Unified products for alternative and pre-alternative algebras

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Abstract

The theory of unified product and extending structures for alternative and pre-alternative algebras are developed. It is proved that the extending structures of these algebras can be classified by using some non-abelian cohomology and deformation map theory.

1 Introduction

As a generalization of associative algebras, alternative algebra has been studied from different aspects by mathematicians. The first well-known non-associative example of alternative algebra is Cayley’s octonion numbers. The structure theory of the finite-dimensional alternative algebras was studied by M. Zorn in [16]. The theory of representations of alternative algebras was given by R. D. Schafer in [13] and N. Jacobson in [12]. Alternative bialgebra and pre-alternative bialgebra are investigated in [8] and [6]. For more general alternative algebra theory, see [14, 15, 7].

The extending structures problems for groups, associative algebras, Hopf algebras, Lie algebras, Leibniz algebras, left-symmetric algebras and Lie conformal algebras have been studied in [1, 2, 3, 4, 5, 9, 10, 11] respectively.

In this paper, we study extending structures for alternative and pre-alternative algebras. This paper is organized as follows. In Section 2, some preliminaries about alternative algebras are recalled. In Section 3, we introduce the concept of unified product $A\#V$ of alternative algebras associated with an extending datum $\Omega(A, V)$. The sufficient and necessary condition to ensure that $A\#V$ with a given canonical product is an alternative algebra is given. Then, we show that there exists an alternative algebra structure on $E$ such that $A$ is a subalgebra of $E$ if and only if $E$ is isomorphic to a unified product of $A$ and $V$. In Section 4, some special cases of unified products are given. In Section 5, we study flag extending structures of unified products. In Section 6, the classifying complements problem for alternative algebras is studied.

Throughout this paper, all vector spaces are assumed to be over an algebraically closed field $K$ of characteristic not equal to 2 and 3. Let $V$ be a vector space. The identity map from $V$ to $V$ is denoted by $id_V$ or $id$.

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2 Preliminaries

In this section, we will recall some basic definitions and facts about alternative algebras and pre-alternative algebras.

Definition 2.1. An alternative algebra is a vector space $A$ with a multiplication $\circ : A \times A \to A : (x, y) \mapsto x \circ y$ such that the following identities hold:

$$(x, y, z) = -(y, x, z), \quad (x, y, z) = -(z, x, y) \quad (2.1)$$

where $(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)$ is the associator of the elements $x, y, z \in A$.

Note that the above identity is equivalent to the following identity:

$$(x \circ y) \circ z - x \circ (y \circ z) + (y \circ x) \circ z - y \circ (x \circ z) = 0, \quad (2.2)$$

$$(x \circ y) \circ z - x \circ (y \circ z) + (x \circ z) \circ y - x \circ (y \circ z) = 0. \quad (2.3)$$

Definition 2.2. Let $A$ be an alternative algebra, $V$ be a vector space. A bimodule of $A$ over the vector space $V$ is a pair of linear maps $\triangleright : A \times V \to V, (x, v) \mapsto x \triangleright v$ and $\triangleleft : V \times A \to V, (v, x) \mapsto v \triangleleft x$ such that the following conditions hold:

$$\begin{align*}
(x \circ y) \triangleright v - x \triangleright (y \circ v) + (y \circ x) \triangleright v - y \triangleright (x \circ v) &= 0, \quad (2.4) \\
(v \triangleleft x) \triangleleft y - v \triangleleft (x \circ y) + (v \circ y) \triangleleft x - v \circ (y \circ x) &= 0, \quad (2.5) \\
v \triangleleft (x \circ y) - (v \triangleleft x) \triangleleft y + x \triangleright (v \circ y) - (x \triangleright v) \triangleleft y &= 0, \quad (2.6) \\
(x \circ y) \triangleright v - x \triangleright (y \circ v) + (x \circ v) \triangleleft y - x \circ (v \circ y) &= 0. \quad (2.7)
\end{align*}$$

hold for all $x, y \in A$ and $v \in V$.

Proposition 2.3. Let $A$ be an alternative algebra, $V$ be a vector space. Then $V$ is an $A$-bimodule if and only if $A \oplus V$ is an alternative algebra under the following multiplication:

$$(x, u) \circ (y, v) \triangleq (x \circ y, x \triangleright v + u \triangleleft y), \quad (2.8)$$

for all $x, y \in A$ and $u, v \in V$.

Definition 2.4. A pre-alternative algebra is a quadruple $(A, \triangleleft, \triangleright)$ in which $A$ is a vector space, $\triangleleft, \triangleright : A \otimes A \to A$ are bilinear maps such that for all $x, y, z \in A$,

$$\begin{align*}
(x \circ y) \triangleright z - x \triangleright (y \triangleright z) + (y \circ x) \triangleright z - y \triangleright (x \triangleright z) &= 0, \quad (2.9) \\
(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) + (x \triangleleft z) \triangleleft y - x \triangleleft (z \triangleleft y) &= 0, \quad (2.10) \\
(x \triangleright y) \triangleleft z - x \triangleright (y \triangleleft z) + (y \triangleright x) \triangleleft z - y \triangleright (x \triangleleft z) &= 0, \quad (2.11) \\
(x \triangleright y) \triangleleft z - x \triangleright (y \triangleleft z) + (x \triangleleft z) \triangleright y - x \triangleright (z \triangleright y) &= 0. \quad (2.12)
\end{align*}$$

where we denote by $x \circ y = x \triangleright y + x \triangleleft y$.

Theorem 2.5. Let $(A, \triangleleft, \triangleright)$ be a pre-alternative algebra. If we define the operation

$$x \circ y \triangleq x \triangleleft y + x \triangleright y,$$

Then $(A, \circ)$ is an alternative algebra, which is called the associated alternative algebra of $(A, \triangleleft, \triangleright)$ and denoted by $\text{Alt}(A) = (A, \circ)$. 

2
**Definition 2.6.** Let \((A, \prec, \succ)\) be a pre-alternative algebra. An \(A\)-bimodule is a vector space \(V\) together with four bilinear maps \(\prec, \succ: A \times V \to V, (x, v) \mapsto x \prec v, (v, x) \mapsto v \succ x\) satisfying the following conditions:

\[
(x \circ y + y \circ x) \succ v = x \succ (y \succ v) + y \succ (x \succ v),
\]
\[
(x \circ v + v \circ x) \succ y = x \succ (v \succ y) - v \succ (x \succ y),
\]
\[
(v \prec x) \prec y + (x \succ v) \prec y = v \prec (x \circ y) + x \succ (v \prec y),
\]
\[
(x \prec v) \prec y + (v \succ x) \prec y = x \prec (v \circ y) + v \succ (x \prec y),
\]
\[
(y \prec v) \prec x + (y \circ x) \prec v = y \prec (v \prec x) + y \succ (v \prec x),
\]
\[
(y \succ v) \prec x + (y \circ x) \succ v = y \succ (v \prec x) + y \succ (v \succ x),
\]
\[
(v \succ x) \prec y + (v \prec y) \prec x = v \prec (x \circ y + y \circ x),
\]
\[
(x \prec v) \prec y + (x \prec y) \prec v = x \prec (v \circ y + y \circ v),
\]

where \(x \circ y = x \prec y + x \succ y, x \circ v = x \prec v + x \succ v, v \circ x = v \prec x + v \succ x\).

**Definition 2.7.** Let \(A\) be an alternative algebra, \(E\) a vector space such that \(A\) is a subspace of \(E\) and \(V\) a complement of \(A\) in \(E\). For a linear map \(\varphi: E \to E\), the following diagram is considered:

\[
\begin{array}{ccc}
A & \xrightarrow{i} & E \\
\downarrow{\text{id}} & & \downarrow{\pi} \\
A & \xrightarrow{i} & E
\end{array}
\]

where \(\pi: E \to V\) is the natural projection of \(E = A \oplus V\) onto \(V\) and \(i: A \to E\) is the inclusion map. We say that \(\varphi: E \to E\) stabilizes \(A\) (resp. co-stabilizes \(V\)) if the left square (resp. the right square) of the above diagram is commutative.

Let \(\circ\) and \(\circ'\) be two alternative structures on \(E\) both containing \(A\) as an alternative subalgebra. If there exists an alternative algebra isomorphism \(\varphi: (E, \circ) \to (E, \circ')\) which stabilizes \(A, \circ\) and \(\circ'\) are called equivalent, which is denoted by \((E, \circ) \equiv (E, \circ')\).

If there exists an alternative algebra isomorphism \(\varphi: (E, \circ) \to (E, \circ')\) which stabilizes \(A\) and co-stabilizes \(V\), then \(\circ\) and \(\circ'\) are called cohomologous, which is denoted by \((E, \circ) \approx (E, \circ')\).

Obviously, \(\equiv\) and \(\approx\) are equivalence relations on the set of all alternative algebra structures on \(E\) containing \(A\) as an alternative subalgebra. Denote by \(\text{Extd}(E, A)\) (resp. \(\text{Extd}'(E, A)\)) the set of all equivalence classes via \(\equiv\) (resp. \(\approx\)). Thus, \(\text{Extd}(E, A)\) is the classifying object of the extending structures problem and \(\text{Extd}'(E, A)\) provides a classification of the extending structures problem from the point of view of the extension problem. In addition, it is easy to see that there exists a canonical projection \(\text{Extd}(E, A) \to \text{Extd}'(E, A)\).
3 Unified products for alternative algebras

In this section, we will introduce the concept of unified product for alternative algebras and give a theoretical answer to the extending structures problem.

Definition 3.1. Let \((A, \circ)\) be an alternative algebra and \(V\) a vector space. An extending datum of \(A\) by \(V\) is a system \(\Omega(A, V)\) consisting of four linear maps

\[\triangleright : A \times V \rightarrow V, \quad \triangleleft : V \times A \rightarrow V, \quad \rightarrow : V \times A \rightarrow A, \quad \leftarrow : A \times V \rightarrow A,\]

and two bilinear maps

\[\ast : V \times V \rightarrow V, \quad \omega : V \times V \rightarrow A.\]

Let \(\Omega(A, V) = (\triangleright, \triangleleft, \rightarrow, \leftarrow, \ast, \omega)\) be an extending datum. Denote by \(A \natural V\) the direct sum vector space \(A \oplus V\) with multiplication:

\[(x, u) \bullet (y, v) = (x \circ y + u \rightarrow y + x \leftarrow v + \omega(u, v), u \ast v + x \triangleright v + u \triangleleft y).\]

for all \(x, y \in A, u, v \in V\). Then \(A \natural V\) is called the unified product of \(A\) and \(V\) if it is an alternative algebra with the multiplication given by above. In this case, the extending datum \(\Omega(A, V)\) is called a extending structure of \(A\) by \(V\).

Theorem 3.2. Let \(A\) be an alternative algebra, \(V\) be a vector space and \(\Omega(A, V)\) an extending datum of \(A\) by \(V\). Then \(A \natural V\) is a unified product if and only if the following conditions hold for all \(x, y \in A, u, v \in V:\)

\[(A1) \quad (u \rightarrow x) \circ z + (x \leftarrow u) \circ z + (u \triangleleft x) \rightarrow z + (x \triangleright u) \leftarrow z = u \rightarrow (x \circ z) + x \circ (u \rightarrow z) + x \leftarrow (u \triangleleft z),\]

\[(A2) \quad (u \triangleleft x) \triangleleft z + (x \triangleright u) \triangleleft z = u \triangleleft (x \circ z) + x \triangleright (u \triangleleft z),\]

\[(A3) \quad \omega(u, v) \circ z + \omega(v, u) \circ z + (u \ast v) \rightarrow z + (v \ast u) \leftarrow z = u \rightarrow (v \rightarrow z) + v \rightarrow (u \rightarrow z) + \omega(u, v \triangleleft z) + \omega(v, u \triangleleft z),\]

\[(A4) \quad (u \ast v) \triangleleft z + (v \ast u) \triangleleft z = u \ast (v \triangleleft z) + v \ast (u \triangleleft z) + u \triangleleft (v \rightarrow z) + v \triangleleft (u \rightarrow z),\]

\[(A5) \quad (u \rightarrow x) \leftarrow w + (x \leftarrow u) \leftarrow w + \omega(u \triangleleft x, w) + \omega(x \triangleright u, w).\]
\[ u \mapsto (x \leftarrow w) + x \leftarrow (u * w) + x \circ (\omega(u, w)) + \omega(u, x \triangleright w), \]  
\[ (A6) \]

\[ (u \triangleleft x) * w + (x \triangleright u) * w + (u \mapsto x) \triangleright w + (x \leftarrow u) \triangleright w \]
\[ = u * (x \triangleright w) + u \triangleleft (x \leftarrow w) + x \triangleright (u * w), \]
\[ (A7) \]

\[ (x \circ y) \leftarrow w + (y \circ x) \leftarrow w \]
\[ = x \circ (y \leftarrow w) + y \circ (x \leftarrow w) + x \leftarrow (y \triangleright w) + y \leftarrow (x \triangleright w), \]
\[ (A8) \]

\[ (x \circ y) \triangleright w + (y \circ x) \triangleright w = x \triangleright (y \triangleright w) + y \triangleright (x \triangleright w), \]
\[ (A9) \]

\[ \omega(u, v) \leftarrow w + \omega(v, u) \leftarrow w + \omega(u * v, w) + \omega(v * u, w) \]
\[ = u \mapsto \omega(v, w) + v \mapsto \omega(u, w) + \omega(u, v * w) + \omega(v, u * w), \]
\[ (A10) \]

\[ (u * v) * w + (v * u) * w + \omega(u, v) \triangleright w + (\omega(v, u)) \triangleright w \]
\[ = u * (v * w) + v * (u * w) + u \triangleleft (\omega(v, w)) + v \triangleleft (\omega(u, w)), \]
\[ (A11) \]

\[ (u \mapsto y) \circ z + (u \triangleleft y) \rightarrow z + (u \rightarrow z) \circ y + (u \triangleright z) \rightarrow y \]
\[ = u \mapsto (y \circ z) + u \rightarrow (z \circ y), \]
\[ (A12) \]

\[ (u \triangleleft y) \circ z + (u \triangleright z) \circ y = u \triangleleft (y \circ z) + u \triangleright (z \circ y), \]
\[ (A13) \]

\[ (x \leftarrow v) \circ y + (x \triangleright v) \rightarrow y + (x \circ y) \leftarrow v \]
\[ = x \circ (v \rightarrow y) + x \leftarrow (v \circ y) + x \circ (y \leftarrow v) + x \leftarrow (y \triangleright v), \]
\[ (A14) \]

\[ (x \triangleright v) \triangleleft y + (x \circ y) \triangleright v = x \triangleright (v \triangleleft y) + x \triangleright (y \triangleright v), \]
\[ (A15) \]
\[(A16)\]
\[(u \ast v) \triangleleft y + (u \triangleleft y) \ast v + (u \rightarrow y) \triangleright v = u \ast (v \triangleleft y) + u \ast (y \triangleright v) + u \triangleleft (v \rightarrow y),\]

\[(A17)\]
\[(x \leftarrow v) \leftarrow w + (x \leftarrow w) \leftarrow v + \omega(x \triangleright v, w) + \omega(x \triangleright v, w) = x \circ \omega(w, v) + x \circ \omega(v, w) + x \leftarrow (w \ast v) + x \leftarrow (v \ast w),\]

\[(A18)\]
\[(x \triangleright v) \triangleright w + (x \triangleright w) \triangleright v + (x \leftarrow v) \triangleright w = x \triangleright (w \ast v) + x \triangleright (v \ast w),\]

\[(A19)\]
\[(u \ast v) \ast w + (u \ast w) \ast v + (u \triangleright v) \triangleright w + (u \triangleright v) \triangleright w = u \ast (v \ast w) + u \ast (w \ast v) + u \triangleleft (\omega(v, w)) + u \triangleleft (\omega(w, v)).\]

**Proof.** Define
\[R((x, u), (y, v), (z, w)) = ((x, u), (y, v), (z, w)) + ((y, v), (x, u), (z, w)),\]
where \((x, u), (y, v), (z, w) \in A \oplus V\). Note that \(A \ast V\) is an alternative algebra if and only if
\[R((x, u), (y, v), (z, w)) = 0\]
for all \(x, y, z \in A\) and \(u, v, w \in V\). The proof is by direct but tedious computations. \(\square\)

**Remark 3.3.** In fact, from conditions (A2), (A8), (A12) and (A14), one show that \(V\) is a bimodule of \(A\).

Given an extending structure \(\Omega(A, V)\). It is obvious that \(A\) can be seen as an alternative subalgebra of \(A \ast V\). Conversely, we will prove that any alternative algebra structure on a vector space \(E\) containing \(A\) as a subalgebra is isomorphic to a unified product.

**Theorem 3.4.** Let \((A, \circ)\) and \((E, \circ)\) be alternative algebras such that \(E\) containing \(A\) as a subalgebra of \(E\). Then, there exists an extending structure \(\Omega(A, V)\) of \(A\) by a subspace \(V\) of \(E\) and an isomorphism of alternative algebras \((E, \circ) \cong A \ast V\) which stabilizes \(A\) and co-stabilizes \(V\).

**Proof.** Note that there is a natural linear map \(p : E \to A\) such that \(p(x) = x\) for all \(x \in A\). Set \(V = \text{Ker}(p)\) which is a complement of \(A\) in \(E\). Then, we define the extending datum \(\Omega(A, V)\) of \(A\) by a subspace \(V\) of \(E\) as follows:
\[-\to: V \times A \to A, \quad u \to x := p(u \circ x), \quad \triangleleft : V \times A \to V, \quad u \leftarrow x := u \circ x - p(u \circ x)\]
\[-\leftarrow: A \times V \to A, \quad x \leftarrow u := p(x \circ u), \quad \triangleright : A \times V \to V, \quad x \triangleright u := x \circ u - p(x \circ u)\]
\[-\omega : V \times V \to A, \quad \omega(u, v) := p(u \circ v), \quad \ast : V \times V \to V, \quad u \ast v := u \circ v - p(u \circ v)\]
for all \( x, y \in A, u, v \in V \). It is easy to see that \( \varphi : A \times V \to E \) defined as \( \varphi(x, u) = x + u \) is a linear isomorphism, whose inverse is as follows: \( \varphi^{-1}(e) := (p(e), e - p(e)) \) for all \( e \in E \). Next, we should prove that \( \Omega(A, V) \) is an extending structure of \( A \) by \( V \) and \( \varphi : A \sharp V \to E \) is an isomorphism of alternative algebras that stabilizes \( A \) and co-stabilizes \( V \). In fact, if \( \varphi : A \times V \to E \) is an isomorphism of alternative algebras, there exists a unique left-symmetric product given by

\[
(x, u) \circ (y, v) = \varphi^{-1}(\varphi(x, u) \circ \varphi(y, v)).
\]

(3.1)

Therefore, for completing the proof, we only need to check that the product defined by above is just the one given in the above extending system \( \Omega(A, V) \). Indeed, we have

\[
(x, u) \circ (y, v) = \varphi^{-1}(\varphi(x, u) \circ \varphi(y, v)) = \varphi^{-1}((x + u) \circ (y + v))
\]

\[
= \varphi^{-1}(x \circ y + x \circ v + u \circ y + u \circ v)
\]

\[
= (x \circ y + p(x \circ v) + p(u \circ y) + p(u \circ v),
\]

\[
x \circ v + u \circ y + u \circ v - p(x \circ v) - p(u \circ y) - p(u \circ v))
\]

\[
= (x \circ y + u \Rightarrow y + x \leftarrow v + \omega(u, v), u \ast v + x \triangleright v + u \triangleright y)
\]

\[
= (x, u) \bullet (y, v).
\]

for all \( x, y \in A, u, v \in V \).

Therefore, \( \varphi : A \sharp V \to E \) is an isomorphism of alternative algebras and the following diagram is commutative

\[
\begin{array}{ccc}
A & \overset{i}{\longrightarrow} & A \sharp V \\
\downarrow{\text{id}} & & \downarrow{\varphi} \\
A & \overset{i}{\longrightarrow} & E \\
\downarrow{\text{id}} & & \downarrow{\pi} \\
& & V
\end{array}
\]

where \( q : A \sharp V \to V \) and \( \pi : E \to V \) are the natural projections. The proof is finished. \( \Box \)

Next, by the above Theorem, for classifying all alternative algebra structures on \( E \) containing \( A \) as a subalgebra, we only need to classify all unified products \( A \sharp V \) associated to all alternative algebra structures \( \Omega(A, V) \) for a given complement \( V \) of \( A \) in \( E \).

**Lemma 3.5.** Let \( \Omega(A, V) \) and \( \Omega'(A, V) \) be two extending structures of \( A \) by \( V \) and \( A \sharp V \), \( A \sharp V \) be the corresponding unified products. Then, there is a bijection between the set of all morphisms of alternative algebras \( \varphi : A \sharp V \to A \sharp' V \) which stabilizes \( A \) and the set of pairs \( (r, s) \), where \( r : V \to A, s : V \to V \) are linear maps satisfying the following conditions:

\[
s(u \triangleleft x) = s(u) \triangleright x,
\]

(3.2)

\[
s(x \triangleright u) = x \triangleright s(u),
\]

(3.3)

\[
s(u \ast v) = r(u) \triangleright s(v) + s(u) \triangleright r(v) + s(u) \triangleright s(v),
\]

(3.4)

\[
r(u \triangleleft x) = r(u)x - u \rightarrow x + s(u) \triangleright x,
\]

(3.5)

\[
r(x \leftarrow u) = x \circ r(u) - x \leftarrow u + x \leftarrow v(u),
\]

(3.6)

\[
r(u \ast v) = r(u) \circ r(v) + \omega'(s(u), s(v))
\]

\[7\]
In addition, $\varphi$ should prove that $s$ costabilizes $V$ if and only if $s : V \to V$ is an isomorphism, and $\varphi_{r,s}$ costabilizes $V$ if and only if $s = id_V$.

**Proof.** Let $\varphi : A^*_V \to A^*_V$ be a homomorphism of alternative algebras. Since $\varphi$ stabilizes $A$, $\varphi(x,0) = (x,0)$. Moreover, we can set $\varphi(0,u) = (r(u),s(u))$, where $r : V \to A$, $s : V \to V$ are two linear maps. Therefore, we get $\varphi(x,u) = (x+r(u),s(u))$. Then, we should prove that $\varphi$ is a homomorphism of alternative algebras if and only if the above conditions hold. It is enough to check that

$$\varphi((x,u) \bullet (y,v)) = \varphi(x,u) \bullet' \varphi(y,v)$$

holds for all generators of $A^*_V$. Obviously, this equation holds for the pair $(x,0), (y,0)$. Then, we consider the pair $(x,0), (0,u)$. According to

$$\varphi((x,0) \bullet (0,u)) = \varphi(x \leftarrow u, x \triangleright u)$$

$$= (x \leftarrow u + r(x \leftarrow u), s(x \triangleright u),$$

and

$$\varphi(x,0) \bullet \varphi(0,u) = (x,0) \circ (r(u),s(u))$$

$$= (x \circ r(u) + x \leftarrow s(u), x \triangleright s(u),$$

we get that this equation holds for the pair $(x,0), (0,u)$ if and only if the first two conditions hold. Similarly, it is easy to check that the other conditions hold for the pair $(0,u), (x,0)$ and $(0,u), (0,y)$.

Assume that $\varphi_{r,s}$ is bijective. It is obvious that $s$ is surjective. Then, we only need to prove that $s$ is injective. Let $x \in V$ such that $s(u) = 0$. Then, we get $\varphi_{r,s}(-r(u),x) = (-r(u) + r(u),s(u)) = (0,0)$. Thus, $x = 0$, i.e. $s$ is injective. Conversely, assume that $s : V \to V$ is bijective. Then, $\varphi_{r,s}$ has the inverse given by $\varphi_{r,s}^{-1}(x,u) = (x - r(s^{-1}(u)), s^{-1}(u))$. Thus, by the first part, $\varphi_{r,s}$ is an isomorphism. Therefore, $\varphi_{r,s}$ is an isomorphism if and only if $s : V \to V$ is an isomorphism. Finally, it is obvious that $\varphi_{r,s}$ costabilizes $V$ if and only if $s = id_V$. The proof is finished. \hfill $\Box$

**Definition 3.6.** Two $\Omega(A,V)$ and $\Omega'(A,V)$ are called equivalent if there exists a pair of linear maps $(r,s)$ where $r : V \to A$, $s : V \to V$ such that the following conditions hold:

$$u \triangleleft x = s^{-1}(s(u) \triangleleft' x),$$

$$x \triangleright u = s^{-1}(x \triangleright' s(u)),$$

$$u \circ v = s^{-1}(r(u) \triangleright' s(v)) = s(u) \triangleleft' r(v) + s(u) \triangleright' s(v),$$
\[ u \bowtie x = r^{-1}(r(u)x - u \rightarrow x + v(u) \rightarrow' x), \quad (3.12) \]
\[ x \leftarrow u = r^{-1}(rr(u) - x \leftarrow u + x \leftarrow' s(u)), \quad (3.13) \]
\[ u \circ v = r^{-1}(r(u)r(v) + \omega'(s(u), s(v)) - \omega(u, v) + r(u) \leftarrow' s(v) + s(u) \rightarrow' r(v)), \quad (3.14) \]

for all \( x, y \in A, u, v \in V \), then \( \Omega(A, V) \) and \( \Omega'(A, V) \) are called equivalent and we denote it by \( \Omega(A, V) \equiv \Omega'(A, V) \).

Moreover, in case \( s = id_V \), the above conditions are reduced to
\[
\begin{align*}
    u \rightarrow x &= r(u) \cdot x + u \rightarrow' x - r(u' \cdot x), \\
    x \leftarrow u &= x \cdot r(u) + x \leftarrow' u - r(x \rightarrow' u), \\
    u \circ v &= r(u) \rightarrow' v + u \leftarrow' r(v) + u \circ v, \\
    \omega(u, v) &= r(u) \cdot r(v) + r(u) \leftarrow' v + u \rightarrow' r(v) + \omega'(u, v)
\end{align*}
\]
\[ (3.15) \]
\[ (3.16) \]
\[ (3.17) \]
\[ (3.18) \]
\[ (3.19) \]

\( \Omega(A, V) \) and \( \Omega'(A, V) \) are called cohomologous and we denote it by \( \Omega(A, V) \approx \Omega'(A, V) \).

Then, by the above discussion, the answer for the extending structures problem of alternative algebras is given as follows:

**Theorem 3.7.** Let \( A \) be an alternative algebra, \( E \) a vector space that contains \( A \) as a subspace and \( V \) a complement of \( A \) in \( E \). Then, we get:

1. Denote \( H^2_A(V, A) := \mathfrak{S}(A, V)/\equiv \). Then, the map
\[
    H^2_A(V, A) \rightarrow Ext(E, A), \quad \overline{\Omega(A, V)} \rightarrow (A \oplus V, \circ)
\]
\[ (3.20) \]

is bijective, where \( \overline{\Omega(A, V)} \) is the equivalence class of \( \Omega(A, V) \) under \( \equiv \).

2. Denote \( H^2(V, A) := \mathfrak{S}(A, V)/\approx \). Then, the map
\[
    H^2(V, A) \rightarrow Ext'(E, A), \quad \overline{\Omega(A, V)} \rightarrow (A \oplus V, \circ)
\]
\[ (3.21) \]

is bijective, where \( \overline{\Omega(A, V)} \) is the equivalence class of \( \Omega(A, V) \) under \( \approx \).

Finally, we give the definition of unified product for pre-alternative algebras.

**Definition 3.8.** Let \((A, \prec, \succ)\) be a pre-alternative algebra and \( V \) be a vector space. An extending datum of \( A \) by \( V \) is a system consisting eight linear maps
\[
\begin{align*}
    \prec, \succ &: A \times V \rightarrow V, \\
    \prec, \succ &: V \times A \rightarrow V, \\
    \prec, \succ &: V \times A \rightarrow A, \\
    \prec, \succ &: A \times V \rightarrow A
\end{align*}
\]

and four bilinear maps
\[
\begin{align*}
    \prec, \succ &: V \times V \rightarrow V, \\
    \omega_\prec, \omega_\succ &: A \times A \rightarrow V.
\end{align*}
\]

Let \( \Omega(A, V) \) be an extending datum. Denote by \( A \oplus V \) the direct sum vector space \( A \oplus V \) with the products
\[
    (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[ (x, u) \ll (y, v) = (x \bowtie y + x \bowtie v + u + y + \omega_\prec(u, v), u + x \bowtie v + u \bowtie y),
\]
\[(x, u) \triangleright (y, v) = (x \succ y + x \succ v + u > y + \omega_{\succ}(u, v), u \succ v + x > v + u > y).\]

for all \(x, y \in A, u, v \in V\). Then \(A \# V\) is called the unified product of \(A\) and \(V\) if it is a pre-alternative algebra with the products given by above. In this case, the extending datum \(\Omega(A, V)\) is called an extending structure of \(A\) by \(V\).

The conditions for a unified product to be a pre-alternative algebra are in the Appendix of this paper.

4 Special cases of unified products

In this section, we show that crossed products and matched pairs of two alternative algebras are both special cases of unified products.

4.1 Crossed products

**Definition 4.1.** Let \((A, \circ)\) and \((B, \ast)\) be two alternative algebras. Then \((A, B)\) is called a crossed system if there exists bilinear maps

\[\rightarrow: B \times A \rightarrow A, \quad \leftarrow: A \times B \rightarrow A, \quad \omega: B \times B \rightarrow A\]

such that the following products on the direct sum space \(A \oplus B\):

\[(x, u) \bullet (y, v) = (x \circ y + u \leftarrow y + x \leftarrow v + \omega(u, v), u \ast v)\]  \hspace{1cm} (4.1)

define an alternative algebra structure. This alternative algebra is called the crossed product of \(A\) and \(B\) and we denote this alternative algebra by \(A \# \omega B\).

**Theorem 4.2.** Let \(A\) and \(B\) be two alternative algebras. Then \((A, B)\) is a crossed products if and only if the following conditions hold:

\[(u \rightarrow x) \circ z + (x \leftarrow u) \circ z = u \rightarrow (x \circ z) + x \circ (u \rightarrow z)\]  \hspace{1cm} (4.2)

\[(x \circ y) \leftarrow w + (y \circ x) \leftarrow w = x \circ (y \leftarrow w) + y \circ (x \leftarrow w),\]  \hspace{1cm} (4.3)

\[(u \rightarrow y) \circ z + (u \rightarrow z) \circ y = u \rightarrow (y \circ z) + u \rightarrow (z \circ y),\]  \hspace{1cm} (4.4)

\[(x \leftarrow v) \circ y + (x \circ y) \leftarrow v = x \circ (v \leftarrow y) + x \circ (y \leftarrow v),\]  \hspace{1cm} (4.5)

\[(u \rightarrow x) \leftarrow w + (x \leftarrow u) \leftarrow w = u \rightarrow (x \leftarrow w) + x \leftarrow (u \ast w) + x \circ \omega(u, w),\]  \hspace{1cm} (4.6)

\[u \rightarrow (v \leftarrow y) + u \rightarrow (y \leftarrow v) = (u \ast v) \rightarrow y + (u \rightarrow y) \leftarrow v + \omega(u, v) \circ y,\]  \hspace{1cm} (4.7)
\[(x \leftarrow v) \leftarrow w + (x \leftarrow w) \leftarrow v\]
\[= x \circ \omega(v,w) + x \circ \omega(v,w) + x \leftarrow (w \ast v) + x \leftarrow (v \ast w), \quad (4.8)\]

\[u \rightarrow (v \rightarrow z) + v \rightarrow (u \rightarrow z)\]
\[= \omega(u,v) \circ z + \omega(v,u) \circ z + (u \ast v) \rightarrow z + (v \ast u) \rightarrow z, \quad (4.9)\]

\[\omega(u,v) \leftarrow w + (\omega(v,u)) \leftarrow w + \omega(u \ast v,w) + \omega(v \ast u,w)\]
\[= u \rightarrow \omega(v,w) + v \rightarrow \omega(u,w) + \omega(u,v \ast w) + \omega(v,u \ast w), \quad (4.10)\]

### 4.2 Matched pair and the factorization problem

**Definition 4.3.** Let \((A, \circ)\) and \((B, \ast)\) be two alternative algebras. Then \((A, B)\) is called a matched pair if there exists bilinear maps

\[\triangleright : A \times B \rightarrow B, \quad \triangleleft : B \times A \rightarrow B, \quad \rightarrow : B \times A \rightarrow A, \quad \leftarrow : A \times B \rightarrow A\]

such that the following multiplication on the direct sum space \(A \oplus B\):

\[(x, u) \bullet (y, v) = (x \circ y + u \rightarrow y + x \leftarrow v, v \ast v + x \leftarrow v + u \triangleleft v) \quad (4.11)\]

define an alternative algebra structure. The above multiplication is called bicrossed product of \(A\) and \(B\). We will denote it by \(A \bowtie B\).

**Theorem 4.4.** Let \(A\) and \(B\) be two alternative algebras. Then \((A, B)\) is a matched pair if and only if the following conditions hold:

\[(u \triangleright x + u \triangleleft x) \leftarrow y + (u \rightarrow x + x \leftarrow u) \circ y\]
\[= u \rightarrow (x \circ y) + x \leftarrow (u \triangleleft y) + x \circ (y \leftarrow u), \quad (4.12)\]

\[(x \circ y + y \circ x) \leftarrow u\]
\[= x \leftarrow (y \triangleright u) + x \circ (y \leftarrow u) + y \leftarrow (x \triangleright u) + y \circ (x \leftarrow u), \quad (4.13)\]

\[(x \circ y) \leftarrow u + (x \triangleright u) \rightarrow y + (x \leftarrow u) \circ y\]
\[= x \leftarrow (u \triangleleft y + y \triangleright u) + x \circ (u \rightarrow y + y \leftarrow u), \quad (4.14)\]

\[u \rightarrow (x \circ y + y \circ x)\]
\[= (u \rightarrow x) \circ y + (u \triangleleft x) \rightarrow y + (u \rightarrow y) \circ x + (u \triangleleft y) \rightarrow x, \quad (4.15)\]

\[(u \rightarrow x + x \leftarrow u) \triangleright v + (x \triangleright u + u \triangleleft x) \ast v\]
\[= x \triangleright (u \ast v) + u \triangleleft (x \leftarrow v) + u \ast (x \triangleright v), \quad (4.16)\]

\[(u \ast v + v \ast u) \triangleleft x\]
\[= u \triangleleft (v \rightarrow x) + u \ast (v \triangleleft x) + v \triangleleft (u \rightarrow x) + v \ast (u \triangleleft x), \quad (4.17)\]

\[(u \ast v) \triangleleft x + (u \rightarrow x) \triangleright v + (u \triangleleft x) \ast v\]
\[= x \triangleleft (v \leftarrow x + x \rightarrow v) + u \ast (v \triangleright x + x \triangleleft v), \quad (4.18)\]

\[x \triangleright (u \ast v + v \ast u)\]
\[= (x \triangleright u) \ast v + (x \leftarrow u) \triangleright v + (x \triangleright v) \ast u + (x \leftarrow v) \triangleright u \quad (4.19)\]

for all \(x, y \in A\) and \(u, v \in V\).
**Definition 4.5.** Let \((A, \prec_A, \succ_A)\) and \((B, \prec, \succ)\) be two pre-alternative algebras. Then \((A, B)\) is called a matched pair if there exits eight bilinear maps

\[
\prec: A \times B \to B, \quad \succ: B \times A \to B, \quad \prec: B \times A \to A, \quad \succ: A \times B \to A
\]

such that the following products on the direct sum space \(A \oplus B\):

\[
(x, u) \ll (y, v) = (x \prec y + x \prec v + u \prec y), \quad (4.20)
\]
\[
(x, u) \gg (y, v) = (x \succ y + x \succ v + u \succ y) \quad (4.21)
\]
define a pre-alternative algebra structure. We denote this pre-alternative algebra by \(A \bowtie \bowtie B\).

### 4.3 Classifying complements for alternative algebras

In this subsection, we will study the classifying complements problem for alternative algebras using the concept of deformation map. An alternative subalgebra \(B\) of \((E, \circ)\) is called an \(A\)-complement of \((E, \circ)\) if \(E = A \oplus B\). If \(B\) is an \(A\)-complement in \((E, \circ)\), then we get \(E \cong A \bowtie B\) for some bicrossed product of \(A\) and \(B\). For an alternative subalgebra \(A\) of \((E, \circ)\), denote \(\mathcal{F}(A, E)\) the set of the isomorphism classes of all \(A\)-complements in \(E\).

**Definition 4.6.** Let \((A, B)\) be a matched pair of alternative algebras. A linear map \(r: B \to A\) is called a deformation map of the matched pair \((A, B)\) if \(r\) satisfies the following condition for all \(u, v \in B\):

\[
r(u \circ v) - r(u) \circ r(v) = u \rightarrow r(v) + r(u) \leftarrow v - r(r(u) \triangleright v + u \triangleleft r(v)). \quad (4.22)
\]

Denote by the set of all deformation maps of the matched pair \((A, B)\) by \(\mathcal{DM}(B, A)\).

**Theorem 4.7.** Let \(A\) be an alternative subalgebra of \((E, \circ)\), \(B\) a given \(A\)-complement of \(E\) with the associated matched pair \((A, B)\).

1. Given a deformation map \(r: B \to A\). Let \(f_r: B \to E = A \bowtie B\) be the linear map defined as

\[
f_r(u) = (r(u), u)
\]

for all \(u \in B\). Then \(\hat{B} := \text{Im}(f_r)\) is an alternative subalgebra of \(E = A \bowtie B\).

Let \(r: B \to A\) be a deformation map of the matched pair. Then, \(B_r := B\) as a vector space is an alternative algebra with the new product given as follows: \(\forall u, v \in B:\)

\[
u \circ_r v := u \circ v + r(v) \triangleright v + u \triangleleft r(v)). \quad (4.23)
\]

\(B_r\) is called the \(r\)-deformation of \(B\). Moreover, \(B_r\) is an \(A\)-complement of \(E\).

2. \(\overline{B}\) is an \(A\)-complement of \(E\) if and only if \(\overline{B}\) is isomorphic to \(B_r\) for some deformation map \(r: B \to A\) of the matched pair \((A, B)\).
Proof. (1) Given a deformation map $r : B \to A$. By the definition of deformation map, we get that for all $u, v \in B$,
\[
(r(u), u) \bullet (r(v), v) = \left( r(u) \circ r(v) + u \rightharpoonup r(v) + r(u) \rightharpoonup v, u \circ v + u \triangleleft r(v) + r(u) \triangleright v \right)
\]
Therefore, $[(r(u), u), (r(v), v)] \in \text{Im}(f_r)$. Thus, $\text{Im}(f_r)$ is an alternative subalgebra of $E = A \bowtie B$. It is easy to see that $A \cap \text{Im}(f_r) = \{0\}$ and $(x, u) = (x - r(u), 0) + (r(u), u) \in A + B$ for all $x \in A, u \in B$. Hence, $\text{Im}(f_r)$ is an $A$-complement of $E = A \bowtie B$.

Next we prove that $B_r$ and $\text{Im}(f_r)$ are isomorphic as alternative algebras. Denote by $\tilde{f}_r : B \to \text{Im}(f_r)$ the linear map induced by $f_r$. Obviously, $\tilde{f}_r$ is a linear isomorphism. Now we prove that $\tilde{f}_r$ is an alternative algebra homomorphism if the product of $B$ is given by the equation in \([123]\). For any $u, v \in B$, we get
\[
\tilde{f}_r(u \circ_r v) = \tilde{f}_r(u \circ v + u \triangleleft r(v) + r(u) \triangleright v) = \left( r(u) \circ r(v) + u \rightharpoonup r(v) + r(u) \rightharpoonup v, u \circ v + u \triangleleft r(v) + r(u) \triangleright v \right) = (r(u), u) \bullet (r(v), v) = \tilde{f}_r(u) \bullet \tilde{f}_r(v)
\]
Thus, $B_r$ is an alternative algebra.

(2) The proof is similar as in \([3\) Theorem 5.3] so we omit the details. \hfill \square

Definition 4.8. Let $(A, B)$ be a matched pair of alternative algebras. For two deformation maps $r, r' : B \to A$, if there exists $\sigma : B \to B$ a linear automorphism of $B$ such that for all $u, v \in B$:
\[
\sigma(u \cdot v) - \sigma(u) \cdot \sigma(v) = \sigma(u) \triangleleft r'(\sigma(v)) + r'(\sigma(u)) \rightharpoonup \sigma(v) - \sigma(u \triangleleft r(v)) - \sigma(r(u) \triangleright v).
\]
Then $r$ and $r'$ are called equivalent. Denote it by $r \sim r'$.

Theorem 4.9. Let $A$ be an alternative subalgebra of $E$, $B$ an $A$-complement of $E$ and $(A, B)$ the associated matched pair. Then, $\sim$ is an equivalence relation on the set $\mathcal{DM}(B, A)$ and the map
\[
\mathcal{HC}^2(B, A) := \mathcal{DM}(B, A)/ \sim \to \mathcal{F}(A, B), \ \mathfrak{r} \mapsto B_r,
\]
is a bijection between $\mathcal{HC}^2(B, A)$ and the isomorphism classes of all $A$-complements of $E$. In particular, we define the factorization index of $A$ in $E$ as $[E : A] := |\mathcal{F}(A, E)| = |\mathcal{HC}^2(B, A)|$.

Proof. It is easy to see that two deformation maps $r$ and $r'$ are equivalent if and only if the corresponding alternative algebras $B_r$ and $B_{r'}$ are isomorphic. Thus we obtain the result. \hfill \square
5 Flag extending structures

In this section, we mainly study the extending structure of $A$ by a 1-dimensional vector space $V$ which is called flag extending structures.

**Definition 5.1.** Let $A$ be an alternative algebra. A flag datum of $A$ consists of the following datum: $\lambda, \mu : A \to k$ are algebraic maps and $D, T : A \to A$ are linear maps satisfying for $x, y \in A, x_0 \in A$ and $k_0 \in K$:

(C1) \[ \mu(D(x)) = 0, \quad \lambda(T(x)) = 0, \]

(C2) \[ T(x_0) = D(x_0), \quad \lambda(x_0) = \mu(x_0), \]

(C3) \[ \mu(x \circ y) = \mu(x)\mu(y), \quad \lambda(x \circ y) = \lambda(x)\lambda(y), \]

(C4) \[ D^2(x) + \mu(x)x_0 = x_0 \circ x + k_0D(x), \]

(C5) \[ T^2(x) + \lambda(x)x_0 = x \circ x_0 + k_0T(x_0), \]

(C6) \[ \mu(x)k_0 + \lambda(D(x)) = \lambda(x)k_0 + \mu(T(x)) + \nu(D(x)), \]

(C7) \[ \lambda(x)k_0 + \mu(T(x)) = \nu(x)k_0 + \lambda(D(x)) + \lambda(T(x)), \]

(C8) \[ x \circ D(y) + \mu(y)T(x) + D(x \circ y) = D(x) \circ y + T(x) \circ y + \mu(x)D(y) + \lambda(x)D(y), \]

(C9) \[ T(x) \circ y + \lambda(x)D(y) + T(x \circ y) = x \circ D(y) + \mu(y)T(x) + x \circ T(y) + \lambda(y)T(x), \]

(C10) \[ T(x \circ y + y \circ x) = x \circ T(y) + y \circ T(x) + \lambda(y)T(x) + \lambda(x)T(y), \]
\( D(x \circ y + y \circ x) = D(x) \circ y + D(y) \circ x + \mu(x)D(y) + \mu(y)D(x), \)

(C12)

\[ T^2(x) + T(D(x)) + \mu(x)x_0 = D(T(x)) + k_0T(x) + x \circ x_0, \]

(C13)

\[ D^2(x) + D(T(x)) + \lambda(x)x_0 = k_0D(x) + T(D(x)) + x_0 \circ x. \]

Denote by \( \mathcal{F}(A) \) the set of all flag datums of \( A \).

**Proposition 5.2.** Let \( A \) be an alternative algebra and \( V \) a vector space of dimension 1 with basis \( \{u\} \). Then there exists a bijection between the set \( \text{Exd}(A,V) \) of all extending structures of \( A \) through \( V \) and the set \( \mathcal{F}(A) \) of all flag datums of \( A \).

Through the above bijection, the unified product corresponding to \( (\Lambda, \lambda, D, d, a_0, u) \in \mathcal{F}(A) \) will be denoted by \( A \kappa_{(\Lambda, \lambda, D, d, a_0, u)} \{u\} \) and has the multiplication given for any \( x, y \in A \) by:

\[
(x, u) \bullet (y, u) := \left( x \circ y + T(x) + D(y) + x_0, \lambda(x)u + \mu(y)u + k_0u \right) \tag{5.1}
\]

That is \( A \kappa_{(\Lambda, \lambda, D, d, a_0, u)} \) \( x \) is the algebra generated by the algebra \( A \) and \( x \) subject to the relations:

\[
u \bullet u = x_0 + k_0u, \quad x \bullet u = T(x) + \lambda(x)u, \quad u \bullet x = D(x) + \mu(x)u. \tag{5.2}
\]

**Definition 5.3.** Let \( A \) be a pre-alternative algebra. A **flag datum** of \( A \) is a 4-tuple:

\[
(\lambda_\prec, \lambda_\succ, \mu_\prec, \mu_\succ, D_\prec, D_\succ, T_\prec, T_\succ, x_0, y_0, k_0, l_0)
\]

where \( \lambda_\prec, \lambda_\succ, \mu_\prec, \mu_\succ : A \to K \) are linear functions, \( D_\prec, D_\succ, T_\prec, T_\succ : A \to A \) are linear maps and \( x_0, y_0 \in A, k_0, l_0 \in K \) satisfying for all \( x, y \in A \):

(P1)

\[ T_\succ(x_0) = D_\succ(x_0) + D_\succ(y_0) + l_0 x_0, \]

(P2)

\[ \mu_\succ(x_0) = l_0k_0 + \lambda_\succ(x_0) + \lambda_\succ(y_0), \]

(P3)

\[
D_\prec(x \succ y) + T_\prec(x) \succ y + \mu_\prec(x)T_\prec(y) + \lambda_\prec(x)T_\succ(y) \\
= \mu_\succ(y)T_\succ(x) + T_\succ(x \succ y) + x \succ T_\succ(y).
\]

(P4)

\[ \mu_\succ(y) \mu_\prec(x) + \lambda_\prec(x) \mu_\succ(y) = \mu_\succ(x \succ y), \]
Denote by $F$ a given for any

Proposition 5.4. Let $A$ be a pre-alternative algebra and $V$ a vector space of dimension 1 with basis $\{u\}$. Then there exists a bijection between the set $\text{Exd}(A, V)$ of all extending structures of $A$ through $V$ and the set $\mathcal{F}(A)$ of all flag datums of $A$.

Through the above bijection, the unified product corresponding to $(\lambda_\prec, \lambda_\succ, \mu_\prec, \mu_\succ, D_\prec, D_\succ, T_\prec, T_\succ, x_0, y_0, k_0, l_0) \in \mathcal{F}(A)$ will be denoted by $A_2\{u\}$ and has the multiplication given for any $a, b \in A$ by:

\[
\begin{align*}
(x + u) \ll (y + u) &= (x \ll y + D_\prec(x) + T_\prec(y) + y_0) + (l_0 u + \lambda_\prec(x) u + \mu_\prec(y) u), \\
(x + u) \gg (y + u) &= (x \gg y + D_\succ(x) + T_\succ(y) + x_0) + (k_0 u + \lambda_\succ(x) u + \mu_\succ(x) u).
\end{align*}
\]

That is $A_2\{u\}$ is the pre-alternative generated by $A$ and $\{u\}$ subject to the relations:

\[
\begin{align*}
u \ll u &= y_0 + l_0 u, & u \gg u &= x_0 + k_0 u, \\
x \ll u &= D_\prec(x) + \lambda_\prec(x) u, & x \gg u &= D_\succ(x) + \lambda_\succ(x) u, \\
u \ll x &= T_\prec(x) + \mu_\prec(x) u, & u \gg x &= T_\succ(x) + \mu_\succ(x) u.
\end{align*}
\]
6 Appendix: Matched pair and unified product condition for pre-alternative algebras

Theorem 6.1. Let $A$ and $B$ be two pre-alternative algebras. Then $(A, B)$ is a matched pair if and only if the following conditions hold $\forall x, y \in A$ and $u, v \in V$:

\[
(x \circ u) > y + (u \circ x) > y + (u \circ x) > y + (x \circ u) > y = x > (u > y) + u > (x > y) + x > (u > y), 
\]

(6.1)

\[
(u \circ x) > y + (x \circ u) > y = x > (u > y) + u > (x > y), 
\]

(6.2)

\[
(x \circ u) > v + (u \circ x) > v = x > (u > v) + u > (x > v), 
\]

(6.3)

\[
(x \circ u) > v + (u \circ x) > v + (x \circ u) > v + (u \circ x) > v = x > (u > v) + u > (x > v) + x > (u > v), 
\]

(6.4)

\[
(x \circ y) > u + (y \circ x) > u = x > (y > u) + y > (x > u) + y > (x > u) + y > (y > u), 
\]

(6.5)

\[
(x \circ y) > u + (y \circ x) > u = x > (y > u) + y > (x > u), 
\]

(6.6)

\[
(v \circ u) > x + (u \circ v) > x = u > (v > x) + v > (u > x), 
\]

(6.7)

\[
(u \circ v) > x + (v \circ u) > x = v > (u > x) + u > (v > x) + u > (v > x) + v > (u > x), 
\]

(6.8)

\[
x < (y \circ u) + x < (u \circ y) + x < (y \circ u) + x < (u \circ y) = (x < u) < y + (x < y) < u + (x < u) < y, 
\]

(6.9)

\[
x < (u \circ y) + x < (y \circ u) = (x < u) < y + (x < y) < u, 
\]

(6.10)

\[
u < (y \circ x) + u < (x \circ y) = (u < y) < x + (u < y) < x + (u < x) < y + (u < x) < y, 
\]

(6.11)

\[
u < (y \circ x) + u < (x \circ y) = (u < y) < x + (u < x) < y, 
\]

(6.12)

\[
x < (v \circ u) + x < (u \circ v) = (x < v) < u + (x < u) < v, 
\]

(6.13)
\[
x < (u \circ v) + x < (v \circ u)
= (x < u) \circ v + (x < v) < u + (x < u) < v + (x < v) < u,
\]

(6.14)

\[
u < (x \circ v) + u < (v \circ x) = (u < x) < v + (u < v) < x,
\]

(6.15)

\[
u < (x \circ v) + u < (v \circ x) + u < (v \circ x) + u < (x \circ v)
= (u < x) < v + (u < v) < x + (u < x) < v,
\]

(6.16)

\[
(x > u) < y + (u < x) < y + (x > u) < y + (u < x) < y
= x > (u < y) + x > (u < y) + u < (x \circ y),
\]

(6.17)

\[
(x > u) < y + (u < x) < y = x < (u \circ y) + u > (x < y),
\]

(6.18)

\[
(u > x) < y + (u < x) > y + (x < u) < y + (x < u) < y
= x < (u \circ y) + u > (x < y) + x < (u \circ y),
\]

(6.19)

\[
x > (y < u) + y < (x \circ u) + y < (x \circ u) + x > (y < u)
= (x > y) < u + (y < x) < u,
\]

(6.20)

\[
(x > y) < u + (y < x) < u = x > (y < u) + y < (x \circ u),
\]

(6.21)

\[
(x > u) < v + (u < x) < v = x > (u < v) + u < (x \circ v),
\]

(6.22)

\[
(x > u) < v + (u < x) < v + (x > u) < v + (u < x) < v
= x > (u < v) + u < (x \circ v) + u < (x \circ v),
\]

(6.23)

\[
(u > x) < v + (x < u) < v = u > (x < v) + x < (u \circ v),
\]

(6.24)

\[
(u > x) < v + (x < u) < v + (u > x) < v + (x < u) < v
= x < (u \circ v) + u > (x < v) + u > (x < v),
\]

(6.25)

\[
(u < v) < x + (v < u) < x = u > (v < x) + v < (u \circ x),
\]

(6.26)

\[
(u > v) < x + (v < u) < x
\]
\[ (x > u) < y + (x ∘ y) > u + (x ∘ u) < y \]
\[ = x > (u < y) + x > (u < y) + x > (u < y) + x > (u < y), \quad (6.29) \]

\[ (x > u) < y + (x ∘ y) > u = x > (u < y) + x > (y > u), \quad (6.30) \]

\[ (u > x) < y + (u ∘ y) > y + (u ∘ y) > x + (u ∘ y) > x \]
\[ = u > (x < y) + u > (y > x), \quad (6.31) \]

\[ (x > y) < y + (x ∘ y) > y = x > (u > y) + x > (y < u), \quad (6.32) \]

\[ (x > u) < y + (x ∘ v) > u = x > (u < v) + x > (v > u), \quad (6.33) \]

\[ (x > u) < v + (x ∘ v) > u + (x > u) < v + (x ∘ v) > u \]
\[ = x > (v > u) + x > (u < v), \quad (6.34) \]

\[ (u > x) < v + (u ∘ v) > x = u > (v > x) + u > (v > x), \quad (6.35) \]

\[ (u ∘ v) > x + (u ∘ x) < v + (u > x) < v \]
\[ = u > (x < v) + u > (v > x) + u > (v > x) + u > (x < v), \quad (6.36) \]

\[ (u > v) < x + (u ∘ x) > v = u > (v < x) + u > (v < x), \quad (6.37) \]

\[ (u > v) < x + (u ∘ x) > v \]
\[ = u > (x < v) + u > (x < v) + u > (v < x) + u > (x < v). \quad (6.38) \]

**Theorem 6.2.** Let \( A \) be a pre-alternative algebra, \( V \) be a vector space and \( Ω(A, V) \) an extending datum of \( A \) by \( V \). Denote by

\[ x ∘ y = x < y + x > y, \quad x ∘ v = x < v + x > v, \quad u ∘ y = u < y + u > y, \]
\[ u ∘ v = u < v + u > v, \quad x ∘ v = x < v + x > v, \quad u ∘ y = u < y + u > y, \]
\[ \omega_\circ(x, y) = \omega_\circ(u, v) + \omega_\circ(u, v). \]

Then \( A \otimes V \) is a unified product if and only if the following conditions hold:

\[ \omega_\circ(u, v) > w + \omega_\circ(v, u) > w + \omega_\circ(v \circ u, w) + \omega_\circ(u \circ v, w) \]
\[ = u > \omega_\circ(v, w) + v > \omega_\circ(u, w) + \omega_\circ(u, v > w) + \omega_\circ(v, u > w), \quad (6.41) \]

\[ (u \circ v) > w + (v \circ u) > w + \omega_\circ(u, v) > w + \omega_\circ(v, u) > w \]
\[ = u > \omega_\circ(v, w) + v > \omega_\circ(u, w) + u > (v > w) + v > (u > w), \quad (6.42) \]

\[ (x \circ u) > y + (u \circ x) > y + (u \circ x) > y + (x \circ u) > y \]
\[ = x > (u > y) + u > (x > y) + x > (u > y), \quad (6.43) \]

\[ (u \circ x) > y + (x \circ u) > y = x > (u > y) + u > (x > y), \quad (6.44) \]

\[ (u \circ x) > v + (x \circ u) > v + \omega_\circ(u \circ x, v) + \omega_\circ(x \circ u, v) \]
\[ = x > (u > v) + u > (x > v) + \omega_\circ(u, x > v) + x > \omega_\circ(u, v), \quad (6.45) \]

\[ (x \circ u) > v + (u \circ x) > v + (x \circ u) > v + (u \circ x) > v \]
\[ = x > (u > v) + u > (x > v) + u > (x > v), \quad (6.46) \]

\[ (x \circ y) > u + (y \circ x) > u \]
\[ = x > (y > u) + y > (x > u) + x > (y > u) + y > (x > u), \quad (6.47) \]

\[ (x \circ y) > u + (y \circ x) > u = x > (y > u) + y > (x > u), \quad (6.48) \]

\[ (v \circ u) > x + (u \circ v) > x + \omega_\circ(u, v) > x + \omega_\circ(v, u) > x \]
\[ = u > (v > x) + v > (u > x) + \omega_\circ(u, v > x) + \omega_\circ(v, u > x), \quad (6.49) \]

\[ (u \circ v) > x + (v \circ u) > x \]
\[ = v > (u > x) + u > (v > x) + u > (v > x) + v > (u > x), \quad (6.50) \]

\[ \omega_\circ(u < v, w) + \omega_\circ(u < w, v) + \omega_\circ(u, w) < v + \omega_\circ(u, v) < w \]
\[ = \omega_\circ(u, w \circ v) + u < \omega_\circ(v, w) + \omega_\circ(u, v \circ w) + u < \omega_\circ(v, w), \quad (6.51) \]

\[ (u < v) < w + (u < w) < v + \omega_\circ(u, w) < v + \omega_\circ(u, v) < w \]
\[ = u < (v \circ w) + u < (w \circ v) + u < \omega_\circ(v, w) + u < \omega_\circ(w, v), \quad (6.52) \]
\[(x < u) < y + (x < y) < u + (x < u) < y\]
\[= x < (y \circ u) + x < (u \circ y) + x < (y \circ u) + x < (u \circ y), \quad (6.53)\]

\[(x < u) < y + (x < y) < u = x < (u \circ y) + x < (y \circ u), \quad (6.54)\]

\[(u < x) < y + (u < x) < y + (u < y) < x + (u < y) < x\]
\[= u < (x \circ y) + (u < (y \circ x)), \quad (6.55)\]

\[(u < x) < y + (u < y) < x = u < (x \circ y) + u < (y \circ x), \quad (6.56)\]

\[(x < u) < v + (x < v) < u + \omega_<(x < u, v) + \omega_<(x < u, v)\]
\[= x < (v \circ u) + x < (u \circ v) + x < \omega_<(v, u) + x < \omega_<(v, u), \quad (6.57)\]

\[(x < u) < v + (x < v) < u + (x < v) < u + (x < u) < v\]
\[= x < (u \circ v) + x < (v \circ u), \quad (6.58)\]

\[(u < v) < x + (u < x) < v + \omega_<(u < x, v) + \omega_<(u, v) < x\]
\[= u < (x \circ v) + u < (v \circ x) + \omega_<(u, x \circ v) + \omega_<(u, v \circ x), \quad (6.59)\]

\[(u < x) < v + (u < v) < x + (u < x) < v\]
\[= u < (x \circ v) + u < (v \circ x) + u < (v \circ x) + u < (x \circ v), \quad (6.60)\]

\[\omega_<(u, v) < w + \omega_<(u < v, w) + \omega_<(v < u, w) + \omega_<(v, u) < w\]
\[= u > \omega_<(v, w) + v < \omega_<(u, w) + \omega_<(u, v < w) + \omega_<(v, u \circ w), \quad (6.61)\]

\[(u > v) < w + (v < u) < w + \omega_>(u, v) < w + \omega_>(v, u) < w\]
\[= v < (u \circ w) + u > (v < w) + u > \omega_<(v, w) + v < \omega_<(v, w), \quad (6.62)\]

\[(x > u) < y + (u < x) < y + (x > u) < y + (u < x) < y\]
\[= x > (u < y) + x > (u < y) + u < (x \circ y), \quad (6.63)\]

\[(x > u) < y + (u < x) < y = u < (x \circ y) + x > (u < y), \quad (6.64)\]

\[(u > x) < y + (u > x) < y + (x < u) < y + (x < u) < y\]
\[= x < (u \circ y) + u > (x < y) + x < (u \circ y), \quad (6.65)\]
\[ (u \succ x) < y + (x < u) < y = x < (u \circ y) + u \succ (x < y), \] \hspace{1cm} (6.66)

\[ x > (y < u) + y < (x \circ u) + y < (x < u) + x \succ (y < u) \]
\[ = (y < x) < u + (x > y) < u, \] \hspace{1cm} (6.67)

\[ (x > y) < u + (y < x) < u = y < (x \circ u) + x \succ (y < u), \] \hspace{1cm} (6.68)

\[ (x > u) < v + (u < x) < v + \omega_<(x > u, v) + \omega_<(u < x, v) \]
\[ = x > (u < v) + u < (x \circ v) + x \succ \omega_<(u, v) + \omega_<(u, x \circ v), \] \hspace{1cm} (6.69)

\[ (u < x) < v + (x > u) < v + (x > u) < v + (u < x) < v \]
\[ = x > (u < v) + u < (x \circ v) + u < (x \circ v), \] \hspace{1cm} (6.70)

\[ (u > x) < v + (x < u) < v + \omega_<(u > x, v) + \omega_<(x < u, v) \]
\[ = u > (x < v) + x < (u \circ v) + \omega_>(u, x < v) + x < \omega_>(u, v), \] \hspace{1cm} (6.71)

\[ (u > x) < v + (x < u) < v + (u > x) < v + (x < u) < v \]
\[ = x < (u \circ v) + u > (x < v) + u > (x < v), \] \hspace{1cm} (6.72)

\[ (u < v) < x + (v < u) < x + \omega_>(u, v) < x + \omega_<(v, u) < x \]
\[ = u > (v < x) + v < (u \circ x) + \omega_>(u, v) + \omega_<(u, v \circ x), \] \hspace{1cm} (6.73)

\[ (u > v) < x + (v < u) < x \]
\[ = u > (v < x) + v < (u \circ x) + u > (v < x) + v < (u \circ x), \] \hspace{1cm} (6.74)

\[ \omega_>(u, w) > v + \omega_>(u \circ w, v) + \omega_>(u, v) < w + \omega_<(u > v, w) \]
\[ = \omega_>(u, v < w) + u > \omega_>(w, v) + \omega_>(u, w > v) + u > \omega_<(v, w), \] \hspace{1cm} (6.75)

\[ (u > v) < w + (u \circ w) > v + \omega_>(u, v) < w + \omega_<(u, w) > v \]
\[ = u > (v < w) + u > (w > v) + u > \omega_>(v, w) + u > \omega_>(w, v), \] \hspace{1cm} (6.76)

\[ (x > u) < y + (x \circ y) > u + (x > u) < y \]
\[ = x > (u < y) + x > (u < y) + x > (y > u) + x > (y > u), \] \hspace{1cm} (6.77)

\[ (x > u) < y + (x \circ y) > u = x > (u < y) + x > (y > u), \] \hspace{1cm} (6.78)
\[(u > x) < y + (u > x) < y + (u \diamond y) \succ x + (u \circ y) > x = u > (x < y) + u > (y > x), \quad (6.79)\]

\[(u > x) < y + (u \circ y) > x = u > (y > x) + u > (x < y), \quad (6.80)\]

\[(x > y) < u + (x \circ u) > y + (x \diamond u) > y = x > (u > y) + x > (y > u) + x > (y < u), \quad (6.81)\]

\[(x > y) < u + (x \circ u) > y = x > (u > y) + x > (y < u), \quad (6.82)\]

\[(x > u) < v + (x \circ v) > u + \omega_{\circ}(x > u, v) + \omega_{\diamond}(x \circ v, u) = x > (u < v) + x > (v > u) + x > \omega_{\circ}(u, v) + x > \omega_{\diamond}(v, u), \quad (6.83)\]

\[(x > u) < v + (x \circ v) > u + (x > u) < v + (x \circ v) > u = x > (v > u) + x > (u < v), \quad (6.84)\]

\[(u > x) < v + (u \circ v) > x + \omega_{\circ}(u > x, v) + \omega_{\diamond}(u, v) > x = u > (x < v) + u > (v > x) + \omega_{\circ}(u, x < v) + \omega_{\diamond}(u, v > x), \quad (6.85)\]

\[(u \circ v) > x + (u \succ x) < v + (u > x) < v = u > (x < v) + u > (v > x) + u > (v > x) + u > (x < v), \quad (6.86)\]

\[(u > v) < x + (u \circ x) > v + \omega_{\circ}(u, v) < x + \omega_{\diamond}(u \circ x, v) = u > (v < x) + u > (x > v) + \omega_{\circ}(u, v < x) + \omega_{\diamond}(u, x > v), \quad (6.87)\]

\[(u > v) < x + (u \circ x) > v = u > (x > v) + u > (x > v) + u > (v < x) + u > (v < x). \quad (6.88)\]

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This is a primary edition, some example will be added in the future.

References
[1] A. L. Agore, G. Militaru, Extending structures for Lie algebras, Monatsh. Math. 174(2014), 169–193. [arXiv:1301.5442].

[2] A. L. Agore, G. Militaru, Extending structures, Galois groups and supersolvable associative algebras, Monatsh. Math. 181 (2016), 1–33. [arXiv:1305.6022].
[3] A. L. Agore, G. Militaru, Unified products for Leibniz algebras. Applications, Linear Algebra Appl. 439 (2013), 2609–2633. [arXiv:1307.2540].

[4] A. L. Agore, G. Militaru, Bicrossed products, matched pair deformations and the factorization index for Lie algebras, Symmetry Integrability Geom. Methods Appl. 10 (2014), 065, 16 pages.

[5] A.L. Agore, G. Militaru, Ito’s theorem and metabelian Leibniz algebras, Linear Multilinear Algebra 63 (2015), 2187–2199.

[6] C.M. Bai, X. Ni, Pre-alternative algebras and pre-alternative bialgebras, Pacific J. Math. 248(2010), 355–390.

[7] A. Elduque and H. C.Myung, Mutations of alternative algebras, Mathematics and its Applications 278, Kluwer, Dordrecht, 1994.

[8] M. E. Goncharov, The classical Yang-Baxter equation on alternative algebras: The alternative D-bialgebra structure on the Cayley-Dickson matrix algebra, Sibirsk. Mat. Zh. 48(5)(2007), 1008–1024.

[9] Y. Hong, Extending structures and classifying complements for left-symmetric algebras, Results Math., 74(2019), 32. [arXiv:1511.08571].

[10] Y. Hong, Extending structures for associative conformal algebras, Linear Multilinear Algebra, 67(2019), 196–212. [arXiv:1705.02827].

[11] Y. Hong and Y. Su, Extending structures for Lie conformal algebras, Algebr. Represent. Theor. 20 (2017), 209–230.

[12] N. Jacobson, Structure of alternative and Jordan bimodules, Osaka J. Math. 6(1954), 1–71.

[13] R. D. Schafer, Representation of alternative algebras, Trans. Am. Math. Soc.,72(1952), 1–17.

[14] R.D. Schafer, An introduction to nonassociative algebras, Pure and Applied Mathematics, vol. 22, Academic Press, New York, London, 1966.

[15] K. A. Zhevlakov, A. M. Slin’ko, I. P. Shestakov, and A. I. Shirshov, Rings that are nearly associative, Academic Press, New York, 1982.

[16] M. Zorn, Theorie der alternativen Ringe, Abh. Math. Sem. Univ. Hamburg. 8(1930), 123–147.
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