ANALYTIC MODULI OF PLANE BRANCHES AND HOLOMORPHIC FLOWS

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Abstract. We study the behaviour (in the infinitesimal neighbourhood of the singularity) of a singular plane branch under the action of holomorphic flows. The techniques we develop provide a new elementary, geometric and dynamical solution to Zariski’s moduli problem for singular branches in \((\mathbb{C}^2, 0)\). Furthermore, we study whether elements of the same class of analytic conjugacy are conjugated by a holomorphic flow; in particular we show that there exists an analytic class that is not complete: meaning that there are two elements of the class that are not analytically conjugated by a local diffeomorphism embedded in a one-parameter flow.

1. Introduction

The “moduli problem for plane branches”, as posed by Zariski [18] and recently solved in an algebraic way [10], has been found to have a particularly elementary solution when set in the context of holomorphic flows [7]. In this context, one may ask whether two analytically equivalent branches are also equivalent under a holomorphic flow, thus comparing the analytic and the holomorphic-flow moduli. This leads in a natural way to studying how a plane singular branch behaves under the action of holomorphic flows, which is the topic of the present work.

Roughly speaking, in the analytic classification of plane branches, these are reduced to normal form and two are equivalent if they share the same normal form [10]. This normal form is obtained by making coefficients of the Puiseux parametrization of the curve equal to 0 working jet by jet. The coefficients that may be turned into 0 are determined by the set of orders of contact of Kähler differentials with the curve. We show that such data is equivalent to providing the set of orders of tangency of germs of holomorphic vector fields defined in a neighborhood of 0 in \(\mathbb{C}^2\) with the curve (Corollary 2). Since vector fields are dynamical objects that generate one-parameter groups, it is natural to consider exponentials of germs of vector fields (with a certain order of tangency with the curve) as normalizing transformations. This is the point of view of the first author in [7]. It has been expanded in this work where the relation between orders of tangency of local vector fields with a curve and the reduction to normal form of its Puiseux parametrization is made explicit in Theorem 3. As a consequence, replacing the set of orders of contact of Kähler differentials with the set of orders of tangency removes the need of interpreting the former set in dynamical terms.

Let us be more precise. Consider a singular holomorphic vector field \(X\) (so that \((0, 0)\) is an equilibrium point of \(X\)) defined in an open neighbourhood \(U\) of \((0, 0) \subset \mathbb{C}^2\) and an irreducible germ of analytic curve \(Γ\) contained in the same open
set \( U \), say \( \Gamma \equiv (f = 0) \) for some \( f \in \mathcal{O}(\mathbb{C}^2, 0) \). Let \( \{ \psi_t \}_{t \in \mathbb{C}} \) be the one-parameter group whose infinitesimal generator is \( X \). Let \( \epsilon \in \mathbb{C} \); consider the curve

\[
\psi_{-\epsilon}(\Gamma) := \Gamma_\epsilon \equiv (f \circ \psi_\epsilon(x, y) = 0).
\]

By expanding \( f \circ \psi_\epsilon \) as a Taylor power series in the variable \( \epsilon \) we obtain

\[
\Gamma_\epsilon \equiv \left( \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^n(f)(x, y) = 0 \right)
\]

where \( X^0(f) = f \) and we define \( X^{j+1}(f) = X(X^j(f)) \) for \( j \geq 0 \) recursively. We shall call \( \{ \Gamma_\epsilon \} \) the \textit{holomorphic deformation} of \( \Gamma \) by \( X \) (or by \( \{ \psi_t \} \)). The coefficient of \( x^i y^j \) for \( f \circ \psi_\epsilon \) is an entire function of \( \epsilon \) for any \( i + j \geq 0 \). Assume for simplicity that the tangent cone of \( \Gamma \) is not \( x = 0 \). The curve \( \Gamma_\epsilon \) has a Puiseux parametrization of the form \( (t^n, \sum_{j=n}^{\infty} a_j(\epsilon)t^j) \) for any \( \epsilon \) in a small neighborhood of \( 0 \) in \( \mathbb{C} \) where \( n \) is the multiplicity of \( \Gamma \) at \( (0, 0) \). Assume also that \( \Gamma \) is not invariant by \( X \) and let \( k \) be the first index such that \( a_k(\epsilon) \) is not a constant function. We denote \( (X, \Gamma)_{(0,0)} = k \). The combination of Corollary 2 and Theorem 3 implies

\[
k = (X, \Gamma)_{(0,0)} = (X(f), f)_{(0,0)} - n - c + 1
\]

where \( (X(f), f)_{(0,0)} \) is the intersection multiplicity of \( f \) and \( X(f) \), or in other words the tangency order of \( X \) with \( \Gamma \), and \( c \) is the conductor of \( \Gamma \). As a consequence, the reduction to normal form depends in a straightforward way on the set of tangency orders of holomorphic vector fields with the curve \( \Gamma \).

Property 2 is not obvious: a priori the value of \( (X, \Gamma)_{(0,0)} \) could have depended on other terms of the Taylor power series expansion of \( f \circ \psi_\epsilon \). As an example of a situation in which further terms of the Taylor power series expansion are relevant, consider the intersection multiplicity \( \Gamma, \Gamma_\epsilon \) of \( \Gamma \). It is equal to \( \min \{ (X^n(f), f)_{(0,0)} : n \geq 1 \} \) for \( \epsilon \in \mathbb{C}^* \) in a small neighborhood of \( 0 \) by Equation (1). The minimum may be realized for \( n > 1 \) as is the case for \( \Gamma = (y^2 - x^3 = 0) \) and \( X = x \frac{\partial}{\partial y} \) where \( (X(f), f)_{(0,0)} = 5 \) and \( (X^2(f), f)_{(0,0)} = 4 \).

The previous discussion motivates the study of the action of one-parameter groups on irreducible curves. Consider an equivalence class \( \mathcal{C} \) for the equivalence relation given by the analytic conjugacy of plane branches. We say that two curves \( \Gamma_1, \Gamma_2 \in \mathcal{C} \) are \textit{connected by a geodesic} if they are conjugated by the time 1 flow \( \exp(X) \) of a germ of holomorphic singular vector field. We say that \( \mathcal{C} \) is \textit{complete} if given any two curves \( \Gamma_1, \Gamma_2 \in \mathcal{C} \) they are connected by a geodesic. The term complete is motivated by analogy with the case of finite dimensional Lie groups \( G \) that have a bi-invariant metric where geodesics are of the form \( t \mapsto \exp(tX) \cdot g \) where \( X \) belongs to the Lie algebra of \( G \). \( g \in G \) and \( t \) varies in \( \mathbb{R} \). An example of a complete class \( \mathcal{C} \) is the class of smooth curves (Proposition 4).

A priori, we could define a notion of formal completeness in which \( X \) is a formal vector field, i.e. a derivation of \( \mathbb{C}[[x, y]] \) that preserves its maximal ideal. The definitions are, in fact, equivalent.

**Theorem 1.** Let \( \mathcal{C} \) be a class of analytic conjugacy of plane branches. Then \( \mathcal{C} \) is complete if and only if \( \mathcal{C} \) is formally complete.

The analytic classification [10] relies, as an intermediate step, in the classification of plane branches modulo unipotent diffeomorphisms, i.e. germs of biholomorphism \( \varphi \) such that the linear part \( D_0 \varphi \) at the origin is a unipotent linear transformation. Since unipotent diffeomorphisms are always embedded in the one-parameter group of a formal vector field (cf. Remark 5), such classes are complete by Theorem 1. Moreover since any analytic conjugacy \( \varphi \) between curves \( \Gamma_1 \) and \( \Gamma_2 \) may be written in the form \( D_0 \varphi \circ \psi \) where \( \psi := (D_0 \varphi)^{-1} \circ \varphi \) has linear part equal to the identity (see Corollary 12), we deduce that \( \Gamma_1 \) and \( \Gamma_2 \) can be connected by two
“segments of geodesics.” More precisely, there exist germs of singular holomorphic vector fields $X$, $Y$ such that $(\exp(Y) \circ \exp(X))(\Gamma_1) = \Gamma_2$ (Corollary 10). A class of analytic conjugacy $C$ of a plane branch $\Gamma$ is identified with the set of left cosets of $\Diff(C^2, 0)/\Stab(\Gamma)$ where $\Diff(C^2, 0)$ is the group of germs of diffeomorphisms defined in a neighborhood of $0 \in C^2$ and $\Stab(\Gamma) = \{ \varphi \in \Diff(C^2, 0) : \varphi(\Gamma) = \Gamma \}$ is the stabilizer of $\Gamma$. It is known that there exist local biholomorphisms that can not be embedded in the flow of a formal vector field (see [19] and [14]) but to show that a class is not complete, we need to prove a stronger result, namely that there exists a left coset $\varphi \circ \Stab(\Gamma)$ in $\Diff(C^2, 0)/\Stab(\Gamma)$ such that none of its elements can be embedded in the flow of a formal vector field. We will show that there exist local biholomorphisms $\varphi_0$ such that any $\varphi \in \Diff(C^2, 0)$ sharing the same second jet as $\varphi_0$ is not embedded in the flow of a formal vector field. Then, we shall prove that there are plane branches $\Gamma$ such that its stabilizer is small: any element of $\Stab(\Gamma)$ has second jet equal to the identity map. Combining these two results we obtain that no element of $\varphi_0 \circ \Stab(\Gamma)$ is embedded in the flow of a formal vector field. By following the previous ideas we obtain

**Proposition 1.** Let $\Gamma'$ be the plane branch with Puiseux parametrization $(t^6, t^7 + t^{10} + t^{11})$. Then the class $C$ of analytic conjugacy of $\Gamma'$ is non-complete.

We can provide a topology in the class $C$ of a plane branch $\Gamma$ by considering a topology in $\Diff(C^2, 0)$ and the corresponding quotient topology in the set $\Diff(C^2, 0)/\Stab(\Gamma)$. A natural choice is the Krull topology (also called $m$-adic topology, where $m$ is the maximal ideal of $C[[x, y]]$) where the sets $S_{k, \varphi}$ of elements of $\Diff(C^2, 0)$ whose $k$-jet coincides with the $k$-jet of $\varphi$ provide a base of open sets of the topology by varying $\varphi$ in $\Diff(C^2, 0)$ and $k \in N$. Proposition 1 can be reinterpreted as a genericity property in the class $C$.

**Proposition 2.** Let $C$ be the analytic class of the plane branch $\Gamma$ with Puiseux parametrization $(t^6, t^7 + t^{10} + t^{11})$. Denote

$$C' = \{ \Gamma' \in C : \Gamma' \text{ and } \Gamma \text{ are connected by a geodesic} \}$$

Then $C \setminus C'$ contains an open set of $C$ for the Krull topology. In particular $C'$ is not dense in $C$.

The previous result does not hold for other natural topologies.

**Proposition 3.** Let $\Gamma$ be a plane branch and $C$ its analytic class of conjugacy. Let $\Gamma' \in C$. Then there exist a holomorphic deformation $\Gamma'_t$ of $\Gamma'$ by a vector field, defined in a neighborhood of $\epsilon = 0$, $\Gamma'_0 = \Gamma'$ and a simple continuous curve $\gamma : [0, 1] \to C$ such that $\gamma(0) = 0$ and $\Gamma'$ is connected by a geodesic to $\Gamma^{\gamma(t)}$ for any $t \in [0, 1]$. 

Despite the similarities with the tools of reduction of singularities of vector fields (as in [2], [3], for instance), our technique is different: the reduction of singularities seeks a “simple form” for the underlying foliation associated to a vector field (and hence, uses techniques based on invariants like those for foliations as in [15] or [3]) whereas we are mostly interested in the behaviour of a vector field under bi-rational maps (blow-ups) and being able to modify it (by multiplication by a function) in order that the associated flow behaves in a specific way on an analytic set (the branch). This is, to our knowledge, the first time this kind of study has been undertaken and we hope to extend it to other contexts.

Notice that in [8], the author provides an algorithm for computing the dimension of the generic component of the analytic moduli of a plane branch, using the dual graph of its desingularisation. Finally, our techniques are quite different from those of classical deformation theory [9]: in this, one is concerned with deformations by
adding a “small” parameter to the equation of the curve and the aim is to study the geometric and topological properties of the moduli so obtained. We are specifically concerned with deformations caused by flows, so that (in a rough sense) we are adding the parameter at all the orders of the equation.

2. Notation and Definitions

Our base ring is \( \mathcal{O} = \mathcal{O}_P \), the ring of germs of holomorphic functions in a neighborhood of a point \( P \) of a two-dimensional complex-analytic manifold, whose base “set” we shall usually denote, as is the custom, \( (\mathbb{C}^2,0) \). The maximal ideal of \( \mathcal{O} \) will be denoted \( \mathfrak{m}_P \) or simply \( \mathfrak{m}_0 \) when no confusion arises. Assume \( P = (0,0) \in \mathbb{C}^2 \) for simplicity. We denote \( \hat{\mathcal{O}} = \mathbb{C}[[x,y]] \) and let \( \mathfrak{m} \) be the maximal ideal of \( \hat{\mathcal{O}} \).

**Definition 1.** We say that \( f, g \in \hat{\mathcal{O}} \) have the same \( k \)-jet and we denote \( j^k f = j^k g \) if \( f - g \in \mathfrak{m}^{k+1} \).

Let \( (f_k)_{k \geq 1} \) be a sequence in \( \hat{\mathcal{O}} \). Then it converges to \( f \in \hat{\mathcal{O}} \) in the \( \mathfrak{m} \)-adic topology (or also the Krull topology) if for any \( l \geq 1 \) there exists \( k_0 \geq 1 \) such that \( j^l f_k = j^l f \) for any \( k \geq k_0 \).

**Definition 2.** We say that \( X \) is a *vector field* if is a \( \mathbb{C} \)-derivation \( X : \mathcal{O}_P \to \mathcal{O}_P \) continuous for the \( \mathfrak{m}_P \)-adic topology. It is *singular* if \( X(\mathfrak{m}_P) = \mathfrak{m}_P \) and *regular* otherwise. In the case \( P = (0,0) \in \mathbb{C}^2 \) we write

\[
X = A(x,y) \frac{\partial}{\partial x} + B(x,y) \frac{\partial}{\partial y},
\]

where \( A := X(x) \) and \( B := X(y) \) belong to \( \mathcal{O} \). Analogously by replacing \( \mathcal{O} \), \( \mathfrak{m}_0 \) with \( \hat{\mathcal{O}}, \mathfrak{m} \), we can define formal vector fields.

**Definition 3.** Let \( X \) be a formal singular vector field. We say that \( X \) is *nilpotent* if its linear part is a nilpotent vector field.

We shall also say that \( P \) is a singular point for \( X \) (especially, but not only, when \( X \) can be understood as a vector field on a larger analytic manifold). Finally, \( X \) is *truly singular* at \( P \) (or \( P \) is a true singularity of \( X \)) if it is singular and there do not exist a regular vector field \( Y \) and a regular holomorphic function \( f \in \mathcal{O}_P \) such that \( X = f^m Y \) for some positive integer \( m \) (this is related to what is called a strictly singular point in [13]). Note that all these definitions are given for the local case: we shall be explicit when dealing with non-local situations.

The *multiplicity* of a formal vector field \( X \) is the largest non-negative integer \( m \) such that \( X(\mathfrak{m}) \subseteq \mathfrak{m}^m \). Thus, a non-singular vector field has multiplicity 0 and, in general, if \( X = a(x,y) \frac{\partial}{\partial x} + b(x,y) \frac{\partial}{\partial y} \), then the multiplicity of \( X \) is the smallest of the multiplicities of \( a(x,y) \) and \( b(x,y) \).

An analytic branch (simply branch) at \( P \) is any reduced and irreducible curve \( \Gamma \subseteq (\mathbb{C}^2,0) \). Unless otherwise specified, all our curves will be analytic branches and they will be defined either by a reduced and irreducible holomorphic function \( f \in \mathfrak{m}_P \) or by a Puiseux expansion \( \varphi(t) = (x(t), y(t)) \) when local coordinates at \( P \) are already chosen. All the results related to desingularisation of plane branches (and, as a requirement, finite sequences of point blow-ups, exceptional divisors, etc.) and their topological (not analytic) structure are assumed known: two good modern references are [5] and [10].

Consider a point \( P \) belonging to a two-dimensional complex analytic manifold \( M \). Denote by \( M_P \) the germ of \( M \) at \( P \) (which is, essentially, the same thing

\[^1\text{As a matter of fact, the expression should be "P is an equilibrium point of X" but we are indulging the custom.} \]
as $(\mathbb{C}^2,0)$. As our work is based on the process of point blow-ups, we need the following

**Definition 4.** Let $X$ be a singular vector field at $P$ and let $\pi: \mathcal{X} \to \mathcal{M}_P$ be the blow-up with centre $P$. The unique holomorphic vector field $\overline{X}$ on the whole $\mathcal{X}$ such that $\pi_* (\overline{X}) = X$ outside of the exceptional divisor $\pi^{-1}(P)$ is called the pull-back of $X$ to $\mathcal{X}$.

The fact that $\overline{X}$ exists is due to the singularity of $X$ at $P$: otherwise, $\overline{X}$ is not defined (it has “poles” on the exceptional divisor).

**Remark.** Notice that we are taking the “true” pull-back of $X$ on $\mathcal{X}$: we are interested in the dynamics of $X$, not just in the geometric structure of its integral curves. Thus, if $(x,y)$ are local coordinates at $P$ and one looks at the chart of $\pi$ with equations $x = \overline{x}, y = \overline{y}$ and

$$X = a(x,y) \frac{\partial}{\partial x} + b(x,y) \frac{\partial}{\partial y}$$

for some $a(x,y), b(x,y) \in \mathfrak{m}_P$, then on the chart $(\overline{x}, \overline{y})$, the local equation of $\overline{X}$ is given by

$$\overline{X} = a(\overline{x}, \overline{y}) \frac{\partial}{\partial \overline{x}} + \frac{1}{\overline{x}} (-a(x,y) + b(x,y)) \frac{\partial}{\partial \overline{y}},$$

expression which shows why $X$ must have a singularity at $P$ in order to admit a pull-back to $\mathcal{X}$. As the reader will have noticed, we do not eliminate the possible common factor $\overline{x}$ in the expression of $\overline{X}$. This implies that, usually, the pull-back of a singular vector field will not be truly singular: it will have some true singularities on the exceptional divisor but most of the points will be just equilibrium points such that, near them, $\overline{X}$ is of the form $\overline{x}^m Y$ for some non-negative integer $m$ and non-singular vector field $Y$.

The reader familiar with the theory of plane holomorphic foliations will notice the similarity and the differences between our approach and the one common in those works. This difference is exactly what makes our technique useful for studying deformations.

Anyway, we can consider the desingularisation of the underlying foliation of a singular vector field. The following result is a restatement of the main one in [15].

**Theorem 2** (cf. [15]). Let $X$ be a singular vector field at $P \in \mathcal{M}_P$. There is a finite sequence of blow-ups $\pi: \mathcal{X} \to \mathcal{M}_P$:

$$\mathcal{X} = \mathcal{X}_N \xrightarrow{\pi_{N-1}} \mathcal{X}_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} \mathcal{X}_1 \xrightarrow{\pi_0} (\mathbb{C}^2,0)$$

$\pi = \pi_0 \circ \cdots \circ \pi_{N-1}$ whose centres $(P_i)_{i=0}^{X-1}$ are singular points for the respective pull-back of $X$ and such that the pull-back $\overline{X}$ of $X$ on $\mathcal{X}$ has a finite number of true singularities and at any of these, say $Q, \overline{X}$ admits an expression of the form

$$x^a y^b \left( \mu_x \frac{\partial}{\partial x} + \lambda_y \frac{\partial}{\partial y} + h.o.t. \right)$$

where $(x,y)$ are local coordinates at $Q$, the exceptional divisor is included in $xy = 0$, $\mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}_{\geq 0}$. The shortest non-empty sequence of blow-ups for which this happens is called the minimal reduction of singularities of $X$.

Let $\Gamma$ and $X$ be an analytic branch and a singular vector field at $(\mathbb{C}^2,0)$. Let $\pi_i: \mathcal{X}_{i+1} \to \mathcal{X}_i$ be the infinite sequence of blow-ups with centre $P_i$, the intersection of the strict transform $\overline{\Gamma}$ of $\Gamma$ with the corresponding exceptional divisor (with $\mathcal{X}_0 = (\mathbb{C}^2,0)$ and $\overline{\Gamma}_0 = \Gamma$). The next result follows easily from the fact that $\Gamma$ is analytic:
Proposition 4. With the notation of the last paragraph, $\Gamma$ is invariant by $X$ if and only if $P_i$ is a singular point of the pull-back $\overline{X}_i$ of $X$ to $X_i$ for any $i \geq 0$. In particular, if $\Gamma$ is not invariant by $X$ then there exists $i_0 \geq 0$ such that $P_{i_0}$ is a singular point of $\overline{X}_{i_0}$ of $X$ for any $0 \leq i \leq i_0$ but $P_{i_0+1}$ is a regular point of $\overline{X}_{i_0+1}$.

This result provides the following

Definition 5. The path shared by a non-invariant analytic branch $\Gamma$ and a singular vector field $X$ is the sequence $(P_0, P_1, \ldots, P_{i_0+1})$ given by Proposition 4. Notice that we include in the shared path the point at which the pull-back of $X$ is non-singular.

Remark. The last point shared by $X$ and $\Gamma$ could be a singular point of the strict transform of $\Gamma$: we only require it to be a regular point for the pull-back of $X$.

The following result will be important in the study of the relation between a curve and its deformation:

Lemma 1. Let $(P_i)_{i=0}^N$ be the shared path between $\Gamma$ and a singular vector field $X$. The last point $P_N$ is not a corner of the exceptional divisor.

Proof. This is because after blowing up a singular point, the exceptional divisor is always invariant for the pull-back. If $P_N$ were a corner, then the pull-back $\overline{X}$ at $P_N$ would possess at least two invariant curves: both components of the exceptional divisor. This would imply that $P_N$ is singular for $\overline{X}$, which contradicts the definition. $\square$

We introduce now our main object of study:

Definition 6. Given a singular germ of analytic vector field $X$ and an irreducible germ of analytic plane curve $\Gamma$ at $(C^2, 0)$ with $\Gamma \equiv (f = 0)$ for $f \in \mathcal{O}$, we define the deformation of $\Gamma$ caused by $X$ or by the flow associated to $X$ as the family $\Gamma_\epsilon \equiv (\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^n(f) = 0) \equiv (f + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} X^n(f) = 0)$.

We shall refer either to the whole family or to any of its elements as “the deformation of $\Gamma$”.

Notice that, because $X$ is singular, if its multiplicity is greater than 1, then the local equation of $\Gamma_\epsilon$ is, roughly speaking, a higher order deformation of the local equation of $\Gamma$, in the sense that the terms added to $f$ are of order at least one more than the vanishing order of $f$. In any case, it is clear that the deformation of a non-singular analytic branch by a singular vector field is non-singular for $\epsilon$ small enough.

The following consequence of the formula for the higher derivative of a product is what makes blow-ups a sensible tool for studying deformations caused by vector fields:

Lemma 2. Let $X$ be a singular vector field at $(C^2, 0)$, $\pi : X \to (C^2, 0)$ be the blow-up with centre $(0, 0)$ and $\overline{X}$ the pull-back of $X$ by $\pi$. If $\Gamma \equiv (f = 0)$ is an analytic branch through $(0, 0)$ and $\overline{\Gamma}$ is its strict transform by $\pi$, then $\overline{\Gamma}_\epsilon = \Gamma_\epsilon$, that is: the strict transform of the deformation of $\Gamma$ by $X$ is the deformation of the strict transform $\overline{\Gamma}$ by $\overline{X}$. This generalises to any finite sequence of blow-ups with centres singular points of $X$ and its successive pull-backs.

Finally, the one-parameter group of diffeomorphisms of a vector field and the one of its pull-back are essentially the same object:
Lemma 3. Let $X$ be a singular vector field at $(\mathbb{C}^2, 0)$ and $(\psi_{X,s}(z))_s$ its one-parameter group of germs of diffeomorphisms. Let $\pi : X \to (\mathbb{C}^2, 0)$ be a sequence of blow-ups whose centres are singular points of each pull-back of $X$ and let $\pi^{-1}(0,0)$ be the pull-back of $X$ to $X$. The diffeomorphism associated to $\pi^{-1}(0,0)$ is the unique holomorphic extension $\psi_{X,s}$ to the whole $X$ of the diffeomorphism $\pi^{-1} \circ \psi_{X,s} \circ \pi$ defined on $X \setminus \pi^{-1}(0,0)$.

3. Main results

The deformation of an analytic branch $\Gamma$ caused by a singular vector field $X$ has a nice behaviour due to Cauchy-Kowalewski’s Theorem:

Proposition 5. Let $X$ be a singular analytic vector field at $(\mathbb{C}^2, 0)$ and let $\Gamma$ be an analytic plane branch which is not invariant for $X$. Then $\Gamma$ and $\Gamma_\epsilon$ are analytically conjugated and they share the same path with $X$ except possibly the last point: for $\epsilon$ small enough, the last shared point is certainly different.

Proof. Let $(P_i)_{i=0}^N$ be the path shared by $X$ and $\Gamma$. By Lemma 1, $P_N$ is not a corner of the exceptional divisor. By definition, the vector field $\pi^{-1}(0,0)$ is non-singular at $P_N$ and it is tangent to the exceptional divisor $E = \pi^{-1}(0,0)$. This implies that $\Gamma_\epsilon$ meets the exceptional divisor away from $P_N$ for $\epsilon$ small enough. □

For the sake of clarity let us recall the definition of intersection multiplicity.

Definition 7. Let $\Delta \equiv (g(x, y) = 0)$ be an analytic curve in $(\mathbb{C}^2, 0)$ which does not contain $\Gamma$. The intersection multiplicity $(\Gamma \cap \Delta)_{(0,0)}$ (also denoted by $(f, g)_{(0,0)}$) of $\Gamma$ and $\Delta$ at $(0,0)$ is the (finite) number

$$(\Gamma \cap \Delta)_{(0,0)} = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} / (f, g).$$

In the case we are dealing with, where $\Gamma$ is a branch, this number can be computed as

$$(\Gamma \cap \Delta)_{(0,0)} = \operatorname{ord}_{\epsilon}(g(\varphi(t)))$$

where $\varphi(t)$ is any irreducible Puiseux parametrization of $\Gamma$. The sub-index $(0,0)$ is usually omitted.

A direct consequence of Proposition 5 is

Corollary 1. Let $\Gamma \equiv (f = 0)$ be a (possibly singular) analytic branch at $(\mathbb{C}^2, 0)$ that is not invariant by a singular analytic vector field $X$. If $n_0, n_1, \ldots, n_N$ is the sequence of multiplicities of $\Gamma$ at the points of the path it shares with $X$, then the intersection multiplicity of $\Gamma$ and $\Gamma_\epsilon$ is given by:

$$(\Gamma, \Gamma_\epsilon)_{(0,0)} = \sum_{i=0}^{N-1} n_i^2 = \min\{ (X^n(f), f)_{(0,0)} : n \geq 1 \}$$

for $0 < |\epsilon| \ll 1$.

Proof. As the sequence of infinitely near points shared by $\Gamma$ and $\Gamma_\epsilon$ is the shared path between $X$ and $\Gamma_\epsilon$ except the last point (for $\epsilon \ll 1$), Noether’s formula (see, for example [3]) gives

$$(\Gamma, \Gamma_\epsilon)_{(0,0)} = \sum_{i=0}^{N-1} n_i n_{P_i}(\Gamma_\epsilon)$$

where $n_{P_i}(\Gamma_\epsilon)$ denotes the multiplicity of the strict transform of $\Gamma_\epsilon$ at $P_i$. Since $\Gamma$ and $\Gamma_\epsilon$ are topologically equivalent (as they are analytically conjugated), their sequence of multiplicities at their infinitely near points are the same: $n_{P_i}(\Gamma_\epsilon) = n_i$, and the first equality follows. Notice that we need set $\epsilon \ll 1$ because $\Gamma$ and
that there exists what we shall call a prepared Puiseux parametrization \(x,y\) by some local diffeomorphism of the form \(\phi(t)\), called an irreducible Puiseux parametrization \(\phi\) where \((\phi, (0,0))\) is what we need, by Corollary 1.

The tangency order between \(X\) and \(\Gamma\) is defined as \(\text{tang}_{(0,0)}(X,\Gamma) = (X(f),f)_{(0,0)}\) (see \([1]\)).

Lemma 4. We have \(\text{tang}_{(0,0)}(X,\Gamma) = (\Gamma,\Gamma_e)\) for \(0 < \epsilon \ll 1\) when \(\Gamma\) is non-singular.

Proof. We can assume that \(\Gamma\) is not invariant by \(X\), since otherwise \(\text{tang}_{(0,0)}(X,\Gamma) = \infty\) and \(\Gamma = \Gamma_e\) for any \(\epsilon \in \mathbb{C}\). As \(\Gamma\) is non-singular and it is not invariant for \(X\), after a change of coordinates, we may assume \(f = y\) and \(X(y) \not\equiv (0)\). Writing

\[X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}\]

we obtain \(B(x,y) = a(x^k + \text{h.o.t.}) + y(\mathcal{T}(x,y))\) for some \(k > 0\) and \(a \neq 0\). As \(A(0,0) \neq 0\), an easy inductive argument implies that \(\text{ord}_x(X^k(y)(x,0)) \geq k\) which is what we need, by Corollary \([1]\).

3.1. Vector fields, differential forms and curves. Consider now a branch \(\Gamma\) which, for the sake of simplicity, we assume tangent to the \(OX\) axis, so \(\Gamma \equiv (f(x,y) = 0)\) with \(f(x,y) = y^n + \text{h.o.t.}\). It is well known that \(\Gamma\) admits what is called an irreducible Puiseux parametrization

\[\varphi(t) \equiv (x(t),y(t)) = \left(t^n, \sum_{i \geq n} a_it^i\right)\]

where \(\varphi(t)\) is not of the form \(\varphi(t^k)\) for any \(k \geq 2\); the greatest common divisor of \(n\) and the exponents appearing in \(y(t)\) is 1. Up to replacing \(\Gamma\) with its conjugate by some local diffeomorphism of the form \((x,y) \mapsto (x,y + a(x))\), one easily deduces that there exists what we shall call a prepared Puiseux parametrization:

Definition 9. A prepared Puiseux parametrization of \(\Gamma\) is an irreducible Puiseux parametrization such that \(m > n\) and \(n \nmid m\).

Before proceeding any further, let us recall some definitions:

Definition 10. The semigroup \(S_\Gamma\) (or simply \(S\)) associated to \(\Gamma\) is the set

\[S_\Gamma = \{(\Gamma \cap \Delta)_{(0,0)} : \Delta \equiv (f(x,y) = 0), f(x,y) \in \mathbb{C}\{x,y\}, \Gamma \not\subset \Delta\}\]

It is a sub-semigroup of \(\mathbb{N}\). It is well known (due to the fact that \(\Gamma\) is a branch) that there is \(c \in S_\Gamma\) such that \(p \geq c\) implies \(p \in S_\Gamma\). The least \(c\) satisfying this property is called the conductor of \(\Gamma\).
Given a differential form \( \omega \in \Omega^1_X \), say \( \omega = a(x, y)dx + b(x, y)dy \), the contact of \( \omega \) with \( \Gamma \) is defined (as in [18]) as
\[
\nu_{\Gamma}(\omega) = \text{ord}_t (a(x(t), y(t)) \dot{x}(t) + b(x(t), y(t)) \dot{y}(t)) + 1,
\]
which does not depend on the parametrization of \( \Gamma \). On the other hand, given a vector field \( X \), say \( X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} \), let us calculate \( \text{tang}_{(0, 0)}(X, \Gamma) \). We have \( \frac{\partial f}{\partial y} \neq 0 \) and (certainly) \( \dot{x}(t) \neq 0 \). Since \( f(x(t), y(t)) = 0 \), we deduce
\[
(4) \quad \frac{\partial f}{\partial x} \dot{x}(t) + \frac{\partial f}{\partial y} \dot{y}(t) = 0,
\]
which can be rewritten as
\[
\frac{\partial f}{\partial x} = -\frac{\dot{y}(t)}{\dot{x}(t)} \frac{\partial f}{\partial y},
\]
so that, when computing the tangency order \( \text{tang}_{(0, 0)}(X, \Gamma) \), one gets
\[
X(f)(x(t), y(t)) = A(x(t), y(t)) \frac{\partial f}{\partial x}(x(t), y(t)) + B(x(t), y(t)) \frac{\partial f}{\partial y}(x(t), y(t)),
\]
which substituting (4), gives
\[
X(f)(x(t), y(t))\dot{x}(t) = \frac{\partial f}{\partial y}(x(t), y(t))(−A(x(t), y(t))\dot{y}(t) + B(x(t), y(t))\dot{x}(t)),
\]
that leads to the following valuative formula:
\[
\text{ord}_t \left( X(f)(x(t), y(t)) \right) + \text{ord}_t (\dot{x}(t)) = \text{ord}_t \left( \frac{\partial f}{\partial y}(x(t), y(t)) \right) + \nu_{\Gamma}(\overline{\omega}) - 1,
\]
for \( \overline{\omega} = B(x, y)dx - A(x, y)dy \). It is well-known (see, for example [18]) that
\[
\text{ord}_t \left( \frac{\partial f}{\partial y}(x(t), y(t)) \right) = c + n - 1
\]
where \( c \) is the conductor of \( S_{\Gamma} \). Hence, we get
\[
\text{tang}_{(0, 0)}(X, \Gamma) + (n - 1) = c + (n - 1) + \nu_{\Gamma}(\overline{\omega}) - 1,
\]
that is:
\[
\text{tang}_{(0, 0)}(X, \Gamma) = \nu_{\Gamma}(\overline{\omega}) + c - 1.
\]
Thus, we might define \( \nu_{\Gamma}(X) \coloneqq \text{tang}_{(0, 0)}(X, \Gamma) - c + 1 \) and obtain, in a natural way:
\[
\nu_{\Gamma}(X) = \nu_{\Gamma}(\overline{\omega}).
\]
Dually, we obtain the following formula for the conductor:

**Corollary 2.** Let \( \Gamma \equiv (f = 0) \) be a singular branch at \((\mathbb{C}^2, 0)\) and \( X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} \) any singular vector field. If \( \omega = -B(x, y)dx + A(x, y)dy \) then
\[
c = \text{tang}_{(0, 0)}(X, \Gamma) - \nu_{\Gamma}(w) + 1.
\]

The next result follows from a simple (classical) computation:

**Lemma 5.** Let \( \omega \) be a singular differential form in \((\mathbb{C}^2, 0)\) and \( \pi : X \to (\mathbb{C}^2, 0) \) be the blow-up of \((\mathbb{C}^2, 0)\) with centre \((0, 0)\) with equations \( x = \overline{\omega}, y = \overline{\overline{\omega}} \). Let \( \Gamma \) be a branch (singular or not) at \((\mathbb{C}^2, 0)\) whose tangent cone is not \( x = 0 \). Consider the differential form in \( X \) given by \( \overline{\omega} = (\pi^* \omega)/\overline{\omega} \) (which is the dual form of the pull-back of \( X \) to \( X \)) and the strict transform \( \overline{\Gamma} \) of \( \Gamma \), whose intersection with \( \pi^{-1}(0, 0) \) is \( P \). If \( n \) is the multiplicity of \( \Gamma \) at \((0, 0)\), then
\[
\nu_{\Gamma}(\omega) = \nu_{\overline{\Gamma}}(\overline{\omega}) + n.
\]
Corollary 3. Let $\Gamma$ be an analytic branch at $(\mathbb{C}^2, 0)$ that is not invariant by a singular analytic vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$. Let $\omega = -B(x, y)dx + A(x, y)dy$ be the "dual" differential form of $X$. Then

$$v_T(\omega) = n_{N-1} + \sum_{j=0}^{N-1} n_j$$

where $n_0, n_1, \ldots, n_N$ is the sequence of multiplicities of $\Gamma$ at the points of the path it shares with $X$. It depends only on $\Gamma$ and $N$.

Proof. Let $(P_j)_{j=0}^N$ be the path shared by $X$ and $\Gamma$ and $n_j$ the multiplicity of the strict transform $\Gamma_j$ of $\Gamma$ at $P_j$. For each $j = 0, \ldots, N$, if we denote by $\overline{X}_j$ the pull-back of $X$ to the respective space (so that $\overline{X}_0 = X$) and $\overline{\omega}_j$ its dual form. We have

$$v_{\overline{\Gamma}_j}(\overline{\omega}_j) = v_{\overline{\Gamma}_{j-1}}(\overline{\omega}_{j-1}) - n_{j-1}, \text{ for } j = 1, \ldots, N-1,$$

by Lemma 5. As $\overline{X}_{N-1}$ does not preserve the tangent cone of $\overline{\Gamma}_{N-1}$ (because $P_N$ is not a singular point of $\overline{X}_N$), we may assume that $\overline{\Gamma}_{N-1} = (t^{n_{N-1}}, t^q + h.o.t.)$ with $q > n_{N-1}$. The form $\overline{\omega}_{N-1}$ can be written

$$\overline{\omega}_{N-1} = (ax + by + h.o.t.)dx + (cx + dy + h.o.t.)dy$$

with $a \neq 0$, which gives $v_{\overline{\Gamma}_{N-1}}(\overline{\omega}_{N-1}) = 2n_{N-1}$ and, from Lemma 5, we get

$$v_{\overline{\Gamma}_N}(\overline{\omega}_N) = n_{N-1}$$

and then

$$n_{N-1} = v_{\overline{\Gamma}_N}(\overline{\omega}_N) = v_T(\omega) - n_0 - n_1 - \cdots - n_{N-1}$$

and the result follows. \hfill \Box

3.2. The shared path and Puiseux's expansion. The concept of the path shared by a singular vector field and an analytic branch is deeply related to the Puiseux expansion of the branch and the contact between the branch and the vector field (or the branch and its deformation).

Start with an analytic branch $\Gamma$ at $(\mathbb{C}^2, 0)$ which is not tangent to the $OY$ axis, so that it admits a Puiseux expansion\footnote{We always assume the parametrizations to be irreducible.} of the form

$$\Gamma \equiv \varphi(t) = (t^n, \sum_{i \geq n} a_it^i),$$

where $\Gamma$ is regular if and only if $n = 1$. Let $X$ be a singular vector field at $(\mathbb{C}^2, 0)$.

We need a technical result foremost:

Lemma 6. Let $a(z) = \sum_{i \geq n} a_i(z)t^i$ be a power series with integer exponents such that $n \geq 1$ and each $a_i(z)$ is a holomorphic function in $z$ (each with its own radius of convergence), with $a_n(0) \neq 0$. If $r(z) = \sum_{i \geq 1} r_i(z)t^i$ is such that

$$r(z)^n = a(z),$$

then $r_i(z)$ are also holomorphic functions in $z$ and $r_1(0) \neq 0$.

Proof. The proof is done by induction on $i$ or, what amounts to the same, by the method of indeterminate coefficients. Actually, one can prove that there exist polynomials $P_j(z)$ in $j-1$ variables such that

$$r(z)^n = \left( r_1(z)^n t^n + \sum_{j=2}^{n} (n^j r_1(z)^{n-1} r_j(z) + P_j(r_1(z), \ldots, r_{j-1}(z))) t^{n+j-1} \right)$$

from which the result follows. \hfill \Box
Definition 11. Assume that \( X \) is a singular vector field. Let \( P \) and \( Q \) be the points defined in the divisor of the blow-up of the origin by the tangent cone of \( \Gamma \) and \( x = 0 \) respectively. We say that \( X \) is prepared relatively to \( \Gamma \) if either \( P \) or \( Q \) is a singular point of \( X \).

Remark 1. Given a vector field \( X \) we may assume that it is prepared up to a linear change of coordinates that preserves the tangent cone of \( \Gamma \) at \( (0,0) \), which is assumed to be \( y = 0 \). The preparation guarantees that no curve of the form \( \Gamma_\epsilon \) has the \( OY \) axis as its tangent cone. Given a vector field \( X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \), it is prepared relatively to \( \Gamma \) if and only if \( \frac{\partial a}{\partial y}(0,0) = 0 \) or \( \frac{\partial b}{\partial x}(0,0) = 0 \). The transform \( X \) of a singular vector field is of the form

\[
\overline{X} = xA(x,y)\frac{\partial}{\partial x} + (\mu_0 + \mu_1y + \mu_2y^2 + xB(x,y))\frac{\partial}{\partial y},
\]

and \( X \) is prepared if and only if \( \mu_0 = 0 \) or \( \mu_2 = 0 \).

Consider the deformation \( \Gamma_\epsilon \) of \( \Gamma \) by \( X \).

Proposition 6. Assume that \( X \) is prepared relatively to \( \Gamma \). The deformation \( \Gamma_\epsilon \) admits an irreducible Puiseux parametrization:

\[
\Gamma_\epsilon \equiv \varphi(t) = \left(t^n, \sum_{i \geq n} a_i(t^i)\right).
\]

with \( a_i(\epsilon) \) being an entire function in \( \epsilon \) and \( a_i(0) = a_i \) for all \( i \geq n \).

Proof. Let \( \{\psi_s\}_{s \in \mathbb{C}} \) the one-parameter group associated to \( X \). Let \( (t^n, \sum a_i(t^i)) \) be a Puiseux parametrization. We define the map

\[
(\gamma_1(t), \gamma_2(t)) = \psi_s(\gamma_1(t), \gamma_2(t)).
\]

It is well-defined and holomorphic in a neighborhood of \( t = 0 \). As a consequence \( \gamma_1(t), \gamma_2(t) \) are of the form \( \gamma_1(s,t) = \sum_{j=1}^\infty b_j(s)t^j \) and \( \gamma_2(s,t) = \sum_{j=1}^\infty c_j(s)t^j \) where \( b_j \) and \( c_j \) are entire functions for any \( j \geq 1 \). Since the multiplicity of every curve \( \Gamma_\epsilon \) is equal to \( n \) for \( s \in \mathbb{C} \), all coefficients \( b_j \) and \( c_j \) with \( j < n \) are identically 0. Moreover \( b_n \) is a never vanishing entire function, otherwise the tangent cone of \( \Gamma_\epsilon \) would be \( x = 0 \) for some \( s \in \mathbb{C} \). Lemma \( \ref{lemma} \) implies that there exists \( \beta(s,t) = \sum_{j=1}^\infty \beta_j(s)t^j \) such that \( \beta(s,t)^n = \gamma_1(s,t) \) where \( \beta_j \) is an entire function for any \( j \geq 1 \) and \( \beta_1 \) is never vanishing. Denote by \( (s,\alpha(s,t)) \) the inverse map of \( (s,\beta(s,t)) \). It is well-defined in a neighborhood of \( t \) since \( \beta_1 \) never vanishes. The parametrization that we are looking for is \( (t^n, \gamma_2(s,\alpha(s,t))) \). \( \square \)

As a corollary we obtain the following result.

Corollary 4. For \( \epsilon \) small enough, the deformation \( \Gamma_\epsilon \) admits an irreducible Puiseux parametrization:

\[
\Gamma_\epsilon \equiv \varphi(t) = \left(t^n, \sum_{i \geq n} a_i(t^i)\right),
\]

with \( a_i(\epsilon) \) holomorphic in \( \epsilon \) and \( a_i(0) = a_i \) for all \( i \geq n \).

The algebraic counterpart to the geometric concept of the shared path is the contact exponent:

Definition 12. The contact exponent of a singular vector field \( X \) with an analytic branch \( \Gamma \) at a point \( P \) of a complex analytic surface \( \mathcal{M} \), denoted \( (X,\Gamma)_P \), is the least \( i \) such that \( a_i(\epsilon) \) is not constant in Proposition \( \ref{prop} \) (for any irreducible Puiseux parametrization of \( \Gamma \)).
Remark 2. The contact exponent is independent of the choice of coordinates. Consider a local biholomorphism \( \phi \in \text{Diff}(\mathbb{C}^2,0) \) such that the linear part \( D_0\phi \) at the origin does not send the tangent line to \( \Gamma \) at 0 to the \( OY \) axis. It can be shown that \( (X,\Gamma)(0,0) = (\phi,X,\phi(\Gamma))(0,0) \) by a simple calculation.

Remark 3. Assume \( \Gamma \) has an irreducible Puiseux expansion \( (t^n, at^m + \text{h.o.t.}) \) with \( a \neq 0 \) and \( n < m \). If \( j = (X, \Gamma)(0,0) < m \) then \( j \) must be a multiple of \( n \); otherwise the topological types of \( \Gamma \) and \( \Gamma_j \) would be different, which is impossible because they are analytically equivalent.

One has a formula analogue to that of Lemma 5 which provides the relation between the shared path and the contact exponent:

Lemma 7. Assume \( \Gamma \) is not invariant for \( X \) and let \( \pi : X \to (\mathbb{C}^2,0) \) be the blow-up with centre \((0,0)\) and \( \overline{\Gamma} \) the strict transform of \( \Gamma \) by \( \pi \), which meets \( \pi^{-1}(0,0) \) at \( P \). Let \( \overline{X} \) be pull-back of \( X \) to \( \overline{X} \). Let \( n \) be the multiplicity of \( \Gamma \) at \((0,0)\) and \( \overline{\pi} \) that of \( \overline{\Gamma} \) at \( P \). Then:

- Either \( \overline{X} \) is non-singular at \( P \) and \((X,\Gamma)(0,0) = n \)
- Or \( \overline{X} \) is singular at \( P \) and

\[
(X,\Gamma)(0,0) = (\overline{X},\overline{\Gamma})_P + \overline{\pi}.
\]

Proof. If \( \overline{X} \) is non-singular at \( P \), the result is straightforward as \( X \) does not fix the tangent cone of \( \Gamma \). Assume, then, that \( P \) is singular for \( \overline{X} \).

Take a prepared irreducible Puiseux parametrization of \( \Gamma_j \):

\[
\Gamma_j \equiv \varphi_j(t) = \left( t^n, \sum_{m \leq i < j} a_i t^i + \alpha_j(t) t^j + \text{h.o.t.} \right)
\]

with \( j = (X, \Gamma)(0,0) \), as in Proposition 6. Let \( \pi : X \to (\mathbb{C}^2,0) \) be the blow-up with centre \((0,0)\) with equations \( x = \overline{x}, y = \overline{y} \), for which \( \overline{\Gamma} \) meets \( \pi^{-1}(0,0) \) at \( \overline{y} = 0 \). There are two cases:

- If \( m \geq 2n \), the curve \( \Gamma_j \) has the same Puiseux parametrization as \( \varphi_j \), except that the \( \overline{y} \)-coordinate has all the exponents subtracted by \( n \). The multiplicity of \( \Gamma_j \) (and hence \( \overline{\Gamma} \)) is \( n \) and the result follows.
- If \( n < m < 2n \), then, \( a_m \neq 0 \) and, by Remark 3, we have \( j \geq m \) (otherwise, \( j = n \) and \( \overline{X} \) would not be singular at \( P \)) and we can write

\[
\Gamma_j \equiv \overline{\varphi}_j(t) = (t^n, \sum_{m \leq i < j} a_i t^i + \alpha_j(t) t^j + \text{h.o.t.})
\]

(with either \( j > m \) and \( a_m \neq 0 \) or \( j = m \) and \( a_j(0) \neq 0 \) which is not of irreducible Puiseux type (as \( m - n < n \)). In order to transform it to an irreducible Puiseux parametrization, one needs to extract \( m - n \)-th roots of the second coordinate:

\[
u = \left( \sum_{i=m}^{\infty} a_i(\epsilon) t^{i-n} \right)^{-m-n}.
\]

Notice that such root is a holomorphic function defined in a neighborhood of \((\epsilon, t) = (0,0)\) since \( a_m(0) \neq 0 \). We obtain

\[
t = \sum_{1 \leq i < j - m + 1} a_i u^i + \alpha_j(\epsilon) u^{i-m+1} + \text{h.o.t.}
\]
where \( \alpha_k(0) \neq 0 \) if \( j = m \). From this, an irreducible Puiseux parametrization of \( \Gamma_\varepsilon \) is given by
\[
\Gamma_\varepsilon \equiv \bar{\Gamma}_\varepsilon(u) = \left( \sum_{n \leq i < j - (m-n)} \bar{\pi}_i u^i + \bar{\pi}_j(\varepsilon) u^{j-(m-n)} + h.o.t., u^{m-n} \right).
\]
with \( \bar{\pi}_j(\varepsilon) \) not constant. Hence, \( (X, \Gamma)(0,0) = (\bar{X}, \bar{\Gamma})_p + m - n \) and \( \bar{\pi} \) is, in this case, \( m - n \), which finishes the proof.

\[\Box\]

Lemma 8 states that if \( P_0 = (0,0), P_1, \ldots, P_N \) is the shared path between \( X \) and \( \Gamma \), then
\[
(X, \Gamma)(0,0) = \begin{cases} \nu_0(\Gamma) & \text{if } N = 1 \\ \nu_{P_i}(\Gamma_1) + \overline{(X_1, \Gamma_1)}_{P_i} & \text{otherwise} \end{cases}
\]
where \( \overline{X_1} \) and \( \Gamma_1 \) are, respectively, the pull-back of \( X \) and the strict transform of \( \Gamma \) at \( P_1 \). Hence:

**Corollary 5.** Let \( P_0, \ldots, P_N \) be the shared path between \( X \) and \( \Gamma \). Then
\[
(X, \Gamma)(0,0) = n_{N-1} + \sum_{j=1}^{N-1} n_j
\]
where \( n_0, n_1, \ldots, n_N \) is the sequence of multiplicities of \( \Gamma \) at the points of the path it shares with \( X \). So that the contact order between \( X \) and \( \Gamma \) depends only on \( \Gamma \) and \( N \).

Furthermore, the contact order of a vector field \( X \) with a branch \( \Gamma \) is essentially that of the dual differential form with \( \Gamma \):

**Theorem 3.** Let \( X = A(x,y) \frac{\partial}{\partial x} + B(x,y) \frac{\partial}{\partial y} \) be a singular vector field at \((\mathbb{C}^2,0)\) and \( \Gamma \) be an analytic branch at \((\mathbb{C}^2,0)\) with multiplicity \( n \), not invariant for \( X \). Let \( \omega = -B(x,y)dx + A(x,y)dy \) be the “dual” differential form of \( X \). Then
\[
\nu_\Gamma(\omega) = (X, \Gamma)(0,0) + n.
\]

The result is an immediate consequence of Corollaries 3 and 5.

**Corollary 6** below will essentially provide the analytic classification of plane branches except for Zariski’s invariant, which requires a specific definition. Consider an analytic branch \( \Gamma \) having irreducible Puiseux expansion
\[
\Gamma \equiv \varphi(t) = (x(t), y(t)) = \left( t^n, \sum_{i \geq n} a_i t^i \right)
\]
and let \( X \) be a singular vector field at \((\mathbb{C}^2,0)\) such that \( (X, \Gamma)(0,0) = j \). Let \( \pi : X \to (\mathbb{C}^2,0) \) be the sequence of blow-ups producing the path \( (P_i)^{N}_{i=0} \) shared by \( X \) and \( \Gamma \), where each \( P_i \) (for \( i = 1, \ldots, N \)) belongs to the irreducible component \( E_i \) of the exceptional divisor \( \pi^{-1}(0,0) \). We require some lemmas. The next result can be seen as a corollary of the Poincaré-Hopf formula.

**Lemma 8.** Let \( X \) be a singular analytic vector field at \((\mathbb{C}^2,0)\) and \( \pi : \mathcal{X} \to (\mathbb{C}^2,0) \) a finite sequence of blow-ups whose centres are singular points for each pull-back of \( X \). If \( E \) is an irreducible component of the exceptional divisor in \( \pi^{-1}(0,0) \) which is not composed of singular points of the pull-back \( \bar{X} \) of \( X \) by \( \pi \), then there are exactly two singular points for \( \bar{X} \) in \( E \) counting multiplicities.

**Lemma 9.** If the pull-back \( \bar{X} \) of \( X \) to \( \mathcal{X} \) has a single singularity in \( E_N \), then \( \bar{X}|_{E_N} \) is a constant vector field away from that singular point.
Corollary 6. The last divisor of the shared path.

Proof. The vector field \( \overline{X}_{|\mathbb{C}^2} \) has a singular point with multiplicity 2 by Lemma 8. It is analytically conjugated to \( \partial/\partial z \) where \( z \) is a complex coordinate in the chart \( \mathbb{P}^1 \mathbb{C} \setminus \{ \infty \} \) of \( \mathbb{P}^1 \mathbb{C} \).

\[ \square \]

Lemma 10. With the setting above, assume \( N > 1 \) (or \( (X, \Gamma)_{(0,0)} > n \), which is the same thing). Let \( \tilde{\Gamma} \) be another singular branch at \( (\mathbb{C}^2, 0) \), topologically equivalent to \( \Gamma \), admitting a parametrization

\[ \tilde{\Gamma} \equiv \tilde{\varphi}(t) = (\tilde{x}(t), \tilde{y}(t)) = \left( t^n, \sum_{i \geq n} \tilde{a}_i t^i \right). \]

Let \( (P_n)_{n=0}^N \), \( (\tilde{P}_n)_{n=0}^\tilde{N} \) be the paths shared by \( X \) and \( \Gamma, \tilde{\Gamma} \), respectively. Then: \( \tilde{a}_i = a_i \xi^i \) for some \( \xi \in \mathbb{C} \) with \( \xi^n = 1 \) and any \( n \leq i < j = (X, \Gamma)_{(0,0)} \) if and only if \( N = \tilde{N} \) and also \( P_k = \tilde{P}_k \) for \( k = 0, \ldots, N - 1 \).

Proof. The result is easily proved using an inductive argument similar to that of Lemma 7, as the coefficients in the Puiseux parametrization of a branch are determined by its infinitely near points. Let us clarify the role of the property \( N = \tilde{N} \). The divisor \( E_N \) has exactly a corner point that is necessarily the unique singular point of \( \overline{X} \) in \( E_N \). The condition on the coefficients of \( \Gamma \) and \( \tilde{\Gamma} \) implies that none of these curves meet \( E_N \) in the corner point and hence \( P_N \) and \( \tilde{P}_N \) are regular points of \( \overline{X} \) and \( N = \tilde{N} \).

Next, we see that if \( (X, \Gamma)_{(0,0)} = j \) then the term of order \( j \) can be eliminated from a Puiseux parametrization of \( \Gamma \) as long as \( X \) has a single singularity on the last divisor of the shared path.

Corollary 6. With the same setting, if \( \overline{X} \) has a single singularity in \( E_N \) then \( \Gamma \) is analytically equivalent via a unique local diffeomorphism in the one parameter group generated by \( X \) to a branch \( \tilde{\Gamma} \) such that

\[ \tilde{\Gamma} \equiv \tilde{\varphi}(t) = (\tilde{x}(t), \tilde{y}(t)) = \left( t^n, \sum_{i \geq n} \tilde{a}_i t^i \right) \]

with \( \tilde{a}_i = a_i \) for \( n \leq i < j \) and \( \tilde{a}_j = 0 \).

Proof. Let \( R \) be the corner point of \( E_N \). Let \( \{ \psi_s \}_{s \in \mathbb{C}} \) be the one parameter group associated to \( X \). We denote \( \Gamma_s = \psi_s(\Gamma) \). The Puiseux parametrization of \( \Gamma_s \) is of the form

\[ \Gamma_s \equiv \varphi_s(t) = (x_s(t), y_s(t)) = \left( t^n, \sum_{n \leq i < j} a_i t^i + \sum_{i \geq j} a_j(s) t^i \right). \]

where \( j = (X, \Gamma)_{(0,0)} \) and \( a_j(s) \) is not a constant function. Given \( s, s' \in \mathbb{C} \) such that \( s \neq s' \), the strict transforms of \( \Gamma_s \) and \( \Gamma_{s'} \) pass through different points of \( E_N \setminus \{ R \} \) by Lemma 9. As a consequence the function \( a_j \) is injective and hence is also surjective. Thus there exists a unique \( s_0 \in \mathbb{C} \) such that \( a_j(s_0) = 0 \). The curve \( \tilde{\Gamma} \) is the curve \( \Gamma_{s_0} \).

\[ \square \]

3.3. The moduli problem and holomorphic flows. As an example of the relevance of our tools, we give a solution to Zariski’s moduli problem [15] using flows instead of just analytic diffeomorphisms. We first need some elementary results on the type of singularities arising after a sequence of blow-ups of a singular analytic vector field. Specifically, nilpotent vector fields only become regular on what are called free exceptional divisors. Notice that any vector field of multiplicity at least 2 is nilpotent since it has vanishing linear part. Then we shall compute Zariski’s \( \lambda \) invariant [17] as some minimum of contacts between the branch and vector fields.
In this subsection, we fix a singular vector field $X$ at $(\mathbb{C}^2,0)$ and a chain of blow-ups

$$X = X_N \xrightarrow{\pi_{N-1}} X_{N-1} \xrightarrow{\pi_{N-2}} \ldots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} (\mathbb{C}^2,0)$$

each $\pi_i$ having centre $P_i$ belonging to $E_i = \pi_{i-1}^{-1}(P_{i-1})$, the exceptional divisor corresponding to the blow-up of $P_{i-1}$. We call $X_i$ the pull-back of $X$ to $X_i$, which we assume is singular at $P_i$ for $i = 0, \ldots, N-1$ (writing $X_0 = X$ and $P_0 = P$) and we assume $P_N$ is a non-singular point of $X_N$ in $E_N = \pi_{N-1}^{-1}(P_{N-1})$. We know that all the exceptional divisors $E_1, \ldots, E_N$ are invariant for $X_N$.

**Definition 13.** A divisor $E_i$ is free if either $i = 1$ or $P_{i-1} \in E_k$ implies $k = i - 1$. In other words, if $E_i$ meets only one other exceptional divisor in $X_i$, or what amounts to the same, if $P_{i-1}$ is not the intersection of two exceptional divisors.

We need several technical results:

**Lemma 11.** If $Y$ is a nilpotent singular vector field at $(\mathbb{C}^2,0)$ admitting two transverse non-singular invariant curves then its multiplicity is strictly greater than 1.

**Proof.** Since $Y$ has two transverse non-singular invariant curves, its linear part must be diagonalisable. Since this linear part is nilpotent by hypothesis, it must be zero, i.e. $Y$ has multiplicity at least 2. □

Nilpotent vector fields become regular only at free divisors:

**Lemma 12.** With the previous notation, assume $X$ is nilpotent. Then $E_N$ (the divisor containing $P_N$, point at which $X_N$ is regular) is a free divisor. Even more, there is only one singularity of $X_N$ in $E_N$ and if $N > 1$, it is the intersection of $E_N$ with the only other divisor it meets (actually, $E_{N-1}$).

**Proof.** If $N \leq 2$, then $E_N$ is automatically free and the statement holds. Assume then that $N > 2$. A simple computation shows that $X_i$ has nilpotent linear part at $P_i$ for all $i = 1, \ldots, N-1$. If $E_N$ were not free, then $P_{N-1}$ would belong to $E_{N-1}$ and another $E_k$ for $k \neq N-1$. As all the exceptional divisors are invariant, $P_{N-1}$ would be a singular point for $X_{N-1}$ with two transverse non-singular invariant curves. We know that $X_{N-1}$ has nilpotent linear part, hence $X_{N-1}$ would have multiplicity at least 2, by Lemma 11. This prevents $P_N$ from being regular for $X_N$, as the multiplicity of a singular vector field decreases at most by one after a single blow-up.

The existence of a single singularity in $E_N$ is a consequence of the existence of a single eigenvector for the linear part of $X_{N-1}$ at $P_{N-1}$. If $N > 1$ then the intersection of $E_N$ with the only other divisor it meets (which is, of necessity, $E_{N-1}$) must be a singular point for $X_N$, as there are two invariant varieties through that point. □

A straightforward application of Lemma 12 gives:

**Proposition 7.** Let $\Gamma$ be a branch through $(\mathbb{C}^2,0)$ and $X$ be a nilpotent singular vector field at $(\mathbb{C}^2,0)$ for which $\Gamma$ is not invariant. Assume $(P_i)_{i=0}^N$ is the path shared by $\Gamma$ and $X$. Then $P_N$ is a non-singular point of $E_N$, $X_N$ has a single singular point in $E_N$ and if $N > 1$ then this singular point is $E_N \cap E_{N-1}$.

**Remark 4.** Let $\Gamma$ be an analytic branch with irreducible Puiseux parametrization

$$\Gamma \equiv \varphi(t) = (x(t), y(t)) = \left(t^n, \sum_{i \geq n} a_it^i\right).$$

We may assume $a_n = 0$ up to replacing $\Gamma$ with $\exp \left( -a_n x \frac{\partial}{\partial y} \right)(\Gamma)$. Furthermore, since $(\Gamma, x^k)_{(0,0)} = (x^k \partial/\partial y, \Gamma)_{(0,0)} = kn$ for $k \geq 1$, we may also assume that the
Puiseux parametrization of $\Gamma$ is prepared by conjugating it with diffeomorphisms embedded in the one-parameter groups of the nilpotent vector fields $x^2 \frac{\partial}{\partial y}, x^3 \frac{\partial}{\partial y}, \ldots$ by Corollary 6 and Proposition 7. So in order to transform $\Gamma$ to normal form by using diffeomorphisms embedded in the flows of nilpotent vector fields, we may assume that the Puiseux parametrization is prepared.

And, applying Corollary 6, we get the first elimination criterion (in the sense of Zariski [18] and Hefez-Hernandes [10]):

**Corollary 7.** Let $\Gamma$ be an analytic branch with prepared irreducible Puiseux parametrization $\Gamma \equiv \varphi(t) = (x(t), y(t)) = \left(t^n, \sum_{i \geq m} a_i t^i\right)$ and let $X$ be a nilpotent singular analytic vector field at $(\mathbb{C}^2, 0)$. Assume the contact exponent $j = (X, \Gamma)_{(0,0)}$ between $X$ and $\Gamma$ is greater than $m$. Then $\Gamma$ is analytically equivalent via a diffeomorphism in the holomorphic flow associated to $X$ to a branch $\tilde{\Gamma}$ with parametrization

$$\tilde{\Gamma} \equiv \tilde{\varphi}(t) = (\tilde{x}(t), \tilde{y}(t)) = \left(t^n, \sum_{i \geq m} \tilde{a}_i t^i\right)$$

with $\tilde{a}_i = a_i$ for $i < j$ and $\tilde{a}_j = 0$.

Thus, we can eliminate any finite number of coefficients (at least those up to the conductor) from the Puiseux expansion of a curve $\Gamma$ as long as their exponents correspond to the contact with a nilpotent vector field. In order to give the complete classification, we only need to study what happens if $X$ has multiplicity 1 and if we can eliminate an infinite family of exponents (the tail of the Puiseux expansion) with a single vector field. Previously, though, note that the terms whose exponent belongs to the semigroup of $\Gamma$ can be removed from a Puiseux expansion via a holomorphic flow:

**Corollary 8.** Let $\Gamma$ and $\varphi(t)$ be as in Corollary 7. If $j > m$ is the intersection multiplicity of a singular analytic curve $\Delta$ with $\Gamma$ then there exists a singular vector field $X$, with vanishing linear part, such that $(X, \Gamma)_{(0,0)} = j$. In particular the same conclusion as in Corollary 7 holds (i.e. the term $a_j$ can be eliminated from the parametrization $\varphi$ without affecting the previous ones).

**Proof.** Consider $f \in \mathbb{C}[x,y]$ such that $j = (\Gamma, f)_{(0,0)}$. We obtain that the multiplicity at the origin is greater than 1 since otherwise $(\Gamma, f)_{(0,0)} \leq m$. The vector field $X = f(x,y) \frac{\partial}{\partial y}$ has vanishing linear part at $(0, 0)$. By Theorem 3 this $X$ has contact exponent $(X, \Gamma)_{(0,0)} = j$. Applying Corollary 7 we are done. □

From all the previous discussions, we may assume that, after a finite composition of local diffeomorphisms embedded in flows (including a linear one, intended to make the tangent cone of $\Gamma$ at $(0, 0)$ different from the $OX$ axis), $\Gamma$ has a prepared Puiseux expansion of the form

$$(7) \quad \Gamma \equiv \varphi(t) = (x(t), y(t)) = \left(t^n, \sum_{i \geq m} a_i t^i\right)$$

where $n < m$, $n \nmid m$, $a_m \neq 0$ and if $i$ is in the semigroup associated to $\Gamma$ and $m < i \leq c$ where $c$ is the conductor of $\Gamma$, then $a_i = 0$. We may also assume that $a_i = 0$ if $i \leq c$ is the contact exponent with a nilpotent vector field. We shall deal with the co-final terms later on (it is well known that they may be eliminated with a single analytic diffeomorphism anyway).
Proposition 8. In the conditions of the last paragraph, let $\lambda$ be the least exponent $\lambda > m$ such that $a_\lambda \neq 0$ and $c$ the conductor of $\Gamma$. Assume $\lambda < c$. Let $X$ be a non-nilpotent singular vector field. Then $(X, \Gamma)(0,0) \leq \lambda$. Moreover $(X, \Gamma)(0,0) < \lambda$ implies that $(X, \Gamma)(0,0)$ is of the form $(pn + qm) - n$ where $p \geq 0$, $q \geq 0$ and $m - n \neq (pn + qm) - n \geq n$.

Proof. Let $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$. Write $A(x,y) = a_{10}x + a_{01}y + \overline{A}(x,y)$ and $B(x,y) = b_{10}x + b_{01}y + \overline{B}(x,y)$. Consider $\varphi^*\omega$, where $\omega$ is the dual form of $\omega = -B(x,y)dx + A(x,y)dy$:

$$\varphi^*\omega = -(b_{10}t^n + b_{01}(am^n + a_\lambda t^\lambda + h.o.t.) + \overline{B}(\varphi(t)))nt^{n-1}dt +$$

$$(a_{10}t^n + a_{01}(am^n + a_\lambda t^\lambda + h.o.t.) + \overline{A}(\varphi(t)))\left(mam^n + a_\lambda t^\lambda + h.o.t.\right)dt.$$

Let $j = (X, \Gamma)(0,0)$, so that $j + n = \nu_\Gamma(\omega)$ by Theorem 3. We may assume $j \notin \{n, m\}$ since $n = (2n + 0 - m) - n$ and $m = (n + m) - n$.

We have $b_{10} = 0$ since otherwise $\nu_\Gamma(\omega) = 2n$ and we would have $j = n$. The property $\nu_\Gamma(\omega) < n + m$ implies that $\nu_\Gamma(\omega)$ is a multiple of $n$. In particular we get $\nu_\Gamma(\omega) \neq m$ and $j \neq m - n$. Moreover $j$ is a multiple of $n$ greater or equal than $n$. So we may assume $\nu_\Gamma(\omega) \geq n + m$ from now on. Indeed we obtain $\nu_\Gamma(\omega) > n + m$ and $j > m$ since $j \neq m$. This implies $b_{10}n = a_{10}m$. Since $X$ is non-nilpotent, it follows that $a_{10} \neq 0$ and $b_{01} \neq 0$. The pull-back $\varphi^*\omega$ satisfies

$$t\varphi^*\omega = (g(t) + (a_{10} - b_{01}n)a_\lambda t^{n+\lambda} + O(t^{n+\lambda+1}))dt$$

where $a_{10} - b_{01}n \neq 0$ and the exponents of all monomials with non-vanishing coefficients of the Taylor power series expansion of $g(t)$ belong to the semigroup

$$S' := \{pn + qm : p \geq 0, q \geq 0, p + q \geq 1\}.$$

We claim that $n + \lambda$ does not belong to $S'$. Otherwise $n + \lambda = pn + qm$. If $p \geq 1$ then $\lambda$ belongs to $S$, a contradiction. If $p = 0$ then $q \geq 2$ and $\lambda$ is the contact order of $(y^{q-1}\partial/\partial x, \overline{\Gamma})(0,0)$ of $\Gamma$ with a nilpotent vector field, again a contradiction.

Since $\nu_\Gamma(\omega) \notin S'$ and $a_{10} - b_{01}n \neq 0$, it follows that $m < j \leq \lambda$. Moreover $j < \lambda$ implies $j \in S' - n$. \qed

Theorem 4. In the same conditions as above, if $j > \lambda$ is the contact exponent of $\Gamma$ with an analytic vector field $X$, then the term of order $j$ can be eliminated from a prepared Puiseux expansion via a diffeomorphism in a nilpotent holomorphic flow.

Proof. Let $X$ be a singular vector field such that $(X, \Gamma)(0,0) = \lambda$. Since $j > \lambda$, $X$ is nilpotent by Proposition 3. Apply Corollary 4 to finish the proof. \qed

We end this section with a characterization of Zariski’s $\lambda$ invariant of a plane branch $\Gamma$ in terms of tangency orders (or contact orders) of vector fields with $\Gamma$.

Theorem 5 (Zariski’s $\lambda$ invariant). In the conditions of Proposition 3, let $\lambda$ be the least exponent $\lambda > m$ such that $a_\lambda \neq 0$ and $c$ the conductor of $\Gamma$. Then $\lambda + n = \nu_\Gamma(mydx - nxdy)$ holds. Indeed if $\lambda < c$ then $m$ and $\lambda$ are the unique positive integers $j$ such that $j$ is the contact exponent of a singular vector field with $\Gamma$ but is not the contact exponent of a nilpotent vector field with $\Gamma$. As a consequence, $\lambda$ is an analytic invariant of $\Gamma$ if $\lambda < c$.

Proof. Let $\lambda < c$ be as in the statement and $\omega = mydx - nxdy$. By direct substitution:

$$\nu_\Gamma(\omega) = \text{ord}_t \left( \left( \sum_{i \geq m} mana_i t^{i+n-1} \right) - \left( \sum_{i \geq m} nia_i t^{i+n-1} \right) \right) + 1$$
so that
\[ v_1(\omega) = \left( nx \frac{\partial}{\partial x} + my \frac{\partial}{\partial y}, \Gamma \right)_{(0,0)} = \text{ord}_t((mn - n\lambda)a_m^\lambda n^{-1} + h.o.t.) + 1 = \lambda + n, \]
which gives the first part of the statement. Moreover \( m \) is also a contact exponent since \( m = (x \partial / \partial x, \Gamma)_{(0,0)} \). It can not be expressed as a contact with a nilpotent vector field, since the coefficient of \( t^m \) in \( g(t) \) can not be erased (Corollary \[4\]).

Assume that there exists \( j \not\in \{m, \lambda \} \) satisfying the hypotheses. We obtain \( j < \lambda \) and \( j + n = (pn + qm) \geq 2n \) for some \( p \geq 0 \) and \( q \geq 0 \) with \( (p, q) \not\in \{(1,0), (0,1), (1,1)\} \) by Proposition \[5\]. The vector field \( x^{p-1} y^q \partial / \partial y \) is nilpotent if \( p \geq 1 \) since \( (p, q) \not\in \{(1,0) \} \) and \( (p, q) \not\in \{(1,1) \} \). It satisfies \( x^{p-1} y^q \partial / \partial y, \Gamma \) \((0,0) = j \) and so we get a contradiction. Analogously if \( q \geq 1 \) the vector field \( x^p y^{q-1} \partial / \partial x \) is nilpotent and \( (x^p y^{q-1} \partial / \partial x, \Gamma)_{(0,0)} = j \) holds, providing a contradiction. \[ \square \]

4. Analytic classes and their completeness

Now we focus on whether curves in the same class of analytic conjugacy of a given plane branch are conjugated by local diffeomorphisms in a one-parameter group. Let us give some definitions.

**Definition 14.** We say that \( \rho(t) \) (where \( \rho(t) \) belongs to the maximal \( m_1 \) ideal of \( \mathbb{C}[t] \)) is a formal diffeomorphism if its linear part is non-vanishing (or in other words if \( \rho(t) \in m_1 \setminus m_2^2 \)). If, in addition to the previous properties, \( \rho(t) \) belongs to \( \mathbb{C}[t] \) then is a local biholomorphism defined in the neighborhood of the origin by the inverse function theorem.

We say that \( \psi(x, y) = (a(x, y), b(x, y)) \in m \times m \) is a formal diffeomorphism and we denote \( \psi \in \text{Diff}(\mathbb{C}^2, 0) \) if the linear part of \( \psi \) at the origin is a linear isomorphism.

**Definition 15.** The Krull topology for formal diffeomorphisms or vector fields is defined by considering them as \( n \)-uples of formal power series and the induced product topology in \( \mathbb{C}[x, y]^n \) (cf. Definition \[4\]).

**Definition 16.** Let \( \psi \) be a formal diffeomorphism. We say that \( \psi \) is **unipotent** if its linear part is a unipotent linear map.

**Definition 17.** The time 1 flow \( \text{exp}(X) \) of a singular vector field \( X \) is
\[ \text{exp}(X)(x, y) = \left( \sum_{j=0}^{\infty} \frac{X^j(x)}{j!}, \sum_{j=0}^{\infty} \frac{X^j(y)}{j!} \right). \]
Given a formal vector field \( X \) we use the previous formula to define the formal diffeomorphism \( \text{exp}(X) \). The formula is well-defined: indeed, if \( (X_k)_{k \geq 1} \) is a sequence of vector fields that converges to \( X \) in the \( m \)-adic topology then \( \text{exp}(X_k) \) converges to \( \text{exp}(X) \) in the Krull topology when \( k \to \infty \).

First let us provide an example of a complete class.

**Proposition 9.** The class of analytic conjugacy of all smooth plane branches is complete.

**Proof.** Consider two smooth curves \( \Gamma, \Gamma' \). Up to a change of coordinates we may assume \( \Gamma \equiv (y = 0) \) and that \( \Gamma' \) is not tangent to \( x = 0 \). Thus, \( \Gamma \) and \( \Gamma' \) admit Puiseux parametrizations \((t, 0)\) and \((t, a(t))\), respectively. As a consequence, the local diffeomorphism \( \psi(x, y) = (x, y + a(x)) \) conjugates \( \Gamma \) and \( \Gamma' \). Since \( \psi = \text{exp}(a(x) \partial / \partial y) \) we are done. \[ \square \]

The following theorem implies Theorem \[4\].
Theorem 6. Let \( \Gamma \) and \( \Gamma' \) be two plane branches that are conjugated by the exponential \( \exp(\hat{X}) \) of a singular formal vector field. Then they are conjugated by a local diffeomorphism embedded in the flow of a singular holomorphic vector field \( X \). Moreover if \( X \) is nilpotent we may assume that \( X \) is nilpotent.

Proof. Assume \( \Gamma \neq \Gamma' \) since the result is trivial otherwise. Up to a linear change of coordinates we may assume that none of the tangent cones of the curves \( \Gamma \) and \( \Gamma' \) is the \( \mathcal{O}Y \) axis. The curves \( \Gamma \) and \( \Gamma' \) have Puiseux parametrizations \( \alpha(t) = (t^n, \alpha(t)) \) and \( \beta(t) = (t^n, b(t)) \) respectively where \( n \) is the common multiplicity at the origin. By hypothesis, there exists \( \rho \in \text{Diff}(\mathbb{C}, 0) \) such that \( (\exp(\hat{X}) \circ \alpha)(t) \equiv (\beta \circ \rho)(t) \).

Write \( \hat{X} = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y} \). Consider a sequence \( (X_k)_{k \geq 1} \) of singular vector fields that converge to \( \hat{X} \) in the \( \mathbb{m} \)-adic topology. For instance this can be obtained by defining \( X_k = A_k(x, y)\frac{\partial}{\partial x} + B_k(x, y)\frac{\partial}{\partial y} \) where \( A_k \) (resp. \( B_k \)) is the polynomial of degree less or equal than \( k \) such that \( \hat{A} - A_k \in \mathbb{m}^{k+1} \) (resp. \( \hat{B} - B_k \in \mathbb{m}^{k+1} \)) for \( k \geq 1 \). Analogously we choose a sequence \( (\rho_k)_{k \geq 1} \) in \( \text{Diff}(\mathbb{C}, 0) \) converging to \( \rho \) in the Krull topology. We define the curve \( \Gamma_k = \exp(X_k)(\Gamma) \) and denote \( (x_k(t), y_k(t)) = (\exp(X_k) \circ \alpha \circ \rho_k^{-1})(t) \) for \( k \geq 1 \). The sequence \( (x_k(t), y_k(t)) \) converges to \( (t^n, b(t)) \) in the Krull topology. Consider \( \sigma_k(t) \) the holomorphic function such that \( \sigma_k(t)^{\ast} \equiv x_k(t) \) and \( \sigma_k'(0) = 1 \) for \( k \gg 1 \). Since \( (\sigma_k')'(0) \neq 0 \), it is a local diffeomorphism and its inverse \( \sigma_k^{-1} \) exists. The sequence \( (\sigma_k)_{k \geq 1} \) converges to \( t \) in the Krull topology. Thus \( (t^n, b_k(t)) := (x_k, y_k) \circ \sigma_k^{-1}(t) \) is a parametrization of \( \Gamma_k \) that converges to \( (t^n, b(t)) \) in the Krull topology when \( k \to \infty \).

Since \( \Gamma \) and \( \Gamma_k \) are conjugated by a local diffeomorphism contained in a one-parameter flow, it suffices to show that for fixed \( k \gg 1 \) there exists a local diffeomorphism \( \theta \in \text{Diff}(\mathbb{C}^2, 0) \) such that \( \theta(\Gamma) = \Gamma \) and \( \theta(\Gamma_k) = \Gamma_k \). Indeed then \( \exp(\theta^{\ast}X_k)(\Gamma) = \Gamma' \). Moreover, if \( \hat{X} \) is nilpotent then \( X_k \) is nilpotent for any \( k \geq 1 \), since \( \hat{X} \) and \( X_k \) have the same linear part at \( (0, 0) \) and \( \theta^\ast X_k \) is nilpotent as a conjugate of \( X_k \).

Thus, we need to prove that there exists \( \theta \) with

\[
\begin{align*}
(\theta \circ \alpha)(t) &\equiv \alpha(t) \\
\theta(t^n, b(t)) &\equiv (t^n, b_k(t)).
\end{align*}
\]

Let \( h(x, y) = 0 \) be a local (irreducible) equation of \( \Gamma \) and define

\[\theta(x, y) = (x, y + h(x, y)\gamma(x, y))\]

If we prove that there exists a holomorphic function \( \gamma(x, y) \) for which the conditions on \( \theta \) are satisfied, we are done. The fact that \( \theta(\alpha(t)) \equiv \alpha(t) \) is obvious by construction. The other condition, \( \theta(t^n, b(t)) \equiv (t^n, b_k(t)) \) is equivalent to

\[\gamma(t^n, b(t)) \equiv \frac{b_k(t) - b(t)}{h(t^n, b(t))} \tag{8} \]

The denominator \( h(t^n, b(t)) \) is not identically 0 since \( \Gamma \neq \Gamma' \). The right hand side of Equation (8) converges to 0 in the Krull topology when \( k \to \infty \). Thus there exists a solution of Equation (8) for some \( k > 1 \), which finishes the proof. \( \square \)

Corollary 9. Let \( \Gamma, \Gamma' \) be two plane branches conjugated by a unipotent formal diffeomorphism \( \psi \in \text{Diff}(\mathbb{C}^2, 0) \). Then \( \Gamma \) and \( \Gamma' \) are conjugated by a local diffeomorphism embedded in a one-parameter group generated by a nilpotent vector field.

We will use the next well-known result.

Remark 5 (cf. [6, 12]). The exponential provides a bijection between the set of formal nilpotent singular vector fields and the set of unipotent formal diffeomorphisms. Moreover, it particularizes to a bijection between the Lie algebra of
formal vector fields with vanishing linear part at the origin and the group of formal
diffeomorphisms with identity linear part.

Corollary 9 is an immediate consequence of Theorem 6 and Remark 5.

**Corollary 10.** Let \( \Gamma \) and \( \Gamma' \) be two plane branches in the same class of analytic
conjugacy. There exists a nilpotent vector field \( X \) and a linear vector field \( Y \) such that
\((\exp(Y) \circ \exp(X))(\Gamma) = \Gamma'\).

**Proof.** Let \( \psi \in \text{Diff}(\mathbb{C}^2,0) \) such that \( \psi(\Gamma) = \Gamma' \). Then we have \( \psi = L \circ \sigma \) where
\( L \) is the linear part of \( \psi \) at the origin and the linear part of \( \sigma \) at the origin is the
identity. Denote \( \overline{\Gamma} = \sigma(\Gamma) \). We have \( L(\overline{\Gamma}) = \Gamma' \). There exists a nilpotent vector field \( X \) such that \( \exp(X)(\overline{\Gamma}) = \overline{\Gamma} \) by Corollary 9. Moreover, \( L \) is of the form \( \exp(Y) \)
for some linear vector field. Therefore we obtain \((\exp(Y) \circ \exp(X))(\Gamma) = \Gamma'\). \( \square \)

We have obtained the analytic reduction of holomorphic branches \([10]\) to short
parametrizations.

**Definition 18.** Let \( \Gamma \) be a germ of plane branch. We denote by \( \Lambda \) the set of
contact exponents between \( \Gamma \) and singular vector fields. Notice that \( \Lambda + n \) is the
set of orders of contact of Kähler differentials with \( \Gamma \) by Theorem 3.

**Corollary 11.** Let \( \Gamma \) be a branch at \((\mathbb{C}^2,0)\) with prepared irreducible Puiseux
parametrization
\[ \Gamma \equiv \varphi(t) = (t^n, \sum_{i \geq m} a_it^i). \]

Let \( \lambda \) be its Zariski invariant (or \( \lambda = \infty \)) and \( c > \lambda \) the conductor of the semigroup
associated to \( \Gamma \). There is a nilpotent singular vector field \( X \) such that
\[ \exp(X)(\Gamma) \equiv \overline{\varphi}(t) = (t^n, a_m t^m + a_\lambda t^\lambda + \sum_{i > \lambda} \overline{a}_i t^i) \]
with \( \overline{a}_i = 0 \) for \( i \in \Lambda \).

A parametrization like in Corollary 11 is called, a short parametrization.

**Proof.** In order to simplify \( \Gamma \) we remove step by step coefficients of \( t^j \) in the second
component of the Puiseux parametrization of \( \Gamma \) for some \( j > n \). The normalizing map is of the form \( \exp(X_j) \) where \( (X_j, \Gamma)_{(0,0)} = j \) and \( X_j \) is a nilpotent singular vector field by Corollary 9 and Theorem 5. Indeed it is easy to see that we may
assume that \( X_j \to 0 \) when \( j \to \infty \) in the \( m \)-adic topology. Moreover the tangent
cone of \( \Gamma \) defines a singular point \( P \) of \( (X_j) \) in the divisor \( E_1 \) of the blow-up of the
origin for \( j > n \), since otherwise \( n < j = (X_j, \Gamma)_{(0,0)} = n \). We deduce that \( \exp(X_j) \)
is a unipotent diffeomorphism whose linear part has matrix
\[ \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix} \]
where \( c_j = 0 \) if \( j \gg 1 \). The limit of the composition of these exponentials, in the
appropriate order, is a well-defined formal unipotent diffeomorphism \( \psi \in \text{Diff}(\mathbb{C}^2,0) \)
conjugating \( \Gamma \) with a curve with Puiseux parametrization of the form \( \overline{\varphi}(t) \). The
result is a consequence of Corollary 9. \( \square \)

**Remark 6.** The reduction to normal form in \([10]\) is obtained via the action of
unipotent diffeomorphisms. We have just restated this fact in the context of holomorphic
flows.

The expression of Corollary 11 can be simplified further by means of another
flow (corresponding to a linear change of coordinates and a change of parameter):
Lemma 13 ([18]). A branch $\Gamma$ whose short parametrization is

$$\Gamma \equiv \varphi(t) = (t^n, a_m t^m + a_\lambda t^\lambda + \sum_{i>\lambda} a_i t^i)$$

is analytically equivalent to

$$\Gamma' \equiv (t^n, t^m + t^\lambda + \sum_{i>\lambda} a_i t^i)$$

where there exist $u, v \in \mathbb{C}^*$ such that $\overline{a}_i = v^m u^{-i} a_i$.\[\square\]

Proof. We define $\psi(x, y) = (u^n x, v^m y)$ for some $u, v \in \mathbb{C}^*$ to be specified later on. We have

$$(\psi \circ \varphi)(t) = (u^n t^n, v^m a_m t^m + v^m a_\lambda t^\lambda + \sum_{i>\lambda} v^m a_i t^i).$$

Define the parameter $s = ut$. The curve $\psi(\Gamma)$ has parametrization

$$(s^n, v^m u^{-m} a_m s^m + v^m u^{-\lambda} a_\lambda s^\lambda + \sum_{i>\lambda} v^m u^{-i} a_i s^i).$$

It suffices to consider $u, v \in \mathbb{C}^*$ such that $v^m u^{-m} = a_m^{-1}$ and $v^m u^{-\lambda} = a_\lambda^{-1}$.\[\square\]

Combining Corollary 11 and Lemma 13 we obtain the following result.

Corollary 12. Let $\Gamma$ be a singular branch in $(\mathbb{C}^2, 0)$ having conductor $c$. Let $(x, y)$ be a local system of coordinates. There exist a local diffeomorphism $\psi$ embedded in the flow of a nilpotent vector field, a linear map $G$ and a reparametrization $\tau \in \text{Diff}(\mathbb{C}, 0)$ such that

$$(G \circ \psi)(\Gamma) \equiv G \circ \psi \circ \varphi \circ \tau)(t) = \left(t^n, t^m + t^\lambda + \sum_{\lambda<i<c} a_i t^i\right)$$

where $\varphi(t)$ is the parametrization of $\Gamma$ with $a_i = 0$ if $i < c$ and $i \in \Lambda \setminus \{\lambda\}$.

Proof. There exists a linear automorphism $H(x, y)$ such that the tangent cone to $\Gamma^* := H(\Gamma)$ at the origin is the axis $y = 0$. There exists a local diffeomorphism $J(x, y) = (x, y + c(x))$ for some $c(x) \in \mathbb{C}\{x\}$ of vanishing order at least 2 such that $J(\Gamma')$ has a prepared irreducible Puiseux parametrization. We apply Corollary 11 to $J(\Gamma')$ to obtain a unipotent diffeomorphism $\phi \in \text{Diff}(\mathbb{C}^2, 0)$ such that $\phi(J(\Gamma'))$ has a short parametrization. Finally we apply Lemma 13 to $\phi(J(\Gamma'))$ to obtain a linear isomorphism $K$ such that $\Gamma'' := K(\phi(J(\Gamma')))$ has the desired parametrization. The diffeomorphism $\phi \circ J$ is unipotent since the linear part $D_0 J$ of $J$ at the origin is the identity map. The conjugate $H^{-1} \circ (\phi \circ J) \circ H$ of $\phi \circ J$ is a unipotent diffeomorphism $\rho \in \text{Diff}(\mathbb{C}^2, 0)$ and then we obtain $\Gamma'' = (G \circ \rho)(\Gamma)$ where $G = K \circ H$ is a linear map. Since $\Gamma$ and $\rho(\Gamma)$ are conjugated by a unipotent local diffeomorphism, it follows that they are conjugated by a local diffeomorphism $\psi$ embedded in the flow of a nilpotent vector field. We obtain $\Gamma'' = (G \circ \psi)(\Gamma)$.\[\square\]

The parametrization provided by Corollary 12 is called a canonical parametrization by Zariski [18] and the normal form of $\Gamma$ by Hefez-Hernandes [19]. We shall use the latter terminology. Moreover, if $\Gamma$ is another branch whose normal form has coefficients $\overline{a}_i$, one can prove (see [18] and [19]) that they are analytically equivalent if and only if there exists $u$ such that $u^{\lambda - m} = 1$ and $\overline{a}_i = u^{i - m} a_i$, which describes the complete moduli of $\Gamma$.
5. Non-complete analytic classes

We provide examples of non-complete analytic classes. Whether or not a single formal diffeomorphism is embedded in the flow of a formal singular vector field is deeply related to the spectrum of its linear part and more precisely to the resonances among its eigenvalues. For the sake of completeness we recall these concepts along with some results. We work in dimension 2 because that is the case we are interested in but the results concerning resonances are valid for any dimension (cf. [11]).

**Definition 19.** Consider a formal singular vector field $X$ whose linear part is in Jordan normal form, in particular $X$ is of the form

$$X = \left( \lambda_1 x + \delta y + \sum_{i+j \geq 2} a_{ij} x^i y^j \right) \frac{\partial}{\partial x} + \left( \lambda_2 y + \sum_{i+j \geq 2} b_{ij} x^i y^j \right) \frac{\partial}{\partial y}.$$

We say that the monomial $x^i y^j \partial / \partial x$ with $i \geq 0$, $j \geq 0$, $i + j \geq 1$ and $(i,j) \neq (0,0)$ is resonant if $i \lambda_1 + j \lambda_2 = \lambda_1$. Analogously we say that the monomial $x^i y^j \partial / \partial y$ with $i \geq 0$, $j \geq 0$, $i + j \geq 1$ and $(i,j) \neq (0,1)$ is resonant if $i \lambda_1 + j \lambda_2 = \lambda_2$.

**Definition 20.** Consider a formal diffeomorphism $\psi$ whose linear part is in Jordan normal form, in particular $\psi$ is of the form

$$\psi(x,y) = \left( \lambda_1 x + \delta y + \sum_{i+j \geq 2} a_{ij} x^i y^j, \lambda_2 y + \sum_{i+j \geq 2} b_{ij} x^i y^j \right).$$

We say that the monomial $x^i y^j e_1 := (x^i y^j, 0)$ with $i \geq 0$, $j \geq 0$, $i + j \geq 1$ and $(i,j) \neq (1,0)$ is resonant if $\lambda_1^2 \lambda_2^2 = \lambda_1$. Analogously we say that the monomial $x^i y^j e_2 := (0, x^i y^j)$ with $i \geq 0$, $j \geq 0$, $i + j \geq 1$ and $(i,j) \neq (0,1)$ is resonant if $\lambda_1^2 \lambda_2 = \lambda_2$. A formal diffeomorphism is non-resonant if there are no resonant monomials.

**Remark 7.** The property of being non-resonant depends only on the eigenvalues of the linear part.

The next result is Poincaré’s linearisation map for formal diffeomorphisms. As is customary, we denote $D_0 \psi$ the linear part of a local diffeomorphism $\psi$.

**Proposition 10** (cf. [11] Theorem 4.21). Let $\psi \in \hat{\text{Diff}}(\mathbb{C}^2, 0)$ be a non-resonant formal diffeomorphism. Then $\psi$ is conjugated by a formal diffeomorphism to $(x,y) \mapsto (\lambda_1 x, \lambda_2 y)$ where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the linear part $D_0 \psi$ of $\psi$ at $(0,0)$.

**Corollary 13.** Let $\psi \in \hat{\text{Diff}}(\mathbb{C}^2, 0)$ be a non-resonant formal diffeomorphism. Then there exists a formal singular vector field $X$ such that $\psi = \exp(X)$.

**Proof.** The formal diffeomorphism $\psi$ is formally conjugated to a linear diagonal map by Proposition 10. Since the latter map is embedded in the flow of a singular vector field, it follows that $\psi$ is embedded in the flow of a formal vector field. □

Let us consider the problem of embedding resonant diffeomorphisms in formal flows. Let $\psi \in \text{Diff}(\mathbb{C}^2, 0)$ and assume for simplicity that $(D_0 \psi)(x,y) = (\lambda_1 x, \lambda_2 y)$. The equation $\psi = \exp(X)$ implies $D_0 \psi = \exp(D_0 X)$. Notice that if $\lambda_1 \neq \lambda_2$ then the choice of the eigenvalues $\lambda_1$, $\lambda_2$ determines completely $D_0 X$.

**Definition 21.** Consider the above setting. We say that a resonance $x^i y^j e_1$ (resp. $x^i y^j e_2$) of $\psi$ is strong if the monomial $x^i y^j \partial / \partial x$ (resp. $x^i y^j \partial / \partial y$) is a resonant monomial of the vector field $\lambda_1 x \partial / \partial x + \lambda_2 y \partial / \partial y$, i.e. if $\lambda_1^2 \lambda_2^2 = \lambda_1$ and $i \log \lambda_1 + j \log \lambda_2 = \log \lambda_1$ (resp. $\lambda_1^2 \lambda_2 = \lambda_2$ and $i \log \lambda_1 + j \log \lambda_2 = \log \lambda_2$).

A resonance of $\psi$ that is not strong will be called weak.
We will use the next particular case of [14 Proposition 1.5].

**Proposition 11.** Let \( \psi \in \hat{\text{Diff}}(\mathbb{C}^2, 0) \) be such that \((D_0\psi)(x, y) = (\lambda_1 x, \lambda_2 y)\). Let \( B : \mathbb{C}^2 \to \mathbb{C}^2 \) be a linear map such that \( \exp(B) = D_0\psi \). Assume that \( j^k\psi = D_0\psi + f_k \) where both components of \( f_k \) are homogeneous polynomials of degree \( k \). Furthermore assume that \( f_k \) contains non-vanishing weakly resonant monomials. Then \( \psi \) is not embedded in the flow of any formal vector field \( X \) such that \( D_0X = B \).

We can return to the problem of determining non-complete classes. The results regarding completeness of analytic classes \( \mathcal{C} \) depend on the topology that we consider for the infinitely dimensional space \( \mathcal{C} \). First, we see that in some sense being connected by a geodesic is a dense property.

**Proof of Proposition 3.** Let \( \psi \) be a local diffeomorphism conjugating \( \Gamma \) and \( \Gamma' \). Up to a linear change of coordinates we may assume that the linear part \( D_0\psi \) of \( \psi \) at the origin is in Jordan normal form, in particular its matrix is of the form

\[
\begin{pmatrix}
u & w \\ 0 & \nu
\end{pmatrix}
\]

where \( u, v \in \mathbb{C}^* \). Consider the family \( \sigma_\epsilon(x, y) = (e^{\epsilon a} x, e^{\epsilon b} y) \) for some \( a, b \in \mathbb{C} \) that are linearly independent over \( \mathbb{Q} \). The map \( \sigma_\epsilon \) converges to \( \text{Id} \) when \( \epsilon \to 0 \). Let us define the family \((\Gamma'_\epsilon)'\) by \( \Gamma'_\epsilon = (\sigma_\epsilon \circ \psi)(\Gamma) \). The map \( D_0(\sigma_\epsilon \circ \psi) \) has eigenvalues \( u e^{\epsilon a} \) and \( v e^{\epsilon b} \).

Denote \( F_{p,q}(\epsilon) = u^p v^q e^{(ap+bq)\epsilon} - 1 \) and \( T_{p,q} = F_{p,q}^{-1}(0) \) for \( (p, q) \in \mathbb{Z} \times \mathbb{Z} \). Resonances between the eigenvalues of \( D_0(\sigma_\epsilon \circ \psi) \) are obtained when there exists \( (p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\} \) such that \((u e^{\epsilon a})^p (v e^{\epsilon b})^q = 1 \). This equation is equivalent to \( \epsilon \in T_{p,q} \). Since \( ap+bq \neq 0 \) the function \( F_{p,q} \) is not constant and \( T_{p,q} \) is a countable closed set for any \( (p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\} \). We deduce that \( T = \bigcup_{(p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}} T_{p,q} \) is countable and hence there exists \( \epsilon_0 \in \mathbb{C}^* \) such that \( \{ t \in \mathbb{R}^* : t^\epsilon = 0 \} \). We define the path \( \gamma : [0, \infty) \to \mathbb{C} \) by \( \gamma(t) = \epsilon_0 \). The map \( \sigma_\epsilon \circ \psi \) is embedded in the flow of a formal vector field for any \( \epsilon \not\in T \) by Corollary 13. Therefore \( \Gamma \) and \( \Gamma'_\epsilon \) are connected by a geodesic for any \( \epsilon \in (0, \infty) \) by Theorem 6. \( \square \)

Let us show that the analytic class \( \mathcal{C}_0 \) of the plane branch \( \Gamma_0 \) with Puiseux parametrization \((t^6, t^7 + t^{10} + t^{11})\) is non-complete. First, we study the stabilizer group \( \text{Stab}(\Gamma_0) = \{ \psi \in \hat{\text{Diff}}(\mathbb{C}^2, 0) : \psi(\Gamma_0) = \Gamma_0 \} \) of \( \Gamma_0 \).

**Lemma 14.** The linear part at the origin of any element \( \psi \) of \( \text{Stab}(\Gamma_0) \) is the identity map.

**Proof.** The linear part \( D_0\psi \) is a map of the form \((x, y) \mapsto (ax + by, cx + dy)\). Since \( D_0\psi \) preserves the tangent cone of \( \Gamma_0 \), we deduce \( c = 0 \). In particular \( ad \neq 0 \) because \( \psi \) is a formal diffeomorphism. We have

\[
\psi(t^6, t^7 + t^{10} + t^{11}) \equiv (at^6 + bt^7 + bt^{10} + bt^{11} + O(t^{12}), dt^7 + dt^{10} + dt^{11} + O(t^{12})).
\]

Consider a formal power series \( \sigma(t) \) such that \( \sigma(t)^6 \equiv (x \circ \psi)(t^6, t^7 + t^{10} + t^{11}) \). It must admit the expression \( \sigma(t) \equiv a^{1/6}t + (b/6)a^{-5/6}t^2 + O(t^3) \). Moreover, it is a formal diffeomorphism in one variable and its inverse \( \sigma^{-1}(t) \) satisfies \( \sigma^{-1}(t) \equiv a^{-1/6}t - (b/6)a^{-4/3}t^2 + O(t^3) \). A simple calculation leads us to

\[
\psi(t^6, t^7 + t^{10} + t^{11}) \circ \sigma^{-1}(t) \equiv \left(t^6, da^{-7/6}t^7 - \frac{7}{6} bda^{-7/3}t^8 + O(t^9) \right).
\]

Since \( \psi \) belongs to \( \text{Stab}(\Gamma_0) \), \( bda^{-7/3} \) vanishes. We deduce \( b = 0 \) as a consequence of \( ad \neq 0 \). Thus the formal diffeomorphisms \( \sigma \) and \( \sigma^{-1} \) are of the form \( t \mapsto a^{1/6}t + O(t^7) \) and \( t \mapsto a^{-1/6}t + O(t^7) \) respectively. We obtain

\[
\psi(t^6, t^7 + t^{10} + t^{11}) \circ \sigma^{-1}(t) \equiv (t^6, a^{-7/6}dt^7 + a^{-10/6}dt^{10} + a^{-11/6}dt^{11} + O(t^{12})).
\]
Since $\psi(\Gamma_0) = \Gamma_0$ there exists $\xi \in \mathbb{C}$ such that $\xi^6 = 1$ and $a^{-7/6}d = \xi^7$, $a^{-11/6}d = \xi^{10}$ and $a^{-11/6}d = \xi^{11}$. We get $a^{1/6} = \xi^{-1}$ by dividing the last two equations and then $a = (a^{1/6})^6 = \xi^{-6} = 1$. By plugging $a^{-1/6} = \xi$ into $a^{-7/6}d = \xi^7$, we get $d = 1$. Hence $G_0\psi$ is the identity map. \hfill \Box

**Proposition 12.** Let $X$ be a formal vector field that preserves $\Gamma_0$. Then $X$ has vanishing second jet.

**Proof.** Since $X$ preserves $\Gamma_0$, it follows that the time $s$ flow $\exp(sX)$ of $X$ preserves $\Gamma_0$ for any $s \in \mathbb{C}$. All the formal diffeomorphisms $\exp(sX)$ in the one-parameter group of $X$ have identity linear part at the origin by Lemma 14. In particular $X$ has vanishing linear part. We write

$$X = \left( \sum_{i+j \geq 2} a_{ij} x^i y^j \right) \frac{\partial}{\partial x} + \left( \sum_{i+j \geq 2} b_{ij} x^i y^j \right) \frac{\partial}{\partial y}. $$

Consider the dual form

$$\omega = - \left( \sum_{i+j \geq 2} b_{ij} x^i y^j \right) dx + \left( \sum_{i+j \geq 2} a_{ij} x^i y^j \right) dy.$$

Since $X$ preserves $\Gamma_0$, it follows that $(t^6, t^7 + t^{10} + t^{11})^*\omega = 0$. We have

$$\nu_\Gamma(x^2 dx) = 18, \quad \nu_\Gamma(x y dx) = 19, \quad \nu_\Gamma(y^2 dx) = 20, \quad \nu_\Gamma(x^2 dy) = 19, \quad \nu_\Gamma(x y dy) = 20, \quad \nu_\Gamma(y^2 dy) = 21$$

and $\nu_\Gamma(x^i y^j dx) \geq 24 \leq \nu_\Gamma(x^i y^j dy)$ for $i + j \geq 3$. Since $(t^6, t^7 + t^{10} + t^{11})^*\omega = 0$, we deduce $b_{20} = 0$. We get

$$-6b_{11}t^{11}(t^7 + t^{10} + t^{11}) - 6b_{02}t^5(t^7 + t^{10} + t^{11})^2 + a_{20}t^{12}(7t^6 + 10t^9 + 11t^{10}) + a_{11}t^8(t^7 + t^{10} + t^{11})(7t^6 + 10t^9 + 11t^{10}) + a_{02}(t^7 + t^{10} + t^{11})^2(7t^6 + 10t^9 + 11t^{10}) + O(t^{23}) = 0.$$

We write the linear system of equations satisfied by the coefficients of $t^{18}, t^{19}, t^{20}, t^{21}$ and $t^{22}$:

$$
\begin{align*}
-6b_{11} &+ 7a_{20} = 0 \\
-6b_{02} &+ 7a_{11} = 0 \\
-6b_{11} &+ 10a_{02} = 0 \\
-6b_{11} &- 12b_{02} + 11a_{20} + 17a_{11} = 0.
\end{align*}
$$

The matrix of the system is regular, hence $b_{11} = b_{02} = a_{20} = a_{11} = a_{02} = 0$. In particular $X$ has a vanishing second jet. \hfill \Box

**Proposition 13.** Let $\psi \in \text{Stab}(\Gamma_0)$. Then $\psi$ and the identity map have the same second jet.

**Proof.** The linear part of $\psi$ is the identity map by Lemma 14. Thus $\psi$ is of the form $\exp(X)$ for some unique formal nilpotent vector field $X$ (in fact $X$ has vanishing linear part) by Remark 5. Let $f = 0$ be an irreducible equation of $\Gamma_0$. Notice that $f \circ \exp(sX) = \sum_{j=0}^\infty \frac{X^j(f)}{j!}$ by Taylor’s formula and that $X^j(f) \in m^{j+1}$ for any $j \geq 1$. Therefore $f \circ \exp(sX)$ belongs to $\mathbb{C}[s][[x, y]]$ and then

$$G(s, t) := f \circ \exp(sX) \circ (t^6, t^7 + t^{10} + t^{11})$$

belongs to $\mathbb{C}[s][[t]]$. Moreover $G(s, t)$ vanishes for $s \in \mathbb{C}$ since $\{\exp(sX) : s \in \mathbb{Z}\}$ is the cyclic group $\langle \psi \rangle$ and $\langle \psi \rangle$ is contained in $\text{Stab}(\Gamma_0)$. Since the coefficients of $t^j$ of $G(s, t)$ are polynomials that vanish at $z$, we deduce $G \equiv 0$. In particular the elements of the one-parameter group generated by $X$ preserve $\Gamma_0$ and hence $X$
preserves \( \Gamma_0 \). By Proposition 12 the vector field \( X \) has vanishing second jet and hence \( j^2 \psi \equiv \text{Id}. \)

We just completed the first step of the proof of Proposition 1. Now we want to construct 2-jets of diffeomorphisms such that any local diffeomorphism with such a 2-jet is not embedded in the flow of a formal vector field.

**Lemma 15.** Let \( \psi \in \hat{\text{Diff}}(C^2, 0) \) such that its second jet is equal to \((x, y) \mapsto (x + x^2 + y^2, -y)\). Then \( \psi \) is not embedded in the flow of a formal vector field.

*Proof.* Assume by contradiction that \( \psi \) is of the form \( \exp(X) \) for some formal nilpotent vector field. The eigenvalues of the linear part of \( X \) at the origin are \( \alpha \) and \( \beta \) with \( e^{\alpha} = 1 \) and \( e^{\beta} = -1 \). We claim that for any choice of \( \alpha \) and \( \beta \) at least one of the resonances \( x^2 e_1 \) or \( y^2 e_1 \) is weak. Otherwise we obtain

\[
2\alpha - \alpha = 0 \quad 2\beta - \alpha = 0 \quad \Rightarrow \quad \alpha = \beta = 0 \quad \Rightarrow \quad e^{\beta} = 1
\]

and since \( e^{\beta} = -1 \) this is a contradiction. Hence, the formal diffeomorphism \( \psi \) is not embedded in a formal flow by Proposition 11.

**Lemma 16.** Let \( \psi \in \hat{\text{Diff}}(C^2, 0) \) be such that its second jet is equal to \((x, y) \mapsto (e^{2\pi i/3} x + y^2, e^{4\pi i/3} y + x^2)\). Then \( \psi \) is not embedded in the flow of a formal vector field.

*Proof.* Assume by contradiction that \( \psi \) is of the form \( \exp(X) \) for some formal nilpotent vector field. The eigenvalues of the linear part of \( X \) at the origin are \( \alpha \) and \( \beta \) with \( e^{\alpha} = e^{2\pi i/3} \) and \( e^{\beta} = e^{4\pi i/3} \). We claim that for any choice of \( \alpha \) and \( \beta \) at least one of the resonances \( x^2 e_2 \) or \( y^2 e_1 \) is weak. Otherwise we have \( 2\alpha = \beta \) and \( 2\beta = \alpha \). This implies \( \alpha = \beta = 0 \), contradicting \( e^{\alpha} = e^{2\pi i/3} \). Therefore the formal diffeomorphism \( \psi \) is not embedded in a formal flow by Proposition 11.

*Proof of Proposition 1.* Consider the diffeomorphism

\[
\psi(x, y) = (x + x^2 + y^2, -y) \quad \text{or} \quad \psi(x, y) = (e^{2\pi i/3} x + y^2, e^{4\pi i/3} y + x^2)
\]

and the curve \( \Gamma = \psi(\Gamma_0) \). Any formal diffeomorphism \( \sigma \) conjugating \( \Gamma_0 \) and \( \Gamma \) is of the form \( \psi \circ \rho \) where \( \rho \in \text{Stab}(\Gamma_0) \). Since \( j^2 \rho \equiv \text{Id} \) by Proposition 12 we deduce \( j^2 (\sigma \circ \rho) \equiv j^2 \psi \) and hence \( \sigma \circ \rho \) is not embedded in the flow of a formal vector field for any \( \rho \in \text{Stab}(\Gamma_0) \) by Lemmas 15 and 16. Therefore the analytic class \( C_0 \) is non-complete.

*Proof of Proposition 2.* Consider the subset \( T \) of \( \text{Diff}(C^2, 0) \) of diffeomorphisms whose second jet is equal to \((x + x^2 + y^2, -y)\) (instead we could choose \((e^{2\pi i/3} x + y^2, e^{4\pi i/3} y + x^2)\) too). The set \( T \) is open in the Krull topology. Moreover, since \( \text{Stab}(\Gamma_0) \) consists of formal diffeomorphisms with trivial second jet, it follows that \( T \) is a union of left cosets of \( \text{Diff}(C^2, 0)/\text{Stab}(\Gamma_0) \). As a consequence its projection \( \tilde{T} \) in \( \text{Diff}(C^2, 0)/\text{Stab}(\Gamma_0) \sim C_0 \) is an open set in the induced quotient topology. Every plane branch \( \Gamma \) in \( \tilde{T} \) is of the form \( \sigma(\Gamma_0) \) where \( \sigma \in \text{Diff}(C^2, 0) \) satisfies \( j^2 \sigma \equiv j^2(x + x^2 + y^2, -y) \). Therefore \( \Gamma_0 \) is not connected to \( \Gamma \) by a geodesic by the proof of Proposition 1. We just obtained an open subset \( \tilde{T} \) of \( C_0 \) whose elements are not connected to \( \Gamma_0 \) by a geodesic.

*Remark 8.* Notice that in the examples in the proof of Proposition 1 the curves \( \Gamma_0 \) and \( \Gamma \) have the same tangent cone.

*Remark 9.* Let us focus in the case where \( \psi(x, y) = (x + x^2 + y^2, -y) \). Zariski’s \( \lambda \) invariant of \( \Gamma_0 \) is \( \lambda = 10 \). Let \( \Gamma \) be the curve of parametrization

\[
\psi(t^6, t^7 + t^{10} + t^{11}) = (t^6 + O(t^{12}), -t^7 - t^{10} - t^{11}).
\]
Up to a change of parameter $t \mapsto ut$ with $u^6 = 1$, the curve $\Gamma$ is of the form
\[
\left( t^6 + O(t^{12}), \frac{t^7}{u^7} - \frac{t^{10}}{u^{10}} - \frac{t^{11}}{u^{11}} \right)
\]
and the coefficient of $t^7$ is equal to 1 if and only if $u = -1$. Then
\[
\left( t^6 + O(t^{12}), t^7 - t^{10} + t^{11} \right)
\]
parametrises $\Gamma$. Every parametrization of $\Gamma$ of the form $(t^6, t^7 + ct^{10} + O(t^{11}))$ satisfies $c = -1$. Thus the curves $\Gamma_0$ and $\Gamma$ are not connected by a geodesic but have the same tangent cone and their parametrizations coincide up to (but not including) the term corresponding to Zariski’s $\lambda$ invariant.

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