PARTIALLY HYPERBOLIC DYNAMICS IN DIMENSION 3

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ABSTRACT. Some of the guiding problems in partially hyperbolic diffeomorphisms in 3-manifolds are the following: (1) ergodicity (2) dynamical coherence (3) classification. A few years ago, the last three authors posed conjectures concerning these subjects. In this paper we survey the state of the art in these problems, and pose new questions.

1. INTRODUCTION

The purpose of this survey is to present the state of the art in the study of partially hyperbolic diffeomorphisms in 3-manifolds. This area has been quite active in the recent years, and we intend to review the newest advances in ergodicity, integrability and classification of these systems.

Partial hyperbolicity was introduced in the late sixties as a generalization to the classical hyperbolic systems. In hyperbolic systems, the tangent bundle splits into two directions, which are invariant under the derivative, one, the stable direction, is contracting, and the other, the unstable direction, is expanding. Namely, a diffeomorphism of a compact manifold \( f : M \to M \) is a hyperbolic diffeomorphism if the tangent bundle splits as \( TM = E_s \oplus E_u \), where \( Df(x)E^s_x = E^s_{f(x)} \) and \( Df(x)E^u_x = E^u_{f(x)} \), and for each pair of unit vectors \( v^s \in E^s_x \) and \( v^u \in E^u_x \) we have:

\[
\|Df(x)v^s\| < 1 < \|Df(x)v^u\|
\]

Partially hyperbolic diffeomorphisms, in turn, allow one extra, center direction, which is neither as expanding as the unstable one nor as contracting as the stable one. Precisely, the tangent bundle splits as \( TM = E^s \oplus E^c \oplus E^u \), where \( Df(x)E^u_x = E^u_{f(x)} \), \( Df(x)E^c_x = E^c_{f(x)} \) and \( Df(x)E^s_x = E^s_{f(x)} \), and for each unit vector \( v^s \in E^s_x \) and \( v^c \in E^c_x \) and \( v^u \in E^u_x \) we have:

\[
\|Df(x)v^s\| < 1 < \|Df(x)v^u\|
\]

\[
\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|
\]

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The set of partially hyperbolic diffeomorphisms is $C^1$-open in $\text{Diff}^1(M)$. See, for instance, Theorem 2.15 in [HPS77]. Therefore, a $C^1$-perturbation of a partially hyperbolic diffeomorphism, is a partially hyperbolic diffeomorphism.

The following are examples of partially hyperbolic diffeomorphisms:

1.1. **Examples.**

1.1.1. **Time-one maps of Anosov flows:** Let us consider an Anosov flow in a 3-manifold $M$. That is, a flow $\phi_t : M \to M$ such that the tangent bundle of $M$ splits into 3 sub-bundles, invariant under $D\phi_t: TM = E^s \oplus X \oplus E^u$, where $X$ is the direction tangent to the flow, and such that for each unit vector $v^s \in E^s$ and $v^u \in E^u$

$$\|D\phi_tv^s\| < 1 < \|D\phi_tv^u\|.$$ Then the time-one map of the flow $\phi_t$ is a partially hyperbolic diffeomorphism (exercise).

An example of this kind of partially hyperbolic diffeomorphism is the following: Consider a compact hyperbolic surface $S$ (i.e., a surface of constant negative sectional curvature) and the corresponding geodesic flow $g_t : T_1S \to T_1S$. Then $f = g_1$ is partially hyperbolic: its center bundle is tangent to the direction of the flow, while the stable and unstable bundles are tangent to the stable and unstable horocycles. See [KH95].

In this case the manifold $M = T_1S$ is a homogeneous space (a quotient of $PSL_2(\mathbb{R})$ by a subgroup of hyperbolic isometries).

As a different type of example, consider the suspension of an Anosov map $A : \mathbb{T}^2 \to \mathbb{T}^2$ by a constant roof function. We remark that these examples are truly different: while the second one the distribution $E^u \oplus E^s$ is integrable, for the first one is not.

Note that in both examples the corresponding Anosov Flow is transitive, but there also exist non transitive Anosov Flows [FWS80]. The time-one maps obtained from these type of flows are, of course, different from the ones obtained in the transitive setting.

1.1.2. **Skew-products:** A partially hyperbolic skew-product is a diffeomorphism of a circle bundle over the two torus of the form $f(x, \theta) = (Ax, h(x, \theta))$ where $A$ is a hyperbolic automorphism of the 2-torus and $h(x, \theta)$ are circle rotations. The resulting ambient manifold is a 3-nilmanifold. In case the product is direct, the ambient manifold is the 3-torus.

1.1.3. **DA-diffeomorphisms:** A DA- partially hyperbolic diffeomorphism is one that is isotopic to an Anosov one. In particular, they are semi-conjugate to Anosov diffeomorphisms. The prototypical example in this class is Mañé’s example [Man78].
It is obtained by taking a linear Anosov map in $T^3$ with eigenvalues $\lambda^{ss} < 1 < \lambda^u < \lambda^{uu}$, and making a bifurcation of the origin into three points along the weak unstable direction (See fig. 1.1.3). The resulting map is a dynamically coherent (robustly) transitive partially hyperbolic diffeomorphism.

2. Open problems

In this survey, we shall focus on three main topics of partially hyperbolic diffeomorphisms in 3-manifolds: ergodicity, dynamical coherence (integrability of the center bundle), and classification. Our claim basically states that almost every volume preserving partially hyperbolic diffeomorphism is ergodic, except for specific ones in some specific manifolds, which can be classified. We also claim that almost every partially hyperbolic diffeomorphism is dynamically coherent, except for some specific ones, that are generalizations of the example presented in Section 4. Finally, we claimed that all partially hyperbolic diffeomorphisms in three-manifolds are leafwise conjugate to a perturbation of some of the examples presented above, in case it is dynamically coherent, or a generalization of the above mentioned example presented in Section 4 in case it is not. This claim has recently been proved wrong. Let us describe these three topics more extensively in the following subsections:

2.1. Ergodicity. One problem in partially hyperbolic dynamics is studying the ergodicity of conservative systems, that is, of $C^r$-diffeomorphisms preserving a smooth volume. We shall denote conservative systems by $\text{Diff}^r_m(M)$, and $m$ will denote the probability arising from this smooth volume. A system $f$ is ergodic if any measurable set satisfying $f(A) = A$ must also satisfy either $m(A) = 1$ or $m(A) = 0$. An ergodic system is one not admitting an invariant measurable set with intermediate measure.

An example of a non-ergodic partially hyperbolic diffeomorphism is, for instance $f = A \times id$, where $A$ is the automorphism on the 2-torus generated by the matrix
\[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix},
\]
and \(id\) is the identity on the circle. Indeed, any interval of the circle times the 2-torus is an invariant set with intermediate measure.

One can easily perturb this system to obtain an \textit{ergodic} partially hyperbolic one. For instance \(g = A \times \text{(irrational rotation)}\). However, this new system \(g\) can be easily perturbed to obtain again non-ergodic diffeomorphisms. It is, in fact approximated by \(g_n = A \times \text{(rational rotation)}\), all of which are non-ergodic.

Pugh and Shub were the first to conjecture that ergodicity, in fact, was very abundant in the partially hyperbolic world, a conjecture that remains open until today. As a matter of fact, we were told by Keith Burns that this conjecture was made public for the first time in Montevideo, in 1995 [PS96]. They considered not only ergodicity, but a stronger concept: \textit{stable ergodicity}.

\textbf{Definition 2.1.} A conservative \(C^2\) diffeomorphism \(f : M \to M\) is stably ergodic (in \(\text{Diff}_c^1(M)\)) if there exists \(U \subset \text{Diff}_c^1(M)\) neighborhood of \(f\) such that every \(g \in U\) of class \(C^2\) is ergodic.

Until 1994, the only known examples of stably ergodic diffeomorphisms were Anosov diffeomorphisms, that is, hyperbolic ones. In 1994, Grayson, Pugh and Shub found the first non-hyperbolic example of a stably ergodic diffeomorphism, which was a partially hyperbolic diffeomorphism, the time-one map of an Anosov flow (see Example 1, in page 2).

A year later, Pugh and Shub conjectured the following:

\textbf{Conjecture 2.2} (Pugh-Shub (1995)[PS96]). \textit{Stable ergodicity is \(C^r\)-dense among volume preserving partially hyperbolic diffeomorphisms, for all \(r \geq 2\)}

This conjecture remains open until today, even though there have been advances, like Burns-Wilkinson [BW10], Burns-Dolgopyat-Pesin [BDP02], Dolgopyat-Wilkinson [DW03], among others. In 2008, Hertz-Hertz-Ures proved the conjecture true for partially hyperbolic diffeomorphisms with one-dimensional center bundle [HHU08b]. So in particular, we have:

\textbf{Theorem 2.3} (Hertz-Hertz-Ures). \textit{Stable ergodicity is \(C^\infty\)-dense among volume preserving partially hyperbolic diffeomorphisms in 3-manifolds.}

That is, the vast majority of partially hyperbolic diffeomorphisms is ergodic, moreover, stably ergodic. Hence, one can ask the following: can we classify the \textit{non-ergodic} partially hyperbolic diffeomorphisms? And also: are there manifolds where all partially hyperbolic diffeomorphisms are ergodic? Can we classify all the 3-manifolds admitting non-ergodic diffeomorphisms?

A first evidence in this direction was obtained in [HHU08a]:

\[
\left( \begin{array}{cc}
2 & 1 \\
1 & 1
\end{array} \right),
\]
**Theorem 2.4** (Hertz-Hertz-Ures). If $N$ is a 3-nilmanifold other than $\mathbb{T}^3$, then all conservative partially hyperbolic diffeomorphisms are ergodic.

The proof of this theorem involved a careful study of accessibility classes. Two points $x$ and $y$ are in the same accessibility class if there is a path piece-wise tangent to $E^s$ or $E^u$. The diffeomorphism $f$ has the accessibility property if there is only one accessibility class. Accessibility implies ergodicity in 3-dimensional manifolds. A proof of this, which was conjectured by Pugh and Shub in [PS96], can be found in [BW10] and in [HHU08b]. This fact is very interesting, since it allows to convert an ergodic problem into a geometric problem: that of studying the set of accessibility classes. In Section 3, we give a better description of this set in 3-manifolds.

From our study of the set of accessibility classes, which is more detailed in Section 3, we got the impression that the only obstruction to ergodicity is the existence of a compact accessibility class:

**Conjecture 2.5** (Non-ergodic conjecture: Hertz-Hertz-Ures (2008)). If a conservative partially hyperbolic diffeomorphism of a 3-manifold is non-ergodic, then there is a 2-torus tangent to $E^s \oplus E^u$.

The importance of this kind of hyperbolic sub-dynamics will become apparent later on, see Section 6. As we will see, these tori seem to be “behind” all non-typical behavior in partially hyperbolic dynamics. Moreover, it is not every orientable manifold that supports a 2-dimensional dynamics like the one appearing in Conjecture 2.5:

**Theorem 2.6** ([HHU11]). If a partially hyperbolic diffeomorphism $f : M^3 \to M^3$ has a 2-dimensional embedded torus $T$ which is tangent either to $E^s \oplus E^u$, $E^c \oplus E^u$ or $E^c \oplus E^s$, then the ambient manifold $M^3$ can only be one of the following possibilities:

1. the 3-torus $\mathbb{T}^3$
2. the mapping torus of $-\text{id} : \mathbb{T}^2 \to \mathbb{T}^2$
3. the mapping tori of hyperbolic automorphisms on 2-tori

![Figure 2](image1.png)  
**Figure 2.** (1) The 3-torus (2) The mapping torus of $-\text{Id}$ (3) The mapping torus of an hyperbolic automorphism.
Therefore, in case Conjecture 2.5 were true, there would be effectively very few manifolds supporting non-ergodic partially hyperbolic dynamics. We state it explicitly as a weaker conjecture:

**Conjecture 2.7 (Weak non-ergodic conjecture: Hertz-Hertz-Ures (2008)).** The only orientable 3-manifolds that admit a non-ergodic conservative partially hyperbolic diffeomorphisms are:

1. the 3-torus $T^3$
2. the mapping torus of $-id: T^2 \rightarrow T^2$
3. the mapping tori of hyperbolic automorphisms on 2-tori

The following is also open:

**Problem 2.8.** Let $f$ be a conservative non-ergodic partially hyperbolic diffeomorphism in any of the manifolds (1), (2), (3) stated in Theorem 2.6. Prove there exists a torus tangent to $E^s \oplus E^u$.

In Section 3 we collect the advances in this conjecture to the best of our knowledge, and a description of the set of accessibility classes.

### 2.2. Dynamical coherence.

In a partially hyperbolic dynamics, the strong bundles, $E^s$ and $E^u$ are always integrable, that is, there are invariant foliations $W^s$ and $W^u$, the *stable* and *unstable* foliations, that are tangent to each of the strong bundles (see, for instance [BP74, HPS77]). However, the center bundle, $E^c$, can be integrable or not. Notice that $E^c$ is a priori only Hölder. In fact, as Wilkinson noticed in [Wil98], the Anosov example of A. Borel, mentioned by Smale in [Sma67], when viewed as a partially hyperbolic dynamics, has a non-integrable center bundle. The Borel-Smale example is explained in detail in [BW08]. For the sake of completeness, let us briefly mention what it is about.

#### 2.2.1. A non-dynamically coherent example.

The aforementioned example is, in fact, a non-toral Anosov automorphism on a six-dimensional nilmanifold. Let $G_1$ and $G_2$ be copies of the three dimensional simply connected nonabelian nilpotent Lie group. And consider bases $X_i, Y_i, Z_i$ of the corresponding Lie algebras $G_i$, $i = 1, 2$, with the bracket condition

$$[X_1, Y_1] = Z_1 \quad [X_2, Y_2] = Z_2 \quad (2.3)$$

Consider now a hyperbolic automorphism $A \in SL(2, \mathbb{Z})$, and let $\lambda > 1$ be one of its eigenvalues. Consider $f$ such that the derivative acts over the Lie algebra as follows:

$$X_1 \mapsto \lambda X_1 \quad X_2 \mapsto \lambda^{-1} X_2$$
$$Y_1 \mapsto \lambda^2 Y_1 \quad Y_2 \mapsto \lambda^{-2} Y_2$$
$$Z_1 \mapsto \lambda^3 Z_1 \quad Z_2 \mapsto \lambda^{-3} Z_2$$
The next step (which we will not explain) is to find a lattice \( \Gamma \) of \( G = G_1 \times G_2 \), such that it is invariant, so that the whole construction yields a diffeomorphism over the six-dimensional nilmanifold \( G/\Gamma \) (the coset space).

This example is, as a matter of fact, an Anosov diffeomorphism. But we may also look at it as a partially hyperbolic diffeomorphism, such that \( E^s \) is the space generated by \( Z_2, E^u \), the space generated by \( Z_1 \), and the center bundle, \( E^c \), is the space generated by \( X_1, X_2, Y_1 \) and \( Y_2 \).

Note that \( X_1, X_2, Y_1 \) and \( Y_2 \) do not satisfy the Frobenius condition, due to (2.3), therefore \( E^c \) is not integrable, that is, there is no invariant foliation tangent to \( E^c \).

After examining this example, we may ask: is the lack of Frobenius condition the only reason for non-integrability of the center bundle? What about the case of one-dimensional \( E^c \), where Frobenius condition is always trivially satisfied? The question of whether a partially hyperbolic diffeomorphism existed with a one-dimensional non-integrable center bundle remained open since the 70’s.

Let us define the concept of integrability of the center bundle.

**Definition 2.9.** A partially hyperbolic diffeomorphism is dynamically coherent if the following two conditions are satisfied:

1. There is an invariant foliation tangent to the distribution \( E^c \oplus E^u \)
2. There is an invariant foliation tangent to the distribution \( E^s \oplus E^c \).

In 2009, Hertz-Hertz-Ures \[HHU14\] provided the first example of a partially hyperbolic diffeomorphism with a non-integrable one-dimensional center-bundle. It is a non-dynamically coherent example in a 3-torus. Let us mention that this example strongly contrasts with a result by Brin, Burago and Ivanov \[BBI09\], which proves that all absolutely partially hyperbolic diffeomorphisms of the 3-torus are dynamically coherent. Absolute partial hyperbolicity is more restrictive than partial hyperbolicity and requires the bound in (1.2) to be uniform, namely: that there exist \( \lambda < 1 < \mu \) such that for all \( x \in M \) and unit vectors \( v^\sigma \in E^\sigma_x \), \( \sigma = s, c, u \) we have

\[
\|Df(x)v^s\| \leq \lambda \leq \|Df(x)v^c\| \leq \mu \leq \|Df(x)v^u\| \tag{2.4}
\]

In \[HHU12\] it was shown that if there is an invariant foliation tangent to the distribution \( E^c \oplus E^u \), then this foliation cannot contain a compact leaf, that is, a 2-torus tangent to \( E^c \oplus E^u \) (see Theorem [4.4]). Note that, in principle, a \( cu \)-torus could coexist with a \( cu \)-foliation, but cannot be part of it.

As a matter of fact, this result later inspired us to construct the non-dynamically coherent example \[HHU14\]. In this example, the distribution \( E^c \oplus E^u \) is uniquely integrable at every point of \( \mathbb{T}^3 \) minus a 2-torus, which is tangent to \( E^c \oplus E^u \). This shows that no invariant foliation exists which is tangent to \( E^c \oplus E^u \), whence the example is non-dynamically coherent. In fact, we proved the following:
Theorem 2.10. [HHU14] There exist a partially hyperbolic diffeomorphism \( f : \mathbb{T}^3 \to \mathbb{T}^3 \) such that

1. There is no invariant foliation tangent to the distribution \( E^c \oplus E^u \)
2. There is an invariant 2-dimensional torus \( T \) tangent to the distribution \( E^c \oplus E^u \)

Moreover, there is a \( C^1 \)-open neighborhood \( U \) of \( f \) such that all \( g \) in \( U \) satisfy conditions (1) and (2).

Again, a 2-dimensional periodic torus appears. How relevant is this object? We conjecture that these tori might be the only obstruction to dynamical coherence.

Conjecture 2.11 (Non-dynamical coherence conjecture: Hertz-Hertz-Ures (2009)). If a partially hyperbolic diffeomorphism of a 3-manifold is not dynamically coherent, then there is a 2-torus tangent to either \( E^c \oplus E^u \) or \( E^s \oplus E^c \).

Again, if this conjecture were true, the only manifolds supporting non-dynamically coherent dynamics would be the ones listed in Theorem 2.6. We state this explicitly as a weaker conjecture:

Conjecture 2.12 (Weak non-dynamical coherence conjecture: Hertz-Hertz-Ures (2009)). The only orientable 3-manifolds supporting non-dynamically coherent dynamics are:

1. the 3-torus \( \mathbb{T}^3 \)
2. the mapping torus of \(-id : \mathbb{T}^2 \to \mathbb{T}^2\)
3. the mapping tori of hyperbolic automorphisms on 2-tori

Conjecture 2.11 was proven true in the 3-torus by Potrie in his PhD Thesis [Pot12]. This result was extended to manifolds with solvable fundamental group by Hammerlindl and Potrie in [HP13]; namely, a non-dynamically coherent partially hyperbolic diffeomorphism in a manifold with solvable fundamental group has a 2-torus tangent to either \( E^c \oplus E^u \) or \( E^s \oplus E^c \). That is, Potrie and Hammerlindl Potrie prove that if the Weak conjecture 2.12 is true, then the Strong conjecture 2.11 also holds. This is the best result in this direction up to this date.

In Section 4 we sketch a proof of this fact for the simplest case.

2.3. Classification. The third main topic in the study of partially hyperbolic dynamics in 3-manifolds is their classification. As early as 2001, Enrique Pujals proposed the following conjecture:

Conjecture 2.13 (Classification conjecture: Pujals (2001)). If a partially hyperbolic diffeomorphism of a 3-manifold is transitive, then it is is (finitely covered by) one of the following:

1. a perturbation of the time-one map of an Anosov flow
(2) a skew product
(3) a DA-diffeomorphism

In 2009, Hammerlindl showed in his PhD Thesis \cite{Ham13} that every absolutely partially hyperbolic diffeomorphism of $\mathbb{T}^3$ is \textit{leafwise conjugate} to its linearization. Let us define this concept:

**Definition 2.14 (Leaf conjugacy).** Two dynamically coherent partially hyperbolic diffeomorphisms $f, g : M \to M$ are leafwise conjugate if there exists a homeomorphism $h : M \to M$ carrying center leaves of $f$ into center leaves of $g$, that is, $h(W^c_f(x)) = W^c_g(h(x))$, and such that

$$h(f(W^c_f(x))) = g(h(W^c_f(x)))$$

Note that under the hypothesis of Hammerlindl’s Thesis (absolute partial hyperbolicity in $\mathbb{T}^3$) there is always dynamical coherence, due to a result by Brin, Burago and Ivanov \cite{BBI09}.

Hammerlindl’s Thesis suggested us that perhaps dynamical coherence was a more suitable hypothesis for a classification of 3-dimensional partially hyperbolic dynamics. This, together with our example \cite{HHU14}, made us propose the following:

**Conjecture 2.15 (Classification conjecture: Hertz-Hertz-Ures (2009)).** Let $f$ be a partially hyperbolic diffeomorphism of a 3-manifold.

If $f$ is dynamically coherent, then it is (finitely covered by) one of the following:

1. a perturbation of a time-one map of an Anosov flow, in which case it is leafwise conjugate to an Anosov flow
2. a skew-product, in which case it is leafwise conjugate to a skew-product with linear base, or
3. a DA, in which case it is leafwise conjugate to an Anosov diffeomorphism of $\mathbb{T}^3$.

If $f$ is not dynamically coherent, then there are a finite number of 2-tori tangent either to $E^c \oplus E^u$ or to $E^s \oplus E^c$, and the rest of the dynamics is trivial, as in the non dynamically coherent example \cite{HHU14} (see also Section 7).

Both conjectures are true in solv-manifolds, as it was proven by A. Hammerlindl and R. Potrie \cite{HP13}.

Very recently, C. Bonatti, K. Parwani and R. Potrie found both a dynamically coherent example and a transitive example that are not leaf-wise conjugate to any of the above models, proving both conjectures wrong \cite{BPP14}.

**Question 2.16.** Is it possible to classify partially hyperbolic dynamics in 3-manifolds, modulo leaf conjugacies?
3. Ergodicity

To establish the ergodicity of partially hyperbolic maps, the most general method available is the so called Hopf method. To explain it, we first recall the following.

**Theorem 3.1** (Stable Manifold Theorem). If \( f \in \text{Diff}^r(M) \) is partially hyperbolic then there exist continuous foliations \( W^s = \{ W^s(x) \}_{x \in M}, W^u = \{ W^u(x) \}_{x \in M} \) tangent to \( E^s, E^u \), called the stable and the unstable foliations. Their leaves are \( C^r \)-immersed lines.

See [HPS77], Theorem 4.1. We point out that the foliations \( W^s, W^u \) are seldom differentiable ([Ano67], pag. 201). Their transverse regularity is Hölder ([PSW97]) in general.

It is not hard to see that a conservative diffeomorphism \( f \) is ergodic if and only for every continuous observable \( \varphi : M \to \mathbb{R} \), the Birkhoff average

\[
\tilde{\varphi}(x) = \lim_{|n| \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x)
\]

is almost everywhere constant. But \( \tilde{\varphi}(x) \) coincides almost everywhere with

\[
\varphi^+(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x)
\]

which is constant on stable manifolds (use uniform continuity of \( \varphi \)). Analogously, \( \varphi(x) \) coincides almost everywhere with

\[
\varphi^-(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^{-k}(x)
\]

which is constant on unstable manifolds.

To explain the method we now assume that \( f \) is a \( C^2 \) conservative Anosov (i.e. partially hyperbolic with trivial center bundle) diffeomorphism which is not ergodic. Then there would be a continuous observable \( \varphi \) for which \( \tilde{\varphi} \) is not almost everywhere constant, and thus neither are \( \varphi^+ \) and \( \varphi^- \). Hence there exist two positive measure invariant sets \( A \) and \( B \) and \( \alpha \in \mathbb{R} \) such that \( \varphi^+(x) \geq \alpha \) for all \( x \in A \), and \( \varphi^-(x) < \alpha \) for all \( x \in B \). Note that \( A \) is saturated by stable manifolds while \( B \) is saturated by unstable manifolds.

Let \( x \) be a point of \( A \) such that almost all points \( w \in W^s(x) \) satisfy \( \varphi^-(w) = \varphi^+(w) = \varphi^+(x) \). Such an \( x \) exists since the stable foliation is absolutely continuous [Ano67] and \( \varphi^+(w) = \varphi^-(w) \) almost everywhere. Consider \( y \) a Lebesgue density point of \( B \). Observe that there exists a piecewise \( C^1 \) path \( c \) connecting \( x \) and \( y \) whose tangent is always in \( E^s \) or \( E^u \). This is so because the manifold is connected,
and thus by transversality \(W^s(x) \cap W^\text{loc}_u(y) \neq \emptyset\). Since \(y\) is a density point of \(B\), the 99% of points in a small ball around \(y\) also belong to \(B\), that is there is a set of measure 0.99m\((B_\delta(y))\) in \(B_\delta(y)\) of points belonging to \(B\). The local unstable manifold of all these points intersect the stable manifold of \(x\). Again by absolute continuity of the stable foliation, there exists a point \(z\) in \(B_\delta(y)\) such that the 99% of the points in its local stable manifold belong to \(B\). Call \(T_1\) the local stable manifold of \(z\); the local unstable manifold of \(z\) intersects the stable manifold of \(x\) at a point \(z'\). Hence there is a local stable manifold \(T_2\) of \(z'\) (contained in the stable manifold of \(x\)), such that the unstable holonomy \(h^u: T_1 \to T_2\) is well defined. Using now the absolute continuity of the unstable foliation, and since \(m^s_z(T_1 \cap B) > 0\) we have that
\[
m^s_z(h^u(T_1 \cap B)) > 0
\]
where \(m^s_z, m^s_{z'}\) denote the conditional measures induced by \(m\) on the corresponding local stable manifolds. Since \(z \in B\), \(W^u(z) \subset B\) and in particular, \(h^u(T_1 \cap B) = T_2 \cap B\). So, we have \(m^s_z(T_2 \cap B) > 0\). This means, there is a positive measure set of points in the stable manifold of \(x\) which belong to \(B\). But this is absurd, since \(x\) was chosen so that almost all points in its stable manifold belong to \(A\).

In brief, there are two fundamental ingredients in the Hopf argument for an Anosov map:

1. every pair of points can be joined by a a concatenation of stable and unstable leaves
2. the stable and unstable foliations are absolutely continuous, and completely transversal.

3.1. **Accessibility, a property that implies ergodicity.** We would like to apply the previous method to a general partial hyperbolic system, that is, when there is some non-trivial center direction. To begin with, observe that in general it is not true that any two pair of points can be joined by a concatenation of stable and unstable leaves. We fix \(f : M \to M\) C^2 conservative partially hyperbolic, and call \(c : [0,1] \to M\) an \(su\)-path if it is piecewise C^1 and for every \(t\) where defined, \(c'(t) \in E^s \cup E^u\).

**Definition 3.2.** For a point \(x \in M\), its accessibility class is the set
\[
AC(x) := \{y : \exists c : [0,1] \to M su\text{-path such that } c(0) = x, c(1) = y\}.
\]
The map \(f\) is accessible if the partition by accessibility classes is trivial, and essentially accessible if the partition by accessibility classes is ergodic (i.e. any Borel set saturated by accessibility classes has either volume 0 or 1).

When there is only one accessibility class, we will say that \(f\) has the accessibility property. From now on, let us suppose this is our case. As for (2), absolute continuity
of the strong foliations is also satisfied ([PS72]), but complete transversality is not (due to the presence of the center direction).

This problem can be overcome if the holonomies are regular enough. For instance, Sacksteder used accessibility and Lipschitzness of the stable and unstable holonomies to prove ergodicity of linear partially hyperbolic automorphisms of nil-manifolds [Sac70]. More generally, Brin and Pesin proved that accessibility and Lipschitzness of the stable and unstable foliations imply ergodicity (in fact, Kolmogorov), in the following way [BP74, Theorem 5.2, p.204], see also [GPS94]: if $A$ and $B$ are defined as before, consider a density point $x$ in $A$, and a density point $y$ in $B$. Take an $su$-path joining $x$ and $y$. Call $h$ a global holonomy map from $x$ to $y$, that is, $h$ is a local homeomorphism that takes points in a neighborhood $U$ of $x$, slides them first along a stable segment, then along an unstable, then along a stable again, etc. until reaching a neighborhood $V$ of $y$, all the $su$-paths are near the original $su$-path joining $x$ and $y$. Since $A$ is essentially $su$-saturated, we have that $h(A \cap U) = A \cap V$ modulo a zero set. Since $h$ can be chosen to be Lipschitz, there exists a constant $C > 1$ such that, for each measurable set $E \subset U$, and for each sufficiently small $r > 0$, we have

$$\frac{1}{C}m(E) < m(h(E)) < Cm(E) \quad (3.8)$$

$$B_{C_2}(y) \subset h(B_r(x)) \subset B_{Cr}(y). \quad (3.9)$$

This implies that

$$\frac{m(B_{C_2}(y) \cap A)}{m(B_{Cr}(y) \setminus A)} \geq \frac{m(h(B_r(x) \cap A))}{m(h(B_{C_2r}(x) \setminus A))} \geq \frac{1}{C^2C'} \frac{m(B_r(x) \cap A)}{m(B_r(x) \setminus A)} \to \infty$$

since $m(B_{C_2r}(x) \setminus A) \leq C'm(B_r(x) \setminus A)$ for some positive constant $C'$. From this we get that $y$ is also a density point of $A$. This is absurd, since $y$ was a density point of $B$, complementary to $A$ modulo a zero set.

This is essentially how the Hopf argument would work in the partially hyperbolic setting. However, Lipschitzness of the holonomy maps is a very strong hypothesis, not satisfied for most of the partially hyperbolic diffeomorphisms.

The idea of Grayson, Pugh and Shub [GPS94], later improved by [Wil98, PS00, HHU08b], [BW10] is to show that the stable and unstable holonomies do preserve density points according to another base different from intervals called Juliennes. These sets are dynamically defined, and constitute Vitali bases.

We refer the reader to [PS00] and [HHU08b] for the precise definition and elaboration of the concept of Julienne. Here we content ourselves with the following rough sketch of the ergodicity argument. Consider for a point $x$ a small center segment, and saturate by local unstable leaves; to gain better control in the size of these unstable segments we pre-iterate $n$ times the local unstable manifold of $f^n(x)$ (of a convenient size). The resulting set is then saturated by locally stable manifolds.
This small prism is called \textit{s-julienne}, and denoted by $J_n^{\text{suc}}(x)$. The subscript $n$ essentially tells the size of the Julienne, and in particular everything is chosen so that $m(J_n^{\text{suc}}(x)) \to 0$. An s-julienne density point of a set $E$ is a point $x$ such that:

$$
\lim_{n \to \infty} \frac{m(J_n^{\text{suc}}(x) \cap E)}{m(J_n^{\text{suc}}(x))} = 1 \quad (3.10)
$$

The scheme is to consider the sets $A$ and $B$ we considered above, and prove:

1. the s-julienne density points of $A$ (and of any essentially $u$-saturated set) coincide with the Lebesgue density points of $A$.
2. the s-julienne density points of $A$ (and of any essentially $s$-saturated set) are preserved by stable holonomies.

An analogous statement is proved for $A$ with respect to $u$-julienne density points, which are defined with respect to the local basis obtained by locally saturating a small center segment first in a dynamic way by stable leaves, and then by unstable leaves. As the stable and unstable holonomies preserve the Lebesgue density points of $A$ we have that if the diffeomorphism has the accessibility property then $A$ is equal to $M$ modulo a zero set. This proves the system is ergodic:

**Theorem 3.3.** If $f \in \Diff^2(M^3)$ is partially hyperbolic and satisfies the accessibility property, then it is ergodic.

**Remark 1.** The strongest version of the theorem above is due to K. Burns and A. Wilkinson [BW10], and states that essential accessibility and a property called center bunching implies ergodicity. Center bunching is trivially satisfied if the dimension of the center bundle is one. Essential accessibility is a weaker property implied by accessibility. Burns-Wilkinson paper is a refined version of the techniques introduced by C. Pugh and M. Shub [PS00]. A simpler version for center-dimension one can be found in [HHU08b].

3.2. Properties of Accessibility Classes. We want to precisely describe non-ergodic partially hyperbolic diffeomorphism, and our conjecture is that this only occurs when there is a compact accessibility class (see Conjecture 2.5), that is, when there is a torus tangent to $E^s \oplus E^u$ (in fact we conjecture that there must be two such tori).

Since accessibility implies ergodicity, in order to describe non-ergodic partially hyperbolic diffeomorphisms, it seems reasonable to look at the non-accessible ones. And, even more precisely, we will study the structure of the set of non-open accessibility classes.
Theorem 3.4. [HHU08b] For each \( x \) in \( M^3 \), its accessibility class \( AC(x) \) is either an open set or an immersed surface. Moreover, \( \Gamma(f) \), the set of non-open accessibility classes of \( f \) is a compact codimension-one laminated set whose laminae are the accessibility classes.

Remark 2. This theorem still holds for partially hyperbolic diffeomorphisms with center dimension one.

Let us begin by a local description of open accessibility classes.

Proposition 3.5. For any point \( x \) in \( M \), the following statements are equivalent:

1. \( AC(x) \) is open.
2. \( AC(x) \) has non-empty interior.
3. \( AC(x) \cap W^c_{loc}(x) \) has non-empty interior for any choice of \( W^c_{loc}(x) \).

Proof. (2) \( \Rightarrow \) (1) Let \( y \) be in the interior of \( AC(x) \), and consider any point \( z \) in \( AC(x) \). Then there is an \( su \)-path from \( y \) to \( z \) of the form \( y = x_0, x_1, \ldots, x_N = z \) such that \( x_n \) and \( x_{n+1} \) are either in the same \( s \)-leaf or in the same \( u \)-leaf. Let \( U \) be a neighborhood of \( y \) contained in \( AC(x) \), and suppose that, for instance \( y = x_0 \) and \( x_1 \) belong to the same \( s \)-leaf. Then \( U_1 = W^s(U) \) is an open set contained in \( AC(x) \), that contains \( x_1 \), so \( x_1 \) is in the interior of \( AC(x) \). Indeed, \( W^s \) is a \( C^0 \)-foliation, so the \( s \)-saturation of an open set is open.

Now, \( x_1 \) and \( x_2 \) belong to the same \( u \)-leaf. If we consider \( U_2 = W^u(U_1) \), then \( U_2 \) is an open set contained in \( AC(x) \) and containing \( x_2 \) in its interior. Defining inductively \( U_n \) as \( W^s(U_{n-1}) \) or \( W^u(U_{n-1}) \) according to whether \( x_n \) belongs to the \( s \)- or the \( u \)-leaf of \( x_{n-1} \), we obtain that all \( x_n \) belong to the interior of \( AC(x) \). In particular, \( z \). This proves that \( AC(x) \) is open.

Figure 3. An \( su \)-path from \( y \) to \( z \)

(1) \( \Rightarrow \) (3) Follows directly from the definition of relative topology.
(3) ⇒ (2) Let $V$ be an open set in $AC(x) \cap W^c_{loc}(x)$, relative to the topology of $W^c_{loc}(x)$. Then $W^s(V)$ is contained in $AC(x)$, and contains a disc $D^{sc}$ of dimension $s+c$ transverse to $E^u$. This implies that $W^u(D^{sc})$ is contained in $AC(x)$ and contains an open set. Therefore, $AC(x)$ has non-empty interior. □

Let $O(f)$ be the set of open accessibility classes, which is, obviously, an open set. Then its complement, $\Gamma(f)$ is a compact set. Let us see that is laminated by the accessibility classes of its points.

For any point $x \in M$, consider a local center leaf $W^c_{loc}(x)$. Locally saturate it by stable leaves, that is, take the local stable manifolds of all points $y \in W^c_{loc}(x)$, to obtain a small $(s+c)$-disc $W^{sc}_{loc}(x)$. Now, locally saturate $W^{sc}_{loc}(x)$ by unstable leaves to obtain a small neighborhood $W^{usc}_{loc}(x)$. See Figure 4. On $W^{usc}_{loc}(x)$, consider the map

$$p_{us}: W^{usc}_{loc}(x) \to W^c_{loc}(x)$$

defined in the following way: given $y \in W^{usc}_{loc}(x)$, there exists a unique point $p_u(y)$ in the disc $W^{sc}(x)$ that belongs to the local unstable manifold of $y$. Since $W^{sc}_{loc}(x)$ is the local stable saturation of $W^c_{loc}(x)$, then $p_u(y) \in W^{sc}_{loc}(x)$ is in the local stable manifold of a unique point $p^{us}(y)$ in $W^c_{loc}(x)$. That is, $p_{us}(y)$ is the point obtained by first projecting along unstable manifolds onto $W^{usc}_{loc}(x)$, and then projecting along stable manifolds onto $W^c_{loc}(x)$. Since the local stable and unstable foliations are continuous, $p_{us}$ is continuous.

Let $AC_x(y)$ be the connected component of $AC(y) \cap W^{usc}_{loc}(x)$ that contains $y$. The points of $AC_x(y)$ are the points that can be accessed by $su$-paths from $y$ without getting out from $W^{usc}_{loc}(x)$, see Figures 4 and 5. Then we have the following local description of accessibility classes of points in $\Gamma(f)$:

**Lemma 3.6.** For any $y \in W^c_{loc}(x)$ such that $y \in \Gamma(f)$, we have $AC_x(y) = p^{-1}_{su}(y)$

*Proof.* Let $y$ be a point in $W^c_{loc}(x)$. Then $p^{-1}_{su}(y) = W^u_{loc}(W^s_{loc}(y))$, which is clearly contained in $AC_x(y)$. But also, we have $p_{su}(AC_x(y)) = y$. Indeed, if $p_{su}(z)$ were
Figure 5. An accessibility class in $\Gamma(f)$

different from $y$, for some $z \in AC_x(y)$, we would have a situation as described in Figure 4. For, since $p_{su}$ is continuous, and $AC_x(y)$ is connected, $p_{su}(AC_x(y))$ is connected. If $p_{su}(AC_x(y))$ contained another point, then it would contain a segment, which has non-empty interior in $W^c_{loc}(x)$. Proposition 3.5 then would imply that $AC(y)$ is open, which is absurd, since $y \in \Gamma(f)$. This proves that also $AC_x(y)$ is contained in $p_{su}^{-1}(y)$.

Hence, due to Lemma 3.6 above, we have that, for each $x \in \Gamma(f)$:

$$AC_x(x) = p_{su}^{-1}(x) = W^u_{loc}(W^s_{loc}(x)) \approx W^u_{loc}(x) \times W^s_{loc}(x)$$

$W^s_{loc}(x)$ and $W^u_{loc}(x)$ are (evenly sized) embedded segments that vary continuously with respect to $x \in M$ (see Hirsch, Pugh, Shub [HPS77] chapters 4 and 5). This implies that $\Gamma(f) \ni x \mapsto AC_x(x)$ is a continuous map that assigns to each $x$ an evenly sized 2-disc. To finish the description of accessibility classes, let us introduce the following definition:

**Definition 3.7.** The foliations $W^s$ and $W^u$ are jointly integrable at a point $x \in M$ if there exists $\delta > 0$ such that for each $z \in W^s_\delta(x)$ and $y \in W^u_\delta(x)$, we have

$$W^u_{loc}(z) \cap W^s_{loc}(y) \neq \emptyset$$

See Figure 5 for an illustration of a point of joint integrability of $W^s$ and $W^u$.

Then Lemma 3.6 and discussion above imply the following:

**Proposition 3.8.** A point $x$ belongs to $\Gamma(f)$ if and only if $W^s$ and $W^u$ are jointly integrable at all points of $AC(x)$.

Indeed, if $x$ belongs to $\Gamma(f)$, then for all $y \in AC(x) \subset \Gamma(f)$, we have $p_{su}(AC_y(x)) = \{y\}$. In particular, if $z \in W^s_\delta(y)$ and $w \in W^u_\delta(y)$, then $W^u_{loc}(z) \cap W^s_{loc}(w) \neq \emptyset$. On the other hand, if $W^s$ and $W^u$ are jointly integrable at all points of $AC(x)$, then $AC(x)$ is a lamina, due to the explanation above (the coherence of the charts $\phi_x$ defined above depend only on the joint integrability of $W^s$ and $W^u$). Moreover, this 2-dimensional
Lemma 3.9. [Did03, Lemma 5] If $W^s$ and $W^u$ are jointly integrable at $x$, then the set

$$W^s_{\text{loc}}(x) = \{W^u(z) \cap W^s(y) : z \in W^s_\delta(x) \text{ and } y \in W^u_\delta(x)\}$$

where $\delta > 0$ is as in the definition of joint integrability (Definition 3.7), is a 2-dimensional $C^1$-disc that is everywhere tangent to $E^s \oplus E^u$.

In order to prove Lemma 3.9 we shall use the following result by Journé:

Theorem 3.10. [Jou88] Let $F^h$ and $F^v$ be two transverse foliations with uniform smooth leaves on an open set $U$. If $\eta : U \to M$ is uniformly $C^1$ along $F^h$ and $F^v$, then $\eta$ is $C^1$ on $U$.

Proof of Lemma 3.9. Let $D$ be a small smooth 2-dimensional disc containing $x$ and transverse to $E^c_x$. Consider a one-dimensional smooth foliation of a small neighborhood $N$ of $x$, transverse to $D$. If $D$ is sufficiently small, there is a smooth map $\pi : N \to D$, which consists in projecting along this smooth one-dimensional foliation. Note that $W^s_{\text{loc}}(x)$ can be seen as the graph of a continuous function $\eta : D \to N$.

We produce a grid on $D$ in the following way: the horizontal lines are the projections of the stable manifolds $W^s(y)$, with $y \in W^s_\delta(x)$, that is, the horizontal lines are of the form $\pi(W^s_{\text{loc}}(\eta(v)))$, with $v \in D$. Analogously, the vertical lines are the projections of the unstable manifolds $W^u_{\text{loc}}(z)$, with $z \in W^u_\delta(x)$, that is, the vertical lines are of the form $\pi(W^u_{\text{loc}}(\eta(w)))$, with $w \in D$.

Now, $v \mapsto W^s_{\text{loc}}(\eta(v))$ and $w \mapsto W^u_{\text{loc}}(\eta(w))$ are continuous in the $C^1$-topology, that is, for close $v$ we obtain close $W^s_{\text{loc}}(\eta(v))$ in the $C^1$-topology ($E^s$ is a continuous bundle). Since $\pi$ is smooth, we also obtain that $F^h = \{\pi(W^s_{\text{loc}}(\eta(v)))\}_{v \in D}$, the horizontal partition of $D$, and $F^v = \{W^u_{\text{loc}}(\eta(w))\}_{w \in D}$, the vertical partition of $D$, are transverse foliations continuous in the $C^1$-topology.

But $\eta$ is uniformly $C^1$ along $F^h$, since $\eta$ along a leaf $F^h(v_0) = \pi(W^s_{\text{loc}}(\eta(v_0)))$ is exactly $W^s_{\text{loc}}(\eta(v_0))$. Indeed, $\eta \circ \pi : W^s_{\text{loc}}(x) \to W^s_{\text{loc}}(x)$ is the identity map, and $W^s_{\text{loc}}(\eta(v_0))$ is a smooth manifold. Analogously, we obtain that $\eta$ is uniformly $C^1$ along $F^v$. Hence, by Theorem 3.10 $\eta$ is $C^1$. \hfill $\Box$

3.3. Properties of the lamination $\Gamma(f)$ of non-open accessibility classes. In order to prove the non-ergodic Conjecture 2.5 our strategy is to describe as accurately as possible the lamination $\Gamma(f)$. More precisely, we would like to see that non-accessibility implies the existence of a compact (toral) accessibility class.

This is the state of the art so far:
**Theorem 3.11.** ([HHU08a, Theorem 1.6]) Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism that is not accessible. Then one of the following possibilities holds:

1. there is a compact accessibility class (a torus tangent to $E^s \oplus E^u$)
2. there exists an invariant sublamination $\Lambda \subset \Gamma(f)$ that trivially extends to a (not necessarily invariant) foliation without compact leaves of $M$. Moreover, if $\Lambda \neq M$ the boundary leaves of $\Lambda$ are periodic, have Anosov dynamics and periodic points are dense in each boundary leaf with the intrinsic topology.
3. $\Gamma(f)$ is a minimal foliation

With respect to item (2), a leaf $L$ of a lamination $\Lambda$ is a **boundary leaf** if there is a transverse segment to $L$ containing a subsegment $\alpha$ with an endpoint in $L$ and such that $\alpha \cap \Lambda = \emptyset$. In [HHU08a] it is proven that boundary leaves are periodic in the conservative setting, and, moreover, that periodic points are dense in each boundary leaf with the intrinsic topology.

If Case (1) holds, then Conjecture 2.5 is true. We conjecture that Case (2) is not possible, more precisely, we conjecture that each boundary leaf should be a torus. Answering the following question would rule Case (2) out:

**Question 3.12.** Let $L$ be a complete immersed surface in a 3-manifold, such that there is an Anosov dynamics on $L$ where

1. each stable and unstable manifold is complete, and angles between stable and unstable manifolds are bounded
2. periodic points are dense with the intrinsic topology
3. the stable and unstable manifold of each periodic point are dense in $L$ with the intrinsic topology

Is $L$ the 2-torus?

Case (3) of Theorem 3.11 means that each leaf of $\Gamma(f)$ is dense. We conjecture that in this case, in fact, $f$ is **essentially accessible**, this means that each set which is union of accessibility classes has either measure one or zero. Essential accessibility in dimension 3 implies ergodicity [BW10], [HHU08b]. If this could be established, then Conjecture 2.5 would be proven true.

Since in Case (1) of Theorem 3.11 Conjecture 2.5 follows trivially, we would like to better describe what happens in Cases (2) and (3). The following describes the accessibility classes in these cases:

**Theorem 3.13.** If $f$ has no compact accessibility class, then the $\pi_1$ of each accessibility class injects in $\pi_1(M)$.

**Proof.** The result follows almost directly from the following Theorem by Novikov:
Theorem 3.14 (Novikov). Let $M$ be a compact orientable 3-manifold and $\mathcal{F}$ a transversely orientable codimension-one foliation without Reeb components. Then, for each leaf $L$ in $\mathcal{F}$, $\pi_1(L)$ injects in $\pi_1(M)$.

A Reeb component of a foliation is a solid torus subfoliated by planes, as in Figure 6.

Question 3.15. Does Theorem 3.13 hold without assuming there are no compact accessibility classes?

If $\Gamma(f) = M$, then we are already in the hypothesis of Theorem 3.14, since the fact that $\Gamma(f)$ has no compact leaves precludes the existence of Reeb components. The rest of the theorem follows by proving that if $\Gamma(f)$ has no compact leaves, and is not a foliation $\Gamma(f) \neq M$, then it can be extended to a foliation without Reeb components. This follows almost immediately from Theorem 4.1 of [HHU08a]:

Theorem 3.16 (Hertz-Hertz-Ures). If $\Lambda \subset \Gamma(f)$ is an orientable and transversely orientable $f$-invariant sublamination without compact leaves such that $\Lambda \neq M$, then all closed complementary regions of $\Lambda$ are $I$-bundles.

By taking finite coverings we may assume that $\Gamma(f)$ itself is orientable and transversely orientable. $\Gamma(f) \neq M$ has no compact leaves, therefore its complementary regions are $I$-bundles. This allows us to extend $\Gamma(f)$ to a foliation in a trivial way, by “copying” the boundary leaves. This means, each complementary region is of the form $L \times [0, 1]$, where $L$ is a boundary leaf of $\Gamma(f)$, we foliate each complementary region, by considering leaves of the form $L \times \{t\}$, with $t \in [0, 1]$.

4. Dynamical Coherence

It turns out that ergodicity in our setting is tightly related to integrability of the invariant bundles. As we explained before the bundles $E^s, E^u$ are always integrable.
The integrability of the center bundle $E^c$, on the other hand, cannot be always guaranteed even in our setting. This was a long standing problem and was recently solved in [HHU14], see 2.9.

Let us recall dynamical coherence, which was stated in Definition 2.9. A partially hyperbolic diffeomorphism is dynamically coherent if there is an $f$-invariant foliation tangent to $E^s \oplus E^c$ (the center-stable foliation), and an $f$-invariant foliation tangent to $E^c \oplus E^u$ (the center-unstable foliation). Note that in this case the center bundle is also integrable: an $f$-invariant foliation tangent to $E^c$ is obtained by simply intersecting the center stable and center unstable leaves, and taking connected components. This is called the center foliation.

Proposition 4.1. If $f : M^3 \rightarrow M^3$ is a partially hyperbolic diffeomorphism whose center bundle is $C^1$, then $f$ is dynamically coherent.

Proof. Observe first that $W^c$ is $f$-invariant: if $c : [0,1] \rightarrow W^c(x)$ is a differentiable curve with $c(0) = x$, then $f \circ c : [0,1] \rightarrow M$ is a differentiable curve tangent to $E^c$, and hence by uniqueness of solutions of differential equations, $f \circ c([0,1]) \subset W^c(f \circ c(0)) = W^c(f(x))$.

Theorem 6.1 and Theorem 7.6 in [HPS77] imply that through each leaf $L$ of $W^c$ there exist immersed submanifolds $W^s(L), W^u(L)$ tangent to $E^{cs}, E^{cu}$ respectively, saturated by the corresponding strong foliations. Again using uniqueness of solutions of differential equations, one proves that the families $W^{cs} = \{W^s(L)\}_{L \in W^c}, W^{cu} = \{W^u(L)\}_{L \in W^c}$ are pairwise disjoint, and since their tangent spaces vary continuously, they form foliations. Invariance follows since $W^c, W^s, W^u$ are invariant. □

More details about this can be found in [BW08]. When the center bundle is not differentiable, we still have curves tangent to it as a consequence of Peano’s Theorem. This family of curves, however, is not assembled as a foliation, but it still can contain relevant information. See [HHU12] and [HHU08c].

Problem 4.2. Find an example of a dynamically coherent partially hyperbolic diffeomorphism that is not leafwise conjugate to a $C^1$ dynamically coherent one. Can [BPP14] examples be adjusted to get one?

Other condition that guarantees dynamical coherence in $T^3$ is absolute partial hyperbolicity, a notion stronger than partial hyperbolicity, which was described in Equation (2.4):

Theorem 4.3 (Brin-Burago-Ivanov - [BBI09]). If $f : T^3 \rightarrow T^3$ is absolutely partially hyperbolic, then $f$ is dynamically coherent.

However, this is not the general case, let us see the following:
4.1. A non-dynamical coherent example.

**Theorem 2.9. [HHU14]** There exists a partially hyperbolic diffeomorphism \( f : \mathbb{T}^3 \to \mathbb{T}^3 \) such that

1. There is no invariant foliation tangent to the distribution \( E^c \oplus E^u \)
2. There is an invariant 2-dimensional torus \( T \) tangent to the distribution \( E^c \oplus E^u \)

Moreover, there is a \( C^1 \)-open neighborhood \( \mathcal{U} \) of \( f \) such that all \( g \) in \( \mathcal{U} \) satisfy conditions (1) and (2).

**Sketch.** Let \( A : \mathbb{T}^2 \to \mathbb{T}^2 \) be a hyperbolic linear map with eigenvalues \( \lambda < 1 < 1/\lambda \). Take \( u \) a unit eigenvector corresponding to the eigenvalue \( \lambda \). Consider also a north pole-south pole function \( f : \mathbb{T} \to \mathbb{T} \) such that

\[
\begin{align*}
f(0) &= 0, f(1/2) = 1/2 \\
f'(1/2) &= \sigma < \lambda < 1 < \nu = f'(0) < 1/\lambda
\end{align*}
\]

and a differentiable function \( \phi : \mathbb{T} \to \mathbb{R} \).

Now construct a perturbation \( F \) of the Axiom-A map \( A \times f \) by “pushing” in the stable direction of \( A \), namely

\[
F(x, \theta) = (Ax, f(\theta)) + (\phi(\theta)e_s, 0), \quad \phi(1/2) = 0.
\]

where \( e_s \) is a unit vector in the \( E^s \) direction of \( A \).

Note the strong unstable direction of \( A \times f \) is unaltered by this perturbation, and in particular the strong stable manifold of the perturbation exists and coincides with the strong stable manifold of the unperturbed map. Observe that the unperturbed map is not partially hyperbolic. Now we study the other invariant directions.

We are seeking invariant directions of the derivative of \( F \):

\[
dF_{(x, \theta)}(v, t) = (Av, f'(\theta)t) + (\phi'(\theta)te_s, 0)
\]

An invariant direction (inside the \( e_s \times \mathbb{T} \) cylinder) will be generated by a vector field of the form \( (a(\theta)e_s, 1) \) for some function \( a \), hence we need to solve

\[
a(f(\theta))f'(\theta) = \lambda a(\theta) + \phi'(\theta). \tag{4.12}
\]

We are thus led to find a solution of the cohomological equation

\[
b \circ f = \lambda b + \phi \tag{4.13}
\]

(the solution of (4.12) is just \( a = b' \)). One then checks that the following two functions are solutions,

\[
\eta(\theta) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \phi(f^{-n} \theta) \tag{4.14}
\]
\[ \zeta(\theta) = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^{-n} \phi(f^n \theta) \]  \hspace{1cm} (4.15)

and that the previous assumptions imply that \( \eta \in C^1(T \setminus \{1/2\}) \), \( \zeta \in C^1(T \setminus \{0\}) \).

Let us define
\[ E^c(\theta) = \text{span}(\eta'(\theta)e_s, 1) \quad \text{for} \quad \theta \neq \frac{1}{2} \]  \hspace{1cm} (4.16)

and
\[ E^s(\theta) = \text{span}(\zeta'(\theta)e_s, 1) \quad \text{for} \quad \theta \neq 0 \]  \hspace{1cm} (4.17)

Back to the invariant directions, note that, for generic \( \phi \), \( \eta'(\theta) \) gets bigger as \( \theta \) approaches \( 1/2 \), and thus if we can choose \( \phi \) so that
\[ \lim_{\theta \to 1/2} \eta'(\theta) = \infty \]  \hspace{1cm} (4.18)

we will get continuity for \( E^c \) by defining
\[ E^c(\theta = \frac{1}{2}) = \text{span}\{(e_s, 0)\} = E^s_A \times 0 \]

Arguing similarly, we define
\[ E^s(\theta = 0) = E^s_A \times 0 \]

and we will get a continuous bundle provided that we prove
\[ \lim_{\theta \to 0} \zeta'(\theta) = \infty \]  \hspace{1cm} (4.19)

Assume for now that we have proved that these bundles are continuous. Now we want to show that \( TT^3 = E^s \oplus E^c \oplus E^u \), or what is equivalent, that the angle between \( E^s \) and \( E^c \) are not zero. What we need to show is that \( \eta' \neq \zeta' \) for \( \theta \neq 0, 1/2 \). Note that for \( \theta = 0, 1/2 \) the angle is not zero, and hence it is not zero in a neighbourhood of these points. But by the cohomological equations,
\[ (\eta' - \zeta') \circ f = \lambda(\eta' - \zeta') \]

and using the form of the dynamics of \( f \), we conclude that the sign of \( \eta' - \zeta' \) is constant in \( (0, 1/2) \) and \( (1/2, 1) \), and clearly non zero. The following lemma is proven in detail in [HHU14].

**Lemma 4.3.** There exists \( \phi \) so that

1. Limits (4.18) and (4.19) hold.
2. \( \eta' \) has opposite sign in \( (0, 1/2) \) and \( (1/2, 1) \).

In particular \( F \) is partially hyperbolic (but NOT absolutely partially hyperbolic). Finally we prove that it is not dynamically coherent. Observe that since the bundles only depend on \( \theta \) we obtain the stable, unstable and center manifolds (provided that this last one exists) by translating a given one of the same type.
Consider the function \( h : T^3 \to T^2 \) given by

\[
h(x, \theta) = x - \eta(\theta)e_s
\]

Then \( F \circ h = h \circ A_T \) and \( h \) is clearly surjective, hence it is a semiconjugacy. Note that we have a parametrization \( l_x(\theta) \) of \( h^{-1}(h(x, 0)) \) given by

\[
l_x(\theta) = (x, 0) + (\eta(\theta)e_s, \theta)
\]

and hence, the family of curves \( \{l_x(\theta)\} \) is tangent to \( E_c \) if \( \theta \neq 1/2 \). For \( \theta \neq 1/2 \) the bundle \( E_c \) is uniquely integrable and hence its invariant curves are precisely the \( l_x(\theta) \). But for \( \theta = 1/2 \) \( E_c = E^s_\lambda \times \{0\} \), hence its tangent curves have to be horizontal. Now we use that \( \eta' \) have different signs on the intervals \((0, 1/2)\) and \((1/2, 1)\) to conclude that this family is not a foliation near \( \theta = 1/2 \), hence the bundle \( E_c \) is not integrable. See Figure 7.

\[\square\]

The example previously constructed in fact is robust, meaning that in a neighborhood of it there are no dynamically coherent partially hyperbolic diffeomorphisms, a surprising fact. This is a consequence of the fact that the invariant torus corresponding to \( \theta = 1/2 \) is a \( cu \)-torus, and in particular is a normally hyperbolic torus. The results of [HPS77] imply that this torus persist under perturbations, meaning that any small perturbation of \( f \) has a \( cu \) torus. On the other hand, outside a neighborhood of this \( cu \)-torus, there is a unique center foliation, which is persistent under perturbations, due also to [HPS77]. This foliation, is extended by invariance to all the complement of the \( cu \)-torus. Hence, if there were a foliation, it shoud contain a \( cu \)-torus. This is not possible, due to the following result:
Theorem 4.4 (Hertz-Hertz-Ures). [HHU14] Let $f : M \rightarrow M$ be a dynamically hyperbolic partially hyperbolic diffeomorphisms. Then the center unstable foliation does not have any closed leaf.

**Question:** Does the same result hold in any dimension?

4.2. Non-dynamical coherence conjecture - state of the art. It has been proven by A. Hammerlindl and R. Potrie [HP13] that the Non-dynamical coherence Conjecture 2.11 is true in tori and solv- and nil-manifolds and their finite covers:

**Theorem 4.5** (Hammerlindl-Potrie [HP13]). If $f$ is a non-dynamical coherent diffeomorphism in a 3-manifold with (virtually) solvable fundamental group, then there exists a 2-torus, tangent either to $E^s \oplus E^c$ or $E^c \oplus E^u$. In particular, any partially hyperbolic diffeomorphism with $\Omega(f) = M$ in these manifolds is dynamically coherent.

This is the sharpest result concerning Conjecture 2.11 so far. Let us give a brief sketch of the ideas used to establish this theorem in the simpler case where $M = T^3$.

This result has been proven by R. Potrie in his thesis (see [Pot12]), and we shall follow his arguments. Consider $f : T^3 \rightarrow T^3$ partially hyperbolic, and by passing to a finite covering, it is no loss of generality to assume that the bundles $E^\sigma_f$ are oriented and, furthermore, that $f$ preserves their orientation.

We will rely heavily in the seminal papers of Brin, Burago and Ivanov ([BI08, BB104, BB109]). The starting point is that the action in homology $f_\ast : H_1(T^3) \rightarrow H_1(T^3)$ is partially hyperbolic. Namely, if $A = f_\ast \in SL(3, \mathbb{R})$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ then $|\lambda_1| < 1 < |\lambda_3|$. We have two cases:

1. $|\lambda_2| = 1$; this is the skew-product case.
2. $|\lambda_2| \neq 1$; in this case $f$ is a DA.

It suffices to show the existence of an $f$-invariant foliation tangent to $E^{cs}$ (the other case being analogous). It turns out that there is a natural candidate for $F^{cs}$.

**Theorem 4.6** ([BI08], Thm. 4.1). There exists a family $B^{cs} = \{B^{cs}(x)\}_{x \in M}$ such that

1. Each $B^{cs}(x)$ is an immersed boundary-less surface of class $C^1$ tangent to $E^c$
2. For every $x, x' \in M$ the surfaces $B^{cs}(x), B^{cs}(x')$ do not cross topologically.
3. $B^{cs}$ is $f$-invariant.

The family $B^{cs}$ is what is called a branched foliation, and its elements are called leaves. To prove that $B^{cs}$ is a genuine foliation it suffices to show that it is unbranched: namely, that any $x \in M$ is contained in exactly one leaf of $B^{cs}$. This fact is a direct consequence of Proposition 1.6 of [BW05] and the remark afterwards.
Another important fact is that $E^{cs}$ is almost integrable, that is, there exist a foliation (not necessarily invariant) that is transverse to $E^u$. This concept of almost integrability has been coined by R. Potrie and has proven useful in this context. Almost integrability of $E^{cs}$ for all 3D partially hyperbolic diffeomorphisms with orientable bundles had been established by Burago-Ivanov:

**Theorem 4.7** ([BI08], Key Lemma). *For every $\epsilon > 0$ sufficiently small there exists a true $C^{1,0+}$ (not necessarily invariant) foliation $T^{cs}_\epsilon$ such that the angle $\angle(TT^{cs}_\epsilon, E^{cs})$ is less than $\epsilon$. Moreover, there exists a continuous surjective map $h^{cs}_\epsilon : M \to M$ that is $\epsilon$-close to the identity and sends the leaves of $T^{cs}_\epsilon$ into leaves of $B^{cs}$.***

Potrie then shows that for sufficiently small $\epsilon$, the lifted foliations $\tilde{F}^u, \tilde{T}^{cs}_\epsilon$ to $\mathbb{R}^3$ have global product structure (that is, any two leaves $F \in \tilde{F}^u$ and $T \in \tilde{T}^{cs}_\epsilon$ intersect exactly in one point). This has the following consequence

**Theorem 4.8** ([Pot12] Prop. 8.4). *If $\tilde{F}^u, \tilde{T}^{cs}_\epsilon$ have global product structure then $B^{cs}$ is unbranched.*

The proof of this theorem relies on two geometrical facts:

**Fact 1:** $\tilde{F}^u$ is quasi-isometric: that is, there exist $a, b > 0$ such that for every $x, y \in \mathbb{R}^3$, $y \in \tilde{F}^u(x)$ the length $l(x, y)$ of the interval with endpoints $x, y$ contained in $\tilde{F}^u(x)$ satisfies

$$l(x, y) \leq a|x - y| + b.$$

**Fact 2:** There exists an $A$-invariant plane $P$ and $R > 0$ such that every leaf of $B^{cs}$ is contained in an $R$-neighborhood of a plane parallel to $P$, and either

1. the projection of $P$ in $\mathbb{T}^3$ is dense, and the $R$-neighborhood of every leaf of $B^{cs}$ contains a plane parallel to $P$, or
2. the projection of $P$ in $\mathbb{T}^3$ is a two torus: in this case there exists a 2-torus tangent to $E^s \oplus E^c$

Note that if $f$ is a DA only (1) above can hold. Let us see how the proof goes:

**Proof.** A modification of Proposition 3.7 in [BBH04] gives that a codimension one foliation in $\mathbb{T}^3$ either has a Reeb component, or there exists $R > 0$ and an $A$-invariant plane $P \subset \mathbb{R}^3$ such that every leaf of the lifted foliation is in a $R$-neighborhood of $P$. The projection of a plane in $\mathbb{T}^3$ is either dense or a two-torus: in the former case every leaf of the lifted foliation is parallel to a fixed translate of the plane, while in the later case there is a leaf of the foliation in $\mathbb{T}^3$ which is a torus (see Theorem 5.3 and Proposition 5.6 in [Pot12]).

As $T^{cs}_\epsilon$ is transverse to $W^u$, it cannot have a Reeb component and thus we can consider the plane $P$ as above. Note that by 4.7 the leaves of $\tilde{B}^{cs}$ are $\epsilon$-close to the leaves of $\tilde{T}^{cs}_\epsilon$, hence the first part of the claim follows.
Now, it suffices to observe that if $T^{cs}$ has a torus leaf, by Theorem 4.7 $B^{cs}$ also has a torus leaf, therefore, there is a 2-torus tangent to $E^s \oplus E^c$. □

Once the above facts are established, the proof of the theorem is carried by standard arguments. To finish the proof of the conjecture, one has to analyse the skew-product and the DA case separately. In both cases, with the machinery developed, it is not too hard to check that the absence of tori tangent to $E^s \oplus E^c$ implies the global product structure referred above, thus implying dynamical coherence.

When the manifold $M$ is a solv-manifold, but not a nilmanifold, the proof of Theorem 4.5 becomes technically harder, although some of the guidelines presented are still valid. It relies on more background on foliation theory (codimension-one foliations of compact solv-manifolds are reasonably well understood). Solv-manifolds of this type are covered by torus bundles over the circle (a reasonable geometric object), and then there is a canonical model (isotopic to the identity) to compare the dynamics. For more details on a complete proof, we refer the reader to [HP13].

5. Classification

For many years, the only known examples of partially hyperbolic diffeomorphisms in 3-manifolds were the ones listed in Subsection 1.1, namely: time-one maps of Anosov flows, skew products, DA-diffeomorphisms, and their perturbations. As it was stated in Subsection 2.3, this led E. Pujals, in 2001 to conjecture that for transitive ones, this was the complete list of partially hyperbolic diffeomorphisms:

**Conjecture 2.12** (Pujals (2001)). Any partially hyperbolic diffeomorphism in a 3-manifold is finitely covered by a map which is conjugated either to

1. A perturbation of the time-one map of an Anosov flow.
2. A perturbation of an Skew Product.
3. A DA.

Two different particular cases of this conjecture were verified in the transitive setting by C. Bonatti and A. Wilkinson [BW05].

**Theorem 5.1** (Bonnati-Wilkinson). Let $f : M \to M$ be a transitive partially hyperbolic diffeomorphism.

1. Assume that there exists some embedded circle $c$ such that $f(c) = c$, with the property that for some $\epsilon > 0$ the set

$$\bigcup_{x \in c} W^s_\epsilon(x) \cap \bigcup_{y \in c} W^u_\epsilon(y) \setminus c$$

\footnote{Interestingly enough, the proof does not extend to the case where $c$ is merely periodic}
contains a connected component that is a circle. Then \( f \) is dynamically coherent and finitely covered by a map which is conjugated to a circle extension of an Anosov map (a topological Skew Product).

(2) Assume that \( f \) is dynamically coherent, and that for some \( \epsilon > 0 \) each end of a center leaf contained in

\[ \bigcup_{x \in c} W^s_\epsilon(x) \]

is periodic. Then, the center foliation is fixed under \( f^n \) and it supports a continuous flow conjugate to an expansive transitive flow.

It would be interesting to know if in case two one in fact can take an Anosov flow, and thus settle Pujals's conjecture for that case. This still remains an open problem.

More recently, a new type example of non-dynamically coherent example was presented, the one described in Subsection 4.1. The examples in Subsection 4.1 suggested another possibility:

**Conjecture 2.14 (Classification conjecture: Hertz-Hertz-Ures (2009)).** \( f \) be a partially hyperbolic diffeomorphism of a 3-manifold.

If \( f \) is dynamically coherent, then it is (finitely covered by) one of the following:

1. a perturbation of a time-one map of an Anosov flow, in which case it is leafwise conjugate to an Anosov flow
2. a skew-product, in which case it is leafwise conjugate to a skew-product with linear base, or
3. a DA, in which case it is leafwise conjugate to an Anosov diffeomorphism of \( T^3 \).

If \( f \) is not dynamically coherent, then there are a finite number of 2-tori tangent either to \( E^c \oplus E^u \) or to \( E^s \oplus E^c \), and the rest of the dynamics is trivial, as in the non dynamically coherent example [HHU14].

Both Conjectures have been proven false very recently by Bonatti, Parwani and Potrie, see [BPP14]. However, both conjectures are true in certain manifolds, as it was proven by Hammerlindl and Potrie:

**Theorem 5.2 (Hammerlindl-Potrie [HP13]).** Both Conjecture 2.13 and Conjecture 2.15 are true on 3-manifolds with (virtually) solvable fundamental group.

Theorem 5.2 was first proved in tori by Hammerlindl in his thesis [Ham13], it was later extended to 3-manifolds with (virtually) nilpotent groups by Hammerlindl and Potrie in [HP14]. Finally, it was extended to 3-manifolds with (virtually) solvable groups, by the same authors in [HP13], still in press.
In [HP13] it is proven that, for solvmanifolds, as stated in Conjecture 2.11, the absence of tori tangent to either $E^s \oplus E^c$ or $E^c \oplus E^u$ implies dynamical coherence. Observe that the existence of such a torus implies the existence of either a repelling or an attracting periodic torus (see more details in Section 6). Transitivity precludes this possibility. Therefore, for solvmanifolds, we can assume there is dynamical coherence in both Conjectures 2.13 and 2.15.

Let us give a flavor of how is Theorem 5.2 proved in the case of solvmanifolds with non-virtually nilpotent fundamental group.

**Theorem 5.3 ([HP13]).** If $f : M \to M$ is a dynamically coherent partially hyperbolic diffeomorphism on a solvmanifold whose fundamental group is not virtually nilpotent, then a finite cover of a finite iterate of $f$ is leafwise conjugate to the time-one map of a suspension Anosov flow.

(See Definition 2.14 for the definition of leaf conjugacy)

Firstly, any such manifold is finitely covered by the mapping torus $M_A$ of a hyperbolic automorphism on $\mathbb{T}^2$, that is $M_A = \mathbb{T}^2 \times \mathbb{R} / \sim$ such that $(Ax, t) \sim (x, t + 1)$, where $A$ is a hyperbolic automorphism of $\mathbb{T}^2$. And any diffeomorphism of $M_A$ has a finite iterate that is homotopic to the identity. This is not hard to prove.

Now, on the universal cover of $M_A$, there are model foliations $A^{cs}$ and $A^{cu}$: $(v_1, t_1)$ and $(v_2, t_2)$ belong to the same leaf of the foliation $A^{cs}$ if and only if $v_1 - v_2$ is in the stable eigenspace of the automorphism $A$. Similarly, $(v_1, t_1)$ and $(v_2, t_2)$ belong to the same leaf of the foliation $A^{cu}$ if and only if $v_1 - v_2$ is in the unstable eigenspace of the automorphism $A$. In [HP13] it is seen that the lift to the universal cover of any foliation without compact leaves is almost-parallel to either $A^{cs}$ or $A^{cu}$. Two foliations $F$ and $F'$ are almost-parallel if there is a uniform bound $R > 0$ such that

1. for each leaf $L \in F$ there is a leaf $L' \in F'$ such that $d_H(L, L') < R$
2. for each leaf $L' \in F'$ there is a leaf $L \in F$ such that $d_H(L, L') < R$

where $d_H$ is the Hausdorff distance, that is

$$d_H(L, L') = \max \left\{ \sup_{x \in L} d(x, L'), \sup_{y \in L'} d(y, L) \right\}$$

Now, neither $F^{cs}$ nor $F^{cu}$ contain compact leaves [HHU12], therefore each one is almost parallel to either $A^{cs}$ or $A^{cu}$. They proceed then to show that if $F^{cs}$ is almost parallel to $A^{cs}$, then $F^{cu}$ is almost parallel to $A^{cu}$. This step is more delicate.

Note that the center leaves of the model foliation, that is, the leaves in $A^c$ that are intersection of leaves $A^{sc}$ and $A^{cu}$, correspond to trajectories of an Anosov flow, which is infinitely expansive. Therefore, any two such leaves $A^c_1$ and $A^c_2$ are at infinite Hausdorff distance. This implies that the almost-parallel relation defined above assigns to each center leaf $F^c$ in the intersection of $F^{sc}$ and $F^{cu}$ a unique center leaf in $A^c$. Less trivially, there is a unique leaf in $F^{sc}$ at finite Hausdorff
distance of each leaf in $\mathcal{A}^{sc}$, and a unique leaf in $\mathcal{F}^{cu}$ at finite distance of each leaf in $\mathcal{A}^{cu}$ (Lemma 5.3 of [HP13]). Therefore, any two center leaves $F_1^c$ and $F_2^c$ that are at finite Hausdorff distance of each other, must be in the intersection of a single leaf of $\mathcal{F}^{sc}$ and a single leaf of $\mathcal{F}^{cu}$.

Now, let us assume that the center bundle $E^c$ is orientable, for otherwise we can take a finite cover. Then, there exists a field $X^c$, tangent to $E^c$, defining a flow $\varphi$ on $M_A$. We claim that $\varphi$ is an expansive flow.

Indeed, any two $\varphi$-trajectories that at most $\varepsilon$-apart, correspond to two center leaves that are at finite Hausdorff distance, henceforth, they are in the intersection of a single leaf of $\mathcal{F}^{sc}$ and a single leaf of $\mathcal{F}^{cu}$. This implies either that a stable leaf intersects (at least) twice a center unstable leaf of $\mathcal{F}^{cu}$ or that an unstable leaf intersects (at least) twice a center stable leaf of $\mathcal{F}^{sc}$. A classical argument à la Novikov, implies the existence of a compact leaf either in $\mathcal{F}^{cu}$ or in $\mathcal{F}^{sc}$, a situation precluded by [HHU12].

Finally, M. Brunella establishes in [Bru93] that any expansive flow on a torus bundle is leafwise conjugate to a transitive Anosov suspension, concluding the classification theorem in solvmanifolds.

6. A tool to solve HHU-conjectures: Anosov Tori

In both the non-ergodicity and the non-integrability conjectures it is proposed that the existence of a map with some specific dynamical property leads to very rigid restrictions in the topology of the ambient manifold. The reader may wonder why this is case, and how one can attempt to prove such type of results. We discuss these issues in this section.

The unifying link is, surprisingly, the existence of certain tori embedded in the manifold.

**Definition 6.1.** We say that the manifold $M$ admits an Anosov torus is there exist an embedded torus $T \subset M$ and a diffeomorphism $\phi : M \to M$ such that

1. $\phi(T) = T$.
2. $\phi|T$ is a linear hyperbolic automorphism $^2$

As we shall see below, not every manifold admits an Anosov torus.

Recall that a three manifold is irreducible if every embedded two-sphere bounds a three-ball. We then have the following topological result:

**Theorem 6.2** (Hertz-Hertz-Ures [HHU11]). Assume that $M$ is a compact irreducible 3-manifold supporting an Anosov Torus. Then $M$ is homeomorphic to either

$^2$This is equivalent to the existence of $\phi$ such that $\phi|T$ be isotopic to an Anosov diffeomorphism, which holds if and only if the action on the first homology group of the torus $H_1(T)$ is hyperbolic
(1) A 3-torus.
(2) The mapping torus of \(-Id : T^2 \to T^2\).
(3) The mapping torus of an hyperbolic automorphism of \(T^2\).

We remark that partial hyperbolicity is not required in Theorem 6.2. This theorem just shows that the 3-manifolds admitting Anosov tori are actually very few. Now we apply this result to the partially hyperbolic context. First we note that irreducibility comes for free in this setting.

**Lemma 6.3.** If a three manifold \(M\) supports a partially hyperbolic diffeomorphism, then \(M\) is irreducible.

**Proof.** A 3-manifold admitting a partially hyperbolic diffeomorphism has a codimension-one foliation having neither Reeb components nor spherical leaves [BB104]. This proves the claim since Rosenberg shows in [Ros68] that any codimension-one foliation in a reducible 3-manifold must have a Reeb component or a spherical leaf. See also [Rou71]. \(\square\)

The following Anosov tori arise naturally in partially hyperbolic dynamics, see more details in [HHU12]:

**Proposition 6.4.** Let \(f : M \to M\) be a partially hyperbolic diffeomorphism, and assume that there exists an \(f\)-invariant embedded torus \(T\) tangent to either \(E^c \oplus E^u\), \(E^s \oplus E^c\) or \(E^u \oplus E^s\). Then \(T\) is an Anosov torus.

**Proof.** Let \(g = f|T\). In each of the different cases, \(g\) or \(g^{-1}\) preserves an expanding foliation by lines, so with no loss of generality we will assume that \(\|dg|E^u\| > 1\). It suffices to prove that \(g_* : \pi_1(T) \approx \mathbb{Z}^2 \to \mathbb{Z}^2\) is hyperbolic.

By taking \(g^2\) if necessary we can suppose that \(g\) preserves the orientation of \(E^u|T\). Since \(g\) preserves a foliation without compact leaves, the integral matrix \(g_*\) has an eigenspace of irrational slope. This implies that either \(g_*\) is hyperbolic or \(g_* = Id\). In the second case \(g\) has a lift \(\widehat{g} : \mathbb{R}^2 \to \mathbb{R}^2\) such that \(\widehat{g} = Id + \alpha\) where \(\alpha\) is a periodic, and in particular bounded, function. Hence there exists a constant \(K > 0\) such that given any \(X \subset \mathbb{R}^2\),

\[
\text{diam}(\widehat{g}^n(X)) \leq \text{diam}(X) + nK.
\]

Let \(\gamma\) be an arc contained in a leaf of \(W^u(x), x \in T\). Then, the length of \(\gamma\) grows exponentially and its diameter grows at most linearly. This implies that given a small \(\epsilon > 0\) there exists an iterate of \(g\) that contains a curve of length arbitrarily large and with end points at distance less than \(\epsilon\). Using Poincare-Bendixon we obtain a singularity. This is a contradiction and then, \(g_*\) is hyperbolic. \(\square\)
Definition 6.5. Let $f : M \to M$ be a partially hyperbolic diffeomorphism. An embedded torus tangent to either $E^{cs}, E^{cu}$ or $E^{su}$ will be called a $cs, cu, su$-torus respectively. In these cases, we say that $f$ admits the corresponding torus.

Lemma 6.6. If $f$ admits an $su$-torus, then $M$ admits an Anosov torus

Proof. Assume $f$ admits an $su$-torus, and consider the lamination $\Lambda$ of all $su$-tori of $f$. This is a compact lamination [Hae62]. Therefore, there is a recurrent leaf, that is, there is a torus $T$ and an iterate $n$, such that $d_H(f^n(T), T) < \varepsilon$ for small $\varepsilon$. There exists a diffeotopy $i_t$ on $M$, taking $f^n(T)$ into $T$. Then $\phi = f^n \circ i_1$ fixes $T$ and $\phi|T$ is isotopic to an Anosov diffeomorphism. □

Lemma 6.7. If $f$ admits an $sc$ or a $cu$ torus, then it admits an $f$-periodic $sc$ or $cu$ torus. Therefore, $M$ admits an Anosov torus (by Proposition 6.4)

Proof. Let $T$ be a $cu$-torus, and consider the sequence $f^{-n}(T)$. Since the family of all compact subsets of $M$, considered with the Hausdorff metric $d_H$, is compact, there is a subsequence $f^{-n_k}(T)$ converging to a compact set $K \subset M$. Therefore, for each $\varepsilon > 0$ there are arbitrarily large $N >> L > 0$ such that $d_H(f^{-N}(T), f^{-L}(T)) < \varepsilon$.

Since $T$ is transverse to the stable foliation, the union of all local stable leaves of $T$ forms a small tubular neighborhood of $T$, $U(T)$. Since stable leaves grow exponentially under $f^{-1}$, if $N >> L$ as above are large enough, then $f^{-L}(U(T)) \subset f^{-N}(U(T))$. This implies that $f^{N-L}(U(T)) \subset U(T)$.

Exercise 6.8. Finish the proof by showing that $\cap_{k=0}^{\infty} f^{k(N-L)}(U(T))$ is a periodic $cu$-torus. □

Corollary 6.9. Suppose that $f : M \to M$ is a partially hyperbolic diffeomorphism admitting an $sc, cu$ or $su$ torus. If $M$ is connected, then $M$ is homeomorphic to either

1. A 3-torus.
2. The mapping torus of $-Id : \mathbb{T}^2 \to \mathbb{T}^2$.
3. The mapping torus of an hyperbolic automorphism of $\mathbb{T}^2$.

Observe that an $sc$ or $cu$ torus cannot appear on the conservative setting.

In the following subsection we sketch the proof of Theorem 6.2. We refer the reader to [HHU11] for its complete demonstration.

6.1. Manifolds admitting Anosov tori.

Theorem 6.10 (Hertz-Hertz-Ures [HHU08a]). Anosov tori are incompressible.
Definition 6.11. An embedded orientable surface $S \subset M$ is incompressible if the homomorphism induced by the inclusion map $i_* : \pi_1(S) \to \pi_1(M)$ is injective.

Equivalently, $S$ is incompressible if every embedded disk $D^2 \subset M$ such that $D^2 \cap S = \partial D^2$, is contractible in $S$ (see for instance [Hat07], page 10).

Now, let us assume, as in the hypotheses of Theorem 6.2 that the irreducible 3-manifold $M$ admits an Anosov torus $T$. Since $T$ is incompressible, we can “cut” $M$ along $T$ and obtain a manifold $N$ having incompressible 2-tori as boundary components. Theorem 6.2 then follows from the following theorem:

Theorem 6.12 (Hertz-Hertz-Ures [HHU11]). Let $N$ be a compact orientable irreducible 3-manifold with nonempty boundary such that all the boundary components are incompressible 2-tori. If $N$ admits an Anosov torus, then $N \approx \mathbb{T}^2 \times [0, 1]$.

In order to prove this, we make use of the Jaco-Shalen-Johannson decomposition, or JSJ-decomposition, which states that any manifold in the hypothesis of Theorem 6.12 can be cut by a (unique) family of incompressible tori, so that the remaining pieces have certain characteristics: they are either Seifert or else atoroidal and acylindrical. We provide these definitions below. See also Theorem 6.15.

The proof of Theorem 6.12 consists in showing, on one hand, that any Seifert manifold having incompressible tori as boundary components is $\mathbb{T}^2 \times [0, 1]$, and, on the other hand, that any manifold in the hypothesis of Theorem 6.12 that is atoroidal has an annulus which is properly embedded and is not isotopic to the boundary of the manifold. This last statement contradicts that the manifold is acylindrical, and shows that every component in the JSJ-decomposition must be Seifert, which proves the theorem.

Any compact 3-manifold, with or without boundary, supporting a foliation by circles is a Seifert manifold (see [Eps81]). This was not the original definition, a more descriptive one is the following:

Definition 6.13. A Seifert manifold is one which admits a decomposition into disjoint circles, the fibers, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to either

1. A solid 2-torus foliated by horizontal circles.
2. A solid 2-torus foliated by the fibration obtained by the identification $D^2 \times [0, 1]/x \sim R_{p/q}(x)$, where $R_{p/q} : D^2 \to D^2$ denotes the rotation of angle $p/q$, and $p, q$ are coprime.

If the manifold has boundary, its connected components are required to be tori, which are also fibered by circles.

The circles of the first type are the generic fibers, while the ones of the second type are the singular fibers. For an introduction to Seifert spaces see [Bri93].
Definition 6.14. Let $N$ be a three manifold with boundary.

1. $N$ is atoroidal if every incompressible torus is $\partial$-parallel, that is, isotopic to a subsurface of $\partial N$.
2. $N$ is acylindrical if every incompressible annulus $A$ that is properly embedded (i.e., $\partial A \subset \partial N$) is $\partial$-parallel by an isotopy fixing $\partial A$.

As we mentioned above, any irreducible orientable 3-manifold having incompressible tori as boundary components admits a natural decomposition into Seifert pieces on one side, and atoroidal and acylindrical components on the other.

Theorem 6.15 (JSJ-decomposition - [Hat07]). If $N$ is an irreducible, orientable 3-manifold, having incompressible tori as boundary components, then there exists a finite collection of disjoint incompressible tori $\mathcal{T}$ such that for each component $N_i$ of $N \setminus \mathcal{T}$, either

1. $N_i$ is a Seifert manifold, or
2. $N_i$ is both atoroidal and acylindrical.

Any minimal such collection is unique up to isotopy. This means that if $\mathcal{T}$ is a collection as described above, it contains a minimal subcollection $m(\mathcal{T})$ satisfying the same claim. All collections $m(\mathcal{T})$ are isotopic.

Any minimal family of incompressible tori as described above is called a JSJ-decomposition of $N$. Note that if $N$ is either atoroidal and acylindrical or Seifert, then $\mathcal{T} = \emptyset$.

Let us sketch how Theorem 6.12 is proved for the case of Seifert manifolds, the other case is more delicate and we refer the reader to [HHU11] for the complete proof.

Assume $N$ is a Seifert manifold, so it admits a foliation by circles, which is called a Seifert fibration. We loose no generality in assuming that one of the incompressible tori of the boundary of $N$ is an Anosov torus $T$. By the definition of Anosov torus, there exists a diffeomorphism $\phi$ on $N$ such that it is a linear hyperbolic automorphism on $T$. The image of the Seifert fibration by $\phi$ is another Seifert fibration, which is non-isotopic to the original one on $T$.

But orientable manifolds admitting two Seifert fibrations which are non-isotopic on its boundary are completely classified:

Lemma 6.16. [Hat07] If $N$ admits two Seifert fibrations which are non-isotopic on $\partial N$, then $N$ is homeomorphic to either:

1. The solid torus
2. A twisted I bundle over the Klein bottle.
3. The torus cross the interval
In the first two cases $\partial N$ consists of a single torus, while in last one it consist of two disjoint tori. To finish the proof, it suffices then to discard the first two cases:

**Lemma 6.17.** If $\partial N$ consists of Anosov Tori, then $\partial N$ cannot be a single torus.

**Proof.** Assume that $\partial N$ is a torus $T$, and consider the inclusion map $i : H_1(\partial N) \to H_1(N)$. Let $\ker i$ be the kernel of the map. Then by Lemma 3.5 in [Hat07] we have

$$\text{rank} (\ker i) = \frac{1}{2} \text{rank} (H_1(\partial N))$$

where $\text{rank}$ denotes the number of $\mathbb{Z}$ summands in a direct sum splitting into cyclic groups. If $\partial \tilde{M} = T$, then $\frac{1}{2} \text{rank} (H_1(T)) = 1$, and hence $K = \ker i$ is a one-dimensional subspace of $H_1(T)$. We have then that $f_* (K) = K$, where $f_* : H_1(T) \to H_1(T)$ is the isomorphism induced by any diffeomorphism $f : N \to N$. This implies that $f_*$ has an eigenvalue which is $1$. Hence, $f$ cannot be isotopic to a hyperbolic automorphism on $T$. This implies that $T$ cannot be an Anosov torus. \hfill \Box

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