On the Cycle Space of a 3–Connected Graph

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Abstract

We give a simple proof of Tutte’s theorem stating that the cycle space of a
3–connected graph is generated by the set of non-separating circuits of the graph.

Keywords: graph, cycle, circuit, cycle space, non-separating circuit, strong
isomorphism.

1 Introduction

We consider undirected graphs with no loops and no parallel edges. All notions on
graphs that are not defined here can be found in [1, 8].

Let $G = (V, E, \psi)$ be a graph, where $V = V(G)$ is the set of vertices, $E = E(G)$ is
the set of edges, and $\psi : E \rightarrow (V^2)$ is the edge-vertex incident function.

If $C$ is a cycle of $G$ then $E(C)$ is called a circuit of $G$. If $X, Y \subseteq E$, then let
$X + Y$ denote the symmetric difference of $X$ and $Y$, i.e. $X + Y = (X \cup Y) \setminus (X \cap Y)$. Then
$2^E$ forms a vector space over $\mathbb{F}_2$. Let $\mathcal{C}(G)$ denote the set of circuits of $G$, and so
$\mathcal{C}(G) \subseteq 2^E$. Let $\mathcal{CS}(G)$ denote the subspace of $2^E$ generated by $\mathcal{C}(G)$. This subspace
is called the cycle space of $G$. Obviously $X \in \mathcal{CS}(G)$ if and only if every vertex $v$ in
the subgraph of $G$ induced by $X$ has even degree. In particular, $\emptyset \in \mathcal{CS}(G)$. If $Z \subseteq E$,
then let $G/Z = (G \setminus Z)$ denote the graph obtained from $G$ by contracting (respectively,
deleting) the edges in $Z$. If $A$ and $B$ are subgraphs of $G$, we write, for simplicity, $G/A$ instead of
$E(A) \subseteq G$, $A + B$ instead of $E(A) + E(B)$, and $A \in \mathcal{F}$ instead of $E(A) \in \mathcal{F}$
for $\mathcal{F} \subseteq 2^E$.

A cycle $C$ (the corresponding circuit $E(C)$) in a connected graph $G$ is called separating
if $G/C$ has more blocks than $G$, and non-separating, otherwise. Let $\mathcal{NC}(G)$
denote the set of non-separating circuits of $G$, and so $\mathcal{NC}(G) \subseteq \mathcal{C}(G)$.

Given two graphs $G$ and $F$ with $E(G) = E(F)$, we say that $G$ is strongly isomorphic to $F$ if there is an isomorphism $\nu : V(G) \rightarrow V(F)$ from $G$ to $F$ that induces the
identity map $\epsilon : E \rightarrow E$.

One of the classical Whitney theorems states:

1.1 [9] Let $G$ and $F$ be two graphs such that $E(G) = E(F)$ and $\mathcal{C}(G) = \mathcal{C}(F)$. If $G$
is 3–connected and $F$ has no isolated vertices, then $G$ is strongly isomorphic to $F$.

A very simple proof of 1.1 is given in [2, 3].

In [2] we proved the following strengthening of 1.1.

1.2 Let $G$ and $F$ be two graphs such that $E(G) = E(F)$ and $\mathcal{NC}(G) = \mathcal{NC}(F)$. If $G$
is 3–connected and $F$ has no isolated vertices, then $G$ is strongly isomorphic to $F$. 

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In [5] we gave some other strengthenings of the Whitney theorem 1.1.

The following theorem, due to W. Tutte [7] and, independently, A. Kelmans [2, 3], is an important result in the study of the graph cycle spaces.

1.3 The set of non-separating circuits of a 3–connected graph generates the cycle space of the graph.

The above Theorem is an obvious Corollary of 1.2. On the other hand, 1.2 follows from 1.1 and 1.3.

In [2] we proved the following theorem.

1.4 Suppose that $G$ is a 3–connected graph, $X \subseteq E(G)$ and $G \setminus X$ is a connected graph. Then there exist two distinct non-separating circuits $A, B$ in $G$ such that $|A \cap X| = 1$ and $|B \cap X| = 1$.

We also gave the following simple
Proof of 1.2, and therefore also 1.3, using 1.4 [2]. Let $G$ be a 3-connected graph. It is sufficient to show that the set $\mathcal{K}(G)$ of cocircuits (i.e. minimal edge cuts) of $G$ is uniquely defined by the set $\mathcal{N}C(G)$ of non-separating circuits of $G$. Let $\mathcal{K}'(G)$ be the set of edge subsets $X$ of $G$ such that $X \neq \emptyset$ and $|X \cap C| \neq 1$ for every $C \in \mathcal{N}C(G)$. Obviously $\mathcal{K}(G) \subseteq \mathcal{K}'(G)$. Let $\mathcal{K}''(G)$ be the set of members of $\mathcal{K}'(G)$ minimal by inclusion. By 1.4, if $X \in \mathcal{K}'(G)$, then there is $Y \in \mathcal{K}(G)$ such that $Y \subseteq X$. Since $Y \in \mathcal{K}(G)$, every proper subset of $Y$ is not in $\mathcal{K}(G)$. Therefore $\mathcal{K}(G) \subseteq \mathcal{K}'(G) \Rightarrow \mathcal{K}''(G) = \mathcal{K}(G)$.

There are several other proofs of 1.3 (see, for example, [1, 8]).

In this paper we give a new fairly simple proof of 1.3.

The results of this paper were presented at the Moscow Discrete Mathematics Seminar in 1977 (see also [6]).

2 Proof of 1.3

We call a graph topologically 3–connected, or simply top 3–connected, if it is a subdivision of a 3–connected graph. A subdivision of a graph $G$ is called top $G$.

A thread in $G$ is a path $T$ in $G$ such that the degree of every inner vertex of $T$ is equal to two and the degree of every end-vertex of $T$ is not equal to two in $G$. Obviously if $C$ is a cycle of $G$ and $E(C) \cap E(T) \neq \emptyset$, then $T \subseteq C$. If $T$ is a thread in $G$, we write $G - (T)$ instead of $G - (T - \text{End}(T))$.

A path $P$ with end-vertices $x$ and $y$ is called a path-chord of a cycle $C$ in $G$ if $V(C) \cap V(P) = \{x, y\}$, and $E(C) \cap E(P) = \emptyset$.

We need the following known facts.
2.1 [3] Let $G$ be a top 3–connected graph and $G$ not top $K_4$. Then $G$ has a thread $T$ such that $G - (T)$ is also a top 3–connected graph.

2.2 [3] Let $G$ be a top 3–connected graph, $C$ a cycle of $G$, and $T$ a thread of $G$ which is a path-chord of $C$, and let $R, S$ be the cycles of $C \cup T$ distinct from $C$. If $C$ is a non-separating cycle of $G - (T)$, then $R$ and $S$ are non-separating cycles of $G$.

**Proof.** Let $Q = S - (T)$. Then $G/R$ has a block, say $H$, containing $E(Q)$. Suppose, on the contrary, that $R \notin \mathcal{NC}(G)$, i.e. $G/R$ has a block $B$ distinct from $H$. Then $B$ is also a block of $G/C$. Suppose that $E(H) \neq E(Q)$. Let $P$ be a block of $G/C$ that meets $E(H) \setminus E(Q)$. Then $E(P) \neq E(B)$ and $E(P) \neq E(T)$, and therefore $C \notin \mathcal{NC}(G - (T))$, a contradiction. Thus $E(H) = E(Q)$. Then $Q$ is a thread of $G$ and $Q$ is parallel to $T$. Therefore $G$ is not top 3–connected, a contradiction.

2.3 [2, 3] Let $G$ be a 3–connected graph. Then for every edge $e$ of $G$ there are two non-separating cycles $P$ and $Q$ of $G$ such that $E(P) \cap E(Q) = e$ and $V(P) \cap V(Q) = \psi(e)$.

**Proof** (a sketch). Since $G$ is top 3–connected, there are two cycles $R$ and $S$ such that $R \cap S = T$. Let $\mathcal{C}_R$ be the set of cycles $C$ in $G$ such that $C \cap R = T$, and so $S \in \mathcal{C}_R$. If $C \in \mathcal{C}_R$, then let $\alpha(C)$ be the number of edges of the block of $G/C$ containing $E(R - (T))$. Let $P$ be a cycle in $\mathcal{C}_R$ such that $\alpha(P) = \max\{\alpha(C) : C \in \mathcal{C}_R\}$. It is easy to show that $P$ is a non-separating cycle of $G$.

Applying the above arguments to $R := P$ and $S := R$, we find another non-separating cycle $Q$ of $G$ such that $P \cap Q = T$.

Now we are ready to prove the following equivalent of 1.3.

2.4 Let $G$ be a top 3–connected graph. Then $\mathcal{CS}(G)$ is generated by $\mathcal{NC}(G)$.

**Proof** (uses 2.1, 2.2, and 2.3). We prove our claim by induction on the number $t(G)$ of threads of $G$. If $G$ is top $K_4$, then our claim is obviously true. So let $t(G) \geq 7$. By 2.1, $G$ has a thread $T$ such that $G' = G - (T)$ is top 3–connected. By the induction hypothesis, $\mathcal{CS}(G')$ is generated by $\mathcal{NC}(G')$. Obviously if $Q \in \mathcal{NC}(G')$ and $T$ is not a path-chord of $Q$, then $Q \notin \mathcal{NC}(G)$. By 2.2, if $C \notin \mathcal{NC}(G')$, $T$ is a path-chord of $C$, and $R, S$ are the cycles of $C \cup T$ distinct from $C$, then $R, S \notin \mathcal{NC}(G)$. In this case $C = R + S$. Therefore every cycle in $G'$ is generated by $\mathcal{NC}(G)$. Now let $A$ be a cycle in $G$ but not in $G'$. Then $T \subseteq A$. By 2.3, there are $P, Q \in \mathcal{NC}(G)$ such that $P \cap Q = T$. Since $T \subseteq A$ and $T \subseteq P$, clearly $A + P \in \mathcal{CS}(G')$, and so $A + P$ is generated by $\mathcal{NC}(G)$. Since $(A + P) + P = A$ and $P \in \mathcal{NC}(G)$, clearly $A$ is also generated by $\mathcal{NC}(G)$.

More information on this topic can be found in the expository paper [4].
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