Bandit Labor Training

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On-demand labor platforms aim to train a skilled workforce to serve its incoming demand for jobs. Since limited jobs are available for training, and it is usually not necessary to train all workers, efficient matching of training jobs requires prioritizing fast learners over slow ones. However, the learning rates of novice workers are unknown, resulting in a tradeoff between exploration (learning the learning rates) and exploitation (training the best workers). Motivated to study this tradeoff, we analyze a novel objective within the stochastic multi-armed bandit framework. Given $K$ arms, instead of maximizing the expected total reward from $T$ pulls (the traditional “sum” objective), we consider the vector of cumulative rewards earned from the $K$ arms at the end of $T$ pulls, and aim to maximize the expected highest cumulative reward (the “max” objective). When rewards represent skill increments, this corresponds to the objective of training a single highly skilled worker from a set of novice workers, using a limited supply of training jobs. For this objective, we show that any policy must incur an instance-dependent asymptotic regret of $\Omega(\log T)$ (with a higher instance-dependent constant compared to the traditional objective) and a worst-case regret of $\Omega(K^{1/3}T^{2/3})$.

We then design an explore-then-commit policy featuring exploration based on appropriately tuned confidence bounds on the mean reward and an adaptive stopping criterion, which adapts to the problem difficulty and achieves these bounds (up to logarithmic factors). We then generalize our algorithmic insights to the problem of maximizing the expected value of the average cumulative reward of the top $m$ arms with the highest cumulative rewards, corresponding to the case where multiple workers must be trained. Our numerical experiments demonstrate the efficacy of our policies compared to several natural alternatives in practical parameter regimes.

Key words: Multi-armed bandits, $\mathcal{L}^\infty$ objective, Online labor platforms, Labor training

1. Introduction

The stochastic multi-armed bandit (MAB) problem (Lai and Robbins 1985, Auer et al. 2002) presents a formal framework to study the exploration vs. exploitation tradeoff fundamental to sequential resource allocation in uncertain settings, with wide-ranging applications in areas such as artificial intelligence, adaptive control, economics, marketing, and healthcare. In this problem, given a set of $K$ arms, each of which yields independent and identically distributed (i.i.d.) rewards over successive pulls, the goal is to adaptively choose a sequence of arms to maximize the expected value
of the total reward attained at the end of \( T \) pulls. The critical constraint is that the reward distributions of the different arms are a priori unknown. Any good policy must hence, over time, optimize the tradeoff between choosing arms that are known to yield high rewards (exploitation) and choosing arms whose reward distributions are yet relatively unknown (exploration). Over several years of extensive analysis, this classical problem is now well understood (see Lattimore and Szepesvári (2020), Slivkins (2019), and Bubeck and Cesa-Bianchi (2012) for a survey).

However, there are several sequential allocation problems arising in practice where the classical objective of maximizing the expected cumulative reward is inappropriate. The main contribution of this paper is the introduction and analysis of a new objective in the classical MAB setup: we consider the vector of cumulative rewards that have been earned from the different arms at the end of \( T \) pulls, and instead of maximizing the expectation of their sum, we aim to maximize the expected value of the maximum (max) of these cumulative rewards across the arms. We also address algorithm-design for a generalization of this objective, in which we pull \( m \) arms in each time period and we are interested in maximizing the average of the top \( m \) cumulative rewards across all arms, where \( 1 \leq m \leq K \) (the max objective corresponding to the case of \( m = 1 \)).

These objectives are motivated in the context of worker training in on-demand labor platforms, which have seen a tremendous rise in the economy in recent years. An important operational objective of these platforms is to develop and maintain a reliable pool of high-quality workers to satisfy the demand for jobs. This problem is challenging since (a) workers continuously leave the platform and hence new workers must be trained on an ongoing basis, (b) the number of “training” jobs available to train the novice workers is limited (for instance, this could result from a limited budget for the incentives offered to the clients for choosing novice workers), and (c) the quality of the workers is unknown: some workers are fast learners while some are slow, and efficient allocation of training jobs entails distinguishing between these types. At the core of this challenging operational question is the following problem. Given a limited number of training jobs, the platform must determine a policy to allocate these jobs to a set of novice workers to maximize some appropriate functional of their terminal skill levels. For a platform that seeks to serve a limited demand from the clients, maximizing the average terminal skill levels across all workers may not be appropriate. A more appropriate objective is to maximize the average skill level of top \( q \)th-percentile workers ordered by their terminal skills, where \( q \) is determined by the volume of demand for regular jobs: higher the demand for jobs, the higher the \( q \) needed. Essentially, the skill levels of the lower-skilled workers at the end of training do not matter, since there is not enough demand for regular jobs to assign to them anyway.

To address this problem, we can use the MAB framework: the set of arms is the set of novice workers, the reward of an arm is the random increment in the skill level of the worker after
performing a job, and the number of training jobs available is $T$. The main assumption we make is that $T$ is not too large, so that the random increments may be assumed to be i.i.d. over time. The mean of these increments can be interpreted as the unknown learning rate or the trainability of a worker. Given $K$ workers, the goal is to adaptively allocate the jobs to these workers to maximize the average terminal skill level amongst the top $1 \leq m \leq K$ (where $m \approx qK$) most terminally skilled workers. This motivates the model and the general objective we consider. While our primary focus is on the max objective, i.e., the case of $m = 1$, we use our insights from this setting to design a well-performing algorithm for the case where $m > 1$.

**The max objective.** We first discuss our results for the max objective ($m = 1$). A key assumption we make in the paper is that the rewards for all arms are non-negative; this is motivated by training applications where rewards represent skill increments. Under this assumption, in the full-information setting where the reward distributions of the arms are known, we first show that the optimal policy for the max objective is identical to the one for the sum objective: one always pulls the arm with the highest mean reward (Proposition 1). This additionally implies that the optimal rewards under the two objectives are identical.

A standard approach in MAB problems is to design a policy that minimizes regret, i.e., the quantity of loss relative to the optimal full-information policy for a given objective over time. In the classical setting with the sum objective, it is well known that any policy must incur an instance-dependent asymptotic regret of $\Omega(\sum_{i \neq i^*} (\Delta_i \log T)/d_i)$ as $T \to \infty$ (Lai and Robbins 1985). Here, $\Delta_i = \mu^* - \mu_i$, i.e., it is the difference between the highest mean reward $\mu^*$ belonging to the arm $i^*$ and the mean reward $\mu_i$ of arm $i$; and $d_i$ is a quantity that captures an appropriate notion of divergence between the reward distribution of arm $i$ and the “closest” distribution within the space of possible distributions having a mean that is at least $\mu^*$. Additionally, it is also well-known that any policy must incur an instance-independent regret of $\Omega(\sqrt{KT})$ in the worst-case over the set of possible bandit instances (Auer et al. 2002).

Since the optimal full-information reward is the same under the sum and the max objectives, and since the maximum of a set of non-negative numbers is always at most the sum of the numbers, any lower bound on the regret for the sum objective implies the same lower bound on the max objective. However, a key feature of the max objective is that the rewards earned from arms that do not eventually turn out to be the ones yielding the highest cumulative reward are effectively a waste. Owing to this feature, we show that any policy must incur a higher instance-dependent regret of $\Omega(\sum_{i \neq i^*} (\mu^* \log T)/d_i)$ in this case (Theorem 1). Moreover, we show that an instance-independent regret of $\Omega(K^{1/3}T^{2/3})$ is inevitable in the worst-case (Theorem 2). Both these results rely on novel arguments that are a significant departure from those involved in proving the corresponding lower bounds for the sum objective.
Attaining these lower bounds requires algorithmic innovation. For the sum objective, well-performing policies are typically based on the principle of optimism in the face of uncertainty. A popular policy class is the Upper Confidence Bound (UCB) class of policies (Agrawal 1995, Auer et al. 2002, Auer and Ortner 2010), in which a confidence interval is maintained for the mean reward of each arm, and at each time, the arm with the highest upper confidence bound is chosen. For a standard tuning of these intervals, this policy – termed UCB1 in literature due to Auer et al. (2002) – guarantees an instance-dependent asymptotic regret of $O(\log T)$ and a regret of $O(\sqrt{KT \log T})$ in the worst case. With a more refined tuning, $O(\sqrt{KT})$ can be achieved (Audibert and Bubeck 2009, Lattimore 2018).

For our max objective, directly using one of the above UCB policies can prove to be disastrous. To see this, suppose that all $K$ arms have equal deterministic rewards. Then, UCB1 will pull each of the arms in a round-robin fashion until a total of $T$ pulls, resulting in the highest terminal cumulative reward of $O(T/K)$; whereas, a reward of $\Theta(T)$ is feasible by simply committing to an arbitrary arm from the start. This results in a $\Omega(T)$ regret in the worst case. Introducing randomness in the rewards doesn’t change this observation: the result of a numerical experiment shown in Figure 1 suggests a $\Omega(T)$ regret on using UCB1 for a two-armed bandit problem with both arms having Bernoulli rewards with mean 0.5.

This observation suggests that any good policy must, at some point, stop exploring and permanently commit to a single arm. A natural candidate is the basic explore-then-commit (ETC) policy, which uniformly explores all arms until some time that is fixed in advance, and then commits to the empirically best arm (Lattimore and Szepesvári 2020, Slivkins 2019). When each arm is chosen $(T/K)^{2/3}$ times in the exploration phase, this strategy can be shown to achieve a regret of $O(K^{1/3}T^{2/3} \sqrt{\log K})$ relative to the sum objective (Slivkins 2019). It is easy to argue that it achieves the same regret relative to the max objective. However, this policy is excessively optimized for the worst case where the means of all the arms are within $(K/T)^{1/3}$ of each other. When the arms are easier to distinguish, this policy’s performance is quite poor due to excessive exploration. For example, consider a two-armed bandit problem with Bernoulli rewards and means $(0.5, 0.5 + \Delta)$, where $\Delta > 0$. For this fixed instance, ETC will pull both arms $\Omega(T^{2/3})$ times and hence incur a regret of $\Omega(T^{2/3})$ relative to our max objective. However, it is well known that UCB1 will not pull the suboptimal arm more than $O(\log T/\Delta^2)$ times with high probability (Auer et al. 2002) and hence for this instance, UCB1 will incur an instance-dependent regret of only $O(\log T)$, which could be much smaller if $\Delta$ is large. Thus, although the worst-case regret of UCB1 is $\Omega(T)$ due to perpetual exploration, for a fixed bandit instance, its asymptotic performance can be significantly better than ETC. This observation motivates us to seek a practical policy with a graceful
dependence of performance on the difficulty of the bandit instance, and which will achieve both: the worst-case bound of ETC and the instance-dependent asymptotic bound of $O(\log T)$.

We propose a new policy with an explore-then-commit structure, in which appropriately defined confidence bounds on the means of the arms are utilized to guide exploration, as well as to decide when to stop exploring. We call this policy Adaptive Explore-then-Commit (ADA-ETC). We show that ADA-ETC adapts to the problem difficulty by exploring less, if appropriate, while attaining the same regret guarantee of $O(K^{1/3}T^{2/3}\sqrt{\log K})$ attained by vanilla ETC in the worst case (Theorem 3). In particular, ADA-ETC guarantees an instance-dependent asymptotic regret of $O(\log T)$ as $T \to \infty$, matching our instance-dependent lower bound up to a constant factor. Finally, our numerical experiments demonstrate that ADA-ETC results in significant improvements over the performance of vanilla ETC in easier settings, while never performing worse in difficult ones, thus corroborating our theoretical results. Our numerical results also demonstrate that naive ways of introducing adaptive exploration based on upper confidence bounds, e.g., simply using the upper confidence bounds of UCB1, may lead to no improvement over vanilla ETC for practical values of $T$ and $K$.

The case of $m > 1$. We next consider an extension to settings where one is interested in maximizing the expected average cumulative reward across the top $m$ arms with the highest cumulative rewards. When the decision-maker can pull one arm per time period, this objective, however, is equivalent to the max objective: the best way to maximize the average cumulative reward across the top $m$ arms is to invest all pulls in the best arm. In practice though, such a solution is far from being appropriate. For example, platforms typically want to provide robust service guarantees to the clients, and hence, training a handful of “stars” while most other workers are poorly trained is not a desirable outcome. Moreover, while we expect $T/m$ to be small, $T$ itself could be large, and hence, if one invests all the $T$ training jobs into training a single worker, it may not be reasonable to assume that the skill increments of this worker are i.i.d. over time – there is expected to be a point of diminishing returns.

In order to account for these concerns, we consider a modification of our problem. While the objective remains the same, we assume that there are $T/m$ time periods, and in each period, the decision-maker pulls $m$ distinct arms. This is equivalent to the constraint that the decisions of $T$ pulls are sequentially taken over $T/m$ batches of size $m$, with the additional requirement that the pulls in each batch are distinct. Such batching has the additional benefit that it may significantly reduce the training period. With such a constraint, it is not feasible, let alone optimal, for the decision-maker to invest all $T$ pulls in a single arm. (In the concluding Section 5, we discuss another formulation to achieve this goal.) Extending Proposition 1, we can show that the optimal policy that maximizes the average cumulative reward across the top $m$ arms is the one that always pulls
the $m$ arms with the highest mean in each time period (i.e., in each batch). We then design an adaptive explore-then-commit policy inspired by the max objective (m-ADA-ETC) that achieves a $\tilde{O}(K^{1/3}T^{2/3}/m)$ upper bound on the regret. We also prove a $\Omega(K^{1/3}T^{2/3}/m^{4/3})$ lower bound on the regret in this case for when $2m \leq K < T$. Our extensive numerical tests show that this policy significantly outperforms other natural policies, including the policy of running the optimal algorithm for the max objective independently on $m$ randomly selected sets of arms, each of size $\approx K/m$.

**Organization.** The paper is organized as follows. We discuss relevant literature in Section 1.1. Our model and the max objective is introduced in Section 2. In this section, we also present the analysis of the max objective, where we first prove lower bounds on the regret, and then present the ADA-ETC policy and the corresponding upper bounds that it achieves. In Section 3 we present the results for the extension of our objective for $m > 1$. Our numerical experiments are presented in Section 4. We conclude the paper with a discussion of our results and open questions in Section 5.

1.1. Related literature

We discuss connections of our model and results to three distinct streams of literature.

**Pure exploration in bandits.** Our max objective endogenizes the goal of quickly identifying the arm with approximately the highest mean reward so that a substantial amount of time can be spent earning rewards from that arm (e.g., “training” a worker). This goal is related to the pure exploration problem in multi-armed bandits. Several variants of this problem have been studied, where the goal of the decision-maker is to either minimize the probability of misidentification of the optimal arm given a fixed budget of pulls (Audibert et al. 2010, Kaufmann et al. 2016, Carpentier and Locatelli 2016); or minimize the expected number of pulls to attain a fixed probability of misidentification, possibly within an approximation error (Even-Dar et al. 2002, Mannor and Tsitsiklis 2004, Even-Dar et al. 2006, Karnin et al. 2013, Vaidhiyan and Sundaresan 2017, Jamieson et al. 2014, Kaufmann et al. 2016); or to minimize the expected suboptimality (called “simple regret”) of a recommended arm after a fixed budget of pulls (Bubeck et al. 2009, 2011, Carpentier and Valko 2015). Jun et al. (2016) additionally has studied the pure-exploration problem under batching constraints similar to our $m > 1$ setting. Extensions to settings where multiple good arms are needed to be identified have also been considered (Bubeck et al. 2013, Kalvanakrishnan et al. 2012, Zhou et al. 2014, Kaufmann and Kalvanakrishnan 2013).

The critical difference from these problems is that in our scenario, the budget of $T$ pulls must not only be spent on identifying an approximately optimal arm but also on earning rewards on that arm. Hence, any choice of apportionment of the budget to the identification problem, or a choice for a target for the approximation error or probability of misidentification within a candidate policy, is a priori unclear and must arise endogenously from our primary objective.
Bandits with switching costs and batched bandits. The fact that focusing on one arm in the long run is prudent for our objective thematically relates this work to the literature on bandits with switching costs, where there is a cost incurred for switching from one arm to another (Cesa-Bianchi et al. 2013, Dekel et al. 2014). Another related line of work is on batched bandits, which imposes a constraint that the policy must split the arm pulls into a small number of batches (Perchet et al. 2016, Gao et al. 2019). However, we note that our objective does not simply amount to keeping the number of switches or batches low; it also matters how “spread apart” the switches are. To enforce this point, we note that the algorithm of Cesa-Bianchi et al. (2013), which restricts the number of switches/batches to $O(\log \log T)$ while attaining $\tilde{O}(\sqrt{T} \log T)$ regret for the sum objective, incurs a worst-case regret of $\Theta(T)$ for our max objective: Figure 1 shows this for $K = 2$ arms with Bernoulli$(0.5)$ rewards.

Learning in online platforms. The operational concerns of learning with the goal of efficient matchmaking in the context of online platforms and marketplaces have seen significant attention in recent literature (Johari et al. 2021, Shah et al. 2020, Massoulié and Xu 2018, Hsu et al. 2021, Kamble and Ozbay 2022). The goal of these works is to design effective online learning policies that can be implemented by the problem in the face of capacity constraints induced by the limited demand. Similar to the settings considered in these works, we also consider a learning problem in a market setting with capacities induced by the demand constraints. There are, however, two key differences from this literature. First, this literature typically focuses on the traditional objective of maximizing the total utility generated in the market, while our distinction is the focus on a new objective motivated by the problem of labor training. Second, unlike these settings, where the capacity constraint results from limited demand, there are two types of capacity constraints that we account for: (a) the limited capacity of training jobs, which are distinct from the regular jobs that these workers are being trained for (this constraint determines $T$ in our model), and (b) the
limited capacity of regular jobs due to which not all workers need to be trained (this constraint determines \( m \) in our model). Effectively, our focus is on a learning problem that arises in the “training” phase of arriving cohorts of workers given a limited supply of training jobs.

2. Model and the max objective

Consider the stochastic multi-armed bandit (MAB) problem parameterized by the number of arms, which we denote by \( K \); the length of the decision-making horizon (the number of discrete times/stages), which we denote by \( T \); and the probability distributions for arms \( 1, \ldots, K \), denoted by \( \nu_1, \ldots, \nu_K \), respectively. We assume that the rewards are non-negative and their distributions have a bounded support, assumed to be \([0, 1] \) without loss of generality (although, this latter assumption can be easily relaxed to allow, for instance, \( \sigma \)-Sub-Gaussian distributions with bounded \( \sigma \)). We define \( V \) to be the set of all \( K \)-tuples of distributions for the \( K \) arms having support in \([0, 1] \).

At each time, the decision-maker chooses an arm to play and observes a reward. Let the arm played at time \( t \) be denoted as \( I_t \) and the reward be denoted as \( X_t \), where \( X_t \) is drawn from the distribution \( \nu_{I_t} \), independent from the previous actions and observations. The history of actions and observations at any time \( t \geq 2 \) is denoted as \( H_t = (I_1, X_1, I_2, X_2, \ldots, I_{t-1}, X_{t-1}) \), and \( H_1 \) is defined to be the empty set \( \phi \). A policy \( \pi \) of the decision-maker is a sequence of mappings \((\pi_1, \pi_2, \ldots, \pi_T)\), where \( \pi_t \) maps every possible history \( H_t \) to an arm \( I_t \) to be played at time \( t \). Let \( \Pi_T \) denote the set of all such policies.

For an arm \( i \), we denote \( n_i^t \) to be the number of times this arm is played until and including time \( t \), i.e., \( n_i^t = \sum_{s=1}^{t} \mathbb{1}_{\{I_s = i\}} \). We also denote \( U_i^n \) to be the reward observed from the \( n^{th} \) pull of arm \( i \). \((U_i^n)_{n \in \mathbb{N}}\) is thus a sequence of i.i.d. random variables, each distributed as \( \nu_i \). Note that the definition of \( U_i^n \) implies that we have \( X_t = U_{I_t}^{n_{I_t}} \). We further define \( \overline{U}_t^i = \sum_{n=1}^{n_i^t} U_i^n \) to be the cumulative reward obtained from arm \( i \) until time \( t \).

Once a policy \( \pi \) is fixed, then for all \( t = 1, \ldots, T \), \( I_t \), \( X_t \), and \( n_i^t \) for all \( i \in \{1, \ldots, K\} \), become well-defined random variables. We consider the following notion of reward for a policy \( \pi \):

\[
\mathcal{R}_T(\pi, \nu) = \mathbb{E}_\nu(\max(\overline{U}_T^1, \overline{U}_T^2, \ldots, \overline{U}_T^K)).
\]  

(1)

In words, the objective value attained by the policy is the expected value of the largest cumulative reward across all arms at the end of the decision making horizon.
When the reward distributions \(\nu_1, \ldots, \nu_K\) are known to the decision-maker, then for a large \(T\), the best reward that the decision-maker can achieve is

\[
sup_{\pi \in \Pi_T} \mathcal{R}_T(\pi, \nu).
\]

A natural candidate for a “good” policy when the reward distributions are known is the one where the decision-maker exclusively plays arm 1 (the arm with the highest mean), attaining an expected reward of \(\mu_1 T\). Let us denote \(\mathcal{R}^*_T(\nu) \triangleq \mu_1 T\). One can show that, in fact, this is the best reward that one can achieve in our problem.

**Proposition 1** For any bandit instance \(\nu \in \mathcal{V}\), \(\sup_{\pi \in \Pi_T} \mathcal{R}_T(\pi, \nu) = \mathcal{R}^*_T(\nu)\).

The proof is presented in Section A in the Appendix. This shows that the simple policy of always picking the arm with the highest mean is optimal for our problem. Next, we denote the regret of any policy \(\pi\) to be

\[
\text{Reg}_T(\pi, \nu) = \sup_{\pi \in \Pi_T} \mathcal{R}_T(\pi, \nu) - \mathcal{R}_T(\pi, \nu).
\]

In the rest of this section, we focus on two objectives. The first is to design a policy \(\pi_T \in \Pi_T\), which attains an asymptotically optimal instance-dependent (i.e., \(\nu\) dependent) bound on \(\text{Reg}_T(\pi_T, \nu)\), simultaneously for (almost) all instances \(\nu \in \mathcal{V}\) as \(T \to \infty\). The second objective is to design a policy \(\pi_T \in \Pi_T\), which achieves the smallest regret in the worst-case over all distributions \(\nu \in \mathcal{V}\), i.e., the one that solves the optimization problem:

\[
\text{Reg}^*_T(\nu) \triangleq \inf_{\pi \in \Pi_T} \sup_{\nu \in \mathcal{V}} \text{Reg}_T(\pi, \nu),
\]

where \(\text{Reg}^*_T\) denotes the minmax (or the best worst-case) regret. In the remainder of this section, we design a single policy that attains the first objective to within a constant factor and the second objective to within a logarithmic factor.

### 2.1. Lower Bounds

We first provide an instance-dependent \(\Omega(\log T)\) asymptotic lower bound on the regret. We let \(\mathcal{M}\) be the set of distributions with support in \([0, 1]\). For \(\nu \in \mathcal{M}\), and \(\mu \in [0, 1]\), define \(d_{\text{inf}}(\nu, \mu, \mathcal{M}) = \inf_{\nu' \in \mathcal{M}} \{D(\nu, \nu') : \mu(\nu') > \mu\}\), where \(\mu(\nu)\) denotes the mean of distribution \(\nu\), and \(D(\nu, \nu')\) is the Kullback-Leibler (KL) divergence between the distributions \(\nu\) and \(\nu'\). \(d_{\text{inf}}(\nu, \mu, \mathcal{M})\) is thus the smallest KL divergence between the distribution \(\nu\) and any other distribution in \(\mathcal{M}\) whose mean is at least \(\mu\).

We say that a sequence of policies \((\pi_T)_{T \in \mathbb{N}}\), where \(\pi_T \in \Pi_T\) for all \(T \in \mathbb{N}\), is consistent for a class \(\mathcal{V} = \mathcal{M}^K\) of stochastic bandits, if for all \(\nu \in \mathcal{V}\) such that there is a unique arm with the highest mean reward, and for any \(p > 0\), we have that \(\lim_{T \to \infty} \text{Reg}_T(\pi_T, \nu)/T^p = 0\). We then have the following result.
Theorem 1. Consider a class $\mathcal{V} = \mathcal{M}^K$ of $K$-armed stochastic bandits and let $(\pi_T)_{T \in \mathbb{N}}$ be a consistent sequence of policies for $\mathcal{V}$. Then, for all $\nu \in \mathcal{V}$ such that the optimal arm is unique,

$$
\liminf_{T \to \infty} \frac{\text{Reg}_T(\pi_T, \nu)}{\log(T)} \geq \sum_{i \neq k^*} \frac{\mu^*}{d_{\inf}(\nu_i, \mu^*, \mathcal{M})},
$$

where $k^*$ is the optimal arm with the highest mean $\mu^*$.

The proof of Theorem 1 is presented in Section B in the Appendix. The result has an intuitive explanation. For convenience, we denote $d_i = d_{\inf}(\nu_i, \mu^*, \mathcal{M})$. Similar to the proof of the lower bound for the sum objective (Lai and Robbins 1985), we can show that for any consistent sequence of policies, each suboptimal arm $i$ must be pulled $\Omega(\log T / d_i)$ number of times in expectation. However, unlike the sum objective where each such pull yields a mean reward of $\mu_i$ and results in an expected regret of $\Delta_i$, for the max objective, each such pull is wasteful and results in an expected regret of $\mu^*$. Despite this intuitive explanation of the result, the proof is not straightforward. In particular, showing that each suboptimal arm $i$ must be pulled $\log T / d_i$ times in expectation doesn’t directly allow us to account for a regret contribution of $\mu^* \log T / d_i$ from arm $i$. This is because, in the full-information setting, with a (relatively high) probability of $\log T / (Td_i)$, one can choose to pull a suboptimal arm $i$ for all the $T$ time periods (and pull the optimal arm for $T$ periods with the remaining probability), thus ensuring that it gets pulled $\log T / d_i$ times in expectation and at the same time resulting in an expected reward contribution of $\mu_i \log T / d_i$, and hence a regret contribution of $(\mu^* - \mu_i) \log T / d_i = \Delta_i \log T / d_i$. To show that this regret is not achievable, we prove a stronger result: we show that for each $\alpha \in (0, 1]$, a suboptimal arm $i$ must be pulled $\alpha \log T / d_i$ times in expectation until time $T^\alpha$ (Proposition 3 in the Appendix). We then argue that the probability of a suboptimal arm being the one with the highest cumulative reward cannot be too high for any consistent sequence of policies, and thus the best way to satisfy the stronger set of lower bounds on the number of pulls for the suboptimal arms in terms of minimizing regret is to chalk these pulls as wasted. This allows us to conclude the higher lower bound on the regret.

We next show that for our objective, a regret of $\Omega(K^{1/3}T^{2/3})$ is inevitable in the worst case.

Theorem 2. Suppose that $K < T$. Then, $\text{Reg}_T^* \geq \Omega((K - 1)^{1/3}T^{2/3})$.

The proof is presented in Section C in the Appendix. Informally, the argument for the case of $K = 2$ arms is as follows. Consider two bandits with Bernoulli rewards, one with mean rewards $(1/2 + 1/T^{1/3}, 1/2)$, and the other with mean rewards $(1/2 + 1/T^{1/3}, 1/2 + 2/T^{1/3})$. Then until time $T^{2/3}$, no algorithm can reliably distinguish between the two bandits. Hence, until this time, either $\Omega(T^{2/3})$ pulls are spent on arm 1 irrespective of the underlying bandit, or $\Omega(T^{2/3})$ pulls are spent on...
arm 2 irrespective of the underlying bandit. In both cases, the algorithm incurs a regret of \( \Omega(T^{2/3}) \), essentially because of wasting \( \Omega(T^{2/3}) \) pulls on a suboptimal arm that could have been spent on earning reward on the optimal arm. This latter argument is not entirely complete, however, since it ignores the possibility of always picking a suboptimal arm until time \( T \), in which case spending time on the suboptimal arm in the first \( \approx T^{2/3} \) time periods was not wasteful. However, even in this case, we can argue that one incurs a regret of \( \approx T \times (1/T^{1/3}) = \Omega(T^{2/3}) \). Thus a regret of \( \Omega(T^{2/3}) \) is unavoidable. Our formal proof builds on this basic argument to additionally determine the optimal dependence on \( K \).

### 2.2. Adaptive Explore-then-Commit (ADA-ETC)

We now define an algorithm that we call Adaptive Explore-then-Commit (ADA-ETC), specifically designed for our problem. It is formally defined in Algorithm 1. The algorithm can be simply described as follows. After choosing each arm once, choose the arm with the highest upper confidence bound, until there is an arm such that (a) it has been played at least \( \tau = \left\lceil \frac{T^{2/3}}{K^{2/3}} \right\rceil \) times, and (b) its empirical mean is higher than the upper confidence bounds on the means of all other arms. Once such an arm is found, commit to this arm until the end of the decision horizon.

#### ALGORITHM 1: Adaptive Explore-then-Commit (ADA-ETC)

1. **Input:** \( K \) arms with horizon \( T \).
2. **Define:** \( \tau = \left\lceil \frac{T^{2/3}}{K^{2/3}} \right\rceil \). For \( n \geq 1 \), let \( \bar{\mu}_n^i \) be the empirical average reward from arm \( i \) after \( n \) pulls and it remains fixed after \( \tau \) pulls, i.e., \( \bar{\mu}_n^i = \frac{1}{\min(n, \tau)} \sum_{s=1}^{\min(n, \tau)} U_i^s \). Also, for \( n \geq 1 \), define,
   
   \[ UCB_n^i = \bar{\mu}_n^i + \sqrt{\frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right)} \mathbb{1}_{(n<\tau)}. \]  
   
   (2)

   \[ LCB_n^i = \bar{\mu}_n^i - \bar{\mu}_n^i \mathbb{1}_{(n<\tau)}. \]  
   
   (3)

3. **Procedure:**
   - **Explore Phase:** From time \( t = 1 \) until \( t = K \), pull each arm once. For \( K < t \leq T \):
     1. Identify \( L_t \in \arg \max_{i \in [K]} LCB_{n_{t-1}}^i \), breaking ties arbitrarily. If
        \[ LCB_{n_{t-1}}^{L_t} > \max_{i \in [K]: i \neq L_t} UCB_{n_{t-1}}^i, \]  
        then define \( i^* \triangleq L_t \), break, and enter the Commit phase. Else, continue to Step 2.
     2. Identify \( E_t \in \arg \max_{i \in [K]} UCB_{n_{t-1}}^i \), breaking ties arbitrarily. Pull arm \( E_t \).
   - **Commit Phase:** Pull arm \( i^* \) until time \( t = T \).
The upper confidence bound is defined in Equation 2. In contrast to its definition in UCB1, it is tuned to eliminate wasteful exploration and to allow stopping early if appropriate. We enforce the requirement that an arm is played at least \( \tau \) times before committing to it by defining a trivial “lower confidence bound” (Equation 3), which takes value 0 until the arm is played less than \( \tau \) times, after which both the upper and lower confidence bounds are defined to be the empirical mean of the arm. The stopping criterion can then be simply stated in terms of these upper and lower confidence bounds (Equation 4): stop and commit to an arm when its lower confidence bound is strictly higher than the upper confidence bounds of all other arms (this can never happen before \( \tau \) pulls since the rewards are non-negative).

Note that the collapse of the upper and lower confidence bounds to the empirical mean after \( \tau \) pulls ensures that each arm is not pulled more than \( \tau \) times during the Explore phase. This is because choosing this arm to explore after \( \tau \) pulls would imply that its upper confidence bound = lower confidence bound is higher than the upper confidence bounds for all other arms, which means that the stopping criterion has been met and the algorithm has committed to the arm.

**Remark 1.** A heuristic rationale behind the choice of the upper confidence bound is as follows. Consider a suboptimal arm whose mean is smaller than the highest mean by \( \Delta \). Let \( P_e \) be the probability that this arm is misidentified and committed to in the Commit phase. Then the expected regret resulting from this misidentification is approximately \( P_e \Delta T \). Since we want to ensure that the regret is at most \( O(T^{2/3}K^{1/3}) \) in the worst-case, we can tolerate a \( P_e \) of at most \( \approx K^{1/3}/(\Delta T^{1/3}) \). Unfortunately, \( \Delta \) is not known to the algorithm. However, a reasonable proxy for \( \Delta \) is \( 1/\sqrt{n} \), where \( n \) is the number of times the arm has been pulled. This is because it is right around \( n \approx 1/\Delta^2 \), when the distinction between this arm and the optimal arm is expected to occur. Thus a good (moving) target for the probability of misidentification is \( \delta_n \approx (K^{1/3}n^{1/2})/T^{1/3} \). This necessitates the \( \sqrt{\log(1/\delta_n)} \approx \sqrt{\log(T/(Kn^{3/2}))} \) scaling of the confidence interval in Equation 2. In contrast, our numerical experiments show that utilizing the traditional scaling of \( \sqrt{\log T} \) as in UCB1 results in significant performance deterioration. Our tuning is reminiscent of similar tuning of confidence bounds under the “sum” objective to improve the performance of UCB1; see [Audibert and Bubeck (2009), Lattimore (2018), Auer and Ortner (2010)].

**Remark 2.** Instead of defining the lower confidence bound to be 0 until an arm is pulled \( \tau \) times, one may define a non-trivial lower confidence bound to accelerate commitment, perhaps in a symmetric fashion as the upper confidence bound. However, this doesn’t lead to an improvement in the regret bound. The reason is that if an arm looks promising during exploration, then eagerness to commit to it is imprudent, since if it is indeed optimal then it is expected to be chosen frequently during exploration anyway; whereas, if it is suboptimal then we preserve the option of eliminating it
by choosing to not commit until after $\tau$ pulls. Thus, to summarize, ADA-ETC eliminates wasteful exploration primarily by reducing the number of times suboptimal arms are pulled during exploration through the choice of appropriately aggressive upper confidence bounds, rather than by being hasty in commitment.

Let ADA-ETC$_{K,T}$ denote the implementation of ADA-ETC using $K$ and $T$ as the input for the number of arms and the time horizon, respectively. We characterize the regret guarantees achieved by ADA-ETC$_{K,T}$ in the following result.

**Theorem 3 (ADA-ETC performance).** Let $K < T$. Consider a $\nu \in V$ such that the optimal arm is unique and relabel arms so that $\mu_1 > \mu_2 \geq \cdots \geq \mu_K$. Then the expected regret of ADA-ETC$_{K,T}$ is upper bounded as

$$
\text{Reg}_T(\text{ADA-ETC}_{K,T}, \nu) \\
\leq \mu_1 \sum_{i=2}^{K} \min \left( \frac{11}{\Delta_i^2} + \frac{16}{\Delta_i^2} \log^+ \left( \frac{T \Delta_i^3}{K} \right), \tau \right) + \frac{24}{\Delta_i^2} \sqrt{\log^+ \left( \frac{T \Delta_i^3}{K} \right), \tau} + \mu_1 \tau \sum_{i=2}^{K} \min \left( 2, \frac{648K}{T \Delta_i^3} \right)
$$

- Regret contribution from wasted pulls in the Explore phase

$$
\begin{align*}
\sum_{i=2}^{K} \exp \left( -\frac{\tau \Delta_i^2}{2} \right) T \Delta_i &+ \sum_{i=2}^{K} \min \left( 1, \frac{320K}{T \Delta_i^3} \right) T (\Delta_i - \Delta_{i-1}),
\end{align*}
$$

- Regret contribution from misidentification in the Commit phase

where $\tau = \lceil \frac{T^{2/3}}{K^{1/3}} \rceil$. In the worst case, we have

$$
\sup_{\nu \in V} \text{Reg}_T(\text{ADA-ETC}_{K,T}, \nu) \leq O(K^{1/3}T^{2/3} \sqrt{\log K}).
$$

The proof of Theorem 3 is presented in Section D in the Appendix. Theorem 3 features an instance-dependent regret bound and a worst-case bound of $O(K^{1/3}T^{2/3} \sqrt{\log K})$. The first two terms in the instance-dependent bound arise from the wasted pulls during the Explore phase. Under vanilla Explore-then-Commit, to obtain near-optimality in the worst case, every arm must be pulled $\tau$ times in the Explore phase (Slivkins 2019). Hence, the expected regret from the Explore phase is $\Omega(K\tau) = \Omega(T^{2/3}K^{1/3})$ irrespective of the instance. On the other hand, our bound on this regret depends on the instance and can be significantly smaller than $K\tau$ if the arms are easier to distinguish. In particular, for a fixed $K$ and $\nu$ (with $\Delta_2 > 0$), the regret from exploration (and the overall regret) is $O(\sum_{i \geq 2} 16\mu_1 \log T / \Delta_i^2)$ under ADA-ETC as opposed to $\Omega(T^{2/3}K^{1/3})$ under ETC as $T \to \infty$. This shows that ADA-ETC attains the instance-dependent lower bound on regret of Theorem 1 up to a constant factor.

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$^1$ We define $\log^+(a) = \log(\max(a, 1))$ for $a > 0$. 

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The next two terms in our instance-dependent bound arise from the regret incurred due to committing to a suboptimal arm, which can be shown to be \( O(K^{1/3}T^{2/3} \sqrt{\log K}) \) in the worst case, thus matching the guarantee of ETC. The first of these terms is not problematic since it is the same as the regret arising under ETC. The second term arises due to the inevitably increased misidentifications occurring due to stopping early in adaptive versions of ETC. If the confidence bounds are aggressively small, then this term increases. In ADA-ETC, the upper confidence bounds used in exploration are tuned to be as small as possible while ensuring that this term is no larger than \( O(K^{1/3}T^{2/3}) \) in the worst case (see Remark 1). Thus, our tuning of the Explore phase ensures that the performance gains during exploration do not come at the cost of higher worst-case regret (in the leading order) due to misidentification.

Remark 3. It is possible to show that using the confidence bounds of UCB1 under ADA-ETC results in the same asymptotic instance-dependent regret bound of \( O(\log T) \) and an instance-independent regret bound of \( O(K^{1/3}T^{2/3} \sqrt{\log K}) \) in the worst case. However, for fixed \( T \) and \( K \), the bounds derived for ADA-ETC, as defined, have an improved dependence on the instance owing to the reasons mentioned in Remark 1. As we shall see in Section 4, this results in significant performance gains for practical values of \( T \) and \( K \). Optimizing finite \( T \) performance is particularly important since in training applications, the assumption of skill increments being i.i.d. is not expected to hold when \( T \) is large.

3. The Case of \( m > 1 \)

Building on our observations in \( m = 1 \) case, we extend our problem to settings where \( m > 1 \). We let \( K \) be the number of all available arms and suppose that the objective is to maximize the expected average cumulative reward across the top \( m \) arms. With \( T \) pulls and no additional constraints, the optimal policy in this problem is to always pull the arm with the highest mean – essentially, the objective boils down to the \( m = 1 \) objective. However, as we discussed in Section 1, such a policy is not practical. We thus modify the problem by assuming that there are \( T/m \) decision points, i.e., the time horizon is \( T/m \), and at each time, \( m \) distinct arms must be chosen (amounting to a total of \( T \) pulls). For simplicity of notation, we assume that \( T/m \) is an integer.

We reuse the notation for the distribution of the arms and their means: \( \nu_i \) denotes the probability distribution for arm \( i \in [K] \) and \( \mu_i \) denotes its mean. We label the arms so that \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \) (breaking ties arbitrarily), and we let \( \nu \) and \( \mu \) denote the vector of probability distributions and their means, respectively. We refer to the arms in the set \( [m] \) as the optimal arms and the rest of them as the suboptimal arms. Different from the \( m = 1 \) case, here we define two measures to capture the difference between the means of optimal arms and the suboptimal arms: \( \bar{\Delta}_i = \mu_i - \mu_{m+1} \)
for $i \in [m]$ and $\Delta_j = \mu_m - \mu_j$ for $j \geq m + 1$. That is, $\bar{\Delta}_i$ for each optimal arm $i$ is the difference between the mean of $i$ and the that of the best suboptimal arm. $\Delta_j$, on the other hand, for each sub-optimal arm $j$, is the difference between the mean of $j$ and that of the worst optimal arm. These two measures will be crucial in the analysis of this problem.

At each time, the decision-maker chooses exactly $m$ arms to play and observes a reward from each played arm. With some abusive reuse of notation, we let the vector $I_t$ denote the set of $m$ arms played at time $t$ and let the vector $X_t$ denote the rewards from those arms. With more abuse of notation, we then have $X_t(i) \sim \nu_{I_t(i)}$ for $i \in I_t$, and all these rewards are assumed to be independent from the previous actions and observations.

We again let the history of actions and observations at any time $t \geq 2$ be denoted as $H_t = (I_1, X_1, I_2, X_2, \ldots, I_{t-1}, X_{t-1})$, and define $H_1$ to be the empty set $\phi$. Hence, a policy $\pi$ of the decision-maker is a sequence of mappings $(\pi_1, \pi_2, \ldots, \pi_T)$, in which $\pi_t$ maps every possible history $H_t$ to a set of $m$ arms to be played at time $t$. We let $\Pi$ denote the set of all such policies.

Recall that $n_i^t$ denotes the number of times arm $i$ is played until and including time $t$, i.e., $n_i^t = \sum_{s=1}^t 1_{(i \in I_s)}$, and $U_n^i$ denotes the reward observed from the $n^{th}$ pull of arm $i$. Note that $(U_n^i)_{n \in \mathbb{N}}$ is a sequence of i.i.d. random variables, with each $U_n^i$ distributed as $\nu_i$. Finally, let $U_t^i$ be the cumulative reward obtained from arm $i$ until time $t$.

We now consider the following notion of reward for a policy $\pi$:

$$R_T(\pi, \nu) = E_{\nu}(\Gamma^m(U^1_{T/m}, U^2_{T/m}, \ldots, U^K_{T/m})),$$

where $\Gamma^m(x)$ denotes the average of the largest $m$ elements in $x \in \mathbb{R}^K$, for an integer $m$ with $1 \leq m \leq K$:

$$\Gamma^m(x) \triangleq \frac{1}{m} \sum_{i=1}^m y(i),$$

where the vector $y$ is vector $x$ sorted in non-increasing order (breaking ties arbitrarily).

In other words, the objective value attained by the policy is the expected value of the average of the $m$ largest cumulative rewards across all arms at the end of the decision-making horizon. When the reward distributions $\nu_1, \ldots, \nu_K$ are known to the decision-maker, then for a large $T$, the best reward that the decision-maker can achieve is

$$\sup_{\pi \in \Pi} R_T(\pi, \nu).$$

In a similar spirit to the $m = 1$ case, a natural candidate for a good policy when the reward distributions are known is the one where the decision-maker focuses on the top $m$ arms with the highest means, attaining an expected reward of $\mu_m \frac{T}{m}$, where $\mu_m \triangleq \frac{1}{m} \sum_{i=1}^m \mu_i$, the average mean of the $m$ highest mean arms. Let us denote $R_T^*(\nu) \triangleq \mu_m \frac{T}{m}$. One can show a result similar to Proposition [1] here too: $R_T^*(\nu)$ is the best reward that one can achieve in our problem.
Proposition 2  For any bandit instance \( \nu \in \mathcal{V} \), \( \sup_{\pi \in \Pi} R_T(\pi, \nu) = R_T^*(\nu) \).

The proof is presented in Section E in the Appendix. This shows that the policy that picks the \( m \) arms with the highest means in all periods is optimal. Next, we denote the regret of any policy \( \pi \) to be

\[
\text{Reg}_T(\pi, \nu) = \sup_{\pi \in \Pi} R_T(\pi, \nu) - R_T(\pi, \nu).
\]

We once again focus on finding a policy \( \pi \) that achieves the smallest \( \nu \)-dependent asymptotic regret (as \( T \to \infty \)) simultaneously for all \( \nu \), and also the smallest the worst-case regret over all distributions \( \nu \in \mathcal{V} \) for a fixed \( T \). To the latter end, let \( \text{Reg}_T^* \) denote the minmax (or the best worst-case) regret:

\[
\text{Reg}_T^* \triangleq \inf_{\pi \in \Pi} \sup_{\nu \in \mathcal{V}} \text{Reg}_T(\pi, \nu).
\]

In the remainder of this section, we will show that a regret of \( \Omega\left(\frac{(K-m)^{1/3}T^{2/3}}{m^{1/3}}\right) \) is inevitable in the worst case. We then will design a policy that attains this regret with a mildly weaker dependence on \( m \).

3.1. Lower Bound

It is clear from the results of the \( m = 1 \) case that an \( \Omega(\log T) \) instance-dependent regret and \( \Omega(T^{2/3}) \) instance-independent regret is inevitable for any policy. In the following result, we try to capture the dependence on \( m \) in the lower bound on the optimal instance-independent regret.

**Theorem 4.** Suppose that \( 2m \leq K < T \). Then, \( \text{Reg}_T^* \geq \Omega\left(\frac{(K-m)^{1/3}T^{2/3}}{m^{1/3}}\right) \).

The proof is presented in Section E in the Appendix and it extends the proof for the \( m = 1 \) case while tackling new challenges to capture the dependence on \( m \). However, we conjecture that the dependence on \( m \) in this bound is sub-optimal. In the next section, we will present an algorithm that attains a regret upper bound of \( \widetilde{O}\left(\frac{(K-m)^{1/3}T^{2/3}}{m}\right) \), which we believe is the best achievable. We leave the closure of this gap as an interesting open question for future work.

3.2. Adaptive Explore-then-Commit for General \( m \) (\( m \)-ADA-ETC)

We now present the algorithm we design for the \( m > 1 \) case. This is an extension of the ADA-ETC policy that we call \( m \)-ADA-ETC. It is formally defined in Algorithm 2.

\[\text{Intuitive reasoning for the improved lower bound is as follows. Assume } m \text{ divides } K \text{ and } K/m \geq 2. \text{ Then, an adversary can construct } m \text{ independent bandit sub-problems with } K/m \text{ arms in each problem such that the best arm needs to be identified and exploited in each of the } m \text{ problems to optimize our original objective. If the decision-maker additionally has the constraint that } T/m \text{ pulls can be expended in each sub-problem, then the lower bound from the } m = 1 \text{ case would imply that a regret of } \Omega\left((K/m - 1)^{1/3}(T/m)^{2/3}\right) = \Omega\left(\frac{(K-m)^{1/3}T^{2/3}}{m}\right) \text{ is inevitable in each sub-problem and thus inevitable in the original problem. However, this argument assumes the constraint of } T/m \text{ pulls per sub-problem, which may induce an avoidable loss.} \]
This algorithm shares a similar methodology to its $m = 1$ counterpart: After choosing each arm at least once, pull $m$ arms with the highest upper confidence bounds, until there are $m$ arms such that (a) they all have been played at least $\tau = \lceil T^{2/3} / (K - m)^{2/3} \rceil$ times, and (b) the smallest empirical mean among them is higher than the upper confidence bounds on the means of all other arms. Then, commit to these $m$ arms until the end of the decision-making horizon.

The upper confidence bound defined in Equation 6 is similar to that we define in Equation 2 for $m = 1$ case, with a modified dependence on the problem parameters, and aims to eliminate wasteful exploration by stopping early. Additionally, the design of the lower confidence bound in Equation 7 and the stopping criterion in Equation 8 again enforce the requirement of all arms being played at least $\tau$ times before being committed to by the algorithm. However, different from the $m = 1$ case, some arms may be pulled more than $\tau$ times while the algorithm is still in the Explore phase. Although this doesn’t mean that those pulls are wasteful: we can show that the arms that get pulled more than $\tau$ pulls during exploration are the ones that will be included in the set of exploited arms in the Commit phase (see Lemma 9 in the Appendix). This fact can be interpreted as meaning that a subset of arms may enter the Commit phase earlier than the others. In accordance, we introduce arm-specific exploration stopping times in our proof of performance guarantees of Algorithm 2.

**Remark 4.** The parallel between the designs for the upper confidence bounds, Equation 6 to that of in Equation 2 for the $m = 1$ case, follows from a similar heuristic rationale. Consider the following example: Let $m \leq K/2$. The $m$ optimal arms all have a mean of 1, and the remaining $K - m$ arms have a mean of $1 - \Delta$. Let $P_e$ denote the probability of labeling some fixed sub-optimal arm $j$ as optimal (i.e., one of the top $m$ arms). Then, the expected regret contributed due to this error is approximately $P_e \Delta T / m$, since our objective considers the average reward from top $m$ arms with the highest cumulative rewards. But the event of incorrectly labeling $j$ as optimal can occur by displacing any one of the top $m$ arms. We can thus approximately bound the probability of this event by the probability of the event that the error in the mean estimate of any one of the top $m$ arms is of order $\Delta$ during exploration. If we let $P_e(i)$ be the probability of such an event for an optimal arm $i$, then by a union bound, $P_e \lesssim \sum_{i=1}^{m} P_e(i)$. This implies that the expected regret due to misidentifying 1 arm is approximately upper bounded as $\sum_{i=1}^{m} P_e(i) \Delta T / m$. Now, to ensure that the regret due to this misidentification is at most $O((K - m)^{1/3} / \sqrt{n})$, we have $P_e(i)$ at most $\approx (K - m)^{1/3} / (\Delta T^{1/3})$, $i \in [m]$. Since $\Delta$ is not known to the algorithm, we again use $1/\sqrt{n}$ as a proxy for $\Delta$. Then, the target for this probability of misidentification is $\delta_n \approx ((K - m)^{1/3} n^{1/2})^{1/3}$. Hence, we get the relevant scaling of the confidence bound in Equation 2 i.e., $\sqrt{\log(1/\delta_n)} \approx \sqrt{\log(T / ((K - m)n^{3/2}))}$.

Let $m$-ADA-ETC$_{K,T}$ denote the implementation of $m$-ADA-ETC using $m$, $K$ and $T$ as the input for the number of arms to be selected, the total number of available arms, and the total number of assignments, respectively. We characterize the regret guarantees it achieves in the following result.
**Algorithm 2:** Adaptive Explore-then-Commit for General \(m\) (\(m\)-ADA-ETC)

1. **Input:** \(K\) arms, \(m\) to be ultimately selected, and horizon \(T\).

2. **Define:** \(\tau = \left\lceil \frac{T^{2/3}}{(K-m)^{2/3}} \right\rceil\). For \(n \geq 1\), let \(\mu^*_i\) be the empirical average reward from arm \(i\) after \(n\) pulls and it remains fixed after \(\tau\) pulls, i.e., \(\mu^*_i = \frac{1}{\min_{s=1}^{\min(n, \tau)} T_s} \sum_{s=1}^{\min(n, \tau)} T_s\). Also, for \(n \geq 1\), define,

\[
\text{UCB}_n^i = \bar{\mu}_n^i + \sqrt{\frac{4 \log \left( \frac{T n^{3/2}}{(K-m)n^{3/2}} \right)}{n}}, \quad (6)
\]

\[
\text{LCB}_n^i = \bar{\mu}_n^i - \sqrt{\frac{4 \log \left( \frac{T n^{3/2}}{(K-m)n^{3/2}} \right)}{n}}. \quad (7)
\]

Also, for \(t \geq 1\), let \(n'_t\) be the number of times arm \(i\) is pulled until and including time \(t\).

3. **Procedure:**

   - **Explore Phase:** From time \(t = 1\) until \(t = \left\lceil \frac{K}{m} \right\rceil\), pull each arm once. For \(\left\lceil \frac{K}{m} \right\rceil < t \leq \left\lceil \frac{T}{m} \right\rceil\):

     1. Identify \(U_t\), the \((m+1)\)th largest element of \(\left\{ \text{UCB}_{n_t}^1, \ldots, \text{UCB}_{n_t}^K \right\}\), breaking ties arbitrarily. Define \(E_t = \{ i \in [K] : \text{UCB}_{n_{t-1}}^i > \text{UCB}_{n_{t-1}}^{U_t} \}\).

        \[
        \min_{i \in E_t} \text{LCB}_{n_{t-1}}^i > \max_{j \in [K \setminus E_t]} \text{UCB}_{n_{t-1}}^j, \quad (8)
        \]

        then let \(I^* = E_t\); break, and enter the Commit phase. Else, continue to Step 2.

     2. Pull all arms in \(E_t\) once.

   - **Commit Phase:** Pull all arms in \(I^*\) until time \(t = \left\lceil \frac{T}{m} \right\rceil\).

**Theorem 5 (\(m\)-ADA-ETC performance).** Let \(K < T\). Consider a \(\nu \in \mathcal{V}\) such that there is a unique set of \(m\) optimal arms and relabel arms so that \(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m > \mu_{m+1} \geq \cdots \geq \mu_K\). Then the expected regret of \(m\)-ADA-ETC is upper bounded as:

\[
\text{Reg}_\tau(\text{m-ADA-ETC}_{K,T,\nu}) \leq \frac{1}{m} \sum_{i=1}^{m} \mu_i \min \left( \kappa, \frac{320(K-m)^2 \tau}{m T \Delta_i^3}, \frac{2(K-m)^2 \tau}{m} \right) + \frac{K-m}{m} \tau \min \left( \sum_{j=m+1}^{K} \mu_j \min \left( 1, \sqrt{\frac{320(K-m)}{T \Delta_j^3}} \right), \mu_m \right)
\]

\[
= \frac{T}{m^2} \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\mu_i - \mu_j) \min \left( \min \left( \frac{320(K-m)}{T \Delta_i^3}, \frac{320(K-m)}{T \Delta_j^3} \right), 1 \right) + \frac{3}{2} \exp\left( -\frac{\tau \Delta_i^2}{2} \right) + \frac{3}{2} \exp\left( -\frac{\tau \Delta_j^2}{2} \right), \quad (9)
\]

where \(\kappa = \sum_{j=m}^{K} \frac{11}{\Delta_j^3} + \frac{16}{\Delta_j^3} \log^+ \left( \frac{T \Delta_j}{K-m} \right) + \frac{24}{\Delta_j^3} \sqrt{\log^+ \left( \frac{T \Delta_j}{K-m} \right)}\), and \(\tau = \left\lceil \frac{T^{2/3}}{(K-m)^{2/3}} \right\rceil\). In the worst case, we have

\[
\sup_{\nu \in \mathcal{V}} \text{Reg}_\tau(\text{m-ADA-ETC}_{K,T,\nu}) \leq O\left( \frac{(K-m)^{1/3}T^{2/3} \sqrt{\log(K-m)}}{m} \right).
\]

The proof of Theorem 5 is presented in Section G in the Appendix. Theorem 5 features an instance-dependent \(O(\log T)\) regret bound and a worst-case bound of \(O\left( \frac{(K-m)^{1/3}T^{2/3}}{m} \sqrt{\log(K-m)} \right)\).
The first two terms in the instance-dependent bound arise from the wasted pulls during the Explore phase. The first term is due to exploration of the suboptimal arms and it specifies the pulls lost from the optimal arms while exploring. The second term is similar but it quantifies the loss in the case where the algorithm commits to a suboptimal set of arms. Both of these terms are at most \( \frac{K - m}{m} \tau m \leq \frac{K - m}{m} \tau = O\left( \frac{(K - m)^{1/3}T^{2/3}}{m} \right) \) in the worst case. The next term arises from the regret incurred to committing to some number of suboptimal arms. We can show that this regret is \( O\left( \frac{(K - m)^{1/3}T^{2/3}}{m} \sqrt{\log(K - m)} \right) \) in the worst case.

4. Experiments

In this section, we benchmark our proposed algorithms, ADA-ETC for \( m = 1 \) and \( m \)-ADA-ETC for \( m > 1 \), against candidate algorithms in the literature. To reiterate the value of appropriately designed upper and lower confidence bounds, we also benchmark our algorithms against some simple variations of ADA-ETC that utilize different confidence bounds to guide exploration.

4.1. ADA-ETC

We compare the performance of ADA-ETC with five algorithms described in Table 1. UCB1 never stops exploring and pulls the arm with the highest upper confidence bound at each time step, while ETC pulls arms in a round-robin fashion and commits to the arm with the highest empirical mean after each arm has been pulled \( \tau \) times. NADA-ETC and UCB1-s have the same algorithmic structure as ADA-ETC: they explore based on upper confidence bounds and commit if the lower confidence bound of an arm rises above upper confidence bounds for all other arms. They differ from ADA-ETC in how the upper and lower confidence bounds are defined. Both NADA-ETC and UCB1-s use UCB1’s upper confidence bound, but they differ in their lower confidence bounds. These definitions are presented in Table 1.

SUCC is an adaptation of the well-known “successive elimination” algorithm of Even-Dar et al. (2006) for best-arm identification, which finds the best arm within a probability of error of \( \delta \) in a sample efficient manner. This algorithm proceeds in rounds and samples every active arm once in each round, eliminating arms based on their empirical performance. In our adaptation, we set \( \delta = (K/T)^{1/3} \) so that the expected regret in case of failure is at most \( \delta T = T^{2/3}K^{1/3} \) in the worst-case. We further force the algorithm to commit to the active arm with the highest empirical mean after \( \tau \) rounds have elapsed.

Instances. We let \( \nu_i \sim Bernoulli(\mu_i) \), where \( \mu_i \) is uniformly sampled from \( [\alpha, 1 - \alpha] \) for each arm in each instance. We sample two sets of instances, each of size 500, with \( \alpha \in \{0, 0.4\} \). The regret for an algorithm for each instance is averaged over 500 runs to estimate the expected regret. We vary \( K \in \{2, 5, 10, 15, 20, 25\} \) and \( T \in \{100, 200, 300, 400, 500\} \). The average regret over the 500 instances under different algorithms and settings is presented in Figures 2 and 3.
Table 1 Benchmark Algorithms

| Algorithm      | Function                                                                 |
|----------------|---------------------------------------------------------------------------|
| ADA-ETC        | $\tilde{\mu}_n^i + \sqrt{\frac{1}{n} \log \left( \frac{T}{Kn^3/2} \right)}\mathbb{1}_{\{n < \tau\}}$ |
| ETC            | $\tilde{\mu}_n^i - \tilde{\mu}_n^i \mathbb{1}_{\{n < \tau\}}$          |
| SUCC           | $\tilde{\mu}_n^i$                                                       |
| LCB            | $\tilde{\mu}_n^i - \tilde{\mu}_n^i \mathbb{1}_{\{n < \tau\}}$          |
| UCB            | $\bar{\mu}_n^i$                                                        |
| LCB            | $\bar{\mu}_n^i$                                                        |
| UCB1           | $\tilde{\mu}_n^i + \sqrt{\frac{1}{n} \log (T)}$                        |
| LCB            | $\tilde{\mu}_n^i - \sqrt{\frac{1}{n} \log (T)}\mathbb{1}_{\{n < \tau\}}$ |
| UCB1-s         | $\tilde{\mu}_n^i + \sqrt{\frac{1}{n} \log (T)}\mathbb{1}_{\{n < \tau\}}$ |
| LCB            | $\tilde{\mu}_n^i - \sqrt{\frac{1}{n} \log (T)}\mathbb{1}_{\{n < \tau\}}$ |

Figure 2 Performance comparison of ADA-ETC for varying values of $T$.
Discussion. ADA-ETC shows the best performance uniformly across all settings, although there are settings where its performance is similar to ETC. As anticipated, these are settings where either (a) $\alpha = 0.4$, in which case, the arms are expected to be close to each other and hence adaptivity in exploring has little benefits, or (b) $T/K$ is relatively small, due to which $\tau$ is small. In these latter situations, the exploration budget of $\tau$ is expected to be exhausted for almost all arms under ADA-ETC, yielding in performance similar to ETC, e.g., if $K = 25$ and $T = 100$, then $\tau = \lceil 4^{2/3} \rceil = 3$, i.e., a maximum of only three pulls can be used per arm for exploring. When $\alpha$ is smaller, i.e., when arms are easier to distinguish, or when $\tau$ is large, the performance of ADA-ETC is significantly better than that of ETC. This illustrates the gains from the adaptivity of exploration under ADA-ETC.

![Graphs showing performance comparison of ADA-ETC for varying values of $K$.](image)

Figure 3 Performance comparison of ADA-ETC for varying values of $K$. 

(a) $T = 100$, $\alpha = 0$

(b) $T = 100$, $\alpha = 0.4$

(c) $T = 500$, $\alpha = 0$

(d) $T = 500$, $\alpha = 0.4$
Furthermore, we observe that the performances of SUCC, UCB1-s and NADA-ETC are essentially the same as ETC for the ranges of $T$ and $K$ we consider. This important observation suggests that naively adding adaptivity to exploration, e.g., based on UCB1’s upper confidence bounds, may not improve upon the performance of ETC in finite parameter settings, and appropriate refinement of the confidence bounds is crucial to the gains of ADA-ETC in these settings. Finally, we note that UCB1 performs quite poorly, thus demonstrating the importance of introducing an appropriate stopping criterion for exploration.

4.2. $m$-ADA-ETC

We compare the performance of $m$-ADA-ETC with four algorithms described in Table 2. $m$-ETC pulls arms in a round-robin fashion and commits to the $m$ arms with the highest empirical means after each arm has been pulled $\tau$ times. As before, $m$-NADA-ETC and $m$-UCB1-s have the same algorithmic structure as $m$-ADA-ETC: they explore based on upper confidence bounds and commit if the lower confidence bound of all arms rise above the upper confidence bounds of all other arms. They differ from $m$-ADA-ETC in how the upper and lower confidence bounds are defined. This time we exclude $m$-UCB1-s from our figures for brevity since, as before, its performance is identical to that of $m$-NADA-ETC for the parameter sets we focus on.

| Table 2 Benchmark Algorithms |
|--------------------------------|
| $m$-ADA-ETC                   | UCB$^i_n = \hat{\mu}^i_n + \sqrt{\frac{4}{n} \log \left( \frac{T}{(K-m)n/2} \right) \mathbb{1}_{(n<\tau)}}}$ |
|                               | LCB$^i_n = \hat{\mu}^i_n - \hat{\mu}^i_n \mathbb{1}_{(n<\tau)}$ |
| $m$-ETC                      | UCB$^i_n = *$ |
|                               | LCB$^i_n = *$ |
| $m$-NADA-ETC                 | UCB$^i_n = \hat{\mu}^i_n + \sqrt{\frac{4}{n} \log (T) \mathbb{1}_{(n<\tau)}}$ |
|                               | LCB$^i_n = \hat{\mu}^i_n - \hat{\mu}^i_n \mathbb{1}_{(n<\tau)}$ |
| $m$-UCB1-s                   | UCB$^i_n = \hat{\mu}^i_n + \sqrt{\frac{4}{n} \log (T) \mathbb{1}_{(n<\tau)}}$ |
|                               | LCB$^i_n = \hat{\mu}^i_n - \sqrt{\frac{4}{n} \log (T) \mathbb{1}_{(n<\tau)}}$ |
| RADA-ETC                     | UCB$^i_n = \hat{\mu}^i_n + \sqrt{\frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right) \mathbb{1}_{(n<\tau)}}$ |
|                               | LCB$^i_n = \hat{\mu}^i_n - \hat{\mu}^i_n \mathbb{1}_{(n<\tau)}$ |

Additionally, we introduce a natural benchmark that utilizes ADA-ETC, which we call randomized ADA-ETC (RADA-ETC). RADA-ETC randomly groups $K$ arms into $m$ subsets of size $K/m$ and runs ADA-ETC (until the end of Explore phase) for each subset with a budget of $T/m$ and returns one arm from each subset. Those arms are then pulled until the total number of remaining pulls allows. These definitions are presented in Table 2.

$^3$Further experiments show that increasing $T$ results in UCB1-s and NADA-ETC eventually outperforming ETC as well as SUCC.
Instances. We let \( \nu_i \sim \text{Bernoulli}({\mu}_i) \), where \( {\mu}_i \) is uniformly sampled from \([\alpha, 1 - \alpha]\) for each arm in each instance. We sample two sets of instances, each of size 250, with \( \alpha \in \{0, 0.4\} \). The regret for an algorithm for each instance is averaged over 100 runs to estimate the expected regret. We vary \( m \in \{2, 5, 10\} \), \( K \in \{4, 20\} \) and \( T \in \{100, 200, 300, 400, 500\} \). The average regret over the 250 instances under different algorithms and settings is presented in Figures 4 and 5.

Discussion. While \( m\text{-ADA-ETC} \) shows the best performance uniformly across almost all settings, \( m\text{-ETC} \) or RADA-ETC perform almost as good in some settings.

Similar to the \( m = 1 \) case, \( m\text{-ETC} \) is expected to perform as good as \( m\text{-ADA-ETC} \) when we either have (a) \( \alpha = 0.4 \), that is, the arms are expected to be close to each other and so that adaptivity in exploring is not as beneficial, or (b) \( T/(K - m) \) is small, so that \( \tau \) is small and is likely to be exhausted under \( m\text{-ADA-ETC} \).

We find a mild anomaly to this trend in Figures 5(b) and 5(d). In this case, we notice that even when \( \alpha = 0.4 \), and \( \tau \) in (b) is smaller than \( \tau \) in (d), the performance of \( m\text{-ETC} \) is significantly worse than \( m\text{-ADA-ETC} \) in (b) compared to (d). This can be explained by the fact that \( K - m \) is larger for (b) than for (d), and thus the confidence bounds of \( m\text{-ADA-ETC} \) are relatively smaller.
in (b) compared to those in (d). This encourages an early stop to the exploration compared to $m$-ETC despite the smaller $\tau$.

![Figure 5](image)

**Figure 5** Performance comparison of $m$-ADA-ETC for varying values of $T$.

We then look at the performance of RADA-ETC. The only scenarios where its performance comes close to that of $m$-ADA-ETC are those in Figures 5 (b) and (d). Here, since the size of the random subsets ($K/m$) is relatively large and arm means are close to each other, the probability of the best arm in a random subset being significantly suboptimal is lower.

And finally, as in the $m = 1$ case, $m$-NADA-ETC is not a suitable alternative to $m$-ADA-ETC, thus illustrating the gains from the tuning of the confidence bounds under $m$-ADA-ETC.

5. Discussion and Conclusion

In this paper, we propose and analyze new objectives under the multi-armed bandit framework motivated in the context of labor training in online labor platforms, where one wishes to train only a subset of incoming workers to satisfy the demand for jobs. These objectives have other applications. For example, related to the context of worker training in labor platforms, these objectives are also relevant to the problem of developing advanced talent within a region for participation in external competitions like Science Olympiads, the Olympic games, etc., with limited training resources. In
these settings, only the terminal skill levels of those finally chosen to represent the region matter. The resources spent on others, despite resulting in skill advancement, are effectively wasteful. This feature is not captured by the sum objective, whereas it is effectively captured by the max objective, particularly in situations where one individual will finally be chosen to represent the region.

Another application is in the context of e-commerce platforms, where the platform may want to groom an “attractor” product with a category of similar products. For instance, consider a product like a tablet cover (e.g., for an iPad). Once the utility of a new product of this type becomes established (e.g., the size specifications of a new version of the iPad becomes available), several brands offering close to identical products serving the same purpose proliferate the marketplace. This proliferation is problematic from the perspective of the platform because customers are inundated by choices and may unnecessarily delay their purchase decision, thereby increasing the possibility of leaving the platform altogether (Settle and Golden 1974, Gourville and Soman 2005). Given a budget for incentivizing customers to pick different products in the early exploratory phase where the qualities of the different products are being discovered, a natural objective for the platform is to groom a product to have the highest volume of positive ratings at the end of this phase. This product then becomes a clear choice for the customers. Our $m = 1$ objective effectively captures this goal.

Open questions and future directions. From a theoretical perspective, the main open question is whether the gap in terms of the dependence on $m$ of the upper and lower bounds on the regret can be eliminated in the case of $m > 1$. As we discussed earlier, we conjecture that the lower bound is loose in this regard. Additionally, our assumption that the rewards are i.i.d. over time could be a limitation in the context of worker training in settings where the number of training jobs available is large. It would be interesting to study our objective in settings that allow rewards to decrease over time; such models, broadly termed rotting bandits (Heidari et al. 2016, Levine et al. 2017, Seznec et al. 2019), have attracted recent focus in literature as a part of the study of the more general class of MAB problems with non-stationary rewards (see, for instance, Besbes et al. 2014, 2019). This literature has so far only focused on the traditional sum objective.

In the case of $m > 1$, we impose the constraint that $m$ arms must be chosen at the same time to avoid the practically undesirable solution of pulling only the best arm. However, there are other meaningful problem formulations that address this issue. One natural objective in this regard would be to maximize the expectation of the minimum cumulative reward across the top $m$ arms with the highest cumulative reward, or in other words, maximize the expectation of the $q^{th}$ percentile of the terminal skill levels across workers where $q \approx m/K$. This objective also generalizes the $m = 1$ setting considered in this paper and would be an interesting extension to consider. While such an objective would maximize the lowest quality level offered to the clients, it may excessively focus
on training slower workers in scenarios where the learning rates of the workers are expected to be skewed. In such cases, this objective may result in low terminal quality levels across all the trained workers. A platform may instead prefer to improve focus on training the better workers in such scenarios so as to generate a sufficient capacity of highly trained workers, though perhaps not sufficient to satisfy the entire demand for jobs. The formulation we consider in this paper for $m > 1$ would be a better alternative in such scenarios since it equally allocates training focus across all the top workers rather than focusing on the slowest workers in that set.

Finally, we note that our paper presents the possibility of studying a wide variety of new objectives under existing online learning setups motivated by training applications, where the traditional objective of maximizing the total rewards is inappropriate. A natural generalization of our objective is the optimization of other functionals of the vector of cumulative rewards, e.g., the optimization of $L^p$ norm of the vector of cumulative rewards for $p > 0$, which has natural fairness interpretations in the context of human training. More generally, one may consider multiple skill dimensions, with job types that differ in their impact on these dimensions. In such settings, a similar variety of objectives may be considered driven by considerations such as fairness, diversity, and focus.

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Appendix A: Proof of Proposition 1

For any policy $\pi$, we have that

$$R_T(\pi, \nu) = \mathbb{E} \left( \max_{i \in [K]} \sum_{t=1}^T U_i^T \right)$$

$$= \mathbb{E} \left( \max_{i \in [K]} \sum_{t=1}^T U_{i_{t-1}+1}^i \mathbb{I}(I_t=i) \right)$$

$$\leq \mathbb{E} \left( \sum_{t=1}^T \max_{i \in [K]} \left( U_{i_{t-1}+1}^i \mathbb{I}(I_t=i) \right) \right)$$

$$= \sum_{t=1}^T \mathbb{E} \left( \max_{i \in [K]} \left( U_{i_{t-1}+1}^i \mathbb{I}(I_t=i) \right) \right)$$

$$= \sum_{t=1}^T \mathbb{E} \left( U_{i_{t-1}+1}^{i_{t-1}+1} \mathbb{I}(I_t=i) \right)$$

$$= \sum_{t=1}^T \mathbb{E} \left( \mathbb{E}(U_{i_{t-1}+1}^{i_{t-1}+1}|\mathcal{H}_t) \right)$$

$$\leq \sum_{t=1}^T \mathbb{E}(\mu_i) \leq \mu_1 T.$$

Here, (a) is obtained due to pushing the max inside the sum; (b) is obtained because $U_{i_{t-1}+1}^i \geq 0$ for all $i$; and (c) holds because the reward for an arm in a period is independent of the past history of play and observations. Thus, the reward of $\mu_1 T$ is the highest that one can obtain under any policy. And this reward can, in fact, be obtained by the policy of always picking arm 1. This shows that

$$\sup_{\pi \in \Pi} R_T(\pi, \nu) = R_T^*(\nu).$$
Appendix B: Proof of Theorem 1

The proof of Theorem 1 relies on the following key result.

**Proposition 3** Consider a class $\mathcal{V} = \mathcal{M}^K$ of $K$-armed stochastic bandits and let $(\pi_T)_{T \in \mathbb{N}}$ be a consistent sequence of policies for $\mathcal{V}$. Then, for all $\alpha \in (0, 1]$ and $\nu \in \mathcal{V}$ such that the optimal arm $k^*$ is unique,

$$\liminf_{T \to \infty} \frac{E_{\nu} [n^T_{\nu}] }{ \log(T)} \geq \frac{\alpha}{d_{\text{inf}}(\nu, \mu^*, \mathcal{M})}$$

holds for each suboptimal arm $i \neq k^*$ in $\nu$, where $\mu^*$ is the highest mean.

**Proof of Proposition 3** In what follows, we denote $P_{\nu}$ to be the probability distribution induced by the policy $\pi$ on events until time $T$ under bandit $\nu$, and we let $E_{\nu}$ denote the corresponding expectation.

Let $\text{Reg}_{\text{SUM}, T}(\pi, \nu)$ denote the expected regret of the sum objective after $T$ pulls of policy $\pi$ under the bandit instance $\nu$, which can be defined as

$$\text{Reg}_{\text{SUM}, T}(\pi, \nu) = \mu^* T - E_{\nu} \left( \sum_{t=1}^{T} X_t \right)$$

(9)

$$= \mu^* T - E_{\nu} \left( \sum_{i=1}^{K} U_{iT} \right),$$

(10)

where $X_t = U^i_{n^T_i}$, which is the reward due to the arm pulled at time $t$, and $U_i = \sum_{n=1}^{n^T_i} U^i_n$, which is the cumulative reward obtained from arm $i$ until time $t$. We need the following two lemmas for our proof.

**Lemma 1.** Fix $\alpha \in (0, 1]$ and a policy $\pi$. Consider a $K$-armed bandit instance $\nu$ with $\mu^* \triangleq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K$. Fix a suboptimal arm $i$ and let $A_i = \{ n^T_i > \frac{\Delta_i}{\alpha} \}$. Then,

$$\text{Reg}_{\text{SUM}, T}(\pi, \nu) > P_{\nu}(A_i) \frac{T^\alpha \Delta_i}{2}.$$

**Lemma 2.** Fix $\alpha \in (0, 1]$ and a policy $\pi$. Consider a $K$-armed bandit instance $\nu$ with $\mu^* \triangleq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_K$. Fix a suboptimal arm $i$ and construct another $K$-armed bandit instance $\nu'$ satisfying $\mu_i' > \mu^* = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{i-1} \geq \mu_{i+1} \geq \cdots \geq \mu_K$. Let $A_i' = \{ n^T_i \leq \frac{\Delta_i}{\alpha} \}$. Then,

$$\text{Reg}_{\text{SUM}, T}(\pi, \nu') \geq P_{\nu'}(A_i') \frac{T^\alpha (\mu_i' - \mu^*)}{2}.$$

The proof of Lemma 1 is presented below at the end of this section. The proof of Lemma 2 is similar and hence is omitted.

Fix $\alpha \in (0, 1]$. We proceed by constructing a second bandit $\nu'$. Fix a suboptimal arm $i$, i.e., $\Delta_i > 0$, and let $\nu_j' = \nu_j$ for $j \neq i$ and pick a $\nu_i' \in \mathcal{M}$ such that $D(\nu_i, \nu_i') \leq \Delta_i + \epsilon$ and $\mu_i' > \mu^*$ for some arbitrary $\epsilon > 0$.

Let $\mu_i (\mu_i')$ be the mean of arm $i$ in $\nu (\nu')$ and $d_i \triangleq d_{\text{inf}}(\nu_i, \mu^*, \mathcal{M})$. Recall that $d_{\text{inf}}(\nu, \mu^*, \mathcal{M}) = \inf_{\nu' \in \mathcal{M}} \{ D(\nu, \nu') : \mu(\nu') > \mu^* \}$ where $\mu(\nu)$ denotes the mean of distribution $\nu$.

Since any lower bound on the regret for the sum objective implies the same lower bound on the max objective, using Lemma 1 and Lemma 2 we have the following:

$$\text{Reg}_{T}(\pi, \nu) + \text{Reg}_{T}(\pi, \nu') \geq \text{Reg}_{\text{SUM}, T}(\pi, \nu) + \text{Reg}_{\text{SUM}, T}(\pi, \nu')$$
Here, \( P_\nu \) (\( \overline{P}_\nu \)) is the probability distribution induced by the policy \( \pi \) on events until time \( \lceil T^{\alpha} \rceil \) under bandit \( \nu \) (\( \nu' \)). The equality then results from the fact that the two events \( \{ n_i' > \frac{T^{\alpha}}{2} \} \) and \( \{ n_i' \leq \frac{T^{\alpha}}{2} \} \) depend only on the play until time \( \lceil T^{\alpha} \rceil \). The last inequality follows from using the Bretagnolle-Huber inequality and divergence decomposition (see Theorem 14.2 and Lemma 15.1 in Lattimore and Szepesvári (2020), respectively) combined with the fact that \( D(\nu_i, \nu'_i) \leq d_i + \epsilon \):

\[
\overline{P}_\nu(A_i) + \overline{P}(\nu')(A_i') \geq \frac{1}{2} \exp \left( -D(P_\nu, \overline{P}_\nu) \right) \geq \frac{1}{2} \exp \left( -E_\nu \left[ n_i' \right] (d_i + \epsilon) \right),
\]

where the events \( A_i \) and \( A_i' \) are defined as they have been in Lemmas 1 and 2 for the fixed arm \( i \).

Rearranging Equation (11) we obtain

\[
\frac{E_\nu \left[ n_i' \right]}{\log(T)} > \frac{1}{d_i + \epsilon} \log \left( \frac{T^{\alpha} \min \{ \Delta_i, \mu'_i - \mu^* \}}{2 \operatorname{Reg}_T(\pi, \nu) + 2 \operatorname{Reg}_T(\pi, \nu')} \right),
\]

and taking the limit inferior yields

\[
\liminf_{T \to \infty} \frac{E_\nu \left[ n_i' \right]}{\log(T)} > \frac{1}{d_i + \epsilon} \liminf_{T \to \infty} \log \left( \frac{T^{\alpha} \min \{ \Delta_i, \mu'_i - \mu^* \}}{2 \operatorname{Reg}_T(\pi, \nu) + 2 \operatorname{Reg}_T(\pi, \nu')} \right),
\]

\[
= \frac{1}{d_i + \epsilon} \liminf_{T \to \infty} \log \left( \frac{\alpha \log(T) + \log(\beta_i) - \log(4) - \log(\operatorname{Reg}_T(\pi, \nu) + \operatorname{Reg}_T(\pi, \nu'))}{\log(T)} \right),
\]

\[
= \frac{1}{d_i + \epsilon} \left( \alpha - \limsup_{T \to \infty} \log \left( \frac{\operatorname{Reg}_T(\pi, \nu) + \operatorname{Reg}_T(\pi, \nu')}{\log(T)} \right) \right),
\]

\[
\geq \frac{\alpha}{d_i + \epsilon},
\]

where \( \beta_i = \min \{ \Delta_i, \mu'_i - \mu^* \} \).

Since \( \pi \) is a consistent policy over the class \( \mathcal{V} \), we can find a constant \( c_p \) for any \( p > 0 \) such that \( \operatorname{Reg}_T(\pi, \nu) + \operatorname{Reg}_T(\pi, \nu') \leq c_p T^p \), which implies

\[
\limsup_{T \to \infty} \frac{\log(\operatorname{Reg}_T(\pi, \nu) + \operatorname{Reg}_T(\pi, \nu'))}{\log(T)} \leq \limsup_{T \to \infty} \frac{p \log(T) + \log(c_p)}{\log(T)} = p.
\]

Then, Equation (14) follows from Equation (15) and the fact that \( p > 0 \) is arbitrary. Since \( \epsilon > 0 \) is arbitrary as well, we have

\[
\liminf_{T \to \infty} \frac{E_\nu \left[ n_i' \right]}{\log(T)} \geq \frac{\alpha}{d_i}
\]

for each \( i \neq k^* \), i.e., each suboptimal arm \( i \) in \( \nu \).
Proof of Lemma \[ \text{Recall that } I_t \text{ is the arm pulled at time } t \text{ and } X_t \text{ is the reward due to arm pulled at time } t, \text{i.e., } X_t \sim \nu_{I_t}. \text{ Then, due to, e.g., Lemma 4.5 in Lattimore and Szepesvári (2020), we can decompose the expected regret as}
\]
\[
\text{Reg}_{\text{SUM,T}}(\pi, \nu) = \sum_{j=1}^{K} \Delta_j \nu_j(n^*_j) + \Delta_j \nu_j(n^j).
\]

Due to the non-negativity of expected number of pulls and the suboptimality gaps, we have
\[
\text{Reg}_{\text{SUM,T}}(\pi, \nu) \geq \Delta_j \nu_j(n^*_j).
\]

Now, we look at \( \nu_j(n^*_j) \):
\[
\nu_j(n^*_j) = \nu_j(n^*_j | A_t) P_{\nu}(A_t) + \nu_j(n^*_j | A_t^i) P_{\nu}(A_t^i)
\]
\[
\geq \nu_j(n^*_j | A_t) P_{\nu}(A_t)
\]
\[
> \frac{T^\alpha}{2} P_{\nu}(A_t),
\]
\[
\text{where (a) is due to event } A_t = \{ n^*_t > \frac{T^\alpha}{2} \}. \text{ Finally, we have}
\]
\[
\text{Reg}_{\text{SUM,T}}(\pi, \nu) > P_{\nu}(A_t) \frac{T^\alpha \Delta_j}{2}.
\]

Proof of Theorem \[ \text{Let } k^* \text{ denote the unique optimal arm in } \nu \text{ and, without loss of generality, let } k^* = 1, \text{i.e., } \mu^* = \mu_1. \text{ Let } I^* \text{ denote the arm with the highest cumulative reward after } T \text{ pulls and recall that } n^*_T \text{ denotes the number of pulls spent on arm } i \text{ until time } T. \text{ Since all of the following expectations are over } \nu, \text{ we drop the subscript of } \nu \text{ hereafter. We first look at the expected regret:}
\]
\[
\text{Reg}_T(\pi, \nu) = \mu^* T - E \left[ \max \left( U_{T,1}, U_{T,2}, \ldots, U_{T,K} \right) \right]
\]
\[
\overset{(a)}{=} E \left[ \sum_{t=1}^{T} U_{t}^1 \right] - E \left[ \max \left( U_{T,1}, U_{T,2}, \ldots, U_{T,K} \right) \right] - E \left[ \max \left( U_{T,1}, U_{T,2}, \ldots, U_{T,K} \right) \right]
\]
\[
\overset{(b)}{=} E \left[ \sum_{t=1}^{T} U_{t}^1 \right] + E \left[ \sum_{t=1}^{T} U_{t}^1 \right] - E \left[ \sum_{t=1}^{T} U_{t}^1 \right] - E \left[ \sum_{t=1}^{T} U_{t}^1 \right]
\]
\[
\overset{(c)}{=} E \left[ \sum_{t=1}^{T} U_{t}^1 \right] + E \left[ \sum_{t=1}^{T} U_{t}^1 \right] - E \left[ \sum_{t=1}^{T} U_{t}^1 \right] + E \left[ \sum_{t=1}^{T} U_{t}^1 \right]
\]
\[
\overset{(d)}{=} \mu^* E \left[ \left( T - n^*_T \right) U_{T,1} \right] + E \left[ \sum_{t=1}^{T} U_{t}^1 \right] - E \left[ \sum_{t=1}^{T} U_{t}^1 \right] + E \left[ \sum_{t=1}^{T} U_{t}^1 \right]
\]
\[
\overset{(e)}{=} \mu^* \sum_{t=1}^{T} E \left[ \left( n^*_T - U_{t}^1 \right) U_{T,1} \right] + \sum_{t=1}^{T} E \left[ \sum_{t=1}^{T} U_{t}^1 \right] - E \left[ \sum_{t=1}^{T} U_{t}^1 \right] + E \left[ \sum_{t=1}^{T} U_{t}^1 \right].
\]
Here, (a) is due to the fact that \( \mathbb{E}[U_t^1] = \mu^* \) for \( t \in [T] \). (b) follows from the definition of \( I^* \). (c) results from \( n^i_T \leq T \) for \( i \in [K] \). (d) is due to the fact that the future rewards from the first arm is independent of the past history of play and observations of policy \( \pi \). Finally, (e) follows from the identity \( T = \sum_{i=1}^{K} n^i_T \).

We first focus on bounding the second term in the Expression 27. In order to do that, for each suboptimal arm \( i, i \neq 1 \), define a “good” event
\[
G_i = \left\{ U_T^1 > U_T^i + \frac{T \Delta_i}{2} \right\}.
\]
Notice that, for \( i \neq 1 \), \( \Delta_i > 0 \).

We proceed by showing that event \( G_i \) occurs with high probability. To that end, consider the complement event
\[
P(G_i^c) = P \left( U_T^1 \leq U_T^i + \frac{T \Delta_i}{2} \right) = P \left( \frac{U_T^1 - U_T^i}{T} - (\mu_1 - \mu_i) \leq \frac{\Delta_i}{2} - (\mu_1 - \mu_i) \right).
\]
By Hoeffding’s inequality,
\[
P(G_i^c) \leq \exp \left( -\frac{2T^2 (\Delta_i^2)}{4T} \right) = \exp \left( -\frac{T \Delta_i^2}{8} \right),
\]
since \(-1 \leq U_{t_1}^1 - U_{t_2}^i \leq 1\) for any pair \( t_1, t_2 \in [T] \). We thus also have that
\[
P(I^* = i, G_i) \leq \exp \left( -\frac{T \Delta_i^2}{8} \right). \tag{29}
\]
We then have
\[
E \left[ \left( \sum_{t=1}^{T} U_t^1 - \sum_{t=1}^{T} U_t^i \right) \mathbb{I}_{\{I^* = i\}} \right]
= E \left[ \left( \sum_{t=1}^{T} U_t^1 - \sum_{t=1}^{T} U_t^i \right) | I^* = i, G_i \right] P(I^* = i, G_i) + E \left[ \left( \sum_{t=1}^{T} U_t^1 - \sum_{t=1}^{T} U_t^i \right) | I^* = i, G_i^c \right] P(I^* = i, G_i^c)
\geq \frac{T \Delta_i}{2} P(I^* = i, G_i) - TP(I^* = i, G_i^c)
\geq \frac{T \Delta_i}{2} P(I^* = i, G_i) - O(1) \tag{30}
\geq 0 - O(1). \tag{31}
\]
Thus the second term in 27 is lower bounded by a (instance-dependent) constant.

Next, we bound the first term in 27. To do so, we first need an upper bound on \( P(I^* = i) \) for any \( i \neq 1 \). By consistency of policy \( \pi \), we have that \( \text{Reg}_{\pi}(\pi, \nu) \leq o(T^p) \) for every \( p > 0 \). Thus from 27 and 30 for any \( i \neq 1 \), we have that
\[
o(T^p) \geq E \left[ \left( \sum_{t=1}^{T} U_t^1 - \sum_{t=1}^{T} U_t^i \right) \mathbb{I}_{\{I^* = i\}} \right] \geq \frac{T \Delta_i}{2} P(I^* = i, G_i) - O(1). \tag{32}
\]
This implies that for any \( i \neq 1 \),
\[
P(I^* = i, G_i) \leq o(T^{p-1}), \tag{33}
\]
for every $p > 0$. Finally, \(29\) and \(33\) together imply that, for any $i \neq 1$, $P(I^* = i) = P(I^* = i, G_i) + P(I^* = i, G_c_i) \leq o(T^{p-1})$ for every $p > 0$.

Finally, we are ready to derive a lower bound on the first term in the expression \(27\). For any $\alpha \in (0, 1)$, we have

$$E[n_i^T/\mu_i] \leq E[n_i^T \mathbb{1}_{(I^* = 1)}] + [T^\alpha]P(I^* \neq 1),$$

for every $p > 0$. But then from Proposition \(3\) we have

$$\frac{\alpha}{d_i} \leq \liminf_{T \to \infty} \frac{E[n_i^T \mathbb{1}_{(I^* = 1)}]}{\log T} \leq \liminf_{T \to \infty} \frac{E[n_i^T \mathbb{1}_{(I^* = 1)}]}{\log T} + \liminf_{T \to \infty} \frac{o(T^{\alpha + p - 1})}{\log T}.$$  \(35\)

By choosing a $p$ such that $0 < p < 1 - \alpha$, we have that $\liminf_{T \to \infty} \frac{o(T^{\alpha + p - 1})}{\log T} = 0$. And thus, for every $\alpha \in (0, 1)$, we have

$$\liminf_{T \to \infty} \frac{E[n_i^T \mathbb{1}_{(I^* = 1)}]}{\log T} \geq \frac{\alpha}{d_i},$$

which implies that

$$\liminf_{T \to \infty} \frac{E[n_i^T \mathbb{1}_{(I^* = 1)}]}{\log T} \geq \frac{1}{d_i}.$$ \(38\)

Finally, putting everything together, from \(27\), \(31\), and \(38\), we have

$$\liminf_{T \to \infty} \frac{\text{Reg}_T(\pi, \nu)}{\log T} \geq \liminf_{T \to \infty} \mu^* \sum_{i \neq 1} E[n_i^T \mathbb{1}_{(I^* = 1)}] \frac{1}{\log T} - \liminf_{T \to \infty} \sum_{i \neq 1} \frac{O(1)}{\log T}$$

$$\geq \sum_{i \neq 1} \frac{\mu^*}{d_i}.$$ \(39\)

Plugging in the definition of $d_i$ and substituting $k^*$ back in place give the desired result. \qed
Appendix C: Proof of Theorem 2

First we fix a policy $\pi \in \Pi$. Let $\Delta \triangleq (K-1)^{1/3}/(2T^{1/3})$. We construct two bandit environments with different reward distributions for each of the arms and show that $\pi$ cannot perform well in both environments simultaneously.

We first specify the reward distribution for the arms in the base environment, denoted as the bandit $\nu = \{\nu_1, \ldots, \nu_K\}$. Assume that the reward for all of the arms have the Bernoulli distribution, i.e., $\nu_i \sim \text{Bernoulli}(\mu_i)$. We let $\mu_1 = \frac{1}{2} + \Delta$, and $\mu_i = \frac{1}{2}$ for $2 \leq i \leq K$. We let $P_\nu$ denote the probability distribution induced over events until time $T$ under policy $\pi$ in this first environment, i.e., in bandit $\nu$. Let $E_\nu$ denote the expectation under $P_\nu$.

Define $n^i_{[\Delta T]}$ as the (random) number of pulls spent on arm $i \in \{1, \ldots, K\}$ until time $[\Delta T]$ (note that $\sum_{i=1}^K n^i_{[\Delta T]} = [\Delta T]$) under policy $\pi$. Specifically, $n^i_{[\Delta T]}$ is the total (random) number of pulls spent on the first arm under policy $\pi$ until time $[\Delta T]$. Under policy $\pi$, let $l^* \in \{1\}$ that is pulled the least in expectation until time $[\Delta T]$, i.e., $l^* \in \text{argmin}_{2 \leq i \leq K} E_\nu(n^i_{[\Delta T]})$. Then clearly, we have that $E_\nu(n^i_{[\Delta T]}) \leq \frac{\Delta T}{l^*}$.

Having defined $l^*$, we can now define the second environment, denoted as the bandit $\nu' = \{\nu_1', \ldots, \nu'_K\}$. Again, assume that the reward for all of the arms have the Bernoulli distribution, i.e., $\nu'_i \sim \text{Bernoulli}(\mu'_i)$. We let $\mu'_1 = \frac{1}{2} + \Delta$, $\mu'_i = \frac{1}{2}$ for $2 \leq i \leq K \setminus \{l^*\}$, and $\mu'_i = \frac{1}{2} + 2\Delta$. We let $P_{\nu'}$ denote the probability distribution induced over events until time $T$ under policy $\pi$ in this second environment, i.e., in bandit $\nu'$. Let $E_{\nu'}$ denote the expectation under $P_{\nu'}$.

With some abuse of notation, for any event $B$, we define:

$$\text{Reg}_{\pi}(\pi, \nu, B) = \mu^*TP_\nu(B) - E_\nu(\max (\mathcal{U}_T^1, \mathcal{U}_T^2, \ldots, \mathcal{U}_T^K) \mathbb{1}_B).$$  

(40)

It is then clear that $\text{Reg}_{\pi}(\pi, \nu) = \text{Reg}_{\pi}(\pi, \nu, B) + \text{Reg}_{\pi}(\pi, \nu, B^c)$. We need the following two results for our proof.

**Lemma 3.** Fix a policy $\pi$. Consider the K-armed bandit instance $\nu$ with Bernoulli rewards and mean vector $\mu = \left(\frac{1}{2} + \Delta, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, where $\Delta < \frac{1}{2}$. Consider the event $A = \{n^i_{[\Delta T]} \leq \frac{\Delta T}{2}\}$. Then we have,

$$\text{Reg}_{\pi}(\pi, \nu, A) \geq \frac{\Delta T}{4} P_\nu(A) - 2\sqrt{T \log(KT)} - 2.$$  

The proof of Lemma 3 is presented below in this section. A similar argument shows the following.

**Lemma 4.** Fix a policy $\pi$. Consider the K-armed bandit instance $\nu'$ with Bernoulli rewards and mean vector $\mu' = \left(\frac{1}{2} + \Delta, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} + 2\Delta\right)$, where $\Delta < \frac{1}{5}$. Consider the event $A^c = \{n^i_{[\Delta T]} > \frac{\Delta T}{2}\}$. Then we have,

$$\text{Reg}_{\pi}(\pi, \nu', A^c) \geq \frac{\Delta T}{4} P_{\nu'}(A^c) - 2\sqrt{T \log(KT)} - 2.$$  

The proof of Lemma 4 is omitted since it is almost identical to that of Lemma 3. These two facts result in the following two inequalities:

$$\text{Reg}_{\pi}(\pi, \nu, A) \geq P_\nu \left( n^i_{[\Delta T]} \leq \frac{\Delta T}{2} \right) \Omega(\Delta T), \quad \text{and}$$  

$$\text{Reg}_{\pi}(\pi, \nu', A^c) \geq P_{\nu'} \left( n^i_{[\Delta T]} > \frac{\Delta T}{2} \right) \Omega(\Delta T).$$  

(41)

(42)
Note that here we have ignored the lower order $\sqrt{T \log(KT)}$ terms since $\Delta T = \Theta(T^{2/3}K^{1/3})$. Now, using the Bretagnolle-Huber inequality (see Theorem 14.2 in Lattimore and Szepesvári (2020)), we have,

$$\text{Reg}_T(\pi, \nu, A) + \text{Reg}_T(\pi, \nu', A^c) \geq \Omega(\Delta T) \left( P_\nu \left( n_{[\Delta T]}^l \leq \frac{\Delta T}{2} \right) + P_\nu' \left( n_{[\Delta T]}^l > \frac{\Delta T}{2} \right) \right)$$

(43)

$$= \Omega(\Delta T) \left( P_\nu \left( n_{[\Delta T]}^l \leq \frac{\Delta T}{2} \right) + P_\nu' \left( n_{[\Delta T]}^l > \frac{\Delta T}{2} \right) \right)$$

(44)

$$\geq \Omega(\Delta T) \exp \left( -D(\nu_\nu, \nu_\nu') \right).$$

(45)

Here, $P_\nu (\cdot)$ is the probability distribution induced by the policy $\pi$ on events until time $[\Delta T]$ under bandit $\nu$ ($\nu'$). The first equality then results from the fact that the two events $\{n_{[\Delta T]}^l \leq \frac{\Delta T}{2}\}$ and $\{n_{[\Delta T]}^l > \frac{\Delta T}{2}\}$ depend only on the play until time $[\Delta T]$. In the second inequality, which results from the Bretagnolle-Huber inequality, $D(\nu_\nu, \nu_\nu')$ is the relative entropy, or the Kullback-Leibler (KL) divergence between the distributions $P_\nu$ and $P_\nu'$ respectively. We can upper bound $D(\nu_\nu, \nu_\nu')$ as,

$$D(\nu_\nu, \nu_\nu') = E_\nu(n_{[\Delta T]}^l) D(\nu_{l^*}, \nu_{l^*}) \leq \frac{[\Delta T]}{K - 1} D(\nu_{l^*}, \nu_{l^*}) \lesssim \frac{8\Delta^3 T}{K - 1},$$

(46)

where $\nu_{l^*}$ ($\nu_{l^*}$) denotes the reward distribution of arm $l^*$ in the first (second) environment. The first equality results from divergence decomposition (see Lemma 15.1 in Lattimore and Szepesvári (2020)) and the fact no arm other than $l^*$ offers any distinguishability between $\nu$ and $\nu'$. The next inequality follows from the fact that $E_\nu(n_{[\Delta T]}^l) \leq ([\Delta T])/(K - 1)$, since by definition, $l^*$ is the arm that is pulled the least in expectation until time $[\Delta T]$ in bandit $\nu$ under $\pi$. Now $D(\nu_{l^*}, \nu_{l^*})$ is simply the relative entropy between the distributions Bernoulli(1/2) and Bernoulli(1/2 + $2\Delta$), which, by elementary calculations, can be shown to be at most $8\Delta^2$, resulting in the final inequality. Thus, we finally have,

$$\text{Reg}_T(\pi, \nu, A) + \text{Reg}_T(\pi, \nu', A^c) \geq \Omega(\Delta T) \exp \left( -\frac{8\Delta^3 T}{K - 1} \right).$$

Substituting $\Delta = (K - 1)^{1/3}/(2T^{1/3})$ gives

$$\text{Reg}_T(\pi, \nu, A) + \text{Reg}_T(\pi, \nu', A^c) \geq \Omega \left( (K - 1)^{1/3}T^{2/3} \right).$$

(47)

Equation (47) along with

$$\text{Reg}_T(\pi, \nu, A^c) \geq -O(\sqrt{T \log(KT)})$$

and

$$\text{Reg}_T(\pi, \nu', A) \geq -O(\sqrt{T \log(KT)}),$$

(48)

(49)

imply that

$$\text{Reg}_T(\pi, \nu) + \text{Reg}_T(\pi, \nu') \geq \Omega \left( (K - 1)^{1/3}T^{2/3} \right).$$

(50)

Finally, using $2\max\{a, b\} \geq a + b$ gives the desired lower bound on the regret.

Showing Equations (48) and (49) is an easy exercise:

$$\text{Reg}_T(\pi, \nu, A^c) = \mu^*TP_\nu(A^c) - E_\nu \left( \max \left( \sum_{t=1}^T U_t^1, \sum_{t=1}^T U_t^2, \ldots, \sum_{t=1}^T U_t^K \right) \mathbb{1}_{A^c} \right) \geq \mu^*TP_\nu(A^c) - E_\nu \left( \max \left( \sum_{t=1}^T U_t^1, \sum_{t=1}^T U_t^2, \ldots, \sum_{t=1}^T U_t^K \right) \mathbb{1}_{A^c} \right) \geq \mu^*TP_\nu(A^c) - \mu^*TP_\nu(A^c) - 2\sqrt{T \log(KT)} - 2 \overset{(a)}{=} -2\sqrt{T \log(KT)} - 2.$$

(51)
Here, (a) follows from an argument essentially identical to the one in the proof of Lemma 3 below and we do not repeat it here for brevity. Similarly, we can show that

\[ \text{Reg}_T(\pi, \nu', A) \geq -2 \sqrt{T\log(KT)} - 2. \]  

(52)

\[ \square \]

**Proof of Lemma 3.** We first have that

\[ \mathbb{E}_\nu \left( \max \left( U_T^1, U_T^2, \ldots, U_T^K \right) \right) = \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{n_1} U_t^1, \sum_{t=1}^{n_2} U_t^2, \ldots, \sum_{t=1}^{n_K} U_t^K \right) \right) \]

(53)

\[ \leq \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^1, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^2, \ldots, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^K \right) \right). \]

(54)

Defining \( T_1 = T - \lceil \frac{T_0}{2} \rceil \), and \( T_i = T \) for all \( i > 1 \), consider the “good” event

\[ G = \left\{ \sum_{t=1}^{T_j} |U_t^j - \mu_j| \leq \sqrt{T\log(KT)} \text{ for all } j \right\}. \]

Since \( U_t^j \in [0, 1] \), by Hoeffding’s inequality, we have that for any \( T' \leq T \),

\[ \mathbb{P}_\nu \left( \left| \sum_{t=1}^{T'} U_t^j - \mu_j T' \right| \leq \sqrt{T' \log(KT)} \right) \geq 1 - 2 \exp(-\frac{2(\sqrt{T' \log(KT)})^2}{T'}). \]

\[ \geq 1 - 2 \exp(-\frac{2(\sqrt{T' \log(KT)})^2}{T}) \]

\[ = 1 - \frac{2}{K^2 T^2} \geq 1 - \frac{2}{KT}. \]

Hence, by the union bound we have that \( P(G) \geq 1 - \frac{2}{KT}. \) Thus we finally have,

\[ \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^1, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^2, \ldots, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^K \right) \right) \]

\[ = \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^1, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^2, \ldots, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^K \right) \right) \]

\[ + \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{T-T-[\frac{2}{\Delta}]} U_t^1, \sum_{t=1}^{T-T-[\frac{2}{\Delta}]} U_t^2, \ldots, \sum_{t=1}^{T-T-[\frac{2}{\Delta}]} U_t^K \right) \right) \]

\[ \leq \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^1, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^2, \ldots, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^K \right) \right) \]

\[ \leq \mathbb{E}_\nu \left( \max \left( \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^1, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^2, \ldots, \sum_{t=1}^{T-[\frac{2}{\Delta}]} U_t^K \right) \right) \]

\[ \leq \mathbb{E}_\nu \left( \left| \sum_{t=1}^{\lceil \frac{T_0}{2} \rceil} T_0 \right| \right) \]

\[ \leq \frac{1}{2} \Delta(T - \frac{\Delta T}{2}) \mathbb{P}_\nu(A) + 2 \sqrt{T\log(KT)} + 2. \]

(55)

Here, (a) follows from the fact that \( \Delta < \frac{1}{2} \). Thus, from Equations (53) and (54) we finally have,

\[ \text{Reg}_T(\pi, \nu, A) = \frac{1}{2} + \Delta T \mathbb{P}_\nu(A) - \mathbb{E}_\nu \left( \max \left( U_T^1, U_T^2, \ldots, U_T^K \right) \right) \]

\[ \geq \frac{1}{2} + \Delta T \mathbb{P}_\nu(A) - \frac{1}{2} + \Delta(T - \frac{\Delta T}{2}) \mathbb{P}_\nu(A) - 2 \sqrt{T\log(KT)} - 2 \]

\[ = \frac{1}{2} + \Delta \frac{\Delta T}{2} \mathbb{P}_\nu(A) - 2 \sqrt{T\log(KT)} - 2 \]

\[ \geq \frac{\Delta T}{4} \mathbb{P}_\nu(A) - 2 \sqrt{T\log(KT)} - 2. \]

(56)

\[ \square \]
Appendix D: Proof of Theorem 3

The proof of Theorem 3 utilizes two technical lemmas. The first one is the following.

**Lemma 5.** Let \( \delta \in (0, 1) \), and \( X_1, X_2, \ldots \), be a sequence of independent 0-mean 1-Sub-Gaussian random variables. Let \( \bar{\mu}_t = \frac{1}{t} \sum_{s=1}^{t} X_s \). Then for any \( x > 0 \),

\[
P\left( \exists t > 0 : \bar{\mu}_t + \sqrt{\frac{4}{t} \log^+ \left( \frac{1}{\delta t^{3/2}} \right)} + x < 0 \right) \leq 39 \delta x^3.
\]

Its proof is similar to the proof of Lemma 9.3 in [Lattimore and Szepesvári, 2020](#), which we present below for completeness.

**Proof of Lemma 5.** We have,

\[
P\left( \exists t > 0 : \bar{\mu}_t + \sqrt{\frac{4}{t} \log^+ \left( \frac{1}{\delta t^{3/2}} \right)} + x < 0 \right)
\]

\[
= P\left( \exists t > 0 : t \bar{\mu}_t + \sqrt{4t \log^+ \left( \frac{1}{\delta t^{3/2}} \right)} + tx < 0 \right)
\]

\[
\leq \lim_{\infty} P\left( \exists t \in [2^i, 2^{i+1}] : t \bar{\mu}_t + \sqrt{4t \log^+ \left( \frac{1}{\delta t^{3/2}} \right)} + tx < 0 \right)
\]

\[
\leq \lim_{\infty} P\left( \exists t \in [0, 2^{i+1}] : t \bar{\mu}_t + 2^{i+2} \log^+ \left( \frac{1}{2 \delta 2^{(i+1) 3/2}} \right) + 2^i x < 0 \right)
\]

\[
\leq \sum_{i=0}^{\infty} \left( 2^{i+3/2} \right) \exp \left( \frac{-2^{i+3/2} \log^+ \left( \frac{1}{2 \delta 2^{(i+1) 3/2}} \right)}{2^{i+2}} \right)
\]

where the first inequality follows from a union bound on a geometric grid. The second inequality is used to set up the argument to apply Theorem 9.2 in [Lattimore and Szepesvári, 2020](#) and the third inequality is due to its application. The fourth inequality follows from \((a + b)^2 \geq a^2 + b^2\) for \( a, b \geq 0 \). Then, using a property of unimodal functions (\( \sum_{j=0}^{d} f(j) \leq \max_{i \in [0, d]} f(i) + \int_{0}^{d} f(i) di \)) for a unimodal function \( f \), the term \( 2^{(i+1) 3/2} \exp \left( -2^{i+3/2} x^2 \right) \) can be upper bounded by \( \frac{2^{i+3/2}}{\log(2) x^3} \). Evaluating the integral to \( \frac{x^3}{\log(2) x^3} \), we get

\[
P\left( \exists t > 0 : \bar{\mu}_t + \sqrt{\frac{4}{t} \log^+ \left( \frac{1}{\delta t^{3/2}} \right)} + x < 0 \right) \leq 39 \delta x^3.
\]

**Lemma 6.** [Lattimore and Szepesvári, 2020](#) Let \( X_1, X_2, \ldots \), be a sequence of independent 0-mean 1-Sub-Gaussian random variables. Let \( \bar{\mu}_t = \frac{1}{t} \sum_{s=1}^{t} X_s \). Let \( \epsilon > 0 \), and \( a > 0 \), and define

\[
\kappa = \sum_{t=1}^{T} \mathbb{I} \{ \bar{\mu}_t + \sqrt{\frac{2a}{t}} > \epsilon \}.
\]

Then \( \mathbb{E}[\kappa] \leq 1 + \frac{2}{\epsilon} (a + \sqrt{a\pi} + 1) \).
Proof of Theorem 3. Let 1 denote the first arm and \( i^* \) denote the arm used in the Commit phase of ADA-ETC. We first define a random variable that quantifies the lowest value of the index of arm 1 can take with respect to its true mean across \( T \) pulls.

\[
\Delta \overset{\triangle}{=} \left( \mu_1 - \min_{n \leq \tau} \left( \bar{\mu}_{1_n} + \frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right) \mathbb{1}_{\{n < \tau\}} \right) \right)^+.
\]

The following bound is instrumental for our analysis. For any \( \tau \),

\[
P(\Delta > x) = P \left( \exists n \leq \tau : \bar{\mu}_{1_n} + \frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right) \mathbb{1}_{\{n < \tau\}} < \mu_1 - x \right)
\]

\[
\leq P \left( \exists n < \tau : \bar{\mu}_{1_n} + \frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right) < \mu_1 - x \right) + P \left( \bar{\mu}_1 < \mu_1 - x \right)
\]

\[
\overset{(a)}{\leq} \min(1, \frac{39K}{T x^3} + \exp(-2\tau x^2)) \quad \overset{(b)}{\leq} \min(1, \frac{40K}{T x^3}).
\]

Here, (a) follows from Lemma 5 and Hoeffding’s inequality, and (b) follows by the definition of \( \tau \) and since \( \exp(-2\alpha^{2/3}) \leq 1/\alpha \) for all \( \alpha \geq 0 \).

We next decompose the regret into the regret from wasted pulls in the Explore phase and the regret from committing to a suboptimal arm in the Commit phase. Let \( \omega \) be the random time when the Explore phase ends. Let \( r^*_{\omega} \) be the reward earned from arm \( i \) until time \( \omega \). Then the expected regret in the event that \( \{i^* = i\} \) is bounded by:

\[
E \left( T \mu_1 - (T - \sum_{j \neq i} n^i_{\omega} - n^i_{\omega}) \mu_i - r^i_{\omega} \mathbb{1}_{\{i^* = i\}} \right).
\]

Note that this expression assumes that the cumulative reward of arm \( i \) will be chosen to compete against \( T \mu_1 \) at the end of time \( T \); however, if there is an arm with a higher cumulative reward, then the resulting regret can only be lower. Thus the total expected regret is bounded by:

\[
\sum_{i=1}^{K} E \left( T \mu_1 - (T - \sum_{j \neq i} n^i_{\omega} - n^i_{\omega}) \mu_i - r^i_{\omega} \mathbb{1}_{\{i^* = i\}} \right)
\]

\[
\overset{(a)}{\leq} \sum_{i=1}^{K} E(T \Delta_i \mathbb{1}_{\{i^* = i\}}) + \mu_1 \sum_{i=1}^{K} E(n^i_{\omega} \mathbb{1}_{\{i^* \neq i\}}) + \sum_{i=1}^{K} E(\bar{\mu}_i - r^i_{\omega}) \mathbb{1}_{\{i^* = i\}}
\]

\[
\overset{(b)}{=} \sum_{i=1}^{K} E(T \Delta_i \mathbb{1}_{\{i^* = i\}}) + \mu_1 \sum_{i=1}^{K} E(n^i_{\omega} \mathbb{1}_{\{i^* \neq i\}}) + \sum_{i=1}^{K} E(\bar{\mu}_i - r^i_{\omega}) \mathbb{1}_{\{i^* = i\}}
\]

\[
= \sum_{i=1}^{K} E(T \Delta_i \mathbb{1}_{\{i^* = i\}}) + \mu_1 \sum_{i=1}^{K} E(n^i_{\omega} \mathbb{1}_{\{i^* \neq i\}}) + \sum_{i=1}^{K} P(i^* = i)(\bar{\mu}_i - E(r^i_{\omega} \mid i^* = i))
\]

\[
\overset{(c)}{=} \sum_{i=1}^{K} E(T \Delta_i \mathbb{1}_{\{i^* = i\}}) + \mu_1 \sum_{i=1}^{K} E(n^i_{\omega} \mathbb{1}_{\{i^* \neq i\}}) + \sum_{i=1}^{K} P(i^* = i)(\bar{\mu}_i - \sum_{n=1}^{\tau} E(U^i_n \mid i^* = i))
\]

\[
\overset{(d)}{\leq} \sum_{i=1}^{K} E(T \Delta_i \mathbb{1}_{\{i^* = i\}}) + \mu_1 \sum_{i=1}^{K} E(n^i_{\omega} \mathbb{1}_{\{i^* \neq i\}}).
\]
Here, (a) results from rearranging terms, and from the fact that \( \mu_i \leq \mu_1 \). Both (b) and (c) result from the fact that in the event that \( \{i^* = i\} \), \( n_i^* = \tau \). (d) holds since, by a standard stochastic dominance argument, \( \tau \mu_i \leq \sum_{n=1}^{\tau} E(U_n | i^* = i) \).

We bound these two terms one by one.

**Regret from Explore.** First, note that an instance-independent bound on the regret from Explore is simply \( K \tau = K \lceil \frac{T^{2/3}}{K^{2/3}} \rceil = O(K^{1/3}T^{2/3}) \), which is the maximum number of pulls possible before ADA-ETC enters the Commit phase. Hence, we now focus on deriving an instance-dependent bound. We have that

\[
E(\sum_{i=1}^{K} n_i^* \mathbb{1}_{(i \neq i^*)}) \leq E(\sum_{i=2}^{K} n_i^*) + \tau P(i^* \neq 1)
\]

\[
= E(\sum_{i \geq 2: \Delta \leq \Delta_1} n_i^*) + E(\sum_{i \geq 2: \Delta > \Delta_1} n_i^*) + \tau P(i^* \neq 1).
\]  

(63)

We first bound the first term. Define the random variable

\[
\eta_i = \sum_{n=1}^{\tau} \mathbb{1}_{\left\{ \hat{\mu}_n + \sqrt{\frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right)} \mathbb{1}_{(n < \tau)} \geq \mu_i + \frac{\Delta_i}{2} \right\}}.
\]

Then in the event that \( \Delta \leq \Delta_1 \), we have that \( n_i^* \leq \eta_i \). We also have that \( n_i^* \leq \tau \). And thus in the event that \( \Delta \leq \Delta_1 \), we have \( n_i^* \leq \min(\eta_i, \tau) \). Hence the first term above is bounded as:

\[
\sum_{i=2}^{K} P(\Delta \leq \Delta_1) \min(E(\eta_i, \tau)) \leq \sum_{i=2}^{K} P(\Delta \leq \Delta_1) \min(\eta_i, \tau) \leq \sum_{i=2}^{K} \min(\eta_i, \tau).
\]

We can now bound \( E(\eta_i) \) as follows:

\[
E(\eta_i) \leq 1 + E\left(\sum_{n=1}^{\tau-1} \mathbb{1}_{\left\{ \hat{\mu}_n + \sqrt{\frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right)} \geq \mu_i + \frac{\Delta_i}{2} \right\}}\right)
\]

\[
= 1 + E\left(\sum_{n=1}^{\tau-1} \mathbb{1}_{\left\{ \hat{\mu}_n + \sqrt{\frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right)} \geq \mu_i + \frac{\Delta_i}{2} \right\}}\right)
\]

\[
\leq 1 + \frac{1}{\Delta_i^2} + E\left(\sum_{n=1}^{\tau-1} \mathbb{1}_{\left\{ \hat{\mu}_n + \sqrt{\frac{4}{n} \log \left( \frac{T}{Kn^{3/2}} \right)} \geq \mu_i + \frac{\Delta_i}{2} \right\}}\right)
\]

\[
\leq 2 + \frac{1}{\Delta_i^2} + \frac{8}{\Delta_i^2} \left(2 \log \left( \frac{T \Delta_i^3}{K} \right) + \sqrt{2 \pi \log \left( \frac{T \Delta_i^3}{K} \right)} + 1\right)
\]

\[
\leq \frac{11}{\Delta_i^2} + \frac{16}{\Delta_i^2} \log \left( \frac{T \Delta_i^3}{K} \right) + 24 \Delta_i \sqrt{\log \left( \frac{T \Delta_i^3}{K} \right)}.
\]

(64)

Here, (a) is due to lower bounding \( 1/n^{3/2} \) by \( \Delta_i^3 \), and adding \( 1/\Delta_i^2 \) for the first \( 1/\Delta_i^2 \) time periods where this lower bound doesn’t hold. (b) is due to Lemma 5. The final inequality results from the fact that \( \Delta_i \leq 1 \) and from trivially bounding \( 2\pi \leq 9 \). Thus, we finally have,

\[
E(\sum_{i \geq 2: \Delta \leq \Delta_1} n_i^*) \leq \sum_{i=2}^{K} \min\left(\frac{11}{\Delta_i^2} + \frac{16}{\Delta_i^2} \log \left( \frac{T \Delta_i^3}{K} \right) + 24 \Delta_i \sqrt{\log \left( \frac{T \Delta_i^3}{K} \right)}, \tau\right).
\]

(65)
We now focus on the second term in Equation 65. Note that we have \( n_i^\alpha \leq \tau \), and hence,

\[
E( \sum_{i \geq 2, \Delta > \frac{\Delta_2}{2}} n_i^\alpha ) \leq \tau \sum_{i = 2}^{K} P(\Delta > \frac{\Delta_2}{2}) \leq \tau \sum_{i = 2}^{K} \min(1, \frac{320K}{T\Delta_i^3}).
\]  

(66)

Here the second inequality follows from Equation 60. Next, we focus on the third term in Equation 63. We have:

\[
P(i^* \neq 1) = P(i^* \neq 1 \text{ and } \Delta \leq \frac{\Delta_2}{2}) + P(i^* \neq 1 \text{ and } \Delta > \frac{\Delta_2}{2})
\]

\[
\leq \min \left( 1, \sum_{i = 2}^{K} P(i^* = i \text{ and } \Delta \leq \frac{\Delta_2}{2}) + P(\Delta > \frac{\Delta_2}{2}) \right) \]

\[
\leq \min \left( 1, \sum_{i = 2}^{K} P(i^* = i \text{ and } \Delta \leq \frac{\Delta_2}{2}) + \frac{320K}{T\Delta_i^3} \right).
\]  

(67)

Here the final inequality again follows from Equation 60. Now in the event that \( \Delta \leq \Delta_2/2 \), \( i^* = i \) implies that there is some \( n \leq \tau \) such that \( LCB_i^\tau = \mu_i - \bar{\mu}_n \mathbb{1}_{(n < \tau)} > \mu_i + \Delta_i/2 \). Thus, we have,

\[
\sum_{i = 2}^{K} P(i^* = i \text{ and } \Delta \leq \frac{\Delta_2}{2}) \leq \sum_{i = 2}^{K} P(\exists n \leq \tau : \bar{\mu}_n - \bar{\mu}_n \mathbb{1}_{(n < \tau)} > \mu_i + \Delta_i/2)
\]

\[
= \sum_{i = 2}^{K} P(\bar{\mu}_n > \mu_i + \Delta_i/2)
\]

\[
\leq \sum_{i = 2}^{K} \exp(-\frac{\tau \Delta_i^2}{2}) \leq \sum_{i = 2}^{K} \frac{8K}{T\Delta_i^3}.
\]  

(68)

Here, (a) follows from Hoeffding’s inequality, and (b) follows from the definition of \( \tau \) and the fact that \( \exp(-\alpha^{2/3}/2) \leq 8/\alpha \) for \( \alpha \geq 0 \). Thus we finally have

\[
\tau P(i^* \neq 1) \leq \tau \min(1, \sum_{i = 2}^{K} \frac{8K}{T\Delta_i^3} + \frac{320K}{T\Delta^3_2})
\]

\[
\leq \tau \min(1, \sum_{i = 2}^{K} \frac{328K}{T\Delta_i^3}).
\]  

(69)

Thus, combining Equations 65, 66, and 69, we have that the regret from the Explore phase is bounded by

\[
\mu_1 \sum_{i = 2}^{K} \min \left( \frac{11}{\Delta_i^2} + \frac{16}{\Delta_i^2} \log^+ \left( \frac{T\Delta_i^3}{K} \right), \tau \right) + \mu_1 \tau \sum_{i = 2}^{K} \min(1, \frac{320K}{T\Delta_i^3}) + \mu_1 \tau \min(1, \sum_{i = 2}^{K} \frac{328K}{T\Delta_i^3})
\]

\[
\leq \mu_1 \sum_{i = 2}^{K} \min \left( \frac{10}{\Delta_i^2} + \frac{16}{\Delta_i^2} \log^+ \left( \frac{T\Delta_i^3}{K} \right), \tau \right) + \mu_1 \tau \sum_{i = 2}^{K} \min(2, \frac{648K}{T\Delta_i^3}).
\]  

(70)

Here the inequality results from the fact that \( \min(1, a) + \min(1, b) \leq \min(2, a + b) \) for \( a, b > 0 \). This finishes our derivation of a distribution dependent bound on the regret from the Explore phase. We next focus on the regret arising from misidentification in the Commit phase.
Regret from Commit. This regret is upper bounded by

$$E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}) + E(\sum_{i: \Delta > \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}).$$  \hfill (71)

We now get instance dependent and independent bounds on each of the above two terms.

An instance dependent bound on $E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i})$. In the event that $\Delta \leq \Delta_i/2$, $i^* = i$ implies that there is some $n \leq \tau$ such that $\text{LCB}^*_n = \hat{\mu}^*_n - \hat{\mu}^*_n 1_{(\hat{\Delta} < \tau)} > \mu_i + \Delta_i/2$. Thus, we have,

$$E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}) \leq \sum_{i=2}^K P(\exists n \leq \tau: \hat{\mu}^*_n - \hat{\mu}^*_n 1_{(\hat{\Delta} < \tau)} > \mu_i + \Delta_i/2) T_{\Delta_i}.$$  \hfill (72)

Now, we have,

$$P(\exists n \leq \tau: \hat{\mu}^*_n - \hat{\mu}^*_n 1_{(\hat{\Delta} < \tau)} > \mu_i + \Delta_i/2) = P(\hat{\mu}^*_n > \mu_i + \Delta_i/2) \leq \exp(-\frac{\tau \Delta^2}{2}).$$  \hfill (73)

Here the final inequality follows from Hoeffding’s inequality. Thus we finally have,

$$E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}) \leq \sum_{i=2}^K \exp(-\frac{\tau \Delta^2}{2}) T_{\Delta_i}.$$  \hfill (74)

An instance independent bound on $E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i})$. We have

\begin{align*}
E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}) &= T^{2/3} K^{1/3} \sqrt{2 \log K} + E(\sum_{i: \Delta \leq \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}) \\
&\overset{(a)}{\leq} T^{2/3} K^{1/3} \sqrt{2 \log K} + E(\sum_{i: \Delta_i \geq \frac{K^{1/3} \tau^{2/3}}{T^{1/3}}} 1_{\{i^* = i\}} T_{\Delta_i}) \\
&\overset{(b)}{\leq} T^{2/3} K^{1/3} \sqrt{2 \log K} + T^{2/3} K^{1/3} \sqrt{2 \log K} + \frac{1}{K} T^{2/3} K^{1/3} \sqrt{2 \log K}.
\end{align*}

Here, (a) follows for the same reason as the derivation of the bound in Equation 74. Next, observe that the function $\exp(-\frac{\tau x}{2})x$ is maximized at $x = \sqrt{2/\tau} = \sqrt{2K^{1/3}/T^{1/3}}$. But since $\Delta_i \geq \sqrt{2 \log K} K^{1/3}/T^{1/3} \geq \sqrt{2K^{1/3}/T^{1/3}}$, by the unimodality of $\exp(-\frac{\tau x^2}{2})x$, we have

$$\exp(-\frac{\tau \Delta^2}{2}) T_{\Delta_i} \leq \exp(-\log K) T^{2/3} K^{1/3} \sqrt{2 \log K} = \frac{1}{K} T^{2/3} K^{1/3} \sqrt{2 \log K}.$$  \hfill (75)

Hence (b) follows.

An instance dependent bound on $E(\sum_{i: \Delta > \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i})$.

\begin{align*}
E(\sum_{i: \Delta > \hat{\Delta}} 1_{\{i^* = i\}} T_{\Delta_i}) &\leq E(\max_{i \in [K]} T_{\Delta_i} 1_{\{\Delta > \hat{\Delta}_i\}}) \\
&= P(\Delta > \frac{\Delta K}{2}) T_{\Delta_K} + \sum_{i=1}^{K-1} P(\frac{\Delta_{i+1}}{2} \geq \Delta > \frac{\Delta_i}{2}) T_{\Delta_i} \\
&= P(\Delta > \frac{\Delta K}{2}) T_{\Delta_K} + \sum_{i=1}^{K-1} (P(\Delta > \frac{\Delta_i}{2}) - P(\Delta > \frac{\Delta_{i+1}}{2})) T_{\Delta_i} \\
&= \sum_{i=2}^K P(\Delta > \frac{\Delta_i}{2}) T(\Delta_i - \Delta_{i-1}) \\
&\leq \sum_{i=2}^K \min(1, \frac{320K}{T \Delta_i^2}) T(\Delta_i - \Delta_{i-1}).
\end{align*}

(76)
Here the final inequality again follows from Equation 60.

**An instance independent bound on** $E(\sum_{i: \Delta > \frac{\Delta_i}{BD}} \mathbb{1}_{i^* = i} T\Delta_i)$. We have,

$$E\left(\sum_{i: \Delta > \frac{\Delta_i}{BD}} \mathbb{1}_{i^* = i} T\Delta_i\right) \leq E(2T\Delta \sum_{i=1}^{K} \mathbb{1}_{i^* = i}) = E(2T\Delta) = 2TE(\Delta).$$

(77)

We then look at $E(\Delta)$. We have,

$$E(\Delta) = \int_0^\infty P(\Delta > x) dx \leq \int_0^\infty \min\left(1, \frac{40K}{Tx^3}\right) dx.$$

This integral evaluates to

$$\int_0^{\left(\frac{40K}{T}\right)^{1/3}} dx + \int_{\left(\frac{40K}{T}\right)^{1/3}}^\infty \frac{40K}{Tx^3} dx \leq \frac{2\left(\frac{40K}{T}\right)^{1/3}}{T^{1/3}}.$$

Combining these results, we have

$$E(\Delta) \leq 2\frac{\left(\frac{40K}{T}\right)^{1/3}}{T^{1/3}}.$$  

(78)

Thus we finally have,

$$E\left(\sum_{i: \Delta > \frac{\Delta_i}{BD}} \mathbb{1}_{i^* = i} T\Delta_i\right) \leq 4\left(\frac{40K}{T}\right)^{2/3} T^{2/3}. $$

(79)

The final instance-dependent bound follows from Equations 70, 74, and 76. The instance-independent bound follows from the fact that the regret from the Explore phase is at most $K\tau = O(T^{2/3}K^{1/3})$ and from Equations 75 and 79.  

□
Appendix E: Proof of Proposition 2

Let $I_t$ denote the set of arms pulled in period $t$ (note that $|I_t| = m$ for all $t \in [T/m]$). Then, for any policy $\pi$, we have that

$$R_T(\pi, \nu) = E \left( \Gamma^m \left( \overline{U}_{T/m}^1, \overline{U}_{T/m}^2, \ldots, \overline{U}_{T/m}^K \right) \right)$$

$$= E \left( \Gamma^m \left( \sum_{t=1}^{T/m} U_{n_{t-1}+1}^i \mathbb{1}_{ \{1 \in I_t\} }, \sum_{t=1}^{T/m} U_{n_{t-1}+1}^2 \mathbb{1}_{ \{2 \in I_t\} }, \ldots, \sum_{t=1}^{T/m} U_{n_{K-1}+1}^K \mathbb{1}_{ \{K \in I_t\} } \right) \right)$$

$$(a) \leq E \left( \sum_{t=1}^{T/m} \Gamma^m \left( U_{n_{t-1}+1}^1 \mathbb{1}_{ \{1 \in I_t\} }, U_{n_{t-1}+1}^2 \mathbb{1}_{ \{2 \in I_t\} }, \ldots, U_{n_{K-1}+1}^K \mathbb{1}_{ \{K \in I_t\} } \right) \right)$$

$$(b) = \sum_{t=1}^{T/m} E \left( \Gamma^m \left( U_{n_{t-1}+1}^1 \mathbb{1}_{ \{1 \in I_t\} }, U_{n_{t-1}+1}^2 \mathbb{1}_{ \{2 \in I_t\} }, \ldots, U_{n_{K-1}+1}^K \mathbb{1}_{ \{K \in I_t\} } \right) \right)$$

$$(c) = \sum_{t=1}^{T/m} \frac{1}{m} \mathbb{E} \left( \sum_{i \in I_t} U_{n_{t-1}+1}^i \mid \mathcal{H}_t \right)$$

$$(d) \leq \sum_{t=1}^{T/m} \frac{1}{m} \mathbb{E} \left( \sum_{i \in I_t} \mu_i \mid \mathcal{H}_t \right) \leq \sum_{t=1}^{T/m} \frac{1}{m} \sum_{i=1}^{m} \mu_i = \frac{\mu_m T}{m}. \quad (80)$$

Here (a) is obtained due to pushing the function $\Gamma^m$ inside the sum; (b) is obtained because $U_{n_{t-1}+1}^i \mathbb{1}_{ \{i \in I_t\} } \geq 0$ for all $i$ and exactly $m$ arms are pulled in each period; (c) is obtained because, conditioned on the history and for a given policy, the set of arms that will be pulled in a period is fixed; and (d) holds because the reward for an arm in a period is independent of the past history of play and observations. Thus, the reward of $\frac{\mu_m T}{m}$ is the highest that one can obtain under any policy. And this reward can, in fact, be obtained by the policy of always picking the top $m$ arms. This shows that

$$\sup_{\pi \in \Pi} R_T(\pi, \nu) = R^*_T(\nu).$$
Appendix F: Proof of Theorem 4

First we fix a policy \( \pi \in \Pi \). Let \( \Delta \overset{\Delta}{=} (K - m)^{1/3}/(2m^{1/3}T^{1/3}) \). We construct two bandit environments with different reward distributions for each of the arms and show that \( \pi \) cannot perform well in both environments simultaneously.

We first specify the reward distribution for the arms in the base environment, denoted as the bandit \( \nu = (\nu_1, \ldots, \nu_K) \). Assume that the reward for all of the arms have the Bernoulli distribution, i.e., \( \nu_i \sim \text{Bernoulli}(\mu_i) \). We let \( \mu_1 = \mu_2 = \cdots = \mu_m = \frac{1}{2} + \Delta \), and \( \mu_i = \frac{1}{2} \) for \( m + 1 \leq i \leq K \). We let \( P_{\nu} \) denote the probability distribution induced over events until time \( \frac{T}{m} \) under policy \( \pi \) in this first environment, i.e., in bandit \( \nu \). Let \( E_{\nu} \) denote the expectation under \( P_{\nu} \).

Let \( \tau = \lceil \frac{2m^2}{\Delta} \rceil \) and define \( n_i^\tau \) as the (random) number of pulls spent on arm \( i \in \{1, \ldots, K\} \) until period \( \tau \) (note that \( \sum_{i=1}^{K} n_i^\tau = m\tau \) until period \( \tau \) under policy \( \pi \)). Also, under policy \( \pi \), let \( l^\tau \) denote the set of \( m \) arms in the set \([K] \setminus \{1, \ldots, m\}\) that is pulled the least in expectation until period \( \lceil \frac{2m^2}{\Delta} \rceil \). Then, we must have that \( E_{\nu}(\sum_{i \in l^\tau} n_i^\tau) \leq \frac{\Delta^2 m^2}{K - m} \approx \frac{\Delta^2 m^2}{m} \).

Having defined \( l^\tau \), we can now define the second environment, denoted as the bandit \( \nu' = (\nu_1', \ldots, \nu_K') \). Without loss of generality, for ease of notation, we can let \( l^\tau \) to be the last \( m \) arms, i.e., \( l^\tau = \{k_m, \ldots, K\} \), where \( k_m = K - m + 1 \). Again, assume that the reward for all of the arms have the Bernoulli distribution, i.e., \( \nu_i' \sim \text{Bernoulli}(\mu_i') \). We let \( \mu_1' = \mu_2' = \cdots = \mu_m' = \frac{1}{2} + \Delta \), \( \mu_i' = \frac{1}{2} \) for \( m + 1 \leq i \leq k_m - 1 \), and \( \mu_{k_m}' = \mu_{k_m+1}' = \cdots = \mu_K' = \frac{1}{2} + 2\Delta \). We let \( P_{\nu'} \) denote the probability distribution induced over events until time \( \frac{T}{m} \) under policy \( \pi \) in this second environment, i.e., in bandit \( \nu' \). Let \( E_{\nu'} \) denote the expectation under \( P_{\nu'} \).

With some abuse of notation, for any event \( B \), we define:

\[
\text{Reg}_T(\pi, \nu, B) = \mu_{\text{m}} \frac{T}{m} P_{\nu}(B) - E_{\nu}(\Gamma_m(\nu_{T/m}^1, \nu_{T/m}^2, \ldots, \nu_{T/m}^K) \| B).
\] (81)

It is then clear that \( \text{Reg}_T(\pi, \nu) = \text{Reg}_T(\pi, \nu, B) + \text{Reg}_T(\pi, \nu, B^c) \).

We define event \( A = \{ \sum_{i=1}^{\tau} 1_{(i \in \{1, \ldots, m\}) > (\frac{T}{m})} < \frac{\tau}{2} \} \). In words, \( A \) is the event where, until time \( \tau \), there are at least \( \tau/2 \) periods in which at most half of the first \( m \) arms are pulled. Then, building on the event \( A \), we need the following two results for our proof.

**Lemma 7.** Fix a policy \( \pi \). Consider the \( K \)-armed bandit instance \( \nu \) with Bernoulli rewards and mean vector \( \mu = (\mu_1, \mu_2, \ldots, \mu_K) \) with \( \mu_1 = \mu_2 = \cdots = \mu_m = \frac{1}{2} + \Delta \) and \( \mu_{m+1} = \cdots = \mu_K = \frac{1}{2} \), where \( \Delta < \frac{1}{4} \). Consider the event \( A = \{ \sum_{i=1}^{\tau} 1_{(i \in \{1, \ldots, m\}) > (\frac{T}{m})} < \frac{\tau}{2} \} \). Then we have,

\[
\text{Reg}_T(\pi, \nu, A) \geq \frac{\Delta T}{4m} P_{\nu}(A) - 2 \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2.
\]

The proof of Lemma 7 is presented below in this section. A similar argument shows the following.

**Lemma 8.** Fix a policy \( \pi \). Consider the \( K \)-armed bandit instance \( \nu' \) with Bernoulli rewards and mean vector \( \mu' = (\mu_1', \mu_2', \ldots, \mu_K') \) with \( \mu_1' = \mu_2' = \cdots = \mu_m' = \frac{1}{2} + \Delta \), \( \mu_{m+1}' = \cdots = \mu_{k_m}' = \frac{1}{2} + 2\Delta \), and \( \mu_{k_m+1}' = \cdots = \mu_K' = \frac{1}{2} \), where \( \Delta < \frac{1}{4} \). Consider the event \( A' = \{ \sum_{i=1}^{\tau} 1_{(i \in \{1, \ldots, m\}) > (\frac{T}{m})} \geq \frac{\tau}{2} \} \). Then we have,

\[
\text{Reg}_T(\pi, \nu', A') \geq \frac{\Delta T}{4m} P_{\nu'}(A') - 2 \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2.
\]
The proof of Lemma 8 is omitted since it is almost identical to that of Lemma 7.

These two facts result in the following two inequalities:

\[
\text{Reg}_T(\pi, \nu) \geq \mathbb{P}_\nu \left( \sum_{t=1}^T \mathbb{1}_{\{t \cap (1, \ldots, m) > \lceil T/m \rceil \}} < \frac{T}{2} \right) \Omega \left( \frac{\Delta T}{m} \right), \quad \text{and}
\]

\[
\text{Reg}_T(\pi, \nu') \geq \mathbb{P}_{\nu'} \left( \sum_{t=1}^T \mathbb{1}_{\{t \cap (1, \ldots, m) > \lceil T/m \rceil \}} \geq \frac{T}{2} \right) \Omega \left( \frac{\Delta T}{m} \right).
\]

As above, we let \( A = \{ \sum_{t=1}^T \mathbb{1}_{\{t \cap (1, \ldots, m) > \lceil T/m \rceil \}} < \frac{T}{2} \} \). Now, using the Bretagnolle-Huber inequality (see Theorem 14.2 in \cite{lattimore2020bandit}), we have,

\[
\text{Reg}_T(\pi, \nu, A) + \text{Reg}_T(\pi, \nu', A^c) \geq \Omega \left( \frac{\Delta T}{m} \right) (\mathbb{P}_\nu (A) + \mathbb{P}_{\nu'} (A^c))
\]

\[
= \Omega \left( \frac{\Delta T}{m} \right) (\mathbb{P}_\nu (A) + \mathbb{P}_{\nu'} (A^c))
\]

\[
\geq \Omega \left( \frac{\Delta T}{m} \right) \exp (-D (\mathbb{P}_\nu, \mathbb{P}_{\nu'})).
\]

Here, \( \mathbb{P}_\nu (\mathbb{P}_{\nu'}) \) is the probability distribution induced by the policy \( \pi \) on events until time \( \lceil \frac{\Delta T}{m} \rceil \) under bandit \( \nu (\nu') \). The equality then results from the fact that the two events \( \{ \sum_{t=1}^T \mathbb{1}_{\{t \cap (1, \ldots, m) > \lceil T/m \rceil \}} < \frac{T}{2} \} \) and \( \{ \sum_{t=1}^T \mathbb{1}_{\{t \cap (1, \ldots, m) > \lceil T/m \rceil \}} \geq \frac{T}{2} \} \) depend only on the play until time \( \lceil \Delta T/m \rceil \). In the second inequality, which results from the Bretagnolle-Huber inequality, \( D (\mathbb{P}_\nu, \mathbb{P}_{\nu'}) \) is the relative entropy, or the Kullback-Leibler (KL) divergence between the distributions \( \mathbb{P}_\nu \) and \( \mathbb{P}_{\nu'} \) respectively. We can upper bound \( D (\mathbb{P}_\nu, \mathbb{P}_{\nu'}) \) as,

\[
D (\mathbb{P}_\nu, \mathbb{P}_{\nu'}) = \sum_{i=k_m}^K \mathbb{E}_\nu (n_i^*) D (\nu_i, \nu_i') = D (\nu_K, \nu_K') \sum_{i=k_m}^K \mathbb{E}_\nu (n_i^*)
\]

\[
\leq D (\nu_K, \nu_K') \frac{\Delta T}{K-m} \frac{m^2}{K-m} \approx \frac{8 \Delta^3 T m}{K-m},
\]

where \( \nu_i (\nu_i') \) denotes the reward distribution of arm \( i \) in the first (second) environment. The first equality results from the fact that only the last \( m \) arms differ between \( \nu \) and \( \nu' \). The second equality follows since the reward distribution of the last \( m \) arms are identical. The first inequality follows from the fact that \( \sum_{i=k_m}^K \mathbb{E}_\nu (n_i^*) \leq \frac{\Delta T m^2}{K-m} \). Now, \( D (\nu_K, \nu_K') \) is simply the relative entropy between the distributions Bernoulli(1/2) and Bernoulli(1/2+\( \Delta \)), which, by elementary calculations, can be shown to be at most \( 8 \Delta^2 \), resulting in the final inequality.

Thus, we finally have,

\[
\text{Reg}_T(\pi, \nu, A) + \text{Reg}_T(\pi, \nu', A^c) \geq \Omega \left( \frac{\Delta T}{m} \right) \exp \left( -\frac{8 \Delta^3 T m}{K-m} \right).
\]

Substituting \( \Delta = (K-m)^{1/3}/(2m^{1/3}T^{1/3}) \) gives

\[
\text{Reg}_T(\pi, \nu) + \text{Reg}_T(\pi, \nu') \geq \Omega \left( \frac{(K-m)^{1/3}T^{2/3}}{m^{4/3}} \right).
\]

Equation 86 along with

\[
\text{Reg}_T(\pi, \nu, A) \geq -\tilde{O} \left( \sqrt{T/m} \right) \quad \text{and}
\]

\[
\text{Reg}_T(\pi, \nu', A) \geq -\tilde{O} \left( \sqrt{T/m} \right),
\]

\[
\text{Reg}_T(\pi, \nu, A^c) \geq -\tilde{O} \left( \sqrt{T/m} \right)
\]

\[
\text{Reg}_T(\pi, \nu', A^c) \geq -\tilde{O} \left( \sqrt{T/m} \right)
\]

\[
\text{Reg}_T(\pi, \nu, A) \geq -\tilde{O} \left( \sqrt{T/m} \right)
\]

\[
\text{Reg}_T(\pi, \nu', A) \geq -\tilde{O} \left( \sqrt{T/m} \right)
\]

\[
\text{Reg}_T(\pi, \nu, A^c) \geq -\tilde{O} \left( \sqrt{T/m} \right)
\]

\[
\text{Reg}_T(\pi, \nu', A^c) \geq -\tilde{O} \left( \sqrt{T/m} \right)
\]
imply that
\[ \text{Reg}_T(\pi, \nu) + \text{Reg}_T(\pi', \nu') \geq \Omega \left( \frac{(K-m)^{1/3}T^{2/3}}{m^{2/3}} \right). \]  
(90)

Finally, using \(2\max\{a, b\} \geq a + b\) gives the desired lower bound on the regret.

Showing Equations (88) and (89) is an easy exercise:

\[ \text{Reg}_T(\pi, \nu, A^c) = \frac{T}{m}P_{\nu}(A^c) - E_{\nu}(\Gamma^m(\bar{U}_{T/m}^1, \ldots, \bar{U}_{T/m}^K) \mathbb{1}_{A^c}) \]
\[ \geq \frac{T}{m}P_{\nu}(A^c) - E_{\nu}(\Gamma^m(\sum_{t=1}^{T/m} U_t^1, \ldots, \sum_{t=1}^{T/m} U_t^K) \mathbb{1}_{A^c}) \]
\[ \overset{(a)}{=} \frac{T}{m}P_{\nu}(A^c) - \frac{T}{m}P_{\nu}(A') - 2\sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2 \]
\[ = -2\sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2. \]  
(91)

Here (a) follows from an argument essentially identical to the one in the proof of Lemma 7 below and we do not repeat it here for brevity. Similarly, we can show that

\[ \text{Reg}_T(\pi, \nu', A) \geq -2\sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2. \]  
(92)

**Proof of Lemma 7**  
We first have that

\[ E_{\nu}(\Gamma^m(\bar{U}_{T/m}^1, \ldots, \bar{U}_{T/m}^K) \mathbb{1}_{A}) = E_{\nu}(\Gamma^m(\sum_{t=1}^{n_{T/m}^1} U_t^1, \ldots, \sum_{t=1}^{n_{T/m}^K} U_t^K) \mathbb{1}_{A}) \]
\[ \leq E_{\nu}(\Gamma^m(\sum_{t=1}^{T_1} U_t^1, \ldots, \sum_{t=1}^{T_{\tau/m}} U_t^m, \sum_{t=1}^{T_{\tau/m}+1} U_t^{m+1}, \ldots, \sum_{t=1}^{T_{\tau/m}K} U_t^K) \mathbb{1}_{A}), \]  
(93)

where \(T_i = \frac{T}{m} - t_i\). The event \(A\) states that, until time \(\tau\), there are at least \(\tau/2\) periods in which at most half of the first \(m\) arms are pulled, so we have that \(\sum_{i=1}^{m} t_i \geq \frac{\tau m}{2}\) and \(t_i \leq \tau\) for any \(i \in [m]\).

Consider the event

\[ G = \left\{ \sum_{i=1}^{T_j} U_t^i - \mu_j T_j \leq \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} \text{ for all } j \right\}, \]

for \(T_j \leq T/m\) for all \(j \in [K]\). Since \(U_t^i \in [0, 1]\), by Hoeffding’s inequality, we have that for any \(T' \leq T/m\),

\[ P_{\nu}\left( \left| \sum_{t=1}^{T'} U_t^i - \mu_j T' \right| \leq \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} \right) \geq 1 - 2 \exp\left( -\frac{2 \left( \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} \right)^2}{T'} \right) \]
\[ \geq 1 - 2 \exp\left( -\frac{2 \left( \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} \right)^2}{T/m} \right) \]
\[ = 1 - \frac{2m^2}{K^2 T^2} \geq 1 - \frac{2m}{K}. \]

Hence, by the union bound we have that \(P(G) \geq 1 - \frac{2m}{K}\).

Recall that, for \(i \in [m]\), we have \(T_i = \frac{T}{m} - t_i\) with \(\sum_{i=1}^{m} t_i \geq \frac{\tau m}{2}\) and \(t_i \leq \tau\) for any \(i \in [m]\). Thus we finally have,
\[
\text{Reg}_T(\pi, \nu, A) = \left( \frac{1}{2} + \Delta \right) \frac{T}{m} \nu(A) - \nu\left( \Gamma^m \left( \sum_{t=1}^{T_1} U_1^t, \ldots, \sum_{t=1}^{T_m} U_m^t, \sum_{t=1}^{T/m} U_{m+1}^t, \ldots, \sum_{t=1}^{T/m} U_K^t \right) \mathbb{I}_A \right)
\]
\[
\geq \left( \frac{1}{2} + \Delta \right) \frac{T}{m} \nu(A) - \left( \frac{1}{2} + \Delta \right) \left( \frac{T}{m} - \left\lceil \frac{T}{2} \right\rceil \right) \nu(A) - 2 \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2
\]
\[
= \left( \frac{1}{2} + \Delta \right) \left[ \frac{\Delta T}{2m} \nu(A) - 2 \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2 \right]
\]
\[
\geq \frac{\Delta T}{4m} \nu(A) - 2 \sqrt{\frac{T}{m} \log \left( \frac{KT}{m} \right)} - 2.
\]
Appendix G: Proof of Theorem 5

Let $\Gamma$ denote the arms used in the Commit phase of $m$-ADA-ETC. We first define $m$ random variables, each quantifying the lowest value of the index of arm $i \in [m]$ can take with respect to its true mean across $\tau$ pulls. Recall that the empirical average reward of arm $i$ remains fixed after $\tau$ pulls.

$$\delta_i \triangleq \left( \mu_i - \min_{n \leq \tau} \left( \bar{\mu}_i + \sqrt{\frac{4}{n} \log \left( \frac{T}{(K-m)n^{3/2}} \right)} \mathbb{1}_{\{n < \tau\}} \right) \right)^+.$$

We also define

$$\delta \triangleq \max_{i \leq m} \delta_i.$$

The following bound, which follows from Equation (50) is instrumental for our analysis. For any $x \geq 0$ and $i \in [m]$,

$$P(\delta_i > x) = P \left( \exists n \leq \tau : \bar{\mu}_i + \sqrt{\frac{4}{n} \log \left( \frac{T}{(K-m)n^{3/2}} \right)} \mathbb{1}_{\{n < \tau\}} < \mu_i - x \right) \leq P \left( \exists n < \tau : \bar{\mu}_i + \sqrt{\frac{4}{n} \log \left( \frac{T}{(K-m)n^{3/2}} \right)} < \mu_i - x \right) + P \left( \bar{\mu}_i < \mu_i - x \right)$$

$$\leq \min(1, \frac{39(K-m)}{Tx^3} + \exp(-2\tau x^2)) \tag{97}$$

$$\leq \min(1, \frac{40(K-m)}{Tx^3}). \tag{98}$$

Again, (a) follows from Lemma 6 and Hoeffding’s inequality, and (b) follows by the definition of $\tau$ and since $\exp(-2\alpha^{2/3}) \leq 1/\alpha$ for all $\alpha \geq 0$. Notice that the expression in (98) does not depend on arm $i$.

We then decompose the regret into the regret from wasted pulls in the Explore phase and the regret from committing to one or more suboptimal arms in the Commit phase. In contrast to the $m = 1$ case, not all arms enter the exploitation phase at the same time. If there is a time $t$ for arm $i$ where and $n_{i+1}^t = \tau + 1$, then arm $i$ belongs to the set of exploited arms from time $(t+1)$ onwards, i.e., $i \in \Gamma$ (Lemma 9).

To that end, we define arm specific stopping times. For $i \in \Gamma$, let $\omega_i$ be the time period prior to arm $i$ being pulled $(\tau + 1)$ - st time, i.e., $\omega_i = \min \left\{ t \leq \frac{T}{m} : n_{i+1}^t = \tau + 1 \right\}$. Note that if $i \notin \Gamma$ we set $\omega_i = T/m$.

For $i \in \Gamma$, let $r_{\omega_i}^i$ be the reward earned from arm $i$ during its exploration. Define the number of missed pulls from arm $i$ during its exploration as $n_{i,\text{miss}}^{\omega_i} \triangleq \omega_i - \tau$, $i \in \Gamma$. Then, the expected regret in the event that $\{ \Gamma = I \}$ is bounded by:

$$E \left( \frac{T}{m} \bar{\mu}_m - \frac{1}{m} \sum_{i \in I} \left( \frac{T}{m} - n_{i,\text{miss}}^{\omega_i} \right) \mu_i - \frac{1}{m} \sum_{i \in I} r_{\omega_i}^i \right) \mathbb{1}_{\{\Gamma = I\}} \tag{99}.$$

Note that this expression assumes that the average of the cumulative rewards of the arms in set $I$ will be chosen to compete against $\frac{T}{m} \bar{\mu}_m$ at the end of time $T/m$; however, if there are arms with higher cumulative rewards than the arms in $I$, then the resulting regret can only be lower. Thus the total expected regret is bounded by:

$$\sum_{I \in \binom{[m]}{\alpha}} E \left( \frac{T}{m} \bar{\mu}_m - \frac{1}{m} \sum_{i \in I} \left( \frac{T}{m} - n_{i,\text{miss}}^{\omega_i} \right) \mu_i - \frac{1}{m} \sum_{i \in I} r_{\omega_i}^i \right) \mathbb{1}_{\{\Gamma = I\}}.$$
\[
\begin{aligned}
&\quad \sum_{i \in \{K\}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i \in I^*)}\right) + \frac{1}{m} \sum_{i=1}^{K} E(n_{\omega_i}^{miss} \mu_i \mathbb{1}_{(i \in I^*)}) + \frac{1}{m} \sum_{i=1}^{K} E(n_{\omega_i}^{miss} \mu_i \mathbb{1}_{(i \in I^*)} - r_{\omega_i}^{i} \mathbb{1}_{(i \in I^*)}) \\
\quad \leq &\quad \sum_{i \in \{K\}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i \in I^*)}\right) + \frac{1}{m} \sum_{i=1}^{K} E(n_{\omega_i}^{i, miss} \mu_i \mathbb{1}_{(i \in I^*)}) + \frac{1}{m} \sum_{i=1}^{K} E\left((\tau \mu_i - r_{\omega_i}^{i}) \mathbb{1}_{(i \in I^*)}\right) \\
= &\quad \sum_{i \in \{K\}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i \in I^*)}\right) + \frac{1}{m} \sum_{i=1}^{K} E(n_{\omega_i}^{i, miss} \mu_i \mathbb{1}_{(i \in I^*)}) + \frac{1}{m} \sum_{i=1}^{K} P(i \in I^*) (\tau \mu_i - \sum_{n=1}^{r} E(U_n^{i} | i \in I^*)) \\
\leq &\quad \sum_{i \in \{K\}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i \in I^*)}\right) + \frac{1}{m} \sum_{i=1}^{K} E(n_{\omega_i}^{i, miss} \mu_i \mathbb{1}_{(i \in I^*)}). \tag{100}
\end{aligned}
\]

Here, (a) results from misidentifications in Commit phase. (b) follows from the fact that in the event of \{i \in I^*\}, \(n_{\omega_i}^{i, miss} = \tau\) and \(r_{\omega_i}^{i} = r_{\omega_i}^{i}\) by the definition of \(\omega_i\). And (c) holds since, by a standard stochastic dominance argument, \(\tau \mu_i \leq \sum_{n=1}^{r} E(U_n^{i} | i \in I^*)\). Here, we let \(\Delta_i\) denote \(\mu_m - \frac{1}{m} \sum_{i \in I^*} \mu_i\).

We bound these two terms one by one.

**Regret from Explore.** First, note that an instance-independent bound on the regret from Explore is simply \(\frac{K-m}{m} \tau - \tau = \frac{K-m}{m} \left[\frac{2^{3/2}}{(K-m)^{3/2}}\right] = O(\frac{(K-m)^{1/3} \log^{2/3}}{m})\), which is the maximum number of allotted pulls on arms not in set \(I^*\) before \(m\)-ADA-ETC enters the Commit phase. Hence, we now focus on deriving an instance-dependent bound. We have that

\[
\begin{aligned}
&\quad \frac{1}{m} \sum_{i=1}^{K} E\left(n_{\omega_i}^{i, miss} \mu_i \mathbb{1}_{(i \in I^*)}\right) \leq \frac{1}{m} \sum_{i=1}^{m} \mu_i E(n_{\omega_i}^{i, miss}) + \frac{1}{m} \sum_{j=m+1}^{K} E(n_{\omega_i}^{j, miss} \mu_j \mathbb{1}_{(j \in I^*)}) \\
&\quad \leq \frac{1}{m} \sum_{i=1}^{m} \mu_i E(n_{\omega_i}^{i, miss}) + \frac{K-m}{m} \frac{\tau}{m} \sum_{j=m+1}^{K} \mu_j P(j \in I^*). \tag{101}
\end{aligned}
\]

Here, (a) follows from the fact that the highest number of pulls missed from arm \(j \in I^*\) is \(\frac{K-m}{m} \tau - \tau\).

We first bound the first term in Equation 102. Recall that \(\Delta_j = \mu_m - \mu_j\) for \(j \geq m+1\) and \(\bar{\Delta}_i = \mu_i - \mu_{m+1}\) for \(i \in [m]\). Then,

\[
\begin{aligned}
&\quad \frac{1}{m} \sum_{i=1}^{m} \mu_i E(n_{\omega_i}^{i, miss}) = \frac{1}{m} E\left(\sum_{i \in [m]} \mu_i n_{\omega_i}^{i, miss}\right) + \frac{1}{m} E\left(\sum_{i \in [m]} \mu_i n_{\omega_i}^{i, miss}\right). \tag{102}
\end{aligned}
\]

We now bound the first term in Equation 103. Define the random variable

\[
\kappa_j = \sum_{n=1}^{r} \mathbb{1}_{\left\{\frac{T}{m} \log \left(\frac{T}{(K-m)n^{3/2}}\right) \mathbb{1}_{(n<\tau)} > \mu_j + \frac{\Delta_j}{2}\right\}}. \tag{104}
\]

Then, in the event that \(\delta_i \leq \bar{\Delta}_i/2\), we have that \(n_{\omega_i}^{i, miss} \leq \sum_{j=m+1}^{K} \kappa_j\). From the previous discussion, we also have that \(n_{\omega_i}^{i, miss} \leq \frac{K-m}{m} \tau\). Hence the first term in Equation 103 is bounded as:

\[
\begin{aligned}
&\quad \frac{1}{m} \sum_{i=1}^{m} \mu_i P\left(\delta_i \leq \bar{\Delta}_i/2\right) E(\min\left(\sum_{j=m+1}^{K} \kappa_j, \frac{K-m}{m} \tau\right)) \leq \frac{1}{m} \sum_{i=1}^{m} \mu_i \min\left(\sum_{j=m+1}^{K} E(\kappa_j), \frac{K-m}{m} \tau\right). \tag{105}
\end{aligned}
\]
We can now bound $E(\kappa_j)$ as follows:

$$E(\kappa_j) \leq 1 + E\left(\sum_{n=1}^{\tau-1} \mathbb{1}\left\{ \tilde{\mu}_{n, \tau} + \frac{4}{n} \log \left( \frac{T}{(K-m)n^{3/2}} \right) > \mu_j + \frac{\Delta_j}{2} \right\} \right)$$

$$= 1 + E\left(\sum_{n=1}^{\tau-1} \mathbb{1}\left\{ \tilde{\mu}_{n, \tau} + \frac{4}{n} \log^+ \left( \frac{T}{(K-m)n^{3/2}} \right) > \mu_j + \frac{\Delta_j}{2} \right\} \right)$$

$$\leq 1 + \frac{1}{\Delta_j^2} + E\left(\sum_{n=1}^{\tau-1} \mathbb{1}\left\{ \tilde{\mu}_{n, \tau} + \frac{4}{n} \log^+ \left( \frac{T\Delta_j^3}{K-m} \right) > \mu_j + \frac{\Delta_j}{2} \right\} \right)$$

$$\leq 2 + \frac{1}{\Delta_j^2} + \frac{8}{\Delta_j^2} \left( 2 \log^+ \left( \frac{T\Delta_j^3}{K-m} \right) + \sqrt{2\pi \log^+ \left( \frac{T\Delta_j^3}{K-m} \right) + 1} \right)$$

$$\leq \frac{11}{\Delta_j^2} + \frac{16}{\Delta_j^2} \log^+ \left( \frac{T\Delta_j^3}{K-m} \right) + \frac{24}{\Delta_j^3} \sqrt{\log^+ \left( \frac{T\Delta_j^3}{K-m} \right) \cdot \frac{K-m}{m} \tau}. \quad (106)$$

Here, (a) is due to lower bounding $1/n^{3/2}$ by $\Delta_j^3$, and adding $1/\Delta_j^2$ for the first $1/\Delta_j^2$ time periods where this lower bound doesn’t hold. (b) is due to Lemma 6 and reorganizing terms. The final inequality results from the fact that $\Delta_j \leq 1$ and from trivially bounding $2\pi \leq 9$. Thus, the first term in Equation 108 is bounded by

$$\frac{1}{m} \sum_{i=1}^{m} \mu_i \min\left( \sum_{j=m+1}^{K} \min\left( \frac{11}{\Delta_j^2} + \frac{16}{\Delta_j^2} \log^+ \left( \frac{T\Delta_j^3}{K-m} \right) + \frac{24}{\Delta_j^3} \sqrt{\log^+ \left( \frac{T\Delta_j^3}{K-m} \right) \cdot \frac{K-m}{m} \tau}, \frac{K-m}{m} \tau \right) \right). \quad (107)$$

Note that we have $\kappa_j \leq \tau$ for $j \geq m+1$ from Equation 104.

Finally, we bound the second term in Equation 103. Note that we have $n_{i, \text{miss}} \leq \frac{K-m}{m} \tau$, and hence,

$$\frac{1}{m} E\left( \sum_{i \in [m]: \delta_i > \Delta_j/2} \mu_i n_{i, \text{miss}} \right) \leq \frac{K-m}{m} \tau \sum_{i=1}^{m} \mu_i P\left( \delta_i > \frac{\Delta_j}{2} \right) \leq \frac{K-m}{m} \tau \sum_{i=1}^{m} \mu_i \min(1, \frac{320(K-m)}{T\Delta_j^3}). \quad (108)$$

Here, the last inequality follows from Equation 98.

Thus, combining Equations 107 and 108, we have that the first term in Equation 102 is bounded by

$$\frac{1}{m} \sum_{i=1}^{m} \mu_i \min\left( \sum_{j=m+1}^{K} \min\left( \frac{11}{\Delta_j^2} + \frac{16}{\Delta_j^2} \log^+ \left( \frac{T\Delta_j^3}{K-m} \right) + \frac{24}{\Delta_j^3} \sqrt{\log^+ \left( \frac{T\Delta_j^3}{K-m} \right) \cdot \frac{K-m}{m} \tau}, \frac{K-m}{m} \tau \right) \right) + \frac{K-m}{m} \tau \sum_{i=1}^{m} \mu_i \min(1, \frac{320(K-m)}{T\Delta_j^3}). \quad (109)$$

Next, we focus on the second term in Equation 102. For $j \in \{m+1, \ldots, K\}$,

$$P(j \in \mathcal{I}^*) = P\left( j \in \mathcal{I}^* \text{ and } \frac{\Delta_j}{2} < \bar{\delta} \right) + P\left( j \in \mathcal{I}^* \text{ and } \frac{\Delta_j}{2} \geq \bar{\delta} \right) \leq \min \left( 1, P\left( \frac{\Delta_j}{2} < \bar{\delta} \right) + P\left( j \in \mathcal{I}^* \text{ and } \frac{\Delta_j}{2} \geq \bar{\delta} \right) \right).$$

Recall that $\bar{\delta} = \max_{i \leq m} \delta_i$. Then,

$$P\left( \frac{\Delta_j}{2} < \bar{\delta} \right) \leq P\left( \bigcup_{i=1}^{m} \left\{ \delta_i > \frac{\Delta_j}{2} \right\} \right) \leq \min(1, \sum_{i=1}^{m} P\left( \delta_i > \frac{\Delta_j}{2} \right)) \leq \min(1, \frac{320m(K-m)}{T\Delta_j^3}). \quad (110)$$
Here, the last inequality follows from Equation 93.

The event $\Delta_j/2 \geq \delta_i$ together with the event $j \in \Gamma^*$ for some $j \in \{m+1, \ldots, K\}$ imply that there exists an arm $i \in [m]$ such that $i \not\in \Gamma^*$ and $\Delta_j/2 \geq \delta_i$. Therefore, these two events imply that there is some $n \leq \tau$ such that $\text{LCB}_n^j = \bar{\mu}_n^j - \bar{\mu}_n^i 1_{(n<\tau)} > \mu_j - \frac{\Delta_j}{2} \geq \mu_m - \frac{\Delta_j}{2} = \mu_j + \frac{\Delta_j}{2}$. Hence,

$$P \left( j \in \Gamma^* \text{ and } \frac{\Delta_j}{2} \geq \delta_i \right) \leq P \left( \exists n \leq \tau : \bar{\mu}_n^j - \bar{\mu}_n^i 1_{(n<\tau)} > \mu_j + \Delta_j/2 \right)$$

$$= P \left( \bar{\mu}_n^j > \mu_j + \Delta_j/2 \right)$$

$$\leq \exp(-\frac{\tau \Delta_j^2}{2}) \leq \frac{8(K - m)}{T \Delta_j^3}. \quad (111)$$

Here, (a) follows from Hoeffding’s inequality, and (b) follows from the definition of $\tau$ and the fact that $\exp(-\alpha^2/2) \leq 8/\alpha$ for $\alpha \geq 0$.

Thus, combining Equations 110 and 111 we finally have

$$P(j \in \Gamma^*) \leq \min(1, \frac{320m(K - m)}{T \Delta_j^3} + \frac{8(K - m)}{T \Delta_j^3})$$

$$\leq \min(1, \frac{328m(K - m)}{T \Delta_j^3}), \quad (112)$$

$j \in \{m+1, \ldots, K\}$. Additionally, since we have $\sum_{i=1}^{K} P(\Gamma^* = i) = 1$ and $|\Gamma| = m$, we can bound $\sum_{j=m+1}^{K} P(j \in \Gamma^*)$ by $m$. This is because the latter expression is counting each subset of $m$ arms at most $m$ times. Then,

$$\sum_{j=m+1}^{K} \mu_j P(j \in \Gamma^*) \leq \min \left( \sum_{j=m+1}^{K} \mu_j \min(1, \frac{328m(K - m)}{T \Delta_j^3}), \ m \mu_m \right). \quad (113)$$

Bringing everything together, the regret from the Explore phase is bounded by

$$\frac{1}{m} \sum_{i=1}^{m} \mu_i \min \left( \sum_{j=m+1}^{K} \min \left( \frac{11}{\Delta_j^3} + \frac{16}{\Delta_j^3} \log^+ \left( \frac{T \Delta_j^3}{K - m} \right) + \frac{24}{\Delta_j^3} \sqrt{\log^+ \left( \frac{T \Delta_j^3}{K - m} \right)}, \ K - m \right), \ K - m \right)$$

$$+ \frac{K - m}{m} \frac{\tau}{m} \sum_{i=1}^{m} \mu_i \min(1, \frac{320(K - m)}{T \Delta_j^3})$$

$$+ \frac{K - m}{m} \frac{\tau}{m} \min \left( \sum_{j=m+1}^{K} \mu_j \min(1, \frac{328m(K - m)}{T \Delta_j^3}), \ m \mu_m \right). \quad (114)$$

This finishes our derivation of a distribution dependent bound on the regret from the Explore phase.

The following result will be useful in the coming parts. We aim to bound the probability of misidentifying an optimal arm, under the event $\frac{\Delta_j}{2} \geq \delta_i$. Under event $\frac{\Delta_j}{2} \geq \delta_i$, $i \not\in \Gamma^*$ but $j \in \Gamma^*$ implies that there is some $n \leq \tau$ such that $\text{LCB}_n^i = \bar{\mu}_n^i - \bar{\mu}_n^i 1_{(n<\tau)} > \mu_j + \frac{\Delta_j}{2}$. Hence,

$$P \left( i \not\in \Gamma^*, \ j \in \Gamma^*, \ \frac{\Delta_j}{2} \geq \delta_i \right) \leq P \left( \exists n \leq \tau : \bar{\mu}_n^i - \bar{\mu}_n^i 1_{(n<\tau)} > \mu_j + \frac{\Delta_j}{2} \right)$$

$$\leq P \left( \bar{\mu}_n^j > \mu_j + \frac{\Delta_j}{2} \right)$$

$$\leq \exp(-\frac{\tau \Delta_j^2}{2}). \quad (115)$$
Here, (a) follows from Hoeffding's inequality.

We next focus on the regret arising from misidentification in the Commit phase.

**Regret from Commit.** This regret is upper bounded by

$$\sum_{i \in (K/n)} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^* = 1)}\right).$$

We now get instance dependent and independent bounds on term above.

**An instance dependent bound on** $\sum_{i \in (K/n)} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^* = 1)}\right)$. We have

$$\sum_{i \in (K/n)} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^* = 1)}\right) = \frac{T}{m} \sum_{i \in (K/n)} E\left(\frac{1}{m} \sum_{i = 1}^m \left(\mu_i - \sum_{j = 1}^m \mu_j\right) \mathbb{1}_{(i^* = 1)}\right)$$

$$= \frac{T}{m} \sum_{i = 1}^m \mu_i - \sum_{j = 1}^K \sum_{i = 1}^m \mu_j \mathbb{1}_{(j \in i^*)}.$$ (117)

We focus on the term inside the parenthesis in Expression (117). To that end, we define $s^* \triangleq \min\left(\frac{1}{|\mathbb{I}^\star|}, 1\right)$, the reciprocal of the number of suboptimal arms in the exploitation set $\mathbb{I}^\star$, given that there are any. Otherwise, we set $s^* = 1$. If there is more than one misidentified arm, this definition of $s^*$ will ensure that we are not counting respective arms multiple times in the below expression.

$$E\left(\sum_{i = 1}^m \mu_i - \sum_{j = 1}^K \mu_j \mathbb{1}_{(j \in i^*)}\right) \overset{(a)}{=} E\left(\sum_{i = 1}^m \mu_i \left(s^* \sum_{j = 1}^{m} \mathbb{1}_{(j \in i^*)}\right)\right) - E\left(\sum_{j = 1}^K \mu_j \mathbb{1}_{(j \in i^*)} \left(s^* \sum_{i = 1}^m \mathbb{1}_{(i \in \mathbb{I}^\star)}\right)\right)$$

$$\overset{(b)}{=} E\left(s^* \sum_{i = 1}^m \sum_{j = m+1}^K \mu_i \mathbb{1}_{(j \in i^*)}\right) - E\left(s^* \sum_{j = 1}^K \sum_{i = 1}^m \mu_j \mathbb{1}_{(i \in \mathbb{I}^\star, j \in i^*)}\right) - E\left(s^* \sum_{j = 1}^K \sum_{i = 1}^m \mu_j \mathbb{1}_{(i \notin i^\star, j \in i^*)}\right)$$

$$\overset{(c)}{=} E\left(s^* \sum_{i = 1}^m \sum_{j = m+1}^K \mu_i \mathbb{1}_{(j \in i^*, j \notin i^\star)}\right) - E\left(s^* \sum_{i = 1}^m \sum_{j = m+1}^K \mu_j \mathbb{1}_{(i \notin i^\star, j \in i^*)}\right) - E\left(s^* \sum_{i = 1}^m \sum_{j = m+1}^K \mu_j \mathbb{1}_{(i \notin i^\star, j \notin i^*)}\right)$$

(118)

Here, (a) follows from the definition of $s^*$. Note that, if $\mathbb{I}^\star = [m]$, then $s^* = 1$ but $s^* \sum_{j = m+1}^K \mathbb{1}_{(j \in i^\star)} = s^* \sum_{j = 1}^m \mathbb{1}_{(j \in i^\star)} = 0$. Nevertheless, this does not imply that (a) is invalid. If $\mathbb{I}^\star = [m]$, then we have zero in the left-hand side of Equation (118) too. (b) follows from reorganizing terms. (c) is due to the following fact:

$$E\left(s^* \sum_{i = 1}^m \sum_{j = m+1}^K \mu_i \mathbb{1}_{(i \in \mathbb{I}^\star, j \notin i^\star)}\right) = E\left(s^* \sum_{i = 1}^m \sum_{j = m+1}^K \mu_i \mathbb{1}_{(i \in \mathbb{I}^\star, j \notin i^\star)}\right)$$

$$\overset{(a)}{=} E\left(s^* \sum_{j = 1}^K \mu_j \mathbb{1}_{(j \in i^\star)} \sum_{i = m+1}^m \mathbb{1}_{(i \in \mathbb{I}^\star)}\right)$$

$$\overset{(b)}{=} E\left(s^* \sum_{j = 1}^K \mu_j \mathbb{1}_{(j \in i^\star)} \sum_{i = 1}^m \mathbb{1}_{(i \in \mathbb{I}^\star)}\right)$$

(119)
We swap indices in (a). (b) follows from the fact that the number of the suboptimal arms in $\mathbf{1}^\ast$ must be same as the number of optimal arms missing from $\mathbf{1}^\ast$. Hence, we have that

$$
\sum_{i \in \binom{K}{m}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^\ast = 1)}\right) = \frac{T}{m^2} E\left(s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\mu_i - \mu_j) \mathbb{1}_{(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast)}\right).
$$

Since $s^*$ is at most 1,

$$
\sum_{i \in \binom{K}{m}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^\ast = 1)}\right) \leq \frac{T}{m^2} E\left(s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\mu_i - \mu_j) \mathbb{1}_{(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast)}\right) \leq \frac{T}{m} \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\mu_i - \mu_j) P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast).
$$

We can further break $P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast)$ down as follows:

$$
P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast) = P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast, \delta_i > \frac{\Delta_i}{2}, \delta_j > \frac{\Delta_j}{2})
+ P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast, \delta_i \leq \frac{\Delta_i}{2}, \delta_j > \frac{\Delta_j}{2})
+ P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast, \delta_i > \frac{\Delta_i}{2}, \delta_j \leq \frac{\Delta_j}{2})
+ P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast, \delta_i \leq \frac{\Delta_i}{2}, \delta_j \leq \frac{\Delta_j}{2}).
$$

Then, using Equations (128), (111) and (115) we have

$$
P(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast) \leq \min \left(\min \left(1, \frac{320(K-m)}{T \Delta_i^3}, \frac{320(K-m)}{T \Delta_j^3}\right) + \exp\left(-\frac{\bar{\Delta}_i^2}{2}\right) + \exp\left(-\frac{\bar{\Delta}_j^2}{2}\right) + \min \left(\exp\left(-\frac{\bar{\Delta}_i^2}{2}\right), \exp\left(-\frac{\bar{\Delta}_j^2}{2}\right)\right), 1\right) \leq \min \left(\min \left(\frac{320(K-m)}{T \Delta_i^3}, \frac{320(K-m)}{T \Delta_j^3}\right) + \frac{3}{2} \exp\left(-\frac{\bar{\Delta}_i^2}{2}\right) + \frac{3}{2} \exp\left(-\frac{\bar{\Delta}_j^2}{2}\right), 1\right).
$$

Then, the instance dependent bound on the regret from Commit is

$$
\sum_{i \in \binom{K}{m}} \sum_{j=m+1}^{K} (\mu_i - \mu_j) \min \left(\min \left(\frac{320(K-m)}{T \Delta_i^3}, \frac{320(K-m)}{T \Delta_j^3}\right) + \frac{3}{2} \exp\left(-\frac{\bar{\Delta}_i^2}{2}\right) + \frac{3}{2} \exp\left(-\frac{\bar{\Delta}_j^2}{2}\right), 1\right).
$$

An instance independent bound on $\sum_{i \in \binom{K}{m}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^\ast = 1)}\right)$. Consider Equation (122) again:

$$
\sum_{i \in \binom{K}{m}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^\ast = 1)}\right) = \frac{T}{m^2} E\left(s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\mu_i - \mu_j) \mathbb{1}_{(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast)}\right).
$$

Then,

$$
\sum_{i \in \binom{K}{m}} E\left(\frac{T}{m} \Delta_i \mathbb{1}_{(i^\ast = 1)}\right) \leq \frac{T}{m^2} E\left(s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\mu_i - \mu_j) \mathbb{1}_{(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast)}\right) \leq \frac{T}{m^2} E\left(s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\bar{\Delta}_i + \Delta_j) \mathbb{1}_{(i \notin \mathbf{1}^\ast, j \in \mathbf{1}^\ast)}\right).
$$
This integral evaluates to

\[
\frac{T}{m^2} E \left( s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\bar{\Delta}_i + \Delta_j) 1_{(i \neq j, j \in \mathcal{V}^* \cup \mathcal{V}^*)} \right) \leq \frac{T}{m^2} E \left( 4 \sum_{i=1}^{m} \delta_i \sum_{j=m+1}^{K} s^* 1_{(i \neq j, j \in \mathcal{V}^*)} \right) \leq 4 \frac{T}{m^2} \sum_{i=1}^{m} E(\delta_i). \tag{134}
\]

Here, the second inequality follows from the definition of \(s^*\) since \(\sum_{j=m+1}^{K} s^* 1_{(i \neq j, j \in \mathcal{V}^*)} \leq 1\) for each \(i \in [m]\). We then look at \(E(\delta_i)\) for \(i \in [m]\). We have,

\[
E(\delta_i) = \int_0^\infty P(\delta_i > x) \, dx \leq \int_0^\infty \min \left( 1, \frac{40(K-m)}{Tx^3} \right) \, dx.
\]

This integral evaluates to

\[
\int_0^{\frac{40(K-m)}{2T^{3/2}}} dx + \int_{\frac{40(K-m)}{2T^{3/2}}}^\infty \frac{40(K-m)}{T^3} dx \leq 2 \left( \frac{40(K-m)}{T^{1/3}} \right)^{1/3}.
\]

Combining these results, we have

\[
E(\delta_i) \leq 2 \left( \frac{40(K-m)}{T^{1/3}} \right)^{1/3}, \tag{135}
\]

for each \(i \in [m]\). Thus we have,

\[
\frac{T}{m^2} E \left( s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\bar{\Delta}_i + \Delta_j) 1_{(i \neq j, j \in \mathcal{V}^* \cup \mathcal{V}^*)} \right) \leq 8 \frac{(40(K-m))^{1/3}T^{2/3}}{m}. \tag{136}
\]

Next, we look at the term in Expression (131)

\[
\frac{T}{m^2} E \left( s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\bar{\Delta}_i + \Delta_j) 1_{(i \neq j, j \in \mathcal{V}^* \cup \mathcal{V}^*)} \right) \leq 2 \frac{T}{m^2} E \left( s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} \bar{\Delta}_i 1_{(i \neq j, j \in \mathcal{V}^*)} \right) \tag{137}
\]
\[
= 2 \frac{T}{m^2} E \left( \sum_{i=1}^{m} \sum_{j=m+1}^{K} \Delta_i \sum_{j=m+1}^{K} s^* \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*, \delta_i \leq \Delta_j/2)} \right) \\
+ 2 \frac{T}{m^2} E \left( \sum_{i=1}^{m} \sum_{j=m+1}^{K} \Delta_i \sum_{j=m+1}^{K} s^* \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*, \delta_i \leq \Delta_j/2)} \right) \\
(y) \leq 2 \frac{T}{m^2} \frac{(K-m)^{1/3} \sqrt{2 \log(K-m)}}{T^{1/3}} m \\
+ 2 \frac{T}{m^2} \sum_{i=1}^{m} \sum_{j=m+1}^{K} \Delta_i \sum_{j=m+1}^{K} P(i \notin \Gamma^*, j \in \Gamma^*, \delta_i \leq \Delta_j/2) \\
\leq 2 \frac{(K-m)^{1/3} T^{2/3} \sqrt{2 \log(K-m)}}{m} \\
+ 2 \frac{T}{m^2} (K-m) \sum_{i=1}^{m} \sum_{j=m+1}^{K} \Delta_i \exp \left( -\frac{T^2}{2} \right) \\
\leq 4 \frac{(K-m)^{1/3} T^{2/3} \sqrt{2 \log(K-m)}}{m}. \quad (138)
\]

Here, (a) follows from the conditions of the indicator, that is, \( \Delta_j/2 < \delta_i \leq \Delta_i/2 \). (b) follows from the condition on \( \Delta_i \) and the fact that \( s^* \leq 1 \). Now, (c) follows from
\[
\sum_{i=1}^{m} \sum_{j=m+1}^{K} s^* \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*, \delta_i \leq \Delta_j/2)} \\
\leq \sum_{i=1}^{m} \sum_{j=m+1}^{K} s^* \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*)} \leq m, \quad (139)
\]
that is, we can at most have \( m \) suboptimal arms that we eventually commit to. (d) is due to Equation 115. Finally, (e) follows from the unimodality of \( \exp(-\frac{T^2}{2})x \), an argument similar to that of Equation 75. Note that in the case of \( 2m > K \), we have \( \sum_{i=1}^{m} \sum_{j=m+1}^{K} s^* \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*)} \leq K - m. \)

Notice that we can directly bound the term in Expression 132 as it is symmetric to the term in Expression 131. Hence,
\[
\frac{T}{m^2} E \left( s^* \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\Delta_i + \Delta_j) \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*, \delta_i > \Delta_j/2, \delta_i \leq \Delta_j/2)} \right) \leq 4 \frac{(K-m)^{1/3} T^{2/3} \sqrt{2 \log(K-m)}}{m}. \quad (140)
\]

Finally, we can bound the term in Expression 133 referring to the techniques we used for bounding the term in Expression 131 and Expression 132 as well:
\[
\frac{T}{m^2} E \left( \sum_{i=1}^{m} \sum_{j=m+1}^{K} (\Delta_i + \Delta_j) s^* \mathbb{1}_{(i \notin \Gamma^*, j \in \Gamma^*, \delta_i \leq \Delta_j/2)} \right)
\]
\[
\leq \frac{T}{m^2} E \left( \frac{1}{m} \sum_{i=1}^{K} \sum_{j=m+1}^{K} \Delta_j s^* \mathbb{1}_{(j \in \Gamma, \delta_i \leq \Delta_j/2)} \right) + \frac{T}{m^2} E \left( \frac{1}{m} \sum_{i=1}^{K} \sum_{j=m+1}^{K} \Delta_j s^* \mathbb{1}_{(j \in \Gamma, \delta_i \leq \Delta_j/2)} \right)
\leq 4 \frac{(K-m)^{1/3} T^{2/3} \sqrt{2 \log(K-m)}}{m}.
\] (141)

Here, the final inequality is due to the series of bounds on Expression \[137\]

Combining Equations [136, 138, 140] and [141] we have

\[
\leq \frac{8(40(K-m))^{1/3} T^{2/3} + 12(K-m)^{1/3} T^{2/3}}{m} \sqrt{2 \log(K-m)}
\leq 45 \frac{(K-m)^{1/3} T^{2/3} \sqrt{2 \log(K-m)}}{m}.
\] (142)

The instance independent bound follows from the fact that the regret from the Explore phase is at most

\[\frac{K}{m} \tau - \tau = O\left(\frac{(K-m)^{1/3} T^{2/3}}{m}\right)\]

and from Equation [142]. \(\square\)

**Lemma 9.** Assume that \(\exists t \in \{\tau + 1, \ldots, \lceil T/m \rceil - 1\}\) such that \(n^t_{i-1} = \tau\) and \(n^t_i = \tau + 1\) for some arm \(i \in [K]\).

Then, \(i \in \Gamma\) and \(n^t_{i/m} = \lceil T/m \rceil + \tau + 1 - t\), i.e., arm \(i\) is pulled in all time periods following \(t\).

**Proof of Lemma**

If there is an arm \(i\) satisfying the conditions of the lemma, then

\[\text{UCB}^t_{n^t_{i-1}} = \text{UCB}^t_i = \bar{\mu}^t_i = \text{LCB}^t_i = \text{LCB}^t_{n^t_{i-1}}.\]

Following the definition of the empirical average reward for arm \(j\),

\[\bar{\mu}^{i+s}_r = \bar{\mu}^t_i\]

for all \(j \in [K]\) and \(s \geq 1\). Hence, for arms with at least \(\tau\) pulls on them, the empirical average reward and upper/lower confidence bounds are the same.

Define the set \(l_s = \{j \in [K] : n^s_j \geq \tau + 1\}\), the set of the arms with at least \(\tau + 1\) pulls on them at time \(s\). Now, we have that \(i \in l_s\), and since it is pulled at time \(t\), (1) it is among the \(m\) arms with the highest upper confidence bounds at time \(t\), and (2) its empirical average reward is not updated after it is pulled. In fact, none of the arms in \(l_s\) have their empirical average rewards updated after they are pulled, and all are in \(l_{t+1}\), that is, \(l_s \subset l_{t+1}\) for \(s \in \{1, \ldots, \lceil T/m \rceil - 1\}\), which we prove next.

Let \(I_s\) denote the set of arms pulled at time \(s\). We claim that \(l_s \subset I_s\) for \(s \leq \lceil T/m \rceil\).

Without loss of generality, and for ease of discussion, assume that arm \(i\) is the first element of set \(l_s\), i.e., \(l_{i-1} = \emptyset\) and \(l_i = i\). Therefore we also have that \(n^t_{i-1} = \tau\) and \(n^t_i = \tau + 1\) since \(i \in I_t\). At time \(t\), there are \((m-1)\) other arms that are being pulled. Then, arm \(i\) will be among the \(m\) arms with the highest upper confidence bounds at time \(t+1\) as well. This is because at most \((m-1)\) arms’ upper confidence bounds are updated after the pulls at time \(t\) and might exceed the empirical average reward of arm \(i\). Therefore, \(i \in I_{t+1}\), and repeating the same argument gives \(i \in I_s\) for \(s \geq t + 2\).

Now, assume that \(\exists t > s\) such that \(l_s = i\) and \(\{i, j\} \in l_s\). As before, we have that \(n^t_{i-1} = \tau\) and \(n^t_i = \tau + 1\) since \(j \in I_t\). Per the same argument as above, at time \(t+1\), there are \((m-2)\) other arms that are being pulled.
Then, arms $i$ and $j$ will both be among the $m$ arms with the highest upper confidence bounds at time $(t_1 + 1)$ as well. Therefore, $i, j \in I_{t_1+1}$, and, as before, $i, j \in I_s$ for $s \geq t_1 + 2$.

Following in this fashion, we get $l_s = I_s$ for $s \geq \omega + 1$, where $\omega = \max_{i \in I^*} \omega_i$. Recall that, for $i \in I^*$, we defined $\omega_i$ as the time period prior to arm $i$ being pulled $(\tau+1)$-st time, i.e., $\omega_i = \min \{ t \leq \frac{T}{m} : n_{i+1} = \tau + 1 \}$. That is, when there are exactly $m$ arms with at least $\tau + 1$ pulls on them, $m$-ADA-ETC is surely in the Commit phase. \qed