Higher derivatives of length functions along earthquake deformations

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1 Introduction

Let $S$ be a closed surface of genus $g \geq 2$ and $T(S)$ the associated Teichmüller space of hyperbolic structures on $S$. Given $\gamma \in \pi_1(S)$, let $L_\gamma : T(S) \to \mathbb{R}$ be the associated length function and $T_\gamma : T(S) \to \mathbb{R}$ the associated trace function. The functions $L_\gamma, T_\gamma$ have a simple relation given by

$$T_\gamma = 2 \cosh(L_\gamma/2).$$

(1)

Let $\beta$ be the homotopy class of a simple multicurve (i.e. a union of disjoint simple non-trivial closed curves in $S$) and $t_\beta$ the vector field on $T(S)$ associated with left twist along the geodesic representative of $\beta$ (see [4]). In this paper, we describe a formula to calculate the higher order derivatives of the functions $L_\gamma, T_\gamma$ along $t_\beta$. In particular we will find a formula for

$$t_\beta L_\gamma = t_\beta t_\beta \ldots t_\beta L_\gamma.$$

The formulae we derive generalize formulae for the first two derivatives due to Kerchoff (1st derivative, see [4]) and Wolpert (1st and 2nd derivatives, see [5, 6]).

Kerckhoff and Wolpert both showed that the first derivative is given by

$$t_\beta L_\gamma = \sum_{p \in \beta' \cap \gamma'} \cos \theta_p,$$

(2)

where $\beta', \gamma'$ are the geodesic representatives of $\beta, \gamma$ respectively and $\theta_p$ is the angle of intersection at $p \in \beta' \cap \gamma'$. Kerckhoff further generalized the formula for the case when $\beta, \gamma$ are measured laminations (see [4]).

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In [6], Wolpert derived this formula for second derivative

\[ t_{\alpha}t_{\beta}L_{\gamma} = \sum_{(p,q) \in \beta' \cap \gamma'} e^{l_{pq}} + e^{l_{qp}} \sin \theta_p \sin \theta_q + \sum_{(r,s) \in \beta' \cap \gamma'} e^{m_{rs}} + e^{m_{sr}} \frac{2}{2(e^{L_{\gamma}} - 1)} \sin \theta_r \sin \theta_s. \]

where \( l_{xy} \) is the length along \( \gamma \) between \( x, y \) (similarly \( m_{xy} \) is the length along \( \beta \)).

It follows from Wolpert’s formula that

\[ t_{\beta}^2 L_{\gamma} = t_{\beta} t_{\beta} L_{\gamma} = \sum_{p,q \in \beta' \cap \gamma'} e^{l_{pq}} + e^{l_{qp}} \frac{2}{2(e^{L_{\gamma}} - 1)} \sin \theta_p \sin \theta_q. \] (3)

Our formula generalizes equations 2, and 3 to higher derivatives. Our approach is to derive a formula for the higher derivatives of \( T_{\gamma} \) and then use the functional relation in equation 1 to derive the formula for \( L_{\gamma} \).

2 Higher Derivative Formula

We take the geodesic representatives of \( \beta \) and \( \gamma \). We let the geometric intersection number satisfy \( i(\beta, \gamma) = n \) and we order the points of intersection \( x_1, \ldots, x_n \) by choosing a base point on \( \gamma \). We let \( \theta_i \) be the angle of intersection of \( \beta, \gamma \) at \( x_i \) and \( l_i \) be the length along \( \gamma \) from \( x_1 \) to \( x_i \). This gives us \( n \)-tuples \((l_1, \ldots, l_n)\) and \((\theta_1, \ldots, \theta_n)\).

In order to describe the formula for the higher derivatives, we first introduce some more notation.

Given \( r \), we let \( P(r) \) be the set of subsets of the set \( \{1, \ldots, r\} \). Then for \( I \in P(r) \) will be denoted by \( I = (i_1, \ldots, i_k) \) where \( 1 \leq i_1 < i_2 < \ldots < i_k \leq r \). We then define \( \hat{I} \) to be the complementary subset. We also let \( |I| \) be the cardinality of \( I \).

We define the alternating length \( L_I \) for \( I = (i_i, \ldots, i_k) \) by

\[ L_I = \sum_{j=1}^{k} (-1)^j l_{i_j} = -l_{i_1} + l_{i_2} - l_{i_3} - \ldots + (-1)^k l_{i_k}. \]

We further define a signature for \( I \in P(r) \). For \( I = (i_1, \ldots, i_k) \) we can consider the integers in \( \{1, \ldots, r\} \) in the ordered blocks \([1, i_1], [i_1, i_2], \ldots, [i_k, r] \). We take the sum of the cardinality of the even ordered blocks. Then

\[ s(I) = (i_2 - i_1 + 1) + (i_4 - i_3 + 1) + \ldots (i_k - i_{k-1} + 1) \quad k \text{ even} \]

\[ s(I) = (i_2 - i_1 + 1) + (i_4 - i_3 + 1) + \ldots (r - i_k + 1) \quad k \text{ odd} \]

For \( (\theta_1, \ldots, \theta_n) \) we also define

\[ \cos(\theta_I) = \prod_{j=1}^{k} \cos(\theta_{i_j}) = \cos(\theta_{i_1}) \cos(\theta_{i_2}) \ldots \cos(\theta_{i_k}) \]
and similarly define \( f(\theta_I) \) for \( f \) a trigonometric functions.

We let \( u_j = l_j + i\theta_j \). The function \( F_r \) is given by

\[
F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} \sin(\theta_I) \cos(\theta_I) \left( e^{L/2 - L_I} + (-1)^r e^{L_I - L/2} \right)
\]

or equivalently

\[
F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_I) \cosh(L/2 - L_I)
\]

for \( r \) even and

\[
F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_I) \sinh(L/2 - L_I)
\]

for \( r \) odd.

We let \( C(n, r) \) be the set of the subsets of size \( r \) of the set \( \{1, 2, \ldots, n\} \). It is given by

\[
C(n, r) = \{ I = (i_1, i_2, \ldots, i_r) \mid 1 \leq i_1 < i_2 < \ldots < i_r \leq n \}
\]

Given \( m \in \mathbb{N} \), we let \([m]\) be the parity of \( m \), i.e. \([m] = 0\) if \( m \) is even, and \([m] = 1\) if \( m \) is odd.

**Theorem 1** Let \( \beta \) be a homotopy class of a simple closed multicurve and \( \gamma \) a homotopy class of non-trivial closed curve. Let the geometric intersection number \( i(\beta, \gamma) = n \). Then

\[
t_k^\beta T_\gamma = \frac{1}{2k} \sum_{[r]=[k]} B_{n,k,r} \sum_{I \in C(n,r)} F_r(u_{i_1}, \ldots, u_{i_r}, L_\gamma)
\]

where \( B_{n,k,r} \) are constants described below.

The first two equations correspond to formulae 2 and 3 for the derivatives of length. We use the above, to derive the next case as an example.

**Third Derivative:** We use the above formula to calculate the formula for the third derivative.

\[
t_3^\beta T_\gamma = \frac{1}{8} ((6n - 4) \sinh(L_\gamma/2) \sum_{i=1}^n \cos(\theta_i) + 12 \left( \sum_{i<j<k} \sinh(L_\gamma/2) \cos(\theta_i) \cos(\theta_j) \cos(\theta_k) \right.
\]

\[
+ \sinh(L_\gamma/2 - l_{ij}) \sin(\theta_i) \sin(\theta_j) \cos(\theta_k) - \sinh(L_\gamma/2 - l_{jk}) \sin(\theta_j) \cos(\theta_i) \sin(\theta_k)
\]

\[
+ \sinh(L_\gamma/2 - l_{jk}) \cos(\theta_i) \sin(\theta_j) \sin(\theta_k))
\]
2.1 Constants $B_{n,k,r}$

We let $P(k,n)$ be the collection of partitions of $k$ into $n$ ordered nonnegative integers, i.e.

$$P(k,n) = \left\{ p = (p_1, p_2, \ldots, p_n) \in \mathbb{N}_0^n \mid \sum_{i=1}^{n} p_i = k \right\}$$

For $p \in P(k,n)$, we define $[p] = ([p_1], \ldots, [p_n])$ where $[n]$ is the parity of $n$. We let $|p| = |p_1| + \ldots + |p_n|$. Then $[p]$ is an n-tuple of 0's and 1's with exactly $|p|$ 1's.

Given $p \in P(k,n)$ we define $B(p)$ as a sum of multinomials given by

$$B(p) = \sum_{q \in P(k,n), |q| = |p|} \binom{k}{q}.$$

It is easy to see that $B(p)$ only depends on $n, k$ and $r = |p|$. We therefore define

$$B_{n,k,r} = B(p) \quad \text{for some } p \text{ with } |p| = r$$

In particular if we let $p_r = (1, 1, \ldots, 1, 0, \ldots, 0) \in P(k,n)$, of $r$ 1's followed by $(n-r)$ 0's, we have

$$B_{n,k,r} = \sum_{p \in P(k,n), |p| = |p_r|} \binom{k}{p}.$$

A simple calculation gives

$$B_{n,k,k} = \binom{k}{p_k} = \binom{k}{1,1,1,\ldots,0,0,\ldots,0} = k!$$

3 Twist Deformation

We consider $T(S)$ as the fuchsian locus of the associated quasifuchsian space $QF(S)$. Let $X \in T(S)$ and $X = \mathbb{H}^2/\Gamma$ where $\Gamma$ is a subgroup of $PSL(2, \mathbb{C})$ acting on upper half space $\mathbb{H}^3 = \{(u, v, w) \in \mathbb{R}^3 \mid w > 0\}$ fixing the hyperbolic plane $\mathbb{H}^2 = \{(u, 0, w) \mid w > 0\}$. Let $\Gamma_z$ be the subgroup of $PSL(2, \mathbb{C})$ obtained by complex shear-bend along $\beta$ by amount $z = s + it$, i.e. left shear by amount $s$ followed by bend of $t$. Then for small $z$, $X_z = \mathbb{H}^3/\Gamma_z \in QF(S)$. In the terminology of Epstein-Marden this is a quake-bend deformation. See II.3 of [3] for details on quake bend deformations and II.3.9 for a detailed discussion of derivatives of length along quakebend deformations.

Let $\gamma \in \Gamma$ be a hyperbolic element and let $\gamma(z) \in \Gamma_z$ be the element of the deformed group corresponding to $\gamma$ and $L(z)$ the complex translation length of $\gamma(z)$. To see how $\gamma$ is deformed, by conjugating, we assume that $\gamma$ has axis the geodesic $g$ with endpoints $0, \infty \in \mathbb{C}$ and is given by

$$\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

with $\lambda = e^{L/2}$ where $L > 0$ is the translation length of $\gamma$. 


We consider the lifts of $\beta$ which intersect the axis $g$ of $\gamma$ and normalize to have a lift of $\beta$ labelled $\beta_1$ which intersects axis $g$ at height 1. We enumerate all other lifts by the order of the height of their intersection point with $g$ starting with the intersection point of $\beta_1$. Let $n$ be such that $\gamma \beta_1 = \beta_{n+1}$. Let $R_i(z)$ be the Möbius transformation corresponding to a complex bend about $\beta_i$ of $z$. Then under the complex bend about $\beta$, $\gamma(z)$ given by

$$\gamma(z) = R_1(z)R_2(z)\ldots R_n(z)\gamma.$$  

A similar description of the deformation of an element in the punctured surface case can be given in terms of shearing coordinates (see [2] for details).

Taking traces we have

$$T(z) = Tr(R_1(z)R_2(z)\ldots R_n(z)\gamma) = 2\cosh(L(z)/2).$$

We can find the derivatives of $L(z)$ by differentiating this formula repeatedly. The final formula is obtained by applying symmetry relations on the derivatives and some elementary combinatorics.

We note that both $T(z)$ and $L(z)$ are holomorphic in $z$. Differentiating in the real direction we have

$$t^k_\beta L = \frac{d^kL}{dz^k}(0) = L^{(k)}(0)$$

Also if we let $b_\beta$ be the vector field on $T(S)$ given by pure bending along $\beta$, then we have by analyticity of $L(z)$

$$b^k_\beta L = i^kL^{(k)}(0) = (it_\beta)^k L$$

This corresponds to the observation that $b_\beta = J.t_\beta$ where $J$ is the complex structure on $QF(S)$ (see [1]).

Figure 1: Lift of $\gamma$
3.1 Derivation of First Two Derivatives

We now calculate the first two derivatives and recover Wolpert’s formulae. By the product rule we have

\[
T'(0) = \sum_{i=1}^{n} Tr(R'_i(0) \gamma) \quad T''(0) = \sum_{i=1}^{n} Tr(R''_i(0) \gamma) + 2 \sum_{i,j=1}^{n} Tr(R'_i(0)R'_j(0) \gamma) \quad (4)
\]

We now describe \( R_i(z) \). Let \( \beta_i \) have endpoints \( a_i, b_i \in \mathbb{R} \) where \( a_i > 0 \) and \( b_i < 0 \). We let \( \lambda_i \) be the height at which \( \beta_i \) intersects \( g \). We orient \( \beta_i \) from \( a_i \) to \( b_i \) and \( g \) from 0 to \( \infty \) and let \( \theta_i \) be the angle \( \beta_i \) makes with side \( g \) with respect to these orientations (see figure 1).

Then

\[ \lambda_i = \sqrt{-a_i b_i} \quad \cos \theta_i = -\frac{a_i + b_i}{a_i - b_i} \quad \sin \theta_i = \frac{2\sqrt{-a_i b_i}}{a_i - b_i}. \]

As \( \beta_i \) intersects at height 1, the distance \( l_i \) between the intersection points of \( \beta_1 \) and \( \beta_i \) is given by \( e^{l_i} = \lambda_i \). Then we let \( f_i \in SL(2, \mathbb{R}) \) acting on the upper-half space by \( f_i(z) = (z - a_i)/(z - b_i) \) and let \( S(z) = f_iR_i(z)f_i^{-1} \). Then \( S_i(z) \) is the complex translation given by

\[ S(z) = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}. \]

Thus as \( R_i(z) = f_i^{-1}S(z)f_i \). Taking derivatives we have \( R'_i(0) = f_i^{-1}S'(0)f_i \) and

\[ R'_i(0) = \frac{1}{a_i - b_i} \begin{pmatrix} -b_i & a_i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & -a_i \\ 1 & -b_i \end{pmatrix} = \frac{1}{2(a_i - b_i)} \begin{pmatrix} -(a_i + b_i) & 2a_ib_i \\ -2 & a_i + b_i \end{pmatrix} \]

Therefore

\[ R'_i(0) = \frac{1}{2} \begin{pmatrix} \cos \theta_i & -e^{l_i} \sin \theta_i \\ -e^{-l_i} \sin \theta_i & -\cos \theta_i \end{pmatrix} \]

Also as \( S''(0) = \frac{1}{4}I \) we have \( R''_i(0) = \frac{1}{4}I \). Using this we have that

\[
Tc(R'_i(0) \gamma) = Tr \left( \frac{1}{2} \begin{pmatrix} \cos \theta_i & -e^{l_i} \sin \theta_i \\ -e^{-l_i} \sin \theta_i & \cos \theta_i \end{pmatrix} \begin{pmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{pmatrix} \right) = \sinh(L/2) \cos \theta_i
\]

\[
Tc(R'_i(0)R'_j(0) \gamma) = Tr \left( \frac{1}{4} \begin{pmatrix} \cos \theta_j & -e^{l_j} \sin \theta_j \\ -e^{-l_j} \sin \theta_j & \cos \theta_j \end{pmatrix} \begin{pmatrix} \cos \theta_j & -e^{l_j} \sin \theta_j \\ -e^{-l_j} \sin \theta_j & \cos \theta_j \end{pmatrix} \begin{pmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{pmatrix} \right)
= \frac{1}{4} \left( \cos \theta_i \cos \theta_j (e^{L/2} + e^{-L/2}) + \sin \theta_i \sin \theta_j (e^{L/2+l_i} - e^{-(L/2+l_i)}) \right) \quad (5)
\]

Let \( l_{ij} \) be the distance along \( \gamma \) from \( \beta_i \) to \( \beta_j \) with respect to the orientation of \( \gamma \). Then for \( i < j \) we have \( l_{ij} = l_j - l_i \), and \( l_{ji} = L - l_{ij} \) for \( i > j \).

\[
Tc(R'_i(0)R'_j(0) \gamma) = \frac{1}{2} (\cos \theta_i \cos \theta_j \cosh(L/2) + \sin \theta_i \sin \theta_j \cosh(L/2 - l_{ij})). \quad (6)
\]

Combining these we obtain the first two derivatives of \( T_\gamma \).

\[ T'(0) = \sinh(L/2) \sum_{i=1}^{n} \cos \theta_i \]

6
\[ T''(0) = \sum_{i,j=1 \atop i < j}^{n} (\cos \theta_i \cos \theta_j \cosh(L/2) + \sin \theta_i \sin \theta_j \cosh(L/2 - l_{ij})) + n \frac{\cosh(L/2)}{2} \]

As \( T(z) = 2 \cosh(L(z)/2) \), then \( T'(0) = \sinh(L/2) L'(0) \) giving
\[ L'(0) = \sum_{i=1}^{n} \cos \theta_i \]

Also \( T''(0) = \frac{1}{2} \cosh(L/2)(L'(0))^2 + \sinh(L/2) L''(0) \). Therefore
\[ T''(0) = \frac{\cosh(L/2)}{2} \left( n + 2 \sum_{i,j=1 \atop i < j}^{n} \cos \theta_i \cos \theta_j \right) + \sum_{i,j=1 \atop i < j} \sin \theta_i \sin \theta_j \cosh(L/2 - l_{ij}). \]

We have
\[ n + 2 \sum_{i,j=1 \atop i \neq j}^{n} \cos \theta_i \cos \theta_j = \left( \sum_{i=1}^{n} \cos \theta_i \right)^2 + \sum_{i=1}^{n} \sin^2 \theta_i \]

and
\[ T''(0) = \frac{\cosh(L/2)}{2} \left( (\sum_{i=1}^{n} \cos \theta_i)^2 + \sum_{i=1}^{n} \sin^2 \theta_i \right) + \sum_{i,j=1 \atop i < j} \sin \theta_i \sin \theta_j \cosh(L/2 - l_{ij}). \] (7)

Solving for \( L''(0) \) we obtain
\[ L''(0) = \sum_{i=1}^{n} \frac{\sin^2 \theta_i}{2 \tanh(L/2)} + \sum_{i,j=1 \atop i < j} \frac{\sin \theta_i \sin \theta_j \cosh(L/2 - l_{ij})}{\sinh(L/2)}. \]

As \( l_{ii} = 0 \) we can write
\[ L''(0) = \sum_{i,j=1}^{n} \frac{e^{l_{ij}} - L/2 + e^{L/2 - l_{ij}}}{2(e^{L/2} - e^{-L/2})} \sin \theta_i \sin \theta_j = \sum_{i,j=1}^{n} \frac{e^{l_{ij}} + e^{l_{ji}}}{2(e^L - 1)} \sin \theta_i \sin \theta_j. \]

The above give the formulae 2 and 3 as described.

### 4 Higher Derivatives

We now derive the formula for higher derivatives. We have the formula
\[ T(z) = \text{Tr}(R_1(z)R_2(z) \ldots R_n(z) \gamma). \]
We let \( P(k, n) \) be the collection of partitions of \( k \) into \( n \) ordered nonnegative integers, i.e.

\[
P(k, n) = \left\{ p = (p_1, p_2, \ldots, p_n) \in \mathbb{N}_0^n \mid \sum_{i=1}^n p_i = k \right\}
\]

Then by the product rule, the \( k \)th derivative of \( T \) at zero is,

\[
T^{(k)}(0) = \sum_{p \in P(k, n)} \binom{k}{p} Tr(R_1^{(p_1)}(0)\ldots R_n^{(p_n)}(0)\gamma)
\]

We have from above that \( R_i(z) = f_i^{-1}S(z)f_i \) where

\[
S(z) = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}.
\]

As \( S^{(2)}(z) = \frac{1}{4}S(z) \), we have for \( m \) even

\[
R_i^{(m)}(0) = \frac{1}{2m}I
\]

and for \( m \) odd we have

\[
R_i^{(m)}(0) = \frac{1}{2m-1}R_i'(0) = \frac{1}{2m} \begin{pmatrix} \cos \theta_i & -e^i \sin \theta_i \\ -e^{-i} \sin \theta_i & -\cos \theta_i \end{pmatrix}.
\]

Let \( z = x + iy \) and define

\[
A(z) = \begin{pmatrix} \cos y & -e^x \sin y \\ -e^{-x} \sin y & -\cos y \end{pmatrix}.
\]

We let \( u_j = l_j + i\theta_j \). Then

\[
R_j^{(p)}(0) = \begin{cases} \frac{1}{2^p} A(u_j) & p \text{ odd} \\ \frac{1}{2^p} I & p \text{ even} \end{cases}
\]

Therefore

\[
T^{(k)}(0) = \frac{1}{2^k} \sum_{p \in P(k, n)} \binom{k}{p} Tr(A(u_1)^{[p_1]}\ldots A(u_n)^{[p_n]}\gamma)
\]

where \([m]\) is the parity of \( m \). We define

\[
F_r(z_1, \ldots, z_r, L) = Tr(A(z_1)\ldots A(z_r)\gamma)
\]

Therefore gathering terms we have

\[
T^{(k)}(0) = \frac{1}{2^k} \sum_{r=0}^k B_{n,k,r} \sum_{1 \leq i_1 < \ldots < i_r \leq n} F_r(u_{i_1}, \ldots, u_{i_r}, L)
\]

where \( B_{n,k,r} \) are the coefficients described above. We note that we only get non-zero terms for \([r] = [k]\), so we have \( B_{n,k,r} = 0 \) for \([k] \neq [r]\).
We define the function
\[ G_r(u_1, \ldots, u_n, L) = \sum_{I \in C(n,r)} F_r(u_i, \ldots, u_i, L). \]

Then \( G_r \) is symmetric in \((u_1, \ldots, u_n)\) and we have
\[ t^k \beta^\gamma = \frac{1}{2k} \sum_{r=0}^{k} B_{n,k,r} G_r(u_1, \ldots, u_n, L_\gamma) \]

### 4.1 Function \( F_r \)

We now calculate the formula for \( F_r \).

**Lemma 1** The function \( F_r \) is given by
\[ F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} \sin(\theta_I) \cos(\theta_I) \left( e^{L/2 - L_I} + (-1)^r e^{L_I - L/2} \right). \]

or equivalently
\[ F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_I) \cosh(L/2 - L_I) \]
for \( r \) even and
\[ F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_I) \sinh(L/2 - L_I) \]
for \( r \) odd.

**Proof:** We have
\[ A(u) = \begin{pmatrix} \cos \theta & -e^l \sin \theta \\ -e^{-l} \sin \theta & \cos \theta \end{pmatrix}. \]
Therefore \( F_r(u_1, \ldots, u_r, L) = Tr(A(u_1), \ldots, A(u_r) \gamma) \) has the form
\[ F_r(u_1, \ldots, u_r, L) = \sum_{I \in P(r)} a_I \sin(\theta_I) \cos(\theta_I) \]
for some coefficients \( a_I \). Expanding we have
\[ F_r(u_1, \ldots, u_r, L) = (A(u_1) \ldots A(u_r) \gamma)_1^1 + (A(u_1) \ldots A(u_r) \gamma)_2^2 = e^{L/2}(A(u_1) \ldots A(u_r))_1^1 + e^{-L/2}(A(u_1) \ldots A(u_r))_2^2. \]

Similarly we have
\[ (A(u_1) \ldots A(u_r))_j^i = \sum_{I \in P(r)} a_j^i(I) \sin(\theta_I) \cos(\theta_I) \]
and define
\[(A(u_1) \ldots A(u_r))^j_j(I) = a^j_j(I) \sin(\theta_j) \cos(\theta_j).\]

We prove the lemma by induction. Given \(I = (i_1, \ldots, i_k) \in P(r)\) then \(I_j = (i_1, i_2, \ldots, i_{j-1}) \in P(i_j)\).

The matrix \(A(u)\) has cos terms on the diagonal and sin off diagonal. As \(\sin(\theta_{ik})\) is the last sin term we have in \((A(u_1), \ldots, A(u_r))^j_j(I)\) we have
\[
(A(u_1) \ldots A(u_r))^j_j(I) = (A(u_1) \ldots A(u_{i_k-1}))^j_2(I_k)(A(u_{i_k})^j_2A(u_{i_k+1})^j_1 \ldots A(u_r)^j_1
\]
\[= \cos(\theta_{i_k+1}) \ldots \cos(\theta_r) \left(-e^{-l_{ik}} \sin(\theta_{ik})\right) (A(u_1) \ldots A(u_{i_k-1}))^j_2(I_k),\]

Now iterating, as the next sin is \(\sin(\theta_{ik})\) we have
\[
(A(u_1) \ldots A(u_{i_k-1}))^j_2(I_k) = (A(u_1) \ldots A(u_{i_k-2}))^j_1(I_{k-1})A^j_1(u_{i_k-1})A^j_2(u_{i_k-1}+1)A^j_2(u_{i_k-1}+2) \ldots A^j_2(u_{i_k-1}).
\]

Thus we have
\[
(A(u_1), \ldots, A(u_r))^j_j(I) = (-1)^{i_k-i_k-1+e^{l_{ik-1}}-l_{ik}} \sin(\theta_{i_k-1}) \ldots \cos(\theta_{i_k}) \sin(\theta_{i_k}) \ldots \cos(\theta_{i_k+1}) \ldots \cos(\theta_r).
\]

As each off-diagonal term switches the index, there must be an even number of off-diagonal terms in the trace and therefore \(|I|\) is even. Then by induction
\[
(A(u_1), \ldots, A(u_r)\gamma)^j_j(I) = (-1)^s(I) \sin(\theta_I) \cos(\theta_I)e^{L/2-L_I}
\]
where
\[
s(I) = (i_2 - i_1 + 1) + (i_4 - i_3 + 1) + \ldots (i_k - i_{k-1} + 1)
\]
and
\[
L_I = \sum_{j=1}^k (-1)^j l_{ij} = -l_{i_1} + l_{i_2} - l_{i_3} - \ldots + (-1)^k l_{i_k}
\]

Similarly
\[
\frac{(A(u_1), \ldots, A(u_r))^j_j(I)}{(A(u_1) \ldots A(u_{i_k-1}))^j_2(I_{k-1})} = (-e^{-l_{ik-1}} \sin(\theta_{ik-1})) \ldots (\cos(\theta_{ik-1})) \ldots (\cos(\theta_{r}))(\cos(\theta_{ik}))(\cos(\theta_{ik+1})) \ldots \cos(\theta_r).
\]

Counting negative signs we have \(r - s(I) + |I|\) negative signs.
\[
(A(u_1), \ldots, A(u_r)\gamma)^j_j(I) = (-1)^{r - s(I) + |I|} \sin(\theta_I) \cos(\theta_I)e^{L_I - L/2}
\]
As \(|I|\) is even we get
\[
(A(u_1), \ldots, A(u_r)\gamma)^j_j(I) = (-1)^{r + s(I)} \sin(\theta_I) \cos(\theta_I)e^{L_I - L/2}
\]
giving the result. □
5 Some examples

We have from the calculations in the last section that

\[ F_0(L) = 2 \cosh(L/2) \quad F_1(u, L) = 2 \sinh(L/2) \cos \theta \]

\[ F_2(u_1, u_2, L) = 2(\cos \theta_1 \cos \theta_2 \cosh(L/2) + \sin \theta_1 \sin \theta_2 \cosh(L/2 - l_{12})) \]

Calculating \( F_3 \) we have

\[ F_3(u_1, u_2, u_3, L) = 2 \sinh(L/2) \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + 2 \sinh(L/2 - l_{12}) \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \]

\[ -2 \sinh(L/2 - l_{13}) \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) + 2 \sinh(L/2 - l_{23}) \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \]

Therefore we have

\[ G_0(L) = 2 \cosh(L/2) \quad G_1(u_1, \ldots, u_n, L) = 2 \sinh(L/2) \sum_{i=1}^{n} \cos \theta_i \]

\[ G_2(u_1, \ldots, u_n, L) = 2 \sum_{i<j}^{n} (\cos \theta_i \cos \theta_j \cosh(L/2) + \sin \theta_i \sin \theta_j \cosh(L/2 - l_{ij})) \]

\[ G_3(u_1, \ldots, u_n, L) = 2 \sum_{i<j<k}^{n} (\sinh(L/2) \cos(\theta_i) \cos(\theta_j) \cos(\theta_k) + \sinh(L/2 - l_{ij}) \sin(\theta_i) \sin(\theta_j) \cos(\theta_k) \]

\[ - \sinh(L/2 - l_{ik}) \sin(\theta_i) \cos(\theta_j) \sin(\theta_k) + \sinh(L/2 - l_{jk}) \cos(\theta_i) \sin(\theta_j) \sin(\theta_k)) \]

As the functions \( G_r \) do not depend on \( k \), once we’ve calculated all derivatives less than \( k \), we only need calculate \( G_k \) to find the \( k \)th derivative.

For \( k = 3 \) we have

\[ t_{3}^{3}T_{\gamma} = \frac{1}{8} (B_{n,3,1}G_1(u_1, \ldots, u_n, L_\gamma) + B_{n,3,3}G_3(u_1, \ldots, u_n, L_\gamma)) \]

\[ B_{n,3,3} = 3! = 6 \quad B_{n,3,1} = (n - 1) \left( \begin{array}{c} 3 \\ 1, 2 \end{array} \right) + \left( \begin{array}{c} 3 \\ 3 \end{array} \right) = 3(n - 1) + 1 = 3n - 2 \]

\[ t_{3}^{3}T_{\gamma} = \frac{1}{8} ( (6n - 4) \sinh(L_\gamma/2) \sum_{i=1}^{n} \cos(\theta_i) + 12 \left( \sum_{i<j<k}^{n} \sinh(L_\gamma/2) \cos(\theta_i) \cos(\theta_j) \cos(\theta_k) \]

\[ + \sinh(L_\gamma/2 - l_{ij}) \sin(\theta_i) \cos(\theta_j) \cos(\theta_k) - \sinh(L_\gamma/2 - l_{ik}) \sin(\theta_i) \cos(\theta_j) \sin(\theta_k) \]

\[ + \sinh(L_\gamma/2 - l_{jk}) \cos(\theta_i) \sin(\theta_j) \sin(\theta_k) ) \]
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