Low-energy Compton scattering on the nucleon and sum rules

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(November 13, 2018)

Abstract

The Gerasimov-Drell-Hearn and Baldin-Lapidus sum rules are evaluated in the dressed K-matrix model for photon-induced reactions on the nucleon. For the first time the sum $\alpha + \beta$ of the electric and magnetic polarisabilities and the forward spin polarisability $\gamma_0$ are explicitly calculated in two alternative ways – from the sum rules and from the low-energy expansion of the real Compton scattering amplitude – within the same framework. The two methods yield compatible values for $\alpha + \beta$ but differ somewhat for $\gamma_0$. Consistency between the two ways of determining the polarisabilities is a measure of the extent to which basic symmetries of the model are obeyed.
I. INTRODUCTION

The interest in the low-energy photon-nucleon interaction has been renewed in recent years. A number of experiments have been carried out to study real Compton scattering at low energies – where the amplitude is parametrised by nucleon polarisabilities – and up to the second resonance region. The new global data fit (last Ref. in [1]) has yielded the electric and magnetic polarisabilities \( \alpha = 12.2 \pm 0.7 \text{[stat+syst]} \) and \( \beta = 1.7 \pm 0.7 \text{[stat+syst]} \), and the forward spin-polarisability \( \gamma_0 = -1.87 \pm 0.18 \text{[stat+syst]} \) has been extracted from the recent measurements of the total photoabsorption cross section. (We use the standard units \( 10^{-4} \text{ fm}^3 \) for \( \alpha \) and \( \beta \) and \( 10^{-2} \text{ fm}^4 \) for \( \gamma_0 \).) Dispersion theory has been successfully utilised to extract nucleon polarisabilities from data with a minimum of model assumptions. We quote results of two recent dispersion calculations: \( \alpha = 11.9 \), \( \beta = 1.9 \) (first Ref. in [3]), \( \gamma_0 = -0.8 \) (second Ref. in [3]). General relations between polarisabilities and total photoabsorption cross sections are provided by the Baldin-Lapidus (BL), Gerasimov-Drell-Hearn (GDH) and other sum rules, which have been evaluated in several meson-exchange models.

Nucleon polarisabilities have been calculated in various dynamical frameworks from a low-energy expansion of the Compton scattering amplitude. Next-to-leading order calculations in the chiral perturbation theory yielded for the scalar polarisabilities \( \alpha = 10.5 \pm 2.0 \), \( \beta = 3.5 \pm 3.6 \), and \( \gamma_0 = -3.9 \) (or \(-3.8 \)). The polarisabilities calculated in the dressed K-matrix model are \( \alpha = 12.1 \), \( \beta = 2.4 \) and \( \gamma_0 = 2.4 \).

The low-energy expansion and the sum rules represent two alternative ways of calculating nucleon polarisabilities. This paper is the first attempt to compare these two methods in the same dynamical framework. We use the dressed K-matrix model (expounded in [3]), which is a relativistic, unitary, crossing symmetric and gauge-invariant approach with certain analyticity constraints incorporated through a dressing procedure for propagators and vertices. This dressing with meson loops up to infinite order is the central element of our approach. The model describes pion-nucleon scattering, pion photoproduction and Compton scattering at both intermediate and low energies.

We consider two particular combinations of the polarisabilities which can be extracted from the low-energy expansion of the forward scattering Compton amplitude, on the one hand, and calculated from sum rules, on the other. These are the sum \( \alpha + \beta \) of the electric and magnetic polarisabilities and the forward spin polarisability \( \gamma_0 \). The purpose of this work is to focus attention on the general question of agreement between the polarisabilities extracted from the low-energy expansion of the amplitude and those obtained from the sum rules. Since the properties of unitarity, causality (analyticity of the amplitude), crossing symmetry and gauge invariance put stringent constraints on both low-energy expansion and sum rules, a discrepancy between the two ways of determining the polarisabilities is a rather direct measure of the extent to which the basic symmetries are obeyed in the model. We find agreement in the leading order terms related to the anomalous magnetic moment (the GDH sum rule) and \( \alpha + \beta \) (the BL sum rule), but some violation in the next order related to \( \gamma_0 \). We will show the effects of certain model assumptions.

II. POLARISABILITIES FROM THE LOW-ENERGY COMPTON SCATTERING

The forward Compton scattering amplitude is expanded in a small photon energy \( \omega \) as
\[ T(\omega) = \overrightarrow{\epsilon'} \cdot \overrightarrow{\epsilon} \left[ -\frac{e^2}{4\pi m} + (\alpha + \beta)\omega^2 \right] + i \overrightarrow{\sigma} \times (\overrightarrow{\epsilon'} \times \overrightarrow{\epsilon}) \left[ -\frac{e^2\kappa^2}{8\pi m^2} + \gamma_0\omega^3 \right] + O(\omega^4), \]

where \( \overrightarrow{\epsilon} \) and \( \overrightarrow{\epsilon'} \) are the polarisations of the initial and final photons, respectively, and \( \overrightarrow{\sigma}, m, e \) and \( \kappa \) are the spin vector, mass, charge and anomalous magnetic moment of the nucleon. The constant and linear terms are given by the Born contribution, and the model-dependent subleading terms are determined by the polarisabilities \( \alpha, \beta \) and \( \gamma_0 \).

The polarisabilities calculated in our model from the low-energy expansion of the Compton amplitude are given in the columns labelled “LE” in Table I, where the original calculations are presented in the three upper rows (the lower rows will be explained below). How the polarisabilities are affected by increasing the “amount of dressing” is best explained in terms of the kernel of the calculation, the K-matrix. The bare calculation corresponds to a K-matrix being the sum of the tree-level s- and u-channel diagrams in which bare vertices and free propagators are used. This approximation is used in traditional K-matrix models [10], where only the pole parts of the loop integrals (and no principal-value parts) are included in the T-matrix by iterating the K-matrix, hence the maximal violation of analyticity in this calculation. The second row in Table I is a calculation in which analyticity is partially restored since the K-matrix is now constructed out of dressed propagators and vertices (2- and 3-point functions), thereby including the principal-value parts of a large class of loop diagrams. This restoration of analyticity is not complete, however, since the 4-point \( \gamma\gamma NN \) contact term is not dressed as the 2- and 3-point functions, only its longitudinal part being uniquely determined from gauge invariance. This deficiency is mitigated in the full calculation, where the transverse part of the \( \gamma\gamma NN \) vertex includes a “cusp” term [11] taking into account the principal-value part of the loop diagram where the two photons couple to the intermediate pion.

III. POLARISABILITIES FROM THE SUM RULES

The sum rules [4] relate the low-energy observables to the photoabsorption cross sections corresponding to the total angular momenta \( \frac{1}{2} \) and \( \frac{3}{2} \). The GDH sum rule involves the proton anomalous magnetic moment \( \kappa = 1.79 \),

\[ \frac{m^2}{2\pi e^2} \int_{\omega_{th}}^{\infty} d\omega \frac{\sigma_{1/2} - \sigma_{3/2}}{\omega} = -\frac{\kappa^2}{4}, \]

where \( \omega_{th} \) is the pion-production threshold energy. We also evaluate the BL sum rule for \( \alpha + \beta \) and the sum rule for \( \gamma_0 \), which can be written, respectively, as

\[ \frac{1}{4\pi^2} \int_{\omega_{th}}^{\infty} d\omega \frac{\sigma_{1/2} + \sigma_{3/2}}{\omega^2} = \alpha + \beta, \quad \frac{1}{4\pi^2} \int_{\omega_{th}}^{\infty} d\omega \frac{\sigma_{1/2} - \sigma_{3/2}}{\omega^3} = \gamma_0. \]

The integrand of the GDH sum rule Eq. (2) is shown in Fig. 1, where we display the results of the fully dressed (solid line) and bare (sparse-dotted line) calculations as well as the contribution of \( \pi N \) intermediate states to the full calculation (dense-dotted line). The data points are from [2], and the result of the unitary isobar model (first in Refs. [5]) is shown for comparison (dashed line). While in good overall agreement with experiment, our results deviate from it for photon energies above 700 MeV. The GDH and BL integrals are shown
as functions of the upper limit of integration in Figs. 2 and 3, respectively. Convergence is not achieved below 1 GeV of photon energy (we remark that the data neither support nor rule out the absence of convergence at these energies). The inelastic contributions (beyond the \(\pi N\) intermediate states) become significant above 450 MeV and bring the calculation to a better agreement with the data. The spin-polarisability sum rule converges by 1 GeV, as Fig. 4 shows. Moreover, this convergence is essentially due to the \(\pi N\) channel, a feature consistent with the strong suppression of the integrand of the \(\gamma_0\) sum rule at higher energies.

**IV. COMPARING THE LOW-ENERGY AND SUM-RULE VALUES OF POLARISABILITIES**

The dynamical model used in this calculation and the sum rules are based on the same general principles: unitarity, crossing and CPT symmetry, gauge invariance and analyticity, with the model approximations described above regarding analyticity. Thus, if the calculated T-matrix obeyed the property of analyticity exactly, the polarisabilities extracted from the low-energy expansion would be equal to those obtained from the sum rules.

The polarisabilities obtained from the sum rules are given in the columns labelled “SR” in Table I. The values of the polarisabilities are taken at the energies where the corresponding sum rules converge, \(\approx 1.5\) GeV for \(\alpha + \beta\), and \(\approx 1\) GeV for \(\gamma_0\). The dressing has a significant influence on the low-energy polarisabilities and a minor effect on the sum rules. This trend is especially pronounced for the spin polarisability \(\gamma_0\). The additional dressing of the \(\gamma\gamma NN\) contact term has a large effect on the low-energy values of polarisabilities but is negligible for the sum rules. This is because the cusp contact term affects only the \(f_{EE}^{1-}\) partial wave in the region of the pion-production threshold [9], while the sum rules integrate the contributions of all partial waves. The effects of restoration of analyticity on the amplitude are most pronounced at lower energies. At order \(\sim \omega^2\), i.e. for \(\alpha + \beta\), the agreement between the low-energy and sum-rule results is improved by the dressing of the 2- and 3-point functions and is further refined by the included contribution of the dressed 4-point function. A problem occurs at third order, i.e. for \(\gamma_0\), where there is a disagreement between the low-energy and sum-rule values.

In the formulation of the model attention has been mainly focused on the consistent dressing of the nucleon. Therefore, the disagreement between the sum rules and the low-energy expansion is likely to be related to treating other degrees of freedom not on the same footing with the nucleon. We will discuss two possible extensions of the dressing procedure, concentrating on the spin polarisability since here the discrepancy is most conspicuous.

1. The resonances beyond the \(\Delta\) have not been included in the dressing equations, partly for simplicity and partly because one expects the associated violation of analyticity to be small. To investigate this more explicitly, we did a calculation in which the only resonance kept in the K-matrix was the \(\Delta\). In this case the low-energy values of polarisabilities are not notably affected, and the sum rule for the spin polarisability is given by the dashed line in Fig. 4. It is seen that including the higher resonances in the dressing would not eliminate the disagreement between the low-energy expansion and the sum rule.

2. In the present version of the model, the \(\Delta\) self-energy is computed in a one-loop approximation only, and dressing of the \(\pi N \Delta\) vertex is not considered. However, as was shown in Ref. [9], the multi-loop corrections tend to strongly enhance the scalar part of the nu-
nucleon self-energy. Therefore, we did an exploratory calculation wherein the $\Delta$ self-energy $\Sigma_\Delta(p) = A_\Delta(p^2) \hat{p} + B_\Delta(p^2)m_\Delta$ was modified in a similar way: its scalar part $B_\Delta(p^2)$ was increased below the nucleon mass, i.e. at $p^2 < m^2$, from its original value of $\approx 1.28$ to $\approx 1.35$. This amount of increase was sufficient to match the low-energy and the sum-rule values of the spin polarisability. The polarisabilities obtained in this exploratory calculation are given in the lower three rows of Table I. Since the $\Delta$ self-energy is altered only far from the resonance mass, the amplitudes of pion and photon scattering on the proton, and hence the sum rules, are unaffected.

In summary, we compared proton polarisabilities calculated from the low-energy Compton scattering with those obtained from the corresponding sum rules. Although the BL sum rule appears not converged by an energy of 1 GeV (which we consider an upper limit for the validity of the model), at this energy the sum rule agrees with the low-energy polarisability. Similarly, the GDH sum rule at $\approx 1$ GeV is consistent with the proton anomalous magnetic moment. These agreements show in particular that the model obeys the essential causality constraints. A discrepancy appears only at higher – third – order in photon energy, which is related to the spin polarisability.

We thank Peter Grabmayr, Harold Fearing and Andrew Lahiff for discussions. S.K. is supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The work of O.S. is part of the research program of the “Stichting voor Fundamenteel Onderzoek der Materie” (FOM) with financial support from the “Nederlandse Organisatie voor Wetenschappelijk Onderzoek” (NWO).
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TABLES

TABLE I. Proton polarisabilities calculated from the low-energy Compton amplitude Eq. (1) and from the sum rules Eqs. (3). The different ingredients of the dressing are explained in Section II. The three lower rows are the results of the exploratory calculation described in Section IV.

|               | \( \alpha + \beta \) | \( \gamma_0 \) |
|---------------|----------------------|----------------|
|               | LE       | SR       | LE       | SR       |
| Bare          | 15.8 + 1.4 = 17.2   | 14.5 - 0.9  = -1.2 |
| 2,3-pt. funct.| 8.9 + 2.4 = 11.3   | 13.8 - 0.1  = -0.7 |
| Full dressing | 12.1 + 2.4 = 14.5  | 13.8 - 2.4  = -0.7 |

|               | \( \Delta \text{ self-energy} \) |
|---------------|-----------------------------------|
| Bare          | 15.8 + 1.4 = 17.2                  |
| 2,3-pt. funct.| 8.9 + 1.5 = 10.4                  |
| Full dressing | 12.1 + 1.6 = 13.7                 |
FIG. 1. The energy dependence of the integrand of the GDH sum rule Eq. (2). The curves are explained in Section III. The data in all figures are from [2].

FIG. 2. The GDH sum rule Eq. (2) as a function of the upper limit of integration.

FIG. 3. The BL sum rule, first of Eqs. (3), as a function of the upper limit of integration.
FIG. 4. The $\gamma_0$ sum rule, second of Eqs. (3), as a function of the upper limit of integration. The dashed line is obtained in the full calculation in which the only retained resonance is the $\Delta$. 