Basis of totally primitive elements of \textbf{WQSym}

Hugo MLODECKI *

 Université Paris-Saclay, CNRS, Laboratoire de recherche en informatique, 91405, Orsay, France.

Abstract. By Foissy’s work, the bidendriform structure of the Word Quasisymmetric Functions Hopf algebra (WQSym) implies that it is isomorphic to its dual. However, the only known explicit isomorphism does not respect the bidendriform structure. This structure is entirely determined by so-called totally primitive elements (elements such that the two half-coproducts are 0). In this paper, we construct a basis indexed by a new combinatorial family called biplane forests in bijection with packed words. In this basis, primitive elements are indexed by biplane trees and totally primitive elements by a certain subset of trees. Thus we obtain the first explicit basis for the totally primitive elements of WQSym.

Abstract. Grâce aux travaux de Foissy, on sait que l’algèbre de Hopf WQSym est isomorphe à sa duale car bidendriforme. Cependant, le seul isomorphisme explicite connu ne respecte pas la structure bidendriforme. Cette structure est entièrement déterminée par les éléments totalement primitifs (annulés par les demi co-produits). Dans ce papier, nous construisons une base indexée par une nouvelle famille combinatoire appelée forêt biplanes, en bijection avec les mots tassés. Dans cette base, les éléments primitifs sont indexés par les arbres et les totalement primitifs par un certain sous-ensemble d’arbres. Ainsi, nous obtenons la première base explicite des éléments totalement primitifs dans WQSym.

Keywords: bidendriform Hopf algebras, Word Quasisymmetric Functions, packed words, primitive elements

Introduction

The varied zoo of combinatorial Hopf algebras is much better understood when one considers the extra algebraic structures that each algebra can have. In this light, operad related theories are very useful. Some important examples are the Hopf algebras of non-commutative and quasi-symmetric functions related to the theory of free Lie algebras [1].

More recently, the Hopf algebra of binary trees was identified as the free dendriform algebra on one generator [2].

Closed related examples include the algebras \textbf{FQSym} of permutations of Malvenuto-Reutenauer [3] and the Hopf algebra \textbf{WQSym} of surjections or, equivalently, ordered

* hugo.mlodecki@lri.fr
set partitions [4]. These two can be seen as noncommutative versions of the algebra of quasi-symmetric functions. Though the first one is trivially self-dual, it is only by a deep theorem of Foissy [5] that one can show that the second one is too. In particular, until Vargas’s work [6], no concrete isomorphism was known.

The first study of $WQSym$ structure is due to Bergeron-Zabrocki [7]. They showed that it is free and co-free. Independently, Novelli-Thibon endowed it with a bidendriform bialgebra structure [8]. Recall that a dendriform algebra is an abstraction of a shuffle algebra where the product is split in two half-products. If the coproduct is also split, and certain compatibilities hold, one gets the notion of bidendriform bialgebra [5].

Building on the work of Chapoton and Ronco [9, 10], Foissy [5] showed that the structure of a bidendriform bialgebra is very rigid. In particular, he defined a specific subspace called the space of totally primitive elements, and showed that it characterizes the whole structure. This does not only re-prove the freeness and co-freeness, as well as the freeness of the primitive lie algebra, but also shows that the structure of a bidendriform bialgebra depends only on its Hilbert series (the series of dimensions of its homogeneous components). In particular, any such algebra is isomorphic to its dual. However, Foissy’s isomorphism is not fully explicit and depends on a choice of a basis of the totally primitive elements. To this end, one needs an explicit basis of the totally primitive elements. Foissy described such a construction for $FQSym$ [11]. Our long term goal is to effectively apply Foissy’s construction to $WQSym$ in order to build an explicit isomorphism respecting the bidendriform structure. With this objective in mind, we provide an explicit basis called the totally primitive elements using a bijection with certain families of trees.

We begin with a background section presenting two rigidity structure theorems that prove, among other things, the self-duality of any bidendriform bialgebra (Theorems 2 and 3). We then define the notion of packed word as well as the specific basis of $WQSym$, which will be the starting point of our combinatorial analysis (Equations (1.7) to (1.9)).

Section 2 is devoted to the combinatorial construction of biplane forests (Definition 28) which are our first key ingredient. They record a recursive decomposition of packed words according to their global descents (Lemma 13) and positions of the maximum letter (Lemma 20). We show that the cardinalities of some specific sets of biplane trees match the dimensions of primitive and totally primitive elements (Theorem 29).

Finally in Section 3 we construct a new basis of $WQSym$ which contains a basis for the primitive and totally primitive elements (see Theorem 36). To do so we decompose the space of totally primitive elements as a certain direct sum which matches the combinatorial decomposition of packed words (Lemma 33).
1 Background

1.1 Cartier-Milnor-Moore theorems for Bidendriform bialgebras

A bialgebra is a vector space over a field $K$, endowed with an unitary associative product $\cdot$ and a counitary coassociative coproduct $\Delta$ satisfying a compatibility relation called the Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$. In this paper all bialgebras are assumed to be graded and connected (i.e. the homogeneous component of degree 0 is $K$). They are therefore Hopf algebras, as the existence of the antipode is implied.

We now recall the elements of the definition of bidendriform bialgebras which are useful for the comprehension of this paper. We refer to [5] for the full list of axioms.

First of all, a dendriform algebra $A$ is a $K$-vector space, endowed with two binary bilinear operations $\prec, \succ$ satisfying the following axioms, for all $a, b, c \in A$:

\[(a \prec b) \prec c = a \prec (b \prec c + b \succ c) , \tag{1.1}\]
\[(a \succ b) \prec c = a \succ (b \prec c) , \tag{1.2}\]
\[(a \prec b + a \succ b) \succ c = a \succ (b \succ c) . \tag{1.3}\]

Adding together Equations (1.1) to (1.3) show that the product $a \cdot b := a \prec b + a \succ b$ is associative. Adding a subspace of scalars, this defines a unitary algebra structure on $K \oplus A$. In this paper, all the dendriform algebras are graded and have null 0-degree component so that the associated algebra is connected.

Dualizing, one gets a notion of co-dendriform co-algebra which is a $K$-vector space with two binary co-operations (i.e., linear maps $A \to A \otimes A$) denoted by $\Delta \prec, \Delta \succ$ satisfying the dual axioms of Equations (1.1) to (1.3). The sum of the two half coproducts $\tilde{\Delta}(a) := \Delta \prec(a) + \Delta \succ(a)$ is a reduced coassociative coproduct. On $K \oplus A$, setting $\Delta(a) := 1 \otimes a + a \otimes 1 + \tilde{\Delta}(a)$ defines a co-associative and co-unitary coproduct.

A bidendriform bialgebra is a $K$-vector space which is both a dendriform algebra and a co-dendriform co-algebra satisfying a set of four relations [5] relating respectively $\prec$ and $\succ$ with $\Delta \prec, \Delta \succ$. Adding those four relations shows that $\cdot$ and $\Delta$ as defined above defines a proper bi-algebra.

We recall here the relevant results of Foissy [5] on the rigidity of bidendriform bialgebras based on the works of Chapoton and Ronco [9, 10].

Let $A$ be a bidendriform bialgebra. We define $\text{Prim}(A) := \ker(\tilde{\Delta})$ as the set of primitive elements of $A$. We also denote by $A(z)$ and $P(z)$ the Hilbert series of $A$ and $\text{Prim}(A)$ defined as $A(z) := \sum_{n=1}^{+\infty} \dim(A_n)z^n$ and $P(z) := \sum_{n=1}^{+\infty} \dim(\text{Prim}(A_n))z^n$. The present work is based on two analogues of the Cartier-Milnor-Moore theorems [5] which we present now. The first one is extracted from the proof of [11, Proposition 6]:

**Proposition 1.** Let $A$ be a bidendriform bialgebra and let $p_1 \ldots p_n \in \text{Prim}(A)$. Then the map

\[p_1 \otimes p_2 \otimes \ldots \otimes p_n \mapsto p_1 \prec (p_2 \prec (\ldots \prec p_n) \ldots). \tag{1.4}\]
is an isomorphism of co-algebras from $T^+(\text{Prim}(A))$ (the non trivial part of the tensor algebra with deconcatenation as coproduct) to $A$. As a consequence, taking a basis $(p_i)_{i \in I}$ of Prim$(A)$, the family $(p_{w_1} \prec (p_{w_2} \prec (\cdots \prec p_{w_n}) \cdots))_w$ where $w = w_1 \ldots w_n$ is a non empty word on $I$ defines a basis of $A$. This implies the equality of Hilbert series $A = P/(1 - P)$.

One can further analyze Prim$(A)$ using the so-called totally primitive elements of $A$ defined as $T\text{Prim}(A) = \text{Ker}(\Delta_\prec) \cap \text{Ker}(\Delta_\succ)$ and $T(z) = \sum_{n=1}^{+\infty} \dim(T\text{Prim}(A_n))z^n$. Recall that a brace algebra is a $K$-vector space $A$ together with an $n$-multilinear operation denoted as $\langle \ldots \rangle$ for all $n \geq 2$ which satisfies certain relations (see [9] for details).

**Theorem 2** ([11, Theorem 5]). Let $A$ be a bidendriform bialgebra. Then Prim$(A)$ is freely generated as a brace algebra by $T\text{Prim}(A)$ with brackets given by

$$
\langle p_1, \ldots, p_{n-1}, p_n \rangle := \sum_{i=0}^{n-1} (-1)^{n-1-i} (p_1 \prec (p_2 \prec (\cdots \prec p_i) \cdots)) \succ p_n \prec ((\cdots (p_{i+1} \succ p_{i+2}) \succ \cdots) \succ p_{n-1}).
$$

A basis of Prim$(A)$ is described by ordered trees that are decorated with elements of TPrim$(A)$ where $p_n$ is the root and $p_1, \ldots, p_{n-1}$ are the children (see [9, 10, 11]). This is reflected on their Hilbert series as $T = A/(1 + A)^2$ or equivalently $P = T(1 + A)$.

Using Proposition 1 and Theorem 2 together with a dimension argument, one can show the following theorem:

**Theorem 3** ([11, Theorem 2]). Let $A$ be a bidendriform bialgebra. Then $A$ is freely generated as a dendriform algebra by TPrim$(A)$.

### 1.2 The Hopf algebra of word-quasisymmetric functions WQSym

The algebra WQSym is a Hopf algebra whose bases are indexed by ordered set partitions or equivalently surjections or packed words. In this paper, we use the latter which we define now.

**Definition 4.** A word over the alphabet $\mathbb{N}_{>0}$ is packed if all the letters from 1 to its maximum $m$ appears at least once. By convention, the empty word $\varepsilon$ is packed. For $n \in \mathbb{N}$, we denote by PW$_n$ the set of all packed words of length (also called size) $n$ and PW $= \bigsqcup_{n \in \mathbb{N}}$ PW$_n$ the set of all packed words.

**Definition 5.** The packed word $u := \text{pack}(w)$ associated with a word over the alphabet $\mathbb{N}_{>0}$ is obtained by the following process: if $b_1 < b_2 < \cdots < b_r$ are the letters occurring in $w$, then $u$ is the image of $w$ by the homomorphism $b_i \mapsto i$.

A word $u$ is packed if and only if $\text{pack}(u) = u$. 
Example 6. The word 4152142 is not packed because the letter 3 does not appear while the maximum letter is 5 > 3. Meanwhile \( \text{pack}(4152142) = 3142132 \) is a packed word. Here are all packed words of size 1, 2 and 3 in lexicographic order:

1, 11 12 21, 111 112 121 122 132 211 212 213 221 231 312 321

We will use the following notations and operations on words over the alphabet \( \mathbb{N}_{>0} \):

First, \( \max(w) \) is the maximum letter of the word \( w \) with the convention that \( \max(\epsilon) = 0 \). Then \( |w| \) is the length (or size) of the word \( w \). The concatenation of the two words \( u \) and \( v \) is denoted as \( u \cdot v \). Moreover, \( u/v \) (resp. \( u\backslash v \)) is the left-shifted (resp. right-shifted) concatenation of the two words where all the letters of the left (resp. right) word are shifted by the maximum of the right (resp. left) word: \( 112/3112 = 44543112 \) and \( 1121\backslash3112 = 11215334 \).

Finally \( u \upuparrows v \) is the shuffle product of the two words. It is recursively defined as

\[
ua \upuparrows vb = (u \upuparrows vb) \cdot a + (ua \upuparrows v) \cdot b
\]

where \( u \) and \( v \) are words and \( a \) and \( b \) are letters. Analogously to the shifted concatenation, one can define the right shifted-shuffle \( u \updownarrow v \) where all the letters of the right word \( v \) are shifted by the maximum of the left word \( u \).

Example 7. \( 12 \updownarrow 11 = 12 \updownarrow 33 = 1233 + 1323 + 1332 + 3123 + 3132 + 3312 \).

Bidendriform bialgebra structure Novelli-Thibon [8] proved that \( \text{WQSym} \) is a bidendriform bialgebra. Their products and coproducts involve overlapping-shuffle. However it will be easier for us to chose, among the various bases known in the literature \([4, 7, 8, 6]\) a basis where the shuffle are non-overlapping. Therefore, we take the dual basis denoted \((R_w)_{w \in \text{PW}}\) of \([7, \text{Equation 23}]\), using the classical bijection between ordered partitions and packed words and redefine the bidendriform structure. The Hopf algebra product and reduced coproduct are respectively recovered as the sum of the half products (see \text{Equation (1.7)}) and half coproducts (see \text{Equations (1.8)} and (1.9)).

The recursive definition of the shuffle product (see \text{Equation (1.5)}) contains two summands. We define them respectively as \( \prec \) and \( \succ \):

\[
ua \prec vb := (u \upuparrows vb) \cdot a, \text{ and } \quad ua \succ vb := (ua \upuparrows v) \cdot b.
\]

We define \( \prec, \succ, \Delta_{\prec} \) and \( \Delta_{\succ} \) on \( \text{WQSym}^+ = \text{Vect}(\mathbb{R}_u \mid u \in \text{PW}_{n}, n \geq 1) \) in the following way: for all \( u = u_1 \cdots u_n \in \text{PW}_{n \geq 1} \) and \( v \in \text{PW}_{m \geq 1} \),

\[
\mathbb{R}_u \prec \mathbb{R}_v := \sum_{w \in u \prec v} \mathbb{R}_w, \text{ and } \quad \mathbb{R}_u \succ \mathbb{R}_v := \sum_{w \in u \succ v} \mathbb{R}_w.
\]
\[ \Delta_\prec (R_u) := \sum_{i=k}^{n-1} \mathbb{R}_{\text{pack}}(u_1 \cdots u_i) \otimes \mathbb{R}_{\text{pack}}(u_{i+1} \cdots u_n), \]  
(1.8)

\[ \Delta_\succ (R_u) := \sum_{i=1}^{k-1} \mathbb{R}_{\text{pack}}(u_1 \cdots u_i) \otimes \mathbb{R}_{\text{pack}}(u_{i+1} \cdots u_n), \]  
(1.9)

Example 8.
\[ R_{211} \prec R_{12} = R_{21341} + R_{23141} + R_{23411} + R_{32141} + R_{32411} + R_{34211}, \]
\[ R_{221} \succ R_{12} = R_{21134} + R_{21314} + R_{23114} + R_{32114}, \]
\[ \Delta_\prec (R_{2125334}) = R_{2123} \otimes R_{112} + R_{212433} \otimes R_1, \]
\[ \Delta_\succ (R_{2125334}) = R_{212} \otimes R_{3112}. \]

Theorem 9. [8, Theorem 2.5] \( (\text{WQSym})_+, \prec, \succ, \Delta_\prec, \Delta_\succ \) is a bidendriform bialgebra.

From now on Prim(\text{WQSym}) and TPrim(\text{WQSym}) are respectively abbreviated to Prim and TPrim. Moreover, we denote homogeneous components using indices as in Prim\(_n\). We give the first values of the dimensions \( a_n := \dim(\text{WQSym}_n) \), \( p_n := \dim(\text{Prim}_n) \) and \( t_n := \dim(\text{TPrim}_n) \):

| n  | 1 | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | OEIS |
|----|----|----|----|----|----|----|----|----|----|------|
| \( a_n \) | 1  | 3  | 13 | 75 | 541 | 4 683 | 47 293 | 545 835 | 7 087 261 | A000670 |
| \( p_n \) | 1  | 2  | 8  | 48 | 368 | 3 376 | 35 824 | 430 512 | 5 773 936 | A095989 |
| \( t_n \) | 1  | 1  | 4  | 28 | 240 | 2 384 | 26 832 | 337 168 | 4 680 272 |   |

2 Decorated forests

In this section we will generalize the construction of [11]. We will start by defining some forests called biplane that are labeled by certain lists of integers. We will construct a bijection between these packed forests and packed words thanks to a decomposition of packed words through global descents and removal of maximums. The recursive structure of forests can then be understood as a chaining of operations generating the elements of \text{WQSym}. This will allow us to construct a basis of TPrim by characterizing a subfamily of biplane trees.

2.1 Decompositions of packed words

Definition 10. A global descent of a packed word \( w \) is a position \( c \) such that all the letters before or at position \( c \) are greater than all letters after position \( c \).
Example 11. The global descents of \( w = 54664312 \) are the positions 5 and 6. Indeed, all letters of 54664 are greater than the letters of 312 and this is also true for 546643 and 12.

**Definition 12.** A packed word \( w \) is irreducible if it has no global descent.

**Lemma 13.** Each word \( w \) admits a unique factorization as \( w = w_1/w_2/\ldots/w_k \) such that \( w_i \) is irreducible for all \( i \).

Example 14. The global descent decomposition of \( 54664312 \) is 21331/1/12. The word \( n \cdot n - 1 \cdot \ldots \cdot 1 \) has 1/1/\ldots/1 as global descent decomposition.

**Definition 15.** Fix \( n \in \mathbb{N} \) and \( w \in \text{PW}_n \). We write \( m := \max(w) + 1 \). For any \( p > 0 \) and any subset \( I \subseteq [1, \ldots, n + p] \) of cardinality \( p \), \( \phi_1(w) = u_1 \ldots u_{n+p} \) is the packed word of length \( n + p \) obtained by inserting \( p \) occurrences of the letter \( m \) in \( w \) so that they end up in positions \( i \in I \). In other words \( u_i = m \) if \( i \in I \) and \( w \) is obtained from \( \phi_1(w) \) by removing all occurrences of \( m \).

Example 16. \( \phi_{2,4,7}(1232) = 1424324 \) and \( \phi_{1,2,3}(\epsilon) = 111 \).

Let \( \text{PW}_n^I \) denote the set of packed words of size \( n \) whose maximums are in positions \( i \in I \). This way \( \phi_1(w) \in \text{PW}_n^{I+1} \). The following lemma is immediate.

**Lemma 17.** Let \( n \in \mathbb{N} \) and \( p > 0 \), for any \( I \subseteq [1, \ldots, n + p] \), \( \phi_1 \) is a bijection from \( \text{PW}_n \) to the \( \text{PW}_n^{I+1} \).

Moreover, for any \( W \in \text{PW}_\ell \) where \( \ell > 0 \) there exists a unique pair \( (I, w) \) where \( I \subseteq [1 \ldots \ell] \) and \( w \) is packed, such that \( W = \phi_1(w) \).

**Definition 18.** Let \( u, v \in \text{PW} \) with \( v \neq \epsilon \). By Lemma 17, there is a unique pair \( (I, v') \) such that \( v = \phi_1(v') \). We denote by \( I' \) the set obtained by adding \( |u| \) to the elements of \( I \). We define \( u \uparrow v = \phi_1(u/v') \). In other words, we remove the maximum letter of the right word, perform a left shifted concatenation and reinsert the removed letters as new maximums.

Example 19. \( 2123 \uparrow 322312 = \phi_{1,4,4,4}(43452212) = 4345622612 \).

**Lemma 20.** Let \( w \) be an irreducible packed word. There exists a unique factorization of the form \( w = u \uparrow \phi_1(v) \) which maximizes the size of \( u \). In this factorization

- either \( v = \epsilon \) and \( I = [1, \ldots, p] \) for some \( p \);
- or the factorization \( v = v_1/\ldots/v_r \) of \( v \) into irreducibles satisfies the inequalities \( i_1 \leq |v_1| \) and \( \sum |v_i| + |I| + 1 - i_p \leq |v_r| \).

Example 21. Here are some decompositions according to Lemma 20:

\[
\begin{align*}
21331 &= 1 \uparrow \phi_{2,3}(11) & 1231 &= \epsilon \uparrow \phi_3(121) \\
1233 &= 12 \uparrow \phi_{1,2}(\epsilon) & 111 &= \epsilon \uparrow \phi_{1,2,3}(\epsilon) \\
543462161 &= (1/212) \uparrow \phi_{1,4}(1/11) = (3212) \uparrow \phi_{1,4}(211)
\end{align*}
\]
2.2 Forests from decomposed packed words

We now apply recursively the decomposition of the former section to construct a bijection between packed words and a certain kind of trees that we now define.

**Definition 22.** An unlabeled biplane tree is an ordered tree (sometimes also called a planar) whose children are organized in a pair of two (possibly empty) ordered forests, which we call the left and right forests.

**Example 23.** The biplan trees \( \text{\includegraphics[width=0.2\textwidth]{blplane_tree1}} \), \( \text{\includegraphics[width=0.2\textwidth]{blplane_tree2}} \), and \( \text{\includegraphics[width=0.2\textwidth]{blplane_tree3}} \) are different. Indeed in the first case, the left forest contains two trees and the right forest is empty, in the second case both forests contain exactly one tree while in the third case we have the opposite of the first case. Here is an example of a bigger biplane tree where the root has two trees in both left and right forests \( \text{\includegraphics[width=0.4\textwidth]{blplane_tree4}} \).

**Remark 24.** These biplan trees are counted by the sequence A006013 in OEIS.

In our construction we will deal with labeled biplane trees where the labels are sorted lists of positive integer. For a labeled biplane tree, we denote by Node\((x, f_\ell, f_r)\) the tree whose root is labeled by \(x\) and whose left (resp. right) forest is given by \(f_\ell\) (resp. \(f_r\)). We also denote by \([t_1, \ldots, t_k]\) a forest of \(k\) trees.

**Example 25.** Node\((((1), [], []), \text{\includegraphics[width=0.2\textwidth]{blplane_tree1}})) = 1\), and Node\(((1,3), [], [\text{Node}(1), [], []])) = \text{\includegraphics[width=0.2\textwidth]{blplane_tree5}}\).

We now apply recursively the decompositions of Lemmas 13 and 20 to get an algorithm which takes a packed word and returns a biplane forest where nodes are decorated by lists of integers:

**Definition 26.** The forest \(F(w)\) (resp. tree \(T(w)\)) associated to a packed word (resp. a non empty irreducible packed word) \(w\) are defined in a mutual recursive way as follows:

- \(F(\epsilon) = []\) (empty forest),
- for any packed word \(w\), let \(w_1/w_2/\ldots/w_k\) be the global descent decomposition of \(w\), then \(F(w) := [T(w_1), T(w_2), \ldots, T(w_k)]\).
- for any non empty irreducible packed word \(w\), we define \(T(w) := \text{Node}(I, F(u), F(v))\) where \(w = u \triangleright \phi_1(v)\) and \(u\) is of maximal length.

**Example 27.** Let \(w = 876795343912\), the decomposition of Lemma 13 gives \(w = w_1/w_2\) with \(w_1 = 6545731217\) and \(w_2 = 12\). Now, we decompose \(w_1\) and \(w_2\) using Lemma 20 as

\[w_1 = 3212 \triangleright \phi_{1,6}(3121) = (1/212) \triangleright \phi_{1,6}(1/121), \quad \text{and} \quad w_2 = 1 \triangleright \phi_1(\epsilon).\]
Basis of totally primitive elements of \( \text{WQSym} \)

It gives the following forest:

\[
F(876795343912) = [T(6545731217), T(12)]
\]

\[
= \begin{array}{c}
\begin{array}{c}
1, 6 \\
F(3212)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T(6545731217) \\
F(1)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T(12) \\
F(e)
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
1, 3 \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1, 2 \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1, 2 \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1, 3 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1, 2 \\
1, 2
\end{array}
\end{array}
\]

As we can see in this example, in order to simplify the notation, we do not write the label when it is the list 1. We now characterize the trees obtained this way:

**Definition 28.** Let \( t \) be a labeled biplane tree. We write \( t = \text{Node}(I, f_\ell, f_r) \) where \( I = [i_1, \ldots, i_p] \), \( f_\ell = [\ell_1, \ldots, \ell_g] \) and \( f_r = [r_1, \ldots, r_d] \), which is depicted as follows:

\[
t = \begin{array}{c}
\begin{array}{c}
I \\
\ell_1 \ldots \ell_g
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
r_1 \ldots r_d
\end{array}
\end{array}
\]

The weight of \( t \) is recursively defined by \( \omega(t) = p + \sum_{i=0}^{g} \omega(\ell_i) + \sum_{j=0}^{d} \omega(r_j) \). In particular, if \( t \) is a leaf then \( \omega(t) = p \). The right-weight of \( t \) is defined by \( \omega_r(t) = p + \sum_{j=0}^{d} \omega(r_j) \). We say that \( t \) is a **packed tree** if it satisfies:

- If there are no right children \( (d = 0) \) then \( i_k = k \) for all \( k \).
- Otherwise:
  \[
  \begin{aligned}
  1 \leq i_1 & \leq \omega(r_1), \\
  1 \leq \omega_r(t) + 1 - i_p & \leq \omega(r_d).
  \end{aligned}
  \tag{2.1}
  \]

- \( f_\ell \) and \( f_r \) are packed forests (i.e. lists of packed trees).

For the rest of the article we denote the set of packed forests of weight \( n \) by \( \mathfrak{F}_n \) and the set of packed trees of weight \( n \) by \( \mathfrak{T}_n \). We also denote by \( \mathfrak{F}^I_n \) the set of packed forests that satisfy Equation (2.1) for \( I = (i_1, \ldots, i_p) \) with \( 0 < i_1 < \ldots < i_p \). Finally we denote the set of packed trees of weight \( n \) with an empty left forest by \( \mathfrak{P}_n \).

There is a unique packed forest of weight 1, namely \( \circ \), here are the packed forests of weight 2: \( \circ \circ \), \( \circ , \circ \), \( 1, 2 \). We show below the packed forests of weight 3:

\[
\begin{array}{c}
\begin{array}{c}
\circ \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ , \circ , \circ \\
1, 2
\end{array}
\end{array}
\]

The following theorem is a generalisation of the construction of [11] for \( \text{FQSym} \) and permutations to \( \text{WQSym} \) and packed words.
Theorem 29. For all $n \in \mathbb{N}$ we have the three following equalities:

$$\dim(WQSym_n) = \#\mathfrak{S}_n \quad \text{and} \quad \dim(\text{Prim}_n) = \#\mathfrak{T}_n \quad \text{and} \quad \dim(TPrim_n) = \#\mathfrak{P}_n.$$ 

Proof. The construction of Definition 26 defines a bijection from packed words to packed forests which restrict to a bijection from irreducible packed words to packed trees. Thanks to Proposition 1 this proves the first two equalities. Recall that a basis of primitive elements is given by Theorem 2 as ordered trees decorated by totally primitive elements. If we consider the label $I$ together with the right forest of each node as decoration, we get that packed trees are in bijection with planar trees decorated by $I$ and an element of $\mathfrak{S}_I$ that is an element of $\mathfrak{P}_I$. \qed

3 A basis for totally primitive elements

In this section we construct a basis of primitive and totally primitive elements of $WQSym$. Thanks to Theorem 29 we now have the combinatorial objects to index those basis. To have the linear independency, we need to show that the decomposition through maximum is compatible with the algebraic structure.

3.1 Decomposition through maximums and totally primitive elements

Definition 30. Let $I = (i_1, \ldots, i_p)$ with $0 < i_1 < \ldots < i_p$. We define the linear map $\Phi_I : WQSym \to WQSym$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in PW_n$,

$$\Phi_I(R_w) := \begin{cases} R_{\phi_I(w)} & \text{if } i_p \leq n+p, \\ 0 & \text{if } i_p > n+p. \end{cases} \quad (3.1)$$

Definition 31. Let $I = (i_1, \ldots, i_p)$ with $0 < i_1 < \ldots < i_p$. We define the projector $\tau_I : WQSym \to WQSym$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in PW_n$,

$$\tau_I(R_w) := \begin{cases} R_w & \text{if } w_i = \max(w) \text{ if and only if } i \in I, \\ 0 & \text{else}. \end{cases} \quad (3.2)$$

These are orthogonal projectors in the sense that $\tau_I^2 = \tau_I$ and $\tau_I \circ \tau_I = 0 \ (I \neq J)$.

Lemma 32. For any $I$, we have $\text{Im}(\Phi_I) = \text{Im}(\tau_I)$ where $\text{Im}(f)$ denotes the image of $f$.

Lemma 33. For any $I$, the projection by $\tau_I$ of a totally primitive element is still a totally primitive element, so that $\tau_I(TPrim) = \text{Im}(\tau_I) \cap TPrim$. Moreover,

$$TPrim = \bigoplus_I \text{Im}(\tau_I) \cap TPrim. \quad (3.3)$$

Proof. Let $w$ a packed word. We have $\Delta_\prec(\tau_I(R_w)) = (\tau_I \otimes \text{Id}) \circ \Delta_\prec(R_w)$ by definition of $\tau$ and $\Delta_\prec$. By linearity, for all $p \in TPrim$, we have $\Delta_\prec(\tau_I(p)) = (\tau_I \otimes \text{Id}) \circ \Delta_\prec(p) = 0$. The same argument works on the right so that $\tau_I(p) \in TPrim$. Moreover $\tau_I$ are orthogonal projectors so $TPrim = \bigoplus_I \tau_I(TPrim) = \bigoplus_I \text{Im}(\tau_I) \cap TPrim$. \qed
3.2 The new basis \( \mathcal{P} \)

**Definition 34.** Let \( t_1, \ldots, t_k \in \mathcal{T} \) and \( f_t = [\ell_1, \ldots, \ell_g], f_r \in \mathcal{F} \),

\[
\mathcal{P} := R_1, \\
\mathcal{P}_{t_1, \ldots, t_k} := \mathcal{P}_{t_k} \prec (\mathcal{P}_{t_k-1} \prec (\cdots \prec \mathcal{P}_{t_1}) \cdots), \\
\mathcal{P}_{\text{Node}(I facile f_t)} := \Phi_I(\mathcal{P}_{f_t}), \\
\mathcal{P}_{\text{Node}(I = \ell, \ldots, \ell_g, f_r)} := \langle \mathcal{P}_{t_1}, \mathcal{P}_{t_2}, \ldots, \mathcal{P}_{t_g}; \Phi_I(\mathcal{P}_{f_r}) \rangle. 
\]

**Example 35.**

\[
\mathcal{P} = \mathcal{P} \prec \mathcal{P} = (\mathcal{P} \prec \mathcal{P} \prec \mathcal{P} \prec \mathcal{P} \prec \mathcal{P}) \prec \Phi_{1,3}(\mathcal{P} \prec \mathcal{P}) = R_{14342} + R_{41342} + R_{43142} + R_{43412} - R_{24341} - R_{42341} - R_{43241} - R_{43421}
\]

**Theorem 36.** For all \( n \in \mathbb{N}_{>0} \)

1. \( \langle \mathcal{P}_f \rangle_{f \in \mathcal{F}} \) is a basis of \( \text{WQSym}_n \),

2. \( \langle \mathcal{P}_t \rangle_{t \in \mathcal{P}_n} \) is a basis of \( \text{Prim}_n \),

3. \( \langle \mathcal{P}_t \rangle_{t \in \mathcal{P}_n} \) is a basis of \( \text{TPrim}_n \).

**Proof.** As \( \dim(\text{WQSym}_1) = \dim(\text{Prim}_1) = \dim(\text{TPrim}_1) = 1 \) the base case is trivial. We argue in a mutually recursive way: by **Proposition 1**, **Item 2** up to degree \( n \) implies **Item 1** up to degree \( n \). Similarly, **Theorem 2** shows that **Item 3** up to degree \( n \) implies **Item 2** up to degree \( n \). By induction it is sufficient to show that **Items 1 and 2** up to degree \( n - 1 \) implies **Item 3** for \( n \).

For all \( k \in \mathbb{N} \), let \( \pi_k \) be the projector on the homogeneous component of degree \( k \)

of \( \text{WQSym} \). We define \( \pi_{<k} := \sum_{i=0}^{k-1} \pi_i \). Fix \( I \) of length \( p \). In the coproduct \( \Delta_{<}(\Phi_I(R_u)) \)

all the maximums must be in the left tensor factor, which therefore must be at least of degree \( i_p \). By linearity, this can be used to prove that \( \Delta_{<}(\Phi_I(\mathcal{P}_f)) = 0 \). A similar reasoning applies to \( \Delta_{<}(\Phi_I(x)) \), so that thanks to the conditions of **Equation (2.1)**, \( \mathcal{P}_t \)

with \( t \in \mathcal{P}_n \) is totally primitive.

In other words, if we define \( \text{Prim}_n(i, j) := \text{Ker}(\pi_{<i} \otimes \pi_{<j}) \circ \Delta \) we have proved that the image of the restriction to \( \text{Prim}_n(i_1, n + 1 - i_p) \) of \( \Phi_I \) is included in \( \text{Im}(\tau_I) \cap \text{TPrim}_n \). By **Proposition 1**, \( \{ \mathcal{P}_f | f \in \mathcal{F}_{n-p} \} \) is a basis of \( \text{Prim}_{n-p}(i_1, n + 1 - i_p) \). Since \( \Phi_I \) is injective on \( \text{WQSym}_{n-p} \), then \( \{ \Phi_I(\mathcal{P}_f) | f \in \mathcal{F}_{n-p} \} \) are linearly independent. Then by **Lemma 33** \( \{ \mathcal{P}_t | t \in \mathcal{P}_n \} \) are linearly independent. By **Theorem 29** it is a basis of \( \text{TPrim}_n \). □

**Conclusion**

Our next obvious step is to adapt the decomposition through maximum of **Definitions 30** and 31 to the dual in order to get an explicit bidendriform isomorphism from \( \text{WQSym} \).
to its dual. Another generalization which should be easy is to do the same for the Hopf algebra \textbf{PQSym} of parking functions. Indeed \textbf{PQSym} is a bidendriform bialgebra [8], and thus self-dual, but no explicit isomorphism with the dual is known.

**Acknowledgments**  The computation and tests needed for this research were done using the open-source mathematical software \textsc{Sage} and its combinatorics features developed by the \textsc{Sage-combinat} community. I particularly thank F. Hivert and V. Pons for all comments on the writing.

**References**

[1] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J. Thibon, “Noncommutative Symmetrical Functions,” *Adv. in Math.*, vol. 112, no. 2, pp. 218 – 348, 1995.

[2] J.-L. Loday and M. O. Ronco, “Hopf Algebra of the Planar Binary Trees,” *Adv. in Math.*, vol. 139, no. 2, pp. 293 – 309, 1998.

[3] C. Malvenuto and C. Reutenauer, “Duality between Quasi-Symmetrical Functions and the Solomon Descent Algebra,” *J. of Algebra*, vol. 177, no. 3, pp. 967 – 982, 1995.

[4] F. Hivert, *Combinatoire des fonctions quasi-symétriques*. PhD thesis, Informatique fondamentale Université de Marne-la-Vallée, 1999.

[5] L. Foissy, “Bidendriform bialgebras, trees, and free quasi-symmetric functions,” *Journal of Pure and Applied Algebra*, vol. 209, no. 2, p. 439–459, 2007.

[6] Y. Vargas, *Algèbres de Hopf de mots tassés et de fonctions motifs*. PhD thesis, Université du Québec à Montréal, 2019.

[7] N. Bergeron and M. Zabrocki, “The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and cofree,” 2005.

[8] J. C. Novelli and J. Y. Thibon, “Polynomial realizations of some trialgebras,” 2006.

[9] M. Ronco, “Primitive elements in a free dendriform algebra,” *Contemp. Math.*, vol. 267, 01 2000.

[10] F. Chapoton, “Un théorème de Cartier–Milnor–Moore–Quillen pour les bigèbres dendriformes et les algèbres braces,” *J. of Pure and Applied Algebra*, vol. 168, no. 1, pp. 1 – 18, 2002.

[11] L. Foissy, “Primitive elements of the Hopf algebra of free quasi-symmetric functions,” *Contemporary Mathematics Combinatorics and Physics*, p. 79–88, 2011.