SYMMETRIC MONGE–KANTOROVICH PROBLEMS AND POLAR DECOMPOSITIONS OF VECTOR FIELDS

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Abstract. We address the problem of whether a bounded measurable vector field from a bounded domain $\Omega$ into $\mathbb{R}^d$ is $N$-cyclically monotone up to a measure-preserving $N$-involution, where $N$ is any integer larger than 2. Our approach involves the solution of a multidimensional symmetric Monge–Kantorovich problem, which we first study in the case of a general cost function on a product domain $\Omega^N$. The polar decomposition described above corresponds to a special cost function derived from the vector field in question (actually $N - 1$ of them). The problem amounts to showing that the supremum in the corresponding Monge–Kantorovich problem when restricted to those probability measures on $\Omega^N$ which are invariant under cyclic permutations and with a given first marginal $\mu$, is attained on a probability measure that is supported on a graph of the form $x \rightarrow (x, Sx, S^2x, ..., S^{N-1}x)$, where $S$ is a $\mu$-measure preserving transformation on $\Omega$ such that $S^N = I$ a.e. The proof exploits a remarkable duality between such involutions and those Hamiltonians that are $N$-cyclically antisymmetric.

1 Introduction

Given Borel probability measures $\mu_1, \mu_2, \ldots, \mu_N$ on a domain $\Omega$ of $\mathbb{R}^d$, and a bounded Borel cost function $c : \Omega^N \to \mathbb{R} \cup \{-\infty\}$, the multi-marginal version of the Monge–Kantorovich problem consists of maximizing $\int_{\Omega^N} c(x_1, x_2, \ldots, x_N) d\pi$ among all probability measures $\pi$ on $\Omega^N$ whose $i$th marginal is equal to $\mu_i$ for each $i = 1, \ldots, N$. We shall use the notation

$$MK(c; \mu_1, \ldots, \mu_N) = \sup \left\{ \int_{\Omega^N} c(x_1, x_2, \ldots, x_N) d\pi ; \pi \in \mathcal{P}(\Omega^N), \right.$$ 

$$\text{proj}_i \pi = \mu_i \text{ for } i = 1, \ldots, N \right\}. \tag{1}$$

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In this paper, we are concerned with the following symmetric version of the above problem:

$$\text{MK}_{\text{sym}}(c, \mu) = \sup \left\{ \int_{\Omega^N} c(x_1, x_2, \ldots, x_N) d\pi; \pi \in \mathcal{P}_{\text{sym}}(\Omega^N, \mu) \right\},$$  \hspace{1cm} (2)

where $\mathcal{P}_{\text{sym}}(\Omega^N, \mu)$ denotes the set of Radon probability measures on $\Omega^N$, which are invariant under the cyclic permutation

$$\sigma(x_1, x_2, \ldots, x_N) = (x_2, x_3, \ldots, x_N, x_1),$$

and whose marginals are equal to the same probability measure $\mu$ on $\Omega$. In other words, $\pi \in \mathcal{P}_{\text{sym}}(\Omega^N, \mu)$ if

$$\int_{\Omega^N} f(x_1, x_2, \ldots, x_N) d\pi = \int_{\Omega^N} f(x_2, x_3, \ldots, x_1) d\pi$$

for every $f \in C(\Omega^N)$, \hspace{1cm} (3)

and for every $i = 1, \ldots, N$,

$$\int_{\Omega^N} f(x_i) d\pi = \int_{\Omega} f(x_i) d\mu$$

for every $f \in C(\Omega)$. \hspace{1cm} (4)

Standard results show that under mild conditions on $c$, there exists $\pi_0 \in \mathcal{P}_{\text{sym}}(\Omega^N, \mu)$ where the supremum above is attained. We are interested in cases where the optimal measure $\pi_0$ is necessarily supported on the graph of a function of the form $x \rightarrow (Sx, S^2x, \ldots, S^{N-1}x)$, where $S$ is a $\mu$-measure preserving transformation on $\Omega$ such that $S^N = I$ a.e. We shall denote

$$\mathcal{S}_N(\Omega, \mu) = \{S: \Omega \rightarrow \Omega; \ S \text{ is } \mu \text{-measure preserving and } S^N = I \text{ a.e.} \}.$$

One can easily extend the original approach of Kantorovich to the multi-marginal and cyclically symmetric case to show that (2) is dual to the following minimization problem

$$\text{DK}_{\text{sym}}^1(c, \mu) :=$$

$$\inf \left\{ N \int_{\Omega} u(x) d\mu; u \in L^1(\Omega, \mu) \text{ and } \sum_{j=1}^{N} u(x_j) \geq \frac{1}{N} \sum_{i=0}^{N-1} c(\sigma^i(x_1, \ldots, x_N)) \right\}.$$

(5)

We shall introduce here a new dual problem involving the class $\mathcal{D}_N(\Omega)$ of all bounded Borel $N$-cyclically antisymmetric Hamiltonians on $\Omega^N$, i.e., the functions $H : \Omega^N \rightarrow \mathbb{R}$ that satisfy

$$\sum_{i=0}^{N-1} H(\sigma^i(x)) = 0 \quad \text{for all } x \in \Omega^N.$$