INTEGRAL POINTS ON QUADRATIC TWISTS AND LINEAR GROWTH FOR CERTAIN ELLIPTIC FIBRATIONS

by

Pierre Le Boudec

Abstract. — We prove that the number of rational points of bounded height on certain del Pezzo surfaces of degree 1 defined over \( \mathbb{Q} \) grows linearly, as predicted by Manin’s conjecture. Along the way, we investigate the average number of integral points of small naive height on quadratic twists of a fixed elliptic curve with full rational 2-torsion.

Contents

1. Introduction .................................................. 1
2. Preliminaries .................................................. 5
3. Integral points of small height on quadratic twists .......... 6
4. Rational points on elliptic fibrations .......................... 9
References ..................................................... 10

1. Introduction

1.1. Rational points on elliptic fibrations. — The main goal of this article is to establish sharp bounds for the number of rational points of bounded height on certain del Pezzo surfaces of degree 1 defined over \( \mathbb{Q} \). In their anticanonical embedding, these surfaces are defined by sextic forms in \( \mathbb{P}(3,2,1,1) \). More precisely, they are isomorphic to a surface \( V \) given by an equation of the shape

\[
y^2 = x^3 + F_4(u, v) x + F_6(u, v),
\]

where the coordinates in \( \mathbb{P}(3,2,1,1) \) are denoted by \( (y : x : u : v) \) to highlight the elliptic fibration and where \( F_4, F_6 \in \mathbb{Z}[u, v] \) are respectively a quartic and a sextic form such that \( 4F_4^2 + 27F_6^2 \) is not identically 0.

For \( x = (y : x : u : v) \in \mathbb{P}(3,2,1,1)(\mathbb{Q}) \), we can choose coordinates \( y, x, u, v \in \mathbb{Z} \) such that for every prime \( p \), either \( p \nmid u \) or \( p \nmid v \) or \( p^2 \nmid x \) or \( p^3 \nmid y \). Then we can define an exponential height function \( H : \mathbb{P}(3,2,1,1)(\mathbb{Q}) \to \mathbb{R}_{>0} \) by setting

\[
H(x) = \max \{|y|^{1/3}, |x|^{1/2}, |u|, |v|\}.
\]

2010 Mathematics Subject Classification. — 11D45, 11G05, 14G05.

Key words and phrases. — Elliptic fibrations, rational points, elliptic curves, quadratic twists, integral points.
For any Zariski open subset $U$ of $V$, we can introduce the number of rational points of bounded height on $U$, that is

$$N_{U,H}(B) = \# \{ x \in U(\mathbb{Q}), H(x) \leq B \}. $$

A conjecture of Manin (see [PMT89]) predicts the asymptotic behaviour of $N_{V,H}(B)$ as $B$ tends to $+\infty$, but the current technology is very far from allowing us to approach it for any surface $V$. A weaker version states that $V$ has linear growth, by which we mean that there should exist an open subset $U$ of $V$ such that, for any fixed $\varepsilon > 0$,

$$N_{U,H}(B) \ll B^{1+\varepsilon}. $$

The only authors who have addressed this problem seem to be Munshi (see [Mun07] and [Mun08]) and Mendes da Costa (see [MdC13]).

More precisely, Mendes da Costa established that for any surface $V$ given by an equation of the shape $[1.1]$, there exists $\delta > [1.2]$ such that $N_{V,H}(B) \ll B^{3-\delta}$, where the constant involved in the notation $\ll$ is independent of the forms $F_4$ and $F_6$. This bound is far from the expectation $[1.2]$ but is not at all trivial, which illustrates the difficulty of this problem in general.

As already remarked by Munshi, it is easier to deal with certain specific examples of singular surfaces. The most striking result in Munshi’s works is the following (see [Mun08, Corollary 3]). Let $V_{e,λ,R} \subset \mathbb{P}(3,2,1,1)$ be the surface defined by

$$y^2 = (x - eR(u,v))(x - λR(u,v))(x - \overline{λ}R(u,v)), $$

where $e \in \mathbb{Z}$, $λ$ is a generator of the ring of integers of an imaginary quadratic field, and $R \in \mathbb{Z}[u,v]$ is a positive definite quadratic form. Then we have

$$N_{V_{e,λ,R},H}(B) \ll B^{5/4+\varepsilon}, $$

where $U_{e,λ,R}$ is defined by removing from $V_{e,λ,R}$ the subset defined by $y = 0$. Although impressive, this result is still far from the conjectured upper bound $[1.2]$.

Let $e_1, e_2, e_3 \in \mathbb{Z}$ be three distinct integers and set $e = (e_1, e_2, e_3)$. We also let $Q \in \mathbb{Z}[u,v]$ be a non-degenerate quadratic form. In this article, we are interested in the surfaces $V_{e,Q} \subset \mathbb{P}(3,2,1,1)$ defined by

$$y^2 = (x - e_1Q(u,v))(x - e_2Q(u,v))(x - e_3Q(u,v)). $$

We let $U_{e,Q}$ be the open subset defined by removing from $V_{e,Q}$ the two subsets given by $y = 0$ and $Q(u,v) = 0$. It is straightforward to check that all the surfaces defined by the equations $[1.3]$ or $[1.5]$ have two singularities of type $D_4$ over $\mathbb{Q}$.

Let us note that, all along this article, the constants involved in the notations $\ll$ and $\gg$ may depend on $\varepsilon$, $e$ and $Q$.

The main result of this article is the following.

**Theorem 1.** — Let $\varepsilon > 0$ be fixed. We have the upper bound

$$N_{U_{e,Q},H}(B) \ll B^{1+\varepsilon}. $$

As in the works of Munshi, the proof of Theorem [1] makes use of the natural elliptic fibration to parametrize the rational points on $U_{e,Q}$. This leads us to investigate the average number of integral points of small naive height on quadratic twists of a fixed elliptic curve with full rational 2-torsion. This is the purpose of section [2].

It is worth mentioning that the analysis of the parametrization of the rational points given by Munshi in [Mun08] shows that it should be easy to adapt Lemma [3] (see section [22]) to prove that the surfaces defined by $[1.3]$ and considered by Munshi also have linear growth.
Another interesting problem is to prove sharp lower bounds for $N_{U_e,Q,H}(B)$. A simpler way to state this is to ask what can be said about the quantity

$$\beta_{U_e,Q}(B) = \frac{\log N_{U_e,Q,H}(B)}{\log(B)}.$$  

In the following, we choose to take $Q(u,v) = uv$, even though similar results could be proved for other choices of $Q$. We respectively call $V_e$ and $U_e$ the surface and the open subset corresponding to this choice. We establish the following result.

**Corollary 1.** — The limit of $\beta_{U_e}(B)$ as $B$ tends to $+\infty$ exists and equals 1. More precisely, we have

$$\beta_{U_e}(B) = 1 + O \left( \frac{1}{\log \log B} \right).$$

To prove the lower bound $B(\log B)^8 \ll N_{U_e,H}(B)$, which is conjecturally best possible, a natural idea is to make use of universal torsors above $V_e$. Indeed, this strategy has been successful to establish Manin’s conjecture for several examples of singular del Pezzo surfaces of low degree (see [BB10] and [LB12] for the most striking results). Hausen and Süss [HST10] Example 5.5] have computed the equations of such a torsor and it turns out that proving this lower bound does not seem to be immediate. It would be interesting to solve this problem.

### 1.2. Integral points of small height on quadratic twists

Given a family of non-rational curves, which is reasonable in some sense, a loose general expectation is that most curves do not have any integral point. Proving results of this flavour is expected to be very hard, even in some simple cases. Another expectation is that the smallest (rational point and a fortiori) integral point on the curves having at least one, is most likely not small. This second statement is easier to approach and the aim of this section is to investigate the case of a family of quadratic twists of a fixed elliptic curve with full rational 2-torsion.

For $n \geq 1$, we introduce the elliptic curve $E_{n,e}$ defined by the equation

$$y^2 = (x - e_1n)(x - e_2n)(x - e_3n).$$

We instantly check that the curves $E_{1,e}$ and $E_{n,e}$ are isomorphic over $\mathbb{Q}(\sqrt{n})$. Our interest lies in the set of integral points on $E_{n,e}$, so we set

$$E_{n,e}(\mathbb{Z}) = \{ P \in E_{n,e}(\mathbb{Q}), x(P) \in \mathbb{Z} \},$$

and also

$$E_{n,e}^*(\mathbb{Z}) = \{ P \in E_{n,e}(\mathbb{Z}), y(P) \neq 0 \}.$$  

The elements of $E_{n,e}^*(\mathbb{Z})$ will be referred to as the non-trivial integral points on $E_{n,e}$.

As already explained, a difficult problem is to obtain upper bounds for the number of $n \leq N$ such that $E_{n,e}$ has at least one non-trivial integral point. Following the philosophy described above, it is reasonable to expect that this set has density 0 but the proof of this statement seems to be out of reach.

An easier problem is to investigate properties of integral points of bounded height on the curves $E_{n,e}$ on average over $n$. Given $P \in E_{n,e}(\mathbb{Z})$ with coordinates $(x,y) \in \mathbb{Z}^2$, we define its exponential naive height $H(P)$ by setting

$$H(P) = \max \{|y|^{1/3}, |x|^{1/2}\}.$$  

The following theorem will be the key result in the proof of Theorem 1. It gives lower and upper bounds for the number of non-trivial integral points of bounded height on the curves $E_{n,e}$ on average over $n$. 


Theorem 2. — We have the bounds

\[ B \ll \sum_{n \geq 1} \# \{ P \in E_{n,e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B \} \ll B (\log B)^{\delta_e}, \]

where \( \delta_e = 4 \) if \( e_1 e_2 e_3 \neq 0 \) and \( \delta_e = 6 \) otherwise.

Note that the interest of Theorem 2 mainly lies in the upper bound, and the lower bound implies that it is sharp up to the factor \( (\log B)^{\delta_e} \). Let us note here that with more work, the lower bound could be improved by a factor \( (\log B)^4 \) but this would not change anything in our other results.

It is not hard to check that there exists an integer \( n \gg B^2 \) for which the set \( \{ P \in E_{n,e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B \} \) is not empty. Therefore, the upper bound in Theorem 2 states that most quadratic twists of \( E_{1,e} \) do not have a non-trivial integral point of small height.

To be more specific, the upper bound in Theorem 2 allows us to establish the following density statement.

Corollary 2. — Let \( A > 6 \) be fixed. The set of \( n \geq 1 \) such that every \( P \in E_{n,e}^*(\mathbb{Z}) \) satisfies

\[ \mathcal{H}(P) > n^{1/2} (\log n)^{-A}, \]

has density 1 in the set of \( n \geq 1 \) such that \( E_{n,e}^*(\mathbb{Z}) \neq \emptyset \).

It is worth pointing out that it is easy to check that if \( e_1 e_2 e_3 \neq 0 \) then any \( P \in E_{n,e}^*(\mathbb{Z}) \) has to satisfy \( \mathcal{H}(P) \gg n^{1/2} \), so Corollary 2 actually holds for any \( A > 0 \) in this case.

1.3. Outline of the article. — We start by establishing Theorem 2. The proof of this result goes in two steps. The first step consists in using the fact that \( E_{n,e} \) has full rational 2-torsion to parametrize the integral points on \( E_{n,e} \) using a complete 2-descent. This is achieved in section 2.1. In the second step, we bound the number of non-trivial integral points of bounded height on the curves \( E_{n,e} \) on average over \( n \). To achieve this, we appeal to the recent result of the author \cite{LB13} Lemma 4. This lemma is stated in section 2.2.

Corollary 2 straightforwardly follows from the upper bound in Theorem 2 after noticing that the number of \( n \leq N \) for which the set \( E_{n,e}^*(\mathbb{Z}) \) is not empty is \( \gg N^{1/2} \).

Finally, we prove Theorem \ref{thm:1} using the natural elliptic fibration and the upper bound in Theorem 2. Corollary \ref{cor:1} also follows from this upper bound, together with the lower bound \( B \ll N_{U,e,H}(B) \).

1.4. Acknowledgements. — It is a pleasure for the author to thank Régis de la Bretèche, Timothy Browning, Daniel Loughran, Dave Mendes da Costa, Peter Sarnak, Arul Shankar and Anders Södergren for interesting and stimulating conversations related to the topics of this article.

The financial support and the perfect working conditions provided by the Institute for Advanced Study are gratefully acknowledged. This material is based upon work supported by the National Science Foundation under agreement No. DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
2. Preliminaries

2.1. Descent argument. — In this section, we derive a convenient parametrization of the integral points on $E_{n,e}$ using the fact that $E_{n,e}$ has full rational 2-torsion. We start by proving the following elementary lemma.

**Lemma 1.** — Let $(y, x_1, x_2, x_3) \in \mathbb{Z}_{\neq 0}^4$ be such that $y^2 = x_1x_2x_3$. There exists a unique way to write

$$x_i = d_1d_2w^2a_i^2a_kb_i^2,$$

for $\{i, j, k\} = \{1, 2, 3\}$ and

$$y = d_1d_2d_3w^3a_1^2a_2^2a_3^2b_1b_2b_3,$$

where $(d_1, d_2, d_3, w, a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_{\neq 0}^4 \times \mathbb{Z}_{\neq 0}^6$ is subject to the conditions $|\mu(a_i)| = 1$ and $\gcd(d_ia_jb_j, d_ja_i) = 1$ for $i, j \in \{1, 2, 3\}, i \neq j$, and $d_1d_2d_3 > 0$.

**Proof.** — Let us set $x = \gcd(x_1, x_2, x_3)$ and let us write $x_i = xx_i'$ for $i \in \{1, 2, 3\}$, where $\gcd(x_1', x_2', x_3') = 1$. We see that $x \mid y$ and we can thus write $y = xy'$. We obtain

$$y'^2 = xx_1'x_2'x_3'.$$

Let us now set $d_i = \text{sign}(x_i')\gcd(x_i', x_j')$ for $\{i, j, k\} = \{1, 2, 3\}$. Let us note that we have $d_1d_2d_3 > 0$. We can write $x_i' = d_jd_k\xi_i$ with $\xi_i > 0$ for $\{i, j, k\} = \{1, 2, 3\}$, where $\gcd(d_i\xi_j, d_j\xi_i) = 1$ for $i, j \in \{1, 2, 3\}, i \neq j$. Since $d_1d_2d_3 \mid y'$, we can write $y' = d_1d_2d_3z$. We thus get

$$z^2 = xx_1\xi_2\xi_3.$$

There is a unique way to write $\xi_i = a_ib_i^2$ with $a_i, b_i > 0$ and $|\mu(a_i)| = 1$ for $i \in \{1, 2, 3\}$. We see that $b_1b_2b_3 \mid z$ so we can write $z = b_1b_2b_3z'$. We finally obtain

$$z'^2 = xa_1a_2a_3.$$

Since $a_1, a_2$ and $a_3$ are squarefree and pairwise coprime, this implies that we can write $x = w^2a_1a_2a_3$ and $z' = wa_1a_2a_3$, which completes the proof.

**Lemma 1** immediately implies the following result, which provides us with the desired parametrization of the non-trivial integral points on $E_{n,e}$.

**Lemma 2.** — There is a bijection between the set of non-trivial integral points on $E_{n,e}$ and the set of $(d_1, d_2, d_3, w, a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_{\neq 0}^4 \times \mathbb{Z}_{\neq 0}^6$ satisfying, for $\{i, j, k\} = \{1, 2, 3\}$, the equations

$$(c_i - c_j)n = d_kw^2a_1a_2a_3(d_ia_jb_j^2 - d_ja_i^2),$$

and the conditions $|\mu(a_i)| = 1$ and $\gcd(d_ia_jb_j, d_ja_i) = 1$ for $i, j \in \{1, 2, 3\}, i \neq j$, and $d_1d_2d_3 > 0$. This bijection is given, for $P \in E_{n,e}(\mathbb{Z})$ with coordinates $(x, y) \in \mathbb{Z}^2$, by

$$x = c_in + d_jd_kw^2a_1^2a_kb_i^2,$$

$$y = d_1d_2d_3w^3a_1^2a_2^2a_3^2b_1b_2b_3,$$

for $\{i, j, k\} = \{1, 2, 3\}$. 
2.2. Geometry of numbers. — The following lemma follows from the recent work of the author [34] Lemma 4]. It draws upon both geometry of numbers and analytic number theory tools, and will be the key result in the proof of Theorem [32]

Lemma 3. — Let \( f = (f_1, f_2, f_3) \) be a vector satisfying the conditions \( \gcd(f_i, f_j) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \), and let \( U_i, V_i \geq 1 \) for \( i \in \{1, 2, 3\} \). Let also \( N_f = N_f(U_1, U_2, U_3, V_1, V_2, V_3) \) be the number of vectors \((u_1, u_2, u_3) \in \mathbb{Z}_{\neq 0}^3 \) and \((v_1, v_2, v_3) \in \mathbb{Z}_{\neq 0}^3 \) satisfying \(|u_i| \leq U_i, |v_i| \leq V_i \) for \( i \in \{1, 2, 3\} \), and the equation

\[
 f_1u_1v_1^2 + f_2u_2v_2^2 + f_3u_3v_3^2 = 0,
\]

and such that \( \gcd(u_iv_i, u_jv_j) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \). Let \( \varepsilon > 0 \) be fixed. We have the bound

\[
 N_f \ll_f (U_1U_2U_3)^{2/3}(V_1V_2V_3)^{1/3} M_e(U_1, U_2, U_3),
\]

where

\[
 M_e(U_1, U_2, U_3) = 1 + \max_{\substack{i, j, k = (1, 2, 3)}} (U_iU_j)^{-1/2+\varepsilon} \log 2U_k.
\]

3. Integral points of small height on quadratic twists

3.1. Proof of Theorem [32]. — Let us first prove the upper bound in Theorem [32] Lemma [34] asserts that \((y, x) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z} \) satisfies the equation

\[
 y^2 = (x - e_1n)(x - e_2n)(x - e_3n),
\]

if and only if \( x \) and \( y \) can be written, for \( \{i, j, k\} = \{1, 2, 3\} \), as

\[
 x = e_1n + d_1d_2w^2a_1^2a_2a_3b_1^2b_2b_3,
\]

\[
 y = d_1d_2d_3w^3a_1^2a_2^2a_3^2b_1b_2b_3,
\]

where \((d_1, d_2, d_3, w, a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_{\neq 0}^4 \times \mathbb{Z}_{\neq 0}^6 \) satisfies, for \( \{i, j, k\} = \{1, 2, 3\} \), the equations

\[
 (e_i - e_j)n = d_kw^2a_1a_2a_3(d_ia_jb_j^2 - d_ja_ib_i^2),
\]

and the conditions \(|\mu(a_i)| = 1 \) and \( \gcd(d_ia_ib_i, a_ia_ia_ia_ia_ia_i) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \), and \( d_1d_2d_1 > 0 \). The equations \( (3.2) \) can have a solution \( n \in \mathbb{Z}_{>0} \) only if

\[
 (e_2 - e_3)d_2d_3a_1b_1^2 + (e_3 - e_1)d_1d_3a_2b_2^2 + (e_1 - e_2)d_1d_2a_3b_3^2 = 0.
\]

Moreover, since \( e_1, e_2 \) and \( e_3 \) are distinct, there is at most one such solution \( n \in \mathbb{Z}_{>0} \). The conditions \( \gcd(d_i, d_ia_ib_i) = 1 \) imply that \( d_i | e_j - e_k \) for \( \{i, j, k\} = \{1, 2, 3\} \), so we can write \( e_2 - e_3 = d_1c_1, e_3 - e_1 = d_2c_2 \) and \( e_1 - e_2 = d_3c_3 \). We obtain the equation

\[
 c_1a_1b_1^2 + c_2a_2b_2^2 + c_3a_3b_3^2 = 0.
\]

Let us call \( h = \gcd(c_1, c_2, c_3) \) and let us write \( c_i = hf_i \) for \( i \in \{1, 2, 3\} \). We thus have \( \gcd(f_1, f_2, f_3) = 1 \). From the two relations \( d_1f_1 + d_2f_2 + d_3f_3 = 0 \) and

\[
 f_1a_1b_1^2 + f_2a_2b_2^2 + f_3a_3b_3^2 = 0,
\]

we deduce that \( \gcd(f_i, f_j) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \).

From now on, we use the notation \( f = (f_1, f_2, f_3) \). We let \( N_f(B) \) be the number of \((w, a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\neq 0}^6 \) satisfying the equation \( (3.3) \), the inequality

\[
 |w|^3a_1^2a_2^2a_3^2b_1b_2b_3 \leq B^3,
\]

and the conditions \( \gcd(a_ib_i, a_ia_ia_i) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \). The investigation above shows that

\[
 \sum_{n \geq 1} \#\{P \in E_n^+(\mathbb{Z}), \mathcal{H}(P) \leq B\} \ll_f N_f(B),
\]
where the maximum is taken over \( f \) satisfying \( f_i | e_j - e_k \) for \( \{i, j, k\} = \{1, 2, 3\} \) and \( \gcd(f_i, f_j) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \).

We have thus proved that it is sufficient for our purpose to bound the quantity \( N_f(B) \). To achieve this, for \( i \in \{1, 2, 3\} \), we let \( W, A_i, B_i \geq 1 \) run over the set of powers of 2 and we define \( M_f = M_f(W, A_1, A_2, A_3, B_1, B_2, B_3) \) as the number of \((w, a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\geq 0}^3\) satisfying the equation \( (3.5) \), the conditions \( W < |w| \leq 2W, A_i < a_i \leq 2A_i \) and \( B_i < b_i \leq 2B_i \), and \( \gcd(a_i b_1, a_i b_j) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \). We have

\[
N_f(B) \ll \sum_{W, A_i, B_i \in \{1, 2, 3\}} M_f,
\]

where the sum is over \( W, A_i, B_i \geq 1, i \in \{1, 2, 3\} \), satisfying the inequality

\[
(3.4) \quad W^3 A_i^2 A_2^3 B_1 B_2 B_3 \leq B^3.
\]

Lemma 3 gives the upper bound

\[
M_f \ll W(A_1 A_2 A_3)^{2/3} (B_1 B_2 B_3)^{1/3} M_{\zeta}(A_1, A_2, A_3),
\]

where \( M_{\zeta}(A_1, A_2, A_3) \) is defined in lemma 3. Choosing for instance \( \varepsilon = 1/4 \) and summing over \( W \) using the condition \( (3.4) \), we finally obtain

\[
N_f(B) \ll \sum_{W, A_i, B_i \in \{1, 2, 3\}} W(A_1 A_2 A_3)^{2/3} (B_1 B_2 B_3)^{1/3} M_{1/4}(A_1, A_2, A_3)
\]

\[
\ll B \sum_{A_i, B_i \in \{1, 2, 3\}} M_{1/4}(A_1, A_2, A_3)
\]

\[
\ll B (\log B)^6,
\]

which completes the first part of the proof of the upper bound in Theorem 2.

Now let us assume that \( e_1 e_2 e_3 \neq 0 \) and let us prove that we can take \( \delta_\varepsilon = 4 \) in Theorem 2. If \( n > 2B^2 \) then, since \( x = \varepsilon n + d_j d_3 a_i^2 a_j b_i^2 + d_j d_k w^2 a_i a_j a_k b_i^2 \) for \( \{i, j, k\} = \{1, 2, 3\}, |x| \leq B^2 \) and \( e_1 e_2 e_3 \neq 0 \), we have \( |d_j d_k| w^2 a_i a_j a_k b_i^2 \geq B^2 \) for \( \{i, j, k\} = \{1, 2, 3\} \), but this is in contradiction with \( |y| \leq B^3 \). This implies that \( \{P \in E_2^{\zeta}(\mathbb{Z}), H(P) \leq B\} \) is empty provided that \( n > 2B^2 \) so we can assume that \( n \leq 2B^2 \). Therefore, for \( \{i, j, k\} = \{1, 2, 3\} \), we get the conditions

\[
(3.5) \quad w^2 a_i^2 a_j a_k b_i^2 \leq 3B^2.
\]

We now proceed similarly as in the first case. We let \( N_f(B) \) be the number of \((w, a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\geq 0}^6\) satisfying the equation \( (3.5) \), the inequalities \( (3.5) \) and the conditions \( \gcd(a_i b_1, a_i b_j) = 1 \) for \( i, j \in \{1, 2, 3\}, i \neq j \). Once again, it is sufficient for our purpose to bound \( N_f(B) \), and we have

\[
N_f(B) \ll \sum_{W, A_i, B_i \in \{1, 2, 3\}} W(A_1 A_2 A_3)^{2/3} (B_1 B_2 B_3)^{1/3} M_{1/4}(A_1, A_2, A_3),
\]

where the sum is over \( W, A_i, B_i \geq 1, i \in \{1, 2, 3\} \), running over the set of powers of 2 and satisfying, for \( \{i, j, k\} = \{1, 2, 3\} \), the inequalities

\[
(3.6) \quad W^2 A_i^2 A_2 A_3^2 B_i^2 \leq 3B^2.
\]
It is obvious that the proof of Corollary 2.

3.2. Proof of Corollary 2. — We start by proving the following lemma, which gives a lower bound for the number of non-trivial integral points. The upper bound in Theorem 2 implies that we have decided not to explore this further.

Let us now prove the lower bound in Theorem 2. We can assume by symmetry that \( e_3 > \max\{e_1, e_2\} \).

Proof — As in the proof of the lower bound in Theorem 2, we can assume that \( \sum_{n > 1} \#\{P \in E_{n,e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B\} \geq \mathcal{P}_e(B) \).

It is obvious that \( \mathcal{P}_e(B) \gg B \), which completes the proof of Theorem 2.

3.2. Proof of Corollary — We start by proving the following lemma, which gives a lower bound for the number of \( n \leq N \) such that the curve \( E_{n,e}^* \) has at least one non-trivial integral point.

Lemma 4. — We have the lower bound

\[
\#\{n \leq N, E_{n,e}^*(\mathbb{Z}) \neq \emptyset\} \gg N^{1/2}.
\]

Proof. — As in the proof of the lower bound in Theorem 2, we can assume that \( e_3 > \max\{e_1, e_2\} \) and thus, if \( n \) can be written as \( n = 2(e_3 - e_1 - e_2)w^2 \) for some \( w \in \mathbb{Z}_{>0} \), then the equalities

\[
y = 4(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)w^3,
\]

\[
x = 2(-2e_1e_2 + e_1e_3 + e_2e_3)w^2,
\]

define a non-trivial integral point on \( E_{n,e} \). Noticing that there are \( \gg N^{1/2} \) such integers \( n \leq N \) completes the proof of the lemma.

It seems likely that the lower bound in lemma 4 could be improved by a few \( \log N \) factors, but since this slight improvement would not essentially change the statement of Corollary 2, we have decided not to explore this any further.

Let \( A > 6 \) be fixed. Let \( \mathcal{N}_A(N) \) be the number of \( n \leq N \) such that there exists \( P \in E_{n,e}^*(\mathbb{Z}) \) satisfying

\[
\mathcal{H}(P) \leq n^{1/2}(\log n)^{-A}.
\]

We have

\[
\mathcal{N}_A(N) \leq \sum_{n \leq N} \#\{P \in E_{n,e}^*(\mathbb{Z}), \mathcal{H}(P) \leq N^{1/2}(\log N)^{-A}\}.
\]

The upper bound in Theorem 2 implies

\[
\mathcal{N}_A(N) \ll N^{1/2}(\log N)^{-A+6},
\]

so lemma 4 shows that \( \mathcal{N}_A(N) = o(\#\{n \leq N, E_{n,e}^*(\mathbb{Z}) \neq \emptyset\}) \), which concludes the proof of Corollary 2.
4. Rational points on elliptic fibrations

4.1. Proof of Theorem 1 — Recall that $V_{e, Q} \subset \mathbb{P}(3, 2, 1, 1)$ is defined by the equation

(4.1) \[ y^2 = (x - e_1Q(u,v))(x - e_2Q(u,v))(x - e_3Q(u,v)). \]

Thus, we have

\[
N_{U_{e,Q}, H}(B) \ll \sum_{\substack{|u|, |v| \leq B \\ Q(u,v) \neq 0}} \# \{(y, x) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}, |y| \leq B^3, |x| \leq B^2, (4.1)\} 
\leq \sum_{n \in \mathbb{Z}_{\neq 0}} \# \{(y, x) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}, |y| \leq B^3, |x| \leq B^2, (3.1)\} \sum_{|u|, |v| \leq B} 1.
\]

Since $Q$ is non-degenerate, we have

\[
\# \{(u, v) \in \mathbb{Z}^2, |u|, |v| \leq B, Q(u, v) = n\} \ll B^\varepsilon.
\]

As a result, we get

\[
N_{U_{e,Q}, H}(B) \ll B^\varepsilon \sum_{n \in \mathbb{Z}_{\neq 0}} \# \{P \in E_{n, e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B\}.
\]

We note that the sum in the right-hand side can be rewritten as

\[
\sum_{n \geq 1} \# \{P \in E_{n, e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B\} + \sum_{n \geq 1} \# \{P \in E_{n, -e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B\}.
\]

Therefore, using twice the upper bound in Theorem 2, we obtain

\[
N_{U_{e,Q}, H}(B) \ll B^{1+\varepsilon},
\]

which ends the proof of Theorem 1.

4.2. Proof of Corollary 1 — We proceed exactly as in the proof of Theorem 1.

We have

\[
N_{U_{e}, H}(B) \ll \sum_{n \in \mathbb{Z}_{\neq 0}} \# \{(y, x) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}, |y| \leq B^3, |x| \leq B^2, (3.1)\} \sum_{|u|, |v| \leq B} 1.
\]

Then, if $n \leq B^2$, we have

\[
\# \{(u, v) \in \mathbb{Z}^2, |u|, |v| \leq B, uv = n\} \leq 2\tau(n) \leq n^{1/\log \log n} \ll B^{2/\log \log B},
\]

and this upper bound also holds if $n > B^2$. This shows that

\[
N_{U_{e}, H}(B) \ll B^{2/\log \log B} \sum_{n \in \mathbb{Z}_{\neq 0}} \# \{P \in E_{n, e}^*(\mathbb{Z}), \mathcal{H}(P) \leq B\}.
\]

As in the proof of Theorem 1 using twice the upper bound in Theorem 2, we obtain

(4.2) \[
N_{U_{e}, H}(B) \ll B^{1+3/\log \log B}.
\]

Now let us prove a lower bound for $N_{U_{e}, H}(B)$. Let us assume by symmetry that $e_3 > \max\{e_1, e_2\}$ so that $2e_3 - e_1 - e_2 > 0$, and let us denote by $v_2(m)$ the 2-adic
valuation of an integer \( m \geq 1 \). Let \( \mathcal{R}_a(B) \) be the number of \( (y, x, u, v) \in \mathbb{Z}_p^4 \) such that
\[
\max\{|y|^{1/3}, |x|^{1/2}, |u|, |v|\} \leq B
\]
and which can be written as
\[
y = 4(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)w_1^3w_2^3,
x = 2(-2e_1e_2 + e_1e_3 + e_2e_3)w_1^2w_2^2,
u = 2^{1+v_2(2e_3-e_1-e_2)}w_1^2,
v = (2e_3 - e_1 - e_2)2^{-v_2(2e_3-e_1-e_2)}w_2^2.
\]
where \( (w_1, w_2) \in \mathbb{Z}_{>0}^2 \) satisfies \( \gcd(w_1, (2e_3-e_1-e_2)) = \gcd(w_2, 2) = 1 \). Since \( \gcd(u, v) = 1 \) in the parametrization above, it is immediate to check that
\[
N_{U_e,H}(B) \geq \mathcal{R}_a(B).
\]
Since we clearly have \( \mathcal{R}_a(B) \gg B \), we have obtained the lower bound
\[
B \ll N_{U_e,H}(B).
\]
Let us note that improving this lower bound by a few \( \log B \) factors would not be hard. However, as already explained in the introduction, proving the lower bound of the expected order of magnitude for \( N_{U_e,H}(B) \) does not seem to be immediate.

Recalling the definition (1.6) of \( \beta_{U_e}(B) \), we see that the two bounds (4.2) and (4.3) complete the proof of Corollary 1.

References

[BB10] S. Baier and T. D. Browning, *Inhomogeneous cubic congruences and rational points on del Pezzo surfaces*, J. Reine Angew. Math., to appear, arXiv:1011.3434v2 (2010).

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. 95 (1989), no. 2, 421–435.

[HS10] J. Hausen and H. Süß, *The Cox ring of an algebraic variety with torus action*, Adv. Math. 225 (2010), no. 2, 977–1012.

[LB12] P. Le Boudec, *Affine congruences and rational points on a certain cubic surface*, arXiv:1207.2685v1 (2012).

[LB13] P. Le Boudec, *Density of rational points on a certain smooth bihomogeneous threefold*, Preprint (2013).

[MdC13] D. Mendes da Costa, *Integral points on elliptic curves and the Bombieri-Pila bounds*, arXiv:1301.4116v2 (2013).

[Mun07] R. Munshi, *Density of rational points on elliptic fibrations*, Acta Arith. 129 (2007), no. 1, 63–70.

[Mun08] R. Munshi, *Density of rational points on elliptic fibrations. II*, Acta Arith. 134 (2008), no. 2, 133–140.

Pierre Le Boudec, Institute for Advanced Study, School of Mathematics, Einstein Drive, Simonyi Hall – Office 111, Princeton, NJ 08540, USA  •  E-mail : pleboudec@ias.edu