Unitarity of Compactified Five Dimensional Yang-Mills Theory

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October, 2001

Abstract

Compactified five dimensional Yang-Mills theory results in an effective four-dimensional theory with a Kaluza-Klein (KK) tower of massive vector bosons. We explicitly demonstrate that the scattering of the massive vector bosons is unitary at tree-level for low energies, and analyze the relationship between the unitarity violation scale in the KK theory and the nonrenormalizability scale in the five dimensional gauge theory. In the compactified theory, low-energy unitarity is ensured through an interlacing cancellation among contributions from the relevant KK levels. Such cancellations can be understood using a Kaluza-Klein equivalence theorem which results from the geometric “Higgs” mechanism of compactification. In these theories, the unitarity violation is delayed to energy scales higher than the customary limit through the introduction of additional vector bosons rather than Higgs scalars.

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The visible four-dimensional world may be only part of a higher dimensional space-time structure, with the extra spatial dimensions substantially larger than the traditional Planck length \(10^{-33}\) cm, but too small to have been probed experimentally \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]\). If the gauge particles propagate in the higher-dimensional space, then from the four-dimensional viewpoint each gauge boson is associated with a Kaluza-Klein (KK) tower of massive vector bosons whose mass splittings are characterized by the (inverse) size of the extra dimensions. In this way compactification of higher dimensions leads to a “geometrical” mechanism for producing massive vector states.

The high-energy behavior of the scattering of longitudinally-polarized massive vector bosons is potentially problematic in Yang-Mills theories, and can result in amplitudes growing with energy at tree-level \([13, 14, 15, 16]\). In the four-dimensional (4D) standard model these amplitudes are exactly canceled by the exchange of spin-0 Higgs particle \([14, 17, 18, 19]\). However, such Higgs scalar states do not exist in a compactified pure gauge theory.

In this letter, we discuss the high-energy behavior of massive vector-boson scattering in the compactified five-dimensional (5D) Yang-Mills theory. We explicitly demonstrate that the scattering of the massive vector bosons is unitary at tree-level for low energies, and analyze the relationship between the scales of 4D unitarity violation and the nonrenormalizability of the 5D gauge theory. In the compactified theory we show that the low-energy unitarity is ensured through an interlacing cancellation among contributions from the relevant KK levels. We observe that this cancellation can be understood from a Kaluza-Klein equivalence theorem resulting from the geometric “Higgs” mechanism of compactification. As a consequence, the unitarity violation is delayed to energy scales higher than the customary limit of Dicus-Mathur and Lee-Quigg-Thacker \([14, 17, 18, 19]\) through the introduction of additional vector bosons rather than Higgs scalars.

The Lagrangian for five-dimensional Yang-Mills theory is given by

\[
L_5 = -\frac{1}{2} \text{Tr}(\hat{F}_{MN}\hat{F}^{MN}), \quad \hat{F}^a_{MN} = \partial_M \hat{A}^a_N - \partial_N \hat{A}^a_M + g_5 C^{abc} \hat{A}^b_M \hat{A}^c_N,
\]

where \(a\) is the gauge index, \(C^{abc}\) the structure constant, and \(g_5\) the 5D gauge coupling with dimension of \((\text{mass})^{-1/2}\). The five-dimensional coordinates are labeled by \(M, N \in (\mu, 5)\) with \(\mu \in (0, 1, 2, 3)\).

For convenience, we may consider this 5D theory with a covariant gauge-fixing term \([2]\),

\[
L_{GF} = -\frac{1}{2\xi} (\partial^M \hat{A}_M^a)^2.
\]

We expect this theory to have high-energy behavior similar to that of an effective 4D KK gauge theory. For instance, consider the elastic gauge-boson scattering, \(\hat{A}^a_{j_1} \hat{A}^b_{j_2} \rightarrow \hat{A}^c_{j_3} \hat{A}^d_{j_4}\), where \(j_k \in (1, 2, 3)\) denotes the polarization state of the massless 5D gauge field \(\hat{A}_M\). For an \(SU(m)\) Yang-Mills theory, we may define the spin-0, gauge-singlet two-particle state,

\[
|\Psi_0\rangle = \frac{1}{\sqrt{3}} \sqrt{m^2 - 1} \sum_{j=1}^3 \sum_{a=1}^{m^2-1} |\hat{A}^a_j \hat{A}^a_j\rangle.
\]

The Feynman amplitude for scattering in this spin-0 gauge-singlet channel takes the form,

\[
\mathcal{M}_0 [\Psi_0 \rightarrow \Psi_0] = \frac{2m}{3} g_5^2 \left( \frac{12}{1 - \cos^2 \theta} - 1 \right),
\]
at the tree-level. In four dimensions, such a behavior would be unitary to arbitrarily high energies (so long as \( g_5 \) was not too large) reflecting the renormalizability of 4D Yang-Mills theory. In five dimensions, however, the properly normalized spin-0 gauge-singlet \( s \)-partial wave amplitude \([20, 21]\) is given by,

\[
T_{00} = \frac{\sqrt{s}}{64\pi^2} \int_0^{\pi} d\theta \sin^2 \theta \mathcal{M}_0 = \frac{23m}{192\pi} \left( g_5^2 \sqrt{s} \right),
\]

where \( \sqrt{s} \) is the center-of-mass energy and \( \theta \) the center of mass scattering angle. Unitarity requires that \( |\text{Re}T_{00}| \leq 1/2 \), and hence this amplitude respects tree-level unitarity only for energies

\[
\sqrt{s} = E_{\text{cm}} \leq \frac{96\pi}{23m} \frac{1}{g_5^2}.
\]

This result is a manifestation that five-dimensional Yang-Mills theory is, at best, a low-energy effective theory valid only up to scales of order \( 1/g_5^2 \).

We now show that these results can be recovered in compactified five-dimensional Yang-Mills theory, viewed in four dimensions. For convenience, we consider compactifying the fifth-dimension to a line segment \( 0 \leq x^5 \leq \pi R \). This can be done consistently by an orbifold projection as follows: restrict the fields \( A^M(x^N) \) to those periodic in \( x^5 \) with period \( 2\pi R \) and further impose a \( \mathbb{Z}_2 \) symmetry,

\[
\hat{A}^\mu(x^\nu, x^5) = + \hat{A}^\mu(x^\nu, -x^5), \quad \hat{A}^5(x^\nu, x^5) = - \hat{A}^5(x^\nu, -x^5).
\]

These projections force the gauge-covariant boundary conditions,

\[
\hat{F}^{5N} = \hat{F}^{N5} = 0
\]

at \( x^5 = 0 \) and \( \pi R \). The theory is then invariant under a restricted set of gauge transformations

\[
\hat{A}^M(x) \rightarrow U(x)\hat{A}^M(x)U^\dagger(x) + \frac{i}{g_5} U(x)\partial^M U^\dagger(x),
\]

which respect the orbifold projection conditions, i.e. gauge transformations \( U(x) = \exp \left[ -ig_5 \epsilon^a(x)T^a \right] \) for which \( \epsilon^a(x^\mu, x^5) = +\epsilon^a(x^\mu, -x^5) \).

The four-dimensional content of the theory is most easily seen by expanding \( \hat{A}_\mu \) in a Fourier cosine series

\[
\hat{A}_\mu = \frac{1}{\sqrt{\pi R}} \left[ A_{\mu}^{00}(x_\nu) + \sqrt{2} \sum_{n=1}^{\infty} A_{\mu}^{an}(x_\nu) \cos \left( \frac{n x_5}{R} \right) \right],
\]

and \( \hat{A}_5 \) in a Fourier sine series

\[
\hat{A}_5 = \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} A_{5}^{an}(x_\nu) \sin \left( \frac{n x_5}{R} \right).
\]

\(^1\text{Note that, due to the properties of five-dimensional phase space, there are no infrared singularities.}\)
In terms of the Fourier expansions, the kinetic energy terms in the Lagrangian (1) become

$$\mathcal{L}_{K.E.} = -\frac{1}{4} \left[ (\partial_\mu A^{\mu 0}_0) + \sum_{n=1}^{\infty} (\partial_\mu A^{\mu n}_n)^2 \right] - \frac{1}{2} \sum_{n=1}^{\infty} [M_n A^{\mu n}_n + \partial_\mu A^{\mu n}_n]^2,$$

where $M_n = n/R$ is the mass of the KK state at level-$n$.

The gauge transformations in eqn. (9) allow for the gauge fixing of the 4D gauge theory [with gauge-bosons $A^{\mu 0}_a(x)$], as well as allowing us to choose values for the $A^{\mu n}_5(x)$. We may therefore impose a general $R_5$ gauge-fixing of the form,

$$\mathcal{L}_{GF}' = -\sum_{n=0}^{\infty} \frac{1}{2\xi_n} (\partial^\mu A^{\mu n}_n + \xi_n M_n A_n^0)^2,$$

where the $\{\xi_n\}$ (with $n = 0, 1, 2, \ldots$) are arbitrary gauge parameters. From these expansions, we see that the zero-modes $\{A^{\mu 0}_a\}$ form an adjoint of massless vector bosons as expected, while the $\{A^{\mu n}_5\}$ form a Kaluza-Klein (KK) tower of adjoint vector bosons with mass $M_n = n/R$. The gauge-fixing term eliminates the kinetic-energy mixing between $A^{\mu n}_5$ and $A^{\mu 0}_a$, and we may identify the $A^{\mu n}_5$ modes as the “eaten” Goldstone bosons of a geometrical “Higgs” mechanism where no physical Higgs boson is actually invoked. The $A^{\mu n}_5$ has a gauge-dependent mass $M_n^2 = \xi_n M_5^2$. The appropriate Faddeev-Popov ghost term can be derived, though it is not needed for the analysis below.

The analysis of the compactified theory proceeds most simply in the “unitary” gauge, $\xi_n = \infty$ for $n \geq 1$, in which $\{A^{\mu n}_5\}$ fully decouple since $M_5 \to \infty$. The self-interactions of the zero-mode fields are that of a usual 4D Yang-Mills theory with gauge-coupling $g = g_5/\sqrt{\pi R}$ and covariant gauge-fixing parameter $\xi_0$. The interactions of the KK modes amongst themselves and with the zero mode gauge-bosons are [22, 23],

$$\mathcal{L}_{int} = -g C^{abc} \sum_{n=1}^{N} \left[ \partial_\mu A^{\mu 0}_a A^{bn\mu} A^{cn\nu} + \partial_\mu A^{\mu n}_n \left( A^{bn\mu} A^{cn\nu} + A^{bn\mu} A^{cn\nu} \right) \right]$$

$$-\frac{g}{\sqrt{2}} C^{abc} \sum_{n,m,\ell=1}^{N} \partial_\mu A^{\mu n}_n A^{bn\mu} A^{cn\nu} \Delta_3(n, m, \ell)$$

$$-\frac{g^2}{4} C^{abc} C^{ade} \sum_{n=1}^{N} \left[ A^{bn\mu} A^{cn\nu} A^{dm\mu} A^{en\nu} + \text{all permutations} \right]$$

$$-\frac{g^2}{4\sqrt{2}} C^{abc} C^{ade} \sum_{n,m,\ell=1}^{N} \Delta_3(n, m, \ell) \left[ A^{bn\mu} A^{cn\nu} A^{dm\mu} A^{en\nu} + \text{all permutations} \right]$$

$$-\frac{g^2}{8} C^{abc} C^{ade} \sum_{n,m,\ell,k=1}^{N} A^{bn\mu} A^{cn\nu} A^{dm\mu} A^{en\nu} \Delta_4(n, m, \ell, k),$$

with $(\Delta_3, \Delta_4)$ given by,

$$\Delta_3(n, m, \ell) = \delta(n + m - \ell) + \delta(n - m + \ell) + \delta(n - m - \ell)$$

$$\Delta_4(n, m, \ell, k) = \delta(n + m - \ell - k) + \delta(n - m + \ell - k) + \delta(n - m - \ell + k) + \delta(n + m + \ell - k)$$

$$+ \delta(n + m + \ell + k) + \delta(n - m - \ell - k) - 2 \delta(n + m + \ell + k),$$
\[
\Delta_4(n, m, \ell, k) = \delta(n + m + \ell - k) + \delta(n + m - \ell + k) + \delta(n - m + \ell + k) \\
+ \delta(n + m - \ell - k) + \delta(n - m - \ell + k) + \delta(n - m + \ell - k) + \delta(n - m - \ell - k).
\]

Since the underlying 5D gauge-theory must break down at energy scale \( \Lambda = O\left(1/g_5^2\right) \), we have truncated the KK tower at the level \( N \) such that \( N/R = \Lambda = O\left(1/g_5^2\right) \).

Inspecting eqns. (14) and (15), we see that the KK tower is a set of self-interacting massive vector bosons, with interactions similar to those of a four-dimensional massive Yang-Mills theory with a characteristic coupling \( g \). The usual arguments \([14, 17, 18, 19]\) would suggest that the scattering of longitudinally-polarized vector bosons at level \( n \) will grow with energy and would violate unitarity at an energy scale,

\[
E^* \sim \frac{4\pi M_n}{g} = \frac{4n\pi}{gR} = \frac{4n\pi^2\delta^2}{g_5\sqrt{R}} = \frac{4n\pi^2g}{g_5^2}.
\]

However, this cannot be the case as can be seen in several ways. First, \( g \) could in principle be arbitrarily small by adjusting \( R \), in which case \( E^* \) could be made arbitrarily small\(^2\). In particular, if this were the case, \( E^* \) could be made much smaller than the intrinsic cutoff of the order \( 1/g_5^2 \) (as inferred from our analysis of the 5D Yang-Mills theory). Second, the compactification can be viewed as the imposition of the appropriate boundary conditions on 5D Yang-Mills fields for which, as we have previously argued, tree-level scattering amplitudes do not grow with energy.

In addition, it has recently been shown that the low-energy properties of a compactified five-dimensional gauge theory may be reproduced in a “deconstructed” (or “remodeled”) four-dimensional effective field theory with a replicated gauge group and an appropriate gauge-symmetry breaking pattern \([24, 25]\). These four-dimensional models may be interpreted as theories in which a compactified fifth dimension is discretized with a lattice spacing of order \( a = R/N \), where \( N \) is the number of replicated gauge groups. Furthermore, these theories can be embedded in a variety of renormalizable four-dimensional gauge theories \([24, 25]\), in which case it is not possible that unitarity is violated at any energy. By making \( N \) large (for fixed \( a \)), \( E^* \) can be made arbitrarily small. In particular \( E^* \) can be made smaller than \( 1/a \), the energy scale at which the deconstructed theory deviates significantly from the compactified 5D Yang-Mills theory. Since the deconstructed theory cannot violate unitarity at this energy, neither can the five-dimensional gauge theory\(^3\).

To elucidate this behavior, we consider the elastic scattering of two longitudinally-polarized KK vector bosons with level-\( n \). The relevant Feynman diagrams are depicted in Fig. 1, which includes the exchange of the zero-mode states, the states with level \( 2n \), as well as the four-point contact coupling of level-\( n \) states amongst themselves. A careful analysis of these contributions\(^4\) shows that the individual terms have energy dependences of \( O(E^4) \) and \( O(E^2) \), but due to cancellations among all of these diagrams, the overall scattering amplitude does not grow with energy. Instead, after a lengthy calculation

\(^2\) It is interesting to note that for \( \delta \) extra dimensions the scale \( E^* \) is proportional to \( g^{2/\delta - 1}/g_{5+\delta}^2 \), and for six dimensions or greater is necessarily smaller than the corresponding \( \Lambda \) – the intrinsic scale of the higher dimensional gauge theory – so long as the compactification size is greater than \( 1/\Lambda \).

\(^3\) Scattering in the deconstructed theory deviates from compactified 5D Yang-Mills theory by corrections of order \( 1/N \), some of which grow with energy. Vector boson scattering in these theories will be explored in a forthcoming publication \([25]\).

\(^4\) Details of this and related calculations will be presented elsewhere \([25]\).
we find that the amplitude approaches a constant,

$$\mathcal{T} \left[ A_L^{an} A_L^{bn} \rightarrow A_L^{cn} A_L^{dn} \right] = g^2 \left[ C^{abe} C^{cde} \left( \frac{5}{2} c \right) + C^{ace} C^{dbe} \left( -\frac{8c^2 - 5c + 9}{2(1-c)} \right) + C^{ade} C^{bce} \left( \frac{8c^2 + 5c + 9}{2(1+c)} \right) \right]$$

$$+ \mathcal{O}(M_n^2/E^2),$$

where \( c = \cos \theta. \)

The cancellations in this amplitude arise from the gauge symmetry of the underlying five-dimensional theory (and in particular the Jacobi identity of the structure constants \( C^{abc} \) which ensures the \( \mathcal{O}(E^2) \) cancellation), as well as the particular masses \( (M_n = n/R) \) of the various KK levels. The unitarity of this process depends crucially on the cancellation of contributions from level 0, \( n, \) and \( 2n. \) Unlike the traditional Higgs mechanism \cite{27}, in which the unitarity of massive vector boson scattering is assured through the exchange of a spin-0 Higgs boson, in the current case the unitarity of level-\( n \) scattering occurs through the introduction of a new set (level-\( 2n \)) of vector bosons! Of course, unitarity of level-\( 2n \) scattering would require the addition of higher-level vector bosons and, ultimately, the entire tower of KK states.

The behavior of the high-energy longitudinal KK scattering in the compactified theory can also be understood from examining the corresponding Goldstone amplitude of \( A_5^{an}. \) We observe that, as a consequence of the geometric Higgs mechanism reflected in the \( R_\xi \) gauge-fixing term \cite{2}, the amplitude of \( A_L^{an} \) and that of \( A_5^{an} \) are connected via a Kaluza-Klein Equivalence Theorem (KK-ET) in the high energy limit \( E \gg M_n. \) In analogy with the traditional ET in the standard model for longitudinal weak gauge boson scattering \(1, 18, 23, 30, 31, 32, 8, \) we deduce the relation,

$$\mathcal{T} \left[ A_L^{an}(p_n), A_L^{bn}(p_m), \ldots \right] = C_{\text{mod}} \mathcal{T} \left[ A_5^{an}(p_n), A_5^{bn}(p_m), \ldots \right] + \mathcal{O}(M_{n,m,...}/E),$$

where each external momentum is put on mass-shell, e.g., \( p_n^2 = M_n^2, \) etc, and the radiative modification factor \( C_{\text{mod}} = 1 + \mathcal{O}(\text{loop}) \) arises only at loop-level \(23, 30, 31, 32, 8\) and is irrelevant to the tree-level analysis below. The physical longitudinal amplitude of \( A_L^{an} \) in \( \langle 18 \rangle \) may be computed in any gauge while the Goldstone \( A_5^{an} \)-amplitude only exists in the \( R_\xi \) gauges such as the ’t Hooft-Feynman gauge \((\xi_n = 1)\) or Landau gauge \((\xi_n = 0)\).

In the \( R_\xi \) gauge, there are additional interactions involving the \( A_5^{an} \) Goldstone states \((n.b. \ A_5 = 0)\).
\[ -\mathcal{L}^{\text{int}}_{\text{int}} = +gC^{abc}_n \sum_{n=1}^{N} A^{n_{\mu}}_5 (\partial_{\mu} A^{n_5}_5 + M_n A^{n_{\mu}}_5) + \frac{g^2}{2} C^{abc} C^{ade}_n \sum_{n=1}^{N} A^{n_{\mu}}_5 A^{n_{\mu}}_5 A^{n_5}_5 A^{n_5}_5 \]

\[ + \frac{g}{\sqrt{2}} C^{abc}_n \sum_{n,m,\ell=1}^{N} A^{n_5}_5 (\partial_{\mu} A^{n_5}_5 + M_\ell A^{n_5}_5) \Delta_3(n, m, \ell) \]

\[ + \frac{g^2}{\sqrt{2}} C^{abc} C^{ade}_n \sum_{n,m,\ell=1}^{N} A^{n_5}_5 A^{m_5}_5 A^{n_5}_5 A^{n_5}_5 \Delta_3(n, m, \ell) \]

\[ + \frac{g^2}{4} C^{abc} C^{ade}_n \sum_{n,m,\ell,k=1}^{N} A^{n_5}_5 A^{m_5}_5 A^{n_5}_5 A^{n_5}_5 \Delta_4(n, m, \ell, k) , \]

where

\[
\Delta_3(n, m, \ell) = \delta(n + m - \ell) + \delta(n - m + \ell) - \delta(n - m - \ell) ,
\]

\[
\Delta_4(n, m, \ell, k) = \delta(n + m + \ell - k) + \delta(n + m + \ell + k) + \delta(n - m + \ell - k) + \delta(n - m - \ell + k) - \delta(n - m - \ell - k) .
\]

From this equation we see that the states \(A^{n_5}_5\) interact as a set of color-octet scalar particles, and their cubic (quartic) vertices contain only one (zero) partial derivative and one or two (two) Goldstone fields of \(A^{n_5}_5\). Power-counting therefore shows that the \(A^{n_5}_5\) amplitude is at most of \(O(E^0)\) and is manifestly unitary in four-dimensions. Based upon the equivalence theorem (18), this should reproduce the same high-energy behavior (17) for longitudinal KK scattering. Using (19) we explicitly compute the \(A^{n_5}_5\) amplitude to be

\[
\mathcal{T} \left[ A^{n_5}_5 \rightarrow A^{n_5}_5 \right] = g^2 \left[ C^{abc}_n C^{ade} \left( -\frac{3}{2} \right) + C^{ace} C^{dfe} \left( -\frac{3(3 + c)}{2(1 - c)} \right) + C^{ade} C^{bce} \left( \frac{3(3 - c)}{2(1 + c)} \right) \right] + \mathcal{O}(M_n^2/E^2) .
\]

This differs from (17) only by an overall constant \(-4c\) times the Jacobi identity,

\[
C^{abc}_n C^{ade} + C^{ace} C^{dfe} + C^{ade} C^{bce} = 0 ,
\]

and thus perfectly agrees with the KK-ET in eqn. (18).

While it is reassuring that the low-energy unitarity of elastic scattering in the five-dimensional Yang-Mills theory is reproduced in the four-dimensional compactified theory, it is natural to wonder how the \textit{bad} high-energy behavior of the underlying five-dimensional theory is manifest in the compactified theory. In fact, the bad high-energy behavior of the underlying theory is manifest not in the behavior of a single scattering channel, but rather in a coupled-channel analysis. Consider energies large compared
to the mass of the level-$N_0$ KK modes, and the (normalized) state consisting of equal parts of pairs of longitudinally polarized gauge bosons from all of the first $N_0$ levels,

$$|\Psi^{ab}\rangle = \frac{1}{\sqrt{N_0}} \sum_{\ell=1}^{N_0} |A^{a\ell}_L A^{b\ell}_L\rangle.$$  

(23)

We then compute the inelastic amplitude,

$$\mathcal{T}\left[ A^{a\ell}_L A^{b\ell}_L \to A^{c\ell}_L A^{d\ell}_L \right] = g^2 \left[ C^{a\ell b\ell c\ell d\ell} + C^{a\ell c\ell b\ell d\ell} \frac{3+c}{1-c} + O\left(M_{n,m}^2/E^2\right) \right]$$

(24)

$$= \frac{2}{3} \mathcal{T}\left[ A^{a\ell}_L A^{b\ell}_L \to A^{c\ell}_L A^{d\ell}_L \right] + O\left(M_{n,m}^2/E^2\right),$$

where we have verified again that the longitudinal and Goldstone amplitudes are equivalent in the high energy limit and differ only by terms of $O(M_{n,m}^2/E^2)$. From these, we arrive at

$$\mathcal{T}\left[ |\Psi^{ab}\rangle \to |\Psi^{cd}\rangle \right] \simeq (N_0 - 1) \mathcal{T}\left[ A^{a\ell}_L A^{b\ell}_L \to A^{c\ell}_L A^{d\ell}_L \right] + N_0 \mathcal{T}\left[ A^{a\ell}_L A^{b\ell}_L \to A^{c\ell}_L A^{d\ell}_L \right] + O\left(M_{n,m}^2/E^2\right), \quad \text{(for \ } N_0 \gg 1).$$

(25)

So, for large $N_0$, the normalized four-dimensional gauge-singlet $s$-wave amplitude is,

$$\mathcal{T}_s^{00} = \frac{1}{64\pi} \int_{-1}^{1} d\cos \theta \frac{1}{m^2 - 1} \sum_{a,c=1}^{m^2-1} \mathcal{T}\left[ |\Psi^{aa}\rangle \to |\Psi^{cc}\rangle \right]$$

$$\simeq N_0 \frac{mg^2}{16\pi} \left[ -1 + 2 \ln \frac{s}{M_n^2 - M_\ell^2} \right] = N_0 \frac{mg^2}{16\pi} O(1) = \frac{N_0}{R} \frac{mg_5^2}{16\pi} O(1),$$

(26)

where we have retained the pole masses in the $(t,u)$-channel contributions to the inelastic Goldstone amplitude in order to avoid the infrared singularity in the phase space. The associated logarithmic factor is of $O(1)$. Requiring the $s$-wave amplitude in eqn. (26) to be less than $1/2$, we find that the KK tower must be truncated at the level $N_0 = N$ such that

$$\frac{N}{R} \lesssim \frac{8\pi^2}{m g_5^2}. \quad \text{(27)}$$

As in our discussion of unitarity of the 5D Yang-Mills theory [cf. eqn. (I)], we see that the compactified 4D KK theory must be treated as an effective theory valid only below a scale of the order $1/g_5^2$. Unlike our expectation based on massive 4D Yang-Mills theory [cf. eqn. (II)], the bound has no dependence on the effective 4D gauge coupling $g (= g_5/\sqrt{\pi R})$ and is therefore independent of the radius of compactification.

Acknowledgments
We are grateful to B. Dobrescu, J. Distler, H. Georgi, and S. Nandi for discussions. This work was supported in part by the Department of Energy under grants DE-FG02-91ER40676 and DE-FG03-93ER40757.

Note added: As this work was being completed, we became aware a new preprint [34] which also considered $R_\xi$ gauge-fixing and the resulting Feynman rules in compactified Yang-Mills theory.

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