Splashing, Recoiling and Deposition: Simulating Droplet Impact Dynamics in Ultracold Bose Gases

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Droplet impact on a surface is closely tied to a diverse field of nature and practical applications, while a complete control of its outcomes remains challenging due to various unmanageable factors related to droplet, surface and surrounding environment. In this work, we propose the quantum simulation of droplet impact dynamics in the platform of ultracold atoms. Specifically, we study the quantum-fluctuation-driven dynamics(QFDD) of two-dimensional Bose-Bose mixtures from an initial Townes soliton towards the formation of quantum droplet. By tuning the fluctuation energy of initial Townes state through its size and number, the subsequent QFDD can exhibit distinct outcomes including splashing, recoiling and deposition, thereby well simulating the droplet impact dynamics. We have utilized the Weber number to identify the thresholds of splashing and recoiling, and further established a universal scaling law between the maximum spreading factor and the Weber number in recoiling regime. In addition, we show the residual QFDD in deposition regime can be used to probe the collective breathing modes of quantum droplet. Our results, which can be directly tested in cold atoms experiments, pave the way for exploring the intriguing droplet dynamics in a clean and fully controlled quantum setting.

Introduction. Studies of droplet impact on a surface are practically relevant to a wide variety of areas in both nature and industry, ranging from the aerosol generation and soil erosion by raindrop impact\cite{1, 2} to the inkjet printing\cite{3, 4}, spray coating\cite{5, 6}, cooling\cite{7, 8}, and fabrications of various anti-icing and superhydrophobic materials\cite{9–11}. The first systematic study of droplet impact dynamics can be traced back to the 19th century by Worthington\cite{12, 13}. Since then such intriguing phenomenon has intrigued great attention and various impact outcomes, including splashing, receding/recoiling, rebound and deposition, have been observed successfully in experiments\cite{14–29}. In general, there are two important physical observables in these studies, namely, the maximum spreading factor ($\beta$)\cite{16, 20, 27} and the splashing threshold($K$)\cite{15, 17–19, 21–26, 29}, which have been shown to not only depend on the properties of droplet itself (size, density, surface tension, viscosity, impact velocity), but also closely rely on the surface condition (roughness, wettability) and surrounding gas (pressure, composition). Because of the complexities associated with various unmanageable factors, it is extremely challenging to deterministically parametrize $\beta$, $K$ and fully control the impact outcomes. In this situation, a common practice is to assume an ideal droplet impact (on a smooth solid surface at atmospheric condition) and then quantify the actual dynamics by the Weber and Reynolds numbers, which, respectively, describe the relative strength of droplet inertia with respect to capillary and viscous forces\cite{30, 31}. Various scaling laws between $\beta$, $K$ and these numbers have been proposed in literature\cite{30, 31}, based on different models or empirical fitting from experimental data. The validity of these scalings, however, crucially depends on the actual surface conditions instead of solely on the droplet itself.

In the past few decades, ultracold atoms have emerged as an ideal platform for quantum simulation, given extremely clean environment and high controllability on the species, number, dimension, interaction strength etc\cite{32, 33}. In particular, a recent important achievement of this field is the realization of quantum droplet in both dipolar gas\cite{34–40} and alkali bosonic mixtures\cite{41–45}, with extremely dilute densities ($\sim 10^{14}–10^{15}\text{cm}^{-3}$) that can be 8 orders of magnitude lower than water. In forming these gaseous droplets, quantum fluctuations play an essential role in providing the repulsive force for their stabilization, for which they are called the quantum droplets\cite{46}. To date, the idea of quantum droplet has been successfully extended to various atomic systems including low dimensional ones\cite{47–52}, Bose-Fermi mixtures\cite{53–58} and multi-component dipolar or alkali atomic mixtures\cite{59–61}. The non-equilibrium properties of quantum droplets have also been investigated in terms of their dynamical formations\cite{62–64} and collisions\cite{65, 66}. These developments offer an unprecedented opportunity for simulating droplet impact dynamics in ultracold atoms, particularly, at the microscopic quantum level and in a highly controllable manner.

In this work, we demonstrate the capability of using ultracold droplets for the quantum simulation of droplet impact phenomena. Specifically, we study the dynamical property of a two-dimensional (2D) Bose-Bose mixture with repulsive intra- and attractive inter-species couplings, whose ground state is a quantum droplet. To highlight the quantum effect in the dynamics, we have chosen the initial state as the Townes soliton general-
ized from the single-species case[67], which features a zero mean-field energy with continuous scale invariance as recently confirmed in experiments[68–71]. In this way, the dynamics here is purely driven by quantum fluctuations and thus can be called the quantum-fluctuation-driven dynamics (QFDD). It is found that by tuning the fluctuation energy of initial Townes state through its size($\sigma_0$) and number($N$), the subsequent QFDD can exhibit distinct outcomes including splashing, recoiling and deposition, thereby well simulating the droplet impact dynamics. We have mapped out the dynamical phase diagram in ($\sigma_0$, $N$) plane and employed the Weber number to characterize different phases. The splashing and recoiling thresholds are identified and a universal scaling law is established between the maximum spreading factor and Weber number here is purely driven by quantum fluctuations recently confirmed in experiments[68–71]. In this way, the mean-field energy with continuous scale invariance, and can only exist when the boson number and coupling strength satisfy $N|g| = 5.85/m$. Such a special solution has been successfully observed in both non-linear optics[68] and ultracold atoms[69–71]. In these experiments, the LHY correction takes little effect as it is much smaller than the mean-field part. In the following, we will show that the Townes soliton can be generalized to two-species bosons if equally neglecting the LHY correction.

By omitting the LHY term in (3), we can see that the two GP equations for $\{\phi_1, \phi_2\}$ can support a single-mode solution $\phi_i = \sqrt{N_i} \phi \exp(-i\mu_i t)$ as long as

$$\frac{N_1}{N_2} = \frac{g_{22} - g_{12}}{g_{11} - g_{12}}$$

(4)

where $\mu_1 = \mu_2 \equiv \mu$, and the single mode $\phi$ satisfies:

$$-\frac{\nabla^2}{2m} + N g_{\text{eff}} |\phi|^2 \phi = \mu \phi.$$  

(5)

Here $N = N_1 + N_2$ is the total number, and the effective interaction $g_{\text{eff}}$ is given by

$$g_{\text{eff}} = \frac{g_{11} g_{22} - g_{12}^2}{g_{11} + g_{22} - 2g_{12}}.$$  

(6)

Apparently we have $g_{\text{eff}} < 0$ in the mean-field collapse regime ($g_{12} < -\sqrt{g_{11} g_{22}}$). It is then straightforward to check that under the condition

$$N m |g_{\text{eff}}| = 5.85,$$  

(7)

there exists a sequence of zero-energy eigenstates with continuous scale invariance, i.e., the eigenstate nature and zero-energy property will not change under an arbitrary scaling transformation $\phi(\rho) \rightarrow \lambda \phi(\lambda \rho)$ (accordingly $E \rightarrow \lambda^2 E$). These stationary solutions are the generalized Townes soliton for two-species bosons. Note that similar generalization also works for the case of unequal masses, with slight modifications in Eqs.(4,6)[72].

In Fig.1(a), we confirm the stationary Townes soliton for two-species bosons once the total number $N$ satisfies (7) and the number ratio $N_1/N_2$ satisfies (4). In comparison, if change $N$ to be smaller or larger, the original profile will shrink (Fig.1(b)) or expand (Fig.1(c)) with time. The profile is also unstable if deviate $N_1/N_2$ from (4), see Fig.1(d). In a word, both conditions (7) and (4) are required in supporting a stationary two-species Townes solution.
Quantum-fluctuation driven dynamics. A crucial difference between the single- and two-species bosons is that quantum fluctuations play an important role in the latter, which can lead to the droplet formation as ground state. It then follows that starting from the generalized Townes soliton, which is the mean-field stationary solution for two-species bosons, the quantum fluctuation can destabilize it strongly and drive its time evolution towards the droplet formation. Such dynamics can be called the quantum-fluctuation driven dynamics (QFDD), also in the droplet formation. Such dynamics can be well highlighted and manipulated.

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Weber number and dynamical phase diagram. We now quantify various dynamical phases in QFDD by the Weber number. Note that the Reynolds number is irrelevant here because of the zero viscosity of Bose condensates. According to the original definition in droplet impact dynamics\cite{30,31}, the Weber number measures the relative strength of droplet inertia with respect to capillary force. Similarly, we write down an extended definition of it for the boson dynamics here:

$$ W = \frac{\Delta E}{D\gamma}, $$

where $\Delta E$ is the energy difference between the Townes soliton and the droplet ground state given the same initial parameters $(N, g_{ij})$; $D$ is the droplet diameter; $\gamma = \int d\rho \left[ \mathcal{E}(\rho) - \mu_1|\phi_1|^2 - \mu_2|\phi_2|^2 \right]$ is the surface tension of the droplet, with $\mu_i$ the chemical potential for the $i$-th species.

As shown in the color plot in Fig.2(a), $W$ defined in (9) can well characterize different dynamical phases in QFDD. Namely, a low $W$ corresponds to the deposition dynamics, where the small $\Delta E$ can be well absorbed by the surface change of the droplet; as increasing $W$, the system enters the recoiling regime and finally end up at splashing, where the large $\Delta E$ overwhelms the capacity of droplet surface and causes its drastic change. Hereafter we refer to the critical $W$ at recoiling-splashing boundary as the splashing threshold ($\mathcal{K}_s$), and at deposition-recoiling boundary as the recoiling threshold ($\mathcal{K}_r$).

In Fig.3(a), we map out the dynamical phase diagram in $(\sigma_0, N)$ parameter plane, according to distinct dynamical outcomes of QFDD including deposition ('D'), recoiling ('R') and splashing ('S'). In addition, we show the contour plot of $W$ in $(\sigma_0, N)$ plane, and one can see clearly that the 'D', 'R' and 'S' phases respectively correspond to the small, intermediate and large $W$ regions. Due to the non-monotonic dependence of $W$ on $\sigma_0$ (as shown in Fig.2(a)), there are two recoiling regions in the diagram, as marked by 'R1' and 'R2'. In Fig.3(b), we extract the two recoiling thresholds($\mathcal{K}_{r1}, \mathcal{K}_{r2}$) and the splashing threshold($\mathcal{K}_s$) along the phase boundaries as varying $N$. One can see that $\mathcal{K}_{r1} \approx \mathcal{K}_{r2}$ are given by a constant $\sim 0.12$, while $\mathcal{K}_s$ is a much larger value and continuously increases with $N$.

Maximum spreading factor. Another important physical quantity to characterize the droplet impact dynamics is the maximum spreading factor $\beta = \sigma_{\text{max}}/\sigma_0$, as defined by the ratio between the maximum spreading radius
FIG. 3. (Color online) (a) Dynamical phase diagram and contour plot of Weber number in \((\sigma_0, N)\) parameter plane. Here \(\sigma_0\) and \(N\) are respectively the size and number of the initial Townes state. The splashing, recoiling and deposition phases are respectively marked by ‘S’, ‘R\(_1\)’, ‘R\(_2\)’ and ‘D’ in the diagram. The white squares marks their boundaries. Gray dashed line shows the location of energy minimum for initial Townes state. (b) Splashing threshold \((K_s)\) and two recoiling thresholds \((K_{r1}, K_{r2})\) as functions of \(N\) along the phase boundaries in (a).

(\(\sigma_{\text{max}}\)) and the initial one \((\sigma_0)\). Clearly one has \(\beta \sim 1\) for the deposition dynamics and \(\beta \to \infty\) for splashing. An interesting behavior of \(\beta\) shows up in the recoiling regime, where \(\beta\) is finite and varies sensitively with \(W\).

In Fig. 4, we extract \(\beta\) as a function of \(W\) along the horizontal lines in Fig. 3(a), i.e., by varying \(\sigma_0\) at several fixed \(N\) in the recoiling regime. Apart from the region near phase boundaries (\(\beta \to 1\) or large enough), the data of \(\beta\) for any given \(N\) well follow the scaling relation:

\[
\beta - 1 \propto W^\alpha, \quad \text{with} \quad \alpha = 1.59. \tag{10}
\]

As shown in Fig. 4, such scaling works well for a wide parameter regime with \(N/10^4 \in (0.5, 2)\), \(\beta \in (1.5, 10)\) and \(W \in (0.2, 1)\). Therefore, we expect the scaling law in (10) reflect a very robust intrinsic property of the ultracold fluid during the recoiling QFDD.

**Breathing modes.** Finally, we demonstrate that the deposition regime of QFDD can be used to probe the breathing modes of quantum droplet. As shown in Fig. 2(b1), in the deposition regime the density profile at longer time essentially follows the droplet profile along with an additional periodic oscillation. Such oscillations are synchronous for both species and at different locations (see Fig. 5(a1,a2)), and their Fourier transformations give the same peak frequencies (Fig. 5(b1,b2)). These peaks represent the collective breathing modes of the residual droplet.

In Fig. 5(c), we show the extracted peak frequencies \((\omega_1, \omega_2)\) as a functions of droplet atom number \(N_d\). For each extracted data we have collected 30 samples by varying the real-space locations or the time intervals in Fourier transformation, from which we obtain both the mean value and the variance. One can see that the obtained results match very well with theoretical predictions of collective breathing modes from Bogoliubov analysis\[72\]. We note that a previous study has extracted one branch of collective mode from the formation dynamics of quantum droplet\[62\], while the dynamics there is not QFDD. Moreover, for larger \(N\) in Fig. 5(c), we show that two branches of collective modes can be simultaneously extracted from the residual oscillations in QFDD. The only complexity in this case is the appearance of an additional frequency peak at \(\omega_2 - \omega_1\) due to the beating of two modes (see Fig. 5(b2)), which therefore calls for a careful handling in order to correctly identify the breathing modes.

**Summary and discussion.** In summary, we have demonstrated the quantum simulation of droplet impact dynamics in ultracold boson mixtures. Various dynamical outcomes similar to those in droplet impact dynamics, including splashing, recoiling and deposition, have been revealed. A remarkable difference here is that these dynamics are purely driven by quantum fluctuations, in-
stead of the mechanical impact force in previous studies. Given the easy manipulation of initial states with tunable fluctuation energies, the current cold atoms platform provide a complete control on these dynamics in the microscopic quantum level. To characterize different dynamics, we have introduced the Weber number and the residual density oscillations in QFDD. The energy unit in (b1,b2,c) is $1/(ml_0^2)$ with $l_0 = 1\mu m$.

Our work is in distinct contrast to the previous studies of QFDD in ultracold atoms, where a visible dynamics can only be achieved for small condensates\cite{73–76}. This is because usually the energy difference per particle between the mean-field and the true quantum ground states decays rapidly as the number $N$ increases\cite{77–80}, and thus for large $N$ the driving force of quantum fluctuations takes negligible effect. However, this is not the case for quantum droplet. For a static droplet, quantum fluctuations have been shown to provide an indispensable repulsive force for its stabilization, without which the whole system will collapse \cite{46}. In parallel, our current work reveals the equally significant role of quantum fluctuations played in its non-equilibrium dynamics, regardless of a small or large droplet.

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[1] Y. S. Joung and C. R. Buie, Aerosol generation by raindrop impact on soil, Nat. Commun. 6, 6083 (2015).
[2] A. R. Vaezi, M. Ahmad, and A. Cerdà, Contribution of raindrop impact to the change of soil physical properties and water erosion under semi-arid rainfalls, Sci. Total Environ. 583, 382 (2017).
[3] D. B. van Dam and C. Le Clerc, Experimental study of the impact of an ink-jet printed droplet on a solid substrate, Phys. Fluids 16, 3403 (2004).
[4] T. Lim, S. Han, J. Chung, J. T. Chung, S. Ko, and C. P. Grigoropoulos, Experimental study on spreading and evaporation of inkjet printed pico-liter droplet on a heated substrate, Int. J. Heat Mass Transfer 52, 431–441 (2009).
[5] S. Schiaffino and A. A. Sonin, Molten droplet deposition and solidification at low Weber numbers, Phys. Fluids 9, 3172–3187 (1997).
[6] R. Mcpherson, The relationship between the mechanism...
of formation, microstructure and properties of plasma-sprayed coatings, Thin Solid Films 83, 297–310 (1981).

[7] M. R. O. Panão and A. L. N. Moreira, Intermittent spray cooling: a new technology for controlling surface temperature, Int. J. Heat Fluid Flow 30, 117–130 (2009).

[8] J.M. Tian, B. Chen, D. Li, Z.F. Zhou, Transient spray cooling: similarity of dynamic heat flux for different cryogens, nozzles and substrates, Int. J. Heat Mass Transfer 108, 561–571 (2017).

[9] L. Mishchenko, B. Hatton, V. Bahadur, J. A. Taylor, T. Krupekin, and J. Aizenberg, Design of ice-free nanostructured surfaces based on repulsion of impacting water droplets, ACS Nano 4, 7699 (2010).

[10] B. Bhushan and Y. C. Jung, Natural and biomimetic artificial surfaces for superhydrophobicity, self-cleaning, low adhesion, and drag reduction, Prog. Mater. Sci. 56, 1 (2011).

[11] D. Khojasteh, M. Kazerouni, S. Salarian, and R. Kamali, Droplet impact on superhydrophobic surfaces: A review of recent developments, J. Ind. Eng. Chem. 42, 1 (2016).

[12] A. M. Worthington, On the forms assumed by drops of liquids falling vertically on a horizontal plate, Proc. R. Soc. 25, 261 (1876).

[13] A. M. Worthington, A study of splashes (Longmans, Green and Company, 1908).

[14] R. Rioboo, C. Tropea, and M. Marengo, Outcomes from a Drop Impact on Solid Surfaces, At. Sprays 11, 155 (2001).

[15] L. Xu, W. W. Zhang, and S. R. Nagel, Drop splashing on a dry smooth surface, Phys. Rev. Lett. 94, 184505 (2005).

[16] I. S. Bayer and C. M. Megaridis, Contact angle dynamics in droplets impacting on flat surfaces with different wetting characteristics, J. Fluid Mech. 558, 415–449 (2006).

[17] A. Latka, A. Strandburg-Peshkin, M. M. Driscoll, C. S. Stevens, and S. R. Nagel, Creation of prompt and thin-sheet splashing by varying surface roughness or increasing air pressure, Phys. Rev. Lett. 109, 054501 (2012).

[18] I.V. Roisman, A. Lembach, and C. Tropea, Drop splashing induced by target roughness and porosity: The size plays no role, Adv. Colloid Interface Sci. 222, 615 (2015).

[19] D. G. Aboud and A.-M.Kietzig, Splashing threshold of oblique droplet impacts on surfaces of various wettability, Langmuir 31, 10100 (2015).

[20] C. Tang, M. Qin, X. Weng, X. Zhang, P. Zhang, J. Li, and Z. Huang, Dynamics of droplet impact on solid surface with different roughness, Int. J. Multiph. Flow 96, 56 (2017).

[21] J. Hao, Effect of surface roughness on droplet splashing, Phys. Fluids 29, 122105 (2017).

[22] F. R. Smith, N. C. Buntsma, D. Brutin, Roughness influence on human blood drop spreading and splashing, Langmuir 34, 1143 (2017).

[23] T. C.de Goede, N. Laan, K. De Bruin, and D. Bonn, Effect of wetting on droplet splashing of Newtonian fluids and blood, Langmuir 34, 5163 (2017).

[24] J. Tian and B. Chen, Dynamic behavior of non-evaporative droplet impact on a solid surface: Comparative study of R113, water, ethanol and acetone, Exp. Therm Fluid Sci. 105, 153 (2019).

[25] H. Almohammadi and A. Amirfazli, Droplet impact: Viscosity and wettability effects on splashing, J. Colloid Interface Sci. 553, 22 (2019).

[26] M.A. Quetzeri-Santiago, K. Yokoi, A.A. Castrejón-Pita, and J.R. Castrejón-Pita, Role of the dynamic contact angle on splashing, Phys. Rev. Lett. 122, 228001 (2019).

[27] H. Zhang, X. Zhang, X. Yi, F. He, F. Niu, and P. Hao, Effect of wettability on droplet impact: Spreading and splashing, Exp. Therm Fluid Sci. 124, 110369 (2021).

[28] L. Yang, Z. Li, T. Yang, Y. Chi, and P. Zhang, Experimental Study on Droplet Splash and Receding Breakup on a Smooth Surface at Atmospheric Pressure, Langmuir 37, 10838 (2021).

[29] P. García-Geijo, E. Quintero, G. Riboux, and J. Gordillo, Spreading and splashing of drops impacting rough substrates J. Fluid Mech. 917, A50 (2021).

[30] A. Yarin, Drop Impact Dynamics: Splashing, Spreading, Receding, Bouncing,..., Annu. Rev. Fluid Mech. 38, 159 (2006).

[31] C. Josserand and S. Thoroddsen, Drop Impact on a Solid Surface, Annu. Rev. Fluid Mech. 48, 365 (2016).

[32] I. Bloch, J. Dalibard and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. 80, 885 (2008).

[33] C. Chin, R. Grimm, P. Julienne and E. Tiesinga, Feshbach resonances in ultracold gases, Rev. Mod. Phys. 82, 1225 (2010).

[34] I. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Observation of Quantum Droplets in a Strongly Dipolar Bose Gas, Phys. Rev. Lett. 116, 215301 (2016).

[35] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau, Self-bound droplets of a dilute magnetic quantum liquid, Nature 539, 259 (2016).

[36] I. Ferrier-Barbut, M. Schmitt, M. Wenzel, H. Kadau, and T. Pfau, Liquid quantum droplets of ultracold magnetic atoms, J. Phys. B 49, 214004 (2016).

[37] L. Chomaz, S. Baier, D. Petter, M.J. Mark, F. Wächtl, L. Santos, and F. Ferlaino, Quantum-Fluctuation-Driven Crossover from a Dilute Bose-Einstein Condensate to a Macrodroplet in a Dipolar Quantum Fluid, Phys. Rev. X 6, 041039 (2016).

[38] L. Tanzi, E. Lucioni, F. Fama, J. Catani, A. Fioretti, C. Gabbanini, R. N. Bisset, L. Santos, and G. Modugno, Observation of a Dipolar Quantum Gas with Metastable Supersolid Properties, Phys. Rev. Lett. 122, 130405 (2019).

[39] F. Böttcher, J.-N. Schmidt, M. Wenzel, J. Hertkorn, M. Guo, T. Langen, and T. Pfau, Transient Supersolid Properties in an Array of Dipolar Quantum Droplets, Phys. Rev. X 9, 011051 (2019).

[40] L. Chomaz, D. Petter, P. Ilzhöfer, G. Natale, A. Trautmann, C. Politi, G. Durastante, R.M.W. van Bijnen, A. Patscheider, M. Sörensen, M.J. Mark, and F. Ferlaino, Long-Lived and Transient Supersolid Behaviors in Dipolar Quantum Gases, Phys. Rev. X 9, 021012 (2019).

[41] C.R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Quantum liquid droplets in a mixture of Bose-Einstein condensates, Science 359, 301 (2018).

[42] P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, L. Tarruell, Bright soliton to quantum droplet transition in a mixture of Bose-Einstein condensates, Phys. Rev. Lett. 120, 135301 (2018).

[43] G. Semeghini, G. Ferioli, L. Masì, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, M. Fattori, Self-bound quantum droplets of atomic mixtures in free space, Phys. Rev. Lett. 120, 235301 (2018).
[44] C. D’Errico, A. Burchianti, M. Prevedelli, L. Salasnich, F. Ancilotto, M. Modugno, F. Minardi, and C. Fort, Observation of quantum droplets in a heteronuclear bosonic mixture, Phys. Rev. Res. 1, 033155 (2019).

[45] Z. Guo, F. Jia, L. Li, Y. Ma, J. M. Hutson, X. Cui, and D. Wang, Lee-Huang-Yang effects in the ultracold mixture of $^{23}$Na and $^{87}$Rb with attractive interspecies interactions, Phys. Rev. Res. 3, 033247 (2021).

[46] D.S. Petrov, Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture, Phys. Rev. Lett. 115, 155302 (2015).

[47] D. S. Petrov and G. E. Astrakharchik, Ultradilute Low-Dimensional Liquids, Phys. Rev. Lett. 117, 100401 (2016).

[48] D. Edler, C. Mishra, F. Wächtler, R. Nath, S. Sinha, and L. Santos, Quantum Fluctuations in Quasi-One-Dimensional Dipolar Bose-Einstein Condensates, Phys. Rev. Lett. 119, 050403 (2017).

[49] K. Jachymski and R. Oldziejewski, Nonuniversal beyond mean field properties of quasi-two-dimensional dipolar Bose gases, Phys. Rev. A 98, 043601 (2018).

[50] P. Zin, M. Pylak, T. Wasak, M. Gajda, and Z. Idziaszek, Quantum Bose-Bose droplets at a dimensional crossover, Phys. Rev. A 98, 051603(R) (2018).

[51] T. Ilg, J. Kumlin, L. Santos, D. S. Petrov, and H. P. Büchler, Dimensional crossover for the beyond-mean-field correction in Bose gases, Phys. Rev. A 98, 051604(R) (2018).

[52] X. Cui, Y. Ma, Droplet under confinement: Competition and coexistence with a soliton bound state, Phys. Rev. Res. 3, L012027 (2021).

[53] X. Cui, Spin-orbit coupling induced quantum droplet in ultracold Bose-Fermi mixtures, Phys. Rev. A 98, 023630 (2018).

[54] S. Adhikari, A self-bound matter-wave boson-fermion quantum ball, Laser Phys. Lett. 15, 095501 (2018).

[55] D. Rakshit, T. Karpiku, M. Brewczyk, and M. Gajda, Quantum Bose-Fermi droplets, SciPost Phys. 6, 079 (2019).

[56] D. Rakshit, T. Karpiku, P. Zin, M. Brewczyk, M. Lewenstein, and M. Gajda, Self-bound Bose-Fermi liquids in lower dimensions, New J. Phys. 21, 073027 (2019).

[57] M. Wenzel, T. Pfau and I. Ferrier-Barbut, A fermionic impurity in a dipolar quantum droplet, Phys. Scr. 93, 104004 (2018).

[58] J.-B. Wang, J.-S. Pan, X. Cui, W. Yi, Quantum droplet in a mixture of Bose-Fermi superfluids, Chin. Phys. Lett. 37, 076701 (2020).

[59] J. C. Smith, D. Baillie, and P. B. Blakie, Quantum Droplet States of a Binary Magnetic Gas, Phys. Rev. Lett. 126, 025302 (2021).

[60] R. N. Bisset, L. A. Peña Ardila, and L. Santos, Quantum Droplets of Dipolar Mixtures, Phys. Rev. Lett. 126, 025301 (2021).

[61] Y. Ma, C. Peng, and X. Cui, Borromean droplet in three-component ultracold Bose gases, Phys. Rev. Lett. 127, 043002 (2021).

[62] G. Ferioli, G. Semeghini, S. Terradas-Briansó, L. Masi, M. Fattori, and M. Modugno, Dynamical formation of quantum droplets in a 39K mixture, Phys. Rev. Res. 2, 013269 (2020).

[63] V. Cikojević, L. Vranješ Markić, M. Pi, M. Barranco, F. Ancilotto, and J. Boronat, Dynamics of equilibration and collisions in ultradilute quantum droplets, Phys. Rev. Res. 3, 043139 (2021).

[64] C. Fort and M. Modugno, Self-evaporation dynamics of quantum droplets in a $^{41}$K-$^{87}$Rb mixture, Appl. Sci. 11, 866 (2021).

[65] G. E. Astrakharchik and B. A. Malomed, Dynamics of one-dimensional quantum droplets, Phys. Rev. A 98, 013631 (2018).

[66] G. Ferioli, G. Semeghini, L. Masi, G. Giusti, G. Modugno, M. Inguscio, A. Gallemí, A. Recati, and M. Fattori, Collisions of Self-Bound Quantum Droplets, Phys. Rev. Lett. 122, 090401 (2019).

[67] R. Y. Chiao, E. Garmire, and C. H. Townes, Self-Trapping of Optical Beams, Phys. Rev. Lett. 13, 479 (1964).

[68] K. D. Moll, A. L. Gaeta, and G. Fibich, Self-Similar Optical Wave Collapse: Observation of the Townes Profile, Phys. Rev. Lett. 90, 203902 (2003).

[69] C. A. Chen and C. L. Hung, Observation of Universal Quench Dynamics and Townes Soliton Formation from Modulational Instability in Two-Dimensional Bose Gases, Phys. Rev. Lett. 125, 250401 (2020).

[70] C. A. Chen and C. L. Hung, Observation of Scale Invariance in Two-Dimensional Matter-Wave Townes Solitons, Phys. Rev. Lett. 127, 023604 (2021).

[71] B. Bakkal-Hassani, C. Maury, Y. Q. Zou, É. L. Cerf, R. Saint-Jalm, P. C. M. Castillo, S. Nascimbene, J. Dalibard, and J. Beugnon, Realization of a Townes Soliton in a Two-Component Planar Bose Gas, Phys. Rev. Lett. 127, 023603 (2021).

[72] See supplementary material for more details.

[73] C. K. Law, H. Pu, and N. P. Bigelow, Quantum Spins Mixing in Spinor Bose-Einstein Condensates, Phys. Rev. Lett. 81, 5257 (1998).

[74] X. Cui, Y. P. Wang and F. Zhou, Quantum-fluctuation-driven coherent spin dynamics in small condensates, Phys. Rev. A 78, 050701(R) (2008).

[75] R. Barnett, J. D. Sau and S. Das Sarma, Antiferromagnetic spinor condensates are quantum rotors, Phys. Rev. A 82, 031602 (R) (2010).

[76] J. Heinze, F. Deuretzbacher and D. Pfannkuche, Influence of the particle number on the spin dynamics of ultracold atoms, Phys. Rev. A 82, 023617 (2010).

[77] M. Koashi and M. Ueda, Exact Eigenstates and Magnetic Response of Spin-1 Bose-Einstein Condensates, Phys. Rev. Lett. 84, 1066 (2000).

[78] T.-L. Ho and S. K. Yip, Fragmented and Single Condensate Ground States of Spin-1 Bose Gas, Phys. Rev. Lett. 84, 4031 (2000).

[79] E. J. Mueller, T.-L. Ho, M. Ueda, and G. Baym, Fragmentation of Bose-Einstein condensates, Phys. Rev. A 74, 033612 (2006).

[80] Q. Zhou and X. Cui, Fate of a Bose-Einstein Condensate in the Presence of Spin-Orbit Coupling, Phys. Rev. Lett. 110, 140407 (2013).
In this Supplemental Material, we provide more details on the derivation of generalized Townes soliton with unequal masses, and the collective breathing modes of quantum droplets.

**TOWNES SOLITON FOR TWO-SPECIES BOSONS WITH UNEQUAL MASSES**

Neglecting the LHY correction, we write down the Gross-Pitaevskii(GP) equation for two-species bosons with unequal mass \((m_1, m_2)\):

\[
\begin{align*}
    i\partial_t \phi_1 &= \left( -\frac{\nabla^2}{2m_1} + g_{11} |\phi_1|^2 + g_{12} |\phi_2|^2 \right) \phi_1; \\
    i\partial_t \phi_2 &= \left( -\frac{\nabla^2}{2m_2} + g_{12} |\phi_1|^2 + g_{22} |\phi_2|^2 \right) \phi_2.
\end{align*}
\]

(11)

(12)

In the case \(m_1 \neq m_2\), we define the mass-imbalance parameter as \(w = m_2/m_1\). By comparing the two GP equations, we can see that they can support a single-mode solution \(\phi_i = \sqrt{N_i} \phi \exp(-i\mu_i t)\) as long as

\[
\frac{N_1}{N_2} = \frac{w g_{22} - g_{12}}{g_{11} - w g_{12}},
\]

(13)

where \(\mu_1 = w\mu_2\), and the single mode \(\phi\) satisfies:

\[
\left( -\frac{\nabla^2}{2m_1} + N g_{\text{eff}} |\phi|^2 \right) \phi = \mu_1 \phi.
\]

(14)

Here \(N = N_1 + N_2\) is the total number, and the effective interaction \(g_{\text{eff}}\) is given by

\[
g_{\text{eff}} = \frac{w (g_{11} g_{22} - g_{12}^2)}{g_{11} + w g_{22} - (w + 1) g_{12}}.
\]

(15)

Again we have \(g_{\text{eff}} < 0\) in the mean-field collapse regime \((g_{12} < -\sqrt{g_{11} g_{22}})\), and the Townes solution occurs when

\[
N m_1 |g_{\text{eff}}| = 5.85.
\]

(16)

For the equal mass case \((w = 1)\), these equations automatically reduce to Eqs.(4-7) in the main text.

**BOGOLIUBOV THEORY FOR THE COLLECTIVE MODE OF QUANTUM DROPLET**

To quantitatively determine the collective excitations of the quantum droplet, it is necessary to solve coupled GP:

\[
\begin{align*}
    i\partial_t \phi_1 &= \left( -\frac{\nabla^2}{2m} + g_{11} n_1 + g_{12} n_2 + \frac{\partial E_{\text{LHY}}}{\partial n_1} \right) \phi_1; \\
    i\partial_t \phi_2 &= \left( -\frac{\nabla^2}{2m} + g_{12} n_1 + g_{22} n_2 + \frac{\partial E_{\text{LHY}}}{\partial n_2} \right) \phi_2.
\end{align*}
\]

(17)

Assuming a small fluctuation mode \(\delta \phi_i\) for the i-th species boson, and only keeping the lowest-order fluctuations in (17), we obtain the following equations for \(\{\delta \phi_i\}\):

\[
\begin{align*}
    i\partial_t \delta \phi_1 &= \left( -\frac{\nabla^2}{2m} + g_{11} n_1 + g_{12} n_2 + \frac{\partial E_{\text{LHY}}}{\partial n_1} \right) \delta \phi_1 + g_{11} n_1 (\delta \phi_1 + \delta \phi_1^*) + g_{12} \phi_1 \phi_2 (\delta \phi_2 + \delta \phi_2^*) \\
    &\quad + \frac{\partial^2 E_{\text{LHY}}}{\partial n_1^2} n_1 (\delta \phi_1 + \delta \phi_1^*) + \frac{\partial^2 E_{\text{LHY}}}{\partial n_1 \partial n_2} \phi_1 \phi_2 (\delta \phi_2 + \delta \phi_2^*) \\
    i\partial_t \delta \phi_2 &= \left( -\frac{\nabla^2}{2m} + g_{12} n_1 + g_{22} n_2 + \frac{\partial E_{\text{LHY}}}{\partial n_2} \right) \delta \phi_2 + g_{22} n_2 (\delta \phi_2 + \delta \phi_2^*) + g_{12} \phi_1 \phi_2 (\delta \phi_1 + \delta \phi_1^*) \\
    &\quad + \frac{\partial^2 E_{\text{LHY}}}{\partial n_1 \partial n_2} \phi_1 \phi_2 (\delta \phi_1 + \delta \phi_1^*) + \frac{\partial^2 E_{\text{LHY}}}{\partial n_2^2} n_2 (\delta \phi_2 + \delta \phi_2^*).
\end{align*}
\]

(18)
According to the standard Bogoliubov analysis, we search for solutions of the form:

$$\delta\phi_i = \exp(-i\mu_it) \sum_j (u_{ij}(\rho) \exp(-i\omega_j t) + v_{ij}^*(\rho) \exp(i\omega_j t)),$$

(19)

Here $\omega_j$ is the $j$-th collective (eigen-)mode of the system. These modes can be extracted from the following coupled equations for $u_{ij}(\rho)$ and $v_{ij}(\rho)$:

$$
\begin{pmatrix}
L_1 + M_1 & M_{12} & M_1 & M_{12} \\
M_{12} & L_2 + M_2 & M_{12} & M_2 \\
-M_1 & -M_{12} & -(L_1 + M_1) & -M_{12} \\
-M_{12} & -M_2 & -M_{12} & -(L_2 + M_2)
\end{pmatrix}
\begin{pmatrix} u_{1j}(\rho) \\ u_{2j}(\rho) \\ v_{1j}(\rho) \\ v_{2j}(\rho) \end{pmatrix}
= \omega_j
\begin{pmatrix} u_{1j}(\rho) \\ u_{2j}(\rho) \\ v_{1j}(\rho) \\ v_{2j}(\rho) \end{pmatrix},
$$

(20)

where

$$
L_i = -\frac{\nabla^2 \rho}{2m} + \sum_j g_{ij} |\phi_j|^2 + \frac{\partial E_{LHY}}{\partial n_i} - \mu_i
$$

$$
M_i = g_{ii} |\phi_i|^2 + \frac{\partial^2 E_{LHY}}{\partial n_i^2} \phi_i
$$

$$
M_{12} = g_{12} \phi_1 \phi_2 + \frac{\partial^2 E_{LHY}}{\partial n_1 \partial n_2} \phi_1 \phi_2.
$$

(21)

In our numerics, we have obtained $\omega_j$ by discretizing the real space to pieces and diagonalizing the large matrix shown in the left-hand-side of (20). In Fig.5 of the main text, we have plotted out two collective modes $(\omega_1, \omega_2)$ below the atom emission threshold $(-\mu_1, -\mu_2)$. These fluctuation modes all have zero angular momentum and represent stable breathing excitations of the system.