EQUIVALENCE BETWEEN EXPONENTIAL STABILIZATION 
AND OBSERVABILITY INEQUALITY FOR MAGNETIC 
EFFECTED PIEZOELECTRIC BEAMS WITH TIME-VARYING 
DELAY AND TIME-DEPENDENT WEIGHTS

AOWEN KONG†
School of Mathematics and Statistics 
Nanjing University of Information Science and Technology 
Nanjing, 210044, China

CARLOS NONATO†
Department of Mathematics, Federal University of Bahia 
Salvador, BA, Brazil

WENJUN LIU∗
School of Mathematics and Statistics 
Nanjing University of Information Science and Technology 
Nanjing, 210044, China

MANOEL JEREMIAS DOS SANTOS∗
Faculty of Exact Sciences and Technology, Federal University of Pará 
Manoel de Abreu Street, s/n, 68440-000, Abaetetuba, Pará, Brazil

CARLOS RAPOSO
Department of Mathematics, Federal University of São João del-Rei 
São João del-Rei, 36307-352, Minas Gerais, Brazil

(Communicated by Chunyou Sun)

ABSTRACT. This paper is concerned with system of magnetic effected piezo- 
electric beams with interior time-varying delay and time-dependent weights, 
in which the beam is clamped at the two side points subject to a single dis-
tributed state feedback controller with a time-varying delay. Under appropriate 
assumptions on the time-varying delay term and time-dependent weights, we 
obtain exponential stability estimates by using the multiplicative technique, 
and prove the equivalence between stabilization and observability.

2020 Mathematics Subject Classification. Primary: 35L05, 35L15; Secondary: 93D15. 
Key words and phrases. Time dependent delay, exponential decay, piezoelectric beams, internal 
observability.
† These authors contributed equally to this work. 
∗ Corresponding authors: wjliu@nuist.edu.cn (W. Liu), jeremias@ufpa.br (M. Santos).
1. Introduction. In this paper, we focus our attention on system of magnetic
affected piezoelectric beams with time-varying delay and time-dependent weights

\[ \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta_1(t) v_t(x, t) + \delta_2(t) v_t(x, t - \tau(t)) = 0, \]  
\[ (x, t) \in (0, L) \times (0, T), \]  
\[ (1) \]

\[ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + \delta p_t(x, t) = 0, \]  
\[ (x, t) \in (0, L) \times (0, T), \]  
\[ (2) \]

with the initial and boundary conditions

\[ v(x, 0) = v_0(x), \quad p(x, 0) = p_0(x), \quad x \in (0, L), \]  
\[ (3) \]
\[ v_1(x, 0) = v_1(x), \quad p_1(x, 0) = p_1(x), \quad x \in (0, L), \]  
\[ (4) \]
\[ v_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), \quad (x, t) \in (0, L) \times (0, \tau(0)). \]  
\[ (5) \]
\[ v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \quad t \in (0, T), \]  
\[ (6) \]
\[ p(0, t) = p_x(L, t) - \gamma v_x(L, t) = 0, \quad t \in (0, T), \]  
\[ (7) \]

where \( \rho, \alpha, \gamma, \beta, \mu, \delta_3 \) are positive constants with \( \alpha, \gamma, \beta \) satisfying \( \alpha = \alpha_1 + \gamma^2 \beta \),
and \( \delta_1(t), \delta_2(t) \) are time-dependent weights. Here the functions \( v(x, t) \) and \( p(x, t) \)
are used to denote the transverse displacement of the beam and the total load of
the electric displacement along the transverse direction at each point \( x \) respectively.

In recent years, piezoelectric materials are widely used in designing devices [1, 4, 7, 8, 31, 39, 42]. The ability of piezoelectric structures to generate deformations controlled by electrical field applications and vice versa, attracts the attention of scientists from various areas in order to design mathematical and computational models capable of providing new knowledge and applications of these materials [5, 12, 14].

Due to the fact that magnetic energy has a relatively small effect on the general
dynamics, magnetic effects are neglected in piezoelectric beam models. However, in closed loop, the magnetic effect can cause oscillations in the output which results in the instability of system, this tells us that the magnetic effect can cause a limitation in the performance of system.

In many studies related to piezoelectric structures, the magnetic effect is neglected and only the mechanical and electrical effects are considered. In general the mechanical effects are modeled using Kirchoff, Euler-Bernoulli or Mindlin-Timoshenko assumptions for small displacements [9] and electrical, magnetic effects are added to system generally using electrostatic, quasi-static and fully dynamic approaches [40]. The electrostatic and quasi-static approaches, despite being widely used, completely exclude the magnetic effect as well as its couplings with mechanical and electrical effects. Morris and Özer [23, 24] established the theory of piezoelectric materials, in which they combined mechanical, magnetic, and electrical effects

\[ \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \]  
\[ (8) \]
\[ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \]  
\[ (9) \]

where \( \gamma, \beta, \mu, \alpha, \rho \) denote the piezoelectric coefficient, the beam coefficient of impermeability, the magnetic permeability, the elastic stiffness and the mass density per unit volume, respectively. And they assumed that the beam is fixed at \( x = 0 \) and free at \( x = L \), thus they got (from modeling) the boundary conditions

\[ v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \quad p(0, t) = \beta p_x(L, t) - \gamma v_x(L, t) = -V(t), \]  

where \( t \in \mathbb{R}^+ \). Then the authors considered \( V(t) = p_t(L, t)/h \) (electrical feedback controller) in \((8)-(9)\) and established strong stabilization for almost all system.
parameters and exponential stability for system parameters in a measure-null set. Besides, Ramos et al. [34] inserted a (mechanical) dissipative term $\delta v_t$ in (8), where $\delta$ is a constant and considered the boundary condition

$$v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \quad p(0, t) = p_x(L, t) - \gamma v_x(L, t) = 0, \quad t \in (0, T).$$

The authors showed, by using energy method, that the system’s energy decays exponentially. This means that the friction term and the magnetic effect work together in order to exponentially stabilize the system. Ramos et al. [32] considered the piezoelectric beam with magnetic effect (8)-(9) with boundary conditions given by

$$v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) + \xi_1 \frac{v(L, t)}{h} = 0, \quad t \in (0, T),$$

$$p(0, t) = \beta p_x(L, t) - \gamma \beta v_x(L, t) + \xi_2 \frac{p(L, t)}{h} = 0, \quad t \in (0, T).$$

They showed that the system is exponentially stable regardless of any relationship between system parameters and exponential stability is equivalent to exact observability at the boundary. Besides, Freitas et al. [11] proved the existence of smooth global attractors with finite fractal dimension and the existence of exponential attractors for the associated dynamical system.

Recently, the control of partial differential equations with time-varying delays has become a hot research topic [3, 6, 10, 17, 18, 19, 20, 21, 33, 35, 36, 38]. In [26], under the condition $0 \leq |\delta_2| \leq \sqrt{1 - d}$, Nicaise et al. proved that the heat equation with time-varying boundary delay is exponentially stable

$$u_t - au_{xx} = 0, \quad (x, t) \in (0, \pi) \times (0, \infty),$$

$$u(0, t) = 0, \quad u(\pi, t) = -\delta_1 u(\pi, t) - \delta_2 u(t - \tau(t)), \quad t \in (0, \infty),$$

$$u(x, 0) = u_0(x), \quad x \in (0, \pi),$$

$$u(\pi, t - \tau(0)) = f_0(t - \tau(0)), \quad t \in (0, \tau(0)),$$

and they proved that the wave equation with time-varying boundary delay is also exponentially stable. Kirane et al. [16] considered the one-dimensional Timoshenko beam model with variable delay $\tau(t)$ in the rotation angle equation

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi_x) = 0, \quad (x, t) \in (0, 1) \times (0, \infty),$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi_x) + \delta_1 \psi_t + \delta_2 \psi_t(x, t - \tau(t)) = 0, \quad (x, t) \in (0, 1) \times (0, \infty),$$

$$\varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \quad t \in (0, \infty),$$

$$\varphi(x, 0) = \varphi_0, \quad \varphi_t(x, 0) = \varphi_1, \quad \psi(x, 0) = \psi_0, \quad x \in (0, 1),$$

$$\psi_t(x, 0) = \psi_1, \quad \psi_t(x, t - \tau(0)) = f_0(t - \tau(0)), \quad (x, t) \in (0, 1) \times (0, \tau(0)),$n

the authors showed that if $\frac{\rho_1}{b} = \frac{\rho_2}{b}$ holds, then the system is exponentially stable. Barros et al. [2] considered the wave equation with delay and damping weights depending on the time

$$u_{tt} - uu_{xx} + \delta_1(t)u_t(x, t) + \delta_2(t)u_t(x, t - \tau(t)) = 0, \quad (x, t) \in (0, L) \times (0, \infty),$$

$$u(0, t) = u(L, t) = 0, \quad t \in (0, \infty),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, L),$$

$$u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), \quad (x, t) \in (0, L) \times (0, \tau(0)).$$
Besides, Nonato et al. [27, 28] considered exponential stability for Timoshenko and thermoelastic laminated beam system with nonlinear weights and time-varying delay. Unlike previous works, the dampings $\delta_1$ and $\delta_2$ depend on time $t$. Under appropriate assumptions about the weights of the damping $\delta_1$ and $\delta_2$, the authors obtained the exponential decay of system.

Our intention in mentioning the last three works was to show situations in which time-dependent delay feedback appears $\tau(t)$ as well as to show situations in which the weight of the damping may vary, which make the problem worse, undoubtedly more attractive and challenging. We consider here the result on magnetic effected piezoelectric beam with interior time-varying delay and time-dependent weights. Since the weights are nonlinear, the operator is nonautonomous, we use the Kato variable norm technique [15] to show that the system is well-posed. We use the standard multiplicative method as in [2] to obtain the exponential stabilization. Finally, we give equivalence between exponential stabilization and observability inequality. The novelty of the work is found, basically, in the application of this technique in a relatively new model (piezoelectric beam with magnetic effect).

It is worth mentioning that although (1)-(2) is formed by two coupled wave equations, one of them with a delay mechanism, it is not immediate that for systems of this nature, the exponential decay of the solution happens, once that there are systems formed by coupled wave equations with delay terms whose solution does not decay exponentially [29]. In some cases, such as the Timoshenko model, which is also a system formed by two coupled wave equations, when subjected to damping of the delay type in one of the equations, exponential decay is obtained, provided the equality of speeds occurs [16, 27, 37].

This paper is arranged as follows. In Section 2, we present some notations and prove the dissipative property of the energy. In Section 3, we provide the well-posedness of solution and summarize the proof method. In Section 4, we present the result of exponential stability. In Section 5, we show the inequality of internal observability by means of the multiplicative techniques. In Section 6, we prove our main result dealing with the equivalence between stabilization and observability.

2. Preliminaries. In this section, we propose hypothesis for the time-varying delay and time-dependent weights as in [2, 25].

**Assumption 1.** We assume that there exist positive constants $\tau_0, \tau$ such that

$$0 < \tau_0 \leq \tau(t) \leq \tau. \quad (10)$$

Moreover, we assume that

$$\tau'(t) \leq d < 1, \quad \forall \ t > 0, \quad (11)$$

$$\tau \in W^{2,\infty}([0, T]), \quad \forall \ T > 0, \quad (12)$$

where $d$ is a constant.

**Assumption 2.** We assume that $\delta_1 : \mathbb{R}^+ \to [0, \infty)$ is a non-increasing function of class $C^1(\mathbb{R}^+)$ satisfying

$$\frac{\delta_1'(t)}{\delta_1(t)} \leq M_1, \quad 0 < \delta_0 \leq \delta_1(t), \quad \forall \ t \geq 0, \quad (13)$$

where $\delta_0$ and $M_1$ are constants such that $M_1 > 0$. 
Assumption 3. We assume that \( \delta_2 : \mathbb{R}^+ \rightarrow \mathbb{R} \) is a function of class \( C^1(\mathbb{R}^+) \), which is not necessarily positives or monotones, such that

\[
\begin{align*}
|\delta_2(t)| & \leq \beta_0 \delta_1(t), \quad (14) \\
|\delta_2'(t)| & \leq M_2 \delta_1(t), \quad (15)
\end{align*}
\]

for some \( 0 < \beta_0 < \sqrt{1 - d} \) and \( M_2 > 0 \).

As in Nicaise et al. [25], we transform the time-varying delay term by introducing a new dependent variable

\[
z(x, \rho, t) = v_t(x, t - \tau(t) \rho), \quad x \in (0, L), \rho \in (0, 1), t \in (0, T).
\]

Now, we rewrite the system to be studied

\[
\begin{align*}
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta_1(t) v_t(x, t) + \delta_2(t) z(x, 1, t) &= 0, \quad (x, t) \in (0, L) \times (0, T), \\
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + \delta_3 p_t(x, t) &= 0, \quad (x, t) \in (0, L) \times (0, T), \\
\tau(t) z_t(x, \rho, t) + (1 - \tau'(t) \rho) z_p(x, \rho, t) &= 0, \quad (x, \rho, t) \in (0, L) \times (0, 1) \times (0, T),
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
v(0, t) &= \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \quad t \in (0, T), \\
p(0, t) &= p_x(L, t) - \gamma v_x(L, t) = 0, \quad t \in (0, T), \\
z(0, t) &= v_t(x, t), \quad (x, t) \in (0, L) \times (0, T),
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
v(x, 0) &= v_0(x), \quad p(x, 0) = p_0(x), \quad x \in (0, L), \\
v_t(x, 0) &= v_1(x), \quad p_t(x, 0) = p_1(x), \quad x \in (0, L), \\
z(x, \rho, 0) &= f_0(x, -\rho \tau(0)), \quad (x, \rho) \in (0, L) \times (0, 1).
\end{align*}
\]

**Definition 2.1.** Let \((v, p)\) be the solution of system (17)-(25). We define the energy of system (17)-(25) as

\[
E(t) = \frac{\rho}{2} \int_0^L |v_t|^2 dx + \frac{\mu}{2} \int_0^L |p_t|^2 dx + \frac{\alpha}{2} \int_0^L |v_x|^2 dx + \frac{\beta}{2} \int_0^L |v_x - p_x|^2 dx + \frac{\xi(t)}{2} \int_0^L \int_{t - \tau(t)}^t e^{\lambda(s-t)} v_t^2(x, s) dx ds,
\]

where \(\xi(t) = \bar{\xi} \delta_1(t)\) is a non-increasing function of class \( C^1(\mathbb{R}^+) \). Besides, we can choose and fix the constants \( \bar{\xi}, \lambda \) such that

\[
\begin{align*}
\frac{\beta_0}{\sqrt{1 - d}} < \bar{\xi} & < 2 - \frac{\beta_0}{\sqrt{1 - d}}, \\
\lambda & < \frac{1}{\tau} \left| \log \frac{|\delta_2|}{\sqrt{1 - d}} \right|.
\end{align*}
\]

**Lemma 2.2.** Under the assumptions (13) and (14), we can reach an agreement that the following inequality holds:

\[
E'(t) \leq -C \int_0^L [v_t^2(x, t) + p_t^2(x, t) + v_t^2(x, t - \tau(t))] dx
- C \int_{t - \tau(t)}^t \int_0^L e^{\lambda(s-t)} v_t^2(x, s) dx ds.
\]
Proof. By differentiating (26) and using Cauchy-Schwarz’s inequality, we get

\[
E'(t) \leq -\left( \delta_1(t) - \frac{|\delta_2(t)|}{2\sqrt{1-d}} - \frac{\xi(t)}{2} \right) \int_0^L v_t^2(x, t) dx - \delta_3 \int_0^L p_t^2(x, t) dx
\]

\[
- \left( e^{-\lambda \xi(t)} \frac{\xi(t)}{2} - \frac{|\delta_2(t)|}{2} \right) \int_0^t \int_0^L e^{-\lambda(t-s)} v_t^2(x, s) ds dx
\]

For the first term on the right-hand side of the inequality, from (13), (14) and (27), we find that

\[
- \left( \delta_1(t) - \frac{|\delta_2(t)|}{2\sqrt{1-d}} - \frac{\xi(t)}{2} \right) \leq -\delta_1(t) \left( 1 - \frac{\beta_0}{2\sqrt{1-d}} - \frac{\xi}{2} \right)
\]

\[
\leq -\delta_0 \left( 1 - \frac{\beta_0}{2\sqrt{1-d}} - \frac{\xi}{2} \right) \leq -C_1.
\]

Similarly for the third term, we find that

\[
- \left( e^{-\lambda \xi(t)} \frac{\xi(t)}{2} - \frac{|\delta_2(t)|}{2} \right) \leq -\delta_0 \left( e^{-\lambda \xi(t)} - \frac{\beta_0}{2\sqrt{1-d}} \right) \leq -C_2.
\]

Besides, since \( \xi(t) \) is a non-increasing function, we obtain

\[
- \left( \lambda \xi(t) - \left( \xi'(t) \xi(t) \right) \right) \leq -\lambda \xi(t) = -\lambda \frac{\xi \xi_1(t)}{2} \leq -\lambda \frac{\xi \delta_1(t)}{2} \leq -C_3.
\]

By choosing \( C = \min\{C_1, C_2, C_3, \delta_1\} \), we complete the proof. Besides, we denote \( C \) by various positive constants at different occurrences.

Lemma 2.3. \([2, 27]\) Let \((v, p, z)\) be a solution to system (17)-(25). Then the energy functional defined by (26) satisfies

\[
\int_0^L |v_t|^2 dx < -\frac{1}{\sigma} E'(t),
\]

\[
\int_0^L |p_t|^2 dx < -\frac{1}{\delta_3} E'(t),
\]

where \( \sigma = \delta_0 \left( 1 - \frac{\xi}{2} - \frac{\beta_0}{2\sqrt{1-d}} \right) \).

Proof. From Lemma 2.1, we have that

\[
-E'(t) \geq \delta_1(t) \left( 1 - \frac{\xi}{2} - \frac{\beta_0}{2\sqrt{1-d}} \right) \int_0^L |v_t|^2 dx
\]

\[
+ \delta_1(t) \left( \frac{\xi(1 - \tau'(t))}{2} - \frac{\beta_0 \sqrt{1-d}}{2} \right) \int_0^L z^2(x, 1, t) dx
\]

\[
\geq 0,
\]
and from (13), we obtain
\[
0 \leq \delta_0 \left( 1 - \frac{\xi}{2} - \frac{\beta_0}{2\sqrt{1 - d}} \right) \int_0^L |v_t|^2 dx \\
\leq \delta_1(t) \left( 1 - \frac{\xi}{2} - \frac{\beta_0}{2\sqrt{1 - d}} \right) \int_0^L |v_t|^2 dx \\
\leq -E'(t),
\]

similarly the inequality holds for \( p_t \) and the lemma is proved.

Lemma 2.4. \[13\] Let \( E : \mathbb{R}^+ \to \mathbb{R}^+ \) be a non-increasing function and assume that there are two constants \( \sigma > 1 \) and \( \omega > 0 \) such that
\[
\int_S^{+\infty} E^{1+\sigma}(t) dt \leq \frac{1}{\omega} E^\sigma(0) E(S), \quad 0 \leq S < +\infty.
\]

Then
\[
E(t) = 0 \quad \forall \ t \geq E(0) \frac{1 + \sigma}{1 + \omega \sigma t}, \quad \text{if} \quad -1 < \sigma < 0,
\]
\[
E(t) \leq E(0) \left( \frac{1 + \sigma}{1 + \omega \sigma t} \right)^{1/\sigma} \quad \forall \ t \geq 0, \quad \text{if} \quad \sigma > 0,
\]
\[
E(t) \leq E(0)e^{1-\omega t} \quad \forall \ t \geq 0, \quad \text{if} \quad \sigma = 0.
\]

3. Well-posedness. In this section, we give the existence and uniqueness of solution to system (17)-(25).

Let \( U = (v, f, p, h, z)^T \), then system (17)-(25) can be rewritten as
\[
\begin{cases}
U_t = A(t)U, \\
U(0) = U_0 = (v_0, v_1, p_0, p_1, g_0(\cdot, -\rho \tau(0)))^T,
\end{cases}
\]
where the operator \( A(t) \) is defined by
\[
A(t) = \begin{pmatrix}
\frac{\alpha}{\rho} v_{xx} - \frac{\gamma \beta}{\rho} p_{xx} - \frac{\delta_1(t)}{\rho} f - \frac{\delta_2(t)}{\rho} z(\cdot, 1, \cdot) \\
\frac{\delta_0}{\rho_0} f \\
\frac{\mu}{\rho} h \\
-\frac{\gamma \beta}{\mu} v_{xx} + \frac{\beta}{\mu} p_{xx} - \frac{\delta_4(t)}{\mu} \tau'(t) \rho - 1 \\
-\frac{\tau(t)}{\rho} z_{\rho}
\end{pmatrix},
\]
with domain
\[
D(A(t)) = \{(v, f, p, h, z)^T \in H : f = z(\cdot, 0) \in (0, L), v_x(L) = p_x(L) = 0\},
\]
for \( t > 0 \), where
\[
H = \left( (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L) \right)^2 \times L^2((0, L); H^1(0, L)),
\]
where \( H^1_0(0, L) := \{ f \in H^1(0, L) : f(0) = 0 \}. \)

Observe that the domain of \( A(t) \) is independent of the time \( t > 0 \), i.e.,
\[
D(A(t)) = D(A(0)), \quad t > 0.
\]
Now, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = (H^1_0(0,L) \times L^2(0,L))^2 \times L^2((0,L) \times (0,1)).$$

For $M = (v, f, p, h, z)^T \in \mathcal{H}$ and $N = (\tilde{v}, \tilde{f}, \tilde{p}, \tilde{h}, \tilde{z})^T \in \mathcal{H}$, we define the inner product on $\mathcal{H}$

$$\langle M, N \rangle_{\mathcal{H}} := \rho \int_0^L f \tilde{f} dx + \mu \int_0^L h \tilde{h} dx + \beta \int_0^L (\gamma v_x - p_x)(\gamma \tilde{v}_x$$

$$+ \alpha_1 \int_0^L v_x \tilde{v}_x dx - \tilde{p}_x) dx + \xi(t) \tau(t) \int_0^L \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho dx. $$

Our existence and uniqueness result is

**Theorem 3.1.** Suppose that assumptions in Section 2 hold. Then for any initial data $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C^1([0, \infty), \mathcal{H})$ of system (17)-(25). Moreover, if $U_0 \in D(\mathcal{A}(0))$, then

$$U \in C([0, \infty), D(\mathcal{A}(0))) \cap C^1([0, \infty), \mathcal{H}).$$

**Proof.** It relies on the variable norm technique (see Kato [15]), which ensures that
(i) $D(\mathcal{A}(0))$ is a dense subset of $\mathcal{H}$;
(ii) $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, $\forall t > 0$;
(iii) for all $t \in [0, T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on $\mathcal{H}$ and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is stable with stability constants $C$ and $m$ independent of $t$, i.e., the semigroup $(S_t(s))_{s \geq 0}$ generated by $\mathcal{A}(t)$ satisfies

$$||S_t(s)(u)||_{\mathcal{H}} \leq Ce^{ms}||u||_{\mathcal{H}}, \quad \forall u \in \mathcal{H}, \quad s \geq 0;$$

(iv) $\partial_t \mathcal{A}(t) \in L^\infty([0, T], B(D(\mathcal{A}(0)), \mathcal{H}))$, where $L^\infty([0, T], B(D(\mathcal{A}(0)), \mathcal{H}))$ is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0, T]$ into the set $B(D(\mathcal{A}(0)), \mathcal{H})$ of bounded operators from $D(\mathcal{A}(0))$ into $\mathcal{H}$.

Indeed, from (33), the domain of $\mathcal{A}(t)$ is independent of the time $t > 0$. We can prove (i), (iii) and (iv) by the same technique as in Nicaise et al. [25], and we complete the proof.

4. Exponential stability. In this section, we provide the following exponential stability result.

**Theorem 4.1.** Under the assumptions of (13), (14) and (15). There exists a positive constant $\kappa$ such that

$$E(t) \leq E(0)e^{1-\kappa t},$$

for any solution of system (17)-(25).

**Proof.** Given $0 \leq S < T < \infty$, multiplying (17) by $v$ and then integrating on $(S, T) \times (0, L)$, we get

$$\int_S^T \int_0^L v (\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta_1(t)v_t(x, t) + \delta_2(t)z(x, 1, t)) dx dt = 0.$$
Using the boundary conditions and integrating by parts, we get that

\[
0 = \left[ \int_0^L \rho v_t \, dx \right]_S^T - \int_S^T \int_0^L \rho v_t^2 \, dxdt + \int_S^T \int_0^L \alpha_1 |v_x|^2 \, dxdt \\
- \int_S^T \int_0^L \gamma_2 \beta v_x v \, dxdt + \int_S^T \int_0^L \gamma_2 \beta_2 v_x v \, dxdt \\
+ \int_S^T \int_0^L \delta_1(t) \nu v_t \, dxdt + \int_S^T \int_0^L \delta_2(t) \nu z(x, 1, t) \, dxdt,
\]

then multiplying (18) by \( p \) and integrating on \((S, T) \times (0, L)\), we get

\[
0 = \left[ \int_0^L \mu pp_i \, dx \right]_S^T - \int_S^T \int_0^L \mu p_i^2 \, dxdt - \int_S^T \int_0^L \beta(\gamma v - p)x_p \, dxdt \\
+ \int_S^T \int_0^L \delta_3 p_{pp} \, dxdt.
\]

Adding (36) and (37), we have

\[
0 = \left[ \int_0^L (\rho v_t + \mu p_i) \, dx \right]_S^T - \int_S^T \int_0^L \rho v_t^2 \, dxdt - \int_S^T \int_0^L \mu p_i^2 \, dxdt \\
+ \int_S^T \int_0^L \alpha_1 v_x^2 \, dxdt + \int_S^T \int_0^L \beta \gamma v_x - p_x^2 \, dxdt + \int_S^T \int_0^L \delta_1(t) \nu v_t \, dxdt \\
+ \int_S^T \int_0^L \delta_3 pp_i \, dxdt + \int_S^T \int_0^L \delta_2(t) \nu z(x, 1, t) \, dxdt.
\]

Similarly, we multiply (19) by \( \xi(t)e^{-2\rho t}z(x, \rho, t) \) and integrate on \((0, L) \times (0, 1) \times (S, T)\) to get that

\[
0 = \int_S^T \int_0^L \int_0^1 \xi(t)e^{-2\rho t}z(t)(1 - \rho \rho(t))z(1) \, dx \, dp \, dt \\
= \frac{1}{2} \int_S^T \int_0^L \int_0^1 \xi(t) \rho(t) e^{-2\rho t} \frac{\partial}{\partial \rho} \rho^2 \, dx \, dp \, dt \\
+ \frac{1}{2} \int_S^T \int_0^L \int_0^1 \xi(t) e^{-2\rho t} (1 - \rho \rho(t)) \frac{\partial}{\partial \rho} \rho^2 \, dx \, dp \, dt.
\]

Using integration by parts and the boundary conditions we get

\[
0 = \left[ \frac{\xi(t) \tau(t)}{2} \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 \, dp \, dx \right]_S^T \\
- \frac{1}{2} \int_S^T \xi(t) \rho(t) \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 \, dp \, dx \, dt \\
+ \frac{1}{2} \int_S^T \xi(t) \int_0^L \left[ e^{-2\rho \tau(t)}(1 - \tau'(t))z^2(x, 1, t) - z^2(x, 0, t) \right] \, dx \, dt \\
+ \int_S^T \xi(t) \tau(t) \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 \, dp \, dx \, dt.
\]
Since $\delta_1$ is a non-increasing function of class $C^1(\mathbb{R}^+)$, its derivatives is non-positive, which implies that $\xi'(t) \leq 0$, this results that
\[
\int_S^T \xi'(t) \tau(t) \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 d\rho dx dt \leq 0.
\]

Moreover, we have
\[
-\frac{1}{2} \int_S^T \xi(t) \int_0^L \int_0^1 e^{-2\rho \tau(t)} (1 - \tau'(t)) z^2(x, 1, t) dx dt \leq 0,
\]
then considering the above inequalities, we get that
\[
\int_S^T \xi(t) \tau(t) \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 d\rho dx dt \\ \leq -\left[ \frac{\xi(t) \tau(t)}{2} \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 d\rho dx \right]_S^T + \frac{1}{2} \int_S^T \xi(t) \int_0^L z^2(x, 0, t) dx dt \tag{39}
\]
Using the definition of $E$, (38) and (39), we have
\[
\gamma_0 \int_S^T E(t) dt \leq \left[ \int_0^L (\rho vv_t + \mu p p_t) dx \right]_S^T - \int_S^T \int_0^L \delta_3 p p_t dx dt \\ - \left[ \frac{\xi(t) \tau(t)}{2} \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 d\rho dx \right]_S^T + \frac{1}{2} \int_S^T \int_0^L (\rho v^2_t + \mu p^2_t) dx dt \\ - \int_S^T \int_0^L \delta_1(t) v v_t dx dt - \int_S^T \int_0^L \delta_2(t) v z(x, 1, t) dx dt \\ + \frac{1}{2} \int_S^T \xi(t) e^{-2\rho \tau(t)} \int_0^L z^2(x, 0, t) dx dt, \tag{40}
\]
where $\gamma_0 = 2 \min \{1, e^{-2\tau(S)}\}$.

By using Young’s, Sobolev-Poincaré’s inequalities and Lemma 2.2, we have that
\[
- \left[ \int_0^L (\rho vv_t + \mu p p_t) dx \right]_S^T \leq \int_0^L v(x, S) v_t(x, S) dx - \int_0^L v(x, T) v_t(x, T) dx \\ + \int_0^L p(x, S) p_t(x, S) dx - \int_0^L p(x, T) p_t(x, T) dx \\ \leq CE(S).
\]

We also know that
\[
- \left[ \frac{\xi(t) \tau(t)}{2} \int_0^L \int_0^1 e^{-2\rho \tau(t)} z^2 d\rho dx \right]_S^T \leq \frac{\xi(S) \tau(S)}{2} \int_0^L \int_0^1 e^{-2\rho \tau(S)} z^2(x, \rho, S) d\rho dx \\ \leq C \xi(S) \tau(S) \int_0^L \int_0^1 z^2(x, \rho, S) d\rho dx \\ \leq CE(S).
\]

From Lemma 2.2, we deduce that
\[
\int_S^T \int_0^L (\rho v^2_t + \mu p^2_t) dx dt \leq -C \int_S^T E'(t) dt \leq CE(S).
\]
Now, we get that
\[
\left| \int_S \int_0^L \delta_1(t) \nu v \, dx \, dt \right| + \left| \int_S \int_0^L \delta_3 p \, dx \, dt \right| \\
\leq \delta_1(0) \left| \int_S \int_0^L \nu v \, dx \, dt \right| + \delta_3 \left| \int_S \int_0^L p \, dx \, dt \right| \\
\leq c(\varepsilon_1) \int_S \int_0^L \nu v^2 \, dx \, dt + \varepsilon_1 \int_S \int_0^L \nu v^2 \, dx \, dt \\
+ c(\varepsilon_2) \int_S \int_0^L p^2 \, dx \, dt + \varepsilon_2 \int_S \int_0^L p^2 \, dx \, dt \\
\leq c(\varepsilon_1) \int_S \int_0^L \nu v^2 \, dx \, dt + (\varepsilon_1 + 2\varepsilon_2 \gamma^2) \int_S \int_0^L \nu v^2 \, dx \, dt \\
+ c(\varepsilon_2) \int_S \int_0^L p^2 \, dx \, dt + 2\varepsilon_2 \int_S \int_0^L \nu v \, dx \, dt \\
\leq C \int_S (-E'(t)) dt + \max \left\{ \frac{2(\varepsilon_1 + 2\varepsilon_2 \gamma^2)}{\alpha_1}, \frac{4\varepsilon_2}{\beta} \right\} \int_S E(t) dt \\
\leq CE(S) + \max \left\{ \frac{2(\varepsilon_1 + 2\varepsilon_2 \gamma^2)}{\alpha_1}, \frac{4\varepsilon_2}{\beta} \right\} \int_S E(t) dt, \quad (41)
\]
and from (14) and (15), we obtain that
\[
\left| \int_S \int_0^L \delta_2(t) v(z, x, 1, t) \, dx \, dt \right| \\
\leq \beta_0 \delta_1(0) \left| \int_S \int_0^L v(z, x, 1, t) \, dx \, dt \right| \\
\leq c(\varepsilon_3) E(S) + \varepsilon_3 \int_S E(t) dt. \quad (42)
\]
Finally,
\[
\frac{1}{2} \int_S \int_0^T \xi(t) \int_0^L z^2(x, 0, t) \, dx \, dt \leq \frac{3}{2} \delta_1(0) \int_S \int_0^L |v_t|^2 \, dx \, dt \\
\leq C \int_S (-E'(t)) dt \leq CE(S).
\]
Choosing \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) small enough, we deduce from (41) and (42) that
\[
\int_S E(t) dt \leq \frac{1}{k} E(S).
\]
Since \( E(S) \leq E(0) \) for \( S \geq 0 \), we have that
\[
\int_S E(t) dt \leq \frac{1}{k} E(0).
\]
We conclude from Lemma 2.4 that
\[
E(t) \leq E(0)e^{1-\kappa t}.
\]
The proof is completed. \( \Box \)
5. Internal observability. In this section, we show the inequality of internal observability by means of the multiplicative techniques. We consider the conservative system given by

\[
\begin{align*}
\rho u_{tt} - \alpha u_{xx} + \gamma \beta w_{xx} &= 0, & (x, t) &\in (0, L) \times (0, T), \\
\mu w_{tt} - \beta w_{xx} + \gamma \beta u_{xx} &= 0, & (x, t) &\in (0, L) \times (0, T),
\end{align*}
\]

with the initial and boundary conditions

\[
\begin{align*}
&u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), \quad x \in (0, L), \\
&u_t(x, 0) = u_1(x), \quad w_t(x, 0) = w_1(x), \quad x \in (0, L), \\
&u(0, t) = \alpha u_x(L, t) - \gamma \beta w_x(L, t) = 0, \quad t \in (0, T), \\
&w(0, t) = w_x(L, t) - \gamma u_x(L, t) = 0, \quad t \in (0, T).
\end{align*}
\]

The total energy associated with system (43)-(48) is given by

\[
E(u, u_t, w, w_t; t) = \frac{\rho}{2} \int_0^L |u_t|^2 dx + \frac{\mu}{2} \int_0^L |w_t|^2 dx 
+ \frac{\alpha_1}{2} \int_0^L |u_x|^2 dx + \frac{\beta}{2} \int_0^L |\gamma u_x - w_x|^2 dx,
\]

which satisfies the identity

\[
E(u, u_t, w, w_t; t) = E(u, u_t, w, w_t; 0), \quad \forall \, t > 0.
\]

Then,

\[
E(t) - E(u, u_t, w, w_t; t) = \frac{\xi(t)}{2} \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v_t^2(x, s) dx ds = E_{\text{delay}}(t).
\]

Meanwhile, we assume that \( v = u + \phi \) and \( p = w + \psi \), where \((\phi, \psi)\) is the solution of the auxiliary problem

\[
\begin{align*}
\rho \phi_{tt} - \alpha \phi_{xx} + \gamma \beta \psi_{xx} + \delta_1(t) \psi_t(x, t) 
+ \delta_2(t) \psi_t(x, t - \tau(t)) &= 0, & (x, t) &\in (0, L) \times (0, T), \\
\mu \psi_{tt} - \beta \psi_{xx} + \gamma \beta \phi_{xx} + \delta_3 p_t(x, t) &= 0, & (x, t) &\in (0, L) \times (0, T),
\end{align*}
\]

with the initial and boundary conditions

\[
\begin{align*}
&\phi(x, 0) = \psi(x, 0) = \phi_t(x, 0) = \psi_t(x, 0) = 0, \quad x \in (0, L), \\
&\phi(0, t) = \alpha \phi_x(L, t) - \gamma \beta \psi_x(L, t) = 0, \quad t \in (0, T), \\
&\psi(0, t) = \psi_x(L, t) - \gamma \phi_x(L, t) = 0, \quad t \in (0, T).
\end{align*}
\]

**Lemma 5.1.** For \( T > \max \left\{ \frac{1}{\alpha_1}, \frac{2L^2\mu}{\beta}, \frac{L^2\rho + 2\gamma^2 L^2\mu}{\beta} \right\} \), there exists a constant \( C_0 > 0 \) (depending on \( T \)), for all solutions \((v, p)\) of system (1)-(7) such that

\[
E(0) \leq C_0 \int_0^T \int_0^L \left[ v_t^2(x, t) + p_t^2(x, t) + v_t^2(x, t - \tau(t)) \right] dx dt 
+ C_0 \int_0^T \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v_t^2(x, s) dx ds dt.
\]
Proof. Multiplying (43), (44) by \( u, w \) respectively and integrating the results by parts on \((0, L) \times (0, T)\), we get

\[
0 = \int_0^L \rho u t u dx + \int_0^T \int_0^L \rho u^2_t dx dt + \int_0^T \int_0^L \alpha_1 u_x^2 dx dt
\]
\[
+ \int_0^T \int_0^L \gamma \beta (\gamma u_x - w_x) w_x dx dt,
\]
\[
0 = \int_0^L \mu w_t w dx + \int_0^T \int_0^L \mu w_t^2 dx dt - \int_0^T \int_0^L \beta (\gamma u_x - w_x) w_x dx dt.
\]

Adding the above two equations and rearranging the resulting equality, we obtain

\[
\mathcal{X}_1(t)^T + 2 \int_0^T \mathcal{E}(u, u_t, w, w_t; t) dt = 2 \int_0^T \int_0^L \rho u^2 + \mu w_t^2 dx dt,
\]

where \( \mathcal{X}_1(t) = \int_0^L (\rho u + \mu w_t) dx \).

For the first term of \( \mathcal{X}(t) \), it follows from Young’s inequality that

\[
\left| \int_0^L \rho u t u dx \right| \leq \frac{\rho}{2} \int_0^L u_x^2 dx + \frac{\rho}{2} \int_0^L u^2 dx, \tag{56}
\]
\[
\left| \int_0^L \rho u_t u dx \right| \leq \frac{\mu}{2} \int_0^L u_t^2 dx + \frac{\mu}{2} \int_0^L w_x^2 dx. \tag{57}
\]

By Hölder’s inequality, it is easy to check that the Poincaré’s inequality holds, i.e., \( \|y\|^2 \leq L^2 \|y_x\|^2 \) for \( y \in H^1(0, L) \) with \( y(0, t) = 0 \). Thus, a direct calculation yields

\[
\int_0^L u^2 dx \leq L^2 \int_0^L u_x^2 dx, \tag{58}
\]
\[
\int_0^L w^2 dx \leq L^2 \int_0^L w_x^2 dx \leq 2L^2 \int_0^L \gamma u_x - w_x^2 dx + 2\gamma^2 L^2 \int_0^L u_x^2 dx. \tag{59}
\]

Substituting (58), (59) into (56), (57) respectively, we deduce that

\[
\left| \int_0^L \rho u t u dx \right| \leq \frac{\rho}{2} \int_0^L u_x^2 dx + \frac{\rho L^2}{2} \int_0^L u_x^2 dx, \tag{60}
\]
\[
\left| \int_0^L \rho u_t u dx \right| \leq \frac{\mu}{2} \int_0^L u_t^2 dx + \mu L^2 \int_0^L \gamma u_x - w_x^2 dx + \mu \gamma^2 L^2 \int_0^L u_x^2 dx. \tag{61}
\]

Then from (60) and (61), we get

\[
|\mathcal{X}_1(t)| \leq \max \left\{ 1, \frac{2L^2 \mu}{\beta}, \frac{L^2 \rho + 2\gamma^2 L^2 \mu}{\alpha_1} \right\} \mathcal{E}(u, u_t, w, w_t; t).
\]

Since \( \mathcal{E}(u, u_t, w, w_t; t) = \mathcal{E}(u, u_t, w, w_t; 0) \), we see that

\[
\mathcal{X}_1(t)^T - \mathcal{X}_1(0) \geq -2 \max \left\{ 1, \frac{2L^2 \mu}{\beta}, \frac{L^2 \rho + 2\gamma^2 L^2 \mu}{\alpha_1} \right\} \mathcal{E}(u, u_t, w, w_t; 0),
\]
\[
2 \int_0^T \mathcal{E}(u, u_t, w, w_t; t) dt = 2T \mathcal{E}(u, u_t, w, w_t; 0).
\]
Thus, (55) can be rewritten as
\[
\mathcal{E}(u, u_t, w, w_t; 0) \leq C_4 \int_0^T \int_0^L (u_t^2(x, t) + w_t^2(x, t)) \, dx \, dt,
\]
where
\[
C_4 = \max \left\{ \frac{\rho \mu}{T - \max \left\{ \frac{2L^2 \mu}{\beta}, \frac{L^2 \rho + 2\gamma^2 L^2 \mu}{\alpha_1} \right\}} \right\} = \frac{\sigma_2}{T - \sigma_1}.
\]
Multiplying (49) by \( \phi_t \) and integrating the result by parts on \((0, L) \times (0, S) \) with \( 0 \leq S \leq T \), we get
\[
\frac{\rho}{2} \int_0^S \int_0^L \frac{d}{dt} \phi_t^2 \, dx \, dt + \gamma \beta \int_0^S \int_0^L (\gamma \phi_x - \psi_x) \phi_t \, dx \, dt
\]
\[
+ \int_0^S \int_0^L \delta_1(t) v_t \phi_t \, dx \, dt + \int_0^S \int_0^L \delta_2(t) v_t(x, t - \tau(t)) \phi_t \, dx \, dt
\]
\[
+ \frac{\alpha_1}{2} \int_0^S \int_0^L \frac{d}{dt} \phi_t^2 \, dx \, dt = \int_0^S (\alpha \phi_x - \gamma \beta \psi_x) \phi_t \bigg|_0^L = 0.
\]
In the same way, we have
\[
\frac{\mu}{2} \int_0^S \int_0^L \frac{d}{dt} \psi_t^2 \, dx \, dt - \beta \int_0^S \int_0^L (\gamma \phi_x - \psi_x) \psi_t \, dx \, dt
\]
\[
+ \int_0^S \int_0^L \delta_3 \psi_t \psi_t \, dx \, dt = \beta \int_0^S (\psi_x - \gamma \beta \psi_x) \psi_t \bigg|_0^L = 0.
\]
Adding the above two equations, we arrive at
\[
\int_0^S \frac{d}{dt} \mathcal{E}(\phi, \phi_t, \psi, \psi_t; t) \, dt = \mathcal{E}(\phi, \phi_t, \psi, \psi_t; S) - \mathcal{E}(\phi, \phi_t, \psi, \psi_t; 0)
\]
\[
= - \int_0^S \int_0^L \delta_1(t) v_t \phi_t \, dx \, dt - \int_0^S \int_0^L \delta_3 \psi_t \phi_t \, dx \, dt
\]
\[
- \int_0^S \int_0^L \delta_2(t) v_t(x, t - \tau(t)) \phi_t \, dx \, dt.
\]
Taking into account the initial condition (51), we have \( \mathcal{E}(\phi, \phi_t, \psi, \psi_t; 0) = 0 \). Therefore,
\[
\mathcal{E}(\phi, \phi_t, \psi, \psi_t; S) = - \int_0^S \int_0^L \delta_1(t) v_t \phi_t \, dx \, dt - \int_0^S \int_0^L \delta_2(t) v_t(x, t - \tau(t)) \phi_t \, dx \, dt
\]
\[
- \int_0^S \int_0^L \delta_3 \psi_t \phi_t \, dx \, dt.
\]
It follows from Young’s inequality that
\[
\mathcal{E}(\phi, \phi_t, \psi, \psi_t; S) \leq \varepsilon \int_0^S \int_0^L (\phi_t^2 + \psi_t^2) \, dx \, dt
\]
\[
+ \frac{m}{4 \varepsilon} \int_0^S \int_0^L (v_t^2 + p_t^2 + v_t^2(x, t - \tau(t))) \, dx \, dt,
\]
where \( 0 \leq S \leq T, m = \max\{2\delta_1^2(0), 2\delta_3^2 \delta_1^2(0), \delta_2^2 \}. \)
Multiplying (49) by \( \phi \) and integrating the result by parts on \((0, L) \times (0, T)\), we get
\[
0 = \int_0^L \rho \phi_\tau \phi dx \bigg|_0^T - \int_0^T \int_0^L \rho \phi_\tau^2 dx dt + \int_0^T \int_0^L \alpha_1 \phi_\tau^2 dx dt,
\]
\[
+ \int_0^T \int_0^L \delta_1(t) \nu_i \phi dx dt + \int_0^T \int_0^L \delta_2(t) \nu_i(x, t - \tau(t)) \phi dx dt
\]
\[
+ \int_0^T \int_0^L \gamma \beta (\gamma \phi_x - \psi_x) \phi dx dt.
\]
In the same way, we have
\[
0 = \int_0^L \mu_1 \psi dx \bigg|_0^T - \int_0^T \int_0^L \mu \psi_\tau^2 dx dt
\]
\[
- \int_0^T \int_0^L \beta (\gamma \phi_x - \psi_x) \psi dx dt + \int_0^T \int_0^L \delta_3 p_\tau \psi dx dt.
\]
Adding the above two equations, we arrive at
\[
\mathcal{X}_2(t) = \int_0^L (\rho \phi_\tau \phi + \mu \psi_\tau \psi) dx.
\]
By Young’s inequality, we have
\[
\int_0^T \int_0^L \delta_1(t) \nu_i \phi dx dt + \int_0^T \int_0^L \delta_2(t) \nu_i(x, t - \tau(t)) \phi dx dt + \int_0^T \int_0^L \delta_3 p_\tau \psi dx dt
\]
\[
\leq \sigma_3 \int_0^T \mathcal{E}(\phi, \phi_\tau, \psi, \psi_\tau; t) dt + \frac{m}{4 \varepsilon} \int_0^T \int_0^L \left( \phi_\tau^2 + \psi_\tau^2 \right) dx dt,
\]
where \( \sigma_3 = \max \left\{ \frac{2 \varepsilon L^2}{\alpha_1}, \frac{4 \varepsilon L^2}{\beta} \right\} \). Then, substituting the above inequality and (63) into (64), we deduce that
\[
\sigma_4 \int_0^T \int_0^L \left( \phi_\tau^2 + \psi_\tau^2 \right) dx dt = 2 \min \{ \rho, \mu \} \int_0^T \int_0^L \left( \phi_\tau^2 + \psi_\tau^2 \right) dx dt
\]
\[
\leq 2 \sigma_1 \mathcal{E}(\phi, \phi_\tau, \psi, \psi_\tau; t) + (2 + \sigma_3) \int_0^T \mathcal{E}(\phi, \phi_\tau, \psi, \psi_\tau; t) dt
\]
\[
+ \frac{m}{4 \varepsilon} \int_0^T \int_0^L (\phi_\tau^2 + \psi_\tau^2) dx dt
\]
\[
\leq (2 \sigma_1 + (2 + \sigma_3) \varepsilon) \int_0^T \int_0^L (\phi_\tau^2 + \psi_\tau^2) dx dt
\]
\[
+ \frac{m}{4 \varepsilon} (1 + 2 \sigma_1 + (2 + \sigma_3) \varepsilon) \int_0^T \int_0^L (\phi_\tau^2 + \psi_\tau^2) dx dt.
\]
Thus, a direct calculation yields
\[
\int_0^T \int_0^L (\phi_\tau^2 + \psi_\tau^2) dx dt \leq C_5 \int_0^T \int_0^L (\phi_\tau^2 + \psi_\tau^2) dx dt,
\]
where \( C_5 = \frac{[1 + 2\sigma_1 + (2 + \sigma_3)T]m}{4\varepsilon[\sigma_4 - (2\sigma_1 + (2 + \sigma_3)T)c]} \).

Assuming that \((v_0, v_1, p_0, p_1, u_0, u_1, w_0, w_1)\), \(v = u + \phi\) and \(p = w + \psi\), we obtain

\[
E(0) = E(v, v_t, p, p_t; 0) + E_{\text{delay}}(0) \\
= E(u, u_t, w, w_t; 0) + E_{\text{delay}}(0) \\
\leq C_4 \int_0^T \int_0^L (u^2_t(x, t) + w^2_t(x, t)) \, dx \, dt + E_{\text{delay}}(0) \\
\leq 2C_4 \int_0^T \int_0^L (v^2_t(x, t) + p^2_t(x, t) + \phi^2_t(x, t) + \psi^2_t(x, t)) \, dx \, dt + E_{\text{delay}}(0) \\
\leq (2C_4 + 2C_4C_5) \int_0^T \int_0^L (v^2_t(x, t) + p^2_t(x, t)) \, dx \, dt \\
+ 2C_4C_5 \int_0^T \int_0^L v^2_t(x, t - \tau(t)) \, dx \, dt \\
+ C_6 \int_0^T \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v^2_t(x, s) \, dx \, ds \, dt.
\]

By choosing \( C_0 = \max\{2C_4 + 2C_4C_5, C_6\} \), the proof is completed. \( \square \)

6. Equivalence between stabilization and observability. In this section, we state and prove the main theorem.

**Theorem 6.1.** The following estimates are equivalent.

i) There exists a constant \( C_0 > 0 \) and \( T > T_0 \) such that for all \((v_0, v_1, p_0, p_1, z_0)\) \( \in \mathcal{H} \) we have

\[
E(0) \leq C_0 \int_0^T \int_0^L [v^2_t(x, t) + p^2_t(x, t) + v^2_t(x, t - \tau(t))] \, dx \, dt \\
+ C_0 \int_0^T \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v^2_t(x, s) \, dx \, ds \, dt.
\]

ii) There exist constants \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) such that for all \((v_0, v_1, p_0, p_1, z_0)\) \( \in \mathcal{H} \) we have

\[
E(t) \leq \gamma_1 E(0)e^{-\gamma_2 t}, \quad \forall t > 0.
\]

**Proof.** (i) \( \Rightarrow \) (ii) It follows from (29) and (54) that

\[
E(T) \leq E(0) \leq C_0 \int_0^T \int_0^L [v^2_t(x, t) + p^2_t(x, t) + v^2_t(x, t - \tau(t))] \, dx \, dt \\
+ C_0 \int_0^T \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v^2_t(x, s) \, dx \, ds \, dt.
\]  

Integrating (29) over \((0, T)\), we have

\[
E(T) - E(0) \leq -C \int_0^T \int_0^L [v^2_t(x, t) + p^2_t(x, t) + v^2_t(x, t - \tau(t))] \, dx \, dt \\
- C \int_0^T \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v^2_t(x, s) \, dx \, ds \, dt.
\]
Then we arrive at
\[
\int_0^T \int_0^L \left[ v_t^2(x, t) + p_t^2(x, t) + v_t^2(x, t - \tau(t)) \right] dx dt
+ \int_0^T \int_{t-\tau(t)}^t \int_0^L e^{\lambda(s-t)} v_t^2(x, s) dx ds dt \leq \frac{1}{C} [E(T) - E(0)].
\] (67)

A combination of (66) and (67) gives
\[
E(T) \leq \frac{C_0}{C + C_0} E(0).
\]

The theory of semigroups (see Pazy [30]) implies that \( ||S(t)|| \leq C_0/(C + C_0), \forall T > 0 \). Then given \( t \in (0, T) \) and \( \delta > 0 \) there is \( n \in \mathbb{N} \) such that \( t = n\delta + r \) with \( 0 \leq r < \delta \) and in turn (see also [41])
\[
||S(t)|| \leq ||S(\delta)||^n ||S(r)|| \leq \left( \frac{C_0}{C + C_0} \right)^n \left( \frac{C_0}{C + C_0} \right)^t
= \left( 1 + \frac{C}{C_0} \right)^{-\frac{t}{\delta}} \left( 1 + \frac{C}{C_0} \right)^{-1+\frac{t}{\delta}} = \gamma_1 e^{-\gamma_2 t},
\]
where \( \gamma_1 = \left( 1 + \frac{C}{C_0} \right)^{-1+\frac{r}{\delta}} \) and \( \gamma_2 = \frac{1}{\delta} \ln \left( 1 + \frac{C}{C_0} \right) \). Consequently,
\[
E(t) \leq \gamma_1 E(0) e^{-\gamma_2 t},
\]
which implies the exponential decay.

\((ii) \Rightarrow (i)\) From (29) and Young’s inequality, we obtain
\[
E(0) - E(T) \leq \int_0^T \int_0^L \delta_1(t) v_t^2(x, t) dx dt + \delta_3 \int_0^T \int_0^L p_t^2(x, t) dx dt
+ \int_0^T \int_0^L \delta_2(t) v_t(x, t) v_t^2(x, t - \tau(t)) dx dt
+ \int_0^T \int_t^T \int_0^L \frac{\lambda \xi(t)}{2} e^{\lambda(s-t)} v_t^2(x, s) dx ds dt
+ \int_0^T \int_0^t \int_0^L \frac{\lambda \xi(t)}{2} e^{-\lambda(\tau(t)-\tau(t))} v_t^2(x, t - \tau(t))(1 - \tau'(t)) dx dt
- \int_0^T \int_0^t \int_0^L \frac{\lambda \xi(t)}{2} e^{\lambda(s-t)} v_t^2(x, s) dx ds dt
\leq \int_0^T \int_0^L \left( \delta_1(t) + \frac{\delta_2(t)}{2} \right) v_t^2(x, t) dx dt + \delta_3 \int_0^T \int_0^L p_t^2(x, t) dx dt
+ \int_0^T \int_0^L 1 + \xi(t) e^{-\lambda \tau_0} v_t^2(x, t - \tau(t)) dx dt
+ \int_0^T \int_t^T \int_0^L \left( \frac{\lambda \xi(t)}{2} - \frac{\xi'(t)}{2} \right) e^{\lambda(s-t)} v_t^2(x, s) dx ds dt
\[
\leq C_7 \int_0^T \int_0^L \left[ v_1^2(x,t) + p_1^2(x,t) + v_1^2(x, t - \tau(t)) \right] dx dt
+ C_7 \int_0^T \int_t^T \int_0^L e^{\lambda(s-t)} v_1^2(x,s) dx ds dt,
\]
where \( C_7 = \max \left\{ \delta_1(0) + \frac{\beta_0^2 \delta_3^2}{2}, \delta_3, \frac{1 + \xi \delta_1(0)e^{-\lambda \tau_0}}{2}, \frac{\xi \delta_1(0)(\lambda + M_1)}{2} \right\}. \)

It follows from the hypothesis (ii) that for \( T \geq T^* = \ln \left( \frac{4 \gamma_1}{\gamma_2} \right) / \gamma_2 \) the energy satisfies
\[
E(T) \leq \frac{3}{4} E(0).
\]

By (68) and (69), we arrive at
\[
E(0) \leq 4C_7 \int_0^T \int_0^L \left[ v_1^2(x,t) + p_1^2(x,t) + v_1^2(x, t - \tau(t)) \right] dx dt
+ 4C_7 \int_0^T \int_t^T \int_0^L e^{\lambda(s-t)} v_1^2(x,s) dx ds dt,
\]
which completes the proof.

Acknowledgments. This work was supported by the National Natural Science Foundation of China (Grant No. 11771216), the Key Research and Development Program of Jiangsu Province (Social Development) (Grant No. BE2019725), the Qing Lan Project of Jiangsu Province and the Postgraduate Research and Practice Innovation Program of Jiangsu Province (Grant No. KYC200945).

REFERENCES

[1] H. Y. S. Al-Zahrani, J. Pal, M. A. Migliorato, G. Tse and D. Yu, Piezoelectric field enhancement in III-V core-shell nanowires, \textit{Nano Energy}, 14 (2015), 382–391.

[2] V. Barros, C. Nonato and C. Raposo, Global existence and energy decay of solutions for a wave equation with non-constant delay and nonlinear weights, \textit{Electronic Research Archive}, 28 (2020), 205–220.

[3] A. Benaissa, A. Benguessoum and S. A. Messaoudi, Energy decay of solutions for a wave equation with a constant weak delay and a weak internal feedback, \textit{Electronic Journal of Qualitative Theory of Differential Equations}, 2014 (2014), 13 pp.

[4] A. Blanguernon, F. Léauté and M. Bernadou, Active control of a beam using a piezoceramic element, \textit{Smart Materials and Structures}, 8 (1999), 116–124.

[5] W. G. Cady, \textit{Piezoelectricity: An Introduction to the Theory and Applications of Electrical Phenomena in Crystals}, Dover Publications, New York, 1964.

[6] M. Chen, W. Liu and W. Zhou, Existence and general stabilization of the Timoshenko system of thermo-viscoelasticity of type III with frictional damping and delay terms, \textit{Advances in Nonlinear Analysis}, 7 (2018), 547–569.

[7] D. Damjanovic, Ferroelectric, dielectric and piezoelectric properties of ferroelectric thin films and ceramics, \textit{Reports on Progress in Physics}, 61 (1999), 1267–1324.

[8] G. Davi and A. Milazzo, Multidomain boundary integral formulation for piezoelectric materials fracture mechanics, \textit{International Journal of Solids and Structures}, 38 (2001), 7065–7078.

[9] J. M. Dietl, A. M. Wickenheiser and E. Garcia, A Timoshenko beam model for cantilevered piezoelectric energy harvesters, \textit{Smart Materials and Structures}, 19 (2010), 547–569.

[10] B. Feng and X. G. Yang, Long-time dynamics for a nonlinear Timoshenko system with delay, \textit{Applicable Analysis}, 96 (2017), 606–625.

[11] M. M. Freitas, A. J. A. Ramos, A. Özver and D. S. Almeida Júnior, Long-time dynamics for a fractional piezoelectric system with magnetic effects and Fourier’s law, \textit{Journal of Differential Equations}, 280 (2021), 891–927.
[12] C. Galassi, M. Dinescu, K. Uchino and M. Sayer, Piezoelectric materials: Advances in science, technology and applications, *Nato Science Partnership Subseries 3*, Springer, Berlin, 2000.

[13] A. Haraux, Two remarks on hyperbolic dissipative problems, *Research Notes in Mathematics Pitman*, 122 (1985), 161–179.

[14] H. Kawai, The Piezoelectricity of poly (vinylidene Fluoride), *Japanese Journal of Applied Physics*, 8 (1969), 975–976.

[15] T. Kato, *Linear and Quasi-Linear Equations of Evolution of Hyperbolic Type*, Summer Sch., 72, Springer, Heidelberg, 2011, 125–191.

[16] M. Kirane, B. Said-Houari and M. N. Anwar, Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks, *Communications on Pure and Applied Analysis*, 10 (2011), 667–686.

[17] G. Liu and L. Diao, Energy decay of the solution for a weak viscoelastic equation with a time-varying delay, *Acta Applicandae Mathematicae*, 155 (2018), 9–19.

[18] W. Liu, D. Chen and Z. Chen, Long-time behavior for a thermoelastic microbeam problem with time delay and the Coleman-Gurtin thermal law, *Acta Mathematica Scientia*, 41 (2021), 609–632.

[19] W. Liu and M. Chen, Well-posedness and exponential decay for a porous thermoelastic system with second sound and a time-varying delay term in the internal feedback, *Continuum Mechanics and Thermodynamics*, 29 (2017), 731–746.

[20] W. Liu and H. Zhuang, Global attractor for a suspension bridge problem with a nonlinear delay term in the internal feedback, *Discrete and Continuous Dynamical Systems-Series B*, 26 (2021), 907–942.

[21] S. A. Messaoudi and W. Al-Khulaifi, General and optimal decay for a viscoelastic equation with boundary feedback, *Topological Methods in Nonlinear Analysis*, 51 (2018), 413–427.

[22] S. A. Messaoudi, A. Fareh and N. Doudi, Well posedness and exponential stability in a wave equation with a strong damping and a strong delay, *Journal of Mathematical Physics*, 57 (2016), 13pp.

[23] K. A. Morris and A. Özer, Strong stabilization of piezoelectric beams with magnetic effects, in *52nd IEEE Conference on Decision and Control*, 2013, 3014–3019.

[24] K. A. Morris and A. Özer, Modeling and stabilizability of voltage-actuated piezoelectric beams with magnetic effects, *SIAM Journal on Control and Optimization*, 52 (2014), 2371–2398.

[25] S. Nicaise and C. Pignotti, Interior feedback stabilization of wave equations with time-dependent delay, *Electronic Journal of Differential Equations*, 2011 (2011), 20pp.

[26] S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, *Discrete and Continuous Dynamical Systems-Series S*, 2 (2009), 559–581.

[27] C. Nonato, M. J. Dos Santos, C. Raposo, Dynamics of Timoshenko system with time-varying weight and time-varying delay, *Discrete and Continuous Dynamical Systems-Series B*, in press.

[28] C. Nonato, C. Raposo and B. Feng, Exponential stability for a thermoelastic laminated beam with nonlinear weights and time-varying delay, *Asymptotic Analysis*, in press.

[29] R. L. Oliveira and H. P. Oquendo, Stability and instability results for coupled waves with delay term, *Journal of Mathematical Physics*, 61 (2020), 13pp.

[30] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[31] G. Poulin-Vitrant, C. Oshman, C. Opoku, A. S. Dahiya, N. Camara, D. Alquier, Hae, L. -P., T. H and M. Lethiecq, Fabrication and characterization of ZnO nanowire-based piezoelectric nanogenerators for low frequency mechanical energy harvesting, *Physics Procedia*, 70 (2015), 909–913.

[32] A. J. A. Ramos, M. M. Freitas, D. S. Almeida Jr., S. S. Jesus and T. R. S. Moura, Equivalence between exponential stabilization and boundary observability for piezoelectric beams with magnetic effect, *Zeitschrift Für Angewandte Mathematik Und Physik*, 70 (2019), 14pp.

[33] A. J. A. Ramos, A. Özer, M. M. Freitas, D. S. Almeida Jr. and J. D. Martins, Exponential stabilization of fully dynamic and electrostatic piezoelectric beams with delayed distributed damping feedback, *Zeitschrift Für Angewandte Mathematik Und Physik*, 72 (2021), 20pp.

[34] A. J. A. Ramos, C. S. L. Gonalves and S. S. Corrêa Neto, Exponential stability and numerical treatment for piezoelectric beams with magnetic effect, *ESAIM Mathematical Modelling and Numerical Analysis*, 52 (2018), 255–274.
[35] C. Raposo, J. A. D. Chuquipoma, J. A. J. Avila and M. L. Santos, Exponential decay and numerical solution for a Timoshenko system with delay term in the internal feedback, *International Journal of Analysis and Applications*, 3 (2013), 1–13.

[36] Z. Sabbagh, A. Khemmoudj, M. Ferhat and M. Abdelli, Existence of global solutions and decay estimates for a viscoelastic Petrovsky equation with internal distributed delay, *Rendiconti del Circolo Matematico di Palermo Series 2*, 68 (2019), 477–498.

[37] B. Said-Houari and Y. Laskri, A stability result of a Timoshenko system with a delay term in the internal feedback, *Applied Mathematics and Computation*, 217 (2010), 2857–2869.

[38] P. Wang and J. Hao, Asymptotic stability of memory-type Euler-Bernoulli plate with variable coefficients and time delay, *Journal of Systems Science and Complexity*, 32 (2019), 1375–1392.

[39] H. J. Xiang and Z. F. Shi, Static analysis for multi-layered piezoelectric cantilevers, *International Journal of Solids and Structures*, 45 (2008), 113–128.

[40] J. Yang, A Review of a few topics in piezoelectricity, *Applied Mechanics Reviewes*, 59 (2006), 335–345.

[41] Y. Zheng, W. Liu and Y. Liu, Equivalence between internal observability and exponential stabilization for suspension bridge problem, *Ricerche di Matematica*, in press.

[42] F. Zhu, M. B. Ward, J. F. Li and S. J. Milne, Core-shell grain structures and ferroelectric properties of Na$_{0.5}$K$_{0.5}$NbO$_3$-LiTaO$_3$-BiScO$_3$ piezoelectric ceramics, *Data in Brief*, 4 (2015), 34–39.

Received March 2021; revised April 2021.

E-mail address: aowenkong@126.com  
E-mail address: carlos.mat.nonato@hotmail.com  
E-mail address: wjliu@nuist.edu.cn  
E-mail address: jeremias@ufpa.br  
E-mail address: raposo@ufsj.edu.br