NON-COMMUTATIVE A-G MEAN INEQUALITY

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ABSTRACT. In this paper we consider non-commutative analogue for the arithmetic-geometric mean inequality

\[ a^r b^{1-r} + (r-1)b \geq ra \]

for two positive numbers \( a, b \) and \( r > 1 \). We show that under some assumptions the non-commutative analogue for \( a^r b^{1-r} \) which satisfies this inequality is unique and equal to \( r \)-mean. The case \( 0 < r < 1 \) is also considered. In particular, we give a new characterization of the geometric mean.

1. Introduction

For any two positive numbers \( a, b \) and \( r > 1 \), we have the arithmetic-geometric mean inequality

\[ a^r b^{1-r} + (r-1)b \geq ra. \]

In this paper we consider the non-commutative analogue of this inequality for bounded linear operators on a Hilbert space. In particular, we give a new characterization of the geometric mean. Recently their ingenious paper [5], Carlen and Lieb used a certain non-commutative analogue of this inequality. Their paper is a motivation of our considerations.

There is one obvious non-commutative analogue as follows. For a bounded positive operator \( X \) on a Hilbert space, we always have

\[ X^r + (r-1) \geq rX. \]

For any two positive invertible operators \( A, B \), set \( X = B^{-1/2}AB^{-1/2} \). Then we have

\[ (B^{-1/2}AB^{-1/2})^r + (r-1) \geq rB^{-1/2}AB^{-1/2} \]

and hence

\[ B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} + (r-1)B \geq rA. \]

Thus if we consider \( B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} \) (so-called \( r \)-mean) as non-commutative analogue for \( a^r b^{1-r} \), we get a desired inequality.

We conjecture that there is no other example of non-commutative analogue for the above arithmetic-geometric mean inequality. The main result of this paper is

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as follows. We consider “non-commutative analogue” $M(A, B)$ for $a^r b^{1-r}$. More precisely $M$ is a two variable map and $M(A, B)$ is a positive invertible operator for any two positive invertible operators $A, B$. We assume that

(i) $M(tA, B) = t^r M(A, B)$ for any positive number $t$,
(ii) $M(A, B)^{-1} = M(A^{-1}, B^{-1})$.

For example,

$$A^{r/2} B^{1-r} A^{r/2}, \quad B^{(1+2r)/2}(B^6 A^{-2} B^6)^{-r/2} B^{(1+2r)/2}$$

satisfy these conditions. Under these assumptions, if the inequality

$$M(A, B) \geq rA + (1 - r)B$$

holds, then we will show that

$$M(A, B) = B^{1/2}(B^{-1/2} AB^{-1/2})^r B^{1/2}.$$

Therefore in a certain sense our conjecture is true.

Of course these two assumptions are too strong. For example,

$$(A^3 + 2B)^2 A^{r/2}(A^3 + 2B)^{-2} B^{1-r} (A^3 + 2B)^{-2} A^{r/2}(A^3 + 2B)^2$$

can be considered as non-commutative analogue for $a^r b^{1-r}$. However this does not satisfy our assumptions.

We shall also consider the case $0 < r < 1$ and show a similar result. That is, under the assumptions (i) and (ii), if the inequality

$$M(A, B) \leq rA + (1 - r)B$$

holds, then we will show that

$$M(A, B) = B^{1/2}(B^{-1/2} AB^{-1/2})^r B^{1/2}.$$

Our result can be considered as a characterization of $r$-mean, in particular the geometric mean. In the paper [3] T. Ando and K. Nishio gave a characterization of the harmonic mean.

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2. Main Result

Throughout this paper we assume that the readers are familiar with basic notations and results on operator theory. We refer the readers to Conway’s book [4].

We denote by $\mathcal{H}$ a (finite or infinite dimensional) complex Hilbert space and by $B(\mathcal{H})$ all bounded linear operators on it. For each operator $A \in B(\mathcal{H})$, its
operator norm is denoted by $||A||$. We denote by $B(\mathcal{H})^+$ the set of all positive invertible operators. For two vectors $\xi, \eta \in \mathcal{H}$, their inner product and norm are denoted by $\langle \xi, \eta \rangle$ and $||\xi||$ respectively.

In this paper we consider the map $M(\cdot, \cdot)$ from $B(\mathcal{H})^+ \times B(\mathcal{H})^+$ to $B(\mathcal{H})^+$.

We fix a positive number $r > 0$. For $A, B \in B(\mathcal{H})^+$, define

$$M_r(A, B) = B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2}.$$ 

Here we remark that

$$B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-r} A^{1/2}. \quad (\dagger)$$

(This is well-known for specialists.) Indeed, if $r$ is an integer, direct computations show this equality. Thus for any polynomial $p(t)$ with $p(0) = 0$, we have

$$B^{1/2} \cdot p(B^{-1/2}AB^{-1/2}) \cdot B^{1/2} = A^{1/2} \cdot p((A^{-1/2}BA^{-1/2})^{-1}) \cdot (A^{-1/2}BA^{-1/2}) \cdot A^{1/2}.$$ 

Thus by continuity we get $(\dagger)$. The map $M_r$ is so-called $r$-mean, and usually the case $0 < r < 1$ is considered. (When $0 < r < 1$, $M_r(A, B)$ is one of the so-called operator means. In particular, in the case $r = 1/2$, $M_r(A, B)$ is said to be the geometric mean.)

First we shall consider the case $r > 1$. The following is our main result.

**Theorem 2.1.** Assume $r > 1$. For any $A, B \in B(\mathcal{H})^+$, if the map $M$ satisfies

(i) $M(A, B) \geq rA + (1-r)B,$

(ii) $M(tA, B) = t^r M(A, B)$ for any positive number $t$,

(iii) $M(A, B)^{-1} = M(A^{-1}, B^{-1}),$

then we have $M = M_r$.

We need some preparations to prove this theorem. The following lemma states that under the assumptions (i) and (ii), the map $M_r(A, B)$ is “less” than $M(A, B)$ in a certain sense. (See Remark 2.1.)

**Lemma 2.2.** For any $A, B \in B(\mathcal{H})^+$, we assume that the map $M$ satisfies

(i) $M(A, B) \geq rA + (1-r)B,$

(ii) $M(tA, B) = t^r M(A, B)$ for any positive number $t$.

Then for any unit vector $\xi \in \mathcal{H}$, if $r \geq 2$ we have

$$\langle A^{-1/2}M(A, B)A^{-1/2} \xi, \xi \rangle \langle (A^{-1/2}M_r(A, B)A^{-1/2})^{-1} \xi, \xi \rangle \geq 1.$$ 

On the other hand if $1 < r \leq 2$ we have

$$\langle (A^{-1/2}M(A, B)A^{-1/2})^{1/(r-1)} \xi, \xi \rangle \langle (A^{-1/2}M_r(A, B)A^{-1/2})^{-1/(r-1)} \xi, \xi \rangle \geq 1.$$
Proof. By assumptions we have

\[ t^r M(A, B) \geq rtA + (1 - r)B \]

and hence

\[ A^{-1/2}M(A, B)A^{-1/2} \geq rt^{1-r} + (1-r)t^{-r}A^{-1/2}BA^{-1/2}. \]

For a unit vector \( \xi \in \mathfrak{H} \), set

\[ f(t) = rt^{1-r} + (1-r)t^{-r}\langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle. \]

Here we remark that \( \langle A^{-1/2}M(A, B)A^{-1/2}\xi, \xi \rangle \geq f(t) \). Then it is easy to see that the maximum value of \( f(t) \) on \( (0, \infty) \) is equal to \( \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle^{1-r} \). Thus we get

\[ \langle A^{-1/2}M(A, B)A^{-1/2}\xi, \xi \rangle \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle^{r-1} \geq 1. \]

In the case \( r \geq 2 \), by the Jensen inequality and (†) we have

\[ \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle^{r-1} \leq \langle (A^{-1/2}BA^{-1/2})^{r-1}\xi, \xi \rangle = \langle (A^{-1/2}M_r(A, B)A^{-1/2})^{-1}\xi, \xi \rangle. \]

So we are done.

Next we consider the case \( 1 < r \leq 2 \). Let \( s \) be a positive number such that \( \frac{1}{r} + \frac{1}{s} = 1 \). Then since \( (r-1) = 1/(s-1) \), we have

\[ \langle A^{-1/2}M(A, B)A^{-1/2}\xi, \xi \rangle \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle^{1/(s-1)} \geq 1 \]

and hence

\[ \langle A^{-1/2}M(A, B)A^{-1/2}\xi, \xi \rangle^{s-1} \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle \geq 1. \]

Since \( s \geq 2 \) and \( (s-1) = 1/(r-1) \), we compute as above

\[ \langle (A^{-1/2}M(A, B)A^{-1/2})^{1/(r-1)}\xi, \xi \rangle \langle (A^{-1/2}M_r(A, B)A^{-1/2})^{-1/(r-1)}\xi, \xi \rangle = \langle (A^{-1/2}M(A, B)A^{-1/2})^{s-1}\xi, \xi \rangle \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle \geq \langle (A^{-1/2}M(A, B)A^{-1/2})\xi, \xi \rangle^{s-1} \langle A^{-1/2}BA^{-1/2}\xi, \xi \rangle \geq 1. \]

\[ \square \]

**Theorem 2.3.** For two positive invertible operators \( X, Y \in B(\mathfrak{H})^+ \), if they satisfy

\[ \langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1 \]

and

\[ \langle Y\xi, \xi \rangle \langle X^{-1}\xi, \xi \rangle \geq 1 \]

for any unit vector \( \xi \in \mathfrak{H} \), then we have \( X = Y \).

In order to show this theorem, we need the following lemma.
Lemma 2.4. For two operators \( X, Y \in B(\mathcal{H})^+ \), they satisfy
\[
\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1
\]
for any unit vector \( \xi \in \mathcal{H} \) if and only if we have
\[
tX + (tY)^{-1} \geq 2
\]
for any positive number \( t \).

Proof. Let \( \xi \in \mathcal{H} \) be a unit vector. We set \( f(t) = t\langle X\xi, \xi \rangle + t^{-1}\langle Y^{-1}\xi, \xi \rangle \) for \( t > 0 \). Then it is easy to see that the minimum value of \( f(t) \) is equal to \( 2\sqrt{\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle} \). Hence we are done. \( \square \)

Proof of Theorem 2.3. By the previous lemma we have
\[
tX + (tY)^{-1} \geq 2
\]
and
\[
tY + (tX)^{-1} \geq 2
\]
for any positive number \( t \). Let \( Z = Y^{1/2}XY^{1/2} \). Then we have \( tZ + t^{-1} \geq 2Y \) and \( t + (tZ)^{-1} \geq 2Y^{-1} \). So we get
\[
\frac{2tZ}{t^2Z + 1} \leq Y \leq \frac{t^2Z + 1}{2t}.
\]
(1)

First we assume that the Hilbert space is finite dimensional because in this case the proof becomes simpler. Take any projection \( P \) of rank one which reduces \( Z \), that is, \( ZP = \lambda P \) for some positive number \( \lambda \). (Here we use the fact that \( Z \) is atomic, thanks to finite dimensionality.) Then we get
\[
\frac{2t\lambda}{t^2\lambda + 1} P \leq PYP \leq \frac{t^2\lambda + 1}{2t} P.
\]
Since \( P \) is of rank one, \( PYP \) is of the form \( PYP = \alpha P \) for some \( \alpha > 0 \). Therefore taking the maximum in \( t \) on the left-hand side and the minimum on the right-hand side we have \( PYP = \lambda^{1/2} P \). On the other hand, since we also have
\[
\frac{t^2Z + 1}{2tZ} \geq Y^{-1} \geq \frac{2t}{t^2Z + 1},
\]
(2)
we get
\[
\frac{t^2\lambda + 1}{2t\lambda} P \geq PY^{-1}P \geq \frac{2t}{t^2\lambda + 1} P
\]
and hence \( PY^{-1}P = \lambda^{-1/2} P \) as above.

Let \( \xi \in \mathcal{H} \) be a vector satisfying \( P\xi = \xi \). Then we have
\[
||Y^{1/2}\xi|| \cdot ||Y^{-1/2}\xi|| = \langle PYP\xi, \xi \rangle^{1/2} < PY^{-1}P\xi, \xi \rangle^{1/2} = \langle \lambda^{1/2}\xi, \xi \rangle^{1/2} < \lambda^{-1/2}\xi, \xi \rangle^{1/2}
\]
\[
= ||\xi||^2 = <Y^{1/2}\xi, Y^{-1/2}\xi>.
\]
By the equality condition for the Cauchy-Schwarz inequality, this implies that $Y^{1/2}\xi$ is a scalar-multiple of $Y^{-1/2}\xi$, in other words, $Y\xi$ is a scalar-multiple of $\xi$. So we get $YP = PYP = Z^{1/2}P$. Since by the spectral theory for a positive matrix there are such projections $P_i$ of rank one such that $\sum_i P_i = 1$, we conclude that $Y = Z^{1/2} = (Y^{1/2}XY^{1/2})^{1/2}$ and hence $X = Y$.

Next we shall consider the general case. The following argument is due to a private communication with T. Ando \cite{Ando}. The author would like to thank Professor Ando for permitting the author to include his argument in this paper.

It is easy to see that for any $t > 0$
\[
\frac{2tZ}{t^2Z + 1} \leq Y - Z^{1/2} \leq \frac{t^2Z + 1}{2t} - \frac{2tZ}{t^2Z + 1}
\]
and
\[
\frac{t^2Z + 1}{2tZ} \geq Z^{-1/2} \geq \frac{2t}{t^2Z + 1}.
\]
Combining these with (1) and (2), we have
\[
\frac{2tZ}{t^2Z + 1} - \frac{t^2Z + 1}{2t} \leq Y - Z^{1/2} \leq \frac{t^2Z + 1}{2t} - \frac{2tZ}{t^2Z + 1}
\]
and
\[
\frac{t^2Z + 1}{2tZ} - \frac{2t}{t^2Z + 1} \geq Y^{-1} - Z^{-1/2} \geq \frac{2t}{t^2Z + 1} - \frac{t^2Z + 1}{2tZ}.
\]
Here we compute
\[
\frac{t^2Z + 1}{2t} - \frac{2tZ}{t^2Z + 1} = (t - Z^{-1/2})^2 \frac{Z(t + Z^{-1/2})^2}{2t(t^2Z + 1)}
\]
and
\[
\frac{t^2Z + 1}{2tZ} - \frac{2t}{t^2Z + 1} = (t - Z^{-1/2})^2 \frac{Z(t + Z^{-1/2})^2}{2t(t^2Z + 1)}.
\]
Therefore there is a positive number $\gamma$ such that for any spectrum $\lambda$ of $Z$ and a projection $P$ which reduces $Z$ we have
\[
\|PYP - Z^{1/2}P\| \leq \gamma\|\lambda - Z^{-1/2}\|P\|^2 \tag{3}
\]
and
\[
\|PY^{-1}P - Z^{-1/2}P\| \leq \gamma\|\lambda - Z^{-1/2}\|P\|^2. \tag{4}
\]

Let us use $(PY^{-1}P)^{-1}$ to denote the inverse of $PY^{-1}$ on $P\mathcal{F}$. Then we see that
\[
PY^{-1}P - Z^{-1/2}P = (PY^{-1}P)(Z^{1/2}P - (PY^{-1}P)^{-1})Z^{-1/2}P
\]
and hence by using (4) there is a positive number $\gamma'$ such that
\[
\|((PY^{-1}P)^{-1} - Z^{1/2}P\| \leq \gamma'\|\lambda - Z^{-1/2}\|P\|^2.
\]
Combining this with (3) we conclude that there is a positive number $\gamma''$ such that for any spectrum $\lambda$ of $Z$ and spectral projection $P$ of $Z$

$$\|PYP - (PY^{-1}P)^{-1}\| \leq \gamma''\|\lambda - Z^{-1/2}\|P^2.$$  \hfill (5)

For any integer $n$, take a partition of unity $\{P_i\}_{i=1}^n$ which consists of spectral projections of $Z$ such that there exist spectrums $\{\lambda_i\}_{i=1}^n$ of $Z^{-1/2}$ satisfying

$$\|\lambda_i - Z^{-1/2}\|P_i \| \leq \frac{\|Z^{-1/2}\|}{n}.$$  

Then it follows from (3) that

$$\sum_{i=1}^n \|P_iYP_i - Z^{1/2}P_i\| \leq \frac{\gamma\|Z^{-1/2}\|^2}{n^2}. \hfill (6)$$

Similarly it follows from (5) that

$$\|P_iYP_i - (P_iY^{-1}P_i)^{-1}\| \leq \frac{\gamma''\|Z^{-1/2}\|^2}{n^2}.$$

Recall the following formula, which is so-called Schur complement

$$(P_iY^{-1}P_i)^{-1} = P_iYP_i - P_iYP_i^+(P_i^+YP_i^+)^{-1}P_i^+YP_i$$

where $P_i^+ = 1 - P_i$. Indeed we can show

$$\{P_iYP_i - P_iYP_i^+(P_i^+YP_i^+)^{-1}P_i^+YP_i\} \cdot P_iY^{-1}P_i$$

$$= P_iYP_iY^{-1}P_i - P_iYP_i^+(P_i^+YP_i^+)^{-1}P_i^+YP_iY^{-1}P_i$$

$$= P_iYP_iY^{-1}P_i - P_iYP_i^+(P_i^+YP_i^+)^{-1}(P_i^+Y - P_i^+YP_i^+)Y^{-1}P_i$$

$$= P_iYP_iY^{-1}P_i - P_iYP_i^+Y^{-1}P_i = P_i.$$  

By using this formula we have

$$\|P_i^+YP_i\|^2 = \||(P_i^+YP_i^+)^{1/2}(P_i^+YP_i^+)^{-1/2}P_i^+YP_i\|^2$$

$$\leq \|Y\| \cdot ||(P_i^+YP_i^+)^{-1/2}P_i^+YP_i\|^2$$

$$= \|Y\| \cdot ||P_iYP_i^+(P_i^+YP_i^+)^{-1}P_i^+YP_i\|^2$$

$$= \|Y\| \cdot ||P_iYP_i - (P_iY^{-1}P_i)^{-1}\|$$

$$\leq \frac{\gamma''\|Y\| \cdot \|Z^{-1/2}\|^2}{n^2}.$$
Therefore for each unit vector $\xi \in \mathcal{H}$ by using the Cauchy-Schwarz inequality we see that

$$\left\| \sum_{i=1}^{n} P_i Y P_i \xi \right\| \leq \sum_{i=1}^{n} \left| \left| P_i Y \right| \right| \cdot \left| \left| P_i \xi \right| \right|$$

$$\leq \sqrt{\sum_{i=1}^{n} \left| \left| P_i Y \right| \right|^2} \cdot \sqrt{\sum_{i=1}^{n} \left| \left| P_i \xi \right| \right|^2}$$

$$= \sqrt{\sum_{i=1}^{n} \left| \left| P_i Y \right| \right|^2} \cdot \sqrt{\sum_{i=1}^{n} \left| \left| P_i \xi \right| \right|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} \gamma'' \left| \left| Y \right| \right| \cdot \left| \left| Z^{-1/2} \right| \right|^2} \cdot \sqrt{n}.$$

Thus we get

$$\left| \left| \sum_{i=1}^{n} P_i Y P_i \right| \right| \leq \sqrt{\frac{\gamma'' \left| \left| Y \right| \right| \cdot \left| \left| Z^{-1/2} \right| \right|^2}{n}}.$$  \hfill (7)

By using (6) and (7) we see that

$$\left| \left| Y - Z^{1/2} \right| \right| \leq \left| \left| \sum_{i=1}^{n} P_i Y P_i - Z^{1/2} P_i \right| \right| + \left| \left| \sum_{i=1}^{n} P_i Y P_i \right| \right|$$

$$\leq \gamma \left| \left| Z^{-1/2} \right| \right| + \sqrt{\frac{\gamma'' \left| \left| Y \right| \right| \cdot \left| \left| Z^{-1/2} \right| \right|^2}{n}}.$$

By tending $n \to \infty$ we get $Y = Z^{1/2}$ and hence $X = Y$. 

□

Now we can prove our main result.

**Proof of Theorem 2.1.** First we consider the case $r \geq 2$. Set

$$X = A^{-1/2} M(A, B) A^{-1/2}$$

and $Y = A^{-1/2} M_r(A, B) A^{-1/2}$.

By Lemma 2.2 for any unit vector $\xi \in \mathcal{H}$ we have

$$\langle X \xi, \xi \rangle \langle Y^{-1} \xi, \xi \rangle \geq 1.$$

On the other hand, thanks to the relations $M(A, B)^{-1} = M(A^{-1}, B^{-1})$ and $M_r(A, B)^{-1} = M_r(A^{-1}, B^{-1})$, applying Lemma 2.2 for the pair $(A^{-1}, B^{-1})$ we have

$$\langle X^{-1} \xi, \xi \rangle \langle Y \xi, \xi \rangle$$

$$= \langle A^{1/2} M(A^{-1}, B^{-1}) A^{1/2} \xi, \xi \rangle \langle (A^{1/2} M_r(A^{-1}, B^{-1}) A^{1/2})^{-1} \xi, \xi \rangle \geq 1.$$

Therefore by Theorem 2.3 we get $X = Y$ and hence $M = M_r$. 

□
In the case $1 < r \leq 2$, set

$$X = (A^{-1/2}M(A, B)A^{-1/2})^{1/(r-1)} \quad \text{and} \quad Y = (A^{-1/2}M_r(A, B)A^{-1/2})^{1/(r-1)}.$$ 

Then in the same way we conclude the desired fact. \hfill \square

**Remark 2.1.**

(i) For positive invertible operators $A, B, C$, the block matrix

$$
\begin{pmatrix}
A & B \\
B & C
\end{pmatrix}
$$

is positive if and only if $A \geq BC^{-1}B$. Therefore for two positive invertible operators $X, Y$, the block matrix

$$
\begin{pmatrix}
X & 1 \\
1 & Y^{-1}
\end{pmatrix}
$$

is positive if and only if $X \geq Y$. On the other hand for any unit vector $\xi \in H$ the matrix

$$
\begin{pmatrix}
\langle X\xi, \xi \rangle & 1 \\
1 & \langle Y^{-1}\xi, \xi \rangle
\end{pmatrix}
$$

is positive if and only if $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$. Thus the condition $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$ is weaker than $X \geq Y$. We do not know whether the condition $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$ define new order $X^{\bowtie} \geq Y$ or not. The author guess that this relation does not satisfy transitivity. Here we remark that if $X^{\bowtie} \geq Y$, then we have $X^{2^{\bowtie}} \geq Y^2$. Indeed if we have $\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1$, then we get

$$
\langle X^{2\xi}, \xi \rangle \langle Y^{-2}\xi, \xi \rangle \geq (\langle X\xi, \xi \rangle)^2 \langle Y^{-1}\xi, \xi \rangle^2 \geq 1.
$$

Thus this relation is not equivalent to usual order. Theorem 2.3 states that if we have $X^{\bowtie} \geq Y$ and $Y^{\bowtie} \geq X$, then we conclude $X = Y$ (reflexivity).

(ii) We would like to conjecture that Theorem 2.1 holds by replacing the condition (iii) with

$$(iii)' \quad M(A, B) = A^rB^{1-r} \quad \text{if} \ A \text{ commutes with} \ B.$$ 

Finally we shall prove the analogue in the case $0 < r < 1$ for Theorem 2.1.

**Theorem 2.5.** Assume $0 < r < 1$. For any $A, B \in B(\mathfrak{H})^+$, if the map $M$ satisfies

(i) $M(A, B) \leq rA + (1-r)B$, 
(ii) $M(tA, B) = t^rM(A, B)$ for any positive number $t$, 
(iii) $M(A, B)^{-1} = M(A^{-1}, B^{-1})$,

then we have $M = M_r$. 

Proof. The proof is essentially same as that of Theorem 2.1. So we would like to
give the sketch of the proof.

By assumptions for any positive number $t$ we have
$$M(A, B) \leq rt^{r-1} A + (1 - r)t^r B$$
and
$$M(A, B)^{-1} \leq rt^{1-r} A^{-1} + (1 - r)t^{-r} B^{-1}.$$ 

Set
$$Y = B^{-1/2} M(A, B) B^{-1/2} \text{ and } Z = B^{-1/2} AB^{-1/2}.$$ 

Then we have
$$\frac{t^r Z}{rt + (1 - r)Z} \leq Y \leq \frac{rz + (1 - r)t}{t^{1-r}}.$$ 

Then by the almost same arguments as those in the proof of Theorem 2.3, we
can show $Y = Z^r$. \hfill \square

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