Research Article

Computing Minimal Doubly Resolving Sets and the Strong Metric Dimension of the Layer Sun Graph and the Line Graph of the Layer Sun Graph

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Let $G$ be a finite, connected graph of order of, at least, 2 with vertex set $V(G)$ and edge set $E(G)$. A set $S$ of vertices of the graph $G$ is a doubly resolving set of $G$ if every two distinct vertices of $G$ are doubly resolved by some two vertices of $S$. The minimal doubly resolving set of vertices of graph $G$ is a doubly resolving set with minimum cardinality and is denoted by $\psi(G)$. In this paper, first, we construct a class of graphs of order $2n + \sum_{i=1}^{k-1}mn'$, denoted by $LSG(n, m, k)$, and call these graphs as the layer Sun graphs with parameters $n, m,$ and $k$. Moreover, we compute minimal doubly resolving sets and the strong metric dimension of the layer Sun graph $LSG(n, m, k)$ and the line graph of the layer Sun graph $LSG(n, m, k)$.

1. Introduction

In this paper, suppose $G$ is a finite, simple connected graph of order of, at least, 2, with vertex set $V(G)$ and edge set $E(G)$. If $x$ and $y$ are vertices in the graph $G$, then the distance $x$ from $y$ in $G$ is denoted by $d_G(x, y)$ or simply $d(x, y)$, where $d(x, y)$ is the length of the shortest path from $x$ to $y$. The line graph of a graph $G$ is denoted by $L(G)$, with vertex set $V(L(G)) = E(G)$ and where two edges of $G$ are adjacent in $L(G)$ if and only if they are incident in $G$, see [1]. Vertices $x, y$ of the graph $G$ are said to doubly resolve vertices $u, v$ of $G$ if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A set $S$ of vertices of the graph $G$ is a doubly resolving set of $G$ if every two distinct vertices of $G$ are doubly resolved by some two vertices of $S$. The minimal doubly resolving set of vertices of graph $G$ is a doubly resolving set with minimum cardinality and is denoted by $\psi(G)$. In this paper, we introduce the notion of a strong metric resolving set and a strong metric dimension problem set of vertices of the graph $G$ was introduced by A. Sebő and E. Tannier [3] and further investigated by O. R. Oellermann and Peters-Fransen [4]. The minimal doubly resolving sets for jellyfish and cocktail party graphs have been obtained in [5]. For more results related to these concepts, see [6–16]. In this paper, first, we construct a class of graphs of order $2n + \sum_{i=1}^{k-1}mn'$, denoted by $LSG(n, m, k)$, and call these graphs as the layer Sun graphs with parameters $n, m,$ and $k$, which is defined as follows:

Let $n, m, k$ be integers such that $n, k \geq 3, m \geq 2$ and $G$ be a graph with vertex set $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$, where $V_1, V_2, \ldots, V_k$ are called the layers of $G$ such that $V_1 = V(C_n) = \{1, 2, \ldots, n\}, V_2 = \{v_1, v_2, \ldots, v_m\}$, and for $l \geq 3$, we have $V_l = \{B_{l, 0}^{(l)}, B_{l, 1}^{(l)}, \ldots, B_{l, m_l-1}^{(l)}, B_{l, m_l}^{(l)}, \ldots, B_{l, m_l}^{(l)}\}$, and let $B_{ij}^{(l)} = \bigcup_{r=1}^{m_l} (v_{ij}, t)^l$, such that every $(v_{ij}, t)^l$ is a vertex in the layer $V_l$, and $B_{ij}^{(l)} \equiv$
Proposition 1. Let $G$ be a graph. It is well known that a doubly resolving set is also a resolving set and $\beta(G) \leq \psi(G)$. In particular, every strong resolving set is a resolving set and $\beta(G) \leq \text{sdim}(G)$.

3. Main Results

3.1. Minimal Doubly Resolving Sets and the Strong Metric Dimension for the Layer Sun Graph $LSG(n, m, k)$

Theorem 1. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that $n, m, k$ are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the metric dimension of $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$.

Proof. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$, where $V_1, V_2, \ldots, V_k$ are the layers of vertices in the layer Sun graph $LSG(n, m, k)$, which is defined already. It is clear that if $W$ is an ordered subset of the layers $V_1 \cup V_2 \cup \ldots \cup V_{k-1}$, then $W$ is not a resolving set in $LSG(n, m, k)$. We may assume that the layer $V_k$ is equal to

$$V_k = \left\{ B_{11}^{(k)}, B_{12}^{(k)}, \ldots, B_{1m_{k-3}}^{(k)}; B_{21}^{(k)}, B_{22}^{(k)}, \ldots, B_{2m_{k-3}}^{(k)}; \ldots; B_{n1}^{(k)}, B_{n2}^{(k)}, \ldots, B_{nm_{k-3}}^{(k)} \right\},$$

where $B_{ij}^{(k)} = \{ \bigcup_{i=1}^{m} (v_{ij}, t) \}$, $1 \leq i \leq n$, $1 \leq j \leq m^{k-3}$. In the following cases, it can be shown that the metric dimension of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$.

Case 1: let $W$ be an ordered subset of the layer $V_k$ in the layer Sun graph $LSG(n, m, k)$ such that

$$W = \left\{ B_{11}^{(k)}, \ldots, B_{1m_{k-3}}^{(k)}, B_{21}^{(k)}, B_{22}^{(k)}, \ldots, B_{2m_{k-3}}^{(k)}; \ldots; B_{n1}^{(k)}, B_{n2}^{(k)}, \ldots, B_{nm_{k-3}}^{(k)} \right\}.$$ 

Hence,

$$V(G) - W = \left\{ V_1, V_2, \ldots, V_{k-1}, B_{11}^{(k)} \right\}.$$

3.2. Complexity

We know that the cardinality of $W$ is $nm^{k-2} - m$ because $|B_{ij}^{(k)}| = m^{k-2} - 1$, $1 \leq i \leq n$, $1 \leq j \leq m^{k-3}$. Therefore, the metric representation of all the vertices $(v_{ij}, 1)^k$, $(v_{ij}, 2)^k$, \ldots, $(v_{ij}, m)^k$ in the component $B_{ij}^{(k)}$ is the same as $nm^{k-2} - m$-vector with respect to $W$. Thus, $W$ is not a resolving set in $LSG(n, m, k)$.

Case 2: let $W$ be an ordered subset of the layer $V_k$ in the layer Sun graph $LSG(n, m, k)$ such that

$$W = \left\{ B_{11}^{(k)} - \{ (v_{ij}, 1)^k, (v_{ij}, 2)^k \}; B_{21}^{(k)}, B_{22}^{(k)}, \ldots, B_{2m_{k-3}}^{(k)}; \ldots; B_{n1}^{(k)}, B_{n2}^{(k)}, \ldots, B_{nm_{k-3}}^{(k)} \right\}.$$ 

Hence,

$$V(G) - W = \left\{ V_1, V_2, \ldots, V_{k-1}, (v_{ij}, 1)^k, (v_{ij}, 2)^k \right\}.$$
We know that $|W| = nm^{k-2} - 2$. Therefore, the metric representation of two vertices $(v_1, 1)^k$, $(v_2, 2)^k$ in the component $B^{(k)}_{t_i}$ is the same as $nm^{k-2} - 2$-vector with respect to $W$. Thus, $W$ is not a resolving set in $LSG(n, m, k)$.

Case 3: let $W$ be an ordered subset of the layer $V_k$ in the layer Sun graph $LSG(n, m, k)$ such that

$$W = \left\{ B^{(k)}_{t_1} - (v_1, 1)^k, B^{(k)}_{t_2} - (v_2, 1)^k, \ldots, B^{(k)}_{t_{n-3}} - (v_{n-3}, 1)^k \right\}.$$

$$= \left\{ (v_1, 1)^k, (v_2, 1)^k, \ldots, (v_{n-3}, 1)^k \right\}.$$

Hence,

$$V(G) - W = \left\{ V_1, V_2, \ldots, V_{k-1}, (v_1, 1)^k, \ldots, (v_{n-3}, 1)^k \right\}.$$

We know that $|W| = nm^{k-2} - nm^{k-3}$. We can show that all the vertices in $V(G) - W$ have different representations with respect to $W$. Let $u$ be the vertex of the layer $V_1 = V(C_n) = \{1, 2, \ldots, n\}$. We can assume without loss of generality that $u = i$, $1 \leq i \leq n$. Hence, $d(u, (v_j, t)^k) = k - 1$. Now, let $u \in V_j = \{v_1, v_2, \ldots, v_n\}$. We can assume without loss of generality that $u = v_i$, $1 \leq i \leq n$. Hence, $d(u, (v_j, t)^k) = k - 2$, where $(v_j, t)^k \in B^{(k)}_{t_i}$, $1 \leq t \leq m$, $1 \leq j \leq m^{k-3}$; otherwise, if $u \neq v_i$, then $d(u, (v_j, t)^k) = k - 2$. In a similar way, we can show that all the vertices in the layers $V_3, \ldots, V_{k-1}$ have different representations with respect to $W$. In particular, for every vertex $u \in \left\{ (v_1, 1)^k, \ldots, (v_{n-3}, 1)^k \right\}$, we have $d(u, (v_j, t)^k) = 2$, $1 \leq t \leq m$, if $u = (v_j, 1)^k$; otherwise, if $u \in \left\{ (v_1, 1)^k, \ldots, (v_{n-3}, 1)^k \right\}$ and $u \neq (v_j, 1)^k$, then $d(u, (v_j, t)^k) > 4$. Therefore, all the vertices in $V(G) - W$ have different representations with respect to $W$. This implies that $W$ is a resolving set in $LSG(n, m, k)$. From the above-mentioned cases, we can be concluded that the minimum possible cardinality of a resolving set in $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$.

**Theorem 2.** Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that $n, m, k$ are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the cardinality of minimum doubly resolving set of the $LSG(n, m, k)$ is $nm^{k-2}$.

**Proof.** In the following cases, it can be shown that the cardinality of minimum doubly resolving set of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2}$.

Case 1: we know that the ordered subset $W$ of vertices in equation (8) in the layer $V_k$ of $LSG(n, m, k)$ is a resolving set for $LSG(n, m, k)$ of cardinality $nm^{k-2} - nm^{k-3}$. We show that this subset is not a doubly resolving set for
LSG$((n, m, k))$. Because if the vertex $u = (v_1, 1)^k \in B^{(k)}_l \in V_{k^l}, 1 \leq i \leq n, 1 \leq j \leq m^{k^l}$ is adjacent to a vertex $v \in V_{k^l-1}$, then for every $x, y \in W$, we have $d((u, x)) - d(u, y) = d((v, x)) - d((v, y))$.

$$W = \left\{ B^{(k)}_1, (v_1, 1)^k, B^{(k)}_1, (v_1, 1)^k, \ldots, B^{(k)}_1, (v_1, 1)^k, B^{(k)}_2, (v_2, 1)^k, \ldots, B^{(k)}_2, (v_2, 1)^k, \ldots, B^{(k)}_{m^k}, (v_{m^k}, 1)^k \right\}$$

(8)

Case 2: now, let the subset of vertices in LSG$((n, m, k))$ be

$$W = \left\{ B^{(k)}_1, (v_1, 1)^k, B^{(k)}_1, (v_1, 1)^k, \ldots, B^{(k)}_1, (v_1, 1)^k, B^{(k)}_2, (v_2, 1)^k, \ldots, B^{(k)}_2, (v_2, 1)^k, \ldots, B^{(k)}_{m^k}, (v_{m^k}, 1)^k \right\}$$

(9)

In a similar fashion as in Case 3 of Theorem 1, we can show that all the vertices in the layers $V_{1}, V_{2}, \ldots, V_{k-1}$ of LSG$((n, m, k))$ and the vertex $(v_1, 1)^k$ in the layer $V_k$ of LSG$((n, m, k))$ have different representations with respect to $W$. So, $W$ is a resolving set in LSG$((n, m, k))$ of cardinality $nm^{k-2} - 1$. Note that, in this case, by a similar way as in Case 1, we can show that this subset is not a doubly resolving set for LSG$((n, m, k))$.

Case 3: finally, let the subset of vertices in LSG$((n, m, k))$ be

$$W = \left\{ B^{(k)}_1, B^{(k)}_2, \ldots, B^{(k)}_{m^k}, \ldots, B^{(k)}_n, B^{(k)}_n, \ldots, B^{(k)}_{m^k} \right\}$$

(10)

In a similar fashion as in Theorem 1, we can show that all the vertices in the layers $V_{1}, V_{2}, \ldots, V_{k-1}$ of LSG$((n, m, k))$ have different representations with respect to $W$. So, this subset is also a resolving set in LSG$((n, m, k))$ of cardinality $nm^{k-2} - 1$. We show that this subset is a doubly resolving set for LSG$((n, m, k))$. It is sufficient to prove that for two vertices $u$ and $v$ in LSG$((n, m, k))$, there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Consider two vertices $u$ and $v$ in LSG$((n, m, k))$. Then, we have the following.

Case 3.1: suppose that both vertices $u$ and $v$ lie in the layer $V_{l}$. Hence, there are $r, s \in \{1, 2, \ldots, n\}$ such that $u = (v_r, 1)^k \in B^{(k)}_l \in V_{k^l}, 1 \leq i \leq n, 1 \leq j \leq m^{k^l}$ is adjacent to a vertex $v \in V_{k^l-1}$, then for every $x, y \in W$, we have $d((u, x)) - d(u, y) = d((v, x)) - d((v, y))$.

$$d((u, x)) - d(u, y) = d((v, x)) - d((v, y))$$

Case 3.3: suppose that both vertices $u$ and $v$ lie in the layer $V_{l}, l \geq 3$ such that these vertices lie in the one component of the layer $V_{l}$, say $B^{(k)}_l, 1 \leq i \leq n, 1 \leq j \leq m^{k^l-3}$. In this case, $d((u, v)) = 2$. Moreover, we know that the layer Sun graph LSG$((n, m, k))$ has the property that, for each vertex $x \in B^{(k)}_l$ in the layer $V_{l}$, there is a component of the layer $V_{l}$, say $B^{(k)}_l, 1 \leq i \leq n, 1 \leq r \leq m^{k^l-3}$ such that for any vertex $x \in B^{(k)}_l$, $d((u, x)) = k - l$. In the same way, for the vertex $v \in B^{(k)}_l$ in the layer $V_{l}$, there is a component of the layer $V_{l}$, say $B^{(k)}_l, 1 \leq i \leq n, 1 \leq \ell \leq m^{k^l-3}, \ell \neq s$ such that, for any vertex $y \in B^{(k)}_l$, we have $d((v, y)) = k - l$. Thus, $d((u, x)) - d((u, y)) = d((v, x)) - d((v, y))$ because $d((u, y)) = k - l + 2$ and $d((v, x)) = k - l + 2$.

Case 3.4: suppose that both vertices $u$ and $v$ lie in the layer $V_{l}, l \geq 3$ such that these vertices lie in the two distinct components of the layer $V_{l}$. We can assume without loss of generality that $u \in B^{(k)}_{p^2}$ and $v \in B^{(k)}_{q^2}, 1 \leq p, q \leq n$, and $1 \leq j_1, j_2 \leq m^{k^l-3}$. Moreover, we know that the layer Sun graph LSG$((n, m, k))$ has the property that, for each vertex $x \in B^{(k)}_{p^2}$ in the layer $V_{l}$, there is a component of the layer $V_{l}$, say $B^{(k)}_{p^2}, 1 \leq i \leq n, 1 \leq \ell \leq m^{k^l-3}$ such that for any vertex $x \in B^{(k)}_{p^2}$, we have $d((u, x)) = k - l$. In the same way, for the vertex $v \in B^{(k)}_{q^2}$ in the layer $V_{l}$, there is a component of the layer $V_{l}$, say $B^{(k)}_{q^2}, 1 \leq s \leq m^{k^l-3}$ such that for any vertex $y \in B^{(k)}_{q^2}$, we have $d((v, y)) = k - l$. Thus, $d((u, x)) - d((u, y)) = d((v, x)) - d((v, y))$.

Case 3.5: suppose that vertices $u$ and $v$ lie in distinct layers $V_{a}, V_{b}$, respectively. Note that if $a = 1$ and $b = 2$, $a = 2$ and $b = 2$, or $a = 2$ and $b = 2$, there is nothing to do. Now, let $3 \leq a < b$. Hence, there is a component of the layer $V_{a}$, say $B^{(k)}_{a^l}, 1 \leq i \leq n, 1 \leq j \leq m^{a^l-3}$ such that $u \in B^{(k)}_{a^l}$. Also, there is a component of the layer $V_{b}$, say $B^{(k)}_{b^l}, 1 \leq p \leq n, 1 \leq q \leq m^{b^l-3}$ such that $v \in B^{(k)}_{b^l}$. In particular, there is a component of the layer $V_{b}$, say $B^{(k)}_{b^l}, 1 \leq p \leq n, 1 \leq \ell \leq m^{b^l-3}$ such that for any vertex $x \in B^{(k)}_{b^l}$, we have $d((u, x)) = k - a$. Now, let $i = p$; if we consider $y \in B^{(k)}_{b^l}, z \neq i, 1 \leq s \leq n$, and $1 \leq \ell \leq m^{k^l-3}$, then we have $d((u, x)) - d((u, y)) = d((v, x)) - d((v, y))$. Because $d((u, y)) = k + a - 2 + d_{c_{l}}(i, z)$, $d((v, y)) = k + b - 2 + d_{c_{l}}(i, z)$ and $d((v, x)) = d((u, y))$. Note that if $i \neq p$, then there is a component of the layer $V_{b}$, say $B^{(k)}_{b^l}, 1 \leq p \leq n, 1 \leq \ell \leq m^{b^l-3}$ such that for any vertex $y \in B^{(k)}_{b^l}$, we have $d((v, y)) = k - b$, and then, we have
Let \( n, k \) which is defined already. Suppose that \( \gamma(LSG) \) is the cardinality of the minimum strongly resolving set of the layer Sun graph \( LSG(n, m, k) \) is \( nm^{k-2} \).

**Theorem 3.** Let \( G = LSG(n, m, k) \) be the layer Sun graph which is defined already. Suppose that \( n, m, k \) are integers such that \( n, k \geq 3 \) and \( m \geq 2 \). Then, the strong metric dimension of \( LSG(n, m, k) \) is \( nm^{k-2} - 1 \).

**Proof.** In the following cases, it can be seen that the cardinality of the minimum strongly resolving set of the layer Sun graph \( LSG(n, m, k) \) is \( nm^{k-2} - 1 \).

\[
W = \left\{ B_{i_1}^{(k)} - (v_1, 1)^k, B_{i_2}^{(k)} - (v_2, 1)^k, \ldots, B_{i_{mk-3}}^{(k)} - (v_{mk-3}, 1)^k \right\}
\]

(11)

Case 2: on the other hand, we can show that the subset \( W \) of vertices in equation (12) in the graph \( G \) is a resolving set for graph \( G \). We show that this subset is a strong resolving set in graph \( G \). It is sufficient to prove that every two distinct vertices \( u, v \in V(G) \) are strongly resolved by a vertex \( w \in W \). Then, we have the following:

\[
W = \left\{ B_{i_1}^{(k)} - (v_1, 1)^k, B_{i_2}^{(k)} , \ldots, B_{i_{mk-3}}^{(k)} , B_{i_{mk-2}}^{(k)} \right\}
\]

(12)

Case 1: we know that the ordered subset \( W \) of vertices in equation (11) in the layer \( V_k \) of the layer Sun graph \( LSG(n, m, k) \) is a resolving set for \( LSG(n, m, k) \) of cardinality \( nm^{k-2} - nm^{k-3} \). Now, let \( N = V_k - W = \left\{ (v_1, 1)^k, \ldots, (v_{mk-3}, 1)^k, \ldots, (v_{mk-1}, 1)^k \right\} \). By considering distinct vertices \( u, v \in N \), we can show that there is not a vertex \( w \in W \) such that \( u \) belongs to a shortest \( v - w \) path or \( v \) belongs to a shortest \( u - w \) path because the valency of every vertex in the layer \( V_k \) is one. So, this subset is not a strong resolving set for \( G \). Thus, we can be conclude that if \( W \) is a strong resolving set for graph \( G \), then \( |W| \geq nm^{k-2} - 1 \) because \( |N| \) must be less than 2.

Case 2.1: suppose that both vertices \( u \) and \( v \) lie in the layer \( V_1 \). Hence, there are \( r, s \in \{1, 2, \ldots, n\} \) such that \( u = r \) and \( v = s \). Moreover, we know that the layer Sun graph \( LSG(n, m, k) \) has the property that, for each vertex \( r \) in the layer \( V_1 \), there is a component \( B_{ir}^{(k)} \), \( 1 \leq j \leq m^{k-3} \) in the layer \( V_k \) such that, for every vertex such as \( w \in B_{ir}^{(k)} \), we have \( d(u, w) = k - 1 \) and \( d(v, w) > k - 1 \), and hence, \( u \) belongs to a shortest \( w - v \) path.

Case 2.2: now suppose that both vertices \( u \) and \( v \) lie in the layer \( V_1 \). In a similar way as in Case 2.1, we can show that the vertices \( u \) and \( v \) are strongly resolved by a vertex \( w \in W \).

Case 2.3: suppose that both vertices \( u \) and \( v \) lie in the layer \( V_l \), \( l \geq 3 \) such that these vertices lie in the one component of the layer \( V_l \), say \( B_{i_l}^{(k)} \), \( 1 \leq i \leq n \), \( 1 \leq j \leq m^{k-3} \). In this case, \( d(u, v) = 2 \). Moreover, we know that the layer Sun graph \( LSG(n, m, k) \) has the property that, for each vertex \( u \in B_{i_l}^{(k)} \) in the layer \( V_l \), there is a component of the layer \( V_k \), say \( B_{i_l}^{(k)} \), \( 1 \leq i \leq n \), \( 1 \leq r \leq m^{k-3} \) such that for any vertex \( w \in B_{i_l}^{(k)} \), we have \( d(u, w) = k - l \), and hence, \( u \) belongs to a shortest \( w - v \) path.

Case 2.4: suppose that both vertices \( u \) and \( v \) lie in the layer \( V_l \), \( l \geq 3 \) such that these vertices lie in the two distinct components of the layer \( V_l \). We can assume without loss of generality that \( u \in B_{i_l}^{(k)} \) and \( v \in B_{j_l}^{(k)} \), \( 1 \leq p, q \leq n \), and \( 1 \leq j_l \leq m^{k-3} \). Moreover, we know that the layer Sun graph \( LSG(n, m, k) \) has the property that, for each vertex \( u \in B_{i_l}^{(k)} \) in the layer \( V_l \), there is a component of the layer \( V_k \), say \( B_{i_l}^{(k)} \), \( 1 \leq r \leq m^{k-3} \) such that for any vertex \( w \in B_{i_l}^{(k)} \), we have \( d(u, w) = k - l \), and hence, \( u \) belongs to a shortest \( w - v \) path.

Case 2.5: suppose that vertices \( u \) and \( v \) lie in distinct layers \( V_k, V_l \) respectively. Note that if \( a = 1 \) and \( b = 2 \), \( a = 1 \) and \( b = 2 \), or \( a = 2 \) and \( b > 2 \), there is nothing to do. Now, let \( 3 \leq b < a \). Hence, there is a component of the layer \( V_k \), say \( B_{i_k}^{(k)} \), \( 1 \leq i \leq n \), \( 1 \leq j \leq m^{k-3} \) such that \( u \in B_{i_k}^{(k)} \). Also, there is a component of the layer \( V_l \), say \( B_{i_l}^{(k)} \), \( 1 \leq p, q \leq n \), \( 1 \leq j_l \leq m^{k-3} \) such that \( v \in B_{i_l}^{(k)} \). In particular, there is a component of the layer \( V_k \), say \( B_{i_k}^{(k)} \), \( 1 \leq i \leq n \), \( 1 \leq r \leq m^{k-3} \) such that, for any vertex \( w \in B_{i_k}^{(k)} \), we have \( d(u, w) = k - a \), and hence, \( u \) belongs to a shortest \( w - v \) path.
Case 2.6: let $u, v$ be two distinct vertices in $V(G) - W$ such that $u = (v_1, 1)^k \in V_k$ and $v \in V_i$. So, there is a component $B(i, j)$ in the layer $V_i$, $1 \leq j \leq n$ and $1 \leq i \leq m^{l-3}$ such that $v \in B(i, j)$. Thus, there is some vertex in the component $B(i, j)$, $1 \leq j \leq m^{k-3}$ say $w$ such that $d(w, v) = k - 1$, $d(w, u) > k - 1$, and $v$ belongs to a shortest $w - u$ path.

Case 2.7: let $u, v$ be two distinct vertices in $V(G) - W$ such that $u = (v_1, 1)^k \in V_k$ and $v \in V_i$. So, there is some $i \in V_i = \{1, 2, \ldots, n\}$ such that $v = i$. If $i = 1$, indeed $d(u, v) = k - 1$, and then, there is a component $B(i, j)$ in the layer $V_k$, $r 
eq 1$ and $1 \leq j \leq m^{k-3}$ such that, for every vertex such as $w$ in the component $B(i, j)$, we have $d(w, v) > k$, $d(w, u) > 2k - 1$, and $v$ belongs to a shortest $w - u$ path. Now, let $i 
eq 1$; indeed, $d(u, v) > k$, and hence, there is a component $B(i, j)$ in the layer $V_k$, $1 \leq j \leq m^{k-3}$ such that, for every vertex such as $w$ in the components $B(i, j)$, we have $d(w, v) = k - 1$, $d(w, u) = 2k - 1$, and $v$ belongs to a shortest $w - u$ path.

Thus, from the abovementioned cases, we can be concluded that the cardinality of minimum strong resolving set of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - 1$. $\square$

3.2. Minimal Doubly Resolving Sets and the Strong Metric Dimension for the Line Graph of Layer Sun Graph $LSG(n, m, k)$

Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Now, let $H$ be a graph with vertex set $V(H) = U_1 \cup U_2 \cup \ldots \cup U_k$, where $U_1, U_2, \ldots, U_k$ are called the layers of $H$ which is defined as follows:

Let $U_1 = V(C_n) = \{1, 2, \ldots, n\}$ and $U_2 = \{u_1, u_2, \ldots, u_n\}$, and for $l \geq 3$, we have

$$U_l = \left\{ D_1^{(l)}, D_2^{(l)}, \ldots, D_{m^{l-3}}^{(l)}, D_1^{(l)}, D_2^{(l)}, \ldots, D_2^{(l)}, \ldots, D_{m^{l-3}}^{(l)} \right\},$$

and let $D_9^{(l)} = \left\{ \bigcup_{i=1}^{m^{l-3}} (u_i, t) \right\}$ such that every $(u_i, t)$ is a vertex in the layer $U_l$ and $D_9^{(l)} \equiv K_m$ in the layer $U_l$, $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$, $1 \leq t \leq m$, where $K_m$ is the complete graph on $m$ vertices. Now, suppose that every vertex $i \in [2, 3, \ldots, n]$ in the cycle $C_n$ or the layer $U_l$ is adjacent to exactly two vertices in the layer $U_2$ say $u_i, u_{i-1} \in U_2$. In particular, for the vertex $1$, the layer $U_1$, we have $1$ adjacent to exactly two vertices in the layer $U_2$, say $u_1, u_n \in U_2$. Also, every vertex $u_i$ in the layer $U_2$ is adjacent to exactly $m$ vertices $(u_i, 1)^k, (u_i, 2)^k, \ldots, (u_i, m)^k, (u_i, 1)^k \in D_9^{(l)} \in U_1$, in particular for $l \geq 3$, every vertex $(u_i, t)^k \in D_9^{(l)} \equiv U_1$ is adjacent to exactly $m$ vertices $\bigcup_{i=1}^{m^{l-3}} (u_i, t)^k \in D_9^{(l)} \in U_1$, and then, the resulting graph is isomorphic with the line graph of the layer Sun graph $LSG(n, m, k)$ with parameters $n, m$, and $k$; in fact, $L(G) \equiv H$. Note that simply we use refinement of the natural labelling of the line graph of the graph $LSG(n, m, k)$. Also, for $i \geq 3$, we recall $D_9^{(l)}$ as the components of $U_l$, $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In particular, we say that two components $D_9^{(l)}$, $D_9^{(l)} \leq i, r \leq n$, $1 \leq j, s \leq m^{l-3}$ are fundamental if $i = r$ and $j \neq s$. It is natural to consider its vertex set of the line graph of the layer Sun graph $LSG(n, m, k)$ is also as partitioned into $k$ layers. The layers $U_1$ and $U_2$ consist of the vertices $\{1, 2, \ldots, n\}$ and $\{u_1, u_2, \ldots, u_n\}$, respectively. In particular, each layer $U_l$ ($l \geq 3$) consists of the $nm^{l-2}$ vertices. Note, that for each vertex $i$ in the layer $U_1$ and every vertex $x \in D_9^{(l)} \in U_1$, $1 \leq j \leq m^{l-3}$, we have $d(i, x) = 1 - 1$. In this section, we consider the problem of determining the cardinality $\gamma(L(G))$ of minimal doubly resolving sets of the line graph of the layer Sun graph $LSG(n, m, k)$. We find the minimal doubly resolving set for the line graph of the layer Sun graph $LSG(n, m, k)$, and in fact, we prove that if $n, k \geq 3$ and $m \geq 2$, then the minimal doubly resolving set of the line graph of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$. Figure 2 shows the line graph of the graph $LSG(3, 3, 4)$. Note that simply we use refinement of the natural labelling of the line graph of the graph $LSG(3, 3, 4)$.

Theorem 4. Let $G = LSG(3, 3, 4)$ be the layer Sun graph which is defined already. Suppose that $n, m, k$ are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the cardinality of minimum doubly resolving set in the line graph of the graph $G$ is $nm^{k-2} - nm^{k-3}$.

Proof. Let $W$ be an ordered subset of the layer $U_1$ in the line graph of the graph $G$ such that

$$W = \left\{ D_1^{(k)} - (u_1, 1)^k, D_2^{(k)} - (u_1, 1)^k, \ldots, D_{m^{k-3}}^{(k)} - (u_1, 1)^k \right\}$$

$$\left\{ (u_{1,2}, 1)^k, \ldots, (u_{1,2}, 1)^k, (u_{1,2}, 1)^k \right\}$$

Hence,

$$V(L(G)) = W = \left\{ U_1, U_2, \ldots, U_{m^{k-3}}, (u_1, 1)^k, \ldots, (u_{m^{k-3}}, 1)^k \right\}$$

We know that $|W| = nm^{k-2} - nm^{k-3}$. In a similar way as in Theorem 1, we can show that this subset is a minimal resolving set for the line graph of the graph $G$. We prove that this subset is a doubly resolving set for the line graph of the graph $G$, and hence, $\beta(L(G)) = \gamma(L(G))$. It is sufficient to prove that, for any two vertices $u$ and $v$ in $L(G)$, there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) = d(v, x) - d(v, y)$. Consider two vertices $u$ and $v$ in $L(G)$. Then, we have the following:

Case 1: suppose that both vertices $u$ and $v$ lie in the layer $U_1$. Hence, there are $r, s \in \{1, 2, \ldots, n\}$ such that $u = r$ and $v = s$. Moreover, we know that the line graph of the graph $G$ has the property that, for each vertex $r$ in
the layer $U_1$ there is some vertex such as $x = (u_r, t)^k$ in the component $D^{(k)}_{i_1}$, $1 \leq j \leq m^{k-3}$ in the layer $U_k$, at distance $k - 1$ from $u$, and in fact, $d(u, x) = k - 1$. In the same way, there is some vertex such as $y = (u_s, t)^k$ in the component $D^{(k)}_{i_2}$, in the layer $U_k$, at distance $k - 1$ from $v$. In particular, it is easy to prove that $d(u, x) - d(u, y) > 0$ because $d(u, y) \geq k$. Also, $d(v, x) - d(v, y) > 0$ because $d(v, y) \geq k$.

Case 2: now, suppose that both vertices $u$ and $v$ lie in the layer $U_2$. In a similar way as in Case 1, we can show that there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Case 3: suppose that both vertices $u$ and $v$ lie in the layer $U_p$, $l \geq 3$ such that these vertices lie in the one component of the layer $U_p$, say $D^{(l)}_{i_1}$, $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In this case, $d(u, v) = 1$. Moreover, we know that the line graph of the graph $G$ has the property that, for each vertex $u \in D^{(l)}_{i_1}$ in the layer $U_p$, there is a component of the layer $U_k$, say $D^{(k)}_{i_2}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that, for every vertex $x \in D^{(k)}_{i_2}$, we have $d(u, x) = k - l$. In the same way, for the vertex $v \in D^{(l)}_{i_1}$ in the layer $V_p$, there is a component of the layer $V_k$, say $D^{(k)}_{i_2}$, $1 \leq i \leq n$, $1 \leq s \leq m^{k-3}$, $r \neq s$ such that, for every vertex $y \in D^{(k)}_{i_2}$, we have $d(v, y) = k - l$. Thus, $d(u, x) - d(v, y) = d(v, x) - d(v, y)$ because $d(u, y) = k - l + 1$ and $d(v, x) = k - l + 1$.

Case 4: suppose that both vertices $u$ and $v$ lie in the layer $U_j$, $l \geq 3$ such that these vertices lie in the two distinct components of the layer $U_j$. We can assume without loss of generality that $u \in D^{(l)}_{i_1}$ and $v \in D^{(l)}_{i_2}$, $1 \leq p, q \leq n$, and $1 \leq j_1, j_2 \leq m^{l-3}$. Moreover, we know that the line graph of the graph $G$ has the property that, for each vertex $u \in D^{(l)}_{i_1}$ in the layer $U_j$, there is a component of the layer $U_k$, say $D^{(k)}_{i_2}$, $1 \leq r \leq m^{k-3}$ such that, for every vertex $x \in D^{(k)}_{i_2}$, we have $d(u, x) = k - l$. In the same way, for the vertex $v \in D^{(l)}_{i_2}$ in the layer $V_j$, there is a component of the layer $V_k$, say $D^{(k)}_{i_2}$, $1 \leq s \leq m^{k-3}$, such that, for every vertex $y \in D^{(k)}_{i_2}$, we have $d(v, y) = k - l$.

In the following, let two components $D^{(l)}_{i_1}$ and $D^{(l)}_{i_2}$ be fundamental; indeed, $p = q$. Hence, $d(u, v) = 2l - 4$, $d(u, y) = d(v, x) = k - l - 5$. Thus, $d(u, x) - d(u, y) = d(v, x) - d(v, y)$. Now, let two components $D^{(l)}_{i_1}$ and $D^{(l)}_{i_2}$ not be fundamental; indeed, $p \neq q$. Hence, $d(u, v) = 2l - 4 + d_{ij}(u, p, u_q)$ and $d(u, y) = d(v, x) = k - l - 4 + d_{ij}(u, p, u_q)$. Thus, $d(u, x) - d(u, y) = d(v, x) - d(v, y)$.

Case 5: suppose that vertices $u$ and $v$ lie in distinct layers $U_a, U_p$, respectively. Note that if $a = 1$ and $b = 2$, $a = 1$ and $b > 2$, or $a = 2$ and $b > 2$, there is nothing to do. Now, let $3 \leq a < b$. Hence, there is a component of the layer $U_a$, say $D^{(a)}_{i_1}$, $1 \leq i \leq n$, $1 \leq j \leq m^{a-3}$ such that $u \in D^{(a)}_{i_1}$. Also, there is a component of the layer $U_b$, say $D^{(b)}_{i_2}$, $1 \leq p \leq n$, $1 \leq q \leq m^{b-3}$ such that $v \in D^{(b)}_{i_2}$. In particular, there is a component of the layer $U_k$, say $D^{(k)}_{i_4}$, $1 \leq i \leq n$, $1 \leq s \leq m^{k-3}$ such that, for any vertex $x \in D^{(k)}_{i_4}$, we have $d(u, x) = k - a$. Now, let $i = p$; if we consider $y \in D^{(k)}_{i_4}$, $z \neq i$, $1 \leq z \leq n$, and $1 \leq s \leq m^{k-3}$, then we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, y) = k + a - 4 + d_{ij}(u, u_z)$, $d(v, y) = k + b - 4 + d_{ij}(u, u_z)$, and $d(u, x) \neq d(v, x)$. Note that if $i \neq p$, then there is a component of the layer $U_k$, say $D^{(k)}_{i_5}$, $1 \leq p \leq n$, $1 \leq z \leq m^{k-3}$ such that, for any vertex $y \in D^{(k)}_{i_5}$, we have $d(u, y) = k - b$, and then, we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Thus, from the abovementioned cases, we conclude that the cardinality of minimum doubly resolving set in the line graph of the graph $G$ is $nm^{k-2} - nm^{k-3}$. □
Theorem 5. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that $n, m, k$ are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the strong metric dimension in the line graph of the graph $G$ is $nm^{k-2} – 1$.

Proof. In a similar way as in the proof of Theorem 3, we can show that the subset

$$W = \left\{ D_{i_1}^{(k)}(v_{i_1}, 1)^k, D_{i_2}^{(k)}, \ldots, D_{i_m}^{(k)}, D_{i_m}^{(k)}(v_{i_m}, 1)^k \right\}$$

(16)

of vertices in the line graph of the graph $G$ is a minimal resolving set of $L(G)$.

4. Conclusions

In this paper, we have constructed a layer Sun graph $LSG(n, m, k)$, discussed this graph, and computed the minimum cardinality of the doubly resolving set and strong resolving set of layer Sun graph $LSG(n, m, k)$ and the line graph of the layer Sun graph $LSG(n, m, k)$. We deduce that, by this way, we can construct a layer jellyfish graph $JFG(n, m, k)$, of order $n + 2 \sum_{k=1}^{m} nm$, where the jellyfish graph is $JFG(n, m)$, which is defined in [5], and by a similar way, we can obtain and compute the minimum cardinality of doubly resolving set and strong resolving set of layer jellyfish graph $JFG(n, m, k)$ and the line graph of the layer jellyfish graph $JFG(n, m, k)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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