A proposal for a conditional definition of the Swap Gate

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In order to realize quantum logical operations, Quantum Computation (QC) requires that its basic tools and concepts obey the laws of physics. One of the fundamental concepts in QC is the conditional quantum dynamics, some times called controlled-unitary operation, which is established by the conditional “If-Then” sentence. The best know example is the c-not gate, which operates on the computational basis as follows: If the control qubit is set to $|1\rangle$, then apply the single qubit quantum NOT gate on the target. Otherwise, if the control qubit is set to $|0\rangle$, then the target qubit is unchanged. This gate represents the paradigm for the conditional quantum dynamics, where the flipping of the target qubit is conditioned to the state of the control qubit. Other gates have been defined in a similar way of conditional evolution; for instance, the control phase gates. However, to the best of our knowledge, such conditional quantum dynamics has not been yet used to define the SWAP gate. Here we propose a possible conditional definition, in the form of If-Then sentence, to construct a SWAP gate in the case of two qubits. This definition suggests a classification in two classes, which depends upon the number of qubits that have to undergo a conditional quantum dynamics.

I. INTRODUCTION

The SWAP gate on two qubits is usually defined as:

$$\hat{U}_{SWAP}|a\rangle_1 |b\rangle_2 = |b\rangle_1 |a\rangle_2,$$

(1)

where the principal task of the SWAP operation is to exchange the state of the first qubit to second qubit, and vice versa. Then, there has to be a physical process that makes the exchange of states. This phenomenon is analogous to the classical example of a perfect elastic collision between two particles of equal masses, where exchange of momentum occurs. Therefore, the physical process involved in the SWAP operation defined by equation (1) is the interchange of states of a composite system. It is worthwhile mentioning that the SWAP operation is one of the most useful gate in quantum computation. Its usefulness range from establishing the universality of two qubit gates, programmable gate arrays, quantum teleportation, the basic implementation of other quantum gates using SWAP operations (the corresponding implementation have been realized with the $\sqrt{SWAP}$) to the construction of optimal quantum circuits.

On the other hand, as it is well known, three CNOT gates can implement the SWAP gate. From this result there seem to be a different realization of the SWAP operation, i. e. a realization in terms of a flipping process. Therefore we can ask the following question: Is there a quantum gate definition, which, by using both conditional evolution and flipping process, produces the swapping of two unknown qubits? The answer to this question is affirmative. To show this case, let us proceed as follows.

First, let us define the SWAP gate, $\hat{U}_{SWAP}$, as:

**Definition 1.** If the states of a two parties system are not equal, then apply the single qubit quantum NOT gate, $\hat{U}_{NOT}$, on each subsystem. Otherwise, left them unchanged.

Where the states of a two parties system are equal in the case $|0\rangle_1 |0\rangle_2$ or $|1\rangle_1 |1\rangle_2$; that is, both in the ground state or both in the excited state. The single qubit quantum NOT gate flips the basic computational states of single qubits: $\hat{U}_{NOT}|0\rangle = |1\rangle$ and $\hat{U}_{NOT}|1\rangle = |0\rangle$. 

DEFINITION 1 is a conditional sentence of the form "if-then", which involves a crucial difference between both definitions of the SWAP gate. In the case of definition given by equation (1) the exchange of states is not a conditional operation, whereas the case stated above in DEFINITION 1 is a conditional operation where the condition is on the global state of the bipartite system. Aside from being distinct of a conventional controlled operation where the condition is on a single part which plays the role of a control qubit. This result constitute a counterpart of the conditional quantum dynamics stated in references, where one partie of a composite system undergoes a coherent evolution that depends on the state of another partie. In the present case, we will obtain a conditional quantum dynamics in which a two partie system undergoes a coherent evolution that depends on the overall state of the whole system. In the former case we can thought the conditional evolution as \( U(\alpha) \), in the latter case the conditional evolution is \( U(-\hat{A}|i\rangle|j\rangle) \), where \( U \) is a unitary evolution operator, which satisfy \( U^\dagger U = 1 \). In fact, there are another conditional two-qubits gate whose conditional evolution depends on the global state of the whole system, see the relative phase gate, \( U_{\text{Phase}}^{\text{relative}} \), in references.

From the conditional DEFINITION 1 of SWAP gate enunciated, it operates on two qubit computational basis as follows:

\[
\begin{align*}
\hat{U}_{\text{SWAP}} |0\rangle_1 |0\rangle_2 &= |0\rangle_1 |0\rangle_2, \\
\hat{U}_{\text{SWAP}} |1\rangle_1 |1\rangle_2 &= |1\rangle_1 |1\rangle_2, \\
\hat{U}_{\text{SWAP}} |0\rangle_1 |1\rangle_2 &= \hat{U}_{\text{NOT}} |0\rangle_1 \hat{U}_{\text{NOT}} |1\rangle_2 = |1\rangle_1 |0\rangle_2, \\
\hat{U}_{\text{SWAP}} |1\rangle_1 |0\rangle_2 &= \hat{U}_{\text{NOT}} |1\rangle_1 \hat{U}_{\text{NOT}} |0\rangle_2 = |0\rangle_1 |1\rangle_2.
\end{align*}
\] (2)

This gate is reversible in the sense that the output is a unique function of the input\(^{20} \). The main difference with the equation (1), is that the physical implementation does not require a swapping on the states but it requires a flipping process.

Secondly, if we begin with two unknown qubits and apply the equation (2), we obtain:

\[
\hat{U}_{\text{SWAP}} (a_1 |0\rangle + b_1 |1\rangle) (a_2 |0\rangle + b_2 |1\rangle) = (a_2 |0\rangle + b_2 |1\rangle) (a_1 |0\rangle + b_1 |1\rangle).
\] (3)

Therefore, by means of a flipping process we obtain a conditional swapping between two unknown qubits.

On the other hand, DEFINITION 1 suggests the following gate classification:

**Gate Class 1** In this class there is a conditional evolution of only one qubit of the whole system or its evolution can be implemented by the evolution of a single system, i.e. there is an application of an unitary operator \( \hat{U} \) only on a single qubit when the condition is satisfied, for example \( |i\rangle \hat{U} |j\rangle \). Therefore, this class requires the manipulation of only a single qubit. The c-not, \( \hat{U}_{\text{Phase}}^{\text{Relative}} \) and \( \hat{U}_{\text{Phase}}^{\text{Absolute}} \) gates belong to this class.

**Gate Class 2** In this class there is a conditional evolution of both qubits of the whole system and its evolution can not be implemented by the evolution of a single system, i.e. there is an application of unitary operators to both subsystems when the condition is satisfied, for example \( \hat{U} |i\rangle_1 \hat{U} |j\rangle_2 \). Therefore, this class requires the manipulation of two qubits. The SWAP (DEFINITION 1) and Double c-not gates\(^{21} \) belong to this class.

In this classification, we consider that the whole system’s evolution can be implemented by a single system evolution when \( \hat{U} |i\rangle_1 \hat{U} |j\rangle_2 = |i\rangle_1 |j\rangle_2 \). For example, \( \hat{U}_{\text{Phase}}^{\text{Relative}} |i\rangle_1 \hat{U}_{\text{Phase}}^{\text{Relative}} |j\rangle_2 = |i\rangle_1 |j\rangle_2 \) or \( \hat{U}_{\text{Phase}}^{\text{Absolute}} |i\rangle_1 |j\rangle_2 \). On the other hand, an example of overall system’s evolution that can not be simulated by a single system evolution is \( \hat{U}_{\text{NOT}} |i\rangle_1 \hat{U}_{\text{NOT}} |j\rangle_2 \neq |i\rangle_1 |j\rangle_2 \) or \( \hat{U}_{\text{Phase}}^{\text{Absolute}} |i\rangle_1 |j\rangle_2 \neq \hat{U}_{\text{Phase}}^{\text{Absolute}} |i\rangle_1 |j\rangle_2 \). Therefore, you can not find one-qubit gate that could satisfy a equality for this equation, precisely because you need to manipulate both subsystems.

At this stage, we want to remark the usefulness of this result on quantum computation. As it is well known, the practical problems of construction of a quantum computer can be solved by means of a distributed quantum computer, i.e. a quantum communication network in which each node can act as a sender or receiver and contains only a small number of qubits. These results guided to the necessity to establish optimal implementations of nonlocal gates only using local operations and classical communication (LOCC) and shared entanglement\(^{22,23} \). For instance A. Barenco et. al.\(^{12} \) proposed the first implementation of a nonlocal SWAP gate on two unknown qubits using two shared ebits and four bits of classical communications (proposed as the fourth application of the c-not gate in references). A slightly different implementation of the nonlocal SWAP gate was proposed by L. Vaidman\(^{14,15} \). Independently, D. Collins et. al.\(^{24} \) and J. Eisert et. al\(^{25,26} \) have demonstrated that the nonlocal implementation of
the SWAP gate requires as necessary and sufficient resources four bits of classical communication together with two shared qubits, besides to shown that a nonlocal c-not gate consumes one bit of classical communications and one shared ebit. A similar work regarding the amount of separability was made by A. Chefles et. al. Also, Zheng et. al. shown a protocol to implement the nonlocal SWAP operator on two entangled pairs. An additional study of the resources needed to implement nonlocal operators was carried out by K. Hammerer et al. and by J. I. Cirac et. al. These protocols have reached the experimental realization. Now, the gate’s classification given above and Definition 1, allows us to explain why the SWAP gate uses more amount of resources than others. In the following we tailor this explanation.

Eisert et. al. have established a protocol to realize an arbitrary control-U gate where the unitary operator \( \hat{U} \) is applied to the target qubit. If we analyze this protocol from the point of view of gate’s classification and Definition 1, the control-U gate belongs to Gates Class 1 too. Following their protocol, we consider the case when Alice holds the qubit \( (\alpha|0\rangle + \beta|1\rangle) \), Bob holds \( (\gamma|0\rangle_B + \delta|1\rangle_B) \) and they share the ebit \( (|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \). Then, the first steps of the protocol correlates the state of system \( A \) with the state of system \( B_1 \), in such a way that if the state of \( A \) is \( |0\rangle_A \) (or \( |1\rangle_A \)) then the state of \( B_1 \) is \( |0\rangle_{B_1} \) (or \( |1\rangle_{B_1} \)). The state of \( A \) and \( B_1 \) are correlated too, then the operator application is realized as if a conditional quantum evolution between \( |i\rangle_A \) and \( |i\rangle_B \) were existed. However, this protocol does not enable to apply an unitary operator on the qubit \( |i\rangle_A \).

Therefore, sharing only one ebit and using two bits of classical communications we can manipulate only one of the qubits of Alice and Bob, because these resources are not sufficient to apply unitary operators to both qubits. Therefore, the nonlocal implementation of Gates Class 2 uses more amounts of resources than the Gates Class 1 because in the former class it is necessary to apply two unitary operators to both qubits of the whole system.

THREE SWAP GATE

To analyze the three qubits swap case, firstly we need to define what does mean the swapping between three qubits. We propose that a three qubits swapping is implemented as similar as a three axis rotation, that is:

\[
\hat{U}^3_{SWAP} (\alpha_1|0\rangle_1 + \beta_1|1\rangle_1) \otimes (\alpha_2|0\rangle_2 + \beta_2|1\rangle_2) \\
\otimes (\alpha_3|0\rangle_3 + \beta_3|1\rangle_3) \\
= (\alpha_3|0\rangle_1 + \beta_3|1\rangle_1) \otimes (\alpha_1|0\rangle_2 + \beta_1|1\rangle_2) \\
\otimes (\alpha_2|0\rangle_3 + \beta_2|1\rangle_3)
\]

(4)

However, a conditional definition similar to DEFINITION 1 does not work, because it does not produce the required change in the computational basis of three qubits.

FREDKIN GATE

On the other hand, it is possible to construct a conditional definition of the Fredkin gate as follows. The Fredkin gate acts in the computational basis of three qubits, where the first qubit plays the roll of control qubit.

DEFINITION 2. If the control qubit is set to \( |1\rangle \), then apply a single qubit quantum NOT gate on both qubits if they are not equal. Otherwise, leave them unchanged.

In conclusion, DEFINITION 1 can throw new light on the seemingly paradoxical situation of uniquely using two ebits when implementing a nonlocal SWAP gate compared with three ebits when implementing the SWAP gate using three c-not gates.

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