Pulling back torsion line bundles to ideal classes

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Abstract

We prove results concerning the specialisation of torsion line bundles on a variety $V$ defined over $\mathbb{Q}$ to ideal classes of number fields. This gives a new general technique for constructing and counting number fields with large class group.

1 Introduction

The purpose of this paper is to study the specialisation of torsion line bundles, on a variety $V$ defined over $\mathbb{Q}$, to ideal classes of number fields, and to make this operation optimal, that is, to make the kernel of specialisation as small as possible.

The main motivation for introducing this technique is the construction of number fields whose class group has large torsion from varieties (over $\mathbb{Q}$) whose Picard group has large torsion. For instance, we show how the problem of constructing quadratic number fields with large class group can be reduced to the problem of finding a hyperelliptic curve with a rational Weierstrass point and a large rational torsion subgroup in its Jacobian (see Corollary 3.2). This technique is an abstraction and generalisation of the method used in [10], which used certain superelliptic curves to obtain new constructions of number fields with a large ideal class group. Further results on the construction of large ideal class groups will appear in a separate paper by the second author. We mention also the closely related geometric techniques used by Mestre in [11, 12, 13, 14] to construct number fields with a large ideal class group.

On the other hand, the question of pulling-back line bundles on arithmetic varieties to nontrivial ideal classes of number fields has been raised by Agboola and Pappas in [1]. Our result positively answers their question in the case of torsion line bundles on a hyperelliptic curve (see Corollary 3.8 for the precise statement).

A consequence of our main result is the following:

Theorem 1.1. Let $V$ be a smooth projective variety over $\mathbb{Q}$, and let $m > 1$ be an integer. Let $S$ be the set of places of bad reduction of $V$, and let $V \rightarrow \text{Spec}(\mathbb{Z}_S)$ be a smooth
projective model of $V$. Then there exists an infinity of number fields $K$ with a point $P \in V(K)$ such that the specialisation map

$$P^* : \text{Pic}(V)[m] = \text{Pic}(\mathcal{V})[m] \to \text{Cl}(\mathcal{O}_{K,S})[m]$$

satisfies

$$\text{rk}_m \text{Im}(P^*) \geq \text{rk}_m \text{Pic}(V)[m] + \#S - \text{rk} \mathcal{O}_{K,S}^\times.$$

More precisely, given a generically finite rational map $V \to \mathbb{A}^1$ of degree $d$, it is possible to find an infinity of such $K$ of degree $d$.

If $A$ is a finitely generated abelian group and $m \geq 2$ an integer, $\text{rk}_m A$ denotes the largest integer $r$ such that $A$ has a subgroup isomorphic to $(\mathbb{Z}/m\mathbb{Z})^r$. See §2.1 for further notation.

Of course, results analogous to those given here can also be established when the variety $V$ is defined over any number field instead of $\mathbb{Q}$. Another generalisation would be to replace line bundles, i.e. $\mathbb{G}_m$-torsors, by torsors under other group schemes, like tori.

Let us briefly describe the proof of Theorem 1.1. We first choose $r$ linearly independent elements of order $m$ in the group $\text{Pic}(V) = \text{Pic}(\mathcal{V})$ (see Lemma 2.7). By Kummer theory over $V$, it is possible to lift these $r$ elements to $r$ elements in the flat (fppf) cohomology group $H^1(V, \mu_m)$. These classes in $H^1(V, \mu_m)$ are represented by $\mu_m$-torsors $\mathcal{X}_i \to \mathcal{V}$. Let $\mathcal{X}_i \to V$ be the restriction of $\mathcal{X}_i$ to $V$, and let $X \to V$ be the fiber product over $V$ of the $\mathcal{X}_i$. We choose a generically finite rational map $V \to \mathbb{A}^1$ of degree $d$ and apply Hilbert’s irreducibility theorem to the composite cover $X \to V \to \mathbb{A}^1$. Thus, we get an infinity of points in $\mathbb{A}^1$ whose inverse image in $V$ is a point $P \in V(K)$, with $[K: \mathbb{Q}] = d$. Moreover, the images of the $\mathcal{X}_i$ by the map

$$P^* : H^1(V, \mu_m) \to \text{Cl}(\mathcal{O}_{K,S}, \mu_m)$$

are $r$ linearly independent elements of order $m$. Now, we look at the image of these elements by the natural map $c : H^1(\mathcal{O}_{K,S}, \mu_m) \to \text{Cl}(\mathcal{O}_{K,S})$. By Kummer theory (see §2.1), the kernel of this map is $\mathcal{O}_{K,S}^\times/m$, but we can improve the situation by asking that the subgroup generated by the $P^* \mathcal{X}_i$ has trivial intersection with the subgroup $\mathbb{Z}/m\mathbb{Z}$. It is also possible to ensure that $K$ is linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_{2m})$. Under these conditions, the kernel of the map $c \circ P^*$ is isomorphic to a subgroup of $\left(\mathbb{Z}/m\mathbb{Z}\right)^{\text{rk} \mathcal{O}_{K,S}^\times - \#S}$, which proves the result after invoking Lemma 2.6.

The plan of this paper is as follows. In section 2, after recalling effective versions of Hilbert’s irreducibility theorem, we prove our main result, Theorem 2.4 involving a variety $V$ that satisfies some special properties with respect to some set $S$ of prime numbers. Then, using the theory of integral models, we give consequences of our result in the case when $V$ is smooth and $S$ is the set of primes of bad reduction of $V$. In section 3, using superelliptic and hyperelliptic curves, we give applications of our technique to the construction of number fields with large class group. Finally, we explain how these results can be applied, in the case of torsion line bundles on hyperelliptic curves, to the question of Agboola and Pappas.
2 Specialisation of $\mu_m$-torsors

2.1 Notation and definitions

By a variety over a field $k$, we will mean a geometrically connected separated scheme of finite type over $k$.

If $A$ is an abelian group, and if $m > 1$ is an integer, we will denote by $A[m]$ the kernel of multiplication by $m$ in $A$ (that is, the $m$-torsion of $A$), and by $A/m$ the cokernel of multiplication by $m$ in $A$. Also, if $A$ is a finitely generated abelian group, the $m$-rank of $A$, which we denote by $\text{rk}_m A$, is the largest integer $r$ such that $A$ has a subgroup isomorphic to $(\mathbb{Z}/m)^r$.

In the following, $S$ will denote a finite set of prime numbers. If $K$ is a number field, we will denote by $\mathcal{O}_{K,S}$ the ring of $S_K$-integers of $K$, where $S_K$ is the set of primes of $K$ lying above primes in $S$. Namely, if we denote by $\mathcal{M}_K$ the set of prime ideals of $K$,

$$\mathcal{O}_{K,S} = \{ x \in K \mid \forall \mathfrak{p} \in \mathcal{M}_K \setminus S_K, v_{\mathfrak{p}}(x) \geq 0 \}.$$

In the particular case when $K = \mathbb{Q}$, we denote this ring by $\mathbb{Z}_S$ instead of $\mathcal{O}_{\mathbb{Q},S}$. Let us note that $\mathbb{Z}_S$ is nothing else than the ring of fractions $\mathbb{Z}[S^{-1}]$.

We recall that, by Kummer theory, we have a canonical isomorphism

$$H^1(K, \mu_m) \simeq K^x/m.$$ 

Under this isomorphism, one identifies the first cohomology group $H^1(\mathcal{O}_{K,S}, \mu_m)$, computed using the fppf topology, as the following subgroup of $H^1(K, \mu_m)$:

$$H^1(\mathcal{O}_{K,S}, \mu_m) = \{ z \in K^x/m \mid \forall \mathfrak{p} \in \mathcal{M}_K \setminus S_K, v_{\mathfrak{p}}(z) \equiv 0 \pmod{m} \}.$$ 

In particular, we have the Kummer exact sequence

$$0 \longrightarrow \mathcal{O}_{K,S}/m \longrightarrow H^1(\mathcal{O}_{K,S}, \mu_m) \longrightarrow \text{Cl}(\mathcal{O}_{K,S})[m] \longrightarrow 0 \quad (1)$$

which follows from fppf Kummer theory over $\text{Spec}(\mathcal{O}_{K,S})$. Let us note that all the groups involved here are $m$-torsion.

Unless specified, all our cohomology groups are computed with respect to the fppf topology.

2.2 Hilbert’s Irreducibility Theorem

An essential tool in our proofs is Hilbert’s Irreducibility Theorem, which we now recall in a suitably general, quantitative form due to Cohen [5] (see also [17, Ch. 9]).

For a point $t = (t_1, \ldots, t_n) \in \mathbb{A}^n(\mathbb{Z})$, define the height $H(t) = \max_j |t_j|$.
Theorem 2.1 (Hilbert Irreducibility Theorem). Let \( V \) be a variety over \( \mathbb{Q} \) of dimension \( l \). Let \( \phi : V \to \mathbb{A}^l \) be a generically finite rational map. For \( t \in \phi(V)(\mathbb{Q}) \), let \( P_t = \phi^{-1}(t) \). Let \( k \) be a number field. Then for all but \( O\left(N^{l-\frac{1}{2}} \log N\right) \) points \( t \in \mathbb{A}^l(\mathbb{Z}) \) with \( H(t) \leq N \), we have \( t \in \phi(V) \) and \( P_t = \text{Spec}(\mathbb{Q}(P_t)) \), where \([k(P_t) : k] = \deg \phi\). If \( l = 1 \), \( O\left(\sqrt{N} \log N\right) \) can be replaced by \( O\left(\sqrt{N}\right) \) in the above.

A very natural problem that arises in connection with Hilbert’s Irreducibility Theorem is to determine the number of distinct number fields \( \mathbb{Q}(P_t) \) with \( H(t) \leq N \) in Theorem 2.1. For \( l = 1 \) this problem has been studied by Dvornicich and Zannier. Let \( C \) be a curve over \( \mathbb{Q} \) and let \( \phi : C \to \mathbb{P}^1 \) be a morphism. For each integer \( t \in \mathbb{A}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{Q}) \), let \( P_t = \phi^{-1}(t) \). Dvornicich and Zannier [6, 7] studied the degree of the field extension \( \mathbb{Q}(P_1, \ldots, P_N) \). Their results imply in particular a useful result on the number of isomorphism classes of number fields in the set \( \{ \mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N) \} \).

Theorem 2.2 (Dvornicich, Zannier). Let \( C \) be a curve over \( \mathbb{Q} \). Let \( \phi : C \to \mathbb{P}^1 \) be a morphism with \( \deg \phi > 1 \). For each integer \( t \), let \( P_t = \phi^{-1}(t) \). Let \( g(N) \) denote the number of isomorphism classes of number fields in the set \( \{ \mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N) \} \). Then \( g(N) \gg \frac{N}{\log N} \).

Remark 2.3. The implied constants in Theorems 2.1 and 2.2 are effective and can be explicitly computed.

2.3 The main result

Let \( V \) be a variety over \( \mathbb{Q} \). By Kummer theory, we have an exact sequence

\[
0 \longrightarrow \mathbb{G}_m(V)/m \longrightarrow H^1(V, \mu_m) \longrightarrow \text{Pic}(V)[m] \longrightarrow 0.
\]

Given \( L \in \text{Pic}(V)[m] \), we say that a \( \mu_m \)-torsor on \( V \) is a lift of \( L \) if its image by the right-hand side map is equal to \( L \).

Theorem 2.4. Let \( V \) be a variety of dimension \( l \) over \( \mathbb{Q} \), and let \( m > 1 \) be an integer. Assume that there exists a finite set \( S \) of prime numbers such that, for any \( L \in \text{Pic}(V)[m] \), there exists a \( \mu_m \)-torsor \( T \to V \) lifting \( L \) such that, for any number field \( K \) and for any point \( P \in V(K) \), the pull-back \( P^*T \) belongs to \( H^1(K, \mathcal{O}_{K,S}, \mu_m) \).

Let \( \phi : V \to \mathbb{A}^l \) be a generically finite rational map. For \( t \in \phi(V)(\mathbb{Q}) \), let \( P_t := \phi^{-1}(t) \). Then for all but \( O\left(N^{l-\frac{1}{2}} \log N\right) \) \( (O(\sqrt{N}) \text{ if } l = 1) \) points \( t \in \mathbb{A}^l(\mathbb{Z}) \) with \( H(t) \leq N \), we have \( t \in \phi(V) \), \( P_t = \text{Spec}(\mathbb{Q}(P_t)) \) where \( \mathbb{Q}(P_t) \) is a number field of degree \( \deg \phi \), and

\[
\text{rk}_m \text{Cl}(\mathcal{O}_{\mathbb{Q}(P_t), S}) \geq \text{rk}_m \text{Pic}(V)_{\text{tors}} + \#S - \text{rk} \mathcal{O}_{\mathbb{Q}(P_t), S}^\times.
\]

(2)
Proof. Let \( r = \text{rk}_m \text{Pic}(V)_{\text{tors}} \). Using the hypotheses of the theorem, we obtain \( r \) torsors \( X_i \to V \) that generate a subgroup isomorphic to \((\mathbb{Z}/m)^r\) in \( H^1(V, \mu_m) \), and that satisfy the key property: for any number field \( K \) and for any point \( P \in V(K) \), the pull-back \( P^*X_i \) belongs to \( H^1(O_{K,S}, \mu_m) \).

Let \( X \) be the product of the \( X_i \) over \( V \). Then \( f : X \to V \) is irreducible of degree \( m^r \), because the \( X_i \to V \) are irreducible and linearly independent in \( H^1(V, \mu_m) \). Let \( k = \mathbb{Q}[\sqrt[2]{u} \mid u \in \mathbb{Z}_S^\times] \). One can apply Hilbert’s irreducibility theorem to the composition \( X \to V \to \mathbb{A}^l : \) for all but \( O \left( N^{l/2} \log N \right) \) \((O(\sqrt{N}) \text{ if } l = 1)\) points \( t \in \phi(V) \cap \mathbb{A}^n(\mathbb{Z}) \) with \( H(t) \leq N \), we have \((\phi \circ f)^{-1}(t) = \text{Spec}(F)\) where \( F \) is a field and \([F : \mathbb{Q}] = [kF : k] = (\deg \phi)(\deg f)\). Now, if \( t \) is such a point, then \( \phi^{-1}(t) = P \in V(K) \) with \( K \subseteq F \) and \([K : \mathbb{Q}] = \deg(\phi)\). Then, pulling back the torsors \( X_i \to V \) along \( P \), we get \( r \) independent elements in \( H^1(K, \mu_m) \) that, by the key property, belong to the subgroup \( H^1(O_{K,S}, \mu_m) \). In other words,

\[
\text{rk}_m H^1(O_{K,S}, \mu_m) \geq r.
\]

In fact, since \( F \) is linearly disjoint from \( k \), we have

\[
\text{rk}_m H^1(O_{K,S}, \mu_m)/(\mathbb{Z}_S^\times/m) \geq r,
\]

where we identify \( \mathbb{Z}_S^\times/m \) with its image under the composite map \( \mathbb{Z}_S^\times/m \to O_{K,S}^\times/m \to H^1(O_{K,S}, \mu_m) \). Note that \( K \) is also linearly disjoint from \( k \), and in particular is linearly disjoint from the cyclotomic field \( \mathbb{Q}(\zeta_{2m}) \), which is contained in \( k \). So the map \( \mathbb{Z}_S^\times/m \to O_{K,S}^\times/m \) is injective and

\[
(O_{K,S}^\times/m)/(\mathbb{Z}_S^\times/m) \cong (\mathbb{Z}/m)^{\text{rk} O_{K,S}^\times - \#S}.
\]

On the other hand, starting from the Kummer exact sequence \((\ref{Kummer})\), we deduce an exact sequence

\[
0 \to (O_{K,S}^\times/m)/(\mathbb{Z}_S^\times/m) \to H^1(O_{K,S}, \mu_m)/(\mathbb{Z}_S^\times/m) \to \text{Cl}(O_{K,S})[m] \to 0.
\]

This implies, by the second statement of Lemma \((\ref{lemma2.6})\) that

\[
\text{rk}_m \text{Cl}(O_{K,S})[m] = \text{rk}_m H^1(O_{K,S}, \mu_m)/(\mathbb{Z}_S^\times/m) - (\text{rk} O_{K,S}^\times - \#S) \geq r + \#S - \text{rk} O_{K,S}^\times,
\]

hence the result. \( \square \)

Remark 2.5. (i) In the theorem, the set \( S \) a priori depends on \( V \) and on \( m \). But, in the applications we give, \( S \) only depends on \( V \).

(ii) The inequality \((\ref{inequality2})\) also holds for the group \( \text{Cl}(\mathbb{Q}(P_i)) \), because \( \text{Cl}(O_{\mathbb{Q}(P_i),S}) \) is a quotient of that group.
Lemma 2.6. Let

\[ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \]  

be an exact sequence of finite \( m \)-torsion abelian groups. Then

\[ \text{rk}_m B \geq \text{rk}_m A + \text{rk}_m C. \]

Moreover, if \( A \) or \( C \) is isomorphic to \((\mathbb{Z}/m)^n\) for some \( n \), then

\[ \text{rk}_m B = \text{rk}_m A + \text{rk}_m C. \]

Proof. Let us first note that \( m \)-torsion abelian groups are modules over the ring \( \mathbb{Z}/m \). It follows from the definition that, if \( M \) is a finite \( m \)-torsion abelian group, and if \( N \) is a subgroup or a quotient of \( M \), then \( \text{rk}_m N \leq \text{rk}_m M \).

We prove the last statement first. Let us assume that \( C \) is isomorphic to \((\mathbb{Z}/m)^n\) for some \( n \). This means that \( C \) is a free \((\mathbb{Z}/m)\)-module of finite rank, thus is a projective object in the category of \((\mathbb{Z}/m)\)-modules. Therefore, by basic homological algebra, the exact sequence (3) splits, that is,

\[ B \cong (\mathbb{Z}/m)^n \oplus A. \]

Now, using the structure theorem for finite abelian groups, it is easy to see that, given a finite \( m \)-torsion abelian group \( M \) and an integer \( r \), the following two conditions are equivalent :

a) \( r = \text{rk}_m M \)

b) one has \( M \cong (\mathbb{Z}/m)^r \oplus M' \) where \( M' \) does not contain any element of order \( m \).

This (easy) fact allows us to conclude: using that a) \( \Rightarrow \) b), we can decompose \( A \) as \((\mathbb{Z}/m)^{\text{rk}_m A} \oplus A'\). This in turn gives us a decomposition of \( B \) as \((\mathbb{Z}/m)^{n+\text{rk}_m A} \oplus A'\) which, by b) \( \Rightarrow \) a), implies that \( \text{rk}_m B = n + \text{rk}_m A \), hence the result.

The case when \( A \) is isomorphic to \((\mathbb{Z}/m)^n\) for some \( n \) is deduced from the previous one by replacing all the groups involved by their duals in the category of \((\mathbb{Z}/m)\)-modules. In the general case, let \( n = \text{rk}_m C \) and let us choose a subgroup of \( C \) isomorphic to \((\mathbb{Z}/m)^n\) (we note that such a subgroup is not unique in general). Then, pulling-back the sequence (3) by the inclusion \((\mathbb{Z}/m)^n \hookrightarrow C\), we get an exact sequence

\[ 0 \longrightarrow A \longrightarrow B_1 \longrightarrow (\mathbb{Z}/m)^n \longrightarrow 0 \]

where \( B_1 \) is some subgroup of \( B \). In particular, \( \text{rk}_m B \geq \text{rk}_m B_1 \). Now, by the second statement of the Lemma, we have \( \text{rk}_m B_1 = \text{rk}_m A + n = \text{rk}_m A + \text{rk}_m C \), from which the result follows.

We note that, when working with abelian groups that are not \( m \)-torsion, the conclusion of Lemma 2.6 no longer holds. For example, consider the 2-ranks in the exact sequence

\[ 0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \]
2.4 Corollaries

Let $V$ be a smooth projective variety over $\mathbb{Q}$. We say that $V$ has good reduction at a prime $p$ if there exists a smooth projective model of $V$ over $\text{Spec}(\mathbb{Z}(p))$. Otherwise, we say that $V$ has bad reduction at $p$.

It is well-known that the set $S$ of primes of bad reduction of $V$ is finite. Moreover, it is possible to construct a smooth projective model $V \to \text{Spec}(\mathbb{Z}_S)$.

Lemma 2.7. Let $V$ be a smooth projective variety over $\mathbb{Q}$. Let $S$ be the set of primes of bad reduction of $V$. Then there exists a smooth projective model $V \to \text{Spec}(\mathbb{Z}_S)$ of $V$. Moreover, any such model satisfies

$$V(\mathcal{O}_{K,S}) = V(K)$$

for any number field $K$, and

$$\text{Pic}(V) = \text{Pic}(V).$$

Proof. The existence of a smooth projective model $V \to \text{Spec}(\mathbb{Z}_S)$ was noted above. The projectivity of $V$ ensures us that any morphism $P : \text{Spec}(\overline{K}) \to V$ extends to a morphism $\text{Spec}(\mathcal{O}_{K,S}) \to V$ (valutative criterion of properness).

Because $V$ is regular, any divisor on $V$ can be extended (by scheme-theoretic closure) into a divisor on $\mathcal{V}$. Thus, the restriction map

$$\text{Pic}(V) \to \text{Pic}(V)$$

is surjective. The kernel of this map is the subgroup of $\text{Pic}(V)$ generated by fibral (or “vertical”) divisors. By assumption $V$ is geometrically connected, so by Zariski’s connectedness theorem $V$ has connected fibers. Therefore, $V$ has irreducible fibers (because smooth and connected implies irreducible), so any fibral divisor is a sum of fibers. In other words, any fibral divisor is the inverse image of a divisor on $\text{Spec}(\mathbb{Z}_S)$. Such a divisor is principal, because $\mathbb{Z}$ is principal. So the subgroup generated by fibral divisors is trivial, and the restriction map is bijective.

Corollary 2.8. Let $V$ be a smooth projective variety over $\mathbb{Q}$. Then, for any $m > 1$, the hypotheses of Theorem 2.4 hold for $V$ and $m$ when taking $S$ to be the set of primes of bad reduction of $V$.

Proof. Let $V \to \text{Spec}(\mathbb{Z}_S)$ be the model of $V$ as in Lemma 2.7. By Kummer theory, we have a diagram with exact lines

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_m(V)/m & \longrightarrow & H^1(V, \mu_m) & \longrightarrow & \text{Pic}(V)[m] & \longrightarrow & 0 \\
0 & \longrightarrow & G_m(V)/m & \longrightarrow & H^1(V, \mu_m) & \longrightarrow & \text{Pic}(V)[m] & \longrightarrow & 0
\end{array}
$$

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where the vertical lines are obtained by restriction to the generic fiber. So, starting from \( L \in \text{Pic}(V)[m] \), we get a \( \mu_m \)-torsor \( T \to V \) whose generic fiber \( T \to V \) is a lift of \( L \).

Now, if \( P : \text{Spec}(K) \to V \) is a point, we have a commutative diagram

\[
\begin{array}{ccc}
H^1(V, \mu_m) & \xrightarrow{P^*} & H^1(O_{K,S}, \mu_m) \\
\downarrow & & \downarrow \\
H^1(V, \mu_m) & \xrightarrow{P^*} & H^1(K, \mu_m)
\end{array}
\]

where the horizontal lines are pull-backs along \( P \). This proves that \( P^*T \) belongs to the subgroup \( H^1(O_{K,S}, \mu_m) \).

**Remark 2.9.** If \( A \) is an abelian variety over \( \mathbb{Q} \), then \( A(\mathbb{Q}) = \text{Pic}^0(A^t) \) where \( A^t \) is the dual abelian variety of \( A \). It seems interesting to apply Corollary 2.8 to this situation (i.e. taking \( V = A^t \)). But then one has to bound the degree of a generically finite rational map \( A^t \to \mathbb{A}^1 \), where \( l = \dim(A) \).

Let us recall that, if \( C \) is a curve defined over \( \mathbb{Q} \), the index \( \text{ind}(C) \) of \( C \) is, by definition, the quantity

\[
\text{ind}(C) = \gcd\{[K : \mathbb{Q}] \mid C(K) \neq \emptyset\} = \gcd\{\deg(D) \mid D \text{ is a rational divisor on } C\}.
\]

Of course, if \( C(\mathbb{Q}) \neq \emptyset \), then \( \text{ind}(C) = 1 \). The converse is false.

**Lemma 2.10.** Let \( C \) be a smooth projective curve over \( \mathbb{Q} \). Then, for all \( m > 1 \),

\[
\text{rk}_{m'} \text{Pic}^0(C)_{\text{tors}} \geq \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}},
\]

where \( m' = m / \gcd(m, \text{ind}(C)) \).

**Proof.** Let \( d \) be the index of \( C \). According to [2, Proposition 11 p. 242], the image of the natural map

\[
\text{Pic}^0(C) \to \text{Jac}(C)(\mathbb{Q})
\]

contains the image of the multiplication by \( d \) map. The result follows. \( \square \)

**Corollary 2.11.** Let \( C \) be a smooth projective curve over \( \mathbb{Q} \), and let \( m > 1 \). Let \( S \) be the set of primes of bad reduction of \( \text{Jac}(C) \), and let

\[
m' = \frac{m}{\gcd(m, \text{ind}(C))}.
\]

Let \( \phi : C \to \mathbb{A}^1 \) be a rational map. Then (with the notation of Theorem 2.4), for all but \( O(\sqrt{N}) \) values \( i = 1, \ldots, N \), we have \([\mathbb{Q}(P_i) : \mathbb{Q}] = \deg \phi\) and

\[
\text{rk}_{m'} \text{Cl}(\mathbb{Q}(P_i)) \geq \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}} + \#S - \text{rk} \mathcal{O}_{\mathbb{Q}(P_i), S}^\times.
\]

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Proof. For simplicity, we assume that \( C(\mathbb{Q}) \neq \emptyset \). In this case, \( m = m' \) (in the general case, we would have to invoke Lemma 2.10). Also, we choose a point \( Q \in C(\mathbb{Q}) \) and consider the morphism

\[ f : C \longrightarrow \text{Jac}(C) \]

sending a point \( P \) to \([P - Q]\) as usual. Then the pull-back map

\[ f^* : \text{Pic}^0(\text{Jac}(C)) \longrightarrow \text{Pic}^0(C) \]

is an isomorphism. Now, let \( L \in \text{Pic}(C)[m] \), and let \( L' \in \text{Pic}^0(\text{Jac}(C))[m] \) be such that \( f^*L' = L \). If \( T' \to \text{Jac}(C) \) is a \( \mu_m \)-torsor lifting \( L' \), then obviously \( T := f^*T' \) is a \( \mu_m \)-torsor lifting \( L \), and for any \( P \in C(K) \), we have

\[ P^*T = (f \circ P)^*T'. \]

Thus, in order to prove that \( P^*T \) belongs to \( H^1(\mathcal{O}_{K,S}, \mu_m) \), it suffices to prove that, for any \( R \in \text{Jac}(C)(K), R^*T' \) belongs to \( H^1(\mathcal{O}_{K,S}, \mu_m) \).

Now, we have to choose a suitable \( \mu_m \)-torsor \( T' \to \text{Jac}(C) \) lifting \( L' \). Let \( J \to \text{Spec}(\mathbb{Z}_S) \) be the Néron model of \( \text{Jac}(C) \) over \( \text{Spec}(\mathbb{Z}_S) \). Then \( J \) is an abelian scheme and, by Lemma 2.7, we have

\[ \text{Pic}(J) = \text{Pic}(\text{Jac}(C)). \]

Therefore, the same arguments as in the proof of Corollary 2.8 give us a \( \mu_m \)-torsor \( T' \to J \) whose generic fiber has the required properties. Finally, we just apply Theorem 2.4 to this situation.

\[ \square \]

Remark 2.12. The main interest of Corollary 2.11 compared to Corollary 2.8 is that in general the set of places of bad reduction of \( \text{Jac}(C) \) is strictly smaller than the set of places of bad reduction of \( C \), even in the case when \( C \) has a rational point.

3 Applications

3.1 Curves

In this section we consider some applications and further refinements of Corollary 2.11. Note that if we fix the degree of \( \phi \), the quantity \( \text{rk} \mathcal{O}_{\mathbb{Q}(P_i), S} - \#S \) that appears in Corollary 2.11 is minimized if every prime in \( S \) is totally ramified in \( \mathbb{Q}(P_i) \) and \( \mathbb{Q}(P_i) \) has the maximum possible number of complex places for a number field of degree \( \phi \). In this case, we have the equalities

\[ \text{rk} \mathcal{O}_{\mathbb{Q}(P_i), S} - \#S = \text{rk} \mathcal{O}_{\mathbb{Q}(P_i)} = \left\lfloor \frac{\deg \phi - 1}{2} \right\rfloor. \]

By choosing appropriate maps \( \phi \) in Corollary 2.11 we show that this advantageous situation can be achieved for hyperelliptic or, more generally, superelliptic curves.
Corollary 3.1. Let \( C \) be (a smooth projective model of) the plane curve defined by
\[
y^n = f(x), \quad f(x) \in \mathbb{Q}[x], \quad n > 1,
\]
with \((\deg f, n) = 1\). Let \( m > 1 \) be an integer. Then there exist \( X^{(n-1)/\deg f}/\log X \) number fields \( k \) of degree \([k : \mathbb{Q}] = n \) with discriminant \( d_k, |d_k| < X \), and
\[
\text{rk}_m \text{Cl}(k) \geq \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}} - \left\lfloor \frac{n-1}{2} \right\rfloor.
\]

Proof. Let \( t = \deg f \) and let \( f = \sum_{i=0}^{t} a_i x^i \). By rescaling \( x \) and \( y \) we can assume that \( a_i \in \mathbb{Z} \) for all \( i \) and, using \((t, n) = 1\), that \( a_t = -1 \). Let \( S \) be the set of primes of bad reduction of \( \text{Jac}(C) \). Let \( M = \prod_{p \in S} p \). Let \( \phi \) be the rational function on \( C \) defined by \( \phi = x - \frac{1}{M} \). For \( i \in \mathbb{Z} \), let \( P_i = \phi^{-1}(i) \). Then \( \mathbb{Q}(P_i) \cong \mathbb{Q}\left( \sqrt{f(\frac{iM+1}{M})} \right) \). Let \( p \in S \). Since \( a_t = -1 \), we have that \( v_p (f(\frac{iM+1}{M})) = -t \). Since \((t, n) = 1\), this implies that \( p \) totally ramifies in \( \mathbb{Q}(P_i) \). Note also that \([\mathbb{Q}(P_i) : \mathbb{Q}] = n \). So every prime in \( S \) is totally ramified in \( \mathbb{Q}(P_i) \). The condition \((t, n) = 1\) also implies that \( y^n = f(x) \) has a rational point at infinity. Then Corollary \ref{cor:3.1} applied to \( \phi \) and \( C \) implies that for all but \( O(\sqrt{N}) \) values \( i = 1, \ldots, N \), we have \([\mathbb{Q}(P_i) : \mathbb{Q}] = n \) and
\[
\text{rk}_m \text{Cl}(\mathbb{Q}(P_i)) \geq \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}} - \text{rk} \mathcal{O}^*_\mathbb{Q}(P_i).
\]
Since \( a_t = -1 \), for \( i \gg 0 \), \( f(\frac{iM+1}{M}) \) is negative. It follows that for \( i \gg 0 \), \( \mathbb{Q}(P_i) \) has exactly one real place if \( n \) is odd and no real places if \( n \) is even. So by Dirichlet’s theorem, \( \text{rk} \mathcal{O}^*_\mathbb{Q}(P_i) = \left\lfloor \frac{n-1}{2} \right\rfloor \) for \( i \gg 0 \). It follows from Theorem \ref{thm:2.1} that there are \( \gg N/\log N \) distinct number fields in the set \( \{\mathbb{Q}(P_1), \ldots, \mathbb{Q}(P_N)\} \). An easy calculation shows that \( |d_{\mathbb{Q}(P_i)}| = O(i^{(n-1)\varepsilon}) \). Combining the above statements then gives the corollary.

Of particular interest is the case where \( C \) is a hyperelliptic curve.

Corollary 3.2. Let \( C \) be a smooth projective hyperelliptic curve over \( \mathbb{Q} \) with a rational Weierstrass point. Let \( g \) denote the genus of \( C \). Let \( m > 1 \) be an integer. Then there exist \( X^{\frac{1}{2g+1}}/\log X \) imaginary quadratic number fields \( k \) with
\[
\text{rk}_m \text{Cl}(k) \geq \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}}, \quad |d_k| < X,
\]
and \( \gg X^{\frac{1}{2g+1}}/\log X \) real quadratic number fields \( k \) with
\[
\text{rk}_m \text{Cl}(k) \geq \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}} - 1, \quad d_k < X.
\]

Proof. Since \( C \) has a rational Weierstrass point, \( C \) is birational to a plane curve defined by \( y^2 = f(x) \), for some \( f(x) \in \mathbb{Q}[x] \) with \( \deg f = 2g + 1 \). The statement on imaginary quadratic fields now follows from Corollary \ref{cor:3.1}. The statement for real quadratic fields follows similarly from the proof of Corollary \ref{cor:3.1} using points \( P_i, i < 0 \), with the extra \(-1\) term coming from the rank one unit group of a real quadratic field.
A classical result of Nagell [15, 16] states that for any positive integer \( m \) there exist infinitely many imaginary quadratic number fields with an element of order \( m \) in the ideal class group. Weinberger [18] and Yamamoto [19], independently, extended Nagell’s result to real quadratic fields and Yamamoto [19] improved Nagell’s result for imaginary quadratic fields to class groups of \( m \)-rank two. As a sample application of Corollary 3.2, we show that it easily implies (largely known) quantitative versions of the results of Weinberger [18] and Yamamoto [19].

**Lemma 3.3.** Let \( b, c \in \mathbb{Q} \), with \( c \neq 0 \), \( b^2 
eq 4c^2 \), and let \( m > 1 \) be an integer. Let \( C \) be the (smooth projective model of the) hyperelliptic curve \( y^2 = x^{2m} + bx^m + c^2 \). Then \( \text{rk}_m \text{Jac}(C)(\mathbb{Q})_{\text{tors}} \geq 2 \).

**Proof.** The curve \( C \) has two \( \mathbb{Q} \)-rational points at infinity that we’ll denote by \( \infty \) and \( \overline{\infty} \). Let \( P, \overline{P} \in C \) be the two points with coordinates \( P = (0, c) \) and \( \overline{P} = (0, -c) \). Then, after possibly switching \( \infty \) and \( \overline{\infty} \), we have

\[
\text{div}(y - x^m - c) = m(P - \infty),
\]

\[
\text{div}(y - x^m + c) = m(\overline{P} - \infty).
\]

Let \( i, j \in \{0, \ldots, m-1\} \) with \( i \geq j, i > 0 \). Then, since \( P + \overline{P} \sim \infty + \overline{\infty} \),

\[
i(P - \infty) + j(\overline{P} - \infty) \sim (i-j)P + j(P + \overline{P}) - (i+j)\infty \\
\sim (i-j)P + j\overline{\infty} - i\infty.
\]

Since the genus of \( C \) is \( m - 1 \) and \( \infty \) is not a Weierstrass point of \( C \), we have \( l(i\infty) = 1 \) as \( i \leq m - 1 \). Then \( i(P - \infty) + j(\overline{P} - \infty) \) is not a principal divisor and it follows that the divisors classes of \( P - \infty \) and \( \overline{P} - \infty \) generate a subgroup \((\mathbb{Z}/m)^2\) in \( \text{Jac}(C)(\mathbb{Q}) \). \( \square \)

Note that there are hyperelliptic curves as in Lemma 3.3 with a rational Weierstrass point (e.g., curves with \( b = -1 - c^2 \)). Then from Corollary 3.2 we immediately obtain the following result.

**Corollary 3.4.** Let \( m > 1 \) be an integer. There exist \( \gg X^{\frac{1}{m-1}}/\log X \) imaginary quadratic number fields \( k \) with \( |d_k| < X \) and \( \text{rk}_m \text{Cl}(k) \geq 2 \) and \( \gg X^{\frac{1}{m-1}}/\log X \) real quadratic number fields \( k \) with \( d_k < X \) and \( \text{rk}_m \text{Cl}(k) \geq 1 \).

If \( m \) is odd, then Byeon [3] and Yu [20] have proved, for imaginary and real quadratic fields, respectively, the better lower bound of \( \gg X^{1/m-\epsilon} \). If \( m \) is even, in the real quadratic case a lower bound of \( \gg X^{1/m} \) was proved by Chakraborty, Luca, and Mukhopadhyay [4]. The imaginary quadratic case of Corollary 3.4 with \( m \) even appears to be new.

Using the results presented here, it is in fact possible to derive quantitative versions of many of the known results on constructing number fields with class groups of large rank. This aspect of the results will be pursued in a future paper.

Corollary 3.2 raises the following natural question.
Question 3.5. Let $p$ be an odd prime. Do there exist hyperelliptic curves $C$ over $\mathbb{Q}$ with $\text{rk}_p \text{Jac}(C)(\mathbb{Q})_{\text{tors}}$ arbitrarily large?

More generally, one can ask whether there exist curves $C$ over $\mathbb{Q}$ of gonality $n$ with $\text{rk}_p \text{Jac}(C)(\mathbb{Q})_{\text{tors}}$ arbitrarily large, where $p$ is a prime not dividing $n$ (in the case $p|n$, a positive answer is easily obtained from curves of the form $y^n = f(x)$, where $f(x)$ splits over $\mathbb{Q}$). This question does not appear to have been extensively investigated. Previous papers studying the problem of constructing Jacobians of curves with large rational torsion subgroups have primarily focused on either curves of low genus (e.g., [9]), or on producing a rational torsion point of large order in the Jacobian of a curve of genus $g$, for every genus $g$ (e.g., [3]).

3.2 Torsion line bundles on arithmetic varieties

We consider here the question formulated in [1], namely:

Question 3.6. Let $\mathcal{L}$ be a nontrivial line bundle over an arithmetic variety $X$. Is it possible to find a section $P$ of $X$ over some number field such that the pull-back of $\mathcal{L}$ by this section is also nontrivial?

Here “arithmetic variety” means a normal scheme whose structural morphism to $\text{Spec}(\mathbb{Z})$ is proper and flat. Let us note that, under these assumptions, the generic fiber of $X$ need not be geometrically connected.

Corollary 3.7. Let $X$ be an arithmetic variety whose generic fiber $X$ is smooth and geometrically connected. Let $S$ be the set of places of bad reduction of $X$, and let $X_S := X \times_{\mathbb{Z}} \text{Spec}(\mathbb{Z}_S)$. Then, given an integer $m > 1$, there exists an infinity of number fields $K$ with a point $P \in X(K)$ such that the image of the restriction map

$$P^* : \text{Pic}(X_S)[m] \longrightarrow \text{Cl}(\mathcal{O}_{K,S})[m]$$

satisfies

$$\text{rk}_m \text{Im}(P^*) \geq \text{rk}_m \text{Pic}(X_S)[m] + \#S - \text{rk} \mathcal{O}_{K,S}^*.$$

If $X$ is the normalization of a plane curve defined by

$$y^n = f(x)$$

with $(\deg(f), n) = 1$, then the same holds with

$$\text{rk}_m \text{Im}(P^*) \geq \text{rk}_m \text{Pic}(X_S)[m] - \left[ \frac{n - 1}{2} \right].$$
Proof. Let us note here that our \( S \) is \textit{a priori} larger than the set of places of bad reduction of \( X \), but the arguments in the proof of Lemma 2.7 still hold, and give us the equality \( \text{Pic}(X_S) = \text{Pic}(X) \). Then Corollary 2.8 can be applied with the same set \( S \), and the first statement can be extracted from the proof of Theorem 2.4. The second statement follows from the proof of Corollary 3.1.

Corollary 3.8. Let \( X \) be an arithmetic variety whose generic fiber is a smooth hyperelliptic curve with a rational Weierstrass point. Let \( \mathcal{L} \) be a torsion line bundle over \( X \) whose generic fiber is nontrivial. Then there exists an infinity of imaginary quadratic fields \( K \) with a section \( P \) of \( X \) over \( \mathcal{O}_K \) such that the pull-back \( P^* \mathcal{L} \) is nontrivial.

Proof. Obviously, it suffices to prove the result when \( \mathcal{L} \) has prime order \( p \). In this case, the equality \( \text{rk}_p \text{Im}(P^*) = \text{rk}_p \text{Pic}(X_S)[p] \) implies that \( \text{ker}(P^*) = 0 \). Therefore, the second statement in Corollary 3.7 implies that, for an infinity of imaginary quadratic fields \( K \), the map

\[
P^* : \text{Pic}(X_S)[p] \longrightarrow \text{Cl}(\mathcal{O}_{K,S})[p]
\]

is injective. On the other hand, we know that \( \mathcal{L} \) has a nontrivial generic fiber, so the restriction of \( \mathcal{L} \) to \( X_S \) is also nontrivial. This proves the result.

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