On Stochastic Stabilization via Nonsmooth Control Lyapunov Functions

Pavel Osinenko, Grigory Yaremenko, Member, IEEE, and Georgiy Malaniya

Abstract—Control Lyapunov function is a central tool in stabilization. It generalizes an abstract energy function—a Lyapunov function—to the case of controlled systems. It is a known fact that most control Lyapunov functions are nonsmooth—so is the case in nonholonomic systems, like wheeled robots and cars. Frameworks for stabilization using nonsmooth control Lyapunov functions exist, such as Dini aiming and steepest descent. This work generalizes the related results to the stochastic case. As the groundwork, a sampled control scheme is chosen in which control actions are computed at discrete moments in time using discrete measurements of the system state. In such a setup, special attention should be paid to the sample-to-sample behavior of the control Lyapunov function. A particular challenge here is a random noise acting on the system. The central result of this work is a theorem that states, roughly, that if there is a, generally nonsmooth, control Lyapunov function, the given stochastic dynamical system can be practically stabilized in the sample-and-hold mode meaning that the control actions are held constant within sampling time steps. A particular control method chosen is based on Moreau–Yosida regularization, in other words, inf-convolution of the control Lyapunov function, but the overall framework is extendable to further control schemes. It is assumed that the system noise be bounded almost surely, although the case of unbounded noise is briefly addressed.

Index Terms—Asymptotic stability, control systems, lyapunov stability, system analysis and design, stability criteria.

NOMENCLATURE

\( X \) State space \( \mathbb{R}^n \), where \( n \in \mathbb{N} \).

\( U \) Control set, which is a subset of \( \mathbb{R}^m \), where \( m \in \mathbb{N} \).

\( \mathbb{R}_+ \) \([0, +\infty)\).

\( E(\cdot) \) Expected value.

\( \mathbb{V} (\cdot) \) Variance.

\( \mathbb{P}(\cdot) \) Probability measure.

\( \| \cdot \|_1 \) Euclidian norm \( \| \cdot \|_2 \).

\( \| \cdot \|_{op} \) Operator norm.

\( \mathcal{K}_\infty \) Class of kappa-infinity functions [5].

\( B_R \) Closed ball of radius \( R \) centered at the origin.

\( \text{Lip}_n(\mathbb{R}) \) Lipschitz constant of \( h(\cdot) \) in \( B_R \).

\( a \wedge b \) \( \min (a, b) \).

\( \langle \cdot, \cdot \rangle \) Scalar product.

\( \text{tr}(\cdot) \) Matrix trace.

\( \mathcal{N}(x) \) Normal distribution with mean \( x \) and variance \( y \).

\( \mathcal{N}(x, y, z) \) Truncated normal distribution with mean \( x \), variance \( y \) and maximal deviation \( z \).

\( \nabla \cdot \) Gradient as a row-vector.

\( \hat{h} \) Lower convex envelope of function \( h \); \( \hat{h} = \sup\{g \mid g \text{ is convex, } g(x) \leq f(x)\} \).

\( \bar{h} \) Upper concave envelope of function \( h \); \( \bar{h} = \inf\{g \mid g \text{ is concave, } g(x) \geq f(x)\} \).

\( \mathcal{D}_f(x) \) Lower directional Dini derivative [25] of \( f(\cdot) \) at \( x \) in direction \( v \). l-fold iteration of function \( h(\cdot) \), like so \( h(h(h(\ldots))) \).

\( \mathbb{I}_A(\cdot) \) Indicator function of set \( A \).

\( a \cdot b \) \( a \) multiplied by \( b \), where \( a \) and \( b \) are scalars.

\( [x] \) \( x \) rounded to the closest greater or equal integer.

\( [x] \) \( x \) rounded to the closest less or equal integer.

\( \text{mod} \) Remainder of \( a \) when divided by \( b \).

I. INTRODUCTION

Stochastic stability theory can be traced back to the works of Khasminskii [1], Kushner [2], Mao [3], and Deng et al. [4], who extended the classical results and translated them into the language of \( \mathcal{K} \)-functions of Khalil [5], convenient for control engineers. Stochastic stability analyses were applied to various types of systems including discrete systems [6], cascaded systems [7], delayed systems [8], systems with input saturation [9], systems with state-dependent switching [10], nonlinear stochastic dynamic systems with singular perturbations [11], hybrid systems [12], hybrid retarded systems [13], linear systems with randomly jumping parameters [14], etc. Practical stochastic stability was addressed in, e.g., [15], [16], and [17]. When it comes to stochastic stabilization, most of the existing works assume continuous application of the control, whereas sampled control schemes were considered in rather specific contexts, e.g., based on approximate discrete-time models [18], [19] or in the event-triggered mode [20]. Stabilization of stochastic logical systems [21],[22] is yet another example of stochastic stabilization performed in discrete time; however, the latter setting also implies a finite state-space, which is not suitable for certain applications, e.g., in robotics.

The importance of sampled control frameworks is dictated by two facts. First, most modern controllers are implemented on digital media whence control actions are naturally computed at discrete moments in time, and based on discrete measurement of the state, rather than continuous. Second, the need for sampled controls is motivated by the fact that the majority of dynamical systems are not even stabilizable by means of feedback laws that depend continuously on the state [23], [24], [25], [26], [27], [28], [29]. There is an intimate connection between discontinuous feedback controls and nonsmooth control Lyapunov functions [30]. In particular, nonexistence of a smooth control Lyapunov function leads to nonexistence of a continuous stabilizing feedback law. It should be noted here that not only do control Lyapunov functions appear mostly nonsmooth, numerical routines for their calculation commonly produce nonsmooth functions—for particular methods, refer, e.g., to [31], [32], [33], [34]. A major problem that arises under discontinuous feedback laws is defining the system trajectory. But even resorting to such generalized notions of solutions as that of

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Filippov does not remedy the situation. It was the prominent work by Clarke et al. [25] that derived a general framework for stabilization by means of nonsmooth control Lyapunov functions by means of sampled controls. Stabilization here was meant as practical stabilization, i.e., the system state could be stabilized into any desirable vicinity of the equilibrium, provided that the sampling time be sufficiently small. In turn, the control law computation was based on the Moreau–Yosida regularization, in other words, inf-convolution of the control Lyapunov function involved. Further methods are applicable—for surveys, refer, e.g., to [29] and [35].

A brief remark should be made on the case of smooth control Lyapunov functions. Here, one can exploit universal formulas for stabilizing controls [36], [37]. Their generalizations to the stochastic case exist. For instance, Florchinger [38] suggested constructions of stochastically stabilizing controls via Lie theory methods. Smooth control Lyapunov functions were also assumed in the stochastic stabilization work of Deng et al. [4] and Gao et al. [20], which addressed event-triggered digital implementation of a given feedback law that stabilizes the system in second moment. In this work, in contrast, nonsmooth control Lyapunov functions are considered.

There is no extension of the Clarke’s general framework for stabilization to the case of stochastic systems. Whereas Deng et al. [4] considered the continuous control case, the work [39] extended on these results in the case of sampled controls. However, the control Lyapunov function involved was smooth, whereas the nonsmooth case was tackled only briefly, specifically, under bounded noise. This work elaborates on the matter of stochastic stabilization via sampled controls computed from nonsmooth control Lyapunov functions to a full technical extent.

The engineering background of the proposed framework for stochastic stabilization is motivated by the two facts. First, as also mentioned above, most modern controllers are realized in digital devices that naturally lead to a sampled control scheme where control actions are computed at discrete, usually evenly distributed, moments in time upon receiving state measurements at discrete moments in time as well. In this work, the control law is treated explicitly in the sample-and-hold mode, whereas the underlying system dynamics model is considered as time-continuous, to be precise, in the form of a stochastic differential equation. Second, the formalism of stochastic differential equations allows accounting for random uncertainty that may be related to model imperfection, system, measurement and/or actuator noise, etc. Effects of noise magnitude and sampling time choice are demonstrated in an experimental case study with a mobile robot in Section V.

Contribution. Informally, the contribution of this work is summarized as follows. In Theorem 1, it is shown that if a given nonlinear stochastic control system with bounded noise has a, generally nonsmooth, control Lyapunov function, a sample-and-hold policy can be produced that stabilizes the system’s equilibrium. The theorem explicitly provides the aforementioned policy and estimates the quality of resulting stabilization for a given configuration of noise, sampling rate, and computation error. Unlike previously mentioned results in stochastic stabilization, Theorem 1 concerns the case of a potentially nonsmooth control Lyapunov function and at the same time accounts for imperfections peculiar to a digital controller, namely control update latency and computation error.

Structure of the article: For full proofs and relevant lemmas, see the arXiv version of the article. ¹ Section II presents the technical preliminaries. Section III presents the main theorem on stochastic stabilization by means of nonsmooth control Lyapunov functions under almost surely bounded driving noise. Section IV presents the discussion of the case of unbounded driving noise. Section V presents the demonstrative experimental study. Finally, Section VI concludes this article.

II. PRELIMINARIES

The aim of this work is to address stabilization of stochastic control systems of the class

\[ \frac{dX_t}{dt} = f(X_t, t) dt + \sigma(X_t, t) dW_t \]  

where \( \{X_t\}_t \) and \( \{U_t\}_t \) are the state and control stochastic processes, \( \mathbb{X} \) and \( U \)-valued, respectively; \( f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n, \sigma : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{n \times d} \); \( \{Z_t\}_t \) is a \( d \)-dimensional random process that is measurable with respect to \( t \). The technical goal is to study stabilization of (1) by Markov control policies in sampled mode. “Sampled mode” here means that the controller computes control actions based on measurements of state at discrete equidistant moments in time and keeps these actions constant between the said moments in time. This is stated mathematically precise in Section III.

Now, proceed to the necessary definitions for stability. Consider a general stochastic system

\[ \frac{dX_t}{dt} = f(X_t, t) dt + \sigma(X_t, t) dW_t \]  

where \( dW_t \) is a placeholder for either \( Z_t \, dt \) or \( dB_t \) with \( \{B_t\}_t \) being a standard Brownian motion. The latter case will be used in the discussion on the unbounded noise case of Section IV. Observe that if we were to apply a concrete policy \( V_t \) to the control system (1) by asserting \( U_t := V_t \), we would get a system of the kind (2) provided the driving noise is of the form \( Z_t \, dt \). Now, we are ready to state the key definitions for stability.

Definition 1 (Semiasymptotic stability in probability): The origin of a stochastic system (2) is said to be semiasymptotically stable in probability in \( R > 0 \) until \( r \geq 0 \) if

\[ x_0 \in B_R \Rightarrow P \left\{ \lim \sup_{t \to \infty} \|X_t\| \leq r \right\} = 1. \]

This notion is similar to (almost sure) local asymptotic stability. The only difference is that with asymptotic stability the attractor is a single point—the origin, whereas with semiasymptotic stability the attractor is a ball centered at the origin. Thus, \( r \) is the radius of the attractor and \( R \) is the radius of the basin of attraction. If a stochastic system is said to be semiasymptotically stable, this kind of attraction will occur almost surely. The need to introduce this notion arises from how adding noise to an asymptotically stable deterministic system impacts the said system’s stability. Unless the magnitude of added noise is too large, the system will generally preserve its attractive properties within some basin of attraction \( (B_R) \), but the attraction will cease in some small neighborhood of the origin \( (B_r) \).

Definition 2 (Semiasymptotic stability on average): The origin of a stochastic system (2) is said to be semiasymptotically stable on average in \( R > 0 \) until \( r \geq 0 \) if

\[ x_0 \in B_R \Rightarrow \lim_{t \to \infty} \mathbb{E} \left\{ \|X_t\| \right\} \leq r. \]

The latter definition is analogous to Definition 1, but describes stability in mean. Here, the expected value of the state is guaranteed to get arbitrarily close to \( B_R \) and stay there permanently, provided that the initial state was within \( B_R \).

The following definitions are required to state and prove Theorem 1, which is the central result of this work.
**Definition 3:** A nonnegative locally Lipschitz-continuous function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a radial growth bound of system (14) iff
\[
\rho(||x||) ||x|| \geq x^T f(x,u) + ||f(x,u)|| \bar{Z} \quad \forall u \in U.
\]
(5)

**Definition 4:** A two-variable function \( y(R;t) \) is called a radial forecast of (1) iff there exists such a radial growth bound \( \rho \) that for each \( R \geq 0 \), \( y(R;\cdot) \) equals to the (local) solution of
\[
\begin{cases}
  \dot{y} = \rho(y) \\
  y(0) = R.
\end{cases}
\]
(6)

**Remark 1:** Note that for a strictly positive radial growth bound \( \rho \),
\[
y(R; t) = y(0; t + y^{-1}(0; R))
\]
where the inversion \( y^{-1} \) is with respect to the second argument.

**Definition 5:** A function \( L \) is called a control Lyapunov function for (1) if it satisfies
\[
\forall x \in X \quad \inf_{\lambda \in \overline{\text{lip}}_t (f(x,u))} D_{\lambda} L(x) \leq -\alpha_1(||x||)
\]
\[
\alpha_1(||x||) \leq L(x) \leq \alpha_2(||x||), \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}_\infty.
\]
(7)

**Definition 6:** Let there be a control Lyapunov function \( L \) for (1) and let \( y_R \) be a radial forecast of (1), then a partial function \( \bar{r} : \mathbb{R}^+_2 \to \mathbb{R} \) is called an attraction function of (1) if
\[
\bar{r}(R, \bar{Z}, \lambda, \delta, \eta) := \alpha_3^{-1} \left( \eta + \sqrt{2\alpha_2(R)} \right) \left( 2Lip_{\lambda} \left( R + \lambda \sqrt{2\alpha_2(R)} \right) \right)
\]
\[
\times \left( \lambda \sqrt{2\alpha_2(R)} \right)^2 + \frac{\delta}{2} Lip_{\lambda} (y(R; \delta))
\]
\[
\times \left( f(y(R; \delta)) + \sigma(y(R; \delta)) \bar{Z} + \bar{\sigma}(y(R; \delta)) \bar{Z} \right)
\]
\[
\times \left( \frac{\lambda}{2} \left( f(y(R; \delta)) + \sigma(y(R; \delta)) \bar{Z} + \bar{\sigma}(y(R; \delta)) \bar{Z} \right)^2 \right)
\]
\[
\times \left( f(y(R; \delta)) + \sigma(y(R; \delta)) \bar{Z} + \bar{\sigma}(y(R; \delta)) \bar{Z} \right)^2
\]
\[
\times \lambda \sqrt{2\alpha_2(R)}
\]
(8)

where for each \( r \geq 0 \) we have
\[
\bar{f}(r) \geq ||f(x,u)|| \quad \forall x \in B_r, \quad u \in U
\]
\[
\bar{\sigma}(r) \geq ||\sigma(x,u)|| \quad \forall x \in B_r, \quad u \in U
\]
\[
||f(x,u) - f(y,u)|| \leq Lip_{\lambda} (r) ||x - y|| \quad \forall x, y \in B_r, \quad u \in U
\]
\[
||L(x) - L(y)|| \leq Lip_{\lambda} (r) ||x - y|| \quad \forall x, y \in B_r
\]
\[
\text{Lip}_{\lambda}(\cdot) \text{ and Lip}_{\bar{y}}(\cdot), \bar{f}(\cdot), \bar{\sigma}(\cdot) \text{ are nondecreasing and upper-semicontinuous functions, respectively.}
\]
(9)

For brevity, we denote \( \bar{r}_{\rho, Z, \lambda, \delta, \eta}(r) := \bar{r}(r, \bar{Z}, \lambda, \delta, \eta) \).

We define \( L_{\lambda}(\cdot) \) as the infimal convolution with a parameter \( \lambda > 0 \) of \( L(\cdot) \) as follows:
\[
L_{\lambda}(x) := \inf_{y \in \mathbb{R}^2} \left( L(y) + \frac{||y - x||^2}{2\lambda^2} \right)
\]
\[
\frac{1}{2} \alpha_1(||x||) \leq L_{\lambda}(x) \leq \frac{1}{2} \alpha_2(||x||), \quad \alpha_1, \alpha_2 \in \mathbb{K}_\infty.
\]
(10)

Here, we used the upper-left index \( \lambda \) for notation purposes to stress the relation to the infimal convolution with a parameter \( \lambda \). One way to stress to obtain \( \frac{1}{2} \alpha_1(\cdot) \) and \( \frac{1}{2} \alpha_2(\cdot) \) is to compute the infimal convolutions of \( \alpha_1(\cdot) \) and \( \alpha_2(\cdot) \).

**Definition 7:** A function \( \bar{r} : \mathbb{R}^2_+ \to \mathbb{R} \) is called a mean attraction function of (1) if
\[
\bar{r}(r, \bar{r}, \bar{\sigma}, \lambda, \delta, \eta) := \alpha_3^{-1} \left( \eta + \sqrt{2\lambda \alpha_2(r)} \right) \left( 2\lambda \text{Lip}_{\lambda} (r + \lambda \sqrt{2\alpha_2(r)}) \right)
\]
\[
\times \text{Lip}_{\lambda} (r + \lambda \sqrt{2\alpha_2(r)}) + \frac{\delta}{2} \text{Lip}_{\lambda} (r) \bar{f}(r)
\]
\[
\times \lambda \sqrt{2\alpha_2(r)}
\]
(11)

where \( \alpha_3(\cdot) \) is the lower convex envelope of \( \alpha_3 \) on \([0, r] \). For brevity, let us denote \( \bar{r}_{\rho, Z, \lambda, \delta, \eta}(r) := \bar{r}(r, \bar{r}, \bar{\sigma}, \lambda, \delta, \eta) \).

**Remark 2:** The domain of definition of \( \bar{r}_{\rho, Z, \lambda, \delta, \eta}(\cdot) \) depends on \( \delta \); however, for each \( R \), \( \bar{r}(R, \bar{Z}, \lambda, \delta, \eta) \) exists, provided that \( \delta \) is sufficiently small. Unlike \( \bar{r} \), the domain of definition of \( \bar{r} \) is not restricted.

Additionally, we assume that some bounds for moments of \( ||Z_t|| \) are known, in particular, there exist numbers \( \bar{\mu}, \bar{\sigma} \) such that
\[
\forall t \bar{\mu} \geq E ||Z_t||
\]
\[
\forall t \bar{\sigma}^2 \geq \mathbb{V} ||Z_t||.
\]
(12)

**III. MAIN THEOREM**

The sample-and-hold mode introduces a latency between a change in the systems state and the controller’s response to that change. Considering such a setting yields a practical advantage: unlike feedback of the kind \( U_t = \mu(X_t) \), the sample-and-hold mode accurately describes policies that digital controllers can implement. Furthermore, it provides other benefits. A system produced by asserting \( U_t := \mu(X_t) \) may have no solutions, e.g.,
\[
\begin{cases}
  \text{d}X_t = U_t \text{d}t, \quad X_0 = 1 \\
  U_t := 1 - 2\mathbb{E}_{\mathbb{B}_+}(X_t)
\end{cases}
\]
(13)

fails to admit a solution. Such issues cannot occur in the sample-and-hold mode, unless \( f(\cdot, U_t) \) or \( \sigma(\cdot, U_t) \) is discontinuous. It is fairly intuitive that a control Lyapunov function constructed for a deterministic system will partially preserve its properties if some noise were to be added, unless the magnitude of that noise is too large. However, naturally, the existence of a control Lyapunov function generally speaking does not imply much about stabilizability of systems described by (1), since the noise term \( \sigma(X_t, U_t) Z_t \text{d}t \) can just entirely overwhelm the deterministic dynamics of the system. A way to remedy this problems is to consider bounded noise models.
Now, consider a special case of (1):
\[
\begin{align*}
\frac{dX_t}{dt} &= f(X_t, U_t) dt + \sigma(X_t, U_t) Z_t dt, \quad X_0 = x_0 \\
f : &\mathbb{R}^n \times U \to \mathbb{R}^n, \quad \sigma : \mathbb{R}^n \times U \to \mathbb{R}^n \\
Z_t &\text{ is a measurable random process w.r.t. } t \\
\forall t \in \mathbb{R} \quad |Z_t| \leq \bar{Z} \quad (\text{bounded noise}) \\
U_t &\defeq \mu(X_{t- (t \text{ mod } 3)}) \quad (\text{sample and hold policy})
\end{align*}
\]
where \( dt \) implies Lebesgue integration.

We obtained the above system from (1) by assuming bounded noise and asserting that a sample-and-hold policy is in place.

On the first glance it may seem like \( |Z_t| \leq \bar{Z} \) imposes a significant restriction on physical objects that this model can represent, since even Gaussian white noise fails to be described by it. However, upon a closer inspection, it becomes clear that this theoretical loss of generality does not have much of a negative impact in practice. Note that noise represented by the following model can very closely mimic Brownian noise:
\[
Z_t dt = \xi_t dt \\
\text{where } \xi_t \sim \mathcal{N}(0, \sigma^2, \bar{Z}), \{\xi_t\} \text{ are independent.}
\]
Indeed, for real-world objects, some values of noise are large enough to be considered impossible as opposed to merely improbable. Therefore, an adequate reduction of support may in fact make the noise model more accurate.

There are also a number of common noise models that imply this kind of boundedness [40], in particular the following.

1) The Doering–Cai–Lin noise
\[
dZ_t = -\frac{1}{\theta} Z_t dt + \sqrt{\frac{1 - Z_t^2}{\theta(\gamma + 1)}} dB_t,
\]
with parameters \( \gamma > -1, \theta > 0 \).

2) The Tsallis–Stanuelio–Borland noise
\[
dZ_t = \frac{1}{\bar{Z}_t} \frac{Z_t}{1 - Z_t} dt + \sqrt{\frac{1 - q}{\theta}} dB_t,
\]
with \( \theta > 0, q < 1 \) parameters.

3) Kessler–Sørensen noise
\[
dZ_t = -\frac{\theta}{\pi\theta} \tan\left(\frac{\pi}{2} Z_t\right) dt + \sqrt{\frac{2}{\pi\sqrt{\theta(\gamma + 1)}}} dB_t
\]
with \( \theta > 0, \gamma \geq 0, \theta = \frac{2^{\gamma+1}}{\gamma+1} \) parameters.

The following theorem demonstrates how a control Lyapunov function can be used to stabilize (14) in a sample-and-hold mode. The theorem explicitly provides a stabilizing policy \( \mu(x) \) for a given control Lyapunov function \( L(x) \). The theorem also provides estimates for the quality of resulting attraction.

**Theorem 1:** Let the following assumptions hold for system (14).

(A1) Bounds as per (9) exist.

(A2) There exists a control Lyapunov function \( L(x) \):
\[
\forall x \in \mathcal{X} \quad \inf_{\mu(x), t \in [0, T]} \mathcal{P}_\mu L(x) \leq -\alpha_3 ||x||
\]
\[
\forall x \in \mathcal{X} \quad \alpha_1 ||x|| \leq L(x) \leq \alpha_2 ||x||
\]
\[
\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K}_{\text{ex}}, \quad \mathcal{L} \mathcal{X} \to \mathbb{R}_+
\]

(A3) The policy \( \mu(x) \) is chosen so as to satisfy
\[
\begin{align*}
\left\langle \frac{x - \frac{\alpha_1}{\lambda^2} f(x, \mu(x))}{\lambda^2}, f(x, \mu(x)) \right\rangle \\
\leq \inf_{u \in \mathcal{U}} \left\langle \frac{x - \frac{\alpha_1}{\lambda^2} f(x, u)}{\lambda^2}, f(x, u) \right\rangle + \eta
\end{align*}
\]
where \( \alpha \defeq \arg\min \{2\lambda^2 L(x^*) + \|x^* - x\| : \lambda > 0 \}. \]

Then, a unique global Carathéodory solution almost surely exists and the following can be claimed.

(C1) \( \lim_{t \to \infty} r_t = r^* \) exists
\[
\forall r_t \defeq \tilde{r}(r_{t-1}, \bar{Z}, \lambda, \delta, \eta), \quad r_0 = R.
\]

(C2) The origin of the system is semiasymptotically stable in probability in \( R \) until \( r^* \)
\[
\forall r \defeq \frac{\alpha_1}{\lambda^2} \left( \alpha_2(\gamma(r^*; \delta)) \right)
\]

(C3) The origin of the system is semiasymptotically stable on average in \( R \) until \( r^* \)
\[
\forall r \defeq \frac{\alpha_1}{\lambda^2} \left( \frac{\gamma}{2} \hat{r}(r, \mu, \bar{Z}, \lambda, \delta) + \left( \tilde{f}(r) + \tilde{F}(r) \right) \right)
\]

\( \hat{\alpha} \) and \( \frac{\gamma}{2} \hat{\alpha} \) are envelopes over \([0, r]\).
Corollary 2: For arbitrary $0 < r < R$, system (14) is semiasymptotically stable in probability in $R$ until $r$, provided that $\bar{Z}, \delta$, and $\eta$ are sufficiently small.

Corollary 3: For arbitrary $0 < r < R$, system (14) is semiasymptotically stable on average in $R$ until $r$, provided that $\mathbb{E}[\|Z_t\|^2]$; $\delta$, and $\eta$ are sufficiently small.

Once an attraction function is obtained for a system of the kind (14), it can be used to inspect the influence of noise, latency, and optimization error on stability. This implies a number of possible applications as follows.

1) Verification. Known values of noise, latency, and optimization error can be used to infer (average) semiasymptotic stability for given radii, producing a formal guarantee. Computing $r = \lambda_{\alpha}^{-1}(\alpha_2(y(r); \delta))$ for some $I$ will yield a radius in which the state will remain permanently once it gets there. Likewise, one can compute $\bar{r} = \lambda_{\alpha}^{-1}(\alpha_2(\bar{r}(r, \bar{\mu}, \bar{\sigma}, \lambda, \bar{\delta}, \eta) + (\bar{f}(r) + \bar{\bar{\sigma}(r))\delta))$ to determine a substantially smaller radius in which the mean state will stay.

2) Tuning. The attraction function can be used to discover values of parameters that ensure desired quality of stabilization. If we were to consider $r$ and $\bar{r}$ as functions of ($\bar{Z}, \delta, \eta$) and ($\bar{\mu}, \bar{\sigma}, \lambda, \bar{\delta}, \eta$), respectively, then both of those functions would be strictly increasing with respect to each one of their arguments. Thus, suitable values of parameters can be identified through a simple grid search.

3) Analysis of robustness. Given a Lyapunov function with a corresponding policy for a deterministic control system, one could investigate its robustness by inspecting how introducing noise, approximation error, and response latency affects the quality of stabilization. This way, evaluating $r$ and $\bar{r}$ for various sets of parameters will show in which settings this controller performs sufficiently well. The latter will reveal suitable implementations and environments, for which the system is guaranteed to function as intended.

4) Safe reinforcement learning. If semiasymptotic stability has been inferred over $R$, then any controller that implements $\mu(x)$ when near $R$ is guaranteed to stay within $\lambda^{-1}(\alpha_2(R))$ regardless of the policy used in other areas of the state space. This allows us to utilize numerical optimization of reinforcement learning, while maintaining formal guarantees of Lyapunov theory (see, e.g., [43]).

Remark 6: Note that there are no extra restrictions on $f(x, \cdot), \sigma(x, \cdot)$ and $U$, other than the ones imposed by (A1). For the above theorem to hold, $f(x, \cdot), \sigma(x, \cdot)$, and $U$ do not even have to be measurable. By using approximate minimizers, we circumvent having to rely on the extreme value theorem.

In the next section, we briefly discuss the case when the system trajectory is an Itô process driven by a Brownian motion hence unbounded noise.

IV. Itô PROCESSES

The case of unbounded noise can too be considered if assumptions of a different kind were to be made. Consider the following Itô drift–diffusion process, driven by standard Brownian motion $B_t$:

\[
\begin{align*}
\text{d}X_t &= f(X_t, U_t)\, \text{d}t + \sigma(X_t, U_t)\, \text{d}B_t \\
U_t &= \mu(X_{t-} \mod \delta) \quad \text{(sample and hold policy)}.
\end{align*}
\] (25)

The generator of the stochastic differential equation of (25), for a smooth function $L$, is defined as follows:

\[
\mathcal{A}^\mu L(x) = \nabla L(f(x, \mu(x))) + \frac{1}{2} \text{tr} \left( \left( \sigma(x, \mu(x)) \right)^\top \nabla^2 L(x) \sigma(x, \mu(x)) \right)
\] (26)

where $\nabla L$ is the gradient vector and $\nabla^2 L$ is the Hessian, i.e., the matrix of second-order derivatives. We define the operator $\Gamma^L_R$ as follows:

\[
\Gamma^L_R \mathcal{A}^\mu L := \int_R^\mu \left( \text{Lip}_CL(R) \int_R^\mu \text{Lip}_F(R) + \text{Lip}_L(R) \int_R^\mu \text{Lip}_R(R) \right)
\] (27)

where

\[
\begin{align*}
\bar{f}^\mu_R := \sup_{x \in B_R} \|f(x, \mu(x))\|, & \quad \bar{\sigma}^\mu_R := \sup_{x \in B_R} \|\sigma(x, \mu(x))\| \\
b^R := \sup_{x \in B_R} \|\nabla L(x)\|, & \quad b^R := \sup_{x \in B_R} \|\nabla^2 L(x)\|.
\end{align*}
\] (28)

Now that we can no longer rely on the boundedness of noise, to assert stability we have to demand the existence of a stronger version of the classical Lyapunov function.

Definition 8 (Stochastic Lyapunov pair): A stochastic Lyapunov pair $(L, \mu)$ is, for the system (25), a pair of functions if: $L \in C^2$; there exist $\alpha_1, \alpha_2 > 0$, and $\alpha_3 \in \mathcal{K}_{\infty}$ with $\alpha_3(\sqrt{\|x\|})$ convex; there exists $\alpha_4 \in \mathcal{K}_{\infty}$ s.t. $\alpha_4(\sqrt{R})$ is concave (monotone condition, cf., [18], [19], [44]); there exists $K > 0$, $K_\mu$ s.t. $\forall x, \mu(x) \in B_{K_\mu}$ and $\forall x, u \in B_{K_\mu}$, $x^\top f(x, u) + \frac{1}{2} \|\sigma(x, u)\|^2 \leq K(1 + \|x\|^2)$; the following properties hold:

\[
\forall x \quad \alpha_1 \|x\|^2 \leq L(x) \leq \alpha_3 \|x\|^2
\] (29)

\[
\forall x \quad \mathcal{A}^\mu L(x) \leq -\alpha_3(\|x\|) + \bar{L}, \bar{L} > 0.
\] (30)

Remark 7: The number $\bar{L}$ above in Definition 8 is related to the noise and differentiates the stochastic case from a deterministic one, which typically possesses a decay condition of the kind $\langle \nabla L(x), f(x, \mu(x)) \rangle \leq -\alpha_3(\|x\|)$.

Remark 8: In general, for a function $\varphi$, the Jensen’s gap, i.e., $\mathbb{E}[\varphi(X)] - \varphi(\mathbb{E}[X])$ can be arbitrarily large. To relate various expected values in the analysis of Theorem 2, the Jensen’s inequality has to be utilized, which motivates the stated conditions. Notice, e.g., [45, Lemma 2.1] also used quadratic bounding functions of $L$. A condition $\mathcal{A}^\mu L(x) = -cL + \bar{L}, c > 0$ (cf., [4, Th. 4.1]) also fits the assumptions since $-cL \leq -c\alpha_3 \|x\|^2$ and $\alpha_3$ is thus effectively $\alpha_3(\sqrt{\|x\|})$ and so $\alpha_3(\sqrt{\|x\|})$ is convex. The function $\alpha_3(\sqrt{\|x\|})$ being concave is satisfied if, e.g., $L$ is quadratic, $f(\cdot, \mu(\cdot))$ is Lipschitz and of linear growth, and $\sigma(\cdot, \mu(\cdot))$ is Lipschitz and bounded. This condition may be seen as restrictive, although, e.g., linear growth is often assumed in stochastic stabilization in mean (see, for instance, [18], [19], [46], [47], [48]). The monotone condition is in the style of [44] and is weaker than those in related works on sampled stochastic stabilization (see, e.g., [18], [19]), whereas it should be noted that universal formulas with bounded controls are known [37]. Furthermore, this condition will secure global existence of strong solutions [44], which is unavoidable in case of sample and hold (S&H) mode, and this is in contrast to the “standard” Lyapunov techniques in stochastic systems [1]. The reason is that, in the latter, decay of the subject Lyapunov function is ensured for all times, whereas in the herein considered case, there are necessarily time intervals in which the said decay cannot be guaranteed.

Theorem 2 (Itô process semiasymptotically stable on average): Consider a stochastic system (25). Suppose there exists a stochastic Lyapunov pair $(L, \mu)$ and $U_t = \mu(X_{t-} \mod \delta)$. Then, for each $R > 0$ and $\alpha_3 > 0$, there exists a sufficiently small $\delta > 0$, which (25) is
semiasymptotically stable on average in $R$ until $\rho$, where

$$\rho = \inf \left\{ \alpha_3 \left( r' \right) - \sum \right\} .$$  \hspace{1cm} (31)

Proof: See [39, Th. 1].

Corollary 4: Suppose that (30) has the form

$$\forall x \in R \left( x, L \right) \leq -\alpha_3 \left( x \right) .$$  \hspace{1cm} (32)

In other words, $(L, \mu)$ is a Lyapunov pair for the noiseless system $x = f(x, u)$. If it holds that

$$\begin{align*}
\exists \delta > 0 & \quad \forall \delta \in B_R \nonumber \\
\alpha_3 (\| x \| ) & \geq \frac{1}{2} \left( \| \sigma (x, u) \|^2 L (x) \sigma (x, u) \right) \nonumber \\
\end{align*}$$

then assuming $U_1 = \mu (x_{\delta} (t) \mod \delta)$, for each $R$ there exists a sufficiently small $\delta > 0$, that (25) is semiasymptotically stable on average in $\rho$ over $R$. In particular, (33) holds if $\| \sigma \|$ is uniformly bounded and $\| \nabla L (x) \|$ has a growth rate lower than that of $\alpha_3$ everywhere except for a vicinity of the origin.

Remark 9: The growth condition (33) in Corollary 4 may be justified as follows. Roughly speaking, taking derivatives decreases the growth rate. That is, one would normally expect that, outside some vicinity of the origin, $\| \nabla L (x) \|$ grows slower than $\| \nabla L (x) \|$. Such is the case when $L$ is, e.g., polynomial. The diffusion function $\sigma$ describes the noise magnification depending on the state and control action. It may be justified in some applications to assume this term to be bounded uniformly in $x, u$. All in all, Corollary 4 gives a particular hint on transferring a Lyapunov pair from a nominal, noiseless, to a noisy system.

V. EXPERIMENTAL STUDY.

An experimental study of mobile robot parking was performed to demonstrate the effects of the developed theory (see Fig. 1). Experiments were performed under practical truncated white Gaussian noise of varying power introduced into the system.

We used the above-described method of inf-convolutions and the model $\left( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \right) = \left( u_1, u_2, u_3 \right)$ with a corresponding nonsmooth control Lyapunov function $L(x) = x_1^2 + x_2^2 + 2x_3^2 - 2x_1 \sqrt{x_1^2 + x_2^2}$ [49], which is global and the control actions are confined to $[-1, 1]^2$. Notice that for a fixed noise power parameter, the decrease of the sampling time has a limited effect due to the noise.

Remark 10: Theorem 1 bears a fairly general character as physical systems are most adequately described by stochastic differential equations and controllers are commonly implemented in a sampled mode. Further examples of such settings include, e.g., mechanical and robotic systems driven by digital microcontrollers, stock market bets with time-discrete decisions to buy or sell, medical devices like insulin pumps with scheduled administration, greenhouse with digital climate control, etc. The nature of noise in such examples can be interpreted in various ways, e.g., as unmodeled mechanical disturbance, say, due to friction or wear; noisy sensors; random fluctuation of stock prices; fluctuations in homeostasis; fluctuations in plant respiration, etc.

VI. CONCLUSION

This work was concerned with stabilization of nonlinear dynamical systems in the sample-and-hold framework. A novel theoretical result was derived that enables synthesis, verification, tuning, and robustness testing of digital stabilizers for stochastic systems with bounded noise.

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