GROUND STATE SOLUTION OF FRACTIONAL SCHRÖDINGER EQUATIONS WITH A GENERAL NONLINEARITY*

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Abstract. In this paper, we study the following fractional Schrödinger equation:
\[
\begin{aligned}
(-\Delta)^s u + mu &= f(u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^s(\mathbb{R}^N), \quad u > 0 \quad \text{on } \mathbb{R}^N,
\end{aligned}
\]
where \( m > 0, \ N > 2s, \ (-\Delta)^s, \ s \in (0, 1) \) is the fractional Laplacian. Using minimax arguments, we obtain a positive ground state solution under general conditions on \( f \) which we believe to be almost optimal.

**Key words**: ground state solution; fractional Schrödinger equation; critical growth.

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1. Introduction and Main Result

We consider the following fractional Schrödinger equation:
\[
\begin{aligned}
(-\Delta)^s u + mu &= f(u) \quad \text{in } \mathbb{R}^N, \\
u &\in H^s(\mathbb{R}^N), \quad u > 0 \quad \text{on } \mathbb{R}^N,
\end{aligned}
\]  
(1.1)

where \( m > 0, \ N > 2s, \ (-\Delta)^s, \ s \in (0, 1) \) is the fractional Laplacian. The nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function. Since we are looking for positive solutions, we assume that \( f(t) = 0 \) for \( t < 0 \). Furthermore, we need the following conditions:

\begin{enumerate}
\item[(f_1)] \( \lim_{t \to 0^+} f(t)/t = 0; \)
\item[(f_2)] \( \lim_{t \to +\infty} f(t)/t^{2^*_s - 1} = 1 \) where \( 2^*_s = 2N/(N - 2s); \)
\item[(f_3)] \( \exists \lambda > 0 \) and \( 2 < q < 2^*_s \) such that \( f(t) \geq \lambda t^{q-1} + t^{2^*_s - 1} \) for \( t \geq 0. \)
\end{enumerate}

Note that, for the case \( s = 1 \), \((f_1)-(f_3)\) were first introduced by J. Zhang, Z. Chen and W. Zou [25]. This hypothesis can be regarded as an extension of the celebrated Berestycki-Lions' type nonlinearity (see [5, 6]) to the fractional Schrödinger equations with critical growth.

Equation (1.1) has been derived as models of many physical phenomena, such as phase transition, conservation laws, especially in fractional quantum mechanics, etc., [16]. (1.1) was introduced by N. Laskin [19, 20] as an extension of the classical nonlinear Schrödinger
equations $s = 1$ in which the Brownian motion of the quantum paths is replaced by a Lévy flight. We refer to [15] for more physical backgrounds.

In recent years, the study of fractional Schrödinger equations has attracted much attention from many mathematicians. In [9] [10] [23], L. Caffarelli, L. Silvestre et al investigated free boundary problems of fractional Schrödinger equations and obtained some regularity estimates. In [7] [8], X. Cabré and Y. Sire studied the existence, uniqueness, symmetry, regularity, maximum principle and qualitative properties of solutions to the fractional Schrödinger equations in the whole space. For more results, we refer to [1] [3] [4] [12] [15] [16] [18] [22].

Our main result is as follows:

**Theorem 1.1.** Assume that the nonlinearity $f$ satisfies $(f_1)-(f_3)$. If $N \geq 4s$, $2 < q < 2^*_s$ or $2s < N < 4s$, $4s/(N - 2s) < q < 2^*_s$, then for every $\lambda > 0$, (1.1) possesses a positive ground state solution. Moreover, the same conclusion holds provided that $2s < N < 4s$, $2 < q \leq 4s/(N - 2s)$ and $\lambda > 0$ sufficiently large.

We note that, to the best of our knowledge, there is no result on the existence of positive ground state solutions for fractional Schrödinger equation under $(f_1)-(f_3)$.

The proof of Theorem 1.1 is based on variational method. The main difficulties lie in two aspects: (i) The facts that the nonlinearity $f(u)$ does not satisfy (AR) condition and the function $f(s)/s$ is not increasing for $s > 0$ prevent us from obtaining a bounded Palais-Smale sequence (PS sequence in short) and using the Nehari manifold respectively. (ii) The unboundedness of the domain $\mathbb{R}^N$ and the nonlinearity $f(u)$ with critical growth lead to the lack of compactness.

To complete this section, we sketch our proof.

To treat the nonlocal problem (1.1), we use the L. Caffarelli and L. Silvestre extension method [11] to study a corresponding extension problem

\begin{equation}
\begin{cases}
-\text{div}(y^{1-2s}\nabla w) = 0 \text{ in } \mathbb{R}^{N+1}, \\
-k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = -mw + f(w) \text{ on } \mathbb{R}^N \times \{0\}.
\end{cases}
\end{equation}

with the corresponding functional

$$I_m(w) = \frac{k_s}{2} \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \frac{m}{2} \int_{\mathbb{R}^N} w^2(x, 0) dx - \int_{\mathbb{R}^N} F(w(x, 0)) dx, \ w \in X^{1,s}(\mathbb{R}^{N+1}).$$

where $F(s) := \int_0^s f(t) dt$ and $X^{1,s}(\mathbb{R}^{N+1})$ is defined as the completion of $C_0^\infty(\mathbb{R}^{N+1})$ under the norm

$$\|w\|_{X^{1,s}(\mathbb{R}^{N+1})} = \left( \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \int_{\mathbb{R}^N} w^2(x, 0) dx \right)^{1/2}.$$

Motivated by J. Hirata, N. Ikoma and K. Tanaka [17], by applying the General Minimax principle (Theorem 2.8 of [24]) to the composite functional

$$I_m \circ \Phi(\theta, w) := I_m(w(e^{-\theta}x, e^{-\theta}y)), \ (\theta, w) \in \mathbb{R} \times X^{1,s}(\mathbb{R}^{N+1}),$$

we conclude that...
we construct a bounded (PS)$_{c_m}$ sequence $\{w_n\}_{n=1}^{\infty} \subset X^{1,s}(\mathbb{R}^{N+1}_+)$ with an extra property $P_m(w_n) \to 0$ as $n \to \infty$ where $c_m$ is the mountain pass level of $I_m$ and $P_m(w) = 0$ is the Pohozaev’s identity of (1.2) (Proposition 3.2 below). Proceeding by standard arguments, the existence of ground state solutions for (1.2) follows.

This paper is organized as follows, in Section 2, we give some preliminary results. In Section 3, we prove the main result Theorem 1.1.

2. Preliminaries

In this section, we collect some preliminary results. Recall that for $s \in (0,1)$, $D^s(\mathbb{R}^N)$ is defined by the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo norm

$$
\|u\|_{D^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2}
$$

and the embedding $D^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is continuous, that is

$$
\|u\|_{L^{2^*_s}(\mathbb{R}^N)} \leq C(N,s)\|u\|_{D^s(\mathbb{R}^N)}
$$

by Theorem 1 of [21]. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right\}
$$

endowed with the norm

$$
\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{D^s(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}.
$$

For $N > 2s$, we see from Lemma 2.1 of [1] that

$$
H^s(\mathbb{R}^N) \text{ is continuously embedded in } L^p(\mathbb{R}^N) \text{ for } p \in [2, 2^*_s].
$$

An important feature of the operator $(-\Delta)^s(0 < s < 1)$ is its nonlocal character. A common approach to deal with this problem was proposed by L. Caffarelli and L. Silvestre [11], allowing to transform (1.1) into a local problem via the Dirichlet-Neumann map in the domain $\mathbb{R}^{N+1}_+ := \{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$. For $u \in D^s(\mathbb{R}^N)$, the solution $w \in X^s(\mathbb{R}^{N+1}_+)$ of

$$
\begin{cases}
-\text{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
w = u & \text{on } \mathbb{R}^N \times \{0\}
\end{cases}
$$

is called $s$-harmonic extension of $u$, denoted by $w = E_s(u)$. The $s$-harmonic extension and the fractional Laplacian have explicit expressions in terms of the Poisson and the Riesz kernels, respectively

$$
w(x,y) = P_y^s * u(x) = \int_{\mathbb{R}^N} P_y^s(x - \xi) u(\xi) d\xi,
$$

where

$$
P_y^s(x) := c(N,s) \frac{y^{2s}}{(|x|^2 + y^2)^{(N+2s)/2}}
$$
with a constant $c(N, s)$ such that $\int_{\mathbb{R}^N} P_{1}^s(x) dx = 1$ (see [18]).

Here, the space $X^s(\mathbb{R}^{N+1}_+)$ is defined as the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ under the norm

$$
\|w\|_{X^s(\mathbb{R}^{N+1}_+)} := \left( \int_{\mathbb{R}^{N+1}_+} k_s y^{2s} |\nabla w|^2 dy \right)^{1/2}.
$$

From [4], the map $E_s(\cdot)$ is an isometry between $D^s(\mathbb{R}^N)$ and $X^s(\mathbb{R}^{N+1}_+)$, i.e. for $w = E_s(u)$,

$$
\|u\|_{D^s(\mathbb{R}^N)} = \|w\|_{X^s(\mathbb{R}^{N+1}_+)}, \quad (2.3)
$$

On the other hand, for a function $w \in X^s(\mathbb{R}^{N+1}_+)$, we shall denote its trace on $\mathbb{R}^N \times \{0\}$ as $u(x) := \text{Tr}(w) = w(x, 0)$. This trace operator is also well defined and it satisfies

$$
\|u\|_{D^s(\mathbb{R}^N)} \leq \|w\|_{X^s(\mathbb{R}^{N+1}_+)}, \quad (2.4)
$$

**Lemma 2.1.** (Theorem 2.1 of [4]) For every $w \in X^s(\mathbb{R}^{N+1}_+)$, it holds that

$$
S(s, N) \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{2/2s} \leq \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w|^2 dy,
$$

where $u = \text{Tr}(w)$. The best constant takes the exact value

$$
S(s, N) = \frac{2\pi^s \Gamma((N + 2s)/2)\Gamma(N/2)^{2s/N}}{\Gamma(s)\Gamma((N - 2s)/2)\Gamma(N)^{2s/N}}
$$

and it is achieved when $u_\delta$ takes the form

$$
u_\delta(x) = \delta^{(N-2s)/2}(|x|^2 + \delta^2)^{-(N-2s)/2}
$$

for some $\delta > 0$ and $w_\delta = E_s(u_\delta)$.

### 3. Proof of the main results

In view of [11], (11) can be transformed into

$$
\begin{cases}
-\text{div}(y^{1-2s} \nabla w) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+,

-k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = -mw + f(w) \quad \text{on} \quad \mathbb{R}^N \times \{0\}
\end{cases}
$$

with the corresponding functional

$$
I_m(w) = \frac{k_s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w|^2 dy + \frac{m}{2} \int_{\mathbb{R}^N} w^2(x, 0) dx - \int_{\mathbb{R}^N} F(w(x, 0)) dx, \quad w \in X^{1,s}(\mathbb{R}^{N+1}_+).
$$

In view of [12] [22], if $w \in X^{1,s}(\mathbb{R}^{N+1}_+)$ is a weak solution to (3.1), the following Pohozaev’s identity holds:

$$
P_m(w) = \frac{k_s(N-2s)}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w|^2 dy + \frac{mN}{2} \int_{\mathbb{R}^N} w^2(x, 0) dx - N \int_{\mathbb{R}^N} F(w(x, 0)) dx = 0. \quad (3.2)$$
Lemma 3.1. \( I_m \) possesses the Mountain-Pass geometry (see \( [2] \)), i.e.

(i) There exist \( \rho_0, \alpha_0 > 0 \) such that \( I_m(w) \geq \alpha_0 \) for all \( w \in X^{1,s}(\mathbb{R}^{N+1}_+) \) with \( \|w\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} = \rho_0 \).

(ii) \( \exists w_0 \in X^{1,s}(\mathbb{R}^{N+1}_+) \) such that \( I_m(w_0) < 0 \).

Proof. (i) By \( (f_1) \) and \( (f_2), \forall \delta > 0, \exists C_\delta > 0 \) such that

\[
f(w) \leq \delta |w|^2 + C_\delta |w|^{2*} - 1 \quad \text{and} \quad F(w) \leq \delta |w|^2 + C_\delta |w|^{2*}. \tag{3.3}
\]

Choosing \( \delta = m/4 \) in \( (3.3) \), we see from Lemma 2.1 that

\[
I_m(w) \geq \frac{1}{4} \|w\|^2_{X^{1,s}(\mathbb{R}^{N+1}_+)} - C \|w\|^2_{X^{1,s}(\mathbb{R}^{N+1}_+),}
\]

then taking \( \rho_0, \alpha_0 > 0 \) small, \( i) \) holds.

(ii) For \( R > 0, T > 0 \), we define

\[
w_R,T(x, y) = \begin{cases} T, & \text{if } (x, y) \in B_{R}^{+}(0), \\ T(R + 1 - (|x|^2 + y^2)^{1/2}), & \text{if } (x, y) \in B_{R+1}^{+}(0) \setminus B_{R}^{+}(0), \\ 0, & \text{if } (x, y) \in \mathbb{R}^{N+1}_+ \setminus B_{R+1}^{+}(0), \end{cases}
\]

then \( w_R \in X^{1,s}(\mathbb{R}^{N+1}_+) \). By \( (f_3) \) and the polar coordinate transformation, we have

\[
I_m(w_{R,T}(x, \theta, y/\theta)) = \frac{k_s \theta^{-2s}}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_{R,T}|^2 dx + \theta^N \left[ \frac{m}{2} \int_{\mathbb{R}^N} w_{R,T}^2(x, 0) dx - \int_{\mathbb{R}^N} F(w_{R,T}(x, 0)) dx \right] 
\]

\[
\leq \frac{k_s \theta^{-2s} T^2}{2} \int_{B_{R+1}^{+}(0) \setminus B_{R}^{+}(0)} y^{1-2s} dx + \theta^N \left[ \frac{m}{2} \int_{B_{R+1}^{+}(0)} w_{R,T}^2(x, 0) dx - \frac{1}{2^*} \int_{B_{R}^{+}(0)} w_{R,T}^{2*}(x, 0) dx \right] 
\]

\[
\leq C \theta^{-2s} T^2 \int_{R}^{R+1} r^{N+1-2s} dr + \theta^N \left[ C \left( \frac{m}{2} T^2 - \frac{1}{2^*} T^{2*} \right) R^N + C T^2 ((R + 1)^N - R^N) \right] 
\]

\[
\leq C T^2 R^{N+1-2s} \theta^{-2s} + \left( C \frac{m}{2} T^2 - \frac{1}{2^*} T^{2*} \right) R^N + C T^2 R^{N-1} \theta^N.
\]

Choosing a large \( T_0 > 0 \) such that \( \frac{m}{2} T^2_0 - \frac{1}{2^*} T^{2*}_0 < 0 \), then we can choose a large \( R_0 > 0 \) such that \( C \left( \frac{m}{2} T^2_0 - \frac{1}{2^*} T^{2*}_0 \right) R_0^N + C T^2 R_0^{N-1} < 0 \), at last, we select a large \( \bar{\theta} > 0 \) to ensure that \( I_m(w_{R_0, \bar{\theta}}(x/\bar{\theta}, y/\bar{\theta})) < 0 \), \( w_{R_0, \bar{\theta}} \) is the desired \( w_0 \).

Hence we define the Mountain-Pass level of \( I_m \):

\[
c_m := \inf_{\gamma \in \Gamma_m} \sup_{t \in [0,1]} I_m(\gamma(t)), \tag{3.4}
\]

where the set of paths is defined as

\[
\Gamma_m := \{ \gamma \in C([0,1], X^{1,s}(\mathbb{R}^{N+1}_+)) : \gamma(0) = 0 \text{ and } I_m(\gamma(1)) < 0 \} \tag{3.5}
\]

By Lemma 3.1(i), we see that \( c_m > 0 \). Moreover, we denote

\[
b_m := \inf \{ I_m(w) : w \in X^{1,s}(\mathbb{R}^{N+1}_+) \setminus \{0\} \text{ be a nontrivial solution of } (3.1) \}.
\]
Next, we will construct a (PS) sequence \( \{w_n\}_{n=1}^{\infty} \) for \( I_m \) at the level \( c_m \) that satisfies 

\[ P_m(w_n) \to 0 \quad \text{as} \quad n \to \infty, \]

i.e.

**Proposition 3.2.** There exists a sequence \( \{w_n\}_{n=1}^{\infty} \) in \( X^{1,s}(\mathbb{R}_+^{N+1}) \) such that, as \( n \to \infty, \)

\[ I_m(w_n) \to c_m, \quad I_m'(w_n) \to 0, \quad P_m(w_n) \to 0. \]  

**Proof.** Define the map \( \Phi: \mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1}) \to X^{1,s}(\mathbb{R}_+^{N+1}) \) for \( \theta \in \mathbb{R}, \ w \in X^{1,s}(\mathbb{R}_+^{N+1}) \) and \( (x, y) \in \mathbb{R}_+^{N+1} \) by \( \Phi(\theta, w) = w(e^{-\theta}x, e^{-\theta}y) \). For every \( \theta \in \mathbb{R}, \ w \in X^{1,s}(\mathbb{R}_+^{N+1}) \), the functional \( I_m \circ \Phi \) is computed as

\[
I_m \circ \Phi(\theta, w) = \frac{k_s}{2} e^{(N-2)s\theta} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 \, dx \, dy + \frac{m}{2} e^{N\theta} \int_{\mathbb{R}^N} w^2(x, 0) \, dx
- e^{N\theta} \int_{\mathbb{R}^N} F(w(x, 0)) \, dx.
\]

By Lemma 3.1, \( (I_m \circ \Phi)(\theta, w) > 0 \) for all \( (\theta, w) \) with \( |\theta|, \|w\|_{X^{1,s}(\mathbb{R}_+^{N+1})} \) small and \( (I_m \circ \Phi)(0, w_0) < 0 \), i.e. \( I_m \circ \Phi \) possesses the Mountain-Pass geometry in \( \mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1}) \). The Mountain-Pass level of \( I_m \circ \Phi \) is defined by

\[
\bar{c}_m := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_m} \sup_{t \in [0,1]} (I_m \circ \Phi)(\tilde{\gamma}(t)),
\]

where the set of paths is

\[
\tilde{\Gamma}_m := \{ \tilde{\gamma} \in C([0,1], \mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1})): \tilde{\gamma}(0) = (0,0) \text{ and } (I_m \circ \Phi)(\tilde{\gamma}(1)) < 0 \}. \tag{3.8}
\]

As \( \tilde{\Gamma}_m = \{ \Phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}_m \} \), the Mountain-Pass levels of \( I_m \) and \( I_m \circ \Phi \) coincide, i.e. \( c_m = \bar{c}_m \).

By the General Minimax principle (Theorem 2.8 of [24]), there exists a sequence \( \{(\theta_n, v_n)\}_{n=1}^{\infty} \) in \( \mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1}) \) such that as \( n \to \infty, \)

\[
(I_m \circ \Phi)(\theta_n, v_n) \to c_m, \tag{3.9}
\]

\[
(I_m \circ \Phi)'(\theta_n, v_n) \to 0 \quad \text{in} \quad (\mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1}))^{-1}, \tag{3.10}
\]

\[
\theta_n \to 0. \tag{3.11}
\]

Indeed, set \( \varepsilon = \varepsilon_n := 1/n^2, \delta = \delta_n := 1/n \) in Theorem 2.8 of [24], then (3.9), (3.10) are direct conclusions from (a), (c) in Theorem 2.8 of [24]. By (3.4) and (3.5), for \( \varepsilon = \varepsilon_n := 1/n^2, \)

\[
\exists \gamma_n \in \Gamma_m, \text{ such that } \sup_{t \in [0,1]} I_m(\gamma_n(t)) \leq c_m + 1/n^2.
\]

Set \( \gamma_n(t) = (0, \gamma_n(t)) \), then

\[
\sup_{t \in [0,1]} (I_m \circ \Phi)(\gamma_n(t)) = \sup_{t \in [0,1]} I_m(\gamma_n(t)) \leq c_m + 1/n^2.
\]

From (b) in Theorem 2.8 of [24], \( \exists (\theta_n, v_n) \in \mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1}) \) such that \( \text{dist}((\theta_n, v_n), (0, \gamma_n(t))) \leq 2/n, \) then (3.11) holds.

For every \( (h, w) \in \mathbb{R} \times X^{1,s}(\mathbb{R}_+^{N+1}), \)

\[
\langle (I_m \circ \Phi)'(\theta_n, v_n), (h, w) \rangle = \langle I_m'(\Phi(\theta_n, v_n)), \Phi(\theta_n, w) \rangle + P_m(\Phi(\theta_n, v_n))h.
\]

Taking \( h = 1, \ w = 0 \) in (3.12), we have

\[
P_m(\Phi(\theta_n, v_n)) \to 0(n \to \infty).
\]
For every \( v \in X^{1,s}(\mathbb{R}_+^{N+1}) \), set \( w(x,y) = v(e^{th}x, e^{th}y) \), \( h = 0 \) in (3.12), by (3.11), we get
\[
\langle I_m' (\Phi(\theta_n, v_n)), v \rangle = o(1)\|v(e^{th}x, e^{th}y)\|_{X^{1,s}(\mathbb{R}_+^{N+1})} = o(1)\|v\|_{X^{1,s}(\mathbb{R}_+^{N+1})}.
\] (3.14)

Denote \( w_n := \Phi(\theta_n, v_n) \) in (3.9), (3.13) and (3.14), we get (3.6).

**Lemma 3.3.** Every sequence \( \{w_n\}_{n=1}^\infty \) satisfying (3.6) is bounded in \( X^{1,s}(\mathbb{R}_+^{N+1}) \).

**Proof.** By (3.6),
\[
c_m + o(1) = I_m(w_n) - \frac{1}{N} P_m(w_n) = \frac{s}{N} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dxdy,
\] (3.15)
we get the upper bound of \( \|w_n\|_{X^{1,s}(\mathbb{R}_+^{N+1})} \), then by Lemma 2.1 we see that \( \{w_n(x,0)\} \) is bounded in \( L^{2^*_s}(\mathbb{R}^N) \). From (3.6) and (3.3), we see that
\[
\frac{k_s(N-2s)}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dxdy + \frac{mN}{2} \int_{\mathbb{R}^N} w_n^2(x,0)dx
\] = \[ N \int_{\mathbb{R}^N} F(w_n(x,0))dx + o(1) \leq \frac{mN}{4} \int_{\mathbb{R}^N} w_n^2(x,0)dx + C \int_{\mathbb{R}^N} w_n^{2^*_s}(x,0)dx + o(1),
\] (3.16)
hence \( \{w_n\} \) is bounded in \( X^{1,s}(\mathbb{R}_+^{N+1}) \).

For the Mountain-Pass level \( c_m \), we have the following estimate:

**Lemma 3.4.** If \( N \geq 4s, 2 < q < 2^*_s \) or \( 2s < N < 4s, 4s/(N-2s) < q < 2^*_s \), then for all \( \lambda > 0, c_m < \frac{\lambda}{N}(k_s S(s, N))^{N/(2s)}. \) Moreover, if \( 2s < N < 4s \) and \( 2 < q \leq 4s/(N-2s) \), then for \( \lambda > 0 \) sufficiently large, the same conclusion holds.

**Proof.** Let \( \phi_0 \in C^\infty(\mathbb{R}_+) \) satisfying
\[
\phi_0(t) = 1 \text{ if } 0 \leq t \leq 1, \quad \phi_0(t) = 0 \text{ if } t \geq 2, \quad 0 \leq \phi_0(t) \leq 1 \text{ and } |\phi_0'(t)| \leq C
\]
and denote \( \phi_s \) the \( s \)-harmonic extension of \( u_\delta \) in Lemma 2.1. Denote
\[
v_\delta(x,y) = \frac{\phi_0(||x||^2 + y^2^{1/2}) u_\delta(x,y)}{\|\phi_0(||x||u_\delta(x)||_{L^{2^*_s}(\mathbb{R}^N)}},
\] by (3.14), we see that
\[
\|v_\delta\|_{X^{1,s}(\mathbb{R}_+^{N+1})}^2 = k_s S(s, N) + O(\delta^{N-2s}),
\] (3.17)
and for any \( p \in [2, 2^*_s) \),
\[
\|v_\delta(x,0)\|_{L^p(\mathbb{R}^N)}^p = \begin{cases}
O(\delta^{(2N-(N-2s)p)/2}), & \text{if } p > N/(N-2s), \\
O(\delta^{N/2} \log \delta), & \text{if } p = N/(N-2s), \\
O(\delta^{(N-2s)p/2}), & \text{if } p < N/(N-2s).
\end{cases}
\] (3.18)
By \((f_3)\),
\[
I_m(v_{\delta}(x/t)) \leq g_{\delta}(t)
\]
\[
:= \frac{k_s}{2} t^{N-2s} \int_{\mathbb{R}^{N+1}} y^{-2s} |\nabla v_{\delta}|^2 dxdy - \frac{1}{2} t^N + \frac{m}{2} t^N \int_{\mathbb{R}^N} v_{\delta}^2(x,0)dx - \frac{\lambda}{q} t^N \int_{\mathbb{R}^N} v_{\delta}^q(x,0)dx
\]
\[
:= h_{\delta}(t) + \frac{m}{2} t^N \int_{\mathbb{R}^N} v_{\delta}^2(x,0)dx - \frac{\lambda}{q} t^N \int_{\mathbb{R}^N} v_{\delta}^q(x,0)dx.
\]
In view of \((3.17)\) and \((3.18)\), for \(\delta > 0\) small, \(g_{\delta}(t)\) has a unique critical point \(t_{\delta} > 0\) which corresponds to its maximum. Therefore, we check from \((3.17)\) and \((3.18)\) that
\[
t_{\delta} = \frac{\left(\frac{N-2s}{2}\right)^{1/2s} \|v_{\delta}\|^{1/s}_{X_s^+(\mathbb{R}^{N+1})}}{\left(\frac{N-2s}{2} - \frac{mN}{2} \|v_{\delta}(x,0)\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda N}{q} \|v_{\delta}(x,0)\|^q_{L^q(\mathbb{R}^N)}\right)^{1/2s}},
\]
then we see from \((3.17)\) and \((3.18)\) that
\[
0 < C_1 \leq t_{\delta} \leq C_2 \text{ for } \delta > 0 \text{ small.}
\]
By \((3.17)\) and \((3.18)\), we get
\[
\max_{t \geq 0} h_{\delta}(t) = h_{\delta}(t_{\delta}') = \frac{s}{N} (k_s S(s, N))^{N/2s} + O(\delta^{N-2s}),
\]
where \(t_{\delta}' = \|v_{\delta}\|^{1/s}_{X_s^+(\mathbb{R}^{N+1})}\) is the maximum point of \(h_{\delta}(t)(t > 0)\).

By \((3.19)\) and \((3.20)\), we have
\[
c_m \leq \sup_{t > 0} I_m(v_{\delta}(x/t)) \leq \sup_{t > 0} g_{\delta}(t)
\]
\[
\leq h_{\delta}(t_{\delta}) + \frac{m}{2} t_{\delta}^N \int_{\mathbb{R}^N} v_{\delta}^2(x,0)dx - \frac{\lambda}{q} t_{\delta}^N \int_{\mathbb{R}^N} v_{\delta}^q(x,0)dx
\]
\[
\leq \frac{s}{N} (k_s S(s, N))^{N/(2s)} + O(\delta^{N-2s}) + C \|v_{\delta}(x,0)\|^2_{L^2(\mathbb{R}^N)} - C \lambda \|v_{\delta}(x,0)\|^q_{L^q(\mathbb{R}^N)}.
\]
Next, we distinguish the following cases:
(i) If \(N > 4s\), then \(q > 2 > N/(N-2s)\), by \((3.18)\) and \((3.21)\), we get
\[
c_m \leq \frac{s}{N} (k_s S(s, N))^{N/(2s)} + O(\delta^{N-2s}) + O(\delta^{2s}) - \lambda \cdot O(\delta^{(2N-(N-2s)q)/2}).
\]
In view of \((2N-(N-2s)q)/2 < 2s < (N-2s)\), we get the conclusion for \(\delta > 0\) small.
(ii) If \(N = 4s\), then \(q = 2 > N/(N-2s)\), by \((3.18)\) and \((3.21)\), we have
\[
c_m \leq \frac{s}{N} (k_s S(s, N))^{N/(2s)} + O(\delta^{2s}(1 + |\log \delta|)) - \lambda \cdot O(\delta^{4s-sq}).
\]
Since \(4s - sq < 2s\), we get the conclusion for \(\delta > 0\) small.
(iii) If \(2s < N < 4s\) and \(N/(N-2s) < q < 2^*_s\), we see from \((3.18)\) and \((3.21)\) that
\[
c_m \leq \frac{s}{N} (k_s S(s, N))^{N/(2s)} + O(\delta^{N-2s}) - \lambda \cdot O(\delta^{(2N-(N-2s)q)/2}).
\]
If \(4s/(N - 2s) < q < 2^*_s\), then \((N - 2s) > (2N - (N - 2s)q)/2\), we get the conclusion for \(\delta > 0\) small. If \(N/(N - 2s) < q \leq 4s/(N - 2s)\), then \((N - 2s) \leq (2N - (N - 2s)q)/2\), we choose \(\lambda = \delta^{-\theta}\) with \(\theta > (2N - (q + 2)(N - 2s))/2 > 0\), we still get the conclusion for \(\delta > 0\) small.

(iv) If \(2s < N < 4s\) and \(q = N/(N - 2s)\), (3.18) and (3.21) yield

\[
c_m \leq \frac{s}{N}(k_s S(s, N))^{N/(2s)} + O(\delta^{N - 2s}) - \lambda \cdot O(\delta^{N/2} |\log \delta|).
\]

Since \((N - 2s) < N/2\), we choose \(\lambda = \delta^{-\theta}\) with \(\theta > 2s - (N/2)\), we get the conclusion for \(\delta > 0\) small.

(v) If \(2s < N < 4s\) and \(2 < q < N/(N - 2s)\), (3.18) and (3.21) show that

\[
c_m \leq \frac{s}{N}(k_s S(s, N))^{N/(2s)} + O(\delta^{N - 2s}) - \lambda \cdot O(\delta^{N - 2s}q/2).
\]

We choose \(\lambda = \delta^{-\theta}\) with \(\theta > (q - 2)(N - 2s)/2\), we get the conclusion for \(\delta > 0\) small. □

**Lemma 3.5.** There is a sequence \(\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N\) and \(R > 0, \beta > 0\) such that

\[
\int_{\Gamma_R(x_n)} w_n^2(x, 0)dx \geq \beta,
\]

where \(\{w_n\}_{n=1}^\infty\) is the sequence given in (3.6).

**Proof.** Assuming on the contrary that the lemma does not hold, then by Lemma 2.2 of [16], it follows that

\[
\int_{\mathbb{R}^N} |w_n(x, 0)|^p dx \to 0 \text{ as } n \to \infty \text{ for all } 2 < p < 2^*_s.
\]

Since \(\langle I'_m(w_n), w_n \rangle = o(1)\) and \(I_m(w_n) \to c_m\), by (f1) and (f2), we get

\[
\|w_n\|_{X^s(\mathbb{R}^N)}^2 + m\|w_n(x, 0)\|_{L^2(\mathbb{R}^N)}^2 \|w_n(x, 0)\|_{L^{2^*_s}(\mathbb{R}^N)}^2 = o(1),
\]

\[
\frac{1}{2}\|w_n\|_{X^s(\mathbb{R}^N)}^2 + \frac{m}{2}\|w_n(x, 0)\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2s}\|w_n(x, 0)\|_{L^{2^*_s}(\mathbb{R}^N)}^2 = c_m + o(1). \tag{3.22}
\]

Let \(l \geq 0\) be such that

\[
\|w_n\|_{X^s(\mathbb{R}^N)}^2 + m\|w_n(x, 0)\|_{L^2(\mathbb{R}^N)}^2 \to l \text{ and } \|w_n(x, 0)\|_{L^{2^*_s}(\mathbb{R}^N)}^2 \to l. \tag{3.23}
\]

It is trivial that \(l > 0\), otherwise \(\|w_n\|_{X^s(\mathbb{R}^N)} \to 0\) as \(n \to \infty\) which contradicts \(c_m > 0\). By (3.22), we get

\[
c_m = \frac{s}{N}l. \tag{3.24}
\]

By Lemma 2.1, we see that

\[
\|w_n\|_{X^s(\mathbb{R}^N)}^2 + m\|w_n(x, 0)\|_{L^2(\mathbb{R}^N)}^2 \geq k_s S(s, N)\|w_n(x, 0)\|_{L^{2^*_s}(\mathbb{R}^N)}^2. \tag{3.25}
\]

Letting \(n \to \infty\) in (3.25), we get \(l \geq (k_s S(s, N))^{N/2s}\), then by (3.24), \(c_m \geq \frac{s}{N}(k_s S(s, N))^{N/(2s)}\), which contradicts Lemma 3.3. □
Proof of Theorem 1.1. Let \( \{w_n\}_{n=1}^{\infty} \) be the sequence given in (3.6) and denote \( \tilde{w}_n(x, y) = w_n(x + x_n, y) \), where \( \{x_n\}_{n=1}^{\infty} \) is the sequence given in Lemma 3.3. By Lemma 3.3 and Lemma 3.5, we see that, up to a subsequence, \( \exists \tilde{w}(x, y) \in X^{1,s}(\mathbb{R}_+^{N+1}) \setminus \{0\} \) such that \( \tilde{w}_n(x, y) \to \tilde{w}(x, y) \) in \( X^{1,s}(\mathbb{R}_+^{N+1}) \), \( \tilde{w}_n(0, x) \to \tilde{w}(0, x) \) in \( L_{loc}^p(\mathbb{R}^N) \) (1 ≤ \( p \leq 2^*_s \)), \( \tilde{w}_n(x, 0) \to \tilde{w}(x, 0) \) a.e. in \( \mathbb{R}^N \) and \( \tilde{w} \) satisfies (3.1). Hence

\[
 b_m \leq I_m(\tilde{w}) = I_m(\tilde{w}) - \frac{1}{N} P_m(\tilde{w}) = \frac{S}{N} \|\tilde{w}\|^2_{X^s(\mathbb{R}_+^{N+1})} \leq \lim_{n \to \infty} \frac{S}{N} \|\tilde{w}_n\|^2_{X^s(\mathbb{R}_+^{N+1})}
\]

(3.26)

For any \( \tilde{w} \in X^{1,s}(\mathbb{R}_+^{N+1}) \setminus \{0\} \) a solution of (3.1), we set the path

\[
 \tilde{\gamma}(t) = \begin{cases} 
 \tilde{w}(x/t, y/t), & \text{if } t > 0, \\
 0, & \text{if } t = 0.
\end{cases}
\]

Since

\[
 I_m(\tilde{\gamma}(t)) = I_m(\tilde{\gamma}(t)) - \frac{1}{N} t^N P_m(\tilde{w}) = \left( \frac{1}{2} t^{N-2s} - \frac{N-2s}{2N} t^N \right) \|\tilde{w}\|^2_{X^s(\mathbb{R}_+^N)},
\]

(3.27)

there exists a \( T_0 > 0 \) large such that \( I_m(\tilde{\gamma}(T_0)) < 0 \) and \( I_m(\tilde{\gamma}(t)) \) achieve the strict global maximum at \( t = 1 \). By the definition of \( I_m(\tilde{w}) \), we see that \( I_m(\tilde{w}) \geq c_m \). Since \( \tilde{w} \) is arbitrary, we see that \( b_m \geq c_m \). Hence, we conclude from (3.26) that \( I_m(\tilde{w}) = c_m = b_m \) and \( I_m(\tilde{w}) = 0 \).

Arguing as Proposition 4.1.1 of [13], we see that \( \tilde{w} \in L^\infty(\mathbb{R}^N) \). Since \( \tilde{w} \) is nonnegative and nontrivial and \( f \) is continuous, we can apply the Harnack’s inequality in Lemma 4.9 of [7] to conclude that \( \tilde{w} \) is positive, that is, \( \tilde{w} \) is in fact a positive ground state solution of (3.1), hence, \( u(x) := \tilde{w}(x, 0) \) is a positive ground state solution of (1.1).

\[
 \square
\]

References

[1] G. Autuoria, P. Pucci, Elliptic problems involving the fractional Laplacian in \( \mathbb{R}^N \), J. Differential Equations 255 (2013) 2340-2362.
[2] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical points theory and applications, J. Funct. Anal. 14 (1973) 349-381.
[3] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, On some critical problems for the fractional Laplacian operator, J. Differential Equations 252 (2012) 6133-6162.
[4] B. Brändle, E. Colorado, A. de Pablo, U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, Proc. Roy. Soc. Edinburgh. Sect. A 143 (2013) 39-71.
[5] H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I existence of a ground state, Arch. Rational Mech. Anal. 82 (1983) 313-345.
[6] H. Berestycki, P. L. Lions, Nonlinear scalar field equations, II existence of infinitely many solutions, Arch. Rational Mech. Anal. 82 (1983) 347-375.
[7] X. Cabrè, Y. Sire, Nonlinear equations for fractional Laplacians, I: regularity, maximum principles, and Hamiltonian estimates, Ann. Inst. H. Poincaré Non Linéaire 31 (2014) 23-53.
[8] X. Cabrè, Y. Sire, Nonlinear equations for fractional Laplacians, II: existence, uniqueness, and qualitative properties of solutions, Trans. Amer. Math. Soc. 367 (2015) 911-941.
[9] L. Caffarelli, J. M. Roquejoffre, Y. Sire, Variational problems for free boundaries for the fractional Laplacian, J. Eur. Math. Soc. 12 (2010) 1151-1179.
[10] L. Caffarelli, S. Salsa, L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, Invent. Math. 171 (2008) 425-461.

[11] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differential Equations 32 (2007) 1245-1260.

[12] X. Chang, Z. Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, Nonlinearity 26 (2013) 479-494.

[13] S. Dipierro, M. Medina, E. Valdinoci, Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^N$, arXiv:1506.01748v1 (2015).

[14] J. M. do O, O. H. Miyagaki, M. Squassina, Critical and subcritical fractional problems with vanishing potentials, arXiv:1410.0843 (2014).

[15] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521-573.

[16] P. Felmer, A. Quaas, J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, Proc. Roy. Soc. Edinburgh. Sect. A 142 (2012) 1237-1262.

[17] J. Hirata, N. Ikoma, K. Tanaka, Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches, Topol. Methods Nonlinear Anal. 35 (2010) 253-276.

[18] T. Jin, Y. Y. Li, J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, J. Eur. Math. Soc. 16 (2014) 1111-1171.

[19] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66 (2002) 056108-056114.

[20] N. Laskin, Fractional quantum mechanics and Levy path integrals, Phys. Lett. A 268 (2000) 298-305.

[21] V. Maz’ya, T. Shaposhnikova, On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Funct. Anal. 195 (2002) 230-238.

[22] X. Ros-Oton, J. Serra, The Pohozaev identity for the fractional laplacian, Arch. Rational Mech. Anal. 213 (2014) 587-628.

[23] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Commun. Pure Appl. Math. 60 (2006) 67-112.

[24] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

[25] J. Zhang, Z. Chen, W. Zou, Standing waves for nonlinear Schrödinger equations involving critical growth, J. London Math. Soc. 90 (2014) 827-844.