Role of acceleration in the expansion of the universe and its influence on an early-universe modified version of the Heisenberg uncertainty principle

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Abstract. From first principles, we examine what adding acceleration does, and does not do, as to an early-universe modified version of the Heisenberg uncertainty principle. In doing so, we examine a Friedmann equation for the evolution of the scale factor, using two cases explicitly—when the acceleration of the expansion of the scale factor is kept in and when it is out—and the intermediate case of when the acceleration and scale factors are important but not dominant. In doing so we tie this discussion into earlier work done on the HUP.

1. Introduction
We will be examining a Friedmann equation for the evolution of the scale factor, using two cases explicitly—when the acceleration of the expansion of the scale factor is kept in and when it is out—and the intermediate cases of when the acceleration and scale factors are important but not dominant. In doing so, we will be tying this discussion into earlier work done on the early-universe modified version of the Heisenberg uncertainty principle (HUP), but from the context of how the acceleration term will affect the HUP, and making sense of [1]

\[ \left\langle (\delta g_{tt})^2 \right\rangle \geq \frac{h^2}{V^2} \text{ and } \delta g_{rt} \approx \delta g_{\theta\theta} \approx \delta g_{\phi\phi} \approx 0. \]  

(1)

Namely, we will be working with [1]

\[ \delta t \Delta E = \frac{h}{\delta g_{tt}} = \frac{h}{a^2(t) \cdot \phi} \ll h \]

\[ \Leftrightarrow S_{\text{initial}}(\text{with} \delta g_{tt}) = (\delta g_{tt})^{-3} S_{\text{initial}}(\text{without} \delta g_{tt}) \gg S_{\text{initial}}(\text{without} \delta g_{tt}). \]  

(2)

That is, the fluctuation of \( \delta g_{tt} \ll 1 \) dramatically boosts initial entropy. It is not what it would be if \( \delta g_{tt} \approx 1 \). The next question to ask would be how could one actually have \( \delta g_{tt} \approx 1 \)? (3)
In short, we would require an enormous ‘inflaton’ style $\phi$-valued scalar function and $a^2(t) \approx 10^{-110}$. How could $\phi$ be initially quite large? Within Planck time, the following holds for mass, as a lower bound [1].

$$m_{\text{graviton}} \geq \frac{2h^2}{(6g_\mu)T_p} \cdot \frac{E - V}{\Delta T^2}$$

(4)

Here, [1]

$$K.E. \approx (E - V) \approx \dot{\phi} \propto a^{-6}.$$  

(5)

Then [1],

$$\dot{\phi} \approx a^{-3} \iff \dot{\phi} \approx t \cdot a^{-3} + H.O.T.$$  

(6)

The question to ask, now, is about the acceleration of the scale factor, due to time. This will be the subject of our inquiry in the next section of this paper.

2. How could anyone get the acceleration of the universe factored into our scale factor?

Begin by looking at material from pages 483–485 of [2]:

$$\left(\frac{\dot{a}}{a}\right)^3 - 3 \cdot \left(\frac{\dot{a}}{a}\right)^2 - 2 \cdot \left(\frac{\ddot{a}}{a}\right) \cdot \left(\frac{\dot{a}}{a}\right) + \left[\frac{-1}{64\pi G \cdot t^2} + \frac{\Lambda}{2}\right] = 0.$$  

(7)

Then, consider two cases of what to do with the ratio $\frac{\dot{a}}{a}$ and solve the above as a cubic equation.

2.1. What if $\frac{\ddot{a}}{a}$ offers a vanishingly small contribution (low acceleration)?

$$\left(\frac{\dot{a}}{a}\right)^3 - 3 \cdot \left(\frac{\dot{a}}{a}\right)^2 - 2 \cdot \left(\frac{\ddot{a}}{a}\right)^2 \approx 0.$$  

(8)

Then, using the idea of a repressed cubic, we will have the following solution for $\frac{\dot{a}}{a}$ [3].

$$\frac{\dot{a}}{a} = \text{Solution} = \xi$$

(9)

2.1.1. Solutions for Eq. (8), in reduced cubic form.

$$\xi = A + B, \quad \frac{-3}{2} \cdot (A - B) - \left(\frac{A + B}{2}\right), \quad \frac{-3}{2} \cdot (A - B) - \left(\frac{A + B}{2}\right)$$

(10)

$$A = \left[\left(\frac{-1}{128\pi G \cdot t^2} + \frac{\Lambda}{4}\right) + \sqrt{\frac{1}{4} \cdot \left(\frac{-1}{64\pi G \cdot t^2} + \frac{\Lambda}{2}\right)^2 + \frac{1}{8}}\right]^{1/3}$$

$$B = -\left[\left(\frac{1}{128\pi G \cdot t^2} - \frac{\Lambda}{4}\right) + \sqrt{\frac{1}{4} \cdot \left(\frac{-1}{64\pi G \cdot t^2} + \frac{\Lambda}{2}\right)^2 + \frac{1}{8}}\right]^{1/3}.$$  

(11)

Then

$$\Theta = \frac{1}{8} \left[2 \cdot \left(\frac{-1}{64\pi G \cdot t^2} + \frac{\Lambda}{2}\right)^2 - 1\right].$$  

(12)

$$\Theta > 0 \Rightarrow \xi \text{ has one real and two imaginary roots.}$$

$$\Theta = 0 \Rightarrow \xi \text{ has three real roots, two of which are equal.}$$

$$\Theta < 0 \Rightarrow \xi \text{ has three real roots, all of which are unique.}$$  

(13)
The situation to watch is when the time, \( t \), is extremely small. Then, one must work with the situation where \( \Theta > 0 \Rightarrow \xi \) has one real and two imaginary roots. That is, the situation is then dominated with one real root and two imaginary roots. We will comment on value of what happens to \( \dot{a} = \text{Solution} = \xi \) if there is one real and two imaginary roots? What would be a possible constraint if we had, for undimensioned units,

\[
2 \cdot \left( -\frac{1}{64\pi G \cdot t^2} + \frac{\Lambda}{2} \right)^2 - 1 \approx 0 \Leftrightarrow \left( -\frac{1}{64\pi G \cdot t^2} + \frac{\Lambda}{2} \right) \approx \frac{1}{\sqrt{2}} \Leftrightarrow \Lambda \approx \sqrt{2} + \frac{1}{32\pi G \cdot t^2} \quad (14)
\]

That is, for the case that one uses undimensioned units, we would have, then,

\[
\Theta \leq 0 \Leftrightarrow \Lambda \geq \sqrt{2} + \frac{1}{32\pi G \cdot t^2} \cdot (15)
\]

How likely is this to happen in the pre-Planckian regime? Not likely. Secondly, we get

\[
\Theta \leq 0 \Rightarrow \xi \text{ has three real roots}. \quad (16)
\]

So if we neglect having the acceleration of the scale factor, by abandoning \( \ddot{a} \) acceleration, we get a weird family of solutions for Eq. (8).

2.1.2. Solutions for Eq. (7), in cubic form, gained by not abandoning \( \ddot{a} \). Following [2, 3], look first at

\[
\tilde{a}_1 = -2 \cdot \left( -\frac{\dot{a}}{a} + \frac{3}{2} \right) \\
\tilde{b}_1 = -\frac{1}{4} \cdot \left[ 1 + 4 \cdot \left( -\frac{\dot{a}}{a} + \frac{1}{256\pi G \cdot t^2} - \frac{\Lambda}{8} \right) \right]. \quad (17)
\]

Our approximation is to set \( \ddot{a} \) as a nonzero constant. In that case, set \( \ddot{a} \) as a nondimensional but very large quantity. Then, a solution exists as given as for a reduced cubic version of Eq. (7), which can be given by

\[
\xi = A_1 + B_1, \frac{\sqrt{-3}}{2} \cdot (A_1 - B_1) - \left( A_1 + B_1 \right), -\frac{\sqrt{-3}}{2} \cdot (A_1 - B_1) - \left( A_1 + B_1 \right) \quad (18)
\]

and

\[
A_1 = \frac{3}{2} \left[ -\frac{\tilde{b}_1}{2} + \sqrt{\left( -\frac{\tilde{b}_1}{2} \right)^2 + \frac{\tilde{a}_1}{27}} \right] \\
B_1 = \frac{3}{2} \left[ -\frac{\tilde{b}_1}{2} + \sqrt{\left( -\frac{\tilde{b}_1}{2} \right)^2 + \frac{\tilde{a}_1}{27}} \right] \quad (19)
\]

When \( \ddot{a} \) is set as a nondimensional constant quantity and possibly quite large, then

\[
\frac{\dot{a}}{a} = \text{Solution} = \xi_1 + \frac{1}{2} \quad (20)
\]
If so, then
\[ \Theta = \frac{\tilde{b}_1^2}{4} + \frac{\tilde{a}_1^3}{27}. \]  
(21)

If \( \ddot{a} \) is constant and very large, the results of the sign of Eq. (21) are as follows.

- \( \Theta > 0 \) \( \Rightarrow \) \( \xi \) has one real and two imaginary roots.
- \( \Theta = 0 \) \( \Rightarrow \) \( \xi \) has three real roots, two of which are equal.
- \( \Theta < 0 \) \( \Rightarrow \) \( \xi \) has three real roots, all of which are unique.  
(22)

Here, with very large constant initial \( \ddot{a} \), we have that the third outcome is by far most likely to happen, in contrast to what would happen in the situation with \( \ddot{a} = 0 \) \[2\], and all of this will make sense of the transition to the “normal” HUP in which we will be considering \[1\]

\[ \delta g_{tt} \approx a^2(t) \cdot \phi \quad \phi \text{ very large} \rightarrow 1. \]  
(23)

In short, we would require an enormous inflaton style \( \phi \)-valued scalar function, and \( a^2(t) \approx 10^{-110} \). We will next discuss the implications of this point in the next section, of a nonzero smallest-scale factor. Secondly, the fact that we are working with a massive graviton will be given some credence when we obtain a lower bound, as will come up in our derivation of modification of the values \[4\]

\[ \langle (\delta g_{uv})^2 (\tilde{T}_{uv}) \rangle \geq \frac{h^2}{V^2}, \quad \langle (\delta g_{tt})^2 (\tilde{T}_{tt}) \rangle \geq \frac{h^2}{V^2} \quad \& \quad \delta g_{rr} \approx \delta g_{\theta\theta} \approx \delta g_{\phi\phi} \approx 0^+. \]  
(24)

The reasons for saying this set of values for the variation of the non-\( g_{tt} \) metric will be presented in the third section: It is due to the smallness of the square of the scale factor in the vicinity of the Planck time interval.

### 3. Nonzero scale factor, initially, and what this tells us physically.

We start with a configuration from Unruh \[5, 6\].

\[ \Delta l \cdot \Delta p \geq \frac{\hbar}{2} \]  
(25)

We will be using the approximation given by Unruh \[5, 6\], of a generalization we will write as

\[ (\Delta l)_{ij} = \frac{\delta g_{ij}}{g_{ij}} \cdot \frac{l}{2} \]  
\[ (\Delta p)_{ij} = \Delta T_{ij} \cdot \delta t \cdot \Delta A. \]  
(26)

We use the following from the Roberson–Walker metric \[7\].

\[ g_{tt} = 1 \]  
\[ g_{rr} = -\frac{a^2(t)}{1 - k \cdot r^2} \]  
\[ g_{\theta\theta} = -a^2(t) \cdot r^2 \]  
\[ g_{\phi\phi} = -a^2(t) \cdot \sin^2 \theta \cdot d\phi^2 \]  
(27)

Following Unruh \[5, 6\], write then, an uncertainty-of-metric tensor, with the following inputs:

\[ a^2(t) \approx 10^{-110}, r \equiv l_p \approx 10^{-35} \text{ m}. \]  
(28)
Then, if $\Delta T_{tt} \approx \Delta \rho,$

$$V^{(4)} = \delta t \cdot \Delta A \cdot r$$

$$g_{rr} \cdot \Delta T_{tt} \cdot \delta t \cdot \Delta A \cdot \frac{r}{2} \geq \frac{h}{2}$$

$$\Leftrightarrow \delta g_{tt} \cdot \Delta T_{tt} \geq \frac{h}{V^{(4)}}.$$ (29)

Eq. (29) is such that we can extract, up to a point, the HUP in time and energy, with one very large caveat: we must use the fluid approximation of space–time [6]

$$T_{ii} = \text{diag}(\rho, -p, -p, -p).$$

Then, Eq. (29) and Eq. (30) together yield

$$\delta t \Delta E \geq \frac{h}{\delta g_{tt}} \neq \frac{h}{2}, \text{ unless } \delta g_{tt} \approx O(1).$$ (31)

How likely is $\delta g_{tt} \approx O(1)$? Not going to happen. Why? The homogeneity of the early universe will keep

$$\delta g_{tt} \neq g_{tt} = 1.$$ (32)

In fact, we have that, from Giovannini [7], if $\phi$ is a scalar function and $a^2(t) \approx 10^{-110},$ then if

$$\delta g_{tt} \approx a^2(t) \cdot \phi \ll 1,$$ (33)

there is no way that Eq. (31) is going to come close to $\delta t \Delta E \geq \frac{h}{2}.$

4. How we can justify writing very small $\delta g_{rr} \approx \delta g_{\theta \theta} \approx \delta g_{\phi \phi} \approx 0^+$ values?

To begin this process, we break it down into the coordinates $rr, \theta \theta,$ and $\phi \phi.$ We will use the fluid approximation, $T_{ii} = \text{diag}(\rho, -p, -p, -p)$ [8], with [1]

$$\delta g_{rr} \geq -\left[ \frac{h \cdot a^2(t) \cdot r^2}{V^{(4)}} \right] \xrightarrow{a \rightarrow 0} 0$$ (34)

$$\delta g_{rr} \geq -\left[ \frac{h \cdot a^2(t)}{V^{(4)} (1 - k \cdot r^2)} \right] \xrightarrow{a \rightarrow 0} 0$$ (35)

$$\delta g_{rr} \geq -\left[ \frac{h \cdot a^2(t) \cdot \sin^2 \theta \cdot d\phi^2}{V^{(4)}} \right] \xrightarrow{a \rightarrow 0} 0$$ (36)

If, as an example, we have negative pressure, with $T_{rr}, T_{\theta \theta},$ and $T_{\phi \phi} < 0,$ and $p = -\rho,$ then the only choice we have is to set $\delta g_{rr} \approx \delta g_{\theta \theta} \approx \delta g_{\phi \phi} \approx 0^+,$ since there is no way that $p = -\rho$ is zero valued. Having said this, the value of $\delta g_{rr}$ being nonzero will be part of how we will be looking at a nonzero lower bound of the graviton mass.

5. Conclusion

To solidify the approach given here in terms of early universe GR theory, we refer to Einstein spaces, via [9], as well as make certain of the stress-energy tensor [10] as we can write it as a modified Einstein field equation. With, then, $\aleph$ as a constant,

$$R_{ij} = \aleph g_{ij}.$$ (37)
Here, the term in the left hand side of the metric tensor is a constant. So then, we can write, with $R$ also a constant [10],

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{ij}} = -\frac{1}{8\pi} \cdot \left[ R - R + \Lambda \right] \cdot g_{ij}. \quad (38)$$

The terms, if we use the fluid approximation, will then tend to a constant energy term on the RHS of Eq. (37) as well as restricting $i$, and $j$ to $t$ and $t$. If so, we have via an Einstein space, which may be more fully developed, in the pre-Planckian regime, a tight justification of Eq. (31) to Eq. (33). We also have the possibility of tying this in with Barbour [11]. This has a tie in with the lower bound to the graviton mass as brought up in Eq. (4), Eq. (5), and Eq. (6), which should be developed more fully later.

Acknowledgments

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