The magnetic susceptibility in QCD

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Outline

1. QCD at finite temperature: diamagnetic or paramagnetic?
2. Why it is not trivial to answer
3. How to answer
4. Numerical results
5. Conclusions & developments
QCD in external magnetic field

External magnetic fields can be relevant for the phenomenology of
- primordial universe
- heavy-ion collisions

From a theoretical point of view they are also interesting as a different way to test the non-perturbative dynamics of QCD.

External E.M. fields couple to QCD through the Dirac matrix. In the unimproved staggered formulation

\[
D_{i,j} = a m \delta_{i,j} + \frac{1}{2} \sum_{\nu=1}^{4} \eta_{\nu}(i) \left[ u_{\nu}(i) U_{\nu}(i) \delta_{i,j-\hat{\nu}} - u_{\nu}^*(i-\hat{\nu}) U^\dagger_{\nu}(i-\hat{\nu}) \delta_{i,j+\hat{\nu}} \right]
\]

\(U_{\nu}(i)\) = non abelian link variables
\(u_{\nu}(i)\) = abelian link variables
The QCD medium

We are interested in the magnetic properties of QCD at finite temperature.

For “small” external magnetic $B$ we can write the free energy as

$$F(B, T) = F(B = 0, T) + F_1(T)B + \frac{1}{2}F_2(T)B^2 + O(B^3)$$

$F_1 \equiv 0$ if no ferromagnetism

$F_2$ is proportional to the magnetic susceptibility (see later)

$F_2 < 0$ paramagnetic medium

$F_2 > 0$ diamagnetic medium

The basic point we want to settle:

is QCD paramagnetic or diamagnetic?
The standard way and a no-go

The determination of magnetic susceptibilities is a standard problem in statistical physics. An estimator of $F_2$ is obtained by using the relation

$$F_2(T) = \left. \frac{\partial^2 F(B, T)}{\partial B^2} \right|_{B=0}$$

which amounts to compute the mean value of some lattice observable at $B = 0$.

In QCD this is not possible: to reduce finite size effects simulations are performed on compact manifold without boundary and as a consequence the possible values of the homogeneous magnetic field are quantized.

$$\frac{\partial}{\partial B}$$ on the lattice is not well defined!
The magnetic field on the lattice

On a compact manifold with no boundary there is an ambiguity in the application of the Stokes theorem: which part of the manifold is the interior and which is the exterior of a path?

For an homogeneous magnetic field $B\hat{z}$ we have

$$\oint A_\mu dx_\mu = AB \quad \oint A_\mu dx_\mu = -(\ell_x \ell_y - A)B$$

This does not affect the motion of a charged particle if we impose

$$\exp(iqBA) = \exp(iqB(A - \ell_x \ell_y)) \quad \Rightarrow \quad qB = \frac{2\pi b}{\ell_x \ell_y} \quad b \in \mathbb{Z}$$

(the $\ell_\mu$'s are the lengths in physical units)
The magnetic field on the lattice (2)

A simple choice of the lattice discretization is

\[ u_y(n) = e^{ia^2 qBn_x} \quad u_x(L_x - 1) = e^{-ia^2 qB L_x n_y} \quad \text{otherwise} \quad u_j(n) = 1 \]

An example for \( L_x = L_y = 4 \).

The E.M. plaquettes are given by

- \( P_{ij} = e^{ia^2 qB} \) for \((i, j) \neq (3, 3)\)
- \( P_{33} = \exp(i a^2 qB + i a^2 qB L_x L_y) \)

Everything is ok if \( a^2 qB L_x L_y = 2\pi b \) with \( b \in \mathbb{Z} \). The idea is the same as the Dirac quantization condition for monopoles (i.e. “invisible” string).
A way to go

We are interested in studying the $B$ dependence of $F$, i.e.

$$\Delta F(B_k, T) = F(B_k, T) - F(0, T)$$

$$a^2 qB_k = \frac{2\pi k}{L_x L_y} \quad k \in \mathbb{Z}$$

$$M(B, T) = \frac{\partial F(B, T)}{\partial b} \text{ is not} \quad \text{the magnetization, but we can evaluate it at non quantized values of } B \text{ in order to get}$$

$$\Delta F(B_k, T) = \int_0^k M(b, T) db$$

All the “periodicity” artefacts that affect $M$ simplify in the integral to give us the correct answer!

We work on finite lattices, so everything is analytic and we adopt the previous expression for the $u_i(n)$ also for non quantized $B$ values. These values of $B$ are non physical but are needed only for the purpose of reconstructing $\Delta F$ for integer $b$. 
Renormalization prescription

The free energy renormalizes additively so it is not enough to fix the sign of \( \Delta F(B_k, T) \) to determine the nature of the magnetic QCD medium.

We can remove the additive renormalization by subtracting the zero temperature value:

\[
(\Delta F)_R(B_k, T) = \Delta F(B_k, T) - \Delta F(B_k, T = 0)
\]

This is motivated by the idea that we want to study the properties of the thermal medium so the zero temperature value has to be subtracted as a normalization.

Our procedure is thus the following:

1. compute the “magnetization” \( M \) for different temperatures and for non quantized \( B \) values
2. integrate \( M \) to get \( \Delta F(B_k, T) \) for the quantized \( B_k \) values
3. compute the renormalized magnetic free energy \( (\Delta F)_R(B_k, T) \)
How $M$ looks like

$M$ computed on a $16^4$ lattice, $N_f = 1 + 1$, $m_\pi \approx 480\text{MeV}$, $a \approx 0.188\text{fm}$. The continuous line is a $3^{rd}$ order spline interpolation.

The numerical integration of $M$ to get $\Delta F$ is performed by means of spline interpolations together with a bootstrap analysis for the error estimation.
Extracting the quadratic term

We now need to estimate $f_2$ defined by $\Delta F(B_k, T) \approx \frac{1}{2} f_2(T) k^2$ ($B_k \propto k$). In order to minimize the error propagation in the integration we fit

$$\Delta F(B_k, T) - \Delta F(B_{k-1}, T) = \int_{k-1}^{k} M(b, T) \, db$$

with the function

$$\frac{1}{2} f_2(T) \left[ k^2 - (k - 1)^2 \right] = \frac{1}{2} f_2(T)(2k - 1)$$

Results for $4 \times 16^3, 4 \times 24^3$ and $16^4$ lattices with $m_\pi \approx 480$MeV and $a \approx 0.188$fm ($T \approx 175$MeV).
A note on the susceptibility

In an usual linear medium we have (in SI units)

\[ M = \chi H \quad B = \mu H \quad \mu = \mu_0(1 + \chi) \]

and the expression for the total free energy (for volume unit) is

\[ F/V = \int H \cdot dB = \frac{1}{2\mu} B^2 = \frac{1}{2\mu_0(1 + \chi)} B^2 \]

In our QCD simulations the magnetic field is quenched and the energy of the electromagnetic field “without QCD” must be subtracted:

\[ F/V = -\int M \cdot dB = -\frac{\chi}{\mu_0(1 + \chi)} \int B \cdot dB \]

moreover there is no back reaction of the medium on the magnetic field, so that \( B = \mu_0 H \) and

\[ F/V = -\frac{\chi}{2\mu_0(1 + \chi)} B^2 \]
The final result

\[ \chi/(1+\chi) \]

Some reference values
- Tungsten: \(7.8 \times 10^{-5}\)
- Platinum: \(2.8 \times 10^{-4}\)
- Liquid Oxygen: \(3.9 \times 10^{-3}\)
- Gadolinium: \(4.8 \times 10^{-1}\)

- \(m_\pi = 195 \text{MeV}, a = 0.188 \text{fm}\)
- \(m_\pi = 275 \text{MeV}, a = 0.17 \text{fm}\)
- \(m_\pi = 480 \text{MeV}, a = 0.141 \text{fm}\)
- \(m_\pi = 480 \text{MeV}, a = 0.188 \text{fm}\)
- \(m_\pi = 480 \text{MeV}, a = 0.24 \text{fm}\)
Check for systematics

- dependence on the volume
- dependence on the spline interpolation and/or the number of points
- dependence on the $B$ field extension out of integers

Systematics are always less than statistical errors
Conclusions & developments

- We presented a simple new strategy to study the magnetic properties of QCD.
- Our results show that the QCD medium near deconfinement behaves as a strong paramagnet.
- A study with improved fermions at physical quark masses is ongoing. Preliminary results are in qualitative and reasonable quantitative agreement with the unimproved staggered result presented here.