On the Validity of Physical Optics for Narrow-band Beam Scattering and Diffraction from the Open Cylindrical Surface

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Abstract—The exact formulas for the induced electric surface current (in the scattering phenomenon) and the equivalent electric surface current (in the diffraction phenomenon) on the open cylindrical surface due to an arbitrary narrow-band beam have been shown in their closed-form expressions within the context of the cylindrical harmonics, which gives information about the validity of the Physical Optics (PO) approximation. Both the Electric Field Integral Equation (EFIE) and the Magnetic Field Integral Equation (MFIE) are used to find the induced (equivalent) electric surface currents in the context of the cylindrical harmonics. The numerical example of the scattering and diffraction of the Hermite Gaussian beam from the open cylindrical surface is shown. The result is useful for the evaluation of the validity of the PO approximation in the cylinder-like surface.

I. Introduction

The Physical Optics (PO) approximation has been extensively used as the approximation of the exact solution in many applications [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], which include microwave imaging, reflector antenna design, and evaluation of Radar Cross Section (RCS) [18, 19, 20, 21]. It is helpful to have an analytical formula to predict the behavior of the PO approximation in order to use it effectively. In this article, the exact closed-form expressions will be shown for the induced (equivalent) electric surface currents on the open cylindrical surface, from which the information of the validity of the PO approximation is obtained for the cylinder-like surface. The scheme used to illustrate the problem is given in Fig. 1. The time dependence $e^{j\omega t}$ ($i = \sqrt{-1}$) has been assumed in this article.

II. The Cylindrical Harmonics

The cylindrical modal expansion of the vector potential $\mathbf{A}(\mathbf{r})$ for the electric surface current $\mathbf{J}_s(\mathbf{r'})$ on an arbitrary surface in the cylindrical coordinate is given as

$$\mathbf{A}(\mathbf{r}) = \mu \int_S \left[ g(\mathbf{r} - \mathbf{r'}) \mathbf{J}_s(\mathbf{r'}) \right] dS' = \frac{\mu}{i8\pi} \int_S \left[ \mathbf{J}_s(\mathbf{r'}) \int_{-\infty}^{\infty} H_0^{(2)}(\Lambda |\mathbf{r} - \mathbf{r'})| e^{-i\mathbf{k}(z-z')} d\mathbf{h}' \right] dS'$$ \hspace{1cm} (1)

where $\mu$ is the permeability of the homogeneous medium. $H_0^{(2)}(\cdot)$ is Hankel function of the second kind of order 0. The scalar Green’s function $g(\cdot \cdot)$ and the transverse wave vector $\Lambda$ are defined as

$$g(\cdot \cdot) = \frac{e^{-ik|\cdot\cdot|}}{4\pi|\cdot\cdot|}, \quad \Lambda = \sqrt{k^2 - h^2}. \hspace{1cm} (2)$$

According to the cylindrical addition theorem,

$$H_0^{(2)}(\Lambda |\mathbf{r} - \mathbf{r'})| = \sum_{m=\infty}^{\infty} \left\{ \begin{array}{ll}
H_m^{(2)}(\Lambda \rho)J_m(\Lambda \rho')e^{im(\phi' - \phi)} & |\rho| > |\rho'|

J_m(\Lambda \rho)H_m^{(2)}(\Lambda \rho')e^{im(\phi - \phi')}& |\rho| < |\rho'|
\end{array} \right\} \hspace{1cm} (3)$$

where $\rho \equiv |\mathbf{r}|$ is the observation coordinate and $\rho' \equiv |\mathbf{r'}|$ is the source coordinate. $J_m(\cdot \cdot)$ is Bessel function of the first kind of integer order $m$ and $H_m^{(2)}(\cdot \cdot)$ is Hankel function of the second kind of integer order $m$. Substituting (3) into (1), the cylindrical modal expansion of $\mathbf{A}(\mathbf{r})$ is obtained,

$$\mathbf{A}(\mathbf{r}) = \text{IFT} \left( \mathbf{g}_S(m, h) H_0^{(2)}(\Lambda \rho) \right) \hspace{1cm} (4)$$

$$\mathbf{g}_S(m, h) = \frac{\mu}{i4} \int_S \left[ \frac{J_m(\Lambda \rho')}{H_m^{(2)}(\Lambda \rho')} \mathbf{J}_s(\mathbf{r'}) e^{i(m\phi + hz')} \right] dS'$$

where $\mathbf{g}_S(m, h)$ is the equivalent source Green’s function.
where, the the superscript “+” denotes $\rho > \rho'$ and the subscript “−” denotes $\rho < \rho'$. The Inverse Fourier Transform (IFT) is defined as

$$\text{IFT} \left( \begin{array} \end{array} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \begin{array} \end{array} \right) e^{-i(m\phi+hz)} dh \right) \right).$$

The electromagnetic field $(E, H)$ is given as

$$E_z(r) = -i\omega A_z^s(r) + \frac{1}{i\omega \mu} \nabla \times [\nabla \times A_z^s(r)], \quad H_z(r) = \frac{1}{\mu} \nabla \times A_z^s(r) \right) \right)$$

III. Exact Formulas for Induced and Equivalent Electric Surface Currents

Due to the fact that $J_s^- = -J_s^+$ (see Fig. [1]), let’s consider the scattering phenomenon and express the incident electromagnetic field $(E', H')$ into the cylindrical harmonics,

$$E'(\rho) = \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[ a_m^h M_m^h(\rho) + b_m^h N_m^h(\rho) \right] dh \right\}$$

$$H'(\rho) = \frac{i}{\eta} \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[ a_m^h N_m^h(\rho) + b_m^h M_m^h(\rho) \right] dh \right\}$$

$$M_m^h(r) = \left[ \frac{\hat{r}}{\rho} \frac{H_m^{(2)}(\Lambda \rho)}{i\kappa} - \frac{m \hat{r}}{kp} H_m^{(2)}(\Lambda \rho) + \frac{2}{k} H_m^{(2)}(\Lambda \rho) \right] e^{-im\phi} e^{-ihz}$$

$$N_m^h(r) = \left[ \frac{\hat{r}}{\rho} \frac{H_m^{(2)}(\Lambda \rho)}{i\kappa} - \frac{m \hat{r}}{kp} H_m^{(2)}(\Lambda \rho) + \frac{2}{k} H_m^{(2)}(\Lambda \rho) \right] e^{-im\phi} e^{-ihz}$$

Similarly, express the scattered electromagnetic field $(E^h, H^h)$ as

$$E^h(\rho) = \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[ a_m^h M_m^h(\rho) + b_m^h N_m^h(\rho) \right] dh \right\}$$

$$H^h(\rho) = \frac{i}{\eta} \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[ c_m^h N_m^h(\rho) + d_m^h M_m^h(\rho) \right] dh \right\}$$

Now the induced electric surface current $J_s^-$ on the cylindrical surface $S$ is given as

$$J_s^-(\rho_0) = \hat{n}^- \times \left[ H^h(\rho_0) + H^h(\rho_0) \right] = \left[ H^h(\rho_0) + H^h(\rho_0) \right] \times \hat{r}_0 \right) \right)$$

$$= \frac{i}{\eta} \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[ (a_m^h + c_m^h) N_m^h(\rho_0) \times \hat{r}_0 + (b_m^h + d_m^h) M_m^h(\rho_0) \right] \hat{r}_0 \right) \right)$$
1. Electric Field Integral Equation (EFIE)

Let’s consider the TM mode ($N^h_m$ for $E$ and $M^h_m$ for $H$) here. From (9) and (13),

\[
J_z^{s,TM}(\rho_0) = \frac{i}{\eta} \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left( b^h_m + d^h_m \right) \Lambda \left( \frac{\partial H^{(2)}_m(\Lambda \rho)}{\partial \rho} \right) \bigg|_{\rho_0} \right. dh \right\}
\]  

(14)

Substituting (14) into (6), the $z$-component of the scattered electric field $E_{z,\phi}^{s,TM}$ on the cylindrical surface is obtained ($A_{z,\phi}^{T,M}$),

\[
E_{z,\phi}^{s,TM}(\rho_0) = -i \omega \left( \frac{\Lambda_0}{k} \right)^2 A_{z,\phi}^{T,M}(\rho_0)
\]  

(15)

\[
= \frac{\pi \rho_0}{2 \eta k} \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left( b^h_m + d^h_m \right) \Lambda^3 J_m(\Lambda \rho_0) H^{(2)}_m(\Lambda \rho_0) \frac{\partial H^{(2)}_m(\Lambda \rho)}{\partial \rho} \bigg|_{\rho_0} \right. \]  

(17)

Now apply the EFIE on the cylindrical surface $E_{z,\phi}^{s,TM}(\rho_0) = -E_{z,\phi}^{i,TM}(\rho_0)$, from (7) and (17),

\[
b^h_m + d^h_m = \frac{2}{\xi} b^h_m, \quad d^h_m = \left[ \frac{2}{\xi} - 1 \right] b^h_m, \quad \xi \equiv i \frac{\pi}{k \Lambda_0} H^{(1)}_m(\Lambda \rho_0) \frac{\partial H^{(2)}_m(\Lambda \rho)}{\partial \rho} \bigg|_{\rho_0}
\]  

(18)

Note that $\xi \rightarrow 1$ for $\rho_0 \rightarrow \infty$, which means that $d^h_m \rightarrow b^h_m$ and the PO approximation reduces to the exact induced electric surface current.

2. Magnetic Field Integral Equation (MFIE)

Let’s also take the TM mode ($N^h_m$ for $E$ and $M^h_m$ for $H$) as an example. From (9) and (13), the $\phi$-component of the scattered magnetic field $H_{\phi,\phi}^{s,TM}(\rho_0)$ on the front side of the cylindrical surface is found as

\[
H_{\phi,\phi}^{s,TM}(\rho_0) = \frac{i}{\eta} \sum_{m=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[ \frac{\xi^*}{2} \left( b^h_m + d^h_m \right) \right] M^h_m(\rho) \right. \]  

(19)

Now apply the MFIE on the cylindrical surface $H_{\phi,\phi}^{s,TM}(\rho_0) + H_{\phi,\phi}^{i,TM}(\rho_0) = -J_{s,\phi}^{z,TM}(\rho_0)$,

\[
b^h_m + d^h_m = \frac{2}{2 - \xi^*} b^h_m, \quad d^h_m = \frac{\xi^*}{2 - \xi^*} b^h_m
\]  

(20)

It is not difficult to show that (18) and (20) are equivalent by using the Wronskian relation,

\[
H^{(2)}_m(\Lambda \rho) \frac{\partial H^{(1)}_m(\Lambda \rho)}{\partial \rho} - H^{(1)}_m(\Lambda \rho) \frac{\partial H^{(2)}_m(\Lambda \rho)}{\partial \rho} = \frac{i4}{\pi \Lambda}
\]  

(21)

3. The Induced and Equivalent Electric Surface Currents

Following the similar procedure, the induced electric surface current for the TE mode ($M^h_m$ for $E$ and $N^h_m$ for $H$) is given as

\[
a^h_m + c^h_m = \frac{2}{\xi^*} a^h_m, \quad c^h_m = \left[ \frac{2}{\xi^*} - 1 \right] a^h_m
\]  

(22)
Figure 2: The PO approximation (dots) Vs. result (lines) from MoM: a) TEM$_{00}$; and b) TEM$_{10}$. Red is for the magnitude; blue is for the real part; and black is for the imaginary part. Results have been normalized.

Substituting (18) and (22) into (13), the total induced and equivalent electric surface currents are obtained,

$$J_s^{-} (\rho_0) = \frac{-i}{\eta} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ a_m^h \mathbf{N}_m^h (\rho_0) \times \hat{\rho}_0 + b_m^h \mathbf{M}_m^h (\rho_0) \times \hat{\rho}_0 \right] \, dh$$

(23)

From (23), it is clear that the exact induced and equivalent electric surface currents only deviate from the PO approximation by a factor of $\frac{1}{\xi}$ for TM mode and $\frac{1}{\xi^*}$ for TE mode.

IV. Numerical Confirmation: the Hermite Gaussian Beam

The incident Hermite Gaussian beam (TEM$_{00}$ and TEM$_{10}$) has been used to test the result given in (23). The TEM$_{mn}$ Hermite Gaussian beam is given as

$$E_{mn} = \hat{z} \sqrt{\frac{\eta}{\pi^{m+n+1} \left( w_y (x) w_z (x) \right)}} H_m \left( \sqrt{\frac{y}{w_y (x)}} \right) H_n \left( \sqrt{\frac{z}{w_z (x)}} \right)$$

$$e^{-\left[ \frac{\left( \frac{1}{w_y (x)} + \frac{1}{w_y (x)} \right)}{w_y (x)} \right] + \left[ \frac{\left( \frac{1}{w_z (x)} + \frac{1}{w_z (x)} \right)}{w_z (x)} \right] e^{-i \left[ k x - \left( m + \frac{1}{2} \right) \arctan \left( \frac{x}{y} \right) - \left( n + \frac{1}{2} \right) \arctan \left( \frac{x}{z} \right) \right]}}$$

(24)

where $H_{m,n}$ is the Hermite polynomial and the following quantities have been defined,

$$w_r (x) = w_{0r} \left[ 1 + \frac{x}{L_r} \right]^\frac{1}{2}, \quad R_r (x) = x + L_r^2 / x, \quad L_r = \frac{k w_{0r}^2}{2}, \quad \tau = y, z$$

(25)

In our numerical computation, both TEM$_{00}$ and TEM$_{10}$ Hermite Gaussian beams are $\hat{z}$-polarized (TM mode only in the cylindrical coordinate). The symmetrical waist radii have been set as $w_{0y} = w_{0z} = 1 \lambda$. The radius of the scattering (diffracting) cylindrical surface is $\rho_0 = 3 \lambda$ and the radius of the observation cylindrical surface is $\rho = 20 \lambda$.

The scattered electric field $E_{s}^z (\phi \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right])$ and the diffracted electric field $E_{d}^z (\phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right])$ calculated from the PO approximation have been plotted (dots) in Fig. 2 together with the result
Figure 3: Result (dots) obtained from Eqn. (23) Vs. result (lines) from MoM: a) TEM$_{00}$ and b) TEM$_{10}$. Red is for the magnitude; blue is for the real part; and black is for the imaginary part. Results have been normalized.

(lines) obtained from the Method of Moment (MoM). Also, the theoretical induced (equivalent) current given in (23) has been used to calculate the scattered electric field $E_s$ and the diffracted electric field $E_d$, which is shown in Fig. 3 (dots), with good agreement with the result from the MoM (lines). All plots are for results on the observation cylindrical surface with radius $\rho = 20\lambda$.

**Conclusion**

The exact formulas for the induced electric surface current in the scattering phenomenon and the equivalent electric surface current in the diffraction phenomenon have been derived, which gives helpful information of the PO approximation in the cylinder-like surface.

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