ADAPTIVE ESTIMATION OF NOISE VARIANCE AND
MATRIX ESTIMATION VIA USVT ALGORITHM

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ABSTRACT. We propose a method for estimating the entries of a large
noisy matrix when the variance of the noise, $\sigma^2$, is unknown without
putting any assumption on the rank of the matrix. We consider the es-
timator for $\sigma$ introduced by Gavish and Donoho [13] and give an upper
bound on its mean squared error. Then with the estimate of the vari-
ance, we use a modified version of the Universal Singular Value Thresh-
olding (USVT) algorithm introduced by Chatterjee [10] to estimate the
noisy matrix. Finally, we give an upper bound on the mean squared
error of the estimated matrix.

1. INTRODUCTION

Consider the statistical estimation problem where the unknown parame-
ters of interest are the entries of a $m \times n$ matrix $M$, where $m \leq n$. Sup-
pose we have observed the $m \times n$ matrix $X$, a noisy version of $M$, that is
$X = M + \sigma A$. The entries of $A$ are i.i.d. with mean zero and variance one
and $\sigma$ is an unknown positive constant. The goal is to recover $M$ from this
noisy observation $X$ and give an upper bound on the mean squared error
(MSE) of its estimate. Given an estimator $\hat{M}$ for $M$, the $\text{MSE}(\hat{M})$ is defined
as

\[
\text{MSE}(\hat{M}) := \mathbb{E}\left[ \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (m_{ij} - \hat{m}_{ij})^2 \right].
\]

where $m_{ij}$ and $\hat{m}_{ij}$ are respectively the $(i,j)$th entries of the matrices $M$
and $\hat{M}$.

The problem of estimating the entries of a large matrix from noisy and/or
incomplete observations has been studied widely. A common approach for
solving this problem is to assume that $M$ is a low-rank matrix. Under the
low-rank assumption there is a huge body of work using spectral methods,
such as [1, 2, 13, 16, 17, 21]. Some other works under certain model assump-
tions are [12, 24].

In a different direction, Emmanuel Candès and his collaborators [5, 6,
8, 9] studied this problem by penalizing the nuclear norm of the matrix

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under convex constraints. Some other notable examples of this penalization approach are [11, 18, 19, 22, 23].

In 2015, Chatterjee [10] proposed a simple estimation procedure, the USVT algorithm. In that work, Chatterjee considered the problem of matrix estimation without putting any assumption on the rank of the matrix $M$. However, in [10] he assumed that the noise entries and therefore the entries of $X$ lie in a bounded interval. Although he added that the results should hold when the entries of $X - M$ are $N(0, \sigma^2)$ when $\sigma$ is known, but the problem of estimating $M$ when $\sigma$ is unknown, remained unsolved.

Later, in [13] Gavish and Donoho proposed an estimator for $\sigma$ based on the observed matrix $X$. In that work, using the Marchenko-Pastur law, they construct their estimator as a function of the median singular value of matrix $X$. They showed that this estimate almost surely converges to $\sigma$ as $n$ goes to infinity under this assumption that the additive noise is orthogonally invariant. In a different work, Nadler and Kritchman [20], proposed an iterative algorithm for estimating the unknown $\sigma$.

In this paper, we consider the problem of matrix estimation when the variance of the noise, $\sigma$, is unknown. We do not put any assumption on the rank of the matrix or boundedness of its entries. However, we consider the mild assumption that entries of the noise matrix $A$ are i.i.d. and sub-Gaussian. First, we give an upper bound on the mean squared error of the estimator of $\sigma$ from [13], $\hat{\sigma}$. Using the estimate of $\sigma$, we modify the USVT algorithm [10] and give an estimate of $M$, $\hat{M}$, based on the observed matrix $X$. At last we give an upper bound on the mean squared error of $\hat{M}$.

In section 2 we study the estimator of the variance. Theorem 2.2 gives an upper bound on the mean squared error of this estimate. Then we give an estimator of $M$ and in Theorem 2.3 give an upper bound on the mean squared error of this estimator of $M$. In section 3 we study a simulated example. Proofs of all theorems and lemmas are in section 4.

2. Set up and Main Results

Consider the $m \times n$ random matrix $X = [x_{ij}]$. Without loss of generality, we let $m \leq n$. Let $X$ be a noisy version of $M$, $X = M + \sigma A$, where $M = [m_{ij}]$ is a deterministic unknown matrix and $A = [\epsilon_{ij}]$ is a random matrix with i.i.d. entries independent of $M$ and $E[\epsilon_{ij}] = 0$ and $E[\epsilon_{ij}^2] = 1$. Noise level $\sigma > 0$ is an unknown deterministic constant. The final goal is to estimate the entries of $M = [m_{ij}] \in \mathbb{R}^{m \times n}$.

Let’s assume that $\sigma$ is known and $|\epsilon_{ij}| \leq 1$, then following the steps of USVT algorithm [10] bellow, we find $\hat{M}$ an estimate of $M$.

1. Let $\sum_{i=1}^{\min(m,n)} \lambda_i(X) u_i v_i^T$ be the singular value decomposition of $X$, where $\lambda_1(X) \geq \ldots \geq \lambda_m(X)$ are the singular values of $X$ and $u_i$ and $v_i$ are its left and right singular vectors.
(2) Choose a small positive number $\eta \in (0, 1]$ and let $S$ be the set of “thresholded singular values”,

$$S_\sigma := \{ i : \lambda_i(X) \geq (2 + \eta)\sigma\sqrt{n} \}.$$ 

(3) Define $\hat{M} := \sum_{i \in S_\sigma} \lambda_i(X)u_i v_i^T$.

Note that set $S_\sigma$ depends on $\sigma$ and also uses the earlier assumption on boundedness of $\epsilon_{ij}$’s.

To continue, first we introduce some notations and definitions.

2.1. Definitions and Notation. Let $X = [x_{ij}]$ be $m \times n$, where $m \leq n$, a random matrix with $x_{ij}$’s i.i.d. and $E[x_{ij}] = 0$ and $E[x_{ij}^2] = 1$. Let

$$Y_n = \frac{1}{n}XX^T,$$

and let $\lambda_1(Y_n) \geq \cdots \geq \lambda_m(Y_n)$ be the singular values of $Y_n$. Define $\mu_m$ as a random counting measure,

$$\mu_m(A) = \frac{1}{m} \# \{ \lambda_j(Y_n) \in A \}, \quad A \subset \mathbb{R}.$$

**Marčenko-Pastur Law:** Let $m, n \to \infty$ such that $m/n \to \gamma \in (0, 1)$. Then $\mu_m \to \mu_{\gamma\text{MP}}$ in distribution, where

$$d\mu_{\gamma\text{MP}}(x) = \frac{1}{2\pi} \frac{\sqrt{(\gamma_+ - x)(x - \gamma_-)}}{\gamma x} 1_{[\gamma_-, \gamma_+]}(x) \, dx$$

with $\gamma_\pm = (1 \pm \sqrt{\gamma})^2$.

Let $F_{\gamma}(x)$ and $F_n$ be the cumulative distribution function (cdf) of $\mu_{\gamma\text{MP}}$ and $\mu_m$ respectively

$$F_{\gamma}(x) = \int_{\gamma_-}^{x} d\mu_{\gamma\text{MP}}(t), \quad F_n(x) = \int_{0}^{x} d\mu_m(t).$$

**Definition 2.1.** For the $m \times n$ matrix $X$ consider the singular value decomposition $X = U\Sigma V^T$, where $\Sigma$ is a $m \times n$ diagonal matrix with diagonal entries $\lambda_1(X) \geq \cdots \geq \lambda_m(X) \geq 0$, and $U$ and $V$ are unitary matrices of size $m \times m$ and $n \times n$ respectively. The nuclear norm of $X$ is defined as the sum of its singular values,

$$\|X\|_* := \sum_{i=1}^{m} \lambda_i(X).$$

2.2. Estimation of $\sigma$. We consider the following estimator of $\sigma$ that was proposed by Donoho and Gavish [13],

$$\hat{\sigma}(X) = \frac{\text{med}(\lambda_i(X))}{\sqrt{m\mu_{\gamma}}},$$

where $\text{med}(\lambda_i(X))$ is the median of $\lambda_i(X)$’s and

$$\mu_{\gamma} = F_{\gamma}^{-1}\left(\frac{1}{2}\right).$$ (2)
In [13], Donoho and Gavish have shown that
\[
\lim_{n \to \infty} \hat{\sigma}(X) = \sigma \quad \text{a.s.}
\]
They have made this conclusion under an asymptotic framework in which they have assumed that the distribution of \( A \) is orthogonally invariant, meaning that for \( R_1 \) and \( R_2 \), \( m \times m \) and \( n \times n \) orthogonal matrices, \( A \) and \( R_1 A R_2 \) have the same distribution. We consider this estimator without the extra invariance assumption on the noise distribution. Theorem 2.2 provides an upper bound on the MSE of \( \hat{\sigma}(X) \).

**Theorem 2.2.** Let \( X = [x_{ij}] \) be a \( m \times n \) random matrix with \( x_{ij} \)’s distributed i.i.d. from a sub-Gaussian distribution such that \( \mathbb{E}(x_{ij}) = m_{ij} \) and \( \text{Var}(x_{ij}) = \sigma^2 \) for unknown values of \( \sigma \) and \( m_{ij} \)’s. For \( \hat{\sigma} = \text{med}(\lambda_i(X)) \sqrt{n} \mu_\gamma \), we have
\[
\mathbb{E}[ (\hat{\sigma} - \sigma)^2 ] \leq 2\|M\|_2^2 \mu_\gamma n^2 + \frac{\sigma^2 C_\gamma}{\mu_\gamma n} + \frac{\sigma^2 C_\gamma,\epsilon}{\mu_\gamma n^{1-\epsilon}},
\]
where \( C_\gamma,\epsilon \) and \( C_\gamma \) are non-negative constants independent of \( n \) and \( \sigma \).

If \( \|M\|_* = o(n) \), Theorem 2.2 implies that the MSE(\( \hat{\sigma} \))

In [20] Kritchman and Nadler suggested an iterative method for estimating \( \sigma \). The strength of \( \hat{\sigma} \) for us is that it’s easy to compute. Also its simple definition made it possible to use random matrix theory to give the upper bound on its MSE.

### 2.3. Modified USVT estimator of \( M \)

With \( \hat{\sigma} \) as the estimator of \( \sigma \), we use the following modified version of the USVT algorithm to find an estimator for \( M \).

1. Let \( \sum_{i=1}^m \lambda_i(X) u_i v_i^T \) be the singular value decomposition of \( X \), where \( \lambda_1(X) \geq \ldots \geq \lambda_m(X) \) are the singular values of \( X \) and \( u_i \) and \( v_i \) are the left and right singular vectors.
2. Choose a small positive number \( \eta \in (0, 1] \) and let \( S \) be the set of “thresholded singular values”
   \[
   S_\hat{\sigma} = \{ i : \lambda_i(X) \geq (2 + \eta)\hat{\sigma}\sqrt{n} \}.
   \]
3. Define \( \tilde{M} = \sum_{i \in S_\hat{\sigma}} \lambda_i(X) u_i v_i^T \).

Theorem 2.3 gives an upper bound on the MSE of \( \tilde{M} \).

**Theorem 2.3.** Let \( X = [x_{ij}] \) be a \( m \times n \) random matrix with \( x_{ij} \)’s distributed i.i.d. from a sub-Gaussian distribution such that \( \mathbb{E}(x_{ij}) = m_{ij} \) and \( \text{Var}(x_{ij}) = \sigma^2 \) for unknown values of \( \sigma \) and \( m_{ij} \)’s. Let \( \hat{\sigma} = \text{med}(\lambda_i(X)) \sqrt{\eta} \mu_\gamma \),
and for \( \eta \in (0, 1) \) define \( \hat{M} \) using the modified USVT algorithm. Then

\[
\text{MSE}(\hat{M}) = \mathbb{E}\left[ \frac{1}{mn} \| \hat{M} - M \|^2_F \right] \\
\leq \frac{C_0 \sigma \| M \|_*}{\gamma n \sqrt{n}} + (2 + \eta)^2 \sigma^2 \left( C_1 + C_2 \frac{\| M \|_4^4}{\sigma^4 \mu_\gamma^2 n^4} \right)^{1/2} \max\left\{ \sqrt{\frac{C_{\epsilon, \gamma} \sigma}{\mu_\gamma \eta^2 n^{1-\epsilon}}}, \sqrt{\frac{\| M \|_2^2}{n^2 \sigma^2 \eta^2 \mu_\gamma}} \right\},
\]

where \( C_0, C_1, \) and \( C_2 \) are constants, and \( C_{\epsilon, \gamma} \) is a constant that depends on \( \epsilon \) and \( \gamma \).

The upper bounds depend on the \( \| M \|_* \) and \( \sigma \), the unknown parameters of the model, and the parameter of choice \( \eta \). Theorem 2.3 implies that if \( \| M \|_* = o(n) \) then \( \text{MSE}(\hat{M}) \) is of order of \( \sigma^2 \).

3. Simulation

In this section we consider a simple simulated example. Let \( n = 1000 \) and \( m = 200 \) and we consider the sequence \( \lambda_1 \geq \cdots \geq \lambda_{200} \) where \( \lambda_i = \exp(3 - i/50) \) for \( i \in \{1, \cdots, 200\} \). Then for each \( r \in \{50, 100, 150, 200\} \) we define the following signal matrix

\[
M_r = U D_r V^T
\]

where \( D_r \) is a \( m \times n \) rectangular diagonal matrix with the \( i \)-th diagonal entry equal to \( \lambda_i \mathbb{I}\{i \leq r\} \), and \( U \) and \( V \) are randomly uniform orthogonal matrices of size \( m \times m \) and \( n \times n \) respectively. For each choice of \( r \in \{50, 100, 150, 200\} \), we generated 100, independent noise matrix \( A \) with \( N(0, 1) \) i.i.d. entries and considered observed matrix \( X = M_r + \sigma A \) for different values of \( \sigma \). In Figure 1 we have plotted the mean squared error of \( \hat{\sigma} \) and \( \hat{M}_r \) for different values of \( r \) and \( \sigma \). We have set \( \eta = 0.02 \) in the USVT algorithm.

4. Proofs

In what follows \( c, C \) and \( C_i \) for \( i \in \mathbb{N} \) are non-negative constants independent of the parameters of the problem, and \( C_\gamma \) and \( C_{\epsilon, \gamma} \) are non-negative constants that only depends on \( \epsilon \) and pair of \( \epsilon \) and \( \gamma \) respectively. For simplicity these values may change from line to line or even in a line. For simplicity in notation and without loss of generality we assume that \( m \) is an even number and therefore use \( \lambda_{m/2} \) instead of \( \text{med}(\lambda_i) \) for the median singular value.

4.1. Proofs of Theorems 2.2 and 2.3.

4.1.1. Proof of Theorem 2.2.
Figure 1. Estimated mean squared error of $\hat{\sigma}$ and $\hat{M}$ for different values of rank, $r = 50, 100, 150,$ and 200. The our error bounds in Theorems 2.2 and 2.3 do not depend on the rank, but it depends on the nuclear norm of the mean matrix $M_r$. In this example we have $\|M_{50}\|_* = 641.193$, $\|M_{100}\|_* = 877.0753$, $\|M_{150}\|_* = 963.851$, and $\|M_{200}\|_* = 995.775$.

Proof. By adding and subtracting $\lambda_m/2(X - M)$ and inequality $(a + b)^2 \leq 2a^2 + 2b^2$ we have

$$
\mathbb{E}[(\hat{\sigma} - \sigma)^2] = \mathbb{E}\left[\left(\frac{\lambda_{m/2}(X)}{\sqrt{n\mu_\gamma}} - \sigma\right)^2\right] \\
\leq \frac{2}{n\mu_\gamma} \mathbb{E}\left[\left(\lambda_{m/2}(X) - \lambda_{m/2}(X - M)\right)^2\right] \quad (3) \\
+ \frac{2\sigma^2}{n\mu_\gamma} \mathbb{E}\left[\left(\lambda_{m/2}\left(\frac{X - M}{\sigma}\right) - \sqrt{n\mu_\gamma}\right)^2\right]. \quad (4)
$$
Using Lemma 4.2 for (4) we have,
\[
\frac{2\sigma^2}{n\mu_\gamma} \mathbb{E}\left[ \left( \lambda_{m/2} \left( \frac{X - M}{\sigma} \right) - \sqrt{n\mu_\gamma} \right)^2 \right] = \frac{2\sigma^2}{\mu_\gamma} \mathbb{E}\left[ \left( \lambda_{m/2} \left( \frac{A}{\sqrt{n}} \right) - \sqrt{\mu_\gamma} \right)^2 \right]
\]
\[
= \frac{2\sigma^2}{\mu_\gamma} \mathbb{E}\left[ \left( \sqrt{F^{-1}_n \frac{1}{2}} - \sqrt{F^{-1}_\gamma \frac{1}{2}} \right)^2 \right]
\]
\[
\leq \frac{\sigma^2 C_{\epsilon,\gamma}}{\mu_\gamma n^{1-\epsilon}}. 
\]  

(5)

To give an upper bound on (3), we use the following inequality from [4] (page 75). For any two \( n \times n \) matrices \( A_1 \) and \( A_2 \) and any two indices \( i \) and \( j \) such that \( i + j \leq n + 1 \),

\[
\lambda_{i+j-1}(A_1 + A_2) \leq \lambda_i(A_1) + \lambda_j(A_2). 
\]  

(6)

For matrices \( X - M \) and \( M \), \( 0 < k \leq \lfloor \sqrt{n} \rfloor \), and indices \( m/2 - k \) and \( m/2 + k + 1 \), (6) gives

\[
\lambda_{m/2}(X) \leq \lambda_{m/2-k}(X - M) + \lambda_{k+1}(M), 
\]  

(7)

and for matrices \( X \) and \( M \), and indices \( m/2 \) and \( k + 1 \), (6) gives

\[
\lambda_{m/2+k}(X - M) \leq \lambda_{m/2}(X) + \lambda_{k+1}(M). 
\]  

(8)

Subtracting \( \lambda_{m/2}(X - M) \) from the left and right sides of (7) and (8) gives

\[
\lambda_{m/2}(X) - \lambda_{m/2}(X - M) \leq \lambda_{m/2-k}(X - M) - \lambda_{m/2}(X - M) + \lambda_{k+1}(M), 
\]

\[
\lambda_{m/2}(X) - \lambda_{m/2}(X - M) \geq \lambda_{m/2+k}(X - M) - \lambda_{m/2}(X - M) - \lambda_{k+1}(M). 
\]

Therefore

\[
(\lambda_{m/2}(X) - \lambda_{m/2}(X - M))^2 
\]

\[
\leq 2 \max\{ (\lambda_{m/2-k}(X - M) - \lambda_{m/2}(X - M))^2, (\lambda_{m/2+k}(X - M) - \lambda_{m/2}(X - M))^2 \} 
\]

\[
+ 2\lambda_{k+1}(M)^2 
\]  

(9)

Note that

\[
\lambda_{k+1}(M) \leq \frac{\|M\|_*}{k+1}. 
\]  

(10)

Taking expectation of (9) and inequality (10) gives

\[
\mathbb{E}[(\lambda_{m/2}(X) - \lambda_{m/2}(X - M))^2] 
\]

\[
\leq 2\mathbb{E}[(\lambda_{m/2-k}(X - M) - \lambda_{m/2}(X - M))^2] 
\]

(11)

\[
+ 2\mathbb{E}[(\lambda_{m/2+k}(X - M) - \lambda_{m/2}(X - M))^2] 
\]

(12)

\[
+ 2\left( \frac{\|M\|_*}{k+1} \right)^2. 
\]

To give an upper bound on (11) we use the following decomposition
\[
\lambda_{m/2-k}(X - M) - \lambda_{m/2}(X - M) = \sigma \sqrt{n} \left( \sqrt{F^{-1}(\frac{1}{2} - \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2} - \frac{k}{n})} \right)
\]

Using inequality \((a + b)^2 \leq 2a^2 + 2b^2\), and Lemma 4.2 we have

\[
E[(\lambda_{m/2-k}(X - M) - \lambda_{m/2}(X - M))^2] \leq 4\sigma^2 n \left[ \left( \sqrt{F^{-1}(\frac{1}{2} - \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2} - \frac{k}{n})} \right)^2 + 4\sigma^2 n \left( \sqrt{F^{-1}(\frac{1}{2})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} \right)^2 + 2\sigma^2 n \left( \sqrt{F_{\gamma}^{-1}(\frac{1}{2} - \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} \right)^2 \right] \leq 2\sigma^2 C_{\epsilon, \gamma} n^{\epsilon} + 2\sigma^2 n \left( \sqrt{F_{\gamma}^{-1}(\frac{1}{2} - \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} \right)^2. \tag{14}
\]

Similarly

\[
E[(\lambda_{m/2+k}(X - M) - \lambda_{m/2}(X - M))^2] \leq 4\sigma^2 n \left[ \left( \sqrt{F^{-1}(\frac{1}{2} + \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2} + \frac{k}{n})} \right)^2 + 4\sigma^2 n \left( \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} \right)^2 + 2\sigma^2 n \left( \sqrt{F_{\gamma}^{-1}(\frac{1}{2} + \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} \right)^2 \right] \leq 2\sigma^2 C_{\epsilon, \gamma} n^{\epsilon} + 2\sigma^2 n \left( \sqrt{F_{\gamma}^{-1}(\frac{1}{2} + \frac{k}{n})} - \sqrt{F_{\gamma}^{-1}(\frac{1}{2})} \right)^2. \tag{15}
\]
Using the mean value theorem, we find the following bound on (14), and (15).

\[
|\sqrt{F^{-1}(\frac{1}{2} \pm \frac{k}{n})} - \sqrt{F^{-1}(\frac{1}{2})}| \leq C_{\gamma} \sqrt{\frac{1}{F^{-1}(1/2) - F^{-1}(1/2)}} - \sqrt{F^{-1}(\frac{1}{2})}
\]

\[
\leq C_{\gamma} \sqrt{n}.
\]

Putting (14), and (15) together with (16) gives

\[
E\left[ \left( \frac{\lambda_{m/n}}{2} (X) - \frac{\lambda_{m/n}}{2} (X - M) \right)^2 \right] \leq \sigma^2 C_{\epsilon, \gamma} n \jmath.
\]

(16)

Inequalities (17), and (10) with \(k = \lceil \sqrt{n} \rceil\) completes the proof,

\[
E\left[ (\hat{\sigma} - \sigma)^2 \right] \leq \frac{2\|M\|_2^2}{\mu_{\gamma} n^2} + \frac{\sigma^2 C_{\gamma}}{\mu_{\gamma} n} + \frac{\sigma^2 C_{\gamma, \epsilon}}{\mu_{\gamma} n^{1 - \epsilon}}.
\]

\[
\square
\]

4.2. Proof of Theorem 2.3.

Proof. Consider the following two sets \(E_1\) and \(E_2\),

\[
E_1 = \{ \|X - M\| \leq (2 + \frac{\eta}{2})\sigma \sqrt{n} \},
\]

\[
E_2 = \{ |\hat{\sigma} - \sigma| \leq \frac{\eta}{20} \sigma \}.
\]

Consider the following decomposition,

\[
E\| \hat{M} - M \|_F^2 = E[\| \hat{M} - M \|_F^2 1\{ (E_1 \cap E_2)^c \}] + E[\| \hat{M} - M \|_F^2 1\{ (E_1 \cap E_2) \}].
\]

Using Lemma 4.1 from [10], which we copy without its proof, we find an upper bound on \(E[\| \hat{M} - M \|_F^2 1\{ (E_1 \cap E_2) \}].\)

Lemma 4.1. Let \(D = \sum_{i=1}^{m} \sigma_i u_i v_i^T\) be the singular value decomposition of \(D\). Fix any \(\delta > 0\) and define

\[
\hat{B} := \sum_{i: \sigma_i > (1 + \delta) \|D - B\|} \sigma_i x_i y_i^T.
\]

Then

\[
\| \hat{B} - B \|_F \leq K(\delta)(\|D - B\| \|B\|_*)^{1/2},
\]

where \(K(\delta) = (4 + 2\delta) \sqrt{2/\delta} + \sqrt{2 + \delta} \).

For \(\delta \geq \eta/5\), we have \(K(\delta) \leq C \sqrt{1 + \delta}\). On the set \(E_1 \cap E_2\), we have \(\delta \geq \eta/5\). Thus by using Lemma 4.1 we have,

\[
\| \hat{M} - M \|_F^2 \leq C(1 + \delta) \|X - M\| \|M\|_* \leq C \sqrt{n} \sigma \|M\|_*.
\]

Therefore

\[
E[\| \hat{M} - M \|_F^2 1\{ (E_1 \cap E_2) \}] \leq C \sqrt{n} \sigma \|M\|_*.
\]

\[
\square
Then by Cauchy-Schwartz inequality we get
\[
\mathbb{E}[\|\hat{M} - M\|^2_2 \mathbb{1}\{(E_1 \cap E_2)^c\}] \leq \sqrt{\mathbb{E}[\|\hat{M} - M\|^4_4]} \mathbb{E}[(E_1 \cap E_2)^c].
\] (18)

Note that
\[
\mathbb{P}[(E_1 \cap E_2)^c] \leq \mathbb{P}(E_1^c) + \mathbb{P}(E_2^c).
\]

By Proposition 2.4 in [26]
\[
\mathbb{P}(E_1^c) \leq Ce^{-cn^2},
\] (19)

and by Chebyshev's inequality
\[
\mathbb{P}(E_2^c) \leq \frac{800\|M\|^2}{\mu \sigma^2 \eta^2 n^2} + \frac{400C\gamma \sigma^2}{\mu \eta^2 n^{1-\epsilon}}.
\] (20)

Inequalities (19) and (20) together give
\[
\mathbb{P}[(E_1 \cap E_2)^c] \leq Ce^{-cn^2} + \frac{800\|M\|^2}{\mu \sigma^2 \eta^2 n^2} + \frac{C\gamma \sigma^2}{\mu \eta^2 n^{1-\epsilon}}.
\] (21)

To finish finding a bound on (18), we find an upper bound on \(\mathbb{E}[\|\hat{M} - M\|^4_4]\). Using inequality \((a + b)^2 \leq 2a^2 + 2b^2\), we have
\[
\|\hat{M} - M\|^4_4 = \sum_{ij} (m_{ij} - \hat{m}_{ij})^2 = \sum_{ij} (m_{ij} - x_{ij} + x_{ij} - \hat{m}_{ij})^2 \leq (2\sum_{ij} (m_{ij} - x_{ij})^2 + (x_{ij} - \hat{m}_{ij})^2)^2 = 4\sum_{ij} (m_{ij} - x_{ij})^2 + \sum_{ij} (x_{ij} - \hat{m}_{ij})^2)^2 \leq 8\sum_{ij} (m_{ij} - x_{ij})^2 + 8\sum_{ij} (x_{ij} - \hat{m}_{ij})^2)^2 = 8\|M - X\|^4_4 + 8\|X - \hat{M}\|^4_4.
\]

Note that
\[
\|\hat{M} - X\|^4_4 = \| \sum_{\lambda_i(X) < (2+\eta)\sigma \sqrt{n}} \lambda_i(X) u_i v_i^T \|^4_4 \\
\leq n^4(2 + \eta)^4 \sigma^4.
\]

Therefore
\[
\mathbb{E}[\|\hat{M} - X\|^4_4] \leq n^4(2 + \eta)^4 \mathbb{E}[\sigma^4],
\] (22)
and Lemma 4.3 gives
\[ \mathbb{E}[\|\hat{M} - X\|^4_F] \leq 4n^4(2 + \eta)^4(4\sigma^4 + \frac{C_{\epsilon,\gamma}\sigma^4}{n^2 - \epsilon} + \frac{64\|M\|_4^4}{n^4\mu_\gamma^2}) \] (23)

Now note that
\[ \|X - M\|^4_F = \sum_{i,j} (x_{ij} - m_{ij})^4 + \sum_{(i,j) \neq (l,k)} (x_{ij} - m_{ij})(x_{lk} - m_{lk})^2 \]
Since \( x_{ij} \)'s are i.i.d sub-Gaussian random variables and thus their forth moment is bounded,
\[ \mathbb{E}[\|X - M\|^4] \leq C\gamma^2\sigma^4n^4. \] (24)
Inequalities (24) and (22) and Lemma 4.3 give
\[ \mathbb{E}[\|\hat{M} - M\|^2_F] \leq C_0\sqrt{n}\sigma\|M\|_* + (2 + \eta)^2\sigma^2\left(C_1 + \frac{C_{\epsilon,\gamma}\sigma}{\sigma^4\mu_\gamma^2n^4}\right)^{1/2}\left(C_{\epsilon,\gamma}\sigma + \frac{800\|M\|_2^2}{\mu_\gamma^2\eta^2n^2} + \frac{C_{\epsilon,\gamma}\sigma}{\mu_\gamma\eta^2n^{1-\epsilon}}\right)^{1/2}. \]
This completes the proof of Theorem 2.3.

4.3. Proofs of the lemmas.

Lemma 4.2. For any \( \epsilon > 0 \) and \( k \in \{0, 1, \ldots, \lfloor \sqrt{n} \rfloor \} \) almost surely
\[ \left| \sqrt{F_{n}^{-1}\left(\frac{1}{2} + \frac{k}{n}\right)} - \sqrt{F_{\gamma}^{-1}\left(\frac{1}{2} + \frac{k}{n}\right)} \right| \leq C_{\epsilon,\gamma}n^{-1/2+\epsilon}, \] (25)
where \( C_{\epsilon,\gamma} \) is a constant that only depends on \( \gamma \) and \( \epsilon \).

Proof. Let \( \Delta_{n,\gamma}^* := \sup_x |F_n(x) - F_\gamma(x)|. \)

In [14], Götze and Tikhomirov have shown that for any \( \epsilon > 0 \), the rate of almost sure convergence of \( \Delta_{n,\gamma}^* \) is at most \( O(n^{-1/2+\epsilon}) \). This means that there exist a constant \( C_\epsilon \) such that \( \Delta_{n,\gamma}^* \leq C_\epsilon n^{-1/2+\epsilon} \) almost surely.
For \( t = \mathbb{F}^{-1}_\gamma \left( \frac{1}{2} + \frac{k}{n} \right) \) and \( \delta > 0 \) such that \([t - \delta, t + \delta] \subset (\gamma_-, \gamma_+)\) we have

\[
\mathbb{F}_\gamma(t + \delta) - \mathbb{F}_\gamma(t - \delta) = \int_{t-\delta}^{t+\delta} d\mu_{MP}(x)
\]

\[
= \frac{1}{2\gamma\pi} \int_{t-\delta}^{t+\delta} \frac{\sqrt{(\gamma+ - x)(x-\gamma-)}}{x} dx
\]

\[
\leq C\gamma \int_{t-\delta}^{t+\delta} \frac{1}{x} dx
\]

\[
= -C\gamma \log(1 - \frac{2\delta}{t + \delta})
\]

\[
= C\gamma \delta.
\]

Note that

\[
|\mathbb{F}_n(\mathbb{F}^{-1}_n(\frac{1}{2} + \frac{k}{n})) - \mathbb{F}_\gamma(\mathbb{F}^{-1}_n(\frac{1}{2} + \frac{k}{n}))| \leq \Delta^*_n, \gamma \leq C\epsilon n^{-1/2+\epsilon}, \quad (26)
\]

where the second inequality is almost sure. Now if \(|\mathbb{F}^{-1}_n(\frac{1}{2} + \frac{k}{n}) - \mathbb{F}^{-1}_\gamma(\frac{1}{2} + \frac{k}{n})| \geq \delta\) then we have

\[
|\mathbb{F}_\gamma(\mathbb{F}^{-1}_n(\frac{1}{2} + \frac{k}{n})) - \mathbb{F}_\gamma(\mathbb{F}^{-1}_\gamma(\frac{1}{2} + \frac{k}{n}))| \geq C\gamma \delta \quad (27)
\]

and this is possible only for \( \delta \leq C\epsilon n^{-1/2+\epsilon} \). Therefore

\[
|\mathbb{F}^{-1}_n(\frac{1}{2} + \frac{k}{n}) - \mathbb{F}^{-1}_\gamma(\frac{1}{2} + \frac{k}{n})| \leq C\epsilon_n^{-1/2+\epsilon} \quad (28)
\]

almost surely. Using the mean value theorem for function \( f(x) = \sqrt{x} \) we have

\[
|\sqrt{\mathbb{F}^{-1}_n(\frac{1}{2} + \frac{k}{n})} - \sqrt{\mathbb{F}^{-1}_\gamma(\frac{1}{2} + \frac{k}{n})}| \leq C\epsilon \gamma n^{-1/2+\epsilon}. \quad (29)
\]

\[
\square
\]

**Lemma 4.3.** Let \( X = [x_{ij}] \) be a \( m \times n \) random matrix with \( x_{ij} \)'s distributed i.i.d. from a sub-Gaussian distribution such that \( \mathbb{E}(x_{ij}) = m_{ij} \) and \( \text{Var}(x_{ij}) = \sigma^2 \) for some unknown value of \( \sigma \). For any arbitrary \( \epsilon > 0 \) and

\[
\hat{\sigma} = \frac{\text{med}(\lambda_i(X))}{\sqrt{n\mu^2}},
\]

we have

\[
\mathbb{E}[\hat{\sigma}^4] \leq 4\sigma^4 + \frac{C\epsilon \gamma \sigma^4}{n^{2-\epsilon}} + \frac{64\|M\|^4}{n^4\mu^2},
\]

where \( C\epsilon, \gamma > 0 \) is a constant independent of \( n \) and \( \sigma \).

**Proof.** Using inequality \((a+b)^2 \leq 2a^2 + 2b^2\) we write

\[
\mathbb{E}[\hat{\sigma}^4] \leq 8\mathbb{E}[(\hat{\sigma} - \sigma)^4] + 8\sigma^4.
\]
By adding and subtracting $\lambda_m/(2\sigma)$ we have
\[
\mathbb{E}[\hat{\sigma} - \sigma]^4 \leq \frac{8}{n^2\mu^-_2} \mathbb{E}[(\lambda_{m/2}(X - M))^4] + \frac{8\sigma^4}{n^2\mu^-_2} \mathbb{E}[(\lambda_{m/2}(X - M)) - \sqrt{n\mu^-_2}]^4].
\]
Note that by Lemma 4.2
\[
\frac{8\sigma^4}{n^2\mu^-_2} \mathbb{E}[(\lambda_{m/2}(X - M)) - \sqrt{n\mu^-_2}]^4 = \frac{8\sigma^4}{\mu^-_2} \mathbb{E} \left[ \left( \sqrt{F^{-1}_n(\frac{1}{2})} - \sqrt{F^{-1}_\gamma(\frac{1}{2})} \right)^4 \right] \leq \frac{\sigma^4 C_{\epsilon,\gamma}}{\mu^-_2 n^{2-\epsilon}}. \quad (30)
\]
Using decomposition (13) we have
\[
\mathbb{E}[(\lambda_{m/2}(X - M)) - \sqrt{n\mu^-_2}]^4 \leq \frac{512n^2\sigma^4}{\mu^-_2} \mathbb{E} \left[ \left( \sqrt{F^{-1}_n(\frac{1}{2})} - \sqrt{F^{-1}_\gamma(\frac{1}{2})} \right)^4 \right] \]
\[
+ \frac{512n^2\sigma^4}{\mu^-_2} \mathbb{E} \left[ \left( \sqrt{F^{-1}_n(\frac{1}{2})} + \sqrt{F^{-1}_\gamma(\frac{1}{2})} \right)^4 \right] \]
\[
+ \frac{1028n^2\sigma^4}{\mu^-_2} \mathbb{E} \left[ \left( \sqrt{F^{-1}_n(\frac{1}{2})} - \sqrt{F^{-1}_\gamma(\frac{1}{2})} \right)^4 \right] \]
\[
+ \frac{64n^2\sigma^4}{\mu^-_2} \mathbb{E} \left[ \left( \sqrt{F^{-1}_\gamma(\frac{1}{2})} - \sqrt{F^{-1}_\gamma(\frac{1}{2})} \right)^4 \right] \]
\[
+ \frac{64n^2\sigma^4}{\mu^-_2} \mathbb{E} \left[ \left( \sqrt{F^{-1}_n(\frac{1}{2})} + \sqrt{F^{-1}_\gamma(\frac{1}{2})} \right)^4 \right] \]
\[
+ \frac{8(\|M\|_*)}{k+1}^4. \quad (31)
\]
Then Lemma 4.2 gives
\[
\mathbb{E}[(\lambda_{m/2}(X) - \lambda_{m/2}(X - M))^4] \leq C_{\epsilon,\gamma} n^\epsilon \sigma^4 + C_{\gamma} \sigma^4 + 8(\|M\|_*)^4. \quad (31)
\]
Therefore (31), and (30) with $k = \lfloor \sqrt{n} \rfloor$ completes the proof,
\[
\mathbb{E}[\hat{\sigma} - \sigma]^4 \leq \frac{C_{\epsilon,\gamma} \sigma^4}{n^{2-\epsilon}} + \frac{C_{\epsilon,\gamma} \sigma^4}{n^{2-\epsilon}\mu^-_2} + \frac{C_{\gamma} \sigma^4}{n^2\mu^-_2} + \frac{64\|M\|_4^4}{n^4\mu^-_2} \leq \frac{C_{\epsilon,\gamma} \sigma^4}{n^{2-\epsilon}} + \frac{64\|M\|_4^4}{n^4\mu^-_2}. \quad (32)
\]
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