Conclusive discrimination between arbitrary quantum states by $N$ sequential receivers

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25 February 2021

Abstract. In the present article, we develop a general framework for the description of an $N$-sequential state discrimination, where each of $N$ receivers always obtains a conclusive result. For this new state discrimination scenario, we derive two mutually equivalent general representations of the success probability and prove that if one of two states, pure or mixed, is prepared by a sender, then the optimal success probability is given by the Helstrom bound for any number $N$ of sequential receivers. Furthermore, we specify receivers' indirect measurements resulting in the optimal $N$-sequential conclusive state discrimination protocol. The developed framework is true for any number $N$ of sequential receivers, any number of arbitrary quantum states, pure or mixed, to be discriminated, and all types of receivers' quantum measurements. The new general results derived within the developed framework are important both from the theoretical point of view and for a successful multipartite quantum communication even in the presence of a quantum noise.

Keywords: sequential state discrimination, the Helstrom bound, indirect measurements, multipartite quantum communication

1. Introduction

Constructing quantum measurements for distinguishing between quantum states with an optimal value of a merit is one of the key problems of quantum information processing which is referred to as a quantum state discrimination [1, 2]. For a quantum state discrimination, various strategies based on the constrains imposed by the quantum measurement theory can be proposed.

In 2013, J. A. Bergou [3] firstly devised the sequential unambiguous state discrimination where a receiver discriminates a sender’s quantum state and a further receiver discriminates posterior states from the previous receiver. However, in [3], every receiver performs the so called sequential unambiguous state discrimination where each receiver’s conclusive result is always confident, even though this receiver can obtain the inconclusive result with a non-zero probability, and until now there have been numerous
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theoretical and experimental developments [4, 5, 6, 7, 8, 9, 10, 11] on only this type of sequential state discrimination.

Unfortunately, the unambiguous state discrimination can be performed if only (i) in the case of pure initial states, they are linearly independent; and (ii) in the case of mixed initial states, they have non-overlapping supports [12]. Therefore, another strategy for a sequential state discrimination should be devised.

Recently, another scenario for a sequential state discrimination has been proposed [13] where each receiver performs a quantum measurement outputting only a conclusive result. This means that receivers’ quantum measurements are designed so as to vanish the possibility for each receiver to obtain a result “I don’t know which quantum state was prepared”. In view of this, we call this scenario sequential conclusive state discrimination.

In the present article, we develop a general framework for the description of an N-sequential conclusive state discrimination. For this new state discrimination scenario, we derive two mutually equivalent general representations of the success probability and prove that in the case where one out of two quantum states, pure or mixed, to be discriminated, is prepared by a sender, the optimal success probability of an N-sequential conclusive state discrimination is given by the Helstrom bound for any number N of sequential receivers. Furthermore, we show that the optimal sequential conclusive state discrimination protocol can be realized via receivers’ indirect measurements of the specific form. The new general results derived within the developed framework are important both from the theoretical point of view and for a successful multipartite quantum communication even in the presence of a quantum noise.

The present article is organized as follows. In Section 2, we present the main issues of the quantum measurement theory, specify the notions of a quantum state instrument and its statistical realizations. In Section 3, we develop a general framework for an N-sequential conclusive state discrimination. In Section 4, we prove that if one out of two states is sent to receivers, then the optimal success probability is given by the Helstrom bound for any number N of sequential receivers. In Section 5, we specify receivers’ indirect measurements realizing the optimal N-sequential conclusive state discrimination protocol. In Section 6, we summarize the main results.

2. Preliminaries

For the description in Section 3 of an N-sequential conclusive state discrimination scenario, let us first recall the main concepts of the quantum measurement theory [14, 15, 2, 16]. Consider a measurement performed on a quantum system described in terms of a complex Hilbert space \( \mathcal{H} \) and with outcomes \( \omega \) in a finite set \( \Omega \). Denote by \( \mathcal{L}(\mathcal{H}) \) the Banach space of all bounded linear operators on \( \mathcal{H} \) and by \( \mathcal{T}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H}) \) – the Banach space of all trace class operators on \( \mathcal{H} \). In case of a finite dimensional space \( \mathcal{H} \), each linear operator is bounded, also, of the trace class.

The complete description of every quantum measurement, projective or generalized,
Conclusive discrimination between arbitrary quantum states by N sequential receivers includes the knowledge of both – statistics of observed outcomes and a family of posterior states, each conditioned by an outcome observed under a single trial of this measurement. In mathematical terms, the complete description of a quantum measurement is specified by the notion of a state instrument \( M(\cdot) \), which is a measure on a measurable space \((\Omega, \mathcal{F}_\Omega)\) with values \( M(F) = \sum_{\omega \in F} M(\omega) \), \( F \subseteq \Omega \), that are completely positive bounded linear maps

\[
M(F)[\cdot] : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}),
\]

satisfying the relation

\[
\sum_{\omega \in \Omega} \text{tr}\{M(\omega)[T]\} = \text{tr}\{M(\Omega)[T]\} = \text{tr}\{T\}, \quad T \in \mathcal{T}(\mathcal{H}).
\]

To each state instrument \( M(\cdot) \), there corresponds the unique observable instrument \( N(\cdot) \) – a measure on \((\Omega, \mathcal{F}_\Omega)\) with values \( N(F) = \sum_{\omega \in F} N(\omega) \), \( F \subseteq \Omega \), that are normal completely positive bounded linear maps

\[
N(F)[\cdot] : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \quad \sum_{\omega \in \Omega} N(\omega)[I_H] = N(\Omega)[I_H] = I_H,
\]

defined to \( M(\cdot) \) via the duality relation

\[
\text{tr}\{M(\omega)[T]Y\} = \text{tr}\{TN(\omega)[Y]\}, \quad \omega \in \Omega,
\]

\[
T \in \mathcal{T}(\mathcal{H}), \quad Y \in \mathcal{L}(\mathcal{H}).
\]

Given a state instrument \( M \) of a quantum measurement and a quantum system state \( \rho \) (density operator) on \( \mathcal{H} \) before this measurement, the probability \( \mu_\rho(F) = \sum_{\omega \in F} \mu_\rho(\omega) \) to observe under this measurement an outcome \( \omega \) in a subset \( F \subseteq \Omega \) has the form

\[
\mu_\rho(F) = \text{tr}\{M(F)[\rho]\} = \text{tr}\{\rho N(F)[I_H]\} = \text{tr}\{\rho M(F)\},
\]

where

\[
M(F) = \sum_{\omega \in F} M(\omega) := N(F)[I_H], \quad M(\Omega) = \sum_{\omega \in \Omega} M(\omega) = I_H,
\]

is the POV measure \( M(\cdot) \) on \((\Omega, \mathcal{F}_\Omega)\), specifying the statistics of the corresponding quantum measurement.

After a single measurement trial, where an outcome \( \omega \in \Omega \) is observed, the state of a quantum system is given by the relation \([17, 18, 19]\)

\[
\rho_{\text{out}}^{(\omega)}(\rho) := \frac{M(\omega)[\rho]}{\mu_\rho(\omega)}
\]

and is called a conditional posterior state. The unconditional posterior state is defined by

\[
\rho_{\text{out}}(\rho) := \rho_{\text{out}}^{(\Omega)}(\rho) = M(\Omega)[\rho].
\]

Every state instrument \( M(\cdot)[\cdot] \) admits the Stinespring-Kraus representation (for details, see \([17, 18]\) and references therein):

\[
M(\omega)[T] = \sum_l K_l(\omega)TK_l^\dagger(\omega), \quad T \in \mathcal{T}(\mathcal{H}), \quad \omega \in \Omega,
\]
Conclusive discrimination between arbitrary quantum states by \( N \) sequential receivers which may be not unique. Here, \( K_l(\omega) \in \mathcal{L}(\mathcal{H}), \omega \in \Omega, l \in \{1, \ldots, L\} \), are bounded linear operators with the operator norms \( \|K_l(\omega)\| \leq 1 \), called the Kraus operators \([20]\) and satisfying the relation
\[
\sum_{\omega, l} K_l^\dagger(\omega) K_l(\omega) = \mathbb{I}_H. \tag{10}
\]
From (4) and (10) it follows that, in terms of Kraus operators, for each \( \omega \in \Omega \),
\[
M(\omega) = \sum_l K_l(\omega) K_l^\dagger(\omega), \tag{11}
\]
\[
\mu_\rho(\omega) = \sum_l \text{tr} \left\{ \rho K_l^\dagger(\omega) K_l(\omega) \right\}, \tag{12}
\]
\[
\rho^{(\omega)}_{\text{out}}(\rho) = \frac{\sum_l K_l(\omega) \rho K_l^\dagger(\omega)}{\mu_\rho(\omega)}. \tag{13}
\]
If, in representation (9), a set \( \{1, \ldots, L\} \) contains only one element, that is, Kraus operators are labeled only by outcomes \( \omega \in \Omega \), then a state instrument \( M \) is called pure, since in this case, mapping \( M(\omega)[] \) "transforms" a pure initial state \( |\psi\rangle\langle\psi| \) to the pure conditional posterior state
\[
\rho^{(\omega)}_{\text{out}}(|\psi\rangle\langle\psi|) = \frac{K(\omega)|\psi\rangle\langle\psi| K^\dagger(\omega)}{\langle\psi| K^\dagger(\omega) K(\omega) |\psi\rangle}, \quad \omega \in \Omega. \tag{14}
\]
2.1. Statistical realizations

According to Ozawa \([21]\), for every observable instrument \( N \), describing a generalized quantum measurement, there exists a statistical realization
\[
\Xi := \{ \tilde{\mathcal{H}}, \sigma, P, U \}, \tag{15}
\]
possibly, not unique, consisting of 4 elements: a complex Hilbert space \( \tilde{\mathcal{H}} \), a density operator \( \sigma \) on \( \tilde{\mathcal{H}} \), a projection-valued measure \( P \) on \( \Omega \) with values \( P(\omega), \omega \in \Omega \), that are projections on \( \mathcal{H} \) and a unitary operator \( U \) on \( \mathcal{H} \otimes \tilde{\mathcal{H}} \), such that, for all \( Y \in \mathcal{L}(\tilde{\mathcal{H}}) \),
\[
N(\omega)[Y] = \text{tr}_{\tilde{\mathcal{H}}} \left\{ (\mathbb{I}_H \otimes \sigma) U^\dagger (Y \otimes P(\omega)) U \right\}, \quad \omega \in \Omega, \tag{16}
\]
where notation \( \text{tr}_{\tilde{\mathcal{H}}} \{ \cdot \} \) means the partial trace over a Hilbert space \( \tilde{\mathcal{H}} \). From relations (4) and (16) it follows that, in terms of the elements of a statistical realization (15), the values \( \mathcal{M}(\omega)[\cdot] \) of a state instrument are given by
\[
\mathcal{M}(\omega)[\rho] = \text{tr}_{\tilde{\mathcal{H}}} \left\{ (\mathbb{I}_H \otimes P(\omega)) U (\rho \otimes \sigma) U^\dagger (\mathbb{I}_H \otimes P(\omega)) \right\}, \quad \omega \in \Omega, \tag{17}
\]
for all density operators \( \rho \) on \( \mathcal{H} \).

Remark 1. The existence for every generalized quantum measurement of a statistical realization \( \Xi = \{ \tilde{\mathcal{H}}, \sigma, P, U \} \) means that each generalized quantum measurement on a state \( \rho \) on \( \mathcal{H} \) can be realized via the indirect measurement, specified by the elements of this statistical realization \( \Xi \). Namely, via a direct measurement \( P(\cdot) \) on some auxiliary quantum system, being initially in a state \( \sigma \) on a Hilbert space \( \tilde{\mathcal{H}} \), after its interaction
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with the original system which results in the composite system state \(U(\rho \otimes \sigma)U^\dagger\) on \(\mathcal{H} \otimes \tilde{\mathcal{H}}\).

Representations (20) and (17) imply that if, for a state instrument \(M\), there exists a statistical realization \(\Xi\) where a state \(\sigma = |b\rangle \langle b|\) is pure while the values of a projection-valued measure \(P\) have the form \(P(\omega) = |\xi_\omega\rangle \langle \xi_\omega|, \quad \omega \in \Omega\) is an orthonormal basis of \(\tilde{\mathcal{H}}\), then, for this state instrument \(M\), there is representation (9) where the Kraus operators are labeled only by an outcome \(\omega \in \Omega\) and are defined via the relations (see Lemma 1 in Section 3.2 of Ref. [18]):

\[
U(|\psi\rangle \otimes |b\rangle) = \sum_\omega K(\omega) |\psi\rangle \otimes |\xi_\omega\rangle, \quad \text{for each } |\psi\rangle \in \mathcal{H},
\]

\[
\langle f| \otimes \langle \xi_\omega| U(|g\rangle \otimes |b\rangle) = \langle f| K(\omega) g\rangle, \quad \text{for all } |f\rangle, |g\rangle \in \mathcal{H}.
\]

In the physical notation,

\[
K(\omega) = \langle \xi_\omega| U|b\rangle.
\]

For more details on the description of a generalized quantum measurement, the notions of an instrument, its statistical realization and the Stinespring–Kraus representation see Sections 2.2–2.5 in Ref. [18]. On the notions of conditional and unconditional posterior states, see also Section 2 in Ref. [19].

3. \(N\)-sequential conclusive state discrimination

For an \(N\)-sequential conclusive state discrimination, consider a general scenario where a sender, say Alice, prepares a quantum system, described in terms of a complex Hilbert space \(\mathcal{H}_A\), possibly infinite-dimensional, in one of states \(\rho_1, ..., \rho_r\), \(r \geq 2\), pure or mixed, with prior probabilities \(q_1, ..., q_r\) and sends her quantum system in the initial state

\[
\rho_{in} = \sum_{i=1,...,r} q_i \rho_i, \quad \sum_i q_i = 1, \quad q_i > 0,
\]

(20)
to the first receiver. On receiving this input state the first receiver performs a conclusive measurement for the discrimination between states \(\{\rho_1, ..., \rho_r\}\), observes an outcome \(i_1 \in \{1, ..., r\}\) and resends the quantum system of Alice in the posterior state, conditioned by this outcome, to a sequential receiver. This procedure is repeated until a conclusive measurement by an \(N\)-th receiver.

Let \(\mathcal{M}_{n}(\cdot)\) \((n = 1, ..., N)\) be a state instrument describing a conclusive quantum measurement with outcomes \(i_n\) in set \(\{1, ..., r\}\) of an \(n\)-th sequential receiver. The corresponding observable instrument \(\mathcal{N}_{n}(\cdot)\) and the POV measure \(M_{n}(\cdot)\) on \(\{1, ..., r\}\) are specified by Eqs. (3)–(4) in Section 2.

The sequence \(A_1 \rightarrow 1 \rightarrow ... \rightarrow k\) of quantum measurements performed by \(k \in \{1, ..., N\}\) receivers, each with an outcome \(i_k \in \{1, ..., r\}\), constitutes a consecutive measurement with an outcome \(\omega = (i_1, ..., i_k) \in \{1, ..., r\}^k\) on the Alice quantum system in an initial state (20) and is described by the state instrument \(\mathcal{M}_{A_1 \rightarrow 1 \rightarrow ... \rightarrow k}(\cdot)\) with
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Values \cite{17}

$$\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k)[\cdot] := \mathcal{M}_k(i_k)[\mathcal{M}_{k-1}(i_{k-1})[\ldots \mathcal{M}_1(i_1)[\cdot]]], \quad (i_1, \ldots, i_k) \in \{1, \ldots, r\}^k.$$  \hspace{1cm} (21)

By Eqs. (3)–(4) the corresponding observable instrument $\mathcal{N}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}$ and the POV measure $\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}$ of $A|\rightarrow 1\rightarrow \ldots \rightarrow k$ consecutive measurement have the forms

$$\mathcal{N}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k)[\cdot] := \mathcal{N}_1(i_1)[\mathcal{N}_2(i_2)[\ldots \mathcal{N}_N(i_k)[\cdot]]], \quad (i_1, \ldots, i_k) \in \{1, \ldots, r\}^k.$$  \hspace{1cm} (22)

$$\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k) = \mathcal{N}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k)[\mathbb{I}_{\mathcal{H}_A}]. \quad (23)$$

From relations (5) and (23) it follows that, for an input state (20) before this consecutive measurement $A| \rightarrow 1 \rightarrow \ldots \rightarrow k$, the probability to receive under this measurement an outcome $\omega = (i_1, \ldots, i_k) \in \{1, \ldots, r\}^k$ is equal to

$$\mu_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k|\rho_i) = \sum_i q_i \text{tr}\{\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k)[\rho_i]\} \quad (24)$$

$$= \sum_i q_i \mu_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k|\rho_i),$$

where

$$\mu_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k|\rho_i) = \text{tr}\{\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k)[\rho_i]\} \quad (25)$$

$$= \text{tr}\{\rho_i \mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i_1, \ldots, i_k)\}.$$  \hspace{1cm} (25)

Therefore, the probability for the first $k \in \{1, \ldots, N\}$ receivers to take the proper decisions on discriminating between input states $\rho_1, \ldots, \rho_r$, given with prior probabilities $q_1, \ldots, q_r$, \textit{i.e. the success probability}, has the form

$$P_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r) = \sum_i q_i \text{tr}\{\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow k}(i, \ldots, i)[\rho_i]\}, \quad (26)$$

$$k \in \{1, \ldots, N\}. $$

For the $N$-sequential conclusive discrimination between states $\rho_1, \ldots, \rho_r$ given with prior probabilities $q_1, \ldots, q_r$, the success probability

$$P_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r) = \sum_i q_i \text{tr}\{\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}(i, \ldots, i)[\rho_i]\} \quad (27)$$

$$= \sum_i q_i \text{tr}\{\rho_i \mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}(i, \ldots, i)\},$$

where $\mathcal{M}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}$ is the POV measure (23) with $k = N$ for the consecutive measurement $A| \rightarrow 1 \rightarrow \ldots \rightarrow N$ with outcomes in $\{1, \ldots, r\}^N$.

Besides representation (27) for the success probability $P_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}^{\text{success}}$, let us also introduce its other mutually equivalent representations. Denote by

$$\rho_k^{(i, \ldots, i)}(\rho_i), \quad k \geq 1,$$  \hspace{1cm} (28)
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denote the posterior state on $H_A$ conditioned by an outcome $(i, \ldots, i) \in \{1, \ldots, r\}^k$ observed under the consecutive measurement $A | \rightarrow 1 \rightarrow \ldots \rightarrow k$ on a state $\rho_i$. In view of relations (7), (21) and (28), we have

$$p_{k-1}^{(i, \ldots, i)}(\rho_i) = \frac{\mathcal{M}_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}(i, \ldots, i)[\rho_i]}{\mu_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}(i, \ldots, i)[\rho_i]}$$

and, therefore,

$$\mathcal{M}_{A|\rightarrow 1 \rightarrow \ldots \rightarrow k}(i, \ldots, i)[\rho_i] = \mathcal{M}_k(i)[\mathcal{M}_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}(i, \ldots, i)[\rho_i]]$$

Since a length of a tuple $(i, \ldots, i) \in \{1, \ldots, r\}^k$ is equal to a number $k$ standing in indices at instrument $\mathcal{M}$, probability $\mu$ and at a posterior state, for short, we below omit the upper decoration at $i, \ldots, i$.

By Eqs. (26) and (30), for all $k \geq 2$, we come to the following representations

$$P_{A|\rightarrow 1 \rightarrow \ldots \rightarrow k}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r)$$

$$= \sum_i q_i \mu_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}(i, \ldots, i)[\rho_i] \cdot \text{tr}\left\{p_{k-1}^{(i, \ldots, i)}(\rho_i)M_k(i)\right\}$$

$$= \sum_i q_i \mu_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r) \sum_i Q_{k-1}^{(i)} \text{tr}\left\{\rho_{k-1}^{(i, \ldots, i)}(\rho_i)M_k(i)\right\},$$

where

$$Q_{k-1}^{(i)} := \frac{q_i \mu_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}(i, \ldots, i)[\rho_i]}{P_{A|\rightarrow 1 \rightarrow \ldots \rightarrow (k-1)}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r)}; \quad \sum_i Q_{k-1}^{(i)} = 1, \quad k \geq 2;$$

are the prior probabilities of the conditional posterior states $p_{k-1}^{(i, \ldots, i)}(\rho_i)$. Relations (26), (31), (32) imply

$$P_{A|\rightarrow 1}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r) = \sum_{i=1}^r q_i \text{tr}\left\{\rho_i M_1(i)\right\}$$

and two mutually equivalent representations for $N \geq 2$:

$$P_{A|\rightarrow 1 \rightarrow \ldots \rightarrow N}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r)$$

$$= \sum_{i=1}^r q_i \text{tr}\left\{\rho_i M_1(i)\right\} \cdot \text{tr}\left\{\rho_1^{[i]}(\rho_i)M_2(i)\right\} \cdot \ldots \cdot \text{tr}\left\{\rho_{N-1}^{[i]}(\rho_i)M_N(i)\right\}$$

$$= P_{A|\rightarrow 1}^{\text{success}}(\rho_1, \ldots, \rho_r|q_1, \ldots, q_r) \times \prod_{n=2}^N P_{(n-1)|\rightarrow n}^{\text{success}}(\rho_1^{(1, \ldots, 1)}(\rho_1), \ldots, \rho_1^{(r, \ldots, r)}(\rho_r)|Q_{n-1}^{(1)}, \ldots, Q_{n-1}^{(r)}).$$
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where

$$\Pr_{(n-1)\to n}^{\text{success}}(\rho_{n-1}^{(1,\ldots,1)}(\rho_1), \ldots, \rho_{n-1}^{(r,\ldots,r)}(\rho_r) | Q_{n-1}^{(1)}, \ldots, Q_{n-1}^{(r)})$$

$$= \sum_{i=1}^{r} Q_n^{(i)} \text{tr} \left\{ \rho_{n-1}^{(i,\ldots,i)}(\rho_i) M_n(i) \right\}$$

(36)

is the success probability for $n$-th receiver to discriminate between posterior states $\rho_{n-1}^{(i,\ldots,i)}(\rho_i)$, each originated from a state $\rho_i$ and conditioned by the outcome “$i$” under the conclusive measurements of all previous receivers.

We stress that, in a general scenario introduced above for an $N$-sequential conclusive state discrimination, we do not impose any constraints either on states $\rho_1, \ldots, \rho_r$ to be discriminated and their prior probabilities $q_1, \ldots, q_r$ or on a structure of conditional posterior states after sequential measurements.

4. Optimal success probability

For only one receiver ($N = 1$), the success probability $\Pr_{A\to 1}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2)$ is given by (33) and in the case where there are only two states $\rho_1, \rho_2$ to be discriminated (i. e. $r = 2$) the Helstrom upper bound [1]

$$\Pr_{A\to 1}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2) = \sum_{i=1,2} q_i \text{tr} \left\{ \rho_i M_1(i) \right\}$$

$$\leq \frac{1}{2} \left( 1 + \| q_1 \rho_1 - q_2 \rho_2 \|_1 \right)$$

(37)

holds for all states $\rho_1, \rho_2$, pure and mixed, all prior probabilities $q_1, q_2$ and any type of a quantum measurement projective or generalized. Moreover, this upper bound is attained if the receiver’s POV measure is given by the projection-valued measure $M_1^{\text{opt}}(i) = P_{\text{opt}}(i)$, $i = 1, 2$ with the values

$$P_{\text{opt}}(1) = \sum_{\lambda_k > 0} E(\lambda_k), \quad P_{\text{opt}}(2) = I_{\mathcal{H}_A} - P_{\text{opt}}(1),$$

$$P_{\text{opt}}^2(j) = P_{\text{opt}}(j), \quad j = 1, 2,$$

$$P_{\text{opt}}(1)P_{\text{opt}}(2) = P_{\text{opt}}(2)P_{\text{opt}}(1) = 0.$$

(38)

Here, $P_{\text{opt}}(1)$ is the orthogonal projection on the invariant subspace of operator $(q_1 \rho_1 - q_2 \rho_2)$, corresponding to its positive eigenvalues and $E(\lambda_k)$ are the spectral projections of the Hermitian operator $q_1 \rho_1 - q_2 \rho_2 = \sum \lambda_k E(\lambda_k)$ on $\mathcal{H}_A$.

From representations (27), (34), (35) for the success probability $\Pr_{A\to 1\to \ldots\to N}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2)$ and the upper bound (37) it immediately follows that, for every $N$-sequential conclusive state discrimination protocol $A\to 1 \to \ldots \to N$,

$$\Pr_{A\to 1\to \ldots\to N}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2) \leq \frac{1}{2} \left( 1 + \| q_1 \rho_1 - q_2 \rho_2 \|_1 \right).$$

(39)

Denote by $\mathcal{M}_N = \{ \mathcal{M}_{A\to 1\to \ldots\to N}(\cdot) \}$ the set of all possible state instruments for $N$-sequential state discrimination protocols.
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**Theorem 1.** Let $\rho_1, \rho_2$ be arbitrary quantum states on a Hilbert space $\mathcal{H}_A$, given with prior probabilities $q_1, q_2$. For an $N$-sequential conclusive discrimination between states $\rho_1, \rho_2$, the optimal success probability is equal to the Helstrom bound

$$P_{opt, success}^{\rho_1, \rho_2} (n) = \max_{\rho_{opt}} \mathbb{P} (\rho_{opt} | q_1, q_2)$$

$$= \frac{1}{2} (1 + \| q_1 \rho_1 - q_2 \rho_2 \|_1),$$

for every number $N \geq 1$ of sequential receivers.

**Proof.** Let the measurement of each $n$-th receiver be described by the state instrument

$$\mathcal{M}_n(i) = P_{opt}(i) M_{opt}(i),$$

and with the POV measure $M_n(i) = P_{opt}(i)$, where $P_{opt}(\cdot)$ is the projection-valued measure \textcolor{blue}{[38]}. In this case, by (21), for each $k \in \{1, ..., N\}$ the corresponding consecutive measurement $A| \rightarrow 1 \rightarrow ... \rightarrow k$ with outcomes in set $\{1, 2\}^k$ is described by the state instrument

$$\mathcal{M}_{A| \rightarrow 1 \rightarrow ... \rightarrow k} (i_1, ..., i_k) := P_{opt}(i_k) \cdot \cdot \cdot P_{opt}(i_1) \cdot \cdot \cdot P_{opt}(i_k),$$

which has the values

$$\mathcal{M}_{A| \rightarrow 1 \rightarrow ... \rightarrow k}^{\text{opt}} (i_1, ..., i_k) = P_{opt}(i) P_{opt}(i), \quad i = 1, 2,$$

$$\mathcal{M}_{A| \rightarrow 1 \rightarrow ... \rightarrow k}^{\text{opt}} (i_1, ..., i_k) = 0, \quad \exists i_{n_1} \neq i_{n_2}.$$

For this state instrument $\mathcal{M}_{A| \rightarrow 1 \rightarrow ... \rightarrow k}^{\text{opt}}$:

(i) the values of the POV measure \textcolor{blue}{[23]} are given by

$$M_{A| \rightarrow 1 \rightarrow ... \rightarrow k} (i, ..., i) = P_{opt}(i), \quad i \in \{1, 2\},$$

$$M_{A| \rightarrow 1 \rightarrow ... \rightarrow k} (i_1, ..., i_k) = 0, \quad \exists i_{n_1} \neq i_{n_2};$$

(ii) for each $k \in \{1, ..., N-1\}$, the conditional posterior states \textcolor{blue}{[27]} take the form

$$\rho_k^{(i_{i},...i)} (\rho_i) = \frac{P_{opt}(i) \rho_i P_{opt}(i)}{\text{tr} \{ \rho_i P_{opt}(i) \} }, \quad i \in \{1, 2\},$$

and satisfy the relation

$$\text{tr} \{ \rho_k^{(i_{i},...i)} (\rho_i) M_{k+1} (i) \} = \text{tr} \{ \rho_k^{(i_{i},...i)} (\rho_i) P_{opt}(i) \} = 1, \quad i \in \{1, 2\};$$

(iii) for $k = 1$

$$P_{A| \rightarrow 1}^{\text{success}} (\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} (1 + \| q_1 \rho_1 - q_2 \rho_2 \|_1);$$

(iv) for each $n \geq 2$, the success probability \textcolor{blue}{[36]} is equal to

$$P_{A| \rightarrow 1}^{\text{success}} (\rho_1, \rho_2 | q_1, q_2) = Q_n^{(1)} \text{tr} \{ \rho_n^{(1)} (\rho_1) M_n (i) \} + Q_n^{(2)} \text{tr} \{ \rho_n^{(2)} (\rho_2) M_n (2) \} = 1$$

Relations \textcolor{blue}{[39], [41]} - \textcolor{blue}{[38]} prove the statement.  

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5. $N$-sequential conclusive state discrimination via indirect measurements

Let, under an $N$-sequential conclusive discrimination between two states, all receivers perform indirect measurements.

**Theorem 2.** There exist receivers’ indirect measurements which implement the optimal $N$-sequential conclusive state discrimination specified in Theorem 1.

**Proof.** Let the indirect measurement of each $n$-th receiver be described by the statistical realization

$$\Xi_n = \left\{ \mathcal{H}, |b_n\rangle\langle b_n|, \tilde{P}_n, U_n \right\}$$

(49)

where $\mathcal{H}$ is a two-dimensional Hilbert space, $|b_n\rangle\langle b_n|$ is a pure state on $\mathcal{H}$, the projection-valued measure $\tilde{P}_n$ has the elements $\tilde{P}_n(1) = |b_n\rangle\langle b_n|$ and $\tilde{P}_n(2) = |b_n^\perp\rangle\langle b_n^\perp| = I_{\mathcal{H}} - |b_n\rangle\langle b_n|$, and $U_n$ is a unitary operator on $\mathcal{H}_A \otimes \mathcal{H}$ which has the specific form

$$U_n = P_{opt}(1) \otimes I_{\mathcal{H}} + P_{opt}(2) \otimes (|b_n^\perp\rangle\langle b_n^\perp| + |b_n\rangle\langle b_n^\perp|),$$

(50)

where the projection-valued measure $P_{opt}(\cdot)$ is introduced in (38). Then by (17), the state instrument $\mathcal{M}_n(\cdot)$ describing the $n$-th receiver’s indirect measurement (49) is given by

$$\mathcal{M}_n(i)[\sigma] = \text{tr}_{\mathcal{H}} \left\{ \mathcal{H} \otimes \tilde{P}_n(i) \left( U (\sigma \otimes |b_n\rangle\langle b_n|) U^\dagger \right) I_{\mathcal{H}} \otimes \tilde{P}_n(i) \right\}$$

(51)

$$= P_{opt}(i)[\sigma] P_{opt}(i), \quad i = 1, 2, \quad \forall \sigma,$$

and is, therefore, equal to the optimal state instrument for each $n$-th receiver specified in (41). This proves the statement. ■

6. Conclusion

In the present article, we have developed a general framework for the description of an $N$-sequential state discrimination, where each of $N$ receivers always obtains a conclusive result. For this new state discrimination scenario, we have derived two mutually equivalent general representations (35), (36) of the success probability and have proved (Theorem 1) that in the case where one of two quantum states, pure or mixed, is prepared by a sender, the optimal success probability is given by the Helstrom bound for any number $N$ of sequential receivers. We have also specified (Theorem 2) receivers’ indirect measurements implementing the optimal $N$-sequential conclusive state discrimination protocol.

The developed framework is true for any number $N$ of sequential receivers, any number of arbitrary quantum states, pure or mixed, to be discriminated, and all types of receivers’ quantum measurements. The new general results derived within the developed framework are important both from the theoretical point of view and for a successful multipartite quantum communication even in the presence of a quantum noise.
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Acknowledgement

The study by E. R. Loubenets in Sections 2, 3 of this work is supported by the Russian Science Foundation under the Grant No 19-11-00086 and performed at the Steklov Mathematical Institute of Russian Academy of Sciences. The study by E. R. Loubenets and Min Namkung in Sections 4, 5 is performed at the National Research University Higher School of Economics.

Min Namkung thanks Prof. Younghun Kwon at Hanyang University (ERICA) for his insightful discussion.

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