THE CLOSED ORBIT CONTROLLABILITY CRITERIUM

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Abstract. We prove that every closed "general" trajectory of the control system $\Sigma_M$ has an open neighborhood on which $\Sigma_M$ is controllable if 1) this orbit contains some point where the Lie algebra rank condition ($LARC$) is satisfied, and 2) the set of control vectors is "involved" at $q$. In particular, for the control systems $\Sigma_M$ on the compact connected manifold $M^n$ with an open control set this gives the following "Closed Orbit Controllability Criterium":

The dynamical system $\Sigma_M$ of the considered type is controllable on $M^n$ if and only if for an arbitrary point $q$ of $M^n$ there exists a closed trajectory of the control system going through this point. We also present examples which show that our conditions are necessary.

Introduction and results

In this note we study the chronological map $G(t_1,\ldots,t_N): R^N \rightarrow M^n$ of the control dynamical system

$$\dot{q}(t) = V_u(q(t)) \quad (\Sigma_M)$$

on a complete manifold $M^n$. We prove that it is an open map in the open domain in $R^N$ for sufficiently big $N$, if the system satisfies the well-known Lie Algebra Rank condition ($LARC$). This implies the controllability of the system in some neighborhood of any, so called "general" closed orbit.

To ensure that an arbitrary trajectory of the system can be approximated by some general trajectory we use the "involvement" condition which, essentially, means that any direction $V_u$ lies in some positive cone generated by some set of $V_{u_i}$ satisfying $LARC$. More precisely, we prove that if the set of control vectors $V_u$ is open in the linear subspace $L$ which it generates (the "open" condition), and its convex hull coincides with $L$ (the "ample" condition), then $\Sigma_M$ is "involved", see definitions below.

For the control dynamical systems $\Sigma_M$ on a compact connected manifold $M^n$ which

1) satisfy the Lie algebra rank condition everywhere and
2) have "involved" control vectors set;

we prove the following "Closed Orbit Controllability Criterium": such system is controllable if and only if through every point of the manifold goes some closed orbit of $\Sigma_M$, see Theorem A below.

One of the applications of the Theorem A is to bilinear systems in the Euclidean space. These are the systems of linear equations

$$\dot{x}(t) = A(u)x(t) = (A + u^1B_1 + \ldots + u^dB_d)x(t), \quad (\Sigma)$$

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where the right-hand side - the linear operator $A(u)$ - is a linear combination of some constant linear operators $A$ and $B_k$ which do not depend on the control parameter $u \in \mathbb{R}^d$. For such systems the "Closed Orbit Controllability Criterium" provides many new conditions for controllability, both necessary and sufficient; see our forthcoming paper [CM]. Here we mention only the simplest one for the systems in three-dimensional Euclidean space $\mathbb{R}^3$, see Theorem B below: if for two control parameters $u$ and $v$ the right-hand side linear operators have complex eigenvalues $\lambda_C(u)$ and $\lambda_C(v)$ correspondingly, then the system $\Sigma$ satisfying LARC is controllable if for the real eigenvalues $\lambda_R(u)$ and $\lambda_R(v)$ of $A(u)$ and $A(v)$ it holds

$$(\lambda_R(u) - Re(\lambda_C(u))) (\lambda_R(v) - Re(\lambda_C(v))) < 0.$$  

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1. Twisted LARC

To introduce notations we remind some basics of the theory of control systems, see [AS].

Let $M^n$ be a complete n-dimensional manifold without boundary. The control system on $M$ is given by the family of dynamical systems

(1) $$\dot{q}(t) = V_u(q(t)), \quad q \in M, u \in U; \quad (\Sigma_M),$$

where the right-hand side is a smooth vector field $V_u$ depending on the control parameter $u$ from some set $U$ of controls (usually the domain in some Euclidean space: $U \subset \mathbb{R}^d$). The solution of $\Sigma_M$ is the trajectory $q(t)$ such that

(2) $$\dot{q}(t) = V_u(t)(q(t)), \quad q \in M, u(t) : R \to U$$

for some admissible control function $u(t)$ (usually, the locally integrable functions $u(t) : R \to U \subset \mathbb{R}^d$).

1.1. Semi-group $G^+$ of $\Sigma_M$.

For a given $u \in U$ denote by $P^t_u$ the one-parameter group of diffeomorphisms of $M$ generated by the vector field $V_u$ on $M$. The standard notations are:

(3) $$P^t_u = P^t_{V_u} = \exp\left(\int_0^t V_u d\tau\right) = e^{tV_u},$$

where $t$ might be positive, negative or zero (and then $P^0_u$ is the identity map $Id$). Compositions of such diffeomorphisms for different vector fields $V_u$, called "chronological products",

(4) $$G(t_1, t_2, \ldots, t_N) = P^{t_1}_{u_1} \circ \ldots \circ P^{t_N}_{u_N}$$

generate the Group $G = G(\Sigma_M)$ of the system, which is the (Frechet-) subgroup of the complete group $Diff(M)$ of all diffeomorphisms of $M$. If we restrict ourself only to nonnegative parameters $t_i$ we have the (positive) semigroup $G^+$ of the system. If we assume that every control function $u(t)$ can be approximated by a piece-wise constant function $u_N(t)$ which equals $u_i$ on the interval $(T_{i-1}, T_i)$ where $T_i = t_N + \ldots + t_i$; and that every trajectory of the control system $\Sigma_M$ can be approximated by trajectories with the piece-wise controls; then the set $O^+(q)$ of points reachable from $q$ by trajectories of $\Sigma_M$ equals the orbit $G^+(q)$ of $q$ under the action of the semigroup $G^+$. 

The system $\Sigma_M$ is controllable if for every two points $q, p$ there exists some trajectory $q(t, u(t))$, $0 \leq t \leq T$ of the system $\Sigma_M$ starting at $q = q(0, u(0))$ and ending at $p = q(T, u(T))$. In other words, the system is controllable if the action of the semi-group $G^+$ is transitive. Since it is easier to handle the group-actions than actions of semi-groups, the question of whether $\Sigma_M$ is controllable or not is, usually, divided into two sub-questions:

1) whether the action of the group $G$ is transitive or not,
2) whether the orbits of the semi-group $G^+$ actually coincide with the orbits of the bigger $G$.

Although in general the first question is not easy, its local version though has the following well-known answer. We may call the system $\Sigma_M$ strongly transitive at $p$ if there exists some open $\epsilon(q)$-neighborhood $B_\epsilon(q)$ such that every point $p$ from this neighborhood if reachable from $q$ by some trajectory $q(t, u(t)), 0 \leq t \leq t(q, \epsilon(q))$ of the $G$-action, where $t(q, \epsilon(q)) \to 0$ as $\epsilon(q) \to 0$. Then the local transitivity in $q$ is assured by the well-known Lie algebra rank condition $(LARC)^1$:

1') $LARC(q)$ The Lie-algebra $LA$ generated by some number of vector fields $V_\alpha$, $i = 1, \ldots, d$ is such that its evaluation at the point $q$ coincide with the whole tangent space $T_q M$ to $M$ at $q$.

Indeed, note that in general the one-parameter diffeomorphisms in the "chronological product" do not commute. Thus, it is possible to move the point by the action of $G$ not only along vector fields $V_\alpha$, for different $i$, but also along their commutators, see the (11) below; and so on. Therefore, the orbit $G(q)$ - by The Orbit Theorem, see [AS] - is some immersed submanifold in $M^n$ with the space of tangent vector fields closed under the Lie bracket. Then from the $LARC$ at $q$ we see that the tangent space of the orbit $G(q)$ equals the tangent space to the whole $M^n$, and that the orbit $G(q)$ contains some open neighborhood of $q$. The last means that the action of $G$ is locally transitive.$^2$

In general, the point $p$ close to $q$ could be reached by "long" trajectories issuing from $q$ so that the $LARC(q)$ at $q$ is not necessary for controllability or transitivity of the $G$-action, see the example 5.3 in [AS]. Actually, slightly generalizing this example it is easy to construct controllable dynamical systems such that the $LARC$ does not hold at any point at all, i.e., such that $LA(q) \neq T_q M^n$ everywhere: say, for arbitrary $n > 2$ we may always have $dim(la(q)) \leq 2$.

**Example 1.** For an arbitrary $n > 2$ there exists a controllable dynamical system $\Sigma_T$ on $n$-dimensional torus $T^n$ such that in all points $dim LA(q) \leq 2$.

**Construction.**

Indeed, take the product $\Pi^n \subset R^n$ of the closed interval $[0, n-1]$ with $(n-1)$-dimensional unit cube $[0, 1]^{n-1}$, and introduce coordinates $\overrightarrow{x} = \{x, y^2, \ldots, y^n\}$ in $\Pi$ correspondingly. Choose some smooth nonnegative function $\omega(r)$ with support inside $[1/3, 2/3]$ and an integral bigger than 1, and define the vector field $V(x, y^2, \ldots, y^n, u)$

1such systems are also called completely nonholonomic or bracket-generating

2Of the actual choice of "generators" $V_\alpha$, $i = 1, \ldots, n$ for systems $\Sigma_M$ which satisfy $LARC$ is not canonical, but depends on the particular form of $V_\alpha$. For instance, when the vector field $V_\alpha$ depends smoothly on $u$ we have

$$V_\alpha = V_{\alpha_0} + dV_{\alpha_0}(u - u_0) + o(\|u - u_0\|) = A + (u - u_0)^t B_t + o(\|u - u_0\|)$$

for $A(q) = V_{\alpha_0}(q)$ and $B_t(q) = \partial V_{\alpha_0}(p)/\partial u^i$ at every $u_0$. Then for the smooth system satisfying $LARC$ we may find $u_0 \in U, U \subset R^d$ such that the Lie-algebra generated by the vector fields $A$ and $B_t$ for $u_t = u_0 + \delta E_i$ for all sufficiently small $\delta$, where $E_i$ is the unit basis vector in $R^d$ with number $i$; coincide with the Lie-algebra generated by vector fields $A$ and $B_t$ at $u_0$. Then, without loss of generality we may assume, that the "generators" above are the vector fields $A, B_t$. We may reformulate this by saying that under the $LARC$ the tangent space $T_q M^n$ to $M^n$ at the point $q$ is generated by vectors $A(q), B_t(q)$ and their commutators: say $C_i(q) = [B_t, B_j](q)$, and so on. This happens in the particular case of so called bilinear systems in Euclidean spaces when $M^n = R^n$;

$$\dot{q}(t) = (A + u^1 B_1 + \ldots + u^d B_d)q(t),$$

which we consider in [CM].
in the parallelepiped $\Pi$ as follows: for $k \leq x \leq k + 1$ where $k = 0, 1, ..., n - 2$ is integer take

\[(\text{exm 1})\]
\[V(x, y^2, ..., y^n, u) = \frac{\partial}{\partial x} + u\omega(x - k)\frac{\partial}{\partial y^{k+2}},\]

where the control $u$ belongs to an open interval $\Omega = (-1, 1)$. Since $V$ is invariant under parallel translations of $R^n$ of the form:

\[V(x, y^2, ..., y^k, ..., y^n, u) = V(x, y^2, ..., y^k + 1, ..., y^n, u)\]

and

\[V(x, y^2, ..., y^k, ..., y^n, u) = V(x + d, y^2, ..., y^k, ..., y^n, u);\]

- in fact, it depends only on $x$ - we see that after identification of opposite sides of the parallelepiped $\Pi$ we obtain the $n$-dimensional torus $T^n$ with a smooth vector field which we denote again by $V(\vec{x}, u)$. All trajectories of $V(\vec{x}, u)$ are curves normal to the sub-torus $T^{n-1}$ defined by $x = 0$ with $\{y^2, ..., y^n\}$ being local coordinates in $T^{n-1}$. From the definition above we see that changing the control $u(t)$ on the interval $t \in [k-1, k], k = 0, ..., n - 2$ leads to the change of the $y^{k+2}$-coordinate, leaving all other coordinates intact. Thus, the point $p$ with arbitrary coordinates $\{y^2, ..., y^n\}$ can be reached from the zero-point $q$ (possibly, after one additional ”rotation” with zero control along $x$-coordinate parallel of the torus $T^n$) with the help of the following piece-wise constant control function $u(t)$:

\[(\text{exm 2})\]
\[u(t) = y^k(\int \omega(r)dr)^{-1}\]

on the interval $k - 1 \leq t \leq k$. We see that $O^+(p) = T^n$, and the control system

\[(\text{exm 3})\]
\[\frac{d}{dt}\vec{x} = V(\vec{x}, u)\quad (\Sigma_T)\]

is controllable, while the dimension of $\mathcal{L}A$ in each point is 1 or 2.

Despite the example above, the controllability implies some generalized $iLARC$ ("integral"). We address this question later, while here we note only the following: if the action of $G$ is transitive then the orbit of any point under this action is the whole manifold $M^n$, i.e., is an open subset in particular. We know that every orbit is an immersed manifold. Therefore, since $M^n$ is a second category set; it has to have the same dimension $n$ as $M^n$. Then compositions (4) ("chronological products") with some fixed $N$ should provide us with an open map (surjection) from the space of $(t_1, ..., t_N)$ to some neighborhood of $q$ in $M^n$. The standard arguments then show that controllability leads to some kind of generalized $LARC$ along all closed trajectories through $q$ compared with the help of corresponding Poincare maps.

In this paper we restrict ourself to the case when the $LARC$ holds at the point $q$ we consider:

\[(5)\]
\[\mathcal{L}A(q) = T_qM^n,\]

and then prove that every ”general” closed orbit through $q$ belongs to the orbit $G^+(q)$ together with some its open neighborhood $B_q^+$, and further, has some open neighborhood $B_q$ on which the system $\Sigma_M$ is controllable.
1.2. Vector fields as operators on $C^\infty(M)$ (after [AS]).

We know that the manifold $M^n$ may be identified with homomorphisms $\phi : C^\infty(M) \to R$ of the algebra of smooth function on $M$; i.e., for each algebra homomorphism $\phi$ there exists some point $q \in M$ such that $\phi$ coincides with the evaluation homomorphism $\hat{q}(a) = a(q)$ at this point, see [AS].

In the same way it is useful to represent diffeomorphisms $P : M \to M$ as automorphisms of the algebra $C^\infty(M)$ of smooth functions on $M$ acting by change of variables: for any given function $a$ on $M$ the action $P$ on $a$ is given by $\hat{P}(a) = a \circ P$. Indeed, every algebra homomorphism $A : C^\infty(M) \to C^\infty(M)$ is some $\hat{P}$ defined by $P(q) = \hat{q}$ where $\hat{q} = \hat{q} \circ A$.

Then the vector field $V$ is given by the derivation of $C^\infty(M)$, i.e., the linear operator $\hat{V} : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule $\hat{a}(a) = \hat{V}(a) + a \hat{V}(b)$. If the vector field $V$ generates the flow $P^t$ then from the definition it follows

$$\frac{d}{dt} \hat{P}^t = \hat{V} \circ P^t = P^t \circ \hat{V}. \tag{6}$$

If $P_*$ denote the differential of $P$ then the vector field $P_*(V)$ equals the derivation $P^{-1} \circ V \circ P$. This justifies the notation $P_* = Ad(P)$, while the calculation

$$\frac{d}{dt} (AdP^t)W_{|t=0} = \frac{d}{dt} (P^t \circ W \circ P^{-t})_{|t=0} = [V, W] \tag{7}$$

shows that the Lie bracket of vector fields $V$ and $W$ is given by the derivative of $Ad(P^t)$ at $t = 0$ which is denoted by $(adV)W = [V, W]$. Since the Lie bracket is preserved under the $P_*$ we conclude $AdP^t[X, Y] = [AdP^tX, AdP^tY]$ which after differentiation at $t = 0$ gives the Jacobi identity

$$(adV)[X, Y] = [(adV)X, Y] + [X, (adV)Y] = [V, [X, Y]] = [[V, X], Y] + [X, [V, Y]].$$

From (6) we may conclude the following asymptotic form for $P^t$, also denoted as

$$exp(V) = \sum_{n=0}^{\infty} \frac{t^n}{n!} V^n, \tag{8}$$

where $V^n = V \circ \ldots \circ V$ (one variable conjugate Taylor formula). If

$$S_m(t) = Id + \sum_{n=1}^{m-1} \int \ldots \int_{\Delta_n(t)} V_{\tau_n} \circ \ldots \circ V_{\tau_1} d\tau_n \ldots d\tau_1 \tag{9}$$

is the partial sum, then for any $a \in C^\infty(M)$, $s \geq 0$ and compact $K \subset M$ it holds

$$\|exp \int_0^t V \tau d\tau - S_m(t)a\|_{s, K} \leq O(t^m), \tag{10}$$

see (2.13) from [AS]. Below we assume that $m$ is big enough, say $m > 2^n$; which is sufficient for estimates below. Using (9-10) for diffeomorphisms generated by different vector fields we deduce the asymptotic representation for an arbitrary chronological product (the conjugate Taylor formula for many variables), or approximation

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3Note that in the proof of this (Gelfand-Neimark type) theorem in the Lemma A1 in [AS] the set $M \setminus K$ can be empty.
formula for the action of the group $G$. For instance, for the curve $q(t) = q \circ P^t_v \circ P^t_w \circ P^{-t}_v \circ P^{-t}_w$ it gives the following expansion:

$$q(t) = q \circ (I_d + t^2 (V \circ W - W \circ V) + o(t^2)),$$

where $V, W$ are generators of the one-parameter groups of diffeomorphisms $P^t_v$ and $P^t_w$. Generally, for the chronological product $G(t_1, ..., t_N) = P^{t_1}_{u_1} \circ ... \circ P^{t_N}_{u_N}$ the same arguments give:

$$q(t_1, ..., t_N) = q \circ G(t_1, ..., t_N) = q \circ (I_d + \sum_{n=1}^{m-1} \frac{t^n}{n!} V_{u_1}^n) \circ ... \circ (I_d + \sum_{n=1}^{m-1} \frac{t^n}{n!} V_{u_N}^n) + O(T^m),$$

where by $T$ we denote $|t_1| + ... + |t_N|$. We may further expand (open all brackets) the above formula (12) to obtain the Taylor type approximation:

$$q(t_1, ..., t_N) = q \circ (I_d + \sum_{k=1}^{k=m-1} \sum_{\alpha^k} \frac{t_1^{\alpha_1} \cdot ... \cdot t_N^{\alpha_N}}{\alpha_1! \cdot ... \cdot \alpha_N!} V^1_{u_1} \circ ... \circ V^N_{u_N}) + O(T^m),$$

where $\alpha^k = (\alpha_1^k, ..., \alpha_N^k)$ is $N$-multi-index of order $k$; i.e., $\|\alpha^k\| = \alpha_1^k + ... + \alpha_N^k = k$; while $V^\alpha_{u_i} = V_{u_i} \circ ... \circ V_{u_i}$ denotes the composition of $\alpha_i$ derivations $V_{u_i}$. For the notational convenience, we consider the vector fields $V_{u_i}$ for $i = 1, ..., N$ to be different, also they may be (and usually are) taken from some finite set $V_i, i = 1, ..., d'$ of vector fields (say, satisfying the LARC). It is also important that, despite the fact that the asymptotic representations of chronological products above in general may not be convergent for some collections of vector fields $V_u$ and on some functions from $C^\infty(M^n)$, in our case - when the set of generators $V_{u_i}$ is finite and we consider the action on coordinate functions - representation above are convergent (and are, actually, nothing more than the Taylor formula).\(^4\)

1.3. Rank of the chronological map.

Due to (6) for the given chronological product $q(t_1, ..., t_N)$ its partial derivatives are:

$$\frac{\partial}{\partial t_i} q(t_1, ..., t_N) = \frac{\partial}{\partial t_i} P^t_{u_1} \circ ... \circ P^t_{u_N} = P^t_{u_1} \circ ... \circ V_{u_i} \circ P^t_{u_i} \circ ... \circ P^t_{u_N},$$

which with the help of (9,10) leads to the following asymptotic representations:

$$\frac{\partial}{\partial t_i} q(t_1, ..., t_N) = \frac{\partial}{\partial t_i} q \circ (I_d + \sum_{n=1}^{m-1} \frac{t^n}{n!} V_{u_i}^n) \circ V_{u_i} \circ (I_d + \sum_{n=1}^{m-1} \frac{t^n}{n!} V_{u_i}^n) \circ ... \circ (I_d + \sum_{n=1}^{m-1} \frac{t^n}{n!} V_{u_N}^n) + O(T^m),$$

or, comparing with (13) we get:

$$\frac{\partial}{\partial t_i} q(t_1, ..., t_N) = q \circ \sum_{k=1}^{k=m-1} \sum_{\alpha^k} \frac{\partial}{\partial t_i} \frac{t_1^{\alpha_1} \cdot ... \cdot t_N^{\alpha_N}}{\alpha_1! \cdot ... \cdot \alpha_N!} V^1_{u_1} \circ ... \circ V^N_{u_N} + O(T^{m-1}).$$

Again, we remind that the chronological map $G : (t_1, ..., t_N) \to q(t_1, ..., t_N) \in M^n$ we understand here as the map from the Euclidean space $R^N$ to algebra homomorphisms of $C^\infty(M)$, i.e., each point of $M$

\(^4\)In fact, we may use these finite approximations for the chronological products as their substitutes to obtain all our results below and the Theorem A in particular. The estimate (10) then guarantee that the claim of the Theorem A is valid for the asymptotic limit under our conditions - openness of $V(q)$ - on the set of control vectors.
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is understood as the evaluation homomorphism. Then partial derivatives of \( q(t) \) above are derivations of \( C^\infty(\mathbb{M}^n) \) corresponding to the vector fields. In a "usual sense" - as sections of the tangent bundle of \( \mathbb{M}^n \) - these partial derivatives are

\[
W_i(q(t_1, ..., t_N)) = d\mathcal{G}(t_1, ..., t_N)(\frac{\partial}{\partial t_i}),
\]

where here the chronological map \( \mathcal{G} \) is the smooth map from the Euclidean space \( \mathbb{R}^N \) to the differential manifold \( \mathbb{M}^n \).

Now we introduce some local coordinates \((x^1, ..., x^n)\) in some small neighborhood which contains both points \( q' = q(t_1, ..., t_N) \) and \( q \) - assuming of course that they are close enough; i.e., that \( |T| \) is small. In this neighborhood we have coordinate functions \( x^j : \mathbb{M}^n \to \mathbb{R} \), which we may assume to be continued outside the coordinate neighborhood in an arbitrary way; i.e., it holds: \( x^j \in C^\infty(\mathbb{M}^n) \). The corresponding coordinate vector fields \( E_i \) are such that

\[
E_i(x^j) \equiv \delta^j_i,
\]

in this coordinate neighborhood. Then for an arbitrary differential operator \( D^\alpha \) of order \( |\alpha| \) strictly bigger than one - not having order zero and one terms - it holds

\[
D^\alpha(x^j) \equiv 0.
\]

Each vector \( W \) tangent to \( \mathbb{M}^n \) at the point \( q' = q(t_1, ..., t_N) \) is defined by its action on coordinate functions as follows:

\[
W = \sum_{i=1}^{n} (W(x^i))E_i.
\]

Rewrite (16) separating differential operators by their orders:

\[
\frac{\partial}{\partial t_i} q(t_1, ..., t_N) = q \circ (\sum_{k=1}^{m-1} d_i^k(t_1, ..., t_N)D^k_\alpha(V_{u_1}, ..., V_{u_n})) + O(T^{m-1}),
\]

where \( d_i^k \) are some polynomials of \( t_i \), and \( D^k_\alpha \) - some finite number of differential operators of order \( k \), i.e., such that in our coordinate system

\[
D^k_\alpha = c^k_{\alpha,s}(x_1, ..., x_n) \frac{\partial^k}{\partial^{s_r}x_r},
\]

where \( c^k_{\alpha,s}(x_1, ..., x_n) \) - some smooth functions, and \( s_r = (s_1, ..., s_n) \) some multi-index of order \( k \) - \( s_1 + ... + s_n = k \). Due to (19) it holds

\[
\frac{\partial}{\partial t_i} q(t_1, ..., t_N)(x^j) = d_i^1(t_1, ..., t_N)D^1_\alpha(x_j) + O(T^{m-1}).
\]

In other words, we see that the differential of the chronological map can be recovered from its action on linear coordinate functions, while this action is determined by its "first order operator part". More precisely, the
differential \(dG(t_1, \ldots, t_N)\) of the chronological map at the point \(q\) on the vector \(\frac{\partial}{\partial t_i}\) equals the derivation - the vector field we denote \(F_i(t_1, \ldots, t_N)\), which action on the coordinate function \(x^j\) is given by:

\[
F_i(t_1, \ldots, t_N)(x^j) = (dG(t_1, \ldots, t_N)(\frac{\partial}{\partial t_i}))^j = \frac{\partial}{\partial t_i} q(t_1, \ldots, t_N)(x^j) = d^1_{\alpha i}(t_1, \ldots, t_N)D^1_{\alpha}(x^j) + O(T^{m-1}).
\]

Examine \(F_i\) more closely. From (16) we conclude:

\[
F_i(t_1, \ldots, t_N) = \sum_{k=1}^{m-1} \sum_{\alpha^k} \frac{\partial}{\partial t_i} (\frac{t_i^{\alpha^k}}{\alpha_1! \cdots \alpha_N!}) D^1_{\alpha} (V_{u_1}^{\alpha_1} \circ \ldots \circ V_{u_N}^{\alpha_N}) + O(T^{m-1}),
\]

where by \(D^1_{\alpha} = D^1(V_{u_1}^{\alpha_1} \circ \ldots \circ V_{u_N}^{\alpha_N})\) we denote the first differential operator part of the compositions of derivations \(V_{u_i}\); which by direct calculations equals

\[
D^1_{\alpha} = D^1(V_{u_1}^{\alpha_1} \circ \ldots \circ V_{u_N}^{\alpha_N}) = (v_{i_1}^{s_1} v_{i_2}^{s_2} \ldots v_{i_{N}}^{s_{N}} \frac{\partial}{\partial x^1} \ldots \frac{\partial}{\partial x^N} (v_{1}^{s_1} \ldots v_{N}^{s_{N}}) \frac{\partial}{\partial t_i}).
\]

where \(v_i^s\) are coordinates of the vector field \(V_{u_i}\):

\[
V_{u_i} = v_i^s \frac{\partial}{\partial x^s}.
\]

Comparing (24) with (25) we see that polynomial coefficients of first order differential operators \(D^1_{\alpha}\) (which (operators) do not depend on \((t_1, \ldots, t_N)\) are the following monomials:

\[
d^1_{\alpha}(t_1, \ldots, t_N) = \frac{\partial}{\partial t_i} (\frac{t_i^{\alpha^k}}{\alpha_1! \cdots \alpha_N!}).
\]

For what follows it is important that such monomials are linearly independent; i.e., if some linear combination \(\sum_{\alpha} \Lambda_\alpha d^1_{\alpha}(t_1, \ldots, t_N)\) is identically zero, then all coefficients \(\Lambda_\alpha\) are zeroes.\(^5\) If some \(D^1_{\alpha}(x^j) \neq 0\) in this linear combination, then the zero set of the action on \(x^j\) of the first order differential part of the differential of the chronological map at the point \(q\) on the vector \(\frac{\partial}{\partial t_i}\) vanishes on the zero set given by the polynomial equation:

\[
\Omega^j_i = \{(t_1, \ldots, t_N) \mid \sum_{\alpha} d^1_{\alpha}(t_1, \ldots, t_N)D^1_{\alpha}(x^j) = 0\}.
\]

As the corollary we conclude the following.

**Lemma 1.** The zero set \(\Omega^j_i \subset R^N\) given by the polynomial equation is closed and has an empty interior.

Next we note that due to our asymptotic expansions (23), (24) for sufficiently small \(|T|\) the differential of the chronological product \(G(t_1, \ldots, t_N) : q \rightarrow q(t_1, \ldots, t_N)\) is not a surjection on \(R^N\) at some point \(q(t')\) for \(t' = (t'_1, \ldots, t'_N)\) if for some system of coordinates of the type we consider, the action of all \(D^1_{\alpha}\) on some particular function \(x^j\) vanish at zero; i.e., such that

\[
q \circ D^1_{\alpha}(x^j) = 0.
\]

\(^5\)To prove that in such combination \(\Lambda_\alpha = 0\) take the partial derivative \(\partial^\alpha / \partial t^\alpha_{i_1} \ldots \) of this linear combination at zero.
Indeed, if the differential of \( G \) is not surjection we may construct (using the Orbit theorem) the coordinate system \( x^1, 1, ..., n \) in some neighborhood of \( q \) such that the first \( n' < n \) coordinate vectors \( E_i, i = 1, ..., n' \) at the point \( q' = q(t_1', ..., t'_N) \) generate the image of the differential of \( G \) at the point \( q \) - some \( n'' \)-dimensional subspace of \( T_q^*M^n \) with \( n'' < n \). Then the derivatives of all \( x^j, j > n'' \) in all direction of this subspace vanish, i.e., (30) holds. If otherwise; i.e., we are given some chronological product (30) is not satisfied for an arbitrary coordinates \( x \), \( T \), then combining our arguments above we come to the following statement.

**Lemma 2 ("PD-LARC").** The chronological map

\[
G_{(u_1, ..., u_N)}(t_1, ..., t_N) = P_{u_1}^1 \circ ... \circ P_{u_N}^N
\]

is a surjection for \( |T| = |t_1| + ... + |t_N| < \epsilon \), where \( \epsilon \) is sufficiently small, and for \( t = (t_1, ..., t_N) \in R^N \setminus \Omega \) where \( \Omega \) closed set with an empty interior, given by some polynomial equation; if there exists \( n \) linearly independent partial derivatives

\[
D_j = D^1(\partial^\alpha(j) \partial_{t_1}^{\alpha_1(j)} \partial_{t_2}^{\alpha_2(j)} ... \partial_{t_N}^{\alpha_N(j)}) G(t_1, ..., t_N)|_{t_1 = ... = t_N = 0}(x^j),
\]

where \( D^1 \) denotes the first order differential part, and \( \alpha(j) \) is some multi-index \( (\alpha(j)_1, ..., \alpha(j)_N) \).

Note, that the Lemma 2 condition is not stronger than the LARC. If the LARC is not satisfied then, using the orbit theorem, we may always construct the coordinate system in \( M^n \), such that the orbit of \( q \) is contained in some coordinate subspace given by, say \( x^n = 0 \). Then all partial derivatives of \( G \) will have the first order differential parts with vanishing action on \( x^n \).

Now we are ready to construct the chronological maps as in the Lemma 2 above. Since any exponential \( P_u^t \) is the identity map when \( t = 0 \), the partial derivative of (32) in (33) equals the same partial derivative of the chronological product only of those \( P_{u_i}^{t_i} \) with \( \alpha(j)_i \neq 0 \). Therefore, our chronological product satisfying conditions of the Lemma 2 may be constructed as the superposition of the chronological products

\[
G^j(t_1^j, ..., t_{k(j)}^j) = P_{u_1}^{t_1^j} \circ ... \circ P_{u_{k(j)}}^{t_{k(j)}}; \quad t^j = (t_1^j, ..., t_{k(j)}^j) \in R^{k(j)},
\]

which partial derivative has the first order differential part equals some \( E_j \) such that \( E_j, j = 1, ..., n \) are linearly independent. Now the LARC guarantees the existence of such \( G^j \).

Indeed, if \( \Sigma_M \) satisfies the LARC we may assume that the basis \( E_j, j = 1, ..., n \) consists of some number of vectors \( V_{u_i}, i = 1, ..., d' \) - which we denote by \( W_i, i = 1, ..., d' \) and their mutual commutators \( W_i, i = d' + 1, ..., n \), where

\[
W_i = [W_{\beta_i^k}, [W_{\beta_i^k-1}, ...]]; i = d' + 1, ..., n
\]

for some \( k \)-multi-index \( \beta^i = (\beta_i^k, ..., \beta_i^1) \), where the order of the commutator \( k \) in turn depends on the index \( i \) of the vector \( W_i \); i.e., \( k = k(i) \). Easy to see that orders of commutators are bounded by \( n \) - the dimension of \( M^n \).

\[\text{6) Hence } m > 2^n \text{ is sufficient.}\]
By definition (35) the derivation $W_i$ is the linear combination of the derivations which are the first order differential parts

$$W_i = \sum_{\sigma} (-1)^{\sigma} D^1(W_{\beta_{\sigma}^{(k)}} \circ W_{\beta_{\sigma^{(k-1)}} \ldots})$$

of differential operators of the form

$$W_{\beta_{\sigma}^{(k)}} \circ W_{\beta_{\sigma^{(k-1)}}} \circ \ldots \circ W_{\beta_{\sigma^{(1)}}},$$

where $\sigma$ some permutations of indexes $\beta_j^i$. An arbitrary derivation in the linear combination (36) is the partial derivative of the chronological product of the type:

$$\prod_{i=1}^n W_{\beta_j^i(t)} \circ W_{\beta_{\sigma^{(k-1)}}} \circ \ldots \circ W_{\beta_{\sigma^{(1)}}},$$

where $t$ some neighborhood $B$ or some number of vectors $E$. An arbitrary derivation in the linear combination (36) is the partial derivative of the chronological product of the type:

$$\prod_{i=1}^n W_{\beta_j^i(t)} \circ W_{\beta_{\sigma^{(k-1)}}} \circ \ldots \circ W_{\beta_{\sigma^{(1)}}},$$

Taking composition of such products for all $i = 1, \ldots, n$ with different arguments $t_i$ we obtain (very long!) chronological products whose partial derivatives realize all monomials in (36). We call such chronological product the general at $q$ relative to the set of generators $\{V_1, \ldots, V_l\}$. They satisfy the Lemma 2, which may be reformulated as follows.

**Lemma 3.** If $\Sigma_M$ satisfies the LARC at $q$ then there exist the general chronological product $G : R^N \rightarrow M^n$ at $q$, such that its differential is a surjection on $R^N \setminus \Omega$ for some closed $\Omega \subset R^N$ with an empty interior which is given by some polynomial equation.

**Definition.** The trajectory $q(t), 0 \leq t \leq \epsilon$ of the control system $\Sigma_M$ is called the general relative to the set of generators $\{V_1, \ldots, V_l\}$ if it contains some interval which goes from $q = q(0)$ to some point $q(\epsilon) = G(t_1, \ldots, t_N)(q)$ with all $t_i$ positive and such that $t = (t_1, \ldots, t_N) \in R^N \setminus \Omega$.

If $q(t), 0 \leq t \leq \epsilon$ is the general trajectory relative to some set of generators of $\Sigma_M$ then by definition some general chronological product $G$ provides the surjection from some neighborhood of zero in $R^N$ over some neighborhood $B(\epsilon)$ of $q(\epsilon)$, which means that all points from this neighborhood are reachable from $q$, or $B(\epsilon) \subset O^+(q)$. If further, this interval $q(t), 0 \leq t \leq \epsilon$ is a part of some closed trajectory $q(t), 0 \leq t \leq t_q$ going through the point $q = q(t_q)$ again under some control $u(t) : [0, t_q] \rightarrow \mathcal{U}$ then by the standard continuous dependence arguments the image of this open neighborhood $B(\epsilon)$ under the flow $F(t, u(t))$ is the open neighborhood of $q(t), \epsilon < t$ also reachable from $q$. The union of all these neighborhoods gives us the open neighborhood $B^+_q$ of the closed trajectory $q(t), 0 \leq t \leq t_q$, which by construction belongs to the positive orbit of $q$: $B^+_q \subset O^+(q)$.

We formulate the obtained results as follows.

**Theorem 1.** If the control dynamical system $\Sigma_M$ satisfies the LARC at the point $q$; i.e., there exists the complete system - the set of generators $-E_j, j = 1, \ldots, n', n' \geq n$ ($E_j$ generate $T_qM^n$) of $T_qM^n$ consisting of some number of vectors $V_{u_i}, i = 1, \ldots, d'$, denoted by $W_i, i = 1, \ldots, d'$; and their mutual commutators $W_i, i = d' + 1, \ldots, n'$

$$W_i = [W_{\beta_{\sigma}^{(k)}}, [W_{\beta_{\sigma^{(k-1)}}}, \ldots]], i = d' + 1, \ldots, n',$$

then the set of points $q(t_1, \ldots, t_N)$ (where all $t_i$ are positive) reachable from $q$ by general trajectories relative to this set of generators is an open subset of $M^n$. If there exists a closed trajectory $q(t), 0 \leq t \leq t_q$ going through the point $q$ then some open neighborhood $B^+_q$ of this trajectory is reachable from $q$:

$$q' \in O^+(q) \quad \text{for all} \quad q' \in B^+_q.$$
Now consider the negative orbit $O^-(q)$ of $q$ - the chronological products $G(t_1, ..., t_N)$ where all $t_i$ are non-negative. By the definition these are the points from which we can reach the point $q$ by chronological products - trajectories of $\Sigma_M$ with piece-wise constant controls; i.e., $q' \in O^-(q)$ if and only if $q \in O^+(q')$. Therefore, applying our Lemma 3 above we see that if some general trajectory goes into the point $q$; i.e., there exists the trajectory $q(t), -\epsilon < t \leq 0, q(0) = q$ of $\Sigma_M$ containing two points also connected by some general chronological product with negative arguments; then the following counterpart of the above theorem is true.

**Theorem 2.** If the control dynamical system $\Sigma_M$ satisfies the LARC at the point $q$; i.e., there exists the complete system - the set of generators - $E_j, j = 1, ..., n', n' \geq n$ ($E_j$ generate $T_qM^n$) of $T_qM^n$ consisting of some number of vectors $V_{u_i}, i = 1, ..., d'$, denoted by $W_i, i = 1, ..., d'$; and their mutual commutators $W_i, i = d' + 1, ..., n'$

$$W_i = [W_{\beta_{i1}}, [W_{\beta_{i-1}}, [...]]; i = d' + 1, ..., n',$$

then the set of points $q(t_1, ..., t_N)$ (where all $t_i$ are negative) from which we can reach $q$ by general trajectories relative to this set of generators is an open subset of $M^n$. If there exists a closed trajectory $q(t), 0 \leq t \leq t_q$ going through the point $q$ then from some open neighborhood $B_q$ of this trajectory we can reach $q$:

$$q \in O^+(q') \quad \text{for all} \quad q' \in B_q^-.$$

2. General trajectories and the involvement condition

Our definition of the general trajectory depends on the set of generators $V_{u_i}, i = 1, ..., d'$. The same trajectory may be general relative to the one set of generators and not - relative to another set.

**Definition.** We say that the system $\Sigma_M$ satisfying the LARC at some point $q$ is involved if for an arbitrary trajectory $q(t)$ going through this point there exists some set of generators $\{V_{u_i}, i = 1, ..., l'\}$ such that the trajectory $q(t)$ is general relative to this set.

In general, the verification of the "involvement" condition is complicated and may require the solution of the classification problem for vector fields on $M^n$. Here we single out the case when the answer is easy.

Denote by $V(q)$ the set of all control vectors $V(q, U) = \{V_u, u \in U\}$ at this point, and by $L(q)$ the linear subspace in $T_qM^n$ generated by $V(q)$. By $\text{Com}(q)$ we denote the convex hull of $V(q)$. We may assume that the dimension ($d'$) of $L(q)$ is locally constant and the family of vector spaces $L(q)$ itself is smooth and locally trivial - the union of all $L(q')$ over the points $q'$ from some small neighborhood $B(q)$ of $q$ - which we denote by $L_B = \{L(q')|q' \in B(q)\}$ - has a natural structure of the direct product $B(q) \times R^{d'}$ (is a locally trivial vector bundle). The set $L_B$ is a subset of the restriction of the tangent bundle $TM$ over $B(q)$, and admits natural coordinates $\{(x^1, ..., x^n, y^1, ..., y^n)\}$ in which the vector $y^i(\partial/\partial x^j)(x)$ tangent to $M$ at the point with coordinates $x = (x^1, ..., x^n)$ has coordinates $(x^1, ..., x^n, y^1, ..., y^n)$.

**Definition.** We say that the set of all control vectors $V_{u_i}, u \in U$ is open at $q$ if the subset $\{V(q')|q' \in B(q)\}$ is an open subset of $L_B$.

Let $V_i, i = 1, ..., N$ be some set of generators. Consider some family of linear transformations of $C(q'): R^N \to R^N$ for $q'$ in a small neighborhood of $q$ given in the basis $\{V_i, i = 1, ..., N\}$ by a matrix $C(q') = (c^k_i(q')); i, k = 1, ..., N$, and define the vector fields $\tilde{V}_k = c^k_iV_i$. Then, since $V(q')$ is open, all vector fields $\tilde{V}_k(q')$ belong to $V(q')$ when $(c^k_i(q'))$ close to the identity matrix; i.e., they are control vectors of our control

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7 Say, is the dependence $V_u$ on $u$ “stable” in some sense or not?
8 For $d' \neq n$ the set $L_B$ is not contained in any coordinate subset of the form $y^i = 0$ in any coordinates.
9 which is stronger than simply to require: “$V(q')$ is an open subset of $L(q')$.”
In particular, if \( C(q', \tau) \) is some continuous family of linear transformations defined in some neighborhood of \( q \) such that \( C(q, 0) \equiv Id \) then for small \( \tau \) vector fields \( \tilde{V}_k(q', \tau) = c^k_i(q', \tau) V_k(q') \) belong to \( V(q') \). We denote by \( \tilde{G}(\tau) (t_1, \ldots, t_N) \) the corresponding chronological product defined with the help of these vector fields:

\[
\tilde{G}(\tau) (t_1, \ldots, t_N) = P^t_1 \tilde{V}_i(\tau) \circ \cdots \circ P^t_N \tilde{V}_N(\tau),
\]

Now we prove that it is possible to move the set of generators in an arbitrary way inside open \( V(q) \), while keeping the equation (29) of the zero set stationary. In order to do so, we define derivatives \( U_i(\tau) \) of \( \tilde{V}_i(\tau) \) in the next lemma; where for simplicity we consider the moment \( \tau = 0 \). The result, of course, is valid for all \( \tau \) sufficiently small.

**Lemma 4.** For an arbitrary vectors \( U_i, i = 1, \ldots, N \) in \( L(q) \) there exists a family of linear transformations \( C(q', \tau) \) in some small neighborhood of \( B(q) \) of \( q \) such that

\[
\delta(\tilde{V}_i(\tau)/\delta \tau)_{|\tau=0} = U_i,
\]

and for all \( i, |\alpha_i| > 1 \) it holds

\[
\frac{\delta V^{\alpha_i}(\tau)(x^j)}{\delta \tau} \bigg|_{\tau=0} = 0,
\]

where \( C(q', \tau) \) is given by the matrix \( (c^k_i(q', \tau)), i, k = 1, \ldots, N, \) and \( \tilde{V}_i(q', \tau) = c^k_i(q', \tau) V_k(q') \).

**Proof.** Indeed, by a direct calculation we see that

\[
\frac{\delta \tilde{V}^{\alpha_i}(\tau)(x^j)}{\delta \tau} \bigg|_{\tau=0} = \sum_{s=0}^{\alpha-1} (\tilde{V}_i^s \circ U_i \circ \tilde{V}_i^{\alpha_i-s-1}).
\]

We find \( c^k_i(\tau) \) for each \( i \) separately as follows: for a given \( i \) find a coordinate system \( (x^1_i, \ldots, x^n_i) \) in a neighborhood of \( q \) such that the particular vector field \( V_i \) with a given \( i \) coincide with the first coordinate field \( \partial/\partial x^1_i \) of this system. In such coordinates the equation (42) takes the form:

\[
\sum_{s=0}^{\alpha-1} \frac{\partial^s}{\partial (x^1_i)^s} \frac{\partial c^k_i}{\partial \tau} \bigg|_{\tau=0} (v^j_k) = 0,
\]

which obviously has the solution \( c^k_i(q', \tau), \) say:

\[
c^k_i(q', \tau) = \delta^k_i + \tau \frac{\delta c^k_i}{\delta \tau} \bigg|_{\tau=0},
\]

and such that the products \( c^k_i v^j_k \) are constant on \( x_i \) coordinates:

\[
\left( \frac{\delta c^k_i}{\delta \tau} \right)_{|\tau=0} (v^j_k)(q') \equiv (U^k_i v^j_k)(q),
\]

where \( v^j_k(q') \) and \( U^k_i \) are coordinates of \( V_k(q') \) and \( U_i(q) \) in the coordinate system \( (x_1^i, \ldots, x_n^i) \). To complete the proof it is sufficient to repeat our arguments for all \( i \), define functions \( c^k_i \) in different coordinate systems \( (x^1_i, \ldots, x^n_i) \), and then rewrite them all (i.e., change coordinates) in the same previously given initial coordinate system \( \Sigma_M \).
system \((x^1, ..., x^n)\). Note, that (45) in particular defines all jets of the functions \(e_i^\alpha\), and (43) holds for all \(\alpha\) with \(|\alpha|\) arbitrarily big.

The Lemma 4 defines the variation on \(\tau\) of the set of generators \(\tilde{V}_i(\tau)\) in such a way that the zero set \(\tilde{\Omega}\) given by (26) for the chronological product (39) is defined by the first members of the approximation sum for \(\tilde{G}(\tau)\): the action of the member \(d_{\alpha}D_{\alpha}^1\) in the asymptotic sum for the differential \(d\tilde{G}(\tau)(\partial/\partial t_i)\) on the coordinate function \(x^j\) does not depend on \(\tau\) if \(|\alpha| > 1\) due to (41) above; we have:

\[
\frac{\partial}{\partial \tau} d\tilde{G}(\tau)(\partial/\partial t_i)(x^j)|_{\tau=0} = \frac{\partial}{\partial \tau} \left( \sum_{\alpha} d_{\alpha}(t_1, ..., t_N)D_{\alpha}^1(x^j) \right)|_{\tau=0} = \frac{\partial}{\partial \tau} \left( \sum_{|\alpha|=1} d_{\alpha}(t_1, ..., t_N)D_{\alpha}^1(x^j) \right)|_{\tau=0} = U_i(x^j).
\]

As we wrote before, the differential of the chronological product may be not a surjection at the point \(t = (t_1, ..., t_N) \in \mathbb{R}^N\) if all partial derivatives \(d\tilde{G}(\tau)(\partial/\partial t_i)\) at this point vanish on some smooth function on \(M^n\), which, without loss of generality we may assume to be some coordinate function \(x^j\). In other words, the image of the differential is contained in the subspace of \(T_{(0)}M^n\), which at this moment \((\tau = 0)\) we may assume to be the tangent space to some coordinate submanifold in \(M^n\) given by \(\{x^j = const.\}\). At the origin \(t = 0\) - the point \(q\) - this image (of \(d\tilde{G}(\tau)\)) always contains vectors \(V_i = V_{u_i}, i = 1, ..., N\). Therefore, we may assume that our coordinate systems \((x^1, ..., x^n)\) are such that

\[
V_i(x^j)(q) = 0,
\]

or, since the derivatives \(U_i\) of \(\tilde{V}_i(\tau)\) at \(q\) also belong to \(V(q)\), that

\[
U_i(x^j)(q) = 0.
\]

Hence, if \(C(q', \tau)\) is defined with the help of the Lemma 4 for all \(0 < \tau < \epsilon\) and depends continuously on \(\tau\) we will have a continuous family of the sets of generators \(\tilde{V}_i(\tau)\) such that in the equation (29) defining the zero set \(\tilde{\Omega}(\tau)\) of the chronological product \(\tilde{G}(\tau)\) has the same coefficients

\[
D_{\alpha}^1(\tilde{V}_1^\alpha_{1}(\tau) \circ ... \circ \tilde{V}_N^\alpha_{N}(\tau)) \equiv const
\]

for all \(\alpha\). Therefore, \(\tilde{\Omega}(\tau)\) does not depend on \(\tau\) and is given in coordinates \((t_1, ..., t_N)\) by the same equation (29). Because of this for an arbitrary point \(t^*\) from \(\Omega(\tau)\) we may find a curve \(t^*(\tau), 0 \leq \tau < \epsilon\) issuing from this point \(t^*(0) = t^*\) and outside \(\tilde{\Omega}(\tau)\) into the open domain \(\mathbb{R}^N\) of points with positive coordinates:

\[
t^*_i(\tau) > 0 \quad \text{and} \quad t^*_i(\tau) \notin \tilde{\Omega}(\tau) \quad \text{for} \quad \tau > 0.
\]

Next we define derivatives \(U_i(\tau)\) by the following equation:

\[
t^*_i(\tau)U_i(\tau)(x^j) = -\frac{\delta}{\delta \tau} t^*_i(\tau)d\tilde{G}(\tau)(t^*(\tau))(\frac{\partial}{\partial t_i})(x^j),
\]

which implies by direct calculations with the help of (49):

\[
\frac{\delta}{\delta \tau} \tilde{G}(\tau)(t^*(\tau)) \equiv 0,
\]

or by continuity that

\[
\tilde{G}(\tau)(t^*(\tau)) \equiv q(t^*).
\]
We see that the continuous family of linear transformations \( C(q', \tau), C(q', 0) \equiv Id \) we just constructed, move the vector fields \( V_i(q') \) - the set of generators we have - into another set \( \tilde{V}_i(q', \tau) = c_i^k(q', \tau)V_k(q') \) of vector fields such that:

1) when \( V(q) \) is open, then for \( \tau \) small enough the vector fields \( \tilde{V}_k(q', \tau) \) belong to our control set \( \{ V_u(q') | u \in U \} \).

2) the point \( q^* = q(t^*) \) which is the image under the chronological product \( G(t^*) \) of some point from the zero set \( \Omega \) now equals the chronological product \( \tilde{G}(\tau)(t^*(\tau)) \) relative to another set of generators, and is the image of the point \( t^*(\tau) \in R^N \) which does not belong to the zero set \( \tilde{\Omega}(\tau) \) of this later chronological product; i.e., the differential of \( d\tilde{G}(\tau)(t^*(\tau)) \) is the surjection on some small neighborhood of \( q^* \).

In particular, all coordinate subspaces in \( R^N \) defined by some number of equalities of the type \( t_i = 0 \) belong to the zero set \( \Omega \). For instance, an arbitrary trajectory \( q(t), 0 \leq t \leq 1 \) which is the solution of \( \dot{q}(t) = V_{u(t)}(q(t)) \), is the image of the first coordinate line \( \{ t_i = 0, i > 1 \} \) in \( R^N \) under the chronological product \( G(t_1, ..., t_N) = P_{V_1}^{t_1} \circ ... \circ P_{V_N}^{t_N} \) - if only we may consider the vector field \( V_{u(t)} \) as the first vector field \( V_1 \) in the construction of our chronological products. If so, then the arguments above implies that \( \Sigma_M \) is involved.

We come also to the same conclusion if trajectories with arbitrary controls \( u(t) \) have ”nice” (e.g., smooth) dependence on \( u \) and may be approximated by trajectories with piece-wise controls; i.e., by chronological products. To treat the general situation we introduce one additional property.

**Definition.** We say that \( V(q) \) is ample if it is open and the convex hull of \( V(q) \) coincide with \( L(q) \).

Now take an arbitrary trajectory \( q(t), 0 \leq t \leq \epsilon \) under control \( u(t) \) issuing from the point \( q \) in which \( \Sigma_M \) satisfies LARC. Since LARC implies local transitivity for \( \epsilon \) sufficiently small \( q(\epsilon) \) equals some chronological product \( G(t_1, ..., t_N) \). Hence, the point \( q \) is connected with \( q(\epsilon) \) by some trajectory \( \tilde{q}(t) \) of \( \Sigma_M \) - possibly different from \( q(t) \) - with piece-wise control function. The part of such trajectory corresponding to the first non-zero \( t_i \) is the solution of \( \Sigma_M \) under some constant control: \( \dot{q} = \pm V_i(q(t)), 0 \leq t \leq |t_i| \). If \( t_i \) is positive, then this part of the trajectory \( \tilde{q}(t) \) belongs to the positive orbit of \( q \). If \( t_i \) is negative, and \( V(q) \) is ample, then \( \tilde{V}_i \) is the linear combination of some other control vectors \( \tilde{V}_k \) with positive coefficients

\[
V_i(q) = \lambda^i \tilde{V}_k(q),
\]

and using the LARC at \( q \) it is not difficult to find a point \( q^* \) on the trajectory \( \tilde{q}(t) \) which equals the chronological product \( \tilde{G}(t_1, ..., t_N) \) defined with the help of the new vectors \( \tilde{V}_k \) and such that the first non-zero argument \( t_k \) is positive. In both cases we obtain the trajectory going from \( q \) to \( q(\epsilon) \) such that some part of it is some coordinate interval with positive coordinates. Now the Lemma 4 claims that \( q(t) \) is the general trajectory relative to some set of generators.

In other words, we just proved the following result.

**Lemma 5.** If \( V(q) \) is ample, then the control system \( \Sigma_M \) satisfying LARC at \( q \) is involved.

Combining this with the claim of the Theorems 1 and 2 we conclude the following.

**Theorem 3.** If for the system \( \Sigma \) the set of control vectors \( V(q) \) is ample at some point \( q \), where the LARC is satisfied, then every closed orbit of \( \Sigma_M \) going through the point \( q \) has some open neighborhood \( B^+_q \) contained in the positive orbit \( O^+(q) \) of \( q \), and another open neighborhood \( B^-_q \) such that for every \( q' \in B^-_q \) the point \( q \) is reachable from \( q' \). Their intersection \( B_q \) is an open neighborhood of the closed trajectory on which the system \( \Sigma_M \) is controllable.

3. The Closed Orbit Controllability Criterion

Now we are ready to prove our main result:
The proof is obvious: every such orbit has an open neighborhood $B_q(p)$ on which the system $\Sigma_M$ is controllable. The union of all $B_q(p)$ covers $M^n$. Since $M^n$ is compact there exists some finite sub-covering

$$M^n = \bigcup_{i=1}^{k} B_{q_i}.$$ 

Because $M^n$ is connected for two arbitrary point $q$ and $p$ we may find a finite sequence $B_{q_1}, ..., B_{q_k}$ from the finite covering (55) such that $q \in B_{q_1}$, $p \in B_{q_k}$ and $B_{q_s} \cap B_{q_{s+1}} \neq \emptyset$. Take $q_0 = q$; some $q_s \in B_{q_s} \cap B_{q_{s+1}}$ and $q_{k+1} = p$. Since $\Sigma_M$ is controllable on each $B_{q_s}$, there exists a trajectory from $q_s$ to $q_{s+1}$ for all $s = 0, ..., k$. Taking the composition of these trajectories we obtain the trajectory of $\Sigma_M$ going from $q$ to $p$. Since $q, p$ arbitrary this proves that $\Sigma_M$ is controllable on $M^n$.

The Closed Orbit Controllability Criterion. Note that if $\Sigma_M$ is controllable on $M^n$ then for an arbitrary point $q$ there exists some closed trajectory of $\Sigma_M$ going through this point. Indeed, since $\Sigma_M$ is controllable there exists some its trajectory $\gamma$ from $q$ to $q(-\epsilon), 0 < \epsilon$ for an arbitrary trajectory $q(t), -\infty < t < \infty$ passing through $q$. The composition of $\gamma$ and the interval $q(t), -\epsilon < t < 0$ gives us the closed non-trivial trajectory of $\Sigma_M$ through $q$. Thus, the Theorem A above provides us with the necessary and sufficient controllability condition for the control systems $\Sigma_M$ on compact manifolds with ample set of control vectors and satisfying the LARC, which we call the closed orbit controllability criterium.

Theorem A. Let $\Sigma_M$ be some control dynamical system on the compact connected manifold $M^n$. If through an arbitrary point $p$ of $M^n$ goes some closed orbit $p(t), 0 \leq t \leq t_p$, which is non-trivial (i.e., $t_p > 0$) and contains some point $q(p)$ where $V(q(p))$ is ample and the LARC is satisfied, than the system $\Sigma_M$ is controllable on $M^n$.

Proof. The proof is obvious: every such orbit has an open neighborhood $B_q(p)$ on which the system $\Sigma_M$ is controllable. The union of all $B_q(p)$ covers $M^n$. Since $M^n$ is compact there exists some finite sub-covering

$$M^n = \bigcup_{i=1}^{k} B_{q_i}.$$ 

Because $M^n$ is connected for two arbitrary point $q$ and $p$ we may find a finite sequence $B_{q_1}, ..., B_{q_k}$ from the finite covering (55) such that $q \in B_{q_1}$, $p \in B_{q_k}$ and $B_{q_s} \cap B_{q_{s+1}} \neq \emptyset$. Take $q_0 = q$; some $q_s \in B_{q_s} \cap B_{q_{s+1}}$ and $q_{k+1} = p$. Since $\Sigma_M$ is controllable on each $B_{q_s}$, there exists a trajectory from $q_s$ to $q_{s+1}$ for all $s = 0, ..., k$. Taking the composition of these trajectories we obtain the trajectory of $\Sigma_M$ going from $q$ to $p$. Since $q, p$ arbitrary this proves that $\Sigma_M$ is controllable on $M^n$.

Theorem A works especially well when applied to control dynamical systems on surfaces, when the trajectory curve is also a hyper-surface; i.e., locally divides the manifold. If, for instance, $M^2$ is a surface and $\Sigma_M$ has the finite set of generator vector fields $V_i$ such that the corresponding dynamical systems $\Sigma_i = \{\dot{q}(t) = V_i(q(t))\}$ are stable; i.e., have non-degenerated zeros, then our closed orbit controllability criterium essentially provides the complete answer to the controllability question. This answer may be given with the help of some diagrams (pictures) describing iterations on the space of orbits of $\Sigma_M$, which we split into some number of intervals, corresponding to the cell representation of the phase portraits of $\Sigma_i$. We describe this in the forthcoming paper [CM], where we pay the particular attention to the projections of bi-linear systems in $R^3$ to the two-dimensional unit sphere $S^2$ of directions in $R^3$. As the example the next theorem gives the sufficient controllability condition for such control systems: let

$$\dot{x}(t) = A(u)x(t) = (A + u^1B_1 + \ldots + u^dB_d)x(t), \quad (\Sigma)$$

be the control systems of linear equations in $R^3$, where the right-hand side - the linear operator $A(u)$ - is a linear combination of some constant linear operators $A$ and $B_k$ which do not depend on the control parameter $u \in R^d$. The bi-linear system $\Sigma$ defines the control system $\Sigma^{pr}$ on the unit sphere $S^2$ in $R^3$, called the projected system, when we consider the evolution of unit directions of solutions $x(t)$ of $\Sigma$. For each control parameter $u$ being fixed the phase portrait of the corresponding projected system admits the natural cell decomposition into, so called Jordan cells. Then the application of our closed orbit controllability criterium provides, among other, the following.

Theorem B. If for two control parameters $u$ and $v$ the right-hand side linear operators $A(u)$ and $A(v)$ have complex eigenvalues $\lambda_C(u)$ and $\lambda_C(v)$ correspondingly, then the system $\Sigma$ satisfying LARC is controllable if for the real eigenvalues $\lambda_R(u)$ and $\lambda_R(v)$ of $A(u)$ and $A(v)$ it holds

$$(\lambda_R(u) - Re(\lambda_C(u))) (\lambda_R(v) - Re(\lambda_C(v))) < 0.$$
We conclude with the example of the control system on the two-dimensional torus which satisfies LARC everywhere and satisfies our closed orbit criterium, but is not controllable since the set of control vectors is not "involved".

**Example 2.** Let \( \{ S^1(\phi), 0 \leq \phi \leq 2\pi \} \subset R^2 \) be the unit circle \( S^1(\phi) = (\cos(\phi), \sin(\phi)) \) in the two-dimensional \( (x,y) \) plane \( R^2 \). Define two vector fields on this circle: \( V_1(\phi) = -\sin(\phi)(\sin(\phi), -\cos(\phi)) \) which equals the unit tangent to \( S^1 \) vector field multiplied by the y-coordinate and has two zeros \( q_1 = (-1,0) \) and \( q_2 = (1,0) \); and \( V_2 \) - any smooth vector field vanishing outside 1/2-neighborhoods of \( q_1 \) and \( q_2 \), but which is not zero at these points. Since these vector fields do not vanish simultaneously and \( S^1 \) is one-dimensional the control system \( q(t) = V_i(q), i = 1,2 \) satisfies LARC. On the other hand it is not controllable because in two point \( p_1 = (0,1) \) and \( p_2 = (0,-1) \) set of all possible directions of trajectories going through these points is the unique vector \((-1,0)\) pointing to the same direction ("to the left"); i.e., any trajectory starting from \( p_1 \) or \( p_2 \) stays in the "left" part of the circle where \( x < 0 \) and can not reach any point with positive \( x \)-coordinate: \( G^+(p_1) \cap \{ x \leq 0 \} \neq S^1 \).

Next we multiply \( S^1(\phi) \) by another circle \( S^1(\psi) \), and denote by \( V_3 \) the unit vector field tangent to this \( S^1(\psi) \). As the result we obtain the two-dimensional torus \( T^2(\phi, \psi) = S^1(\phi) \times S^1(\psi) \) and the control system on it defined by three vector fields \( V_i, i = 1,2,3 \). Easy to see that this system 1) satisfies LARC everywhere, 2) for an arbitrary point \( q \) there exists non-degenerated closed trajectory of the system going through this point - the second factor \( S^1(\psi) \), 3) the system is not controllable.

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