HAIRY GRAPHS TO RIBBON GRAPHS VIA A FIXED SOURCE GRAPH COMPLEX

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ABSTRACT. We show that the hairy graph complex (HGC_{n+1,d}) appears as an associated graded complex of the oriented graph complex (OGC_{n+1,d}), subject to the filtration on the number of targets, or equivalently sources, called the fixed source graph complex. The fixed source graph complex (OGC_{1,d+0}) maps into the ribbon graph complex RGC, which models the moduli space of Riemann surfaces with marked points. The full differential d on the oriented graph complex OGC_{n+1} corresponds to the deformed differential \( d + h \) on the hairy graph complex HGC_{n,d}, where h adds a hair. This deformed complex (HGC_{n,d+h}) is already known to be quasi-isomorphic to standard Kontsevich’s graph complex GC_{2n}. This gives a new connection between the standard and the oriented version of Kontsevich’s graph complex.

1. Introduction

The main motivation for the present work is the explicit morphism of complexes

\[ F : (OGC_{1}, \delta) \to (RGC[1], \delta + \Delta_1) \]

constructed by S. Merkulov and T. Willwacher in [12]. Here (OGC_{1}, \delta) is the oriented version of Kontsevich’s graph complex, and (RGC, \delta + \Delta_1) is a complex of ribbon graphs. Ribbon graphs (sometimes called fat graphs) models Riemann surfaces with marked points [14]. For our ribbon graph complex RGC, we get

\[ H^k(RGC, \delta) \cong \prod_{g,n} \left( H^k_{c}(M_{g,n}, \mathbb{Q}) \otimes \text{sgn}_n \right)^{\mathbb{Z}} \oplus \mathbb{Q} \text{ for } k = 1, 5, 9, \ldots \]

where \( H_{c}(M_{g,n}, \mathbb{Q}) \) is the compact support cohomology of the moduli space of Riemann surfaces of genus \( g \) with \( n \) marked points, see [9] for more details. In this context, the differential \( \delta + \Delta_1 \) constructed in [12] is a deformation of the classical differential \( \delta \) on RGC. A simple observation gives that the same explicit formula \( F \) is also a map of complexes

\[ F : (OGC_{1}, \delta_0) \to (RGC[1], \delta), \]

where it is instead the differential on OGC_{1} that is not standard. The differential \( \delta_0 \) splits vertices of graphs in OGC in a way that preserves the number of target vertices. To the authors best knowledge, the (oriented) fixed target graph complex (OGC_{1,\delta_0}) has not been studied earlier. In this paper, we show that there is a quasi-isomorphism from the oriented graph complex with this new differential to the better known hairy graph complex HGC_{n,d}, studied in e.g. [1], [6].

Theorem 1.1. There is a map of graded vector spaces

\[ G : OGC_{n+1} \to HGC_{n,d} \]

such that the associated morphisms of complexes

\[ G : (OGC_{n+1}, \delta) \to (HGC_{n,d}, \delta). \]

and

\[ G : (OGC_{n+1}, \delta_0) \to (HGC_{n,d}, \delta + \chi) \]

are quasi-isomorphisms. Here \( \chi \) is the extra differential on HGC_{n,d} that adds a hair, considered in [6].

As a corollary, we get a relationship between the ribbon graph complex and the hairy graph complex.

Corollary 1.2. We have an explicit zig-zag of morphisms

\[ (HGC_{n}, \delta) \leftarrow (OGC_{1}, \delta_0) \to (RGC[1], \delta), \]

where the left map is a quasi-isomorphism.

Key words and phrases. Graph Complexes, Hairy Graph complex, Ribbon Graph Complex.
A recent result by M. Chan, S. Galatius and S. Payne states that there exists an embedding
\[ H^k(GC_{g,n}^{marked}, \delta) \to \bigoplus_{g,n} H^k_{g,n+1}(M_{g,n}, \mathbb{Q}). \]
Here, \( GC_{g,n}^{marked} \) is a complex of hairy graphs where each hair is labeled by an integer. However, no explicit map is given. After symmetrizing both sides and using Theorem 1.1, we get that there exists an embedding
\[ H(OGC_1, \delta_0) \to \prod_{g,n} \left( H_*(M_{g,n}, \mathbb{Q}) \otimes \text{sgn}_n \right)^{2g} \oplus \left\{ \begin{array}{ll} \mathbb{Q} & \text{for } k = 1, 5, 9, \ldots, \\ 0 & \text{otherwise,} \end{array} \right. \]
We conjecture that this embedding is given explicitly by the map \( H(F) \). This conjecture is also given in [12], and to support it, it is shown that \( H(F) \) is nontrivial on all loop classes of \( OGC_1 \).
Let \( (SG_{g,n}, d) \) be the graph complex consisting of directed graphs that contain at least one source vertex, with the edge contracting differential \( d \). In [25], the second author showed that the projection
\[ (SG_{g,n}, d) \to (OG_{g,n}, d) \]

Remark 1.3. The graph complex \( (SG_{g,n}, \delta) \) with the vertex splitting differential \( \delta \), is dual of the complex \( (SG_{g,n}, d) \) with the edge contraction differential \( d \). The space \( SG_{g,n} \) is spanned by formal linear combinations of graphs, while its dual space \( SGC_{g,n} \) is spanned by formal series of the same graphs.

Each result regarding a graph complex \( (G, d) \) transfers to a dual result regarding \( (GC, \delta) \).
\[ \text{e.g. the map } G : (OGC_{g,n+1}, \delta_0) \to (HGC_{g,n}, \delta) \text{ has a dual map } \Phi : (HG_{g,n}, d) \to (OG_{g,n+1}, d_0), \]

We consider the relation between oriented graphs and sourced graphs also holds true when we consider the fixed source graph complexes, with differentials \( d_0 \) that preserve the number of source vertices.

**Proposition 1.4.** The projection
\[ (SG_{g,n}, \delta_0) \to (OG_{g,n}, \delta_0), \]
is a quasi-isomorphism.

**Remark 1.5.** In [2], we consider the differential \( \delta_0 \) that preserves the number of target vertices as opposed to source vertices. However, as there is an isomorphism \( \text{Inv} : OGC \to OGC \) that inverts all the edges, thus mapping source vertices to target vertices and vice versa, we may shift freely between considering source vertices and target vertices. To keep notation consistent with what is previously established in [25], we shall mainly consider the differential that preserves the number of source vertices, which we will also denote by \( \delta_0 \).

Let \( GC_{g,n}^2 \) (\( HGC_{g,n}^2 \)) denote the version of Kontsevich’s standard (hairy) graph complex that includes graphs with 2-valent vertices. The following result connects Kontsevich’s graph complex and hairy graph complex.

**Theorem 1.6 (16, 17, 18, 6).** There is a quasi-isomorphism
\[ (GC_{g,n}^2, \delta) \to (HGC_{g,n}^2, \delta + \chi), \]
by summing over all ways to attach a hair to each graph.

Together with Theorem 1.1, we arrive to the following corollary.

**Corollary 1.7.** There is a zig-zag of explicit quasi-isomorphisms
\[ (GC_{g,n}^2, \delta) \to (HGC_{g,n}^2, \delta + \chi) \leftrightarrow (HGC_{g,n}, \delta + \chi) \leftrightarrow (OGC_{g,n+1}, \delta) \leftrightarrow (SG_{g,n+1}, \delta). \]

The previous corollary gives a fourth proof that the cohomology of Kontsevich’s graph complex is equal to the cohomology of the oriented graph complex, together with [12], [21], and [24]. The proof from [24] gave an explicit map
\[ (G, d) \to (OG, d), \]
where the superscript \( \varnothing \) means that the loop graphs are omitted. This proof gives explicit maps too, and this time they are also naturally defined for loops.

**Structure of the paper.** In Section 2 we recall the required definitions and some results. Section 3 introduces the map between the oriented and the hairy graph complexes, showing that it is indeed a map of complexes. Section 4 contains our main results about quasi-isomorphisms. In Section 5 we recall the definition of Ribbon graphs and the map \( F \) of Merkulov and Willwacher from [12], giving sense to our motivation.
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2. **Required definitions and results**

In this section we recall basic notation and several results shown in the literature that will be used in the paper.

2.1. **General notation.** We work over a field \( \mathbb{K} \) of characteristic zero. All vector spaces and differential graded vector spaces are assumed to be \( \mathbb{K} \)-vector spaces.

Graph complexes as vector spaces are generally defined by the graphs that span them. When we say a *single term graph* in a graph complex, we mean the base graph, while any linear combination (or a series) of graphs will be called an *element* of the graph complex.

2.2. **Directed, oriented and sourced graph complexes.** The directed, oriented and sourced graph complexes \( \text{DG}_n, \text{OG}_n \) and \( \text{SG}_n \) are defined in [25]. In this paper, we will only consider single directional complexes. Accordingly, let us recall a simplified definition.

Consider the set of directed connected graphs \( \vec{V}, \vec{E}, \text{grac}^2 \) with \( n \geq 0 \) distinguishable vertices and \( e \geq 0 \) distinguishable directed edges, all vertices being at least 2-valent, and without tadpoles (edges that start and end at the same vertex). We also ban passing vertices, i.e. 2-valent vertices with one incoming and one outgoing edge. This condition will not change the homology, as shown in [25, Subsection 3.2].

Let \( \vec{V}, \vec{E}, \text{Ograc}^2 \subset \vec{V}, \vec{E}, \text{grac}^2 \) be the subset of all graphs without closed paths along the directed edges. For \( s \geq 0 \), let \( \vec{V}, \vec{E}, \text{S}_s \text{grac}^2 \subset \vec{V}, \vec{E}, \text{grac}^2 \) be the subset of graphs that have exactly \( s \) sources, i.e. vertices without incoming edges.

For \( n \in \mathbb{Z} \), let the degree of an element of \( \vec{V}, \vec{E}, \text{grac}^2 \) be \( d = n - vn - (1 - n)e \). Let

\[
\begin{align*}
\vec{V}, \vec{E}, \text{DG}_n & := (\vec{V}, \vec{E}, \text{grac}^2)[n - vn - (1 - n)e], \\
\vec{V}, \vec{E}, \text{OG}_n & := (\vec{V}, \vec{E}, \text{Ograc}^2)[n - vn - (1 - n)e], \\
\vec{V}, \vec{E}, \text{S}_s \text{G}_n & := (\vec{V}, \vec{E}, \text{S}_s \text{grac}^2)[n - vn - (1 - n)e],
\end{align*}
\]

be the vector spaces of formal linear combinations of some elements of \( \vec{V}, \vec{E}, \text{grac}^2 \) with coefficients in \( \mathbb{K} \). They are graded vector spaces with non-zero terms only in degree \( d = n - vn - (1 - n)e \).

There are natural right actions of the group \( S_v \times S_e \) on \( \vec{V}, \vec{E}, \text{grac}^2 \), \( \vec{V}, \vec{E}, \text{Ograc}^2 \) and \( \vec{V}, \vec{E}, \text{S}_s \text{grac}^2 \), where \( S_v \) permutes vertices and \( S_e \) permutes edges. Let \( \text{sgn}_v \) and \( \text{sgn}_e \) be one-dimensional representations of \( S_v \), respectively \( S_e \), where the odd permutation reverses the sign. They can be considered as representations of the product \( S_v \times S_e \).

Let us consider the spaces of invariants:

\[
\begin{align*}
\vec{V}, \vec{E}, \text{DG}_n & := \begin{cases} 
(\vec{V}, \vec{E}, \text{DG}_n \otimes \text{sgn}_v)_{\vec{E}, \text{S}_s \text{G}_n} & \text{for } n \text{ even}, \\
(\vec{V}, \vec{E}, \text{DG}_n \otimes \text{sgn}_v)_{\vec{E}, \text{S}_s \text{G}_n} & \text{for } n \text{ odd},
\end{cases} \\
\vec{V}, \vec{E}, \text{OG}_n & := \begin{cases} 
(\vec{V}, \vec{E}, \text{OG}_n \otimes \text{sgn}_v)_{\vec{E}, \text{S}_s \text{G}_n} & \text{for } n \text{ even}, \\
(\vec{V}, \vec{E}, \text{OG}_n \otimes \text{sgn}_v)_{\vec{E}, \text{S}_s \text{G}_n} & \text{for } n \text{ odd},
\end{cases} \\
\vec{V}, \vec{E}, \text{S}_s \text{G}_n & := \begin{cases} 
(\vec{V}, \vec{E}, \text{S}_s \text{G}_n \otimes \text{sgn}_v)_{\vec{E}, \text{S}_s \text{G}_n} & \text{for } n \text{ even}, \\
(\vec{V}, \vec{E}, \text{S}_s \text{G}_n \otimes \text{sgn}_v)_{\vec{E}, \text{S}_s \text{G}_n} & \text{for } n \text{ odd}.
\end{cases}
\end{align*}
\]

As the group is finite, the space of invariants may be replaced by the space of coinvariants. The underlying vector space of the directed graph complex is given by

\[
\text{DG}_n := \bigoplus_{\nu \geq 1, \nu \geq 0} \vec{V}, \vec{E}, \text{DG}_n.
\]

The underlying vector space of the oriented graph complex is given by

\[
\text{OG}_n := \bigoplus_{\nu \geq 1, \nu \geq 0} \vec{V}, \vec{E}, \text{OG}_n.
\]

The underlying vector space of the sourced graph complex with \( s \) sources is given by

\[
\text{S}_s \text{G}_n := \bigoplus_{\nu \geq 1, \nu \geq 0} \vec{V}, \vec{E}, \text{S}_s \text{G}_n.
\]
and the full sourced graph complex is given by

\[(12) \quad \text{SG}_n := \bigoplus_{i \geq 1} S_i G_n.\]

Dual spaces of those defined above are spanned by the same graphs, but infinite sums are allowed. Since \(V, E, DG_n\) are finitely dimensional, this does not make difference. But duals of total complexes are defined as:

\[(13) \quad \text{DGC}_n := \prod_{i \geq 1, e \geq 0} V_i E_i DG_n,\]

\[(14) \quad \text{OGC}_n := \prod_{i \geq 1, e \geq 0} V_i E_i OG_n,\]

\[(15) \quad S_s \text{GC}_n := \prod_{i \geq 1, e \geq 0} V_i E_i S_s G_n,\]

\[(16) \quad \text{SGC}_n := \prod_{i \geq 1} S_i G_n.\]

Note that every oriented graph is sourced and both of them are directed, so there are inclusions and a projections

\[(17) \quad \text{OGC}_n \hookrightarrow \text{SGC}_n \hookrightarrow \text{DGC}_n, \quad \text{DG}_n \twoheadrightarrow \text{SGC}_n \twoheadrightarrow \text{OGC}_n.\]

2.3. **Hairy graph complex.** The hairy graph complex \(HGC_{m,n}\) is in general defined e.g. in [6]. In this paper we are interested in the case when \(m = n\), that is \(HGC_{m,n}\). For simplicity, we will use the shorter notation \(HGC_n := HGC_{n,n}\).

Let us quickly recall the definition, similar to the one of oriented and sourced graph complexes. Consider the set of directed connected graphs \(\bar{V}, \bar{E}, \bar{H}, \text{grac}^3\) with \(v > 0\) distinguishable vertices, \(e \geq 0\) distinguishable directed edges and \(s \geq 0\) distinguishable hairs attached to some vertices, all vertices being at least 3-valent, and without tadpoles (edges that start and end at the same vertex). In hairy graphs, the valence also includes the attached hairs, i.e. the valence of a vertex is the number of edges and hairs attached to it.

For \(n \in \mathbb{Z}\), let the degree of an element of \(\bar{V}, \bar{E}, \bar{H}, \text{grac}^3\) be \(d = n - vn - (1 - n)e - s\). Let

\[(18) \quad \bar{V}, \bar{E}, \bar{H}_n := (\bar{V}, \bar{E}, \bar{H}, \text{grac}^3)[n - vn - (1 - n)e - s]\]

be the vector space of formal series of \(\bar{V}, \bar{E}, \bar{H}, \text{grac}^3\) with coefficients in \(K\). It is a graded vector space with a non-zero term only in degree \(d = n - vn - (1 - n)e - s\).

There is a natural right action of the group \(S_v \times S_e \times (S_r \ltimes \mathbb{Z}_2^s)^e\) on \(\bar{V}, \bar{E}, \bar{H}, \text{grac}^3\), where \(S_v\) permutes vertices, \(S_e\) permutes hairs, \(S_r\) permutes edges and \(S_r\ltimes \mathbb{Z}_2^s\) changes the direction of edges. Let \(\text{sgn}_v, \text{sgn}_e, \text{sgn}_s\), and \(\text{sgn}_2\) be one-dimensional representations of \(S_v\), respectively \(S_e\), respectively \(S_r\), respectively \(S_2\), where the odd permutation reverses the sign. They can be considered as representations of the whole product \(S_v \times S_e \times (S_r \ltimes \mathbb{Z}_2^s)\). Let us consider the space of invariants:

\[(19) \quad V, E, H_n := \begin{cases} (\text{sgn}_v \otimes \text{sgn}_e \otimes \text{sgn}_s)^{S_v \times S_e \times (S_r \ltimes \mathbb{Z}_2^s)} & \text{for } n \text{ even,} \\ (\text{sgn}_v \otimes \text{sgn}_e \otimes \text{sgn}_s \otimes \text{sgn}_2^{S_v \times S_e \times (S_r \ltimes \mathbb{Z}_2^s)}) & \text{for } n \text{ odd.} \end{cases}\]

Again, because the group is finite, the space of invariants may be replaced by the space of coinvariants. The **hairy graph complex** with \(s\) hairs is

\[(20) \quad H, G_n := \bigoplus_{i \geq 1, e \geq 0} V, E, H_n G_n,\]

and the general one is

\[(21) \quad HG_n := \bigoplus_{i \geq 1} H, G_n.\]

Duals are:

\[(22) \quad H, GC_n := \prod_{i \geq 1, e \geq 0} V, E, H_n G_n,\]

\[(23) \quad HGC_n := \prod_{i \geq 1} H, G_n.\]
2.4. The differential. The standard differential acts by edge contraction:

\[(24) \quad d(\Gamma) = \sum_{a \in E(\Gamma)} \Gamma/a\]

where \(\Gamma\) is a graph from \(\bar{V}, \bar{E}, \text{grac}^2\) or \(\bar{V}, \bar{E}, \bar{H}, \text{grac}^3\). \(E(\Gamma)\) is its set of edges and \(\Gamma/a\) is the graph from \(\bar{V}_{v-1}\bar{E}_{v-1}\text{grac}^2\) or \(\bar{V}_{v-1}\bar{E}_{v-1}\bar{H}_1\text{grac}^3\) respectively, produced from \(\Gamma\) by contracting edge \(a\) and merging its end vertices. If a tadpole or a passing vertex is produced, we consider the result to be zero. Also, for oriented and sourced versions, if a closed path is formed or the last source is removed respectively, we consider the result to be zero. The precise signs and verification that the map can be extended to space of invariants and graph complexes \(OG_n, SG_n, \text{and} HGC\) can be found in [25] Subsection 2.9. In [25] Subsection 2.10] it is shown that \(\delta\) is a differential. Exactly the same arguments hold for hairy graph complex \(HG_n\).

The differential can not produce closed path, so the projection

\[(25) \quad p : (SG_n, d) \rightarrow (OG_n, d)\]

is well defined.

On \(HG_n\) we define another differential \(h\), that deletes a hair, summed over all hairs:

\[(26) \quad h(\Gamma) = \sum_{i \in H(\Gamma)} \Gamma/i\]

where \(\Gamma\) is a graph from \(\bar{V}, \bar{E}, \bar{H}_1\text{grac}^3\). \(H(\Gamma)\) is its set of hairs and \(\Gamma/i\) is the graph from \(\bar{V}, \bar{E}, \bar{H}_{v-1}\text{grac}^3\), produced from \(\Gamma\) by deleting hair \(i\). If it was the last hair or a 2-valent vertex is formed, we consider the result to be zero. We can easily extend it to \(HG_n\) and see that \(h^2 = 0\), so it is a differential. Also, it easily follows that \(hd + hd = 0\), so \(d + h\) is also a differential.

2.5. Loop order and the dual differentials. So far, we have defined complexes \((OG_n, d), (SG_n, d), (HG_n, d)\), \((HG_n, h)\) and \((HG_n, d + h)\). For a (hairy) graph \(\Gamma\), with \(e\) edges and \(v\) vertices, let the loop order of \(\Gamma\) be given by \(e - v\). The differentials \(d\) and \(h\) preserve the loop order, hence all complexes above admit sub-complexes

\[B_bOG_n := \bigoplus_{e - v = b} V_vE_vOG_n, \quad B_bSG_n := \bigoplus_{e - v = b} V_vE_vSG_n, \quad B_bHG_n := \bigoplus_{e - v = b, s \geq 1} V_vE_vH_sG_n,\]

for each \(b \in \mathbb{Z}\). It is clear that

\[OG_n = \bigoplus_{b \geq 0} B_bOG_n, \quad SG_n = \bigoplus_{b \geq 0} B_bSG_n, \quad HG_n = \bigoplus_{b \geq 0} B_bHG_n.\]

Furthermore, complexes \(B_bOG_n, B_bSG_n, B_bHG_n\) are finite dimensional in each homological degree \(k\). Hence there are canonical isomorphisms of vector spaces to their duals

\[B_bOG_n \rightarrow \text{hom}(B_bOG_n, \mathbb{K}) =: B_bOGC_n, \quad B_bSG_n \rightarrow \text{hom}(B_bSG_n, \mathbb{K}) =: B_bSGC_n, \quad B_bHG_n \rightarrow \text{hom}(B_bHG_n, \mathbb{K}) =: B_bHGC_n,\]

identifying a single term graph \(\Gamma\) to the linear map that maps \(\Gamma\) to 1 and all other graphs to 0.

Any differential \(\Delta\) on \(G = OG_n, SG_n\), or \(HG\), that preserves the loop order, is paired with a dual differential \(\nabla\leftrightarrow \Delta\) on the dual space \(GC = OGC_n, SGC_n\), or \(HGC\) such that

\[(GC, \nabla) \cong \prod_b \left( \text{hom}(B_bG, \mathbb{K}), \Delta' \right)\]

We denote the dual differentials of \(d\) and \(h\) by

\[(27) \quad \delta \leftrightarrow d, \quad \chi \leftrightarrow h.\]

The differential \(\delta\) splits a vertex in all possible ways, while \(\chi\) adds a hair in all possible ways. The vertex splitting differential (and adding a hair differential) are often defined as the standard differential(s). For hairy graph complexes both \(\delta\) and \(\chi\) are defined in [6].

The dual of the projection \(p : (SG_n, d) \rightarrow (OG_n, d)\) is the inclusion

\[(28) \quad t : (OGC_n, \delta) \rightarrow (SGC_n, \delta).\]
It is a well known result from homological algebra that the cohomology of a dual complex is dual to the homology of the complex. For this reason, we are free to study any of them, in order to obtain the results regarding (co)homology.

2.6. Skeleton version of directed graph complex. Instead of directed, oriented and sourced graph complexes $(DG_n, d)$, $(OG_n, d)$ and $(SG_n, d)$, it may sometimes be useful to consider their isomorphic skeleton versions: $(D^{sk}G_n, d)$, $(O^{sk}G_n, d)$ and $(S^{sk}G_n, d)$.

They are defined using theory from [25, Section 2] as follows: Consider the set of directed connected graphs $\bar{V}_n \bar{E}_n \bar{G}rac^3$ with $\nu > 0$ distinguishable vertices and $\epsilon \geq 0$ distinguishable directed edges, all vertices being at least 3-valent, without tadpoles. In this context, those graphs are called core graphs because we are going to attach edge type to them. Let $\bar{V}_n \bar{E}_n \bar{G}rac^3 := (\bar{V}_n \bar{E}_n \bar{G}rac^3)$ be the generated vector space. To each edge of a core graph $\Gamma \in \bar{V}_n \bar{E}_n \bar{G}rac^3$, we attach an element of graded $\langle 3 \rangle$ module $\Sigma$ spanned by $\{\rightarrow [n - 1], \leftarrow [n - 1], \rightarrow [n - 2]\}$ with $S_2$ action

\[
(29) \quad \rightarrow \leftrightarrow \leftarrow, \quad \rightarrow \leftrightarrow \leftarrow \mapsto -(-1)^\nu \rightarrow \leftrightarrow \leftarrow
\]

to get $\bar{V}_n \bar{E}_n \bar{G}rac^3 \otimes \Sigma^{\geq \nu}$. We say that the type of an edge is its attached element of $\Sigma$. Let

\[
(30) \quad \bar{V}_n \bar{E}_n \bar{G}rac^3 := \bar{V}_n \bar{E}_n \bar{G}rac^3 \otimes_{S_2} \Sigma^{\geq \nu} = (\bar{V}_n \bar{E}_n \bar{G}rac^{\geq \nu})_{S_2}
\]

where $S_2^{\geq \nu}$ acts by reversing edges in $\bar{V}_n \bar{E}_n \bar{G}rac^3$, as well as on the element from $\Sigma$ attached to the corresponding edge.

Similarly as before, there are natural right actions of the groups $S_\nu$ and $S_\epsilon$ on $\bar{V}_n \bar{E}_n \bar{G}rac^3$, where $S_\nu$ permutes vertices and $S_\epsilon$ permutes edges. Let $sgn_n$ be one-dimensional representations of $S_\nu$, where the odd permutation reverses the sign. The sign of the action of $S_\nu$ depends on the types of the permuted edges, such that switching edges with odd degree types change the sign, c.f. [25, Subsection 2.7]. Then let

\[
(31) \quad V_n E_n D^{sk}G_n := \left\{ \begin{array}{ll}
\left( \bar{V}_n \bar{E}_n \bar{G}rac_{\geq \nu} \right)_{S_\nu} \times \Sigma_{\geq \nu} [n - \nu n] & \text{for even,} \\
\left( \bar{V}_n \bar{E}_n \bar{G}rac^{\geq \nu} \otimes sgn_n \right)_{S_\nu} \times \Sigma_{\geq \nu} [n - \nu n] & \text{for odd.}
\end{array} \right.
\]

This means that for even $n$ switching edges of type $\rightarrow$ or $\leftrightarrow$ change the sign, while for odd $n$ switching vertices and edges of type $\rightarrow \leftrightarrow$ change the sign. Note the degree shift $[n - \nu n]$ that comes from the degrees of vertices, while the degree of edges of particular type is already included in $\Sigma$.

The skeleton version of directed graph complex is

\[
(32) \quad D^{sk}G_n := \bigoplus_{\nu \geq 1, \epsilon \geq 0} V_n E_n D^{sk}G_n.
\]

On graded $\langle 3 \rangle$ module $\Sigma$ there is also a differential defined with

\[
(33) \quad \rightarrow \leftrightarrow \leftarrow \leftarrow \mapsto -(-1)^\nu \rightarrow \leftrightarrow \leftarrow \mapsto
\]

that induces the edge differential $d_E$ on $D^{sk}G_n$, c.f. [25, Subsection 2.6]. The core differential $d_C$ comes from contracting edges of type $\rightarrow$ and $\leftarrow$, c.f. [25, Subsection 2.10]. This enables us to define the combined differential

\[
(34) \quad d := d_C + (-1)^{\nu \deg d_E}
\]

on $D^{sk}G_n$ as in [25, Subsection 2.11].

This skeleton version $D^{sk}G_n$ is defined to be isomorphic to the original version $DG_n$. In short, 3-valent vertices and 2-valent sources in a single term graph in $DG_n$ are called skeleton vertices. Strings of edges and vertices between two skeleton vertices that have to be in the set $\{\rightarrow, \leftarrow, \rightarrow \leftrightarrow \leftarrow \}$, are called skeleton edges. A corresponding graph in $D^{sk}G_n$ is the one with skeleton vertices as vertices and skeleton edges as edges, where $\rightarrow \leftrightarrow \leftarrow \leftarrow \rightarrow$ is mapped to $\rightarrow \leftrightarrow \leftarrow$. One can check that the degrees and parities are correctly defined, and obtain the following result. C.f. [25, Subsection 3.4]. But note that we have more skeleton vertices here because 2-valent sources are also considered skeleton vertices, and therefore there are less skeleton edges.

**Proposition 2.1.** There is an isomorphism of complexes

\[
(35) \quad \kappa : (DG_n, d) \rightarrow (D^{sk}G_n, d).
\]

Since oriented and sourced graph complexes $(OG_n, d)$ and $(SG_n, d)$ are both quotient of directed graph complex $(DG_n, d)$, we can induce skeleton versions of oriented and sourced graph complexes as follows.
Definition/Proposition 2.2. Skeleton versions of oriented and sourced graph complexes are

\( O^{sk}G_n := \kappa (OG_n), \)
\( S^{sk}G_n := \kappa (SG_n), \)

with the differential \( d \) as on \( D^{sk}G_n, \) where forbidden graphs are considered zero.

The quotient maps, by abuse of notation again denoted by \( \kappa, \) are isomorphisms of complexes.

\( \kappa : (OG_n, d) \rightarrow (O^{sk}G_n, d), \)
\( \kappa : (SG_n, d) \rightarrow (S^{sk}G_n, d). \)

3. The map \( \Phi \)

In this section we are going to define the main map \( \Phi : (HG_n, d + h) \rightarrow (OGC_{n+1}, \delta). \) Thanks to Definition/Proposition 2.2, it suffices to give

\( \Phi : (HG_n, d + h) \rightarrow (O^{sk}G_{n+1}, d). \)

The map \( G : OGC_{n+1} \rightarrow HG_n, \)

from Theorem 1.1 is the map dual of \( \Phi. \)

3.1. Forests. Let us pick the number of vertices \( v, \) the number of edges \( e \) and the number of hairs \( s. \) Let \( \Gamma \in \bar{V}v \bar{E}e \bar{H}s G_n \) be a single term graph.

A forest is any sub-graph of \( \Gamma \) that contains all its hairs (and thus all vertices with hairs), that does not contain cycles (of any orientation), and whose every connected component has exactly one hair. Let a spanning forest be a forest that contains all vertices. Let \( F(\Gamma) \) be the set of all spanning forests of \( \Gamma. \) An example of a spanning forest is given in Figure 3.1.

![Figure 3.1. An example of a hairy graph and a spanning forest. Edges of the forest are red, while other edges are dotted.](image)

3.2. Model pairs. For a chosen spanning forest \( \tau \in F(\Gamma), \) our first goal is to define \( \Phi_\tau(\Gamma) \in \bar{V}v \bar{E}e O^{sk}G_{n+1}. \) As our final goal is to define a map

\( \Phi : V_v, E_e H_h G_n \rightarrow V_v, E_e O^{sk}G_{n+1}, \)

the definition of \( \Phi_\tau(\Gamma) \) must be invariant of the action of \( \bar{S}_v \times \bar{S}_e \times (\bar{S}_s \ltimes \bar{S}_2^{se}). \) We will only define \( \Phi_\tau(\Gamma) \) on some pairs \( (\Gamma, \tau) \) that are called models, and we extend the definition to all pairs \( (\Gamma, \tau) \) by the requirement that the map is invariant under the symmetry action. In order to do that correctly, we need to check two conditions:

(i) if there is an element of \( \bar{S}_v \times \bar{S}_e \times (\bar{S}_s \ltimes \bar{S}_2^{se}) \) that sends one model to another model, the definition is invariant under its action, and
(ii) for every pair \( (\Gamma, \tau) \), there is an element of \( \bar{S}_v \times \bar{S}_e \times (\bar{S}_s \ltimes \bar{S}_2^{se}) \) that sends it to a model.

Note that the spanning forest \( \tau \) has \( v - s \) edges. For a pair \( (\Gamma, \tau) \) to be a model we require that:

- edges of a connected component in \( \tau \) are directed away from the hair of that connected component;
- an edge in \( \tau \) has the same label as the vertex it is heading to, labels being in the set \( \{1, \ldots, v - s\}; \)
- a hair labelled by \( x \) stands on the hairy vertex labelled by \( x + v - s. \)
An example of a model is given in Figure 3.2.

![Figure 3.2](image)

Figure 3.2. An example of model with the spanning forest from Figure 3.1. Edges of the forest are red, while other edges are dotted. Labels of vertices and hairs are thick.

It is clear that every pair \((\Gamma, \tau)\) can be using an element of \(S_v \times S_s \times (S_e \ltimes S_2^\infty)\) mapped to a model, so the condition (ii) is fulfilled.

### 3.3. Defining the map for a model

Let us now pick up a model \((\Gamma, \tau)\), consisting of a single term graph \(\Gamma \in \bar{V}_v \bar{E}_e \bar{H}_s G_n\) and \(\tau \in F(\Gamma)\). A graph \(\phi(\Gamma)\) obtained from \(\Gamma\) by deleting all hairs and ignoring the degree is a core graph in \(\bar{V}_v \bar{E}_e \text{Grac}^3\). To its edges that belong to \(E(\tau)\) we attach an edge type \(\to\), and to those that do not belong to \(E(\tau)\) we attach edge type \(\to\) to get an element of \(\bar{V}_v \bar{E}_e \text{Grac}^3\), and then after taking coinvariants and adding the degrees an element

\[
\Phi_\tau(\Gamma) \in V_v E_e O^d G_{n+1}.
\]

It is straightforward to check the following:

- the map is well defined in a sense that if there is an element of \(S_v \times S_s \times (S_e \ltimes S_2^\infty)\) that sends one model to another model, the same result in \(V_v E_e O^d G_{n+1}\) is obtained, i.e. condition (i) from above is fulfilled;
- The degree of \(\Phi_\tau(\Gamma)\) is by 1 greater than the degree of \(\Gamma\);
- The result \(\Phi_\tau(\Gamma)\) is oriented, i.e. it is an element of \(V_v E_e O^d G_{n+1}\).

An example of \(\Phi_\tau(\Gamma)\) is given in Figure 3.3.

![Figure 3.3](image)

Figure 3.3. Oriented graph \(\Phi_\tau(\Gamma)\) for the graph \(\Gamma\) and spanning forest \(\tau\) from Figure 3.1.

### 3.4. The final map

The map is now extended to all pairs \((\Gamma, \tau)\) by invariance under the action of \(S_v \times S_s \times (S_e \ltimes S_2^\infty)\). Then let us define

\[
\Phi: \bar{V}_v \bar{E}_e \bar{H}_s G_n \to V_v E_e O^d G_{n+1}, \quad \Gamma \mapsto \sum_{\tau \in F(\Gamma)} \Phi_\tau(\Gamma).
\]

The invariance under all actions implies that the induced map \(\Phi: V_v E_e H_s G_n \to V_v E_e O^d G_{n+1}\) is well defined. It is then extended to the whole

\[
\Phi: HG_n \to O^d G_{n+1}.
\]

**Proposition 3.4.** The map \(\Phi: (HG_n, d + h) \to (O^d G_{n+1}, d)\) is a map of complexes of degree 1, i.e.

\[
\Phi((d + h)\Gamma) = d\Phi(\Gamma)
\]

for every \(\Gamma \in HG_n\).
Proof. We have already checked that the degree of $\Phi$ is 1. 
For the other claim let us pick a single term graph $\Gamma \in \bar{V}, \bar{E}, G_n$. It holds that

$$\Phi(\Gamma) = \sum_{a \in E(\Gamma)} \Phi(\Gamma) = \sum_{a \in E(\Gamma)} \Phi(\Gamma)$$

where $\Gamma / a$ is contracting an edge $a$ in $\Gamma$. Spanning forests of $\Gamma / a$ are in natural bijection with spanning forests of $\Gamma$ that contain $a$, $\tau / a \leftrightarrow \tau$, so we can write

$$\Phi(\Gamma) = \sum_{a \in E(\Gamma)} \Phi(\Gamma) = \sum_{a \in E(\Gamma)} \Phi(\Gamma / a).$$

Lemma 3.5. Let $\Gamma \in \bar{V}, \bar{E}, \bar{H}, G_n$, $\tau \in F(\Gamma)$ and $a \in E(\tau)$. Then

$$\Phi(\Gamma) = \Phi(\Gamma / a) = \Phi(\Gamma / a)$$

where $\sim$ means that they are in the same class of coinvariants under the action of $S_\tau \times S_e$.

Proof. It is clear that one side is $\pm$ the other side. Careful calculation of the sign is left to the reader. □

The lemma implies that

$$\Phi(\Gamma) = \sum_{\tau \in F(\Gamma)} \Phi(\Gamma / a).$$

Still on the left-hand-side of the claimed relation we have

$$\Phi(\Gamma / i) = \sum_{a \in E(\Gamma)} \Phi(\Gamma / a).$$

where $\Gamma / i$ deletes hair $i$.

Spanning forests in $F(\Gamma / i)$ correspond to a kind of forests of $\Gamma$ where one connected component has two hairs. Therefore, let $FD(\Gamma)$ (double-hair forests) be the set of all subgraphs $\lambda$ of $\Gamma$ that contain all vertices and hairs, have no cycles, whose one connected component has exactly two hairs and whose other connected components have exactly one hair. Let those two hairs be $j(\lambda)$ and $k(\lambda)$. Sets $\{(i, \tau) \mid i \in H(\Gamma), \tau \in F(\Gamma / i)\}$ and $\{|(\lambda, i) \mid i \in H(\lambda), (\lambda, i)\}$ are clearly bijective. So we have

$$\Phi(\Gamma / i) = \sum_{\lambda \in ED(\Gamma)} (\Phi(\Gamma / j(\lambda)) + \Phi(\Gamma / k(\lambda))).$$

On the other side the differential is $d = d_C \pm d_E$. It acts on edges of $\Phi(\Gamma)$. They come from edges in $\Gamma$ and can be split into three sets:

- edges that are in $\tau$ form the set $E(\tau)$;
- edges that connect two connected components of $\tau$ form the set $ED(\tau)$;
- edges that make a cycle in a connected component of $\tau$ form the set $EC(\tau)$.

Only edges of type $\rightarrow$ and $\leftarrow$ can be contracted by $d_C$. They come from the edges in $E(\tau)$ so

$$d_C(\Phi(\Gamma)) = d_C \left( \sum_{\tau \in F(\Gamma)} \Phi(\Gamma) \right) = \sum_{\tau \in F(\Gamma)} d_C \left( \Phi(\Gamma) \right) = \sum_{\tau \in F(\Gamma)} \Phi(\Gamma) / a \sim \Phi(d\Gamma).$$

The edge differential $d_E$ acts on edges of type $\rightarrow$, which are those in the sets $ED(\tau)$ and $EC(\tau)$. We then split

$$d_E(\Phi(\Gamma)) = d_E(\Phi(\Gamma)) + d_E(\Phi(\Gamma)),$$

where

$$d_E(\Phi(\Gamma)) = \sum_{a \in ED(\tau)} d_E(a) \Phi(\Gamma), \quad d_E(\Phi(\Gamma)) = \sum_{a \in EC(\tau)} d_E(a) \Phi(\Gamma),$$

where $d_E(a)$ maps edge $a$ as $\rightarrow \mapsto \leftarrow \mapsto (-1)^{a+1} \leftarrow$.

Lemma 3.6. Let $\Gamma \in \bar{V}, \bar{E}, \bar{H}, G_n$ be a single term graph. Then

$$\sum_{\tau \in F(\Gamma)} d_C(\Phi(\Gamma)) \sim 0.$$
Proof. Let

\[ N(\Gamma) := \sum_{\tau \in FD(\Gamma)} d_{ED}(\Phi_{\tau}(\Gamma)) = \sum_{\tau \in FD(\Gamma)} \sum_{a \in ED(\tau)} d_{E_a}^{(a)}(\Phi_{\tau}(\Gamma)). \]

Terms in the above relation can be summed in another order. Let \( FC(\Gamma) \) (cycled forests) be the set of all subgraphs \( \rho \) of \( \Gamma \) that contain all vertices and hairs, have \( v - s + 1 \) edges, and whose every connected component has exactly one hair. Those graphs are similar to spanning forests, but have one cycle. Let \( C(\rho) \) be the set of edges in the cycle of \( \rho \). Clearly, \( \rho \setminus \{a\} \) for \( a \in C(\rho) \) is a spanning forest of \( \Gamma \) and sets \( \{(\tau, a)|\tau \in F(\Gamma), a \in ED(\tau)\} \) and \( \{(a, \rho)|\rho \in FC(\Gamma), a \in C(\rho)\} \) are bijective, so

\[ N(\Gamma) = \sum_{\rho \in FC(\Gamma)} \sum_{a \in C(\rho)} d_{E_a}^{(a)}(\Phi_{\rho,\{a\}}(\Gamma)). \]

It is now enough to show that

\[ \sum_{a \in C(\rho)} d_{E_a}^{(a)}(\Phi_{\rho,\{a\}}(\Gamma)) \sim 0 \]

for every \( \rho \in FC(\Gamma) \). Let \( y \in V(\Gamma) \) be the vertex in the cycle of \( \rho \) closest to the hair of its connected component (along \( \rho \)). After choosing \( a \in C(\rho) \), the cycle in \( \Phi_{\rho,\{a\}}(\Gamma) \) has the edge \( a \) of type \( \rightarrow \), and other edges of type \( \rightarrow \) or \( \leftarrow \) with direction from \( y \) to the edge \( a \), such as in the following diagram.

After acting by \( d_{E_a}^{(a)} \) this \( \rightarrow \leftarrow \) is replaced by \( \rightarrow + (-1)^n \leftarrow \), such as in the following diagram.

Careful calculation of the sign shows that those two terms are cancelled with terms given from choosing neighbouring edges in \( C(\rho) \), and two last terms which do not have a corresponding neighbour are indeed 0 as they have a cycle along arrows. This concludes the proof that \( N(\Gamma) = 0 \). \( \square \)

The similar study of the action on edges from \( ED(\tau) \) leads to the following lemma.

**Lemma 3.7.** Let \( \Gamma \in \tilde{V}, \tilde{E}, \tilde{H}, G_n \) be a single term graph. Then

\[ \sum_{\tau \in FD(\Gamma)} d_{ED}(\Phi_{\tau}(\Gamma)) \sim \sum_{\lambda \in ED(\Gamma)} (\Phi_{\lambda}(\Gamma/j(\lambda)) + \Phi_{\lambda}(\Gamma/k(\lambda))). \]

**Proof.** It holds that

\[ \sum_{\tau \in FD(\Gamma)} d_{ED}(\Phi_{\tau}(\Gamma)) = \sum_{\tau \in FD(\Gamma)} \sum_{a \in ED(\tau)} d_{E_a}^{(a)}(\Phi_{\tau}(\Gamma)). \]

For \( \lambda \in FD(\Gamma) \) let \( P(\lambda) \) be the set of edges in the path from \( j(\lambda) \) to \( k(\lambda) \). Clearly, \( \lambda \setminus \{a\} \) for \( a \in P(\lambda) \) is a spanning forest of \( \Gamma \) and \( a \) is in \( ED(\Gamma) \) for that spanning forest. One can easily see that sets \( \{(\tau, a)|\tau \in F(\Gamma), a \in ED(\tau)\} \) and \( \{\lambda, a)|a \in FD(\Gamma), a \in P(\lambda)\} \) are bijective, so

\[ \sum_{\tau \in FD(\Gamma)} d_{ED}(\Phi_{\tau}(\Gamma)) = \sum_{\lambda \in ED(\Gamma)} \sum_{a \in P(\lambda)} d_{E_a}^{(a)}(\Phi_{\lambda,\{a\}}(\Gamma)). \]

To finish the proof it is enough to show that

\[ \sum_{a \in P(\lambda)} d_{E_a}^{(a)}(\Phi_{\lambda,\{a\}}(\Gamma)) \sim \Phi_{\lambda}(\Gamma/j(\lambda)) + \Phi_{\lambda}(\Gamma/k(\lambda)) \]

for every \( \lambda \in FD(\Gamma) \). After choosing \( a \in P(\lambda) \) the path from \( j(\lambda) \) to \( k(\lambda) \) along \( \lambda \) in \( \Phi_{\lambda,\{a\}}(\Gamma) \) has the edge \( a \) of type \( \rightarrow \), and the other edges of type \( \rightarrow \) or \( \leftarrow \) with direction from \( j(\lambda) \) or \( k(\lambda) \) to the edge \( a \), such as in the following diagram.
After acting by $d_E^{(0)}$ this $\rightarrow\rightarrow$ is replaced by $\rightarrow\rightarrow + (-1)^n$, such as in the following diagram.

\[
\begin{array}{c}
\bullet \quad j(a) \\
\bullet \quad k(a) \\
\end{array}
\begin{array}{c}
\bullet \quad j(a) \\
\bullet \quad k(a) \\
\end{array} + (-1)^n
\]

Careful calculation of the sign shows that those two terms are cancelled with terms given from choosing neighbouring edges in $P(\lambda)$. The two last terms which does not have corresponding neighbour are exactly $\Phi_\tau(\Gamma/j(a))$ and $\Phi_\delta(\Gamma/k(a))$, which was to be demonstrated.

Equations (47) and (45), and Lemmas 3.6 and 3.7 imply that

$$\Phi(d(\Gamma)) + \Phi(h(\Gamma)) \sim d_C(\Phi(\Gamma)) + \sum_{\delta \in \mathcal{FD}(\Gamma)} \left( \Phi_\tau(h,j(\delta)(\Gamma)) + \Phi_\delta(h,k(\delta)(\Gamma)) \right) \sim d_C(\Phi(\Gamma)) + \sum_{\tau \in \mathcal{FD}(\Gamma)} d_E(\Phi_\tau(\Gamma)) = d_C(\Phi(\Gamma)) + d_E(\Phi(\Gamma)) = d(\Phi(\Gamma)).$$

After taking coinvariants this implies that

$$\Phi(d(\Gamma)) + \Phi(h(\Gamma)) = d(\Phi(\Gamma))$$

for each $\Gamma \in \mathcal{H}_n$. Hence, $\Phi : (\mathcal{H}_n, d + h) \rightarrow O(G_{n+1}, d)$ is a map of complexes.

### 3.5. Differential grading on the number of hairs/sources

Note that the part of $\mathcal{H}_n$ with $s$ hairs is mapped to the part of $O(G_{n+1})$ with $s$ sources, so the map $\Phi : \mathcal{H}_n \rightarrow O(G_{n+1})$ can be split as

$$\Phi : \mathcal{H}_n \rightarrow S_G O_{n+1},$$

where $S_G O_{n+1}$ is the part of $O_{n+1}$ spanned by oriented graphs with exactly $s$ sources.

There are filtrations on $(\mathcal{H}_n, d + h)$ and $(O_{n+1}, d)$ on the number of hairs and sources respectively. Their differential gradings are $(\mathcal{H}_n, d)$ and $(O_{n+1}, d_0)$, where $d_0$ is the part of $d$ that does not destroy a source. We will refer to graph complexes with such a differential $d_0$ as **fixed source graph complexes**.

As the map $\Phi$ maps a graph with $s$ hairs to a sum of graphs all containing $s$ sources, the same map of vector spaces, $\Phi$, is also a map of complexes

$$\Phi : (\mathcal{H}_n, d) \rightarrow (O_{n+1}, d_0).$$

### 3.6. The dual map

As the map $\Phi$ preserves the loop order, we obtain a dual map $\Phi \leftrightarrow G$

$$G : (O_{n+1}, \delta) \rightarrow (H_{G_n}, \delta + \chi) .$$

Similarly, the same map is a map of complexes

$$G : (O_{n+1}, \delta_0) \rightarrow (H_{G_n}, \delta) ,$$

where $\delta_0 \leftrightarrow d_0$ is the part of $\delta$ that does not produce a source by splitting non-source.

Let $\Gamma \in O_{n+1}$ be a single term graph. We define the **hairy skeleton** $hs(\Gamma) \in H_{G_n}$ to be the hairy graph obtained from $\Gamma$ by:

- putting a hair on each source vertex;
- contracting each occurrence of bi-valent target vertices $\rightarrow\rightarrow$ into a single non-oriented edge.

It is evident that

$$G(\Gamma) = \begin{cases} 
hs(\Gamma) & \text{if } \Gamma = \Phi_\tau(hs(\Gamma)) \text{ for some spanning forest } \tau \\
0 & \text{otherwise.} 
\end{cases}$$

**Proposition 3.8.** Let $\Gamma \in O_{n+1}$ be a single term graph whose hairy skeleton $hs(\Gamma)$ contains $v$ vertices, $e$ edges, and $s$ sources. We have that $\Gamma = \Phi_\tau(hs(\Gamma))$ for some spanning forest $\tau$ if and only if $\Gamma$ contains $e - v + s$ bi-valent target vertices $\rightarrow\rightarrow\rightarrow$.

**Proof.** If $\Gamma = \Phi_\tau(hs(\Gamma))$, it is clear that $\Gamma$ must contain $e - v + s$ of bi-valent target vertices $\rightarrow\rightarrow\rightarrow$. Now, let $\Gamma$ be an oriented graph that contains $e - v + s$ bi-valent target vertices $\rightarrow\rightarrow\rightarrow$. Let $\gamma$ be the subgraph of $\Gamma$ obtained by removing all bi-valent targets $\rightarrow\rightarrow\rightarrow$ from $\Gamma$. Note that $\gamma$ is a graph with $v$ vertices and $v - s$ edges. Such a graph must contain at least $s$ different connected components, with equality if and only if $\gamma$ is a forest. Next, note that $\gamma$ is oriented, hence each component must contain a source vertex, and $\gamma$ contains precisely $s$ source vertices.

We conclude that $\gamma$ must be a forest with $s$ components, where each component contains precisely one source vertex. Hence, we get that

$$\Gamma = \Phi_{hs(\Gamma)}(hs(\Gamma)).$$
This proposition gives us an alternative description of the map $G$. 

(55) \[ G(\Gamma) = \begin{cases} h_s(\Gamma) & \text{if } \Gamma \text{ contains } v - e + s \text{ bi-valent target vertices} \\ 0 & \text{otherwise.} \end{cases} \]

4. The map is a quasi-isomorphism

**Theorem 4.1.** The map \( \Phi : (H_G, d) \to (O_G, d) \) and the projection \( p : (SG, d) \to (O_G, d) \) are quasi-isomorphisms.

**Proof.** We prove both claims simultaneously. After splitting by the number of hairs/sources, we need to prove that \( \Phi : (H_G, d) \to (O_G, d) \) are quasi-isomorphisms. Using Definition/Proposition 2.2 it is enough to prove the claim for \( \Phi : (H_G, d) \to (S_k G_{n+1}, d) \leftarrow (S_k G_{n+1}, d) \)

where

(56) \[ S_k O^k G_n := \kappa (S_k G_n) . \]

(57) \[ S_k^2 G_n := \kappa (S_k G_n) . \]

On the mapping cone of them we set up the spectral sequence on the number of vertices. Standard splitting of complexes as the product of complexes with fixed loop number implies the correct convergence. It is therefore enough to prove the claim for the first differential of the spectral sequence (differential grading).

The edge differential does not change the number of vertices, while the core differential does. Therefore, on the first page of the spectral sequences there are mapping cones of the maps

\[ \Phi : (H,G, 0) \to (S_k O^k G_{n+1}, d) \leftarrow (S_k G_{n+1}, d) , \]

where \( d \) is the part of the edge differential \( d_e \) that does not destroy a source.

Complexes are direct sums of those with fixed number of vertices and edges, so it is enough to show the claim for \( \Phi : (V, E, H, G, 0) \to (V, E, S_k O^k G_{n+1}, d) \leftarrow (V, E, S_k G_{n+1}, d) \),

where \( V, E, S_k O^k G_{n+1} \) is the part of \( V, E, O^k G_{n+1} \) with \( s \) sources.

Recall that the skeleton versions of oriented and sourced graph complexes \( V, E, S_k O^k G_{n+1} \) and \( V, E, S_k G_{n+1} \) are spaces of invariants of the action of \( S_k \times S_{e} \), while the hairy graph complex \( V, E, H, G, 0 \) is the space of invariants of the action of \( S_k \times S_{e} \times (S_{e} \times S_{e}^2) \). In order to have the same group acting in a latter case, let us redefine the hairy graph complex in two steps: Let

(58) \[ V, E, H, G := \begin{cases} (V, E, H, G_n \otimes sgn)_{\leq 2}^{2^e} & \text{for } n \text{ even}, \\ (V, E, H, G_n \otimes sgn)_{\leq 2}^{2^e} & \text{for } n \text{ odd}, \end{cases} \]

and then

(59) \[ V, E, H, G := \begin{cases} (V, E, H, G_n \otimes sgn)_{\leq 2}^{2^e} & \text{for } n \text{ even}, \\ (V, E, H, G_n \otimes sgn)_{\leq 2}^{2^e} & \text{for } n \text{ odd}. \]

The action of \( S_e \times S_e \) clearly commutes with the map \( \Phi \). Since the edge differential does not change the number of vertices and edges, taking homology commutes with taking invariants of that action. Therefore, it is now enough to show the claim for

\[ \Phi : (V, E, H, G, 0) \to (V, E, S_k O^k G_{n+1}, d) \leftarrow (V, E, S_k^2 G_{n+1}, d) . \]

Let us pick up a particular single term graph \( \Gamma \in V, E, H, G_n \). Note that in \( \Gamma \) edges are (up to the sign) undirected. Let \( \langle \Omega \rangle \) be the subspace of \( V, E, S_k O^k G_{n+1} \) spanned by oriented graphs of the shape \( \Gamma \) without hairs, but with source exactly where the hairs were. Similarly, let \( \langle S \rangle \) be the subspace of \( V, E, S_k G_{n+1} \) spanned by sourced graphs of the shape \( \Gamma \) without hairs, but with source exactly where the hairs were.
The map $\Phi$ is defined such that $\Phi(\Gamma) \in \langle \Omega \Gamma \rangle$. Also, differential $d_{E0}$ acts within particular subspaces $\langle \Omega \Gamma \rangle$ and $\langle \Omega \Gamma \rangle$. Therefore, we can split the map as a direct sum and it is enough to prove the claim for
\[ \Phi : ((\Gamma), 0) \rightarrow ((\Omega \Gamma), d_{E0}) \leftrightarrow ((\Omega \Gamma), d_{E0}), \]
for every $\Gamma \in V, E, H, G_n$.

In order to prove that, let us choose $v - s$ edges in $\Gamma$, say $a_1, \ldots, a_{v-s}$. Let $F(a_1, \ldots, a_i)$ be the sub-graph of $\Gamma$ that includes those edges, all hairs and all necessary vertices. We require that for every $i = 1, \ldots, v - s$ the subgraph $F(a_1, \ldots, a_i)$ is a forest. Recall that in a forest, every connected component has exactly one hair. Clearly, $F(a_1, \ldots, a_{v-s})$ is a spanning forest.

For every $i = 0, \ldots, v - s$, we form two graph complexes $\langle \Omega \Gamma \rangle$ and $\langle \Omega \Gamma \rangle$ as follows: They are all spanned by graphs with a core graph being un-haired $\Gamma$ with attached edge types from dg $\langle S_2 \rangle$ module $\Sigma$ spanned by $\{ \rightarrow [n], \leftrightarrow [n], \rightarrow [n - 1], \rightarrow [n] \}$ with $S_2$ action
\[ (61) \quad \leftrightarrow \leftrightarrow, \quad \rightarrow \leftrightarrow (-1)^v \rightarrow, \quad \rightarrow \rightarrow (-1)^{v+1} \rightarrow, \]
and the differential
\[ (62) \quad \rightarrow \leftrightarrow \rightarrow \leftrightarrow (-1)^v \leftarrow \leftrightarrow. \]

The complex $\langle \Omega \Gamma \rangle$ is spanned by graphs whose attached edge types fulfill the following conditions:

1. Edges $a_1, \ldots, a_i$ have type $\rightarrow$, and other edges have other types.
2. No vertex in the forest $F(a_1, \ldots, a_i)$ has a neighbouring edge of type $\rightarrow$ or $\leftrightarrow$ heading towards it.
3. Every vertex outside the forest $F(a_1, \ldots, a_i)$ has a neighbouring edge of type $\rightarrow$ or $\leftrightarrow$ heading towards it.

The complex $\langle \Omega \Gamma \rangle$ is spanned by graphs whose attached edge types fulfill the conditions above together with:

4. There are no cycles along arrows on edges of type $\rightarrow$ and $\leftrightarrow$.

Examples of graphs in $\langle \Omega \Gamma \rangle$ and $\langle \Omega \Gamma \rangle$ are shown in Figure 4.2.

![Figure 4.2](image_url)

**Figure 4.2.** Two examples of graphs in $\langle \Omega \Gamma \rangle$, with $\Gamma$ as in Figure 5.1. The forest $F(a_1, a_2, a_3)$ is depicted red. The right example is also in $\langle \Omega \Gamma \rangle$, while the left one is not.

The differential on $\Sigma$ induces the differential $d_{E0}$ on $\langle \Omega \Gamma \rangle$ and $\langle \Omega \Gamma \rangle$, where the forbidden graphs are considered zero. It is straightforward to check that
\[ (63) \quad \langle \Omega \Gamma^0, d_{E0} \rangle = \langle \Omega \Gamma, d_{E0} \rangle, \quad \langle \Omega \Gamma^0, d_{E0} \rangle = \langle \Omega \Gamma, d_{E0} \rangle. \]

Also, it holds that
\[ (64) \quad \langle \Omega \Gamma^{v-i} \rangle = \langle \Omega \Gamma^{v-i} \rangle \]
is one dimensional, spanned by the graph with edges $a_1, \ldots, a_{v-s}$ of type $\rightarrow$ and other edges of type $\leftrightarrow$.

There are natural projections of complexes $\pi_i : \langle \Omega \Gamma \rangle \rightarrow \langle \Omega \Gamma \rangle$ for every $i = 0, \ldots, v - s$. Also for every $i = 1, \ldots, v - s$, there are natural maps
\[ (65) \quad f^i : \langle \Omega \Gamma^{i-1} \rangle \rightarrow \langle \Omega \Gamma \rangle, \quad g^i : \langle \Omega \Gamma^{i-1} \rangle \rightarrow \langle \Omega \Gamma \rangle, \]
that only change the type of the edge $a_i$ as
\[ (66) \quad \rightarrow \leftrightarrow 0, \quad \rightarrow \rightarrow, \quad \leftarrow \leftarrow (-1)^{i+1} \rightarrow, \]
where forbidden graphs are considered zero. The following diagram clearly commutes.

\[ \begin{array}{c}
\langle \Omega \Gamma^{i-1} \rangle \\
\downarrow f^i
\end{array} \quad \begin{array}{c}
\langle \Omega \Gamma^{i-1} \rangle \leftrightarrow \\
\downarrow g^i
\end{array} \]

\[ \begin{array}{c}
\langle \Omega \Gamma^i \rangle \\
\uparrow \pi_i
\end{array} \quad \begin{array}{c}
\langle \Omega \Gamma^i \rangle \leftrightarrow \\
\uparrow \pi_i
\end{array} \]

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Lemma 4.3. For every $i = 1, \ldots, v - s$ maps $f^i : \langle \Omega^{v-1} \rangle \to \langle \Omega^i \rangle$ and $g^i : \langle \Sigma^{v-1} \rangle \to \langle \Sigma^i \rangle$ are quasi-isomorphisms.

Proof. Let us prove the claim for $g^i$. The essential difference between $\langle \Sigma^{v-1} \rangle$ and $\langle \Sigma^i \rangle$ is in the edge $a_i$, it has to be of type $\rightarrow$ in $\langle \Sigma^i \rangle$, and it is of another type in $\langle \Sigma^{v-1} \rangle$. Since $g^i$ does not change types of other edges, it splits as a direct sum of maps between complexes with fixed types of other edges

$$g^i \mid_{\langle \Sigma^{v-1} \rangle} : \langle \Sigma^{v-1} \rangle \to \langle \Sigma^i \rangle$$

where $\langle \Sigma^{v-1} \rangle$ and $\langle \Sigma^i \rangle$ are sub-complexes spanned by graphs with fixed types of all edges other than $a_i$. It is enough to show that each $g^i$ is a quasi-isomorphism.

Here, depending on the choice of fixed edge types, the condition of being sourced can disallow some possibilities for the edge $a_i$ in both $\langle \Sigma^{v-1} \rangle$ and $\langle \Sigma^i \rangle$. We list all cases, showing that the map is a quasi-isomorphism in all of them. Let the vertex that is in the forest $F(a_1, \ldots, a_i)$ but not in the forest $F(a_1, \ldots, a_{i-1})$ be called $x_i$.

1. If there is a vertex in the forest $F(a_1, \ldots, a_{i-1})$ that has a neighboring edge of type $\rightarrow$ or $\leftarrow$ heading towards it, or there is a vertex outside the forest $F(a_1, \ldots, a_i)$ that does not have a neighboring edge of type $\rightarrow$ or $\leftarrow$ heading towards it, both $\langle \Sigma^{v-1} \rangle$ and $\langle \Sigma^i \rangle$ are zero complexes and the map is clearly a quasi-isomorphism.
2. If it is not the case from (1) and $x_i$ has a neighboring edge of type $\rightarrow$ or $\leftarrow$ heading towards it, the edge $a_i$ (that goes from a vertex in the forest $F(a_1, \ldots, a_{i-1})$ towards $x_i$) can have types $\rightarrow$ or $\leftarrow$ in $\langle \Sigma^{v-1} \rangle$, making it acyclic. In $\langle \Sigma^i \rangle$, no type is allowed, so it is again the zero complex. Therefore, the map is again a quasi-isomorphism.
3. If it is not the case from (1) and $x_i$ does not have a neighboring edge of type $\rightarrow$ or $\leftarrow$ heading towards it, the edge $a_i$ must have type $\rightarrow$ in $\langle \Sigma^{v-1} \rangle$. In $\langle \Sigma^i \rangle$ that edge must have type $\rightarrow$, making the map an isomorphism. Thus, it is also a quasi-isomorphism.

The lemma implies that

$$f := f^{v-s} \circ \cdots \circ f^1 : \langle \Omega \rangle \to \langle \Omega^{v-s} \rangle,$$

and

$$g := g^{v-s} \circ \cdots \circ g^1 : \langle \Sigma \rangle \to \langle \Sigma^{v-s} \rangle$$

are quasi-isomorphisms and the following diagram commutes:

$$
\begin{array}{ccc}
\langle \Omega \rangle & \xrightarrow{\pi} & \langle \Sigma \rangle \\
\downarrow & & \downarrow g \\
\langle \Omega^{v-s} \rangle & \xleftarrow{\text{id}} & \langle \Sigma^{v-s} \rangle.
\end{array}
$$

Lemma 4.4. The map $f \circ \Phi : \langle \Gamma \rangle \to \langle \Omega^{v-s} \rangle$ is a quasi-isomorphism.

Proof. Both complexes are one-dimensional, so we only need to prove that $f \circ \Phi \neq 0$. The left-hand side complex has a generator $\Gamma$. It holds that

$$f \circ \Phi(\Gamma) = f \left( \sum_{\tau \in \Gamma(\Gamma)} \Phi_\tau(\Gamma) \right).$$

The map $\Phi_\tau$ gives edges in $E(\tau)$ type $\rightarrow$ or $\leftarrow$, and type $\rightarrow \leftarrow$ to the other edges. After that, the map $f = f^1 \circ \cdots \circ f^{v-s}$ kills all graphs with any of edges $a_1, \ldots, a_{i-1}$ being of type $\rightarrow \leftarrow$. Therefore, $f \circ h_{s,\tau}$ is non-zero only if the forest $\tau$ consist exactly of the edges $a_1, \ldots, a_{i-1}$. Let us call this forest $T$. So

$$f \circ \Phi(\Gamma) = f(\Phi_T(\Gamma)).$$

It is clearly the generator of $\langle \Omega^{v-s} \rangle$, and therefore non-zero.

With this lemma, we have proven that the diagonal map in the following commutative diagram is also a quasi-isomorphism:

$$
\begin{array}{cccc}
\langle \Omega \rangle & \xrightarrow{\Phi} & \langle \Omega \rangle & \xrightarrow{\pi} & \langle \Sigma \rangle \\
\downarrow f \circ \Phi & & \downarrow f & & \downarrow g \\
\langle \Omega^{v-s} \rangle & \xleftarrow{\text{id}} & \langle \Sigma^{v-s} \rangle.
\end{array}
$$
Together with the previous result we conclude that all mentioned maps are quasi-isomorphisms. Which concludes the proof. □

**Corollary 4.5.** The map

\[ \Phi : (HG_n, d + h) \rightarrow (OG_{n+1}, d) \]

and the projection

\[ p : (SG_n, d) \rightarrow (OG_n, d) \]

are quasi-isomorphism.

**Proof.** On the mapping cone of \( \Phi \), we set up a spectral sequence on the number \( s \), which is the number of hairs in \( HG_n \) and number of sources in \( OG_{n+1} \). Standard splitting of complexes as the product of complexes with fixed loop number implies the correct convergence. On the first page of the spectral sequence we have exactly the mapping cone of

\[ \Phi : (HG_n, d) \rightarrow (OG_{n+1}, d_0) \]

which is acyclic according to Theorem 4.1. By the spectral sequence argument, the mapping cone of

\[ \Phi : (HG_n, d + h) \rightarrow (OG_{n+1}, d) \]

is also acyclic, hence \( \Phi \) is here a quasi-isomorphism.

The same argument works for the projection \( p \). However, it has already been proven in [25, Theorem 1.1]. □

**Corollary 4.6.** (Theorem 4.1). The map dual map of \( G \leftrightarrow \Phi \), given explicitly in (54), is a quasi-isomorphism of complexes

\[ G : (OGC_{n+1}, \delta) \rightarrow (HGC_n, \delta + \chi) \]

and of complexes

\[ G : (OGC_{n+1}, \delta_0) \rightarrow (HGC_n, \delta). \]

**Proof.** We may fix the loop order \( b \). As \( B_bOG_{n+1} \) and \( B_bHG_n \) are finite dimensional in each degree, it is clear that each quasi-isomorphism

\[ F : (B_bOG_{n+1}, d) \rightarrow (B_bHG_n, d + h), \quad F : (B_bOG_{n+1}, d_0) \rightarrow (B_bHG_n, d) \]

has a dual quasi-isomorphism

\[ G : (B_bOG_n, \delta) \rightarrow (B_bHGC_n, \delta + \chi), \quad G : (B_bOG_n, \delta_0) \rightarrow (B_bHGC_{n+1}, \delta_0). \]

As \( \Phi \) preserves the loop order, its dual map \( G \) must be a quasi-isomorphism. □

5. **Application to ribbon graphs and the moduli space of curves with punctures**

In this section, we will follow [13] in order to define the ribbon graph complex \((RGC[1], \delta + \Delta_1)\), as well as the map \( F : (OGC_1, \delta) \rightarrow (RGC[1], \delta + \Delta_1)\). This will allow us to make the observation that \( F \) is also a map of complexes

\[ F : (OGC_1, \delta_0) \rightarrow (RGC[1], \delta). \]

5.1. **Ribbon Graphs.**

**Definition 5.1.** A ribbon graph \( \Gamma \) is a triple \((F_\Gamma, \iota_\Gamma, \sigma_\Gamma)\), where \( F_\Gamma \) is a finite set, \( \iota_\Gamma : F_\Gamma \rightarrow F_\Gamma \) is an involution with no fixed points, i.e.

\[ \iota_\Gamma^2 = id, \quad \iota_\Gamma(f) \neq f, \]

for every \( f \in F_\Gamma \), and \( \sigma_\Gamma : F_\Gamma \rightarrow F_\Gamma \) is a bijection.

The elements of \( F_\Gamma \) are called flags or half edges. The orbits of the involution \( \iota_\Gamma \) are called edges, and the set of all edges will be denoted by \( E(\Gamma) \). The orbits of \( \sigma_\Gamma \) are called vertices, and the set of all vertices will be denoted by \( V(\Gamma) \).

**Definition 5.2.** We say that a cyclic ordering on a finite set \( A \) is a \( \mathbb{Z}_{|A|} \) action on \( A \) with precisely 1 orbit.

The difference between an ordinary graph and a ribbon graph is that each vertex in a ribbon graph is equipped with a cyclic ordering of its adjacent (half) edges, given by \( f + 1 = \sigma_\Gamma f \).

We may draw a picture of a ribbon graph \( \Gamma \) in the following way:
(1) For each vertex \((i_1i_2\ldots i_k) \in V(\Gamma)\), draw a dot with clockwise ordered lines labeled by \(i_1, i_2, \ldots, i_k\) connected to the dot

\[
(i_1i_2\ldots i_k) \leftrightarrow \begin{array}{c}
 i_1
 \ldots
 i_k
 \ldots
 i_3
 \end{array}.
\]

(2) For each edge \((iOi) \in E(\Gamma)\), connect the lines labeled by \(i_o\) and \(i_e\).

Two very basic examples of ribbon graphs are

\[
((1, 2), (12), (1)) = \begin{array}{c}
 1 & 2
 \end{array} \quad ((1, 2), (12), (12)) = \begin{array}{c}
 1 \\
 2 \\
 \end{array}.
\]

We call the orbits of the permutation \(\sigma_\Gamma^{-1} \circ \iota_\Gamma\) boundaries of \(\Gamma\), and we denote the set of boundaries by \(B(\Gamma)\).

For example, the ribbon graph

\[
\begin{array}{c}
 1 \\
 2 \\
 \end{array}
\]

has one boundary \((12)\), while the ribbon graph

\[
\begin{array}{c}
 \circ \\
 \end{array}
\]

has two boundaries, \((1)\) and \((2)\).

5.2. A PROP of ribbon graphs. Let \(\text{rgra}_{n,m,k}\) be the set of ribbon graphs with vertex set labeled by \([n]\), boundary set labeled by \([m]\) and edge set labeled by \([k]\). That is ribbon graphs

\[
\Gamma = ([k] \sqcup [k], \iota_k, \sigma_\Gamma),
\]
where \(\iota_k\) is the natural involution on \([k] \sqcup [k]\), together with bijections \(\nu_\Gamma : [n] \to V(\Gamma), b_\Gamma : [m] \to B(\Gamma)\). Here, the edge labeled by \(i \in [k]\) is the pair \((i_1, i_2), i = i_1 \in [k] \sqcup \emptyset, i = i_2 \in \emptyset \sqcup [k]\). We say that the edge \((i_1, i_2)\) is intrinsically oriented from \(i_1\) to \(i_2\).

The group \(G_k := S_k \times S_k^\infty\) acts naturally on \(\text{rgra}_{n,m,k}\) by permuting edges and reversing the intrinsic orientation. Let \(R\text{Gra}(n, m)\) be the vector space

\[
R\text{Gra}(n, m) := \prod_{k \geq 0} \left( \chi(\text{rgra}_{n,m,k})[k] \otimes \text{sgn}_k \right)^{G_k}
\]

The space

\[
R\text{Gra} := \bigoplus_{n,m \geq 1} R\text{Gra}(n, m)
\]

is an \(S\)-bimodule, where \(S_n\) acts on \(R\text{Gra}(n, m)\) by permuting vertex labels, and \(S_m\) acts on \(R\text{Gra}(n, m)\) by permuting boundary labels.

In order to define the properadic composition maps \(\circ : R\text{Gra} \otimes R\text{Gra} \to R\text{Gra}\), we have to make a few definitions.

**Definition 5.3.** Let \(A\) and \(B\) be two finite sets with cyclic orderings. We say that an ordered \(A\)-partition \(p\) of \(B\) is a partition

\[
\bigcup_{a \in A} p_a = B,
\]

where each

\[
p_a = \{b_a^1, \ldots, b_a^k\} \subset B
\]

is ordered with

\[
b_a^i + 1 = \begin{cases} b_a^{i+1} & i < k \\ b_a^{i+1} & i = k, \text{ where } r = \min\{j \in \mathbb{Z} | j \geq 1, p_{a+j} \neq \emptyset\}. \end{cases}
\]

We denote the set of all ordered \(A\)-partitions of \(B\) by \(P(A, B)\).

**Definition 5.4.** Let \(\Gamma = (F, \iota, \sigma)\) be a ribbon graph, and let \(v \in V(\Gamma)\), \(b \in B(\Gamma)\) such that \(v \cap b = \emptyset\). Then, for each ordered partition \(p \in P(b, v)\), we define the \(p\)-grafted ribbon graph

\[
\iota_{p\Gamma} = (F, \iota_{p\Gamma}, \sigma_{p\Gamma})
\]

by letting \(\sigma_{p\Gamma}\) be the unique permutation such that

\[
(1) \quad \sigma_{p\Gamma}|_{F(v \cup b)} = \sigma|_{F(v \cup b)};
\]

\(16\)

(2) for each \( j \in b \)
\[
\circ_p \sigma(j) = \begin{cases} 
\min p_j, & p_j \neq \emptyset, \\
\sigma(j) & p_j = \emptyset 
\end{cases}
\]
(3) for each \( i \in p_j \)
\[
\circ_p \sigma(i) = \begin{cases} 
\sigma(i) & i \neq \max p_j, \\
\sigma(j) & i = \max p_j. 
\end{cases}
\]

For two ribbon graphs \( \Gamma_1 = (F_1, t_1, \sigma_1) \) and \( \Gamma_2 = (F_2, t_2, \sigma_2) \), \( v \in V(\Gamma_1) \), \( b \in B(\Gamma_2) \), and \( p \in P(b, v) \), we set
\[
\Gamma_1 \circ_p \Gamma_2 := \left( F_1 \sqcup F_2, t_1 \sqcup t_2, \circ_p(\sigma_1 \sqcup \sigma_2) \right).
\]

A picture of the ribbon graph \( \circ_p \Gamma \) is obtained from a picture of the ribbon graph \( \Gamma \) by removing the vertex \( v \) and reconnecting its adjacent edges to the corners of the boundary \( b \) according to the partition \( p \).

**Lemma 5.5.** There is a bijection of vertices
\[
p_V : V(\Gamma) \setminus \{v\} \to V(\circ_p \Gamma)
\]
and a bijection of boundaries
\[
p_B : B(\Gamma) \setminus \{b\} \to B(\circ_p \Gamma),
\]
such that \( v' \subseteq p_V(v') \) and \( b' \subseteq p_B(b') \) for every \( v' \in V(\Gamma) \setminus \{v\} \) and \( b' \in B(\Gamma) \setminus \{b\} \).

Furthermore, if \( v' \cap b = \emptyset \) we have equality \( p_V(v') = p_V(v) \). Similarly if \( v \cap b' = \emptyset \), we have \( b' = p_B(b') \).

**Proof.** Pick a boundary \( b' \in B(\Gamma) \setminus \{b\} \), and a flag \( j' \in b' \). Suppose that \( (\circ_p \sigma)^{-1} i(j') \neq \sigma^{-1} i(j') \). Then we must have \( i(j') = \sigma(j) \) or \( i(j') = \min p_j \) for some \( j \in b \). The first case implies that \( \sigma^{-1} i(j') \in b \) and, therefore, we get \( j' \in b \), which contradicts our choice of \( j' \). If \( i(j') = \min p_j \), then \( (\circ_p \sigma)^{-1} i(j') \in b \). Furthermore, for \( r = 1, 2, 3, \ldots \), we have that \( ((\circ_p \sigma)^{-1} i)^r(j') \) until \( i(\sigma^{-1} i)^r(j') = \sigma(j) \), in which case \( (\circ_p \sigma)^{-1} i(j') = \sigma^{-1} i(j') \).

It follows that \( b' \) is a subset of the orbit of \( (\circ_p \sigma)^{-1} i(j') \). Thus, we may define the map
\[
p_B : B(\Gamma) \setminus \{v\} \to B(\circ_p \Gamma)
\]
by setting \( p_B(b') \) to be the \( (\circ_p \sigma)^{-1} i \) orbit of any \( j' \in b' \). From the arguments above, it follows that \( p_B \) is injective and \( b' \subseteq p_B(b') \). Finally, we note that for any \( j \in b \), there must exist an \( r \geq 1 \) such that \( i(\sigma^{-1} i)^r(j) \notin b \). Hence \( p_B \) must also be surjective.

If \( v \cap b' = \emptyset \), then it is clear from the construction that \( b' = p_B(b') \). Similarly, we can define
\[
p_V : V(\Gamma) \setminus \{v\} \to V(\circ_p \Gamma)
\]
by setting \( p_V(v') \) to be the \( \circ_p \sigma \) orbit of any \( i \in v' \). By the same arguments as above, we get that \( p_V \) is well defined, bijective, and \( v' \subseteq p_V(v) \). \( \square \)

This lemma implies that, for mutually disjoint \( v_1, v_2 \in V(\Gamma) \), \( b_1, b_2 \in B(\Gamma) \), and partitions \( p_1 \in P(b_1, v_1) \), \( p_2 \in P(b_2, v_2) \), we may define
\[
\circ_{p_1, p_2} \Gamma := \circ_{p_2}(\circ_{p_1} \Gamma).
\]

It is clear that we have
\[
(70) \quad \circ_{p_1, p_2} \Gamma = \circ_{p_2}(\circ_{p_1} \Gamma) = \circ_{p_1}(\circ_{p_2} \Gamma) = \circ_{p_1, p_2} \Gamma.
\]

For each \( k \leq m_1, n_2 \), we define the prooperadic composition maps
\[
\circ_k : \text{RGr}_d(n_1, m_1) \otimes \text{RGr}_d(n_2, m_2) \to \text{RGr}_d(n_1 + n_2 - k, m_1 + m_2 - k),
\]
composing the boundaries \( m_1 - k + 1, \ldots, m_1 \) of \( \Gamma_1 \) to the vertices \( 1, \ldots, k \) of \( \Gamma_2 \), by
\[
\Gamma_1 \circ_k \Gamma_2 := \prod_{i=1}^k \left( \sum_{p \in P(b_1(n_1 - k + 1), \Gamma_2(i))} \circ_p(\Gamma_1 \sqcup \Gamma_2) \right).
\]
The element \( \Gamma_1 \circ_k \Gamma_2 \) is the sum of all graphs obtained from \( \Gamma_1 \) and \( \Gamma_2 \) by
(1) Remove each vertex 1,...,k from Γ₂;
(2) For each i ∈ [k], reconnecting each half edge in vΓ₂(i) to a corner of the boundary hΓ₁(m₁−k+i), respecting the cyclic orientations.

**Proposition 5.6.** The composition maps ∗ : RGra ⊗ RGra → RGra defines a PROP structure on RGra.

**Proof.** It follows by Lemma 5.5 and (70) that the maps are well defined and well behaved. □

Let LieB₀,₀ be the PROP of (degree shifted) Lie bi-algebras, i.e. the PROP generated by symmetric 3-valent corollas of degree 1

\[
\begin{align*}
1 & \quad 2 \quad 1 \\
\end{align*}
\]

modulo the relations

\[
\begin{align*}
\begin{array}{c}
1 & \quad 2 \\
\end{array} + 
\begin{array}{c}
2 & \quad 3 \\
\end{array} + 
\begin{array}{c}
3 & \quad 1 \\
\end{array} = 0,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
1 & \quad 2 \\
\end{array} + 
\begin{array}{c}
2 & \quad 3 \\
\end{array} + 
\begin{array}{c}
3 & \quad 1 \\
\end{array} = 0,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
1 & \quad 2 \\
\end{array} + 
\begin{array}{c}
2 & \quad 3 \\
\end{array} + 
\begin{array}{c}
3 & \quad 1 \\
\end{array} = 0.
\end{align*}
\]

**Proposition 5.7 ([12]).** There is a map of PROPs

\[
s : \text{LieB}_{0,0} \rightarrow \text{RGra}
\]

given by

\[
\begin{align*}
\begin{array}{c}
1 \\
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
1 \\
\end{array},
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
2 \\
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
2 \\
\end{array},
\end{align*}
\]

**Remark 5.8.** The map s factors through the PROP of involutive Lie bialgebras LieB^!₀,₀ which is generated by the corollas (71) modulo the relations (72), (73), (74) plus the additional relation

\[
\begin{align*}
\begin{array}{c}
\end{array} = 0.
\end{align*}
\]

5.3. **The ribbon graph complex RGC.** Let hoLieB₀,₀ be the quasi-free differential graded PROP generated by skew symmetric corollas

\[
\begin{align*}
\begin{array}{c}
1 & \quad 2 \\
\end{array} \quad \cdots \quad 
\begin{array}{c}
1 & \quad 2 \\
\end{array} \\
\begin{array}{c}
1 & \quad 2 \\
\end{array} \quad \cdots \\
\begin{array}{c}
1 & \quad 2 \\
\end{array}
\end{align*}
\]

of degree 1, with the differential

\[
\delta
\]

\[
\begin{align*}
\begin{array}{c}
1 & \quad 2 \\
\end{array} \quad \cdots \\
\begin{array}{c}
1 & \quad 2 \\
\end{array} \quad \cdots \\
\begin{array}{c}
1 & \quad 2 \\
\end{array}
\end{align*}
\]

\[
\sum_{J_1,J_2} J_1 \cdot J_2
\]
It was shown in [10] that $\text{hoLieB}_{0,0}$ is a minimal resolution of $\text{LieB}_{0,0}$, i.e. the natural projection map $p : \text{hoLieB}_{0,0} \to \text{LieB}_{0,0}$ is a quasi-isomorphism.

We define the ribbon graph complex $(\text{RGC}, \delta + \Delta_1)$ to be the deformation complex

$$(\text{RGC}, \delta + \Delta_1) := \text{Def}(\text{hoLieB}_{0,0} \xrightarrow{\pi p} \text{RGr}).$$

As a graded vector space

$$\text{RGC} \cong \prod_{n,m \geq 1} (\text{RGr}(n,m))^\otimes x^2 \text{Gr}$$

is spanned by ribbon graphs in $\text{RGr}$ coinvariant under the actions of permuting vertices and boundaries.

The differentials $\delta + \Delta_1$ act on a ribbon graph $\Gamma$ with $n$ vertices and $m$ boundaries by

$$\delta \Gamma = \sum_{i=1}^{n} \Gamma_{\delta i}, \quad \Delta_1 \Gamma = \sum_{i=1}^{n} \Gamma_{\Delta_1 i},$$

where

$$\Gamma_{\delta i} = \sum_{j=m+1}^{\infty} \Gamma_{i j}, \quad \Gamma_{\Delta_1 i} = \sum_{j=1}^{m} \Gamma_{i j}.$$

In words, the first part of the differential, $\delta$, splits vertices in a way that respects the cyclic ordering. The other part of the differential, $\Delta_1$, adds an edge between each pair of corners of each boundary.

5.4. The Map $F : \text{OGC}_1 \to \text{RGC}$. Consider the deformation complex

$$\text{Def}(\text{hoLieB}_{0,0} \to \text{LieB}_{0,0}) \cong \prod_{n,m \geq 1} (\text{LieB}_{0,0}(n,m))^\otimes x^2 \text{Gr}.$$ 

It is spanned by oriented graphs with all vertices 3-valent, with ingoing and outgoing hairs, without internal sources or targets, modulo the relations (72), (73), (74). In [13], S. Merkulov and T. Willwacher constructed a quasi-isomorphism

$$F_1 : \text{OGC}_1 \to \text{Def}(\text{hoLieB}_{0,0} \to \text{LieB}_{0,0})[1]$$

$$\Gamma \mapsto F_1(\Gamma).$$

The element $F_1(\Gamma)$ is obtained from a graph $\Gamma \in \text{OGC}_1$ by attaching an incoming leg to each source, an outgoing leg to each target, and setting it to 0 if it contains a vertex that is not 3-valent.

Next, the map $s : \text{LieB}_{0,0} \to \text{RGr}$ from Proposition 5.7 gives us a map of complexes

$$F_2 : \text{Def}(\text{hoLieB}_{0,0} \to \text{LieB}_{0,0})[1] \to (\text{RGC}[1], \delta + \Delta_1)$$

by $F_2(\Gamma) : \text{hoLieB}_{0,0} \to \text{LieB}_{0,0} := s \circ \Gamma : \text{hoLieB}_{0,0} \to \text{RGr}$. We now have our map

$$(76) \quad F := F_2 \circ F_1 : \text{OGC}_1 \to \text{Def}(\text{hoLieB}_{0,0} \to \text{LieB}_{0,0})[1] \to (\text{RGC}[1], \delta + \Delta_1).$$

We note that $F_2$ maps a graph in $\text{LieB}_{0,0}(n,m)^\otimes x^2 \text{Gr}$ to a sum of ribbon graphs with $n$ vertices and $m$ boundaries. We also have that $F_1$ maps a graph $\Gamma \in \text{OGC}_1$ with $n$ sources and $m$ targets to a graph in $\text{LieB}_{0,0}(n,m)^\otimes x^2 \text{Gr}$. 

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Hence $F$ maps a graph $\Gamma$ with $m$ target vertices to a sum of ribbon graphs with $m$ boundaries. We may take a filtration on $\text{RGC}$ by the number of boundaries, and a filtration on $\text{OGC}\_1$ by the number of target vertices. Then
\[
\text{gr}(\text{RGC}, \delta + \Delta_1) = (\text{RGC}, \delta),
\]
and
\[
\text{gr}(\text{OGC}\_1, \delta) = (\text{OGC}\_1, \delta_0).
\]
As $F$ maps a graph with $m$ target vertices to a sum of ribbon graphs with precisely $m$ boundaries, we get that
\[
\text{gr}F : \text{gr}(\text{OGC}\_1, \delta) \to \text{gr}(\text{RGC}[1], \delta + \Delta_1)
\]
is given by the same map of vector spaces $F : (\text{OGC}\_1, \delta_0) \to (\text{RGC}[1], \delta)$.

We have now established both maps mentioned in Corollary 1.7.

**Theorem 5.9** (Corollary 1.7). We have a zig-zag of morphisms
\[
(\text{HGC}_0, \delta) \leftrightarrow (\text{OGC}_1, \delta_0) \to (\text{RGC}[1], \delta),
\]
where the left map is a quasi-isomorphism, given explicitly in (54) and (55), and the right map is given in (76).

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