The Deep Ritz Method for Parametric $p$-Dirichlet Problems

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Abstract

We establish error estimates for the approximation of parametric $p$-Dirichlet problems deploying the Deep Ritz Method. Parametric dependencies include, e.g., varying geometries and exponents $p \in (1, \infty)$. Combining the derived error estimates with quantitative approximation theorems yields error decay rates and establishes that the Deep Ritz Method retains the favorable approximation capabilities of neural networks in the approximation of high dimensional functions which makes the method attractive for parametric problems. Finally, we present numerical examples to illustrate potential applications.

Keywords: Deep Ritz Method, Parametric Problems, Neural Networks, Non-linear Variational Problems.

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I Introduction

In the present work, we study the Deep Ritz Method for parametric $p$-Dirichlet problems both theoretically and numerically. More precisely, for a given open set $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, a given exponent $p \in (1, \infty)$, and a right-hand side $f \in L^p(\Omega)$, we are seeking for a function $u^* \in W^{1,p}(\Omega)$ that solves

$$-\text{div}(|\nabla u^*|^{p-2}\nabla u^*) = f \quad \text{in } \Omega,$$

subjected to various boundary conditions and parametric dependencies. Encoding the boundary conditions and parametric dependencies in a subspace $U$ of $W^{1,p}(\Omega)$, the variational problem (1) is equivalently expressible as a minimization problem which is amenable to the Deep Ritz Method. More precisely, $u^* \in W^{1,p}(\Omega)$ solves the variational problem (1) if and only if it is minimal for the $p$-Dirichlet energy $E : U \rightarrow \mathbb{R}$, defined by

$$E(v) := \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} f \, v \, dx$$

for every $v \in U$. Motivated by recent empirical success in the application of neural network based methods to parametric problems [Hennigh et al. (2021)] as well as their relevance to engineering applications, we include parametric dependencies in our analysis. For example, using one neural network as an ansatz function, we solve simultaneously for a parametrized family of domains. Another example treats the exponent $p \in (1, \infty)$ in the formulation of the $p$-Dirichlet problem as a parameter. We theoretically analyze the error made by this approach also in the parametric setting.

Our theoretical results decompose the error of the Deep Ritz Method into optimization accuracy, expressivity of the ansatz class and – in case of the boundary penalty method for Dirichlet boundary conditions – a term corresponding to the penalization parameter. Combining the error estimates with quantitative approximation results from the literature, we can – at least theoretically – derive error decay rates. Further, we deduce that the potent expressivity of neural networks, especially in high dimensional settings, is retained by the Deep Ritz Method for (parametric) $p$-Dirichlet problems. To the best of our knowledge, our results present the first error estimates of the Deep Ritz Method for non-linear and parametric equations. Finally, we present numerical results illustrating the application of the Deep Ritz Method to parametric $p$-Dirichlet problems.

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Neural Network Based Methods to Solve PDEs Investigating artificial neural networks as ansatz classes for the solution of PDEs, cf. [Li et al. (2020)], inverse or data enhanced problems, cf. [Zhang et al. (2018)] or [Zhu et al. (2019)], and the solution of PDEs in high spatial dimensions, cf. [E and Yu (2018)] or [Han et al. (2018)]. Among the most popular approaches are physics informed neural networks, cf. [Kass et al. (2019)], neural operator methods [Li et al. (2020)] and the Deep Ritz Method, cf. [E and Yu (2018)]. Both, the fact that neural network based methods usually circumvent the necessity of mesh formation and the good approximation capabilities of neural networks for high dimensional functions [Weinan et al. (2020); Wojtowytsch et al. (2020); Jentzen et al. (2018)] motivate the investigation of neural network based methods as an alternative to more traditional numerical schemes, such as finite elements or finite differences for parametric and high dimensional problems.

Parametric Problems In the context of the Deep Ritz Method, we solve PDEs by minimizing their corresponding energy formulation, if available. In this setting, a typical parametric problem is of the form

$$u_p^* = \arg\min_{v \in U(p)} E_p(v),$$

where $p \in \mathcal{P}$ is a fixed parameter from the parameter space $\mathcal{P} \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, and $U(p)$ is a space of functions defined on an open set $\Omega(p) \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, usually realized by a Sobolev space. Typical examples for the parametric dependence of $E_p : U(p) \rightarrow \mathbb{R}$, $p \in \mathcal{P}$, include parametric forcing terms, PDE coefficients and geometries. More explicitly, we consider examples in which $E_p : U(p) \rightarrow \mathbb{R}$, for every $p = (p_1, p_2, p_3)^T \in \mathcal{P} \subseteq \mathbb{R}^N$ and $v \in U(p)$, takes the form

$$E_p(v) = \frac{1}{P_1} \int_{\Omega(p)} |\nabla v|^p \, dx - \int_{\Omega(p)} f(p_3, \cdot) \, v \, dx.$$  

The approach to solve parametric problems with the Deep Ritz Method is to use neural networks that take both a parameter $p = (p_1, p_2, p_3)^T \in \mathcal{P}$ and a spatial variable $x \in \Omega(p_2)$ as an input, i.e., mapping of the particular form $((p, x)^T \mapsto u_0(p, x)) : \bigcup_{p \in \mathcal{P}} (p = (p_1, p_2, p_3)^T) \times \Omega(p_2) \rightarrow \mathbb{R}$. Here, by $\theta \in \Theta$, we denote the neural network’s parameters and by $\Theta$ the neural network’s parameter space. Then, we consider the minimization problem

$$\min_{\theta \in \Theta} \mathcal{L}(\theta) = \min_{\theta \in \Theta} \mathcal{E}(u_0) = \min_{\theta \in \Theta} \int_{\mathcal{P}} E_p(u_0(p, \cdot)) \, d\mu(p),$$

for some suitable measure $\mu$ on $\mathcal{P}$. Solving this minimization problem yields a solution of (2) simultaneously for the whole parameter space $\mathcal{P}$. Incorporating PDE parameters in the above way directly into the ansatz class constitutes a great benefit for engineering applications that often require the exploration of parameter spaces. For an application of industrial scale (in the context of physics informed neural networks), we refer to [Hennigh et al. (2021)], where a parametric geometry was used to determine the optimal design of a heat sink.

1.1 Main Contribution and Related Work

Let the energy of a parametric problem be given, i.e., $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$, where $\mathcal{U}$ is a function space prescribed through the structure of the dependencies to a parameter space $\mathcal{P} \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, for every $v \in \mathcal{U}$ defined by

$$\mathcal{E}(v) = \int_{\mathcal{P}} E_p(v) \, d\mu(p),$$

where $E_p : U(p) \rightarrow \mathbb{R}$, $p \in \mathcal{P}$, is of the form [3]. Our main results are several Céa type estimates for $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$. Denote by $u^* \in \mathcal{U}$, a minimizer of (5) and let $v_{\theta} \in \mathcal{U}$, $\theta \in \Theta$, denote the realization of a neural network with parameter space $\Theta$, then, it holds

$$\rho_1^2(\theta, u^*) \leq \left( \mathcal{E}(v_{\theta}) - \inf_{\psi \in \Theta} \mathcal{E}(v_{\psi}) \right) + \inf_{\psi \in \Theta} \rho_2^2(\theta, v_{\theta}, u^*) =: \delta(v_{\theta}) + \eta(\Theta).$$

Here, $\rho_1^2, \rho_2^2 : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$ are problem-dependent error measures, in the context of the $p$-Dirichlet problem, usually given (up to multiplicative constants) as the so-called natural distance

$$\rho_1^2(u^*, v) \sim \rho_2^2(u^*, v) \sim \int_{\mathcal{P}} \|F_{p_1}(V_x u^*) - F_{p_1}(V_x v)\|_{L^2(\Omega(p_1))}^2 \, d\mu(p),$$

where $F_{p_1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $p = (p_1, p_2, p_3)^T \in \mathcal{P}$, is defined by $F_{p_1}(a) := |a|^{p_1-2}a$ for all $a \in \mathbb{R}^d$, compare to Section II for more details on the natural distance, and by $V_x$ the gradient with respect to the spatial variable $x \in \Omega(p_2)$ only is meant.

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[3] For two functions $g, h : D \rightarrow \mathbb{R}$, where $D$ is an arbitrary set, we write $g \sim h$ if and only if there exist constants $c, C > 0$ such that $cg \leq h \leq Cg$ in $D$. 

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The reasons we are interested in the estimate (6) are the following:

1. It decomposes the error \( p^2(v_0, u') \) into a contribution \( W(v_0) \) capturing the effect of the (usually incomplete) optimization accuracy and a term \( \eta(\Theta) \) that quantifies the expressivity of the ansatz class. This shows the convergence of the Deep Ritz Method given successful optimization and growing ansatz classes.

2. Using results from the approximation theory literature, we employ the estimate (6) to deduce – at least theoretically – error decay rates for the application of the Deep Ritz Method to the \( p \)-Dirichlet problem. Note that for the natural distance no results are known in the literature. Hence, we discuss the relation to Sobolev topologies, where a rich approximation theory is known.

3. Combining the estimate (6) with quantitative universal approximation theorems such as Gühring and Raslan (2021), we show that solving the \( p \)-Dirichlet problem with the Deep Ritz Method retains the favorable approximation capabilities of neural networks for smooth functions, compare to Theorem 3. This is especially useful if the PDE of interest is posed in high spatial dimensions, since here classical solutions schemes are facing the curse of dimensionality. As we do not assume any lower-dimensionality structure on the PDE, it is not possible to obtain a dimension independent result as in Jentzen et al. (2018) or Barron (1993), yet a sufficient amount of smoothness (in the sense of Sobolev spaces) of the solution leads to improved error decay rates. We stress that in all results that break the curse of dimensionality some sort of assumptions are present and we propose the smoothness assumption as yet another.

Further, we also analyze the effect of the boundary penalty method and derive a result similar to the estimate (6), with an additional term accounting for the boundary penalty. The conclusions as above, thus, apply to the boundary penalty method. Finally, we present numerical results indicating that the Deep Ritz Method is well-suited to solve parametric problems of the form analyzed theoretically.

To the best of our knowledge, there are no results in the literature that estimate the error of the Deep Ritz Method for the \( p \)-Dirichlet problem so far. Existing results, such as Müller and Zeinhofer (2021), Xu (2020), Jiao et al. (2021), Duan et al. (2021), treat only linear elliptic equations and none of these works consider \( p \)-Dirichlet exist in the finite element literature, e.g., Diening and Růžička (2007). However, the proofs don’t generalize to the case of the Deep Ritz Method, as the set of neural parametric settings. Error estimates for the Deep Ritz Method for the \( p \)-Dirichlet problem so far. Existing results, such as Müller and Zeinhofer (2021); Xu (2020); Jiao et al. (2021), treat only linear elliptic equations and none of these works consider parametric settings. Error estimates for the \( p \)-Dirichlet exist in the finite element literature, e.g., Diening and Růžička (2007). However, the proofs don’t generalize to the case of the Deep Ritz Method, as the set of neural networks of a given architecture does not possess a vector space structure and, hence, arguments based on optimality criteria – such as Galerkin orthogonality – are not available and need to be circumvented.

II Preliminaries

2.1 Functional analytical notation

For a (real) Banach space \( X \) equipped with norm \( \| \cdot \| : X \to \mathbb{R}_{\geq 0} \), we denote by \( X^\ast \) its topological dual space equipped with the dual norm \( \| \cdot \|_{X^\ast} : X^\ast \to \mathbb{R}_{\geq 0} \), defined by \( \| x^\ast \|_{X^\ast} := \sup_{\| x \| \leq 1} \langle x^\ast, x \rangle_X \) for every \( x^\ast \in X^\ast \). Here, \( \langle \cdot, \cdot \rangle_X : X^\ast \times X \to \mathbb{R}_{\geq 0} \) denotes the duality pairing, defined by \( \langle x^\ast, x \rangle_X := x^\ast(x) \) for every \( x^\ast \in X^\ast, x \in X \).

2.2 Standard function spaces

Throughout the entire section, if not otherwise specified, we denote by \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N} \), a bounded domain, i.e., a bounded, connected and open set.

Lebesgue spaces. For \( p \in [1, \infty] \), we denote by \( L^p(\Omega) \), the space of (Lebesgue-)measurable functions \( u : \Omega \to \mathbb{R} \) that are integrable in \( p \)-th power, i.e., \( \int_{\Omega} |u|^p \, dx < \infty \) if \( p \in [1, \infty) \) and \( \sup_{x \in \Omega} |u(x)| < \infty \) if \( p = \infty \). Endowed with the norm \( \| u \|_{L^p(\Omega)} := \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}} \), if \( p \in [1, \infty) \) and \( \| u \|_{L^\infty(\Omega)} := \sup_{x \in \Omega} |u(x)| \), if \( p = \infty \), the space \( L^p(\Omega) \) forms a Banach space, which is separable if \( p \in [1, \infty) \) and reflexive if \( p \in (1, \infty) \), cf. (Adams and Fournier 2003, Chapter 2).

Sobolev spaces. For \( k \in \mathbb{N} \) and \( p \in [1, \infty] \), we denote by \( W^{k,p}(\Omega) \), the subspace of \( L^p(\Omega) \) of functions with partial distributional derivatives up to \( k \)-th order in \( L^p(\Omega) \). Endowed with the norm \( \| u \|_{W^{k,p}(\Omega)} := \sum_{0 \leq \alpha \leq k} \| D^\alpha u \|_{L^p(\Omega)} \), the space \( W^{k,p}(\Omega) \) forms a Banach space, which is separable if \( p \in [1, \infty) \) and reflexive if \( p \in (1, \infty) \), cf. (Adams and Fournier 2003, Chapter 3). For \( k \in \mathbb{N} \) and \( p \in [1, \infty] \), we denote by \( W^{0,p}(\Omega) \), the closure of all compactly supported smooth functions \( C_0^\infty(\Omega) \) in \( W^{k,p}(\Omega) \). If \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N} \), is a bounded Lipschitz domain, then there exists a linear, continuous trace operator operator \( \text{tr} : W^{1,p}(\Omega) \to L^p(\partial \Omega) \) such that \( \text{tr}(u) = u|_{\partial \Omega} \) for all \( u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega}) \) and \( \text{tr}(u) = 0 \) for all \( u \in W^{1,p}_0(\Omega) \). In particular, we will omit writing ‘\text{tr}’ in this context, e.g., we will employ the abbreviation \( \| u \|_{L^p(\partial \Omega)} := \| \text{tr}(u) \|_{L^p(\partial \Omega)} \). Further, in the context of a penalization scheme, the following Friedrich’s inequality takes a crucial role:
 annoying, we introduce our used notation for the functions represented by a feed-forward neural network. Consider natural numbers $d, m, L, N_0, \ldots, N_l \in \mathbb{N}$ and let

$$\theta = \left( (A_1, b_1), \ldots, (A_L, b_L) \right)^T \in \Theta := \prod_{l=1}^L \mathbb{R}^{N_l \times N_{l-1}} \times \mathbb{R}^{N_l}$$

be a tuple of matrix-vector pairs, where $A_l \in \mathbb{R}^{N_l \times N_{l-1}}$ and $b_l \in \mathbb{R}^{N_l}$ for $l = 1, \ldots, L$. In particular, we always assume that $N_0 = d$ and $N_L = m$. The matrix-vector pairs $(A_l, b_l) \in \mathbb{R}^{N_l \times N_{l-1}} \times \mathbb{R}^{N_l}$, $l = 1, \ldots, L$, induce affine-linear mappings $T_l : \mathbb{R}^{N_{l-1}} \rightarrow \mathbb{R}^{N_l}$, $l = 1, \ldots, L$. Then, a neural network function $u^0_\theta : \mathbb{R}^d \rightarrow \mathbb{R}^m$ with parameters $\theta \in \Theta$ and activation function $g : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$u^0_\theta(x) := T_L(g(T_{L-1}(g(\cdots g(T_1(x)))))) \quad \text{in } \mathbb{R}^m \quad \text{for all } x \in \mathbb{R}^d.$$

The set of all neural network functions of a certain architecture $\Theta$ is then given by $\mathcal{F}_\Theta^d := \{ u^0_\theta \mid \theta \in \Theta \}$. Here, $d$ denotes the input dimension, while $m$ denotes the output dimension of the neural network. Apart from that, $L$ is called the depth and $W = \max_{l=0, \ldots, L} N_l$ the width of the neural network. A neural network is called shallow, if it has depth $L = 2$ and deep otherwise. The total number of parameters and the total number of neurons of such a neural network is given by $\dim(\Theta)$ and $\sum_{l=0}^L N_l$, respectively. Throughout what follows, we restrict to the case $m = 1$ since we only consider scalar functions. If we have $u = u^0_\theta$ for some $\theta \in \Theta$, we say the function $u$ can be realized by the neural network $\mathcal{F}_\Theta^d$. Note that we often drop the superscript $g$ if it is clear from the context.

In the following, we need the square of the ReLU activation function which is defined by $\text{ReLU}^2 := \max(0, \cdot)^2$.

\section{Neural networks}

\subsection{Quantitative Universal Approximation}

Theorem 3 (Quantitative Universal Approximation). Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain. Moreover, let $p \in [1, \infty]$ and $k \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ and every $u \in W^{k,p}(\Omega)$, there exists a fully-connected ReLU$^2$-network $u_n \in W^{k,p}(\Omega)$ with parameter space $\Theta_n$ of dimension $\Theta(n)$ such that, it holds

$$\|u - u_n\|_{W^{k,p}(\Omega)} \leq c(p) \left( \frac{1}{n} \right)^{\frac{1}{d}} \|u\|_{W^{k,p}(\Omega)},$$

where $c(p) > 0$ depends only on $p \in [1, \infty]$ and $d,k \in \mathbb{N}$.

Remark 4. Theorem 3 is a special case of [Gühring and Raslan, 2021, Theorem 4.9]. It is proven there for a wide range of activation functions and higher order Sobolev approximations. Furthermore, it is also shown that the approximation rate is – up to a logarithmic factor – optimal, if one assumes that the weights are encodable. We refer the reader to the original work for details.
III Brief review of the p-Dirichlet problem

In this section, we give a brief review of the p-Dirichlet problem. To keep the presentation fairly simple, we initially restrict ourselves to the p-Dirichlet problem subject to homogeneous Dirichlet boundary conditions. The latter, for a fixed exponent \( p \in (1, \infty) \) and a fixed right-hand side \( f \in W_0^{1,p}(\Omega) \), seeks for a function \( u^* \in W_0^{1,p}(\Omega) \) such that for every \( v \in W_0^{1,p}(\Omega) \), it holds

\[
\int_{\Omega} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx = \langle f, v \rangle_{W_0^{1,p}(\Omega)}.
\]  

(8)

Resorting to the celebrated monotone operator theory, cf. Růžička (2004, Satz 1.39), it is readily seen that (8) admits a unique solution. In what follows, we reserve the notation \( u^* \in W_0^{1,p}(\Omega) \) for this solution. For being amenable to the Deep Ritz Method, the variational problem (8) must be equivalently expressible as a minimization problem. A minimization problem equivalent to (8) is given by the minimization of the weak convergence in \( \Gamma \)

\[ E(v) := \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \langle f, v \rangle_{W_0^{1,p}(\Omega)}. \]  

(9)

Since \( E : W_0^{1,p}(\Omega) \to \mathbb{R} \) is a properly4 strictly convex, weakly coercive5 and lower semi-continuous functional, the direct method in the calculus of variations, cf. Dacorogna (2008), implies the existence of a unique minimizer. More precisely, due to the convexity and Frechét differentiability of \( E : W_0^{1,p}(\Omega) \to \mathbb{R} \), this minimizer coincides with the solution \( u^* \in W_0^{1,p}(\Omega) \) to (8).

In (Dondl et al., 2021, Section 5.2), it has been established that the restrictions \( E_n := E|_{M_n} : M_n \to \mathbb{R}, n \in \mathbb{N} \), where \( (M_n)_{n \in \mathbb{N}} \) is a suitable conformal (i.e., \( M_n \subseteq W_0^{1,p}(\Omega) \) for all \( n \in \mathbb{N} \)) and possibly non-linear sequence of ansatz classes, a class of neural networks, for example, \( \Gamma \)-converges to \( E : W_0^{1,p}(\Omega) \to \mathbb{R} \) with respect to weak convergence in \( W_0^{1,p}(\Omega) \).

We are interested in error estimates for the minimization problem (9) for general classes \( M_n \subseteq W_0^{1,p}(\Omega) \), \( n \in \mathbb{N} \), of ansatz functions, to be realized by neural networks. Due to the potential non-linearity of the ansatz classes \( M_n \subseteq W_0^{1,p}(\Omega) \), \( n \in \mathbb{N} \), we cannot resort to Galerkin orthogonality relations, which usually play a decisive role in the derivation of Céa type lemmata and, thus, error estimates, cf. Diening and Růžička (2007). Instead, we follow a commonly used approach from convex analysis and replace the missing Galerkin orthogonality relations by co-coercivity properties of the strongly convex p-Dirichlet energy. To this end, we identify a suitable measure for the co-coercivity of the p-Dirichlet energy \( E : W_0^{1,p}(\Omega) \to \mathbb{R} \) at \( u^* \in W_0^{1,p}(\Omega) \), i.e., we identify bi-variate, symmetric mappings \( \rho^2_1, \rho^2_2 : W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}_{\geq 0} \) such that for every \( v \in W_0^{1,p}(\Omega) \), it holds

\[
\rho^2_1(v, u^*) \leq E(v) - E(u^*) \leq \rho^2_2(v, u^*). \]  

(10)

Then, the two-sided estimate (10)4 implies a Céa type lemma, which can be used to derive error estimates. An intuitive – but also somewhat naïve – approach is to choose (up to some multiplicative constants) \( \rho^2_1(v, u) = \rho^2_2(v, u) = ||\nabla v - \nabla u||_{W_0^{1,p}(\Omega)} \) for all \( v, w \in W_0^{1,p}(\Omega) \). However, it turned out that this choice is not well-suited for both an a priori and an a posteriori error analysis for the p-Dirichlet energy \( E : W_0^{1,p}(\Omega) \to \mathbb{R} \) (and (8)) as, e.g., one obtains convergence rates that are sub-optimal for a discretization using linear finite element spaces, cf. Barrett and Liu (1993). The optimal choice results from the observation that by the Taylor expansion, cf. (25) for a justification, and the optimality condition \( DE(u^*) = 0 \) in \( W_0^{1,p}(\Omega) \), for every \( v \in W_0^{1,p}(\Omega) \), we have that

\[
E(v) - E(u^*) = \langle DE(u^*), v - u^* \rangle_{W_0^{1,p}(\Omega)} + \int_0^1 D^2 E(\tau v + (1 - \tau) u^*) \, d\tau \]  

(11)

\[
= \int_0^1 D^2 E(\tau v + (1 - \tau) u^*) \, d\tau. \]

With (11) we observe that the optimal distance measures \( \rho^2_1, \rho^2_2 : W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}_{\geq 0} \) must form upper and lower bounds, resp., for the second variation of \( E : W_0^{1,p}(\Omega) \to \mathbb{R} \), i.e., (11)2. To identify such measures,
we make the ansatz that, uniformly with respect to \( v, w \in W^{1,p}_0(\Omega) \), it holds
\[
\|F(\nabla v) - F(\nabla w)\|_{L^p(\Omega)}^2 \sim \int_0^1 D^2E(\tau v + (1 - \tau)w) \cdot [v - w, v - w] (1 - \tau) \, d\tau ,
\]
i.e., \( p_2^2(v, w) \sim p_2^2(v, w) \sim \|F(\nabla v) - F(\nabla w)\|_{L^p(\Omega)}^2 \), for some (possibly non-linear) function \( F : \mathbb{R}^d \to \mathbb{R}^d \) with \( F(0) = 0 \).

The ansatz \((12)\) has the particular advantage that, in terms of Lebesgue norms, we enter a linear level, while all the non-linearity of the \( p \)-Dirichlet energy is covered by the function \( F : \mathbb{R}^d \to \mathbb{R}^d \). But how to identify \( F : \mathbb{R}^d \to \mathbb{R}^d \)? To this end, we consider the case \( w = 0 \in W^{1,p}_0(\Omega) \), so that, uniformly with respect to \( v \in W^{1,p}_0(\Omega) \),
\[
\|F(\nabla v)\|_{L^p(\Omega)}^2 \sim \int_0^1 D^2E(\tau v) \cdot [v, v] (1 - \tau) \, d\tau \sim \|\nabla v\|_{L^p(\Omega)}^p ,
\]
where we used for the second equivalence that \( \min[1, p - 1]\|a\|^{-p} - 2b \leq D^2\phi(a) : b \otimes b \leq \max[1, p - 2]\|a\|^{-p} - 2|b|^2 \) for all \( a \in \mathbb{R}^d \setminus \{0\} \) and \( b \in \mathbb{R}^d \), where \( \phi \in C^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{0\}) \), defined by \( \phi(a) := \frac{1}{2}|a|^2 \) for all \( a \in \mathbb{R}^d \), denotes the \( p \)-Dirichlet density, as well as that \( \int_0^1 \tau^{p-1} (1 - \tau) \, d\tau = \frac{1}{p^2 + 1} \). The equivalence \((13)\), in turn, suggests the choice
\[
F(a) := \|a\|_2 a \quad \text{for all } a \in \mathbb{R}^d .
\]
which guarantees that \( \|F(\nabla v)\|_{L^p(\Omega)}^p = \|\nabla v\|_{L^p(\Omega)}^p \) for all \( v \in W^{1,p}_0(\Omega) \) and, thus, is sufficient for the ansatz \((12)\) for the particular case \( w = 0 \in W^{1,p}_0(\Omega) \). That \((12)\) even holds for all \( v, w \in W^{1,p}_0(\Omega) \) if \( F : \mathbb{R}^d \to \mathbb{R}^d \) is defined by \((14)\) is shown in the subsequent section and for which we will resort to the following key properties of \( F : \mathbb{R}^d \to \mathbb{R}^d \).

**Lemma 5.** Let \( p \in (1, \infty) \) and \( d \in \mathbb{N} \). Then, there exists a constant \( c(p) > 0 \), depending only on \( d \in \mathbb{N} \) and \( p \in (1, \infty) \), such that the following statements apply:

(i) For every \( a, b \in \mathbb{R}^d \), it holds
\[
c(p)^{-1}[F(a) - F(b)]^2 \leq \|\nabla a - \nabla b\|^2_{L^p(\Omega)} \cdot (a - b) \leq c(p)\|F(a) - F(b)\|^2 .
\]

(ii) For every \( a, b \in \mathbb{R}^d \), it holds
\[
c(p)^{-1}[F(a) - F(b)]^2 \leq \|a + b\|^2_{L^p(\Omega)} \cdot (a - b) \leq c(p)\|F(a) - F(b)\|^2 .
\]

**Proof.** See [Diening et al. 2007 Appendix] or [Diening and Ettwein 2008, Appendix].

**Remark 6.** By carefully reviewing the proofs in [Diening et al. 2007 Appendix], it can be found that for the constants \( c(p) > 0 \), \( p \in (1, \infty) \), in Lemma 5 depend continuously on \( p \in (1, \infty) \), i.e., it holds \( (p \mapsto c(p)) \in C^0(1, \infty) \).

Eventually, we introduce the compact notation \( p_2^2 : W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega) \to \mathbb{R} \), for every \( v, w \in W^{1,p}_0(\Omega) \) defined by
\[
p_2^2(v, w) := \|F(\nabla v) - F(\nabla w)\|_{L^p(\Omega)}^2 .
\]

Since \( p_2^2 : W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega) \to \mathbb{R} \) arises naturally from the ansatz \((12)\) and is the optimal distance measure for the \( p \)-Dirichlet problem because of the two-sided estimate \((10)\), it is usually referred to as the natural distance in the literature, cf. [Diening and Růžička 2007; Diening et al. 2007; Diening and Ettwein 2008; Kaltenbach and Růžička 2022].

**Remark 7 (\( \varphi \)-Dirichlet problem).** We could further consider more general convex functions \( \varphi \in C^1(\mathbb{R}^d) \) than the \( p \)-Dirichlet density. For example, we could consider \( \varphi \in C^1(\mathbb{R}^d) \) to be given as \( \varphi(a) := \varphi(|a|) \) for all \( a \in \mathbb{R}^d \), where \( \varphi \in C^2(0, \infty) \) is a balanced \( N \)-function, cf. [Diening and Růžička 2007; Kaltenbach and Růžička 2022], i.e., satisfies the \( \Delta_2 \)- and the \( V_2 \)-condition as well as \( \varphi'(a) \sim \varphi''(a) a \) uniformly with respect to \( a > 0 \). In fact, every result of this section, Section \( \nabla \) and Section \( \nabla \) can be generalized to the \( \varphi \)-Dirichlet problem, i.e., a non-linear Dirichlet problem with so-called Orlicz-structure. To be more precise, for given right-hand side \( f \in W^{1,p}_0(\Omega) \), where \( W^{1,p}_0(\Omega) := \{ v \in W^{1,1}_0(\Omega) \mid \varphi(|\nabla v|) \in L^1(\Omega) \} \) denotes the Orlicz–Sobolev space, the \( \varphi \)-Dirichlet problem seeks for an Orlicz–Sobolev function \( u^* \in W^{1,\varphi}_0(\Omega) \) such that for every \( v \in W^{1,p}_0(\Omega) \), it holds
\[
\int_\Omega A(\nabla u^*) \cdot \nabla v \, dx = \left( f, v \right)_{W^{1,\varphi}_0(\Omega)} ,
\]
where \( A : \mathbb{R}^d \to \mathbb{R}^d \) for every \( a \in \mathbb{R}^d \) is defined by \( A(a) := D\varphi(|a|) \frac{a}{|a|} \). In this case, the natural distance is defined analogously but with \( F : \mathbb{R}^d \to \mathbb{R}^d \) for every \( a \in \mathbb{R}^d \) is defined by \( F(a) := \sqrt{A(a)} \frac{a}{|a|} \)

---

\(^7\) For quadratic matrices \( A = (a_{ij})_{i,j=1,\ldots,d} \), \( B = (b_{ij})_{i,j=1,\ldots,d} \in \mathbb{R}^{d \times d} \), \( A : B := \sum_{i,j=1}^d a_{ij} b_{ij} \) denotes the Frobenius inner product.

\(^8\) For vectors \( a, b \in \mathbb{R}^d \), the matrix \( a \otimes b \in \mathbb{R}^{d \times d} \), defined by \( (a \otimes b)_{ij} = a_i b_j \) for all \( i, j = 1, \ldots, d \), denotes the dyadic product.
Then, the following statements apply:

**Remark 9.**

Moreover, let \( E : U \to \mathbb{R} \) for every \( v \in U \) be defined by

\[
E(v) := \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \langle f, v \rangle_{W^{1,p}(\Omega)}.
\]

Then, the following statements apply:

(i) There exists a unique minimizer \( u^* \in U \) for \( E : U \to \mathbb{R} \).

(ii) There exists a constant \( c(p) > 0 \), depending only on \( p \in (1, \infty) \) and \( d \in \mathbb{N} \) such that for every \( v \in U \), it holds

\[
c(p)^{-1} \rho_2^p(v, u^*) \leq E(v) - E(u^*) \leq c(p) \rho_2^p(v, u^*),
\]

where \( F : \mathbb{R}^d \to \mathbb{R}^d \) is defined by \([14]\). In particular, we can choose \( c(p) > 0 \) such that \( (p \mapsto c(p)) \in C^0(1, \infty) \).

**Remark 10.**

For the closed subspace \( U \) of \( W^{1,p}(\Omega) \), we have, e.g., in mind \( W^{1,p}(\Omega) := \{ v \in W^{1,p}(\Omega) \mid v = 0 \text{ in } \Gamma_0 \} \), where \( \Gamma_0 \subseteq \partial \Omega \) satisfies \( \mathbb{R}^{d-1}(\Gamma_0) > 0 \), or \( W^{1,p}(\Omega)/\mathbb{R} := \{ v \in W^{1,p}(\Omega) \mid \int_{\Omega} v \, dx = 0 \} \), or closed subsets of these spaces.

**Remark 11.**

Theorem 8 also applies for \( U = W^{1,p}(\Omega) \) if \( f \in W^{1,p}(\Omega)^* \) vanishes on constants and if we drop the uniqueness in point (i). More precisely, for \( U = W^{1,p}(\Omega)/\mathbb{R} \), Theorem 8 already implies the existence of a minimizer \( u^* \in W^{1,p}(\Omega)/\mathbb{R} \) of \( E : W^{1,p}(\Omega)/\mathbb{R} \to \mathbb{R} \), cf. Remark 10. Since \( f \in W^{1,p}(\Omega)^* \) vanishes on constants, this implies \( E(v) = E(v - \frac{1}{\Omega} \int_{\Omega} v \, dx) \geq E(u^*) \) for all \( v \in W^{1,p}(\Omega) \), i.e., \( u^* \in W^{1,p}(\Omega)/\mathbb{R} \) is minimal for \( E : W^{1,p}(\Omega)/\mathbb{R} \to \mathbb{R} \). In particular, for every \( c \in \mathbb{R} \), \( u^* + c \in W^{1,p}(\Omega) \) is minimal for \( E : W^{1,p}(\Omega) \to \mathbb{R} \), due to \( E(u^* + c) = E(u^*) \).

An immediate consequence of Theorem 8 is the following Céa type lemma.

**Corollary 12 (Céa Type Lemma).** Let the assumptions of Theorem 8 be satisfied. Moreover, let \( M \subseteq U \) be an arbitrary subset. Then, there exists a constant \( c(p) > 0 \), depending only on \( p \in (1, \infty) \) and \( d \in \mathbb{N} \), such that for every \( v \in M \), it holds

\[
\rho_2^p(v, u^*) \leq c(p) \left( \delta + \inf_{\tilde{\theta} \in M} \rho_2^p(\tilde{\theta}, u^*) \right),
\]

where \( \delta := \delta(v) := E(v) - \inf_{\tilde{\theta} \in M} E(\tilde{\theta}) \). In particular, we can choose \( c(p) > 0 \) such that \( (p \mapsto c(p)) \in C^0(1, \infty) \).

**Remark 13.** For the conformal subset \( M \subseteq U \), we have, e.g., in mind a set of all neural network realizations \( \mathcal{M}_\Theta \) of a certain architecture \( \Theta \subseteq \mathbb{R}^N \), \( N \in \mathbb{N} \), and activation function \( \rho : \mathbb{R} \to \mathbb{R} \) or modifications of this set, e.g., using multiplicative weights to enforce homogeneous Dirichlet boundary conditions on \( \Gamma_D \) or additive integral mean corrections to enforce a vanishing integral mean constraint.

**Proof of Corollary 12.** Let \( v \in M \) be fixed, but arbitrary. Then, by referring to Theorem 8, we find that

\[
c(p)^{-1} \rho_2^p(v, u^*) \leq E(v) - \inf_{\tilde{\theta} \in M} E(\tilde{\theta}) + \inf_{\tilde{\theta} \in M} E(\tilde{\theta}) - E(u^*) \leq \delta + c(p) \inf_{\tilde{\theta} \in M} \rho_2^p(\tilde{\theta}, u^*).\]

The proof of Theorem 8 is based on the justification of the Taylor expansion \([11]\) and, then, to establish the equivalence \([12]\). To trace the later, in the following lemma, we first fall back to the finite dimensional case.
Lemma 14 (Point-wise Estimate). Let $p \in (1, \infty)$ and $d \in \mathbb{N}$. Then, there exits a constant $c(p) > 0$, depending only on $d \in \mathbb{N}$ and $p \in (1, \infty)$, such that for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, we have that

$$c(p)^{-1} |F(a) - F(b)|^2 \leq \int_0^1 D^2 \phi(\tau a + (1 - \tau)b) : (a - b) \otimes (a - b) (1 - \tau) \, d\tau \leq c(p) |F(a) - F(b)|^2,$$

where $\phi \in C^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{0\})$, defined by $\phi(a) := \frac{1}{2} |a|^p$ for all $a \in \mathbb{R}^d$, denotes the $p$-Dirichlet density. In particular, we can choose $c(p) > 0$ such that $(p \mapsto c(p)) \in C^0(1, \infty)$.

Proof. We introduce the abbreviation $\eta^2 : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(0, 0)^T\} \to \mathbb{R}_{\geq 0}$, for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$ defined by

$$\eta^2(a, b) := \int_0^1 D^2 \phi(\tau a + (1 - \tau)b) : (a - b) \otimes (a - b) (1 - \tau) \, d\tau.$$

Using $D^2 \phi(a) : b \otimes b \geq \min\{1, p - 1\}|a|^p - 2|b|^2$ for all $a \in \mathbb{R}^d \setminus \{0\}, b \in \mathbb{R}^d$ (cf. [Růžička 2004, p. 73, ineq. (1.35)]), for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, we obtain

$$\eta^2(a, b) \geq \min\{1, p - 1\} \int_0^1 |\tau a + (1 - \tau)b|^p (1 - \tau) \, d\tau.$$

Apart from that, with the help of Jensen’s inequality applied with respect to the measure $d\mu = (1 - \tau) d\tau$, i.e., in particular, we use that $d\mu([0, 1]) = \frac{1}{2}$, for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, we observe that

$$\left( \int_0^1 |\tau a + (1 - \tau)b|^p (1 - \tau) \, d\tau \right)^{1/p} \leq \int_0^1 |\tau a + (1 - \tau)b|^p \, d\mu(\tau).$$

Then, we continue by incorporating (19) and, thus, find that for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, it holds

$$\eta^2(a, b) \geq \min\{1, p - 1\} \left( 2 \int_0^1 |\tau a + (1 - \tau)b|^p (1 - \tau) \, d\tau \right)^{1/p} \frac{|a - b|^2}{(|a| + |b|)^2}.$$

There exists a constant $c > 0$, depending only on $d \in \mathbb{N}$, such that for every $a, b \in \mathbb{R}^d$, it holds

$$2 \int_0^1 |\tau a + (1 - \tau)b|^p (1 - \tau) \, d\tau \geq c (|a| + |b|),$$

which readily follows from the fact the both sides define norms on $\mathbb{R}^d \times \mathbb{R}^d$ and, thus, need to be equivalent. Using (21) in (20), for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, we deduce that

$$\eta^2(a, b) \geq \min\{1, p - 1\} c^{p} (|a| + |b|)^p - 2 |a - b|^2.$$

Eventually, resorting to Lemma 5, we conclude the existence of a constant $c(p) > 0$, depending only on $d \in \mathbb{N}$ and $p \in (1, \infty)$, with $(p \mapsto c(p)) \in C^0(1, \infty)$, such that for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, it holds

$$\eta^2(a, b) \geq c(p)^{-1} |F(a) - F(b)|^2.$$

On the other hand, since also $D^2 \phi(a) : b \otimes b \leq \max\{1, p - 2\}|a|^{p-2} |b|^2$ for all $a \in \mathbb{R}^d \setminus \{0\}, b \in \mathbb{R}^d$, which, again, follows very similarly to [Růžička 2004, p. 73, ineq. (1.35)], we find that

$$\eta^2(a, b) \geq c(p)^{-1} |F(a) - F(b)|^2.$$

Since, appealing to [Diening et al. 2007, Appendix, Lemma 6.1], there is a constant $c(p) > 0$, depending only on $d \in \mathbb{N}$ and $p \in (1, \infty)$, with $(p \mapsto c(p)) \in C^0(1, \infty)$, such that for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, it holds

$$\int_0^1 |\tau a + (1 - \tau)b|^p (1 - \tau) \, d\tau \leq c(p) (|a| + |b|)^{p-2},$$

we deduce from (22) that for every $a, b \in \mathbb{R}^d$ with $|a| + |b| > 0$, it holds $\eta^2(a, b) \leq \max\{1, p-2\} c(p)(|a| + |b|)^{p-2}|a - b|^2$, which, resorting again to Lemma 5 eventually, completes the proof of Lemma 14.
Now we have it all at our disposal to prove Theorem 8.

Proof of Theorem 8. (i). The p-Dirichlet energy $E: U \to \mathbb{R}$ is proper, strictly convex, continuous and, thus, lower semi-continuous. In addition, the validity of Poincaré’s inequality (17), in a standard manner, i.e., in combination with the $\varepsilon$-Young inequality, cf. (12) or (42), guarantees the weak coercivity of $E: U \to \mathbb{R}$, so that the direct method in the calculus of variations yields, cf. Dacorogna (2008), the existence of a unique minimizer $u^* \in U$ of $E: U \to \mathbb{R}$.

(ii). We proceed similar to Dieng and Kreuzer (2008) Lemma 16.). Again, we employ the notation $\phi \in C^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{0\}$, defined by $\phi(a) := \frac{1}{2} |a|^p$ for all $a \in \mathbb{R}^d$, for the p-Dirichlet density. Since $D\phi \in C^0(\mathbb{R})^d$ with $|D\phi(a)| = |a|^{p-1}$ for all $a \in \mathbb{R}^d$, the p-Dirichlet energy is continuously Frechet differentiable with

$$
\langle DE(u), v \rangle := \int_{\Omega} D\phi(u) \cdot \nabla v \, dx - \langle f, v \rangle_{W^{1,p}(\Omega)}.
$$

for all $u, v \in U$. In particular, due to the minimality of $u^* \in U$, we have that $DE(u^*) = 0$ in $U$, i.e., for every $v \in U$, it holds

$$
\langle DE(u), v \rangle_{U} = 0.
$$

However, $E: U \to \mathbb{R}$ is not twice continuously Frechet differentiable. Therefore, we consider regularizations $(\phi_\varepsilon)_{\varepsilon > 0} \subseteq C^2(\mathbb{R}^d)$, defined by $\phi_\varepsilon(a) := \frac{1}{2} (\varepsilon^2 + |a|^2)^{1/2}$ for every $\varepsilon > 0$ and $a \in \mathbb{R}^d$, having the following properties:

1. $(\phi_\varepsilon)_{\varepsilon > 0} \subseteq C^2(\mathbb{R}^d)$, for all $a \in \mathbb{R}^d$ and $\phi_\varepsilon(a) \leq 2\frac{\varepsilon}{p} (|a|^p + \varepsilon^p)$ for all $a \in \mathbb{R}^d$,
2. $(DE_\varepsilon)(a) \to \phi(a)$ for all $a \in \mathbb{R}^d$ and $|D\phi(a)| \leq 2\frac{\varepsilon}{p} (|a|^p + \varepsilon^p)$ for all $a \in \mathbb{R}^d$,
3. $(D^2\phi_\varepsilon)(a) \to (D^2\phi)(a)$ for all $a \in \mathbb{R}^d \setminus \{0\}$ and $|(D^2\phi)(a)| \leq (p-1)2\frac{\varepsilon}{p} - 2 (e^{p-2} + |a|^{p-2})$ for all $a \in \mathbb{R}^d$.

Inasmuch as $(\phi_\varepsilon)_{\varepsilon > 0} \subseteq C^2(\mathbb{R}^d)$ satisfies (1), (2) and (3), it is easily checked that for every $\varepsilon > 0$, the regularized $p$-Dirichlet energy $E^\varepsilon: U \to \mathbb{R}$, for every $v \in U$ defined by

$$
E^\varepsilon(v) := \int_{\Omega} \phi_\varepsilon(\nabla v) \, dx - \langle f, v \rangle_{W^{1,p}(\Omega)}
$$

is twice continuously Frechet–differentiable. In consequence, using Taylor’s formula and Fubini’s theorem, for every $\varepsilon > 0$ and $v \in U$, we obtain

$$
E(v) - E(u^*) = \langle DE(u^*), v - u^* \rangle_{U} + \int_{0}^{1} D^2E^\varepsilon(\tau v + (1 - \tau)u^*) \, d\tau \cdot (v - u^*) \, d\tau.
$$

Next, given both (1), (2) and (3), it is allowed to apply Lebesgue’s dominated convergence theorem in (24). Hence, by passing for $\varepsilon \to 0$ in (24), using (23) in doing so, for every $v \in U$, we find that

$$
E(v) - E(u^*) = \langle DE(u^*), v - u^* \rangle_{U} + \int_{0}^{1} D^2\phi(\tau v + (1 - \tau)u^*) : \nabla(v - u^*) \, d\tau \cdot (v - u^*) \, d\tau.
$$

Apart from that, resorting to Lemma 14, we deduce the existence of a constant $c(p) > 0$, depending only on $d \in \mathbb{N}$ and $p \in (1, \infty)$, with $(p \mapsto c(p)) \in C^0(1, \infty)$, such that for every $v \in U$, it holds

$$
c(p)^{-1} r_p^2(v, u^*) \leq \int_{\Omega} \int_{0}^{1} D^2\phi(\tau v + (1 - \tau)u^*) : \nabla(v - u^*) \, d\tau \cdot (v - u^*) \, d\tau \leq c(p) r_p^2(v, u^*). \tag{26}
$$

Eventually, by combining (25) and (26), we conclude the assertion of Theorem 8. \qed
In the case of Dirichlet boundary conditions, a common approach is to approximately enforce the latter by a soft penalty. More precisely, to approximate homogeneously Dirichlet boundary conditions, for given a \( f \in W^{1,p}(\Omega)^* \), \( p \in (1, \infty) \), and a (large) penalty parameter \( \lambda > 0 \), we consider the boundary penalized \( p \)-Dirichlet energy \( E_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \), for every \( v \in W^{1,p}(\Omega) \) defined by

\[
E_\lambda(v) := \frac{1}{p} \int_\Omega |\nabla v|^p \, dx + \frac{\lambda}{p} \int_{\partial \Omega} |v|^p \, ds - \langle f, v \rangle_{W^{1,p}(\Omega)}.
\]  

(27)

In the limit \( \lambda \to \infty \), we obtain a homogeneous Dirichlet boundary condition. The natural distance measure in this case, is the boundary penalized natural distance \( \rho^2_{\mathcal{E},\lambda} : W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to \mathbb{R} \), for every \( v, w \in W^{1,p}(\Omega) \) defined by

\[
\rho^2_{\mathcal{E},\lambda}(v, w) := |F(v) - F(w)|^2_{L^p(\Omega)} + \lambda \|F(v) - F(w)\|^2_{L^p(\partial \Omega)}.
\]  

(28)

Let us denote by \( u^* \in W^{1,p}_0(\Omega) \), the solution of the \( p \)-Dirichlet problem with homogeneous Dirichlet boundary condition, i.e., the minimizer of (27) over \( W^{1,p}_0(\Omega) \), and by \( u^*_\lambda \in W^{1,p}(\Omega) \) the minimizer of (27) over \( W^{1,p}(\Omega) \). Then, we can analyze the effect of the penalty.

**Theorem 15 (Boundary Penalty).** Let \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N}, \) be a bounded domain, \( f \in L^p(\Omega), p \in (1, \infty), \) and \( M \subset W^{1,p}(\Omega) \). Moreover, assume that \( \|\nabla u^*\|^p - 2\|u^*\| \cdot n \in L^p(\partial \Omega) \). Then, there exists a constant \( c(p) > 0 \), depending only on \( d \in \mathbb{N} \) and \( p \in (1, \infty) \), such that for every \( v \in M \) and \( \lambda \geq 1 \), it holds

\[
\rho^2_{\mathcal{E},\lambda}(v, u^*) + \|\nabla v\|^p \leq 2 \left( \delta_\lambda + c(p) \left( \inf_{v \in M} \rho^2_{\mathcal{E},\lambda}(v, u^*_\lambda) + \lambda^{-\frac{1}{p}} \right) \right),
\]  

(29)

where \( \delta_\lambda := \delta_\lambda(v) := E_\lambda(v) - \inf_{v \in M} E_\lambda(v) \). In particular, we can choose \( c(p) > 0 \) such that \( (p \mapsto c(p)) \in C^0(1, \infty) \).

**Proof.** We divide the proof into three main steps:

**Step I.** Repeating the regularization arguments in the proof of Theorem 8 we are able to show that for every \( v \in M \) and \( \lambda \geq 1 \), it holds

\[
c(p)^{-1} \rho^2_{\mathcal{E},\lambda}(v, u^*_\lambda) \leq E_\lambda(v) - E_\lambda(u^*_\lambda) \leq c(p) \rho^2_{\mathcal{E},\lambda}(v, u^*_\lambda).
\]  

(30)

**Step II.** Next, we need to estimate the distance of \( u^* \in W^{1,p}_0(\Omega) \) and \( u^*_\lambda \in W^{1,p}(\Omega) \). To this end, let \( \lambda \geq 1 \) be fixed, but arbitrary. Then, the minimality of \( u^*_\lambda \in W^{1,p}(\Omega) \) and \( u^* = 0 \) in \( L^p(\partial \Omega) \) yield

\[
E_\lambda(u^*_\lambda) \leq E_\lambda(u^*) = E(u^*).
\]  

(31)

Thus, using for every \( \epsilon > 0 \), the \( \epsilon \)-Young inequality with constant \( c(p, \epsilon) := \frac{(p\epsilon)^{\frac{1}{p}}}{p} \), we deduce from (31) that

\[
\frac{1}{p} \|\nabla u^*_\lambda\|^p \geq \frac{\lambda}{p} \|u^*_\lambda\|^p \leq E(u^*) + c(p, \epsilon) \|f\|_{L^p(\Omega)} + \epsilon \|u^*_\lambda\|^p.
\]  

(32)

In addition, owing to Friedrich’s inequality (cf. Theorem 1), there exists a constant \( c_{F,\epsilon}(p) > 0 \), only depending on \( d \in \mathbb{N} \) and \( p \in (1, \infty) \), with \( (p \mapsto c_{F,\epsilon}(p)) \in C^0(1, \infty) \), such that

\[
\|u^*_\lambda\|^p \leq c_{F,\epsilon}(p) \left( \|\nabla u^*_\lambda\|^p + \|u^*_\lambda\|^p \right).
\]  

(33)

Hence, choosing \( \epsilon := \frac{1}{2p} \min\{\lambda, 1\} \), i.e., \( \epsilon = \frac{1}{2p} \min\{\lambda, 1\} \) if \( \lambda \geq 1 \), in (32), using (33) in doing so, we find that

\[
\frac{1}{p} \|\nabla u^*_\lambda\|^p + \frac{\lambda}{p} \|u^*_\lambda\|^p \leq E(u^*) + c(p, \epsilon) \|f\|_{L^p(\Omega)} + \frac{1}{2p} \|\nabla u^*_\lambda\|^p + \frac{\lambda}{2p} \|u^*_\lambda\|^p.
\]  

(34)

Absorbing the last two terms on the right-hand side of (34) in the left-hand side, we obtain

\[
\frac{\lambda}{2p} \|u^*_\lambda\|^p \leq \frac{1}{2p} \|\nabla u^*_\lambda\|^p + \frac{\lambda}{2p} \|u^*_\lambda\|^p
\]  

(35)

\[
\leq E(u^*) + c(p, \epsilon) \|f\|_{L^p(\Omega)}.
\]

As \( u^* \in W^{1,p}_0(\Omega) \) is minimal for \( E : W^{1,p}_0(\Omega) \to \mathbb{R} \), which, in turn, is Frechét differentiable, for every \( v \in W^{1,p}_0(\Omega) \), we have that

\[
\int_{\Omega} \|\nabla u^*\|^p - 2\|u^*\| \cdot \nabla v \, dx = \int_{\Omega} f \cdot v \, dx.
\]  

(36)
Due to \( f \in L^p(\Omega) \), from (36), we deduce that \(|\nabla u|^{p^*-2}\nabla u\in W^{1,p}(\text{div}; \Omega)\) with \(-\text{div}(|\nabla u|^{p^*-2}\nabla u) = f\) in \(L^p(\Omega)\). In particular, appealing to Proposition \(\mathcal{P}\) for every \(v \in W^{1,p}(\Omega)\), we have that
\[
\int_\Omega |\nabla u|^{p^*-2}\nabla u\cdot \nabla v \, dx - \int_{\partial \Omega} |\nabla u|^{p^*-2}\nabla u \cdot n \, v \, ds = \int_\Omega f \cdot v \, dx.
\]  (37)

Similarly, as \(u_\lambda^*\in W^{1,p}(\Omega)\) is minimal for \(E_\lambda: W^{1,p}(\Omega) \to \mathbb{R}\), which is Frechét differentiable, for every \(v \in W^{1,p}(\Omega)\), we have that
\[
\int_\Omega |\nabla u_\lambda^*|^{p^*-2}\nabla u_\lambda^* \cdot \nabla v \, dx + \lambda \int_\Omega |u_\lambda^*|^{p^*-2}u_\lambda^* \cdot v \, dx = \int_\Omega f \cdot v \, dx.
\]  (38)

Subtracting (38) from (37), choosing \(v := u^* - u_\lambda^*\in W^{1,p}(\Omega)\), we observe, using that \(u^* = 0\) on \(L^p(\partial \Omega)\) and \(-(\nabla u_\lambda^*)^2\nabla u_\lambda^* = |\nabla u_\lambda^*|^{p^*-2}u_\lambda^*\) \(\leq 0\) and (35), that
\[
\int_\Omega \left(|\nabla u|^{p^*-2}\nabla u^* - |\nabla u_\lambda^*|^{p^*-2}u_\lambda^*\right) \cdot (\nabla u^* - \nabla u_\lambda^*) \, dx = \int_\Omega \left(|\nabla u|^{p^*-2}\nabla u^* - |\nabla u_\lambda^*|^{p^*-2}u_\lambda^*\right) (u^* - u_\lambda^*) \, ds
\]
\[
\quad \leq - \int_\Omega |\nabla u|^{p^*-2}\nabla u^* \cdot n \, u_\lambda^* \, ds
\]
\[
\quad \leq c(p) \left| |\nabla u|^{p^*-2}u^* \cdot n \right|_{L^{p^*}(\partial \Omega)} \left( E(u^*) + \|f\|_{L^{p^*}(\Omega)}^p \right)^{\frac{1}{2}} L^{-\frac{1}{2}}.
\]  (39)

Thus, appealing to Lemma \(\mathcal{L}\), i.e., there exists a constant \(c(p) > 0\), depending only on \(d \in \mathbb{N}\) and \(p \in (1, \infty)\), with \((p \mapsto c(p)) \in C^0(1, \infty)\), such that
\[
c(p)^{-1} p^2(u^*, u_\lambda^*) \leq \int_\Omega \left(|\nabla u|^{p^*-2}\nabla u^* - |\nabla u_\lambda^*|^{p^*-2}u_\lambda^*\right) \cdot (\nabla u^* - \nabla u_\lambda^*) \, dx,
\]
we conclude from (39) that
\[
p^2(u^*, u_\lambda^*) \leq c(p) \left| |\nabla u|^{p^*-2}u^* \cdot n \right|_{L^{p^*}(\partial \Omega)} \left( E(u^*) + \|f\|_{L^{p^*}(\Omega)}^p \right)^{\frac{1}{2}} L^{-\frac{1}{2}}.
\]  (40)

**Step III.** Combining (30), (34) and (40), we obtain a constant \(c(p) > 0\), depending only on \(d \in \mathbb{N}\) and \(p \in (1, \infty)\), with \((p \mapsto c(p)) \in C^0(1, \infty)\), such that for every \(v \in M\) and \(\lambda \geq 1\), it holds
\[
p^2(v, u^*) + \|v\|_{L^{p^*}(\partial \Omega)}^p \leq 2\left( p^2(v, u_\lambda^*) + \|F(v) - F(u_\lambda^*)\|_{L^p(\partial \Omega)}^p + p^2(u^*, u_\lambda^*) + \|F(u_\lambda^*)\|_{L^p(\partial \Omega)}^p \right)
\]
\[
\leq 2\left( p^2(v, u_\lambda^*) + p^2(u^*, u_\lambda^*) + \|u_\lambda^*\|_{L^{p^*}(\partial \Omega)}^p \right)
\]
\[
\leq 2\left( \delta_\lambda + c(p) \left( \inf_{\partial \Omega} \left( \int_{\partial \Omega} |\nabla v_\lambda^*|^2 \, ds \right) + \lambda^{-\frac{p}{2}} \right) \right).
\]

VI. Parametric Problems

In this section, we generalize our results, in particular, Theorem \(\mathcal{T}\) to parametric problems. In principle, the procedure is quite analogous: We establish the existence of a minimizer of our parametric problem. This, again, is closely related to the validity of a corresponding parametric Poincaré inequality. Then, we deduce that the minimizer of our parametric problem for each fixed parameter is minimizer of the respective original \(p\)-Dirichlet problem and resort to Theorem \(\mathcal{T}\).

To start with, we examine a parametric problem with a varying exponent. Meaning that – in the simplest case – we are looking for a function \((c_\lambda, \lambda) \mapsto u^*(c_\lambda, \lambda)) : \mathcal{P} \times \Omega \to \mathbb{R}\) such that \(u^*(c_\lambda, \lambda)\) solves the \(p(c_\lambda, \cdot)\)-Dirichlet problem with exponent \(p(c_\lambda)\). The following proposition formalizes and generalizes this idea, allowing the exponent to be a function \(c : \mathcal{P} \to \mathbb{R}\). Treating a parametric problem of this form as a minimization problem requires non-standard function spaces.

**Proposition 16 (Variable Exponents).** Let \(\Omega \subseteq \mathbb{R}^d, d \in \mathbb{N}\), and \(\mathcal{P} \subseteq \mathbb{R}^N\), \(N \in \mathbb{N}\), be bounded domains and \(p \in L^\infty(\mathcal{P})\) such that there exist \(p^-, p^+ \in (1, \infty)\) with \(p^- \leq p(c) \leq p^+\) for a.e. \(c \in \mathcal{P}\). Moreover, we define the variable exponent Lebesgue space\(^9\)
\[
L^{p(c)}(\mathcal{P} \times \Omega) := \left\{ v \in L^0(\mathcal{P} \times \Omega) \middle| \int_\mathcal{P} \int_\Omega |v(c, \cdot)|^{p(c)} \, dx \, dc < \infty \right\},
\]
\(^9\)Here, \(L^0(\mathcal{P} \times \Omega)\) denotes the space of scalar (Lebesgue–)measurable functions on \(\mathcal{P} \times \Omega\).
and the variable exponent Bochner–Lebesgue space
\[ U := \{ v \in L^p(\mathcal{P} \times \Omega) \mid v(\rho, \cdot) \in W^{1,p(p)}_0(\Omega) \text{ for a.e. } \rho \in \mathcal{P}, |\nabla_x v| \in L^p(\mathcal{P} \times \Omega) \}, \]
where the gradient \( \nabla_x \) for a.e. \( \rho \in \mathcal{P} \) is to be understood with respect to the variable \( x \in \Omega \) only. For fixed \( f \in L^p(\mathcal{P} \times \Omega) \), i.e., \( f \in L^p(\mathcal{P} \times \Omega) \) and \( \int_{\mathcal{P}} \int_{\Omega} |f(\rho, x)|^{p(p)} \, dx \, d\rho < \infty \), where \( p' \in L^\infty(\mathcal{P}) \) is defined by \( p' := \frac{p}{p-1} \) for all \( p \in \mathcal{P} \), we define variable exponent \( p \)-Dirichlet energy \( \mathcal{E} : U \to \mathbb{R} \) for every \( v \in U \) by
\[ \mathcal{E}(v) := \int_{\mathcal{P}} \left[ \frac{1}{p(p)} \int_{\Omega} |\nabla_x v(\rho, \cdot)|^{p(p)} \, dx - \int_{\Omega} f(\rho, \cdot) v(\rho, \cdot) \, dx \right] \, d\rho. \]

Then, the following statements apply:

(i) There exists a unique (parametric) minimizer \( u^* \in U \) of \( \mathcal{E} : U \to \mathbb{R} \).

(ii) For a.e. \( \rho \in \mathcal{P} \), \( u^*(\rho, \cdot) \) is a unique minimizer of \( E_p : W^{1,p(p)}_0(\Omega) \to \mathbb{R} \), for every \( v \in W^{1,p(p)}_0(\Omega) \) defined by
\[ E_p(v) := \frac{1}{p(p)} \int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} f(\rho, \cdot) v \, dx. \]

(iii) For a.e. \( \rho \in \mathcal{P} \) and \( v \in W^{1,p(p)}_0(\Omega) \), it holds
\[ c(p(p))^{-1} \left\| F_p(v) - F_p(v)(\rho, \cdot) \right\|^2_{L^2(\Omega)^d} \leq E_p(v) - E_p(u^*), \]
where \( F_p : \mathbb{R}^d \to \mathbb{R}^d \), \( p \in \mathcal{P} \), for every \( p \in \mathcal{P} \) is defined by \( F_p(a) := |a|^p - a \) for all \( a \in \mathbb{R}^d \) and \( c(p(p)) > 0 \) is the constant from Theorem 8.

Remark 17. (i) For the variable exponent \( p \in L^\infty(\mathcal{P}) \), we actually have in mind the identity mapping, i.e., \( p(\rho) = p_1 \) for all \( \rho \in (p_1, \ldots, p_N)^\circ \). Since, however, Proposition 16 also applies for general \( p \in L^\infty(\mathcal{P}) \) such that there exist \( p^- \in (1, \infty) \), \( p \leq p^- \leq p \) for a.e. \( \rho \in \mathcal{P} \), we immediately consider this case, in order to keep potential future applications within the realm of possibility as well.

(ii) Since \( (p \mapsto c(p)) \in C^1(1, \infty) \) in Theorem 8 as well as \( p \in L^\infty(\mathcal{P}) \) in Proposition 16 from Proposition 16(iii), for every \( v \in U \), it follows that
\[ \inf_{p \in \mathcal{P}} c(p(p))^{-1} \rho_{p, p}^2(v, u^*) \leq \mathcal{E}(v) - \mathcal{E}(u^*) \leq \sup_{p \in \mathcal{P}} c(p(p)) \rho_{p, p}^2(v, u^*), \]
where \( \rho_{p, p}^2(v, u^*) := \int_{\mathcal{P}} \left\| F_p(v) - F_p(u^*) \right\|^2_{L^2(\Omega)^d} \, d\rho \) for all \( v \in U \).

Proof. (i). The space \( U \) equipped with the norm \( \| v \|_{U(p)} := \| v \|_{L^p(\mathcal{P} \times \Omega)} + \| \nabla_x v \|_{L^p(\mathcal{P} \times \Omega)} \), where
\[ \| v \|_{L^p(\mathcal{P} \times \Omega)} := \inf \left\{ \lambda > 0 \mid \int_{\mathcal{P}} \int_{\Omega} \frac{\| v(\rho, x) \|^p}{\lambda} \, dx \, d\rho \leq 1 \right\} \]
denotes the Luxemburg norm, cf. [Diebing et al. 2011], forms a reflexive Banach space, cf. [Kaltenbach 2021], Proposition 3.7 & Proposition 3.9 or [Kaltenbach and Růžička 2021], Proposition 3.6 & Proposition 3.7. Apparently, \( \mathcal{E} : U \to \mathbb{R} \) is strictly convex and continuous. In addition, for every \( v \in U \), due to Poincare's inequality applied for a.e. fixed \( \rho \in \mathcal{P} \), which is allowed since \( v(\rho, \cdot) \in W^{1,p(p)}_0(\Omega) \) for a.e. \( \rho \in \mathcal{P} \), we have that
\[ \int_{\mathcal{P}} \int_{\Omega} |v(\rho, x)|^{p(p)} \, dx \, d\rho \leq (1 + 2diam(\Omega)^p) \int_{\mathcal{P}} \int_{\Omega} |\nabla_x v(\rho, x)|^{p(p)} \, dx \, d\rho, \]
which for every \( v \in U \) and \( \varepsilon \in (0, \frac{1}{p^+}) \), using for a.e. \( \rho \in \mathcal{P} \), the \( \varepsilon \)-Young inequality with \( c(p(p), \varepsilon) := \frac{p(p)}{p(p) - 1} \), implies that
\[ \mathcal{E}(v) \geq \int_{\mathcal{P}} \frac{1}{p(p)} \int_{\Omega} |\nabla_x v(\rho, \cdot)|^{p(p)} \, dx \, d\rho - \int_{\mathcal{P}} \int_{\Omega} c(p(p), \varepsilon)|f(\rho, \cdot)|^{p(p)} - \varepsilon |v(\rho, \cdot)|^{p(p)} \, dx \, d\rho \]
\[ \geq \left( \frac{1}{p^+} - \varepsilon (1 + 2diam(\Omega)^p) \right) \int_{\mathcal{P}} \int_{\Omega} |\nabla_x v(\rho, \cdot)|^{p(p)} \, dx \, d\rho - \frac{(p^-)^{1-p^-}}{(p^+)^p} \int_{\mathcal{P}} \int_{\Omega} |f(\rho, \cdot)|^{p(p)} \, dx \, d\rho. \]

More precisely, these references prove only the case \( N = 1 \), since therein \( \mathcal{P} \) represents a time interval in an unsteady fluid flow problem. However, the proofs can be generalized verbatimly to the case \( N > 1 \), so that we will refrain from proving these results again at this point.
Hence, since \( \int_\Omega |v(p, \cdot)|^{p(p)} + |\nabla_v v(p, \cdot)|^{p(p)} \, dx \, dp \to \infty \) if \( \|v\|_U \to \infty \) (cf. (Diening et al., 2011, Lemma 3.2.4)) from (41) and (42) for \( \varepsilon \in (0, \frac{1}{2}] \) sufficiently small, we conclude that from \( \|v\|_U \to \infty \), i.e., \( \delta : U \to \mathbb{R} \) is weakly coercive, so that the direct method in the calculus of variations, cf. (Dacorogna, 2008), yields the existence of a unique minimizer \( u' \in U \) of \( \delta : U \to \mathbb{R} \).

ad (ii). A standard calculation shows that \( \delta : U \to \mathbb{R} \) is continuously Frechét differentiable with

\[
\langle D\delta(u), v \rangle_U = \int_\Omega \langle DE_p(u(p, \cdot)), v(p, \cdot) \rangle_{W_0^{1,p(p)}(\Omega)} \, dp
\]

for all \( u, v \in U \). Therefore, due to the minimality of \( u' \in U \), for every \( v \in U \), we necessarily have that

\[
0 = \langle D\delta(u'), v \rangle_U = \int_\Omega \langle DE_p(u'(p, \cdot)), v(p, \cdot) \rangle_{W_0^{1,p(p)}(\Omega)} \, dp.
\]  

(43)

Inasmuch as \( W_0^{1,p}((\Omega) \to W_0^{1,p(p)}(\Omega) \) densely for a.e. \( p \in \mathcal{P} \) and \( W_0^{1,p}((\Omega) \) is separable and, thus, contains a countable dense subset \( \langle \psi_k \rangle_{k \in \mathbb{N}} \subseteq W_0^{1,p}((\Omega)) \), the subset \( \langle \psi_k \rangle_{k \in \mathbb{N}} \) lies even densely in \( W_0^{1,p(p)}((\Omega) \) for a.e. \( p \in \mathcal{P} \).

Next, choosing \( v = \varphi \psi_k \in U \) in (43) for arbitrary \( \varphi \in C_0^\infty(\mathcal{P}) \) and \( k \in \mathbb{N} \), we further deduce that

\[
\int_\Omega \langle DE_p(u'(p, \cdot)), \psi_k \rangle_{W_0^{1,p(p)}(\Omega)} \varphi \, dp = 0,
\]  

(44)

so that for each fixed \( k \in \mathbb{N} \), the fundamental lemma of calculus of variations implies that for a.e. \( p \in \mathcal{P} \), it holds

\[
\langle DE_p(u'(p, \cdot)), \psi_k \rangle_{W_0^{1,p(p)}(\Omega)} = 0.
\]  

(45)

As \( \langle \psi_k \rangle_{k \in \mathbb{N}} \) is dense in \( W_0^{1,p(p)}((\Omega) \) for a.e. \( p \in \mathcal{P} \), from (45) we infer that for a.e. \( p \in \mathcal{P} \), it holds for all \( v \in W_0^{1,p(p)}((\Omega) \)

\[
\langle DE_p(u'(p, \cdot)), v \rangle_{W_0^{1,p(p)}(\Omega)} = 0.
\]

Eventually, since for a.e. \( p \in \mathcal{P} \), the \( p(p) \)-Dirichlet energy \( E_p : W_0^{1,p(p)}((\Omega) \to \mathbb{R} \) is strictly convex, for a.e. \( p \in \mathcal{P} \), the slice \( u'(p, \cdot) \in W_0^{1,p(p)}((\Omega) \) is a unique minimizer of \( E_p : W_0^{1,p(p)}((\Omega) \to \mathbb{R} \).

ad (iii). Follows from point (ii) and Theorem 8

\[\square\]

Remark 18. Proposition 16 also applies for the variable exponent Bochner–Lebesgue space

\[U := \{ v \in L^{p(\cdot)}(\mathcal{P} \times \Omega) \mid v(\cdot, \cdot) \in U(p), v(p, \cdot) \in L^{p(\cdot)}(\mathcal{P} \times \Omega) \},\]

where either \( U(p) := W_0^{1,p(p)}((\Omega) \) for \( \Gamma_D \subseteq \partial \Omega \) with \( \omega^{p(p-1)}(\Gamma_D) > 0 \) or \( U(p) := W_0^{1,p(p)}((\Omega) / \mathcal{P} \). In fact, analogous arguments as in (Kaltenbach, 2021, Proposition 3.7 & Proposition 3.9) show that \( U \) equipped with \( \| \cdot \|_U := \| \cdot \|_{L^{p(\cdot)}(\mathcal{P} \times \Omega)} + \| \nabla_v \cdot \|_{L^{p(\cdot)}(\mathcal{P} \times \Omega)} \) forms a reflexive Banach space for these choices and for a.e. \( p \in \mathcal{P} \), a Poincaré inequality with a constant which can be bounded independently of \( p \in \mathcal{P} \) applies. Then, the same arguments in Remark 11 show if \( f \in L^{p(\cdot)}(\mathcal{P} \times \Omega) \) satisfies \( \int_\Omega f(p, \cdot) \, dx = 0 \) for a.e. \( p \in \mathcal{P} \), then Proposition 16 also applies for the variable exponent Bochner–Lebesgue space

\[U := \{ v \in L^{p(\cdot)}(\mathcal{P} \times \Omega) \mid v(\cdot, \cdot) \in W_0^{1,p(p)}((\Omega) \) for a.e. \( p \in \mathcal{P}, v(p, \cdot) \in L^{p(\cdot)}(\mathcal{P} \times \Omega) \},\]

if we drop the uniqueness in point (i) in Proposition 16.

Next, we examine a parametric problem with a varying right hand side.

Corollary 19 (Variable Right-Hand Sides). Let \( \Omega \subseteq \mathbb{R}^d, d \in \mathbb{N}, \) and \( \mathcal{P} \subseteq \mathbb{R}^N, N \in \mathbb{N}, \) be bounded domains and \( p \in (1, \infty) \). Moreover, we define Bochner–Lebesgue space

\[U := L^p(\mathcal{P}, W_0^{1,p}(\Omega)).\]

For fixed \( f \in L^p(\mathcal{P} \times \Omega) \), we define the variable right-hand side \( p \)-Dirichlet energy \( \delta : U \to \mathbb{R} \) for every \( v \in U \) by

\[
\delta(v) := \int_\Omega \left[ \frac{1}{p} \int_\Omega |\nabla_v v(p, \cdot)|^p \, dx - \int_\Omega f(p, \cdot) v(p, \cdot) \, dx \right] \, dp.
\]

Then, the following statements apply:
(i) There exists a unique (parametric) minimizer $u^* \in \mathcal{U} \triangleq \Omega \rightarrow \mathbb{R}$.

(ii) For a.e. $p \in \mathcal{P}$, $u^*(p, \cdot) \in W_0^{1,p}(\Omega)$ is a unique minimizer of $E_p : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$, for every $v \in W_0^{1,p}(\Omega)$ defined by

$$E_p(v) \triangleq \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} f(p, \cdot) \, v \, dx.$$ 

(iii) For a.e. $p \in \mathcal{P}$ and $v \in W_0^{1,p}(\Omega)$, it holds

$$c(p)^{-1} \|F(v) - F(\nabla u^*(p, \cdot))\|^2_{L^p(\Omega)} \leq E_p(v) - E_p(u^*(p, \cdot)) \leq c(p) \|F(v) - F(\nabla u^*(p, \cdot))\|^2_{L^p(\Omega)}$$

where $c(p) > 0$ is the constant from Theorem $[8]$. 

Proof. Follows from Proposition $[16]$ for constant exponent $p(\cdot) = p \in L^\infty(\mathcal{P})$. 

To conclude this section, we examine a parametric problem with a varying domain.

**Proposition 20** (Variable Domains). Let $\Omega \subseteq \mathbb{R}^d$, a bounded Lipschitz domain and $p \in (1, \infty)$. Moreover, let $q_p : \Omega \rightarrow \Omega(p), p \in \mathcal{P} \triangleq (0, T), T > 0$, the induced flow of a smooth, compactly supported vector field $v : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, cf. [11]. For the non-cylindrical domain $Q := \bigcup_{p \in \mathcal{P}} (p) \times \Omega(p)$, we define the variable domain Bochner–Lebesgue space

$$\mathcal{U} := L^p(\mathcal{P}, W_0^{1,p}(\Omega(p))) \triangleq \{ u \in L^p(Q) \mid u(p, \cdot) \in W_0^{1,p}(\Omega(p)) \}$$

where the gradient $\nabla x$ for a.e. $p \in \mathcal{P}$ is to be understood with respect to the variable $x \in \Omega(p)$ only. For fixed $f \in L^p(Q)$, we define the variable domain $p$-Dirichlet energy $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$ for every $v \in \mathcal{U}$ by

$$\mathcal{E}(v) \triangleq \int_{\Omega} \left[ \frac{1}{p} \int_{\Omega(p)} |\nabla v|^p \, dx - \int_{\Omega(p)} f(p, \cdot) \, v \, dx \right] \, dp.$$ 

Then, the following statements apply:

(i) There exists a unique (parametric) minimizer $u^* \in \mathcal{U} \triangleq \Omega \rightarrow \mathbb{R}$.

(ii) For a.e. $p \in \mathcal{P}$, $u^*(p, \cdot) \in W_0^{1,p}(\Omega(p))$ is a unique minimizer of $E_p : W_0^{1,p}(\Omega(p)) \rightarrow \mathbb{R}$, for every $v \in W_0^{1,p}(\Omega(p))$ defined by

$$E_p(v) \triangleq \frac{1}{p} \int_{\Omega(p)} |\nabla v|^p \, dx - \int_{\Omega(p)} f(p, \cdot) \, v \, dx.$$ 

(iii) For a.e. $p \in \mathcal{P}$ and $v \in W_0^{1,p}(\Omega(p))$, it holds

$$c(p)^{-1} \|F(v) - F(\nabla u^*(p, \cdot))\|^2_{L^p(\Omega(p))} \leq E_p(v) - E_p(u^*(p, \cdot)) \leq c(p) \|F(v) - F(\nabla u^*(p, \cdot))\|^2_{L^p(\Omega(p))}$$

where $c(p) > 0$ is the constant from Theorem $[8]$. 

**Remark 21.** (i) For the induced flows $q_p : \Omega \rightarrow \Omega(p), p \in \mathcal{P} \triangleq (0, T), T < 0$, we actually have in mind the expansion mapping, i.e., $q_p(x) = p_1 x$ for all $x \in \Omega$ and $p = (p_1, \ldots, p_n)^T \in \mathcal{P}$, where $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, is star-shaped with respect to a ball containing the origin, e.g., $\Omega \subseteq B(0)$. Since, however, Proposition $20$ applies for general induced flows $q_p : \Omega \rightarrow \Omega(p), p \in \mathcal{P} \triangleq (0, T), T > 0$, we immediately consider this case, in order to keep potential future applications within the realm of possibility.

(ii) Since $(p \mapsto c(p)) \in C^0(1, \infty)$ in Theorem $[8]$ from Proposition $20$ (iii), for every $v \in \mathcal{U}$, it follows that

$$\text{ess inf}_{p \in \mathcal{P}} c(p)^{-1} \rho_{\text{TV}}^2(v, u^*) \leq \mathcal{E}(v) - \mathcal{E}(u^*) \leq \text{ess sup}_{p \in \mathcal{P}} c(p) \rho_{\text{TV}}^2(v, u^*),$$

where $\rho_{\text{TV}}^2(v, u^*) \triangleq \int_{\mathcal{P}} \|F(v(x), \cdot) - F(v(x), u^*(\cdot))\|^2_{L^p(\Omega(p))} \, dp$ for all $v \in \mathcal{U}$.

**Proof.** (i). The space $\mathcal{U}$ equipped with the norm $\|u\|_{\mathcal{U}} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$ forms a reflexive Banachspace, cf. [15], Proposition 3.17 & Corollary 3.25 or [17], Nägele and Křížek (2018). Apparently, $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$ is strictly convex and continuous. Apart from that, for every $v \in \mathcal{U}$, due to Poincaré's inequality applied for each fixed $p \in \mathcal{P}$, which is allowed since $v(p, \cdot) \in W_0^{1,p}(\Omega(p))$ for all $p \in \mathcal{P}$, we have that

$$\int_{\mathcal{P}} \int_{\Omega(p)} |v(p, x)|^p \, dx \, dp \leq \int_{\mathcal{P}} (2\text{diam}(\Omega(p)))^p \int_{\Omega(p)} |\nabla v(p, x)|^p \, dx \, dp$$

$$\leq (1 + 2\sup_{p \in \mathcal{P}} \text{diam}(\Omega(p)))^p \int_{\mathcal{P}} \int_{\Omega(p)} |\nabla v(p, x)|^p \, dx \, dp,$$ (46)
which for any \( v \in \mathcal{U} \) and \( \varepsilon \in (0, 1] \), using for each \( p \in \mathcal{P} \), the \( \varepsilon \)-Young inequality with constant \( c(p, \varepsilon) := \frac{[\varepsilon^{1-p}]}{p'} \), implies that
\[
\delta(v) \geq \frac{1}{p} \int_{\mathcal{P}} \int_{\Omega(p)} |\nabla v(p, \cdot)|^p \, dx \, dp - \int_{\mathcal{P}} \int_{\Omega(p)} c(p, \varepsilon)|f(p, \cdot)|^{p'} \, dx \, dp
\geq \left( \frac{1}{p} - \varepsilon \frac{1}{p} \right) \frac{1}{p} \int_{\mathcal{P}} \int_{\Omega(p)} |\nabla v(p, \cdot)|^p \, dx \, dp - \frac{(pc)^{-p'}}{p'} \int_{\mathcal{P}} \int_{\Omega(p)} |f(p, \cdot)|^{p'} \, dx \, dp.
\]

(47)

From (46) and (47) for \( \varepsilon > 0 \) sufficiently small, using that, by assumption, \( \sup_{p \in \mathcal{P}} \text{diam}(\Omega(p)) < \infty \), we conclude that from \( \|v\|_{\mathcal{U}} \to \infty \), it follows that \( \delta(v) \to \infty \), i.e., \( \delta : \mathcal{U} \to \mathbb{R} \) is weakly coercive, so that the direct method in the calculus of variations, cf. [Dacorogna 2008], yields the existence of a unique minimizer \( u^* \in \mathcal{U} \) of \( \delta : \mathcal{U} \to \mathbb{R} \).

ad (ii). A direct calculation shows that \( \delta : \mathcal{U} \to \mathbb{R} \) is continuously Fréchet differentiable with
\[
\langle D\delta(u), v \rangle_{\mathcal{U}} = \int_{\mathcal{P}} \langle D\mathcal{F}(u(p, \cdot)), v(p, \cdot) \rangle_{W^{1,p}_0(\Omega(p))} \, dp
\]
for all \( u, v \in \mathcal{U} \). Therefore, due to the minimality of \( u^* \in \mathcal{U} \), for every \( v \in \mathcal{U} \), we necessarily have that
\[
0 = \langle D\delta(u^*), v \rangle_{\mathcal{U}} = \int_{\mathcal{P}} \langle D\mathcal{F}(u^*(p, \cdot)), v(p, \cdot) \rangle_{W^{1,p}_0(\Omega(p))} \, dp.
\]

(48)

Since \( W^{1,p}_0(\Omega(0)) \) is separable, there exists a countable dense subset \( (\psi_k)_{k \in \mathbb{N}} \subseteq W^{1,p}_0(\Omega(0)) \). Apart from that, appealing to [Nägele 2015, Lemma 2.1], for any \( p \in \mathcal{P} \), the pull-backs \( (\psi_k^{-1})_k \in \mathcal{P} := (\psi_k \circ \Phi_k^{-1})_{k \in \mathbb{N}} \subseteq W^{1,p}_0(\Omega(p)) \), are dense in \( W^{1,p}_0(\Omega(p)) \). In addition, [Nägele et al. 2017, p. 6 ff.] shows that \( (\psi_k)_{k \in \mathbb{N}} \) is dense in \( \mathcal{U} \). Next, choosing \( v = \Phi_k \psi_k \in \mathcal{U} \) in (48) for arbitrary \( \Phi_k \in C_0^\infty(\mathcal{P}) \) and \( k \in \mathbb{N} \), we further deduce that
\[
\int_{\mathcal{P}} \langle D\mathcal{F}(u^*(p, \cdot)), \psi_k(p) \rangle_{W^{1,p}_0(\Omega(p))} \, dp = 0,
\]
so that, owing to the countability of \( (\psi_k)_{k \in \mathbb{N}} \subseteq \mathcal{U} \), the fundamental lemma of calculus of variations implies that for each \( p \in \mathcal{P} \), it holds for all \( k \in \mathbb{N} \)
\[
\langle D\mathcal{F}(u^*(p, \cdot)), (\psi_k^{-1})_k \rangle_{W^{1,p}_0(\Omega)} = 0.
\]

As \( (\psi_k^{-1})_k \) is dense in \( W^{1,p}_0(\Omega(p)) \) for all \( p \in \mathcal{P} \), we find that for each \( p \in \mathcal{P} \), it holds for all \( v \in W^{1,p}_0(\Omega(p)) \)
\[
\langle D\mathcal{F}(u^*(p, \cdot)), v \rangle_{W^{1,p}_0(\Omega)} = 0.
\]

Eventually, since for each \( p \in \mathcal{P} \), the \( p \)-Dirichlet energy \( E_p : W^{1,p}_0(\Omega(p)) \to \mathbb{R} \) is strictly convex, for each \( p \in \mathcal{P} \), the slice \( u^*(p, \cdot) \in W^{1,p}_0(\Omega(p)) \) is a unique minimizer of \( E_p : W^{1,p}_0(\Omega(p)) \to \mathbb{R} \).

ad (iii). Follows from point (ii) and Theorem 8.

□

Remark 22. Proposition 20 also applies for the variable domain Bochner–Lebesgue space
\[
\mathcal{U} := L^p(\mathcal{P}, \mathcal{U}(\cdot)) := \left\{ v \in L^p(Q) \mid v(p, \cdot) \in \mathcal{U}(p) \text{ for a.e. } p \in \mathcal{P}, \|\nabla v\|_{L^p} \leq \|v\|_{\mathcal{U}} \right\},
\]
where either \( \mathcal{U}(p) := W^{1,p}_0(\Omega(p)) \) for \( \Gamma_D \subseteq \partial \Omega \) with \( H^{-1}(|\Gamma_D|) > 0 \) or \( \mathcal{U}(p) := W^{1,p}(\Omega(p)) \). In fact, analogous arguments as in [Nägele 2015, Proposition 3.17 & Corollary 3.25] show that \( \mathcal{U} \) equipped with \( \|\cdot\|_{\mathcal{U}} := \|\cdot\|_{L^p} + \|\nabla \cdot\|_{L^p} \) forms a reflexive Banach space for these choices and for each \( p \in \mathcal{P} \), a Poincaré inequality that can be bounded independently of \( p \in \mathcal{P} \) applies. Then, the same arguments as in Remark 11 show if \( f \in L^p(\mathcal{U}) \) satisfies \( \int_{\mathcal{U}} f(p, \cdot) \, dx = 0 \) for a.e. \( p \in \mathcal{P} \), then Proposition 16 also applies for the variable domain Bochner–Lebesgue space
\[
\mathcal{U} := L^p(\mathcal{P}, W^{1,p}_0(\Omega(\cdot))) := \left\{ v \in L^p(Q) \mid v(p, \cdot) \in W^{1,p}_0(\Omega(p)) \text{ for a.e. } p \in \mathcal{P}, \|\nabla v\|_{L^p} \leq \|v\|_{\mathcal{U}} \right\},
\]
if we drop the uniqueness in point (i) in Proposition 20.

\footnote{Here, we exploit that there exists \( K > 0 \) such that \( K^{-1} \leq \det(D \Phi(p)) \leq K \) in \( \Omega(p) \) for all \( p \in \mathcal{P} \), cf. [Nägele et al. 2017 (3.1)].}
such that for any \( \theta \)
Assume that \( u^* \) minimizes the \( p \)-Dirichlet energy \( E : U \rightarrow \mathbb{R} \) over the closed subspace \( U \subseteq W^{1,p}(\Omega) \) and \( M \subseteq U \) is an arbitrary subset. Further, \( \rho_p^2 : U \times U \rightarrow \mathbb{R} \), again, denotes the natural distance, \( c(p) > 0 \) is a constant depending (continuously) on \( p \in (1, \infty) \) and \( d \in \mathbb{N} \), and \( \delta := \delta(v) := E(v) - \inf_{\varphi \in M} E(\varphi) \) quantifies the energy mismatch between \( v \in M \) an the energy minimum over \( M \). Note that for parametric problems considered in Section VII, we derived similar estimates, adapting the choice of \( \rho_p^2 : U \times U \rightarrow \mathbb{R} \) and the space \( U \), cf. Proposition 16 and Remark 17 as well as Proposition 20 and Remark 21.

To derive error decay rates from equation (49), we need to estimate the term involving the infimum. Note that, with respect to the natural distance \( \rho_p^2 \), the error decay rate equals the approximation rate with respect to \( \rho_p^2 \) for functions in \( U \). However, in the context of neural networks, the natural distance \( \rho_p^2 \) has not yet been studied from an approximation theoretic viewpoint. Therefore, we require its relation to Sobolev topologies, where approximation results are known, cf. Theorem 3.

**Lemma 23** (Relation between natural distance and \( W^{1,p} \)-semi norm). Let \( \Omega \subseteq \mathbb{R}^d \), \( d \in \mathbb{N} \), be a bounded domain and \( p \in (1, \infty) \). Then, there exists a constant \( c(p) > 0 \), depending only on \( d \in \mathbb{N} \) and \( p \in (1, \infty) \), such that the following relations apply:

\[
\begin{align*}
(i) & \quad \text{if } p \in [2, \infty), \text{ then for every } u, v \in W^{1,p}(\Omega), \text{ it holds} \quad c(p)^{-1} \| u - v \|_{L^p(\Omega)}^p \leq \rho_p^2(u, v) \leq c(p) \left( \| u \|_{L^p(\Omega)}^p + \| v \|_{L^p(\Omega)}^p \right)^{p-2} \| u - v \|_{L^p(\Omega)}^2. \\
(ii) & \quad \text{if } p \in (1, 2), \text{ then for every } u, v \in W^{1,p}(\Omega), \text{ it holds} \quad c(p)^{-1} \rho_p^2(u, v) \leq \| u - v \|_{L^p(\Omega)}^p \leq c(p) \left( \| u \|_{L^p(\Omega)}^p + \| v \|_{L^p(\Omega)}^p \right)^{p-2} \rho_p^2(u, v)^{\frac{p}{p-2}}.
\end{align*}
\]

In particular, we have that \( (p \mapsto c(p)) \in C^0(1, \infty) \).

**Proof.** The proof of this Lemma is deferred to the end of the section. \( \square \)

We are now in the position to derive error decay rates. As a first result, we consider a pure Neumann problem without parametric dependencies. We use a Neumann problem as this corresponds to an unconstrained minimization problem over the space \( W^{1,p}(\Omega) \) and this simplifies the derivation of error decay rates. However, pure Dirichlet boundary conditions via penalization can also be considered using Theorem 15.

**Theorem 24.** Let \( f \in W^{1,p}(\Omega)^* \), \( p \in (1, \infty) \), be such that \( \langle f, c \rangle_{W^{1,p}(\Omega)} = 0 \) for all \( c \in \mathbb{R} \). Moreover, let \( u^* \in W^{1,p}(\Omega) \) a weak solution of the \( p \)-Laplace problem with homogeneous Neumann boundary conditions, i.e., \( u^* \in W^{1,p}(\Omega) \) is minimal for \( E : W^{1,p}(\Omega) \rightarrow \mathbb{R} \), for every \( v \in W^{1,p}(\Omega) \) defined by

\[
E(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \langle f, v \rangle_{W^{1,p}(\Omega)}.
\]

Assume that \( u^* \in W^{k,p}(\Omega) \) for some \( k > 1 \). Then, for every \( n \in \mathbb{N} \), there exists a parameter space \( \Theta_n \) of dimension \( \Theta(n) \) such that for any \( \theta \in \Theta_n \), the corresponding fully connected \( \text{ReLU}^2 \)-network \( u_\theta \in W^{1,p}(\Omega) \) satisfies

\[
\| \nabla u_\theta - \nabla u^* \|_{L^p(\Omega)} \leq c(p) \cdot \begin{cases} \delta_n^{\frac{1}{p}} + \| u^* \|_{W^{1,p}(\Omega)} \left( \frac{1}{n} \right)^{\frac{1}{p-2}} & \text{if } p \in [2, \infty), \\
\delta_n^{\frac{1}{p}} + \| u^* \|_{W^{1,p}(\Omega)} \left( \frac{1}{n} \right)^{\frac{1}{p-2}} & \text{if } p \in (1, 2), \end{cases}
\]

where \( \delta_n := \delta_n(u_\theta) := E(u_\theta) - \inf_{\varphi \in \Theta_n} E(u_\varphi) \) is the optimization error and \( c(p) > 0 \) depends only on \( p \in (1, \infty) \) and \( d \in \mathbb{N} \).
Remark 25 (Implications to High-Dimensional Problems). In the above result we are interested in the error decay rates, especially with respect to the spatial dimension $d \in \mathbb{N}$. Ignoring constants and the contribution $\delta_n$ of inaccurate optimization, we obtain the rates $2/p \cdot (k-1)/d$ and $2/(k-1)/d$ for $p \geq 2$ and $p \leq 2$, respectively. This shows that, up to the factors $2/p$ or $p/2$, the error decay rate is the same as the approximation rate. Thus, the favorable approximation capabilities of neural networks for high dimensional smooth functions are retained by the Deep Ritz Method for $p$-Dirichlet problems.

Remark 26 (Comparison to Finite Element Methods). It is possible to approximate $W^{k,p} (\Omega)$ functions by finite element ansatz functions with the rate $(k-1)/d$. Following the proof of Theorem 24 this yields the same error decay rates as a neural network ansatz class. However, this requires finite element ansatz classes of polynomial degree $k-1$, cf. [Ern and Guermond (2004)]. Using neural networks, one ansatz class realizes the convergence rates of finite element ansatz spaces of arbitrary high order.

Proof. Let $p \in [2, \infty)$. If $p \in [2, \infty)$, then we estimate using the relation of the natural distance to Sobolev norms (cf. Lemma 23, Céa’s Lemma 12) and the Quantitative Universal Approximation Theorem (cf. Theorem 3) that

$$c(p)^{-1} \| \nabla u_0 - \nabla u^* \|_{L^p(\Omega')}^p \leq \rho^2_p(u_0, u^*) \leq c(p) \left( \delta_n + \inf_{\psi \in \Theta_n} \rho^2_p(\psi, u^*) \right)$$

$$\leq c(p) \left( \delta_n + \inf_{\psi \in \Theta_n} \left( \| \nabla \psi \|_{L^p(\Omega')}^p + \| \nabla u^* \|_{L^p(\Omega')}^p \right) \right) \left( \| \nabla \psi \|_{L^p(\Omega')}^2 + \| \nabla u^* \|_{L^p(\Omega')}^2 \right)^{-\frac{1}{2}}$$

$$\leq c(p) \left( \| \nabla u_0 \|_{L^p(\Omega')}^2 + \| \nabla u^* \|_{L^p(\Omega')}^2 \right)^{\frac{1}{2}} \left( \delta_n + \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \right)^{\frac{1}{2}},$$

where $u_0 \in W^{k,p}(\Omega)$ is the ReLU2-network from Theorem 3 which satisfies $\| \nabla u_0 \|_{L^p(\Omega')} \leq c(p) \| \nabla u^* \|_{W^{1,p}(\Omega)}$. Let $p \in (1, 2]$. If $p \in (1, 2]$, then, again, using the relation of the natural distance to Sobolev norms (cf. Lemma 23) and Céa’s Lemma 12 we obtain

$$\| \nabla u_0 - \nabla u^* \|_{L^p(\Omega')} \leq c(p) \left( \| \nabla u_0 \|_{L^p(\Omega')} + \| \nabla u^* \|_{L^p(\Omega')} \right) \left( \delta_n + \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \right).$$

Hence, it remains to estimate the first factor in (51). Using that $f \in W^{1,p}(\Omega')$ vanishes on constant functions, the Poincaré–Wirtinger inequality and the $\epsilon$-Young inequality, for every $v \in W^{1,p}(\Omega)$ and $\epsilon > 0$, it holds

$$E(v) = \frac{1}{p} \| \nabla v \|_{L^p(\Omega')}^p + \left( \epsilon^{\frac{1}{p}} - \int_\Omega v \, dx \right)_{W^{1,p}(\Omega)}^p$$

$$\geq \frac{1}{p} \| \nabla v \|_{L^p(\Omega')}^p - c_p(\epsilon) \| f \|_{W^{1,p}(\Omega')}^p - \epsilon \left( \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \right)$$

$$\geq \left( \frac{1}{p} - \epsilon c_p \right) \| \nabla v \|_{L^p(\Omega')}^p - c_p(\epsilon) \| f \|_{W^{1,p}(\Omega')}^p,$$

where $c(p, \epsilon) := \frac{(p-1)\epsilon}{p}$. Hence, choosing $\epsilon > 0$ sufficiently small in (52), for every $v \in W^{1,p}(\Omega)$, we find that

$$\| \nabla v \|_{L^p(\Omega')} \leq c(p) \left( E(v) + \| f \|_{W^{1,p}(\Omega')}^p \right)^{\frac{1}{2}}.$$

Using that $\inf_{\psi \in \Theta_n} E(\psi) \geq 0$, which follows from the fact that $u_{\psi} = 0$ for $\psi = 0 \in \Theta_n$, and $E(u^*) = 0$, this implies that

$$\| \nabla u_0 \|_{L^p(\Omega')} \leq c(p) \left( E(u_0) + \| f \|_{W^{1,p}(\Omega')}^p \right)^{\frac{1}{2}} \leq c(p) \left( \delta_n \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Employing again (53) and $E(u^*) \leq E(0) = 0$, we get $\| \nabla u^* \|_{L^p(\Omega')} \leq c(p) \| f \|_{W^{1,p}(\Omega')}^{\frac{1}{2}}$, and, consequently, using (54),

$$\left( \| \nabla u_0 \|_{L^p(\Omega')} + \| \nabla u^* \|_{L^p(\Omega')} \right)^{\frac{2}{p}} \leq c(p) \left( \delta_n \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \right)^{\frac{1}{p}} \leq c(p) \left( \delta_n \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \right)^{\frac{1}{p}}.$$

Since $\delta_n \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \leq 2 \delta_n$, assuming $\delta_n \leq 1$, it holds $\delta_n \inf_{\psi \in \Theta_n} \| \nabla \psi - u^* \|_{W^{1,p}(\Omega)}^2 \leq 2 \delta_n^2$, which provides the missing estimate to establish the assertion. □
Theorem 27. Let $p \in L^{\infty}(\mathcal{P})$ be such that $2 \leq p^- \leq p(p) \leq p^+ < \infty$ for a.e. $p \in \mathcal{P}$ and let $f \in L^{p^+}(\mathcal{P} \times \Omega)$ be such that $\int_{\Omega \times \mathcal{P}} f(p, \cdot) \, dx = 0$ for a.e. $p \in \mathcal{P}$, where $\mathcal{P} \subseteq \mathbb{R}^{d_\mathcal{P}}, d_\mathcal{P} \in \mathbb{N}$, is a parameter space and $\Omega \subseteq \mathbb{R}^{d_\Omega}, d_\Omega \in \mathbb{N}$, the physical domain. Moreover, let $u^* \in \mathcal{U} := \{ p \in L^{p^+}(\mathcal{P} \times \Omega) \mid \| \psi(p, \cdot) \|_{L^{p^+}(\mathcal{P} \times \Omega)} \}$ for a.e. $p \in \mathcal{P}, \| \nabla \psi \|_{L^{p^+}(\mathcal{P} \times \Omega)}$ be a weak solution of the parametric metric Laplace problem with homogeneous Neumann boundary conditions and right-hand side $f$, i.e., $u^* \in \mathcal{U}$ is minimal for $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$, for every $v \in \mathcal{U}$ defined by

$$
\mathcal{E}(v) := \int_{\mathcal{P}} \left( -\frac{1}{p(p)} \int_{\Omega} |\nabla v(p, x)|^{p(p)} \, dx + \int_{\Omega} f(p, x) v(p, x) \, dx \right) \, dp.
$$

Assume that $u^* \in W^{k,p^+}(\mathcal{P} \times \Omega)$ for some $k > 1$. Then, for every $n \in \mathbb{N}$, there exists a parameter space $\Theta_n$ of dimension $\Theta(n)$ such that for any $\Theta \in \Theta_n$, the corresponding fully-connected ReLU-network $u_\Theta \in W^{1,p^+}(\mathcal{P} \times \Omega)$ satisfies

$$
\int_{\mathcal{P} \times \Omega} |\nabla u_\Theta(p, \cdot) - \nabla u^*(p, \cdot)|^{p^+} \, dp \leq c(p) \left( \delta_n + \| u^* \|_{W^{k,p^+}(\mathcal{P} \times \Omega)} \left( \frac{1}{n} \right)^{\frac{2(k-1)}{k+p^+}} \right),
$$

where $\delta_n := \delta_n(u_\Theta) := \mathcal{E}(u_\Theta) - \inf_{\Theta \in \Theta_n} \mathcal{E}(u_\Theta)$ is the optimization error and $c(p) > 0$ only depends on $p^-, p^+ \in [2, \infty)$ and $d_\Omega \in \mathbb{N}$.

Proof. Similarly to the proof of Theorem 24, resorting to the relation of the natural distance to Sobolev norms (cf. Lemma 23) first for a.e. $p = p(p)$ and then for $p = p^+$, the Céa’s type lemma for parametric variable exponents (cf. Remark 17 (ii) & Remark 18) and the embedding $L^{p^+}(\Omega) \hookrightarrow L^{p^+}(\mathcal{P} \times \Omega)$ with constant $2(1 + |\Omega|)$ (cf. Diening et al. [2011], Corollary 3.3.4)) valid for a.e. $p \in \mathcal{P}$, we find that

$$
\int_{\mathcal{P} \times \Omega} |\nabla u_\Theta(p, \cdot) - \nabla u^*(p, \cdot)|^{p^+} \, dp \leq c(p) \left( \delta_n + \inf_{\psi \in \Theta_n} \left\{ \int_{\mathcal{P}} \left( \| \nabla u_\Theta(p, \cdot) \|_{L^{p^+}(\mathcal{P} \times \Omega)}^2 + \| \nabla u^*(p, \cdot) \|_{L^{p^+}(\mathcal{P} \times \Omega)}^2 \right) \, dp \right\} \right),
$$

where $u_\Theta \in W^{1,p^+}(\mathcal{P} \times \Omega)$ is the ReLU-network from Theorem 3 satisfying $\| \nabla u_\Theta \|_{L^{p^+}(\mathcal{P} \times \Omega)} \leq c(\Theta) \| u^* \|_{W^{k,p^+}(\mathcal{P} \times \Omega)}^{p^+ - 2}$ and $c(\Theta) > 0$ a constant which depends only on $p, p^+ \in [2, \infty)$, and $d_\Omega \in \mathbb{N}$.

Theorem 28. Let $p \in [2, \infty), \Omega, \mathcal{P} : \Omega \rightarrow \mathcal{P}(\mathcal{P}, p, \rho \in \mathcal{P}, an\ induced\ flow\ and\ f \in L^{p^+}(\Omega)$, where $\Omega := \bigcup_{p \in \mathcal{P}} \{ p \} \times \Omega(p)$, be such that $\int_{\Omega} f(p, \cdot) \, dx = 0$ for a.e. $p \in \mathcal{P}$, where $\mathcal{P} \subseteq \mathbb{R}^{d_\mathcal{P}}, d_\mathcal{P} \in \mathbb{N}$, is a parameter space and $\Omega \subseteq \mathbb{R}^{d_\Omega}, d_\Omega \in \mathbb{N}$, is the physical domain. Moreover, let $u^* \in L^{p^+}(\mathcal{P}, W^{1,p^+}(\Omega))$ be a weak solution of the parametric $p$-Laplace problem with homogeneous Neumann boundary conditions and right-hand side $f$, i.e., $u^* \in L^{p^+}(\mathcal{P}, W^{1,p^+}(\Omega))$ is minimal for $\mathcal{E} : L^{p^+}(\mathcal{P}, W^{1,p^+}(\Omega)) \rightarrow \mathbb{R}$, for every $v \in L^{p^+}(\mathcal{P}, W^{1,p^+}(\Omega))$ defined by

$$
\mathcal{E}(v) := \int_{\mathcal{P}} \left( -\frac{1}{p(p)} \int_{\Omega(p)} |\nabla v(p, x)|^{p(p)} \, dx + \int_{\Omega(p)} f(p, x) v(p, x) \, dx \right) \, dp.
$$

Assume that $u^* \in W^{k,p^+}(\Omega)$ for some $k > 1$. Then, for every $n \in \mathbb{N}$, there exists a parameter space $\Theta_n$ of dimension $\Theta(n)$ such that for any $\Theta \in \Theta_n$, the corresponding fully-connected ReLU-network $u_\Theta \in W^{1,p^+}(\Omega)$ satisfies

$$
\| \nabla u_\Theta - \nabla u^* \|_{L^{p^+}(\Omega)} \leq c(p) \left( \delta_n + \| u^* \|_{W^{k,p^+}(\Omega)} \left( \frac{1}{n} \right)^{\frac{2(k-1)}{k+p^+}} \right),
$$

where $\delta_n := \delta_n(u_\Theta) := \mathcal{E}(u_\Theta) - \inf_{\Theta \in \Theta_n} \mathcal{E}(u_\Theta)$ is the optimization error and $c(p) > 0$ only depends on $p \in [2, \infty)$ and $d_\Omega \in \mathbb{N}$.
Proof. Similarly to the proof of Theorem 24, resorting to the relation of the natural distance to Sobolev norms (cf. Lemma 23 for a.e. \( p \in \mathcal{B} \) and the Céa’s type lemma for parametric variable exponents (cf. Remark 17 (ii) & Remark 18), we find that

\[
\int_Q |\nabla_x u_0(p, \cdot) - \nabla_x u'(p, \cdot)|^p \, d\rho \leq c(p) \left( \int_{\mathcal{B}} \|F(\nabla_x u_0(p, \cdot)) - F(\nabla_x u'(p, \cdot))\|_{L^2(Q(p))}^p \, d\rho \right)
\]

\[
\leq c(p) \left( \delta_n + \inf_{\psi \in \mathcal{E}_n} \left( \int_{Q} \left( \|\nabla u_0(p, \cdot)\|_{L^2(Q(p))}^p + \|\nabla u'(p, \cdot)\|_{L^2(Q(p))}^p \right) \cdot \left( \|\nabla u_0(p, \cdot) - \nabla u'(p, \cdot)\|_{L^2(Q(p))}^p \right) \, d\rho \right)
\]

\[
\leq c(p) \left( \delta_n + \inf_{\psi \in \mathcal{E}_n} \left( \|\nabla u_0\|_{W^{2,p}(Q)}^p + \|\nabla u'\|_{L^p(Q')}^p \right) \cdot \left( \|\nabla u_0 - \nabla u'\|_{L^2(Q')}^p \right) \right)
\]

\[
\leq c(p) \left( \delta_n + \left( \|\nabla u_0\|_{L^p(Q)}^p + \|\nabla u'\|_{L^p(Q')}^p \right) \cdot \left( \|u_0 - u'\|_{W^{1,p}(Q')}^p \right) \right)
\]

\[
\leq c(p) \left( \delta_n + \|u_0\|_{W^{2,p}(Q \times \Omega)}^p \left( \frac{1}{n} \right)^{\frac{2(p-1)}{p}} \right),
\]

where \( u_0 \in W^{1,p}(Q) \) is the ReLU^2-network from Theorem 3 satisfying \( \|\nabla_x u_n\|_{L^p(Q')} \leq c(p) \|\nabla_x u'\|_{W^{1,p}(Q')}^p \) and \( c(p) > 0 \), a constant which depend only on \( p \in [2, \infty) \) and \( d \in \mathbb{N} \). \( \square \)

Proof of Lemma 23 The following proof is inspired by (Nakov and Toulopoulos, 2021, Section 3.1).

**ad (i)** By referring to Lemma 3 (ii), we deduce the existence of a constant \( c(p) > 0 \), depending only on \( d \in \mathbb{N} \) and \( p \in (1, \infty) \), with \( (p \mapsto c(p)) \in C^0(1, \infty) \), such that for every \( u, v \in W^{1,p}(\Omega) \), it holds

\[
\|\nabla u - \nabla v\|_{L^p(\Omega)}^p \leq \int_\Omega |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} \, dx \leq c(p) \rho^2(u, v),
\]

and, using Hölder’s inequality with respect to \( \left( \frac{2}{p}, \frac{p}{p-2} \right) \),

\[
c(p)^{-1} \rho^2(u, v) \leq \int_\Omega |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} \, dx
\]

\[
\leq \left( \int_\Omega |\nabla u - \nabla v|^p \, dx \right)^{\frac{2}{p}} \left( \int_\Omega (|\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{p-2}{p}}
\]

\[
\leq \left( \|\nabla u\|_{L^p(\Omega')}^p + \|\nabla v\|_{L^p(\Omega')}^p \right)^{p-2} \|\nabla u - \nabla v\|_{L^p(\Omega')}^2.
\]

**ad (ii)** By referring to Lemma 3 (ii), we deduce the existence of a constant \( c(p) > 0 \), depending only on \( d \in \mathbb{N} \) and \( p \in (1, \infty) \), with \( (p \mapsto c(p)) \in C^0(1, \infty) \), such that for every \( u, v \in W^{1,p}(\Omega) \), using Hölder’s inequality with respect to \( \left( \frac{2}{p}, \frac{2(p-1)}{p} \right) \), it holds

\[
\|\nabla u - \nabla v\|_{L^p(\Omega')}^p \leq \left( \int_\Omega |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} \, dx \right)^{\frac{p}{2}} \left( \int_\Omega (|\nabla u| + |\nabla v|)^p \, dx \right)^{\frac{p-2}{2}}
\]

\[
\leq \left( \|\nabla u\|_{L^p(\Omega')}^p + \|\nabla v\|_{L^p(\Omega')}^p \right)^{(p-2)p/2} \left( \int_\Omega |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} \, dx \right)^{\frac{p-2}{2}}
\]

\[
\leq c(p) \left( \|\nabla u\|_{L^p(\Omega')} + \|\nabla v\|_{L^p(\Omega')} \right)^{p-2} \rho^2(u, v),
\]

and

\[
c(p)^{-1} \rho^2(u, v) \leq \int_\Omega |\nabla u - \nabla v|^2 (|\nabla u| + |\nabla v|)^{p-2} \, dx
\]

\[
\leq \int_\Omega |\nabla u - \nabla v|^p \frac{|\nabla u - \nabla v|^{2-p}}{|\nabla u| + |\nabla v|^{2-p}} \, dx \leq \|\nabla u - \nabla v\|_{L^p(\Omega')}^p.
\]

\( \square \)
In this section, we present numerical examples of parametric $p$-Dirichlet problems and comment on the practical aspects of the method. To resolve problems of the form \( (4) \) in practice, one needs to choose an ansatz class, an optimization algorithm and a quadrature rule.

**Optimization** In principle, every algorithm to solve unconstrained minimization problems can be used to solve \( (4) \). We use a combination of Adam and L-BFGS. The former is a gradient descent method with adaptive moment estimation (cf. Kingma and Ba (2014)). The latter is a quasi-Newton method (cf. Liu and Nocedal (1989)), which we employ in the later stages of the optimization for its fast local convergence properties.

**Quadrature** In practice, the integrals appearing in \( (4) \) need to be approximated. For lower dimensions \((\leq 2)\), we employ a fine grid of the form \( \prod_{i=1}^{d} \varepsilon_{i} \mathbb{Z}, \varepsilon_{i} > 0, i \in \{1, d\}, d = 1, 2, \) and compute the integrals weighting all points in the grid by the reciprocal of the amount of grid points in the domain \( \Omega \) or \( \mathcal{P} \times \Omega \), respectively. Here, the number of integration points is chosen such that no further improvement can be observed upon refining. We found that this lies well within reasonable computational complexity. For three or more dimensions, we resort to a combination of random integration points that are re-sampled every few iterations, e.g., for the parameter space \( \mathcal{P} \), and a fine grid of the form \( \prod_{i=1}^{d} \varepsilon_{i} \mathbb{Z}, \varepsilon_{i} > 0, i \in \{1, \ldots, d\}, d = 1, 2, \) e.g., for the spatial domain \( \Omega \). In doing so, we deliberately select a coarser grid with respect to the parameter dimension to benefit from transfer learning between the parameters.

**Network Architectures** Our estimate in Corollary 12 applies to any ansatz class and the particular choice of network architecture and activation function enters through the ansatz class’ expressivity and its behavior under the chosen optimizer. We usually use a simple fully-connected architecture, possibly with a random Fourier embedding to mitigate spectral bias Tancik et al. (2020); Hennigh et al. (2021). Further, we frequently encode (homogeneous) Dirichlet boundary conditions directly into the architecture by multiplying the ansatz functions by a fixed smooth function vanishing only on the boundary of the computational domain.

The neural network training is performed employing TensorFlow (version 2.8.2), cf. [Abadi et al. (2015)], on a Colab Pro, i.e., with a single Tesla P100-PCIE-16GB and 13.9GB RAM as well as access to a High-RAM run-time environment. After the neural network training, the trainable variables of the network are extracted and, subsequently, stored in a FEniCS (version 2019.1.0), cf. Logg and Wells (2010), ‘Expression’ class for a straightforward comparison of the trained neural network to exact solutions or (if the latter are not given) to finite element solutions obtained on an adequately refined triangulation, exploiting the access to various quadrature formulas provided by FEniCS that are employed for error computation. All plots are generated using the Matplotlib (version 3.5.1) library, cf. [Hunter (2007)].

### 8.1 Variable Right Hand Side

In this section, we examine a parametric Dirichlet problem, i.e., 2-Dirichlet problem, on a fixed domain \( \Omega := (-1, 1) \subset \mathbb{R} \) with homogeneous Dirichlet boundary condition and a parameter-dependent right-hand side \( f \in L^{2}(\mathcal{P} \times \Omega) \), where \( \mathcal{P} := (0, 6) \), for every \( (\rho, \pi)^{T} \in \mathcal{P} \times \Omega \) defined by

\[
 f(\rho, x) := \rho^{2} \sin(\rho \pi x).
\]

More precisely, we are interested in approximating for each fixed \( \rho \in \mathcal{P} \), the unique minimizer \( u_{\rho} \in W_{0}^{1,2}(\Omega) \) of the Dirichlet energy \( E_{\rho} : W_{0}^{1,2}(\Omega) \to \mathbb{R} \), for every \( v \in W_{0}^{1,2}(\Omega) \) defined by

\[
 E_{\rho}(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^{2} \, dx - \int_{\Omega} f(\rho, \cdot) \cdot v \, dx.
\]

Due to Corollary 19, for this, it suffices to approximate the unique parametric minimizer \( u^{*} \in L^{2}(\mathcal{P}, W_{0}^{1,2}(\Omega)) \) of the variable right-hand side Dirichlet energy \( \mathcal{E} : L^{2}(\mathcal{P}, W_{0}^{1,2}(\Omega)) \to \mathbb{R} \), for every \( v \in L^{2}(\mathcal{P}, W_{0}^{1,2}(\Omega)) \) defined by

\[
 \mathcal{E}(v) := \int_{\rho} \left[ \frac{1}{2} \int_{\Omega} |\nabla x v(\rho, \cdot)|^{2} \, dx - \int_{\Omega} f(\rho, \cdot) v(\rho, \cdot) \, dx \right] \, d\rho.
\]

The unique parametric minimizer \( u^{*} \in L^{2}(\mathcal{P}, W_{0}^{1,2}(\Omega)) \) for every \( (\rho, x)^{T} \in \mathcal{P} \times \Omega \) is given via

\[
 u^{*}(\rho, x) := \frac{1}{\pi^{2}} \left( \sin(\rho \pi x) - \sin(\rho \pi) x \right).
\]
To approximate the parametric minimizer \( u^* \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \), we deploy a fully-connected feed-forward neural network with a Gaussian Fourier embedding to mitigate spectral bias and four hidden layers of width 16 whose realization is denoted by \( v_0 \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \). Then, the total number of trainable variables is 1.393, where 528 variables are associated with the Gaussian Fourier embedding. As activation function, we employ the approximated GELU activation function, cf. [Hendrycks and Gimpel (2016)], i.e., \( g : \mathbb{R} \to \mathbb{R} \), for every \( x \in \mathbb{R} \) defined by

\[
g(x) := \frac{\sqrt{2}}{\pi} \left( 1 + \tanh \left( \frac{\sqrt{2}}{\pi} (x + 0.044715x^3) \right) \right) \approx x\Phi(x),
\]

where \( \Phi \) is the cumulative distribution of a \( N(0, 1) \) random variable. The homogeneous Dirichlet boundary condition is enforced by means of the multiplicative weight \( \eta \in C^\infty(\Omega) \), defined by \( \eta(x) := (1 - x)(1 + x) \) for all \( x \in \Omega \), i.e., we do not employ \( v_0 \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) for the approximation of the parametric minimizer \( u^* \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) but the function \( u_0 := \eta v_0 \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \). The neural network is trained using 20,000 steps of the Adam optimization algorithm with a fixed learning rate of \( \varepsilon := 1e-3 \). At each training step, we employ the same \( n_{\text{int}} = 100,000 \) equi-distant interior points in \( \mathcal{P} \times \Omega \). To be more precise, at each training step, we employ the same Cartesian grid generated by \( n_p = 100 \) equi-distant points \( \{p_1, \ldots, p_{np}\} \) in \( \mathcal{P} \) and \( n_x = 1000 \) equi-distant points \( \{x_1, \ldots, x_{nx}\} \) in \( \Omega \), i.e., we employ \( \{p_1, \ldots, p_{np}\} \times \{x_1, \ldots, x_{nx}\} \). Here, we deliberately select a coarser grid with respect to the parameter dimension to benefit from transfer learning between the parameters.

In Figure 1, we depict the trained parametric neural network realization \( u_0 \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) and the parametric minimizer \( u^* \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \), their gradients and respective point-wise errors. In it, we clearly observe that the error at the limiting parameters \( p = 6 \) is relatively high, which may be traced back to the fact that transfer learning with respect to the parameters in this case is restricted to one direction.

In Figure 2 and Figure 3, for \( p = 2, 3, 4, 5 \), we compare the slice \( u_0(p, \cdot) \in W_0^{1,2}(\Omega) \) of the trained parametric neural network realization \( u_0 \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) to the slice \( u^*_0(p, \cdot) \in W_0^{1,2}(\Omega) \) of the parametric minimizer \( u^* \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \). In it, we observe that the errors are evenly distributed and not concentrated anywhere.

Figure 1: Plots of the trained parametric neural network realization \( u_0 \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) (top left) and its spatial gradient \( \nabla_x u_0 \in L^2(\mathcal{P} \times \Omega) \) (top right), the parametric minimizer \( u^* \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) (middle left) and its spatial gradient \( \nabla_x u^* \in L^2(\mathcal{P} \times \Omega) \) (middle right), and the error \( u_0 - u^* \in L^2(\mathcal{P}, W_0^{1,2}(\Omega)) \) (bottom left) and its spatial gradient \( \nabla_x (u_0 - u^*) \in L^2(\mathcal{P} \times \Omega) \) (bottom right).
Due to Proposition 16, for this, it suffices to approximate the unique parametric minimizer \( u^* \in \mathcal{U} \), where \( \mathcal{U} \) is the variable exponent Bochner–Lebesgue space defined in Proposition 16 of the variable exponent \( p(\cdot) \)-Dirichlet energy \( E_p : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R} \), for every \( v \in W_0^{1,p(\cdot)}(\Omega) \) defined by

\[
E_p(v) := \frac{1}{p(\cdot)} \int_{\Omega} |\nabla v|^{p(\cdot)} \, dx - \int_{\Omega} v \, dx.
\]  

(56)

Due to Proposition [16] for this, it suffices to approximate the unique parametric minimizer \( u^* \in \mathcal{U} \), where \( \mathcal{U} \) is the variable exponent Bochner–Lebesgue space defined in Proposition [16] of the variable exponent \( p(\cdot) \)-Dirichlet energy \( E_{\cdot} : \mathcal{U} \to \mathbb{R} \), for every \( v \in \mathcal{U} \) defined by

\[
E_{\cdot}(v) := \int_{\mathcal{D}} \left[ \frac{1}{p(\cdot)} \int_{\Omega} |\nabla v(\cdot, \cdot)|^{p(\cdot)} \, dx - \int_{\Omega} v(\cdot, \cdot) \, dx \right] \, dp.
\]

The unique parametric minimizer \( u^* \in \mathcal{U} \) for every \( (x, p) \in \mathcal{D} \times \Omega \) is given via

\[
u^*(x, p) := \frac{1}{p(\cdot)} (1 - |x|^p).
\]
To approximate the parametric minimizer \( u^* \in U \), we deploy a fully-connected feed-forward neural network with four hidden layers of width 16. The total number of trainable variables is 881. In accordance with \cite{Li et al. 2020a}, as activation function, we employ the s2ReLU activation function, i.e., \( g : \mathbb{R} \to \mathbb{R} \), for every \( x \in \mathbb{R} \) defined by
\[
g(x) := \sin(2\pi x) \max\{x, 0\} \max(1 - x, 0) .
\]
Similar to Section 8.1 the homogeneous Dirichlet boundary condition is enforced by means of the multiplicative weight \( \eta \in C^\infty(\Omega) \), defined by \( \eta(x) := (1 - x)(1 + x) \) for all \( x \in \Omega \). Then, the resulting neural network realization is again denoted by \( u_\theta \in U \). At each training step, we employ the same \( n_{\text{ini}} = 100,000 \) equi-distant interior points in \( \mathcal{P} \times \Omega \), as in Section 8.1 i.e., a Cartesian grid generated by \( n_p = 100 \) equi-distant points \( \{p_1, \ldots, p_{n_p}\} \) in \( \mathcal{P} \) and \( n_x = 1000 \) equi-distant points \( \{x_1, \ldots, x_{n_x}\} \) in \( \Omega \), with a coarser grid with respect to the parameter dimension to benefit from transfer learning between the parameters.

In Figure 4 and Figure 5, for \( p = 2, 3, 4, 5 \), we compare the slice \( u_\theta(p, \cdot) \in W^{1,p}_0(\Omega) \) of the trained parametric neural network realization \( u_\theta \in U \) to the slice \( u^*(p, \cdot) \in W^{1,p}_0(\Omega) \) of the parametric minimizer \( u^* \in U \).

![Figure 4: Plots of the trained parametric neural network realization \( u_\theta \in U \) (top left) and its spatial gradient \( \nabla_x u_\theta \in L^p(\mathcal{P} \times \Omega) \) (top right), the minimizer \( u \in U \) (middle left) and its spatial gradient \( \nabla_x u^* \in L^p(\mathcal{P} \times \Omega) \) (middle right), and the errors \( u^* - u_\theta \in U \) (bottom left) and its spatial gradient \( \nabla_x u^* - \nabla_x u_\theta \in L^p(\mathcal{P} \times \Omega) \) (bottom right).](image-url)
Due to Proposition 20, for this, it sufficing for each fixed parametric neural network realization $u_0 \in \mathcal{U}$ of the slice $u^* (p, \cdot) \in W_0^{1,p} (\Omega)$ (solid colored line; right).

Figure 6: For $p = 2, 3, 4, 5$, plots of the slice $\nabla_x u_0(p, \cdot) \in L^p(\Omega)$ (solid colored line; left) of the gradient of the trained parametric neural network realization $u_0 \in \mathcal{U}$ of the slice $u^* (p, \cdot) \in L^p(\Omega)$ (dashed black line; left) of the gradient of the parametric minimizer $u^* \in \mathcal{U}$, and the point-wise error $u^* (p, \cdot) - u_0(p, \cdot) \in L^p(\Omega)$ (solid colored line; right).

### 8.3 Variable Domain

In this section, we examine a parametric Dirichlet problem, i.e., 2-Dirichlet problem, on the variable domain $\Omega(p) := (-p, p)$, $p \in \mathcal{P}$, where $\mathcal{P} := \{1, 2\}$, with homogeneous Dirichlet boundary condition, and a fixed right-hand side $f := 1 \in L^2(Q)$, where $Q := \bigcup_{p \in \mathcal{P}} [p] \times \Omega(p)$. More precisely, we are interested in approximating for each fixed $p \in \mathcal{P}$, the unique minimizer $u_p \in W_0^{1,2} (\Omega(p))$ of the Dirichlet energy $E_p : W_0^{1,2} (\Omega(p)) \rightarrow \mathbb{R}$, for every $v \in W_0^{1,2} (\Omega(p))$ defined by

$$E_p(v) := \frac{1}{2} \int_{\Omega(p)} |\nabla v|^2 \, dx - \int_{\Omega(p)} v \, dx.$$ 

Due to Proposition 20, for this, it suffices to approximate the unique parametric minimizer $u^* \in L^2(\mathcal{P}, W_0^{1,2} (\Omega(\cdot)))$, where $L^2(\mathcal{P}, W_0^{1,2} (\Omega(\cdot)))$ is the variable domain Bochner–Lebesgue space defined in Proposition 20 of the variable domain Dirichlet energy $\mathcal{E} : L^2(\mathcal{P}, W_0^{1,2} (\Omega(\cdot))) \rightarrow \mathbb{R}$, for every $v \in L^2(\mathcal{P}, W_0^{1,2} (\Omega(\cdot)))$ defined by

$$\mathcal{E}(v) := \int_{\mathcal{P}} \left[ \frac{1}{2} \int_{\Omega(p)} |\nabla v(p, \cdot)|^2 \, dx - \int_{\Omega(p)} v(p, \cdot) \, dx \right] \, dp.$$ 

The unique parametric minimizer $u^* \in L^2(\mathcal{P}, W_0^{1,2} (\Omega(\cdot)))$ for every $(p, x)^T \in Q$ is given via

$$u^*(p, x) := \frac{p^2 - x^2}{2}.$$
To approximate the parametric minimizer \( u' \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \), we deploy a fully-connected feed-forward neural network with four hidden layers of width 16 and realization \( v_0 \in L^2(\mathcal{P}, W_{1,2}^1(\Omega(\cdot))) \). Then, the total number of trainable variables is 881. As activation function, we employ the approximated GELU activation function, cf. (55). Similar to Section 8.1, the homogeneous Dirichlet boundary condition is enforced by means of the multiplicative weight \( \eta \in C^0(\mathcal{P} \times \Omega) \), defined by \( \eta(p, x) := (p - x)(p + 1)/p^2 \) for all \((p, x)\), and, i.e., we do not use \( v_0 \in L^2(\mathcal{P}, W_{1,2}^1(\Omega(\cdot))) \) for the approximation of the parametric minimizer \( u' \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \). The neural network is trained using 20,000 steps of the Adam optimization algorithm with a fixed learning rate of \( \epsilon := 1e^{-3} \). At each training step, we employ the same \( n_{\text{int}} = 685,608 \) equi-distant interior points in \( \mathcal{P} \times \Omega \). To be more precise, at each training step, we employ the same grid generated by first choosing \( n_{\text{p}} = 100 \) equi-distant interior points \( \{p_1, \ldots, p_{n_{\text{p}}}\} \) in \( \mathcal{P} \) and, then, for each of these points \( p \in \mathcal{P} \) choosing \( n_{\text{q}}(p) = 2000 \) equi-distant interior points \( \{x_1(p), \ldots, x_{n_{\text{q}}(p)}(p)\} \) in \( \Omega(p) \), i.e., we employ \( \bigcup_{i=1}^{n_{\text{p}}} \{p\} \times \{x_1(p), \ldots, x_{n_{\text{q}}(p)}(p)\} \). We deliberately select a coarser grid with respect to the parameter dimension to benefit from transfer learning between the parameters.

In Figure 4, we depict the trained parametric neural network realization \( u_0 \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \) and the parametric minimizer \( u' \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \), their gradients and respective point-wise errors.

In Figure 5 and Figure 6 for \( p = 2, 3, 4, 5 \), we compare the slice \( u_0(p, \cdot) \in W_{1,2}^0(\Omega(p)) \) of the trained parametric neural network realization \( u_0 \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \) to the slice \( u'_p = u'(p, \cdot) \in W_{1,2}^0(\Omega(p)) \) of the parametric minimizer \( u' \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \).

![Figure 7](image-url)  
Figure 7: Plots of the trained parametric neural network realization \( u_0 \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \) (top left) and its spatial gradient \( \nabla u_0 \in L^2(Q) \) (top right), the parametric minimizer \( u' \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \) (middle left) and its spatial gradient \( \nabla u' \in L^2(Q) \), and the error \( u' - u_0 \in L^2(\mathcal{P}, W_{1,2}^0(\Omega(\cdot))) \) (bottom left) and its spatial gradient \( \nabla u' - \nabla u_0 \in L^2(Q) \) (bottom right).
the variable exponent Bochner–Lebesgue space defined in Remark 18, of the variable exponent $u^*(p, \cdot) \in W^{1,2}_0(\Omega(p))$ and the point-wise error $u^*(p, \cdot) - u_0(p, \cdot) \in W^{1,2}_0(\Omega(p))$ (solid colored line; right).

Figure 8: For $p = 1.2, 1.4, 1.6, 1.8$, plots of the slice $u_0(p, \cdot) \in W^{1,2}_0(\Omega(p))$ (solid colored line; left) of the trained parametric neural network realization $u_0(p, \cdot) \in L^2(\mathcal{P}, W^{1,2}_0(\Omega(\cdot)))$, of the slice $u^*(p, \cdot) \in W^{1,2}_0(\Omega(p))$ (dashed black line; left) of the parametric minimizer $u^*(p, \cdot) \in L^2(\mathcal{P}, W^{1,2}_0(\Omega(\cdot)))$, and the point-wise error $u^*(p, \cdot) - u_0(p, \cdot) \in W^{1,2}_0(\Omega(p))$ (solid colored line; right).

Figure 9: For $p = 1.2, 1.4, 1.6, 1.8$, plots of the slice $\nabla u_0(p, \cdot) \in L^2(\Omega(p))$ (solid colored line; left) of the gradient of the trained parametric neural network realization $u_0(p, \cdot) \in L^2(\mathcal{P}, W^{1,2}_0(\Omega(\cdot)))$, of the slice $\nabla u^*(p, \cdot) \in L^2(\Omega(p))$ (dashed black line; left) of the gradient of the parametric minimizer $u^*(p, \cdot) \in L^2(\mathcal{P}, W^{1,2}_0(\Omega(\cdot)))$, and the point-wise error $\nabla u^*(p, \cdot) - \nabla u_0(p, \cdot) \in L^2(\Omega(p))$ (solid colored line; right).

### 8.4 Parametric Right-Hand Side and Exponent

In this section, we examine a 7-dimensional, parametric $p(\cdot)$-Dirichlet problem on a fixed domain $\Omega := B_2(0) \subseteq \mathbb{R}^2$ with a pure Neumann boundary condition, parameter-dependent right-hand side $f \in C^\infty(\mathcal{P} \times \Omega)$, for every $p := (A, \alpha, x_0, y_0, 0)^T \in \mathcal{P} := (\frac{3n}{2}, \frac{3n}{2}) \times (0.2, 0.5) \times (-0.3, 0.3) \times (-0.3, 0.3) \times (1.8, 2.2) \subseteq \mathbb{R}^5$ and $(x, y)^T \in \Omega$ defined by

$$f(p, x, y) := \frac{A}{2\pi\alpha} \exp \left( - \frac{1}{2\alpha^2} \| (x, y)^T - (x_0, y_0)^T \|^2 \right),$$

and parameter-dependent exponent $p \in C^\infty(\mathcal{P})$, defined by $p(p) := p$ for every $p = (A, \alpha, x_0, y_0, 0)^T \in \mathcal{P}$. More precisely, we are interested in approximating for each fixed $p \in \mathcal{P}$, a minimizer $u_p \in W^{1,p(p)}(\Omega)$ of the $p$-Dirichlet energy $E_p : W^{1,p(p)}(\Omega) \to \mathbb{R}$, for every $v \in W^{1,p(p)}(\Omega)$ defined by

$$E_p(v) := \frac{1}{p(p)} \int_{\Omega} |v|^p(p) \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} f(p, \cdot) v \, dx. \quad (57)$$

Very similar to Proposition 16 or Remark 18 for this, it suffices to approximate a minimizer $u^* \in \mathcal{U}$, where $\mathcal{U}$ is the variable exponent Bochner–Lebesgue space defined in Remark 18 of the variable exponent $p(\cdot)$-Dirichlet energy $\delta : \mathcal{U} \to \mathbb{R}$, for every $v \in \mathcal{U}$ defined by

$$\delta(v) := \int_{\mathcal{P}} \left[ \frac{1}{p(p)} \int_{\Omega} |\nabla v(p, \cdot)|^p(p) \, dx + \frac{1}{2} \int_{\Omega} |v(p, \cdot)|^2 \, dx - \int_{\Omega} f(p, \cdot) v(p, \cdot) \, dx \right] \, dp .$$

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To approximate the parametric minimizer $u^* \in \mathcal{U}$, we deploy a fully-connected feed-forward neural network four hidden layers of width 32 and realization $\nu_0 \in \mathcal{U}$. The total number of trainable variables is $3.457$. As activation function, we employ the s2relu activation function, cf. Li et al. (2020a). The neural network is trained using 200 epochs consisting of each 300 steps of the Adam optimization algorithm with a fixed learning rate of $\epsilon = 1e^{-3}$. At each epoch, we employ $n_{\text{int}} = 140.625$ interior points in $\mathcal{P} \times \Omega$. More precisely, at each epoch, we employ a grid generated by the Cartesian product of $n_p = 25$ uniformly random distributed points $\{p_1, \ldots, p_n\}$ in $\mathcal{P}$ and a Cartesian grid of $n_\epsilon = 75 \times 75 = 5.625$ equi-distant points $\{x_1, \ldots, x_n\}$ in $\Omega$, i.e., we employ $\{p_1, \ldots, p_n\} \times \{x_1, \ldots, x_n\}$. Again, we deliberately select a coarser grid with respect to the parameter dimension to benefit from transfer learning between the parameters. Since the authors are not aware of an exact representation formula of the parametric minimizer $u^* \in \mathcal{U}$, to examine the accuracy of the trained neural network realization $u_0 \in \mathcal{U}$, we compare for $n_{\text{rand}} = 1.200$ uniformly randomly sampled parameters $p \in \mathcal{P}_{\text{rand}} = \{p_1, \ldots, p_{n_{\text{rand}}}\}$, the slice $u_0(p_\cdot) \in W^{1,p}(\Omega)$ to the respective continuous Lagrange minimizer $u_0^*(p) \in P^1(\mathcal{H}_0)$ of $E_p : P^1(\mathcal{H}_0) \to \mathbb{R}$, where $\mathcal{H}_0$ is a triangulation of $\Omega$, obtained using gmsh (version 4.6.0), cf. Geuzaine and Remacle (2020), with mesh-size $h = 3.125e-2$, i.e., 8,272 degrees of freedom. For any $p \in \mathcal{P}_{\text{rand}}$, $u_0^*(p) \in P^1(\mathcal{H}_0)$ is approximated deploying the Newton line-search algorithm of PETSc, cf. Balay et al. (2019), with an absolute tolerance of $\tau_{\text{abs}} = 1e^{-8}$ and a relative tolerance of $\tau_{\text{rel}} = 1e^{-10}$. The linear system emerging in each Newton step is solved deploying PETSc’s generalized minimal residual method (GMRES). Using a midpoint (i.e., barycenter) quadrature rule with respect to $\mathcal{H}_0$, we obtain the absolute errors

$$
\varepsilon_{\text{abs}}^{L^p} = \frac{1}{n_{\text{rand}}} \sum_{p=(\alpha,\nu,\theta) \in \mathcal{P}_{\text{rand}}} \|u_0(p) - u_0(p_\cdot)\|_{L^p(\mathcal{H}_0)} = 2.863e^{-2},
$$

$$
\varepsilon_{\text{abs}}^{W^{1,p}} = \frac{1}{n_{\text{rand}}} \sum_{p=(\alpha,\nu,\theta) \in \mathcal{P}_{\text{rand}}} \|\nabla u_0(p) - \nabla u_0(p_\cdot)\|_{L^p(\mathcal{H}_0)} = 3.229e^{-2},
$$

and the relative errors

$$
\varepsilon_{\text{rel}}^{L^p} = \frac{1}{n_{\text{rand}}} \sum_{p=(\alpha,\nu,\theta) \in \mathcal{P}_{\text{rand}}} \frac{\|u_0(p) - u_0(p_\cdot)\|_{L^p(\mathcal{H}_0)}}{\|u_0^*(p)\|_{L^p(\mathcal{H}_0)}} = 2.712e^{-2},
$$

$$
\varepsilon_{\text{rel}}^{W^{1,p}} = \frac{1}{n_{\text{rand}}} \sum_{p=(\alpha,\nu,\theta) \in \mathcal{P}_{\text{rand}}} \frac{\|\nabla u_0(p) - \nabla u_0(p_\cdot)\|_{L^p(\mathcal{H}_0)}}{\|\nabla u_0^*(p)\|_{L^p(\mathcal{H}_0)}} = 9.476e^{-2}.
$$

Figure 10 indicates that the absolute errors, cf. (58), and relative errors, cf. (59), for $n_{\text{rand}} = 1.200$ randomly sampled points from the parameter space $\mathcal{P}$ are already sufficiently accurate, and randomly sampling additional points will change the error value only slightly.

Figure 11 for the generic parameter $p = (2\pi, 0.3, 0, 2) \in \mathcal{P}$, we depict the slice of $u_0(p_\cdot) \in W^{1,p}(\mathcal{H}_0)$ of trained parametric neural network realization $u_0 \in \mathcal{U}$, the continuous Lagrange minimizer $u_0^*(p) \in W^{1,p}(\mathcal{H}_0)$, their gradients and respective point-wise errors. In it, we see that although training on the generic parameter $p = (2\pi, 0.3, 0, 2) \in \mathcal{P}$ was not done directly, high accuracy was already achieved using transfer learning only.

![Figure 10: Plots of the error evolutions of $\varepsilon_{\text{abs}}^{L^p}$, $\varepsilon_{\text{abs}}^{W^{1,p}}$, $\varepsilon_{\text{rel}}^{L^p}$ and $\varepsilon_{\text{rel}}^{W^{1,p}}$, cf. (58) and (59), for increasing number of randomly uniform sampled parameters in $\mathcal{P}$, i.e., for $n_{\text{rand}} \in \{20, \ldots, 2.000\}$. Starting from the first dot that represents the mean error of 20 randomly uniform sampled parameters in $\mathcal{P}$, for $k \in \{1, \ldots, 60\}$, the $k$-th dot represents the mean of the $(k-1)$-th dot and new 20 randomly uniform sampled parameters in $\mathcal{P}$.](image-url)
Figure 11: Plots of the parametric neural network realization $u_\theta(p, \cdot) \in W^{1,p}(\Omega)$ (top left), the modulus of its gradient $|\nabla u_\theta(p, \cdot)| \in L^{p}(\Omega)$ (bottom left), the continuous Lagrange minimizer $u_c^*(p) \in W^{1,p}(\Omega)$ (top middle), the modulus of its gradient $|\nabla u_c^*(p)| \in L^{p}(\Omega)$ (bottom middle), the error $u_c^*(p) - u_\theta(p, \cdot) \in W^{1,p}(\Omega)$ (top right), and the modulus of its gradient $|\nabla u_c^*(p) - \nabla u_\theta(p, \cdot)| \in L^{p}(\Omega)$ (bottom right) for $p = (2 \pi, 0.3, 0, 0, 2)^T \in \varnothing$ and mesh-size $h = 3.125 \times 10^{-2}$, i.e., 8,272 degrees of freedom.

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