From Quantum Link Models to D-Theory: A Resource Efficient Framework for the Quantum Simulation and Computation of Gauge Theories

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Quantum link models provide an extension of Wilson’s lattice gauge theory in which the link Hilbert space is finite-dimensional and corresponds to a representation of an embedding algebra. In contrast to Wilson’s parallel transporters, quantum links are intrinsically quantum degrees of freedom. In D-theory these discrete variables undergo dimensional reduction, thus giving rise to asymptotically free theories. In this way \((1+1)\)-d \(\mathbb{C}P(N-1)\) models emerge by dimensional reduction from \((2+1)\)-d \(SU(N)\) quantum spin ladders, the \((2+1)\)-d confining \(U(1)\) gauge theory emerges from the Abelian Coulomb phase of a \((3+1)\)-d quantum link model, and \((3+1)\)-d QCD arises from a non-Abelian Coulomb phase of a \((4+1)\)-d \(SU(3)\) quantum link model, with chiral quarks arising naturally as domain wall fermions. Thanks to their finite-dimensional Hilbert space and their economical mechanism of reaching the continuum limit by dimensional reduction, quantum link models provide a resource efficient framework for the quantum simulation and computation of gauge theories.

1. Introduction

Gauge theories play a fundamental role in the standard model of particle physics. The strong interaction is described by QCD — the non-Abelian \(SU(3)\) gauge theory of quark and gluon fields. The Abelian \(U(1)\) gauge theory of QED is also relevant in atomic, molecular, and condensed matter physics, and in quantum optics.
Strongly coupled gauge theories confront us with great computational challenges. In particular, simulations of their real-time evolution or of their behavior at non-zero fermion density with classical computers are affected by very severe sign problems [1]. Quantum simulation and computation have emerged as very promising tools which circumvent the sign problem because they work directly with quantum hardware and thus naturally incorporate entanglement and quantum interference. In this way Feynman’s vision [2] of simulating complicated physical systems by other well-controlled quantum systems has become reality [3]. Quantum simulators [4] are special purpose quantum computers which are used as digital [5] or analog [6] devices, for example, using ultracold atoms in optical lattices [7,8], trapped ions [9], photons [10], or superconducting circuits on a chip [11]. A digital quantum simulator is a precisely controllable many-body system that is programmed to execute a sequence of quantum gate operations. The initial state of the simulated system is encoded as quantum information, and the real-time evolution is driven stroboscopically by a sequence of quantum gates. In an analog quantum simulator, on the other hand, the time evolution proceeds continuously. Analog devices are limited to simpler interactions, but they can be scaled up to larger system sizes.

Implementing gauge theories on quantum hardware is a non-trivial challenge [12–14]. Several analog [15–22] as well as digital [23–26] constructions for gauge theory quantum simulators have already been proposed. Experimental realizations of analog or digital quantum simulations or computations of lattice gauge theories, some based upon quantum link models, include [27–35]. Here we discuss quantum link models as a promising resource efficient regularization of Abelian and non-Abelian gauge theories. The goal is to provide a pedagogical introduction to those aspects of this alternative formulation of gauge theories that are most relevant to upcoming quantum simulation or quantum computation applications, rather than reviewing this broad subject as a whole.

2. Abelian Lattice Gauge Theories in the Hamiltonian Formulation

Lattice gauge theories were introduced by Wegner [36] for a $\mathbb{Z}(2)$ gauge symmetry and by Wilson for general Abelian or non-Abelian gauge symmetries [37]. Here we construct an Abelian $U(1)$ gauge theory in the Hamiltonian formulation [38] with the fundamental variables residing on the links of a regular spatial lattice. First, we use quantum mechanical analog “particles” moving around in the $U(1)$ group manifold, which is just a circle $S^1$, as the basic building blocks of the theory. The resulting link Hilbert space is infinite-dimensional. Then we construct quantum link models [39–41] by replacing these basic building blocks by quantum links, i.e. quantum spins endowed with a gauge symmetry, which reside in a finite-dimensional link Hilbert space.

(a) Analog “Particles” Moving in the Group Manifold $U(1) = S^1$

The basic building blocks of Wilson’s lattice gauge theory are group-valued parallel transporters associated with the links connecting neighboring lattice sites. The link variables of an Abelian $U(1)$ lattice gauge theory are hence complex phases $\exp(i\varphi) \in U(1)$. In order to familiarize ourselves with these basic variables, we first consider a simple quantum mechanical analog, a “particle” that is moving in the group manifold $U(1) = S^1$. A quantum mechanical particle of mass $M$ that moves on a circle of radius $R$ has a moment of inertia $I = MR^2$ and is described by its angular position $U = \exp(i\varphi)$, $U^\dagger = \exp(-i\varphi)$, $\varphi \in [-\pi, \pi]$. The particle’s angular momentum operator plays the role of an electric field in the gauge theory and is given by $E = -i\partial_\varphi$ (in units where $\hbar = 1$). The corresponding commutation relations take the form

$$[E, U] = U, \quad [E, U^\dagger] = -U^\dagger, \quad [U, U^\dagger] = 0,$$

(2.1)

The kinetic energy operator $T$ as well as its spectrum are given by

$$T = \frac{E^2}{2I}, \quad [T, E] = 0, \quad T|m\rangle = \frac{m^2}{2I}|m\rangle, \quad \langle\varphi|m\rangle = \frac{1}{\sqrt{2\pi}} \exp(im\varphi), \quad m \in \mathbb{Z}.$$

(2.2)
Since the number of eigenstates is infinite, the Hilbert space is infinite-dimensional.

Let us now consider three “particles” moving on $S^1$. We associate the “particles” with the links 12, 23, and 31 that connect the sites 1, 2, 3 of a triangle. Their angular momenta turn into the electric fields of a gauge theory on a triangular lattice $E_{12} = -i\partial_x \varphi_1$, $E_{23} = -i\partial_x \varphi_2$, $E_{31} = -i\partial_x \varphi_3$.

The corresponding Hamiltonian contains a specific 3-body interaction

$$H = T_{12} + T_{23} + T_{31} + V_{123} = \frac{E_{12}^2}{2I} + \frac{E_{23}^2}{2I} + \frac{E_{31}^2}{2I} - \frac{1}{2e^2} \cos(\varphi_1 + \varphi_2 + \varphi_3),$$

(2.3)

Due to the special form of the 3-body force, the Hamiltonian commutes with the three relative angular momenta of the particles

$$G_1 = E_{12} - E_{31}, \quad G_2 = E_{23} - E_{12}, \quad G_3 = E_{31} - E_{23}, \quad [H,G_1] = [H,G_2] = [H,G_3] = 0.$$  

(2.4)

The relative angular momenta $G_1$, $G_2$, $G_3$ turn into the generators of infinitesimal gauge transformations associated with the lattice sites.

(b) Many “Particles” in $S^1$ Forming a $U(1)$ Lattice Gauge Theory

The quantum mechanical analog “particles” moving in the group space $U(1)$ are used to build a Wilsonian lattice gauge theory. In that case, the “particles” embody parallel transporters $U_{xy} \in U(1)$ associated with the links $(xy)$ connecting neighboring lattice sites $x$ and $y$. A $U(1)$ gauge theory on a triangular lattice is then described by the Hamiltonian

$$H = \frac{e^2}{2} \sum_{(xy)} E_{xy}^2 - \frac{1}{2e^2} \sum_{(xyz)} (U_{xy}U_{yz}U_{zx} + U_{xy}^\dagger U_{yz}^\dagger U_{zx}^\dagger) .$$

(2.5)

Here $(xyz)$ denotes a triangular plaquette. We have identified the moment of inertia $I = 1/e^2$ as a function of the gauge coupling $e$.

The structure of this lattice gauge theory is characterized by the link-based operator algebra

$$[E_t, E_{t'}] = 0, \quad [E_t, U_{l'}] = i\delta_{l,l'} U_t, \quad [E_t, U_{l'}^\dagger] = -i\delta_{l,l'} U_t^\dagger,$$

$$[U_{l}, U_{l'}] = [U_{l}, U_{l'}^\dagger] = [U_{l}, U_{l'}^\dagger] = 0 .$$

(2.6)

In particular, the operators $E_t$ or $U_{l'}$, which reside on different links $l$ and $l'$, commute with each other. As a result of these commutation relations, the Hamiltonian commutes with the generators of gauge transformations associated with the lattice sites $x$

$$G_x = \sum_t (E_{x,x+1} - E_{x-1,x}), \quad [H,G_x] = 0 .$$

(2.7)
Here $\mathbf{i}$ is the unit-vector pointing in one of the three lattice directions, $i \in \{1, 2, 3\}$, of the triangular lattice (cf. Fig. 1).

It is important to note that gauge symmetries are qualitatively different from global symmetries. While global symmetries may give rise to degeneracies in the physical spectrum, gauge symmetries just reflect a redundancy in the description of the physics. Gauge invariance guarantees that the redundancy does not affect physical results. This is a consequence of Gauss’ law, which implies that all physical states $|\Psi\rangle$ must be gauge invariant, $G_x |\Psi\rangle = 0$. Gauge transformations associated with different sites commute with each other, $[G_x, G_y] = \delta_{xy}$, as well as with the Hamiltonian. The eigenstates $|\Psi, Q\rangle$, with $Q = \{Q_x\}$, of the Hamiltonian can be characterized by the eigenvalues $Q_x \in \mathbb{Z}$ of all gauge generators, $G_x |\Psi, Q\rangle = Q_x |\Psi, Q\rangle$. Due to Gauss’ law, the physical Hilbert space is drastically reduced to the states with $Q_x = 0$. Still, one can assign a physical meaning to the states $|\Psi, Q\rangle$ with some $Q_x \neq 0$. Those represent a system in the presence of external static charges $Q_x \in \mathbb{Z}$. As a consequence of the compact nature of the gauge group $U(1)$, the charges are quantized in integer units. The canonical quantum statistical partition function for a gauge theory is

$$Z_Q = \text{Tr}[\exp(-\beta H)P_Q] .$$

(2.8)

Here $P_Q$ is an operator that projects on the appropriate charge sector. It is interesting to investigate a lattice gauge theory in the presence of two opposite external charges $Q_x = 1, Q_y = -1$, located at different lattice sites $x$ and $y$. The potential $V(x-y)$ between the charges is then given by

$$Z_Q = \frac{\exp(-\beta V(x-y))}{V(x-y) \sim \sigma|x-y|} .$$

(2.9)

Generically, at strong gauge coupling $e$ and at low temperature (large $\beta$), lattice gauge theories with a compact gauge group (such as $U(1)$) are confining with a linearly rising charge-anti-charge potential that is characterized by the string tension $\sigma$.

(c) Quantum Spins as Building Blocks of Abelian Quantum Link Models

We now replace the analog “particle” by a quantum spin $S \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, acting in a $(2S+1)$-dimensional Hilbert space, and obeying the standard commutation relations $[S^+, S^-] = i S^z, S^z = S^ \pm \pm i S^2, [S^+, S^+] = S^+^2, [S^-, S^-] = S^-^2, [S^+, S^-] = 2 S^z$. This resembles the commutation relations $[E, U] = U, [E, U^\dagger] = -U^\dagger$ of the analog “particle”, if we identify $S^\pm$ with $E, S^+\pm$ with $U$, and $S^-$ with $U^\dagger$. However, the relation $[S^+, S^-] = 2 S^z$ does not match $[E, U^\dagger] = 0$. This is because the latter only holds in an infinite-dimensional Hilbert space. We now introduce the Hamiltonian

$$H = \frac{(S^\dagger S)^2}{2I}, \quad [H, S^\dagger] = 0 , \quad E_m = \frac{m^2}{2I} , \quad m \in \{-S, -S+1, \ldots, S-1, S\} .$$

(2.10)

For large integer spin, $S \in \mathbb{Z}$, its energy spectrum resembles the one of the “particle” Hilbert space. Interestingly, for half-odd-integer values of $S$ the quantum spin Hamiltonian provides additional opportunities which lead to theories that are inaccessible in the Wilson framework.

(d) $U(1)$ Quantum Link Models

Now we introduce an alternative approach to lattice field theory, which uses intrinsically quantum mechanical degrees of freedom — in this case $U(1)$ quantum links — which are quantum spins endowed with a gauge symmetry. Quantum spins reside in a finite-dimensional Hilbert space and are directly provided by Nature as a natural candidate for quantum hardware. Quantum spins $\frac{1}{2}$ embody the concept of a qubit. The simplest $U(1)$ quantum links are quantum spins $\frac{1}{2}$ residing on the links of a lattice. Quantum link models provide a generalization of Wilson’s lattice gauge theory. In particular, they also provide additional models that are inaccessible in the standard Wilson framework. At the same time, the Wilson theory is contained in the quantum link framework in the “classical” limit $S \rightarrow \infty$. Although quantum links can be viewed as discrete quantum variables, they naturally lead to Hamiltonians with exact continuous
local symmetry. Universality, which relies on symmetries, guarantees that the same continuum limits can be reached as in the standard Wilson framework of lattice field theory.

The Hamiltonian of a $U(1)$ quantum link model on a triangular lattice has the same form of eq. (2.5) as in the Wilson theory, but the operator algebra is modified to

$$
[E_i, E_{i'}) = 0, \quad [E_i, U_{i'}] = i \delta_{i i'} U_i, \quad [E_i, U_{i'}] = -i \delta_{i i'} U_i^+, \quad [U_i, U_{i'}] = [U_i^+, U_{i'}] = 0, \quad [U_i, U_{i'}^+] = 2 \delta_{i i'} E_i.
$$

(2.11)

Only the last commutator differs from the Wilson theory, for which $[U_i, U_{i'}^+] = 0$. This deviation has no effect on the essential commutation relation $[H, G_x] = 0$, because $G_x = \sum_i (E_{x,i+1} - E_{x,i-1})$ does not depend on $U_i$ or $U_i^+$. Consequently, we have constructed an Abelian gauge theory with exact $U(1)$ gauge symmetry from discrete quantum link variables that reside in a finite-dimensional Hilbert space.

Quantum simulator constructions for $U(1)$ quantum link models with dynamical fermions have used, for example, ultracold Bose-Fermi mixtures in optical superlattices [18], while constructions without fermions have been based on Rydberg atoms in optical lattices [20] or on superconducting quantum circuits [21]. Numerous different aspects of $(2 + 1)$-d $U(1)$ quantum link models have been investigated in [42–58].

(e) The $S = \frac{1}{2}$ Quantum Link Model on a Triangular Lattice

Let us consider the $U(1)$ quantum link model on a triangular lattice with the smallest possible 2-dimensional link Hilbert space corresponding to $S = \frac{1}{2}$. Since then $(S^2)^2 = \frac{1}{4}$, the electric field term in the Hamiltonian is a trivial constant, which can be omitted such that

$$
H = -J \sum_{\langle xy \rangle} (U_{xy} + U_{xy}^+ + U_{xy}^+ U_{xy}^2), \quad U_{xy} = U_{xy}^+ U_{xy} U_{xy}^+, \quad J = \frac{1}{2 e^2}.
$$

(2.12)

We have added a term proportional to $\lambda$. This term is analogous to the Rokhsar-Kivelson term [59] in the quantum dimer models of condensed matter physics, which are considered in the context of high-temperature superconductivity. The model of eq. (2.12) has a rich confining dynamics that is not accessible in the Wilson framework. In particular, it has “nematic” confined phases for which the discrete lattice rotation invariance is spontaneously broken [60].

By an exact duality transformation one can construct height variables associated with the hexagonal lattice that is dual to the original triangular lattice. The dual lattice consists of two sublattices $A$ and $B$. The height variables on sublattice $A$ reside at the center $\tilde{x}$ of a triangle and take values $h_{\tilde{x}}^A \in \{0, 1\}$. The height variables on sublattice $B$, on the other hand, take the values $h_{\tilde{x}}^B \in \{-\frac{1}{2}, \frac{1}{2}\}$. The electric flux connecting the sites $\tilde{x} = x + \frac{1}{3}(i - j)$ and $\tilde{x}' = x + \frac{1}{3}(i - k)$, where $j = (i - 1) \mod 3$ and $k = (i + 1) \mod 3$, is given by $E_{\tilde{x}, \tilde{x}'} = \left(h_{\tilde{x}}^A - h_{\tilde{x}'}^B \right) \mod 2 = \pm \frac{1}{2}$. This relation guarantees that the Gauss law is satisfied modulo 2. The full Gauss law results from an additional constraint on the height variables.

The phases of the model are distinguished by two sublattice order parameters

$$
M_A = \frac{2}{L^2} \sum_{\tilde{x} \in A} \left(h_{\tilde{x}}^A - \frac{1}{2}\right), \quad M_B = \frac{2}{L^2} \sum_{\tilde{x} \in B} h_{\tilde{x}}^B, \quad M_A, M_B \in [-1, 1].
$$

(2.13)

The order parameter distributions over the $(M_A, M_B)$ plane are illustrated in Fig. 2. There is a very weak first-order phase transition at $\lambda_c = -0.215(1)$. In the phase at $\lambda < \lambda_c$ both order parameters are non-zero $M_A, M_B \neq 0$, while for $\lambda > \lambda_c$ only one sublattice orders. Both phases are characterized by the spontaneous breakdown of lattice rotation invariance, and are qualitatively new “nematic” confined phases. Similarly, on the square lattice there are “crystalline” confined phases in which lattice translation invariance is spontaneously broken [43, 44]. Both on the triangular and on the square lattice, the phase transition that separates the two bulk confined phases is characterized by a ring-shaped order parameter distribution indicating an emergent,
Figure 2. [Color online] Order parameter distributions for the $U(1)$ quantum link model on the triangular lattice in the $(M_A, M_B)$ plane for $L = 64$ at $\lambda = -0.2156$ (left), $-0.2152 \approx \lambda_c$ (middle), and $-0.2146$ (right).

approximate, global $SO(2)$ symmetry, which is spontaneously broken. The corresponding dual pseudo-Goldstone boson resembles an almost massless photon. However, since $(2 + 1)$-d $U(1)$ gauge theories are always confining, the pseudo-Goldstone boson is dual to a massive “photon-ball”. Since the phase transitions are first order, one cannot take a continuum limit of these particular lattice models.

Figure 3. [Color online] Energy distribution for the strings connecting two charges $\pm 1$ at distance $r = 15\sqrt{3}$ (a), and $\pm 2$ at $r = 26$ (b), with $\lambda = -0.1 > \lambda_c$, as well as $\pm 3$ at distance $r = 15\sqrt{3}$ (c), and $\pm 2$ at $r = 26$ (d), with $\lambda = -0.3 < \lambda_c$.

The energy density of the confining strings that connect external charges $Q_x$ and $Q_y = -Q_x$ located at distant lattice sites $x$ and $y$ are illustrated in Fig.3. Remarkably, the string that connects the external charges fractionalizes into strands, each carrying fractional electric flux $\frac{1}{2}$. The strands are interfaces that separate the different bulk phases. Interestingly, the interior of the strands consists of the bulk phase that is realized on the other side of the phase transition.

The dual height representation has been used to implement the square lattice $U(1)$ quantum link model on a configurable arrays of Rydberg atoms [51]. As illustrated in Fig.4, for the model on the triangular lattice, the use of the dual height variables gives rise to a particularly resource efficient encoding in a quantum circuit [60]. In this way the real-time dynamics of the confining strings is accessible to quantum simulations on near-term devices.

(f) D-Theory: Continuum Physics from Dimensional Reduction

The $(2 + 1)$-d $U(1)$ quantum link models discussed before have first order phase transitions and thus do not give rise to a continuum limit. In the Wilson theory with its infinite-dimensional link Hilbert space, on the other hand, a continuum limit is obtained at a second order phase transition that is reached in the weak coupling limit $\epsilon \to 0$. Polyakov was first to argue that $U(1)$ gauge theories in three space-time dimensions confine at all values of the gauge coupling [61]. This is due to the proliferation of magnetic monopoles, which are instantaneous events in 3-d space-time that play the role of instantons. Göpfert and Mack [62] proved rigorously that the correlation
length, which represents the inverse mass of a confined “photon-ball”, diverges as $\xi \sim \exp(c/e^2)$ in the weak coupling limit, thus showing that confinement persists at all couplings. Taking the continuum limit in the Wilson theory may not be the most practical approach when quantum simulations or computations shall be employed in order to investigate the real-time evolution.

Quantum link models approach the continuum limit in their own way, namely by the dimensional reduction of discrete variables, which is natural in the D-theory framework. In order to understand this way of taking the continuum limit, we start out with the theory in one more spatial dimension. Hence, we consider the $U(1)$ quantum link model on a 3-d spatial lattice. In a 4-d space-time, monopoles are no longer event-like but represent particles that travel along their worldlines. When monopoles condense, they lead to confinement (just as in the lower-dimensional theory). However, in a 4-d space-time $U(1)$ gauge theories also possess Coulomb phases with a massless unconfined photon. In the Wilson formulation of $U(1)$ gauge theory on a 4-d space-time lattice, the Coulomb phase is separated from the confined phase by a weak first order quantum phase transition in the bare gauge coupling $e$.

For concreteness, let us consider the $U(1)$ quantum link model on a 3-d cubic spatial lattice. It is plausible that this model exists in a $(3 + 1)$-d Coulomb phase even when it is realized in the extreme quantum limit with quantum spins $\frac{1}{2}$ on each link [63]. A Coulomb phase is characterized by an infinite correlation length $\xi = \infty$ associated with the massless photon. It is interesting to ask what happens when one compactifies one of the spatial dimensions to a finite extent $L'$. If $\xi$ would remain infinite, the Coulomb phase would persist even in $(2 + 1)$-d. However, $(2 + 1)$-d $U(1)$ gauge theories are known to be always confining. The effective gauge coupling $e'$ of the dimensionally reduced $(2 + 1)$-d theory is related to the gauge coupling $e$ of the $(3 + 1)$-d theory by $1/e'^2 = L'/e^2$, which implies $\xi \sim \exp(c/e^2) = \exp(cL'/e^2) \gg L'$. Interestingly, with increasing extent $L'$ of the finite spatial dimension, the correlation length $\xi$ increases exponentially, and becomes much larger than $L'$ itself. As a result, the theory undergoes dimensional reduction from $(3 + 1)$-d to $(2 + 1)$-d. The dimensional reduction of discrete variables is characteristic of D-theory, which provides a natural way to take the continuum limit in quantum link models. Unlike in the Wilson framework, where one tunes the value of the coupling $e$, in D-theory one just moderately increases the extent of an extra dimension. In practice, the extent of the extra dimension is just a few lattice spacings, because the correlation length $\xi$ responds exponentially to $L'$. In this way, one piles up discrete quantum link variables in an extra dimension, in order to provide the minimal number of degrees of freedom that are necessary to approach the continuum limit in a resource efficient manner.

3. Non-Abelian Hamiltonian Lattice Gauge Theories

Non-Abelian gauge theories play a central role in the standard model of particle physics. In particular, the strong interaction between quarks is mediated by the $SU(3)$ gluon gauge field.
of QCD. Non-Abelian gauge theories are also important in quantum information science, in particular, in the context of topological quantum computation, which is based on \((2 + 1)\)-d Chern-Simons gauge theories [64]. In this section we discuss non-Abelian lattice gauge theories in the Hamiltonian formulation, first with Wilson’s lattice gauge theory and then using quantum link models. Again, via the dimensional reduction of discrete variables, D-theory offers a natural way of approaching the continuum limit.

(a) Analog “Particles” Moving in the Group Manifold \(SU(2) = S^3\)

Let us consider the quantum mechanical analog “particle” for a Wilson-type parallel transporter in an \(SU(2)\) lattice gauge theory. The corresponding group manifold is the sphere \(S^3\). Consequently, the position of the analog “particle” is described by an \(SU(2)\) matrix

\[
U = \cos \alpha + i \sin \alpha \hat{e}_\alpha \cdot \hat{\sigma}, \quad \hat{e}_\alpha = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).
\]

Since \(SU(2)\) is non-Abelian, we distinguish transformations that multiply \(U\) from the left and from the right. The corresponding \(SU(2)_L \times SU(2)_R\) algebra is generated by

\[
L = \frac{1}{2}(J - \vec{K}), \quad \bar{R} = \frac{1}{2}(J + \vec{K}), \quad J_L = \exp(\pm i\varphi) (\pm \partial_\theta + i \cot \theta \partial_\varphi), \quad J_3 = -i\partial_\varphi,
\]

\[
K_\pm = \exp(\pm i\varphi) \left(i \sin \theta \partial_\alpha + i \cot \alpha \cos \theta \partial_\beta \mp \frac{\cot \alpha}{\sin \theta} \partial_\varphi \right), \quad K_3 = i(\cos \theta \partial_\alpha - \cot \alpha \sin \theta \partial_\beta),
\]

which obey the commutation relations \([\bar{R}, U] = U\hat{\sigma}, [\bar{L}, U] = -\hat{\sigma} U\). The kinetic energy of the analog “particle” corresponds to the Laplace-Beltrami operator (the Laplacian) of the group manifold, which together with its energy spectrum takes the form

\[
T = \frac{1}{2I} \left(J^2 + \vec{K}^2\right) = \frac{1}{I} \left(\bar{R}^2 + \vec{L}^2\right), \quad E_i = \frac{j_L(j_L + 1) + j_R(j_R + 1)}{I} = \frac{l(l + 2)}{2I}.
\]

Here \(j_L = j_R\) with \(l = j_L + j_R \in \{0, 1, 2, \ldots\}\) and each state is \((2j_L + 1)(2j_R + 1) = (l + 1)^2\)-fold degenerate. Since the number of eigenstates is infinite, the corresponding Hilbert space is again infinite-dimensional.

Just as in the Abelian case, we again consider three “particles” moving in the group manifold, associating the “particles” with the links 12, 23, and 31 of a triangular plaquette. The corresponding Hamiltonian now takes the form

\[
H = T_{12} + T_{23} + T_{31} + V_{123}\]

\[
= \varepsilon^2 \left(\tilde{R}_{12}^2 + \tilde{R}_{23}^2 + \tilde{R}_{31}^2 + \tilde{L}_{12}^2 + \tilde{L}_{23}^2 + \tilde{L}_{31} + \bar{R}_{31}^2 + \bar{L}_{31} + \bar{L}_{12} + \bar{L}_{23} + \bar{L}_{31}\right) - \frac{1}{4\varepsilon^2} \text{Tr}(U_{12} U_{23} U_{31} + U_{31}^\dagger U_{23}^\dagger U_{12}^\dagger).
\]

The Hamiltonian commutes with the three infinitesimal gauge generators

\[
G_1 = \bar{L}_{12} + \bar{R}_{31}, \quad G_2 = \bar{L}_{23} + \bar{R}_{12}, \quad G_3 = \bar{L}_{31} + \bar{R}_{23}, \quad [H, G_1] = [H, G_2] = [H, G_3] = 0.
\]

Again, by introducing an entire triangular lattice with an analog “particle” associated with each link, we now construct an \(SU(2)\) lattice gauge theory

\[
H = \varepsilon^2 \sum_{(xy)} \left(\tilde{R}_{xy}^2 + \tilde{L}_{xy}^2\right) - \frac{1}{4\varepsilon^2} \sum_{(xyz)} \text{Tr}(U_{12} U_{23} U_{31} + U_{31}^\dagger U_{23}^\dagger U_{12}^\dagger).
\]

In this case, the gauge generators associated with the lattice sites \(x\) obey

\[
G_x = \sum_i (\tilde{L}_{x,x+i} + \tilde{R}_{x-i,x}), \quad [H, G_x] = 0, \quad [G_x^a, G_y^b] = i\delta_{xy} e_{abc} G_z^c.
\]

The non-Abelian Gauss law takes the form \(G_x^a|\Psi\rangle = 0\). Local violations of Gauss’ law manifest themselves as external static non-Abelian gauge charges, which are characterized by an \(SU(2)\) representation \(Q \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\\}\) and one of the \(2Q + 1\) corresponding values \(Q^3 \in \{-Q, -Q + 1, \ldots, Q - 1, Q\}\), such that \(G_x^{2l}|\Psi, Q, Q^3\rangle = Q(Q + 1)|\Psi, Q, Q^3\rangle, G_x^{2l+1}|\Psi, Q, Q^3\rangle = Q^3|\Psi, Q, Q^3\rangle\).
(b) Quantum Links as Building Blocks of Non-Abelian Gauge Theories

We will now replace Wilson-type parallel transporters by non-Abelian quantum links, in order to be able to address the gauge dynamics in a finite-dimensional Hilbert space per link. A Wilson-type parallel transporter \( U \) is a matrix that takes values in the gauge group. Obviously, its matrix elements \( U^{ij} \in \mathbb{C} \) commute with each other, \( [U^{ij}, U^{kl}] = 0 \). When one insists (unnecessarily) on this property, the commutation relations can only be realized in an infinite-dimensional Hilbert space. Just like Wilson-type parallel transporters, non-Abelian quantum links are matrices. However, their matrix elements \( U^{ij} \) are non-commuting operators, such that \( [U^{ij}, U^{kl}] \neq 0 \).

In a non-Abelian quantum link model, there is a link-based embedding algebra which contains the quantum link \( U \) as well as the generators of gauge transformations \( L^a \) and \( R^a \) associated with the left and right end of a link. In addition, there may be an Abelian generator \( E \). These generators obey the same commutation relations as in the Wilson theory

\[
\begin{align*}
[L^a, L^b] &= i f_{abc} L^c, \quad [R^a, R^b] = i f_{abc} R^c, \quad [L^a, R^b] = [L^a, E] = [R^a, E] = 0, \quad [L^a, U] = -\lambda^a U, \quad [R^a, U] = U \lambda^a, \quad [E, U] = U,
\end{align*}
\]

except that the elements of a quantum link matrix do not commute. Here the \( \lambda^a \) are generators of the gauge Lie algebra which satisfy \( [\lambda^a, \lambda^b] = i f_{abc} \lambda^c \). Depending on the gauge group, the commutation relations \( [U^{ij}, U^{kl}] \neq 0 \) are such that the corresponding embedding algebra closes.

In a \( U(N) \) or \( SU(N) \) gauge theory, an \( N \times N \) quantum link matrix is built from \( 2N^2 \) Hermitian operators, which replace the real and imaginary parts of the complex-valued matrix elements \( U^{ij} \in \mathbb{C} \) of a parallel transporter in the Wilson framework [65]. Together with the \( 2(N^2 - 1) \) generators \( L^a \) and \( R^a \), and an Abelian generator \( E \), for a \( U(N) \) gauge theory this yields

\[
U(N): U^{ij}, L^a, R^a, E, 2N^2 + 2(N^2 - 1) + 1 = 4N^2 - 1 \quad SU(2N) \quad \text{generators}.
\]

For a \( U(1) \) gauge theory the embedding algebra \( SU(2N) \) reduces to \( SU(2) \). For an \( SO(N) \) or more precisely \( \text{Spin}(N) \) gauge group, the quantum link has \( N^2 \) elements that replace the real-valued matrix elements \( U^{ij} \in \mathbb{R} \) of the Wilson theory [66]. In addition, there are \( 2 \left( \frac{N(N-1)}{2} \right) \) generators \( L^a \) and \( R^a \), leading to the embedding algebra \( SO(2N) \)

\[
SO(N): U^{ij}, L^a, R^a, N^2 + 2 \left( \frac{N(N-1)}{2} \right) = N(2N-1) \quad SO(2N) \quad \text{generators}.
\]

Finally for the gauge group \( Sp(N) \) there are \( 4N^2 \) Hermitian operators describing the \( 2N \times 2N \) quantum link matrix and \( 2N(2N+1) \) generators \( L^a \) and \( R^a \), leading to the embedding algebra \( Sp(2N) \)

\[
Sp(N): U^{ij}, L^a, R^a, 4N^2 + 2N(2N+1) = 2N(4N+1) \quad Sp(2N) \quad \text{generators}.
\]

Since \( SU(2) = SO(3) = Sp(1) \), an \( SU(2) \) gauge theory can be realized with the embedding algebras \( SU(4) = SO(6) \) or \( Sp(2) = SO(5) \), leading to the simplest \( SU(2) \) quantum link model.

(c) The \( SU(2) \) Quantum Link Model on a Honeycomb Lattice

As a simple example of a non-Abelian quantum link model, let us consider the \( SU(2) \) quantum link model on a honeycomb lattice [67]. The simplest representations of the embedding algebra are the 5-d vector and the 4-d spinor representation of \( SO(5) \), whose weight diagrams are illustrated in Fig.5. Under the \( SU(2)_L \times SU(2)_R \) gauge transformations at the left and right end of a link, they decompose as

\[
\{5\} = \{1,1\} + \{2,2\}, \quad \{4\} = \{2,1\} + \{1,2\}.
\]

The vector representation \( \{5\} \) transforms trivially under the center \( \mathbb{Z}(2) \) of the universal covering group \( \text{Spin}(5) \) of the embedding algebra \( SO(5) \). It contains the state \( \{1,1\} \) which corresponds to vanishing flux, i.e. \( j_L = j_R = 0 \), as well as four states \( \{2,2\} \) with \( j_L = j_R = \frac{1}{2} \). This resembles a truncation of the Wilson theory, which is characterized by \( j_L = j_R \in \{0, \frac{1}{2}, 1, 2, \ldots\} \).
to a 5-d Hilbert space per link. A corresponding flux configuration connecting two external charges \( Q = \frac{3}{2} \) is illustrated in Fig. 6. The spinor representation \( \{4\} \), on the other hand, transforms non-trivially under the center \( \mathbb{Z}(2) \) and is characterized by \( (j_L, j_R) = (\frac{1}{2}, 0) \) or \( (0, \frac{1}{2}) \). Since now \( j_L \neq j_R \), this model is qualitatively different from the Wilson theory. There are two ways of satisfying Gauss’ law at a lattice site, which are again illustrated in Fig. 5. A flux configuration connecting two external charges \( Q = \frac{3}{2} \) that satisfies the Gauss law is illustrated in Fig. 6. The triangular lattice that is dual to the original hexagonal lattice can be divided into four sublattices with corresponding height variables that take values \( \pm 1 \). It is straightforward to extend the construction of a quantum circuit to the \( SU(2) \) quantum link model and to study the corresponding non-Abelian string dynamics on a chip.

(d) \( (1 + 1) \)-d \( CP(N - 1) \) Model from Dimensional Reduction of a \( (2 + 1) \)-d \( SU(N) \) Quantum Spin Ladder

Let us now consider the D-theory approach to the asymptotically free \( (1 + 1) \)-d \( CP(N - 1) \) models [68,69], which result from the dimensional reduction of discrete \( SU(N) \) quantum spin variables [70]. Although these models have only a global \( SU(N) \) symmetry, they share many features with non-Abelian gauge theories. In particular, they are asymptotically free, have a non-perturbatively generated mass gap, as well as a topological charge and \( \theta \)-vacuum states.

Aiming at the \( (1 + 1) \)-d \( CP(2) \) model, let us now consider a 2-d bipartite square lattice of short even extent \( L' \) in the 2-direction with open boundary conditions, as illustrated in Fig. 7. In the continuum limit, the 2-direction will disappear via dimensional reduction, while the 1-direction remains as the physical spatial dimension. We install \( SU(3) \) triplet quantum spins \( T^a_2 \) on the even
A site and anti-triplet spins \(-T^y_a\) on the odd \(B\) sites, in order to realize an anti-ferromagnetic spin ladder Hamiltonian that commutes with the total \(SU(3)\) spin \(T^a\)

\[
H = -J \sum_{\langle xy \rangle} T^a_x T^a_y, \quad \langle T^a_x, T^b_y \rangle = i \delta_{abc} f_{abc} T^c_x, \quad T^a = \sum_{x \in A} T^a_x - \sum_{y \in B} T^a_y, \quad [H, T^a] = 0. \tag{3.13}
\]

We couple a chemical potential to the conserved non-Abelian \(SU(3)\) charge \(T^a\) and obtain the grand canonical partition function \(Z = \text{Tr} \exp(-\beta[H - \mu_3 T^3 - \mu_8 T^8])\).

Let us first consider the system at zero temperature, \(\beta \to \infty\), in the infinite-volume limit, \(L, L' \to \infty\). It turns out that the \(SU(3)\) symmetry then breaks spontaneously to \(U(2)\) \cite{71}, thus leading to \(8 - 4 = 4\) massless Goldstone bosons, whose low-energy dynamics are described by an effective field theory in terms of \(3 \times 3\) matrix fields \(P(x)\) which take values in the coset space \(SU(3)/U(2) = \mathbb{CP}(2)\), i.e. \(P(x)^\dagger = P(x), \quad P(x)^2 = P(x), \quad \text{Tr}(P(x)) = 1\). When we make the extent \(L'\) of the 2-direction finite, the Mermin-Wagner theorem implies that the continuous global \(SU(3)\) symmetry can no longer break spontaneously. Hence, the Goldstone bosons pick up an exponentially small mass \(m = 1/(\xi c)\). Their low-energy effective action takes the form

\[
S[P] = \int_0^\beta d\beta \int_0^L dx_1 \int_0^{L'} dx_2 \rho_s \text{Tr} \left( \partial_\beta P \partial_\beta P + \frac{1}{c^2} D_x P D_x P \right), \quad D_x P = \partial_\beta P - \mu_a [T^a, P]. \tag{3.14}
\]

Here \(\rho_s\) is the spin stiffness and \(c\) is the spinwave velocity. When \(\xi \gg L'\) the field becomes \(x_2\)-independent and the system undergoes dimensional reduction from \((2 + 1)\)-d to \((1 + 1)\)-d with the dimensionless coupling constant \(1/g^2 = L' \rho_s/c\), thus leading to the \(\mathbb{CP}(2)\) model action

\[
S[P] = \int_0^\beta dx_3 \int_0^L dx_1 \frac{1}{g^2} \text{Tr} \left( \partial_\beta P \partial_\beta P + \frac{1}{c^2} D_x P D_x P \right). \tag{3.15}
\]

Due to asymptotic freedom of the \((1 + 1)\)-d \(\mathbb{CP}(2)\) model, the correlation length is exponentially large in \(1/g^2\), i.e. \(\xi \sim \exp(4\pi/3g^2) = \xi \sim \exp(4\pi L' \rho_s/3c)\) (here \(4\pi/\beta\) is the 1-loop coefficient of the \(\beta\)-function). This justifies the assumption that \(\xi \gg L'\) already for moderately large values of \(L'\). Dimensional reduction hence results as a consequence of asymptotic freedom.

The unconventional \((2 + 1)\)-d \(SU(N)\) quantum spin ladder regularization of the \((1 + 1)\)-d \(\mathbb{CP}(N - 1)\) model makes their real-time dynamics accessible to quantum simulation experiments using ultracold alkaline-earth atoms \(^{87}\text{Sr}\) or \(^{173}\text{Yb}\) in optical lattices \cite{72}. Using a worm algorithm in Monte Carlo simulations on a classical computer the phase diagram of the model has been computed as a function of the chemical potentials \(\mu_3\) and \(\mu_8\) (cf. Fig.7) \cite{73}. There are phases...
in which the massive bosons undergo single- or double-species Bose-Einstein condensation, the latter with ferromagnetism. It would be most interesting to perform quantum simulation experiments of the corresponding “condensed matter physics” of the $CP(2)$ model.

(e) D-Theory: Continuum QCD from Dimensional Reduction

Just like Abelian gauge fields in $(3+1)$-d, in $(4+1)$-d non-Abelian $SU(N)$ gauge fields can exist in a Coulomb phase with massless gauge bosons and hence with an infinite correlation length $\xi$. The corresponding low-energy effective theory is a $(4+1)$-d Yang-Mills theory with the action

$$S[G_{\mu}] = \int dt d^3 x \left[ L' \right] \frac{1}{2e^2} \text{Tr} \left( G_{\mu \nu} G_{\mu \nu} + \frac{1}{c^2} G_{\mu \nu} G_{\mu \nu} \right), \quad \mu, \nu \in \{1, 2, 3, 4\}.$$  \hspace{1cm} (3.16)

When the extent $L'$ of the extra dimension becomes finite, due to confinement in $(3+1)$-d, $\xi$ cannot remain infinite. Assuming that $\xi \gg L'$, the theory undergoes dimensional reduction from 4 to 4 dimensions such that

$$S[G_{\mu}] \rightarrow \int dt d^3 x \frac{1}{2e^2} \text{Tr} \left( G_{ij} G_{ij} + \frac{1}{c^2} G_{\mu \nu} G_{\mu \nu} \right), \quad i, j \in \{1, 2, 3\}, \quad \frac{1}{g^2} = \frac{L'}{c^2}, \quad \frac{1}{m} \sim \exp \left( \frac{24\pi^2 L'}{11Nc^2} \right).$$  \hspace{1cm} (3.17)

Again, the extent $L'$ of the extra dimension determines the asymptotically free dimensionless gauge coupling $g$ (here the 1-loop coefficient of the $\beta$-function is $24\pi^2/11N$). Indeed $\xi \gg L'$ because, due to asymptotic freedom, $\xi$ increases exponentially with $L'$ [41].

It is very natural to incorporate Shamir’s variant [74] of Kaplan’s domain wall fermions [75] in this $(4+1)$-d setup, which can be regularized in the D-theory framework with $SU(N)$ quantum links [65]. The corresponding Hamiltonian is given by

$$H = e^2 \sum_{x, \mu} \left( R_{(x, \mu)}^2 + (L_{x, \mu}^2) \right) - \frac{1}{2Nc^2} \sum_{x, \mu \neq \nu} \text{Tr}[U_{x, \mu} U_{x+\hat{\mu}, \nu} U_{x+\hat{\mu}, \nu} U_{x, \nu}]$$

$$- J' \sum_{x, \mu} \left( \det U_{x, \mu} + \det U_{x, \mu} \right) + \frac{1}{2} \sum_{x, \mu} \left[ \Psi_{x+\hat{\mu}}^\dagger \gamma_0 \Psi_{x+\hat{\mu}} - \Psi_{x+\hat{\mu}}^\dagger \gamma_0 \Psi_{x+\hat{\mu}} \right]$$

$$+ M \sum_{x, \mu} \Psi_{x+\hat{\mu}}^\dagger \Psi_{x} + \frac{r}{2} \sum_{x, \mu} \left[ 2\Psi_{x+\hat{\mu}}^\dagger \Psi_{x} - \Psi_{x+\hat{\mu}}^\dagger \Psi_{x} - \Psi_{x+\hat{\mu}}^\dagger \gamma_0 U_{x, \mu} U_{x, \nu} \Psi_{x} \right].$$

At finite extent $L'$, the domain wall fermions have a residual mass $\mu = 2M \exp(-ML')$. For a sufficiently large domain wall mass $M > \frac{24\pi^2}{(11N)c^2}$, the theory reaches the chiral limit together with the continuum limit. The continuum limit is controlled by $\frac{1}{m} \propto \exp \left( \frac{24\pi^2}{(11N)c^2} \right)$ in the presence of $N_f$ flavors of quarks, which are described by the anti-commuting fermion creation and annihilation operators $\Psi_0^\dagger$ and $\Psi_0$. Ultracold alkaline-earth atoms in an optical superlattice can again be used to embody $SU(N)$ quantum links in ultracold matter [19]. This provides a concrete vision for how to ultimately quantum simulate QCD [76].

4. Conclusion

D-theory applied to quantum link models provides a formulation of gauge theories that allows their resource efficient implementation in quantum simulators or quantum computers. The continuum limit is reached naturally (i.e. without fine-tuning) by a moderate increase of the size of an extra spatial dimension. In the near future, close collaborations between theorists and experimentalists hold the promise to realize many different aspects of strongly coupled gauge theories. Even if one works in a lower-dimensional space-time, with a smaller gauge group, a reduced matter content, or away from the continuum limit, once quantum simulations of gauge theories are realized experimentally, they become a very exciting subject in their own right. Exploring their real-time or finite-density dynamics, even at the qualitative level to which one will be limited without systematic error correction, is most interesting along the way towards ultimately quantum simulating QCD [76].
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