A Singularly Perturbed System of Parabolic Equations

A. S. Omuraliev1* and P. Esengul kyzy1**

(Submitted by A. B. Muravnik)

1Kyrgyz-Turkish Manas University, Bishkek, 720044 Kyrgyzstan
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Abstract—The work is devoted to the construction of the asymptotic behavior of the solution of a singularly perturbed system of equations of parabolic type, in the case when the limit equation has a regular singularity as the small parameter tends to zero. The asymptotics of the solution of such problems contains boundary layer functions.

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1. INTRODUCTION

We consider the first boundary-value problem for a system of singularly perturbed parabolic equations

\[ L_\varepsilon u \equiv (\varepsilon + t)\partial_t u - \varepsilon^2 A(x)\partial_x^2 u - D(t)u = f(x, t), \quad (x, t) \in \Omega, \]
\[ u(x, 0, \varepsilon) = h(x), \quad u(0, t, \varepsilon) = u(1, t, \varepsilon) = 0, \]

where \( \Omega = (0, 1) \times (0, T], \varepsilon > 0 \) is small parameter,

\[ u = u(x, t, \varepsilon) = col(u_1(x, t, \varepsilon), u_2(x, t, \varepsilon), u_n(x, t, \varepsilon)), \quad A(x) \in C^\infty([0, 1], \mathbb{C}^n), \]
\[ D(t) \in C^\infty([0, T], \mathbb{C}^n), \quad f(x, t) \in C^\infty([\Omega], \mathbb{C}^n). \]

The work is a continuation of [1], where instead of the matrix-function \( A(x) \), there was a scalar function and an asymptotic of the solution was constructed containing two functions describing the boundary layers along \( x = 0 \) and \( x = 1 \). In this case, the asymptotics contains \( 2m \) parabolic boundary layer functions describing the boundary layers along \( x = 0 \) and \( x = 1 \).

Construction of the asymptotic solution of a singularly perturbed system of parabolic equations is devoted works [2–5]. In [2], a regularized asymptotic is constructed in the case when the matrix of coefficients for the desired function has zero multiple eigenvalue. A similar problem was studied in [3] and an asymptotic of the boundary layer type was constructed. The method of boundary functions in [4] studied the bisingular problem for systems of parabolic equations, which is characterized by the presence of nonsmoothness of the asymptotic terms and a singular dependence on a small parameter. In [5, 6], various problems for split systems of two equations of parabolic type were studied, and asymptotics of the boundary layer type were constructed. The problems of differential equations of parabolic type with a small parameter were studied in [7–9].

*E-mail: asan.omuraliev@manas.edu.kg
**E-mail: peyil.esengul@manas.edu.kg
2. STATEMENT OF THE PROBLEM

We consider the first boundary-value problem (1). The problem is solved under the following assumptions:

1) For $n$-dimensional vector functions $f(x, t)$ and $h(x)$, the inclusions
   \[ f(x, t) \in C^\infty(\Omega, \mathbb{C}^n), \quad h(x) \in C^\infty([0, 1], \mathbb{C}^n), \]
   are fulfilled for $n \times n$-matrix-valued functions $D(t)$ and $A(x)$-inclusions
   \[ D(t) \in C^\infty([0, T], \mathbb{C}^{n \times n}), \quad A(x) \in C^\infty([0, 1], \mathbb{C}^{n \times n}); \]

2) The real parts of all roots $\lambda_i(x), i = \overline{1, n}$, of the equation $\text{det}(A(x) - \lambda E) = 0$ are positive and $\lambda_i(x) \neq \lambda_j(x)$ for all $x \in [0, 1], i \neq j, i, j = \overline{1, n};$

3) The real parts of the eigenvalues $\beta_j(t), j = \overline{1, n}$ of the matrix $D(t)$ are negative, i.e. $\text{Re}\beta_j(0) < 0$ and $\beta_i(0) \neq \beta_j(t) \forall t \in [0, T], i \neq j, i, j = \overline{1, n};$

4) Completed the conditions of approval the initial and boundary conditions $h(0) = h(1) = 0$.

3. REGULARIZATION OF THE PROBLEM

Following [1, p. 316; 5, p. 18], we introduce regularizing variables
\[ \xi_{i,l} = \frac{\varphi_{i,l}(x)}{\sqrt{\varepsilon}}, \quad \varphi_{i,l}(x) = (-1)^{l-1} \int_{t-1}^{t} \frac{d \tau}{\sqrt{\lambda_l(\tau)}}, \quad l = 1, 2, \quad i = \overline{1, n}, \]
\[ \tau = \frac{1}{\varepsilon} \ln \frac{t + \varepsilon}{\varepsilon}, \quad \mu_j = \beta_j(0) \ln \frac{t + \varepsilon}{\varepsilon} \equiv K_j(t, \varepsilon), \quad j = \overline{1, n}, \]
and an extended function such that
\[ \tilde{u}(M, \varepsilon)|_{\xi = (\varphi(x)/\varepsilon) \equiv u(x, t, \varepsilon), \quad M = (x, t, \xi, \tau, \mu), \quad \xi = (\xi_1, \xi_2), \]
\[ \varphi_l(x) = (\varphi_{1,l}, \varphi_{2,l}(x), ..., \varphi_{n,l}(x)), \quad l = 1, 2, \quad \mu = (\mu_1, \mu_2, ..., \mu_n). \]

Based on (2), we find the derivatives from (3):
\[ \partial_t \tilde{u} \equiv \left. \left( \partial_t \tilde{u} + \frac{1}{\varepsilon(t + \varepsilon)} \partial_{\tau} \tilde{u} + \sum_{j=1}^{n} \frac{\beta_j(0)}{t + \varepsilon} \partial_{\mu_j} \tilde{u} \right) \right|_{\theta = \chi(x, t, \varepsilon)}, \]
\[ \partial_x \tilde{u} \equiv \left. \left( \partial_x \tilde{u} + \sum_{l=1}^{2} \sum_{i=1}^{n} \left[ \frac{1}{\sqrt{\varepsilon}} \varphi_{l,i}(x) \partial_{\xi_l,i} \tilde{u} \right] \right) \right|_{\theta = \chi(x, t, \varepsilon)}, \]
\[ \partial_x^2 \tilde{u} \equiv \left. \left( \partial_x^2 \tilde{u} + \sum_{l=1}^{2} \sum_{i=1}^{n} \left[ \frac{1}{\sqrt{\varepsilon}} \left( \varphi_{l,i}(x) \right)^2 \partial_{\xi_l,i}^2 \tilde{u} + \frac{1}{\sqrt{\varepsilon}} \xi^\tau_{l,i} \tilde{u} \right] \right) \right|_{\theta = \chi(x, t, \varepsilon)}, \]
\[ L^\xi_{l,i} \tilde{u} \equiv 2\varphi_{l,i}(x) \partial_{\xi_l,i}^2 \tilde{u} + \varphi''_{l,i}(x) \partial_{\xi_l,i} \tilde{u} + \frac{1}{\sqrt{\varepsilon}} \xi^\tau_{l,i} \tilde{u}, \quad \xi(x, t, \varepsilon) = \left( \frac{\varphi(x)}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}} \ln \frac{t + \varepsilon}{\varepsilon}, K(t, \varepsilon) \right), \]
\[ K(t, \varepsilon) = (K_1(t, \varepsilon), K_2(t, \varepsilon), ..., K_n(t, \varepsilon)), \quad \theta = (\tau, \mu, \xi). \]
According to these calculations, as well as (1) and (3), we pose the following extended problem
\[ L_{\varepsilon} \tilde{u} \equiv \varepsilon \partial_t \tilde{u} + \frac{1}{\varepsilon} T_0 \tilde{u} + T_1 \tilde{u} - \sqrt{\varepsilon} L \xi \tilde{u} - t \partial_t \tilde{u} - \varepsilon^2 L_x \tilde{u} = f(x, t), \]
\[ \tilde{u}|_{t=0} = 0, \quad \tilde{u}|_{x=0, \xi_i=0} = \tilde{u}|_{x=1, \xi_i=0} = 0, \quad i = \overline{1, n}, \]

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In this case, the identity holds
\[
\left( \tilde{L}_\varepsilon \tilde{u}(M, \varepsilon) \right)_{\theta = \chi(x, t, \varepsilon)} \equiv L_\varepsilon u(x, t, \varepsilon).
\]

The solution to the extended problem (4) will be defined as a series
\[
\tilde{u}(M, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{k/2} u_k(M).
\]

Substituting (6) into problem (4) and equating the coefficients for the same powers of \( P \), we obtain the following equations
\[
T_0 u_0 = 0, \quad T_0 u_1 = 0, \quad T_0 u_2 = f(x, t) - T_1 u_0, \quad T_0 u_3 = L_\varepsilon u_0 - T_1 u_1,
\]
\[
T_0 u_k = L_\varepsilon u_{k-3} + L_x u_{k-6} - \partial_t u_{k-4} - T_1 u_{k-2}.
\]

The initial and boundary conditions for them are set in the form
\[
\left. u_k(M) \right|_{t=\mu=\tau=0} = 0, \quad \left. u_k(M) \right|_{x=-l-1, \xi_i, j=0} = 0, \quad k \geq 0, \quad i = \overline{1, n}, \quad l = 1, 2.
\]

4. SOLVABILITY OF ITERATIVE PROBLEMS

Each of problems (7) has innumerable solutions; therefore, we single out a class of functions in which these problems were uniquely solvable. We introduce the following function classes
\[
U_1 = \left\{ V(x, t) : V(x, t) = \sum_{i=1}^{n} v_i(x, t) \psi_i(t), \quad v_i(x, t) \in C^\infty(\Omega) \right\},
\]
\[
U_2 = \left\{ Y(\mathcal{N}) : Y(\mathcal{N}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| y_i^{j}(\mathcal{N}^i_j) \right| < c \exp\left( -\frac{\xi_{i,j}}{8\tau} \right) \right\},
\]
\[
U_3 = \left\{ C(x, t) : C(x, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(x, t) \exp(\mu_j) + p_i(x) \psi_i(t), \quad c_{ij}(x, t) \in C^\infty(\Omega) \right\},
\]
\[
U_4 = \left\{ Z(\mathcal{N}) : Z(\mathcal{N}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| z_{i,j}^{j}(\mathcal{N}^i_j) \right| < c \exp\left( -\frac{\xi_{i,j}^2}{8\tau} \right) \right\},
\]
where \( \mathcal{N}^i_j = (x, \xi_i, \tau, \mu_i), i = 1, 2, \ldots, n, l = 1, 2 \). From these classes of functions we construct a new class as a direct sum: \( U = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \). The function \( u_k(M) \in U \) is representable in vector form
\[
u_k(M) = \Psi(t) \left[ V_k(x, t) + C^k(x, t) \exp(\mu) \right]
\]
\[+ \sum_{i=1}^{l} B(x) \left[ Y^{k,i}(\mathcal{N}) + Z^{k,i}(\mathcal{N}) \right] \exp(\mu), \quad C^k(x, t) = C^k_1(x, t) + \Lambda(P(x)),
\]
\[
C^k_1(x, t) = (c_{ij}(x, t)) = \text{col}(v_{k1}, v_{k2}, \ldots, v_{kn}), \quad Y^{k,i}(\mathcal{N}) = \text{col}(y_1^{k,i}(\mathcal{N}^i_1), y_2^{k,i}(\mathcal{N}^i_2), \ldots, y_n^{k,i}(\mathcal{N}^i_n)), \quad Z^{k,i}(\mathcal{N}) = (z_{ij}^{k,i}(\mathcal{N}^i_j)),
\]
\[
\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_n(t)), \quad B(x) = (b_1(x), b_2(x), \ldots, b_n(x)), \quad \exp(\mu) = \text{col}(\exp(\mu_1), \exp(\mu_2), \ldots, \exp(\mu_n)).
\]
or in coordinate form
\[
\begin{align*}
u_k(M) = & \sum_{i=1}^{n} v_{k,i}(x,t)\psi_i(t) + \sum_{l=1}^{2} \sum_{i=1}^{n} y_i^{k,l}(N^l_i)b_i(x) + \\
& \sum_{i,j=1}^{n} \left\{ c_{i,j}^{k}(x,t)\psi_i(t) + \sum_{l=1}^{2} z_{i,j}^{k,l}(N^l_i)b_i(x) \right\} \exp(\mu_j) + \sum_{i=1}^{n} p_i^{k}(x)\psi_i(t) \exp(\mu_i). \tag{8}\end{align*}
\]

The vector-functions \(b_i(x), \psi_i(t)\) included in these classes are eigenfunctions of the matrices \(A(x)\) and \(D(t)\), respectively
\[
A(x)b_i(x) = \lambda_i(x)b_i(x), \quad D(t)\psi_i(t) = \beta_i(t)\psi_i(t), \quad i = \overline{1,n}. \tag{9}
\]
Moreover, according to condition 1), they are smooth of their arguments.

Along with the eigenvectors \(b_i(x)\) and \(\psi_i(t)\), the eigenvectors \(b_i^{*}(x), \psi_i^{*}(t), i = \overline{1,n}\) of the conjugate matrices \(A^*(x), D^*(t)\) will be used
\[
A^*(x)b_i^{*}(x) = \bar{\lambda}_i(x)b_i^{*}(x), \quad D^*(t)\psi_i^{*}(t) = \bar{\beta}_i(t)\psi_i^{*}(t)
\]
and they are selected biorthogonal
\[
(b_i(x), b_j^{*}(x)) = \delta_{i,j}, \quad (\psi_i(t), \psi_j^{*}(t)) = \delta_{i,j}, \quad i, j = \overline{1,n},
\]
where \(\delta_{i,j}\) is Kronecker symbol.

We calculate the action of the operators \(T_{0}, T_{1}, L_{\xi}, L_{\varepsilon}\) on the function \(u_k(M, \varepsilon)\) from (9), taking into account relations (10) and \(\varphi_{ri}^{2}(x) = 1/\lambda_i(x), i = \overline{1,n}\), we have
\[
T_0 u_k(M) = \sum_{i=1}^{n} \sum_{l=1}^{2} \left\{ \partial_r y_i^{k,l}(N^l_i) - \partial_{\xi_{i,j}}^{2} y_i^{k,l}(N^l_i) \right\} b_i(x) + \\
\sum_{j=1}^{n} \left[ \partial_r z_{i,j}^{k,l}(N^l_i) - \partial_{\xi_{i,j}}^{2} z_{i,j}^{k,l}(N^l_i) \right] \exp(\mu_j) \right\} b_i(x); \tag{10}
\]
or vector form
\[
T_0 u_k(M) = \sum_{i=1}^{n} B(x) \left\{ \partial_r Y^{k,l}(N^l) - \partial_{\xi_{i,j}}^{2} Y^{k,l}(N^l) + \left[ \partial_r Z^{k,l}(N^l) - \partial_{\xi_{i,j}}^{2} Z^{k,l}(N^l) \right] \exp(\mu_j) \right\},
\]

\(Y^{k,l}(N^l)\) is \(n\)-vector, \(Z^{k,l}(N^l)\) is \(n \times n\)-matrix. Here \(B(x)\) is a matrix function \((n \times n)\) whose columns are the eigenvectors \(b_i(x)\) of the matrix \(A(x)\). We calculate
\[
T_1 u_k = t\partial_t u_k + \sum_{j=1}^{n} \beta_j(0)\partial_{\mu_j} u_k - D(t)u_k
\]

\[
= t \sum_{i=1}^{n} \left\{ \partial_t v_{k,i} + \sum_{r=1}^{n} \alpha_{r,i}(t)v_{k,r}(x,t) \right\} \psi_i(t) + \sum_{l=1}^{2} \partial_t y_i^{k,l}(N^l_i)b_i(x) + \\
\sum_{j=1}^{n} \left[ \partial_t c_{i,j}^{k}(x,t) + \sum_{r=1}^{n} \alpha_{r,i}(t)c_{r,j}^{k}(x,t) + \alpha_{j,i}(t)p_j^{k}(x) \right] \psi_i(t) + \\
\sum_{l=1}^{2} \partial_t z_{i,j}^{k,l}(N^l_i)b_i(x) \right\} \exp(\mu_j) + \\
+ \sum_{i,j=1}^{n} \beta_j(0)c_{i,j}^{k}(x,t)\psi_i(t) \exp(\mu_j) + \sum_{i=1}^{n} \beta_i(0)p_i^{k}(x)\psi_i(t) \exp(\mu_i)\]
Given these representations, we calculate
\[ \alpha_{i,r} = \left( \psi_i(t), \psi^*_r(t) \right), \quad \gamma_{i,r}(x, t) = \left( D(t) b_i(x), b^*_r(x) \right). \]

It will be shown below that the scalar functions \( y_i^{k,l}(N_i^l) \) and \( z_{i,j}^{k,l}(N_i^l) \) are representable in the form
\[ y_i^{k,l}(N_i^l) = d_i^{k,l}(x, t) y_i^{k,l}(\xi_i, \tau), \quad z_{i,j}^{k,l}(N_i^l) = \omega_{i,j}^{k,l}(x, t) z_{i,j}^{k,l}(\xi_i, \tau). \]

Given these representations, we calculate
\[
L_{\xi} u_k(M) = \sum_{l=1}^2 A(x) \sum_{i=1}^n \left\{ 2 \varphi_i'(x) \left( b_i(x) d_i^{k,l}(x, t) \right)' + \varphi_i''(x) \left( b_i(x) d_i^{k,l}(x, t) \right) \right\} \partial_{\xi_i} y_i^{k,l}(\xi_i, \tau)
+ \sum_{j=1}^n \left[ 2 \varphi_i'(x) \left( b_i(x) \omega_{i,j}^{k,l}(x, t) \right)' + \varphi_i''(x) \left( b_i(x) \omega_{i,j}^{k,l}(x, t) \right) \right] \partial_{\xi_i} z_{i,j}^{k,l}(\xi_i, \tau) \exp(\mu_j) \right\}
+ L_x u_k(M) = A(x) \left[ \partial^2_x (V_k(x, t)) + \sum_{l=1}^2 \partial^2_x \left( B(x) Y^{l,k}(N_i^l) \right) \right. \\
+ \partial^2_x \left( \Psi(t) C^{k,0}(x, t) \right) \exp(\mu) + \sum_{l=1}^2 \partial^2_x \left( B(x) Z^{l,k}(N_i^l) \right) \exp(\mu) \right].
\]

Satisfy the function (9) of the boundary conditions from (4)
\[
y_i^{k,l}(N_i^l)|_{t=\tau=\mu=0} = 0, \quad z_{i,j}^{k,l}(N_i^l)|_{t=\tau=\mu=0} = 0,
\]
\[
c_i^{k,l}(x, 0) = -v_{i,i}(x, 0) - p_i(x) - \sum_{j \neq i} c_{i,j}^{k,l}(x, 0),
\]
\[
y_i^{k,l}(N_i^l)|_{\xi_i=0} = d_i^{k,l}(x, t), \quad d_i^{k,l}(x, t) b_i(x)|_{x=\tau} = -v_i(l - 1, t) \psi_i(t),
\]
\[
z_{i,j}^{k,l}(N_i^l)|_{\xi_i=0} = \omega_{i,j}^{k,l}(x, t), \quad \omega_{i,j}^{k,l}(x, t) b_i(x)|_{x=\tau} = -[c_{i,j}(l - 1, t) + p_i(l - 1)] \psi_i(t). \]

In general form, the iterative equations (7) are written as
\[ T_0 u_k(M) = h_k(M). \]

**Theorem 1.** Let \( h_k(M) \in U \) and conditions 2), 3) hold. Then equation (14) has a solution \( u_k(M) \in U \), if the equations are solvable
\[
\partial_{\tau} y_i^{k,l}(N_i^l) - \partial^2_{\xi_i} y_i^{k,l}(N_i^l) = h_i^{k,1}(N_i^l) \equiv \tilde{h}_i^{k,1}(x, t) \bar{h}_i^{k,1}(\xi_i, \tau),
\]
\[
\partial_{\tau} z_{i,j}^{k,l}(N_i^l) - \partial^2_{\xi_i} z_{i,j}^{k,l}(N_i^l) = h^{k,2}_{ij}(N_i^l) \equiv \tilde{h}^{k,2}_{ij}(x, t) \bar{h}^{k,2}_{ij}(\xi_i, \tau). \]

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\textbf{Proof.} Let \( h_k(M) = \sum_{i=1}^{n} h_i^{k,1}(N^l_i) + \sum_{j=1}^{n} h_i^{k,2}(N^l_i) \exp(\mu_j) \) \( b_i(x) \in U \). Pose (9) into equation (14), then, on the basis of calculations (12), with respect \( y_i^{k,l}(N^l_i) \) \( z_{i,j}^{k,l}(N^l_i) \), we obtain equations (15). These equations, under appropriate boundary value conditions
\[
y_i^{k,l}(N^l_i)|_{t=\tau=\mu=0} = 0, \quad y_i^{k,l}(N^l_i)|_{\xi_{i,l}=0} = d_{i,l}^{k,l}(x,t), \quad z_{i,j}^{k,l}(N^l_i)|_{\xi_{i,l}=0} = \omega_{i,j}^{k,l}(x,t).
\]
Solutions are represented in the form
\[
y_i^{k,l}(N^l_i) = a_i^{k,l}(x,t) \text{erfc} \left( \frac{\xi_{i,l}}{2\sqrt{\tau}} \right) + h_i^{k,l}(x,t) I_1(\xi_{i,l}, \tau),
\]
\[
z_{i,j}^{k,l}(N^l_i) = \omega_{i,j}^{k,l}(x,t) \text{erfc} \left( \frac{\xi_{i,l}}{2\sqrt{\tau}} \right) + h_{i,j}^{k,2}(x,t) I_2(\xi_{i,l}, \tau),
\]
\[
I_r(\xi_{i,l}, \tau) = \frac{1}{2\sqrt{\pi}} \int_{0}^{\tau} \int_{0}^{\infty} \frac{\tilde{h}_{i,r}(\eta, s)}{\sqrt{\tau-s}} \left[ \exp \left( -\frac{(\xi_{i,l} - \eta)^2}{4(\tau-s)} \right) - \exp \left( -\frac{(\xi_{i,l} + \eta)^2}{4(\tau-s)} \right) \right] d\eta ds, \quad r = 1, 2,
\]
where \( \tilde{h}_{i,r}(x, t) \), \( \tilde{h}_{i,r}(\eta, s) \) are known functions. Evaluation of the integral
\[
|I_r(\xi_{i,l}, \tau)| \leq c \exp \left( -\frac{\xi_{i,l}^2}{8\tau} \right).
\]
\[ \square \]

\textbf{Theorem 2.} Let conditions 1)–4) be satisfied, then equation (7) under additional conditions
\begin{enumerate}
  \item \( u_k|_{t=\mu=\tau=0} = 0, u_k|_{x=t-1, \xi_{i,l}=0} = 0, l = 1, 2 \);
  \item \( -T_1 u_k - \partial_t u_k - L x u_k = 0 \quad u_k, x u_k - 4 \in U_2 \oplus U_4 \);
  \item \( L \xi u_k = 0 \),
\end{enumerate}
has the unique solution in \( U \).

\textbf{Proof.} Satisfying the function \( u_k(M) \in U \) with the boundary conditions from (1) we obtain (13). Based on calculations (10)–(12) we have
\[
-T_1 u_k - \partial_t u_k - L x u_k = -t \sum_{i=1}^{n} \partial_t v_{k,i}(x,t) \psi_i(t)
\]
\[
+ \sum_{i=1}^{n} \alpha_{r,i}(t) v_{k,r}(x,t) \psi_i(t) + \sum_{i=1}^{n} \alpha_{r,i}(t) \psi_i(t) \exp(\mu_r)
\]
\[
+ \sum_{j=1}^{n} \left[ \partial_t c_{i,j}^{k} + \sum_{r=1}^{n} \alpha_{r,i}(t) \xi_{i,j}^{k} \exp(\mu_r) \right] \psi_i(t) \exp(\mu_j)
\]
\[
+ \sum_{l=1}^{n} \left[ \partial_t y_{i}^{k,l}(N^l_i) + \sum_{j=1}^{n} \partial_t z_{i,j}^{k,l}(N^l_i) \exp(\mu_j) \right] b_i(x)
\]
\[
- \sum_{i,j=1}^{n} \psi_i(t) c_{i,j}^{k}(x,t) (\beta_j(0) - \beta_j(t)) \exp(\mu_j)
\]
\[
+ \sum_{l=1}^{n} \sum_{i=1}^{n} \left[ \sum_{r=1}^{n} \gamma_{i,r}(x,t) b_{r,x}(x) y_{i}^{k,l}(N^l_i) + \sum_{j=1}^{n} \sum_{r=1}^{n} \gamma_{i,r}(x,t) b_{r,x}(x) \psi_j^{k,l}(N^l_i) \exp(\mu_j) \right]
\]

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Providing the solvability of equation (17) we set

\[
\begin{align*}
&\sum_{i=1}^{n} \sum_{r=1}^{n} \alpha_{r,i}(t) p^{k-2}_{j}(x) \exp(\mu_{r}) \psi_{i}(t) \\
&+ A(x) \sum_{i=1}^{n} \left\{ \frac{\partial_{x}^{2} v_{k-2,i}}{\alpha_{r,i}(t) v_{k-2,i}} + \sum_{j=1}^{n} \partial_{x} \alpha_{r,j}(t) c^{k-2}_{r,j}(x,t) \exp(\mu_{j}) \right\} \psi_{i}(t) \\
&+ A(x) \sum_{i=1}^{n} \left\{ \frac{\partial_{x}^{2} v_{k-4,i}}{\alpha_{r,i}(t) v_{k-4,i}} + \sum_{j=1}^{n} \partial_{x} \alpha_{r,j}(t) c^{k-4}_{r,j}(x,t) \exp(\mu_{j}) \right\} \psi_{i}(t) \\
&+ A(x) \sum_{i=1}^{n} \left\{ \frac{\partial_{x}^{2} v_{k-4,i}}{\alpha_{r,i}(t) v_{k-4,i}} + \sum_{j=1}^{n} \partial_{x} \alpha_{r,j}(t) c^{k-4}_{r,j}(x,t) \exp(\mu_{j}) \right\} b_{i}(x).
\end{align*}
\]

Providing the condition of Theorem 1, we set

\[
\begin{align*}
t \left[ \partial_{t} V_{k} + A^{T}(t) V_{k} \right] &= -\partial_{t} V_{k-2} - L_{x} V_{k-4}(x,t), \\
\begin{align*}
&\partial_{t} C^{k} + A^{T}(t) \left( C^{k}(x,t) + \Lambda(P^{k}(x)) \right) + \left[ C^{k}(x,t) + \Lambda(P^{k}(x)) \right] \Lambda(\beta(0)) \\
&= -\partial_{t} C^{k-2} - A^{T}(t) \left[ C^{k-2} + \Lambda(P^{k-2}(x)) \right] + L_{x} \left[ C^{k-4} + \Lambda(P^{k-4}(x)) \right],
\end{align*}
\end{align*}
\]

Equation (16) is solved without an initial condition and uniquely determines the function \( V_{k}(x,t) \) [10].

Providing the solvability of equation (17) we set

\[
\begin{align*}
&\sum_{i=1}^{n} \sum_{r=1}^{n} \alpha_{r,i}(t) p^{k-2}_{j}(x) \exp(\mu_{r}) \psi_{i}(t) \\
&+ A(x) \sum_{i=1}^{n} \left\{ \frac{\partial_{x}^{2} v_{k-4,i}}{\alpha_{r,i}(t) v_{k-4,i}} + \sum_{j=1}^{n} \partial_{x} \alpha_{r,j}(t) c^{k-4}_{r,j}(x,t) \exp(\mu_{j}) \right\} \psi_{i}(t) \\
&+ A(x) \sum_{i=1}^{n} \left\{ \frac{\partial_{x}^{2} v_{k-4,i}}{\alpha_{r,i}(t) v_{k-4,i}} + \sum_{j=1}^{n} \partial_{x} \alpha_{r,j}(t) c^{k-4}_{r,j}(x,t) \exp(\mu_{j}) \right\} b_{i}(x).
\end{align*}
\]

or in coordinate form

\[
\begin{align*}
c^{k}_{ij}(x,t)|_{t=0} &= -\frac{1}{\beta_{j}(t) - \beta_{i}(t)} \left[ \partial_{x}^{k-2} + \sum_{r \neq j} \alpha_{r,i}(t) c^{k-2}_{r,j}(x,t) - q_{ij}(x,t) \right] \bigg|_{t=0}, \\
p^{k-2}_{i}(x) &= -\frac{1}{\alpha_{ii}(t)} \left[ \partial_{x}^{k-2} + \alpha_{ii}(t) c^{k-2}_{i,i}(x,t) - q_{ii}(x,t) \right] \bigg|_{t=0},
\end{align*}
\]

where \( q_{ij}(x,t) \) is known functions included in \( A(x) \partial_{x}^{2} [C^{k-4}(x,t) + \Lambda(P^{k-4}(x))] \).

On the basis of (12), condition 3), Theorem 2 is ensured if arbitrary functions \( d^{L,k}_{i}(x,t)b_{i}(x) \), \( \omega^{k,l}_{i,j}(x,t,b_{i}(x) \) are solutions to the problems

\[
\begin{align*}
2\varphi^{i}_{i}(x) \left( d^{L,k}_{i}(x,t)b_{i}(x) \right) + \varphi^{i}_{i,i}(x) \left( d^{L,k}_{i}(x,t)b_{i}(x) \right) &= 0, \\
d^{L,k}_{i}(x,t)b_{i}(x)|_{x=\iota-1} &= -u_{k}(\iota - 1, t) \psi_{i}(t), \quad i = 1, \ldots, n, \\
2\varphi^{i}_{i,i}(x) \left( \omega^{L,k}_{i,j}(x,t)b_{i}(x) \right) + \varphi^{i}_{i,i,i}(x) \left( \omega^{L,k}_{i,j}(x,t)b_{i}(x) \right) &= 0, \\
\omega^{L,k}_{i,j}(x,t)b_{i}(x)|_{x=\iota-1} &= -\left| c^{k}_{ij}(\iota - 1, t) + p_{i}(\iota - 1) \right| \psi_{j}(t).
\end{align*}
\]
Thus, arbitrary functions $d_i^{k,l}(x,t)$, $\omega_{ij}^{k,l}(x,t)$, $v_{ki}(x,t)$, $c_{ij}^{k,l}(x,t)$ included in (9)) are uniquely determined.

Iterative equation (7) for $k = 0, 1$ is homogeneous; therefore, by Theorem 1, it has a solution $u_k(M) \in U$ if the functions $y_i^{k,l}(N_t^l)$, $z_{i,j}^{k,l}(N_t^l)$ are solutions of the equations

$$\partial_t y_i^{k,l}(N_t^l) = \partial_{\xi^l} y_i^{k,l}(N_t^l), \quad \partial_t z_{i,j}^{k,l}(N_t^l) = \partial_{\xi^l} z_{i,j}^{k,l}(N_t^l)$$

for boundary value conditions

$$y_i^{k,l}(N_t^l) |_{t=\tau=0} = 0, \quad y_i^{k,l}(N_t^l) |_{\xi^l=0} = d_i^{k,l}(x,t),$$

$$z_{i,j}^{k,l}(N_t^l) |_{t=\tau=0} = 0, \quad z_{i,j}^{k,l}(N_t^l) |_{\xi^l=0} = \omega_{ij}^{k,l}(x,t).$$

From this problem we find

$$y_i^{0,l}(N_t^l) = d_i^{0,l}(x,t) \text{erfc} \left( \frac{\xi^l t}{2\sqrt{\tau}} \right), \quad z_{i,j}^{0,l}(N_t^l) = \omega_{ij}^{0,l}(x,t) \text{erfc} \left( \frac{\xi^l t}{2\sqrt{\tau}} \right).$$

The functions $d_i^{0,l}(x,t)$, $\omega_{ij}^{l}(x,t)$ are determined from problems (17) which ensure that condition $L^\xi u_0 = 0$ is satisfied. Using calculations (11), the free term of iterative equation (7) at $k = 2$ is written as $F_2(M) = -T_1 u_0(M) + f(x,t)$ by Theorem 1, an equation with such a free term is solvable in $U$, if

$$t \sum_{i=1}^n \left\{ \partial_i v_{0,i}(x,t) + \sum_{r=1}^n \alpha_{ri}(x)v_{0,r}(x,t) \right\} - \beta_i(t)v_{0,i}(x,t) = f(x,t),$$

$$t \sum_{i,j=1}^n \left\{ \partial_{ij} c_{0,i}(x,t) + \sum_{r=1}^n \alpha_{ri}(t)c_{0,r}(x,t) \right\} + \alpha_{ji}(t)p_{j}^0(x)$$

$$+ \sum_{i,j=1}^n [\beta_j(0) - \beta_i(t)]c_{0,i}(x,t) + \sum_{i=1}^n [\beta_i(0) - \beta_i(t)] p_i^0(x) = 0. \tag{18}$$

From (18) we uniquely determine $c_{0,i}(x,t) = 0, \forall i \neq j$ and the function $c_{i,i}(x,t)$ is determined from the equation

$$t \left[ \partial_{i} c_{i,i}(x,t) + \alpha_{ii}(t)c_{i,i}(x,t) \right] + (\beta_i(0) - \beta_i(t))c_{i,i}(x,t) + [\beta_i(0) - \beta_i(t)] p_i^0(x) = 0$$

under the initial condition $c_{i,i}^0(x,0) = -v_{0,i}(x,0) - p_i^0(x)$. The first equation from (18), by virtue of condition 2), has a solution satisfying the condition $|v^0(x,0)| < \infty$. We calculate the free term of equation (7) at $k = 3$: $F_3(M) = -T_1 u_1$, which has the same view as $T_1 u_0$. Providing the solvability of equation $T_0 u_3 = -T_1 u_1$ in $U$, with respect to $c_{i,j}(x,t)$, $v_{1,i}(x,t)$ we obtain equations (18).

In the next step ($k = 4$) the free term has the view $F_4(M) = -T_1 u_2 - \partial_t u_0 + L^\xi u_1$. The functions $d_i^{1,l}(x,t)$, $\omega_{ij}^{1,l}(x,t)$ entering the $u_1(M)$ provide the condition $L^\xi u_1 = 0$. Providing the solvability of the iterative equation at $k = 4$, we set

$$t \left[ \partial_{i} c_{i,j}(x,t) + \sum_{k=1}^n \alpha_{ki}(x)v_{2,k}(x,t) \right] - \beta_i(t)v_{2,i}(x,t) = -\partial_t v_{0,i}(x,t).$$

For $c_{ij}^2(x,t)$ we obtain the same equation of the form (18), but with the right-hand side $\partial_{ij} c_{ij}^2(x,t) + \sum_{k=1}^n \alpha_{ki}(x) \left( c_{k,j}^1(x,t) + \alpha_{j,i}(x)p_j^1(x) \right)$. Taking off the degeneracy of this equation as $t = 0$, we set $p_i^1(x) = -\frac{1}{\alpha_{ii}(t)} \left[ \partial_{i} c_{i,i}^0 + \alpha_{ii}(x)c_{i,i}^0(t=0) \right]$. Further repeating the described process, using Theorems 1 and 2, sequentially determining $u_k(M)$, $k = 0, 1, ..., n$, we construct a partial sum

$$u_{\varepsilon n}(M) = \sum_{k=0}^n \varepsilon^{k/2} u_k(M). \tag{19}$$
For the remainder term
\[ R_\varepsilon(M) = u(M, \varepsilon) - u_\varepsilon(M) = u(M, \varepsilon) - \sum_{k=0}^{n+2} \varepsilon^{k/2} u_k(M) + \sum_{l=1}^{2} \frac{n+1}{l} u_{n+l}, \]
we get the problem
\[ \tilde{\varepsilon} R_\varepsilon(M) = \varepsilon^{n+1/2} g_{\varepsilon}(M), \quad R_\varepsilon(M)|_{\varepsilon=0} = R_\varepsilon|_{x=-1, \varepsilon=0} = 0, \quad l = 1, 2, \]
where
\[ g_{\varepsilon}(M) = -\sum_{l=0}^{1} (T_l u_{n+1+l}) \varepsilon^{l/2} - 3 \varepsilon^{l/2} \partial_t u_{n-1+l} + 5 \varepsilon^{l/2} L_x u_{n-3+l} - \sum_{l=0}^{1} \varepsilon^{l/2} \tilde{\varepsilon} u_{n+1+l}. \]
From the form (9) of the function \( U \), based on conditions 1)–3) and the form of regularizing variables from (2), we conclude that \( \| g_{\varepsilon}(M) \| < C \). In the equation for \( R \), we make the restriction by means of regularizing functions, then, based on (5), we obtain the problem
\[ \tilde{\varepsilon} R_\varepsilon(x, t, \varepsilon) = \varepsilon^{(n+1)/2} g_{\varepsilon}(x, t, \varepsilon), \quad R_\varepsilon|_{t=0} = R_\varepsilon|_{x=0} = R_\varepsilon|_{x=1} = 0. \]
We divide both sides of this equation by \((t + \varepsilon)\), while the properties of the matrices \( A \) are preserved. Therefore, using Theorem 5.5 of [11], we obtain the estimate
\[ \| R_\varepsilon(x, t, \varepsilon) \| < c \varepsilon^{(n+1)/2}. \]
Passing to Euclidean norms, like [12], or the same estimate can be obtained using the maximum principle [11], p. 23.

By narrowing in this problem by means of regularizing functions. Following [11, 12], passing to Euclidean norms, we obtain a problem that is limited by the maximum principle
\[ \| R_\varepsilon(x, t, \varepsilon) \| < c \varepsilon^{n+1/2}. \]  

**Theorem 3.** Let condition 1)–4) be satisfied. Then the restriction of the constructed solution (19) is an asymptotic solution to the problem (1), i.e. \( \forall n = 0, 1, ... \) the estimate (20) holds at \( \varepsilon \to 0. \)

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