Abstract—A general optimization framework is proposed for simultaneously transmitting and reflecting reconfigurable intelligent surfaces (STAR-RISs) with coupled phase shifts, which converges to the Karush–Kuhn–Tucker (KKT) optimal solution under some mild conditions. More particularly, the amplitude and phase-shift coefficients of STAR-RISs are optimized alternately in closed form. To demonstrate the effectiveness of the proposed optimization framework, the throughput maximization problem is considered in a case study. It is rigorously proved that the KKT optimal solution is obtained. Numerical results confirm the effectiveness of the proposed optimization framework compared to baseline schemes.

Index Terms—Coupled phase shifts, Karush–Kuhn–Tucker (KKT), simultaneous transmission and reflection (STAR).

I. INTRODUCTION

RECENTLY, the novel concept of simultaneously transmitting and reflecting reconfigurable intelligent surfaces (STAR-RISs) has been proposed [1], [2]. In contrast to the conventional reflecting-only RISs [3], STAR-RISs can transmit and reflect the incident signals to both sides, thus enabling a full-space smart radio environment.

Due to this unique benefit, STAR-RISs have attracted significant attention. However, most of the existing works on STAR-RISs assumed that the phase shifts of the transmission and reflection coefficients can be independently adjusted, which requires complex RIS hardware. Recently, the authors of [4] have shown that for low-cost passive lossless STAR-RISs, the phase-shift coefficients for transmission and reflection are coupled with each other, which introduces new design challenges. To tackle the coupled phase shifts, an element-wise optimization algorithm was proposed in [5] for a two-user system, and an alternating optimization (AO) algorithm was proposed in [6] for a multi-user system. Noteworthy, for the aforementioned algorithms, the application scenarios are limited and the optimality of the obtained solution cannot be guaranteed. This motivates us to propose a general optimization framework for STAR-RISs with coupled phase shifts.

In this letter, we propose a general penalty-based optimization framework for STAR-RISs with coupled phase shifts, which is capable of obtaining the Karush-Kuhn-Tucker (KKT) optimal solution under some mild conditions. Next, as a case study, we consider a throughput maximization problem subject to the coupled phase-shift constraint. Our numerical results verify the effectiveness of the proposed algorithm.

II. COUPLED PHASE-SHIFT MODEL FOR STAR-RISs

Let $t_n$ denote the incident signal for the $n$-th element of the considered $N$-element STAR-RIS, where $n \in N = \{1, \ldots, N\}$. Then, the corresponding transmitted signal $t_n$ and reflected signal $r_n$ are given by $t_n = \beta_{t,n} \exp(\phi_{t,n} s_n)$ and $r_n = \beta_{r,n} \exp(\phi_{r,n} s_n)$, respectively, where $\beta_{t,n} \in [0, 1]$ and $\beta_{r,n} \in [0, 1]$ denote the real-valued transmitted and reflection amplitudes, and $\phi_{t,n} \in [0, 2\pi)$ and $\phi_{r,n} \in [0, 2\pi)$ denote the corresponding phase shifts [1]. In practice, the values of the amplitudes and phase shifts are determined by the corresponding electric and magnetic impedances of the STAR-RIS. According to the analysis in [4], passive lossless STAR-RISs have to meet the following two constraints:

$$\beta_{t,n}^2 + \beta_{r,n}^2 = 1, \quad (1a)$$
$$\cos(\phi_{t,n} - \phi_{r,n}) = 0. \quad (1b)$$

Specifically, the first constraint stems from the law of energy conservation, and the second constraint referred to as the coupled phase-shift constraint is due to the zero-valued real part of the electric and magnetic impedances of the passive lossless STAR-RIS elements. Note that the coupled phase-shift constraint is a non-convex nonlinear equality constraint implying that the absolute phase-shift difference $|\phi_{t,n} - \phi_{r,n}|$ can only assume values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Using existing methods, it is challenging to transform (1b) into a convex form. To overcome this obstacle, in the following section, a general optimization framework is proposed to handle the coupled phase-shift constraint efficiently.

III. A GENERAL OPTIMIZATION FRAMEWORK FOR STAR-RISs WITH COUPLED PHASE SHIFTS

Consider the following optimization problem:

$$\min_{x \in X, \theta, \theta_r} F(x, \theta; \theta_r) \quad (2a)$$
$$\text{s.t. } \beta_{t,n}^2 + \beta_{r,n}^2 = 1, \forall n \in N, \quad (2b)$$
$$\cos(\phi_{t,n} - \phi_{r,n}) = 0, \forall n \in N, \quad (2c)$$

where $X$ denotes the convex feasible set of optimization variable $x$, vector $\theta = [\beta_{t,1} \exp(\phi_{t,1}), \ldots, \beta_{t,N} \exp(\phi_{t,N})]^T, \forall i \in \{t, r\}$ contains the transmission and reflection coefficients, and $F(x, \theta; \theta_r)$ is the convex objective function or the convex approximation of the original objective function for the specific problem considered. Depending on the application scenario, $F(\cdot)$ can be some utility function such as transmit power, weighted sum-rate, harvested energy, and sensing accuracy, and $x$ can represent the transmit waveforms or resource allocation variables. Note that in the above problem, the constraints $\beta_{t,n} \in [0, 1]$ and $\beta_{r,n} \in [0, 1]$ are omitted without changing the optimal objective value. This is because for any
optimal solution \((\beta^{*}_{t,n}, \phi^{*}_{t,n}), \forall i \in \{t, r\}\), of problem (2) with \(\beta^{*}_{t,n} \in [-1, 0] \), it can be verified that \((-\beta^{*}_{t,n}, \phi^{*}_{t,n} + \pi)\) is also a feasible solution achieving the same objective value.

In problem (2), the non-convexity is caused by non-convex STAR-RIS constraints (2b) and (2c). To tackle this challenge, we define the auxiliary variables \(\tilde{\theta}_{i} = [\tilde{\beta}_{i,1} e^{j\tilde{\phi}_{i,1}}, \ldots, \tilde{\beta}_{i,N} e^{j\tilde{\phi}_{i,N}}]^T, \forall i \in \{t, r\}\), such that \(\tilde{\theta}_{i} = \theta_{i}, \forall i \in \{t, r\}\). Then, problem (2) can be rewritten as:

\[
\min_{x \in \mathcal{X}, \theta, \tilde{\theta}_{t}, \tilde{\theta}_{r}} F(x, \theta_t, \theta_r) \\
\text{s.t. } (3c), (3d),
\]

with low complexity by alternately optimizing the amplitudes and phase shifts. Firstly, the objective function of problem (6) can be reformulated as follows:

\[
\sum_{i \in \{t,r\}} \|\hat{\theta}_i + \theta_i\|^2 = \sum_{i \in \{t,r\}} (\theta_i^H \hat{\theta}_i + \theta_i^H \theta_i + 2\text{Re}(\theta_i^H \hat{\theta}_i)) \\
= \sum_{i \in \{t,r\}} \sum_{n \in \mathcal{N}} \|\tilde{\beta}_{i,n}^2 + \tilde{\beta}_{r,n}^2\|_{2} + \sum_{i \in \{t,r\}} (\theta_i^H \tilde{\theta}_i + \sum_{i \in \{t,r\}} 2\text{Re}(\tilde{\theta}_i^H \tilde{\theta}_i)),
\]

where \(a\) is due to constraint \(\tilde{\beta}_{t,n}^2 + \tilde{\beta}_{r,n}^2 = 1\). In the objective function, only the term \(\sum_{i \in \{t,r\}} 2\text{Re}(\tilde{\theta}_i^H \tilde{\theta}_i)\) involves the optimization variables, while the other terms are constants. Then, we decompose \(\tilde{\theta}_i\) to amplitude vector \(\beta_i = [\tilde{\beta}_{i,1}, \ldots, \tilde{\beta}_{i,N}]^T\) and phase-shift vector \(\phi_i = [\tilde{\phi}_{i,1}, \ldots, \tilde{\phi}_{i,N}]^T\), i.e.,

\[\tilde{\theta}_i = \text{diag}(\beta_i)\phi_i = \text{diag}(\tilde{\psi}_i)\beta_i, \forall i \in \{t,r\}.\]

Consequently, problem (6) can be rewritten as follows:

\[
\min_{\beta_{t}, \tilde{\psi}_{t}, \beta_{r}, \tilde{\psi}_{r}} \sum_{i \in \{t,r\}} 2\text{Re}(\tilde{\theta}_i^H \tilde{\theta}_i) \\
\text{s.t. } (9c), (9d),
\]

where \(\chi_{\tilde{\psi}_n}^*\) denotes a set of a pair of closed-form solutions:

\[
\chi_{\tilde{\psi}_n}^* = \{e^{i(\pi - \angle \tilde{\phi}_n)}, e^{i(\pi - \angle \tilde{\phi}_n)}, e^{i(\pi - \angle \tilde{\phi}_n)}\}. \quad (11)
\]

**Proof**: Please refer to Appendix A.

**Proposition 2 (Closed-Form Optimal Solution for Amplitudes for Given Phase Shifts):** Define \(\tilde{\theta}_i = \text{diag}(\tilde{\psi}_i^H)\theta_i = [\tilde{\theta}_{i,1}, \ldots, \tilde{\theta}_{i,N}]^T\), \(\phi_n^* = \tilde{\phi}_{t,n} + j\tilde{\phi}_{r,n}\), and \(\phi_n^* = \tilde{\phi}_{t,n} - j\tilde{\phi}_{r,n}\). Then, for any given \(\beta_t\) and \(\beta_r\), the optimal solutions for the elements of \(\tilde{\psi}_t\) and \(\tilde{\psi}_r\) are given by

\[
(\hat{\psi}_{n,t}^*, \hat{\psi}_{n,r}^*) = \arg\min_{(\tilde{\psi}_{n,t}, \tilde{\psi}_{n,r})} \text{Re}(\tilde{\beta}_{t,n}^* \tilde{\psi}_{n,t}) + \text{Re}(\tilde{\beta}_{r,n}^* \tilde{\psi}_{n,r}),
\]

where \(\gamma_{\tilde{\phi}_n}^*\) denotes a set of a pair of closed-form solutions:

\[
\gamma_{\tilde{\phi}_n}^* = \{e^{i(\pi - \angle \phi_n^*)}, e^{i(\pi - \angle \phi_n^*)}, e^{i(\pi - \angle \phi_n^*)}\}. \quad (12)
\]

**Proof**: Please refer to Appendix B.

According to **Propositions 1 and 2**, we can further divide block \([\theta_t, \theta_r]\) into two sub-blocks, namely \([\tilde{\theta}_t, \tilde{\theta}_r]\) and \([\beta_t, \beta_r]\). Then, the overall BSUM/BCD algorithm to solve
Algorithm 1 BSUM/BCD Algorithm for Solving Problem (4)

1. Initialize the optimization variables
2. repeat
3. update \(\{x, \theta_i, \rho_i\}\) by solving problem (5)
4. update \(\{
\psi_i, \phi_i\}\) by (10)
5. update \(\{\beta_i, \tilde{\beta}_i\}\) by (12)
6. until convergence

The solution to problem (4) is summarized in Algorithm 1. Since the optimal solution is obtained in each step, the convergence of Algorithm 1 is guaranteed [9]. The complexity of updating \(\{\psi_i, \phi_i\}\) and \(\{\beta_i, \tilde{\beta}_i\}\) is \(O(4N)\) and \(O(2N)\), respectively, where \(O(\cdot)\) is the big-O notation. Moreover, the complexity of updating \(\{x, \theta_i, \rho_i\}\) is determined by the exact form of problem (5) and the methods used to solve it.

IV. CASE STUDY AND NUMERICAL RESULTS

To verify the effectiveness of the proposed general optimization framework, in this section, we use a case study, where we maximize the throughput in a narrowband STAR-RIS-aided communication system.

A. System Model and Problem Formulation

Consider an \(M\)-antenna base station (BS), an \(N\)-element STAR-RIS, and \(K\) single-antenna users, whose indices are collected in \(K\). Without loss of generality, we assume that the users in subset \(K_\ell = \{1, \ldots, K_\ell\}\) are located on the transmission side, and the users in subset \(K_r = \{K_\ell + 1, \ldots, K\}\) are located on the reflection side. The direct BS-user channels are assumed to be blocked. Thus, the received signal at user \(k\), \(\forall k \in K_i, \forall i \in \{\ell, r\}\), is given by

\[
y_k = h_k^H \Theta_i G \sum_{\ell \in K} w_\ell s_\ell + n_k,
\]

where \(h_k \in \mathbb{C}^{N \times 1}\) denotes the STAR-RIS-user-k channel, \(G \in \mathbb{C}^{M \times N}\) denotes the BS-STARS-RIS channel, \(\Theta_i = \text{diag}(\theta_i)\) denotes the transmission/reflection coefficients of the STAR-RIS, \(w_\ell \in \mathbb{C}^{M \times 1}\) denotes the active beamforming vector at the BS for delivering information symbol \(s_\ell \in \mathbb{C}\) to user \(\ell\), and \(n_k \sim \mathcal{CN}(0, \sigma_k^2)\) denotes complex Gaussian noise with power \(\sigma_k^2\). Then, the signal-to-interference-plus-noise ratio (SINR) for decoding \(s_\ell\) at user \(k\), \(\forall k \in K_i, \forall i \in \{\ell, r\}\), is given by

\[
\gamma_k = \frac{|h_k^H \Theta_i G w_\ell|^2}{\sum_{\ell \in K \setminus k} |h_k^H \Theta_i G w_\ell|^2 + \sigma_k^2}.
\]

The corresponding achievable rate is \(R_k = \log_2(1 + \gamma_k)\).

We aim to maximize the throughput of the considered STAR-RIS-aided system subject to a transmit power constraint and coupled STAR-RIS-aided phase-shift and amplitude constraints. The corresponding optimization problem can be formulated as follows:

\[
\max_{W, \theta_i, \rho_i} \sum_{k \in K} R_k
\]

s.t. \(\text{tr}(WW^H) \leq P_t\),

\[
\beta_{t,n}^2 + \beta_{r,n}^2 = 1, \forall n \in \mathcal{N},
\]

\[
\cos(\phi_{t,n} - \phi_{r,n}) = 0, \forall n \in \mathcal{N},
\]

where \(W = [w_1, \ldots, w_K]\) and \(P_t\) denotes the BS transmit power. We note that existing methods for solving throughput-maximization problems cannot be directly applied to problem (16) due to the coupled STAR-RIS phase-shift and amplitude constraints. In the following section, we show that the proposed general optimization framework can be used to effectively solve problem (16).

B. Solution to Problem (16) Using the Proposed Framework

Note that the objective function of problem (16) is non-convex with respect to \(\{W, \theta_i, \rho_i\}\). In order to employ the proposed optimization framework, we first transform the throughput maximization problem into an equivalent weighted mean square error (MSE) minimization problem applying the well-known weighted minimum mean square error (WMMSE) method [10] as follows:

\[
\max_{w, \phi_i} \sum_{k \in K} \omega_k e_k
\]

s.t. \([16b] - [16d]\). (17b)

Here, \(\omega = [\omega_1, \ldots, \omega_K]^{T}\) denotes the vector of weights, and \(e_k, \forall k \in K_i, \forall i \in \{\ell, r\}\), denotes the MSE as follows:

\[
e_k = |v_k|^2 \left( \sum_{\ell \in K} |\theta_{\ell}^T \text{diag}(h_k^H G w_\ell)|^2 + \sigma_k^2 \right) - 2Re\{v_k^T \theta_{\ell}^T \text{diag}(h_k^H G w_\ell)\} + 1,
\]

where \(v = [v_1, \ldots, v_K]^{T}\) are auxiliary variables. According to [10], it can be proved that if \(\{\omega, \phi_i, W, \theta_i, \rho_i\}\) is a KKT optimal solution to problem (17), \(\{W, \theta_i, \rho_i\}\) is also a KKT optimal solution to problem (16). In problem (17), the objective function is block-wise convex with respect to \(\{\omega, \phi_i\}\), \(W\), and \(\{\theta_i, \rho_i\}\). Moreover, the feasible sets \(\{\omega, \phi_i\}\) and \(W\) are also convex. Thus, we have transformed the throughput maximization problem into the canonical form of problem (2), where the corresponding optimization variables, feasible set, and objective function are given by \(x = \{\omega, \phi_i, W, \theta_i, \rho_i\}\) and \(F(x, \theta_i, \rho_i) = \sum_{k \in K} \omega_k e_k\), respectively. Therefore, we can employ the proposed framework to solve problem (16).

By defining \(\theta_i = \theta_r\) and \(\rho_i = \rho_r\), problem (17) can be equivalently reformulated as follows:

\[
\min_{w, \phi_i} \sum_{k \in K} \omega_k e_k
\]

s.t. \(\theta_i = \theta_r\), \(\rho_i = \rho_r\), \(\text{tr}(WW^H) \leq P_t\),

\[
\beta_{t,n}^2 + \beta_{r,n}^2 = 1, \forall n \in \mathcal{N},
\]

\[
\cos(\phi_{t,n} - \phi_{r,n}) = 0, \forall n \in \mathcal{N}.
\]

By moving equality constraint (19b) via a penalty term to the objective function, the following problem is obtained:

\[
\min_{w, \phi_i} \sum_{k \in K} \omega_k e_k + \frac{1}{2\rho} \sum_{i \in \{\ell, r\}} \|\theta_i - \theta_r + \rho \lambda_i\|^2
\]

s.t. \(\text{(19c)} - \text{(19e)}\). (20b)

Then, the above problem can be solved via BCD by alternately optimizing the blocks \{\omega, \phi_i\}, \(W, \{\theta_i, \rho_i\}\), \{\psi_i, \phi_i\}, and \{\beta_i, \tilde{\beta}_i\}.

1) Subproblem With Respect to \(\{\omega, \phi_i\}\): The optimal \(\omega_k\) and \(\nu_k, k \in K_i, \forall i \in \{\ell, r\}\), are given by [10]

\[
\omega_k = 1 + \gamma_k,
\]

\[
u_k = \frac{\theta_i^T \text{diag}(h_k^H G w_\ell) + \sigma_k^2}{\sum_{\ell \in K} |\theta_{\ell}^T \text{diag}(h_k^H G w_\ell)|^2 + \sigma_k^2}.
\]
Algorithm 2 PDD-Based Algorithm for Solving Problem (19)
1: Initialize the optimization variables, and $0 < c < 1$
2: repeat
3:  repeat
4:   update $\{w, v\}$ by (21) and (22)
5:   update $W$ by solving (20) for $W$
6:   update $\{\theta_t, \theta_r\}$ by solving (20) for $\{\theta_t, \theta_r\}$
7:   update $\{\beta_t, \beta_r\}$ by (12)
8: until convergence
9: end if
10: if $\delta \leq \eta$ then set $\lambda_i = \lambda_i + \frac{1}{\rho} (\hat{\theta}_i - \theta_i), \forall i \in \{t, r\}$
11: else set $\rho = c \rho$
12: end if
13: set $\eta = 0.9 \delta$
14: until $\delta$ falls below a predefined threshold

2) Subproblems With Respect to $W$ and $\{\theta_t, \theta_r\}$: Note that the objective function of (19) is convex with respect to $W$ and $\{\theta_t, \theta_r\}$, respectively. Thus, the corresponding optimal solution can be efficiently obtained using existing optimization toolboxes, such as CVX [11].

3) Subproblems With Respect to $\{\psi_t, \psi_r\}$ and $\{\beta_t, \beta_r\}$: The optimal $\{\psi_t, \psi_r\}$ and $\{\beta_t, \beta_r\}$ can directly obtained based on Propositions 1 and 2.

Finally, the dual variables $\{\lambda_t, \lambda_r\}$ and penalty factor $\rho$ can be updated following the PDD framework. As a consequence, the overall algorithm for solving problem (19) is summarized in Algorithm 2, where $\delta = \max \{\|\theta_t - \theta_t\|_\infty, \|\theta_r - \theta_r\|_\infty\}$ denotes the constraint violation function. If the MFCQ condition is satisfied, the PDD framework can obtain a KKT optimal solution to problem (19), c.f. [7]. Thus, we show that the MFCQ condition indeed holds for problem (19).

Proposition 3: MFCQ holds for problem (19) at any feasible point $\{W, \theta_t, \theta_r\}$ with $W \neq 0$.

Proof: Please refer to Appendix C.

According to Proposition 3, we can conclude that a KKT optimal solution to problem (19) can be obtained with Algorithm 2, which is also a KKT optimal solution to the original problem (16).

Remark 1: This case study reveals that once the original problem is transformed into a form for which 1) the objective function is convex or block-wise convex, and 2) the feasible set is convex or block-wise convex except for constraints (2b) and (2c), the proposed optimization framework can be directly used. Typically, such a transformation can be achieved by existing methods such as WMMSE, majorization-minimization, and successive convex optimization (SCA). In most cases, if the transformed problem satisfies the mild MFCQ condition, the KKT optimal solution can be obtained when PDD is employed to update the dual variables and the penalty factor in the proposed framework [7]. Otherwise, at least the convergence of the proposed framework can be guaranteed.

C. Numerical Results

In this section, simulation results are provided to verify the effectiveness of the proposed optimization framework. Here, we assume that the BS with $M = 8$ antennas is 50 meters, under an angle of $20^\circ$, away from the STAR-RIS. The users are located on half-circles centered at the STAR-RIS with a radius of 3 meters. We also assume that half of the users are located on the transmission side and the rest are located on the reflection side. The channels are modeled as Ricean fading channels with a Ricean factor of 3 dB and a path loss exponent of 2.2. The path loss at the reference distance of 1 meter is set to 30 dB. The transmit power of the BS and the noise power at the users are set to 20 dBm and $-110$ dBm, respectively.

In Fig. 1, we first study the convergence of the proposed Algorithm 2 for $N = 20$. As can be seen from Fig. 1(a), the throughput rapidly converges to a stationary value for all considered values of $K$. Moreover, Fig. 1(b) shows the absolute phase-shift difference $|\phi_{t,n} - \phi_{r,n}|$ for all 20 elements. As can be observed, the absolute phase-shift differences converge to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, i.e., $\cos(\phi_{t,n} - \phi_{r,n}) = 0$. Next, we consider the following benchmark schemes for performance comparison. 1) STAR-RIS, Coupled, AO [6]: In this scheme, AO is exploited, where the coefficients of one side are optimized subject to the coupled phase-shift constraints by fixing the coefficients of the other side. 2) STAR-RIS, PS-PSC, T/R: This refers to the heuristic primary-secondary phase-shift configuration (PS-PSC) scheme, where the phase shifts for transmission (T)/reflection (R) side are optimal and those for the other side are adjusted accordingly. 3) STAR-RIS, Independent [2]: In this scheme, the phase shift coefficients of the STAR-RIS can be independently adjusted. 4) Conventional RIS [3]: In this scheme, there are two $\frac{N}{2}$-element reflect-only and transmit-only RISs deployed adjacent to each other.

In Fig. 2, we show the throughput versus the number of elements $N$ for different STAR-RIS optimization schemes when $K = 6$. The results are obtained by averaging over 100 random channel realizations. As can be observed, the proposed PDD-based optimization framework significantly outperforms the AO-based algorithm and the heuristic PS-PSC schemes for coupled phase-shift STAR-RISs. Moreover, Fig. 2 also reveals that the coupled phase-shift model achieves almost the same performance as the independent one and achieves a significant performance gain over the conventional RIS.
We proposed a general optimization framework for STAR-RISs with coupled phase shifts, which gives the provable KKT optimal solution under some mild conditions. Then, as a case study, we investigated throughput maximization based on the proposed optimization framework. Our numerical results confirmed the effectiveness of the proposed optimization method. The proposed framework can be extended to STAR-RIS design in various network architectures.

APPENDIX A
PROOF OF PROPOSITION 1

For any given $\beta_{1}$ and $\beta_{i}$, problem (9) can be decomposed into a series of independent optimization problems for each pair of $(\psi_{i,n},\psi_{r,n})$, which leads to
\[
\min_{\psi_{i,n},\psi_{r,n}} \text{Re}(\tilde{\beta}_{i}^{*}\tilde{\psi}_{i,n}) + \text{Re}(\tilde{\beta}_{r}^{*}\tilde{\psi}_{r,n}) \tag{23a}
\]
subject to
\[
\cos(\tilde{\psi}_{i,n} - \phi_{r,n}) = 0, \tag{23b}
\]
and $|\tilde{\psi}_{i,n}| = 1, |\tilde{\psi}_{r,n}| = 1. \tag{23c}$

Substituting the above constraint into the objective function, problem (23) can be further simplified as
\[
\min_{|\tilde{\psi}_{i,n}|=1} \text{Re}(\tilde{\beta}_{i}^{*}	ilde{\psi}_{i,n}) + j\text{Re}(\tilde{\beta}_{r}^{*}\tilde{\psi}_{r,n}),
\]
the optimal solution to which can be readily obtained as follows:
\[
\tilde{\psi}_{r,n} = e^{j(\pi - \phi_{r,n} + 3j\phi_{r,n})}, \tag{25}
\]
Comparing the objective values for the above two solutions, the optimal solution to problem (23) can be obtained, which completes the proof.

APPENDIX B
PROOF OF PROPOSITION 2

For any given $\tilde{\psi}_{i}$ and $\tilde{\psi}_{r}$, problem (9) can be decomposed into a series of independent optimization problems for each pair of $(\beta_{i},\tilde{\beta}_{r,n})$, which leads to
\[
\min_{\beta_{i},\tilde{\beta}_{r,n}} \text{Re}(\tilde{\beta}_{i}^{*}\tilde{\psi}_{i,n}) + \text{Re}(\tilde{\beta}_{r}^{*}\tilde{\psi}_{r,n}) \tag{26a}
\]
such that $\beta_{i}^{2} + \tilde{\beta}_{r,n}^{2} = 1, 0 \leq \beta_{i}, \tilde{\beta}_{r,n} \leq 1. \tag{26b}$

Since $\tilde{\beta}_{i,n}$ is real-valued, we simplify the objective function as follows:
\[
a_{n}\beta_{i} \tilde{\psi}_{i,n} + b_{n}\tilde{\beta}_{r,n}, \tag{27}
\]
and $b_{n} = |\tilde{\psi}_{r,n}| \cos(\phi_{r,n})$. To solve the simplified problem, polar coordinates are used, where we set $\beta_{i,n} = \sin\omega_{n}$ and $\tilde{\beta}_{r,n} = \cos\omega_{n}$ with $\omega_{n} \in [0, \frac{\pi}{2}]$. Hence, constraint (26b) is automatically satisfied. Based on this transformation, the objective function can be rewritten as
\[
a_{n}\sin\omega_{n} + b_{n}\cos\omega_{n} = \sqrt{a_{n}^{2} + b_{n}^{2}} \cos(\phi_{r,n} + \xi_{n}), \tag{27}
\]
where $\cos\xi_{n} = \frac{a_{n}}{\sqrt{a_{n}^{2} + b_{n}^{2}}}$ and $\sin\xi_{n} = \frac{b_{n}}{\sqrt{a_{n}^{2} + b_{n}^{2}}}$. Then, the optimal solution in (13) for minimizing $\sin(\omega_{n} + \xi_{n})$ with respect to $\omega_{n} \in [0, \frac{\pi}{2}]$ can be readily obtained, which completes the proof.

APPENDIX C

PROOF OF PROPOSITION 3

In this Appendix, we show that the MFCQ condition [8, Appendix C] is satisfied by problem (19). Note that for problem (19), optimizing $\tilde{\theta}$ is equivalent to optimizing its amplitudes $\tilde{\beta}$ and phase shifts $\phi$, which are linear independent. Furthermore, the gradients of $\mu_{i}$ at $t_{i}$ are linear independent if and only if those of $\mu_{i,0}$ are linear independent. Then, according to the definition of MFCQ [8, Appendix C], we are left to show that there exists a matrix $D_{WF}$ and a vector $d_{\omega}$ such that
\[
\text{Re}(\text{tr}(\text{W FD}_{WF}^{H})) < 0, \tag{29a}
\]
and $\nabla_{\mu_{i}}d_{\omega} = 0, \forall i \in \{1, 2, 3, 4\}, \tag{29b}$

where $\text{Re} \mu_{i}$ denotes the Jacobian matrix of $\mu_{i}$ with respect to $\omega$. It can be readily proved that the above equations are satisfied by setting $D_{WF} = -W$ and $d_{\omega} = 0$, which completes the proof.

REFERENCES

[1] Y. Liu et al., “STAR: Simultaneous transmission and reflection for 360° coverage by intelligent surfaces,” IEEE Trans. Wireless Commun., vol. 28, no. 6, pp. 102–109, Dec. 2021.
[2] X. M. et al., “Simultaneously transmitting and reflecting (STAR) RIS aided wireless communications,” IEEE Trans. Wireless Commun., vol. 21, no. 5, pp. 3083–3098, May 2022.
[3] C. Huang et al., “Reconfigurable intelligent surfaces for energy efficiency in wireless communication,” IEEE Trans. Wireless Commun., vol. 18, no. 8, pp. 4157–4170, Aug. 2019.
[4] J. Xu et al., “STAR-RISs: A correlated T&R phase-shift model and practical phase-shift configuration strategies,” IEEE J. Sel. Topics Signal Process., vol. 16, no. 5, pp. 1097–1111, Aug. 2022.
[5] Y. Liu et al., “Simultaneously transmitting and reflecting (STAR)-RIS: A coupled phase-shift model,” in Proc. IEEE Int. Conf. Commun. (ICC), May 2022, pp. 2840–2845.
[6] H. Niu and X. Liang, “Weighted sum-rate maximization for STAR-RIS-aided networks with coupled phase-shifters,” IEEE Syst. J., early access, Apr. 7, 2022, doi: 10.1109/SYST.2022.3159551.
[7] Q. Shi and M. Hong, “Penalty dual decomposition method for non-smooth nonconvex optimization—Part I: Algorithms and convergence analysis,” IEEE Trans. Signal Process., vol. 68, pp. 4108–4122, Jun. 2020.
[8] Q. Shi et al., “Penalty dual decomposition method for non-smooth nonconvex optimization—Part II: Applications,” IEEE Trans. Signal Process., vol. 68, pp. 4242–4257, Jun. 2020.
[9] M. Razaviyayn et al., “A unified convergence analysis of block successive minimization methods for non-smooth optimization,” SIAM J. Optim., vol. 23, no. 2, pp. 1126–1153, 2013.
[10] S. S. Christiansen et al., “Weighted sum-rate maximization using weighted MMSE for MIMO-BC beamforming design,” IEEE Trans. Wireless Commun., vol. 7, no. 12, pp. 4792–4799, Dec. 2008.
[11] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 2.1,” Mar. 2014. [Online]. Available: http://cvxr.com/cvx