Inspired by recent lattice calculations, we model certain aspects of the $\theta$-vacuum using a matrix model with gaussian weights. The vacuum energy exhibits a cusp at $\theta < \pi$ that is sensitive to both the accuracy of the numerical analysis and the maximum density of winding modes present in a finite volume.

Recent lattice study of the $CP^3$ model on the lattice\cite{1} have revealed an intriguing result: the energy as a function of the CP breaking $\theta$ angle levels off at large $\theta$. The result is very interesting, since in that model the string tension for small charges is related to the $\theta$-derivative of the vacuum energy. A plateau indicates the loss of confinement. The result was reexamined in \cite{2} where the authors pointed out to some subtleties regarding the technical methods applied.

In this talk we will address these issues using a random matrix model with the partition function

$$Z(\theta, N_f) = \left\langle \prod_{j=1}^{N_f} \det \begin{pmatrix} \text{im}_\theta e^{i\theta/N_f} & W \\ W^\dagger & \text{im}_\theta e^{-i\theta/N_f} \end{pmatrix} \right\rangle.$$  \hspace{1cm} (1)

where the averaging is carried over the matrices $W$, $W^\dagger$ using the weight $\exp(-n/2TrW^\dagger W)$ and over the matrix size $n$ and $\sigma$, with weights $\exp(-\chi^2/2\chi^* V)$ and $\exp(-\sigma^2/2\sigma^* V)$, respectively. Here $W$ is a complex asymmetric $n_+ \times n_-$ matrix, $n = n_+ + n_-$, and $\sigma \pm \chi = 2n_\pm - \langle n \rangle$. The mean number of zero modes $\langle n \rangle$ is either fixed from the outside or evaluated using the gaussian measure. Here and for simplicity we use the quenched measure without the fermion determinant to fix $\langle n \rangle$. Throughout, we set the value of the quark condensate to $\Sigma = 1$ in the chiral limit.

Eq. (1) is borrowed from the effective instanton vacuum analysis\cite{3} where $n_+$ counts the number of right-handed zero modes, and $n_-$ the number of left-handed zero modes. The number of exact topological zero modes is commen-
surate with the net winding number carried by the instantons and antinstantons. By analogy with \( \chi_* \) and \( \sigma_* \) will refer to the unquenched topological susceptibility and particle compressibility, respectively, with \( \sigma_*^2 = 12n_* / 11N_c \) and \( n_* = \langle n \rangle / V \) the mean density of zero modes. If the compressibility \( \sigma_* \) is assumed small in units of \( \Sigma = 1 \), then typically \( n \sim \langle n \rangle \).

**quenched case:** In the case of the quenched partition function (\( N_f = 0 \)) one probes the nature of the gaussian measure. We start discussing the case where \( \langle n \rangle = \infty \) with no restriction on the value of \( n \), and hence no restriction on the value of \( \chi \). Then, we discuss the case where \( \langle n \rangle \) is large but finite, so that \( |\chi| \leq n \) with typically \( n \sim \langle n \rangle \) for a peaked distribution in \( n \).

When the sum is unrestricted and infinite, using Poisson resummation formula we have

\[
Z_Q(\theta) = \sum_{k=\infty}^{+\infty} e^{-\frac{1}{2}V\chi_*(\theta - 2\pi k)^2} = \theta_3(\theta/2, e^{-\tau}).
\]

(2)

with \( \theta \) being the third elliptic function and \( \tau = 1/(2V\chi_*) \). The result is manifestly \( 2\pi \) periodic. The vacuum energy, \( F_Q(\theta) = -\ln Z_Q(\theta)/V \) as \( V \to \infty \) is simply \( F_Q(\theta) = \min \chi_* (\theta + \text{mod } 2\pi)^2 \) in agreement with the saddle-point approximation. This result is in agreement with the result using large \( N_c \) arguments and recent duality arguments. We observe that the cusp at \( \theta = \pi \) (mod \( 2\pi \)) sets in for \( V = \infty \).

In the matrix model being considered the sum over \( \chi \) is restricted to \( |\chi| < N \), with \( N = \max n \). We denote by \( n = N/V \) the maximum density of winding modes. While in general \( n \neq n_* \), for a peaked distribution in \( n \) (small compressibility \( \sigma_* \)) we expect \( n \sim n_* \). This will be assumed throughout unless indicated otherwise. Hence

\[
Z_Q(\theta) = \sum_{\chi=-(N-1)}^{N-1} e^{i\theta \chi} e^{-\chi^2/2V\chi_*}
\]

(3)

Approximating the sum in (3) by an integral and evaluating it by saddle point we obtain \( Z_Q \sim e^{-V\chi_* \theta^2/2} \) apparently in agreement with the infinite sum. However, the Euler-MacLaurin summation formula shows that deviation from this result in the case of a finite sum is expected for \( \theta > \theta_c \sim n/\chi_* \).

Indeed, this is confirmed by detailed numerical calculations of the vacuum energy as shown in Fig 1 (left). For \( N = 250 \) and \( \chi_* = 1 \), the double precision (16 digit) numerics (circles) breaks away from the saddle point approximation (solid line) at \( \theta/\pi \sim 0.2 \) for both \( n = 1 \) and \( n = 4 \), while the high-precision (64 digits) calculations (dashed line) agree with the saddle point result at
\( n = 4 \) but break away at \( \theta/\pi \sim 0.3 \approx 1/\pi \chi^* \) at \( n = 1 \). On the right we show the numerical result for the \( CP^3 \) model showing a similar behavior.

\[ n = 4 \text{ but break away at } \theta/\pi \sim 0.3 \approx 1/\pi \chi^* \text{ at } n = 1. \]

Figure 1. Free energy in a quenched model (left) and in the \( CP^3 \) model (right). See text.

**unquenched case:** Analytical and numerical studies of the unquenched matrix model for \( N_f = 1 \) and \( N_f = 2 \) lead to results similar to the ones observed in the quenched case. The mean-field result breaks down for \( \theta > \theta_c \sim n/\chi_{\text{top}} \). For small quark masses \( \chi_{\text{top}} \sim m \), the breakdown is observed only for ensembles with very small maximal winding number.

In this talk we have shown that the vacuum energy of a chiral matrix model develops a cusp at \( \theta < \pi \) that is sensitive to the numerical accuracy. The cusp persists at high accuracy if the maximum winding number considered is low, in disagreement with mean-field results. Similar observations in current lattice simulations should therefore be interpreted with care.

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