Brane Universes, AdS/CFT, Hamiltonian Formalism and the Renormalization Group

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Abstract

The AdS/CFT correspondence is developed from classical solutions on AdS$_5$ with two boundaries. The corresponding limits and the reduction of degrees of freedom are discussed, as well as the required renormalization on the field theory side. The Hamiltonian first-order approach towards the solution of coupled gravitational/matter equations of motion is introduced, and the RG interpretation is exposed. Finally we discuss a recent approach towards a naturally vanishing cosmological constant which is based on the AdS/RG correspondence.

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1 Introduction

Recently two conceptually different applications of 5-dimensional universes supplemented with (3 + 1)-dimensional branes have been intensively discussed:

i) Brane universes, where the 5th dimension is a physical (spatial) dimension; the study of its physical consequences resembles to the standard Kaluza-Klein reduction (from 5 to 4 dimensions): One decomposes the fields into modes which solve the equations of motion (including contributions from the actions on the branes) along the 5th dimension. Of particular interest are the gravitational modes and the induced cosmological evolution.

ii) the AdS/CFT-correspondence, which relates - via a holographic principle - classical solutions of a (super-) gravity theory on AdS$_5$ to correlation functions of composite operators of a (3 + 1) dimensional quantum field theory.

The aim of the present note is a discussion of the conceptual differences and common techniques of both approaches. It is based on a lecture at the LPT Orsay.

First, in section 2, we consider the general solution of the equations of motion of a (free massive) scalar field in a 5-dimensional bulk with (negative) cosmological constant, i.e. AdS$_5$. (The corresponding Einstein-equations are not re-derived; to this end we refer to, e.g., [1, 2, 3]. The boundary or “jump” conditions at branes do, however, not yet play a role). Following, to a large extent, [4] we consider the classical action integrated over a part of AdS$_5$. The aim is to see the reduction of “degrees of freedom”, i.e. integration constants, in the limit where the integral extends far inside of AdS$_5$. One recovers the holographic principle, a one-to-one correspondence between the integrated classical action and correlators of a (3+1)-dimensional QFT [5, 6, 4]. The required limits, associated to renormalization on the QFT side, are discussed in some detail. We add a very superficial mini-review on recent applications of the AdS/CFT-correspondence - incomplete and without references.

The next chapter (section 3) is dedicated to the Hamiltonian formalism. Evidently both 5-d cosmology and AdS/CFT-correspondence require solutions of the (classical) combined gravitational and matter equations of motion. We discuss the Hamiltonian first order approach, an identity for the $y$-integrated bulk action, and the Hamiltonian constraint, valid in the presence of gravity. This section is essentially based on [7] (see also [8]). Following [7, 9, 10, 11] we draw the analogy to renormalization group (RG) equations.
In the last section 4 we comment on some recent proposals towards a vanishing cosmological constant without fine tuning. We discuss, in particular, the approach of [9, 11, 10] based on the RG interpretation of the 5-d dynamics - at least the way we understand it.

The purpose of this paper is purely pedagogical. Its aim is to discuss the interface between 5-d cosmology and the AdS/CFT-correspondence; it is not meant to represent a review on any of these two subjects. Hence the corresponding discussions and references are far from complete. We found it appropriate, on the other hand, to discuss the links between these subjects employing common conventions and common techniques.

2 CFT from ADS$_5$ with two boundaries

Let us start, to fix the conventions, with pure gravity in a 5-dimensional bulk with a negative cosmological constant. The corresponding action reads (with $R_{\mu\nu} = R^p_{\mu\nu\rho}$)

$$S_{\text{Grav}} = -\int dy \int d^4x \sqrt{-g_5} \left\{ \frac{1}{2\kappa_5^2} R - \Lambda \right\} + \text{Boundary Terms} . \quad (2.1)$$

The curvature scalar $R$ contains second derivations of the metric; before deriving the gravitational equations of motion, these terms should be transformed into expressions quadratic in first derivatives by means of partial integrations. The boundary terms in (2.1) are chosen such that they cancel precisely the corresponding total derivative terms [12].

We look for a 5-dimensional metric, which solves the Einstein equations following from (2.1) and preserves 4-dimensional Poincaré-invariance, of the form

$$ds^2 = a^2(y)\eta_{ij}dx^i dx^j + b^2(y) \, d^2y \quad , \quad (2.2a)$$

$$\eta_{ij}dx^i dx^j = d\vec{x}^2 - dt^2 \quad . \quad (2.2b)$$

$i, j$ are indices perpendicular to the 5th dimension and take the four values 0 . . . 3; greek indices $\mu, \nu$ will take all five values 0 . . . 3 and 5.

The following expressions for $a(y)$ and $b(y)$, which solve the Einstein equations and describe an AdS space, are most frequently used:
\( a(y) = e^{-y/\lambda}, \quad b(y) = 1 \) \quad \text{with} \quad \lambda^2 = \frac{6}{\Lambda \kappa_5^2}, \quad (2.3a)

or, with \( y' = \lambda \exp(y/\lambda) \) and omitting the prime,

\( a(y) = b(y) = \frac{\lambda}{y} \). \quad (2.3b)

Often the convention \( \lambda = 1 \) is employed.

In the case of \((2.3a)\) \( a(y) \) diverges for \( y \to -\infty \), whereas for \((2.3b)\) \( y \) takes only positive values and \( a(y) \) diverges at \( y = 0 \). In both cases the “warp factor” \( a(y) \) decreases for increasing \( y \). (Sometimes, however, \( y \) is replaced by \( -y \) in \((2.3a)\)).

In the Randall-Sundrum model I \[2\] two flat 3-branes are placed into the bulk: Using the metric \((2.3a)\), brane 1 (with positive tension) is situated at \( y_1 = 0 \), and brane 2 (with negative tension) at \( y_2 = \pi r_c \). “Standard matter fields” are supposed to live on brane 2 where the warp factor is exponentially small compared to brane 1. The orbifold geometry \( S^1/Z_2 \) (with \( a(y), b(y) \) even) can be represented on the entire \( y \) axis with \( a(-y) = a(y), \quad a(y + 2n\pi r_c) = a(y) \) \((n \text{ integer})\), and similarly for \( b(y) \) and all other (even) fields.

In the Randall-Sundrum model II \[3\] “standard matter fields” are supposed to live on brane 1, whereas the location of the brane 2 is pushed to \( y_2 \to +\infty \).

The first scenario is depicted in Fig. 1, where we plot the warp factor \( a(y) \) versus \( y \) for the metric \((2.3a)\). The branes are represented by vertical lines in the positive or negative direction according to their tension. The behavior of \( a(y) \) for \( y < y_1 \) or \( y > y_2 \), in the case of an orbifold geometry, is indicated as dashed lines.

Now we will add matter to the bulk, first in the form of a free massive scalar field \( \varphi \). The action reads

\[ S = S_{\text{Grav}} + S_{\text{matter}}, \]

\[ S_{\text{matter}} = -\int dy \int d^4x \sqrt{-g_5} \left\{ \frac{1}{2} \partial_\mu \varphi \ g_{\mu\nu} \partial_\nu \varphi + \frac{1}{2} m^2 \varphi^2 \right\}, \quad (2.4) \]

and the scalar equation of motion is given by

\[ \left( \partial_\mu \sqrt{-g_5} \ g^{\mu\nu} \partial_\nu - \sqrt{-g_5} \ m^2 \right) \varphi = 0 \quad . \quad (2.5) \]

Subsequently it is somewhat more convenient to employ the metric \((2.3b)\) where eq. \((2.5)\) turns into
Fig. 1: The warp factor $a(y)$ versus $y$ for the metric (2.3a). The branes are represented by vertical lines in the positive or negative direction according to their tension. The behavior of $a(y)$ for $y < y_1$ or $y > y_2$, in the case of an orbifold geometry, is indicated as dashed lines. "UV" and "IR" are the corresponding regimes of a 4-d field theory.
\[
\left( \frac{\partial_y^2}{y} - \frac{3}{y} \partial_y + \eta^{ij} \partial_i \partial_j - \frac{\lambda^2 m^2}{y^2} \right) \phi(y, x^i) = 0 . \tag{2.6}
\]

Next we consider the Fourier transform in the 4-dimensional space \( \{ x^i \} \) perpendicular to \( y \):

\[
\phi(y, x^i) = \int \frac{d^4 p}{(2\pi)^4} e^{i p \cdot x} \phi(y, p) \tag{2.7}
\]

whereupon the equation of motion becomes

\[
\left( \frac{\partial_y^2}{y} - \frac{3}{y} \partial_y - \left( p^2 + \frac{\lambda^2 m^2}{y^2} \right) \right) \phi(y, p) = 0 \tag{2.8}
\]

with \( p^2 = \vec{p}^2 - p_0^2 \).

As a second order differential equation in \( y \) the general solution admits two \((p\text{-dependent})\) integration constants \( C_1, C_2 \), and reads

\[
\phi(y, p) = C_1(p) y^2 K_\nu(py) + C_2(p) y^2 I_\nu(py) \tag{2.9}
\]

with \( \nu = \sqrt{4 + \lambda^2 m^2} \), \( p = \sqrt{p^2} \), and where \( K_\nu \) and \( I_\nu \) are Bessel functions. As boundary conditions we can thus impose the values of \( \phi \) at two different values of \( y \) as

\[
\phi(y_1, p) \overset{!}{=} \phi_1(p) , \quad \phi(y_2, p) \overset{!}{=} \phi_2(p) . \tag{2.10}
\]

These boundary conditions fix \( C_1(p), C_2(p) \) to be of the form

\[
C_1(p) = \frac{1}{N} \left( y_1^{-2} \phi_1(p) I_\nu(py_2) - y_2^{-2} \phi_2(p) I_\nu(py_1) \right) , \quad C_2(p) = \frac{1}{N} \left( y_2^{-2} \phi_2(p) K_\nu(py_1) - y_1^{-2} \phi_1(p) K_\nu(py_2) \right) , \quad N = K_\nu(py_1) I_\nu(py_2) - K_\nu(py_2) I_\nu(py_1) . \tag{2.11}
\]

The next object of our desire is the matter action \( (2.4) \), with the \( y \) integral confined to the interval \( \{ y_1, y_2 \} \):

\[
S_{1,2} = - \int_{y_1}^{y_2} dy \int d^4 x \sqrt{-g} \left\{ \frac{1}{2} \partial_{\mu} \varphi g^{\mu\nu} \partial_{\nu} \varphi + \frac{1}{2} m^2 \varphi^2 \right\} \tag{2.12}
\]

where \( \varphi \) is a solution of the equations of motion. Using again the metric \( (2.3b) \), the Fourier transform \( (2.7) \) and the equations of motion \( (2.8) \), \( S_{1,2} \) can be expressed as
\[ S_{1,2} = - \left[ \frac{\lambda^3}{2y^2} \int \frac{d^4p}{(2\pi)^4} \phi(y, p) \partial_y \phi(y, -p) \right]^{y_2}_{y_1}. \tag{2.13} \]

Clearly, with \( \phi \) given by (2.9) and the integration constants given by (2.11), \( S_{1,2} \) will be a (quadratic) functional of the boundary values \( \phi_1(p) \) and \( \phi_2(p) \). It can be straightforwardly obtained from (2.13), but its explicit expression is somewhat lengthy.

Now we want to push \( y_2 \to +\infty \), keeping \( \phi_2(p) \) finite. From \( I_\nu(py) \sim (2\pi py)^{-1} e^{py} \) for \( y \to \infty \) one finds, from (2.11), that \( C_2(p) \) vanishes. Alternatively, from (2.9) one deduces that \( C_2(p) \) has to vanish, if \( \phi(y, p) \) is required to be bounded for \( y \to +\infty \). Hence the expression (2.9) for \( \phi(y, p) \) assumes the form

\[ \phi(y, p) = \frac{y^2 K_\nu(py)}{y_1^2 K_\nu(py_1)} \phi_1(p), \tag{2.14} \]

i.e. all dependence on \( \phi_2(p) \) has disappeared.

This is a version of the celebrated holographic principle: The space under consideration is AdS\( _5 \) with an inner boundary located at \( y = y_1 \) (the part of the space with \( y < y_1 \) is cut off), and the configuration of \( \phi(y, p) \), for all \( y \geq y_1 \), is determined by its value on \( \phi_1(p) \) on the boundary. We have “lost” one integration constant of the equations of motion through the requirement that \( \phi(y, p) \) is bounded all over AdS\( _5 \) within the boundary, notably for \( y \to +\infty \).

Denoting \( S_{1,2} \), for \( y_2 \to \infty \), by \( S_1 \), we obtain from (2.13) and (2.14) (using a recursion formula for \( K_\nu \))

\[ S_1(\phi_1) = -\frac{\lambda^3}{2y_1^2} \int \frac{d^4p}{(2\pi)^4} \phi_1(p) \phi_1(-p) \left( -2 - \nu + \frac{py_1 K_{\nu+1}(py_1)}{K_\nu(py_1)} \right). \tag{2.15} \]

The next steps are the following: we are interested in the limit \( y_1 \to 0 \) and, most importantly, we interpret \( S_1(\phi_1) \) differently: we identify \( \phi_1 \) (or its Fourier transform \( \varphi_1(x^i) \)) with a source for a (composite) operator \( O \) of some 4-dimensional field theory, and \( S_1(\phi_1) \) (or \( S_1(\varphi_1) \)) as the corresponding generating functional of connected Green functions \([6,4]\). In the Euclidean 4-d theory we assume

\[ e^{-S_1(\varphi_1)} = \langle e^{\varphi_1 \cdot O} \rangle = \int \mathcal{D}\chi \ e^{-S(\chi) + \int \varphi_1 \mathcal{O}(\chi) d^4x} \tag{2.16} \]

where \( S(\chi) \) is the action of some conformal field theory with fundamental fields \( \chi \), and \( \mathcal{O}(\chi) \) a composite operator. Conventionally sources for operators are denoted by \( J \), and one writes
\[ e^{-G(J)} = \int \mathcal{D}\chi \, e^{-S(\chi)} + \int J \mathcal{O}(\chi) d^4x \quad . \tag{2.17} \]

The statement thus reads \( S_1(\phi_1) = G(J = \varphi_1) \).

A remark on the renormalization of the generating functional of Green functions of composite operators is in order: Now, in general, multiplicative renormalization of \( J \) (or the operator \( \mathcal{O}(\chi) \)) is not sufficient to render Green functions with several insertions of \( \mathcal{O}(\chi) \) - i.e. higher powers of \( J \) in \( G(J) \) - UV finite. To this end one has to add a local polynomial in \( J \) and derivatives (with divergent coefficients in the limit where a UV cutoff is removed) to \( G(J) \) (see, e.g., \[13\]). Perturbatively, this polynomial is of finite order in \( J \) and derivatives, if the operator \( \mathcal{O}(\chi) \) is relevant (if the mass dimension of \( J \) is positive), but of infinite order in \( J \) otherwise.

Let us return to the \( y_1 \rightarrow 0 \) limit of \( S_1(\phi_1) \) in (2.15), which is obviously divergent. Within the interpretation (2.16) of \( S_1(\phi_1) \) it is natural to relate \( y_1^{-1} \) to some UV cutoff \( \Lambda_{UV} \) of the 4-dimensional field theory. In view of the invariance of the 5-dimensional classical theory (including gravity) under coordinate reparametrizations it is wiser, however, to associate \( \Lambda_{UV} \) directly to the warp factor \( a(y) \):

\[ \Lambda_{UV} \sim a(y_1) \cdot M \tag{2.18} \]

where \( M \) is some fundamental scale. (Within the present metric (2.3b) we have, of course, \( a(y_1) \sim y_1^{-1} \)).

In order to “renormalize” \( S_1(\phi_1) \) to the order \( \phi_1^2 \) we are thus allowed to add polynomials in \( p^2 \) with divergent coefficients for \( y_1 \rightarrow 0 \). Let us assume that \( \nu \) equals an integer \( n \), at it happens in most cases (see below). The leading behavior of \( K_n(z) \) for \( z \rightarrow 0 \) reads

\[
K_n(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} 2^{2k-n-1} \Gamma(n-k) \ z^{2k-n} \\
+(-1)^{n+1} \sum_{k=0}^\infty \frac{2^{-2k-n}}{k! \Gamma(n+k+1)} \ z^{2k+n} (\ln z + \text{const.}) \quad . \tag{2.19}
\]

Due to the presence of the logarithm in (2.19) we cannot cancel all terms in \( S_1(\phi_1) \), for \( y_1 \rightarrow 0 \), by a polynomial in \( p^2 \). If we rewrite \( S_1(\phi_1) \) in the form

\[ S_1(\phi_1) = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \, \phi_1(p) \, \phi_1(-p) \, F(y_1, p^2) \quad . \tag{2.20} \]
the leading non-analytic term in \( F(y_1, p^2) \) reads

\[
F(y_1, p^2) \sim \lambda^3 y_1^{2n-4} \frac{2^{1-2n}}{(n-1)!} (-p^2)^n \ln p^2 .
\]  

(2.21)

For \( n \neq 2 \) the remaining dependence on \( y_1 \) can be cancelled by multiplicative renormalization:

\[
\phi_1 = y_1^{2-n} \phi_{1,\text{ren}}
\]

(2.22)

Within the interpretation (2.16), and after Fourier-transformation back to ordinary space, (2.20) with (2.21) implies the following behaviour of the 2-point-function of the operator \( \mathcal{O} \) associated to the “source” \( \phi_1 \):

\[
< \mathcal{O}(0) \mathcal{O}(x) > \sim \frac{1}{|x|^{2\Delta}} , \quad \Delta = 2 + n
\]

(2.23)

This is indeed the behaviour of a 2-point-function of an operator of dimension \( \Delta \) in a conformal field theory. (This result can also be derived directly in ordinary space, for arbitrary \( \nu \neq \text{integer} \), provided the required “renormalization” is carefully performed \([3]\)). The fact that (2.23) (and, by the way, the correct power of \( y_1 \) in (2.22), if expressed in terms of \( \Delta \)) relate properties of a 4-dimensional (conformal) quantum field theory to classical solutions of a 5-dimensional field theory on AdS is the “AdS/CFT-miracle”.

Albeit the purpose of the present note is to discuss relations between 4d/5d field theories, we will make some brief remarks on its 10-dimensional origin. The underlying framework considered by Maldacena [4] was type II B string theory with \( N \) D3-branes on top of each other; this gives rise to a 10-dimensional metric of the form

\[
d s^2 = \left(1 + \frac{y^4}{\lambda^4}\right)^{-\frac{1}{4}} \eta_{ij} \, dx^i \, dx^j + \left(1 + \frac{y^4}{\lambda^4}\right)^{\frac{1}{4}} \left(\frac{\lambda^4}{y^4} \, dy^2 + \frac{\lambda^4}{y^2} \, d\Omega_5^2\right) .
\]

(2.24)

d\( \Omega_5^2 \) is the metric of \( S_5 \). \( \lambda \) is related to the gauge coupling \( g_{YM} \) of a \( SU(N) \) \( \mathcal{N} = 4 \) super \( YM \) theory (which appears as the 4-dimensional conformal field theory in this case) and the string scale \( \alpha' \) via

\[
\lambda^4 = 2N g_{YM}^2 \alpha'^2
\]

(2.25)
in the t’Hooft limit \( Ng_{YM}^2 \) large (but \( \alpha' \) is small). In the limit
the metric (2.24) simplifies to
\[ ds^2 = \frac{\lambda^2}{y^2} (\eta_{ij} dx^i dx^j + dy^2) + \lambda^2 d\Omega_5^2 \] (2.27)

First, the part depending on \( x^i, y \) coincides with the AdS\(_5\) metric (2.3b). Second, the radius \( \lambda \) of \( S_5 \) has become \( y \)-independent. The 5-dimensional theory on AdS\(_5\) is gauged \( \mathcal{N} = 8 \) supergravity [14], supplemented with Kaluza-Klein modes from \( S_5 \). Since the masses of the Kaluza-Klein modes are corresponding multiples of the inverse radius \( \lambda \) (\( = \lambda \) as defined in (2.3a)) one finds indeed that \( \nu \), as defined below (2.9), assumes integer values in these cases (as used in (2.19), (2.21)).

Since, for \( \lambda \) fixed, we have
\[ \alpha' = \frac{\lambda^2}{\sqrt{2N g_{YM}^2}} \] (2.28)

one finds that higher orders in the inverse t’Hooft coupling \( N g_{YM}^2 \) correspond to higher orders in the string scale \( \alpha' \).

In view of the limit (2.26) one may wonder why the AdS/CFT correspondence, which seemed to be based on the limit \( y \to 0 \), works at all. However, the crucial point is not the limit \( y \to 0 \), but the “reduction of the degrees of freedom”, i.e. the reduction of integration constants of an arbitrary bulk solution (2.3). The constraint \( C_2(p) = 0 \) in (2.3) arose from pushing the brane on \( y_2 \) to \( y_2 \to +\infty \), and from requiring the bulk solution \( \phi(y, p) \) to remain bounded for \( y \to +\infty \). It is thus the large \( y \) regime (satisfying (2.26)) which leads to the holographic principle, the one-to-one correspondence of a classical bulk solution to a configuration on a brane at \( y_1 \). The replacement of the Bessel function \( K_n(z) \) by its leading behaviour for \( z \to 0 \) in (2.19) requires \( y^2 p^2 \ll 1 \) instead of (2.26); taken together both inequalities require
\[ p^2 \ll \frac{1}{\lambda^2} \] (2.29)

For momenta violating (2.29) higher derivative terms in the supergravity action (associated to higher orders in the string scale \( \alpha' \)) would become relevant.

Let us very briefly - without references - and without pretension to completeness list scenarios to which the 5d/4d correspondence has been applied. First, one can use the trivial vacuum of \( \mathcal{N} = 8 \) supergravity on AdS\(_5\) (with a \( y \)-independent
cosmological constant) as a background. On this background one can consider \( x^i \)-dependent fluctuations of the various fields, i.e. solve the equations of motion (with boundary conditions at \( y_1 \), which fix the bulk solutions uniquely) iteratively to 2nd, 3rd or even 4th order in the fields. The corresponding integrated action \( S_1(\phi_1) \) then allows to obtain 2, 3 or even 4 point functions of composite operators in \( d = 4 \) \( \mathcal{N} = 4 \) \( SU(N) \) Yang-Mills theory.

The various scalars of \( \mathcal{N} = 8 \) supergravity - including Kaluza-Klein modes - can indeed be interpreted as sources for (gauge invariant) local operators of \( \mathcal{N} = 4 \) \( SU(N) \) \( YM \). The various vector fields \( A_\mu \) of \( \mathcal{N} = 8 \) supergravity (for \( \mu \neq 5 \)) can be interpreted as sources for currents \( J_\mu \), associated to global symmetries of \( \mathcal{N} = 4 \) \( SU(N) \) \( YM \). Fluctuations of the traceless components of the graviton (with \( \mu, \nu \neq 5 \)) give correlators of the energy-momentum tensor of \( \mathcal{N} = 4 \) \( SU(N) \) \( YM \); the two-point function allows to obtain the analog of a central charge and a c-theorem. Fluctuations of the dilaton give correlators of the trace of the energy-momentum tensor.

Second, one can look for a non-trivial background solution of \( d = 5 \) \( \mathcal{N} = 8 \) supergravity: A solution of the scalar equations of motion and the Einstein equations with a \( y \)-dependent \( (x^i\)-independent) metric and (some) \( y \)-dependent \( (x^i\)-independent) scalar fields. The AdS\(_5\)/CFT-correspondence is maintained, if i) for small \( y \) (large \( a(y) \)) the scalar fields are in the trivial vacuum; ii) for large \( y \) the scalars (and hence \( a(y) \)) assume a constant, \( y \)-independent value. Hence 5-dimensional space-time is of the AdS\(_5\)-form both for small and large \( y \), but, in general, with different cosmological constants. In order to solve the scalar equations of motion, the values of the scalar fields are at extrema of the scalar potential both at small \( y \) (trivially) and at large \( y \). These configurations are called kink-solutions. One can show that generally \( a(y) \) still decreases in \( y \) for all \( y \).

From the point of view of a 4-dimensional field theory at fixed \( y \), the scalar fields are sources for composite operators or, since the sources are \( x^i \)-independent, masses and couplings. The \( y \)-dependent kink solutions are then interpreted as a renormalization group (RG) flow. The small \( y \) (large \( a(y) \), hence large UV cutoff \( \Lambda_{UV} \)) regime is the “UV region”, the large \( y \) (small \( a(y) \), small \( \Lambda_{UV} \)) regime the “IR region” (cf. Fig. 1). The 4-d theory at small \( y \) is always \( \mathcal{N} = 4 \) \( SU(N) \) \( YM \); the 4-d theory at large \( y \) can be identified by its unbroken symmetries: e.g. “Higgsed” \( \mathcal{N} = 4 \) \( YM \), or \( \mathcal{N} = 1 \) \( YM \), or even \( \mathcal{N} = 0 \) \( YM \). In any case the 4-d theory at large \( y \) is a CFT (with vanishing \( \beta \)-function) as long as the 5-d space is AdS\(_5\) for large
the examples with $\mathcal{N} = 0$ supersymmetry correspond, however, to non-unitary CFTs (the kink-solution does not end up in a local minimum of the scalar potential for $y \to +\infty$).

On any “background” ($y$-dependent, but $x^i$-independent scalars and metric) one can study “fluctuations”: $x^i$-dependent (and $y$-dependent) solutions of the equations of motion of scalars, vectors $A_i$, and the $(i, j)$ components of the 5-d graviton, with prescribed boundary-values at some small $y = y_1$. Since the 5-d space is (again) AdS$_5$ for large $y$, one imposes again decreasing wave functions for large $y$, and the boundary values at $y_1$ determine again uniquely the solutions for all $y$. As before, one starts with the linearized equations of motion (in the fluctuations), and studies, possibly, higher orders in the fluctuations iteratively.

Then, as before, one computes $S_1(\phi_1)$, the bulk action integrated from $y = y_1$ to $y = +\infty$. This functional is at least quadratic in $\phi_1$, and allows to obtain the 2 (or higher)-point functions of the associated operators.

Third, with some courage, one can look for non-trivial background solutions of $d = 5$ $\mathcal{N} = 8$ supergravity, which do not behave like AdS$_5$ for large $y$: If the warp-factor $a(y)$ vanishes for some finite $y$, the corresponding 4-d theory, in the infra-red, is some non-conformal theory.

Support for this conjecture arises, e.g., from the corresponding 2-point function of the dilaton $\varphi$. The associated 4-d operator is the trace of the energy-momentum tensor, which contains the operator $F_{\mu\nu}F^{\mu\nu}$. If one isolates the leading non-analytic behavior of $S_1(\varphi_1)$ to $O(\varphi_1^2)$ as in eqs. (2.20), (2.21) above, one obtains no logarithmic cut in $p^2$, but a sequence of (Regge-) poles in $p^2$. These are interpreted as a glueball Regge trajectory, since glueballs would couple to the corresponding operator. Thus one hopes to describe a confining 4-d gauge theory.

However, the vanishing warp-factor $a(y)$ at finite $y$ implies, in general, a naked singularity in the 5-d bulk. Most importantly, the divergent 5-d curvature at such a singularity leads to a breakdown of the “supergravity approximation”, i.e. the possibility to neglect (stringy) higher powers of $\alpha'$ or higher powers of the curvature tensor in the bulk action.

One possibility is to replace the bulk action near the singularity by the action of the full string theory [15]. In [16] it is proposed that naked singularities are allowed (and the corresponding boundary conditions on the fields can be derived), if they can be approached smoothly in the $T \to 0$ limit of a black hole solution with temperature $T$, which “hides” the singularity behind a horizon (for $T \neq 0$).
It should have become clear that the search for \( y \)-dependent solutions of the coupled scalar/gravitational equations of motion plays an important role in this framework, as well as the computation of the \( y \)-integrated bulk action. Also the interpretation of the \( y \)-dependence as an RG flow merits further clarification: The equations of motion contain obviously second derivatives in \( y \), whereas RG equations are generally first order equations. To both ends a first order formulation of the classical bulk dynamics, i.e. a Hamilton-like approach, proves to be helpful. This will be the subject of the next section.

3 Hamiltonian constraint and RG equations

The approach developed in the present section is based, to a large extent, on [7]. Subsequently we will denote all degrees of freedom in the bulk, scalars or components of the graviton, by \( q_\alpha(y, x^i) \). Furthermore we denote \( y \)-derivatives by primes, \( \partial_y q_\alpha = q'_\alpha \). Concerning the 5-d Lagrangian in the bulk we assume, that all second derivatives stemming from the curvature scalar have been removed by partial integrations, and that the corresponding total derivative terms are omitted resp. cancelled by previously added boundary terms [12]. Thus we can write

\[
L_5 = L_5(q_\alpha, \partial_i q_\alpha, q'_\alpha) \quad .
\] (3.1)

As in section 2 we are interested in the action \( S_{1,2} \) involving a \( y \)-integration from \( y = y_1 \) to \( y = y_2 \):

\[
S_{1,2} = - \int_{y_1}^{y_2} dy \int d^4x \ L_5(q_\alpha, \partial_i q_\alpha, q'_\alpha) \quad .
\] (3.2)

where the fields \( q_\alpha \) are solutions of the equations of motion with prescribed boundary values at \( y = y_1 \) and \( y = y_2 \). Now we apply the action principle of classical mechanics, i.e. we consider the variation of \( S_{1,2} \) with fluctuations \( \delta q_\alpha, \delta \partial_i q_\alpha, \delta q'_\alpha \) around the solutions of the equations of motion:

\[
\delta S_{1,2} = - \int_{y_1}^{y_2} dy \int d^4x \left( \frac{\delta L_5}{\delta q_\alpha} \delta q_\alpha + \frac{\delta L_5}{\delta \partial_i q_\alpha} \delta \partial_i q_\alpha + \frac{\delta L_5}{\delta q'_\alpha} \delta q'_\alpha \right) \quad .
\] (3.3)

As usual one writes the variations of derivatives of \( q_\alpha \) as derivatives of \( \delta q_\alpha \) and uses partial integration. Then one uses that \( q_\alpha \) solves the equations of motion,
\begin{equation}
\frac{\delta L_5}{\delta q_\alpha} - \partial_i \frac{\delta L_5}{\delta \partial_i q_\alpha} - \partial_y \frac{\delta L_5}{\delta q_\alpha} = 0 ,
\end{equation}
and that the space is assumed to be compact in the $x^i$-directions. Hence boundary terms arise only from the partial integration of $\partial_y \delta q_\alpha$, and one obtains
\begin{equation}
\delta S_{1,2} = \int d^4 x \frac{\delta L_5}{\delta q_\alpha} \bigg|_{y_1} \delta q_\alpha(y_1, x^i) - \int d^4 x \frac{\delta L_5}{\delta q_\alpha} \bigg|_{y_2} \delta q_\alpha(y_2, x^i) .
\end{equation}
Let us now concentrate on the dependence of $S_{1,2}$ on the boundary values of $q_\alpha$ at $y_1$, which we denote by $\hat{q}_\alpha$:
\begin{equation}
q_\alpha(y_1, x^i) \equiv \hat{q}_\alpha(x^i) .
\end{equation}
From (3.5) one obtains
\begin{equation}
\delta S_{1,2} = \int d^4 x \frac{\delta L_5}{\delta q_\alpha} \bigg|_{y_1} \delta q_\alpha(y_1, x^i) - \int d^4 x \frac{\delta L_5}{\delta q_\alpha} \bigg|_{y_2} \delta q_\alpha(y_2, x^i) .
\end{equation}
Next we use that the 5-d (classical) Lagrangian $L_5$ can generally be written as
\begin{equation}
L_5(q_\alpha, \partial_i q_\alpha, q'_\alpha) = \frac{1}{2} q'_\alpha G^{\alpha\beta}(q) q^\beta + \bar{\mathcal{L}}_5 ,
\end{equation}
\begin{equation}
\bar{\mathcal{L}}_5 = \frac{1}{2} \partial_i q_\alpha \bar{G}^{\alpha\beta}(q) \partial_i q_\beta + V(q) .
\end{equation}
Thus one finds
\begin{equation}
\frac{\delta L_5}{\delta q_\alpha} = G^{\alpha\beta}(q) q'_\beta
\end{equation}
or, from (3.7) (using (3.9) at $y = y_1$, where (3.6) holds),
\begin{equation}
\hat{q}'_\alpha = \hat{G}^{-1}_{\alpha\beta}(\hat{q}) \frac{\delta S_{1,2}}{\delta \hat{q}_\beta} .
\end{equation}
This relation will be used below.

Now we turn to the Hamiltonian constraint. First we note that a solution $q_\alpha$ of the equations of motion, with boundary values at $y_1$ and $y_2$, will be of the general form $q_\alpha(y, q(y_1), q(y_2))$. Inserting the solutions into $L_5$ in (3.2), $S_{1,2}$ appears to be of the form
\[ S_{1,2} = S_{1,2}(y_1, y_2, q(y_1), q(y_2)) \] \hspace{1cm} (3.11)

However, since \( \mathcal{L}_5 \) contains gravity, invariance under general coordinate transformations guarantees that \( S_{1,2} \) does not depend explicitly on \( y_1, y_2 \); reparametrizations of \( y \) can be compensated for by corresponding variations of the metric (and the fields). From (3.11) this implies

\[ 0 = \frac{\partial S_{1,2}}{\partial y_1} = \frac{dS_{1,2}}{dy_1} - \int d^4x \frac{\delta S_{1,2}}{\delta q_\alpha(y_1, x^i)} q'_\alpha(y_1, x^i) \] \hspace{1cm} (3.12)

On the other hand, using (3.2) the total derivative of \( S_{1,2} \) w.r.t. \( y_1 \) is given by \( \mathcal{L}_5 \) at \( y = y_1 \). Using (3.2) and (3.6) in the first line, and (3.7) in the second line, one obtains

\[ 0 = \int d^4x \left\{ \mathcal{L}_5(\hat{q}) - \frac{\delta S_{1,2}}{\delta \hat{q}_\alpha} \hat{q}_\alpha' \right\} = \int d^4x \mathcal{H}_5(\hat{q}) \] \hspace{1cm} (3.13)

This is a version of the Hamiltonian constraint (\( \mathcal{H} = 0 \) on solutions) in general relativity. Here, however, \( \mathcal{H}_5 \) is the generator of translations in \( y \) and not, as usual, in \( t \). Actually, also a local version (in \( x^i \)) of (3.13) can be derived [7], if one would allow for a \( x^i \)-dependence of the boundary value \( y_1 \) (which would require, however, a \( x^i \)-dependent metric \( \eta_{ij} \) in (2.2)).

If one uses (3.8a) and (3.10) in the first identity in (3.13), it can be rewritten as

\[ 0 = \int d^4x \left\{ -\frac{1}{2} \frac{\delta S_{1,2}}{\delta \hat{q}_\alpha} G^{-1}_{\alpha\beta}(\hat{q}) \frac{\delta S_{1,2}}{\delta \hat{q}_\beta} + \tilde{\mathcal{L}}_5(\hat{q}) \right\} \] \hspace{1cm} (3.14)

This version of the Hamiltonian constraint allows to obtain informations on the functional form of \( S_{1,2} \) in terms of the component \( \tilde{\mathcal{L}}_5 \) of the bulk Lagrangian, cf. (3.8b). However, (3.14) does not lead to any constraints beyond the equations of motion; it is just a convenient way of expressing their consequences.

Let us return to the parametrization (2.2a) of the metric in the bulk. In addition we confine ourselves to \( x^i \)-independent configurations of scalar fields \( \varphi_i \). On these configurations the formal results (3.10) and (3.14) can be used most easily in order to construct solutions of the equations of motion. Now the bulk Lagrangian reads
where $G^{ij}(\varphi)$ denotes a sigma model metric, and $V(\varphi)$ includes a possible cosmological constant. With $q_\alpha = \{a, \varphi_i\}$ ($b'$ does not appear in $\mathcal{L}_5$ or the Hamiltonian $\mathcal{H}_5$) one can read off $\mathcal{G}^{\alpha\beta}$, as defined in (3.8a), from (3.15): $\mathcal{G}^{aa} = -12a^2/b\kappa_5^2$ (where the indices $a$ of $\mathcal{G}^{aa}$ correspond to the warp factor $a(y)$) and $\mathcal{G}^{ij} = (a^4/b)G^{ij}$. Thus eqs. (3.10) become

\[
\hat{a}' = -\frac{b\kappa_5^2}{12a^2}\frac{\partial S_{1,2}}{\partial a}, \quad \hat{\varphi}_i' = \frac{b}{a^4}G_{ij}^{-1}(\hat{\varphi})\frac{\partial S_{1,2}}{\partial \hat{\varphi}_j},
\]

where the hats indicate again the fields at $y = y_1$. With $\mathcal{G}^{\alpha\beta}$ as above, and $\tilde{\mathcal{L}}_5(\hat{q}) = V(\hat{q}) = \hat{a}^4bV(\hat{\varphi})$, equation (3.14) assumes the form, omitting the $d^4x$-integral and dividing by $\hat{b}$,

\[
0 = \frac{\kappa_5^2}{24a^2}\left(\frac{\partial S_{1,2}}{\partial a}\right)^2 - \frac{1}{2a^4} \frac{\partial S_{1,2}}{\partial \hat{\varphi}_i} G_{ij}^{-1}(\hat{\varphi}) \frac{\partial S_{1,2}}{\partial \hat{\varphi}_j} + \hat{a}^4 V(\hat{\varphi}).
\]

Reparametrization invariance in $d = 4$ suggests the following ansatz for $S_{1,2}$ proportional to $\sqrt{-g_4(y_1)} = \hat{a}^4$:

\[
S_{1,2} = \hat{a}^4W(\hat{\varphi}) + \ldots
\]

where the dots denote terms independent of $\hat{a}$, $\hat{\varphi}$ arising, possibly, from the upper end $y = y_2$ of the $y$-integration. With this ansatz eqs. (3.16) become

\[
\frac{\hat{a}'}{a} = -\frac{\kappa_5^2}{3} W, \quad \hat{\varphi}_i' = \hat{b} G_{ij}^{-1} W_j ,
\]

and eq. (3.17) can be brought into the form

\[
\frac{2\kappa_5^2}{3} W^2 - \frac{1}{2} W_i G_{ij}^{-1} W_j + V(\hat{\varphi}) = 0
\]

with $W_i = \partial W/\partial \hat{\varphi}_i$.

In \cite{8} (in the metric (2.3a), where $\hat{b} = 1$, and with $G_{ij} = \delta_{ij}$ and different conventions in the gravitational sector) eqs. (3.19) and (3.20) have been proposed as a “short cut” towards the search for solutions of the equations of motion in the bulk: Instead of solving the coupled second order differential equations (3.4) for $a(y)$, $\varphi_i(y)$ one first tries to find a “superpotential” $W(\varphi)$ which solves (3.20) (with $V(\varphi)$
given). Then one is left with the integration of the remaining first order equations (3.19). The number of integration constants matches: For each scalar field there is one from (3.20) and one from the second of eqs. (3.19), in agreement the analysis in section 2. The combined Einstein equations and Bianchi identities also allow for just one integration constant for the warp factor, in agreement with the first of eqs. (3.20).

It should be emphasized, however, that not all solutions can be written in the form (3.18)-(3.20). A counter example is given by the iterative solution (2.9) in section 2, with

$$C_1(p) = c_1 p^\nu, \quad C_2(p) = c_2 p^{-\nu}$$

and $p \to 0$, if both $c_1$ and $c_2$ are non-zero. Only for $c_1 = 0$ or $c_2 = 0$ the solution can be written in the form (3.20). Generally, solutions of the form (3.18)-(3.20) have the particular property of preserving $N = 1$ supersymmetry [17, 18, 8].

Let us now return to 5-d brane universes, and re-install branes at $y_1$, $y_2$ with actions $S^{(1)}(a, b, \phi)$ and $S^{(2)}(a, b, \phi)$ respectively. These imply jump conditions [1, 2, 3] to the right of $y_1$ of the form

$$a'(y_1) = \frac{1}{2} \left[ -b \left( \frac{\partial S^{(1)}}{\partial a} \right) \right]_{y_1}, \quad \phi'_i(y_1) = \frac{1}{2} \left[ \frac{b}{a^4} G_{ij}^{-1} \frac{\partial S^{(1)}}{\partial \phi_j} \right]_{y_1}, \quad (3.21)$$

and jump conditions to the left of $y_2$ of the form

$$a'(y_2) = \frac{1}{2} \left[ \frac{b}{12a^3} \frac{\partial S^{(2)}}{\partial a} \right]_{y_2}, \quad \phi'_i(y_2) = \frac{1}{2} \left[ -\frac{b}{a^4} G_{ij}^{-1} \frac{\partial S^{(2)}}{\partial \phi_j} \right]_{y_2}. \quad (3.22)$$

Here we assumed orbifold boundary conditions. If, instead, the branes indicate "ends of the world", the factors 1/2 in (3.21) and (3.22) have to be omitted.

Clearly, eqs. (3.21) and (3.22) have to be consistent with eq. (3.19) at $y = y_1$, $y_2$, respectively, in order to allow for a global solution. If these equations are inconsistent, $x^i$-independent solutions for $a$, $b$ and $\phi_i$ do not exist; in particular this implies an $x^i$-dependent warp factor $a(y, x^i)$ on "our" brane in contradiction to the (practically) static and homogeneous observed universe. The argument can be turned around: assuming a static and homogeneous universe, and

$$S^{(1)} = a^4 W^{(1)}, \quad S^{(2)} = a^4 W^{(2)}, \quad (3.23)$$

one can derive $W^{(2)} = -W^{(1)} = \pm 2W$ where $W$ has to satisfy (3.20) [19].
Supersymmetry is not involved in deriving these constraints, which coincide with the ones employed in [2, 3], but supersymmetry helps to satisfy them (see, e.g., [18, 20]).

Finally we turn to the interpretation of \( S_{1,2} (\hat{\varphi}) \) as the generating functional of connected Green functions of composite operators, and of eqs. (3.16) as RG equations for the “sources” (= couplings) \( \hat{a}, \hat{\varphi} \), at \( y = y_1 \). As already stated in section 2 it is not reasonable to identify \( y \) directly with a RG scale; as in eq. (2.18) we should rather associate an UV cutoff (which we identify with a RG scale subsequently; the following RG equations have to be interpreted correspondingly) with the warp factor \( a(y) \). With

\[
\Lambda_{UV} \frac{\partial}{\partial \Lambda_{UV}} = a(y) \frac{\partial}{\partial a(y)}
\]  

(3.24)

and

\[
\frac{\partial}{\partial y} = a' \frac{\partial}{\partial a} = \frac{a'}{a} \Lambda_{UV} \frac{\partial}{\partial \Lambda_{UV}}
\]  

(3.25)

we can rewrite the second of eqs. (3.16) as (omitting the hats in the following)

\[
\Lambda_{UV} \frac{\partial \varphi_i}{\partial \Lambda_{UV}} = -\frac{12}{\kappa_{5}^{2}} \left( G_{ij}^{-1} \frac{\partial S_{1,2}}{\partial \varphi_j} \right)^{-1} \left( a \frac{\partial S_{1,2}}{\partial a} \right)
\]

\[
\equiv \beta_i (\varphi)
\]  

(3.26)

In particular, with the ansatz (3.18) for \( S_{1,2} \), one obtains

\[
\beta_i (\varphi) = -\frac{3}{\kappa_{5}^{2}} \left( G_{ij}^{-1} (\varphi) \frac{W_j}{W} \right)
\]  

(3.27)

Hence the holographic RG-flow is derived from a potential \([1],[8]\), i.e. it is proportional to the gradient of a c-function \( W \) \([8]\).

However, \( S_{1,2} \) does not necessarily have to be of the form (3.18). Let us assume, following \([4],[11]\), that \( S_{1,2} \) is given by

\[
S_{1,2} = a^4 \left( W(\varphi) + \tilde{W}(a, y) \right)
\]  

(3.28)

where \( \tilde{W} \) is a small correction. Inserting (3.28) into (3.17) and using (3.20) one obtains, to first order in \( \tilde{W} \),
\[
\left( a \frac{\partial}{\partial a} + 4 - \frac{3}{\kappa_a^2} \frac{W_i}{W} G^{-1}_{ij} \frac{\partial}{\partial \varphi_j} \right) \tilde{W} = 0 \quad .
\] (3.29)

With (3.24) and (3.27) this becomes indeed an RG equation for \( \tilde{W} \):

\[
\left( \Lambda_{UV} \frac{\partial}{\partial \Lambda_{UV}} + 4 + \beta_i \frac{\partial}{\partial \varphi_i} \right) \tilde{W} = 0 \quad (3.30)
\]

Further consequences of the RG interpretation of the Hamiltonian approach to the \( y \)-dependence are derived in \[7\]. (It should be noted, however, that the step from (3.30) towards a RG equation involving an infra-red scale \( \mu \) with \( \Lambda_{UV} \partial / \partial \Lambda_{UV} = -\mu \partial / \partial \mu \) holds only for scale invariant theories; otherwise \( \mu \) has no physical significance).

In \[9\] an equation is derived, which resembles Polchinski’s exact renormalization group equation \[21\]. To this end one defines

\[
S = S_{UV} + S_{IR} = S_{y_1,y} + S_{y,y_2} \quad (3.31)
\]

with, possibly, \( y_1 \to -\infty \) (in the metric (2.2a) with \( b(y) = 1 \)). Here

\[
S_{UV} = S_{y_1,y} \quad (3.32)
\]

is interpreted as an effective action in the Wilsonian sense, where degrees of freedom with momenta \( p^2 \) with \( M^2 a^2(y_1) > p^2 > M^2 a^2(y) \) have been integrated out. \( M^2 \) is some fundamental scale.) Similarly,

\[
S_{IR} = S_{y,y_2} \quad (3.33)
\]

with \( a(y_2) = 0 \), corresponds to an effective action involving the path integral over modes with \( M^2 a^2(y) > p^2 > 0 \). Hence \( S \) is the full quantum effective action, which is splitted into its UV and IR part in (3.31). The "split-point" corresponds to a scale \( \mu \) with \( \mu^2 = M^2 a^2(y) \).

In analogy to (3.24) one defines \( \beta \)-functions as

\[
\beta_i = -\frac{12 \kappa_a^2}{\kappa_a^2} G^{-1}_{ij} \frac{\partial S_{UV}}{\partial \varphi_j} \left( a(y) \frac{\partial S_{UV}}{\partial a(y)} \right)^{-1} .
\] (3.34)

Returning to \( G^{ij} = (a^4/b)G^{ij} \) and defining

\[
\gamma = -\frac{b \kappa_a^2}{12a^4} \left( a(y) \frac{\partial S_{UV}}{\partial a(y)} \right)
\] (3.35)
eq. (3.34) can be expressed as

\[ \gamma \beta_i = G^{-1}_{ij} \frac{\partial S_{UV}}{\partial \varphi_j(y)} \cdot (3.36) \]

(The minus sign in (3.35) is related to the fact that now we take the variation of \( S_{UV} \) at the upper limit of the \( y \) integral.) Defining the \( \beta \) function of the warp factor as \( \beta_a = a \) eq. (3.36) holds again for all fields \( q_\alpha = \{a, \varphi_i\} \):

\[ \gamma \beta_\alpha = G^{-1}_{\alpha\beta} \frac{\partial S_{UV}}{\partial q_\beta(y)} \cdot (3.37) \]

Multiplying (3.37) by \( \partial S_{UV}/\partial q_\alpha(y) \) one obtains

\[ \gamma \beta_\alpha \frac{\partial S_{UV}}{\partial q_\alpha} = \frac{\partial S_{UV}}{\partial q_\alpha} G^{-1}_{\alpha\beta} \frac{\partial S_{UV}}{\partial q_\beta} \cdot (3.38) \]

Eq. (3.38) shows some formal similarity to Polchinski’s exact renormalization group equation [21]. A similar equation can be derived for \( S_{IR} \). In [9] (below eq. (24)) the relevance of (3.38) for \( S = S_{UV} + S_{IR} \) is emphasized. Since, however, classical solutions \( q_\alpha \) extremize the full effective action \( S \) (the contributions from \( S_{UV} \) and \( S_{IR} \) in (3.31) cancel) it becomes now a trivial identity.

4 Vanishing cosmological constant

As discussed before, the problem of the vanishing 4-d cosmological constant in a 5-d brane universe can be phrased as the problem of obtaining \( x^i \)-independent solutions for the warp factor \( a(y) \). Given the jump conditions at the branes (and, in addition, continuity of the fields across the branes) one typically ends up with more constraints than available integration constants. Thus, as in 2-brane universes with orbifold boundary conditions [1, 2, 3], the parameters of the actions in the bulk and on the branes have to be fine tuned relatively to each other.

Recent proposals to avoid such fine tunings are: In [22] a 1-brane/2-bulks scenario (with orbifold-like boundary conditions) is considered, where a scalar field - which couples like a dilaton - ensures the compatibility of the jump conditions with the bulk equations of motion. In [23] a 1-brane/2-bulks scenario (without orbifold-like boundary conditions) is considered, where the number of constraints does not exceed the number of integration constants. In both cases, however, a naked singularity (where \( a(y) = 0 \)) is encountered at finite values of \( y \).
The proposal in [9, 10, 11] is quite different: It is based on a 2-brane/1-bulk scenario which, a priori, seems to require fine tunings among the actions on the branes and in the bulk. However, the RG interpretation of the $y$-dependence (or $a(y)$-dependence) of integrated bulk action is taken literally, motivated by the AdS/CFT correspondence: First, the small-$y$ region (where $a(y)$ is large) is identified with the UV regime of field theory/string theory. Since supersymmetry is valid in this regime, an action on a brane at small $y_1$ allows for jump conditions consistent with the bulk action. (In the approximation of $x^i$-independent field configurations as considered near the end of section 3, the superpotential $W^{(1)}$ in (3.23) is related to the bulk potential $V$ via (3.20), due to supersymmetry at the “large scale” $y_1$. Hence eqs. (3.19) and eqs. (3.21) (without factors 1/2) are consistent due to supersymmetry.) Then one considers $S_{y_1,y}$ given by

$$S_{y_1,y} = - \int_{y_1}^{y} dy \int d^4x \mathcal{L}_5 \quad .$$

(4.1)

which is interpreted (as discussed below eq. (3.31)) as a Wilsonian effective action. Recall that, for $y \to y_2$ with $a(y_2) = 0$, $S_{y_1,y_2}$ corresponds to the full quantum effective action. It is assumed that at some intermediate scale dynamical supersymmetry breaking takes place. The brane where we live on is situated at $y_2$. In agreement with previous AdS/FT results the integrated action $S_{y_1,y_2}$ corresponds to a non-conformal field theory (as our world), since the bulk space-time near the naked singularity at $y_2$ is not AdS$_5$. The fact that the jump conditions match at $y_2$ is considered as a tautology: The action $S^{(2)}$ on the brane 2 is the full quantum effective action (generating a possible cosmological constant on brane 2), but $S_{y_1,y_2}$ is the same effective action:

$$S_{y_1,y_2} = S^{(2)} \quad .$$

(4.2)

Hence a combined solution of (3.16) and (3.22) (after a trivial change of sign of $S^{(2)}$, and without the factors 1/2) with $a(y)$ independent from $x^i$, for all $y$ including the IR regime $y_2$, is possible. Although $a(y)$ is no longer a component of the metric from the 4-d point of view, but rather a source for an operator, this argument indicates that a flat brane – consistent with our (practically) static and homogenous universe – is a solution of the combined eqs. of motion and jump-conditions at any value of $y$.

The equality (4.2) is supported by the results in [7, 9, 10, 11] and sketched
near the end of the previous section which indicate, that \( S_{y_1,y} \) satisfies a RG-like flow equation with respect to \( a(y) \) which is of the same form as the RG equation satisfied by a cutoff quantum effective action \( S^{(2)} \). In \([9, 10, 11, 24]\) arguments are put forward which should support the identification of \( S_{y_1,y} \) with the quantum effective action beyond the previously employed approximations, as \( x^i \)-independent field configurations and, notably, a classical action in the bulk. To our opinion it still remains to be shown, however, whether a cutoff quantum effective action - with some concrete definition of the cutoff, which also remains to be found - can be written as an integrated bulk action, without contradicting any of our knowledge of local quantum field theory. After all these concepts should remain valid far below the string scale.

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