NOTE ON EXTENSIONS OF THE BETA FUNCTION

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Abstract. The classical beta function \( B(x, y) \) is one of the most fundamental special functions, due to its important role in various fields in the mathematical, physical, engineering and statistical sciences. Useful extensions of the classical Beta function has been considered by many authors. In the present paper, our main objective is to study the convergence of extensions of classical beta function and introduce modified extension of classical beta function. It is interpreted numerically and geometrically in the view of convergence, further properties and integral presentations are established.

1. Introduction and Preliminaries

In many areas of applied mathematics, various types of special functions become essential tools for scientists and engineers. The (Euler’s) classical beta function \( B(x, y) \) is one of the most fundamental special functions, because of its important role in various fields in the mathematical, physical, engineering and statistical sciences. During last four decades or so, several interesting and useful extensions of the familiar special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) have been considered by many authors. In the present work, we are concerned with various generalizations of the classical beta function, which can be found in the literature (see [1, 2, 3, 4]).

Recently the generalized beta function \( B_p^{\delta, \zeta; \kappa, \mu}(\alpha, \beta) \) is introduced by Srivastava et al., which is the most generalized extension of classical beta function and is defined as (see [1]):

\[
B_p^{\delta, \zeta; \kappa, \mu}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, _1F_1 \left( \delta; \zeta; -\frac{p}{t^\kappa (1-t)^\mu} \right) \, dt, \tag{1.1}
\]

where \( \Re(p) \geq 0; \min[\Re(\alpha), \Re(\beta), \Re(\delta), \Re(\zeta)] > 0; \min[\Re(\kappa), \Re(\mu)] > 0 \) and \( _1F_1(.) \) is confluent hypergeometric function, which is special case of the well known generalized hypergeometric series \( pF_q(.) \).

The generalized hypergeometric series \( pF_q(p, q \in \mathbb{N}) \) is defined as (see [5, p.73]) and [6 pp. 71-75]:

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\[ \left[ \alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z \right]_p = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{n!} = pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \]

where \((\lambda)_n\) is the Pochhammer symbol defined (for \(\lambda \in \mathbb{C}\)) by (see[6, p.2 and p.5]):

\[ (\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \]

\[ = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \]

and \(\mathbb{Z}_0^-\) denotes the set of Non-positive integers and \(\Gamma(\lambda)\) is familiar Gamma function.

When \(\kappa = \mu\), (1.1) reduces to the generalized extended beta function \(B^{(\delta,\zeta;\mu)}_p(\alpha, \beta)\) defined by (see[2, p. 37])

\[ B^{(\delta,\zeta;\mu)}_p(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} F_1(\delta; \zeta; -\frac{p}{t(1-t)^\mu}) \, dt, \]

where \(\Re(p) \geq 0; \min[\Re(\alpha), \Re(\beta), \Re(\delta), \Re(\zeta)] > 0; \Re(\mu) > 0.\)

The special case of (1.5), when \(\mu = 1\) reduces to the generalized beta type function as follows (see[3, p.4602])

\[ B^{(\delta,\zeta)}_p(\alpha, \beta) = B^{(\delta,\zeta;1)}_p(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} F_1(\delta; \zeta; -\frac{p}{t(1-t)^\mu}) \, dt, \]

where \(\Re(p) \geq 0; \min[\Re(\alpha), \Re(\beta), \Re(\delta), \Re(\zeta)] > 0.\)

The further special case of (1.6) when \(\delta = \zeta\) is given due to Choudhary et. al.[4] by

\[ B_p(\alpha, \beta) = B(\alpha, \beta; p) = B^{(\delta,\zeta)}_p(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \exp \left(-\frac{p}{t(1-t)^\mu}\right) \, dt, \]

where \(\Re(p) \geq 0).\)

When we choose \(p = 0\), all the above extensions reduces to the classical beta function \(B(x, y)\), which is defined by
\[ B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} dt, \quad (\Re(\alpha) > 0, \Re(\beta) > 0). \]

It is clear to see the following relationship between the classical beta function \( B(\alpha, \beta) \) and its extensions:

\[ B(\alpha, \beta) = B_0(\alpha, \beta) = B^{(0,0)}(\alpha, \beta) = B^{(0,0)}_\infty(\alpha, \beta). \]  

(1.9)

In particular, Euler’s beta function \( B(x, y) \) has a close relationship to his gamma function,

\[ B(\alpha, \beta) = B(\beta, \alpha) = \Gamma(\alpha)\Gamma(\beta) / \Gamma(\alpha + \beta). \]  

(1.10)

2. Extensions of Classical Beta Function do not convergent

We claim that the above extensions of classical beta function in equations (1.1), (1.5), (1.6), and (1.7) are not convergent. We prove it mathematically as well as numerically tested in Section 4.

First we prove that the extension in equation (1.7) is not convergent as follows.

**Claim 1.** If \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0 \), then the extensions of classical beta function in equation (1.7) does not convergent.

**Proof.** To prove our claim, we write the exponential function in series form, the equation (1.7) reduces to the following form

\[ B(\alpha, \beta; p) = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} \int_0^\infty t^{\alpha-n-1} (1 - t)^{\beta-n-1} dt, \quad (2.1) \]

Further using the definition of the beta function in the above equation (2.1), we have

\[ B(\alpha, \beta; p) = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} B(\alpha - n, \beta - n) \]

\[ = B(\alpha, \beta) + \frac{-p}{1!} B(\alpha - 1, \beta - 1) + ... \]

(2.2)

In the above equation (2.2), \( B(\alpha, \beta; p) \) is an power series involving \( B(\alpha - n, \beta - n) \) (where \( n = 0, 1, 2, ... \)). The series \( B(\alpha, \beta; p) \) will be convergent if \( B(\alpha - n, \beta - n) \) is convergent and \( B(\alpha - n, \beta - n) \) will be convergent only if \( \Re(\alpha - n) > 0 \) and \( \Re(\beta - n) > 0 \).
But this is not possible as $\alpha$ and $\beta$ are finite and $n \to \infty$. The series $B(\alpha, \beta; p)$ contains many terms, which does not exist. Moreover $B(\alpha, \beta; p)$ is the sum of terms involving $B(\alpha - n, \beta - n)$, which are not convergent, which implies that $B(\alpha, \beta; p)$ is not convergent. □

Example 1. If we choose $\alpha = 5, \beta = 7, p = 3$, then from equation (2.2), we have

$$B(5, 7; 3) = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} B(5 - n, 7 - n),$$

(2.3)

The above series $B(5, 7; 3)$ is the sum of terms involving the beta function $B(5 - n, 7 - n)$ and $B(5 - n, 7 - n)$ does not exist for $n > 5$. Therefore $B(5, 7; 3)$ is not convergent.

In the following section, we introduce modified extension of classical beta function in equation (3.1). Further, the extension of classical beta function (1.7), Modified extension of classical beta function (3.1) and the classical beta function are compared and tested numerically to test the convergence of extensions of the classical beta function.

3. Modified extension of beta function

In this section, we introduce modified extension of classical beta function (MECBF). Its convergence is proved mathematically, then numerical results are established and compared the results with that of the classical beta function and extension of classical beta function.

We introduce modified extension of classical beta function as follows

$$B_m(\alpha, \beta) = B(\alpha, \beta; m) = \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} e^{mt(1-t)} dt,$$

(3.1)

where $\Re(\alpha) > 0, \Re(\beta) > 0, m \in \mathbb{C}; |m| < M$ (where $M$ is positive finite real number).

In our investigation to test the convergence of the above extension of the classical beta function the following definitions are required.

**Definition 1** (Ratio Test[9]). Let $\sum_{n=1}^{\infty} p_n$ be a series of positive terms, and suppose that $\frac{p_{n+1}}{p_n} \to$ a limit $l$ as $n \to \infty$. Then (i) If $l < 1$ the series $\sum p_n$ is convergent, (ii) If $l > 1$, then the series is divergent. If $l = 1$, the ratio test fails, and the question of convergence of $\sum p_n$ must be investigated by some other methods.

**Definition 2** (Leibniz’s Test or Alternating Series Test[9]). The series $\sum_{n=1}^{\infty} (-1)^n a_n$ (where $a_n > 0$) is convergent provided (i) $a_n$ is a decreasing sequence, (ii) $a_n \to 0$ as $n \to \infty$.

**Theorem 1.** If $\Re(\alpha) > 0, \Re(\beta) > 0, m \in \mathbb{C}; |m| < M$ (where $M$ is positive finite real number), then the modified extension of the classical beta function in equation (3.1) is convergent.
Proof. We can write the above equation (3.1) as follows

\[ B(\alpha, \beta; m) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \sum_{n=0}^{\infty} \frac{[mt(1-t)]^n}{n!} dt \]

\[ = \sum_{n=0}^{\infty} \frac{(m)^n}{n!} \int_0^1 t^{\alpha+n-1}(1-t)^{\beta+n-1} dt, \]  

(3.2)

Further, using the definition of beta function, the above equation (3.2) reduces to

\[ B(\alpha, \beta; m) = \sum_{n=0}^{\infty} \frac{(m)^n}{n!} B(\alpha + n, \beta + n). \]  

(3.3)

In the above equation, \( B(\alpha, \beta; m) \) is in series form involving \( B(\alpha + n, \beta + n) \) (where \( n = 0, 1, 2, \ldots \)) and in each term of the series, \( B(\alpha + n, \beta + n) \) is convergent since \( \Re(\alpha + n) > 0 \) and \( \Re(\beta + n) > 0 \) for \( \Re(\alpha), \Re(\beta) > 0 \), which implies that each term of the series (3.3) exist.

Now we shall prove that \( B(\alpha, \beta; m) \) is convergent. \( m \) may be greater than or less than 0, so there are two cases as follows.

Case 1. If \( m > 0 \), then to prove \( B(\alpha, \beta; m) \) is convergent.

The equation (3.3) can be written as

\[ B(\alpha, \beta; m) = \sum_{n=0}^{\infty} a_n, \]  

(3.4)

where \( a_n = \frac{(m)^n}{n!} B(\alpha + n, \beta + n). \)

Further

\[ \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \infty > 1. \]  

(3.5)

By ratio test, \( B(\alpha, \beta; m) \) is convergent for \( m > 0 \).

Case 2. If \( m < 0 \), then to prove that the extension of classical beta function \( B(\alpha, \beta; m) \) is convergent.

To prove this case, let \( m = -p \) (where \( p > 0 \)), then equation (3.3)

\[ B(\alpha, \beta; m) = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} B(\alpha + n, \beta + n), \]  

(3.6)
the above equation (3.6) can be written as

\[ B(\alpha, \beta; m) = - \sum_{n=0}^{\infty} (-1)^{n+1} b_n, \]

where \( b_n = \frac{(p)^n}{n!} B(\alpha + n, \beta + n). \)

The series (3.7) is an alternating series, therefore

1. \( a_n > 0, \forall p > 0, \Re(\alpha, \beta) > 0 \)
2. \( a_n - a_{n+1} = \frac{p^n}{n!} B(\alpha + n, \beta + n) \times \left[ 1 - \frac{(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 1)} \right] > 0 \)
3. \( \lim_{n \to \infty} a_n = 0 \) if \( p \leq 2 \) \( (p^n/n! \to 0 \) as \( n \to \infty \) only if \( p \leq 2 \) and \( B(\alpha + n, \beta + n) \to 0 \) as \( n \to \infty \)

All the conditions of Leibniz’s test have been satisfied, therefore \( B(\alpha, \beta; m) \) is convergent for \( 0 < p \leq 2 \) i.e. \( -2 \leq m < 0 \).

From the above two cases 1 and 2 implies that the power series in equation (3.3) is convergent.

4. Comparison of numerical values of extended beta function, modified extension of the classical beta function and classical beta function

In Table 1, the columns B1, B2 are the values of the classical beta function \( B(x, y) \); the columns EB1, EB2, EB3, EB4 are the values of the extension of classical beta function; the columns MB1, MB2, MB3, MB4 are values of modified extension of the classical beta function, where \( B_1 = B(x, y) \), \( EB_1 = B(x, y; 0.01) \), \( MB_1 = B(x, y; 0.01) \), \( B_2 = B(x, 0.25) \), \( EB_2 = B(x, 0.25; 0.00) \), \( MB_2 = B(x, 0.25; 0.00) \), \( EB_3 = B(x, 0.25; 0.01) \), \( MB_3 = B(x, 0.25; 0.01) \), \( EB_4 = B(x, 0.25; 0.01) \), and \( MB_4 = B(x, 0.25; 0.01) \).
Table 2. Numerical Values of MECBF

| x   | y   | B(x,y,-2.0335) | B(x,y,-2.0) | B(x,y,-1.0) | B(x,y,0.0) | B(x,y,1.0) | B(x,y,2.0) | B(x,y,2.0335) |
|-----|-----|----------------|-------------|-------------|------------|------------|------------|----------------|
| 0.00| 0.00| NaN            | NaN         | NaN         | NaN        | NaN        | NaN        | NaN            |
| 0.01| 0.01| 198.27177610  | 198.2958642 | 199.06411204| 199.96575732| 201.03554301| 202.30476477| 202.35127432  |
| 0.02| 0.02| 98.27460320   | 98.29781890 | 99.05065967 | 99.93608768 | 100.9830172 | 102.22848838| 102.27412810  |
| 0.03| 0.03| 64.94418565   | 64.96691911 | 65.70435720 | 66.57217010 | 67.59910183 | 68.82077895 | 68.86556756  |
| 0.04| 0.04| 48.28052079   | 48.30783821 | 49.02591017 | 49.87579792 | 50.88290361 | 52.08158267 | 52.12538368  |
| 0.05| 0.05| 38.23605999   | 38.30541007 | 39.0134459  | 39.84694542 | 40.83466875 | 42.01084757 | 42.05398809  |
| 0.06| 0.06| 31.62010537   | 31.64146162 | 32.33487366 | 33.15225447 | 34.12102693 | 35.27519021 | 35.31753282  |
| 0.07| 0.07| 26.86144500   | 26.88236422 | 27.56179291 | 28.36312969 | 29.31337104 | 30.4459075  | 30.48755229  |
| 0.08| 0.08| 23.29333673   | 23.31382939 | 23.97960405 | 24.76526226 | 25.69738092 | 26.80891675 | 26.84971362  |
| 0.09| 0.09| 20.51895277   | 20.53902903 | 21.1946950  | 21.96180446 | 22.87617970 | 23.96709764 | 24.00714582  |
| 0.10| 0.10| 18.3019555    | 18.31986529 | 18.95928227 | 19.71463949 | 20.61169459 | 21.68239456 | 21.7270964  |
| 1.0  | 1.0 | 0.72102562    | 0.72477846  | 0.84887277  | 1.00000000  | 1.18459307 | 1.41066813 | 1.41909001   |
| 2.0  | 2.0 | 0.11166177    | 0.11238923  | 0.13665458  | 0.16666667  | 0.20385173 | 0.25000000 | 0.25172603   |
| 3.0  | 3.0 | 0.02163870    | 0.02171912  | 0.02692731  | 0.03333333  | 0.04133360 | 0.05133577 | 0.05171118   |
| 4.0  | 4.0 | 0.00455690    | 0.00459058  | 0.00572283  | 0.00714286  | 0.00892526 | 0.01116423 | 0.01124846   |
| 5.0  | 5.0 | 0.00100158    | 0.00100918  | 0.00126512  | 0.00158730  | 0.00199311 | 0.00250456 | 0.00252383   |
| 6.0  | 6.0 | 0.00022592    | 0.00022766  | 0.00028649  | 0.00036075  | 0.00045452 | 0.00057299 | 0.00057574   |
| 7.0  | 7.0 | 0.00000518    | 0.000005225 | 0.00006594  | 0.00008325  | 0.00010515 | 0.00013287 | 0.00013932   |
| 8.0  | 8.0 | 0.000001205   | 0.000001214 | 0.00001536  | 0.00001943  | 0.00002458 | 0.00003112 | 0.00003137   |
| 9.0  | 9.0 | 0.000000283   | 0.000000285 | 0.00000361  | 0.00000457  | 0.00000579 | 0.00000734 | 0.00000740   |
| 10.0 | 10.0| 0.00000067    | 0.00000067  | 0.00000085  | 0.00000108  | 0.00000174 | 0.00000176 | 0.00000176   |

MB3 = B(x, 0.25; 0.01), EB4 = B(x, y; 0.00); MB4 = B(x, y; 0.00) and x, y = 0 : 1 : 10.

These values are computed by employing Matlab (R2012a). The values of the extension of the classical beta function are computed by using the equation (2.2) and those of modified extension of the classical beta function from equation (3.3). Both the extension in equations (2.2) and (3.3) are in series form involving beta functions B(x − n, y − n) (exist for Re(x − n > 0, y − n) > 0) and B(x + n, y + n) (exist for Re(x + n > 0, y + n) > 0) respectively, and the values can be computed easily. Therefor the values of extension of the classical beta function (2.2) are computed by taking sum of the first five terms of the series in the view of existing values of B(x − n, y − n) to find some numerical values and in the case of modified extension of classical beta function, sum of first 1000 terms have been taken for x, y = 0 : 1 : 10. The numerical results of the same are established in Table 1.

From Table 1, it is easily observed that the values of extended beta function in columns EB1, EB4 does not exist for x, y = 0 : 1 : 5 (since Re(x − n < 0, y − n) < 0); rest values of the same exist for x, y = 6 : 1 : 10 (since Re(x − n) > 0, Re(y − n) < 0). In columns...
EB2, EB3 the values of extended beta function do not exist since in these cases we choose \( y = 0.25 \) therefore \( \Re(y - n) \) becomes less than zero, which implies if \( 0 < x, y < 1 \) the values of the extended beta function do not exist even if sum of first two terms of the series (2.2) to be taken. If we take large sum of the terms of the series (2.2) (say \( n = 1000 \)) the extended beta function will not converge for any value of \( x, y \in [0, 1000] \) from which we conclude that the extended beta function does not exists for any value of \( x \) and \( y \), since it is series having infinite terms and \( \Re(x), \Re(y) \) can not be infinite.

In the case of modified extension of classical beta function, the columns B1, MB4 are identical, which implies that if we choose \( m = 0 \), we have the values of classical beta function from the modified extension of the classical beta function. Column B2, MB2 are also identical since \( m = 0 \) and at the fix value of \( y = 0.25 \). If we compare the values of B1 with MB1 and those of B2 with MB3, we can see the impact of the value of \( m \) on the classical beta function.

**Note 1.** It is also noted that if \( p = 0 \), the extension of classical beta function should be reduced to the classical beta function i.e. \( B(\alpha, \beta; 0) = B(\alpha, \beta) \). But the same not depict in the graph cited in [3] Fig. 1, p. 31. The graph depicts the values of extension of the classical beta function \( B(1, 2.25; 0) = B(1, 0.25) = 13.75 \) when \( p = 0, x = 1, y = 0.25 \) and \( B(10, 2.25; 0) = B(10, 0.25) = 2.18 \) (approximately) when \( p = 0, x = 10, y = 0.25 \), which are not the correct values of the classical beta function. The correct values of the classical beta function at these point are \( B(1, 0.25) = 4 \) and \( B(10, 0.25) = 2.0582 \). It also can be easily observed from the figure that when \( p \neq 0 \) the graph of extended beta function totally different from that of the classical beta function. i.e. the behavior and nature of the curves differ from that of classical beta function.

**Note 2.** From the above discussion, it is very clear that the extension of classical beta function in equation (1.7) do not convergent. The extensions in equations (1.1), (1.5) and (1.6) are further extension of the extended beta function (1.7), therefor all these extensions also can not be convergent.

**Note 3.** Due to the lack of the convergence of the extension of the classical beta function, all the associated results established by Chaudhary et al. [4] do not convergent.

5. **Numerical results and their interpretation of modified extension of the classical beta function**

The numerical results of modified extension of classical beta function (MECBF) have been calculated in this section by employing the Matlab. We choose the values of variable \( x, y \) and parameter \( m \) as \( x, y \in [0, 10] \) and \( m \in [-2.0335, 2.0335] \). All the numerical values of MECBF are presented in Table 1, from which we can easily observe that \( B(x, y; m) \) does not exist at \( x = y = 0 \) and it is also tested that \( B(x, y; m) \) does not exist for \( m < -2.0335 \) and \( m > 2.0335 \). It can be easily investigated that \( B(x, y; m) \to \infty \) as \( x, y \to 0 \) and \( B(x, y; m) \to 0 \) as \( x, y \to \infty \), which implies that the behavior of modified extension of classical beta function is the same as that of classical beta function, which can be observed from Table 2 and Figure 1. Also \( B(x, y; 0) \) represents the values of classical beta function \( B(x, y) \).
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6. GEOMETRICAL INTERPRETATION OF MODIFIED EXTENSION OF THE CLASSICAL BETA FUNCTION

If $\beta = 1$, $m \in [-2, 2]$ and $x \in [0, 1]$, then we observe that the graph of $B(x, y; m)$ in Figure 1 is decreasing. In Figure 2, we choose $m = 2$ and $x, y \in [0, 25]$, which depicts that as $x, y \to 0$, $B(x, y; m) \to \infty$ and $B(x, y; m) \to 0$ as $\alpha, \beta \to \infty$. We also check the effect of $m$ on the modified extension of classical beta function. For this purpose, we fix the values of $x$ and $y$ as shown in Figure 3, then we plot the graph which depicts that $B(x, y; m)$ is an increasing function as the values of $m$ increase. It is very clear from Figure 1 that for the graph of classical beta function and modified extension of classical beta function remains concave upward (or convex downward) for different values of $\alpha, \beta$ and $m$. The value of $m$ does not effect the nature of classical beta function, the main effect of the value of $m$ is that it just push the curve up or drag down the curve from the curve of the classical beta function as shown in the Figure 1 the same behavior can be observed from the Table 2.

From the above proof of radius of convergence of series and further more numerically investigation of the power series in Table 1, we find that the interval of convergence of the series is $(-2.0336, 2.0336)$, which implies that $B(x, y; m)$ is convergent for $|m| < M$ where $M$ is positive real number slightly greater than 2.0335.
Figure 2. 3D presentation of of MECBF.

Note 4. From the above discussion, it is easy to conclude that the value of $\Re(m)$ lies in the interval $[-2.0335, 2.0335]$ i.e. $-2.0335 \leq \Re(m) \leq 2.0335$.

Note 5. In the sequel of this paper, $|m| < M$ represents the circle of convergence and $M$ is the radius of convergence of equation (3.1), where $M$ is sightly grater than from 2.0335.

7. Integral representation of the MECBF function

The integral representation of the modified extended beta function is important to check both the extension is natural and simple and for use later. It is also important to investigate the relationship between the classical beta function and the modified extension of the classical beta function. In this connection we first provide a relationship between them. The following integral formula is useful for further investigation [7]

$$\int_0^{\infty} x^m \exp(-\beta x^n)dx = \frac{\Gamma(\gamma)}{n\beta^n}, \quad (\gamma = \frac{m+1}{n}). \quad (7.1)$$

Theorem 2 (Relation between modified extension of the classical beta function and the classical beta function). If $\Re(\alpha + s) > 0$, $\Re(\beta + s) > 0$, $m \in \mathbb{C}$; $|m| < M$ (where $M$ is positive finite real number slightly greater than 2.0335), then we have the following
relation
\[ \int_0^\infty m^{s-1} B(\alpha, \beta; m) \, dm = (-1)^s \Gamma(s) B(\alpha + s, \beta + s). \] (7.2)

Proof. Multiplying both sides of equation (3.1) by \( m^{s-1} \), then integrate w.r.t. \( m \) from \( m = 0 \) to \( m = \infty \), we have
\[ \int_0^\infty m^{s-1} B(\alpha, \beta; m) \, dm 
= \int_0^\infty m^{s-1} \left[ \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \exp(mt(1-t)) \, dt \right] \, dm, \] (7.3)
interchanging the order of integration, the above equation (7.3), reduces to
\[ \int_0^\infty m^{s-1} B(\alpha, \beta; m) \, dm 
= \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \left[ \int_0^\infty m^{s-1} \exp(mt(1-t)) \, dm \right] \, dt, \] (7.4)

further using the formula given in equation (7.1), after simplification the above equation (7.4) reduces to

\[ \int_0^\infty m^{s-1} B(\alpha, \beta; m) \, dm = (-1)^s \Gamma(s) B(\alpha + s, \beta + s). \] (7.2)

Figure 3. Graph of MECBF for fixed value of \( x, y \) and \( m = -2 : .5 : 2. \)
\[
\int_0^\infty m^{s-1} B(\alpha, \beta; m) \, dm = (-1)^s \Gamma(s) \int_0^1 t^{\alpha + s - 1} (1 - t)^{\beta + s - 1} \, dt, \tag{7.5}
\]

using the definition of classical beta function, we have the required result. \(\square\)

**Remark 1.** By setting \(s = 1\), the result in equation (7.2) reduces to

\[
\int_0^\infty B(\alpha, \beta; m) \, dm = -B(\alpha + 1, \beta + 1), \tag{7.6}
\]

\(\Re(\alpha) > -1, \Re(\beta) > -1,\)

gives the interesting relation between classical beta function and modified extended beta function.

**Remark 2.** All the derivatives of the modified extension of classical beta function (MECBF) with respect to the parameter \(m\) can be expressed in terms of the function as

\[
\frac{\partial^n}{\partial^n m} B(\alpha, \beta; m) = B(\alpha + n, \beta + n), \tag{7.7}
\]

\(\Re(\alpha + n) > 0, \Re(\beta + n) > 0.\)

**Theorem 3** (Integral representations of the modified extension of the classical beta function). If \(\Re(x) > 0, \Re(y) > 0, m \in \mathbb{C}; |m| < M\) (where \(M\) is positive finite real number slightly greater than 2.0335), then we have the following relation

\[
B(\alpha, \beta; m) = 2 \int_0^{\pi/2} \cos^{2\alpha - 1} \theta \sin^{2\beta - 1} \theta \times \exp(m \cos^2 \theta \sin^2 \theta) \, d\theta, \tag{7.8}
\]

\[
B(\alpha, \beta; m) = \int_0^\infty \frac{u^{a-1}}{(1 + u)^{a + \beta}} \exp(mu/(1 + u)^2) \, du, \tag{7.9}
\]

\[
B(\alpha, \beta; m) = \frac{1}{2} \int_0^\infty \frac{u^{a-1} + u^{\beta-1}}{(1 + u)^{a + \beta}} \times \exp(mu/(1 + u)^2) \, du, \tag{7.10}
\]

\[
B(\alpha, \beta; m) = (c - a)^{1-\alpha-\beta} \int_a^c (u - a)^{\alpha-1}(c - u)^{\beta-1} \times \exp\left(\frac{m(u - a)(c - u)}{(c - a)^2}\right) \, du, \tag{7.11}
\]

\[
B(\alpha, \beta; m) = 2^{1-\alpha-\beta} \int_{-1}^1 (1 + t)^{\alpha-1}(1 - t)^{\beta-1} \times \exp(m(1 - t^2)/4) \, du, \tag{7.12}
\]
\[ B(\alpha, \beta; m) = 2^{1-\alpha-\beta} \times \int_{-\infty}^{\infty} \exp \left[ (\alpha - \beta)x + \frac{m}{4 \cosh^2 x} \right] \frac{dx}{(\cosh x)^{\alpha+\beta}}, \]  
(7.13)

\[ B(\alpha, \beta; m) = 2^{2-\alpha-\beta} \int_{0}^{\infty} \cosh((\alpha - \beta)x) \times \exp \left[ \frac{m}{4 \cosh^2 x} \right] \frac{dx}{(\cosh x)^{\alpha+\beta}}, \]  
(7.14)

\[ B(\alpha, \beta; m) = 2^{1-\alpha-\beta} \times \int_{-\infty}^{\infty} \exp \left[ \frac{1}{2} (\alpha - \beta)x + \frac{m}{2 \cosh x} \right] \frac{dx}{(\cosh x/2)^{\alpha+\beta}}, \]  
(7.15)

\[ B(\alpha, \beta; m) = 2^{2-\alpha-\beta} \int_{0}^{\infty} \cosh((\alpha - \beta)x/2) \times \exp \left[ \frac{m}{2 \cosh x} \right] \frac{dx}{(\cosh x/2)^{\alpha+\beta}}. \]  
(7.16)

Proof. The result (7.8) can be easily obtained by setting \( t = \cos^2 \theta \), to prove (7.9) choose \( t = u/(1 + u) \), (7.10) can be easily obtained by applying the symmetric property in equation (7.9) then adding new one and (7.9), the result in equation (7.11) is obtained by taking \( t = (u - a)/(c - a) \), setting \( a = -1, c = 1 \) in equation (7.11) gives the result in equation (7.12) and to prove the result in equation (7.13) put \( u = \tanh x \) in (7.12). The results in equation (7.14), (7.15) and (7.16) can be easily obtained from the result (7.13). \( \square \)

Remark 3 (Useful inequalities). If \( \Re(\alpha) > 0, \Re(\beta) > 0 \), then we have the following inequality

\[ |B(\alpha, \beta; m)| \leq 1.6626 B(\alpha, \beta) \]  
(7.17)

follows from the integral representation (7.9). Since the function \( \exp(mu/(1+u)^2) \) attains its maximum value 1.6626 at \( u = 1 \) and \( m = 2.0335 \). The equality is verified with the help of numerical results by using Matlab.

8. Properties of the modified extension of the classical beta function

Theorem 4 (Functional Relation). If \( \Re(\alpha) > 0, \Re(\beta) > 0, m \in \mathbb{C}; |m| < M \) (where \( M \) is positive finite real number slightly greater than 2.0335), then we have the following relation

\[ B(\alpha, \beta + 1; m) + B(\alpha + 1, \beta; m) = B(\alpha, \beta; m). \]  
(8.1)
Proof. LHS of the above equation (8.1) is equal to
\[
\int_0^1 \{t^{\alpha-1}(1-t)^\beta + t^{\alpha}(1-t)^{\beta-1}\}e^{mt(1-t)}\,dt
\]  
(8.2)
after simplification the above equation (8.2) reduced to
\[
\int_0^1 \{t^{\alpha-1}(1-t)^{\beta-1}\}e^{mt(1-t)}\,dt = B(\alpha, \beta; m).
\]  
(8.3)

If we choose \( m = 0 \), we get the usual relation for the beta function from (8.1).

**Theorem 5** (Symmetry). If \( \Re(\alpha) > 0, \Re(\beta) > 0, m \in \mathbb{C}; |m| < M \) (where \( M \) is positive finite real number slightly greater than 2.0335), then we have the following relation
\[
B(\alpha, \beta; m) = B(\beta, \alpha; m).
\]  
(8.4)

**Proof.** From equation (3.3) we have
\[
B(\alpha, \beta; m) = \sum_{n=0}^{\infty} \frac{(m)^n}{n!} B(\alpha + n, \beta + n)
\]  
(8.5)

Since usual beta function is symmetric i.e. \( B(\alpha, \beta) = B(\beta, \alpha) \). Using this property in the right hand side of the equation (8.5), then we have
\[
B(\alpha, \beta; m) = \sum_{n=0}^{\infty} \frac{(m)^n}{n!} B(\beta + n, \alpha + n) = B(\beta, \alpha; m).
\]  
(8.6)

**Theorem 6** (First Summation Relation). If \( \Re(\alpha) > 0, \Re(1 - \beta) > 0, m \in \mathbb{C}; |m| < M \) (where \( M \) is positive finite real number slightly greater than 2.0335), then we have the following relation
\[
B(\alpha, 1 - \beta; m) = \sum_{n=0}^{\infty} \frac{(\beta)^n}{n!} B(\alpha + n, 1; m).
\]  
(8.7)

**Proof.** The LHS of the equation (8.7) can be written as
\[
B(\alpha, 1 - \beta; m) = \int_0^1 t^{\alpha-1}(1-t)^{-\beta} \exp(mt(1-t))\,dt,
\]  
(8.8)

using the binomial series expansion \((1-t)^{-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)^n}{n!} t^n\) in the above equation (8.8) and then interchanging the order of summation and integration, the above result (8.8) reduced to the following form
\[ B(\alpha, 1 - \beta; m) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \int_0^1 t^{\alpha+n-1} \exp(mt(1-t))dt \]  
\[ \Rightarrow B(\alpha, 1 - \beta; m) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} B(\alpha + n, 1; m). \]  

**Theorem 7** (Second Summation Relation). If \( \Re(\alpha) > 0, \Re(\beta) > 0, m \in \mathbb{C}; |m| < M \) (where \( M \) is positive finite real number slightly greater than 2.0335), then we have the following relation

\[ B(\alpha, \beta; m) = \sum_{n=0}^{\infty} B(\alpha + n, \beta + 1; m). \]  

**Proof.** The LHS of the equation (8.7) can be written as

\[ B(\alpha, \beta; m) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \exp(mt(1-t))dt, \]  

using the binomial series expansion \((1-t)^{\beta-1} = (1-t)^{\alpha} \sum_{n=0}^{\infty} t^n, \ (|t| < 1)\) and interchanging the order of summation and integration, the above equation (8.12) reduces to

\[ B(\alpha, \beta; m) = \sum_{n=0}^{\infty} \int_0^1 t^{\alpha+n-1} (1-t)^{\beta} \exp(mt(1-t))dt \]  

\[ \Rightarrow B(\alpha, \beta; m) = \sum_{n=0}^{\infty} B(\alpha + n, \beta + 1; m). \]  

**Theorem 8** (Separation). If \( \Re(\alpha) > 0, \Re(\beta) > 0, m \in \mathbb{C}; |m| < M \) (where \( M \) is positive finite real number slightly greater than 2.0335), then \( B(\alpha, \beta; m) \) can be separated into real and imaginary part of \( m \) as follows

\[ B(\alpha, \beta; r \cos \theta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \exp(r t(1-t) \cos \theta)dt, \]  

\[ B(\alpha, \beta; r \sin \theta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \exp(r t(1-t) \sin \theta)dt, \]
where \( r = \sqrt{x^2 + y^2} = |m| < M \) and \( \theta = \tan^{-1} y/x \).

Proof. Since \( m \in \mathbb{C} \), so let \( m = x + iy \) where \( x, y \in \mathbb{R} \) and also let \( x+iy = r \cos \theta + ir \sin \theta \) \( \Rightarrow r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1} y/x \), then from equation (3.1), we have

\[
B(\alpha, \beta; x + iy) = \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1} \exp(rt(1 - t) \cos \theta) dt,
\]

(8.17)

after simplification the above equation (8.17) reduces to

\[
B(\alpha, \beta; r \cos \theta + ir \sin \theta) = \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1} \exp(rt(1 - t) \cos \theta) dt \\
+ i \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1} \exp(rt(1 - t) \sin \theta) dt,
\]

(8.18)

equating the real and imaginary parts of \( m \) only, we have the required results. \( \square \)

9. The modified extended beta distribution

It is expected that there will be many applications of the modified extension of the classical beta function, like there were of the generalized gamma function. One application that springs to mind is to Statistics. For example, the conventional beta distribution can be extended, by using our modified extension of the classical beta function, to variables \( p \) and \( q \) with an infinite range. It appears that such an extension may be desirable for the project evaluation and review technique used in some special cases.

We define the extended beta distribution by

\[
f(t) = \begin{cases} 
\frac{t^{p-1}(1 - t)^{q-1} \exp(mt(1 - t))}{B(p, q; m)}, & 0 < t < 1 \\
0, & \text{otherwise.}
\end{cases}
\]

(9.1)

A random variable \( X \) with probability density function (pdf) given in equation (9.1) will be said to have the extended beta distribution with parameters \( p \) and \( q \), \(-\infty < p, q < \infty \) and \( |m| < M \) where \( M \) is positive real number slightly greater than 2.0335. If \( \nu \) is any real number \([8]\), then

\[
E(X^\nu) = \frac{B(p + \nu, q; m)}{B(p, q; m)}.
\]

(9.2)

In particular, for \( \nu = 1 \),

\[
\mu = E(X) = \frac{B(p + 1, q; m)}{B(p, q; m)}
\]

(9.3)
represents the mean of the distribution and
\[
\sigma^2 = E(X^2) - (E(X))^2
= \frac{B(p, q; m)B(p + 2, q; m) - B^2(p + 1, q; m)}{B^2(p, q; m)}
\] (9.4)
is a variance of the distribution.

The moment of generating function of the distribution is
\[
M(t) = \sum_{n=0}^{\infty} \frac{t^n E(X^n)}{n!} = \frac{1}{B(p, q; m)} \sum_{n=0}^{\infty} \frac{t^n}{n!} B(p + n, q; m).
\] (9.5)
The commutative distribution of (9.1) can be written as
\[
F(x) = \frac{B_x(p, q; m)}{B(p, q; m)}
\] (9.6)
where
\[
B_x(p, q; m) = \int_0^x t^{p-1}(1 - t)^{q-1} \exp(mt(1 - t))dt,
\] (9.7)
is the modified extended incomplete beta function. For \(m = 0\), we must have \(p, q > 0\) in (9.7) for convergence, and \(B_x(p, q; 0) = B_x(p, q)\), where \(B_x(p, q)\) is the incomplete beta function [7] defined as
\[
B_x(p, q) = B_x(p, q; 0) = \frac{x^p}{p} _2F_1(p, 1 - q; p = 1; x).
\] (9.8)

It is to be noted that the problem of expressing \(B_x(p, q, m)\) in terms of other special functions remains open. Presumably, this distribution should be useful in extending the statistical results for strictly positive variables to deal with variables that can take arbitrarily large negative values as well.

10. CONCLUSION

The (Euler’s) classical beta function \(B(x; y)\) play an important role in various fields in the mathematical, physical, engineering and statistical sciences. The extensions of the classical beta function is also important for further investigation in the respective areas, provided these extensions of beta function should be convergent for each parameter involving. The modified extension of classical beta function is convergent and if we choose \(m = 0\), the classical beta function is obtained. It is very clear from Figure 1 that
for the graph of classical beta function and modified extension of classical beta function remains concave upward (or convex downward) for different values of $\alpha, \beta$ and $m$. The value of $m$ does not affect the nature of classical beta function, the main effect of the value of $m$ is that it just push the curve up or drag down the curve from the curve of the classical beta function as shown in the Figure 1 the same behavior can be observed from the Table 2. Both modified extension of classical beta function and classical beta function \( i.e. B(\alpha, \beta; m) \) & \( B(\alpha, \beta) \to 0 \) as \( x, y \to \infty \); \( B(\alpha, \beta; m) \) & \( B(\alpha, \beta) \to \infty \) as \( x, y \to 0 \) and both \( B(\alpha, \beta; m) \) & \( B(\alpha, \beta) \) does not exist at \( x = y = 0 \).

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