the grid-tied inverter network dynamics (12) satisfying:

\[
\|\tilde{v}_{\text{ODQ}}^* - \tilde{w}\|_{C,\infty} \leq \frac{1}{2} \|\tilde{w}\|_{C,\infty};
\]

(ii) if additionally, we have \( |T_{\text{PLL}}| > |\tau_{\text{PLL}}| \) and the \( 2n \times 2n \) matrix

\[
M := -D(\tilde{v}_{\text{ODQ}}^*)+D'(\tilde{\eta}_{\text{ODQ}}^*)\tilde{y}_{\text{red}}^{-1})R(-\delta^*)H[\tau_s^* \otimes I_2]^{-1}
\]

is Hurwitz, then, there exists an \( \varepsilon^* > 0 \) such that, for every \( \varepsilon \leq \varepsilon^* \), the reference-tracking trajectory \( (\tilde{x}_{\tilde{s}}, \tilde{\eta}_{\text{ODQ}}^*) \) is locally exponentially stable.

**PROOF.** Regarding part (i), the proof follows from combining Lemma 17 and Lemma 2. Regarding part (ii), consider the dimensionless grid-tied inverter network dynamics (12). Using (16), and the change of variables \( \tilde{\eta}_s = R(-\delta)H\tilde{\phi}_s \) and defining the variable \( \Delta x \in \mathbb{R}^{13n} \) by \( \Delta x = x - x^*_s \), we get a three-time-scale decomposition of the system. We define the states \( z_1 \in \mathbb{R}^{6n}, z_2 \in \mathbb{R}^{6n} \), and \( y \in \mathbb{R}^{2n} \) as follows:

\[
\begin{align*}
z_1 &= \left( \Delta \hat{\gamma}_{\text{ODQ}} \Delta \hat{\delta}_{\text{ODQ}} \Delta \hat{\nu}_{\text{ODQ}} \right)^T, \\
z_2 &= \left( \Delta \hat{\nu}_{\text{PLL}} \Delta \hat{\phi}_{\text{PLL}} \Delta \delta \Delta \hat{\nu}_{\text{avg}} \right)^T, \\
y &= \Delta \hat{\eta}_s,
\end{align*}
\]

where \( z_1 \) is faster than \( z_2 \) and \( z_2 \) is faster than \( y \). The corresponding time-scales are given by \( \tau = \frac{\tau_1}{\varepsilon} \) and \( \tau' = \frac{\tau_2}{\varepsilon} \). The quasi-steady-state manifold for the time-scale \( \tau \) is given by:

\[
\begin{align*}
\Delta \hat{\gamma}_{\text{ODQ}} &= \hat{Y}_{\text{red}}^{-1} \Delta \hat{\eta}_s, \\
\Delta \hat{\delta}_{\text{ODQ}} &= \Delta \hat{\eta}_s, \\
\Delta \hat{\nu}_{\text{ODQ}} &= \hat{Y}_{\text{red}}^{-1} \Delta \hat{\eta}_s,
\end{align*}
\]

Since the quasi-steady state manifold is an isolated manifold, the singular perturbation problem is in standard form and therefore, the boundary-layer dynamics are:

\[
\begin{align*}
\frac{d\Delta \hat{\gamma}_{\text{ODQ}}}{d\tau} &= -\varepsilon^2 |\tau_c \otimes I_2|^{-1} \Delta \hat{\delta}_{\text{ODQ}}, \\
\frac{d\Delta \hat{\delta}_{\text{ODQ}}}{d\tau} &= \varepsilon^2 |\tau_{\text{LC}} \otimes I_2|^{-1} \left( \Delta \hat{\gamma}_{\text{ODQ}} - \Delta \hat{\nu}_{\text{ODQ}} \right) - |\tau_c \otimes I_2|^{-1} [T_c \otimes I_2] \Delta \hat{\delta}_{\text{ODQ}}, \quad (19a) \\
\frac{d\Delta \hat{\nu}_{\text{ODQ}}}{d\tau} &= \varepsilon^2 |\tau_{\text{LC}} \otimes I_2|^{-1} \left( \Delta \hat{\delta}_{\text{ODQ}} - \Delta \hat{\nu}_{\text{ODQ}} \right) - |\tau_c \otimes I_2|^{-1} [T_c \otimes I_2] \Delta \hat{\nu}_{\text{ODQ}}. \quad (19c)
\end{align*}
\]

We first show that for the boundary-layer dynamics (19a), the origin is the exponentially stable equilibrium point. Note that the boundary-layer dynamics (19a) are linear and can be expressed in the form \( \dot{z}_1 = (N \otimes I_2)z_1 \), where \( N \in \mathbb{R}^{6n \times 6n} \) is given by

\[
N = \begin{pmatrix}
0_{2n \times 2n} & -\varepsilon^2 |\tau_c|^{-1} & 0_{2n \times 2n} \\
\varepsilon^2 |\tau_{\text{LC}}|^{-1} & -\varepsilon^2 |\tau_{\text{LC}}|^{-1} |T_c|^{-1} & -\varepsilon^2 |\tau_{\text{LC}}|^{-1} \\
0_{2n \times 2n} & \varepsilon^2 |\tau_{\text{LC}}|^{-1} & -\varepsilon^2 |\tau_{\text{LC}}|^{-1} \hat{Y}_{\text{red}}
\end{pmatrix}
\]

Since \( -(\hat{Y}_{\text{red}}^T + \hat{Y}_{\text{red}}) \) is negative definite, using Lemma 16, matrix \( N \), and therefore matrix \( N \otimes I_2 \) is Hurwitz. Hence, the origin is the locally exponentially stable point of the boundary-layer dynamics (19a). Similarly, one gets the quasi-steady-state manifold for time-scale \( \tau' \) as follows:

\[
\begin{align*}
\Delta \hat{\nu}_{\text{PLL}} &= 0_n, \\
\Delta \hat{\phi}_{\text{PLL}} &= 0_n, \\
\Delta \delta &= \arg(\hat{Y}_{\text{red}}^{-1} \hat{\eta}_s) - \arg(\hat{v}_{\text{ODQ}}), \\
\Delta \hat{\nu}_{\text{avg}} &= \frac{3}{2} D'(\Delta \hat{\eta}_s) \left( \hat{Y}_{\text{red}}^{-1} \Delta \hat{\eta}_s \right) + \frac{3}{2} D'(\hat{\eta}_s) \left( \hat{Y}_{\text{red}}^{-1} \Delta \hat{\eta}_s \right), \\
&+ \frac{3}{2} D'(\Delta \hat{\eta}_s) \left( \hat{Y}_{\text{red}}^{-1} \hat{\eta}_s^* \right).
\end{align*}
\]
Since the quasi-steady state manifold is an isolated manifold, the singular perturbation problem is in standard form and the corresponding boundary-layer dynamics are:

\[
\frac{d\Delta \hat{v}_{PLL}}{dt} = \epsilon [\tau_{PLL}]^{-1} (\Delta \hat{v}_{od} - \Delta \hat{v}_{PLL}), \quad (20a)
\]

\[
\frac{d\Delta \hat{\phi}_{PLL}}{dt} = -\epsilon [\tau_{PLL}]^{-1} \Delta \hat{v}_{PLL}, \quad (20b)
\]

\[
\frac{d\Delta \delta}{dt} = \epsilon [\tau'_{PLL}]^{-1} (\Delta \hat{\phi}_{PLL} - \Delta \hat{v}_{PLL}), \quad (20c)
\]

\[
\frac{d\Delta s_{avg}}{dt} = -\epsilon [\tau_s \otimes I_2]^{-1} \Delta s_{avg}. \quad (20d)
\]

We show that the boundary-layer dynamics (20) are stable around the origin. Note that the linearized boundary-layer equation dynamics around the origin have the form \(\Delta z_2 = S \Delta z_2\), where \(S \in \mathbb{R}^{5n \times 5n}\) has the upper block triangular form \(S = \begin{pmatrix} S_{11} & S_{12} \\ 0_{2n \times 3n} & S_{22} \end{pmatrix}\) with:

\[
S_{11} = \begin{pmatrix} -\epsilon [\tau_{PLL}]^{-1} & 0_{n \times n} & \epsilon [\tau_{PLL}]^{-1} \hat{v}_{on}^* \\ -\epsilon [\tau'_{PLL}]^{-1} & 0_{n \times n} & 0_{n \times n} \end{pmatrix},
\]

\[
S_{22} = -\epsilon [\tau_s \otimes I_2]^{-1}.
\]

Since \([T_{PLL}] > [\tau_{PLL}]\), by Lemma 16, matrix \(S\) is Hurwitz. Therefore, the origin is the locally exponentially stable point of the boundary-layer equations (20). Now, consider the reduced-order dynamics. Noting the fact that \(\hat{\phi}_{PLL} = \hat{v}_{PLL}\) on the quasi-steady-state manifold, we get

\[
\Delta \hat{n}_s = R(-\delta)H[\tau_s' \otimes I_2]^{-1} \left( \hat{s}^* - \frac{3}{2} D' (\eta_s) (\hat{Y}_{red}^{-1} \eta_s + \hat{w}) \right),
\]

where \(\delta = \delta(\hat{n}_s) = \arg \left( \hat{Y}_{red}^{-1} \hat{n}_s + \hat{w} \right)\). Linearizing this equation, we get \(\Delta \hat{y} = M' \Delta \hat{y}\), where \(M'\) is given by

\[
M' := -\frac{3}{2} R(-\delta^*)H[\tau_s' \otimes I_2]^{-1} \left( D(\hat{v}^*_{oDQ}) + D'(\hat{i}^*_{oDQ}) \hat{Y}_{red}^{-1} \right).
\]

Remark 8 (1) As it is well-known in the singular perturbation analysis, the largest value of \(c^*\) for which the statement of Theorem 7(ii) holds is hard to compute. However, for the case when the frequency of the grid is constant, a standard lower bound on \(c^*\) can be obtained using techniques in [15, Lemma 2.2].

(2) The statements in Theorem 7 can be interpreted as follows: If conditions (i) and (ii) hold, then one can tune the parameter \(c\) of the inverters to guarantee local exponential stability of the reference-tracking trajectory \((\hat{x}_{oDQ}^*, \hat{i}_{oDQ}^*)\).

(3) If the grid frequency is constant, then \((\hat{x}_{oDQ}^*, \hat{i}_{oDQ}^*)\) is an equilibrium point for the full-order dimensionless grid-tied inverter network dynamics (12) and one can use Theorem 7(ii) to guarantee the local exponential stability of this equilibrium point for small enough \(c\).

(4) While the sufficient condition in Theorem 7(ii) still requires checking that a 2n-dimensional matrix is Hurwitz, our approach eliminates the dynamics of the current controller, PLL, and LC filter from the small-signal stability analysis. As a result, it reduces the computational complexity by order \(1/6\). In fact, checking small-signal stability of the full-order dynamics (12) requires linearizing the system and finding eigenvalues of a 13n-dimensional linearized matrix.

(5) By part (i), there exists a family of equilibrium points \((\hat{x}_{oDQ}^*, \hat{i}_{oDQ}^*)\) for the grid-tied inverter network dynamics (12). However, in part (ii), we focus on the equilibrium point \((\hat{x}_{oDQ}^*, \hat{i}_{oDQ}^*)\). The reason is that this equilibrium point is locally exponentially stable when there is no power injection for the inverters (see Corollary 10).

(6) Theorem 7(ii) brings out the role of network topology, the power injections/demand structure, and inverter parameters in small-signal stability of the networks of grid-following inverters. In the expression for the matrix \(M\), the term \(\hat{Y}_{red}\) reveals the role of network topology. The terms \(\hat{i}_{oDQ}^*, \hat{v}_{oDQ}^*\) and \(\delta^*\) (which can be obtained by solving the power flow equations (13)) reveal the role of network topology as well as power injection/demand structure of the network. Finally, the term \(\tau_s^*\) reveals the role of inverters’ internal dynamics.

(7) In the proof of Theorem 7, for every time-scale, we provide an explicit form for the quasi-steady state manifold of the system. This is a crucial in singular perturbation analysis since the stability analysis is feasible only if the quasi-steady state manifold is isolated.

4.2 Corollaries

We now present a suite of corollaries that may be applicable in different settings and shed more light on to
Using simple algebraic manipulations, we get
\[ \det(M) \geq \left| \frac{V_g^2}{R^2 + \omega_g^2 L^2} \right|. \]  
\[ (21) \]
then the following statements hold:

(i) there exist two reference-tracking trajectories
\( (\hat{x}_0, \hat{i}_{DQ}) \) and \( (\hat{x}_1, \hat{i}_{DQ}) \) satisfying
\[ \|\hat{v}_{DQ} - (\hat{v}_0)\|_{c, \infty} \leq \frac{1}{2}; \]

(ii) if we have \( |T_{PLL}| > |\tau_{PLL}| \), then there exists a \( \epsilon > 0 \) such that, for every \( \epsilon \leq \epsilon^* \), \( (\hat{x}_0, \hat{i}_{DQ}) \) is locally exponentially stable.

\textbf{PROOF.} Regarding part (i), we define the dimensionless resistance \( \hat{R} = V_g^{-2}s_{nom} R \) and dimensionless inductance \( \hat{L} = V_g^{-2}s_{nom} L \). Therefore, we get
\[ \hat{v}_{DQ} = \left( \begin{array}{c} \hat{R} - \omega_g \hat{L} \\ \omega_g \hat{L} \end{array} \right). \]
Then, (21) is equivalent to (17) and the result follows from Theorem 7(i). Regarding part (ii), note that we have \( \hat{v}_{DQ} = (\hat{R}_{L} + \omega_g \hat{L}) \hat{I}_{DQ} = (\hat{v}_0) \). Therefore, matrix \( M \) in (18) has the following form:
\[ M = - (D(\hat{v}_{DQ}) + D'(\hat{i}_{DQ})\hat{v}_{red}) = (-\delta^*)H(\tau_s \otimes I_2)^{-1} \]
\[ = - \left( \begin{array}{c} 2\hat{R}_{i_{DQ}} \\ 1 + 2\hat{R}_{i_{DQ}} \end{array} \right) R(-\delta^*)H(\tau_s)^{-1}. \]
Using simple algebraic manipulations, we get
\[ \det(M) = (2\hat{v}_{DQ} - 1)(\tau_s)^{-1} \] and \( \text{tr}(M) = -2(\hat{v}_{DQ})(\tau_s)^{-1}. \) Since \( \|\hat{v}_{DQ} - (\hat{v}_0)\|_2 \leq \frac{1}{2}, \) we have \( 2\hat{v}_{DQ} \geq 1. \) This implies that \( \det(M) \geq 0. \) Moreover, for the reference-tracking trajectories \( (\hat{x}_0, \hat{i}_{DQ}) \), we have \( \hat{v}_{DQ} > 0. \) This implies that \( \text{tr}(M) < 0 \) and \( M \) is Hurwitz. Thus, by Theorem 7(ii), \( (\hat{x}_0, \hat{i}_{DQ}) \) is locally exponentially stable.

In microgrids with high penetrations of solar energy generation, it is usually not possible to guarantee a steady flow of power at all times. This is due to the fact that the energy generation in solar panels depends on the amount of sunlight they receive. In particular, at night when there is no sunlight, the solar panels would not generate any power. Therefore, it is important to study the special case where we have no power injection for the inverters.

\textbf{Corollary 10 (No power injection from inverters)} Consider the dynamics (12) with zero reference power injections. Then, there exists a family of equilibrium points \( (\hat{x}_n, 0_{2n}) \). Moreover, if \( |T_{PLL}| > |\tau_{PLL}| \), then the reference-tracking trajectory \( (\hat{x}_n, 0_{2n}) \) of the system (12) is locally exponentially stable.

\textbf{PROOF.} Since \( \hat{s}^* = 0_n \), we get \( \hat{i}_{DQ}^* = 0_n \). Also, by Theorem 7(i), there exists a family of reference-tracking trajectories \( (\hat{x}_n, 0_{2n}) \) for the grid-tied inverter network dynamics (12) such that
\[ \|\hat{v}_{DQ} - \hat{\tilde{w}}\|_{c, \infty} \leq \frac{1}{2}\|\hat{\tilde{w}}\|_{c, \infty}. \]
This implies that, for every \( k \in \{1, \ldots, n\} \), we have \( \hat{\tilde{v}}_{DQ} > 0. \) Moreover, matrix \( M \) in (18) can be written as:
\[ M = \frac{(D(\hat{v}_{DQ}^*) + D'(\hat{i}_{DQ})\hat{v}_{red})}{R(-\delta^*H[\tau_s \otimes I_2]^{-1} \]
\[ = \frac{D(\hat{v}_{DQ}^*)R(-\delta^*)H[\tau_s \otimes I_2]^{-1}}{-[(\hat{v}_{DQ}^*) \otimes I_2][\tau_s \otimes I_2]^{-1}, \]
where the second equality is using the fact that \( \hat{i}_{DQ} = 0_n \) and the third equality is using the fact that \( D(\hat{v}_{DQ}^*)R(-\delta^*) = D(\hat{v}_{DQ}^*) \). Since the matrix \( -[(\hat{v}_{DQ}^*) \otimes I_2] \) is always diagonal and Hurwitz, and \( [\tau_s \otimes I_2] \) is diagonal with positive diagonal elements, the result follows from Theorem 7(ii).

\textbf{Corollary 11 (Resistive network of inverters)} Consider the dynamics (12). Suppose the lines and loads are purely resistive and the reference power injections and demands are purely active. The following statements hold:

(i) if
\[ \|\hat{\tilde{u}}\hat{\tilde{v}}_{red}^{-1}\hat{\tilde{u}}^{-1}[\hat{\tilde{p}}^*]\|_{c, \infty} \leq \frac{3}{8}, \]
where \( \hat{\tilde{u}} = \hat{\tilde{v}}_{red}^{-1}\hat{\tilde{v}}_g \), then there exists a family of reference-tracking trajectories \( (\hat{x}_n, \hat{i}_{DQ}) \) for the dynamical system (12) with the property that
\[ \|\hat{v}_{DQ} - \hat{\tilde{w}}\|_{c, \infty} \leq \frac{1}{2}\|\hat{\tilde{w}}\|_{c, \infty}. \]

(ii) if additionally we have \( |T_{PLL}| > |\tau_{PLL}| \) and the Metzler matrix
\[ N := -[(\hat{v}_{DQ}^*)^{-1} + (\hat{v}_{DQ}^*)^{-1}] \]
(22)
is Hurwitz then there exists a $\epsilon^* > 0$ such that, for every $\epsilon \leq \epsilon^*$, the reference-tracking trajectory $(\hat{x}_0, \hat{\mathbf{i}}_{\text{DQ}}^*)$ is locally exponentially stable.

PROOF. Regarding part (i), since the network is resistive and power injections/demands are purely active, we know that

$$\hat{Y}_\text{red}^{-1} = \hat{L}_\text{red}^{-1} \otimes I_2, \quad \hat{\mathbf{s}}^* = \hat{\mathbf{p}}^* \otimes (1) \quad \hat{\mathbf{w}} = \hat{\mathbf{u}} \otimes I_2.$$

Therefore:

$$\|D'(\hat{\mathbf{w}})\hat{Y}_\text{red}^{-1}(D'(\hat{\mathbf{w}}))^{-1}D'(\hat{\mathbf{s}}^*)\|_{C, \infty} = \|\hat{\mathbf{u}}\hat{L}_\text{red}^{-1}[\hat{\mathbf{u}}^{-1}[\hat{\mathbf{p}}^*]]\|_{C, \infty}.$$

The result then follows from Theorem 7(i). Regarding part (ii), since there are no reactive-power injections from the inverters, we have $\hat{\mathbf{i}}_{\text{DQ}}^* = \mathbf{0}_n$. Since the input voltage for the PLL and power controller is the output voltage of the LC filter (i.e., $\hat{\mathbf{v}}_{\text{AD}}^*$), and the network is purely resistive, we have $\hat{\mathbf{v}}_{\text{AD}}^* = \mathbf{0}_n$ and as a result $\hat{\mathbf{d}}^* = \mathbf{0}_n$.

Alternatively, this observation can be proved rigorously as follows. Since the network is purely resistive, then the admittance matrix $Y_{\text{red}}$, and the vector $\mathbf{w}$ have the form given in Remark 1. Moreover, when the power injections are purely active, the power injection vector $\mathbf{s}^*$ has the form $\mathbf{s}^* = \mathbf{p}^* \otimes (1)$. Therefore, starting from the initial condition $\mathbf{v}^{(0)} = \hat{\mathbf{w}} = \hat{\mathbf{u}} \otimes (1)$, the $k$th iteration in Lemma 17(ii) has the form $\mathbf{v}^{(k)} = \mathbf{u}^{(k)} \otimes (1)$, for every integer $k \in \mathbb{Z}_0^+$. This implies that, in the limit, we have $\hat{\mathbf{v}}_{\text{AD}}^* = \mathbf{0}_n$ and as a result $\hat{\mathbf{d}}^* = \mathbf{0}_n$. Therefore, the matrix $\hat{M}$ in condition (18) simplifies as shown below:

$$\hat{M} = -\left((\hat{\mathbf{v}}_{\text{oq}}^* \otimes I_2 - \hat{\mathbf{i}}_{\text{oq}}^* \hat{L}_\text{red}^{-1} \otimes I_2)[\tau_s^* \otimes I_2]\right)^{-1}$$

$$= -[\hat{\mathbf{v}}_{\text{oq}}^*][\tau_s^*]^{-1} \otimes I_2 - \hat{\mathbf{i}}_{\text{oq}}^* \hat{L}_\text{red}[\hat{\mathbf{v}}_{\text{oq}}^*]^{-1} \otimes I_2.$$

Using Lemma 15, the matrix $\hat{M}$ is Hurwitz if and only if the matrices

$$-\left((\hat{\mathbf{v}}_{\text{oq}}^* \pm \hat{\mathbf{i}}_{\text{oq}}^* \hat{L}_\text{red}^{-1})[\tau_s^*]^{-1}\right)$$

are Hurwitz. Note that the active-power injections from the inverters to the grid are non-negative in steady state. This implies that $\hat{\mathbf{i}}_{\text{oq}}^* \geq \mathbf{0}_n$. Thus, the matrices (23) are similar to the following matrices:

$$-[\tau_s^*]^{-1/2}[\hat{\mathbf{v}}_{\text{oq}}^*][\tau_s^*]^{-1/2} \pm [\tau_s^*]^{-1/2}[\hat{\mathbf{i}}_{\text{oq}}^*]^{-1/2} \hat{L}_\text{red}[\hat{\mathbf{i}}_{\text{oq}}^*]^{-1/2}[\tau_s^*]^{-1/2}.$$

Since the matrix $[\hat{\mathbf{v}}_{\text{oq}}^*]$ is positive definite and the matrix $[\hat{\mathbf{i}}_{\text{oq}}^*]^{-1/2} \hat{L}_\text{red}[\hat{\mathbf{i}}_{\text{oq}}^*]^{-1/2}$ is positive semidefinite, the matrix

$$-[\tau_s^*]^{-1/2}[\hat{\mathbf{v}}_{\text{oq}}^*][\tau_s^*]^{-1/2} \pm [\tau_s^*]^{-1/2}[\hat{\mathbf{i}}_{\text{oq}}^*]^{-1/2} \hat{L}_\text{red}[\hat{\mathbf{i}}_{\text{oq}}^*]^{-1/2}[\tau_s^*]^{-1/2}$$

is Hurwitz. Moreover, note that the matrix $L_{\text{red}}$ is a grounded Laplacian matrix and by $[4, E. 9.10]$ its inverse $L_{\text{red}}^{-1}$ is non-negative. Also, the matrices $[\tau_s^*], [\hat{\mathbf{i}}_{\text{oq}}^*]$, and $[\hat{\mathbf{v}}_{\text{oq}}^*]$ are all diagonal with non-negative diagonal elements. This implies that the matrix $-\hat{\mathbf{v}}_{\text{oq}}^*[\tau_s^*]^{-1} + \hat{\mathbf{i}}_{\text{oq}}^* \hat{L}_\text{red}[\tau_s^*]^{-1}$ has non-negative off-diagonal elements and therefore, it is Metzler. The proof of Corollary (11) then follows from Theorem 7.

□

Remark 12 (Computational complexity) Using Corollary 11 for small-signal stability of resistive networks with purely active power injections, it is sufficient to show that the matrix $N$ given in (22) is Hurwitz. It is worth mentioning that, for Metzler matrices, there are computationally efficient methods for checking Hurwitzness. In particular, one can reformulate the Hurwitzness of (22) as the following linear programming:

\begin{align*}
\text{minimize} & \quad 0^T \xi, \\
\text{subject to} & \quad (\mathcal{N}_i) \xi < 0, \quad \forall i \{1, \ldots, n\} \\
& \quad \xi_i > 0, \quad \forall i \{1, \ldots, n\}. \quad (24)
\end{align*}

A solution to the the linear programming problem (24) can be found using distributed methods, where each node runs an algorithm involving only local variables and information exchange with its neighbors [27]. As a result, one can check the Hurwitzness of $N$ using the linear programming (24) in $O(n)$ time where $m$ is the number of nonzero elements in $N$ [27].

5 Numerical Simulations

In this section, we present numerical simulation results for radial network (see Fig. 2) with identical inverters and reference-power setpoints with the goal of answering the following questions:

- Over what range of values for $\epsilon$ does Theorem 7(ii) guarantee small-signal stability?
- How do the changes in other parameters of the inverters affect this range?
- How conservative is the condition (18) in Theorem 7(ii) to analyze small-signal stability?

We will see that the first and the second questions can be interpreted as network design problems, and the third question can be interpreted as a network monitoring problem. Solutions to these problems are provided in Sections 5.1 and 5.2, respectively. Our test case is a family of radial networks $\{\mathcal{G}(n)\}_{n \in \mathbb{N}}$, where $\mathcal{G}(n)$ is the weighted undirected connected graph with the node

1 All the numerical simulations are carried in MATLAB R2016a performed on a computer with Intel Core i5 processor @ 1.6 GHZ CPU and 4 GB RAM.
set (buses) \( N_0 = \{0, \ldots, n\} \), the edge set (branches) \( E_n = \{(i, i+1) \mid i = 0, \ldots, n-1\} \), and the matrix-valued weights (admittances) \( a_{ik} = a_{kj} = (RI_2 + \omega_c LJ)^{-1} \), for every \((j, k) \in E_n\), where the line resistance is \( R = 0.2 \Omega \) and line inductance is \( L = 2 \cdot 10^{-5} \text{ H} \). We assume that the grid voltage \( v_g = i(120\sqrt{2}) \text{ V (peak)} \), and frequency \( \omega_g = 120\pi \text{ rad/s} \). For every radial network \( G(n) \), we assume that node 0 corresponds to the grid bus (see Fig. 2), nodes \( \{1, \ldots, n\} \) are the inverters with \( s_{\text{nom}} = 1000 \text{ VA} \), and

\[
\tau_{\text{PLL}} = \epsilon^2, \quad \tau_{\text{PLL}}' = \frac{\epsilon^2}{V_g}, \quad T_{\text{PLL}} = \epsilon, \quad T_s = \epsilon
\]

\[
\tau_s' = (0.1)V_g, \quad T_s = 10\epsilon, \quad \tau_c = \frac{V_g}{s_{\text{nom}}}\epsilon^2, \quad T_c = \epsilon^2
\]

\[
\tau_{\text{LC}} = V_g^{-2}s_{\text{nom}}\epsilon^2, \quad \tau_{\text{LC}}' = V_g^{-2}s_{\text{nom}}^{-1}\epsilon^2.
\]

We first define the notion of stability margin.

**Definition 13 (Stability margin)** Given a family of networks \( \{G(n)\}_{n \in \mathbb{N}} \) and uniform reference power injection \( s^* \in \mathbb{C} \), the Stability Margin (SM) of \( \{G(n)\}_{n \in \mathbb{N}} \) is the maximum \( N \in \mathbb{N} \) such that the dimensionless grid-tied inverter network dynamics (12) with underlying graph \( G(N) \) and the reference power injection \( s^* = s^*_{\text{nom}} \) has a locally stable equilibrium point.

**Remark 14** (1) For networks with large penetration of renewable energy units, the notion of stability margin is of practical importance. In particular, this notion can be used to study the robustness of the grid with respect to penetration of inverters.

(2) For our test case network, SM of the network depends on the parameter \( \epsilon \);

(3) One can use the matrix \( M \) defined in (18) to estimate the SM of the network; from Theorem 7(ii), there exists \( \epsilon^* > 0 \) such that, for every \( \epsilon \leq \epsilon^* \), SM of the network is larger than this estimate.

### 5.1 Designing Grid-tied Inverter Networks

Considering the problem of designing a network of inverters, it is crucial to examine the efficiency of the our analytic results for variable operating conditions. In this part, we study the problem of finding SM for the radial network shown in Fig 2 and we numerically investigate 1) the largest range of \( \epsilon \) for which Theorem 7(ii) holds 2) the effect of other parameters of the system on this range. In order to carry out the these task, we first study the effect of parameter \( \epsilon \) for different active power injections on the SM of the system. The result is shown in Fig. 4.

Note that the dashed line in Figure 4 are the estimates of SM based on the sufficient condition in Theorem 7(ii). Therefore, according to the data in Figure 4, the largest domains of the parameter \( \epsilon \) for which Theorem 7(ii) holds are \((0, 0.005], (0, 0.004], \) and \((0, 0.0035] \), for the active power injections \( p^* = 1000 \text{ W} \), \( p^* = 1500 \text{ W} \), and \( p^* = 2000 \text{ W} \), respectively. It is interesting that, in these intervals, the dashed line and the solid lines coincide with curves defined in (18) to estimate the SM of the network.

![Fig. 4. Stability margins for different active power injections. The dashed lines show the estimates of SM computed using Hurwitzness of matrix \( M \).](image)

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![Fig. 5. Stability margins for different power factors. The dashed lines show the estimates of SM computed using Hurwitzness of matrix \( M \).](image)

Finally, we numerically investigate the effect of changes
in other parameters of the network on the range of \( \epsilon \) for which Theorem 7(ii) holds. The experiment setup is as follows. Given a time-constant \( \tau \) in the dimensionless grid-tied inverter-network dynamics (12), we scale it using a scaling factor \( \kappa \) and study the effect of \( \kappa \) on the largest \( \epsilon^\star \) for which Theorem 7(ii) holds. For the radial network with 10 inverters (see Fig.2), the results are shown in Figure 6 for the time-constants \( \tau_{LC}, \tau_{LC}^\prime, \tau_t, T_{PLL} \). It is interesting to note that, starting from \( \kappa = 1 \), increasing the scaling factor will decreases \( \epsilon^\star \), for every time-constant. For the \( LC \) filter time-constants \( \tau_{LC}, \tau_{LC}^\prime \), decreasing \( \kappa \) will not decrease the value of \( \epsilon^\star \). However, for the current controller time-constant \( \tau_t \) and for the \( PLL \) time-constants \( T_{PLL} \), decreasing the scaling factor will decrease \( \epsilon^\star \).

![Figure 6](image)

Fig. 6. Effect of parameter scaling on largest \( \epsilon^\star \) in Theorem 7(ii) for the radial network (see Fig 2) with 10 inverters.

5.2 Monitoring Grid-tied Inverter Networks

In this section, we study the conservativeness and computational efficiency of condition (18) in Theorem 7(ii) for monitoring the small-signal stability of the network. For our test case, we pick \( \epsilon = 0.001 \) and therefore based on Section 5.1, we are in the range of applicability of Theorem 7(ii). Assume that we have uniform reference active-power injections \( \hat{\mathbf{s}}^\star = \hat{\mathbf{P}}_{lin} \) for the network. We denote the threshold of the reference active-power injection for which the full-order linearized system is stable by \( \hat{\mathbf{P}}_{lin} \) and the threshold of the reference active-power injection for which matrix \( M \) in (18) is Hurwitz by \( \hat{\mathbf{P}}_{test} \). Similarly, we denote the computational time for checking the small-signal stability using the linearization test by \( T_{lin} \) and the computation time for checking the Hurwitzness of matrix \( M \) in (18) by \( T_{test} \). Table 1 compares these thresholds and computation times for the test case mentioned above with different number of nodes. From Table 1, it is clear that condition (18) gives accurate estimates for threshold of stability and the corresponding computation times are approximately \( \frac{1}{5} \) of the computation time to perform a full-order eigenvalue analysis of the linearized system.

| Number of inverters | \( T_{test} \) (s) | \( T_{lin} \) (s) | \( \hat{P}_{test} \) | \( \hat{P}_{lin} \) |
|---------------------|-------------------|-----------------|-----------------|-----------------|
| 60                  | 26.14             | 132.94          | 1.68            | 1.69            |
| 70                  | 27.38             | 136.92          | 1.24            | 1.24            |
| 80                  | 26.90             | 137.25          | 0.95            | 0.96            |
| 90                  | 25.14             | 130.17          | 0.76            | 0.76            |
| 100                 | 16.46             | 109.91          | 0.61            | 0.62            |
| 110                 | 21.65             | 144.65          | 0.51            | 0.51            |
| 120                 | 21.05             | 146.61          | 0.43            | 0.43            |

Table 1

Comparing the computation time and accuracy of condition (18) for small-signal stability. The unit of the quantities \( T_{test} \) and \( T_{lin} \) is second and the quantities \( \hat{P}_{test} \) and \( \hat{P}_{lin} \) are dimensionless.

6 Conclusion

We studied small-signal stability of grid-tied networks of grid-following inverters and loads. Using a time-scale analysis and by suitable choice of a family of parameters for the inverters, we presented an analytic sufficient condition for local exponential stability. We showed that, compared to the eigenvalue analysis for small-signal stability, this sufficient condition has the advantages of reducing the computational complexity of the problem. Finally, using numerical simulations in MATLAB, we studied the efficiency and range of applicability of our sufficient condition. As the use of inverters are becoming more and more wide-spread in the power grids, it is of growing interest to understand how the interaction among the grid-following inverters can affect the stability and reliability of power networks. We believe that our results in this paper is a first step toward a rigorous and comprehensive analysis of small-signal stability in low-inertia networks consist of large number of grid-following inverters.

A Table of variables and parameters

Table (A.1) collects the variables and their symbols for grid-following inverter model and table (A.2) collects the parameter values for this class of inverters in the literature.
Lemma 15 Let \( A, B \in \mathbb{C}^{n \times n}, C \in \mathbb{C}^{m \times m} \) and \( \eta_1, \ldots, \eta_m \) be eigenvalues of the matrix \( C \). Then the following conditions are equivalent:

(i) \( \lambda \) is an eigenvalue of \( A \otimes I_m + B \otimes C \);

(ii) \( \lambda \) is an eigenvalue of \( A + \eta_k B \) for some \( k \in \{1, \ldots, m\} \).

**PROOF.** Using Schur’s Theorem [13, Theorem 2.3.1], there exist a unitary matrix \( U \in \mathbb{C}^{m \times m} \) and an upper triangular matrix \( \tilde{H} \) with eigenvalues of \( H \) on its diagonal such that \( C = U H U^{-1} \). Note that the matrix \( A \otimes I_m + B \otimes C \) is similar to the matrix

\[
(I_n \otimes U)(A \otimes I_m + B \otimes C)(I_n \otimes U^{-1}) = A \otimes I_m + B \otimes H,
\]

where for the last equality we used the identity \((X \otimes Y)(Z \otimes W) = (XZ \otimes YW)\). Matrix \( A \otimes I_m + B \otimes H \) has the following block form:

\[
A \otimes I_m + B \otimes H = \begin{pmatrix}
A + \eta_1 B & * & \cdots & * \\
0_{n \times n} & A + \eta_2 B & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0_{n \times n} & 0_{n \times n} & \cdots & A + \eta_m B
\end{pmatrix}.
\]

Thus, \( \lambda \) is an eigenvalue of \( A \otimes I_m + B \otimes C \) if and only if it is an eigenvalue of \( A + \eta_k B \), for some \( k \in \{1, \ldots, n\} \).

Lemma 16 Let \( \Gamma, \Pi, \Xi, \Sigma \in \mathbb{R}^{n \times n} \) be diagonal matrices with positive diagonal entries such that \( \Gamma \succ \Sigma \) and \( N \in \mathbb{R}^{n \times n} \) be such that \( N + N^T \) is a negative definite matrix. Then the following matrices are Hurwitz:

\[
A = \begin{pmatrix}
-\Gamma & 0_{n \times n} & \Sigma \\
-\Sigma & 0_{n \times n} & 0_{n \times n} \\
-\Xi & \Xi & 0_{n \times n}
\end{pmatrix}, \quad B = \begin{pmatrix}
0_{n \times n} & -\Gamma & 0_{n \times n} \\
\Xi & -\Sigma & -\Xi \\
0_{n \times n} & \Pi & \Pi N
\end{pmatrix}.
\]

**PROOF.** Note that the characteristic polynomial of the matrix \( A \) is

\[
\lambda^3 I_n + \lambda^2 \Gamma + \lambda \Xi \Sigma + \Xi \Sigma \Xi = 0.
\]

Since all matrices \( \Gamma, \Sigma, \Xi, \) and \( \Sigma \) are diagonal, \( \lambda \) is an eigenvalue of \( A \) if and only if

\[
\lambda^3 I_n + \lambda^2 \Gamma_i(\iota) + \lambda \Xi(\iota) i(\Xi)(\Sigma)(\iota) = 0.
\]

The diagonal elements of the matrices \( \Gamma, \Sigma, \Xi, \) and \( \Sigma \) are positive. Therefore, using the Routh–Hurwitz criteria, \( \lambda_i \in \mathbb{R}_- \) if and only if \( \Gamma_i(\iota)(\Xi)(\Sigma)(\iota) > 0 \). Thus, \( A \) is Hurwitz if and only if \( \Gamma \succ \Sigma \). To show that the matrix \( B \) is Hurwitz, we use LaSalle’s invariance principle [14, Theorem 4.4]. Consider the dynamical system \( \dot{x} = Bx \), where \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^{3n} \). We define the Lyapunov function \( V : \mathbb{R}^{3n} \rightarrow \mathbb{R} \) by

\[
V(x_1, x_2, x_3) = \frac{1}{2} \begin{pmatrix} x_1^T \Gamma^{-1} x_1 + x_2^T \Xi^{-1} x_2 + x_3^T \Pi^{-1} x_3 \end{pmatrix}.
\]

Then, it is easy to check that \( \dot{V}(x_1, x_2, x_3) = -x_1^T \Xi^{-1} x_2 + x_2^T (N^T + N) x_3 \). Therefore, by LaSalle’s invariance principle, the trajectories of the system \( \dot{x} = Bx \) converge to the largest invariant set inside \( S = \{ x \in \mathbb{R}^{3n} | V(x) = 0 \} \). Since \( N \) is negative definite, it is easy to see that \( S = \{ x \in \mathbb{R}^{3n} | x_2 = x_3 = 0 \} \). Let us denote the largest invariant set inside \( S \) by \( L \). Our goal is to show that \( L = \{ 0_{3n} \} \). Suppose that \( \gamma : t \mapsto (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \) is a trajectory which belongs identically to \( S \). Then we have \( \gamma_3(t) = \gamma_2(t) = 0_n \). Therefore, we see that \( \dot{\gamma}_2(t) = 0_n \) \( \Rightarrow \Xi \gamma_1(t) = 0_n \). This implies that \( \gamma_1(t) = \gamma_2(t) = \gamma_3(t) = 0_n \). Thus, the only invariant set inside \( S \) is \( \{ 0_{3n} \} \). This implies that \( B \) is Hurwitz, and this completes the proof of the lemma. □

C Solutions to power flow equation

Consider (13) with \( \hat{w} = -\hat{Y}_{\text{red}}^{-1}\hat{Y}_{0g} \).

Lemma 17 Suppose

\[
\|D(\hat{w})\hat{Y}_{\text{red}}^{-1}(D(\hat{w}))^{-1}D'(S^*)\|_{C, \infty} \leq \frac{2}{\varepsilon}.
\]

Then the following statements hold:

---

Table A.2: Dimensionless parameters of the inverter model.
(i) the power flow equations (13) has a unique solution \( \hat{\mathbf{v}}_{oDQ}, \hat{\mathbf{i}}_{oDQ} \) with \( \hat{\mathbf{v}}_{oDQ} \in \Omega \), where

\[
\Omega = \{ \mathbf{y} \in \mathbb{R}^{2n} \mid \| \mathbf{y} - \hat{\mathbf{w}} \|_{C,\infty} \leq \frac{1}{2} \| \hat{\mathbf{w}} \|_{C,\infty} \};
\]

(ii) for every \( \mathbf{v}^0 \in \Omega \), the iteration procedure

\[
\mathbf{v}^{k+1} = \mathbf{v} + \frac{2}{3} \hat{\mathbf{v}}_r \mathbf{D}(\mathbf{v}^k) - \mathbf{i}^* \Rightarrow \mathbf{v}^k, \quad \forall k \in \mathbb{N},
\]

converges to \( \hat{\mathbf{v}}_{oDQ} \), where \( (\hat{\mathbf{v}}_{oDQ}, \hat{\mathbf{v}}_r(\hat{\mathbf{v}}_{oDQ} - \mathbf{v})) \) is the unique solution to (13).

**Proof.** By considering \( \mathbb{R}^{2n} \cong \mathbb{C}^n \), part (i) and (ii) are straightforward generalization of [38, Theorem 1]. One should note the fact that the nodal variables in [38, Theorem 1] are average power injections/demands and therefore the power flow equations has the form \( S = V \mathbf{T} \). However, in this paper, the nodal variables are instantaneous power injections/demands and the power flow equations read \( s = \frac{2}{3} \mathbf{D}(v_{oDQ}) i_{oDQ} \).

\[
\Box
\]

References

[1] U.S. Energy Information Administration. EIA forecasts renewables will be fastest growing source of electricity generation. Technical report, Jan 2019.

[2] J. L. Agorreta, M. Borrega, J. López, and L. Marroyo. Modeling and control of N-paralleled grid-connected inverters with LCL filter coupled due to grid impedance in PV plants. *IEEE Transactions on Power Electronics*, 26(3):770–785, 2011. doi:10.1109/TPEL.2010.2095429

[3] S. Bolognani and S. Zampieri. On the existence and approximation to AC power flow in rectangular coordinates. In *IEEE Transactions on Power Electronics*, 31(1):163–172, 2016. doi:10.1109/TPWRS.2015.2395452

[4] F. Bullo. *Lectures on Network Systems*. CreateSpace, 1 edition, 2018. With contributions by J. Cortés, F. Dörfler, and S. Martínez. URL: http://motion.me.ucsb.edu/book-lns

[5] S. Y. Caliskan and P. Tabuada. Uses and abuses of the swing equation model. In *IEEE Conf. on Decision and Control*, pages 6662–6667, Osaka, Japan, December 2015. doi:10.1109/CDC.2015.7403268

[6] S. Curi, D. Groß, and F. Dörfler. Control of low inertia power grids: A model reducution approach. In *IEEE Conf. on Decision and Control*, pages 5708–5713, Melbourne, Australia, December 2017. doi:10.1109/CDC.2017.8264521

[7] S. V. Dhople, S. S. Guggilam, and Y. C. Chen. Linear approximations to AC power flow in rectangular coordinates. In *Allerton Conf. on Communications, Control and Computing*, pages 211–217, September 2015. doi:10.1109/ALLERTON.2015.7447008

[8] S. V. Dhople, B. B. Johnson, F. Dörfler, and A. O. Hamadeh. Synchronization of nonlinear circuits in dynamic electrical networks with general topologies. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 61(9):2677–2690, 2014. doi:10.1109/TCSI.2014.2332250

[9] F. Dörfler and F. Bullo. Kron reduction of graphs with applications to electrical networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 60(1):150–163, 2013. doi:10.1109/TCSI.2012.2215780

[10] P. Fairley. 800,000 Microinverters Remotely Retrofitted on Oahu—in One Day. *IEEE Spectrum*, 5 Feb 2015.

[11] D. Groß, C. Arghir, and F. Dörfler. On the steady-state behavior of a nonlinear power system model. *Automatica*, 90:248–254, 2018. doi:10.1016/j.automatica.2017.12.057

[12] J. He, Y. W. Li, D. Bosnjak, and B. Harris. Investigation and active damping of multiple resonances in a parallel-inverter-based microgrid. *IEEE Transactions on Power Electronics*.

[13] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2nd edition, 2012.

[14] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3 edition, 2002.

[15] P. V. Kokotović, H. K. Khalil, and J. O’Reilly. Singularity Perturbation Methods in Control: Analysis and Design. SIAM, 1999. doi:10.1137/1.9781611971118

[16] P. Kundur. *Power System Stability and Control*. McGraw-Hill, 1994.

[17] Y. Lin, B. Johnson, V. Gevorgian, V. Purba, and S. Dhople. Stability assessment of a system comprising a single machine and inverter with scalable ratings. In *North American Power Symposium*, pages 1–6, September 2017. doi:10.1109/NAPS.2017.8107355

[18] P. C. Loh and D. G. Holmes. Analysis of multiloop control strategies for LC/LCL/LCL-filtered voltage-source and current-source inverters. *IEEE Transactions on Industry Applications*, 41(2):644–654, 2005. doi:10.1109/TIA.2005.844956

[19] M. Lu, X. Wang, P. C. Loh, and F. Blaabjerg. Resonance Interaction of Multiparallel Grid-Connected Inverters With LCL Filter. *IEEE Transactions on Power Electronics*, 32(2):894–899, Feb 2017. doi:10.1109/TPEL.2016.2585547

[20] M. Lu, Y. Yang, B. Johnson, and F. Blaabjerg. An interaction-admittance model for multiple-inverter grid-connected systems. *IEEE Transactions on Power Electronics*, 2018. To appear. doi:10.1109/TPEL.2018.2881139

[21] L. Luo and S. V. Dhople. Spatiotemporal model reduction of inverter-based islanded microgrids. *IEEE Transactions on Energy Conversion*, 29(4):823–832, 2014. doi:10.1109/TEC.2014.2348716

[22] C. A. Plet, M. Graovac, T. C. Green, and R. Iravani. Fault response of grid-connected inverter dominated networks. In *IEEE Power & Energy Society General Meeting*, pages 1–8, July 2010. doi:10.1109/PES.2010.5589981

[23] N. Pogaku, M. Prodanovic, and T. C. Green. Modeling, analysis and testing of autonomous operation of an inverter-based microgrid. *IEEE Transactions on Power Electronics*, 22(2):613–625, 2007. doi:10.1109/TPEL.2006.890003

[24] M. Prodanovic and T. C. Green. Control and filter design of three-phase inverters for high power quality grid connection. *IEEE Transactions on Power Electronics*, 18(1):373–380, 2003. doi:10.1109/TPEL.2002.807166

[25] V. Purba, S. V. Dhople, S. Jafarpour, F. Bullo, and B. Johnson. Network-cognizant model reduction of grid-tied three-phase inverters. In *Allerton Conf. on Communications, Control and Computing*, October 2017. doi:10.1109/ALLERTON.2017.8262722

[26] V. Purba, S. Jafarpour, B. Johnson, F. Bullo, and S. V. Dhople. Reduced-order structure-preserving model for
parallel-connected three-phase grid-tied inverters. In *IEEE Workshop on Control and Modeling for Power Electronics*, Stanford, USA, July 2017. [10.1109/COMPEL.2017.8013389]

[27] A. Rantzer. Scalable control of positive systems. *European Journal of Control*, 24:72–80, 2015. [10.1016/j.ejcon.2015.04.004]

[28] M. Rasheduzzaman, J. A. Mueller, and J. W. Kimball. An accurate small-signal model of inverter-dominated islanded microgrids using dq reference frame. *IEEE Journal of Emerging and Selected Topics in Power Electronics*, 2(4):1070–1080, 2014. [10.1109/JESTPE.2014.2338131]

[29] M. Rasheduzzaman, J. A. Mueller, and J. W. Kimball. Reduced-order small-signal model of microgrid systems. *IEEE Transactions on Sustainable Energy*, 6(4):1292–1305, 2015. [10.1109/TSTE.2015.2433177]

[30] V. R. Saksena, J. O’Reilly, and P. V. Kokotović. Singular perturbations and time-scale methods in control theory: Survey 1976-1983. *Automatica*, 20(3):273–293, 1984. [10.1016/0005-1098(84)90044-X]

[31] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi. Conditions for stability of droop-controlled inverter-based microgrids. *Automatica*, 50(10):2457–2469, 2014. [10.1016/j.automatica.2014.08.009]

[32] J. Schiffer, D. Zonetti, R. Ortega, A. M. Stanković, T. Sezi, and J. Raisch. A survey on modeling of microgrids — From fundamental physics to phasors and voltage sources. *Automatica*, 74:135–150, 2016. [10.1016/j.automatica.2016.07.036]

[33] J. W. Simpson-Porco, F. Dörfler, and F. Bullo. Synchronization and power sharing for droop-controlled inverters in islanded microgrids. *Automatica*, 49(9):2603–2611, 2013. [10.1016/j.automatica.2013.05.018]

[34] J. W. Simpson-Porco, Q. Shaﬁee, F. Dörﬂer, J. M. Vasquez, J. M. Guerrero, and F. Bullo. Secondary frequency and voltage control of islanded microgrids via distributed averaging. *IEEE Transactions on Industrial Electronics*, 62(11):7025–7038, 2015. [10.1109/TIE.2015.2436879]

[35] M. Sinha, F. Dörﬂer, B. B. Johnson, and S. V. Dhople. Uncovering droop control laws embedded within the nonlinear dynamics of Van der Pol oscillators. *IEEE Transactions on Control of Network Systems*, 4(2):347–358, 2017. [10.1109/TCSN.2015.2503558]

[36] L. A. B. Törres, J. P. Hespanha, and J. Moehlis. Synchronization of identical oscillators coupled through a symmetric network with dynamics: A constructive approach with applications to parallel operation of inverters. *IEEE Transactions on Automatic Control*, 60(12):3226–3241, 2015. [10.1109/TAC.2015.2418400]

[37] E. Twining and D. G. Holmes. Grid current regulation of a three-phase voltage source inverter with an LCL input filter. *IEEE Transactions on Power Electronics*, 18(3):888–895, 2003. [10.1109/TPEL.2003.810838]

[38] C. Wang, A. Bernstein, J. Y. Le Boudec, and M. Paolone. Explicit conditions on existence and uniqueness of load-flow solutions in distribution networks. *IEEE Transactions on Smart Grid*, 9(2):953–962, 2018. [10.1109/TSG.2016.2572060]

[39] A. Yazdani and R. Iravani. *Voltage-Sourced Converters in Power Systems: Modeling, Control, and Applications*. IEEE Press, 2010.

[40] C. Yu, X. Zhang, F. Liu, F. Li, H. Xu, R. Cao, and H. Ni. Modeling and resonance analysis of multiparallel inverter system under asynchronous carriers conditions. *IEEE Transactions on Power Electronics*, 32(4):3192–3205, 2017. [10.1109/TPEL.2016.2576585]