Geometric aspects of negative refraction

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Abstract. Using Ptolemy's theorem and inequality, it is shown that the law of negative refraction of an optical ray on the interface between an ordinary isotropic medium and an electromagnetic metamaterial with negative permittivity and negative permeability corresponds to the Fermat’s principle of least time. The optical path length of the ray between two corresponding points at positive and negative refraction is compared.

1. Introduction
The interest in metamaterials – artificial electromagnetic substances with values of material parameters not peculiar to natural materials and media, manifested in the early 2000s, actualized the interest in the little-known phenomenon of “negative” refraction [1]. With negative refraction on the smooth surface of the separation of the two media, the refracted ray is located on the same side relative to the normal to the boundary of the media as the incident ray. An unusual deflection of the ray, as L.I. Mandel’shtam had first shown [2], is explained by the fact that the incident and refracted waves differ in the type of energy transfer: one of them has the direction of the wave energy flow coincides with the direction of the phase velocity (forward wave), while the other is directly opposite (backward wave). In macroscopically isotropic and transparent electromagnetic metamaterials, backward normal waves exist if the effective material parameters (dielectric permittivity and magnetic permeability) are simultaneously negative values. As it turned out, the scalar form of Snell’s law is also suitable for the case of negative refraction, if the refractive index of the “left-handed” medium with an backward wave is considered a negative value [3]. This assumption, however, does not agree with the definition of the refractive index as the modulus of the refractive vector [4] and may be the cause of misunderstandings.

2. Negative refraction and Fermat’s principle of least time
In particular, the negative refractive index requires a revision of the Fermat’s principle of least time [5], but this position was not further confirmed [6, 7]. In this paper, we propose another version of the proof of the correlation of negative refraction to Fermat’s principle, which is based on the Ptolemy’s theorem and the Ptolemy’s inequality for a quadrangle inscribed in a circle. The elegant geometric proof of the law of positive refraction, carried out on this basis, was first published over fifty years ago [8] and subsequently repeatedly reproduced [9, 10].

The vector form of the Snell–Descartes laws is derived in relation to the wave vectors (or refraction vectors) of plane waves interacting with the surface of different media, but not to their Umov–Poynting’s vectors [4]. In case of negative refraction, the refractive vector of the refracted wave, as well as the refractive vector of the reflected wave, is directed from the boundary into the same medium in which the incident wave propagates [11], and the scalar form of the refractive law has the form

\[ n_i \sin \theta_i = n_f \sin (\pi - \theta_f), \]
where $n_i$ and $n_r$ are the refractive indices of the medium with the incident wave and the medium with the refracted wave, respectively. The oppositely directed flow of wave energy (refracted geometric-optical ray), as is evident, forms with the direction of the normal to the boundary negative angle $-\theta_i$.

The refractive index $n$ and the phase velocity $v$ of the wave are related by $n = c/v$, where $c$ is the speed of light in vacuum. Therefore, formula (1) can be rewritten as $v_i \sin \theta_i = v_r \sin \theta_r = K = \text{const}$. Figure 1 shows the trajectory $P_1Q_2P$ of the refraction vectors, the refractive point of the rays $Q$ is located on the interface of the media so that the angle of incidence $\theta_i$ and the angle of refraction $\theta_r$ are in agreement with the formula (1). Through the three endpoints of the ray path, you can draw a circle that intersects with the normal line to the boundary of the media at the point $A$. As a result of the construction, a quadrangle $PQ_2PA$ is formed, for which Ptolemy's theorem is valid. The theorem states that the product of the diagonals of a quadrangle is equal to the sum of the products of the opposite sides:

$$AQ \cdot P_2 = P_1Q \cdot AP_2 + P_2Q \cdot AP_1.$$  \hspace{1cm} (2)

\[ \text{Figure 1. The course of rays at negative refraction and the quadrangle of Ptolemy's theorem} \]

The sides of the quadrangle $AP_1$ and $AP_2$ are the chords on which the inscribed angles $\angle P_1QA = \theta_i$ and $\angle P_2QA = \theta_r$ rest, therefore $AP_1 = 2r \sin \theta_i = 2rK/v_i$ and $AP_2 = 2r \sin \theta_r = 2rK/v_r$. Formula (2) takes the form

$$AQ \cdot P_2 = 2rK \left( \frac{P_1Q}{v_i} + \frac{P_2Q}{v_r} \right),$$ \hspace{1cm} (3)

where the expression in parentheses is the time it takes the ray to traverse the path $P_1Q_2$.

Let's assume that on the boundary of the media there is another point $Q^*$ instead of a point $Q$, for which the time of the route is the shortest. Since the point $Q^*$ does not lie on the circle of radius $r$, according to Ptolemy's inequality, instead of equality (2), the inequality

$$AQ^* \cdot P_1P_2 < 2rK \left( \frac{P_1Q^*}{v_i} + \frac{P_2Q^*}{v_r} \right).$$ \hspace{1cm} (4)
In a triangle $\triangle AQQ^*$, the side $AQ^*$ is the hypotenuse, therefore $P_1P_2 \cdot AQQ^* > P_1P_2 \cdot AQ$. Combining this inequality with inequality (4) and taking into account (3), we obtain that

$$\frac{P_1Q}{v_i} + \frac{P_2Q}{v_f} < \frac{P_1Q^*}{v_i} + \frac{P_2Q^*}{v_f},$$

that is, the Fermat’s principle corresponds to the point $Q$, and not $Q^*$.

In contrast to [8–10], here the circle and the inscribed quadrilateral are located above the interface line. This circumstance makes it possible to extend this version of the proof of compliance with Fermat’s principle using Ptolemy’s inequality to the case of specular reflection by putting $n_t = n_i$.

3. **Optical path length at positive and negative refraction**

The principle of least time is applied separately to the cases of positive and negative refraction. In order to compare the optical path length between two points in both refractive variants, we turn to Figure 2.

![Figure 2. Ray pattern of positive and negative refraction](image)

It has a straight line segment $P_1CP_2$ which corresponds to the trajectory of the ray in a homogeneous medium, and the point $C$ is on the boundary line of the media. If there is a refractive index contrast, the refractive point does not coincide with the point $C$, and the boundary line is divided into areas that differ in the type of refraction. Positive refraction corresponds to the area $AB$, the length of which is equal to the projection of the ray path $P_1O_+P_2$ on the boundary of the media. The position of the refractive point $O_+$ within the segment $AC$ corresponds to the refraction from a denser to a less dense medium, and the position on the site $CB$ (as shown in Figure 2) – refraction into a denser medium. With negative refraction, the refractive point $O_-$ of the ray path $P_1O_-P_2$ is located to the right of the point $B$ (refraction into a denser medium) or to the left of the point $A$ (refraction into a denser medium).

Let $l_i$ ($l_f$) be the geometric length of the section $P_1O_+$ ($P_1O_-$) of the positive (negative) refraction path for the incident ray. Similarly, $l_i$ ($l_f$) is the geometric length of the section $O_+P_2$ ($O_-P_2$) of the positive (negative) refraction path for the refracted ray. The relative refractive index $n$ is related to the angles of incidence and refraction by ratio

$$n = \frac{n_i}{n_t} = \frac{\sin \varphi_i}{\sin \varphi_f} = \frac{\sin \theta_i}{\sin \theta_f}.$$  \hspace{1cm} (5)
When considering triangles $\triangle_{\text{I}}O_1O_2$ and $\triangle_{\text{II}}O_1O_2$ taking into account (5) we find:

$$L_i = l_i \frac{\cos \varphi_i}{\cos \theta_i}, \quad L_t = l_i \frac{\cos \varphi_t}{\cos \theta_t}.$$  \hfill (6)

From Figure 2 it follows that $\varphi_i < \theta_i$ and $\varphi_t < \theta_t$, therefore, not only $L_i > l_i$, $L_t > l_i$, but in general $L_i + nL_t > l_i + nL_t$ – the optical path length at negative refraction is greater than at positive refraction.

In particular, for the value $n = 1$ that corresponds to the negative refraction at the boundary of the vacuum and the “anti-vacuum”, we have $L_i + nL_t = L = l \frac{\cos \varphi}{\cos \theta}$ (here $l$ – the distance $P_1C_{P_2}$, $\varphi$ – the angle of inclination of the line $P_1C_{P_2}$, $\theta$ is the incident angle under negative refraction).

4. Summary

The negative refraction of waves on the plane interface of isotropic media with two positive and two negative dielectric permittivity and magnetic permeability obeys the Fermat’s principle of least time, as well as the usual (positive) refraction. The proof of this proposition, made using Ptolemy’s theorem and inequality, extends directly to the case of specular reflection.

The optical path length of the light ray between two points located on different sides of the interface of the media, with negative refraction is greater than with positive refraction.

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References

[1] Schuster A 1904 An Introduction to the Theory of Optics (London: Edward Arnold)
[2] Mandel’shtam L I 1972 Lectures in Optics, Relativity Theory and Quantum Mechanics (Moscow: Nauka)
[3] Veselago V G 1967 Soviet Physics, Solid State 8 2854–2856
[4] Fedorov F I 1958 Optics of Anisotropic Media (Minsk: Izd. AN BSSR)
[5] Veselago V G 2002 Physics – Uspekhi 45(10) 1097–1099 doi: 10.1070/PU2002v045n10ABEH001223
[6] Fisanov V V 2015 Izv. vuzov. Fizika 58 24–27
[7] Fisanov V V 2017 Izv. vuzov. Fizika 60 8–10
[8] Pedoe D 1964 The American Mathematical Monthly 71(5) 543–544, doi: 10.2307/2312600
[9] Pedoe D 1970 A Course of Geometry for Colleges and Universities (Cambridge: Cambridge University Press) 464 p.
[10] Glassner A 1998 IEEE Computer Graphics and Applications 18 (2) 104–108 doi: 10.1109/38.656793
[11] Fisanov V V 2014 Russian Physics Journal 57(5) 691–696 doi: 10.1007/s11182-014-0292-9